Number Theory Problems in Mathematical Competitions (2015 - 2016)

Demo Version

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Introduction

0.1 Preface

This is a demo to a question bank containing number theory problems chosen from mathematical competitions around the world in the $2015-2016$ school year. The original version of this problem set contains over 300 problems and can be downloaded from this link:

`https://parvardi.com/downloads/NT2016/`

I have classified the problems into 7 categories, and ordered them alphabetically based on the competition name in each category. So, the level of difficulty of the problems do not match each other. Do not be surprised if you see a super easy problem after solving an IMO one. This gives, in my humble opinion, a good opportunity to the problem-solver to be able to evaluate the difficulty of a question before solving it completely. I have selected the first 10 problems in each category and put them in this demo file so you can have a taste of the main theme of the problem set.

Some of the problems have been taken from the contests page at AoPS, and you can (hopefully) find the solutions to those problems in the provided link. The other problems have been mostly taken from the website of the corresponding country/competition.

If you find any errors in this set, please do not hesitate to send me an email at `parvardi@math.ubc.ca` and I will make sure that the error is fixed in the next edition. The credit would be, of course, yours.

I will add hints and possibly solutions to (the original version of) this problem set in the next editions. I just wanted to publish the set before the problems get old and would add the solutions gradually. If you want to help me in writing solutions to the problems, I would be very happy to receive an email from you.

Enjoy problem solving!

Amir Hossein Parvardi,
May 2018.

0.2 Problem Categories

In what follows, I'm going to say a few words about each of the 7 saint category of the problems.

Divisibility: do not expect these problems to be easy only because they belong to the first section: you will find extremely difficult problems among them. Problems in this category include but are not limited to conventional divisibility questions, modular arithmetic and congruence equations, and finding the remainder upon division by a specific number.

Diophantine Equations: I think you already know what to expect to see in this section. The main theme is solving Diophantine equations over integers or natural numbers. Equations involving primes, reciprocals of numbers, and factorials are among my favorites in this category. There are also a few interesting problems which ask you to solve equations containing binomial coefficients.

Arithmetic Functions: mainly includes problems about the number-of-divisors and sum-of-divisors functions. Other problems involve functions from naturals to naturals ($\mathbb{N} \to \mathbb{N}$), sum of digits in base n, and Euler's totient function $\varphi(n)$.

Polynomials: it is always interesting to find applications of algebra in number theory. The problems in this category are mostly taken from serious math competitions and tend to be difficult.

Digits: I tried to separate all the problems which are somehow related to the presentation of a number in some base and put them in this category. You will find very interesting problems in this section. If you are a fan of solving unconventional and new types of number theory problems, try this section!

Sequences: contains problems about integer, natural, or rational sequences. It is surprising to see how often these problems happen in Olympiad competitions. Problems have very different tastes in this section, including but not limited to finding a specific term in the sequence, divisibility issues, and weird sums on sequence terms.

Miscellaneous: contains problems which did not fit into other categories.

Chapter 1

Problems

1.1 Divisibility

Problem 1 (Austria National Competition Final Round)**.** Let a, b, and c be integers such that

$$\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a}$$

is an integer. Prove that each of the numbers

$$\frac{ab}{c}, \frac{ac}{b}, \text{ and } \frac{bc}{a}$$

is an integer.

Problem 2 (Azerbaijan Balkan Math Olympiad Third TST)**.** Find all natural numbers n for which there exist primes p and q such that the following conditions are satisfied:

1. $p + 2 = q$, and
2. $2^n + p$ and $2^n + q$ are both primes.

Link

Problem 3 (Azerbaijan Junior Mathematical Olympiad)**.** Given

$$34! = 295232799039a041408476186096435b0000000,$$

in decimal representation, find the numbers a and b. Link

Problem 4 (Azerbaijan Junior Mathematical Olympiad)**.** A quadruple (p, a, b, c) of positive integers is called a *good quadruple* if

(a) p is an odd prime,

(b) a, b, and c are distinct,

(c) $ab + 1, bc + 1$, and $ca + 1$ are divisible by p.

Prove that for all good quadruples (p, a, b, c),

$$p + 2 \leq \frac{a + b + c}{3},$$

and show the equality case.

Link

Problem 5 (Balkan)**.** Find all monic polynomials f with integer coefficients satisfying the following condition: there exists a positive integer N such that p divides $2(f(p)!) + 1$ for every prime $p > N$ for which $f(p)$ is a positive integer.
Note. A monic polynomial has a leading coefficient equal to 1. Link

Problem 6 (Baltic Way)**.** Let n be a positive integer and let a, b, c, d be integers such that $n|a + b + c + d$ and $n|a^2 + b^2 + c^2 + d^2$. Show that

$$n|a^4 + b^4 + c^4 + d^4 + 4abcd.$$

Link

Problem 7 (Bay Area Olympiad)**.** The distinct prime factors of an integer are its prime factors listed without repetition. For example, the distinct prime factors of 40 are 2 and 5. Let $A = 2^k - 2$ and $B = 2^k \cdot A$, where k is an integer ($k \geq 2$). Show that for every integer k greater than or equal to 2,

1. A and B have the same set of distinct prime factors, and

2. $A + 1$ and $B + 1$ have the same set of distinct prime factors.

Link

Problem 8 (Belgium Flanders Math Olympiad Final Round)**.** Find the smallest positive integer n which does not divide 2016!.

Problem 9 (Benelux)**.** Find the greatest positive integer N with the following property: there exist integers $x_1, x_2, \dots, x_N$ such that $x_i^2 - x_i x_j$ is not divisible by 1111 for any $i \neq j$. Link

Problem 10 (Bulgaria National Olympiad)**.** Find all positive integers m and n such that

$$\left(2^{2^m} + 1\right)\left(2^{2^n} + 1\right)$$

is divisible by mn.

1.2 Diophantine Equations

Problem 11 (APMO)**.** A positive integer is called *fancy* if it can be expressed in the form

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}},$$

where $a_1, a_2, \ldots, a_{100}$ are non-negative integers that are not necessarily distinct. Find the smallest positive integer n such that no multiple of n is a fancy number.

Link

Problem 12 (Argentina Intercollegiate Olympiad First Level)**.** Find all positive integers a, b, c, and d, all less than or equal to 6, such that

$$\frac{a}{b} = \frac{c}{d} + 2.$$

Problem 13 (Argentina Intercollegiate Olympiad Third Level)**.** Find a number with the following conditions:

1. it is a perfect square,
2. when 100 is added to the number, it equals a perfect square plus 1, and
3. when 100 is again added to the number, the result is a perfect square.

Problem 14 (Austria Regional Competition)**.** Determine all positive integers k and n satisfying the equation

$$k^2 - 2016 = 3^n.$$

Problem 15 (Azerbaijan TST)**.** The set A consists of positive integers which can be expressed as $2x^2 + 3y^2$, where x and y are integers (not both zero at the same time).

1. Prove that there is no perfect square in A.
2. Prove that the product of an odd number of elements of A cannot be a perfect square.

Link

Problem 16 (Baltic Way)**.** For which integers $n = 1, 2, \ldots, 6$ does the equation

$$a^n + b^n = c^n + n$$

have a solution in integers? Link

Problem 17 (Bay Area Olympiad)**.** Find a positive integer N and $a_1, a_2, \ldots, a_N$, where $a_k \in \{1, -1\}$ for each $k = 1, 2, \ldots, N$, such that

$$a_1 \cdot 1^3 + a_2 \cdot 2^3 + a_3 \cdot 3^3 + \cdots + a_N \cdot N^3 = 20162016,$$

or show that this is impossible. Link

Problem 18 (Belgium National Olympiad Final Round)**.** Solve the equation

$$2^{2m+1} + 9 \cdot 2^m + 5 = n^2$$

for integers m and n.

Problem 19 (Bosnia and Herzegovina TST)**.** Determine the largest positive integer n which cannot be written as the sum of three numbers bigger than 1 which are pairwise coprime. Link

Problem 20 (Bulgaria National Olympiad)**.** Determine whether there exists a positive integer $n < 10^9$ such that n can be expressed as a sum of three squares of positive integers in more than 1000 distinct ways.

1.3 Arithmetic Functions

Problem 21 (Austria Federal Competition for Advanced Students Final Round)**.** Determine all composite positive integers n with the following property: If $1 = d_1 < d_2 < \ldots < d_k = n$ are all the positive divisors of n, then

$$(d_2 - d_1) : (d_3 - d_2) : \cdots : (d_k - d_{k-1}) = 1 : 2 \cdots : (k-1).$$

Problem 22 (Austria Beginners' Competition)**.** Determine all nonnegative integers n having two distinct positive divisors with the same distance from $n/3$.

Problem 23 (Azerbaijan Balkan Math Olympiad First TST)**.** Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) = n + 2015,$$

for all $n \in \mathbb{N}$. Link

Problem 24 (Benelux)**.** Let n be a positive integer. Suppose that its positive divisors can be partitioned into pairs (i.e. can be split in groups of two) in such a way that the sum of each pair is a prime number. Prove that these prime numbers are distinct and that none of these are a divisor of n. Link

Problem 25 (CCA Math Bonanza)**.** Compute

$$\sum_{k=1}^{420} \gcd(k, 420).$$

Link

Problem 26 (China South East Mathematical Olympiad)**.** Let n be a positive integer and let D_n be the set of all positive divisors of n. Define

$$f(n) = \sum_{d \in D_n} \frac{1}{1+d}.$$

Prove that for any positive integer m,

$$\sum_{i=1}^{m} f(i) < m.$$

Link

Problem 27 (China TST)**.** Set positive integer $m = 2^k \cdot t$, where k is a nonnegative integer, t is an odd number, and let $f(m) = t^{1-k}$. Prove that for any positive integer n and for any positive odd number $a \le n$, $\prod_{m=1}^{n} f(m)$ is a multiple of a. Link

Problem 28 (China TST)**.** For any two positive integers x and $d > 1$, denote by $S_d(x)$ the sum of digits of x taken in base d. Let a, b, b', c, m, and q be positive integers, where $m > 1, q > 1$, and $|b - b'| \geq a$. It is given that there exists a positive integer M such that

$$S_q(an + b) \equiv S_q(an + b') + c \pmod{m}$$

holds for all integers $n \geq M$. Prove that the above equation is true for all positive integers n. Link

Problem 29 (Estonia National Olympiad Eleventh Grade)**.** Let n be a positive integer. Let $\delta(n)$ be the number of positive divisors of n and let $\sigma(n)$ be their sum. Prove that

$$\sigma(n) > \frac{(\delta(n))^2}{2}.$$

Problem 30 (European Mathematical Cup Juniors)**.** Let $d(n)$ denote the number of positive divisors of n. For a positive integer n we define $f(n)$ as

$$f(n) = d(k_1) + d(k_2) + \cdots + d(k_m),$$

where $1 = k_1 < k_2 < \cdots < k_m = n$ are all divisors of the number n. We call an integer $n > 1$ *almost perfect* if $f(n) = n$. Find all almos perfect numbers.

1.4 Polynomials

Problem 31 (Brazil National Math Olympiad)**.** Consider the second-degree polynomial $P(x) = 4x^2 + 12x - 3015$. Define the sequence of polynomials

$$P_1(x) = \frac{P(x)}{2016} \quad \text{and} \quad P_{n+1}(x) = \frac{P(P_n(x))}{2016}$$

for every integer $n \geq 1$.

(a) Show that exists a real number r such that $P_n(r) < 0$ for every positive integer n.

(b) Find how many integers m are such that $P_n(m) < 0$ for infinite positive integers n.

Link

Problem 32 (Canada National Olympiad)**.** Find all polynomials $P(x)$ with integer coefficients such that $P(P(n)+n)$ is a prime number for infinitely many integers n. Link

Problem 33 (China National Olympiad)**.** Let p be an odd prime and $a_1, a_2, \dots, a_p$ be integers. Prove that the following two conditions are equivalent:

1. There exists a polynomial $P(x)$ with degree $\leq \frac{p-1}{2}$ such that $P(i) \equiv a_i \pmod{p}$ for all $1 \leq i \leq p$.

2. For any natural $d \leq \frac{p-1}{2}$,

$$\sum_{i=1}^{p} (a_{i+d} - a_i)^2 \equiv 0 \pmod{p},$$

where indices are taken modulo p.

Link

Problem 34 (ELMO)**.** Big Bird has a polynomial P with integer coefficients such that n divides $P(2^n)$ for every positive integer n. Prove that Big Bird's polynomial must be the zero polynomial. Link

Problem 35 (International Olympiad of Metropolises)**.** Let $r(x)$ be a polynomial of odd degree with real coefficients. Prove that there exist only finitely many (or none at all) pairs of polynomials $p(x)$ and $q(x)$ with real coefficients satisfying the equation $(p(x))^3 + q(x^2) = r(x)$. Link

Problem 36 (Iran Third Round National Olympiad)**.** Let F be a subset of the set of positive integers with at least two elements and P be a polynomial with integer coefficients such that for any two elements of F like a and b, the following two conditions hold:

(i) $a+b \in F$, and

(ii) $\gcd(P(a), P(b)) = 1$.

Prove that $P(x)$ is a constant polynomial. Link

Problem 37 (Iran Third Round National Olympiad)**.** Let P be a polynomial with integer coefficients. We say P is *good* if there exist infinitely many prime numbers q such that the set

$$X = \{P(n) \bmod q : n \in \mathbb{N}\}$$

has at least $\frac{q+1}{2}$ members. Prove that the polynomial $x^3 + x$ is good. Link

Problem 38 (Iran Third Round National Olympiad)**.** We call a function g *special* if $g(x) = a^{f(x)}$ (for all x) where a is a positive integer and f is polynomial with integer coefficients such that $f(n) > 0$ for all positive integers n.
A function is called an *exponential polynomial* if it is obtained from the product or sum of special functions. For instance, $2^x 3^{x^2+x-1} + 5^{2x}$ is an exponential polynomial.
Prove that there does not exist a non-zero exponential polynomial $f(x)$ and a non-constant polynomial $P(x)$ with integer coefficients such that

$$P(n) | f(n)$$

for all positive integers n. Link

Problem 39 (Korea Winter Program Practice Test)**.** Let $p(x)$ be an irreducible polynomial with integer coefficients, and q a fixed prime number. Let a_n be the number of solutions of the equation

$$p(x) \equiv 0 \pmod{q^n}.$$

Prove that we can find M such that $\{a_n\}_{n \geq M}$ is constant. Link

Problem 40 (Pan-African Mathematical Olympiad)**.** For any positive integer n, we define the integer $P(n)$ by

$$P(n) = n(n+1)(2n+1)(3n+1)\dots(16n+1)$$

Find the greatest common divisor of the integers $P(1), P(2), P(3), \dots, P(2016)$.
Link

1.5 Digits

Problem 41 (Argentina Intercollegiate Olympiad Second Level)**.** Find all positive integers x and y which satisfy the following conditions:

1. x is a 4−digit palindromic number, and
2. $y = x + 312$ is a 5−digit palindromic number.

Note. A palindromic number is a number that remains the same when its digits are reversed. For example, 16461 is a palindromic number.

Problem 42 (Bundeswettbewerb Mathematik)**.** A number with 2016 zeros that is written as $101010\ldots0101$ is given, in which the zeros and ones alternate. Prove that this number is not prime. Link

Problem 43 (Caltech Harvey Mudd Math Competition (CHMMC) Fall)**.** We say that the string $d_k d_{k-1}\cdots d_1 d_0$ represents a number n in base -2 if each d_i is either 0 or 1, and $n = d_k(-2)^k + d_{k-1}(-2)^{k-1} + \cdots + d_1(-2) + d_0$. For example, 110_{-2} represents the number 2. What string represents 2016 in base -2? Link

Problem 44 (CentroAmerican)**.** Find all positive integers n that have 4 digits, all of them perfect squares, and such that n is divisible by $2, 3, 5$, and 7. Link

Problem 45 (Croatia IMO TST, Bulgaria TST)**.** Let $p > 10^9$ be a prime number such that $4p+1$ is also a prime. Prove that the decimal expansion of $\frac{1}{4p+1}$ contains all the digits $0, 1, \ldots, 9$. Link

Problem 46 (Germany National Olympiad Second Round Eleventh/Twelfth Grade)**.** The sequence $x_1, x_2, x_3, \ldots$ is defined as $x_1 = 1$ and

$$x_{k+1} = x_k + y_k \quad \text{for } k = 1, 2, 3, \ldots$$

where y_k is the last digit of decimal representation of x_k. Prove that the sequence $x_1, x_2, x_3, \ldots$ contains all powers of 4. That is, for every positive integer n, there exists some natural k for which $x_k = 4^n$.

Problem 47 (Germany National Olympiad Fourth Round Tenth Grade[1])**.** A sequence of positive integers $a_1, a_2, a_3, \ldots$ is defined as follows: a_1 is a 3 digit number and a_{k+1} (for $k \geq 1$) is obtained by

$$a_{k+1} = a_k + 2 \cdot Q(a_k),$$

where $Q(a_k)$ is the sum of digits of a_k when represented in decimal system. For instance, if one takes $a_1 = 358$ as the initial term, the sequence would be

$$\begin{aligned}
a_1 &= 358,\\
a_2 &= 358 + 2 \cdot 16 = 390,\\
a_3 &= 390 + 2 \cdot 12 = 414,\\
a_4 &= 414 + 2 \cdot 9 = 432,\\
&\vdots
\end{aligned}$$

[1]Thanks to Arian Saffarzadeh for translating the problem.

Prove that no matter what we choose as the starting number of the sequence,

(a) the sequence will not contain 2015.

(b) the sequence will not contain 2016.

Problem 48 (IberoAmerican). Let k be a positive integer and suppose that we are given $a_1, a_2, \ldots, a_k$, where $0 \leq a_i \leq 9$ for $i = 1, 2, \ldots, k$. Prove that there exists a positive integer n such that the last $2k$ digits of 2^n are, in the following order, $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$, for some digits $b_1, b_2, \ldots, b_k$. Link

Problem 49 (India IMO Training Camp). Given that n is a natural number such that the leftmost digits in the decimal representations of 2^n and 3^n are the same, find all possible values of the leftmost digit. Link

Problem 50 (Middle European Mathematical Olympiad). A positive integer n is called *Mozart* if the decimal representation of the sequence $1, 2, \ldots, n$ contains each digit an even number of times. Prove that:

1. All Mozart numbers are even.

2. There are infinitely many Mozart numbers.

Link

1.6 Sequences

Problem 51 (Argentina Intercollegiate Olympiad Third Level)**.** Let $a_1, a_2, \ldots, a_{15}$ be an arithmetic progression. If the sum of all 15 terms is twice the sum of the first 10 terms, find $\dfrac{d}{a_1}$, where d is the common difference of the progression.

Problem 52 (Baltic Way)**.** Prove or disprove the following hypotheses.

(a) For all $k \geq 2$, each sequence of k consecutive positive integers contains a number that is not divisible by any prime number less than k.

(b) For all $k \geq 2$, each sequence of k consecutive positive integers contains a number that is relatively prime to all other members of the sequence.

Link

Problem 53 (Baltic Way)**.** Let $p > 3$ be a prime such that $p \equiv 3 \pmod 4$. Given a positive integer a_0 define the sequence $a_0, a_1, \ldots$ of integers by $a_n = a_{n-1}^{2^n}$ for all $n = 1, 2, \ldots$. Prove that it is possible to choose a_0 such that the subsequence $a_N, a_{N+1}, a_{N+2}, \ldots$ is not constant modulo p for any positive integer N. Link

Problem 54 (Bosnia and Herzegovina TST)**.** We call an infinite sequence $a_1 < a_2 < a_3 < \cdots$ of positive integers *nice* if for every positive integer n, we have

$$a_{2n} = 2a_n.$$

Prove the following statements:

(a) If there is given a nice sequence $\{a_n\}_{n=1}^{\infty}$ and a prime number $p > a_1$, then there exists some term of the sequence which is divisible by p.

(b) For every prime number $p > 2$, there exists a nice sequence such that no terms of the sequence are divisible by p.

Link

Problem 55 (China Second Round Olympiad Second Test)**.** Let $p > 3$ and $p + 2$ be prime numbers and define the sequence $\{a_n\}_{n=1}^{\infty}$ by $a_1 = 2$ and

$$a_n = a_{n-1} + \left\lfloor \frac{pa_{n-1}}{n} \right\rfloor, \quad n = 2, 3, \ldots$$

show that for any all n with $3 \leq n \leq p - 1$, we have $n | pa_{n-1} + 1$. Link

Problem 56 (China South East Mathematical Olympiad)**.** Let $\{a_n\}$ be a sequence consisting of positive integers such that

$$n^2 \mid \sum_{i=1}^{n} a_i \quad \text{and} \quad a_n \leq (n + 2016)^2$$

for all $n \geq 2016$. Define $b_n = a_{n+1} - a_n$. Prove that the sequence $\{b_n\}$ is eventually constant. Link

Problem 57 (China TST)**.** Let $c \geq 2$ and $d \geq 2$ be positive integers. Let $\{a_n\}$ be the sequence satisfying $a_1 = c$ and $a_{n+1} = a_n^d + c$ for $n = 1, 2, \ldots$. Prove that for any $n \geq 2$, there exists a prime number p such that $p \mid a_n$ and $p \nmid a_i$ for $i = 1, 2, \ldots, n-1$. Link

Problem 58 (Croatia First Round Competition)**.** A sequence $\{a_n\}_{n=1}^{\infty}$ is defined by $a_1 = a_2 = 1$, and

$$a_{n+1} = \frac{a_2^2}{a_1} + \frac{a_3^2}{a_2} + \cdots + \frac{a_n^2}{a_{n-1}} \quad \text{for } n \geq 2.$$

Find a_{2016}.

Problem 59 (ELMO)**.** Cookie Monster says a positive integer n is *crunchy* if there exist $2n$ real numbers $x_1, x_2, \ldots, x_{2n}$, not all equal, such that the sum of any n of the x_i's is equal to the product of the other n ones. Help Cookie Monster determine all crunchy integers. Link

Problem 60 (Germany TST, Taiwan TST First Round, Slovakia TST)**.** Determine all positive integers M such that the sequence $a_0, a_1, a_2, \ldots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term. Link

1.7 Miscellaneous

Problem 61 (Azerbaijan Junior Mathematical Olympiad)**.** Prove that if for a real number a, $a + \frac{1}{a}$ is an integer, then $a^n + \frac{1}{a^n}$ is also an integer for any positive integer n. Link

Problem 62 (Bundeswettbewerb Mathematik)**.** There are $\frac{n(n+1)}{2}$ distinct sums of two distinct numbers, if there are n numbers. For which n ($n \geq 3$) do there exist n distinct integers, such that those sums are $\frac{n(n-1)}{2}$ consecutive numbers? Link

Problem 63 (Chile[2])**.** Find all prime numbers that do not have a multiple ending in 2015.

Problem 64 (Chile)**.** Find the number of different numbers of the form $\lfloor \dfrac{i^2}{2015} \rfloor$, where $i = 1, 2, \ldots, 2015$.

Link

Problem 65 (China South East Mathematical Olympiad)**.** Define the sets

$$
\begin{aligned}
A &= \{a^3 + b^3 + c^3 - 3abc : a, b, c \in \mathbb{N}\},\\
B &= \{(a + b - c)(b + c - a)(c + a - b) : a, b, c \in \mathbb{N}\},\\
P &= \{n : n \in A \cap B, 1 \leq n \leq 2016\}.
\end{aligned}
$$

Find the number of elements of P. Link

Problem 66 (China TST)**.** Does there exist two infinite sets S and T of positive integers such that any positive integer n can be uniquely expressed in the form

$$n = s_1t_1 + s_2t_2 + \cdots + s_kt_k,$$

where k is a positive integer dependent on n, $s_1 < s_2 < \cdots < s_k$ are elements of S, and $t_1, t_2, \ldots, t_k$ are elements of T? Link

Problem 67 (Croatia First Round Competition)**.** Find all pairs (a, b) of positive integers such that $1 < a, b \leq 100$ and

$$\frac{1}{\log_a 10} + \frac{1}{\log_b 10}$$

is a positive integer.

Problem 68 (Croatia Final Round National Competition)**.** Determine the sum

$$\frac{2^2 + 1}{2^2 - 1} + \frac{3^2 + 1}{3^2 - 1} + \cdots + \frac{100^2 + 1}{100^2 - 1}.$$

[2] Thanks to Kamal Kamrava and Behnam Sajadi for translating the problem.

Problem 69 (Estonia IMO TST First Stage)**.** Prove that for every positive integer $n \geq 3$,

$$2 \cdot \sqrt{3} \cdot \sqrt[3]{4} \cdots \sqrt[n-1]{n} > n.$$

Problem 70 (Estonia Regional Olympiad Twelfth Grade)**.** Determine whether the logarithm of 6 in base 10 is larger or smaller than $\frac{7}{9}$.