Problem set 1

FE

1) Let S be the set of positive real numbers. Find all functions $f\colon S^3 \to S$ such that, for all positive real numbers x, y, z and k, the following three conditions are satisfied:
  (a) $xf(x,y,z) = zf(z,y,x)$,
  (b) $f(x,ky,k^2z) = kf(x,y,z)$,
  (c) $f(1,k,k+1) = k+1$.

2) Consider function $f : R \to R$ which satisfies the conditions for any mutually distinct real numbers a, b, c, d satisfying $\frac{a-b}{b-c} + \frac{a-d}{d-c} = 0$, $f(a), f(b), f(c), f(d)$ are mutully different and $\frac{f(a)-f(b)}{f(b)-f(c)} + \frac{f(a)-f(d)}{f(d)-f(c)} = 0$. Prove that function f is linear

3) Let $f : R^2 \to R$ be a function such that $f(x,y) + f(y,z) + f(z,x) = 0$ for real numbers x, y, and z. Prove that there exists a function $g : R \to R$ such that $f(x,y) = g(x) - g(y)$ for all real numbers x and y.

4) Find all differentiable functions $f : R \to R$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

5) Find all functions from positive integers to itself such that $f(a+b) = f(a) + f(b) + f(c) + f(d)$ for all $c^2 + d^2 = 2ab$

Graph Theory

1) Prove that a finite simple planar graph has an orientation so that every vertex has out-degree at most 3.

2) Let G be a tournoment such that it's edges are colored either red or blue. Prove that there exists a vertex of G like v with the property that, for every other vertex u there is a mono-color directed path from v to u.

3) A communications network consisting of some terminals is called a 3-connector if among any three terminals, some two of them can directly communicate with

each other. A communications network contains a windmill with n blades if there exist n pairs of terminals $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}$ such that each x_i can directly communicate with the corresponding y_i and there is a hub terminal that can directly communicate with each of the $2n$ terminals $x_1, y_1, \ldots, x_n, y_n$. Determine the minimum value of $f(n)$, in terms of n, such that a 3 -connector with $f(n)$ terminals always contains a windmill with n blades.

NT Poly

1) Find all polynomials f with integer coefficient such that, for every prime p and natural numbers u and v with the condition:

$$p \mid uv - 1$$

we always have $p \mid f(u)f(v) - 1$

2) p is a polynomial with integer coefficients and for every natural n we have $p(n) > n$. x_k is a sequence that: $x_1 = 1, x_{i+1} = p(x_i)$ for every N one of x_i is divisible by N. Prove that $p(x) = x + 1$

3) For integral m, let $p(m)$ be the greatest prime divisor of m. By convention, we set $p(\pm 1) = 1$ and $p(0) = \infty$. Find all polynomials f with integer coefficients such that the sequence

$$\{p\left(f\left(n^2\right)\right) - 2n\}_{n \geq 0}$$

is bounded above. (In particular, this requires $f\left(n^2\right) \neq 0$ for $n \geq 0$.)