Anti-Problems

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DMW-ANTIPROBS, OTIS*

§1 Lecture notes

This is a unit dedicated to problems whose solutions make you groan. It'll be great.

§1.1 About anti-problems

It's hard to define an anti-problem, but "you know it when you see it", I guess.

After EGMO 2019, I dissected this further into three categories.

- **Bait**, aka *wtf that actually works?*. (Example: "really, just take mod 2?") These problems are often misleading about what to try rather than having a completely unexpected solution path. If you miss such a problem, the solution path might have crossed your mind, but you didn't pursue it because it didn't feel promising, or sometimes just felt too stupid.
- **Contrived**, which often feature unattractive artificial statements. But the solutions to these are often short and sometimes even elegant. The defining characteristic of these problems is that once you see the solution, you know that the problem statement was essentially rigged to make that solution method work.

  Often, these have the property you either solve quickly or not at all. If you miss such a problem, then when you look at the solution, you feel you were very far off.
- **Derp**, aka *wait, that's it?*. Problems where even after you read the solution, you still don't feel like you know why the problem is true.

This is not a great trichotomy; in particular, they are pretty correlated and one can often argue a problem fits more than one label. However, I hope it's better than nothing.

§1.2 Heuristics for all anti-problems

- Always try the simplest things first if they won't take too much time. Even if they don't work, seeing why they fail is often useful.

- One of the most important and difficult skills to learn about olympiads is when you are making progress on a problem. As an extreme example, if you've been trying to use the Chebotarev density theorem on JMO2 for the last half hour without new progress, you're *probably* barking up the wrong tree.

  In many of this unit's problems, you might go down a solution path that turns out to be a dead end. It's really important to learn to start over in these cases. Otherwise, in addition to not solving #2, you might run out of time for #3.

- In particular: sometimes you just won't see it. (These problems are inherently noisy.) Cut your losses and bail.

- These problems are often actually hard anyways. When contest organizers review problems, they may think the solution "looks" easy or simple, and thus underestimate its difficulty. The windmill problem was a C3!

- Dangerous advice: problem number is more correlated with the *length* of the solution than actual difficulty. (Notable counterexample: both P1 and P4 on USAMO 2016. I think it's more true of IMO.)

§1.3 Advice for bait anti-problems

For problems in this unit, there will often be a general approach that looks "too easy to possibly work", but does actually work if you push it hard enough. However there will often also be an approach which does *not* work but is more tempting. Here are examples of what I'm talking about:

- You only need to look at size, and it works.
- You only need to look mod 2, mod 3, etc, and it just works.
- You have a yes/no question and you can just construct it.

Here is a common theme: **throw away things**!

If you've never done it before, it is now time to work on the following famous problem (which admittedly is more of a Process problem than bait).

Example 1.1 (IMO 2011/2)

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.

Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Some hints: ignore the conclusion "infinitely many ..." and look for an invariant.

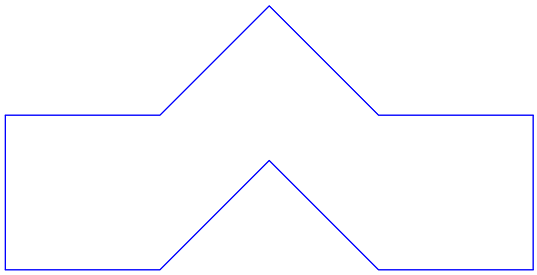
§1.4 Advice for contrived anti-problems

In computer science there's a principle called "keep it simple, stupid", usually abbreviated as **KISS**. This will be the broad theme of this lecture: problems that have solutions that are *short and simple*, but which are easy to make complicated.

Example 1.2 (Communicated by Ashwin Sah)

Let $ABCDEZYXWV$ be an equilateral decagon with interior angles $\angle A = \angle V = \angle E = \angle Z = \angle C = 90^\circ$, $\angle W = \angle Y = 135^\circ$, $\angle B = \angle D = 225^\circ$, and $\angle X = 270^\circ$. Determine whether or not one can dissect $ABCDEZYXWV$ into four congruent polygons.

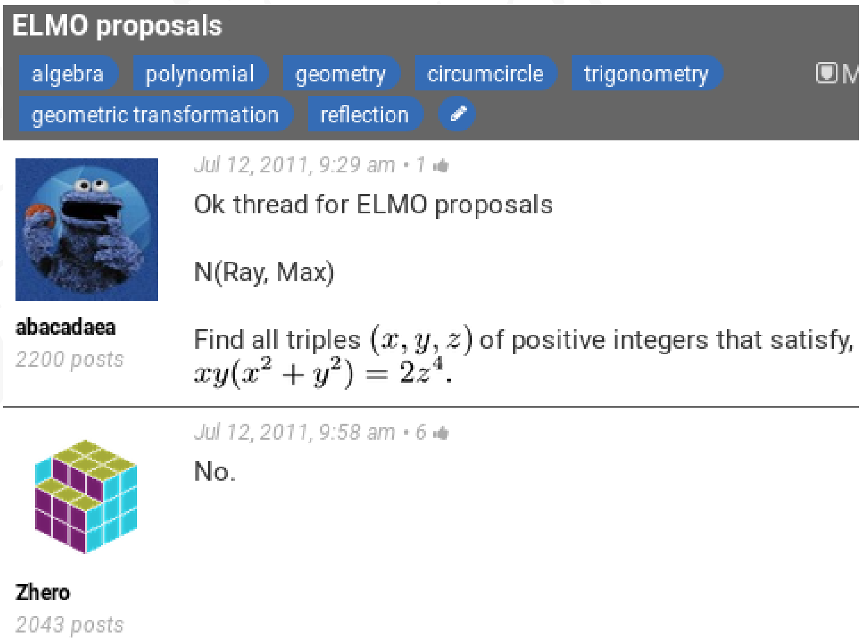
Walkthrough. Stare at the shape until you either give up or figure out what's going on.



Example 1.3 (Rejected from ELMO Shortlist)

Find all positive integers x, y, z satisfying $xy(x^2 + y^2) = 2z^4$.

Walkthrough. You can actually solve the problem eventually if you work hard enough with standard methods, but try to find the short one-line solution that makes this the kind of problem to give your enemies.



Example 1.4 (EGMO 2019/6)

On a circle, Alina draws 2019 chords, the endpoints of which are all different. A point is considered marked if it is either

(i) one of the 4038 endpoints of a chord; or

(ii) an intersection point of at least two chords.

Of the 4038 points meeting criterion (i), Alina labels 2019 points with a 0 and the other 2019 points with a 1. She labels each point meeting criterion (ii) with an arbitrary integer (not necessarily positive).

Along each chord, Alina considers the segments connecting two consecutive marked points. (A chord with k marked points has $k-1$ such segments.) She labels each such segment in yellow with the sum of the labels of its two endpoints and in blue with the absolute value of their difference. Alina finds that the $N+1$ yellow labels take each value $0, 1, \dots, N$ exactly once. Show that at least one blue label is a multiple of 3.

Walkthrough. This problem is famous in my notes as one of the most artificial statements that I have ever seen, and was responsible for the creation of this unit. (It has some "bait" attributes, too.)

Let G be the obvious graph. Assume for contradiction no blue labels are divisible b 3. Let e_{ij} denote the number of segments joining $i \pmod 3$ to $j \pmod 3$. We will work only modulo 3; the actual numbers are less relevant.

(a) How many odd-degree vertices are labeled 0?

(b) How many odd-degree vertices are labeled 1?

(c) Write down a system of equations which gives $e_{ij} \pmod 2$, by global ideas.

(d) Show that $e_{02} \equiv e_{12} \equiv 1 - e_{01} \pmod 2$.

(e) Get a contradiction with the choice of $0, \dots, N$.

§1.5 Advice for derp anti-problems

Evan is the kind of person that seeks order in the world, and is upset when he can't impose it.

So like, usually, when I work on an olympiad problem, I like to be able to analyze it when I'm done. I should be able to say something like,

- "this is a great global problem", or
- "I was able to solve this problem because I considered the $n = 2^a 3^b 5^c$ case and thought about its iterated φ, and then it became clear what was happening", or
- "I missed this problem because I was trying hard to construct an example satisfying $a_i a_{i+1} \le 2i+1$, but I should have just required $a_i a_{i+1} = 2i+1$ instead".

You know, something like that.

And then sometimes I run into problems which I can't solve, and then I look at the solution and all I can do is stare at the screen in annoyance, because I'm not actually sure what to say.

Here are several such problems. I think the unifying theme in them is that the solutions are usually *unsatisfying*: you were asked to prove that X is true, and then you read the proof and understand each line and still have no idea why X is true. It's the complete opposite of the "rigid" unit, except it's not "free", either, it's just... a proof.

Example 1.5 ("Warmup 2", from Sasha's class at MOP 2017)
Do there exist positive integers $a_1, \dots, a_{100}$ integers such that for each $k = 1, \dots, 100$, the number $a_1 + \cdots + a_k$ has exactly a_k divisors?

Example 1.6 (Friends in Taiwan)
An $n \times n$ matrix is said to be a *magic square* if the $2n + 2$ sums of rows, columns, and diagonals are all equal. Determine all integers $n \geq 1$ for which one can construct a magic square all of whose entries are pairwise distinct prime numbers.

§1.6 The just-do-it approach

Tim Gowers has called certain problems amenable to the "just-do-it" approach, which he describes at https://gowers.wordpress.com/2008/08/16/just-do-it-proofs/.

Basically, sometimes there is actually *nothing stopping you* from solving the problem, and you just need to plow it in the dumbest way possible; the problem becomes harder the more clever you try to be.

Token example:

Example 1.7 (Math Prize 2017/4)
Define a *lattice line* as a line containing at least 2 lattice points. Is it possible to color every lattice point red or blue such that every lattice line contains exactly 2017 red points?

Walkthrough. Let L_1, L_2, ... denote the countably many lattice lines, in some order. It is not hard to do the "finite" step:

(a) Show that for every integer n, we can construct a set T_n of lattice points such that each line L_1, ..., L_n passes through exactly 2017 points in T_n.

(b) Make sure your solution to (a) works. Is it possible to get stuck because you accidentally colored 2018 points on L_N already for some N in the future?

The issue is that we need a set T_∞ that works for all lines at once: there is a difference between "unbounded" and "infinite"! Put another way, we have proven the statement $P(n)$ that "there exists a set T_n as in (a)" for every $n = 1, 2, \dots$, by induction of the usual shape $P(n) \implies P(n+1)$ but we really need the statement $P(\infty)$, which we cannot reach by using a normal induction. Thus, we need to do a little more work.

(c) Modify your approach to (a) such that we have the additional property $T_1 \subseteq T_2 \subseteq \dots$. (For some people, no additional modification is needed.)

(d) Prove that

$$T_\infty = \bigcup_{n \geq 1} T_n$$

fits the bill.

(e) Why was part (c) necessary? (In other words, what goes wrong if you try to fix over-red lines retroactively?)

As an aside, this is sort of a simple case of a "transfinite induction": the last step breaks the realm of normal induction and brings us into the world of statements $P(\alpha)$ for infinite ordinals α. In set theory, transfinite induction proves a statement $P(-)$ for any *ordinal* α, and this proof typically involves both a successor case $P(\alpha) \implies P(\alpha+1)$, as well as a limit case similar to the above.

To formalize, it helps to understand transfinite induction (ordinals).

Theorem 1.8 (Transfinite induction)

Given a statement $P(-)$, suppose that

- $P(0)$ is true, and
- If $P(\alpha)$ is true for all $\alpha < \beta$, then $P(\beta)$ is true.

Then $P(\alpha)$ is true for every ordinal α.

This is often split in the following way.

- (Zero Case) First, resolve $P(0)$.
- (Successor Case) Show that from $P(\alpha)$ we can get $P(\alpha+1)$.
- (Limit Case) For λ a limit ordinal, show that $P(\lambda)$ holds given $P(\alpha)$ for all $\alpha < \lambda$.

Conceptually, I think this is quite related to greedy algorithms. But at a human level it feels somewhat different; in particular, often the solutions here feel grossly inefficient.

Here's another example for practice.

Example 1.9

Show that there exists a set $A \subset \mathbb{R}^2$ such that for every real number $r > 0$, there is exactly one pair of points in A which are a distance r apart.

Walkthrough. This is just-do-it, but you'll need ordinals.

Let λ be the smallest ordinal such that $|\lambda| = |\mathbb{R}|$. Enumerate the positive real numbers $r_{<\lambda}$. Then, just-do-it!

§2 Practice problems

Instructions: Solve [32♣]. If you have time, solve [42♣]. Problems with red weights are mandatory.

> Michael, come here. This problem has been bugging me. Every time I sit down I see it and I think about it and it bothers me. I know it's true and I can get it for 6 or 7 but not 100
>
> Zuming Feng (apocryphal) to Michael Ren, on Warmup 2

Each section in alphabetical order.

§2.1 Bait

[2♣] **Problem 1.** Consider $n \geq 1$ nonempty open intervals $I_1 = (a_1, b_1)$, $I_2 = (a_2, b_2)$, …, $I_n = (a_n, b_n)$. Prove that one can color them red or blue such that for any point $x \in \mathbb{R}$, the number of red segments containing x and the number of blue segments containing x differ by at most 1.

[2♣] **Problem 2** (Saudi Arabia TST)**.** Prove that there exists integers $a_0, a_1, a_2, ..., a_{100} \geq 2$ which satisfy

$$a_0! = a_1! a_2! \dots a_{100}!.$$

Here $a! = 1 \times 2 \times \dots \times a$ as usual.

[2♣] **Problem 3** (IMO 2009/6)**.** Let $a_1, a_2, \dots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + \dots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \dots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

[2♣] **Problem 4** (IMO 2012/2)**.** Let $a_2, a_3, \dots, a_n$ be positive reals with product 1, where $n \geq 3$. Show that

$$(1 + a_2)^2 (1 + a_3)^3 \dots (1 + a_n)^n > n^n.$$

[2♣] **Problem 5** (Iran TST 2017/2)**.** There are 13 students who have just completed a 6-problem IMO team selection test. For each problem, the submissions of each student have been ranked, with no ties. To select the six-person IMO team, the national committee wants to start by choosing a permutation of the six problems, and then select the student with the highest score on each problem who is not already on the team. Is it possible that all 13 students have a chance of being a team member?

[2♣] **Problem 6** (Shortlist 2010 A3)**.** Consider nonnegative real numbers $x_1, \dots, x_{100}$ such that $x_i + x_{i+1} + x_{i+2} \leq 1$ for $i = 1, \dots, 100$ (with indices taken modulo 100). Find the maximum possible value of

$$S = \sum_{i=1}^{100} x_i x_{i+2} = x_1 x_3 + x_2 x_4 + x_3 x_5 + \dots + x_{99} x_1 + x_{100} x_2.$$

[2♣] **Problem 7** (Shortlist 2014 C1)**.** Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. We dissect R into smaller rectangles with sides parallel to R so that none of the n points lie in the interior of a rectangle in the dissection. Prove that we need to dissect R into at least $n + 1$ rectangles.

[2♣] **Problem 8** (Shortlist 2016 C2)**.** Find all positive integers n for which it is possible to arrange all positive divisors of n (including 1 and n) in a rectangular grid of some size (with all cells filled) such that

- each divisor appears exactly once,
- all columns have equal sum,
- all rows have equal sum.

[2♣] **Problem 9** (SPARC 2017, communicated by Chelsea Voss)**.** There are $2n$ pairwise distinct points $A_1, \dots, A_n$ and $B_1, \dots, B_n$ in $\mathbb{R}^d$ such that $|A_iB_j| = 1$ for $i, j = 1, \dots, n$. Find the smallest possible value of d as a function of n.

[2♣] **Problem 10** (Taiwan Quiz 2014/1J/2)**.** Determine whether there exist ten sets $A_1, A_2, \dots, A_{10}$ such that

(i) each set is of the form $\{a, b, c\}$, where $a \in \{1, 2, 3\}$, $b \in \{4, 5, 6\}$, $c \in \{7, 8, 9\}$,

(ii) no two sets are the same,

(iii) if the ten sets are arranged in a circle $(A_1, A_2, \dots, A_{10})$, then any two adjacent sets have no common element, but any two non-adjacent sets intersect. (Note: A_{10} is adjacent to A_1.)

[2♣] **Problem 11** (USAMO 1985/1)**.** Determine whether or not there are any positive integral solutions of the simultaneous equations

$$\begin{aligned} x_1^2 + x_2^2 + \dots + x_{1985}^2 &= y^3 \\ x_1^3 + x_2^3 + \dots + x_{1985}^3 &= z^2 \end{aligned}$$

with distinct integers $x_1, x_2, \dots, x_{1985}$.

[2♣] **Required Problem 12** (USAMO 2002/6)**.** I have an $n \times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of the sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are real constants c and d such that

$$\frac{1}{7}n^2 - cn \le b(n) \le \frac{1}{5}n^2 + dn$$

for all $n > 0$.

[2♣] **Problem 13** (Iran TST 2010/5)**.** Circles Ω_1 and Ω_2 intersect at points P and K. Line XY is the common tangent of the two circles which is nearer to P with X on Ω_1 and Y on Ω_2. Line XP intersects Ω_2 for the second time at C, and line YP intersects Ω_1 again at B. Let A be intersection of lines BX and CY. Prove that if Q is the second intersection point of circumcircles of ABC and AXY, then $\angle QXA = \angle QKP$.

§2.2 Contrived

[2♣] **Problem 14.** For a given $n \geq 1$, is it possible to divide the numbers $\{1, 2, \dots, 2n\}$ into n pairs, such that the sum of each pair is a prime number?

[2♣] **Required Problem 15** (At MOP 2018, ostensibly for my 22nd birthday)**.** Let a_1, a_2, ... be a sequence of nonnegative integers which satisfies the recurrence

$$a_{n+2} = a_n a_{n+1} + 1$$

for $n = 1, 2, \dots$. Prove that $a_{2018} - 22$ is not a prime.

[2♣] **Required Problem 16** (Putnam 2018 B6)**.** Prove that the number of length 2018-tuples whose entries are in $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860, is at most

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$

[2♣] **Problem 17** (Tournament of Towns 2013)**.** There is an 8×8 chessboard drawn in a plane with alternating black and white squares. First, Peter secretly selects some square of the chessboard and then selects some point in the interior of that square. Then, Basil can draw a (simple) polygon in the plane and ask Peter whether the chosen point is inside or outside the polygon. What is the minimum number of questions sufficient for Basil to determine whether the point is black or white?

[2♣] **Problem 18** (Tournament of Towns, Fall 2013 Senior A-6)**.** There are 11 piles consisting of 10 stones each. Two players Alice and Bob take turns playing a game. On Alice's turn, she picks a pile and removes up to three stones (inclusive). On Bob's turn, he picks up to three nonempty piles and removes one stone from each. Both players are required to remove at least one stone, and the player who removes the last stone wins. Determine which player has the winning strategy.

[2♣] **Required Problem 19** (Tristan Shin)**.** Find all integers a, b, c which satisfy

$$(a-b)(a^2-b^2)(a^3-b^3) = 3c^3.$$

[2♣] **Problem 20** (TSTST 2012/6)**.** Positive real numbers x, y, z satisfy $xyz + xy + yz + zx = x + y + z + 1$. Prove that

$$\frac{1}{3}\left(\sqrt{\frac{1+x^2}{1+x}} + \sqrt{\frac{1+y^2}{1+y}} + \sqrt{\frac{1+z^2}{1+z}}\right) \leq \left(\frac{x+y+z}{3}\right)^{5/8}.$$

§2.3 Derp

[2♣] **Problem 21.** Let p_n denote the nth prime number (so $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on). Prove that

$$\frac{p_n + p_{n+1}}{2}$$

is not a prime for any $n \geq 1$.

[2♣] **Problem 22.** Consider a standard game of chess (as described in `www-math.bgsu.edu/~zirbel/chess/BasicChessRules.pdf`, say). We make the following modification: on a turn, if a player makes a move which neither captures a piece nor puts their opponent's king in check, then they may make a second move on the same turn with the same properties. (This does not stack, so at most two moves are made per turn.) Can Black guarantee victory in a finite number of turns?

[2♣] **Problem 23.** Let $n > 2017$ be an integer. Show that it is possible to find a collection $\mathcal{F}$ of subsets of $\{1, \dots, n\}$, such that $|\mathcal{F}| > 1.001^n$ and whenever X and Y are distinct elements of $\mathcal{F}$, we have

$$|X \cup Y| - |X \cap Y| \ge \frac{n}{3}.$$

[2♣] **Required Problem 24** (BrMO 2010/1)**.** A math camp has 2010^{2010} students, and each student has at most three friends (friendship is symmetric). The camp director wishes to arrange them in a line such that any pair of friends is at most 2010 students apart. Is this always possible?

[2♣] **Problem 25** (BAMO, for middle schoolers!)**.** A finite number of people form several committees. Every committee has at least two people and any two committees have a common member. Prove that we can give each person a red, green, or blue hat so that every committee has two members with differently colored hats.

[2♣] **Problem 26** (China TST 2016/3/2)**.** In the coordinate plane the points with both coordinates being rational numbers are called rational points. For any positive integer n, is there a way to use n colours to colour all rational points, every point is coloured one colour, such that any line segment with both endpoints being rational points contains the rational points of every colour?

[2♣] **Problem 27** (EGMO 2017/3)**.** There are 2017 lines in the plane such that no three of them go through the same point. Turbo the snail sits on a point on exactly one of the lines and starts sliding along the lines in the following fashion: she moves on a given line until she reaches an intersection of two lines. At the intersection, she follows her journey on the other line turning left or right, alternating her choice at each intersection point she reaches. She can only change direction at an intersection point. Can there exist a line segment through which she passes in both directions during her journey?

[2♣] **Problem 28** (ELMO 2009/4)**.** Let n be a positive integer. Given n^2 points in a unit square, prove that there exists a broken line of length $2n + 1$ that passes through all the points.

[2♣] **Problem 29** (Shortlist 2009 N2)**.** A positive integer N is *balanced* if it is the product of an even number of not necessarily distinct prime divisors. Given positive integers a and b, define $P(x) = (x + a)(x + b)$.

(a) Show that there exists distinct positive integers a and b such that the numbers $P(1)$, ..., $P(50)$ are all balanced.

(b) Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$.

[1♣] **Mini Survey.** At the end of your submission, answer the following questions.

(a) About how many hours did the problem set take?

(b) Name any problems that stood out (e.g. especially nice, instructive, boring, or unusually easy/hard for its placement).

Any other thoughts are welcome too. Examples: suggestions for new problems to add, things I could explain better in the notes, overall difficulty or usefulness of the unit.

§3 Solutions to the walkthroughs

§3.1 Solution 1.1, IMO 2011/2

Orient ℓ in some direction, and color the plane such that its left half is red and right half is blue. The critical observation is that:

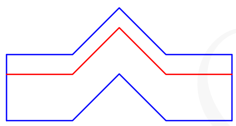
> **Claim** — The number of points on the red side of ℓ does not change, nor does the number of points on the blue side (except at a moment when ℓ contains two points).

Thus, if $|\mathcal{S}| = n+1$, it suffices to pick the initial configuration so that there are $\lfloor n/2 \rfloor$ red and $\lceil n/2 \rceil$ blue points. Then when the line ℓ does a full 180° rotation, the red and blue sides "switch", so the windmill has passed through every point.

(See official shortlist for verbose write-up; this is deliberately short to make a point.)

§3.2 Solution 1.2, Communicated by Ashwin Sah

Make a stack of four copies of the same polygon except with $AV = EZ = 1/4$.



§3.3 Solution 1.3, Rejected from ELMO Shortlist

The answer is $x = y = z$, which all work. To see these are the only solutions, the equation can be rewritten as

$$(x+y)^4 = (2z)^4 + (x-y)^4.$$

Fermat's Last Theorem then implies $x - y = 0$ or $x = y = z$.

§3.4 Solution 1.4, EGMO 2019/6

Only the labels mod 3 matter at all.

Assume for contradiction no blue labels are divisible by 3. Let e_{ij} denote the number of segments joining $i \pmod 3$ to $j \pmod 3$. By double-counting (noting that points in (ii) are counted an even number of times but points in (i) are counted once) we derive that

$$\begin{aligned} e_{01} + e_{02} &\equiv 2019 \pmod 2 \\ e_{01} + e_{12} &\equiv 2019 \pmod 2 \\ e_{02} + e_{12} &\equiv 0 \pmod 2 \end{aligned}$$

which gives

$$e_{02} \equiv e_{12} \equiv 1 - e_{01} \pmod 2.$$

However, one can check this is incompatible with the hypothesis that the yellow labels are $0, 1, \dots, N$.

> **Remark.** In addition, we can replace the points/segments by any graph G for which there are
>
> - an odd number of leaves (or just odd-degree vertices) labeled 0,
> - an odd number of leaves (or just odd-degree vertices) labeled 1,

- and the remaining vertices have even degree.

Thus the geometry of the problem is smoke and mirrors, too.

§3.5 Solution 1.5, "Warmup 2", from Sasha's class at MOP 2017

Answer: yes.

The idea is to define the partial sums instead as follows. Let $d(n)$ denote the divisor function. Let s_N be suitably large, then define by downwards recursion

$$s_n = s_{n+1} - d(s_{n+1})$$

with N set such that $s_0 = 1$. Then $s_k = a_1 + \dots + a_k$ works fine.

All we need is to ensure this sequence stays positive. But $x - d(x) \ge x/2$ for $x \ge 8$, so it's enough to take $n = 2^{103}$.

Remark. Lore: Zuming didn't solve this problem. Neither did I. Michael Ma also didn't solve this problem during the class (but tried using Green-Tao). Blue got it more quickly.

Second-hand from Michael Ren:

> "Michael, come here. This problem has been bugging me. Every time I sit down I see it and I think about it and it bothers me. I know it's true and I can get it for 6 or 7 but not 100". — Z

§3.6 Solution 1.6, Friends in Taiwan

The answer is all $n \neq 2$. For $n \ge 3$, use **Green-Tao theorem** on *any* integer magic square of size n.

(The construction for $n = 1$ is obvious while any 2×2 magic square must have all entries equal.)

§3.7 Solution 1.7, Math Prize 2017/4

Let $L_1, L_2, \dots$ denote the countably many lattice lines, in some order. We construct by induction a set T_n of lattice points (for each $n \ge 1$) such that each line $L_1, \dots, L_n$ passes through exactly 2017 points in T_n.

To do this, at the nth step, we take T_{n-1} and add in between 2015 and 2017 red points on L_n such that

- no red point we add is on any of $L_1, \dots, L_{n-1}$, and
- no red point we add is collinear with any two red points in T_{n-1}. (This ensures that at future steps of the algorithm, each line passes through at most two red points already).

Finally, note that our construction has the property that $T_1 \subseteq T_2 \subseteq \dots$; thus the union

$$T_\infty = \bigcup_{n \ge 1} T_n$$

satisfies the construction.

Remark. One incorrect approach is to try and edit the choice of red points retroactively if the line L_n is already full. This makes it impossible to take the union at the last step.

§3.8 Solution 1.9

This solution is from https://gowers.wordpress.com/2008/08/16/just-do-it-proofs/, but see also the comment at the very bottom. Also, I think the indexing of this needs to have $\mathbb{R}$ map to a cardinal in order for this to work; if you map to e.g. a successor ordinal then there is no way you can add the last real number.

Let λ be the smallest ordinal such that $|\lambda| = |\mathbb{R}|$. We enumerate the positive real numbers $r_{<\lambda}$.

Idea: define a set A_α such that every distance $r_{<\alpha}$ appears in A_α (and no distance appears twice). The process is:

- Let $A_0 = \{(0, 0)\}$.
- Given A_α, define $A_{\alpha+1}$ by:
  - If r_α already present, done; $A_{\alpha+1} = A_\alpha$.
  - Otherwise, want to add a point P a distance r_α away from the origin (this locus is a circle Γ), and avoiding:
    1. Can't be r_β away from any existing point. Eliminates at most $|2 \times \alpha|$.
    2. Avoid perpendicular bisectors of any segments. Eliminates $|2 \times \alpha \times \alpha|$ possibilities.

    So in total, $|2 \times \alpha + 2 \times \alpha \times \alpha| = |\alpha|$. On the other hand, $|\Gamma| = |\mathbb{R}|$, and λ was smallest possible with size $|\mathbb{R}|$.
- For limit ordinals μ, let $A_\mu = \bigcup A_{<\mu}$.