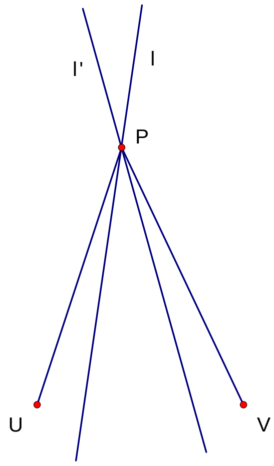
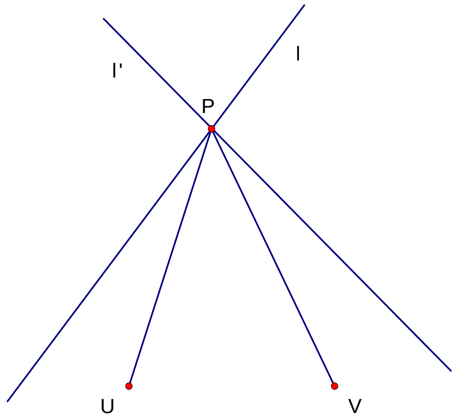
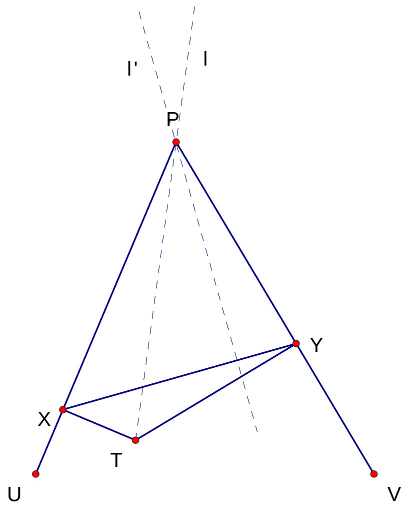
Isogonal Conjugate

Definition 1. Given an acute angle $\angle UPV$ and a line l passing through P, the isogonal of l wrt $\angle UPV$ is defined to be the line l' symmetric to l about the internal angle bisector of $\angle UPV$.

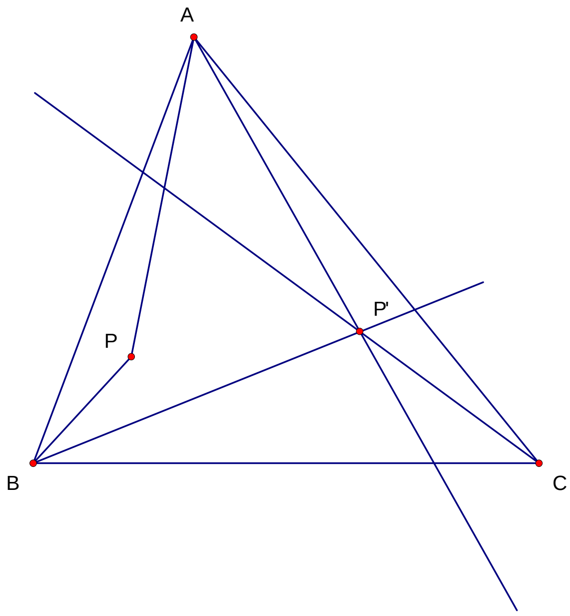




Theorem 1. If X, Y is the orthogonal projections of T on lines UP, VP respectively, then the isogonal of line TP is perpendicular to line XY.



Theorem 2. (Isogonal Conjugate Theorem) Let ABC be a triangle and P be a point distinct from A, B, C. Then the isogonal of lines AP, BP, CP wrt $\angle CAB, \angle ABC, \angle BCA$ respectively are concurrent. The concurrent point P' is called the isogonal conjugate of P wrt to $\vee ABC$.

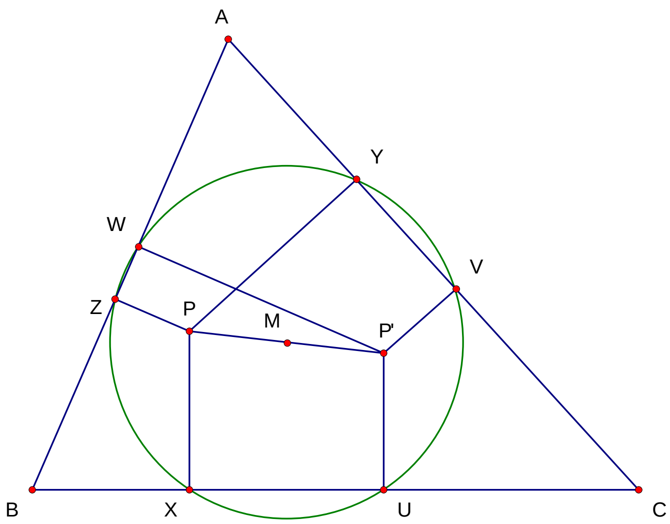


Theorem 3. The isogonal conjugate P' of P is the circumcenter of $\triangle XYZ$ where X,Y,Z are the reflections of P about BC, CA, AB respectively.

Theorem 4. P is a point at infinity if and only if P' lies on the circumcircle of $\triangle ABC$.

Theorem 5. (Pedal Circle Theorem) Let M be the midpoint of P and its isogonal conjugate P'. Let X,Y,Z be the orthogonal projections of P and U,V,W be the orthogonal projections of P' on BC, CA, AB respectively. Then U,V,W,X,Y,Z all lies on a circle centered at M. Moreover

$$PX \cdot P'U = PY \cdot P'V = PZ \cdot P'W .$$



Theorem 6. Let P and P' be two isogonal conjugates wrt $\triangle ABC$. Then the isogonal of AP wrt $\angle BPC$ and the isogonal of AP' wrt $\angle BP'C$ are symmetric to each other about BC.

Theorem 7. Let P be a point in the plane of $\triangle ABC$, and let D, E, F be the reflections of P about the perpendicular bisectors of BC, CA, AB respectively. Then the points P, D, E, F lie on a circle centered at the circumcenter O of $\triangle ABC$. Moreover $\triangle DEF \sim \triangle ABC$.

Definition 2. If G is the centroid of $\triangle ABC$ and T is an arbitrary point in the plane, then the image of the point T under the homothety with center G and factor $-\frac{1}{2}$ is called the complent of T wrt $\triangle ABC$.

Theorem 8. Let P be a point in the plane of $\triangle ABC$, and let D, E, F be the reflections of P about the perpendicular bisectors of BC, CA, AB respectively. Denote by A_M, B_M, C_M the midpoints of BC, CA, AB and by D_M, E_M, F_M the midpoints of EF, FD, DE. Let Q' be the complement of P' wrt $\triangle ABC$. Then the lines $A_M D_M, B_M E_M, C_M F_M$ pass through the point Q'.

Reference:

1. Isogonal Conjugation with respect to a triangle --- *Darij Grinberg*