# Two "Hyacinthos" messages related to my note "New Insight on the Ninepoint Circle" / Darij Grinberg

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**"Hyacinthos" message #6127**

**From:** darij_grinberg
**Date:** Tue Dec 10, 2002 1:49pm
**Subject:** Questions on Feuerbach Theorem, Feuerbach points etc.

Here is one of my geometry-college messages again:

http://www.mathforum.org/epigone/geometry-college/cleehehzim

Questions on Feuerbach Theorem, Feuerbach points etc., 27.9.2002

--- 1. INTRODUCTION ---

The Feuerbach circle and related theorems are a very hard but engaging chapter of triangle geometry. It seems that one can look at them from a lot of points of view and discover new theorems again and again. In the following, I want to share some results and questions.

The notations will be constant: ABC is the reference triangle, O is the circumcenter, I is the incenter, H is the orthocenter, G is the centroid and F is the center of the Feuerbach (nine-point) circle which passes through the midpoints of the sides of ABC and the feet of the altitudes.
The points A', B', C' are the midpoints of the sides BC, CA, AB, and Ha, Hb, Hc are the feet of the altitudes of ABC.

--- 2. THE EXISTENCE OF THE FEUERBACH CIRCLE ---

The first wonderful theorem on the Feuerbach circle was known to Euler:

THEOREM 1. The circle which passes through the midpoints of the sides of ABC also passes through the feet of the altitudes and through the midpoints of the segments AH, BH and CH.

I. e.: The points A', B', C', Ha, Hb, Hc and the midpoints of AH, BH, CH are concyclic.

There is a lot of proofs of this theorem. Some proofs use an equivalent variant of Theorem 1:

THEOREM 2. The Feuerbach circle and the circumcircle have got two centers of similtude: The outer center of similtude is H and the inner one is G.

I have found a proof of Theorem 1 which is some more different. The triangles AB'C', A'BC', A'B'C and A'B'C' are congruent (that are the four triangles in which the lines A'B', B'C', C'A' divide triangle ABC). Let us take, for example, triangles AB'C' and A'B'C'. The triangles are congruent; so their circumcircles ka and k are congruent. (ka is the circumcircle of AB'C', k the circumcircle of A'B'C'.) So k is the image of ka in the reflection on the line B'C'; so the image of A in this reflection must lie on k.

One can easily see that this image is Ha. So, Ha lies on k; similarly, Hb and Hc lie on k. The fact that Ha, Hb, Hc, A', B' and C' lie on a circle is established. By applying the fact to triangles AHB and BHC, we show easily that the midpoints of AH, BH, CH lie on the same circle. Theorem 1 is proved.

--- 3. ON THE TRIGONOMETRIC PROOF ---

Following facts are called Feuerbach's theorems:

THEOREM 3. The Feuerbach circle touches the incircle (externally).

THEOREM 4. The Feuerbach circle touches each excircle (internally).

There are two ways of proving these facts analytically (we will only do it for Theorem 3). If R is the circumradius and r is the inradius of ABC, and r' is the inradius of the orthic triangle, then we have the formulas

$$OI^2 = R^2 - 2Rr; \quad (A1)$$
$$HO^2 = R^2 - 4Rr'; \quad (A2)$$
$$HI^2 = 2r^2 - 2Rr'. \quad (A3)$$

Applying the well-known formula for the median of a triangle, we easily get an expression for the median IF (F is the midpoint of the side HO of triangle HOI) and find IF^2 = (R/2 + r)^2, and IF = R/2 + r. This means that the Feuerbach circle and the incircle touch each other, qed.

The problem is to prove (A1), (A2) and (A3). While formula (A1), the so-called Euler's formula, is well-known, the both other formulas are quite hard.

For proving (A2), I apply (A1) to the orthic triangle. In the orthic triangle, the circumcenter is F and the incenter is H (well-known properties). The circumradius is R/2 and the inradius is r'. After (A1), we get

$HF^2 = (R/2)^2 - 2(R/2)r' = (R^2)/4 - Rr'$,

and $HO^2 = (2*HF)^2 = 4 * [(R^2)/4 - Rr'] = R^2 - 4Rr'$, qed.

I haven't found any easy proof for (A3).

That was the first way; the second way is a bit different. Instead of r', we use the angles of triangle ABC:

$$OI^2 = R^2 - 2Rr; \qquad \text{(B1)=(A1)}$$
$$HO^2 = R^2 - 8R^2 \cos A \cos B \cos C; \qquad \text{(B2)}$$
$$HI^2 = 2r^2 - 4R^2 \cos A \cos B \cos C. \qquad \text{(B3)}$$

These formulas are equivalent to (A1), (A2), (A3), since

$$r' = 2R \cos A \cos B \cos C.$$

So, there is no real difference between the (A1)-(A3) and the (B1)-(B3) formula blocks. The formula (B3) is still hard to prove, but I have noticed that the orthocenter H divides the altitude AHa in two segments $AH = 2R\cos A$ and $HHa = 2R\cos B\cos C$. So

$AH * HHa = 4R^2 \cos A \cos B \cos C$.

(Similarly, $BH * HHb = CH * HHc$ = the same, which shows that $AH * HHa = BH * HHb = CH * HHc$.)

The formula (B3) now can be reduced to

$HI^2 = 2r^2 - AH * HHa$,

or

$$AH * HHa = 2r^2 - HI^2. \qquad \text{(B3a)}$$

This remembers me on a theorem on circle powers which was proven in Coxeter/Greitzer "Revisited Geometry" 2 §4:

The product AH * HHa is the the power of the orthocenter H with respect to each circle which has a cevian as a diameter.

I tried to use this fact without any success.