13

Combinatorics

In a very loose sense, combinatorics is the area of mathematics concerned with counting. By this, we don't mean calling out the numbers $1, 2, 3, \ldots$ or anything like that. Combinatorics is rather about counting in a clever way, or counting without counting. Actually, this isn't a particularly good description of combinatorics either. Pretty much anything that doesn't quite fit into another mathematical category tends to be described as combinatorial. This makes combinatorics a treasure trove of mathematical delights!

13.1 Addition and multiplication

Suppose that the Irish restaurant chain O'Donald's has five types of burger and three types of drink. If you would like a burger *or* a drink, then you have $5 + 3 = 8$ choices. On the other hand, if you would like a burger *and* a drink, then you have $5 \times 3 = 15$ choices.

Many combinatorics problems boil down to applying these addition and multiplication principles in various rather ingenious ways. To make sure that you have some familiarity with these ideas, consider the following basic examples of combinatorial questions. If necessary, you should read over these carefully until you fully understand what is going on—especially because what goes on here, though basic, is extremely useful.

Problem In how many ways can n people stand in a line?

Solution There are n choices of position for the first person to stand in the line. After the first position has been chosen, there are $n - 1$ positions remaining for the second person. And after the second position has been chosen, there are $n - 2$ positions remaining for the third person, and so on.

Therefore, the number of ways to choose the order in which n people stand in a line is

$$n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

You probably already know that this number is written as $n!$ and is called 'n factorial'. $\square$

Problem How many three-digit numbers contain no zeros or nines as digits?

Solution There are 8 choices for the first digit, 8 choices for the second digit and 8 choices for the third digit, that is, any number from 1 to 8.

Hence, there are $8^3 = 512$ such numbers. $\square$

Problem Each letter in Morse code is a sequence of at most four dots and dashes. How many letters are possible?

Solution There are $2^1 = 2$ letters with 1 signal, $2^2 = 4$ letters with 2 signals, $2^3 = 8$ letters with 3 signals, and $2^4 = 16$ letters with 4 signals.

Hence, the answer is $2 + 4 + 8 + 16 = 30$. □

So Morse code only just copes with the 26 letters of the English alphabet!

13.2 Subtraction and division

Sometimes the best way to solve a counting problem is to count too much—a technique often known as *overcounting*. If you have counted k too many objects, then the answer is obtained by subtracting k. On the other hand, if you have counted each object k times, then the answer is obtained by dividing by k.

Many combinatorics problems boil down to applying these subtraction and division principles in conjunction with the addition and multiplication principles we discussed in the previous section.

Problem How many five-digit positive integers have at least two digits the same?

Solution The number of five-digit positive integers is $99999 - 10000 + 1 = 90000$.

The number of five-digit positive integers with no two digits the same is $9 \times 9 \times 8 \times 7 \times 6 = 27216$. This is because there are 9 ways to choose the first digit (because it cannot be zero), 9 ways of choosing the second digit because it cannot be the same as the first, 8 ways of choosing the third digit because it cannot be equal to either of the first two digits, and so on.

Hence, the answer is $90000 - 27216 = 62784$. □

Problem In how many ways can you choose a four-flavour combination from 10 ice-cream flavours?

Solution Well, if you actually cared about the order in which you chose them, then the answer would simply be $10 \times 9 \times 8 \times 7$. But since the order doesn't matter, we have overcounted each combination by a factor of $4 \times 3 \times 2 \times 1$—the number of ways of reordering the four flavours that we have chosen.

Therefore, the answer is

$$\frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210.$$

This can be conveniently expressed using factorial notation as

$$\binom{10}{4} = \frac{10!}{4!\,6!}.$$

□

Problem In how many ways is it possible to arrange the letters of the word *recurrence*?

Solution There are 10! ways to rearrange the letters, but again we have overcounted. The three occurrences of the letter r can be rearranged in 3! ways among themselves. In general, if a letter occurs k! times, then there are k! ways of rearranging the occurrences of that letter among themselves. Since the letters r and e occur three times, the letter c twice and the letters u and n once, the answer is

$$\binom{10}{3,3,2,1,1} = \frac{10!}{3!\,3!\,2!\,1!\,1!} = 50400. \qquad \square$$

13.3 Binomial identities

A lot of fun can be had playing around with binomial coefficients. These lie in the intersection of algebra, combinatorics, polynomials and even calculus, giving rise to many problems with multiple solutions from different areas of mathematics.

Recall the following basic identities.

- **Symmetry**
$$\binom{n}{k} = \binom{n}{n-k}$$
- **Binomial theorem**
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$
- **Addition formula**
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$
- **In-and-out formula**
$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

Problem For every positive integer n, prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0.$$

Solution We provide two very distinct proofs of this fact. In the next section we provide a third proof! This is a rather common feature of binomial identities and it pays to be comfortable with the various types of proof.

- Method 1: Pascal's triangle
Recall that we can obtain the nth row of Pascal's triangle from the $(n-1)$th row by adding the two terms above each element of the nth row as per the addition formula above. In particular, with the convention that $\binom{n}{k} = 0$ if $k < 0$ or $k > n$, we have

$$\binom{n}{0} = \binom{n-1}{-1} + \binom{n-1}{0}$$

$$\binom{n}{2} = \binom{n-1}{1} + \binom{n-1}{2}$$

$$\binom{n}{4} = \binom{n-1}{3} + \binom{n-1}{4}$$

$$\vdots$$

and so $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$ is equal to the sum of all the binomial coefficients on the $(n-1)$th row of Pascal's triangle. Similarly,

$$\binom{n}{1} = \binom{n-1}{0} + \binom{n-1}{1}$$

$$\binom{n}{3} = \binom{n-1}{2} + \binom{n-1}{3}$$

$$\binom{n}{5} = \binom{n-1}{4} + \binom{n-1}{5}$$

$$\vdots$$

and so $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$ is also equal to the sum of all the binomial coefficients on the $(n-1)$th row of Pascal's triangle. Therefore,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots,$$

which is equivalent to what we wanted to prove. □

- Method 2: Algebra
  The *binomial theorem* states that

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots + \binom{n}{n}x^ny^0.$$

All we need to do now is substitute $x = -1$ and $y = 1$ to obtain the desired result. □

Problem For every positive integer n, prove that

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

Solution

- Method 1: Algebra
  We use the in-and-out formula to write $k\binom{n}{k} = n\binom{n-1}{k-1}$. Then the left-hand side becomes

$$n\binom{n-1}{0}+n\binom{n-1}{1}+\cdots+n\binom{n-1}{n-1}=n2^{n-1},$$

where $\displaystyle\sum_{k=0}^{n-1}\binom{n-1}{k}=(1+1)^{n-1}$ is a consequence of the binomial theorem.

- Method 2: Calculus
  Substituting $y=1$ into the binomial formula, we obtain

$$(x+1)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\cdots+\binom{n}{n}x^n.$$

  Now look at what happens when we differentiate both sides.

$$n(x+1)^{n-1}=\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+\cdots+n\binom{n}{n}x^{n-1}$$

  All we need to do now is substitute $x=1$ to obtain the desired result. $\square$

13.4 Bijections

Another way to prove combinatorial identities is to look for combinatorial interpretations for both sides of the identity. This is really a good technique and a useful skill to have. Here we discuss two problems we considered earlier but present bijective solutions.

Problem For every positive integer n, prove that

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}=0.$$

Solution The identity is equivalent to the fact that, for every positive integer n,

$$\sum_{k \text{ odd}}\binom{n}{k}=\sum_{k \text{ even}}\binom{n}{k}.$$

If S is a set with n elements, the left-hand side counts the number of subsets of S with an odd number of elements while the right-hand side counts the number of subsets of S with an even number of elements.

Now that we have a combinatorial interpretation for both sides of the equation, it makes sense to look for a one-to-one correspondence—a *bijection* to use more precise terminology—between the objects described by the left-hand side and the objects described by the right-hand side.

In other words, we want to find a rule for turning a subset of S with an odd number of elements into a subset of S with an even number of elements, and vice versa.

We start by picking some fixed $s \in S$. The rule is to add the element s to your set if it doesn't already contain it, and to remove the element s from your set if it does already contain it. Since we are either adding or subtracting a single element, this rule certainly turns a subset of S with an odd number of elements into a subset of S with an even number of elements, and vice versa. It is a bijection because the rule can be reversed. In fact, the rule is its own inverse![1] This completes our combinatorial proof. $\square$

Problem For every positive integer n, prove that

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

Solution The idea is to find a set of objects which can be counted in two ways—one of which produces the left-hand side while the other produces the right-hand side. In this particular case, the form of the left-hand side gives a strong hint as to what that set might be.

Each term of the left-hand side is of the form $k\binom{n}{k}$ which suggests that, from a set of n people, we would like to choose a committee of k people. This can be done in $\binom{n}{k}$ ways. Furthermore, we would like to choose a president from this committee. This can be done in k ways. As k ranges from 1 up to n, we are choosing committees of every possible size. So what we have shown is that the left-hand side counts the number of ways to choose a committee, with one person designated the president, from a set of n people.

Of course, all that remains is to show that the number of ways to choose a committee, with one person designated the president, from a set of n people also happens to be $n2^{n-1}$. Whereas we earlier counted by choosing the committee first and then the president, we will now count by choosing the president first and then the remainder of the committee. But this is easy, because there are n choices for the president and for each of the remaining $n-1$ people, there are two choices. Each person is either in the committee or is not. Thus the number of ways of choosing a committee with president is $n2^{n-1}$. This completes our combinatorial proof. □

13.5 The supermarket principle

This is a common trick in counting problems. Although it is a well-defined technique, it does not seem to have a well-defined name but is another example of bijective combinatorics. We shall call it the *supermarket principle* because of the following problem.

Problem The supermarket has an unlimited supply of apple tarts, chocolate muffins and cheesecakes.

How many ways are there to buy seven cakes from the supermarket?

Solution You buy your seven cakes and take them to the checkout, where you put them on the conveyor belt. Just to make things easy, you place all the apple tarts together first, then the chocolate muffins, and then the cheesecakes. To make things really easy for the stressed-out checkout attendant, you place some of those conveyor belt dividers between each type of cake.

Now there will be nine objects on the conveyor belt: seven cakes and two dividers. (If you buy no apple tarts, for example, the first object will be a divider.) Choosing which of these nine objects are the dividers uniquely determines what cakes you have bought. So the answer is $\binom{9}{2} = 36$. □

In this question, we established a bijective correspondence.

$$\left\{\begin{array}{c}\text{ways of buying 7}\\ \text{cakes of 3 types}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{ways of placing 2}\\ \text{dividers among 7 cakes}\end{array}\right\}$$

You should convince yourself that this is actually correct. That is, for each way of buying the prescribed cakes, there is a way of placing the dividers; and conversely, for each way of placing the dividers, there is a way of buying cakes so that the dividers are placed there. Both of these are quite obvious once you understand what is going on, but often this is difficult to express!

Now for a less supermarket-oriented and slightly more subtle example. There are certainly other solutions to it, but perhaps this is the most elegant.

Problem Find the number of three-digit numbers whose digit sum is 10.

Solution Represent a number by a collection of 'units' (1s) and 'dividers' (Δs). For example, 325 is represented by

$$1\,1\,1\,\Delta\,1\,1\,\Delta\,1\,1\,1\,1\,1.$$

Now every such three-digit number corresponds to a way of placing two dividers among ten 1s. However *the correspondence is not bijective!* This is because the 'conveyor belt'

$$\Delta\,1\,\Delta\,1\,1\,1\,1\,1\,1\,1\,1\,1$$

would correspond to the number 019, which is not a three-digit number. You can't have a divider in the first place. But provided that you start with a 1, the correspondence *is* bijective. (You should make sure of this.)

Ignoring the initial 1, the answer is the number of ways of placing two dividers amongst the remaining nine 1s. There are 11 objects in total and so the number of ways of nominating two of them to be dividers is simply $\binom{11}{2} = 55$. □

13.6 Pigeonhole principle

Although the pigeonhole principle was introduced in sections 1.9 and 1.10, it is a powerful combinatorial tactic which we showcase here.

Problem Given 27 distinct odd positive integers less than 100, prove there is a pair of them whose sum is 102.

Solution Consider the following sets of odd positive integers.

$$\{1, 101\}, \{3, 99\}, \ldots, \{49, 53\}, \{51\}$$

Note that each of the first 25 sets consists of two numbers whose sum is 102, while the 26th set consists of a single number. Furthermore, the 26 sets form a partition of all the odd positive integers less than 100. So given 27 numbers (pigeons), some set (pigeonhole) has two different numbers (pigeons) chosen (living) from (in) it! Those two numbers add to 102. □

Problem The prime factorisations of $r+1$ positive integers $a_1, \dots, a_{r+1}$ together involve only r primes $p_1, \dots, p_r$.

Prove that there is a non-empty subset of these integers whose product is a perfect square.

Solution Recall that a positive integer is a perfect square if and only if the exponent of every prime in its prime factorisation is even. So given a product of powers of these primes $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, define the *parity pattern* of n to be $(\alpha_1, \dots, \alpha_r)$, where the α_i are considered modulo 2.

Clearly when you multiply two numbers, you add their parity patterns term by term modulo 2, and if you can divide, then you subtract their parity patterns modulo 2. We just need to show there is a non-empty subset of our $r+1$ integers whose product has parity pattern $(0, \dots, 0)$.

So, consider all 2^{r+1} subsets of our $r+1$ integers, and consider their parity patterns. But there are only 2^r possible parity patterns. By the pigeonhole principle there are two distinct subsets X and Y of $\{a_1, \dots, a_{r+1}\}$ whose products have the same parity pattern $\mathbf{v}$.

If X and Y were disjoint, then the product of the elements of $X \cup Y$ would have parity pattern $2\mathbf{v} = (0, \dots, 0)$, as required.

If X and Y overlap, let $D = X \cap Y$. We claim that the product of the elements of $X \cup Y \setminus D$ is a perfect square. Indeed consider the equality,

$$\left(\prod_{x \in X} x\right)\left(\prod_{y \in Y} y\right) = \left(\prod_{x \in X \setminus D} x\right)\left(\prod_{y \in Y \setminus D} y\right)\left(\prod_{d \in D} d^2\right).$$

The LHS has parity pattern $(0, \dots, 0)$ because $\prod_{x \in X} x$ and $\prod_{y \in Y} y$ have the same parity pattern. Also $\prod_{d \in D} d^2$ has parity pattern $(0, \dots, 0)$. It follows that

$$\left(\prod_{x \in X \setminus D} x\right)\left(\prod_{y \in Y \setminus D} y\right)$$

also has parity pattern $(0, \dots, 0)$ and so the product of the elements of $X \cup Y \setminus D$ is a perfect square, as required. $\square$

13.7 Principle of inclusion–exclusion

Here is a basic idea of set theory.[2]

$$|X \cup Y| = |X| + |Y| - |X \cap Y|$$
$$|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |Y \cap Z| - |Z \cap X| + |X \cap Y \cap Z|$$

This can be generalised as per the following theorem.

Principle of inclusion–exclusion

$$|X_1 \cup \cdots \cup X_n| = \sum_{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,n\}} (-1)^{k-1}\,|X_{i_1} \cap \cdots \cap X_{i_k}|.$$

It can also be inverted, since

$$(S \setminus X_1) \cup \cdots \cup (S \setminus X_n) = S \setminus (X_1 \cap \cdots \cap X_n)$$

and

$$(S \setminus X_1) \cap \cdots \cap (S \setminus X_n) = S \setminus (X_1 \cup \cdots \cup X_n),$$

which yield

$$|X_1 \cap \cdots \cap X_n| = \sum_{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,n\}} (-1)^{k-1}\,|X_{i_1} \cup \cdots \cup X_{i_k}|.$$

Problem An absent-minded postman has n letters to deliver to n different addresses.

In how many ways can he deliver the mail, one letter to each address, so that no letter is delivered to its correct address?

Solution This problem really involves permutations, which we consider to be bijections from the set $\{1, 2, \ldots, n\}$ to itself or equivalently, arrangements of the numbers $1, 2, \ldots, n$. What we are seeking is the number D_n of derangements of the set $\{1, 2, \ldots, n\}$. A *derangement* is a permutation of a set which leaves no element fixed.

For $i = 1, 2, \ldots, n$, let X_i denote the set of permutations which fix the number i. The number of elements in X_i is easy to count and it is simply $(n-1)!$. Furthermore, we know that $X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_k}$ is simply the set which fixes the number i_1, fixes the number i_2, and so on, up to the number i_k. This is a set which is also easy to count and its number of elements is simply $(n-k)!$. Note that the total number of ways of choosing k of the X_i is equal to $\binom{n}{k}$.

What does the set $X_1 \cup X_2 \cup \cdots \cup X_n$ represent? It simply represents the number of permutations that fix 1 or 2 or 3, and so on, up to n. In other words, it is the set of permutations that fix at least one of $1, 2, \ldots, n$. So the number of elements it contains is simply $n! - D_n$. Therefore,

$$\begin{aligned}
|X_1 \cup \cdots \cup X_n| &= \sum_{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,n\}} (-1)^{k-1}\,|X_{i_1} \cap \cdots \cap X_{i_k}| \\
\Rightarrow \qquad n! - D_n &= \sum_{k=1}^{n} (-1)^{k-1}(n-k)!\binom{n}{k} \\
\Rightarrow \qquad D_n &= n! - \sum_{k=1}^{n} (-1)^{k-1}\frac{n!}{k!} \\
&= n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right).
\end{aligned}$$

□

13.8 Double counting

Double counting is short for counting the same thing in two different ways.

Problem At a party each person knew exactly 22 others. For any pair of people X and Y who knew one another, there was no other person at the party whom they both knew. For any pair of people X and Y, who did not know one another, there were exactly 6 other people whom they both knew.

How many people were at the party?

Solution This problem has an obvious graph theory interpretation, where vertices represent people and edges represent a mutual acquaintanceship between two persons. Define a *vee* to be a triple of persons such that exactly two of the three pairs of acquaintances know each other. We count the number of vees in two different ways.[3]

Suppose there are n people at the party. Concentrating on vertices we see that each vertex contributes $\binom{22}{2} = 231$ vees since each vertex has 22 edges emanating from it. Thus the total number of vees is $231n$.

On the other hand adding the degrees of each vertex gives $22n$, but this overcounts the number of edges by a factor of two. Therefore, the total number of edges is $11n$. This means that the total number of pairs of vertices not connected by an edge is

$$\binom{n}{2} - 11n.$$

Each such non-edge makes a vee with 6 other vertices. Thus the total number of vees is

$$6\left(\binom{n}{2} - 11n\right).$$

Equating the two expressions for the number of vees yields

$$231n = 6\left(\binom{n}{2} - 11n\right).$$

This is easily solved for n and yields $n = 100$. □

Problem Let $p_n(k)$ be the number of permutations of the set $\{1, 2, \ldots, n\}$ which have exactly k fixed points.

Prove that

$$\sum_{k=0}^{n} k p_n(k) = n!.$$

Solution We find a combinatorial interpretation for the sum. We are counting each permutation, but counting each with multiplicity equal to the number of its fixed points. So permutations without any fixed points, that is, derangements, are not counted at all: permutations with one fixed point are counted once: those with two fixed points are counted twice: and so on. Consider a permutation as the numbers 1 to n, written in some order.

For each permutation with one or more fixed points, colour in one of the fixed points as our 'favourite'. Then the set of 'permutations with favourite fixed points' has precisely $\sum_{k=0}^{n} kp_n(k)$ elements. This is a combinatorial interpretation for the sum.

For example, here are the 'permutations with favourite fixed points' for the case $n = 3$.

$$\mathbf{1}23, \quad 1\mathbf{2}3, \quad 12\mathbf{3}, \quad \mathbf{1}32, \quad 21\mathbf{3} \quad \text{and} \quad 3\mathbf{2}1.$$

How many of these objects are there? Well, with **1** as the favourite fixed point, there are $(n-1)!$ permutations—one for each permutation of the other objects. Similarly, with **2** as the favourite fixed point, there are also $(n-1)!$ permutations. There are $(n-1)!$ permutations for each individual favourite fixed point.

This gives $n \times (n-1)! = n!$ of them overall.[4] □

13.9 Injections

Problem A permutation $(x_1, x_2, \ldots, x_{2n})$ of the set $\{1, 2, \ldots, 2n\}$, where n is a positive integer, is said to be *good* if

$$|x_i - x_{i+1}| = n$$

for at least one i in $\{1, 2, \ldots, 2n-1\}$, and is said to be *bad* otherwise.

Show that, for each n, there are more good permutations than bad permutations.

Solution Since we want to show there are *more* of one thing than another, we don't construct a bijection, but an *injection*! We find a map from bad permutations to good permutations which is injective but not surjective: this will show there are more good permutations.

First think about what a good permutation means. For given $x \in \{1, \ldots, 2n\}$, there is only one y in the set for which $|x - y| = n$. So a permutation is good if and only if we see at least one of the pairs

$$(1, n+1), (2, n+2), \ldots, (n, 2n)$$

occurring as adjacent numbers in the permutation (in any order). A bad permutation is one where we see none of these pairs of numbers adjacent. In what follows, it will be convenient to define the notation $i * n$ as follows.

$$i * n = \begin{cases} i + n & \text{if } 1 \leq i \leq n \\ i - n & \text{if } n + 1 \leq i \leq 2n \end{cases}$$

Now take any bad permutation. Here is how to make a good one out of it. Suppose $x_1 = i$. Then $i * n$ can't occur as x_2, but must occur later on. Thus the permutation looks like

$$(i, A, i * n, B),$$

where A is a non-empty sequence, and B is another (possibly empty) sequence, neither of which contains a pair of the form $(j, j * n)$. We make our permutation good by putting i and $i * n$ together to make

$$(A, i, i * n, B).$$

Thus we have a map ϕ from the set of bad permutations to the set of good permutations given by

$$\phi(i, A, i * n, B) = (A, i, i * n, B).$$

We first show our map ϕ is not surjective. Any good permutation in the image of ϕ is of the form $(A, i, i * n, B)$, where A is non-empty and the only adjacent pair of the form $(j, j * n)$ occurs if $j = i$. Clearly there are more good permutations than these. For example, any permutation of the form $(1, n+1, \ldots)$ is a good permutation that is not in the image of ϕ.

Now we show ϕ is injective. Suppose that $\phi(i_1, A_1, i_1 * n, B_1) = \phi(i_2, A_2, i_2 * n, B_2)$. Then from the definition of ϕ we have

$$(A_1, i_1, i_1 * n, B_1) = (A_2, i_2, i_2 * n, B_2.)$$

However from our earlier observation the only adjacent pair of the form $(j, j * n)$ on the LHS occurs at $j = i_1$. Similarly, the only adjacent pair of the form $(j, j * n)$ on the RHS occurs at $j = i_2$. This allows us to deduce that $i_1 = i_2$. Then it follows that $A_1 = A_2$ and $B_1 = B_2$. Therefore, ϕ is injective. □

13.10 Recursion

As a first example of the technique of *recursion*, we solve the problem about the absent-minded postman that we saw in section 13.7.

Problem An absent-minded postman has n letters to deliver to n different addresses.

In how many ways can he deliver the mail, one letter to each address, so that no letter is delivered to its correct address?

Solution Suppose that D_n is the number of derangements of n objects. The trick here is to relate D_n to some of the previous values $D_1, D_2, \ldots, D_{n-1}$.

- We will take a derangement of $n-1$ letters and construct $n-1$ distinct derangements of n letters. Suppose that in our derangement of $n-1$ letters, letter 1 is delivered to address j. Then we adjust this by delivering letter 1 to address n and letter n to address j. This gives a derangement of n letters. In general, we could take any $i \in \{1, 2, \ldots, n-1\}$ and suppose that in our derangement of $n-1$ letters, letter i is delivered to address j. Then we adjust this by delivering letter i to address n and letter n to address j. This gives a derangement of n letters.

  So given a derangement of $n-1$ letters and a number $i \in \{1, 2, \ldots, n-1\}$, we have constructed a derangement of n letters. You can check that these are indeed all different. Thus we have constructed

  $$(n-1)D_{n-1}$$

  derangements of n letters.

- Unfortunately, we have not accounted for every possible derangement of n letters. The ones we are missing are precisely those where letter n is delivered to address j and letter j is delivered to address n for some $j \in \{1, 2, \ldots, n-1\}$. But in this case, notice that the remaining letters and addresses form a derangement on $n-2$ letters. Therefore, for each j there are D_{n-2} ways in which this can happen. But since we can take any $j \in \{1, 2, \ldots, n-1\}$ there are

  $$(n-1)D_{n-2}$$

  cases in total, where the letter n has swapped addresses with another letter.

Putting our two pieces of information together we have

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

This is a nice simple recursion for D_n which we can manipulate as follows.

$$D_n - nD_{n-1} = -(D_{n-1} - (n-1)D_{n-2})$$

Letting

$$E_n = D_n - nD_{n-1},$$

for $n = 1, 2, \ldots$, we see that

$$E_n = -E_{n-1} = E_{n-2} = \cdots = (-1)^{n-2}E_2 = (-1)^{n-2}(D_2 - 2D_1) = (-1)^n.$$

Here we have used $D_2 = 1$ and $D_1 = 0$.

Therefore,

$$D_n - nD_{n-1} = (-1)^n,$$

and so,

$$\frac{D_n}{n!} - \frac{D_{n-1}}{(n-1)!} = \frac{(-1)^n}{n!}.$$

Since this is true for all integers $n \geq 1$ we also have the same equation for n replaced successively by $n-1, n-2, \ldots, 1$. Upon summing these equations, all the middle terms on the left-hand side cancel out leaving us with

$$\frac{D_n}{n!} - \frac{D_0}{0!} = \frac{(-1)^n}{n!} + \frac{(-1)^{n-1}}{(n-1)!} + \cdots + \frac{(-1)^1}{1!},$$

which is equivalent to the result we obtained earlier in section 13.7. □

Problem In town A there are n girls and n boys such that each girl knows each boy. In town B there are n girls and $2n-1$ boys such that girl k knows boys $1, 2, 3, \ldots, 2k-1$, and only these boys. Let $A(n, r)$ denote the number of different ways in which r girls from town A can dance with r boys from town A, forming r pairs where the girl knows the boy. Similarly, let $B(n, r)$ denote the number of different ways in which r girls from town B can dance with r boys from town B, forming r pairs where the girl knows the boy.

Prove that $A(n, r) = B(n, r)$, for $r = 1, 2, \ldots, n$.

Solution We can calculate the number $A(n, r)$ very easily. The number of ways of choosing r girls is $\binom{n}{r}$ and the number of ways of choosing r boys is $\binom{n}{r}$. The number of ways of pairing them up is $r!$. Therefore, we have

$$A(n, r) = \binom{n}{r}^2 r! = \frac{n!^2}{(n-r)!^2\, r!}.$$

To show that

$$B(n, r) = \frac{n!^2}{(n-r)!^2\, r!}$$

directly is a very difficult task indeed. However, what we can do is relate $B(n,r)$ to the values $B(n-1,r-1)$ and $B(n-1,r)$ in the following way.

We establish a recurrence relation for $B(n,r)$. Let $n \geq 2$ and $2 \leq r \leq n$. There are two cases for a desired selection of r pairs of girls and boys.

- Case 1: Girl n is dancing.
  The remaining $r-1$ girls can choose their partners in $B(n-1,r-1)$ ways while girl n can choose her partner from any of the unchosen $2n-r$ boys.

  This contributes $(2n-r)B(n-1,r-1)$ to the value of $B(n,r)$.

- Case 2: Girl n is not dancing.
  This case is easy because there are simply $B(n-1,r)$ possible choices.

So for every $n \geq 2$ and $2 \leq r \leq n$, we have the recursion

$$B(n,r) = (2n-r)B(n-1,r-1) + B(n-1,r),$$

with initial conditions

$$\begin{aligned} &B(n,r) = 0, \quad \text{if } r > n \\ &B(n,1) = 1 + 3 + 5 + \cdots + (2n-1) = n^2. \end{aligned}$$

It is directly verified that the numbers $A(n,r)$ satisfy the same initial conditions and recurrence relations, from which it follows that $A(n,r) = B(n,r)$ for all n and $r \leq n$. We leave it to the reader to do the algebra that verifies this. □

13.11 Double counting via tables

Often a problem can be interpreted in terms of a table. The information is somehow recorded in the cells of the table. This gives a really clear way of understanding what is happening. Then by examining the situation in the table horizontally on the one hand and vertically on the other hand, we have a natural way to do some double counting.

Problem In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either *pass* or *fail*. Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants.

Prove that

$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

Solution It is advantageous to organise the information of this problem in a table. We do so by having one row for each contestant and one column for each judge. We place a 1 or a 0 in a cell if the judge corresponding to that column has given the contestant corresponding to that row a pass or a fail, respectively.

The tactic now is to find something in the table which we can double count. What we will count is the number of matches between two judges' ratings. More precisely, we are counting the number of instances where two different judges J_x and J_y agree on the result of contestant C. Let the number of these instances be N.

	J_1	J_2	J_3	J_4	J_5	J_6	J_7
C_1	0	1	0	0	0	0	1
C_2	1	0	1	0	1	0	1
C_3	1	1	1	0	0	0	0
C_4	1	1	0	0	1	1	0
C_5	0	1	0	1	0	1	1

- *Think vertically.*
For each pair of judges, we know that they agree in at most k places. Since there are $\binom{b}{2}$ pairs of judges, we conclude that

$$N \leq k\binom{b}{2}.$$

- *Think horizontally.*
Suppose a contestant has received m passes and n fails, where $m+n=b$. The m judges who awarded the contestant a pass contribute $\binom{m}{2}$ agreements, while the n judges who awarded the contestant a fail contribute $\binom{n}{2}$ agreements.

Therefore, looking along a row, the number of agreements is

$$\begin{aligned}\binom{m}{2}+\binom{n}{2} &= \frac{m(m-1)}{2}+\frac{n(n-1)}{2}\\ &= \frac{m^2+n^2}{2}-\frac{b}{2}.\end{aligned}$$

Since $m+n=b$ is constant, m^2+n^2 is minimised when m and n are as close to each other as possible. Since b is odd this occurs when $\{m,n\}=\{\frac{b-1}{2},\frac{b+1}{2}\}$. So we have

$$\begin{aligned}\frac{m^2+n^2}{2}-\frac{b}{2} &\geq \frac{1}{2}\left[\left(\frac{b-1}{2}\right)^2+\left(\frac{b+1}{2}\right)^2\right]-\frac{b}{2}\\ &= \left(\frac{b-1}{2}\right)^2.\end{aligned}$$

Summing over the number of rows, we obtain

$$N \geq a\left(\frac{b-1}{2}\right)^2.$$

All that remains is to put these two pieces of information together. What we have shown is that

$$a\left(\frac{b-1}{2}\right)^2 \leq N \leq k\binom{b}{2}.$$

It follows that

$$\begin{aligned}\frac{k}{a} &\geq \left(\frac{b-1}{4}\right)^2 \Big/ \binom{b}{2} \\ &= \frac{b-1}{2b}.\end{aligned}$$

□

13.12 Combinatorial reciprocal principle

The *what* principle, I hear you say? Yes, this is another principle to which we gave a name, because it seemed really interesting at the time, and because nobody else seems to have named it. Here's the principle: if you have a set S of objects falling into k different categories, then

$$\sum_{x \in S} \frac{1}{\text{number of objects in the same category as } x\text{, including } x} = k.$$

To help you understand what this is saying, write down some examples and then try to prove the principle. Once it is well understood, this principle can solve some otherwise difficult, or even apparently unapproachable, problems.

Problem Students from 13 different countries participated in the 491st International Mathematics Bonanza. Each student belonged to one of five different age groups.

Prove that there were at least nine participants in the Bonanza who had more fellow participants in his or her age group than fellow participants from his or her own country.

Solution Given a student x, let A_x denote the number of students (including x) in the same age group as x, and let C_x denote the number of students from the same country. So by the combinatorial reciprocal principle,

$$\sum_x \frac{1}{A_x} = 5 \quad \text{and} \quad \sum_x \frac{1}{C_x} = 13.$$

Thus

$$\sum_x \frac{1}{C_x} - \frac{1}{A_x} = 8.$$

Since A_x and C_x are positive integers, we have

$$\frac{1}{C_x} - \frac{1}{A_x} < 1$$

for each x.

So the number of x for which $\frac{1}{C_x} - \frac{1}{A_x} > 0$ is more than 8. That is, there are at least 9 students for which $\frac{1}{C_x} > \frac{1}{A_x}$, that is, $A_x > C_x$. Thus we are done. □

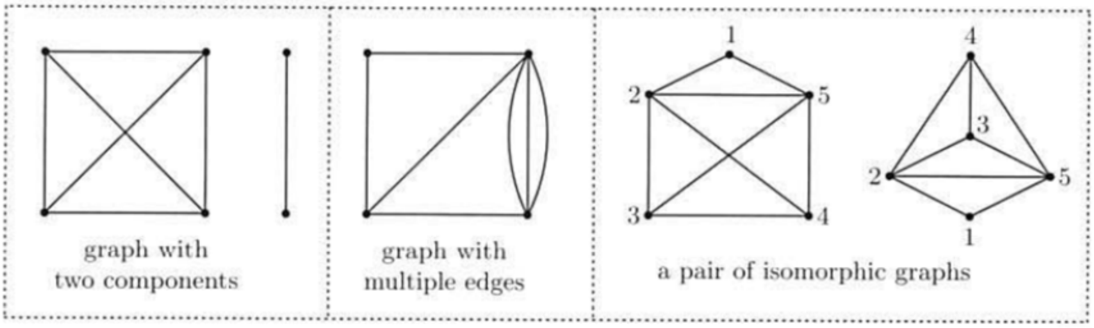
14

Graph theory

Graph theory is a truly remarkable area of mathematics. In one sense, the concept of a graph is so easy that every human being has some innate understanding of it. On the other hand, graph theory offers some notoriously difficult problems and is the site of much active mathematical research today. In between the very simple and the very complicated lies a vast region of interesting and amazing results, some of which we'll soon see.

When we use the word *graph* in this chapter, we don't mean a complicated representation of a function on a set of coordinate axes or anything like that. We simply mean a diagram obtained by joining dots.

In the usual graph theory terminology, the dots are called *vertices* while the lines connecting them are called *edges*. The edges of a graph merely represent relationships between the vertices. In particular, we usually don't care how a graph is drawn in the plane. In fact, we consider two graphs G and H to be the same—the technical word is *isomorphic*—if there is a one-to-one correspondence between their vertices, such that two vertices are connected by an edge in G if and only if the two corresponding vertices are connected by an edge in H. Note that a graph is allowed to consist of more than one connected component.



For most of our purposes, we will assume that a graph is finite and has no loops (a loop is an edge connecting a vertex with itself) or multiple edges (two or more edges connecting the same pair of vertices). However, there are situations where considering loops and multiple edges is useful.

14.1 Degree

Let's start with a few fundamental definitions in graph theory. If one of the endpoints of the edge e is the vertex v, then we say that e and v are *incident*. If two vertices u and v are incident to the same edge, then we say that u and v are *adjacent*. Now define the *degree* of a vertex v to be the number of edges incident to v and denote it by $\deg(v)$.

Problem Show that at any party, there are always at least two people with exactly the same number of friends at the party.

This is the first of many party problems which we will examine. In fact, you can think of every graph theory problem as a party problem in disguise. Vertices represent people, while edges represent mutual acquaintance. For the time being, we won't deal with the case of celebrities or forgetful people, that is, where A knows B but B doesn't know A. Furthermore, we don't count people as knowing themselves, so that the graph has no loops, and we don't allow people to know each other twice over, so that the graph has no multiple edges. The degree of a vertex represents the number of people at the party that a person knows so, in some sense, degree is a popularity index!

Throughout the chapter, we will switch between party language and graph language, depending on the circumstances. This particular problem can be translated into graph theory terminology in the following way.

> Show that in any graph, there exist two vertices with the same degree.

Solution With the goal of obtaining a contradiction, suppose that each person knows a different number of people at the party.

If there are n partygoers, then they can know $0, 1, 2, \ldots, n-1$ people, and these are the only possibilities. Since there are n people and n possibilities for the number of people they know, there must be one person who knows 0 people, one person who knows 1 person, and so on, up to one person who knows all $n-1$ other people at the party. However, it's impossible for there to be two people at the party, one who knows no one else and one who knows everybody else. This is the desired contradiction, so we can conclude that there exist two people who know the same number of people at the party. □

At any party, if you ask everyone in the room, including yourself, how many hands they shook, and add up all of the answers, then you will always end up with an even number. In fact, the sum will be twice the number of handshakes that have occurred during the party. We can state this in the language of graph theory in the following way.

Handshaking lemma In any graph, the sum of the degrees of all the vertices is equal to twice the number of edges.

This is true because each edge contributes two to the sum of the degrees, one for each vertex incident to it. A simple corollary of the handshaking lemma is the fact that the number of vertices of odd degree in any graph must be even.

Problem Is it possible to build a house with exactly eight rooms, each with three doors, and such that exactly three of the house's doors lead outside?

Solution If you could build such a house, then you could construct a corresponding graph in the following way. Let there be nine vertices, one for each room and one to represent the outside of the house. Place an edge between two vertices if there is a door between the two corresponding areas. The conditions of the problem assert that every vertex of our graph has degree three.

Since there are nine vertices, the sum of the degrees is $9 \times 3 = 27$. But the handshaking lemma tells us that there should be $13\frac{1}{2}$ edges in our graph, which is clearly impossible! $\square$

14.2 Directed graphs

There are many graph theory problems which involve tournaments, one-way roads and the like. These are easily represented by a *directed graph*. This is a graph in which every edge has a direction, usually indicated by an arrow. For instance, in the case of a tournament, players can be represented by vertices and matches can be represented by directed edges, where the direction of an edge points from the winner to the loser.

Each vertex of a directed graph has both an *indegree*, which counts the number of incoming edges, as well as an *outdegree*, which counts the number of outgoing edges. We will denote the indegree and outdegree of a vertex v by $\mathrm{indeg}(v)$ and $\mathrm{outdeg}(v)$, respectively. Sometimes, we may want to refer to the total degree of a vertex, which is simply $\mathrm{indeg}(v) + \mathrm{outdeg}(v)$.

Problem Consider a squash tournament in which each of n people plays against every other person exactly once. Let L_k and W_k be the number of losses and wins of the kth player, respectively.

Prove that

$$L_1^2 + L_2^2 + \cdots + L_n^2 = W_1^2 + W_2^2 + \cdots + W_n^2.$$

Solution We rearrange the given equation and seek to prove that

$$(L_1 + W_1)(L_1 - W_1) + (L_2 + W_2)(L_2 - W_2) + \cdots + (L_n + W_n)(L_n - W_n) = 0.$$

In the directed graph which represents this tournament, the total degree for the kth player is

$$\mathrm{indeg}(v) + \mathrm{outdeg}(v) = L_k + W_k = n - 1.$$

Since this is a constant, we can divide the previous equation through by $n - 1$ to obtain

$$(L_1 - W_1) + (L_2 - W_2) + \cdots + (L_n - W_n) = 0.$$

So our aim is to prove that

$$L_1 + L_2 + \cdots + L_n = W_1 + W_2 + \cdots + W_n.$$

But this is now quite simple, since the left-hand side represents the sum of the indegrees, which is equal to the number of edges in the graph. Similarly, the right-hand side represents the sum of the outdegrees, which is also equal to the number of edges in the graph. $\square$

14.3 Connected graphs, cycles and trees

There's quite a bit of terminology in graph theory, but thankfully, the vast majority of it seems to make perfect sense.

- A *connected graph* is one in which it's possible to walk between any two vertices along the edges. In other words, the entire graph consists of only one piece.
- A *cycle* is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ with v_1 adjacent to v_2, v_2 adjacent to v_3, and so on, with v_n adjacent to v_1.
- A *tree* is a connected graph which has no cycles.

Problem If G is a graph with V vertices and E edges, prove that G is a tree if and only if it is connected and $E = V - 1$.

Solution First, we will prove that if G is a tree, then G is connected and $E = V - 1$. Of course, the fact that G is connected follows immediately from the definition of a tree, so it's only necessary to prove that $E = V - 1$.

We will proceed by induction on the number of vertices.

The base case is rather trivial, since you can see for yourself that the statement is true if G has 1 or 2 vertices.

Suppose now that the statement holds for all trees with n vertices and consider an arbitrary tree with $n + 1$ vertices. The main idea is to prove that there exists a vertex with degree 1.

Start at a random vertex and start walking along a path which never visits the same vertex twice. Sooner or later, you must get stuck in one of two ways: either you return to a vertex you already visited, or you reach a dead end. The former case implies that G contains a cycle, which is a contradiction. The latter case gives us the vertex of degree 1 that we are looking for. Now we simply remove this vertex and the single edge incident to it. This leaves us with a tree with n vertices and, by the inductive hypothesis, $n - 1$ edges. Hence, our original tree must have $n + 1$ vertices and n edges and so satisfies $E = V - 1$.

Now we aim to prove the converse, that if G is connected and $E = V - 1$, then G is a tree. Of course, the fact that G is connected is immediate, so it's only necessary to prove that G has no cycles.

Again, we will proceed by induction on the number of vertices.

The base case is rather trivial, since you can see for yourself that the statement is true if G has 1 or 2 vertices.

Suppose now that the statement holds for all G with n vertices and consider an arbitrary graph with $n + 1$ vertices which is connected and satisfies $E = V - 1$. Once again, the main idea is to prove that there exists a vertex with degree 1.

From the handshaking lemma, we can deduce that if every vertex in the graph has degree at least 2, then $V \leq E$, which is a contradiction. Furthermore, since G is connected, there are no vertices with degree 0. This gives us the vertex of degree 1 that we are looking for. Now we simply remove this vertex and the single edge incident to it. This leaves us with a graph which is connected with n vertices and $n - 1$ edges. By the inductive hypothesis, such a graph has no cycles. Furthermore, adding in a new edge incident to a vertex with degree 1 cannot create any cycles. In other words, G is a tree. $\square$

The solution to the previous problem should give you some ideas which you can use to prove the following important results.

Theorem Consider a graph with V vertices and E edges.

- If $E \leq V - 2$, then the graph is not connected.
- If $E \geq V$, then the graph contains a cycle.

14.4 Complete graphs and bipartite graphs

There are certain types of graph which seem to pop up in all sorts of problems.

- The *complete graph* K_n is the graph consisting of n vertices with an edge between every pair of vertices.
- A *bipartite graph* is one whose vertices can be coloured black and white such that every edge is incident to one black vertex and one white vertex.
- The *complete bipartite graph* $K_{m,n}$ is the graph consisting of m black vertices and n white vertices with an edge between each black and each white vertex.

The best way to remember jargon like this is to put it to good use!

Problem In the parliament of Frenemia each member is friends with exactly one other member and enemies with exactly one other member.

Prove that the members of parliament can be divided into two chambers so that no chamber contains a pair of mutual friends or a pair of mutual enemies.

Solution As is often the case, our first step will be to phrase this in graph theory language.

> In a graph, the edges are coloured red and blue in such a way that each vertex is incident to one red edge and one blue edge. Prove that the graph is bipartite.

We will give a procedure for colouring the vertices black and white such that every edge is incident to one black vertex and one white vertex.

If you try drawing some graphs which satisfy the conditions of the problem, then you'll find that they all consist of a bunch of cycles, each one with an even number of vertices.

With this in the back of our minds, let's start by taking any vertex and colouring it black. Now take another vertex adjacent to it and colour it white. And take another vertex adjacent to this one and colour it black. Continue walking around the graph, alternately colouring vertices black and white until you get stuck. Clearly, this can only happen in one of two ways: either you reach a dead end, or you meet a vertex which has already been coloured. The former case actually never arises, since every vertex has degree two while a dead end is a vertex of degree one. In the second case—and you should carefully think about why this is so—we must return to the vertex at which we started.

In summary, we have traversed a cycle, alternately colouring vertices black and white. Now a problem arises if the cycle has an odd number of vertices. But we've been told that the edges alternate between red and blue, so there must be an even number of vertices along the cycle. At this stage, we have successfully managed to colour some of the vertices black and white. If we have coloured every vertex, then we are done, but if not, then we simply repeat the process, starting at an uncoloured vertex.

This recipe will eventually colour every vertex in such a way that each edge is incident to one black vertex and one white vertex, so the graph is certainly bipartite. □

14.5 Pigeonhole principle

Let's return to the graph theory party scene! We will assume that at a graph theory party, every pair of people either know each other, in which case we call them friends, or they do not know each other, in which case we call them strangers. We will use vertices to represent people, red edges to represent friends and blue edges to represent strangers. Therefore, every party is simply a complete graph with each edge coloured red or blue.

Problem Prove that, at any party with six people, there must exist three mutual friends or three mutual strangers.

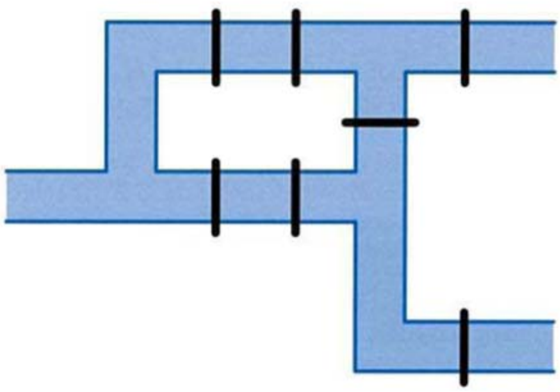
Solution In graph theory terminology, this problem translates to the following.

> Prove that given a complete graph on six vertices with each edge coloured red or blue, there exists a monochromatic[2] triangle.

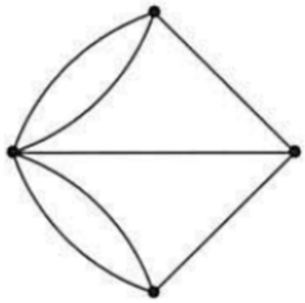
Consider a random partygoer A. Of the five edges incident to A, the pigeonhole principle guarantees that at least three of them are the same colour. Without loss of generality, let these edges be red and let them join A to the party people B, C and D. If BC is red, then triangle ABC is red. If CD is red, then triangle ACD is red. If DB is red, then triangle ADB is red. So to avoid a red triangle, the edges BC, CD and DB must all be blue, which forces triangle BCD to be blue. So we simply cannot avoid having a monochromatic triangle. □

14.6 Euler trails

In the eighteenth century, the inhabitants of Königsberg, now known as Kaliningrad, liked to walk along the Pregel River. They would use the city's seven bridges to cross over to one of the two islands in the river or to the other bank of the river. And they often wondered whether or not it was possible to design a walking tour which crossed each bridge exactly once. Here is a map of the situation.



The people tried in vain for many years until the great mathematician Leonhard Euler proved that it was impossible. His first move was to reduce the map to a graph, with each vertex representing a land mass and each edge a bridge as in the following diagram.



In this section only, we will allow a graph to have loops and multiple edges. So what we would like to know is whether it's possible to walk around the graph, traversing every edge exactly once. Such a walk is known as an *Euler trail*.

Problem Prove that the Königsberg graph has no Euler trail.

Solution Let's suppose that the Königsberg graph has an Euler trail and hope for a contradiction.

We'll start with the obvious fact that it must start somewhere and end somewhere. Between the start and end, each time we visit a vertex, we must have walked along two edges incident to it, one going in and one coming out. Thus every vertex other than the start and end must have even degree. So if there exists an Euler trail, then there can be at most two vertices of odd degree. But you can see for yourself that the graph has four vertices of odd degree, a contradiction. □

This proof shows that, if a graph has an Euler trail, then it must have at most two vertices of odd degree. However, by the handshaking lemma, we know that any graph has an even number of vertices of odd degree. So for a graph to have an Euler trail, the number of vertices of odd degree must be either 0 or 2. It turns out that the converse of this statement is true as long as the graph is connected.

Problem Prove that a graph has an Euler trail if and only if it is connected and has 0 or 2 vertices of odd degree.

Nailing down the details of this proof requires quite a bit of work. We will only present the main ideas of the proof and leave it up to you, dear reader, to sort out all of the details!

Solution Suppose that a connected graph has 0 or 2 vertices of odd degree. If there are 2 vertices of odd degree, call one of them S and one of them F. If there are 0 vertices of odd degree, pick any vertex and call it both S and F. We will prove that there exists an Euler trail which starts at S and finishes at F.

We will proceed by strong induction on the number of edges.

For a connected graph with one edge, it's easy to find an Euler trail!

Now assume that we have a graph with E edges and that the result is true for any graph with fewer than E edges. The idea is to start at S and commence walking randomly, never traversing the same edge twice. Since there are only finitely many edges, sooner or later you will get stuck somewhere. It's impossible to get stuck at a vertex with even degree, because each time we can walk in, there will always be an edge along which we can walk out. In fact, it turns out (and you should think about why) that the only place we can get stuck is at F.

If our walk so far has traversed every edge of the graph, then we are done! Otherwise, we've missed some of the edges and these remaining edges must form a number of smaller connected graphs $G_1, G_2, \ldots, G_n$. Each of these components G_k must have all its vertices of even degree (think carefully about why this must be so). And each such G_k intersects our previously constructed walk at some vertex v_k. Furthermore, each G_k has fewer than E edges, so the inductive hypothesis guarantees that there is an Euler trail for each G_k which starts and finishes at v_k.

Now we modify our original walk as follows. Start the same way but for each k whenever you reach one of the vertices v_k, take a detour along the Euler trail for G_k. Our new walk now traverses all the edges from the original walk, along with all the edges from the G_k. So it's an Euler trail for the original graph. □

That was quite a difficult argument, so take some time to think about it carefully. The idea of the random walk which cannot be stopped turns out to be a very useful one in graph theory.

14.7 Paths

A *walk* is any sequence of vertices $v_1, v_2, \ldots, v_n$ with v_1 adjacent to v_2, v_2 adjacent to v_3, and so on, with v_{n-1} adjacent to v_n.

A *path* is any walk with distinct vertices.

Problem In Eulerland, there are 100 cities and two airlines, Air Gauss and Air Jordan. For any two cities in Eulerland, exactly one of the companies provides direct flights in both directions between them. It's known that there are two cities a and b such that it is impossible to travel from a to b using only Air Jordan flights.

Prove that it's possible to travel between any two cities in Eulerland using only Air Gauss flights.

Solution We can use vertices to represent cities and edges to represent flights, but how can we represent the two airlines? Simple! We use red edges to represent Air Gauss and blue edges to represent Air Jordan. It turns out that a whole variety of graph theory problems involve colouring things in. In graph theory language, the problem translates to the following.

> Suppose that each edge of the complete graph on 100 vertices is coloured red or blue. It's known that there are two vertices a and b such that there is no red path from a to b.
>
> Prove that there is a blue path between any two vertices.

Let A denote the set of all vertices which are connected to a by a red path, including a itself. Similarly, let B denote the set of all vertices which are connected to b by a red path, including b itself. And finally, let C denote the set of remaining vertices. (It is possible that C is empty.) We have the following observations.

- Every edge between a vertex in A and a vertex in B must be blue—for otherwise, there would be a red path from a to b.
- Every edge between a vertex in A and a vertex in C must be blue—for otherwise, that vertex in C could be reached by a red path from a.
- Every edge between a vertex in B and a vertex in C must be blue—for otherwise, that vertex in C could be reached by a red path from b.

We will now prove that for two arbitrary vertices u and v, there exists a blue path between them. We've already established that there exists a blue edge, and hence a blue path, between any two vertices which are in different groups. So all that remains is to consider when u and v lie in the same group.

- If u and v lie in A, consider the blue path $u \to b \to v$.
- If u and v lie in B, consider the blue path $u \to a \to v$.
- If u and v lie in C, consider the blue path $u \to a \to v$.

Therefore, in all possible cases, there is a blue path between u and v. □

14.8 Extremal principle

In chapter 1, we discussed the many advantages of using the extremal principle, that is, considering the minimum or maximum of some value. The extremal principle is very useful in graph theory as you can see for yourself in the following problems!

Problem In the country of König, it's possible to travel by plane between any two of the cities, although you might have to take several flights, stopping at other intermediate cities along the way. By a journey, we mean a sequence of flights which never visits the same city twice. Let m be the maximum possible number of flights on a journey between two cities in König.

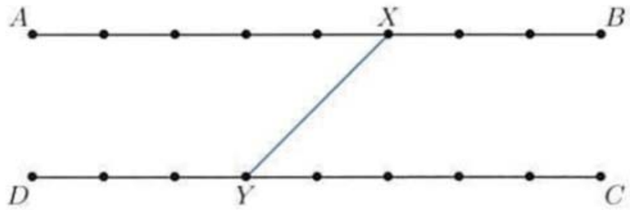
Prove that any two journeys of length m must have at least one city in common.

Solution Clearly, we can interpret cities as vertices, flights as edges, and journeys as paths. If we define the *length* of a path to be the number of edges that it traverses, then the problem can be restated as follows.

> If the longest path in a connected graph has length m, prove that any two paths of length m in the graph must share a vertex.

To obtain a contradiction, assume there exist two paths P_1 and P_2 of length m in the graph that don't share a vertex. Let P_1 join vertex A to vertex B and let P_2 join vertex C to vertex D. The aim is to find a path with length greater than m, which will then contradict the fact that m is the maximum length of a path.

How might we construct such a path? Well, since the graph is connected, there must be a path from a vertex on P_1 to a vertex on P_2. In fact, let X be a vertex on P_1 and Y a vertex on P_2 such that there exists a path from X to Y of minimal length. Using such a path of minimal length ensures that it is a path which cannot pass through any other vertices of P_1 or P_2, because otherwise it would not be of minimal length.



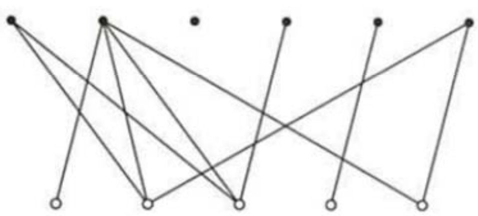
Note that either the path from A to X or the path from B to X has length at least $\frac{m}{2}$. Without loss of generality, let it be from A to X. Similarly, either the path from C to Y or the path from D to Y has length at least $\frac{m}{2}$. Without loss of generality, let it be from C to Y. Now the path from A to X to Y to C passes through no vertex twice, and has length strictly greater than m. However, this contradicts the fact that m is the maximum length of a path. Therefore, any two paths of length m in the graph must share a vertex. □

Problem There are several boys and girls at a party, where each girl dances with at least one boy but no boy dances with every girl.

Prove that there exist two boys B_1 and B_2 and two girls G_1 and G_2 such that B_1 dances with G_1, B_2 dances with G_2, B_1 does not dance with G_2 and B_2 does not dance with G_1.

It's quite difficult to attack this problem directly. As we'll soon see, the best approach is to consider the boy who dances with the most girls. Like some other proofs in this book, this may appear as surprising as a magician pulling a rabbit out of a hat! But the point, dear reader, is for you to carefully read over the proof and ask how you could have thought of it yourself. What features of the problem lead you to think about the extremal principle? And why would you apply it in this particular way? If you understand the answers to these questions, then you will soon be ready to start pulling rabbits out of your own hat!

Solution Of course, vertices will represent people while edges represent couples who dance together. It's useful to use black vertices for the boys and schematically place them at the top of the diagram and to use white vertices for the girls and schematically place them at the bottom of the diagram.



We'll start by considering one of the boys, Max say, who dances with the maximum number of girls. Now there is a girl, Anne say, who doesn't dance with Max. And Anne must dance with some boy, Bob say, who isn't Max. So if one of Max's dance partners, of whom there are maximally many, does not dance with Bob, then we're done!

But if Bob danced with all of Max's dance partners, as well as with Anne, then Bob would have danced with more girls than Max. This contradiction means that there must be some girl, Carly say, who dances with Max but not Bob.

So we can take B_1, B_2, G_1 and G_2 to be Max, Bob, Carly and Anne, respectively. $\square$

14.9 Count and count again

The handshaking lemma asserts that, in any graph, the sum of the degrees of all the vertices is equal to twice the number of edges. One way to prove this is to consider the number of endpoints of edges. Counting these in one way, we obtain the sum of the degrees of all the vertices and counting these in another way, we obtain twice the number of edges.

We have already witnessed the virtues of double counting in section 13.8 and here, we'll see that it can be particularly useful when dealing with graphs.

Problem In a senate, there are 30 senators and each pair of them are either mutual friends or mutual enemies. Each senator has exactly 6 enemies and every group of 3 senators forms a commission.

Find the total number of commissions whose members are either all mutual friends or all mutual enemies.

Solution This problem can be paraphrased purely in graph theory terminology.

> Each edge of the complete graph K_{30} is coloured either red or blue. Each vertex is incident to exactly 6 blue edges.
>
> Find the total number of monochromatic triangles.

Let X be the number of monochromatic triangles. We define a *vee* to be a pair of edges which are incident to the same vertex. The trick here is to count *colourful vees*—that is, those consisting of one red edge and one blue edge—in two different ways.

- If we concentrate on one particular vertex, we see that it's incident to 6 blue edges and 23 red edges. This means that there are $6 \times 23 = 138$ colourful vees formed from the edges incident to one particular vertex.

  Since there are 30 vertices altogether, the total number of colourful vees is

  $$30 \times 138 = 4140.$$

- Note that there are $\binom{30}{3} = 4060$ triangles in our graph. Since X of these are monochromatic, we know that $4060 - X$ of these are not. But in a monochromatic triangle, there are no colourful vees, while in a triangle which is not monochromatic, there are two colourful vees.

  Therefore, the total number of colourful vees is

  $$2(4060 - X) = 8120 - 2X.$$

Now we simply need to equate these two results and we end up with

$$\begin{aligned} 4140 &= 8120 - 2X \\ \Rightarrow \qquad X &= 1990. \end{aligned}$$

$\square$

14.10 Planar graphs

When trying to solve graph theory problems, obviously you draw graphs. Unfortunately, the edges sometimes cross and that just looks plain ugly. It would be nice to draw our graphs without any edges crossing and, when this is possible, we call the graph *planar*.

A planar graph drawn without edges crossing will always divide the plane into regions, one of which is infinitely large, which we call *faces*. One of the most useful theorems concerning planar graphs is the following.

Euler's formula For a connected planar graph with V vertices, E edges and F faces, $V - E + F = 2$. When using Euler's formula, we always include the infinitely large face on the outside.

Problem Suppose a planar graph with E edges divides the plane into F faces.

Prove that $3F \leq 2E$.

Solution Let's count the number E of edges. Imagine cutting the planar graph along its edges. We get a collection of polygons and one figure which might be called an 'anti-polygon' corresponding to the outside face. Since each original edge splits into two edges, the total number of newly formed edges is $2E$. However, each of our F newly formed polygons, including the outside anti-polygon, has at least three edges. Thus the number of newly formed edges is at least $3F$. It follows that $3F \leq 2E$. □

Problem Prove that K_5, the complete graph on five vertices, is not planar.

Solution To obtain a contradiction, let's suppose that K_5 is planar.

Given that $V = 5$ and $E = 10$, Euler's formula tells us that if we could draw K_5 in the plane without edges crossing, then we would have $F = 7$. But from the previous problem, $2E \geq 3F$ and so $20 \geq 21$. Contradictions don't come more blatantly than that! □

Problem If a graph is planar, prove that it has at least one vertex of degree less than or equal to five.

Solution In the hope of finding a contradiction, let's start with the assumption that there exists a planar graph, all of whose vertices have degree at least six.

Suppose that this graph has V vertices, E edges and divides the plane into F faces. Then the sum of the degrees is at least $6V$, so the handshaking lemma asserts that $6V \leq 2E$ or equivalently,

$$V \leq \frac{E}{3}.$$

From the previous problem, we know that $F \leq \frac{2E}{3}$.

Substituting these two inequalities into Euler's formula, we obtain

$$2 = V - E + F \leq \frac{E}{3} - E + \frac{2E}{3} = 0,$$

an obvious contradiction. □

14.11 Polyhedra

If you take a bunch of polygons and glue them together so that no side is left unglued, then the resulting object is usually called a *polyhedron*. Typical examples include the tetrahedron, the square pyramid, the cube and the soccer ball with its pentagonal and hexagonal patches. The corners of the polygons are called *vertices*, the sides of the polygons are called *edges* and the polygons themselves are called *faces*. We say that a polyhedron is *convex* if, for each plane which lies along a face, the polyhedron lies on one side of that plane. So, for example, the cube is convex while the polyhedron formed by gluing three congruent cubes together to form an L shape is not.

The fact that every polyhedron is a graph is a rather simple statement. This is because if you have a polyhedron and simply ignore the faces, then what you have left over is just a bunch of vertices connected in pairs by edges, or in other words, a graph.

A more interesting statement is the following fact. Every convex polyhedron corresponds to a planar graph. So why is this true? Well, suppose that your polyhedron is made from some sort of rubbery material, like a balloon. If you pop the balloon by removing one of the faces, then what remains is a rubbery sheet with the vertices and edges still drawn on it. Now just stretch this out flat onto a table and there you have your planar graph. Note that this planar graph has the same number of vertices, edges and faces as the original polyhedron. The face that we removed from the polyhedron now corresponds to the infinitely large face of the planar graph. In particular, Euler's formula applies to all convex polyhedra.

Problem Suppose that you have a convex polyhedron and you are told that each face is a quadrilateral or a hexagon and that three faces meet at every vertex. Furthermore, every quadrilateral face shares an edge with four hexagonal faces, while every hexagonal face shares an edge with three quadrilateral faces and three hexagonal faces.

Deduce the number of quadrilateral faces and the number of hexagonal faces of the polyhedron.

Solution Let V be the number of vertices, E the number of edges, Q the number of quadrilateral faces, and H the number of hexagonal faces of the polyhedron. We will deduce these four values by writing down four equations that they satisfy and solving them.

- The first equation comes from applying the handshaking lemma to the polyhedron. Since three faces meet at every vertex, every vertex of the polyhedron has degree three. Therefore, the sum of the degrees is simply $3V$ and we obtain the equation

$$3V = 2E. \tag{1}$$

- The second equation comes from double counting the edges via face contributions. Each quadrilateral contributes four edges and each hexagon contributes six edges. But each edge has been double counted due to the fact that it is adjacent to two faces so we find that $4Q + 6H = 2E$, or equivalently,

$$E = 2Q + 3H. \tag{2}$$

- The third equation comes from a clever double counting argument. The trick here is to count the number of times a quadrilateral face shares an edge with a hexagonal face. This happens four times for each quadrilateral face. In other words, the number is $4Q$. Arguing in a different way, we can say that this happens three times for each hexagonal face. In other words, the number is $3H$. Of course, these numbers must be the same, so we must have

$$4Q = 3H. \tag{3}$$

- The fourth equation is simply Euler's formula. Since $F = Q + H$ we may write

$$V - E + Q + H = 2. \tag{4}$$

It is not hard to solve equations (1)–(4). For example, using (3) we have $H = \frac{4}{3}Q$. Put this into (2) to find $E = 6Q$. Put this into (1) to find $V = 4Q$. Finally put everything into (4) to obtain $Q = 6$. We may then go on to find $H = 8$, $E = 36$ and $V = 24$. □

14.12 Graph theory and inequalities

This nice intersection of mathematical topics doesn't appear too often, but is very interesting indeed.

Problem Given a graph with n vertices and E edges, prove that the number of triangles is at least

$$\frac{4E^2}{3n} - \frac{nE}{3}.$$

Solution The basic idea of the solution is rather simple and elegant. Start by labelling the vertices $v_1, v_2, \ldots, v_n$, and suppose that they have degrees $d_1, d_2, \ldots, d_n$, respectively.

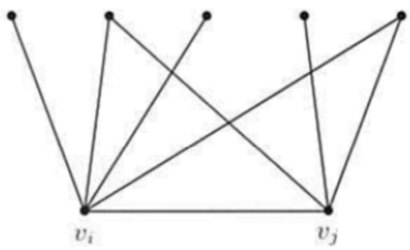
Next take two vertices v_i and v_j which are connected by an edge. Then there are $d_i - 1$ edges which emanate from v_i to the rest of the graph and $d_j - 1$ edges which emanate from v_j to the rest of the graph. Since there are $n - 2$ other vertices, then by the pigeonhole principle a triangle will be created if

$$(d_i - 1) + (d_j - 1) > n - 2.$$

In fact, we can guarantee that the edge between v_i and v_j will be involved in at least

$$(d_i - 1) + (d_j - 1) - (n - 2) = d_i + d_j - n$$

triangles.



If we sum over all possible edges, then we will count each triangle at most three times. So, letting T denote the total number of triangles, we have the inequality

$$3T \geq \sum_{\text{edges}} (d_i + d_j - n) = \sum_{\text{edges}} (d_i + d_j) - nE.$$

Since we are aiming for the result $3T \geq \frac{4E^2}{n} - nE$, all that remains to be proved is the inequality

$$\sum_{\text{edges}} (d_i + d_j) \geq \frac{4E^2}{n}.$$

Note that in this sum over edges, the number d_i appears once for each edge that is incident to the vertex v_i. That is, d_i occurs in the sum d_i times. So we can write this inequality equivalently as

$$\sum_{i=1}^{n} d_i^2 \geq \frac{4E^2}{n} = \frac{(2E)^2}{n} = \frac{1}{n}\left(\sum_{i=1}^{n} d_i\right)^2.$$

Here, we've used the handshaking lemma to express $2E$ as the sum of the degrees.

We've finally arrived at a point where the problem is reduced to algebra. One way to finish off the problem is to invoke the power means inequality[3], the QM–AM in particular.

From the power means inequality, we know

$$\sqrt{\frac{d_1^2 + d_2^2 + \cdots + d_n^2}{n}} \geq \frac{d_1 + d_2 + \cdots + d_n}{n}$$

and this is exactly what we need to complete the proof. $\square$

15

Games and invariants

Games? Who said games? We always seem ready to play a game. Why? Because games are fun! Games often turn up in mathematics problems and if there is one important point you should remember, it is this:

Play the game!

The reason is that you gain insight into what the problem is all about. In fact you might need to play the game for quite a while before you notice something that just might be the key element in solving the problem.

15.1 Number invariants

There are lots of fun problems which look something like this.

> You are given the configuration A and a set of legal moves which change the configuration. Can you use these moves to end up with the configuration B?

If the answer is *yes*, then you could prove this by simply demonstrating a sequence of legal moves which takes configuration A to configuration B. But if the answer is *no*, then you have to be much trickier. More often than not, you will need to use the idea of an invariant. An *invariant* is something—for example, a number—which we can associate to every configuration such that performing a legal move doesn't change it. So if the value of the invariant for configuration A differs from its value for configuration B, then the task is impossible to achieve, no matter how hard you try. All of this probably sounds quite cryptic and will remain so until we see some examples in action.

Problem Given some numbers, we may choose two of them, say a and b, and replace them with the single number $a + b$.

Prove that if we start with the numbers

$$1, 2, 3, \ldots, 100$$

and apply the operation 99 times, we always end up with the same final number.

Solution It should be obvious that, after applying the operation 99 times, we will be left with only one number. And it should seem intuitively clear that, no matter what order we choose to perform the additions, the final number will be the sum of the original numbers. This is certainly true, but we can state it in the language of invariants by associating to the numbers $a_1, a_2, \ldots, a_n$ the sum

$$I = a_1 + a_2 + \cdots + a_n.$$

The number I is an invariant because it doesn't change when we apply the operation. Therefore, it must be the same for both the initial and final configurations. Indeed, we have

$$1 + 2 + \cdots + 100 = 5050,$$

so the final number will always be 5050. $\square$

Problem Given some numbers, we may choose two of them, say a and b, and replace them with the single number $ab + a + b$.

Prove that if we start with the numbers

$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{100}$$

and apply the operation 99 times, we always end up with the same final number.

Solution Again, it should be obvious that, after applying the operation 99 times, we will be left with only one number. Although this problem is more difficult than the previous one, it's still easily solved once you stumble upon the correct invariant. We simply associate to the numbers $a_1, a_2, \ldots, a_n$ the value

$$I = (a_1 + 1)(a_2 + 1) \cdots (a_n + 1).$$

Removing the numbers a and b from our list divides the value of I by

$$(a + 1)(b + 1).$$

However, adding the number $ab + a + b$ to our list multiplies the value of I by

$$ab + a + b + 1 = (a + 1)(b + 1).$$

So, the number I is an invariant because it doesn't change when we apply the operation.

The value of I for the initial configuration can be determined using the following veritable feast of cancellation.

$$I = \left(\frac{1}{1} + 1\right)\left(\frac{1}{2} + 1\right)\left(\frac{1}{3} + 1\right) \cdots \left(\frac{1}{100} + 1\right) = \frac{\cancel{2}}{1} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{4}}{\cancel{3}} \cdots \frac{101}{\cancel{100}} = 101$$

Since the invariant must be the same for both the initial and final configurations, the final number will always be 100. $\square$

Of course, the difficulty in this solution lies in finding the actual invariant to use. A good problem will often leave clues which can guide you in the right direction. For example, an observation that might lead you to discover the correct invariant for this problem is the fact that $ab + a + b$ almost factorises as $(a + 1)(b + 1) - 1$.

15.2 Parity

Another extremely useful invariant is *parity*, which essentially means oddness or evenness. You should always keep your eyes open for a parity argument.

Problem Given some numbers, we may choose two of them, say a and b, and replace them with the difference $|a-b|$. Suppose that we start with the numbers

$$1, 2, 3, \ldots, 2n,$$

for some odd positive integer n.

If we apply the operation $2n-1$ times, show that we always end up with an odd number.

Solution Since we always end up with an odd number, a likely candidate for an invariant is the parity of the sum of the numbers. Initially, we have

$$1+2+3+\cdots+2n = \frac{2n(2n+1)}{2} = n(2n+1),$$

which is odd.

To show that the parity of the sum of the numbers doesn't change after each operation, suppose that we remove the numbers a and b, where we assume without loss of generality that $a \geq b$. Then the sum of the numbers would change by

$$-a-b+(a-b) = -2b,$$

which is an even number.

So the parity of the sum of the numbers is indeed an invariant. This shows that the final number must be odd. □

Problem You are given an 8×8 chessboard where the squares are coloured black and white in the usual way. You are allowed to switch the colours of all the squares in a row or column.

Can you end up with exactly one black square on the board?

Solution The first thing you should do is take out some pen and paper, draw a chessboard, and try to obtain exactly one black square. Playing around with the problem in this way should convince you, sooner or later, that you probably cannot do it. In fact, you might even be able to conjecture that it's impossible to obtain an odd number of black squares. So let's try to show that the parity of the number of black squares is an invariant.

We note that if a row or column has x black squares, then after switching the colours, it will have $8-x$ black squares. So the change in the number of black squares is

$$-x+(8-x) = 8-2x,$$

obviously an even number.

Therefore, the parity of the number of black squares is indeed an invariant and, since there are initially 32 black squares, it's impossible to end up with exactly one black square on the board. □

15.3 Modular arithmetic invariants

Parity is the same thing as considering an integer modulo 2. But in the world of invariants, sometimes modulo 2 just isn't good enough and you might need to consider integers modulo other numbers.

Problem A magic lolly machine has the property that if two of Stephen's Stupendous Smarties are inserted, then three of Justin's Jumbo Jaffas come out. Also, if three of Stephen's Stupendous Smarties are inserted, then two of Justin's Jumbo Jaffas come out. The reverse also occurs, that is, if two of Justin's Jumbo Jaffas are inserted, then three of Stephen's Stupendous Smarties come out. And if three of Justin's Jumbo Jaffas are inserted, then two of Stephen's Stupendous Smarties come out.

(a) If I want to turn two Jaffas into exactly 61 Jaffas, what is the minimum number of Smarties that I also end up with?

(b) Can I turn one Jaffa and one Smartie into 10 Jaffas and no Smarties?

Solution

(a) The first thing you should do is take out some pen and paper, pretend you have a magic lolly machine, and work out how many Jaffas and Smarties you can obtain. For example, you might come up with the following possibilities, created from just two Jaffas.

Jaffas	2	0	3	1	7	1	13	1	28	0	63	61
Smarties	0	3	1	4	0	9	1	19	1	43	1	4
Difference	2	−3	2	−3	7	−8	12	−18	27	−43	62	57

It seems that if I want to turn two Jaffas into 61 Jaffas, then I might have to create at least four Smarties.

If we're looking for an invariant, we could think about the sum of the number of Jaffas and Smarties, but this doesn't seem to be very useful.

The *difference*, on the other hand, is very interesting indeed. In fact, it seems that

$$J - S \pmod 5$$

is an invariant, where J is the number of Jaffas and S is the number of Smarties.

Let's prove this by considering the effect of one operation of the magic lolly machine. The pair (J, S) can become any of

$$(J-2, S+3), \quad (J-3, S+2), \quad (J+3, S-2) \quad \text{or} \quad (J+2, S-3).$$

In all of these cases, the difference changes from $J - S$ to $J - S \pm 5$, so we can now conclude that $J - S \pmod 5$ is indeed an invariant.

Initially, we only have two Jaffas and $J - S \equiv 2 \pmod 5$. So to end up with 61 Jaffas, we need at least four Smarties in order to maintain $J - S \equiv 2 \pmod 5$.

But be careful! While the invariant allows us to decide that certain (J, S) combinations are *not* possible, it doesn't necessarily allow us to decide which ones actually *are* possible. So we still need to show that one can obtain 61 Jaffas and four Smarties, but this we have already accomplished in the table above. □

(b) Consider the problem of turning one Jaffa and one Smartie into 10 Jaffas and no Smarties. Our invariant does not exclude this as a possibility, since $J - S \equiv 0 \pmod 5$ for both cases. However, observing that we need at least two Jaffas or at least two Smarties to use the magic lolly machine tells us that this task is impossible. $\square$

15.4 Colouring invariants

You may have come across the following classic puzzle before. The idea behind its beautifully simple solution can be generalised to solve far more difficult problems.

Problem Consider an 8×8 chessboard, where the top-right and bottom-left squares have been removed.

Is it possible to tile this mutilated chessboard with 2×1 rectangles?

Solution The first thing you should do is take out some pen and paper, draw a mutilated chessboard, and try to tile it with 2×1 rectangles. However, I can tell you right now that you will fail, not because your tiling skills are poor, but because the task is impossible!

Perhaps surprisingly, the key to this problem is the standard black-and-white colouring of the chessboard. This is because a 2×1 rectangle will always occupy two adjacent squares on the chessboard and hence, cover one black square and one white square. Therefore, any part of the chessboard that can be tiled with 2×1 rectangles must have the same number of black and white squares.

Now we note that the standard chessboard has 32 squares of each colour, while the mutilated chessboard is obtained by removing two squares of the same colour. Since there are now 30 squares of one colour remaining and 32 squares of the other colour, it is impossible to tile the mutilated chessboard with 2×1 rectangles. $\square$

We were lucky in this problem, because the standard 8×8 chessboard came with a colouring which helped our cause, free of charge. But sometimes, as in the next problem, you have to invent your own colouring.

Problem Show that a 10×10 chessboard cannot be tiled with 4×1 rectangles.

Solution The tactic is to find a colouring of the chessboard such that any 4×1 rectangle on the board occupies one square of each colour. Of course, this means that we require four colours, which we will call 0, 1, 2 and 3. Working along the bottom row of the chessboard, we may as well label the first four squares 0, 1, 2 and 3, in that order. After that, every square in the row must be coloured according to the repeating pattern $0, 1, 2, 3, 0, 1, 2, 3,$ and so on. If we apply the same argument along the columns, we might end up with the following colouring.

1	2	3	0	1	2	3	0	1	2
0	1	2	3	0	1	2	3	0	1
3	0	1	2	3	0	1	2	3	0
2	3	0	1	2	3	0	1	2	3
1	2	3	0	1	2	3	0	1	2
0	1	2	3	0	1	2	3	0	1
3	0	1	2	3	0	1	2	3	0
2	3	0	1	2	3	0	1	2	3
1	2	3	0	1	2	3	0	1	2
0	1	2	3	0	1	2	3	0	1

We call this a modulo 4 colouring, because if we label the rows and columns $0, 1, 2, \ldots$, then the square in row i and column j is coloured $i + j$ modulo 4.

This colouring certainly obeys the rule that a 4×1 rectangle on the board always occupies one square of each colour. Of course, we're hoping that there are not the same number of squares of each colour. One way to verify this is to simply count them and you would indeed find that this is true. However, that is rather pedestrian, so let's use a slicker, more stylish, approach.

We simply note that it is quite easy to demonstrate a tiling of the entire board except for the 2×2 square in the top-right corner. The tiled part of the board must certainly contain the same number of squares of each colour, otherwise we wouldn't have been able to tile it. However, the remaining part of the board does not because there is one square coloured 0, two squares coloured 1, one square coloured 2 and no squares coloured 3. Hence there cannot be the same number of squares of each colour on the entire chessboard. We conclude that a 10×10 chessboard cannot be tiled with 4×1 rectangles. $\square$

For other problems, you might need to use a modulo n colouring for some other positive integer n. Or something completely different might be needed! Using the notation (i, j) to represent the square in row i and column j, other useful colourings include the following.

- Colour (i, j) according to $i \pmod 2$. This yields a striped pattern.
- Colour (i, j) according to $(i \pmod 2, j \pmod 2)$. This uses four colours but is different from the modulo 4 colouring used in the above solution.

15.5 Monovariants

An invariant is something which doesn't change when you perform a particular move. On the other hand, a *monovariant* is a value which always gets larger or always gets smaller when you perform a particular move. For example, if you keep spending money without ever earning any, then you will never again have as much money as when you started spending. Here the monovariant is obviously the amount of money that you have. This idea is crucial to solving many problems, including the following.

Problem Given some numbers, we may choose two of them, say a and b, and replace them with the numbers

$$a + \frac{b}{2} \quad \text{and} \quad b - \frac{a}{2}.$$

If we start with a set of non-zero numbers S and keep applying the operation, show that we can never again obtain the set S.

Solution Let the numbers be $a_1, a_2, \ldots, a_n$ and consider the sum of squares

$$M = a_1^2 + a_2^2 + \cdots + a_n^2.$$

We will determine the change in M after we replace two of the numbers, say a and b.

$$\text{change in } M = \left(a + \frac{b}{2}\right)^2 + \left(b - \frac{a}{2}\right)^2 - a^2 - b^2 = \frac{a^2}{4} + \frac{b^2}{4} \geq 0$$

So M is not an invariant but a monovariant, because it never decreases. In fact, since S consists of non-zero numbers, M is guaranteed to increase after the first operation. So the same value of M can never again be obtained by applying the operation. Therefore, we can never again obtain the set S. □

For some reason, squares often feature in monovariant problems, as they did in the previous one. Next, we will see how the idea of a monovariant can help when it doesn't seem like it should.

Problem A unit fraction is a number of the form $\frac{1}{n}$, where n is a positive integer.

Prove that every rational number between 0 and 1 can be expressed as a sum of finitely many distinct unit fractions.

Solution We will show that this can be achieved using the greedy algorithm.[3] You should convince yourself that the solution to the problem follows immediately once we have proven the following statement.

> Given a rational number $0 < r < 1$, subtract the largest unit fraction less than or equal to r. Prove that, if we continue to do this, we must eventually reach the number 0 after finitely many subtractions. Furthermore, we never subtract the same unit fraction more than once.

The idea is to show that the numerator of the rational number is a monovariant which decreases until we reach 0.

Write the number r as

$$r = \frac{a}{b},$$

where a and b are relatively prime positive integers.

Let the largest unit fraction less than or equal to r be $\frac{1}{m}$, so that

$$\frac{1}{m} \leq \frac{a}{b} < \frac{1}{m-1}.$$

Therefore, after one step, we have the fraction

$$\frac{a}{b} - \frac{1}{m} = \frac{am - b}{bm}.$$

However, the inequality $\frac{a}{b} < \frac{1}{m-1}$ implies that $am - b < a$.

This means that the numerator of the fraction strictly decreases after each step[4], until we eventually reach the number 0.

It should be clear that we never subtract the same unit fraction more than once because if $\frac{1}{m}$ is the largest unit fraction less than or equal to r, then $r - \frac{1}{m}$ is too small to be able to subtract $\frac{1}{m}$ again. □

15.6 Invariants as cost

In our increasingly capitalistic world, we regularly think about money. So it's sometimes useful to think of an invariant as the cost of something, like we do in the following example.

Problem Initially, there is a pawn placed in each square in the bottom four rows of an 8×8 chessboard. If two pawns are in adjacent squares of the same row, you are allowed to remove them and add a pawn in the row above.

Is it possible to place a pawn in the top row of the chessboard?

Solution Let's number the rows in order so that 1 is the lowest row while 8 is the highest row. The idea behind this problem is as follows: since two pawns in row R can become one pawn in row $R+1$, we should consider the cost of a pawn in row $R+1$ to be twice the cost of a pawn in row R.

So suppose that pawns in row 1 cost \$1, pawns in row 2 cost \$2, pawns in row 3 cost \$4, and so on, so that pawns in row 8 cost \$128. The total value of the pawns initially on the chessboard is

$$8 \times (\$1 + \$2 + \$4 + \$8) = \$120.$$

Since this cost is invariant, it's impossible to place a pawn in the top row of the chessboard, which would cost \$128. □

15.7 Permutation parity

A *permutation* is just a rearrangement of objects. More formally a permutation of a set S is a bijection $f: S \to S$.

For finite sets there is a convenient two-line notation that represents a permutation. For example, if $S = \{1, 2, 3, 4\}$, then the notation

$$\begin{pmatrix} 1\,2\,3\,4 \\ 2\,3\,1\,4 \end{pmatrix}$$

represents the permutation $f(1) = 2$, $f(2) = 3$, $f(3) = 1$ and $f(4) = 4$.

A shorter one-line notation is just to write

$$(2\,3\,1\,4)$$

to mean the same thing.

We can combine permutations by applying them successively. Thus if we applied $(2\,3\,1\,4)$ and then followed this by $(1\,2\,4\,3)$, the net result is $(2\,4\,1\,3)$. This is written as

$$(1\,2\,4\,3)(2\,3\,1\,4) = (2\,4\,1\,3) \quad \text{or} \quad \begin{pmatrix} 1\,2\,3\,4 \\ 1\,2\,4\,3 \end{pmatrix}\begin{pmatrix} 1\,2\,3\,4 \\ 2\,3\,1\,4 \end{pmatrix} = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,4\,1\,3 \end{pmatrix}.$$

Note that because permutations are functions, we write the order of composition of permutations from right-to-left, as shown in the above example.

It turns out that permutations have a parity invariant which is highly useful.

Parity of a permutation The *parity* of a permutation is defined as the parity of the number of inversions in the permutation.

A pair $i < j$ is said to be an *inversion* if j occurs before i in the permutation. For example, the permutation $(2\,3\,1\,4)$ has inversions $(2, 1)$ and $(3, 1)$. Thus $(2\,3\,1\,4)$ is an *even* permutation. The permutation $(1\,2\,4\,3)$ has $(4, 3)$ as its only inversion and so it is an *odd* permutation. The composition $(2\,4\,1\,3)$ has three inversions so it is an odd permutation.

The important thing you need to know about permutation parity is the following.

Parity of compositions of permutations When composing two permutations, their parities behave in the same way as ordinary parity does for addition.

In particular we have the following.

- The composition of two even permutations is an even permutation.
- The composition of two odd permutations is an even permutation.
- The composition of an even permutation and an odd permutation is an odd permutation.

These can be proved via the following exercises which we leave for you to do.

- A *transposition* is a permutation which swaps two elements. Prove that applying a transposition to a permutation changes its parity.
- Prove that any permutation can be written as a composition of transpositions.

- Thus conclude that the parity of a permutation is simply the parity of the number of transpositions which can be used to express it.[5]

The following illustrates how permutations and parity may be used to solve a problem quickly.

Problem Andrew, Brenda and Chris are competing in a three-person race. At some point in the race Andrew is winning, with Brenda coming second and Chris third. From here on until the end of the race it is noted that there are 36 times when their relative order changes. The race ends with Brenda finishing third.

If at no point in time did all three draw level, who won the race?

Solution The original order may be notated as [A,B,C] (Andrew first, Brenda second, Chris third). Each change in the order is a transposition which is an odd permutation. Since there are 36 such permutations, the net result is an even permutation of [A,B,C].

We are given that B comes last, so the order is either [A,C,B] or [C,A,B]. But [A,C,B] is an odd permutation, whereas [C,A,B] is an even permutation. Thus the final order must be Chris first, Andrew second and Brenda third. □

15.8 Combinatorial games

For the remainder of this chapter, we consider combinatorial games, which are games that satisfy the following conditions.

- There are two players who take turns to move.
- There is no luck involved.
- The game always ends after a finite number of moves.
- Each player's moves are known to the other player.
- Either one player wins and the other loses or the game results in a draw.

The most important result concerning combinatorial games is the following.

Fundamental theorem of combinatorial games In a combinatorial game, either there exists a winning strategy for one of the players or both of them can force a draw.

Problem There are two piles on a table, one containing 1000 coins and the other containing 1001 coins. Two players take turns to remove at least one coin from one of the piles. The winner is the player who removes the last coin from the table. Since a draw is impossible, the fundamental theorem of combinatorial games guarantees that there exists a winning strategy for one of the players.

Which player has a winning strategy?

Solution The easiest way to determine which player has a winning strategy is simply to come up with a winning strategy which works!

In this game, the first player has a winning strategy which we can describe as follows: take coins from the larger pile to leave two piles of the same size.

For this strategy to work, the first player must always see two piles of different sizes on their move. This is certainly true for their first move since the piles contain 1000 and 1001 coins. Subsequently, the second player is always faced with two piles of the same size which they must turn into two piles of different sizes. So the first player continues to see two piles of different sizes when it is their turn to move.

But we still have to show why the second player has no chance of winning against this strategy. The reason for this is that, as we have already mentioned, the second player always leaves two piles of different sizes. However, to win the game, you must leave two piles of equal size, both with zero coins. Since the second player can never win the game as long as the first player follows this plan, what we have described is a winning strategy for the first player. □

15.9 Position analysis

Often, a winning strategy doesn't present itself very easily but can still be found by using a technique called *position analysis*. If you are about to move and you have a strategy which allows you to win, then we call the current position of the game a *winning position*. On the other hand, if you are about to move and you do not have a strategy which allows you to win, then we call the current position of the game a *losing position*. One corollary of the fundamental theorem of combinatorial games is the following extremely useful result.

Theorem In a combinatorial game with no draws, every position of the game can be categorised as winning or losing. Furthermore, it must be the case that

- from a winning position, it is *possible* to move to a losing position, and
- from a losing position, it is *impossible* to move to a winning position.

The idea of position analysis is to use this result to analyse enough small cases until we see can some general patterns. If all goes well, then we should be able to describe all of the winning positions, all of the losing positions, and perhaps even a winning strategy.

Problem Initially, there are n coins on a table and two players take turns to remove 1, 2, 3 or 4 coins. A player wins if he removes the last coin.

Find the winning and losing positions.

Solution In this problem, we can describe the position of the game by the number of coins on the table. It is clear that 1, 2, 3 and 4 are all winning positions because we can simply remove all of the coins on the table if it is our move.

But what happens when there are 5 coins on the table? Well, the only possibilities are for us to leave 1, 2, 3 or 4 coins on the table, thereby leaving our opponent in a winning position. So 5 must be a losing position.

But if 5 is a losing position, then 6, 7, 8 and 9 must all be winning positions. That is because from these positions, we can remove 1, 2, 3 or 4 coins to leave 5 coins on the table, thereby leaving our opponent in a losing position.

Continuing this argument gives us the following table.

Winning	1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, ...
Losing	5, 10, 15, 20, ...

At this stage, it seems like a safe bet that the losing positions are simply the multiples of 5. To prove that this is true, all we need to do is demonstrate that

- from a non-multiple of 5, it is possible to move to a multiple of 5, and
- from a multiple of 5, it is impossible to move to a multiple of 5.

Both of these statements are obvious! For the first, we use the fact that a non-multiple of 5 must be of the form $5k+1, 5k+2, 5k+3$ or $5k+4$ and subtracting 1, 2, 3 or 4 from each of these, respectively, leaves a multiple of 5.

For the second, we note that the difference between two multiples of 5 is still a multiple of 5 and certainly cannot be 1, 2, 3 or 4. □

Note that not only have we determined the winning and losing positions, we have also uncovered a simple winning strategy: always reduce the number of coins to a multiple of 5.

15.10 The copycat strategy

The notion of symmetry is a most powerful one in mathematics. Here, we will see how to use it to beat your friends at certain games, simply by being a copycat.

Problem Next to a square table is a pile of circular coins, all the same size. Two players take turns putting a coin on the table so that it doesn't touch any other coin. The player who cannot do so loses the game.

Show that the first player can always win.

Solution The first player can win by placing the first coin at the centre of the table. Now, wherever the second player places their coin, the first player simply copies them by placing their coin symmetrically opposite.

After each of the first player's moves, the configuration of coins is symmetric under a 180° rotation of the table. Thus, whenever the second player can place a coin on the table, the first player can also do so by placing their coin symmetrically opposite.[6] So by using this copycat strategy, the first player can always win. □

15.11 Pairing strategies

Many games involve the movement of pieces into various positions. Sometimes, a strategy can be given by pairing up the positions so that wherever the first player can go, the second player should go in the position that it is paired with.

Problem Alice and Bob play a game on a large[7] grid where they take turns to choose a square and mark it. Alice moves first and marks squares with an **X** while Bob marks squares with an **O**. They play until one of the players marks a row or a column of five consecutive squares, and this player wins the game. If no player marks a row or column of five consecutive squares, then the game is declared a draw.

Show that Bob can prevent Alice from winning.

Solution Label the board as shown in the diagram, repeating the pattern in all directions.

1	2	3	3	1	2	3	3
1	2	4	4	1	2	4	4
3	3	1	2	3	3	1	2
4	4	1	2	4	4	1	2
1	2	3	3	1	2	3	3
1	2	4	4	1	2	4	4
3	3	1	2	3	3	1	2
4	4	1	2	4	4	1	2

Each square is paired with the neighbouring square that contains the same label.

Suppose that whenever Alice plays, Bob plays in the neighbouring square with the same label. In this way, Alice can never occupy both squares of such a pair.

But the pairing was chosen very carefully so that any block of five consecutive squares in a row or column contains such a pair. Needless to say, you should check that this is true for yourself. Hence, Alice can never mark a row or a column of five consecutive squares. □

Since Bob can prevent Alice from winning, the fundamental theorem of combinatorial games states that he must have a winning strategy or both players can force a draw. In the next section, we will use a sneaky technique to show that the latter is the case.

15.12 Strategy stealing

In the previous problem we showed that Bob could prevent Alice from winning. But as unlikely as it seems, could Bob force a win for himself? This is where strategy stealing shows its usefulness.

Problem Alice and Bob play a game on a large grid where they take turns to choose a square and mark it. Alice moves first and marks squares with an **X** while Bob marks squares with an **O**. They play until one of the players marks a row or a column of five consecutive squares, and this player wins the game. If no player marks a row or column of five consecutive squares, then the game is declared a draw.

Show that Alice can prevent Bob from winning.

Solution The main idea is that Alice's extra move at the start of the game can never hurt her chances of winning. However, this intuition doesn't constitute a proof in itself, but can be turned into one by using the concept of strategy stealing.

In order to obtain a contradiction, suppose that Bob has a winning strategy. In other words, it is possible to write a book which describes how Bob can win, no matter how Alice plays. Suppose now that Alice manages to steal this second player's strategy book and plays as follows. She simply places her first move randomly on the grid, ignores the fact that she has moved, and then pretends that she is the second player. She continues to do this until the book tells her to move in the square where she moved first. Since she has already done this, she can use this move to mark a different random square on the grid. Continuing in this way, we see that any book which provides a winning strategy for the second player can be used to provide a winning strategy for the first player! This blatantly contradicts the fundamental theorem of combinatorial games. So we must conclude that there is no winning strategy for the second player at all. In other words, Alice can prevent Bob from winning. □

Strategy stealing is brimming with trickery, so have a read through the above solution again until it's well understood. After that, you may have a look at the following example of strategy stealing at its best.

Problem In the game *Double Chess*, the rules of chess are changed so that White and Black alternately make two legal moves at a time.

Show that Black doesn't have a winning strategy.

Solution In order to obtain a contradiction, let us suppose to the contrary that Black does have a winning strategy. Then it must be the case that whatever White does on the first move, Black can win from the resulting position.

So what would happen if White started by moving a knight out and then back to the square that it was originally on? At this stage, the board looks exactly the same as it did initially, but it's now Black's turn to move. Remember that, by assumption, Black can force a win from this position. But if that is the case, couldn't White have just mirrored Black's strategy from the very beginning of the game? Of course this is possible and so White also seems to have a winning strategy. This contradicts the fundamental theorem of combinatorial games, because White and Black cannot both have winning strategies. So our original assumption must have been wrong and we conclude that Black doesn't have a winning strategy. □

16

Combinatorial geometry

Combinatorial geometry is an exotic hybrid. Usually a combinatorial problem is posed in a geometric setting. The usual ideas in combinatorics are often insufficient to take into account extra constraints that come from the geometric setting. But in this chapter we will see ideas that can be used for such problems.

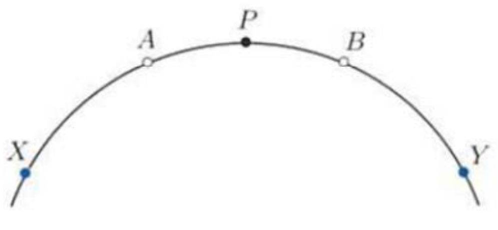
16.1 Proof by contradiction

This is one of the most basic techniques in mathematics in general. Recall from section 1.5 that the method is to show that if the proposition is false, then it follows that some nonsense is true. Of course this is a situation we cannot tolerate and so the proposition must be true.

Problem Is it possible to colour every point on a circle using the two colours white and blue so that there is no isosceles triangle whose vertices all have the same colour?

Solution After experimenting for a while you may be convinced that it is not possible. So assume for the sake of contradiction that it is possible.

Then certainly we can pick two points A and B of the same colour, say white. If X is the point on the circle such that the arc lengths XA and AB are equal, then triangle AXB is isosceles and so X must be blue. Similarly the point Y is blue where Y satisfies $AB = BY$.



Consider now the point P, say, on the minor arc of the circle halfway between A and B. Since $AP = PB$, P cannot be white. Furthermore, since $XP = YP$, point P cannot be blue.

Thus P cannot be any colour, which is a contradiction. □

This proof is not quite complete because it may occur that the points A, B, X, Y and P are not all distinct. For instance, if ABX forms an equilateral triangle, then $X = Y$.

We leave it for you to think about how to overcome this problem.

Actually, there is a really short solution to this problem as follows.

Solution Let $ABCDE$ be any regular pentagon inscribed in the circle. Then by the pigeonhole principle, three of its vertices must be the same colour.

However any three points of a regular pentagon form an isosceles triangle! □

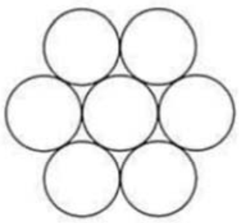
16.2 Extremal principle

Sometimes picking out something that is extremal in some sense can be just what needs to be focused on to solve a problem completely.

Problem We are given a set of discs in the plane with pairwise disjoint interiors. Each disc is tangent to at least six other discs of the family.

Prove that there are infinitely many discs in the set.

Solution Assume that the family is finite. Then there is a disc, D say, of minimal radius r. Thus there are at least six discs around D and of radius at least r. However there is only room for there to be exactly six discs, all of radius equal to r around D.



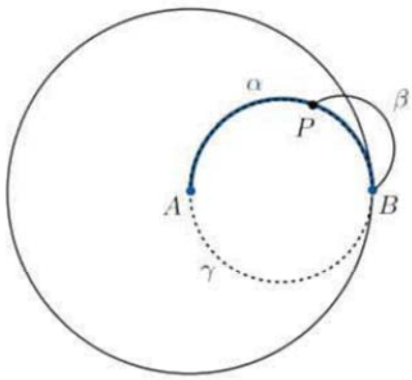
We may apply the same argument to each of these discs, thus generating infinitely many discs of radius r at ever increasing distances from D. This is a contradiction. □

Problem A closed[6] and bounded[7] shape S in the plane has the property that any two points of S can be connected by a semicircular arc which lies completely in S.

Find all possibilities for the figure S.

Solution Since S is closed and bounded, there exist points $A, B \in S$ such that the distance $d = AB$ is maximal.[8] Let α be the semicircular arc lying in S which joins A to B and let γ be the full circle defined by α.

Let P be any point on α. Then we know that there is a semicircular arc β lying in S which joins B to P. If β lies outside of γ, then there exists a point on β, whose distance from A is greater than d.[9] This is a contradiction.



Thus β lies inside γ. This is true for all points $P \in \alpha$. Since the set of such β covers the entire interior of γ, we see that the interior of γ is a subset of S.

Since S is closed, it follows that the boundary of γ is also a subset of S. Finally, since any point lying outside of γ would give rise to a distance in S greater than d, we conclude that S is in fact the closed disc described by γ. □

Problem Given n points in the plane, no three of which are collinear, show it is possible to join them up in sequence so that we have a broken line consisting of $n-1$ segments, no two of which cross each other.

Solution There does not seem to be anything obviously extremal to look at here. However, if you drew a really long path through the points, it seems likely that the path would contain lots of intersections. A shorter path probably would contain fewer intersections. The shortest path hopefully would contain none. Let us prove that this is indeed the case.

Let $A_1A_2\ldots A_n$ be a path of minimal length. Suppose that A_jA_{j+1} crosses A_iA_{i+1} $(i<j)$. Then consider the quadrilateral $A_iA_jA_{i+1}A_{j+1}$. The diagonals intersect at a point P inside the quadrilateral.

From the triangle inequality we have

$$A_iP + A_jP > A_iA_j \quad \text{and} \quad A_{i+1}P + A_{j+1}P > A_{i+1}A_{j+1}.$$

Adding these two equations yields

$$A_iA_{i+1} + A_jA_{j+1} > A_iA_j + A_{i+1}A_{j+1}.$$

Thus the path

$$A_1A_2\ldots A_iA_jA_{j-1}\ldots A_{i+2}A_{i+1}A_{j+1}A_{j+2}\ldots A_n$$

is of shorter length than $A_1A_2\ldots A_n$, which is a contradiction. □

16.3 Perturbation

Sometimes objects such as lines or points may be not quite in the position that you want them. A miniscule jiggle of the configuration can sometimes rectify this. We illustrate this with another way to approach the previous problem.

Problem Given n points in the plane, no three of which are collinear, show it is possible to join them up in sequence so that we have a broken line consisting of $n-1$ segments, no two of which cross each other.

Solution The points lie in the x-y plane. So if we label the points $P_1, P_2, \ldots, P_n$ according to increasing x-coordinate, then we could simply join P_1 to P_2, P_2 to P_3, and so on.

However, it may not be the case that the x-coordinates are all distinct. Surely we can rotate the configuration in the plane so that all x-coordinates are distinct! Indeed all we have to do is rotate the configuration so that the y-axis is not parallel to any of the lines formed by joining all $\binom{n}{2}$ pairs of points. □

16.4 Induction

More than likely, a problem for which we can build larger examples out of smaller examples can be approached by mathematical induction.

Problem Given 1002 distinct points in the plane, we join every pair of points with a line segment and colour its midpoint red.

Show that there are at least 2001 red points.

Solution We prove by the induction the more general statement that for $n \geq 2$ points there are at least $2n-3$ red points.

The result is clearly true for $n = 2$.

Suppose now that the result is true for $n = 2, 3, \ldots, m$ where $m \geq 2$. Consider an arrangement of $m+1$ points. By using a perturbation argument we may assume that all points have distinct x-coordinates. Label them as $A_1, A_2, \ldots, A_m, A_{m+1}$ by increasing x-coordinate. By the inductive assumption we have at least $2m-3$ red points from the midpoints of $A_1, A_2, \ldots, A_m$. Can we find two more red points by using A_{m+1}? Yes! The midpoints of $A_{m+1}A_{m-1}$ and $A_{m+1}A_m$ are distinct and both are to the right of all red points considered so far. Thus we have at least $2m-1 = 2(m+1)-3$ red points in all. This completes the induction. □

As an extension, can you determine where equality occurs in the above problem?

16.5 Discrete intermediate value theorem

The ordinary *intermediate value theorem* states that if $f(x)$ is a continuous function defined on some interval which achieves the value $f(a)$ somewhere and the value $f(b)$ somewhere else, then $f(x)$ achieves all values between $f(a)$ and $f(b)$. In particular, if $f(x)$ is positive somewhere and negative somewhere else, then it is necessarily zero somewhere in between.

There is a discrete analogue of this that involves sequences of integers.

Discrete intermediate value theorem If $a_1, a_2, \ldots, a_n$ is a sequence of integers with the property that $|a_i - a_{i+1}| \leq 1$ for $i = 1, 2, \ldots, n-1$ and is such that $a_i < 0$ and $a_j > 0$ for some $1 \leq i, j \leq n$, then $a_k = 0$ for some $1 \leq k \leq n$.

The following problem illustrates how useful this can be.

Problem We are given $2n+1$ blue points in the plane such that no three are collinear and no four are concyclic.

For every pair of blue points A, B, show that there exists a circle passing through A, B with $n-1$ blue points inside it, three points on its boundary and $n-1$ blue points outside it.

Solution Consider any two blue points A and B and orient the plane so that AB is vertical. For each point P on the perpendicular bisector of AB there is an associated circle Γ_P having centre P and passing through A and B. The idea is to consider what happens as P varies from far to the left of AB to far to the right of AB.

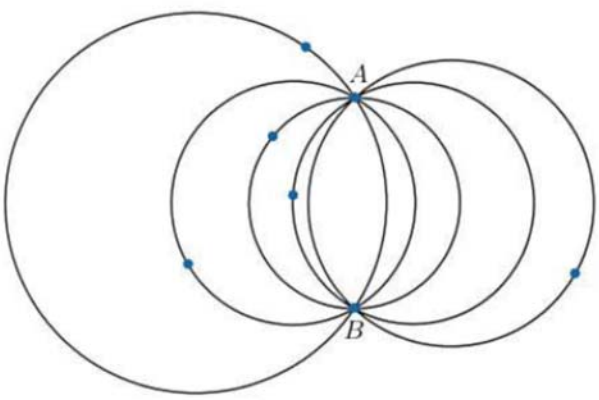
Suppose there are a blue points to the left of the line AB and b blue points to the right of the line AB. WLOG $a \geq b$. Since $a + b = 2n - 1$, a and b are of opposite parity and so we must have $a > b$.

There is a point P_{left}, such that the corresponding circle contains in its interior all of the a blue points located to the left of AB but none of the b blue points to the right of AB. There is also a point P_{right}, such that the corresponding circle contains in its interior all of the b blue points located to the right of AB but none of the a blue points to the left of AB.[10]

As P varies continuously between P_{left} and P_{right} the circle Γ_P will meet the remaining $2n-1$ blue points one by one. When Γ_P meets the ith such blue point, let a_i be given by

$$a_i = I(\Gamma_P) - O(\Gamma_P),$$

where $I(\Gamma_P)$ is the number of blue points lying inside Γ_P and $O(\Gamma_P)$ is the number of blue points lying outside Γ_P. Note that a_i is even because $I(\Gamma_P)$ and $O(\Gamma_P)$ have the same parity due to $I(\Gamma_P) + O(\Gamma_P) = 2n - 2$.



If the first point that Γ_P meets is to the left of AB, then $a_1 = a - 1 - b$. If it is to the right of AB, then $a_1 = a - (b-1)$. Either way we have $a_1 \geq a - b - 1 \geq 0$ because $a > b$. Similarly, if the last point that Γ_P meets is to the left of AB, then $a_{2n-1} = b - (a-1)$. If it is to the right of AB, then $a_{2n-1} = b - 1 - a$. Either way we have $a_{2n-1} \leq b - a + 1 \leq 0$ because $b < a$. To summarise we have $a_1 \geq 0$ and $a_{2n-1} \leq 0$.

What happens when Γ_P goes from meeting the ith blue point to meeting the $(i+1)$th blue point?

- If both such points are to the left of AB, then $I(\Gamma_P)$ decreases by 1 while $O(\Gamma_P)$ increases by 1. Thus $a_{i+1} = a_i - 2$.
- If both points are to the right of AB, then $I(\Gamma_P)$ increases by 1 while $O(\Gamma_P)$ decreases by 1. Thus $a_{i+1} = a_i + 2$.
- If the points are on opposite sides of AB, then $I(\Gamma_P)$ and $O(\Gamma_P)$ remain unchanged. Thus $a_{i+1} = a_i$.

In all cases we have $|a_{i+1} - a_i| \leq 2$. Finally, applying the discrete intermediate value theorem to the sequence $\frac{1}{2}a_1, \frac{1}{2}a_2, \ldots, \frac{1}{2}a_{2n-1}$ guarantees that $a_i = 0$ for some i. Then the circle corresponding to this i satisfies the conclusion of the problem. □

16.6 Convex hull

Consider a wooden board with some nails hammered into it. What happens when you take a rubber band and stretch it tightly around the area occupied by the nails? You end up with a polygon whose angles are all less than or equal to 180°. The area occupied by this polygon is called the *convex hull* of the nails.

The notion of convexity is defined as follows. A set S is *convex* if for any two points A and B in S, the whole line segment AB lies entirely in S. It is easily shown that the intersection of convex sets is also a convex set. The convex hull of a set T is defined to be the intersection of all convex sets containing T. It is in fact the smallest convex set containing T.

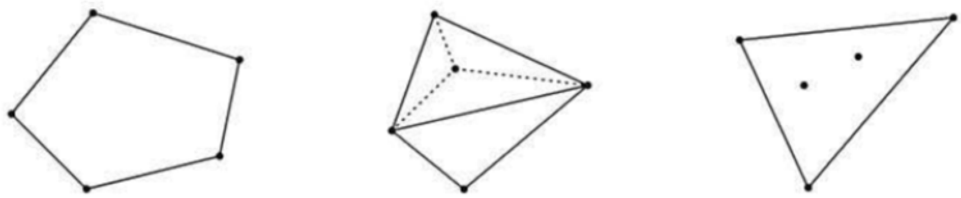
Problem Let S be a set of five distinct points in the plane.

Show there exist three points A, B and C of S such that

$$108^\circ \leq \angle ABC \leq 180^\circ.$$

Solution It is tempting to oversimplify by asserting that the five points form a pentagon and so one of its interior angles is at least 108°. Although this is true, that interior angle might also be more than 180°. This is where the convex hull becomes useful.

Consider the perimeter of the convex hull. If it contains all five points, then we may use the above argument since all interior angles are at most 180°.



If the convex hull is a quadrilateral, divide the quadrilateral into two triangles as shown. The fifth point must be inside one of those triangles. Using line segments, join that fifth point to the vertices of the triangle in which it lies. Since these segments define three angles around the fifth point, none of which exceeds $180°$, and whose sum is $360°$, it follows that one of those angles is between $120°$ and $180°$.

If the convex hull is a triangle, again we find a triangle with a point (in fact two points) in its interior leading to an angle between $120°$ and $180°$.

If the convex hull is a segment, then all points are collinear and so three of them form a $180°$ angle. □

16.7 Euler's formula

Euler's formula was discussed in sections 14.10 and 14.11. Recall that for convex polyhedra Euler's formula tells us

$$V - E + F = 2,$$

where V, E and F are the respective numbers of vertices, edges and faces. Recall further that this is also valid for planar graphs remembering that we count the infinitely large face on the outside as one of the faces.

Problem The square $ABCD$ contains n points $P_1, P_2, \ldots, P_n$ in its interior such that no three of the $n+4$ points $A, B, C, D, P_1, \ldots, P_n$ are collinear.

(a) Show that it is possible to subdivide the square into triangles in such a way that the vertices of each triangle are among the $n+4$ given points.

(b) Show that the number of resulting triangles in (a) is always the same no matter how the subdivision is carried out.

Solution

(a) It is easy to establish using induction that a subdivision is always possible. Indeed each extra point would land inside some triangle. Subdividing this triangle into three further triangles by joining the interior point to the three vertices loses the original triangle but creates three smaller triangles. This construction yields $2n+2$ triangles. □

Note that such a proof by induction that *all* subdivisions as described in the problem lead to the same number of triangles is faulty. This is because not all such subdivisions can be constructed inductively in this way from subdivisions with fewer triangles.[11] So instead for part (b) we resort to Euler's formula along with a counting argument.

(b) Let T be the number of triangles. Then $F = T + 1$ because of the infinitely large outside face that has four sides. Each edge belongs to two faces. Since each triangle has three edges and the outside face has four edges we have

$$E = \frac{3T + 4}{2}.$$

Substituting this into Euler's formula where $V = n + 4$ yields

$$n + 4 - \frac{3T + 4}{2} + T + 1 = 2.$$

Thus $T = 2n + 2$. □

16.8 Pigeonhole principle

Problem Six points are given inside an equilateral triangle of area 4.

Prove that among the nine points which include the three vertices of the triangle and the six given points, three of these form a triangle of area at most 1.

Solution Divide the triangle up into four equilateral triangles of area 1. Since we have nine points in total, by the pigeonhole principle at least three of these points lie inside or on the boundary of one of these four triangles and thus define a triangle of area at most 1. □

In fact more is true! We can sharpen the result from 1 to $\frac{4}{13}$ as follows.

Suppose that we place one of the six points inside the triangle, and use this point to subdivide the triangle into three smaller triangles. Next place a second point of the six points. This will fall inside one of the three smaller triangles and we can use our second point to subdivide this smaller triangle into three even smaller triangles, making five triangles in all. Continuing in this fashion placing one point at a time until all six points are placed results in the original triangle being subdivided into 13 triangles.

Thus one of these triangles has area at most $\frac{4}{13}$.

16.9 Colouring

Questions about colouring points in the plane often use the pigeonhole principle.

Problem Let S be a disc. The points of S are painted in finitely many colours.

Show that for every $n \geq 3$ there exist infinitely many congruent polygons with n sides contained in S such that all of them have their vertices painted in the same single colour.

Solution This problem initially seems quite daunting. The number of colours and the value of n are allowed to be quite large. The best way to start is to examine simple cases. We begin with the simplest case where we have only two colours and our congruent polygons are triangles.

Consider any regular pentagon P, lying entirely in the interior of S. By the pigeonhole principle three of the vertices of P have the same colour. However, there are infinitely many disjoint translates of the set of five vertices of P which also lie in S. By the preceding argument each of these contains a monochromatic triangle. Thus we have infinitely many such monochromatic triangles.

But the triangles we are considering only come in one of two congruence types. Thus applying the infinite pigeonhole principle, one of these congruence types contains infinitely many monochromatic triangles.

We now have infinitely many congruent monochromatic triangles. Finally, a second application of the infinite pigeonhole principle allows us to conclude that one of the two colours contains infinitely many such congruent monochromatic triangles.

We leave it to the reader to work out how to solve the problem in its full generality as stated. For m colours, the issue is basically to find a number k so that whenever P has k vertices there are always n vertices of P of the same colour. $\square$

A couple of comments are in order here. First, P did not have to be regular. In the case P is not regular, we have perhaps up to $\binom{5}{3} = 10$ different congruence types of triangle for each translate of P. But that is still finitely many. Second, why did we choose P to be a pentagon? The answer is that a pentagon has enough vertices so that if we colour them using two colours, a monochromatic triangle always appears. So P could have had any number of vertices greater than or equal to 5 and the proof would still work.