A GENERALIZATION OF THE PROBLEM 2

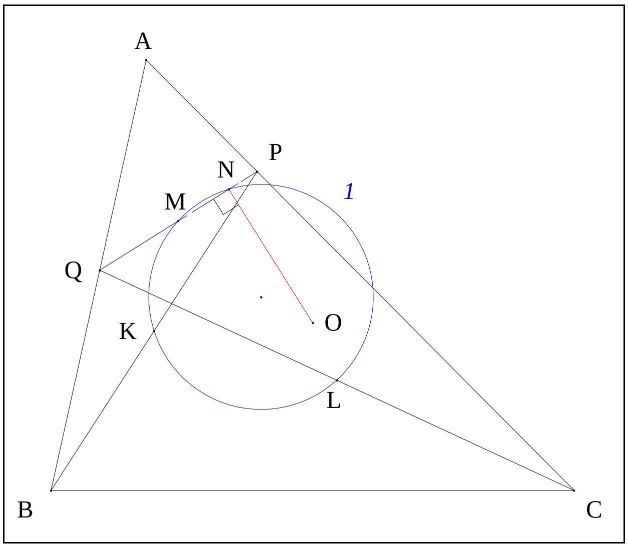
50^{st} I.M.O

BREMEN GERMANY 2009 [1]

AN UNEXPECTED PROOF WITH THE MIDCIRCLE

†

Jean - Louis AYME



Abstract. The author presents an aesthetic proof of a generalization of the I.M.O. 2009 problem 2, based on the midcircle.
The figures are all in general position and all the theorems quoted can be proved synthetically.

Summary

[1] St.-Denis, Île de la Réunion (France), le 20/10/2010.

DAY 1 JULY 15, 2009

PROBLEM 2

50-th International Mathematical Olympiad

Bremen, Germany, July 10–22, 2009

First Day – July 15

2. Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the midpoints of the segments BP, CQ, and PQ, respectively, and let Γ be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

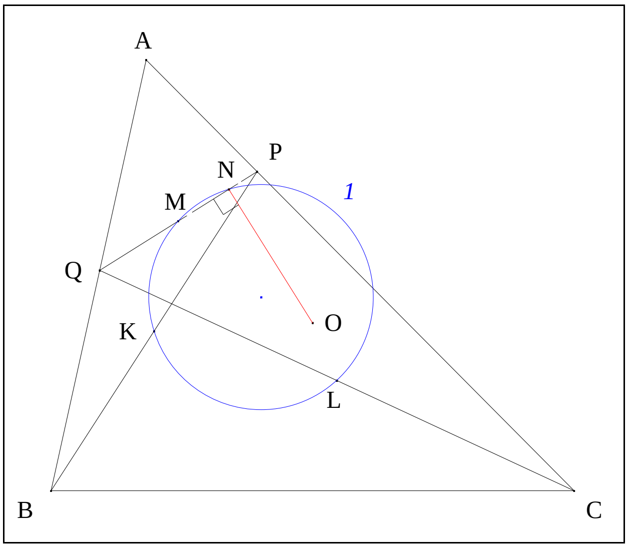
Historical note : this IMO problem [2] has been proposed by Sergei Berlov (Russia).
The IMO 2009 which take place in Bremen (Federal Republic of Germany) from the 10 of July until the 22 of that month, assemble 104 states and 565 competitors of whom 59 girls.
The subjects have been proposed in 55 languages.

[2] IMO 2009, Problem 2 generalized, *Art of Problem Solving* du 15/07/2009 ; http://www.artofproblemsolving.com/Forum/viewtopic.php?t=288839.
Pour john_john et pappus… et les autres !, *Les Matématiques.net* ; http://www.les-mathematiques.net/phorum/read.php?8,536647.

A GENERALIZATION OF THE PROBLEM 2

VISION

Figure :



Features :	ABC	a triangle,
	O	the center of the circumcircle of ABC,
	P, Q	two interior points of the sides CA, AB resp.,
	K, L, M	the midpoints of the segments BP, CQ, PQ resp.,
	1	the circle passing through K, L, M,
and	N	the second intersection of *1* and PQ.

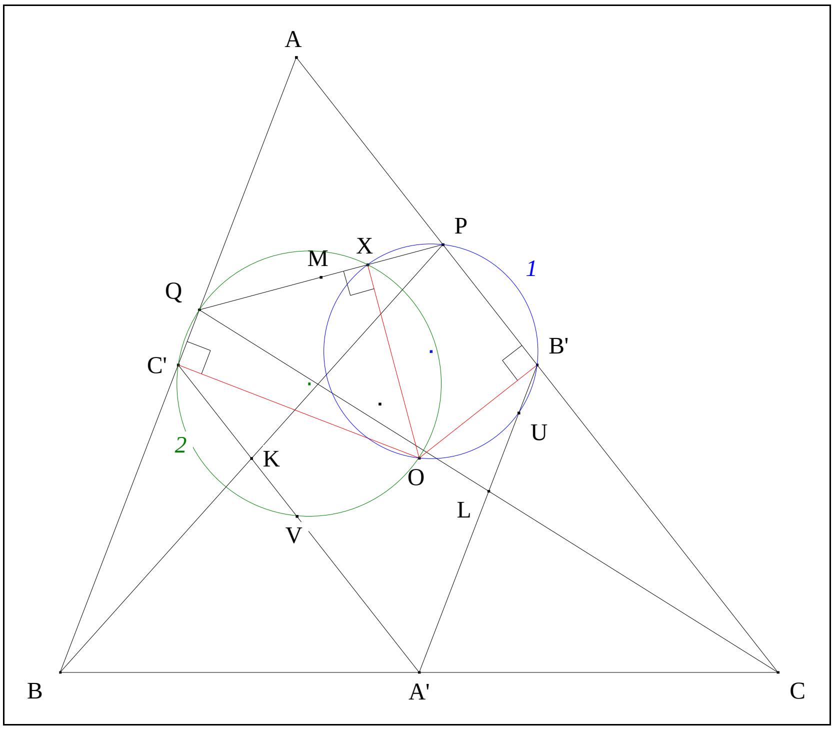
Given : ON is perpendicular to PQ. [3]

Historical note : this problem 2 has been generalized by Ta hong son (Hanoi, Vietnam) who is known under the pseudonym "April" on the site *Art of Problem Solving* and proposed [4] again in 2010 on the same site.

A SYNTHETIC PROOF OF THE AUTHOR

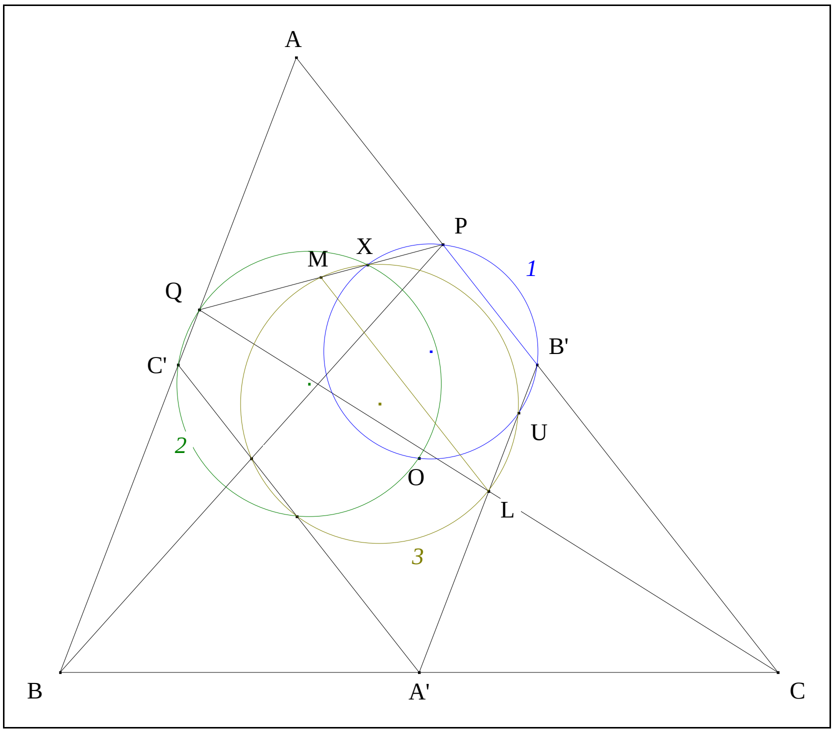
[3] IMO 2009, Problem 2, # 13, *Art of Problem Solving* du 15/07/2009 ;
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=288839.
Pour john_john et pappus… et les autres !, *Les Matématiques.net* ;
http://www.les-mathematiques.net/phorum/read.php?8,536647.

[4] Hard Geometry, *Art of Problem Solving* du 16/10/2010 ;
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=372184.



- Note A', B', C', X the feet of the perpendiculars through O on (BC), (CA), (AB), (PQ),
  1 the circle with diameter OP ; it goes through B' and X ;
  and *2* the circle with diameter OQ ; it goes through C' and X.

- **Remarks :** **(1)** A', B' and L are collinear
  (2) A', C' and K are collinear.

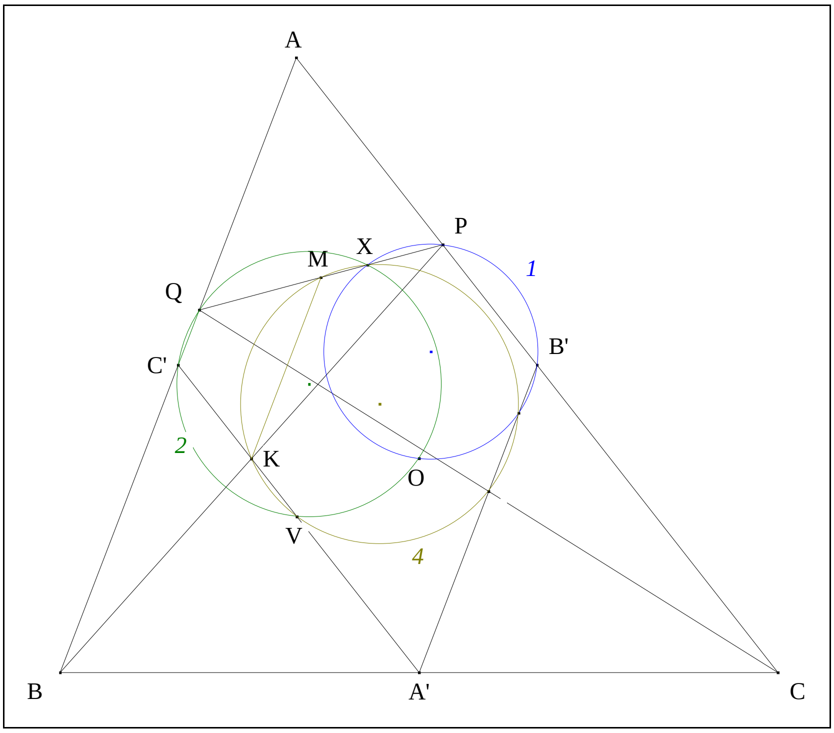


- Note U the second point of intersection A'B' with *1*.

- **Remark :** PB' // ML.

- The circle *1*, the basic points X and U, the borning monians PXM and B'UL, the parallels PB' and ML, lead to the Reim's theorem **0"** ;
consequently, X, U, M and L are concyclic.

- Note *3* this circle.

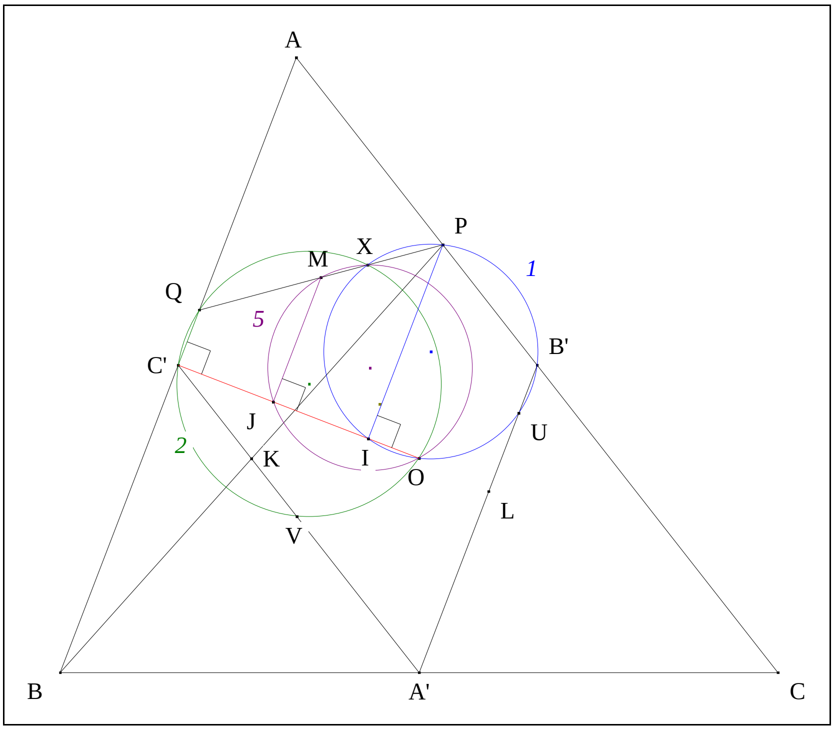


- Note V the second point of intersection A'C' with *2*.

- **Remark :** QC' // MK.

- The circle *2*, the basic points X and V, the borning monians QXM and C'VK, the parallels QC' and MK, lead to the Reim's theorem **0"** ;
consequently, X, V, M and K are concyclic.

- Note *4* this circle.



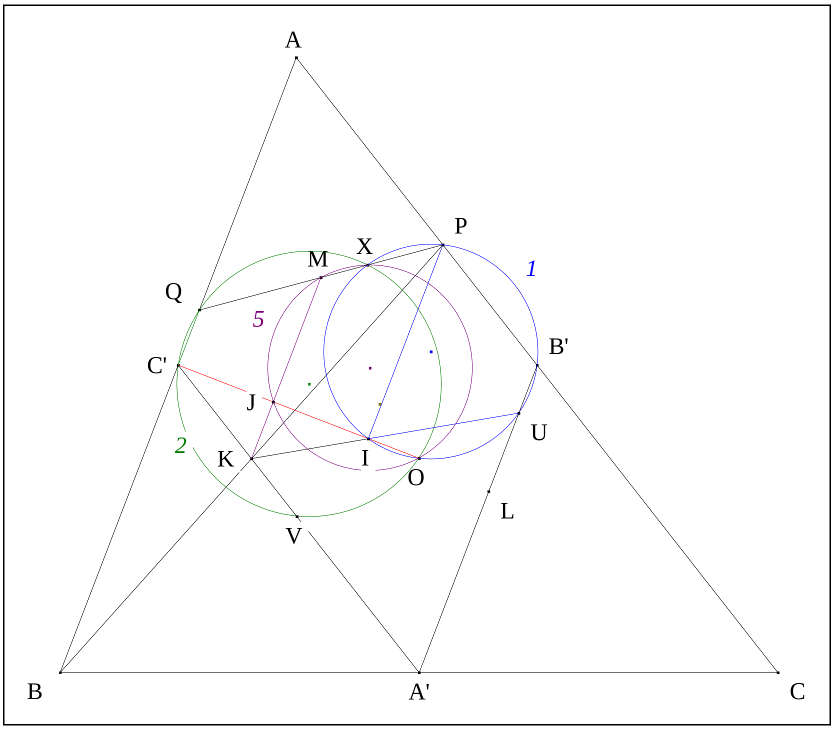
* Note 5 the midcircle of *1* and *2* (Cf. Annexe 1) ;
  and I, J the second points of intersection of OC' with *1*, *5* resp..

* The circle *1* and *5*, the basic points O and X, the monians IOJ and PXM, lead to the Reim's theorem **0** ;
  consequently, IP // JM.

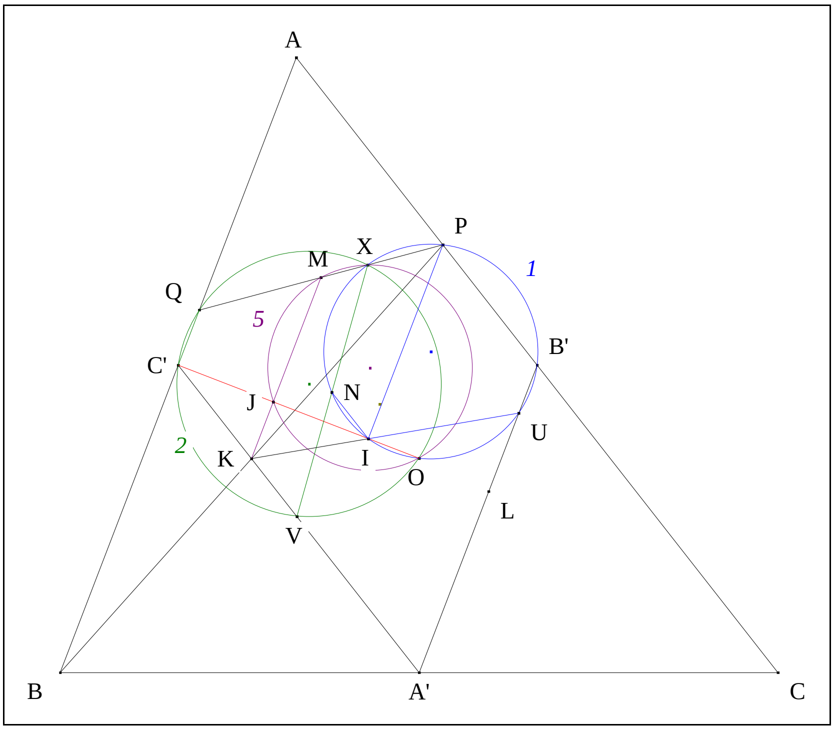
* The circle *5* and *2*, the basic points O and X, the monians JOC' and MXQ, lead to the Reim's theorem **0** ;
  consequently, JM // C'Q.

* According to "The midpoint theorem" applied to ABC, C'Q // B'U.

* **Partial conclusion :** IP, JM, C'Q and B'U are together parallel and perpendicular to OC'.



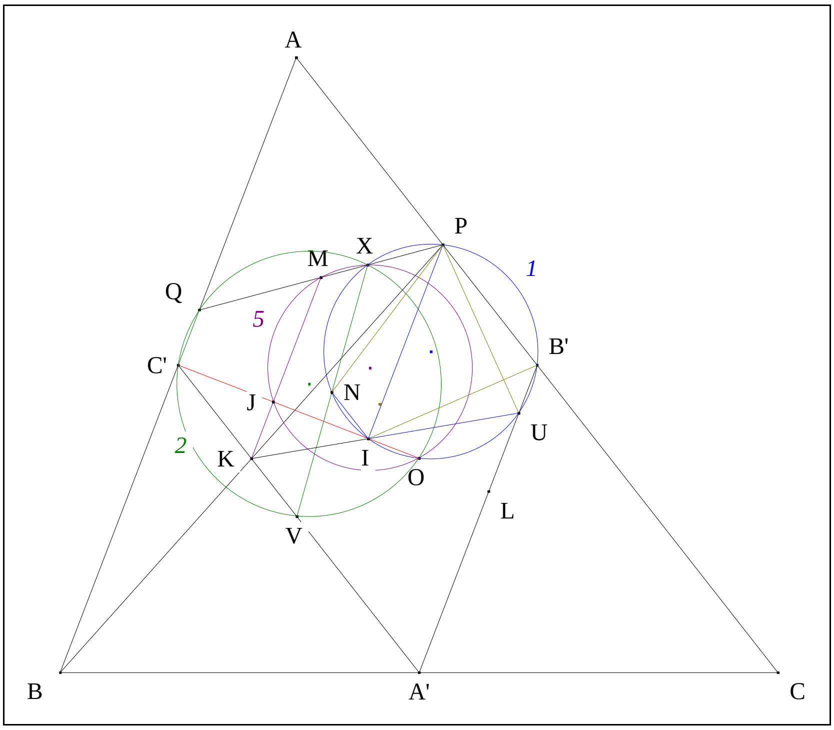
- **Remark :** IP, JM and C'Q are together parallel and perpendicular to OC'.
- According to "The midpoint theorem" applied to the triangle PQB, M, J and K are collinear.
- According to "Midcircle theorem" (Cf. Annexe 1), J is the midpoint of the segment IC'.
- According to "The perpendicular bisector theorem", the triangle KIC' is K-isoceles.
- **Partial conclusion :** KJ is the K-interior bisector of KIC'.
- **Commentar :** we have to prove that K, I and U are collinear.



- Note N the second point of intersection of XV with *1*.

- The circle *1* and *2*, the basic points O and X, the monians IOC' and NXV, lead to the Reim's theorem **0** ; consequently, IN // C'V.

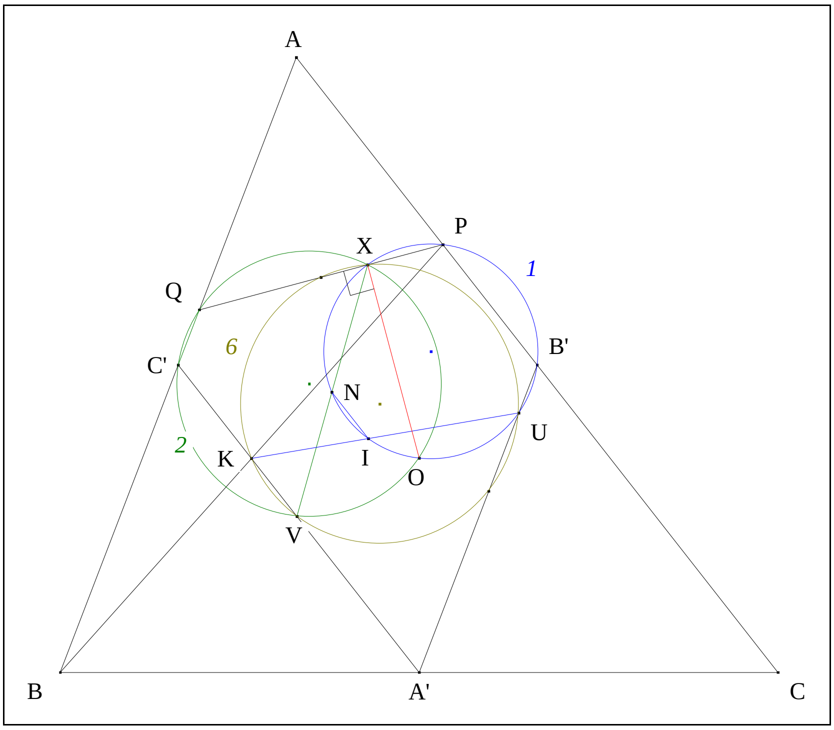
- According to "The midpoint theorem" applied to ABC, C'V // B'P ;
  by transitivity of the relation //, IN // B'P.



- The trapeze PIUB' being cyclic, PU = IB' ;
  the trapeze PNIB' being cyclic, IB' = NP ;
  by transitivity of the relation //, PU = PN ;

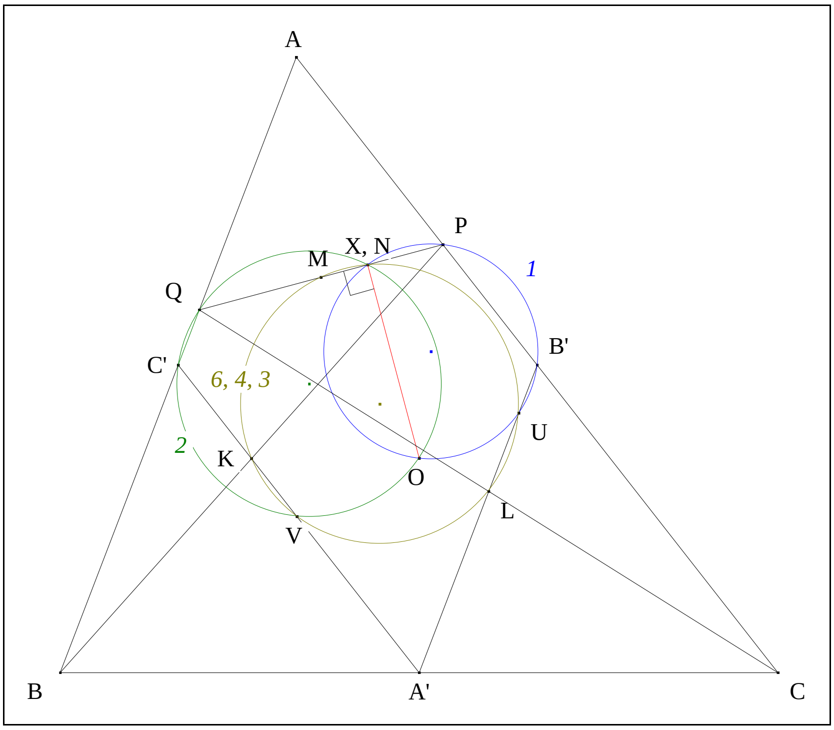
consequently, IP is the I-interior bisector of the triangle IUN.

- **Partial conclusion :** K, I and U are collinear.



- The circle *1*, the basic points X and U, the borning monians NXV and IUK, the parallels NI and VK, lead to the Reim's theorem **0"** ;
  consequently, X, U, V and K are concyclic.

- Note *6* this circle.



- The circle *6* and *4* having the three points X, V and K in common are identical ;

consequently, X, U, V, K and M are concyclic.

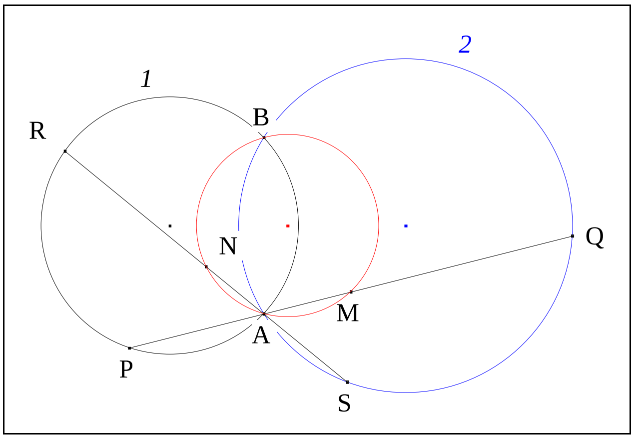
- The circle *4* and *3* having the three points X, U and M in common are identical ;
  consequently, X and N are identical.

- **Conclusion :** ON is perpendicular to PQ.

Remark : when M is identical to N, PQ is tangent to *6* and
we conclude that OM is the perpendicular bisector of the segment PQ.

ANNEXE

1. The midcircle



Features : *1*, *2* two intersecting circles,
A, B the points of intersection of *1* and *2*,
P, R two points on *1*,
Q, S the second points of intersection of AP, AR with *2* resp.,
et M, N the midpoints of the segment PQ, RS resp..

Given : M, N, A et B are concyclic.