The Euler point of a cyclic quadrilateral / Darij Grinberg

This note is an answer to Floor van Lamoen's message #6140 of Hyacinthos. I prove some properties of a cyclic quadrilateral.

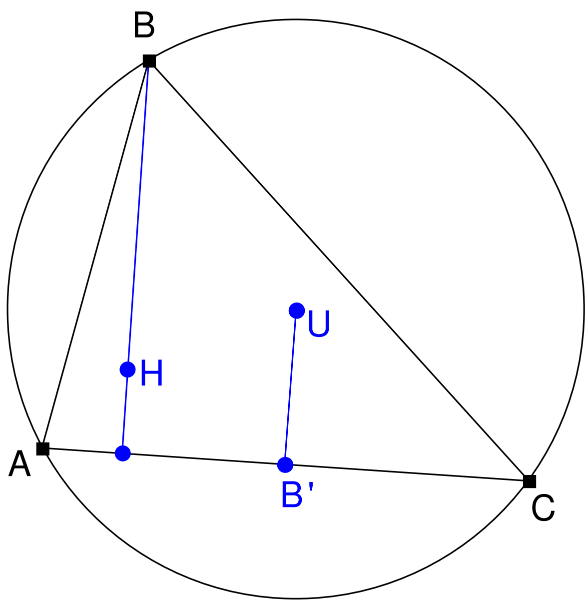

Fig. 1

At first, we repeat an important theorem (Fig. 1). Let ABC be a triangle with the orthocenter H and the circumcenter U. Denote by B' the midpoint of CA. Then $BH = 2 \bullet UB'$.

A proof can be found in [1]; the perhaps simplest proof is the following one: The segment BH joins the vertex B of triangle ABC with its orthocenter H; the segment $B'U$ joins the vertex B' of the medial triangle $A'B'C'$ with its orthocenter U [since the circumcenter of a triangle is the orthocenter of the medial triangle]. The two segments are corresponding segments in the two triangles. Since the medial triangle is similar to triangle ABC with factor 1/2, we thus have $B'U = 1/2 \bullet BH$, qed.

A corollary is:

Theorem 1: If two triangles AB_1C and AB_2C are inscribed in the same circle and have a common side CA, then the segment H_1H_2 joining their orthocenters is equal, parallel and equally directed to the segment B_1B_2 (Fig. 2).

Hereby, H_1 denotes the orthocenter of ΔAB_1C, and H_2 the orthocenter of ΔAB_2C.

Proof (Fig. 3): For both triangles, the length UB' is common; thus, both segments B_1H_1 and B_2H_2 are equal to $2 \bullet UB'$. These two segments B_1H_1 and B_2H_2 are further parallel (as they are both orthogonal to CA). Thus, $B_1H_1H_2B_2$ is a parallelogram, and consequently, $H_1H_2 \parallel B_1B_2$ and $H_1H_2 = B_1B_2$. Furthermore, the segments B_1B_2 and H_1H_2 are equally directed (this will be important!).

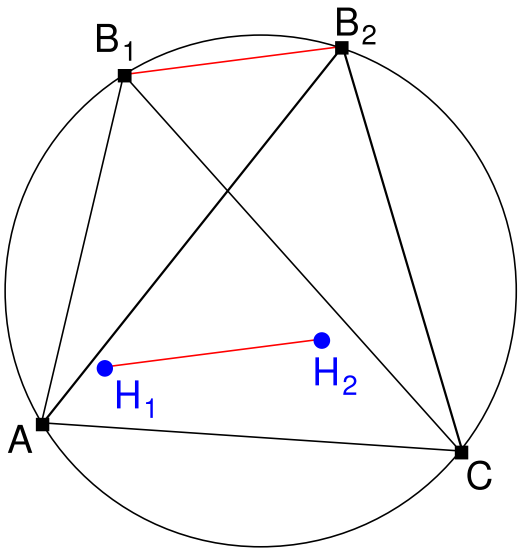

Fig. 2

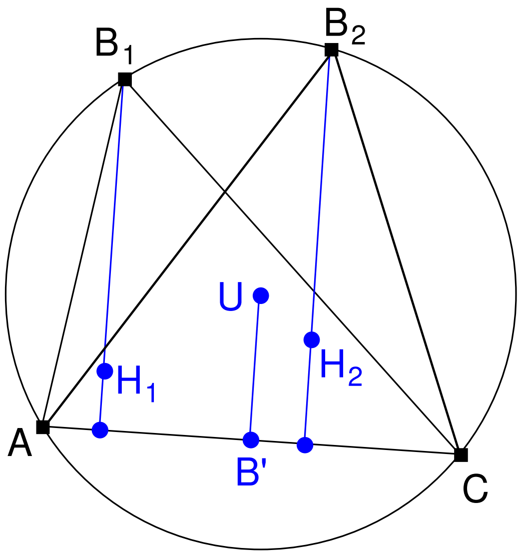

Fig. 3

Now consider a cyclic quadrangle *ABCD* and the orthocenters H_a, H_b, H_c, H_d of triangles *BCD*, *CDA*, *DAB*, *ABC* respectively (Fig. 4). After Theorem 1, we have now: The segment H_aH_b joins the orthocenters of triangles *BCD* and *ACD*; thus, it is equal, parallel and equally directed with the segment *BA*. This means that the segment H_aH_b is equal, parallel and *oppositely directed* with the segment *AB* (mind the difference!). Analogously, the segment H_bH_c is equal, parallel and oppositely directed with the segment *BC*, the segment H_aH_c is equal, parallel and oppositely oriented with the segment *AC*, etc.. Therefore, the quadrangle $H_aH_bH_cH_d$ is *congruent and homothetic* with the quadrangle *ABCD*, the factor of homothety being -1 (not $+1$ due to the opposite orientation). Consequently, the homothety mapping quadrangle *ABCD* to quadrangle $H_aH_bH_cH_d$ is a central symmetry. We summarize:

Theorem 2: The quadrangles *ABCD* and $H_aH_bH_cH_d$ are congruent and centrally symmetric. After the properties of central symmetry, we have:

The center *E* of the symmetry mapping *ABCD* to $H_aH_bH_cH_d$ lies on the four segments AH_a, BH_b, CH_c, DH_d and bisects each of them (Fig. 5).

I call *E* the **Euler point** of the cyclic quadrangle *ABCD*.

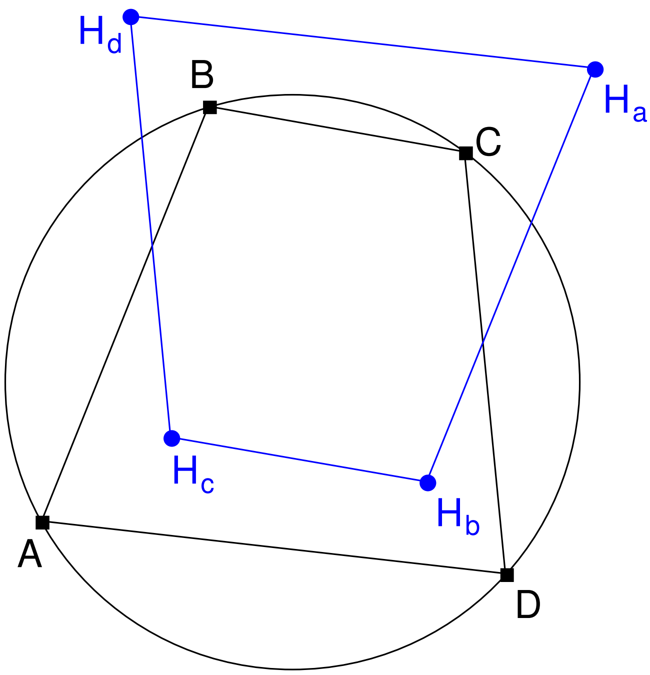

Fig. 4

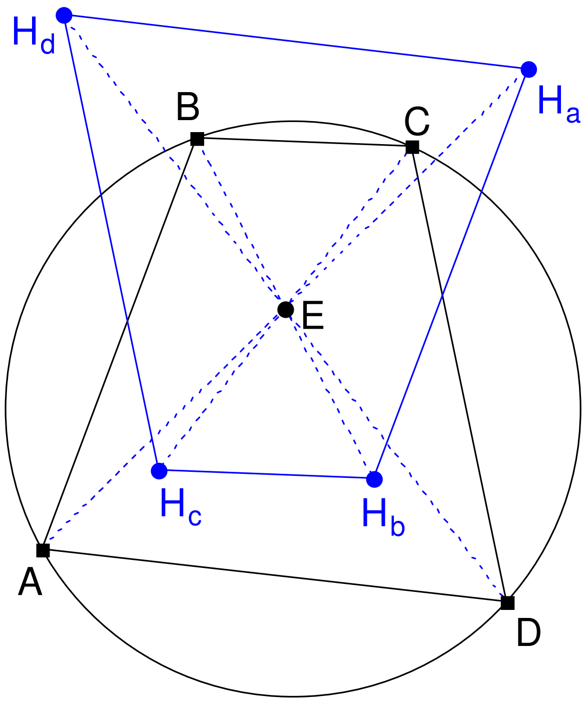

Fig. 5

But this Euler point has even more properties. Again, we need a lemma from triangle geometry (Fig. 6):

The nine-point circle of a triangle ABC is the image of the circumcircle in the homothety centered at the orthocenter H and having factor 1/2. In other words: If P is a point on the circumcircle of ΔABC, then the midpoint of the segment PH lies on the nine-point circle of ΔABC.

Proof: We remember that the midpoints X, Y, Z of the segments AH, BH, CH respectively lie on the nine-point circle. The homothety centered at H and having factor 1/2 maps A, B, C to X, Y, Z; consequently, it maps the circumcircle of ΔABC to the circumcircle of ΔXYZ (i. e. to the nine-point circle of ΔABC).

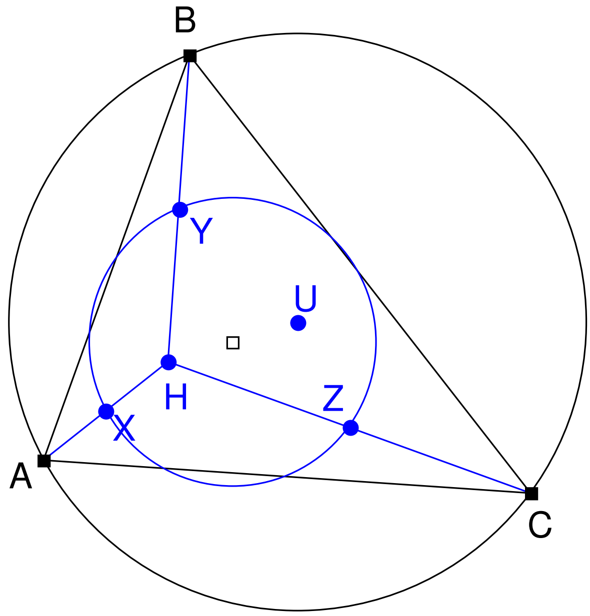

Fig. 6

If P is a point on the circumcircle of ΔABC, then the midpoint of the segment PH lies on the nine-point circle of ΔABC. After another well-known theorem ([1], [2] and [3]), this point also lies on the Simson line of the point P with respect to ΔABC.

Apply this to our cyclic quadrangle $ABCD$. The point E is the midpoint of segment AH_a joining the orthocenter of triangle BCD with the point A on its circumcircle. Thus, E lies on the Simson line of the point A with respect to ΔBCD and on the nine-point circle of ΔBCD. Similarly, we prove that E lies on the Simson line of the point B with respect to ΔCDA and on the nine-point circle of ΔCDA; that E lies on the Simson line of the point C with respect to ΔDAB and on the nine-point circle of ΔDAB; that E lies on the Simson line of the point D with respect to ΔABC and on the nine-point circle of ΔABC. We sum up:

Theorem 3: In the configuration of Theorem 2, the Euler point E lies on all the four Simson lines which result if three of the four points A, B, C, D are treated as the vertices of a triangle and the fourth one as a point on its circumcircle (the red lines on Fig. 7).

The Euler point also lies on the nine-point circles of triangles BCD, CDA, DAB, ABC (the red circles on Fig. 8).

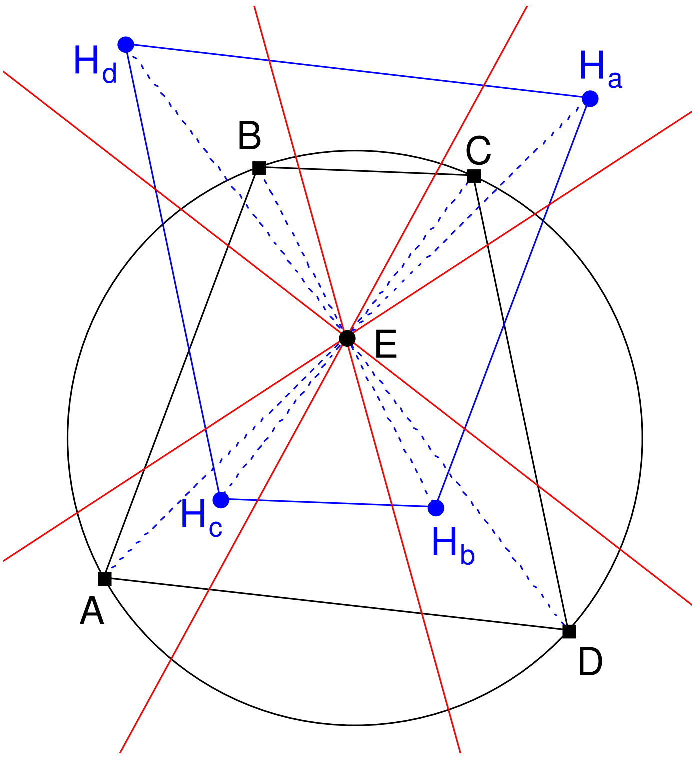

Fig. 7

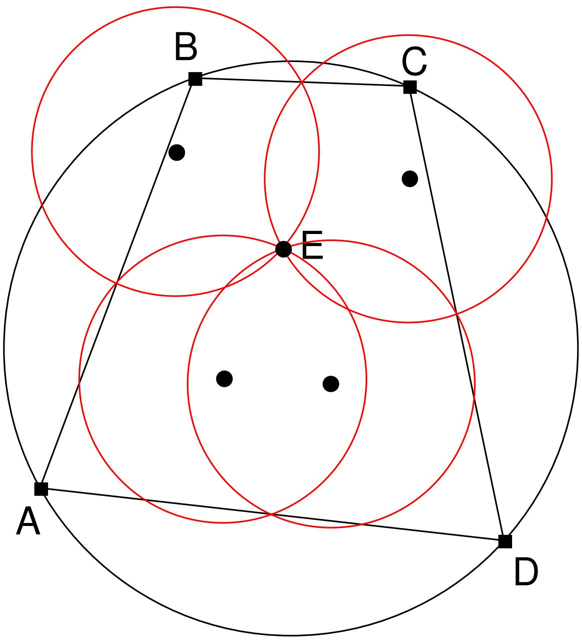

Fig. 8

We continue with another property. Let U be the circumcenter and r the circumradius of the cyclic quadrangle $ABCD$. Since the radius of the nine-point circle of a triangle is half the circumradius, the nine-point circles of the triangles BCD, CDA, DAB, ABC have the same radius $r/2$.

The nine-point center of a triangle is (after well-known properties) the midpoint between the orthocenter and circumcenter. This yields that the nine-point center is the image of the orthocenter in the homothety centered at the circumcenter with factor 1/2.

Thus, the nine-point centers F_a, F_b, F_c, F_d of triangles BCD, CDA, DAB, ABC are the images of the orthocenters H_a, H_b, H_c, H_d in the homothety centered in the common circumcenter U (the circumcenter of the cyclic quadrangle $ABCD$) with factor 1/2. This yields that the quadrangle $F_aF_bF_cF_d$ is similar to the quadrangle $H_aH_bH_cH_d$ with factor 1/2. As $H_aH_bH_cH_d$ is congruent to $ABCD$, the quadrangle $F_aF_bF_cF_d$ is similar to $ABCD$ with factor 1/2. As $ABCD$ is a cyclic quadrangle with circumradius r, also $F_aF_bF_cF_d$ is cyclic with circumradius $r/2$ (Fig. 9). We summarize:

Theorem 4: The nine-point circles of the triangles BCD, CDA, DAB, ABC all have the equal radius $r/2$. The quadrilateral formed by their centers is similar to $ABCD$ and also cyclic; the circumradius of this quadrilateral is $r/2$.

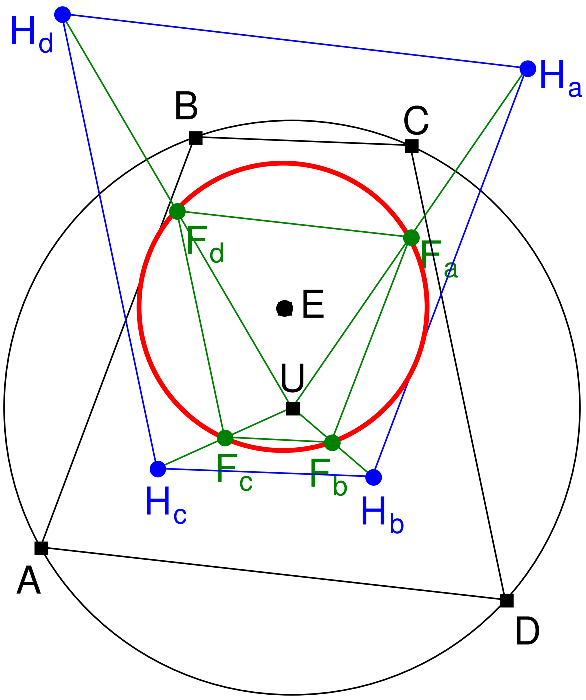

Fig. 9

But return to the Euler point E. It is the center of symmetry of the quadrangles $ABCD$ and $H_aH_bH_cH_d$. Thus, the central symmetry in E maps every point of the quadrangle $ABCD$ to the corresponding point of the quadrangle $H_aH_bH_cH_d$. Thus, the Euler point E of quadrangle $ABCD$ is mapped to the Euler point E' of quadrangle $H_aH_bH_cH_d$; but E is the center of the symmetry, i. e. a fixed point; thus, $E = E'$, and we get:

Theorem 5: The Euler point E of quadrangle $ABCD$ is also the Euler point E' of $H_aH_bH_cH_d$. After Theorem 3, we get: The Euler point E lies on all four Simson lines which result by treating three of the four points H_a, H_b, H_c, H_d as the vertices of a triangle and the fourth one as a point on its circumcircle (the quadrilateral $H_aH_bH_cH_d$ is cyclic, since it is the symmetrical image of a cyclic quadrialateral).

The Euler point also lies on the nine-point circles of triangles $H_bH_cH_d$, $H_cH_dH_a$, $H_dH_aH_b$, $H_aH_bH_c$.

We conclude the note with a surprise. Let M_{XY} be the midpoint of an arbitrary segment XY. We conjecture on Fig. 10 that the segment $M_{AB}E$ is orthogonal to CD. This is proven by the simple argument that the segment $M_{AB}E = M_{AB}M_{BH_b}$ (since after Theorem 2, we have $E = M_{BH_b}$) is a midparallel of ΔABH_b, therefore parallel to segment AH_b; but AH_b is an altitude in ΔCDA, thus orthogonal to CD. This yields that the segment $M_{AB}E$ is orthogonal to CD; we rewrite this in the following way: The perpendicular from M_{AB} to CD passes through E. Similarly, the perpendiculars from M_{CD} to AB, from M_{BC} to DA, from M_{AC} to BC etc. pass through E. We get the following theorem:

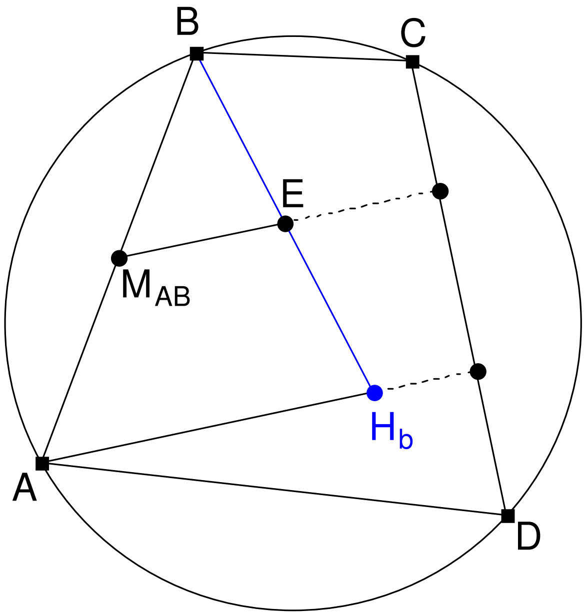

Fig. 10

Theorem 6: The four perpendiculars dropped from the midpoint of a side of the cyclic quadrangle $ABCD$ to the opposite side, pass through E. These are the perpendiculars from M_{AB} to CD, from M_{BC} to DA, from M_{CD} to AB, from M_{DA} to BC.

The two perpendiculars dropped from the midpoint of a diagonal of the cyclic quadrangle $ABCD$ to the other diagonal, pass through E. These are the perpendiculars from M_{AC} to BD and from M_{BD} to AC.

These six perpendiculars are the red lines on Fig. 11.

From Theorem 6, we immediately get:

Theorem 7: The Euler point E of a cyclic quadrilateral $ABCD$ coincides with the anticenter defined in [1]!

After [1], the point E is the symmetric image of the circumcenter U in the centroid S of the cyclic quadrangle $ABCD$.

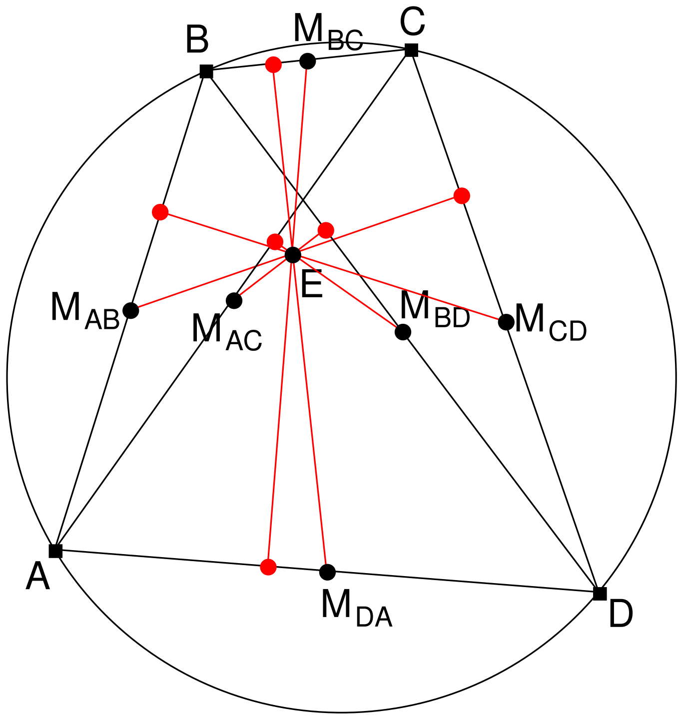

Fig. 11

Literatur

[1] R. Honsberger: *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, USA 1995.

[2] J. Kratz: *Geometrische Streifzüge im Umfeld der Simson-Geraden*, Didaktik der Mathematik 1/1990 pages 1-14.

[3] J. Brejcha: *Die Wallaceschen Geraden und die Feuerbachschen Kreise in einem Sehnenviereck*, Elemente der Mathematik ???.