Anti-Steiner points with respect to a triangle

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We begin with a result by S. N. Collings ([1]):

Given a line g passing through the orthocenter H of a triangle ABC, we denote by a', b', c' the reflections of g in the sidelines BC, CA, AB, respectively. Then, the lines a', b', c' meet at one point, and this point lies on the circumcircle of ΔABC (Fig. 1).

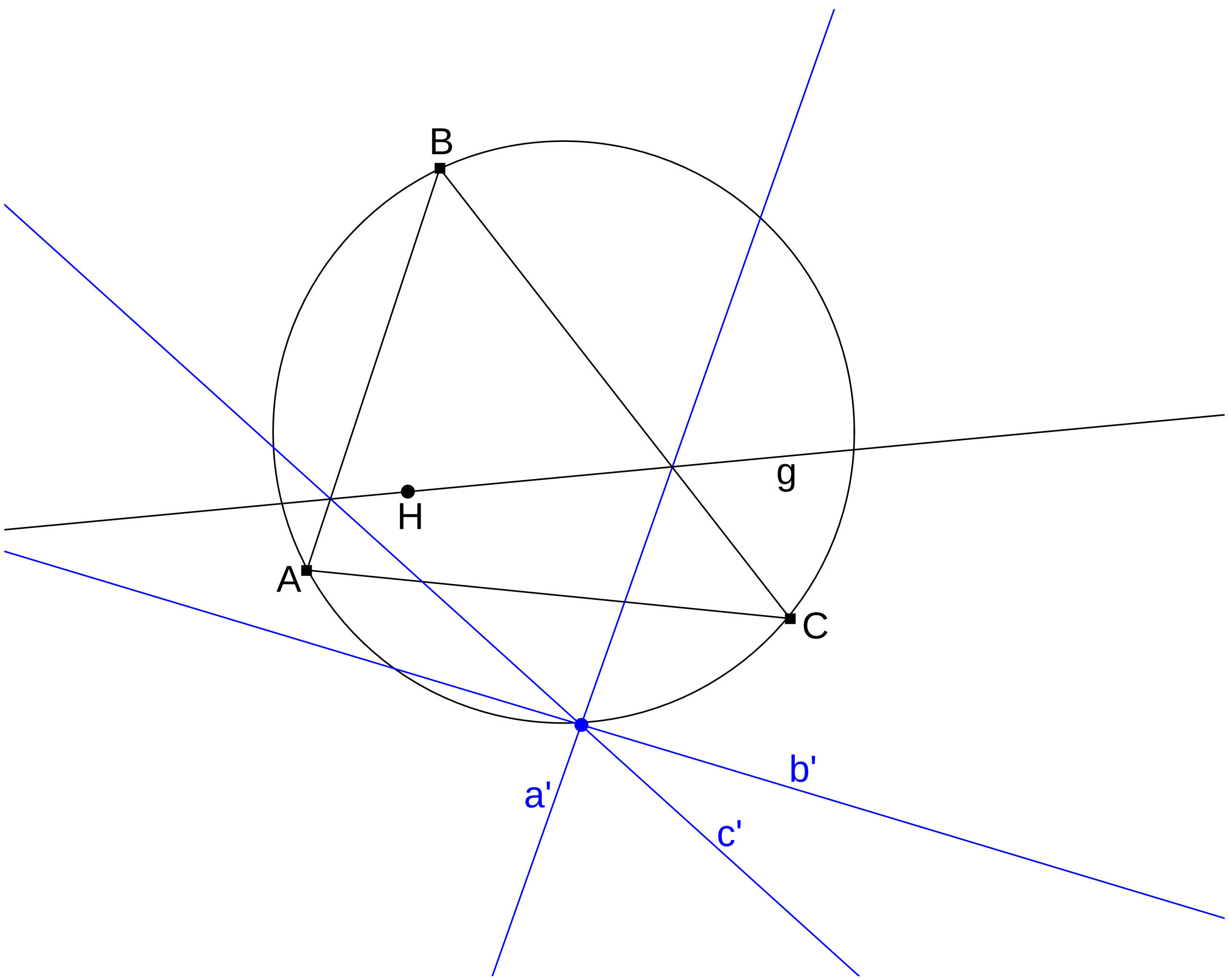

Fig. 1

Proof. Let P be any point on g different from H, and let X, Y, Z be the reflections of P in the sidelines BC, CA, AB. Since P lies on g, X lies on a', Y on b', and Z on c'.

Next, we denote by A', B', C' the reflections of the orthocenter H in the sidelines BC, CA, AB. Since H lies on g, A' lies on a', B' on b', and C' on c'.

Hence, our lines a', b', c' can be written as $a' = XA'$, $b' = YB'$, $c' = ZC'$.

Hereafter, we will use directed angles modulo 180°, also called crosses. See, e. g., [3], [4], [5] for these angles; in [3], directed angles modulo 180° are the Winkeltyp 4. This kind of angles has the very powerful advantage to provide the possibility to prove many results without referring to a picture and independently of the arrangement of points. In this note, the drawings are made for the sake of illustration only; all proofs work independently of these drawings.

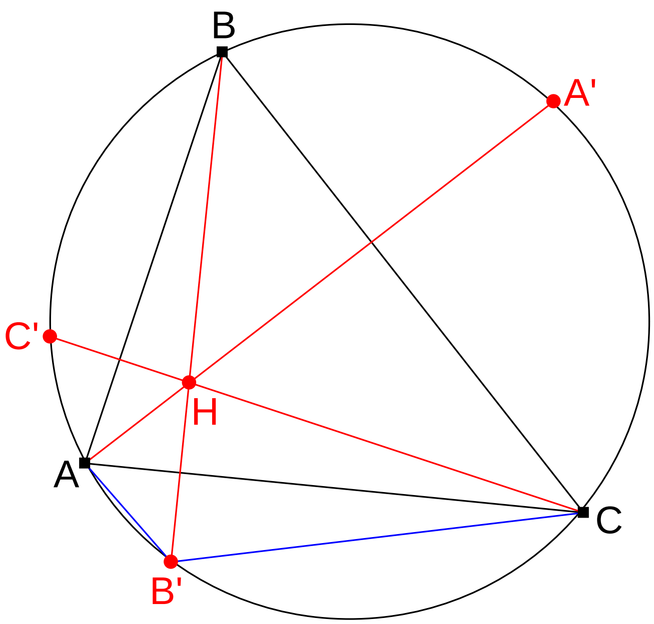

Fig. 2

We shall prove the following familiar lemma first:

Lemma 1. The points A', B', C' lie on the circumcircle of triangle ABC.

Proof. The lines AA', BB', CC' are the altitudes of ΔABC. We have

$$\measuredangle(AA';\ CC') = \measuredangle(BC;\ AB)\,, \tag{1}$$

since

$$\begin{aligned}\measuredangle(AA';\ CC') &= \measuredangle(AA';\ BC) + \measuredangle(BC;\ AB) + \measuredangle(AB;\ CC') \\ &= 90^\circ + \measuredangle(BC;\ AB) + 90^\circ = 180^\circ + \measuredangle(BC;\ AB) = \measuredangle(BC;\ AB)\,.\end{aligned}$$

Now, as B' is the reflection of H in CA, the lines AB' and $B'C$ are the reflections of the lines AH and HC in CA. Reflection in a line switches the sign of an angle; hence

$$\measuredangle(AB';\ B'C) = -\measuredangle(AH;\ HC) = -(AA';\ CC') = -\measuredangle(BC;\ AB) \qquad \text{(from (1))},$$

hence $\measuredangle(AB';\ B'C) = \measuredangle(AB;\ BC)$. Consequently, B' lies on the circumcircle of triangle ABC. Similar reasoning shows the same for A' and C', and Lemma 1 is proven.

We have

$$\measuredangle(B'A;\ AA') = \measuredangle(AB';\ AA') = \measuredangle(AB';\ CA) + \measuredangle(CA;\ AA')\,.$$

Since the line AB' is the reflection of AH in CA, we have $\measuredangle(AB';\ CA) = \measuredangle(CA;\ AH) = \measuredangle(CA;\ AA')$; hence,

$$\begin{aligned}
\measuredangle(B'A;\ AA') &= \measuredangle(CA;\ AA') + \measuredangle(CA;\ AA') = 2 \cdot \measuredangle(CA;\ AA') \\
&= 2 \cdot (\measuredangle(CA;\ BC) + \measuredangle(BC;\ AA')) = 2 \cdot \measuredangle(CA;\ BC) + 2 \cdot \measuredangle(BC;\ AA') \\
&= 2 \cdot \measuredangle(CA;\ BC) + 2 \cdot 90^\circ = 2 \cdot \measuredangle(CA;\ BC) + 180^\circ = 2 \cdot \measuredangle(CA;\ BC),
\end{aligned}$$

and

$$\measuredangle(B'A;\ AA') = 2 \cdot \measuredangle ACB. \qquad (2)$$

Now, let the lines a' and b' meet at Φ (Fig. 3). Then,

$$\measuredangle(B'\Phi;\ \Phi A') = \measuredangle(b';\ a') = \measuredangle(b';\ CA) + \measuredangle(CA;\ g) + \measuredangle(g;\ BC) + \measuredangle(BC;\ a').$$

Since b' is the reflection of g in CA, we have $\measuredangle(b';\ CA) = \measuredangle(CA;\ g)$, and since a' is the reflection of g in BC, we have $\measuredangle(BC;\ a') = \measuredangle(g;\ BC)$. Thus,

$$\begin{aligned}
\measuredangle(B'\Phi;\ \Phi A') &= \measuredangle(CA;\ g) + \measuredangle(CA;\ g) + \measuredangle(g;\ BC) + \measuredangle(g;\ BC) \\
&= 2 \cdot (\measuredangle(CA;\ g) + \measuredangle(g;\ BC)) = 2 \cdot \measuredangle(CA;\ BC) = 2 \cdot \measuredangle ACB,
\end{aligned}$$

and (2) yields $\measuredangle(B'\Phi;\ \Phi A') = \measuredangle(B'A;\ AA')$. Hence, the point Φ lies on the circle through the points B', A, A', i. e. on the circumcircle of triangle ABC (cf. Lemma 1). But Φ is defined as $a' \cap b'$. Hence, we can state that the point of intersection of the line a' with the circumcircle different from A' lies on the line b'. Similarly, this point of intersection lies on c'. Hence, the lines a', b', c' meet at one point on the circumcircle of ΔABC, qed..

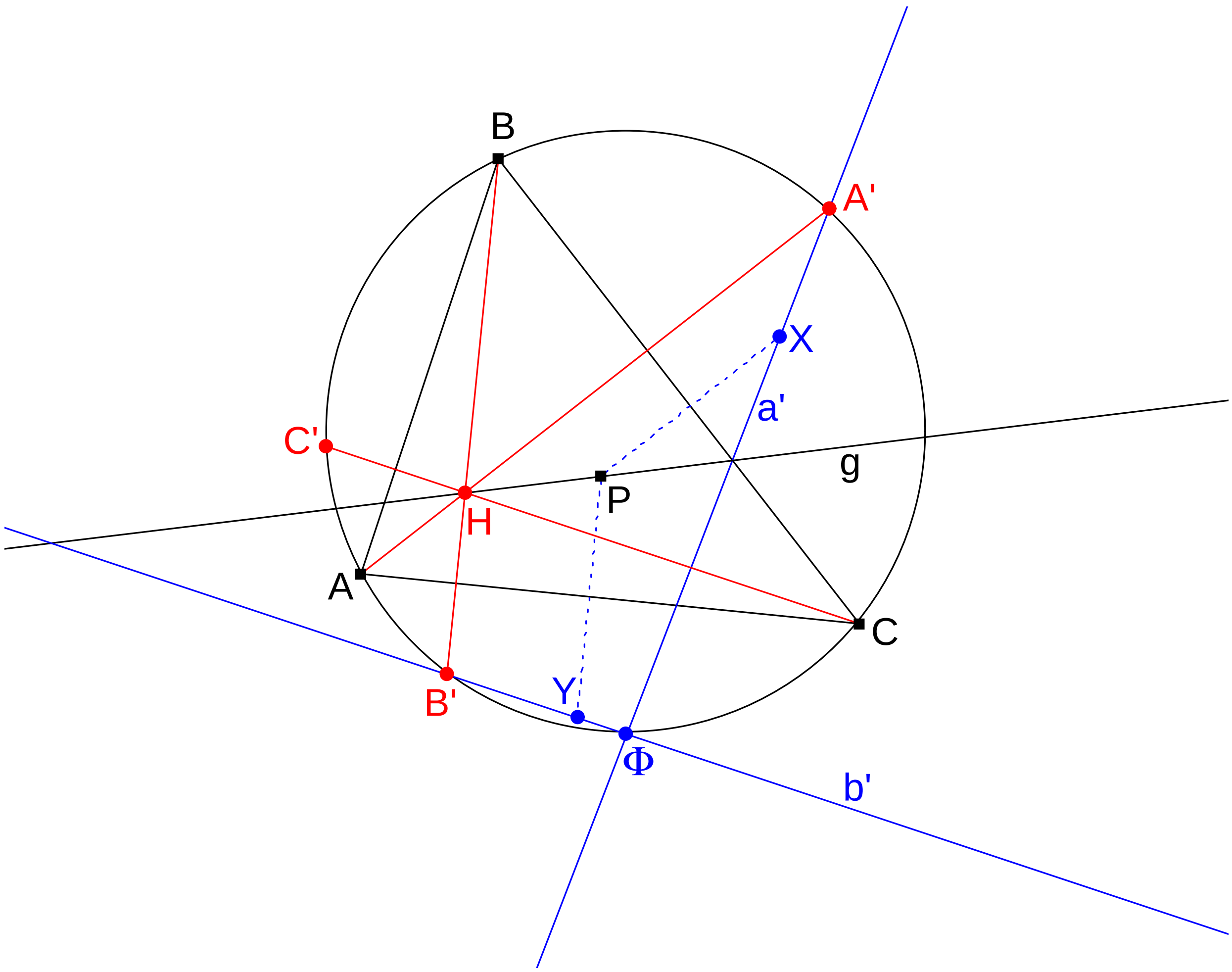

Fig. 3

Notes.

1. The point Φ where the lines a', b', c' meet will be called **Anti-Steiner point** of the line g with respect to triangle ABC in this note. The reason for this naming is the following: The reflections of a point R lying on the circumcircle of a triangle ABC in the sidelines BC, CA, AB are known to lie on one line, which also passes through the orthocenter H of triangle ABC. This line is the so-called **Steiner line** of R with respect to ΔABC. Now we have:

Corollary 2. In our configuration, g is the Steiner line of Φ.

Proof (Fig. 4). Since Φ lies on a', the reflection of Φ in BC lies on the reflection of a' in BC, i. e. on g. Similarly, the reflections of Φ in CA and AB lie on g, too. Hence, the Steiner line of Φ is the line g, qed..

This justifies the term "Anti-Steiner point". (The name "Steiner point" is preserved for a particular point related to the first Brocard triangle, X_{99} in Clark Kimberling's ETC [6].)

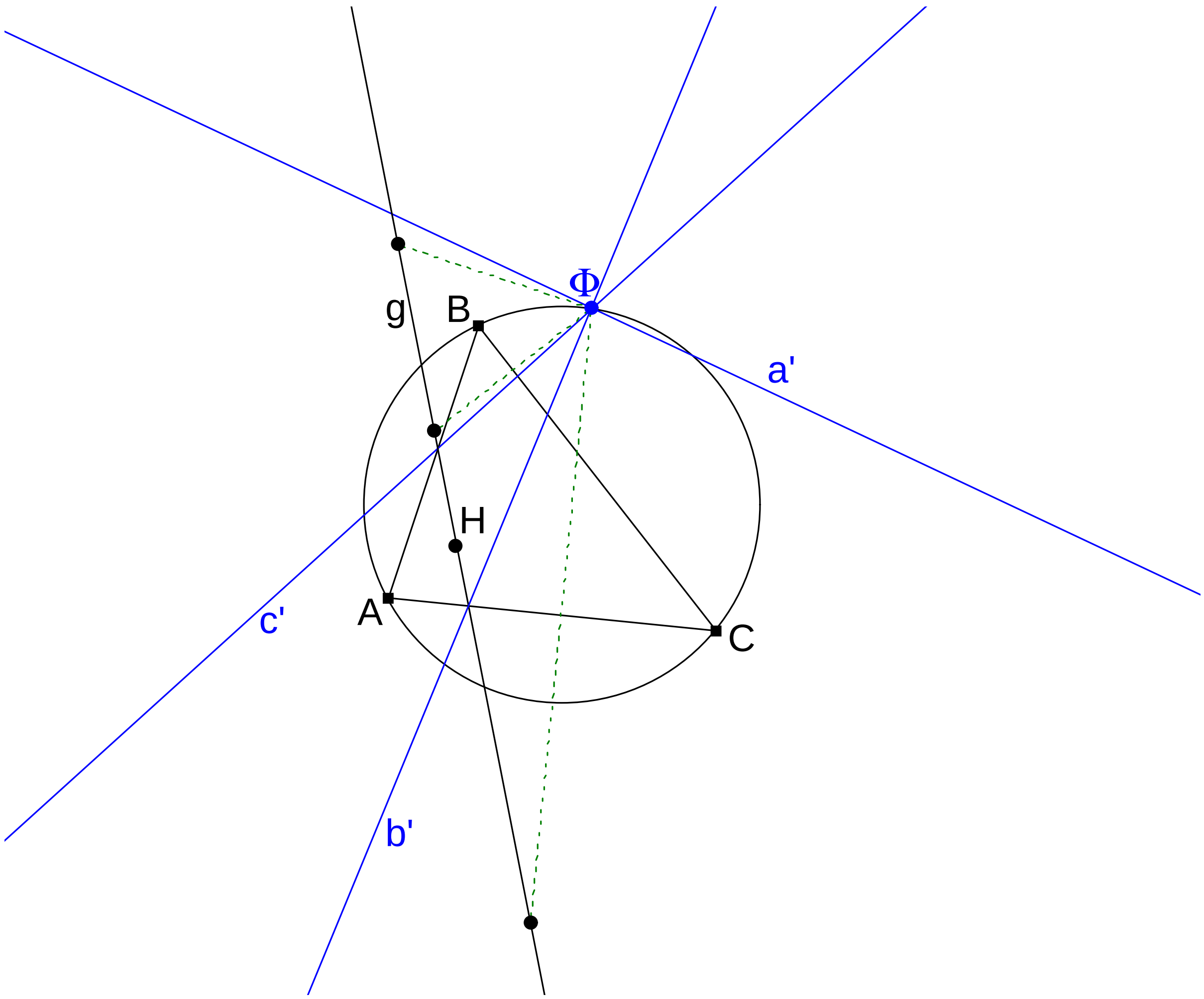

Fig. 4

2. An interesting corollary found by S. N. Collings and mentioned by M. S. Longuet-Higgins ([2]) states:

Corollary 3. The Anti-Steiner point Φ of a line g passing through the orthocenter H lies on the circumcircles of triangles AYZ, BZX, CXY, where X, Y, Z are the reflections of an arbitrary point P lying on g in the sidelines BC, CA, AB.

Proof. If P is the orthocenter H of ΔABC, we get $X = A'$, $Y = B'$, $Z = C'$, and the circumcircles of triangles AYZ, BZX, CXY coincide with the circumcircle of triangle ABC (since A', B', C' lie on the circumcircle of ΔABC, see Lemma 1), and Φ certainly lies on this circumcircle.

We are going to consider the case $P \neq H$ now. We have shown before that $\measuredangle (B'\Phi;\ \Phi A') = 2 \cdot \measuredangle ACB$, i. e. $\measuredangle Y\Phi X = 2 \cdot \measuredangle ACB$. On the other hand, for Y is the reflection of P in CA, we have $\measuredangle YCA = \measuredangle ACP$; for X is the reflection of P in BC, we get $\measuredangle BCX = \measuredangle PCB$. Hence,

$$\begin{aligned} \measuredangle YCX &= \measuredangle YCA + \measuredangle ACP + \measuredangle PCB + \measuredangle BCX \\ &= \measuredangle ACP + \measuredangle ACP + \measuredangle PCB + \measuredangle PCB \\ &= 2 \cdot (\measuredangle ACP + \measuredangle PCB) = 2 \cdot \measuredangle ACB. \end{aligned}$$

We infer that $\measuredangle Y\Phi X = \measuredangle YCX$, and Φ lies on the circumcircle of triangle CXY. Similarly, Φ lies on the circumcircles of triangles AYZ and BZX. Corollary 3 is proven.

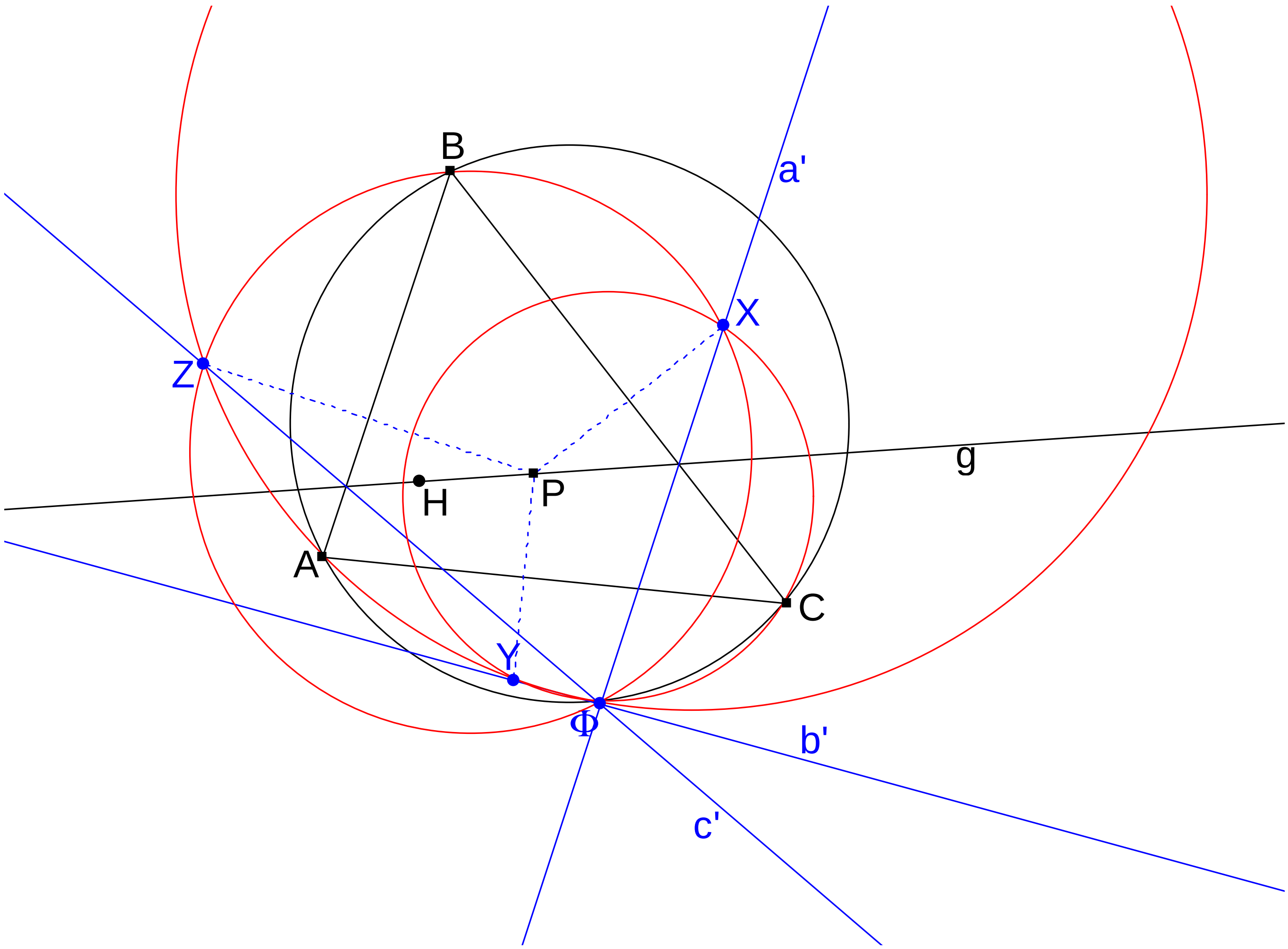

Fig. 5

3. Any line through the orthocenter H has an Anti-Steiner point. Inasmuch as the most familiar lines through H are the altitudes h_a, h_b, h_c from A, B, C and the Euler line e of triangle ABC, I will mention their Anti-Steiner points now.

- The Anti-Steiner point of the altitude h_a is the vertex A, since the line h_a passes through A, and hence its reflections in CA and AB pass through A, too, i. e. the three reflections concur in A. Analogously, the Anti-Steiner points of the altitudes h_b and h_c are B and C.

- The Anti-Steiner point of the Euler line e of ΔABC is a remarkable point of the triangle. In Clark Kimberling's ETC [6], it is the triangle center X_{110}, with trilinear coordinates
$$X_{110}\left(\frac{a}{b^2-c^2} : \frac{b}{c^2-a^2} : \frac{c}{a^2-b^2}\right) = X_{110}\left(\csc(\beta-\gamma) : \csc(\gamma-\alpha) : \csc(\alpha-\beta)\right).$$
This point X_{110} is the focus of the Kiepert parabola and can also be called the **Euler reflection point** of triangle ABC. Hence, we can state the following result:

The reflections of the Euler line of a triangle in the sidelines concur at one point on the circumcircle of the triangle. It is called the **Euler reflection point** of the triangle.
Moreover, applying Corollary 3 with the Euler line e as g and the circumcenter of triangle ABC as P, we obtain the following:
If X, Y, Z are the reflections of the circumcenter of a triangle ABC in the sidelines BC, CA, AB, then the Euler reflection point of ΔABC lies on the circumcircles of triangles AYZ, BZX, CXY.

References

[1] S. N. Collings: *Reflections on a triangle 1*, Mathematical Gazette 1973, pages 291-293.

[2] M. S. Longuet-Higgins: *Reflections on reflections 1*, Mathematical Gazette 1974, pages 257-263.

[3] Eberhard M. Schröder: *Ein neuer Winkelbegriff für die Elementargeometrie?*, Praxis der Mathematik 9/1982, pages 257-269.

[4] J. v. Yzeren: *Pairs of Points: Antigonal, Isogonal, and Inverse*, Mathematics Magazine 5/1992, pages 339-347.

[5] R. A. Johnson: *Advanced Euclidean Geometry*, New York 1960.

[6] Clark Kimberling: *Encyclopedia of Triangle Centers*, `http://faculty.evansville.edu/ck6/`