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Brocard Points, Circulant Matrices, and Descartes' Folium

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In the flourishing days of triangle geometry, many special points were discovered and investigated. Apart from well-known points like the centroid (or median point), the orthocenter, and the circumcenter, 'new' triangle points were studied, points called the symmedian point (or point of Lemoine) and the points of Gergonne, Nagel, Torelli, and Brocard, to name but a few. There is little doubt in my mind that the Brocard points rank amongst the most interesting of these special points associated with the triangle. Although general interest has long since waned and results once regarded as important have sunk into oblivion, it might still be worth our while to revive some of the gems of triangle geometry. In the brilliant light of modern knowledge we might even discover new and interesting insights.

In the literature on Euclidean geometry some books can be singled out that deal exclusively with the geometry of the triangle and the circle. An excellent monograph is [4], and for those with a smattering of German [3] gives much information, too; [5] is of a more general nature, but this work also contains many pages devoted to the triangle and its associated points. Finally, the Brocard configuration is the singular topic of Emmerich's treatise [2], recommendable for its proverbial 'Gründlichkeit.'

In the rich field of Brocardian geometry, our attention shall be focused on the set of triangles equibrocardal to a given triangle (T). In order to explain the terminology, our first concern should be with the reader who wishes to be introduced to the Brocard points and the Brocard angle of a plane triangle.

Well then, given a triangle (T) with vertices A_1, A_2, and A_3, notation: $(T) = A_1A_2A_3$, the first (or positive) Brocard point of (T) is the unique point Ω such that the angles $\angle\Omega A_1A_2$, $\angle\Omega A_2A_3$, and $\angle\Omega A_3A_1$ are equal. The second (or negative)

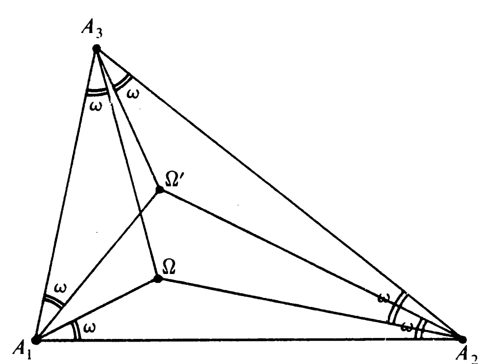

FIGURE 1.
The Brocard points Ω and Ω' of triangle $(T) = A_1A_2A_3$.

Brocard point Ω' of $(T) = A_1A_2A_3$ is the first Brocard point of triangle $(T)' = A_1A_3A_2$, which is obtained from (T) by changing its orientation (see FIGURE 1). As it happens, the angles $\omega := \angle\Omega A_1A_2 = \angle\Omega A_2A_3 = \angle\Omega A_3A_1$ and $\omega' := \angle\Omega' A_1A_3 = \angle\Omega' A_3A_2 = \angle\Omega' A_2A_1$ coincide; the common value is known as the Brocard angle of (T). These and other useful facts shall be discussed in the next section.

The main questions we shall be concerned with in this note are:

(i) How can one describe in a systematic way all plane triangles with Brocard angle equal to that of a given triangle (T)? Such triangles are known as equibrocardal.

(ii) If one restricts the positions of equibrocardal triangles in some natural way so as to avoid duplications by translation, rotation, similarity, etc., how can one describe the locus of the Brocard points of these triangles?

Answers to these questions shall be given in subsequent sections, but first we have to introduce some simple facts about the Brocard configuration.

Some Basic Facts

First suppose that the interior of triangle $(T) = A_1A_2A_3$ contains a point Ω such that $\angle\Omega A_1A_2 = \angle\Omega A_2A_3 = \angle\Omega A_3A_1$. Then the line joining A_2 and A_3 is tangent to the circle (c_2) through A_1, A_2, and Ω. This can be seen by observing that $\angle\Omega A_1A_2$ inscribed in the arc $A_2\Omega$ of (c_2) equals $\angle\Omega A_2A_3$ (see FIGURE 2). This means that Ω is a point common to three circles, each tangent to one side of (T) at different vertices and passing through a second vertex of (T). Conversely, it is not really difficult to see that the three circles (c_1), (c_2), and (c_3), where (c_i) is tangent to A_iA_{i+1} at A_i and passing through A_{i+2}, are concurrent. Here the indices i of A_i are taken modulo 3, which means that A_i and A_j are identical whenever $i \equiv j \pmod 3$. The point of intersection Ω of these circles is necessarily interior to (T). Clearly, Ω is completely determined by this construction.

Having established the existence and uniqueness of the Brocard points Ω and Ω', we turn our attention to an interesting analytical identity, which may serve as a defining expression for the Brocard angle ω. In order to convince ourselves of its validity, we need a little trigonometry.

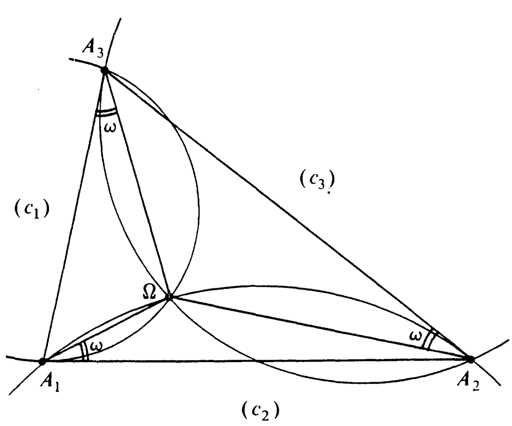

FIGURE 2.
Concurrent circles $(c_1), (c_2)$, and (c_3) meeting at Ω.

Let α_1, α_2, and α_3 denote the angles of (T) at the vertices A_1, A_2, and A_3 with opposite sides a_1, a_2, and a_3, respectively. Applying the rule of sines successively in the triangles $A_1A_3\Omega$, $A_2A_3\Omega$ and $A_1A_2A_3$ (see FIGURE 2) yields

$$A_3\Omega/\sin(\alpha_1-\omega)=a_2/\sin\alpha_1,$$
$$A_3\Omega/\sin\omega=a_1/\sin\alpha_3,$$

and

$$a_1/\sin\alpha_1=a_2/\sin\alpha_2,$$

respectively. Eliminating a_1, a_2 and $A_3\Omega$ from these expressions gives

$$\sin(\alpha_1-\omega)\sin\alpha_2\sin\alpha_3=\sin^2\alpha_1\sin\omega,$$

and dividing through by $\sin\alpha_1\sin\omega$, using $\alpha_1+\alpha_2+\alpha_3=\pi$, results in

$$(\cot\omega-\cot\alpha_1)\sin\alpha_2\sin\alpha_3=\sin(\alpha_2+\alpha_3).$$

The reader is invited to deduce the elegant identity

$$\cot\omega=\cot\alpha_1+\cot\alpha_2+\cot\alpha_3. \tag{1}$$

On squaring (1) and observing that, because $\alpha_1+\alpha_2+\alpha_3=\pi$,

$$\cot\alpha_1\cot\alpha_2+\cot\alpha_2\cot\alpha_3+\cot\alpha_3\cot\alpha_1=1,$$

we obtain the equivalent relation

$$\cot^2\omega=\cot^2\alpha_1+\cot^2\alpha_2+\cot^2\alpha_3+1,$$

which is easily rewritten as

$$1/\sin^2\omega=1/\sin^2\alpha_1+1/\sin^2\alpha_2+1/\sin^2\alpha_3. \tag{2}$$

From the obvious inequalities

$$0<\omega<\min(\alpha_1,\alpha_2,\alpha_3)<\pi/2,$$

it follows immediately that (1)—and hence also (2) by equivalence—uniquely determine ω. Also, by symmetry, we find that $\cot\omega=\cot\omega'$ which implies $\omega=\omega'$, an assertion made before but unproven so far.

Many pleasing relations between the Brocard angle ω and other triangle quantities can be established in a similar way. We refer to [**2**] and [**6**] for details. Particularly useful is the following relation between ω, the sides a_i, and the area Δ of (T):

$$4\Delta\cot\omega=a_1^2+a_2^2+a_3^2. \tag{3}$$

To prove this, we recall that

$$2\Delta=a_2a_3\sin\alpha_1.$$

By the rule of cosines in triangle $A_1A_2A_3$ we also have

$$a_1^2=a_2^2+a_3^2-2a_2a_3\cos\alpha_1.$$

Combining these expressions yields

$$4\Delta\cot\alpha_1=-a_1^2+a_2^2+a_3^2.$$

Because of symmetry, similar formulae exist for $\cot \alpha_2$ and $\cot \alpha_3$. Substitution into (1) of the expressions for $\cot \alpha_i$ thus obtained immediately gives (3).

In a later section we shall construct an analytic formula for the exact position of Ω in relation to the positions of the vertices A_1, A_2, and A_3 of the given triangle (T). The formulae (1), (2), and (3) play a significant part in that construction.

An obvious inequality for the Brocard angle ω—we mentioned it before—is

$$\omega < \min(\alpha_1, \alpha_2, \alpha_3).$$

One may ask instead for an absolute upper bound for ω, i.e., an upper bound independent of (T), and preferably one that is least, so that for each value below this bound a triangle exists with Brocard angle agreeing with that value. Again we shall use some trigonometry. From (1) we deduce

$$\begin{aligned}\sin(\alpha_1 + \omega)/\sin \omega &= \sin \alpha_1 \cot \omega + \cos \alpha_1 \\ &= \sin \alpha_1(\cot \alpha_1 + \cot \alpha_2 + \cot \alpha_3) + \cos \alpha_1 \\ &= \sin(\alpha_1 + \alpha_2)/\sin \alpha_2 + \sin(\alpha_1 + \alpha_3)/\sin \alpha_3 \\ &= \sin \alpha_3/\sin \alpha_2 + \sin \alpha_2/\sin \alpha_3,\end{aligned}$$

so that, by the rule of sines in (T),

$$\sin(\alpha_1 + \omega)/\sin \omega = a_3/a_2 + a_2/a_3.$$

This shows that

$$\sin(\alpha_1 + \omega)/\sin \omega \geqslant 2,$$

with equality if and only if $a_2 = a_3$. Consequently

$$2 \sin \omega \leqslant \sin(\alpha_1 + \omega) \leqslant 1,$$

and hence

$$0 < \omega \leqslant \pi/6, \tag{4}$$

with equality if and only if triangle (T) is equilateral.

When asking for a triangle (T), by construction or otherwise, with a prescribed Brocard angle $\omega \leqslant \pi/6$, one is actually asking for the possible values of the angles α_1, α_2 and α_3 of (T). In other words, one wants to find $\alpha_1, \alpha_2, \alpha_3$ with $\alpha_i > 0$, $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ and such that $\cot \alpha_1 + \cot \alpha_2 + \cot \alpha_3$ has a prescribed value $\geqslant \sqrt{3} = \cot(\pi/6)$.

Picking up our interrupted argument again, define the function

$$h(\delta) = \sin(\delta + \omega) - 2 \sin \omega \tag{5}$$

in the variable δ, where ω has a fixed value in the range $0 < \omega \leqslant \pi/6$. Restricting δ to values between 0 and π, it is obvious that the function $h(\delta)$ has precisely two zeros δ_1 and δ_2. Clearly, $\delta_1 + \delta_2 = \pi - 2\omega$. Moreover, observing that

$$\cot \delta + \cot \omega = \sin(\delta + \omega)/(\sin \delta \sin \omega) = 2 \sin \omega/(\sin \delta \sin \omega) = 2/\sin \delta,$$

we see that the equation $h(\delta) = 0$ may be rewritten as

$$\cot^2(\delta/2) - 2 \cot \omega \cot(\delta/2) + 3 = 0.$$

Hence

$$\cot(\delta_1/2) = \cot\omega + \sqrt{\cot^2\omega - 3}$$

and (6)

$$\cot(\delta_2/2) = \cot\omega - \sqrt{\cot^2\omega - 3}\,.$$

Note that the condition (4) is necessary and sufficient for the existence of δ_1 and δ_2. Since

$$h(\alpha) = \sin(\alpha + \omega) - 2\sin\omega \geqslant 0$$

for any one of the angles α of a triangle (T) with Brocard angle ω, it follows immediately that

$$\delta_1 \leqslant \alpha \leqslant \delta_2. \tag{7}$$

Apparently, δ_1 and δ_2 give the minimal and maximal values respectively that any angle of a triangle with prescribed Brocard angle ω can possibly attain. Conversely, given $\omega \in (0, \pi/6]$, choose $\alpha_1 = \alpha$ satisfying (7). Then the expression

$$\sin(\alpha + \omega)/(2\sin\omega)$$

uniquely determines the ratio of the sides a_2 and a_3, provided we prescribe the sign of $a_2 - a_3$. The resulting triangles are all similar and are equibrocardal with Brocard angle ω.

FIGURE 3 below gives some values of ω and corresponding values of δ_1 and δ_2.

ω	5.00	10.00	15.00	20.00	25.00	27.50	29.00	29.50	29.75	30.00
δ_1	5.04	10.32	16.17	23.16	32.70	39.94	46.84	50.51	53.20	60.00
δ_2	164.96	149.68	133.83	116.84	97.30	85.06	75.16	70.49	67.30	60.00

FIGURE 3.
Minimal value δ_1 and maximal value δ_2 for the angles of triangles with prescribed Brocard angle ω measured in degrees.

The Neuberg Circles

Having set the scene, and preparations being complete, we can now embark on the investigation of the set of triangles equibrocardal to a given triangle (T).

To begin with, let us agree to the following restrictions, which can be made without losing generality. Usually, we shall only consider triangles with the same orientation as the given triangle $(T) = A_1A_2A_3$, i.e., the vertices are numbered counterclockwise. Further, it is clearly sufficient to choose only one representative from each class of directly similar triangles. Two triangles are called directly similar if the one is homothetic to the image of the other after a suitable translation and/or rotation. So, directly similar means similar, but orientation preserving.

The restrictions imposed so far are clarified by the correspondence between triangles and points $(\alpha_1, \alpha_2, \alpha_3)$ in 3-space, situated in the plane $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. In other words, each triangle, up to similarity, is given by an ordered 3-tuple of angles. In this way a restricted set of equibrocardal triangles with prescribed Brocard angle may be visualized as a closed curve in the plane $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ in 3-space. However the points $(\alpha_1, \alpha_2, \alpha_3)$, $(\alpha_3, \alpha_1, \alpha_2)$, and $(\alpha_2, \alpha_3, \alpha_1)$ correspond to directly similar triangles. So only one third of the ω-curve, namely the part contained in the shaded region (see FIGURE 4), corresponds to all triangles with the same Brocard angle as (T), but not directly similar to (T). In FIGURE 4 the position of triangle (T) of FIGURE 1 is shown as a point on the ω-curve. We also should make a positional choice, that is to say, to some extent we are free to prescribe the positions of the triangles in the Euclidean plane. This may be done in various ways. Two natural possibilities present themselves, namely, the position of one side could be fixed for all triangles, or they could be required to have a common point, like the centroid. Recall that the centroid of a triangle is the point at which the medians meet. From the last lines of the previous section it is clear that for any given $\omega \in (0, \pi/6]$, each triple (α_1, A_1, A_2), where α_1 is chosen in size between δ_1 and δ_2, and $A_1 \neq A_2$, uniquely determines the third vertex A_3 of triangle $(T) = A_1A_2A_3$ with Brocard angle ω and prescribed orientation. Thus, if we restrict the positions of the equibrocardal triangles by fixing their base A_1A_2—or any other side—it should be possible to describe the locus of the third vertex A_3. In FIGURE 5 triangle (T) has Brocard angle ω, and N lies on the perpendicular bisector of A_1A_2 at a distance $l = \frac{1}{2}a_3 \cot \omega$ from the middle M of the base A_1A_2, where a_3 is the length of A_1A_2. Further, we denote by m the length of the median from A_3 and by ϕ the angle $\angle A_3MN$. Finally, r designates the length of the third side of triangle A_3MN. We intend to prove that r only depends on ω and a_3, to be precise

$$2r = a_3\sqrt{\cot^2\omega - 3}\,. \tag{8}$$

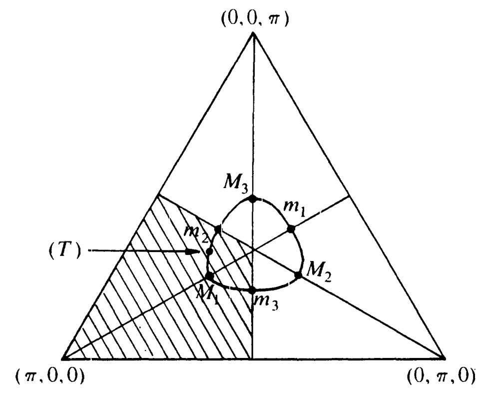

FIGURE 4.
ω-curve ($\omega \approx 26.9$ deg).

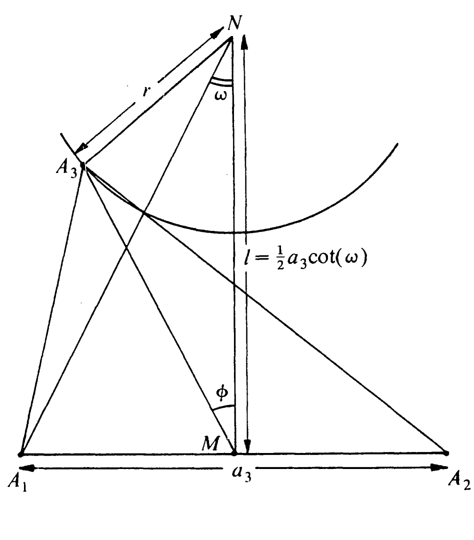

FIGURE 5.
Locus of A_3 for fixed ω and base A_1A_2 of length a_3.

We leave it to the reader to verify that

$$4m^2 = 2a_1^2 + 2a_2^2 - a_3^2 \quad \text{and} \quad 2\Delta = ma_3 \cos\phi.$$

Here, as usual, Δ denotes the area of triangle $A_1A_2A_3$. By the law of cosines in triangle A_3MN we obtain

$$\begin{aligned} r^2 &= l^2 + m^2 - 2lm\cos\phi = l^2 + m^2 - 2\Delta\cot\omega \\ &= \frac{1}{4}a_3^2\cot^2\omega + \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2 - \frac{1}{4}a_3^2 - \frac{1}{2}\left(a_1^2 + a_2^2 + a_3^2\right), \end{aligned}$$

according to (1). Consequently, $4r^2 = a_3^2(\cot^2\omega - 3)$ as required.

Since N is fixed—indeed, l depends on a_3 and ω only—the geometric interpretation of this result is that A_3 describes a circle with centre N and radius r, given by (8). This circle and its two associates, each obtained by fixing one side of the given triangle, are called the Neuberg circles, after their discoverer.

All triangles having their base of length b, their Brocard angle ω as well as their orientation in common with a given triangle (T), have their third vertex on a so-called Neuberg circle of diameter $2r = b\sqrt{\cot^2\omega - 3}$.

In FIGURE 6 below, triangle $A_1A_2A_3$ is pictured again with the Neuberg circle with centre N. Suppose A_3' is the second point of intersection of A_1A_3 with the Neuberg circle. Then triangles $A_1A_2A_3$ and $A_1A_2A_3'$ are (indirectly) similar. This is because one angle (in this case α_1) and the Brocard angle ω together completely determine the shape of the triangle. As a consequence the lengths of the tangents from A_1 and A_2 to the Neuberg circle are both equal to the length a_3 of the base, or $A_1R_2 = A_1R_1 = A_1A_2 = a_3$. Moreover, $\angle A_2A_1R_1 = \delta_1$ and $\angle A_2A_1R_2 = \delta_2$, the smallest and largest value, respectively, any angle of a triangle with Brocard angle ω can possibly attain. Comparing FIGURES 4 and 6, we observe that running through the Neuberg circle clockwise corresponds to running through the ω-curve of FIGURE 4 counter-clockwise. For instance, the points R_1 and m_1 correspond to the same triangles and so do R_2 and M_1. Finally, what is the locus of the Brocard points Ω and Ω' when A_3 runs

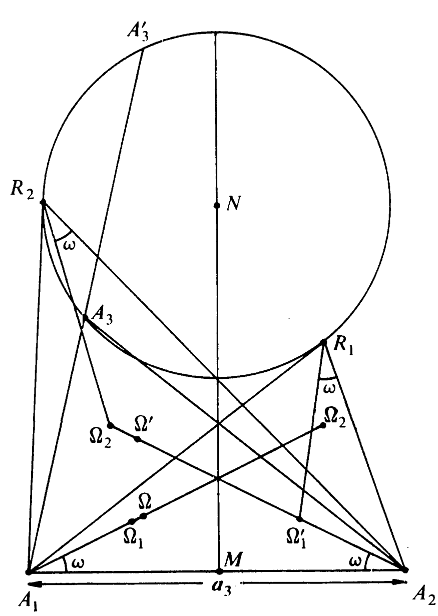

FIGURE 6.
Locus of Ω and Ω' for fixed A_1A_2, Brocard angle ω and orientation.

through the Neuberg circle? Apparently, Ω' runs through the line segment $\Omega_1'\Omega_2'$ twice. Here Ω_i' is the second Brocard point of triangle (T_i), where $(T_1) = A_1A_2R_1$ and $(T_2) = A_1A_2R_2$. A similar line segment $\Omega_1\Omega_2$ is obtained as the locus of Ω by reflection of $\Omega_1'\Omega_2'$ in the line through M and N. It may be verified that both line segments have length

$$\frac{4}{3}a_3\sqrt{1-4\sin^2\omega}.$$

The locus of the Brocard point Ω of a triangle with fixed base of length a_3 and Brocard angle ω, as its third vertex A_3 runs through a Neuberg circle, is a line segment of length $\frac{4}{3}a_3\sqrt{1-4\sin^2\omega}$.

A Circulant Matrix

In the previous section we chose a given line segment as the positional fixture for equibrocardal triangles associated with (T), namely, the base of (T). Next we shall fix only one point common to all triangles to be considered. It turns out that the centroid is a good choice. Hence from now on all triangles shall have their centroids coinciding with that of the given triangle (T).

Up to this point, we have used methods and arguments of a geometric and trigonometric nature only. But we shall see that complex numbers and a little linear algebra also prove to be particularly useful.

Let triangle (T) be situated in the complex plane, so that its vertices are given by complex numbers z_1, z_2, and z_3. For obvious reasons we shall use capital Z's instead of capital A's to indicate these vertices. Also we shall change (T) into (Z). Since the centroid of (Z) has an important role to play, we choose it to coincide with the origin O. This means that

$$z_1 + z_2 + z_3 = 0.$$

It would be nice if we could find transformations transforming (Z) into equibrocardal triangles, leaving its centroid fixed and such that triangles of all different shapes with the same Brocard angle and orientation as (Z) appear as images under these transformations. Obvious examples of transformations with these properties are those given by the even permutations of the vertices and by rotations about O over a fixed angle, and also homothetic transformations with centre O share these properties. All of these may be regarded as linear or matrix transformations A, transforming complex vectors $z = (z_1, z_2, z_3)$ in unitary space $\mathbb{C}^3$ into complex vectors $w = (w_1, w_2, w_3) \in \mathbb{C}^3$ by means of the vector relation

$$Az = w.$$

For our purpose it suffices to choose only real matrices A. The complex vector z, usually written as a column instead of a row of complex numbers, is associated with the triangle (Z) and w is associated with the triangle (W). For instance, the transformations permuting the vertices of (Z) without changing its orientation are given by the three permutation matrices

$$P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that $P_2 = P_1^2$ and $P_0 = P_1^3$. In general, the orientation of (Z) remains unchanged by the transformation A provided $\det(A) > 0$. Further, because of symmetry, it is reasonable to require that

$$A = P_i^{-1} A P_i,$$

for $i = 0, 1, 2$. This means that two triangles (Z) and (W), corresponding by the linear relation $Az = w$, remain so after the same permutation is applied to their vertices. Thus restricted, the matrix A becomes what is known as a circulant matrix, i.e., a matrix of type (see [1])

$$A = \begin{pmatrix} s & t & r \\ r & s & t \\ t & r & s \end{pmatrix}. \tag{9}$$

Recall that the reason for considering matrices or matrix transformations was to find representatives for all differently shaped triangles with the same Brocard angle as the basic triangle $(Z) = Z_1 Z_2 Z_3$. Thus, if we define

$$F(z) = \left(|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_1|^2\right)/\Delta(z), \tag{10}$$

where, not surprisingly $\Delta(z)$ denotes the area of (Z), we would like to find out which conditions have to be imposed on A to guarantee that

$$F(Az) = F(z)$$

for all z. Indeed, it follows from (3) that $F(z) = 4\cot\omega$, provided z is the vector associated with the vertices of (Z). First of all, the transformation $w = Az$ causes the area of (Z) to be multiplied by a factor $\det(A)$. Hence $\Delta(w) = \det(A)\Delta(z)$. Note that for real r, s, and t, because (9) holds,

$$\det(A) = r^3 + s^3 + t^3 - 3rst,$$

and, in particular, $\det(A) > 0$. Secondly, on putting $u_i = z_i - z_{i+1}$, where the indices i are taken modulo 3, we get

$$\begin{aligned}\Delta(w)F(w) &= |su_1 + tu_2 + ru_3|^2 + |ru_1 + su_2 + tu_3|^2 + |tu_1 + ru_2 + su_3|^2 \\ &= \left(r^2 + s^2 + t^2 - rs - st - tr\right)\left(|u_1|^2 + |u_2|^2 + |u_3|^2\right),\end{aligned}$$

because of the relation $u_1 + u_2 + u_3 = 0$. We leave the somewhat tedious calculations to the reader. So

$$\begin{aligned}F(z)/F(w) &= \det(A)/(r^2 + s^2 + t^2 - rs - st - tr) \\ &= (r^3 + s^3 + t^3 - 3rst)/(r^2 + s^2 + t^2 - rs - st - tr) \\ &= r + s + t.\end{aligned}$$

We conclude that, if A is given by (9) and if $\det(A) > 0$, then $w = Az$ and z are associated with equibrocardal triangles if and only if $r + s + t = 1$.

Further, as rotations about O and homothetic transformations with centre O do not affect the Brocard angle, we may also, without loss of generality, prescribe a particular position for one of the vertices of the image triangle (W). Let us choose W_1 on the line through Z_1 and Z_2. This choice, being equivalent with $r = 0$, forces W_i to be incident with the line through Z_i and Z_{i+1} for each i. Moreover, these points W_i divide the sides of (Z) into equal ratios (see FIGURE 7).

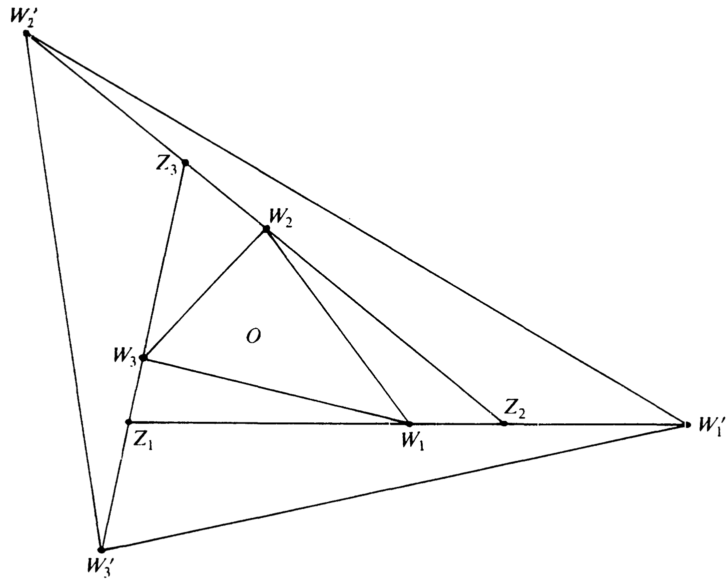

FIGURE 7.
Equibrocardal triangles in the complex plane: $w = Az$, $w' = A'z$; $t = 3/4$ for A, $t = 3/2$ for A'; see (11).

The final form of the circulant A with the required properties now is

$$A = \begin{pmatrix} 1-t & t & 0 \\ 0 & 1-t & t \\ t & 0 & 1-t \end{pmatrix} \quad \text{with } t \in \mathbb{R}. \tag{11}$$

Note that

$$\det(A) = t^3 + (1-t)^3 = t^2 - t(1-t) + (1-t)^2 = 3t^2 - 3t + 1$$

for all choices of $t \in \mathbb{R}$.

So far, we have found a large set of differently shaped equibrocardal triangles as images of a given triangle (Z) under special circulant matrix transformations. But does this set exhaust all possibilities up to orientation and similarity? Surprisingly, the answer is yes, it does. To prove this, we consider a certain triangle transformation σ_λ in the complex plane, which resembles a sort of distorted reflection in the real axis (see FIGURE 8). For each triangle (Z) in the upper half plane H, let $\sigma_\lambda(z) = w$ be associated with the triangle in the lower half-plane, derived from (Z) by multiplying the imaginary parts of z_i by a constant factor $-\lambda$ $(0 < \lambda \leqslant 1)$. In other words, $\mathrm{Re}(w_i) = \mathrm{Re}(z_i)$ and $\mathrm{Im}(w_i) = -\lambda\,\mathrm{Im}(z_i)$ for $i = 1, 2, 3$. The same effect is obtained by the orthogonal projection in 3-space of a triangle in a given plane onto a second plane. What effect does this transformation σ_λ have on the Brocard angle? To find out, we reconsider the function $F(z)$ of (10). As before, $u_i = z_i - z_{i+1}$. A straightforward calculation shows that

$$2\Delta(w)F(w) = (1+\lambda^2)\Delta(z)F(z) + (1-\lambda^2)\,\mathrm{Re}\big(u_1^2 + u_2^2 + u_3^2\big).$$

Since $\Delta(w) = \lambda\Delta(z)$, we may also write

$$2\lambda F(w)/F(z) = 1 + \lambda^2 + (1-\lambda^2)E(z),$$

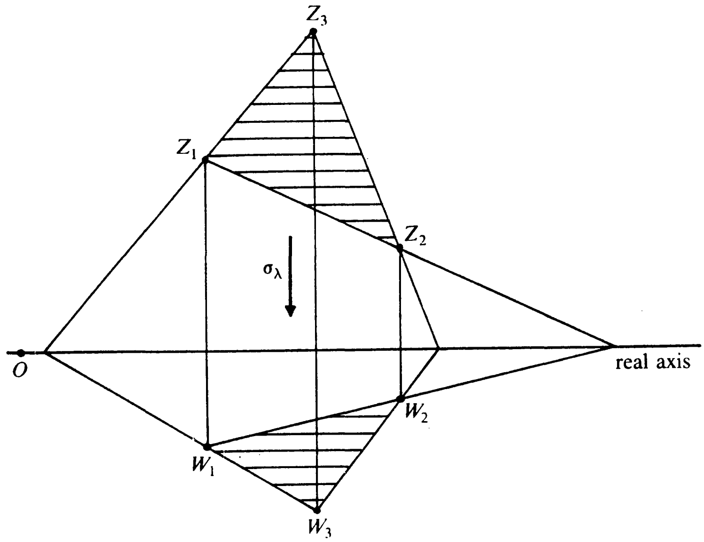

FIGURE 8.
Distorted reflection $\sigma_\lambda(\lambda = 1/2)$: $\mathrm{Re}(w_i) = \mathrm{Re}(z_i), \mathrm{Im}(w_i) = -\lambda \mathrm{Im}(z_i)$.

or

$$2\cot \omega_\lambda / \cot \omega = \lambda^{-1} + \lambda + (\lambda^{-1} - \lambda) E(z),$$

where ω_λ is the Brocard angle of (W) and the expression $E(z)$, defined by

$$E(z) = \mathrm{Re}\{(u_1^2 + u_2^2 + u_3^2)/(|u_1|^2 + |u_2|^2 + |u_3|^2)\},$$

depends on the shape of (Z) only. In case of an equilateral triangle (Z), the expression $E(z)$ vanishes. To prove this, assume that $|u_1| = |u_2| = |u_3|$ and define $v_i = u_i/|u_i|$ for $i = 1,2,3$. Then $E(z)$ satisfies

$$3E(z) = \mathrm{Re}(v_1^2 + v_2^2 + v_3^2).$$

Now $v_1 + v_2 + v_3 = 0$ as $u_1 + u_2 + u_3 = 0$. Further, v_1, v_2, and v_3 lie on the unit circle, which implies that v_2/v_1 and v_3/v_1 are cubic roots of unity with $v_2/v_1 + v_3/v_1 = -1$. Hence $v_2/v_1 = \rho$ and $v_3/v_1 = \rho^2$ so that

$$v_1^2 + v_2^2 + v_3^2 = v_1^2(1 + \rho + \rho^2) = 0$$

as required. Moreover,

$$2\cot \omega_\lambda = (\lambda^{-1} + \lambda)\sqrt{3},$$

as $\cot(\pi/6) = \sqrt{3}$.

This shows that the resulting Brocard angle ω_λ merely depends on the multiplication factor λ! As a consequence, any two equibrocardal triangles (W^1) and (W^2) in the lower half-plane may be seen as the images of two equilateral triangles (Z^1) and (Z^2), respectively, in H by the same transformation σ_λ:

$$\sigma_\lambda(z^i) = w^i, \qquad i = 1,2.$$

Without loss of generality we may assume that the centroids of (W^1) and (W^2)

coincide. Then also (Z^1) and (Z^2) have the same centroid C. Clearly, there is a homothetic transformation (such a transformation multiplies the distance between every two points by the same constant factor) with centre C transforming (Z^2) into $(Z^2)'$ such that the vertices of the latter triangle are incident with the sides of the former, one vertex of $(Z^2)'$ on each side of (Z^1). See FIGURE 9 below.

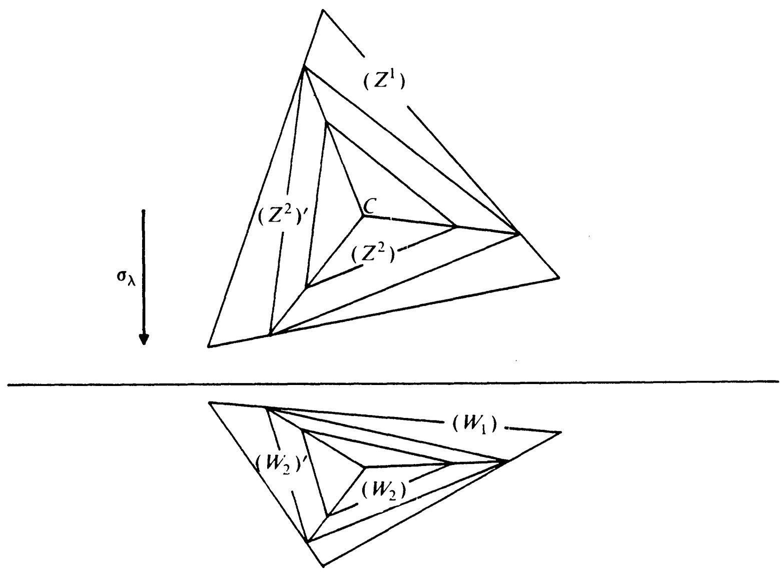

FIGURE 9.
Equilateral triangles are mapped onto equibrocardal triangles by the distorted reflection σ_λ.

Since both (Z^1) and $(Z^2)'$ are equilateral, the vertices of $(Z^2)'$ divide the sides of (Z^1) into equal ratios. The corresponding triangles (W^1) and $(W^2)'$ have the same property, because the transformation σ_λ preserves ratios. This proves that any triangle, agreeing in both orientation and Brocard angle with a given triangle (Z), is directly similar to the triangle onto which (Z) is mapped by a suitable circulant matrix transformation of type (11).

Triangles, the vertices of which divide the sides of a given triangle with Brocard angle ω cyclically in the same ratios, are equibrocardal. Except for similarity and orientation of vertices these triangles exhaust all possible triangles with Brocard angle ω.

Finally, we would like to know in what way the points of the ω-curve of FIGURE 4 correspond to the triangles (W), where $w = Az$, A is a circulant matrix of type (11) and (Z) is the given triangle with Brocard angle ω. Clearly, when t runs through $\mathbb{R}$, the point on the ω-curve corresponding to t, runs through this curve in a counter-clockwise fashion; the point indicated by (T) corresponds to $t = 0$. The part of the ω-curve between the points m_2 and m_3—note that it is contained in the shaded area of FIGURE 4—corresponds approximately to the t-interval $-0.23 \leqslant t \leqslant 0.46$.

Descartes' Folium

This final section is devoted to a description of the locus of the Brocard points as their associated triangles move through the Brocard configuration discussed in the previous section. Because of the beauty of the final result, going through the (sometimes tedious) derivations is certainly worth the trouble.

In FIGURE 10 we have gathered the necessary information obtained in the foregoing sections. For convenience we take the positive Brocard point Ω of (Z) to be the origin O of the complex plane. The complex number associated with the positive Brocard point of triangle (W) shall be denoted by ω_t, so that $\omega_0 = 0$.

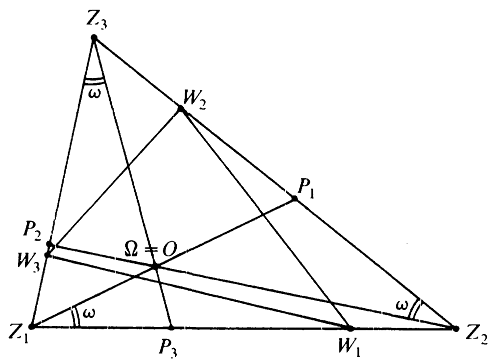

FIGURE 10.
$|z_i - w_i| = t|z_i - z_{i+1}|$.

It is obvious that every complex number α can be written in one way only as a 'convex' combination of z_1, z_2 and z_3, to be precise, $\alpha = c_1z_1 + c_2z_2 + c_3z_3$ for a unique triple of real numbers c_1, c_2, and c_3 with $c_1 + c_2 + c_3 = 1$. In particular

$$0 = \omega_0 = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 \tag{12}$$

with $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Since Z_3, Ω, and P_3 are collinear (see FIGURE 10), there is a real number c such that

$$\omega_0 = (1-c)p_3 + cz_3.$$

Naturally, p_3 is the complex number associated with the point P_3. Also

$$p_3 = c_1z_1 + c_2z_2$$

with $c_1 + c_2 = 1$, because Z_1, Z_2, and P_3 are collinear. Consequently,

$$\omega_0 = (1-c)c_1z_1 + (1-c)c_2z_2 + cz_3.$$

Comparing this with (12), we may deduce that $c = \lambda_3$, because of the uniqueness of this expression. The number c has an obvious interpretation, namely, as the ratio of the directed line segments $P_3\Omega$ and P_3Z_3.

To calculate c and hence λ_3, we observe that the triangles $Z_1\Omega P_3$ and $Z_3Z_1P_3$ are similar, which implies

$$A_1P_3/A_3P_3 = P_3\Omega/P_3A_1 \quad \text{or} \quad c = (A_1P_3/A_3P_3)^2.$$

The latter expression may be written, by the rule of sines in triangle $Z_1P_3Z_3$, as

$$(\sin\omega/\sin\alpha_1)^2 = (\sin\omega/\Delta(z))^2|(z_1 - z_2)(z_3 - z_1)|^2/4.$$

Hence (12) can be rewritten as

$$(2\Delta(z)/\sin\omega)^2\omega_0 = |u_1u_2|^2z_1 + |u_2u_3|^2z_2 + |u_3u_1|^2z_3, \tag{13}$$

because of symmetry. Recall that $u_i = z_i - z_{i+1}$ and that $\Delta(z)$ signifies the area of triangle (Z). The corresponding formula for (W) may be derived analogously. Before giving this formula explicitly, let us renew the habit of taking indices modulo 3. Let us also agree to the following abbreviated notation:

$$\Sigma_i e_i = e_1 + e_2 + e_3,$$

where the sum Σ_i extends over $i = 1,2,3$. Hence, the right-hand side of (13) in abbreviated form looks like

$$\Sigma_i |u_i u_{i+1}|^2 z_i.$$

The promised formula for (W) now may be written as

$$(2\Delta(w)/\sin\omega)^2 \omega_t = \Sigma_i |(w_i - w_{i+1})(w_{i+1} - w_{i+2})|^2 w_i. \tag{14}$$

As we want to make explicit the dependence of (14) on the parameter t, we substitute

$$w_i = (1-t)z_i + tz_{i+1}.$$

Clearly, $|w_i - w_{i+1}|^2$ is a quadratic polynomial in t. This allows us to define

$$p_i(t) = |w_i - w_{i+1}|^2 = a_i t^2 + b_i t(1-t) + c_i(1-t)^2. \tag{15}$$

A few simple properties of the coefficients of $p_i(t)$ are readily established. For instance

$$a_i = c_{i+1} = |u_{i+1}|^2 \quad \text{and} \quad a_i + b_i + c_i = a_{i+1}. \tag{16}$$

The former is the result of the substitutions $t = 1$ and $t = 0$, and the latter follows from the substitution $t = 1/2$ in conjunction with $\Sigma_i u_i = 0$. Also

$$(2\Delta(z)/\sin\omega)^2 = 4\Delta^2(z)\Sigma_i(1/\sin^2\alpha_{i+1}) = \Sigma_i a_i c_i,$$

because of (2) and the observation that for each i

$$4\Delta^2(z) = a_i c_i \sin^2\alpha_{i+1}.$$

All these notational simplifications are intended to give (13) and (14) a less complicated appearance. Thus (13) becomes

$$0 = \omega_0 \Sigma_i a_i c_i = \Sigma_i a_i c_i z_i \tag{17}$$

and (14) eventually looks like

$$\{\Delta(w)/\Delta(z)\}^2 \omega_t \Sigma_i a_i c_i = \Sigma p_i(t) p_{i+1}(t)(1-t)z_i + tz_{i+1} \tag{18}$$

or

$$\{\Delta(w)/\Delta(z)\}^2 \omega_t \Sigma_i a_i c_i = \Sigma_i \{(1-t)p_i(t)p_{i+1}(t) + tp_i(t)p_{i+2}(t)\} z_i.$$

Now the left-hand side of (18) is the product of ω_t and a quartic polynomial in t, because

$$\{\Delta(w)/\Delta(z)\}^2 = \{\det(A)\}^2 = (3t^2 - 3t + 1)^2. \tag{19}$$

Concentrating on the right-hand side of (18), which we shall denote by $P(t)$, we see that it is a polynomial of degree 5 in t with complex coefficients. Obviously, $P(0)=P(1)=0$ because of (17). Hence, as a polynomial in $\mathbb{C}[t]$, $P(t)$ is divisible by $t(1-t)$. It can also be shown that $P(t)$ is divisible by the polynomial $\det(A)=3t^2-3t+1$. In fact,

$$P(t)=t(1-t)(3t^2-3t+1)\{\alpha t+\beta(1-t)\}\Sigma_i a_i c_i, \tag{20}$$

where the complex numbers α and β are defined by

$$\alpha\Sigma_i a_i c_i=\Sigma_i\left(a_i^2+b_i c_i\right)z_i, \qquad \beta\Sigma_i a_i c_i=\Sigma_i\left(c_i^2+a_i b_i\right)z_i.$$

In establishing this result, frequent use is made of (17).

Inserting our findings (19) and (20) into (18) yields

$$\begin{aligned}\omega_t&=\left\{t^2(1-t)\alpha+t(1-t)^2\beta\right\}/(3t^2-3t+1)\\&=\left(\tau^2\alpha+\tau\beta\right)/(1+\tau^3),\end{aligned}$$

where $\tau=t/(1-t)$. We are nearly through, because on putting

$$X=\tau^2/(1+\tau^3), \qquad Y=\tau/(1+\tau^3),$$

so that

$$\omega_t=X\alpha+Y\beta,$$

and letting t run through all real values $\neq 1$, an unexpected curve emerges as the locus of Ω_t, namely the curve given by

$$X^3+Y^3=XY, \tag{21}$$

in the (X,Y)-coordinate system with basis $\{\alpha,\beta\}$. We recognize this curve, usually given by (21) relative to an orthogonal coordinate system, as the famous Folium of Descartes. Information on this cubic curve can be found in most classical texts on analytic geometry or on plane algebraic curves.

Note that

$$\alpha+\beta=2\omega_{1/2}=\Sigma_i z_i,$$

which shows that the centroid lies on the line segment $\Omega\Omega_{1/2}$ at two-thirds of its length as seen from $\Omega=0$. Also the points on the closed loop of the curve correspond to the t-values between 0 and 1. In FIGURE 11 below we see the original triangle (Z) and the corresponding locus of Ω_t.

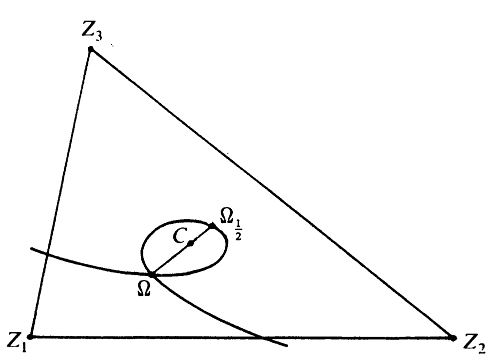

FIGURE 11.
The locus of the Brocard point Ω_t of (W) as in FIGURE 10.

If t runs through $\mathbb{R}$, *each of the Brocard points of the triangle with vertices given by* $w = Az$, *where A has the form* (11) *and z corresponds to the vertices of the given triangle, runs through a twisted version of the Folium of Descartes.*

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A Nonconstructible Isomorphism

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Very often we need to make a careful distinction between those mathematical objects which merely exist and those objects which are actually constructible. Metaphysically speaking, "God" is, of course, one such object. But it is rather difficult to find simple and concrete examples of such objects suitable for a relatively lower (say, at an advanced undergraduate) level. In this note, I would like to provide one such down-to-earth example of an isomorphism which can be explained to any class that has had a dose of elementary linear algebra and calculus. Apart from being a nondescriptive isomorphism, this example also demonstrates (i) the importance of dimension, (ii) the so-called cardinality arguments, (iii) the intricacy of primes, rationals, and irrationals, and finally (iv) the use of 'external' machinery (here vector spaces, to prove a result within group theory) to dig this truth from the "deep well" of mathematics (see J. Larmor [1]).

FACT 1: There is an isomorphism from the additive group $\mathbb{R} = \langle R; + \rangle$ of all reals to the multiplicative group $\mathbb{R}^* = \langle R^+; \cdot \rangle$ of all positive reals.

Proof. The exponential map $x \to e^x$ is an isomorphism because $e^{x+y} = e^x \cdot e^y$ and e^x is one-to-one, onto, and always positive.

FACT 2: There is no isomorphism from the additive group $\mathbb{Q} = \langle Q; + \rangle$ of all rationals to the multiplicative group $\mathbb{Q}^* = \langle Q^+; \cdot \rangle$ of all positive rationals.