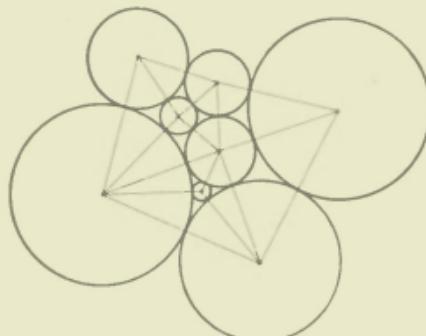
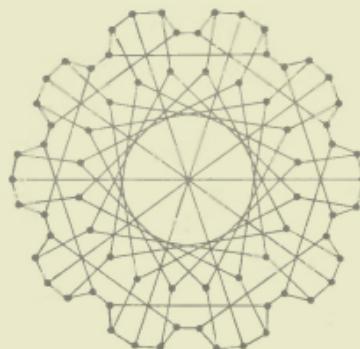


PEARLS *in* GRAPH THEORY

*A
Comprehensive
Introduction*



Nora Hartsfield
Gerhard Ringel



PEARLS *in* GRAPH THEORY

A Comprehensive Introduction

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Foreword

What is a pearl? It could be a graph, theorem, proof, conjecture, or exercise that provokes thought, causes surprise, stimulates interest, or inspires further research. We have collected as many pearls as possible in this book with the hope of maximizing the pleasure for both the teacher and the student. Consequently, students will find themselves receptive to new and challenging mathematical concepts. Historically, many areas of graph theory have developed from recreational mathematics. This entertainment factor can be used as a stimulant to the student and provide him or her with a new view of mathematics.

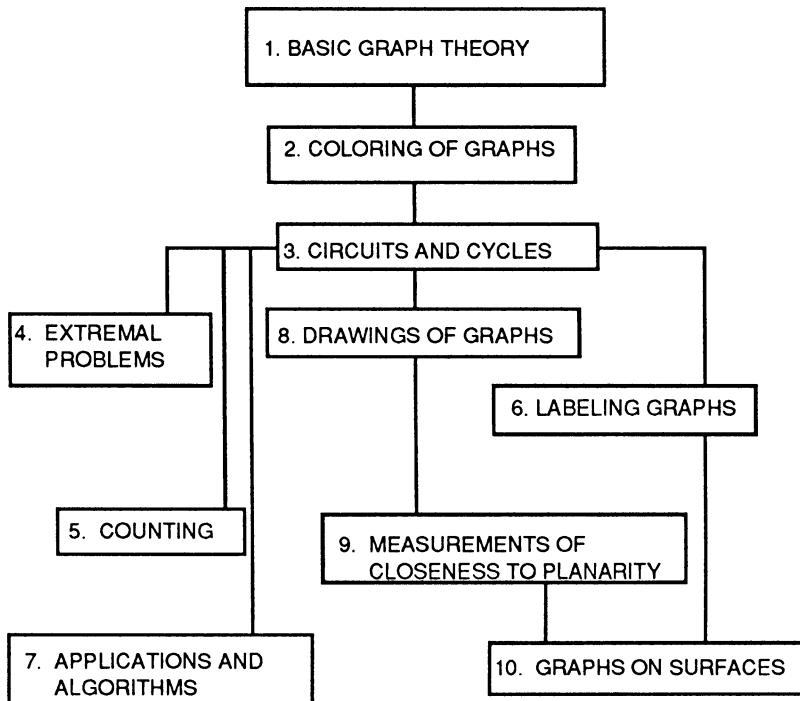
Combinatorics and graph theory are fast growing fields. Increasing numbers of courses are being taught at the undergraduate and graduate levels, and the wide variety of applications has increased the popularity of graph theory in other fields, including the natural sciences, engineering, genetics, and even in such social sciences as psychology and sociology. Mathematicians also find that employing graphs in their work facilitates the understanding and solving of problems encountered in their research.

Gerhard Ringel has taught courses in combinatorics at the University of California at Santa Cruz since 1970. At first his students consisted only of mathematics majors but, as the years passed, more and more computer science students joined his courses. The study of graph theory is now becoming essential for computer science majors. It is of fundamental importance to the understanding of abstract data structures and of the complexity and analysis of algorithms.

Prerequisites for this book are a strong interest in the subject and a good foundation in high school mathematics. A course in discrete mathematics is recommended but not necessary.

The exercises following each section have been designed to implement the concepts of the section and, we hope, to motivate further study. The starred exercises are considered particularly challenging.

The following chart explains the relationship among the chapters of the book. Chapters 1, 2, and 3 are considered basic to the understanding of the remainder of the text. For a more mathematically intensive course, we suggest that Chapters 6, 8, 9, and 10 be studied. If there are computer science majors in the class, Chapter 7 would be of prime importance. If time is available, Chapters 4 and/or 5 are recommended. The authors believe that Chapter 8 is the most interesting and should not be omitted from any course.



We thank Dorothy Hollinger for excellent work in typing, correcting and formatting; also Linda Valdés and Gnanalakshmi Vanniasegaram for repeated careful proofreading.

Santa Cruz, California, March, 1990

Nora Hartsfield
Gerhard Ringel

Chapter 1

BASIC GRAPH THEORY

1.1 Graphs and Degrees of Vertices

Before we give any definitions and theory, let us consider the following example. There is a basket containing an apple, a banana, a cherry and a date. Four children named Erica, Frank, Greg and Hank are each to be given a piece of the fruit. Erica likes cherries and dates; Frank likes apples and cherries; Greg likes bananas and cherries; and Hank likes apples, bananas, and dates. Figure 1.1.1 describes the situation. The problem is to give each child a piece of fruit that he or she likes. The reader is invited to find a solution. One can see the advantage of using Figure 1.1.1 as an aid to solving the problem.

Figure 1.1.1 is, in fact, an example of a *graph*. Another example with which we are all familiar is a road map. The map in Figure 1.1.2 is greatly simplified, of course. It shows some different ways of driving from San Jose to San Francisco.

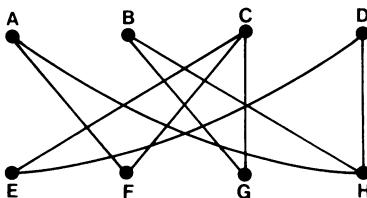


Figure 1.1.1

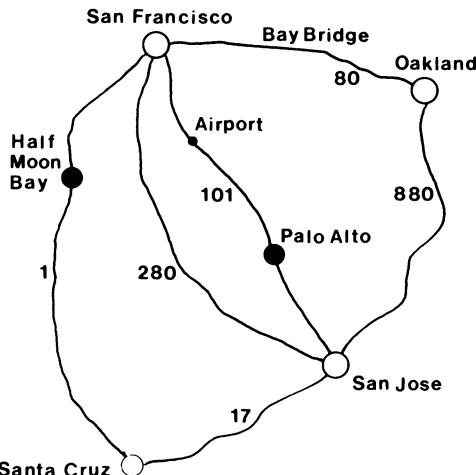
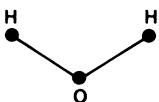
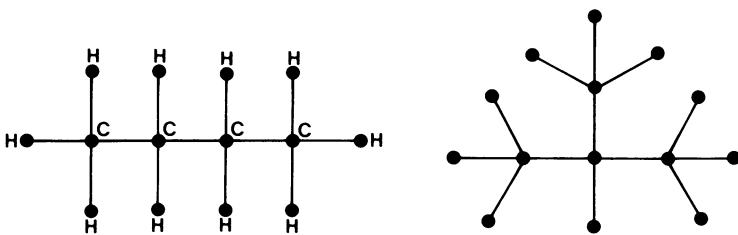


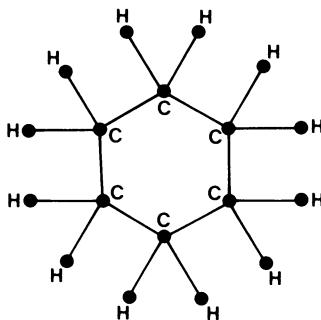
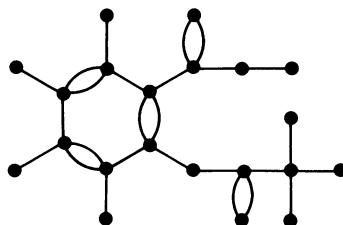
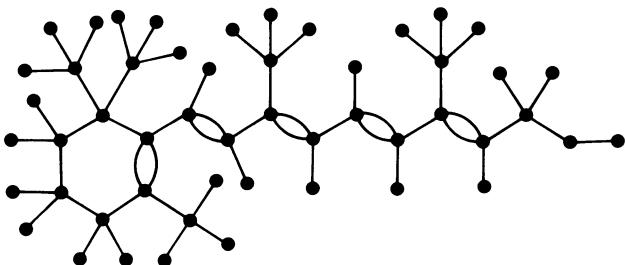
Figure 1.1.2

Chemists use diagrams to picture molecules, and these diagrams are also graphs. Usually the hydrogen atoms are omitted from the diagrams by the chemists using shorthand structure, but the Kekulé structure includes them.

Figure 1.1.3. Water, H_2O .Figure 1.1.4. Butane and isobutane, C_4H_{10} .

In Figures 1.1.6 and 1.1.7 the labels for carbon, hydrogen, and oxygen have been left out.

Graph theory is used in designing printed circuits for use in electronics devices. Circuits printed on silicon chips must be designed differently from

Figure 1.1.5. Cyclohexane, C_6H_{12} .Figure 1.1.6. Aspirin, $C_9H_8O_4$.Figure 1.1.7. Vitamin A, $C_{20}H_{30}O$.

those using insulated wires, since the conducting portions cannot cross one another. The graph of Figure 1.1.8 shows part of a printed circuit used in a computer.

Algorithms can also be described by graphs. The graph of Figure 1.1.9 is a chart of the steps used in an algorithm for solving a certain problem using a computer.

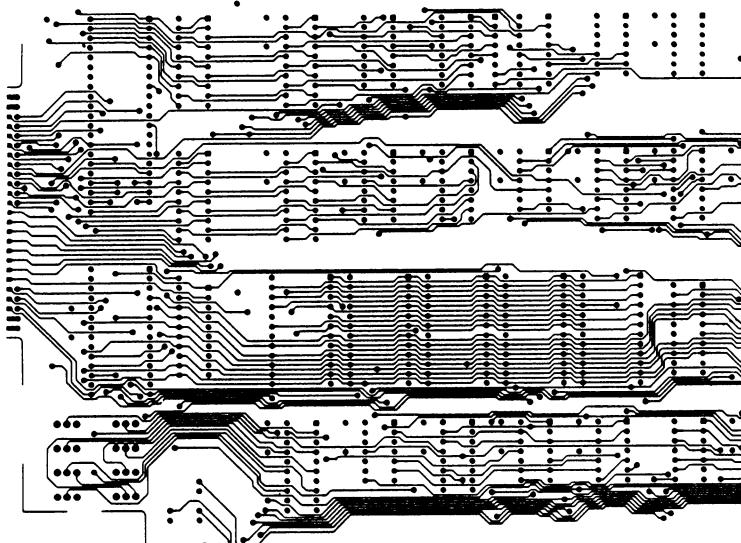


Figure 1.1.8. Portion of a printed circuit

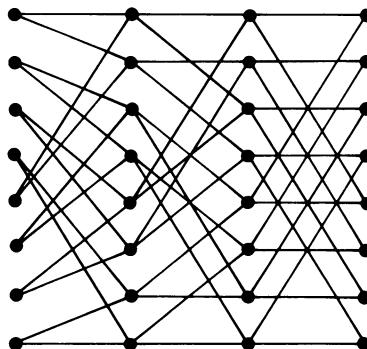


Figure 1.1.9

In the study of lattices and Boolean algebras, graphs arise as diagrams of these structures. Those who have studied set theory will recognize the diagrams of Figures 1.1.10 and 1.1.11 as lattices of subsets. For convenience, we use shorter notation for sets; for example, we write ABD for the set $\{A, B, D\}$. The

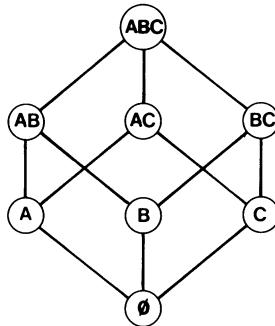
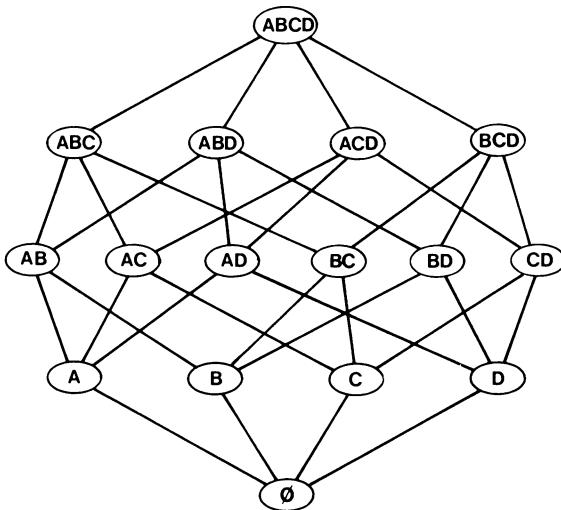
Figure 1.1.10. The lattice of subsets of ABC .

Figure 1.1.11

diagrams of Figures 1.1.10 and 1.1.11 show all subsets of the set at the top. If one set is derived from another by deleting one element, then the two sets are connected by a line in the diagram.

The different factorizations of the integer 60 are shown in the diagram of Figure 1.1.12. Some more examples of graphs are shown in Figures 1.1.13, 1.1.14, 1.1.15, 1.1.16, and 1.1.17.

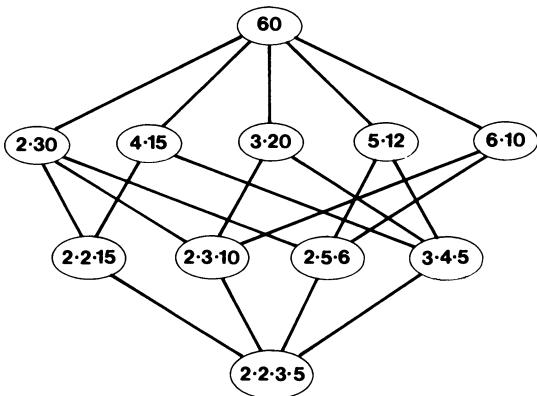


Figure 1.1.12. The factorizations of 60.

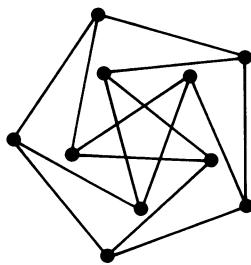


Figure 1.1.13

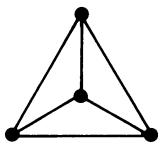


Figure 1.1.14

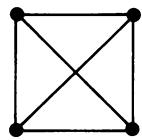


Figure 1.1.15

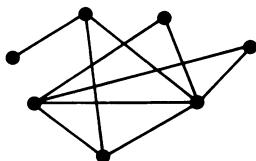


Figure 1.1.16

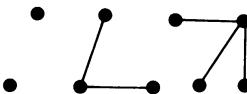


Figure 1.1.17

A *graph* G is a pair of sets (V, E) where V is nonempty, and E is a (possibly empty) set of unordered pairs of elements of V . The elements of V are called the *vertices* of G and the elements of E are called the *edges* of G . Sometimes we will write $V(G)$ for the vertices of G and $E(G)$ for the edges of G . Usually we represent the vertices by points in the plane, but the vertices can be any objects, as in the first example, where the vertices are fruit and children. Similarly, an edge is usually represented by a line connecting two vertices in the plane. The line might be straight or curved. However, an edge might be a road between two cities, a telephone line between two houses, or a preference between a child and a fruit.

A graph may be finite or infinite, depending on whether the set V is finite or infinite. For the most part we shall consider finite graphs. The reader should assume that a graph G is finite unless we say G is an infinite graph.

Sometimes when we draw a graph in the plane we cannot avoid having edges cross. Can you guess which of the graphs in Figures 1.1.1-1.1.17 can be drawn in the plane with no edges crossing?

Notice that the graph in Figure 1.1.17 is not in "one piece," that is, it is not connected. The other graphs we have seen so far are connected. Later we shall give an exact definition of a connected graph.

In a graph it is not allowed that two vertices are joined by more than one edge. If we allow such a thing, the structure is called a *multigraph*, as in Figure 1.1.18. Thus the structure in Figure 1.1.18 is not a graph. The one in Figure 1.1.19 has what we call a *loop*, which is an edge incident with only one vertex. If we allow loops, we call the structure a *pseudograph*. Most of the time we will work only with graphs. So every graph is a multigraph, but not conversely, and every multigraph is a pseudograph, but not conversely. The graphs for aspirin and vitamin A, Figures 1.1.6 and 1.1.7, are examples of multigraphs.

If x and y are vertices of a graph G , we say x is *adjacent* to y if there is an edge between x and y . We shall denote such an edge by xy . We may also say x and y are *neighbors*. We say that a vertex x is *incident* with an edge e if x is an endpoint of e . We also say that e is incident with x whenever x is an endpoint of e .

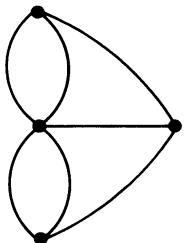


Figure 1.1.18

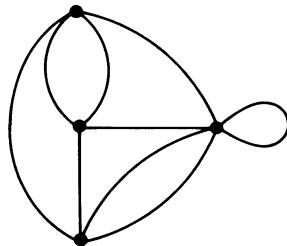


Figure 1.1.19

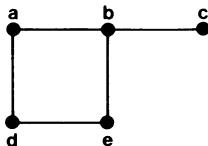


Figure 1.1.20

In Figure 1.1.20, a is adjacent to b and d ; b is adjacent to a , c , and e ; c is adjacent to b ; e is adjacent to b and d ; and d is adjacent to a and e . The vertex a is incident with the edges ab and ad ; b is incident with the edges ab , bc , and be ; and so on.

We normally denote the number of vertices in a graph G by p and the number of edges in G by q .

Look again at Figures 1.1.13, 1.1.14, and 1.1.15. Every vertex is incident with three edges. We say that each vertex has degree 3. Thus the *degree* of a vertex v is the number of edges incident with v . The graph in Figure 1.1.17 has two vertices of degree 0. We call such a vertex an *isolated vertex*. A vertex of degree 1 is called an *end vertex*.

Let us add the degrees of the vertices in the graph shown in Figure 1.1.16.

$$5 + 4 + 3 + 3 + 2 + 2 + 1 = 20.$$

Now let us count the edges in the graph. The number of edges is $q = 10$. So in this particular graph, when we add up the degrees of all the vertices, the answer we get is twice the number of edges. This should not be a big surprise because when we add the degrees of all the vertices we are adding the number of edges incident with each vertex. But we have added each edge twice since each edge is incident with two vertices. We have just proved

Theorem 1.1.1. Let v_1, v_2, \dots, v_p be the vertices of a graph G , and let d_1, d_2, \dots, d_p be the degrees of the vertices, respectively. Let q be the number of edges of G . Then

$$d_1 + d_2 + \cdots + d_p = 2q.$$

Theorem 1.1.1 could be stated as follows.

$$\sum_{i=1}^p d_i = 2q \quad \text{or} \quad \sum_{v \in V} \deg v = 2q.$$

The Σ notation is used to denote a sum.

Example. Is there a graph with seven vertices, each of which has degree 5? We can answer this question by using Theorem 1.1.1. If there is such a graph, then the sum of all the degrees equals five times seven, which is 35. By Theorem 1.1.1, this number must be an even number, so no such graph exists.

Go back to Figure 1.1.16. We list all the degrees of the vertices

$$5, 4, 3, 3, 2, 2, 1.$$

We can do the same for Figure 1.1.17.

$$3, 2, 1, 1, 1, 1, 0, 0.$$

These sequences of non-negative integers are the degree sequences of graphs, namely the graphs in Figures 1.1.16 and 1.1.17. A sequence of non-negative integers is called *graphic* if there exists a graph whose degree sequence is precisely that sequence. Consider the problem, which of the following sequences are graphic:

- i) 5, 4, 3, 2, 2, 1
- ii) 5, 5, 4, 4, 0
- iii) 6, 5, 5, 4, 3, 3, 2, 2, 2
- iv) 6, 6, 6, 6, 4, 3, 3, 0.

For i) we see immediately that the sum is an odd number, so sequence i) is not graphic. For sequence ii) the sum is even, so we must do something else. Notice that the first number is a five. This means that if there is a graph with this sequence, it has a vertex which is adjacent to five other vertices. But then the graph must have at least six vertices, and there are only five numbers in the sequence. Thus ii) is not graphic. Sequences iii) and iv) cannot be decided that quickly. However, there is an algorithm that works for every sequence. The basic idea behind the algorithm is that one can tell if a given sequence is graphic

by looking at another, simpler sequence. In the case of sequence iii) we get the following sequences; how we get them we shall explain later.

- (1) $6 \underline{5} \underline{5} \underline{4} \underline{3} \underline{3} \underline{2} \underline{2} \underline{2}$
- (2) $4 \underline{4} \underline{3} \underline{2} \underline{2} \underline{1} \underline{2} \underline{2}$
- (3) $4 \underline{4} \underline{3} \underline{2} \underline{2} \underline{2} \underline{2} \underline{1}$
- (4) $3 \underline{2} \underline{1} \underline{1} \underline{2} \underline{2} \underline{1}$
- (5) $3 \underline{2} \underline{2} \underline{2} \underline{1} \underline{1} \underline{1}$
- (6) $1 \underline{1} \underline{1} \underline{1} \underline{1} \underline{1}$

Sequence (6) is graphic. It is the degree sequence of the graph in Figure 1.1.21.



Figure 1.1.21

To get sequence (2) from sequence (1), we omit the first number, and since it is a 6, we subtract 1 from each of the next six numbers. To get sequence (3) from sequence (2) we rearrange in descending order. From sequence (3) to sequence (4) we again omit the first number, which is now a 4, then we subtract 1 from each of the next four numbers. We rearrange the order to obtain (5). We omit the first number, which is a 3 and subtract 1 from each of the next three numbers to obtain sequence (6) which is graphic, so we stop. The rule will be that since we obtain a graphic sequence by these steps, then the given sequence (1) is also graphic. Theorem 1.1.2 guarantees that this works.

Theorem 1.1.2. (Havel, Hakimi) *Consider the following two sequences and assume sequence (1) is in descending order.*

$$(1) \quad s, \quad t_1, \quad t_2, \dots, \quad t_s, \quad d_1, \dots, d_n$$

$$(2) \quad t_1-1, \quad t_2-1, \dots, \quad t_s-1, \quad d_1, \dots, d_n$$

Then sequence (1) is graphic if and only if sequence (2) is graphic.

Proof. First assume that sequence (2) is graphic. Then there is a graph G_2 with degree sequence equal to sequence (2). We construct G_1 from G_2 , by adding a vertex and connecting it to the vertices whose degrees are $t_1-1, t_2-1, \dots, t_s-1$. Then G_1 is a graph with degree sequence equal to sequence (1), hence sequence (1) is graphic.

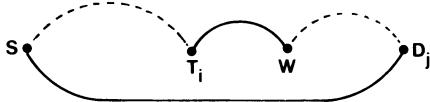
Now we suppose that sequence (1) is graphic. Then there is a graph H , with degree sequence (1). Let the vertex S have degree s , T_i have degree t_i , D_j have degree d_j . If S is adjacent to T_1, T_2, \dots, T_s , then we simply remove this vertex and the edges incident with it, thus obtaining a graph with degree sequence equal to sequence (2). However, this may not be the case. Suppose that for some i with $1 \leq i \leq s$ vertex S is not adjacent to vertex T_i . Then vertex S is adjacent to some vertex D_j with $j \geq 1$.



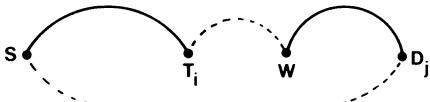
Since the sequence is arranged in descending order, $t_i \geq d_j$. If $t_i = d_j$, we simply exchange T_i and D_j .



If $t_i > d_j$, then T_i has more neighbors than D_j has, so there is a vertex W such that T_i is adjacent to W and D_j is not adjacent to W . We do not know whether W is one of the T -vertices or one of the D -vertices, but it does not matter.



In this case we remove the edges SD_j and T_iW and add the edges ST_i and D_jW to obtain the graph H_2 , still having the degree sequence (1).



If S is now adjacent to T_1, T_2, \dots, T_s , as before we remove S to obtain a graph with sequence (2). If not we repeat the above procedure. Since this is a finite graph, for some m , H_m is the graph we seek. Thus sequence (2) is graphic. ■

We now use the procedure to determine whether sequence iv) is graphic.

$$\begin{array}{r}
 \text{iv. } 6 \underline{6} \underline{6} \underline{6} 4 \underline{3} \underline{3} 0 \\
 5 \underline{5} \underline{5} 3 \underline{2} \underline{2} 0 \\
 4 \underline{4} \underline{2} \underline{1} \underline{1} 0 \\
 3 1 0 0 0
 \end{array}$$

Since there is no graph with one vertex of degree 3, one vertex of degree 1 and three vertices of degree 0, we see that the last sequence is not graphic, hence neither is sequence iv.

Exercises 1.1

- 1.1.1. Seven students go on vacations. They decide that each will send a postcard to three of the others. Is it possible that every student receives postcards from precisely the three to whom he sent postcards?
- 1.1.2.
 - a. Prove that for every even number $n \geq 4$ there exists a graph with n vertices, all of which have degree 3, without using Theorem 1.1.2.
 - b. Prove that for every odd number $n \geq 5$ there exists a graph with $n+1$ vertices such that exactly n vertices have degree 3.
- 1.1.3. Prove that for every number $n \geq 5$ there exists a graph with n vertices, all of which have degree 4.
- 1.1.4. Which of the following sequences are graphic?
 - a. 5 5 4 4 3 2 2 1 1
 - b. 6 5 4 3 2 2 2 2
 - c. 4 4 4 4 3 3
 - d. 7 6 5 4 4 3 2 1
- 1.1.5. Show that in a graph the number of vertices of odd degree is even.
- 1.1.6. Use induction to prove that
 $(n, n, n-1, n-1, \dots, 4, 4, 3, 3, 2, 2, 1, 1)$
 is always graphic.
- 1.1.7. Prove that no graph has all degrees different; that is, prove that in a degree sequence there is at least one repeated number.
- 1.1.8. Is either of the following sequences graphic?
 - a) (6, 6, 5, 5, 2, 2, 2, 2)
 - b) (6, 6, 5, 5, 3, 3, 3, 3).

- 1.1.9. Draw a graph with degree sequence
 a) (4, 3, 2, 2, 1)
 b) (4, 3, 3, 3, 1).
- 1.1.10. Figure 1.1.12 shows the multiplication diagram for the number 60.
 Construct the multiplication diagram for the number 48.
- 1.1.11. Draw a graph with eight vertices, four of which have degree 4 and four of which have degree 3.

1.2 Subgraphs, Isomorphic Graphs

A *subgraph* of a graph G is a graph H such that every vertex of H is a vertex of G , and every edge of H is an edge of G also. In other words, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Figure 1.2.1 shows a graph G with some of its subgraphs.

When are two graphs "the same"? If we look at Figure 1.1.14 and Figure 1.1.15, the drawings look different, but the structure of the graphs is the same. The graphs in Figure 1.2.2 are also the same.

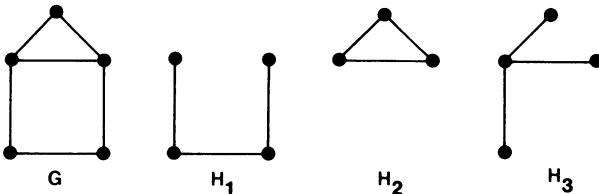


Figure 1.2.1

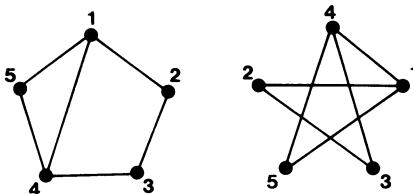


Figure 1.2.2

Two graphs G_1 and G_2 with p vertices are said to be *isomorphic* if the vertices of G_1 and G_2 can be labeled with the numbers from 1 to p such that whenever vertex i is adjacent to vertex j in G_1 , then vertex i is adjacent to vertex j in G_2 and conversely. Such a labeling is the same as a one-to-one correspondence between $V(G_1)$ and $V(G_2)$ that preserves adjacency.

In general, for large graphs it is difficult to tell whether two graphs are isomorphic. If two graphs are isomorphic, then they must have the same degree sequence. Thus, if we are asked to determine if two graphs are isomorphic, the first thing we should do is check whether they have the same degree sequence. If the sequences are the same, then we have to do some more checking. For example, the two graphs in Figure 1.2.3 have the same degree sequence, yet they are not isomorphic.

In the graph on the right, each vertex of degree 4 is adjacent to two other vertices of degree 4. In the graph on the left, each vertex of degree 4 is adjacent to only one other vertex of degree 4. Thus they are not isomorphic.

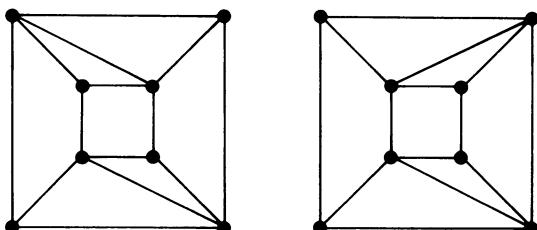


Figure 1.2.3

Some graphs are so famous that they have names. We now introduce you to some of them.

The graph in Figure 1.1.13 is called the Petersen graph. It has many interesting properties, and we shall refer to it later on.

The graph in Figure 1.1.14 or Figure 1.1.15 is called the complete graph on four vertices, denoted K_4 . In general, K_n is the graph with n vertices, and every vertex is adjacent to every other vertex. So K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. K_n has n subgraphs isomorphic to K_{n-1} .

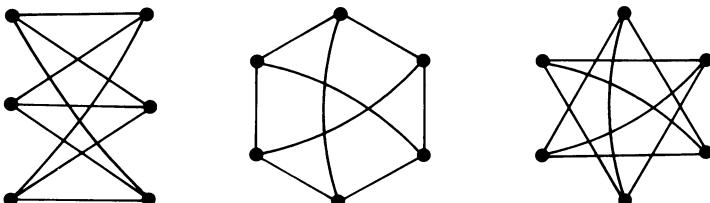


Figure 1.2.4

The first graph on the left in Figure 1.2.4 is a graph called $K_{3,3}$. In general, the graph $K_{m,n}$ is the *complete bipartite graph*, which means that $K_{m,n}$

has $m+n$ vertices divided into two sets, say m red vertices and n blue vertices. Every red vertex is adjacent to every blue vertex, and no two vertices of the same color are adjacent. Figure 1.2.5 has two different pictures of $K_{4,4}$.

K_4 is the same as the graph consisting of the vertices and edges of the tetrahedron. If we consider the vertices and edges of the cube, we get the graph called Q_3 . Likewise, there are an octahedral graph, a dodecahedral graph, and an icosahedral graph. (See Figure 1.2.5).

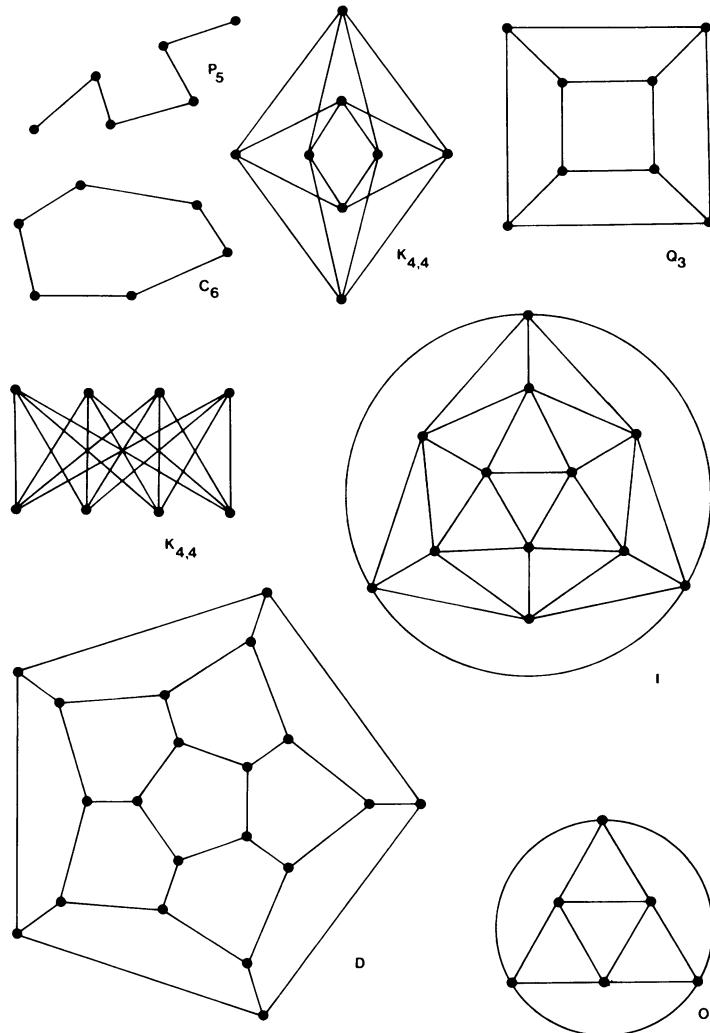


Figure 1.2.5

The graph with $n+1$ vertices labeled x_0, x_1, \dots, x_n and edges $x_0x_1, x_1x_2, x_2x_3, \dots, x_{n-1}x_n$ is called a *path of length n*, denoted P_n . We call x_0 and x_n the *end vertices* of the path, and we say the vertices x_0 and x_n are *connected by the path P_n* . In Fig. 1.2.5 there is a picture of P_5 .

The *cycle of length n*, C_n , is the graph with n vertices x_0, x_1, \dots, x_{n-1} and the edges $x_0x_1, x_1x_2, x_2x_3, \dots, x_{n-1}x_0$. Fig. 1.2.5 shows a picture of C_6 .

The graphs P_n and C_n as graphs are not so interesting, but as subgraphs of larger graphs they are very interesting.

Exercises 1.2

- 1.2.1. Find a graph with five vertices and with exactly
 - a. one cycle
 - b. three cycles
 - c. six cycles.
- 1.2.2. Find a graph with five vertices and exactly 22 cycles.
- 1.2.3. Find a graph with five vertices and exactly 13 cycles.
- 1.2.4. Find a graph G with six vertices and seven edges such that G does not contain a subgraph isomorphic to C_4 . (There are several solutions.)
- 1.2.5. Find a graph G with $p = 6$ and $q = 12$ such that G has no subgraph isomorphic to K_4 .
- 1.2.6. How would you communicate the graph of Figure 1.1.16 over the telephone?
- 1.2.7. Prove that the two graphs in Figure 1.2.6 are isomorphic.
- 1.2.8. Consider the graphs in Figure 1.2.4. Are any two of them isomorphic? Prove that your answer is correct.

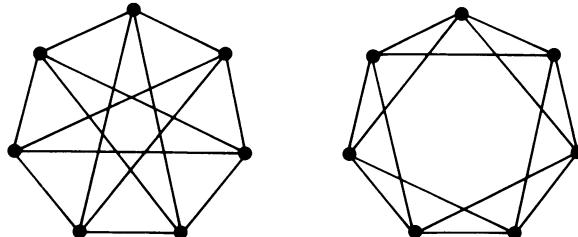


Figure 1.2.6

- 1.2.9. Are any two of the graphs in Figure 1.2.7 isomorphic? Prove that your answer is correct.

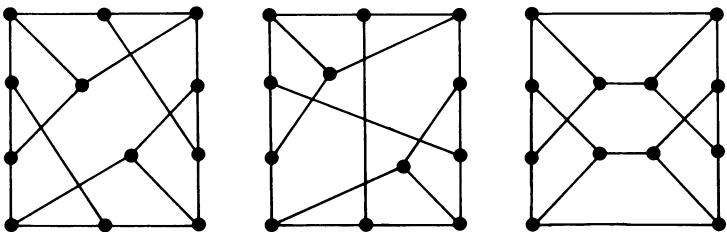


Figure 1.2.7

1.2.10. Prove that the graphs of Figures 1.1.13 and 1.2.8 are isomorphic.

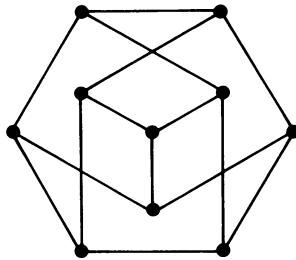


Figure 1.2.8

1.3 Trees

A graph G is *connected* if for any two vertices a and b there is a path from a to b . Notice that if a connected graph contains a cycle, removing an edge from the cycle will not disconnect the graph.

Theorem 1.3.1. *If G is a connected graph with p vertices and q edges, then $p \leq q+1$.*

Proof. The proof is by induction on the number of edges in G . If G has only one or two edges then the theorem is true. Assume the theorem is true for each graph with fewer than n edges. Let G be a given connected graph with n edges and p vertices. We consider two cases.

1) If G contains a cycle then we remove one edge of the cycle. We obtain a graph H , and H is still connected and has $n-1$ edges. The number of vertices of H is still p , and by the induction hypothesis $p \leq (n-1)+1$. Thus $p \leq n$, and certainly $p \leq n+1$.

2) If G does not contain a cycle, then we find a longest path in G . Let a and b be the vertices at the end of the path. The vertex a must be of degree 1;

otherwise the path could be made longer, or there would be a cycle in G . We remove the vertex a and the edge incident with a . We obtain a graph H ; H is still connected, and H has $p-1$ vertices and $n-1$ edges. Hence by the induction hypothesis, $p-1 \leq (n-1) + 1$. It follows that $p \leq n+1$, and therefore Theorem 1.3.1 is true for G also. ■

A *tree* is a connected graph that contains no subgraph isomorphic to a cycle. For instance, a path of length n , P_n , is a tree. A *forest* is a graph that contains no cycle. Every connected subgraph of a forest is a tree. Every subgraph of a forest is a forest, and strangely enough, all subgraphs of trees are forests. Notice that every tree is also a forest.

Theorem 1.3.2. *If G is a tree with p vertices and q edges, then $p = q+1$.*

Proof. The proof is by induction on the number of edges. If G is a tree with one edge, then the theorem is true for G . Assume that the theorem is true for all trees with fewer than n edges. Let G be a tree with n edges. Again we select a longest path in G with a and b the ends of the path. Vertex a must have degree 1, since otherwise the path could be made longer or there would be a cycle in G . Then we subtract vertex a from G together with the edge incident with a . We obtain a tree H with $p-1$ vertices and $n-1$ edges. By the induction hypothesis, $p-1 = (n-1) + 1$. It follows that $p = n+1$. So Theorem 1.3.2 is also true for G . ■

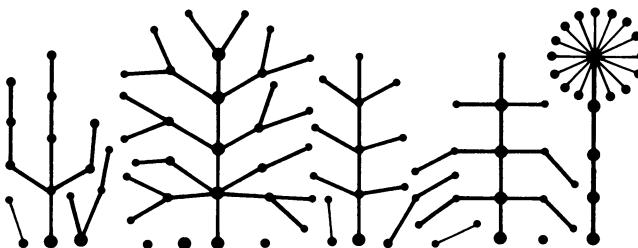


Figure 1.3.1. A forest

Theorem 1.3.3. *If G is connected, and $p = q+1$, then G is a tree.*

Proof. Assume the theorem is not true. Then there must be a connected graph G with $p = q+1$ such that G is not a tree. If G is not a tree, then G contains a

cycle, and thus we can subtract an edge from G and obtain a graph H that is still connected and has p vertices and $q-1$ edges. By Theorem 1.3.1, $p \leq (q-1) + 1$, so $p \leq q$, which is a contradiction since we assumed $p = q+1$. Therefore G is a tree. ■

Theorem 1.3.4. *Every tree with at least one edge has at least two end vertices.*

The proof is left as an exercise.

There are many interesting problems concerning trees and cycles. In order to help the reader, we now do one.

Example. Let the average degree of a connected graph G be greater than two. Prove that G has at least two cycles.

Solution. Let G be a connected graph, and let d_1, d_2, \dots, d_p be the degree sequence of G . Then if the average degree is greater than two, we have

$$2 < \frac{d_1 + d_2 + \dots + d_p}{p}.$$

By theorem 1.1.1, the sum equal $2q$, so

$$2 < \frac{2q}{p} \quad \text{or} \quad 2p < 2q \quad \text{or} \quad p < q, \quad (1.1)$$

thus G is not a tree by Theorem 1.3.2, and hence G has at least one cycle. We subtract an edge of the cycle from G . We obtain a connected graph G' with $p' = p$ vertices and $q' = q-1$ edges. By (1.1), $p' \leq q'$, and by Theorem 1.3.2 G' is not a tree, so it contains a cycle. Since we destroyed a cycle in G by removing an edge to obtain G' , G must have contained at least two cycles.

Theorem 1.3.5. *A graph G is a tree if and only if there exists exactly one path between any two vertices.*

Proof. We must prove two things since the theorem says "if and only if." First we assume G is a tree. Let v_1 and v_2 be vertices of G . Since trees are connected there is a path from v_1 to v_2 . Suppose there are two paths from v_1 to v_2 , $P_1 = v_1 u_1 u_2 \cdots u_n v_2$ and $P_2 = v_1 w_1 w_2 \cdots w_m v_2$. If u_1 is distinct from w_1 , then we follow P_1 until we find a vertex contained in P_1 that is also in P_2 . (This may be v_2 .) Then we have a cycle. If $u_1 = w_1$, then we look at u_2 . For some i , $u_i \neq w_i$, since there are two $v_1 v_2$ paths by assumption. Then we follow P_1 from

u_{i-1} until we find a vertex contained in P_1 that is also in P_2 , and then take P_2 back to u_{i-1} , and again we obtain a cycle. But G is a tree, so there are no cycles. Thus our assumption that there are two v_1v_2 paths is false.

Now we assume G is a graph with exactly one path between any two vertices. First we observe that G is connected. Suppose that G contains a cycle, $v_1v_2 \cdots v_nv_1$. There are clearly two paths from v_1 to v_n . This is a contradiction, since G has exactly one path between any two vertices. Hence G contains no cycles, and G is a tree. ■

A *spanning subgraph* of a graph G is a subgraph H of G such that H contains all the vertices of G ; in other words, $V(H) = V(G)$. A *spanning tree* of G is a spanning subgraph of G that is a tree.

Theorem 1.3.6. *Every connected graph G contains a spanning tree.*

Proof. If G is a tree, there is nothing to prove, since G is then a spanning tree of G . If G is not a tree, then G contains a cycle. Let e_1 be an edge of the cycle, and let $H_1 = G - e_1$, that is, the graph obtained from G by deleting the edge e_1 . If H_1 is a tree, then we are done. If not, then H_1 contains a cycle. Let e_2 be an edge of the cycle, and let $H_2 = H_1 - e_2$. We continue in this manner; since G is finite, for some i , H_i will be a tree. Thus G contains a spanning tree.

■

Exercises 1.3

- 1.3.1. Let T be a tree with vertices of degree only 3 or 1. If T has 10 vertices of degree 3, how many vertices of degree 1 are in T ?
- 1.3.2. Let G be a connected graph with n vertices and n edges. How many cycles does G have?
- 1.3.3. Prove theorem 1.3.4.
- 1.3.4. Find two nonisomorphic trees that have the same degree sequence.
- 1.3.5. Let the average degree of the vertices of a connected graph G be less than two. How many cycles does G have?
- 1.3.6. Let the average degree of the vertices of a connected graph G be equal to two. How many cycles does G have?
- 1.3.7. The degree sequence of a tree is 5, 4, 3, 2, 1, . . . , 1. Determine the number of 1's in the sequence.

- 1.3.8. Show that if G is a tree, and all the degrees of vertices in G are odd, then the number of edges is odd.
- 1.3.9. Show by an example that the statement of Problem 1.3.8 is not true if G is not a tree.
- *1.3.10. Prove that if $q > \binom{p-1}{2}$, then G is connected.
- 1.3.11. Prove that P_n is the only tree with $n+1$ vertices and only two vertices of degree 1.
- 1.3.12. Let F be a forest with 100 vertices and 90 edges. How many new edges must be added without adding vertices to obtain a tree?
- 1.3.13. Is there a disconnected graph with degree sequence $(4, 4, 3, 3, 3, 3, 3, 3)$?
- 1.3.14. The only connected graphs having more vertices than edges are the _____.
- 1.3.15. There are only two nonisomorphic graphs with degree sequence $(3, 3, 3, 3, 3, 3, 6)$. Find them.
- 1.3.16. Which of the graphs in Figure 1.3.2 is *not* isomorphic to the cube graph of Figure 1.2.5? Prove your answer.

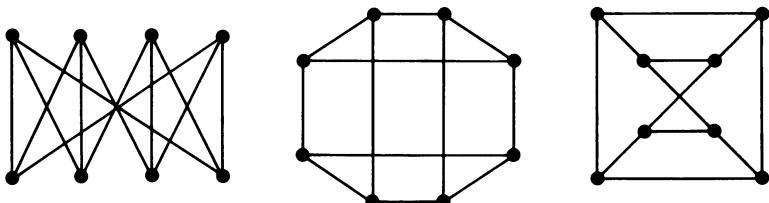


Figure 1.3.2

- 1.3.17. Which of the graphs in Figure 1.2.3 is isomorphic to the graph of Figure 1.3.3? Prove your answer.
- 1.3.18. Which of the graphs in Figure 2.1.1 is isomorphic to the graph of Figure 1.3.4? Prove your answer.
- 1.3.19. Which of the graphs in Figure 1.3.5 are isomorphic to graphs in Figure 1.2.4? Prove your answer.

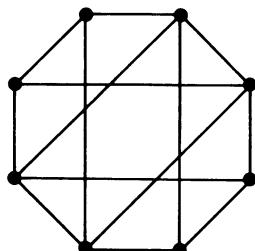


Figure 1.3.3

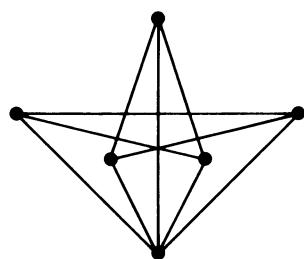


Figure 1.3.4

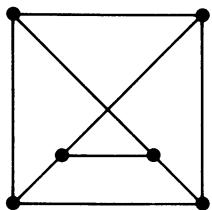


Figure 1.3.5

- 1.3.20. The average degree of the vertices of a tree is 1.99. How many edges does the tree have?

Chapter 2

COLORINGS OF GRAPHS

2.1 Vertex Colorings

The first coloring problem in graph theory is 150 years old. It is the famous four-color problem, that we shall discuss in a later chapter.

The *wheel with n spokes*, W_n , is the graph that consists of an n -cycle and one additional vertex that is adjacent to all the vertices of the cycle. W_3 is isomorphic to K_4 . In Figure 2.1.1 we display W_4 and W_5 .

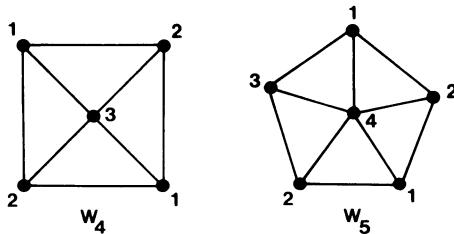


Figure 2.1.1

In Figure 2.1.1 we have assigned to each vertex a number. We shall call these numbers *colors*. Notice that no two adjacent vertices have the same color. Given a graph G , we define a *coloring of G* to be an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color.

One of the questions one can ask is, given a graph G , how many colors are necessary for a coloring of G ? Clearly, if G has p vertices we can color G with p colors. The interesting question is, what is the *smallest* number of colors with which G can be colored? For a given graph G , we denote the minimum

number of colors necessary to color G by $\chi(G)$, the *chromatic number* of the graph G . To repeat the definition, $\chi(G) = n$ means the graph G is colorable by n colors, and G is *not* colorable by $n-1$ colors. For example, in Figure 2.1.1, $\chi(W_4) = 3$ and $\chi(W_5) = 4$. The reader should convince himself or herself that these graphs cannot be colored with fewer colors.

The only graph with p vertices and chromatic number equal to p is K_p .

Suppose we wish to determine the chromatic number of a given graph G . This is really two problems. If we claim that $\chi(G) = n$, we must show that G cannot be colored with $n-1$ colors and show that it is possible to color G with n colors. One way of doing this is to exhibit a coloring of G by n colors. For instance, what is the chromatic number of the graph G in Figure 2.1.2? We see that there is a subgraph isomorphic to K_4 . Thus $\chi(G) \geq 4$.

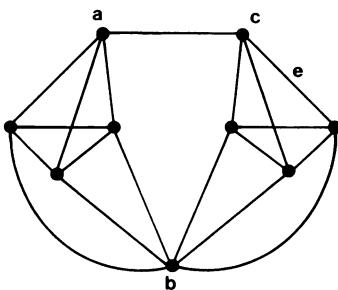


Figure 2.1.2

Now we try to color G with four colors. When we try this, we see that vertices a and b must receive the same color, since each lies on a K_4 subgraph, and the K_4 subgraphs share a common triangle. For the same reason, vertices b and c must receive the same color. But vertices a and c are adjacent, so they cannot have the same color. Thus $\chi(G) \geq 5$. Now it is easy to color the graph with five colors. Vertices b and c receive color 5; vertex a receives color 4; and colors 1, 2, and 3 are used to color the remaining two triangles in the middle.

Let G be a graph, and let v be a vertex of G . If we remove v and all the edges incident with v from G , the resulting graph is called $G - v$. Similarly, if e is any edge of G , the graph obtained from G by removing e is $G - e$. We leave all the vertices in G when we remove an edge. If v is a vertex of K_p , then $K_p - v$ is isomorphic to K_{p-1} . If G is the graph in Figure 2.1.2, $G - a$ is the graph in Figure 2.1.3, and $G - e$ is the graph in Figure 2.1.4.

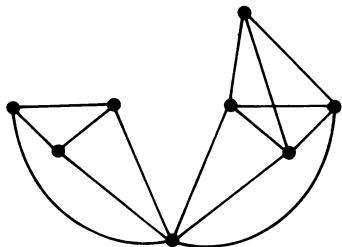


Figure 2.1.3

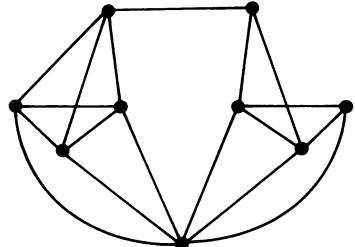


Figure 2.1.4

Let G be a graph. If H is a subgraph of G , and $H \neq G$, then H is called a *proper subgraph* of G . If $\chi(H) < \chi(G)$ for every proper subgraph H of G , then we say that G is *critical*. The graphs of Figures 2.1.2 and 2.1.5 are critical, and the complete graph K_n is also critical for every n .

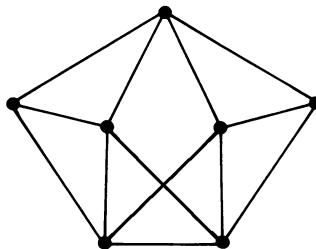


Figure 2.1.5

The following theorem is easy to verify.

Theorem 2.1.1. *Every critical graph is connected.*

Theorem 2.1.2. *Every graph G contains a critical subgraph H such that $\chi(H) = \chi(G)$.*

Proof. If G is critical, then the theorem is true for $H = G$. If G is not critical, then there exists a proper subgraph of H_1 of G with $\chi(H_1) = \chi(G)$. If H_1 is critical, then we are done; set $H = H_1$. If H_1 is not critical, then there exists a proper subgraph H_2 of H_1 such that $\chi(H_2) = \chi(H_1) = \chi(G)$. We continue in this manner. Since G is finite, for some k we must obtain a subgraph H_k such that H_k is critical. Then $H = H_k$. ■

The wheel W_5 , shown in Figure 2.1.1, is critical with $\chi(W_5) = 4$. Notice that the degree of each vertex of W_5 is at least three.

Theorem 2.1.3. *If G is critical with chromatic number four, then the degree of each vertex is at least three.*

Proof. Suppose that the theorem is false. Then there exists a critical graph G with chromatic number four and a vertex v of G such that the degree of v is at most two. Since G is critical, $G - v$ can be colored with only three colors. So we color $G - v$ with three colors. We put the vertex v back; v is adjacent to at most two vertices, so there is at least one of the three colors left to color v . But this is a contradiction, since G has chromatic number four. Thus the degree of each vertex is at least three. ■

We now state the theorem again in more generality.

Theorem 2.1.3. *If G is critical with chromatic number χ , then the degree of each vertex is at least $\chi - 1$.*

The proof is the same; instead of four we substitute χ . Another example that illustrates the more general theorem is the critical graph of Figure 2.1.2, where $\chi = 5$, and the degree of each vertex is at least four.

Theorem 2.1.4. *If G is a critical graph with p vertices and q edges, and G has chromatic number χ , then the relation*

$$(\chi - 1)p \leq 2q$$

holds.

Proof. By Theorem 2.1.3, the degree of each vertex of G is at least $\chi - 1$, and there are p vertices. So the sum of the degrees of the vertices of G is at least $(\chi - 1)p$. By Theorem 1.1.1, the sum of the degrees of the vertices of G is equal to $2q$. Hence Theorem 2.1.4 is proved. ■

We know that if a graph G contains a subgraph isomorphic to K_4 or W_n for n odd, then $\chi(G) \geq 4$. The graphs in Figures 2.1.6 and 2.1.7 also have chromatic number 4, but they contain no triangles, and that makes it hard to check that the two graphs are not colorable by three colors.

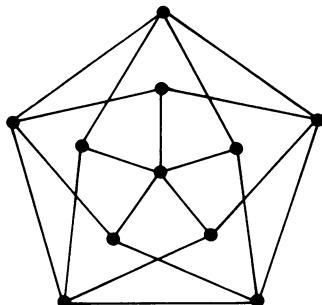


Figure 2.1.6. The Grötzsch graph.

The *girth* of a graph is the length of a shortest cycle in the graph. The girth of the graphs in Figure 2.1.6 and Figure 2.1.7 is four. The graph in Figure 2.1.7 is also critical. There exist graphs of arbitrarily high girth and arbitrarily high chromatic number [7], [17].

Theorem 2.1.5 (Erdős-Lovász). *For every two integers $m, n \geq 2$, there exists a graph with chromatic number n whose girth exceeds m .*

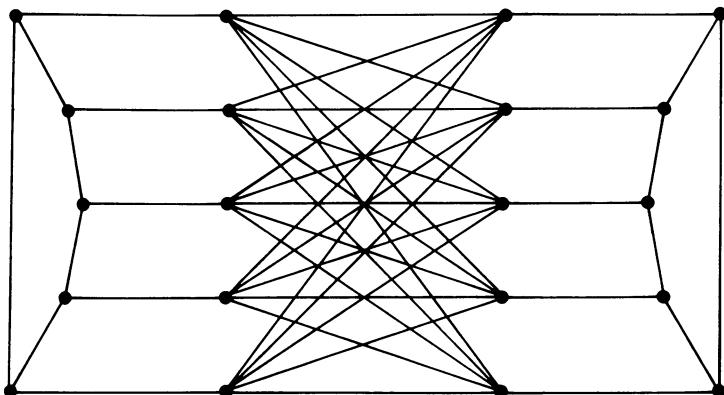


Figure 2.1.7

An unsolved coloring problem is the following conjecture of Lovász. If G is a graph such that $\chi(G - v - w) = \chi(G) - 2$ for every pair of adjacent vertices v and w , then G is a complete graph. Exercise 2.1.3 is a modified version of this problem. It differs by only one word, but it is much easier to prove.

A graph G is called *bipartite* if $\chi(G) \leq 2$. A special example of a bipartite graph, the complete bipartite graph $K_{m,n}$, was defined in Chapter 1.

The cube graph Q_3 of Figure 1.2.5 is another example of a bipartite graph, but the dodecahedron graph is not.

Let G be a graph, and let x and y be vertices of G . The *distance from x to y* in G , denoted by $d(x,y)$, is the length of a shortest path from x to y . If there is no path from x to y , we say $d(x,y) = \infty$. In Figure 2.1.2, $d(a,b) = 2$, $d(a,c) = 1$.

Theorem 2.1.6. *A graph G is bipartite if and only if every cycle in G has even length.*

Proof. Assume that G is bipartite. Suppose that G contains an odd cycle C . Then $\chi(C) = 3$. Thus $\chi(G) \geq 3$, but this is a contradiction since G is bipartite. Hence G cannot contain an odd cycle.

Now assume that G is a graph with no odd cycles. Without loss of generality, we assume that G is connected. (Why can we do this?) Select a vertex x_0 of G . We color G as follows. If x is a vertex of G , we color x red if $d(x_0,x)$ is even, and we color x blue if $d(x_0,x)$ is odd. (So for instance, since $d(x_0,x_0) = 0$, x_0 is colored red because 0 is even.) Since every distance is either even or odd, every vertex is now colored. Now we must show that no two adjacent vertices have the same color. We consider two adjacent vertices x and y .

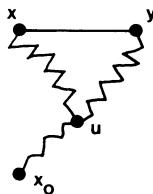


Figure 2.1.8

We choose a shortest path from x_0 to x and a shortest path from x_0 to y . Let u be the last common vertex in these shortest paths. (See Figure 2.1.8.) The vertex u may be equal to x_0 , or u may also be x or y . Now we consider $d(u,x)$ and $d(u,y)$. If u is one of x or y , then either $d(u,x) = d(u,y) + 1$ or $d(u,x) = d(u,y) - 1$. In either case, one of the distances is odd and one is even. We say they have different *parity*. If u is not one of x or y , we now compute the length of the cycle in Figure 2.1.8. It is $d(u,x) + 1 + d(u,y)$. We know this is even. Hence $d(u,x)$ and $d(u,y)$ have different parity. Since

$$d(x_0,x) = d(x_0,u) + d(u,x)$$

and

$$d(x_0, y) = d(x_0, u) + d(u, y)$$

$d(x_0, x)$ and $d(x_0, y)$ also have different parity. Thus x and y receive different colors, and $\chi(G) \leq 2$, so G is bipartite. ■

Theorem 2.1.6 shows that trees are bipartite. Every cycle in a tree has even length, since there are no cycles in a tree. In other words, if there is a cycle in a tree, then the cycle has even length. An "if-then" statement is always true if there is no circumstance that makes the "if" part true. Such a statement is *vacuously true*. We give you a strange example. All my violins have length $\sqrt{-1}$. This is true since I have no violins. See also Exercise 2.1.8.

The *diameter* of a graph G is the maximum distance between any two vertices of G . For instance, the diameter of P_n is n ; the diameter of W_n is 2 if $n > 3$; the diameter of K_n is 1; and the diameter of $K_{m,n}$ is 2 if m or n is at least two. In applications of graph theory to communications networks, diameter is important. One wishes to have a graph with small diameter and as few edges as possible, together with certain other restrictions. This is a very active area of research now.

Exercises 2.1

- 2.1.1. Determine $\chi(C_n)$. (The answer depends on n .)
- 2.1.2. Determine $\chi(W_n)$.
- 2.1.3. Prove that if G is a graph with $\chi(G - v - w) = \chi(G) - 2$ for every pair of vertices v and w in G , then G is complete. (Hint—assume that G is not complete.)
- 2.1.4. Determine the chromatic number of the graphs of Figure 1.2.3, and find a critical subgraph for each of them.

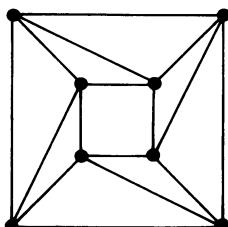


Figure 2.1.9

- 2.1.5. Repeat exercise 2.1.4 for Figure 2.1.9.
- 2.1.6. Color the vertices of the dodecahedron graph (Figure 1.2.5) with five colors such that any two vertices with distance at most two have different colors.
- 2.1.7. How many colors do you need to do Exercise 2.1.6 for the Petersen graph (Figure 1.1.13)?
- 2.1.8. A *cubic tree* is a tree all of whose vertices have either degree 3 or degree 1. Someone says every cubic tree with an even number of edges has a cycle. Is this true?
- 2.1.9. Determine the chromatic number of the graphs of Figure 1.2.5.
- 2.1.10. For each graph in Figure 1.2.5 find a critical subgraph with the same chromatic number as the given graph.
- 2.1.11. Find the graph with six vertices and the most edges that has no cycle of odd length as a subgraph.
- 2.1.12. Prove that if T is a tree, then $\chi(T) \leq 2$, without using Theorem 2.1.6. (Hint—use induction on p , the number of vertices of T .)
- 2.1.13. How many different ways can three ladies and three gentlemen sit around a round table with six seats such that only gentlemen sit by ladies? In graph theory language, how many subgraphs isomorphic to C_6 does the complete bipartite graph $K_{3,3}$ have?
- 2.1.14. Find a graph G with 6 vertices and 12 edges such that G has no subgraph isomorphic to $K_{3,3}$.
- 2.1.15. a) Find a graph G with 6 vertices and 9 edges such that G has no subgraph isomorphic to K_4 .
b) Find a graph G with 7 vertices and 9 edges such that G has no subgraph isomorphic to C_4 .
- 2.1.16. Let T be a tree with n vertices and diameter three. How many end vertices are in T ?
- 2.1.17. Color the vertices of C_n in such a way that any two vertices with distance at most two are different colors. How many colors are needed? (The answer depends on whether n is divisible by 3.)
- 2.1.18. How many cycles of length n does the wheel W_n contain?
- 2.1.19. How many subgraphs isomorphic to W_5 does the icosahedron graph have? (Figure 1.2.5.)

2.1.20. Prove Theorem 2.1.1.

2.2 Edge Colorings

Two edges in a graph are said to be *adjacent* if they are both incident with the same vertex. An *edge coloring* of a graph G is an assignment of colors to the edges of G . Notice the difference from vertex colorings, where adjacent vertices must receive different colors. For an edge coloring, no restriction is made on the colors of adjacent edges. A *proper edge coloring* is an edge coloring with the additional property that no two adjacent edges receive the same color. The reader could easily properly edge color the edges of K_4 , with three colors, and she or he will have such opportunities in the exercises. Figure 2.2.1 shows a proper edge coloring of a graph.

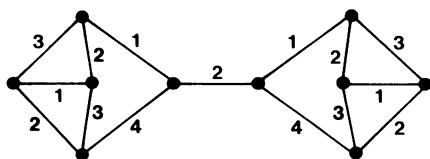


Figure 2.2.1

Theorem 2.2.1. *Let G be a graph. The number of colors required for a proper edge coloring of G is greater than or equal to the maximum degree of any vertex of G .*

Proof. If v is a vertex which is incident with t edges, then there must be at least t colors in a proper edge coloring of these edges, since they are adjacent to each other. ■

A graph G is said to be *regular of degree k* , or *k -regular*, if every vertex of G has degree equal to k . The graph in Figure 2.2.1 is regular of degree 3. The complete graph K_n is regular of degree $n-1$. The *edge chromatic number* of a graph G is the minimum number of colors required for a proper edge coloring of G . In some sense, "most" 3-regular graphs have edge chromatic number three. The dodecahedron graph and the cube graph have edge chromatic number three. Those 3-regular graphs that have edge chromatic number four are so interesting that they have a special name. They are called *snarks*. The Petersen graph (Figure 1.1.3) is an example of a snark.

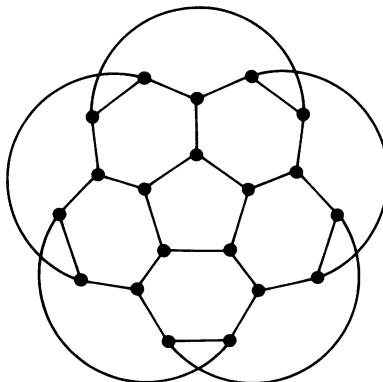


Figure 2.2.2. A snark.

Theorem 2.2.2 (Vizing) [24]. *The edge chromatic number of a k -regular graph is either k or $k+1$.*

We shall not prove this. Vizing's Theorem tells us that any *cubic* (3-regular) graph can be properly edge-colored with at most four colors.

The graph K_n is regular of degree $n-1$. By Vizing's Theorem it is properly edge colorable by either n or $n-1$ colors. In fact, if n is even, then the edge chromatic number of K_n is $n-1$. If n is odd, then the edge chromatic number of K_n is n .

Theorem 2.2.3. *The edge chromatic number of K_{2n} is $2n-1$.*

Proof. The proof of this theorem uses a technique that we shall call the turning trick. We shall see it again. We denote the vertices of K_{2n} by $0, 1, 2, \dots, 2n-2, x$, and we arrange the numbered vertices as a regular $(2n-1)$ -gon. (See Figure 2.2.3 for $2n = 10$.) Then the vertex x is placed outside. Let $C_1, C_2, \dots, C_{2n-1}$ denote $2n-1$ distinct colors. The edges we shall color C_1 are the following

$$(C_1) \quad 0x, 12n-2, 22n-3, \dots, n n-1.$$

Notice that these edges, except the edge $0x$, are parallel. To get the edges for the color C_2 , we take the edge $1x$ as the outside edge, then we "turn" everything else one unit to the right. (See the dotted lines in Figure 2.2.3.) The edges for the rest of the colors are

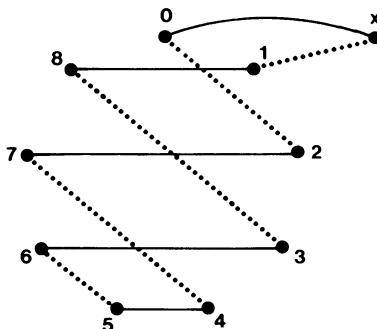


Figure 2.2.3

$$\begin{array}{ll}
 (C_1) & 0x, \quad 1\ 2n-2, \quad 2\ 2n-3, \quad \dots, \quad n\ n-1 \\
 (C_2) & 1x, \quad 2\ 0, \quad 3\ 2n-2, \quad \dots, \quad n+1\ n \\
 (C_3) & 2x, \quad 3\ 1, \quad 4\ 0, \quad \dots, \quad n+2\ n+1 \\
 & \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
 & \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
 (C_{2n-1}) & 2n-2\ x, \quad 0\ 2n-3, \quad 1\ 2n-4, \quad \dots, \quad n-1\ n-2.
 \end{array}$$

This gives a proper edge coloring of K_{2n} with $2n-1$ colors. By Theorem 2.2.1 at least $2n-1$ colors are required; thus Theorem 2.2.3 is proved. ■

Theorem 2.2.4. *The edge chromatic number of K_{2n-1} is $2n-1$.*

Proof. Certainly we can properly color the edges of K_{2n-1} with $2n-1$ colors, since by Theorem 2.2.3 we can properly edge color K_{2n} with $2n-1$ colors, then remove a vertex to obtain K_{2n-1} without disturbing the propriety of the coloring. What we must show is that we cannot properly edge color K_{2n-1} using only $2n-2$ colors. To see this, recall that there are $\binom{2n-1}{2} = (n-1)(2n-1)$ edges in K_{2n-1} . If these edges are colored by only $2n-2$ colors, then some color goes to at least n edges. But this forces two adjacent edges to have the same color. Therefore $2n-1$ colors are required. ■

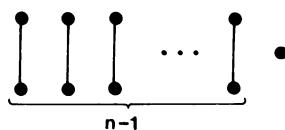


Figure 2.2.4

A spanning subgraph H of a graph G is called a 1-factor of G if H is regular of degree 1. An r -factor of a graph G is an r -regular spanning subgraph of G .

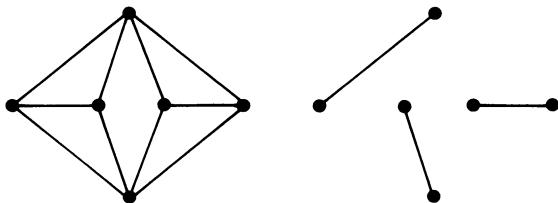


Figure 2.2.5. A 1-factor of a graph.

The very first example in Chapter 1 concerns finding a 1-factor in a graph, pictured in Figure 1.1.1, whose vertices are children and fruit.

Of course a graph with an odd number of vertices never has a 1-factor. (Why?) But there are also some connected graphs with an even number of vertices that have no 1-factors. Two such graphs are pictured in Figure 2.2.6.

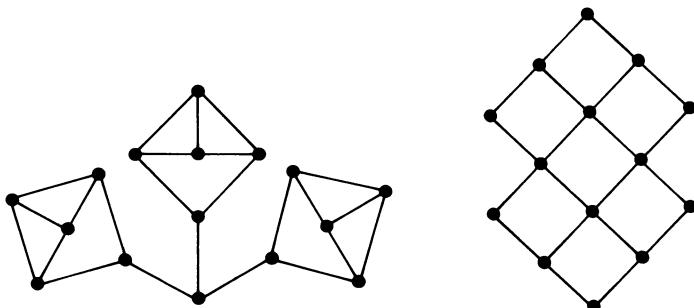


Figure 2.2.6

In the proof of Theorem 2.2.3, we used the turning trick to obtain a proper edge coloring of K_{2n} using $2n-1$ colors. Notice that the edges colored with a particular color are actually a 1-factor of K_{2n} . We shall discuss this further in the next section.

Exercises 2.2

- 2.2.1. Show that if the edges of K_6 are colored with red and blue, then there will be a red triangle or a blue triangle.

- 2.2.2. Show that Exercise 2.2.1 is not true for K_5 .
- 2.2.3. Determine the edge chromatic number of $K_{m,n}$.
- 2.2.4. Determine the edge chromatic number of the graph in Figure 2.1.6.
- 2.2.5. Is the graph K_4 a snark? Prove your answer.
- 2.2.6. For which values of m and n does $K_{m,n}$ have a 1-factor?
- 2.2.7. Properly color the edges of the cube and dodecahedron with three colors.
- 2.2.8. Find all solutions to the equations $C_x = K_{y,z}$, $C_x = K_y$, $K_x = K_{y,z}$, $K_x = W_y$, and $P_x = K_{y,z}$.
- 2.2.9. There are four nonisomorphic connected graphs with the property that the product $pq = 20$. Find all of them.
- 2.2.10. There are two connected graphs with the property that the sum $p+q = 8$. Find them.

2.3 Decompositions and Hamilton Cycles

We present the following definition in an intuitive manner. If we color the edges of a graph G with two colors, red and blue, no restrictions, then the red edges form a subgraph H_1 of G , and the blue edges form a subgraph H_2 of G . Taken together, H_1 and H_2 are all of G . We say that G is *decomposed* or *decomposable* into H_1 and H_2 . More generally, a graph G is said to be decomposable into subgraphs H_1, H_2, \dots, H_t if any two subgraphs H_i and H_j have no edges in common, and the union of all the subgraphs H_i is G . Notice that all the vertices could appear several times. The union refers to an edge union. In Figure 1.2.1 the subgraphs H_1 and H_2 form a decomposition of G . H_2 and H_3 do *not* form a decomposition of G , since they have edges in common.

At the end of Section 2.2, we pointed out that the edges of K_{2n} that are colored with one particular color in the proof of Theorem 2.2.3 form a 1-factor. In fact, taking all $2n-1$ colors, we actually have a decomposition of K_{2n} into 1-factors.

This decomposition has an immediate application to round-robin tournaments in chess, tennis, etc. Say we have ten players. The director would either look in his calendar where usually a table of pairings is given, or he could

look at Figure 2.2.3 and identify the players by the numbers $0, 1, \dots, 8, x$. Then he reads the pairings from the picture this way:

First round	0	x	1	8	2	7	3	6	4	5
Second round	1	x	2	0	3	8	4	7	5	6

and so on.

If the number of players is odd, then we know that in each round one player has a bye. The same strategy works, but now x is an imaginary player, so whichever player is paired with x has a bye.

Going back to graph theory, we know that K_{2n-1} has no 1-factor, since the number of vertices is odd. But by Theorem 2.2.4, K_{2n-1} has a decomposition into $2n-1$ subgraphs, each isomorphic to the graph of Figure 2.2.4. These subgraphs could be pictured without the isolated point, because the decomposition refers to edges only.

Recall that the *girth* of a graph is the length of a shortest cycle in the graph. A longest cycle in a graph has no name, but if a longest cycle happens to include every vertex of the graph, then the cycle itself has a name. A *Hamilton cycle* in a graph G is a cycle containing every vertex of G .

Figure 2.3.1 shows a plan of a zoo. Every vertex represents a cage with interesting animals. The edges are pathways. A visitor wants to enter the zoo, say at vertex 1, and visit each cage exactly once, then at the end return to vertex 1. This is precisely a Hamilton cycle. The vertices of the cycle in order are 1, 2, 3, and so on. It happens that in this graph there are some edges, for example

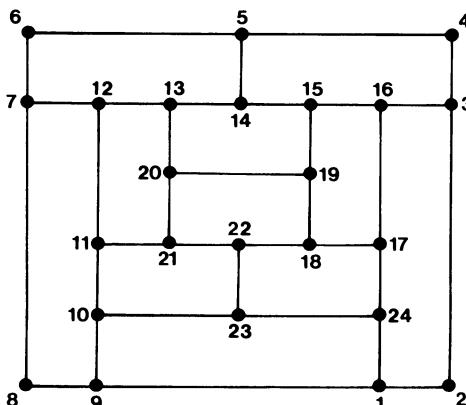


Figure 2.3.1. A Hamilton cycle in a zoo.

the edge from 1 to 9, that are not contained in any Hamilton cycle. In other words, you cannot find a Hamilton cycle going through the edge from 1 to 9.

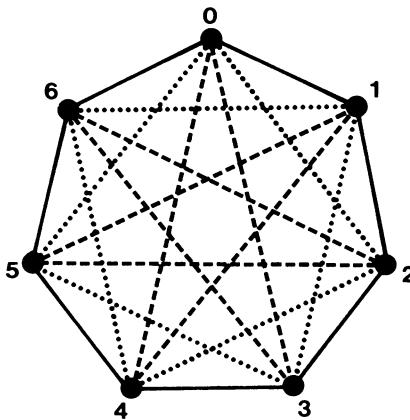


Figure 2.3.2

Figure 2.3.2 is a picture of the complete graph K_7 . It is easy to see that K_7 contains a Hamilton cycle. The following is a Hamilton cycle in K_7 .

$$0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6.$$

Then we can find another one,

$$0 \ 2 \ 4 \ 6 \ 1 \ 3 \ 5,$$

that contains no edges of the first cycle. There is one more, that contains no edges of the first two. We leave it to the reader to find it. What we have discovered is that K_7 is decomposable into three Hamilton cycles, or we can say K_7 is decomposable into three cycles of length seven. This seems a natural way to decompose K_n into Hamilton cycles. Clearly, if n is even this is impossible, since then each vertex has odd degree. But the reader should try the above method for K_9 . It will not work. But it can be done differently, using the turning trick.

Theorem 2.3.1 (Lucas) *The complete graph K_{2n+1} has a decomposition into n Hamilton cycles.*

We illustrate the method of proof with the special case of K_9 (Figure 2.3.3). The vertices are denoted by 0, 1, 2,..., 7, x . The first Hamilton cycle is

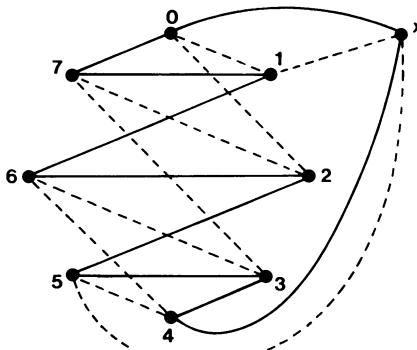


Figure 2.3.3

$0\ 7\ 1\ 6\ 2\ 5\ 3\ 4\ x$, which is the solid zigzag line in Figure 2.3.3. The second Hamilton cycle we get by turning the first one by one unit clockwise. This corresponds to adding 1 to each number and taking the remainder after division by 8. The next two cycles are obtained the same way, by turning farther.

For K_{2n+1} , the vertices are denoted by $0, 1, 2, \dots, 2n-1, x$, and we choose the cycles

(C_1)	0	$2n-1$	1	$2n-2$	2	$2n-3$	\cdots	$n-1$	n	x
(C_2)	1	0	2	$2n-1$	3	$2n-2$	\cdots	n	$n+1$	x
(C_3)	2	1	3	0	4	$2n-1$	\cdots	$n+1$	$n+2$	x
.
.
.
(C_n)	$2n-1$	$2n-2$	0	$2n-3$	1	$2n-4$	\cdots	$n-2$	$n-1$	x

Theorem 2.3.2. K_{2n} has a decomposition into $n-1$ Hamilton cycles and a 1-factor.

The proof of Theorem 2.3.2 uses the 1-factors presented in the proof of Theorem 2.2.3. Take the first two 1-factors, C_1 and C_2 , to form the first Hamilton cycle; C_3 and C_4 for the second; and so on. Then the 1-factor C_{2n-1} will be left over.

A *Hamilton path* in a graph G is a path that contains every vertex of G . Since every path is also a tree, a Hamilton path in G is a spanning tree of G . Clearly, if a graph has a Hamilton cycle, it also has a Hamilton path, but the

converse is not true. There are graphs that have a Hamilton path but no Hamilton cycle. A trivial example is the path P_n . A more interesting example is the Petersen graph. We have seen that K_{2n} has no decomposition into Hamilton cycles. However, it does have a decomposition into Hamilton paths!

Theorem 2.3.3. K_{2n} has a decomposition into n Hamilton paths.

To prove this quite elegantly, consider K_{2n+1} , that we know has a decomposition into n Hamilton cycles. Now we remove a vertex from K_{2n+1} , thus obtaining the desired decomposition of K_{2n} into n Hamilton paths. See Figure 2.3.3, and ignore the vertex x .

We can use Theorem 2.3.3 to show that K_{10} can be decomposed into a P_1 , a P_2 , a P_3, \dots , a P_8 , and a P_9 . By Theorem 2.3.3, K_{10} has a decomposition into five Hamilton paths, each of which is a P_9 . We take one as it is for the P_9 , then split the rest into P_1 and P_8 , P_2 and P_7 , P_3 and P_6 , P_4 and P_5 .

Theorem 2.3.4. The complete graph K_{2n} has a decomposition into $2n-1$ paths consisting of one path of each length k for $k = 1, 2, 3, \dots, 2n-1$.

The example for $2n = 10$ generalizes easily to a proof of the general theorem.

Now we mention an unsolved problem.

Conjecture (Gyárfás, Lehel) [9]. Given $T_1, T_2, \dots, T_{2n-1}$, where each T_i is a tree with i edges, it has been conjectured that K_{2n} can be decomposed into these given trees.

Recall that a snark is a cubic graph with edge chromatic number four.

Theorem 2.3.5. A snark has no Hamilton cycle.

Proof. Let G be a snark, and suppose that G has a Hamilton cycle. Color the edges of the cycle with colors 1 and 2 alternating. This is a proper edge coloring of the cycle, since the number of vertices is even. Color the remaining edges in the graph by the color 3. At each vertex there are two edges of the cycle, one of which is colored 1 and one of which is colored 2, and a third edge that is colored 3. Thus we have a proper edge coloring of G with only three colors, which contradicts the fact that G is a snark. Hence our assumption that G has a Hamilton cycle must be false. ■

The converse of Theorem 2.3.5 is not true. The graphs of Figures 2.3.4 and 2.3.5 are properly edge colorable by three colors, but have no Hamilton cycle. They do have Hamilton paths.

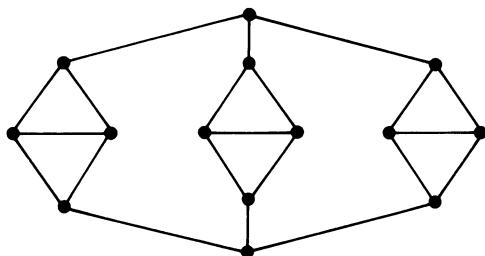


Figure 2.3.4

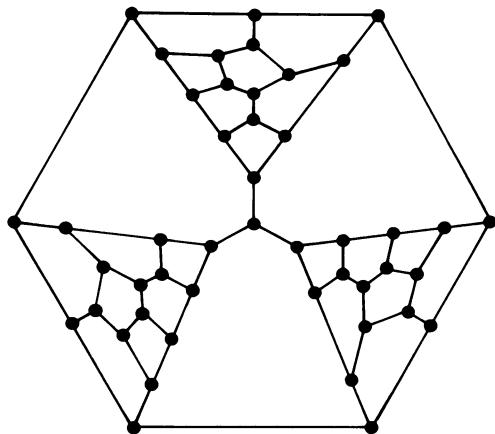


Figure 2.3.5. The Tutte Graph

In general it is very difficult to tell if a particular graph has a Hamilton cycle or a Hamilton path.

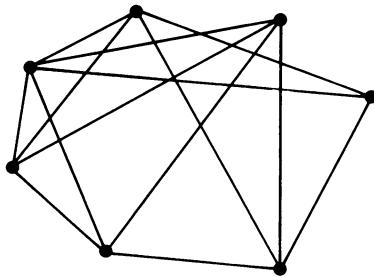


Figure 2.3.6

The graph of Figure 2.3.6 contains a Hamilton cycle. One of the unsolved problems in graph theory is to find a theorem of the form "A graph G contains a Hamilton path (cycle) if and only if"

Exercises 2.3.

- 2.3.1. At the end of Section 2.1, we defined the diameter of a graph. Find spanning trees in the wheel W_n , one with diameter two, one with diameter three, and in general, one with diameter h for each number $2 \leq h \leq n$.
- 2.3.2. How many subgraphs isomorphic to the path P_n are in the wheel W_n ?
- 2.3.3. Find a spanning tree in the graph of Figure 2.3.7 and the graph of Figure 2.3.8 that is
 - a. a path.
 - b. not a path

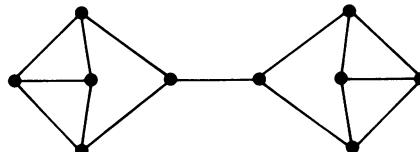


Figure 2.3.7

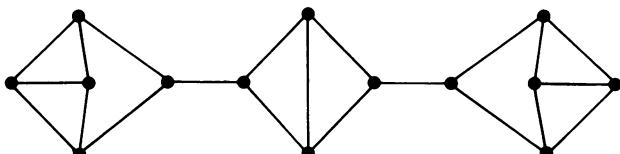


Figure 2.3.8

- 2.3.4. A tree has only vertices of degree 5 and degree 1. If the tree has $4n+2$ vertices, how many have degree 5?
- 2.3.5. Find the girths of the graphs (except P_5) in Figure 1.2.5.
- 2.3.6. Find the diameter of each of the graphs in Figure 1.2.5.
- 2.3.7. Find a Hamilton cycle and a Hamilton path in each of the graphs in Figure 1.2.5, if they exist.

- 2.3.8. Thirteen people who are not superstitious wish to have dinner together at a round table for a few nights so that each person has different neighbors every night. For how many nights can they do this?
- 2.3.9. Find a Hamilton path and a Hamilton cycle in each of the graphs in Figure 1.2.7.
- 2.3.10. Find a Hamilton path in the graphs in Figure 2.3.4 and 2.3.5.
- 2.3.11. The method of decomposing K_n into Hamilton cycles by first taking consecutive vertices, then taking every second vertex, then every third, and so on, works for K_7 and K_{11} . Why does it not work for K_9 ?
- 2.3.12. Fill in the blanks:
- The complete bipartite graph $K_{m,n}$ has a Hamilton cycle if and only if _____.
 - The complete bipartite graph $K_{m,n}$ has a Hamilton path if _____.
- 2.3.13. How many subgraphs isomorphic to C_4 are there in the complete bipartite graph $K_{m,n}$?
- 2.3.14. How many subgraphs isomorphic to C_4 are in the wheel W_n ?
- 2.3.15. How many Hamilton cycles are in the wheel W_n ?
- 2.3.16. There are nine nonisomorphic spanning trees in the graph on the left in Figure 2.2.6. Find them all. What are their diameters?
- 2.3.17. First draw a tree T with degree sequence $(3, 3, 1, 1, 1, 1)$. Then find a decomposition of K_6 into three subgraphs isomorphic to T .
- 2.3.18. Decompose $K_{3,4}$ into two Hamilton paths.
- 2.3.19. Decompose $K_{2,n}$ into two isomorphic trees, each having diameter
- 2.
 - 3.
 - 4.

2.4. More Decompositions

So far we have only used the turning trick to decompose K_n into isomorphic subgraphs. In fact, these subgraphs were either 1-factors, Hamilton cycles, or Hamilton paths. Now we consider some more examples of decompositions using the turning trick.

The Petersen graph has 15 edges, so we could ask if it is possible to decompose the graph into five isomorphic subgraphs. Consider the shaded P_3 in Figure 2.4.1. Turning gives us a decomposition of the Petersen graph into five subgraphs isomorphic to P_3 . However, if we ask for a decomposition of the Petersen graph into three isomorphic subgraphs, we must experiment or use a different drawing of the Petersen graph, as in Figure 1.2.8. (See Exercise 2.4.14.)

In Figure 2.4.2 the shaded subgraph can be turned to obtain a decomposition of Q_3 into four subgraphs isomorphic to P_3 . Figure 2.4.3 shows two decompositions of K_9 into nine isomorphic subgraphs. In the first example each subgraph is isomorphic to a particular tree, and in the second example, each subgraph is isomorphic to C_4 .

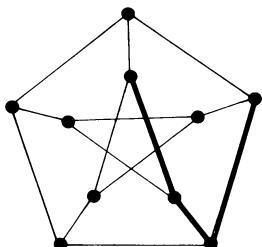


Figure 2.4.1

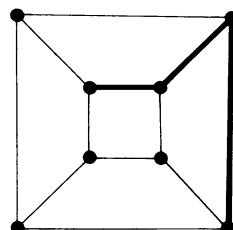


Figure 2.4.2

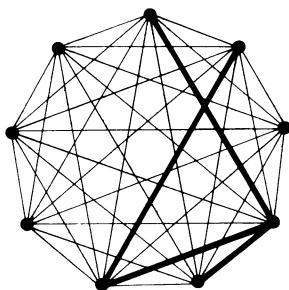
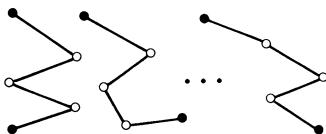


Figure 2.4.3

The number of edges in Q_3 is 12, so one might try to decompose Q_3 into three P_4 subgraphs. If you try this, you will soon see that it is impossible. In the following more general example we prove this fact.

Let G be a cubic graph. Then G has no decomposition into subgraphs, each of which is isomorphic to a path of length four.

To see this, suppose that G is a cubic graph with $2m$ vertices. Then G has $3m$ edges, by Theorem 1.1.1. If we wanted to have any hope of decomposing G into paths of length four, certainly $3m$ must be divisible by 4, so $m = 4k$ for some k . Then G has $8k$ vertices. Assume there is such a decomposition. There will be $3k$ paths of length four.



None of the white vertices can be identical, or we would have a vertex of degree 4. There are then at least $3 \cdot 3k = 9k$ vertices in G , but G has only $8k$ vertices. Thus no such decomposition exists.

A *bridge* in a connected graph G is an edge of G whose removal disconnects G into two subgraphs. All trees have bridges. In fact, every edge of a tree is a bridge, and trees are the only connected graphs with this property.

Theorem 2.4.1. *Let G be a connected graph. Then G is a tree if and only if every edge of G is a bridge.*

The proof is left as an exercise.

We call the two subgraphs left after removing a bridge the *banks* of the bridge. One or both of the banks could be an isolated vertex. In a disconnected graph, the subgraphs that are connected and not contained in any larger connected subgraph are called *components*. Hence, removing a bridge from a connected graph leaves a graph with two components.

In Figure 2.4.4 we reconsider the graph of Figure 2.3.6. Figure 2.4.4 shows a decomposition of this graph into paths of length two, each of which is designated by a number. Such a decomposition is *always* possible whenever the graph is connected and has an even number of edges!

Theorem 2.4.2. *A connected graph with an even number of edges is decomposable into subgraphs each isomorphic to a path of length two.*

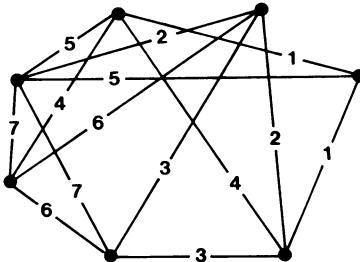
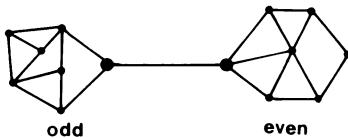


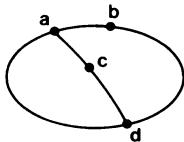
Figure 2.4.4

Proof. The proof is by induction on the number of edges in the graph. If a connected graph has two edges, the theorem is true. Assume that the theorem is true for connected graphs with fewer than $2n$ edges. Given a connected graph G with $2n$ edges, we consider several cases.

1. If G has a bridge ab such that b has degree greater than one then one bank has an even number of edges, and the other has an odd number of edges. Let a be the end of the bridge that is on the odd bank and b the other one. We consider the subgraphs H_1 consisting of the odd bank plus the bridge ab and H_2 consisting of the even bank. Both subgraphs are connected, and each has fewer than $2n$ edges. Thus, by the induction hypothesis, each has a decomposition into subgraphs isomorphic to P_2 . The union of these decompositions is a decomposition of G .

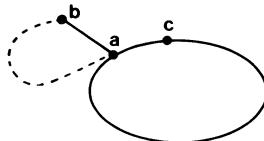


2. Case 2 is a special case that does not get included in any other case. If G is the graph $K_{1,t}$ (this graph is sometimes called a *star* or an *umbrella*), then the decomposition is clear. Any two edges form a P_2 , and the number of edges is even whenever t is even.
3. Let C be a longest cycle in G , and assume that C has a diagonal path, that is, a path from one vertex a of the cycle to a vertex d of the cycle, such that the path is not part of the cycle.



The path could be just a single edge, in which case c is equal to d in the picture. Since C is the longest cycle in G , d is not equal to b . In this case we remove the edge ab and the edge ac from G . The resulting graph is still connected; it has $2n-2$ edges; and, by the induction hypothesis, it has a decomposition into subgraphs isomorphic to P_2 . The path bac can then be added back to obtain the desired decomposition of G .

4. Let C be a longest cycle in G , and assume C has no diagonal path. If C contains only vertices of degree 2, then C is all of G , and the decomposition is obvious. Suppose that C contains a vertex a of degree at least three. If there is a bridge incident with a , either this is already covered in case 1, or the bridge has a vertex of degree 1. If there is no bridge incident with a , then there is a cycle containing a .



We consider G minus the edges ab and ac . It is possible that b may be an isolated point in the resulting graph, but this does not affect the argument. The resulting graph has $2n-2$ edges in the connected portion, thus by the induction hypothesis it has a decomposition into subgraphs isomorphic to P_2 . We then add the path bac to obtain a decomposition of G . ■

Exercises 2.4

- 2.4.1. Decompose the complete graph K_6 into three subgraphs, each isomorphic to a path of length five.

- 2.4.2. Decompose K_5 into five isomorphic subgraphs.
- 2.4.3. Decompose K_7 into three isomorphic subgraphs.
- 2.4.4. Prove that neither K_6 nor K_7 can be decomposed into two isomorphic subgraphs.

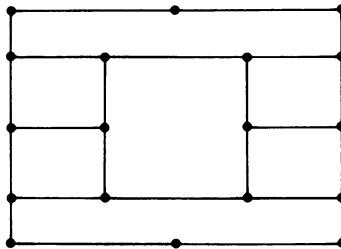


Figure 2.4.5. The tennis graph

- 2.4.5. Does the tennis graph of Figure 2.4.5 have a Hamilton cycle? If yes, is there a Hamilton cycle through any given edge of the tennis graph? (One picture should be enough.)
- 2.4.6. The tennis graph has 24 edges. 24 is divisible by 2, 3, 4, 6, 8, and 12. For each of these divisors d of 24, find a decomposition of the tennis graph into d isomorphic subgraphs. (Of course you are allowed to use the theorems of the text.)
- 2.4.7. Is the tennis graph of Figure 2.4.5 bipartite? If no, what is the smallest number of edges you need to remove to obtain a bipartite graph?
- 2.4.8. Find a 2-factor in the tennis graph.
- 2.4.9. Decompose the graphs in Figures 2.3.7 and 2.3.8 into subgraphs isomorphic to P_3 .
- 2.4.10. Prove that the cubic graph of Figure 2.2.6 is not decomposable into subgraphs isomorphic to P_3 .
- 2.4.11. For $n = 3$ and $n = 4$, decompose the wheel W_n into two isomorphic subgraphs.
- 2.4.12. A cubic graph G is decomposable into trees isomorphic to $K_{1,3}$ if and only if G is _____.
- 2.4.13. Find a graph with six edges that cannot be decomposed into two isomorphic subgraphs.

- 2.4.14. Decompose the Petersen graph of Figure 1.2.8 into three isomorphic subgraphs. (A graph or subgraph does not need to be connected.)
- 2.4.15. Decompose the cube graph Q_3 into four subgraphs with three edges each such that no two of the subgraphs are isomorphic. (Do not consider isolated vertices when determining if they are isomorphic.)
- 2.4.16. The complete bipartite graph $K_{3,4}$ has 12 edges, as does Q_3 . Find the same kind of decomposition for $K_{3,4}$ as for Q_3 in Exercise 2.4.15.
- 2.4.17. Decompose $K_{3,4}$ into two isomorphic subgraphs. (Several solutions.)
- 2.4.18. Prove that if n is even, the wheel W_n is decomposable into two isomorphic subgraphs.
- 2.4.19. Prove that if n is odd, and $n \geq 5$, then the wheel W_n is not decomposable into two isomorphic subgraphs.
- 2.4.20. Let n be divisible by s . Prove that the wheel W_n is decomposable into s isomorphic subgraphs. (Several solutions.)
- 2.4.21. Find a decomposition of K_{2n} into two subgraphs G_1 and G_2 such that

$$\chi(G_1) + \chi(G_2) = n+2.$$

(Many solutions.)

- 2.4.22. Prove theorem 2.4.1.
- 2.4.23. Decompose K_{11} into pentagons (C_5). (Hint – consider Figure 2.4.3.)
- 2.4.24. Prove that the following statements are equivalent.
- a) A graph T is connected and has no cycles.
 - b) Every edge of the graph T is a bridge.
 - c) A graph T is connected and has $q = p - 1$.
 - d) There is a unique path between any two vertices of T .
- (Equivalent means that if one of the statements is true, then the other three are also true.)
- 2.4.25. How many components does a forest with p vertices and q edges have?

Chapter 3

CIRCUITS AND CYCLES

3.1 Eulerian Circuits

The very first problem in graph theory was the Königsberg bridge problem. In Königsberg there are seven bridges over the delta area of the Pregel River. The Pregel divides Königsberg into two islands and two banks. The problem was to begin at any bridge, cross all seven bridges exactly once, and return to the starting point. No one ever succeeded in doing this, and Leonhard Euler gave a proof that this was impossible. In 1736 he wrote the first graph theory paper, about the Königsberg bridge problem.

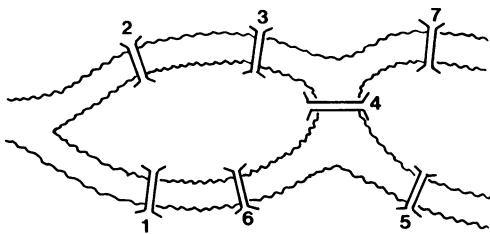


Figure 3.1.1. The Königsberg Bridges

In this section we are dealing with multigraphs and pseudographs as well as graphs. The multigraph in Figure 3.1.2 describes the Königsberg bridge problem. The problem is then to start with a pencil at one vertex and draw the entire figure without repeating any edges or lifting the pencil from the paper.

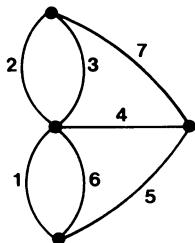


Figure 3.1.2

When we work with graphs, it is sufficient to label the vertices only, because edges can be identified using the vertices that are endpoints. Such an identification is well-defined, since there is at most one edge between any two vertices. Working with pseudographs and multigraphs, we must label edges as well if we wish to describe paths, cycles, and so on. In this section we shall represent vertices by capital letters and edges by lower-case letters.

With these conventions, a *walk* in a pseudograph G is an alternating sequence

$$(1) \quad A_1 e_1 A_2 e_2 A_3 \cdots A_{n-1} e_{n-1} A_n$$

of vertices and edges of G , with the property that every edge e_i is incident with A_i and A_{i+1} , and $A_i \neq A_{i+1}$ if e_i is not a loop. If $A_1 \neq A_n$, then the walk is an *open walk*, and if $A_1 = A_n$, then it is a *closed walk*. Vertices and edges may be repeated in a walk. The name "walk" is a good descriptive name, since one is "walking around the graph" in such a way that one must finish an edge once one has stepped on it.

A *trail* in a pseudograph G is a walk in G with the property that no edge is repeated.

A *path* in a pseudograph G is a trail in G with the property that no vertex is repeated. Notice that if G happens to be a graph, this is the same definition we have already given for a path.

For example, in Figure 3.1.3, a walk from A to C is

$$A \ t \ B \ f \ D \ u \ D \ f \ B \ j \ C.$$

This is not a trail, since the edge f is repeated. A trail from A to C is

$$A \ t \ B \ f \ D \ h \ C \ g \ D \ e \ A \ l \ C,$$

which is not a path since the vertices A , C , and D each occur more than once in

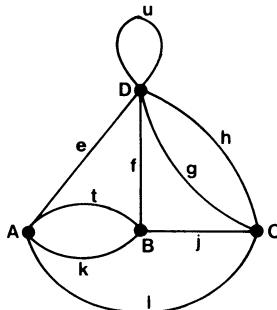


Figure 3.1.3

the trail. Finally, a path from A to C is

$$A \ t \ B \ f \ D \ h \ C.$$

The *length* of a walk W is the number of edges in W . The length of the walk (1) is $n - 1$.

A *closed trail*, also called a *circuit*, is a trail whose endpoints are the same vertex. A *closed path*, usually called a *cycle*, is a path whose endpoints are the same vertex. The shortest cycle possible in a graph has length three, and we often call this a *triangle*. In a pseudograph, however, it is possible to have a cycle of length one or two. The first is called a *loop*, and the second is often called a *lune*. The *degree* of a vertex V in a pseudograph is the number of edges incident with V , but loops are counted twice. Thus in Figure 3.1.3, vertex D has degree 6, comprised of four edges and a loop.

An *Eulerian circuit* in a pseudograph G is a circuit that contains every edge of G . For example, an Eulerian circuit in the pseudograph of Figure 3.1.3 is the circuit

$$A \ e \ D \ u \ D \ h \ C \ g \ D \ f \ B \ j \ C \ l \ A \ k \ B \ t \ A.$$

This one is not unique; there are many Eulerian circuits in the pseudograph.

The pseudograph in Figure 3.1.2, the Königsberg pseudograph, has no Eulerian circuit. Euler asked himself this question in 1736. He found the following theorem:

Theorem 3.1.1 (Euler). *If a pseudograph G has an Eulerian circuit, then G is connected and the degree of every vertex is even.*

Notice that every vertex of the pseudograph in Figure 3.1.3 has even degree, while the pseudograph in Figure 3.1.2 has some vertices of odd degree. It is easy to prove Theorem 3.1.1 by considering that if such a circuit exists, and if A is a given vertex of G , then each time A occurs in the circuit there is an edge to get in and an edge to get out. That means that if A occurs h times in the circuit, the degree of A must be $2h$. ■

More than 100 years later, Hierholzer proved that the converse of Theorem 3.1.1 is also true.

Theorem 3.1.2 (Hierholzer). *If a pseudograph G is connected, and the degree of every vertex of G is even, then G has an Eulerian circuit.*

Before we prove Theorem 3.1.2, we need the following lemma.

Lemma 3.1.3. *If every vertex in a pseudograph G has positive even degree, then any given vertex of G lies on some circuit of G .*

Proof. Suppose A is a vertex in G . We begin to walk, starting at A and taking any edge incident with A . At any vertex we come to, we can always leave by an edge we have not yet used, since if we arrive at a vertex B , and we have already traveled through B h times, this is a total of $2h + 1$ edges at the moment of arrival at B . But the degree of B is even, so there must be a way out. Since G is finite, eventually we come back to A . Thus we have found a circuit in G through A . ■

Now we are ready to prove Theorem 3.1.2.

Proof. Let G be a connected pseudograph such that the degree of every vertex of G is even. Let C be a longest circuit in G . If C contains every edge of G , then C is an Eulerian circuit, and the theorem is true. If C does not contain every edge of G , we subtract the edges of C from G and call the resulting pseudograph H . All vertices of H have even degree. Since G is connected, H and C have a vertex A in common. By Lemma 3.1.3, A lies on a circuit C_1 in the component of H that contains A . We then have

$$(C) \quad \cdots e_1 A e_2 \cdots$$

$$(C_1) \quad \cdots f_1 A f_2 \cdots .$$

We now extend C at A by inserting C_1

$$(C_2) \quad \cdots e_1 A f_2 \cdots f_1 A e_2 \cdots.$$

See also Figure 3.1.4.

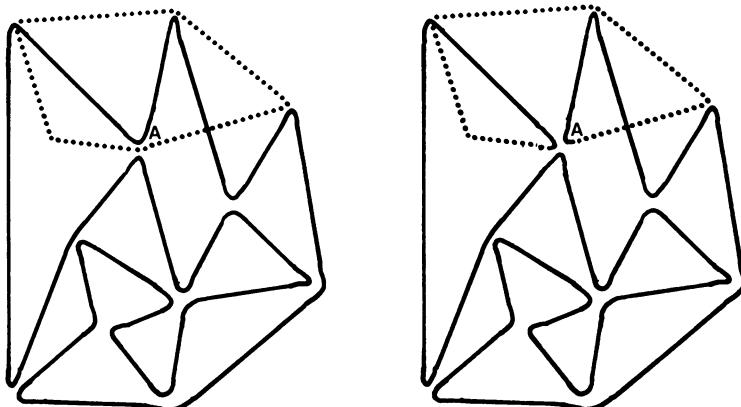


Figure 3.1.4

Then C_2 is longer than C , which is a contradiction, since we assumed C was a longest circuit in G . Thus G has an Eulerian circuit. ■

Notice that one can tell very quickly if a pseudograph has an Eulerian circuit. In contrast, it is extremely difficult to tell if a graph has a Hamilton cycle. While we are proving the results concerning Eulerian circuits for pseudographs, they are also true for graphs and are perhaps more interesting.

Recall that a 2-factor of a graph G is a spanning subgraph of G that is regular of degree 2. In other words, a 2-factor consists of one or more cycles. This definition extends to pseudographs with no problems.

The octahedron graph in Figure 3.1.5 is decomposed into two 2-factors in two different ways. In the first, each of the 2-factors consists of a hexagon. In the second, one 2-factor is a hexagon, and the other consists of two disjoint triangles. Another example of a graph that can be decomposed into two 2-factors is $K_{4,4}$. We leave the decomposition as an exercise.

These two examples are not accidents. We have a theorem that tells us that they have such decompositions.

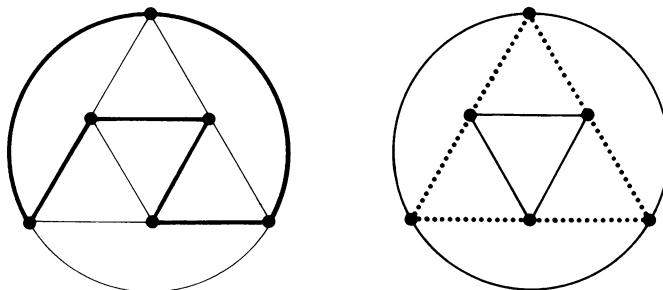


Figure 3.1.5

Theorem 3.1.4. *If a pseudograph G is regular of degree 4, then G has a decomposition into two 2-factors.*

Proof. Let G be a pseudograph that is regular of degree 4. We assume G is connected, otherwise we could consider each connected portion separately. Since every vertex of G has even degree, G has an Eulerian circuit. The length of the circuit is q , the number of edges in G . Since

$$q = \frac{4p}{2} = 2p,$$

there are an even number of edges in the circuit. We color the edges of the circuit, alternating red and blue. Then every vertex is incident with two red edges and two blue edges, unless the vertex is incident with a loop, in which case the loop forms a cycle of length one. The red edges form a 2-factor of G , and the blue edges form another 2-factor of G . ■

In case we have a pseudograph where all the vertices have even degree, we have another decomposition theorem. Look first at Figure 3.1.6. It shows a pseudograph that is decomposable into cycles. The way it is drawn, this decomposition is immediately apparent, since here a cycle looks like a circle in the drawing.

Theorem 3.1.5. *A pseudograph G has a decomposition into cycles if and only if every vertex of G has even degree.*

Proof. Assume that G has a decomposition into cycles. Consider one vertex A of G . If A belongs to h of these cycles, its degree must be $2h$.

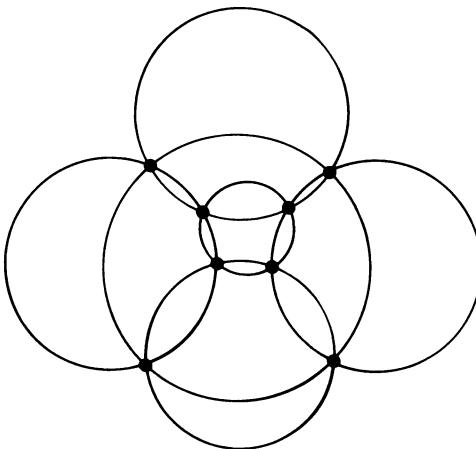


Figure 3.1.6. The Miquel configuration.

Now we assume that every vertex of G has even degree. The proof of this direction is by induction on the number of edges in G . If G has all even degree vertices and only one or two edges, the theorem is true. Suppose that the theorem is true for pseudographs with fewer than n edges and all degrees of vertices even. Let G be a pseudograph with n edges and all even degrees. By Lemma 3.1.3, there exists a circuit in G . We take C to be a shortest circuit in G . Then C must be a cycle or we could shorten it by deleting all edges between repeated vertices. Now consider G minus the edges of C . The resulting graph, denoted by $G - C$, still has all even degrees and has fewer than n edges, so by the induction hypothesis $G - C$ has a decomposition into cycles. We now add back the cycle C , thus obtaining a decomposition of G into cycles. ■

What can we say if the pseudograph has some vertices of odd degree? We certainly know that the number of vertices of odd degree is even (see Exercise 1.1.5). An *Eulerian trail* in a pseudograph G is a trail that contains every edge and every vertex of G . If G has precisely two vertices of odd degree, we have the following theorem.

Theorem 3.1.6. *A pseudograph G has an Eulerian trail if and only if G is connected and has precisely two vertices of odd degree.*

Proof. Suppose that G has an Eulerian trail T . Clearly G is connected. Let the trail begin at a vertex A and end at a vertex B . We add an edge e from A to

B . There may already be one or more edges from A to B in G , but we can still add an edge since we allow pseudographs. Then $T + e$ is an Eulerian circuit in the pseudograph $G + e$, and by Theorem 3.1.1 every vertex of $G + e$ has even degree. Thus G has exactly two vertices of odd degree, namely A and B .

Now we suppose that G is connected and has exactly two vertices A and B of odd degree. As before, we add an edge e from A to B . Then $G + e$ is connected, and every vertex has even degree. By Theorem 3.1.2 $G + e$ has an Eulerian circuit C . Now we are finished, since $C - e$ is an Eulerian trail in G .

■

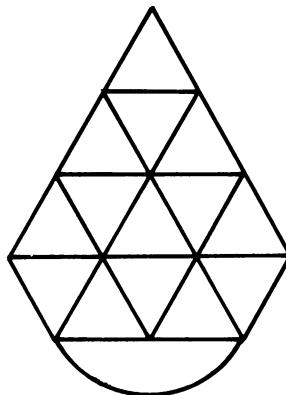


Figure 3.1.7

There is an immediate application of Theorem 3.1.6. If an artist wishes to make an ornament like the one in Figure 3.1.7, and she wants to make it from a single piece of wire, she knows she can do this if she starts at a vertex of odd degree and ends up at a vertex of odd degree. Thus this is possible when there are exactly two vertices of odd degree.

We generalize the last theorem as follows.

Theorem 3.1.7. (Listing). *If G is a connected pseudograph with precisely $2h$ vertices of odd degree, $h \neq 0$, then there exist h trails in G such that each edge of G is in exactly one of these trails. Furthermore, fewer than h trails with this property cannot be found.*

Proof. Consider the $2h$ vertices of odd degree in G , and add h new edges in such a way that a pseudograph H is obtained that has only vertices of even degree. Then H has an Eulerian circuit. If we then subtract these edges from

the circuit, the Eulerian circuit is interrupted h times, so that what is left are h trails that contain every edge of G . To see that no fewer than h such trails contain every edge of G , consider that every vertex of odd degree must be the end vertex of a trail. Thus there must be at least h trails. ■

An application of Theorem 3.1.7 can be found on the tennis court. The tennis graph (Figure 2.4.5) has 12 vertices of odd degree. If the groundskeeper for a tennis court is using his machine to put the chalk lines on the tennis court, he must interrupt the machine at least six times. If he thinks that it is smart to start at the corner, he is wrong. The corners have even degree! He should begin at a vertex of odd degree.

Exercises 3.1

- 3.1.1. Which graphs of Figure 1.2.5 are Eulerian? Find the Eulerian circuit in them.
- 3.1.2. Find a decomposition into cycles of $K_{4,6}$.
- 3.1.3. Decompose $K_{4,4}$ into two 2-factors.

In the exercises below, the vertices of odd degree play an important role.

- 3.1.4. What is the length of the longest trail in each of the graphs in Figure 1.2.5?
- 3.1.5. Find a decomposition of the Grötzsch graph (Figure 2.1.6) into three paths. (Of course they cannot be of equal length.)
- 3.1.6. Prove that a regular graph of degree 5 cannot be decomposed into subgraphs, each isomorphic to a path of length six.
- 3.1.7. If a cubic graph is decomposed into paths, then the average length of these paths is at most three. Prove it.
- 3.1.8. Referring to Problem 3.1.7, is it always possible to find a decomposition into paths with average length exactly three?
- 3.1.9. Decompose the octahedron graph of Figure 1.2.5 into three paths of length four.
- 3.1.10. Find an example of a graph regular of degree 4 that is decomposable into paths with average length greater than four.
- 3.1.11. Five international delegations meet at a round table on one evening. There are two Americans, two Frenchmen, two Germans, two

Englishmen, and two Italians. Each possible pair of nationalities must occur in the seating. That is, there must be an American seated next to a Frenchman, a German seated next to an Italian, and so on. How would you arrange the seating order? (Hint — find an Eulerian circuit in K_5 .)

- 3.1.12. The same situation occurs as in 3.1.11, but the table has an obstruction between two of the seats, and you know that the two Frenchman do not like the two Italians and vice versa. Now how do you arrange the seating order?
- *3.1.13. Show that if every edge of a pseudograph G lies on an odd number of cycles, then G has an Eulerian circuit. (McKee).
- 3.1.14. The craftsman wishes to make the ornament in Figure 3.1.7 from one piece of wire, and he does not want the wire to cross itself. Is this possible? If so, show how.
- 3.1.15. Fill in the blank. $K_{m,n}$ is decomposable into cycles if and only if _____.
- 3.1.16. A graph G has only vertices of degree 3 and degree 4 and is decomposable into two spanning trees. How many vertices of degree 3 are in G ?

3.2. The Oberwolfach Problem

Exercise 2.3.8 was a special case of the following problem. In a meeting there are $2n + 1$ participants, and they have dinner at a round table n times with different neighbors each time. Normally, however, they do not have such a big table. In fact, at Oberwolfach, in Germany, there is a mathematical institute, and most of the tables have six places. Every week they have meetings at Oberwolfach on different mathematical subjects. About every two years there is a meeting on graph theory. They have actually tried to have everyone sit next to everyone else at least once for the various meals at these meetings. At the meetings, this problem has not actually been solved. Ideally, the problem is, given

$$2n + 1 = t_1 + t_2 + \cdots + t_s$$

and s round tables T_1, T_2, \dots, T_s such that table T_i seats t_i people, $2n + 1$ people should have dinner for n nights so that after a meeting everyone has sat next to everyone else. Gerhard Ringel gave this problem the name

"Oberwolfach Problem." It is not completely solved, but many papers have been published about it, and many cases are solved. For example, if $s = 3$ the problem is solved.

In graph theoretical language, the Oberwolfach Problem is as follows. Given a graph T with $2n + 1$ vertices that is regular of degree 2 (we know this is a graph consisting of one or more cycles of various sizes), decompose the complete graph K_{2n+1} into n subgraphs isomorphic to T . We are using " T " to denote the table graph.

If there is only one table, the graph theoretical version is that K_{2n+1} has a decomposition into Hamilton cycles. This is solved (see Theorem 2.3.1).

We present some examples of solutions to the problem. Each is labeled by the symbol (t_1, t_2, \dots, t_s) , where there are t_i seats at table T_i . Each of the following examples uses the turning trick.

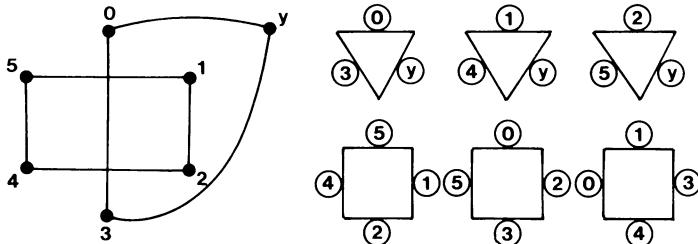


Figure 3.2.1

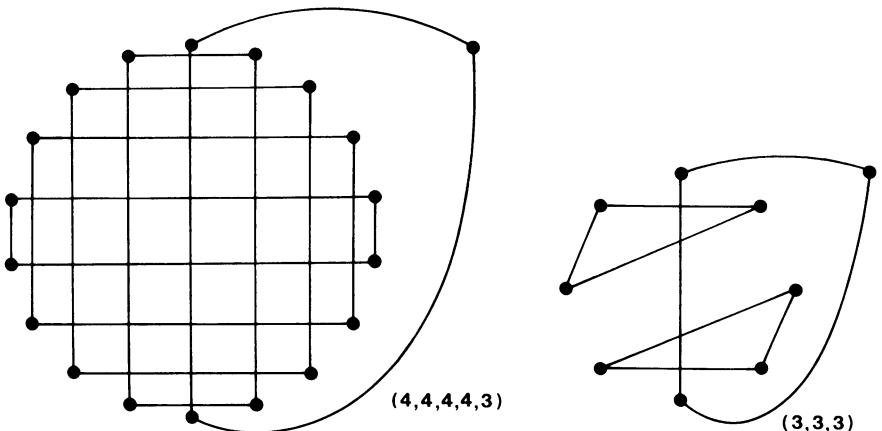


Figure 3.2.2

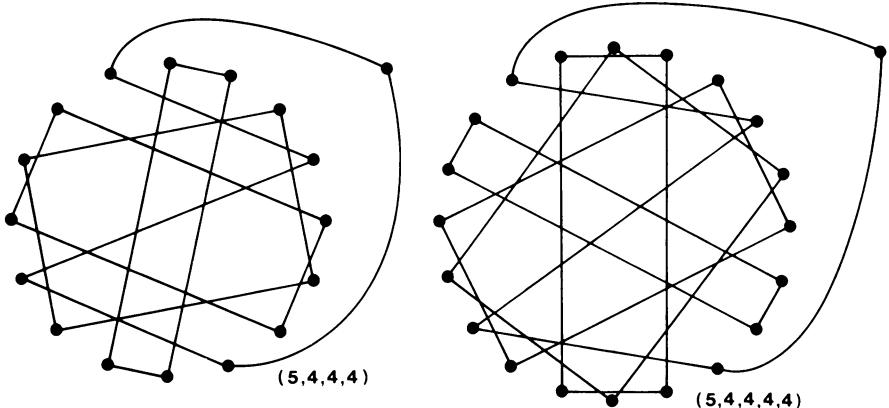


Figure 3.2.3

The solutions in Figure 3.2.1 and the left of Figure 3.2.2 generalize as follows. If there are k tables that seat four and one table that seats three, the first seating is

$$\begin{aligned}
 (T_1) \quad & (0, 2k+1, \dots, y) \\
 (T_2) \quad & (k, k+1, 3k+1, 3k+2) \\
 (T_3) \quad & (k-1, k+2, 3k, 3k+3) \\
 \cdot & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 \cdot & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 \cdot & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 (T_{k+1}) \quad & (1, 2k, 2k+2, 4k+1).
 \end{aligned}$$

The next seating is obtained by turning by one unit, which means add one to each number, y stays the same, and, after adding one, replace $4k + 2$ by 0.

Regular graphs.

Now we wish to speak a little bit about regular graphs. Recall that a bridge in a connected graph G is an edge whose removal disconnects G .

Theorem 3.2.1. *A regular graph of even degree has no bridge.*

Proof. Suppose that the theorem is not true. Then there is a graph G that is regular of degree $2h$, and G has a bridge b . Then $G - b$ has two banks, and

each bank has exactly one vertex of odd degree, which is impossible by Exercise 1.1.5. Thus G has no bridge. ■

Remark. More generally, a graph that has all even degree vertices has no bridge.

We have seen examples of cubic graphs with bridges, such as the graph of Figures 2.3.7 and 2.3.8.

Theorem 3.2.2. *A cubic graph that contains a bridge is not decomposable into three 1-factors.*

Proof. If such a decomposition existed, the bridge would lie in precisely one of the 1-factors. If we delete the 1-factor containing the bridge, each of the other two 1-factors consists of a 1-factor of the first bank together with a 1-factor of the second bank. But each bank has an odd number of vertices and cannot possibly have a 1-factor. Thus no such decomposition exists. ■

The cubic graph of Figure 2.2.6 does not even have a 1-factor. By these examples, we see that it is difficult to make positive statements about cubic graphs that have bridges. The situation is quite different for cubic bridgeless graphs.

Recall that a snark is a cubic graph with edge chromatic number four, which means that a snark is not decomposable into three 1-factors. However, a bridgeless snark always has a 1-factor.

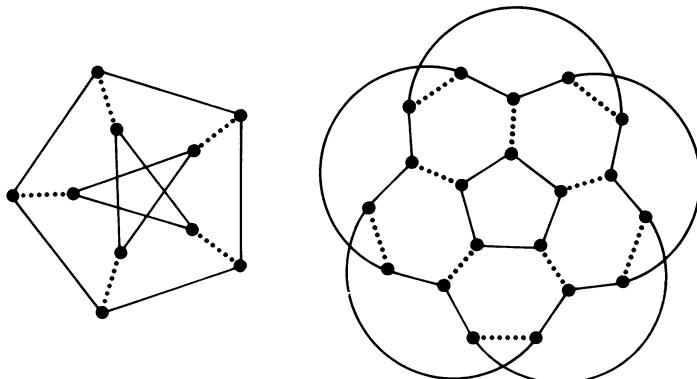


Figure 3.2.4

Theorem 3.2.3 (Petersen) [18]. A cubic bridgeless graph G has a decomposition into a 1-factor and a 2-factor.

We know that a 2-factor is a union of cycles. If all the cycles in the above decomposition are of even length, then G has a decomposition into three 1-factors.

We shall not prove Petersen's Theorem. We shall, however, see it again when we discuss the Four Color Problem. Petersen's Theorem tells us that any cubic bridgeless graph can be constructed by taking some cycles and connecting every vertex with one other vertex, either from the same cycle or from some other cycle. Figure 3.2.5 is an example. Petersen's Theorem could be written in the form *Every cubic bridgeless graph has a 2-factor*.

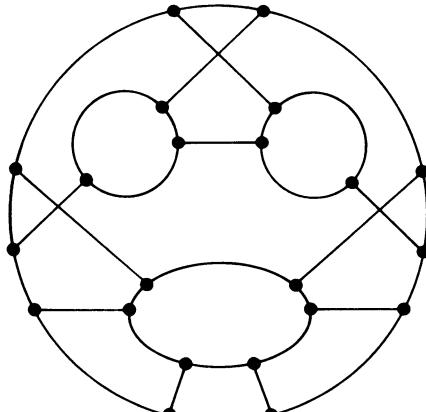


Figure 3.2.5

An interesting application of Petersen's Theorem is the proof of the following.

Theorem 3.2.4. Every cubic bridgeless graph is decomposable into paths of length three.

As an example, see Figure 2.4.1 where we used the turning trick to decompose the Petersen graph into P_3 subgraphs.

Proof. Given a cubic bridgeless graph G , Petersen's Theorem tells us that the edges can be colored red and blue so that every vertex is incident with one blue edge and two red edges. The example of Figure 3.2.5 is drawn in such a way

that the blue edges appear to be straight lines, while the red edges appear to be curved. We now number the blue edges 1, 2, and so on in any order. See Figure 3.2.6. Then we consider a cycle formed by red edges. We travel around the cycle, numbering each red edge in the cycle by the same number as the blue edge incident with the beginning of that red edge. If we do this with every red cycle, every number appears on three edges, forming a P_3 . Thus we obtain a decomposition of G into subgraphs isomorphic to P_3 . ■

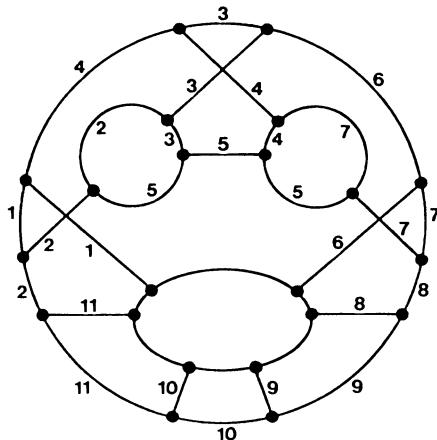


Figure 3.2.6

A famous conjecture of Berge was recently proved by Zhang Limin.

Theorem 3.2.5 (Berge, Zhang) [25]. *Every 4-regular graph contains a 3-regular subgraph.*

The proof is quite long, even harder than the proof of Petersen's theorem. We would like to mention some easy examples. K_5 is 4-regular and contains a subgraph isomorphic to K_4 , that is 3-regular. The octahedron graph O of Figure 3.2.7 has a 1-factor and therefore also has a 3-factor that is a cubic subgraph. Figure 3.2.8 is a larger example.

Exercises 3.2

- 3.2.1. Find two different decompositions of the cube graph Q_3 into a 2-factor and a 1-factor.
- 3.2.2. Find a decomposition as promised by Petersen's Theorem for the graph of Figure 3.2.9.

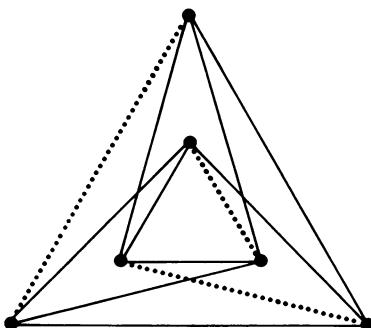


Figure 3.2.7

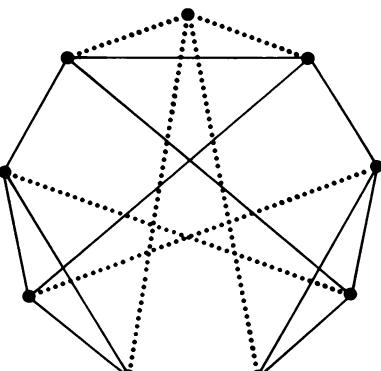


Figure 3.2.8

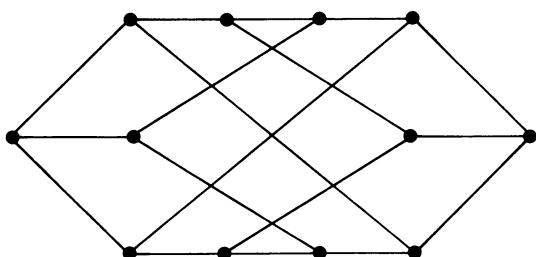


Figure 3.2.9

- 3.2.3. The following is a method for arranging eight players for an evening of bridge. The first seating is assigned, so we assume that at Table I players 0 and 1 are partners, and players 2 and 3 are partners. At Table II players 4 and 5 are partners, and players 6 and 7 are partners.

After some number of hands are played, the high-scoring team from each table moves to the other table and changes partners for the next round. How many rounds can be played before someone must have the same partner? (How many disjoint 1-factors in K_8 ?)

- 3.2.4. Nine players wish to play bridge for an evening. One player must sit out for each round. Is there an equitable way of arranging the seating? (See problem 3.2.3.)
- 3.2.5. Design a method for twelve players to play an evening of bridge at three tables, and answer the question of problem 3.2.3.
- *3.2.6. Solve the Oberwolfach Problem for 11 people and two tables, one seats three and one seats eight.
- *3.2.7. Solve the Oberwolfach Problem for 27 people, where all the tables seat three.
- 3.2.8. Show that Theorem 3.2.4 is not true if the word "bridgeless" is omitted. (Hint – look at cubic graphs that have bridges.)
- 3.2.9. Find a decomposition of K_{11} into 11 C_5 subgraphs.

3.3. Infinite Lattice Graphs

This is the only section where we consider infinite graphs. Define the infinite graph L_2 in the following way. The vertices are all the points in the plane whose coordinates are both integers, and two vertices are adjacent if they have geometric distance equal to one. A portion of L_2 is pictured in Figure 3.3.1. We shall not include a picture of the whole graph.

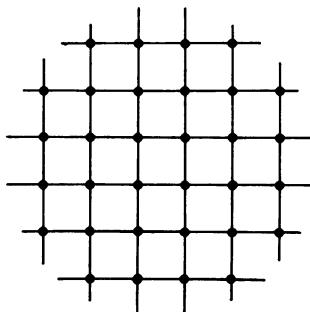


Figure 3.3.1

In an infinite graph we cannot have a Hamilton cycle as defined in Section 2.3 because a cycle has only finitely many vertices. If, however, we think of a

Hamilton cycle as being a connected 2-factor of a graph, this makes sense for infinite graphs as well as for finite graphs. For finite graphs, a connected 2-factor is a Hamilton cycle, and for infinite graphs, it is a spanning path with no end. In other words, the path is infinite in both directions. We shall call such an infinite spanning path a *Hamilton line*.

There are many different ways to find a Hamilton line in L_2 . One of these ways is pictured in Figure 3.3.2. Figure 3.3.3 shows a decomposition of L_2 into two Hamilton lines.

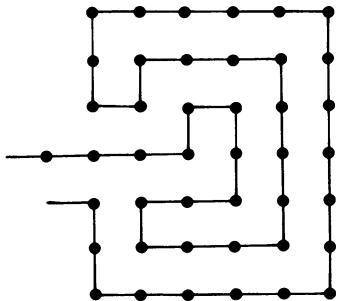


Figure 3.3.2

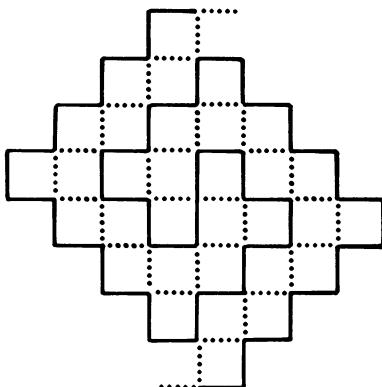


Figure 3.3.3

An Eulerian circuit must also be modified slightly if we wish to extend the definition for infinite graphs. An *Eulerian line* is an Eulerian trail that is infinite in both directions. In other words, it is a trail that contains every edge and every vertex of the infinite graph and has no beginning vertex or ending vertex. In more practical terms, an infinite graph with an Eulerian line can be manufactured out of one infinite piece of wire.

Figure 3.3.4 shows a simple example of an Eulerian line in L_2 . The Eulerian line in Figure 3.3.5, however, is quite sophisticated. It is, in fact, central-symmetric. The center is in the middle of the middle edge. That means that if you turn the figure 180° about the center, the picture will still look the same. Figure 3.3.3 is also central-symmetric about the vertex where the two Hamilton lines cross. If you turn only 90°, the two Hamilton lines change places.

Both Eulerian circuits and Eulerian trails are defined for finite graphs. We have seen how to translate the first to infinite graphs. An Eulerian trail has two

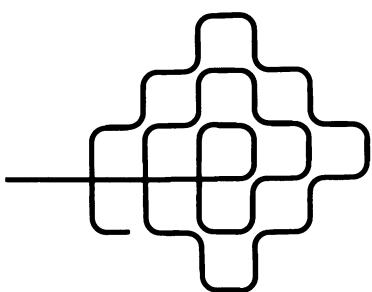


Figure 3.3.4

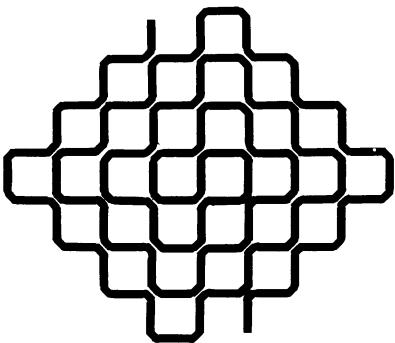


Figure 3.3.5

endpoints in a finite graph, and in an infinite graph we can save one of them. A trail in an infinite graph that begins at one vertex y , has infinite length, and includes every edge of the graph exactly once, is called a *one-way Eulerian trail*. It is clear that the vertex y must have odd degree, and the rest of the vertices must have even degree. However, this is not sufficient. See Exercise 3.3.17. The reader will have an opportunity to find a one-way Eulerian trail in the infinite spiderweb graph in Figure 3.3.6.

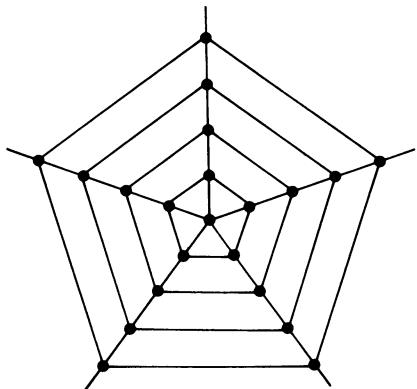


Figure 3.3.6

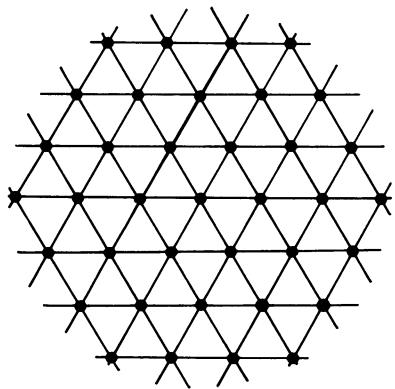


Figure 3.3.7

A *one-way Hamilton path* in an infinite graph is a path of infinite length, starting at a vertex z and passing through every vertex of the graph. The degree of z in the graph does not matter.

Exercises 3.3

- 3.3.1. In the graph L_2 , a Hamilton line is a spanning tree with vertices only of degree 2. Find a spanning tree T of L_2 such that T has *no* vertices of degree 2.
- 3.3.2. Find a spanning subgraph in L_2 where each vertex has degree 3.
- 3.3.3. Find a finite connected subgraph H in L_2 such that the vertices of H have an average degree of three. (There are many solutions, but one is nice.)
- 3.3.4. What is the chromatic number of L_2 ?
- 3.4.5. What is the girth of L_2 ?
- 3.3.6. Find a decomposition of L_2 into four 1-factors.
- 3.3.7. Find a decomposition of L_2 into subgraphs isomorphic to
- C_4
 - C_8
 - C_{12}
 - In general, C_{4h} .
- 3.3.8. Prove that it is not possible to decompose L_2 into subgraphs isomorphic to C_6 .
- 3.3.9. Given an integer $t \geq 1$, prove that L_2 is decomposable into subgraphs isomorphic to P_t , the path of length t .
- 3.3.10. Decompose L_2 into
 - three connected subgraphs.
 - *three isomorphic connected subgraphs.
- 3.3.11. Let r, s be a fixed pair of integers. L_2 is decomposed into subgraphs, each isomorphic to $K_{r,s}$. For what pairs r, s is this possible? Find all such pairs, and draw the decompositions.
- 3.3.12. Find a finite subgraph of L_2 with a maximum number of edges but with diameter
 - $d = 3$.
 - $d = 4$. (Two solutions.)
 - $d = 5$.
- 3.3.13. Define M_2 to be the lattice comprised of integer points in the plane such that points with geometric distance $\sqrt{2}$ are adjacent. Is M_2 connected?

- 3.3.14 Define N_2 to be the lattice comprised of integer points in the plane such that points with geometric distance $\sqrt{5}$ are adjacent. Is N_2 connected?
- 3.3.15 Let v be any vertex of L_2 . Find a Hamilton line in L_2-v .
- 3.3.16 Find a decomposition into Hamilton lines of the infinite triangular lattice graph pictured in Figure 3.3.7.
- 3.3.17 Show by an example that there is an infinite graph with all degrees even and no Eulerian line.
- 3.3.18 Find a one-way Hamilton path in L_2 .
- 3.3.19 Find a one-way Eulerian trail in the graph of Figure 3.3.6.
- 3.3.20 Find a one-way Hamilton path in the graph of Figure 3.3.7.

Chapter 4

EXTREMAL PROBLEMS

4.1 A Theorem of Turan

Extremal problems are problems that ask what is the largest or smallest graph with a specified property. Usually we shall use *largest* to mean the graph with the most edges, given the number of vertices. The number of vertices in a graph G is called the *order* of G . The number of edges in G is called the *size* of G . An *induced subgraph* of a graph G is a subgraph of G obtained by taking a subset W of the vertices of G together with *every* edge of G that has both endpoints in W . In Figure 1.2.1, the subgraph H_2 is induced.

Problem. Find a largest graph G with n vertices and chromatic number two.

Since G has chromatic number two, the vertices are colored by two colors, red and blue. If there is a red vertex that is not adjacent to a blue vertex, we can add this edge and increase the number of edges. Thus we know every blue vertex is adjacent to every red vertex, and we have a complete bipartite graph. Suppose there are m_1 blue and m_2 red vertices, with $m_1 + m_2 = n$. We shall show that m_1 and m_2 are as close to equal as possible, which means that they are either equal or only one apart. In other words, $|m_1 - m_2| \leq 1$.

Suppose that $m_2 \leq m_1$. If $m_2 + 2 \leq m_1$, then we choose another complete bipartite graph \hat{G} with $m_2 + 1$ red vertices and $m_1 - 1$ blue vertices. Then G has $m_1 m_2$ edges, and \hat{G} has $(m_1 - 1)(m_2 + 1)$ edges. We have

$$(m_1 - 1)(m_2 + 1) - m_1 m_2 = m_1 - m_2 - 1.$$

Since we assumed $m_2 \leq m_1 - 2$, $m_1 - m_2 - 1 \geq 1$, and thus \hat{G} has at least one more edge than G , is still colorable by two colors, and has n vertices. This shows that m_1 and m_2 differ by at most one. Referring to the first problem, where $n = 2h$, it means m_1 and m_2 are both equal to h .

There is a way to write the result without reference to two cases, odd and even. Using Gauss' symbol for the greatest integer function

$$\lfloor x \rfloor = \text{the greatest integer } \leq x.$$

This notation is now outdated, and we use the notation

$$\lfloor x \rfloor = \text{the greatest integer } \leq x.$$

So, for example,

$$\lfloor 5 \rfloor = 5, \lfloor \pi \rfloor = 3, \lfloor \frac{1}{2} \rfloor = 0, \lfloor \frac{200}{99} \rfloor = 2.$$

We shall refer to $\lfloor x \rfloor$ as the *floor symbol*. The symbol $\lceil x \rceil$ is defined

$$\lceil x \rceil = \text{the smallest integer } \geq x.$$

This is called the *ceiling symbol*. For example,

$$\lceil 5 \rceil = 5, \quad \lceil \pi \rceil = 4, \quad \lceil \frac{1}{2} \rceil = 1, \quad \lceil \frac{200}{99} \rceil = 3.$$

Using this notation, the graph G that is the solution to our problem is the graph K_{m_1, m_2} with $m_1 = \lfloor \frac{n}{2} \rfloor$ and $m_2 = \lceil \frac{n+1}{2} \rceil$.

The following discussion is a preparation for solving a more general problem.

Suppose that n is some positive integer, and we wish to have $n = n_1 + n_2 + \dots + n_k$, with all the n_i 's as close in value as possible. Clearly if n is divisible by k , they will all be equal. If n is not divisible by k , then any two of them will differ by no more than one. That is, $|n_i - n_j| \leq 1$ for all i and j . For example, for $n = 15$ and $k = 4$,

$$15 = 3 + 4 + 4 + 4$$

is the solution. Again, using the greatest integer symbol,

$$15 = \left\lfloor \frac{15}{4} \right\rfloor + \left\lfloor \frac{16}{4} \right\rfloor + \left\lfloor \frac{17}{4} \right\rfloor + \left\lfloor \frac{18}{4} \right\rfloor$$

or in general

$$n = \left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{n+1}{k} \right\rfloor + \left\lfloor \frac{n+2}{k} \right\rfloor + \cdots + \left\lfloor \frac{n+k-1}{k} \right\rfloor.$$

This notation is rather clumsy, so we shall use $n = n_1 + n_2 + \cdots + n_k$, with $|n_i - n_j| \leq 1$.

Problem. *Find the largest graph G with n vertices and chromatic number k .*

Since G is k -chromatic, the vertices of G can be colored with k colors. Let m_i be the number of vertices colored with color i for $i = 1, 2, \dots, k$. Then $n = m_1 + m_2 + \cdots + m_k$. Since G has a maximum number of edges, every pair of vertices that are colored with different colors is adjacent. G is then the complete k -partite graph K_{m_1, m_2, \dots, m_k} .

If we denote the number of edges in a k -partite graph K_{m_1, m_2, \dots, m_k} by $A(m_1, m_2, \dots, m_k)$, then

$$A(m_1, m_2, m_3, \dots, m_k) = m_1 m_2 + A(m_1 + m_2, m_3, \dots, m_k).$$

Suppose that in G any two of these numbers differ by more than one, say $m_2 + 2 \leq m_1$. Then we consider a new graph $\hat{G} = K_{m_1-1, m_2+1, m_3, \dots, m_k}$. The number of edges in \hat{G} is

$$A(m_1-1, m_2+1, m_3, \dots, m_k) = (m_1-1)(m_2+1) + A(m_1+m_2, m_3, \dots, m_k).$$

The number of edges in \hat{G} minus the number of edges in G is

$$(m_1-1)(m_2+1) - m_1 m_2 = m_1 - m_2 - 1.$$

Since $m_1 \geq m_2 + 2$, we have $m_1 - m_2 - 1 \geq 1$. So \hat{G} has at least one more edge than G , hence G could not have the maximum number of edges. The assumption that $m_2 + 2 \leq m_1$ leads to a contradiction, thus any two of the numbers m_i and m_j differ by at most one.

We have just proved

Theorem 4.1.1. *The largest graph with n vertices and chromatic number k is a complete k -partite graph K_{n_1, n_2, \dots, n_k} with $n = n_1 + n_2 + \cdots + n_k$ and $|n_i - n_j| \leq 1$.*

Now we are ready to discuss Turan's Theorem.

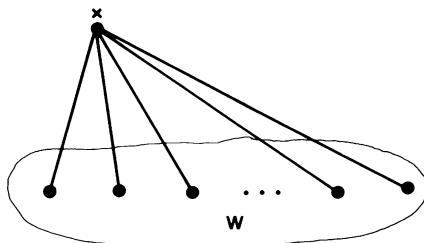
Theorem 4.1.2 (Turan). *The largest graph with n vertices that contains no subgraph isomorphic to K_{k+1} is a complete k -partite graph K_{n_1, n_2, \dots, n_k} with $n = n_1 + n_2 + \dots + n_k$ and $|n_i - n_j| \leq 1$.*

The conclusion of Theorem 4.1.2 is the same as that of Theorem 4.1.1, but the assumption is weaker. The assumption of Theorem 4.1.1 implies the assumption in Theorem 4.1.2, but not conversely. Certainly no graph with chromatic number k contains a subgraph isomorphic to K_{k+1} , but, for instance, a cycle of length five contains no K_3 and still has chromatic number three.

First we consider the special case of Turan's Theorem where k is equal to two.

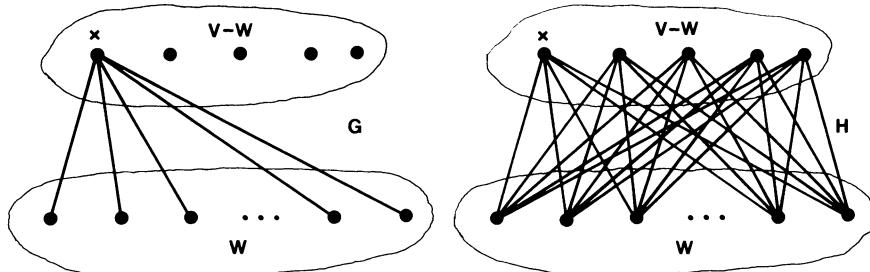
Theorem 4.1.2*. *The largest graph with n vertices that contains no triangle is the complete bipartite graph K_{n_1, n_2} , with $n = n_1 + n_2$, and $|n_1 - n_2| \leq 1$.*

Proof. Let G be a graph with n vertices that does not contain a triangle. Let V be the vertex set of G . We choose a vertex x of G with the largest degree in G , that is, $\deg_G(x)$ is maximal. Consider the set W of vertices of G that are adjacent to x .



W is sometimes called the *neighborhood of x* or the *neighbor set of x* . No two vertices in W can be adjacent, or G would contain a triangle.

Define a new graph H with the same set of vertices V as G , and let W be the same subset of V as above, such that in H every vertex in $V - W$ is adjacent to every vertex in W , and no other edges are in H . In other words, H is a complete bipartite graph.



If z is a vertex in $V - W$, then

$$\deg_H(z) = \deg_H(x) = \deg_G(x) \geq \deg_G(z),$$

since we chose x to be a vertex of maximal degree in G . Suppose that the number of vertices in W is w . Then if z is a vertex in W ,

$$\deg_H(z) = n - w \geq \deg_G(z),$$

since we showed that no two vertices in W could be adjacent in G , or G would contain a triangle. Thus we have found a new graph H such that for every vertex z of G , $\deg_G(z) \leq \deg_H(z)$. This means that the total number of edges in H must be at least the number of edges in G . H is a complete bipartite graph, and we have seen in the proof of Theorem 4.1.1 that the largest complete bipartite graph with n vertices is K_{n_1, n_2} with $n = n_1 + n_2$ and $|n_1 - n_2| \leq 1$. ■

The proof of the more general theorem requires the following lemma.

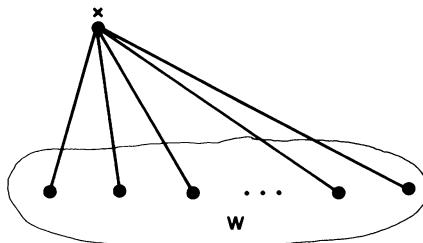
Lemma 4.1.3. *If G is a graph on n vertices that contains no K_{k+1} then there is a k -partite graph H with the same vertex set as G such that*

$$\deg_G(z) \leq \deg_H(z)$$

for every vertex z of G .

Proof. The proof is by induction on k . The case $k = 2$ is contained in the proof of Theorem 4.1.2*.

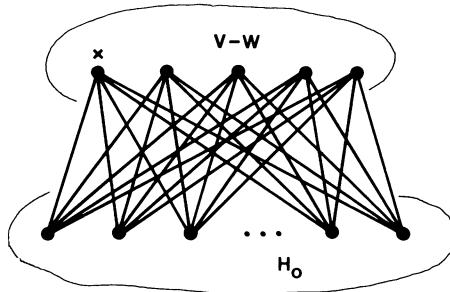
Suppose that the lemma is true for all values less than k , and suppose that G is a graph with n vertices not containing K_{k+1} . Let V be the vertex set of G , and let x be a vertex of G with $\deg_G(x)$ maximal. Let W be the set of vertices adjacent to x in G , and let G_0 be the subgraph induced by the set W .



G_0 cannot contain a subgraph isomorphic to K_k ; otherwise G would contain a subgraph isomorphic to K_{k+1} , since x is adjacent to all the vertices of W . Thus the lemma holds for G_0 , and there exists a $(k-1)$ -partite graph H_0 such that

$$\deg_{G_0}(z) \leq \deg_{H_0}(z)$$

for every vertex z in W . Take the vertices in $V - W$ and connect each of them to every vertex in H_0 to form a new graph H .



For every vertex z in $V - W$ we have

$$\deg_G(z) \leq \deg_G(x),$$

since x was the vertex of maximal degree in G , and

$$\deg_G(x) = \deg_H(x) = \deg_H(z),$$

because we connected every vertex in $V - W$ to every vertex in W to construct H . Thus for every vertex z in $V - W$

$$\deg_G(z) \leq \deg_H(z).$$

Now let z be a vertex in W . Let w be the number of vertices in W . Then

$$\deg_G(z) \leq \deg_{G_0}(z) + n - w,$$

since z can be adjacent to at most $n - w$ of the vertices in $V - W$ in G . Also,

$$\deg_{G_0}(z) + n - w \leq \deg_{H_0}(z) + n - w$$

by the induction hypothesis. Finally

$$\deg_{H_0}(z) + n - w = \deg_H(z),$$

because we constructed H by connecting each vertex in W to every vertex in $V - W$. Thus

$$\deg_G(z) \leq \deg_H(z)$$

for all vertices z in W . Since the above inequality holds for all z in V , the lemma is proved. ■

In the proof of Theorem 4.1.1, we showed that the largest k -partite graph with n vertices is the complete k -partite graph K_{n_1, n_2, \dots, n_k} with $|n_i - n_j| \leq 1$.

Theorem 4.1.2 is now proved as follows. Lemma 4.1.3 shows that if G is a graph with n vertices that does not contain a subgraph isomorphic to K_k , then G can be replaced by a k -partite graph without decreasing the number of edges. Since the largest k -partite graph with n vertices is the complete k -partite graph above, the theorem is proved.

If n and k are given, this largest graph is called the *Turan graph* $T_{n,k}$.

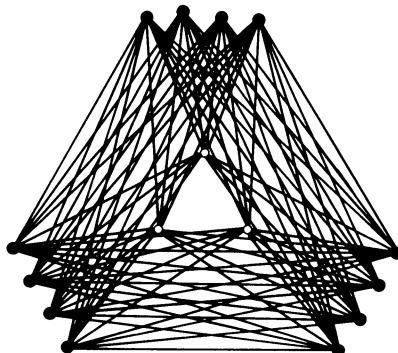


Figure 4.1.1. The Turan graph $T_{15,4}$.

The octahedron graph of Figure 1.2.5 is also the graph $T_{6,3}$, the largest graph on six vertices that contains no K_4 .

Exercises 4.1

- 4.1.1. Find a largest graph with eight vertices and girth six.

- 4.1.2. Find a largest graph with six vertices and diameter four.
- 4.1.3. Find a largest graph with seven vertices and diameter four.
- 4.1.4. Find a largest graph with seven vertices and no cycle of length greater than three.
- 4.1.5. Find a largest graph with seven vertices and girth four.
- 4.1.6. Find a largest graph with ten vertices and exactly one triangle (cycle of length three).
- 4.1.7. Find a largest graph with ten vertices and exactly one cycle of odd length. (No proof is expected; just make a good guess.)
- *4.1.8. Let G be a graph with n vertices. Prove that G has $2^n - 1$ induced subgraphs.

4.2. Cages

Problem. *Find the smallest cubic graph G with girth g .*

If g is three, then $G = K_4$, and if g is four, then $G = K_{3,3}$. For larger values of g , the answers are no longer so easy to obtain, and some very interesting graphs arise. A smallest cubic graph with girth g is called a g -cage. Thus K_4 is a 3-cage, and $K_{3,3}$ is a 4-cage. In fact, K_4 is the unique 3-cage, which means that any 3-cage is isomorphic to K_4 . $K_{3,3}$ is the unique 4-cage.

Theorem 4.2.1. *The Petersen graph is the unique 5-cage.*

Proof. Let G be any 5-cage. Fix a vertex of G , that lies on a 5-cycle, and call it 1. Since G is cubic, 1 has three neighbors; call them 2, 3, and 4. Each of these vertices has two additional neighbors, as shown in Figure 4.2.1. Now none of vertices 5, 6, 7, 8, 9, or 10 can be equal to any of vertices 2, 3, or 4, since then G would contain a triangle. Moreover, all ten vertices must be distinct, or G would contain a cycle of length at most four, and thus there are at least ten vertices in G . We shall now try assuming that G has no more than ten vertices. Consider vertex 5. It must be adjacent to two more vertices, and it cannot be adjacent to vertex 6, or we get a triangle. It can only be adjacent to one of 7 or 8 and one of 9 or 10. Without loss of generality, we assume 5 is adjacent to 7 and 9. Similarly, vertex 6 cannot be adjacent to 7 or 9, or we get a 4-cycle, and thus 6 is adjacent to 8 and 10. Vertex 7 cannot be adjacent to 8 or

9, or we get a triangle. Thus 7 must be adjacent to 10, and finally, 8 must be adjacent to 9. The resulting cubic graph has girth five. We showed that a 5-cage must have at least ten vertices, and here is a cubic graph of girth five with ten vertices. Thus G is a 5-cage. We must now show that it is unique. We redraw Figure 4.2.2, and we can recognize that G is isomorphic to the Petersen graph. Since each step of the construction was in broadest generality, the Petersen graph is the unique 5-cage; any other 5-cage is isomorphic to the Petersen graph.

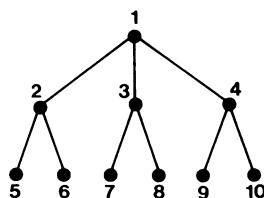


Figure 4.2.1

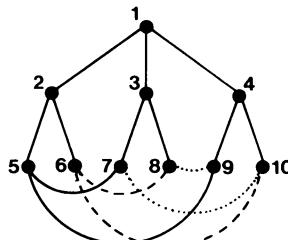


Figure 4.2.2

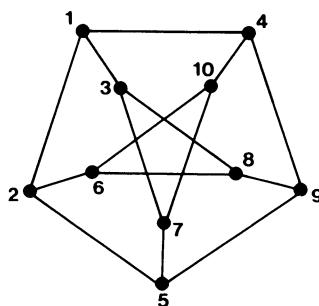


Figure 4.2.3. The Petersen graph.

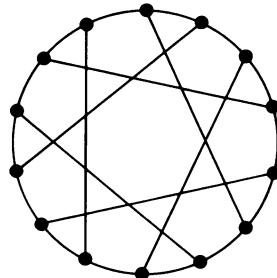


Figure 4.2.4. The Heawood graph (1890).

Theorem 4.2.2. *The Heawood graph of Figure 4.2.4 is the unique 6-cage.*

Proof. Let G be any 6-cage. Fix any edge e of G , and label its endpoints 1 and 2. Since G is cubic, each endpoint is adjacent to two distinct vertices, as in Figure 4.2.5. None of the vertices 3, 4, 5, 6 can be adjacent to each other, since otherwise there would be a cycle of length smaller than six. In fact, all 14 vertices in Figure 4.2.5 must be distinct for the same reason. Thus a 6-cage has at least 14 vertices. Assume that there is a 6-cage with exactly 14 vertices. Vertex 7 should be adjacent to vertices 11 and 13, 8 should be adjacent to 12 and 14, 9 should be adjacent to 11 and 14 and 10 should be adjacent to 12 and 13. We leave it to the reader to complete the proof by completing Figure 4.2.5 and labeling the Heawood graph accordingly.

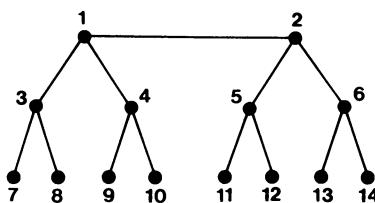


Figure 4.2.5

The unique 7-cage is the McGee graph of Figure 4.2.6. The unique 8-cage is the Tutte-Coxeter graph shown in Figure 4.2.7. The 9-cage is not known. There are three distinct 10-cages, one of which is pictured in Figure 4.2.8. The 12-cage has 126 vertices. These are the only known cages.

There are also some results concerning minimal v -regular graphs of girth g .

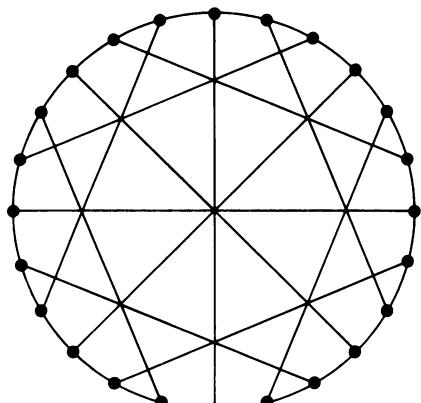


Figure 4.2.6. The McGee Cage

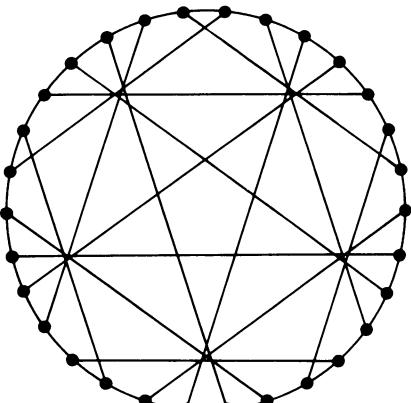


Figure 4.2.7. The Tutte-Coxeter Cage

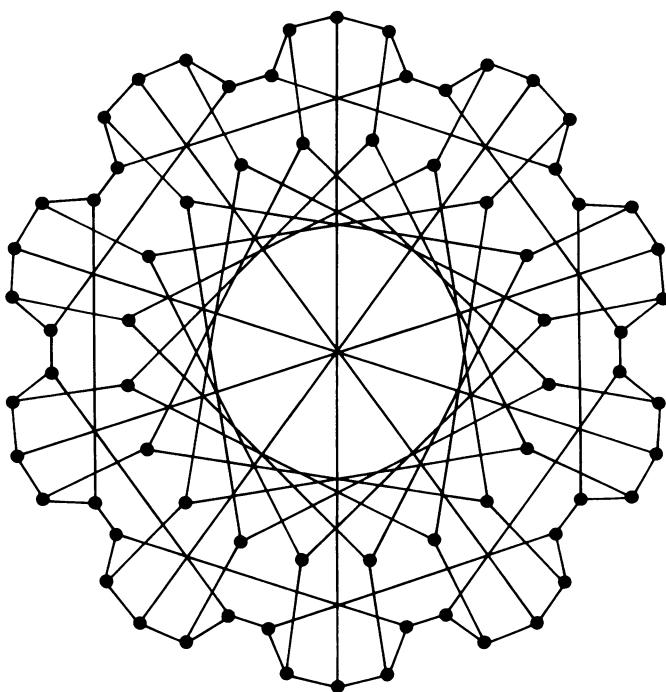


Figure 4.2.8. Balaban's 10-Cage

Exercises 4.2

- 4.2.1. Which of the cages shown have Hamilton cycles?
- 4.2.2. Find the largest graph with n vertices and diameter
 - a) $d = 1$.
 - b) $d \neq 1$.
 - c) $d = 3$.
- 4.2.3. Find a largest graph with ten vertices, such that each vertex has even degree.
- 4.2.4. Find a graph G with eight edges and a minimal number of vertices, such that G has no triangle. (There are two solutions.)
- 4.2.5. Find a largest graph with twelve vertices and diameter
 - a) $d = 10$.
 - b) $d = 9$. (Many solutions.)
- 4.2.6. Given $K_{m,n}$ with $n \leq m$, how long is the longest
 - a) path?
 - b) cycle?
 - *c) trail?
 - d) walk?
- 4.2.7. $K_{2,2h}$ is decomposable into cycles. How long are these cycles?
- 4.2.8. Complete the proof of Theorem 4.2.2.

4.3. Ramsey Theory

In Exercise 2.2.1, the reader was asked to show that any edge-coloring of K_6 with two colors will contain a monochromatic triangle. The result of Exercise 2.2.2 shows that K_6 is minimal with this property. We now do these exercises.

Problem. *Show that if the edges of K_6 are colored with two colors, then there will be a monochromatic triangle. Also show that K_6 is minimal with this property.*

Given a coloring of the edges of K_6 with red and blue, let v be any vertex of K_6 . Either there are at least three red edges incident with v or at least three blue edges incident with v , since v has degree 5. We assume there are three red edges. If any of the dotted edges in Figure 4.3.1 is red, there is a red triangle. If

all are blue, they form a blue triangle. Thus any edge-coloring of K_6 by two colors contains a monochromatic triangle.

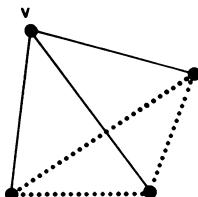


Figure 4.3.1

Figure 4.3.2 shows a coloring of K_5 with two colors and no monochromatic triangle.

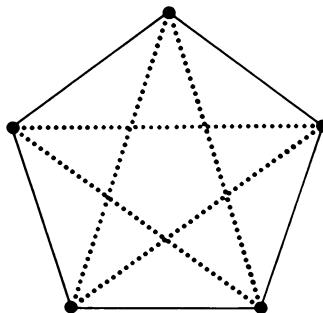


Figure 4.3.2

It is also known that if the edges of K_{18} are colored with red and blue, there must be a red K_4 or a blue K_4 and that K_{18} is minimal with this property.

Theorem 4.3.1 (Ramsey) [19]. *For every number n , there is a number $r(n)$ such that any edge-coloring of the complete graph with $r(n)$ vertices using red and blue must contain either a red K_n or a blue K_n .*

In other words, if we take a large enough complete graph and color the edges with red and blue, we can be sure of having either a red K_n or a blue K_n . To prove Theorem 4.3.1 directly is difficult but it is easier to prove a generalization of Theorem 4.3.1, namely Theorem 4.3.2, which will be formulated later.

Problem. *Show that if the edges of K_9 are colored with red and blue, then there will be a red K_3 or a blue K_4 . Also show that K_9 is minimal with this property.*

Assume that the edges of K_9 are colored with red and blue. If at any vertex of K_9 , there are four red edges, as in Figure 4.3.3, then there will be a red triangle or a blue K_4 . This must be true since if any of the dotted edges is red, there is a red K_3 . If all of the dotted edges are blue, they form a blue K_4 .

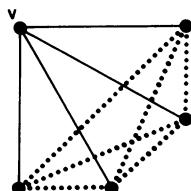


Figure 4.3.3

On the other hand, if at any vertex v of K_9 , there are six blue edges, the other endpoints of these edges form a K_6 . We have already seen that if the edges of K_6 are colored with two colors, there will be a monochromatic triangle. If the triangle is red, we are done. If the triangle is blue, together with the blue edges to the vertex v we have a blue K_4 . Thus at any vertex of K_9 , if the result is not true, there can be at most three red edges and at most five blue edges. Hence there must be exactly three red edges and five blue edges. So the red edges form a cubic spanning subgraph of K_9 , and the blue edges form a spanning subgraph of K_9 that is regular of degree 5. But this is impossible, since no graph with nine vertices can be regular of odd degree. Hence, any coloring of the edges of K_9 with red and blue contains either a red K_3 or a blue K_4 .

Figure 4.3.4 shows a coloring of the edges of K_8 with red and blue such that there is no red K_3 and no blue K_4 . Thus, K_9 is minimal with this property.

The *Ramsey number* $r(m,n)$ is the smallest number with the property that any edge-coloring of the complete graph with $r(m,n)$ vertices using red and blue must contain a red K_m or a blue K_n . We have just shown that $r(3,4) = 9$. Very few of the Ramsey numbers are actually known.

Notice that $r(1,n) = 1$, since any edge-coloring of K_1 with two colors contains a red K_1 or a blue K_n . This is true because K_1 has no edges, so all the edges of K_1 are red. Also, $r(2,n) = n$, because any edge-coloring of K_n with red and blue contains a red K_2 , that is, it contains a red edge, or it contains a blue K_n . The known Ramsey numbers are

$$\begin{array}{ll}
 r(1,n) = 1 & r(2,n) = n \\
 r(3,3) = 6 & r(3,4) = 9 \\
 r(3,5) = 14 & r(3,6) = 18 \\
 r(3,7) = 23 & r(3,9) = 36 \\
 r(4,4) = 18. &
 \end{array}$$

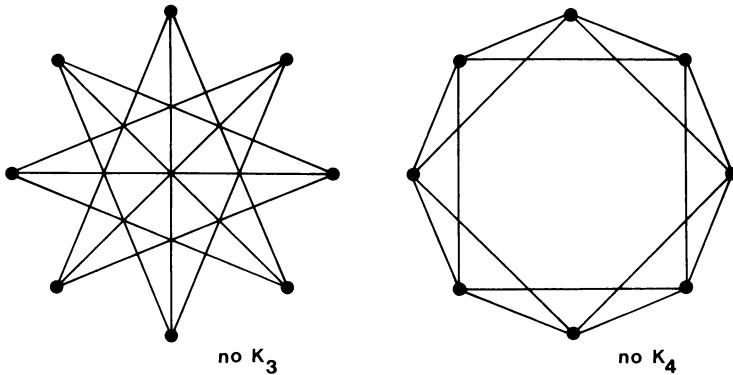


Figure 4.3.4

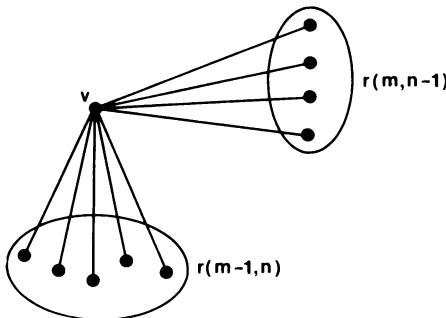
Theorem 4.3.2. For every m and n , there exists the Ramsey number $r(m,n)$ such that any edge-coloring of $K_{r(m,n)}$ with red and blue contains a red K_m or a blue K_n . Furthermore, $r(m,n)$ satisfies the inequality

$$r(m,n) \leq r(m-1,n) + r(m,n-1).$$

Proof. We proceed by induction on $k = m+n$. The smallest value that makes sense is $k = 4$, for $m = 2$ and $n = 2$. Then

$$r(2,2) = 2 \leq 1+1 = r(1,2) + r(2,1).$$

Now suppose that the theorem is true for all values less than k . Let G be the complete graph with $r(m-1,n) + r(m,n-1)$ vertices. Assume that the edges of G are colored with red and blue. Let v be any vertex of G . At v either there are $r(m-1,n)$ red edges or $r(m,n-1)$ blue edges.



To see that this is true, consider that the degree of v is $r(m-1,n) + r(m,n-1) - 1$. If there are fewer than $r(m-1,n)$ red edges, then there must be at least $r(m,n-1)$ blue edges.

Case 1. If there are $r(m-1,n)$ red edges at v , the subgraph H induced by the other endpoints of these edges is a complete graph with $r(m-1,n)$ vertices that is edge-colored with red and blue. Thus either there is a red K_{m-1} in H , that together with v forms a red K_m in G , or there is a blue K_n in H and hence also in G . Therefore, if there are $r(m-1,n)$ red edges at v , either G contains a red K_m or a blue K_n .

Case 2. If there are $r(m,n-1)$ blue edges at v , the subgraph I induced by the other endpoints of these edges is a complete graph with $r(m,n-1)$ vertices that is edge-colored with red and blue. Thus either there is a red K_m in I and hence in G , or there is a blue K_{n-1} in I , that, together with v , forms a blue K_n in G . Therefore, if there are $r(m-1,n)$ blue edges at v , either G contains a red K_m or a blue K_n , and we have established that

$$r(m,n) \leq r(m-1,n) + r(m,n-1).$$

■

If we define $r(n,n) = r(n)$, Theorem 4.3.1 is now a special case of Theorem 4.3.2.

Problem. Show that if the edges of $K_{5,5}$ are colored with two colors, there will be a monochromatic $K_{2,2}$.

There are 25 edges in $K_{5,5}$, so one of the colors will go to at least 13 edges. We focus on the subgraph S with these edges. Since each edge has one endpoint in each of the partite sets, we can look at the number of edges incident

with the five vertices in one of the sets. Notice that the coloring will contain a monochromatic $K_{2,2}$ precisely when two of these vertices have two neighbors in common. There are three cases.

Case 1. One vertex v has degree 5 in S . Since the average degree of the remaining four vertices is two, at least one has degree at least two; call it w . Then v and w have two neighbors in common, and there is a $K_{2,2}$.

Case 2. One vertex v has degree 4 in S . Since there are still at least nine edges, at least one remaining vertex w has degree 3 in S , and there is a $K_{2,2}$.

Case 3. At least three vertices have degree at least three in S , and there is a $K_{2,2}$.

Exercises 4.3

- 4.3.1. Color the edges of $K_{3,3}$ by two colors so that there is no monochromatic $K_{2,2}$.
- 4.3.2. Color the edges of $K_{4,4}$ by two colors so that there is no monochromatic $K_{2,2}$.
- 4.3.3. Color the edges of K_6 by three colors so that there is no monochromatic cycle.
- 4.3.4. Show that any edge-coloring of K_{2h+1} by h colors contains a monochromatic cycle.
- 4.3.5. Show that $r(3,5) \leq 14$.

Chapter 5

COUNTING

5.1 Counting 1-Factors

Often we are interested to know how many of a certain mathematical object may exist. This type of problem is called a *counting problem*. Many of the counting problems in graph theory are unsolved. For instance, how many nonisomorphic graphs are there with n vertices? How many nonisomorphic connected graphs are there with n vertices? How many nonisomorphic trees are there with n vertices? In this chapter we shall discuss first a few easy questions and then a couple of more challenging problems.

Problem. *How many 1-factors are there in $K_{n,n}$?*

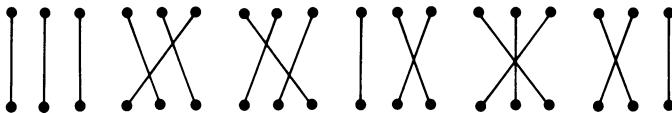


Figure 5.1.1

Figure 5.1.1 shows the six 1-factors of $K_{3,3}$. To compute in general how many 1-factors are in $K_{n,n}$, number the red vertices by 1, 2, 3, . . . , n . We begin at red vertex number 1. To choose a 1-factor, we can choose from n blue vertices to be adjacent to vertex number 1. Once we have chosen a blue vertex, we now have $n - 1$ blue vertices left to choose for the red vertex number 2. Then

there will be $n-2$ possibilities left for number 3. If we continue in this manner, we see that there are

$$n(n-1)(n-2) \cdots (3)(2)(1)$$

possibilities. This number is usually called n factorial, denoted by $n!$.

Problem. *How many subgraphs isomorphic to P_k are in the graph K_n ?*

Recall that P_k is a path of length of k . It has $k+1$ vertices. Clearly if $k \geq n$, the answer is zero. So we suppose that $k < n$. Now we count the number of P_k subgraphs in K_n . If we begin at some vertex, there are $n-1$ choices for the next vertex in the path and $n-2$ choices for the vertex following that, until we reach the $(k+1)$ st vertex. This is a total of

$$n(n-1)(n-2) \cdots (n-k).$$

But we have counted everything twice, since beginning at vertex $k+1$ we could get the same path as beginning at vertex 1 in the other direction. So we divide by 2, and the number of subgraphs of K_n that are isomorphic to P_k is

$$\frac{n(n-1)(n-2) \cdots (n-k)}{2} = \frac{n!}{2(n-k-1)!}.$$

Problem. *How many subgraphs isomorphic to $K_{1,3}$ are in K_n ?*

First we need to choose four vertices from n vertices. There are n choices for the first, $n-1$ choices for the second, $n-2$ for the third, and $n-3$ for the fourth. That means

$$n(n-1)(n-2)(n-3)$$

possibilities. However, choosing vertices 1, 2, 3, 4 is the same as choosing 1, 3, 2, 4 and all the other permutations of four objects. So we must divide by $4!$ to get the correct amount

$$\frac{n(n-1)(n-2)(n-3)}{4!}.$$

This quantity is called n choose four and is denoted by $\binom{n}{4}$. We have found that $\binom{n}{4}$ is the number of ways to choose four objects from a set of n objects. This is also called the number of 4-combinations of an n -set. In general, we have $\binom{n}{k}$ is

the number of k -combinations of an n -set, where

$$\binom{n}{k} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

To avoid having troubles with this notation, we define

$$0! = 1.$$

For every choice of four vertices, we have four possibilities to form a $K_{1,3}$ subgraph. (See Figure 5.1.2.) Thus the number of subgraphs isomorphic to $K_{1,3}$ in K_n is $4\binom{n}{4}$.

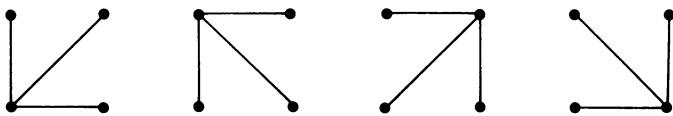


Figure 5.1.2

We now solve the same problem in a different way. In $K_{1,3}$ there is one vertex of degree 3, and there are three vertices of degree one. If we first choose the vertex of degree 3, there are n ways to choose. Then we can choose the three vertices of degree 1 in $\binom{n-1}{3}$ ways. Thus the number of $K_{1,3}$ subgraphs in K_n is

$$n\binom{n-1}{3}.$$

We leave it as an exercise to show that

$$4\binom{n}{4} = n\binom{n-1}{3}.$$

Problem. How many 1-factors are in K_{2h} ?

Figure 5.1.3 shows that there are three 1-factors in K_4 . To find a general formula, we shall denote the number of 1-factors of K_{2h} by $g(2h)$. Then we choose one vertex x . There are $2h-1$ choices for an edge containing the vertex x . Once we have chosen such an edge, we can now disregard the two ends of the edge and consider how many 1-factors can be made from the remaining $2h-2$ vertices. Thus we get the formula

$$g(2h) = (2h-1)g(2h-2).$$

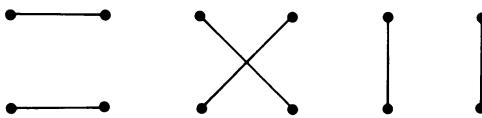


Figure 5.1.3

Notice that

$$g(2h-2) = (2h-3)g(2h-4).$$

We can substitute this quantity for $g(2h-2)$ and get

$$g(2h) = (2h-1)(2h-3)g(2h-4).$$

If we continue to substitute we get the formula

$$g(2h) = (2h-1)(2h-3)(2h-5) \cdots (3)(1).$$

This almost looks like a factorial. If we multiply and divide by

$$(2h-2)(2h-4)(2h-6) \cdots (4)(2) = 2^{h-1}(h-1)!,$$

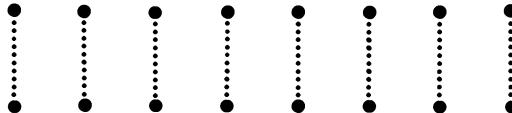
we will get

$$g(2h) = \frac{(2h-1)!}{2^{h-1}(h-1)!}.$$

Problem. How many 1-factors are in the graph $G = K_{n,n}$ minus a 1-factor?

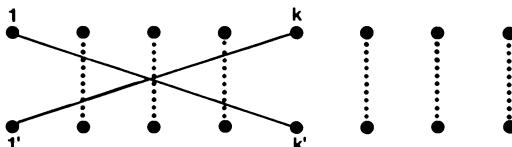
The graph G is obtained by taking out n edges that form a 1-factor of $K_{n,n}$. This problem is famous and is called the *derangements* of n objects. In different terms, the problem can be formulated as the Hatchet Problem and the Letter Problem. The Hatchet Problem is as follows. A group of n men go to a restaurant and check their hats. How many ways can their hats be returned to them so that no man gets his own hat back? The Letter Problem is this. A secretary has typed n letters and addressed n envelopes for the letters. The janitor knocks the letters and envelopes off the desk. How many ways can the janitor pick them up so that in every envelope is a wrong letter?

Let D_n denote the number of derangements of n objects, that is, D_n is the number of 1-factors in the graph $G = K_{n,n} - \text{(1-factor)}$. Clearly $D_1 = 0$ and $D_2 = 1$. If we take $K_{3,3}$ and subtract the first 1-factor shown in Figure 5.1.1, the resulting graph will have as 1-factors only those 1-factors of $K_{3,3}$ that contain no vertical edge. Thus $D_3 = 2$.

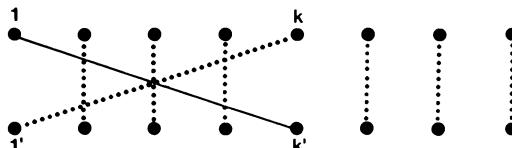


The dotted lines indicate that the edge is not supposed to be used.

We can derive an expression for D_n in terms of D_{n-1} and D_{n-2} . First, we count the number of 1-factors that contain the edges $1k'$ and $k1'$.



There are D_{n-2} such 1-factors for each value of k , and k can assume $n-1$ values. Thus this gives us $(n-1)D_{n-2}$ possibilities. Now we count those 1-factors that contain the edge $1k'$ but not the edge $k1'$. There are D_{n-1} such 1-factors for each value of k , and again k can assume $n-1$ values.



Thus we have shown that

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

Such an expression is called a *recurrence* or a *recursive formula*. A computer can compute D_n for any n given initial values and using the recurrence, but a human would get bored. We are more interested in a direct expression that does not depend on previous values of D_n .

We can write the above recurrence as

$$D_n - nD_{n-1} = -[D_{n-1} - (n-1)D_{n-2}].$$

Since this is true for every n , it is also true if we write it for $n-1$, as

$$D_{n-1} - (n-1)D_{n-2} = -[D_{n-2} - (n-2)D_{n-3}].$$

We can now substitute for $D_{n-1} - (n-1)D_{n-2}$ and obtain

$$D_n - nD_{n-1} = (-1)^2[D_{n-2} - (n-2)D_{n-3}].$$

If we continue to substitute in this manner, we obtain

$$D_n - nD_{n-2} = (-1)^{n-2}[D_2 - 2D_1] = (-1)^{n-2}$$

since $D_1 = 0$ and $D_2 = 1$. Since $(-1)^{n-2} = (-1)^n$, we have

$$D_n = nD_{n-1} + (-1)^n.$$

This is again a recurrence, but already a better one, since now there is only one previous value. We divide by $n! = (n-1)!n$

$$\frac{D_n}{n!} = \frac{D_{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}.$$

Now we write this formula for $n = 2, 3, \dots$.

$$\begin{array}{rclcrcl} \frac{D_2}{2!} & = & \frac{D_1}{1!} & + & \frac{1}{2!} \\ \frac{D_3}{3!} & = & \frac{D_2}{2!} & - & \frac{1}{3!} \\ \frac{D_4}{4!} & = & \frac{D_3}{3!} & + & \frac{1}{4!} \\ \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots \end{array}$$

$$\frac{D_{n-1}}{(n-1)!} = \frac{D_{n-2}}{(n-2)!} + \frac{(-1)^{n-1}}{(n-1)!}$$

$$\frac{D_n}{n!} = \frac{D_{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}$$

Now we add them all up. Notice that each term on the left side in one equation appears on the right side in the following equation, so this term cancels from each up until $\frac{D_n}{n!}$. Then we have

$$\frac{D_n}{n!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}.$$

So the number D_n of 1-factors in $K_{n,n} - (1\text{-factor})$ is

$$D_n = n! \left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!} \right].$$

Those readers who are familiar with the exponential function

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots$$

will recognize that

$$D_n \approx \frac{n!}{e}.$$

So if there are 10 questions and 10 answers on an exam, for instance, out of all $10!$ ways of matching questions and answers, the probability of matching all of them wrong is

$$\frac{D_{10}}{10!} \approx \frac{1}{e} \approx \frac{1}{3}.$$

Hence it is more likely that at least one question and answer match correctly.

Exercises 5.1

- 5.1.1. How many subgraphs isomorphic to P_2 are in $K_{m,n}$?
- 5.1.2. How many 4-cycles are in $K_{m,n}$?
- 5.1.3. How many subgraphs isomorphic to P_3 are in $K_{m,n}$?
- 5.1.4. Prove that $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$.
- 5.1.5. Which is larger, $450! + 453!$ or $451! + 452!?$
- 5.1.6. Show that $4\binom{n}{4} = n\binom{n-1}{3}$.
- 5.1.7. Find a third way to count the number of subgraphs isomorphic to $K_{1,3}$ in K_n , and show that your answer is equal to $4\binom{n}{4}$.
- * 5.1.8. How many nonisomorphic induced subgraphs with k vertices are in $K_{m,n}$, where $k \leq m \leq n$?
- * 5.1.9. Show that $\binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \cdots + \binom{m}{k-1}\binom{n}{1} + \binom{m}{k}\binom{n}{0} = \binom{m+n}{k}$.
- * 5.1.10. Do Problem 5.1.8, omitting the word nonisomorphic.
- 5.1.11. Show that $\binom{s}{2} + \binom{t}{2} + st = \binom{s+t}{2}$.
- 5.1.12. How many 1-factors does the wheel W_n have?

5.2. Cayley's Spanning Tree Formula

Recall that a spanning tree of a graph G is a subgraph of G that is a tree and that contains all the vertices of G . In 1889 Cayley proved the following theorem [6].

Theorem 5.2.1. *The number of spanning trees in K_n is*

$$s(K_n) = n^{n-2}.$$

Many different proofs have been given for this very interesting result. One of the simplest and most elegant is due to Prüfer, and this is the proof we shall present here. First let us recognize that the formula is true for $n = 3$ and $n = 4$ by direct inspection. If $n = 3$, there are three spanning trees that can be obtained by deleting one edge of K_3 for each tree. For $n = 4$, Figure 5.2.1 shows the 16 spanning trees of K_4 . They are all different, but many of them are isomorphic. In fact there are only two nonisomorphic trees with four vertices,

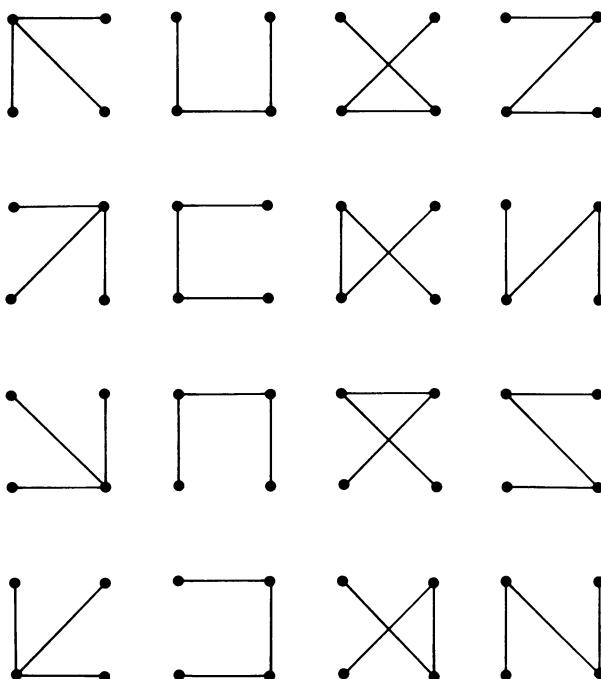


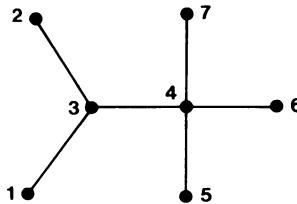
Figure 5.2.1

namely $K_{1,3}$ and P_3 . But we are not counting nonisomorphic trees on n vertices; this problem is unsolved. The problem of counting the spanning trees of K_n is really the same problem as counting the labeled trees with n vertices, which means that the vertices are labeled by $1, 2, 3, \dots, n$.

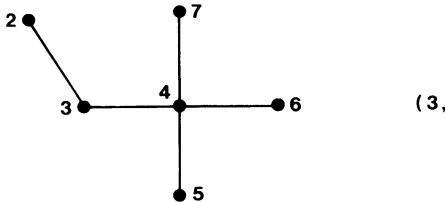
Before we present Prüfer's proof of Cayley's formula, we consider the following, apparently unrelated problem. How many difference sequences $(b_1, b_2, b_3 \dots b_{n-2})$ of length $n-2$ are there whose elements are $1, 2, 3, \dots, n$, where repetition is allowed? There are n possibilities for b_1 , n possibilities for b_2 , n possibilities for b_3 , and so on. So the total number is

$$n \cdot n \cdot n \cdot \dots \cdot n = n^{n-2}.$$

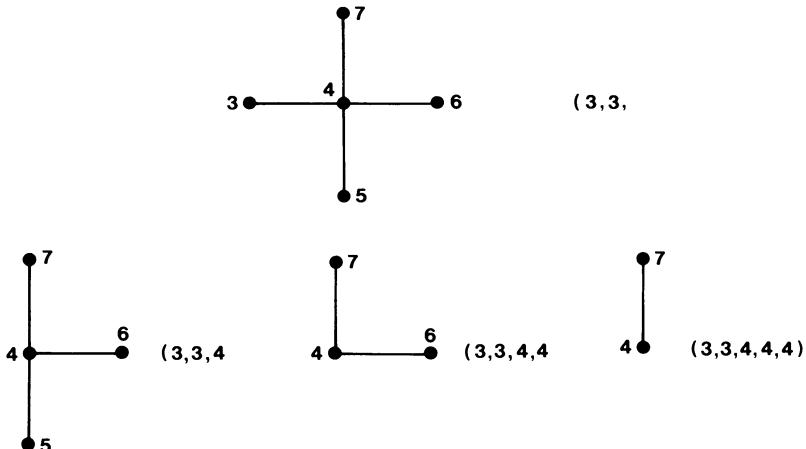
We realize that this is the same as the number in the theorem. This gives us the idea to find a one-to-one correspondence between the set of spanning trees of K_n and the set of sequences of length $n-2$ whose elements are $1, 2, 3, \dots, n$. That is, we describe every tree by a sequence in such a way that no two trees are described by the same sequence.



As an example for $n = 7$, given the tree T above where the vertices are labeled, we first look for the vertex of degree 1 with the smallest label. This is vertex number 1. The first element of the sequence will be the *neighbor* of vertex 1, namely vertex 3. Then we consider the tree T_1 that results from deleting the vertex 1 from the tree T .

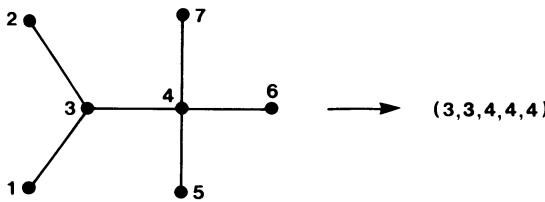


We do the same step again. This time the smallest label on a vertex of degree 1 is 2. The neighbor of vertex 2 is vertex 3, which will be the second element of the sequence.



Notice that the degree of vertex 3 in the original tree T was three, and the number 3 appears twice in the sequence. The degree of vertex 4 was four, and the 4 appears three times in the sequence. In fact it is easy to see that if a vertex v has degree k in the tree, the number v will appear $k-1$ times in the sequence. So the vertices of degree 1 do not appear in the sequence.

Thus far we have taken a tree T and associated a sequence $(3,3,4,4,4)$ with T .

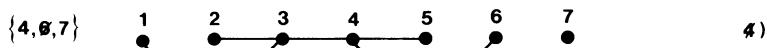
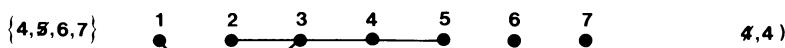
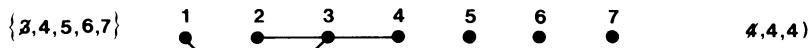


Now suppose we have the sequence $(3,3,4,4,4)$. We construct a tree T by the following procedure. We look for the smallest element of the set $V = \{1,2,3,4,5,6,7\}$ that does *not* appear in the sequence. It will be 1.

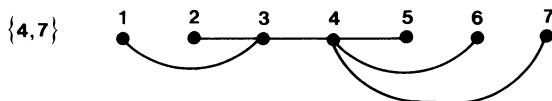
We connect vertex 1 to vertex 3 in our tree. Now we look at $V - \{1\}$



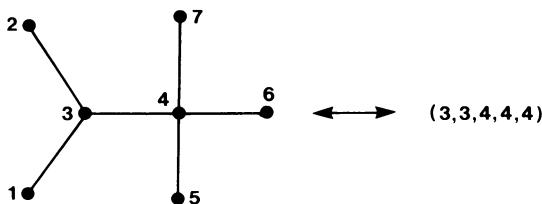
and the sequence with the first 3 removed. Now 2 is the smallest, so we connect vertex 2 to vertex 3. Then we continue in this manner.



Finally, we connect the last two numbers left in V , 4 and 7.

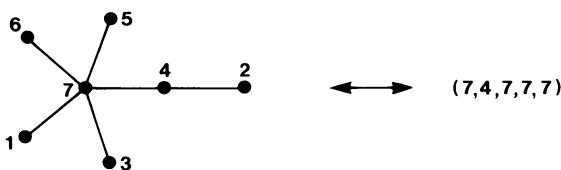
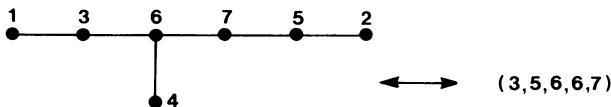


We realize this is the same labeled tree T that we started with.



This double arrow is what we mean by a one-to-one correspondence.

For more examples, the reader is invited to associate the following two pairs of trees and sequences.



We have described a procedure to go from a labeled tree T_0 to a sequence S and a procedure to go from S to a labeled tree T_1 . What is left to prove is that $T_0 = T_1$. To see this, consider that the procedures both consist of a sequence of single steps. A single step in the first procedure is the reverse of a single step in the second procedure. So, for instance, a step in the first procedure is to find the smallest labeled vertex of degree one in the tree, delete this vertex, and write down the label on its neighbor. This is precisely the reverse of a step in the second procedure, which is to take the first number in the sequence as a vertex label and to label a vertex of degree 1 adjacent to it by the smallest number of $1, 2, \dots, n$ that has not yet been used and does not appear in the sequence.

Exercises 5.2

- 5.2.1. Find the sequence associated by the Prüfer method with the tree in Figure 5.2.2.

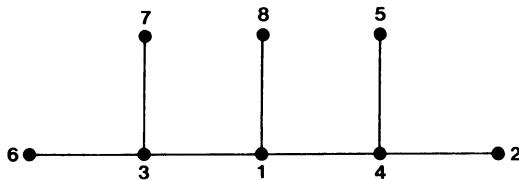


Figure 5.2.2

- 5.2.2. What sequence is associated with a
 a. path?
 b. $K_{1,n}$?
- 5.2.3. Given a sequence, can you determine the maximum degree of the associated tree by just looking at the sequence?
- 5.2.4. Find the trees associated with the sequences
 a. (6, 1, 1, 4)
 b. (1, 7, 2, 2, 2, 2)
- 5.2.5. We know that there are 125 spanning trees in K_5 . Find how many nonisomorphic trees there are with five vertices.
- 5.2.6. Do problem 5.2.5 for six vertices.
- *5.2.7. A regular graph is decomposed into spanning trees. Prove that the graph is complete.
- 5.2.8. How many spanning trees are in the cycle C_n ?

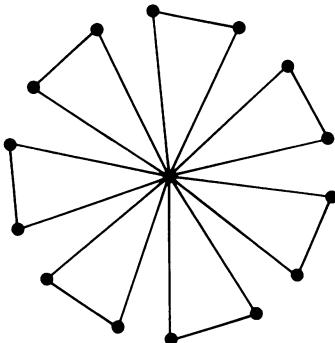


Figure 5.2.3

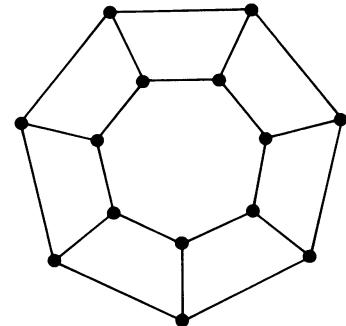


Figure 5.2.4

- 5.2.9. If we take n triangles and assume that they have exactly one vertex in common, we obtain the windmill graph M_n . Figure 5.2.3 shows M_7 . How many spanning trees are in the graph M_n ?
- 5.2.10. The graph in Figure 5.2.4 could be called the 7-sided prism graph. If there are n sides, it will be the n -sided prism graph. How many Hamilton cycles are in the n -sided prism graph? (Hint – consider whether n is even or odd.)
- 5.2.11. How many Hamilton paths (spanning paths) are in the wheel W_n ?

5.3. More Spanning Trees

There is a nice formula for the number of spanning trees in the complete bipartite graph $K_{m,n}$, namely

$$s(K_{m,n}) = m^{n-1} n^{m-1}.$$

The proof of the formula in general is difficult, but we shall prove two special cases, the case $m = 2$, and n is arbitrary and the case $m = 3$, with n arbitrary.

Theorem 5.3.1. *The number of spanning trees in the graph $K_{2,n}$ is $n 2^{n-1}$.*

Proof. The graph $K_{2,n}$ has two blue vertices, a and b , and n red vertices as in Figure 5.3.1. In any spanning tree of $K_{2,n}$, the unique path between a and b has length two, as in the picture. We can now count, there are n ways to choose x , and for each of the remaining $n-1$ red vertices, there are two choices, either to be adjacent to a or to b . This gives us a total of

$$n 2^{n-1}$$

different spanning trees. ■

Theorem 5.3.2. *The number of spanning trees in the graph $K_{3,n}$ is $n^2 3^{n-1}$.*

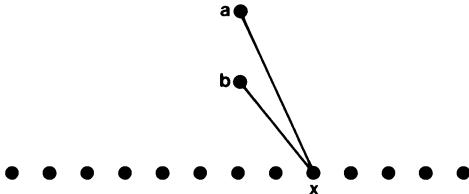


Figure 5.3.1

Proof. For $K_{3,n}$ there are three blue vertices a , b , and c , and n red vertices.

In any spanning tree of $K_{3,n}$ either each of the pairs a, b ; a, c ; and b, c has distance two as in Figure 5.3.2, or one of the pairs has distance four as in Figure 5.3.3.

We count the number of spanning trees as in Figure 5.3.2. There are n ways to choose the vertex x . For each of the remaining $n-1$ red vertices, there are three possibilities. Each is either adjacent to a , to b , or to c . Thus there are

$$n 3^{n-1}$$

spanning trees with each pair a, b ; a, c ; and b, c having distance two.

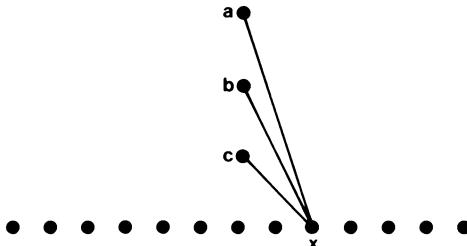


Figure 5.3.2

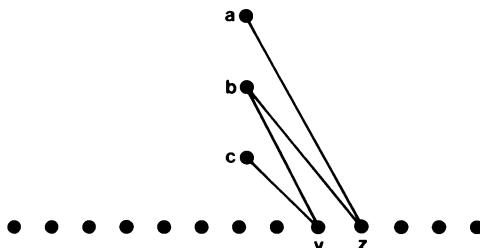


Figure 5.3.3

For Figure 5.3.3, if the points y and z are chosen, there are six possibilities for the path of length four, since there are three ways to choose the center point b , and y and z can be interchanged. There are $\binom{n}{2} = \frac{n(n-1)}{2}$ ways to choose the pair y and z , and for each of the remaining $n-2$ red vertices there are again three possibilities. Hence for Figure 5.3.3 there are

$$6 \frac{n(n-1)}{2} 3^{n-2} = n(n-1)3^{n-1}$$

different spanning trees. So the total number is

$$n 3^{n-1} + n(n-1)3^{n-1} = n^2 3^{n-1}. \quad \blacksquare$$

The formula for the number of spanning trees of the wheel with n spokes, W_n , is difficult to derive, so we simply present the formula without proof. The number $s(W_n)$ satisfies the recurrence

$$s(W_n) = 3s(W_{n-1}) - 3s(W_{n-3}) + s(W_{n-4}).$$

After solving the recurrence, one obtains the formula

$$s(W_n) = \left[\frac{3 + \sqrt{5}}{2} \right]^n + \left[\frac{3 - \sqrt{5}}{2} \right]^n - 2.$$

Since $W_3 = K_4$, the reader is invited to check the formula for $n = 3$.

Exercises 5.3

- 5.3.1. Let a be a vertex in K_n . How many spanning trees are there in K_n with the property that the vertex a is an end vertex in the tree?
- 5.3.2. Given a tree T with n vertices, how many subgraphs of T are isomorphic to a path?
- 5.3.3. Decompose K_6 into three spanning trees such that
 - a) each of these trees is isomorphic to a path.
 - b) *not* each of these trees is isomorphic to a path.
 - c) each of these trees is *not* isomorphic to a path.
- 5.3.4. Let G be a connected graph with exactly one cycle. The length of the cycle is r . What is the number $s(G)$ of spanning trees of G ?
- 5.3.5. Prove that the number of nonisomorphic spanning trees in the graph $K_{2,n}$ is $\left\lfloor \frac{n+1}{2} \right\rfloor$.

Chapter 6

LABELING GRAPHS

6.1 Magic Graphs and Graceful Trees

In this section we consider graphs and the possibility of assigning a number to every vertex or every edge or both, with certain conditions. This is different from coloring problems because we shall use some properties and structures of numbers such as ordering and addition that are not properties of colors. We begin with a curiosity. Stewart found a labeling of the edges of the cube that is illustrated in Figure 6.1.1. Notice that at each vertex the sum of the labels on the edges incident with that vertex is 83. Moreover, the edge labels are all distinct, and they are all primes!

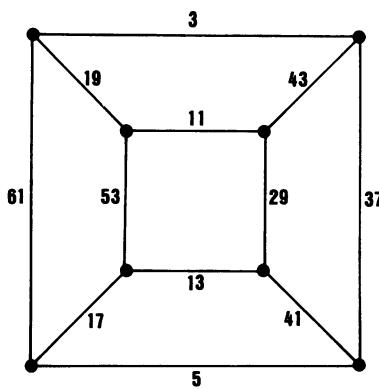


Figure 6.1.1. Stewart's cube.

Suppose that G is a graph with q edges. We shall say that G is *magic* if the edges of G can be labeled by the numbers $1, 2, 3, \dots, q$ so that the sum of the labels of all the edges incident with any vertex is the same. Figure 6.1.2 shows two examples of magic graphs. In the literature what we have defined as magic is referred to as super-magic.

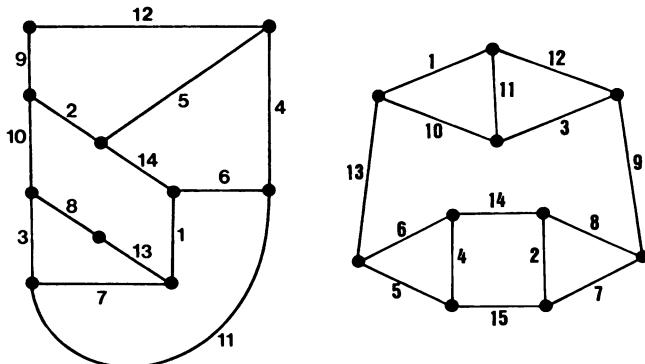


Figure 6.1.2

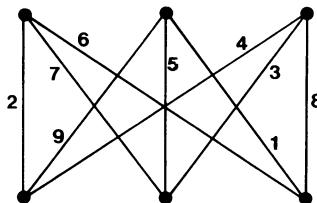


Figure 6.1.3

In Figure 6.1.3 we have the graph $K_{3,3}$. The edges are labeled with the numbers $1, 2, 3, \dots, 9$ in such a way that the sum at each vertex is 15. This labeling is derived from the well-known magic square in Figure 6.1.4, as follows. Call the rows in the square red vertices and the columns blue vertices. Label the edges between a red vertex and a blue vertex by the label on the square that is common to the row and column of the magic square. It is known that for every $n \geq 3$ there is an $n \times n$ magic square [32]. Figure 6.1.5 shows a 4×4 magic square. The labeling of $K_{4,4}$ in Figure 6.1.6 is magic.

Theorem 6.1.1. $K_{n,n}$ is magic for $n \neq 2$.

2	7	6
9	5	1
4	3	8

Figure 6.1.4

3	11	14	6
8	5	9	12
10	2	7	15
13	16	4	1

Figure 6.1.5

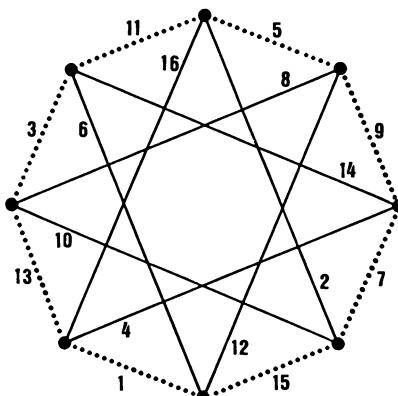


Figure 6.1.6

Notice that $K_{4,4}$ is decomposable into two Hamilton cycles. The labels on the edges of one of the Hamilton cycles in Figure 6.1.6 are

$$15, 1, 13, 3, 11, 5, 9, 7.$$

The edges of the other Hamilton cycle are labeled by

$$2, 16, 4, 14, 6, 12, 8, 10.$$

The graph of Figure 6.1.7 is another example of a bipartite graph that is decomposable into two Hamilton cycles and is magic. The labeling strategy is the same as in Figure 6.1.6 and in the proof of the following theorem.

Theorem 6.1.2. *If a bipartite graph G is decomposable into two Hamilton cycles, then G is magic.*

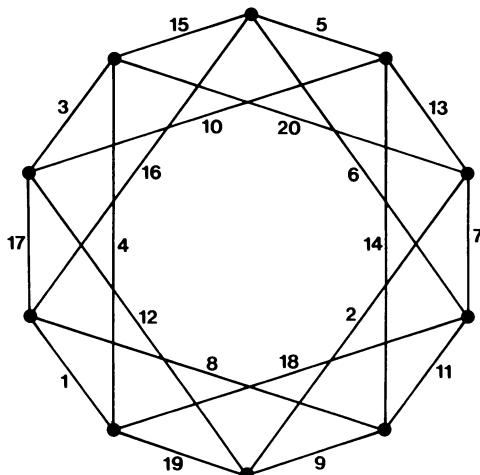


Figure 6.1.7

Proof. Since G is bipartite, the length of the Hamilton cycle is even, say $2n$. Thus the number of edges in G is $q = 4n$. As in the previous two examples, we label one Hamilton cycle with even numbers and the other with odd numbers. We choose a vertex a and label the edges of the first Hamilton cycle starting at a by

$$4n-1, 1, 4n-3, 3, \dots, 2n+1, 2n-1.$$

Then we label the edges of the second Hamilton cycle, starting at a , by

$$2, 4n, 4, 4n-2, \dots, 2n, 2n+2.$$

Since G is bipartite, the vertices can be colored red and blue with no two adjacent vertices of the same color. If a is blue, then the sum of the odd-numbered edges at all blue vertices except a is $4n-2$. This can be checked by looking at the row of odd numbers above. The sum of the even-numbered edges at all blue vertices except a is $4n+4$. The sum of all edges at a is $8n+2$, by direct inspection. Thus, the sum of all edges at any blue vertex is $8n+2$. The sum of the odd-numbered edges at each red vertex is $4n$. The sum of the even-numbered edges at each red vertex is $4n+2$. Hence the sum of all edges at any red vertex is $8n+2$, and G has been proved to be magic. ■

Figure 6.1.8 shows the four-dimensional cube graph. It is decomposable into two Hamilton cycles and is bipartite, so by Theorem 6.1.2 it is magic. See Exercise 6.1.4.

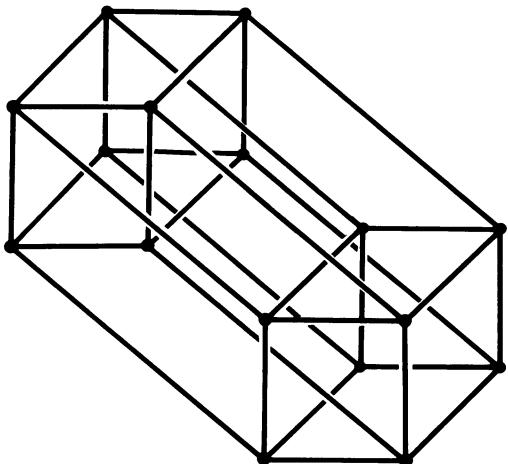


Figure 6.1.8

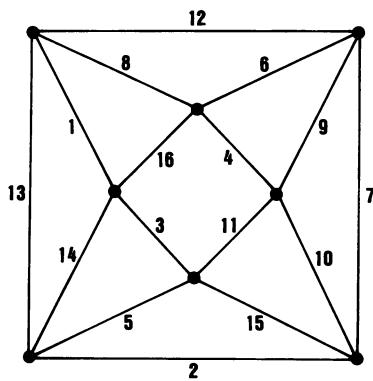


Figure 6.1.9

Figure 6.1.9 is an example of a graph that is decomposable into two Hamilton cycles. However, the graph is not bipartite, so the labeling method of Theorem 6.1.2 does not work. The labeling in Figure 6.1.9 is done in an *ad hoc* manner.

In Exercise 6.1.6 it is possible to show that if n is a multiple of four, then K_n is not magic, but it is known that for every other $n \geq 6$, K_n is magic [22]. Figure 6.1.10 shows a magic labeling of K_7 .

Theorem 6.1.3. *If a graph G is decomposable into two magic spanning subgraphs G_1 and G_2 where G_2 is regular, then G is magic.*

Proof. Let q_1 and q_2 denote the number of edges of G_1 and G_2 , respectively. Consider a magic labeling of G_1 and a magic labeling of G_2 . To each label of G_2 , we add q_1 . Since G_2 is regular, we have added the same amount at each vertex. We now have the edges of G labeled with the integers $1, 2, 3, \dots, q_1, q_1+1, \dots, q_1+q_2$, and the sum of the labels at each vertex is the same. Hence G is magic. ■

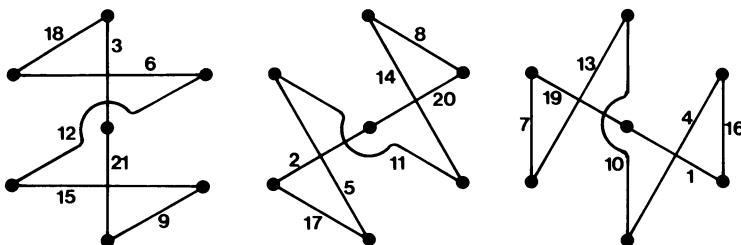


Figure 6.1.10

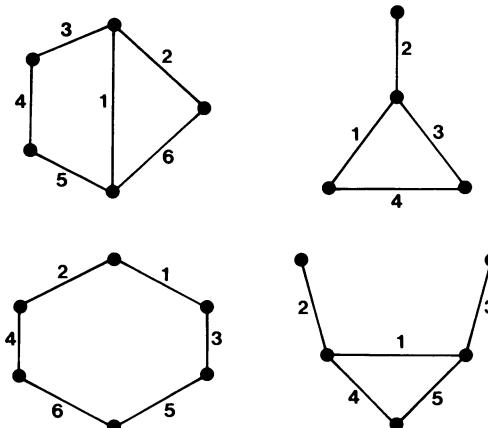


Figure 6.1.11

Figure 6.1.11 shows some examples of graphs whose edges are labeled with the integers $1, 2, \dots, q$ so that the sum of the labels at any given vertex is different from the sum of the labels at any other vertex, that is, no two vertices have the same sum. We shall call a graph that can be so labeled *antimagic*.

Conjecture. Every connected graph different from K_2 is antimagic.

Even if we were to restrict ourselves to trees, we do not know for sure if all except K_2 are antimagic.

Conjecture. *Every tree different from K_2 is antimagic.*

In the exercises the reader will have the opportunity to prove the antimagic conjecture for some classes of graphs.

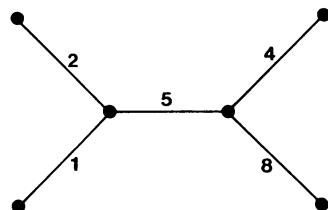


Figure 6.1.12. The Leech tree.

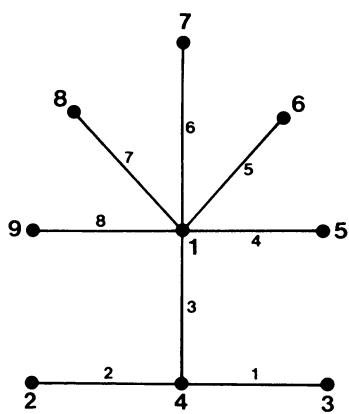


Figure 6.1.13

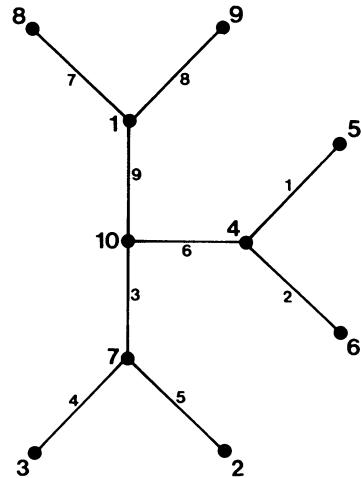


Figure 6.1.14

Figure 6.1.12 is a picture of a labeled tree. The tree has six vertices, thus there are $\binom{6}{2}$ pairs of vertices, and every pair of vertices is connected by a unique path. If you add the numbers on the edges for each of these paths, each

number from 1 to $\binom{6}{2}$ appears as exactly one of these sums. For which values of n is such a labeling of a tree on n vertices possible? John Leech gave the answer for $n \leq 6$, including the example of Figure 6.1.12 and the fact that this is impossible for $n = 5$. It is not known if this is possible for $n > 6$.

If a tree has n vertices, then it has $n-1$ edges. If it is possible to label the vertices by $1, 2, 3, \dots, n$ and the edges by $1, 2, 3, \dots, n-1$ so that the label on any edge equals the difference between the labels of the two endpoints, then the tree is said to be *graceful*. Figures 6.1.13 and 6.1.14 show graceful labelings of trees. Gerhard Ringel has conjectured that all trees are graceful. The conjecture has been proved for several classes of trees, but it has not been proved in general.

Exercises 6.1.

- 6.1.1. What is the sum at each vertex in a magic labeling of $K_{n,n}$?
- 6.1.2. Let G be a magic graph with p vertices and q edges. What is the sum at each vertex in a magic labeling of G ?
- 6.1.3. Find a magic labeling of the octahedron graph of Figure 1.2.5.
- 6.1.4. Find a magic labeling of the four-dimensional cube of Figure 6.1.8.
- 6.1.5. If K_n is magic, what is the sum at each vertex?
- 6.1.6. Prove that if $n = 4k$ for some integer k , then K_n is not magic.
- 6.1.7. Prove that the wheel W_n is not magic for $n \geq 9$. (This is actually also true for smaller values.)
- 6.1.8. Show that $K_{3,3}$ is antimagic.
- 6.1.9. Show that the cube graph of Figure 1.2.5 is antimagic.
- 6.1.10. Prove that if n is odd, then C_n is antimagic.
- 6.1.11. Prove that if n is even, then C_n is antimagic.
- 6.1.12. Prove that W_n is antimagic.
- *6.1.13. Prove that K_n is antimagic for $n \geq 3$.
- 6.1.14. Prove that the path P_n is antimagic for $n \geq 2$.
- 6.1.15. Show that $K_{1,n}$ is antimagic for $n \geq 2$.
- 6.1.16. Show that $K_{2,n}$ is antimagic.
- 6.1.17. Find a connected graph G with 10 vertices such that *every* labeling of the edges with the numbers $1, 2, 3, \dots, q$ is an antimagic labeling.

- 6.1.18. Label the edges of a tree with four vertices so that $1, 2, 3, \dots, 6$ appear as the sums of the six paths, as in the Leech tree of Figure 6.1.12.
- 6.1.19. Prove that the path P_n is graceful.
- 6.1.20. Prove that the star $K_{1,n}$ is graceful.
- 6.1.21. In the forest of Figure 1.3.1, prove that the following trees are graceful.
- The first tree on the far right.
 - The tree near the center with 11 vertices.
 - The second tree from the left with 10 vertices.

6.2 Conservative Graphs

Up until now, we have considered only undirected graphs. Recall that a graph G is defined to be a pair of sets (V, E) , where V is nonempty, and E is a (possibly empty) set of unordered pairs of elements of V . A *directed graph* D is a pair of sets (V, A) where V is nonempty, and A is a (possibly empty) set of *ordered* pairs of elements of V . In other words, D is a graph where each edge has a direction, usually indicated by an arrow. These directed edges are usually called *arcs*.

When we look at magic labelings of graphs, the label is added on each of the two ends of an edge. In a labeling of a directed graph, the label is added on one end of the arc and subtracted on the other end, depending on which direction the arrow is pointing. If the arrow goes toward a vertex, we add. If the label goes away from a vertex, we subtract.

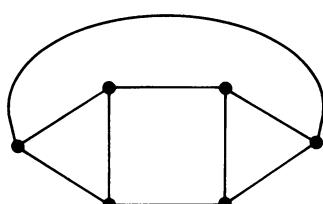


Figure 6.2.1

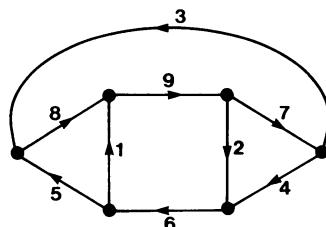


Figure 6.2.2

The edges of the graph in Figure 6.2.1 can be directed and labeled with the numbers $1, 2, 3, \dots, 9$ so that the sum of the labels on the arcs coming into a vertex is the same as the sum of the labels on the arcs going out of that vertex.

A graph with q edges that can be so directed and labeled with the numbers $1, 2, 3, \dots, q$ is called *conservative*. So the undirected graph of Figure 6.2.1 is conservative, since the directed, labeled graph of Figure 6.2.2 shows it. The above property at each vertex has been known for a long time as Kirchhoff's Current Law.

Kirchhoff's Current Law. *In a directed labeling of a conservative graph, the sum of the incoming labels is equal to the sum of the outgoing labels at each vertex.*

Sometimes, for brevity, we might say the *directed sum* at each vertex is zero.

Originally Kirchhoff had in mind the flow of water through a system of pipes or the flow of electricity through a circuit. For instance, if the arcs in Figure 6.2.2 are water pipes, and if the label on an arc represents liters per second, then if Kirchhoff's Current Law holds at each vertex, the water can actually flow through the system.

The dodecahedron graph of Figure 1.2.5 is conservative. Figure 6.2.3 shows a conservative labeling of the dodecahedron. The labeling was found by Sharon Cabaniss, using a computer.

Theorem 6.2.1. *If G is decomposable into two Hamilton cycles, then G is conservative.*

Proof. Assume that G has n vertices. Choose a vertex a and traverse the edges of one Hamilton cycle, orienting them (placing arrows on them) in the same direction as you go. Label the edges beginning at a by $1, 3, 5, \dots, 2n-1$. See Figure 6.2.4 for the example $n = 7$. Then traverse and orient the edges of the second Hamilton cycle as before, and label the edges beginning at a by $2n, 2n-2, 2n-4, \dots, 4, 2$. Notice that in the first row, at each vertex except a there is a net increase of two, and in the second row at each vertex except a there is a net decrease of two. At a there are the labels $2n$ and 1 going out and $2n-1$ and 2 coming in. Thus Kirchhoff's Current Law holds at every vertex. ■

We checked that Kirchhoff's Current Law holds at the vertex a in the proof of Theorem 6.2.1, but this was not really necessary, because there is the following theorem.

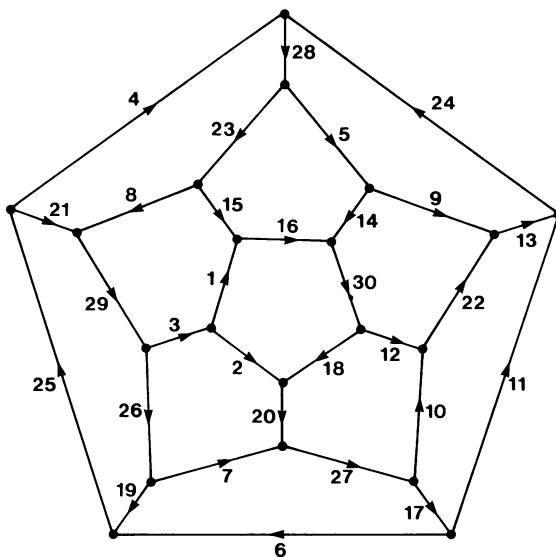


Figure 6.2.3. The Cabaniss dodecahedron

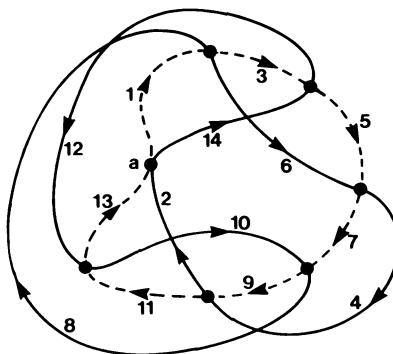


Figure 6.2.4

Theorem 6.2.2 (Kirchhoff's Global Current Law). *If G is a labeled, directed graph such that Kirchhoff's Current Law holds at every vertex of G except a particular vertex a , then Kirchhoff's Current Law also holds at the vertex a .*

To a physicist who was thinking of the flow of water or electricity in the system, this theorem would seem quite natural, but a mathematician wants a proof. The proof is actually very simple but not easy to write down.

Proof. Suppose that the degree of a is $h+f$, where the incoming edges at a are labeled c_1, c_2, \dots, c_h , the outgoing edges at a are labeled b_1, b_2, \dots, b_f , and the other labels in the graph will be denoted by d_i . Let S be the set of all vertices except a . By assumption, Kirchhoff's Current Law holds at every vertex in S . So at each vertex in S , if we add the labels going in and the labels coming out, the directed sum will be zero. The sum of all these sums is as follows:

$$d_1 - d_1 + d_2 - d_2 + \cdots + b_1 + b_2 + \cdots + b_f - c_1 - c_2 - \cdots - c_h = 0.$$

Each d_i occurs twice, once as incoming and once as outgoing, so they all cancel, and we are left with

$$b_1 + b_2 + \cdots + b_f = c_1 + c_2 + \cdots + c_h,$$

and therefore Kirchhoff's Current Law holds at a . ■

Notice that if we add the same number, for example 100, to every label in Figure 6.2.4, Kirchhoff's Current Law still holds at every vertex. If we try this with Figure 6.2.2, it will not work. This leads us to the following definition. A graph G with q edges is called *strongly conservative* if for every number h there exists a directed labeling of the edges of G with the numbers $h+1, h+2, \dots, h+q$ such that Kirchhoff's Current Law holds at every vertex. Thus the graph in Figure 6.2.4 is strongly conservative, while the graph in Figures 6.2.1 and 6.2.2 may or may not be strongly conservative. We cannot say at the moment.

The proof of the following theorem is actually easier than the proof of the corresponding theorem for magic graphs.

Theorem 6.2.3. *If G is decomposable into two subgraphs H_1 and H_2 , and if H_1 is conservative, and H_2 is strongly conservative, then G is conservative. Moreover, if both H_1 and H_2 are strongly conservative, then G is strongly conservative.*

Proof. Let q_1 be the number of edges in H_1 and q_2 the number of edges in H_2 , so the number of edges in G is $q_1 + q_2$. Take a conservative labeling of H_1 using the numbers $1, 2, 3, \dots, q_1$ and a strongly conservative labeling of H_2 using the numbers $q_1 + 1, q_1 + 2, \dots, q_1 + q_2$. In both these labelings Kirchhoff's Current Law holds, so the directed sum at each vertex is zero in G . Hence we have exhibited a conservative labeling of G .

Now we suppose that both H_1 and H_2 are strongly conservative, and h is a given number. Take a strongly conservative labeling of the edges of H_1 using the numbers $h+1, h+2, \dots, h+q_1$, and a strongly conservative labeling of the edges of H_2 using the numbers $h+q_1+1, h+q_1+2, \dots, h+q_1+q_2$. Again, the directed sum at each vertex is zero, and we obtain a strongly conservative labeling of the edges of G using the numbers $h+1, h+2, \dots, h+q_1+q_2$. Thus G is strongly conservative. ■

Notice that the graph of Theorem 6.2.1 is actually strongly conservative, since the conservative labeling consists of two edges directed in and two edges directed out at each vertex. Hence, adding a constant h to the label of each edge does not affect the directed sum at each vertex.

Theorem 6.2.1*. *If G is decomposable into two Hamilton cycles, then G is strongly conservative.*

Theorem 6.2.4. *If G is a graph with n vertices, where n is odd, and G is decomposable into three Hamilton cycles, then G is strongly conservative.*

Proof. Call the three Hamilton cycles H_1 , H_2 , and H_3 . We direct the edges of each of H_1 , H_2 , and H_3 in such a way that a traveler going around one of these cycles finds all the arrows going in the same direction. See Figures 6.2.5 and 6.2.6. We choose a vertex a and label the edges of H_3 , beginning at a , by

$$(H_3) \quad a \quad 3n, 3n-6, \dots, 15, 9, 3, \quad b \quad 3n-3, \dots, 18, 12, 6.$$

Then we label the edges of H_1 , beginning at a , by

$$(H_1) \quad a \quad 1, 4, 7, \dots, 3n-5, 3n-2.$$

For H_2 , we do not begin at a . We begin at the vertex b between the edge labeled 3 and the edge labeled $3n-3$ in H_3 .

$$(H_2) \quad \dots, 3n-1, \quad b \quad 2, 5, 8, \dots$$

Then in every vertex except a and b there is a net decrease of six in H_3 , a net increase of three in H_1 , and a net increase of three in H_2 . Hence Kirchhoff's Current Law holds at every vertex other than a or b . At vertex a , coming in we have the labels 6, $3n-2$, and t , for some number t . Going out we have the

labels $3n$, 1, and $t+3$. The total in is $3n + t + 4$, which is equal to the total out. Thus Kirchhoff's Current Law holds at a . By Theorem 6.2.2, Kirchhoff's Current Law must hold at b . Of course one can also check it directly. Since at each vertex there are three arrows coming in and three arrows going out, a constant h can be added to each label, and Kirchhoff's Current Law will still hold. Hence G is strongly conservative. ■

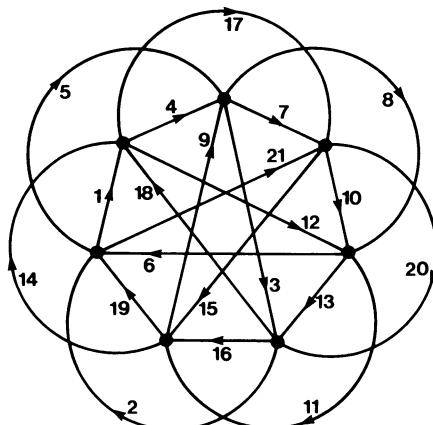


Figure 6.2.5

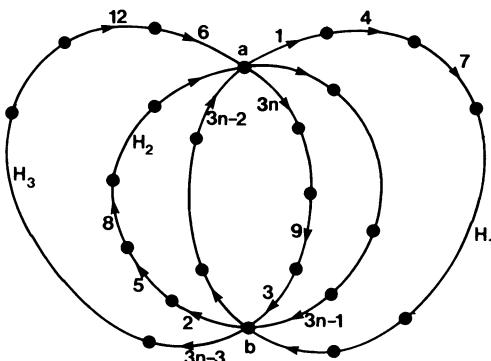


Figure 6.2.6

Theorem 6.2.5. If n is odd, $n \geq 5$, then K_n is conservative.

Proof. If n is odd, K_n is decomposable into $\frac{n-1}{2}$ Hamilton cycles by Theorem 2.3.1. The number $\frac{n-1}{2}$ can be written as $2+2+2+\cdots+2$ or $2+2+2+\cdots+3$. By Theorem 6.2.4, the union of three Hamilton cycles is strongly conservative. By Theorem 6.2.1*, the union of two Hamilton cycles is strongly conservative. So we can decompose K_n , for odd $n \geq 5$, into some number of strongly conservative subgraphs. By applying Theorem 6.2.3 several times, K_n is shown to be conservative for n odd, $n \geq 5$. ■

Theorem 6.2.6. For $n \geq 3$, the wheel with n spokes, W_n , is conservative.

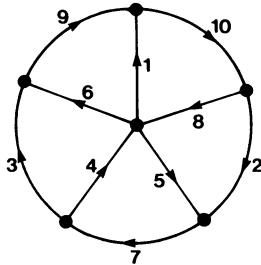


Figure 6.2.7

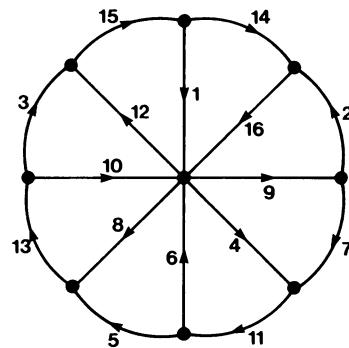


Figure 6.2.8

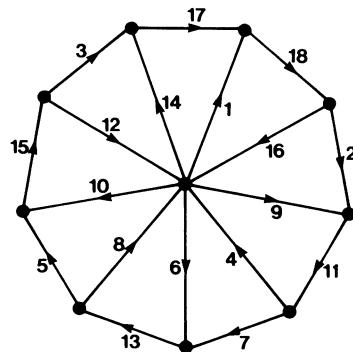


Figure 6.2.9

Proof. For n odd, direct the edges of the outer cycle in a clockwise direction. Choose the labels for the cycle and the spokes as in Figure 6.2.10, where the curve at the top represents the cycle. See Figure 6.2.9 for $n = 9$.

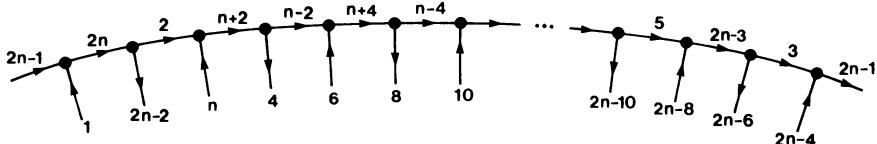


Figure 6.2.10

One can check that every number from 1 to $2n$ occurs precisely once. It is helpful to observe that all the numbers of the cycle are odd except $2n$ and 2 and that all the numbers on the spokes are even except 1 and n . One can also check that Kirchhoff's Current Law holds at each vertex of degree 3. Then, by Theorem 6.2.2, Kirchhoff's Current Law also holds at the center vertex, that is of degree n .

If n is even, we refer to Figure 6.2.11. See Figure 6.2.8 for $n = 8$. Notice that the edge labeled 2 is directed differently. Again, one can check that every number from 1 to $2n$ occurs exactly once and that Kirchhoff's Current Law holds at each vertex. Thus W_n is conservative for $n \geq 3$. ■

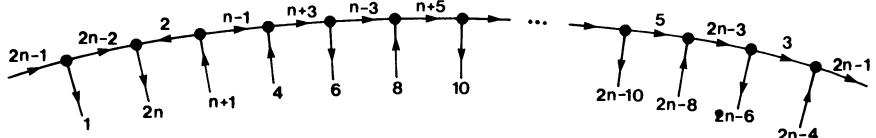


Figure 6.2.11

Theorem 6.2.7. If n is even, $n \geq 4$, then K_n is conservative.

Proof. The general proof works for $n \neq 6$. For the case $n = 6$, Figure 6.2.12 shows K_6 is conservative. We write K_n as the union of W_n and $K_{n-1} - C_{n-1}$, that is, K_{n-1} minus the edges of a Hamilton cycle. W_n is conservative, by Theorem 6.2.6, and $K_{n-1} - C_{n-1}$ is strongly conservative by Theorems 6.2.1*, 6.2.3, and 6.2.4. And so, by Theorem 6.2.3, K_n is conservative for n even, $n \geq 4$. ■

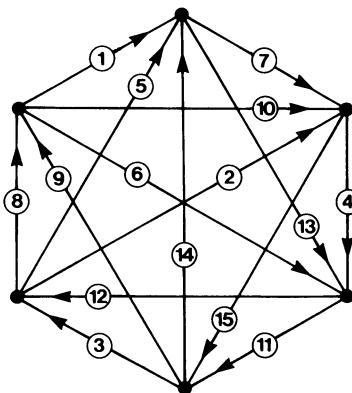


Figure 6.2.12

Exercises 6.2

- 6.2.1. Find a conservative labeling of the graph of Figure 6.2.13.

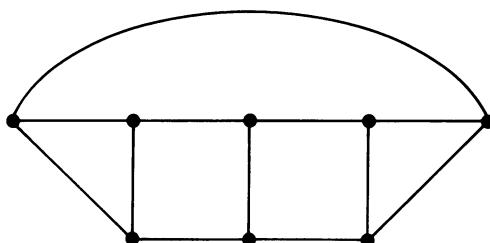


Figure 6.2.13

- 6.2.2. Find conservative labelings of
- the octahedron graph of Figure 1.2.5.
 - the cube graph of Figure 1.2.5.
 - the complete bipartite graph $K_{4,4}$.
 - the complete graph K_5 .
- 6.2.3. First prove that $K_{4,4}$ is strongly conservative, then use this fact to show that $K_{4m,4n}$ is conservative for all $m,n \geq 1$. (Hint – use Theorem 6.2.3.)
- 6.2.4. Show that the cycle C_n is not conservative.

- 6.2.5. Find the smallest graph, other than K_2 , that is magic but not conservative.
- 6.2.6. Find the smallest graph that is conservative but not magic.
- 6.2.7. Find the smallest graph that is both conservative and magic.
- 6.2.8. $K_{3,3}$ is not conservative, but it is possible to label the edges with the numbers 2, 3, 4, . . . , 10 so that Kirchhoff's Current Law holds at each vertex. Do so.

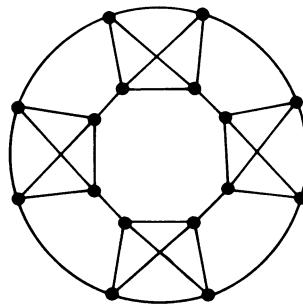


Figure 6.2.14

- 6.2.9. Is the graph of Figure 6.2.14
- conservative?
 - strongly conservative?

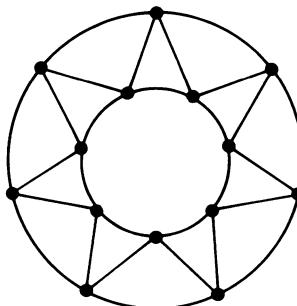


Figure 6.2.15

- 6.2.10. Is the graph of Figure 6.2.15
- conservative?
 - strongly conservative?
- 6.2.11. Prove that if a graph G has a vertex of odd degree, then G is not strongly conservative. (Hint – choose h sufficiently large.)

- 6.2.12. Prove that $K_{n,n}$ is never conservative if n is odd. (Hint – assume $K_{n,n}$, n odd, is conservative, then consider the subgraph consisting of all odd labeled edges.)
- *6.2.13. Prove that if $\binom{mn+1}{2}$ is odd, then $K_{m,n}$ is not conservative.

Chapter 7

APPLICATIONS AND ALGORITHMS

7.1 Spanning Tree Algorithms

In Section 1.1 we presented an algorithm for determining whether a given sequence of nonnegative integers is graphic. The proof of Theorem 3.1.2 actually contains an algorithm, although we did not call it an algorithm. There are many problems in graph theory for which algorithms are useful; for example, finding a spanning tree in a connected graph, determining whether a given graph is Hamiltonian, finding the diameter, the girth, the chromatic number of a given graph, and many more. There are problems of these types in the exercises of this book, but the graphs are small enough that we can solve these problems by inspection or by experiment. If, however, we try to solve such a problem for a graph with, say, 1000 vertices, then we need a formal procedure that we are assured always works. Such a formal procedure is called an *algorithm*.

Notice that for any of the problems mentioned above, algorithms always exist; that is, one can always find a procedure that works. For example, to determine whether a sequence of nonnegative integers of length p is graphic, one could draw every graph with p vertices and then check to see if any of these graphs has the given sequence as its degree sequence. Certainly the method presented in Chapter 1 involves much less work. So another feature of algorithms with which we are concerned is efficiency.

Problem. *Given a connected graph, find a spanning tree of the graph.*

The algorithm we present consists of three steps. We assume that the graph has at least three vertices.

Step 1. Select any edge e of the graph, label the endpoints of e by v_1 and v_2 , and begin growing the tree with e . Set $i = 1$; $j = 3$.

Step 2. Check whether the neighbors of v_i in the graph are all in the existing tree.

Step 3. If there is a neighbor of v_i in the graph that is not yet in the tree, label this neighbor v_j , and add the edge $v_i v_j$ to the tree. If j is equal to the number of vertices in the graph, stop; you now have a spanning tree. Otherwise, set $j = j+1$. If all neighbors of v_i in the graph are in the tree, set $i = i+1$, and return to Step 2.

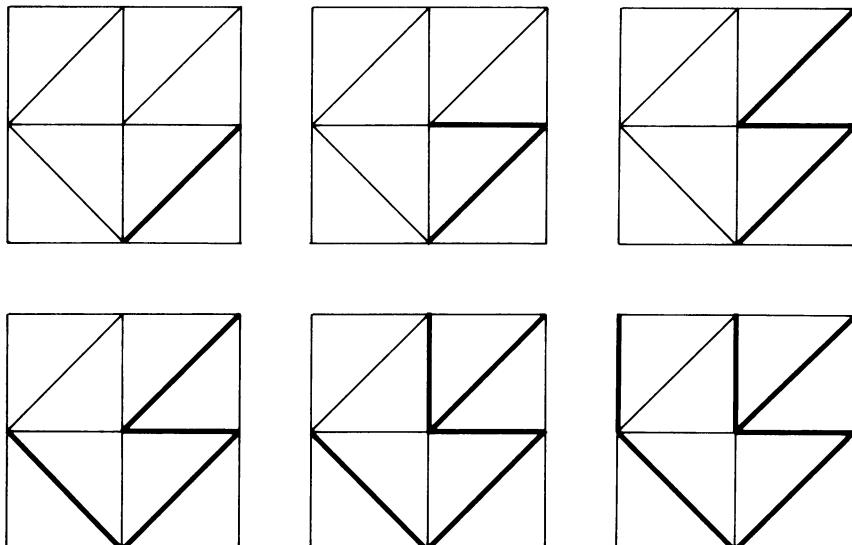


Fig. 7.1.1. Growing a tree.

We illustrate the use of the algorithm by using it to find a spanning tree for the graph of Figure 7.1.2.

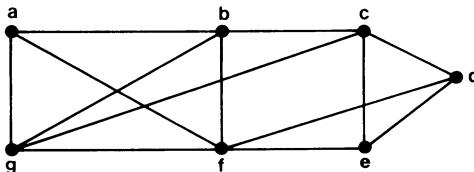
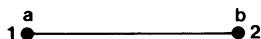


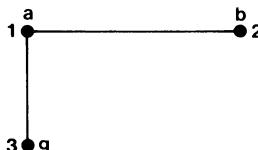
Figure 7.1.2

Step 1. Select the edge ab to begin growing the tree. Label the endpoints v_1 and v_2 , set $i = 1$ and $j = 3$.

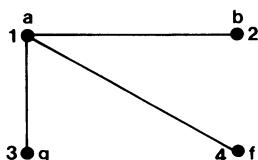


Step 2. Is there a neighbor of vertex a in the graph that is not yet in the tree? Yes, g is such a neighbor.

Step 3. Add vertex g and the edge ag to the tree. Label vertex g by 3. Since $3 < 7$, set $j = 4$, and return to Step 2.



Step 2. Vertex f is a neighbor of a in the graph that is not yet in the tree.



The rest of the procedure is shown in Figure 7.1.3.

In order to establish that the above procedure is indeed an algorithm for finding a spanning tree in a connected graph, we need to assure ourselves that the procedure always works. So we should consider what would happen if the procedure did not work. Then it would be possible to follow the above steps for some connected graph and not obtain a spanning tree of the graph. That is, the

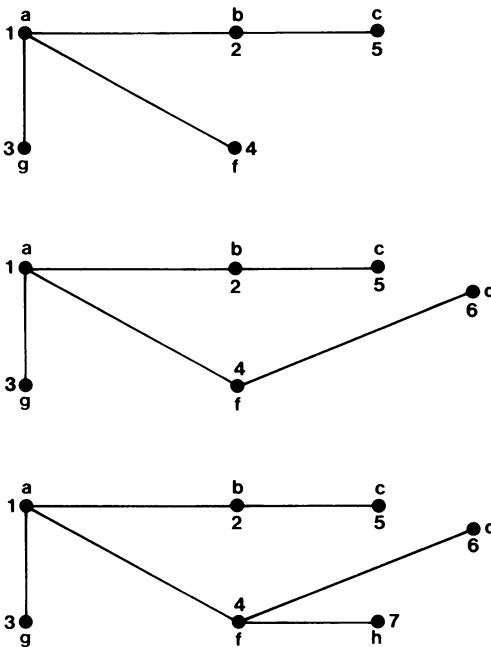


Figure 7.1.3

subgraph we obtained would not span the graph or would contain a cycle. The fact that the resulting subgraph contains no cycle is easy to see, because any time we add an edge to the tree, the edge is incident with only one vertex already in the tree. To see that the tree must span the graph, suppose that it does not. Then there is a vertex v in the given graph that does not get included in the tree. Since the original graph is connected, there is a path from v to a vertex in the tree. The vertices in the path are added in Step 3 of the algorithm until v itself is actually a neighbor in the graph of a vertex in the tree. Then v will be added in Step 3. Thus the procedure actually works, and we are justified in calling it an algorithm.

An analysis of the efficiency of this algorithm is beyond the scope of this book, but the reader is invited to compare it with other spanning tree algorithms such as removing an edge from a cycle in the original graph, then continuing to remove edges from cycles until no cycles remain. One should remember that in a very large graph it may be very complicated to determine whether the graph contains a cycle.

A problem that is closely related to the above has many applications to real-world situations. Here, we shall consider graphs whose edges are labeled with numbers, called *weights*.

Suppose that an amusement park is to be built. The sites of the various rides are determined beforehand, and the builder must also construct a paved walkway that provides access to all the rides. The builder would like to minimize the amount of paving, thus reducing his costs. In the graph of Figure 7.1.4, the vertices represent the rides, and the edges are labeled with the distances between the rides. Where edges are missing, there is some obstruction.

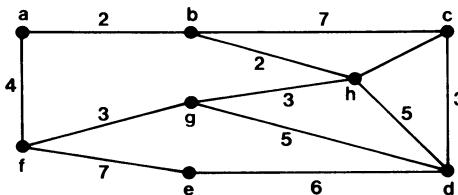


Figure 7.1.4

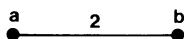
To solve this problem, we need to find a spanning tree that has the smallest possible sum of the weights on the edges; that is, we need to find a *minimum-weight spanning tree*. The algorithm we present here is due to Kruskal.

Step 1. Of all the edges in the graph, select one with smallest weight. (There may be more than one edge with this weight, and the choice is then arbitrary.) Begin growing the tree with this edge.

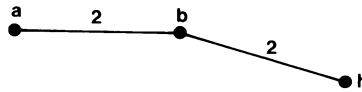
Step 2. Again select an edge with smallest weight that is not yet in the existing forest, and add it, provided that the addition does not create a cycle.

Step 3. Check to see if you now have a spanning tree. If so, stop. If not, repeat Step 2.

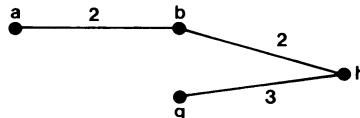
We now employ the algorithm to find a minimum-weight spanning tree for the graph of Figure 7.1.4. There are two edges with the smallest weight 2. We select the edge ab .



Of the edges remaining, the one with smallest weight is bh . Adding it to the existing forest does not create a cycle, so we do so.



Now an edge with smallest weight remaining is hg with weight 3. Again we can add it without creating a cycle.



The rest of the additions are shown in Figure 7.1.5.

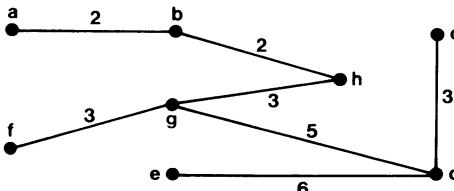
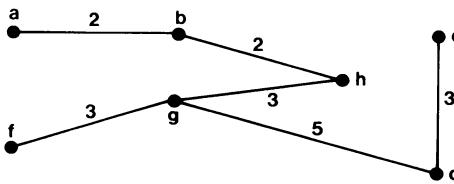
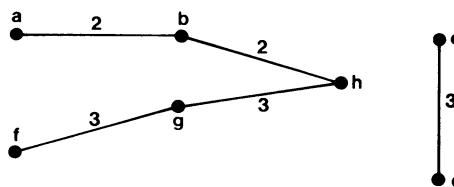
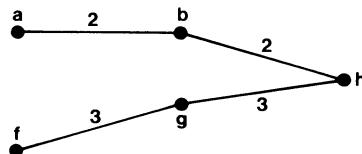


Figure 7.1.5

A proof that Kruskal's algorithm actually produces a minimum-weight spanning tree can be found in most operations research texts and various discrete mathematics texts [31]. Instead of presenting the proof here, we shall consider how to handle the problem of determining whether the addition of a particular edge produces a cycle. If the original graph is relatively small, this is of course simple to determine by inspection. For large graphs, however, one needs a more efficient method.

Recall that a *component* of a graph is a maximal connected subgraph of the graph. Now, notice that if the endpoints of an edge under consideration are in different components of the existing forest, adding the edge does not create a cycle. If they lie in the same component, adding the edge would create a cycle. Hence, if we mark the vertices that lie in the same component of the forest, we can be sure that adding an edge will not create a cycle if its endpoints have distinct component labels. Then when an edge is added between two components, we relabel the vertices of one of the components to correspond to the labels in the other.

Exercises 7.1.

- 7.1.1. Find a spanning tree in the graph of
 - a. Figure 2.2.2.
 - b. Figure 4.2.8.
 - c. Figure 6.1.8
- 7.1.2. Use Kruskal's algorithm to find a minimum weight spanning tree for each of the graphs of Figure 7.1.6.
- 7.1.3. What is the weight of a minimum weight spanning tree for the graph of Figure 6.1.7?
- 7.1.4. Find a minimum weight spanning tree for the graph of Figure 3.2.6, assuming that the unlabeled edges all have weight 5.
- 7.1.5. In this section a real-world application for Kruskal's algorithm is given. Give a real-world application for the spanning tree algorithm and construct a graph that describes the situation.
- 7.1.6. Prove that if T_1 and T_2 are two different spanning trees of a connected graph, and if e_1 is an edge of T_1 that is not in T_2 , then there is an edge e_2 that is in T_2 but not in T_1 with the property that $T_2 + e_1 - e_2$ is a spanning tree of the graph.

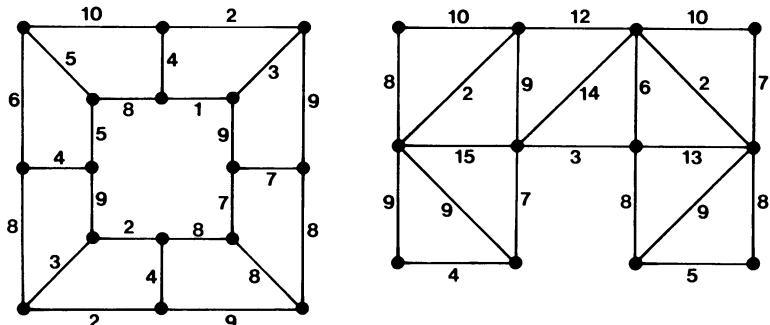


Figure 7.1.6

- * 7.1.7. Using the result of problem 7.1.6, prove that Kruskal's algorithm produces a minimum weight spanning tree for a graph G whose edges are labeled with distinct weights.
- 7.1.8. A minimum weight spanning tree for the graph of Figure 7.1.7 has weight 17. Rearrange the weights 1, 2, ..., 9 on the edges in such a way that the weight of a minimum weight spanning tree is 19.

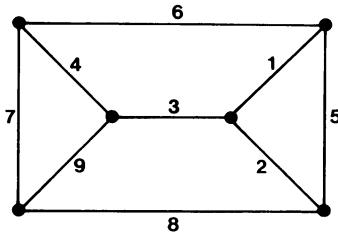


Figure 7.1.7

- 7.1.9. Label the edges of the wheel W_6 with the weights 1, 1, 2, 2, 3, 3, ..., 6, 6 in such a way that the number of minimum weight spanning trees is
 - one.
 - more than one.

- 7.1.10. Label the edges of the cube graph Q_3 with four 1's, four 2's, and four 3's in such a way that a minimum weight spanning tree has weight
 a) 10.
 b) 11.

7.2. Matchings in Graphs, Scheduling Problems

A *matching* in a bipartite graph is a subgraph that is regular of degree 1. That is, a matching is a set of edges with no endpoints in common. A *maximum matching* is a matching that has the most edges possible. If in the bipartite graph under consideration there are equal numbers of red and blue vertices, then a maximum matching could be a 1-factor. Certainly if the graph contains a 1-factor, then the 1-factor will be a maximum matching.

At the very beginning of Chapter 1 is an example of a bipartite graph whose vertices are labeled by children and fruit. The problem is to give each child a piece of fruit he or she likes. In the terms we have just defined, the problem is to find a maximum matching in the bipartite graph of Figure 1.1.1.

Another example of a matching problem concerns assigning workers to tasks. This is known as a *scheduling problem*. Suppose that a foreman at a construction site has seven workers and eight jobs to be done. The graph of Figure 7.2.1 illustrates the situation. The lettered vertices represent the workers, the numbered vertices represent the jobs, and there is an edge between a worker vertex and a job vertex when the worker's skills are suited to that job.

An efficient algorithm for finding a maximum matching in a bipartite graph is called the Hungarian algorithm, developed by Kuhn [15] and arising from the work of König and Egerváry. We illustrate the procedure with the graph in Figure 7.2.1. The first step is to find any matching; one easy way of doing so is to take the lettered vertices in order, and for each select an edge whose numbered endpoint is not yet included in an edge of the matching, as long as this is possible. Figure 7.2.2 pictures one such initial selection.

Our initial matching gives an assignment of five workers to five jobs, leaving two workers and three jobs unmatched. Of course, even in a maximum matching one job will be left out. We now try to improve the matching so that more vertices are matched. We notice that vertices 1 and g are not yet included. Vertex 1 is not adjacent to vertex g , or we could simply add the edge between them to our matching. But vertex g is adjacent to vertex 5, and 1 is adjacent to c . See Figure 7.2.3.

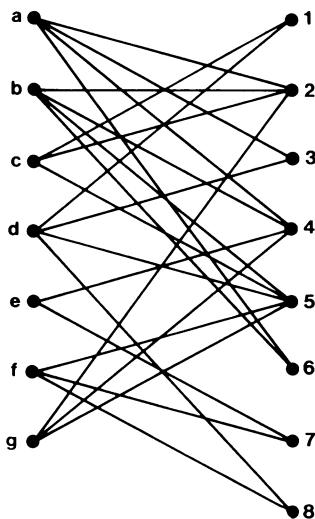


Figure 7.2.1

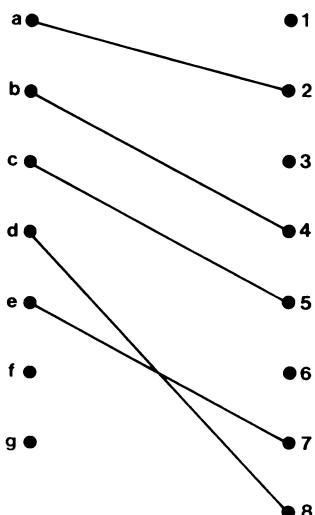


Figure 7.2.2

We can improve the matching by adding the edges from c to 1 and from g to 5 and removing the edge from c to 5. This gives us the new matching shown in Figure 7.2.4.

Given a matching M in a bipartite graph, an M -alternating path in the graph is a path whose edges alternate between edges in M and edges not in M . The path between g and 1 shown in Figure 7.2.3 is an M -alternating path. An M -alternating path that joins two unmatched vertices is called an M -augmenting path, because the matching can be improved by deleting from M the edges of M that are in the path and adding the remaining edges of the path to M .

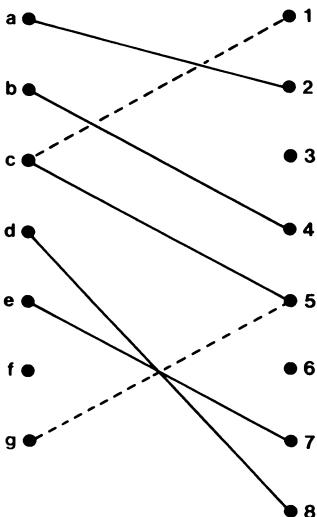


Figure 7.2.3

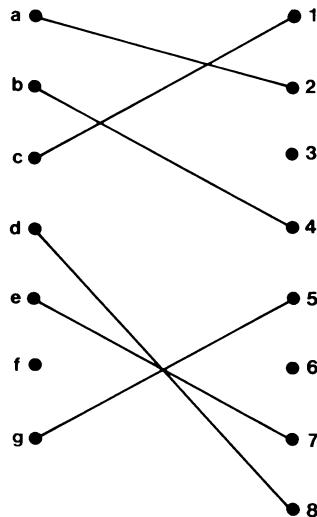


Figure 7.2.4

In the Hungarian algorithm, the method for finding M -alternating paths is as follows. We begin with an unmatched vertex and grow a tree from that vertex that includes the M -alternating paths from the vertex to other vertices, if any exist. The procedure we shall use to grow the tree is what is known as a *depth-first search*. This means that we go as far as possible along one M -alternating path before branching. A tree for vertex f of our example is shown in Figure 7.2.5.

We can stop growing the tree at this point because it contains an M -augmenting path from f to 3. Deleting the edges from 5 to g and from 2 to a

from our matching and adding the edges from f to 5, from g to 2, and from a to 3 we obtain a maximum matching, shown in Figure 7.2.6.

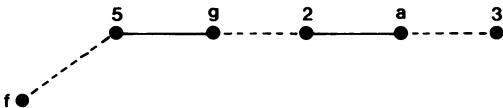


Figure 7.2.5

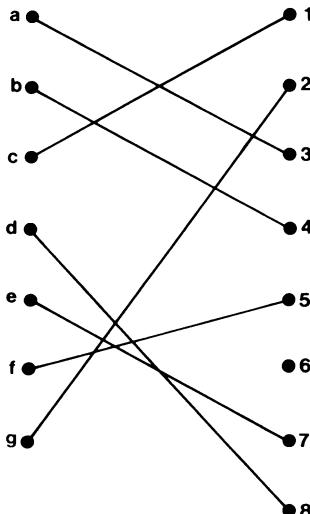


Figure 7.2.6

Now that we have seen how the individual steps are implemented, we can write the Hungarian algorithm in a more formal way. Denote the red vertices by letters and the blue vertices by numbers. We assume that there are m red vertices and n blue vertices and that $m \leq n$.

The Hungarian algorithm.

Step 1. Start with any matching. Mark all red vertices eligible.

Step 2. Are there any unmatched, eligible red vertices? If not, stop; the matching is maximum. If so, go to Step 3.

Step 3. If α is an unmatched, eligible red vertex, grow an M -alternating tree from α . If the tree contains an M -augmenting path, delete the M -edges of the path from M , and add the other edges of the path to M . Mark all red vertices eligible, and go to Step 2. If the tree contains no M -augmenting path, mark α ineligible, and go to Step 2.

We present one more example, using the Hungarian algorithm to find a maximum matching for the graph of Figure 7.2.7.

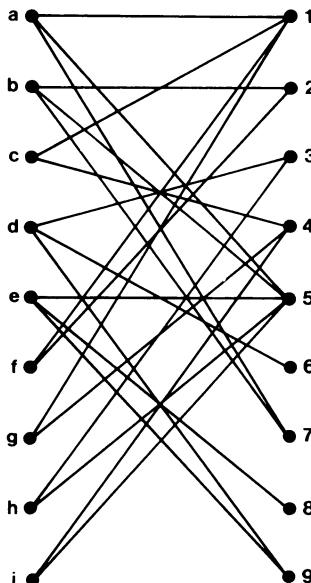
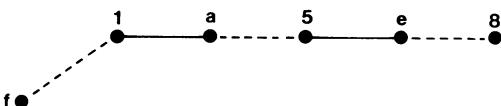


Figure 7.2.7

Step 1. $M = \{a 1, b 2, c 4, d 3, e 5\}$.

Step 2. The vertex f is an unmatched, eligible red vertex.

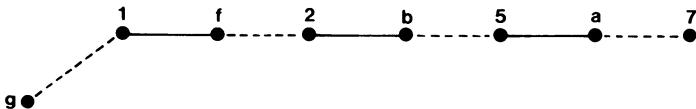
Step 3. Grow the tree from f .



There is an M -augmenting path from f to 8. Our new matching is $M = \{a 5, b 2, c 4, d 3, e 8, f 1\}$.

Step 2. The vertex g is an unmatched, eligible red vertex.

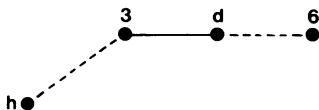
Step 3. Grow the tree from g .



There is an M -augmenting path from g to 7. Set $M = \{a7, b5, c4, d3, e8, f2, g1\}$.

Step 2. The vertex h is an unmatched, eligible red vertex.

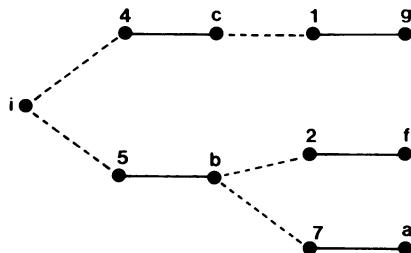
Step 3. Grow the tree from h .



There is an M -augmenting path from h to 6. Set $M = \{a7, b5, c4, d6, e8, f2, g1, h3\}$.

Step 2. The vertex i is the only unmatched, eligible red vertex.

Step 3. Grow the tree from i .



No M -augmenting path exists from i to an unmatched blue vertex. The matching is maximum, so we stop.

Exercises 7.2.

- 7.2.1. Six people arrive late at a restaurant and discover that there is one serving left of each of seven dinners, Pork Chops, Prime Rib, Vegetarian Curry, Trout, Linguini with Clam Sauce, Macaroni and Cheese, and Barbecued Pork Spare Ribs. Mr. A cannot eat pork; Mrs.

A eats no meat at all; Mr. B is allergic to curry, clams, and barbecue sauce; Mr. C, by one of those quirks that make life stranger than fiction, will not eat anything that starts with the letter C; Mrs. C prefers to eat pasta, but will settle for beef or fish; and Miss D will eat anything. Can the six people all get a dinner they will eat?

- 7.2.2. An accounting firm has eight CPA's available and seven audits to be scheduled. The table shows which of the CPA's are qualified for the various audits; there is an x under audit n if the CPA is qualified. Find an assignment of qualified CPA's to the audits.

Audit:	1	2	3	4	5	6	7
CPA							
a			x	x	x		
b	x			x		x	
c			x			x	x
d		x	x	x			
e	x	x	x				x
f		x		x		x	
g		x		x	x		
h			x	x	x		

- 7.2.3. Mrs. A operates a dance school for girls and boys. The parents are coming to a dance recital, and Mrs. A wants the dancing couples to be evenly matching by height, that is, she wants to pair boys with girls whose height is within one inch of the boy's height. The heights are shown below. Can Mrs. A achieve this pairing?

Girl	Height	Boy	Height
1	4'10"	a	4'7"
2	5'	b	4'9"
3	4'8"	c	5'1"
4	5'6"	d	4'10"
5	5'1"	e	5'4"
6	5'3"	f	5'
7	4'11"	g	5'5"
8	5'5"	h	5'2"
9	5'1"	i	5'2"
10	5'	j	4'11"

7.3. Binary Trees and Prefix Codes

In Chapter 6 we defined directed graphs. In a directed graph, at any vertex there is some number of arcs directed toward, or coming into, that vertex and some number of arcs directed away from, or coming out of, that vertex. The number of arcs coming into a vertex v is called the *indegree* of v . The number of arcs coming out of a vertex v is called the *outdegree* of v . If there is an arc from vertex v to vertex w , we say that w is *adjacent from* v and v is *adjacent to* w .

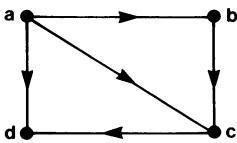


Figure 7.3.1

In the directed graph of Figure 7.3.1, vertex a has indegree 0 and outdegree 3, b has indegree 1 and outdegree 1, c has indegree 2 and outdegree 1, and d has indegree 2 and outdegree 0.

A *rooted tree* is a directed tree that has precisely one vertex r that has indegree 0. The vertex r is called the *root* of the rooted tree. A *binary tree* is a rooted tree with the additional property that every vertex has outdegree 0, 1, or 2. Those vertices with outdegree 0 are called *leaves*. The rooted tree of Figure 7.3.2 is a binary tree. Usually the arrows are omitted from the drawing, as in Figure 7.3.3, and we assume that the arcs are directed downward. Thus, unlike trees in nature, binary trees have the root at the top and the leaves at the bottom.

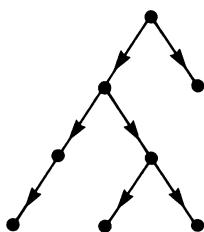


Figure 7.3.2

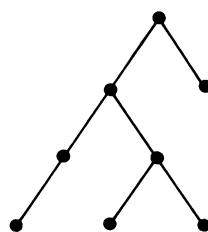


Figure 7.3.3

Define an *alphabet* A to be a finite set of symbols. For our examples here we shall assume that $A = \{0, 1\}$. A *code* is a set C of sequences of elements A . The elements of C will be called *codewords*. Since we are assuming that

$A = \{0, 1\}$, our codewords will be binary sequences. One example of such a code is $C = \{0, 00, 10, 101\}$. Since computers are inherently binary, it is very natural to use codes consisting of binary sequences to store information in a computer.

Suppose that we wish to represent the message

MADAM, I'M ADAM.

using a code. That is, we wish to *encode* the message. We can see that we need a codeword to represent *M*, one for *A*, one for *,*, one for *D*, and so on. We would like to have the encoded message as short as possible, so we might select the code

$$C = \{0, 1, 00, 01, 10, 11, 000\},$$

representing the symbols

$$\begin{array}{ccccccc} M & A & D & , & I & ' & . \\ 0 & 1 & 00 & 01 & 10 & 11 & 000 \end{array}.$$

When we encode our message we get the string

$$010010011011010010000$$

of 0's and 1's.

We can see that our code is not very satisfactory, because when we attempt to return to the original message, we encounter some problems. We do not know where the spaces between words occur, and when we see, for instance, 01001, there is ambiguity. Does this represent *MADA*, or *MI*, or *,DA*? The first difficulty is easily resolved; we simply designate an additional codeword to represent a space. For example, we could have space = 111. For the second problem, we must do a little more work. The difficulty is arising from the fact that 0 is a codeword, 1 is a codeword, and 01 is a codeword. We say that 0 is a *prefix* of 01. A *prefix code* is a code C with the property that no codeword in C is the prefix of another codeword in C . Hence our example is not a prefix code, but $C_1 = \{000, 001, 010, 0111\}$ is a prefix code.

We can use binary trees to construct prefix codes. Starting from the root, at each vertex we label the arc going to the left by a 0 and the one going to the right by a 1. Such a labeling of the binary tree in Figure 7.3.3 is shown in Figure 7.3.4.

If we now label the leaves of the binary tree by the symbols for which we want codewords, we can obtain a prefix code for these symbols by recording the

labels on the arcs in the path from the root to each leaf. Hence, if the leaves in the tree of Figure 7.3.4 are labeled by M , A , D , and I respectively from left to right, we get the codewords

$$\begin{array}{cccc} M & A & D & I \\ 000 & 010 & 011 & 1 \end{array},$$

and since none of these codewords is the prefix of another, we can use them without ambiguity. We actually need a larger binary tree, since we have eight symbols to represent. Thus we might derive a prefix code for our message by using the binary tree shown in Figure 7.3.5.

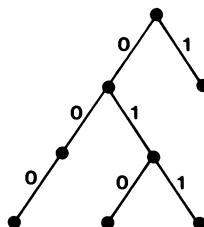


Figure 7.3.4

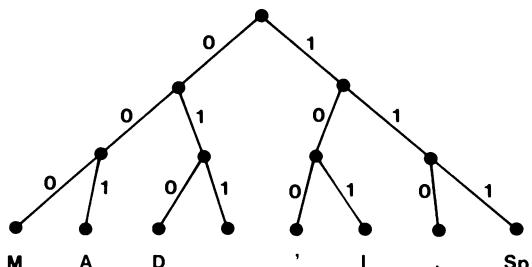


Figure 7.3.5

The prefix code we derive using this binary tree is

$$\begin{array}{ccccccc} M & A & D & , & ' & I & space \\ 000 & 001 & 010 & 011 & 100 & 101 & 110 \quad 111 \end{array}.$$

Using these codewords to encode our message, we get the string

000001010001000011111101100000111001010001000110.

We have resolved the problem of ambiguity, but we may have sacrificed economy. We would like to have both if possible. One way of doing so would be to notice that certain letters occur more frequently than others in our original

message. The letter M occurs four times, while the letter I occurs only once. It would make sense to have shorter codewords for those symbols that occur the most frequently. To develop a method for obtaining a code that is optimal with this property, we need a few more definitions.

We shall use *weight* to mean a nonnegative integer, in particular, the number of times that a symbol occurs will be called a weight. A *weighted binary tree* T is a binary tree whose leaves are labeled by weights. For each leaf b in T , we denote the weight of b by $w(b)$, and let $l(b)$ be the length of the path from the root to b . Then if L is the set of leaves of T , we can define the *weight of the binary tree* T to be

$$w(T) = \sum_{b \in L} l(b)w(b).$$

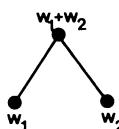
If we observe that the length of a codeword derived from a binary tree is $l(b)$, where b is the leaf labeled with the represented symbol, when we have also labeled b with the weight of the symbol, we can see that the contribution of this codeword to the length of the encoded message is exactly $l(b)w(b)$. Thus our message will have shortest length precisely when $w(T)$ is minimal.

Define a *multiset* to be a collection of elements with repetitions allowed. For instance, $M = \{2, 2, 3, 4, 4\}$ is a multiset. Given a multiset M of weights, a *minimum-weight binary tree* for the multiset M is a binary tree T whose leaves are labeled by the elements of M , such that $w(T)$ is minimal.

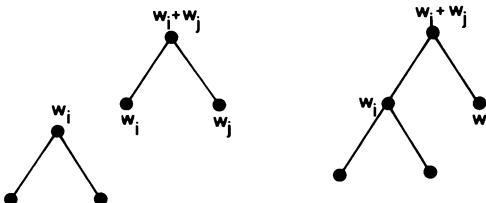
There is an elegant algorithm due to David Huffman [12] for finding a minimum-weight binary tree for a given multiset M of weights. We shall first state the algorithm and then illustrate its use by an example.

Huffman's algorithm.

Step 1. Select two smallest weights w_1 and w_2 in M . (There may be more than one way to do so, and the choice is then arbitrary.) Make these the labels on two leaves whose arcs are incident with the same vertex, and begin growing the tree with this configuration. Label the vertex of outdegree 2 with the sum $w_1 + w_2$, cross out w_1 and w_2 in M , and add $w_1 + w_2$ as an element of M .



Step 2. Select two smallest weights remaining in M . Make these the labels on two leaves whose arcs are incident with the same vertex, and add this configuration to the existing directed forest. If one or both of the weights used is a sum that labels a vertex with outdegree 2 and indegree 0, replace the leaf labeled with the sum by the existing vertex labeled by that sum, retaining the rest of the configuration. Label the new vertex of outdegree 2 by the sum



of the two weights, cross out the two weights in M , and replace them by the single weight that is their sum.

Step 3. Does M have more than one element? If so, go to Step 2. If not, stop; T is a minimum-weight binary tree for the original M , and $w(T)$ is the remaining element in T .

We illustrate the use of Huffman's algorithm by developing a prefix code for our message.

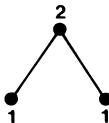
MADAM, I'M ADAM.

The weights of the symbols are shown below.

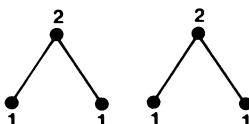
Symbol	Weight
<i>M</i>	4
<i>A</i>	4
<i>D</i>	2
,	1
<i>I</i>	1
,	1
.	1
space	2

Hence our multiset $M = \{4, 4, 2, 1, 1, 1, 1, 2\}$.

Two smallest weights in M are 1 and 1. We make these the labels on two leaves, cross out two 1's in M and replace them by a 2.

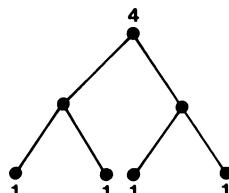


Our new multiset is $M = \{4, 4, 2, 2, 1, 1, 2\}$. The two smallest weights in M are again 1 and 1. They are not labels of vertices with outdegree 2 and indegree 0, so we again make them the labels on leaves adjacent from a common vertex, cross out the 1's in M and replace them by a 2.

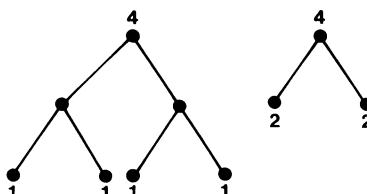


The new multiset is $M = \{4, 4, 2, 2, 2, 2\}$. It has more than one element, so we return to Step 2.

Two smallest weights are now 2 and 2. These occur as sums labeling vertices of outdegree 2 and indegree 0 already in the forest, so we add arcs so that these two vertices are adjacent from a common vertex and label that vertex by the sum of the two weights. We cross out two 2's in M and replace them by a 4.



Notice that we no longer need the labels on the vertices that are not actually leaves in the tree unless they have indegree 0. Our new multiset is $M = \{4, 4, 4, 2, 2\}$. It has more than one element. The two smallest weights in M are 2 and 2. These are not labels of existing vertices with indegree 0, so we make them the labels of leaves.



Continuing in this manner we obtain the successive forests shown in Figure 7.3.6.

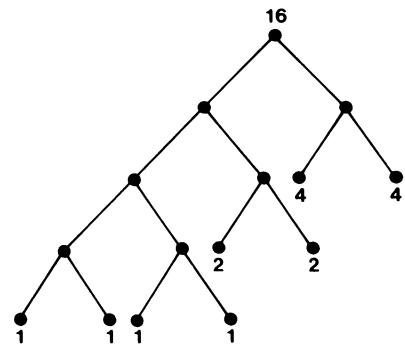
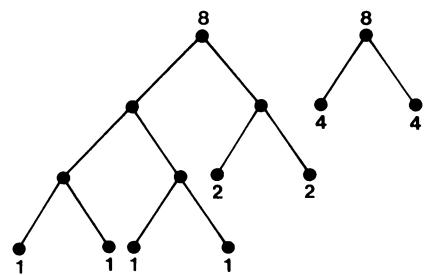
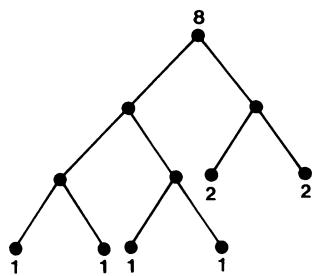


Figure 7.3.6

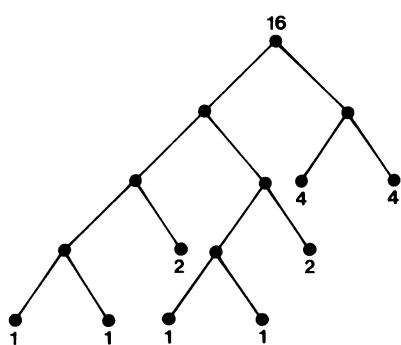
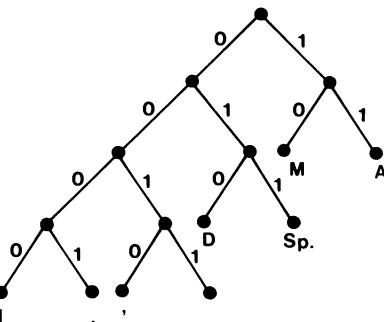


Figure 7.3.7

Notice that the binary tree produced by the algorithm may not be unique. What is unique is the weight of the tree; that is, any tree obtained by using Huffman's algorithm for a given multiset M of weights will have the same weight, and that weight is minimal. Figure 7.3.7 shows a different binary tree for our multiset M .

To see that Huffman's algorithm produces a minimum-weight binary tree for a multiset M of weights, consider that if M has only two elements, w_1 and w_2 , then certainly the minimum-weight binary tree for M will be the binary tree T consisting of the root and two leaves labeled w_1 and w_2 . Then $w(T) = w_1 + w_2$. Then assume that Huffman's algorithm produces a minimum-weight binary tree for any multiset of weights with $n-1$ elements, and let M be a multiset of weights containing n elements. If w_1 and w_2 are two smallest elements of M , the first part of the tree constructed will be as shown in Figure 7.3.8. If we now consider \hat{M} to be the multiset obtained by removing w_1 and w_2 from M and adding $w_1 + w_2$ to M , then \hat{M} has $n-1$ elements, and by our assumption Huffman's algorithm will produce a minimum-weight binary tree for \hat{M} . Then replacing the leaf labeled by $w_1 + w_2$ in this tree by the configuration of Figure 7.3.8 gives a minimum-weight binary tree for the multiset M .

We now label the arcs and leaves of the binary tree in Figure 7.3.6 to derive a prefix code for our message.



Our new code is

M	A	D	$Sp.$	I	$,$	$.$	
10	11	010	011	0000	0001	0010	0011

and the new encoding of the message is

10110101110000101100000010100111101011100011.



Figure 7.3.8

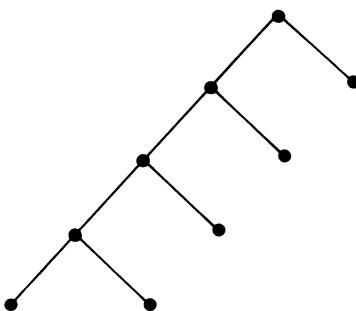


Figure 7.3.9

Exercises 7.3.

- 7.3.1. Why is a code derived from a binary tree as shown in this section always a prefix code?
- 7.3.2. A *regular binary tree* is a binary tree with all vertices having outdegree 0 or 2. Show that a minimum-weight binary tree for a given multiset M of weights is necessarily a regular binary tree.
- 7.3.3. Develop a prefix code and encode the message
TEASE THE FLEAS EASILY, PLEASE.
- 7.3.4. Construct a minimum-weight binary tree for $M = \{4, 4, 4, 4, 4, 4, 4, 4\}$.
- 7.3.5. Find M such that the minimum weight binary tree for M is the binary tree of Figure 7.3.9. (More than one solution is possible.)
- 7.3.6. Suppose that all symbols occur with the same frequency in a message. If there are n symbols, how long will the binary codewords in a prefix code for the message be, assuming that the length of the encoded message is minimal?
- 7.3.7. For which multisets M is the minimum-weight binary tree for M uniquely determined?

Chapter 8

DRAWINGS OF GRAPHS

8.1 Planar Graphs

In all the previous chapters, we dealt with the area of pure, or abstract, graph theory. That means that the vertices are elements of an abstract set, and the edges are pairs of elements of the set. However, every graph theorist in the world visualizes the graphs as structures where the vertices are points and the edges are lines. So it is very *convenient* to see a graph as a geometric structure. In this chapter and the following, we shall work in the area of topological graph theory. There it is *required* to see the vertices as points and the edges as lines in the plane. In other words, we work with *drawings* of graphs. Our treatment will not be a rigorous topological discussion. We shall define what a simple curve is intuitively. The curve on the left in Figure 8.1.1 is simple; the one on the right is not, because it crosses itself. We shall use, but not prove, the Jordan Curve Theorem, which states that any simple closed curve divides the plane into two regions, the inside and the outside.

Some graphs cannot be drawn in the plane without edges crossing. The reader is invited to try drawing K_5 in the plane without any edges crossing.

A *planar graph* is a graph that can be drawn in the plane with no edges crossing.

A *plane drawing* of a planar graph is a drawing of the graph in the plane with no crossings. We wish to emphasize that the graph itself is completely independent of any drawing.

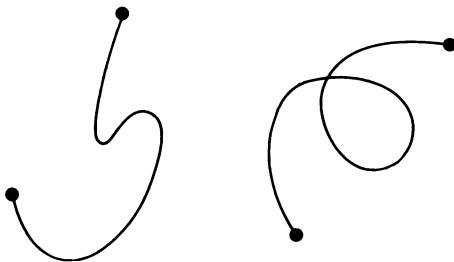


Figure 8.1.1

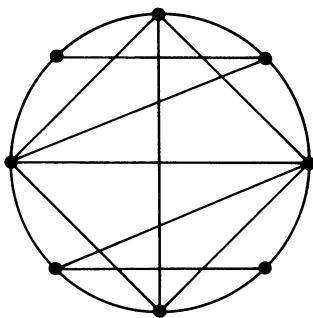


Figure 8.1.2. A planar graph

The plane drawing of the octahedron graph in Figure 1.2.5 divides the plane into eight triangles. One of the triangles has infinite area. We call this one the *outside* triangle. These triangles are called faces and, in fact, the word *octahedron* means eight faces. In previous chapters we used the word triangle to mean the graph K_3 , but now K_3 will be only the boundary of a triangle. In general, the polygons formed by the edges in a plane drawing of a planar graph will be called *faces* or *regions*. In the cube graph, the faces are all quadrangles. If the graph has end vertices, then a face could look like the faces in the drawing of Figure 8.1.3.

Remarkably enough, drawing a graph in the plane with no crossings is the same as drawing the graph on the surface of the sphere with no crossings. Then there is no need to single out a region as an outside region.

We now consider some of the nice planar graphs we have encountered. Let r denote the number of regions in a plane drawing and, as usual, p and q will be the number of vertices and edges. Then from Figures 1.1.14 and 1.2.5, we can make the following table.

	p	q	r
Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

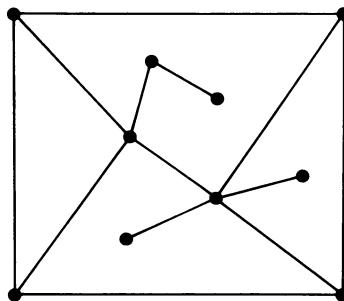


Figure 8.1.3

Euler looked at these numbers in 1750 and discovered that in these examples

$$p - q + r = 2$$

is true. Then he discovered that this formula holds for every plane drawing of a connected graph. For instance, the formula is true for a tree. In a tree we know that $r = 1$; $q = p - 1$ by Theorem 1.3.2, so

$$p - q + r = p - (p - 1) + 1 = 2.$$

We can check that the formula holds for the plane drawing in Figure 8.1.3.

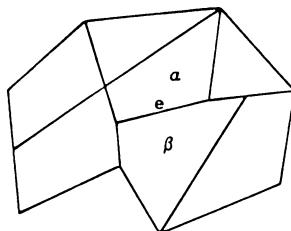
Theorem 8.1.1. (Euler's Polyhedral Formula). *If a plane drawing of a connected graph with p vertices and q edges has r regions, then*

$$p - q + r = 2.$$

Proof. The proof is by induction on the number of cycles in the graph. If a connected graph G has no cycles, then G is a tree, $q = p - 1$, and in a plane drawing of G there is one region. So as we computed above, $p - q + r = 2$.

Now we assume that the formula holds for all plane drawings of

connected graphs with fewer than n cycles. Given a plane drawing of a connected graph G with n cycles, p vertices, q edges, and r regions, consider one cycle C of G . By the Jordan Curve Theorem C divides the plane into an inside and an outside. Let e be an edge of C . This edge lies on exactly two distinct regions. One is part of the outside, and one is part of the inside.



If we remove e , these two regions will become one. The plane drawing of $G - e$ will have p vertices, $q-1$ edges, and $r-1$ regions. Since $G - e$ has fewer than n cycles, by the induction hypothesis,

$$p - (q-1) + (r-1) = 2.$$

If we cancel the ones, we get the formula for G . ■

A planar graph is called *maximal planar* if adding an edge between any two nonadjacent vertices results in a nonplanar graph. The cube graph of Figure 1.2.5 is not maximal planar because we can add a diagonal edge in any one of the square regions, and the resulting graph will still be planar. The octahedron graph is maximal planar. The graph $K_5 - e$ is maximal planar, as is the graph K_4 . In these examples, all the regions are triangles. This is true in general, since if there is a region with more than three sides, we can add edges between nonadjacent vertices on the boundary of this region and still have a planar graph.

Theorem 8.1.2. *If G is a maximal planar graph with p vertices and q edges, $p \geq 3$, then $q = 3p - 6$.*

Proof. Consider a plane drawing of G with r regions. Every region is bounded by three edges, and every edge lies on two regions. This gives us the equation $3r = 2q$. To see that this is true, we count these things in two ways. Suppose that we count the number h of pairs (t, e) where t is a triangle, and e is an edge of t . Since each triangle contains three edges, and there are r triangles, we have

that $h = 3r$. Alternately, since each edge is on the boundary of two triangles, and there are q edges, we have that $h = 2q$. Hence $r = \frac{2}{3}q$, and by Theorem 8.1.1, $q = 3p - 6$. ■

The next theorem is an immediate consequence of Theorem 8.1.2.

Theorem 8.1.3. *A planar graph with p vertices, $p \geq 3$, has no more than $3p - 6$ edges.*

Theorem 8.1.3 tells us that in a planar graph, for a given number of vertices, the number of edges is limited.

Theorem 8.1.4. *The graph K_5 is not planar.*

Proof. Suppose that K_5 is planar. K_5 has $p = 5$ and $q = 10$, then by Theorem 8.1.3, if K_5 is planar, then

$$10 \leq 3 \cdot 5 - 6,$$

which is not true. Thus K_5 is not planar. ■

A bipartite graph has no triangles. So a planar bipartite graph is never maximal planar. The cube graph is an example of a planar bipartite graph in which the addition of an edge between any two nonadjacent vertices would either result in a nonplanar graph or a nonbipartite graph or both.

Theorem 8.1.5. *If G is a planar bipartite graph with p vertices and q edges, $p \geq 3$, then $q \leq 2p - 4$.*

Proof. The proof is the same as that of Theorem 8.1.2, except that now, instead of a triangle, the region with the fewest sides possible is a quadrangle so that $2q \geq 4r$. ■

Theorem 8.1.6. *The complete bipartite graph $K_{3,3}$ is not planar.*

The proof of Theorem 8.1.6 is left as an exercise.

Theorem 8.1.7. *Every planar graph contains at least one vertex of degree ≤ 5 .*

Proof. If the theorem is not true, then there exists a planar graph G with p vertices and q edges and every vertex of degree ≥ 6 . Then in G

$$2q = \sum \deg(v) \geq 6p, \text{ so}$$

$$q \geq 3p,$$

which is a contradiction by Theorem 8.1.3. ■

In fact, as the next theorem will show, if G is a connected planar graph with at least four vertices, then G has at least four vertices of degree at most five.

Theorem 8.1.8. *Suppose G is a maximal planar graph with p vertices and q edges, $p \geq 4$. Let p_i denote the number of vertices of degree i . Then*

$$3p_3 + 2p_4 + p_5 = 12 + p_7 + 2p_8 + 3p_9 + 4p_{10} + \dots$$

Proof. We write p as the sum of the p_i 's, and $2q$ as the sum of the degrees of the vertices of G .

$$p = p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 + p_{10} + \dots, \text{ and}$$

$$2q = 3p_3 + 4p_4 + 5p_5 + 6p_6 + 7p_7 + 8p_8 + 9p_9 + 10p_{10} + \dots$$

By Theorem 8.1.2, $q = 3p - 6$, so that if we multiply the first equation by six and subtract the second equation, we obtain $3p_3 + 2p_4 + p_5 - p_7 - 2p_8 - 3p_9 - 4p_{10} - \dots = 12$, and thus

$$3p_3 + 2p_4 + p_5 = p_7 + 2p_8 + 3p_9 + 4p_{10} + \dots + 12. \quad \blacksquare$$

Exercises 8.1.

- 8.1.1. Draw the graph of Figure 8.1.2 in the plane with no crossings.
- 8.1.2. Draw the graph of Figure 8.1.2 in the plane with no crossings *and* such that every edge is a straight line.
- 8.1.3. Show that every graph with at most three cycles is planar.
- 8.1.4. Is the graph of Figure 2.1.9 maximal planar?
- 8.1.5. Are the graphs of Figure 4.3.4
 - a. planar?
 - b. maximal planar?

- 8.1.6. Prove Theorem 8.1.6.
- 8.1.7. Find the smallest planar graph that is regular of degree 4.
- 8.1.8. Prove that the icosahedron graph of Figure 1.2.5 is not the only planar graph that is regular of degree 5.
- 8.1.9. Is the graph of Figure 8.1.4
- planar?
 - maximal planar?
 - If not maximal planar, either remove or add edges, or both, to make it maximal planar.

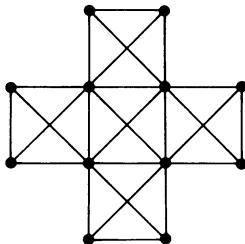


Figure 8.1.4

- 8.1.10. Suppose that a graph G has 1000 vertices and 3000 edges. Is G planar?
- 8.1.11. Find a graph G with degree sequence $(4, 4, 4, 4, 3, 3)$ such that
- G is planar.
 - G is not planar.
- 8.1.12. Given a drawing of a planar graph regular of degree 4 with 16 vertices, if each region is either a triangle or a square, how many triangles and how many quadrangles are there?
- 8.1.13. Given a tree with p vertices, how many edges must you add without adding vertices to obtain a maximal planar graph?

8.2. The Four Color Theorem

Recall that in a multigraph two vertices may be connected by more than one edge. A plane drawing of a multigraph may contain regions that have only two edges, called *lunes*.

A plane drawing of a connected, bridgeless, planar multigraph is called a *map*. If the multigraph is cubic in addition to the above properties, the drawing is called a *normal map*. Figure 8.2.1 shows an example of a normal map. The plane drawing of a graph in Figure 8.2.2 is not a map, because the graph has a bridge.

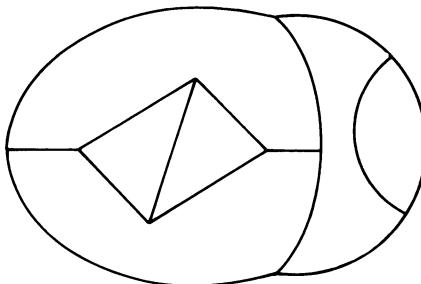


Figure 8.2.1

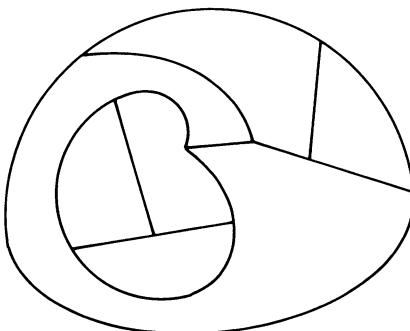


Figure 8.2.2

In a map, the regions are called *countries*. Two countries are said to be *adjacent* if they have an edge in common. The map of Figure 8.2.1 has two countries that are adjacent twice, but it is still a map.

A *coloring* of a map M is an assignment of a color to each of the countries of M so that no two adjacent countries have the same color. Recall that two edges in a graph are said to be adjacent if they are incident with the

same vertex and that a proper edge coloring of a graph G is an assignment of a color to each edge of G so that no two adjacent edges receive the same color.

Theorem 8.2.1. (Tait, 1880). *If a normal map has a coloring of the countries by four colors, then the edges of the map can be properly colored by three colors.*

Proof. Given a normal map colored by four colors, if the colors are 1, 2, 3, 4, we shall color the edges with the three colors a, b, c by the following recipe.

If an edge lies between countries colored	color the edge
1 and 2	a
3 and 4	a
1 and 3	b
2 and 4	b
1 and 4	c
2 and 3	c .

In Figure 8.2.4 we have all the ways, up to turnings and reflections, that three countries in a normal map colored with four colors can meet at a vertex. In each case, the recipe gives a proper edge coloring. ■

Figure 8.2.3 shows the recipe. The edges are now colored by three colors. We must show that this is a proper coloring.

This was only half of Tait's Theorem. He also proved the converse, which is much harder. Before we prove the converse, we shall need a lemma.

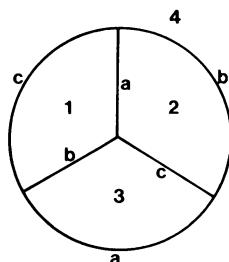


Figure 8.2.3

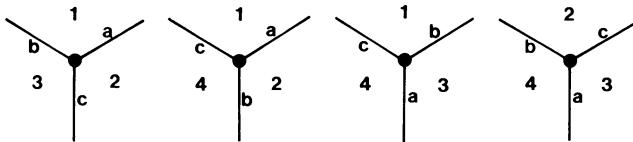


Figure 8.2.4

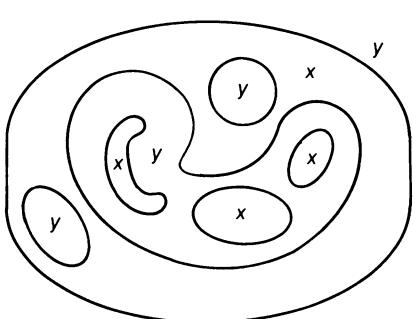


Figure 8.2.5

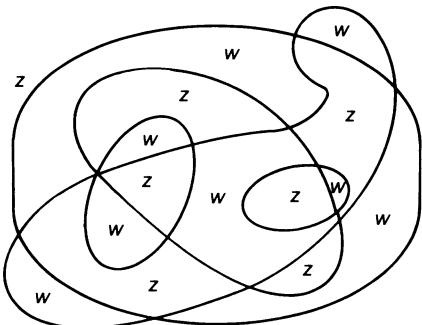


Figure 8.2.6

Lemma 8.2.2. *Given a finite number of simple closed curves (cycles) in the plane, the regions defined by these curves are colorable by two colors.*

Proof. We first remark that the lemma is true whether the curves intersect each other, as in Figure 8.2.6, or not, as in Figure 8.2.5.

Each cycle divides the plane into an inside and an outside. If a region is inside an even number of curves, color it x . If it is inside an odd number of curves, color it y . Since the boundary between any two of these regions is part of one of the simple closed curves, one of the regions will be inside exactly one more curve than the other. Hence one will be inside an even number of curves, and the other will be inside an odd number of curves, so they will receive different colors. ■

Theorem 8.2.3 (Tait). *If the edges of a normal map can be properly colored by three colors, then the countries of a map can be colored by four colors.*

Proof. Given a normal map M and a proper coloring of the edges of M by three colors a, b, c , first we consider the subgraph of M that consists of all the edges labeled a or b . The subgraph consists of a set of cycles. By Lemma 8.2.2, the regions defined by these cycles are colorable by two colors, x and y . In this coloring, every country of M receives either x or y . Now we consider the subgraph of M that consists of all the edges labeled a or c . This subgraph consists of a set of cycles, and again, by Lemma 8.2.2, the regions are colorable by two colors, z and w . In this coloring every country of M receives either z or w . Now each country of M has received one of the four pairs xz, xw, yz , or yw . We consider these four pairs as four colors. Consider two adjacent countries in M . If the edge between these countries is labeled b , then one has an x and the other has a y , so their colors are different. If the edge between is labeled c , then one has a z and the other has a w , so their colors are different. If the edge between is labeled a , then one has an x and the other has a y , so again their colors are different. Hence M is colorable by four colors. ■

Now we can even find an interesting labeling of the vertices by two symbols, black and white.

Theorem 8.2.4. *If in a normal map the edges are properly colored by three colors, then the vertices can be labeled black and white so that around any given country, the number of black vertices minus the number of white vertices is always a multiple of three.*

Proof. Here is the recipe for labeling the vertices. If the edges a, b, c occur clockwise around a vertex, label it black ●. If they occur counterclockwise around a vertex, label that vertex white ○. See Figure 8.2.7.

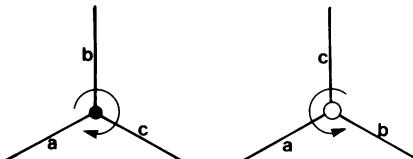


Figure 8.2.7

Now we must show that the labeling given by the above recipe actually has the property that around a given country, the number of black vertices minus the number of white vertices is a multiple of three.

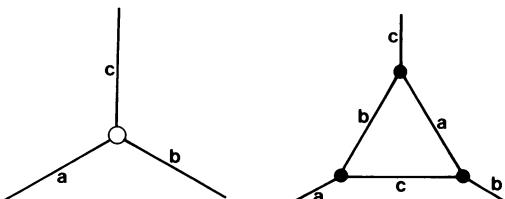


Figure 8.2.8

We construct a new map M' from M by replacing each white vertex of M by a triangle with black vertices as in Figure 8.2.8. Now we consider a given country A of M' .

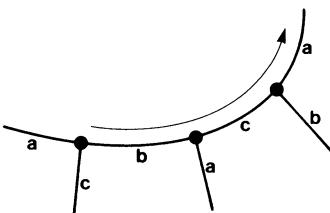


Figure 8.2.9

The edges and vertices around A are as in Figure 8.2.9. Notice that the number of black vertices around A is a multiple of three, since going counterclockwise around the inside of A , an a is always followed by a b , a b is always followed by a c , and a c is always followed by an a . Thus in M' , all the vertices are black, and the number of vertices around every country is a multiple of three. Now we return to M , and every time that we restore a white vertex, we shrink a triangle. Around a particular country, this results in one more white vertex and two fewer black vertices. Then the number of black vertices minus the number of white vertices around this country is still a multiple of three. ■

Theorem 8.2.5. *If in a normal map M , the vertices can be labeled by black and white so that around each country in M the number of black vertices minus the number of white vertices is a multiple of three, then the edges of M can be properly colored by three colors.*

Proof. Given a normal map M and a labeling of the vertices by black and white with the above property, we construct the normal map M' by blowing up every

white vertex into a triangle with three black vertices. So we have a map M' such that all the vertices are black, and all the countries are triangles, hexagons, nonagons, etc. We now color the edges of M' by three colors as follows. We begin at any country and walk counterclockwise around the inside of the country and color the edges by a, b, c, a, b, c, \dots as we come to them. The number of edges around each country is a multiple of three, so the coloring is proper so far. Now we color a region adjoining the first region by the same recipe; one edge will already be colored, so the coloring is determined. To see that the coloring is proper, consider Figure 8.2.10.

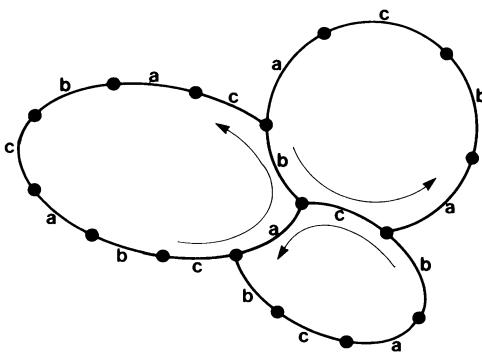


Figure 8.2.10

Now we return to the map M from M' . As in Figure 8.2.8, when we restore a white vertex, we shrink a triangle with three black vertices, and we keep the coloring of the remaining edges. This is a proper coloring, since we shrink a triangle with edges colored a, b , and c , and the edges remaining around the white vertex must be colored a, b , and c . ■

We now come to the statement of the Four Color Theorem. This theorem was a conjecture for over a hundred years, from 1850 to 1976, when Appel and Haken proved it by means of a computer program [2]. The program took a long time to run, and no human can read the entire proof, because it is too long.

Theorem 8.2.6 (Four Color Theorem). *In every normal map the countries are colorable by four colors.*

According to the previous theorems of this section, the Four Color Theorem can be stated in the following equivalent form.

Theorem 8.2.7. *In every planar cubic bridgeless graph, the edges are properly colorable by three colors.*

Given a normal map M we construct a new multigraph $G(M)$ from M . The vertices of $G(M)$ are the countries of M . Two vertices of $G(M)$ are adjacent if the countries are adjacent in M . This multigraph is planar and we obtain a plane drawing of $G(M)$ by placing a vertex in the middle of each country of M , call it the capital of the country, and connect two vertices by an edge if the countries are adjacent. This drawing of $G(M)$ is called the *dual* of M .

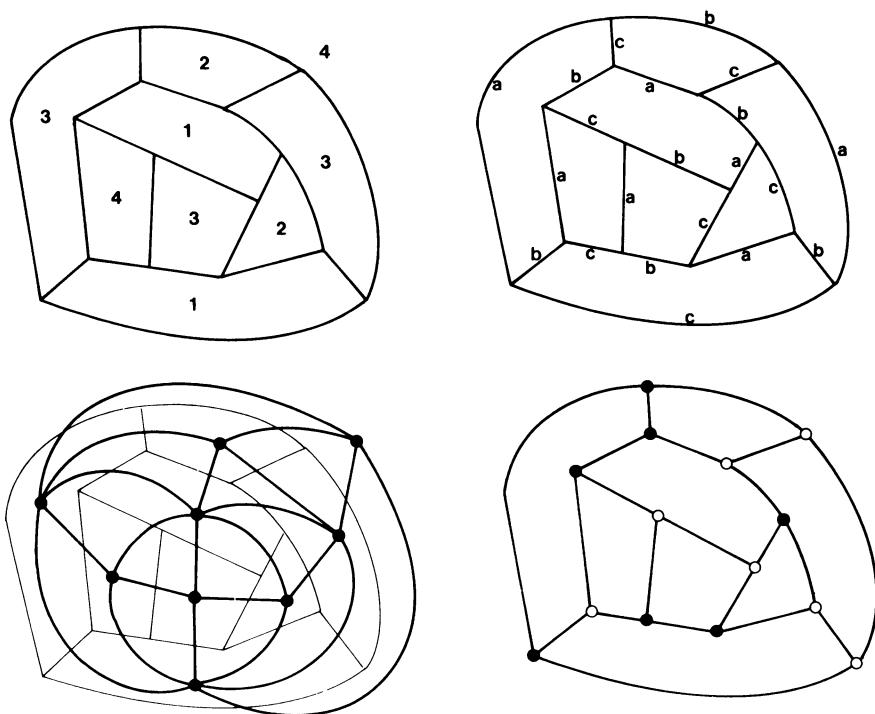


Figure 8.2.11

The bottom left of Figure 8.2.11 shows the dual of the normal map above it. Since every vertex of M has degree 3, every region in the resulting drawing of $G(M)$ is a triangle. We can generalize the idea of a dual for any map. Under this generalization, the dual of $G(M)$ is again M . Figure 8.2.12 shows another example of a plane drawing of a graph and its dual.

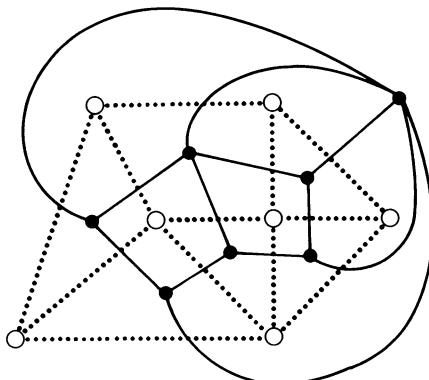


Figure 8.2.12

If D is a plane drawing of a maximal planar graph with at least four vertices, the dual $G(D)$ is a normal map. To see this, consider that D has only triangular regions, so $G(D)$ has only vertices of degree 3. Since D is a plane drawing of a connected graph, $G(D)$ has no bridges or loops.

Now that we have the definition of dual, we can state the Four Color Theorem in a graph theoretical version.

Theorem 8.2.8. *The vertices of every maximal planar graph with at least four vertices can be colored with at most four colors.*

Of course we can leave out the condition that the graph has at least four vertices and the maximal planar condition since *every* planar graph is a subgraph of a maximal planar graph. So the most general version is as follows.

Theorem 8.2.9. *The vertices of every planar graph can be colored by at most four colors.*

Recalling the definition of chromatic number of a graph, we could now say that if G is a planar graph, then the chromatic number of G is at most four. Since there is no short elegant proof of the Four Color Theorem, one can still say there is a Four Color Problem, namely to find a comfortable proof. However, as we shall see in the next section, one can prove a Five Color Theorem for the plane.

Exercises 8.2.

- 8.2.1. Take the plane drawing of the dodecahedron in Figure 1.2.5 and find the coloring of the countries, the proper coloring of the edges, and the labeling of the vertices by black and white guaranteed by the theorems in this section.
- 8.2.2. The plane drawings of the graphs K_4 (the tetrahedron), the cube, the octahedron, the dodecahedron, and the icosahedron, are sometimes called the five regular solids. Find the duals of each of these drawings. One of the interesting properties of the five regular solids is that the dual of a regular solid is a regular solid. Which are duals of each other?
- 8.2.3. Why is the map of the United States not a normal map?
- 8.2.4. Prove that the states in the map of the United States are not colorable by three colors.
- *8.2.5. Prove that the icosahedron graph of Figure 1.2.5 is the only maximal planar graph that is regular of degree 5. (Hint. Use the results of Problem 8.2.2 and Theorem 8.1.1.)

8.3. The Five Color Theorem

As in Section 8.2, we shall again concentrate on normal maps. If we can show that all normal maps in the plane are colorable by five colors, then it is not hard to see that all planar maps can be colored by five colors.

Lemma 8.3.1. *No map in the plane has five mutually adjacent countries.*

Proof. If such a map existed, the dual of the map would contain a subgraph isomorphic to K_5 . But K_5 is not planar, by Theorem 8.1.4. ■

Theorem 8.3.2. (The Five Color Theorem). *The countries of every normal map in the plane can be colored by five colors.*

Proof. We shall prove this theorem indirectly. Assume that the theorem is not true. Then there must be a normal map that is not colorable by five colors. Of all such maps, we select one with a minimal number of countries. Call this map M .

- a) Assume that in M there is a lune, which is a country with exactly two edges, as in Figure 8.3.1.



Figure 8.3.1

Notice that a and b must be distinct, or M would contain a bridge, and this would not be a normal map. We consider the map M' that has the configuration of Figure 8.3.1 replaced by an edge from a to b and is the same as M everywhere else. M' is still normal and the number of countries in M' is less than the number of countries in M , hence M' is colorable by five colors. In a coloration of M' by five colors, only two colors meet at the edge in consideration. Hence there is a color left for coloring the lune in M . So we arrive at a contradiction if we assume that M contains a lune.

- b) Assume that in M there is a triangular country.

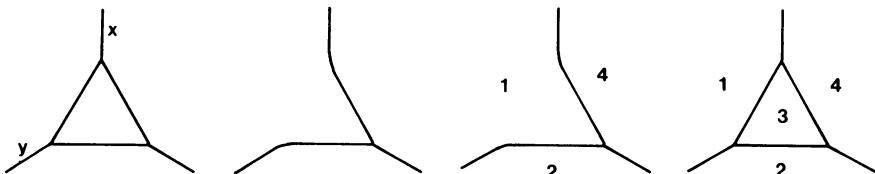


Figure 8.3.2

First we notice that x and y are distinct edges, as in Figure 8.3.2, since otherwise M would contain a lune, and we showed in a) that this gives a contradiction. We consider the map M' that is the same as M except that the triangular country is replaced by the second configuration in Figure 8.3.2. M' has fewer countries than M and is therefore colorable by five colors as illustrated in the third part of Figure 8.3.2. Only three colors meet here, so when we restore the triangular country, there is a color left to color it. Thus if M contains a triangular country we reach a contradiction.

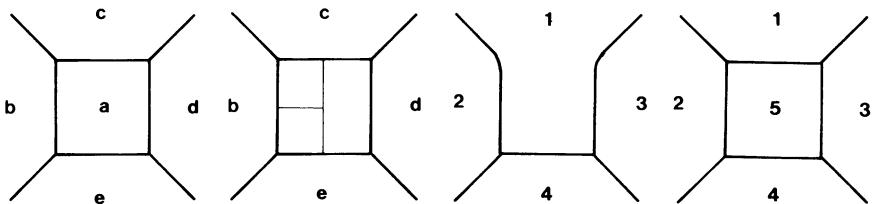


Figure 8.3.3

c) Assume that M contains a country a that is a quadrilateral. This country a cannot have only two different countries adjacent to it. In Figure 8.3.3, either b and d are different, or c and e are different. If not, we divide a into three countries, as in the second picture in Figure 8.3.3. Then we have five mutually adjacent countries in a planar map, which is not possible by Lemma 8.3.1. Assume that c and e are different. Then we consider the map M' that is the same as M except that the edge between a and c is deleted, as in Figure 8.3.3. Then M' is still a normal map, has fewer countries than M , and hence can be colored by five colors. The third picture in Figure 8.3.3 shows the coloring in M' . When we restore the quadrilateral country a , we will have a color left to color it. Thus we get a contradiction if M contains a quadrilateral country.

d) Assume that M contains a country that is a pentagon and is adjacent to five different countries. There must be a pair of these countries that is not adjacent, since otherwise there would be five mutually adjacent countries, which is impossible by Lemma 8.3.1. Assume b is not adjacent to d . We consider the map M' that is the same as M except that the edges between a and b and between a and d are deleted, as in the second picture in Figure 8.3.4. M' has fewer countries than M and can be colored with five colors. The third picture in Figure 8.3.4 shows the coloring in the area under consideration. Only four colors appear there, so there is a color remaining to color the region a . Hence if we assume that M contains a pentagon that is adjacent to five different countries, we arrive at a contradiction.

e) Assume that M contains a country that is a pentagon and is adjacent to four different countries, as in Figure 8.3.5. There must be a pair of these countries that is not adjacent, since otherwise these four countries together with a would be five mutually adjacent countries. Assume b is not adjacent to d . Then we proceed as in case d.

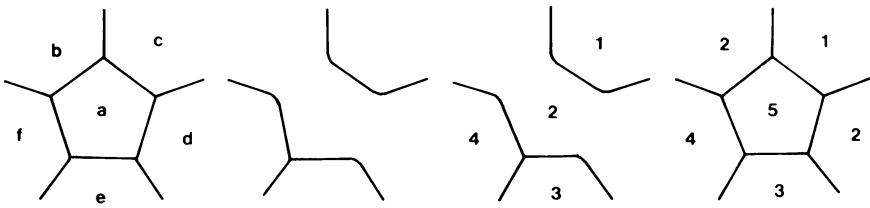


Figure 8.3.4

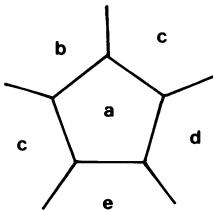


Figure 8.3.5

f) Assume M contains a country that is a pentagon and is adjacent to three different countries, as in Figure 8.3.6. Then, if we divide a into two countries a_1 and a_2 , as in the right of Figure 8.3.6, we have five mutually adjacent countries, which is impossible by Lemma 8.3.1. Thus M does not contain this configuration.

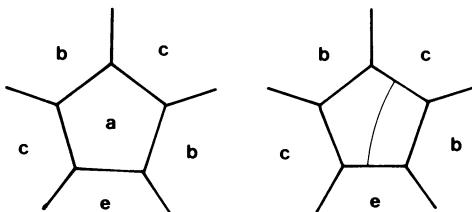


Figure 8.3.6

We have shown that the map M contains no country smaller than a hexagon. Now in M there are p vertices, q edges, and r countries. Since every vertex has degree 3,

$$3p = 2q.$$

Since every country has at least six edges, and every edge lies on two countries,

$$2q \geq 6r.$$

By Theorem 8.1.1, $p - q + r = 2$. So we have

$$6r = 2q + 12.$$

This gives us $2q < 6r$, which is a contradiction. So the map M cannot exist, and therefore the theorem is proved. ■

Exercises 8.3.

- 8.3.1. Prove that any normal map with the property that the number of edges of each country is a multiple of three can be colored by four colors.
- 8.3.2. How many colors are required to color the map that is a plane drawing of the wheel W_n ?
- 8.3.3. What is the dual of a plane drawing of the wheel W_n ?
- 8.3.4. Draw the dual of the graph in Figure 2.3.5.
- 8.3.5. How many colors are required to color the *vertices* of the graph of
 - a. Figure 3.3.6?
 - b. Figure 3.3.7?
- 8.3.6. Consider the graph of Figure 3.3.6 with the center vertex removed. How many colors are required to color the *vertices* of this graph?
- 8.3.7. State the following theorem of Heawood in the dual form:
Every plane triangulation where all vertices have even degree is vertex colorable by three colors.

8.4 Graphs and Geometry

The graph in Figure 8.4.1 is a planar graph. There are various ways to draw the graph in the plane with no crossings.

In Figures 8.4.2 and 8.4.3 are two different plane drawings of the graph of Figure 8.4.1, and the vertices have not even been moved.

If we are allowed to move the vertices, we can even draw the graph with no crossings so that all the edges are straight lines, as in Figures 8.4.4 and 8.4.5.

In 1936, Wagner proved that such a drawing of a planar graph is always possible. The result was proved independently by Fary.

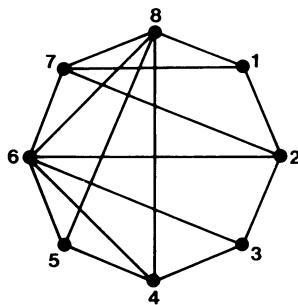


Figure 8.4.1

Theorem 8.4.1 (Wagner). *Every planar graph has a plane drawing where every edge is a straight line.*

A region in a plane drawing of a planar graph is called *convex* if whenever any two points on the boundary of or in the region are connected by a straight line segment, the straight line segment lies entirely inside the region. In Figure 8.4.2 there are six regions that are not convex, five finite ones and the infinite outside region. The outside region is never convex. We shall say that the planar graph G is *stretchable* if G has a plane drawing where all edges are straight line segments. Hence, Wagner's Theorem says every planar graph is stretchable.

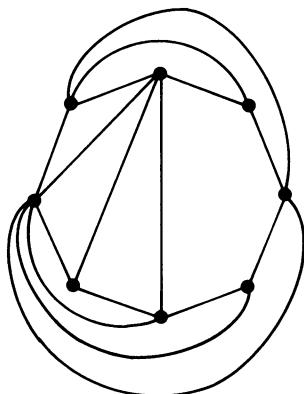


Figure 8.4.2

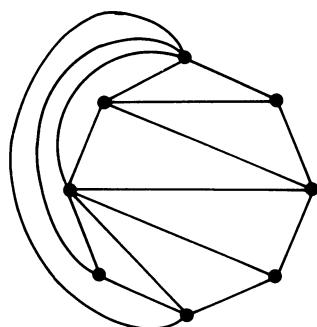


Figure 8.4.3

Proof. The proof is by induction on the number of vertices in the graph.

The base for the induction is the graph K_4 . K_4 and all of its subgraphs are stretchable.

We assume that all planar graphs with fewer than p vertices are stretchable, and let G be a planar graph with p vertices, $p > 4$. If G is not maximal planar, we can add edges without adding vertices to make G maximal planar. So we assume that G is maximal planar. By Theorem 8.1.8, G has at least four vertices of degree at most five. All the regions in a drawing of G are triangles, including the outside region. Thus one of the four vertices of degree at most five is not on the boundary of the outside region. Call this vertex h , and consider the neighborhood of h . Since the degree of h is either three, four, or five as shown in the first line of Figure 8.4.6, h is incident with three, four, or five triangles, as in the second line of Figure 8.4.6. Now consider the graph $G - h$ and the particular region shown in the third line of Figure 8.4.6. The graph $G - h$ is planar and has fewer vertices than G , hence, by our assumption, $G - h$ is stretchable, as in the fourth line of Figure 8.4.6. We now have a plane drawing of $G - h$ with all edges straight line segments and containing a particular region R that is either a triangle, quadrangle, or pentagon. If R is a convex region, as in the fourth line of Figure 8.4.6, we simply replace h in the center of R and connect h by straight line segments to the vertices on the boundary, as in the fifth line of Figure 8.4.6. If R is a quadrangle or a pentagon, R may not be convex, and we shall have to place h more carefully. If R is a quadrangle and R is not convex, then R must be as in Figure 8.4.7. Then placing h anywhere inside the shaded region gives the desired result. If R is a pentagon, and R is not convex, then R could be either of the shapes in Figure 8.4.8. Again, placing h inside the shaded region, we can connect h by straight line segments to the vertices on the boundary of R . This completes the induction, and the theorem is proved. ■

When we look at Figures 8.4.4 and 8.4.5, we realize that the drawings are still very flexible in the sense that we can move vertices and still have all straight line edges. This gives us the idea that it might be possible to fulfill another geometric requirement.

On the left of Figure 8.4.9, the graph of Figure 8.4.1 is now drawn in a very special way. It is now possible to draw circles around every vertex so that the vertex is the midpoint of the circle, and so that the two circles of any two adjacent vertices touch at precisely one point. Geometers refer to circles that touch at precisely one point as *tangent circles*, from the Latin *tangere*, which

means "touch." It is conjectured that every planar graph can be drawn in such a fashion.

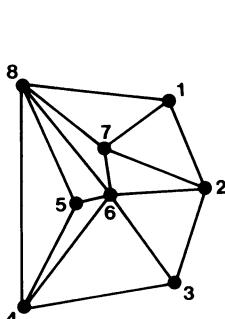


Figure 8.4.4

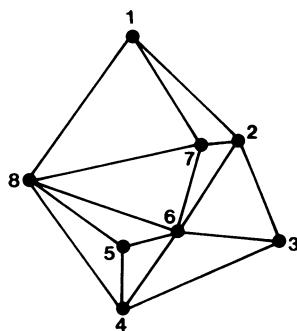


Figure 8.4.5

Given a finite set of nonoverlapping circles in the plane, call these circles coins. Construct a graph by considering every coin as a vertex and declaring that two vertices are adjacent if and only if the coins touch. If a graph can be constructed in this way, then it is called a *coin graph*.

Conjecture. *Every planar graph is a coin graph.*

If the conjecture is true, then it is probably very hard to prove. If the conjecture is false, a single counterexample will disprove it, but probably a complicated one.

If all the coins have the same size, the coin graph is called a *penny graph*. The reader will have the opportunity to construct some penny graphs in the exercises.

We would now like to consider another very special class of planar graphs. In addition to having edges drawn as straight lines, these graphs are regular of degree d , and every edge has the same geometric length. If $d = 1$, then K_2 is the only connected graph having all these properties. If $d = 2$, the smallest connected graph with these properties is K_3 , but cycles of any length $l \geq 3$ can be drawn in this way. For $d = 3$, Figure 8.4.10 shows an example of such a graph.

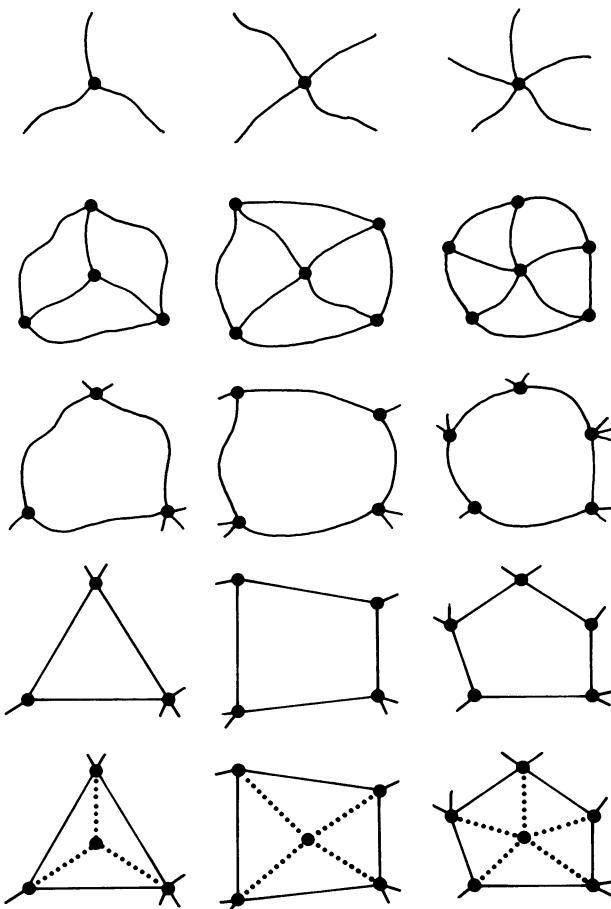


Figure 8.4.6

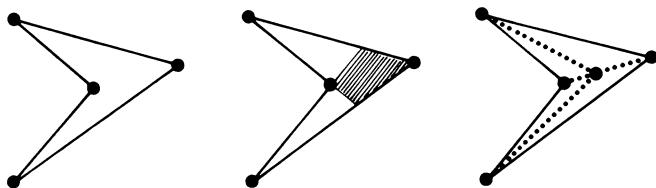


Figure 8.4.7

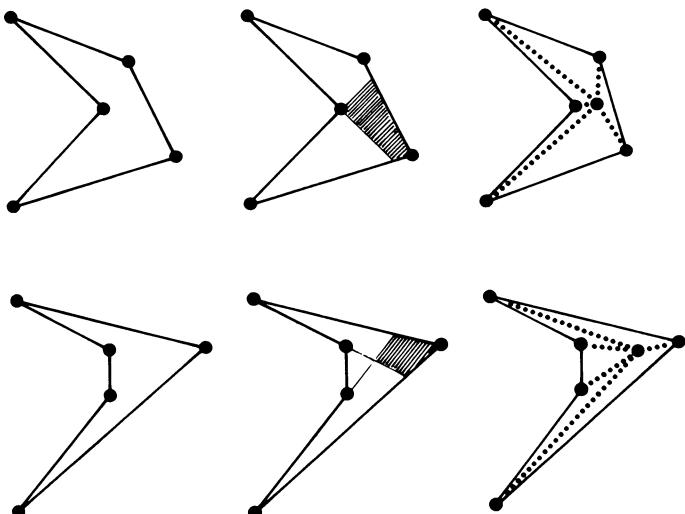


Figure 8.4.8

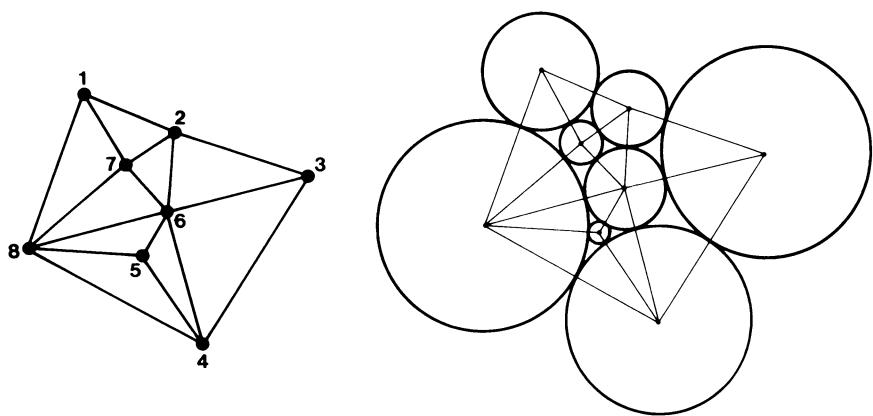


Figure 8.4.9

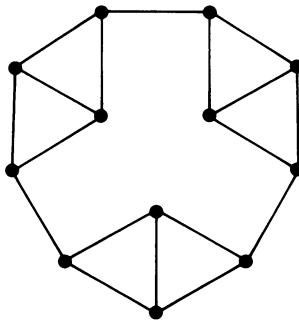


Figure 8.4.10

The graph of Figure 8.4.10 is not the smallest example for $d = 3$. The reader will have the opportunity to find the smallest example.

For $d = 4$, examples are much more difficult to find.

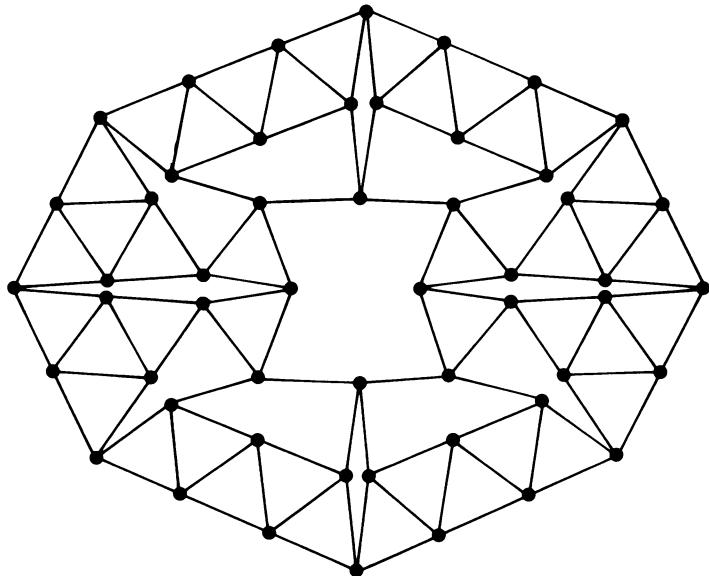


Figure 8.4.11. Harborth's graph.

The graph of Figure 8.4.11 was constructed by Heiko Harborth. It is not known if this is the smallest example for $d = 4$.

For $d \geq 5$, such graphs do not exist. The proof of this fact will be an exercise.

If we consider the same problem, except that now we allow crossings and nonplanar graphs, the situation is different and still interesting. Now if $d = 3$, there is a drawing of a graph with six vertices such that every edge has the same geometric length. See Exercise 8.4.10.

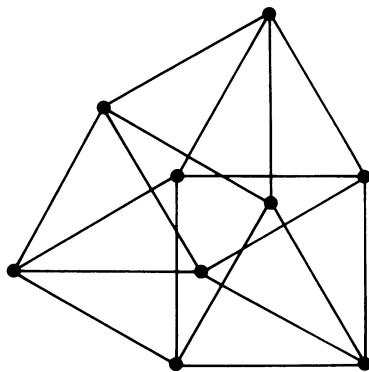


Figure 8.4.12

Harborth studied this problem, and he found the graphs shown in Figures 8.4.12 and 8.4.13 for $d = 4$.

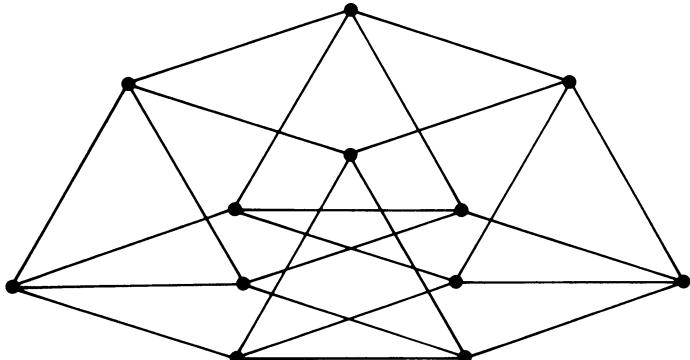


Figure 8.4.13

It is not difficult to prove that such a graph exists for every value of d . The proof goes as follows. Given a solution for $d - 1$, construct a solution for d by taking a copy of the solution for $d - 1$ and drawing another copy of this solution so that each vertex is unit distance from its counterpart in the first copy and so that no two vertices coincide. Then connect corresponding vertices in

the two copies. The graphs of Figures 8.4.12 and 8.4.13 were not, however, constructed this way.

Harborth has conjectured that every planar graph can be drawn in the plane with no crossings such that each edge is a straight line of integer geometric length. Figure 8.4.14 is a drawing of K_4 in the plane such that each edge has integer geometric length. The edges are labeled with the geometric lengths. Figure 8.4.15 is a drawing of the dodecahedron graph with 10 edges of geometric length 2 and 20 edges of length 1.

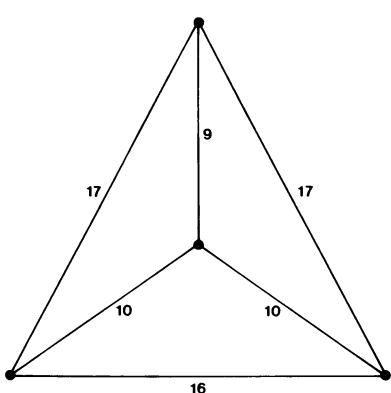


Figure 8.4.14

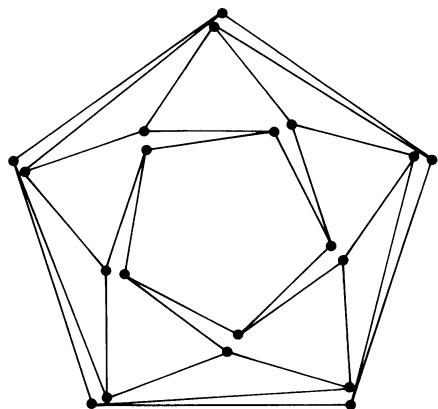


Figure 8.4.15

G. Ringel observed that many planar graphs have a planar drawing where every edge is a straight line segment and each finite region has the same area. Figure 8.4.15a is an example. So the question is: *Is this true for every planar graph?* The answer is no, we found a counterexample of a planar graph with 33 edges and 13 vertices.

Exercises 8.4

- 8.4.1. Find a plane drawing of the octahedron graph O of Figure 1.2.5 where all the edges are straight lines.
- 8.4.2. Find a plane drawing of the graph on the right of Figure 4.3.4 where all the edges are straight lines.
- 8.4.3. Find a plane drawing of the graph in Figure 8.4.16 where all the edges are straight lines.

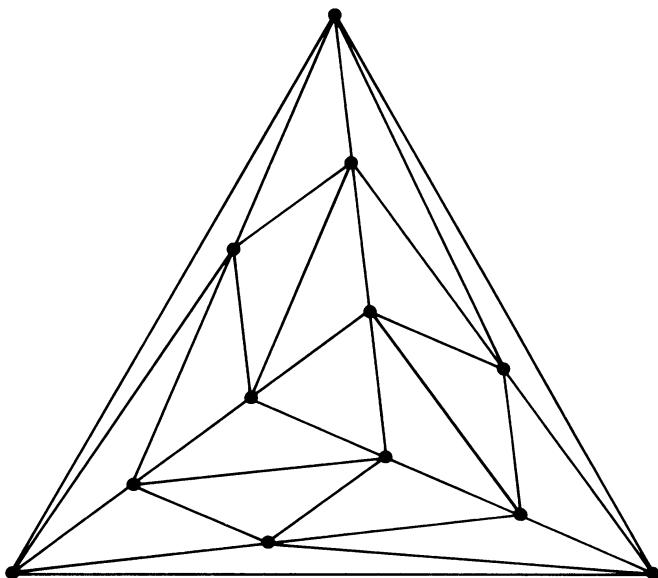


Figure 8.4.15a. Ringel's Icosahedron. All 19 triangles have the same area.

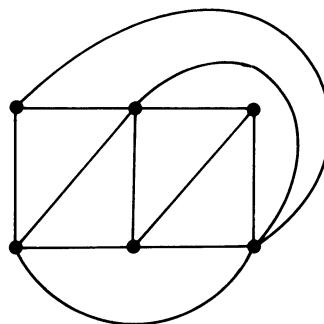


Figure 8.4.16

- 8.4.4. For what numbers n is $K_{1,n}$ a penny graph?
- 8.4.5. For what numbers n is the cycle C_n a penny graph?
- 8.4.6. For what numbers n is the wheel W_n a penny graph?
- 8.4.7. Show that there is a penny graph that requires four colors.
- 8.4.8. Show that four colors is sufficient to color any penny graph without using the Four Color Theorem. (Hint – use proof by contradiction;

look at the two pennies with the largest geometric distance, and use the idea of a critical graph.)

- 8.4.9. Find the smallest graph that is planar, regular of degree 3, and every edge has the same geometric length.
- 8.4.10. Do Exercise 8.4.9, omitting the requirement of planarity.
- 8.4.11. Connect A to a , B to b , C to c , and D to d in Figure 8.4.17 by edges that are inside the rectangle and do not cross each other.

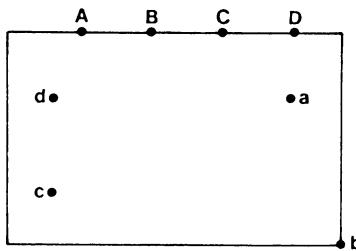


Figure 8.4.17. Dudeney's Puzzle.

- 8.4.12. Draw the cube graph Q_3 of Figure 1.2.5 in the plane so that the geometric length of every edge is either one or two.
- 8.4.13. Verify that Figure 8.4.14 is geometrically correct by using Pythagoras' Theorem.
- 8.4.14. Find a planar graph with 8 edges that has no plane drawing in which every finite region is convex.

Chapter 9

MEASUREMENTS OF CLOSENESS TO PLANARITY

9.1 Crossing Number

In Section 8.1, we proved that K_5 and $K_{3,3}$ are not planar. Figure 9.1.1 is not a picture of K_5 , but it is a picture of a *subdivision* of K_5 , which means that some vertices of degree 2 may have been added to the edges of K_5 . Similarly, the graph in Figure 9.1.2 is a subdivision of $K_{3,3}$. A graph is also considered to be a subdivision of itself.

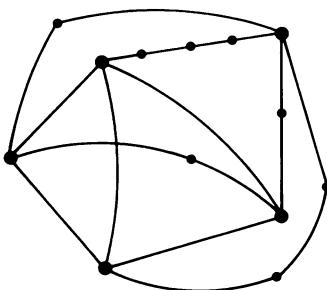


Figure 9.1.1

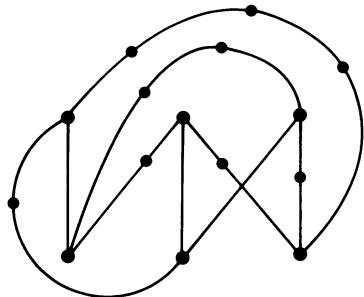


Figure 9.1.2

If a graph G is not planar, then a subdivision of G is not planar. If a graph G contains a nonplanar subgraph, then G is not planar. Taking these two facts into consideration, we can state the following theorem.

Theorem 9.1.1. *If a graph G contains a subgraph that is a subdivision of K_5 or of $K_{3,3}$, then G is not planar.*

Remarkably enough, not only is it true that if a graph G contains a subdivision of K_5 or of $K_{3,3}$, then G is not planar. It is also true that every nonplanar graph contains a subdivision of one of these two graphs. This result is due to Kuratowski [16], and the proof is difficult.

Theorem 9.1.2. (Kuratowski). *If G is a nonplanar graph, then G contains a subgraph that is a subdivision of K_5 or of $K_{3,3}$.*

The following statement is actually equivalent to Kuratowski's Theorem.

Assume that G is a graph with the property that G is not planar, and for every edge e of G , the graph $G - e$ is planar. Then G is either a subdivision of K_5 or a subdivision of $K_{3,3}$.

A graph can be drawn in many different ways. Figures 9.1.3 and 9.1.4 show two different drawings of $K_{3,3}$ in the plane.

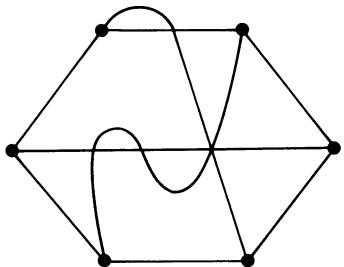


Figure 9.1.3

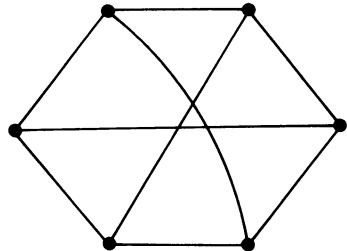


Figure 9.1.4

A *simple drawing* of a graph in the plane is a drawing in which

- a) any two distinct edges have at most one crossing.
- b) any two edges incident with the same vertex do not cross.
- c) no three edges cross in a common point.

In Figure 9.1.3, all three conditions are violated. Thus the drawing of Figure 9.1.3 is not simple, but the drawing of Figure 9.1.4 is simple.

The top row of Figure 9.1.5 shows the three non-simple configurations; the bottom row shows their three simple counterparts. In a drawing of a graph in the plane, the three non-simple configurations can be avoided by very slight

changes that do not affect the drawing of the rest of the graph. This has been done in the second row of Figure 9.1.5. Every finite graph thus has a simple drawing in the plane.

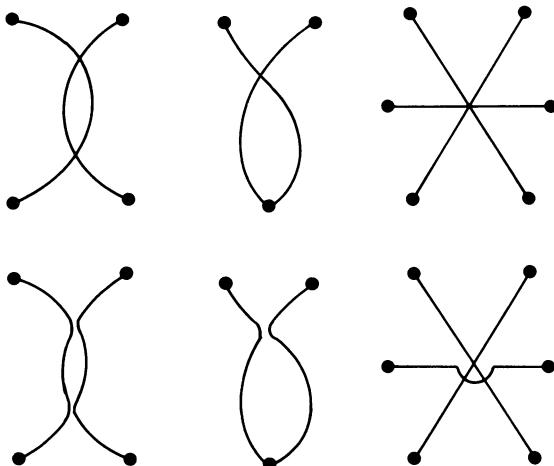


Figure 9.1.5

If more than two edges cross at the same point, as in the top right of Figure 9.1.5, we call this a *multiple crossing*.

Let G be a graph. In general, there are many ways to draw G in the plane with no multiple crossings. Of all these drawings, there will be at least one that has a minimum number of crossings. This minimum number is called the *crossing number* of G , denoted $cr(G)$. Such a drawing with a minimum number of crossings is always a simple drawing. Consider Figure 9.1.5. Of course, the crossing number of any planar graph is zero. Also, $cr(K_5) = 1$ and $cr(K_{3,3}) = 1$.

Problem. Find the crossing number of the graph in Figure 9.1.6.

Solution. First we try to draw the graph with as few crossings as we can. We cannot seem to get rid of the one crossing in Figure 9.1.7, so we begin to think that the crossing number is probably one. According to Theorem 9.1.2, if the crossing number is greater than zero, then the graph contains either a subdivision of K_5 or a subdivision of $K_{3,3}$. Figure 9.1.8 shows the subdivision of $K_{3,3}$ contained in the graph. Thus the crossing number of this graph is equal to one.

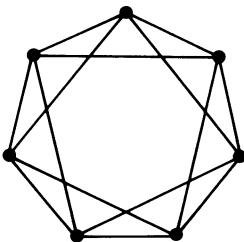


Figure 9.1.6

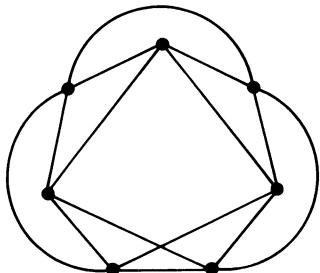


Figure 9.1.7

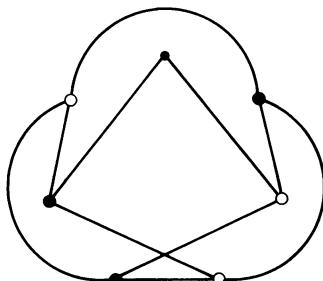


Figure 9.1.8

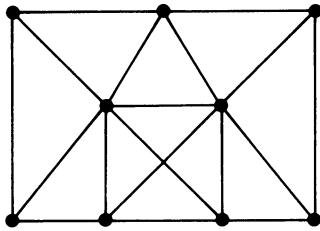


Figure 9.1.9

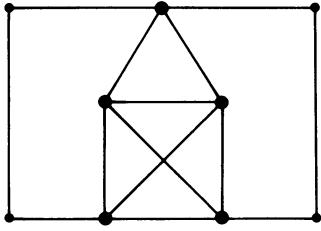


Figure 9.1.10

Problem. Find the crossing number of the graph in Figure 9.1.9.

Solution. We again try to get rid of as many crossings as possible, but one is always left. So we apply Theorem 9.1.2, and we find the subdivision of K_5 shown in Figure 9.1.10.

Theorem 9.1.3. Every simple drawing of K_4 in the plane has either zero or one crossing.

Proof. Figure 9.1.11 shows drawings of K_4 in the plane that have one crossing and zero crossings, respectively.

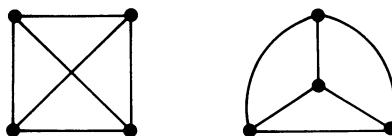


Figure 9.1.11

To prove the theorem, we must show that no more than one crossing is possible. Assume there is already one crossing. We focus on this crossing in Figure 9.1.12; the edges (12) and (34) cross. If the edge (14) is to cross any other edge, that edge must be the edge (23), since otherwise the drawing would no longer be simple. The boundary of the shaded triangle divides the plane into an inside area and an outside area. If the edge (14) crosses the edge (23) only once, the end of the edge will be in the inside area, since crossing the other two edges is not permitted. Since the same argument applies for a crossing of the edges (13) and (24), we see that it is not possible to have two crossings in a simple drawing of K_4 . ■

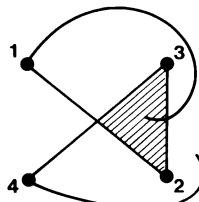


Figure 9.1.12

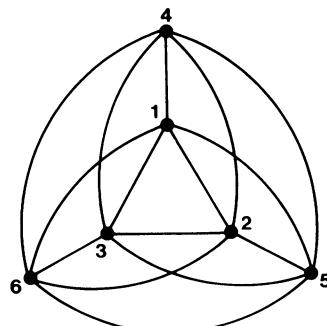


Figure 9.1.13

The drawing in Figure 9.1.13 is a simple drawing of K_6 in the plane with three crossings. Each crossing can be uniquely described by four vertices, which are the endpoints of the edges that cross. In this particular case, we have the quadruples (1245) , (1346) , and (2356) . By Theorem 9.1.3, in a simple drawing four vertices cannot determine two crossings.

Theorem 9.1.4. *The crossing number of K_6 is*

$$cr(K_6) = 3.$$

First proof. By the drawing in Figure 9.1.13 we have that

$$cr(K_6) \leq 3.$$

Suppose we have a different drawing of K_6 with only two crossings. We identify the vertices by numbers and describe the two crossings by two quadruples, say (1234) and (3456) . Consider the drawing we obtain from this new drawing by removing the vertex 4 together with all the edges incident with 4. The result is a plane drawing of K_5 , which is impossible. This tells us that

$$cr(K_6) \geq 3, \text{ and hence}$$

$$cr(K_6) = 3. \blacksquare$$

Second proof. Assume that we can draw K_6 with only two crossings. Then there are two edges in K_6 whose removal results in a planar graph. The resulting planar graph has six vertices and thirteen edges. By Theorem 8.1.3, in a planar graph with p vertices and q edges, $q \leq 3p - 6$. Hence we have

$$13 \leq 3 \cdot 6 - 6,$$

which is not true. So the assumption that we can draw K_6 with only two crossings must be false. ■

Recall the ceiling and floor symbols from Chapter 4.

Richard Guy [8] showed that for the complete graph the following inequality holds.

$$cr(K_n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

Guy also conjectured that equality holds, but this has not been proved for $n > 10$, and in fact the proofs for $7 \leq n \leq 10$ are very uncomfortable.

We now consider the crossing number of the complete bipartite graph $K_{m,n}$. In order to get an upper bound for $cr(K_{m,n})$, we put the m blue vertices on the vertical axis and the n red vertices on the horizontal axis, distributing as evenly as possible between left and right and between top and bottom, as in Figure 9.1.14.

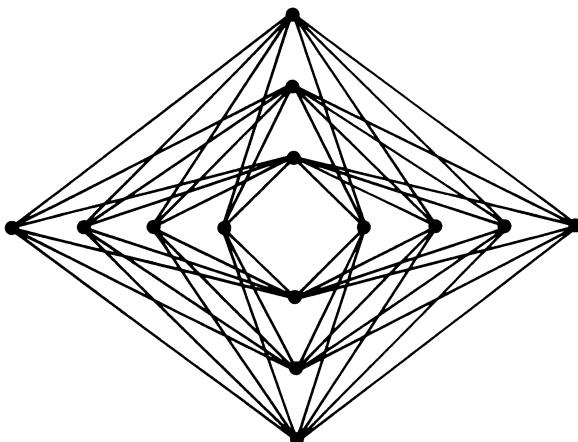


Figure 9.1.14

First we assume that m and n are both even, say $m = 2t$ and $n = 2s$. Then there are t blue vertices on the top and t blue vertices on the bottom, and there are s red vertices on the left and s red vertices on the right. In Figure 9.1.14, t is three, and s is four. We connect each red vertex to every blue vertex with a straight line. As before, no multiple crossings at the same point are allowed, so if one occurs we simply move one or more vertices up or down a small amount. We can count the number of crossings in this drawing. In the first quadrant, there is a crossing determined by each pair of red vertices together with each pair of blue vertices. Thus there are $\binom{t}{2}\binom{s}{2}$ crossings in the first quadrant. Since all four quadrants have the same number because m and n are both even, the total number of crossings is

$$4\binom{t}{2}\binom{s}{2} = 4 \frac{t(t-1)}{2} \frac{s(s-1)}{2},$$

and hence we have

$$cr(K_{2t,2s}) \leq t(t-1)s(s-1).$$

If we do not require that m and n both be even, the situation is a little different, but the idea is the same. We consider two more cases. Suppose that m and n are both odd; $m = 2t+1$ and $n = 2s+1$. Then there are $t+1$ blue vertices above and t blue vertices below. There are $s+1$ red vertices on the right and s red vertices on the left. The number of crossings when all edges are drawn is

$$\begin{aligned} & \binom{t+1}{2} \binom{s+1}{2} + \binom{t+1}{2} \binom{s}{2} + \binom{t}{2} \binom{s+1}{2} + \binom{t}{2} \binom{s}{2} \\ &= \left[\binom{t+1}{2} + \binom{t}{2} \right] \left[\binom{s+1}{2} + \binom{s}{2} \right] \\ &= \left[\frac{(t+1)t}{2} + \frac{t(t-1)}{2} \right] \left[\frac{(s+1)s}{2} + \frac{s(s-1)}{2} \right] \\ &= \left[\frac{t(t+1+t-1)}{2} \right] \left[\frac{s(s+1+s-1)}{2} \right] \\ &= t^2 s^2. \end{aligned}$$

Thus

$$cr(K_{2t+1,2s+1}) \leq t^2 s^2.$$

The last case is where m is odd and n is even; $m = 2t+1$ and $n = 2s$.

Then

$$cr(K_{2t+1,2s}) \leq t^2 s(s-1).$$

The proof is left as an exercise. We summarize all three cases in Theorem 9.1.5.

Theorem 9.1.5. *The crossing number of $K_{m,n}$ satisfies the inequality*

$$cr(K_{m,n}) \leq \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Equality has been shown if $m \leq 6$ and n is arbitrary. It is conjectured that equality holds for all m and n . In this book we present a nice proof for $m = 3$ and n arbitrary.

Theorem 9.1.6. *The crossing number of $K_{3,n}$ is*

$$cr(K_{3,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof. By Theorem 9.1.5, we know that $cr(K_{3,n}) \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$. We must show that in any simple drawing of $K_{3,n}$ in the plane there are at least $\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ crossings. Assume that we have a simple drawing of $K_{3,n}$ in the plane with a minimal number c of crossings. We construct a new graph G from this drawing. There are three blue vertices and n red vertices in $K_{3,n}$. The n red vertices will be the vertices of G . Now consider any two red vertices of $K_{3,n}$. Together with the three blue vertices, they form a $K_{3,2}$. If there is no crossing in this $K_{3,2}$ in the drawing, we draw a green edge between the two red vertices in G . Thus G will have n vertices and at least $\binom{n}{2} - c$ edges. G has no triangles, since then there would be a drawing of $K_{3,3}$ in the plane with no crossings, which is impossible. By Turan's Theorem, Theorem 4.1.2*, the number of edges in G is at most $\left\lfloor \frac{n+1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$. Thus

$$\binom{n}{2} - c \leq \left\lfloor \frac{n+1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \quad \text{or} \quad \binom{n}{2} - \left\lfloor \frac{n+1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \leq c.$$

The reader can verify that $\binom{n}{2} - \left\lfloor \frac{n+1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ by considering the two cases $n = 2s$ and $n = 2s+1$. Hence we have

$$\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \leq cr(K_{3,n}) \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \text{ or}$$

$$cr(K_{3,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor. \quad \blacksquare$$

We can use the result of Theorem 9.1.6 to prove that equality holds in Theorem 9.1.5 when $m = 4$.

Theorem 9.1.7. *The crossing number of $K_{4,n}$ is*

$$cr(K_{4,n}) = 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof. By Theorem 9.1.5, we have

$$cr(K_{4,n}) \leq 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

We must show that any simple drawing of $K_{4,n}$ in the plane has at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ crossings. Given a simple drawing of $K_{4,n}$ in the plane with a minimal number of crossings, we label the four blue vertices 1, 2, 3, 4. We shall denote the number of crossings in the drawing that involve vertices 1 and 2 by h_{12} , those that involve 1 and 3 by h_{13} , and so on. Then we have

$$h_{12} + h_{13} + h_{14} + h_{23} + h_{24} + h_{34} = cr(K_{4,n}).$$

If we remove the vertex 1 from the drawing, we are left with a simple drawing of $K_{3,n}$ in the plane, with $h_{23} + h_{24} + h_{34}$ crossings. Similarly, removing one of vertices 2, 3, or 4 leaves us a simple drawing of $K_{3,n}$ in the plane. We obtain the inequalities

$$\begin{aligned} h_{23} + h_{24} + h_{34} &\geq cr(K_{3,n}) \\ h_{13} + h_{14} + h_{34} &\geq cr(K_{3,n}) \\ h_{12} + h_{14} + h_{24} &\geq cr(K_{3,n}) \\ h_{12} + h_{13} + h_{23} &\geq cr(K_{3,n}) \end{aligned}$$

$$2(h_{12} + h_{13} + h_{14} + h_{23} + h_{24} + h_{34}) \geq 4cr(K_{3,n})$$

and hence

$$cr(K_{4,n}) \geq 2cr(K_{3,n}).$$

Thus we have shown that

$$cr(K_{4,n}) = 2\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor. \quad \blacksquare$$

Exercises 9.1.

- 9.1.1. a) Find the crossing number of the graph in Figure 9.1.15.
 b) Add one edge to the graph of Figure 9.1.15, and increase the crossing number.

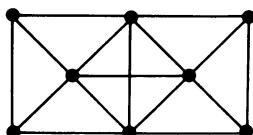


Figure 9.1.15

9.1.2. Show that $cr(K_{2t+1,2s}) \leq t^2s(s-1)$.

9.1.3. Let G be the graph of Figure 9.1.16. Show that $cr(G) = 5$.

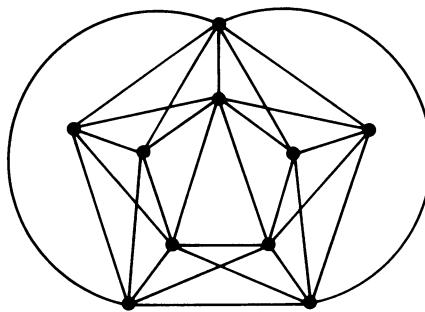


Figure 9.1.16

9.1.4. What is the crossing number of the graph in Figure 9.1.17?

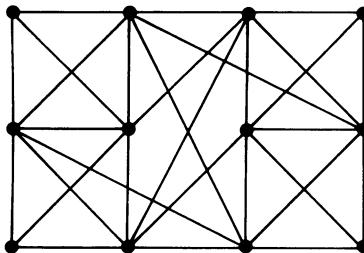


Figure 9.1.17

9.1.5. What is the crossing number of the graph in Figure 9.1.18?

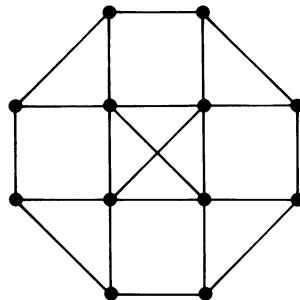


Figure 9.1.18

9.1.6. The crossing number of the Petersen graph is two. Draw the Petersen graph with two crossings.

- 9.1.7. Determine the crossing number of the graph in Figure 9.1.19.

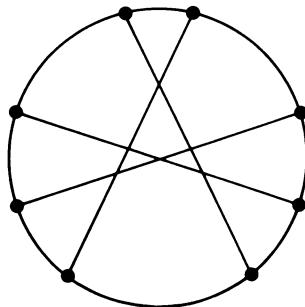


Figure 9.1.19

- 9.1.8. Find the graph with the smallest number of vertices and crossing number two.
- 9.1.9. Find the graph with the smallest number of edges and crossing number two.
- 9.1.10. Given a simple drawing of K_n with a cycle outside and all the other edges inside, as in Figure 2.3.2, how many crossings are there?
- 9.1.11. Use Kuratowski's Theorem to show that the Petersen graph is not planar.
- 9.1.12. What is the maximum number of crossings possible in any simple drawing of P_n ?
- 9.1.13. Find a simple drawing of P_6 with the maximum number of crossings.
- *9.1.14. Find a simple drawing of the cycle C_6 with nine crossings.

9.2. Thickness and Splitting Number

Figure 9.2.1 shows a decomposition of K_8 into two planar subgraphs. We leave it to the reader to show that they are actually planar.

We shall say that the *thickness* of K_8 is two, because K_8 is decomposable into two planar subgraphs, and two is the smallest number for which this is possible. It is not necessary that the subgraphs be isomorphic. In general, the *thickness* of a graph G , denoted by $\theta(G)$, is the minimum number of planar subgraphs in a decomposition of G into planar subgraphs. So $\theta(G) = t$ means there is a decomposition of G

$$G = H_1 \cup H_2 \cup H_3 \cup \cdots \cup H_t,$$

where H_i is planar for each i , and there is no decomposition of G into $t-1$ planar subgraphs.

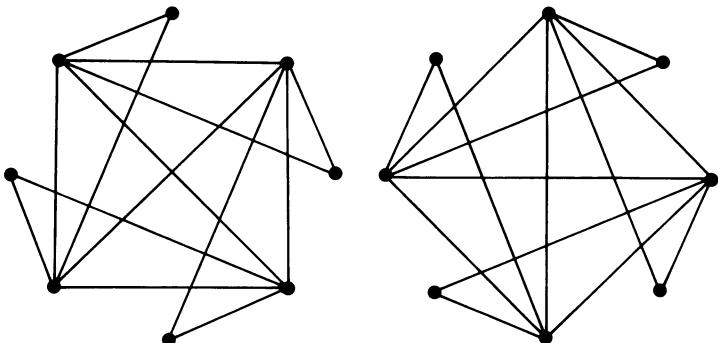


Figure 9.2.1

Theorem 9.2.1. *If G is a graph with p vertices and q edges, then*

$$\theta(G) \geq \frac{q}{3p-6}.$$

Proof. Suppose that $\theta(G) = t$. Then there is a decomposition of G into t planar subgraphs H_i , and for each of these subgraphs the number of edges q_i is bounded as in Theorem 8.1.3,

$$q_i \leq 3p - 6.$$

Since $q_1 + q_2 + \dots + q_t = q$, this gives us

$$q \leq t(3p - 6). \quad \blacksquare$$

In Theorem 9.2.1, we made no restrictions on G , but if G happens to be bipartite, we can prove the following lower bound.

Theorem 9.2.2. *If G is a bipartite graph with p vertices and q edges, then*

$$\theta(G) \geq \frac{q}{2p-4}.$$

The proof of Theorem 9.2.2 is left as an exercise.

A formula for the thickness of the complete graph has been proved after a long struggle and with contributions from Beineke, Harary, Vasak, Alekseev, and Gonchakov [1], [3], [4], [23].

Theorem 9.2.3. *The thickness of the complete graph is*

$$\theta(K_n) = \begin{cases} \left\lfloor \frac{n+7}{6} \right\rfloor & \text{if } n \neq 9, 10 \\ 3 & \text{if } n = 9, 10. \end{cases}$$

By using Theorem 9.2.1, one can derive that

$$\theta(K_n) \geq \left\lceil \frac{n(n-1)}{6(n-2)} \right\rceil = \left\lfloor \frac{n+7}{6} \right\rfloor,$$

but the calculation is lengthy and boring so we omit it here, and we will not give it as an exercise either.

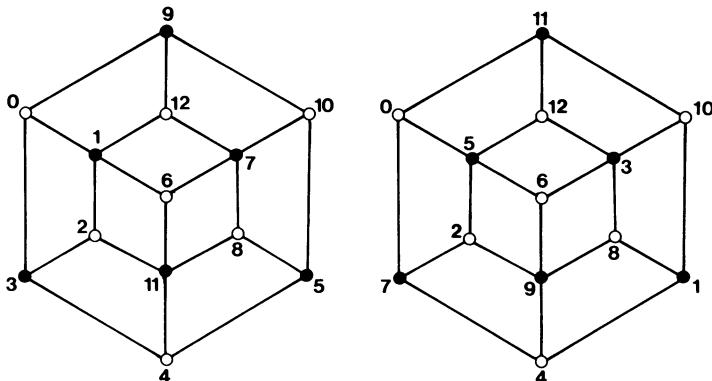


Figure 9.2.2

There are only partial results known for the complete bipartite graph. Figure 9.2.2 shows a decomposition of $K_{6,7}$ into two (isomorphic) planar subgraphs (notice that every odd-numbered vertex is adjacent to every even-numbered vertex). Hence the thickness of $K_{6,7}$ must be two since $K_{6,7}$ is not planar. One can derive a lower bound using Theorem 9.2.2,

$$\theta(K_{m,n}) \geq \frac{mn}{2(m+n-2)}.$$

Beineke, Harary, and Moon [5] have shown that

$$\theta(K_{m,n}) = \left\lceil \frac{mn}{2(m+n-2)} \right\rceil$$

if mn is even and for some other cases also. It is conjectured that equality holds for all m and n .

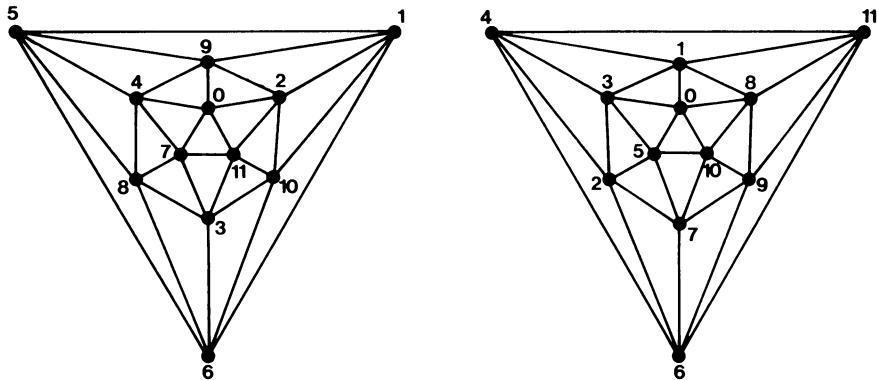


Figure 9.2.3

As a final example, Figure 9.2.3 shows a decomposition of $K_{2,2,2,2,2}$ into two isomorphic planar graphs, each of which is an icosahedron. The reader can check that every pair of numbers is adjacent with the exceptions of $(0, 6)$, $(1, 7)$, $(2, 8)$, $(3, 9)$, $(4, 10)$, and $(5, 11)$.

The crossing number and the thickness are examples of measurements of "closeness to planarity" for graphs. Some other such parameters have been studied also.

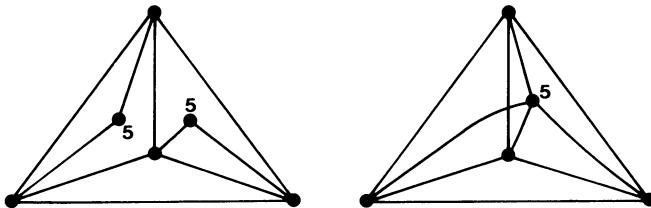


Figure 9.2.4

The left of Figure 9.2.4 shows a planar graph with two vertices labeled 5. If we identify (glue together) the two vertices labeled 5, we get the graph K_5 shown on the right of the figure.

Let G be a graph, and let u and v be vertices of G . We construct a new graph G' by replacing u and v by a single new vertex w , where w is adjacent to

all the vertices in G' that are adjacent to either u or v in G . We call this process *identifying* two vertices. Usually if two vertices are to be identified, we denote them by the same symbol.

If we replace a single vertex w of G by two vertices u and v so that u is adjacent to some of the vertices that were adjacent to w , and v is adjacent to the rest of the vertices that were adjacent to w , we say that we have *split* the vertex w . The *splitting number* $\sigma(G)$ of a graph G is the smallest k such that G can be obtained from a planar graph by k vertex identifications. In other words, the splitting number is the minimum number of vertex splittings required to transform G into a planar graph. According to Figure 9.2.4, the splitting number of K_5 is $\sigma(K_5) = 1$.

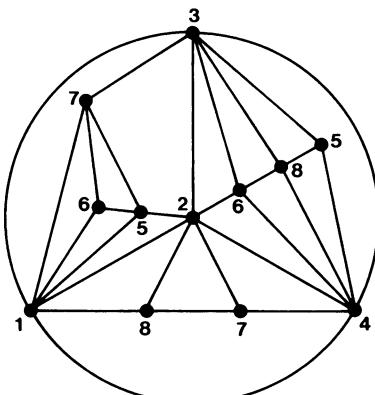


Figure 9.2.5

Figure 9.2.5 shows a planar graph H with 12 vertices. If we identify vertices that are labeled with the same number, we obtain K_8 . We say H is a *planar splitting* of K_8 . Thus $\sigma(K_8) \leq 4$. On the other hand, it must be that $\sigma(K_n) \geq n-4$, since if we could transform K_n into a planar graph by only $n-5$ splittings, then there are five vertices that are not split, and these form a K_5 , which we know is not planar. Therefore $\sigma(K_8) = 4$.

If we remove the two vertices labeled 8 from Figure 9.2.5, we obtain a planar splitting of K_7 . Then if we remove the two vertices labeled 7 from the resulting graph, we obtain a planar splitting of K_6 . Thus $\sigma(K_7) = 3$ and $\sigma(K_6) = 2$.

The planar graph of Figure 9.2.6 is a planar splitting of K_9 . Thus $\sigma(K_9) \leq 6$. The proof that $\sigma(K_9) \geq 6$ involves checking many cases, and M. Jungerman has verified it using a computer.

Theorem 9.2.4. (Hartsfield, Jackson, Ringel) [10]. *If $n \geq 10$,*

$$\sigma(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil.$$

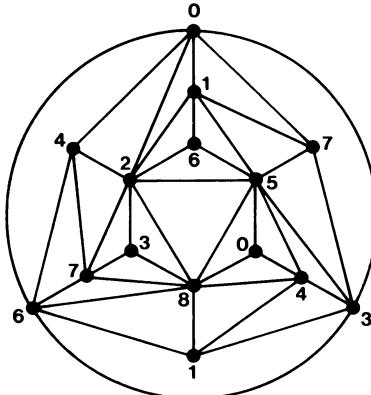


Figure 9.2.6

We shall only prove the easy part here, which is that

$$\sigma(K_n) \geq \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil.$$

To see that this is true, suppose that we have a splitting H of a graph G , with $\sigma(G) = s$ splittings. Then if G has p vertices and q edges, H has $p+s$ vertices and q edges. Since H is planar, by Theorem 8.1.3,

$$q \leq 3(p+s)-6.$$

Hence,

$$\frac{q-3p+6}{3} \leq s = \sigma(G).$$

If $G = K_n$, then $p = n$ and $q = \binom{n}{2}$ so

$$\sigma(K_n) \geq \frac{\frac{n(n-1)}{2} - 3n + 6}{3} = \frac{(n-3)(n-4)}{6}.$$

Theorem 9.2.5. (Jackson, Ringel) [14]. *The splitting number of $K_{m,n}$ is*

$$\sigma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil \text{ for } m, n \geq 2.$$

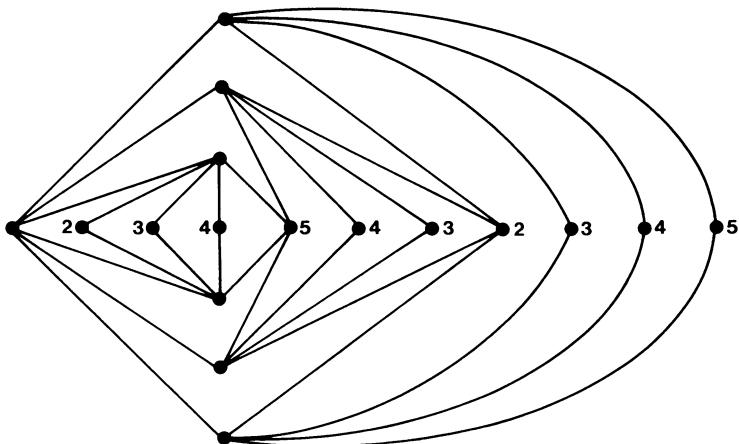


Figure 9.2.7. A splitting of $K_{5,6}$.

The graph of Figure 9.2.7 is a splitting of $K_{5,6}$. The number of splittings for each vertex is equal to the number of times the vertex appears in the picture minus one. In Figure 9.2.7 the vertex 4 appears three times, so you must do two vertex identifications to get one vertex labeled 4. Thus the reader can check that there are six splittings in the graph of Figure 9.2.7, which is

$$\frac{(5-2)(6-2)}{2}.$$

The proof of the lower bound for Theorem 9.2.5 is left as an exercise.

Exercises 9.2.

- 9.2.1. Draw the graph of Figure 9.2.1 in the plane with no crossings.
- 9.2.2. Show that $\theta(K_7) = 2$.
- 9.2.3. Show that for the graph $K_{3,3,3}$ shown in Figure 9.2.8,

$$\theta(K_{3,3,3}) = 2.$$

- 9.2.4. Show that if G is regular of degree 4, then $\theta(G) \leq 2$.

9.2.5. Show that $\sigma(G) \leq (\theta(G)-1)p$.

9.2.6. Prove that

$$\sigma(K_{m,n}) \geq \frac{(m-2)(n-2)}{2}.$$

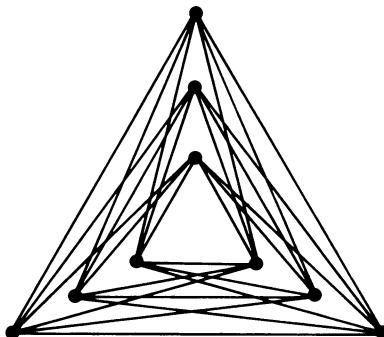


Figure 9.2.8

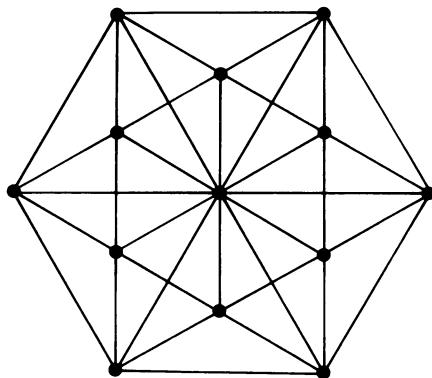


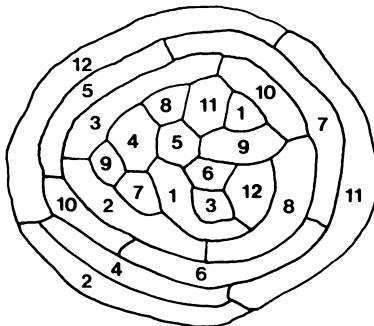
Figure 9.2.9

9.2.7. What is the splitting number of the graph of Figure 9.2.9?

9.3. Heawood's Empire Problem

In 1889, Heawood wanted to solve the Four Color Problem, but he could not do so. He proved the Five Color Theorem that we discussed in Chapter 8. He also considered maps that we now call *empire maps*. If we were to look at a map of the world before World War II, where all the countries that paid

allegiance to Great Britain were the same color, and likewise for those countries with allegiance to Germany, France, Spain, Belgium, and so on, such a world map would be an example of an empire map. A country is assumed to be connected. On the other hand, in a map we call a certain collection of m disjoint countries an *empire*. If we wish to specify that an empire consists of m countries, we call it an m -*pire*. Two empires are adjacent if they share a common boundary edge.



9.3.1. Heawood's map of twelve mutually adjacent 2-pires.

The map of Figure 9.3.1 is a map of twelve mutually adjacent 2-pires that was found by Heawood. He apologized because the map is not symmetric. But it took about 80 years before anyone could find a symmetric one. Scott Kim was the first to find a symmetric map of twelve mutually adjacent 2-pires, the one in Figure 9.3.2.

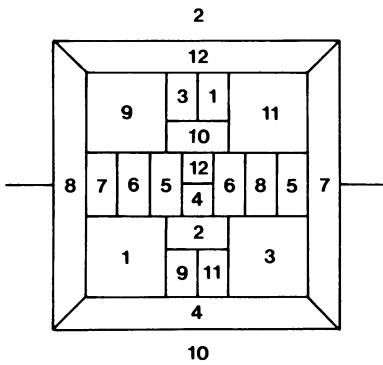


Figure 9.3.2. Kim's symmetric map of twelve mutually adjacent 2-pires.

Heawood asked himself how many colors would be needed to color every m -pire map in the plane for a given value of m . If m is equal to one, then

the problem is the Four Color Problem. For m greater than one, the problem is more general. Heawood solved the problem completely for the case when m is two. Figure 9.3.1 shows that there exists a 2-pire map that requires twelve colors, and Heawood also proved that no 2-pire map requires more than twelve colors.

Theorem 9.3.1 (Heawood, 1890) [11]. *Every 2-pire map can be colored by twelve colors, and there exist 2-pire maps that require twelve colors.*

Proof. Recall that a graph G with chromatic number χ is called *critical* if any smaller subgraph of G has chromatic number less than χ . Given a 2-pire map M , we consider the graph G_2 that is the graph with the 2-pires as vertices, where two vertices are adjacent if the two corresponding 2-pires are adjacent. We shall call G_2 the *empire graph*. We can assume that G_2 is critical with chromatic number χ . If not, then we can change M until G_2 is critical by either removing empires or by changing M as in Figure 9.3.3 as many times as needed.

Now we consider the graph G_1 that has as vertices the countries of M , such that two vertices are adjacent in G_1 if the countries are adjacent in M . We call G_1 the *country graph*. If G_2 has p vertices, then G_1 has $2p$ vertices. G_2 has q_2 edges, and G_1 has q_1 edges with $q_2 \leq q_1$. Since the country graph is planar, we have

$$q_2 \leq q_1 \leq 6p - 6,$$

by Theorem 8.1.3. By Theorem 2.1.4, since G_2 is critical with chromatic number $\chi(G_2) = \chi$,

$$(\chi-1)p \leq 2q_2, \text{ and so}$$

$$(\chi-1)p \leq 12p - 12, \text{ or}$$

$$\chi-1 \leq 12 - \frac{12}{p}, \text{ or}$$

$$\chi-1 < 12, \text{ and hence}$$

$$\chi \leq 12. \blacksquare$$

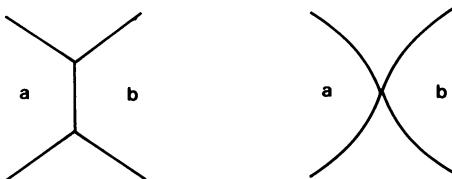


Figure 9.3.3

Theorem 9.3.2 (Heawood) [11]. *Every m -pire map can be colored by $6m$ colors.*

The proof of Theorem 9.3.2 follows the same idea as the proof of Theorem 9.3.1.

Heawood conjectured the following in 1890, and it was proved by Jackson and Ringel 94 years later.

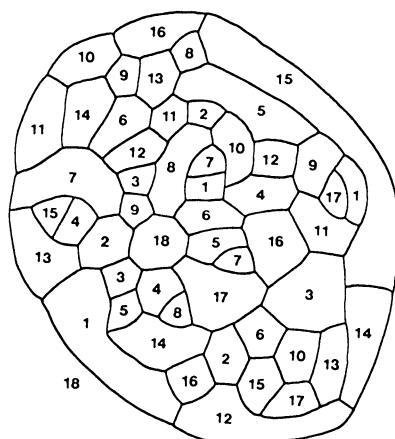


Figure 9.3.4. Taylor's map of 18 mutually adjacent 3-pires.

Theorem 9.3.3 (Heawood, Jackson, Ringel) [13]. *For every $m > 1$, there exists a map in the plane of $6m$ mutually adjacent m -pires.*

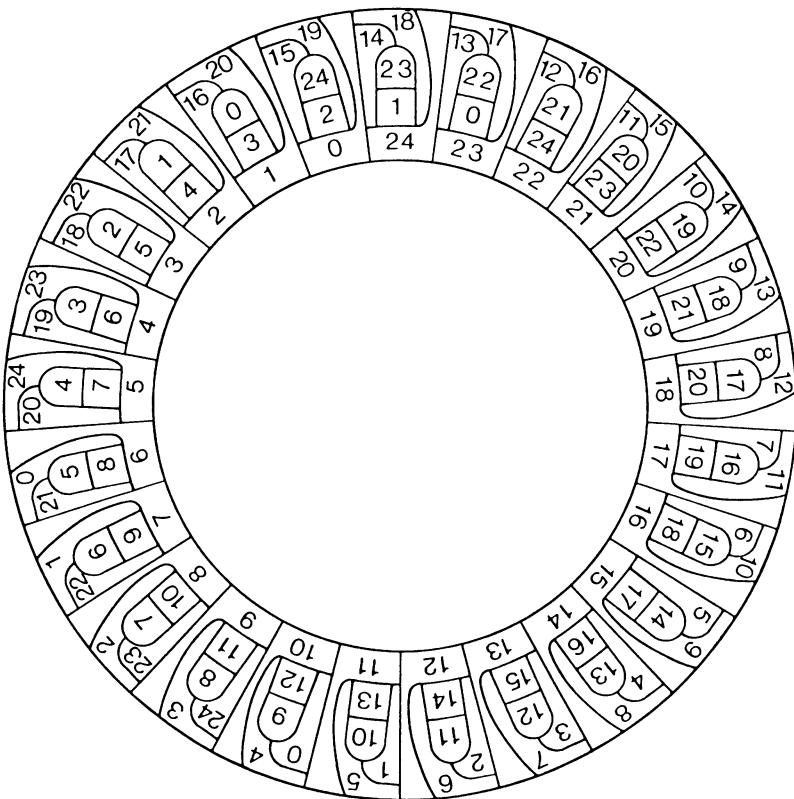


Figure 9.3.5. A symmetric map of 25 mutually adjacent 5-pires.

The map in Figure 9.3.4 was constructed by Herbert Taylor. One of the 3-pires is actually a 2-pire. The map still verifies that there is a 3-pire map that requires 18 colors since we can add another country to the 2-pire without destroying any adjacencies. Taylor also constructed the map of 24 mutually adjacent 4-pires used in the proof of Theorem 9.3.3.

To prove the theorem for all $m \geq 5$, more sophisticated techniques were needed. In particular, a map with a great deal of symmetry had to be constructed. The smallest example of one of these maps is the map of 30 mutually adjacent 5-pires shown in Figure 9.3.6. Before we consider the map, however, we present the map in Figure 9.3.5, which is a symmetric map of 25 mutually adjacent 5-pires.

The map in Figure 9.3.6 was constructed from the map in Figure 9.3.5. Five more 5-pires are added in an irregular way. The idea can be generalized and leads to the proof of Theorem 9.3.3.

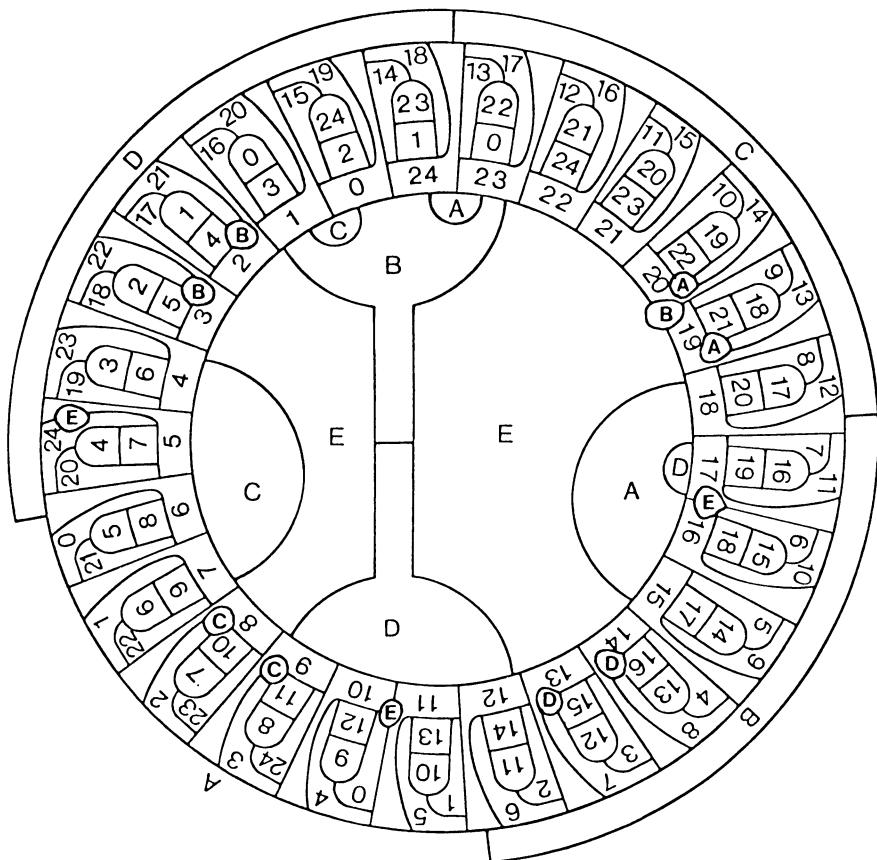


Figure 9.3.6. A map of 30 mutually adjacent 5-pires.

We would like to make a few remarks about the splitting number of K_n . The map of Figure 9.3.1 can serve as a proof that the splitting number of K_{12} is 12, since the dual of the map is a planar splitting of K_{12} . By identifying vertices with the same number, one obtains K_{12} .

The map of Figure 9.3.7 is the dual of a planar splitting of K_{22} .

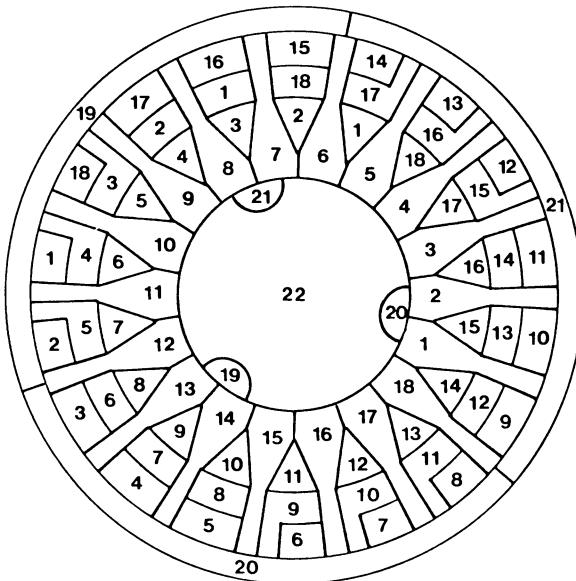


Figure 9.3.7. The dual of a splitting of K_{22} .

One can check that every number is adjacent to every other number. The numbers 1 through 18 are all 4-pires; thus there are three splittings for each. The numbers 19, 20, and 21 are 2-pires; there is one splitting for each. The map shows that

$$\sigma(K_{22}) \leq 57.$$

By Theorem 9.2.4

$$\sigma(K_{22}) \geq \frac{(22-3)(22-4)}{6} = 57.$$

This is the only symmetric splitting of K_n that we know of for $n > 12$.

Another coloring problem was proposed by Ringel in 1949. It is known as the Earth-Moon Problem. The idea is similar to the idea of a 2-pire map, but now one of the countries of each 2-pire is on one sphere, and the other is on a second sphere. It is as if each country on earth has a colony on the moon. The question is, what is the smallest number k_2 of colors so that every earth-moon map can be colored by k_2 colors? By the same proof we gave for the 2-pire

planar maps, one can show that $k_2 \leq 12$ and, since $\theta(K_9) = 3$, there is no map of nine mutually adjacent earth-moon 2-pires. Thus one can say immediately that

$$8 \leq k_2 \leq 12.$$

For quite a time, this was all that was known about k_2 . Then Sulanke constructed the earth-moon map of Figure 9.3.8.

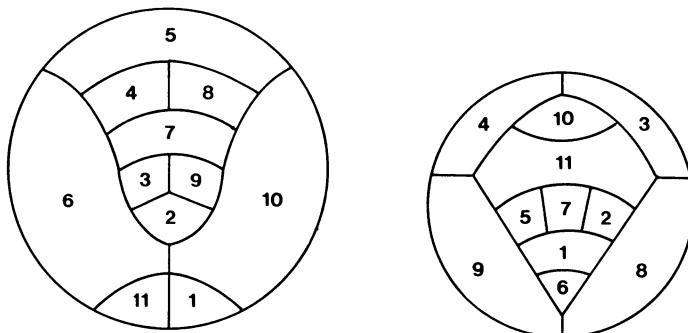


Figure 9.3.8. Sulanke's earth-moon map.

There are 11 earth-moon 2-pires in the map. The empire graph is K_{11} minus C_5 . Thus the following pairs of empires are not adjacent: (1, 3), (3, 5), (5, 2), (2, 4), and (4, 1). The empire graph contains a subgraph that is a complete graph on the vertices of 6, 7, 8, 9, 10, and 11. This requires six colors. The vertices 1, 2, 3, 4, and 5 are adjacent to all of the other six, and they form a C_5 , which requires three colors. Hence nine colors are required to color the map. The problem is still open, since we know k_2 is either 9, 10, 11, or 12.

If we generalize the problem to more than two spheres, we know a little more. For the three planet problem, k_3 is either 16, 17, or 19. For the four planet problem, k_4 is either 22, 23, or 24. For the m planet problem, k_m is either $6m-2$, $6m-1$, or $6m$. The proof of the upper bound is the same as that for the m -pire problem, and for the lower bound, the thickness of the complete graph K_{6m-2} is

$$\theta(K_{6m-2}) = \left\lfloor \frac{6m+5}{6} \right\rfloor = m,$$

so there are m -planet m -pire maps that require $6m-2$ colors.

We now do an example of an exercise involving m -pairs.

Example. Find a 2-pire map in the plane with the largest number of 2-pires such that every pair of 2-pires is adjacent twice.

Solution. Let n be the number of 2-pires in the map. Then the dual of the map has $2n$ vertices and $2\binom{n}{2} = n(n-1)$ edges. We know by Theorem 8.1.3 that $q \leq 3p - 6$, so in this instance

$$\begin{aligned} n(n-1) &\leq 6n-6, \text{ or} \\ n^2 - 7n + 6 &\leq 0, \text{ or} \\ (n-1)(n-6) &\leq 0. \end{aligned}$$

Since $n-1$ is positive, $n-6 \leq 0$, hence $n \leq 6$.

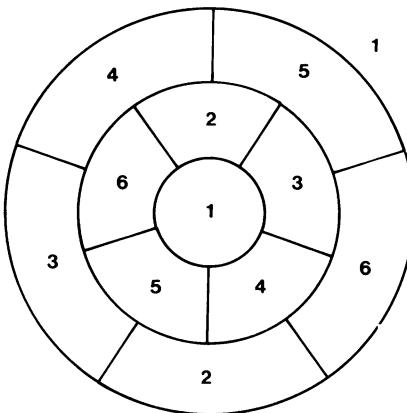


Figure 9.3.9

Now that we know n is at most six, we first try to achieve n exactly equal to six. Then the map will have exactly 12 countries. There exists a nice map with 12 countries, namely the dodecahedron. We find that if we consider opposite countries as 2-pires, we obtain the map in Figure 9.3.9, and this map has the desired properties. A second solution, derived from the cube, is pictured in Figure 9.3.10.

Exercises 9.3.

- 9.3.1. Prove that $\sigma(G) \leq cr(G)$.
- 9.3.2. According to Theorem 9.2.4, $\sigma(K_{13}) \geq 15$. Prove that $\sigma(K_{13}) = 15$ by modifying Figure 9.3.1 or Figure 9.3.2 appropriately.

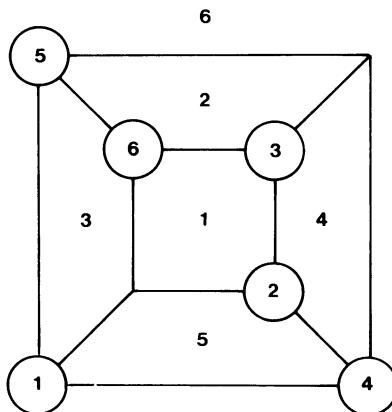


Figure 9.3.10

- 9.3.3. Find an earth-moon map which requires eight colors.
- 9.3.4. Show that the splitting number of the Petersen graph is at most two by finding a splitting of the Petersen graph.
- 9.3.5. Show that the splitting number of the Heawood graph of Figure 4.2.4 is at most three.
- 9.3.6. Find an empire map in the plane with 11 mutually adjacent empires such that 10 of them are 2-pires and one is a 1-pire. (Hint. Consolidate two carefully selected countries in Heawood's map of Figure 9.3.1.)
- 9.3.7. Prove that it is not possible to construct an empire map in the plane with 11 mutually adjacent empires such that nine of them are 2-pires and two are 1-pires.
- 9.3.8. Find an empire map in the plane with 10 mutually adjacent empires such that eight of them are 2-pires and two are 1-pires.
- 9.3.9. Find an empire map in the plane with nine mutually adjacent empires such that six of them are 2-pires and three are 1-pires. (Hint. One of the pictures in this chapter might be very helpful.)
- 9.3.10. Find an empire map in the plane with eight mutually adjacent empires such that four of them are 2-pires and four are 1-pires. (Hint. See the hint for Problem 9.3.9.)
- 9.3.11. The splitting number of K_{12} is 12. Can this be shown by an empire map with one 1-pire, ten 2-pires, and one 3-pire?

- 9.3.12. An *island map* is a map in which one region is designated as the ocean, and every country in the map is adjacent to the ocean. What is the chromatic number of an island map? (The ocean is not colored.)
- 9.3.13. Which of the m -pires in the map of Figure 9.3.4 is actually a 2-pire?

Chapter 10

GRAPHS ON SURFACES

10.1 Rotations of Graphs

In Section 8.2, we discussed briefly a labeling of the vertices of a cubic graph by either black or white according to whether the edges around the vertex were colored a , b , c in a clockwise or counterclockwise direction. Now we shall not be concerned with edge colorings but with another application of labeling the vertices by black and white. We shall be working with drawings of graphs in the plane but not necessarily planar graphs. There are no restrictions on the black and white coloration of the vertices; all such colorings are allowed. Such a coloration of the vertices of a cubic graph G is called a *rotation* of G , and we shall sometimes denote a particular rotation of G by the Greek letter ρ . We shall still consider black to indicate clockwise and white to indicate counterclockwise, as in Figure 10.1.1.

The left of Figure 10.1.2 shows a rotation of a cubic graph.

Now we must explain what is shown on the right of Figure 10.1.2. Suppose that a traveler starts walking along an edge of the graph in either of the two directions possible. When the traveler comes to a black vertex, he turns left and walks along the edge that he encounters in that direction. When he comes to a white vertex, he turns right and walks along that edge. Figure 10.1.1 shows how the traveler turns at each vertex. If he returns to his starting point, and the next edge would be the first edge repeated in the same direction, then he has completed a *circuit*. We say that the circuit is *induced* by the rotation. The right

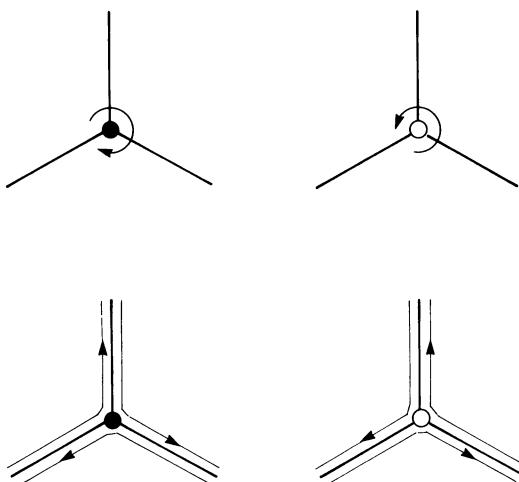


Figure 10.1.1

of Figure 10.1.2 shows one of the circuits induced by the rotation ρ shown on the left. The other two circuits induced by the rotation ρ are shown in Figure 10.1.3. Notice that one of the three circuits is a circuit only and not a cycle, since one edge is used twice, and that no circuits other than these three are induced by this rotation. Also notice that each edge is traversed in each direction in at least one of the circuits.

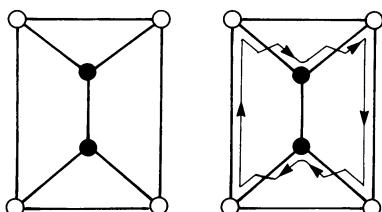


Figure 10.1.2

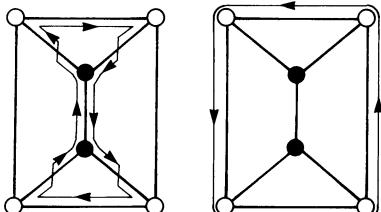


Figure 10.1.3

The rotation ρ induces three circuits. We shall denote the number of circuits induced by a rotation ρ by $r(\rho)$. So in this case $r(\rho) = 3$.

Figure 10.1.4 shows a different rotation ρ_1 of the same graph G of Figure 10.1.2. The rotation ρ_1 induces one circuit only. We call such a rotation a *circular rotation*. Finally, a rotation ρ_2 that induces five circuits is shown in Figure 10.1.5. Altogether the graph has 2^6 different rotations, since each vertex can be either black or white, and there are six vertices. We have selected three

of these rotations for discussion. No matter which rotation we choose, the number of circuits induced by the rotation will be either one, three, or five. Later we shall see why this must be so.

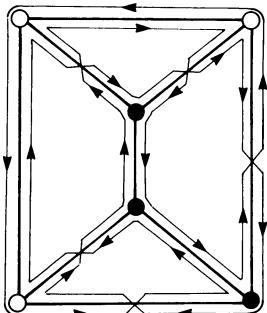


Figure 10.1.4

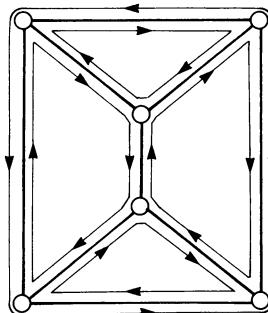


Figure 10.1.5

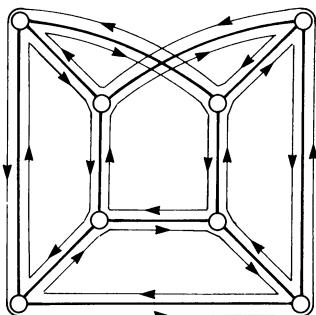


Figure 10.1.6

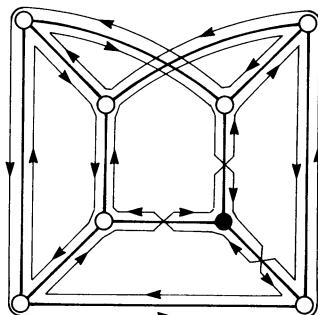


Figure 10.1.7

Figures 10.1.6 and 10.1.7 show another cubic graph and two rotations of this graph. This time, the number of circuits for any rotation ρ of the graph is either $r(\rho) = 4$, as in Figure 10.1.6, or $r(\rho) = 2$ as in Figure 10.1.7.

We could describe the graph of Figure 10.1.2 by numbering the vertices and listing for each vertex the vertices adjacent to that vertex. We call such a listing a *scheme*.

The scheme for the numbering of the graph in Figure 10.1.8 is as follows:

1. 2, 3, 6
2. 1, 3, 5
3. 1, 2, 4
4. 3, 5, 6
5. 2, 4, 6
6. 1, 4, 5 .

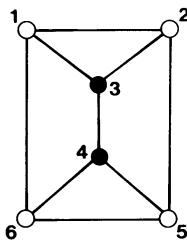


Figure 10.1.8

The first row tells us that vertex 1 is adjacent to vertices 2, 3, and 6; the second row tells us that vertex 2 is adjacent to vertices 1, 3, and 5; and so on. Every graph can be described by such a scheme. Every scheme that describes a graph has the following property:

if in row $i \dots j \dots$,

then in row $j \dots i \dots$,

because if i is adjacent to j , then j is adjacent to i . The converse is also true. Every scheme with the above property describes a graph.

If we use a scheme simply to describe a graph, then order is not important in each row. But if we consider order, then the notion of a scheme also gives us a chance to describe a rotation of a graph. So for instance, in Figure 10.1.8 the rotation around vertex 1 is described by the row

$$1. \ 6 \ 3 \ 2$$

which means that after 6 comes 3, after 3 comes 2, and after 2 comes 6. In other words, the row is written cyclically. Thus the following listings are equivalent:

$$1. \ 6 \ 3 \ 2$$

$$1. \ 2 \ 6 \ 3$$

$$1. \ 3 \ 2 \ 6 .$$

The entire scheme for the rotation shown in Figure 10.1.8 is

1. 6 3 2
2. 1 3 5
3. 1 2 4
4. 3 5 6
5. 2 4 6
6. 5 4 1.

Notice that a scheme gives us a way of describing a rotation of a graph without reference to any drawing of the graph.

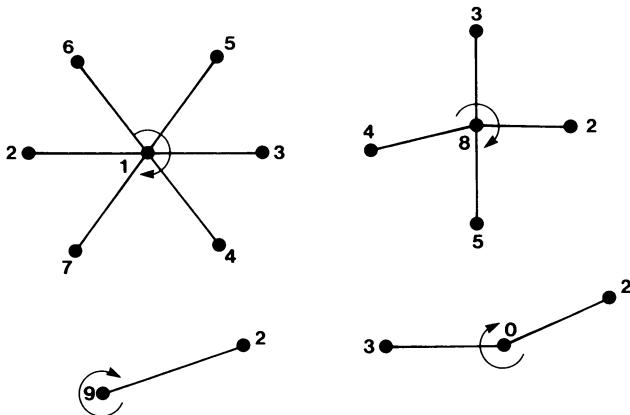


Figure 10.1.9

A *rotation* for a particular vertex of a graph is an ordered cyclic listing of the vertices adjacent to that vertex. We can consider rotations for vertices with any degree, not only for vertices with degree 3. For Figure 10.1.9, we see a vertex 1 with the neighbors in the order

$$1. \ 2 \ 6 \ 5 \ 3 \ 4 \ 7.$$

For a vertex of degree 6, there are 120 different rotations of that vertex. Another rotation shown in Figure 10.1.9 is

$$8. \ 2 \ 5 \ 4 \ 3.$$

For a vertex of degree 4, there are six different rotations. Finally, in Figure 10.1.9 are the rotations

$$9. \ 2 \quad \text{and}$$

$$0. \ 2 \ 3.$$

So in the first row, there is only one vertex adjacent to vertex 9, thus 2 comes after 2. For vertices of degree 1 or 2, only one rotation is possible.

Theorem 10.1.1. *If a vertex v of a graph has degree d , then there are $(d-1)!$ different rotations of v .*

Proof. We fix a neighbor a_1 of v . Then in the scheme we have

$$v. \quad a_1 \ a_2 \ a_3 \cdots a_d.$$

There are $d-1$ choices for a_2 , $d-2$ choices for a_3 , and so on, down to two choices for a_{d-1} and one choice for a_d . Hence the total number is $(d-1)!$. ■

A rotation of a graph consists of rotations for every vertex of the graph.

The scheme

$$\begin{array}{l} 0. \quad 1 \ 3 \ 2 \ 4 \\ 1. \quad 3 \ 0 \ 2 \ 4 \\ 2. \quad 1 \ 4 \ 0 \ 3 \\ 3. \quad 2 \ 4 \ 1 \ 0 \\ 4. \quad 1 \ 0 \ 3 \ 2 \end{array}$$

is a scheme for a rotation of K_5 . We shall now explain how to use the scheme to find the circuits induced by a rotation without drawing a picture of the graph. We begin with any edge in any direction, say 0 1. That means we arrive at vertex 1, coming from vertex 0. So we look in row 1, and we see that after 0 there is a 2. So we have 0 1 2 thus far. We are now at vertex 2, coming from 1. We look in row 2 and see that after 1 comes 4. Now we have 0 1 2 4. In row 4, 2 is followed by 1, so we get 0 1 2 4 1. Continuing in this manner, we obtain

$$0 \ 1 \ 2 \ 4 \ 1 \ 3 \ 0 \ 2 \ 3 \ 4 \ 2 \ 0 \ 4 \ 3 \ 1 \ 0 \ 3 \ 2 \ 1 \ 4.$$

In row 4, 1 is followed by 0, and in row 0, 4 is followed by 1. If we continue to write the vertices down, we repeat for the first time the same edge in the same direction. This tells us that we have the entire circuit, so we do not need to write any more. We can also determine that there are no more circuits, since the length of this circuit is 20, and every edge is used in both directions.

The following scheme determines another interesting rotation of K_5 .

0. 1 3 4 2
1. 2 4 0 3
2. 3 0 1 4
3. 4 1 2 0
4. 0 2 3 1.

The circuits determined by this rotation are

- 1 0 3 4
- 2 1 4 0
- 3 2 0 1
- 4 3 1 2
- 0 4 2 3.

This time there are five circuits, each of which has length four.

We would like to characterize the idea of a planar graph in purely graph theoretical terms, without appealing to the Jordan Curve Theorem and without using terms such as drawing a graph in the plane. We shall give a new definition of what it means for a graph to be planar, independent of the ideas in Chapter 8, and the reader should look to see how the concepts are analogous.

Notice that in the graph of Figure 10.1.2, $p = 6$, $q = 9$ and $r(\rho)$ is either 1, 3, or 5. So for this graph, $p - q + r(\rho) \leq 2$ for all rotations ρ . The following theorem looks similar to Theorem 8.1.1. In the proof we shall use both cycles and circuits, so we recall the definitions here. A cycle has no repeated edges and no repeated vertices. A circuit induced by a rotation may have repeated vertices and edges used in both directions.

Theorem 10.1.2. *Given a connected graph with p vertices and q edges, and a rotation ρ which induces $r(\rho)$ circuits, the inequality*

$$p - q + r(\rho) \leq 2$$

holds. Furthermore, the alternating sum

$$p - q + r(\rho)$$

is even.

Proof. The proof is by induction on the number of cycles in the graph. If there are no cycles in the graph, then it is a tree, since it is connected. Any rotation of

a tree induces precisely one circuit. (See Figure 10.1.10.) In a tree $q = p - 1$, and the inequality becomes

$$p - (p-1) + 1 \leq 2 \text{ or}$$

$$2 \leq 2,$$

and since two is an even number, the theorem is true if there are no cycles. Now we assume that the theorem is true for all connected graphs that have n or fewer cycles. Let G be a graph with $n+1$ cycles, and let ρ be a rotation of G . We select an edge e that lies on a cycle of G . The rotation ρ induces circuits in G ,

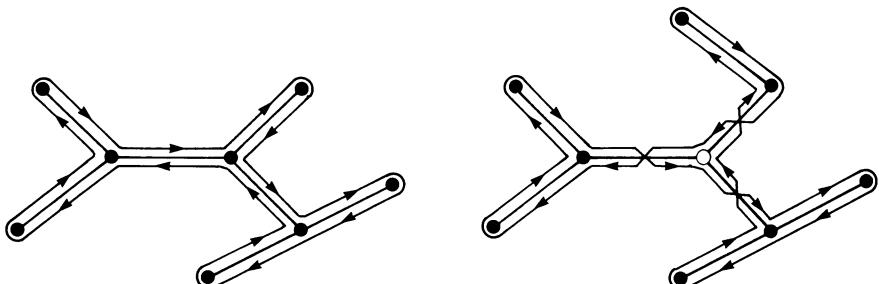


Figure 10.1.10

and either one of these circuits uses the edge e twice, as in the top of Figure 10.1.11, or the edge e occurs in two different circuits as in the top of Figure 10.1.12. We consider the graph $G-e$, and choose the rotation $\hat{\rho}$ which is the same as ρ everywhere except at the endpoints of e and at the endpoints of e simply has e deleted. If e occurred twice in one circuit of G , given the rotation ρ , then the one circuit of G will be replaced by two circuits in $G-e$, as in Figure 10.1.11 on the bottom. Then $r(\rho) = r(\hat{\rho}) - 1$. If e occurred in two different circuits in G , given the rotation ρ , then in $G-e$ the two circuits will be replaced by one circuit, as in Figure 10.1.12. Then $r(\rho) = r(\hat{\rho}) + 1$. So in either of these cases

$$r(\rho) = r(\hat{\rho}) \pm 1.$$

The graph $G-e$ has n or fewer cycles. If G has p vertices and q edges, $G-e$ has p vertices and $q-1$ edges. By the induction hypothesis,

$$p - (q-1) + r(\hat{\rho}) \leq 2, \text{ and}$$

$$p - (q-1) + r(\hat{\rho})$$

is even. Thus

$$\begin{aligned} p - q + r(\rho) &= p - (q-1) - 1 + r(\hat{\rho}) \pm 1 \\ &\leq p - (q-1) + r(\hat{\rho}) \leq 2. \end{aligned}$$

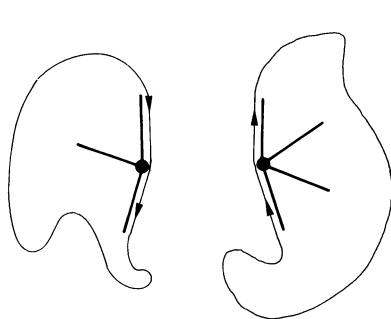
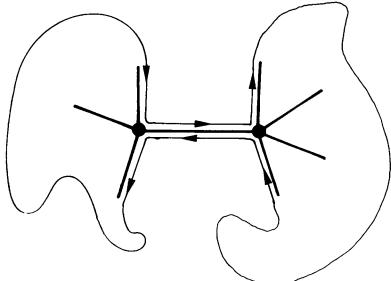


Figure 10.1.11

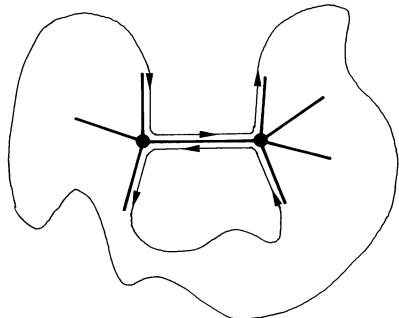


Figure 10.1.12

Since the quantity -1 ± 1 either equals zero or minus two, and since $p - (q-1) + r(\hat{\rho})$ is even, $p - q + r(\rho)$ is also even. ■

Exercises 10.1

- 10.1.1. Find a rotation of the cube graph of Figure 1.2.5 inducing four circuits, each of length six.
- 10.1.2. Find a rotation of the cube graph inducing two circuits.
- 10.1.3. Find a circular rotation of the wheel W_4 .
- 10.1.4. Find a rotation of the wheel W_5 inducing two circuits.

- 10.1.5. Find a connected graph G with a rotation ρ such that there are four induced circuits and their lengths are 3, 4, 5, and 6. (Hint: There might be vertices of degree 2.)
- 10.1.6. Prove that it is not possible to find a graph G with a rotation ρ such that the lengths of the five induced circuits are 3, 4, 5, 5, and 6.
- 10.1.7. How many different rotations does a cubic graph with $2n$ vertices have?
- 10.1.8. Suppose that a graph G has p vertices with degrees $d_1, d_2, d_3, \dots, d_p$. How many different rotations does G have?
- 10.1.9. List the circuits induced by the following rotation of $K_{3,3}$.

0. 1 3 5
1. 2 4 0
2. 1 5 3
3. 2 4 0
4. 1 3 5
5. 2 0 4

- 10.1.10. List the circuits induced by the following rotation of K_5 .

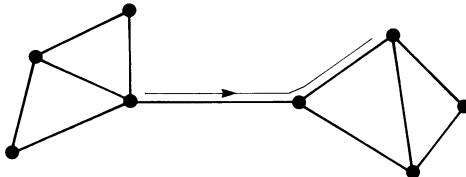
0. 1 2 4 3
1. 0 2 4 3
2. 0 1 4 3
3. 0 1 4 2
4. 0 2 3 1

- 10.1.11. Find a circular rotation of $K_{3,3}$.

10.2. Planar Graphs Revisited

In this chapter we present a new definition of planar graphs using the idea of rotations. We wish to consider now the interesting case where equality holds in Theorem 10.1.2. If a graph G is connected, and there exists a rotation ρ of G so that $p - q + r(\rho) = 2$, then we call G a *connected planar graph*. The rotation ρ will be called a *planar rotation of G* . If G is not connected, then we consider each connected subgraph of G . If each such subgraph satisfies these conditions, then the entire graph is called *planar*.

Theorem 10.2.1. *If a graph G has a bridge, then for any rotation ρ of G , the bridge occurs in both directions in one circuit induced by ρ .*



Proof. Since circuits are closed, if we begin a circuit on one endpoint of the bridge, the only way to come back to the bank on which this endpoint lies is to use the bridge again in the other direction. ■

Theorem 10.2.2. *If H is a subgraph of a planar graph G , then H is planar.*

Proof. Let G be a planar graph with p vertices and q edges, and assume that ρ is a planar rotation of G . It is sufficient to prove the theorem for G connected and for $H = G - e$ for some edge e of G , since we can obtain any subgraph of G by removing enough edges and throwing out isolated vertices when necessary. We consider three cases.

Case 1. If one of the endpoints of e , say v , has degree 1, then $G - e$ equals $G - v$ plus an isolated vertex. $G - v$ has $p - 1$ vertices and $q - 1$ edges. Let $\hat{\rho}$ be the rotation of $G - v$ that is the same as ρ everywhere but at the other endpoint of e , and there has e deleted. Then $r(\hat{\rho}) = r(\rho)$. Hence for $G - v$,

$$\begin{aligned} & (p-1) - (q-1) + r(\hat{\rho}) \\ &= p - q + r(\rho) = 2, \end{aligned}$$

and so $G - v$ is planar, hence $G - e$ is planar, since an isolated vertex is also planar.

Case 2. If the edge e is a bridge, then on one bank there are p_1 vertices, q_1 edges, and $r_1(\hat{\rho})$ circuits, where $\hat{\rho}$ is the rotation that is the same as ρ everywhere except at the endpoints of e , and at the endpoint of e has e deleted, and on the other bank there are p_2 vertices, q_2 edges, and $r_2(\hat{\rho})$ circuits, and $p_1 + p_2 = p$, $q_1 + q_2 = q - 1$, and $r_1(\hat{\rho}) + r_2(\hat{\rho}) = r(\rho) + 1$. Also

$$p - q + r(\rho) = 2,$$

$$p - (q - 1) + r(\rho) + 1 = 4, \text{ and}$$

$$p_1 + p_2 - (q_1 + q_2) + r_1(\hat{\rho}) + r_2(\hat{\rho}) = 4, \text{ so}$$

$$p_1 - q_1 + r_1(\hat{\rho}) + p_2 - q_2 + r_2(\hat{\rho}) = 4.$$

Since by Theorem 10.1.2,

$$p_1 - q_1 + r_1(\hat{\rho}) \leq 2, \text{ and}$$

$$p_2 - q_2 + r_2(\hat{\rho}) \leq 2,$$

it must be true that

$$p_1 - q_1 + r_1(\hat{\rho}) = 2, \text{ and}$$

$$p_2 - q_2 + r_2(\hat{\rho}) = 2.$$

Hence both connected portions of $G - e$ are planar, so $G - e$ is planar.

Case 3. Suppose e is not a bridge. The graph $G - e$ has p vertices and $q - 1$ edges. If $\hat{\rho}$ is the rotation of $G - e$ that is the same as ρ everywhere except at the endpoints of e and at the endpoints of e simply has e deleted, then as in Figures 10.1.11 and 10.1.12, we see that in $G - e$ either one circuit is replaced by two, or two circuits are replaced by one. Thus either $r(\hat{\rho}) = r(\rho) - 1$ or $r(\hat{\rho}) = r(\rho) + 1$. If $r(\hat{\rho}) = r(\rho) - 1$, then

$$p - (q - 1) + r(\hat{\rho})$$

$$= p - q + 1 + r(\rho) - 1 = 2,$$

and thus $G - e$ is planar. If $r(\hat{\rho}) = r(\rho) + 1$, then

$$p - (q - 1) + r(\hat{\rho})$$

$$= p - q + 1 + r(\rho) + 1 = 4,$$

which contradicts Theorem 10.1.2, since $G - e$ is connected. Hence $r(\hat{\rho})$ does not equal $r(\rho) + 1$, and we see that in all three cases, $G - e$ is planar. ■

Theorem 10.2.3. *The complete bipartite graph $K_{3,3}$ is not planar.*

Proof. If $K_{3,3}$ is planar, then there is a rotation ρ of $K_{3,3}$ such that

$$p - q + r(\rho) = 2.$$

For $K_{3,3}$, $p = 6$, and $q = 9$, so $r(\rho) = 5$. Then the average length of a circuit induced by ρ is

$$\frac{18}{5} = 3\frac{3}{5}$$

since each edge is used twice. But this implies that at least one of the circuits has length three. Since the shortest circuit in $K_{3,3}$ has length four, no rotation induces five circuits, and $K_{3,3}$ is not planar. ■

Theorem 10.2.4. *The complete graph K_5 is not planar.*

The proof of Theorem 10.2.4 is left as an exercise.

A *maximal planar graph* is a planar graph with the property that if any pair of nonadjacent vertices are connected by an edge, the resulting graph is no longer planar. The graphs K_3 and K_4 are maximal planar, and they have no nonadjacent pairs of vertices. The graph K_5-e is maximal planar for every edge e . The graph $K_{3,3}-e$ is *not* maximal planar.

Theorem 10.2.5. *A maximal planar graph G with three or more vertices is connected and has no bridge.*

Proof. Assume that G is a maximal planar graph with p vertices and q edges. The proof that G is connected is very similar to the proof of Theorem 10.2.2, Case 2, where we removed a bridge. If G is not connected we can add an edge between two connected portions. This edge will be a bridge in the new graph, two circuits will be replaced by one, and the new graph has one more edge than G . So we only need to show that G cannot have a bridge.

If G has a bridge, since G has at least three vertices, one bank has more than one vertex. Denote the endpoints of the bridge by 1 and 2, and assume that the vertex 3 adjacent to 1 is on the same circuit as the bridge in a planar rotation ρ of G , as in Figure 10.2.1.

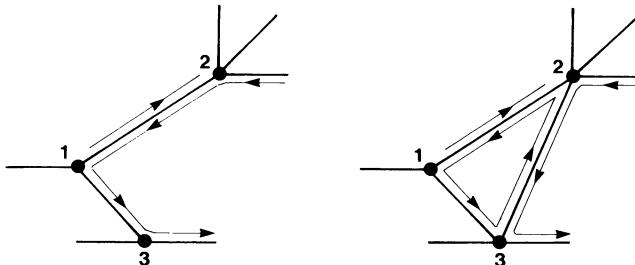


Figure 10.2.1

We now add an edge e between 2 and 3. We choose the rotation $\hat{\rho}$ of $G+e$ that is the same as ρ everywhere except at 2 and at 3, and if ρ has

$$\begin{aligned} 2. \quad & \cdots x \ 1 \ y \ \cdots \\ 3. \quad & \cdots z \ 1 \ w \ \cdots, \end{aligned}$$

then $\hat{\rho}$ has

$$\begin{aligned} 2. \quad & \cdots x \ 3 \ 1 \ y \ \cdots \\ 3. \quad & \cdots z \ 1 \ 2 \ w \ \cdots. \end{aligned}$$

Then $r(\hat{\rho}) = r(\rho) + 1$, as shown in Figure 10.2.1, where the one circuit containing the bridge becomes two circuits, one of which has length three. The graph $G+e$ has $q+1$ edges, so

$$\begin{aligned} p - (q+1) + r(\hat{\rho}) \\ = p - q - 1 + r(\rho) + 1 \\ = p - q + r(\rho) = 2. \end{aligned}$$

So we have added an edge between two nonadjacent vertices in G , and we still have a planar graph. This contradicts our assumption that G was maximal planar; hence G contains no bridge. ■

Theorem 10.2.6. *Every planar rotation ρ of a maximal planar graph has the property that every circuit induced by ρ has length three.*

Proof. Suppose that the theorem is not true. Then there is a maximal planar graph G and a planar rotation ρ of G with the property that at least one circuit induced by ρ has length greater than three. Assume that G has p vertices and q edges. We consider three cases.

Case 1. There is a circuit of length five or longer induced by ρ with all vertices distinct (a cycle), as in Figure 10.2.2. Then at least one pair of vertices on the cycle is not adjacent, since otherwise the given graph would contain a subgraph isomorphic to K_5 , and K_5 is not planar. Assume that 1 is not adjacent to 4. We add the edge e between 1 and 4, obtaining a new graph $G+e$, and we define the rotation $\hat{\rho}$ of $G+e$ to be the same as ρ everywhere except at 1 and 4. At 1 and 4 the rotation $\hat{\rho}$ is as shown in Figure 10.2.2. Then $G+e$ has $q+1$ edges and $r(\hat{\rho}) = r(\rho) + 1$. Then

$$p - (q+1) + r(\hat{\rho})$$

$$= p - q - 1 + r(\rho) + 1$$

$$= p - q + r(\rho) = 2,$$

and $G+e$ is planar. This contradicts our assumption that G was maximal planar.

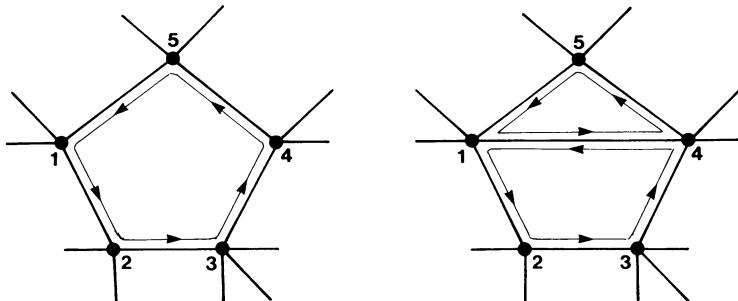


Figure 10.2.2

Case 2. There is a circuit induced by ρ of length four that is a cycle. If any two vertices of the cycle are not adjacent, then we add an edge as in Case 1 and arrive at a contradiction. So we suppose that all four vertices are mutually adjacent, as in Figure 10.2.3.

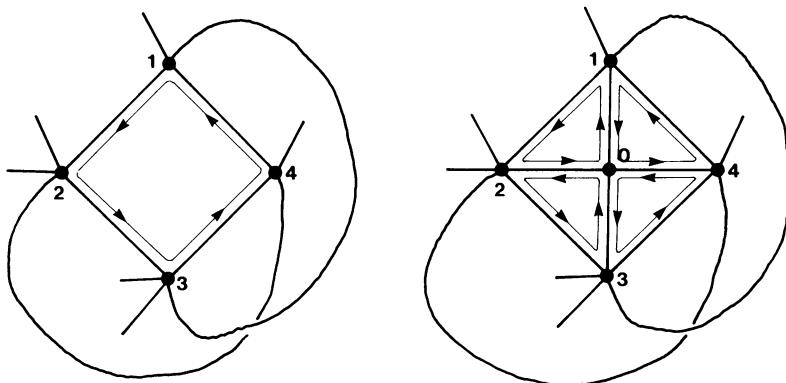


Figure 10.2.3

Then we construct a new graph \hat{G} from G by adding a vertex 0 and connecting it to all four vertices of the cycle. We define the rotation $\hat{\rho}$ of \hat{G} to be the same as ρ everywhere except at vertices 1, 2, 3, and 4. The rotation $\hat{\rho}$ will be as in Figure 10.2.3 at vertices 1, 2, 3, 4, and 0. So \hat{G} has $p+1$ vertices, $q+4$ edges, and $r(\hat{\rho}) = r(\rho) + 3$. Thus for \hat{G} ,

$$\begin{aligned}
 & (p+1) - (q+4) + r(\hat{\rho}) \\
 &= p - q - 3 + r(\rho) + 3 \\
 &= p - q + r(\rho) = 2,
 \end{aligned}$$

and so \hat{G} is planar. But \hat{G} contains a subgraph isomorphic to K_5 since the vertices 0, 1, 2, 3, and 4 are all mutually adjacent. This is a contradiction because K_5 is not planar.

Case 3. G contains a circuit of length at least eight with a repeated edge, as in Figure 10.2.4.

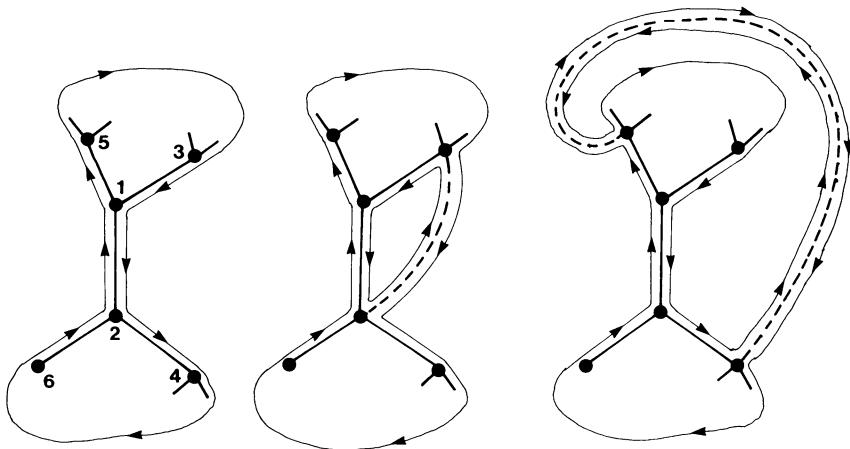


Figure 10.2.4

If any odd-numbered vertex is not adjacent to any even-numbered vertex, we add the edge and still have a planar graph, as in Case 1. Thus all odd-numbered vertices are adjacent to all even-numbered vertices. But then G contains a subgraph isomorphic to $K_{3,3}$, and this is a contradiction because $K_{3,3}$ is not planar. So we have shown that all circuits induced by ρ must have length three. ■

Theorem 10.2.7. *In a maximal planar graph with p vertices, $p \geq 3$, and q edges*

$$q = 3p - 6.$$

Proof. For any planar rotation ρ of a maximal planar graph,

$$3r(\rho) = 2q,$$

by Theorem 10.2.6. Hence

$$p - q + r(\rho) = p - q + \frac{2}{3}q = 2,$$

and thus

$$q = 3p - 6. \quad \blacksquare$$

Theorem 10.2.8. Suppose that G is a maximal planar graph with p vertices, $p \geq 4$. Let p_i denote the number of vertices of G with degree i . Then

$$3p_3 + 2p_4 + p_5 = 12 + p_7 + 2p_8 + 3p_9 + \cdots.$$

The proof of Theorem 10.2.8 is the same as the proof of Theorem 8.1.8.

In Chapter 10 we have presented a one-dimensional theory of planar graphs, without using two-dimensional concepts such as "plane" or "face." We still were able to recover all the theorems for planar graphs. We would like to prove that the two definitions are equivalent; however, we can only prove one direction without going deeply into topology.

Theorem 10.2.9. A connected graph that can be drawn in the plane with no crossings has a planar rotation. That is, graphs that are planar under the definition in Chapter 8 are also planar under the definition in Chapter 10.

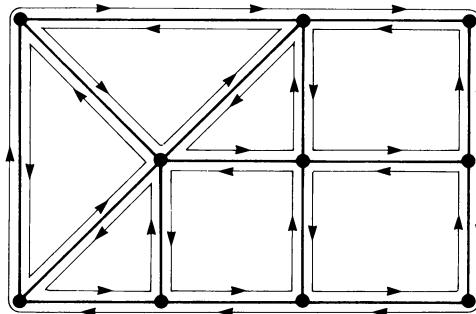


Figure 10.2.5

Proof. Let G be a connected graph with p vertices and q edges that can be drawn in the plane with no crossings. We take a plane drawing of G with r regions and assign the rotation ρ to G that results from giving every vertex in the drawing a clockwise orientation. See Figure 10.2.5 for an example. Then

the circuits induced by ρ are the boundaries of the regions in the drawing, hence $r(\rho) = r$. ■

We introduced the combinatorial treatment of planar graphs not only because of its beauty, but also because it leads us to a natural generalization of the idea of planar graphs, that is, graphs on other surfaces.

Exercises 10.2.

- 10.2.1. Show that the graph of Figure 8.1.2 is planar under the definition of Chapter 10.
- 10.2.2. Prove that a tree is planar under the definition of Chapter 10.
- 10.2.3. Prove or disprove: Given a graph G , if every subgraph of G that is different from G is planar, then G is planar.
- 10.2.4. Prove Theorem 10.2.4.
- 10.2.5. Find a planar rotation of the graph $K_5 - e$.
- 10.2.6. Find a planar rotation of the graph $K_{3,3} - e$.
- 10.2.7. Suppose that ρ is a rotation of K_n . Is $r(\rho)$ odd or even?

10.3. The Genus of a Graph

If a finite graph can be drawn in the plane with no crossings, then it can be drawn on the sphere with no crossings. The sphere is an example of what is called a *closed surface* or a *surface without boundary*. For the sake of brevity, we shall refer to a closed surface as a surface. We shall not define surface rigorously, but we shall give some examples of surfaces.

Figure 10.3.1 shows a surface that is called the *torus*. We can obtain a different representation of this surface by gluing a handle onto a sphere. The surface shown in Figure 10.3.2 is a generalization of the torus. Instead of one handle, this surface has three handles. We call the sphere S_0 ; it has zero handles. We call the torus S_1 ; the torus has one handle. The surface in Figure 10.3.2 is called S_3 and, in general, the surface obtained by attaching g handles to a sphere is called S_g . The official name of S_g is the *orientable surface of genus g* .

The question we wish to consider is, given a graph G and a surface S_g , can G be drawn on S_g with no edges crossing? That is, can G be *embedded* in S_g ?

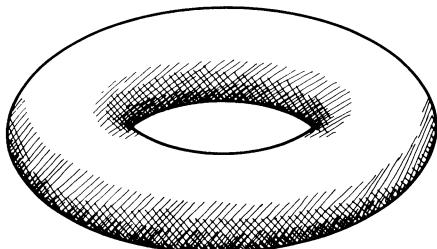


Figure 10.3.1

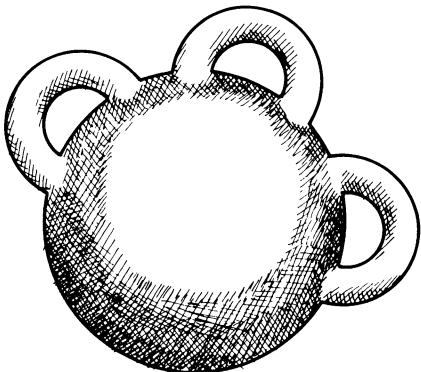


Figure 10.3.2

It is clear that if a graph G is embeddable into S_g , then G is also embeddable into S_{g+1} . The most interesting question we can ask is what is the minimal value of g for which G is embeddable in S_g ? This g will be called the *genus* of G , denoted $\gamma(G)$. The statement $\gamma(G) = g$ means that G is embeddable in S_g , and G is not embeddable in S_{g-1} . So if a graph is planar, then its genus is zero. Similarly, the genus of K_5 is one, and the genus of $K_{3,3}$ is one.

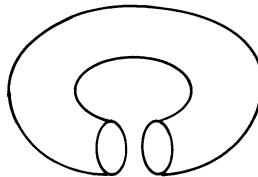
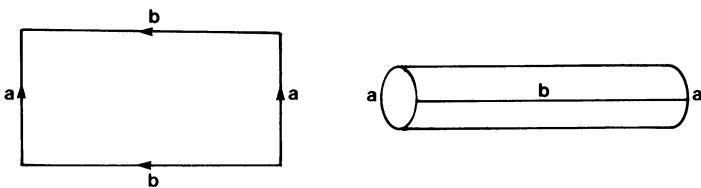


Figure 10.3.3

In Figure 10.3.3, we make a torus out of a rectangle. First we glue the edges labeled b together to get a cylinder, then we bend the cylinder around and identify the edges labeled a .

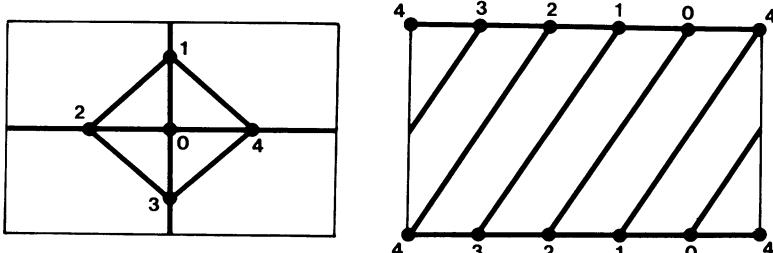


Figure 10.3.4

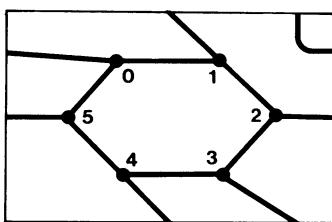


Figure 10.3.5

Figure 10.3.4 shows two different embeddings of K_5 in the torus. Figure 10.3.5 shows $K_{3,3}$ embedded in the torus.

In order to present the more interesting results concerning the genus of a graph, we either need the entire topological theory of closed surfaces, which would be another book, or we need a different approach, using rotations. For those readers who are interested in the theory of closed surfaces, we suggest either Ringel's book *Map Color Theorem* [48] or Frechet and Fan's book *Initiation to the Combinatorial Topology* [43].

Given a connected graph G with p vertices and q edges, and a rotation ρ of G that induces $r(\rho)$ circuits, by Theorem 10.1.2 we know that

$$p - q + r(\rho) \leq 2,$$

and $p - q + r(\rho)$ is even. When $r(\rho)$ is as large as possible, we say that ρ is a *maximal rotation* of G . Clearly, if a graph has a planar rotation, such a rotation is also maximal. An example of a maximal rotation is the following rotation of K_5 , where $r(\rho) = 5$. We know no rotation of K_5 with $r(\rho) = 7$ is possible. Why?

0.	1	4	3	2
1.	0	2	3	4
2.	0	3	4	1
3.	0	4	1	2
4.	0	1	2	3.

The circuits induced by this rotation are precisely the boundaries of the faces of the embedding shown in Figure 10.3.4 on the left. This is not the only maximal rotation of K_5 . Another maximal rotation was presented in Section 10.1. The circuits induced by that rotation are precisely the boundaries of the faces of the embedding on the right of Figure 10.3.4. This embedding of K_5 is particularly pleasing because every face is a quadrangle.

The embedding of $K_{3,3}$ shown in Figure 10.3.5 also reflects a maximal rotation, namely the following.

0.	1	5	3
2.	1	5	3
4.	1	5	3
1.	0	4	2
3.	0	4	2
5.	0	4	2.

The proofs of the following theorems are left as exercises.

Theorem 10.3.1. *If every circuit induced by a rotation ρ of a graph G has length three, then ρ is a maximal rotation of G .*

Theorem 10.3.2. *In a bipartite graph G , if every circuit induced by a rotation ρ of G has length four, then ρ is a maximal rotation of G .*

Two more examples of aesthetically pleasing embeddings of graphs on the torus are the embeddings of $K_{4,4}$ and K_7 shown in Figures 10.3.6 and 10.3.7, respectively. Vertices with the same label on the edges and corners of the rectangle are actually the same vertices, that is, the two vertices labeled 2 in Figure 10.3.6 are the same, as are the two vertices labeled 4 and the four vertices labeled 0.

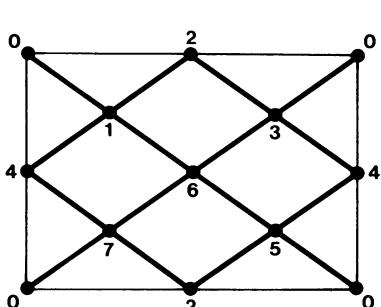


Figure 10.3.6

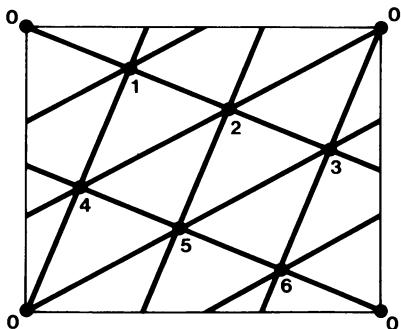


Figure 10.3.7

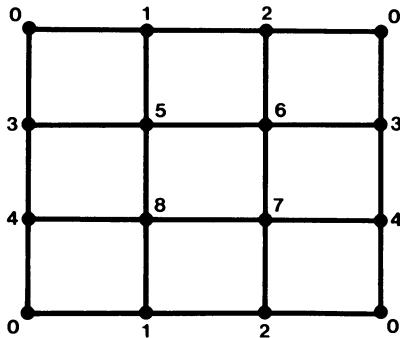


Figure 10.3.8

The graph of Figure 8.4.12 is embeddable into the torus; Figure 10.3.8 shows an embedding of the graph in the torus. We leave it to the reader to verify that this is actually the same graph. The reader should notice that opposite vertices drawn on the edges of the rectangle are identical. The four vertices on the corner are actually one vertex.

Figures 10.3.6 and 10.3.7 show that $\gamma(K_{4,4}) = 1$, and $\gamma(K_7) = 1$.

We now give another definition for the genus of a graph. Let G be a connected graph, and let ρ be a maximal rotation of G . Then $\gamma(G) = g$, where

$$p - q + r(\rho) = 2 - 2g.$$

Since by Theorem 10.1.2, $p - q + r(\rho) \leq 2$ and $p - q + r(\rho)$ is even, g will be a nonnegative integer. Notice that if g is zero, then $p - q + r(\rho) = 2$, and this is consistent with the definition of a planar graph given in Section 10.2. Referring to the examples of Figures 10.3.4, 10.3.5, 10.3.6, and 10.3.7, since these are embeddings of graphs in the torus, we expect to find that $p = q + r(\rho) = 0$ in all these four cases. For instance, in Figure 10.3.7, $p = 7$, $q = 21$, and $r(\rho) = 14$, so

$$p - q + r(\rho) = 7 - 21 + 14 = 0.$$

This means that the genus of each of these graphs must be one, since they are not planar.

Theorem 10.3.3. *For the complete bipartite graph $K_{m,n}$,*

$$\gamma(K_{m,n}) \geq \frac{(m-2)(n-2)}{4}.$$

Proof. Since $K_{m,n}$ is a bipartite graph, for any rotation ρ of $K_{m,n}$, the shortest circuit induced by ρ has length at least four. Since each edge appears twice in circuits, this means that

$$2q \geq 4r(\rho).$$

In $K_{m,n}$, $p = m+n$, and $q = mn$, so the above becomes

$$\frac{mn}{2} \geq r(\rho).$$

If we denote $\gamma(K_{m,n})$ by g , since $p - q + r(\rho) = 2 - 2g$, we have

$$m + n - mn + \frac{mn}{2} \geq 2 - 2g, \text{ or}$$

$$g \geq \frac{mn - 2m - 2n + 4}{4}, \text{ and so}$$

$$\gamma(K_{m,n}) \geq \frac{(m-2)(n-2)}{4}. \blacksquare$$

The following scheme is a scheme for a maximal rotation of $K_{4,6}$, where the four red vertices are denoted by odd numbers, and the six blue vertices are denoted by even numbers.

0.	1	3	5	7		
2.	7	5	3	1		
4.	1	3	5	7		
6.	7	5	3	1		
8.	1	3	5	7		
10.	7	5	3	1		
1.	0	2	4	6	8	10
3.	10	8	6	4	2	0
5.	0	2	4	6	8	10
7.	10	8	6	4	2	0.

This rotation of $K_{4,6}$ induces twelve circuits of length four. Two of the circuits induced by this rotation are 1 0 3 10 and 3 4 5 6. The reader can check that every circuit induced by the rotation has length four, and hence this is a maximal rotation of $K_{4,6}$, by Theorem 10.3.2. In $K_{4,6}$, $p = 10$, $q = 24$, and for the given rotation ρ , $r(\rho) = 12$. Thus

$$p - q + r(\rho) = 10 - 24 + 12 = -2 = 2 - 2g$$

and so $g = 2$. Thus $\gamma(K_{4,6}) = 2$.

The above scheme can be generalized easily for the special case when m and n are both even. In this special case, equality holds in Theorem 10.3.3. For the even vertices, rows 0 and 2 are given below, and row $i+4$ is the same as row i .

0.	1	3	5	...	2n-1
2.	2n-1	...	5	3	1.

For the odd vertices, rows 1 and 3 are given below, and row $i+4$ is the same as row i .

1.	0	2	4	...	2m-2
3.	2m-2	...	4	2	0.

The resulting scheme is a scheme for a maximal rotation of $K_{m,n}$. Thus, when m and n are both even,

$$\gamma(K_{m,n}) = \frac{(m-2)(n-2)}{4},$$

since in this case all the inequalities in the proof of Theorem 10.3.3 are equalities, as shown by the rotation given above, where all induced circuits have length four. (See Exercise 10.3.5.)

With a little more effort [20], one can prove the following theorem, where m and n are arbitrary.

Theorem 10.3.4 (Ringel). *The genus of the complete bipartite graph is*

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

We now consider the problem of the genus of the complete graph. We can read off the rotation of K_7 shown in Figure 10.3.7.

0.	1	3	2	6	4	5
1.	2	4	3	0	5	6
2.	3	5	4	1	6	0
3.	4	6	5	2	0	1
4.	5	0	6	3	1	2
5.	6	1	0	4	2	3
6.	0	2	1	5	3	4.

We can see by the picture that the genus of K_7 is one, but we can also see this by using the scheme, since every induced circuit has length three. For larger graphs it is often difficult to draw a picture, so we must work with schemes. Those readers who wish to see how difficult it is should try drawing K_8 with no crossings on a sphere with two handles. The above scheme satisfies the following rule.

Rule Δ^* . If in row $i. \dots j, k \dots$,
then in row $k. \dots i, j \dots$.

If a scheme for a rotation of a graph satisfies Rule Δ^* , then every circuit induced by the rotation has length three. To see that this is true, consider Figure 10.3.9. Assume that in row $i. \dots j, k \dots$. Then the left of Figure 10.3.9 applies. Because of Rule Δ^* , in row $k. \dots i, j \dots$, so the second picture in Figure 10.3.9 holds. Now we apply the rule again, so that in row $j. \dots k, i \dots$, so we get the third picture in Figure 10.3.9. This gives us a circuit of length three as in the last picture.

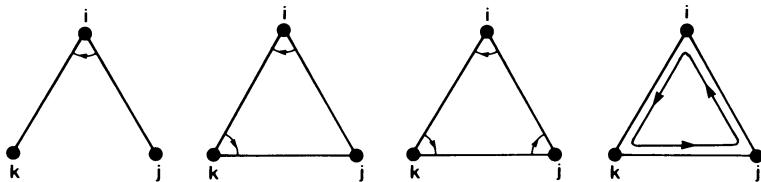


Figure 10.3.9

The following scheme is a scheme for a maximal rotation of K_8 . Not all of the induced circuits have length three; two of them have length four. But it is not possible to find a rotation of K_8 that induces only circuits of length three, since K_8 has 28 edges, and 28 is not divisible by three.

0.	2	7	3	1	4	5	6
2.	4	1	5	3	6	7	0
4.	6	3	7	5	0	1	2
6.	0	5	1	7	2	3	4
1.	7	6	5	2	4	0	3
3.	1	0	7	4	6	2	5
5.	3	2	1	6	0	4	7
7.	5	4	3	0	2	6	1.

The two induced circuits of length four are 0 2 4 6 and 7 5 3 1. In K_8 , $p = 8$, $q = 28$, and $r(\rho) = 18$ for this maximal rotation ρ , since $56 = 16 \cdot 3 + 2 \cdot 4$. Hence by definition of genus

$$p - q + r(\rho) = 2 - 2g, \text{ or}$$

$$8 - 28 + 18 = -2 = 2 - 2g,$$

and thus $g = \gamma(K_8) = 2$.

Theorem 10.3.5. *The genus of the complete graph satisfies the inequality*

$$\gamma(K_n) \geq \frac{(n-3)(n-4)}{12}.$$

Proof. If ρ is a maximal rotation for a graph G with p vertices and q edges, then the genus g of G is given by

$$p - q + r(\rho) = 2 - 2g.$$

Since the shortest circuit possible has length three, and every edge is used twice in circuits, we have

$$2q \geq 3r(\rho),$$

and so

$$p - q + \frac{2}{3}q \geq 2 - 2g, \text{ or}$$

$$2g \geq 2 - p + \frac{1}{3}q.$$

For the graph $G = K_n$, $p = n$ and $q = \binom{n}{2}$, so

$$2g \geq 2 - n + \frac{n(n-1)}{6}, \text{ and hence}$$

$$g \geq \frac{(n-3)(n-4)}{12}. \quad \blacksquare$$

The following scheme, found by Heffter in 1891, gives a rotation of K_{12} satisfying Rule Δ^* . Thus every induced circuit has length three, and the rotation is maximal. Also, all the inequalities in the proof of Theorem 10.3.5 are equalities in this case, and so

$$\gamma(K_{12}) = 6.$$

0.	5	7	2	1	11	8	4	3	9	6	10
4.	9	11	6	5	3	0	8	7	1	10	2
8.	1	3	10	9	7	4	0	11	5	2	6
1.	8	6	11	0	2	5	9	10	4	7	3
5.	0	10	3	4	6	9	1	2	8	11	7
10.	4	2	7	8	10	1	5	6	0	3	11
2.	3	6	8	5	1	0	7	9	4	10	11
6.	7	10	0	9	5	4	11	1	8	2	3
10.	11	2	4	1	9	8	3	5	0	6	7
3.	2	11	9	0	4	5	10	8	1	7	6
7.	6	3	1	4	8	9	2	0	5	11	10
11.	10	7	5	8	0	1	6	4	9	3	2

Heffter used the following rule:

Rule R*. If in row i . \cdots, j, k, l, \cdots ,
then in row k . \cdots, l, i, j, \cdots .

The reader will have the opportunity to show that Rule Δ^* and Rule R^* are equivalent. (The reason for the stars is that there also exist Rules Δ and R , which are used for a different purpose, detailed in *Map Color Theorem* [48].)

We shall now prove that equality holds in the formula of Theorem 10.3.5 for the genus of K_{12t+7} . First we look at the smallest case, where $t = 0$. The scheme for the rotation of K_7 shown in Figure 10.3.7 has the delightful property that all the information needed to construct the entire scheme is given in row 0. To obtain row i from row 0, we add i to every entry of row 0, subtracting seven from any resulting entry that is then greater than six. In other words, the arithmetic is considered *modulo* 7. Such a scheme is called *additive*. The scheme shown above for K_{12} is not additive.

In order to construct row 0 of the schemes for K_{12t+7} , we shall use conservative graphs that are called current graphs. Figure 10.3.10 shows the current graph used for K_{19} .

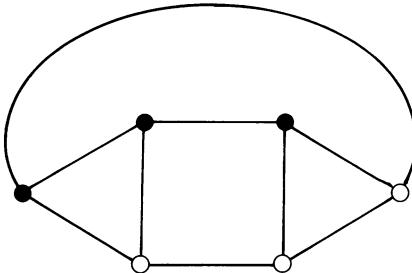


Figure 10.3.10

Notice that the given rotation of the graph in Figure 10.3.10 is circular, as shown in Figure 10.3.11.

We record the *log* of the circuit by beginning on any edge in any direction, say the edge labeled 8 in the positive direction, that is, in the direction of the arrow. If we go in the direction opposite to that of the arrow we record a negative number. The log of the circuit for the current graph of Figure 10.3.12 is

$$8, 9, 7, 4, -2, -9, -1, 5, -3, -7, 2, 6, 1, -8, -5, -6, -4, 3.$$

We replace each negative number $-i$ by $19-i$, and the log becomes

8 9 7 4 17 10 18 5 16 12 2 6 1 11 14 13 15 3.

We choose this row as the zero row of the scheme. Then the scheme is

$$\begin{array}{cccccccccccccccccc}
 0. & 8 & 9 & 7 & 4 & 17 & 10 & 18 & 5 & 16 & 12 & 2 & 6 & 1 & 11 & 14 & 13 & 15 & 3 \\
 1. & 9 & 10 & 8 & 5 & 18 & 11 & 0 & 6 & 17 & 13 & 3 & 7 & 2 & 12 & 15 & 14 & 16 & 4 \\
 2. & 10 & 11 & 9 & 6 & 0 & 12 & 1 & 7 & 18 & 14 & 4 & 8 & 8 & 13 & 16 & 15 & 17 & 5 \\
 \cdot & & & & & & & & & & & & & & & & & & \\
 \cdot & & & & & & & & & & & & & & & & & & \\
 18. & 7 & 8 & 6 & 3 & 16 & 9 & 17 & 4 & 15 & 11 & 1 & 5 & 0 & 10 & 13 & 12 & 14 & 2.
 \end{array}$$

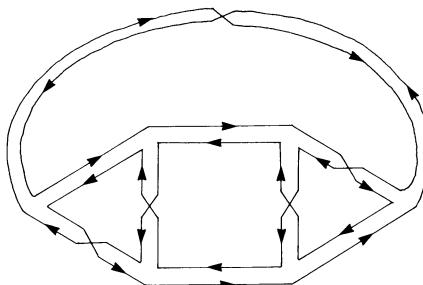


Figure 10.3.11

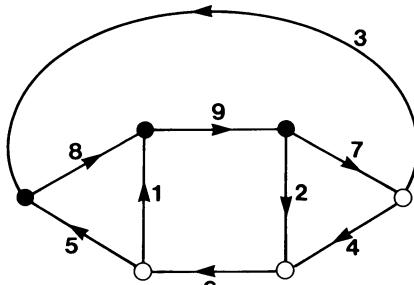


Figure 10.3.12

In the scheme row i is obtained from row 0 by adding i to each entry of row 0, where the arithmetic is considered modulo 19.

We shall now prove that the above scheme satisfies Rule Δ^* . Assume that in the scheme we have

$$i. \quad \dots, j, k, \dots$$

Since the i^{th} row is obtained by adding i to each entry in row 0, we have in row 0

$$0. \quad \dots, j-i, k-i, \dots$$

This means that somewhere in the current graph we have the vertex shown in Figure 10.3.13. Then elsewhere in the zero row we have

$$0. \cdots i-k, j-k \cdots .$$

Thus in row k there must be

$$k. \cdots i, j, \cdots ,$$

and hence the above scheme satisfies Rule Δ^* . This scheme shows that $\gamma(K_{19}) = 20$.

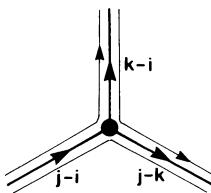


Figure 10.3.13

Figure 10.3.14 shows the current graph used to generate row 0 of a scheme for a rotation of K_{31} where the induced circuits are all triangles.

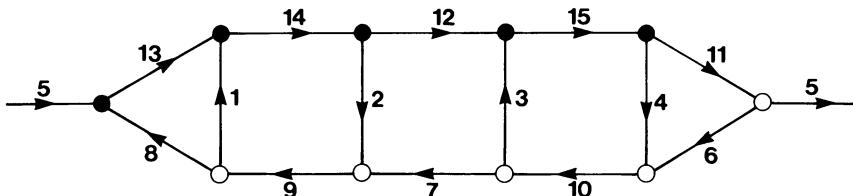


Figure 10.3.14

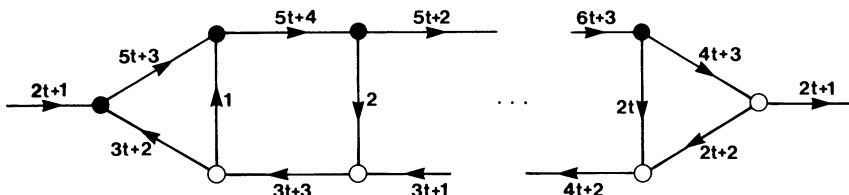


Figure 10.3.15

The generalization of the current graph for K_{12t+7} is shown in Figure 10.3.15. The maximal rotations of K_{12t+7} obtained from these current graphs show that equality holds in Theorem 10.3.5 for $n = 12t + 7$. Hence we have proved Theorem 10.3.6 for the case $n = 12t + 7$. The theorem has been proved

for all $n \geq 3$. The proof for all n was a massive endeavor, and contributions were made by Terry, Welch, and Gustin in addition to Ringel and Youngs [21].

Theorem 10.3.6 (Ringel, Youngs). *The genus of the complete graph is*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil,$$

for $n \geq 3$.

Recall that a graph G has chromatic number $\chi(G)$ if the vertices of G can be colored by $\chi(G)$ colors, and the vertices of G cannot be colored by $\chi(G) - 1$ colors. Also recall that a graph G is critical with chromatic number χ if G has chromatic number χ and every subgraph of G other than G itself has chromatic number less than χ .

Theorem 10.3.7 (Heawood). *If G is critical, and $\gamma(G) \leq g$, where $g \geq 1$, then*

$$\chi(G) \leq \left\lceil \frac{7 + \sqrt{1+48g}}{2} \right\rceil.$$

(Notice that if g is zero, this is no longer Heawood's Theorem, but is instead the famous Four Color Theorem.)

Proof. Let G be critical with chromatic number χ . Then by Theorem 2.1.4

$$(\chi - 1)p \leq 2q.$$

Since $\gamma(G) \leq g$, there exists a maximal notation ρ of G such that

$$p - q + r(\rho) \geq 2 - 2g, \text{ or}$$

$$q - r(\rho) \leq p - (2 - 2g), \text{ or}$$

$$3q - 3r(\rho) \leq 3p - 3(2 - 2g).$$

Since $2q \geq 3r(\rho)$,

$$q \leq 3p - 6 + 6g.$$

Combining inequalities, we obtain

$$(\chi - 1)p \leq 6p - 12 + 12g, \text{ or}$$

$$\chi - 1 \leq 6 + \frac{12g - 12}{p}.$$

Since $g \geq 1$ and $p \geq \chi$, we can replace p by χ and multiply by χ without disturbing the inequality, and thus

$$\chi^2 - \chi \leq 6\chi + 12g - 12, \text{ or}$$

$$\chi^2 - 7\chi - (12g - 12) \leq 0.$$

Using the quadratic formula, we can factor the inequality in the form

$$\left[\chi - \frac{7 + \sqrt{1+48g}}{2} \right] \left[\chi - \frac{7 - \sqrt{1+48g}}{2} \right] \leq 0.$$

Since $g \geq 1$, it follows that $\sqrt{1+48g} \geq 7$. Since $\chi \geq 1$ the second factor is always positive. It follows that the first factor is at most zero. ■

The next theorem follows from the Ringel-Youngs Theorem and the Heawood Theorem.

Theorem 10.3.8 (Map Color Theorem). *For every $g \geq 1$, there exists a critical graph G and $\gamma(G) \leq g$ and*

$$\chi(G) = \left\lfloor \frac{7 + \sqrt{1+48g}}{2} \right\rfloor.$$

What Theorems 10.3.7 and 10.3.8 say is that every graph G that has a rotation ρ so that

$$p - q + r(\rho) = 2 - 2g,$$

for $g \geq 1$, can be vertex colored by at most $\left\lfloor \frac{7 + \sqrt{1+48g}}{2} \right\rfloor$ colors. So for instance, any graph on the torus ($g = 1$) can be colored by at most seven colors, and there are graphs that can be drawn on the torus with no crossings that require seven colors.

Exercises 10.3.

- 10.3.1. Without adding any vertices, what is the maximum number of edges that can be added to the cube graph of Figure 1.2.5 so that the resulting graph is still planar?
- 10.3.2. Verify that the graphs pictured in Figure 8.4.12 and 10.3.8 are isomorphic.
- 10.3.3. Prove Theorem 10.3.1.

- 10.3.4. Prove Theorem 10.3.2.
- 10.3.5. Show that if a scheme for rotation ρ of a graph G satisfies the following
- Rule Q*. If in row $i. \dots j,k \dots$,
and in row $k. \dots i,l \dots$,
then in row $j. \dots l,i \dots$,
and in row $l. \dots k,j \dots$,
then every circuit induced by ρ has length four.
- 10.3.6. Show that Rules Δ^* and R^* are equivalent. (That is, show that a scheme satisfies Rule Δ^* if and only if it satisfies Rule R^* .)
- 10.3.7. Find the genus of the 6-cage of Figure 4.2.4.
- 10.3.8. Determine a lower bound for the genus of the 10-cage of Figure 4.2.8.
- 10.3.9. Using Theorem 10.3.6, determine n so that $\gamma(K_n) = n$
- 10.3.10. Using Theorem 10.3.6, determine n so that $\gamma(K_n) = n+1$.

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