D. Classical Hölder Inequality

In this appendix we discuss the simplest version of Hölder's inequality. This form is used in Chapters II and IV.

LEMMA D.1. *Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, and let u and v be two non-negative numbers, at least one being non-zero. Then the function $f : [0,1] \to \mathbb{R}$ defined by*

$$f(t) = ut + v(1-t^q)^{\frac{1}{q}}, \ \ t \in [0,1],$$

has a unique maximum point at

$$s = \left[\frac{u^p}{u^p + v^p}\right]^{\frac{1}{q}}. \tag{1}$$

The maximum value of f is

$$\max_{t\in[0,1]} f(t) = (u^p + v^p)^{\frac{1}{p}}. \tag{2}$$

PROOF. If $v = 0$, then $f(t) = tu$, $\forall\, t \in [0,1]$ (with $u > 0$), and in this case the Lemma is trivial. Likewise, if $u = 0$, then $f(t) = v(1-t^q)^{\frac{1}{q}}$, $\forall\, t \in [0,1]$ (with $v > 0$), and using the inequality

$$(1-t^q)^{\frac{1}{q}} < 1, \ \ \forall\, t \in (0,1],$$

we immediately get

$$f(t) < f(0),, \ \ \forall\, t \in (0,1],$$

and the Lemma again follows.

For the remainder of the proof we are going to assume that $u, v > 0$. We concentrate on the first assertion. Obviously f is differentiable on $(0,1)$, so the "candidates" for the maximum points are 0, 1, and the solutions of the equation

$$f'(t) = 0. \tag{3}$$

Let s be defined as in (1), so under the assumption that $u, v > 0$, we clearly have $0 < s < 1$. We are going to prove first that s is the unique solution in $(0,1)$ of the equation (3) We have

$$f'(t) = u + v \cdot \frac{1}{q}(1-t^q)^{\frac{1}{q}-1} \cdot q \cdot t^{q-1} = u - v\left(\frac{t^q}{1-t^q}\right)^{\frac{1}{p}}, \ \ t \in (0,1), \tag{4}$$

so the equation (3) reads

$$u - v\left(\frac{t^q}{1-t^q}\right)^{\frac{1}{p}} = 0.$$

Equivalently we have

$$\left(\frac{t^q}{1-t^q}\right)^{\frac{1}{p}} = u/v, \qquad \frac{t^q}{1-t^q} = (u/v)^p, \qquad t^q = \frac{(u/v)^p}{1+(u/v)^p} = \frac{u^p}{u^p+v^p},$$

which gives $t = s$.

Having shown that the "candidates" for the maximum point are 0, 1, and s, let us show that s is the only maximum point. For this purpose, we go back to (4) and we observe that f' is also continuous on $(0,1)$. Since

$$\lim_{t\to 0+} f'(t) = u > 0 \text{ and } \lim_{t\to 1-} f'(t) = -\infty,$$

and the equation (3) has exactly one solution in $(0,1)$, namely s, this forces

$$f'(t) > 0,\ \ \forall\, t \in (0,s) \text{ and } f'(t) < 0,\ \ \forall\, t \in (s,1).$$

This means that, f is increasing on $[0,s]$ and decreasing on $[s,1]$, and we are done.

The maximum value of f is then given by

$$\max_{t\in[0,1]} f(t) = f(s),$$

and the fact that $f(s)$ equals the value in (2) follows from an easy computation. □

Theorem D.1 (Hölder's inequality). *Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be non-negative numbers. Let $p, q > 1$ be real number with the property $\frac{1}{p} + \frac{1}{q} = 1$. Then:*

$$\sum_{j=1}^{n} a_j b_j \le \left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^{n} b_j^q\right)^{\frac{1}{q}}. \tag{5}$$

Moreover, one has equality only when the sequences $(a_1^p, \dots, a_n^p)$ and $(b_1^q, \dots, b_n^q)$ are proportional.

Proof. The proof will be carried on by induction on n. The case $n = 1$ is trivial.

Case $n = 2$.

Assume $(b_1, b_2) \ne (0,0)$. (Otherwise everything is trivial). Define the number

$$r = \frac{b_1}{(b_1^q + b_2^q)^{1/q}}.$$

Notice that $r \in [0,1]$, and we have

$$\frac{b_2}{(b_1^q + b_2^q)^{1/q}} = (1 - r^q)^{1/q}.$$

Notice also that, upon dividing by $(b_1^q + b_2^q)^{1/q}$, the desired inequality

$$a_1 b_1 + a_2 b_2 \le (a_1^p + a_2^p)^{\frac{1}{p}} (b_1^q + b_2^q)^{\frac{1}{q}} \tag{6}$$

reads

$$a_1 r + a_2 (1 - r^q)^{1/q} \le (a_1^p + a_2^p)^{1/p}. \tag{7}$$

It is obvious that this is an equality when $a_1 = a_2 = 0$. Assume $(a_1, a_2) \ne (0,0)$, and set up the function

$$f(t) = a_1 t + a_2 (1 - t^q)^{1/q},\ \ t \in [0,1].$$

We now apply Lemma D.1, which immediately gives us (7). Let us examine when equality holds. If $a_1 = a_2 = 0$, the equality obviosuly holds, and in this case (a_1, a_2) is clearly proportional to (b_1, b_2). Assume $(a_1, a_2) \ne (0,0)$. Again by Lemma D.1, we know that equality holds in (7), exactly when

$$r = \left[\frac{a_1^p}{a_1^p + a_2^p}\right]^{\frac{1}{q}},$$

that is,

$$\frac{b_1}{(b_1^q + b_2^q)^{\frac{1}{q}}} = \left[\frac{a_1^p}{a_1^p + a_2^p}\right]^{\frac{1}{q}},$$

or equivalently

$$\frac{b_1^q}{b_1^q + b_2^q} = \frac{a_1^p}{a_1^p + a_2^p}.$$

Obviously this forces

$$\frac{b_2^q}{b_1^q + b_2^q} = \frac{a_2^p}{a_1^p + a_2^p},$$

so indeed (a_1^p, a_2^p) and (b_1^q, b_2^q) are proportional.

Having proven the case $n = 2$, we now proceed with the proof of:

The implication: Case $n = k \Rightarrow$ *Case* $n = k + 1$.

Start with two sequences $(a_1, a_2, \dots, a_k, a_{k+1})$ and $(b_1, b_2, \dots, a_k, b_{k+1})$. Define the numbers

$$a = \left(\sum_{j=1}^{k} a_j^p\right)^{\frac{1}{p}} \text{ and } b = \left(\sum_{j=1}^{k} b_j^q\right)^{\frac{1}{q}}.$$

Using the assumption that the case $n = k$ holds, we have

$$\sum_{j=1}^{k+1} a_j b_j \le \left(\sum_{j=1}^{k} a_j^p\right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^{k} b_j^q\right)^{\frac{1}{q}} + a_{k+1}b_{k+1} = ab + a_{k+1}b_{k+1}. \tag{8}$$

Using the case $n = 2$ we also have

$$ab + a_{k+1}b_{k+1} \le (a^p + a_{k+1}^p)^{\frac{1}{p}} \cdot (b^q + b_{k+1}^q)^{\frac{1}{q}} = \left(\sum_{j=1}^{k+1} a_j^p\right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^{k+1} b_j^q\right)^{\frac{1}{q}}, \tag{9}$$

so combining with (8) we see that the desired inequality (5) holds for $n = k + 1$.

Assume now we have equality. Then we must have equality in both (8) and in (9). On the one hand, the equality in (8) forces $(a_1^p, a_2^p, \dots, a_k^p)$ and $(b_1^q, b_2^q, \dots, b_k^q)$ to be proportional (since we assume the case $n = k$). On the other hand, the equality in (9) forces (a^p, a_{k+1}^p) and (b^q, b_{k+1}^q) to be proportional (by the case $n = 2$). Since

$$a^p = \sum_{j=1}^{k} a_j^p \text{ and } b^q = \sum_{j=1}^{k} b_j^q,$$

it is clear that $(a_1^p, a_2^p, \dots, a_k^p, a_{k+1}^p)$ and $(b_1^q, b_2^q, \dots, b_k^q, b_{k+1}^q)$ are proportional. $\square$