Iranian Mathematical Olympiad

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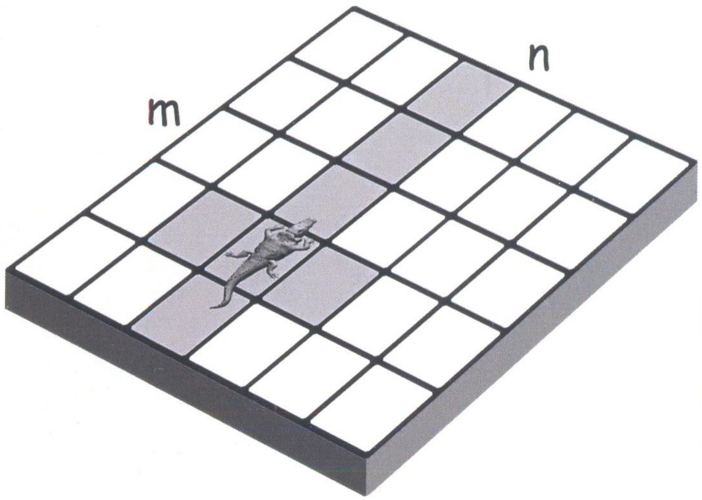
Iranian Mathematical Olympiad



2004-2005
Young Scholars Club
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22^{nd}

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Contents:

Problems

First Round Time: 2×4.5 hours

1. Let ABC be a triangle with $\angle A = 90^0$. D is the intersection of angle bisector of $\angle A$ and the line BC. I_a is the center of the outer inscribed circle of ABC with respect to A. Prove that
$$\frac{AD}{DI_a} \leq \sqrt{2} - 1$$

2. Assume that $f : [0, +\infty) \mapsto \mathbb{R}$ is a function such that $f(x) - x^3$ and $f(x) - 3x$ are both increasing. Prove that $f(x) - x^2 - x$ is also increasing.

3. The transportation ministry has just decided to pay 80 companies to repair 2400 roads. These roads are between 100 cities. Each road is between two cities and there is at most one road between any two cities. We know that each company must repair exactly 30 roads. For a company to repair a road, it is necessary to have stations at both cities on its endpoints. Prove that there is a city such that at least 8 companies have stations there.

4. Find all functions $f : \mathbb{N} \mapsto \mathbb{N}$ such that for all $m, n \in \mathbb{N}$
$$f(m) + f(n) \mid m + n$$

5. The angle bisector of $\angle A$ from the triangle ABC meets the line BC and the circumcircle of ABC respectively at D and M. A line through D meets the circle with center M and radius MB at X and Y. Prove that AD bisects the angle XAY.

6. We have a $m \times n$ table where $m \geq 4$. The crocodile nut can attack all its column cells and its right and left cells (a total of $m + 2$ cells). What is the minimum number of crocodiles needed to attack all the table.

Second Round **Time:** 2×4.5 **hours**

1. Solve the following equation in prime numbers

$$p^3 = p^2 + q^2 + r^2$$

2. We have $n > 1$ natural numbers $a_1, a_2, \dots, a_n$ not all of them are equal. Prove that there exist infinitely many primes p such that

$$p \mid a_1^k + a_2^k + \dots + a_n^k$$

for some number k.

3. We have two finite sets of points A, B in the plane. For every four distinct points of $A \cup B$ there is a line such that it separates the points of A and B among these four. Prove that there is a line which separates A and B.

4. $P(n)$ is the number of ways we can travel from the point $(0,0)$ to the point $(n,0)$ with three moves $(x,y) \to (x+1, y-1)$ and $(x,y) \to (x+1,y)$ and $(x,y) \to (x+1,y+1)$ such that we are always above the line $y=0$.

   a) Find a recursive formula for the sequence $\{P(n)\}_{n=0}^{\infty}$.

   b) Suppose that C_i is the ith Catalan number. Prove that

$$P(n) = \sum_{i=0}^{\infty} \binom{n}{2i} C_i$$

   c) Prove that

$$C_n = \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} P(2n-i)$$

5. Prove that all the roots of

$$P(x) = x^n - 5x^{n-1} + 12x^{n-2} - 15x^{n-3} + a_{n-4}x^{n-4} + \dots + a_0$$

can not be Real and positive.

6. Let ABC be a triangle. Let O be its circumcenter. Let A' be the midpoint of BC. Let A'' be the intersection of AA' with the circumcircle. Let Q_a be on AO such that $A'Q_a \perp AO$. Let P_a be the intersection of $A'Q_a$ with the tangent to circumcircle at A''. We similarly construct P_b and P_c. Prove that P_a, P_b, P_c are collinear.

Third Round **Time:** 2×4.5 **hours**

1. Suppose $a_1 \le a_2 \le \dots \le a_n$ are positive real numbers such that

$$\frac{a_1 + a_2 + \dots + a_n}{n} = m \quad , \quad \frac{a_1^2 + a_2^2 + \dots + a_n^2}{n} = 1$$

Prove that for any $1 \le i \le n$,if $a_i \le m$ then

$$n - i \ge n(m - a_i)^2$$

2. Assume ABC is an isosceles triangle that $AB = AC$ Suppose P is a point on the extension of side BC. X and Y are points on AB and AC such that:

$$PX \| AC \quad , \quad PY \| AB$$

Let T be the midpoint of arc BC. Prove that $PT \perp XY$.

3. Suppose there are 18 lighthouses on the Persian Gulf. Each of the lighthouses lightens an angle with size 20 degrees. Prove that we can choose the directions of the lighthouses such that the whole blue (always) Persian Gulf is lightened.

4. Find all $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that there exists $k \in \mathbb{N}$ and a prime p that for all $n \ge k$, $f(n+p) = f(n)$ and also if $m|n$ then $f(m+1)|f(n)+1$.

5. n points are fixed on the plane. We lay a disc on the plane in such a way that at least one of the point lies inside it. At each step we move the center of the disc to the barycenter of all points which were inside the disc in the previous step. Prove that after some step the disc doesn't move any more.

6. We have a tetrahedron $ABCD$ in the space. H_a,H_b,H_c,H_d are orthocenters of triangles BCD,ACD,ABD,ABC. Prove that AH_a,BH_b,CH_c,DH_d are concurrent if and only if the following equality holds

$$AB^2 + CD^2 = AC^2 + BD^2 = AD^2 + BC^2$$

Solutions

First Round Solutions

1. Let H and H' be the points on AB and AC respectively such that $I_aH \perp AB$ and $I_aH' \perp AC$. Let h be the length of the altitude from A to BC.
$$\frac{AD}{DI_a} = \frac{h}{r_a}$$
where r_a is the radius of the outer inscribed circle with respect to A.
$$h \cdot BC = 2S_{ABC} = AB \cdot AC \longrightarrow h = \frac{AB \cdot AC}{BC}$$
Now because $\angle I_aHA = \angle I_aH'A = \angle HAH' = 90^0$ and $I_aH = I_aH'$ we have that I_aHAH' is a square so
$$r_a = AH = \frac{AB + AC + BC}{2}$$
therefore
$$\frac{AD}{DI_a} = \frac{2AB \cdot AC}{AB \cdot BC + AC \cdot BC + BC^2} = \frac{2AB \cdot AC}{AB \cdot BC + AC \cdot BC + AB^2 + AC^2}$$
Now notice that
$$BC = \sqrt{AB^2 + AC^2} \geq \frac{AB + AC}{\sqrt{2}}$$
Using the fact that $(AB + AC)^2 \geq 4AB \cdot AC$ and putting this in the main equation it gives us
$$\frac{AD}{DI_a} \leq \frac{2AB \cdot AC}{2\sqrt{2}AB \cdot AC + AB^2 + AC^2}$$
Now noticing the fact that $AB^2 + AC^2 \geq 2AB \cdot AC$ we get
$$\frac{AD}{DI_a} \leq \frac{1}{\sqrt{2} + 1} = \sqrt{2} - 1$$

2. We will show that $f(x) - x^2 - x$ is increasing in $[0, 1]$ and $[1, +\infty)$ so we get the desired result. Assume $x \geq y$ and $x, y \in [0, 1]$ then
$$f(x) - f(y) \geq 3(x - y)$$
Now $3 \geq x + y + 1$ so
$$f(x) - f(y) \geq (x + y + 1)(x - y) = x^2 + x - y^2 - y$$
Now suppose that $x, y \in [1, +\infty)$ then
$$f(x) - f(y) \geq (x^2 + xy + y^2)(x - y)$$
And since $x^2 + xy + y^2 \geq x + 1 + y$ one has
$$f(x) - f(y) \geq (x + 1 + y)(x - y) = x^2 + x - y^2 - y$$

3. First note that each company must have at least 9 stations. Because $\binom{8}{2} = 28 \leq 30$ so if it has 8 stations or less it can repair at most 28 roads. Now notice that there are at least $80 \times 9 = 720$ stations and by the pigeonhole principle there are at least 8 stations in a city.

4. Let $m = n$ in the hypothesis to obtain

$$2f(m) \mid 2m \longrightarrow f(m) \mid m \longrightarrow f(m) \leq m$$

Let p be a prime greater than m. Put $n = p - m$ to get

$$f(m) + f(n) \mid p$$

So we have $f(m) + f(n) = 1$ or $f(m) + f(n) = p$. The first case can not occur because $f(m) + f(n) \geq 2$. So $f(m) + f(n) = p = m + n$. But since $f(m) \leq m$ and $f(n) \leq n$ equality occurs only if $f(m) = m$ and $f(n) = n$.

5. $BXCY$ is cyclic so $DB \cdot DC = DX \cdot DY$. $ABMC$ is cyclic so $DB \cdot DC = DA \cdot DM$ so $DX \cdot DY = DA \cdot DM$ and hence $AXMY$ is cyclic. But $MX = MY$ so M is the midpoint of the arc XMY. So AM is the angle bisector of $\angle XAY$.

6. Obviously n crocodiles are enough (just put one crocodile on every column.). We show that n crocodiles are necessary to attack the table. First assume that $m \geq 5$. Assume the cases with minimum crocodiles and among them choose the one with minimum number of empty columns. If this case has no empty columns we have at least n crocodiles. If it has an empty column then that column's cells must be attacked by m crocodiles on the right and left of that column. Delete these crocodiles and put 5 crocodiles in the 2-left column,left column,this column, right column and 2-right column. Then clearly the table is still attacked and the number of empty columns is reduced by one, contradiction! So we must have at least n crocodiles. Now assume that $m = 4$. Again choose the case with minimum number of empty columns. If there is no empty column, again, we have at least n crocodiles. If not, consider one of its empty columns. The 2-left column and 2-right column must be empty otherwise we can omit one of those 4 crocodiles. So by induction we can show that even columns are empty and odd ones are not. Sum of crocodiles of every two consecutive odd columns must be 4. So by analyzing a few cases we can deduce that we have at least n crocodiles.

Second Round Solutions

1. If $p = 2$ then the equation becomes

$$q^2 + r^2 = 4$$

which obviously has no solution. So suppose that p is odd. Consider the equation modulo p

$$q^2 + r^2 \overset{p}{\equiv} 0$$

So we have $p \mid q, r$ or $p = 4k + 1$. In the first case since q, r are primes we have $p = q = r$ so the equation becomes $p^3 = 3p^2$ which has the solution $p = q = r = 3$. In the second case Consider the equation modulo 4

$$q^2 + r^2 \overset{4}{\equiv} 0$$

and hence we have $2 \mid q, r$ so $q = r = 2$. Thus $p^3 - p^2 = 8$ forcing p to be 2. But it is not a solution. Finally we got the only solution $p = q = r = 3$.

2. We may assume that $a_1, \ldots, a_n$ are relatively prime otherwise let $d = (a_1, \ldots, a_n)$. Then let $a_i' = \frac{a_i}{d}$. Clearly if $p \mid {a_1'}^k + \ldots + {a_n'}^k$ then also $p \mid {a_1}^k + \cdots + {a_n}^k$. Now suppose that the set of prime devisors of ${a_1}^k + \ldots + {a_n}^k$ is finite say $\{p_1, \ldots, p_s\}$. Let t be a number such that ${p_i}^t \nmid j$ for $j = 1, \ldots, n$. Now let $u = \varphi((p_1 p_2 \ldots p_s)^t)$. Then let a be such that $b = au > t$. Now consider the number $c = {a_1}^b + \ldots + {a_n}^b$. Let q be one of the p_i's. Then if a_i is divisible by q then $a_i^b \overset{q^t}{\equiv} 0$ because $b > t$. If a_i is not divisible by q then $a_i^b \overset{q^t}{\equiv} 1$. So c modulo q^t is equivalent to one of $0, 1, 2, \ldots, n$. It can not be 0 because not all a_is are divisible by q. But according to the choice of t, c is not divisible by q^t. Hence $c \leq (p_1 \ldots p_s)^t$ but we can make b so large that c becomes large enough (Because not all of a_is are 1)Contradiction.

3. Let $conv(X)$ denote the convex hull of X. If $conv(A) \cap conv(B) = \emptyset$ then by separation theorem there is such a line. So $conv(A) \cap conv(B)$ is not empty. Now assume that $x \in conv(A) \cap conv(B)$. By the theorem of Caratheodory there are $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$ such that $x \in a_1a_2a_3 \cap b_1b_2b_3$. If the edges of $a_1a_2a_3$ and $b_1b_2b_3$ do not intersect then one of the triangles is inside the other so for example $a_1a_2a_3$ is inside $b_1b_2b_3$. But then b_1, b_2, b_3, a_1 can not be separated by a line. So the edges intersect. For instance a_1a_2 and b_1b_2 intersect in the point y. If a line separates a_1, a_2, b_1, b_2 then y is not in neither of the half-planes because if it's in the half-planes say that of B then it can not be on the segment a_1a_2 Contradiction.

4. a) Consider the first time when we reach the line $y = 0$. For example $(x, 0)$. Then between $(1, 1)$ and $(x - 1, 1)$ there is a path which never goes under $y = 1$ and also there is a path between $(x, 0)$ and $(n, 0)$ which never goes under $y = 0$ so there are $P(x - 2)P(n - x)$ ways. Varying x from 1 to n we get $P(n) = \sum_{i=1}^{n} P(i - 2)P(n - i)$. Here we assume that $P(-1) = P(0) = 1$.

b) Let C_i be the i-th Catalan number. Delete all of the paths like $(x, y) \to (x + 1, y)$. This forms one of the C_i paths as in the catalan problem. Conversely we can add straight paths between the moves of C_i. But in how many ways? We must insert $n - 2i$ objects into $2i + 1$ positions. This can be done by $\binom{n}{2i}$ ways. So

$$P(n) = \sum_{i=0}^{\infty} \binom{n}{2i} C_i$$

c) Let T be the set of all paths we count by P_{2n}. And let A_i be the paths of T which have a move like $(i-1,y) \to (i,y)$. Then

$$C_n = \mid T - \bigcup_{i=1}^{2n} A_i \mid$$

Observe that $\mid A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_s} \mid = P(2n-s)$ because we can delete straight moves. Now by inclusion-exclusion principle we have

$$C_n = \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} P(2n-i)$$

5. Suppose that all the roots are positive reals. Name them as $a_1, \ldots, a_n$. We compute $\sum {a_i}^3$. We have

$$\sum a_i = 5 \quad , \quad \sum a_i a_j = 12 \quad , \quad \sum a_i a_j a_k = 15$$

So

$$(\sum a_i)^3 = \sum {a_i}^3 + 3\sum {a_i}^2 a_j + 6\sum a_i a_j a_k$$

and

$$(\sum a_i)(\sum a_i a_j) = \sum {a_i}^2 a_j + 3\sum a_i a_j a_k$$

So we have

$$\begin{aligned} 0 \ \le \ & \sum {a_i}^3 = (\sum a_i)^3 - 3(\sum a_i)(\sum a_i a_j) + 3\sum a_i a_j a_k \\ = \ & 125 - 180 + 45 \\ = \ & -10 < 0 \end{aligned}$$

Contradiction.

6. We show that they are all on the radial axis of the circumcircle and nine point circle. By a homothety ABC goes to $A'B'C'$. In this homothety AO goes to the $A'N$ where N is the center of nine point circle. So $A'N \parallel AO$ and hence $A'P_a \perp A'N$. Therefore $A'P_a$ is tangent to the nine point circle. So we must prove that $A'P_a = A''P_a$. We prove that $\angle P_a A'' A' = \angle P_a A' A''$. $\angle P_a A'' A' = \angle A'AB + \angle C$ and $\angle P_a A' A'' = 90^\circ - \angle A'AO$. Thus we must prove that $\angle BAA' + \angle A'AO + \angle C = 90^\circ$. This comes up to prove that $\angle OAB + \angle C = 90^\circ$ which is obvious.

Third Round Solutions

1. Define $b_k = m - a_k$. Obviously $b_1 \ge b_2 \ge \cdots \ge b_n$. and since a_k is positive, $b_k \le m$. Now

$$\sum_{k=1}^{n} b_k = 0 \quad , \quad \sum_{k=1}^{n} b_k^2 = n(1-m^2)$$

So we have $b_i \ge 0$ and we must prove that $b_i^2 \le \frac{n-i}{n}$. First notice that $b_1 + b_2 + \cdots + b_i \ge ib_i$. i.e. $b_{i+1} + b_{i+2} + \cdots + b_n \le -ib_i$. By Cauchy-Shwartz inequality one has

$$b_1^2 + b_2^2 + \cdots + b_i^2 \ge \frac{(b_1 + b_2 + \cdots + b_i)^2}{i} \quad , \quad b_{i+1}^2 + b_{i+2}^2 + \cdots + b_n^2 \ge \frac{(b_{i+1} + b_{i+2} + \cdots + b_n)^2}{n-i}$$

And adding these two inequalities:

$$n(1-m^2) = \sum_{k=1}^{n} b_k^2 \ge \frac{inb_i^2}{n-i}$$

So $b_i^2 \le \frac{(n-i)(1-m^2)}{i}$. Now using the hypothesis, $b_i^2 \le m$

$$b_i^2 \le \min\left\{m^2, \frac{(n-i)(1-m^2)}{i}\right\} \le \frac{n-i}{n}$$

The last inequality is obvious and we're done.

2. We will prove that P is on the radial axis of the circles with diameters TX and TY. Suppose we have proved this If K is the other intersection of circles with diameter TX and TY, then T, K and P are collinear. And $\widehat{YKT} = \widehat{XKT} = \frac{\pi}{2}$. So Y, X and K are collinear and $PT \perp XY$ and we're done. Suppose M and N are orthogonal projections of T on PX and PY. Now it's enough to prove $PX \cdot PM = PN \cdot PY$, Because M and N are on circles with diameters TX and TY. Now you see that

$$\frac{PY}{AB} = \frac{PC}{BC} \quad , \quad PN = PB \sin\frac{A}{2}$$

So $PN \cdot PY = PB \cdot PC \cdot \frac{AC}{BC} \sin\frac{A}{2}$ Similarly $PM \cdot PX = PB \cdot PC \frac{AB}{BC} \sin\frac{A}{2}$. But $AB = AC$ so the problem is proved.

3. Consider a big circle containing the Persian Gulf. Suppose its center is not collinear with any two of the lighthouses. Now there is a diameter ℓ of circle that 9 lighthouses are at each side of it. Now using 9 lighthouses of each side we enlighten the other side. We can do it this way: consider the 18-gon $A_1A_2 \ldots A_{18}$ that is circumscribed on circle with $\ell \| A_1A_2$. Now suppose we want to cover one semicircle with the points that are not at the same side that A_1. Suppose from the points on the other semicircle the nearest to A_2A_3 is O_1. Now from O_1 draw 2 lines one parallel to A_1A_2 and the other parallel to A_2A_3 (that is the region it covers). Also suppose from rest of the 9 points O_2 is the nearest to A_3A_4. From O_2 draw two lines one parallel to A_2A_3 and the other parallel to A_3A_4. You see on each line parallel to ℓ and in the semicircle we want to cover it, regions O_1 and O_2 cover have intersection. Now define $O_3, O_4, \ldots, O_9$ like this. Now you can see that these points cover every point on every line parallel to ℓ.

4. Suppose $n \ge k$ and p doesn't divide $n-1$.You see there is k that $n-1 | n+kp$. So $f(n) | f(n+kp)+1$ but we know that $f(n) = f(n+kp)$, Therefore $f(n)|1$ and $f(n) = 1$. Consider an arbitrary $n \ge 1$, We know that $n-1 | (n-1)kp$, So $f(n) | f((n-1)kp) + 1 = 2$. So for every $n \ne 1$, $f(n) \in \{1, 2\}$. Now we have 2 cases:

a) $f(n) = 2$ for all $n \geq k$ and $p|n-1$, Consider n that p doesn't divide $n-1$. There is m that $n-1|m$ and $p|m-1$. So $f(n)|f(m)+1 = 3$ and $f(n) = 1$. So $f(n) = 1$ for all $n \geq k$ or m doesn't divide n; and is arbitarily defined for every $n < k$ and $p|n-1$.

b) $f(n) = 1$, for all $n \geq k$ and $p|n-1$, In this case $f(n) = 1$, for all $n \geq k$, and if we suppose $S = \{a|f(a) = 2\}$ then it doesn't exist $m, n \in S$ that $m-1|n$. So all of functions with the problem properties are defined like this: Suppose S is a finite subset of natural numbers that there doesn't exist $m, n \in S$ that $m-1|n$ and for $n > 1$, $f(n) = 2$ if and only if $n \in S$. $f(1)$ is arbitarily defined with this condition that $f(2)|f(1)+1$

5. We define a function that at each turn decreases or is constant. At each step let A be the set of points in the disc and B the set of points not in the disc. Define

$$f(O) = \left(\sum_{X \in A} OX^2\right) + |B|r^2$$

We know that for points $X_1, X_2, \ldots, X_k$ the sum $\sum_{i=1}^{k} OX_i^2$ is minimum when O is baricenter of the points $X_1, X_2, \ldots, X_k$. So suppose after one turn A changes to A' and B changes to B' and O goes to O'. Then

$$\sum_{X \in A} OX^2 \geq \sum_{X \in A} O'X^2 = \sum_{X \in A \cap A'} O'X^2 + \sum_{X \in A \cap B'} O'X^2 \geq \sum_{X \in A \cap A'} O'X^2 + |A \cap B'|r^2$$

$$|B|r^2 = |B \cap A'|r^2 + |B \cap B'|r^2 \geq \sum_{X \in B \cap A'} O'X^2 + |B \cap B'|r^2$$

Adding the 2 inequalities one has

$$\left(\sum_{X \in A} OX^2\right) + |B|r^2 \geq \left(\sum_{X \in A'} OX^2\right) + |B'|r^2$$

Now notice that the set of points that O can go to them is finite. So since $f(O)$ doesn't increase at each turn, it'll be constant after some turns. It means that location of O is constant after some turns.

6. First we prove that concurrency implies the equality: If AH_a, DH_d intersect at H then we have plane P_1 which passes through the points A, D, H_a, H_d. Then the lines AH_d, DH_a which are altitudes of triangles ABC, DBC are concurrent. Because these lines belong to the planes ABC, DBC their intersection point must lie on BC and hence AH_d, DH_a, BC are concurrent in a point say E. But with locus property of altitude we have the equality

$$AB^2 - AC^2 = BE^2 - CE^2 = DB^2 - DC^2$$

which implies

$$AB^2 + CD^2 = AC^2 + BD^2$$

So one part is done. For the other part we notice that locus of the points like M for which we have $MA^2 - MB^2 = k$ (where A, B, k are fixed) is a plane perpendicular to AB. Now assume that $AB^2 + CD^2 = AC^2 + BD^2$ then we have $AB^2 - AC^2 = DB^2 - DC^2$ and hence A, D lie on a plane perpendicular to BC. So A, D, H_a, H_d are on that plane and thus AH_a, DH_d are concurrent. Similarly BH_b intersects with AH_a, DH_d. If these three lines are not concurrent then they are pairwise concurrent and hence they all lie in one plane. But then A, B, D, H_a, H_b, H_d and so H_c and finally C lie on a plane but then $ABCD$ will be degenerate. So AH_a, BH_b, DH_d and similarly CH_c are concurrent and the problem is solved.