On the feet of the incenter on the perpendicular bisectors / Darij Grinberg

The following result I came across while searching for analogues of the first Brocard triangle with other points instead of the symmedian point.

Theorem. From the incenter O of a triangle ABC, draw perpendiculars OX, OY and OZ to the perpendicular bisectors of the sides BC, CA and AB, respectively. Then:

a) Triangle XYZ is oppositely similar to triangle ABC.

b) The lines AX, BY and CZ pass through the Nagel point N of triangle ABC.

c) One of the segments OX, OY and OZ equals to the sum of the two others. More precisely: If $a \leq b \leq c$ or $c \leq b \leq a$, then $OY = OZ + OX$.

Note. It can be shown that the points O and U are the triangle centers X_{100} and X_{104} of triangle XYZ, respectively. (The X_{100} point of a triangle is the anticomplement of the Feuerbach point; it lies on the circumcircle. The X_{104} point is the point diametrically opposite to X_{100} on the circumcircle.) See also Hyacinthos messages #6440, #6441, #6442, #6443.

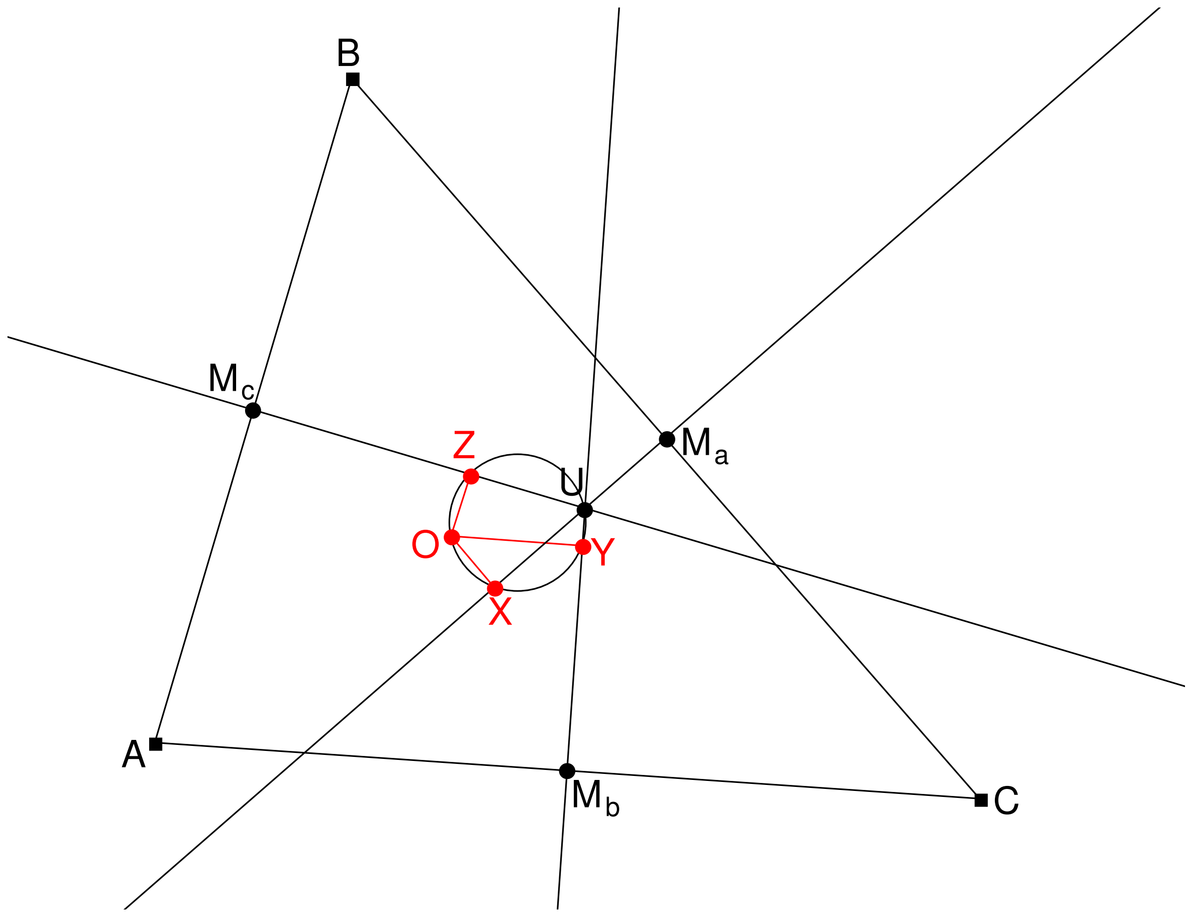

Fig. 1

Proof of Theorem.

a) We will prove a generalization of **a**):

From an arbitrary point P in the plane of a triangle ABC, construct the perpendiculars PX_P, PY_P and PZ_P to the perpendicular bisectors of the sides BC, CA and AB. Then triangle $X_PY_PZ_P$ is oppositely similar to triangle ABC.

Proof. Let U be the circumcenter of ΔABC, i. e. the intersection of the three perpendicular bisectors. Then $\measuredangle PX_PU = 90°$, $\measuredangle PY_PU = 90°$ and $\measuredangle PZ_PU = 90°$; so that the points X_P, Y_P and Z_P lie on the circle with diameter PU. This means that the points P, U, X_P, Y_P and Z_P are concyclic (Fig. 2). Hence, $\measuredangle Y_PZ_PX_P = \measuredangle Y_PUX_P$ (with directed angles). On the other hand, $\measuredangle Y_PUX_P = 180° - \measuredangle M_aUM_b$, where M_a, M_b and M_c are the midpoints of the sides BC, CA and

AB, respectively. Thus, $\measuredangle Y_P Z_P X_P = 180° - \measuredangle M_a U M_b$.

But for $\measuredangle UM_aC = 90°$ and $\measuredangle UM_bC = 90°$, the points M_a and M_b lie on the circle with diameter UC; consequently, UM_aCM_b is a cyclic quadrilateral, and therefore $\measuredangle M_bCM_a = 180° - \measuredangle M_aUM_b$, what yields $\measuredangle ACB = 180° - \measuredangle M_aUM_b$. Together with the equation $\measuredangle Y_PZ_PX_P = 180° - \measuredangle M_aUM_b$ which was already proven, this results in $\measuredangle ACB = \measuredangle Y_PZ_PX_P$. Analogously, $\measuredangle BAC = \measuredangle Z_PX_PY_P$ and $\measuredangle CBA = \measuredangle X_PY_PZ_P$. Hence, triangles ABC and $X_PY_PZ_P$ have oppositely equal angles; they are oppositely similar. This proves the generalization of **a**).

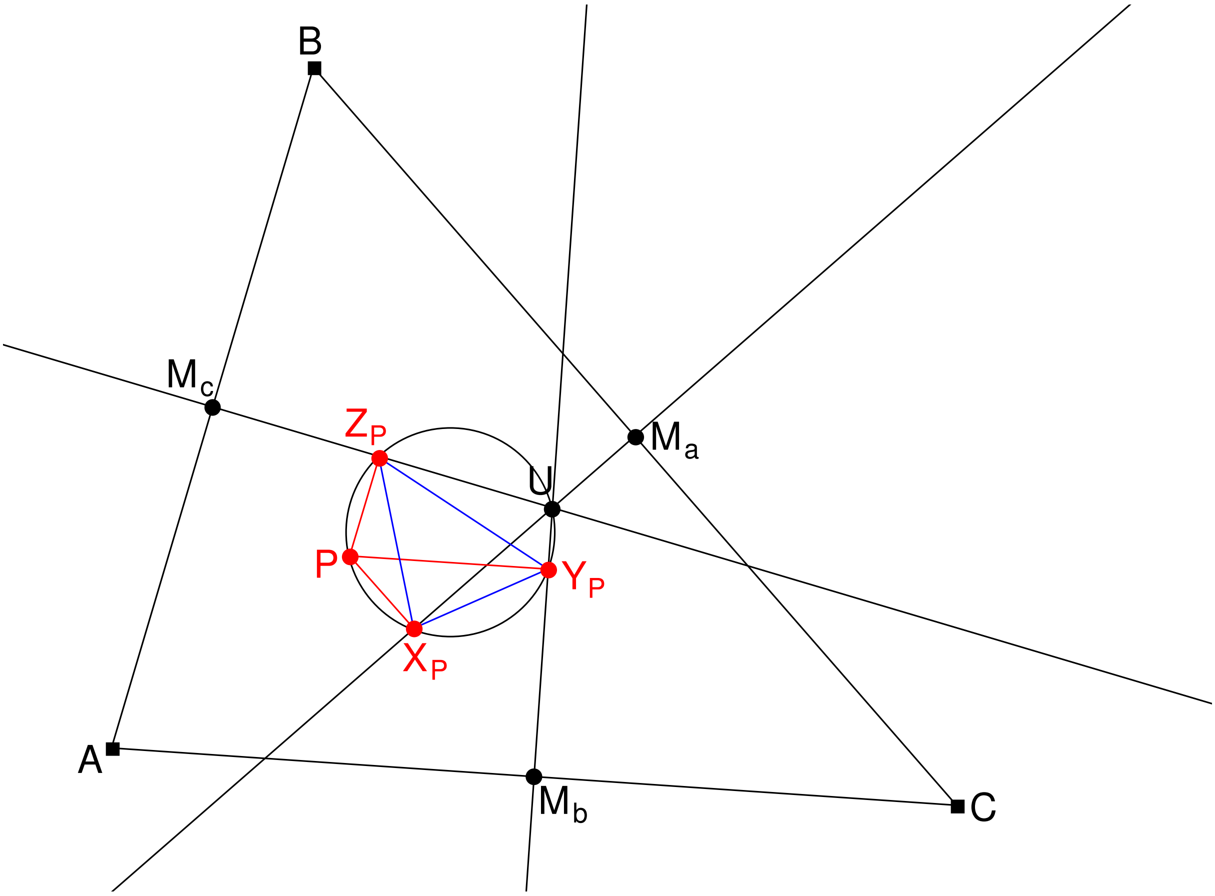

Fig. 2

b) The Nagel point N of triangle ABC is defined as the intersection of the lines AA_a, BB_b and CC_c, where A_a is the point of tangency of the a-excircle with BC, and B_b and C_c are similarly defined.

On the other hand, let E be the intersection of BY and CA. We intend to show that the points E and B_b coincide; this will entail that the Nagel point N, lying on the line BB_b, must lie on the line BY; and analogously, N will lie on the lines CZ and AX.

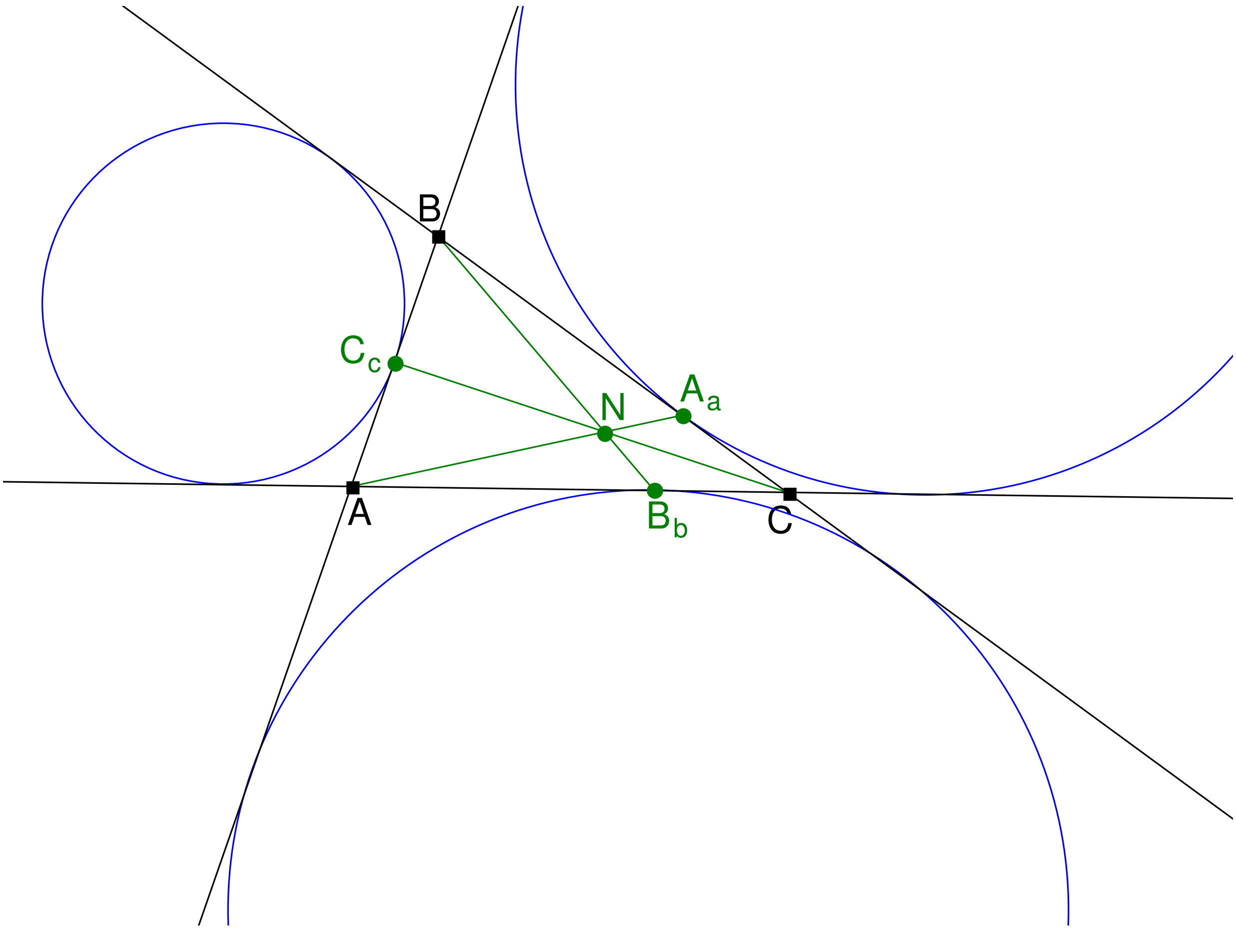

Fig. 3

Here is the proof in detail. Consider the point of tangency B_b of the b-excircle with CA. It is well-known that $AB_b = s - c$, where $s = \frac{1}{2}(a+b+c)$.

Let H_b be the foot of the B-altitude of triangle ABC; and let M_b be the midpoint of CA.

Then

$$\frac{H_bB_b}{M_bB_b} = \frac{AB_b - AH_b}{AB_b - AM_b} = \frac{(s-c) - c\cos\alpha}{(s-c) - \frac{b}{2}} = \frac{(s-c) - c \bullet \frac{b^2+c^2-a^2}{2bc}}{(s-c) - \frac{b}{2}}$$

$$= \frac{\frac{a+b-c}{2} - c \bullet \frac{b^2+c^2-a^2}{2bc}}{\frac{a+b-c}{2} - \frac{b}{2}} \qquad (\text{since } s - c = \frac{a+b+c}{2} - c = \frac{a+b-c}{2})$$

$$= \frac{\frac{a+b-c}{2} - \frac{b^2+c^2-a^2}{2b}}{\left(\frac{a-c}{2}\right)} = \frac{(a+b-c) - \frac{b^2+c^2-a^2}{b}}{a-c} = \frac{\left(\frac{(a+b-c)b - (b^2+c^2-a^2)}{b}\right)}{a-c}$$

$$= \frac{(a+b-c)b - (b^2+c^2-a^2)}{b(a-c)} = \frac{ab + b^2 - cb - b^2 - c^2 + a^2}{b(a-c)}$$

$$= \frac{ab - cb - c^2 + a^2}{b(a-c)} = \frac{(a-c)b + (a^2-c^2)}{b(a-c)} = \frac{(a-c)b + (a-c)(a+c)}{b(a-c)}$$

$$= \frac{(a-c)(a+b+c)}{b(a-c)} = \frac{a+b+c}{b} = \frac{2s}{b}.$$

If Δ is the area of triangle ABC, then it is well-known that $\Delta = \frac{1}{2}b \bullet BH_b$. So we have $b = 2\Delta/BH_b$. Another canonical formula of Triangle Geometry says $\Delta = \rho s$, where ρ is the inradius of ΔABC. Thus, $s = \Delta/\rho$. Hence we get

$$\frac{H_bB_b}{M_bB_b} = \frac{2s}{b} = \frac{2\Delta/\rho}{2\Delta/BH_b} = \frac{BH_b}{\rho}. \tag{1}$$

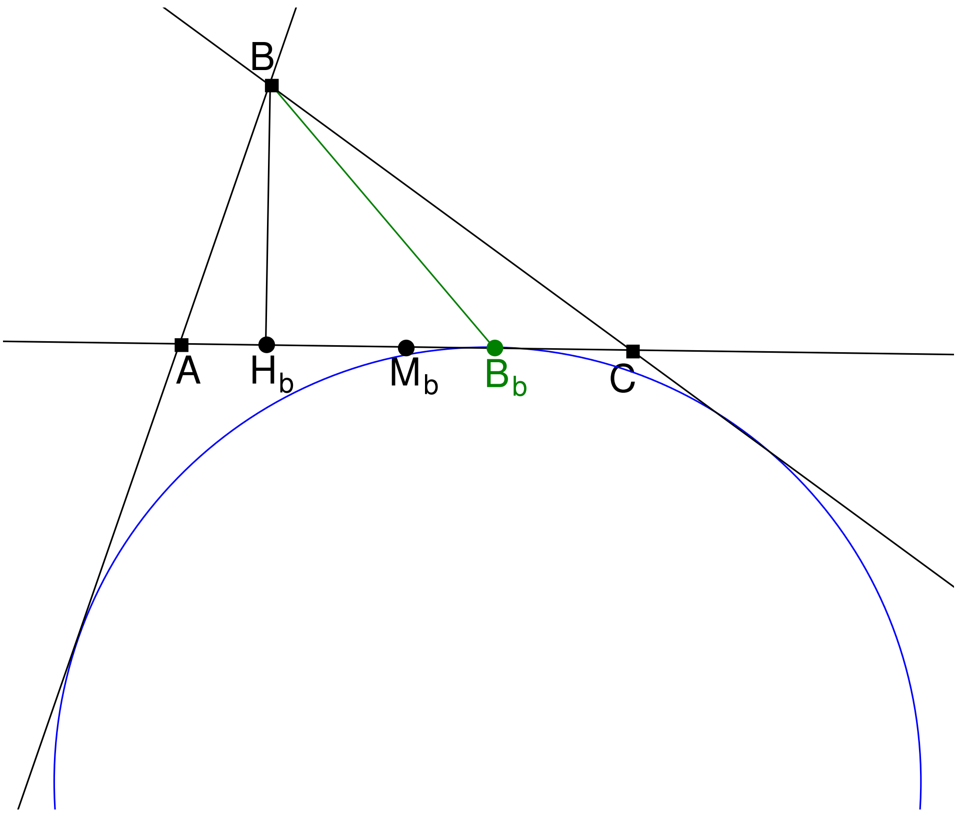

Fig. 4

Now consider the intersection E of BY and CA (Fig. 5). The distance from the incenter O to the side CA is the inradius ρ; but as OY is orthogonal to the perpendicular bisector of CA, i. e. parallel to CA, the distance from Y to CA is also ρ. I. e., we have $YM_b = \rho$.

The lines BH_b and YM_b are parallel (both being orthogonal to CA); this yields

$$\frac{H_bE}{M_bE} = \frac{BH_b}{YM_b} = \frac{BH_b}{\rho}.$$

Comparing with (1), we find

$$\frac{H_bE}{M_bE} = \frac{H_bB_b}{M_bB_b}, \qquad \text{i. e.} \qquad \frac{H_bE}{EM_b} = \frac{H_bB_b}{B_bM_b}.$$

Thus, the points E and B_b must be identical. Since the Nagel point N lies on the line BB_b, it therefore lies on the line BE, i. e. on the line BY. Analogously, N lies on the lines CZ and AX (Fig. 6).

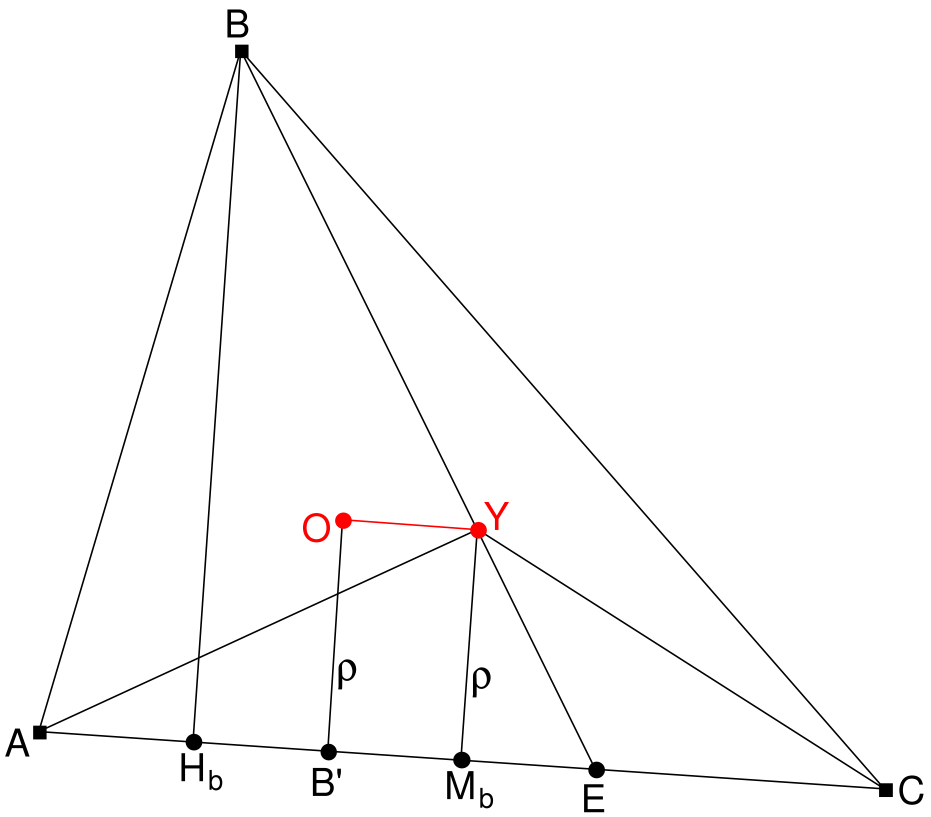

Fig. 5

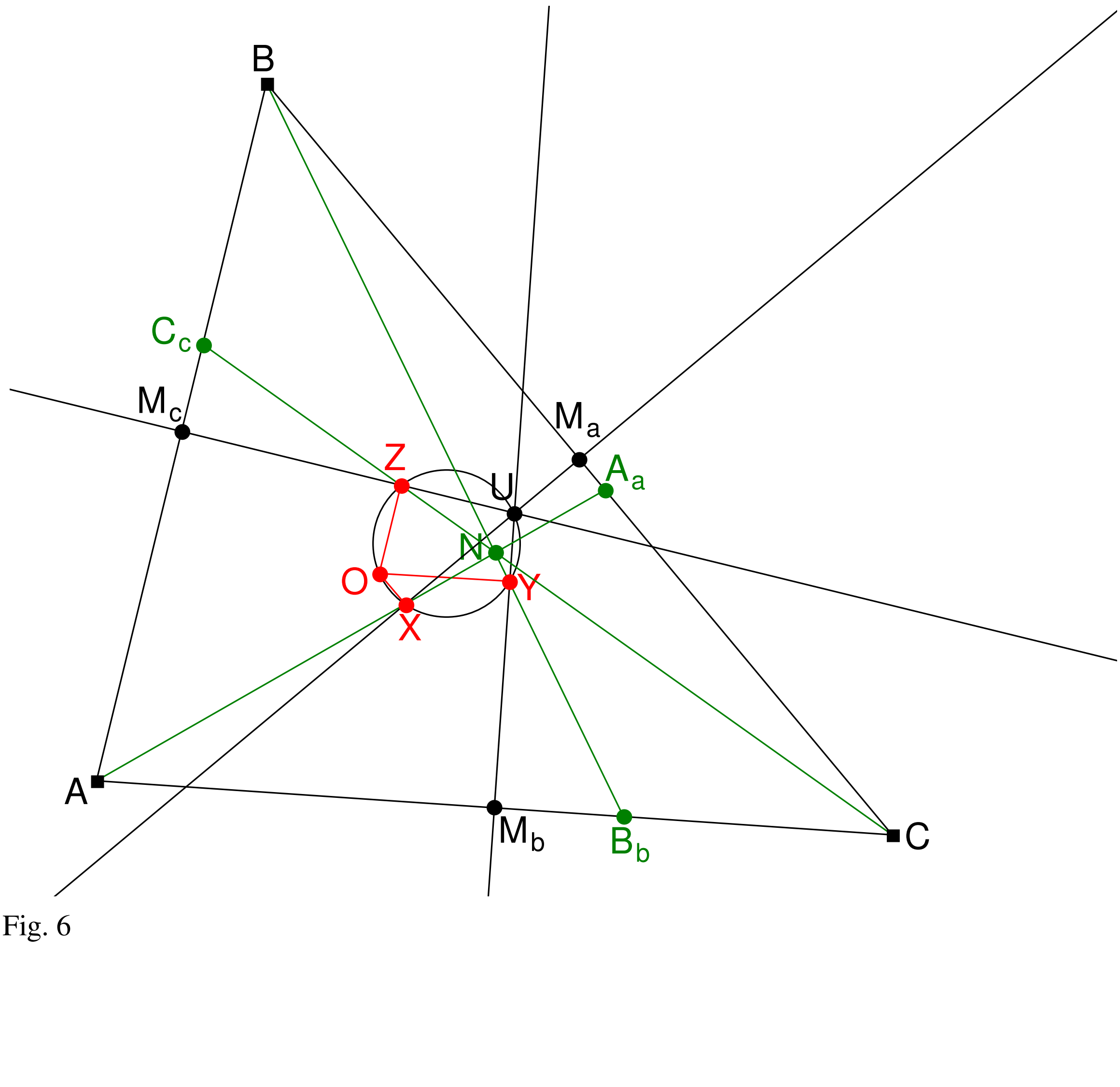

Fig. 6

c) Without loss of generality take $c \le b \le a,$ as we have on Fig. 5. The foot B' of the perpendicular from O to CA is the point of tangency of the incircle with CA. Consequently, $AB' = s - a$ (well-known relation).

Having three right angles, the quadrilateral OYM_bB' is a rectangle. This leads to $OY = B'M_b$. Thus,

$$OY = B'M_b = AM_b - AB' = \frac{b}{2} - (s - a) = \frac{b}{2} - \left(\frac{a+b+c}{2} - a\right)$$
$$= \frac{b}{2} - \left(\frac{-a+b+c}{2}\right) = \frac{a-c}{2}.$$

In the same manner, we can show $OX = \frac{b-c}{2}$ and $OZ = \frac{a-b}{2};$ finally, this yields

$$OY = \frac{a-c}{2} = \frac{a-b}{2} + \frac{b-c}{2} = OZ + OX,$$

qed.