Functional Equations (Version W)

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§1 Reading

Read §3.1-§3.5 of *The OTIS Excerpts*. Alternatively, you can use *Introduction to Functional Equations* from my website (entire handout).

§2 Lecture notes

Example 2.1 (IMO 2017/2)
Solve over $\mathbb{R}$ the functional equation

$$f\left(f(x)f(y)\right) + f(x+y) = f(xy).$$

Walkthrough. This problem is sort of divided into two parts. One is the "standard" part, which is not easy *per se*, but which experienced contestants won't find surprising. However, the argument in the final part is quite nice and conceptual, and much less run-of-the-mill.

We begin with some standard plug/chug.

(a) Find all three linear solutions and convince yourself there are no other polynomial solutions.

(b) Check that if f is a solution, then so is $-f$.

(c) Show that $f(z) = 0$ for some z.

(d) Show that if $f(0) = 0$ then $f \equiv 0$. So we henceforth assume $f(0) \neq 0$.

(e) Using the cancellation trick, prove that if $f(z) = 0$ (and $f(0) \neq 0$) for some z, then $z = 1$. Using the proof of (c), deduce that $f(0) = \pm 1$.

From (b) and (e), we assume $f(0) = 1$, $f(1) = 0$ in what follows, and will try to show $f(x) \equiv 1 - x$. This lets us plug in some more stuff.

(f) Show that $f(x+1) = f(x) - 1$ and compute f on all integer values.

(g) Show that $f(f(x)) = 1 - f(x)$. Thus if f was surjective we would be done. However, this seems hard to arrange, since the original equation has everything wrapped in f's.

(h) Using the triple involution trick, prove that $f(1 - f(x)) = f(x)$. Thus if f was injective, we would also be done.

So we will now prove f is injective: this is the nice part. Assume $f(a) = f(b)$; we will try to prove $a = b$.

(i) Show that if N is a sufficiently large integer, then we can find x and y such that $x + y = a + N$ and $xy = b + N$. Use this to prove that $f(f(x)f(y)) = 0$ for that pair (x, y) and hence thus $f(x)f(y) = 1$.

(j) The previous part shows us how we might think about using the cancellation trick. However, it is basically useless since $f(x)f(y) = 1$ is not really a useful condition.

However, modify the approach of (i) so that instead the conclusion end ups as $f(x)f(y) = 0$ instead. Deduce that $1 \in \{x, y\}$ in that case.

(k) Using the argument in (j) prove that $a = b$.

Some historical lore about this problem: this was shortlisted as A6, and in my opinion too hard for the P2 position, despite being nice for a functional equation. Most countries did poorly, with USA and China having only two solves, but the Korean team had an incredibly high five solves. However, an unreasonably generous 4 points was awarded for progress up to part (h), thus cancelling a lot of the advantage from the Korean team. Thus I was relieved that the Korean team still finished first.

Example 2.2 (USAMO 2002/4)
Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

Walkthrough. This is a classic example of getting down to a Cauchy equation, and then pushing just a little harder.

(a) Find all linear solutions and show there are no higher-degree polynomial ones.

(b) Show that f is odd and hence $f(0) = 0$.

(c) Show that f is additive and $f(x^2) = xf(x)$.

(d) Optionally: prove that the problem statement is *equivalent* to the relations in (c). Hence we can more or less ignore the given equation now.

(e) Prove that f is linear, by inserting $x = a + b$ into $f(x^2) = xf(x)$.

Example 2.3 (ELMO 2014, Evan Chen)
Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$\begin{aligned} f(x+f(y)) &= g(x)+h(y) \\ g(x+g(y)) &= h(x)+f(y) \\ h(x+h(y)) &= f(x)+g(y) \end{aligned}$$

for all real numbers x and y.

Walkthrough. This is a *system* of functional equations. So, just like we eliminate variables from a system of equations, we will try to eliminate functions from our system.

(a) Find first all (injective) linear solutions. (One approach: show first that if f, g, h are linear then the slopes are all 1.)

(b) Let $a = f(0)$, $b = g(0)$, $c = h(0)$. Show that

$$f(x+a) = g(x) + c$$

and similarly. This allows one to rewrite any function in terms of the others.

(c) Use (b) to rewrite $f(x+f(y)) = g(x) + h(y)$ in terms of only the function f and the constants a, b, c.

(d) Use the cancellation trick on the result in (c). Deduce that $f(x) \equiv x + a$.

(e) Show that the functions you found in (a) were the only ones.

Example 2.4 (USAMO 2016/4)
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x)+xy)\cdot f(x-3y) + (f(y)+xy)\cdot f(3x-y) = (f(x+y))^2.$$

Walkthrough. This is sort of a famous example of an unreasonably cruel pointwise trap. It is not hard to get to that point:

(a) Show that $f(0) = 0$.

(b) Prove that f is even (be careful here).

(c) Conclude that for each x, either $f(x) = x^2$ or $f(4x) = 0$.

Isn't that terrible? It takes some work to even reduce that to a “normal” pointwise trap.

(d) Show that $f(z) = 0 \iff f(2z) = 0$. (This is tricky. Take your time.)

(e) Prove that for each x, either $f(x) = x^2$ or $f(x) = 0$.

Now assume there is an $a > 0$ for which $f(a) = 0$. We will now prove that $f \equiv 0$, for any other given $b > 0$. The cleanest approach to this requires the use of an inequality: note that part (e) implies $f(x) \geq 0$ for all x. Thus we will try to force the right-hand side of the given equation to be zero.

(f) Show that we can assume WLOG that $a > b$, by using (d).

(g) Show that we can find positive (x, y) now such that $x - 3y = b$ and $x + y = a$.

(h) Using inequalities, deduce $f(b) = 0$.

Historical lore: everything up to (e) was worth 0 points (harsh even by my standards). Not a forgiving way to start problem 4.

§3 Practice problems

Instructions: Solve [40$\clubsuit$]. If you have time, solve [50$\clubsuit$]. Problems with red weights are mandatory.

> Well, they're laughing at us anyway, we might as well get paid.
>
> P. T. Barnum in *The Greatest Showman*

[2$\clubsuit$] **Problem 1.** Solve $f(m+n) = f(m) + f(n) + mn$ for $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$.

[2$\clubsuit$] **Problem 2** (Iran TST 1996)**.** Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 + y) = f(f(x) - y) + 4f(x)y$$

for all real numbers x and y.

[3$\clubsuit$] **Problem 3** (IMO 2008/4)**.** Find all functions f from the positive reals to the positive reals such that

$$\frac{f(w)^2 + f(x)^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z satisfying $wx = yz$.

[2$\clubsuit$] **Problem 4** (ELMO SL 2010 A3)**.** Solve over $\mathbb{R}$ the functional equation

$$f(x+y) = \max(f(x), y) + \min(f(y), x).$$

[3$\clubsuit$] **Required Problem 5** (IMO 1977/6)**.** The function $f \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ satisfies

$$f(n+1) > f(f(n))$$

for every positive integer n. Show that f is the identity.

[2$\clubsuit$] **Problem 6.** Solve the functional equation $f(f(n)) + f(n)^2 = n^2 + 3n + 3$ for $f \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$.

[3$\clubsuit$] **Problem 7** (EGMO 2017/2)**.** Find the smallest positive integer k for which there exists a coloring of the positive integers $\mathbb{Z}_{>0}$ with k colors and a function $f \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with the following two properties:

(i) For all positive integers m, n of the same color, $f(m+n) = f(m) + f(n)$.

(ii) There are positive integers m, n such that $f(m+n) \neq f(m) + f(n)$.

[3$\clubsuit$] **Problem 8** (Shortlist 2015 A2)**.** Determine all functions $f \colon \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

[3$\clubsuit$] **Problem 9** (TSTST 2018, Evan Chen and Yang Liu)**.** As usual, let $\mathbb{Z}[x]$ denote the set of single-variable polynomials in x with integer coefficients. Find all functions $\theta \colon \mathbb{Z}[x] \to \mathbb{Z}$ such that for any polynomials $p, q \in \mathbb{Z}[x]$,

- $\theta(p+1) = \theta(p) + 1$, and
- if $\theta(p) \neq 0$ then $\theta(p)$ divides $\theta(p \cdot q)$.

[2♣] **Problem 10** (INMO 2015/3). Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ we have

$$f(x^2 + yf(x)) = xf(x+y).$$

[3♣] **Problem 11** (USMCA 2019/4). Solve over $\mathbb{R}$ the functional equation

$$[f(f(x)+y)]^2 = (x-y)\,(f(x)-f(y)) + 4f(x)f(y).$$

[5♣] **Problem 12** (USAMO 2014/2). Find all $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf\left(2f(y)-x\right) + y^2 f\left(2x - f(y)\right) = \frac{f(x)^2}{x} + f\left(yf(y)\right)$$

for all $x, y \in \mathbb{Z}$ such that $x \neq 0$.

[3♣] **Problem 13** (USAMO 2012/4; also Balkan 2012). Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(n!) = f(n)!$ for all positive integers n and such that $m-n$ divides $f(m)-f(n)$ for all distinct positive integers m, n.

[3♣] **Problem 14** (IMO 2004/2). Find all polynomials P with real coefficients such that for all reals a, b, c such that $ab+bc+ca = 0$, we have

$$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c).$$

[5♣] **Required Problem 15** (IMO 2013/5). Suppose a function $f : \mathbb{Q}_{>0} \to \mathbb{R}$ satisfies:

(i) If $x, y \in \mathbb{Q}_{>0}$, then $f(x)f(y) \geq f(xy)$.

(ii) If $x, y \in \mathbb{Q}_{>0}$, then $f(x+y) \geq f(x) + f(y)$.

(iii) There exists a rational number $a > 1$ with $f(a) = a$.

Prove that f is the identity function.

[5♣] **Problem 16** (IMO 2009/3). Suppose that $s_1, s_2, s_3, \dots$ is a strictly increasing sequence of positive integers such that the sub-sequences $s_{s_1}, s_{s_2}, s_{s_3}, \dots$ and $s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \dots$ are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \dots$ is itself an arithmetic progression.

[5♣] **Problem 17** (Korea 2019 camp). Find all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ which obey the following condition: if a, b, c are the side lengths of a triangle with inradius r, then $f(a)$, $f(b)$, $f(c)$ are the side lengths of a triangle with inradius $f(r)$.

[9♣] **Required Problem 18** (Shortlist 2010 A6). Prove that if two functions $f, g\colon \mathbb{N} \to \mathbb{N}$ obey

$$f(g(n)) = f(n) + 1 \quad \text{and} \quad g(f(n)) = g(n) + 1$$

for each positive integer n, then $f = g$.

[1♣] **Mini Survey.** At the end of your submission, answer the following questions.

(a) About how many hours did the problem set take?

(b) Name any problems that stood out (e.g. especially nice, instructive, boring, or unusually easy/hard for its placement).

Any other thoughts are welcome too. Examples: suggestions for new problems to add, things I could explain better in the notes, overall difficulty or usefulness of the unit.

§4 Solutions to the walkthroughs

§4.1 Solution 2.1, IMO 2017/2

The only solutions are $f(x) = 0$, $f(x) = x - 1$ and $f(x) = 1 - x$, which clearly work.
Note that

- If f is a solution, so is $-f$.
- Moreover, if $f(0) = 0$ then setting $y = 0$ gives $f \equiv 0$. So henceforth we assume $f(0) > 0$.

Claim — We have $f(z) = 0 \iff z = 1$. Also, $f(0) = 1$ and $f(1) = 0$.

Proof. For the forwards direction, if $f(z) = 0$ and $z \neq 1$ one may put $(x, y) = \left(z, z(z-1)^{-1}\right)$ (so that $x + y = xy$) we deduce $f(0) = 0$ which is a contradiction.
For the reverse, $f(f(0)^2) = 0$ by setting $x = y = 0$, and use the previous part. We also conclude $f(1) = 0$, $f(0) = 1$. □

Claim — If f is injective, we are done.

Proof. Setting $y = 0$ in the original equation gives $f(f(x)) = 1 - f(x)$. We apply this three times on the expression $f^3(x)$:

$$f(1 - f(x)) = f(f(f(x))) = 1 - f(f(x)) = f(x).$$

Hence $1 - f(x) = x$ or $f(x) = 1 - x$. □

Remark. The result $f(f(x)) + f(x) = 1$ also implies that surjectivity would solve the problem.

Claim — f is injective.

Proof. Setting $y = 1$ in the original equation gives $f(x+1) = f(x) - 1$, and by induction

$$f(x + n) = f(x) - n. \tag{1}$$

Assume now $f(a) = f(b)$. By using (1) we may shift a and b to be large enough that we may find x and y obeying $x + y = a + 1$, $xy = b$. Setting these gives

$$\begin{aligned} f(f(x)f(y)) &= f(xy) - f(x + y) = f(b) - f(a + 1) \\ &= f(b) + 1 - f(a) = 1 \end{aligned}$$

so $f(x)f(y) = 0$ by the claim, hence $1 \in \{x, y\}$. But that implies $a = b$. □

Remark. Jessica Wan points out that for any $a \neq b$, at least one of $a^2 > 4(b-1)$ and $b^2 > 4(a-1)$ is true. So shifting via (1) is actually unnecessary for this proof.

> **Remark.** One can solve the problem over $\mathbb{Q}$ using only (1) and the easy parts. Indeed, that already implies $f(n) = 1 - n$ for all n. Now we induct to show $f(p/q) = 1 - p/q$ for all $0 < p < q$ (on q). By choosing $x = 1 + p/q$, $y = 1 + q/p$, we cause $xy = x + y$, and hence $0 = f\left(f(1 + p/q)f(1 + q/p)\right)$ or $1 = f(1 + p/q)f(1 + q/p)$.
> By induction we compute $f(1 + q/p)$ and this gives $f(p/q + 1) = f(p/q) - 1$.

§4.2 Solution 2.2, USAMO 2002/4

The answer is $f(x) = cx$, $c \in \mathbb{R}$ (these obviously work).

First, by putting $x = 0$ and $y = 0$ respectively we have

$$f(x^2) = xf(x) \quad \text{and} \quad f(-y^2) = -yf(y).$$

From this we deduce that f is odd, in particular $f(0) = 0$. Then, we can rewrite the given as $f(x^2 - y^2) + f(y^2) = f(x^2)$. Combined with the fact that f is odd, we deduce that f is additive (i.e. $f(a + b) = f(a) + f(b)$).

> **Remark** (Philosophy)**.** At this point we have $f(x^2) \equiv xf(x)$ and f additive, and everything we have including the given equation is a direct corollary of these two. So it makes sense to only focus on these two conditions.

Then

$$\begin{aligned} f((x+1)^2) &= (x+1)f(x+1) \\ \implies f(x^2) + 2f(x) + f(1) &= (x+1)f(x) + (x+1)f(1) \end{aligned}$$

which readily gives $f(x) = f(1)x$.

§4.3 Solution 2.3, ELMO 2014, Evan Chen

Let a, b, c denote the values $f(0)$, $g(0)$ and $h(0)$. Notice that by putting $y = 0$, we can get that

$$\begin{aligned} f(x+a) &= g(x) + c \\ g(x+b) &= h(x) + a \\ h(x+c) &= f(x) + b. \end{aligned}$$

Thus the given equation may be rewritten in the form

$$f(x + f(y)) = [f(x+a) - c] + [f(y-c) + b].$$

At this point, we may set $x = y - c - f(y)$ and cancel the resulting equal terms to obtain

$$c - b = f\left(y + a - c - f(y)\right).$$

Since f is injective, this implies that $y + a - c - f(y)$ is constant, so that $y - f(y)$ is constant. Thus, f is linear, and $f(y) \equiv y + a$. Similarly, $g(x) \equiv x + b$ and $h(x) \equiv x + c$.

Finally, we just need to notice that upon placing $x = y = 0$ in all the equations, we get $2a = b + c$, $2b = c + a$ and $2c = a + b$, whence $a = b = c$.

So, the family of solutions is $f(x) = g(x) = h(x) = x + c$, where c is an arbitrary real. One can easily verify these solutions are valid.

Authorship comments I had wanted a system of functional equations for a long time (seeing that we already had functional equations). Initially I had some trivial equation $f(x+y) = g(x) + h(y)$ which dies upon setting $x = y = 0$ everywhere and then just $y = 0$. Then, I tried $f(g(x+y)) = g(x) + h(y)$ for f, g, h injective. I thought I had gotten this to work, but it turns out my solution was actually wrong, which made me very sad.

After a few more attempts I got this, which I spent some time solving. Once I succeeded, I proposed the problem to ELMO.

§4.4 Solution 2.4, USAMO 2016/4

We claim that the only two functions satisfying the requirements are $f(x) \equiv 0$ and $f(x) \equiv x^2$. These work.

First, taking $x = y = 0$ in the given yields $f(0) = 0$, and then taking $x = 0$ gives $f(y)f(-y) = f(y)^2$. So also $f(-y)^2 = f(y)f(-y)$, from which we conclude f is even. Then taking $x = -y$ gives

$$\forall x \in \mathbb{R} : \qquad f(x) = x^2 \qquad \text{or} \qquad f(4x) = 0 \qquad (\bigstar)$$

for all x.

Now we claim

Claim — $f(z) = 0 \iff f(2z) = 0 \qquad (\spadesuit)$.

Proof. Let $(x, y) = (3t, t)$ in the given to get

$$\left(f(t) + 3t^2\right) f(8t) = f(4t)^2.$$

Now if $f(4t) \neq 0$ (in particular, $t \neq 0$), then $f(8t) \neq 0$. Thus we have $(\spadesuit)$ in the forwards direction.

Then $f(4t) \neq 0 \overset{(\bigstar)}{\implies} f(t) = t^2 \neq 0 \overset{(\spadesuit)}{\implies} f(2t) \neq 0$ implies the reverse direction, the last step being the forward direction $(\spadesuit)$. $\square$

By putting together $(\bigstar)$ and $(\spadesuit)$ we finally get

$$\forall x \in \mathbb{R} : \qquad f(x) = x^2 \qquad \text{or} \qquad f(x) = 0 \qquad (\heartsuit)$$

We are now ready to approach the main problem. Assume there's an $a \neq 0$ for which $f(a) = 0$; we show that $f \equiv 0$.

Let $b \in \mathbb{R}$ be given. Since f is even, we can assume without loss of generality that $a, b > 0$. Also, note that $f(x) \geq 0$ for all x by $(\heartsuit)$. By using $(\spadesuit)$ we can generate $c > b$ such that $f(c) = 0$ by taking $c = 2^n a$ for a large enough integer n. Now, select $x, y > 0$ such that $x - 3y = b$ and $x + y = c$. That is,

$$(x, y) = \left(\frac{3c+b}{4}, \frac{c-b}{4}\right).$$

Substitution into the original equation gives

$$0 = (f(x) + xy) f(b) + (f(y) + xy) f(3x - y) = (f(x) + f(y) + 2xy) f(b)$$

Since $f(x) + f(y) + 2xy > 0$, if follows that $f(b) = 0$, as desired.