Grinding

aka the casework unit

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DMY-GRINDING, OTIS*

§1 Reading

Explanation of name: https://en.wikipedia.org/wiki/Grinding_(gaming).
(Not actually required reading.)

§2 Lecture notes

None of the problems that follow have hard ideas. But they require significant care, and many involve detailed case analysis. Often these problems have pathological answers, and so missing certain cases will lead to major deductions.

Whenever you see a problem that has this feeling, my overall advice is: be *extremely neat*. If you miss an answer, it will not end well for your score.

Some particular advice along these lines:

- When complex bashing, I often give the advice to do the calculation neatly on official paper, the first time. The same goes for many of the grinding problems: if you know what you're doing and recognize that you have a finite case check left, then do it neatly on official paper the first time.
- If you find yourself doing a similar/repeated calculation several times, you should take the effort to organize all of them, so that you avoid re-doing the same calculation, and so that you can really make sure you've gotten all of them. (Analogous to keeping a **cache** in computer science.)
- If you have a function $f(n)$ or $f(a,b)$ that you are evaluating at many places, consider making a table. (You'll see many of these in the official solutions.)
- When doing casework (think "case 2.3.1"), consider using nested or bulleted lists. The structure of a casework solution is essentially a rooted tree, and using a nested list makes it visually easy to see where you are in said tree. I also recommend giving your cases headings like "case 1", "case 2.3.1", et cetera. (You'll see many of these in the official solutions.)

In terms of working neatly, you can find my scratch paper from the AIME 2013 at the following link, which may or may not be helpful: https://web.evanchen.cc/static/AIME-2013-scratch.pdf.

Example 2.1 (China TST 1995)

Find the smallest prime number not of the form $|2^a - 3^b|$, where a and b are nonnegative integers.

Walkthrough. The main lesson from this problem is to *get the answer right*, after the proof is easy. This can be done methodically.

(a) Make a table of values of $|2^a - 3^b|$ for $\max(2^a, 3^b) < 100$.

(b) Come up with a guess for the answer, and then sanity-check it to make sure that guess is right.

(c) Once you have the answer, prove it using standard techniques.

Example 2.2 (Shortlist 2003 N2)

Each positive integer a is subjected to the following procedure, yielding the number $d = d(a)$:

- The rightmost digit of a is moved to the leftmost position, yielding a number b.
- The number b is squared, yielding $c = b^2$.
- The leftmost digit of c is moved to the rightmost position, yielding a number d.

For example if $a = 2003$ we have $b = 3200$, $c = 10240000$, and $d = 2400001 = d(2003)$.

Find all positive integers a such that $d(a) = a^2$.

Walkthrough. In what follows, we assume a has $n+1$ digits, meaning $10^n \le a < 10^{n+1}$. In particular, $10^{2n} \le d < 10^{2n+2}$. Let us write

$$\begin{aligned} a &= \overline{\dots x} \\ b &= \overline{x \dots} \\ c &= b^2 = \overline{y \dots} \\ d &= \overline{\dots y} \end{aligned}$$

so that $y = x^2 \pmod{10}$ (here $\dots$ denotes some irrelevant digits).

(a) Find the three possible pairs (x, y) by brute-force.

(b) Use the result of (a) to conclude that $y = x^2$ and that $10^{2n} \le c < 10^{2n+1}$.

(c) Let $a = 10m + x$, and work through the calculation to show that $m = 2x \cdot \frac{10^n - 1}{9}$.

(d) Determine which pairs (x, n) lead to working values of a. You should find one infinite family of solutions and two sporadic ones.

Not much to say here... just be methodical.

Example 2.3 (China TST 2001)
Find all integers x such that

$$f(x) = x^6 + 15x^5 + 85x^4 + 225x^3 + 274x^2 + 120x + 1$$

is a perfect square.

Walkthrough. What an excellent problem!

(a) Evaluate $f(x)$ for $|x| \le 4$. In particular, you should find quite a few values of x for which $f(x)$ is actually a square.

(b) Using your results from (a), figure out how to rewrite f in a manageable way (i.e. explain where the seemingly random coefficients come from).

(c) List all eight values of x for which $f(x)$ is a square.

We will now show these are the only values.

(d) Consider $256f(x)$. Rewrite everything in terms of the variable $y = 2x + 5$, so that one gets an even polynomial in y. From now on, we assume WLOG that $y > 0$.

(e) Show that $256f(x) < (2y^3 - 35y)^2$ holds always.

(f) Prove that $256f(x) > (2y^3 - 35y - 3)^2$ for large enough odd y.

(g) Treat the edge cases not handled in (f) by hand, in whichever way you see fit.

(h) Solve $256f(x) = (2y^3 - 35y - 1)^2$. (For parity reasons, this is the only case we need to consider.)

(i) Conclude.

§3 Practice problems

Instructions: Solve [28♣]. If you have time, solve [40♣]. Problems with red weights are mandatory.

> It's just, I gotta say, it becomes kinda hard to love your job when no one seems to like you for doing it.
>
> Ralph from *Wreck-It Ralph*

[2♣] **Problem 1** (TSTST 2017)**.** Find all nonnegative integer solutions to $2^a+3^b+5^c = n!$.

[2♣] **Problem 2** (IMO 2012/4)**.** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

[2♣] **Problem 3** (Putnam 2017 A1)**.** Find the smallest set S of positive integers such that

(a) $2 \in S$,

(b) $n \in S$ whenever $n^2 \in S$,

(c) $(n + 5)^2 \in S$ whenever $n \in S$.

(The set S is "smallest" in the sense that S is contained in any other such set.)

[2♣] **Problem 4** (Croatia TST 2016/2/4)**.** Find all primes (p, q) such that

$$p(p^2 - p - 1) = q(2q + 3).$$

[5♣] **Problem 5** (JMO 2017/4)**.** Are there any triples (a, b, c) of positive integers such that $(a - 2)(b - 2)(c - 2) + 12$ is a prime number that properly divides the positive number $a^2 + b^2 + c^2 + abc - 2017$?

[5♣] **Problem 6** (ELMO SL 2013 N1)**.** Find all ordered triples of non-negative integers (a, b, c) such that $a^2 + 2b + c$, $b^2 + 2c + a$, and $c^2 + 2a + b$ are all perfect squares.

[3♣] **Problem 7** (PUMaC 2014)**.** Find all positive integers a, b, c, d such that $a(a-1)+s$, $b(b - 1) + s$, $c(c - 1) + s$, $d(d - 1) + s$ are all perfect squares, where $s = a + b + c + d$.

[5♣] **Problem 8** (Shortlist 2001 C5)**.** Find all finite sequences $x = (x_0, \dots, x_n)$ such that for each $k = 0, \dots, n$, the number k appears in x exactly x_k times.

[9♣] **Problem 9** (ELMO SL 2018 N2)**.** Call a number n *good* if it can be expressed in the form $2^x + y^2$ where x and y are nonnegative integers.

(a) Prove that there exist infinitely many sets of 4 consecutive good numbers.

(b) Find all sets of 5 consecutive good numbers.

[3♣] **Problem 10** (Shortlist 2000 N2)**.** Find all positive integers n such that $d(n)^3 = 4n$, where $d(n)$ is the number of positive divisors of n.

[3♣] **Problem 11** (Shortlist 2001 G6)**.** Let ABC be a triangle and P a point outside it. Let $D = \overline{AP} \cap \overline{BC}$, $E = \overline{BP} \cap \overline{CA}$, $F = \overline{CP} \cap \overline{AB}$. Suppose that the areas of triangles PBD, PCE, PAF are equal to each other. Prove that each of these areas is equal to the area of triangle ABC itself.

[5♣] **Problem 12** (Shortlist 2008 N3). Let a_0, a_1, ... be a sequence of positive integers satisfying $\gcd(a_i, a_{i+1}) > a_{i-1}$ for every positive integer i. Prove that $a_i \ge 2^i$ for every positive integer i.

[5♣] **Problem 13** (Putnam 2017 B5). A line in the plane of a triangle T is called an equalizer if it divides T into two regions having equal area and equal perimeter. Find positive integers $a > b > c$, with a as small as possible, such that there exists a triangle with side lengths a, b, c that has exactly two distinct equalizers.

[5♣] **Problem 14** (IMO 2015/2). Find all positive integers a, b, c such that each of $ab - c$, $bc - a$, $ca - b$ is a power of 2 (possibly including $2^0 = 1$).

[5♣] **Problem 15** (China TST 2006). For a positive integer M, if there exists integers a, b, c, d such that $ad = bc$ and

$$M \le a < b \le c < d \le M + 49$$

then we say that M is *good*, otherwise we say that M is *bad*. Determine the greatest good number and the smallest bad number.

(Story: Wenying Gao made the 2006 IMO China team by solving this problem, and shared it at MOP in 2014.)

[1♣] **Mini Survey.** At the end of your submission, answer the following questions.

(a) About how many hours did the problem set take?

(b) Name any problems that stood out (e.g. especially nice, instructive, boring, or unusually easy/hard for its placement).

Any other thoughts are welcome too. Examples: suggestions for new problems to add, things I could explain better in the notes, overall difficulty or usefulness of the unit.

§4 Solutions to the walkthroughs

§4.1 Solution 2.1, China TST 1995

The answer is 41. First, we construct the following large table and observe that each of the ten primes less than 40 appears:

$2^a - 3^b$	1	2	4	8	16	32	64
1	0	1	3	7	15	31	63
3	-2	-1	1	5	13	29	61
9	-8	-7	-5	-1	7	23	55
27	-26	-25	-23	-19	-11	5	37
81	-80	-79	-77	-73	-65	-49	-17

Now, we prove that 41 can't be achieved. This is two routine Diophantine equations.

First, if $2^a - 3^b = 41$, then clearly $a \geq 3$ according to the table above, but then mod 8 gives a contradiction as we have $-3^b \equiv 41 \pmod 8$, or $3^b \equiv 7 \pmod 8$, which never happens.

Secondly, suppose for contradiction that $2^a - 3^b = -41$, and again assume $a \geq 3$, so modulo 8 gives b is even. Assume $b > 0$ as well. Taking modulo 3 now gives $2^a \equiv 1 \pmod 3$, so a is even. Thus we have

$$\left(2^{a/2} - 3^{b/2}\right)\left(2^{a/2} + 3^{b/2}\right) = -41$$

which is absurd.

> **Remark.** Makes much more sense to do breadth-first search using a table above once one realizes the answer is big.
>
> It seems most of the difficulty of this problem is obtaining the correct answer, after which the proof is easy.

§4.2 Solution 2.2, Shortlist 2003 N2

In what follows, we assume a has $n+1$ digits, meaning $10^n \leq a < 10^{n+1}$. In particular, $10^{2n} \leq d < 10^{2n+2}$.

First, we analyze the last digit of a, say x. Suppose

$$\begin{aligned} a &= \overline{\dots x} \\ b &= \overline{x \dots} \\ c = b^2 &= \overline{y \dots} \\ d &= \overline{\dots y} \end{aligned}$$

so that $y = x^2 \pmod{10}$ (here $\dots$ denotes some irrelevant digits).

We now make a large table by noting that if we know the leading digit of b, we can find all possible leading digits of $c = b^2$, hence all possible final digits of d.

x	1	2	3	4	5	6	7	8	9	0
$x^2 \pmod{10}$	1	4	9	6	5	6	9	4	1	0
possible first digits of $\overline{x\dots}^2$	123	45678	91	12	23	34	456	678	89	123456789

From this, we conclude that $x \in \{1, 2, 3\}$, and we must have $y = x^2$. In particular, $10^{2n} \leq c < 10^{2n+1}$, hence $10^{2n} \leq d < 10^{2n+1}$.

Now we proceed to do the calculation: set

$$\begin{aligned} a &= 10m + x \\ b &= 10^n x + m \\ c &= 10^{2n} x^2 + 2 \cdot 10^n \cdot mx + m^2 \\ d &= 10\left(2 \cdot 10^n \cdot mx + m^2\right) + x^2 \end{aligned}$$

Since $a^2 = d$, we have

$$\begin{aligned} 100m^2 + 20mx + x^2 &= 10\left(2 \cdot 10^n \cdot mx + m^2\right) + x^2 \\ 10m + 2x &= 2 \cdot 10^n \cdot x + m \\ m &= 2x \cdot \frac{10^n - 1}{9} = 2x \cdot \underbrace{1 \dots 1}_{n} \end{aligned}$$

In other words, a should be one of the following shapes: $a = 2 \dots 21$, $a = 4 \dots 42$, $a = 6 \dots 63$. A quick check now shows the answers are:

$$a = \underbrace{2 \dots 2}_{n \ge 0} 1, \qquad a = 2, \qquad a = 3.$$

§4.3 Solution 2.3, China TST 2001

The answers are $x \in \{-7, -5, -4, \dots, 0, 2\}$. First note

$$f(x) = x(x+1)(x+2)(x+3)(x+4)(x+5) + 1.$$

From here it is not hard to see all the claimed answers work. We prove they are the only ones.

Let $y = 2x + 5$ be an odd integer. We seek y such that

$$\begin{aligned} 256f(x) &= 4[(y^2-1)(y^2-9)(y^2-25) + 64] \\ &= 4y^6 - 140y^4 + 1036y^2 - 644 \end{aligned}$$

is a perfect square. Since f is even, we may assume $y > 0$ is positive without losing generality.

First, note we have unconditionally

$$256f(x) < 4y^6 - 140y^4 + 1225y^2 = (2y^3 - 35y)^2.$$

Next, we claim that:

> **Claim** — We have $f(x)$ is not a square for any odd $y \ge 17$.

Proof. First, we contend that for these y,

$$256f(x) > (2y^3 - 35y - 3)^2.$$

Indeed, if we expand and rearrange we get

$$\begin{aligned} & 256f(x) > 4y^6 + 140y^4 - 12y^3 + 1225y^2 + 21y + 9 = (2y^3 - 35y - 3)^2 \\ \iff\ & 0 < 12y^3 - 189y^2 - 210y - 653 = y\left(y(12y - 189) - 210\right) - 653 \end{aligned}$$

which is true.

Following this, we consider two more situations:

- If $256f(x) = (2y^3-35y-1)^2$ then $4y^3-189y^2-70y-645 = 0$. Note $y \mid 645 = 3\cdot5\cdot43$, but that polynomial is positive for $y > 30$.
- The case $256f(x) = (2y^3-35y-2)^2$ is impossible for parity reasons.

Hence there are no solutions in which $y \ge 17$. $\square$

> **Remark.** Alternatively, one can check $256f(x) > (2y^3-35y-7)^2$ for $y > 9$, which gives a different set of cases.

Hence remains to check $y \in \{7, 9, 11, 13, 15\}$ manually. These correspond to $x \in \{1, 2, 3, 4, 5\}$. Write:

- $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 + 1 = 721$.
- $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 + 1 = 5041 = 71^2$.
- $3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 + 1 = 20161 \equiv 11 \pmod{13}$.
- $4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 + 1 = 60481 \equiv 5 \pmod{13}$.
- $5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 + 1 = 151201 \equiv 6 \pmod{11}$.

This completes the proof.

For comedic value, here is a chat message I received in 2011, lightly edited.

`here's a problem for you`

`find all integer` x `such that` $x^6+15x^5+85x^4+225x^3+274x^2+120x+1$ `is a perfect square`

```
and now I will announce that this problem is absolutely terrible,
tell you that it was ACTUALLY on a REAL CHINA TST

and tell you that when I saw the solution, I cried

and that was the day I knew we HAD to beat China at the IMO

and then we didn't

and it made me sad

and now, I will tell you that you shouldn't work on this problem
and you should rather have me just tell you the solution because
the problem is really bad

ASDFASDFASDF

okay

just thinking about this problem

fills me with both anger and sadness

I am sorry for forcing this problem on you

okay

now I will tell you the solution

the answers are 0, -1, -2, -3, -4, and -5

I'm sure plugging in -5 shouldn't be too difficult
```

`anyway` $y = 2x + 5$`, then`

`let` k `be the square root of the value of this polynomial`

`sorry if you don't want me to tell you the solution, I think it is the best apology I can possibly give for forcing this problem upon you`

`then` $16k^2$ `is sandwiched between` $(2y^3 - 35y)^2$ `and` $(2y^3 - 35y - 1)^2$

`for all but 80 values of y which are easily checkable by mod 5`

`and mod 8`