

# Circles

Yufei Zhao

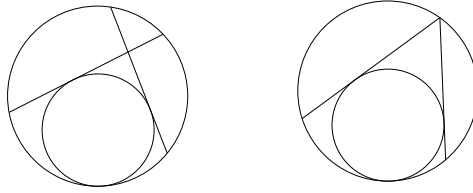
yufeiz@mit.edu

## 1 Warm up problems

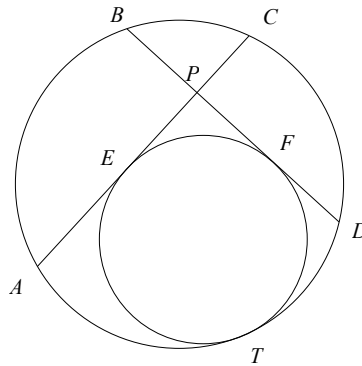
- Let  $AB$  and  $CD$  be two segments, and let lines  $AC$  and  $BD$  meet at  $X$ . Let the circumcircles of  $ABX$  and  $CDX$  meet again at  $O$ . Prove that triangles  $OAB$  and  $OCD$  are similar.
- (Miquel's theorem) Let  $ABC$  be a triangle. Points  $X, Y$ , and  $Z$  lie on sides  $BC, CA$ , and  $AB$ , respectively. Prove that the circumcircles of triangles  $AYZ, BXZ, CXY$  meet at a common point.
- (Also Miquel's theorem) Let  $a, b, c, d$  be four lines in space, no two parallel, no three concurrent. Let  $\omega_a$  denote the circumcircle of the triangle formed by lines  $b, c, d$ . Similarly define  $\omega_b, \omega_c, \omega_d$ .
  - Prove that  $\omega_a, \omega_b, \omega_c, \omega_d$  pass through a common point. This is called the Miquel point.
  - Continuing the above notation, prove that the centers of  $\omega_a, \omega_b, \omega_c, \omega_d$ , and the Miquel point all lie on a common circle.
- (Simson line) Let  $ABC$  be a triangle, and let  $P$  be another point on its circumcircle. Let  $X, Y, Z$  be the feet of perpendiculars from  $P$  to lines  $BC, CA, AB$  respectively. Prove that  $X, Y, Z$  are collinear.
- Let  $\angle AOB$  be a right angle,  $M$  and  $N$  points on rays  $OA$  and  $OB$ , respectively. Let  $MNPQ$  be a square such that  $MN$  separates the points  $O$  and  $P$ . Find the locus of the center of the square when  $M$  and  $N$  vary.
- Let  $ABCD$  be a convex quadrilateral such that the diagonals  $AC$  and  $BD$  are perpendicular, and let  $P$  be their intersection. Prove that the reflections of  $P$  with respect to  $AB, BC, CD, DA$  are concyclic.
- An interior point  $P$  is chosen in the rectangle  $ABCD$  such that  $\angle APD + \angle BPC = 180^\circ$ . Find  $\angle DAP + \angle BCP$ .
- Let  $AB$  be a chord in a circle and  $P$  a point on the circle. Let  $Q$  be the projection of  $P$  on  $AB$  and  $R$  and  $S$  the projections of  $P$  onto the tangents to the circle at  $A$  and  $B$ . Prove that  $PQ^2 = PR \cdot PS$ .
- Let  $ABC$  be an acute triangle. The points  $M$  and  $N$  are taken on the sides  $AB$  and  $AC$  respectively. The circles with diameters  $BN$  and  $CM$  intersect at points  $P$  and  $Q$ . Prove that  $P, Q$ , and the orthocenter  $H$  are collinear.
- Among the points  $A, B, C, D$  no three are collinear. The lines  $AB$  and  $CD$  intersect at  $E$ , and  $BC$  and  $DA$  intersect at  $F$ . Prove that either the circles with diameters  $AC, BD, EF$  pass through a common point, or no two of them have any common point. (The line through the midpoints of  $AC, BD, EF$  is called the *Newton-Gauss line*.)

## 2 Tangent circles

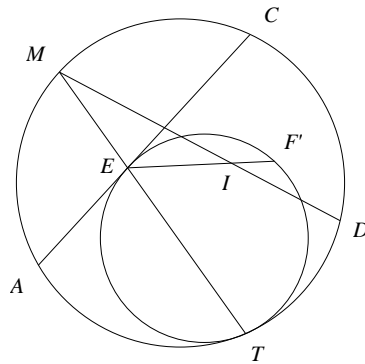
In this section, we explore the following two configurations.<sup>1</sup>



1. Let chords  $AC$  and  $BD$  of a circle  $\omega$  intersect at  $P$ . A smaller circle  $\omega_1$  is tangent to  $\omega$  at  $T$  and to segments  $AP$  and  $DP$  at  $E$  and  $F$  respectively.

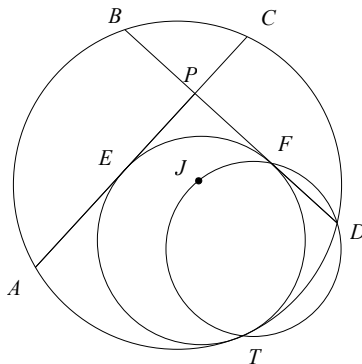


- (a) Prove that ray  $TE$  bisects arc  $ABC$  of  $\omega$ .
- (b) Let  $I$  be the incenter of triangle  $ACD$  and  $M$  be the midpoint of arc  $ABC$  of  $\omega$ . Prove that  $MA = MI = MC$ .
- (c) Let  $F'$  be the common point of  $\omega_1$  and line  $EI$  other than  $E$ . Prove that  $I, F', D, T$  are concyclic.



- (d) Prove that  $DF'$  is tangent to  $\omega_1$ . This means that  $F = F'$ , so that  $E, F, I$  are collinear. (Remember this fact.)
- (e) Let  $J$  be the incenter of triangle  $APD$ . Prove that  $T, D, F, J$  are concyclic.

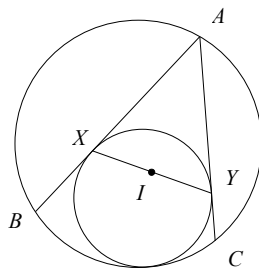
<sup>1</sup>Thanks to Oleg Golberg for providing some of these problems.



(f) Prove that ray  $TJ$  bisects angle  $ATD$ .

2. Now we consider a special case of the part (d) in the previous problem. Try to find a short proof of the following result. (Hint: use Pascal's theorem)

Let  $ABC$  be a triangle and  $I$  its incenter. Let  $\Gamma$  be the circle tangent to sides  $AB$ ,  $AC$ , as well as the circumcircle of  $ABC$ . Let  $\Gamma$  touch  $AB$  and  $AC$  at  $X$  and  $Y$ , respectively. Show that  $I$  is the midpoint of  $XY$ .



3. Let  $\Omega$  be the circumcircle of  $ABC$ . A circle  $\omega$  is tangent to sides  $AB$ ,  $AC$  and circle  $\Omega$  at points  $X, Y$  and  $Z$ , respectively. Let  $M$  be the midpoint of the arc  $BC$  of  $\Omega$  which does not contain  $A$ . Prove that lines  $XY$ ,  $BC$  and  $ZM$  have a common point.  
(Can you prove the result when  $\Omega$  only passes through  $B$  and  $C$  and contains  $A$  in the interior?)
4. A circle  $\omega$  is tangent to sides  $AB$ ,  $AC$  of triangle  $ABC$  and to its circumcircle at points  $X, Y$  and  $Z$ . Segments  $AZ$  and  $XY$  meet at  $T$ . Prove that  $\angle BTX = \angle CTY$ .
5. (Sawayama-Thébault) Let  $ABC$  be a triangle with incenter  $I$ . Let  $D$  a point on side  $BC$ . Let  $P$  be the center of the circle that touches segments  $AD$ ,  $DC$ , and the circumcircle of  $ABC$ , and let  $Q$  be the center of the circle that touches segments  $AD$ ,  $BD$ , and the circumcircle of  $ABC$ . Show that  $P, Q, I$  are collinear.

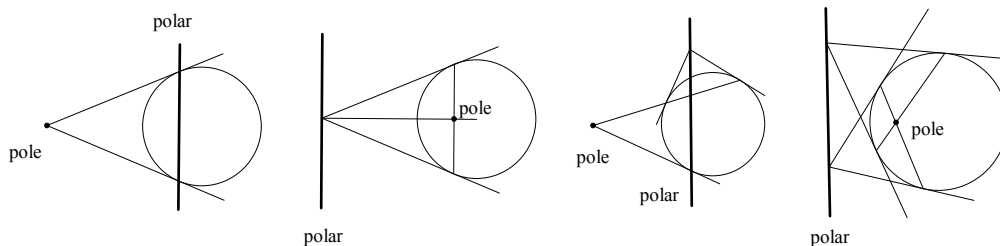
### 3 Projective quickies

*Warning: The introductory text in this section is very rough. It is not meant for you to learn from. Rather, I assume that you already know the basics of projective geometry, and this should serve as a quick review of the important concepts.*

You should be familiar with the notion of *pole* and *polar*. A quick definition is as follows:

**Definition.** Suppose that  $\omega$  is a circle with center  $O$ , and  $P$  and  $P'$  are inverses respect to  $\omega$  (i.e.,  $P'$  lies on ray  $OP$  such that  $OP \cdot OP' = r^2$ , where  $r$  is the radius of  $\omega$ ). Let  $\ell$  be the line through  $P'$  perpendicular to  $OP$ . Then we say that  $\ell$  is the *polar* of  $P$  and  $P$  is the *pole* of  $\ell$ .

Here are a few diagrams that may help you to remember the common setups of poles and polars.



Make sure that you understand the *duality* behind poles and polars. If you need to prove that something is true, it suffices to prove its polar dual. The polar map transforms points to lines, lines to points. It transforms the intersection of two lines to the line joining the two points, and vice-versa. For instance:

- To show that line  $\ell$  passes through point  $P$ , it suffices to show that the pole of  $\ell$  lies on the polar of  $P$ .
- To show that three points are collinear, it suffices to show that their poles are concurrent.

The cross ratio of four collinear points  $A, B, C, D$  is defined as

$$(A, B; C, D) = \frac{AC \cdot BD}{AD \cdot BC}$$

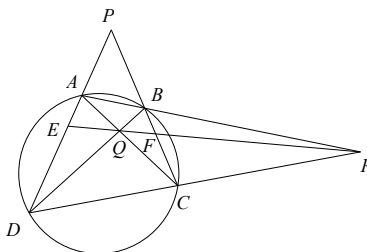
where the lengths are directed. *Cross ratios are preserved under projective transformations.* (Besides coincidence, this is pretty much the only thing that's projective-invariant.)

The most significant case is when  $(A, B; C, D) = -1$ , in which case we say that  $(A, B; C, D)$  is *harmonic*. This notion of cross ratios and harmonic quadruples is not limited to collinear points, but also applies to pencils of lines, and concyclic points. Harmonic quadruples arise frequently, so learn to recognize them!

Possibly the most useful fact from polar geometry is the *self-polarity* of the diagonal triangle of a cyclic quadrangle. (Part of the reason why it's powerful is because it is not easy to prove simply using Euclidean geometry.)

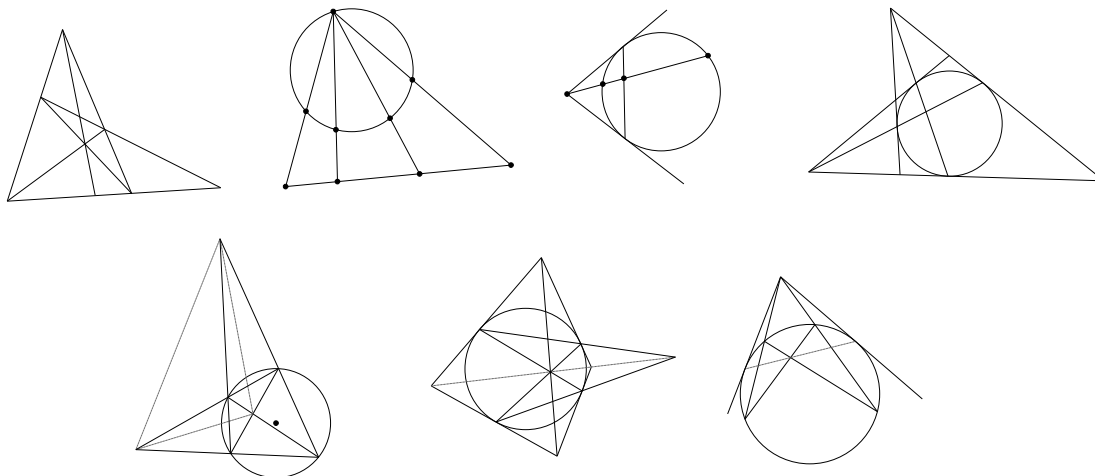
**Theorem.** Let  $ABCD$  be a cyclic quadrilateral with circumcircle  $\omega$ , and let  $AD \cap BC = P$ ,  $AC \cap BD = Q$ ,  $AB \cap CD = R$ . Then  $PQR$  is self-dual with respect to  $\omega$ . That is,  $P$  is the pole of  $QR$ ,  $Q$  is the pole of  $PR$ , and  $R$  is the pole of  $PQ$ .

In particular, we see that  $O, P, Q, R$  forms an orthocentric quadruple, where  $O$  is the center of  $\omega$ . (This means that any of the four points is the orthocenter of the triangle formed by the other three).



Here is a quick sketch of the proof. Let line  $QR$  meet  $AD$  and  $BC$  at  $E$  and  $F$  respectively. Then from the configuration of lines, we see that  $(A, D; P, E)$  and  $(C, B; P, F)$  are both harmonic. It follows the polar of  $P$  is  $EF$ , which coincides with  $QR$ .  $\square$

You should develop the skill of recognizing when to use projective geometry. Here is a gallery of diagrams showing the patterns that tend to hint the use of projective geometry, in particular the use of poles and polars. Try to figure out the projective significance of each diagram. Ask Yufei if you need help.



- Let  $ABC$  be a triangle and  $I$  be its incenter. Let the incenter of  $ABC$  touch sides  $BC, CA, AB$  at  $D, E, F$  respectively. Let  $S$  denote the intersection of lines  $EF$  and  $BC$ . Prove that  $SI \perp AD$ .
- Let  $UV$  be a diameter of a semicircle, and let  $P, Q$  are two points on the semicircle. The tangents to the semicircle at  $P$  and  $Q$  meet at  $R$ , and lines  $UP$  and  $VQ$  meet at  $S$ . Prove that  $RS \perp UV$ .
- A circle is inscribed in quadrilateral  $ABCD$  so that it touches sides  $AB, BC, CD, DA$  at  $E, F, G, H$  respectively.
  - Show that lines  $AC, EF, GH$  are concurrent. In fact, they concur at the pole of  $BD$ .
  - Show that lines  $AC, BD, EG, FH$  are concurrent.
- (China 1996) Let  $H$  be the orthocentre of triangle  $ABC$ . From  $A$  construct tangents  $AP$  and  $AQ$  to the circle with diameter  $BC$ , where  $P, Q$  are the points of tangency. Prove that  $P, H, Q$  are collinear.
- (China 1997) Let quadrilateral  $ABCD$  be inscribed in a circle. Suppose lines  $AB$  and  $DC$  intersect at  $P$  and lines  $AD$  and  $BC$  intersect at  $Q$ . From  $Q$ , construct the two tangents  $QE$  and  $QF$  to the circle where  $E$  and  $F$  are the points of tangency. Prove that the three points  $P, E, F$  are collinear.
- (Butterfly theorem) Let  $\omega$  be a circle and let  $XY$  be a chord. Let  $M$  be the midpoint of  $XY$ , and let  $AB$  and  $CD$  be two chords of  $\omega$ , both passing through  $M$ . Let  $XY$  meet chords  $AD$  and  $BC$  at  $P$  and  $Q$  respectively. So that  $MP = MQ$ .
- Points  $C, M, D$  and  $A$  lie on line  $\ell$  in that order with  $CM = MD$ . Circle  $\omega$  is tangent to line  $\ell$  at  $A$ . Let  $B$  be the point on  $\omega$  that is diametrically opposite to  $A$ . Lines  $BC$  and  $BD$  meet  $\omega$  at  $P$  and  $Q$ . Prove that the lines tangent to  $\omega$  at  $P$  and  $Q$  and line  $BM$  are concurrent. (Hint: what is the significance of midpoints in terms of harmonic conjugates?)

8. **A very important fact.** Let  $ABC$  be a triangle and  $\Gamma$  its circumcircle. Let the tangent to  $\Gamma$  at  $B$  and  $C$  meet at  $D$ . Then  $AD$  coincides with a symmedian of  $\triangle ABC$ . (The *symmedian* is the reflection of the median across the angle bisector, all through the same vertex.)  
(Hint: consider a reflection about the angle bisector of  $\angle A$ .)
9. Let  $ABCD$  be a quadrilateral (not necessarily cyclic), and let  $X$  be the intersection of diagonals  $AC$  and  $BD$ . Suppose that  $BX$  is a symmedian of triangle  $ABC$  and  $DX$  is a symmedian of triangle  $ADC$ . Prove that  $AX$  is a symmedian of triangle  $ABD$ .
10. Let  $ABC$  be a triangle and tangent at point  $C$  to the circumcircle of  $ABC$  meets  $AB$  in  $M$ . The line perpendicular to  $OM$  in  $M$  intersects  $BC$  and  $AC$  in  $P$  and  $Q$  respectively. Prove that  $MP = MQ$ .
11. (Iberoamerican 1998) The incircle of triangle  $ABC$  is tangent to sides  $BC, CA, AB$  at  $D, E, F$ , respectively. Let segment  $AD$  meet the incircle again at  $Q$ . Show that the line  $EQ$  passes through the midpoint of segment  $AF$  if and only if  $AC = BC$ .
12. Let  $ABCD$  be a convex quadrilateral (not necessarily cyclic). Let lines  $AB$  and  $CD$  meet at  $E$ ,  $AD$  and  $BC$  meet at  $F$ , and  $AC$  and  $BD$  meet at  $P$ . Let  $M$  be the projection of  $P$  onto  $EF$ .
- (a) Show that  $\angle AMP = \angle CMP$ .  
(b) Show that  $\angle BMC = \angle AMD$ .
- (Hint: what do internal and external angle bisectors have to do with harmonic divisions?)
13. Let  $ABCD$  be a circumscribed about a circle quadrilateral with the incenter  $I$ . Let  $M$  be the projection of  $I$  onto  $AC$ . Prove that  $\angle AMB = \angle AMD$ .
14. Let  $ABCD$  be a convex quadrilateral (not necessarily cyclic). Let lines  $AB$  and  $CD$  meet at  $E$ ,  $AD$  and  $BC$  meet at  $F$ . Suppose  $X$  is a point inside the quadrilateral such that  $\angle AXE = \angle CXF$ . Prove that  $\angle AXB + \angle CXD = 180^\circ$ .  
(Hint: apply a *polar transformation* centered at  $X$  and see what happens. And yes, you don't start with any circle. It's kind of weird I know ...)
15. Let  $ABCD$  be a cyclic quadrilateral with circumcenter  $O$ . Let lines  $AD$  and  $BC$  meet at  $E$  and lines  $AB$  and  $CD$  meet at  $F$ . Let  $M$  be the projection of  $O$  onto line  $EF$ . Prove that  $MO$  bisects  $\angle BMD$ .
16. (IMO 1985) A circle with center  $O$  passes through the vertices  $A$  and  $C$  of triangle  $ABC$ , and intersects the segments  $AB$  and  $BC$  again at distinct points  $K$  and  $N$ , respectively. The circumscribed circles of the triangle  $ABC$  and  $KBN$  intersect at exactly two distinct points  $B$  and  $M$ . Prove that angle  $\angle OMB = 90^\circ$ .
17. Let  $ABCD$  be a cyclic quadrilateral with circumcenter  $O$ . Let lines  $AB$  and  $CD$  meet at  $E$ ,  $AD$  and  $BC$  meet at  $F$ , and  $AC$  and  $BD$  meet at  $P$ . Furthermore, let  $EP$  and  $AD$  meet at  $K$ , and let  $M$  be the projection of  $O$  onto  $AD$  be  $M$ . Prove that  $BCMK$  is cyclic.
18. Let  $ABC$  be a triangle with incenter  $I$ . Fix a line  $\ell$  tangent to the incircle of  $ABC$  (not  $BC$ ,  $CA$  or  $AB$ ). Let  $A_0, B_0, C_0$  be points on  $\ell$  such that

$$\angle AIA_0 = \angle BIB_0 = \angle CIC_0 = 90^\circ$$

Show that  $AA_0, BB_0, CC_0$  are concurrent.

19. Let  $ABC$  be a triangle with incenter  $I$ . Fix a line  $\ell$  tangent to the incircle of  $ABC$  (not  $BC$ ,  $CA$  or  $AB$ ). Let  $\ell$  intersect the sides of the triangle at  $M, N, P$ . At  $I$ , erect perpendiculars to  $IM, IN$  and  $IP$  and let them intersect the corresponding sides of the triangle at  $M_0, N_0$ , and  $P_0$  respectively. Show that  $M_0, N_0, P_0$  lie on a line tangent to the incircle.
20. Let  $ABC$  be a triangle, and  $\omega$  its incircle. Let  $\omega$  touch the sides  $BC, CA, AB$  at  $D, E, F$  respectively. Let  $X$  be a point on  $AD$  and inside  $\omega$ . Let segments  $BX$  and  $CX$  meet  $\omega$  at  $Q$  and  $R$  respectively. Show that lines  $EF, QR, BC$  are concurrent.

**Food for thought.** Let  $ABC$  be a triangle with inscribed circle  $\omega$ . There exists a projective transformation that preserves  $\omega$  but sends  $ABC$  to any other desired triangle inscribed in  $\omega$ ! Can you prove this?

If a geometry problem is completely projective-invariant, then perhaps sending  $ABC$  to an equilateral triangle or an isosceles right triangle might be a good idea. Note that when the circumcircle is not present, we can simply apply an affine transformation to get what we want.