On the Lemoine circumcevian triangle / Darij Grinberg

Let L be the symmedian point of an arbitrary triangle ΔABC. The circumcircle of ΔABC intersects AL at X, BL at Y and CL at Z. Then, the triangle XYZ is the circumcevian triangle of the point L; we call it the **Lemoine circumcevian triangle**. Obviously, we have:

Theorem 1. The triangle ABC and the Lemoine circumcevian triangle XYZ have the same circumcenter.

We intend to prove another theorem ([1]):

Theorem 2. The triangle ABC and the Lemoine circumcevian triangle XYZ have the same symmedian point, i. e. the point L is also the symmedian point of ΔXYZ.

First, we note:

Theorem 3. The triangles ΔALC and ΔZLX are similar.

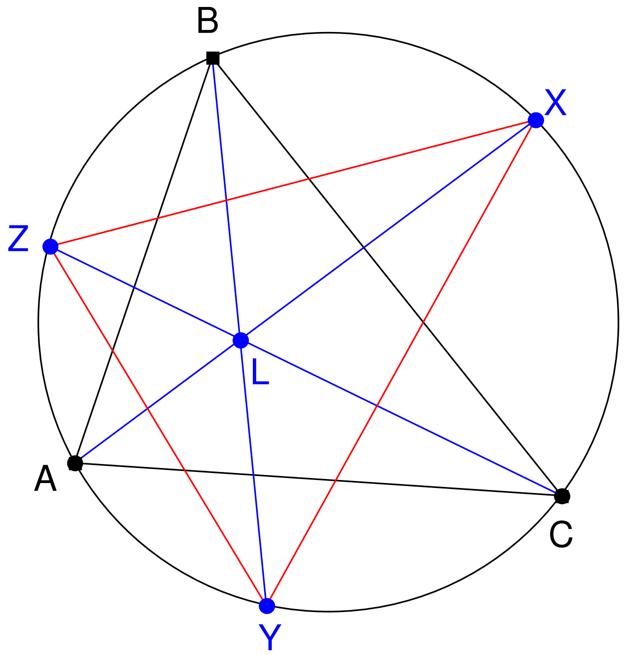

Fig. 1

In fact,

$$\measuredangle ALC = \measuredangle ZLX \quad \text{and}$$
$$\measuredangle LCA = \measuredangle ZCA = \measuredangle ZXA \quad \text{(cyclic)}$$
$$= \measuredangle LXZ,$$

what gives $\Delta ALC \sim \Delta ZLX$.

From this similarity, we obtain that the altitudes of triangles ΔALC and ΔZLX are proportional to the corresponding sides. Hence, if we denote by $d(P;\ g)$ the distance of an arbitrary point P from a line g, then we have

$$\frac{d(L;\ ZX)}{d(L;\ CA)} = \frac{ZX}{CA},$$

i. e.

$$d(L;\ ZX) = ZX \bullet \frac{d(L;\ CA)}{b}.$$

But we know that the symmedian point L has homogeneous trilinear coordinates $L(a : b : c)$ with respect to the original triangle, i. e. there exists a real k for which

$$d(L;\ BC) = ka; \qquad d(L;\ CA) = kb; \qquad d(L;\ AB) = kc.$$

Thus,

$$d(L;\ ZX) = ZX \bullet \frac{d(L;\ CA)}{b} = ZX \bullet k.$$

Analogously, $d(L;\ XY) = XY \bullet k$ and $d(L;\ YZ) = YZ \bullet k$. Thus, the point L has homogeneous trilinear coordinates $L(YZ : ZX : XY)$ with respect to ΔXYZ. Consequently, L is the symmedian point of ΔXYZ, what completes the proof.

Referring to this property, the triangle XYZ is called **cosymmedian triangle** of ΔABC.

As a corollary, we get:

Theorem 4. The triangle ABC and the Lemoine circumcevian triangle XYZ have a common Brocard axis.

Indeed, the two triangles have a common circumcenter and a common symmedian point, and therefore they have a common Brocard axis (since the Brocard axis joins the circumcenter with the symmedian point).

Now we are going to show another property:

Theorem 5. Let

$$1 = YZ \cap CA; \qquad 2 = YZ \cap AB;$$
$$3 = ZX \cap AB; \qquad 4 = ZX \cap BC;$$
$$5 = XY \cap BC; \qquad 6 = XY \cap CA.$$

Then, the lines $14,\ 25$ and 36 pass through the point L (Fig. 2).

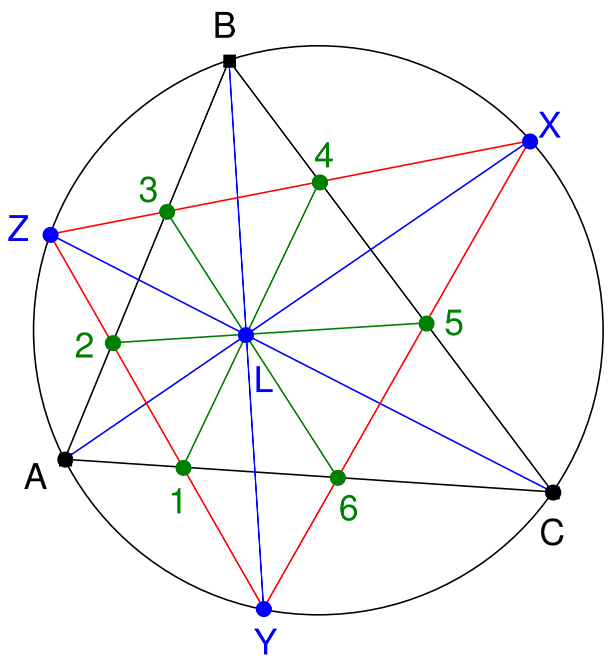

Fig. 2

After a bit of thinking, this result turns out to be quite simple and independent from the presumption that L is the symmedian point. In fact, L can be an arbitrary point. The proof (Fig. 3) uses the Pascal Theorem, applied to the inscribed hexagon $ABCZYX$, yielding that the intersections of opposite sides, i. e. the points

$$AB \cap ZY = 2; \qquad BC \cap YX = 5; \qquad CZ \cap XA = L$$

are collinear. Hence, L lies on 25; analogously, L lies on 14 and on 36.

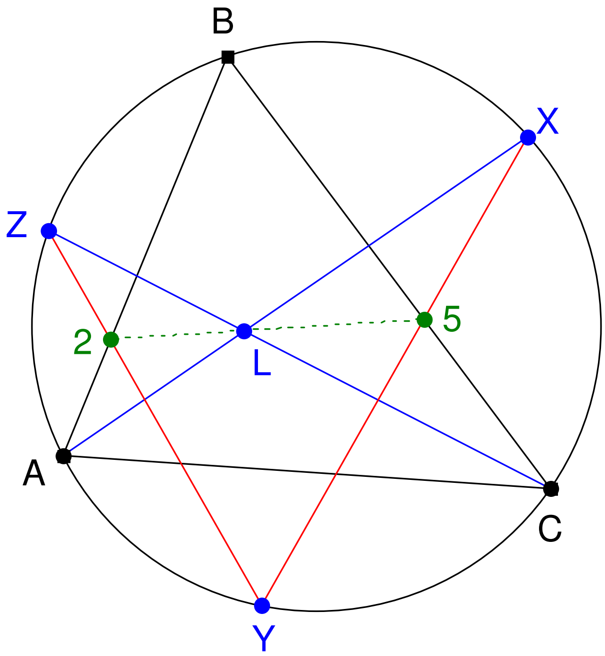

Fig. 3

References

[1] Ross Honsberger: *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, USA 1995.