A CIRCLE TANGENT TO THE INCIRCLE

FIRST SYNTHETIC PROOF [1]

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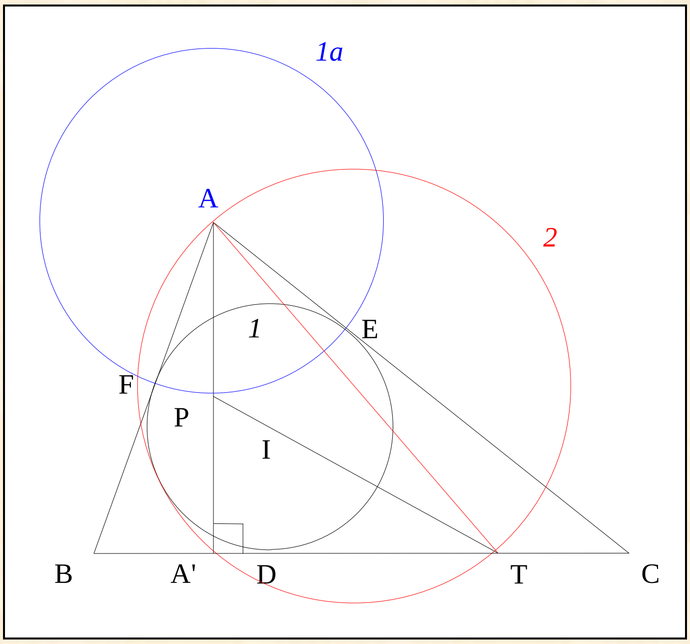
Jean - Louis AYME



Abstract. We present the first synthetic proof of a particular circle tangent to the incircle of a triangle.
The figures are all in general position and all the theorems quoted can all be proved synthetically.

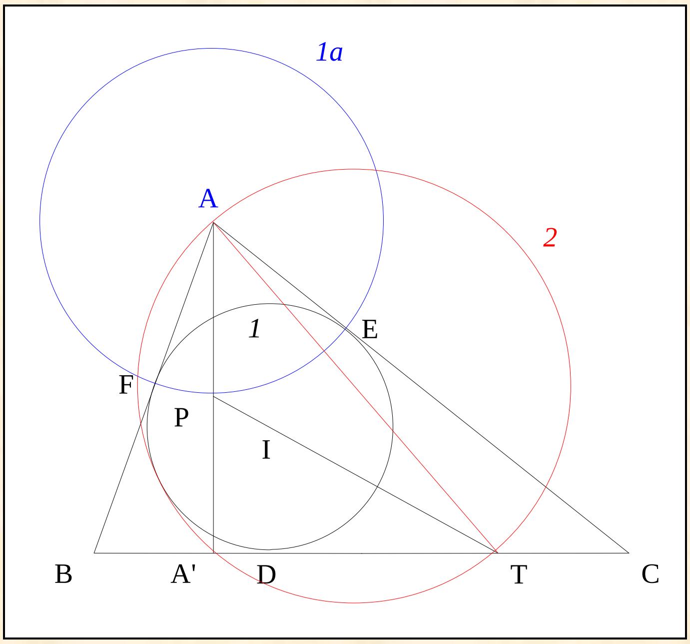
Sommaire

[1] 24/09/2010.

A. THE PROBLEM

VISION

Figure :



Features : ABC a triangle,
1 the incircle of ABC,
I the center of *1*,
DEF the contact triangle of ABC,
A' the foot of the A-altitude of ABC,
1a the cercle of center A, passing through E,
P the point of intersection in ABC of AA' with *1a*,
T the point of intersection of PI and BC
and *2* the circle with diameter AT.

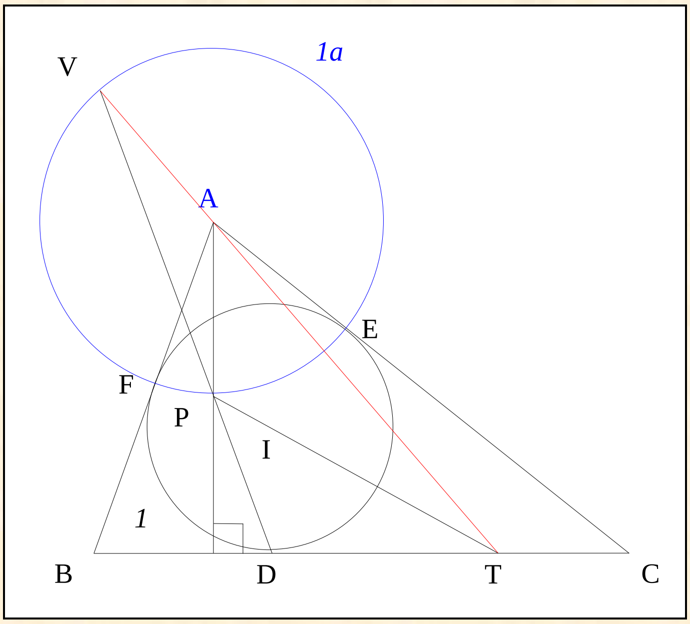
Given : *2* is tangent to *1*.[2]

[2] Tangent circles, Art of Problem Solving (17/09/2010) ; http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=367279.

B. A LEMMA AND ITS CONSEQUENCES

VISION

Figure :

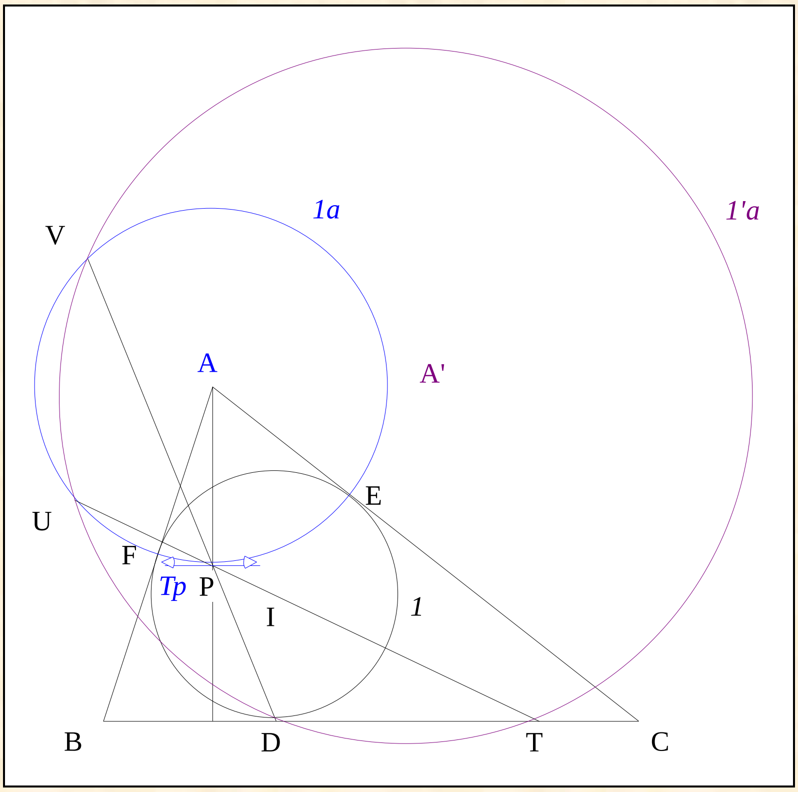


Features :	ABC	a triangle,
	2	the incircle of ABC,
	I	the center of *1*,
	DEF	the contact triangle of ABC,
	1a	the circle of center A, passing through E,
	P	the point of intersection inside ABC of the A-altitude of ABC with *1a*,
	T	the point of intersection of PI and BC,
and	V	the second point of intersection of PD with *1a*.

Given : V, A and T are collinear. [3]

VISUALIZATION

[3] Ayme J.-L., Three collinear points, Art of Problem Solving (22/09/2010) ; http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=367758.

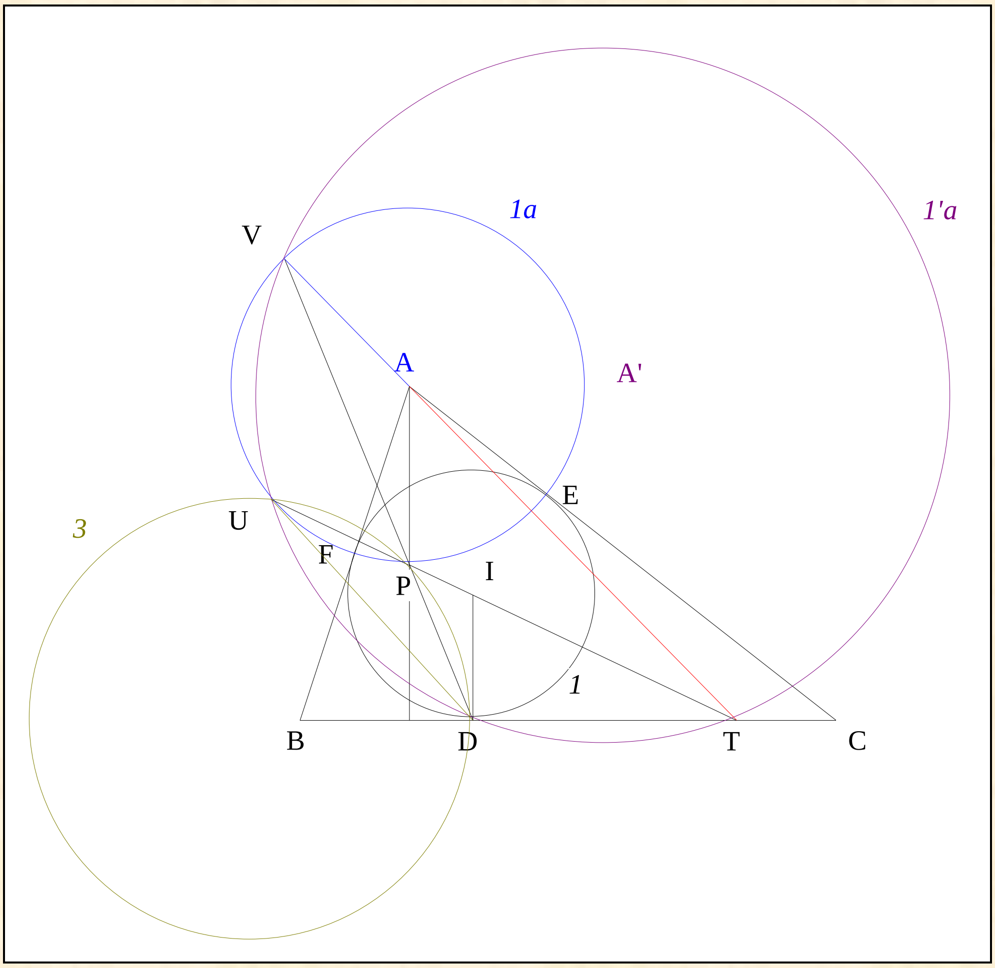


- Note U the second point of intersection of PI with *1a*.
  and *Tp* the tangent to *1a* at P.

- **Remark :** *Tp* // TD.

- **Partial conclusion :** the circle *1a*, the basic points U and V, the monians PUT and PVD, the parallels *Tp* and TD, lead to the Reim's theorem **1"** ;
  in consequence, U, V, T and D are concyclic.

- Note *1'a* this circle
  and A' the center of *1'a*.



- Note $P_{1a}(\mathrm{I})$ the power of I wrt *2*.

- We have : $P_{1a}(\mathrm{I}) = \overline{IP}.\overline{IU} = \mathrm{IE}^2 = \mathrm{ID}^2$;
  in consequence, the circle passing through U, P and D is tangent to ID at D.

- Note *3* this circle.

- An angle chasing modulo 2. π

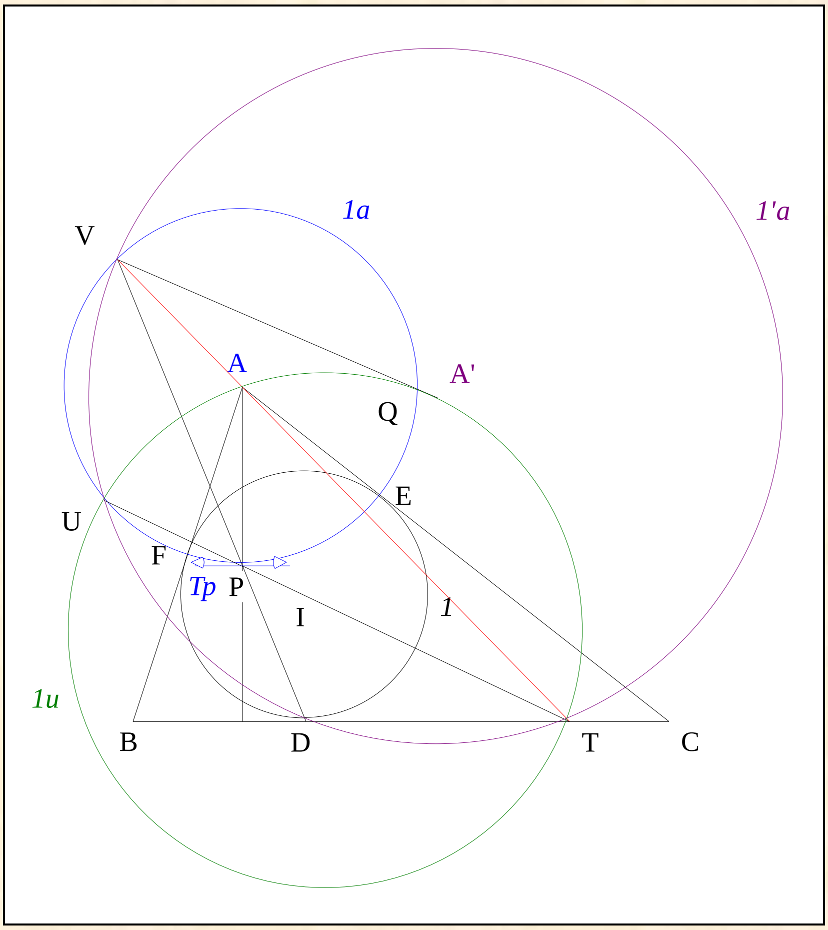
  according to "The chordal angle theorem", <DVT = <DUT ;
  according to another writing, <DUT = <DUP
  according to "The tangent-chordal angle theorem", <DUP = <IDP ;
  according to "Angles with parallel sides", <IDP = <APV ;
  the triangle APV being A-isoceles, <APV = <PVA ;
  according to another writing, <PVA = <DVA ;
  according to the transitivity of the relation =, <DVT = <DVA ;

  in consequence, VT // VA ;
  according to the Euclide's postulate, VT = VA.

- **Conclusion :** V, A and T are collinear.

Historical note : the proof above is inspired from those of the Russian "skytin" and the Greek Architect Kostas Vittas.

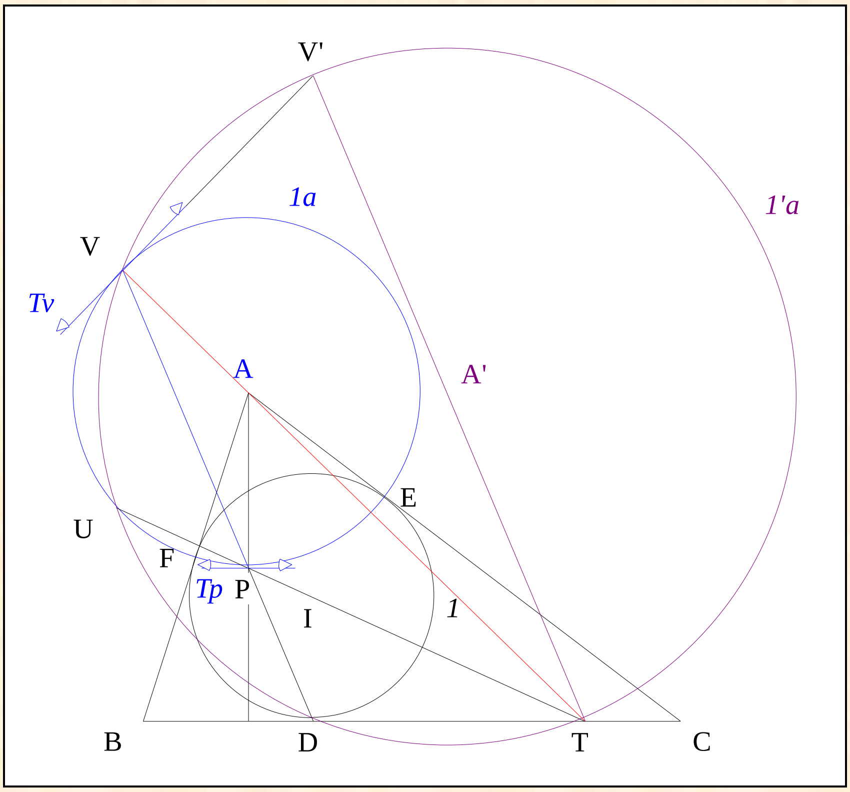
Consequences : **(1)** a Morley's circle



- Note Q the second point of intersection of VA' with *1a*.
- **Conclusion :** according to "The Morley's circle" [4] applied to *1a* and *1'a*, U, A, A', T and Q are concyclic.
- Note *1u* this circle.

(2) A tangent

[4] Ayme J.-L., Les cercles de Morley…, G. G. G. vol. 2 ; http://jl.ayme.pagesperso-orange.fr/.



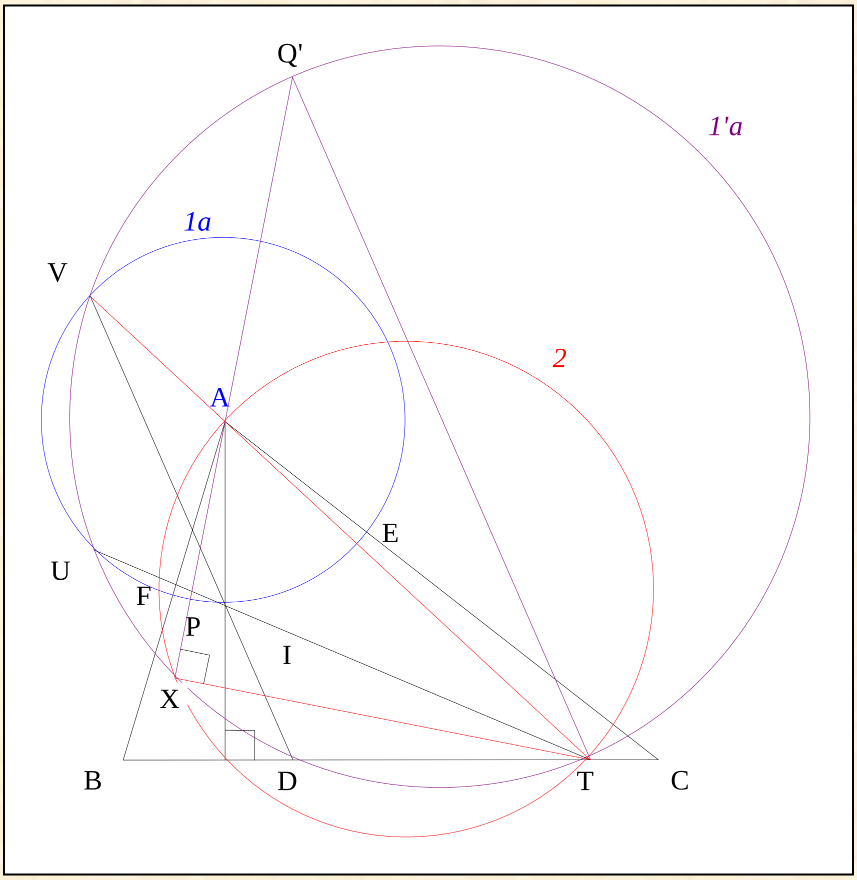
- Note *Tv* the tangent to *1a* at V
  and V' the second point of intersection of *Tv* with *1'a*.

- The circles *1a* and *1'a*, the basic points U and V, the monians PUT and VVV',
  lead to the Reim's theorem **0** ; in consequence, PV // TV'.

- **Remark :** VV' ⊥ YT.

- **Conclusion :** according to Thalès "Inscribed triangle in a half circle", TV' is a diameter of *1'a*.

(3) Three collinear points

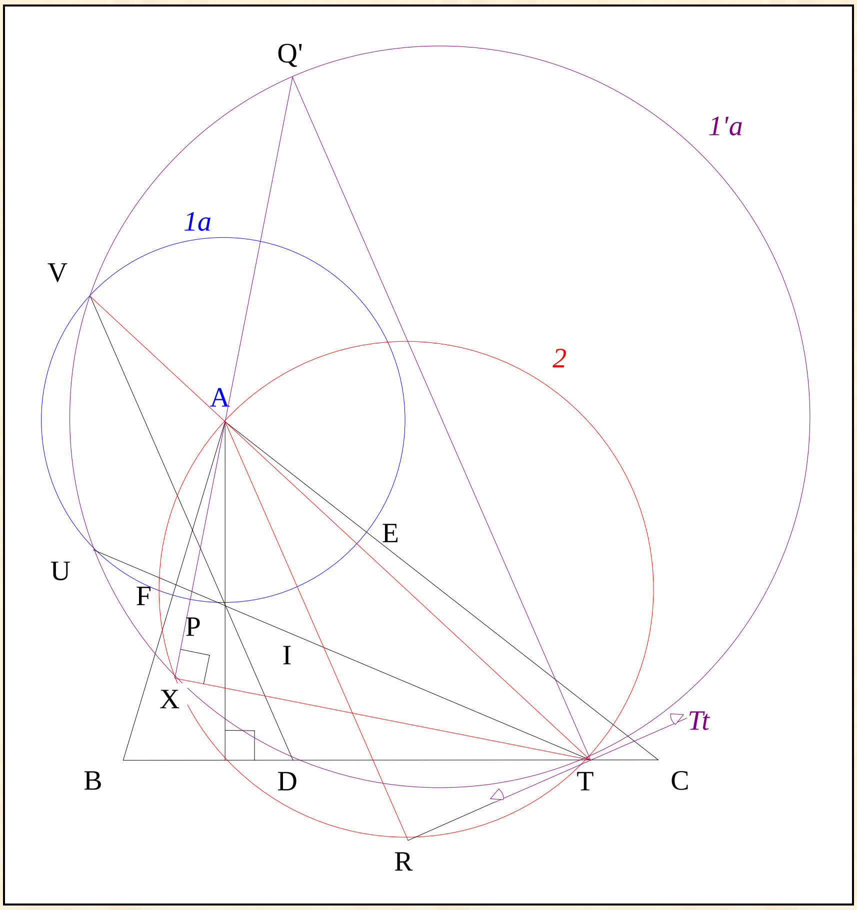


- Note 2 the circle with diameter AT
  and X the second point of intersection of *1'a* and *2*.

- According to Thalès "Triangle inscriptible in a half circle", XA ⊥ XT ;
  XT ⊥ XQ' ;
  according to the axiom IVa of perpendicularity, XA // XQ' ;
  according to Euclide's postulate, XA = XQ'.

- **Conclusion :** X, A and Q' are collinear.

(4) Again a parallel

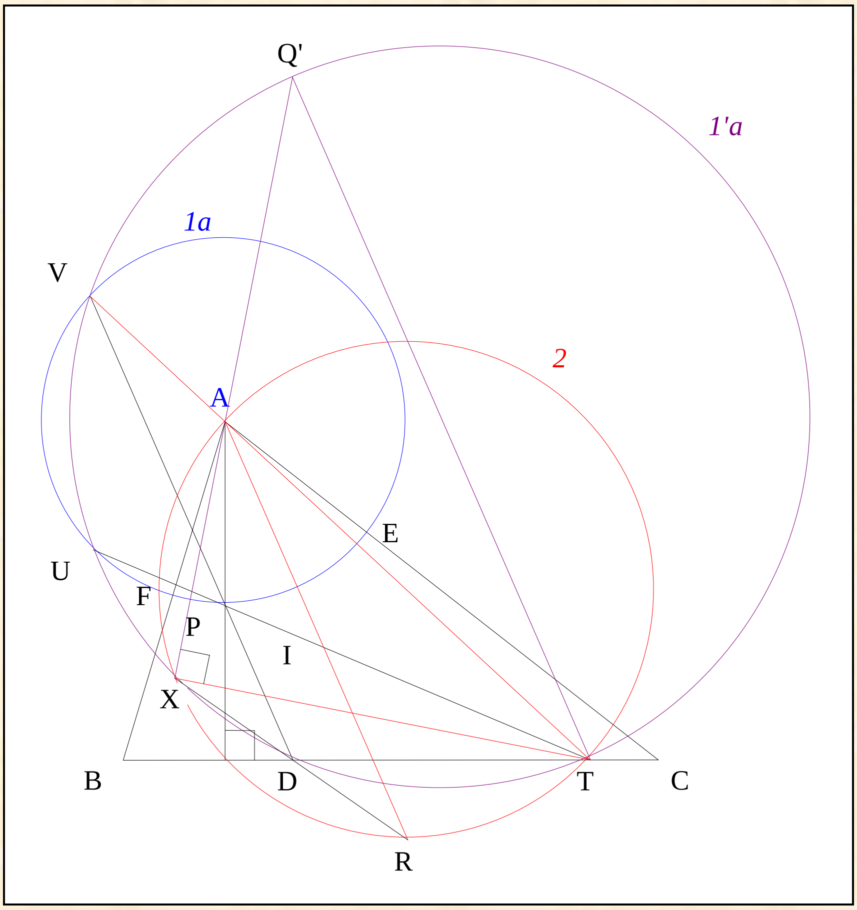


- Note *Tt* the tangent to *1'a* at T
  and R the second point of intersection of *Tt* with *2*.

- The circles *2* and *1'a*, the basic points X and T, the monians AXQ' and RTT, lead to the Reim's theorem **3** ; in consequence, AR // Q'T.

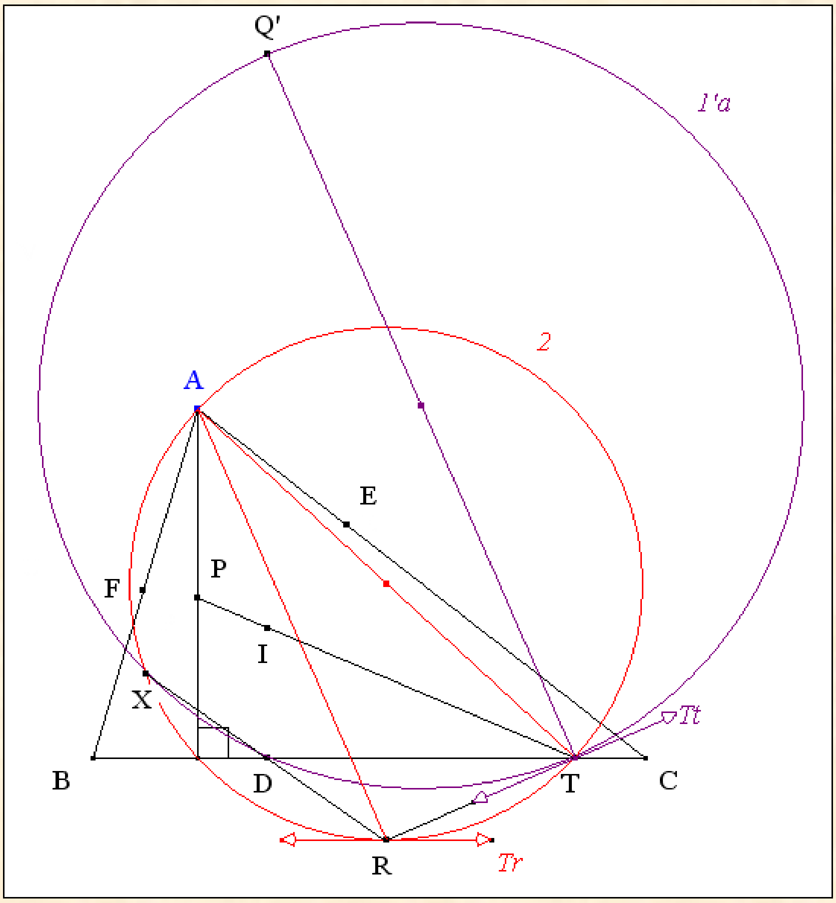
- **Conclusion :** DV, Q'T and AR are together parallel.

(5) Again three collinear points



- **Conclusion :** the circles *1'a* and *2*, the basic points X and T, the monian VTA, the parallels VD and AR, lead to the Reim's theorem **3'** ; in consequence, D, X and R are collinear.

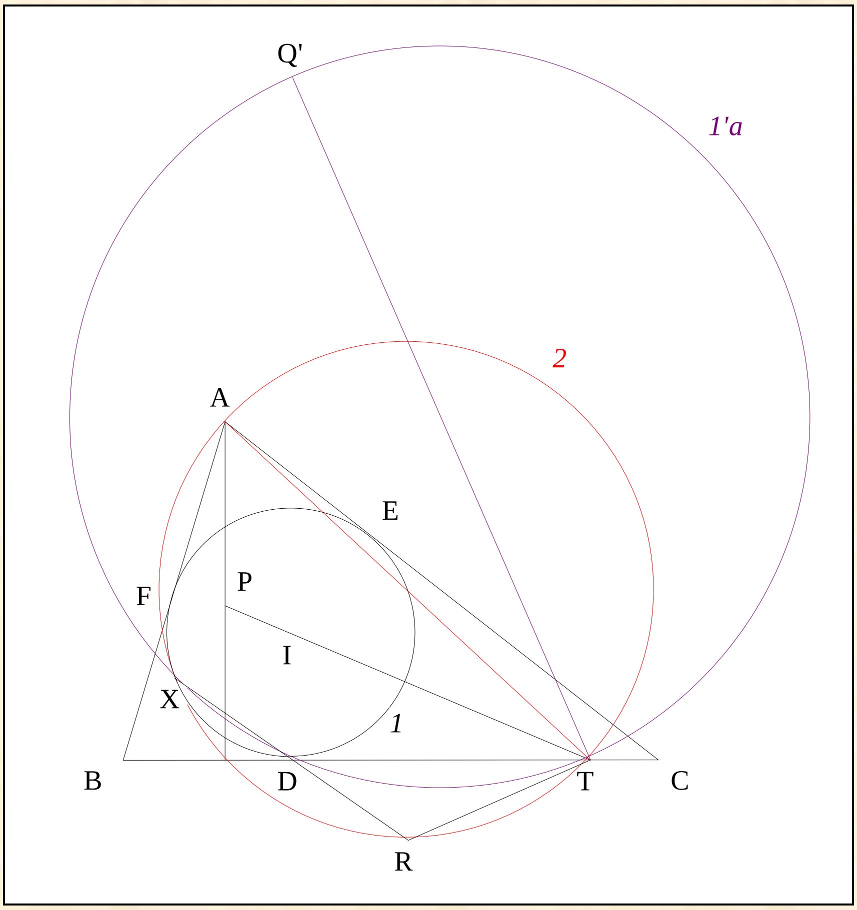
(6) A parallel tangent



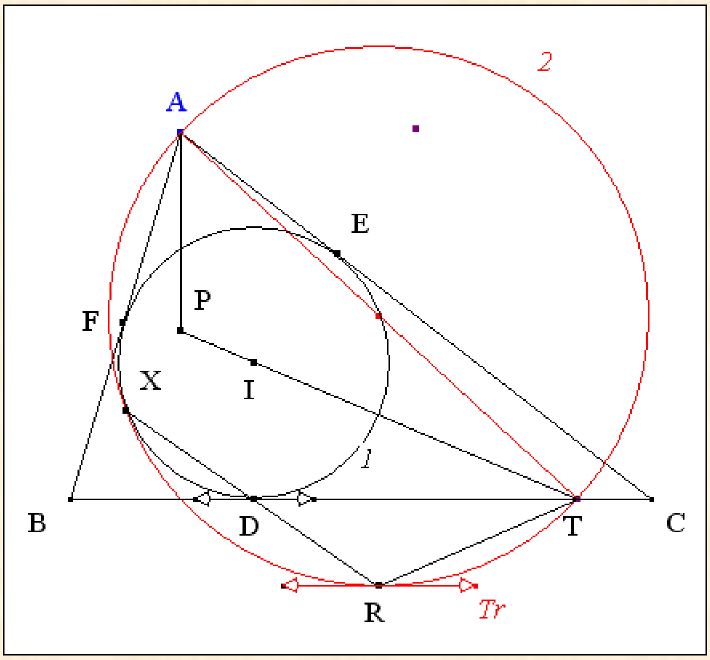
- Note *Tr* the tangent to *1'a* at R.

- **Conclusion :** the circles *2* and *1'a*, the basic points X and T, the monians RXD and RTT, lead to the Reim's theorem **2** ; in consequence, *Tr* // DT.

C. THE FIRST SYNTHETIC PROOF



- According to the pivot theorem applied to the triangle DTR with D on DT, T on TR and X on RD, *1*, *1'a* and *2* are concurrent at X.



- **Remark :** *Tr* // BC.

- **Conclusion :** the circles *2* and *1*, the basic points X, the monian RXD, the parallels *Tr* and BC, lead to the Reim's theorem **8** ; in consequence, *2* is tangent to *1* at X.

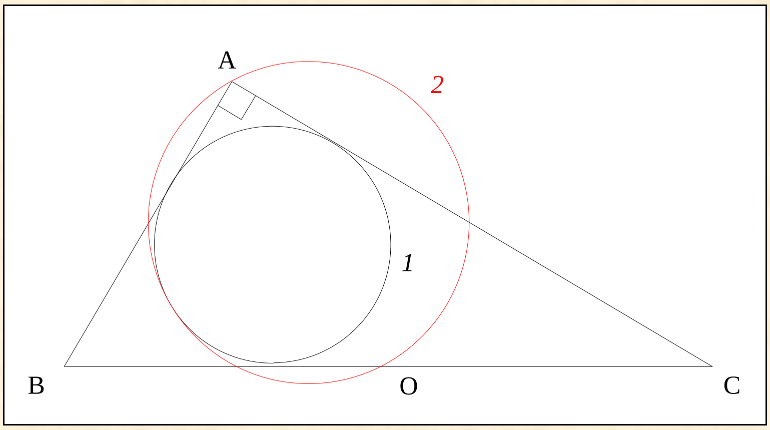
Historical note : this problem proposed on the site *Art of Problem Solving* was found by the Russian "skytin" during the resolution of another problem.

Comment : this result can also be in relation with a remarkable problem of Vladimir Protassov[5]. Consequently, the circle *1'a* goes through the B-excenter of the triangle BTA.

D. A PARTICULAR CASE

VISION

Figure :



Features : ABC a right triangle at A,
1 the incircle of ABC,
O the midpoint of BC
and *2* the circle with diameter AO.

Given : *2* is tangent to *1*.[6]

Comments : directly, *2* is the Euler's circle of ABC and is tangent to the incircle at the Feuerbach's point according to the Feuerbach's theorem[7].
Indirectly, we can made a link with our general problem : the line joining O to the incenter I of ABC passes through our point P.

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5 Ayme J.-L., Un remarquable résultat de Vladimir Protassov, G.G.G. vol. 2 ; http://perso.orange.fr/jl.ayme.
6 Ayme J.-L., Two tangent circles, Art of Problem Solving (25/09/2010) ; http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=368116.
7 Ayme J.-L., Le théorème de Feuerbach, G.G.G. vol. 1 ; http://perso.orange.fr/jl.ayme.