Isogonal conjugation with respect to a triangle
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version 23 September 2006

1. Trivialities on isogonal lines

The aim of this note is to provide a rather detailled and general introduction into isogonal conjugation with respect to a triangle. No new results of the author will be presented, but some results published elsewhere will be proven in a (presumably) new way.

Throughout this work, we will make use of directed angles modulo 180°. This is the kind of angles referred to as "directed angles" in [1], 1.7, and we refer the reader to [1] for their basic properties. (Also, [2], [3] and [4] provide introductions to this type of angles.)

Two preliminary conventions are to be made at first:

- "Wrt" is an abbreviation for "with respect to".
- The A**-altitude** of a triangle ABC will mean the altitude of triangle ABC issuing from its vertex A. Similarly, the A**-median** of triangle ABC will mean the median of triangle ABC issuing from its vertex A.

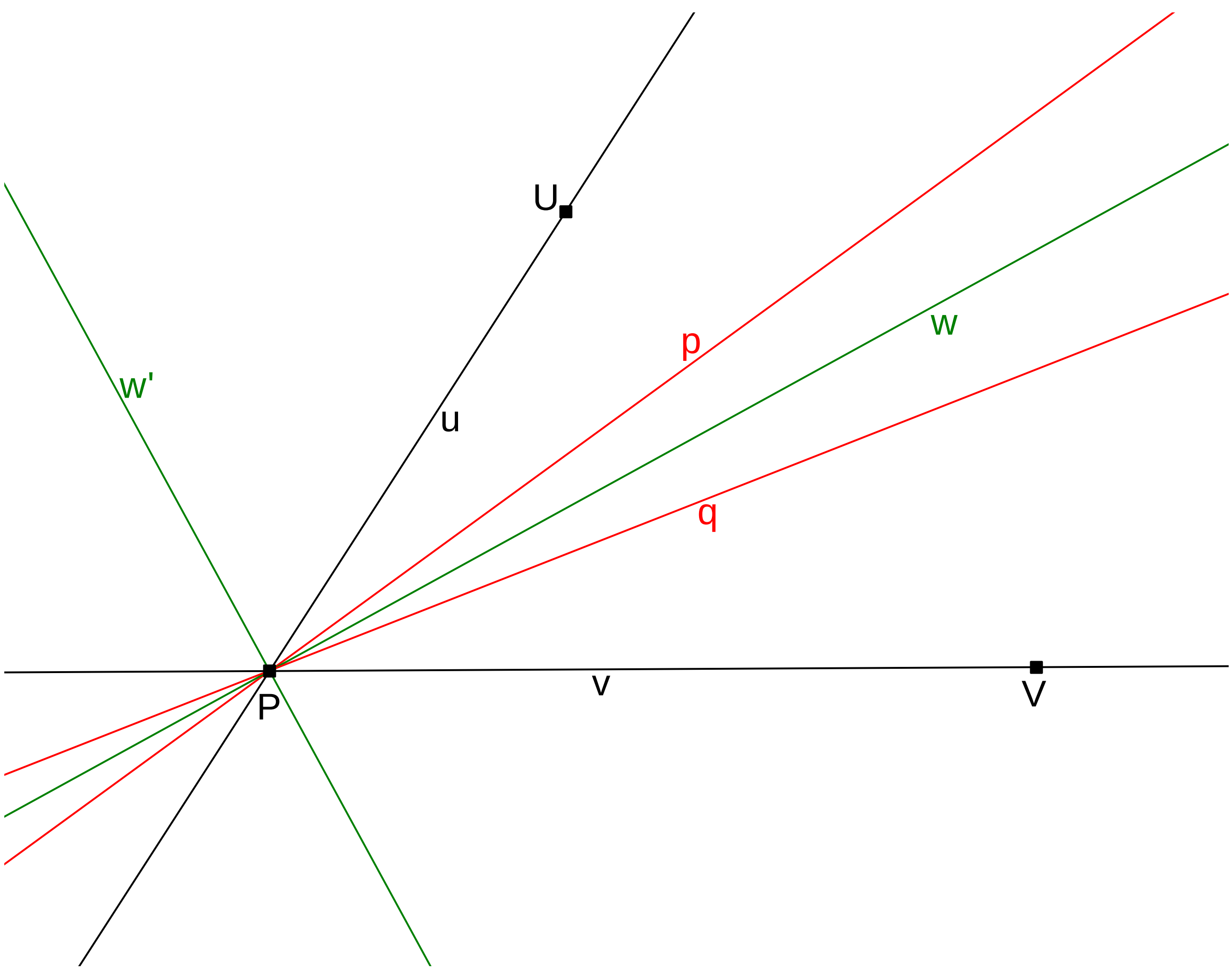

Fig. 1

We start with a simple property of lines (Fig. 1):

Theorem 1. Let u and v be two lines which intersect at an Euclidean (i. e. not infinite) point P. Let p and q be two more lines through the point P. Let w and w' be the two angle bisectors of the angles formed by the lines u and v. Then, the following four assertions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ are pairwisely equivalent:

Assertion $\mathcal{A}_1$: We have $\measuredangle(u;\ q) = -\measuredangle(v;\ p)$.

Assertion $\mathcal{A}_2$: We have $\measuredangle(v;\ q) = -\measuredangle(u;\ p)$.

Assertion $\mathcal{A}_3$: The lines p and q are symmetric to each other wrt the line w.

Assertion $\mathcal{A}_4$: The lines p and q are symmetric to each other wrt the line w'.

Proof of Theorem 1. This proof is almost trivial; we are giving it here only for the sake of completeness.

First, if assertion $\mathcal{A}_1$ holds, i. e. if $\measuredangle(u;\ q) = -\measuredangle(v;\ p)$, then

$$\measuredangle(v;\ q) = \measuredangle(v;\ u)+\measuredangle(u;\ q) = \measuredangle(v;\ u)+(-\measuredangle(v;\ p)) = -(\measuredangle(v;\ p) - \measuredangle(v;\ u)) = -\measuredangle(u;\ p),$$

so assertion $\mathcal{A}_2$ also holds. Similarly, we show the converse: if assertion $\mathcal{A}_2$ holds, then assertion $\mathcal{A}_1$ also holds. Thus, the assertions $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent.

Now we will prove the equivalence of assertions $\mathcal{A}_1$ and $\mathcal{A}_3$: Since the line w is an angle bisector of one of the angles between the lines u and v, we have $\measuredangle(u;\ w) = -\measuredangle(v;\ w)$. Now, assertion $\mathcal{A}_1$ states that $\measuredangle(u;\ q) = -\measuredangle(v;\ p)$; this rewrites as $\measuredangle(u;\ w)+\measuredangle(w;\ q) = -(\measuredangle(v;\ w) + \measuredangle(w;\ p))$, what, in view of $\measuredangle(u;\ w) = -\measuredangle(v;\ w)$, simplifies to $\measuredangle(w;\ q) = -\measuredangle(w;\ p)$. But since the lines w, p, q all pass through the point P, this equation is equivalent to stating that the lines p and q are symmetric to each other wrt the line w. This, however, is assertion $\mathcal{A}_3$. Thus we have shown that the assertions $\mathcal{A}_1$ and $\mathcal{A}_3$ are equivalent; similarly we can prove the assertions $\mathcal{A}_1$ and $\mathcal{A}_4$ to be equivalent. Hence, the equivalence of all four assertions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ is established, and Theorem 1 is proven.

Based on Theorem 1 we make a definition: If one of the four assertions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ holds - and therefore, according to Theorem 1, the three others hold as well -, then we say that the line q is **isogonal** to the line p wrt the lines u and v.

For two lines p and q through the point P, we can easily see that the line q is isogonal to the line p wrt the lines u and v if and only if the line p is isogonal to the line q wrt the lines u and v. (In fact, the line q is isogonal to the line p wrt the lines u and v if and only if $\measuredangle(u;\ q) = -\measuredangle(v;\ p)$; this is equivalent to $\measuredangle(v;\ p) = -\measuredangle(u;\ q)$, and this holds if and only if the line p is isogonal to the line q wrt the lines u and v.) Hence, the assertions "the line q is isogonal to the line p wrt the lines u and v " and "the line p is isogonal to the line q wrt the lines u and v " are equivalent; thus, instead of any of these assertions, we can simply say that "the lines p and q are **isogonal to each other** wrt the lines u and v ".

Instead of saying "isogonal wrt the lines u and v ", we will often say "isogonal wrt the angle UPV ", where U is a point on the line u (distinct from P) and V is a point on the line v (distinct from P).

For each line p through the point P, there exists one and only one line q through the point P which is isogonal to the line p wrt the lines u and v; in fact, this line q must satisfy the equation $\measuredangle(u;\ q) = -\measuredangle(v;\ p)$, and this holds for one and only one line q through the point P (this line can be constructed as a line through a given point which forms a given angle with another given line). This line q which is isogonal to the line p wrt the lines u and v is called the **isogonal** (or **isogonal line**) of the line p wrt the lines u and v, or, equivalently, the **isogonal** (or **isogonal line**) of the line p wrt the angle UPV.

2. Isogonals and perpendicular bisectors

The next properties of isogonals we are going to show are not much harder to prove, but turn out to be of remarkable usefulness:

Theorem 2. Let u and v be two lines intersecting at an Euclidean point P. Let T be a point in the plane.

a) Let X and Y be the orthogonal projections of the point T on the lines u and v. Then, the line XY is perpendicular to the isogonal of the line PT wrt the lines u and v. (See Fig. 2.)

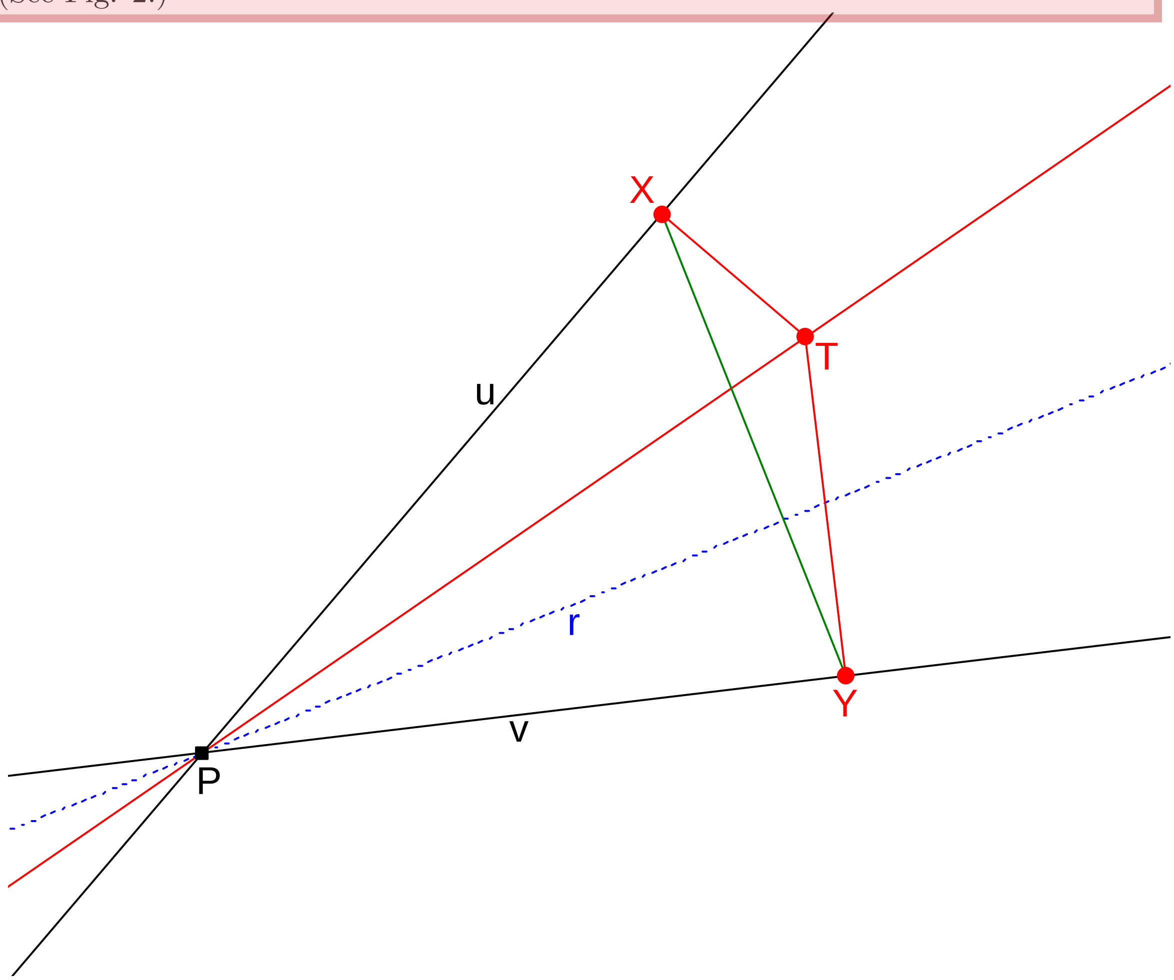

Fig. 2

b) Let X' and Y' be the reflections of the point T in the lines u and v. Then, the perpendicular bisector of the segment $X'Y'$ is the isogonal of the line PT wrt the lines u and v. (See Fig. 3.)

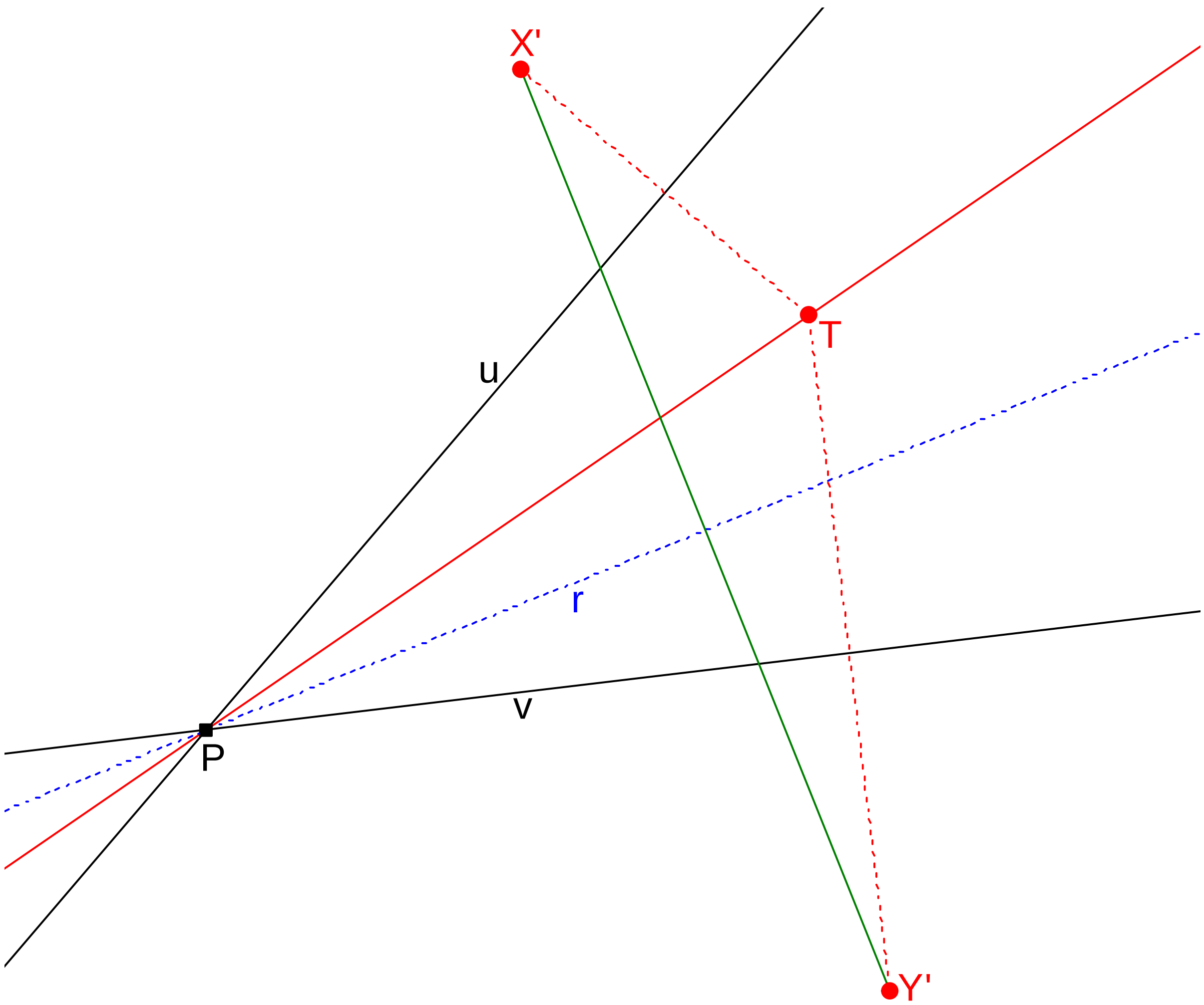

Fig. 3

Proof of Theorem 2. Let r be the isogonal of the line PT wrt the lines u and v. Then, $\measuredangle(u;\ r) = -\measuredangle(v;\ PT)$ and $\measuredangle(v;\ r) = -\measuredangle(u;\ PT)$.

a) (See Fig. 4.) Since $\measuredangle PXT = 90^\circ$ and $\measuredangle PYT = 90^\circ$, the points X and Y lie on the circle with diameter PT. Thus, $\measuredangle YXP = \measuredangle YTP$, so that $\measuredangle(XY;\ u) = \measuredangle(TY;\ PT)$. But $TY \perp v$ yields $\measuredangle(TY;\ v) = 90^\circ$, so that $\measuredangle(XY;\ u) = \measuredangle(TY;\ PT) = \measuredangle(TY;\ v) + \measuredangle(v;\ PT) = 90^\circ + \measuredangle(v;\ PT)$.

Hence, $\measuredangle(XY;\ r) = \measuredangle(XY;\ u) + \measuredangle(u;\ r) = (90^\circ + \measuredangle(v;\ PT)) + (-\measuredangle(v;\ PT)) = 90^\circ$. Thus, the line XY is perpendicular to the line r, that is, to the isogonal of the line PT wrt the lines u and v. This proves Theorem 2 **a)**.

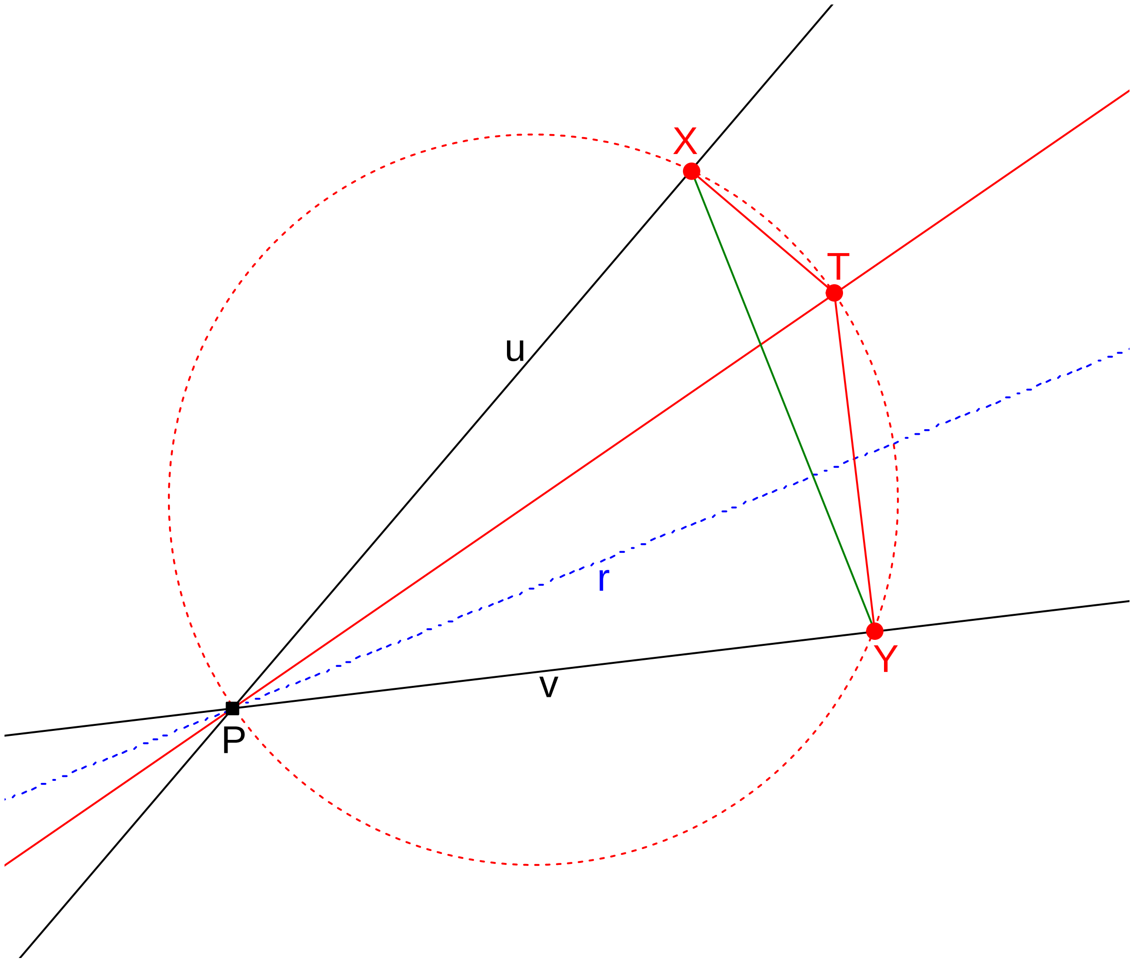

Fig. 4

b) (See Fig. 5.) Since X' is the reflection of the point T in the line u, we have $PX' = PT$; similarly, $PY' = PT$. Thus, $PX' = PT = PY'$, what entails that the point P is the center of the circle through the points X', T, Y'. Thus, on the one hand, the central angle theorem for directed angles modulo 180° yields $\measuredangle PY'X' = 90° - \measuredangle X'TY'$. On the other hand, $PX' = PY'$ implies that P lies on the perpendicular bisector of the segment $X'Y'$.

Since Y' is the reflection of the point T in the line v, we get $\measuredangle(PY';\ v) = -\measuredangle(PT;\ v)$ and $TY' \perp v$; the latter yields $\measuredangle(v;\ TY') = 90°$. Similarly, $\measuredangle(u;\ TX') = 90°$. Therefore,

$$\begin{aligned}\measuredangle(PY';\ X'Y') &= \measuredangle PY'X' = 90° - \measuredangle X'TY' = 90° - \measuredangle(TX';\ TY') = \measuredangle(v;\ TY') - \measuredangle(TX';\ TY') \\ &= \measuredangle(v;\ TX') = \measuredangle(v;\ u) + \measuredangle(u;\ TX') = \measuredangle(v;\ u) + 90°.\end{aligned}$$

Consequently,

$$\begin{aligned}\measuredangle(v;\ X'Y') &= \measuredangle(PY';\ X'Y') - \measuredangle(PY';\ v) = (\measuredangle(v;\ u) + 90°) - (-\measuredangle(PT;\ v)) \\ &= (\measuredangle(PT;\ v) + \measuredangle(v;\ u)) + 90° = \measuredangle(PT;\ u) + 90°,\end{aligned}$$

so that

$$\begin{aligned}\measuredangle(r;\ X'Y') &= \measuredangle(v;\ X'Y') - \measuredangle(v;\ r) = (\measuredangle(PT;\ u) + 90°) - (-\measuredangle(u;\ PT)) \\ &= (\measuredangle(PT;\ u) + 90°) - \measuredangle(PT;\ u) = 90°.\end{aligned}$$

This means that the line r is perpendicular to the line $X'Y'$. Now, the perpendicular bisector of the segment $X'Y'$ is also perpendicular to the line $X'Y'$. Hence, the perpendicular bisector of the segment $X'Y'$ is parallel to the line r. But since the perpendicular bisector of the segment $X'Y'$ and the line r have a common point (namely P), they can only be parallel if they coincide. Hence we see that the perpendicular bisector of the segment $X'Y'$ coincides with the line r, that is, with the isogonal of the line PT wrt the lines u and v. This proves Theorem 2 **b)** and thus completes the proof of Theorem 2.

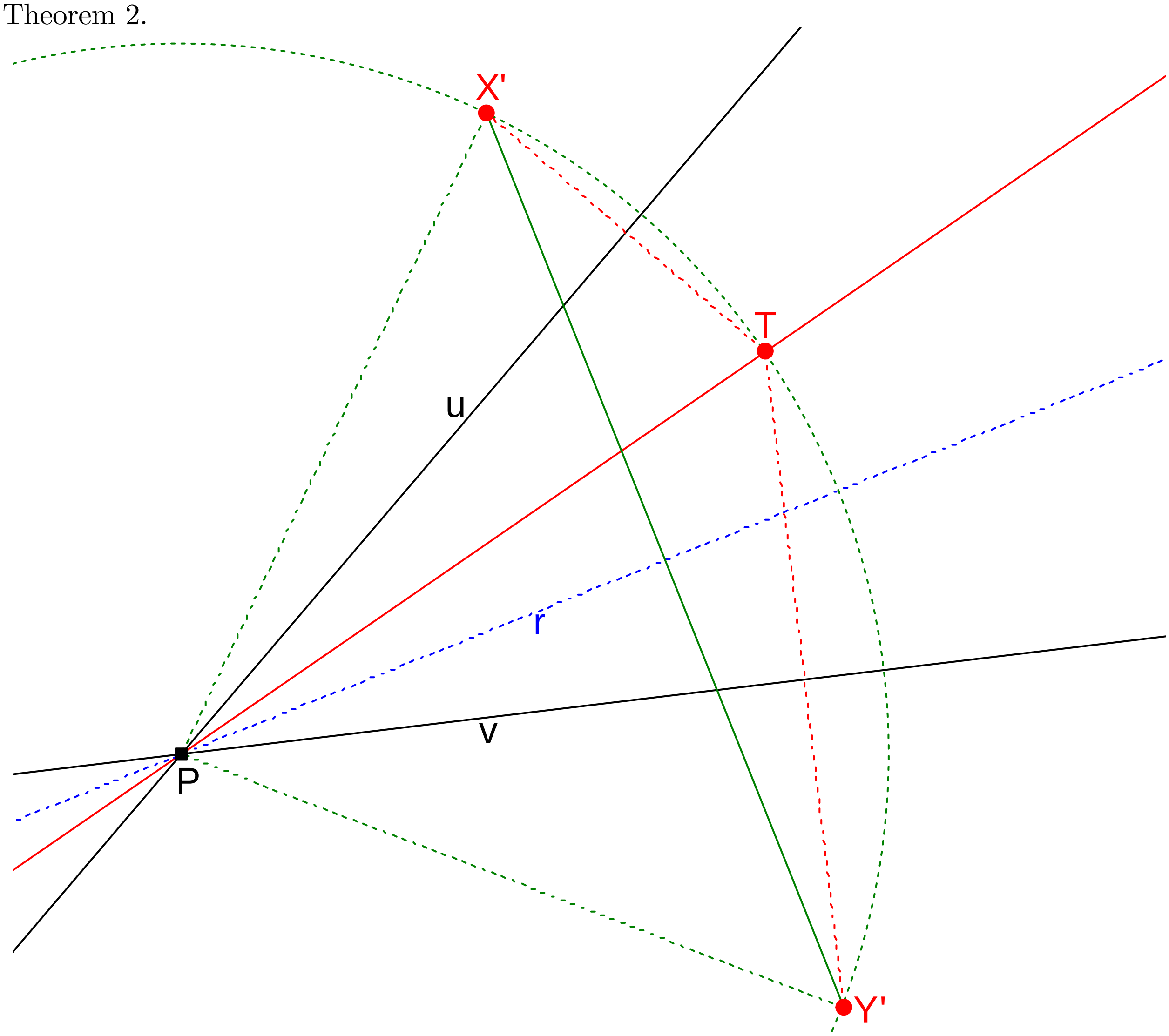

Fig. 5

Remark. (See Fig. 6.) As X is the orthogonal projection of the point T on the line u, whereas X' is the reflection of T in u, we see that X is the midpoint of the segment TX'. Similarly, Y is the midpoint of the segment TY'. Thus, the line XY is a midparallel in triangle $X'TY'$ and thus parallel to its side $X'Y'$. Hence, the assertion that $X'Y' \perp r$ (the crucial assertion in the proof of Theorem 2 **b)**) is equivalent to the assertion that $XY \perp r$ (this is the assertion of Theorem 2 **a)**). This shows that Theorems 2 **a)** and 2 **b)** can be derived from each other.

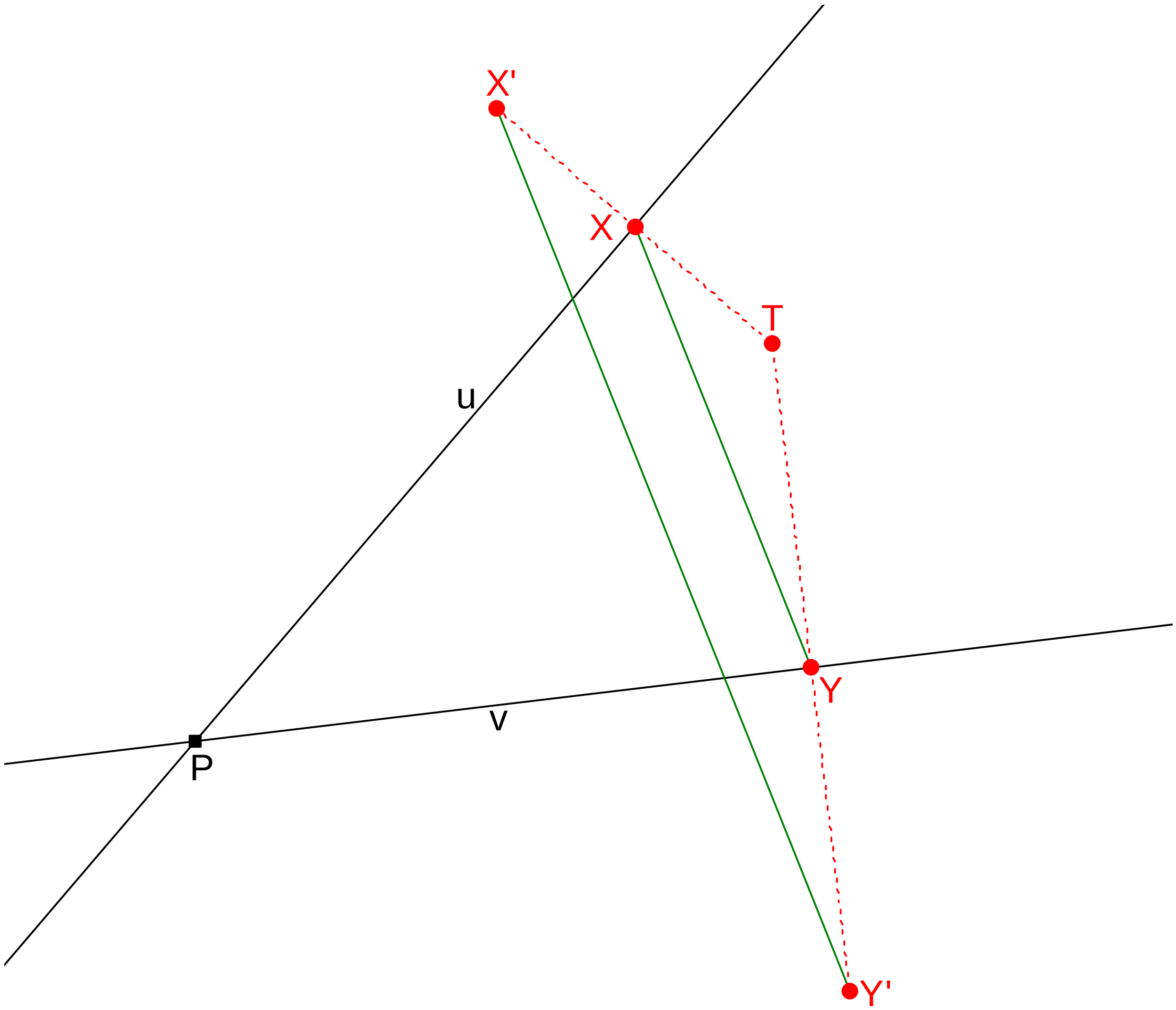

Fig. 6

Note that Theorem 2 will not only turn out useful to us in our study of isogonal conjugates, but it can also be applied to olympiad problems like the IMO 2004 problem 5 ([8], post #2).

3. Isogonal conjugation wrt triangles

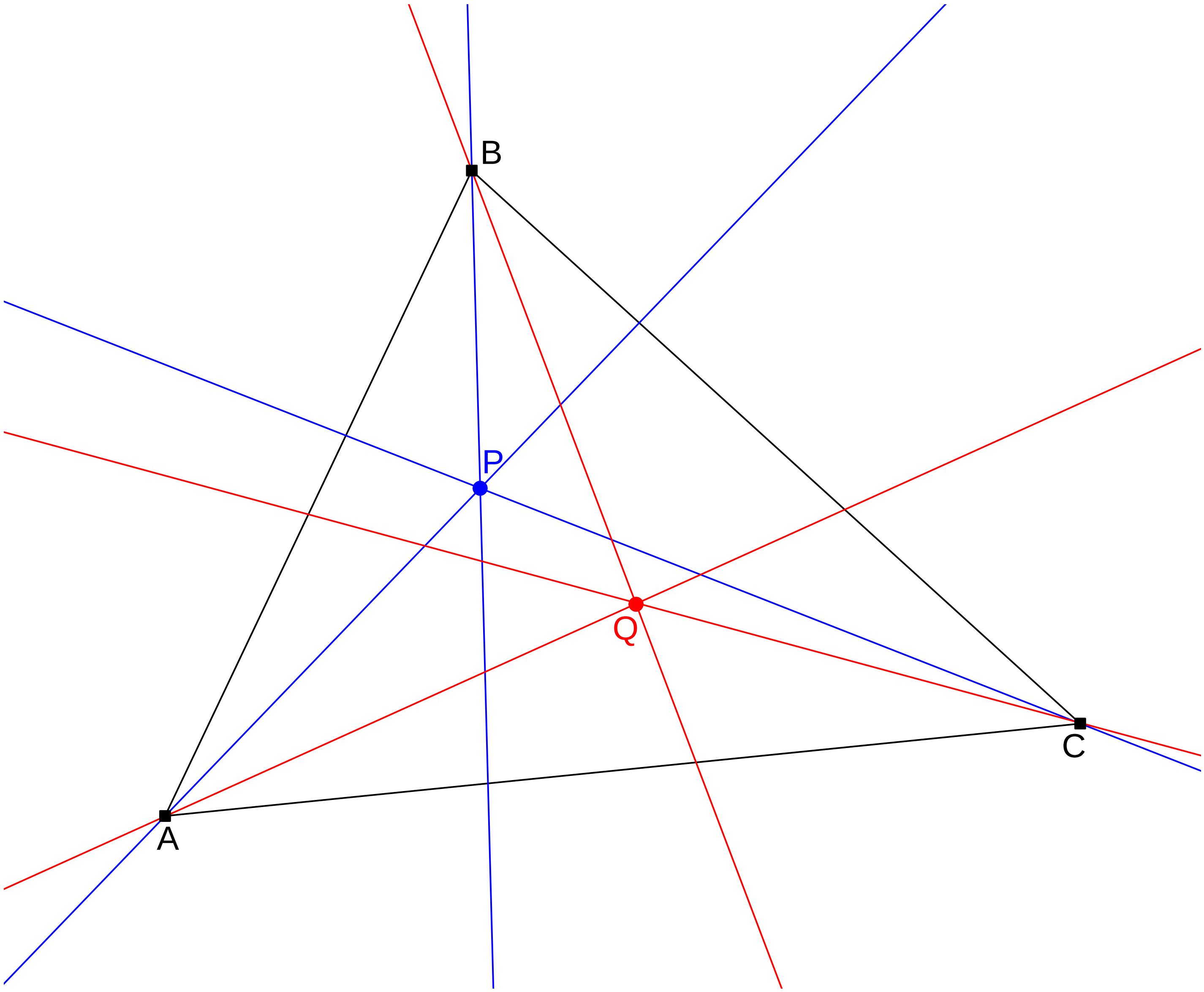

Fig. 7

Now we show the first serious result on isogonals, the **isogonal conjugate theorem**:

Theorem 3. Let ABC be a triangle and P a point (distinct from the vertices A, B, C). Then, the isogonals of the lines AP, BP, CP wrt the angles CAB, ABC, BCA concur at one point Q. (See Fig. 7.)

It has to be noticed that this theorem is formulated for the projective plane - this means that the point P can be an Euclidean or an infinite point, and that the isogonals of the lines AP, BP, CP wrt the angles CAB, ABC, BCA can concur at an Euclidean or at an infinite point as well.

Proof of Theorem 3.[1] We distinguish between two cases:

Case 1: The point P is an Euclidean (i. e. not infinite) point.

(See Fig. 8.) Let X', Y', Z' be the reflections of the point P in the lines BC, CA, AB.

[1] The following proof is basically the proof given in [9], apart from the difference that [9] doesn't use directed angles.

Since Y' and Z' are the reflections of the point P in the lines CA and AB, according to Theorem 2 **b)** it follows that the perpendicular bisector of the segment $Y'Z'$ is the isogonal of the line AP wrt the lines CA and AB, that is, the isogonal of the line AP wrt the angle CAB. Similarly, the perpendicular bisectors of the segments $Z'X'$ and $X'Y'$ are the isogonals of the lines BP and CP wrt the angles ABC and BCA. Thus, the isogonals of the lines AP, BP, CP wrt the angles CAB, ABC, BCA are the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$ and thus concur at one point - at the circumcenter of triangle $X'Y'Z'$. This proves Theorem 3 in Case 1.

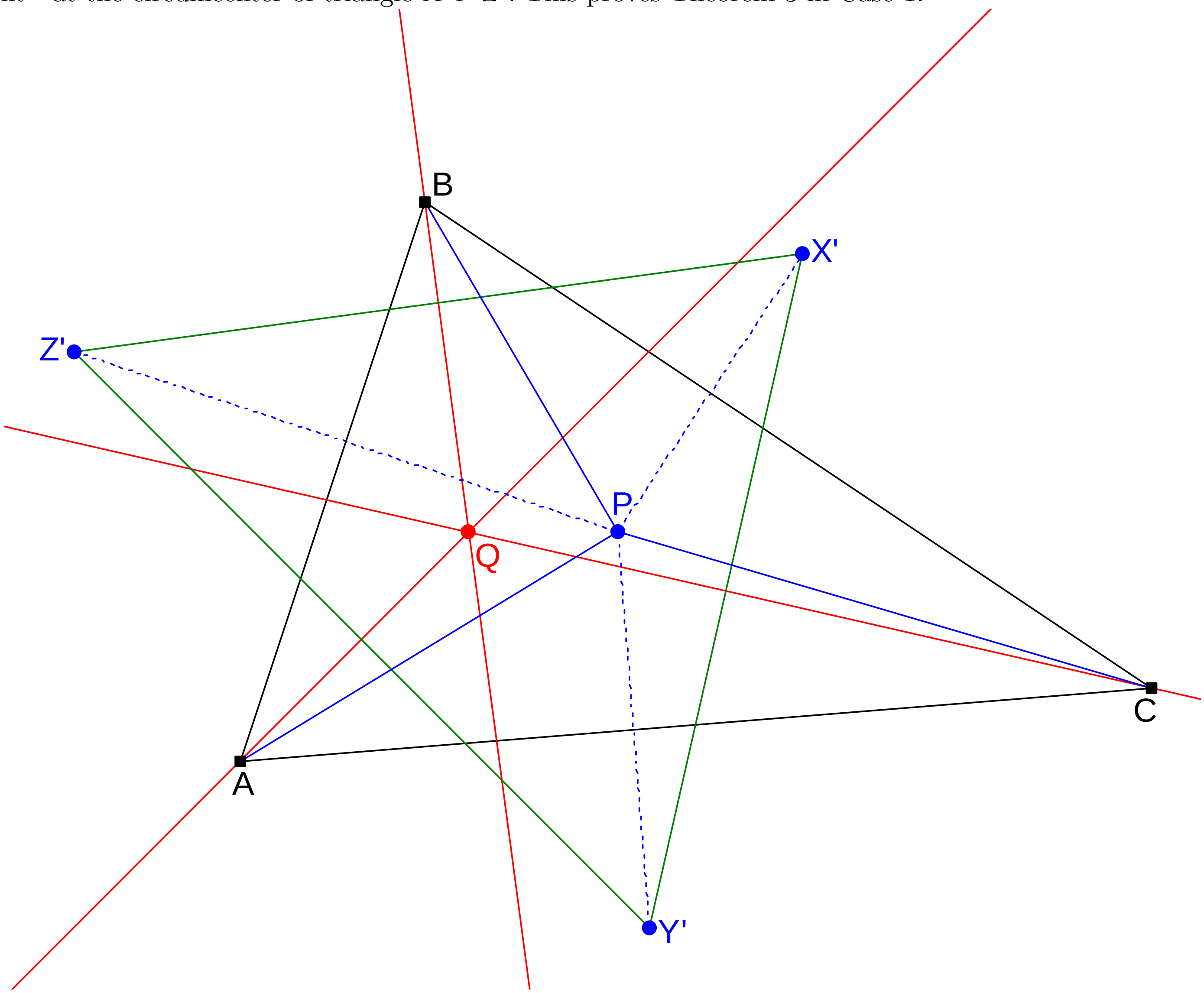

Fig. 8

Case 2: The point P is an infinite point. Then, the lines AP, BP, CP are parallel. (See Fig. 9.) Let the isogonal of the line AP wrt the angle CAB meet the circumcircle of triangle ABC at a point Q_1 (apart from A). Since the lines AP and AQ_1 are isogonal to each other wrt the angle CAB, we have $\measuredangle (CA;\ AQ_1) = -\measuredangle (AB;\ AP)$. Since Q_1 lies on the circumcircle of triangle ABC, we have $\measuredangle AQ_1C = \measuredangle ABC$, thus

$\measuredangle(AQ_1;\ CQ_1) = \measuredangle(AB;\ BC)$. Hence,

$$\begin{aligned}\measuredangle(CA;\ CQ_1) &= \measuredangle(CA;\ AQ_1) + \measuredangle(AQ_1;\ CQ_1) = -\measuredangle(AB;\ AP) + \measuredangle(AB;\ BC) \\ &= -(\measuredangle(AB;\ AP) - \measuredangle(AB;\ BC)) = -\measuredangle(BC;\ AP).\end{aligned}$$

But $AP \parallel CP$ yields $\measuredangle(BC;\ AP) = \measuredangle(BC;\ CP)$, and thus we obtain $\measuredangle(CA;\ CQ_1) = -\measuredangle(BC;\ CP)$. This means that the line CQ_1 is the isogonal of the line CP wrt the angle BCA. In other words, the point Q_1 lies on the isogonal of the line CP wrt the angle BCA. Similarly, the point Q_1 lies on the isogonal of the line BP wrt the angle ABC. We already know that the point Q_1 lies on the isogonal of the line AP wrt the angle CAB. Thus, the isogonals of the lines AP, BP, CP wrt the angles CAB, ABC, BCA concur at one point - namely, at the point Q_1. Herewith we have not only proved Theorem 3 in Case 2, but we have also shown that in this case - i. e. in the case when the point P is infinite -, the point of intersection of the isogonals of the lines AP, BP, CP wrt the angles CAB, ABC, BCA lies on the circumcircle of triangle ABC (in fact, this point of intersection is our point Q_1 and lies, by its definition, on the circumcircle of triangle ABC).

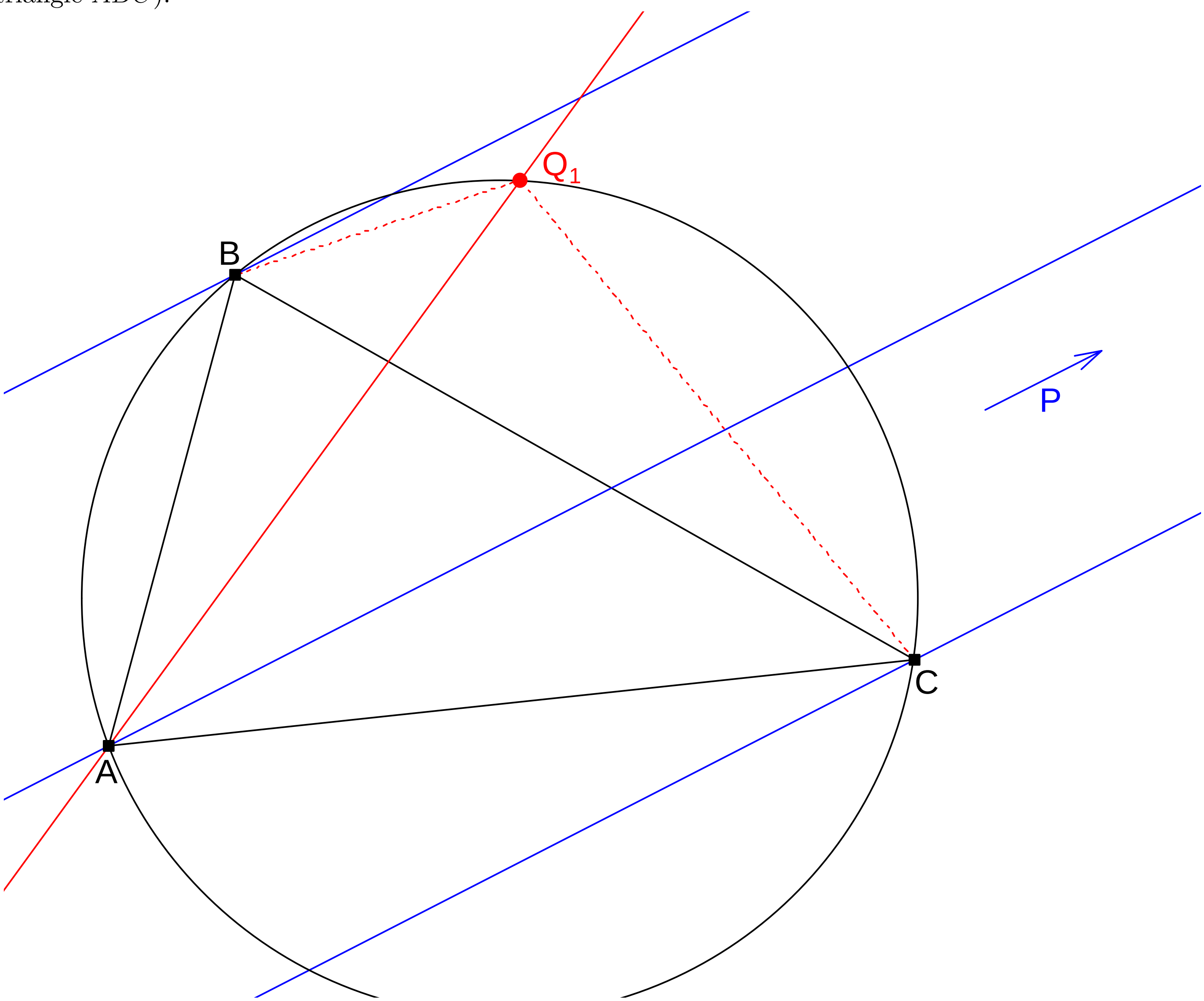

Fig. 9

Now Theorem 3 is completely proven. Based on Theorem 3, we introduce a notion:

The point of intersection of the isogonals of the lines AP, BP, CP wrt the angles CAB, ABC, BCA is called the **isogonal conjugate** of the point P wrt the triangle ABC.

If Q is the isogonal conjugate of a point P wrt the triangle ABC, then the line AQ is the isogonal of the line AP wrt the angle CAB. This means that the lines AP and AQ are isogonal to each other wrt the angle CAB. This, in turn, shows that the line AP is the isogonal of the line AQ wrt the angle CAB. Similarly, the lines BP and CP are the isogonals of the lines BQ and CQ wrt the angles ABC and BCA. Now, the point P is the point of intersection of the lines AP, BP, CP, so it therefore must be the point of intersection of the isogonals of the lines AQ, BQ, CQ wrt the angles CAB, ABC, BCA. Hence, the point P is the isogonal conjugate of the point Q wrt the triangle ABC (as long as the point P doesn't lie on any of the lines BC, CA, AB, since otherwise, the point Q is one of the vertices A, B, C of triangle ABC, and thus the isogonal conjugate of Q is not defined).

Thus we have shown that if Q is the isogonal conjugate of a point P wrt a triangle ABC, then, in turn, the point P is the isogonal conjugate of the point Q wrt the triangle ABC (as long as the point P doesn't lie on any of the lines BC, CA, AB). Thus, instead of saying that "the point Q is the isogonal conjugate of the point P wrt the triangle ABC " or that "the point P is the isogonal conjugate of the point Q wrt the triangle ABC ", we can say that "the points P and Q are **isogonally conjugate points** wrt the triangle ABC ".

Some first properties of isogonal conjugates can be obtained by harvesting our proof of Theorem 3. We start with the following fact:

Theorem 4. *Let P and Q be two isogonally conjugate points wrt a triangle ABC. Then, the point P is an infinite point if and only if the point Q lies on the circumcircle of triangle ABC. (See Fig. 10.)*

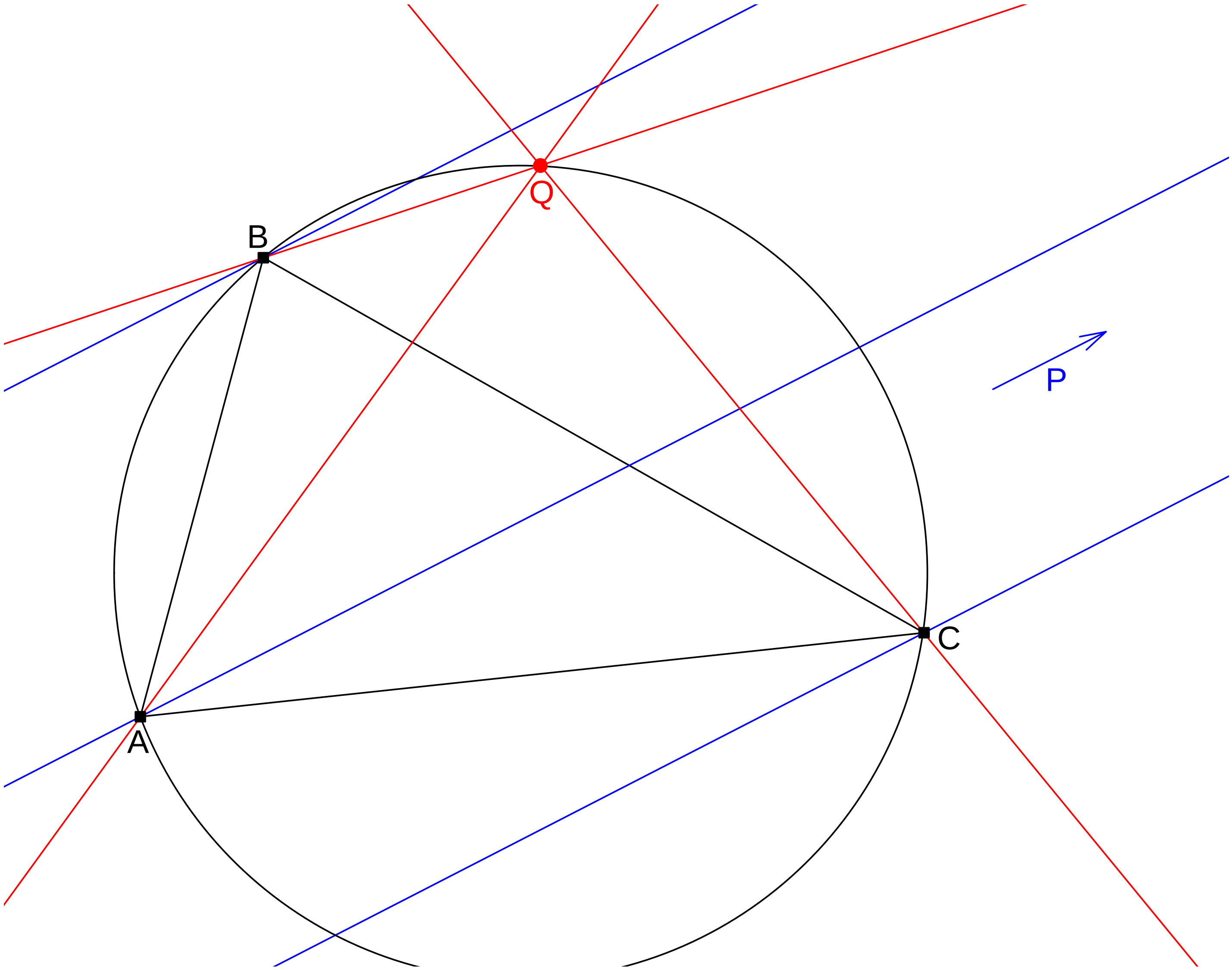

Fig. 10

Proof of Theorem 4. In order to show Theorem 4, we have to prove two assertions:

Assertion 1: If the point P is an infinite point, then the point Q lies on the circumcircle of triangle ABC.

Assertion 2: If the point Q lies on the circumcircle of triangle ABC, then the point P is an infinite point.

Proof of Assertion 1. The point Q is the isogonal conjugate of P wrt triangle ABC, that is, the point of intersection of the isogonals of the lines AP, BP, CP wrt the angles CAB, ABC, BCA. But since the point P is an infinite point, this point of intersection must lie on the circumcircle of triangle ABC, as we saw in the proof of Theorem 3 in Case 2. Thus, Q lies on the circumcircle of triangle ABC, and Assertion 1 is proven.

First proof of Assertion 2. Let P_1 be the point of intersection of the line AP with the line at infinity. Let Q_1 be the isogonal conjugate of this point P_1 wrt triangle ABC. Then, the line AQ_1 is the isogonal of the line AP_1 wrt the angle CAB. But since the point Q is the isogonal conjugate of the point P wrt triangle ABC, the line AQ is the isogonal of the line AP wrt the angle CAB. Since the lines AP_1 and AP coincide, their isogonals wrt the angle CAB must also coincide; i. e., the lines AQ_1 and AQ coincide. This means that the point Q_1 lies on the line AQ. Since P_1 is an infinite point, the already established Assertion 1 shows that its isogonal conjugate Q_1 lies on the circumcircle of triangle ABC. Thus, the point Q_1 is the point of intersection of the line AQ with the circumcircle of triangle ABC (different from A). But since the point Q lies on the circumcircle of triangle ABC, the point Q itself is the point of intersection of the line AQ with the circumcircle of triangle ABC (different from A). Hence, the points Q_1 and Q coincide. Thus, the isogonal conjugates of these points Q_1 and Q wrt triangle ABC must also coincide; but the isogonal conjugate of Q_1 is the point P_1 (since we have defined Q_1 as the isogonal conjugate of P_1 wrt triangle ABC), and the isogonal conjugate of Q is the point P. Hence, the points P_1 and P coincide. Since P_1 is an infinite point, it follows that the point P is an infinite point, and Assertion 2 is proven.

Second proof of Assertion 2. We will show the following assertion, which is nothing but Assertion 2 with P and Q interchanged:

Assertion 2': If the point P lies on the circumcircle of triangle ABC, then the point Q is an infinite point.

This assertion 2' can be derived from our proof of Theorem 3 as follows: Since the point P lies on the circumcircle of triangle ABC, it is an Euclidean point; thus, according to the proof of Theorem 3 in Case 1, the isogonals of the lines AP, BP, CP wrt the angles CAB, ABC, BCA are the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$. Since the point Q is the isogonal conjugate of the point P wrt triangle ABC, it is the point of intersection of these isogonals, thus the point of intersection of the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$. Now, as the point P lies on the circumcircle of triangle ABC, according to a well-known fact (Steiner line theorem), the reflections X', Y', Z' of the point P in the lines BC, CA, AB lie on one line; the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$ are all perpendicular to this line $X'Y'Z'$ and thus all parallel to each other. The point Q is the point of intersection of these parallel perpendicular bisectors, therefore an infinite point (actually, it is the infinite point of intersection of all lines perpendicular to the line $X'Y'Z'$ and can be regarded as the "circumcenter" of the degenerate triangle $X'Y'Z'$). Thus, Assertion 2' is proven, and therefore Assertion 2 as well. This completes our proof of Theorem 4.

4. Reflections and pedal circles

Another almost trivial consequence of the proof of Theorem 3 is the following fact (Fig. 11):

Theorem 5. If P is an Euclidean point in the plane of triangle ABC, and X', Y', Z' are the reflections of the point P in the lines BC, CA, AB, then the isogonal conjugate Q of the point P wrt the triangle ABC is the circumcenter of triangle $X'Y'Z'$, and the lines AQ, BQ, CQ are the perpendicular bisectors of the sides $Y'Z'$, $Z'X'$, $X'Y'$ of this triangle.

Proof of Theorem 5. Since P is an Euclidean point, we can retrieve from our proof of Theorem 3 in Case 1 the observation that the isogonals of the lines AP, BP, CP wrt the angles CAB, ABC, BCA are the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$. But as the point Q is the isogonal conjugate of P wrt triangle ABC, the lines AQ, BQ, CQ are the isogonals of the lines AP, BP, CP wrt the angles CAB, ABC, BCA. Hence, the lines AQ, BQ, CQ are the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$. The point Q, being the point of intersection of the lines AQ, BQ, CQ, is therefore the point of intersection of the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$, thus the circumcenter of triangle $X'Y'Z'$. This proves Theorem 5.

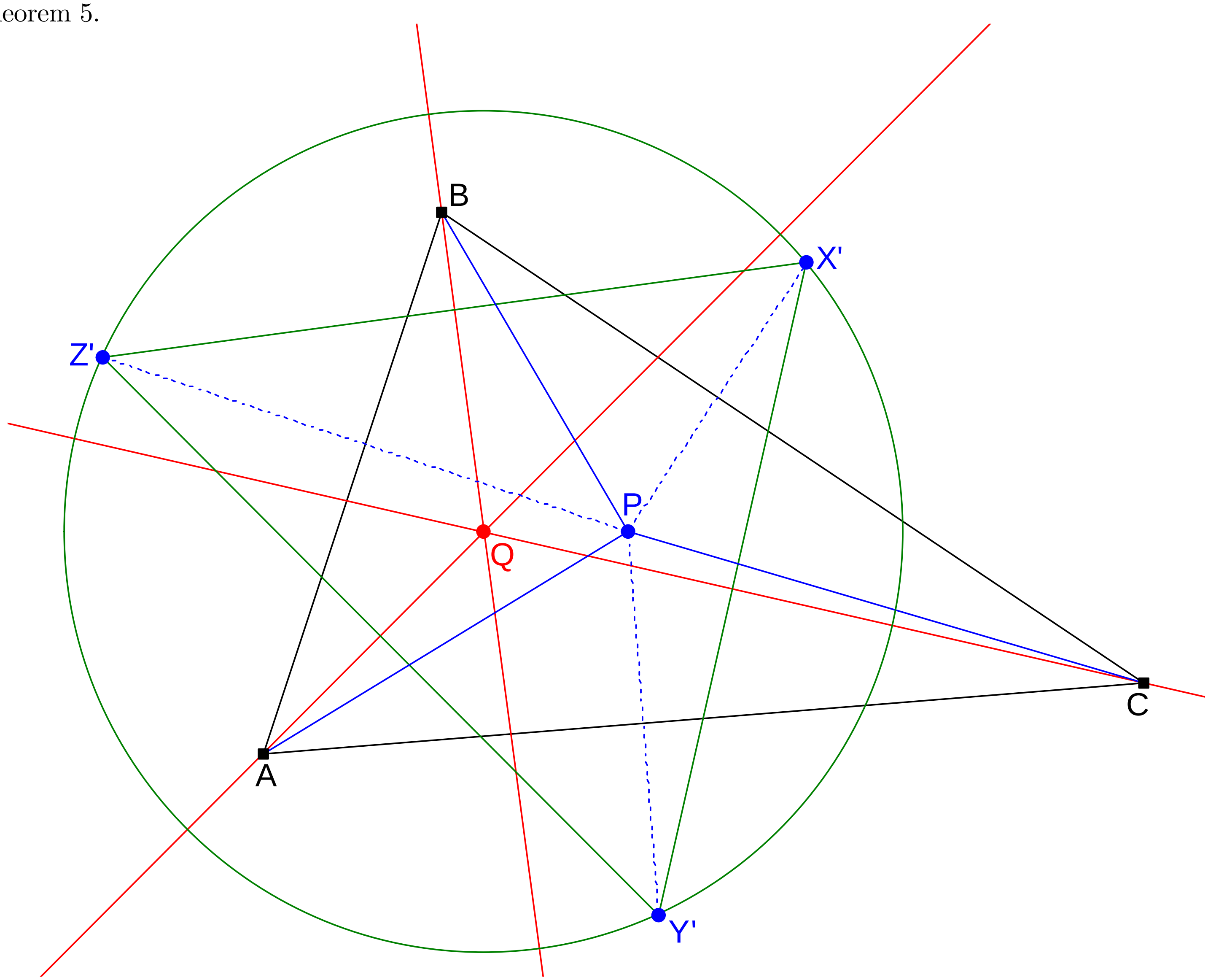

Fig. 11

Another easily accessible property of isogonals is the following one (Fig. 12):

Theorem 6. If P is an Euclidean point in the plane of a triangle ABC, if X, Y, Z are the orthogonal projections of the point P on the lines BC, CA, AB, and if Q is the isogonal conjugate of the point P wrt the triangle ABC, then $AQ \perp YZ$, $BQ \perp ZX$, $CQ \perp XY$.

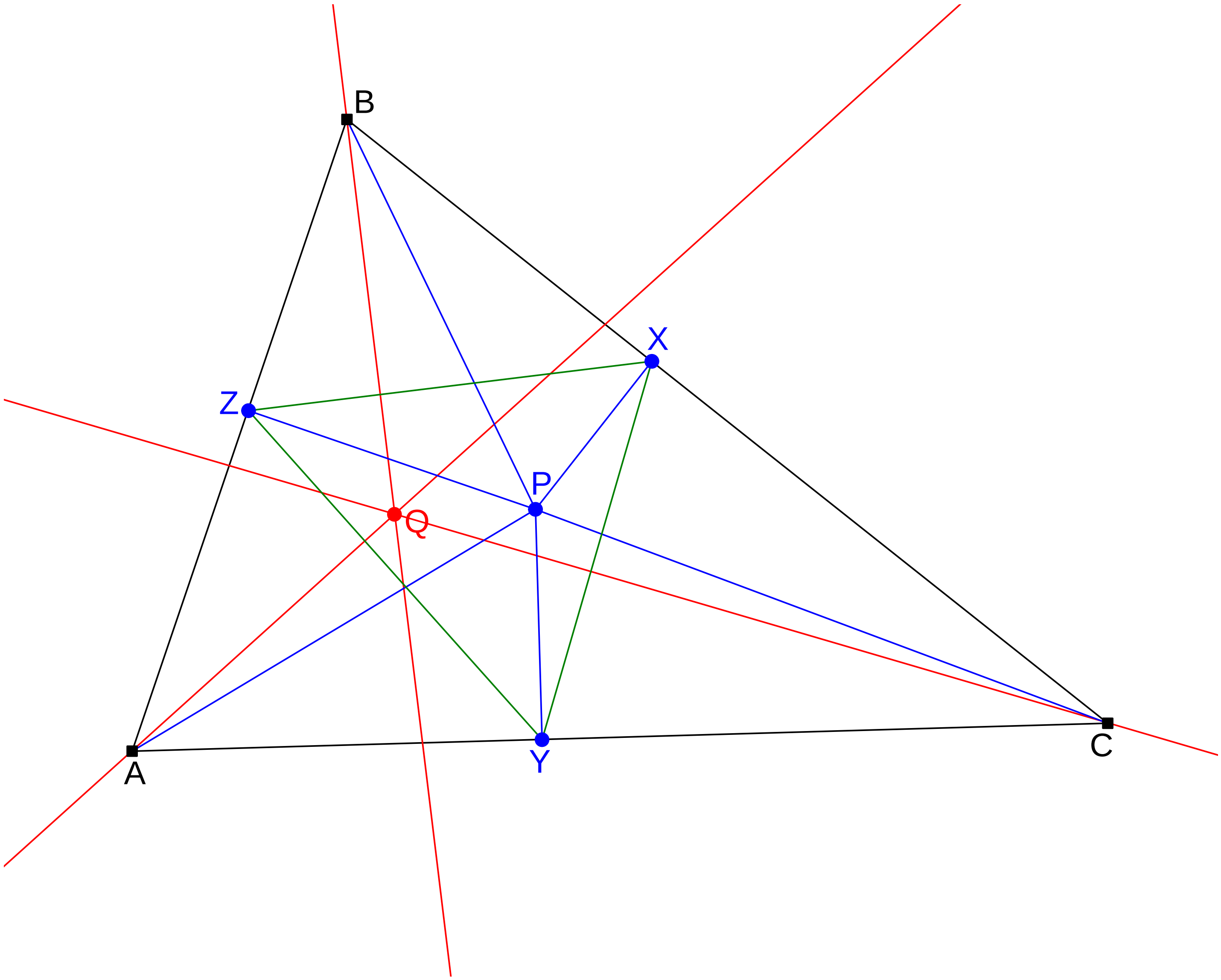

Fig. 12

Proof of Theorem 6. Since the points Y and Z are the orthogonal projections of the point P on the lines CA and AB, according to Theorem 2 **a)** we see that the line YZ is perpendicular to the isogonal of the line AP wrt the lines CA and AB, that is, to the isogonal of the line AP wrt the angle CAB. But as the point Q is the isogonal conjugate of P wrt triangle ABC, the line AQ is the isogonal of the line AP wrt the angle CAB. Hence, the line YZ is perpendicular to the line AQ. Similarly, the lines ZX and XY are perpendicular to the lines BQ and CQ, and Theorem 6 is verified.

We slowly move to deeper waters and show two well-known but less obvious properties of isogonal conjugates (Fig. 13):

Theorem 7. Let P and Q be two Euclidean isogonally conjugate points wrt a triangle ABC. Let X, Y, Z be the orthogonal projections of the point P on the lines BC, CA, AB, and let U, V, W be the orthogonal projections of the point Q on the lines BC, CA, AB.

a) We have $PX \cdot QU = PY \cdot QV = PZ \cdot QW$.

b) The points X, Y, Z, U, V, W lie on one circle centered at the midpoint M of the segment PQ.

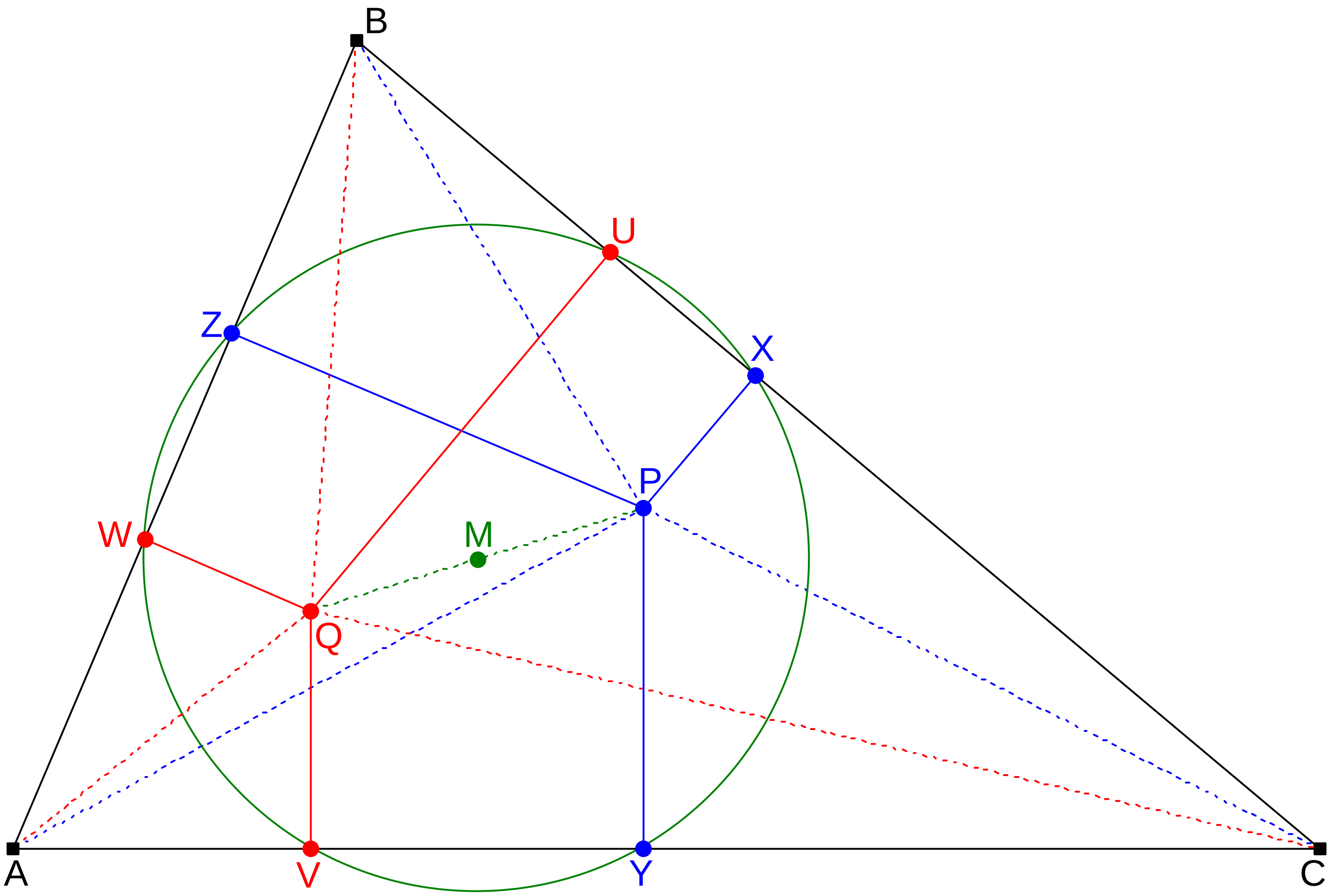

Fig. 13

Proof of Theorem 7. **a)** Since P is the isogonal conjugate of Q wrt the triangle ABC, the line AP is the isogonal of the line AQ wrt the angle CAB. Thus, $\measuredangle(CA;\ AP) = -\measuredangle(AB;\ AQ)$, that is, $\measuredangle YAP = -\measuredangle WAQ$. Furthermore, $\measuredangle AYP = -\measuredangle AWQ$ (since $\measuredangle AYP = 90^\circ$ and $\measuredangle AWQ = 90^\circ$, and since we are working with directed angles modulo 180°, we have $90^\circ = -90^\circ$). Thus, triangle AYP is oppositely similar to triangle AWQ. Similarly, triangle AZP is oppositely similar to triangle AVQ. Thus, the quadrilateral $AYPZ$, being formed by the triangles AYP and AZP, is oppositely similar to the quadrilateral $AWQV$, being formed by the triangles AWQ and AVQ. This similitude yields $PY : PZ = QW : QV$, hence $PY \cdot QV = PZ \cdot QW$. Analogous considerations lead to $PX \cdot QU = PY \cdot QV$. Thus, $PX \cdot QU = PY \cdot QV = PZ \cdot QW$, and Theorem 7 **a)** is proven.

b) (See Fig. 14.) The point Q is the isogonal conjugate of P wrt the triangle ABC. Thus, according to Theorem 5, the point Q is the circumcenter of triangle $X'Y'Z'$, where X', Y', Z' are the reflections of the point P in the lines BC, CA, AB. Hence, $QX' = QY' = QZ'$.

Since X is the orthogonal projection of the point P on the line BC, while X' is the reflection of P in BC, we see that X is the midpoint of the segment PX'. On the other hand, M is the midpoint of the segment PQ. Thus, the segment MX is a midparallel in triangle PQX', so that $MX = \frac{QX'}{2}$. Similarly, $MY = \frac{QY'}{2}$ and $MZ = \frac{QZ'}{2}$. Therefore, $QX' = QY' = QZ'$ entails $MX = MY = MZ$.

As the configuration is symmetric in the points P and Q, we similarly obtain $MU = MV = MW$.

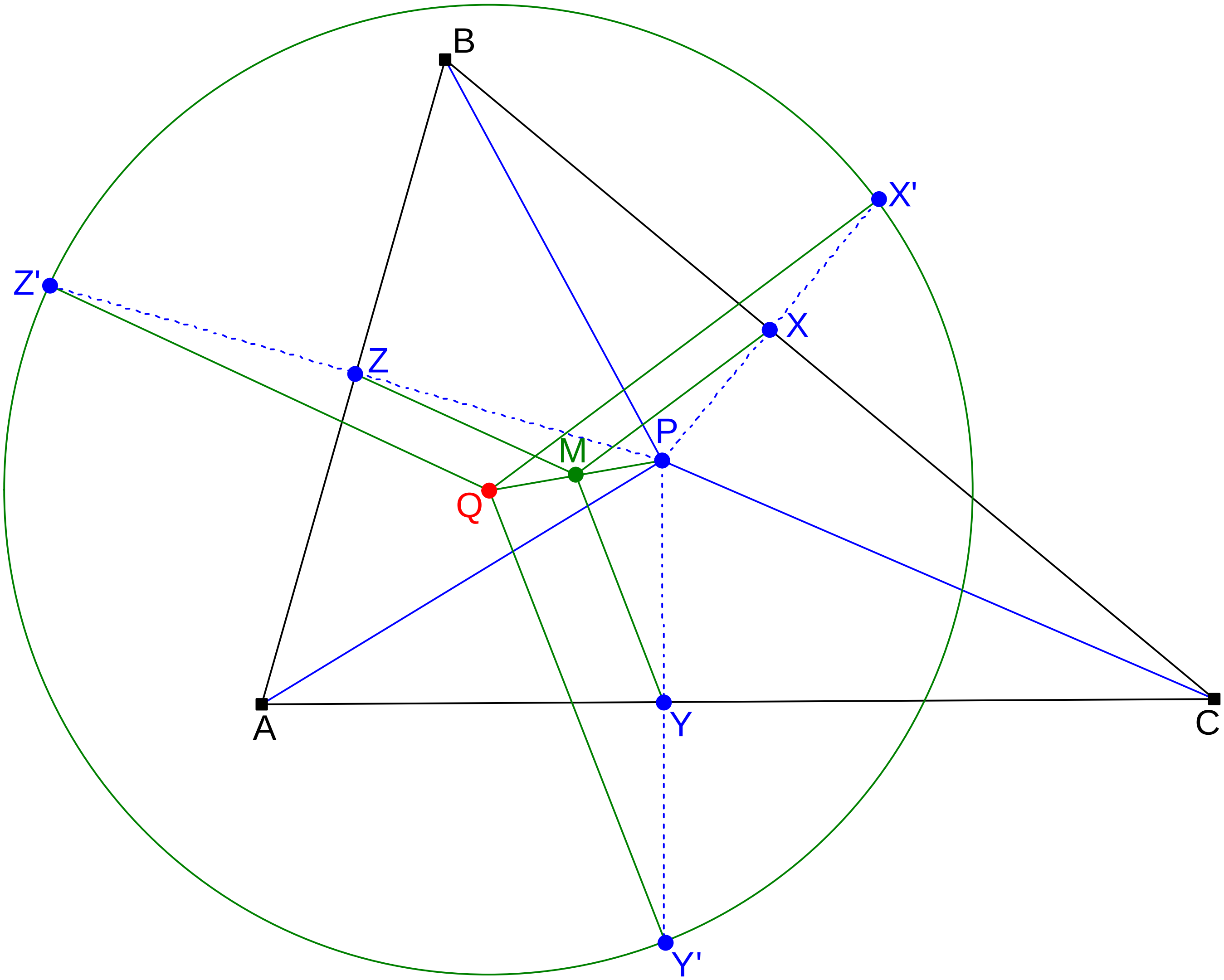

Fig. 14

(See Fig. 15.) Since $PX \perp UX$ and $QU \perp UX$, we have $PX \parallel QU$; thus, the quadrilateral $PXUQ$ is a trapezoid with the bases PX and QU. Hence, the line through the midpoints of its legs PQ and UX is the midparallel of this trapezoid, therefore is parallel to its base PX, and thus, since $PX \perp UX$, perpendicular to the line UX. Hence, this midparallel passes through the midpoint of the segment UX and is perpendicular to the line UX; thus, it is the perpendicular bisector of the segment UX. Since the midpoint of the segment PQ lies on this midparallel, we thus obtain that the midpoint of the segment PQ lies on the perpendicular bisector of the segment UX. The midpoint of the segment PQ is M; thus, the point M lies on the perpendicular bisector of the segment UX. This yields $MU = MX$. Together with $MX = MY = MZ$ and $MU = MV = MW$, this yields $MX = MY = MZ = MU = MV = MW$; in other words, the points X, Y, Z, U, V, W lie on one circle centered at M. This proves Theorem 7 **b)**.

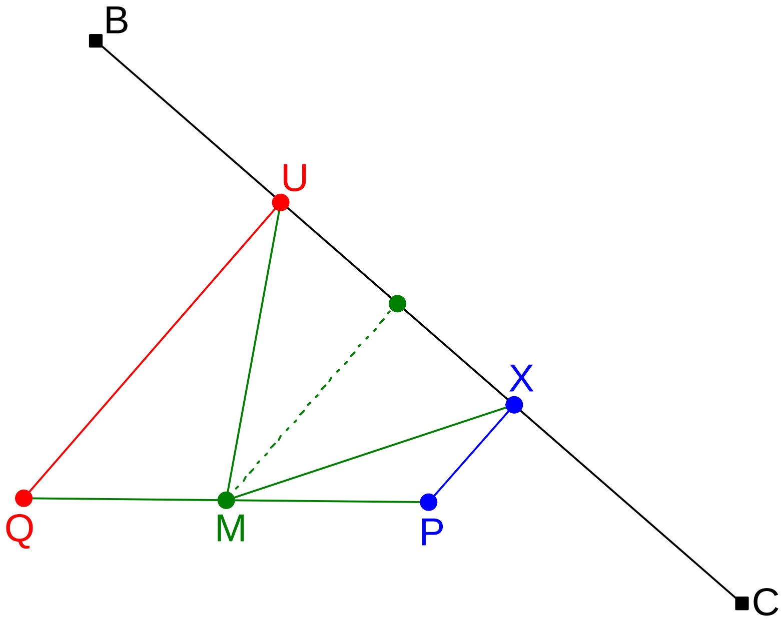

Fig. 15

Theorem 7 **b)** is known as the **pedal circle theorem**, and the proof we have given here is apparently new.

5. A result by Hatzipolakis, Yiu and Ehrmann

The following observations are mostly due to Jean-Pierre Ehrmann ([6]) and, to smaller amounts, to Peter Scholze and me. We start with an interesting result of Ehrmann (Fig. 16):

Theorem 8. Let P and Q be two Euclidean isogonally conjugate points wrt a triangle ABC. Then, the isogonal of the line AP wrt the angle BPC and the isogonal of the line AQ wrt the angle BQC are symmetric to each other wrt the line BC.

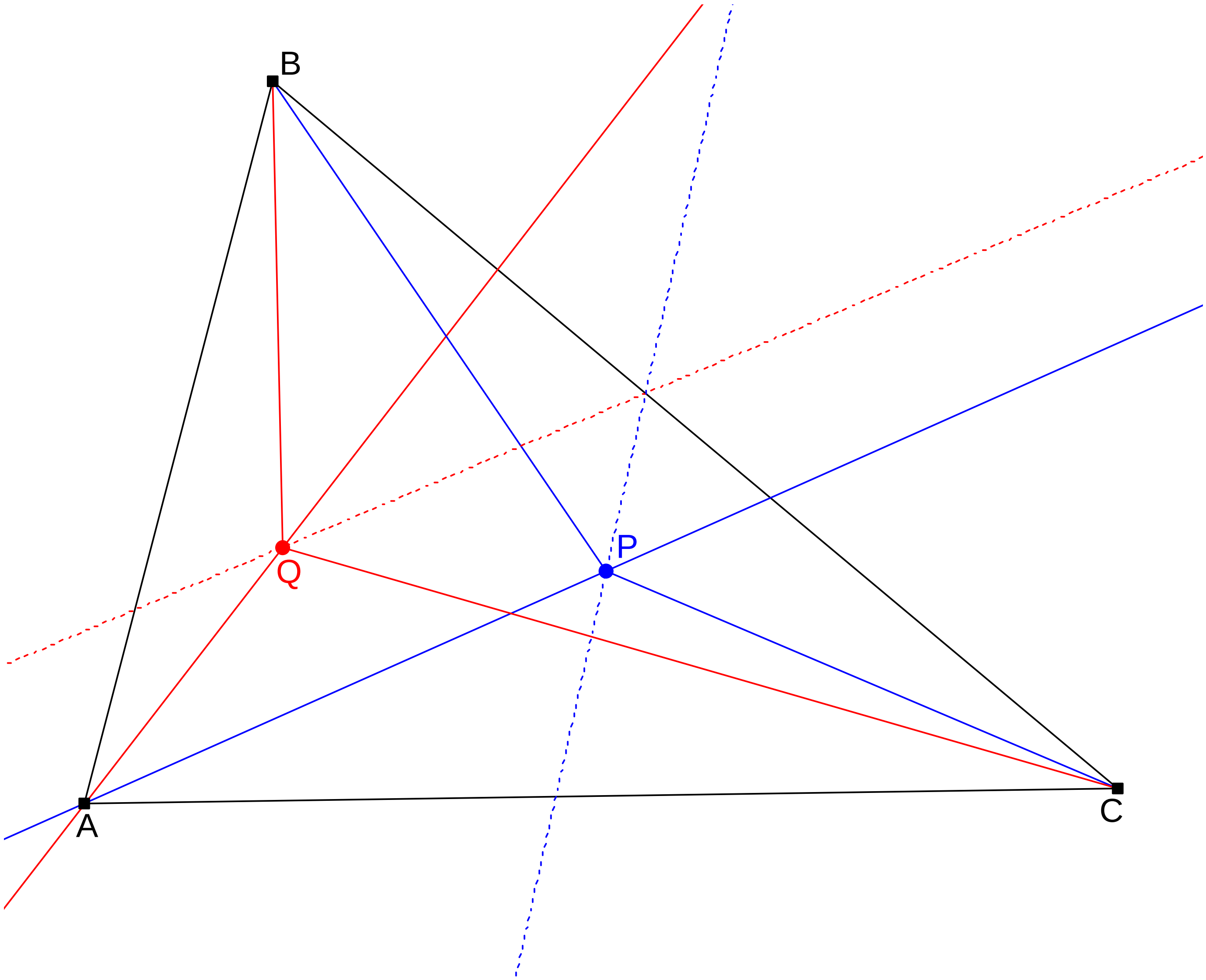

Fig. 16

Proof of Theorem 8. (See Fig. 17.) Let X', Y', Z' be the reflections of the point P in the lines BC, CA, AB, and let U' be the reflection of the point Q in the line BC. Then, the point Q is, in turn, the reflection of U' in the line BC. Since the points Q and X' are the reflections of the points U' and P in the line BC, the line QX' is the reflection of the line $U'P$ (or, in other words, of the line PU') in the line BC. This means that the lines PU' and QX' are symmetric to each other wrt the line BC.

According to Theorem 5, the point Q is the circumcenter of triangle $X'Y'Z'$, thus the center of the circle through the points X', Y', Z'. Hence, by the central angle theorem for directed angles modulo 180°, we have $\measuredangle QX'Z' = 90^\circ - \measuredangle Z'Y'X'$. In other words, $\measuredangle (QX';\ Z'X') = 90^\circ - \measuredangle (Y'Z';\ X'Y')$. Furthermore, Theorem 5 tells us that the lines AQ, BQ, CQ are the perpendicular bisectors of the segments $Y'Z'$, $Z'X'$, $X'Y'$; this yields $AQ \perp Y'Z'$, $BQ \perp Z'X'$, $CQ \perp X'Y'$, and thus $\measuredangle (Y'Z';\ AQ) = 90^\circ$,

$\measuredangle(BQ;\ Z'X') = 90°$ and $\measuredangle(X'Y';\ CQ) = 90°$. Hence,

$$\begin{aligned}
\measuredangle(BQ;\ QX') &= \measuredangle(BQ;\ Z'X') - \measuredangle(QX';\ Z'X') = 90° - (90° - \measuredangle(Y'Z';\ X'Y')) \\
&= \measuredangle(Y'Z';\ X'Y') = \measuredangle(Y'Z';\ AQ) - \measuredangle(X'Y';\ AQ) \\
&= 90° - \measuredangle(X'Y';\ AQ) = \measuredangle(X'Y';\ CQ) - \measuredangle(X'Y';\ AQ) = \measuredangle(AQ;\ CQ) \\
&= -\measuredangle(CQ;\ AQ).
\end{aligned}$$

Thus, the line QX' is the isogonal of the line AQ wrt the angle BQC. Similarly, the line PU' is the isogonal of the line AP wrt the angle BPC. Since we know that the lines PU' and QX' are symmetric to each other wrt the line BC, we have thus proven that the isogonal of the line AP wrt the angle BPC and the isogonal of the line AQ wrt the angle BQC are symmetric to each other wrt the line BC. Theorem 8 is thus established.

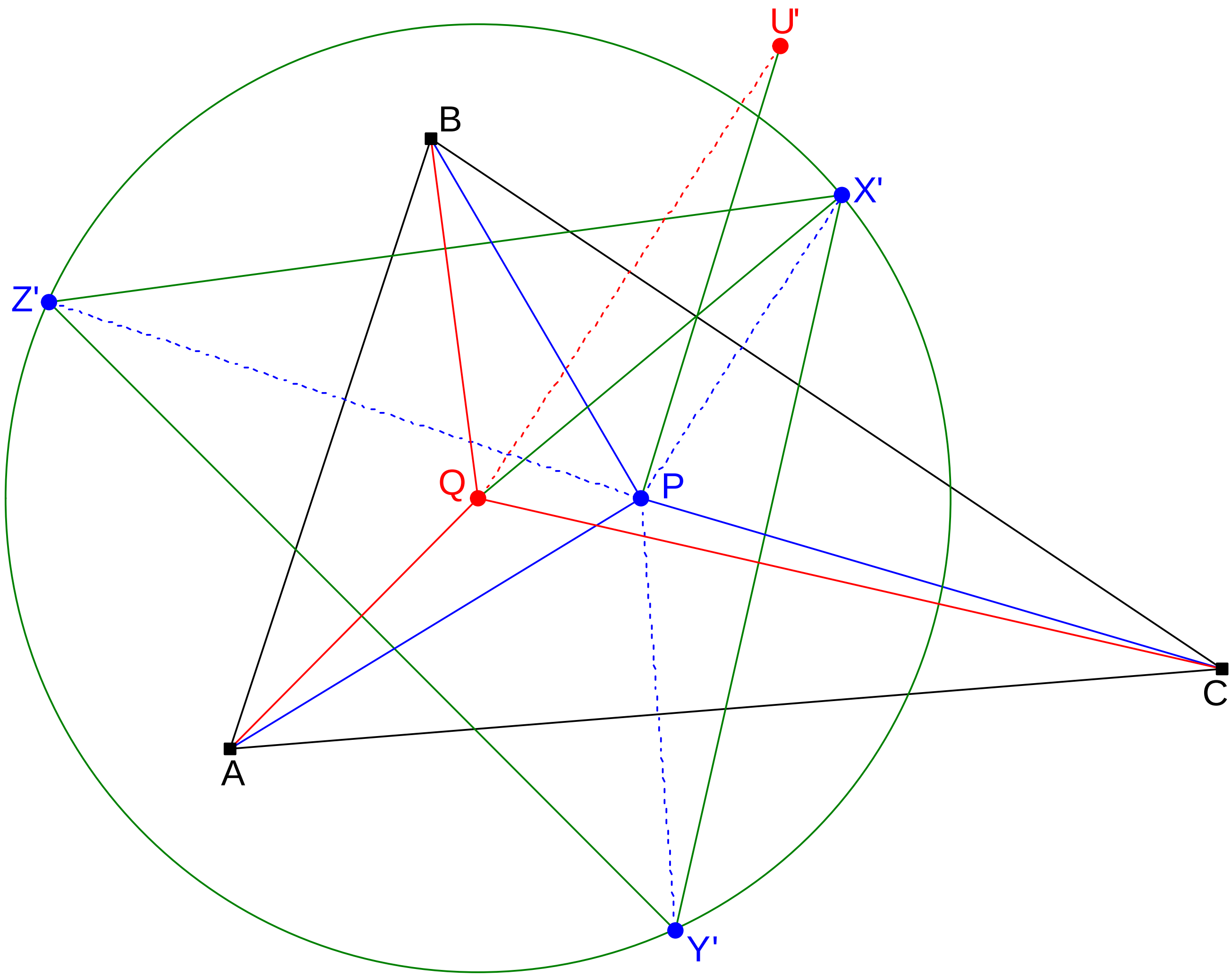

Fig. 17

The preceeding theorem is a crucial lemma in the proof of a result that was conjectured by Antreas P. Hatzipolakis, verified using computer algebra by Paul Yiu ([7]) and proven synthetically by Jean-Pierre Ehrmann ([6]):

Theorem 9. Let P be a point in the plane of a triangle ABC. The lines AP, BP, CP intersect the lines BC, CA, AB at the points A', B', C'. Let Q be the isogonal conjugate of the point P wrt the triangle ABC. Then, the reflections of the lines AQ,

BQ, CQ in the lines $B'C'$, $C'A'$, $A'B'$ concur at one point. (See Fig. 18.)

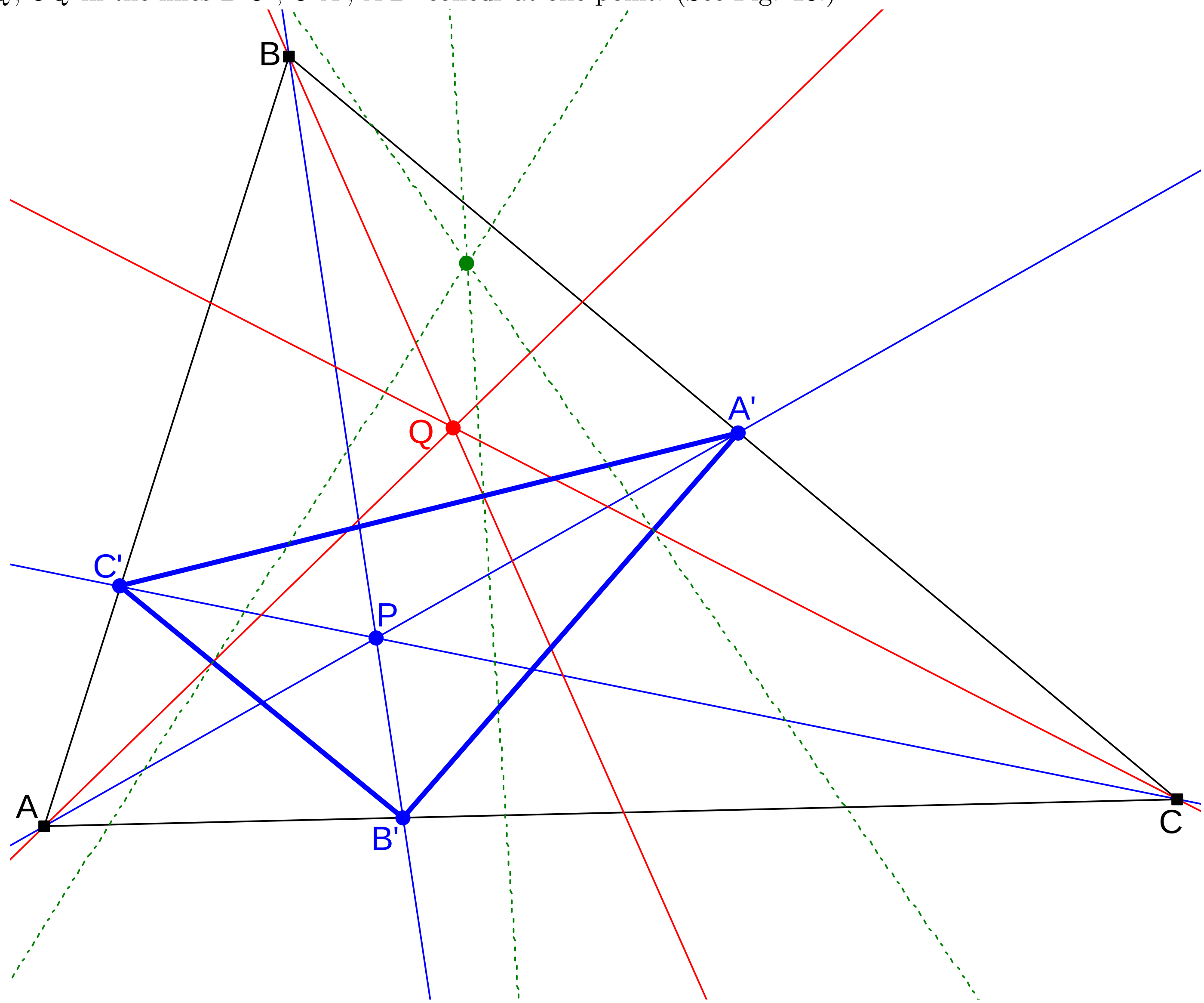

Fig. 18

Proof of Theorem 9. Again, we follow the proof given by Ehrmann in [6], rewriting it in a more elementary fashion.

(See Fig. 19.) Let A_1, B_1, C_1, P_1 be the isogonal conjugates of the points A, B, C, P wrt the triangle $A'B'C'$.

Since P_1 is the isogonal conjugate of P wrt triangle $A'B'C'$, the line $A'P_1$ is the isogonal of the line $A'P$ wrt the angle $C'A'B'$. Since A_1 is the isogonal conjugate of A wrt triangle $A'B'C'$, the line $A'A_1$ is the isogonal of the line $A'A$ wrt the angle $C'A'B'$. Since the lines $A'P$ and $A'A$ coincide, their isogonals wrt the angle $C'A'B'$ must also coincide; i. e., the lines $A'P_1$ and $A'A_1$ coincide. Hence, the points A', A_1, P_1 are collinear. Similarly, the points B', B_1 and P_1 are collinear, and the points C', C_1 and P_1 are collinear.

Since B_1 is the isogonal conjugate of B wrt triangle $A'B'C'$, the line $A'B_1$ is the isogonal of the line $A'B$ wrt the angle $C'A'B'$. Since C_1 is the isogonal conjugate of C wrt triangle $A'B'C'$, the line $A'C_1$ is the isogonal of the line $A'C$ wrt the angle $C'A'B'$. Since the lines $A'B$ and $A'C$ coincide, their isogonals wrt the angle $C'A'B'$ must also coincide; this means that the lines $A'B_1$ and $A'C_1$ coincide. In other words, the points A', B_1, C_1 are collinear. Similarly, the points B', C_1, A_1 are collinear, and the points C', A_1, B_1 are collinear.

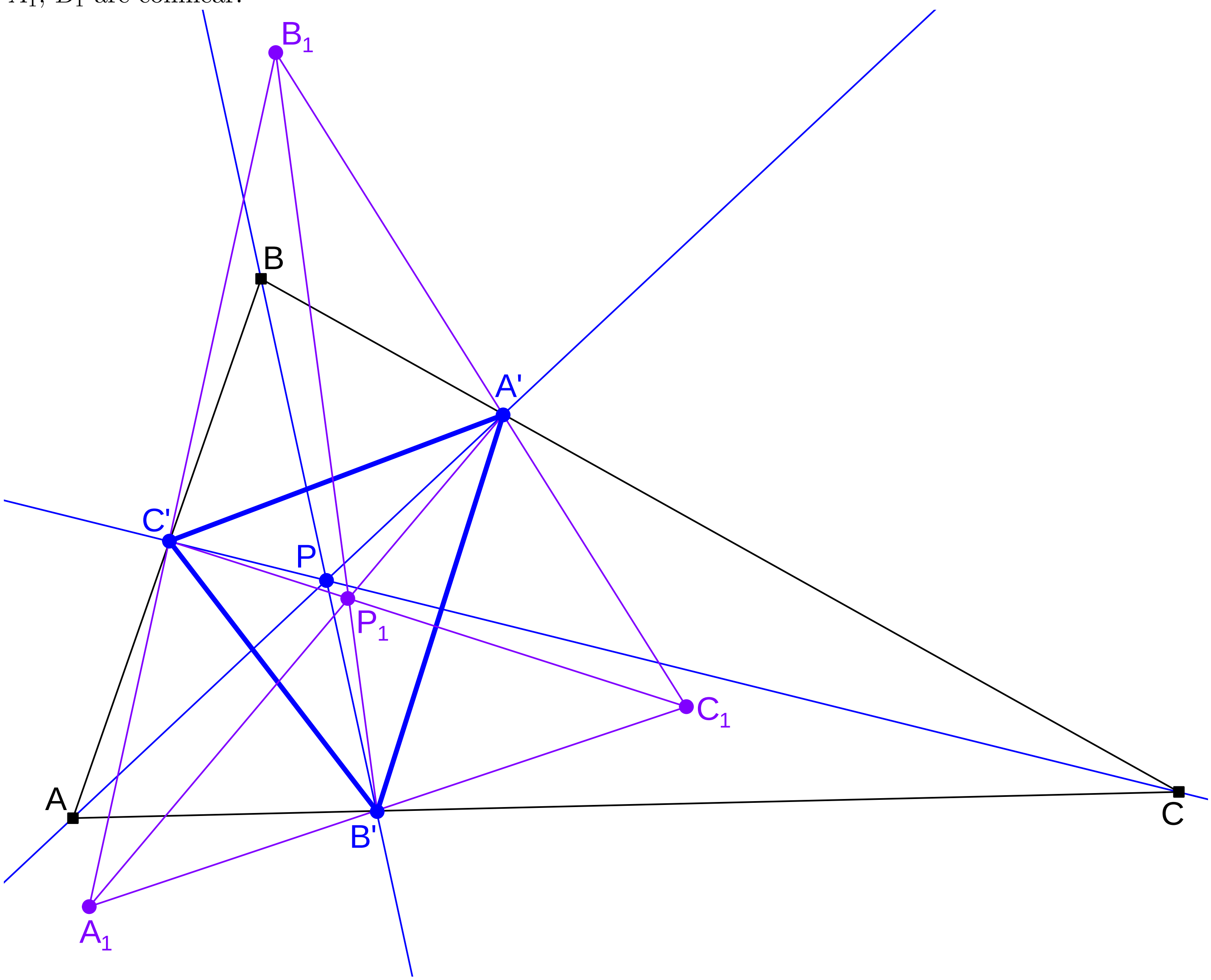

Fig. 19

Since Q is the isogonal conjugate of the point P wrt triangle ABC, the line AQ is the isogonal of the line AP wrt the angle CAB.

(See Fig. 20.) Since A and A_1 are isogonally conjugate points wrt the triangle $A'B'C'$, Theorem 8 yields that the isogonal of the line $A'A$ wrt the angle $B'AC'$ and the isogonal of the line $A'A_1$ wrt the angle $B'A_1C'$ are symmetric to each other wrt the line $B'C'$. Now, the isogonal of the line $A'A$ wrt the angle $B'AC'$ is the isogonal of the line AP wrt the angle CAB (since the line $A'A$ is the line $A'P$, and the angle $B'AC'$ is the angle CAB), and this is the line AQ. Further, the isogonal of the line $A'A_1$ wrt the angle $B'A_1C'$ is the isogonal of the line A_1P_1 wrt the angle $C_1A_1B_1$ (since the line $A'A_1$ is the line A_1P_1, and the angle $B'A_1C'$ is the angle $C_1A_1B_1$). Thus we obtained that the line AQ and the isogonal of the line A_1P_1 wrt the angle $C_1A_1B_1$ are symmetric to each other wrt the line $B'C'$. In other words, the reflection of the line AQ in the line $B'C'$ is the isogonal of the line A_1P_1 wrt the angle $C_1A_1B_1$. Similarly, the reflections of the lines BQ and CQ in the lines $C'A'$ and $A'B'$ are the isogonals of the lines B_1P_1 and C_1P_1 wrt the angles $A_1B_1C_1$ and $B_1C_1A_1$. Altogether, the reflections of the lines AQ, BQ, CQ in the lines $B'C'$, $C'A'$, $A'B'$ are the isogonals of the lines A_1P_1, B_1P_1, C_1P_1 wrt the angles $C_1A_1B_1$, $A_1B_1C_1$, $B_1C_1A_1$, and thus they concur at one point - at the isogonal conjugate of the point P_1 wrt the triangle $A_1B_1C_1$. Theorem 9 is proven.

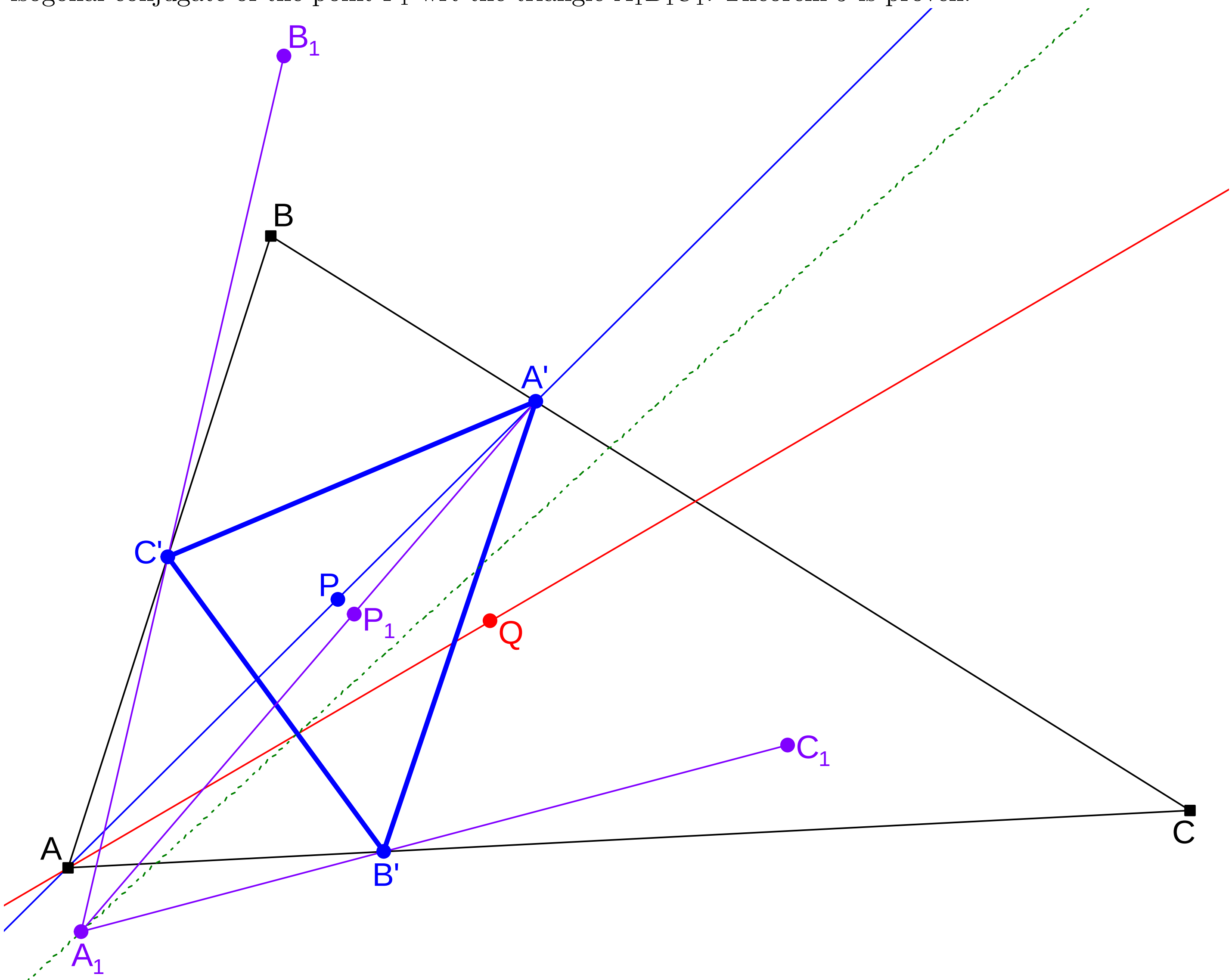

Fig. 20

6. Reflections in perpendicular bisectors

As an application of isogonal conjugates, we are now going to prove some properties of the reflections of a point in the perpendicular bisectors of a triangle noted by José Carlos Chávez Sandoval in [5]. We start with some trivial facts:

Theorem 10. Let P be an Euclidean point in the plane of a triangle ABC, and let D, E, F be the reflections of this point P in the perpendicular bisectors of the segments BC, CA, AB.

a) The points P, D, E, F lie on one circle centered at the circumcenter O of triangle ABC.

b) Triangle DEF is oppositely similar to triangle ABC. (See Fig. 21.)

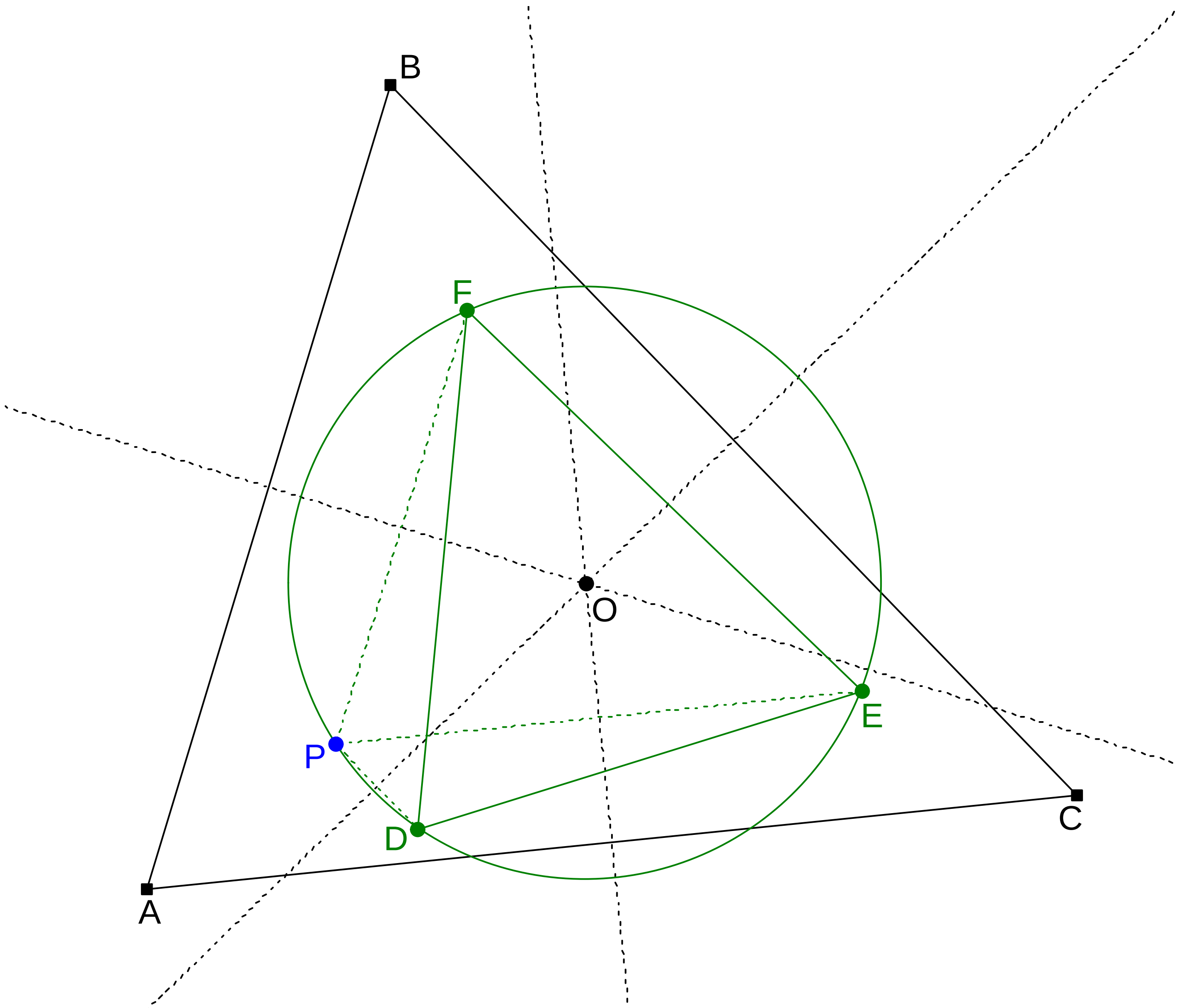

Fig. 21

Proof of Theorem 10. **a)** The circumcenter O of triangle ABC lies on the perpendicular bisector of its side BC. The point D is the reflection of the point P in this perpendicular bisector. Thus, $OD = OP$. Similarly, $OE = OP$ and $OF = OP$. Consequently, $OP = OD = OE = OF$, so that the points P, D, E, F lie on one circle centered at O, and Theorem 10 **a)** is proven.

b) As the points P, D, E, F lie on one circle, we have $\measuredangle FDE = \measuredangle FPE$. But as the point E is the reflection of P in the perpendicular bisector of CA, the line PE is perpendicular to the perpendicular bisector of CA. In turn, the perpendicular bisector of CA is perpendicular to the line CA. Hence, the line PE is parallel to the line CA. Similarly, the line PF is parallel to the line AB. Thus, $\measuredangle(PF;\ PE) = \measuredangle(AB;\ CA)$, what becomes $\measuredangle FPE = \measuredangle BAC$. Hence, $\measuredangle FDE = \measuredangle FPE = \measuredangle BAC = -\measuredangle CAB$. Similarly, $\measuredangle DEF = -\measuredangle ABC$. Consequently, the triangles DEF and ABC are oppositely similar, and Theorem 10 **b)** is proven.

Now we come to a nontrivial property of triangle DEF. Before we formulate it we define a traditional notion in triangle geometry:

If S is the centroid of triangle ABC, and T is an arbitrary point in the plane, then the image of the point T under the homothety with center S and factor $-\dfrac{1}{2}$ is called the **complement** of the point T wrt the triangle ABC.

Now we show a theorem by José Carlos Chávez Sandoval ([5]):

Theorem 11. Let P be an Euclidean point in the plane of a triangle ABC, and let D, E, F be the reflections of this point P in the perpendicular bisectors of the segments BC, CA, AB. Denote by A_M, B_M, C_M the midpoints of the segments BC, CA, AB, and by D_M, E_M, F_M the midpoints of the segments EF, FD, DE. Let Q be the isogonal conjugate of the point P wrt the triangle ABC, and let Q' be the complement of the point Q wrt the triangle ABC. Then, the lines A_MD_M, B_ME_M, C_MF_M pass through the point Q'. (See Fig. 22.)

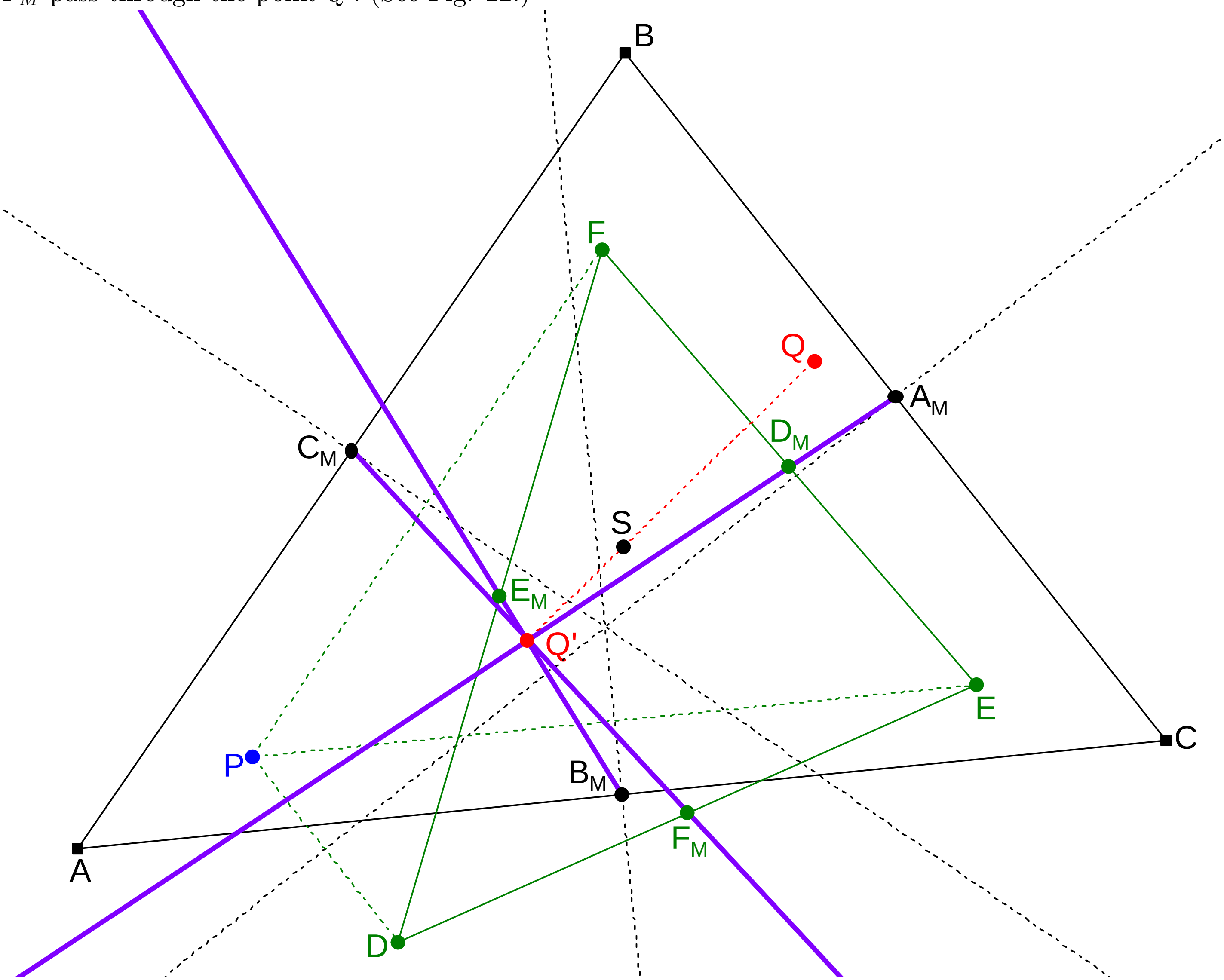

Fig. 22

Proof of Theorem 11. (See Fig. 23.) Since A_M is the midpoint of the side BC of triangle ABC, the segment AA_M is the A-median of this triangle; thus, it passes through the centroid S of triangle ABC and is divided by this centroid in the ratio $2:1$. This means, we have $\dfrac{AS}{SA_M} = 2$ (with directed segments). On the other hand, the point Q' is the complement of the point Q wrt triangle ABC, thus the image of the point Q under the homothety with center S and factor $-\frac{1}{2}$; this means that the point Q' lies on the line SQ and satisfies $SQ' = -\frac{1}{2} \cdot SQ$, so that $2 \cdot SQ' = -SQ = QS$, and $\frac{QS}{SQ'} = 2$. Comparing this with $\frac{AS}{SA_M} = 2$, we get $\frac{AS}{SA_M} = \frac{QS}{SQ'}$, so that, by Thales, $A_MQ' \parallel AQ$.

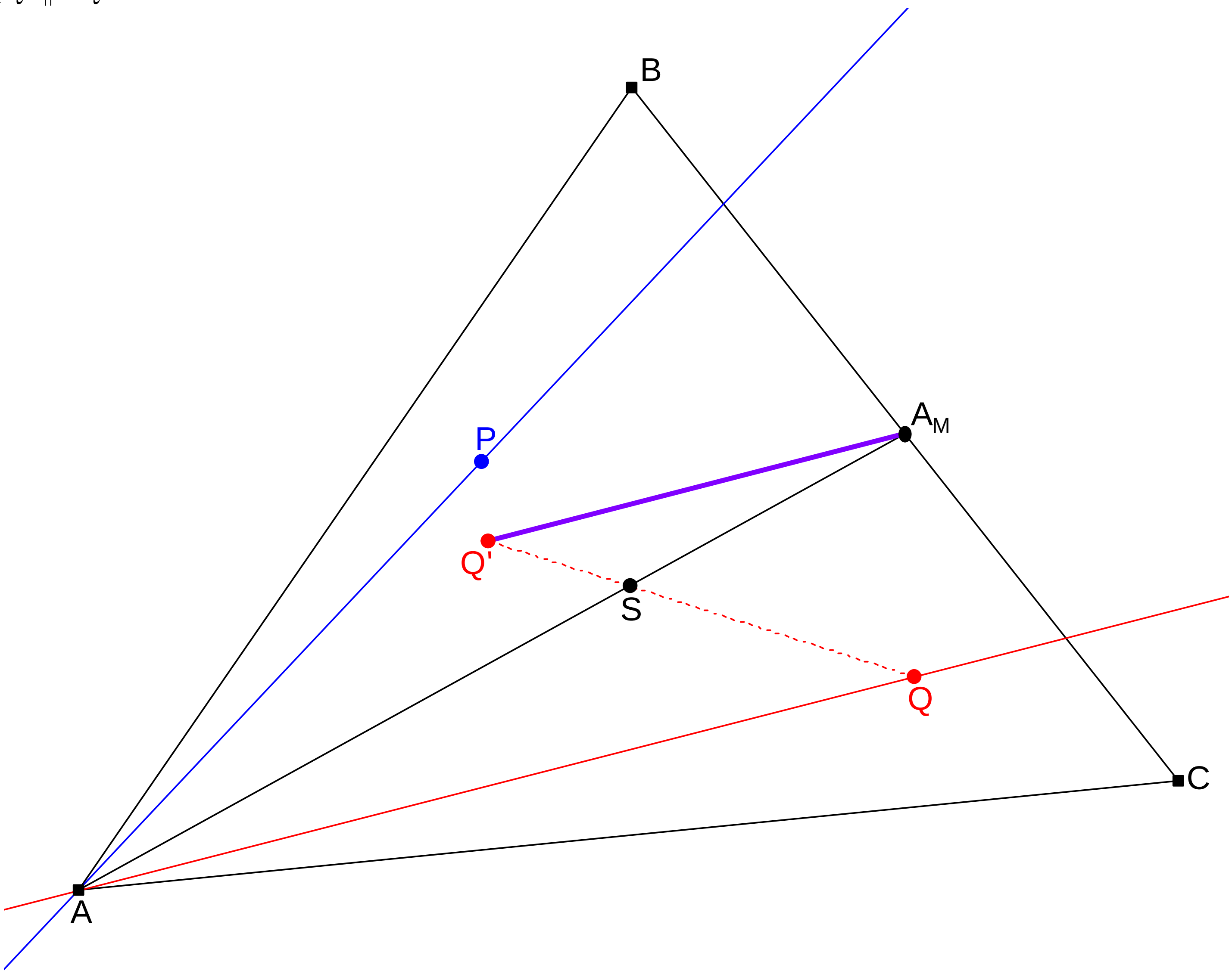

Fig. 23

(See Fig. 24.) Now, let Z' be the reflection of the point P in the line AB. Then, $\measuredangle Z'AB = -\measuredangle PAB$. On the other hand, the reflection wrt the perpendicular bisector of the segment AB maps the points A and B to the points B and A, respectively, and the point P to the point F (because the point F was defined as the reflection of the point P in the perpendicular bisector of the segment AB). Since reflection wrt a line changes the sign of angles (but leaves them invariant in other respects), we thus have $\measuredangle FBA = -\measuredangle PAB$. Together with $\measuredangle Z'AB = -\measuredangle PAB$, this results in $\measuredangle FBA = \measuredangle Z'AB$, that means, $\measuredangle (BF;\ AB) = \measuredangle (AZ';\ AB)$. Therefore, $BF \parallel AZ'$. Similarly, $AF \parallel BZ'$. Thus, the quadrilateral $AFBZ'$ is a parallelogram. Using vectors, this rewrites as $\overrightarrow{AZ'} = \overrightarrow{FB}$.

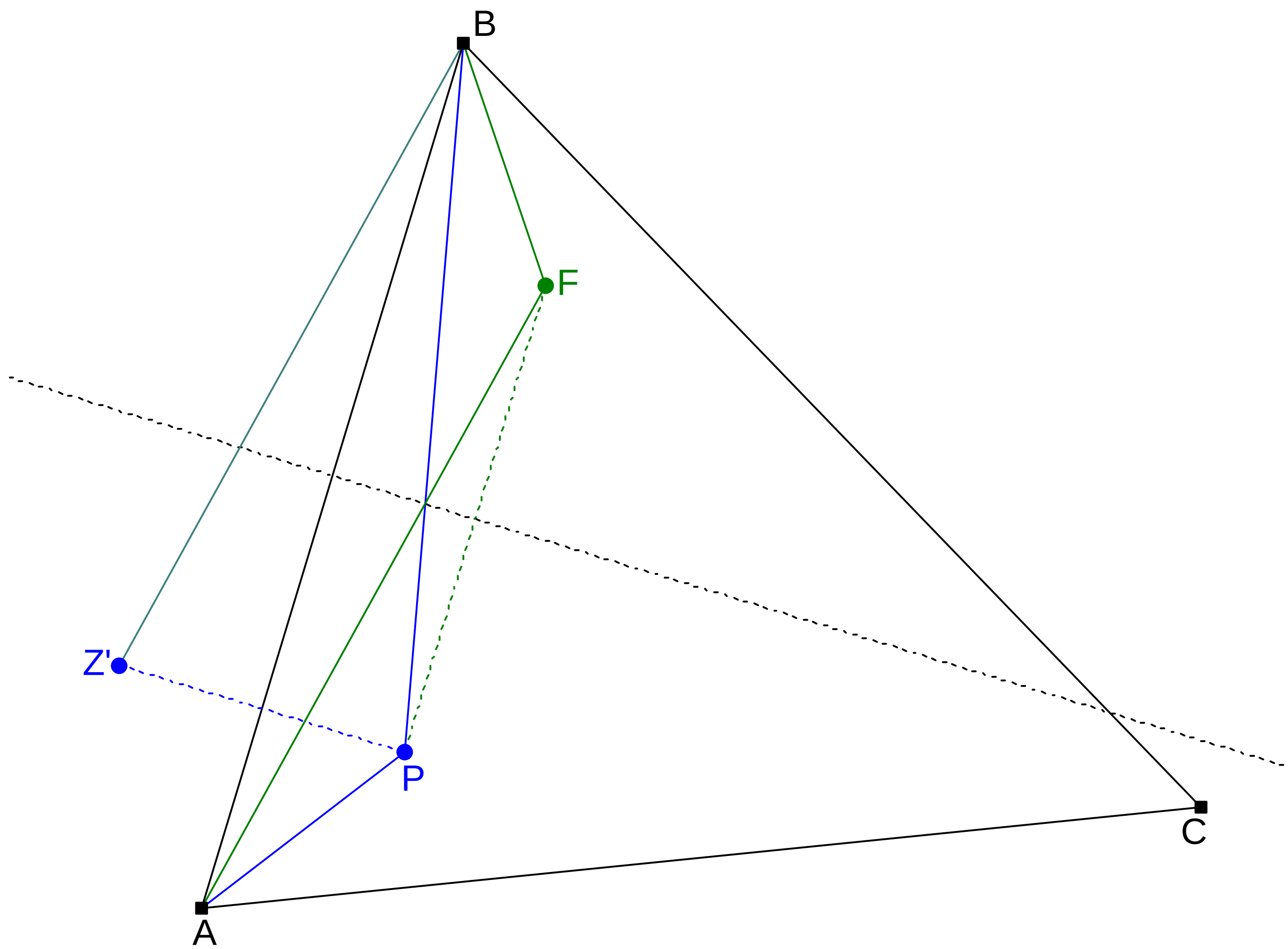

Fig. 24

(See Fig. 25.) After we have introduced the reflection Z' of the point P in the line AB, we also denote by Y' the reflection of the point P in the line CA. Similarly to $\overrightarrow{AZ'} = \overrightarrow{FB}$, we can then show $\overrightarrow{AY'} = \overrightarrow{EC}$. Furthermore, according to Theorem 5, the line AQ is the perpendicular bisector of the segment $Y'Z'$; consequently, the midpoint X'_M of the segment $Y'Z'$ lies on AQ.

On the other hand, since X'_M is the midpoint of the segment $Y'Z'$, we have $\overrightarrow{AX'_M} = \frac{\overrightarrow{AY'} + \overrightarrow{AZ'}}{2}$. Since $\overrightarrow{AY'} = \overrightarrow{EC}$ and $\overrightarrow{AZ'} = \overrightarrow{FB}$, this becomes $\overrightarrow{AX'_M} = \frac{\overrightarrow{EC} + \overrightarrow{FB}}{2}$.

Since A_M is the midpoint of the segment BC, we have

$$\overrightarrow{D_M A_M} = \frac{\overrightarrow{D_M B} + \overrightarrow{D_M C}}{2} = \frac{\left(\overrightarrow{D_M F} + \overrightarrow{FB}\right) + \left(\overrightarrow{EC} - \overrightarrow{ED_M}\right)}{2}.$$

But since D_M is the midpoint of the segment EF, we have $\overrightarrow{D_M F} = \overrightarrow{ED_M}$, so this simplifies to

$$\overrightarrow{D_M A_M} = \frac{\left(\overrightarrow{ED_M} + \overrightarrow{FB}\right) + \left(\overrightarrow{EC} - \overrightarrow{ED_M}\right)}{2} = \frac{\overrightarrow{EC} + \overrightarrow{FB}}{2} = \overrightarrow{AX'_M}.$$

Thus, $D_M A_M \parallel AX'_M$. Since the line AX'_M coincides with the line AQ, we thus get $D_M A_M \parallel AQ$. On the other hand, we know that $A_M Q' \parallel AQ$. Therefore, $D_M A_M \parallel$

A_MQ'. Now, the lines D_MA_M and A_MQ' have a common point (the point A_M), and thus can only be parallel if they coincide. As a consequence, the lines D_MA_M and A_MQ' coincide, i. e., the point Q' lies on the line A_MD_M. Similarly, we may show that the point Q' lies on the lines B_ME_M and C_MF_M, and the proof of Theorem 11 is finished.

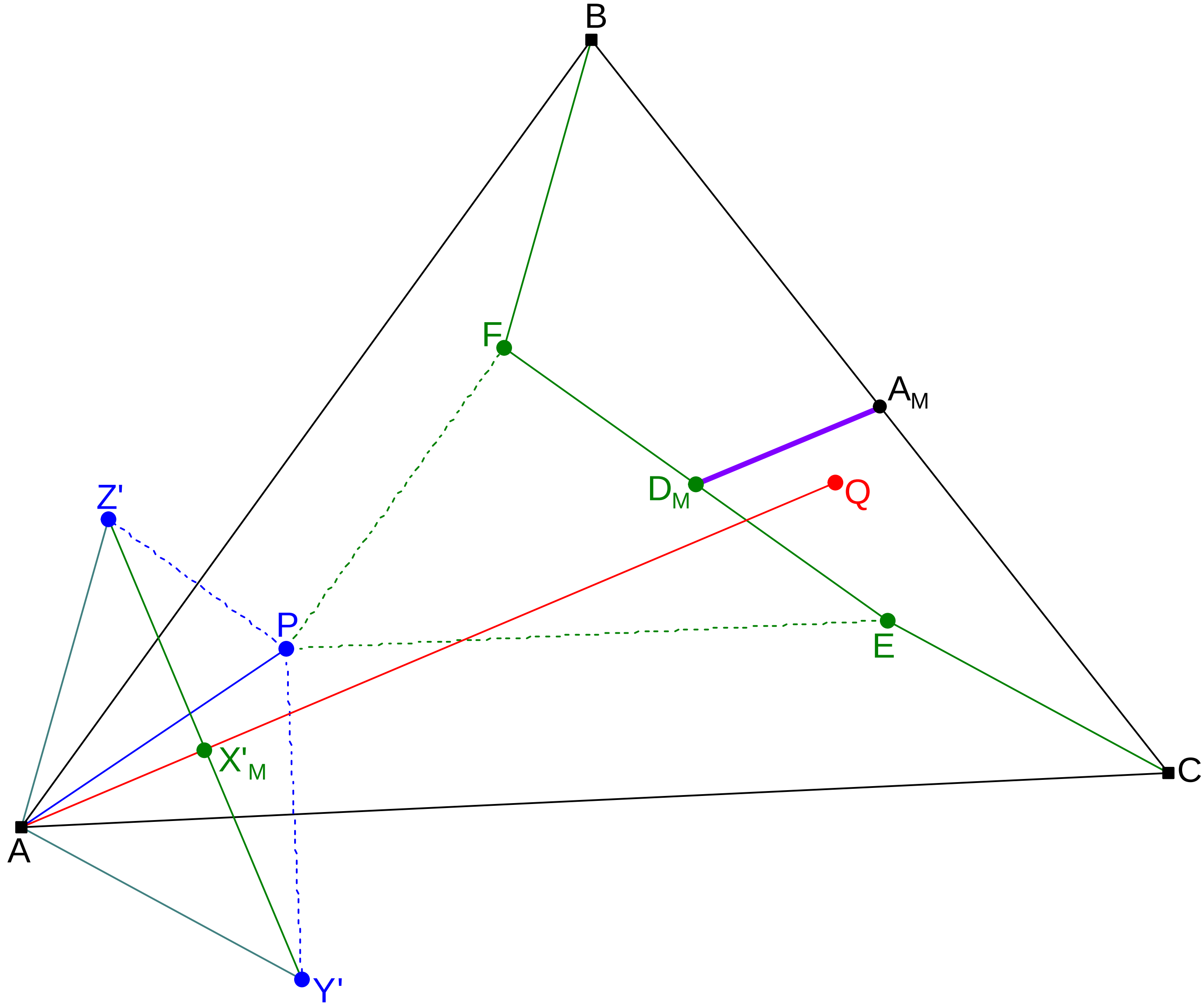

Fig. 25

7. More on isogonal conjugates and reflections

The following theorem sheds light on a familiar configuration - reflections of a point in the sides of a triangle and its isogonal conjugate, see Theorem 5 - from a new, symmetric viewpoint:[2]

Theorem 12. Let A, B, C, D be four distinct points in the plane, and let A', B', C', D' be four distinct points in the plane. Then, the following four assertions $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$ are pairwisely equivalent:

[2] Of course, the point D in the following has nothing to do with the point D from Theorems 10 and 11, and the points A', B', C' in the following have nothing to do with the points A', B', C' from Theorem 9.

Assertion $\mathcal{B}_1$: The points A, B, C, D are the circumcenters of triangles $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$.

Assertion $\mathcal{B}_2$: The lines AB, BC, CD, DA, AC, BD are the perpendicular bisectors of the segments $C'D'$, $D'A'$, $A'B'$, $B'C'$, $B'D'$, $A'C'$.

Assertion $\mathcal{B}_3$: The points A', A', A', B', B', B', C', C', C', D', D', D' are the reflections of the points B', C', D', C', D', A', D', A', B', A', B', C' in the lines CD, DB, BC, DA, AC, CD, AB, BD, DA, BC, CA, AB.

Assertion $\mathcal{B}_4$: The points A', B', C', D' are the isogonal conjugates of the points A, B, C, D wrt the triangles BCD, CDA, DAB, ABC.

(See Fig. 26.)

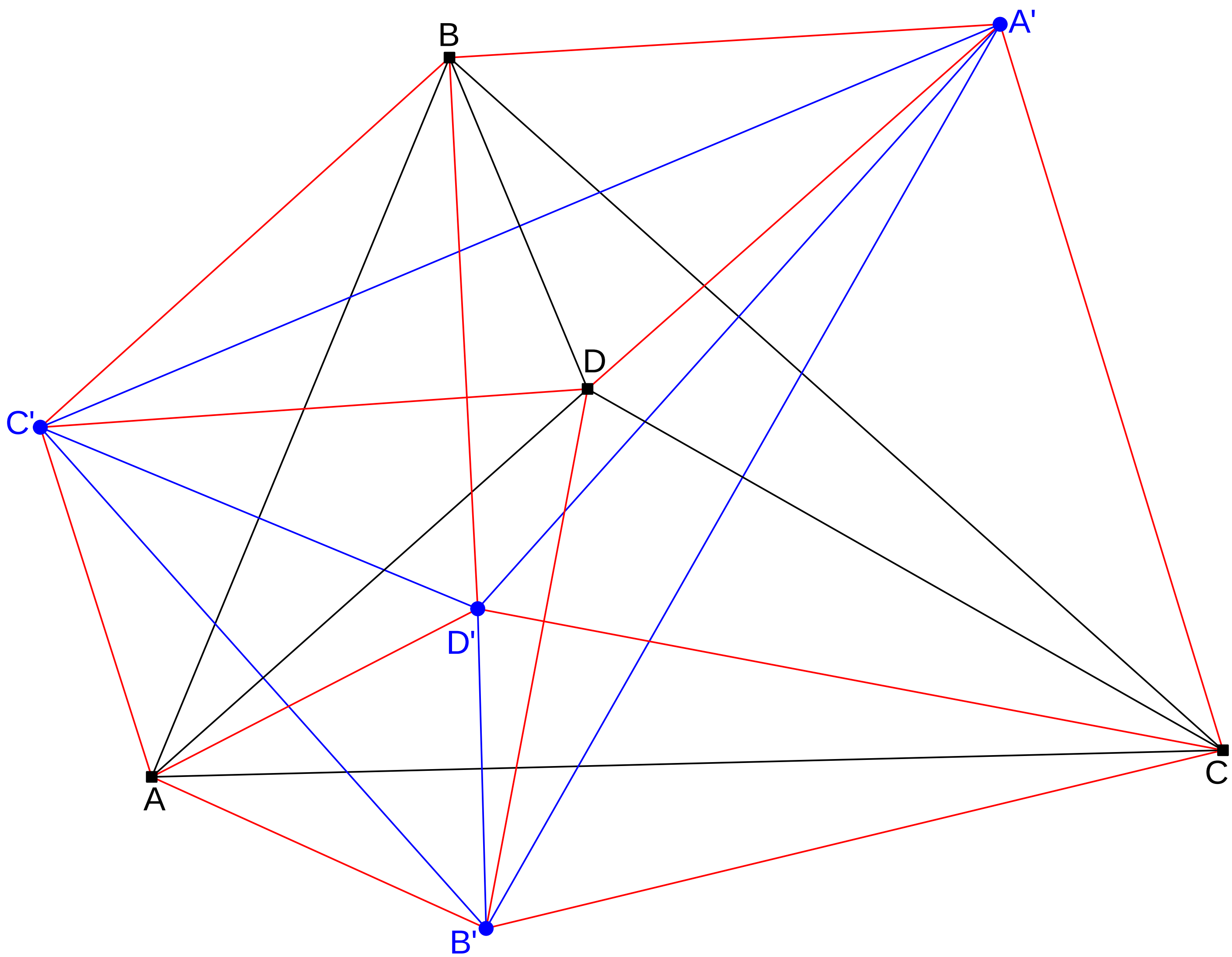

Fig. 26

Proof of Theorem 12. First, we show the equivalence of the assertions $\mathcal{B}_1$ and $\mathcal{B}_2$:

If Assertion $\mathcal{B}_1$ holds, then the points A, B, C, D are the circumcenters of triangles $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$. Then, since the circumcenter of a triangle lies on the perpendicular bisectors of its sides, the point A, being the circumcenter of triangle $B'C'D'$, must lie on the perpendicular bisector of the segment $C'D'$. Similarly, the point B must lie on the perpendicular bisector of the segment $C'D'$. Hence, the line AB is the perpendicular bisector of the segment $C'D'$. Similarly, the lines BC, CD,

DA, AC, BD are the perpendicular bisectors of the segments $D'A'$, $A'B'$, $B'C'$, $B'D'$, $A'C'$, and thus Assertion $\mathcal{B}_2$ is fulfilled.

Conversely, if Assertion $\mathcal{B}_2$ holds, then the lines AB, BC, CD, DA, AC, BD are the perpendicular bisectors of the segments $C'D'$, $D'A'$, $A'B'$, $B'C'$, $B'D'$, $A'C'$. Hence, the point A, being the point of intersection of the lines AB, AC, DA, must be the point of intersection of the perpendicular bisectors of the segments $C'D'$, $B'D'$, $B'C'$, and thus the circumcenter of the triangle $B'C'D'$. Similarly, the points B, C, D are the circumcenters of the triangles $C'D'A'$, $D'A'B'$, $A'B'C'$. Thus, Assertion $\mathcal{B}_1$ must hold.

Hence, we have shown that the assertions $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent.

The equivalence of the assertions $\mathcal{B}_3$ and $\mathcal{B}_2$ evidently follows from the following obvious fact: If P_1 and P_2 are two distinct points and g_1 is a line, then the point P_2 is the reflection of the point P_1 in the line g_1 if and only if the line g_1 is the perpendicular bisector of the segment P_1P_2.

Altogether we have now shown that the assertions $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are pairwisely equivalent. In order to verify the equivalence of all four assertions $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$, it remains to prove that the assertions $\mathcal{B}_3$ and $\mathcal{B}_4$ are equivalent. In order to prove this, we have to establish two auxiliary results:

Auxiliary result 1. If Assertion $\mathcal{B}_4$ holds, then so does Assertion $\mathcal{B}_3$.

Auxiliary result 2. If Assertion $\mathcal{B}_3$ holds, then so does Assertion $\mathcal{B}_4$.

Proof of Auxiliary result 1. Assume that Assertion $\mathcal{B}_4$ holds. Then, particularly, the point B' is the isogonal conjugate of the point B wrt triangle CDA. Thus, the line AB' is the isogonal of the line AB wrt the angle DAC. Hence, $\measuredangle(AC;\ AB') = -\measuredangle(DA;\ AB)$. Equivalently, $\measuredangle CAB' = -\measuredangle DAB$. Similarly, $\measuredangle CAD' = -\measuredangle BAD$. Thus, $\measuredangle CAB' = -\measuredangle DAB = -(-\measuredangle BAD) = -\measuredangle CAD'$. Now, if B_1' is the reflection of the point D' in the line AC, then $\measuredangle CAB_1' = -\measuredangle CAD'$, so that $\measuredangle CAB_1' = \measuredangle CAB'$. Hence, the point B_1' lies on the line AB'. Similarly, the point B_1' lies on the line CB'. But the lines AB' and CB' have only one point in common, namely the point B'. Thus, since the point B_1' lies on both of these lines, we must have $B_1' = B'$. Since we have introduced the point B_1' as the reflection of the point D' in the line AC, we thus conclude that the point B' is the reflection of the point D' in the line AC. Similarly, the points A', A', A', B', B', C', C', C', D', D', D' are the reflections of the points B', C', D', C', A', D', A', B', A', B', C' in the lines CD, DB, BC, DA, CD, AB, BD, DA, BC, CA, AB. In other words, Assertion $\mathcal{B}_3$ holds. This proves Auxiliary result 1.

First proof of Auxiliary result 2. Assume that Assertion $\mathcal{B}_3$ is valid. Since the assertions $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are equivalent, it thus follows that Assertions $\mathcal{B}_1$ and $\mathcal{B}_2$ hold as well; i. e., the points A, B, C, D are the circumcenters of triangles $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$, and the lines AB, BC, CD, DA, AC, BD are the perpendicular bisectors of the segments $C'D'$, $D'A'$, $A'B'$, $B'C'$, $B'D'$, $A'C'$.

Since the lines DA, AB, AC are the perpendicular bisectors of the segments $B'C'$, $C'D'$, $B'D'$, we have $DA \perp B'C'$, $AB \perp C'D'$, $AC \perp B'D'$, so that $\measuredangle(DA;\ B'C') = 90^\circ$, $\measuredangle(AB;\ C'D') = 90^\circ$, $\measuredangle(B'D';\ AC) = 90^\circ$. Thus,

$$\begin{aligned}\measuredangle(DA;\ AB) &= \measuredangle(DA;\ B'C') + \measuredangle(B'C';\ AB) = 90^\circ + \measuredangle(B'C';\ AB)\\ &= \measuredangle(AB;\ C'D') + \measuredangle(B'C';\ AB) = \measuredangle(B'C';\ C'D') = \measuredangle B'C'D'.\end{aligned}$$

On the other hand, since A is the circumcenter of triangle $B'C'D'$, thus the center of a circle through the points B', C', D', the central angle theorem yields $\measuredangle D'B'A =$

$90^\circ - \measuredangle B'C'D'$. Hence,

$$\begin{aligned}\measuredangle(AC;\ AB') &= \measuredangle(B'D';\ AB') - \measuredangle(B'D';\ AC) = \measuredangle D'B'A - 90^\circ \\ &= (90^\circ - \measuredangle B'C'D') - 90^\circ = -\measuredangle B'C'D'.\end{aligned}$$

Comparing this with $\measuredangle(DA;\ AB) = \measuredangle B'C'D'$, we infer $\measuredangle(AC;\ AB') = -\measuredangle(DA;\ AB)$. Thus, the line AB' is the isogonal of the line AB wrt the angle DAC. Similarly, the lines CB' and DB' are the isogonals of the lines CB and DB wrt the angles ACD and CDA. Thus, the point B' is the point of intersection of the isogonals of the lines CB, DB, AB wrt the angles ACD, CDA, DAC. In other words, the point B' is the isogonal conjugate of the point B wrt the triangle CDA. Similarly, the points C', D', A' are the isogonal conjugates of the points C, D, A wrt the triangles DAB, ABC, BCD. Hence, Assertion $\mathcal{B}_4$ holds; this proves Auxiliary result 2.

Second proof of Auxiliary result 2. It is particularly easy to prove Auxiliary result 2 basing on Theorem 5:

Assume that Assertion $\mathcal{B}_3$ holds. Then, in particular, the point A' is the reflection of the point B' in the line CD. Hence, $DA' = DB'$. Similarly, $DB' = DC'$. Thus, $DA' = DB' = DC'$, and this signifies that the point D is the circumcenter of triangle $A'B'C'$.

Since we assumed Assertion $\mathcal{B}_3$ to hold, the points A', B', C' are the reflections of the point D' in the lines BC, CA, AB. Thus, after Theorem 5, the isogonal conjugate of the point D' wrt triangle ABC is the circumcenter of triangle $A'B'C'$. But as we know that the point D is the circumcenter of triangle $A'B'C'$, it follows that the point D is the isogonal conjugate of the point D' wrt triangle ABC. Hence, in turn, the point D' is the isogonal conjugate of the point D wrt triangle ABC. Similarly, the points A', B', C' are the isogonal conjugates of the points A, B, C wrt triangles BCD, CDA, DAB. Therefore, Assertion $\mathcal{B}_4$ holds. This again proves Auxiliary result 2.

This completes the proof of Theorem 12.

An easy consequence of Theorem 12 is (Fig. 27):

Theorem 13. Let P be an Euclidean point in the plane of triangle ABC, and let X', Y', Z' be the reflections of this point P in the lines BC, CA, AB. Let Q be the isogonal conjugate of the point P wrt triangle ABC. Then, the points X', Y', Z' are the isogonal conjugates of the points A, B, C wrt the triangles BQC, CQA, AQB.

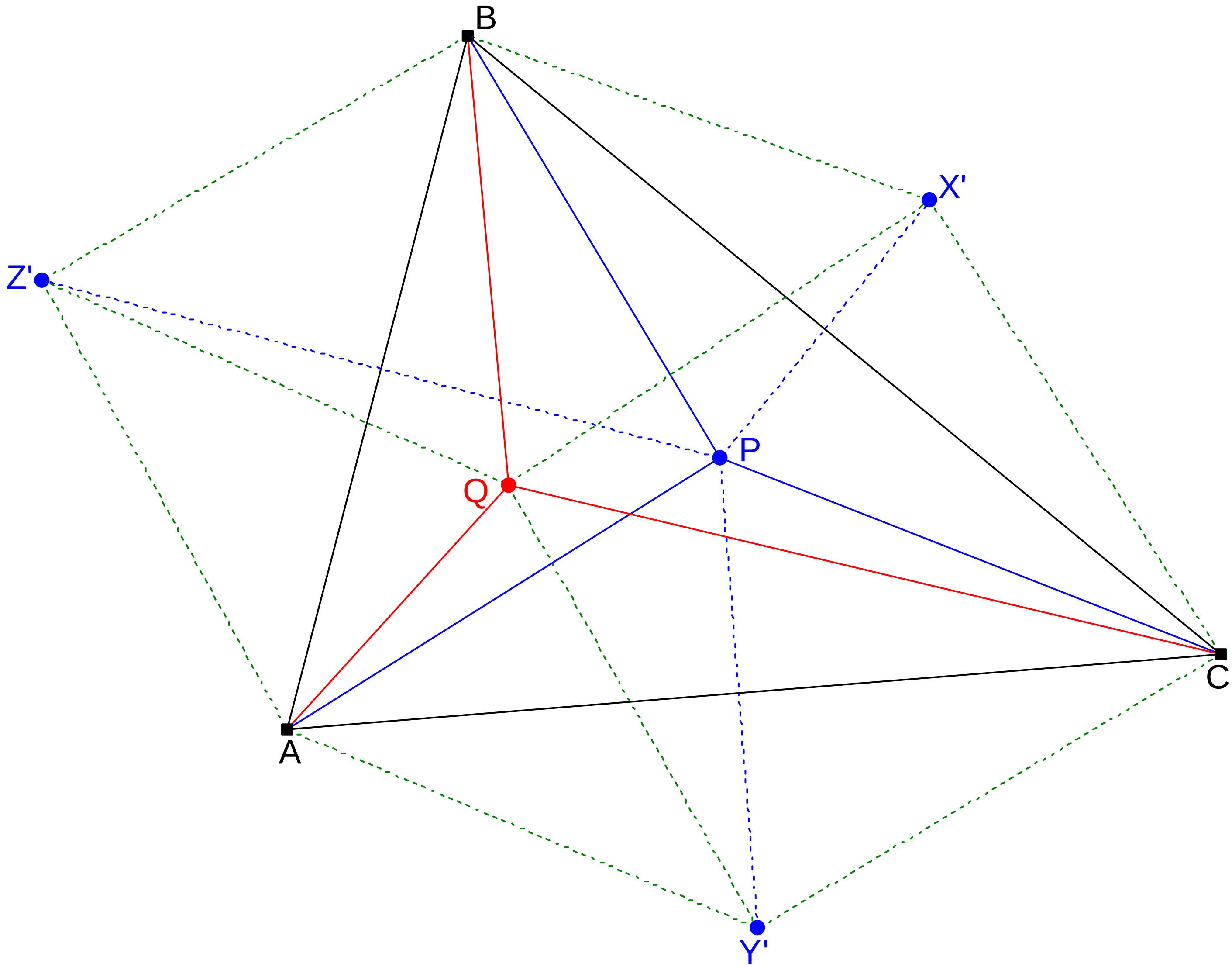

Fig. 27

Proof of Theorem 13. (See Fig. 28.) Since Y' is the reflection of the point P in the line CA, we have $AY' = AP$. Similarly, $AZ' = AP$. Therefore, $AY' = AP = AZ'$; thus, the point A is the circumcenter of triangle $Y'Z'P$. Similarly, $BZ' = BP = BX'$ and $CX' = CP = CY'$, what yields that the points B and C are the circumcenters of triangles $Z'PX'$ and $PX'Y'$. Further, according to Theorem 5, the point Q is the circumcenter of triangle $X'Y'Z'$. Altogether, we see that the points A, B, C, Q are the circumcenters of triangles $Y'Z'P$, $Z'PX'$, $PX'Y'$, $X'Y'Z'$. Thus, the four distinct points A, B, C, Q and the four distinct points X', Y', Z', P fulfill the Assertion $\mathcal{B}_1$ of Theorem 12. But since, according to Theorem 12, the Assertion $\mathcal{B}_1$ is equivalent to Assertion $\mathcal{B}_4$, these points must therefore also satisfy Assertion $\mathcal{B}_4$. In other words, the points X', Y', Z', P are the isogonal conjugates of the points A, B, C, Q wrt the triangles BCQ, CQA, QAB, ABC. Equivalently, the points X', Y', Z', P are the isogonal conjugates of the points A, B, C, Q wrt the triangles BQC, CQA, AQB, ABC. This implies Theorem 13.

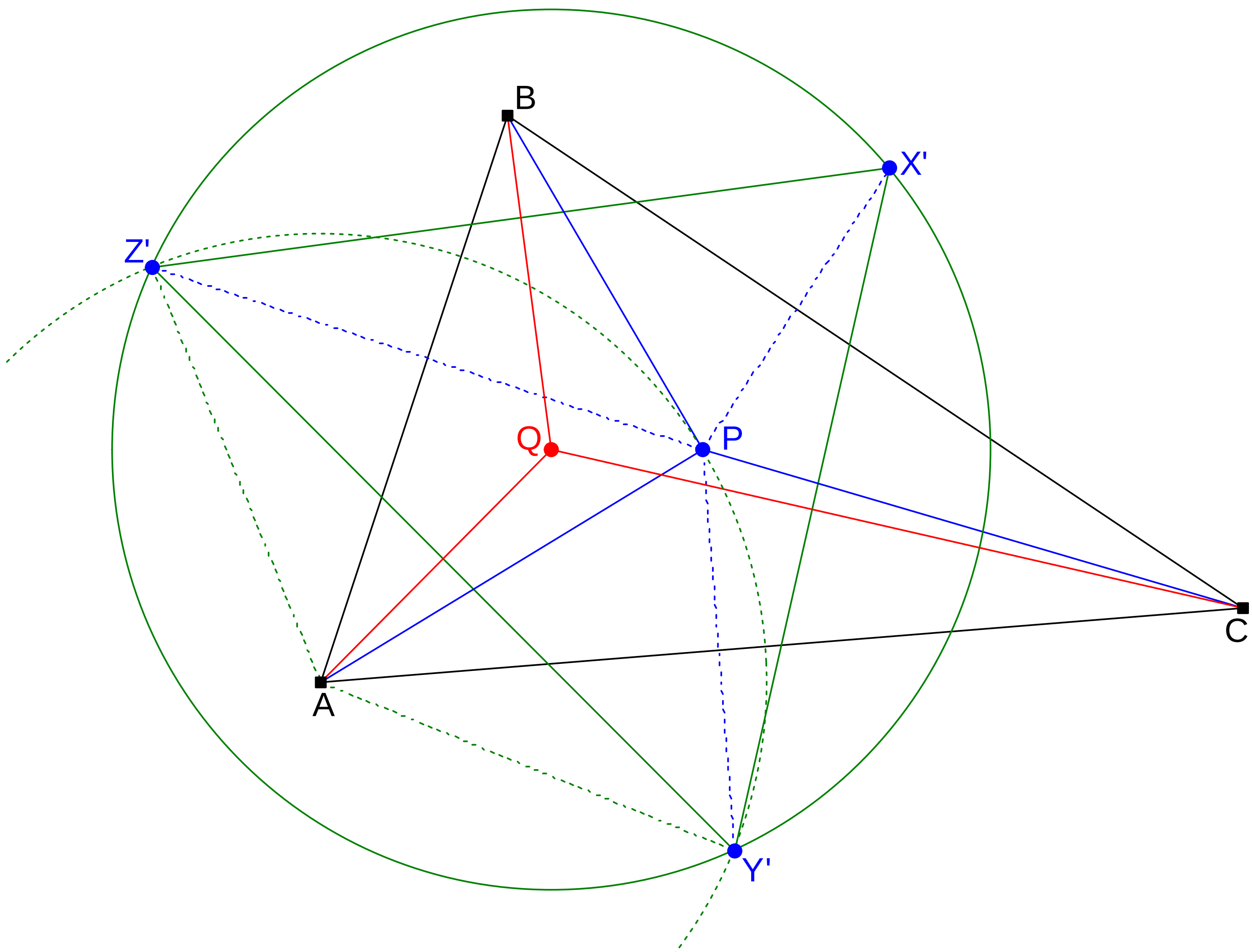

Fig. 28

Theorem 13 simplifies our proof of Theorem 8 given above. In fact, according to Theorem 13, the point X' is the isogonal conjugate of the point A wrt triangle BQC, what immediately yields that the line QX' is the isogonal of the line QA (that is, the line AQ) wrt the angle BQC; this was a crucial result in the proof of Theorem 8. [Thanks to Marcello Tarquini for reminding me of this shortcut.]

8. Isogonal conjugates of basic centers

It is useful to identify the isogonal conjugates of known triangle centers. We start with a trivial fact:

Theorem 14. Let P be a point in the plane of a triangle ABC. Then, the isogonal conjugate of the point P wrt triangle ABC coincides with the point P if and only if the point P is the incenter or one of the three excenters of triangle ABC. (See Fig. 29.)

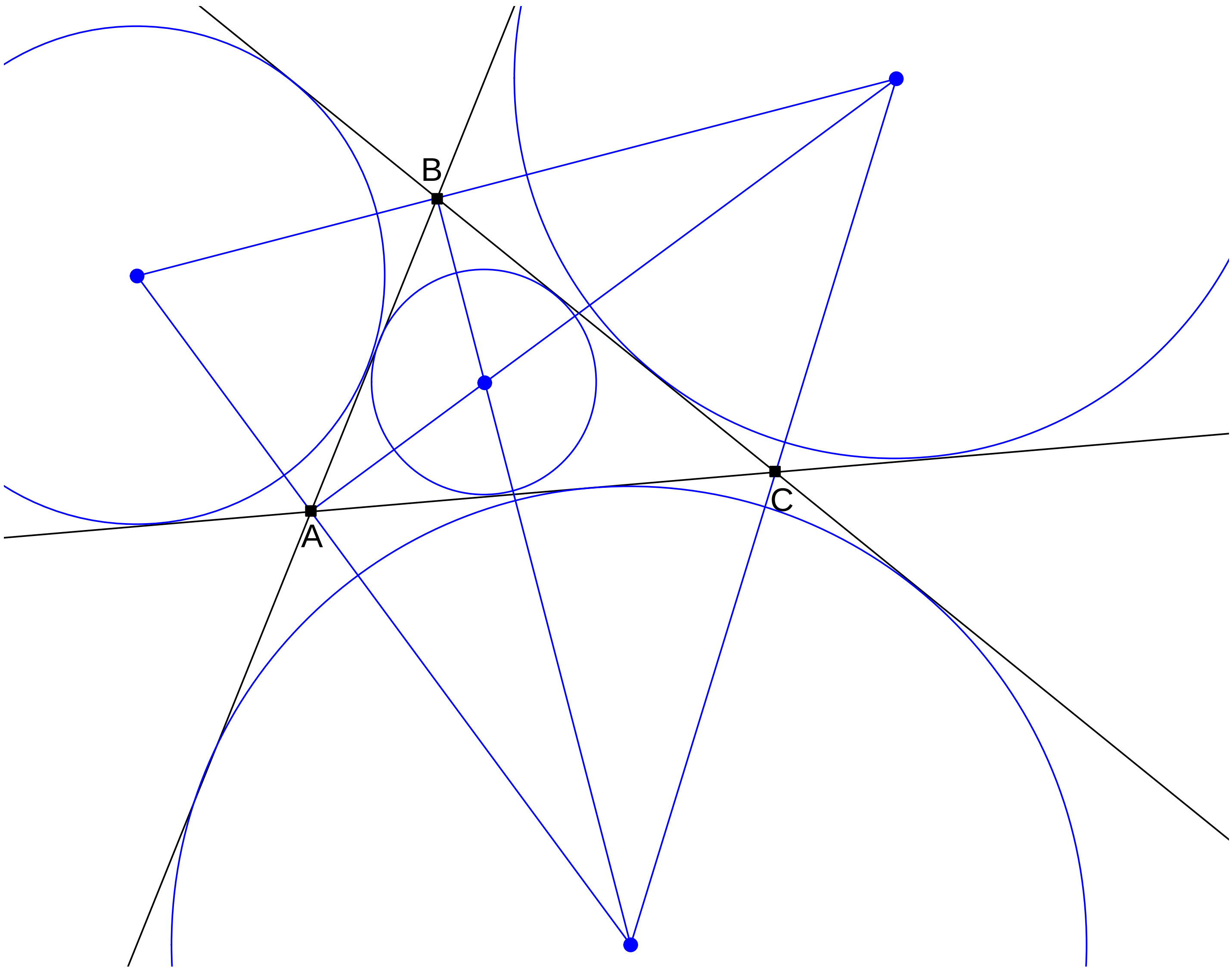

Fig. 29

Proof of Theorem 14. According to the definition of isogonals, the isogonal of the line AP wrt the angle CAB coincides with the line AP if and only if $\measuredangle (AB;\ AP) = -\measuredangle (CA;\ AP)$. But this is equivalent to the line AP being either the internal or the external angle bisector of the angle CAB. Hence, the isogonal of the line AP wrt the angle CAB coincides with the line AP if and only if the line AP is either the internal or the external angle bisector of the angle CAB. Similarly, the isogonal of the line BP wrt the angle ABC coincides with the line BP if and only if the line BP is either the internal or the external angle bisector of the angle ABC, and the isogonal of the line CP wrt the angle BCA coincides with the line CP if and only if the line CP is either the internal or the external angle bisector of the angle BCA.

The isogonal conjugate of the point P wrt triangle ABC is the point of intersection of the isogonals of the lines AP, BP, CP wrt the angles CAB, ABC, BCA. Consequently, the isogonal conjugate of the point P wrt triangle ABC coincides with the point P if and only if the isogonals of the lines AP, BP, CP wrt the angles CAB, ABC, BCA intersect at the point P, i. e. if and only if the isogonals of the lines AP, BP, CP wrt the angles CAB, ABC, BCA coincide with the lines AP, BP, CP. As we know, this is equivalent to the lines AP, BP, CP being the internal or external angle bisectors of the angles CAB, ABC, BCA. But there are exactly four points P in the plane of triangle ABC such that the lines AP, BP, CP are the internal or external angle bisectors of the angles CAB, ABC, BCA, and these points are the incenter and the three excenters of triangle ABC. Thus it is proven that the isogonal conjugate of the point P wrt triangle ABC coincides with the point P if and only if the point P is the incenter or one of the three excenters of triangle ABC. This proves Theorem 14.

A slightly less trivial fact is the following (Fig. 30):

Theorem 15. Let O be the circumcenter and H the orthocenter of a triangle ABC. Then, the points O and H are isogonally conjugate points wrt triangle ABC.

In brief: The circumcenter and the orthocenter of any triangle are isogonally conjugate points wrt this triangle.

First proof of Theorem 15. Being the circumcenter of triangle ABC, the point O is the center of a circle through the points A, B, C. Thus, according to the central angle theorem, $\measuredangle CAO = 90° - \measuredangle ABC$. On the other hand, since H is the orthocenter of triangle ABC, we know that $AH \perp BC$, so that $\measuredangle (AH;\ BC) = 90°$, and therefore

$$\begin{aligned} \measuredangle (AB;\ AH) &= \measuredangle (AB;\ BC) - \measuredangle (AH;\ BC) = \measuredangle ABC - 90° = -(90° - \measuredangle ABC) \\ &= -\measuredangle CAO = -\measuredangle (CA;\ AO) . \end{aligned}$$

Hence, the line AH is the isogonal of the line AO wrt the angle CAB. Similarly, the lines BH and CH are the isogonals of the lines BO and CO wrt the angles ABC and BCA. Thus, the point H, being the point of intersection of these lines AH, BH, CH, must be the isogonal conjugate of the point O wrt triangle ABC. This proves Theorem 15.

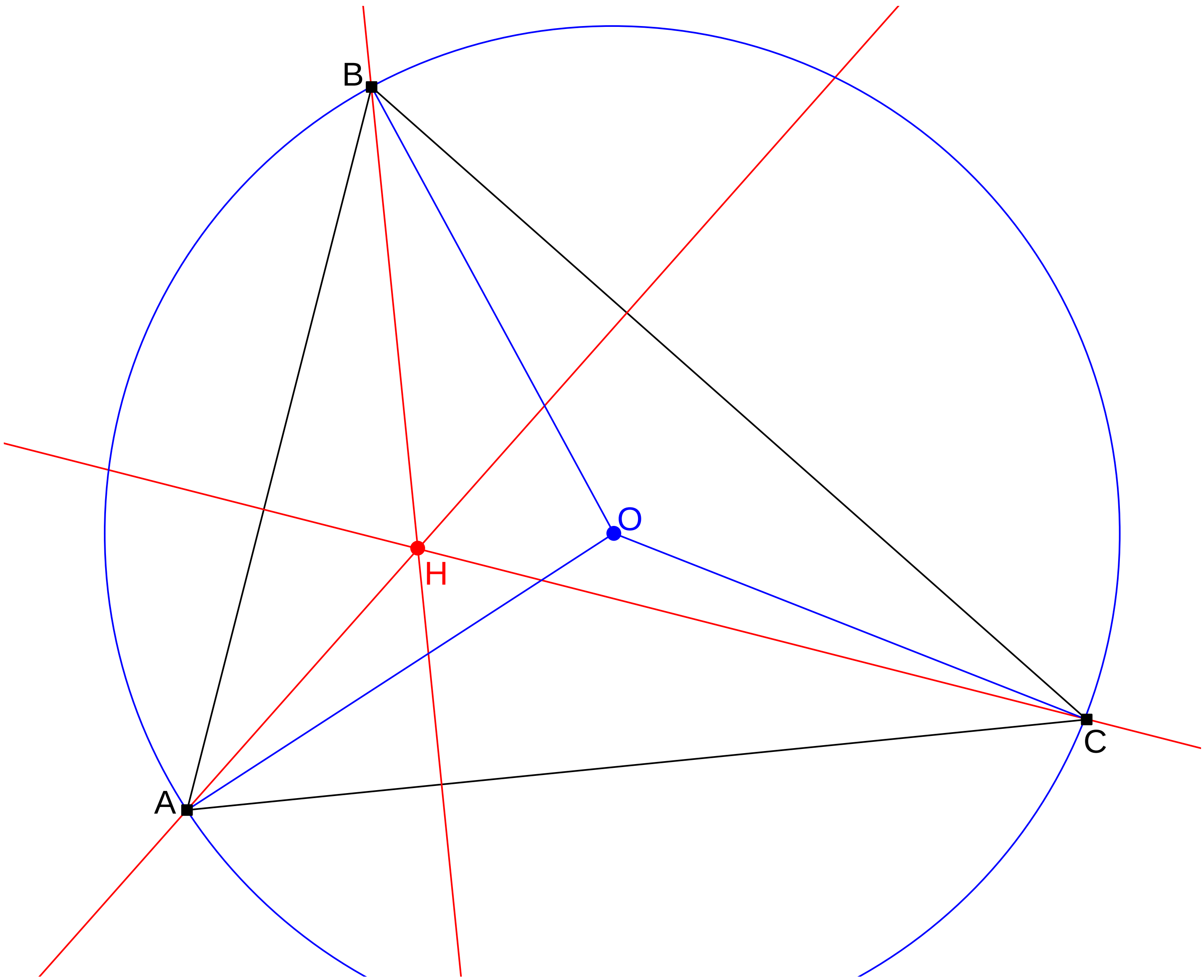

Fig. 30

Second proof of Theorem 15. (See Fig. 31.) Let B_M and C_M be the midpoints of the sides CA and AB of triangle ABC. Then, $B_MC_M \parallel BC$. On the other hand, since the circumcenter O of triangle ABC lies on the perpendicular bisectors of its sides CA and AB, the midpoints B_M and C_M of these sides CA and AB must be the orthogonal projections of the point O on the lines CA and AB. Hence, by Theorem 2 **a)** (applied to the two lines CA and AB and the point O in the plane), the line B_MC_M is perpendicular to the isogonal of the line AO wrt the lines CA and AB, i. e. wrt the angle CAB. In other words: The isogonal of the line AO wrt the angle CAB is perpendicular to the line B_MC_M. Since $B_MC_M \parallel BC$, this isogonal must therefore be perpendicular to the line BC, and since this isogonal passes through the point A, we can conclude that it is the A-altitude of triangle ABC and thus passes through its orthocenter H. So the point H lies on the isogonal of the line AO wrt the angle CAB. Similarly, the point H lies on the isogonals of the lines BO and CO wrt the angles ABC and BCA. Thus, the point H is the point of intersection of these isogonals, i. e. the isogonal conjugate of the point O wrt triangle ABC. Once again Theorem 15 is proven.

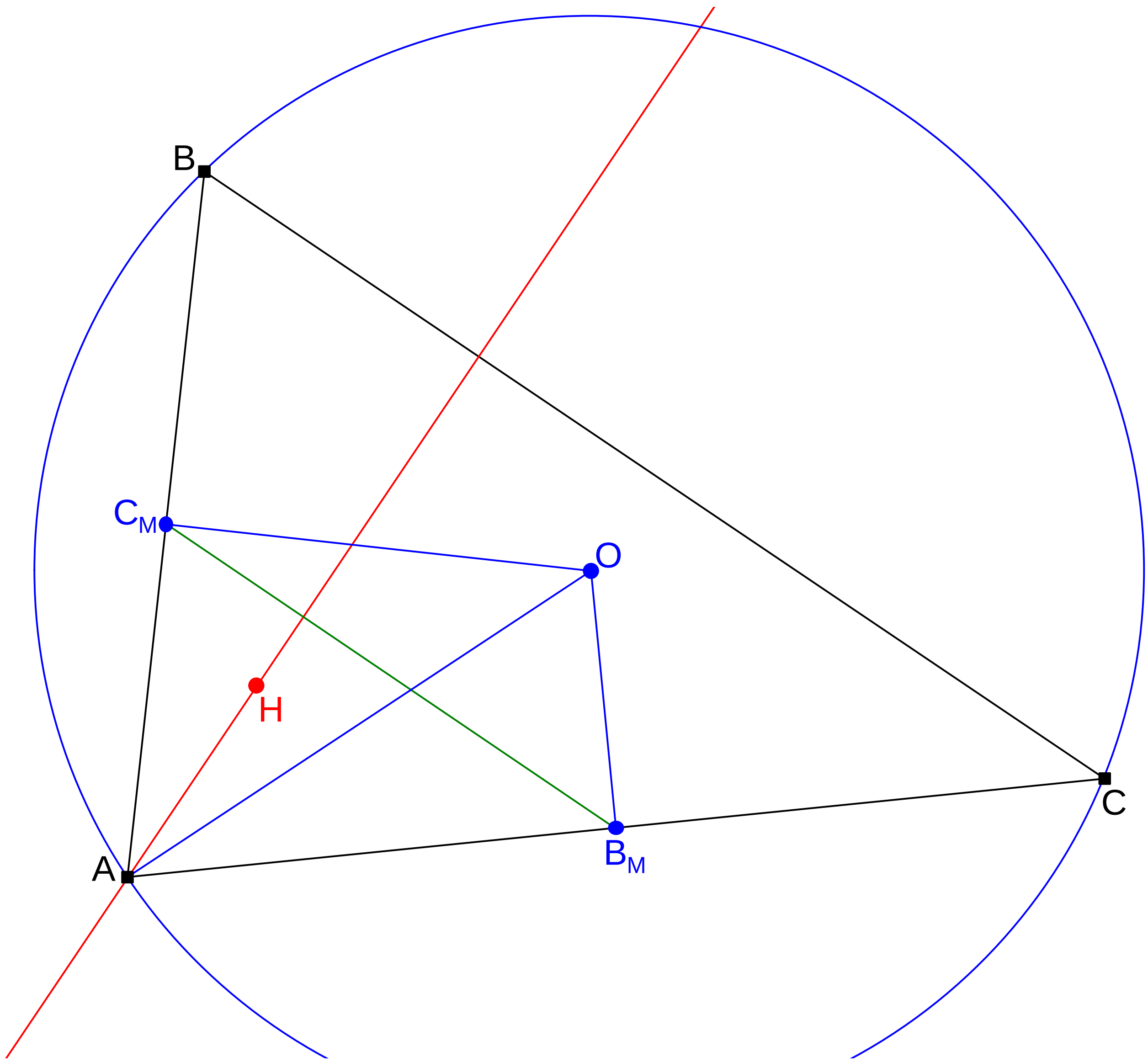

Fig. 31

At this point, we mention an easy consequence of Theorem 15 which will be of use to us later (Fig. 32):

Theorem 16. Let A_H, B_H, C_H be the feet of the altitudes of triangle ABC from the vertices A, B, C. Let O be the circumcenter of triangle ABC. Then, $AO \perp B_H C_H$, $BO \perp C_H A_H$, $CO \perp A_H B_H$.

Proof of Theorem 16. Obviously, the orthocenter H of triangle ABC lies on the altitude of triangle ABC from the vertex A. The point A_H is the foot of this altitude. Thus, the point A_H is the orthogonal projection of the point H on the line BC. Similarly, the points B_H and C_H are the orthogonal projections of the point H on the lines CA and AB. By Theorem 15, the point O is the isogonal conjugate of the point H wrt triangle ABC. Thus, applying Theorem 6 to the point H in the plane of triangle ABC, we get $AO \perp B_H C_H$, $BO \perp C_H A_H$, $CO \perp A_H B_H$. This proves Theorem 16.

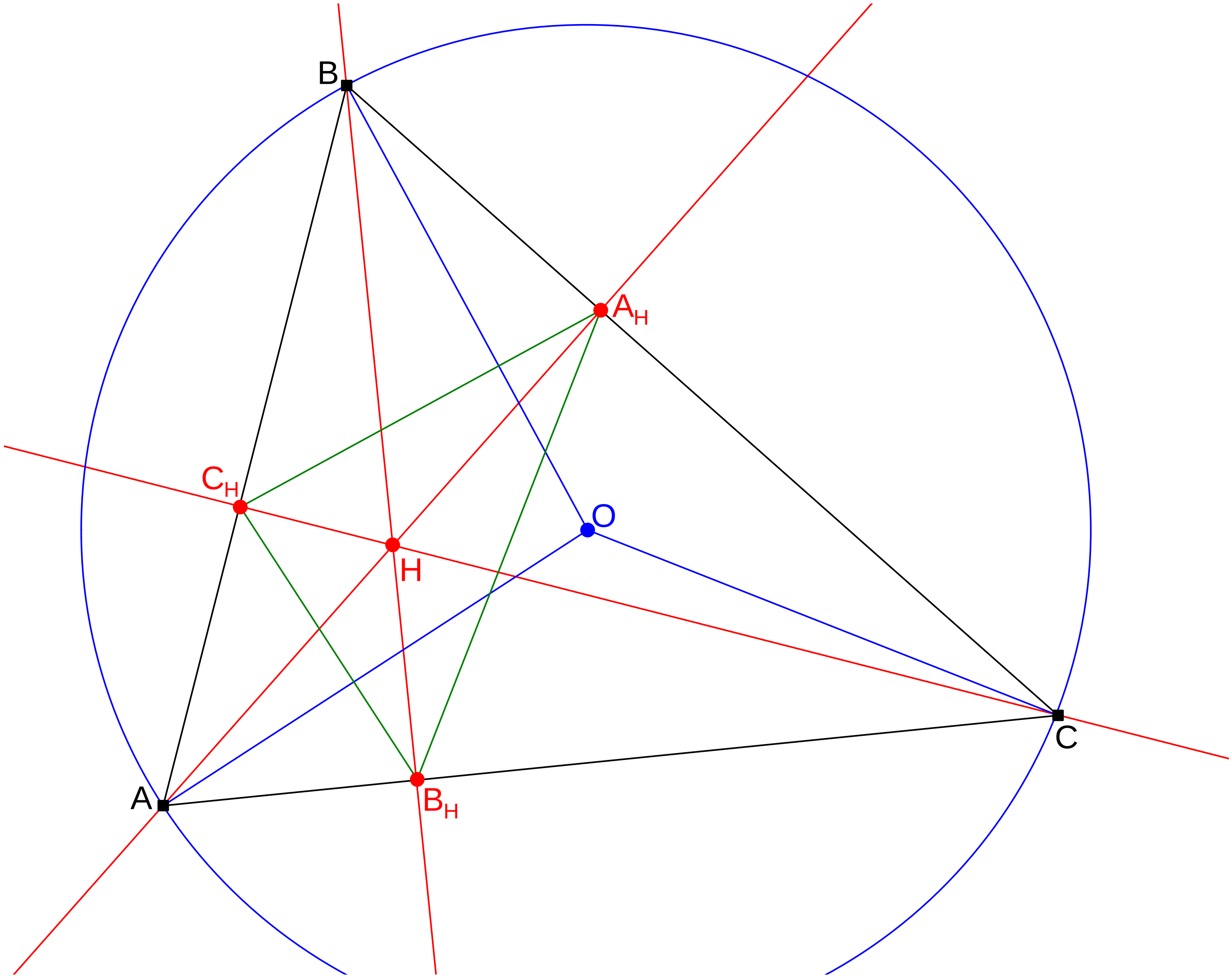

Fig. 32

Theorem 16 can also be directly shown through an angle chase.

9. Symmedians and antiparallels

(See Fig. 33.) Let S be the centroid of triangle ABC. Then, the lines AS, BS, and CS are the A-median, the B-median, and the C-median of triangle ABC, respectively. The isogonals of these medians AS, BS, CS wrt the angles CAB, ABC, BCA are called the **A-symmedian**, **B-symmedian**, **C-symmedian** of triangle ABC, respectively, and altogether referred to as the three **symmedians** of triangle ABC. As a consequence of their definition, these symmedians of triangle ABC intersect at one point, namely at the isogonal conjugate of the point S wrt triangle ABC. This isogonal conjugate of the point S wrt triangle ABC is called the **symmedian point** of triangle ABC and will be denoted by K in the following. Then, as the A-symmedian, the B-symmedian, and the C-symmedian of triangle ABC intersect at the point K, they are the lines AK, BK, and CK.

In brief: The symmedian point of a triangle is the point of intersection of its symmedians, and it is the isogonal conjugate of the centroid of this triangle wrt this triangle.

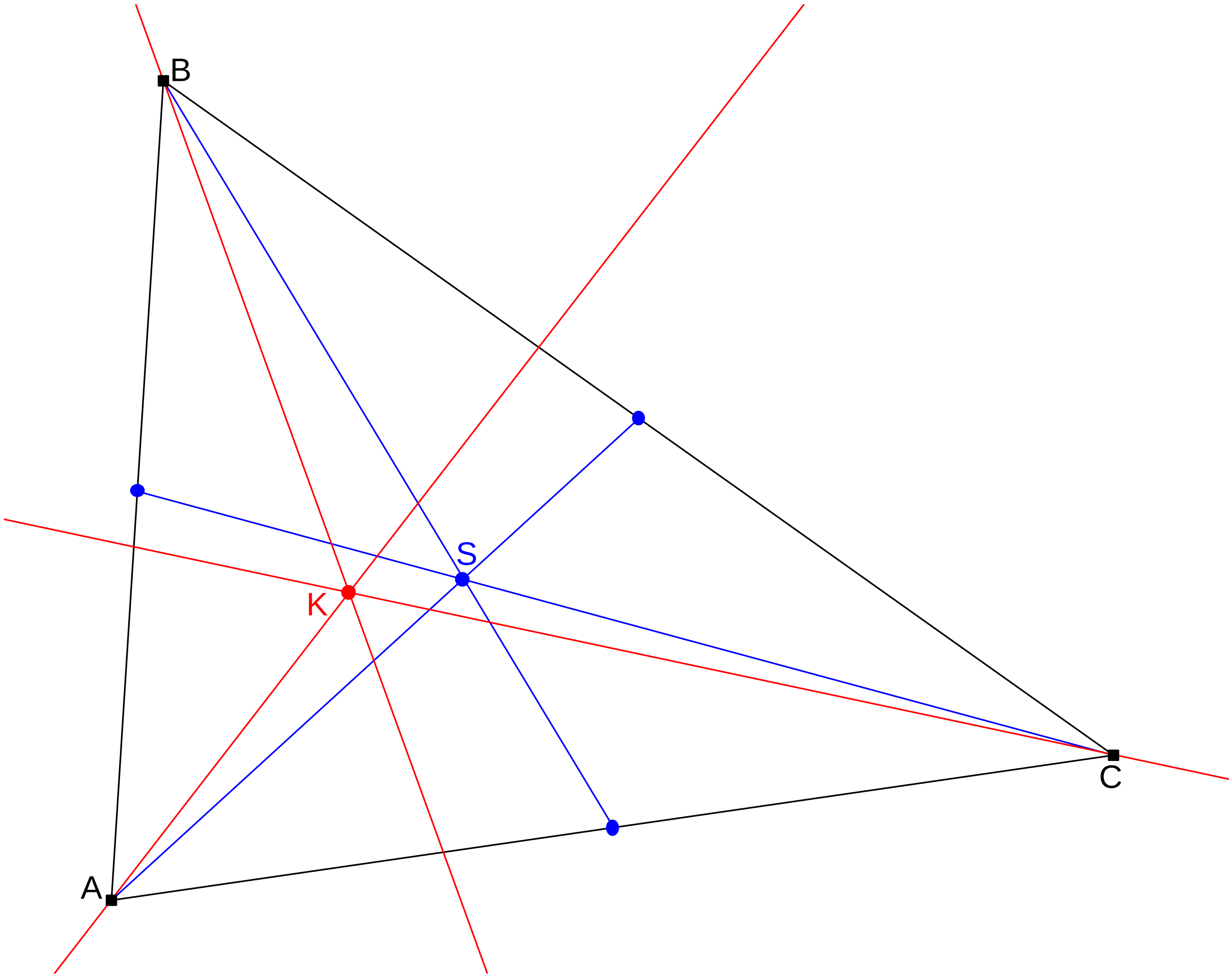

Fig. 33

We won't discuss further properties of the symmedian point here, but we use the occasion to give an introduction into the often utilized concept of antiparallels wrt a side of a triangle:

Let ABC be a triangle, and g a line in the plane. The line g is said to be **antiparallel** to BC wrt the triangle ABC if and only if it is parallel to the tangent to the circumcircle of triangle ABC at the point A. In this case, one also uses to say that the line g is an **antiparallel** to the side BC of triangle ABC.

Similarly, we define when a line is called antiparallel to CA or to AB wrt triangle ABC, or, in other words, when it is an antiparallel to the side CA or to the side AB of triangle ABC.

An important remark about this notion of antiparallelism is that the meaning of the word "antiparallel", in contrast to the meaning of the word "parallel", depends on the triangle ABC: Just to say that two lines are antiparallel to each other doesn't make sense; one can only say that a line is antiparallel to a side of a triangle wrt this triangle. Accordingly, in the formulation "the line g is an antiparallel to the side BC of triangle ABC ", one cannot omit the clause "of triangle ABC ", since it specifies the triangle of reference for the notion "antiparallel".

Clearly, through any point there exists exactly one antiparallel to the side BC of triangle ABC (in fact, by its definition, an antiparallel to the side BC of triangle ABC means a parallel to the tangent to the circumcircle of triangle ABC at the point A, and through any point there exists exactly one parallel to this tangent). Similarly, through any point there exists exactly one antiparallel to the side CA of triangle ABC and exactly one antiparallel to the side AB of triangle ABC.

The basic advantage of the notion of antiparallelism are the many equivalent criteria for a line to be antiparallel to a side of a triangle. Some of these criteria are given by the following theorem[3]:

Theorem 17. Let ABC be a triangle, and let g be a line in the plane. Consider the following ten assertions:

Assertion $\mathcal{E}_0$: The line g is antiparallel to BC wrt triangle ABC.

Assertion $\mathcal{E}_1$: The line g is parallel to the tangent to the circumcircle of triangle ABC at the point A.

Assertion $\mathcal{E}_2$: The line g is perpendicular to the line AO, where O is the circumcenter of triangle ABC.

Assertion $\mathcal{E}_3$: We have $\measuredangle(CA;\ g) = \measuredangle(BC;\ AB)$.

Assertion $\mathcal{E}_4$: We have $\measuredangle(AB;\ g) = \measuredangle(BC;\ CA)$.

Assertion $\mathcal{E}_5$: The line g is parallel to the line B_HC_H, where B_H and C_H are the feet of the B-altitude and the C-altitude of triangle ABC. [4]

Assertion $\mathcal{E}_6$: If P and Q are the points of intersection of the line g with the lines CA and AB, then the triangles APQ and ABC are oppositely similar.

Assertion $\mathcal{E}_7$: If P and Q are the points of intersection of the line g with the lines CA and AB, then there exists a circle which meets the line CA at the points C and P and meets the line AB at the points B and Q. [5]

Assertion $\mathcal{E}_8$: If P and Q are the points of intersection of the line g with the lines CA and AB, then $AB \cdot AQ = AC \cdot AP$, where the segments are directed.

Assertion $\mathcal{E}_9$: If P and Q are the points of intersection of the line g with the lines CA and AB, then the midpoint of the segment PQ lies on the A-symmedian of triangle

[3] Of course, the points P and Q from Theorem 17 have nothing to do with the points P and Q from the above results on isogonal conjugates.

[4] If triangle ABC is right-angled at A, then these feet B_H and C_H both coincide with the vertex A; in this case, the line B_HC_H is to be understood as the tangent to the circumcircle of triangle ABC at the point A.

[5] Hereby, the following convention applies:

If a circle k touches a line g at a point T, then we say that the circle k meets the line g at the points T and T.

This convention clarifies how the Assertion $\mathcal{E}_7$ is to be understood if e. g. the point P coincides with the point C. In this case, Assertion $\mathcal{E}_7$ states that there exists a circle which meets the line CA at the points C and C (that is, touches the line CA at the point C) and meets the line AB at the points B and Q.

This is the reason why the formulation "there exists a circle which meets the line CA at the points C and P and meets the line AB at the points B and Q " is superior to the shorter formulation "the points B, C, P, Q lie on one circle". In fact, if $P \neq C$ and $Q \neq B$, these formulations are equivalent, but e. g. in the case when the point P coincides with the point C, the points B, C, P, Q always lie on one circle; hence, if we had used the formulation "the points B, C, P, Q lie on one circle" for Assertion $\mathcal{E}_7$, then this Assertion $\mathcal{E}_7$ would not be equivalent to $\mathcal{E}_0$ in the case $P = C$ (and similarly in the case $Q = B$ as well).

ABC.

Then, we have:

a) The six assertions $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$ and $\mathcal{E}_5$ are pairwisely equivalent.

b) If the line g doesn't pass through the point A, then the ten assertions $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$, $\mathcal{E}_5$, $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$ and $\mathcal{E}_9$ are pairwisely equivalent.

(See Fig. 34 for Assertions $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$, Fig. 36 for Assertion $\mathcal{E}_5$, Fig. 37 for Assertions $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$, and Fig. 38 for Assertion $\mathcal{E}_9$.)

Theorem 17 yields, altogether, nine criteria for a line to be antiparallel to BC wrt triangle ABC. Similar criteria hold for antiparallelism to CA or to AB.

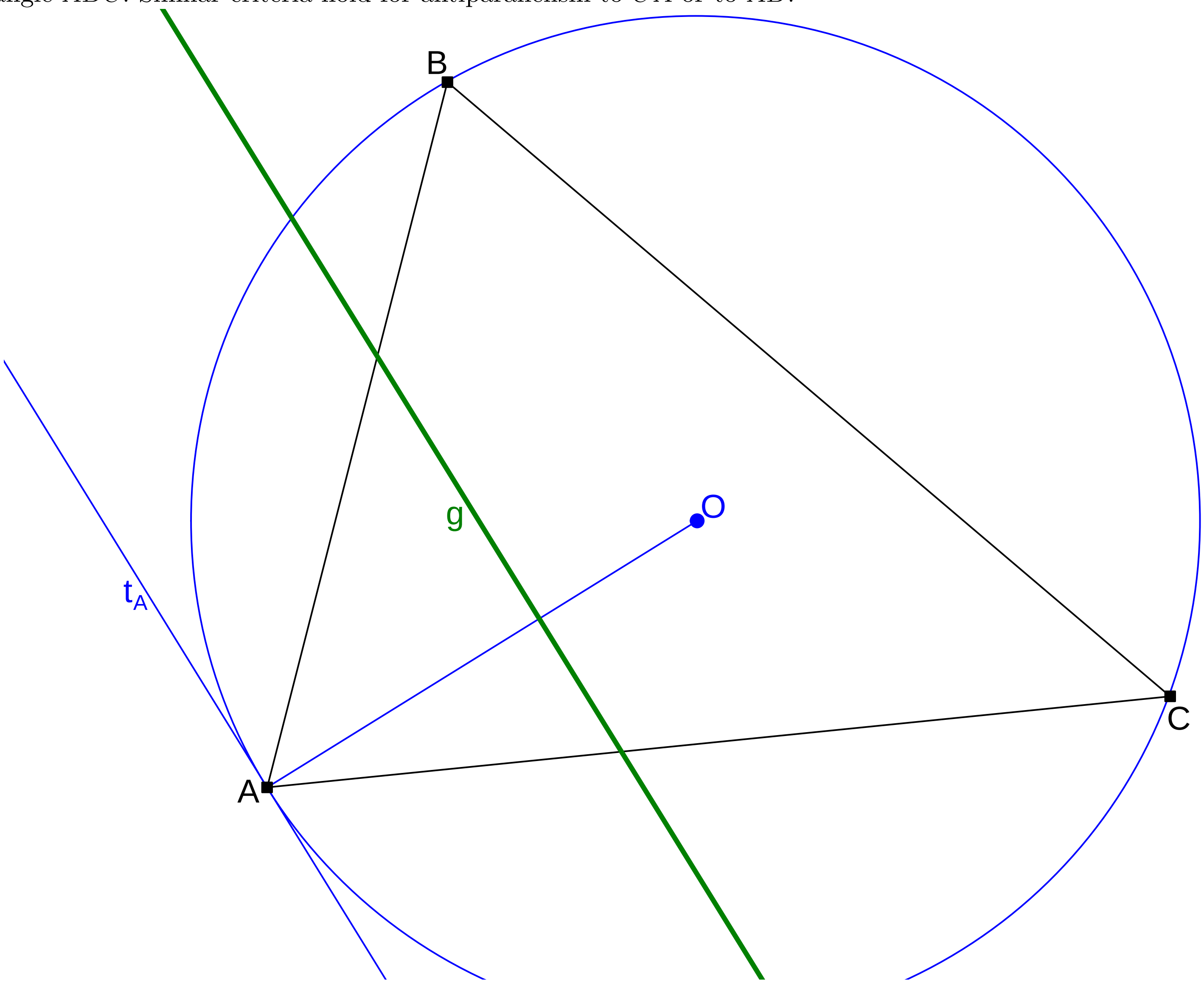

Fig. 34

Our proof of Theorem 17 will require a fact which slightly extends the intersecting chords theorem, the intersecting secants theorem, and the intersecting secant and tangent theorem:

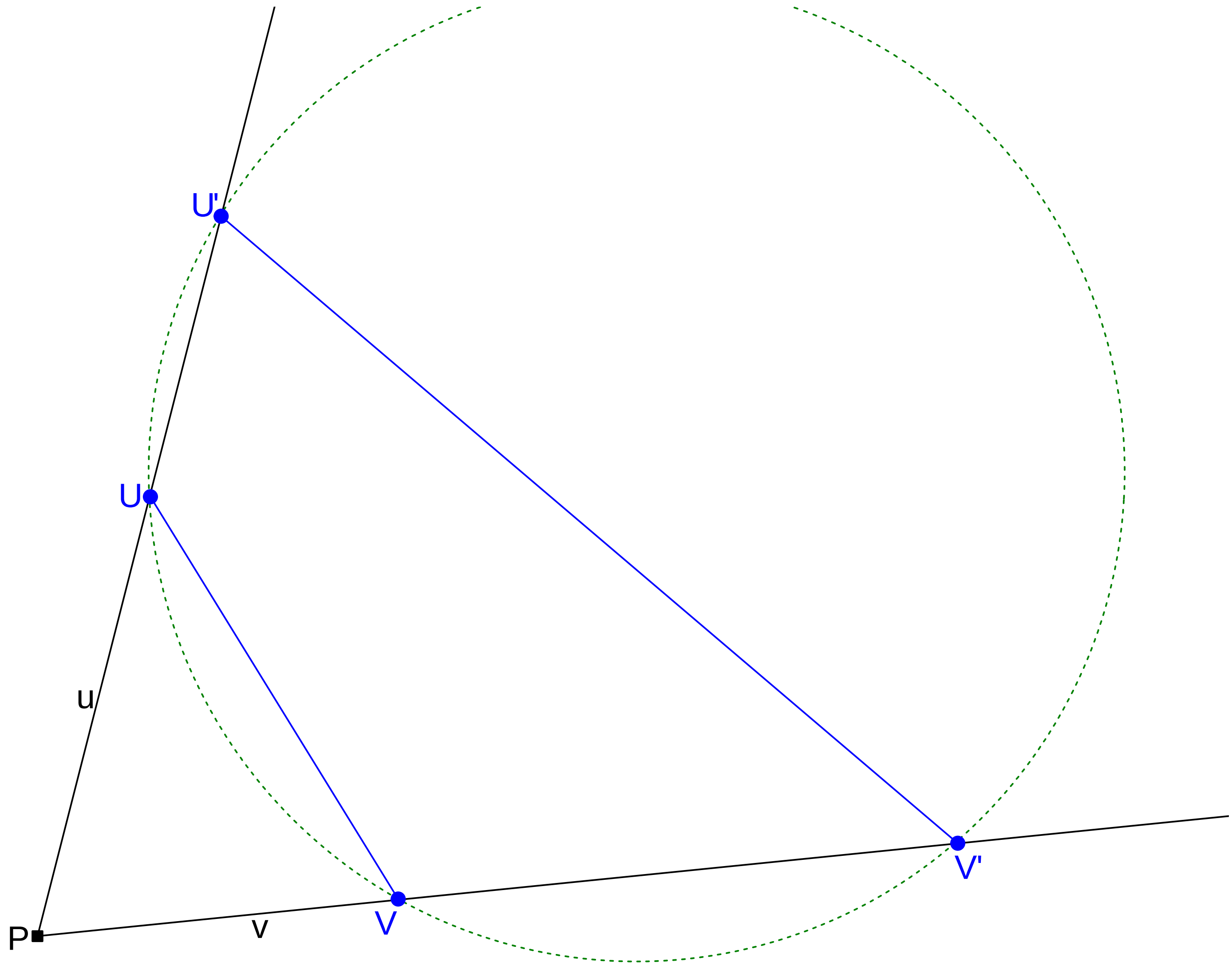

Fig. 35

Theorem 18. Let u and v be two lines which intersect at an Euclidean point P. Let U and U' be two points on the line u distinct from P, and let V and V' be two points on the line v distinct from P. Then, the following three assertions $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are pairwisely equivalent:

Assertion $\mathcal{D}_1$: The triangles PUV and $PV'U'$ are oppositely similar.

Assertion $\mathcal{D}_2$: There exists a circle which meets the line u at the points U and U' and meets the line v at the points V and V'. [6]

Assertion $\mathcal{D}_3$: We have $PU \cdot PU' = PV \cdot PV'$, where the segments are directed.

(See Fig. 35.)

Proof of Theorem 18.[7] We consider only the case when $U \neq U'$ and $V \neq V'$. The

[6] Hereby, the same convention applies as in Assertion $\mathcal{E}_7$ of Theorem 17.

[7] This proof is given here but for the sake of consequence in our application of directed angles modulo 180°. In fact, the equivalence of Assertions $\mathcal{D}_1$ and $\mathcal{D}_3$ is a trivial corollary of a well-known similitude criterion which states that two triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are oppositely similar if and only if $\angle P_1P_2P_3 = -\angle Q_1Q_2Q_3$ und $P_1P_2 : P_2P_3 = Q_1Q_2 : Q_2Q_3$. But here, the sign $\angle$ stands for directed angles modulo 360°; such a similitude criterion cannot hold for directed angles modulo 180°. As we aim at using directed angles modulo 180° throughout this paper, we will prove this equivalence in a different way.

cases $U = U'$ and $V = V'$ can be handled in the same way, with the only difference that here and there instead of chordal angles, we have angles between a chord and a tangent.

First we show the equivalence of the Assertions $\mathcal{D}_1$ and $\mathcal{D}_2$:

Assertion $\mathcal{D}_1$ states that triangles PUV and $PV'U'$ are oppositely similar. This is equivalent to $\measuredangle PUV = -\measuredangle PV'U'$ (in fact, if triangles PUV and $PV'U'$ are oppositely similar, then $\measuredangle PUV = -\measuredangle PV'U'$, and conversely, if $\measuredangle PUV = -\measuredangle PV'U'$, then, together with $\measuredangle UPV = -\measuredangle V'PU'$, this yields the opposite similarity of triangles PUV and $PV'U'$). But $\measuredangle PUV = -\measuredangle PV'U'$ rewrites as $\measuredangle U'UV = \measuredangle U'V'V$, and this equation holds if and only if the points U, V, U' and V' lie on one circle. Since $U \neq U'$ and $V \neq V'$, the points U, V, U' and V' lie on one circle if and only if there exists a circle which meets the line u at the points U and U' and meets the line v at the points V and V'. The latter is Assertion $\mathcal{D}_2$. Combining these steps, we realize that we have proven the equivalence of Assertions $\mathcal{D}_1$ and $\mathcal{D}_2$.

Now we are going to prove the equivalence of Assertions $\mathcal{D}_2$ and $\mathcal{D}_3$:

If Assertion $\mathcal{D}_2$ holds, then there exists a circle which meets the line u at the points U and U' and meets the line v at the points V and V'. The power of the point P wrt this circle equals $PU \cdot PU'$ on the one hand, and equals $PV \cdot PV'$ on the other hand. Thus, $PU \cdot PU' = PV \cdot PV'$, so that Assertion $\mathcal{D}_3$ is valid.

Conversely: If we assume Assertion $\mathcal{D}_3$ to hold, then $PU \cdot PU' = PV \cdot PV'$. Now, let V_1' be the point of intersection of the circle through the points U, U' and V with the line v different from V. [8] Then, this circle meets the line u at the points U and U' and meets the line v at the points V and V_1'. Consequently, the power of the point P wrt this circle equals $PU \cdot PU'$ on the one hand, and equals $PV \cdot PV_1'$ on the other hand. Thus, $PU \cdot PU' = PV \cdot PV_1'$. Comparison with $PU \cdot PU' = PV \cdot PV'$ yields $PV \cdot PV_1' = PV \cdot PV'$, thus $PV_1' = PV'$ (since $PV \neq 0$). Since the points V_1' and V' both lie on the line v and since we use directed segments, this entails that the points V_1' and V' coincide. The fact that the circle through the points U, U' and V meets the line u at the points U and U' and meets the line v at the points V and V_1' can now be rewritten as follows: The circle through the points U, U' and V meets the line u at the points U and U' and meets the line v at the points V and V'. Thus, Assertion $\mathcal{D}_2$ is fulfilled.

Hence, we have shown the equivalence of Assertions $\mathcal{D}_2$ and $\mathcal{D}_3$. Altogether, we subsume that all three Assertions $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are equivalent, and Theorem 18 is proven.

Proof of Theorem 17. **a)** The equivalence of Assertions $\mathcal{E}_0$ and $\mathcal{E}_1$ is a paraphrase of the definition of antiparallelism.

In order to show the equivalence of Assertions $\mathcal{E}_1$ and $\mathcal{E}_2$, it is enough to show that the tangent to the circumcircle of triangle ABC at the point A is perpendicular to the line AO. But this is clear, since O is the center of this circumcircle.

[8] If this circle happens to touch the line v, then we set $V_1' = V$.

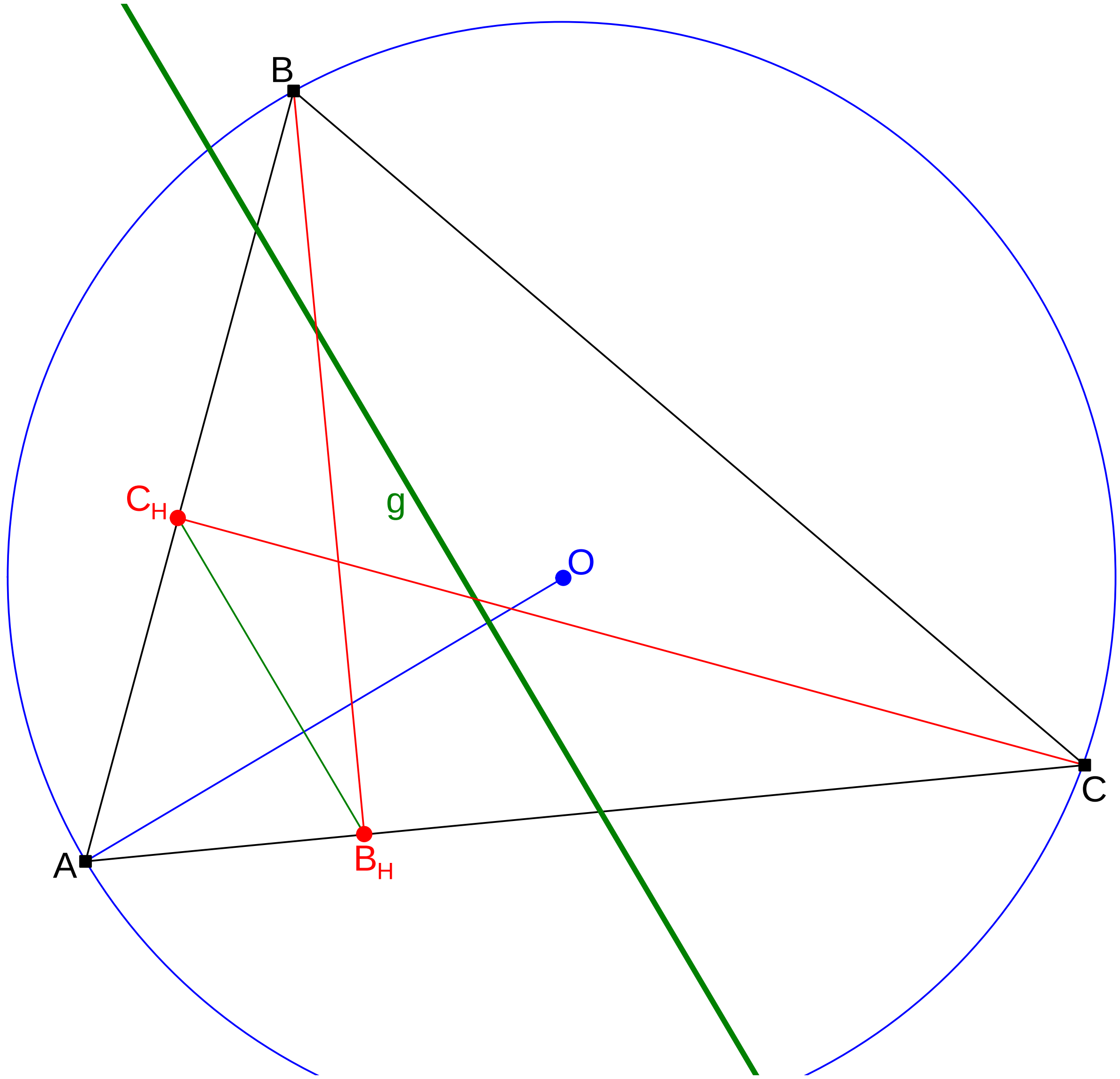

Fig. 36

The equivalence of Assertions $\mathcal{E}_1$ and $\mathcal{E}_3$ can be proven as follows: (See Fig. 35.) Let t_A be the tangent to the circumcircle of triangle ABC at the point A. Then, Assertion $\mathcal{E}_1$ states that $g \parallel t_A$. This is equivalent to $\measuredangle(CA;\ g) = \measuredangle(CA;\ t_A)$. But since t_A is the tangent to the circumcircle of triangle ABC at the point A, while $\measuredangle CBA$ is the chordal angle of the chord CA in this circumcircle, we have $\measuredangle(CA;\ t_A) = \measuredangle CBA$, what rewrites as $\measuredangle(CA;\ t_A) = \measuredangle(BC;\ AB)$. Hence, the equation $\measuredangle(CA;\ g) = \measuredangle(CA;\ t_A)$ is equivalent to the equation $\measuredangle(CA;\ g) = \measuredangle(BC;\ AB)$. But this equation is Assertion $\mathcal{E}_3$. Hence it is shown that Assertion $\mathcal{E}_1$ is equivalent to Assertion $\mathcal{E}_3$.

An analogous argument shows the equivalence of the Assertions $\mathcal{E}_1$ and $\mathcal{E}_4$.

In order to prove the equivalence of the Assertions $\mathcal{E}_2$ and $\mathcal{E}_5$, it is obviously sufficient to verify that $AO \perp B_HC_H$. But this follows from Theorem 16.[9]

Altogether, we have proven all six Assertions $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$ and $\mathcal{E}_5$ to be equivalent. Thus, the proof of Theorem 17 **a)** is complete.

[9] The relation $AO \perp B_HC_H$ also holds in the case when triangle ABC is right-angled at A. In fact, in this case we have specified that the line B_HC_H is the tangent to the circumcircle of triangle ABC at the point A, and this tangent is perpendicular to the line AO (as we already saw).

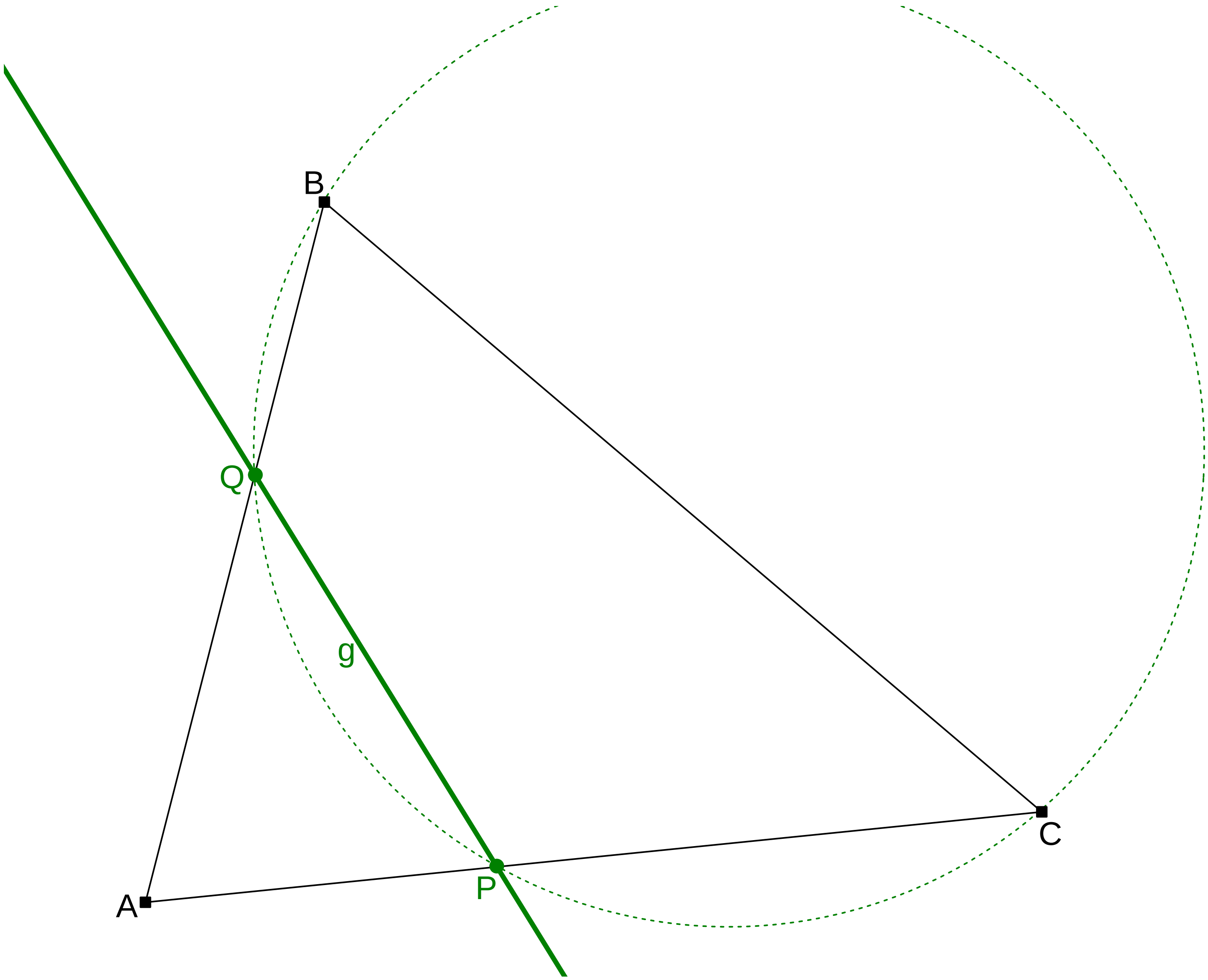

Fig. 37

b) As Theorem 17 **a)** is already demonstrated, we know that the Assertions $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$ and $\mathcal{E}_5$ are all equivalent. It just remains to prove the equivalence of the Assertions $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$ and $\mathcal{E}_9$ to these Assertions in the case when the line g doesn't pass through the point A.

First we establish the equivalence of Assertions $\mathcal{E}_3$ and $\mathcal{E}_6$:

If Assertion $\mathcal{E}_3$ holds, then $\measuredangle(CA;\ g) = \measuredangle(BC;\ AB)$; this rewrites as $\measuredangle APQ = -\measuredangle ABC$. On the other hand, $\measuredangle PAQ = -\measuredangle BAC$. Hence, the triangles APQ and ABC are oppositely similar, so that Assertion $\mathcal{E}_6$ is valid.

Conversely: If Assertion $\mathcal{E}_6$ holds, then triangles APQ and ABC are oppositely similar, what yields $\measuredangle APQ = -\measuredangle ABC$. In other words, $\measuredangle(CA;\ g) = \measuredangle(BC;\ AB)$. Hence, Assertion $\mathcal{E}_3$ is valid.

Thus we have shown the equivalence of the Assertions $\mathcal{E}_3$ and $\mathcal{E}_6$.

The equivalence of the Assertions $\mathcal{E}_6$, $\mathcal{E}_7$ and $\mathcal{E}_8$ follows from Theorem 18, applied to the two lines CA and AB which intersect at the Euclidean point A, the two points P and C on the line CA distinct from A, and the two points Q and B on the line AB distinct from A.

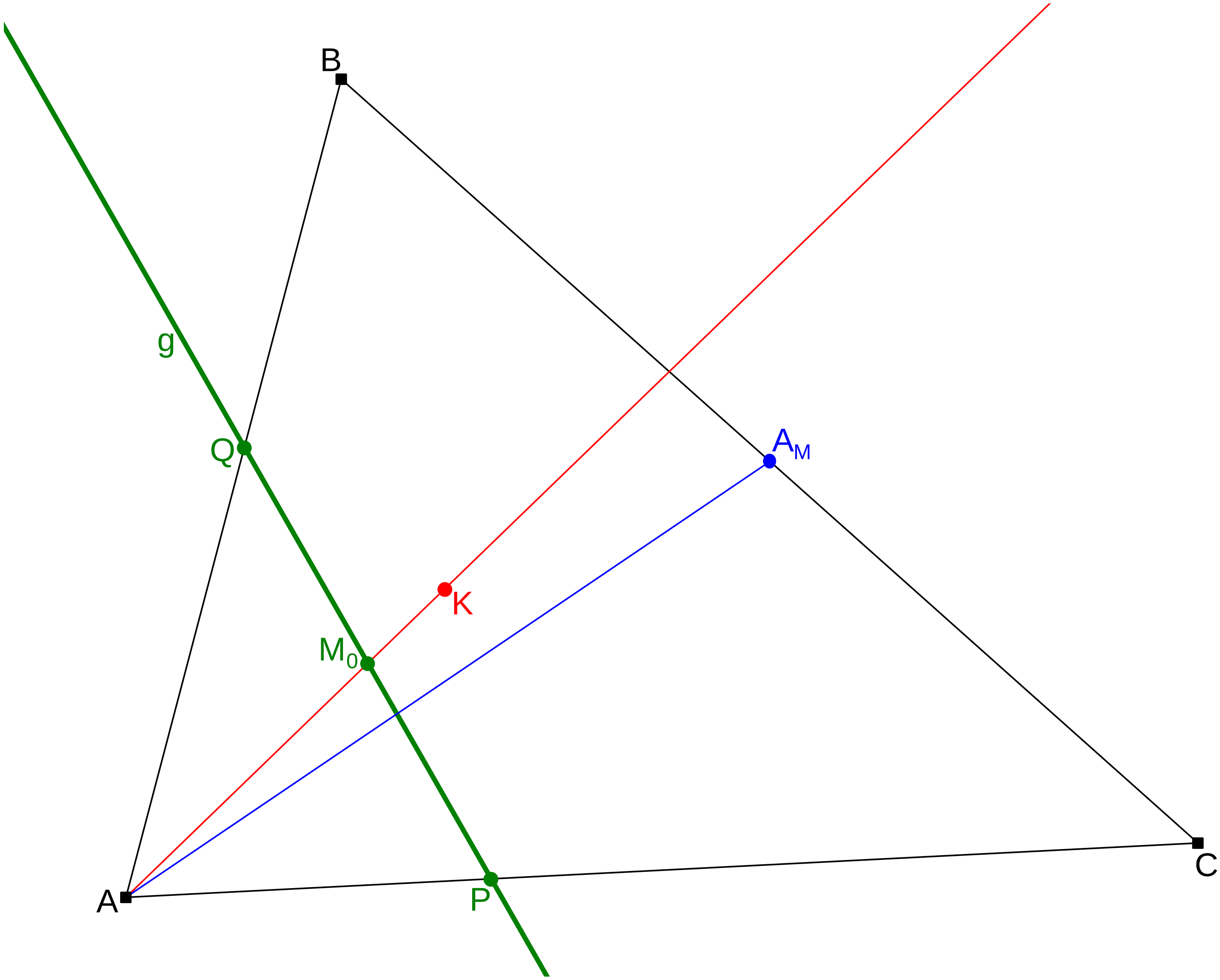

Fig. 38

Altogether, we now know that the nine Assertions $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$, $\mathcal{E}_5$, $\mathcal{E}_6$, $\mathcal{E}_7$ and $\mathcal{E}_8$ are all pairwisely equivalent. In order to prove the equivalence of all ten Assertions $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$, $\mathcal{E}_5$, $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$ and $\mathcal{E}_9$, it therefore suffices to prove the equivalence of the Assertions $\mathcal{E}_0$ and $\mathcal{E}_9$. This proof is obtained by showing the following two auxiliary results:

Auxiliary result 1. If Assertion $\mathcal{E}_0$ holds, then Assertion $\mathcal{E}_9$ holds.

Auxiliary result 2. If Assertion $\mathcal{E}_9$ holds, then Assertion $\mathcal{E}_0$ holds.

Proof of Auxiliary result 1. (See Fig. 38.) Assume that Assertion $\mathcal{E}_0$ holds. As we already know that Assertion $\mathcal{E}_0$ is equivalent to Assertion $\mathcal{E}_6$, it thus follows that Assertion $\mathcal{E}_6$ holds as well. That is, the triangles APQ and ABC are oppositely similar. In oppositely similar triangles, corresponding points form oppositely equal angles. Let M_0 be the midpoint of the segment PQ, and A_M the midpoint of the segment BC. Then, the points M_0 and A_M are corresponding points in the oppositely similar triangles APQ and ABC (being the midpoints of their respective sides PQ and BC). Hence, they form oppositely equal angles; particularly, $\measuredangle PAM_0 = -\measuredangle BAA_M$. In other words, $\measuredangle (CA;\ AM_0) = -\measuredangle (AB;\ AA_M)$. Hence, the line AM_0 is the isogonal of the line AA_M

wrt the angle CAB. But since A_M is the midpoint of the segment BC, the line AA_M is the A-median of triangle ABC. Therefore, the line AM_0 is the isogonal of the A-median of triangle ABC wrt the angle CAB, hence the A-symmedian of triangle ABC. Consequently, the point M_0 lies on the A-symmedian of triangle ABC. Since M_0 is the midpoint of the segment PQ, this is exactly Assertion $\mathcal{E}_9$. Thus, Auxiliary result 1 is proven.

Proof of Auxiliary result 2. (See Fig. 39. This figure is intentionally drawn wrong in order not to tempt to unfounded conclusions.) Assume that Assertion $\mathcal{E}_9$ is valid. In other words, the midpoint M_0 of the segment PQ lies on the A-symmedian of triangle ABC.

Let the antiparallel to the side BC of triangle ABC through the point M_0 intersect the lines CA and AB at the points P' and Q'. Since the line $P'Q'$ is antiparallel to BC wrt triangle ABC, it fulfills Assertion $\mathcal{E}_0$ of Theorem 17. As we have already shown that Assertion $\mathcal{E}_0$ implies Assertion $\mathcal{E}_9$ (this was our Auxiliary result 1), we thus conclude that this line $P'Q'$ fulfills Assertion $\mathcal{E}_9$. That is: The midpoint of the segment $P'Q'$ lies on the A-symmedian of triangle ABC.

Hence, the midpoint of the segment $P'Q'$ is the point of intersection of the line $P'Q'$ with the A-symmedian of triangle ABC. But this point of intersection is the point M_0. Therefore, the midpoint of the segment $P'Q'$ must be the point M_0.

Since M_0 is the midpoint of the segment PQ, the reflection with respect to the point M_0 maps the point P to the point Q. Since M_0 is the midpoint of the segment $P'Q'$, the reflection with respect to the point M_0 maps the point P' to the point Q'. Now, if the points P and P' were distinct, then the points Q and Q' would therefore be distinct as well (since a reflection maps distinct points to distinct points), and we would have $QQ' \parallel PP'$ (since a reflection maps a line to a parallel line); but this cannot be true, since the lines QQ' and PP' are the lines AB and CA, and we have $AB \nparallel CA$. Hence, the points P and P' cannot be distinct, i. e. we must have $P = P'$. Similarly, $Q = Q'$. As we know that the line $P'Q'$ is antiparallel to BC wrt triangle ABC, we can thus conclude that the line PQ is antiparallel to BC wrt triangle ABC. This means that Assertion $\mathcal{E}_0$ holds. Thus, Auxiliary result 2 is proven.

This completes the proof of Theorem 17.

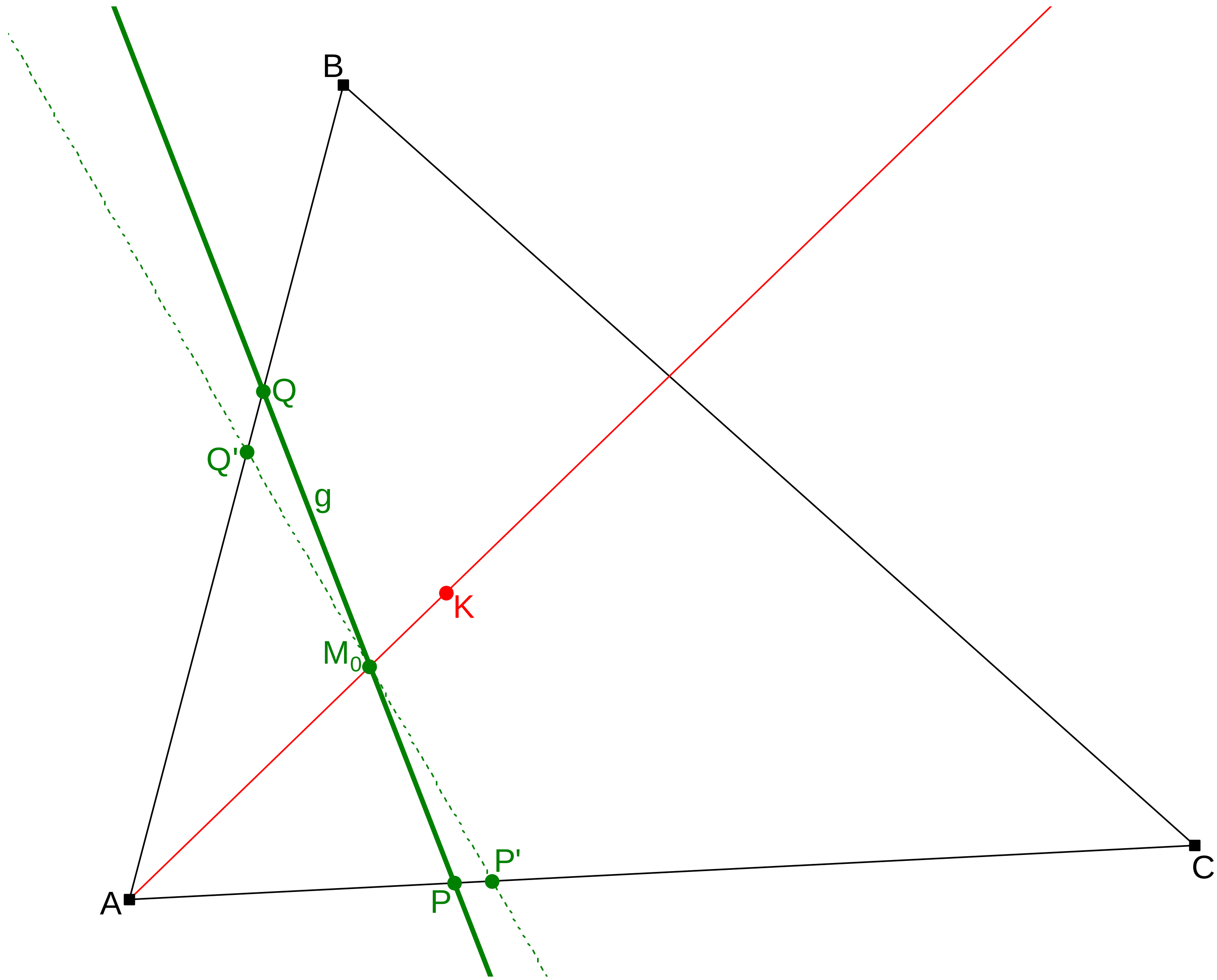

Fig. 39

The notion of antiparallelism is mostly applied in the theory of Tucker circles, but can also often be used to simplify some standard ways of conclusion in elementary geometry.

10. Isogonal conjugates of Kiepert points

A means for identification of isogonal conjugates is the following fact (Fig. 40):

Theorem 19. Let ABC be a triangle, and let P and Q be two points on the perpendicular bisector of the segment BC.

a) The following five assertions $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ and $\mathcal{F}_5$ are pairwisely equivalent:

Assertion $\mathcal{F}_1$: We have $\measuredangle ABP = \measuredangle ACQ$.

Assertion $\mathcal{F}_2$: We have $\measuredangle ACP = \measuredangle ABQ$.

Assertion $\mathcal{F}_3$: We have $\measuredangle BCP + \measuredangle BCQ = \measuredangle BAC$.

Assertion $\mathcal{F}_4$: We have $\measuredangle PBC + \measuredangle QBC = \measuredangle BAC$.

Assertion $\mathcal{F}_5$: The points P and Q are inverse to each other wrt the circumcircle of triangle ABC.

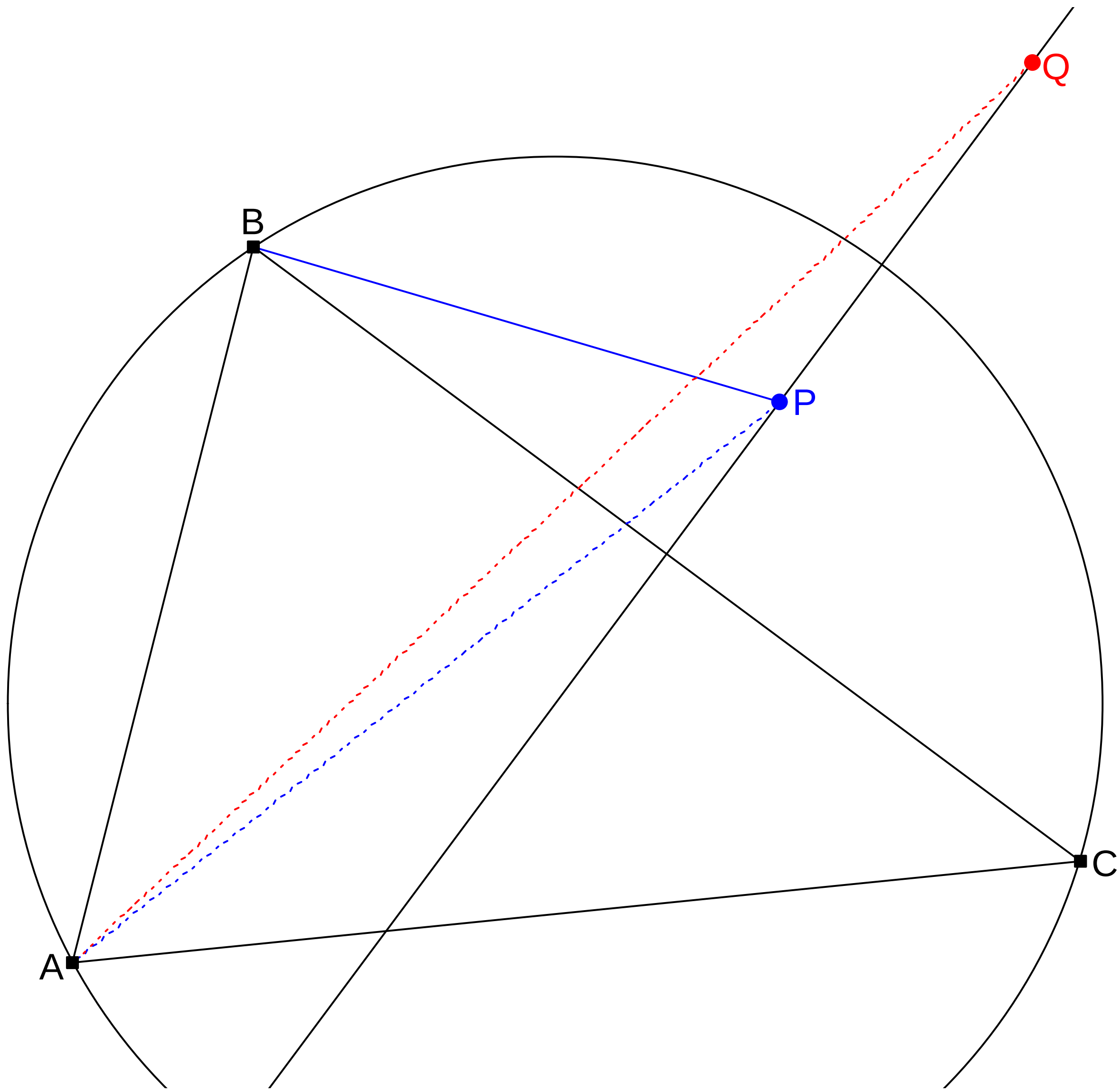

Fig. 40

b) If one of the five Assertions $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ and $\mathcal{F}_5$ holds, then the lines AP and AQ are isogonal to each other wrt the angle CAB.

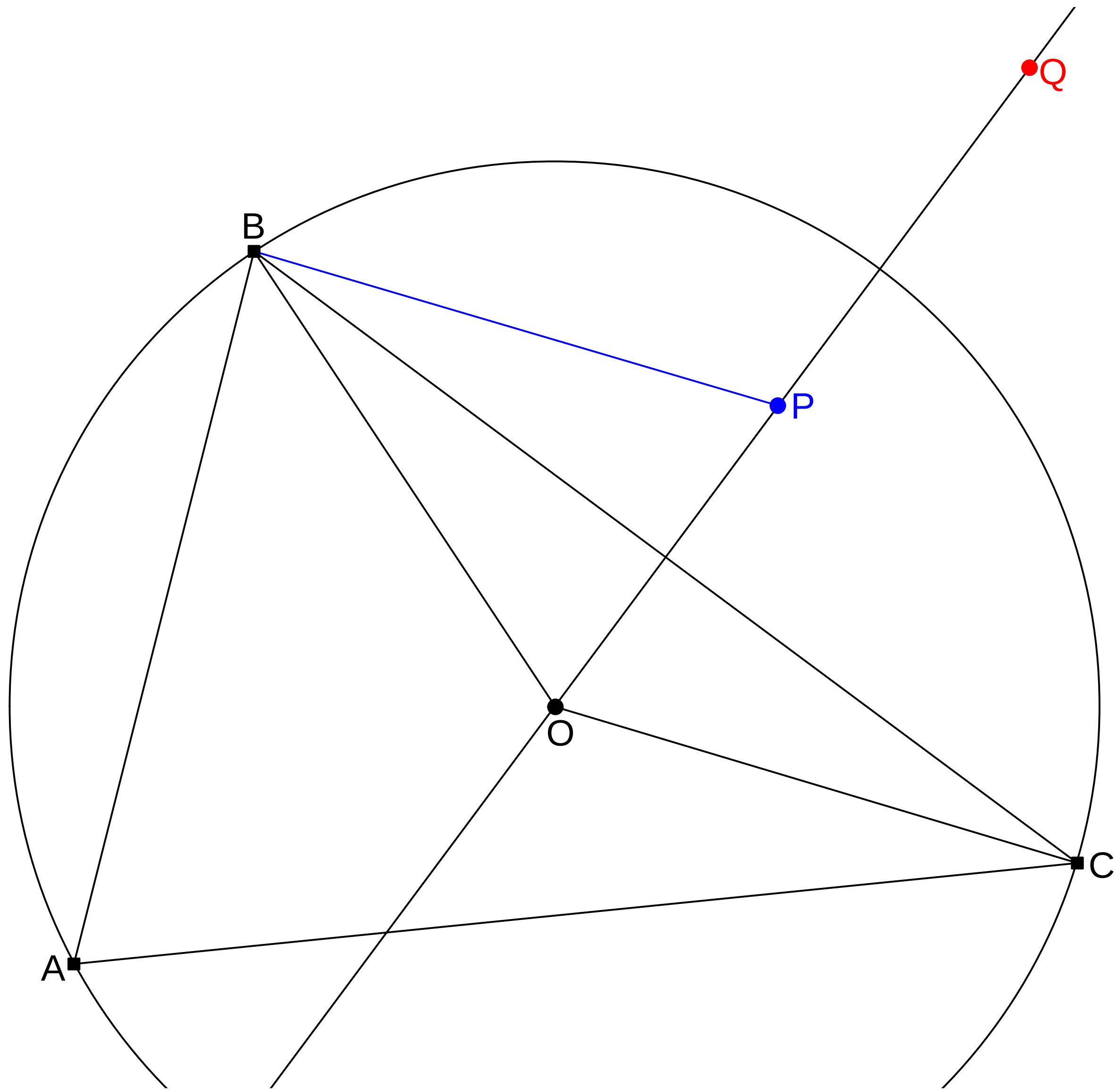

Fig. 41

Proof of Theorem 19. **a)** (See Fig. 41.) Let O be the center of the circumcircle of triangle ABC. Then, O is the center of a circle through the points A, B, C; hence, by the central angle theorem, $\measuredangle ABO = 90^\circ - \measuredangle BCA$.

The circumcenter O of triangle ABC obviously lies on the perpendicular bisector of its side BC. On the other hand, we know that the points P and Q lie on this perpendicular bisector. Hence, the points O, P, Q lie on one line, namely on the perpendicular bisector of the segment BC. Thus, $OQ \perp BC$, so that $\measuredangle (OQ;\ BC) = 90^\circ$. Hence,

$$\measuredangle BQO = \measuredangle (BQ;\ OQ) = \measuredangle (BQ;\ BC) - \measuredangle (OQ;\ BC) = \measuredangle QBC - 90^\circ,$$

thus $\measuredangle QBC = \measuredangle BQO + 90^\circ$. Since the point Q lies on the perpendicular bisector of the segment BC, we have $BQ = CQ$; this means that triangle BQC is isosceles, and satisfies $\measuredangle QBC = \measuredangle BCQ$. Hence, $\measuredangle QBC = \measuredangle BQO + 90^\circ$ becomes $\measuredangle BCQ =$

$\measuredangle BQO + 90°$. Consequently,

$$\begin{aligned}\measuredangle ACQ &= \measuredangle BCQ - \measuredangle BCA = (\measuredangle BQO + 90°) - \measuredangle BCA \\ &= (90° - \measuredangle BCA) + \measuredangle BQO = \measuredangle ABO + \measuredangle BQO = \measuredangle ABO - \measuredangle OQB.\end{aligned}$$

Consider the two lines OP and OB which intersect at the Euclidean point O; further, consider the two points P and Q on the line OP distinct from O, and the two points B and B on the line OB distinct from O. We can apply Theorem 18 to these lines and points. Assertion $\mathcal{D}_1$ of Theorem 18 then states that the triangles OPB and OBQ are oppositely similar, and Assertion $\mathcal{D}_3$ states that $OP \cdot OQ = OB \cdot OB$ (where the segments are directed). As Theorem 18 ensures that the Assertions $\mathcal{D}_1$ and $\mathcal{D}_3$ are equivalent, we thus get:

Auxiliary result 1. The triangles OPB and OBQ are oppositely similar if and only if $OP \cdot OQ = OB \cdot OB$.

Now we can show the equivalence of the Assertions $\mathcal{F}_1$ and $\mathcal{F}_5$:

If Assertion $\mathcal{F}_1$ holds, then $\measuredangle ABP = \measuredangle ACQ$. Since $\measuredangle ABP = \measuredangle ABO + \measuredangle OBP$ and $\measuredangle ACQ = \measuredangle ABO - \measuredangle OQB$, this becomes $\measuredangle ABO + \measuredangle OBP = \measuredangle ABO - \measuredangle OQB$, so that $\measuredangle OBP = -\measuredangle OQB$. Furthermore, it is evident that $\measuredangle BOP = -\measuredangle QOB$. Hence, the triangles OPB and OBQ are oppositely similar. According to Auxiliary result 1, this entails $OP \cdot OQ = OB \cdot OB$, that is, $OP \cdot OQ = OB^2$. Now, we know that the points O, P, Q lie on one line, and that O is the center and OB is the radius of the circumcircle of triangle ABC. Hence, the equation $OP \cdot OQ = OB^2$ signifies that the points P and Q are inverse to each other wrt the circumcircle of triangle ABC. Thus, Assertion $\mathcal{F}_5$ must hold.

Conversely: Assume that Assertion $\mathcal{F}_5$ holds. This means that the points P and Q are inverse to each other wrt the circumcircle of triangle ABC. Since O is the center and OB is the radius of this circumcircle, this yields $OP \cdot OQ = OB^2$. In other words, $OP \cdot OQ = OB \cdot OB$. According to Auxiliary result 1, we thus conclude that the triangles OPB and OBQ are oppositely similar, so that $\measuredangle OBP = -\measuredangle OQB$. Hence, $\measuredangle ABP = \measuredangle ABO + \measuredangle OBP = \measuredangle ABO - \measuredangle OQB = \measuredangle ACQ$, and thus Assertion $\mathcal{F}_1$ is valid.

Thus we have shown that Assertions $\mathcal{F}_1$ and $\mathcal{F}_5$ are equivalent. In a similar way we can prove that Assertions $\mathcal{F}_2$ and $\mathcal{F}_5$ are equivalent.

As seen above, $\measuredangle QBC = \measuredangle BCQ$. Thus,

$$\measuredangle (BQ;\ CP) = \measuredangle (BC;\ CP) + \measuredangle (BQ;\ BC) = \measuredangle BCP + \measuredangle QBC = \measuredangle BCP + \measuredangle BCQ,$$

so that

$$\begin{aligned}\measuredangle ACP - \measuredangle ABQ &= \measuredangle (AC;\ CP) - \measuredangle (AB;\ BQ) = (\measuredangle (AC;\ BQ) + \measuredangle (BQ;\ CP)) - \measuredangle (AB;\ BQ) \\ &= \measuredangle (BQ;\ CP) - (\measuredangle (AB;\ BQ) - \measuredangle (AC;\ BQ)) = \measuredangle (BQ;\ CP) - \measuredangle (AB;\ AC) \\ &= (\measuredangle BCP + \measuredangle BCQ) - \measuredangle BAC.\end{aligned}$$

Consequently, we have $\measuredangle ACP = \measuredangle ABQ$ if and only if $\measuredangle BCP + \measuredangle BCQ = \measuredangle BAC$. In other words, Assertion $\mathcal{F}_2$ is equivalent to Assertion $\mathcal{F}_3$.

We have $\measuredangle QBC = \measuredangle BCQ$ and similarly $\measuredangle PBC = \measuredangle BCP$. Thus, the equation $\measuredangle BCP + \measuredangle BCQ = \measuredangle BAC$ is equivalent to the equation $\measuredangle PBC + \measuredangle QBC = \measuredangle BAC$. In other words, Assertion $\mathcal{F}_3$ is equivalent to Assertion $\mathcal{F}_4$.

Altogether, we have proven the equivalence of all five Assertions $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ and $\mathcal{F}_5$; thus, Theorem 19 **a)** is verified.

b) Assume that one of the five Assertions $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ and $\mathcal{F}_5$ holds. Since, according to Theorem 19 **a)**, all these five Assertions are equivalent, we can therefore conclude that Assertion $\mathcal{F}_1$ holds, i. e. we have $\measuredangle ABP = \measuredangle ACQ$.

(See Fig. 42.) Since the point P lies on the perpendicular bisector of the segment BC, we have $PB = PC$. The circle with center P and radius $PB = PC$ passes through the points B and C; let B_P and C_P be the points of intersection of this circle with the lines AB and CA different from B and C. Then, $PB_P = PC_P = PB = PC$.

Since $PB_P = PB$, the triangle BPB_P is isosceles, so that $\measuredangle BB_PP = \measuredangle PBB_P$. Hence, $\measuredangle AB_PP = \measuredangle BB_PP = \measuredangle PBB_P = -\measuredangle ABP = -\measuredangle ACQ$.

On the other hand, the points B, C, B_P and C_P lie on one circle (namely, on the circle with center P and radius $PB = PC$); thus, $\measuredangle BB_PC_P = \measuredangle BCC_P$. This yields $\measuredangle AB_PC_P = \measuredangle BB_PC_P = \measuredangle BCC_P = -\measuredangle ACB$. Similarly, $\measuredangle AC_PB_P = -\measuredangle ABC$. Consequently, the triangles AC_PB_P and ABC are oppositely similar.

From $\measuredangle AB_PP = -\measuredangle ACQ$ and $\measuredangle AB_PC_P = -\measuredangle ACB$, it follows that

$$\measuredangle C_PB_PP = \measuredangle AB_PP - \measuredangle AB_PC_P = (-\measuredangle ACQ) - (-\measuredangle ACB) = -(\measuredangle ACQ - \measuredangle ACB) = -\measuredangle BCQ.$$

Similarly, $\measuredangle B_PC_PP = -\measuredangle CBQ$. Thus, the triangles PC_PB_P and QBC are oppositely similar.

Since triangle AC_PB_P is oppositely similar to triangle ABC, and triangle PC_PB_P is oppositely similar to triangle QBC, the quadrilateral AC_PPB_P formed by the triangles AC_PB_P and PC_PB_P is oppositely similar to the quadrilateral $ABQC$ formed by the triangles ABC and QBC. Consequently, $\measuredangle C_PAP = -\measuredangle BAQ$. In other words, $\measuredangle(CA;\ AP) = -\measuredangle(AB;\ AQ)$. Thus, the line AP is the isogonal of the line AQ wrt the angle CAB. This proves Theorem 19 **b)**, and thus concludes the proof of Theorem 19.

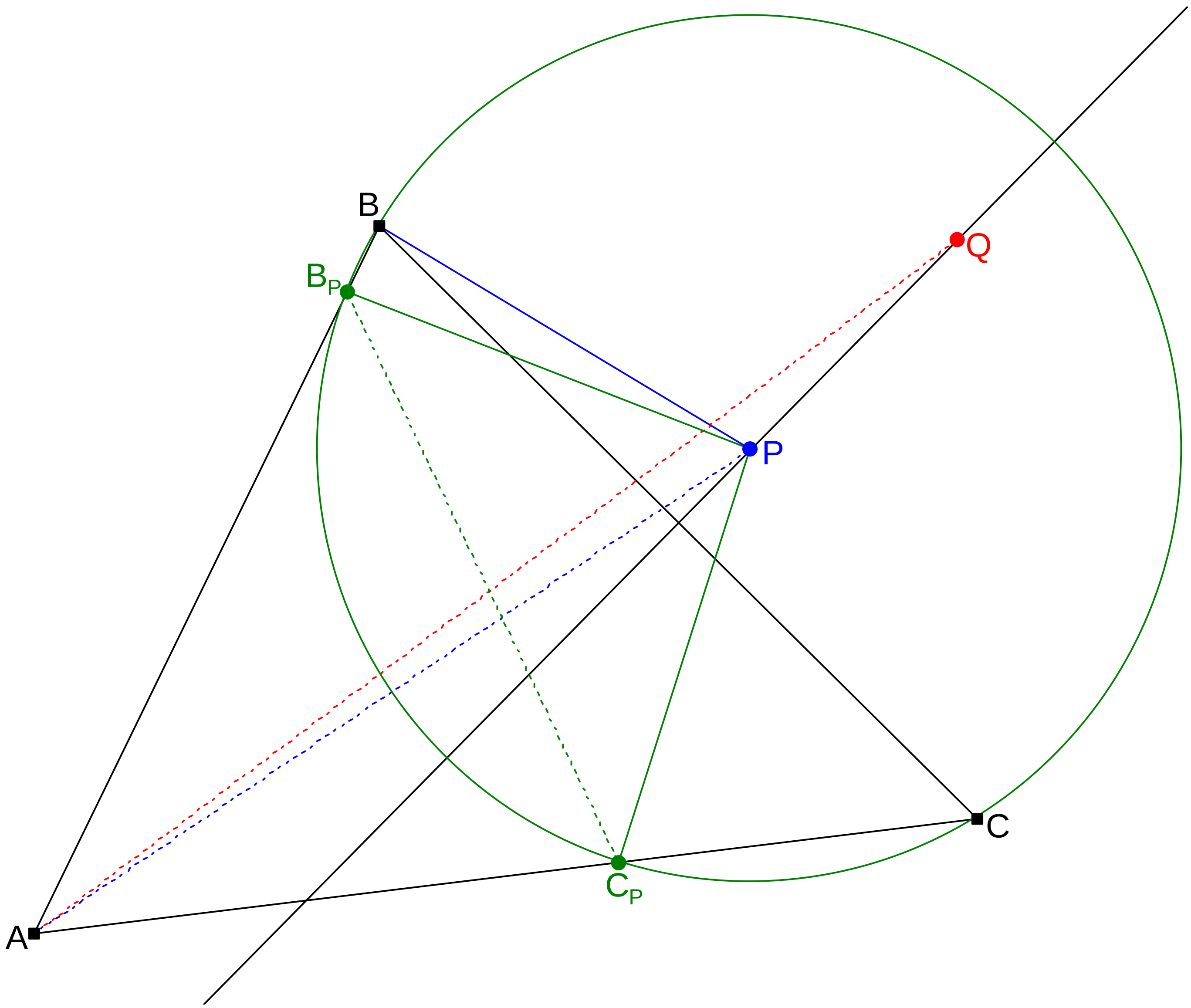

Fig. 42

Theorem 19 can be used to obtain a result about the isogonal conjugates of the so-called *Kiepert points* of the triangle. These points are defined as follows:

(See Fig. 43.) Let φ be an arbitrary angle. On the sides BC, CA, AB of triangle ABC, we erect isosceles triangles $BA_\varphi C$, $CB_\varphi A$, $AC_\varphi B$ with the bases BC, CA, AB and the equal base angle

$$\measuredangle BCA_\varphi = \measuredangle A_\varphi BC = \measuredangle CAB_\varphi = \measuredangle B_\varphi CA = \measuredangle ABC_\varphi = \measuredangle C_\varphi AB = \varphi.$$

The **Kiepert theorem** states that the lines AA_φ, BB_φ, CC_φ concur at one point. We denote this point by K_φ, and call it the φ**-Kiepert point** of triangle ABC. Occasionally, triangle $A_\varphi B_\varphi C_\varphi$ is referred to as the φ**-Kiepert triangle** of triangle ABC.

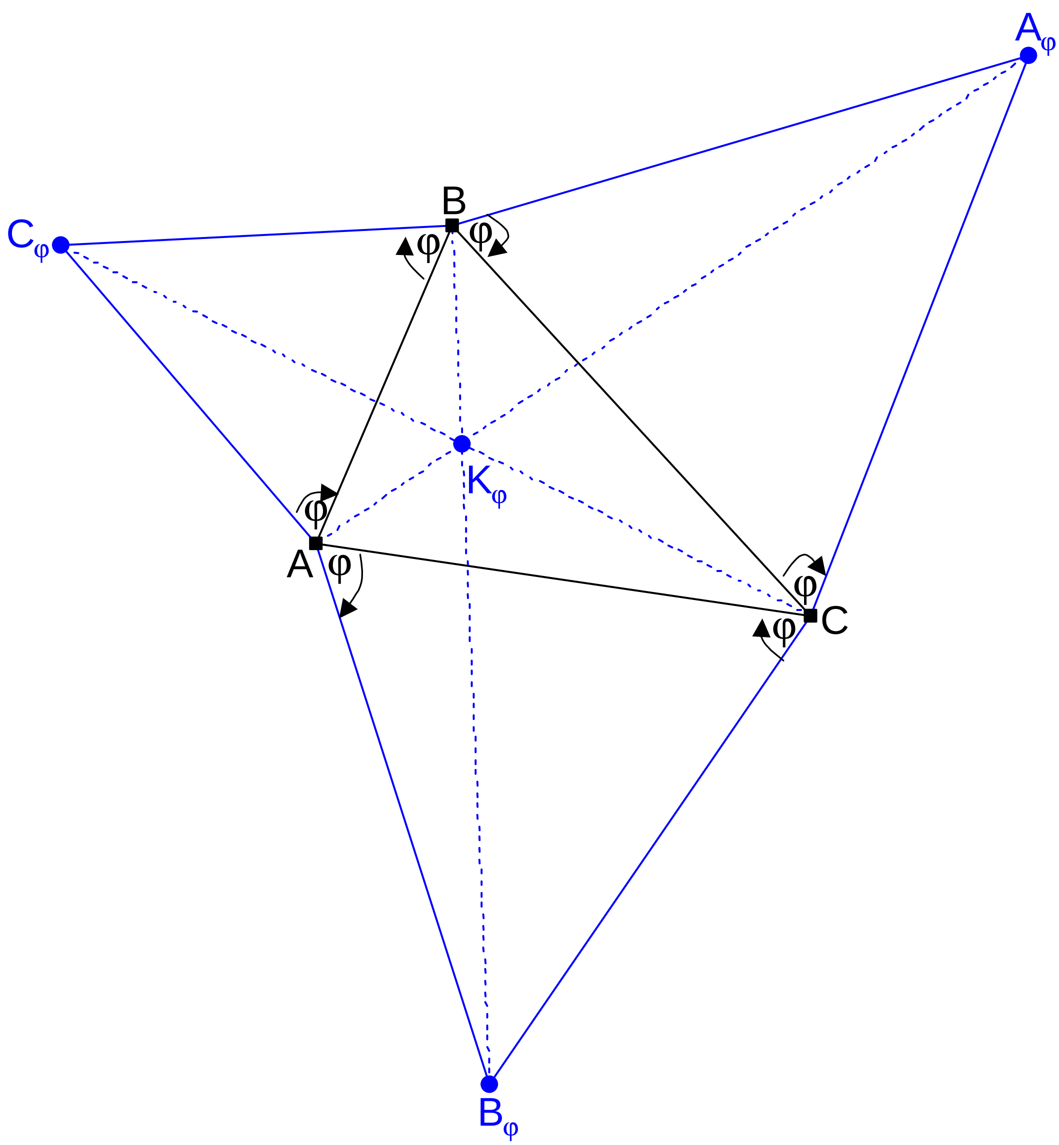

Fig. 43

(See Fig. 44.) Let A'_φ, B'_φ, C'_φ be the inverses of the points A_φ, B_φ, C_φ in the circumcircle of triangle ABC. Let O be the center of this circumcircle.

Since triangle $BA_\varphi C$ is isosceles with base BC, we have $BA_\varphi = CA_\varphi$; thus, the point A_φ lies on the perpendicular bisector of the segment BC. The circumcenter O of triangle ABC also lies on the perpendicular bisector of its side BC. Thus, the line OA_φ is the perpendicular bisector of the segment BC. Now, the point A'_φ is the inverse of the point A_φ in the circumcircle of triangle ABC and therefore lies on the line OA_φ (since O is the center of this circumcircle). Thus, the point A'_φ lies on the perpendicular bisector of the segment BC.

Now as we know that the points A_φ and A'_φ both lie on the perpendicular bisector of the segment BC, we can apply Theorem 19 to them. Since the points A_φ and A'_φ are inverse to each other wrt the circumcircle of triangle ABC, they fulfill Assertion $\mathcal{F}_5$ of Theorem 19. Since, according to Theorem 19 **a)**, the Assertions $\mathcal{F}_3$ and $\mathcal{F}_5$ are equivalent, they therefore also fulfill Assertion $\mathcal{F}_3$; that is, we have $\measuredangle BCA_\varphi + \measuredangle BCA'_\varphi = \measuredangle BAC$. Furthermore, according to Theorem 19 **b)**, the validity of Assertion $\mathcal{F}_5$ implies that the lines AA_φ and AA'_φ are isogonal to each other wrt the angle CAB.

Since $\measuredangle BCA_\varphi = \varphi$, the equation $\measuredangle BCA_\varphi + \measuredangle BCA'_\varphi = \measuredangle BAC$ becomes $\varphi +$

$\measuredangle BCA'_\varphi = \measuredangle BAC$, thus $\measuredangle BCA'_\varphi = \measuredangle BAC - \varphi$. Since the point A'_φ lies on the perpendicular bisector of the segment BC, we have $BA'_\varphi = CA'_\varphi$; thus, triangle $BA'_\varphi C$ is isosceles with base BC, and this yields $\measuredangle BCA'_\varphi = \measuredangle A'_\varphi BC$. Combining this with $\measuredangle BCA'_\varphi = \measuredangle BAC - \varphi$, we obtain $\measuredangle BCA'_\varphi = \measuredangle A'_\varphi BC = \measuredangle BAC - \varphi$. Similarly, triangle $CB'_\varphi A$ is isosceles with base CA and fulfills $\measuredangle CAB'_\varphi = \measuredangle B'_\varphi CA = \measuredangle CBA - \varphi$, and triangle $AC'_\varphi B$ is isosceles with base AB and fulfills $\measuredangle ABC'_\varphi = \measuredangle C'_\varphi AB = \measuredangle ACB - \varphi$.

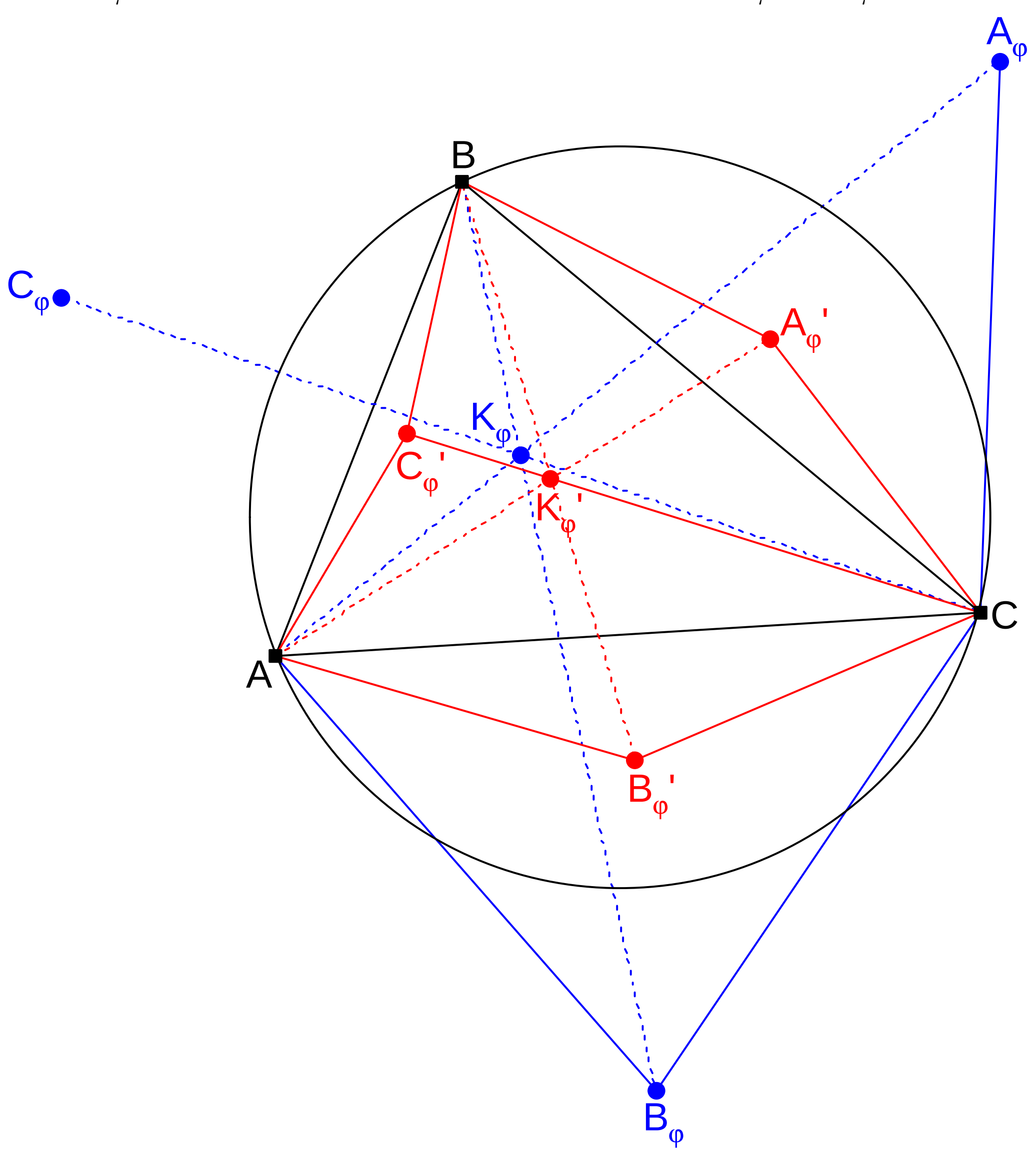

Fig. 44

Since the lines AA_φ and AA'_φ are isogonal to each other wrt the angle CAB, the line AA'_φ is the isogonal of the line AA_φ wrt the angle CAB. Now, the line AA_φ is the line AK_φ. Hence, the line AA'_φ is the isogonal of the line AK_φ wrt the angle CAB. Similarly, the lines BB'_φ and CC'_φ are the isogonals of the lines BK_φ and CK_φ wrt the angles ABC and BCA. Thus, altogether, the lines AA'_φ, BB'_φ, CC'_φ are the isogonals of the lines AK_φ, BK_φ, CK_φ wrt the angles CAB, ABC, BCA, and hence they concur at one point, namely at the isogonal conjugate of the point K_φ wrt triangle ABC.

Summing up, we see:

Theorem 20. Let ABC be a triangle, and φ an arbitary angle. On the sides BC, CA, AB of triangle ABC, we erect isosceles triangles $BA_\varphi C$, $CB_\varphi A$, $AC_\varphi B$ with the bases BC, CA, AB and the equal base angle

$$\measuredangle BCA_\varphi = \measuredangle A_\varphi BC = \measuredangle CAB_\varphi = \measuredangle B_\varphi CA = \measuredangle ABC_\varphi = \measuredangle C_\varphi AB = \varphi.$$

Then, the lines AA_φ, BB_φ, CC_φ concur at one point: the φ-Kiepert point K_φ of triangle ABC.

Let A'_φ, B'_φ, C'_φ be the inverses of the points A_φ, B_φ, C_φ in the circumcircle of triangle ABC. Then, the triangles $BA'_\varphi C$, $CB'_\varphi A$, $AC'_\varphi B$ are isosceles with the bases BC, CA, AB and the base angles $\measuredangle BCA'_\varphi = \measuredangle A'_\varphi BC = \measuredangle BAC - \varphi$, $\measuredangle CAB'_\varphi = \measuredangle B'_\varphi CA = \measuredangle CBA - \varphi$, $\measuredangle ABC'_\varphi = \measuredangle C'_\varphi AB = \measuredangle ACB - \varphi$. The lines AA'_φ, BB'_φ, CC'_φ concur at one point, namely at the isogonal conjugate of the point K_φ wrt the triangle ABC.

Using the theory of Tucker circles, we can show that this isogonal conjugate lies on the Brocard axis of triangle ABC, which is the line joining the circumcenter and the symmedian point of triangle ABC.

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