War-Hammer
662 posts

Aug 11, 2013, 11:24 pm PM #1109

> pco wrote:
>
> > War-Hammer wrote:
> > Haven't you seen my edit ???
>
> You are welcome. Glad to have helped you.
>
> Answer to your question is "No". I'm sorry but I spent last hours to solve your second version of problem.
> In the mean time, you invented a third one.
> I suppose that, while we'll try to solve it, you'll discover a fourth version...
>
> You should take more care when copying problems statements.
>
> I stop any further personal effort. But I'm sure a lot of users will continue to solve your successive erroneous problems.
> Enjoy mathlink.

Sorry for my different version of the problem.

This is my mistake , I've edited before your solution , but it was late.

Again sorry for my many edited.

And indeed thank you for complete solution for the previous version.

---

aktyw19
1315 posts

Aug 22, 2013, 11:22 pm PM #1110

Problem
Find all monotonic function $f:\mathbb{Z}\to\mathbb{Z}$ such that for $x,y\in\mathbb{Z}$, $f\left(x^{2009}+y^{2009}\right)=[f\left(x\right)]^{2009}+[f\left(x\right)]^{2009}$

---

randomuse...
985 posts

Aug 22, 2013, 11:53 pm PM #1111

Do you mean $f(x^{2009}+y^{2009})=f(x)^{2009}+f(y)^{2009}$?

---

aktyw19
1315 posts

Aug 24, 2013, 1:54 pm PM #1112

> randomusername wrote:
> Do you mean $f(x^{2009}+y^{2009})=f(x)^{2009}+f(y)^{2009}$?

yes

---

xxp2000
520 posts

Aug 24, 2013, 7:02 pm PM #1113

> aktyw19 wrote:
>
> > randomusername wrote:
> > Do you mean $f(x^{2009}+y^{2009})=f(x)^{2009}+f(y)^{2009}$?
>
> yes

Let $a=2009$.
$P(0,0): f(0)=0$.
$P(x,-x): f(-x)=-f(x)$
$P(1,0): f(1)=\pm1$
Consider $f(1)=1$.
$P(1,1): f(2)=2$.
$P(n,0): f(n^a)=f(n)^a$. So $f(x)=x$ when $x=2^a,2^{a^2},\cdots$.
If $f(n)=n, f(m)=m, n<m$, monotonicity implies $f(x)=x, n\ge x\ge m$.
So $f(x)=x, x\ge 0$. Since f is odd, $f(x)=x$.

For the case $f(1)=-1$, we get another solution $f(x)=-x$.

---

oty
870 posts

Aug 26, 2013, 3:25 am PM #1114

Wonderfull solution Dear **xxp2000** congrat's . Let's
Problem 344
Let $n\geq 3$ be a positive integer . Find all continious functions , $f:[0,1]\rightarrow\mathbb{R}$ for which

$$f(x_1)+f(x_2)+...+f(x_n)=1$$

whenever : $x_1;x_2,...,x_n\in[0,1]$ and $x_1+x_2+...+x_n=1$

---

xxp2000
520 posts

Aug 26, 2013, 6:06 am PM #1115

> oty wrote:
> Wonderfull solution Dear **xxp2000** congrat's . Let's
> **Problem 344**
> Let $n\geq 3$ be a positive integer . Find all continious functions , $f:[0,1]\rightarrow\mathbb{R}$ for which
>
> $$f(x_1)+f(x_2)+...+f(x_n)=1$$
>
> whenever : $x_1;x_2,...,x_n\in[0,1]$ and $x_1+x_2+...+x_n=1$

Obviously we have $f(x_1+x_2)+f(0)=f(x_1)+f(x_2)$ when $x_1+x_2\le 1$.
Let $F(x)=f(x)-f(0)$, $F(x+y)=F(x)+F(y), x+y\le 1, x,y\ge 0$.
$F(1)=mF(\frac{1}{m})$ and $F(\frac{k}{m})=kF(\frac{1}{m})=\frac{k}{m}F(1)$, where $0<k<m$ are integers.
So $F(r)=rF(1)$ for rational $r\in[0,1]$.
By continuity, $F(x)=xF(1)$.

Now $f(x)=ax+b$ with constants a,b. The f.e. implies $a+b=1$ and $f(\frac{1}{n})=\frac{1}{n}$.
So the only solution is $f(x)=x$.

aktyw19
1315 posts

Aug 29, 2013, 4:21 pm PM #1116

Problem
Find all non-increasing or non-decreasing functions $f: R^+ \cup \{0\} \to R$ such that
$f(x+y)-f(x)-f(y)=f(xy+1)-f(xy)-f(1)$ for all $x, y \geq 0$, and $f(3)+3f(1)=3f(2)+f(0)$.

xxp2000
520 posts

Aug 31, 2013, 3:24 am PM #1117

> aktyw19 wrote:
> Problem
> Find all non-increasing or non-decreasing functions $f: R^+ \cup \{0\} \to R$ such that
> $f(x+y)-f(x)-f(y)=f(xy+1)-f(xy)-f(1)$ for all $x, y \geq 0$, and $f(3)+3f(1)=3f(2)+f(0)$.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=410339&hilit=ukraine

aktyw19
1315 posts

Aug 31, 2013, 4:18 am PM #1118

Problem
Let $r \geq 2$. Find all function $f: [0,1] \to [0,1]$ satisying $(rx-(r-1)f(x)) \in [0,1]$ and $f(rx-(r-1)f(x))=x$.

xxp2000
520 posts

Sep 1, 2013, 5:02 am PM #1119

> aktyw19 wrote:
> Problem
> Let $r \geq 2$. Find all function $f: [0,1] \to [0,1]$ satisying $(rx-(r-1)f(x)) \in [0,1]$ and $f(rx-(r-1)f(x))=x$.

$(rx-(r-1)f(x)) \in [0,1]$ implies $\frac{rx-1}{r-1} \leq f(x) \leq \frac{rx}{r-1}$.
Suppose $a_n x + b_n \leq f(x) \leq c_n x$, the f.e. implies
$a_n(rx-(r-1)f(x)) + b_n \leq x \leq c_n(rx-(r-1)f(x))$,
or $a_{n+1}x + b_{n+1} \leq f(x) \leq c_{n+1}x$, where
$a_{n+1} = \frac{a_n r - 1}{a_n(r-1)}, b_{n+1} = \frac{b_n}{a_n(r-1)}, c_{n+1} = \frac{c_n r - 1}{c_n(r-1)}$.
With $r \geq 2$, we can show $\lim a_n = 1, \lim b_n = 0, \lim c_n = 1$.
So $f(x) = x$.

aktyw19
1315 posts

Sep 1, 2013, 10:24 am PM #1120

Problem
Find all continuous functions $f: [0;1] \to R$ such that: $f(x) = \frac{1}{2}(f(\frac{x}{2}) + f(\frac{x+1}{2}))$.

War-Hammer
662 posts

Sep 1, 2013, 9:55 pm PM #1121

Please submit the number of the problem.

xxp2000
520 posts

Sep 1, 2013, 11:37 pm PM #1122

> aktyw19 wrote:
> Problem
> Find all continuous functions $f: [0;1] \to R$ such that: $f(x) = \frac{1}{2}(f(\frac{x}{2}) + f(\frac{x+1}{2}))$.

Let $M = \max f(x)$. Since f is continuous, we can find $f(a) = M$. The f.e. implies $f(\frac{a}{2}) = f(\frac{a+1}{2}) = M$.
We can show $f(\frac{a}{2^n}) = M, n \in \mathbb{N}$ by induction. So $M = \lim_{n\to\infty} f(\frac{a}{2^n}) = f(0)$.
Similarly we can show $f(0) = \min f(x)$. So the only f is the constant function.

hofamo
58 posts

Sep 2, 2013, 1:29 am PM #1123

> xxp2000 wrote:
>
> > aktyw19 wrote:
> > Problem
> > Find all continuous functions $f: [0;1] \to R$ such that: $f(x) = \frac{1}{2}(f(\frac{x}{2}) + f(\frac{x+1}{2}))$.
>
> Let $M = \max f(x)$. Since f is continuous, we can find $f(a) = M$. The f.e. implies $f(\frac{a}{2}) = f(\frac{a+1}{2}) = M$.
> We can show $f(\frac{a}{2^n}) = M, n \in \mathbb{N}$ by induction. So $M = \lim_{n\to\infty} f(\frac{a}{2^n}) = f(0)$.
> Similarly we can show $f(0) = \min f(x)$. So the only f is the constant function.

why f has maximum.

xxp2000
520 posts

Sep 2, 2013, 4:06 am PM #1124

> hofamo wrote:
> why f has maximum.

http://en.wikipedia.org/wiki/Extreme_value_theorem

aktyw19
1315 posts

Sep 2, 2013, 6:08 pm PM #1125

Problem 347
Find all the functions $f: \mathbb{R} \to \mathbb{R}$ satisfying

for all $x \in \mathbb{R}$ $f(x+f(x)) + f(x-f(x)) = 2x$.

War-Hammer
662 posts

Sep 3, 2013, 12:58 am PM #1126

> aktyw19 wrote:
> Problem
> Find all the functions $f: \mathbb{R} \to \mathbb{R}$ satisfying
>
> for all $x \in \mathbb{R}$ $f(x+f(x)) + f(x-f(x)) = 2x$.

Please and please follow the rules.

socrates
1872 posts

Sep 30, 2013, 1:47 am PM #1127

Problem 348 (easier version of a longlisted one (2012))
Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+f(x)+2y)=f(2x)+y+f(y),$$

for all $x,y\in\mathbb{R}$

hofamo
58 posts

Oct 4, 2013, 4:07 am PM #1128

at first with Y=-x-f(x) we have x+f(x)=f(2x) . now f(f(2x)+2Y)=f(2x)+f(2y) or f(f(x)+y)=f(x)+f(y) (name of this equation is A) . now with y=z-f(x) we have f(z)-f(x)=f(z-f(x)) .we have f(z)-f(x) for all of (z,x) from R^2 is in R(f) . it s meaning f(2x)-f(x) is in R(f) or x is in R(f) for all x . R=R(f) . now we can put f(x)=t in A . (t can has all of numbers in R) now f(t+y)=t+f(y)=y+f(t) and f(x)-x=C and C is fix. f(x)=x+C . and it s right in first equation.

socrates
1872 posts

Oct 27, 2013, 4:30 pm PM #1129

Problem 349
Let $\alpha\in\mathbb{Z}_{\geq 0}$.
Find all functions $f:[0,+\infty)\to[0,+\infty)$ such that

$$f(f(x)+\alpha y+f(y))=x+(\alpha+1)f(y),\forall x,y\in[0,+\infty)$$

What can we say if $\alpha\in\mathbb{R}_{\geq 0}$.

jjax
108 posts

Oct 27, 2013, 7:39 pm • 2 PM #1130

We first assume that $\alpha\neq 0$.
Let $P(x,y)$ denote the proposition that $f(f(x)+\alpha y+f(y))=x+(\alpha+1)f(y)$.

Varying x shows that for some M, f takes all values above M (partial surjectivity).
Also, if $f(a)=f(b)$ then $P(a,y),P(b,y)$ show that $a=b$ (injectivity).

$P(x,p)$ and $P(x,q)$ give $f(f(x)+\alpha p+f(p))-f(f(x)+\alpha q+f(q))=(\alpha+1)(f(p)-f(q))$.
Writing $d=\alpha p+f(p)-\alpha q+f(q)$ and $e=(\alpha+1)(f(p)-f(q))$ we get that $f(z+d)=f(z)+e$ for all sufficiently large z.

Choose y large in $P(x,y)$. Adding e to both sides gives
$(x+e)+(\alpha+1)f(y)=f(f(x)+\alpha y+f(y)+d)$.
Comparing that to $P(x+e,y)$ gives $f(f(x)+\alpha y+f(y)+d)=f(f(x+e)+\alpha y+f(y))$.
Injectivity gives $f(x+e)=f(x)+d$ for all x.

$P(x,y+d)$ gives $x+(\alpha+1)f(y)+(\alpha+1)e=f(f(x)+\alpha y+f(y)+\alpha d+e)$.
Thus $f(f(x)+\alpha y+f(y))+(\alpha+1)e=f(f(x)+\alpha y+f(y)+\alpha d)+d$.
Thus for sufficiently large z, we have $f(z+\alpha d)=f(z)+(\alpha+1)e-d$.
Likewise, $f(z+\alpha e)=f(z)+(\alpha+1)d-e$ for all large z.

$P(x+\alpha d,y)$ gives $x+(\alpha+1)f(y)+\alpha d=f(f(x)+(\alpha+1)e-d+\alpha y+f(y))$.
Thus $f(f(x)+\alpha y+f(y))+\alpha d=f(f(x)+\alpha y+f(y))+(\alpha+1)d-e+d-e$ and so $d=e$.
That is, $\alpha p+f(p)-\alpha q+f(q)=(\alpha+1)(f(p)-f(q))$, so $f(p)-p=f(q)-q$ for all p,q.
Thus, $f(x)=x+k$ for some nonnegative constant k.

Testing, we see that when $\alpha\neq 2$ we must have $f(x)=x$ for all x, and when $\alpha=2$ we can have $f(x)=x+k$ for any nonnegative k.

Click to reveal hidden text

socrates
1872 posts

Oct 27, 2013, 10:32 pm PM #1131

Problem 350
Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+f(x)+\frac{y}{2})=f(\frac{x}{2})+y+f(y),$$

for all $x,y\in\mathbb{R}$

socrates
1872 posts

Oct 27, 2013, 10:37 pm PM #1132

Problem 351
Determine all functions $f:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ such that

$$3f(x+y+z)-f(-x+y+z)-f(x-y+z)-f(x+y-z)=8\left(\sqrt{f(x)f(y)}+\sqrt{f(y)f(z)}+\sqrt{f(z)f(x)}\right),$$

for all $x,y,z\in\mathbb{R}_{\geq 0}$ such that $-x+y+z,x-y+z,x+y-z\geq 0$.

socrates
1872 posts

Oct 28, 2013, 3:43 pm PM #1133

Problem 352
Determine all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that

$$f(f(n+1)+3)=n,$$

for all $n\in\mathbb{Z}$

pco
14052 posts

Oct 28, 2013, 3:55 pm PM #1134

> *socrates wrote:*
>
> New Problem
> Determine all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that
>
> $$f(f(n+1)+3)=n,$$
>
> for all $n\in\mathbb{Z}$

Setting $g(n)=f(n+3)$, we get $g(g(n))=n+2$ and so $g(n+2)=g(n)+2$ and so :
$g(2p)=2p+a$
$g(2p+1)=2p+1+b$

If a is even, we get $g(g(2p))=2p+2a$ and so $a=1$, odd.

So a is odd and then :

$g(g(2p)) = 2p + a + b$ and so $a + b = 2$ and b is odd too. Then :
$g((g(2p+1)) = 2p + 1 + a + b = 2p + 3$ is OK

So :
$g(2p) = 2p + 2u + 1$
$g(2p+1) = 2p + 2 - 2u$

Hence the result :
$f(2n) = 2n - 2 - 2u$
$f(2n+1) = 2n + 2u - 1$
Whatever is $u \in \mathbb{N}$
(notice that $u = 0$ gives the trivial solution $f(n) = n - 2$)

---

socrates
1872 posts

Oct 29, 2013, 12:17 am PM #1135

New Problem
a) Determine all injective functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)y + x) = f(x)f(y) + f(x),$$

for all $x, y \in \mathbb{R}$.

b) Determine all injective functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(f(x)y + x) = f(x)f(y) + f(x),$$

for all $x, y \in \mathbb{Z}$.

---

pco
14052 posts

Oct 29, 2013, 1:20 am • 1 PM #1136

> *socrates wrote:*
> Problem 353
> a) Determine all injective functions $f : \mathbb{R} \to \mathbb{R}$ such that
>
> $$f(f(x)y + x) = f(x)f(y) + f(x),$$
>
> for all $x, y \in \mathbb{R}$.

Let $P(x, y)$ be the assertion $f(f(x)y + x) = f(x)f(y) + f(x)$

$P(0,0) \implies f(0) = 0$

Let $x \neq 0$, then, since injective, $f(x) \neq 0$ and $P(x, -\frac{x}{f(x)}) \implies f(-\frac{x}{f(x)}) = -1$

So, since injective, $-\frac{x}{f(x)} = c$ and so $\boxed{f(x) = ax}$ which indeed is a solution, whatever is $a \in \mathbb{R} \setminus \{0\}$

---

pco
14052 posts

Oct 29, 2013, 9:06 pm PM #1137

> *socrates wrote:*
> **Problem 354**
> b) Determine all injective functions $f : \mathbb{Z} \to \mathbb{Z}$ such that
>
> $$f(f(x)y + x) = f(x)f(y) + f(x),$$
>
> for all $x, y \in \mathbb{Z}$.

Let $P(x, y)$ be the assertion $f(f(x)y + x) = f(x)f(y) + f(x)$

$P(0,0) \implies f(0) = 0$ and so, since injective, $a = f(1) \neq 0$

$P(1, x) \implies f(ax + 1) = a(f(x) + 1)$
$P(1,1) \implies f(a + 1) = a(a + 1)$
$P(ax + 1, 1) \implies f(a(f(x) + 1) + ax + 1) = a(f(x) + 1)(a + 1)$
$P(a + 1, x) \implies f(a(a + 1)x + a + 1) = a(a + 1)(f(x) + 1)$

And so, since injective : $a(f(x) + 1) + ax + 1 = a(a + 1)x + a + 1$ and so :

$\boxed{f(x) = ax}$ $\forall x$ whatever is $a \in \mathbb{Z} \setminus \{0\}$, which indeed is a solution.

---

aktyw19
1315 posts

Oct 30, 2013, 1:00 am PM #1138

Problem 355
Find all functions $f : \mathbb{R} \to \mathbb{R}$ wchich satisfy equation $f(x) = f(2x) + (f(x))^2$ for all x

---

fractals
3027 posts

Oct 30, 2013, 11:49 am PM #1139

Solution:
First suppose that there exists $a \leq \frac{1}{2}$ with $f(x) \leq a$ for all $x \in \mathbb{R}$. Then certainly from $f\left(\frac{x}{2}\right)^2 - f\left(\frac{x}{2}\right) + f(2x) = 0$ we have $f\left(\frac{x}{2}\right) = \frac{1 \pm \sqrt{1 - 4f(2x)}}{2}$. So now certainly the smaller root must be at most a, so $\frac{1 - \sqrt{1 - 4f(2x)}}{2} \leq a$ and thus $0 \leq 1 - 2a \leq \sqrt{1 - 4f(2x)}$ so $f(2x) \leq a - a^2$ after squaring and rearranging. Thus we have

Lemma. $f(x) \leq a$ for all $x \in \mathbb{R}$ implies $f(x) \leq a - a^2$ for all $x \in \mathbb{R}$ if $a \leq \frac{1}{2}$.

Now notice that from $f(x)^2 - f(x) + f(2x) = 0$, the discriminant is nonnegative, so we must have $1 - 4f(2x) \geq 0$ so $f(x) \leq \frac{1}{4}$ for all $x \in \mathbb{R}$. Let $a_0 = \frac{1}{4}$, and let $a_{n+1} = a_n - a_n^2$. First notice that $0 < a_n < \frac{1}{4}$ for $n > 0$ is simple. And also $a_{n+1} < a_n$ is obvious. Thus this sequence has a limit, and thus if the limit is L we must have $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (a_n - a_n^2) = L - L^2$ so $L = 0$. So now clearly as $f(x) \leq a_n$ for all $n \geq 0$ from the Lemma, we must have $f(x) \leq \lim_{n\to\infty} a_n = L = 0$. So now $f(x) \leq 0$ for all $x \in \mathbb{R}$ so in particular $f(x) = f\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right)^2 \geq 0$ since $f\left(\frac{x}{2}\right) \leq 0$. Thus $f(x) \geq 0$ for all $x \in \mathbb{R}$ so combining we find $f(x) = 0$ for all x real.

---

aktyw19
1315 posts

Oct 30, 2013, 4:51 pm PM #1140

Problem 356
Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xy) + f(x + y) = f(xy + x) + f(y)$.

---

socrates
1872 posts

Oct 31, 2013, 3:12 am PM #1141

> *fractals wrote:*

> Solution:
>
> ...
>
> So now $f(x) \leq 0$ for all $x \in \mathbb{R}$ so in particular $f(x) = f\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right)^2 \geq 0$ since $f\left(\frac{x}{2}\right) \leq 0$. Thus $f(x) \geq 0$ for all $x \in \mathbb{R}$ so combining we find $f(x) = 0$ for all x real.

Why?

---

socrates
1872 posts

Oct 31, 2013, 3:18 am PM #1142

> *aktyw19 wrote:*
> Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xy) + f(x+y) = f(xy+x) + f(y)$.

http://www.mathematica.gr/forum/viewtopic.php?f=111&t=18283

---

aktyw19
1315 posts

Oct 31, 2013, 12:14 pm PM #1143

Find all functions that $f : \mathbb{Q} \to \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y) + xy(x+y)$

---

socrates
1872 posts

Oct 31, 2013, 6:47 pm PM #1144

> *aktyw19 wrote:*
> Find all functions that $f : \mathbb{Q} \to \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y) + xy(x+y)$

The function $f(x) - \frac{x^3}{3}$ is Cauchy (additive), so $f(x) - \frac{x^3}{3} = cx$

thus $f(x) = \frac{x^3}{3} + cx$, where c is any constant.

---

socrates
1872 posts

Oct 31, 2013, 10:55 pm PM #1145

New Problem
Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(a+b)^3 - f(a)^3 - f(b)^3 = 3f(a)f(b)f(a+b),$$

for all $a, b \in \mathbb{Z}$.

---

pco
14052 posts

Nov 1, 2013, 12:02 am • 1 PM #1146

> *socrates wrote:*
> New Problem
> Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that
>
> $$f(a+b)^3 - f(a)^3 - f(b)^3 = 3f(a)f(b)f(a+b),$$
>
> for all $a, b \in \mathbb{Z}$.

Let $P(x,y)$ be the assertion $f(x+y)^3 - f(x)^3 - f(y)^3 = 3f(x)f(y)f(x+y)$
Let $a = f(1)$

$P(0,0) \implies f(0) = 0$
$P(x,-x) \implies f(-x) = -f(x)$

$P(1,1) \implies (f(2) - 2a)(f(2) + a)^2 = 0$ and so either $f(2) = 2a$, either $f(2) = -a$

1) If $f(2) = 2a$
==============
Suppose that $f(k) = ka \; \forall k \in \{1, 2, ..., n\}$ for some $n \geq 2$
$P(n,1) \implies (f(n+1) - a(n+1))(f(n+1)^2 + a(n+1)f(n+1) + a^2(n^2 - n + 1)) = 0$

If $a = 0$, this implies $f(n+1) = 0 = a(n+1)$
If $a \neq 0$, discriminant of the quadratic is $-3a^2(n-1)^2 < 0$ and so $f(n+1) = a(n+1)$

So $\boxed{f(n) = an}$ $\forall n \in \mathbb{Z}$ (using $f(-x) = -f(x)$) which indeed is a solution, whatever is $a \in \mathbb{Z}$

2) If $f(2) = -a$
==============
$P(2,1) \implies f(3)(f(3)^2 + 3a^2) = 0$ and so $f(3) = 0$

$P(n,3) \implies f(n+3) = f(n)$

And so $\boxed{f(3n) = 0 \text{ and } f(3n+1) = a \text{ and } f(3n+2) = -a}$ which indeed is a solution, whatever is $a \in \mathbb{Z}$

---

aktyw19
1315 posts

Nov 1, 2013, 12:08 am PM #1147

find all functions $f : \mathbb{R} \to \mathbb{R}$ such that: $f(x^2 + f(y)) = y + (f(x))^2$

---

socrates
1872 posts

Nov 1, 2013, 1:34 am PM #1148

> *aktyw19 wrote:*
> find all functions $f : \mathbb{R} \to \mathbb{R}$ such that: $f(x^2 + f(y)) = y + (f(x))^2$

IMO 92
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=366399&sid=611f74b9321e3785c7d53f415399ec20#p366399

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aktyw19
1315 posts

Nov 1, 2013, 1:42 am PM #1149

Find all functions $f : \mathbb{R} \to \mathbb{Z}$ such that

$$f(x) + f(y) \leq f(x+y)$$

for all reals x, y, where equality is achieved if and only if

$$x - f(x) + y - f(y) < 1.$$

socrates
1872 posts

Nov 1, 2013, 1:50 am PM #1150

Another new problem:
How many functions $f:\mathbb{N}_0\to\mathbb{N}_0$ satisfy $f(0)=2011,\ f(1)=111$, and

$$f(\max\{x+y+2,xy\})=\min\{f(x+y),f(xy+2)\},$$

for all $x,y\in\mathbb{N}_0$?

pco
14052 posts

Nov 1, 2013, 4:26 pm • 1 PM #1151

> socrates wrote:
> Another new problem:
> How many functions $f:\mathbb{N}_0\to\mathbb{N}_0$ satisfy $f(0)=2011,\ f(1)=111$, and
>
> $$f(\max\{x+y+2,xy\})=\min\{f(x+y),f(xy+2)\},$$
>
> for all $x,y\in\mathbb{N}_0$?

Let $P(x,y)$ be the assertion $f(\max(x+y+2,xy))=\min(f(x+y),f(xy+2))$
Let $f(0)=2011$, $f(1)=111$ and $a=f(2)$

Let $x\ge 2$: $P(x-2,1)\implies f(x+1)=\min(f(x-1),f(x))$

So :
1) If $111\ge a$
===============
Then $f(\mathbb{N}_0)=\{2011,111,a,a,a,a,...\}$ with $a\le 111$ and it is easy to see that this is sufficient.
And so 112 such functions

2) If $111<a$
==============
$f(\mathbb{N}_0)=\{2011,111,a,111,111,111,111,...\}$
$P(2,0)\implies f(4)=f(2)$ and so $a=111$, impossible in this bloc 2)

Hence the answer : $\boxed{112}$ such solutions.

nbh
18 posts

Nov 4, 2013, 8:25 am PM #1152

problem:find all decrease functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$10^x f(2x)=f(x)\forall x,y\in\mathbb{R}^+$$

socrates
1872 posts

Nov 8, 2013, 9:15 pm PM #1153

New Problem
Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y^3+f(y))=f(x),$$

for all $x,y\in\mathbb{R}$

pco
14052 posts

Nov 8, 2013, 9:36 pm • 1 PM #1154

> socrates wrote:
> New Problem
> Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that
>
> $$f(x+y^3+f(y))=f(x),$$
>
> for all $x,y\in\mathbb{R}$

Let $P(x,y)$ be the assertion $f(x+y^3+f(y))=f(x)$

If $f(x)+x^3=c$ is constant $\forall x$, we get $c=0$ and the solution $\boxed{f(x)=-x^3}$ $\forall x$
If $f(x)+x^3$ is not constant, $\exists a,b$ such that $f(a)+a^3-(f(b)+b^3)=u>0$

$P(x-(b^3+f(b)),a)\implies f(x+u)=f(x-(b^3+f(b)))$
$P(x-(b^3+f(b)),b)\implies f(x)=f(x-(b^3+f(b)))$
and so $f(x+u)=f(x)\forall x$

$P(x-y^3-f(y),y+u)\implies f(x+3y^2u+3yu^2+u^3)=f(x-y^3-f(y))$
$P(x-y^3-f(y),y)\implies f(x)=f(x-y^3-f(y))$
and so $f(x+3y^2u+3yu^2+u^3)=f(x)\forall x,y$
And since $3y^2u+3yu^2+u^3$ may take any value $z\ge u^3$ we want, we got $f(x+z)=f(x)\forall x,\forall z\ge u^3$

Let then $x>y$:
$f(x)=f(x+u^3)$
$f(y)=f(y+(x-y+u^3))$
And so $f(x)=f(y)$ and the solution
$\boxed{f(x)=c}$ $\forall x$, which indeed is a solution.

socrates
1872 posts

Nov 9, 2013, 11:55 pm PM #1155

a) Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)\,f(yf(x))=f(y)\,,$$

for all $x,y\in\mathbb{R}$.

b) Determine all continous functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$f(x)\,f(yf(x))=f(y)\,,$$

for all $x,y\in\mathbb{R}^+$.

c) (open question) Determine all monotone functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$f(x)\,f(yf(x))=f(y)\,,$$

for all $x, y \in \mathbb{R}^+$.

alibez
357 posts

Nov 17, 2013, 8:43 pm PM #1156

> socrates wrote:
> c) (open question) Determine all monotone functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
> $$f(x) f(yf(x)) = f(y),$$
> for all $x, y \in \mathbb{R}^+$.

What is your definition of a monotone?

$1) \quad x > y \to f(x) > f(y)$

$2) \quad x > y \to f(x) \geq f(y)$

IDMasterz
1409 posts

Nov 25, 2013, 7:55 am PM #1157

> arkanm wrote:
>
> > socrates wrote:
> > a) Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that
> > $$f(x) f(yf(x)) = f(y),$$
> > for all $x, y \in \mathbb{R}$.
>
> Let $p(x, y) \implies f(x)f(yf(x)) = f(y)$, then:
>
> $p(x, 0) \implies f(x)f(0) = f(0) \Leftrightarrow f(0)(f(x) - 1) = 0$. So either $f(x) = 1 \ \forall x \in \mathbb{R}$ or $f(0) = 0$. In the second case,
>
> $p(0, y) \implies f(y) = 0 \ \forall y \in \mathbb{R}$.
>
> :? is this wrong?

No its not, but the problem is then that his next questions have no solutions...

tenniskidp...
2376 posts

Nov 25, 2013, 9:03 am PM #1158

That's not a true statement, IDMasterz, because $f(y) = \frac{1}{y}$ is a solution. He plugs in $y = 0$ which you can't do when you're working in $\mathbb{R}^+$.

jjax
108 posts

Nov 28, 2013, 12:45 am PM #1159

> socrates wrote:
> b) Determine all continous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
> $$f(x) f(yf(x)) = f(y),$$
> for all $x, y \in \mathbb{R}^+$.

Observe that if the function is constant, then $f(x) = 1$ for all x. Consider now a nonconstant solution.

Define $S = f(\mathbb{R}^+)$ to be the set of values s such that for some x we have $f(x) = s$.

The equation $sf(ys) = f(y)$ tells us that if $s, t \in S$ then $st \in S$, and also $\frac{s}{t} \in S$. Thus S contains values that are arbitrarily large and values that are arbitrarily close to zero (choose s^k, s^{-k}).
By the intermediate value theorem we see that f is surjective, taking all positive real values.

Thus, substituting $z = f(x)$ in $f(x) f(yf(x)) = f(y)$ we get $zf(yz) = f(y)$.
Putting $y = 1$ we get $f(z) = \frac{a}{z}$ where $a = f(1)$ is an arbitrary constant. It's trivial to verify that this is indeed a solution.

Thus, either $f(x) = 1$ or $f(x) = \frac{a}{x}$.

jjax
108 posts

Nov 28, 2013, 1:22 am • 1 PM #1160

> socrates wrote:
> c) (open question) Determine all monotone functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
> $$f(x) f(yf(x)) = f(y),$$

If it's the latter, here's a simple construction of a strange function. I'd say it's unlikely that there is a nice characterization of all solutions.

Consider any integer k. For all x in the interval $(2^k, 2^{k+1}]$ define $f(x) = 2^{-k-1}$, and extend this definition over all such intervals. This function is basically a "stepped" version of the function $\frac{1}{x}$.

Now, the only values in $f(\mathbb{R}^+)$ are powers of 2, so the equation is $2^a f(y2^a) = f(y)$. If $y \in (2^k, 2^{k+1})$ then clearly $y2^a \in (2^{k+a}, 2^{k+a+1}]$. Substituting into the equation, we see that the identity does hold.

lehungviet...
1043 posts

Nov 30, 2013, 5:02 pm • 1 PM #1161

Problem 367
Find all functions $f : \mathbb{N}^* \to \mathbb{N}^*$ which satisfy the following conditions:
a) $f(f(n)) = n, \forall n \in \mathbb{N}^*$
b) $n | (f(1) + f(2) + ... + f(n)), \forall n \in \mathbb{N}^*$

Where $\mathbb{N}^* = \{1, 2, 3, ..., \}$ or $0 \notin \mathbb{N}^*$

pco
14052 posts

Dec 2, 2013, 7:37 pm • 1 PM #1162

> lehungvietbao wrote:
> **New problem**
> Find all functions $f : \mathbb{N}^* \to \mathbb{N}^*$ which satisfy the following conditions:
> a) $f(f(n)) = n, \forall n \in \mathbb{N}^*$
> b) $n | (f(1) + f(2) + ... + f(n)), \forall n \in \mathbb{N}^*$
>
> Where $\mathbb{N}^* = \{1, 2, 3, ..., \}$ or $0 \notin \mathbb{N}^*$

This problem is quite surprising. Is it a real olympiad exercise ?
There are infinitely many solutions and I dont think there is a general form for these solutions.

Hereunder is an algorithm in order to build infinitely many solutions (but it is not a general algorithm giving all the solutions).

Let $a \geq 3$
Set $f(1) = a$ and $f(a) = 1$
Let $A = \{1, a\}$ (A is the set of all positive integers n for which value of $f(n)$ is currently known)

Situation at each beginning of a new step :
Let $c(A) = \max\{p \in \mathbb{N}$ such that $[1, p] \subseteq A\}$
So $k \in A \; \forall k \in [1, c(A)]$ and $c(A) + 1 \notin A$.
Condition 1) $\forall k \in [1, c(A)]$, $f(k) \in A$ and $f(f(k)) = k$
Condition 2) $\forall k \in [1, c(A)]$, $k | \sum_{i=1}^{k} f(i)$
Condition 3) $\forall k > c(A)$: if $k \in A$, then $k + 1 \notin A$

It's easy to check that conditions are verified at the beginning of the first step. Then :
Let $n = c(A)$
Let $S = \sum_{k=1}^{n} f(k)$
Let $u \in [0, n]$ such that $-S \equiv u \pmod{n+1}$

If $n + 2 \notin A$
Choose any $m \in \mathbb{N}$ such that $u + m(n+1) > \max(A) + 1$ and
Set $f(n+1) = u + m(n+1)$
Set $f(u + m(n+1)) = n + 1$ (possible since $u + m(n+1) \notin A$)
Set $A \to A \cup \{n+1, u + m(n+1)\}$
Then :
$c(A) = n + 1$ and conditon 1) is still true
$\sum_{i=1}^{n+1} f(i) = S + u + m(n+1) \equiv 0 \pmod{n+1}$ and so condition 2) is still true
$u + m(n+1) > \max(A) + 1$ implies that condition 3) is still true
Next step

If $n + 2 \in A$
Then condition 3) implies $n + 3 \notin A$
Let then $v \in [0, n+1]$ such that $v \equiv S + u + f(n+2) \pmod{n+2}$
Choose any $m \in \mathbb{N}$ such that $u + v(n+1) + m(n+1)(n+2) > \max(A) + 1$ and :
Set $f(n+1) = u + v(n+1) + m(n+1)(n+2)$
Set $f(u + v(n+1) + m(n+1)(n+2)) = n + 1$ (possible since $u + v(n+1) + m(n+1)(n+2) \notin A$)
Set $A \to A \cup \{n+1, u + v(n+1) + m(n+1)(n+2)\}$
Then :
$c(A) = n + 2$ (due to condition 3)) and conditon 1) is still true
$\sum_{i=1}^{n+1} f(i) = S + u + v(n+1) + m(n+1)(n+2) \equiv 0 \pmod{n+1}$
$\sum_{i=1}^{n+1} f(i) = S + u + v(n+1) + m(n+1)(n+2) + f(n+2) \equiv 0 \pmod{n+2}$ and so condition 2) is still true
$u + v(n+1) + m(n+1)(n+2) > \max(A) + 1$ implies that condition 3) is still true
Next step

And since each step increases $c(A)$, we clearly define $f(n)$ for any n

---

lehungviet...
1043 posts

Dec 5, 2013, 4:44 pm • 1 PM #1163

Problem 368

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2), \forall x, y \in \mathbb{R}$

---

pco
14052 posts

Dec 5, 2013, 7:29 pm • 1 PM #1164

> lehungvietbao wrote:
> **Problem 368**
>
> Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$P(\frac{x+1}{2}, \frac{x-1}{2}) \implies g(x) = xg(1)$

Hence the result : $\boxed{f(x) = x^3 + ax}$ $\forall x$, which indeed is a solution, whatever is $a \in \mathbb{R}$

---

lehungviet...
1043 posts

Dec 6, 2013, 7:25 am • 2 PM #1165

Problem 369
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that :

$$f(x-y)f(x+y) = [f(x)+f(y)]\left[f(x) - xy + yf\left(\frac{x+y}{y}\right) - x + y\right], \; \forall x, y \in \mathbb{R}$$

---

hungkg
135 posts

Dec 6, 2013, 11:23 am PM #1166

> lehungvietbao wrote:
> **Problem 369**
> Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that :

$$f(x-y)f(x+y) = [f(x)+f(y)]\left[f(x) - xy + yf\left(\frac{x+y}{y}\right) - x + y\right],\ \forall x, y \in \mathbb{R}$$

Let P(x,y) be assertion $f(x-y)f(x+y) = [f(x)+f(y)]\left[f(x) - xy + yf\left(\frac{x+y}{y}\right) - x + y\right],\ \forall x, y \in \mathbb{R}/$

$$P(x,x) \Rightarrow f(0)f(2x) = [2f(x)]\left[f(x) - x^2 + xf(2)\right]$$

Give $x = 0 \Rightarrow f^2(0) = 2f^2(0) \Rightarrow f(0) = 0.$

$$P(0,x) \Rightarrow f(-x)f(x) = f(x)\left[xf(1) + x\right]$$

.

Give $x = -x \Rightarrow f(-x)\left[xf(1) + x + f(x)\right] = 0.$
Therefore, we have $f(-x) = 0$ or $f(x) = -(f(1)+1)x.$

hungkg
135 posts

Dec 6, 2013, 2:10 pm • 1 *PM #1167*

Problem 370

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x^2\right) - f\left(y^2\right) = f\left(x+y\right)f\left(x-y\right), \forall x, y \in \mathbb{R}.$$

[Moderator says: Please learn $\LaTeX$]

Tan
431 posts

Dec 16, 2013, 10:37 am *PM #1168*

Click to reveal hidden text

DannyKoz
101 posts

Dec 18, 2013, 10:13 am *PM #1169*

Why is this post still here this is here for more than three years

lehungviet...
1043 posts

Dec 28, 2013, 8:39 pm • 1 *PM #1170*

Problem 371
Find all continous functions $f : \mathbb{R} \to \mathbb{R}$ such that:

$$f(x+2002)(f(x)+\sqrt{2003}) = -2004 \quad \forall x \in \mathbb{R}$$

pco
14052 posts

Dec 28, 2013, 9:09 pm • 1 *PM #1171*

> *lehungvietbao wrote:*
> **Problem 371**
> Find all continous functions $f : \mathbb{R} \to \mathbb{R}$ such that:
>
> $$f(x+2002)(f(x)+\sqrt{2003}) = -2004 \quad \forall x \in \mathbb{R}$$

$f(x) \notin \{0, -\sqrt{2003}\}\, \forall x$ and so, since continuous :

Either $f(x) > 0\, \forall x$, but then $f(x+2002)(f(x)+\sqrt{2003}) > 0 > -2004$, impossible

Either $f(x) < -\sqrt{2003}\, \forall x$ but then $f(x+2002)(f(x)+\sqrt{2003}) > 0 > -2004$, impossible

Either $0 > f(x) > -\sqrt{2003}\, \forall x$, but then $-\frac{1}{\sqrt{2003}} > \frac{1}{f(x+2002)}$ and so $\frac{2004}{\sqrt{2003}} < -\frac{2004}{f(x+2002)} = f(x) + \sqrt{2003}$
And so $f(x) > \frac{2004}{\sqrt{2003}} - \sqrt{2003} > 0$, impossible.

So __no such function__.

lehungviet...
1043 posts

Dec 29, 2013, 4:45 pm • 1 *PM #1172*

Problem 372
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f\left(0\right) \neq 0$ and

$$f\left(x+y\right)f\left(x-y\right) = \left(f(x)\right)^2 - \sin^2 y \quad \forall x, y \in \mathbb{R}$$

lehungviet...
1043 posts

Dec 29, 2013, 4:47 pm • 1 *PM #1173*

Problem 373
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

1043 posts

Problem 374
Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$\begin{cases} f(0) = 2 \\ f(x + f(x+2y)) = f(2x) + f(2y) \quad \forall x, y \in \mathbb{Z} \end{cases}$$

.

pco
14052 posts

Dec 29, 2013, 6:02 pm *PM #1175*

> *lehungvietbao wrote:*
> **Problem 374**
> Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that
>
> $$\begin{cases} f(0) = 2 \\ f(x + f(x+2y)) = f(2x) + f(2y) \quad \forall x, y \in \mathbb{Z} \end{cases}$$
>
> .

Let $P(x,y)$ be the assertion $f(x+f(x+2y))=f(2x)+f(2y)$

If $f(2n)=2n+2$ for some $n\ge 0$, then $P(0,n)\implies f(2n+2)=2n+4$ and so we got $f(2x)=2x+2\,\forall x\ge 0$

Let $x\le 0$: $P(-2x,x)\implies f(2-2x)=f(-4x)+f(2x)$ and so $2-2x+2=-4x+2+f(2x)$ and so $f(2x)=2x+2\,\forall x\in\mathbb Z$

If $f(2k+1)=2u$ for some $k,u\in\mathbb Z$: $P(2k+1-2u,u)\implies 2u=f(2k+1)=4k-2u+6$ and so $2u=2k+3$, impossible.

So x odd $\implies f(x)$ odd and $x+f(x)$ even and so $f(x+f(x))=x+f(x)+2$. So $P(x,0)\implies f(x)=x+2$

And so $\boxed{f(x)=x+2}\ \forall x\in\mathbb Z$ which indeed is a solution.

amatysten
73 posts

Dec 29, 2013, 11:23 pm • 2 PM #1176

> *lehungvietbao wrote:*
> **Problem 373**
> Find all functions $f:\mathbb R\to\mathbb R$ such that
> $$f(x+y)+f(x-y)-2f(x)f(1+y)=2xy(3y-x^2)\quad\forall x,y\in\mathbb R$$

1. Putting $y=0$ into the equation we get $2f(x)(1-f(1))=0$. Since $\forall x f(x)=0$ isn't a solution, we get $f(1)=1$.
2. Now we put $x=1$ and get $f(1-y)-f(1+y)=2y(3y-1)$. Notice that LHS is an odd function, but RHS is not.
3. The answer is: no such function.

pco
14052 posts

Dec 30, 2013, 8:26 pm PM #1177

> *lehungvietbao wrote:*
> **Problem 372**
> Find all functions $f:\mathbb R\to\mathbb R$ such that $f(0)\neq 0$ and
> $$f(x+y)f(x-y)=(f(x))^2-\sin^2 y\quad\forall x,y\in\mathbb R$$

Let $P(x,y)$ be the assertion $f(x+y)f(x-y)=f(x)^2-\sin^2 y$
Let $a=f(0)$

1) If $f(u)=0$ for some u
=================
$P(x,x-u)\implies f(x)^2=\sin^2(x-u)$ and so $f(x)=e(x)\sin(x-u)$ for some function $e(x)$ from $\mathbb R\to\{0,1\}$

Plugging this in original equation, we get $(e(x+y)e(x-y)-1)(\cos 2y-\cos(2x-2u))=0$
And so $e(x+y)=e(x-y)\forall x,y$ such that $\cos 2y\neq\cos(2x-2u)$
And so $e(x)=e(y)\forall x,y$ such that $\cos(x-y)\neq\cos(x+y-2u)$

And so $e(x)=e(y)\forall x,y\neq u+k\pi$ and $e(u+k\pi)$ may be any value but this is of poor importance since then $f(x)=0$

Hence two solutions :
$\boxed{f(x)=\sin(x-u)}\ \forall x$ and $\boxed{f(x)=-\sin(x-u)}\ \forall x$ which indeed are solutions, whatever is $u\neq k\pi$

2) If $f(x)\neq 0\forall x$
===============
$$P(0,x)\implies f(-x)=\frac{a^2-\sin^2 x}{f(x)}$$

So $P(-x,-y)\implies \dfrac{a^2-\sin^2(x+y)}{f(x+y)}\dfrac{a^2-\sin^2(x-y)}{f(x-y)}=\dfrac{(a^2-\sin^2 x)^2}{f(x)^2}-\sin^2 y$

And so $\dfrac{(a^2-\sin^2(x+y))(a^2-\sin^2(x-y))}{f(x)^2-\sin^2 y}=\dfrac{(a^2-\sin^2 x)^2}{f(x)^2}-\sin^2 y$

And so $f(x)^4-2f(x)^2(a^2(1-2sin^2x)+\sin^2 x)+(a^2-\sin^2 x)^2=0$ (the terms in y cancel themselves)

Discriminant is $4a^2\sin^2 x\cos^2 x(1-a^2)$ and so $a^2\le 1$

But $a^2\le 1$ implies that $\exists t$ such that $a^2=\sin^2 t$ and then $P(0,t)\implies f(t)f(-t)=0$, impossible in this paragraph

So no other solution.

lehungviet...
1043 posts

Dec 31, 2013, 4:36 pm • 1 PM #1178

Problem 375
Find all functions $f:\mathbb Z\to\mathbb Z$ satisfy all the following conditions
a) $f(f(n))=f(n)$
b) $f(f(m)+f(n))=f(m+n)$
c) f has infinity values

$$f(m)+f(n)=f\left(\frac{m+n}{2}\right)+f(3m)\quad\forall m,n\in\mathbb Z$$

lehungviet...
1043 posts

Dec 31, 2013, 4:39 pm • 1 PM #1180

Problem 377
Prove that for all functions $f:\mathbb Z\to\mathbb Z$ then exist $x_0\in\mathbb Z$ such that
$$f(f(x_0))\neq 1-x_0^4$$

amatysten
73 posts

Jan 1, 2014, 12:56 am PM #1181

Seemingly forgotten problem. Tan's solution wasn't correct.

> *Quote:*
> **Problem 370**
> Find all functions $f:\mathbb R\to\mathbb R$ such that
> $$f\left(x^2\right)-f\left(y^2\right)=f(x+y)\,f(x-y)\,,\forall x,y\in\mathbb R.$$

1. $\boxed{\forall x f(x)=0}$ is a solution. We'll be looking for others. $R(x,y)$ as always.
2. $R(0,0)\Rightarrow f(0)=0; R(x,0)\Rightarrow f(x^2)=f^2(x)\Rightarrow \forall t\geq 0[f(t)\geq 0]$.
3. $x\geq y\geq 0\Rightarrow f(x)-f(y)=f(\sqrt{x}+\sqrt{y})f(\sqrt{x}-\sqrt{y})\geq 0\Rightarrow f$ is non-decreasing.
4. $f^2(1)=f(1)\in\{0;1\}$. $f(1)=0\Rightarrow \forall k\in\mathbb{Z}[f(k)=0]$ by induction.
5. $f^2(1/3)=f^2(2/3)=f^2(4/3)=f(1/9)=f(4/9)=f(16/9)$.
6. $f^2(7/9)=f^2(2/9)\Rightarrow -f^2(2/9)=f^2(1)-f^2(7/9)=f(2/9)f(16/9)$.
7. If $f(2/9)=0\Rightarrow f((2/9)^{2^n})=0$ and by induction $f(m(2/9)^{2^n})=0$. Since $2/9<1, m(2/9)^{2^n}$ are dense in $\mathbb{R}$ $\Rightarrow \forall x f(x)=0$, since f is non-decreasing.
8. $f(2/9)\neq 0\Rightarrow -f(2/9)=f(16/9)\Rightarrow f^2(1/3)-f^2(1/9)=f(4/9)f(2/9)=$ $=-f^2(1/9)\Rightarrow f(1/9)=0\Rightarrow \forall x f(x)=0$ as in 7.
9. $f(1)=1\Rightarrow f^2(2)-1=f(3)\Rightarrow (f^2(2)-1)^2-1=f^3(2)\Rightarrow f(2)\in\{0;-1;2\}$.
10. $f(2)=0\Rightarrow f^2(3/2)=f^2(1/2)\Rightarrow 1-f^2(1/2)=f(3/2)f(1/2)=\pm f^2(1/2)\Rightarrow f^2(1/2)=1/2=f(1/4)$.
11. $f(1/4)-f^2(1/4)=f^2(1/2)-f^2(1/4)=f(3/4)f(1/4)$. $f(1/4)\neq 0$ as in 7. $\Rightarrow f(3/4)=1-f(1/4)\Rightarrow 0=1-2f(1/4)=f^2(3/4)-f^2(1/4)=f(1/2)$. A contradiction.
12. $f(2)=-1\Rightarrow f^2(3/2)=f^2(1/2)-1\Rightarrow -f^2(3/2)=1-f^2(1/2)=$ $=f(1/2)f(3/2)\Rightarrow f(3/2)=0, f^2(1/2)=1=f^2(5/2)$.
13. $0=f^2(1)-f^2(1/4)=f(5/4)f(3/4)$. $f(5/4)\neq 0$, otherwise $f(5m/4)=0\Rightarrow f(5/2)=0$. Thus, $f(3/4)=0\Rightarrow \forall x f(x)=0$ as in 7.
14. $f(2)=2\Rightarrow 1-2f^2(1/2)=1-2f(1/4)=f(1/2)$ as in 11. $\Rightarrow f(1/2)\in\{-1;1/2\}$.
15. $f(1/2)=-1\Rightarrow f(1/4)=1\Rightarrow f(3/4)=0$. Thus $f(1/2)=1/2, f(1/4)=1/4$.
16. Let $f(m/2^n)=m/2^n, \forall n\leq 2k, \forall m\Rightarrow f^2(5/2^{2k+2})=f^2(4/2^{2k+2})+$ $+f(9/2^{2k+2})f(1/2^{2k+2})=f^2(1/2^{2k})+f^2(3/2^{k+1})f^2(1/2^{k+1})=(5/2^{2k+2})^2$.
17. $f^2(3/2^{2k+2})=\dfrac{(f^2(4/2^{2k+2})-f^2(1/2^{2k+2}))^2}{f^2(5/2^{2k+2})}=\cdots=(3/2^{2k+2})^2$.
18. $f(2/2^{2k+2})=\dfrac{f^2(3/2^{2k+2})-f^2(1/2^{2k+2})}{f(4/2^{2k+2})}=\cdots=2/2^{2k+2}$.
19. We have $f(1/2^{2k+2})=1/2^{2k+2}$ and $f(1/2^{2k+1})=1/2^{2k+1}$, so we can prove by induction, that $f(m/2^{2k+2})=m/2^{2k+2}$. Thus, $f(m/2^n)=m/2^n, \forall m,n\in\mathbb{Z}$ and, since they are dense in $\mathbb{R}$ and f is non-decreasing, $\boxed{f(x)=x,\forall x}$. Those two fit and there're no other functions.

P.S. If anyone has a shorter solution, could you please post it.
P.P.S Corrections are welcome. I had to banish induction to make the text a bit shorter.

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amatysten
73 posts

Jan 1, 2014, 1:12 am PM #1182

> *lehungvietbao wrote:*
> **Problem 375**
> Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ satisfy all the following conditions
> a) $f(f(n))=f(n)$
> b) $f(f(m)+f(n))=f(m+n)$
> c) f has infinity values

1. f is injective, since if there exist $m_1\neq m_2[f(m_1)=f(m_2)]\Rightarrow f(m_1+n)=f(m_2+n),\forall n$. It means that f is periodic and cannot yield infinitely many values.
2. From injectivity we get $f(m)+f(n)=m+n\Rightarrow f(m)=m+C$. Putting it into b) we get $C=0$.
3. The only solution is $\boxed{f(m)=m,\forall m}$.

P.S It's strange that we haven't even used a). There must be an error somewhere

This post has been edited 1 time. Last edited by amatysten, Jan 1, 2014, 1:46 pm

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amatysten
73 posts

Jan 1, 2014, 1:36 am • 1 PM #1183

> *lehungvietbao wrote:*
> **Problem 376**
> Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that
> $$f(m)+f(n)=f\left(\frac{m+n}{2}\right)+f(3m)\quad\forall m,n\in\mathbb{Z}$$

1. Putting $m=0, n=1$ we get $f(1/2)$ undefined. The answer is: no such function.
The statement of the problem seems incorrect. There should be

> *The Great Corrector wrote:*
> **Problem 376.1**
> Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $\forall m,n\in\mathbb{Z}$ satisfying $\dfrac{m+n}{2}\in\mathbb{Z}$
> $$f(m)+f(n)=f\left(\frac{m+n}{2}\right)+f(3m)$$

1. Putting $m=n=k$ we get $f(k)=f(3k),\forall k$.
2. Putting $m=0, n=2k$ we get $f(2k)=f(k),\forall k$.
3. Putting $n=0, m=2k$ we get $f(2k)+f(0)=f(k)+f(6k)\Rightarrow f(6k)=C=f(3k)=f(k)$.

> **Problem 377**
> Prove that for all functions $f:\mathbb{Z}\to\mathbb{Z}$ then exist $x_0\in\mathbb{Z}$ such that
> $$f(f(x_0))\neq 1-x_0^4$$

1. Let's assume, that $ff(x)=1-x^4,\forall x$. Let $f(0)=a\Rightarrow ff(0)=f(a)=1\Rightarrow ff(a)=f(1)=1-a^4\Rightarrow$ $ff(1)=f(1-a^4)=0\Rightarrow 1-(1-a^4)^4=ff(1-a^4)=f(0)=a$.
2. $1-a=(1-a)^4(1+a)^4(1+a^2)^4\Rightarrow a\in\{1;0\}$. $a=0\Rightarrow ff(0)=0\neq 1; a=1\Rightarrow ff(1)=1\neq 0$. We come to contradiction.

---

lehungviet...
1043 posts

Jan 1, 2014, 4:32 pm • 1 PM #1185

Problem 378
Let $f(x)$ be continuous functions on $[0;1]$ satisfy all the following conditions :
a) $f(0)=0, f(1)=1$
b) $5f\left(\dfrac{3x+y}{4}\right)=4f(x)+f(y)\forall x,y\in n\,[0;1]$ and $x\geq y$
Calculate $f(\dfrac{27}{\ }$

Calculate $f(55)$

lehungviet...
1043 posts

Jan 1, 2014, 4:42 pm • 1 PM #1186

Problem 379
Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $4f(x)f(y)=f(2xy)$ and $f(0)=0$ for all $x,y\in\mathbb{R}$

lehungviet...
1043 posts

Jan 1, 2014, 4:43 pm • 1 PM #1187

Problem 380

$f(x;y)$ defined for all natural numbers x,y such that
$f(0,y)=y+1;$
$f(x+1,0)=f(x,1);$
$f(x+1,y+1)=f(x,f(x+1,y))$
Calculate $f(4;2004)$

lehungviet...
1043 posts

Jan 1, 2014, 4:48 pm • 1 PM #1188

Problem 381
Let $P=\{1,2,\ldots,n\}$ be a set. The function $f:P\to\{1,2,\ldots,m\}$ satisfy :$(f(A\cap B))=\min\{f(A),f(B)\}$.
Find the relation between the functions f satisfy above condition and $\sum_{j=1}^{m} j^n$

pco
14052 posts

Jan 3, 2014, 4:05 pm • 1 PM #1189

> lehungvietbao wrote:
> **Problem 379**
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $4f(x)f(y)=f(2xy)$ and $f(0)=0$ for all $x,y\in\mathbb{R}$

Let $P(x,y)$ be the assertion $4f(x)f(y)=f(2xy)$
Let $a=f(1)$. Note that $P(\frac{\sqrt2}{2},\frac{\sqrt2}{2})\implies a\ge 0$

If $f(u)=0$ for some $u\ne 0$, then $P(\frac{x}{2u},u)\implies\boxed{f(x)=0}\ \forall x$, which indeed is a solution
Let us from now consider $f(x)\ne 0\ \forall x\ne 0$ (and so $a>0$

Let $g(x)=\frac{f(x)}{a}$ so that $P(x,y)$ becomes $4ag(x)g(y)=g(2xy)$ with $g(1)=1$ and $g(x)=0\iff x=0$

$P(xy,1)\implies 4ag(xy)=g(2xy)$ and so $g(xy)=g(x)g(y)$ with $g(2)=4a$ and so two families of solutions :

$g(0)=0$ and $g(x)=e^{h(\ln|x|)}\forall x\ne 0$ where $h(x)$ is any additive function such that $h(\ln 2)=\ln 4a$.
$g(0)=0$ and $g(x)=\text{sign}(x)e^{h(\ln|x|)}\forall x\ne 0$ where $h(x)$ is any additive function such that $h(\ln 2)=\ln 4a$.

Hence the solutions of original problem :

S1 : $f(x)=0\forall x$

S2 : $f0)=0$ and $f(x)=ae^{h(\ln|x|)}\forall x\ne 0$ where $a>0$ and $h(x)$ is any additive function such that $h(\ln 2)=\ln 4a$

S3 : $f0)=0$ and $f(x)=a.\text{sign}(x)e^{h(\ln|x|)}\forall x\ne 0$ where $a>0$ and $h(x)$ is any additive function such that $h(\ln 2)=\ln 4a$

Note that we can also write $h(x)=u(x)+\frac{\ln 4a-u(\ln 2)}{\ln 2}x$ where $u(x)$ is any additive function.

pco
14052 posts

Jan 3, 2014, 4:36 pm • 2 PM #1190

> lehungvietbao wrote:
> **Problem 380**
>
> $f(x;y)$ defined for all natural numbers x,y such that
> $f(0,y)=y+1;$
> $f(x+1,0)=f(x,1);$
> $f(x+1,y+1)=f(x,f(x+1,y))$
> Calculate $f(4;2004)$

Let $P(x)$ be the assertion $f(0,x)=x+1$
Let $Q(x)$ be the assertion $f(x+1,0)=f(x,1)$
Let $R(x,y)$ be the assertion $f(x+1,y+1)=f(x,f(x+1,y))$

Adding $Q(0)$ and $P(1)$, we get $f(1,0)=2$
Addind $P(f(1,x))$ and $R(0,x)$, we get $f(1,x+1)=f(1,x)+1$
And immediate induction gives $f(1,x)=x+2$

$Q(1)\implies f(2,0)=3$
$R(1,x)\implies f(2,x+1)=f(2,x)+2$
And immediate induction gives $f(2,x)=2x+3$

$Q(2)\implies f(3,0)=5$

And obviously no simple induction gives a general closed form for this $f(4,x)=\underbrace{2^{2^{\cdots^{2^{16}}}}}_{x\text{ times}}-3$

Since, as usual, this is a serious real olympiad exercise you got in a real contest/exam, there is certainly a nice olympiad level form for $f(4,2004)$

And, as usual, I'm quite sure you'll never post this nice solution when your teacher will give it to you :(

lehungviet...
1043 posts

Jan 8, 2014, 8:32 am PM #1191

Problem 382
Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that

$$f(x+y^2+z^3)=f(x)+f^2(y)+f^3(z)\quad\forall x,y,z\in\mathbb{N}$$

Where $\mathbb{N}$ is set of natural numbers (i.e $\mathbb{N}=\{0,1,2,..\}$)

lehungviet...

Jan 8, 2014, 8:33 am PM #1192

1043 posts

Problem 383
Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that

$$f(x^4+5y^4+10z^4)=f^4(x)+5f^4(y)+10f^4(z)\quad\forall x,y,z\in\mathbb{N}$$

Where $\mathbb{N}$ is set of natural numbers (i.e $\mathbb{N}=\{0,1,2,..\}$)

---

lehungviet...
1043 posts

Jan 8, 2014, 9:38 am PM #1193

Problem 384
Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
$$\frac{f(1)-(f(x)+f(y)+f(z))+(f(xy)+f(yz)+f(zx))-f(xyz)}{(1-x^2)(1-y^2)(1-z^2)}=2011\quad\forall x,y,z\neq\pm1$$

---

lehungviet...
1043 posts

Jan 8, 2014, 9:47 am PM #1194

Problem 385
Let k be a given positive number . Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$f(x)f(y)=kf(x+yf(x))$$

for all $x,y>0$.

---

pco
14052 posts

Jan 8, 2014, 1:57 pm • 1 PM #1195

> lehungvietbao wrote:
> **Problem 382**
> Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that
> $$f(x+y^2+z^3)=f(x)+f^2(y)+f^3(z)\quad\forall x,y,z\in\mathbb{N}$$
> Where $\mathbb{N}$ is set of natural numbers (i.e $\mathbb{N}=\{0,1,2,..\}$)

I supposed that $f^2(x)$ means $f(x)f(x)$ and not $f(f(x))$

Let $P(x,y,z)$ be the assertion $f(x+y^2+z^3)=f(x)+f(y)^2+f(z)^3$

$P(0,0,0)\implies f(0)=0$
$P(x,1,0)\implies f(x+1)=f(x)+f(1)^2$ and so $f(x)=f(1)^2x$

Plugging this back in original equation, we find **two solutions** :
$f(x)=0\forall x$
$f(x)=x\forall x$

---

pco
14052 posts

Jan 8, 2014, 2:31 pm • 1 PM #1196

> lehungvietbao wrote:
> **Problem 384**
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
> $$\frac{f(1)-(f(x)+f(y)+f(z))+(f(xy)+f(yz)+f(zx))-f(xyz)}{(1-x^2)(1-y^2)(1-z^2)}=2011\quad\forall x,y,z\neq\pm1$$

Let $P(x,y,z)$ be the assertion $\frac{f(1)-(f(x)+f(y)+f(z))+(f(xy)+f(yz)+f(zx))-f(xyz)}{(1-x^2)(1-y^2)(1-z^2)}=2011$

$P(x,0,0)\implies f(x)=2011x^2+f(1)-2011\forall x\neq1$, still true when $x=1$

And so $\boxed{f(x)=2011x^2+a}\ \forall x$, which indeed is a solution, whatever is $a\in\mathbb{R}$

---

pco
14052 posts

Jan 8, 2014, 5:01 pm • 2 PM #1197

> lehungvietbao wrote:
> **Problem 385**
> Let k be a given positive number . Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that
> $$f(x)f(y)=kf(x+yf(x))$$
> for all $x,y>0$.

Let $P(x,y)$ be the assertion $f(x)f(y)=kf(x+yf(x))$

1) Then only injective solution is $f(x)=ax+1$ when $k=1$
==
Suppose $f(x)$ be an injective solution.

$P(x,1)\implies f(x)f(1)=kf(x+f(x))$
$P(1,x)\implies f(1)f(x)=kf(1+xf(1))$
So $f(x+f(x))=f(1+xf(1))$ and so, since injective, $x+f(x)=1+xf(1)$ and $f(x)=x(f(1)-1)+1$

2) The only non injective solution is $f(x)=k\ \forall x$
==================================
Let $A=f^{-1}(\{k\})$
Let $B=f(\mathbb{R}^+)$

2.1) $A\neq\emptyset$

If $f(x)$ is not injective, let $u>v$ such that $f(u)=f(v)$

$P(u,\frac{v-u}{f(u)})\implies f(\frac{v-u}{f(u)})=k$ and so $A\neq\emptyset$

2.2) $u\in B\implies\frac{u^2}{k}\in B$ and $\frac{k^2}{u}\in B$

Let $f(x)=u$:

$P(x,x) \implies f(x(u+1)) = \frac{u}{k}$ and so $\frac{u}{k} \in B$

Let $a,b \in A$:$P(a,b) \implies a+kb \in A$ and simple induction gives $(1+nk)a \in A$
Let then $n \in \mathbb{N}$ such that $(1+nk)a > x$

$P(x, \frac{(1+nk)a-x}{u}) \implies f(\frac{(1+nk)a-x}{u}) = \frac{k^2}{u}$ and so $\frac{k^2}{u} \in B$
Q.E.D.

2.3) $f(x) = k \forall x$

Suppose $\exists u \in B$ such that $u \neq k$.
If $u < k$, let $v = u$ so that $v < k$ and $v \in B$
If $u > k$, let $v = \frac{k^2}{u}$ so that $v < k$ and $v \in B$

$v < k \in B \implies \frac{v^2}{k} < v < k$ and so, repeatedly, we get some $w < 1$ and $w < k$ such that $w \in B$

Let then z such that $f(z) = w$: $P(z, \frac{z}{1-w}) \implies w = k$, impossible

So no such u
So $\boxed{f(x) = k}$ $\forall x$, which indeed is a solution.
Q.E.D.

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lehungviet...
1043 posts

Jan 10, 2014, 5:43 pm PM #1198

Problem 386
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(y - f\left(x\right)\right) = f\left(x^n - y\right) - \left(n-1\right)yf\left(x\right)$$

, $\forall x, y \in \mathbb{R}$ and $n > 1$ is a given positive integer .

---

lehungviet...
1043 posts

Jan 10, 2014, 8:12 pm PM #1199

Problem 387
Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(\sqrt[n]{x^n+y^n}) = f(x) + f(y) \quad \forall x, y \in \mathbb{R}$$

Uwhere n is a given positive integer .

---

pco
14052 posts

Jan 11, 2014, 2:25 am PM #1200

> *lehungvietbao wrote:*
> **Problem 386**
> Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
>
> $$f\left(y - f\left(x\right)\right) = f\left(x^n - y\right) - \left(n-1\right)yf\left(x\right)$$
>
> , $\forall x, y \in \mathbb{R}$ and $n > 1$ is a given positive integer .

Let $P(x,y)$ be the assertion $f(y-f(x)) = f(x^n-y) - (n-1)yf(x)$

$P(x, \frac{x^n+f(x)}{2}) \implies f(x)(f(x)+x^n) = 0$ and so $\forall x$, either $f(x) = 0$, either $f(x) = -x^n$

$f(x) = -x^n \forall x$ is not a solution. So Let $u \neq 0$ such that $f(u) = 0$
Let $v \neq 0$ such that $f(v) = -v^n$:

$P(u,v) \implies f(u^n - v) = -v^n \neq 0$ and so $v^n = (u^n-v)^n$ and so $v = \frac{u^n}{2}$

So at most one such v but then, choosing any other u (which exists, since at most on v), we get a contradiction on v

So $\boxed{f(x) = 0}$ $\forall x$, which indeed is a solution.

---

pco
14052 posts

Jan 11, 2014, 2:38 am PM #1201

> *lehungvietbao wrote:*
> **Problem 387**
> Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that
>
> $$f(\sqrt[n]{x^n+y^n}) = f(x) + f(y) \quad \forall x, y \in \mathbb{R}$$
>
> Uwhere n is a given positive integer .

Let $g(x)$ from $[0,\infty) \to \mathbb{R}$ defined as $g(x) = f(\sqrt[n]{x})$
Equation is $g(x^n+y^n) = g(x^n)+g(y^n)$ and so, since continuous, $g(x) = ax$

And so $\boxed{f(x) = ax^n}$ $\forall x$, which indeed is a solution.

---

lehungviet...
1043 posts

Jan 11, 2014, 5:24 pm PM #1202

Problem 388
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^3) + f(y^3) = (x+y)(f(x^2) + f(y^2) - f(xy)) \quad \forall x, y \in \mathbb{R}$$

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lehungviet...
1043 posts

Jan 11, 2014, 7:43 pm PM #1203

Problem 389
Find all continuous functions f defined on $\mathbb{R}$ such that

\[f\left(\sqrt{\frac{x^2+y^2}{2}}\right)=\sqrt{f(x)f(y)} \quad \forall x,y\in\mathbb R}\]

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lehungviet...
1043 posts

Jan 11, 2014, 7:50 pm PM #1204

Problem 390
Find all continuous functions $f(x)$ defined on $\mathbb{R}$ such that

`\[{\begin{cases}f(x+y)+f(x-y)=2f(x)f(y)\\f(0)=1, \exists x_{0}\in\mathbb {R}: f(x_{0}})>1\end{cases}\quad \forall x,y\in\mathbb R\]`

lehungviet...
1043 posts

Jan 11, 2014, 7:55 pm PM #1205

Problem 391
Let $b>0$. Find all continuous functions $f(x)$ defined on

$$D:=\{x+2bk : x\in(-b,b), k\in\mathbb{Z}\}$$

and such that

$$f(x+y)=\frac{f(x)+f(y)}{1-f(x)f(y)}\quad \forall x,y,x+y\in D$$

lehungviet...
1043 posts

Jan 11, 2014, 8:00 pm PM #1206

Problem 392
Find all continuous functions $f(x),g(x)$ defined on $\mathbb{R}$ such that

$$\begin{cases}f(x+y)=f(x)g(y)+f(y)g(x)\\ g(x+y)=g(x)g(y)-f(x)f(y)\end{cases}\quad \forall x,y\in\mathbb{R}$$

pco
14052 posts

Jan 12, 2014, 5:31 pm PM #1207

> *lehungvietbao wrote:*
> **Problem 388**
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
> $$f(x^3)+f(y^3)=(x+y)(f(x^2)+f(y^2)-f(xy))\quad \forall x,y\in\mathbb{R}$$

Let $P(x,y)$ be the assertion $f(x^3)+f(y^3)=(x+y)(f(x^2)+f(y^2)-f(xy))$
Let $a=f(1)$

$P(0,0)\implies f(0)=0$
Comparing $P(x,0)$ and $P(-x,0)$, we get $f(-x^3)=-f(x^3)$ and so $f(x)$ is odd.

(1) : $P(x,0)\implies f(x^3)=xf(x^2)$
(2) : $P(x,1)\implies f(x^3)=xf(x^2)+ax-xf(x)+f(x^2)-f(x)$
(3) : $P(x,-1)\implies f(x^3)=xf(x^2)+ax+xf(x)-f(x^2)-f(x)$
(2)+(3)-2X(1) : $\boxed{f(x)=ax}$ $\forall x$, which indeed is a solution, whatever is $a\in\mathbb{R}$

pco
14052 posts

Jan 12, 2014, 6:49 pm • 1 PM #1208

> *lehungvietbao wrote:*
> **Problem 389**
> Find all continuous functions f defined on $\mathbb{R}$ such that
> `\[ f\left(\sqrt{\frac{x^2+y^2}{2}\right)=\sqrt{f(x)f(y)} \quad \forall x,y\in\mathbb R}\]`

Let $P(x,y)$ be the assertion $f(\sqrt{\frac{x^2+y^2}{2}})=\sqrt{f(x)(f(y)}$

If $f(u)=0$ for some u, then $P(u,u)\implies f(|u|)=0$
Then $P(\sqrt{2x^2-u^2},|u|)\implies f(x)=0\,\forall x\ge\frac{|u|}{\sqrt{2}}$
Simple induction implies then $f(x)=0\forall x>0$ and continuity implies $f(x)=0\,\forall x\ge 0$
Then $P(x,x)\implies |f(x)|=f(|x|)=0$ and so the solution $\boxed{f(x)=0}$ $\forall x$

If $f(x)\ne 0\,\forall x$, $f(x)$ has a constant sign (since continuous) and so $f(x)>0\,\forall x$
$P(x,x)\implies f(|x|)=|f(x)|=f(x)$ and so $f(x)$ is even.

Let then $g(x)=\ln(f(\sqrt{|x|}))$ and functional equation becomes $g(\frac{x^2+y^2}{2})=\frac{g(x^2)+g(y^2)}{2}$

This is a very classical equation whose solution (with continuity) is $g(x^2)=ax^2+b$

Hence the solution $\boxed{f(x)=e^{ax^2+b}}$ $\forall x$, which indeed is a solution, whatever are $a,b\in\mathbb{R}$

pco
14052 posts

Jan 12, 2014, 7:26 pm • 2 PM #1209

> *lehungvietbao wrote:*
> **Problem 390**
> Find all continuous functions $f(x)$ defined on $\mathbb{R}$ such that

It's a rather classical school-case (part of D'alembert functional equation).

Let $P(x,y)$ be the assertion $f(x+y)+f(x-y)=2f(x)f(y)$
$P(x,x)\implies f(2x)=2f(x)^2-1$

Let the sequence a_n defined as $a_0=x_0$ and $a_{n+1}=2a_n^2-1$. We easily get $a_n=f(2^nx_0)$ and $\lim_{n\to+\infty}a_n=+\infty$
So $f(x)$ is not upper-bounded.

If $\exists t$ such that $f(t)\le 0$, continuity implies that $\exists w\ne 0$ such that $f(w)=0$
$P(x+w,w)\implies f(x+2w)=f(x)$ and so $f(x)$ is periodic, and so bounded, in contradicton with the previous sentence.
So no such t and $f(x)>0\forall x$

$P(0,x)\implies f(-x)=f(x)$ and so $f(x)$ is even and WLOG $x_0>0$
Let $u>0$ such that $f(x_0)=\cosh u$
$P(\frac{x}{2},\frac{x}{2})\implies f(\frac{x}{2})=\sqrt{\frac{1+f(x)}{2}}$ (we used the fact that $f(x)>0\forall x$)
Simple induction implies then $f(\frac{x_0}{2^n})=\cosh\frac{u}{2^n}$

$P((k+1)\frac{u}{2^n},\frac{u}{2^n}) \implies f((k+2)\frac{u}{2^n})+f(k\frac{u}{2^n})=2f((k+1)\frac{u}{2^n})f(\frac{u}{2^n})$

Simple induction implies then $f(k\frac{x_0}{2^n})=\cosh k\frac{u}{2^n}\forall k,n\in\mathbb{N}$

And continuity allows us to write $f(x)=\cosh\frac{u}{x_0}x\;\forall x\ge 0$

And, since even, we get $\boxed{f(x)=\cosh ax}\;\forall x$, which indeed is a solution, whatever is $a\in\mathbb{R}$

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mathdebam
356 posts

Jan 12, 2014, 8:08 pm PM #1210

Problem 393
Prove that there does not exists ant function from integers to itself such that $f(m+f(n))=f(m)-n$.

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pco
14052 posts

Jan 12, 2014, 8:21 pm PM #1211

> *mathdebam wrote:*
> **Problem 393**
> Prove that there does not exists ant function from integers to itself such that $f(m+f(n))=f(m)-n$.

Let $P(x,y)$ be the assertion $f(x+f(y))=f(x)-y$
$f(x)$ is bijective.
If $f(u)=0$, then $P(x,u)\implies u=0$ and so $f(0)=0$

$P(0,x)\implies f(f(x))=-x$ and so $f(-x)=-f(x)$

$P(x,-f(y))\implies f(x+y)=f(x)+f(y)$ and so $f(x)=f(1)x$

Plugging this back in equation, we get $f(1)^2=-1$

Hence the result.

---

pco
14052 posts

Jan 12, 2014, 9:02 pm • 1 PM #1212

> *lehungvietbao wrote:*
> **Problem 391**
> Let $b>0$. Find all continuous functions $f(x)$ defined on
> $$D:=\{x+2bk: x\in(-b,b),k\in\mathbb{Z}\}$$
> and such that
> $$f(x+y)=\frac{f(x)+f(y)}{1-f(x)f(y)}\quad\forall x,y,x+y\in D$$

Let $P(x,y)$ be the assertion $f(x+y)=\frac{f(x)+f(y)}{1-f(x)f(y)}$, true $\forall x,y,x+y\ne(2k+1)b$

$P(x,0)\implies f(0)=0$
$P(x,-x)\implies f(-x)=-f(x)\forall x\ne(2k+1)b$ and so $f(x)$ is odd.

If $f(t)=0$ for some $t\in(0,b)$:
$P(x,t)\implies f(x+t)=f(x)\forall x,x+t\ne(2k+1)b$
$P(\frac{t}{2},\frac{t}{2}f(\frac{t}{2})=0$
So $f(x)=0\,\forall x\ne(2k+1)b$

Let us from now consider $f(x)\ne 0\forall x\in(0,b)$
Since $f(x)$ solution implies $-f(x)$ solution, WLOG $f(x)>0\forall x\in(0,b)$

Let $u\in(0,b)$ and $a=\arctan f(u)\in(0,\frac{\pi}{2})$ such that $f(u)=\tan a$
Let $x\in(0,b)$: $P(\frac{x}{2},\frac{x}{2})\implies f(\frac{x}{2})=\frac{\sqrt{f(x)^2+1}-1}{f(x)}$ (remember $f(x)>0$)
Simple induction implies $f(\frac{u}{2^n})=\tan a2^n$

Let n,k such that $(k+1)\frac{u}{2^n}<b$:

Using $P(k\frac{u}{2^n},\frac{u}{2^n})$ and $f(\frac{u}{2^n})=\tan a2^n$, simple induction gives $f(k\frac{u}{2^n})=\tan k\frac{a}{2^n}$

And so (continuity) $f(x)=\tan\frac{a}{u}x\,\forall x\in(0,b)$
And so (odd) $f(x)=\tan\frac{a}{u}x\forall x\in(-b,b)$ (as a first consequence : $\frac{au}{b}\ge\frac{\pi}{2}$)

πu πu

And its simple to extend to $\boxed{f(x)=\tan\frac{\pi}{2b(2k+1)}x}\;\forall x\in D$, which is indeed a solution

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pco
14052 posts

Jan 13, 2014, 12:42 am • 3 PM #1213

> *lehungvietbao wrote:*
> **Problem 392**
> Find all continuous functions $f(x),g(x)$ defined on $\mathbb{R}$ such that
> $$\begin{cases}f(x+y)=f(x)g(y)+f(y)g(x)\\ g(x+y)=g(x)g(y)-f(x)f(y)\end{cases}\quad\forall x,y\in\mathbb{R}$$

Let $S(x,y)$ be the assertion $f(x+y)=f(x)g(y)+f(y)g(x)$
Let $C(x,y)$ be the assertion $g(x+y)=g(x)g(y)-f(x)f(y)$

Let $h(x)=f(x)^2+g(x)^2$

Functionnal equations imply $h(x+y)=h(x)h(y)$ which very easily, since continuous, give solutions :
Either $h(x)=0\forall x$ and the solution $\boxed{f(x)=g(x)=0}$ $\forall x$
Either $h(x)=e^{ax}\forall x$

Note then that (f,g) solution implies $(\pm f(x)e^{-ax},g(x)e^{-ax})$ is solution too and so WLOG $f(x)^2+g(x)^2=1\forall x$

1) Solutions where $f(x)$ or $g(x)$ is constant
===================================
If one is constant, $f^2+g^2=1$ and continuity implies that both are constant.
Plugging back in equations we get $f(x)=0\,\forall x$ and $g(x)=1\forall x$

And so the solution $\boxed{f(x)=0\ \forall x\text{ and }g(x)=0\ \forall x}$

2) Solutions where neither $f(x)$ nor $g(x)$ is constant
===================================
2.1) $f(0)=0$ and $g(0)=1$

If $g(0)\neq 1$, $S(x,0)$ implies $f(x)=\alpha g(x)$ for some $\alpha\neq 0$ and $f^2+g^2=1$ implies f,g constant, impossible.
So $g(0)=1$ and $f(0)=0$
Q.E.D.

2.2) $\exists t>0$ such that $g(t)=0$ and $f(t)=1$ and $g(x)>0\forall x\in(0,t)$

$C(x,x)\implies g(2x)=g(x)^2-f(x)^2=2g(x)^2-1$

If $g(u)>1$ for some u, the sequence $g(2^nu)$ is positive unbounded anf $f^2(x)+g^2(x)=1$ is not possible. So $g(x)\le 1\forall x$
If $g(u)>0\,\forall x$ and non constant, $\exists u$ such that $0<g(u)<1$ and it's immediate to see then that $g(2^nu)<0$ for some n, impossible.

So $g(x)\le 0$ for some x and so (continuity), $\exists x$ such that $g(x)=0$

So $f(x)=\pm 1$ and $S(x,-x)\implies g(-x)=0$
let $A=\{x>0$ such that $g(x)=0\}$. In this paragraph, $A\neq\emptyset$ and so $\exists t=\inf(A)$
Since $g(x)$ is continuous, $g(t)=0$ and $t>0$ (since $g(0)=1$)

Since (f,g) solution implies $(-f,g)$ solution, WLOG $f(t)=1$

2.3) symetry and periodicity

$S(x,t)\implies f(x+t)=g(x)$
$C(x,t)\implies g(x+t)=-f(x)$
From there, we get :
$f(x+2t)=-f(x)$
$g(x+2t)=-g(x)$
$f(x+4t)=f(x)$
$g(x+4t)=g(x)$

2.4) Solution

$g(x)>0\,\forall x\in(0,t)$ (by definition of t)
If $f(u)=0$ for some $u\in(0,t)$, then $g(u-t)=f(u)=0$ and so $g(t-u)=0$ (see 3) above), impossible.
So $f(x)>0\,\forall x\in(0,t)$

$S(\frac{x}{2},\frac{x}{2})\implies f(x)=2f(\frac{x}{2})g(\frac{x}{2})$
$C(\frac{x}{2},\frac{x}{2})\implies g(x)=g(\frac{x}{2})^2-f(\frac{x}{2})^2$
If $x\in(0,t)$ and using the fact that $f(x)>0$ and $g(x)>0$ over $(0,t)$, this allows unique determination of $f(\frac{x}{2})$ and $g(\frac{x}{2})$ from $f(x)$ and $g(x)$

Starting from $f(t)=1$ and $g(t)=0$, a simple induction from these formulas implies $f(\frac{t}{2^n})=\sin\frac{\pi}{2^{n+1}}$ and $g(\frac{t}{2^n})=\cos\frac{\pi}{2^{n+1}}$

$S(k\frac{t}{2^n},\frac{t}{2^n})\implies f((k+1)\frac{t}{2^n})=f(k\frac{t}{2^n})g(\frac{t}{2^n})+f(\frac{t}{2^n})g(k\frac{t}{2^n})$
$C(k\frac{t}{2^n},\frac{t}{2^n})\implies g((k+1)\frac{t}{2^n})=g(k\frac{t}{2^n})g(\frac{t}{2^n})-f(k\frac{t}{2^n})f(\frac{t}{2^n})$
Starting from $f(\frac{t}{2^n})=\sin\frac{\pi}{2^{n+1}}$ and $g(\frac{t}{2^n})=\cos\frac{\pi}{2^{n+1}}$, a simple induction from these forulas implies :
$f(k\frac{t}{2^n})=\sin k\frac{\pi}{2^{n+1}}$ and $g(k\frac{t}{2^n})=\cos k\frac{\pi}{2^{n+1}}$

And continuity ends the process : $f(x)=\sin\frac{\pi}{2t}x$ and $g(x)=\cos\frac{\pi}{2t}x\,\forall x\ge 0$

$\boxed{f(x)=\pm e^{ax}\sin bx\text{ and }g(x)=e^{ax}\cos bx}$

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lehungviet...
1043 posts

Jan 13, 2014, 12:14 pm PM #1214
Problem 394
Find all functions $f,g:\mathbb{R}\to\mathbb{R}$ such that
$$f(x+g(y))=xf(y)-yf(x)+g(x)\quad\forall x,y\in\mathbb{R}$$

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lehungviet...
1043 posts

Jan 13, 2014, 12:18 pm PM #1215
Problem 395
Find all functions f,g defined on $\mathbb{R}$ such that f is odd function and
$$f(x^2)-f(y^2)=(x-y)g(x+y)\quad\forall x,y\in\mathbb{R}$$

---

lehungviet...
1043 posts

Jan 13, 2014, 12:31 pm PM #1216
Problem 396
Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f\left(\frac{x+y}{1+xy}\right)=\frac{f(x)f(y)}{|1+xy|}\quad \forall x,y\in\mathbb{R}\quad 1+xy\neq 0$$

pco
14052 posts

Jan 13, 2014, 5:31 pm • 1 PM #1217

> lehungvietbao wrote:
> **Problem 395**
> Find all functions f,g defined on $\mathbb{R}$ such that f is odd function and
> $$f(x^2)-f(y^2)=(x-y)g(x+y)\quad \forall x,y\in\mathbb{R}$$

Let $P(x,y)$ be the assertion $f(x^2)-f(y^2)=(x-y)g(x+y)$
Note that $f(x)$ odd implies $f(0)=0$

$$P(\frac{x+1}{2},\frac{x-1}{2})\implies f((\frac{x+1}{2})^2)-f((\frac{x-1}{2})^2)=g(x)$$

$$P(\frac{x-1}{2},\frac{1-x}{2})\implies f((\frac{x-1}{2})^2)-f((\frac{1-x}{2})^2)=(x-1)g(0)$$

$$P(\frac{1-x}{2},\frac{x+1}{2})\implies f((\frac{1-x}{2})^2)-f((\frac{x+1}{2})^2)=-xg(1)$$

Adding these three lines, we get $g(x)=(g(1)-g(0))x+g(0)$

Plugging $g(x)=ax+b$ in $P(x,0)$, we get $f(x^2)=ax^2+bx$ and so $b=0$ and $f(x)=ax\forall x\geq 0$ and, since odd, $f(x)=ax\forall x$

Hence the solutions $\boxed{f(x)=g(x)=ax}$ $\forall x$, which indeed is a solution, whatever is $a\in\mathbb{R}$

lehungviet...
1043 posts

Jan 13, 2014, 8:50 pm PM #1218

Problem 397
Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that :

$$\begin{cases} f(4)=4\\ f(2m)=2f(m),\ \forall m\equiv 1\ (mod\ 2)\\ f(m)<f(n),\ \forall m,n\in\mathbb{N}:m<n\end{cases}$$

Where $\mathbb{N}$ is set of natural number (i.e $\mathbb{N}=\{0,1,2,..\}$)

pco
14052 posts

Jan 13, 2014, 9:04 pm PM #1219

> lehungvietbao wrote:
> **Problem 396**
> Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that
> $$f\left(\frac{x+y}{1+xy}\right)=\frac{f(x)f(y)}{|1+xy|}\quad \forall x,y\in\mathbb{R}\quad 1+xy\neq 0$$

Let $P(x,y)$ be the assertion $f(\frac{x+y}{1+xy})=\frac{f(x)f(y)}{|1+xy|}$

1) Solutions over $(-1;+1)$
================
Let $g(x)=\frac{f(\tanh x)}{1-\tanh x}$: $g(x)$ is a continuous function from $\mathbb{R}\to\mathbb{R}$

$P(\tanh x,\tanh y)\implies g(x+y)=g(x)g(y)\forall x,y$ and so :

Either $g(x)=0\forall x$, and so $f(x)=0\forall x\in(-1,1)$
Either $g(x)=e^{2ax}\forall x$ and for some $a\in\mathbb{R}$, and so $f(x)=(1+x)^a(1-x)^{1-a}\forall x\in(-1,1)$ and for some $a\in\mathbb{R}$

2) Solutions over $(-\infty,-1)\cup(+1,+\infty)$
============================
Let $x,y\neq 0$ such that $1+xy\neq 0$:
Comparaison of $P(x,y)$ and $P(\frac 1x,\frac 1y)\implies|xy|f(\frac 1x)f(\frac 1y)=f(x)f(y)$

And so we quickly get either $f(\frac 1x)=\frac{f(x)}{|x|}\forall x\neq 0$, either $f(\frac 1x)=-\frac{f(x)}{|x|}\forall x\neq 0$.

Note that continuity implies $f(\frac 1x)=\frac{f(x)}{|x|}$ is $a\in\{0,1\}$

If $a\in(0,1)$, then $f(1)=f(-1)=0$
If $a=0$, then $f(1)=0$ and $f(-1)=2$
If $a=1$, then $f(1)=2$ and $f(-1)=0$

4) Synthesis of solutions
==============
So we got :

S1 :
$f(x)=0\forall x$

S2 :
$f(x)=|1+x|^a|1-x|^{1-a}\forall x$ and for any $a\in[0,1]$

S3 :
$f(x)=|1+x|^a|1-x|^{1-a}\forall x\in[-1,+1]$
$f(x)=-|1+x|^a|1-x|^{1-a}\forall x\notin[-1,+1]$
Where $a\in(0,1)$

This post has been edited 1 time. Last edited by pco, Jan 14, 2014, 2:41 pm

pco
14052 posts

Jan 13, 2014, 11:28 pm PM #1220

> lehungvietbao wrote:
> **Problem 397**
> Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that :
> $$\begin{cases} f(4) = 4 \\ f(2m) = 2f(m), \ \forall m \equiv 1 \ (mod\ 2) \\ f(m) < f(n), \ \forall m, n \in \mathbb{N} : m < n \end{cases}$$
> Where $\mathbb{N}$ is set of natural number (i.e $\mathbb{N} = \{0, 1, 2, ..\}$)

Let $A = \{x \in \mathbb{N}$ such that $f(x) = x\}$. $4 \in A$
$f(n)$ is strictly increasing and so $u \in A$ implies $n \in A\,\forall n \le u$ and so :
either $A = \mathbb{N}$, either $A = [0, M]$ for some $M \ge 4$

If $A = [0, M]$ and $M \ge 4$ odd, then $f(2M) = 2M$ and so $2M > M \in A$, and so contradiction
If $A = [0, M]$ and $M \ge 4$ even, then $f(2(M-1)) = 2(M-1)$ and so $2M - 2 > M \in A$, and so contradiction

So no such M and $A = \mathbb{N}$ and $\boxed{f(n) = n}\,\forall n$, which indeed is a solution.

CanVQ
72 posts

Jan 14, 2014, 9:36 pm • 1 PM #1221

> lehungvietbao wrote:
> **Problem 395**
> Find all functions f, g defined on $\mathbb{R}$ such that f is odd function and
> $$f(x^2) - f(y^2) = (x - y)g(x + y) \quad \forall x, y \in \mathbb{R} \quad (1)$$

Replacing y by $-y$, we get
$$(x - y) \cdot g(x + y) = (x + y) \cdot g(x - y), \quad \forall x,\, y \in \mathbb{R}. \quad (2)$$

Now, we substitute $u = x + y$ and $v = x - y$ in (2) to get

\[\frac{g(u)}{u}=\frac{g(v)}{v},\quad \forall u,\,v \in \R \setminus \{0\}.\]

So $g(x) = ax, \ \forall x \in \mathbb{R} \setminus \{0\}$. On the other hand, by replacing $x = 1$ and $y = -1$ in (1), we get $g(0) = 0$ and so
$$g(x) = ax, \quad \forall x \in \mathbb{R}.$$
From this, we get
$$f(x^2) - f(y^2) = a(x^2 - y^2),$$
or
$$f(x^2) - ax^2 = f(y^2) - ay^2, \quad \forall x,\, y \in \mathbb{R}.$$
It follows that
$$f(x) = ax + c, \quad \forall x \ge 0.$$
And since f is odd, we must have $f(0) = 0$, so $c = 0$ and $f(x) = ax, \ \forall x \ge 0$. Again, since f is odd, we have
$$f(x) = ax, \quad \forall x \in \mathbb{R}.$$

Finally, we have $f(x) = ax$ and $g(x) = ax$.

pco
14052 posts

Jan 15, 2014, 11:21 pm • 1 PM #1222

> lehungvietbao wrote:
> **Problem 394**
> Find all functions $f, g : \mathbb{R} \to \mathbb{R}$ such that
> $$f(x + g(y)) = xf(y) - yf(x) + g(x) \quad \forall x, y \in \mathbb{R}$$

Let $P(x, y)$ be the assertion $f(x + g(y)) = xf(y) - yf(x) + g(x)$
Let $a = f(0)$
Let $b = g(0)$
$P(0, 0) \implies f(b) = b$

1) If $a = 0$, the only solution is $f(x) = g(x) = 0\,\forall x$

$P(0, x) \implies f(f(x)) = 0$

$P(x, f(x)) \implies \boxed{f(x) = 0}\,\forall x$, which indeed is a solution
Q.E.D

2) If $a \ne 0$, then $f(x) = \dfrac{t}{t+1}(x - t)$ and $g(x) = t(x - t)$
=====================================
$P(0, x) \implies f(g(x)) = b - ax$ and so $f(x)$ is surjective and $g(x)$ is injective

2.1) If $f(u) = f(v) = 0$, then $u = v$

$P(0, u) \implies f(g(u)) = b - au$
$P(g(u), v) \implies f(g(u) + g(v)) = auv - vb + g(g(u))$
Swapping u, v wz get $f(g(u) + g(v)) = auv - ub + g(g(v))$
Subtracting, we get : $g(g(u)) + ub = g(g(v)) + vb$

$P(g(u),0) \implies f(g(u)+b) = au + g(g(u))$
$P(b,u) \implies f(b+g(u)) = -bu + g(b)$
Subtracting, we get $g(g(u)) + bu = -au + g(b)$

And so $g(g(u)) + ub = g(g(v)) + vb$ implies $u = v$
Q.E.D.

2.2) $\exists t$ such that $g(t) = 0$

Let $u = g(\frac{b}{a})$: $P(0, \frac{b}{a}) \implies f(u) = 0$ and so $u \neq 0$
Since $f(x)$ is surjective and $u \neq 0$, let t such that $f(t) = -\frac{g(u)}{u}$

$P(u,t) \implies f(u + g(t)) = 0$ and so, using 2.1 above $u + g(t) = u$ and $g(t) = 0$
Q.E.D.

2.3) $f(x) = \frac{t}{t+1}(x-t) \text{ and } g(x) = t(x-t)$

Let t such that $g(t) = 0$

$P(t,t) \implies f(t) = 0$
$P(x,t) \implies (t+1)f(x) = g(x)$ and so $b = (t+1)a$

$P(x,y)$ becomes then $f(x + (t+1)f(y)) = xf(y) + (t+1-y)f(x)$

$P(x,0) \implies f(x + (t+1)a) = ax + (t+1)f(x)$

$P(x+(t+1)a, y) \implies f(x+(t+1)a+(t+1)f(y)) = (x+(t+1)a)f(y) + (t+1-y)f(x+(t+1)a)$
$= (x+(t+1)a)f(y) + (t+1-y)(ax+(t+1)f(x))$
$P(x+(t+1)f(y), 0) \implies f(x+(t+1)f(y)+(t+1)a) = a(x+(t+1)f(y)) + (t+1)f(x+(t+1)f(y))$
$= a(x+(t+1)f(y)) + (t+1)(xf(y) + (t+1-y)f(x))$

So $(x+(t+1)a)f(y) + (t+1-y)(ax+(t+1)f(x)) = a(x+(t+1)f(y)) + (t+1)(xf(y)+(t+1-y)f(x))$

So $f(y) = -\frac{a}{t}y + a$ and $f(x) = -\frac{a}{t}x + a$ and $g(x) = -a\frac{t+1}{t}x + a(t+1)$

Plugging this back in original equation, we get $a = -\frac{t^2}{t+1}$

And so the final result : $\boxed{f(x) = \frac{t}{t+1}(x-t) \text{ and } g(x) = t(x-t)}$ $\forall x$, whatever is $t \neq -1$
(we kept the value $t = 0$ in order to cover the case 1) but normaly, in this paragraph, we should say $t \neq 0$)
Q.E.D.

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lehungviet...
1043 posts

Jan 17, 2014, 10:06 am PM #1223

Problem 398
Find all functions $f(x)$ defined on $\mathbb{R}$ such that

$$f\left(\frac{x^2+x+1}{x}\right) + f\left(\frac{x^2-x+1}{x}\right) = 2\left(x^2 + \frac{1}{x} + 3\right) \quad \forall x \neq 0$$

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lehungviet...
1043 posts

Jan 17, 2014, 10:09 am PM #1224

Problem 399
Let $q(x)$ be a given function. Find all functions $f(x)$ defined on $\mathbb{R}$ such that

$$f(x) + f(-x) + f\left(\frac{1}{x}\right) + f\left(\frac{-1}{x}\right) = q(x) \quad \forall x \neq 0$$

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lehungviet...
1043 posts

Jan 17, 2014, 10:20 am PM #1225

Problem 400
Let $p(x), q(x)$ be given period functions (additive) defined on $\mathbb{R}$ (with period $T = 2$)
Find all functions $f(x)$ defined on $\mathbb{R}$ such that

$$\begin{cases} f(x+4) = f(x) \\ p(x)f(x+2) + f(x) = q(x) \end{cases} \quad \forall x \in \mathbb{R}$$

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lehungviet...
1043 posts

Jan 17, 2014, 12:57 pm PM #1226

Problem 401
Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(f(x) + y) = xf(1 + xy) \quad \forall x, y \in \mathbb{R}^+$$

Problem 398
Find all functions $f(x)$ defined on $\mathbb{R}$ such that

$$f\left(\frac{x^2+x+1}{x}\right) + f\left(\frac{x^2-x+1}{x}\right) = 2\left(x^2 + \frac{1}{x} + 3\right) \quad \forall x \neq 0$$

Setting $x = 2$ in the functional equation, we get $f(\frac{7}{2}) + f(\frac{3}{2}) = 15$

Setting $x = \frac{1}{2}$ in the functional equation, we get $f(\frac{7}{2}) + f(\frac{3}{2}) = \frac{21}{2}$

So contradiction and no such function.

---

pco
14052 posts

Jan 19, 2014, 11:13 pm PM #1228

> lehungvietbao wrote:
> **Problem 399**

Let $q(x)$ be a given function. Find all functions $f(x)$ defined on $\mathbb{R}$ such that

$$f(x)+f(-x)+f\left(\frac{1}{x}\right)+f\left(\frac{-1}{x}\right)=q(x) \quad \forall x\neq 0$$

If $q(x)\neq q(-x)\forall x$ or $q(x)\neq q(\frac{1}{x})\forall x\neq 0$: no solution.

If $q(x)=q(-x)\forall x$ and $q(x)=q(\frac{1}{x})\forall x\neq 0$: $f(x)=\frac{1}{4}q(x)+g(x)$ where $g(x)$ is any solution of :

$g(x)+g(-x)+g(\frac{1}{x})+g(-\frac{1}{x})=0\forall x$ which is rather easy to solve :

1) either with piece per piece construction

2) either in more simple way $g(x)=u(x)+u(-x)+u(\frac{1}{x})-3u(-\frac{1}{x})$ whatever is $u(x)$

---

pco
14052 posts

Jan 19, 2014, 11:31 pm • 1 PM #1229

> *lehungvietbao wrote:*
> **Problem 400**
> Let $p(x),q(x)$ be given period functions (additive) defined on $\mathbb{R}$ (with period $T=2$)
> Find all functions $f(x)$ defined on $\mathbb{R}$ such that
>
> $$\begin{cases} f(x+4)=f(x) \\ p(x)f(x+2)+f(x)=q(x) \end{cases} \quad \forall x\in\mathbb{R}$$

In my humble opinion, you really should be able now to solve these exercises alone (you were helped for nearly ten similar exercises up to now and should be able to learn from our help). If you are unable to learn from the help obtained thru forum, maybe you should abandon olympiad target.

We get :
$p(x)f(x+2)+f(x)=q(x)$
$p(x)f(x)+f(x+2)=q(x)$

If $\exists u$ such that $p(u)=-1$ and $q(u)\ne 0^$, then no solution
If $p(u)=-1$ implies $q(x)=0$, then solution is :

Definition of $f(x)$ over $[0,4)$:
$\forall x\in[0,2)$ such that $|p(x)|=1$: $f(x)$ takes any value you want and $f(x+2)=q(x)-f(x)$
$\forall x\in[0,2)$ such that $|p(x)|\neq 1$: $f(x)=\frac{q(x)}{p(x)+1}$ and $f(x+2)=f(x)$

And extend these definition over $\mathbb{R}$ using $f(x+4)=f(x)$

And it's easy to see that this is a general solution.

---

pco
14052 posts

Jan 20, 2014, 1:37 am PM #1230

> *lehungvietbao wrote:*
> **Problem 401**
> Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that
>
> $$f(f(x)+y)=xf(1+xy) \quad \forall x,y\in\mathbb{R}^+$$

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=533460

War-Hammer
662 posts

Jan 28, 2014, 3:27 am • 1 PM #1231

Problem 402 :

Find all function $f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ such that :

$$f(\frac{x+y}{2})=\frac{2f(x)f(y)}{f(x)+f(y)}$$

---

pco
14052 posts

Jan 28, 2014, 3:56 am PM #1232

> *War-Hammer wrote:*
> **Problem 402 :**
>
> Find all function $f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ such that :

Let $g(x)=\frac{1}{f(x)}$ and we get $g(\frac{x+y}{2})=\frac{g(x)+g(y)}{2}$

It's then very classical to conclude $g(x)=ax+b$ with ($a>0$ and $b\geq 0$) or ($a=0$ and $b>0$). We dont need continuity since codomain is $\mathbb{R}^+$

So $f(x)=\frac{1}{ax+b}$ with a,b matching the above requirements.

CanVQ
72 posts

Jan 28, 2014, 4:05 pm • 1 PM #1233

A new solution for problem 394: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=38&t=148108&p=3374014#p3374014

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amatysten
73 posts

Jan 30, 2014, 8:26 pm PM #1234

Problem 403
Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$f(xf(y))=f(xy)+x \quad \forall x,y\in\mathbb{R}^+$$

amatysten
73 posts

Jan 30, 2014, 8:28 pm PM #1235

Problem 404
Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(f(x)+y)=2x+f(f(y)-x)\quad \forall x,y\in\mathbb{R}$$

---

amatysten
73 posts

Jan 30, 2014, 8:40 pm PM #1236

Problem 405
Find all real a for which there exists a function $f:\mathbb{R}\to\mathbb{R}$ such that

$$x+af(y)\le y+ff(x)\quad \forall x,y\in\mathbb{R}$$

---

pco
14052 posts

Jan 30, 2014, 9:34 pm PM #1237

> *amatysten wrote:*
> **Problem 403**
> Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that
>
> $$f(xf(y))=f(xy)+x\quad \forall x,y\in\mathbb{R}^+$$

Let $P(x,y)$ be the assertion $f(xf(y))=f(xy)+x$

Note that if $u\in f(\mathbb{R})$, then $\exists v>0$ such that $f(v)=u$ and $P(x,\frac{v}{x})\implies u+x\in f(\mathbb{R})$
So $\exists a\ge 0$ such that $f(\mathbb{R})=[a,+\infty)$ for $(a,+\infty)$

Then $P(1,x)\implies f(f(x))=f(x)+1$ and so $f(x)=x+1\,\forall x>a$

Let then $y>a$ and $x>\frac{a}{y+1}$ so that $f(y)=y+1$ and $f(xf(y))=xf(y)+1=xy+x+1$:
$P(x,y)\implies f(xy)=xy+1$ and so $f(x)=x+1\,\forall x>\frac{a^2}{a+1}$

And so, repeating this operation, we get $\boxed{f(x)=x+1}$ $\forall x>0$, which indeed is a solution.

---

pco
14052 posts

Jan 30, 2014, 10:22 pm PM #1238

> *amatysten wrote:*
> **Problem 404**
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
>
> $$f(f(x)+y)=2x+f(f(y)-x)\quad \forall x,y\in\mathbb{R}$$

Let $P(x,y)$ be the assertion $f(f(x)+y)=2x+f(f(y)-x)$

$P(\frac{f(0)-x}{2},-f(\frac{f(0)-x}{2}))\implies x=f(\text{...something...})$ and so $f(x)$ is surjective

Let then t such that $f(t)=0$

a) : $P(x,t)\implies f(f(x)+t)=2x+f(-x)$

b) : $P(x,\frac{f(-x)-f(x)}{2})\implies f(\frac{f(x)+f(-x)}{2})=2x+f(f(\frac{f(-x)-f(x)}{2})-x)$

c) : $P(\frac{f(-x)-f(x)}{2},-x)\implies f(f(\frac{f(-x)-f(x)}{2})-x)=f(-x)-f(x)+f(\frac{f(x)+f(-x)}{2})$

a-b-c) : $f(f(x)+t)=f(x)$

And so, since surjective : $\boxed{f(x)=x-t}$ $\forall x$, which indeed is a solution, whatever is $t\in\mathbb{R}$

CanVQ
72 posts

Jan 30, 2014, 10:48 pm • 2 PM #1239

> *amatysten wrote:*
> **Problem 403**
> Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that
>
> $$f(xf(y))=f(xy)+x\quad \forall x,y\in\mathbb{R}^+\quad (1)$$

Replacing x by $f(x)$ in (1), we get

$$f\big(f(x)\cdot f(y)\big)=f\big(y\cdot f(x)\big)+f(x)=f(xy)+y+f(x),\quad \forall x,\,y\in\mathbb{R}^+.\quad (2)$$

$f(x)=x+1$ is the solution of our problem.

CanVQ
72 posts

Jan 30, 2014, 11:43 pm • 2 PM #1240

> *amatysten wrote:*
> **Problem 404**
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
>
> $$f(f(x)+y)=2x+f(f(y)-x)\quad \forall x,y\in\mathbb{R}\quad (1)$$

Replacing $y=-f(x)$ in (1), we get

$$f\Big(f\big(-f(x)\big)-x\Big)=f(0)-2x,\quad \forall x\in\mathbb{R}.\quad (2)$$

This result shows that f is surjective. Now, we will prove that f is injective. Assume that there are $a,\ b$ such that $f(a)=f(b)$. By taking $y=a$ and $y=b$ in (1) respectively, we get

$$f\big(f(x)+a\big)=f\big(f(x)+b\big),\quad \forall x\in\mathbb{R}.\quad (3)$$

Since f is surjective, the identity (3) implies that

$$f(x)=f(x+t)=f(x-t),\quad \forall x\in\mathbb{R},\quad (4)$$

where $t=a-b$. Replacing x by $x+t$ in (1) and using (4), we get

$$f\big(f(x)+y\big)=2(x+t)+f\big(f(y)-x\big),\quad \forall x,\, y\in\mathbb{R}.\quad (5)$$

Replacing x by $x-t$ in (1) and using (4), we get

$$f\big(f(x)+y\big)=2(x-t)+f\big(f(y)-x\big),\quad \forall x,\, y\in\mathbb{R}.\quad (6)$$

Comparing (5) with (6), we deduce that $t=0$, or $a=b$. So f is injective.

Now, replacing $x=0$ in (1), we get

$$f\big(f(0)+y\big)=f\big(f(y)\big),\quad \forall y\in\mathbb{R}.\quad (7)$$

Since f is injective, it follows that $f(y)=y+f(0),\ \forall y\in\mathbb{R}$. It is easy to check that the function $f(x)=x+c$ satisfies the given condition.

---

hal9v4ik
368 posts

Jan 31, 2014, 10:38 am PM #1241

Problem 405
It is trivial that for $a=0$ there are no solutions.
Now see that $af(y)-y$ and $x-f(f(x))$ are bounded, i.e. there are A and B such that $A>af(y)-y$ (1) and $B>x-f(f(x))$ (2).
from (1) we can get that $A/a|>f(f(x))-f(x)/a$ combining by (2) we get some const $C>x-f(x)/a=>C*a^2>a^2*y-af(y)$, summing this with (1) we get some const $D>(a^2-1)*x$. Which possible only when $a=1,-1$. For $a=1$ and $a=-1$ we have answers $f(x)=x$ and $f(x)=-x$, respectively.

---

amatysten
73 posts

Jan 31, 2014, 8:11 pm PM #1242

@hal9v4ik
It seems your solution works only when $a>0$. When you divide by $a<0$ you'll get $\frac{A}{a}<ff(x)-\frac{f(x)}{a}$.

---

jjax
108 posts

Jan 31, 2014, 10:32 pm • 1 PM #1243

> *amatysten wrote:*
> **Problem 405**
> Find all real a for which there exists a function $f:\mathbb{R}\to\mathbb{R}$ such that
> $$x+af(y)\le y+ff(x)\quad \forall x,y\in\mathbb{R}$$

Let $P(x,y)$ denote the proposition that $x+af(y)\le y+ff(x)$.

We will show that either $a=1$ or a is negative. In either case we will construct a suitable function.

Case 1: $a>0$. We will show that $a=1$.
$P(0,x)$ gives $af(x)\le x+f(f(0))$, and keeping in mind the sign of a we obtain $f(x)\le\frac{1}{a}x+c$ where c is some constant.
Thus, we obtain $f(f(x))\le\frac{1}{a}f(x)+c\le\frac{1}{a^2}x+d$ where d is constant.
Thus, the original equation yields $x+af(y)\le y+f(f(x))\le y+\frac{1}{a^2}x+d$, and rearranging gives $af(y)-y-d\le x(\frac{1}{a^2}-1)$. Allowing x to vary, we see that we must have $\frac{1}{a^2}-1=0$, or $a=1$. It's trivial to see that the identity function satisfies the condition.

Case 2: $a=0$. This case is clearly impossible, as you may vary y in the inequality $x\le y+f(f(x))$.

Case 3: $a<0$. Set $b=-a>0$. We will construct a function f satisfying $x\le y+bf(y)+f(f(x))$ for each b.
Subcase: $b\ge 1$. Choose $f(x)=|x|$, the absolute value function. Clearly $x\le y+|y|+||x||\le y+b|y|+||x||$ so we are done.
Subcase: $b<1$. Choose $f(x)=\frac{1}{b}|x|$. Clearly $x\le y+b\times\frac{1}{b}|y|+\frac{1}{b^2}|x|$ so we are done.

---

jjax
108 posts

Jan 31, 2014, 10:38 pm • 1 PM #1244

> *amatysten wrote:*
> **Problem 404**
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
> $$f(f(x)+y)=2x+f(f(y)-x)\quad \forall x,y\in\mathbb{R}$$

I realise multiple solutions have been posted, but here's a short one:
Let $P(x,y)$ denote the proposition that $f(f(x)+y)=2x+f(f(y)-x)$.

First we note that f is surjective. Indeed, $P(x,-f(x))$ gives $f(0)=2x+f(f(-f(x))-x)$ and varying x gives surjectivity.
Thus the function has a root u with $f(u)=0$.

72 posts

Problem 406. Find all function $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f\big(y+f(x)\big)=f(x)\cdot f(y)+f\big(f(x)\big)+f(y)-xy,\quad \forall x,\, y\in\mathbb{R}.$$

---

amatysten
73 posts

Feb 1, 2014, 7:16 pm PM #1246

Quite nice

> *CanVQ wrote:*
> **Problem 406.** Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying
> $$f\big(y+f(x)\big)=f(x)\cdot f(y)+f\big(f(x)\big)+f(y)-xy,\quad \forall x,\, y\in\mathbb{R}.$$

1. Denote the given equation by $P(x,y)$. $P(x,0)$ yields $f(0)=0$, since $f(x)=const$ doesn't match.
$P_{LHS}(x,f(y))=P_{LHS}(y,f(x))\Rightarrow f(x)ff(y)=(x+ff(x))f(y)-f(x)y$.
$\forall x f(x)=0$ isn't a solution, so $\exists a f(a)\ne 0\Rightarrow ff(y)=\frac{a+ff(a)}{f(a)}f(y)-y=Cf(y)-y,\forall y$.

$f(a)$

3. $P(x,y)$ rewrites as $f(y+f(x))=f(y)(f(x)+1)+Cf(x)-x(y+1)$.
$P(x,x):f(x+f(x))=f^2(x)+Cf(x)+f(x)-x-x^2$.
$P(f(x),x):f(Cf(x))=Cf^2(x)-2xf(x)+C^2f(x)-Cx$.

4. $P_{LHS}(f(x),x+f(x))=P_{LHS}(x,Cf(x))\Rightarrow$
$\Rightarrow f(x+f(x))(Cf(x)-x+1)+C^2f(x)-Cx-f(x)(x+f(x)+1)=$
$=f(Cf(x))(f(x)+1)+Cf(x)-x(Cf(x)+1)$.
Using 3. we get (after long computations) $\forall x\neq 0 \quad f^2(x)-Cxf(x)+x^2=0$. It mean that if such function exists, then $\forall x\neq 0$ either $f(x)=k_1x$ or $f(x)=k_2x$, where $k_1,k_2\neq 0$ are roots of $t^2-Ct+1=0, k_1k_2=1$.

5. WLOG set $\{x\neq 0|f(x)=k_1x\}$ is infinite. $P(x,x):k_x(x+k_1x)=k_1^2x^2+l_xk_1x+k_1x-x^2$ or $k_x(1+k_1)-l_xk_1-k_1=(k_1^2-1)x$. Since $k_x,l_x\in\{k_1,k_2\}$ LHS can take only finite number of values. But RHS takes infinitely many values, unless $k_1^2=1\Rightarrow k_2=k_1=\pm 1$. They both match.

6. The answer is $\boxed{f(x)=x,\forall x}$ and $\boxed{f(x)=-x,\forall x}$.

CanVQ
72 posts

Feb 2, 2014, 4:58 pm • 1 PM #1247

> *CanVQ wrote:*
> I will propose a new problem:
>
> **Problem 406.** Find all function $f:\mathbb{R}\to\mathbb{R}$ satisfying
> $$f\big(y+f(x)\big)=f(x)\cdot f(y)+f\big(f(x)\big)+f(y)-xy,\quad \forall x,\,y\in\mathbb{R}.\quad (1)$$

This is my solution:

Replacing y by $y+f(z)$ in (1), we have
$$\begin{aligned}f\big(y+f(z)+f(x)\big)&=\big[1+f(x)\big]\cdot f\big(y+f(z)\big)+f\big(f(x)\big)-x\big[y+f(z)\big]\\&=\big[1+f(x)\big]\Big\{\big[1+f(z)\big]\cdot f(y)+f\big(f(z)\big)-yz\Big\}+f\big(f(x)\big)-x\big[y+f(z)\big]\\&=\big[1+f(x)\big]\big[1+f(z)\big]\cdot f(y)+f\big(f(x)\big)+f\big(f(z)\big)-y(x+z)\\&\quad+f(x)\cdot f\big(f(z)\big)-yz\cdot f(x)-x\cdot f(z).\quad (2)\end{aligned}$$

Changing the position of x and z in (2), we get
$$f(x)\cdot f\big(f(z)\big)-yz\cdot f(x)-x\cdot f(z)=f(z)\cdot f\big(f(x)\big)-yx\cdot f(z)-z\cdot f(x),\quad \forall x,\,y,\,z\in\mathbb{R}.\quad (3)$$

Since (3) holds for any $y\in\mathbb{R}$, we must have
$$x\cdot f(z)=z\cdot f(x),\quad \forall x,\,z\in\mathbb{R},\quad (4)$$

from which it follows that $f(x)=kx,\,\forall x\in\mathbb{R}$. Plugging this result into the original equation, we get $k=\pm 1$. So $f(x)=x$ or $f(x)=-x$. These functions satisfy our problem.

amatysten
73 posts

Feb 2, 2014, 5:55 pm PM #1248

Problem 407. Find all functions $f:\mathbb{R}^+\to\mathbb{R}$ satisfying two conditions:
$$(i)\quad f(x)+f(y)\le\frac{f(x+y)}{2}\quad \forall x,\,y\in\mathbb{R}^+$$
$$(ii)\quad \frac{f(x)}{x}+\frac{f(y)}{y}\ge\frac{f(x+y)}{x+y}\quad \forall x,\,y\in\mathbb{R}^+.$$

CanVQ
72 posts

Feb 2, 2014, 10:19 pm • 1 PM #1249

> *amatysten wrote:*
> **Problem 407.** Find all functions $f:\mathbb{R}^+\to\mathbb{R}$ satisfying two conditions:
> $$(i)\quad f(x)+f(y)\le\frac{f(x+y)}{2}\quad \forall x,\,y\in\mathbb{R}^+$$
> $$(ii)\quad \frac{f(x)}{x}+\frac{f(y)}{y}\ge\frac{f(x+y)}{x+y}\quad \forall x,\,y\in\mathbb{R}^+.$$

Replacing $x=y$ in (i) and in (ii) respectively, we get
$$f(2x)=4\cdot f(x),\quad \forall x\in\mathbb{R}^+.\quad (1)$$

From this result and (ii), we can easily prove by induction that
$$f(2^nx)=2^{2n}\cdot f(x),\quad \forall x\in\mathbb{R}^+,\,n\in\mathbb{N}\quad (2)$$

Also, using (ii), we can establish the following inequality
$$\frac{f(x_1+\cdots+x_m)}{x_1+\cdots+x_m}\le\frac{f(x_1)}{x_1}+\cdots+\frac{f(x_m)}{x_m},\quad \forall x_1,\,\ldots,\,x_m\in\mathbb{R}^+,\,m\in\mathbb{N}^*.\quad (4)$$

Now, for any $n\in\mathbb{N}^*$, we choose $k\in\mathbb{N}^*$ such that $2^k>n$. Take $m=2^k+1-n,\ x_1=x_2=\cdots=x_{m-1}=x$ and $x_m=nx$ in (4), we get
$$\frac{f(2^kx)}{2^kx}\le\frac{(2^k-n)\cdot f(x)}{x}+\frac{f(nx)}{nx}.\quad (5)$$

Since $f(2^kx)=2^{2k}\cdot f(x)$, we can easily deduce that
$$f(nx)\ge n^2\cdot f(x),\quad \forall x\in\mathbb{R}^+,\,n\in\mathbb{N}^*.\quad (4)$$

From (3) and (4), we have
$$f(nx)=n^2\cdot f(x),\quad \forall x\in\mathbb{R}^+,\,n\in\mathbb{N}^*.\quad (5)$$

Setting $g(x)=\dfrac{f(x)}{x}$, then we have

$$g(nx) = n \cdot g(x), \quad \forall x \in \mathbb{R}^+,\ n \in \mathbb{N}^*. \quad (6)$$

We will prove that g is decreasing, from which it will follows that $g(x) = kx$. From (i) and (ii), we have

$$g(x)+g(y) \geq \frac{f(x+y)}{x+y} \geq \frac{2\cdot f(x)+2\cdot f(y)}{x+y} = \frac{2x\cdot g(x)+2y\cdot g(y)}{x+y},$$

or

$$(x-y)\big[g(x)-g(y)\big] \leq 0.$$

From this, it is easy to see that $g(x)$ is decreasing. And hence, in combination with (6), we get $g(x) = kx,\ \forall x \in \mathbb{R}^+$ where $k \leq 0$ is a given constant. So $f(x) = kx^2$.

aktyw19
1315 posts

Feb 3, 2014, 4:21 am PM #1250

Problem 408
Find (if it exists) such $f: R \mapsto R$: $f(f(x)-x) = 2x$ for $x \in R$ other than $f(x) = 2x$.

amatysten
73 posts

Feb 3, 2014, 10:49 am PM #1251

$\boxed{f(x) = -x, \forall x}$. I'm not sure if I understood correctly. We just need to show one such function, right?

amatysten
73 posts

Feb 4, 2014, 1:44 pm PM #1252

We can make a little change and...

Problem 408 (a)
Given a constant $a \in \mathbb{R}$. Find all functions $f: [a; +\infty) \to \mathbb{R}$ satisfying

$$f(f(x)-x) = 2x, \quad \forall x \geq a$$

pco
14052 posts

Feb 4, 2014, 2:16 pm • 1 PM #1253

> *amatysten wrote:*
> We can make a little change and...
>
> **Problem 408 (a)**
> Given a constant $a \in \mathbb{R}$. Find all functions $f: [a; +\infty) \to \mathbb{R}$ satisfying
> $$f(f(x)-x) = 2x, \quad \forall x \geq a$$

Functional equation implies $f(x) - x \geq a\ \forall x \geq a$
And so $f(f(x)-x) - (f(x)-x) \geq a$ wich is $3x - f(x) \geq a$ and so $f(x) \leq 3x - a\ \forall x \geq a$

So $3x - a \geq f(x) \geq x + a\ \forall x \geq a$

Suppose now that we have $a_nx + b_na \geq f(x) \geq c_nx + d_na\ \forall x \geq a$ and for some $a_n, c_n > 0$

This implies $a_n(f(x)-x) + b_na \geq 2x \geq c_n(f(x)-x) + d_na\ \forall x \geq a$

And so $(1+\frac{2}{c_n})x - \frac{d_n}{c_n}a \geq f(x) \geq (1+\frac{2}{a_n})x - \frac{b_n}{a_n}a\ \forall x \geq a$

And it's easy to show that starting with $(a_0, b_0, c_0, d_0) = (3, -1, 1, 1)$ we get :
$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} c_n = 2$
$\lim_{n\to+\infty} b_n = \lim_{n\to+\infty} b_n = 0$

And so $\boxed{f(x) = 2x}\ \forall x \geq a$, which indeed is a solution.

amatysten
73 posts

Feb 4, 2014, 10:06 pm PM #1254

That was fast
My own solution, just to show a little different approach.

> *amatysten wrote:*
> **Problem 408 (a)**
> Given a constant $a \in \mathbb{R}$. Find all functions $f: [a; +\infty) \to \mathbb{R}$ satisfying
> $$f(f(x)-x) = 2x, \quad \forall x \geq a$$

1) We'll denote a new function $g(x) = f(x) - x$ and notice that it must be bounded from below by $a \in \mathbb{R}$.

2) Then $f(x) = g(x) + x$ and $fg(x) = gg(x) + g(x)$ and the given equation rewrites as $gg(x) + g(x) - 2x = 0$.

3) Fixing x and denoting $a_0 = x, a_n = gg\dots g(x)$ (n times) we get a recursive equation $a_{n+1} + a_n - 2a_{n-1} = 0, n \geq 1$.

4) It's solution $a_n = C + (-2)^n \frac{a_1 - a_0}{3}$. For very big n a_n takes arbitrary big negative values (yielding a contradiction, since it

aktyw19
1315 posts

Problem 409
Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that $f(x + f(x) + \frac{y}{2}) = f(\frac{x}{2}) + y + f(y)$

alibez
357 posts

Feb 8, 2014, 8:41 pm PM #1256

hint

shatlykimo
70 posts

Feb 10, 2014, 8:49 pm PM #1257

Problem 410
Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + y + f(y)) = 2y + (f(x))^2 \quad \forall x, y \in \mathbb{R}$$

Inequalities
33 posts

Feb 10, 2014, 10:34 pm PM #1258

see http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=575372&p=3390848#p3390848

aktyw19

Feb 10, 2014, 10:56 pm PM #1259

Problem 411

1315 posts

Find all the functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

for all $x\in\mathbb{R}$ $f(x+f(x))+f(x-f(x))=2x$

---

hal9v4ik
368 posts

Feb 11, 2014, 9:14 am PM #1260

I think that for this question we have infinitely many solutions, pco's help needed.

---

lehungviet...
1043 posts

Feb 24, 2014, 12:57 pm PM #1261

Problem 412

Find all functions $f(x)$ and $g(x)$ defined, continuous on $\mathbb{R}$ such that

$$\begin{cases} f(x-y)=f(x)g(y)-f(y)g(x) \\ g(x-y)=g(x)g(y)+f(x)f(y) \end{cases} \qquad \forall x,y\in\mathbb{R}$$

---

lehungviet...
1043 posts

Feb 24, 2014, 1:00 pm • 1 PM #1262

Problem 413

Find all functions $f(x)$ and $g(x)$ defined, continuous on $\mathbb{R}$ such that

$$\begin{cases} (f(x)+g(x))^2=1+f(2x) \\ f(0)=0 \\ f(x+y)+f(x-y)=2f(x)g(y) \end{cases} \qquad \forall x,y\in\mathbb{R}$$

MffM
139 posts

Mar 7, 2014, 6:13 am • 3 PM #1263

> lehungvietbao wrote:
>
> **Problem 413**
>
> Find all functions $f(x)$ and $g(x)$ defined, continuous on $\mathbb{R}$ such that
>
> $$\begin{cases} (f(x)+g(x))^2=1+f(2x) \\ f(0)=0 \\ f(x+y)+f(x-y)=2f(x)g(y) \end{cases} \qquad \forall x,y\in\mathbb{R}$$

Let´s make some considerations:

01)

In the third equation, let's put: $y=x\Rightarrow f(2x)+f(0)=2f(x)g(x)\Rightarrow f(2x)=2f(x)g(x)\,(*)$.

From the first equation and using $(*)$: $f^2(x)+2f(x)g(x)+g^2(x)=1+f(2x)\Rightarrow f^2(x)+g^2(x)=1\Rightarrow |g(x)|\le 1$.

02)

Let $\{U_n(\alpha)\}$ $(n\in\mathbb{N})$ a sequence such that: $U_0(\alpha)=0, U_1(\alpha)=1\wedge U_{n+2}(\alpha)=2\alpha\cdot U_{n+1}(\alpha)-U_n(\alpha)$.

If $|\alpha|<1$, $\{U_n(\alpha)\}$ will have the following characteristic equation: $r^2-2\alpha\cdot r+1=0$, whose roots are $\text{cis}(\theta)$ and $\text{cis}(-\theta)$, where $\theta=\arccos(\alpha)$.

Therefore, the general form of the sequence is: $U_n(\alpha)=A\cdot\text{cis}(n\theta)+B\cdot\text{cis}(-n\theta)$. Since $U_0(\alpha)=0\Rightarrow B=-A\Rightarrow U_n(\alpha)=2Ai\cdot\sin(n\theta)$.

Since $U_1(\alpha)=1\Rightarrow 2Ai\cdot\sin(\theta)=1\Rightarrow U_n(\alpha)=\dfrac{\sin(n\theta)}{\sin(\theta)}$.

03)

Let's assume that: $f(x)\neq 0$.

In the third equation, let's put: $x\leftrightarrow 0\wedge y\leftrightarrow x$. Thus: $f(x)+f(-x)=2f(0)g(x)=0\Rightarrow f(-x)=-f(x)$.

In $(*)$, let's put: $x\leftrightarrow -x$. Thus: $f(-2x)=2f(-x)g(-x)\Rightarrow -2f(x)=-2f(x)g(-x)\,(**)$.

Adding $(*)$ and $(**)$: $0=f(x)g(x)-f(x)g(-x)\Rightarrow g(-x)=g(x)$.

Solution:

Clearly $(f\equiv 0, g\equiv 1)\wedge(f\equiv 0, g\equiv -1)$ are trivial solutions. Let's search the others.

Let´s take $x\in\mathbb{R}$ such that $|g(x)|\neq 1\Rightarrow f(x)\neq 0$.

Let $\{U_n(\alpha)\}$ the sequence defined in 02). Let´s take $\alpha=g(x)\Rightarrow|\alpha|<1$ [from 01)] and we can consider $\theta=\arccos\alpha$.

We have already shown that: $U_n(\alpha)=\dfrac{\sin(n\theta)}{\sin(\theta)}=\dfrac{\sin(n\theta)}{|f(x)|}\,(I)$.

Proof:

To do this proof, we will use the stronger version of Finite Induction Principle (FIP).

First of all, clearly we have: $f(0\cdot x)=0=f(x)\cdot U_0(\alpha)\wedge f(1\cdot x)=f(x)\cdot U_1(\alpha)$.

We will assume that the statement is true $\forall n\le N\ \ (N\ge 2)$ and, then, prove that it's true for $N+1$. In the third equation, let's put: $x\leftrightarrow Nx\wedge y\leftrightarrow x$. Therefore:

$f[(N+1)x]+f[(N-1)x]=2\alpha f(Nx)$. Remembering the definition of $\{U_n(\alpha)\}$, we obtain:

$f[(N+1)x]=2\alpha f(x)\cdot U_N(\alpha)-f(x)\cdot U_{N-1}(\alpha)=f(x)\cdot U_{N+1}(\alpha)$. c.q.d

---

From (I) and (II): $f(nx)=f(x)\cdot\dfrac{\sin(n\theta)}{|f(x)|}\,(\forall n\in\mathbb{N})\,(M)$.

In the third equation, we can take: $y \to nx$: $f\left[(n+1)x\right] - f\left[(n-1)x\right] = 2f(x)\,g(nx)$. Using (M), we'll have:

$$\frac{f(x)}{|f(x)|} \cdot \left[\sin(n+1)\theta - \sin(n-1)\theta\right] = 2f(x)\,g(nx) = \frac{f(x)}{|f(x)|} \cdot 2\sin(\theta)\cos(n\theta).$$

Simplifying this previous equation, we have: $g(nx) = \cos(n\theta)\ (\forall n \in \mathbb{N})\ (N)$.

In the equation (M), let's put: $x \leftrightarrow -x \Rightarrow f(-nx) = f(-x) \cdot \dfrac{\sin(n\theta)}{|f(-x)|}$, where $\theta = \arccos\left[g(-x)\right] = \arccos\left[g(x)\right]$.

Thus: $f(-nx) = -f(x) \cdot \dfrac{\sin(n\theta)}{|f(x)|} = f(x) \cdot \dfrac{\sin(-n\theta)}{|f(x)|} \Rightarrow f(zx) = f(x) \cdot \dfrac{\sin(z\theta)}{|f(x)|}$ $(\forall z \in \mathbb{Z})$.

Let's consider $q \in \mathbb{N}^*$. In the equation (N), we can take: $n = q,\ x \to \dfrac{x}{q} \wedge \theta = \arccos g\left(\dfrac{x}{q}\right) \Rightarrow g(x) = cos(q\theta)$.

From the previous equation: $\arccos g(x) = \arccos\cos(q\theta) \Rightarrow arccosg(x) = q \cdot arccosg\left(\dfrac{x}{q}\right)\ (III)$.

In the equation (M), we can take: $n = q,\ x \to \dfrac{x}{q} \wedge \theta = \arccos g\left(\dfrac{x}{q}\right) \Rightarrow f(x) = f\left(\dfrac{x}{q}\right) \cdot \dfrac{\sin(q\theta)}{\left|f\left(\frac{x}{q}\right)\right|}$.

From (III): $\sin(q\theta) = \sin\left[q \cdot arccosg\left(\dfrac{x}{q}\right)\right] = \sin\left[\arccos g(x)\right] = |f(x)|$.

Therefore, we have the following property: $\dfrac{f\left(\frac{x}{q}\right)}{\left|f\left(\frac{x}{q}\right)\right|} = \dfrac{f(x)}{|f(x)|}\ (IV)$.

In equation (M), let's put: $x \to \dfrac{x}{q} \wedge \theta = \arccos g\left(\dfrac{x}{q}\right)$: $f\left(\dfrac{n}{q} \cdot x\right) = f\left(\dfrac{x}{q}\right) \cdot \dfrac{\sin(n\theta)}{\left|f\left(\frac{x}{q}\right)\right|}$.

Using (III) and (IV): $f\left(\dfrac{n}{q} \cdot x\right) = f(x) \cdot \dfrac{\sin\left(\frac{n}{q}\theta\right)}{|f(x)|}$, where $\theta = \arccos g(x)$.

We have extended the result in (M) for all rational numbers. To extend to the irrationals, we must use continuity. Therefore, we have:

$f(ux) = f(x) \cdot \dfrac{\sin(u\theta)}{|f(x)|}\ (\forall u, x \in \mathbb{R}, f(x) \neq 0 \wedge \theta = \arccos g(x))$. Let's take $x = K, u \to \dfrac{u}{K} \wedge \beta = \dfrac{\arccos g(K)}{K}$:
$f(u) = \dfrac{f(K)}{|f(K)|} \cdot \sin(\beta u) = \pm\sin(\beta u)$.

Using $(*)$, we have: $g(u) = \cos(\beta u)$.

Our answer is valid only if $|g(K)| \neq 1$. To extend for all reals, we use continuity again.

MffM
139 posts

Mar 8, 2014, 4:55 am • 1 PM #1264

> *lehungvietbao wrote:*
> **Problem 412**
> Find all functions $f(x)$ and $g(x)$ defined, continuous on $\mathbb{R}$ such that
> $$\begin{cases} f(x-y) = f(x)g(y) - f(y)g(x) \\ g(x-y) = g(x)g(y) + f(x)f(y) \end{cases} \qquad \forall x, y \in \mathbb{R}$$

If $f(x) = 0\ (\forall x \in \mathbb{R})$, let´s consider: $P_0(x,y) : g(x-y) = g(x)g(y)$. Let´s search other solutions than the trivial ones.

$P_0(0,0) : 1 = g(0) = g^2(0) \Rightarrow g(0) = 0 \vee g(0) = 1$.

$P_0(0,x) : g(-x) = g(0)g(x)$. If $g(0) = 0 \Rightarrow g(x) = 0\ (\forall x \in \mathbb{R}) \Rightarrow g(0) = 1$.

$P_0(x,x) : 1 = g(0) = g^2(x) \Rightarrow g(x) = -1 \vee g(x) = 1\ (\forall x \in \mathbb{R})$. Since g is continuous, we must have only the constant solutions and $g(x) = 1\ (\forall x \in \mathbb{R})$ is the only that satisfies $P_0(x,y)$.

Let´s assume that exists $x \in \mathbb{R}$ such that $f(x) \neq 0$.

Let´s consider the following functional equations:

$$\begin{cases} P_1(x,y) : f(x-y) = f(x)g(y) - f(y)g(x) \\ P_2(x,y) : g(x-y) = g(x)g(y) + f(x)f(y) \end{cases} \qquad \forall x, y \in \mathbb{R}$$

$P_1(0,x) : f(0-x) = f(0)g(x) - f(x)g(0) \Rightarrow f(-x) = -f(x)g(0) = -f(x)\ \ (*)$.

$P_2(0,x) : g(0-x) = g(0)g(x) + f(0)f(x) \Rightarrow g(-x) = g(x)\ \ (**)$.

$P_2(x,x) : g(x-x) = g(x)g(x) + f(x)f(x) \Rightarrow g^2(x) + f^2(x) = 1\ \ (II)$.

Using $(*)$, $(**)$ and considering $P_1(y,-x)$:

$f(y+x) = f(y)g(-x) - f(-x)g(y) \Rightarrow f(x+y) = f(y)g(x) + f(x)g(y)\ \ (***)$.

Adding $(***)$ with $P_1(x,y)$: $f(x-y) + f(x+y) = 2f(x)g(y)\ \ (III))$.

(I), (II) and (III) give us the same solutions as in problem 413. c.q.d

---

gobathegreat
401 posts

Mar 9, 2014, 4:55 pm PM #1265

I think 411 is too hard for this level so i will post a new one so marathon does not stop
Problem 414
Let f be a strictly increasing function defined on N such that $f(f(n)) = 3n$. Find value of $f(2014)$

Bigwood
383 posts

Mar 9, 2014, 5:29 pm PM #1266

If $f(1)=1$, $f(f(1))=1$, contradiction. If $f(1)>2$, contradicts to increasing. Then $f(1)=2$, $f(2)=3$. Inductively, we get $f(3^k)=2\cdot 3^k$ and $f(2\cdot 3^k)=f(3^{k+1})$. $f(729)=1458$ and $f(1458)=2187$ means $f(729+i)=1458+i$ $(i=1\dots729)$. Then we get $f(2014-729)=2014$. Hence we see $f(2014)=3\cdots(2014-729)=3855$.

Problem 415

$$f(x)+f(\frac{1}{1-x})=\frac{1}{x}$$

for f from $\mathbb{R}-\{0,1\}$ to itself.

---

gobathegreat
401 posts

Mar 9, 2014, 8:15 pm PM #1267

> Bigwood wrote:
> If $f(1)=1$, $f(f(1))=1$, contradiction. If $f(1)>2$, contradicts to increasing. Then $f(1)=2$, $f(2)=3$. Inductively, we get $f(3^k)=2\cdot 3^k$ and $f(2\cdot 3^k)=f(3^{k+1})$. $f(729)=1458$ and $f(1458)=2187$ means $f(729+i)=1458+i$ $(i=1\dots729)$. Then we get $f(2014-729)=2014$. Hence we see $f(2014)=3\cdots(2014-729)=3855$.
>
> Problem 415
>
> $$f(x)+f(\frac{1}{1-x})=\frac{1}{x}$$
>
> for f from $\mathbb{R}-\{0,1\}$ to itself.

Let $P(x)$ be assertion of

$$f(x)+f(\frac{1}{1-x})=\frac{1}{x}$$

$$P(x): f(x)+f(\frac{1}{1-x})=\frac{1}{x}(1)$$

$$P(\frac{x-1}{x}): f(\frac{x-1}{x})+f(x)=\frac{x}{x-1}(2)$$

$$P(\frac{1}{1-x}): f(\frac{1}{1-x})+f(\frac{x-1}{x})=1-x\,(3)$$

$$\frac{(1)+(2)-(3)}{2}: f(x)=\frac{x^3-x^2+2x-1}{x(x-1)}$$

Problem 416

Find all functions f defined on Z that satisfy
1) If p divides $m-n$ then $f(m)=f(n)$
2) $f(mn)=f(m)f(n)$ where p is fixed prime for all integers m and n.

---

lehungviet...
1043 posts

Mar 10, 2014, 4:26 pm PM #1268

Problem 410 and 411 are unsolve.

---

gobathegreat
401 posts

Mar 10, 2014, 4:54 pm PM #1269

410 is solved and 411 is I think too hard for our level (nobody solved it for a month)

---

meho96
23 posts

Mar 28, 2014, 3:29 am PM #1270

Find all functions $f:\mathbb{Q}_{>0}\to\mathbb{Q}_{>0}$ that satisfy:
f(x) + f(1/x)= 1
f(f(x))=f(x+1)/f(x).

Mod:Do not double post http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=582781

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gobathegreat
401 posts

Mar 29, 2014, 12:59 am PM #1271

416 is posted and not solved

---

meho96
23 posts

Apr 4, 2014, 11:02 pm • 1 PM #1272

i think its a pity to end this marathon, so here i post a nice problem :
find all functions $f:\mathbb{R}\to\mathbb{R}$ if f(x² + y + f(y)) = 2y + f²(x) .

---

gobathegreat
401 posts

Apr 4, 2014, 11:47 pm PM #1273

This one is famous: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=38&t=147421&p=833525&hilit=iran+tst+2007#p833525
Try another one

---

368 posts

If f is nonconstant then it is obvious that f is surjective.For $y=a$ such that $f(a)=0$ we get that f is injective.Now we finish proof by putting $x=1$ that $y+1=f(y)+f(1)$ that means that $f(x)=x+c$,it is easy to find that $c=0$.

---

aktyw19
1315 posts

Apr 5, 2014, 1:25 pm PM #1276

Problem

Find all functions $f:\mathbb{R}_+\to\mathbb{R}_+$ such that for all $x>0$ and $0<y<1$ then $(1-y)f(x)=f(f(yx)\frac{1-y}{y})$

---

socrates
1872 posts

Apr 6, 2014, 6:19 pm • 1 PM #1277

New problem:

Determine all functions $f:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ such that

$$f(x+f(x+y))=f(2x)+y,$$

for all $x,y\in\mathbb{R}_{>0}$

---

hal9v4ik
368 posts

Apr 6, 2014, 7:23 pm • 1 PM #1278

Firstly, note that f is injective.Suppose that there are some $f(a)=f(b)$.Take some $x<a,b$.Putting $P(x,a-x)$ and $P(b-x,x)$ will give that $a=b$.

We will prove that $f(x) \geq x$.Suppose contrary that $f(a) = b$ and $a > b$.$P(a-b, b)$ gives $a - b = 0$ contradiction.
From equation we by last result we find that $f(x+y) - x - y \leq f(2x) - 2x$.For any $2 * a \geq b \geq a$ we can find x, y such that $x + y = a$ and $2x = b$ which gives $f(a) - a \leq f(b) - b$.Also there are some x' and y' such that $2x' = a$ and $x' + y' = b$ so we get $f(a) - a \geq f(b) - b$.
So we have got that $f(a) - a = f(b) - b$ for all $2a \geq b \geq a$.We can easily say that $f(x) - x$ is constant for all x.So $f(x) = x$.

amatysten
73 posts

May 24, 2014, 7:50 pm • 2 PM #1279

Problem 422
Find all functions $f(x) : \mathbb{Q} \to \mathbb{Z}$ such that

$$f\left(\frac{f(x)+a}{b}\right) = f\left(\frac{x+a}{b}\right) \quad \forall x \in \mathbb{Q}, \forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}^+$$

This post has been edited 1 time. Last edited by amatysten, May 24, 2014, 10:18 pm

shmm
597 posts

May 24, 2014, 8:05 pm PM #1280

a , b are parametres?

amatysten
73 posts

May 24, 2014, 10:12 pm PM #1281

They are variables, like x. You can choose them.

nguyenqua...
7 posts

May 25, 2014, 10:11 am PM #1282

find all function $f : \mathbb{N}^* \to \mathbb{N}^*$ that satisfy:
a. $f(1) = 2$;
b. $f(f(n)) = f(n) + n, \forall n \in \mathbb{N}^*$;
c. $f(n) < f(n+1), \forall n \in \mathbb{N}^*$.

This post has been edited 1 time. Last edited by nguyenquangnhat9, May 25, 2014, 5:13 pm

shmm
597 posts

May 25, 2014, 4:51 pm PM #1283

problem 422 is not solved.

amatysten
73 posts

May 26, 2014, 12:24 pm PM #1284

@nguyenquangnhat9
Are you sure about the statement? It seems there is a set of solutions that's too difficult to describe.
For example we can have $f(4) = 6$ or $f(4) = 7$ every one of which is giving a different solution.
And then if $f(4) = 6$, we can have $f(7) = 11$ or $f(7) = 12$ not yielding any contradiction again. And so on...
This set can be inductively described, but that's just too ugly.

There's a possibility that I'm an error, as always, but everything seems to be ok.

hal9v4ik
368 posts

May 26, 2014, 11:01 pm PM #1285

> *amatysten wrote:*
> **Problem 422**
> Find all functions $f(x) : \mathbb{Q} \to \mathbb{Z}$ such that
>
> $$f\left(\frac{f(x)+a}{b}\right) = f\left(\frac{x+a}{b}\right) \quad \forall x \in \mathbb{Q}, \forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}^+$$

This one was on our TST

nguyenqua...
7 posts

May 26, 2014, 11:51 pm PM #1286

> *amatysten wrote:*
> @nguyenquangnhat9
> Are you sure about the statement? It seems there is a set of solutions that's too difficult to describe.
> For example we can have $f(4) = 6$ or $f(4) = 7$ every one of which is giving a different solution.
> And then if $f(4) = 6$, we can have $f(7) = 11$ or $f(7) = 12$ not yielding any contradiction again. And so on...
> This set can be inductively described, but that's just too ugly.
>
> There's a possibility that I'm an error, as always, but everything seems to be ok.

I understand your comments. The exactly problem is "Determine if there exists a function". But I want to find all function and I try to do it. It's very hard.

amatysten
73 posts

May 27, 2014, 7:30 am PM #1287

> *nguyenquangnhat9 wrote:*
> I want to find all function and I try to do it.

1. You can easily construct all such functions. First you've got $f(1) = 2, f(2) = 3, f(3) = 5$, etc.
2. Then, you choose the least yet undefined number, $f(4)$ in our case and determine
it's possible values, 6 or 7 in our case. You can choose any.
3. Let, for ex, $f(4) = 6$. It'll give a new chain of definitions: $f(6) = 10, f(10) = 16$, etc.
4. GOTO part 2.
5. One can prove, that during this process there will always remain undefined numbers,

nguyenqua...
7 posts

May 27, 2014, 9:19 am PM #1288

> *amatysten wrote:*
> > *nguyenquangnhat9 wrote:*
> > I want to find all function and I try to do it.
>
> 1. You can easily construct all such functions. First you've got $f(1) = 2, f(2) = 3, f(3) = 5$, etc.
> 2. Then, you choose the least yet undefined number, $f(4)$ in our case and determine
> it's possible values, 6 or 7 in our case. You can choose any.
> 3. Let, for ex, $f(4) = 6$. It'll give a new chain of definitions: $f(6) = 10, f(10) = 16$, etc.
> 4. GOTO part 2.
> 5. One can prove, that during this process there will always remain undefined numbers,
> there will always be possible values for it and whatever value you choose,
> the forthcoming chain will not overlap with previous ones.
> 6. But clearly these functions cannot be described easier, then by this very algorithm.
> They are just too wild and numerous.

Therefor, we can prove that there are infinite function

Therefor, we can prove that there are infinite function

YESMAths
817 posts

Jun 2, 2014, 6:12 pm PM #1289

> hal9v4ik wrote:
>
> > amatysten wrote:
> > **Problem 422**
> > Find all functions $f(x) : \mathbb{Q} \to \mathbb{Z}$ such that
> > $$f\left(\frac{f(x)+a}{b}\right) = f\left(\frac{x+a}{b}\right) \quad \forall x \in \mathbb{Q}, \forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}^+$$
>
> This one was on our TST

@hal9v4ik: Were you able to solve it? Could you please provide us with a solution? Err.. because still it is unsolved and we are working on it

manuel153
257 posts

Jun 2, 2014, 11:22 pm PM #1290

For problem 422:
Does anybody have a non-constant solution?

pco
14052 posts

Jun 3, 2014, 12:40 am • 1 PM #1291

> manuel153 wrote:
> For problem 422:
> Does anybody have a non-constant solution?

For example $f(x) = \lceil x \rceil$

fclvbfm934
732 posts

Jun 3, 2014, 6:25 am • 3 PM #1292

Partial Solution to 422

I found the solutions are $f(x) = \lfloor x \rfloor$, $f(x) = \lceil x \rceil$ or $f(x) = c$ for some $c \in \mathbb{Z}$. The latter clearly works. To test the first solution, let $x + a = bq + r$ where $0 \le r < b$. Therefore,

$$f\left(\frac{bq + \lfloor r \rfloor}{b}\right) = q$$

which is clearly true. The ceiling function can be checked similarly.

Assume that $f(0) \neq 0$. Then, by letting $x = 0$, we have

$$f(\frac{f(0)+a}{b}) = f(\frac{a}{b})$$

Now substitute $(a, b) \to (af(0), bf(0))$. Now we have

$$f(\frac{a}{b} + \frac{1}{b}) = f(\frac{a}{b}) = f(\frac{a}{b} + 2/b) = \cdots = f(\frac{a}{b} + k/b)$$

I now claim that $f(x) = f(y)$ for all $x, y \in \mathbb{Q}$. Let $x = m_1/n_1$ and $y = m_2/n_2$. Take $b = n_1n_2$ and $a = m_1n_2$ and $k = m_2n_1 - m_1n_2$. We see that $f(x) = f(y)$ for all $x, y \in \mathbb{Q}$. Hence, $f(x) = c$ for all $c \neq 0$ is a solution.

So now assume that $f(0) = 0$. Suppose there exist $x, y \in \mathbb{Z}$ such that $x \neq y$ and $f(x) = f(y)$. Notice then that

$$f(\frac{x+a}{b}) = f(\frac{y+a}{b})$$

Taking $b = 1$ gives us

$$f(x+a) = f(y+a)$$

and taking $a = -x$ we have $f(0) = f(y - x) = 0$. Now let x be $y - x$ in the original functional equation:

$$f(a/b) = f(\frac{a}{b} + \frac{y-x}{b})$$

And because we $y - x \neq 0$, we see that f is constant by similar reasoning to that above. Hence $f(x) = 0$.

Otherwise, assume that if $f(x) = f(y)$, then $x = y$ when $x, y \in \mathbb{Z}$. Therefore, letting $b = 1$, we see that

$$f(x) + a = x + a$$

So $f(x) = x$ for all $x \in \mathbb{Z}$. Now assume that x isn't necessarily an integer. We have

In our equation, plug in $a = -f(x)$ and we get that

$$f(\frac{x - f(x)}{b}) = 0$$

So, we see that either $f(x) = c$ is a constant function or we have:
$f(n) = n$ for all integers n
$f(x + a) = f(x) + a$ for all integers a and rational x
$f(\frac{x - f(x)}{b}) = 0$ for all $x \in \mathbb{Q}$ and $b \in \mathbb{Z}^+$

shmm
597 posts

Jun 4, 2014, 1:28 am • 1 PM #1293

Problem 423
If $f : N - - - - > N$ and m, n are natural numbers , then find all functions
$2f(mn) \geq f(m^2 + n^2) - f(m)^2 - f(n)^2 \geq 2f(m)f(n)$

Particle
179 posts

Jun 4, 2014, 10:00 am • 4 PM #1294

Continuing after fclvbfm934.
Suppose $P(x,a,b)$ implies the main equation.

Solution
$f\left(\frac{1}{2}\right)\in\{0,1\}$
Proof

Let $f(\frac{1}{2})=c$. If $c>0$, then $2c-1>0$. Substitute $x=\frac{1}{2}, a=c-1$ and $b=2c-1$ in the main equation. We get

$$f\left(\frac{2c-1}{2c-1}\right)=f\left(\frac{2c-1}{2(2c-1)}\right)\implies c=1$$

And if $c\le 0$, then $1-2c>0$. So substitute $x=\frac{1}{2}, a=-c, b=1-2c$. We'll get $c=0$.

If $f\left(\frac{1}{2}\right)=0$

1. For integer $n>1, f(\frac{1}{n})=0$
Proof

$P(\frac{1}{2},0,b)\implies f\left(\frac{1}{2b}\right)=0.$
$P(\frac{1}{2n},1,2n+1)\implies f\left(\frac{1}{2n+1}\right)=f\left(\frac{1}{2n}\right)=0.$

Now we inductively prove for $0\le k<2^n, f\left(\frac{k}{2^n}\right)=0$
Proof

The base case is true for $n=1$. Now if $k<2^{n-1}$, then $f\left(\frac{k}{2^{n-1}}\right)=0$. Now $P(\frac{k}{2^{n-1}},0,2)$ implies $f\left(\frac{k}{2^n}\right)=0$.

For $k=2^{n-1}$ this is obvious. Now assume $k>2^{n-1}$. Let $k=2^{n-1}+m$. Note that $f(\frac{k}{2^{n-1}})=1+f(\frac{m}{2^{n-1}})=1$. Now $P(\frac{k}{2^{n-1}},0,2)\implies f\left(\frac{k}{2^n}\right)=0$

Let $p,q\in\mathbb{N}$ with $p<q$. Then $f(\frac{p}{q})=0$.
Proof

Suppose $f(\frac{p}{q})=c$.
$P(\frac{p}{q},q+1,p)\implies f\left(\frac{c+p}{1+q}\right)=f\left(\frac{p}{q}\right)$. Applying this repeatedly, we get

$c=f\left(\frac{p}{q}\right)=f\left(\frac{c+p}{1+q}\right)=f\left(\frac{2c+p}{2+q}\right)=\cdots=f\left(\frac{nc+p}{q+n}\right)$ for all $n\in\mathbb{N}\quad(1)$

So $f\left(\frac{p-qc}{n+q}\right)=f\left(\frac{nc+p}{n+q}-c\right)=0$, since $c\in\mathbb{Z}$. Hence $p-qc\neq 0$ because otherwise it will mean $c=\frac{p}{q}$ =non-integer. If $c>1$, then $p-qc\le p-2q<-q$. So taking $n=qc-p-q$ gives $f(-1)=0$ which is not possible. And if $c<0$, then $p-qc\ge p+q>q$. So taking $n=p-qc-q$ gives $f(1)=0$ which is not possible either. So $c\in\{0,1\}$
If $c=1$, then (1) implies $1=f(\frac{p}{q})=f(\frac{1+p}{1+q})=\cdots=f(\frac{r}{2^k})$ for some r and k which is a contradiction. So $c=f(\frac{p}{q})=0$.

Since $f(a+\frac{p}{q})=a+f(\frac{p}{q})$ for $a\in Z$, we get $f(x)=\lfloor x\rfloor\forall x\in\mathbb{Q}$.

If $f\left(\frac{1}{2}\right)=1$

hal9v4ik
368 posts

Jun 4, 2014, 9:00 pm PM #1295

Problem 423 is SRMC 2014 problem.
It was posted here :http://www.artofproblemsolving.com/Forum/viewtopic.phpf=57&t=583283&hilit=srmc+2014+SRMC+2014
Solution in that thread much more easy than that was on official paper.

shmm
597 posts

Jun 4, 2014, 9:04 pm PM #1296

Yes, official solution is very ugly

YESMAths
817 posts

Jun 5, 2014, 7:00 pm • 1 PM #1297

Problem 424

mad
53 posts

Jun 9, 2014, 6:10 pm PM #1298

Find all functions f from integer to integer such that $f(x+f(y))=f(x)+f(y)$.

pco
14052 posts

Jun 9, 2014, 6:58 pm PM #1299

mad wrote:

Find all functions f from integer to integer such that $f(x+f(y))=f(x)+f(y)$.

Let $A=\{y\in\mathbb Z\text{ such that }f(x+y)=f(x)+y\forall x\in\mathbb Z\}$
Note that, trivially, $f(\mathbb Z)\subseteq A$ and A is an additive subgroup of $\mathbb Z$
Then $x\sim y\iff x-y\in A$ is an equivalence relation.
Choosing then $r(x)$ as any fonction which associates to an integer x a representant (unique per class) of its equivalence class, we get :
$f(x)=f(r(x)+x-r(x))=f(r(x))+x-r(x)$ since $x-r(x)\in A$

From there, it's easy to show that the general form of silution is :

Let A any additive subgroup of $\mathbb Z$
Let $\sim$ the equivalence relation $x-y\in A$
Let $r(x)$ from $\mathbb Z\to\mathbb Z$ any fonction which associates to an integer x a representant (unique per class) of its equivalence class
Let $g(x)$ any function from $\mathbb Z\to A$

Then $\boxed{f(x)=g(r(x))+x-r(x)}$

And since the only additive subgroups of $\mathbb Z$ are $\{0\}$ and $k\mathbb Z$ where $k\in\mathbb N$ we get :

If $A=\{0\}$: $\boxed{f(x)=0}\forall x$

If $A=k\mathbb Z$, choosing $r(x)=x-k\left\lfloor\frac xk\right\rfloor$, we get : $\boxed{f(x)=kh(x-k\left\lfloor\frac xk\right\rfloor)+k\left\lfloor\frac xk\right\rfloor}\forall x$, which indeed is a solution, whatever is $h(x)$ from $\mathbb Z\to\mathbb Z$

For example :
$k=1\implies\boxed{f(x)=x+c}\forall x$ which indeed is a solution, whatever is $c\in\mathbb Z$

$k=2\implies f(2n)=2n+2a$ and $f(2n+1)=2n+2b$ which indeed is a solution, whatever are $a,b\in\mathbb Z$

$k=10\implies f(x)=10\left(\left\lfloor\frac{x}{10}\right\rfloor+\left\lfloor200\sin\frac{\pi}{5}x\right\rfloor\right)$

And a lot of other solutions.

---

pco
14052 posts

Jun 9, 2014, 7:37 pm PM #1300

> *YESMAths wrote:*
> Problem 424
> Let $f:\mathbb R\to\mathbb R$ satisfy $f(xf(y))=yf(x)$ for all real x and y.
> $a)$Show that f is an odd function.
> $b)$Determine f, given that f has exactly one discontinuity.

Let $P(x,y)$ be the assertion $f(xf(y))=yf(x)$

$f(x)=0\forall x$ is an odd solution (and not a solution of b). So let us consider from now that $f(x)$ is not the allzero function.

$P(0,1)\implies f(0)=0$
Let $u\ne0$ such that $f(u)\ne0$: $P(u,x)\implies f(uf(x))=xf(u)$ and so $f(x)$ is bijective.

$P(x,1)$+ bijection $\implies f(1)=1$
$P(1,x)\implies f(f(x))=x$
$P(-1,f(-1))\implies f(-1)^2=1$ and so, since bijective, $f(-1)=-1$

$P(x,f(-1))\implies f(-x)=-f(x)$ and so $f(x)$ is odd

$P(x,f(y))\implies f(xy)=f(x)f(y)$ and we immediately get that the only involutive multiplicative function with exacly one discontinuity point is :
$f(0)=0$ and $f(x)=\frac1x\forall x\ne0$ which indeed is a solution

---

Mikasa
56 posts

Jun 9, 2014, 8:30 pm PM #1301

Problem 426:

Find all continuous functions $f:\mathbb R\to\mathbb R$ such that $f(xy)=xf(y)+yf(x)$.

P.S. I don't have a complete solution of this problem. 🙁

---

pco
14052 posts

Jun 9, 2014, 8:44 pm • 1 PM #1302

> *Mikasa wrote:*
> **Problem 426:**
>
> Find all continuous functions $f:\mathbb R\to\mathbb R$ such that $f(xy)=xf(y)+yf(x)$.
>
> P.S. I don't have a complete solution of this problem. 🙁

$g(x)=c\ln x$

$P(-1,-1)\implies f(-1)=0$ and so $P(x,-1)\implies f(-x)=-f(x)$

Hence the solution $\boxed{f(0)=0\text{ and }f(x)=cx\ln|x|\ \forall x\neq0}$, which indeed is a solution.

Mikasa
56 posts

Jun 9, 2014, 11:05 pm PM #1303

Thanks to pco 🙂 Let's proceed to **Problem 427:**
Find all functions $u:\mathbb R\to\mathbb R$ for which there exists a strictly monotone function $f::\mathbb R\to\mathbb R$ such that,
$f(x+y)=f(x)u(y)+f(y)\forall x,y\in\mathbb R$.

---

pco
14052 posts

Jun 10, 2014, 12:23 am • 1 PM #1304

> *Mikasa wrote:*
> Thanks to pco 🙂 Let's proceed to **Problem 427:**
> Find all functions $u:\mathbb R\to\mathbb R$ for which there exists a strictly monotone function $f::\mathbb R\to\mathbb R$ such that,
> $f(x+y)=f(x)u(y)+f(y)\forall x,y\in\mathbb R$.

Let $P(x,y)$ be the assertion $f(x+y)=f(x)u(y)+f(y)$

Since $f(x)$ is strictly monotone, exists t such that $f(t)\neq 0$. Then :
$P(x,t) \implies f(x+t)=f(x)u(t)+f(t)$
$P(t,x) \implies f(x+t)=f(t)u(x)+f(x)$
Subtracting, we get $u(x)=af(x)+1$ where $a=\dfrac{u(t)-1}{f(t)}$

If $a=0$, we get $\boxed{\text{S1 : } u(x)=1\ \forall x}$ which indeed is a solution (choose for example $f(x)=x$)

If $a\neq 0$, $u(x)$ is strictly monotone and original equation may be written $u(x+y)=u(x)u(y)$ and so $u(x)=e^{cx}$ (since strictly monotone) for some $c\neq 0$

And so $\boxed{\text{S2 : } u(x)=e^{cx}\ \forall x}$ which indeed is a solution (choose for example $f(x)=e^{cx}-1$), whatever is $c\neq 0$

hal9v4ik
368 posts

Jun 10, 2014, 2:37 pm PM #1305

> *nguyenquangnhat9 wrote:*
> find all function $f:\mathbb{N}^*\to\mathbb{N}^*$ that satisfy:
> a. $f(1)=2$;
> b. $f\left(f\left(n\right)\right)=f\left(n\right)+n, \forall n\in\mathbb{N}^*$;
> c. $f\left(n\right)<f(n+1), \forall n\in\mathbb{N}^*$.

This one is same as IMO 1993 P5 but we must find all functions.Do you have solution? I've found at least 2 completely different functions.

YESMAths
817 posts

Jun 10, 2014, 3:37 pm PM #1306

> *hal9v4ik wrote:*
>
> > *nguyenquangnhat9 wrote:*
> > find all function $f:\mathbb{N}^*\to\mathbb{N}^*$ that satisfy:
> > a. $f(1)=2$;
> > b. $f\left(f\left(n\right)\right)=f\left(n\right)+n, \forall n\in\mathbb{N}^*$;
> > c. $f\left(n\right)<f(n+1), \forall n\in\mathbb{N}^*$.
>
> This one is same as IMO 1993 P5 but we must find all functions.Do you have solution? I've found at least 2 completely different functions.

Hello, hal9v4ik! See here http://www.artofproblemsolving.com/Forum/viewtopic.php?p=372306&sid=5958af00fbddb587b9631d3d3c19df42#p372306
And here:
Official Solution

Notice that for $\alpha=\dfrac{1+\sqrt5}{2}$, $\alpha^2\cdot n=\alpha\cdot n+n$ for all $n\in\mathbb{N}$. We shall show that $f(n)=[\alpha n+\frac12]$ (the closest integer to αn) satisfies the requirements. Observe that f is strictly increasing and $f(1)=2$. By definition of f, $|f(n-\alpha n|\leq\frac12$ and $f(f(n))-f(n)-n$ is an integer. On the other hand,
$|f(f(n))-f(n)-n|=|f(f(n))-f(n)-\alpha^2n+\alpha n|=|f(f(n))-\alpha f(n)+\alpha f(n)-\alpha^2n-f(n)+\alpha n|=|(\alpha-1)(f(n)-\alpha n)+(f(f(n))-\alpha f(n))|\leq(\alpha-1)|f(n)-\alpha n|+|(f(f(n))-\alpha f(n))|\leq\frac12(\alpha-1)+\frac12=\frac12\alpha<1$,
which implies that $f(f(n))-f(n)-n=0$.

Let's now see what is nguyenquangnhat9's solution.

Utkarsh99
10 posts

Jun 17, 2014, 5:04 pm PM #1307
He wants all functions and the IMO problem asked to prove the existence.
Infact @pco showed (not clearly) there are infinitely many functions satisfying the equation

mad
53 posts

Jun 18, 2014, 6:53 pm PM #1308
Problem 427
Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(xf(y)+f(x))=2f(x)+xy$$

Mikasa
56 posts

Jun 20, 2014, 10:32 am • 1 PM #1309

> *mad wrote:*
> **Problem 427**
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
>
> $$f(xf(y)+f(x))=2f(x)+xy$$

Case 1: $f(0)=1$
$P(0,0)\Rightarrow f(1)=2$
$P(-1,-1)\Rightarrow f(-1)=0$
$P(-1,1)\Rightarrow f(-2)=-1$

$P(x,-2)\Rightarrow f(f(x)-x)=2(f(x)-x)$. Now by surjectivity $\exists u_x\in\mathbb{R}$ such that $f(u_x)=f(x)-x$.
So we get that $f(f(u_x))=2f(u_x)$. But,
$P(u_x,-1)\Rightarrow f(f(u_x))=2f(u_x)-u_x$. So $u_x=0$ and,
$f(x)=x+f(0)=x+1\forall x\in\mathbb{R}$.

Case 2: $c=0$ i.e. $f(0)=0$
$P(x,0)\Rightarrow f(f(x))=2f(x)$ and thus by surjectivity $f(x)=2x\forall x\in\mathbb{R}$ which is clearly not a solution.

So the only solution to the equation is $f(x)=(x+1)\forall x\in\mathbb{R}$

Please feel free to post the next problem, as I don't have any to post right now.

Jun 20, 2014, 12:21 pm PM #1310

shmm
597 posts

Ok. New problem.
Find all functions $f: R-->R$ such that $f(sinx+cosx)=f(sinx)+f(cosx)$

---

shmm
597 posts

Jun 27, 2014, 3:38 pm PM #1311

Well, one more problem: find all functions $f: R-->R$ such that $(x-y)(f(x^2)+f(y^2)+\frac{1}{2}f(2xy))=f(x^3)-f(y^3)$

---

gobathegreat
401 posts

Jun 27, 2014, 6:48 pm PM #1312

> shmm wrote:
> Well, one more problem: find all functions $f: R-->R$ such that $(x-y)(f(x^2)+f(y^2)+\frac{1}{2}f(2xy))=f(x^3)-f(y^3)$

Here. Do not post new problem if you have already thread for it.

---

pco
14052 posts

Jun 28, 2014, 1:19 pm PM #1313

> shmm wrote:
> Ok. New problem.
> Find all functions $f: R-->R$ such that $f(sinx+cosx)=f(sinx)+f(cosx)$

Is it a serious real olympiad problem ? or just a fake personal poor invention ?

There are trivially infinitely many solutions and I would be very surprised if a general form exists for all of them ...

---

shmm
597 posts

Jul 12, 2014, 7:57 pm PM #1314

Yes , it was Olympiad problem

YESMAths
817 posts

Jul 12, 2014, 11:23 pm PM #1315

@ shmm

---

shmm
597 posts

Jul 13, 2014, 12:37 am PM #1316

District Olympiad Turkmenistan

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pco
14052 posts

Jul 13, 2014, 2:32 pm PM #1317

> shmm wrote:
> Yes , it was Olympiad problem

Dont hesitate to post here the official general form for the infinitely many solutions when you'll get it :)

---

socrates
1872 posts

Jul 13, 2014, 8:24 pm PM #1318

New Problem
Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^3+f(y))=f^3(x)+y$$

---

socrates
1872 posts

Jul 13, 2014, 8:37 pm PM #1319

(Another) **New Problem**
Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(f(x)-y^2)=f(x^2)+y^2f(y)-2f(xy).$$

---

xxp2000
520 posts

Jul 14, 2014, 2:37 am PM #1320

> socrates wrote:
> (Another) **New Problem**
> Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that
> $$f(f(x)-y^2)=f(x^2)+y^2f(y)-2f(xy).$$

Obviously $f(x)=0$ is the only contant solution. Now consider non-constant solution.

1) $f(0)=0$ and $f(x)\neq 0, x\neq 0$.
$P(1,1)$ shows there exists $f(b)=0$.
$P(b,0): f(b^2)=3a$
Let $a=f(0)$. $P(0,y): f(a-y^2)=y^2f(y)-a$ gives $y^2f(-y)=y^2f(y)$ thus f is even.
$P(b,b): f(b^2)=0$. So $a=0$ and $f(y^2)=y^2f(y)$.
Suppose $f(b)=0, b\neq 0$. Then $f(b^2)=0$.
Compare $P(0,y)$ and $P(b,y)$, $f(by)=0$. Absurd!

2) $f(x)=x^2$.
Since $f(y^2)=y^2f(y)$, $P(y,y): f(f(y)-y^2)=0$ thus $f(y)=y^2$ by 1).

> **New Problem**
> Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that
> $$f(x^3+f(y))=f^3(x)+y$$

Obviously f is bijective. Let $f(0)=a$ and $f(b)=0$.
1) $f(x+y)=f(x)+f(y)$
$P(x,b): f(x^3)=f(x)^3+b$
$P(0,y): f(f(y))=a^3+y$. In particular, $a=a^3+b$.
$P(x,f(y)): f(x^3+y+a^3)=f(x^3)-a+a^3+f(y)$, or $f(x+y+a^3)=f(x)+f(y)+a^3-a$.
Let $x=0$ above, $f(y+a^3)=f(y)+a^3$.
So we have $f(x+y)=f(x)+f(y)-a$.
Now we will show $a=0$. We notice $f(-1)+f(1)=2a$.
Since $f(x^3)=f(x)^3+a-a^3, y-y^3=a-a^3$ has three distinct roots $f(0),f(1),f(-1)$.
By Vieta's theorem, $f(1)+f(-1)+a=0=3a$. So $a=0$ and Cauchy equation holds for f.

2) $f(x)=x$ and $f(x)=-x$ are the solutions

Let $f(1)=c$. For rational r, $f((rx+1)^3)=f(rx+1)^3$ becomes
$r^3f(x^3)+3r^2f(x^2)+3rf(x)+c=r^3f(x)^3+3cr^2f(x)^2+3c^2rf(x)+c^3$.
Treat both sides as polynomial of r and compare coefficients,
$c^2=1, f(x^2)=cf(x)^2$.
With Cauchy equation, it is easy to show the only solutions are $f(x)=x$ and $f(x)=-x$.

shmm
597 posts

Jul 14, 2014, 11:46 am PM #1322

xxxp2000 post new problem , please

ws5188
79 posts

Jul 17, 2014, 11:05 am PM #1323

Find all tame solutions of the equation f(x+y)=f(x)f(y)/(f(x)+f(y)).

Find all polynomial functions f(x) such that f(x)f(x+1)=f(x^2+x+1) for all real numbers x.

shmm
597 posts

Jul 17, 2014, 11:24 am PM #1324

You mean: $f(x+y)=\dfrac{f(x)f(y)}{f(x)+f(y)}$

pco
14052 posts

Jul 17, 2014, 2:58 pm PM #1325

> *ws5188 wrote:*
> Find all tame solutions of the equation f(x+y)=f(x)f(y)/(f(x)+f(y)).

Without any precision, I consider that domain of function is $\mathbb{R}$, codomain of function is $\mathbb{R}$ and domain of functional equation is $\mathbb{R}^2$.

If so, setting $y=0$ in functional equation, we get $f(x)=0\forall x$ which obviously is not a solution.

So no solution for this functional equation.

rod16
248 posts

Jul 17, 2014, 4:42 pm PM #1326

Find all function $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+f(\frac{1}{x}))+f(y+f(\frac{1}{y}))=f(x+f(\frac{1}{y}))+f(y+f(\frac{1}{x}))$$

with $x,y\in\mathbb{R}$ and $xy\neq 0$ with one of the following conditions:
*) f is monotonic.
**) f don't have any condition more.

ws5188
79 posts

Jul 17, 2014, 6:26 pm PM #1327

Did anybody solve the polynomial function problem?
Shmm and pco: thanks for fixing the format and providing a solution to the first problem.

YESMAths
817 posts

Jul 17, 2014, 7:00 pm • 1 PM #1328

Click to reveal hidden text

> Hey guys! Let's not forget to number the problems.
>
> ws5188
>
> rod16

bnmh
14 posts

Jul 19, 2014, 2:29 pm PM #1329

Problem 436 :find all functions $f:\mathbb{N}^*\to\mathbb{N}^*$ such that $\forall n\in\mathbb{N}^*$ we have:
1) $(f(n))^2<nf(n+1)$
2) $f(2n+1)\leq 4nf(n)$
3) $f(2n)\leq 2(2n-1)f(n)$

gobathegreat
401 posts

Jul 19, 2014, 4:38 pm • 1 PM #1330

> *bnmh wrote:*
> **Problem 436** :find all functions $f:\mathbb{N}^*\to\mathbb{N}^*$ such that $\forall n\in\mathbb{N}^*$ we have:
> 1) $(f(n))^2<nf(n+1)$
> 2) $f(2n+1)\leq 4nf(n)$
> 3) $f(2n)\leq 2(2n-1)f(n)$

Do not post new problems until the last problem is solved.

xxp2000
520 posts

Jul 20, 2014, 4:32 am • 1 PM #1331

> *YESMAths wrote:*
> Hey guys! Let's not forget to number the problems.
>
> > *ws5188 wrote:*
> > Find all tame solutions of the equation f(x+y)=f(x)f(y)/(f(x)+f(y)).

> **Problem 434.** Find all polynomial functions $f(x)$ such that $f(x)f(x+1)=f(x^2+x+1)$ for all real x.

Here is the solution.

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=3196694#p3196694

nessre
1 post

Sep 6, 2014, 10:13 pm PM #1332

xxp2000 pose a new problem and don't forget to number it

YESMAths
817 posts

Sep 7, 2014, 12:01 am • 1 PM #1333

> *nessre wrote:*
> xxp2000 pose a new problem and don't forget to number it

I suppose he can't, because Problem 435 and Problem 436 are still unsolved.
So, if you can provide a correct solution for 435 as well as 436, then we can proceed with a new problem.

bappa1971
168 posts

Oct 3, 2014, 3:44 pm • 1 PM #1334

> bnmh wrote:
> **Problem 436** :find all functions $f:\mathbb{N}^*\to\mathbb{N}^*$ such that $\forall n\in\mathbb{N}^*$ we have:
> 1) $(f(n))^2<nf(n+1)$
> 2) $f(2n+1)\le 4nf(n)$
> 3) $f(2n)\le 2(2n-1)f(n)$

Assuming $\mathbb{N}*$ as the set of positive integers.
from (1) and (3) it follows that $f(1)^2<f(2)\le 2f(1) \implies f(1)<2 \implies f(1)=1$
Now for any $n\ge 1$, let $f(n)\ge n$, then from (1), $n^2\le f(n)^2<nf(n+1) \implies n<f(n+1) \implies f(n+1)\ge n+1$
Now take $g(n)=f(n)-n$, then $g(n)\ge 0$ for all n.
Now from (1),
$(n+g(n))^2=n^2+2ng(n)+g(n)^2<n(n+1+g(n+1))$
$\implies n+2g(n)\le n+2g(n)+\dfrac{g(n)^2}{n}<n+1+g(n+1)$
$\implies 2g(n)-1<g(n+1)$
$\implies 2g(n)\le g(n+1)$
$\implies 2^ag(n)\le 2^{a-1}g(n+1)\le 2^{a-2}g(n+2)\le\cdots\le g(n+a)$ for all n
Now assume that for some n, $g(n)>0$. Than for any $N>n$ we have $g(N)\ge 2^{N-n}g(n)>0$. Also note that $g(2N)\ge 2^Ng(N)$
Then from (3) it follows $2^Ng(N)+2N\le g(2N)+2N=f(2N)\le(4N-2)f(N)=(4N-2)(N+g(N))$
Which is certainly not true for N large enough and $g(N)>0$.
Hence $g(n)=0$ for all n and so $f(n)=n$

mihirb
1818 posts

Oct 11, 2014, 9:18 pm PM #1335

Problem 437

> Let $f:\mathbb{N}\to\mathbb{N}$ be a strictly increasing function that satisfies $f(f(n))=3n$ for every natural number n. Determine $f(2006)$.

utkarshgupta
2019 posts

Oct 12, 2014, 9:09 am PM #1336

> mihirb wrote:
> Problem 437
>> Let $f:\mathbb{N}\to\mathbb{N}$ be a strictly increasing function that satisfies $f(f(n))=3n$ for every natural number n. Determine $f(2006)$.

If I remember right this is quite famous.
We prove by induction $f(3^n)=2\cdot 3^n$ and $f(2\cdot 3^n)=3^{n+1}$

utkarshgupta
2019 posts

Oct 12, 2014, 9:12 am PM #1337

> rod16 wrote:
> Find all function $f:\mathbb{R}\to\mathbb{R}$ such that
> $$f(x+f(\frac{1}{x}))+f(y+f(\frac{1}{y}))=f(x+f(\frac{1}{y}))+f(y+f(\frac{1}{x}))$$
> with $x,y\in\mathbb{R}$ and $xy\neq 0$ with one of the following conditions:
> *) f is monotonic.
> **) f don't have any condition more.

What do you mean by monotonic *non decreasing* or *strictly increasing* :)

Wiokito
75 posts

Nov 19, 2014, 10:13 pm PM #1338

> utkarshgupta wrote:
>> mihirb wrote:
>> Problem 437
>
> If I remember right this is quite famous.
> We prove by induction $f(3^n)=2\cdot 3^n$ and $f(2\cdot 3^n)=3^{n+1}$

Hello , i'm waiting for all the answer of the exercice of $f(2006)$

TripteshBis...
162 posts

Dec 11, 2014, 1:09 pm • 1 PM #1339

> mihirb wrote:
> Problem 437

It is very easy to do by induction $f(3^n)=2\cdot 3^n$ and $f(2\cdot 3^n)=3^{n+1}$
There are 3^n-1 integers between 3^n and $2\cdot 3^n$ and 3^n-1 integers between $2\cdot 3^n$ and 3^{n+1}

TripteshBis...
162 posts

PROBLEM 438
Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ that satisfy the relation
$f(m+n)+f(m)f(n)=f(mn+1)$

YESMAths
817 posts

Dec 11, 2014, 10:02 pm • 1 PM #1341

> TripteshBiswas wrote:
> PROBLEM 438
> Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ that satisfy the relation
> $f(m+n)+f(m)f(n)=f(mn+1)$

Hi Triptesh! :) Please note that Problem 435 is still unsolved. So, it is better not to post a new problem. :)

fabian17458
3 posts

Dec 12, 2014, 3:18 am PM #1342

> TripteshBiswas wrote:
> PROBLEM 438

Find all functions $f:\mathbb{Z}\rightarrow\mathbb{Z}$ that satisfy the relation
$f(m+n)+f(m)f(n)=f(mn+1)$

Solution:
Click to reveal hidden text

TripteshBis...
162 posts

Dec 12, 2014, 1:04 pm • 2

PM #1343

> *fabian17458 wrote:*
>
> > *TripteshBiswas wrote:*
> >
> > PROBLEM 438
> > Find all functions $f:\mathbb{Z}\rightarrow\mathbb{Z}$ that satisfy the relation
> > $f(m+n)+f(m)f(n)=f(mn+1)$
>
> Solution:
> Click to reveal hidden text
>
> $m=0, n=1$ yields $f(0)=0$. Putting $m=0$ gets $f(n)=f(1)=c$. By controlling the solutions we find $c=0$, so the only solution is $f(n)=0$.

You have just done a trivial case.
Other solutions are :-
$f(x)=x^2-1$
$f(x)=x-1$
$f(x)=(x+1)mod2$
$f(x)=(xmod3)-1$
$f(x)=((x+1)mod2)((xmod4)+1)$
$f(x)=(xmod3)^2-1$

This post has been edited 1 time. Last edited by TripteshBiswas, Dec 13, 2014, 11:13 am

fabian17458
3 posts

Dec 13, 2014, 5:04 am

PM #1344

Oh yes, what a silly mistake! Putting $m=0, n=1$, we get $f(0)=0$ or $f(1)=0$.

USJL
64 posts

Jan 4, 2015, 5:45 pm • 3

PM #1345

$(0,1):f(0)f(1)=0$

If $f(0)=0$,

$(0,n):f(n)=f(1)$. Hence f is a constant, which is equal to 0.

So suppose that $f(0)\neq 0$, then $f(1)=0$.

$(0,0):(f(0)+1)f(0)=0$. Since $f(0)\neq 0$, $f(0)=-1$.

$(-1,2):f(-1)f(2)=f(-1)$.

Case 1 : $f(-1)=0$

Click to reveal hidden text

Case 2 : $f(-1)\neq 0$

Click to reveal hidden text

In conclusion, there are 7 solutions in total, which I don't really want to rewrite again...

This is very complicated......

chiekh
179 posts

Feb 24, 2015, 1:53 am

PM #1346

Find all functions f:\mathbb{Z}\rightarrow\mathbb{Z} that satisfy the relation
$f(m^2+n)+f(m)f(n^3)=\frac{f(mn^5)}{f(n)^2}+f(n)+m^2$

USJL
64 posts

Feb 26, 2015, 5:57 pm

PM #1347

Hello chiekh:

I think it isn't appropriate to post a new problem while Problem 435 is still unsolved.

By the way, it seems like you just forget that 0 shouldn't appear at the denominator, which results to the fact that $f(x)=x$ isn't a solution.

In fact, there is no solution. So maybe you should be sure that you don't make mistakes when proposing a question or checking whether there is any unsolved problem.

Find all functions $f:\mathbb{R}_+\rightarrow\mathbb{R}_+$ such that for all $x>0$ and $0<y<1$ then $(1-y)f(x)=f(f(yx)\frac{}{y})$

it not yet solve , any solution? if not i will solve it two year

lebathanh
393 posts

Sep 16, 2016, 7:32 pm

PM #1350

new problem intersecting: is any fucntion can be express a(x)-b(x) with a(x) is a even fucntion and b(x) is a odd fucntion?

lebathanh
393 posts

Sep 16, 2016, 7:49 pm

PM #1351

> *Amir Hossein wrote:*
>
> I'm posting next problem.
>
> **Problem 43 :**
> Let f be a real function defined on the positive half-axis for which $f(xy)=xf(y)+yf(x)$ and $f(x+1)\leq f(x)$ hold for every positive x and y. Show that if $f(1/2)=1/2$, then
>
> $$f(x)+f(1-x)\geq -x\log_2 x-(1-x)\log_2(1-x)$$

for every $x \in (0,1)$.

the problem not solved

spacewalker
109 posts

Sep 16, 2016, 7:51 pm

PM #1352

For your 2nd problem:

The functions we want are:

$$a(x) = \frac{f(x)+f(-x)}{2} \text{ and } b(x) = \frac{f(-x)-f(x)}{2}$$

It is easy to verify that $a(x)$ is even, $b(x)$ is odd and

$$a(x)-b(x) = \frac{f(x)+f(-x)}{2} - \frac{f(-x)-f(x)}{2} = \frac{2f(x)}{2} = f(x)$$

so we are done.

Quick Reply