Geometrical Gems

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You must always invert.

- Carl Gustav Jacobi

§ Morley's theorem

Theorem

If the internal angle trisectors of a $\triangle ABC$ meet at X, Y, Z, i.e. $\measuredangle ZAB = \measuredangle YAB = \frac{\measuredangle A}{3}$ then $\triangle XYZ$ is equilateral.

Well, this looks like a theorem that is too good to be true. Why not, since all we have in there are some trisectors and we get a beautiful and cute equilateral triangle there.

Let's try proving this.[1]

Let's just work backward. Kind of cheating with the problem, but I know.
Let $\overline{BZ} \cap \overline{CY} = P$. Define Q, R similarly. This helps us as we immediately get X as the incenter of $\triangle BPC$.

So from angle chasing we see that if $\measuredangle A = 3\alpha$ etc, then

$$\measuredangle BPC = 60^\circ + 2\alpha$$

. Also, $\overline{PX}$ bisects $\measuredangle BPC$. What if we had $PY = PZ$? That could give us $\triangle PZX \cong \triangle PYX$ and then $XY = XZ$. By similar relations, we'd be done.

So we just need $\measuredangle PYZ = 60^\circ - \alpha$.
Seeing no other way out using pure angle chasing [2], we resort to "reversing the paradigm"[3], i.e., we construct $\triangle ABC$ from $\triangle XYZ$! So we just construct isosceles triangles $\triangle PYZ$ (with base angle $60^\circ - \alpha$) etc on sides of $\triangle XYZ$. Also we let $\overline{QZ} \cap \overline{RY} = A'$ etc. Now we can just angle chase to show that $\overline{PX}$ bisects $\measuredangle YPZ$ and

$$\measuredangle B'XC' = 90^\circ + \frac{\measuredangle B'PC'}{2}$$

which gives us that X is the incenter of $B'PC'$ and upon further angle chasing, we see that the angles of $A'B'C'$ are $3\alpha, 3\beta, 3\gamma$, and thus we're done. ∎

[1] Part of the reason why is that although I've seen many proofs, there have been none in my sight which gave the motivation behind it.
[2] Atleast we assume we do not have a way out right now that is easily seen
[3] See the quotation just below the title of this article

§ Adam's circle

Theorem

Let the parallels to the sides through the Gergonne point G meet the remaining sides at P, Q, R, S, T, U, with $P, Q \in \overline{BC}, R, S \in \overline{CA}, T, U \in \overline{AB}$ resp. Then, P, Q, R, S, T, U are concyclic with center I, the incenter of ABC.

What a beautiful theorem! We would like to show that each of the six points P, Q, R, S, T, U is at the same distance from the incenter I. Clearly the inradii ID, IE, and IF meet the sides of ABC at right angles. Thus at D, for example, the right triangles IDP and IDQ have a common leg $ID = r$. If we can show that PD and DQ are equal, then these right triangles are congruent and $IP = IQ$. More generally, if we can show that all six of the little segments PD, DQ, RE, ES, TF, and FU are equal, then six congruent right triangles result, and their six equal hypotenuses yield the desired conclusion. Therefore let us proceed to show that

$$PD = DQ = RE = ES = TF = FU$$

The equal tangents CD and CE make CDE isosceles, and the parallel lines DE and PS make triangles CDE and CPS similar. Thus CPS is also isosceles, and the differences PD and SE in the lengths of the arms of these isosceles triangles are equal:

$$PD = SE$$

Similarly $ER = FU$ and $FT = DQ$. Thus, if we can show that $PD = DQ$, similar arguments will yield $RE = ES$ and $TF = FU$, and

$$PD = SE = ER = FU = FT = DQ$$

Let us conclude, then, by showing that $PD = DQ$. To this end, let extensions of DE and DF meet the line through A parallel to BC at X and Y . Triangles AXE and CDE have equal corresponding angles and therefore are similar; and since CDE is isosceles, so is AXE. Similarly, AYF is similar to isosceles triangle BDF and, with the equal tangents AE and AF, we have altogether that

$$AX = AE = AF = AY,$$

revealing that DA is a median of DXY . Now a dilatation sends the equal segments YA and AX to equal images MG and GN which lie along a segment of MGN parallel to YX, and therefore also parallel to PQ . Thus, in parallelograms $MGQD$ and $GNDP$, we have $MG = DQ$ and $GN = PD$, and since $MG = GN$, then $PD = DQ$, completing the proof. ■

§ The Centroid, the Circumcenter and the Symmedian point

Theorem

Let each of the circles $\odot(P), \odot(Q), \odot(R)$ pass through the centroid G of $\triangle ABC$ and two of the vertices. Prove that the circumcenter O and the centroid G of $\triangle ABC$ are, respectively, the centroid and symmedian point of the triangle $\triangle PQR$ which is determined by the centers of these circles.

Firstly, $\odot(P)$ goes through B, G, and C, and therefore $\measuredangle BPG$ at the center is twice the angle $\measuredangle BCG$ at the circumference. But PR bisects this angle $\measuredangle BPG$ (the perpendicular PR to the chord BG from the center bisects the central angle $\measuredangle BPG$), and we have

$$\measuredangle RPG = \measuredangle GCB = \measuredangle TPS$$

(where T is the midpoint of BC and S is $\overline{BC} \cap \overline{PQ}$), which is $\measuredangle QPO$, making $\overline{GP}$ and $\overline{OP}$ a pair of isogonal lines at P in $\triangle PQR$ and similar relations making G and O a pair of isogonal conjugates of $\triangle PQR$.
Now construct the reflection of G over the midpoint of AC and call it H. Then, it is not difficult to see by angle chasing and Thales' theorem that

$$\triangle AHG \cup \{G'\} \sim \triangle QPR \cup \{O\}$$

with G' being the centroid of $\triangle AHG$. This gives us that O is the centroid of $\triangle PQR$, as claimed. $\blacksquare$.

§ The Fuhrmann Circle

Theorem

Let X', Y', Z' be the midpoints of the arcs BC, CA, AB not containing A, B, C respectively. Let their reflections over the sides be X, Y, Z. Then, the circle $\odot(XYZ) \equiv \odot F$ contains the orthocenter H, and the Nagel point M, with $\overline{HM}$ being a diameter of the circle. This circle also contains the points T, U, V on $\overline{AH}, \overline{BH}, \overline{CH}$ resp, with $AT = BU = CV = 2r$ with r being the inradius of ABC.

Here we provide a sketch of a proof. Since the reflection of H wrt BC lies on $\odot(ABC)$, upon reflection in BC, we see $HX \parallel AL$ where L is the antipode of X' in $\odot(ABC)$. $HX \parallel AL \perp AX'$ and thus $HX \perp AX'$. Homothety $\mathbb{H}(G, -2)$ at G, the centroid, with ratio -2 takes ABC to its antimedial triangle $A'B'C'$ and sends I to M, and thus $A'M \parallel AX' \perp HX$. X, A' are also the reflection of X', A resp in the midpoint of BC, so $A'X \parallel AX' \parallel A'M$ an thus X, M, A' are collinear. Thus the circle with diameter HM passes through X, and similarly Y, Z. This completes the proof of the first part. As M is the incenter of $A'B'C'$, we have $\operatorname{dist}(M, B'C') = 2r$ and thus $\measuredangle ATM = 90°$. This completes the proof of the second part. $\blacksquare$.

§ Problems

1. If parallels through the symmedian point of a triangle meet the opposite sides at P, Q, R, S, T, U, with $P, Q \in \overline{BC}, R, S \in \overline{CA}, T, U \in \overline{AB}$ resp. Then, P, Q, R, S, T, U are concyclic with center the midpoint of the segment joining the symmedian point and the circumcenter.

2. If antiparallels through the symmedian point of a triangle meet the opposite sides at P, Q, R, S, T, U, with $P, Q \in \overline{BC}, R, S \in \overline{CA}, T, U \in \overline{AB}$ resp. Then, P, Q, R, S, T, U are concyclic with center the symmedian point.

3. Let P be a point and $A'B'C'$ be its circumcevian triangle. Then reflections of A' over BC etc are concyclic with the orthocenter of ABC.

4. Let P be a point and $A'B'C'$ be its circumcevian triangle. Then reflections of A' over the foot of perpendicular from P to BC, etc are concyclic with the orthocenter of ABC.

5. Prove that if $A'B'C'$ is the cevian triangle of P wrt ABC, then $B'C' \cap BC$ etc are collinear at a line called the trilinear polar of P wrt ABC. If P is the orthocenter, the trilinear polar is the radical axis of the circumcircle and the nine point circle, and hence perpendicular to the Euler line.

6. Consider the circle with diameter GH (orthocentroidal circle) with G the centroid of ABC and H the orthocenter. It turns out that I always lies inside this circle. Plenty of other triangle centers also have a position fixed relative to this circle, i.e., they either always lie inside, or on, or outside the circle. Find some of them.

7. Let 2 perpendicular lines through the orthocenter meet BC, CA, AB at $A_1, A_2, B_1, B_2, C_1, C_2$ resp. Then prove that the midpoints of A_1A_2, B_1B_2, C_1C_2 are collinear. (Droz Farny's Theorem).

8. Let the incircle touch BC, CA, AB at D, E, F resp and let the $A-$ excircle touch them at D_1, E_1, F_1 resp. Then prove that if A_1 is the foot of $\perp$ from D to EF and if A_2 is the foot of $\perp$ from D_1 to E_1F_1, then AA_1, AA_2 are isogonal wrt A.

9. With the same notations as the previous problem, prove that I (the incenter), the midpoint of AD, the midpoint of BC are collinear.

10. Also prove that ID_1 and I_AD meet on the midpoint of the A-altitude, where I_A is the $A-$excenter.

11. Prove that the circumcevian triangle and the pedal triangle of a point are similar

12. The A-mixtilinear is the circle internally tangent to $\odot(ABC)$ and AB, AC. Find all properties you can and prove them. For references, see Evan Chen's "A Guessing Game: Mixtilinear Incircles" and Forum Geometricorum's 2006 article "On Mixtilinear Incircles and Excircles" by Khoa Lu Nguyen and Juan Carlos Salazar.

13. Prove that the midpoint of BC, the midpoint of the A-altitude and the symmedian point are collinear.

14. Prove that if $\{X, X'\}$ and $\{Y, Y'\}$ are pairs of isogonal conjugates, then $XY' \cap X'Y$ and $XY \cap X'Y'$ are also isogonal conjugates.

15. Prove that the Gergonne point of the medial triangle is the perspector of the excentral triangle and the medial triangle.
    The following notations are useful for the next 2 problems:
    G is the centroid, O the circumcenter, T_i the i^{th} Fermat-Torricelli point, Ap_i the i^{th} Apollonius point, and L the symmedian point of a triangle ABC.

16. Prove that the lines T_1T_2 and OAp_1Ap_2 meet on the symmedian point of $\triangle ABC$.

17. Prove that $T_1Ap_2 \cap T_2Ap_1 = G$.

18. Prove that G, I, S_p, N_a are collinear where S_p, N_a are the incenter of the medial triangle (Spieker center) and the Nagel point resp. Also find the ratio in which thay divide the line segment joining each other.

19. Prove that the Brocard points lie on the circle with diameter OK where K is the symmedian point, and that $OBr_1 = OBr_2$.

20. Prove that the first Brocard point is the second Brocard point of its circumcevian triangle and vice versa.

21. Prove that the symmedian point is the symmedian point of its circumcevian triangle too.

22. Prove that the two Soddy points lie on the line joining the incenter and the Gergonne point of $\triangle ABC$.

23. Prove that the orthology centers of ABC and $I_AI_BI_C$ are the incenter and the Bevan point (the circumcenter of $I_AI_BI_C$), and the line passing through them contains the circumcenter of ABC.

24. Prove that OI is the Euler line of the intouch triangle of ABC.

25. Prove that the nine point center of the cevian triangle of the incenter lies on OI.

26. Prove that the symmedian point of a triangle is the unique point in is plane which the the centroid of its pedal triangle.

27. Prove that the pedal triangle of P wrt the pedal triangle of P wrt the pedal triangle of a P wrt $\triangle ABC$ is similar to $\triangle ABC$. For n-gons, the statement is repeated n times, i.e., the n^{th} pedal n-gon of a point wrt an n-gon is similar to it.

28. If the internal bisectors of $\measuredangle B$ and $\measuredangle C$ meet the opposite sides at B_1, C_1 then prove $B_1C_1 \perp OI_A$.

29. Prove that the incircle, excircles are tangent to the nine-point circle. Also prove as many properties of the tangency points as you can. The points are called the Feuerbach point, A, B and C-Feuerbach points respectively.

30. Prove that the cevian circle of I passes through F_e where F_e is the Feuerbach point of ABC.

31. Let DEF be the intouch triangle and let DE, DF intersect the line parallel to BC and through A at X, Y. Prove that the foot of D on the Euler line of DXY is F_e.

32. Find out about the orthopole and the orthotransversal and the orthocorrespondent!

33. Prove that the exsimilicenter and the insimilicenter of the incircle and circumcircle are isogonally conjugate to the Gergonne and the Nagel point. (Find the order of isogonal conjugacy!).

34. Prove that the reflection of any line through H about the sides of ABC meet on the circumcircle of ABC. In fact, the point of concurrence P has the property that that line is the Steiner line of $ABCP$. P is called the Anti-Steiner point of the line wrt ABC. When the line is the Euler line of ABC, then $P \equiv E_r$ is the Euler reflection point of ABC.

35. Prove that F_e is the concurrency point reflection of the line OI in the sides of the intouch triangle, i.e., the Euler reflection point of the intouch triangle.

36. Let $ABCD$ be a quadrilateral, with the intersections of diagonals AC, BD being P. Then Euler lines of PAB, PBC, PCD, PDA are concurrent $\iff ABCD$ is cyclic.

37. Given a triangle ABC and a point P, suppose, A_0, B_0, C_0 are the reflections of P over BC, CA, AB. Prove that AA_0, BB_0, CC_0 are concurrent iff PP^* is parallel to the Euler line of ABC where P^* is the isogonal conjugate of P wrt ABC.

38. Prove that isotomic conjugate of the orthocenter is the symmedian point of the antimedial triangle.

39. Prove that the Euler line is perpendicular to the line joining the isotomic conjugates of the 2 Brocard points.

40. Let N be the nine-point center, N^* be the isogonal conjugate of N (Kosnita point), n' be the isotomic conjugate of N and N'' be the isogonal conjugate of N'. Prove that $NN'' \parallel ON^*$ where O is the circumcenter.

41. (Telv Cohl) Let D_e be the de-Longchamps point of ABC, i.e., the reflection of H in O. Then prove that the Feuerbach point of $\triangle ABC$ lies on the radical axis of $\odot(ABC)$ and $\odot(IGD_e)$

42. (Sondat's theorem) Given two triangles such that they are orthologic and perspective, prove that the two centers of orthology are collinear with the center of perspectivity.

43. (Poncelet's theorem) Let ω and Ω be the incircle and the circumcircle of an n-gon. Then for any point A on Ω, there exists an n-gon with A as its vertex and ω and Ω as its incircle and circumcircle resp.

44. For the special case of Poncelet's theorem for $n = 3$, let T be a triangle having ω and Ω as the incircle and the circumcircle resp, and let P be any point. Then prove that the locus of the isogonal conjugate of P wrt T is a circle.