Linearity of Expectation and the Probabilistic Method

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1 Linearity of Expectation

1. What is the expected number of fixed points of a permutation of $(1, 2, \cdots, n)$ chosen uniformly at random from the set of all permutations of $(1, 2, \cdots n)$?

2. What is the expected number coin flips required to get a head if the probability of getting a head on a flip is p?

3. What is the expected number of times you have to pick a number with replacement from the set $\{1, 2, 3\}$ uniformly at random until you see every number at least once?

4. Generalize the previous problem. This is known as the coupon collector's problem.

5. You and your friend flip a (fair) coin until you get a HH (in which case you win) or your friend gets a HT (in which case your friend wins). Is this a fair game? (This is not a trick question. The answer really is as simple as it seems.)

6. Same rules but this time you both have your own *separate* coin. You're racing to see who can get their desired sequence first. Is this still a fair game?

   - What is the expected number of flips required to get a HT?
   - What is the expected number of flips required to a get a HH? Can you give an intuitive explanation as to why the answer to this question and the previous one is different?

7. Let $p_k(n)$ be the number of permutations of $(1, 2, \cdots, n)$ with exactly k fixed points. What is $\sum_{k=0}^{n} k^3 p_k(n)$?

2 The Probabilistic Method

1. Prove that any spanning graph of $K_{n,n}$ with at least $n^2 - n + 1$ edges has a perfect matching. Is $n^2 - n + 1$ tight? Can you find a spanning subgraph of $K_{n,n}$ with $n^2 - n$ edges with no perfect matching?

2. (IMC 2002) 200 participants take part in a math test consisting of six problems. If every problem is solved by at least 120 participants, show that there exists a pair of participants such that each problem is solved by at least one of the participants from the pair.

3. (IMO Shortlist 1987) Show that it is possible to color the elements of the set $\{1, 2, \cdots, 1987\}$ using four colors so that there does not exist a monochromatic arithmetic progression of length ten.

4. (Russia 1996, paraphrased) Let $A = \{1, 2, \cdots, 1600\}$ and $A_1, A_2, \cdots, A_{16000}$ are all subsets of A with $|A_i| = 80$ for $1 \leq i \leq n$. Show that there exist i, j with $i \neq j$ such that $|A_i \cap A_j| \geq 4$.

5. (Iran TST 2006) Suppose 799 teams participate in a tournament in which every pair of teams plays against each other exactly once. Prove that there exist two disjoint groups A and B of 7 teams each such that every team from A defeated every team from B.

6. Show that there exists a tournament on n vertices with at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

7. A tournament is said to have property S_k if given any set of k vertices, there exists someone that beats everyone in that set. Show that tournaments with property S_k exist for every k.

8. A *dominating set* of a graph is a subset D of the vertices of the graph such that for every vertex v, either $v \in D$ or some neighbor of v is in D. The size of the smallest dominating set of a graph is called its domination number (which is usually denoted by γ). Deduce an upper bound for the domination number of a graph with minimum degree δ. First, construct a random subset S of the vertices by selecting each vertex independently with probability p. Then for every vertex that is not in S, check if any of its (at least δ) neighbors are in S. If not, put it in a set T. $S \cup T$ is a dominating set. What is its expected size?

9. In a class, each boy is friends with at least one girl. Show that there exists a group of at least half of the students, such that each boy in the group is friends with an odd number of girls in the group.

10. (USAMO 1985) (was left as an exercise) There are n people at a party. Prove that there are two people such that, of the remaining $n - 2$ people, there are at least $\lfloor n/2 \rfloor - 1$ of them, each of whom knows both or else knows neither of the two. Assume that "know" is a symmetrical relation; $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

11. (not discussed in class) Here's a way to get a better lower bound on the largest independent set of a graph (in terms of its average degree). Take a random permutation $v_1, v_2, \cdots, v_n$ of the vertices. Construct a set S probabilistically as follows: insert v_i into S if none of its neighbors with a smaller index is in S. Is S an independent set? What is its expected size? You might have to use Jensen's inequality for this.

12. (some old Putnam problem) Let $P(z) = z^2 + az + b$ be a polynomial with complex coefficients. If $|z| = 1$, $|P(z)| = 1$. Show that a and b are necessarily zero.

3 Things to do

1. Please walk yourself through the solutions to the problems you saw today at some later time. That really helps in consolidating ideas.

2. Look up "Unexpected Uses of Probability". You should find a note by some guy named Ravi Boppana.

3. Look up "Expected Uses of Probability". You should find a note by Evan Chen.

4. Google `po shen loh probabilistic method`. You should find a note with lots of citations.

5. If you really want to learn the probabilistic method, read Noga Alon's book.

6. Buy our book. Help us beat "HSC Hacks: Short-cut Suggestion".