Chapter 2

Basic Concepts

Contents

In this chapter, we explore the basic concepts and definitions in the theory of functional equations. As an appetizer, let's start with notations.

2.1 Notation and Definition

- $\mathbb{C}$: the set of complex numbers.
- $\mathbb{R}$: the set of real numbers.
- $\mathbb{R}^{\geq 0}$: the set of non-negative real numbers.
- $\mathbb{R}^{+}$: the set of positive real numbers.
- $\mathbb{R}^{-}$: the set of negative real numbers.
- $\mathbb{Q}$: the set of rational numbers.
- $\mathbb{Q}^{+}$: the set of positive rational numbers.
- $\mathbb{Q}^{\geq 0}$: the set of non-negative rational numbers.
- $\mathbb{Z}$: the set of integers.
- $\mathbb{N}$: the set of positive integers. Also denoted by $\mathbb{Z}^{+}$.
- $\mathbb{N}_0$: defined as $\mathbb{N} \cup \{0\}$. Also denoted by $\mathbb{Z}^{\geq 0}$.
- $\gcd(a, b)$: the greatest common divisor of a and b. Also denoted by (a, b).
- $\operatorname{lcm}(a, b)$: the least common multiple of a and b. Also denoted by $[a, b]$.
- *WLOG*: Without Loss Of Generality.
- *RHS* and *LHS*: Right Hand Side and Left Hand Side.
- *Assertion*: an expression in a few variables. You constantly see phrases like

  "Let $P(x, y)$ be the assertion $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$."

  This is because we don't want to write the equation $f(x+y) = f(x)+f(y)$ every time we want to plug in new values of x and y into the equation. For instance, instead of writing

"By plugging in $x = 1$ and $y = 2$ into the equation
$f(x + y) = f(x) + f(y)$, we find that..."

we write

"Using $P(1, 2)$, we get..."

- *General solution*: a family of solutions to a functional equation which respects these two conditions:
  - Any function in the given form indeed is a solution, and
  - Any solution can be written in the given form.

Read more examples about this in section 2.3.3 of chapter 2.

2.2 Concepts

2.2.1 Injection, Surjection, and Bijection

If you are reading this sentence, you already know what a function is. So, I'm not going to define that. However, the definitions of an injective or surjective function might not be obvious for the reader. Instead of giving formal definitions, I would like to explain these concepts in examples.

- When we write $f : A \to B$ is a function, we mean that the domain of f is A and its codomain is B. The domain of f is the set of values that f can act on. For instance, if the domain of f is the interval $[1, 2]$, it means that $f(x)$ *is defined* for any $x \in [1, 2]$.

- The codomain of f, on the other hand, is a set that contains all the values that the output of f can get. Please stop here and read the last sentence again. With this definition, we do not require all elements in the codomain of f to be the image of some element in the domain of f. Let me give you an example to make this clear. Suppose that $f : [1, \infty) \to \mathbb{R}$ is given such that

$$f(x) = \frac{1}{x}, \quad \forall x \in [1, \infty). \tag{2.1}$$

  Here, $\mathbb{R}$ is the codomain of f. However, there is no $x \in [1, \infty)$ for which $f(x) = 2$. In other words, there is an element in the codomain of f which is not *admitted* by any input.

- It now makes sense to define the set of all possible outputs of f. This set is called the *image* of f and is denoted (usually) by $\mathrm{Im}(f)$. In other words,

$$\mathrm{Im}(f) = \{f(x) : x \text{ is in the domain of } f\}.$$

  For instance, in the example given in (2.1), the image of f is $(0, 1]$. This is because the reciprocal of a number $x > 1$ is always positive and < 1.

- So far, we have found out that the image of a function is not necessarily the same as its codomain.

- A very natural question that comes up here is the following: when is the codomain of f equal to $\mathrm{Im}(f)$? A function with this latter property is called a *surjection*, a *surjective* function, or sometimes an *onto* function. Consider the example in (2.1) again (assuming the codomain is $\mathbb{R}$). This function is not a surjection. However, if we set the codomain to be $(0, 1]$, the discussion in the previous part implies that the new function is surjective. Formally, a function $f : A \to B$ is surjective if for any $b \in B$, there is some $a \in A$ such that $f(a) = b$.

- When talking about surjectivity, for any b in the codomain of f, we only care about the *existence* of an element a in the domain such that $f(a) = b$. If such an element exists, then the function is surjective. However, in order for the function to be *injective*, we want each element of the codomain of f to be mapped to by **at most** one element in the domain. That is, there should not exist any two different elements in the codomain which are mapped to by the same element in the domain. An injective function is sometimes called *one-to-one*. Formally, $f : A \to B$ is injective if and only if $f(a_1) = f(a_2)$ yields $a_1 = a_2$ for all $a_1, a_2 \in A$. The function $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$ defined by $f(x) = x^2$ is surjective (why?) but not injective. The reason is that for any non-zero $x \in \mathbb{R}$, we have $f(x) = x^2 = (-x)^2 = f(-x)$ but $x \neq -x$.

- A function that is both injective and surjective is called a *bijective* function or a *bijection*. Some people tend to call a bijection a *one-to-one correspondence*, but not me. Can you think of a bijective function now?

Now, let's see an example of how we prove surjectivity or injectivity in a given functional equation. Consider first the following problem from Vietnam National Olympiad 2017:

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f(xf(y) - f(x)) = 2f(x) + xy, \quad \forall x, y \in \mathbb{R}.$$

Call this assertion $P(x, y)$. We aim to prove that f is surjective. That is, we want to show that for any $b \in \mathbb{R}$, there exists some $a \in \mathbb{R}$ such that $f(a) = b$. Using $P(1, y)$, we arrive at the equation

$$f(f(y) - f(1)) = 2f(1) + y, \quad \forall y \in \mathbb{R}.$$

Consider $b \in \mathbb{R}$. Notice that the above equation holds for all real values of y. So, if we choose y so that $2f(1) + y = b$, or equivalently $y = b - 2f(1)$, we would have $f(a) = b$, where $a = f(y) - f(1) = f(b - 2f(1)) - f(1)$. This is exactly what we wanted to prove! Hence, f is surjective.

The following example is due to Dan Schwarz (mavropnevma), whom I truly miss.

Example 2. Given $f : \mathbb{R} \to \mathbb{R}$ that satisfies

$$\left(f(2^{x^3+x})\right)^2 - f(2^{2x}) \le 2$$

and

$$\left(f(2^{2x})\right)^3 - 3f(2^{x^3+x}) \ge 2$$

for all real x, we want to show that f is not injective. The point here is to look for reals x such that $2^{x^3+x} = 2^{2x}$. One can easily find by a simple search for such numbers that this happens for $x \in \{-1, 0, 1\}$. In fact, if we let $a = -1$, $b = 0$, and $c = 1$, then

$$\begin{aligned} u &= 2^{a^3+a} = 2^{2a} = \frac{1}{4}, \\ v &= 2^{b^3+b} = 2^{2b} = 1, \\ w &= 2^{c^3+c} = 2^{2c} = 4. \end{aligned}$$

Therefore, for any $z \in \{u, v, w\}$, using the given inequalities in the problem, we find that

$$\begin{aligned} f(z)^2 - f(z) \le 2 &\implies (f(z)+1)(f(z)-2) \le 0 \\ &\implies f(z) \in [-1, 2], \\ f(z)^3 - 3f(z) \ge 2 &\implies (f(z)+1)^2 (f(z)-2) \ge 0 \\ &\implies f(z) \in \{-1\} \cup [2, \infty). \end{aligned}$$

Since we want both $f(z) \in [-1, 2]$ and $f(z) \in \{-1\} \cup [2, \infty)$ to happen simultaneously, this means that $f(z) \in \{-1, 2\}$. Hence,

$$f\left(\frac{1}{4}\right), f(1), f(4) \in \{-1, 2\},$$

and so two of these values are equal to each other. This directly implies that f is not injective.

2.2.2 Monotone Functions

Definition 1 (Increasing Function)**.** A function $f : A \to B$ is called *increasing* on an interval $I \subseteq A$ if for any $a_1, a_2 \in I$ with $a_1 > a_2$, we have $f(a_1) \ge f(a_2)$.

Consider the function $f : [1, \infty) \to \mathbb{R}$ defined by $f(x) = x^3 - x$. This function is increasing. Can you prove it? One way is to use the definition given above. The other way, which is much simpler, is to use calculus: $f'(x) > 0$ for all x in the domain.

Notice that if for all $a_1, a_2 \in I$ with $a_1 > a_2$, we have $f(a_1) > f(a_2)$, then the function is called *strictly increasing* (or *monotone increasing*) on I. Try to show that the function given in the example above ($f(x) = x^3 - x$, $x \in [1, \infty)$) is a strictly increasing function.

Definition 2 (Decreasing Function)**.** A function $f : A \to B$ is called *decreasing* on an interval $I \subseteq A$ if for any $a_1, a_2 \in I$ with $a_1 > a_2$, we have $f(a_1) \le f(a_2)$.

You should now be able to guess the definition of a strictly decreasing function. Sometimes people use the word *nonincreasing function* to emphasize that the function is decreasing, but not strictly decreasing. The same goes for nondecreasing functions.

Definition 3 (Monotone Function)**.** A function is called *monotone* if it is either always increasing or always decreasing.

So far, the examples we have seen where functions with domain either $\mathbb{R}$ or $\mathbb{Q}$. Let's see an example of an *arithmetic function* now. An arithmetic function is any function with domain $\mathbb{N}$. That is, a function that acts on natural numbers $1, 2, 3, \ldots$. We usually denote arithmetic functions with $f(n)$ (instead of $f(x)$) because **n**aturally, n feels more like an integer. Note that arithmetic functions need not give integer values in their output. That is, the codomain of an arithmetic function is not necessarily $\mathbb{N}$. For instance, the function $f : \mathbb{N} \to \mathbb{Q}$ defined by $f(n) = 1/n$ for all naturals n is an arithmetic function, which is strictly decreasing (why?). Another example is $f : \mathbb{N} \to \mathbb{C}$, given by $f(n) = \log n$ for all $n \in \mathbb{N}$.

Example 3. An arithmetic function $f : \mathbb{N} \to \mathbb{Z}$ satisfies the following inequality for all positive integers n:

$$(f(n + 1) - f(n))(f(n + 1) + f(n) + 4) \le 0.$$

We want to prove that f is not injective. First, notice that the expresion may be written as

$$\begin{aligned}
&(f(n+1)-f(n))(f(n+1)+f(n)+4) \\
&\qquad = (f(n+1))^2-(f(n))^2+4(f(n+1)-f(n)) \\
&\qquad = \Big((f(n+1))^2+4f(n+1)\Big)-\Big((f(n))^2+4f(n)\Big) \\
&\qquad = \Big((f(n+1))^2+4f(n+1)+4\Big)-\Big((f(n))^2+4f(n)+4\Big) \\
&\qquad = (f(n+1)+2)^2-(f(n)+2)^2 .
\end{aligned}$$

So, the given inequality becomes $(f(n+1)+2)^2-(f(n)+2)^2 \le 0$, or

$$(f(n+1)+2)^2 \le (f(n)+2)^2 .$$

Define a new arithmetic function $g:\mathbb{N}\to\mathbb{N}$ by $g(n)=(f(n)+2)^2$. Then, we get $g(n+1)\le g(n)$ for all $n\in\mathbb{N}$. This means that the function g is decreasing. To see this, notice that if $a>b$, then

$$g(a)\le g(a-1)\le g(a-2)\le\cdots\le g(b+1)\le g(b).$$

So, $g(a)\le g(b)$ and g is a decreasing function. But since the output of g are positive integers (why?), the function g must be eventually a constant function because the smallest $g(n)$ can get is 1. So, let $g(n)=(f(n)+2)^2=c^2$ for all $n\ge n_0$, where c and n_0 are positive integers. This means that $f(n)$ can take only the values $-2-c$ and $-2+c$, and hence is not injective (see Example (2) for a reasoning).

2.2.3 Even and Odd Functions

Example 4. Consider the functional equation

$$f(xf(y)-y)+f(xy-x)+f(x+y)=2xy,$$

where the domain and codomain of f are supposed to be $\mathbb{R}$. Let's call the latter assertion $P(x,y)$. Then, $P(0,0)$ implies

$$f(0)+f(0)+f(0)=0,$$

or simply $f(0)=0$. Also, $P(x,0)$ gives

$$f(-x)+f(x)=0,$$

which means that $f(x)=-f(-x)$ holds for all reals x. In this case, we call f an *odd* function.

Example 5. Now, consider the functional equation

$$f(x+y)+f(x-y)=2\max(f(x),f(y)),$$

where f is defined to be from $\mathbb{Q}$ to $\mathbb{Q}$. Let's call the given assertion $P(x,y)$. Take any two rationals x and y, and suppose, WLOG, that $f(x) \geq f(y)$. Then, $P(x,y)$ gives

$$f(x+y)+f(x-y)=2f(x).$$

On the other hand, $P(y,x)$ yields

$$f(x+y)+f(y-x)=2f(x).$$

Comparing, we see that $f(x-y)=f(y-x)$ for all $x,y \in \mathbb{Q}$. Now, let $a=x-y$ to obtain $f(a)=f(-a)$ for all $a \in \mathbb{Q}$. In this scenario, we say that f is an *even* function.

For easier referencing, I'm including the accurate definitions of even and odd functions here:

Definition 4 (Even Function)**.** Suppose that A is a set with the property that if $a \in A$, then $-a \in A$. Let $f : A \to B$ be a function such that $f(x)=f(-x)$ for all $x \in A$. Then, we call f an *even* function.

Definition 5 (Odd Function)**.** Suppose that A is a set with the property that if $a \in A$, then $-a \in A$. Let $f : A \to B$ be a function such that $f(x)=-f(-x)$ for all $x \in A$. Then, we call f an *odd* function.

If f is odd, then we can easily find by plugging $x=0$ that $f(0)=0$.

2.2.4 Involutive Functions

Definition 6. An *involutive* function, or simply an *involution*, is a function $f : A \to B$ such that $f(f(x))=x$ happens for all $x \in A$. Notice that this directly implies $A=B$ and so f must be a surjection.

Example 6. If $f : A \to A$ is an involution, then f is a bijection because an involution is in fact a permutation, and hence f is a one-to-one and onto map. However, the converse is not necessarily true. Take for instance $f : \{0,2,4\} \to \{0,2,4\}$ defined by $f(n)=n+2 \pmod 4$, for all $n \in A$. It is clear that $f(0)=2$ and $f(f(0))=f(2)=4 \neq 0$.

Example 7. Suppose that $f : \mathbb{N} \to \mathbb{N}$ is an arithmetic function satisfying $f(x + f(y)) = f(x) + y$ for all positive integers x, y. Let's show that f is involutive. Let the given assertion be denoted by $P(x, y)$. Then, $P(x, y)$ and $P(y, x)$ give

$$f(x + f(y)) = y + f(x), \tag{2.2}$$
$$f(y + f(x)) = x + f(y). \tag{2.3}$$

Now, if you look closely:

$$x + f(y) \overset{f}{\mapsto} y + f(x) \overset{f}{\mapsto} x + f(y).$$

Now, if we can prove that every positive integer is representable as $x + f(y)$ for some $x \in \mathbb{Z}^{\geq 0}$ and $y \in \mathbb{N}$, then it implies that f is involutive. This is very easy for $n \geq 2$: to get $n = x+f(y)$, choose $y = 1$ and $x = n-f(1)$ (note that we want $x \geq 1$ to be in the domain of the function and that's why we should care about n being at least 2). So, we only need to prove that $f(1) = 1$. We now use an idea which is due to Gabriel (harazi). Let $P(x, y)$ be the assertion given in (2.2). Computing $P(x, 1)$ gives $f(x+f(1)) = 1+f(x)$ and $P(f(1), x)$ gives $f(f(1)+f(x)) = x+f(f(1))$ for all $x \geq 1$. Now, $P(f(x), 1)$ implies

$$f(f(x) + f(1)) = 1 + f(f(x)).$$

Combining the latter two identities, we get

$$x + f(f(1)) = 1 + f(f(x)).$$

Hence, to show that $f(f(1)) = 1$, it suffices to find a natural x such that $f(f(x)) = x$. We have actually proved a much stronger thing: $f(f(x)) = x$ for all $x \geq 2$. So, $f(f(1)) = 1$ and we arrive to the conclusion that f is involutive, i.e., $f(f(x)) = x$ for all $x \geq 1$. Now, if we check $P(x, f(y))$, we see that f satisfies $f(x+y) = f(x)+f(y)$, i.e., f is an *additive* function. We discuss these functions as well as their solutions in the general case (when the domain is $\mathbb{Q}$ or $\mathbb{R}$) in section 2.3.

To sum it up, any additive involutive function f satisfies the functional equation in this example.

2.2.5 Functions Related to Integers

I didn't name this section *arithmetic functions* because, as we saw in section (2.2.2), an arithmetic function has the domain of only natural numbers. In this section, we study functions with a more general domain, say $\mathbb{R}$, that are somehow connected with integers. For instance,

Chapter 4

Selected Solutions

Contents

4.1 Functions Over $\mathbb{C}$

Problem 1. Let

$$f(x) = \begin{cases} z, & \text{if } \Re(z) \geq 0, \\ -z, & \text{if } \Re(z) < 0, \end{cases}$$

be a function defined on $\mathbb{C}$. A sequence $\{z_n\}$ is defined as $z_1 = u$ and for all $n \geq 1$,

$$z_{n+1} = f\left(z_n^2 + z_n + 1\right).$$

Given $\{z_n\}$ is periodic, find all possible values of u.

Solution (by pco)**.** Note that $\Re(f(x)) \geq 0$ whatever x is, and so $\Re(z_n) \geq 0$ for all integers $n \geq 2$. Let $z_n = a + ib$ for some $n \geq 2$ (so that $a \geq 0$). Then,

$$z_n^2 + z_n + 1 = (a^2 - b^2 + a + 1) + i(2ab + b),$$

and therefore,

$$\begin{aligned} |z_n^2 + z_n + 1|^2 &= (a^2 - b^2 + a + 1)^2 + (2ab + b)^2 \\ &= a^2 + b^2 + (a^2 - b^2 + 1)^2 + 4a^2b^2 + 2a(a^2 + b^2 + 1) \\ &\geq a^2 + b^2. \end{aligned}$$

Since $|f(x)| = |x|$, we get

$$|z_{n+1}| \geq |z_n|, \quad \forall n \geq 2.$$

So, periodicity implies $|z_{n+1}| = |z_n|$ and thus $a = 0$ and $b = \pm 1$. Hence, $z_n = \pm i$ for all $n \geq 2$. This means that $u \in \{-i, +i\}$.

Problem 2. Consider the functional equation $af(z) + bf(w^2 z) = g(z)$, where w is a complex cube root of unity, a and b are fixed complex numbers, and $g(z)$ is a known complex function. Prove that the complex function $f(z)$ can be uniquely determined if $a^3 + b^3 \neq 0$

Solution (by pco)**.** Note that

$$\begin{aligned} af(z) + bf(w^2 z) &= g(z), \\ af(wz) + bf(z) &= g(wz), \\ af(w^2 z) + bf(wz) &= g(w^2 z). \end{aligned}$$

If $a = 0$ and $b \neq 0$, first equation uniquely defines $f(z)$. If $a \neq 0$ and $b = 0$, then the first equation uniquely defines $f(z)$. Finally, if $ab \neq 0$, it is easy to cancel $f(wz)$ and $f(w^2 z)$ amongst the three equations and we get

$$(a^3 + b^3) f(z) = \text{some expression not depending on } f.$$

4.2 Functions Over $\mathbb{R}$

4.2.1 Cauchy-type and Jensen-type

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$,

$$f(x) + f(x + f(y)) = y + f(f(x) + f(f(y))).$$

Solution (by pco)**.** Let $P(x, y)$ be the given assertion. Let $a = f(0)$ and $b = f(a)$. Subtracting $P(f(x), 0)$ from $P(a, x)$, we get $f(f(x)) = b - x$. Note that this implies that $f(x)$ is bijective. Then $P(f(x), y)$ implies

$$f(f(x) + f(y)) = x + y - b + f(2b - x - y).$$

And $P(a, x + y)$ implies

$$f(f(0) + f(x + y)) = x + y - b + f(2b - x - y).$$

And so, since f is injective, $f(x + y) + a = f(x) + f(y)$ and $f(x) - a$ is additive. Using this additivity back in original equation, it is easy to show that no such function satisfies the given condition.

Problem 6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for some $a, b \in \mathbb{R}$,

$$f(x)f(y) = x^a f\left(\frac{y}{2}\right) + y^b f\left(\frac{x}{2}\right)$$

holds for all reals x and y.

Solution (by TuZo)**.** If $a = b$, we can denote

$$\frac{f(x)}{x^a} = g(x),$$

so that

$$g(x)g(y) = g(x) + g(y), \quad \forall x, y \in \mathbb{R},$$

and here use the new function $h(x) = g(e^x)$ to get $h(x + y) = h(x) + h(y)$. This is the classic Cauchy equation with the solution $h(x) = kx$ for all real x and any real k.

Problem 7. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $f(xf(y)) = yf(x)$ for all positive reals x and y, and $\lim_{x\to\infty} f(x) = 0$.

Solution (by pco)**.** Clearly, $f(x)$ is bijective: if $f(a) = f(b)$ then

$$bf(x) = f(xf(b)) = f(xf(a)) = af(x)$$

and so $a = b$. We also have

$$f\left(xf\left(\frac{y}{f(x)}\right)\right) = y$$

and so $f(x)$ is surjective. We have $f(xf(1)) = f(x)$ and so, since f is bijective, $f(1) = 1$. Plugging $x = 1$ and $y = t$ then gives $f(f(t)) = t$ for all positive t and so

$$f(xy) = f(xf(f(y))) = f(x)f(y), \quad \forall x, y \in \mathbb{R}^+.$$

Setting then $f(x) = e^{g(\ln x)}$, we get

$$g(x + y) = g(x) + g(y), \quad \forall x, y \in \mathbb{R}.$$

Now, $\lim_{x\to+\infty} g(x) = -\infty$ and so $g(x)$ is upperbounded from a given point and so is linear. Plugging this back in original equation, we get $g(x) = -x$ and so

$$f(x) = \frac{1}{x}, \quad \forall x > 0.$$

4.2.2 Continuity

Problem 9. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y)f(x - y) = (f(x)f(y))^2, \quad \forall x, y \in \mathbb{R}.$$

Solution (by pco)**.** Let $P(x, y)$ be the assertion

$$f(x + y)f(x - y) = f(x)^2 f(y)^2.$$

$P(0, 0)$ gives us $f(0) \in \{-1, 0, 1\}$. So, we break the problem into two cases.

1. If $f(0) = 0$, then $P(x, 0)$ implies $f \equiv 0$, which is indeed a solution.

2. If $f(0) \neq 0$, then $f(x)$ being a solution implies $-f(x)$ is a solution too. So, WLOG, suppose that $f(0) = 1$. If $f(a) = 0$ for some a, then

$$\begin{aligned} P\left(\frac{a}{2}, \frac{a}{2}\right) &\implies f\left(\frac{a}{2}\right) = 0 \\ &\implies f\left(\frac{a}{2n}\right) = 0, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Continuity would then imply $f(0) = 0$. So, no such a exists and we have $f(x) > 0$ for all real x. Let then $g(x) = \ln f(x)$ so that functional equation becomes a new assertion $Q(x, y)$:

$$g(x + y) + g(x - y) = 2g(x) + 2g(y),$$

where g is continuous and $g(0) = 0$. Now,

$$\begin{aligned} Q(0, x) &\implies g(-x) = g(x), \\ Q(x, x) &\implies g(2x) = 4g(x), \\ Q(2x, x) &\implies g(3x) = 9g(x). \end{aligned}$$

Easy induction on n for $Q(nx, x)$ gives

$$g(nx) = n^2 g(x), \quad \forall x \in \mathbb{R}, n \in \mathbb{N}.$$

From there we easily get $g(x) = x^2 g(1)$ for all rational x and continuity allows us to conclude that $g(x) = cx^2$. Hence, $f(x) = e^{cx^2}$ for any real c, and one can easily check that this is indeed a solution. Since we assumed $f(0) = 1$ in the beginning, we must also consider the other solution $f(x) = -e^{cx^2}$.

Problem 11. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that:

$$f(x - f(y)) = f(x) - y$$

for all real numbers x, y.

Solution (by pco)**.** Let $P(x, y)$ be the given assertion.

$$P(f(x), x) \implies f(f(x)) = x + f(0),$$

and so $f(x)$ is bijective. Let then u such that $f(u) = 0$.

$$P(x, u) \implies u = 0,$$

and so $f(0) = 0$. Therefore, $f(f(x)) = x$. Now, $P(x + y, f(y))$ implies that $f(x+y) = f(x)+f(y)$. It is now easy to conclude that problem is equivalent to find all involutive (functions with $f(f(x)) = x$) and additive functions. Now, because of continuity, additivity implies linearity and involutivity implies two solutions: $f(x) = x$ and $f(x) = -x$ for all x.

Problem 13. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xy) + f(x + y) = f(xy + x) + f(y)$$

for all real numbers x, y.

Solution (by pco)**.** Let $P(x, y)$ be the given assertion and let $c = f(1)$ and $d = f(-1)$. Note that if $f(x)$ is a solution then $f(x) + c$ is also a solution. So, WLOG, assume that $f(0) = 0$. Let $a, b > 0$ and define the sequences $\{x_n\}$ and $\{y_n\}$ recursively so that $x_1 = a$ and $y_1 = b$ and

$$x_{n+1} = \frac{x_n}{y_n + 1} \quad \text{and} \quad y_{n+1} = \frac{(x_n + y_n + 1)y_n}{y_n + 1}$$

for all $n \geq 1$. It is easy to show that

$$\lim_{n\to+\infty} = 0 \quad \text{and} \quad \lim_{n\to+\infty} y_n = a + b.$$

Subtracting $P(x_n/y_n + 1, y_n)$ from $P(y_n, x_n/y_n + 1)$, we get

$$f(x_n) + f(y_n) = f(x_{n+1}) + f(y_{n+1}),$$

and so, setting $n \to +\infty$ and using continuity,

$$f(a) + f(b) = f(a + b).$$

And so, since f is continuous,

$$f(x) = cx, \quad \forall x \geq 0.$$

Let then $x \leq -1$. In this case, $P(-x, -1)$ implies $f(x) = cx + c + d$ for all $x \leq -1$. Let then $x \in (-1, 0)$ and choose $y > \max(-x, -\frac{1}{x})$ so that $y(x + 1) > 0$, $x + y > 0$, and $xy < -1$.

$$P(y, x) \implies f(x) = cx + c + d, \quad \forall x \in (-1, 0).$$

Continuity at 0 gives $c + d = 0$ and so $f(x) = cx$, which indeed is a solution. Hence the general solution is $f(x) = ax + b$, $\forall x$ which indeed is a solution, whatever are $a, b \in \mathbb{R}$.