33^{rd} Iranian Mathematical Olympiad

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Selected Problems with Solutions

This booklet is prepared by Goodarz Mehr, Mohammad PourMohammadi and Hesameddin Rajabzadeh.
With special thanks to Yeganeh Alimohammadi, Sepideh Azhdari, Mohsen Jamaali, Erfan Salavati and Mojtaba Zare.

Iranian Team Members at the 57^{th} IMO (Hong Kong - Hong Kong)

From left to right:

- Mohammad Reza Aminian
- Farbod Ekbatani
- Amin Rakhsha
- Amirmojtaba Sabour
- Mahbod Majid
- Alireza Hediehloo

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Preface

The 33$^{\text{rd}}$ Iranian National Mathematical Olympiad consisted of four rounds. The First Round was held on 20$^{\text{th}}$ January 2015 nationwide. The exam consisted of 30 multiple-choice questions and a time of 3.5 hours. In total, more than 10000 students participated in the exam and more than 1500 of them were admitted for participation in the next round.

The Second Round was held on 7$^{\text{th}}$ and 8$^{\text{th}}$ May 2015. In each day, participants were given 3 problems and 4.5 hours to solve them. After this round, the top 63 students were selected to participate in the Third Round.

The examination of the Third Round consisted of four separate exams, and a Final Exam with 8 problems and a specified time for each problem. At the end of this round, 26 students were awarded a bronze medal, 24 students were awarded a silver medal, and the top 13 students were awarded a gold medal. The following list represents the names of the gold medalists:

1. Morteza Abolghasemi
2. Mohammad Reza Aminian
3. Farbod Ekbatani
4. Alireza Hediehloo
5. Mahbod Majid
6. Arad Nassiri
7. Amin Rakhsha
8. Farhoud Rostamkhani
9. Amirmojtaba Sabour
10. Yousef Shakiba
11. Sina Taslimi
12. Soroush Taslimi
13. Hossein Zakerinia

The Team Selection Test was held on 6 days, having the same structure as the International Mathematical Olympiad (IMO). In the end, the top 6 participants were selected to become members of the Iranian Team at the 57$^{\text{th}}$ IMO.

In this booklet, we present the 6 problems of the Second Round, the 8 problems of the Final Exam of the Third Round, and 7 proposed problems of the Team Selection Test, together with their solutions.

It's a pleasure for the authors to offer their grateful appreciation to all the people who have contributed to the conduction of the 33$^{\text{rd}}$ Iranian Mathematical Olympiad, including the National Committee of Mathematics Olympiad, problem proposers, problem selection groups, exam preparation groups, coordinators, editors, instructors and all those who have shared their knowledge and effort to increase the Mathematics enthusiasm in our country, and assisted in various ways to the conduction of this scientific event.

Problems

Second Round

1. (Omid Naghshineh Arjmand) Arash and Bahram have divided a circular cake into inequal pieces. Initially, Arash can take one of the pieces. Then, Bahram can only take one of the two pieces next to the previous piece, and moving forward in this manner, each person in his turn can take one of those pieces of the cake that at least one of its neighboring pieces has already been taken. Prove that if initially the cake is divided into five pieces and Arash knows the weight of each piece, he can play in such a way that he gets at least half of the cake by the end of the game. (→ p.17)

2. (Ali Khezeli, Erfan Salavati) A special computer can store algebraic expressions in its memory, which is unlimited and initially only expression x is stored in it. This computer can perform the following operations:

- If algebraic expression f is stored in the computer's memory, $\frac{1}{f}$ can be stored in it too (assuming that f is not equivalent to zero).
- If algebraic expressions f and g are stored in the computer's memory, $f+g$ and $f-g$ can be stored in it too (f and g can be equal).

For example, the following expressions can be stored in the computer's memory: $\frac{1}{x}$, $x-\frac{1}{x}$, $\frac{1}{x-\frac{1}{x}}$, $\frac{1}{x-\frac{1}{x}}+\frac{1}{x}$ and ... .

Find all natural numbers n for which x^n or an expression equivalent to it can be stored in the computer's memory (two algebraic expressions on variable x are called equivalent if for each value of x belonging to the domain of both expressions, they are equal). (→ p.18)

3. (Amir Saeedi) A circle passing through vertices B and C of triangle ABC intersects sides AC and AB at points D and E, respectively. If P is the intersection point of BD and CE, H is the foot of the perpendicular

line from P to AC and M and N are the midpoints of BC and AP, prove that triangles MNH and CAE are similar. ($\rightarrow$ p.19)

4. (Ali Zamani) In quadrilateral $ABCD$, AC is the bisector of angle A and $\angle ADC = \angle ACB$. X and Y are the feet of the altitudes from A to BC and CD, respectively. Prove that the orthocenter of triangle AXY is located on line BD (A triangle's orthocenter is the intersection point of its altitudes). ($\rightarrow$ p.20)

5. (Amir Hossein Gorzi) The perimeter of a circle is divided by $2n$ points into $2n$ equal sections. $n+1$ intervals with lengths of $1, 2, \ldots$ and $n+1$ have been placed on this circle with their endpoints from those $2n$ points. Prove that one of these intervals is completely inside another one. ($\rightarrow$ p.20)

6. (Mohsen Jamali) $n \geq 50$ is a natural number. Prove that n can be written as the sum of two natural numbers whose prime factors do not exceed $\sqrt{n}$. For example, 94 can be written as $80 + 14$, none of which has a prime factor greater than $\sqrt{94}$. ($\rightarrow$ p.21)

Third Round

1. (Erfan Salavati, Mostafa Eynollahzadeh) **Co-Algebraic Sequences!**

Suppose that $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two sequences of real numbers. These sequences are said to be Co-Algebraic if a non-zero two-variable polynomial $P(x, y)$ with real coefficients exists such that for each natural number n, $P(a_n, b_n) = 0$.

a) Prove that sequences n and 2^n (for each natural number n) are not Co-Algebraic.

b) Are sequences 2^n and 3^n (for each natural number n) Co-Algebraic?

c) Suppose that $f(x, y)$ is a non-zero two-variable polynomial with real coefficients. Prove that there exists a natural number n such that $f(2^n, 3^n)$ is not divisible by 5^n.

150 minutes (→ p.23)

2. (Morteza Saghafian) **Repeating Functions, Function Repetition!**

Let $f_1 : \mathbb{N} \to \mathbb{N}$ be a function. Function f_2 is constructed from f_1 such that for each $k \in \mathbb{N}$, $f_2(k)$ is the number of times k has appeared in the range of f_1, i.e. $f_2(k)$ is the number of natural numbers n for which $f_1(n) = k$.

Suppose that the range of f_1 is such that f_2 can be defined over all natural numbers, so no number appears in the range of f_1 an infinite or zero number of times. Similarly, function f_3 can be constructed from function f_2 and consequently for each natural number i, function f_{i+1} can be constructed from function f_i.

a) Prove that for each natural number T, there exists a function f_1 such that for each natural number i, f_i is definable and also the sequence $f_1, f_2, f_3, \ldots$ is periodic with period T, i.e. T is the smallest number such that $f_{T+1} = f_1$.

b) Does there exist a function f_1 such that for each natural number i, f_i is definable and these functions are mutually unequal?

60 minutes (→ p.24)

3. (Morteza Saghafian) **Perpendicular Tetrahedra!**

Two line segments in space are called perpendicular if they intersect and are perpendicular to each other at the intersection point.

a) Prove that there exist two regular tetrahedra such that each edge of one is perpendicular to an edge of the other.

b) For an arbitrary regular tetrahedron, prove that there exists a unique tetrahedron with the above property.

90 minutes (→ p.26)

4. (Mostafa Eynollahzadeh) **Computable Functions!**

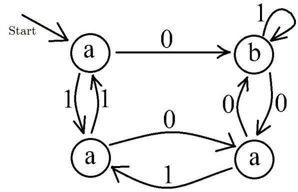
In this question, functions on natural numbers that are computable by a computer (which has a finite memory and output) for arbitrary large numbers are investigated. These functions are called Computable Functions. Since a computer has a finite number of outputs, the investigation is restricted only to those functions that have a finite range (which is equivalent to all possible finite outputs of a computer). Next, the method by which a number is given to a computer needs to be clarified. A common way is using the representation of numbers in a fixed base k.

By determining the inputs and outputs of a computer, a mathematical model can be derived to describe (i.e. to model) its operation. The main idea here is that a computer has a finite number of states (considering the states of all inner components) and it changes after receiving each input digit according to a predefined program. Consequently, the output is a function of the computer's state. Using these concepts a precise definition of a k-Computable sequence can be given.

First, a Machine needs to be defined. A Machine with input set A and output set B (sets A and B are finite) comprises a finite number of states and an algorithm that determines the state of the Machine after an input (which is an arbitrary member of A) is entered. Also, each state of the Machine determines a member of B as the output. Moreover, a specific state, called the Starting State, is used to start the Machine. A Machine of this kind can determine a member of B as the output for a finite sequence of entries from A. This can be done by setting the Machine to its Starting State and then entering the inputs from the sequence one by one. The output of the final state is considered as the computed value.

Let k be a natural number greater than 1. A sequence $a_1, a_2, a_3, \ldots$ with values in the finite set B is called k-Computable if there exists a Machine with input set $\{0, 1, \ldots, k-1\}$ and output set B such that when digits of an arbitrary number n in base k are entered to the Machine from right to left, respectively, the Machine's output is a_n.

As an example, consider the following Machine for $k = 2$ (circles represent states and the output of each state is written on it. This Machine receives 0 and 1 as inputs).



Using this Machine the sequence $a, b, a, a, a, b, a, b, \dots$ can be computed. For example, for $n = 10 = (1010)_2$, 0, 1, 0 and 1 should be given to the Machine starting from its Starting State. If this is done, the lower left state is reached which gives a as the output. According to the definition, the sequence $a, b, a, a, a, b, a, b, \dots$ is 2-Computable.

a) Suppose m is a natural number. For each natural number n, define (a_n) as the remainder of n when divided by m (so values of this sequence are all in the finite set $\{0, 1, \dots, m-1\}$). Prove that for each natural number k, this sequence is k-Computable.

b) Prove that if sequence (a_n) is k-Computable, so are sequences (b_n) and (c_n) that are defined as $b_n = a_{n+1}$ and $c_n = a_{2n}$.

c) Define sequence (a_n) to be 1 for perfect square natural numbers and 0 for the rest. Is this sequence 3-Computable?

d) The definition of Computable sequences can be altered by assuming that the digits are given to the Machine from left to right. These sequences are called k-Left Computable. Prove that this definition is equivalent to the previous one, i.e. a sequence is k-Computable if and only if it is k-Left Computable.

150 minutes ($\rightarrow$ p.28)

5. (Morteza Saghafian) **Ugly Duck!**

Ali is given a piece of paper shaped like an equilateral triangle. He cuts the paper into two pieces with one cut. Then, he puts the pieces arbitrarily on a table and cuts them both with one cut to get four pieces of paper. Lastly, he puts all four pieces on the table and cuts them all with one cut to get eight pieces. In the end, he will have eight pieces of paper using three cuts.

Ali wants to do this in such a way that in the end, seven pieces are equal to each other but not to the last one.

a) Prove that if seven pieces are equal, then they are either triangles or quadrilaterals.

b) Prove that the seven equal pieces cannot be quadrilaterals.

c) Is it possible for the seven equal pieces to be triangles and not equal to the last one?

90 minutes ($\rightarrow$ p.31)

6. (Yeganeh Alimohammadi, Sepideh Azhdari) **Rookland!**

For two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in $\mathbb{R}^2$, their distance can be defined as,

$$d(A, B) = |x_1 - x_2| + |y_1 - y_2|.$$

a) The perpendicular bisector of two points in the plane is the locus of points equally spaced from those two points. Determine the perpendicular bisector of two arbitrary points according to the new definition.

b) The Apollonius circle of two points in the plane is the locus of points for which the ratio of their distances to those two points is equal to a constant value m $(m \neq 1)$. Determine the Apollonius circle of two arbitrary points according to the new definition.

c) The distance from point A to line l is defined as

$$\min\{d(A, P) | P \in l\}.$$

In Euclidean Geometry, a parabola in the plane is the locus of points equally distanced from a point and a line. Determine how a parabola will look like according to the new definition.

d) Let ABC be a triangle in the plane, and X_a a point on side BC such that,

$$d(A, B) + d(B, X_a) = d(A, C) + d(C, X_a).$$

X_b and X_c are defined similarly. Are AX_a, BX_b and CX_c concurrent?

e) Is it possible to have an infinite number of points in the plane such that their mutual distances are squares of natural numbers?

90 minutes (→ p.32)

7. (Erfan Salavati) **Almost Symmetric Coloring!**

Vertices of a regular n-gon have been colored blue and red such that for each rotation of the n-gon, the number of vertices that have different colors before and after the rotation is less than 32 percent of the number of vertices. Prove that either the number of blue vertices or the number of red vertices is less than 20 percent of the number of vertices, n.

60 minutes (→ p.36)

8. (Ali Golmakani) **The Lord of the Rings!**

Three non-co-planar rings are located in the space.

a) Is it always possible to find a circle that passes through all three circles?

b) Is it always possible to find a square that passes through all three circles?

60 minutes (→ p.37)

Team Selection Test

1. (Navid Safaei) Suppose a, b, c and d are positive real numbers such that $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} = 2$. Prove that

$$\sqrt{\frac{a^2+1}{2}} + \sqrt{\frac{b^2+1}{2}} + \sqrt{\frac{c^2+1}{2}} + \sqrt{\frac{d^2+1}{2}} \geq 3(\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}) - 8.$$

(→ p.39)

2. (Omid Naghshineh Arjmand) Suppose that a council consists of five members and that decisions in this council are made according to a method based on the positive or negative vote of its members. The method used by this council has the following two properties:

- **Ascension**: If the presumptive final decision is favorable and one of the opposing members changes his/her vote, the final decision will still be favorable.
- **Symmetry**: If all of the members change their vote, the final decision will change too.

Prove that the council uses a weighted decision-making method, i.e. nonnegative weights $w_1, w_2, \ldots, w_5$ can be assigned to members of the council such that the final decision is favorable if and only if sum of the weights of those in favor is greater than sum of the weights of the rest. (→ p.39)

3. (Morteza Saghafian) A real function has been assigned to every cell of an $n \times n$ table. Prove that a function can be assigned to each row and each column of this table such that the function assigned to each cell is equivalent to the combination of functions assigned to the row and the column containing it. (→ p.41)

4. (Hooman Fattahi) Points X and Y are located on sides AB and AC of triangle ABC ($X, Y \neq A$) such that the reflection of line BC with respect to XY is tangent to the circumcircle of triangle AXY. If O denotes the circumcenter of triangle ABC, prove that the circumcircle of triangle AXY is tangent to the circumcircle of triangle BOC. (→ p.41)

5. (Mahyar Sefidgaran, Mohyeddin Motevassel) $p \neq 13$ is a prime number in the form of $8k+5$ for some natural number k, and 39 is a non-residue modulo p. Prove that equation $x_1^4 + x_2^4 + x_3^4 + x_4^4 \overset{p}{\equiv} 0$ has a solution in the set of integers such that $p \nmid x_1 x_2 x_3 x_4$. (→ p.42)

6. (Ali Sayyadi) Let ABC be a triangle with altitudes AD, BF and CE. Let P and Q be two points on line EF such that $EP = DF$ (E is between P and F) and $QF = DE$ (F is between E and Q). Assuming that the perpendicular bisector of DQ intersects AB at X and the perpendicular bisector of DP intersects AC at Y, prove that the midpoint of BC lies on line XY. (→ p.43)

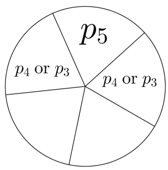
7. (Morteza Saghafian, Mohammad Firouzi) Let P and P' be two unequal regular n-gons and A and A' two points inside P and P', respectively. Suppose $\{d_1, d_2, \ldots, d_n\}$ are the distances from A to the vertices of P and $\{d'_1, d'_2, \ldots, d'_n\}$ are the distances from A' to the vertices of P'. Is it possible for $\{d'_1, d'_2, \ldots, d'_n\}$ to be a permutation of $\{d_1, d_2, \ldots, d_n\}$? (→ p.43)

Solutions

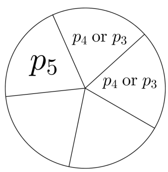
Second Round

1. First, label the pieces using $\{p_1, p_2, p_3, p_4, p_5\}$, such that the lightest piece is p_1 and the heaviest one is p_5. If Arash manages to get p_5 and p_4 or p_5 and p_3 then he automatically wins (i.e. gets at least half of the cake by the end of the game), since Bahram will have exactly two pieces with a total weight less than half the weight of the whole cake. Depending on the arrangement of the pieces one of the four cases below may occur. In each case, Arash has a winning strategy as follows:

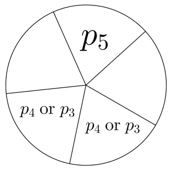
- p_5 is adjacent to both p_3 and p_4. For Arash to win, he should take p_5 in his first turn. In his next turn he should take p_3 or p_4, whichever is possible.



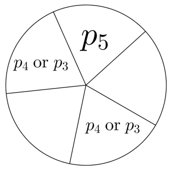
- p_5, p_4 and p_3 are adjacent and p_5 is not the middle one. Arash can win by employing a strategy similar to the previous case (i.e. first taking p_5 and then taking p_3 or p_4, whichever possible).



- p_5 is not adjacent to p_3 or p_4. Arash can win using the same strategy as before.



- p_5 and exactly one of the pieces p_3 or p_4 are adjacent.



  The winning strategy in this case depends on the total weight of pieces p_3 and p_4.

  If the total weight is less than half the weight of the whole cake, Arash can win by taking p_5, which means Bahram can at best take p_3 and p_4, which have a total weight less than half the weight of the whole cake.

  If the total weight is more than half the weight of the whole cake, Arash can win by taking the piece (from p_3 or p_4) which is not adjacent to p_5, and then in his upcoming turn, by taking either p_5 or the other piece (from p_3 or p_4).

By employing the strategy described above, Arash can win regardless of the cakes arrangement and Bahram's moves.

2. For simplicity, $\mathcal{S}$ is used to denote the set of all algebraic expressions that can be stored in the computer's memory. It is claimed that $x^n \in \mathcal{S}$ if and only if n is odd.

To prove one part of the assertion made above, it is claimed that all of the expressions in $\mathcal{S}$ are odd, i.e. $f(-x) = -f(x)$ for all expressions $f \in \mathcal{S}$. Since x is an odd expression; and since $f \pm g$ and $\frac{1}{f}$ are all odd expressions if f and g are odd, it can be concluded that every expression in $\mathcal{S}$ is odd. Using this claim and by noting that for even natural number n, x^n is even, it can be concluded that $x^n \notin \mathcal{S}$ for all even natural numbers n.

For the other part, it will be shown by induction on k that for every integer $k \geq -1$, $x^{2k+1} \in \mathcal{S}$.

The base cases $k = -1, 0$ are trivial ($\frac{1}{x}, x \in \mathcal{S}$). Suppose that $x^{2k-3}, x^{2k-1} \in \mathcal{S}$. The goal is to show that $x^{2k+1} \in \mathcal{S}$.

$$\left.\begin{array}{rcr} x^{2k-3}, x^{2k-1} \in \mathcal{S} & \Rightarrow & \frac{1}{x^{2k-3}+x^{2k-1}} \in \mathcal{S} \\ x^{2k-1} \in \mathcal{S} & \Rightarrow & \frac{1}{x^{2k-1}} \in \mathcal{S} \end{array}\right\} \Rightarrow$$

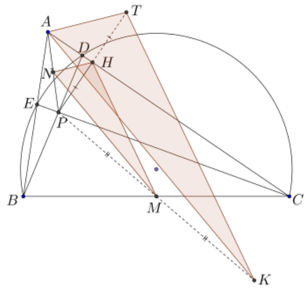
$$\frac{1}{x^{2k-1}} - \frac{1}{x^{2k-3}+x^{2k-1}} \in \mathcal{S},$$

$$\frac{1}{x^{2k-1}} - \frac{1}{x^{2k-3}+x^{2k-1}} \equiv \frac{1}{x^{2k-1}+x^{2k+1}} \in \mathcal{S}.$$

This means that $x^{2k-1}+x^{2k+1} \in \mathcal{S}$ and since $x^{2k-1} \in \mathcal{S}$, it is concluded that $x^{2k+1} \in \mathcal{S}$. This completes the induction process, proving that $x^n \in \mathcal{S}$ for all odd natural numbers n.

Remark. It can be proved that $\mathcal{S}$ is the set of all odd rational expressions in x with rational coefficients.

3. Let K and T be the reflections of P with respect to M and H, respectively. According to Thales' Theorem, triangles AKT and MNH are similar. On the other hand, for triangles ABD and AEC, $\angle EBD = \angle ECD$ and $\angle BAC$ appears in both triangles; therefore, these two triangles are similar. Now it suffices to prove that triangles AKT and ABD are similar, i.e. it must be shown that $\angle BAD = \angle KAT$ and $\frac{AB}{AD} = \frac{AK}{AT}$.



Note that these relations are equivalent to the similarity of triangles ADT and ABK. But T is the reflection of P with respect to line AC, so triangles ADT and ADP are congruent and it suffices to show that triangles APD and ABK are similar.

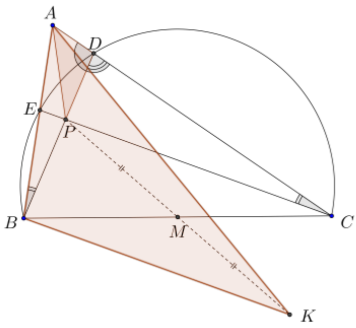
Since diameters of quadrilateral $BPCK$ bisect each other, it is a parallelogram. Thus $BK \parallel CP$. This implies that

$$\angle ABK = \angle AEC = 180^\circ - \angle BEC = 180^\circ - \angle BDC = \angle ADB.$$

Furthermore, since $BPCK$ is a parallelogram, $BK = CP$.

In order to complete the proof it has to be shown that

$$\frac{AD}{DP} = \frac{AB}{BK} = \frac{AB}{CP}.$$



To prove this, the law of sines can be used in triangles PDC and ABD. It is sufficient to prove that

$$\frac{\sin(\angle ADB)}{\sin(\angle ABD)} = \frac{\sin(\angle CDP)}{\sin(\angle DCP)}.$$

But $\angle ABD = \angle DCP$ and $\angle ADB = 180^\circ - \angle CDP$, which means the equation above holds, and this completes the proof.

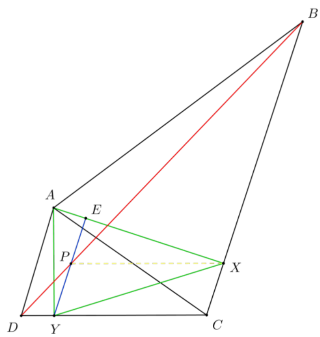
4. Let E be the foot of the perpendicular line from Y to AX and P be the intersection point of YE and BD. It is sufficient to prove that $XP \perp AY$, or equivalently, $XP \parallel CD$. Note that

$$YE \parallel CB \Rightarrow \frac{DY}{YC} = \frac{DP}{PB}. \qquad (1)$$

On the other hand, triangles ADC and ACB are similar, so

$$\frac{DY}{YC} = \frac{CX}{XB}. \qquad (2)$$

(1) and (2) show that $\frac{DP}{PB} = \frac{CX}{XB}$, and this implies $XP \parallel CD$, as desired.



5. Consider the interval with length 1. If this interval is inside another one the assertion is proved. Therefore, assume that no other interval contains it. Removing this interval from the circle (i.e. cutting the circle and transforming it into a line) changes the problem to this:

"n intervals $I_2 = [a_2, b_2], \ldots, I_{n+1} = [a_{n+1}, b_{n+1}]$ are inside the interval $[0, 2n-1]$ such that their endpoints are all integers and the length of I_j is j. Prove that there exist $i \neq j$ such that $I_i \subseteq I_j$."

Assume to the contrary that no interval is inside another one, and define function f as follows,

$$f : \{2, \ldots, n\} \to \Big(\{0, 1, \ldots, 2n-1\} - \{a_{n+1}, a_{n+1}+1, \ldots, b_{n+1}\}\Big).$$

- If $a_j < a_{n+1}$, define $f(j) := a_j$.
- If $a_j \geq a_{n+1}$, define $f(j) := b_j$.

Since no interval is completely inside I_{n+1}, for each $2 \leq j \leq n$, if $a_j \geq a_{n+1}$, then $b_{n+1} < b_j$. Therefore, function f is well-defined. f is injective, because if $f(i) = f(j)$ for some $i < j$, I_i and I_j have a common endpoint, so $I_i \subseteq I_j$ which contradicts the assumption.

Note that the number of elements in $\{0, 1, \ldots, 2n-1\} - \{a_{n+1}, a_{n+1}+1, \ldots, b_{n+1}\}$ is $2n-(n+2) = n-2$, so there are no injective functions from $\{2, \ldots, n\}$ to this set, which is a contradiction and proves the assertion.

6. The notation $x \sqsubseteq y$ is used to show that none of prime factors of x exceed y. Obviously if $r \leq m$, $r \sqsubseteq m$; and if $r \sqsubseteq m$ and $s \sqsubseteq m$, then $rs \sqsubseteq m$.

Suppose that $[\sqrt{n}] = m$. Therefore, $n = m^2 + r$, where $0 \leq r \leq 2m$ ($n \geq 50$ so $m \geq 7$). Two cases are possible depending on the parity of m.

- m is odd, which means $m+1$ is even. For every number $t \leq m+1$, $t \sqsubseteq m$ $(*)$, because if $1 \leq t \leq m$, obviously $t \sqsubseteq m$; and if $t = m+1 = 2(\frac{m+1}{2})$, again $t \sqsubseteq m$ $(2, \frac{m+1}{2} \leq m)$.

  Let $1 \leq j \leq m+1$ be such that $j \overset{m+1}{\equiv} n$. It is claimed that

  $$n = \underbrace{(n-j)}_{a} + \underbrace{j}_{b}$$

  is a desired decomposition. By $(*)$, it can be concluded that $b = j \sqsubseteq m$. On the other hand, $n - j = (m+1)(\frac{n-j}{m+1})$, where $\frac{n-j}{m+1} < m+1$. Since $m+1 \sqsubseteq m$ and $\frac{n-j}{m+1} \sqsubseteq m$, $a = n-j \sqsubseteq m$.

- m is even. Like the previous case, let $-1 \leq j \leq m$ be such that $n \overset{m+2}{\equiv} j$. If $1 \leq j \leq m$,

  $$n = \underbrace{(n-j)}_{a} + \underbrace{j}_{b};$$

  and if $j = 0$,

  $$n = \underbrace{\frac{1}{2}n}_{a} + \underbrace{\frac{1}{2}n}_{b}.$$

Using a similar argument, it is easy to see that in both cases $a \sqsubseteq m$ and $b \sqsubseteq m$.

Finally, if $j = -1$, then $m + 2 | n + 1$. Since $m^2 \leq n \leq m^2 + 2m$, then $n + 1 = (m - 1)(m + 2) = m^2 + m - 2$ or $n + 1 = m(m + 2)$.

In the first case, since $m - 3 < m$,

$$n = \underbrace{m^2}_{a} + \underbrace{m - 3}_{b},$$

is a desired decomposition. In the second case,

$$n = (m + 1)^2 - 2 = (m - 2)(m + 4) + 7 = \underbrace{2(m - 2)(\tfrac{m+4}{2})}_{a} + \underbrace{7}_{b}.$$

Since $m \geq 7$, $\frac{m+4}{2} \leq m$, so $a \sqsubseteq m$ and $b \sqsubseteq m$.

This argument shows that a natural number $n \geq 50$ can be written as the sum of two natural numbers with prime factors not exceeding $\sqrt{n}$.
Remark. All integers $n \geq 8$, except for 23, have such a decomposition.

Third Round

1. a) Assume to the contrary that there exists a non-zero polynomial $P(x,y) \in \mathbb{R}[x,y]$ such that for every $n \in \mathbb{N}$, $P(n, 2^n) = 0$. Let d be the degree of P with respect to its second variable, y. $P(x,y)$ can be written as,

$$P(x,y) = p_d(x)y^d + \cdots + p_1(x)y + p_0(x),$$

where p_i's are polynomials in x and p_d is not zero.

For every $\epsilon > 0$, there exists $N_1 > 0$ such that for every $n > N_1$, $|p_d(n)| > \epsilon$. Furthermore, there exists $N_2 > 0$ such that for every $n > N_2$,

$$|p_0(n)|, \ldots, |p_{d-1}(n)| < (\sqrt{2})^n = 2^{\frac{n}{2}}.$$

Now for every $n > \max\{N_1, N_2\}$,

$$\begin{aligned}
|P(n,2^n)| &= |p_d(n)2^{dn} + \cdots + p_1(n)2^n + p_0(n)| \\
&\geq |p_d(n)2^{dn}| - \Big(|p_{d-1}(n)2^{(d-1)n}| + \cdots + |p_1(n)2^n| + |p_0(n)|\Big) \\
&> \epsilon 2^{dn} - 2^{\frac{n}{2}}(1 + 2^n + \cdots + 2^{(d-1)n}) \\
&> \epsilon 2^{dn} - 2^{(d-1)n+\frac{n}{2}+1} \\
&= 2^{dn}(\epsilon - 2^{-\frac{n}{2}+1}).
\end{aligned}$$

Since $2^{-\frac{n}{2}+1}$ approaches zero as n approaches infinity, $|P(n,2^n)|$ approaches infinity and cannot be zero for large values of n.

Remark. This proof shows that for every non-zero polynomial with real coefficients $P(x,y)$, $P(n,2^n)$ can be zero for only a finite number of times.

b) No! Let $\alpha = \log_2 3$ (i.e., $2^\alpha = 3$). It is easy to prove that α is irrational. Suppose that the two mentioned sequences are Co-Algebraic and that there exists a non-zero polynomial $P(x,y)$ such that for every positive natural number n, $P(2^n, 3^n) = 0$. This polynomial is the sum of monomials of the form $c_{k,l}x^k y^l$, where k and l are nonnegative integers and $c_{k,l} \neq 0$ is a real number. Thus $P(2^n, 3^n)$ is the sum of expressions of the form $c_{k,l}2^{nk}3^{nl} = c_{k,l}2^{n(k+\alpha l)}$. Since α is irrational, different pairs (k,l) of integers lead to different values for $k+\alpha l$. Let β be the maximum value of $k + \alpha l$ among all of the pairs (k, l) for which $c_{k,l}$ is not zero. Therefore, $P(2^n, 3^n)$ can be written as

$$P(2^n, 3^n) = c2^{\beta n} + c_1 2^{\beta_1 n} + \cdots + c_k 2^{\beta_k n}, \tag{$*$}$$

where $c \neq 0$ and $\beta_i < \beta$ for $1 \leq i \leq k$. Now, dividing both sides of $(*)$ by $2^{\beta n}$ results in

$$\frac{1}{2^{\beta n}} P(2^n, 3^n) = c + c_1 2^{(\beta_1 - \beta)n} + \cdots + c_k 2^{(\beta_k - \beta)n}.$$

The left hand side approaches zero and the right hand side approaches c as n approaches infinity. This is a contradiction because it is assumed that $c \neq 0$, and shows that two sequences 2^n and 3^n are not Co-Algebraic.

c) According to part (b), there exists $m \in \mathbb{N}$ such that $f(2^m, 3^m) \neq 0$. Let k be the largest natural number such that $5^k | P(2^m, 3^m)$. It is claimed that if $n := m + 4 \times 5^k$, then $f(2^n, 3^n)$ is not divisible by 5^{k+1}, and since $k + 1 < n$, not divisible by 5^n either. In order to prove it, note that $\varphi(5^{k+1}) = 4 \times 5^k$, and by Euler's theorem

$$2^{4\times 5^k} \overset{5^{k+1}}{\equiv} 3^{4\times 5^k} \overset{5^{k+1}}{\equiv} 1.$$

Hence

$$f(2^n, 3^n) = f(2^{m+4\times 5^k}, 3^{m+4\times 5^k}) \overset{5^{k+1}}{\equiv} f(2^m, 3^m) \overset{5^{k+1}}{\not\equiv} 0.$$

2. a) Consider a table with $T+1$ rows and an infinite number of columns (towards right) such that the first column is filled by numbers $1, 2, \ldots, T$ and 1. An algorithm is introduced for filling the cells of the table, then $f_i(j)$ is defined as the value of the intersection cell of the i^{th} row and the j^{th} column of the table.

1				$\cdots$
2				$\cdots$
3				$\cdots$
$\vdots$				$\cdots$
$T-1$				$\cdots$
T				$\cdots$
1				$\cdots$

Label the cells of the table, except those of the first row, by $a_1, a_2, a_3, \ldots$, like the following table.

a_1	a_3	a_6	a_{10}	$\cdots$
a_2	a_5	a_9		$\cdots$
a_4	a_8			$\cdots$
a_7				$\cdots$
$\vdots$				$\cdots$

In each step of the algorithm, some cells of the table are filled in a way that after the j^{th} step a_j is filled. Suppose that a_j is at the intersection of the k^{th} row and the l^{th} column.

i. If a_j was empty before the j^{th} step, the smallest natural number that has not yet appeared in the table is written in a_j at that step.

ii. The number in a_j should be equal to the number of repetitions of l in the $(k-1)^{\text{th}}$ row of the table $(*)$. For this reason, enough l's should be put in the first empty cells of the $(k-1)^{\text{th}}$ row so that the number of l's becomes equal to the number written in a_j. Using this algorithm, if $l > T$, before this step no l exists in the $(k-1)^{\text{th}}$ row; and if $l \leq T$, at most one such l exists (the one in the first column). The reason is that every number that has been written in the $(k-1)^{\text{th}}$ row before this step, except the first number, is written when the algorithm was used for the first $l-1$ cells of the row, and in those steps only numbers less than l were written. Since the number in a_j is positive, it was not less than the number of l's in the $(k-1)^{\text{th}}$ row (before this step), so a sufficient number of l's (maybe zero) can always be added to satisfy $(*)$.

iii. If a_j is in the second row, after the previous step the numbers in the first row should be copied into the cells of the last row.

Using this algorithm, all of the cells of the table are filled with natural numbers, and if $f_i(j)$ is defined to be the number in the j^{th} cell of the i^{th} row, f_i's satisfy all of the problem's conditions.

b) Yes! To construct such functions an approach similar to the previous part is used, utilizing a table which is infinite from right and below. Like part (a), j is written in the j^{th} cell of the first column and the cells of the table, except those of the first row, are labeled by $a_1, a_2, \ldots$ like the following table.

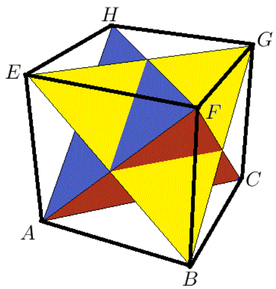
a_1	a_3	a_6	a_{10}	$\cdots$
a_2	a_5	a_9	a_{14}	$\cdots$
a_4	a_8	a_{13}		$\cdots$
a_7	a_{12}			$\cdots$
a_{11}				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

1					$\cdots$
2					$\cdots$
3					$\cdots$
4					$\cdots$
5					$\cdots$
6					$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The table can be filled using the algorithm of part (a) (since no bottom row exists in this table, part iii of the algorithm is not necessary). Like before, it can be showed that a_j is filled at the j^{th} step and if a_j is the intersection of the k^{th} row and the l^{th} column, the number in a_j is not less than the number of l's in the $(k-1)^{\text{th}}$ row.

Again, if $f_i(j)$ is defined as the number in the intersection of the i^{th} row and the j^{th} column, f_i's satisfy all of the problem's conditions; and since the numbers in the first column are all distinct, functions f_i are mutually unequal.

3. a) The following figure of a cube $ABCDEFGH$ shows that the two regular tetrahedra $ACHF$ and $BDEG$ are perpendicular.



b) Consider two perpendicular tetrahedra $ABCD$ and $XYZT$. One of the following cases involves those three edges of $ABCD$ that are perpendicular to edges XY, XZ and XT of $XYZT$:

- These edges form a triangle in $ABCD$.
- They have a common vertex.
- They form an open path.

It will be shown that the second case is not possible. Assume to the contrary that the edges connected to X in $XYZT$ are perpendicular to the edges connected to (for instance) A in $ABCD$, so the edges of triangles BCD and YZT should be perpendicular. Therefore, these two triangles must be in the same plane. On the other hand, the vertices of triangle YZT are on the perpendicular lines from X to AB, AC and AD, which is a contradiction to the fact that YZT is on the same plane as BCD.

Next, it is claimed that the third case is possible for at most two vertices of $XYZT$. Assume that those edges of $ABCD$ perpendicular to the edges connected to X in $XYZT$ form an open path. It is obvious that there exist exactly two faces of $ABCD$ (say P_{1X} and P_{2X}) such that each one contains exactly two edges of this path.

Lemma. *Tetrahedron $ABCD$ and vertex X must be on different sides of P_{iX}, for $i = 1$ and 2.*

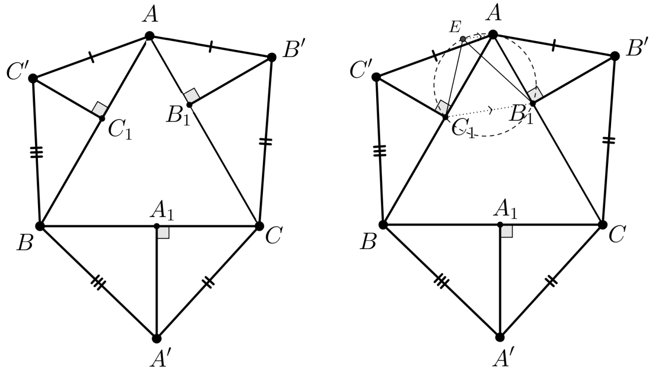
Proof. Assume to the contrary that $ABCD$ is on the same side of P_{iX} as X, for $i = 1$ or 2. Let f be the face of $ABCD$ contained in P_{iX}. Two vertices of $XYZT$ lie on the extensions of perpendiculars from X to edges of f and so lie on the opposite side of X. On the other hand, these vertices create an edge that must be perpendicular to an edge of $ABCD$. This is impossible, however, since they lie on opposite sides, and proves the lemma.

Using the lemma it can be concluded that X is in the smaller region between P_{1X} and P_{2X} and outside $ABCD$. This lemma can be applied to any other vertex, say Y, for which the edges connected to it are perpendicular to an open path. It is claimed that there can be at most two such vertices. To prove it, assume to the contrary that there are more than two such vertices. Then using the pigeonhole principle, two vertices, say X and Y, have a common associated plane, i.e. $P_{iX} = P_{jY}$. So according to the lemma, X and Y must be on the same side of this plane and apart from $ABCD$. This is impossible, however, because XY must intersects an edge of $ABCD$.

It can be concluded that the third case is possible for at most two vertices of $XYZT$, so the first case is possible for at least two vertices of that tetrahedron. Using the following lemma, the place of these two vertices can be uniquely determined by $ABCD$. Now, perpendicular lines from these two vertices to three edges of $ABCD$ determine the lines on which 5 edges of $XYZT$ are located, so all of the vertices of $XYZT$ are uniquely determined.

Lemma. *Consider an equilateral triangle ABC and a point D outside of the plane of ABC such that the angle between mutual perpendicular lines from D to the sides of ABC is $60°$. Prove that B_1, C_1 and A_1 (feet of perpendicular lines from D to sides AC, AB and BC, respectively) are the midpoints of sides AC, AB and BC, respectively.*

Proof. Form the net of $ABCD$ like the following pictures, where $AB' = AC' = AD$, $BA' = BC' = BD$ and $CA' = CB' = CD$.



Point E is located on the plane such that $EC_1 = C'C_1$ and $EB_1 = B'B_1$. First, suppose that $E \neq A$. Then two triangles EB_1C_1 and DB_1C_1 are obviously congruent, so $\angle C_1EB_1 = \angle C_1DB_1 = 60°$. Therefore, quadrilateral EAB_1C_1 is cyclic. Using Pythagoras' Theorem for segments AC' and AB' results in

$$AC_1^2+EC_1^2 = AC_1^2+C'C_1^2 = AC'^2 = AB'^2 = AB_1^2+B'B_1^2 = AB_1^2+EB_1^2.$$

On the other hand, using the Cosine Theorem for side AE in triangles AEC_1 and AEB_1 results in

$$AB_1^2 + EB_1^2 - 2AB_1.EB_1 \cos\alpha = AC_1^2 + EC_1^2 - 2AC_1.EC_1 \cos\alpha.$$

Since $\cos\alpha \neq 0$ ($E \neq A$), $AC_1.EC_1 = AB_1.EB_1$. This implies that the area of triangles AEC_1 and AEB_1 is equal, so $AE \parallel B_1C_1$. Now, using the fact that EAB_1C_1 is a cyclic quadrilateral, it can be concluded that $AB_1 = EC_1 = C'C_1$ and $AC_1 = EB_1 = B'B_1$.

By defining points F and G similar to E and following the argument above, it can be concluded that $A'A_1 = AB_1 = AC_1$, which implies that $E = A$.

Now, assuming that $E = A$, $AC_1 = C'C_1$ and $AB_1 = B'B_1$. Since $AC' = AB'$ and $AB_1 = AC_1 = \frac{\sqrt{2}}{2}AC'$, it can be concluded that $CB_1 = BC_1$. Therefore, $B'C = C'B$ which results in $BA' = C'B = B'C = CA'$. This implies that A_1 is the midpoint of BC. Continuing this argument by using points F and G, it is easy to show that B_1 is the midpoint of AC and C_1 is the midpoint of AB, as desired.

4. a) The idea here is that a computer (with sufficiently large but finite memory) can compute the remainder of arbitrary large natural numbers

when divided by m. The following describes a Machine for computing the remainder of a given number modulo m in base k:

- **Inputs:** Set $\{0, 1, \ldots, k-1\}$.
- **States:** A state exists for each pair (i, j) of $\{0, 1, \ldots, m-1\}$.
- **Outputs:** The output of state (i, j) is j.
- **Starting State:** $(1, 0)$.
- **Transition Function:** When r is entered, the state of the Machine changes from (i, j) to (i', j'), where i' and j' are the remainders of ki and $j + ri$ when divided by m, respectively.

b) Let X be the Machine that computes (a_n) in base k. To introduce a Machine that computes (b_n), where $b_n = a_{n+1}$, in base k, first a Machine is designed that takes an r-digit number n as input and gives (a, b) as output, where a and b are the r^{th} and $(r+1)^{\text{th}}$ digits of $n+1$ in base k (from right). Note that $n+1$ has at most $r+1$ digits, and if it has only r digits, then b will be zero. The following Machine will perform this operation:

- **Inputs:** Set $\{0, 1, \ldots, k-1\}$.
- **States:** A state exists for each pair (a, b) of integers, where $0 \leq a < k$ and $b \in \{0, 1\}$.
- **Outputs:** The output of state (a, b) is (a, b) itself.
- **Starting State:** $(1, 0)$.
- **Transition Function:** When r is entered, the state of the Machine changes from (a, b) to (a', b'), where a' is the remainder of $r + b$ when divided by m, and b' is 0 for $r + b < k$ and 1 elsewhere.

Call this Machine B. Now, make Machine Y from B and X to compute (b_n). Denote the set of states and the Starting State of X by S_X and s_X, respectively. S_B and s_B are defined similarly. The following Machine will perform the desired operation:

- **Inputs:** Set $\{0, 1, \ldots, k-1\}$.
- **States:** $S_X \times S_B$.
- **Outputs:** The output of state (x, y), where $x \in S_X$ and $y \in S_B$, is defined as follows: Let (a, b) be the output of state y in B. Now, if $b = 0$, define the output of (x, y) (in Y) to be equal to the output of x (in X); and if $b \neq 0$, define it to be the output of X after entering b when X is in state x.

- **Starting State:** (s_X, s_B).

- **Transition Function:** When r is entered, the state of the Machine changes from (x, y) to (x', y'), where y' is the state of B after entering r when it is in state y. Now, if (a, b) is the output of y', x' will be the state of X after entering a when it is in state x.

To compute $c_n = a_{2n}$, first define a Machine C similar to B for the purpose of multiplication by 2. Then use C instead of B in the definition of Y above to get the desired result. The Machine C is described as follows:

- **Inputs:** Set $\{0, 1, \ldots, k-1\}$.

- **States:** A state exists for each pair (a, b) of integers, where $0 \leq a < k$ and $b \in \{0, 1\}$.

- **Outputs:** The output of state (a, b) is (a, b) itself.

- **Starting State:** $(0, 0)$.

- **Transition Function:** When r is entered, the state of the Machine changes from (a, b) to (a', b'), where a' and b' are the quotient and remainder of $2r + b$ when divided by k, respectively.

c) No! Assume to the contrary that X is a Machine that computes (a_n) in base 3. Let N be the number of states of X. Note that for a sufficiently large natural number m, the number of m-digit perfect square numbers in base 3 is more than N. So there are two perfect squares a and b such that after entering digits of these two numbers to X (from right) the same state is reached. Therefore, entering numbers of the form $a + 3^m n$ and $b + 3^m n$ results in the same output. Hence if (a_n) is 3-Computable, then for each natural number n, $a + 3^m n$ is a perfect square if and only if $b + 3^m n$ is a perfect square. But this is impossible, because the difference of these numbers is $b - a$ (which is constant), whereas the difference of consecutive perfect square numbers approaches infinity.

d) Suppose that X is a Machine with inputs from the set $\{0, 1, \ldots, k-1\}$. It will be shown that each k-Computable sequence is k-Left Computable, the converse can be proved similarly. To do this, it suffices to introduce a Machine Y with the same input set as X such that the output of Y after entering digits of a number from left to right equals to the output of X after entering digits of that same number from right to left. Suppose that $\{s_1, s_2, \ldots, s_m\}$ is the state set of X, s_1 its Starting State and x_i the output of state s_i. The following Machine will perform the desired operation:

- **Inputs:** Set $\{0, 1, \ldots, k-1\}$.
- **States:** All m-tuples $(A_1, A_2, \ldots, A_m)$ of partitions of the set $\{s_1, s_2, \ldots, s_m\}$.
- **Outputs:** The output of state $(A_1, A_2, \ldots, A_m)$ will be A_i, where $s_1 \in A_i$.
- **Starting State:** $(\{s_1\}, \{s_2\}, \ldots, \{s_m\})$.
- **Transition Function:** When r is entered, the state of the Machine changes from $(A_1, A_2, \ldots, A_m)$ to $(B_1, B_2, \ldots, B_m)$, where B_i is the set of all those states of X that change to a state in A_i after entering r in X.

5. a) Note that every polygon produced during this process is convex. That said, the idea here is to find an upper bound for the sum of angles of them. After cutting a polygon P into two polygons P_1 and P_2, three possibilities arise: ($\sigma(Q)$ denotes the sum of angles of polygon Q)

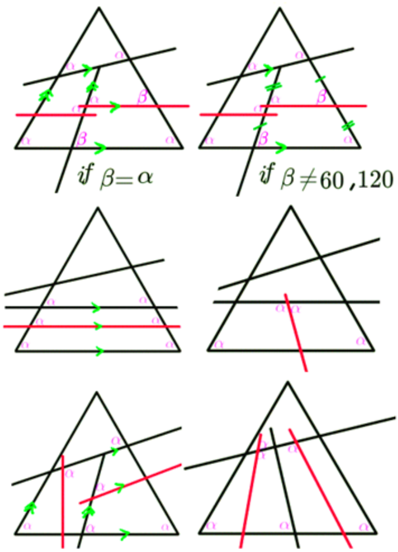
- The cutting line connects two vertices of P. In this case $\sigma(P) = \sigma(P_1) + \sigma(P_2)$.
- The cutting line connects a vertex of P to an inner point of a side. In this case $\sigma(P_1) + \sigma(P_2) = \sigma(P) + \pi$.
- The cutting line connects two inner points from two sides of P. In this case $\sigma(P_1) + \sigma(P_2) = \sigma(P) + 2\pi$.

The first polygon is a triangle and the sum of its angles is π. Therefore, after the first cut the sum of angles of the resulting polygons is at most $\pi + 2\pi = 3\pi$. After the second cut each of these two polygons are divided into two new polygons. Therefore, after the second cut the sum of angles of these polygons is at most $3\pi + 2 \times 2\pi = 7\pi$. Similarly, after the third cut the sum of angles of the resulting eight polygons is at most $7\pi + 4 \times 2\pi = 15\pi$. Now, if the 7 equal pieces have at least 5 sides, then the sum of angles of each polygon is at least 3π and the sum of angles of the last piece is at least π. Therefore,

$$22\pi = \pi + 7 \times 3\pi \leq \text{sum of angles of all pieces} \leq 15\pi.$$

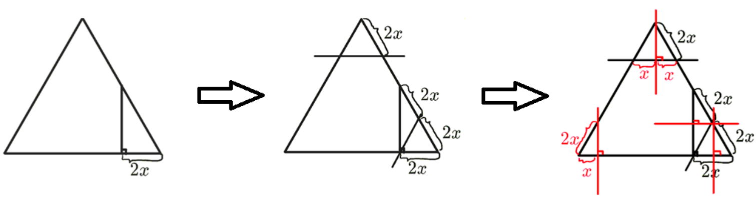
This contradiction shows that those seven equal pieces must be either triangles or quadrilaterals.

b) According to the previous part, if these equal pieces are quadrilaterals, then the last piece should be a triangle. Furthermore, the cutting line of each polygon connects two inner points from two sides of the polygon. Assuming that such a cutting process is possible, a result is that the seven congruent quadrilaterals cannot be cyclic; because if they are cyclic, one of the following cases happens, and it is easy to show that each case results in a contradiction.



Assuming that the quadrilaterals are not cyclic, it is easy to show that they must be parallelograms. Like before, it is easy to check the resulting case and verify that it is impossible.

c) Yes! Let x be a very small positive number. Then the cutting process can be done as instructed below:



6. In the solution to this problem, for two points A and B in the plane, AB will be used for their Euclidean distance and $d(A, B)$ for their newly defined distance.

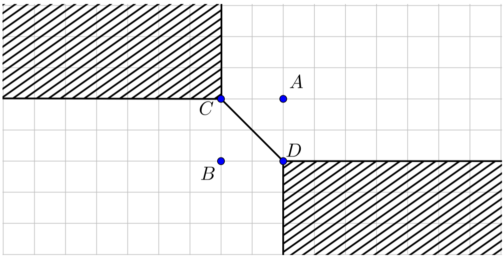
a) Assume $A = (x_a, y_a)$ and $B = (x_b, y_b)$ are two arbitrary points in the plane. The new perpendicular bisector of A and B comprises points

(x, y) satisfying

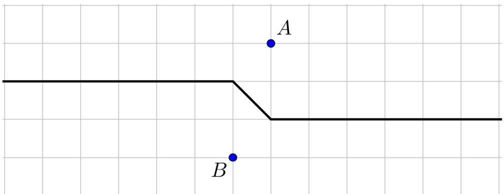
$$\left|x - x_b\right| + \left|y - y_b\right| = \left|x - x_a\right| + \left|y - y_a\right|.$$

Draw two lines from each of the points A and B parallel to the coordinate axes to create a rectangle $ACBD$. Based on the slope of AB two cases are possible:

- The slope of AB is 1 or -1. In this case the perpendicular bisector of A and B is the union of segment CD and the hatched parts in the figure below (The figure is drawn for the case where the slope is 1. The other case is similar):



- The slope of AB is not ± 1. Assume that this slope is positive and greater than one. The following figure shows the perpendicular bisector of A and B in this case. Other cases are similar.



b) First, a lemma is presented.

Lemma. *For every point Z on line XY,* $\frac{d(X,Z)}{d(Y,Z)} = \frac{XZ}{YZ}$.

Proof. It can be derived from Thales' Theorem easily.

Consider points C and D on line AB such that C is inside segment AB, D is outside of it and $\frac{AC}{CB} = \frac{AD}{DB} = m$. Obviously, points satisfying these properties are unique and according to the lemma, $\frac{d(A,C)}{d(C,B)} = \frac{d(A,D)}{d(D,B)} = m$.

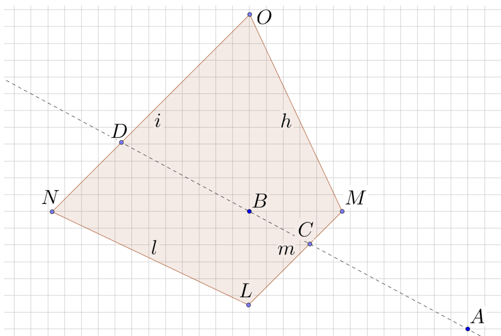
By definition, the intersection of the circle with center A and radius $d(A,C)$ and the circle with center B and radius $d(B,C)$ lies on the

Apollonius circle (note that circles in this new metric are Euclidean squares with diagonals parallel to the coordinate axes). This intersection is a segment which is denoted by ML (as can be seen in the following figure). The intersection of similar circles for D (instead of C) determines another part of the Apollonius circle and is denoted by ON (as can be seen in the following figure). In what that follows, without loss of generality assume that $m > 1$.

An easy calculation reveals that the slope of segment LN is $\frac{m-1}{m+1}$. For an arbitrary point Z on this segment,

$$d(A, Z) = d(A, N) - \Big|x_Z - x_N\Big| - \Big|y_Z - y_N\Big|,$$
$$d(B, Z) = d(B, N) - \Big|x_Z - x_N\Big| + \Big|y_Z - y_N\Big|.$$

Since $\frac{|y_Z - y_N|}{|x_Z - x_N|} = \frac{m-1}{m+1}$ and $d(A, N) = md(B, N)$, it can be deduced that $d(A, Z) = md(B, Z)$. Therefore, all of the points of segment LN are on the Apollonius circle. A similar argument will show that all of the points of segment OM are on the Apollonius circle, too. For an arbitrary point P of the plane not on the boundry of $OMLN$, the quotient $\frac{d(A,P)}{d(B,P)}$ is greater or less than m. Therefore, the Apollonius circle of points A and B in this new metric comprises points on the boundary of quadrilateral $OMLN$.



c) The goal is to find the parabola created by point $F = (0, 0)$ and line l with equation $y = mx + c$. For an arbitrary point P in the plane, draw two lines from P parallel to the coordinate axes. These two lines intersect l at two points, say X and Y. It can be seen that $\min(PX, PY)$ is the distance from P to l.

Without loss of generality, assume that $m \geq 1$ and $c > 0$. Therefore, the distance from a point (x, y) in the plane to l is equal to $|x - \frac{y-c}{m}|$ (length of the horizontal segment from (x, y) to l). So for finding the points of the parabola the following equation must be solved,

$$\left|x\right| + \left|y\right| = \left|x - \frac{y-c}{m}\right| + \left|y - y\right| = \left|x - \frac{y-c}{m}\right|,$$

or equivalently,

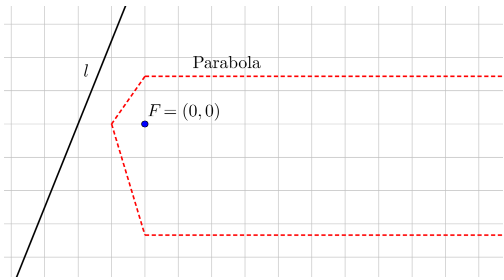
$$m|x| + m|y| = |mx + c - y|.$$

For an arbitrary point (x, y) in the plane above line l $(y > mx + c)$,

$$m|x| + m|y| \geq m|x| + |y| \geq -mx + y > y - mx - c = |y - mx - c|.$$

Therefore, all of the points of the parabola lie below l $(y < mx + c)$. Evaluating cases resulting from the sign of x and y, it can be seen that the parabola is the union of the following four segments (Similar to the following figure):

$$\begin{cases} y = \frac{c}{m+1}, & \text{for } x > 0, y > 0 \\ y = \frac{c}{1-m}, & \text{for } x > 0, y < 0 \\ 2mx - (m+1)y + c = 0, & \text{for } x < 0, y > 0 \\ 2mx + (m-1)y + c = 0, & \text{for } x < 0, y < 0 \end{cases}$$



d) Yes. To prove it, a lemma is needed.

Lemma. *Let A', B' and C' be three points on sides BC, AC and AB of triangle ABC, respectively, such that*

$$\frac{d(B, A')}{d(C, A')} \frac{d(C, B')}{d(A, B')} \frac{d(A, C')}{d(B, C')} = 1.$$

Then lines AA', BB' and CC' are concurrent.

Proof. According to the assumption and the lemma in part (b),

$$\frac{BA'}{CA'} \frac{CB'}{AB'} \frac{AC'}{BC'} = 1.$$

Therefore, due to Ceva's Theorem in Euclidean geometry these lines are concurrent.

Let $p = \frac{1}{2}(d(A,B) + d(B,C) + d(C,A))$. So $d(B,X_a) + d(A,B) = d(C,X_a)+d(A,C) = p$. Therefore, $d(B,X_a) = p-d(A,B)$ and $d(C,X_a) = p - d(A,C)$. Similar equalities for X_b and X_c imply

$$\frac{d(B,X_a)}{d(C,X_a)} \frac{d(C,X_b)}{d(A,X_b)} \frac{d(A,X_c)}{d(B,X_c)} = 1.$$

Now according to the lemma, it is easy to see that these lines are concurrent.

e) No. Assume to the contrary that S is an infinite set of points in the plane such that the mutual distances of its point are all perfect squares.

Lemma. *There is no infinite subset $\{P_1 = (x_1, y_1), P_2 = (x_2, y_2), \ldots\}$ of S for which the two sequences $\{x_i\}$ and $\{y_i\}$ are both monotone.*

Proof. Assume to the contrary that there exists an infinite set of points $P_i = (x_i, y_i)$ in S such that both sequences $\{x_i\}$ and $\{y_i\}$ are increasing sequences (other cases can be proven similarly). For each natural number $j > 2$, let $a_j = \sqrt{d(P_1, P_j)} \in \mathbb{N}$ and $b_j = \sqrt{d(P_2, P_j)} \in \mathbb{N}$. As both sequences $\{x_j\}$ and $\{y_j\}$ are increasing, for every natural number $j > 2$,

$$a_j^2 = d(P_1, P_j) = d(P_1, P_2) + d(P_2, P_j) = d(P_1, P_2) + b_j^2.$$

Since sequences a_j and b_j are increasing, $d(P_1, P_2)$ can be written in infinitely many ways as the difference of two perfect squares ($d(P_1, P_2) = a_j^2 - b_j^2$). This is impossible, however, and proves that S cannot exist.

By a translation it can be assumed that S contains the origin of the coordinate. It is claimed that the number of points of S in the first coordinate quadrant is finite and because of symmetry, the number of points in each quadrant is finite. Therefore, S is a finite set.

If set $\{y \geq 0 | \exists x \geq 0; (x, y) \in S\}$ is unbounded, then an infinite subset $\{P_1 = (x_1, y_1), P_2 = (x_2, y_2), \ldots\}$ of S can be found such that $y_1 < y_2 < \cdots$. Therefore, a subsequence $\{P_{n_i}\}_i$ of P_n can be founded such that the sequence x_{n_i} is monotone, which is impossible according to the lemma. This means that the second coordinate of points of S in the first coordinate quadrant is bounded. Similarly, it can be concluded that the first coordinate of such points is bounded. Therefore, there exists $M > 0$ such that all points of S in the first coordinate quadrant lie in the square $[0, M] \times [0, M]$. Since each two points of S have a distance of at least 1, only a finite number of points of S can be in this square, which means that overall, S is finite.

7. Obviously, there are exactly $n - 1$ different nontrivial rotations of a regular n-gon (rotations of $\frac{2\pi i}{n}$ angles, for $1 \leq i \leq n - 1$). Denote the

number of blue and red vertices of the polygon by b and r, respectively ($a+b=n$), and the number of vertices that have different colors before and after the rotation of $\frac{2\pi j}{n}$ angle by t_j. According to the problem statement, $t_j < \frac{32}{100}n$ for $j=1,2,\ldots,n-1$. On the other hand, during the $n-1$ rotations each blue vertex coincides with each red vertex exactly once and conversely, each red vertex coincides with each blue vertex exactly once. Therefore, from one hand the total number of pair of points with opposite colors after all of the nontrivial rotations is $\sum_{j=1}^{n-1} t_j$, and $2rb = 2r(n-r)$ from another. Hence,

$$2r(n-r) = \sum_{j=1}^{n-1} t_j < (n-1) \times \frac{32}{100}n < \frac{32}{100}n^2,$$

or equivalently,

$$r^2 - nr + \frac{16}{100}n^2 > 0 \Leftrightarrow (r-\frac{n}{2})^2 > \frac{9}{100}n^2 \Leftrightarrow |r-\frac{n}{2}| > \frac{3}{10}n,$$

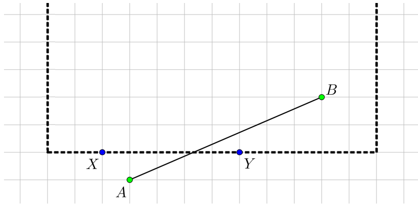
which completes the proof.

8. a) No! As a counterexample, consider three small equal circles on the circumference of a sphere. Each of these circles divides the circumference of the sphere into two parts. Call the smaller part for each circle its "inner domain", and assume that the inner domains of these circles are mutually disjoint. Now, if there exists a circle that passes through all three of these circles, it must intersect the sphere in at least three points (it must intersect each inner domain at one point) which is impossible, since any circle that is not on the circumference of a sphere meets it in at most two points.

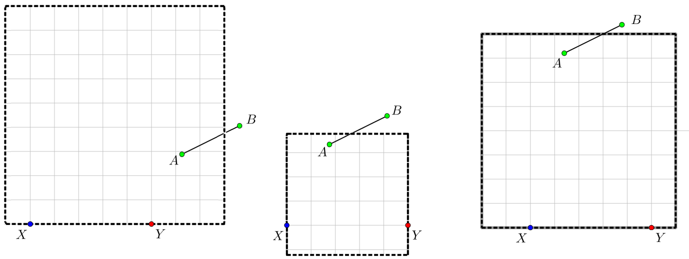
b) Yes. Consider three circles C_R, C_B and C_G in the space such that no two of them are co-planar. Denote the plane containing C_i by π_i (for $i \in \{R, B, G\}$). For each circle C_i, let X_i be a point in the plane π_i lying inside C_i, and choose them so that these three points are not collinear, so they form a plane, say π. Due to our assumption plane π meets each circle at exactly two points. Denote these intersection points by $\{G_1, G_2\}$, $\{B_1, B_2\}$ and $\{R_1, R_2\}$, respectively. Now, the goal is to find a square in π that intersects each of the three segments G_1G_2, B_1B_2 and R_1R_2 at exactly one inner point (not an endpoint). Obviously, since this square separates each pair of points, it must pass through all three circles and hence is the desired square. To show its existence, first a lemma is proved.

Lemma. *For points X, Y, A and B in the plane where XY is not parallel to AB, a square exists that passes through points X and Y and separates A from B.*

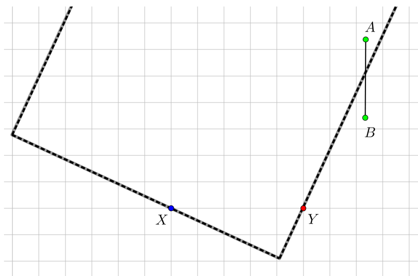
Proof. First, suppose that XY is not perpendicular to AB. If line XY separates A from B, a large square with X and Y both on one of its sides works.



On the other hand, if both points A and B lie on one side of line XY, depending on the distance from A and B to XY and the position of the feet of perpendicular lines from these two points to line XY, a suitable square as depicted in the following figures can be found.



If $XY \perp AB$, the argument must be changed only when the intersection point of AB and XY lies outside of both segments. In this case, a suitable square can be found like the following figure.



Since points R_1, R_2, B_1 and B_2 are not collinear, points X on R_1R_2 and Y on B_1B_2 exist such that line XY does not contain any point from R_1R_2 and B_1B_2 other than X and Y and also, G_1G_2 is not parallel to XY. According to the lemma, a square passing through X and Y (and no other point on R_1R_2 and B_1B_2) and separating G_1 from G_2 exist, which completes the proof.

Team Selection Test

1. The following equality holds,

$$\sum\left(\sqrt{\frac{a^2+1}{2}}-\sqrt{a}\right)=\sum\frac{\frac{a^2+1}{2}-a}{\sqrt{\frac{a^2+1}{2}}+\sqrt{a}}=\frac{1}{2}\sum\frac{(a-1)^2}{\sqrt{\frac{a^2+1}{2}}+\sqrt{a}}.$$

According to Cauchy-Schwarz inequality, $\sqrt{\frac{a^2+1}{2}}+\sqrt{a}\leq\sqrt{2(\frac{a^2+1}{2}+a)}=a+1$, hence

$$\frac{1}{2}\sum\frac{(a-1)^2}{\sqrt{\frac{a^2+1}{2}}+\sqrt{a}}\geq\frac{1}{2}\sum\frac{(a-1)^2}{a+1}.$$

On the other hand, $\frac{(a-1)^2}{a+1}=(a-3)+\frac{4}{a+1}$; therefore,

$$\frac{1}{2}\sum\frac{(a-1)^2}{a+1}=\frac{1}{2}\sum(a-3)+2\underbrace{\sum\frac{1}{a+1}}_{2}=\frac{1}{2}\left(\sum a\right)-2.$$

Now, it suffices to prove that $\frac{1}{2}\left(\sum a\right)-2\geq 2\left(\sum\sqrt{a}\right)-8$, or equivalently, $3+\frac{1}{4}\sum a\geq\sqrt{a}$, which can be deduced by the following AM-GM inequality

$$3+\frac{1}{4}\sum a=\underbrace{\sum\frac{a}{a+1}}_{2}+\sum\frac{a+1}{4}=\sum\left(\frac{a}{a+1}+\frac{a+1}{4}\right)\geq\sum\sqrt{a}.$$

2. Call a family $\mathcal{F}$ that comprises subsets of $X=\{1,2,3,4,5\}$ "a decision family" if the following properties are satisfied:

i. If $A\in\mathcal{F}$ and $A\subseteq B$, then $B\in\mathcal{F}$.

ii. $A\in\mathcal{F}$ if and only if $A^c\notin\mathcal{F}$ $(A^c=X-A)$.

Now, the problem can be paraphrased using families of subsets as follows:
New Statement. *For every decision family $\mathcal{F}$ of $\{1,2,3,4,5\}$, nonnegative weights $w_1, w_2, \ldots, w_5$ exist such that*

$$A \in \mathcal{F} \quad \Leftrightarrow \quad \sum_{i \in A} w_i > \sum_{i \in A^c} w_i.$$

In order to prove this new assertion, note that according to condition (ii), those members of $\mathcal{F}$ that have at most two elements determine $\mathcal{F}$ uniquely. By using complements, it can be deduced from condition (i) that,

i$'$. If $B \notin \mathcal{F}$ and $A \subseteq B$, then $A \notin \mathcal{F}$.

Obviously, $\emptyset \notin \mathcal{F}$, because if $\emptyset \in \mathcal{F}$, $X \notin \mathcal{F}$ (according to (i)), so $\emptyset \notin \mathcal{F}$ (according to (i$'$)), which is a contradiction.

Assume that $A, B \in \mathcal{F}$. If $A \cap B = \emptyset$, then $A \in B^c$. Since $B \in \mathcal{F}$, $B^c \notin \mathcal{F}$, so $A \notin \mathcal{F}$ (according to (i$'$)) which is a contradiction. This shows that any two members of $\mathcal{F}$ have a non-empty intersection, which evidently implies that $\mathcal{F}$ has at most one single-element member. By considering different cases it can be seen that, up to permutations of $\{1,2,3,4,5\}$, for single-element and two-element members of $\mathcal{F}$ only the following cases are possible:

- No member of $\mathcal{F}$ has at most two elements.

  In this case, assuming $w_1 = w_2 = \cdots = w_5 = 1$ satisfies the two conditions.

- Members of $\mathcal{F}$ that have at most two elements are $\{1,2\}, \{1,3\}$ and $\{2,3\}$.

  w_4 and w_5 can be set to zero. Next, note that inequalities $w_1 + w_2 > w_3$, $w_2 + w_3 > w_1$ and $w_3 + w_1 > w_2$ must hold for w_1, w_2 and w_3. Therefore, w_1, w_2 and w_3 are sides of a triangle. For instance, setting $w_1 = 3$, $w_2 = 4$ and $w_3 = 5$ satisfies the two conditions.

- The only member of $\mathcal{F}$ that has at most two elements is $\{1,2\}$.

  Due to symmetry, set $w_1 = w_2 = a$ and $w_3 = w_4 = w_5 = b$. a and b should satisfy $2a > 3b$, $a > 0$ and $b > 0$. Obviously, setting $a = 2$ and $b = 1$ satisfies this condition.

- Members of $\mathcal{F}$ that have at most two elements are $\{1,2\}$ and $\{1,3\}$.

  A similar argument shows that, for instance, $(w_1, w_2, w_3, w_4, w_5) = (3,2,2,1,1)$ works.

- Members of $\mathcal{F}$ that have at most two elements are $\{1,2\}, \{1,3\}$ and $\{1,4\}$.

  In this case, setting $(w_1, w_2, w_3, w_4, w_5) = (5,2,2,2,0)$ works.

- Members of $\mathcal{F}$ that have at most two elements are $\{1,2\}, \{1,3\}$, $\{1,4\}$ and $\{1,5\}$.

  In this case, setting $(w_1, w_2, w_3, w_4, w_5) = (5, 2, 2, 2, 2)$ works.

- Members of $\mathcal{F}$ that have at most two elements are $\{1\}, \{1,2\}, \{1,3\}$, $\{1,4\}$ and $\{1,5\}$.

  In this case, setting $(w_1, w_2, w_3, w_4, w_5) = (5, 1, 1, 1, 1)$ works.

Upshot.

Members of $\mathcal{F}$ that have at most two elements	w_1	w_2	w_3	w_4	w_5
No such member	1	1	1	1	1
$\{1,2\}, \{1,3\}$ and $\{2,3\}$	3	4	5	0	0
$\{1,2\}$	2	2	1	1	1
$\{1,2\}$ and $\{1,3\}$	3	2	2	1	1
$\{1,2\}, \{1,3\}$ and $\{1,4\}$	5	2	2	2	0
$\{1,2\}, \{1,3\}, \{1,4\}$ and $\{1,5\}$	5	2	2	2	2
$\{1\}, \{1,2\}, \{1,3\}, \{1,4\}$ and $\{1,5\}$	5	1	1	1	1

3. First, a lemma is proved.

Lemma. *For each natural number j, there exists a bijection h_j from $\mathbb{R}$ to interval $(j, j+1)$.*

Proof. Obviously, $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is a bijection, as is $l_j(x) = \frac{1}{\pi}x + j + \frac{1}{2}$ from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $(j, j+1)$, so $h_j(x) = l_j(\arctan(x))$ is the desired fuction.

Denote by $f_{i,j}$ the fuction in cell (i, j) of the table. For each $x \in (1,2) \cup (2,3) \cup \cdots \cup (n, n+1)$ and each $1 \le i, j \le n$, define $g_i(x) := f_{i,j}\big(h_j^{-1}(x)\big)$, and for other values of x define $g_i(x)$ arbitrarily. Therefore, for each real number x,

$$h_j(x) \in (j, j+1) \;\Rightarrow\; g_i(h_j(x)) = f_{i,j}(h_j^{-1}(h_j(x))) = f_{i,j}(x).$$

Comment. The statement is also true for integer functions (i.e., functions from $\mathbb{Z}$ to $\mathbb{Z}$). A similar solution works, since for each $1 \le i \le n$, there exist a bijection between $\mathbb{Z}$ and integers equivalent to i modulo n, and these sets are all disjoint.

4. Let ω be the circumcircle of triangle AXY. Since the reflection of line BC with respect to XY is tangent to ω, the reflection of ω with respect to XY is tangent to line BC at a point that is denoted by T.

Let P be the second intersection point of circumcircles of triangles BXT and CYT (other than T). It is claimed that circumcircles of triangles BOC and AXY are tangent at P.

First, note that since T lies on the reflection of ω with respect to XY, $\angle XTY = \angle XAY = \angle BAC$. Therefore,

$$\begin{aligned}\angle BPC &= \angle BPT + \angle CPT = \angle BXT + \angle CYT\\ &= 360^\circ - \angle AXT - \angle AYT = \angle XAY + \angle XTY\\ &= 2\angle BAC = \angle BOC.\end{aligned}$$

This implies that P lies on the circumcircle of triangle BOC. On the other hand,

$$180^\circ - \angle BAC = \angle XBT + \angle YCT = \angle XPY.$$

Therefore, P lies on the circumcircle of triangle AXY. Now it suffices to show that these two circles have the same tangent line at P, or equivalently, $\angle BPX = \angle BCP + \angle XAP$. Since $\angle BCP = \angle TYP$ and $\angle XAP = \angle XYP$, $\angle BCP + \angle XAP = \angle TYP + \angle XYP = \angle XYT$. However, according to the assumptions, $\angle XYT = \angle XTB = \angle BPX$, which completes the proof.

5. Note that

$$\sum_{1\le x_i\le p-1} 1 - (x_1^4 + x_2^4 + x_3^4 + x_4^4)^{p-1} \overset{p}{\equiv}$$

$$(p-1)^4 - (p-1)^4 \overbrace{\sum_{\frac{p-1}{4}|\theta_i;\sum\theta_i = p-1} \binom{p-1}{\theta_1,\theta_2,\theta_3,\theta_4}}^{A}.$$

To solve the problem, it is enough to show that $A \overset{p}{\not\equiv} 1$.

$$\begin{aligned}A &\overset{p}{\equiv} 4\binom{p-1}{p-1} + 12\binom{p-1}{\frac{p-1}{4}, 3\frac{p-1}{4}} + 12\binom{p-1}{\frac{p-1}{4}, \frac{p-1}{4}, \frac{p-1}{2}} + \binom{p-1}{\frac{p-1}{4}, \frac{p-1}{4}, \frac{p-1}{4}, \frac{p-1}{4}}\\ &\overset{p}{\equiv} 4 + \frac{12(p-1)!}{(\frac{p-1}{4})!(\frac{3(p-1)}{4})!} + \frac{6(p-1)!}{(\frac{p-1}{2})!^2} + \frac{12(p-1)!}{(\frac{p-1}{2})!(\frac{p-1}{4})!^2} + \frac{(p-1)!}{(\frac{p-1}{4})!^4}\\ &\overset{p}{\equiv} 4 + \frac{12(p-1)!}{(-1)^{\frac{p-1}{4}}(p-1)!} + \frac{6(p-1)!}{-1} + \frac{12(p-1)!}{(\frac{p-1}{2})!(\frac{p-1}{4})!^2} + \frac{(p-1)!}{(\frac{p-1}{4})!^4}\\ &\overset{p}{\equiv} 4 - 12 + 6 + 12\frac{(\frac{p-1}{2})!}{(\frac{p-1}{4})!^2} + \frac{-1}{(\frac{p-1}{4})!^2}.\end{aligned}$$

If $a \overset{p}{\equiv} (\frac{p-1}{2})!$ and $b \overset{p}{\equiv} \frac{1}{(\frac{p-1}{4})!^2}$, then $A \overset{p}{\equiv} -2 + 12ab - b^2$; therefore,

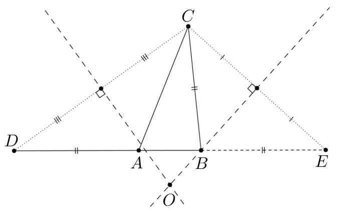
$$-A + 1 \overset{p}{\equiv} b^2 - 12ab + 3 \overset{p}{\equiv} (b - 6a)^2 - 36a^2 + 3 \overset{p}{\equiv} (b - 6a)^2 + 39.$$

And according to the assumption, $\left(\frac{-39}{p}\right) = -1$, which completes the proof.

6. It is claimed that XY is the perpendicular bisector of EF. To show it, first a lemma is proved.

Lemma. *Let D be a point on the extension of side AB of triangle ABC such that A is between B and D, and $AD = BC$. Let O be the intersection point of external angle bisector of vertex B in triangle ABC and the perpendicular bisector of DC. Then O lies on the perpendicular bisector of AB.*

Proof. Let E be a point on the extension of side AB of triangle ABC such that B is between E and A and $BE = BC$. Since triangle EBC is isosceles, the external angle bisector of B is the perpendicular bisector of EC. Therefore, O is the intersection point of perpendicular bisectors of AC and CE, which means O is the circumcenter of triangle DEC, so O lies on the perpendicular bisector of DE. Since $BE = AD$, this line is the perpendicular bisector of AB as well, which completes the proof.



Note that AE and AF are the external angle bisectors of triangle DEF. Using the lemma for triangle DEF and point Q on the extension of EF implies that X lies on the perpendicular bisector of EF. By a similar argument, Y lies on it too, which means that XY is the perpendicular bisector of EF. On the other hand, since points E, F, C and B lie on a circle with diameter BC ($\angle BEC = \angle BFC = 90°$), the midpoint of BC is the center of this circle and lies on the perpendicular bisector of EF. Therefore, the midpoint of BC lies on line XY.

7. Suppose that $P = P_1P_2 \dots P_n$, $P' = P'_1P'_2 \dots P'_n$ and that AP_i's are a permutation of $A'P'_i$'s. Let O and O' be the centers of P and P', respectively. Without loss of generality it can be assumed that A is

inside or on the perimeter of triangle OP_1P_2 and $AP_1 \leq AP_2$; A' is inside or on the perimeter of $O'P_1'P_2'$ and $A'P_1' \leq A'P_2'$. Using the following lemma AP_i's and $A'P_i'$'s can be sorted in ascending order.

Lemma. *In a situation as described above, if n is odd, then*

$$\begin{aligned} &AP_1 \leq AP_2 \leq AP_n \leq AP_3 \leq AP_{n-1} \leq AP_4 \leq \cdots \\ &\qquad\qquad \leq AP_{\frac{n+3}{2}-1} \leq AP_{\frac{n+3}{2}+1} \leq AP_{\frac{n+3}{2}}, \end{aligned}$$

and if n is even, then

$$AP_1 \leq AP_2 \leq AP_n \leq AP_3 \leq \cdots \leq AP_{\frac{n}{2}+2} \leq AP_{\frac{n}{2}+1}.$$

Proof. The assertion can be easily proved by looking at the position of A with respect to the perpendicular bisector of P_2 and P_n (which is OP_1), then the perpendicular bisector of P_n and P_3, then the perpendicular bisector of P_3 and P_{n-1} and so on.

It can be assumed that similar inequlities hold for $A'P_i'$'s. So the problem statement is equivalent to,

$$AP_1 = A'P_1' \leq AP_2 = A'P_2' \leq AP_n = A'P_n' \leq AP_3 = A'P_3' \leq \cdots .$$

Assume that P is greater than P'. For each $1 \leq i \leq n$, ($P_{n+1} = P_1$, $P_{n+1}' = P_1'$)

$$\begin{aligned} &AP_i = A'P_i',\ AP_{i+1} = A'P_{i+1}', P_iP_{i+1} > P_i'P_{i+1}' \\ &\qquad\qquad \Rightarrow \angle P_iAP_{i+1} > \angle P_i'A'P_{i+1}'. \end{aligned}$$

Hence

$$360^\circ = \sum_{i=1}^{n} \angle P_iAP_{i+1} > \sum_{i=1}^{n} \angle P_i'A'P_{i+1}' = 360^\circ,$$

which is a contradiction. This shows that P and P' are equal.