High School Olympiads

Concurrent and cyclic

Reply

buratinogigle

#1 Sep 23, 2011, 12:13 am • 2

Let ABC be triangle. (O) is a circle which passes through B, C. AB, AC cut (O) again at F, E. BE cuts CF at D.

a) Prove that tangent at E, F of (O) and AD are concurrent at T

b) DA cuts EF at G, BG cuts TC at M, CG cuts TB at N. Prove that M, N lie on (O).

This is generalization of following problem

Let ABC be a triangle. Altitude BE, CF intersect at H. HA cuts EF at G. T is midpoint of HA. Prove that G is orthocenter of triangle TBC.

Luis González

#2 Sep 23, 2011, 1:44 am • 3

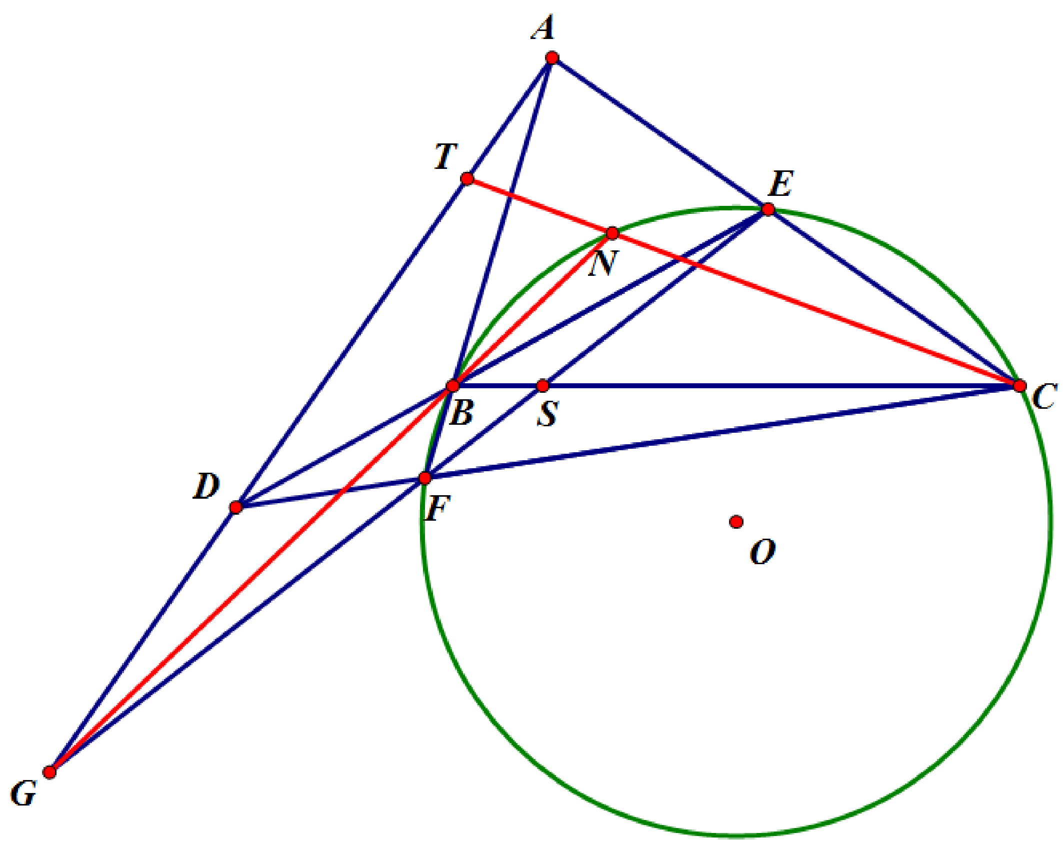
Let $K \equiv BC \cap EF.$ Pencil $A(B,C,D,K)$ is clearly harmonic $\Longrightarrow$ AD is the polar of K with respect to (O) $\Longrightarrow$ Tangents of (O) at E,F meet on $AD.$ Let TC cut (O) again at M' and $P \equiv TC \cap EF.$ Since EF is the polar of T with respect to $(O),$ then $(M',C,P,T)=-1,$ but the pencil $G(B,C,F,D)$ is obviously harmonic $\Longrightarrow (M,C,P,T)=-1 \Longrightarrow$ $M \equiv M',$ i.e. M lies on $(O).$ Analogously, N lies on $(O).$

Love_Math1994

#3 Sep 23, 2011, 12:50 pm • 1

Hope i correct .
a,Easy to see that S is pole of AD (from Brocard theorem) so EF pass thru pole of AD,so AD pass thru pole of EF is T.
b,Denote N is intersection of BG and (O).CN intersect AG at T.We will prove that N in EE.
Use Pascal theorem for FBNEEC we have the intersection of (CN,EE),(BE,CF),(BN,EF) is collinear. Or T in EE.
Done .

Attachments:



Quick Reply

High School Olympiads

Three collinear points

Source: (own)

jayme
#1 Sep 21, 2011, 11:19 pm

Dear Mathlinkers,
1. ABC a triangle
2. (O) the circumcircle of ABC,
3. P, Q two points
4. P1P2P3 the P-cevian triangle of ABC
5. Q1Q2Q3 the Q-cevian triangle of ABC
6. (Oa) the circumcircle of AP1Q1 and circularly
7. A* the second intersection point of (O) and (Oa), and circularly
8. A' the point of intersection of AA* and BC, and circularly
Prouve that A', B' and C' are collinear.
Sincerely
Jean-Louis

Luis González
#2 Sep 22, 2011, 11:36 am

Since AA' is the radical axis of $(O),(O_a)$, then $\overline{A'B}\cdot\overline{A'C}=\overline{A'P_1}\cdot\overline{AQ_1}$, i.e. B,C and P_1,Q_1 are pairs of inverse points under an inversion with pole A'. Hence, by inversion properties, we have

$$\frac{\overline{CQ_1}}{\overline{BP_1}}=\frac{\overline{A'P_1}\cdot\overline{A'Q_1}}{\overline{A'B}\cdot\overline{A'P_1}}=\frac{\overline{A'Q_1}}{\overline{A'B}}\ ,\ \frac{\overline{BQ_1}}{\overline{CP_1}}=\frac{\overline{A'P_1}\cdot\overline{A'Q_1}}{\overline{A'C}\cdot\overline{A'P_1}}=\frac{\overline{A'Q_1}}{\overline{A'C}}$$

$$\Longrightarrow\frac{\overline{A'B}}{\overline{A'C}}=\frac{\overline{BP_1}}{\overline{P_1C}}\cdot\frac{\overline{BQ_1}}{\overline{Q_1C}}\Longrightarrow A(B,C,A',P\cdot Q)=-1$$

Analogously, we have $B(C,A,B',P\cdot Q)=-1$ and $C(A,B,C',P\cdot Q)=-1$. Therefore, A',B',C' lie on the trilinear polar of $P\cdot Q$ WRT $\triangle ABC$.

jayme
#3 Sep 22, 2011, 5:00 pm

Dear Luis and Mathlinkers,
thank dear Luis for your interest in this problem. My synthetic proof avoid inversion and will be seen next on my site.
Sincerely
Jean-Louis

jayme
#4 Nov 24, 2011, 4:56 pm

Dear Mathlinkers,
an article concerning the “Ayme's theorem” and this question have been put on my website.
http://perso.orange.fr/jl.ayme vol. 20 p. 10
You can use Google translator
Sincerely
Jean-Louis

High School Olympiads

P is on radical axis (oWn)

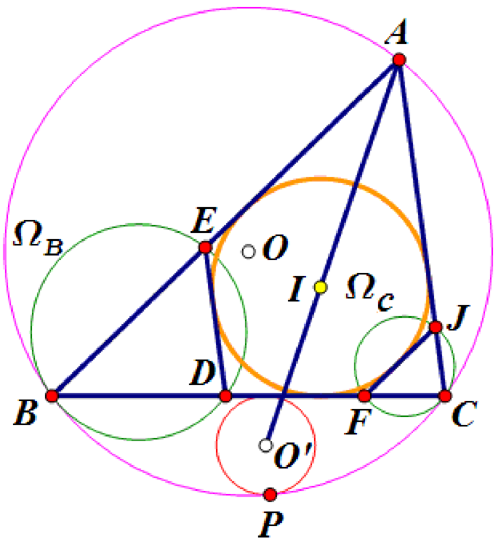
lym
#1 Sep 21, 2011, 3:00 am

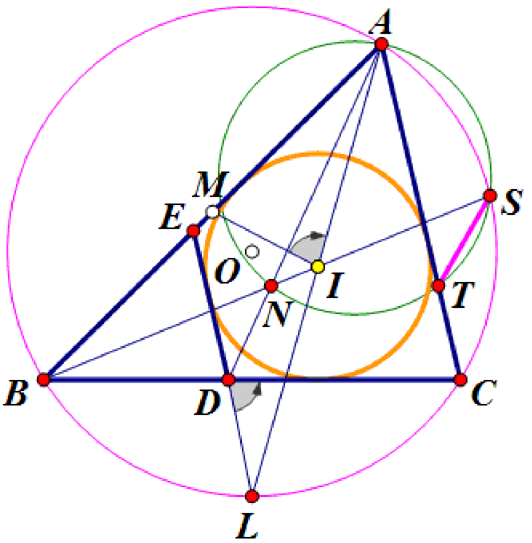
Given a $\triangle ABC$ with its incircle (I) and circumcircle (O)□$DE \parallel AC$□$FJ \parallel AB$□DE□FJ are both the tangents of (I)
AI intersects (O) at L□M is the midpoint of AB□$BI \cap AD = N$□BI intersects (O) at S□AC intersects $\odot ANS$ at T
O' is on AI□circle (O') is inscribed in (O) at P and (O') is also tangent to BC . Prove that

□1□□$\angle CDL = \angle AIM$□TS pass throught the A-excenter of $\triangle ABC$

□2□□P is on the radical axis of $\odot BDE$ and $\odot CFJ$ Click to reveal hidden text

Attachments:





Luis González
#2 Sep 21, 2011, 8:48 am

2) Let $V \equiv IA \cap BC$ be the foot of the internal bisector of $\angle BAC$. V is the insimilicenter of $(I) \sim (O')$ and P is the exsimilicenter of $(O) \sim (O)' \Longrightarrow PV$ passes through the insimilicenter X_{55} of $(I) \sim (O)$. Clearly, $U \equiv DE \cap FJ$ is the reflection of A about I, since $AEUJ$ is a rhombus with incirle (I), hence $\frac{VD}{VC} = \frac{VU}{VA} = \frac{VF}{VB} \Longrightarrow VB \cdot VD = VC \cdot VF$ $\Longrightarrow V$ has equal power WRT Ω_B and Ω_C, i.e. V lies on the radical axis of Ω_B, Ω_C. On the other hand, we know that X_{55} is the pole of the anti-orthic axis of $\triangle ABC$ WRT (O), thus, if V' is the foot of the external bisector of $\angle BAC$, then $\tau_a \equiv \overline{PVX_{55}}$ is the polar of V' WRT $(O) \Longrightarrow$ Tangents of (O) at B, C meet at a point X lying on τ_a, i.e. X has equal power WRT $\Omega_B, \Omega_C \Longrightarrow \tau_a$ is the radical axis of Ω_B, Ω_C and the proof is completed.

skytin
#3 Sep 21, 2011, 6:20 pm

Solution :
(1) Reflect C wrt L and get point C'
Angle C'IC = 90 = DIC , so C' is on DI
L is center of (CC'BI)
Angle CC'D = IBM , DCC' = BAI , so IBMA ~ LC'DC angle LDC = IMA
See my solution here :
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=431955
ST intersect (ABC) at points S , K
K is tangent point of some B - mixlinear excircle , well known that I a , K , S are on same line , done

K is tangent point of some B - mixlinear excircle , well known that I_a , K , S are on same line . done

skytin

#4 Sep 21, 2011, 8:08 pm

Solution :
(2) DE intersect FJ at point A'
Tangents thru B , C to (ABC) intersects at point K
W is intersection point of IA and (ABC)
Let w is circle with center at W and tangent to KB , KC , BC
AI intersect BC at point L
After Monge's theorem we get that KL goes thru H = homotety center of (O) , (I) and PL goes thru H too
Not hard to prove that LD*BL = LF*LC , so P is on KLH = radical center of (DBE) , (CFJ) . done

Quick Reply

High School Olympiads

How to prove this one?

Reply

m1234a

#1 Sep 20, 2011, 2:11 pm

ABC is a tirangle
Let symetric points of point P to side AB and AC be F, E ,AND O, H are the circumcentre and othercentre , prove

$S_{FHE} = \frac{R^2 - OP^2}{2} \sin 2A$.\

R is the radius of circumcircle of ABC

Luis González

#2 Sep 21, 2011, 4:21 am

Y, Z are the projections of P onto AC, AB (midpoints of PE,PF) and θ denotes the angle between the directions AH and $YZ \parallel EF$. Then, we have

$$[HEF] = [HEAF] - [FEA] = AH \cdot YZ \cdot \sin\theta - \tfrac{1}{2}PA^2 \cdot \sin 2A \Longrightarrow$$

$$\frac{2[HEF]}{\sin 2A} = \frac{2R \cdot YZ \cdot \sin\theta}{\sin A} - PA^2 = PA(2R \cdot \sin\theta - PA)$$

Let AP cut (O) at D. Then $\measuredangle DCA = \measuredangle DCB + \measuredangle C = \measuredangle PYZ + \measuredangle C = \theta \pmod{\pi} \Longrightarrow AD = 2R \cdot \sin\theta$. Thus

$$\frac{2[HEF]}{\sin 2A} = PA(AD - PA) = PA \cdot PD = |R^2 - PO^2|.$$

TelvCohl

#3 Feb 6, 2015, 3:39 pm

My solution:

Let $D = AP \cap \odot(ABC), T = AE \cap \odot(HCA)$.

Since $AE = AF = AP$,
so $\angle AEF = 90^\circ - \angle BAC = \angle ACH = \angle ATH \Longrightarrow EF \parallel TH$,

hence $[HEF] = [TEF] = \frac{1}{2} \cdot EF \cdot ET \cdot \cos A$

$= \frac{1}{2} \cdot (2 \cdot AE \cdot \sin A) \cdot ET \cdot \cos A = \frac{1}{2} \cdot (AP \cdot PD) \cdot \sin 2A = \frac{|\mathrm{R}^2 - OP^2|}{2} \cdot \sin 2A$.

Q.E.D

Quick Reply

High School Olympiads

Orthocentre

Locked

FoolMath

#1 Sep 20, 2011, 11:22 pm

Let triangle ABC, H is orthocentre of ABC and M is midpoint of BC. Let (I) is incircle, (I) touch with AB, AC at E, F. Prove that M, I, H are collinear if and only if E, F, H are collinear

Luis González

#2 Sep 20, 2011, 11:37 pm

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=5397
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=240137
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=306898

High School Olympiads

Triangle of the three centres is obtuse

Source: Romanian TST 1997

WakeUp
#1 Sep 17, 2011, 9:02 pm

We are given in the plane a line ℓ and three circles with centres A,B,C such that they are all tangent to ℓ and pairwise externally tangent to each other. Prove that the triangle ABC has an obtuse angle and find all possible values of this this angle.

Mircea Becheanu

Luis González
#2 Sep 18, 2011, 1:29 pm • 1

Let a,b,c be the radii of the circles $(A),(B),(C),$ respectively. Assuming that (C) is the smallest circle, then we have the relation $\sqrt{ab}=\sqrt{ac}+\sqrt{ab}$ $(\star)$ (well-known). Label $\angle ACB=\gamma.$ Then by cosine law in $\triangle ABC,$ we have

$$\cos\gamma=\frac{c(a+b+c)-ab}{(a+c)(b+c)}\Longrightarrow \gamma>90^\circ \Longleftrightarrow \ c(a+b+c)-ab<0$$

Thus, using the relation $(\star),$ the latter inequality is equivalent to

$$c\left[c-2\sqrt{ab}+\left(\sqrt{a}+\sqrt{b}\right)^2\right]<c\left(\sqrt{a}+\sqrt{b}\right)^2\Longleftrightarrow c<2\sqrt{ab}$$

Which is clearly true, since the length of the common tangent between $(A),(B)$ is greater that the radius of $(C).$ Now, using $\cos\gamma$ from the latter expression, we get

$$\sin\frac{\gamma}{2}=\sqrt{\frac{1-\cos\gamma}{2}}=\sqrt{\frac{ab}{(a+c)(b+c)}}=\sqrt{\frac{1}{\left(1+\frac{c}{a}\right)\left(1+\frac{c}{b}\right)}}$$

So $\sin\dfrac{\gamma}{2}$ is maximum $\Longleftrightarrow \Delta=\left(1+\dfrac{c}{a}\right)\left(1+\dfrac{c}{b}\right)$ is minimum.

Setting $x=\sqrt{\frac{c}{a}},y=\sqrt{\frac{c}{b}},$ we shall minimize $\Delta=(1+x^2)(1+y^2)$ with the constraint $x+y=1,$ coming from the equation $(\star).$ Therefore, letting $\omega=xy,$ we obtain $\Delta=(1+x^2)(1+y^2)=\omega^2-2\omega+2$ where $\frac{1}{4}\ge\omega>0,$ due to $\omega=xy=x(1-x)=x-x^2$ with $1>x>0.$ The quadratic $\omega^2-2\omega+2$ is strictly decreasing in the interval $\frac{1}{4}\ge\omega>0,$ so the minimum value of Δ occurs when $\omega=\frac{1}{4}\Longrightarrow x=y=\frac{1}{2}\Longrightarrow \sin\frac{\gamma}{2}$ reaches its maximum value when $a=b,$ i.e. $(A)\cong(B).$

$$\sin\frac{\gamma}{2}=\sqrt{\frac{a^2}{(a+c)^2}}=\frac{a}{a+\frac{a}{4}}=\frac{4}{5}\Longrightarrow \ 90^\circ<\gamma\le 2\sin^{-1}\left(\tfrac{4}{5}\right)\approx 106,26^\circ.$$

High School Olympiads

radical axis

Reply

hungnsl
#1 Sep 18, 2011, 8:28 am

Let ABC be a triangle and $(Ib), (Ic)$ be its excircles with respect to B and C. Let (Ob) and (Oc) be the symmetric circles of (Ib) and (Ic) with respect to the midpoints of AC and AB. Show that the radical axis of (Ob) and (Oc) bisects the perimeter of triangle ABC.

P/S: i have an extremely heavy computational proof for this using power of a point, but i hope someone will help me with a (much) simpler solution.

Luis González
#2 Sep 18, 2011, 10:25 am

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=35317
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=163607
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=142767

Quick Reply

High School Olympiads

2011 Lusophon Mathematical Olympiad - Problem 5

Reply

Source: 2011 Lusophon Mathematical Olympiad - Problem 5

nunoarala

#1 Sep 17, 2011, 8:35 pm • 1

Consider two circles, tangent at T, both inscribed in a rectangle of height 2 and width 4. A point E moves counterclockwise around the circle on the left, and a point D moves clockwise around the circle on the right. E and D start moving at the same time; E starts at T, and D starts at A, where A is the point where the circle on the right intersects the top side of the rectangle. Both points move with the same speed. Find the locus of the midpoints of the segments joining E and D.

This post has been edited 1 time. Last edited by nunoarala, Sep 21, 2011, 7:53 pm

Luis González

#2 Sep 18, 2011, 9:35 am

Let $(O_1,1) \cong (O_2,1)$ denote the given circles touching the top side of the rectangle at $A,B.$ Their common internal tangent (symmetry axis) cuts $\overline{AB}$ at $P.$ Since D,E move with equal angular velocities, then $\angle AO_1D=-\angle TO_2E \Longrightarrow$ $\mathcal{L}: E \mapsto D$ is the composition of the axial symmetries about $PT, PO_1, AO_1,$ in that order. Thus, $\mathcal{L}$ is equivalent to the composition of an axial symmetry about certain axis ℓ and translation parallel to $\ell \Longrightarrow$ midpoint M of $\overline{DE}$ lies on $\ell.$ Now, if Q denotes the antipode of B WRT $(O_2),$ we have $\mathcal{L}: T \mapsto A, Q \mapsto T \Longrightarrow \ell \equiv AT.$

Note that M is always an ordinary point (not lying at infinity), so the locus of M will be actually a segment contained in the line $AT,$ precisely, the diameter of the circle $(T,1)$ intersected by the line $AT.$

yetti

#3 Sep 19, 2011, 7:44 pm

Circle $(T,1)$ touches top rectangle side at P. PTO_1A is a square.
Translate $D \in (O_1,1)$ on the right by $\overrightarrow{O_1T}$ to $D' \in (T,1)$ and $E \in (O_2,1)$ on the left by $\overrightarrow{O_2T}$ to $E' \in (T,1)$. This takes A,T to P,O_1.
$DD'EE'$ is parallelogram $\Longrightarrow$ segments $DE, D'E'$ have common midpoint M.
$\measuredangle D'TP = \measuredangle DO_1A = -\measuredangle EO_2T = -\measuredangle E'TO_1 \Longrightarrow D'E' \parallel PO_1 \Longrightarrow M$ is on diameter of $(T,1)$ perpendicular to PO_1 (or parallel to AT).

Number1

#4 Sep 19, 2011, 9:25 pm • 1

We can get D form E with compositon reflection about O_1O_2, rotation about O_2 with rotation angle 90 and translation $\overrightarrow{O_1O_2}$.

Then use http://en.wikipedia.org/wiki/Hjelmslev's_theorem

nunoarala

#5 Sep 20, 2011, 2:06 am

Both solutions seem to be correct.

Besides, I'd like to know how to post those problems from Lusophon Mathematical Olympiad in the Contest section, since this contest isn't on the list. Could someone tell me how to do it?

Quick Reply

High School Olympiads

Concurrence

Reply

huyhoang

#1 Sep 15, 2011, 7:57 pm

Let ABC be a scalene triangle and I is the incenter of triangle ABC. Let the incircle of triangle IBC touches IB, IC at A_1, A_2. Denote B_1, B_2, C_1, C_2 analogously. Denote $A_3 = B_1B_2 \cap C_1C_2$. B_3 and C_3 are defined analogously. Prove that AA_3, BB_3, BB_3 are concurrence.

Luis González

#2 Sep 17, 2011, 9:52 pm • 1

Incircles $(I_1), (I_2), (I_3)$ of $\triangle IBC, \triangle ICA, \triangle IAB$ touch BC, CA, AB at X, Y, Z. Lines A_1A_2, B_1B_2, C_1C_2 cut BC, CA, AB at D, E, F. Since the cross ratio (B, C, X, D) is harmonic, we have

$$\frac{DB}{DC} = \frac{BX}{XC} = \frac{XI_1 \cdot \cot \widehat{XBI_1}}{XI_1 \cdot \cot \widehat{XCI_1}} = \frac{\cot \frac{B}{4}}{\cot \frac{C}{4}}$$

Thus, multiplying the symmetric expressions together yields

$$\frac{DB}{DC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = \frac{\cot \frac{B}{4}}{\cot \frac{C}{4}} \cdot \frac{\cot \frac{C}{4}}{\cot \frac{A}{4}} \cdot \frac{\cot \frac{A}{4}}{\cot \frac{B}{4}} = 1$$

By the converse of Menelaus' theorem, D, E, F are collinear, i.e. $\triangle ABC$ and the triangle $\triangle A_3B_3C_3$ bounded by the lines A_1A_2, B_1B_2, C_1C_2 are perspective through $\overline{DEF}$. Hence, by Desargues theorem, AA_3, BB_3, CC_3 concur.

Quick Reply

High School Olympiads

diagonals

Reply

elegant

#1 Sep 16, 2011, 6:32 pm

In a cyclic quadrilateral $ABCD$, lines AD and BC meet at E, and diagonals AC and BD meet at F. If M and N are the midpoints of AB and CD, prove that

$$\frac{MN}{EF}=\frac{1}{2}\left|\frac{AB}{CD}-\frac{CD}{AB}\right|$$

Moderator edit: Topic LaTeXed

Luis González

#2 Sep 16, 2011, 10:13 pm

elegant, please use $\LaTeX$, why is it so difficult for you?. You've been asked to use $\LaTeX$ several times before, but you seem not interested in improving your posts. Please check the AoPS LaTeX guide for further explanations. As for the problem, the expression holds for any 4 concyclic points, thus the quadrilateral ABCD can be either simple or complex. See involving absolute value and [Synthetic solution desired [2 MN/EF = |AC/BD - BD/AC|]

elegant

#3 Sep 17, 2011, 4:05 pm

I am using an easier software than LATEX and LATEX seems to me a bother.

Quick Reply

High School Olympiads

Cyclic quadrilateral

Reply

Source: Me

borislav_mirchev
#1 Jul 21, 2011, 12:50 am • 1

It is given a quadrilateral with intersection point of the diagonals - P that is inscribed in circle with center O. M and N are the feets of the perpendiculars from P to AD and BC respectively. K and L are the intersection points of PM with BC and PN with AD respectively. Q is the intersection point of KL and OP. Prove that Q is the middle of KL.

Luis González
#2 Jul 21, 2011, 10:26 am • 2

From $\angle CPK = \angle APM = 90^\circ - \angle PAD = 90^\circ - \angle PBC$ we deduce that the circumcenter F of $\triangle BPC$ lies on PK. Analogously, circumcenter E of $\triangle APD$ lies on PL. Since OE, OF are the perpendicular bisectors of $\overline{AD}, \overline{BC}$, then $OE \parallel PF$ and $OF \parallel PE \Longrightarrow PFOE$ is a parallelogram $\Longrightarrow OP$ bisects $\overline{EF}$. But from $\triangle APD \sim \triangle BPC$, we get $\frac{PE}{PL} = \frac{PF}{PK} \Longrightarrow EF \parallel LK$. Thus, OP also bisects $\overline{LK}$.

buratinogigle
#3 Jul 21, 2011, 3:22 pm • 3

An easy generalization

Let $ABCD$ be cyclic quadrilateral inscribed (O). AC cuts BD at P. M, L lie on AD, N, K lie on BC such that $MNKL$ is cyclic and MK, NL pass through P. Prove that circumcenter O^* of $(MNKL)$ lies on OP.

borislav_mirchev
#4 Jul 21, 2011, 5:24 pm

Thank you for the excellent solution and the nice generalisation.

Luis González
#5 Sep 15, 2011, 10:20 pm • 2

> buratinogigle wrote:
> Let $ABCD$ be cyclic quadrilateral inscribed (O). AC cuts BD at P. M, L lie on AD, N, K lie on BC such that $MNKL$ is cyclic and MK, NL pass through P. Prove that circumcenter O^* of $(MNKL)$ lies on OP.

Since $\angle PML = \angle PNK$ and $\angle PLM = \angle PKN$, then $\triangle PAD$ and $\triangle PBC$ are similar with corresponding cevians PM, PN and PL, PK, respectively. Therefore, the midpoints U, V of $\overline{ML}, \overline{NK}$ (orthogonal projections of O^* onto AD,BC) are then homologous $\Longrightarrow \frac{AU}{UD} = \frac{BV}{VC}$. Now, according to the problem nice lemma on cyclic quadrilateral, O^* lies on OP, as desired.

Quick Reply

High School Olympiads

Search locus orthocentre

Reply

mathlife2011

#1 Sep 14, 2011, 7:58 pm

For any triangle ABC.Consider a circle (O) changes go through A non-exposed to the line AB, AC and O lies on the line BC.M, N is the second intersection point (O) with AB, AC.Find the locus orthocentre of triangle AMN

Luis González

#2 Sep 15, 2011, 2:04 am

Clearly, the reflection P of A about BC lies on $(O),$ i.e. Circles (O) pass through the fixed points $A,P.$ Steiner line of P with respect to $\triangle AMN$ is fixed, i.e. the orthocenter of $\triangle AMN$ runs through the line connecting the reflections of P about $AB,AC.$

P.S. For a proof of the latter assertion, see the topic Bisect Line.

Quick Reply

High School Olympiads

A perpendicular of a cevian passing the incenter

Source: Indonesian Mathematics Olympiad 2011 Unproposed Problems

chaotic_iak

#1 Sep 13, 2011, 2:48 pm • 1

The incircle of an acute triangle ABC is tangent to BC at D, and is centered at I. A line passing I and perpendicular to AD intersects the incircle at P and Q, where P is on the same side as B on respect to AD. Prove that BP intersects CQ on AD.

Luis González

#2 Sep 14, 2011, 12:21 am

BP cuts AD at K and the incircle (I) touches CA, AB at E, F, respectively. BC, EF are the polars of D, A with respect to $(I) \Longrightarrow AD$ is the polar of $M \equiv BC \cap EF$ with respect to $(I) \Longrightarrow IM \perp AD$, i.e. $P, Q \in IM \Longrightarrow K(P, Q, D, M) = -1$. But, since $(B, C, D, M) = -1$, then $K(P, C, D, M) = -1 \Longrightarrow KQ$ and KC coincide i.e. BP, CQ, AD concur at K, as desired.

Lyub4o

#3 Sep 14, 2011, 1:39 am

What do you mean with $(B, C, D, M) = -1$,

r1234

#4 Sep 15, 2011, 11:11 pm

> Lyub4o wrote:
> What do you mean with $(B, C, D, M) = -1$,

see harmonic division:http://http://www.artofproblemsolving.com/Forum/blog.php?u=51470

math_explorer

#5 Sep 16, 2011, 11:07 am

Suppose BP and CQ intersect AD at K, L respectively.

Let ω be the incircle of $\triangle ABC$.

Construct a tangent to ω through P, cutting AB and BC at R and S respectively.
Then ω is also the B-excircle of $\triangle BRS$, touching RS at P.
Since PQ is perpendicular to AD, we know RS and AD are parallel; thus, there exists a homothecy centered at B which maps RS to AD, and this homothecy maps P to K. Thus the B-excircle of $\triangle ABD$ touches AD at K. We can now compute $KD - AK = AB - BD$ by properties of the excircle.

Analogously $LD - AL = AC - CD$, and by properties of the incircle $AC - CD = AB - BD$, so $KD - AK = LD - AL$ or $2KD - AD = 2LD - AD$, so K and L are the same point.

High School Olympiads

Extended of Newton line

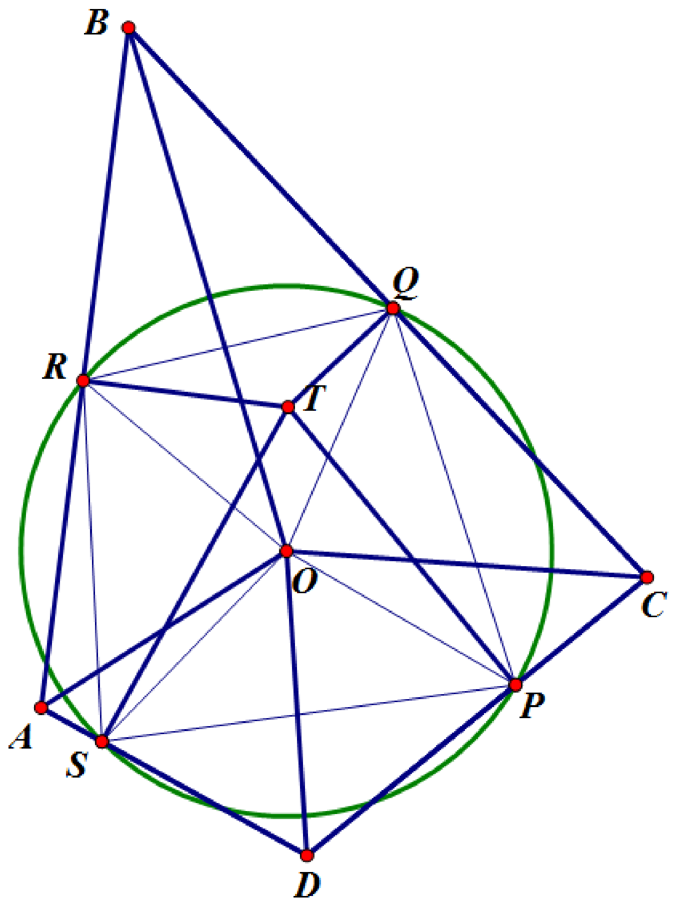
Reply

Love_Math1994

#1 Sep 13, 2011, 10:13 pm

Given (O) is cicrumcircle of $PQRS$ and T is inside (O).Contruct perpendicular line at P, Q, R, S of TP, TQ, TR, TS,four line intersect at A, B, C, D as you seen in fig.Prove that $S_{OAB} + S_{OCD} = S_{OAD} + S_{OBC}$
Easy to see that from this we have the midpoints of AC, BD and O is colinear.(Same as proof of Newton line)
If you can prove midpoints of AC, BD and O is colinear but not use $S_{OAB} + S_{OCD} = S_{OAD} + S_{OBC}$,plz show me.Thanks.

Attachments:



Luis González

#2 Sep 13, 2011, 11:36 pm • 1

This follows from the fact that AB, BC, CD, DA are tangent to a central conic $\mathcal{K}$ with pedal circle (O) and foci T and T' (the reflection of T about O). See the previous lemma in the problem Concurrent in hexagon. If T lies inside (O), as the problem states, then $\mathcal{K}$ is an ellipse. Thus, there exist parallel projections taking the quadrilateral $ABCD$ circumscribed about $\mathcal{K}$ into a quadrilateral $A'B'C'D'$ circumscribed about a circle with center O'. Midpoints go to midpoints under the parallel projection $\Longrightarrow$ Newton line n of $ABCD$ goes to Newton line n' of $A'B'C'D'$. Since $O' \in n'$ (well-known), then $O \in n$ and the conclusion follows.

P.S. Note that the result is still true for points T outside (O) if the areas are oriented appropriately, but we cannot use parallel projection straightforwardly.

Lyub4o

#3 Sep 13, 2011, 11:45 pm

I would be happy to see the proof of Newton's line if someone can give a link.

Quick Reply

High School Olympiads

three circles

Locked

Source: by Slavko Moconja

feraligater

#1 Sep 12, 2011, 9:44 pm

Given a triangle ABC, with circumcircle l. Let Q be the point inside $\triangle ABC$ such that $QB=QC$. Consider a circle $k(Q,QB)$ and let $\{M\}=k\cap AB$ $\{N\}=k\cap AC$. Consider circle r throw points A,M and N. If $r\cap l=\{P\}$ prove that $\angle APQ=90$.

Luis González

#2 Sep 12, 2011, 10:13 pm

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=549
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=60787
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=84104
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=251993

High School Olympiads

Point on OI

Reply

skytin

#1 Sep 11, 2011, 8:46 pm • 1

Let (I) is incircle of triangle ABC
X' , Y' , Z' are midpoint's of smalest arc's BC , AC , AB
reflect X' , Y' , Z' wrt OI and get points X , Y , Z
Prove that AX , BY , CZ intersects on OI

Luis González

#2 Sep 11, 2011, 11:30 pm • 1

Let AX cut IO at P. Since OI bisects $\angle XPX' \equiv \angle APX$ and $OA = OX'$, then we deduce that O is the midpoint of the arc AX' of $\odot(APX')$. Hence, $\angle OPA = \angle OAI$ yields $OA^2 = OI \cdot OP \Longrightarrow P$ is the inverse of the incenter I in the circumcircle (O) of $\triangle ABC$, i.e. P is X_{36} of $\triangle ABC$. Similarly, BY and CZ pass through X_{36}.

buratinogigle

#3 Sep 12, 2011, 10:21 am

An extension

Let ABC be a triangle inscribed (O) and point P. PA, PB, PC intersect (O) again at X, Y, Z. X', Y', Z' are reflections of X, Y, Z through OP. Prove that AX', BY', CZ' and OP are concurrent.

Luis González

#4 Sep 12, 2011, 10:42 am • 1

> *buratinogigle wrote:*
> An extension
>
> Let ABC be a triangle inscribed (O) and point P. PA, PB, PC intersect (O) again at X, Y, Z. X', Y', Z' are reflections of X, Y, Z through OP. Prove that AX', BY', CZ' and OP are concurrent.

AX',BY',CZ' and OP concur at the inverse of P in (O). The reasoning is exactly the same as the former proof.

Quick Reply

High School Olympiads

Show Two Segments Meet on AD

Source: 2011 MMO Problem #4

bluecarneal

#1 Sep 11, 2011, 11:21 pm

Let D be the foot of the internal bisector of the angle $\angle A$ of the triangle ABC. The straight line which joins the incenters of the triangles ABD and ACD cut AB and AC at M and N, respectively.
Show that BN and CM meet on the bisector AD.

hatchguy

#2 Sep 12, 2011, 1:48 am

Let I be the incenter of triangle ABC, O_1 and O_2 the incenters of triangles ABD and ACD respectively.

Clearly, $B - O_1 - I$ and $C - O_2 - I$.

By the angle bisector theorem applied in triangle ABI with bisector AO_1 we get

$$AI = \frac{AB \cdot IO_1}{BO_1}$$

Similarly, in triangle ACI we get

$$AI = \frac{AC \cdot IO_2}{CO_2}$$

Hence

$$\frac{AB \cdot IO_1}{BO_1} = \frac{AC \cdot IO_2}{CO_2} => \frac{BO_1 \cdot IO_2}{IO_1 \cdot CO_2} = \frac{AB}{AC} = \frac{BD}{DC}$$

By ceva's theorem, the problem is equivalent to showing

$$\frac{AM \cdot BD \cdot CN}{BM \cdot CD \cdot AN} = 1 \leftrightarrow \frac{BD}{DC} = \frac{BM \cdot AN}{AM \cdot CN}$$

Note that

$$\frac{AN}{AM} = \frac{\sin \angle AMN}{\sin \angle ANM}$$

Also, notice that $\angle BO_1M = \angle IO_1O_2$ and $\angle NO_2C = \angle IO_2O_1$ and therefore by sine law on triangles BMO_1 and CNO_2 we get

$$BM = \frac{\sin \angle IO_1O_2 \cdot BO_1}{\sin \angle AMN}$$

and

and

$$CN = \frac{\sin\angle IO_2O_1 \cdot CO_2}{\sin\angle ANM}$$

and therefore we obtain

$$\frac{BM}{CN} = \frac{\sin\angle IO_1O_2 \cdot BO_1 \cdot \sin\angle ANM}{\sin\angle IO_2O_1 \cdot CO_2 \cdot \sin\angle AMN} = \frac{IO_2 \cdot BO_1 \cdot \sin\angle ANM}{IO_1 \cdot CO_2 \cdot \sin\angle AMN}$$

Hence

$$\frac{BM \cdot AN}{AM \cdot CN} = \frac{IO_2 \cdot BO_1}{IO_1 \cdot CO_2} = \frac{BD}{DC}$$

and we are done.

Luis González

#3 Sep 12, 2011, 3:02 am • 1

Let I, U, V be the incenters of $\triangle ABC, \triangle ABD, \triangle ACD.$ Internal bisectors of $\angle DAB, \angle DAC$ cut BC at P, Q and external bisector of $\angle BAC$ cuts BC at $E.$ Let EU cut AB, AC, AQ at $M', N', V'.$ Since AI, AE also bisect $\angle UAV'$, it follows that $A(U, V', I, E) = -1 \Longrightarrow I(U, V', D, E) = -1.$ But, $I(B, C, D, E) = -1,$ thus $IV' \equiv IC' \Longrightarrow V \equiv V',$ $M \equiv M',$ and $N \equiv N'.$ Therefore, if $K \equiv BN \cap CM,$ we have $A(B, C, K, E) = -1 \Longrightarrow K \in AD.$

vittasko

#4 Sep 12, 2011, 4:35 am • 1

We denote as $I,\ U,\ V,$ the incenters of the triangles $\triangle\ ABC,\ \triangle\ ABD,\ \triangle\ ACD$ respectively and let be the point $T \equiv AD \cap MN.$

Because of DU bisects the angle $\angle ADB$ and $DU\perp DV,$ we conclude that the points $S \equiv BC \cap MN,\ U,\ T,\ V,$ are in Harmonic Division.

So, the pencil $I.SUTV$ is also Harmonic and then, we have that the points $S,\ B,\ D,\ C,$ are in Harmonic Division.

Hence, from the complete quadrilateral $AMKNBC,$ where $K \equiv BN \cap CM,$ we conclude that K lies on AD and the proof is completed.

Kostas Vittas.

campos

#5 Sep 22, 2011, 6:01 am

this problem was shortlisted for the Iberoamerican olympiad 2004, held in Spain... i guess the spanish guys submitted it again for this olympiad :(

shatlykimo

#6 Jan 17, 2014, 7:16 pm

where i can find Mediterran Mathematical Olympiad 2013?

jayme

#7 Jan 17, 2014, 7:44 pm

Dear Mathlinkers,
see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=515962
Sincerely
Jean-Louis

High School Olympiads

An orthopolar-like configuration

Reply

Source: My notes

Luis González

#1 Sep 11, 2011, 12:33 pm

$\triangle ABC$ is scalene and P is an ordinary point in its plane (not lying at infinity). P^* is the isogonal conjugate of P WRT $\triangle ABC$. D, E, F are the midpoints of BC, CA, AB and the perpendiculars to PA, PB, PC through P cut EF, FD, DE at $A_0, B_0, C_0,$ respectively. Show that A_0, B_0, C_0 are collinear, if and only if the line PP^* passes through the centroid G of $\triangle ABC$.

P.S. This is a generalization of the problem another collinearity.

Quick Reply

High School Olympiads

ratio in incircle

Reply

jgnr

#1 Nov 30, 2010, 8:04 pm

The incircle ω of $\triangle ABC$ touches sides AB and BC at F and D, respectively. Segments AD and CF meet ω at H and K, respectively. Prove that

$$\frac{FD\times HK}{FH\times DK}=3.$$

oneplusone

#2 Dec 1, 2010, 6:50 pm

Let AC touch the incircle at E. Let G be the centroid of $\triangle DEF$. Let M be the midpoint of DE. Note that $FEKD, DEHF$ are harmonic, so $\angle EFM=\angle KFD$ and $\angle FEM=\angle FKD$, so $\triangle FEM\sim\triangle FKD$, so $\frac{FD}{DK}=\frac{FM}{ME}$. Now since $\angle MDG=\angle HDF=\angle HKF$ and $\angle GMD=180-\angle GME=180-\angle FDK=\angle FHK$, we have $\triangle GMD\sim\triangle FHK$. Thus $\frac{KH}{HF}=\frac{DM}{MG}$. Multiply both fractions together, we have $\frac{KH\cdot DF}{KD\cdot FH}=\frac{FM}{MG}\frac{DM}{ME}=3$.

Luis González

#3 Sep 11, 2011, 9:17 am

ω touches AC at E and BE cuts ω again at P. Since $G_e\equiv AD\cap BE\cap CF$ is always inside ω, then there exist homologies taking ω into a circle ω' with center G_e', the image of G_e. Thus if we denote the projected points with primes, the projected $\triangle A'B'C'$ is equilateral with incircle $(G_e')\equiv\omega'$. Clearly, FE and KP are the F- and K- symmedian of $\triangle FDH$ and $\triangle KDH$, thus if FE, KP cut DH at U, V, we have

$$\frac{UD}{UH}=\left(\frac{FD}{FH}\right)^2,\ \frac{VH}{VD}=\left(\frac{HK}{DK}\right)^2\Longrightarrow\left(\frac{FD\cdot HK}{FH\cdot DK}\right)^2=\frac{UD}{UH}\cdot\frac{VH}{VD}$$

In the equilateral $\triangle A'B'C'$, it's clear that $(D',H',U',V')=9\Longrightarrow$

$$(D',H',U',V')=(D,H,U,V)=\frac{UD}{UH}\cdot\frac{VH}{VD}=9\Longrightarrow\frac{FD\cdot HK}{FH\cdot DK}=3.$$

Quick Reply

High School Olympiads

hexagon and midpoints

Reply

anonymouslonely
#1 Sep 10, 2011, 7:56 pm

Let $ABCDEF$ a convex hexagon with $AB = EF, BC = DE$ and the angles ABC and DEF are congruent and acute. Prove that the midpoints of AF, BE, CD are on a line.

Luis González
#2 Sep 10, 2011, 9:52 pm

Denote by M, N, L the midpoints of AF, BE, CD and P, Q, R, S the midpoints of BF, BD, EC, EA. From $AB = EF$ and $BC = DE$, we deduce that the parallelograms $MPNS$ and $LQNR$ are rhombi $\Longrightarrow NM$ and NL bisect $\angle PNS$ and $\angle QNR$. But from $BA \parallel NS, BC \parallel NR, ED \parallel NQ$ and $EF \parallel NP$, we have $\angle ABC = \angle SNR$ and $\angle DEF = \angle QNP \Longrightarrow \angle SNR = \angle QNP \Longrightarrow$ Bisectors of $\angle PNS$ and $\angle QNR$ coincide, thus M, N, L are collinear, as desired.

Jeroen
#3 Sep 10, 2011, 10:05 pm

From the conditions it follows easily that triangle ABC en FED are congruent.
For $ABCDEF$ is convex, those triangles must have opposite orientation.
This means there exists a congruence (more specifically, a glide reflection) composed of a reflection and a translation parallel to the reflection axis that sends triangle ABC to triangle FED.
Now it is clear that the midpoints of AF, BE and CD all lie on the reflection axis.

Quick Reply

High School Olympiads

Two surprising parallels

Source: (own)

jayme
#1 Sep 8, 2011, 4:37 pm

Dear Mathlinkers,
P and P* are two isogonal points wrt ABC, (1a) the circle passing through P, B, C,
P' the second point of intersection of AP with (1a),
Q' the symmetric of P' wrt BC,
(O) the circumcircle of ABC, O the center of (O),
U, V the second points of intersection of AQ', AP* with (O)
and W the symmetric of V wrt OP*.
Prouve : UW is parallel to BC.
Sincerely
Jean-Louis

Luis González
#2 Sep 10, 2011, 6:18 am • 1

From the topics Antigonal points and Another proof, we deduce that the antigonal conjugate X of P lies on AQ' and its isogonal conjugate X^* is the inverse of P^* with respect to (O). Therefore, $OA^2 = OV^2 = OP^* \cdot OX^*$ yields $\angle OAV = \angle OVA = \angle OX^*V \Longrightarrow A, O, V, X^*$ are concyclic and OX^* bisects $\angle AX^*V \Longrightarrow$ Reflection W of V about OP^*X^* lies on AX^*. Now, since $AU \equiv AX$ and $AW \equiv AX^*$ are isogonals with respect to $\angle A$, then $UW \parallel BC$.

lym
#3 Sep 10, 2011, 7:27 pm

Let AP intersect BC at D□M the midpoint of AP'□$P'Q'$ intersect BC at E□F is the feet of A about $P'Q'$
Then $AF \parallel BC$□$\triangle MP'F \sim \triangle OVA$□and $AP^* \cdot AP' = AB \cdot AC = AD \cdot AV$□so $DP^* \parallel P'V$
So $\frac{EP'}{EF} = \frac{DP'}{DA} = \frac{P^*V}{P^*A}$□so $\angle EMP' = \angle P^*OV$□i.e. $\angle Q'AP' = \angle VAW$□so $UW \parallel BC$.

skytin
#4 Sep 11, 2011, 2:18 pm

This nice problem has solution with using only similarity

skytin
#5 Sep 11, 2011, 5:53 pm

Solution :
Reflect A wrt CB and get point A'
construct point O' , such that triangle COA ~ P*O'A
COA ~ P*O'A , so easy to see that CP*A ~ P*O'A , P*O'A ~ A'BA
after easy angle chasing we get that P*O' is perpendicular to VC and OP* ... to VW , so angle O'P*O = CVW
P*OO'A ~ A'P'BA , so angle Q'AB = BA'P' = O'P*O = CVW = CAW , BC || UW . done

High School Olympiads

Another proof

Reply

jayme

#1 Sep 8, 2011, 4:22 pm

Dear Mathlinkers,
I research a nother proof of:
let E be the isogonal conjugate of a point D with respect to a triangle ABC, let E' be the inverse of the point E in the circumcircle of triangle ABC. Then, the isogonal conjugate of the point E' with respect to triangle ABC is the antigonal conjugate D' of the point D with respect to triangle ABC.
Sincerely
Jean-Louis

Luis González

#2 Sep 8, 2011, 11:01 pm • 1

We use oriented angles (mod 180 deg) throughout the proof. We let D' be the isogonal conjugate of E' WRT ABC. By properties of isogonal conjugates we have then

$$\angle(EB, EC) = \angle(DC, DB) + \angle(AB, AC)$$

$$\angle(E'B, E'C) = \angle(D'C, D'B) + \angle(AB, AC)$$

On the other hand, let (O) be the circumcircle of $\triangle ABC$. Inversion WRT (O) takes B, C into themselves and E into E', therefore, by inversion properties we have that $\angle(EB, EC) = -\angle(E'B, E'C) + \angle(OB, OC), (\star)$. Substituting $\angle(EB, EC)$ and $\angle(E'B, E'C)$ from the first two expressions into $(\star)$ yields

$$\angle(DC, DB) + \angle(AB, AC) = -\angle(D'C, D'B) - \angle(AB, AC) + 2\angle(AB, AC)$$

$\Longrightarrow \angle(DC, DB) = -\angle(D'C, D'B)$, which means that D' lies on the reflection of $\odot(DBC)$ about BC. By similar reasoning, D' lies on the reflection of $\odot(DCA)$ about $CA \Longrightarrow D'$ is antigonal conjugate of D WRT $\triangle ABC$.

armpist

#3 Sep 9, 2011, 12:50 am

> jayme wrote:
> Dear Mathlinkers, I research another proof of:

Dear Jean-Louis,

What proof do you have?

M.T.

jayme

#4 Sep 9, 2011, 10:47 am

Dear Armpist, Luis and Mathlinkers,
I have only this reference
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=50&t=109112&p=618838
Dear Luis , I will studied your proof
Many thanks.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

concurrent lines formed by incenter and excenters

Reply

Source: Iran 3rd round 2011-geometry exam-p5

goodar2006

#1 Sep 6, 2011, 4:42 pm • 2

Given triangle ABC, D is the foot of the external angle bisector of A, I its incenter and I_a its A-excenter. Perpendicular from I to DI_a intersects the circumcircle of triangle in A'. Define B' and C' similarly. Prove that AA', BB' and CC' are concurrent.

proposed by Amirhossein Zabeti

This post has been edited 1 time. Last edited by goodar2006, Sep 14, 2011, 11:28 pm

mahanmath

#2 Sep 6, 2011, 5:55 pm • 1

Actually they are concurrent at OI
Hint

Luis González

#3 Sep 7, 2011, 12:53 am • 3

Let I_b, I_c be the excenters of $\triangle ABC$ againts $B, C.$ Incenter I and circumcicle (O) of $\triangle ABC$ become orthocenter and 9 point circle of $\triangle I_aI_bI_c \Longrightarrow (O)$ cuts $\overline{I_bI_c}$ again at its midpoint $U. I_aD$ is the polar of I WRT the circumcircle (U) of BCI_bI_c $\Longrightarrow IU \perp DI_a,$ i.e. A' lies on the line connecting I with the midpoint U of the arc BAC of $(O) \Longrightarrow A'$ is the tangency point of the A-mixtilinear incircle ω_A of $\triangle ABC$ with $(O).$ For a proof, see problem 2.20 of 2005 mosp 2.20(g) 4.44(g). Thus, A and A' are the exsimilicenters of $\omega_A \sim (I)$ and $\omega_A \sim (O) \Longrightarrow AA'$ passes through the exsimilicenter X_{56} of $(I) \sim (O).$ Likewise, BB' and CC' pass through $X_{56}.$

jayme

#4 Sep 7, 2011, 12:24 pm • 1

Dear Mathlinkers,
If I am not wrong , the point of concurs is the Schroder point.
We can see
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=283185
Sincerely
Jean-Louis

shoki

#5 Sep 7, 2011, 3:07 pm • 1

if L is the midpoint of arc BAC then $AI.AI_a = AB.AC = AL.AD$ (similarity) and so we have $AIL \sim ADI_a$ and so $LI \perp DI_a$ but it is well-known that LI passes thru the tangent point of A-mixtillinear circle with (ABC).the rest is trivial.

r1234

#6 Sep 7, 2011, 3:09 pm • 1

The concurrency point is also the isogonal conjugate of Nagel Point of $\triangle ABC$.

TelvCohl

#7 Oct 21, 2014, 6:05 pm • 1

My solution :

Since I is the intersection of BI_b and CI_c ,
so I_aD is the polar of I with respect to $\odot(BCI_bI_c)$,
hence the line through I and perpendicular to I_aD passes through the midpoint of I_bI_c .
i.e. A' is the tangency point of the A-mixtillinear circle with $\odot(ABC)$

Similarly, B', C' is the tangency point of B- mixtillinear circle, C- mixtillinear circle with $\odot(ABC)$,
so we conclude that AA', BB', CC' are concurrent at the exsimilicenter of $\odot(I) \sim \odot(O)$ (X_{56} in ETC) .

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Mar 19, 2016, 7:42 am

nikolapavlovic

#8 Mar 19, 2016, 5:16 am

Claim 1
Let D be the midpoint of arc CAB and I the incenter of the triangle.
Than D,I and the point of touch of the A-mixtlinear circle are collinear

---

Proof:
Let the mixtlinear circle touch AB,AC in P, Q and the circumcircle in
S.
It s well know that I is the midpoint of PQ
The homothety taking the A-mixtlinear circle to circumcircle (at S) takes P,Q to the midpoints of arcs
AB, AC,(X,Y respectively) and clearly it takes L to A.
Now by inspection SL is the symmedian in $\triangle PQS \implies AS$ is the symmedian in $\triangle SXY$.Now let SI sut the circumcircle in
$D^{'}$.We will prove $D \equiv D^{'}$
From the properties of the median and symmedian
$AP = D^{'}Q$ but $AP = DQ$ so we are done.

Claim 2:
Let the respective mixtlinear touch the $\odot ABC$ in $A^{'}, B^{'}, C^{'}$ than $AA^{'}$,$BB^{'}$,$CC^{'}$are concurrent

---

Proof
$A^{'}$ is the center of positive homothety that takes A-mixtliniear circle to $\odot ABC$ A is the center of positive homothety that takes the aforementioned circle to the inscribed circle thus by Monge -De' Alambert
$AA^{'}$ passes thru the positive center of homothety that takes the incircle to the $\odot ABC$ thus we are done

---

Now let I_A, I_B, I_C be the respective excenters.
Let the midpoint of arcCAB be F
Its well known that $\triangle ABC$ is the orthic triangle of $\triangle I_AI_BI_C$
By La -Hire s theorem I_AD is the polar of I wrt $\odot I_CI_BBC$ so FI(F is the midpoint of I_BI_C) is perpendicular to I_AD and so we are done by Claim1 and Claim2

This post has been edited 6 times. Last edited by nikolapavlovic, Mar 19, 2016, 5:31 am

Quick Reply

High School Olympiads

Ah, don't bother me with this much point (show AE prp KL)

Reply

Source: Middle European Mathematical Olympiad 2011 - Individuals I-3

Amir Hossein
#1 Sep 6, 2011, 5:25 pm

In a plane the circles $\mathcal{K}_1$ and $\mathcal{K}_2$ with centers I_1 and I_2, respectively, intersect in two points A and B. Assume that $\angle I_1AI_2$ is obtuse. The tangent to $\mathcal{K}_1$ in A intersects $\mathcal{K}_2$ again in C and the tangent to $\mathcal{K}_2$ in A intersects $\mathcal{K}_1$ again in D. Let $\mathcal{K}_3$ be the circumcircle of the triangle BCD. Let E be the midpoint of that arc CD of $\mathcal{K}_3$ that contains B. The lines AC and AD intersect $\mathcal{K}_3$ again in K and L, respectively. Prove that the line AE is perpendicular to KL.

Luis González
#2 Sep 7, 2011, 12:08 am • 2

$\angle(BA, BC) = \angle BAC + \angle ACB = \angle BAC + \angle DAB = \angle DAC.$ Similarly, we have $\angle(BD, BA) = \angle DAC \Longrightarrow \angle DBC = 2\angle DAC.$ Thus, $ED = EC$ and $\angle DEC = \angle DBC = 2\angle DAC$ imply that E is the circumcenter of $\triangle ACD.$ Since KL is antiparallel to CD with respect to the lines $AC, AD,$ then it follows that KL is perpendicular to the A-circumdiameter AE of $\triangle ACD,$ as desired.

RSM
#3 Sep 7, 2011, 12:42 am • 1

Under inversion wrt A suppose B, C, D goes to B', C', D'. $AC'B'D'$ is a parallelogram. Suppose, E goes to E'. Clearly, $\frac{CE'}{DE'} = \frac{AC}{AD}$. Since E' lies on $\odot B'C'D'$, so it is the reflection of A on $C'D'$. So $AE' \perp C'D'$. So $AE \perp KL$,since KL is anti-parallel to CD.

vprasad_nalluri
#4 Sep 7, 2011, 11:19 pm • 1

$\angle KAL = \angle KAB + \angle BAL = \angle ACB + \angle ADB = \angle BDL + \angle ADB = \angle ADL \Longrightarrow \angle I_1DL = \angle I_1AL = 90^\circ.$ So AD is the polar of L wrt circle (I_1) and $I_1L \perp AD$ (or AK), $I_2K \perp AL.$ Follows E is the orthocentre of $\triangle AKL.$ Hence, $AE \perp KL.$

[Moderator edit: LaTeXed]

littletush
#5 Nov 27, 2011, 11:51 am

it suffices to prove that A is the orthocenter of triangle EKL and by angle substitutions it's easy to obtain $AK\perp EL, AL\perp EK$.

Quick Reply

High School Olympiads

tangent iff OM=ON

Reply

Source: Iran 3rd round 2011-geometry exam-p2

goodar2006

#1 Sep 6, 2011, 4:28 pm • 3

In triangle ABC, ω is its circumcircle and O is the center of this circle. Points M and N lie on sides AB and AC respectively. ω and the circumcircle of triangle AMN intersect each other for the second time in Q. Let P be the intersection point of MN and BC. Prove that PQ is tangent to ω iff $OM = ON$.

proposed by Mr.Etesami

This post has been edited 1 time. Last edited by goodar2006, Sep 14, 2011, 11:25 pm

Luis González

#2 Sep 6, 2011, 11:15 pm • 3

Since Q is the Miquel point of $\triangle ABC \cup MN$, it follows that P, Q, M, B are concyclic. Let $R \equiv PQ \cap AC$ and assume that PQR is tangent to the circumcircle (O). Then $\angle RQA = \angle QBM = \angle QPM \Longrightarrow QA \parallel PN.$ Hence, the cyclic quadrilateral $AQMN$ is an isosceles trapezoid with bases AQ and $MN \Longrightarrow$ perpendicular bisector of $\overline{AQ}$ is also the perpendicular bisector of $\overline{MN} \Longrightarrow OM = ON.$ The converse can be proved with exactly the same arguments.

sayantanchakraborty

#3 Sep 1, 2014, 8:16 pm

This problem is nice.Here's a pure synthetic solution.

1.PQ tangent to $\omega \Rightarrow OM = ON$.

Since PQ is tangent to ω we have $\angle CQP = \angle QBC = \angle QBP$.Also note that $\angle CQP = \angle QAC = \angle QAN = \angle QMN = \angle QMP$.Thus $\angle QMP = \angle QBP \implies QMBP$ is concyclic.So $\angle NCQ = \angle ACQ = \angle ABQ = \angle QBM = \angle QPM = \angle QPN \implies QNCP$ is cyclic.Hence $\angle CQP = \angle CNP = \angle ANM = \angle QAN$.So $AQ \parallel MN \implies AQNM$ is an isosceles trapezoid.Thus $QM = AN$ and $AM = QN$.

Now note that $\angle AMQ = \angle ANQ \Rightarrow \angle BMQ = \angle CNQ$.Also $\angle MBQ = \angle ABQ = \angle ACQ = \angle NCQ$.Thus $\triangle BMQ \sim \triangle CNQ \implies \frac{BM}{CN} = \frac{MQ}{NQ} \implies BM \times NQ = CN \times QM \implies BM \times AM = CN \times AN$.
So since power of M and N are equal wrt ω we have $OM = ON$.

2.$OM = ON \Rightarrow PQ$ is tangent to ω.

As with the above situation it follows that $\triangle BMQ \sim \triangle CNQ \Rightarrow \frac{BM}{CN} = \frac{MQ}{NQ} \Rightarrow BM \times NQ = CN \times MQ$.
Also by hypothesis we have $BM \times AM = CN \times AN$.Combining these yeilds $AN \times NQ = AM \times MQ \Rightarrow [AMQ] = [ANQ] \Rightarrow [AMZ] = [QNZ] \Rightarrow AZ \times MZ = QZ \times NZ$.
We also have $\frac{AZ}{MZ} = \frac{QZ}{NZ}$.Summing these it follows that $AN = QM$ or $AQNM$ is an isosceles trapezoid.

Also note that as with the previous situation we similarly have $QNCP$ cyclic.Thus $\angle PQC = \angle PNC = \angle ANM = \angle NAQ = \angle CAQ$ or PQ is tangent to ω as required.

Quick Reply

High School Olympiads

O,I,D collinear if D,X,Y collinear

Source: Iran 3rd round 2011-geometry exam-p3

goodar2006
#1 Sep 6, 2011, 4:33 pm • 2

In triangle ABC, X and Y are the tangency points of incircle (with center I) with sides AB and AC respectively. A tangent line to the circumcircle of triangle ABC (with center O) at point A, intersects the extension of BC at D. If D,X and Y are collinear then prove that D,I and O are also collinear.

proposed by Amirhossein Zabeti

This post has been edited 1 time. Last edited by goodar2006, Sep 14, 2011, 11:27 pm

Luis González
#2 Sep 6, 2011, 10:04 pm • 14

Let the incircle (I) touch BC at $Z.$ XY and BC are the polars of A,Z with respect to $(I) \Longrightarrow D \equiv XY \cap BC$ is the pole of AZ with respect to $(I) \Longrightarrow ID \perp AZ.$ Since DA is tangent to the circumcircle (O) and the cross ratio (B,C,D,Z) is harmonic, then we deduce that AZ is also the polar of D with respect to $(O) \Longrightarrow OD \perp AZ.$ Therefore, O,I,D lie on a perpendicular to $AZ.$

goodar2006
#3 Sep 6, 2011, 10:22 pm

really beautiful luis, I couldn't believe it the moment I was reading because I used a 3page complete brute force for solving it

Virgil Nicula
#4 Sep 17, 2011, 6:19 am • 4

An interesting metrical remark.

IRAN, 2011. In $\triangle ABC$ let X and Y be the tangent points of incircle $w=C(I,r)$ with AB and AC respectively. The tangent at A to the circumcircle

$C(O,R)$ of $\triangle ABC$ intersects BC at D . Prove that $D\in XY\iff D\in OI\iff IL\parallel BC\iff IO\perp AZ$ $\iff a(b+c)=b^2+c^2$

$\iff (s-a)^2=(s-b)(s-c)$ where $2s=a+b+c$, $Z\in BC\cap w$ and L is the **Lemoine**'s point (symmedian center) .

Proof. Suppose w.l.o.g. $b\ne c$.

$\blacktriangleright$ $\boxed{D\in XY}\iff \frac{DB}{DC}=\frac{ZB}{ZC}\iff \frac{c^2}{b^2}=\frac{s-b}{s-c}\iff \frac{b^2-c^2}{b^2+c^2}=\frac{b-c}{a}\iff \frac{b+c}{b^2+c^2}=\frac 1a\iff$ $\boxed{a(b+c)=b^2+c^2}$.

$\blacktriangleright$ $\boxed{IL\parallel BC}\iff [BLC]=[BIC]\iff \frac{a^2}{a^2+b^2+c^2}=\frac{a}{2s}\iff a^2(a+b+c)=a\left(a^2+b^2+c^2\right)\iff$ $\boxed{a(b+c)=b^2+c^2}$.

$\blacktriangleright$ $\boxed{(s-a)^2=(s-b)(s-c)}\iff a^2-2a(b+c)+(b+c)^2=a^2-(b-c)^2\iff \boxed{a(b+c)=b^2+c^2}$.

▶ $\boxed{IO \perp AZ}$ $\iff OA^2 - OZ^2 = IA^2 - IZ^2 \iff ZB \cdot ZC = IA^2 - IX^2 \iff$ $\boxed{(s-b)(s-c) = (s-a)^2}$.

▶ $\boxed{D \in XY}$ $\iff Z$ is the conjugate of D w.r.t. $\{B, C\}$ $\iff AZ$ is polar line of D $\iff DO \perp AZ$ $\iff$ $\boxed{D \in OI}$.

This post has been edited 7 times. Last edited by Virgil Nicula, Jan 29, 2016, 3:42 am

erfan_Ashorion

#5 Apr 3, 2012, 3:24 am

oh..nice problem...and nice solution by me

lemma 1:suppose that AZintersect incircle at Pdraw tangent to incircle at Pproof that Dlie on this tangent..!proof:its not hard and im sure that see it in mathlink if U need say me i message it

lemma 2$\frac{AB^2}{AC^2} = \frac{BZ}{ZC}$that Z is the point that incircle is tangent to BC

proof:

$\frac{BZ}{ZC} = \frac{BD}{DC} = \frac{DA^2}{DC^2} = \frac{\sin C^2}{\sin DAC^2} = \frac{\sin C^2}{\sin B^2} = \frac{AB^2}{AC^2}$

proof or problem:

we proof $IDC = ODC$ first we have:($H$is the perpendicular of Aon $BC$and M is midpoint of BC

$IDC = IPZ = IZP = ZAH$

for second we have:

$ODC = OAM$

we know that:

$MAC = ZAB$(because that we proof by lemma 2 $AZ$is symmedian!)and$OAC = HAB \to OAM = ZAH \to IDC = ODC$

sayantanchakraborty

#6 Sep 1, 2014, 11:53 pm

If D, X, Y are collinear,then simple angle chasing yeilds that $\angle ADC$ is bisected by DXY.So we have $\frac{DB}{DA} = \frac{BX}{AX} = \frac{s-b}{s-a}$ and $\frac{DC}{DA} = \frac{CY}{AY} = \frac{s-c}{s-a}$.Multiplying these two relations and noting that $DA^2 = DB \times DC$ we get $(s-a)^2 = (s-b)(s-c) \Rightarrow \triangle^2 = (s-a)^3$.Now let $IZ \perp DC$ and $OK \perp DC$ with Z, K on BC.Then sine rule in $\triangle DBA$ gives

$$\frac{DB}{c} = \frac{sinC}{sin(B-C)} \Rightarrow DB = \frac{csinC}{sin(B-C)} = \frac{\frac{c^2}{2R}}{\frac{b}{2R}\frac{a^2+b^2-c^2}{2ab} - \frac{c}{2R}\frac{a^2+c^2-b^2}{2ac}} = \frac{c^2a}{b^2-c^2}.$$

$$\frac{IZ}{OK} = \frac{r}{RcosA} = \frac{\frac{\triangle}{s}}{\frac{abc}{4\triangle}\frac{c^2+b^2-a^2}{2bc}} = \frac{2(b+c-a)^3}{a(c^2+b^2-a^2)(a+b+c)}.$$(Here we have used the fact that $\triangle^2 = (s-a)^3$).

Now $\frac{DZ}{DK} = \frac{IZ}{OK} \Leftrightarrow \frac{DB+(s-b)}{DB+\frac{a}{2}} = \frac{2(b+c-a)^3}{a(c^2+b^2-a^2)(a+b+c)} \Leftrightarrow DB = \frac{c^2a}{b^2-c^2}$

which is true.So $\frac{DZ}{DK} = \frac{IZ}{OK}$ and D, I, O are collinear.

I think the statement can be 'if and only if' because both the conditions yeild $(s-a)^2 = (s-b)(s-c)$.Anyways credit goes to Luis for his projective approach.

Quick Reply

High School Olympiads

fixed incircle and circumcircle

Source: Iran 3rd round 2011-geometry exam-p4

goodar2006
#1 Sep 6, 2011, 4:37 pm • 3

A variant triangle has fixed incircle and circumcircle. Prove that the radical center of its three excircles lies on a fixed circle and the circle's center is the midpoint of the line joining circumcenter and incenter.

proposed by Masoud Nourbakhsh

This post has been edited 1 time. Last edited by goodar2006, Sep 14, 2011, 11:27 pm

Luis González
#2 Sep 6, 2011, 9:25 pm • 5

Denote by (I,r) and (O,R) the fixed incircle and circumcircle of $\triangle ABC$. $\triangle DEF$ and $\triangle A_0B_0C_0$ are the medial and antimedial triangle of $\triangle ABC$ with incenters $S,N,$ the Spieker point and Nagel point of $\triangle ABC,$ respectively. It's well-known that S is the radical center of the three excircles of $\triangle ABC$ (see Hard geometry problem), further, N is the reflection of I about $S.$ Since the incircle $(N,2r)$ and 9 point circle (O,R) of $\triangle A_0B_0B_0$ are internally tangent, we obtain $ON=R-2r.$ So if M is the midpoint of $\overline{IO},$ we have $MS=\frac{1}{2}ON=\frac{1}{2}R-r=\text{const} \Longrightarrow$ Locus of S is then a circle with center M and radius $\frac{1}{2}R-r.$

shoki
#3 Sep 7, 2011, 3:16 pm • 1

in fact if O,I are fixed then H which is the orthocenter lies on a fixed circle which center is the reflection of O wrt I.this was in a crux magazine ... but unfortunately i don't remember the vol :(and this quickly solve the problem as the radical center is I' the incenter of the median triangle collinear with G,I and G collinear with O,H.

High School Olympiads

Orthogonal circles (Own)

skytin
#1 Sep 5, 2011, 6:03 pm • 2

Given quadrilateral ABCD and if there exist :
Circle (I_1) is tangent to segments AB , BC , CD , AD
Circle (I_2) is tangent to segments AC , BD and lines BC , AD
Prove that circles (I_1) , (I_2) are orthogonal

Luis González
#2 Sep 6, 2011, 1:44 pm • 2

(I_1) touches AB, BC, CD, DA at P, Q, R, S and (I_2) touches BD, CA, AD, BC at $M, N, L, K.$ By Newton's theorem in the tangential quadrilateral $ABCD,$ the lines BD, AC, QS and PR concur at $X \Longrightarrow \overline{QXS} \perp I_1I_2$ is the polar of $H \equiv MN \cap I_1I_2$ WRT $(I_2) \Longrightarrow X(M, N, H, Q) = -1.$ But if $E \equiv BC \cap AD,$ then $\triangle PQS$ becomes the E-extouch triangle of $\triangle ABE \Longrightarrow X(B, A, P, Q) = -1,$ i.e. $X(M, N, P, Q) = -1,$ which yields $XH \equiv XP \Longrightarrow$ $H \equiv PR \cap MN \cap I_1I_2 \, (\star).$

On the other hand, $Y \equiv AB \cap MN \cap KL$ is the pole of EX WRT $(I_2) \Longrightarrow E(B, A, X, Y) = -1 \Longrightarrow Y \in \overline{CRD}.$ Hence, together with $(\star),$ we deduce that $\overline{YLK} \perp I_1I_2$ is the polar of H WRT $(I_1).$ As a result, the polar of K WRT $(I_1),$ the polar of Q WRT (I_2) and I_1I_2 concur at $H \Longrightarrow HQ$ is perpendicular to KI_1 through U and HK is perpendicular to QI_2 through $V.$ Then $J \equiv QI_2 \cap KI_1$ is the orthocenter of $\triangle HQK \Longrightarrow HJ \parallel I_2K \parallel I_1Q,$ which yields $(I_1, I_2, H, E) = -1$ $\Longrightarrow H$ is the insimilicenter of $(I_1) \sim (I_2).$ Thereby, KH, QH pass through the antipodes Q', K' of Q, K WRT $(I_1), (I_2),$ respectively $\Longrightarrow V \in (I_1)$ and $U \in (I_2).$ Now, since $(V, Q', K, H) = -1,$ the circle $\odot(KUH)$ with diameter $\overline{KH}$ is then orthogonal to $(I_1) \Longrightarrow {r_1}^2 = \overline{I_1U} \cdot \overline{I_1K} = p(I_1, (I_2)) \Longrightarrow (I_1) \perp (I_2).$

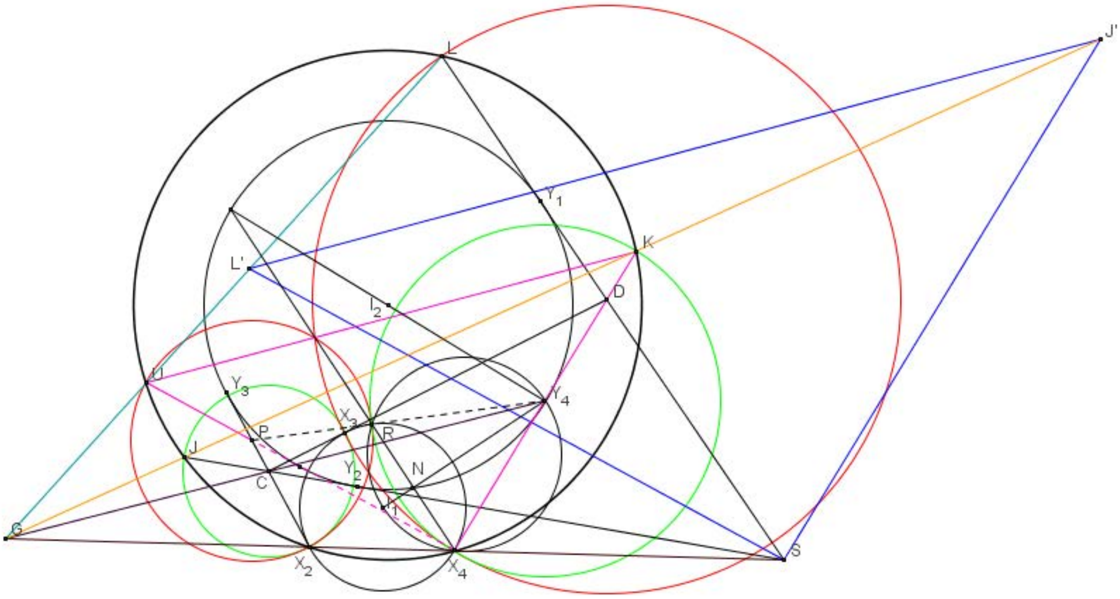
skytin
#3 Sep 7, 2011, 6:26 pm

Solution :
Lemma :
Given two circles a , b
poits A_1 , A_2 and B_1 , B_2 are on a and b
given that A_1A_2B_1B_2 is cyclic and A_1B_1 goes thru homotety center of a , b
then not hard to prove that A_2B_2 goes thru this point too

Let (I_1) is tangent to AB , BC , CD , AD at points X_1 , X_2 , X_3 , X_4
(I_2) is tangent to AD , AC , CB , DB at points Y_1 , Y_2 , Y_3 , Y_4
Well known that AC intersect DB at point S on X_2X_4
Reflect X_4 wrt I_1Y_1 and get point R
RY_1 intersect X_2Y_3 at point P
Let circle (P) has center P and radius PR , circle (Y_1) has center Y_1 and radius RY_1
circle (C) has center C and radius CX_3 , circle (D) has center D and radius X_3D
Reflect X_4 wrt DI_2 , I_2Y_1 and get poins L , K
K is on (Y_1) , L is on (D)
reflect X_2 wrt CI_2 , PI_2 and get points J , U
U is on (P) , J is on (C)
Easy to see that I_2 is center of (LX_4X_2UJK)
from Newton's theorem we get that CY_1 intersect PD at point G on X_2X_4
not hard to prove that X_2X_4 goes thru homotety center of (C) , (Y_1) and of (P) , (D)
After Lemma we get that G is intersection point of UL , JK
anngle CJK = JKD and PUL = ULD
SJ = SY_2 + X_2Y_3 = SY_4 + X_4Y_1 = SL
let circle with center at S and radius SJ intersect UL , JK at second points L' , J'

after Reim's theorem we get that L'J' || UK
angle SL'U = ULS = PUL , so PU || SL' , like the same we get that SJ' || X_4K
Let PU intersect AD at point X_4'
G is homotety center of triangles Y_1X_4P and J'SL' , so X_4' is on GS , X_4' = X_4
X_4P is tangent to (I_2)
X_4P + PR = X_4Y_1 + Y_1R , so X_4-excircle of Y_1X_4P is tangent to PY_1 at point R
Let Y_1'Y_1 is diameter of (I_2)
After Monge's theorem we get that X_4R goes thru homotety center of (I_1) , (I_2) , so Y_1' is on X_4R
X_4R intersect I_1Y_1 at point N
angle Y_1NR = 90 = Y_1NY_1' , so N is on (I_2)
I_1N*I_1Y_1 = I_1X_4*Y_1X_4 , so (I_1) is orthogonal to (I_2) . done

Attachments:

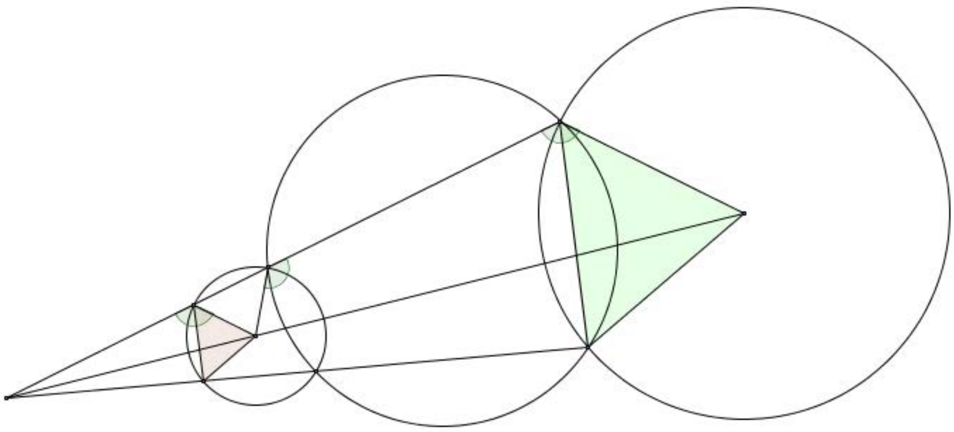


skytin

#4 Sep 7, 2011, 9:23 pm

Solution to Lemma :
see picture

Attachments:



skytin
#5 Sep 10, 2011, 6:18 pm • 1

Another solution :
Let (I_1) is tangent to AB , BC , CD , AD at points X_1 , X_2 , X_3 , X_4
(I_2) is tangent to AD , AC , CB , DB at points Y_1 , Y_2 , Y_3 , Y_4
Well known that AC intersect DB at point S on X_2X_4
Let BY_1 intersect X_2X_4 at point G
GA intersect BC at point Q
from Newton's theorem we get that Y_1Q is tangent to (2)
let's use the converse Brianchon theorem for Y_1ASBQX_4 , we get that QX_4 is tangent to (2) at point K
Y_1Q + QK = Y_1H + Y_3X_2 = Y_1X_4 + KX_4 , so K is tangent point of Y_1 excircle of Y_1QX_4 with X_4Q
Let X_4L is diameter of (I_1)
After Monge's theorem we get that Y_1K goes thru homotety center of (I_1) , (I_2) , so L is on Y_1K
I_2X_4 intersect Y_1L at point M
Angle X_4ML = 90 , so M is on (1)
I_2M*I_2X_4 = I_1K*I_1K , so (I_1) is orthogonal to (I_2) . done

Quick Reply

High School Olympiads

1982 USAMO #5

Reply

Mrdavid445

#1 Aug 17, 2011, 1:39 am

A, B, and C are three interior points of a sphere S such that AB and AC are perpendicular to the diameter of S through A, and so that two spheres can be constructed through A, B, and C which are both tangent to S. Prove that the sum of their radii is equal to the radius of S.

Luis González

#2 Sep 5, 2011, 6:04 am • 1

Let $\Omega_1(O_1,r_1)$ and $\Omega_2(O_2,r_2)$ denote the spheres through A,B,C internally tangent to $\mathcal{S}(O,R)$ at $U,V.$ Since $AB \perp OA$ and $AC \perp OA,$ then the plane ABC is perpendicular to $OA \Longrightarrow O_1O_2 \parallel OA \Longrightarrow O_1,O_2,O,A$ lie on a same plane $\delta,$ which obviously contains $U,V,$ because of $U \in OO_1$ and $V \in OO_2.$ This plane δ cuts then $\Omega_1,\Omega_2,\mathcal{S}$ into great circles $(O_1,r_1),(O_2,r_1),(O,R)$ such that $A \equiv (O_1) \cap (O_2)$ and (O) touches $(O_1),(O_2)$ internally at $U,V.$

WLOG assume that $r_1 > r_2.$ Then $OO_2-OO_1=R-r_2-(R-r_1)=r_1-r_2=AO_1-AO_2 \Longrightarrow O$ and A lie on different branches of the hyperbola $\mathcal{H}$ with foci $O_1,O_2,$ major axis length (r_1-r_2) and conjugate axis the perpendicular bisector ℓ of $\overline{O_1O_2}.$ Since $\mathcal{H}$ is symmetric about $\ell,$ then $OA \parallel O_1O_2$ implies that O is the reflection of A about $\ell,$ i.e. AOO_1O_2 is an isosceles trapezoid with legs $OO_1=AO_2 \Longrightarrow R-r_1=r_2.$

Quick Reply

High School Olympiads

Prove that circumradii of triangles are the same

Reply

Source: Tournament of Towns 2007 - Fall - Senior A-Level - P2

Amir Hossein

#1 Sep 4, 2011, 1:09 pm

Let K, L, M and N be the midpoints of the sides AB, BC, CD and DA of a cyclic quadrilateral $ABCD$. Let P be the point of intersection of AC and BD. Prove that the circumradii of triangles PKL, PLM, PMN and PNK are equal to one another.

Luis González

#2 Sep 4, 2011, 8:44 pm

$\triangle PAD$ and $\triangle PBC$ are similar with corresponding medians PN and $PL \Longrightarrow \triangle PND$ and $\triangle PLC$ are similar $\Longrightarrow \angle DPN = \angle CPL$. But from $NK \parallel BD$ and $LK \parallel AC$, we have $\angle DPN = \angle KNP, \angle CPL = \angle KLP \Longrightarrow \angle KNP = \angle KLP \Longrightarrow \odot(PKL) \cong \odot(PNK)$. By similar reasoning, we obtain $\odot(PKL) \cong \odot(PLM)$, $\odot(PLM) \cong \odot(PMN)$ and the conclusion follows.

Quick Reply

High School Olympiads

Lines symmetric to AT,BT,CT wrt BC,CA,AB are concurrent

Reply

Source: Tournament of Towns 2007 - Spring - Senior A-Level - P7

Amir Hossein
#1 Sep 3, 2011, 1:39 pm

T is a point on the plane of triangle ABC such that $\angle ATB = \angle BTC = \angle CTA = 120^\circ$. Prove that the lines symmetric to AT, BT and CT with respect to BC, CA and AB, respectively, are concurrent.

Luis González
#2 Sep 3, 2011, 9:26 pm

This is a well known problem (see the links below). It's also interesting to note that there are more remarkable points P, besides the Fermat points, whose cevians AP,BP,CP reflected about BC,CA,AB concur, namely the points P lying on the Isocubic K060 with pivot X(265) and pole X(1989).

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=304719
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=175130

genxium
#3 Sep 7, 2011, 9:14 am

[geogebra]144ec5f21f51a926bb17a3b94d6ef9aaacdc61a1[/geogebra]

They concurrent at one of the isodynamic points , which is one the intersections of 3 Apollonius circles(each circle has takes the intersection of internal/external bisectors and the respective side of $\triangle ABC$ as a diameter , you can refer to here), construct the isodynamic point first and then prove each symmetric line passes through it, by and ratio method you like, I did with lots of trigonometry calculations which I don't think is proper to post ^_^

You may see basic features(with proofs) of the isodynamic points in this pdf file

simplependulum
#4 Sep 7, 2011, 4:56 pm • 1

Notice that AT bisects $\angle BTC$, consider the isogonal conjugate of A w.r.t. ΔBTC , P , we have$\angle BCP = \angle ACT = \angle BCT^*$, $\angle CBP = \angle ABT = \angle CBT^*$, where T^* is the isogonal conjugate of T w.r.t. ΔABC , implying that P is the reflection of T^* in BC but we find that A, T, P are collinear , the reflection of AT in BC is therefore a line passing through T^* .

Quick Reply

High School Olympiads

USA tst 2006

Reply

paul1703

#1 Sep 2, 2011, 5:24 pm

In acute triangle ABC, segments AD;BE, and CF are its altitudes, and H is its orthocenter. Circle w, centered at X, passes through A and H and intersects sides AB and AC again at Q and P (other than A), respectively. The circumcircle of triangle XPQ is tangent to segment BC at R. Prove that $\frac{CR}{BR}=\frac{ED}{FD}$

Luis González

#2 Sep 3, 2011, 5:44 am

Clearly $\triangle HPQ \sim \triangle HEF$ are spirally similar with center H and rotational angle $\angle AHX$. Hence, if XH cuts BC at R^*, then $\triangle PQR^* \sim \triangle EFD$ are spirally similar with center H, their common incenter. Thus, from $XP=XQ$, we deduce that X is the midpoint of the arc PQ of $\odot(PQR^*)$, i.e. $R^* \in \odot(XPQ)$. Now, when $\odot(XPQ)$ is tangent to BC, the intersections $\odot(XPQ) \cap BC$ are identical $\Longrightarrow R \equiv R^* \Longrightarrow \triangle PQR \sim \triangle EFD$. Hence,

$$\angle BQR = 180^\circ - \angle QPR - \angle B = 180^\circ - \angle FED - \angle B = \angle B.$$

Which means that $\triangle BQR$ is isosceles with legs $BR=QR$. By similar reasoning, $\triangle CPR$ is isosceles with legs $CR=PR$. Hence, $\frac{CR}{BR}=\frac{PR}{QR}=\frac{ED}{FD}$, as desired.

P.S. Similar solutions can be seen in the topic acute triangle and its altitudes.

Quick Reply

High School Olympiads

Determine the acute angle between the lines AC and BD

Reply

Source: Tournament of Towns 2007 - Spring - Junior A-Level - P6

Amir Hossein
#1 Sep 3, 2011, 12:04 am

In the quadrilateral $ABCD$, $AB = BC = CD$ and $\angle BMC = 90^\circ$, where M is the midpoint of AD. Determine the acute angle between the lines AC and BD.

Luis González
#2 Sep 3, 2011, 1:38 am

Let N, U, V be the midpoints of BC, AC, BD. Since $NU = NV = \frac{1}{2}AB$, then the parallelogram $MUNV$ is obviously a rhombus. But from $\angle BMC = 90^\circ$, we obtain $NM = \frac{1}{2}BC = NV = NU \Longrightarrow \triangle MUN$ and $\triangle MVN$ are both equilateral $\Longrightarrow \angle UNV = 120^\circ$. Thus, if $Q \equiv AB \cap DC$, we have $\angle BQC = 120^\circ$. Let P be the intersection of the diagonals AC, BD. Then simple angle chase yields

$$\angle CPD = \angle CBD + \angle BCA = \frac{1}{2}(\angle BCQ + \angle CBQ) = 30^\circ.$$

nima1376
#3 May 23, 2014, 10:09 pm

another solution:
let point X on CM such that $CM = MX. \Rightarrow AXDC$ is parallelogram $\Rightarrow AX = CD$,
$CM = MX, BM \perp CX \Rightarrow BC = BX = BA = CD = AX$.
so B is circumcenter of CAX. $\widehat{ABX} = 60 \Rightarrow \widehat{ACX} = 30$.
similar we find $\widehat{MBD} = 30$.
let AC meets BD at O.
$\widehat{CAM} = \widehat{CMD} - 30, \widehat{BDM} = \widehat{BMA} - 30, \widehat{BMA} + \widehat{CMD} = 90 \Rightarrow AOD = 150$
so we are done.

Quick Reply

High School Olympiads

Prove that the line AA' passes through the midpoint of BB'

Reply

Source: Tournament of Towns 2007 - Spring - Senior O-Level - P3

Amir Hossein
#1 Sep 3, 2011, 12:17 am

B is a point on the line which is tangent to a circle at the point A. The line segment AB is rotated about the centre of the circle through some angle to the line segment $A'B'$. Prove that the line AA' passes through the midpoint of BB'.

Luis González
#2 Sep 3, 2011, 12:50 am • 1

Let $P \equiv AB \cap A'B'$ and $M \equiv AA' \cap BB'$. By Menelaus' theorem for $\triangle PBB'$ cut by $\overline{AA'M}$, keeping in mind that $\overline{AB} \cong \overline{A'B'}$ and $\overline{PA} \cong \overline{PA'}$, we get

$$\frac{BM}{MB'} = \frac{A'P}{B'A'} \cdot \frac{AB}{AP} = \frac{AP}{AP} \cdot \frac{AB}{AB} = 1 \Longrightarrow M \text{ is the midpoint of } \overline{BB'}.$$

Goutham
#3 Sep 3, 2011, 1:30 am • 1

Let O be the centre of the circle and AA' meet BB' at M. We have <AOA'=2<BAA' but since the line segment AB is rotated to A'B' about centre O, <AOA'=<BOB' and hence, <BOB'=2<BAA'. Consider triangle OBB'. It is isosceles as OB=OB'. We already know <BOB'=2<BAA' and so, the remaining angles <OBB', <OB'B are both 90-<BAA'. If we consider triangle AOB, <OAB is 90 degrees as AB is tangent to the circle (O) and so, <ABO=90-<AOB. Take triangle ABM. <MAB=<BAA'(they are essentially the same angle), <ABM=<ABO+<OBM=90-<AOB+<90-<BAA'(as derived earlier). So, we can find <AMB. We see that it is equal to <AOB which proves that the quadrilateral AOMB is cyclic. Now, one of the angles of the quadrilateral <OAB is 90 degrees and hence, so is <OMB or in other words, OM is perpendicular to AB. Now, in isosceles triangle OBB', M lies on BB' and OM is perpendicular to BB' implies that M is the midpoint of BB'.

sunken rock
#4 Sep 4, 2011, 1:41 am • 1

Take the reflections of B in A and respectively B' in A', the problem is obvious then.

Best regards,
sunken rock

genxium
#5 Sep 5, 2011, 8:43 am • 1

centred at O: $A \sim A', B \sim B'$.

Take $M = AA' \cap BB'$, $\triangle OAA' \sim \triangle OBB'$, $\quad MOBA$ is cyclic.

This post has been edited 1 time. Last edited by Amir Hossein, Sep 5, 2011, 1:43 pm
Reason: Fixed LaTeX.

Quick Reply

High School Olympiads

quadrilateral

Reply

hungnsl
#1 Sep 1, 2011, 9:15 pm

A circle (I) is inscribed in a quadrilateral $ABCD$. Prove that:

$$IB^2+\frac{IA.IB.IC}{ID}=AB.BC$$

mathfighter
#2 Sep 2, 2011, 10:01 am

HI Is $ABCD$ Tangential quadrilateral?

hungnsl
#3 Sep 2, 2011, 10:06 am

Yes, it can also be called "circumscribed quadrilateral". I think the statement of the question was clear and there's no "vocabulary tricks".

Luis González
#4 Sep 2, 2011, 1:03 pm

Let the incircle (I,r) touch AB,BC,CD,DA at P,Q,R,S. Then $K\equiv PQ\cap IB, L\equiv QR\cap IC, M\equiv ID\cap RS$, $N\equiv SP\cap IA$ are the midpoints of PQ,QR,RS,SP. Inversion WRT (I,r) takes A,B,C,D into N,K,L,M, due to $IN\cdot IA=r^2$, etc. Further, by inversion properties, we have

$$AB=NK\cdot\frac{r^2}{IN\cdot IK}\ ,\ BC=KL\cdot\frac{r^2}{IK\cdot IL}$$

Substituting these relations into the given expression we obtain

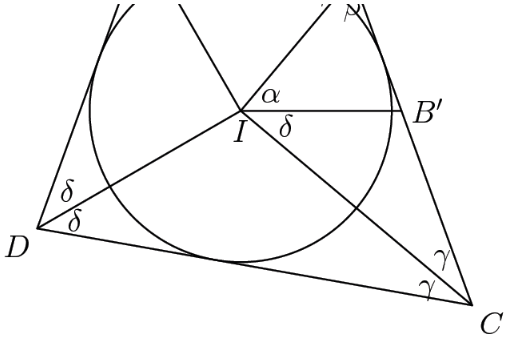
$$\frac{r^4}{IK^2}+\frac{IM}{r^2}\cdot\frac{r^6}{IN\cdot IK\cdot IL}=\frac{r^4\cdot NK\cdot KL}{IK^2\cdot IN\cdot IL}\Longrightarrow$$

$IN\cdot IL+IM\cdot IK=NK\cdot KL\ (\star)$ (to be proved)

Translate I by $\overrightarrow{LK}$ into E. Since $KLMN$ is a parallelogram (Varignon parallelogram of PQRS), then it follows that $\triangle ENK\cong\triangle IML\Longrightarrow\angle NEK=\angle MIL=\angle NPK\Longrightarrow E$ lies on the circumcircle of the quadrilateral $INPK$. By Ptolemy's theorem for $INEK$, we get then $IN\cdot EK+EN\cdot IK=NK\cdot EI$. But $EK=IL, EN=IM$ and $EI=KL\Longrightarrow IN\cdot IL+IM\cdot IK=NK\cdot KL$, which is the relation $(\star)$ we were supposed to prove.

nsato
#5 Sep 2, 2011, 11:01 pm • 2

Let $\alpha=\angle IAB=\angle IAD$, $\beta=\angle IBC=\angle IBA$, $\gamma=\angle ICD=\angle ICB$, and $\delta=\angle IDA=\angle IDC$. Then $2\alpha+2\beta+2\gamma+2\delta=2\pi$, so $\alpha+\beta+\gamma+\delta=\pi$.



Take point B' on BC so that $\angle BIB' = \alpha$ and $\angle CIB' = \delta$. Then triangles $BB'I$ and ABI are similar, so

$$\frac{BI}{AB} = \frac{B'I}{AI} = \frac{BB'}{BI}.$$

Also, triangles CIB' and CDI are similar, so

$$\frac{CI}{CD} = \frac{B'I}{DI} = \frac{B'C}{CI}.$$

Then $BI^2 = AB \cdot BB'$, and

$$\frac{AI \cdot BI \cdot CI}{DI} = \frac{AB \cdot B'I \cdot CI}{DI} = AB \cdot B'C,$$

so

$$BI^2 + \frac{AI \cdot BI \cdot CI}{DI} = AB \cdot BB' + AB \cdot B'C = AB \cdot (BB' + B'C) = AB \cdot BC.$$

crazyfehmy

#6 Sep 2, 2011, 11:27 pm • 1

It is equivalent to the well-known identity below:

$$\sin\alpha \sin\gamma + \sin\beta \sin\delta = \sin(\alpha+\beta)\sin(\alpha+\delta)$$

according to the **nsato**'s drawing.

Quick Reply

High School Olympiads

fixed point

Reply

Gleek-00

#1 Aug 31, 2011, 9:59 pm

Let $\triangle ABC$ be an isoceles triangle $(AB = AC)$.

And let P be a point on (BC). Let M be a point on (AC) such that $(MP) \parallel (AB)$, and let N be a point on (AB) such that $(PN) \parallel (AC)$

Show that the bisector of $[MN]$ passes throught a fixed point.

Luis González

#2 Aug 31, 2011, 10:29 pm

Since the quadrilateral $PNAM$ is a parallelogram and $\triangle BNP$ is clearly isosceles with legs $NB = NP$, then we have $AM + AN = AN + NB = AB = \text{const}$. Thus, if D, E denote the midpoints of AC, AB, according to Fixed point, the arcs MN and DE of $\odot(AMN)$ and $\odot(ADE)$ have the same midpoint O, i.e. the circumcenter O of $\triangle ABC$. Thus, the perpendicular bisector of $\overline{MN}$ always passes through O.

Luis González

#3 Sep 2, 2011, 8:40 am

If $\triangle ABC$ is non-isosceles, the perpendicular bisector ℓ of $\overline{MN}$ does not pass through a fixed point but it touches a fixed parabola with focus the projection of the circumcenter O on the A-symmedian and directrix the A-median. Using the same notations of the previous post, $L \equiv DE \cap MN \cap AP$ is the center of the parallelogram $PNAM$. Then by Menelaus' theorem for $\triangle ADE$ cut by $\overline{MLN}$, we get

$$\frac{NE}{MD} = \frac{NA}{AM} \cdot \frac{LE}{DL} = \frac{PM}{PN} \cdot \frac{PB}{PC} = \frac{AB}{AC} \cdot \frac{PC}{PB} \cdot \frac{PB}{PC} = \frac{AB}{AC}$$

Since the circles $\odot(AED)$ and $\odot(AMN)$ meet at A and the center U of the spiral similarity taking $\overline{NE}$ into $\overline{MD}$, it follows that $\frac{UE}{UD} = \frac{NE}{MD} = \frac{AB}{AC} = \frac{AE}{AD} \Longrightarrow AU$ is the A-symmedian of $\triangle ADE \Longrightarrow U$ is the orthogonal projection of O on the A-symmedian s_a of $\triangle ABC$. Reflection V of U about ℓ clearly lies on $\odot(AMN)$ and since $UVMN$ is an isosceles trapezoid with legs $UN = VM$, we have $\angle UAN = \angle VAM$, i.e. AU, AV are isogonal lines WRT $\angle BAC \Longrightarrow V$ moves on the A-median m_a of $\triangle ABC$. Therefore, the perpendicular bisector ℓ of $\overline{UV}$ and $\overline{MN}$ always touches the parabola with focus U and directrix m_a.

Quick Reply

High School Olympiads

Equality for acute triangle, and tangents of circumcircle

Reply

Source: Problem 6 of Russian Regional Olympiad 2011, grade 11

wavelet3000

#1 Sep 1, 2011, 8:07 am

ω is the circumcirle of an acute triangle ABC. The tangent line passing through A intersects the tangent lines passing through points B and C at points K and L, respectively. The line parallel to AB through K and the line parallel to AC through L intersect at point P. Prove that $BP = CP$.
(Author: P. Kozhevnikov)

Luis González

#2 Sep 1, 2011, 9:52 pm

O is the center of ω and $M \equiv AC \cap OL, N \equiv AB \cap OK$ are the midpoints of $AC, AB.$ Since AB is the polar of K with respect to $\omega,$ then $PK \perp ON$ is the polar of N with respect to $\omega.$ Likewise, PL is the polar of M with respect to $\omega.$ Therefore, P is the pole of the A-midline MN with respect to $\omega \Longrightarrow P$ lies on the perpendicular from O to $MN,$ i.e. the perpendicular bisector of $\overline{BC} \Longrightarrow PB = PC.$

Quick Reply

High School Olympiads

Convex Quadrilateral

Locked

Source: 1996 Bulgaria, Problem 5

youarebad

#1 Sep 1, 2011, 9:34 am

A convex quadrilateral $ABCD$ is given for which $\angle ABC + \angle BCD < 180^\circ$. The common point of the lines AB and CD is E. Prove that $\angle ABC = \angle ADC$ if and only if $AC^2 = CD \cdot CE - AB \cdot AE$.

Luis González

#2 Sep 1, 2011, 10:53 am

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=56472
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=220854
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=278455
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http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=352471

High School Olympiads

Question about triangle II

Reply

Source: OBM - 2009

proglote

#1 Sep 1, 2011, 4:50 am

$\triangle ABC$ is given with $\angle A = 120$ and $BC = 12$. Let D and E be the intersection points of the incircle with sides AC and AB respectively. Line DE intersects the circumference with diameter BC at points K and L. Find the distance between the midpoint of KL and the midpoint of BC.

I tried this without success and the official solution is very frustrating, as in "how would I remotely consider that". I would like to see your solutions, because there certainly is a better way :)

Luis González

#2 Sep 1, 2011, 6:20 am • 1

Let the incircle (I) touch BC at F, let M, N denote the midpoints of BC, KL and U, V the orthogonal projections of B, C onto KL. It's clear that MN is the perpendicular bisector of $\overline{KL}$, i.e. $\overline{MN}$ is the median of the right trapezoid $BUVC$. Thus, keeping in mind that $\angle UBA = \angle IAB = \angle IAC = \angle VCA = \frac{1}{2}\angle A$, we have

$$MN = \frac{1}{2}(UB + CV) = \frac{1}{2}\cos\frac{A}{2}(BE + CD) = \frac{1}{2}\cos\frac{A}{2}(BF + CF) =$$

$$= \frac{1}{2}\cos\frac{A}{2}\cdot BC = \frac{12}{2}\cdot\cos 60^\circ = 3.$$

Quick Reply

High School Olympiads

lb lc r

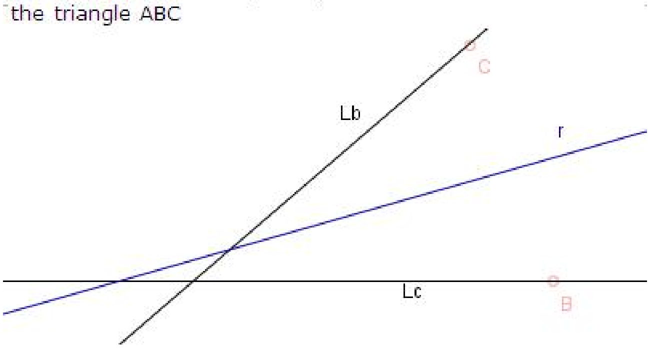
Reply

jrrbc

#1 Aug 8, 2009, 7:50 pm

Attachments:

Given the sidelines **Lb** and **Lc** and the line **r** connecting the incenter **I** and the Gergonne point **Ge**, find the vertices of the triangle ABC



Luis González

#2 Aug 31, 2011, 9:15 pm • 2

Before going into the solution of the problem, we solve two previous constructions.

Construction 1: Given four points M, N, P, Q, no three of them collinear, and a line m through M, constuct the conic $\mathcal{H}$ through M, N, P, Q tangent to m.

Solution: Construct the homologous P', Q', N' of P, Q, N in a circle γ tangent to m through M, which is the center of the homology mapping the wanted conic $\mathcal{H}$ into γ. Three pairs of homologous points (P, P'), (Q, Q') and (N, N') are sufficient to construct the axis e and limit line ℓ of the homology. Construct the pole O' of ℓ WRT γ and the conjugate points K, K' on ℓ WRT γ such that $MK \perp MK'$. This is, construct $U \equiv \ell \cap m$ and the circle (U) with radius UM cuts ℓ at K, K'. Then MK, MK' are parallel to the axes of $\mathcal{H} \Longrightarrow$ lines $O'K, O'K'$ cut γ at A', B' and C', D', i.e. images of the vertices A, B and C, D of $\mathcal{H}$. Thus, it remains to construct the homologous points A, B, C, D of A', B', C', D' under the referred homology to get the axes AB, CD and the center $O \equiv AB \cap CD$ of $\mathcal{H}$. If the limite line ℓ cuts γ (say at two points), then the wanted conic $\mathcal{H}$ is a hyperbola and we might consider the construction of its asymptotes.

Construction 2: Given an ellipse $\mathcal{E}$ with foci D, F and semi-major axis legth a and a line ℓ, which cuts the ellipse, then find the intersections $\mathcal{E} \cap \ell$ with ruler and compass.

Solution: Label $P_1, P_2 \equiv \mathcal{E} \cap \ell$ the desired intersections. Reflections U, V of D about the tangents of $\mathcal{E}$ at P_1, P_2 lie on the focal circle $\mathcal{F} \equiv (F, 2a) \Longrightarrow$ circles $(P_1), (P_2)$ that pass through D, F are internally tangent to $\mathcal{F}$ at U, V. Thus, if D' is the reflection of D about ℓ, then we shall construct the two circles that pass through D, D' and touch $\mathcal{F}$ internally. Draw two arbitrary circles ω_1, ω_2 passing through D, D', then the radical axes τ_1 and τ_2 of $\mathcal{F}, \omega_1$ and $\mathcal{F}, \omega_2$ meet at the radical center R of the pencil of circles through D, D'. Therefore, tangents from R to $\mathcal{F}$ touch $\mathcal{F}$ at $U, V \Longrightarrow P_1 \equiv \ell \cap FU$ and $P_2 \equiv \ell \cap FV$.

Solution to the problem: Clearly, the internal bisector of $\angle(\mathcal{L}_b, \mathcal{L}_c)$ cuts r at the incenter I, thus the incircle (I) of $\triangle ABC$ is that with center I tangent to $\mathcal{L}_b, \mathcal{L}_c$ at Y, Z. Project any line through A (not cutting the incircle) to infinity and the incircle (I) into a circle (I'). If we denote the projected points with primes, then Y, Z go to Y', Z' diametrically opposed in (I'), $\mathcal{L}_b, \mathcal{L}_c$ go to the tangents $\mathcal{L}'_b, \mathcal{L}'_c$ of (I') at Y', Z' and variable tangents $BC \equiv \mathcal{L}_a$ of (I) at X go to variable tangents $\mathcal{L}'_a$ of $(I)'$ at X', cutting $\mathcal{L}'_b, \mathcal{L}'_c$ at C', B'. Thus, Gergonne point $G_e \equiv BY \cap CZ$ of $\triangle ABC$ goes to $G_e{}' \equiv B'Y' \cap C'Z'$ and $P \equiv YZ \cap AX$ goes to $P' \equiv Y'Z' \cap A'X'$

$P = YZ \cap AX$ goes to $P' = Y'Z' \cap A'X'$.

Now, for convenience, we get rid of the primes in the projected figure. Then, we have $PG_e \parallel \mathcal{L}_b \parallel \mathcal{L}_c,$ i.e. $G_eP \perp YZ$ and since the cross ratio (P, X, G_e, A) is harmonic, it follows that $\overline{G_eP} = -\overline{G_eX} \Longrightarrow$

$$\overline{G_eP}^2 = \frac{\overline{XP}^2}{4} = \frac{\overline{PY} \cdot \overline{PZ}}{4} = \frac{\overline{IY}^2 - \overline{PI}^2}{4} \Longrightarrow 4 \cdot \frac{\overline{G_eP}^2}{\overline{IY}^2} + \frac{\overline{PI}^2}{\overline{IY}^2} = 1.$$

This means that G_e moves on the ellipse with pedal circle $(I),$ focal axis YZ and semi-minor axis length $\frac{1}{2}IY.$ Hence, in the original figure, G_e moves on an ellipse $\mathcal{E}$ tangent to $\mathcal{L}_b, \mathcal{L}_c$ at $Y, Z,$ since G_e is always an ordinary point (not at infinity), consequently, G_e will be the intersection $\mathcal{E} \cap r.$ Two arbitrary tangents $\mathcal{L}_1$ and $\mathcal{L}_2$ to (I) determine two points G_1 and G_2 lying on $\mathcal{E},$ namely the Gergonne points of $\triangle(\mathcal{L}_b, \mathcal{L}_c, \mathcal{L}_1)$ and $\triangle(\mathcal{L}_b, \mathcal{L}_c, \mathcal{L}_2).$ Using the **Construction 1**, we locate the axes and foci of the ellipse passing through Y, Z, G_1, G_2 tangent to $\mathcal{L}_b$ and subsequently, using the **Construction 2**, we find the intersections $G_e \equiv \mathcal{E} \cap r \Longrightarrow B \equiv YG_e \cap \mathcal{L}_c$ and $C \equiv ZG_e \cap \mathcal{L}_b,$ which completes $\triangle ABC.$ Note that r and $\mathcal{E}$ have at most 2 intersections, which, in general, yields two possible triangles $\triangle ABC.$

Quick Reply

High School Olympiads

TRIANGLE PROBLEM

Locked

Source: Challenge and Thrill of Pre-College Mathematics

arjun sengupta

#1 Aug 31, 2011, 8:31 pm

In ABC,BD and CE are the bisectors of <B,<C cuttingCA,AB at D,E respectively.If <BDE=24 degree and <CED=18 degree,find the angle of ABC.

Luis González

#2 Aug 31, 2011, 9:01 pm • 1

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=127460
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=68096
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=220396
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=386177

High School Olympiads

Circle (Give the picture, pls !!)

Locked

Source: Bulgaria 1996, problem 2

youarebad

#1 Aug 31, 2011, 11:16 am

Please solve and give the picture for this problem :

The circles k_1 and k_2 with respective centers O_1 and O_2 are externally tangent at the point C, while the circle k with center O is externally tangent to k_1 and k_2. Let l be the common tangent of k_1 and k_2 at the point C and let AB be the diameter of k perpendicular to l. Assume that O and A lie on the same side of l. Show that the lines AO_2, BO_1, l have a common point.

Luis González

#2 Aug 31, 2011, 12:23 pm

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=875026
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=278448
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=412936
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=422580

High School Olympiads

Three nice concurrent lines

Reply

Source: (own)

jayme

#1 Aug 30, 2011, 10:04 pm

Dear Mathlinkers,
1. ABC be a triangle, A'B'C' the median triangle
2. M a point
3. P, Q, R the symmetrics of M wrt A', B', C',
4. (1), (2), (3) the resp. circumcircles of PBC, QCA, RAB
5. A" a point on (1), B" the second point of intersection of A"C with (2), C" the second point of intersection of B"A with (3).

Prouve : A"P, B"Q, C"R are concurrent.

Sincerely
Jean-Louis

Luis González

#2 Aug 30, 2011, 11:16 pm

Let T be the 2nd intersection of the circles $\odot(QCA), \odot(RAB)$. Keeping in mind that $BMCP, CMAQ, AMBR$ are all parallelograms, then simple angle chase gives $\angle(TB,TC) = \angle(MA,MB) - \angle(MA,MC) = \angle(MC,MB) \Longrightarrow T \in \odot(PBC)$. Thus, by the converse of Miquel theorem, it follows that $C'' \in BA''$, i.e. T is the Miquel point of $\triangle A''B''C'' \cup ABC$. If R_A denotes the radius of $\odot(PBC)$, we have

$$\frac{\sin \widehat{PA''B''}}{\sin \widehat{PA''C''}} = \frac{2R_A}{2R_A} \cdot \frac{PC}{PB} = \frac{PC}{PB} = \frac{MB}{MC}$$

By similar reasoning, we obtain the expressions

$$\frac{\sin \widehat{QB''C''}}{\sin \widehat{QB''A''}} = \frac{MC}{MA}, \quad \frac{\sin \widehat{RC''A''}}{\sin \widehat{RC''B''}} = \frac{MA}{MB}$$

$$\Longrightarrow \frac{\sin \widehat{PA''B''}}{\sin \widehat{PA''C''}} \cdot \frac{\sin \widehat{QB''C''}}{\sin \widehat{QB''A''}} \cdot \frac{\sin \widehat{RC''A''}}{\sin \widehat{RC''B''}} = \frac{MB}{MC} \cdot \frac{MC}{MA} \cdot \frac{MA}{MB} = 1$$

By the converse of Trig-Ceva in $\triangle A''B''C''$, we get that PA'', QB'', RC'' concur.

lym

#3 Aug 31, 2011, 2:15 am

By using a basic lemma of mine□we can get T is on the conic which pass through $ABCPQR$.

Quick Reply

High School Olympiads

nice problem

m1234a

#1 Aug 29, 2011, 9:21 am

The tangent line to the circumcircle of triangleABC at point A intersect BC at point P , the angle bisector of $\angle BPA$ intersects AC at E ,AB at D , CD and BE intersect at Q , O is the circumcentre of the circumcircle of ABC , prove : O,Q,P are collinear if and only if $\angle CAB=60$(degrees) ..

Luis González

#2 Aug 30, 2011, 8:16 am

WLOG assume that $\angle B>\angle C,$ thus B lies between C and $P.$ Since the pencil $A(B,C,Q,P)$ is harmonic, then we deduce that AQ is the polar of P WRT (O) (AQ is the A-symmedian of ABC) $\Longrightarrow AQ \perp OP.$ Assume that O,Q,P are collinear $\Longrightarrow PB \cdot PC=PA^2=PQ \cdot PO \Longrightarrow O,Q,B,C$ are concyclic. Further, O is the midpoint of the arc $BQC \Longrightarrow$ OQ,AQ bisect $\angle DQE \ (\star).$ On the other hand, simple angle chase yields

$$\angle ADE=\angle PAD+\angle APD=\angle C+\tfrac{1}{2}(\angle B-\angle C)=\tfrac{1}{2}(\angle B+\angle C)$$

Thus $\triangle ADE$ is isosceles with apex $A,$ i.e. $AD=AE.$ Together with $(\star),$ we deduce that A,D,Q,E are concylic $\Longrightarrow$ $\angle BOC=2\angle A=\angle BQC=180^{\circ}-\angle A \Longrightarrow \angle A=60^{\circ}.$ The proof of converse is immediate.

High School Olympiads

Right angle

Reply

Source: Me (probably discovered before)

borislav_mirchev

#1 Aug 30, 2011, 4:28 am

The quadrilateral ABCD is inscribed in the circle k(O). E is the intersection point of the diagonals. F is the intersection point of the circumcircles of the triangles AOD and BOC. Prove that <EFO is a right angle.

Luis González

#2 Aug 30, 2011, 4:54 am • 1

Inversion with respect to the circumcircle (O,R) of $ABCD$ takes the circles $\odot(ODA)$ and $\odot(OCB)$ into the straight lines DA and $CB,$ respectively $\Longrightarrow P \equiv AD \cap BC$ is the inverse of $F \Longrightarrow F$ lies on $OP.$ Moreover, if OF cuts (O,R) at $U,V,$ we have $R^2=OU^2=OV^2=OF \cdot OP \Longrightarrow$ Cross ratio (U,V,F,P) is harmonic $\Longrightarrow EF$ is the polar of P with respect to $(O,R) \Longrightarrow EF \perp OF,$ as desired.

Quick Reply

High School Olympiads

Antigonal points

Reply

jayme

#1 Aug 28, 2011, 3:57 pm

Dear Mathlinkers,
P and Q are two antigonal points wrt ABC, (1a) the circle passing through P, B, C,
(1'a) the symmetric of (1a) wrt BC ; it passes through Q ;
P' the second point of intersection of AP with (1a),
Q' the second point of intersection of AQ with (1'a).
Prouve that P'Q' is perpendicular to BC.
Sincerely
Jean-Louis

Luis González

#2 Aug 28, 2011, 11:33 pm

Antigonal conjugate Q of P WRT $\triangle ABC$ is the common point of the reflections of $\odot(PBC), \odot(PCA), \odot(PAB)$ about $BC, CA, AB,$ respectively. Hence, by simple angle chase we have $\angle(QC, QQ') = \angle(PA, PC) = \angle(PP', PC),$ $\pmod{\pi}$. Thus, since $\odot(QCQ') \cong \odot(PCP')$ are symmetric about $BC,$ then their chords $\overline{CQ'} \cong \overline{CP'}$ are also symmetric about $BC \Longrightarrow P'Q' \perp BC,$ as desired.

Quick Reply

High School Olympiads

Beautiful Geometry

Reply

Source: Me

borislav_mirchev

#1 Aug 28, 2011, 3:01 am

Trangle ABC with incenter I is inscribed in circle k. AL is the angle bisector of the angle <A (L is on BC). A circle with radii AI intersects k at the points D and E. F is the intersection point of AL and DE. Prove that IL=IF.

Luis González

#2 Aug 28, 2011, 5:12 am • 2

AI cuts the circumcircle k again at the midpoint P of its arc BC and let the tangents from P to (I) cut k again at D', E'. By Poncelet Porism, we get that $D'E'$ is tangent to $(I) \Longrightarrow I$ is also the incenter of $\triangle PD'E' \Longrightarrow AD' = AE' = AI$, i.e. $D \equiv D, E \equiv E'$. Let PD, PE cut BC at U, V. From $PB^2 = PU \cdot PD = PV \cdot PE$, we get that $DUVE$ is cyclic. Thus, if $M \equiv BC \cap DE$, then by simple angle chase we have

$$\angle IPM + \angle IMP = \tfrac{1}{2}(\angle DMU + \angle DPE) + \pi - \angle UVE$$

$$\angle DMU + \angle DPE = (\angle UVE - \angle DUV) + (\angle UVE - \angle DEV) = 2\angle UVE - \pi$$

$$\Longrightarrow \angle IPM + \angle IMP = \tfrac{1}{2}(2\angle UVE - \pi) + \pi - \angle UVE = \frac{\pi}{2}$$

Therefore, $MI \perp LF \Longrightarrow \triangle MFL$ is isosceles with apex $M \Longrightarrow \overline{IF} = -\overline{IL}.$

Quick Reply

High School Olympiads

The inscribed sphere of a tetrahedron

Reply

Source: Serbia and Montanagro 2003 \grade 11\second day

ehsan2004

#1 Aug 3, 2005, 4:48 pm

The inscribed sphere of a tetrahedron $ABCD$ touches ABC, ABD, ACD and BCD at D_1, C_1, B_1 and A_1 respectively. Consider the plane equidistant from A and plane $B_1C_1D_1$ (parallel to $B_1C_1D_1$) and the three planes defined analogously for the vertices B, C, D. Prove that the circumcenter of the tetrahedron formed by these four planes coincides with the circumcenter of tetrahedron of $ABCD$.

Luis González

#2 Aug 28, 2011, 12:29 am • 1

O is the circumcenter of $ABCD$ and $\ell_a, \ell_b, \ell_c, \ell_d$ denote the perpendiculars dropped from O to the faces $DBC, DCA, DAB, ABC,$ respectively. Label δ_A the plane equidistant from A and $B_1C_1D_1$ (parallel to this). Planes $\delta_B, \delta_C, \delta_D$ are denoted analogously. Then $\delta_A, \delta_B, \delta_C$ are the radical planes of the insphere (I) of $ABCD$ and the spheres $(A), (B), (C)$ with zero radii, respectively $\Longrightarrow D' \equiv \delta_A \cap \delta_B \cap \delta_C$ is the radical center of $(A), (B), (C), (I) \Longrightarrow D'$ has equal power with respect to $(A), (B), (C), (I)$, i.e. $AD' = BD' = CD' \Longrightarrow D' \in \ell_d$. Likewise, $A' \equiv \delta_B \cap \delta_C \cap \delta_D$, $B' \equiv \delta_C \cap \delta_A \cap \delta_D$ and $C' \equiv \delta_A \cap \delta_B \cap \delta_D$ lie on ℓ_a, ℓ_b, ℓ_c, respectively. Now, tetrahedra $A_1B_1C_1D_1 \sim A'B'C'D'$ are homothetic and $IA_1 \parallel OA', IB_1 \parallel OB', IC_1 \parallel OC', ID_1 \parallel OD'$ means that I, O are homologous points. Thus, since I is the circumcenter of $A_1B_1C_1D_1$, then O is the circumcenter of $A'B'C'D'$, as desired.

Quick Reply

High School Olympiads

bulgaria 1998

Reply

elegant

#1 Aug 25, 2011, 11:06 am

A convex pentagon ABCDE is inscribed in a circle with radius R. Let r(XYZ) denote the inradius of a triangle XYZ. Prove that (a) cos□CAB + cos□ABC + cos□BCA = 1 + r(ABC)/R; (b) if r(ABC) = r(AED) and r(ABD) = r(AEC), then □ABC=□AED.

(b)?

Luis González

#2 Aug 26, 2011, 12:34 am

The relation (a) is very well known (Hint: Carnot's theorem). Let $r_1, r_2, r_3, r_4, r_5, r_6$ be the inradii of $\triangle ABC, \triangle AED,$ $\triangle BCD, \triangle EDC, \triangle AEC, \triangle ABD,$ respectively. By Mikami and Kobayashi theorem, which is easy to prove using Carnot's theorem, we get $r_2+r_3+r_6=r_1+r_4+r_5.$ If $r_1=r_2$ and $r_5=r_6,$ then $r_3=r_4.$ But, since any scalene triangle is unambiguosly defined by its incircle, one angle and its opposite side, then we deduce that $\triangle BCD \cong \triangle EDC \Longrightarrow$ $BC=ED.$ Then, by the same reason, we have $\triangle ABC \cong \triangle AED.$

Quick Reply

High School Olympiads

geometry

Reply

nama2

#1 Aug 22, 2011, 6:29 pm

let ABC be a triangle.An arbitary line cuts . the lines BC , CA , AB at M ,N ,P respectively . let X ,Y , Z be repectively the centroids of triangle ANP ,BPM , CMN .prove that

$$S_{XYZ} = \frac{2}{9} S_{ABC}$$

yetti

#2 Aug 23, 2011, 4:24 am • 2

Suppose $\triangle ABC$ is fixed and line $k \equiv MNP$ with fixed direction moves with time t, in {u,v)-coordinate system, according to $v = mu + pt + q$. Centroids X, Y, Z of $\triangle ANP, \triangle BPM, \triangle CMN$ then move on their fixed A-, B-, C-median lines with constant velocities. Area of $\triangle XYZ$ equal to $S_{XYZ} = \frac{u_X(v_Y - v_Z) + u_Y(v_Z - v_X) + u_Z(v_X - v_Y)}{2}$ is therefore at most quadratic function of time t. When the line k passes through A, then $A \equiv N \equiv P \equiv X$, B-, C-medians of $\triangle BPM, \triangle CMN$ meet at the midpoint of $AM \Longrightarrow YZ = \frac{1}{3}BC$ and distance $\text{d}(X, YZ) = \frac{2}{3}\text{d}(A, BC) \Longrightarrow$ $S_{XYZ} = \frac{1}{2}YZ \cdot \text{d}(X, YZ) = \frac{2}{9} \cdot \frac{1}{2}BC \cdot \text{d}(A, BC) = \frac{2}{9}S_{ABC}$. Likewise, $S_{XYZ} = \frac{2}{9}S_{ABC}$, when the line k passes through B or $C \Longrightarrow S_{XYZ}$ is constant independent of time t. This results holds for any slope m of the line k (different from the slopes of BC, CA, AB).

This post has been edited 1 time. Last edited by yetti, Nov 22, 2011, 6:20 am

Luis González

#3 Aug 23, 2011, 11:32 am • 3

Let D, E, F, H be the midpoints of PB, NC, AP, MC, respectively, then we have

$[DFEH] = [FEP] + [PEH] + [PHD] = \frac{1}{2}[AEP] + [CHE] + [DHB]$ (1)

$\frac{1}{2}[PECB] = [CHE] + [DHB]$ (2)

From (1) and (2), we get $[DFEH] = \frac{1}{2}[PECB] + \frac{1}{2}[AEP] = \frac{1}{2}[ABC]$ (3)

On the other hand, $\frac{MY}{MD} = \frac{MZ}{ME} = \frac{NX}{NF} = \frac{NZ}{NH} = \frac{2}{3}$ yields $ZX = \frac{2}{3}FH, ZY = \frac{2}{3}DE$ and $ZX \parallel FH, ZY \parallel DE$.

$\Longrightarrow [XYZ] = \frac{1}{2}ZX \cdot ZY \cdot \sin \angle XZY = \frac{1}{2} \cdot \frac{4}{9}DE \cdot FH \cdot \sin \angle(DE, FH)$

But, from (3), we get $[DFEH] = \frac{1}{2}DE \cdot FH \cdot \sin \angle(DE, FH) = \frac{1}{2}[ABC]$

$\Longrightarrow [XYZ] = \frac{4}{9} \cdot \frac{1}{2}[ABC] = \frac{2}{9}[ABC]$.

nama2

#4 Aug 24, 2011, 11:14 am

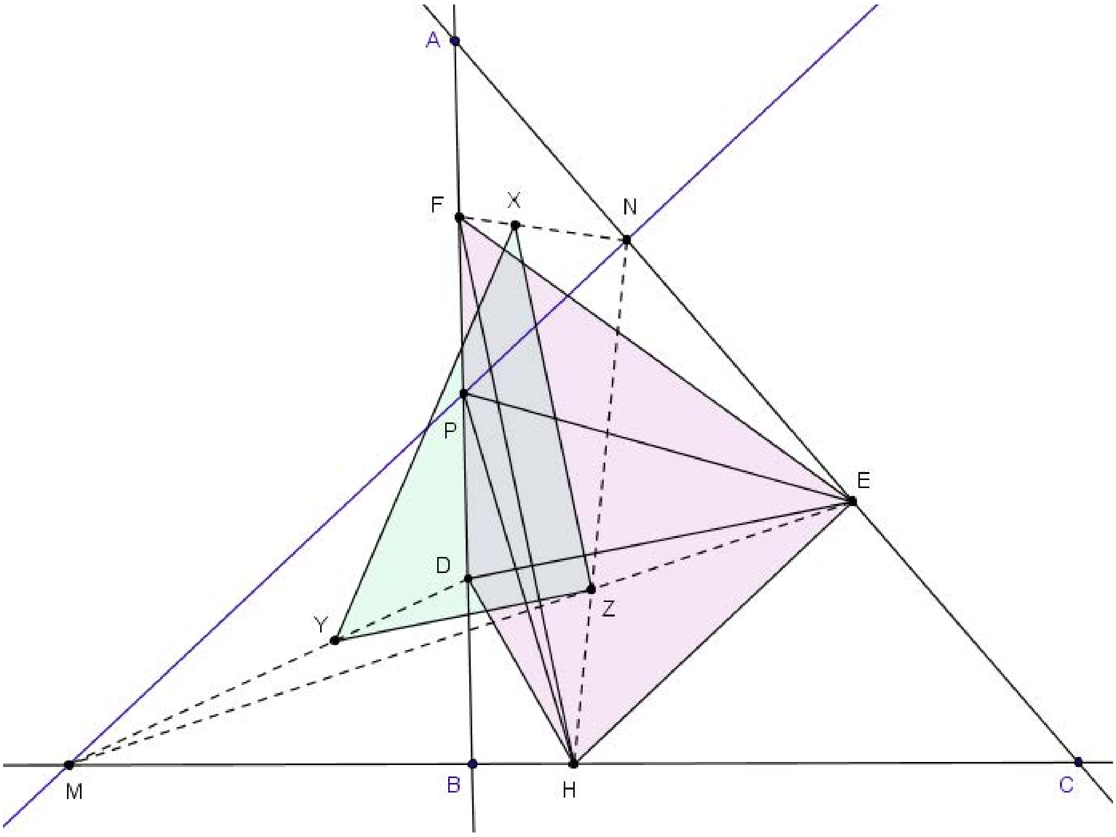
arbitary line .so DFEH quadrilateral can not exist .

Luis González

#5 Aug 25, 2011, 1:53 am • 1

nama2, the addition of areas depends on the configuration. Please, consider the following sketch

Attachments:



AnhIsGod

#6 Nov 21, 2011, 6:53 pm

problem in M&Y magazine.

Quick Reply

High School Olympiads

equilateral

Reply

r1234
#1 Aug 24, 2011, 4:56 pm

In a triangle the kosnita point lies on the euler line.Show that the triangle is isosceles.(can it be extended to equilateral??)

Luis González
#2 Aug 25, 2011, 12:13 am • 2

Let O,H,N,K be the circumcenter, orthocenter, 9 point center and Kosnita point of $\triangle ABC$ (the isogonal conjugate of N). Assume that $K \in OH.$ Isogonal conjugation WRT $\triangle ABC$ takes O,H into each other and N into $K \Longrightarrow O,H,K$ lie on the isogonal conjugate of the euler line $OH,$ i.e. a rectangular circum-hyperbola $\mathcal{K}.$ But the collinearity of O,H,K implies that $\mathcal{K}$ degenerates into either $AH \cup BC, BH \cup CA$ or $CH \cup AB \Longrightarrow$ either $b=c, c=a$ or $a=b.$

Alternate approach

RSM
#3 Aug 25, 2011, 12:41 am

It can be taken to equilateral very easily. Just add the condition that the Kosnita point doesn't lie inside the segment OH where O,H are the circumcenter and orthocenter of ABC and its proof is very simple. So I am leaving it.

Quick Reply

High School Olympiads

problem

Locked

gold46

#1 Aug 22, 2011, 1:10 am

Let D and E points on sides AB and BC of $\triangle ABC$ such that $ADEC$ is concyclic quadrilateral. Let ω be circle tangents AB, BC and circumcircle of $ADEC$, externally. If M is the tangency point of ω and circumcircle of $ADEC$, prove that incenter of $\triangle ABC$ lies on bisector of $\angle AMC$.

Luis González

#2 Aug 22, 2011, 2:09 am

gold46, please, next time use meaningful subjects. This is a rather old problem

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=6086
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=41667
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=253207
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=407366

High School Olympiads

Constant area of a triangle.

Reply

Virgil Nicula

#1 Aug 18, 2011, 6:49 pm

Let ABC be a triangle, a line d so that $d\parallel BC$, $A\not\in d$ and a mobile point $M\in d$.
Denote $N\in AC$ for which $NB\parallel MA$. Prove that the area of the triangle CMN is constant.

Luis González

#2 Aug 22, 2011, 12:37 am • 1

Let $P \equiv d \cap AB, Q \equiv d \cap AC$ and $D \equiv AM \cap BC.$ Then we have

$$\frac{[CMN]}{[CMA]}=\frac{CN}{CA} \ , \ \frac{[CMA]}{[CPA]}=\frac{QM}{QP} \Longrightarrow [CMN]=[CPA] \cdot \frac{QM}{QP} \cdot \frac{CN}{CA}$$

But $d \parallel BC, DA \parallel NB$ yield $\frac{QM}{QP}=\frac{CD}{CB}=\frac{CA}{CN} \Longrightarrow [CMN]=[CPA]=\text{const}$

Quick Reply

High School Olympiads

Three colinear points:

Akhenaten

#1 Aug 21, 2011, 8:43 pm

Let ABC be a triangle and D and E are points on $[AB]$ and $[AC]$, respectively, as (DE) is parallel to (BC). Let P be any point inside the triangle ADE, and F and G are the intersections of (DE) with (BP) and (CP) respectively. Q is the second intersection of the circumcircles to triangles CEO and PFE.
Prove that A, P and Q are collinear.

Luis González

#2 Aug 21, 2011, 10:51 pm

There's definitely a typo here, as the point O is not previously defined. In fact, I think Q is the second intersection of the circumcircles of $\triangle PGD$ and $\triangle PFE$.

If AP cuts DE and BC at M, N, then from $DE \parallel BC$ we deduce that $\frac{MF}{MG} = \frac{NB}{NC} = \frac{MD}{ME}$, that can be written as $MG \cdot MD = MF \cdot ME \Longrightarrow M$ has equal power WRT $\odot(PGD)$ and $\odot(PFE) \Longrightarrow APM$ is therefore the radical axis of $\odot(PGD)$ and $\odot(PFE) \Longrightarrow A, P$ and Q are collinear.

High School Olympiads

A point on the Soddy line

jayme
#1 Aug 20, 2011, 7:18 pm

Dear Mathlinkers,
ABC a triangle,
DEF the contact triangle,
(1), (2), (3) the circles centered at A, B, C and passing resp. through E and F, F and D, D and E,
(4) the external Soddy circle,
P, Q, R the contact of (4) wrt (1), (2), (3)
and XYZ the tangential triangle of PQR.
Prouve : the intersection of AX and DP lie on the Soddy line.
Sincerely
Jean-Louis

Luis González
#2 Aug 21, 2011, 1:06 am • 1

Relabel $(A),(B),(C)$ the circles centered at A,B,C passing through $E,F,D,$ respectively and (S) the external Soddy circle. Let EF,FD,DE cut BC,CA,AB at $U,V,W,$ respectively. From the topic On the line through Incentre and Gergonne Point (see post #5), we know that S lies on the Soddy line of $\triangle ABC$ and that $U \equiv BC \cap EF \cap RQ$ is the radical center of $(A),(S)$ and the incircle (I) of $\triangle ABC \Longrightarrow UD = UP.$ Thus, if $\mathcal{C}_A$ is the circle tangent to UD,UP through $D,P,$ then D,P are the exsimilicenters of $(I) \sim \mathcal{C}_A$ and $(S) \sim \mathcal{C}_A \Longrightarrow DP$ passes through the exsimilicenter K of $(I) \sim (S).$ Analogously, EQ,FR pass through $K.$ On the other hand, since $V \equiv AE \cap XQ$ and $W \equiv AF \cap XR,$ then $\triangle AEF$ and $\triangle XQR$ are perspective through the trilinear polar $\overline{UVW}$ of the Gergonne point of $\triangle ABC.$ Hence, by Desargues theorem, the lines AX,EQ,FR concur at $K,$ i.e. AX,DP and the Soddy line SI concur at the exsimilicenter K of $(I) \sim (S),$ which is the 1st Eppstein point of $\triangle ABC.$

jayme
#3 Aug 21, 2011, 12:27 pm

Dear Luis and Mathlinkers,
thank you for your suggestion to consider the circle Ca which will made my approach more easy.
This circle is a gate...
This point K has been discovered by Adrian Oldknow in 1996 and I don't understand why he is never mentioned or associated with Eppstein. Do you have an explication?
Sincerely
jean-Louis

Luis González
#4 Aug 21, 2011, 10:23 pm

Dear Jean-Louis, according to Mathworld this point was originally called the inner Oldknow point (1996), the perspector of ABC and the tangential triangle of the outer Soddy triangle. Surprisingly, Eppstein (2001) missed the fact that his 1st Eppstein point (the perspector of the intouch triangle of ABC and the outer Soddy triangle) coincides with the inner Oldknow point.

High School Olympiads

Googo's theorem

littletush

#1 Aug 19, 2011, 12:38 pm

Circle O is the incirle of quadrilateral $ABCD$, line OX is perpendicular to AC at X. Prove that $\angle BXC = \angle DXC$.

Luis González

#2 Aug 21, 2011, 8:11 am

Let P,Q,R,S be the tangency points of the incircle (O) with $AB,BC,CD,DA,$ respectively. Then PS and QR are the polars of A and C with respect to $(O) \Longrightarrow U \equiv PS \cap QR$ is the pole of AC with respect to $(O) \Longrightarrow U \in OX$ and the pencil $C(Q,R,U,X)$ is harmonic. Hence, if $Y \equiv AC \cap BD,$ then the cross ratio (B,D,U,Y) is hamonic. Together with $XY \perp XU,$ we deduce that OX, AC bisect $\angle BXD \Longrightarrow \angle BXC = \angle DXC.$

High School Olympiads

Skew edges in a tetrahedron are perpendicular

Reply

Source: 2008 St. Petersburg Math. Olympiad, Round II, Grade 11

Shu

#1 Aug 19, 2011, 9:41 am

Each of the faces of tetrahedron $ABCD$ is an acute triangle. In triangles ABC and ABD, let K and L, respectively, denote the feet of the altitudes from A. Suppose that C, K, L and D are concyclic. Prove that AB and CD are perpendicular.

Luis González

#2 Aug 19, 2011, 10:57 am

If P is the orthogonal projection of A on the plane BCD, then $PK \perp BC$ and $PL \perp BD$. Let B' be the antipode of B WRT the circumcircle of $\triangle BDC$, thus from the cyclic quadrilateral $CKLD$, we get $\angle BKL = \angle BDC$ and $\angle BLK = \angle BCD$. Therefore, $B'C \perp BC$ and $B'D \perp BD$ yield $BLPK \sim BCB'D \Longrightarrow \angle KBP = \angle DBB' \Longrightarrow P$ lies on the B-altitude of $\triangle BCD$. Consequently

$$AD^2 - AC^2 = PD^2 - PC^2 = BD^2 - BC^2 \implies AB \perp CD.$$

Quick Reply

High School Olympiads

triangle

Reply

ngocduy

#1 Aug 19, 2011, 7:13 am

Give a triangle ABC ($AB \neq BC, BC \neq CA, CA \neq AB$). Draw $BH \perp AC$ $(H \in AC)$. $B_1 \in AC$ where $B_1A = B_1C$. Draw bisectrix BD of B. E is the midpoint of BH. $EB_1 \cap BD = M$. N is the midpoint of BD. Prove $\widehat{BCN} = \widehat{DCM}$

Luis González

#2 Aug 19, 2011, 8:14 am

BD cuts the circumcircle of $\triangle ABC$ again at the midpoint P of its arc AC. $B_1P \perp AC$ is the perpendicular bisector of $\overline{AC}$. Since E is the midpoint of $\overline{BH}$ and $B_1P \parallel BH$, it follows that the pencil $B_1(B,H,E,P)$ is harmonic $\Longrightarrow$ Cross ratio (B,D,M,P) is harmonic. But, PC is tangent to $\odot(BCD)$, due to $\angle CBP = \angle PCA = \frac{1}{2}\angle ABC$, thus CM is the C-symmedian of $\triangle BCD \Longrightarrow CM, CN$ are isogonals WRT $\angle BCD$.

ngocduy

#3 Aug 19, 2011, 10:25 am

Thanks, But I don't know what the C-symmedian of a triangle be. Help me please

Luis González

#4 Aug 19, 2011, 10:43 am

In the triangle ABC with circumcircle (O), let the tangents of (O) at B,C meet at A'. Define B',C' similarly. Then AA',BB',CC' are known as the symmedians of ABC. They are the reflections of the medians of ABC about their corresponding angle bisectors, thus they concur at the isogonal conjugate of the centroid of ABC. Also, note that if B'C' cut BC at T, then AA' is the polar of T with respect to (O). Hence B,C,T and the intersection of AA' with BC are harmonically separeted.

ngocduy

#5 Aug 19, 2011, 3:50 pm

OK. Thanks

Quick Reply

High School Olympiads

A triad of circles tangent to the circumcircle

Luis González

#1 Jul 23, 2011, 11:33 am • 1

$\triangle ABC$ is scalene with circumcircle (O). D,E,F are the midpoints of BC,CA,AB and let $\omega_A,\omega_B,\omega_C$ be the circles tangent to BC,CA,AC through D,E,F and the arcs BC,CA,AB of (O) (not containing A,B,C). Then prove the following propositions:

1) τ_A is the common external tangent of $\omega_B,\omega_C,$ such that their centers O_B,O_C and A lie on the same side of τ_A. Define similarly the lines τ_B,τ_C. Then τ_A,τ_B,τ_C bound a triangle $\triangle A_0B_0C_0$ homothetic to $\triangle ABC$ with homothety coefficient $-\frac{1}{4}$ and homothetic center on the line connecting the incenter I and centroid G of $\triangle ABC$.

2) Let $\delta_{BC},\delta_{CA},\delta_{AB}$ denote the lengths of the common external tangents between $(\omega_B,\omega_C),(\omega_C,\omega_A)$ and (ω_A,ω_B). Then $\delta_{BC}=\delta_{CA}=\delta_{AB}=\frac{1}{4}(a+b+c)$.

3) The radical center of $\omega_A,\omega_B,\omega_C$ is the midpoint between the incenter I and Mittenpunkt of $\triangle ABC$.

4) The Apollonius circle ω externally tangent to $\omega_A,\omega_B,\omega_C$ is also tangent to the incircle of $\triangle ABC$ through its Feuerbach point F_e. Its center I_0 divides the segment $\overline{IN}$ connecting the incenter I and 9-point center N of $\triangle ABC$ in the ratio $2:1$, and its radius ϱ equals a third of the sum of the inradius and circumradius of $\triangle ABC$.

5) If S_1,S_2,S_3 denote the insimilicenters of the pairs $(\omega_B,\omega_C),(\omega_C,\omega_A),(\omega_A,\omega_B)$, then the lines AS_1,BS_2,CS_3 concur at a point lying on the diacentral line IO.

P.S. My proofs to propositions 1,2,3 and 4 can be seen in the paper Luis González, On a triad of circles tangent to the circumcircle and the sides at their midpoints, Forum Geometricorum. Proposition 5 is an additional result not included in the referred article.

Luis González

#2 Aug 19, 2011, 6:28 am

> *Quote:*
> **5)** If S_1,S_2,S_3 denote the insimilicenters of the pairs $(\omega_B,\omega_C),(\omega_C,\omega_A),(\omega_A,\omega_B)$, then the lines AS_1,BS_2,CS_3 concur at a point lying on the diacentral line IO.

We use barycentric coordinates with respect to $\triangle ABC$. Thus, coordinates of the midpoints D,E,F of $\overline{BC},\overline{CA},\overline{AB}$ and the midpoints P,Q,R of the circumcircle arcs $BC,CA,AB,$ are given by

$$D\ (0:1:1)\ ,\ E\ (1:0:1)\ ,\ F\ (1:1:0)$$

$$P\ (-a^2:b(b+c):c(b+c))$$

$$Q\ (a(c+a):-b^2:c(c+a))$$

$$R\ (a(a+b):b(a+b):-c^2)$$

Coordinates of the midpoints O_B,O_C of $\overline{EQ},\overline{FR}$ are then

$$O_B\ (3a^2+c^2+4ac-b^2:-2b^2:3c^2+a^2+4ac-b^2)$$

$$O_C\ (3a^2+b^2+4ab-c^2:3b^2+a^2+4ab-c^2:-2c^2)$$

If R_C,R_B denote the radii of $\omega_B,\omega_C,$ we have then

$$\frac{R_C}{R_B}=\frac{RF}{QE}=\frac{AF\cdot\tan\frac{C}{2}}{AE\cdot\tan\frac{B}{2}}=\frac{c\cdot\tan\frac{C}{2}}{b\cdot\tan\frac{B}{2}}=\frac{c(c+a-b)}{b(a+b-c)}$$

Thus, after simplifications, the coordinates of the point S_1 dividing $\overline{O_C O_B}$ in the ratio $\frac{O_c S_1}{S_1 O_B}=\frac{R_C}{R_B}$ can be written as

$$S_1\left(f(a,b,c):\frac{b(c+a+3b)}{c+a-b}:\frac{c(a+b+3c)}{a+b-c}\right)$$

Cyclic permutations of a,b,c give the coordinates of S_2,S_3. Hence, we conclude that $\triangle ABC$ and $\triangle S_1S_2S_3$ are perspective through a point lying on the diacentral line of $\triangle ABC$, namely

$$X_{3361}\left(\frac{a(b+c+3a)}{b+c-a}:\frac{b(c+a+3b)}{c+a-b}:\frac{c(a+b+3c)}{a+b-c}\right)$$

P.S. I'd be very grateful if anyone could provide a simpler solution to this proposition.

jayme

#3 Aug 19, 2011, 12:49 pm

Dear Luis and Mathlinkers,
nice article dear Luis...
Only for make a link with my article
http://perso.orange.fr/jl.ayme vol. 4 La promesse-le tour-le prestige or magic geometry...
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Collinear problem

Source: China South East Mathematical Olympiad 2011

Issl

#1 Aug 18, 2011, 5:05 pm

In triangle ABC , AA_0,BB_0,CC_0 are the angle bisectors , $A_0,B_0,C_0$are on sides BC,CA,AB, draw $A_0A_1//BB_0,A_0A_2//CC_0$,A_1 lies on AC ,A_2 lies on AB , A_1A_2 intersects BC at A_3.B_3 ,C_3 are constructed similarly.Prove that : A_3,B_3,C_3 are collinear.

Luis González

#2 Aug 18, 2011, 10:53 pm

The result is still true for any P-cevian triangle $\triangle A_0B_0C_0.$ Let A_0A_1 cut B_0C_0 at $U.$ From the harmonic pencil $B_0(A_0,C_0,P,A),$ we deduce that A_1 is midpoint of $\overline{UA_0}.$ Similarly, if A_0A_2 cuts B_0C_0 at $V,$ then A_2 is midpoint of $\overline{VA_0}$ $\Longrightarrow A_1A_2$ is the A_0 -midline of $\triangle A_0UV.$ If $X_0 \equiv B_0C_0 \cap BC,$ then A_3 is midpoint of $\overline{A_0X_0}.$ Defining Y_0,Z_0 on CA,AB similarly as $X_0,$ we get that B_3,C_3 are midpoints of $\overline{B_0Y_0},\overline{C_0Z_0}.$ X_0,Y_0,Z_0 are collinear on the trilinear polar τ of P WRT $\triangle ABC,$ hence A_3,B_3,C_3 lie on the Newton's line of the complete quadrangle bounded by the sidelines of $\triangle A_0B_0C_0$ and $\tau,$ i.e. $\overline{A_3B_3C_3}$ is the trilinear polar of $P^2.$

erfan_Ashorion

#3 Oct 27, 2011, 2:01 am

oh i have a solution but it is 11:25 in iran and i want to go sleep

but i say a small of my solution and compleat it tomorrow!

i proof that :

$$\frac{BA3}{A3C}=\frac{BC0.CB0}{(AC-BC0).(AB-CB0)}$$

continuation of it tomorrow!

High School Olympiads

mixtilinear property

Locked

hungnsl
#1 Aug 18, 2011, 8:54 pm

Let points A and B lie on the circle (O), and let C be a point inside the circle. Suppose that there is a circle tangent to segments AC,BC and circle (O) at P,R,Q, respectively. Denote by I the incenter of triangle ABC. Prove that APIQ and BRIQ are cyclic.

Luis González
#2 Aug 18, 2011, 9:51 pm

This configuration and its variations are already discussed, e.g.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=407366
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=336809
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=253207
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=41667

High School Olympiads

Spherical Triangles

Reply

Source: USAMO 1979 Problem 2-edited to change problem statement

Binomial-theorem

#1 Aug 17, 2011, 1:46 am

Let S be a great circle with pole P. On any great circle through P, two points A and B are chosen equidistant from P. For any *spherical triangle* ABC (the sides are great circles ares), where C is on S, prove that the great circle are CP is the angle bisector of angle C.

Note. A great circle on a sphere is one whose center is the center of the sphere. A pole of the great circle S is a point P on the sphere such that the diameter through P is perpendicular to the plane of S.

This post has been edited 1 time. Last edited by Binomial-theorem, Aug 28, 2011, 5:30 am

Luis González

#2 Aug 17, 2011, 11:17 am

Reflection C' of C about N also lies on the equator and since there are infinitely many great circles through C, C', then we deduce that C, C', A and C, C', B lie on a great circle, respectively. $NC = NC', NA = NB$ and $\measuredangle ANC = \measuredangle BNC'$ imply that the spherical triangles ANC and BNC' are congruent by SAS criterion $\Longrightarrow$ N-spherical altitudes of ANC and BNC' are congruent, i.e. N is equidistant from the great circles AC and $BC' \Longrightarrow CN$ bisects the spherical lune formed by the great circles CB, CA.

Quick Reply

High School Olympiads

USAMO 1983 Problem 4 - Constructing altitude of terahedron

Reply

Source: USAMO 1983 Problem 4

Binomial-theorem

#1 Aug 17, 2011, 3:54 am

Six segments S_1, S_2, S_3, S_4, S_5, and S_6 are given in a plane. These are congruent to the edges AB, AC, AD, BC, BD, and CD, respectively, of a tetrahedron $ABCD$. Show how to construct a segment congruent to the altitude of the tetrahedron from vertex A with straight-edge and compasses.

Luis González

#2 Aug 17, 2011, 7:30 am

Label the edges $BC = a, CA = b, AB = c, DA = d, DB = e, DC = f.$ Construct in the plane ABC triangles $\triangle UAB$ and $\triangle VAC$, such that $AU = AV = d, UB = e$ and $VC = f$. Then the projections M, N of U, V on AB, AC coincide with the projections of D on AB, AC and $P \equiv UM \cap VN$ is the orthogonal projection of D on the plane ABC. This follows from the fact that $\triangle UAB$ and $\triangle VAC$ are obtained by rotating the faces $\triangle DAB$ and $\triangle DAC$ about AB, AC. Thus if we construct a right triangle with hypotenuse equal to UM and one leg equal to MP, then the other leg gives the measure of the D-altitude of $ABCD$.

Quick Reply

High School Olympiads

Q and R are collinear, maximize \frac{1}{PQ}+\frac{1}{PR}

Reply

Source: USAMO 1979 Problem 4-edited to proper wording

Binomial-theorem

#1 Aug 17, 2011, 1:54 am

Show how to construct a chord BPC of a given angle A through a given point P such that $\dfrac{1}{BP}+\dfrac{1}{PC}$ is a maximum.

This post has been edited 1 time. Last edited by Binomial-theorem Aug 28, 2011, 5:37 am

Luis González

#2 Aug 17, 2011, 5:50 am

Perform the inversion with center P and radius $\overline{PO}$. Lines OA, OB go to the circles $(O_1), (O_2)$ passing through P, O and the line QR cuts $(O_1), (O_2)$ again at the inverses Q', R' of Q, R. Hence

$$\frac{1}{PQ}+\frac{1}{PR}=\frac{PQ'+PR'}{PO^2}=\frac{Q'R'}{PO^2}$$

Thus, it suffices to find the line through P that maximizes the length of the segment $\overline{Q'R'}$. If M, N are the midpoints of PQ', PR', i.e. the projections of O_1, O_2 onto QR, then from the right trapezoid O_1O_2NM, we deduce that $O_1O_2 \ge MN = \frac{1}{2}Q'R'$. Consequently, $2 \cdot O_1O_2$ is the greatest possible length of $Q'R'$, which obviously occurs when O_1O_2NM is a rectangle. Hence, Q, R are the intersections of OA, OB with the perpendicular to PO at P.

Virgil Nicula

#3 Aug 17, 2011, 8:52 am

See **here** two methods.

Quick Reply

High School Olympiads

Quadratic equation with geometric taste

Reply

Source: Mathesis 1890

Luis González

#1 Apr 13, 2009, 1:40 am • 1

Solve the equation

$$\sqrt{abx(x-a-b)}+\sqrt{bcx(x-b-c)}+\sqrt{cax(x-c-a)} = \sqrt{abc(a+b+c)}$$

Solution

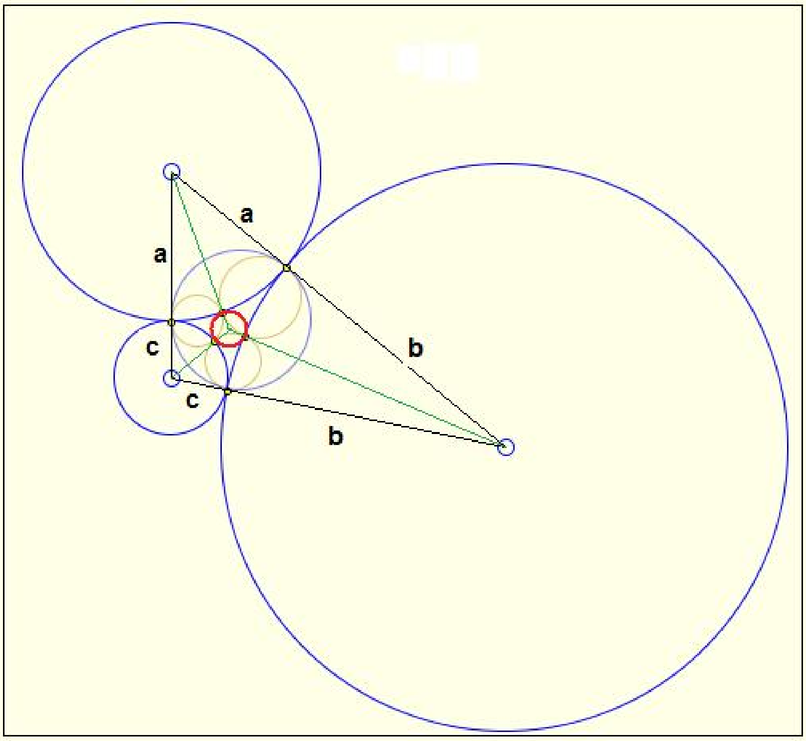
spanferkel

#2 Oct 12, 2009, 9:29 pm • 3

What a nice example of how geometry can facilitate things which would be arkward with algebra or calculus! I've found it.

hint: Herons formula

Attachments:



Luis González

#3 Aug 17, 2011, 12:05 am • 1

Thank you spanferkel for your nice approach. Now, I present a detailed solution

$$\alpha = \sqrt{bcx(x-b-c)}\ ,\ \beta = \sqrt{cax(x-c-a)}$$

$$\gamma = \sqrt{abx(x-a-b)}\ ,\ \delta = \sqrt{abc(a+b+c)}$$

By Heron's formula, we figure out that the numbers $\alpha, \beta, \gamma, \delta$ represent twice the area of the triangles with side lengths $(b+c, x-b, x-c), (c+a, x-c, x-a), (a+b, x-a, x-b), (b+c, c+a, a+b)$, respectively. Now, we consider the triangle $\triangle ABC$ with side lengths $(b+c, c+a, a+b)$. From $\alpha+\beta+\gamma=\delta$, we deduce that there exists a point P, such that $PA=x-a, PB=x-b, PC=x-c$. Using the Briggs formulas, we get

$$\sin\tfrac{1}{2}\widehat{CAP}=\sqrt{\frac{a(x-a-c)}{(x-a)(c+a)}}\ ,\ \cos\tfrac{1}{2}\widehat{CAP}=\sqrt{\frac{cx}{(x-a)(c+a)}}$$

$$\sin\tfrac{1}{2}\widehat{BAP}=\sqrt{\frac{a(x-a-b)}{(x-a)(a+b)}}\ ,\ \cos\tfrac{1}{2}\widehat{BAP}=\sqrt{\frac{bx}{(x-a)(a+b)}}$$

$$\sin\tfrac{1}{2}\widehat{CAB}=\sqrt{\frac{bc}{(a+b)(a+c)}}$$

Substituting these values into the trigonometric identity

$$\sin\tfrac{1}{2}\widehat{CAB}=\sin\tfrac{1}{2}\widehat{CAP}\cdot\cos\tfrac{1}{2}\widehat{BAP}+\cos\tfrac{1}{2}\widehat{CAP}\cdot\sin\tfrac{1}{2}\widehat{BAP}$$

yields $bc(x-a)=b\beta+c\gamma$. By similar reasoning, we get the cyclic expressions $ca(x-b)=c\gamma+a\alpha$ and $ab(x-c)=a\alpha+b\beta$. From these three we obtain

$$x(bc+ca+ab)-abc=2a\alpha=2a\sqrt{bcx(x-b-c)}$$

$$[(bc+ca+ab)^2-4bca^2]x^2+2abc(bc+ca+ab)x+a^2b^2c^2=0$$

$$[x(bc+ca+ab)+abc]^2=4\delta^2x^2 \implies$$

$$x=-\frac{abc}{ab+bc+ca\pm 2\delta}$$

Quick Reply

High School Olympiads

minimum of side length

Reply

ivanbart-15
#1 Aug 15, 2011, 11:20 pm

In right angled $\triangle ABC$ is given $\angle ACB = 90^\circ$, $\angle ABC = 30^\circ$ and $AC = 1$ and an equilateral triangle is inscribed such that every side of $\triangle ABC$ contents one of its vertices. Find, with proof, a minimum of the side length in equilateral triangle and find out for which value it's attained.

This post has been edited 1 time. Last edited by ivanbart-15, Aug 16, 2011, 6:12 pm

Luis González
#2 Aug 16, 2011, 6:08 am • 1

According to IMO Shortlist 1993 p13, the side length of this equilateral triangle is

$$d=\frac{2\sqrt{2}[ABC]}{\sqrt{a^2+b^2+c^2+4\sqrt{3}[ABC]}}=\frac{2\sqrt{2}\cdot\frac{\sqrt{3}}{6}}{\sqrt{1+\frac{1}{3}+\frac{4}{3}+4\sqrt{3}\cdot\frac{\sqrt{3}}{6}}}=\frac{1}{\sqrt{7}}$$

ivanbart-15
#3 Aug 16, 2011, 7:17 pm

your solution is correct, but I made a little mistake: $AC=1$, not $BC=1$. so the minimum is $DE=\sqrt{\frac{3}{7}}$. Did you find for which value the minimum is attained?

My proof:

$\triangle DEF$ is inscribed in right angled $\triangle ABC$ such that $D\in BC, E\in AC, F\in AB$. Let G be a point on AB such that $DG \parallel AC$ and G lies on a circumcircle of $\triangle DEF$. $\angle BGD = 90^\circ - 30^\circ = 60^\circ$. $\angle DGE = \angle DFE = 60^\circ$, which gives that $\angle AGE = 60^\circ$, too. So, $\triangle EAG$ is equilateral.
Let H be a point on AC, such that GH is altitude of $\triangle EAG$ and let $AE = x$,then $EC = 1-x$. $DC = GH = \sqrt{3}\cdot\frac{x}{2}$.
From $\triangle EDC$, we obtain the following:
$DE^2 = (\frac{\sqrt{7}}{2}x - \frac{2}{\sqrt{7}})^2 + \frac{3}{7} \geq \frac{3}{7}$. So $DE = \sqrt{\frac{3}{7}}$ and it's attained for $x = \frac{4}{7}$.

Quick Reply

High School Olympiads

Q is on the smaller arc AC of circumcircle of ABC

Reply

Source: Vietnam Northern Summer Camp of Mathematics, 2011 - P3

Amir Hossein

#1 Aug 15, 2011, 4:12 pm

Given an acute triangle ABC such that $\angle C < \angle B < \angle A$. Let I be the incenter of ABC. Let M be the midpoint of the smaller arc BC, N be the midpoint of the segment BC and let E be a point such that $NE = NI$. The line ME intersects circumcircle of ABC at Q (different from A, B, and C). Prove that

(i) The point Q is on the smaller arc AC of circumcircle of ABC.

(ii) $BQ = AQ + CQ$

Luis González

#2 Aug 15, 2011, 10:43 pm

> *Amir Hossein wrote:*
>
> Given an acute triangle ABC such that $\angle C < \angle B < \angle A$. Let I be the incenter of ABC. Let M be the midpoint of the smaller arc BC, N be the midpoint of the segment BC and let E be a point such that $NE = NI$. The line ME intersects circumcircle of ABC at Q (different from A, B, and C). Prove that
>
> **(i)** The point Q is on the smaller arc AC of circumcircle of ABC.
>
> **(ii)** $BQ = AQ + CQ$

Amir, shouldn't the point E be specifically the reflection of I about N ?. See Prove that NB=NA+NC.

Quick Reply

High School Olympiads

Tetrahedron and sphere - TT 2009 Senior-A3

Reply

Amir Hossein
#1 Sep 3, 2010, 3:13 pm

Every edge of a tetrahedron is tangent to a given sphere. Prove that the three line segments joining the points of tangency of the three pairs of opposite edges of the tetrahedron are concurrent.

(7 points)

Luis González
#2 Aug 15, 2011, 10:06 am • 1

Label the tetrahedron $ABCD.$ The edges BC,CA,AB,DB,DC,DA are tangent to the sphere at $P,Q,R,S,T,U,$ respectively. According to the topic Tangent points of a quadrilateral to a sphere are coplanar, we get that P,T,U,R an R,S,T,Q are coplanar $\Longrightarrow PU \cap RT \neq \emptyset$ and $QS \cap RT \neq \emptyset.$ PU,QS,RT are projected from D on the plane ABC into $AP,BQ,CR,$ but since $\odot(PQR)$ is the incircle of $\triangle ABC$ (it could also be an excircle) then AP,BQ,CR are the Gergonne cevians of $\triangle ABC$ concurring at its Gergonne point $G.$ Therefore, intersections $PU \cap RT$ and $QS \cap RT$ lie on $DG \Longrightarrow PU,QS,RT$ concur.

Quick Reply

High School Olympiads

Geometric Inequalities Marathon

Locked

BigSams

#1 Apr 21, 2011, 6:07 am • 7

I like Geometry. I like Inequalities. Hence I love Geometric Inequalities. Let's have a Marathon.

The topic is rather niche, but there is a good (increasing) number of problems to keep us busy.
Algebraic, geometric and hybrid methods are welcome. This means geometric theorems, known lemmas, trigonometric identities, well-known algebraic inequalities such as AM-GM and Cauchy-Schwarz, etc. are probably going to be common.

Rules are mostly the same as that of other marathons:
1) Do not post a new problem until a correct solution for the previous one has been solved or it is mutually agreed that we should move on.
2) Hide your solutions, but do not hide the problems.
3) Provide complete solutions in $\LaTeX$. No partial progresses until after a full solution is given and no hints or outlines.
4) If you know the source, do not post a link. Allow members with less encyclopaedic experience to try it. At the same time, do not post very well known problems.
5) Do not post a problem unless you have a solution at hand.
6) Finally, make sure that the problems you post cannot be solved immediately by a well-known formula/technique, but requires uncommon insight.
7) If 48 hours (2 days) have passed since a problem has been proposed, and no complete solution has been posted, then the problem proposer is required to post an appropriate hint (meaning not too revealing but not too cryptic). If there is no complete solution for another 24 hours (1 day), then the problem proposer is required to post a complete solution. (this is a new rule added later after reccommended by **Thalesmaster**)

This is not really a rule for all marathons in general, but always state what your variables mean. For example: Let s be the semiperimeter of $\triangle ABC$.

Let's start off with a classic and increase the level as we go on.
Problem 1.
Let the circumradius and inradius of a triangle be R and r respectively. Prove that $R \geq 2r$.

Have fun.

1=2

#2 Apr 22, 2011, 7:55 am

Is this an adequate proof?

This post has been edited 1 time. Last edited by 1=2, Apr 22, 2011, 7:56 am

tonypr

#3 Apr 22, 2011, 7:55 am

Solution to problem 1

Problem 2: Let P be a point inside the triangle ABC. Prove that

$$PA+PB+PC<AB+BC+AC$$

1=2

#4 Apr 22, 2011, 8:11 am

solution for 2

I'm sorry, I do not have any inequality problems to post =(

BigSams

#5 Apr 22, 2011, 8:14 am • 1

My solution to Problem 1

Problem 3.
Let a, b, c be the side lengths of a triangle. Let s be the semiperimeter and S be the area of the triangle. Prove that
$\frac{1}{s(s-a)} + \frac{1}{s(s-b)} + \frac{1}{s(s-c)} \le \frac{ab+bc+ca}{4S^2}$.

Goutham

#6 Apr 22, 2011, 2:30 pm • 1

Solution to problem 3:

Problem 4:
Let r, R, Δ denote the inradius, circumradius, area of a triangle respectively, prove that

$$\sqrt{3}\Delta \le r(4R+r)$$

Mateescu Constantin

#7 Apr 22, 2011, 3:03 pm • 2

Solution to problem 4

Problem 5. Prove that in any triangle ABC with circumradius R and inradius r , the following inequality holds :
$\cos\frac{B-C}{2} \ge \sqrt{\frac{2r}{R}}$.

Thalesmaster

#8 Apr 23, 2011, 1:26 am • 1

That's a nice idea **BigSams** :)
Solution to problem 5

Problem 6 :
Let I be the incentre of $\triangle ABC$ and let R and r be its circumradius and inradius respectively.
Prove that :
$6r \le AI + BI + CI \le \sqrt{12(R^2 - Rr + r^2)}$

FantasyLover

#9 Apr 23, 2011, 6:32 am • 1

Solution

Problem 7.

Suppose quadrilateral $ABCD$ is tangential in which a circle with center I is inscribed. Prove the following inequality:

$$AB + BC + CD + DA \ge \sqrt{2}(AI + BI + CI + DI).$$

tonypr

#10 Apr 23, 2011, 7:09 am • 2

Alternate solution to the left hand side of problem 6

Thalesmaster

#11 Apr 23, 2011, 5:57 pm • 2

Alternate solution to the right hand side of problem 6

FantasyLover

#12 Apr 25, 2011, 6:25 am

Hint for Problem 7, as requested by BigSams

BigSams

#13 Apr 26, 2011, 7:09 am • 1

Yeah I requested FantasyLover to post a hint last night because it seemed that people were having a hard time with it. Thanks, FantasyLover.
See if this is right: (Btw, what is the source?)
Solution to Problem 7 based on FantasyLover's hint

I am getting the feeling that not many people are enthusiastic about Geometric Inequalities, so let's deal with some more classics to develop familiarity.
Problem 8.
In a triangle with side lengths a, b, c and circumradius R, prove that $9R^2 \geq a^2 + b^2 + c^2$.

RSM

#14 Apr 26, 2011, 7:49 am

Moderator Edit: solution hidden.
Solution to Problem 8

Redeem

#15 Apr 26, 2011, 7:58 am

Problem 9:
Prove that for any non-degenerate quadrilateral with sides a, b, c, d we have:
$$\frac{a^2+b^2+c^2}{d^2} \geq \frac{1}{3}$$

High School Olympiads

Line tangent to a circle

Locked

hatchguy

#1 Aug 14, 2011, 11:15 am • 1

Let C_1 and C_2 be two circles internally tangent to circle C at M and N, respectively, such that C_1 passes through the center of C_2. X and Y are the points of intersection of C_1 and C_2. A and B are the points of intersection of XY and C. Let E and F be the points of intersection of MA and MB with C_1, respectively. Show that EF is tangent to C_2.

Luis González

#2 Aug 14, 2011, 12:09 pm

Posted several times before, e.g.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=1945
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=19775
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=49138
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=358885

High School Olympiads

Set of lines tangent to a sphere

Reply

Source: 2007 St. Petersburg Math. Olympiad, Round III, Grade 11

Shu

#1 Aug 13, 2011, 2:01 am

Let A and B be two points outside of a given sphere in space. Prove that the set of all points M for which the lines AM and BM are tangent to the sphere is contained in the union of two planes.

Luis González

#2 Aug 14, 2011, 4:07 am • 1

For each line ℓ passing through A and tangent to the sphere $\omega,$ there exists two points M satisfying the desired property. Namely, the intersections M_1,M_2 of ℓ with the cone $\mathcal{B}$ with vertex B circumscribed around $\omega.$ Let X,Y,Z be the tangency points of AM_1,BM_1,BM_2 with ω and $P \equiv XY \cap AB, Q \equiv XZ \cap AB.$ By Menelaus' theorem for $\triangle ABM_1$ and $\triangle ABM_2$ cut by XY and $XZ,$ we deduce that

$$\frac{\overline{PA}}{\overline{PB}}=\frac{\overline{YM_1}}{\overline{BY}} \cdot \frac{\overline{XA}}{\overline{M_1X}}=\frac{\overline{XA}}{\overline{BY}}=\frac{\overline{XA}}{\overline{XM_2}} \cdot \frac{\overline{ZM_2}}{\overline{BZ}}=-\frac{\overline{QA}}{\overline{QB}}$$

Ratio $\frac{\overline{XA}}{\overline{BY}}$ is obviously constant for any $\ell,$ thus P,Q are fixed points on the line $AB.$ Since the polar plane of M_1 WRT ω passes through $P,$ then the polar plane τ_1 of P WRT ω passes through $M_1.$ Likewise, M_2 lies on the polar plane τ_2 of Q WRT $\omega.$ Therefore, not only do M_1,M_2 lie on the fixed planes $\tau_1,\tau_2,$ they describe two conics, namely the intersections of τ_1,τ_2 with the cone $\mathcal{B}.$

Quick Reply

High School Olympiads

Circle through orthocenter and two vertices of a triangle

Reply

Source: 2007 St. Petersburg Math. Olympiad, Round III, Grade 10 & 11

Shu

#1 Aug 13, 2011, 1:44 am • 1

A circle passing through the orthocenter of an acute triangle ABC and the vertices A and C, intersects the lines AB and BC at points X and Y, respectively. On the side AC, let Z and T denote the points where $ZX = ZY$ and $ZA = TC$. Prove that BT is perpendicular XY.

Luis González

#2 Aug 13, 2011, 6:56 am • 1

Without loss of generality, assume that Y lies on the segment $\overline{BC}$ and X lies on the extension of $BA.$ Since $\angle CXA = 180^\circ - \angle CHA = \angle CBA,$ it follows that $\triangle BCX$ is isosceles with apex $C \Longrightarrow CH$ bisects $\angle XCY \Longrightarrow H$ is the midpoint of the arc XAY of $\odot(ACX) \Longrightarrow HZ$ is the perpendicular bisector of $\overline{XY}.$ But XY is antiparallel to AC WRT $BA, BC \Longrightarrow HZ$ is parallel to the B-circumdiameter of $\triangle ABC.$ Since the reflection of H about the midpoint M of AC coincides with the reflection of B about the circumcenter $O,$ then the reflection T of Z about M is the intersection of BO with $AC \Longrightarrow \overline{BOT} \perp XY.$

genxium

#3 Aug 16, 2011, 4:49 pm

Assume O is the circumcentre of $\triangle BAC$ and O_a is the circumcentre of $\triangle HAC$, then O_a, O are symmetric wrt. AC.

$ZX = ZY$ implies that Z is on the $perpendicular - bisector$ of XY, so $ZO_a \perp XY$, and Z, T are symmetric wrt. OO_a, or saying , $O \sim O_a, Z \sim T$ wrt. the midpoint of AC .

Assume $\angle BAC = \alpha, \angle BCA = \gamma$.

$\angle O_aZT = 2\pi - \dfrac{\pi}{2} - (\pi - \alpha) - \gamma = \dfrac{\pi}{2} - \gamma + \alpha$, hence $\angle OTZ = \angle O_aZT = \dfrac{\pi}{2} - \gamma + \alpha$, so B, O, T conlinear, finally $BT \perp XY$, as expected

sunken rock

#4 Aug 16, 2011, 9:01 pm

Observation: H is the circumcenter of $\triangle BXY$ - very easy to prove.

Best regards,
Sunken rock

Quick Reply

High School Olympiads

orthocenter and circumcenter

Locked

Source: France 1999 http://www.imomath.com/othercomp/Fra/FraMO99.pdf

JLapin

#1 Aug 13, 2011, 5:20 am

Hello
It's from a French maths contest and I couldn't find any solution in the forum

Prove that the points symmetric to the vertices of a triangle with respect to the opposite side are collinear if and only if the distance from the orthocenter to the circumcenter is twice the circumradius.

Thanks in advance

Luis González

#2 Aug 13, 2011, 5:43 am

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=2791
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=18080
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=170578
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=353274

High School Olympiads

Midpoint between orthocenter and circumcenter in a triangle

Source: 2011 Czech & Slovak Mathematical Olympiad, Rnd. III, Cat. A

Shu

#1 Aug 12, 2011, 10:23 pm • 1

In acute triangle ABC, which is not equilateral, let P denote the foot of the altitude from C to side AB; let H denote the orthocenter; let O denote the circumcenter; let D denote the intersection of line CO with AB; and let E denote the midpoint of CD. Prove that line EP passes through the midpoint of OH.

Luis González

#2 Aug 12, 2011, 11:16 pm • 2

Circumcircle (E) of $\triangle PCD$ is obviously tangent to the circumcircle (O) of $\triangle ABC$ through C. Inversion with center H and power $\overline{HC} \cdot \overline{HP}$ takes (E) into itself and takes (O) into the 9-point circle (N) of $\triangle ABC$. Thus, by conformity, (E) is also tangent to (N) through the inverse P of $C \Longrightarrow P, E$ and N (midpoint of OH) are collinear.

jevgeniy

#3 Aug 12, 2011, 11:31 pm • 1

[My solution]

sunken rock

#4 Aug 13, 2011, 12:28 am • 1

A solution close to **jevgeniy**'s:
Call F, K the second intersections of CO, CP with the circumcircle and take M on PE so that $MH \parallel CD$ and I shall prove that $MH = OE$ $(*)$.
E being the midpoint of $CD \implies \frac{CE}{OE} = \frac{CD}{DF}$ (1). It is well known that $PK = PH$ and $FK \parallel AB$, $(\angle ACD = \angle PCB)$ so $\frac{FD}{CD} = \frac{KP}{PC}$ (2). On the other hand: $\frac{MH}{CE} = \frac{PH}{PC} = \frac{KP}{PC}$ (3); with (1) and (2), $\frac{MH}{CE} = \frac{OE}{CE}$, done.

Best regards,
sunken rock

High School Olympiads

bulgaria 2004 grade 10

Reply

elegant

#1 Aug 11, 2011, 5:39 pm

Let M be the centroid of □ABC. Prove that: (a) cot□AMB =(BC² + CA² − 5AB²)/12[ABC]; (b) cot□AMB + cot□BMC + cot□CMA ≤ −√3.

Luis González

#2 Aug 12, 2011, 7:58 am

$$\cot \widehat{AMB} = \frac{MA^2 + MB^2 - AB^2}{4[MAB]} = \frac{\frac{1}{9}(2b^2 + 2c^2 - a^2 + 2a^2 + 2c^2 - b^2) - c^2}{4 \cdot \frac{1}{3}[ABC]}$$

$$\Longrightarrow \cot \widehat{AMB} = \frac{a^2 + b^2 - 5c^2}{12[ABC]}$$

Adding the cyclic expressions together yields

$$\cot \widehat{AMB} + \cot \widehat{BMC} + \cot \widehat{CMA} = -\frac{a^2 + b^2 + c^2}{4[ABC]} \le -\sqrt{3}$$

The last implication follows from Weitzenböck inequality $a^2 + b^2 + c^2 \ge 4\sqrt{3}[ABC]$

Quick Reply

High School Olympiads

AREA No.5 (create using geogebra)

Reply

krenkovr
#1 Aug 11, 2011, 8:54 pm

Let triangle MNP is inscribe in triangle ABC (M lie on line AB, N on a line BC and P on a line AC). Let H_1, H_2 and H_3 is orthocenters on a triangles AMP, MBN and NCP, appropriate. Prove that area of triangle MNP is equal with area of triangle $H_1H_2H_3$.

Luis González
#2 Aug 11, 2011, 9:44 pm

Note that lines PH_1, NH_2 are both perpendicular to AB, lines MH_2, PH_3 are both perpendicular to BC and lines NH_3, MH_1 are both perpendicular to CA, i.e. opposite sidelines of the hexagon $PH_1MH_2NH_3$ are parallel. Thus, according to the topic UG002 - Geometry (see the previous lemma), we have $|\triangle MNP| = |\triangle H_1H_2H_3|$.

P.S. For an alternate proof of the lemma see the topic equal of area.

krenkovr
#3 Aug 11, 2011, 10:28 pm

Thank you luisgeometra. Intrsting Lemma.

Quick Reply

High School Olympiads

Bicentric Quadrilateral 3

Source: Me

juancarlos

#1 May 27, 2005, 12:42 am

Let $ABCD$ be a bicentric quadrilateral, I its the incenter and M,N are the midpoints of AC,BD, also EF is the third diagonal. If P is the cut point of AC,BD further the line IP cut EF at Q . Prove that: $[ABCD]=2\cdot MN\cdot\sqrt{EQ\cdot FQ}$

pestich

#2 May 27, 2005, 8:41 am

Hello JC,

From Casey Book I, Prop.7 area of EMN (or FMN) is 1/4 of areas of ABCD.
Line MN is the Gauss line , goes thru midpoint of EF. A circle on diameter EF goes thru I and IQ is Geo-Mean of EQ and FQ and it is equal to the height of each of EMN or FMN to side MN.

Maj. Pestich

Luis González

#3 Aug 11, 2011, 1:25 am

Let us put some substance to pestich's proof.

If O is the circumcenter of $ABCD$, then I,O,P lie on a perpendicular to EF, since EF is the polar of P WRT (O) and (I). By Newton's theorem, we know that MN passes through I and the midpoint L of EF. Circle (L), whose diameter connects the conjugate points E,F WRT (O), is then orthogonal to (O) and the circle $\odot(PMN)$ with diameter $\overline{PO}\Longrightarrow$ $LE^2=LM\cdot LN\Longrightarrow LE$ is tangent to $\odot(EMN)$, i.e. (L) is the E-Apollonius circle of $\triangle EMN$. But $\triangle EAC\sim\triangle EBD$ yields $\triangle EMC\sim\triangle END\Longrightarrow\angle MEC=\angle NED\Longrightarrow EI$ bisects $\angle NEM$, thus $I\in(L)$. If T denotes the projection of E on MN, then $LI=LE$ implies that $ET=IQ=\sqrt{EQ\cdot FQ}$.

According to Ratio of areas, the area of $\triangle EMN$ is 1/4 of the area of $ABCD$, thus

$$[ABCD]=4\cdot[EMN]=2\cdot MN\cdot ET=2\cdot MN\cdot\sqrt{EQ\cdot FQ}.$$

High School Olympiads

Congruent angles at a midpoint of a side in obtuse triangle

Reply

Source: Spanish Mathematical Olympiad, National Stage, 2011

Shu
#1 Aug 10, 2011, 2:06 am • 2

In triangle ABC, $\angle B = 2\angle C$ and $\angle A > 90^\circ$. Let D be the point on the line AB such that CD is perpendicular to AC, and let M be the midpoint of BC. Prove that $\angle AMB = \angle DMC$.

Luis González
#2 Aug 10, 2011, 8:00 am • 1

Perpendicular bisector of $\overline{BC}$ cuts AB at P. Then $\angle PCB = \angle ABC = 2\angle ACB \Longrightarrow CA$ is the internal bisector of $\angle PCB \Longrightarrow CD \perp CA$ is the external bisector of $\angle PCB \Longrightarrow$ cross ratio (P, B, A, D) is harmonic. Hence, from $MP \perp MB,$ we deduce that MP, BC bisect $\angle AMD$ internally and externally $\Longrightarrow \angle AMB = \angle DMC.$

professordad
#3 Aug 10, 2011, 9:46 pm • 2

Another way

official solution

This post has been edited 3 times. Last edited by professordad, Jan 27, 2013, 1:32 am

hatchguy
#4 Aug 12, 2011, 10:51 am • 1

Let the angle bisector of $\angle ABC$ intersect AM and AC at N and K respectively. Let $L \equiv DM \cap AC$. Let $\angle ACB = \alpha$.

We show that $BN = CL$ which will imply triangles BNM and CLM to be congruent which will give the desired result $(\angle NBM = \angle \frac{B}{2} = C)$

By menelaus theorem applied to the points M, N, A in triangle BKC we have:

$$\frac{NK}{BN} = \frac{AK}{AC} => \frac{BK}{BN} = \frac{AK+AC}{AC} => BN = \frac{AC \cdot BK}{AK+AC}$$

By menelaus theorem applied to points M, L, D in triangle ABC we have:

$$\frac{LA}{CL} = \frac{AD}{BD} => \frac{AC}{CL} = \frac{AD+BD}{BD} => CL = \frac{AC \cdot BD}{AD+BD}$$

Hence we must prove $\frac{BK}{AK+AC} = \frac{BD}{AD+BD} \leftrightarrow \frac{AK+AC}{BK} = \frac{BD+AD}{BD}$

Since $BK = KC$ the last is equivalent to $\frac{2AK+KC}{KC} = \frac{BD+AD}{BD} \leftrightarrow \frac{2AK}{KC} = \frac{AD}{BD}$

By the angle bisector theorem and sine law we have $\frac{2AK}{KC} = \frac{2AB}{BC} = \frac{2\sin\alpha}{\sin 3\alpha}$

By sine law in BCD we have $BD = \frac{\sin(90+\alpha)\cdot CD}{\sin 2\alpha}$ and also $AD = \frac{CD}{\sin 3\alpha}$ and therefore $\frac{AD}{BD} = \frac{\sin 2\alpha}{\cos\alpha \cdot \sin 3\alpha}$

Hence we must prove $\frac{\sin 2\alpha}{\cos\alpha\cdot\sin 3\alpha}=\frac{2\sin\alpha}{\sin 3\alpha}\leftrightarrow 2\sin\alpha\cdot\cos\alpha=\sin 2\alpha$ which is obviously true.

littletush

#5 Oct 9, 2011, 11:33 am

symmetric methods.
let T be a point on MD,and AC bisects angle BCT,Q be the bisector foot on AC.it's not hard to prove B,Q,T are collinear

Quick Reply

High School Olympiads

Concurrent: line thru orthocenter & circumcenter, BC & other

Reply

Source: XV Rioplatense Mathematical Olympiad (2006), Level 3

Shu

#1 Aug 10, 2011, 2:09 am • 1

The acute triangle ABC with $AB \neq AC$ has circumcircle Γ, circumcenter O, and orthocenter H. The midpoint of BC is M, and the extension of the median AM intersects Γ at N. The circle of diameter AM intersects Γ again at A and P. Show that the lines AP, BC, and OH are concurrent if and only if $AH = HN$.

Luis González

#2 Aug 10, 2011, 7:36 am • 1

Let T be the antipode of A WRT $\Gamma.$ Since TC, BH are both perpendicular to AC and TB, CH are both perpendicular to AB, then $HCTB$ is parallelogram $\Longrightarrow T$ is the reflection of H about $M.$ Thus, $\angle APM = 90^\circ$ implies that M, H, P are collinear. If $D \equiv AP \cap BC$, then H is also the orthocenter of $\triangle ADM.$ Therefore, $OH \in D \Longleftrightarrow OH \perp AM \Longleftrightarrow OH$ is the perpendicular bisector of $\overline{AN} \Longleftrightarrow AH = HN.$

Quick Reply

High School Olympiads

Ruler-&-compass construction of triangle with minimum area

Source: XII Rioplatense Mathematical Olympaid (2003), Level 3

Shu

#1 Aug 9, 2011, 7:07 am • 1

Inside right angle XAY, where A is the vertex, is a semicircle Γ whose center lies on AX and that is tangent to AY at the point A. Describe a ruler-and-compass construction for the tangent to Γ such that the triangle enclosed by the tangent and angle XAY has minimum area.

Luis González

#2 Aug 9, 2011, 9:46 am • 1

If the tangent cuts AX, AY at M, N, then the area $[AMN]$ is minimum, if and only if $\angle MNA = 60^\circ$. This follows from the well known property: Among all triangles with fixed incircle, the equilateral one has the least area.

Proof: Using standard notations for the scalene triangle $\triangle ABC$, by AM-GM we have:

$$\sqrt[3]{(s-a)(s-b)(s-c)} \leq \frac{(s-a)+(s-b)+(s-c)}{3} = \frac{s}{3} \Longrightarrow$$

$$s^4 \geq 27s(s-a)(s-b)(s-c) \Longrightarrow \ s^2 \geq 3\sqrt{3}[ABC] \Longrightarrow [ABC] \geq 3\sqrt{3}r^2$$

The right side of the latter inequality is indeed the area of an equilateral triangle with inradius r.

High School Olympiads

bulgaria 2007

Reply

elegant

#1 Aug 9, 2011, 4:22 am

A circle with center I is inscribed in a quadrilateral ABCD with □BAD+ □ADC > 180°. A line through I meets AB and CD at points X and Y, respectively. Prove that if IX = IY then AX • DY = BX • CY.

Luis González

#2 Aug 9, 2011, 5:46 am

Let $P \equiv AB \cap CD.$ Hence $\overline{IX} = -\overline{IY}$ implies that $\triangle PXY$ is isosceles with apex $P.$ Assuming that I becomes the P-excenter of $\triangle PBC,$ then it follows that $\angle CIY = \angle IBX.$ Thus, $\angle PXI = \angle PYI$ implies that $\triangle BIX$ and $\triangle ICY$ are similar $\Longrightarrow \frac{BX}{IY} = \frac{IX}{CY} \Longrightarrow BX \cdot CY = \frac{1}{4}XY^2.$ Analogously, $AX \cdot DY = \frac{1}{4}XY^2$ and the conclusion follows.

Quick Reply

High School Olympiads

Euler lines are concurrent.

Reply

Morleyiquel

#1 Aug 9, 2011, 12:24 am

$ABCD$ is a cyclic quadrilateral and P is the intersection of its diagonals AC,BD, Prove that Euler lines of $\triangle PAB$, $\triangle PBC$, $\triangle PCD$ and $\triangle PDA$ concur

Luis González

#2 Aug 9, 2011, 1:03 am

This "infamous" problem has been posted several times before, e.g.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=107997
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=187520
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=204979
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=302860 (see the lemma)

Quick Reply

High School Olympiads

Five concyclic points

Source: own

livetolove212

#1 Aug 7, 2011, 10:04 am

Given 5 points A_1, A_2, A_3, A_4, A_5 in the plane and an arbitrary point P such that P is not lie on any circumcircle of these points.
1. Prove that the pedal circles of P wrt triangles $A_1A_2A_3, A_2A_3A_4, A_1A_3A_4, A_1A_2A_4$ concur at a point X_5.
2. Similarly we define X_1, X_2, X_3, X_4. Prove that X_1, X_2, X_3, X_4, X_5 are concyclic.

Luis González

#2 Aug 8, 2011, 4:45 am

Let $A, B, C, D, E, F, G, H, I, J$ be the projections of P onto $A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_1, A_1A_3, A_2A_4, A_3A_5, A_4A_1, A_5A_2$. $X_1 \equiv \odot(BCG) \cap \odot(CDH), X_2 \equiv \odot(CDH) \cap \odot(DEI), X_3 \equiv \odot(DEI) \cap \odot(EAJ), X_4 \equiv \odot(EAJ) \cap \odot(ABF), X_5 \equiv \odot(ABF) \cap \odot(BCG)$. In this topic, we've proved that the pedal circles of P WRT $\triangle A_1A_2A_3, \triangle A_2A_3A_4, \triangle A_1A_3A_4, \triangle A_1A_2A_4$ concur at X_5. The same reasoning yields that C, F, I, X_2, X_5 are concyclic. Then

$$\angle(X_5X_4, X_5X_2) = \angle(IF, IX_2) - \angle(AF, AX_4).$$

But, on the other hand we have

$$\angle(IF, IX_2) = \angle(IF, IE) - \angle(X_3X_2, X_3E) = -\angle(AE, AF) - \angle(X_3X_2, X_3E)$$

$$\angle(X_5X_4, X_5X_2) = -\angle(AE, AF) - \angle(AF, AX_4) - \angle(X_3X_2, X_3E) \implies$$

$$\angle(X_5X_4, X_5X_2) = -\angle(X_3X_2, X_3E) - \angle(X_3E, X_3X_4) = -\angle(X_3X_2, X_3X_4)$$

This implies that X_2, X_3, X_4, X_5 are concyclic. Analogous reasoning yields that X_1 lies on the same circle and the proof is completed then.

High School Olympiads

bulgaria 2004 grade 9

Reply

elegant
#1 Aug 5, 2011, 2:51 pm

Let A_1 and B_1 be points on the sides AC and BC of $\triangle ABC$ such that $4 \cdot AA_1 \cdot BB_1 = AB^2$. If $AC = BC$, prove that the line AB and the bisectors of $\angle AA_1B_1$ and $\angle BB_1A_1$ are concurrent.

[Moderator: Please, use $\LaTeX$ to make your post readable. Topic edited.]

Luis González
#2 Aug 6, 2011, 12:27 am

If O is the midpoint of AB, then the given condition is equivalent to $\frac{AO}{AA_1} = \frac{BB_1}{BO}$. Thus, $\angle OAA_1 = \angle OBB_1$ implies that $\triangle OAA_1 \sim \triangle B_1BO$ are similar by SAS criterion $\Longrightarrow \angle BB_1O = \angle AOA_1$, this yields

$$\angle A_1OB_1 = 180^\circ - \angle AOA_1 - \angle BO_1B = \angle ABC = 90^\circ - \tfrac{1}{2}\angle A_1CB_1$$

Thus, O is the C-excenter of $\triangle CA_1B_1 \Longrightarrow$ Bisectors of $\angle AA_1B_1$ and $\angle BB_1A_1$ pass through the midpoint of AB.

Quick Reply

High School Olympiads

Three midpoints are collinear

Reply

Source: own

livetolove212

#1 Aug 5, 2011, 2:01 pm

We start from the theorem of Morley about six circles:
Given 4 points A, B, C, D on the circle (O). Denote (O_1) the circle passes through A, B; (O_2) the circle passes through B, C; (O_3) the circle passes through C, D; the circle passes through D, A. Let A', B', C', D' be the second intersections of 4 circles $(O_1), (O_2), (O_3), (O_4)$. Then $A'B'C'D'$ is a cyclic quadrilateral.

Now let I be the circumcenter of $A'B'C'D'$. Then prove that the midpoints of O_1O_3, O_2O_4 and OI are collinear.

Luis González

#2 Aug 5, 2011, 11:29 pm

Let $U \equiv AA' \cap BB'$. Since $O_1O_2 \perp BB', O_1O_4 \perp AA', OO_1 \perp AB$ and $IO_1 \perp A'B'$, it follows that $\angle OO_1O_4 = \angle UAB = \angle BB'A' = \angle IO_1O_2 \Longrightarrow OO_1, IO_1$ are isogonal lines WRT $\angle O_2O_1O_4$. By similar reasoning, OO_2, IO_2 are isogonal lines WRT $\angle O_1O_2O_3 \Longrightarrow O, I$ are isogonal conjugates WRT the triangle bounded by $O_1O_2, O_1O_4, O_2O_3 \Longrightarrow$ There exists a conic $\mathcal{K}$ with foci O, I tangent to O_1O_2, O_1O_4, O_2O_3. Similarly, $\mathcal{K}$ touches O_3O_4. Now, by Newton's theorem, the center of the conic $\mathcal{K}$ inscribed in the quadrangle $O_1O_2O_3O_4$ lies on its Newton line, i.e. midpoints of OI, O_1O_3 and O_2O_4 are collinear.

livetolove212

#3 Aug 6, 2011, 6:28 am

Dear Luis,
The idea of this problem is to use Newton's line of $O_1O_2O_3O_4$. But it can be solved sinthetically. Can you avoid conic?

Quick Reply

High School Olympiads

Circle tangent to circumcircle and Euler circle

Reply

Source: own

MariusStanean

#1 Aug 5, 2011, 8:17 pm

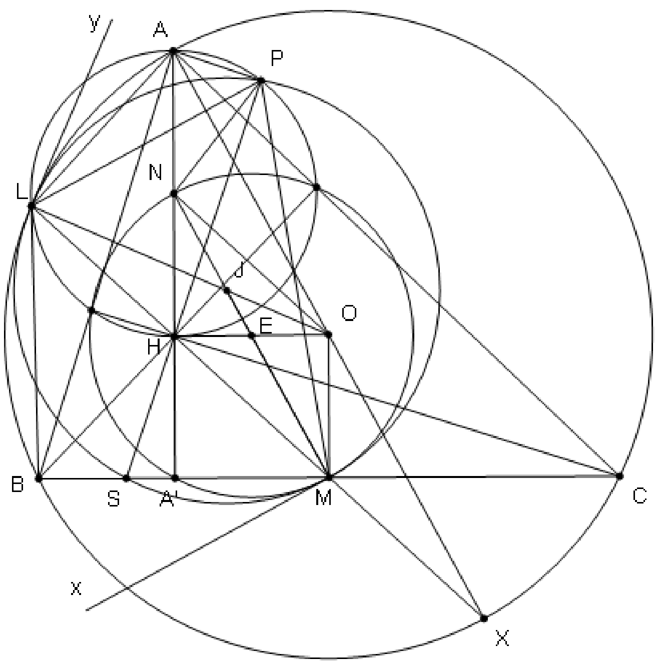
In $\triangle ABC$, let H be the orthocenter and (O), ω are circumcircle and Euler circle, respectively. Let M be the midpoint of BC, S the feet of the simedian from H of $\triangle BHC$ and P is the orthogonal projection of A onto HS. Prove that the circumcircle of $\triangle MPS$ is tangent to (O) and ω.

livetolove212

#2 Aug 5, 2011, 9:16 pm • 2

Without lost of general we assume that $\angle B > \angle C$.
Let N be the midpoint of AH, E be the center of 9-point circle, A' be the projection of A on BC.
We have $\angle SHA' = \angle BHA' - \angle BHS = \angle C - \angle MHC$.
Since $MN//AO$ we get $\angle HMN = \angle A'HM - \angle HNM = \angle A'HC - \angle MHC - \angle HAO$
$= \angle B - (\angle B - \angle C) - \angle MHC = \angle C - \angle MHC$.
Therefore $\angle SHA' = \angle HMN$.
But N is the midpoint of AH so $NH = NP$, which follows that $\angle SHA' = \angle NPH$.
So $NHMP$ is a cyclic quadrilateral.
Construct the tangent Mx of (E).
We have $\angle xMB = \angle A'NM = \angle SPM$. Then Mx is also the tangent of (SPM).
Therefore (SPM) is tangent to (E).
On the other side, let L be the intersection of MH and (O), X be the reflection of A wrt O.
We have M, H, X are collinear then $\angle ALH = 90^o$.
So $ALA'M$ is cyclic. We get $HL.HM = HA.HA' = HP.HS$. This mean L lies on (SPM).
Construct the tangent Ly of (SPM).
We have $\angle yLP = \angle LMP$ then $\angle yLA = \angle yLP - \angle ALP = \angle LMP - \angle AHP = \angle NMH$
$= \angle AXH = \angle LBA$.
So Ly is also the tangent of (O).
Therefore (SPM) is tangent to (O). We are done.

Attachments:



Luis González

#3 Aug 5, 2011, 10:17 pm • 3

Tangent of $\odot(BHC)$ at H cuts BC at $Q.$ Circle Ω with center Q and radius QH is the H-Apollonius circle of $\triangle BHC \Longrightarrow \Omega$ is orthogonal to the pencil of circles through B, C $(\star).$ On the other hand, the reflection of H about M is the antipode T of A WRT $(O),$ thus if the ray MH cuts (O) at $R,$ we have $\angle HRA = 90^\circ \Longrightarrow \angle BRH = \angle ARB - 90^\circ = \angle HBM \Longrightarrow MB^2 = MH \cdot MR.$ Together with $(\star),$ it folllows that $R \in \Omega$ and the tangent of (O) at R passes through $Q.$ Hence, from $(B, C, Q, S) = -1,$ we deduce that $QR^2 = QH^2 = QB \cdot QC = QS \cdot QM \Longrightarrow \odot(RMS)$ is tangent to $(O).$ Since R, P are the inverses of M, S under the inversion with center H that swaps (O) and $\omega,$ then $\odot(RMS)$ passes through P and touches $\omega.$

jayme

#4 Aug 6, 2011, 11:09 am

Dear Mathlinkers,
by considering the tangents at A, M, A', S a more simple proof can be find...
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Concurrent lines thru orthocenters of Δs w/ in- & excenter

Reply

Source: LXXIV St. Petersburg Mathematical Olympiad (2008), Round III

Shu

#1 Aug 4, 2011, 10:13 pm • 1

In triangle ABC, let I denote the incenter and let I_A denote the excenter corresponding to A. Line ℓ_A passes through the orthocenters of triangles BIC and BI_AC. Define ℓ_B and ℓ_C similarly. Prove that ℓ_A, ℓ_B, and ℓ_C are concurrent.

Luis González

#2 Aug 4, 2011, 11:41 pm • 1

Let U_1, V_1 be the orthocenters of $\triangle IBC, \triangle I_ABC.$ Since BV_1, CI are both perpendicular to CI_A and CV_1, BI are both perpendicular to $BI_A,$ then IBV_1C is a parallelogram $\Longrightarrow V_1$ is the reflection of I about the midpoint of $BC.$ Analogously, U_1 is the reflection of I_A about the the midpoint of $BC.$ If $\triangle A_0B_0C_0$ is the antimedial triangle of $\triangle ABC,$ whose vertices are the reflections of A, B, C about the midpoints of $BC, CA, AB,$ then obviously V_1, U_1 are the incenter and A_0 -excenter of $\triangle A_0BC \Longrightarrow \ell_A$ passes through the incenter of $\triangle A_0B_0C_0$ (Nagel point of ABC). Likewise, ℓ_B, ℓ_C pass through the Nagel point of $\triangle ABC.$

Quick Reply

High School Olympiads

Concurrent diagonals and tangents to circles thru quad

Reply

Source: LXXIV St. Petersburg Mathematical Olympiad (2008), Round III

Shu

#1 Aug 4, 2011, 8:17 pm

Let $ABCD$ be a cyclic quadrilateral. Let s_1 be the circle passing through A and B and tangent to line AC, and let s_2 be the circle passing through C and D and tangent to line AC. Prove that lines AC, BD, and the other internal tangent to s_1 and s_2 are concurrent.

Luis González

#2 Aug 4, 2011, 10:45 pm

O is the circumcenter of $ABCD$ and $P \equiv AC \cap BD.$ The centers U,V of s_1, s_2 are the intersections of the perpendicular bisectors of AB, CD with the perpendiculars to AC through $A,C.$ If E denotes the intersection of the perpendicular bisector of CD with the perpendicular to BD at $D,$ then it follows that $PAUB \sim PDEC$ due to $\triangle PAB \sim \triangle PDC.$ Hence, $\angle APU = \angle DPE \ (\star).$ Circles $\odot(PDE)$ and $\odot(PCV)$ with diameters PE, PV cut EV at the orthogonal projection M of P onto $EV \Longrightarrow \angle CPV = \angle CMV = \angle DMV = \angle DPE.$ Together with $(\star),$ we obtain $\angle APU = \angle CPV \Longrightarrow$ U,V,P are collinear and the conclusion follows.

Quick Reply

High School Olympiads

Concylic: a vertex, two midpoints, & point inside triangle

Reply

Source: LXXIV St. Petersburg Mathematical Olympiad (2008), Round III

Shu

#1 Aug 4, 2011, 12:09 am

Inside acute triangle ABC, a point K is chosen such that $\angle AKC = 2\angle ABC$ and $AK/KC = (AB/BC)^2$. Let A_1 and C_1 denote the midpoints of BC and AB, respectively. Prove that K lies on the circumcircle of triangle A_1BC_1.

Luis González

#2 Aug 4, 2011, 1:11 am • 2

Let the B-symmedian of $\triangle ABC$ cut the circumcircle (O) of $\triangle ABC$ again at $D.$ Then the tangents of (O) at B, D and AC concur at the center U of the B-Apollonius circle of $\triangle ABC \Longrightarrow \frac{UA}{UC} = \frac{AB^2}{BC^2}.$ $P \equiv UO \cap BD$ is the orthogonal projection of O onto $BD,$ which lies on the circle $\odot(BA_1C_1)$ with diameter $BO.$

$UD^2 = UB^2 = UA \cdot UC = UP \cdot UO \Longrightarrow A, C, P, O$ are concyclic.

Hence, it suffices to show that $K \equiv P.$ Indeed, from $\angle AKB = 2\angle ABC,$ it follows that $K \in \odot(OAC)$ and from $\frac{AK}{KC} = \frac{AB^2}{BC^2},$ we deduce that the external bisector KO of $\angle AKC$ passes through $U \Longrightarrow K \equiv P.$

horizon

#3 Feb 9, 2012, 9:54 pm

here is my solution
let the circumcenter of triangle of triangle ABC be O,the circumcenter of triangle AOI be P,let T be the reflection of O about point T,KT intersects AC at D_1,and BT interects AC at D_2,$\frac{AD_1}{D_1C} = (\frac{AB}{BC})^2$
$\frac{AD_2}{D_2C} = \frac{AB}{BC} * \frac{sinC}{sinA} = (\frac{AB}{BC})^2$
so B, K, T are collinear
hences $\angle BKO = \angle OKT = 90$ so B, K, C_1, a_1 are cyclic

genxium

#4 Feb 11, 2012, 12:59 pm

This proposition is directly from a well-known Lemma.

Approach 1: Assume $U = \odot BO \cap \odot OAC, V = \odot OAC \cap BU, W = BV \cap AC$ by $OU \perp AU$ knows that OV is the diameter, then figure out $\frac{AU}{UC} = \frac{AW}{WC} = \frac{S_{ABV}}{S_{CBV}} = \frac{AB^2}{BC^2}$. Hence $U = K$.

Approach 2: Assume KV bisects $\angle AKC$ and insects $\odot OAC$ at V, then $OK \perp BK$.

I firstly solved it with approach 1, but obviously 2nd is better.

Quick Reply

High School Olympiads

Equal segments in two excircles, midpts, reflections diagram

Reply

Source: LXXIV St. Petersburg Mathematical Olympiad (2008), Round III

Shu

#1 Aug 3, 2011, 11:35 pm • 1

Two excircles of triangle ABC are tangent to AB and AC at points P and Q, respectively. Let L and M denote the midpoints of PQ and BC, respectively. Let L_1 and L_2 be the reflections of point L about the midpoints of BM and CM, respectively. Prove that $L_1P = L_2Q$.

Luis González

#2 Aug 4, 2011, 12:55 am • 1

This is true for any pair of points P, Q on $\overline{BA}, \overline{CA}$, such that $BP = CQ$. Denote by U, V the midpoints of MB, MC and let P_1, Q_1 be the reflections of P, Q about U, V. Then we have $MP_1 = BP = CQ = MQ_1$, i.e. $\triangle MP_1Q_1$ is isosceles with base P_1Q_1 $(\star)$. On the other hand, let D, E be midpoints of BQ, CP, which clearly lie on MQ_1, MP_1. Since $BP = CQ$, then $LD = LE \Longrightarrow$ parallelogram $LDME$ is a rhombus $\Longrightarrow ML$ bisect $\angle DME \equiv \angle Q_1MP_1$, together with $(\star)$, it follows that L lies on the perpendicular bisector of $P_1Q_1 \Longrightarrow PL_1 = LP_1 = LQ_1 = QL_2$.

genxium

#3 Aug 6, 2011, 12:03 am

BP=CQ=(b+c-a)/2, Let D be the midpoint of BQ, E be the midpoint of CP, MDLE is a rhombus.

So $\angle DML = \frac{\angle DME}{2} = \frac{\angle BAC}{2}$,$\angle DMB = \angle ACB$,$\angle LMB = \frac{\angle BAC}{2} + \angle ACB$, denote I the in-center of $\triangle ABC$, then $LM//AI$, finally $\triangle BPL_1$ and $\triangle CQL_2$ are congruent(SAS).

This post has been edited 2 times. Last edited by genxium Feb 12, 2012, 2:21 am

horizon

#4 Feb 9, 2012, 9:40 pm

it's easy by using vectors.

by $\left|\overrightarrow{L_1P}\right| = \left|\overrightarrow{PB} + \overrightarrow{LM}\right|$

$\left|\overrightarrow{L_2Q}\right| = \left|\overrightarrow{QC} + \overrightarrow{LM}\right|$,

so we only need to prove that

$\left|3\overrightarrow{PB} + \overrightarrow{QC}\right| = \left|3\overrightarrow{QC} + \overrightarrow{PB}\right|$ by $PB = QC$ it's trivial

Quick Reply

High School Olympiads

Prove that OA and RA are perpendicular

Source: USA TSTST 2011/2012 P4

MellowMelon
#1 Jul 27, 2011, 3:32 am

Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $OA \perp RA$.

Luis González
#2 Jul 27, 2011, 4:07 am • 1

It's well known that the orthocenter H of $\triangle ABC$ is the center of negative inversion that swaps its 9-point circle and circumcircle (O). Hence, P, Q are the inverses of N, M under the referred inversion $\Longrightarrow M, N, P, Q$ lie on a circle ω. Hence, R is the radical center of ω, (O) and $\odot(AMN) \Longrightarrow R$ lies on the radical axis of (O) and $\odot(AMN)$, which is the tangent of (O) through A, since (O) and $\odot(AMN)$ are internally tangent through A.

buratinogigle
#3 Jul 30, 2011, 5:21 pm

There is generalization for this

Let ABC be a triangle and XYZ is pedal triangle of a point P with respect to ABC. P' is isogonal conjugate of P. (O, R) is circumcircle of triangle XYZ. $\mathcal{C}$ is circle $(P, 2R)$. Ray YP', ZP' intersects $\mathcal{C}$ at M, N, resp. MN cuts YZ at R. T is projection of R on PA. Prove that T is inversion of A with respect to $\mathcal{C}$.

Luis González
#4 Jul 31, 2011, 12:24 pm

OK, here is a proof of the generalization:

WLOG assume that P is inside $\triangle ABC$, then so is its isogonal conjugate P', which is the exsimilicenter of the pedal circle $\mathcal{K}$ of (P, P') and $\mathcal{C}$, since the the dilatation with center P' and factor 2 carries the center of $\mathcal{K}$ (midpoint of PP') into P. Consequently, P' is also the center of the negative inversion that swaps $\mathcal{K}$ and $\mathcal{C} \Longrightarrow M, N$ are the inverses of Y, Z under this inversion $\Longrightarrow X, Y, M, N$ lie on a same circle $\omega \Longrightarrow R \equiv YZ \cap MN$ is the radical center of $\mathcal{C}, \omega$ and the circle $\odot(AYZ)$ with diameter $\overline{AP} \Longrightarrow RT \perp AP$ is the radical axis of $\odot(AYZ)$ and $\mathcal{C}$. Therefore if AP cuts $\mathcal{C}$ at U, V, we have $\overline{TU} \cdot \overline{TV} = \overline{TA} \cdot \overline{TP}$. But since P is the midpoint of $\overline{UV}$, then we deduce that $(U, V, A, T) = -1 \Longrightarrow \overline{PU}^2 = \overline{PA} \cdot \overline{PT} \Longrightarrow T$ is the inverse of A WRT $\mathcal{C}$.

SnowEverywhere
#5 Aug 5, 2011, 5:29 pm • 1

Let the points diametrically opposite to B and C be R and S, respectively. Since R is diametrically opposite to B, $RC \perp BC$ and $RA \perp BA$ which imply that $RC \| AH$ and that $RA \| CH$. Hence $HARC$ is a parallogram which implies that R, N and H are collinear. By the same argument, S, M and H are collinear. Now note that $RSBC$ is a rectangle and hence that $RS \| BC \| MN$. Hence $\angle QPM = \angle QPS = \angle QRS = \angle QNM$ which implies that $PQNM$ is cyclic. Let ω_1, ω_2 and ℓ denote the circumcircle of AMN, the circumcircle of $PQNM$ and the tangent line to ω at A. Note that ℓ is also tangent to ω_1 at A since there is a homothety with center A taking AMN to ABC. Hence pairwise ℓ, MN and PQ are the radical axes between the circles ω, ω_1 and ω_2. Hence these line concur at R which implies that $OA \perp RA$.

v_Enhance
#6 Apr 13, 2012, 1:08 pm • 3

OK I think I am getting the hang of this.

Diagram

Vieta FTW

Babai
#7 Apr 15, 2012, 12:55 am

@v_ enhance can you please say what is EFFT??

v_Enhance
#8 Apr 15, 2012, 8:42 am • 1

Whoops, I copy-pasted this from an article I was writing about barycentrics, so I didn't realize I hadn't defined it. Sorry about that.

EFFT is short for Evan's Favorite Forgotten Trick; the statement of the standard and strong version is as follows:
Click to reveal hidden text

It's actually not really necessary here; there are synthetic and analytic means of finding the equation of the tangent (although Strong EFFT is, to my knowledge, the fastest).

EDIT: Sometime around this coming Tuesday, I'll be releasing the article I mentioned on AoPS. Shameless plug there. :>

jayme
#9 Apr 15, 2012, 5:18 pm • 1

Dear Mathlinkers,
1. B' the symmetric of B wrt O
2. C' the symmetric of C wrt O
3. It is known that P, H, M and C' are collinear, Q, H, N and B' are collinear
4. L the point of intersection of PC and AB'
5. Note that B'C' // BC // MN
5. According to Pascal theorem applied to ABCPC'B'A, ML // BC ;
6. M, N, L and R are collinear
7. Note Ta the tangent at A
8.According to a reciprocity of Pascal theorem, NL is the pascal line of the degenerated hexagon PQB'A Ta CP ; consequently, Ta goes through R.

and we are done

Sincerely
Jean-Louis

v_Enhance
#10 Apr 17, 2012, 9:49 am • 1

> *v_Enhance wrote:*
> EDIT: Sometime around this coming Tuesday, I'll be releasing the article I mentioned on AoPS. Shameless plug there. :>

In case anyone's interested, the article I mentioned earlier is now available here.

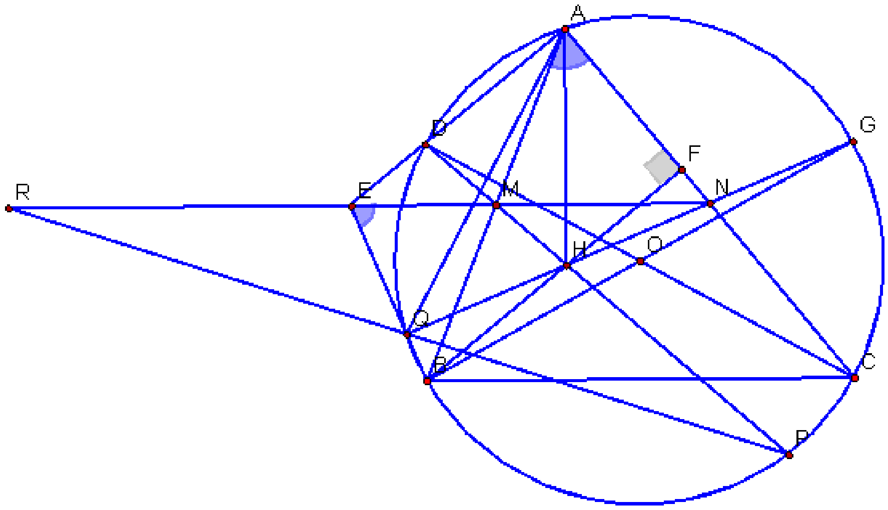
thecmd999
#11 Aug 31, 2013, 8:19 am

Solution

vslmat
#12 Oct 7, 2013, 2:56 pm

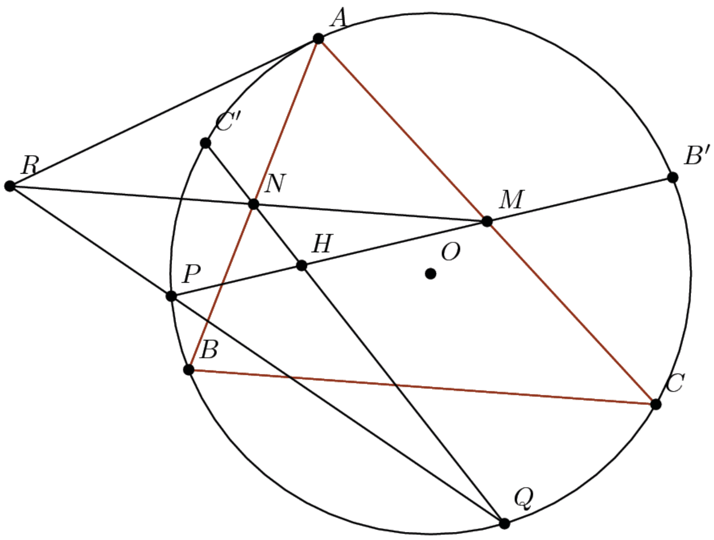
Let BO cut w at G, then it is well known fact that G, N, H, Q are collinear, similarly, if CO cuts w at D than D, M, H, P are collinear. Let AD cut QB at E. As $EANQ$ is cyclic, $\angle QEN = \angle QAB = 180^\circ - \angle EBC$, this means that $EN \parallel BC$ and thus R, E, M, N collinear. By Pascal in $AABQPD$ we have $R' \equiv AA \cap QP; M \equiv AB \cap PD; E \equiv AD \cap BQ$ collinear, than $R' \equiv R$, RA is the tangent to w at A and $OA \perp RA$.

Attachments:



AnonymousBunny

#13 Jul 4, 2014, 12:43 am • 1



Let B' and C' be the antipodes of B, C respectively on ω. Since $B'C$ and AH are both perpendicular to BC, they are parallel to each other. Also, since $B'A$ and CH are both perpendicular to AB, they are parallel. It follows that $AHCB'$ is a parallelogram. Since the diagonals of a parallelogram bisect each other, H, M, B' are collinear. Similarly, H, N, C' are also collinear. Now, since $C'B$ and $B'C$ are both perpendicular to $BC, C'B \parallel B'C$. Also, their diagonals BB' and CC' bisect each other at O. As a result, $C'BAB'$ is a parallelogram, implying $B'C' \parallel BC$. Both these lines are parallel to MN by the midpoint theorem. Now, note that

$$\angle QPM = \angle NMP = \angle C'B'P = 180^\circ - \angle C'NM = \angle QNM,$$

so quadrilateral $PQMN$ is cyclic. Note that MN and PQ are the radical axes of $(PQMN), (AMN)$ and $(PQMN), (ABC)$ respectively. Hence, their point of intersection, R must be their radical center. Consequently, R lies on the radical axis of (AMN) and (ABC). Trivially A also lies on this radical axis, so RA is the radical axis of (AMN) and (ABC). However, since $OM \perp AB$ and $ON \perp AC$, the circumcenter of $\triangle AMN$ is the midpoint of OA. Since the radical axis of two circles is perpendicular to the line joining their centers, it follows that $RA \perp OA$. $\blacksquare$

Sardor

#14 Dec 3, 2014, 11:27 pm

Let $HM \cap w = T$ and $HN \cap w = S$.It's easy to see that $HN = NS$ and $TM = MH$.We have $OH \cdot HS = RH \cdot HT \implies OH \cdot HN = RH \cdot MH$ thus $ORNM$ is cyclic.We know that (AMN) and w are tangent at A.By radical axis theorem (for $w, (AMN), (QRNM)$) we have MN, QR, AA are concurrent at R , hence RA is common tangent of (AMN) and w, so $OA \perp RA$.

K6160

#15 Mar 25, 2016, 12:17 pm

Nice problem :). First we claim that $MNPQ$ is cyclic. This follows immediately from angle chasing since the reflections of H across M and N lie on the circumcircle. Since $MNPQ$ is cyclic, R is the radical center of ω, (AMN), and $(MNPQ)$. So RA is the radical axis of ω and (AMN), which is just the tangent at A. $\square$

Kezer

#16 Apr 27, 2016, 12:37 am

> K6160 wrote:
> This follows immediately from angle chasing since the reflections of H across M and N lie on the circumcircle.

How exactly does it immediately follow from angle chasing?

AMN300

#17 May 14, 2016, 9:50 am

This is a terrific problem! nevermind, it is a pretty simple angle chase. see **thecmd999**'s post for details

solution

This post has been edited 1 time. Last edited by AMN300, May 14, 2016, 9:54 am
Reason: blargh

Quick Reply

High School Olympiads

On the reflections of a line wrt the sidelines of a triangle

Reply

Source: me

pohoatza

#1 Jul 30, 2011, 6:53 am

Let d be an arbitrary line in the plane of a given triangle ABC. The reflections d_a, d_b, d_c of d in the sidelines of ABC bound a triangle, namely XYZ.
(a) Prove that the incenter of triangle XYZ lies on the circumcircle of triangle ABC.
(b) If $U,\ V$ are the incenter, and the circumcenter of triangle XYZ, respectively, prove that the (second) intersection of the line UV with the circumcircle of ABC is a fixed point, independent of the position of d. Denote this point by E. Moreover, prove that E is the Euler reflection point of ABC (i.e. E is the concurrency point of the reflections of the Euler line of ABC into the triangle's sidelines).

Luis González

#2 Jul 30, 2011, 8:55 am • 1

Let $X \equiv d_b \cap d_c, Y \equiv d_c \cap d_a, Z \equiv d_a \cap d_b. O, H$ are the circumcenter and orthocenter of $\triangle ABC. AH, BH, CH$ cut the circumcircle (O) of $\triangle ABC$ again at $P, Q, R.$ From the topics The incenter lies on circumcircle [Iran Second Round 95] and A conjecture on OIM 2011 Problem 6 we know that $U \in (O), \triangle PQR \sim \triangle XYZ$ and $PU \parallel YZ.$ As a result, the angle between the diacentral line OH of $\triangle PQR$ with QR equals the angle between the diacentral line UV of $\triangle XYZ$ with $YZ.$ Thus, $\angle(OH,QR) = \angle(UV,YZ) = \angle PUV.$ But the isogonal conjugate E^* of E WRT $\triangle PQR$ is the infinite point of OH $\Longrightarrow \angle PQE = \angle RQE^* = \angle(OH,QR).$ Hence $\angle PQE = \angle PUV \Longrightarrow V \in EU.$

Quick Reply

High School Olympiads

Show that O1O3 is perpendicular to O2O4

Reply

Source: Iran Third Round Problems 1993 – Poblem 5

Amir Hossein

#1 Jul 29, 2011, 7:41 pm

In a convex quadrilateral $ABCD$, diagonals AC and BD are equal. We construct four equilateral triangles with centers O_1, O_2, O_3, O_4 on the sides sides AB, BC, CD, DA outside of this quadrilateral, respectively. Show that $O_1O_3 \perp O_2O_4$.

Luis González

#2 Jul 30, 2011, 5:12 am

The problem and its generalization have been discussed before, e.g.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=405634
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=364771
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=349911
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=289497
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=1370

Quick Reply

High School Olympiads

Collinear (diagonal intersect perp bisector)'s in a heaxgon

Reply

Source: XVIII Tuymaada Mathematical Olympiad (2011), Senior Level

Shu

#1 Jul 29, 2011, 11:46 pm

In a convex hexagon $AC'BA'CB'$, every two opposite sides are equal. Let A_1 denote the point of intersection of BC with the perpendicular bisector of AA'. Define B_1 and C_1 similarly. Prove that A_1, B_1, and C_1 are collinear.

Luis González

#2 Jul 30, 2011, 4:12 am • 1

D is the midpoint of AA' and M, N are the orthogonal projections of C, B onto AA'. From $DA_1 \parallel CM \parallel BN$, we get $\frac{A_1B}{A_1C} = \frac{DN}{DM}$ $(\star)$. On the other hand, from $|AB^2 - A'B^2| = |AN^2 - A'N^2|$ and $|AC^2 - A'C^2| = |AM^2 - A'M^2|$ we get

$$DN = \frac{|AB^2 - A'B^2|}{2 \cdot AA'} \ , \ DM = \frac{|AC^2 - A'C^2|}{2 \cdot AA'}$$

Substituting DN, DM into $(\star)$ yields $\frac{A_1B}{A_1C} = \frac{|AB^2 - A'B^2|}{|AC^2 - A'C^2|}$

Multiplying the cyclic expressions together gives

$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = \frac{|AB^2 - A'B^2|}{|AC^2 - A'C^2|} \cdot \frac{|BC^2 - B'C^2|}{|BA^2 - B'A^2|} \cdot \frac{|CA^2 - C'A^2|}{|CB^2 - C'B^2|} = 1$$

By Menelaus' theorem in $\triangle ABC$, we conclude that A_1, B_1, C_1 are collinear.

Quick Reply

High School Olympiads

Lines thru the midpoint of the common chord of two circles

Reply

Source: XVIII Tuymaada Mathematical Olympiad (2011), Senior Level

Shu

#1 Jul 29, 2011, 10:56 pm

Circles ω_1 and ω_2 intersect at points A and B, and M is the midpoint of AB. Points S_1 and S_2 lie on the line AB (but not between A and B). The tangents drawn from S_1 to ω_1 touch it at X_1 and Y_1, and the tangents drawn from S_2 to ω_2 touch it at X_2 and Y_2. Prove that if the line X_1X_2 passes through M, then line Y_1Y_2 also passes through M.

Luis González

#2 Jul 30, 2011, 1:03 am • 2

Let O_1, O_2 be the centers of $\omega_1, \omega_2.$ Assume that X_1X_2 passes through $M.$ If $U \equiv X_1Y_1 \cap O_1O_2,$ then $BS_1 \perp O_1O_2$ is the polar of U WRT $\omega_1 \Longrightarrow$ Pencil $M(X_1, Y_1, S_1, U)$ is harmonic. Since $MS_1 \perp MU,$ then it follows that AB, O_1O_2 bisect $\angle X_1MY_1,$ i.e. MY_1 is the reflection of X_1X_2 across $AB.$ By similar reasoning, MY_2 is the reflection of X_1X_2 across AB $\Longrightarrow Y_1Y_2$ passes through $M.$

anonymouslonely

#3 Aug 25, 2011, 10:35 pm • 1

$O_2MS_2X_2$ and $MS_1X_1O_1$ are cyclic quadrilaterals because O_2M is perpendicular on AB and O_2X_2 is perpendicular on S_2X_2 and for the another is a similar proof.
because X_1, M, X_2 are on a line then the angle $S_2O_2X_2$ and the angle $S_1O_1X_1$ are congruent.
this also is the condition for the congruence of the angles S_2MY_2 and S_1MY_1 which means M in on the line Y_1Y_2.

Quick Reply

High School Olympiads

Circumcenter ratio

Reply

zero.destroyer

#1 Jul 29, 2011, 12:24 am

Let M be a point in triangle ABC. The interior angle bisectors of BMC, AMC, and AMB intersect the sides BC, AC, AB at respective points D,E, F.
Show that the lines AD, BE, CF are concurrent at P, and
(PA*PB*PC)/(PD*PE*PF)=8 if and only if M is the circumcenter of ABC.

Luis González

#2 Jul 29, 2011, 1:12 am

$\frac{MB}{MC}=\frac{DB}{DC}$, $\frac{MC}{MA}=\frac{EC}{EA}$, $\frac{MA}{MB}=\frac{FA}{FB}$. (Angle bisector theorem)

$\frac{DB}{DC}\cdot\frac{EC}{EA}\cdot\frac{FA}{FB}=\frac{MB}{MC}\cdot\frac{MC}{MA}\cdot\frac{MA}{MB}=1 \Longrightarrow AD, BE, CF$ concur.

By Van Aubel's theorem, we have

$$\frac{PA}{PD}=\frac{EA}{EC}+\frac{FA}{FB}=\frac{MA}{MC}+\frac{MA}{MB}=MA\left(\frac{MB+MC}{MB\cdot MC}\right)$$

Therefore, multiplying the cyclic expressions together yields

$$\frac{PA}{PD}\cdot\frac{PB}{PE}\cdot\frac{PC}{PF}=\frac{(MB+MC)(MC+MA)(MA+MB)}{MA\cdot MB\cdot MC}\geq 8$$

The latter inequality follows from the well-known $(x+y)(y+z)(z+x)\geq 8xyz, (x,y,z>0)$, which is equivalent to $[2,1,0]\geq[1,1,1]$ (Muirhead's theorem). Therefore, the equality holds $\Longleftrightarrow MA=MB=MC \Longleftrightarrow M$ is the circumcenter of $\triangle ABC$.

Quick Reply

High School Olympiads

Problem (Own?)

Reply

skytin
#1 Jul 28, 2011, 11:24 am

Given triangle ABC
Incircle (I) is tangent to it's sides at points X , Y , Z
H is orthocenter of triangle AIC
G is intersection point of AX and CZ
N is midpoint of ZX
Prove that H is on GN

Luis González
#2 Jul 28, 2011, 12:33 pm

Let $U \equiv YZ \cap BC$ and $V \equiv XZ \cap CA$. M, L are the midpoints of BC, AB and the external bisector ℓ_b of $\angle ABC$ cuts ML at P. Since $BP \parallel XZ$, then $\triangle BPM$ and $\triangle XVC$ are homothetic with homothetic center lying on their common sideline BC. But from $(B, C, X, U) = -1$, we get $\frac{UB}{UX} = \frac{UM}{UC} \Longrightarrow U$ is the homothetic center of $\triangle BPM$ and $\triangle XVC$ $\Longrightarrow U, P, V$ are collinear, i.e. the external bisector ℓ_b of $\angle ABC$, the B-midline ML of $\triangle ABC$ and the trilinear polar UV of the Gergonne point G WRT $\triangle ABC$ concur. Hence, their respective poles N, H, G WRT (I) are collinear.

skytin
#3 Jul 28, 2011, 1:36 pm

See my solution here :
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=419535

buratinogigle
#4 Jul 28, 2011, 11:49 pm

Nice problem, here is an generalization

Let ABC be a triangle inscribed (O). Bisector of $\angle BAC$ intersects (O) again at D. P is a point on AD. E is isogonal conjugate of D with respect to triangle PBC. Q, R are projections of P on AC, AB, resp. BQ cuts CR at F. Prove that EF passed through midpoint of QR.

Is this post Collinear relative ?

Quick Reply

High School Olympiads

Excircle (I_a)

Reply

Source: own

MariusStanean
#1 Jul 22, 2011, 1:06 pm

In triangle ABC, the excircle (I_a) touches sides AB, AC at D, E respectively. Let $\{J\} = BE \cap CD$, and X, Y are symmetric of B, C with respect to midpoint of CD, BE. Prove that $JX = JY$.

This post has been edited 1 time. Last edited by MariusStanean, Jul 27, 2011, 8:58 pm

skytin
#2 Jul 26, 2011, 9:08 pm

Do you mean that XB = YC ?

Luis González
#3 Jul 27, 2011, 10:34 pm • 2

Incircle (I) of $\triangle ABC$ touches BC, CA, AB at P, Q, R. $BCXD$ is clearly a parallelogram, thus $CX = BD = CP$ and $DX = BC = DR \Longrightarrow C, D$ have equal powers WRT (I) and the circle (X) with zero radius $\Longrightarrow CD$ is the radical axis of $(I), (X)$. Likewise, BE is the radical axis of (I) and the circle (Y) with zero radius $\Longrightarrow J \equiv BE \cap CD$ is the radical center of $(I), (X), (Y) \Longrightarrow J$ lies on the radical axis of $(X), (Y)$, which is the perpendicular bisector of $\overline{XY}$, i.e. $JX = JY$, as desired.

skytin
#4 Jul 28, 2011, 11:01 am

Easy to see that EYDX is parallelogram and midpoint of DE (N) is on YX
So wee need to prove that NJ is perpendicular to YX
Let make reflections of points X , Y wrt midpoint of BC and get points X' , Y'
X' is on BA , Y' is on CA , X'Y' || XY
Let H is orthocenter of triangle I_aBC
(I_a) is tangent to BC at point F
F' is on BC , such that FC = CF'
tangent from B to BHF || EF' , so after Reim's theorem (EFF') intersect (BHF) at points G and F , were G is on EB
EY' is diameter of (EFF') , so angle EGY' = BGH = 90 , so G is on HY'
HX intersect DC at point L
HL*HX' = HF*HF = HG*HY' , so GLX'Y' is cyclic
Let line thru X and || EJ intersect line thru Y and || DC at point U
U is orthocenter of Y'HX'
J is radical center of (GLX'Y') , (JY') , (JX') , so HJ is height of YHX
U is on HJ
Reflect U wrt H and get point U'
Easy to see that EU'DJ is parallelogram , so N is on HJ
MJ is perpendicular to X'Y'
X'Y' || XY
So problem done.

Quick Reply

High School Olympiads

BD,CE,OP concurrent iff PBD and PCE have same incenter

Source: USA TST 2011 P7

MellowMelon
#1 Jul 27, 2011, 3:20 am • 1

Let ABC be an acute scalene triangle inscribed in circle Ω. Circle ω, centered at O, passes through B and C and intersects sides AB and AC at E and D, respectively. Point P lies on major arc BAC of Ω. Prove that lines BD, CE, OP are concurrent if and only if triangles PBD and PCE have the same incenter.

Luis González
#2 Jul 27, 2011, 5:30 am • 5

Assume that $\triangle PBD$ and $\triangle PCE$ share the same incenter $\Longrightarrow \angle BPD$ and $\angle CPE$ share angle bisectors $\Longrightarrow$ $\angle BPE = \angle CPD$. Since $\angle PBA = \angle PCA$, then we deduce that $\triangle PEB$ and $\triangle PDC$ are directly similar $\Longrightarrow$ $\angle PEA = \angle PDA \Longrightarrow A, D, E, P$ are concyclic. AP, DE, BC concur at the radical center R of $\Omega, (O)$ and $\odot(AED)$. If $U \equiv BD \cap CE$, then AU, AR are the polars of R, U WRT $(O) \Longrightarrow AU$ is perpendicular to OR at V and OU is perpendicular to AR at P^*. Whence, $RE \cdot RD = RO \cdot RV = RA \cdot RP^*$ implies that A, D, E, P^* are concyclic $\Longrightarrow P$ and P^* are identical and the conclusion follows.

The converse can be proved with similar arguments. Let $U \equiv BD \cap CE$ and $R \equiv BC \cap DE$. If OP passes through U, then P coincides with the orthogonal projection of O onto AR and since the pencils $P(B, D, U, R)$ and $P(E, C, U, R)$ are harmonic, PU bisects $\angle BPD$ and $\angle CPE \Longrightarrow O$ becomes the midpoint of the arcs DB and EC of $\odot(PBD)$ and $\odot(PEC)$. Thus, the intersection $\overline{UP} \cap (O)$ is the common incenter of $\triangle PBD$ and $\triangle PCE$.

djmathman
#3 Jun 18, 2014, 2:23 am • 4

Three hours for the 'if' part, two for the 'only if'. Oops. (Then again I haven't tackled a difficult geo in a while so I guess I'm going to be a bit rusty.)

First, tackle the 'if' direction. Note that the condition of concurrent incenters implies that the angle bisectors of $\angle BPD, \angle CPE$ must be the same line, so $\angle BPE = \angle CPD$. Additionally, $\angle PBA = \angle PCD$ since $APBC$ is cyclic, so $\triangle PEB \sim \triangle PDC$. Looking at the spiral similarity sending the former to the latter gives $\triangle PED \sim \triangle PBC$, and it is well known that this implies $P = (AED) \cap \Omega$.

Now let M and N be the midpoints of $\overline{BE}$ and $\overline{CD}$ respectively. By MGT (or spiral similarity if you prefer) $\triangle PED \sim \triangle PMN$ so $\triangle PEM \sim \triangle PDN \implies \angle PMA = \angle PNA \implies APMN$ is cyclic. But note that since $\angle OMA = \angle ONA = 90^\circ$ we have $AMON$ cyclic as well. Hence $APMO$ is too cyclic (heh APMO), so $\angle APO = \angle AMO = 90^\circ$.

Denote $X = BD \cap CE$ and $T = DE \cap BC$. Brokard gives O as the orthocenter of $\triangle AXT \implies OX \perp AT$. But since $OP \perp AT$ we must have O, P, X collinear $\implies BD, CE, OP$ concurrent.

Similar reasoning can be applied in the opposite direction. Let P' denote the point that AT intersects Ω (where $T = DE \cap BC$ is defined as before). Since DE is the radical axis of (AED) and ω and BC is the radical axis of ω and Ω, T is the radical center of all three circles. Hence $AT \equiv AP'$ is the radical axis of Ω and $(AED) \implies APED$ is cyclic. Using the same reasoning as before gives $AP' \perp P'O$ and $AT \perp XO \equiv P'O$. Since two nonparallel lines can only intersect in one point, P' is the point at which OX intersects $\Omega \implies P' \equiv P$. Thus $\triangle PED \sim \triangle PBC$, so spiral similarity gives $\triangle PEB \sim \triangle PDC$.

Now I claim that this implies the incenters are identical. To prove this, let I and I' be the incenters of $\triangle PDB$ and $\triangle PEC$ respectively. Note that since

$$\pi - \angle BPD = \angle PDB + \angle PBD$$

$$= (\angle PDE + \angle EDB) + (\angle PBE + \angle EBD)$$
$$= (\angle PCB + \angle ECB) + (\angle PCD + \angle ECD)$$
$$= 2\angle C = \angle BOD,$$

we have that $PDOB$ is cyclic. Hence the bisector of $\angle BPD$ passes through O and by Fact 5 we have $IO = ID = IB$. Similarly, we have $PEOC$ cyclic so $I'O = I'E = I'C$. But since it's obvious that I, I', and O are collinear, it must be true that $I \equiv I'$. ■

TelvCohl

#4 Nov 19, 2014, 11:22 pm • 2

If BD, CE, OP are concurrent :

Let OP cuts arc DE of ω at I and let $F \equiv BD \cap CE, X \equiv BC \cap DE$. Since AX is the polar of F WRT ω, so $OF \perp AX$ $\Longrightarrow P \equiv OF \cap \Omega$ is the Miquel point of complete quadrilateral $\{DE, EB, BC, CD\}$. Since P is the projection of O on AX, so from $P(B, D; F, X) = -1, P(E, C; F, X) = -1$ we get PO bisect $\angle BPD$ and $\angle EPC$.

Since $\angle OEC = \frac{1}{2}(180^\circ - \angle COE) = 90^\circ - \angle CBE = 90^\circ - \angle ADE = 90^\circ - \angle APE = \angle EPO = \angle OPC$, so we get O is the midpoint of arc CE in (PCE), hence I is the incenter of $\triangle PCE$. Similarly, we can get I is the incenter of $\triangle PBD$, so $\triangle PBD$ and $\triangle PCE$ have the same incenter I .

If $\triangle PBD$ and $\triangle PCE$ have common incenter :

Let I be the common incenter of $\triangle PBD, \triangle PCE$ and let $F \equiv BD \cap CE, X \equiv BC \cap DE$. Since $\angle BPI = \angle IPD$ and $\angle EPI = \angle IPC$, so $\angle BPE = \angle CPD$. Since $\angle PBE = \angle PCD$, so $\triangle PBE \sim \triangle PCD \Longrightarrow P$ is the Miquel point of complete quadrilateral $\{DE, EB, BC, CD\}$, hence we get P is the projection of O on AX and $OP \perp AX$ **... (1)**. Since F is the pole of AX WRT ω, so we get $OF \perp AX$ **... (2)**.

From **(1)** and **(2)** we get O, F, P are collinear. i.e. BD, CE, OP are concurrent

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Jan 13, 2016, 7:39 am

JuanOrtiz

#5 May 29, 2015, 6:32 am

Restatement: "ABCD is a cyclic quad, AB and CD cut at P, BC and AD cut at Q, BD and AC cut at R, O is center of ABCD. M is a point such that MPAD is cyclic. Prove ORM collinear iff MBD, MAC have the same incenter".

Proof: if they have the same incenter then <BMC=<AMD and thus M is the miquel point, and by a well known fact, R is the inverse of M, done.

If ORM are collinear then M is the miquel point, and therefore PMBC, PMAD, MBAQ, MCQD are collinear. This implies MO bisects <BMD and <AMC. Notice BRD are collinear and thus by inversion OBMD are concyclic, and thus the incenter of MBD is $\omega \cap OR$. Same for MAC, so done.

toto1234567890

#6 Jan 12, 2016, 1:48 pm

If we see that P is on the circumference of the $\triangle OBD$ and $\triangle OCE$ then, the rest is easy.

K6160

#7 Apr 10, 2016, 1:05 am

We will first prove the if statement. If PBD and PCE have the same incenter, then $\angle BPE = \angle CPD$. Also, since $\angle PBA = \angle PAC$, it follows that there is a spiral similarity at P that sends BE to CD. Hence, P is the Miquel Point of $BCDE$ and it is a well known fact that the line through the center of the $\odot(BCDE)$ and the Miquel Point passes through $BD \cap EC = K$.

Now the only if statement. P is the Miquel Point of $BCDE$ since it is the unique point that lies on the circumcircle of ABC and line OK. Therefore, $PBOD$ and $PEOC$ are cyclic, and by Fact 5, the incenters of PBE and PCE are equal to $\omega \cap PO$. □

Quick Reply

High School Olympiads

A perpendicular to AO

Reply

Source: own

jayme

#1 Jul 26, 2011, 6:00 pm

Dear Mathlinkers,
ABC a triangle, O the center of the circumcircle of ABC, XYZ the A-excontact triangle, X' the symmetric of X wrt YZ and U the point of intersection of BC and YZ.
Prove : X'U is perpendicular to AO.
Sincerely
Jean-Louis

Luis González

#2 Jul 26, 2011, 8:35 pm

I_a is the A-excenter of $\triangle ABC$ and without loss of generality assume that $AC > AB$. Then simple angle chase gives $\angle XUY = \angle XYZ - \angle XZY = \frac{1}{2}(\angle XI_aZ - \angle XI_aY) = \frac{1}{2}(\angle B - \angle C) \Longrightarrow \angle XUX' = 2\angle XUY = \angle B - \angle C$ $\Longrightarrow UX'$ is antiparallel to BC WRT AB, AC.

P.S. Likewise, the incircle (I) touches BC, CA, AB at X, Y, Z. If $U \equiv YZ \cap BC$ and X' is the reflection of X about YZ, then UX' is antiparallel to BC WRT AB, AC.

jayme

#3 Aug 4, 2011, 3:26 pm

Dear Mathlinkers,
an article concerning the “Schiffler point and a result by L. and T. Emelyanov” and this result has been put on my website with a new approach.
http://perso.orange.fr/jl.ayme vol. 9
You can use Google translator
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Disk centered at A rolling on BC

Reply

Source: Own

sunken rock

#1 Jul 25, 2011, 11:00 pm

A circular disk centered at vertex A of the equilateral triangle $\triangle ABC$ of side length l rests on BC (i.e. its radius equals the altitude of the triangle). A random movement on BC will bring its center at A', with $AA' \parallel BC$ and $AA' < \frac{l}{2}$.
If, at its initial position the disk intersects AB, AC at M, N respectively and at its final position the intersections with the same sides are P, Q respectively (M, P on AB), then $AA' = MP + NQ$.

Best regards,
sunken rock

Luis González

#2 Jul 26, 2011, 1:05 am

Since AA' bisects $\angle PAQ \equiv \angle BAC$ externally and A' is equidistant from P, Q, it follows that A' is the midpoint of the arc PAQ of $\odot(PAQ)$. By Ptolemy's theorem for $APQA'$, we get $AA' \cdot PQ + AP \cdot QA' = AQ \cdot PA'$. But $\triangle PQA'$ is clearly equilateral, then $PQ = PA' = QA'$ implies that $AA' = AQ - AP$, $(AQ > AP)$. Let D be the tangency point of (A, AM) with BC. Then substituting $AQ = AD + NQ$ and $AP = AD - MP$ into the latter expression gives $AA' = MP + NQ$.

sunken rock

#3 Jul 26, 2011, 12:36 pm

My solution has been the same however, after after taking another look at it I came out with the following:
Extend AB to cut again the circles A and A' at N' and Q' respectively and call D the foot of perpendicular from A' to AB.
Obviously, $AM = AN' = l$ $PD = DQ'$ and $N'Q' = NQ$, so $PD = AP + AD = AM - MP + AD$ and $DQ' = PN' + Q'N' = AN' - AD + Q'N' = AM - AD + QN$; from $PD = DQ' \implies AM - MP + AD = AM - AD + NQ$ or $2AD = MP + NQ$, but from $\triangle AA'D$, $2AD = AA'$, done.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Concurrent 13

buratinogigle

#1 Jul 25, 2011, 8:59 pm

Let AB, CD, EF be chords of circle (O) such that segment EF cuts segments AB, CD at M, N and A, C are in the same side with EF. (O_1) touches ME, MB and (O) internally, (O_2) touches NF, ND and (O) internally. P, Q are contact points of EF with $(O_1), (O_2)$, R, S are contact points of (O) with $(O_1), (O_2)$. Prove that angle bisector of $\angle PO_1R, \angle QO_2S, \angle O_1OO_2$ are concurrent.

I found it from the post The angle bisectors concurrency thank to Petry.

Luis González

#2 Jul 25, 2011, 11:23 pm

It's well known that RP and SQ bisect $\angle ERF$ and $\angle ESF$, thus RP, SQ pass through the midpoint M of the arc EF of (O), which does not contain R, S. Further, $MP \cdot MR = MQ \cdot MS = ME^2$ implies that P, Q, R, S lie on a circle (I) $\Longrightarrow$ angle bisectors of $\angle PO_1R, \angle QO_2S$ (perpendicular bisectors of PR, QS) and perpendicular bisector of PQ (midline of the parallels $PO_1 \parallel QO_2$) concur at I. Consequently, I is equidistant from RO_1, PO_1, QO_2 and $SO_2 \Longrightarrow$ angle bisector of $\angle O_1OO_2$ passes through I.

High School Olympiads

Find all possible values of angle A

Reply

Source: USAMO 1987 Problem 2

Binomial-theorem
#1 Jul 25, 2011, 5:26 am

The feet of the angle bisectors of ΔABC form a right-angled triangle. If the right-angle is at X, where AX is the bisector of $\angle A$, find all possible values for $\angle A$.

Luis González
#2 Jul 25, 2011, 12:58 pm • 1

This problem has been discussed many times before, e.g.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=114998
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=207066
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=304972
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=358388

Quick Reply

High School Olympiads

Perpendicularity in incircle/circumcircle/arc midpt diagram

Reply

Source: XVII Olimpíada Matemática Rioplatense (2008)

Shu

#1 Jul 25, 2011, 9:55 am

In triangle ABC, where $AB < AC$, let X, Y, Z denote the points where the incircle is tangent to BC, CA, AB, respectively. On the circumcircle of ABC, let U denote the midpoint of the arc BC that contains the point A. The line UX meets the circumcircle again at the point K. Let T denote the point of intersection of AK and YZ. Prove that XT is perpendicular to YZ.

Luis González

#2 Jul 25, 2011, 11:32 am • 1

T is the orthogonal projection of X onto YZ and ray AT cuts the circumcircle (O,R) of $\triangle ABC$ at K. Then, we shall show that K, X and the midpoint U of the arc BAC of (O) are collinear. Since the pencil $T(B,C,X,Y)$ is clearly harmonic and $XT \perp YZ$, then TX, YZ bisect $\angle BTC$. Thus, $\angle BZT = \angle CYT$ implies that $\triangle TZB$ and $\triangle TYC$ are similar. Hence, we deduce that

$$\frac{KB}{KC} = \frac{2R \cdot \sin \widehat{ZAT}}{2R \cdot \sin \widehat{YAT}} = \frac{AY}{AZ} \cdot \frac{TZ}{TY} = \frac{BZ}{CY} = \frac{XB}{XC}$$

$\Longrightarrow KX$ bisects $\angle BKC$ internally, i.e. K, X, U are collinear, as desired.

littletush

#3 Dec 4, 2011, 11:21 am

again Kevin Yu's theorem:
in cyclic quadrilateral ABCD,AC intersects BD at E,then
$\frac{AE}{EC} = \frac{AB*AD}{BC*DC}$
is useful.

ThirdTimeLucky

#4 Mar 29, 2013, 2:20 am

Sorry for reviving, This sure can be proved directly too..

Let $YZ \cap BC = L$. Then we have $T\{BC, XL\}$ is harmonic. So $K\{BC, XL\}$ is harmonic too.

By angle chasing, we have $\angle TKX = \angle AKU = \angle ACU = \angle UCB - \angle ACB = \frac{B-C}{2}$ and also $\angle TLX = \angle ZLB = 180^\circ - \angle LZB - \angle LBZ = 180^\circ - (90^\circ - \frac{A}{2}) - (180^\circ - B) = \frac{B-C}{2}$.

So $ZLKX$ is cyclic. Since KU bisects $\angle BKC$, it follows that KL is the external angle bisector of $\angle BKC$ and hence $\angle XKL = \angle XTL = 90^\circ$ That is $XT \perp YZ$.

Quick Reply

High School Olympiads

Midpoint on Nagel line (Own)

Reply

skytin
#1 Jul 21, 2011, 6:04 pm

Given triangle ABC , A'B'C' is it's extouch triangle , XYZ is midarctriangle Mid-Arc Triangle
Circles (AA'X) , (BB'Y) , (CC'Z) intersects at point P
Fe is Feuerbach point of ABC
Prove that midpoint of segment FeP is on the Nagel line of ABC

Luis González
#2 Jul 22, 2011, 12:59 pm • 1

Let $\triangle I_aI_bI_c$ be the excentral triangle of $\triangle ABC.$ Since the incenter I and circumcenter O of $\triangle ABC$ become orthocenter and 9-point center of $\triangle I_aI_bI_c,$ then the circumcenter U of $\triangle I_aI_bI_c$ is the reflection of I about $O.$ (O) cuts I_aI_b again at its midpoint $D,$ i.e. the orthogonal projection of U onto $I_bI_c.$ Since $UDCA'$ is cyclic, due to the right angles at D, A', we have $\overline{I_aU} \cdot \overline{I_aA'} = \overline{I_aD} \cdot \overline{I_aC} = \overline{I_aX} \cdot \overline{I_aA},$ which implies that $\odot(AXA')$ passes through U and the inverse P of U under the inversion with center I and power $\overline{IA} \cdot \overline{IX} = -2Rr.$ Likewise, $\odot(BYB')$ and $\odot(CZC')$ pass through $U, P.$ Now, let G, N be the centroid and 9-point center of $\triangle ABC.$ IG cuts PF_e at $M.$

$$\frac{\overline{MP}}{\overline{MF_e}} = \frac{\overline{GO}}{\overline{GN}} \cdot \frac{IP}{IF_e} \cdot \frac{IN}{IO} = -2 \cdot \frac{2Rr}{IU \cdot r} \cdot \frac{R-2r}{2 \cdot IO} = -\frac{R^2-2Rr}{IO^2} = -1$$

Hence, M is the midpoint of $\overline{F_eP},$ i.e. midpoint of $\overline{F_eP}$ lies on the Nagel line $IG.$

skytin
#3 Jul 22, 2011, 1:29 pm

To Luisgeometra and other mathlinkers who solved previous problem :
Let X , Y , Z are midpoints of segments AI_a BI_b CI_c
P' is isogonal conjecture of point P wrt XYZ
Prove that PP' || Nagel line of ABC

TelvCohl
#4 Nov 22, 2015, 9:49 pm • 1

Lemma 1 : Nagel point N_a, de Longchamps point J, Bevan point T of $\triangle ABC$ are collinear.

Proof : Let τ be the Gergonne line of $\triangle ABC$. From Sondat theorem we know TN_a is perpendicular to the trilinear polar ϱ of N_a WRT $\triangle ABC$, so notice the Newton line $\mathcal{N}$ of the complete qual $\{\triangle ABC, \tau\}$ is the complement of ϱ WRT $\triangle ABC$ we get $TN_a \perp \mathcal{N}$. Since the Steiner line of the complete qual $\{\triangle ABC, \tau\}$ passes through the incenter I and the orthocenter H of $\triangle ABC$, so the anticomplement N_aJ of IH WRT $\triangle ABC$ is perpendicular to $\mathcal{N} \Longrightarrow N_a, J, T$ are collinear.

Lemma 2 : Let I, H, F_e be the incenter, orthocenter, Feuerbach point of $\triangle ABC$, resp. Then IH is the orthotransversal of F_e WRT the intouch triangle $\triangle DEF$ of $\triangle ABC$.

Proof : Let O be the circumcenter of $\triangle ABC$ and let D_1 be the antipode of D in $\odot(I)$. Since F_e is the anti-Steiner point of OI WRT $\triangle DEF$ (well-known), so $D_1(D, F_e; E, F) = (D, F_e; E, F)$ (cross ratio on $\odot(I)$) $= I(A, O; B, C)$, hence notice OI is tangent to the Feuerbach hyperbola $\mathcal{F}$ of $\triangle ABC$ at I we get $I(A, O; B, C) = H(A, I; B, C)$ (cross ratio on $\mathcal{F}$) $\Longrightarrow$

$$D_1(D, F_e; E, F) = H(A, I; B, C) = I(D, H; E, F)$$

$\Longrightarrow D_1F_e, EF, IH$ are concurrent. i.e. the intersection of EF and the orthotransversal of F_e WRT $\triangle DEF$ lies on IH.

Similarly, we can prove IH passes through the intersection of the orthotransversal of F_e WRT $\triangle DEF$ with FD, DE, resp, so IH is the orthotransversal of F_e WRT $\triangle DEF$.

Lemma 3 : Given a $\triangle ABC$ and a point P. Let $\triangle DEF$ be the cevian triangle of P WRT $\triangle ABC$ and let Q be the Miquel point of D, E, F WRT $\triangle ABC$. Let X be the Miquel point of the complete quadrilateral $\{CA, AB, BP, CP\}$ and define Y, Z similarly. Then **(1)** $\odot(AQD), \odot(BQE), \odot(CQF)$ are coaxial and the second intersection W of these three circles is the isogonal conjugate (WRT $\triangle ABC$) of the antigonal conjugate of P WRT $\triangle ABC$. **(2)** The isogonal conjugate T of W WRT $\triangle XYZ$ is the complement (WRT $\triangle ABC$) of the antigonal conjugate of P WRT $\triangle ABC$.

Proof : Let $M_A, M_a, M_B, M_b, M_C, M_c$ be the midpoint of BC, AP, CA, BP, AB, CP, respectively. For the proof of **(1)** you can see Coaxal circles (post #5) **...** $(\bigstar)$. Moreover, from the proof in $(\bigstar)$ we know T is the intersection of DX, EY, FZ. Since Y is the center of the spiral similarity of $\overline{AM_aP} \mapsto \overline{BM_AC}$, so Y lies on $\odot(DM_AM_a)$. Analogously, we can prove $Z \in \odot(DM_AM_a) \Longrightarrow D, M_A, M_a, Y, Z$ are concyclic. Similarly, we can prove E, M_B, M_b, Z, X are concyclic and F, M_C, M_c, X, Y are concyclic, so if T_1 is the second intersection of $\odot(EM_BM_b)$ and $\odot(FM_CM_c)$ then

$$\measuredangle YT_1Z = \measuredangle YFX + \measuredangle XEZ = \measuredangle YFC + \measuredangle CFX + \measuredangle XEB + \measuredangle BEZ.$$

Notice $\measuredangle YFC = \measuredangle YBC, \measuredangle BEZ = \measuredangle BCZ$ and $\measuredangle CFX + \measuredangle XEB = \measuredangle CAX + \measuredangle XAB = \measuredangle CAB$ we get

$$\measuredangle YT_1Z = \measuredangle YBA + \measuredangle ACZ = \measuredangle YDP + \measuredangle PDZ = \measuredangle YDZ \Longrightarrow T_1 \in \odot(DM_AM_a).$$

From $\triangle YZD \sim \triangle YFX$ (see $(\bigstar)$) we get $\measuredangle YT_1X = \measuredangle YFX = \measuredangle YZD = \measuredangle YT_1D$, so T_1 lies on DX. Similarly, we can prove T_1 lies on EY and FZ, so DX, EY, FZ are concurrent at $T_1 \Longrightarrow T_1 \equiv T$.

Finally, since $\measuredangle M_BTM_C = \measuredangle M_BTX + \measuredangle XTM_C = \measuredangle M_BEX + \measuredangle XFM_C = \measuredangle CPX + \measuredangle XPB = \measuredangle CPB$, so we get the complement T_2 (WRT $\triangle ABC$) of the antigonal conjugate of P WRT $\triangle ABC$ lies on $\odot(TM_BM_C)$. Analogously, we can prove T_2 lies on $\odot(TM_CM_A)$ and $\odot(TM_AM_B)$, so $T_2 \equiv T \Longrightarrow T$ is the complement (WRT $\triangle ABC$) of the antigonal conjugate of P WRT $\triangle ABC$.

Back to the main problem :

Let I, O, N, H, N_a, J, T be the incenter, circumcenter, 9-point center, orthocenter, Nagel point, de Longchamps point, Bevan point of $\triangle ABC$, resp. Let I_a, I_b, I_c be the A-excenter, B-excenter, C-excenter of $\triangle ABC$, resp. Let $\triangle DEF$ be the intouch triangle of $\triangle ABC$. From $\triangle I_aBC \cup A' \cup X \sim \triangle I_aI_bI_c \cup A \cup T$ we get $I_aA \cdot I_aX = I_aA' \cdot I_aT$, so T lies on $\odot(AA'X)$. Similarly, we can prove $T \in \odot(BB'Y)$, and $T \in \odot(CC'Z)$, so from **Lemma 3** **(1)** $\Longrightarrow \odot(AA'X)$, $\odot(BB'Y), \odot(CC'Z)$ are coaxial and the second intersection P of these three circles is the Bevan-Schroder Point of $\triangle ABC$.

Let I^* be the image of I under the Inversion $\mathbf{I}(\odot(O))$. Since A, I^*, O, X are concyclic, so from $IO \cdot II^* = IA \cdot IX = IT \cdot IP \Longrightarrow P$ is the midpoint of II^*, hence notice the image of I^* under the Inversion $\mathbf{I}(\odot(I))$ is the 9-point center of $\triangle DEF$ and $(\infty, P; I, I^*) = -1$ we get the image of P under the Inversion $\mathbf{I}(\odot(I))$ is the orthocenter V of $\triangle DEF$.

Since the Steiner line of the second intersection of VF_e with $\odot(I)$ WRT $\triangle DEF$ is perpendicular to the orthotransversal IH (from **Lemma 2**) of F_e WRT $\triangle DEF$, so we get $\measuredangle(IH, OI) = \measuredangle IF_eV$ ($\because F_e$ is the anti-Steiner point of OI WRT $\triangle DEF$) $= \measuredangle F_ePI \Longrightarrow F_eP \parallel IH \parallel N_aT$ ($\because N_aT \equiv N_aJ$ (from **Lemma 1**) is the anticomplement of IH WRT $\triangle ABC$). Since the reflection T^* of T in N_a lies on IF_e ($\because IT^* \parallel ON_a \parallel NI$ (complement of ON_a WRT $\triangle ABC$) $\equiv IF_e$), so the Nagel line IN_a of $\triangle ABC$ passes through the midpoint of F_eP.

> *skytin wrote:*
>
> Let X , Y , Z are midpoints of segments AI_a BI_b CI_c
> P' is isogonal conjecture of point P wrt XYZ
> Prove that PP' || Nagel line of ABC

Problem : Given a $\triangle ABC$ with orthic triangle $\triangle DEF$. Let X, Y, Z be the midpoint of AD, BE, CF, respectively. Let H, N_a, J be the incenter, Nagel point, Bevan-Schroder Point of $\triangle DEF$, respectively. Let K be the isogonal conjugate of J WRT $\triangle XYZ$. Prove that $HN_a \parallel JK$.

Proof : Let $\triangle D_1E_1F_1$ be the cevian triangle of N_a WRT $\triangle DEF$ and let $\triangle D_2E_2F_2$ be the medial triangle of $\triangle DEF$. From $\triangle DEA \sim \triangle DHF \Longrightarrow \triangle DEX \sim \triangle DHE_2$, so notice $EE_1 \parallel HE_2$ we get $\measuredangle DE_1E = \measuredangle DE_2H = \measuredangle DXE \Longrightarrow X$ lies on $\odot(DEE_1)$. Similarly, we can prove $X \in \odot(DFF_1)$, so X is the Miquel point of the complete quadrilateral formed by FD, DE, EN_a, FN_a. Analogously, we can prove Y, Z is the Miquel point of the complete quadrilateral $\{DE, EF, FN_a, DN_a\}, \{EF, FD, DN_a, EN_a\}$, respectively, so from **Lemma 3** **(2)** we know K is the complement (WRT $\triangle DEF$) of the antigonal conjugate of N_a WRT $\triangle DEF$. Rewriting the problem as following **:**

Given a $\triangle ABC$ with incenter I, Nagel point N_a and Bevan-Schroder Point J. Let K be the complement (WRT $\triangle ABC$) of the antigonal conjugate L of N_a WRT $\triangle ABC$. Prove that $IN_a \parallel JK$.

Proof : Let O, N, F_e be the circumcenter, 9-point center, Feuerbach point of $\triangle ABC$, respectively. Clearly, F_e is the midpoint of N_aL, so we get $IK \overset{\parallel}{=} \frac{1}{2}LN_a \overset{\parallel}{=} F_eN_a \Longrightarrow IKN_aF_e$ is a parallelogram, hence $N_aK = IF_e = r$. On the other hand, ON_a is the anticomplement of $NI \equiv IF_e$, so O, N_a, K are collinear and $ON_a = R - 2r$. Since the reflection of I in J is the image of I under the Inversion $\mathbf{I}(\odot(O))$, so we conclude that

$$IJ = \frac{1}{2}\left(\frac{R^2}{\sqrt{R^2-2Rr}} - \sqrt{R^2-2Rr}\right) = \frac{Rr}{\sqrt{R^2-2Rr}} \Longrightarrow \frac{OI}{IJ} = \frac{R-2r}{r} = \frac{ON_a}{N_aK} \Longrightarrow IN_a \parallel JK.$$

Quick Reply

High School Olympiads

A conjecture on OIM 2011 Problem 6

Reply

Source: own

jayme

#1 Jul 21, 2011, 8:45 pm

Dear Mathlinkers,
1. ABC a triangle
2. (O) the circumcircle of ABC
3. HaHbHc the H-circumtriangle of ABC
4. (T) a tangent to (O)
5. A*B*C* the triangle determined by the reflexion of (T) wrt BC, CA, AB
6. I* the incenter of A*B*C*
7. (Tb) the Hb-isogonal of HbI* wrt HaHbHc
Prouve : (Tb) is parallel to (T).
Sincerely
Jean-Louis

Luis González

#2 Jul 22, 2011, 12:20 am

The line τ doesn't necessarily have to be tangent to the circumcircle $(O).$ So, this configuration has more to do with The incenter lies on circumcircle [Iran Second Round 95]. From that topic, we know that $I^* \equiv AA^* \cap BB^* \cap CC^*$ lies on (O) and easy angle chasing reveals that $\triangle A^*B^*C^*$ is similar to $\triangle H_aH_bH_c.$ Since H is the incenter of $\triangle H_aH_bH_c,$ we obtain $\angle B^*I^*H_a = \angle BH_bH_a = \angle C^*B^*I^* \Longrightarrow B^*C^* \parallel H_aI^*.$ Therefore, τ is parallel to the reflection of H_aI^* access $AH_a,$ i.e. τ is parallel to the isogonal of H_aI^* WRT $\angle H_bH_aH_c \Longrightarrow$ isogonal conjugate of I^* WRT $\triangle H_aH_bH_c$ is the infinity point of $\tau.$

Quick Reply

High School Olympiads

Tangent line to circumcircle

Reply

Source: own

MariusStanean

#1 Jul 21, 2011, 11:25 am

Let ABC be a triangle with $AB\neq AC$. Denote by D and E feet of the altitude from B and C. Let M,N and P by the midpoints of the segments BC,MD and ME respectively. If $\{S\}=NP\cap BC$ and DE intersect the line through A parallel to BC in T, prove that ST is tangent to circumcircle of ADE.

Luis González

#2 Jul 21, 2011, 12:09 pm

It's well known that the circle with diameter $\overline{BC}$ is orthogonal to $\omega \equiv \odot(ADE)$ (this can be proved even by simple angle chase) $\Longrightarrow ME,MD$ are tangents of ω. Let K be the second intersection of ω with AM. Since the parallel to BC through A is obviously tangent to ω, then T is the pole of AM WRT $\omega \Longrightarrow$ Tangent of ω at K passes through $T \Longrightarrow \triangle TAK$ is isosceles with legs $TA=TK$. If $S' \equiv TK \cap BC$, from $AT \parallel MS'$, it follows that $\triangle S'KM$ is isosceles with legs $S'M=S'K \Longrightarrow S'$ has equal power WRT ω and $M \Longrightarrow S'$ lies on the radical axis PN of $\omega, M \Longrightarrow S \equiv S'$. Thus, ST is tangent to ω through K.

buratinogigle

#3 Jul 21, 2011, 4:30 pm

Very nice problem and solution, I think we can generalize it as following

Let ABC be a triangle and a circle pass thourgh B,C cuts AB,AC at F,E, resp. M is midpoint of BC. MT_1,MT_2 are tangent of (AEF) at T_1,T_2, resp. I,K are midpoint of MT_1,MT_2, resp. IK cuts BC at T. EF cuts a line pass through A parallel to BC at S. Prove that ST is tangent to circumcircle (AEF).

skytin

#4 Jul 21, 2011, 4:57 pm • 1

> *buratinogigle wrote:*
> Very nice problem and solution, I think we can generalize it as following
>
> Let ABC be a triangle and a circle pass thourgh B,C cuts AB,AC at F,E, resp. M is midpoint of BC. MT_1,MT_2 are tangent of (AEF) at T_1,T_2, resp. I,K are midpoint of MT_1,MT_2, resp. IK cuts BC at T. EF cuts a line pass through A parallel to BC at S. Prove that ST is tangent to circumcircle (AEF).

Good problem

Quick Reply

High School Olympiads

IMO 2011 Problem 6

Reply

WakeUp

#1 Jul 19, 2011, 6:12 pm • 11

Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

Proposed by Japan

This post has been edited 1 time. Last edited by WakeUp, Jul 24, 2011, 6:12 pm

theSA

#2 Jul 19, 2011, 7:18 pm • 5

the only pure geometry in the IMO 2011 is P6!
Weird!

buratinogigle

#3 Jul 19, 2011, 7:36 pm • 10

There is a generalition for this

Let ABC be a triangle and a point P. A line pass through P intersect circumcircle $(PBC), (PCA), (PAB)$ again at P_a, P_b, P_c, resp. Let ℓ_a, ℓ_b, ℓ_c, be tangets of circumcircle $(PBC), (PCA), (PAB)$ at P_a, P_b, P_c, resp. Prove that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b, ℓ_c is tangent to the circumcircle (ABC).

When $P \equiv H$ orthocenter we have problem 6. I don't have solution, yet. I think we can use invension...

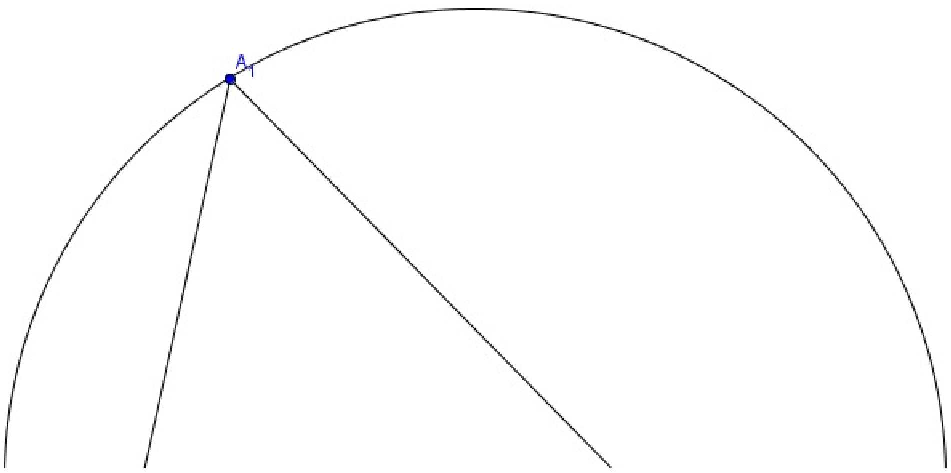
vulalach

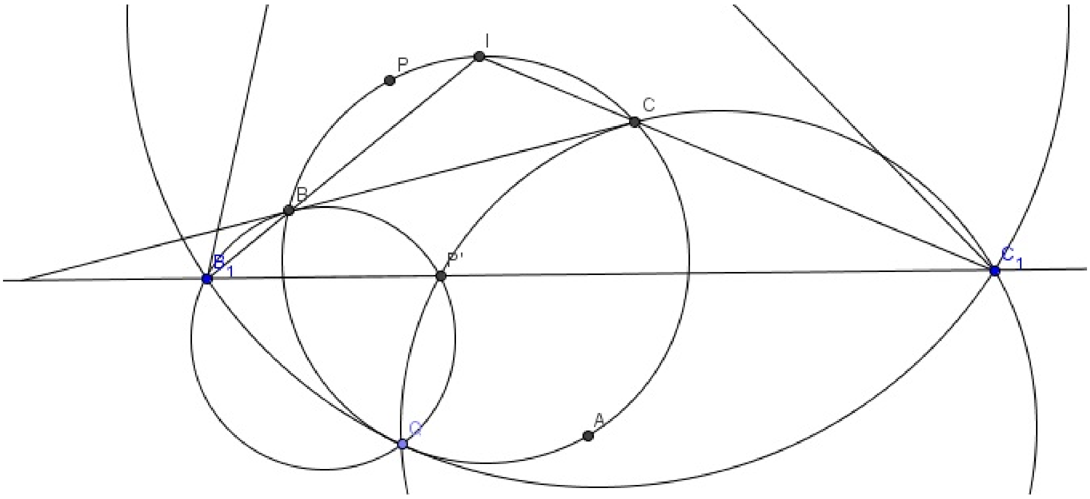
#4 Jul 19, 2011, 7:53 pm • 3

Some idea.
App steiner line wrt P and Migel theorem.

Nazar_Serdyuk

#5 Jul 19, 2011, 8:58 pm • 13





Denote the vertices of triangle determined by the lines ℓ_a, ℓ_b and ℓ_c by A_1, B_1, C_1 respectively.
Denote the intersection points of l with lines BC, CA, AB by A', B', C' respectively.
Denote the incenter of $A_1B_1C_1$ by I.
Denote the circumcircle of triangle ABC by w.
Denote by P tangent point of l and w.
WLOG points C' and B' lies on the sides of $\triangle A_1B_1C_1$ and point A' lies on extension of B_1C_1.

Point A is the excenter of triangle $A_1B'C' \Longrightarrow A_1A$ is the bisector of $\angle B_1A_1C_1$, similarly B_1B and C_1C - are the bisectors of angles $\angle A_1B_1C_1$ and $\angle A_1C_1B_1$ respectively, then

(1) AA_1, BB_1, CC_1 - passes though I.

$$\angle B_1IC_1 = 90^\circ + \frac{1}{2}\angle B_1A_1C_1; \angle C'AB' = 90^\circ - \frac{1}{2}\angle B'A_1C' \Longrightarrow \angle BIC + \angle BAC = 180^\circ \Longrightarrow$$

(2) A, B, C, I lie on a circle w.

Let P' - be a reflection of P with respect to the line BC. Then P' lies on B_1C_1. Let Q - be the intersection point of circumcircles of triangles B_1BP' and C_1CP'. Let $\angle IB_1C_1 = \alpha$, $\angle IC_1B_1 = \beta$, $\angle BCP' = x$, $\angle CBP' = y$. Then $\angle BQC = \angle BQP' + \angle P'QC = \angle BB_1P' + \angle P'C_1C = 180^\circ - \angle B_1IC_1 \Longrightarrow$

(3) points I, B, C, Q lie on a circle w.

$\angle B_1QC_1 = \alpha + \beta + x + y = 2\alpha + 2\beta$, because $\angle BP'C = \angle B_1IC_1 \Longrightarrow$

(4) points A_1, B_1, C_1, Q lie on a circumcircle of triangle $A_1B_1C_1$.

$$\angle B_1QB + \angle QC_1B_1 = x + \angle QC_1B_1 = \angle QCP' + \angle P'CB = \angle QCB \Longrightarrow$$

(5) $\angle B_1QB + \angle QC_1B_1 = \angle QCB$.

Let t - tangent line from Q to the circumcircle of triangle $A_1B_1C_1$. Then $\angle(g, QB) = \angle(g, QB_1) + \angle(QB_1, QB) = \angle(QC_1, C_1B_1) + x = \angle(QC, CB) \Longrightarrow t$ is the tangent line from Q to w $\Longrightarrow$

(6) w tangents to circumcircle of triangle $A_1B_1C_1$ at Q.

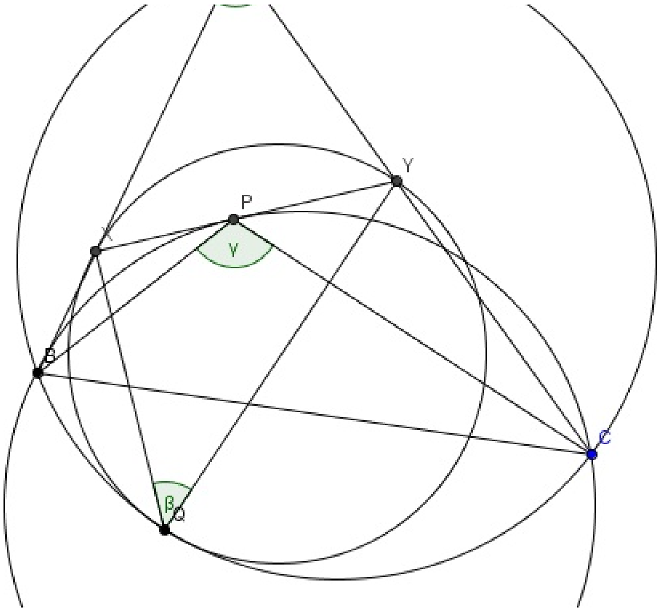
Nazar_Serdyuk

#6 Jul 20, 2011, 12:34 am • 7

Very good classic geometry problem for IMO, In fact there are two main parts here:

First is to prove that AA_1, BB_1, CC_1 - passes though I and that A, B, C, I lie on a circle w, that is not hard.

Second part is to find the way to prove that two particular circles tangent to each other while exactly point of tangency is not given from the beginning. I know only two ways here - inversion and a trick with two circles when the point of tangency is defined as an intersection point of two other circles. The second way works here. In particular case the following theorem holds:



Given a triangle ABC with circumcircle w. Points X and Y are on segments AB, AC respectively, $\angle BAC = \alpha$. Then there exist an arc of measure β constructed on XY (in other half-plane then point A) that tangent to w if and only if there exist an arc of measure $\alpha + \beta$ constructed on BC (in the same half-plane with point A) that tangent to XY.

This theorem can be applied in this problem for triangle IBC and segment B_1C_1

It is not too hard to repeat the trick with two circles but it is hard to reinvent such trick during the contest.

---

math154

#7 Jul 20, 2011, 1:51 am • 25

Yes, of course it's a very nice problem if you do it synthetically.

But taking this at home (not working that efficiently, either, e.g. thinking about the fact that Harry Potter is over), I was able to solve this in less than two hours with very straightforward complex numbers calculations (i.e. showing a discriminant is zero ...), which was extremely disappointing given the relatively simple nature of problems 4 and 5. There's only one "grind" at the end which some people like Yi Sun would not even call a grind in the first place.

In the future, I feel like we need more problems like IMO 2008 #6 which are both very hard to bash (in the context of the other two problems in that day) and very nice. (No, I did not bash it. It's one of my favorite geometry problems.)

Also, on a random note, WHERE IS THE GOOD, CHALLENGING NUMBER THEORY IN COMPETITIONS NOW?!!! Sorry, I just felt like saying that.

Too easy...

---

livetolove212

#8 Jul 20, 2011, 4:19 am • 8

> *WakeUp wrote:*
>
> Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

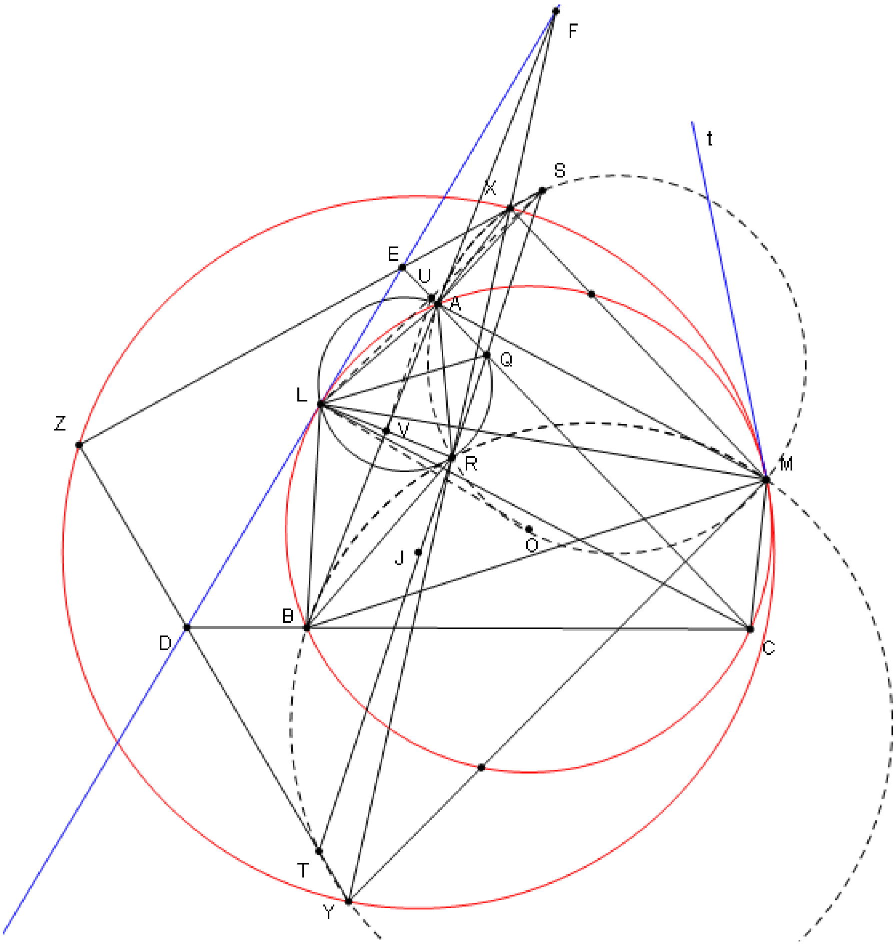
Denote D, E, F the intersections of l and BC, CA, AB, respectively; X, Y, Z the intersections of 3 lines l_a, l_b, l_c.
Let L be the point of contact of l and Γ; R, S, T be the reflections of L wrt AB, AC, BC; M be the Miquel point of the completed quadrilateral $XSRYTZ$.
$\angle ZXY = \angle FEX + \angle EFX = 180^o - 2\angle LEC + 2\angle LFA = 180^o - 2\angle BAC.$
We have the distances from A to l, DE, FY are equal so XA is the bisector of angle ZXY.
We get $\angle ZXA = \frac{1}{2}\angle ZXY = 90^o - \angle BAC.(1)$
On the other side, let U, V be the projections of L onto AC, AB then UV passes through the midpoint of LQ.
$\angle ARL = \angle ALR = 90^o - \angle LAV = 90^o - \angle LUV = \angle UQL$, which follows that L, A, Q, R are concyclic. We get

$\angle ARQ = \angle LRQ - \angle LRA = 180^o - \angle LAQ - 90^o + \angle LAB = 90^o - \angle BAC.(2)$
From (1) and (2) we obtain $A \in (XRS)$. Similarly with B, C.
So $\angle AMB = \angle XMY - \angle XMA - \angle BMY = 2\angle ACB - \angle XRA - \angle BRY$
$= 2\angle ACB - 180^o + \angle ARB = 2\angle CAB - 180^o + \angle ALB = \angle ACB$.
Therefore $M \in \Gamma$.
Construct a tangent Mt of (XYZ). We will show that Mt is also a tangent of Γ iff $\angle tMA = \angle ABM$.
$\Leftrightarrow \angle AMX + \angle XMt = \angle ABR + \angle RBM(3)$
But $\angle XMt = \angle XYM = \angle RBM, \angle AMX = ARX = \angle ALE = \angle ABL = \angle ABR$.
Hence (3) is true. We are done.

Another generalization: Given a triangle ABC with its circumcenter O. Let P be an arbitrary point in the plane. The line through P intersects $(BPC), (CPA), (APB)$ again at A_1, B_1, C_1. Let l_a be the tangent line through A_1 of $(BPC), l'_a$ be the reflection of l_a wrt BC. Similarly we define l'_b, l'_c. Then the circumcircle of the triangle formed by l'_a, l'_b, l'_c is tangent to (O).

When P lies on (O) we have IMO Pro.6

Attachments:



This post has been edited 2 times. Last edited by livetolove212, Jul 20, 2011, 5:51 pm

Fedor Petrov

#9 Jul 20, 2011, 3:15 pm • 7

> *buratinogigle wrote:*
> There is a generalition for this
>
> Let ABC be a triangle and a point P. A line pass through P intersect circumcircle $(PBC), (PCA), (PAB)$ again at P_a, P_b, P_c, resp. Let ℓ_a, ℓ_b, ℓ_c, be tangets of circumcircle $(PBC), (PCA), (PAB)$ at P_a, P_b, P_c, resp. Prove that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b, ℓ_c is tangent to the circumcircle (ABC).
>
> When $P \equiv H$ orthocenter we have problem 6. I don't have solution, yet. I think we can use invension...

Yes, we really may do it, and after inversion the corresponding circle will touch the circle ABC in the Miquel point of lines AB, BC, AC and our line passing through P. It may be easily checked by some direct angle-chasing.

goodar2006
#10 Jul 20, 2011, 10:50 pm • 3

> Fedor Petrov wrote:
>
> > buratinogigle wrote:
> > There is a generalition for this
> >
> > Let ABC be a triangle and a point P. A line pass through P intersect circumcircle $(PBC), (PCA), (PAB)$ again at P_a, P_b, P_c, resp. Let ℓ_a, ℓ_b, ℓ_c, be tangets of circumcircle $(PBC), (PCA), (PAB)$ at P_a, P_b, P_c, resp. Prove that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b, ℓ_c is tangent to the circumcircle (ABC).
> >
> > When $P \equiv H$ orthocenter we have problem 6. I don't have solution, yet. I think we can use invension...
>
> Yes, we really may do it, and after inversion the corresponding circle will touch the circle ABC in the Miquel point of lines AB, BC, AC and our line passing through P. It may be easily checked by some direct angle-chasing.

what you said seems really intresting, would you please give us a detailed solution?? Thanks

nsato
#11 Jul 20, 2011, 11:24 pm • 8

Interestingly, the fact that the incenter of the determined triangle lies on the circumcircle of triangle ABC appeared on a 1995 Iran olympiad:
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=379391
(And it works for any line.)

darij grinberg
#12 Jul 20, 2011, 11:41 pm • 7

livetolove212's generalization is exactly the same as buratino's. Congratulations, buratino and Fedor, for the first correct solution using inversion. (Nobody gave that in the exam. Then again, apparently only inversions with center T were attempted, where T is the point of tangency of ℓ with Γ.)

Also there is a nice solution by a contestant whose name and country I don't remember, which shows a bit more: If the altitudes of triangle ABC from the vertices A, B, C intersect Γ a second time at X, Y, Z, respectively, and if X', Y', Z' are the reflections of T (the point where ℓ touches Γ) in BC, CA, AB, then the lines XX', YY', ZZ' concur at the point where Γ touches Γ'. Try to reconstruct the solution from this.

Also there exists an approach which uses Casey's theorem (with three circles being degenerate to the vertices A', B', C', and the fourth circle being Γ). I think IRN1 found it, and maybe others too.

shoki
#13 Jul 21, 2011, 12:28 am • 3

yeah ... darij is right ... she found that solution

Luis González
#14 Jul 21, 2011, 12:33 am • 9

> buratinogigle wrote:
> Let ABC be a triangle and a point P. A line pass through P intersect circumcircle $(PBC), (PCA), (PAB)$ again at P_a, P_b, P_c, resp. Let ℓ_a, ℓ_b, ℓ_c, be tangets of circumcircle $(PBC), (PCA), (PAB)$ at P_a, P_b, P_c, resp. Prove that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b, ℓ_c is tangent to the circumcircle (ABC).

Perform an inversion with center P and arbitrary power. Label inverse points with primes. The given line ℓ through P is double and circles $\odot(PBC), \odot(PCA)$ and $\odot(PAB)$ go to the sidelines $B'C', C'A', A'B'$ of $\triangle A'B'C'$. ℓ cuts $B'C', C'A'$, $A'B'$ at P_a', P_b', P_c'. Thus, tangents ℓ_a, ℓ_b, ℓ_c of $\odot(PBC), \odot(PCA), \odot(PAB)$ through P_a, P_b, P_c go to the circles $\omega_a, \omega_b, \omega_c$ passing through P and tangent to $B'C', C'A', A'B'$ through P_a', P_b', P_c'. Pairwise circles $\omega_a, \omega_b, \omega_c$ meet at

D', E', F', the inverses of the vertices of $\triangle(\ell_a, \ell_b, \ell_c) \equiv \triangle DEF$. Hence, in order to show that $\odot(DEF)$ is tangent to $\odot(ABC)$, we shall show that $\odot(D'E'F')$ is tangent to $\odot(A'B'C')$.

For convenience, drop the primes from the new figure. Let M be the Miquel point of $\triangle ABC \cup \ell$. Henceforth, we'll use oriented angles (mod 180). From the tangencies of ω_b, CA and ω_c, AB we get $\angle AP_bP_c = \angle PDP_b$ and $\angle AP_cP_b = \angle PDP_c \Longrightarrow D \in \odot(AP_bP_c)$. Likewise, we have $E \in \odot(BP_cP_a)$ and $F \in \odot(CP_aP_b)$. Now, the rest is just simple angle chase using inscribed angles spanning the same arcs.

$\angle EFD = \angle PFD + \angle PFE = \angle P_cP_bD + \angle P_cP_aE = \angle DMP_c + \angle EMP_c$

$\Longrightarrow \angle EFD = \angle EMD \Longrightarrow M \in \odot(DEF)\ (\star)$.

$\angle MFE = \angle EFP_a + \angle MFP_a = \angle EP_aB + \angle MCB = \angle EMB + \angle MAB$.

$\Longrightarrow 90^\circ - \angle MFE + \angle EMB = 90^\circ - \angle MAB \Longrightarrow$ Circumcenters of $\triangle MEF$ and $\triangle MBA$ are collinear with M. Together with $(\star)$, we deduce that $\odot(ABC)$ and $\odot(DEF)$ are tangent through M, as desired.

This post has been edited 2 times. Last edited by Luis González, Jul 12, 2012, 12:43 am

Swistak

#15 Jul 21, 2011, 3:06 am • 2

> *darij grinberg wrote:*
>
> Also there exists an approach which uses Casey's theorem (with three circles being degenerate to the vertices A', B', C', and the fourth circle being Γ). I think IRN1 found it, and maybe others too.

Could you say more about it? I thought about it few seconds after I read this problem, but I couldn't compute $d_X, d_Y, d_Z, XY, XZ, YZ$, where X, Y, Z are vertices of triangle given in this problem.

livetolove212

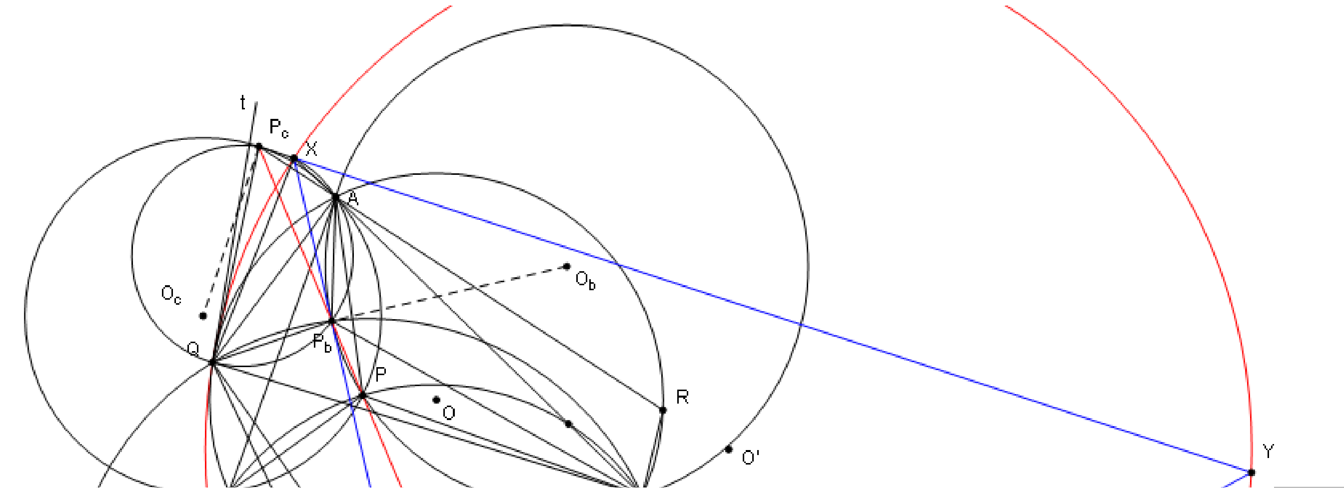
#16 Jul 21, 2011, 8:22 am • 3

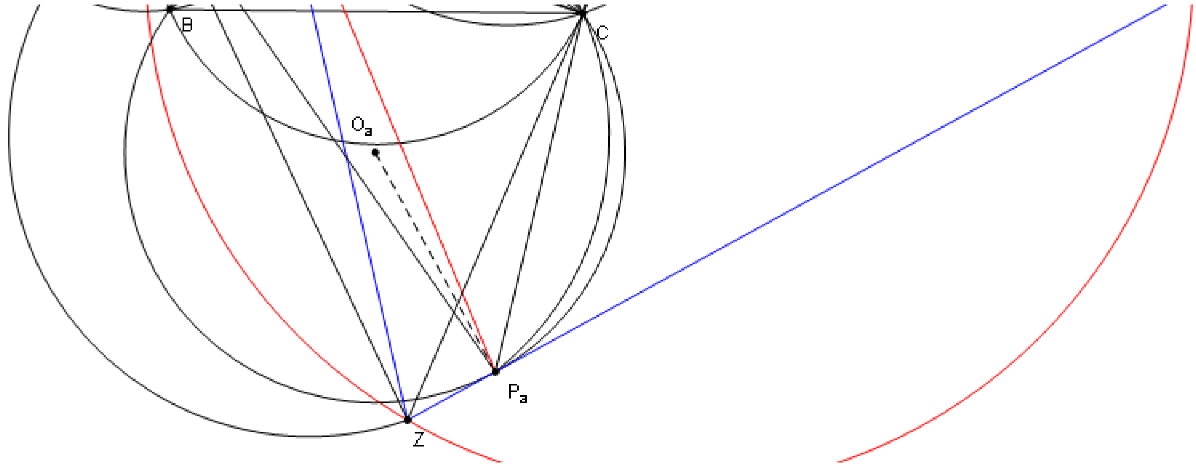
We can use angle chasing to prove Mr.Hung's generalization. The idea is the same as my proof for problem 6.

Let X, Y, Z be the triangle formed by l_a, l_b, l_c; Q be the Miquel point of the completed quadrilateral $XYZP_aP_bP_c$; R be the intersection of AP_c and CP_a.
$\angle AP_cX = \angle P_cPA = \angle P_bCA = \angle XP_bA$ then $A \in (XP_bP_c)$. Similarly with B, C.
We will show that Q lies on (ABC).
We have $\angle P_cAB + \angle P_aCB = \angle P_cPB + \angle P_aPB = 180^o$ then A, B, C, R are concyclic.
We conclude that $\angle AQC = \angle AQP_b + \angle CQP_b = \angle P_bP_cA + \angle P_bP_aC = 180^o - \angle ARC = \angle ABC$. Therefore $Q \in (ABC)$.
Construct the tangent Qt of (XYZ). The idea is to show that Qt is also the tangent of (ABC), iff $\angle tQA = \angle ACQ$.
$\Leftrightarrow \angle tQX + \angle XQA = \angle ACP_b + \angle P_bCQ(*)$
But $\angle tQX = \angle QZX = \angle QCP_b, \angle XQA = \angle XP_bA = \angle QCP_b$. So $(*)$ is true. We are done.

We have a new **problem**:
Given a triangle ABC with its circumcircle (O). Let (O') be a circle tangent to (O). P is an arbitrary point on (O). Three rays PA, PB, PC (or AP, BP, CP) intersect (O') again at A_1, B_1, C_1. Let $A_2B_2C_2$ be the triangle formed by the reflections of A_1B_1 wrt AB, B_1C_1 wrt BC, C_1A_1 wrt CA. Then $(A_2B_2C_2)$ is also tangent to (O).

Attachments:





This post has been edited 2 times. Last edited by livetolove212, Aug 19, 2011, 8:13 am

stani95

#17 Jul 21, 2011, 4:50 pm • 1

Try something I did at home - prove that $OO' = R' - R$, where O' is the curcumcenter of $A'B'C'$ and R' is the curcumradius of $A'B'C'$. I did it using triangle $OO'I$ and applying the cosine Th, where I is the incenter of $A'B'C'$. It works but you have to spend a lot of time calculating it!

armpist

#18 Jul 22, 2011, 12:17 am • 3

Dear MLs

If we reflect not just the line l, but also the circle (ABC) in
the sides, we get get a triangle with sides tangent to Karnot
circles. By Bobillier theorem this triangle remains similar when
tangency point of l changes.
A*B*C* is triangle determined by the lines l_a, l_b, l_c
The tangency points of triangle A*B*C* with Karnot
circles are collinear.

From this it should be possible to quickly show that
(ABC) and (A*B*C*) are incircle and 9pc of some triangle,
and are internally tangent at F point.

M.T.

Ichserious

#19 Jul 22, 2011, 4:26 pm • 4

Among the 6 contestants who got 7 in this problem, two of them are bronze medalists (and one from HKG!) interesting!

jayme

#20 Jul 22, 2011, 5:31 pm • 4

Dear Mathlinkers,
having the same beginning ideas as Armpist, a synthetic proof without any calculation is possible and will appears next on my site.
Sincerely
Jean-Louis

armpist

#21 Jul 22, 2011, 9:06 pm • 1

Dear MLs and J-L

It is also worth mentioning that triangle A*B*C* takes
it's extreme dimensions when line l tangency points

it's extreme dimensions when line l tangency points
are on Euler line of ABC.

M.T.

This post has been edited 1 time. Last edited by armpist, Jul 23, 2011, 5:13 am

alikhezeli

#22 Jul 23, 2011, 2:17 am • 4

I have a solution using Casey's theorem. If the powers of A', B', C' with respect to Γ are p_a, p_b, p_c respectively, it suffices to prove

$$\pm B'C'.\sqrt{p_a} \pm C'A'.\sqrt{p_b} \pm A'B'.\sqrt{p_c} = 0.$$

We try to compute $B'C'.\sqrt{p_a}$. In doing so, we omit constants that are the same for the other two terms.
Let d_a, d_b, d_c be the distances of A, B, C from l respectively. By considering the angles of $\Delta A'B'C'$ and the fact that I, the incenter of $\Delta A'B'C'$, lies on Γ, we have:

$$A'A = \frac{d_a}{\cos A}.$$

$$A'I \sim \frac{1}{\sin \frac{A'}{2}} \sim \frac{1}{\cos A}.$$

So we have

$$\sqrt{p_a} \sim \frac{\sqrt{d_a}}{\cos A} \sim \frac{AP}{\cos A}$$

where P is the point of tangency of l with Γ. Also, we have

$$B'C' \sim \sin A' \sim \sin 2A.$$

So

$$B'C'.\sqrt{p_a} \sim AP.\sin A \sim AP.BC.$$

Now the assertion follows by the Ptolemy's theorem.

This post has been edited 1 time. Last edited by alikhezeli, Jul 26, 2011, 5:59 pm

provacc

#23 Jul 25, 2011, 4:20 pm • 1

I have done this problem after much effort, not realizing that I have reproved the Miquel point's existence along the way. Thus my solution is exactly like livetolove212's.
Also, I find Nak's theorem especially useful. If you do not like angle chasing, here's a proof using Pascal's theorem.

For easy explanation, the arc measure beta constructed on XY is a part of the w2 circle.
Denote the contact point of the two circles L. LX and LY intersect for the second time with (w) at X' and Y'.
Call P' the intersection of BY' and CX'. According to Pascal, X,P' and Y are collinear (1).
Also, it's obvious that XY//X'Y'. Thus we can prove angle P'BC = angle YP'C, leading to the fact that (BP'C) contacts XY (2)
From (1) and (2) it can be concluded that P' = P.
The rest is to prove that angle BPC = alpha + beta, which is fairly easy.

bork

#24 Jul 26, 2011, 12:01 am • 1

To luisgeometra

$\angle MFE = \angle EFP_a + \angle MFP_a = \angle EP_aB + \angle MCB = \angle EMB + \angle MAB.$

Can you explain why is that so?

Andy Loo

#25 Jul 26, 2011, 6:51 am • 1

> Swistak wrote:
>
> > darij grinberg wrote:
> >
> > Also there exists an approach which uses Casey's theorem (with three circles being degenerate to the vertices A', B', C', and the fourth circle being Γ). I think IRN1 found it, and maybe others too.
>
> Could you say more about it? I thought about it few seconds after I read this problem, but I couldn't compute $d_X, d_Y, d_Z, XY, XZ, YZ$, where X, Y, Z are vertices of triangle given in this problem.

One bronze medalist from Hong Kong used Casey's theorem to solve it too~

alikhezeli

#26 Jul 26, 2011, 6:05 pm • 1

> Swistak wrote:
>
> > darij grinberg wrote:
> >
> > Also there exists an approach which uses Casey's theorem (with three circles being degenerate to the vertices A', B', C', and the fourth circle being Γ). I think IRN1 found it, and maybe others too.
>
> Could you say more about it? I thought about it few seconds after I read this problem, but I couldn't compute $d_X, d_Y, d_Z, XY, XZ, YZ$, where X, Y, Z are vertices of triangle given in this problem.

I have written my solution using casey's theorem above (http://www.artofproblemsolving.com/Forum/viewtopic.php?p=2369673#p2369673). Previously, I didn't use the name of the theorem. I corrected it now.

Lei Lei

#27 Jul 28, 2011, 10:41 am • 1

> darij grinberg wrote:
>
> ....
>
> Also there is a nice solution by a contestant whose name and country I don't remember, which shows a bit more: If the altitudes of triangle ABC from the vertices A, B, C intersect Γ a second time at X, Y, Z, respectively, and if X', Y', Z' are the reflections of T (the point where ℓ touches Γ) in BC, CA, AB, then the lines XX', YY', ZZ' concur at the point where Γ touches Γ'. Try to reconstruct the solution from this.
>
> ...

one of contestants from Singapore showed it in his solution. Not to mention the fact that whole proof only employed angel chasing.

Lei^2

jayme

#28 Aug 4, 2011, 3:20 pm • 1

Dear Mathlinkers,
an article concerning the “52nd I.M.O. Problem 6” has been put on my website
with a new approach based on the Mannheim’s circle
http://perso.orange.fr/jl.ayme vol. 9
You can use Google translator
Sincerely
Jean-Louis

malcolm

#29 Aug 5, 2011, 12:03 pm • 3

Another complex number solution:

Click to reveal hidden text

benimath

#30 Aug 5, 2011, 3:10 pm • 1

> *armpist wrote:*
> Dear MLs
>
> If we reflect not just the line l, but also the circle (ABC) in
> the sides, we get get a triangle with sides tangent to Karnot
> circles. By Bobillier theorem this triangle remains similar when
> tangency point of l changes.
> A*B*C* is triangle determined by the lines l_a, l_b, l_c
> The tangency points of triangle A*B*C* with Karnot
> circles are collinear.
>
> From this it should be possible to quickly show that
> (ABC) and (A*B*C*) are incircle and 9pc of some triangle,
> and are internally tangent at F point.
>
> M.T.

Can you give more details please? This seems like an interesting new solution, but I can't figure out the details. Thank you.

Gleek-00

#31 Aug 6, 2011, 4:42 am • 1

I wonder wich method Lisa Sauermann used to solve this problem ?

neergard

#32 Aug 13, 2011, 7:22 am • 4

> *buratinogigle wrote:*
> There is a generalition for this
>
> Let ABC be a triangle and a point P. A line pass through P intersect circumcircle $(PBC), (PCA), (PAB)$ again at P_a, P_b, P_c, resp. Let ℓ_a, ℓ_b, ℓ_c, be tangets of circumcircle $(PBC), (PCA), (PAB)$ at P_a, P_b, P_c, resp. Prove that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b, ℓ_c is tangent to the circumcircle (ABC).
>
> When $P \equiv H$ orthocenter we have problem 6. I don't have solution, yet. I think we can use invension...

We need neither inversion nor Miquel:

Throughout the following, angles are assumed to be oriented and are considered modulo $180°$. Let $A' = \ell_b \cap \ell_c$ etc. Since $\angle ABP_c = \angle APP_c = \angle APP_b = \angle ACP_b$, the point $A'' = BP_c \cap CP_b \in (ABC)$. Similarly B'' and C''. Since $\angle CB''C'' = \angle CBC'' = \angle CBP_a = \angle CP_aP$, we have $B''C'' \parallel \ell_a$. Similarly for the other sides of triangle $A''B''C''$. Thus the triangles $A'B'C'$ and $A''B''C''$ are related by a homothety with centre $K = B'B'' \cap C'C''$. We prove $K \in (ABC)$. Then the circumcircles $(A'B'C')$ and (ABC) of these triangles touch each other in K.

Let $K_b = B'B'' \cap (ABC)$ and $K_c = C'C'' \cap (ABC)$. Then $\angle BK_bB' = \angle BK_bB'' = \angle BC''B'' = \angle BP_aB'$. Similarly $\angle BK_bB' = \angle BP_cB'$, so B, B', K_b, P_a, P_c are concyclic. Similarly C, C', K_c, P_a, P_b are concyclic. Now we have $\angle P_aK_bB = \angle P_aP_cB = \angle PP_cB = \angle PAB$. Similarly $\angle P_aK_cC = \angle PAC$, so $\angle P_aK_cC - \angle P_aK_bB = \angle PAC - \angle PAB = \angle BAC = \angle BK_bC = \angle P_aK_bC - \angle P_aK_bB$, whence we get $\angle P_aK_cC = \angle P_aK_bC$. We infer $K_b \in (CC'K_c)$, and similarly $K_c \in (BB'K_b)$. Since the circles $(BB'K_b)$ and $(CC'K_c)$ have at most one common point different from P_a, we get $K_b = K_c = K$, so $K \in (ABC)$, and we are done.

Swistak

#33 Mar 13, 2012, 2:28 am • 1

There's also pretty nice solution using very simple angle chasing, Pascal and homothety.

Image not found

Γ is the circumcircle of ABC and Ω is the circumcircle of EFG.

J is the intersection of GA and FB. We know that J is the incenter of triangle EFG and that J lies on Γ. Let I and H be points lying on Γ such that $DA = AI$ and $DB = BH$. By lemma proved here: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=469237 by JustN, we know that FH and GI intersect on Γ. Let M be the point of their intersection. $\angle IMH = 2\angle ACB$ since M lies on Γ. We also know that $\angle FEG = 180^o - 2\angle ACB$, so M also lies on Ω (since $\angle GMF = \angle IMH = 2\angle ACB = 180^o - \angle GEF$). But IH is parallel to FG so by homothety we know that these circles must be tangent in M.

IDMasterz

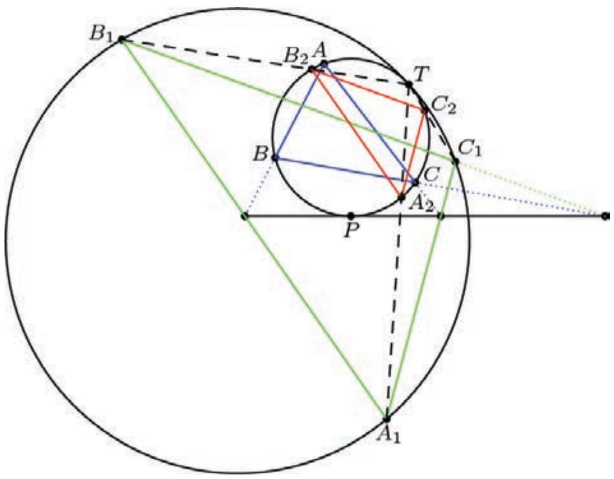
#34 Jul 13, 2013, 4:16 pm

I posted a proof here: http://www.artofproblemsolving.com/Forum/blog.php?u=118092&b=90745

v_Enhance

#35 Jun 12, 2014, 10:22 am • 11

> *math154 wrote:*
>
> But taking this at home (not working that efficiently, either, e.g. thinking about the fact that Harry Potter is over), I was able to solve this in less than two hours with very straightforward complex numbers calculations (i.e. showing a discriminant is zero ...), which was extremely disappointing given the relatively simple nature of problems 4 and 5. There's only one "grind" at the end which some people like Yi Sun would not even call a grind in the first place.



It turns out that it's even easier if you make the right claims. If you can guess the second intersection A_2 below, then this is a 15-minute computation. Of course, you make such guesses by drawing a REALLY good diagram.

Complex numbers with ω the unit circle and $p = 1$. Let $A_1 = \ell_B \cap \ell_C$, and let $a_2 = a^2$ (in other words, A_2 reflection of P across the diameter of ω through A). Define B_1, C_1, B_2, C_2 similarly. Simple angle chasing shows that $\overline{B_1C_1} \parallel \overline{B_2C_2}$; it follows that $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$. Now the claim is that $\overline{A_1A_2}$, $\overline{B_1B_2}$, $\overline{C_1C_2}$ concur at a point T on ω, which solves the problem.

First, we need to compute A_1. If we reflect the points $1+i$ and $1-i$ over $\overline{AB}$, then we get two points Z_1, Z_2 with

$$z_1 = a + b - ab(1-i) = a + b - ab + abi$$
$$z_2 = a + b - ab(1+i) = a + b - ab - abi$$
$$\implies z_1 - z_2 = 2abi \text{ and } \overline{z_1}z_2 - \overline{z_2}z_1 = -2i\left(a + b + \frac{1}{a} + \frac{1}{b} - 2\right).$$

Anyways, ℓ_C is $\overline{Z_1Z_2}$, and so doing the same thing with ℓ_B and using the intersection formula,

$$a_1 = \frac{-2i\left(a+b+\frac{1}{a}+\frac{1}{b}-2\right)(2aci) - -2i\left(a+c+\frac{1}{a}+\frac{1}{c}-2\right)(2abi)}{\left(-\frac{2}{ab}i\right)(2aci) - \left(-\frac{2}{ac}i\right)(2abi)}$$
$$[c-b]\,a^2 + \left[\frac{c}{b} - \frac{b}{c} - 2c + 2b\right]a + (c-b)$$

$$= \frac{\quad\cdots\quad}{\frac{c}{b}-\frac{b}{c}}$$
$$= a + \frac{(c-b)\left[a^2-2a+1\right]}{(c-b)(c+b)/bc}$$
$$= a + \frac{bc}{b+c}(a-1)^2.$$

Then the second intersection of $\overline{A_1A_2}$ with ω is given by

$$\frac{a_1-a_2}{1-a_2\overline{a_1}} = \frac{a+\frac{bc}{b+c}(a-1)^2-a^2}{1-a-a^2\cdot\frac{(1-1/a)^2}{b+c}}$$
$$= \frac{a+\frac{bc}{b+c}(1-a)}{1-\frac{1}{b+c}(1-a)}$$
$$= \frac{ab+bc+ca-abc}{a+b+c-1}.$$

As this is symmetric in a, b, c we are done. ■

Never thought the day would come when I could solve this~

Wolstenholme

#36 Nov 1, 2014, 6:24 am • 1

Let ℓ be tangent to Γ at T and let $A_1 = \ell_b \cap \ell_c$ and $B_1 = \ell_c \cap \ell_a$ and $C_1 = \ell_a \cap \ell_b$. Let $A_2 = \ell \cap \ell_a$ and $B_2 = \ell \cap \ell_b$ and $C_2 = \ell \cap \ell_c$. WLOG assume that the order of points on line ℓ is C_2, T, B_2, A_2 (to avoid configuration issues). Let I be the incenter of $\triangle A_1B_1C_1$.

Lemma 1: $AA_1 \cap BB_1 \cap CC_1 = I$

Proof: Consider $\triangle A_2B_1C_2$. It is clear that line AB is the internal angle bisector of $\angle B_1C_2A_2$ and it is also clear that line BC is the internal angle bisector of $\angle B_1A_2C_2$. Therefore B is the incenter of this triangle; hence, line BB_1 bisects $\angle A_1B_1C_1$. Similarly line AA_1 bisects $\angle B_1A_1C_1$ so we have the desired result.

Lemma 2: $I \in \Gamma$

Proof: Note that $\angle BIC = \angle B_1IC_1 = 90 + \dfrac{\angle B_1A_1C_1}{2} = 180 - \dfrac{\angle A_1C_2B_2}{2} - \dfrac{\angle A_1B_2C_2}{2} =$ $\angle AC_2B_2 + \angle A_{B2}C_2 = 180 - \angle BAC$ which implies the desired result.

Now Let O be the center of Γ and let T_B and T_C be the reflections of T about OB and OC respectively. Clearly T_B and T_C are on Γ.

Lemma 3: $T_BT_C \parallel B_1C_1$

Proof: Note that $\angle C_1A_2T = \angle COT - \angle BOT$ which immediately implies the desired result.

Letting T_A be the reflection of T about OA we have that $T_AT_B \parallel A_1B_1$ and $T_CT_A \parallel C_1A_1$ as well. Therefore there exists a homothety taking $\triangle T_AT_BT_C$ to $\triangle A_1B_1C_1$. It suffices to show that the center of this homothety lies on Γ, because then the homothety will take Γ to the circumcircle of $\triangle A_1B_1C_1$ and so will be the tangency point between these two circles.

Now let Q be the reflection of T about BC. Cleary $Q \in B_1C_1$. Since $\angle TBQ = 2\angle TBC = \angle TBT_C$ we have that $Q \in BT_C$ and similarly that $Q \in CT_B$. Therefore $Q = B_1C_1 \cap BT_C \cap CT_B$. Now let P be the second intersection of line B_1T_B and Γ and let $X = PT_C \cap IC_1$. It suffices to show that $X = C_1$. By Pascal's Theorem on cyclic hexagon T_CBICT_BP we have that Q, B_1, X are collinear so X lies on line B_1C_1 which implies that $X = C_1$ so P is the desired tangency point and we are done.

TelvCohl

#37 Feb 26, 2015, 10:18 pm • 2

My solution :

Lemma:

Let ℓ be a line cut $\odot(ABC)$ at T .

Let H be the orthocenter of $\triangle ABC$ and S be the Anti-Steiner point of TH WRT $\triangle ABC$

Let H be the orthocenter of $\triangle ABC$ and S be the Anti-Steiner point of TH WRT $\triangle ABC$.
Let $\triangle A^*B^*C^*$ be the triangle formed by the reflection of ℓ in BC, CA, AB, respectively .

Then $S \in \odot(A^*B^*C^*)$.

Proof:

Let X, Y, Z be the reflection of T in BC, CA, AB, respectively .

It's well-known that X, Y, Z, H are collinear at the Steiner line of T WRT $\triangle ABC$,
so from symmetry we get $\angle C^*XS = \angle(TH, \ell) = \angle C^*YS \Longrightarrow S \in \odot(C^*XY)$.

Similarly we can prove $S \in \odot(A^*YZ)$ and $S \in \odot(B^*XZ)$,
so S is the Miquel point of complete quadrilateral $\{B^*C^*, C^*A^*, A^*B^*, XYZ\} \Longrightarrow S \in \odot(A^*B^*C^*)$.

From the lemma we get following property :

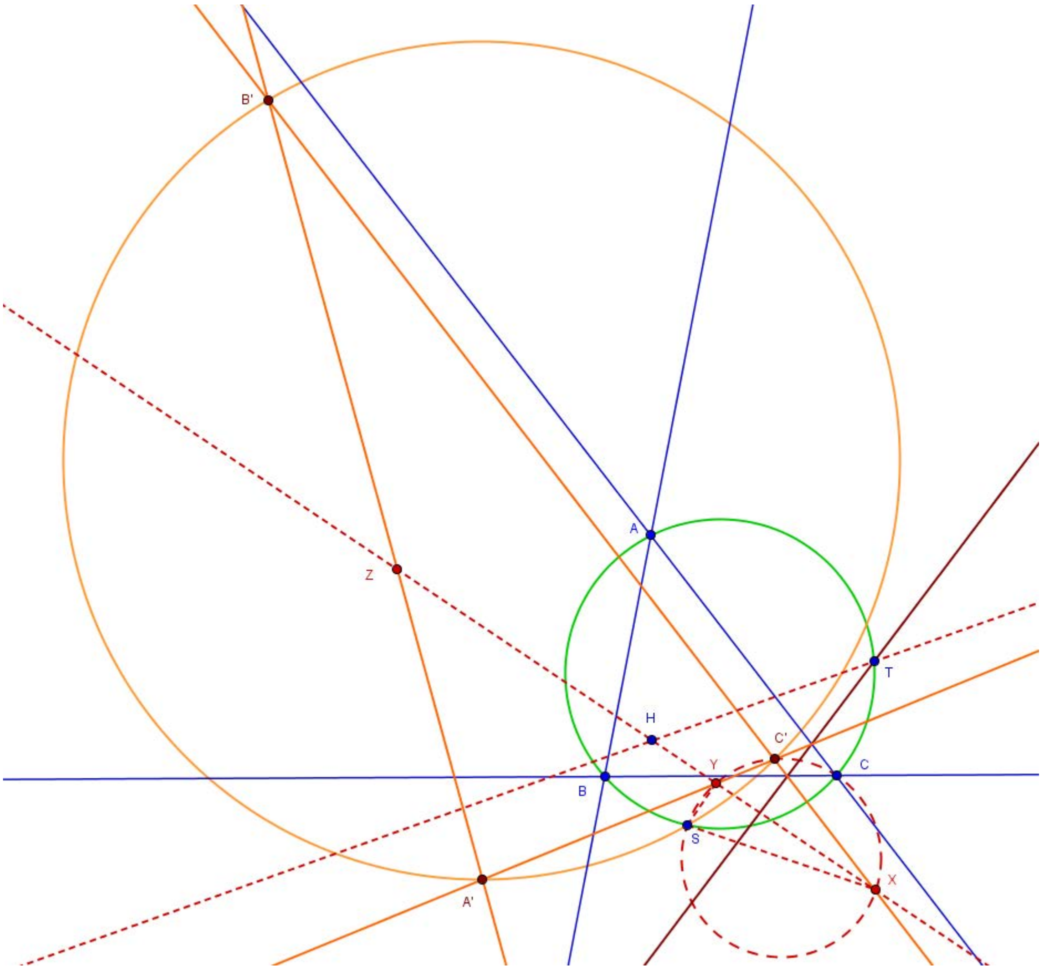
Let ℓ be a line cut $\odot(ABC)$ at P, Q and H be the orthocenter of $\triangle ABC$.
Let P^*, Q^* be the Anti-Steiner point of HP, HQ WRT $\triangle ABC$, respectively .
Let $\triangle A^*B^*C^*$ be the triangle formed by the reflection of ℓ in BC, CA, AB, respectively .

Then $\{P^*, Q^*\} = \odot(ABC) \cap \odot(A^*B^*C^*)$.

When P coincide with Q we get the original problem

P.S. I posted another generalization of 2011 IMO P6 at here

Attachments:



pi37

#38 May 21, 2015, 6:07 am

We use complex numbers, denoting the complex values of points by lowercase.

Let $p = 1$, and let a, b, c be on the unit circle.

If X is the intersection of the reflection of the tangent to P across AC and AB, with Y, Z defined similarly, and let

$d=a^2, e=b^2, f=c^2$. We claim that DEF and XYZ are homothetic centered about a point on the unit circle.

The reflection of a point x over ac is $a+c-ac\bar{x}$, and a point lies on the tangent of P iff the the sum of itself and its conjugate is 2. So if x corresponds to X, then

$$a+c-ac\bar{x}+\frac{1}{a}+\frac{1}{c}-\frac{x}{ac}=2$$

and

$$b+c-bc\bar{x}+\frac{1}{b}+\frac{1}{c}-\frac{x}{bc}=2$$

Subtracting these and dividing by $a-b$ yields

$$1-c\bar{x}-\frac{1}{ab}+\frac{x}{abc}=0 \Rightarrow ac\bar{x}=a-\frac{1}{b}+\frac{x}{bc}$$

so

$$c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-2=\frac{x}{ac}+\frac{x}{bc}$$

and

$$x=\frac{abc^2+ab+bc+ca-2abc}{a+b}$$

Let

$$s=\frac{abc-ab-bc-ca}{1-a-b-c}$$

Then

$$s-f=\frac{abc-ab-bc-ca}{1-a-b-c}-c^2=\frac{abc-ab-bc-ca-c^2+ac^2+bc^2+c^3}{1-a-b-c}=\frac{(c-1)(a+c)(b+c)}{1-a-b-c}$$

and

$$s-x=\frac{abc^2+ab+bc+ca-2abc}{a+b}-c^2=\frac{abc^2+ab+bc+ca-2abc-ac^2-bc^2}{a+b}=\frac{(c-1)(abc-ab-bc-ca)}{a+b}$$

so

$$\frac{s-f}{s-x}=\frac{(a+b)(a+c)(b+c)}{(1-a-b-c)(abc-ab-bc-ca)}$$

This ratio is symmetric, so it is the same for permutations of the variables. We claim that it is also real. This is true because its conjugate is

$$\frac{(1/a+1/b)(1/a+1/c)(1/b+1/c)}{(1-1/a-1/b-1/c)(1/abc-1/ab-1/bc-1/bc)}=\frac{(a+b)(a+c)(b+c)}{(abc-ab-bc-ca)(1-a-b-c)}$$

which is equal to the original. So S is the homothety center mapping DEF to XYZ. But the conjugate of s is

$$\frac{1/abc-1/ab-1/bc-1/ca}{1-1/a-1/b-1/c}=\frac{1-a-b-c}{abc-ab-bc-ca}$$

which is its inverse, so s lies on the unit circle as desired.

This post has been edited 1 time. Last edited by pi37, May 21, 2015, 6:08 am

JuanOrtiz

#39 Jun 28, 2015, 6:17 am

Let $\ell \cap \ell_a = D, \ell \cap \ell_b = E$, and $\ell \cap \ell_c = F$. Let $\ell_b \cap \ell_c = X, \ell_c \cap \ell_a = Y$, and $\ell_a \cap \ell_b = Z$. Let P_A be the reflection of P across side BC. Let $\triangle X'Y'Z'$ be the triangle inscribed in Γ homothetic to XYZ, and let $Q = YY' \cap \Gamma$.

Firstly, notice that C is the incenter of $\triangle DEZ$, and thus ZC bisects $\angle YZX$. By the analogous statements, we conclude $ZC \cap YB \cap XZ = H$, the incenter of $\triangle XYZ$. (Note: $H \in \Gamma$, since $\angle YHZ = 90 + (\angle X)/2 = 90 + (180 - 2\angle A)/2 = 180 - \angle A$).

Secondly, notice that

$$\widehat{\text{AX'}} - \widehat{\text{CZ'}} = \angle(AC, X'Z') = \angle(\ell, AC) = \widehat{\text{AP}} - \widehat{\text{PC}},$$
$$\widehat{\text{CZ'}} - \widehat{\text{BY'}} = \angle(BC, Y'Z') = \angle(\ell, BC) = \widehat{\text{PC}} - \widehat{\text{PB}}, \text{ and}$$
$$\widehat{\text{BY'}} - \widehat{\text{AX'}} = \angle(AB, X'Y') = \angle(\ell, AB) = \widehat{\text{PB}} - \widehat{\text{PA}},$$

which implies that $\widehat{\text{AX'}} = \widehat{\text{AP}}, \widehat{\text{BY'}} = \widehat{\text{BP}}$, and $\widehat{\text{CZ'}} = \widehat{\text{CP}}$.

Now, notice that $CP = CP_A = CZ'$ and $BP = BP_A = BY'$. Hence,

$$\angle PP_AZ' = 180 - \frac{\angle PCZ'}{2} = 90 - \widehat{\text{PC}} = 90 - \frac{\angle PBP_A}{2} = \angle BP_AP,$$

which implies that B, P_A, and Z' are collinear. Analogously, C, P_A, and Y' are also collinear.

Finally, apply Pascal's Theorem to the cyclic hexagon $Y'QZ'BHC$. Since $Y'Q' \cap BH = Y$, $Z'B \cap Y'C = P_A$, and $YP_A \cap HC = Z$, we obtain that Q, Z', and Z are collinear.

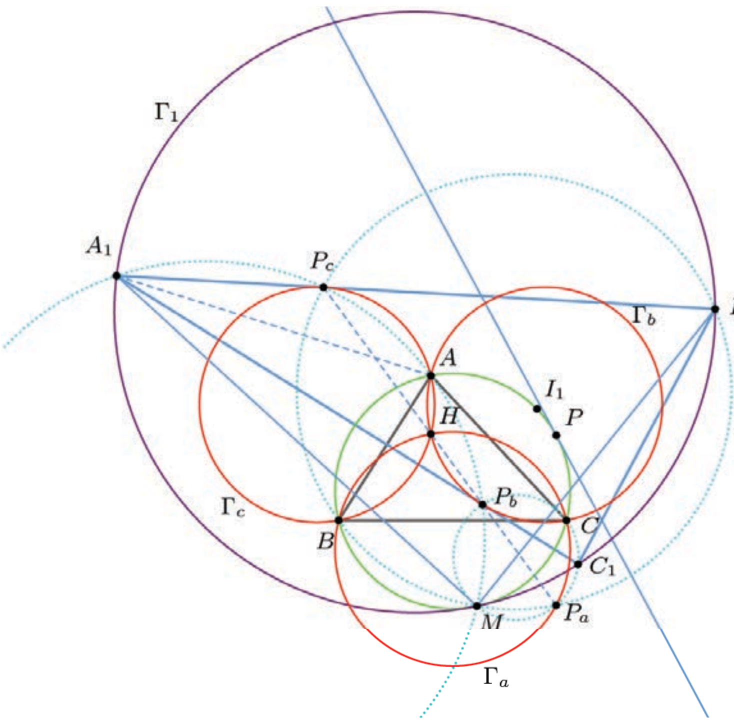
Analogously, Q, X', and X are collinear, which implies that Q is the center of the homothety from $\triangle X'Y'Z'$ to $\triangle XYZ$. Since $Q \in \odot(X'Y'Z') = \Gamma$, this immediately implies that $Q \in \odot(XYZ)$, and so Γ and $\odot(XYZ)$ are tangent at Q.

Q.E.D.

This post has been edited 5 times. Last edited by JuanOrtiz, Jun 28, 2015, 10:23 am

liberator

#40 Jul 22, 2015, 7:38 pm • 1



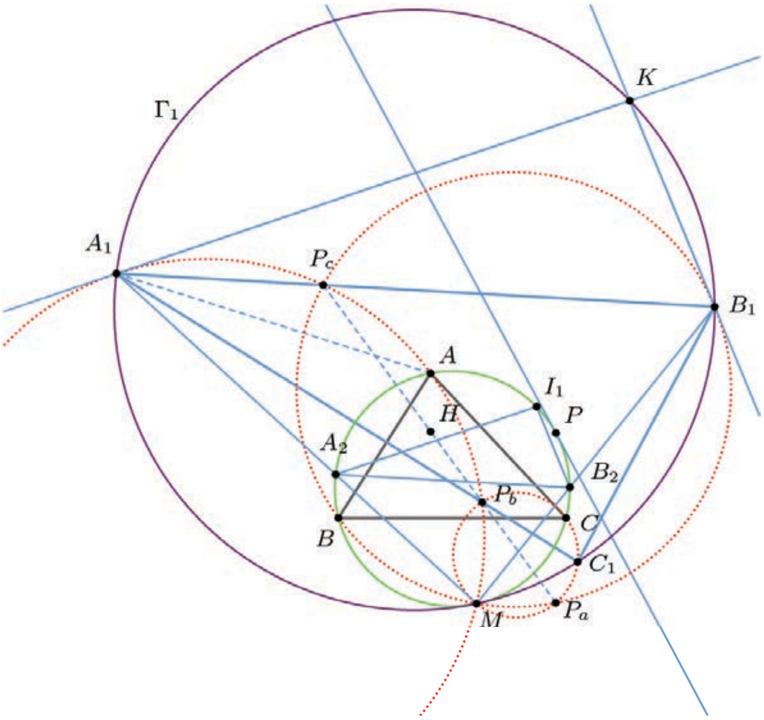
Let $A_1 = \ell_b \cap \ell_c$, and define B_1, C_1 analogously; let Γ_1 be the circumcircle of $\triangle A_1B_1C_1$. Reflect in BC so that $\Gamma \mapsto \Gamma_a, P \mapsto P_a$, with similarly named objects for reflections in CA, AB. Note that ℓ_a is tangent to Γ_a, with analogous results for Γ_b, Γ_c, and $\overline{P_aP_bP_c}$ is the Steiner line of P w.r.t $\triangle ABC$, and it passes through the orthocenter H. Since the reflections of H over BC, CA, AB lie on Γ, it follows that $\Gamma_a, \Gamma_b, \Gamma_c$ concur at H.

Lemma. (Iran MO 2nd Round 1995 Q2) Let I_1 be the incenter of $\triangle A_1B_1C_1$. Then I_1 lies on Γ.

Proof. If O_b, O_c are the centers of Γ_b, Γ_c then $\triangle AP_bP_c \sim \triangle AO_bO_c \implies AP_b = AP_c$. By Reim's theorem on Γ_b, Γ_c, we have $A_1B_1 \parallel \overline{\{P_cA \cap \Gamma_b\}P_b}$; by a converse of Reim's theorem, we have $A_1 \in (AP_bP_c)$. Denote this circle as ω_a: we can similarly obtain circles ω_b, ω_c.

$\therefore \measuredangle AA_1P_c = \measuredangle P_bA_1A$, so AA_1 is the internal bisector of $\measuredangle C_1A_1B_1$. Similarly, we get BB_1, CC_1 are the internal bisectors of $\measuredangle A_1B_1C_1, \measuredangle B_1C_1A_1$ respectively. Hence AA_1, BB_1, CC_1 concur at I_1.

Define M as the Miquel point of the line $\overline{P_aP_bP_c}$ w.r.t $\triangle A_1B_1C_1$, which lies on $\Gamma_1, \omega_a, \omega_b, \omega_c$. Now observe that $(ABMCI_1)$ is a Mannheim circle, so I_1 lies on Γ. $\square$



Suppose the tangents to ω_a, ω_b at A_1, B_1 respectively intersect at K. By Reim's theorem on ω_a, ω_b we have $\overline{\{B_1M \cap \omega_a\}A_1} \parallel B_1K$; by a converse of Reim's theorem, we have $K \in \Gamma_1$.

By Reim's theorem on Γ, ω_a, we have $A_1K \parallel A_2I_1$. Similarly, $B_1K \parallel B_2I_1$.

By the converse of Miquel's theorem in $\triangle B_1B_2P_c$, B_2, A, P_c are collinear.

By Reim's theorem on Γ, ω_a, we have $A_2B_2 \parallel A_1P_c$, so $\triangle A_2B_2I_1$ and $\triangle A_1B_1K$ are homothetic.

$\therefore$ By a converse of Reim's theorem, Γ, Γ_1 are tangent at M, as required.

Quick Reply

High School Olympiads

Parallelogram, 3 circumcenters

Reply

Source: ARO 2011 11-2

Bugi

#1 Jul 19, 2011, 5:36 pm

T is a point on side BC of parallelogram $ABCD$, $\angle A < 90^\circ$ such that $\triangle ATD$ is acute. Let O_1, O_2 and O_3 be the circumcenters of triangles ABT, DAT and CDT respectively. Prove that the orthocenter of $O_1O_2O_3$ lies on AD.

Luis González

#2 Jul 19, 2011, 11:43 pm

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=406967

All the geometry problems from the last ARO are already posted

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=151&t=404325
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=406956
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=406956
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=406963
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=404946

Quick Reply

High School Olympiads

Concurrent 12

Reply

buratinogigle
#1 Jul 19, 2011, 7:59 pm

Let p,q,r be three lines concur at O, a_1,a_2,a_3 be three three lines concur at A. p intersect a_1,a_2,a_3 at P_1,P_2,P_3, q intersect a_1,a_2,a_3 at Q_1,Q_2,Q_3, r intersect a_1,a_2,a_3 at R_1,R_2,R_3, resp. P_3Q_2 cuts r at K. L is a point such that $(P_2Q_2AL)=-1$. Assume that Q_1R_2,P_2R_3 and OA are concurrent. Prove that R_1P_2,Q_2R_3 and KL are concurrent.

Luis González
#2 Jul 19, 2011, 11:12 pm • 1

It suffices to consider the configuration where OA goes to infinity through a central projection. $p \parallel q \parallel r$ and $a_1 \parallel a_2 \parallel a_3$. Thus, L becomes the midpoint of P_2Q_2 and $Q_1R_2 \parallel P_2R_3$. Since $\triangle P_2P_3R_3$ and $\triangle Q_1Q_2R_2$ are centrally similar through K, then $P_2, \underline{Q_1, K}$ are collinear. Let $U \equiv P_2R_1 \cap P_3Q_2$ and $V \equiv R_3Q_2 \cap P_2Q_1$. By Menelaus' theorem for $\triangle KR_2Q_2, \overline{P_2UR_1}$ and $\triangle KP_2R_2, \overline{Q_2VR_3}$, we get

$$\frac{P_2Q_2}{P_2R_2} \cdot \frac{R_2R_1}{R_1K} \cdot \frac{KU}{UQ_2} = 1, \quad \frac{R_3R_2}{R_3K} \cdot \frac{KV}{VP_2} \cdot \frac{P_2Q_2}{Q_2R_2} = 1$$

Since $\frac{P_2R_2}{Q_2R_2} = \frac{R_2K}{R_1K}$, $\frac{R_3R_2}{R_3K} = \frac{P_2Q_1}{P_2K} = \frac{R_2R_1}{R_2K}$, from the latter expressions we get

$\frac{KU}{UQ_2} = \frac{KV}{VP_2} \Longrightarrow R_1P_2$ and Q_2R_3 intersect on the median KL of $\triangle KP_2Q_2$.

buratinogigle
#3 Jul 21, 2011, 2:51 pm

Thanks, this problem is come from the post Square-square when we use projective transformation, the problem in that post is also true for two similar parallelogram.

Quick Reply

High School Olympiads

Lengths of the angle bisectors of a triangle with perimeter

Reply

Source: VAIMO 2, German Pre-TST 2002

orl

#1 Jul 17, 2011, 6:43 am

Prove: If x, y, z are the lengths of the angle bisectors of a triangle with perimeter 6, than we have:

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \geq 1.$$

Luis González

#2 Jul 18, 2011, 4:22 am

$x^2 = \frac{4bcs(s-a)}{(b+c)^2}$, $y^2 = \frac{4cas(s-b)}{(c+a)^2}$, $z^2 = \frac{4abs(s-c)}{(a+b)^2}$

By AM-GM $(b+c)^2 \geq 4bc$, $(c+a)^2 \geq 4ca$, $(a+b)^2 \geq ab$

$$\Longrightarrow \ s \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) \geq \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}$$

By AM-HM $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \geq \frac{9}{(s-a)+(s-b)+(s-c)} = \frac{9}{s}$

$$\Longrightarrow \ \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \geq \frac{9}{s^2} = 1.$$

Quick Reply

High School Olympiads

AIMO 5/1, German TST 2010

Reply

Source: AIMO 5/1, German TST 2010

orl

#1 Jul 17, 2011, 5:40 am

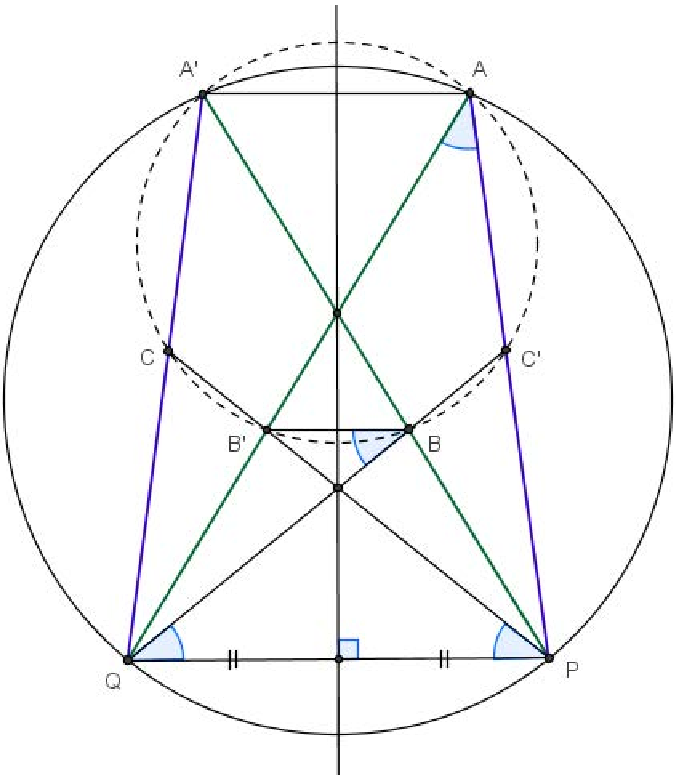
In the plane we have points P, Q, A, B, C such triangles APQ, QBP and PQC are similar accordantly (same direction). Then let A' (B', C' respectively) be the intersection of lines BP and CQ (CP and AQ; AP and BQ, respectively.) Show that the points A, B, C, A', B', C' lie on a circle.

Luis González

#2 Jul 18, 2011, 1:13 am • 2

Figuring out the sketch takes more time than solving the problem.

Attachments:



Quick Reply

High School Olympiads

Incenter and center of Euler circle

Reply

vntbqpqh234
#1 Jul 16, 2011, 6:57 pm

Let triangle ABC, and I,E are incenter and center of its Euler circle.
BE meets AC at M,CE meets AB at N.
If I lies on MN. Find all condition for triangle ABC.(*)

I am doing a problem and need slove (*), hope somebody can wirte a nice lemmas for triangle ABC
thanks.

Luis González
#2 Jul 17, 2011, 3:52 am

Let BI,CI cut CA,AB at B',C'. According to the topic Collinear, $I \in MN \Longleftrightarrow$ The isogonal conjugate of E WRT $\triangle ABC$ (the Kosnita point K_0 of ABC) lies on $B'C' \Longleftrightarrow$ The sum of the oriented distances from K_0 to AB, AC equals the oriented distance from K_0 to BC. The trilinears of K_0 are $(\sec(B-C) : \sec(C-A) : \sec(A-B))$, therefore $I \in MN \Longleftrightarrow \sec(B-C) = \sec(C-A) + \sec(A-B)$.

Quick Reply

High School Olympiads

With the de Longchamps point

jayme
#1 Aug 23, 2008, 2:57 pm

Dear Mathlinkers,
let ABC be a triangle, L the de Longchamps point of ABC, A' the point of intersection of AL and BC, A'b, A'c the feet of the perpendiculars issue from A' onto AC, AB.
Prove synthetically that AL, BA'b and CA'c are concurrent.
Sincerely
Jean-Louis

Luis González
#2 Jul 16, 2011, 12:01 pm

We prove a stronger version of the problem.

Proposition. P is a point on the side BC of $\triangle ABC.$ X,Y are the projections of P onto $AC,AB.$ Then AP,BX,CY concur $\Longleftrightarrow$ either $P \equiv B$ or $P \equiv C,$ or P is the cevian trace of the De Longchamps point X_{20} on $BC.$

AP,BX,CY concur, i.e. pedal triangle $\triangle PXY$ of P WRT $\triangle ABC$ is perspective with $\triangle ABC \Longleftrightarrow P$ lies on the Darboux cubic $\mathcal{Q}$ of $\triangle ABC.$ Hence P is the trace of $\mathcal{Q}$ on $BC.$ Barycentric equation of $\mathcal{Q}$ is given by

$$\mathcal{Q} \equiv \sum_{\text{cyclic}} x(a^2S_A-S_BS_C)(b^2z^2-c^2y^2)=0 \ (\star)$$

Thus, setting $x=0$ in the equation $(\star)$ gives the coordinates of P

$$a^2(b^2S_B-S_CS_A)yz^2+a^2(c^2S_C-S_AS_B)zy^2=0$$

$$yz[z(b^2S_B-S_CS_A)-y(c^2S_C-S_AS_B)]=0$$

Therefore, the lines AP,BX,CY concur $\Longleftrightarrow$ either $yz=0,$ i.e. $P \equiv B \equiv (0:1:0)$ or $P \equiv C \equiv (0:0:1),$ or $P \equiv (0:b^2S_B-S_CS_A:c^2S_C-S_AS_B),$ which is the cevian trace of X_{20} on $BC.$

P.S. For a derivation of $(\star)$ see the topic Locus.

skytin
#3 Jul 16, 2011, 7:59 pm

Lemma
Lat HA is height of ABC and N is it's midpoint .
Then angle NA'bA = HAB , angle AA'cN = CAH
I prove it here :
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=398778&hilit=Find+relation
Let A'A'b and A'A'c intersect line thru point A and || to BC at Y and X
Not hard to prove using Cheva theorem that if YA = AX then this problem done
Circle (A'YA) intersect A'bN at points P and A'b
Angle NPA = A'bA = CAH = AA'cN and angle PNA = A'cNA , so triangle PNA = A'cNA
A'cA = AP , angle PYA = NA'bA = NAA'c = AXA'c
TTriangle PYA = A'cAX , so YA = AX

High School Olympiads

canada 1990

Locked

elegant

#1 Jul 16, 2011, 6:01 am

The diagonals AC and BD of a cyclic quadrilateral ABCD meet at X. Let A', B', C', D' be the feet of the perpendiculars from X to AB, BC, CD, DA, respectively. Prove that $A'B' + C'D' = A'D' + B'C'$.

Luis González

#2 Jul 16, 2011, 10:33 am

This problem and its converse have been discussed before, e.g.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=358747
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=393615
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=396514

High School Olympiads

On a particular case of Thebault's theorem

Source: Jean-Pierre Ehrmann and me

pohoatza

#1 Apr 25, 2008, 5:42 pm

Let P be an arbitrary point inside a triangle ABC, with cevian triangle $A'B'C'$ (i.e. A', B', C' are the intersections of the lines AP, BP, CP with the sides BC, CA, and AB, respectively). Denote by $\mathcal{T}_A^b$, $\mathcal{T}_A^c$ the circles tangent simultanously to AA', $A'B$, Γ, and AA', $A'C$, Γ, respectively, where Γ is the circumcircle of triangle ABC. Similarly, define the circles $\mathcal{T}_B^c$, $\mathcal{T}_B^a$, $\mathcal{T}_C^a$, $\mathcal{T}_C^b$. Prove that the six circles $\mathcal{T}_A^b$, $\mathcal{T}_A^c$, $\mathcal{T}_B^c$, $\mathcal{T}_B^a$, $\mathcal{T}_C^a$, $\mathcal{T}_C^b$ are congruent if and only if P is the Nagel point of triangle ABC.

pohoatza

#2 May 7, 2008, 2:44 pm • 1

Better version:

Let A' be an arbitrary point on the side BC of a triangle ABC. Denote by $\mathcal{T}_A^b$, $\mathcal{T}_A^c$ the circles simultanously tangent to AA', $A'B$, Γ and AA', $A'C$, Γ, respectively, where Γ is the circumcircle of ABC. Prove that $\mathcal{T}_A^b$, $\mathcal{T}_A^c$ are congruent if and only if AA' passes through the Nagel point of triangle ABC.

Luis González

#3 Jul 14, 2011, 10:57 am • 1

Let (U) be the Thebault circle tangent to $\overline{AA'}, \overline{A'B}, \Gamma$ and (V) the Thebault circle tangent to $\overline{AA'}, \overline{A'C}, \Gamma$. $(U), (V)$ touch $\overline{BC}$ at D, E. UV passes through the incenter I of $\triangle ABC$ and $ID \perp UA', IE \perp VA'$. If $(U) \cong (V)$, then it follows that $\overline{UIV}$ is parallel to BC, i.e. $UDEV$ is a rectangle. Thus, if X is the tangency point of the incircle (I) with BC, then $IUDX$ and $IVEX$ are rectangles $\Longrightarrow ID = XU, IE = XV \Longrightarrow \triangle IDE \cong \triangle XUV \Longrightarrow$ $\angle UXV = \angle DIE = \angle UA'V = 90^\circ \Longrightarrow UVA'X$ is cyclic. Further, $UV \parallel BC$ implies that $UVA'X$ is indeed an isosceles trapezoid with legs $UX = VA' \Longrightarrow DX = EA'$. Since $BD = CE$, due to obvious symmetry, then $BX = CA'$ $\Longrightarrow A'$ is the tangency point of the A-excircle with BC.

jayme

#4 May 31, 2012, 5:59 pm

Dear Mathlinkers,
for this difficult and nice problem, you can see

http://perso.orange.fr/jl.ayme vol. 20 "Shape and Mouvement" p. 20

Sincerely
Jean-Louis

High School Olympiads

Tetrahedron 1.4

Reply

Source: 54th Polish 2003

tuan119

#1 Jul 12, 2011, 7:38 pm

Let $ABCD$ is a tetrahedron. The insphere touches the face ABC at H. The exsphere opposite D (which also touches the face ABC and the three planes containing the other faces) touches the face ABC at O. If O is the circumcenter of ABC, show that H is the orthocenter of ABC.

Luis González

#2 Jul 13, 2011, 12:02 am

Let $\mathcal{C}$ be the cone with vertex D tangent to the insphere and D-exsphere of $ABCD$ (both spheres lie inside the cone). The planes DBC, DCA, DAB are clearly tangent to $\mathcal{C}$ along generatrices of $\mathcal{C}$. The intersection $\mathcal{K}$ of $\mathcal{C}$ with the plane ABC is then a conic section inscribed in $\triangle ABC$ with foci O, H (Dandelin theorem). Hence, orthogonal projections Q, R of H onto AC, AB and orthogonal projections E, F of O onto AC, AB lie on the pedal circle of $\mathcal{K}$. Thus, $\angle EQR = \angle EFR$ implies that $\angle AHR = \angle EOA \Longrightarrow \angle HAB = \angle OAC$, i.e. AO, AH are isogonal WRT $\angle BAC$. Analogously, BO, BH are isogonal WRT $\angle CBA \Longrightarrow O, H$ are isogonal conjugates WRT $\triangle ABC$.

Quick Reply

High School Olympiads

8th Stevanovic point in a Feuerbach-like configuration

Reply

Source: me

pohoatza

#1 Jun 8, 2011, 3:50 am

If A_1, B_1, C_1 are the feet of the internal angle bisectors, then the reflections of the line passing through the orthocenter H_1 of $A_1B_1C_1$ and through the incenter of ABC in the sidelines of $A_1B_1C_1$ are concurrent at F', the second intersection of the incircle with the circle passing through the feet of the internal angle bisectors (the point different from the Feuerbach point).

Note (justification of the Subject). An ETC-verification gives this point to be X_{3024}, the 8th Stevanovic point.

Luis González

#2 Jul 11, 2011, 2:25 am • 1

Incircle (I) of $\triangle ABC$ touches BC, CA, AB at D, E, F and let $\triangle A_0B_0C_0$ be the antimedial triangle of $\triangle DEF$. Since B_0C_0, C_0A_0, A_0B_0 are respectively perpendicular to IA, IB, IC, it follows that A_1, B_1, C_1 are the poles of B_0C_0, C_0A_0, A_0B_0 WRT $(I) \Longrightarrow A_0, B_0, C_0$ are the poles of B_1C_1, C_1A_1, A_1B_1 WRT (I). Segments $\overline{IA}, \overline{IB}, \overline{IC}$ cut the circumcircle ω of $\triangle A_1B_1C_1$ at P, Q, R and X, Y, Z are the reflections of I across B_1C_1, C_1A_1, A_1B_1. Inversion WRT (I) takes ω into the pedal circle ω' of I WRT $\triangle A_0B_0C_0$, thus $\overline{IA}, \overline{IB}, \overline{IC}$ cut ω' at the inverses P', Q', R' of P, Q, R and midponts X', Y', Z' of IA_0, IB_0, IC_0 are the inverses of X, Y, Z.

According to the thread Intersect on circle (see post #4), $P'X', Q'Y', R'Z'$ meet at the Poncelet point F' of $A_0B_0C_0I$, which lies on ω'. But since their inverses $\odot(IXP), \odot(IYQ)$ and $\odot(IZR)$ concur at the anti-Steiner point of I WRT $\triangle A_1B_1C_1$, then $\omega \cap (I)$ is the anti-Steiner point F' of I WRT $\triangle A_1B_1C_1$ and the Feuerbach point X_{11} of $\triangle ABC$.

If O_0 is the circumcenter of $\triangle A_0B_0C_0$, then according to the thread Two Yango's problem (see problem #1), F' is, in general, different from the anti-Steiner point of IO_0 (Euler line of DEF) WRT $\triangle DEF$. Indeed, the Euler's reflection point of $\triangle DEF$ is the Feuerbach point X_{11} of $\triangle ABC$ (well-known). Hence, we conclude that the anti-Steiner point F' of IH_1 WRT $\triangle A_1B_1C_1$ is the intersection $(I) \cap \omega$, different from X_{11}, i.e. the 8th Stevanovic point X_{3024} of $\triangle ABC$ and the proof is completed $\square$.

P.S. The problem can be generalized, which is no surprise, but unfortunately I haven't made much progress on the synthetic proof.

Generalization

RSM

#3 May 7, 2012, 2:44 pm • 4

I have posted a synthetic proof of the generalisation here:-
http://www.artofproblemsolving.com/blog/69529

IDMasterz

#4 Feb 22, 2015, 6:44 pm

I have posted a synthetic proof of the generalisation proposed by Luis on my blog. For convenience, here is the proof in its entirety.

Problem: Prove that the circumcircle of the cevian triangle meets the pedal circle of a point P at the Poncelet point of P w.r.t. ABC and the anti-steiner point of P w.r.t. the cevian triangle.

Note that a proof of the first proposition can be found here. From here, define $A'B'C'$ and $P_AP_BP_C$ to be the pedal and cevian triangle of P w.r.t. ABC.

We prove three lemmas:

Lemma 1: In triangle ABC, let the anti-steiner point of P be T. Let the hyperbola through $ABCHP$, where H is the orthocentre of ABC, meet $\odot ABC$ at U. Then P, T, U are collinear.

Proof: Let HP meet the sides of the triangle ABC at X, Y, Z. From the paper linked before, we know U is the isogonal conjugate of the point at infinity on OP^*, where P^* is the isogonal conjugate of P w.r.t. ABC. Let the circumcevian triangle of H w.r.t. ABC be $H_AH_BH_C$. Note that X, T, H_A are collinear. If the isogonal conjugate of U' is $U*$, where U' is the intersection of $TP \cap \odot ABC$, then;

$$O(A, B; C, U^*) = T(H_A, H_B; H_C, U') = T(X, Y; Z, P) = O(A, B; C, P^*)$$

So, $U' = U$.

Lemma 2: Consider the configuration given in lemma 1.3 from the about-to-be published paper. Then, the two conics $\mathcal{C}, \mathcal{C}'$ are homothetic.

Proof: Consider the point at infinity ∞ on $\mathcal{C}$. Then $A'\infty, B'\infty, C'\infty$ concur on $\mathcal{C}'$, so done.

Lemma 3: Consider the two triangles defined for Theorem 1.2 in the paper. Let any line through Q meet the conics $\mathcal{C}, \mathcal{C}'$ again at X, Y. Let the antipodes of X, Y w.r.t. their respective conics be X', Y'. Then, X', Y', Q are collinear.

Proof: Since $X' \mapsto Y'$ preserves cross ratio, we will prove the assertion for the points A, B, C. By dilating about D with factor $\frac{1}{2}$ and same about A, it is evident that if M_1, M_2 are the midpoint of QA, QA', then we must show that, if the centres of the conics are O, O', that $OM_1 \parallel O'M_2$. But, when we reflect OM_1 over an asymptote of $\mathcal{C}$, it is parallel to $AQ \equiv AA'$. By symmetry, it follows $OM_1 \parallel O'M_2$ as desired.

Main Proof: Let the circumcevian triangle of P w.r.t. $A'B'C'$ be XYZ and the anti-pedal triangle of P w.r.t XYZ be $A_1B_1C_1$. Let the second intersections of the circle $\odot A'B'C'$ with the sides of $A_1B_1C_1$ be $A_2B_2C_2$.

Let $PP_A \cap B_1C_1 = T$ and the point at infinity perpendicular to AP be P_∞. Then $P(T, P_\infty; B_1, C_1) = A'(B, A; B', C')$ so T bisects B_1C_1. Under inversion with pole the power of P w.r.t. $A'B'C'$, we get that if $\odot P_AP_BP_C \cap \odot A'B'C' = Q$, then $QP \cap \odot TUV \cap \odot A_2B_2C_2$ is a point Q' where TUV is the medial triangle of $A_1B_1C_1$.

But, Q' is the centre of the rectangular hyperbola through $A_1B_1C_1H'P$ where H' is the orthocentre of $A_1B_1C_1$. Denote the conics $A_1B_1C_1H'P$ and $A'B'C'HP$ as $\mathcal{C}', \mathcal{C}$ where H is the orthocentre of $A'B'C'$. Note that P is the common orthology centre of $A_1B_1C_1$ and $A'B'C'$. Further, it isn't hard to see that HP is tangent to $\mathcal{C}'$ and similarly $H'P$ is tangent to $\mathcal{C}$ (this follows by definition). Let the antipodes of P in both conics be P_1, P_2. Then by lemma 3 it follows that P_1P_2T are collinear.

Now, $\angle TP_1P = \angle H'PT = \angle TP_2H'$, so $H'P_2 \parallel PP_1$. Similarly, $HP \parallel PP_2$, so the anti-pode of H w.r.t. $\mathcal{C}$ (call it H_1) lies on $PP_2 \equiv PQ'$. Since H_1 lies on $\odot A'B'C'$, it follows $H_1 \equiv Q'$!

Hence, by lemma 1, Q must be the anti-steiner point of HP w.r.t. $A'B'C'$.

We are done

This post has been edited 1 time. Last edited by IDMasterz, Jul 8, 2015, 10:42 pm
Reason: typo

Quick Reply

High School Olympiads

Incircle Problem

Reply

Source: Zhao Yu Fei - circles # 20

jsc

#1 Jul 10, 2011, 9:27 pm

Let ABC be a triangle, and ω its incircle. Let ω touch the sides BC, CA, AB at D, E, F respectively. Let X be a point on AD and inside ω. Let segments BX and CX meet ω at Q and R respectively. Show that lines EF, QR, BC are concurrent.

Many apologies if this is too easy for the Olympiad Section. Thanks for any help.

Luis González

#2 Jul 10, 2011, 10:34 pm

This configuration has been discussed many times before. For instance, according to Triangle and intersecting lines (post #2), BR, CQ, XD concur. Thus, QR cuts BC at the harmonic conjugate of D WRT $B, C \Longrightarrow EF, QR, BC$ concur.

Alternate proof: There exists a central projection taking ω into another circle and the line through A and $EF \cap BC$ into a line at infinity. If we denote projected points with primes, then the projected triangle $\triangle A'B'C'$ is A'-isosceles with incircle ω' and due to obvious axial symmetry, the lines $E'F', Q'R', B'C'$ are parallel $\Longrightarrow EF, QR, BC$ also concur.

Quick Reply

High School Olympiads

Proving that five points are coplanar

Reply

Source: ToT 2003 SA-6

Goutham

#1 Jul 3, 2011, 10:57 am • 1

Let O be the center of insphere of a tetrahedron $ABCD$. The sum of areas of faces ABC and ABD equals the sum of areas of faces CDA and CDB. Prove that O and midpoints of BC, AD, AC and BD belong to the same plane.

mihai miculita

#2 Jul 3, 2011, 2:26 pm • 1

1). Ussing barycentric coordonate, if denote by S_x area of the face opose of vertex $X\in\{A,B,C,D\}$,
then have: $A(1;0;0;0); B(0;1;0;0); C(0;0;1;0); D(0;0;0;1); O(S_a;S_b;S_c;S_d)$
and if M_{xy} midpoint of edge $[XY]\in\{[BC];[AD];[AC];[BD]\}$, then:
$M_{bc}(0;1;1;0); M_{ad}(1;0;01); M_{ac}(1;0;1;0); M_{bd}(0;1;0;1)$
2). Or $M_{bc}M_{ac}M_{ad}M_{bd}\text{-parallogram}\Rightarrow M_{bd}\in\text{plane }(M_{bc}M_{ac}M_{ad})\Rightarrow$

$$O\in\text{plane }(M_{bc}M_{ac}M_{ad})\Leftrightarrow 0=\begin{vmatrix} S_a & S_b & S_c & S_d\\ 0 & 1 & 1 & 0\\ 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 1\end{vmatrix}=$$

$$=\begin{vmatrix} S_a-S_d & S_b & S_c-S_b & S_d\\ 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{vmatrix}=\begin{vmatrix} S_a-S_d & S_c-S_b\\ 1 & 1\end{vmatrix}\Leftrightarrow$$

$$\Leftrightarrow S_a-S_d=S_c-S_b\Leftrightarrow \boxed{S_a+S_b=S_c+S_d}.$$

Luis González

#3 Jul 9, 2011, 11:34 am • 1

Let M,N,L,K be the midpoints of $BC,AD,AC,BD.$ Since $MLNK$ is a parallelogram, then M,L,N,K lie on a same plane $\delta.$ It's known that δ divides the tetrahedron into two pentahedra with equal volumen, hence:

$$\frac{1}{2}[ABCD]=[ODCMK]+[ODCLN]+[OCML]+[OKDN]\pm[OMLNK]$$

But $[ODCMK]=\frac{_3^{^4}[OCDB],$ $[ODCLN]=\frac{3}{4}[OCDA], [OCML]=\frac{1}{4}[OABC]$ and $[OKDN]=\frac{1}{4}[OABD].$ Thus, substituting the volumen $[ABCD]$ (in the first equation) as the sum of the volumes $[OCDB],[OCDA],[OABC],[OABD],$ yields

$$[OABC]+[OABD]-[OCDA]-[OCDB]=\pm 4\cdot[OMLNK] \ (\star)$$

Since $OABC,OABD,OCDA,OCDB$ have equal O-altitudes (inradius of ABCD), then the given condition is equivalent to $[OABC]+[OABD]=[OCDA]+[OCDB].$ Together with $(\star),$ we get $[OMLNK]=0 \Longrightarrow O\in\delta.$

Quick Reply

High School Olympiads

Find angle measure in convex quadrilateral

Source: Turkey IMO TST 1995 #4

GlassBead
#1 Jul 8, 2011, 8:58 pm

In a convex quadrilateral $ABCD$ it is given that $\angle CAB = 40^\circ$, $\angle CAD = 30^\circ$, $\angle DBA = 75^\circ$, and $\angle DBC = 25^\circ$. Find $\angle BDC$.

Luis González
#2 Jul 9, 2011, 5:45 am

$\triangle ABC$ is clearly isosceles with base angle $\angle BAC = \angle BCA = 40^\circ$. Let the perpendicular from B to AC cut AD at E. $\triangle AEC$ is isosceles with apex angle $\angle AEC = 120^\circ \Longrightarrow \angle CEB = \angle DEC = 60^\circ$, i.e. ED bisects $\angle BEC$ externally. Since $\angle ABE = 50^\circ$, then $\angle EBD = 75^\circ - 50^\circ = 25^\circ \Longrightarrow BD$ bisects $\angle EBC$ internally. Thus, D is the B-excenter of $\triangle BEC \Longrightarrow \angle BDC = \frac{1}{2}\angle BEC = 30^\circ$.

Number1
#3 Jul 9, 2011, 9:12 pm

Here is another solution:

Let E be such point that ΔBCE is equilateral.
Since $BA = BC = BE$ triangle ABE is isosceles with $\angle BAE = 70^\circ$.
This means that E lies on side AD.
But now ΔBDE is isosceles ($\angle EBD = \angle BDE = 35^\circ$) so E is center of circle circumscribed for ΔBCD.
Thus $\angle BDC = \dfrac{1}{2}\angle BEC = 30^\circ$.

sunken rock
#4 Jul 10, 2011, 12:32 am

Take F so that ACF is an equilateral triangle, B and F on one side and the other of AC. Obviously, D is on the perpendicular bisector of AF and, as Luis proved, BD is angle bisector of $\angle CBF$, that means $BCDF$ is cyclic, hence $\angle BFC = \angle BDC = 30^\circ$.

Best regards
sunken rock

High School Olympiads

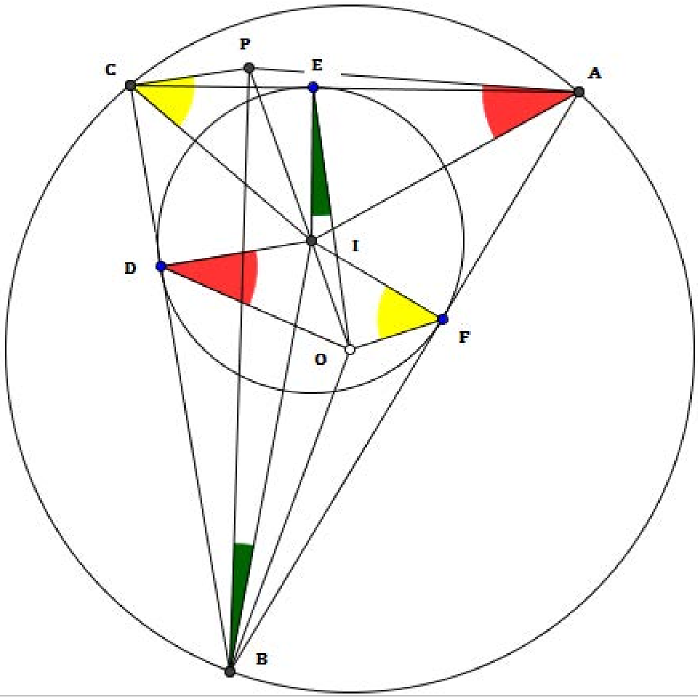
New point on OI ? (Own)

skytin

#1 Jul 5, 2011, 2:53 pm • 1

Given triangle ABC
(I) is incircle on ABC
DEF is intouch triangle of ABC
O is circumcircle of ABC
Let point P is given such that angle IDO = IAP , PCI = OFI
Prove that angle IBP = IEO too

Attachments:



This post has been edited 2 times. Last edited by skytin, Jul 12, 2011, 5:05 pm

Luis González

#2 Jul 7, 2011, 9:34 am

Let the A-mixtilinear incircle ω_a of $\triangle ABC$ touch its circumcircle (O,R) at T. Thus, A,T are the exsimilicenters of $(I,r) \sim \omega_a$ and $(O,R) \sim \omega_a \Longrightarrow U \equiv AT \cap IO$ is the exsimilicenter of $(I) \sim (O)$. By Stewart theorem for the cevian AU of $\triangle AOI$, we deduce that

$$AU^2 = AI^2 \cdot \frac{UO}{IO} - R^2 \cdot \frac{IU}{IO} + UO^2 - IO \cdot UO =$$

$$= AI^2 \cdot \frac{R}{R-r} - R^2 \cdot \frac{r}{R-r} + UO^2 - IO^2 \cdot \frac{R}{R-r} =$$

$$= AI^2 \cdot \frac{R}{R-r} + UO^2 - \frac{(R^2-2Rr)R + R^2 r}{R-r} =$$

$$= AI^2 \cdot \frac{R}{R-r} + UO^2 - R^2 = AI^2 \cdot \frac{R}{R-r} - AU \cdot UT$$

$$AU \cdot (AU + UT) = AI^2 \cdot \frac{R}{R-r} \Longrightarrow AU = \frac{R}{R-r} \cdot \frac{AI^2}{AT} \ (\star)$$

Let M, N be the midpoints of the arcs BC and BAC of (O). Then T, I, N are collinear. For a proof, see problem 2.20 of 2005 mosp 2.20(g) 4.44(g). Thus, from $\triangle AIT \sim \triangle NIM$, we obtain

$$\frac{NI}{AI} = \frac{NM}{AT} = \frac{2R}{AT} \Longrightarrow \ AT = \frac{2R \cdot AI}{NI}$$

Substituting AT from this expression into $(\star)$ gives $\dfrac{AI}{AU} = \dfrac{2(R-r)}{NI}$

If L is the midpoint of AI and K lies on $\overline{ON}$ such that $DIKO$ is a parallelogram, then the latter expression translates to $\frac{AL}{AU} = \frac{NK}{NI}$. Together with $\angle TAM = \angle TNM$, it follows that $\triangle ALU \sim \triangle NKI$ (SAS) $\Longrightarrow$ $\angle ILU = \angle MKI = \angle IDO$. If P is the reflection of I about U, then $UL \parallel PA \Longrightarrow \angle IAP = \angle ILU = \angle IDO$. Hence, by similar reasoning, we conclude that there is a point P satisfying $\angle IAP = \angle IDO, \angle IBP = \angle IEO$ and $\angle ICP = \angle IFO$ (angles oriented according to the given sketch). This point is the reflection of I about the exsimilicenter of $(I) \sim (O)$, i.e. P is Kimberling center X_{46} of $\triangle ABC$.

skytin

#3 Jul 7, 2011, 3:14 pm

My solution :
Let I_a I_b I_c is centers of excircles
Let points D' E' F' there tangent points with the oposite sides of ABC
Easy to see that I is orthocenter of I_aI_bI_c and O is 9 point center of this triangle
Let point O' is center of circle (I_aI_bI_c)
O is midpoint of IO'
Let point X' is midpoint of I_aA
X is midpoint of I_aO'
Let D_1 is point on X'D' such that FO is perpedicular to BC
Make reflection of point I wrt midpoint of BC and get point I'
Easy to see that I' is orthocenter of BI_aC
Triangle CBI_a ~ I_aI_bI_c , so I_aI'/I_aD' = I_aI/I_aA = I_aN/I_aX' were N is midpoint of I_aI
So NI' || X'D'
Easy to see that DI = I'D' = ND_1
D_1O = NO - DI = R - r
Let point S is on X'D' such that S is on XO too
Na is Nagel point of ABC
Well known that ANa || TI and ANa = 2*TI were T is midpoint of BC
So II' = ANa and II' || ANa
NaI' || I_aA
Let X'D' intersect II' at point D_1' , easy to see that NaD_1' || I'D' and NaD_1' = I'D'
Like the same construct points F_1 and F_1' on Z'F
And E_1 , E_1' on Y'E
O is circumcenter of D_1E_1F_1
Na is circumcenter of D_1'E_1'F_1'
X'D' Y'E Z'F are concurrent at homotety center of D_1E_1F_1 and D_1'E_1'F_1' at point W on ONa
So WNa / WO = r/(R-r)
NaI' intersect X'D' at point S'
S' is midpoint of NaI' and NaS' = IA/2
NaS' || I_aA || SO
Like the same construct points H and H' on YE
And K , K' on ZF
Easy to see that triangles S'H'K' and SHK are homotetive with center W and coefficient r/(R-r) , so FO = F'Na*((R-r)/r) = IA*((R-r)/2r)
Let circle (O'SOD') intersect line BC at points D and D"
Angle D"OO' = 90 and O'O = OI , so D"O is simmetry line of (O'SOD') and (OD"DI)
Easy to see that after translation all points on vector OI then (O'SOD') -> (OD"DI)
So S -> S" and S" is on (OD"DI)
S"I || SO || IA , so S" is on IA
Like the same construct points H" , K" on (EOI) , (FOI)
Easy to see that triangle S"H"K" = SHK and homotetive
SHKO is homotetive to ABCI
S"H"K" is homotetive to ABC
So S"I/IA = 2r/(R-r)
Construct point P such that S"H"K"O is homotetive to ABCP , then P is on OI , OI/IP = 2r/(R-r) and angle PAI = OS"I = ODI . like the same ather angles . done

skytin
#4 Jul 7, 2011, 4:06 pm

Now we will prove this problem :
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=416306
Use same notations as in my previous post
Easy to see that triangle Y'D'Z' = Z'AZ' ~ I_aI_bI_c
Like the same Y'F'X' ~ I_aI_bI_c ~ Y'D'Z'
Not hard to prove that X'D'Y' ~ Z'Y'F' , so angle X'DY' = Y'Z'F'
D'Y'Z'W is cyclic
Angle F'WD' = 180 - D'Y'Z' = 180 - I_aI_bI_c = X'IZ'
Easy to see that if we make homotety of Poncelet point
of X'Y'Z'I wrt point I and coefficient 2 then we get point W
Let W' is midpoint of IW
See my post here :
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=398778
Let N' is midpont of ONa
N is midpoint of N'I
J is midpoint of AO
JN' = ANa/2 = IT , so N'JIT is parallelogram
N is midpoint of JT
Not hard to prove that N is 9 point center of midpoint triangle of midpoint triangle of ABC
After angle chasing not hard to prove that O is center of (D_1E_1F_1W) , so OW = D_1O = R-r
Na is center of (D_1'E_1'F_1'W) , so NaW = r
2N'W = r + R - r
NW' = R/4 , so W' is on 9 point circle of midpoint triangle of ABC
I'' is incenter ofmidpoint triangle of ABC
I'' is midpoint of NaI , so I''W' = r/2
W' is on incircle (I'')
So (N) is tangent to (I'') at point W' , so W' is Feuerbach point of midpoint triangle of ABC and W' is Poncelet point of X'Y'Z'I

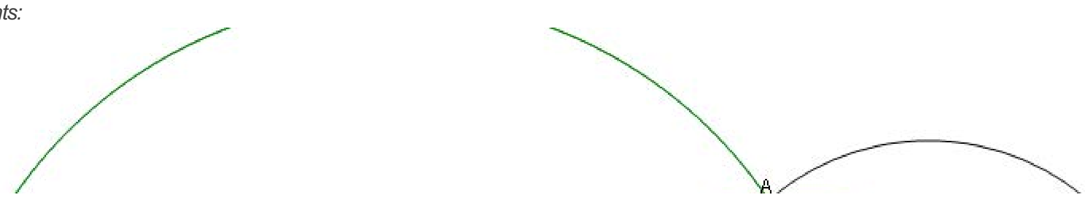
skytin
#5 Jul 7, 2011, 4:12 pm

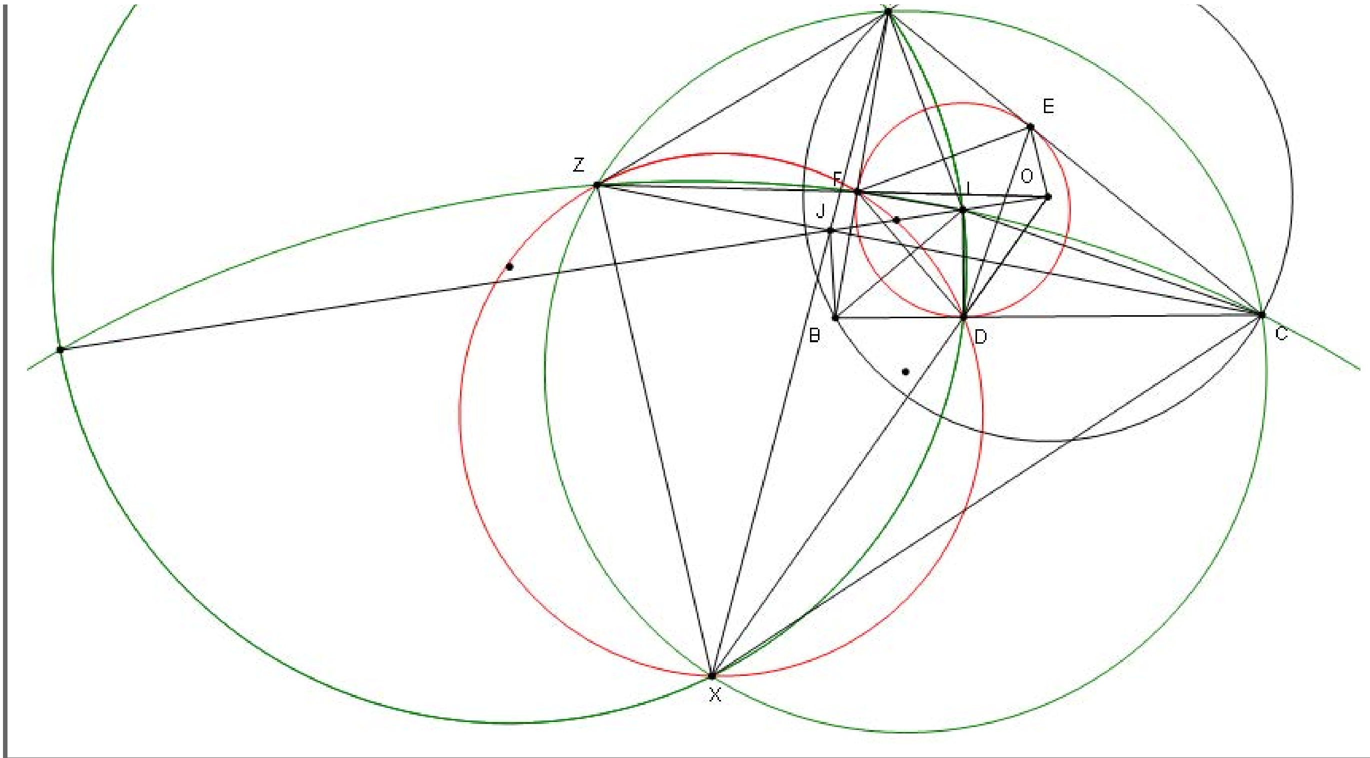
Is this problem well known ?

livetolove212
#6 Jul 11, 2011, 3:50 pm • 2

Lemma: Given a triangle ABC and its incircle (I).(I) touches BC, CA, AB at A_1, B_1, C_1, respectively. Then 3 circles $(AIA_1), (BIB_1), (CIC_1)$ are coaxal.
Proof for the generalization: http://www.artofproblemsolving.com/Forum/viewtopic.php?t=315748
Back to the problem:
Let X be the intersection of OD and (AID). Then $\angle XAI = \angle IDO$. Similarly we define Y, Z. The main idea is to show that AX, BY, CZ concur at a point which lies on OI.
We will show that $XZAC$ is cyclic iff $\angle XZC = \angle XAC$. (1)
Applying the lemma above, we get $OD.OX = OF.OZ$, which follows that $XZFD$ is cyclic.
Then $\angle XZC = \angle XZF - \angle FZC = \angle FDO - (180^o - \angle FIC)$
$\angle XAC = \angle XAI + \dfrac{\angle A}{2} = \angle IDO + \dfrac{\angle A}{2}$.
So $(1) \Leftrightarrow \angle FDO + \angle FIC - 180^o = \angle IDO + \dfrac{\angle A}{2}$
$\Leftrightarrow \angle FDI + \angle FID + \angle DIC - 180^o = \dfrac{\angle A}{2}$
$\Leftrightarrow \dfrac{\angle B}{2} - \angle B + 90^o - \dfrac{\angle C}{2} = \dfrac{\angle A}{2}$ (right!)
Therefore $XZAC$ is cyclic. Let J be the intersection of AX and CZ then $JA.JX = JC.JZ$, or J lies on the radical axis of (AID) and (CIF) or J lies on OI. Similarly we get AX, BY, CZ concur at J.

Attachments:





skytin

#7 Jul 11, 2011, 4:38 pm

Nice solution !

skytin

#8 Jul 12, 2011, 8:56 pm

See my another solution here :
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=417739

TelvCohl

#9 Nov 26, 2015, 10:47 pm • 1

Let T^* be the image of T (arbitrary point) under the Inversion $\mathbf{I}(\odot(I))$. Let V be the complement of O^* WRT $\triangle DEF$. From $A^*V \parallel DO^* \Longrightarrow \measuredangle IDO = \measuredangle DO^*I = \measuredangle A^*VI = \measuredangle IAV^*$. Analogously, we can prove $\measuredangle IEO = \measuredangle IBV^*$ and $\measuredangle IFO = \measuredangle ICV^*$, so $V^* \equiv P$ and P lies on OI ($\because OI$ (Euler line of $\triangle DEF$) passes through the Centroid of $\triangle DEF$).

XmL

#10 Nov 26, 2015, 11:58 pm

Let $(AIE) \cap OD = X, (CIE) \cap OE = Y$, thus $AX \cap CY = P$ It is well known that OI is the radical axis of the two circles, so $P \in OI \iff ACXY$ is cyclic.

Let $AI \cap OD = X', CI \cap OE = Y'$, it is not hard to show that $X'Y'||ED$, hence $ACX'Y'$ is cyclic. Note that $\angle OAX' = \angle OXA$, thus X, Y are the images of X', Y' under the inversion about (ABC) and hence $ACXY$ is cyclic

Telv: how do you revive these old problem? where do you find them?

This post has been edited 1 time. Last edited by XmL, Nov 27, 2015, 12:00 am
Reason: ..

Quick Reply

High School Olympiads

Square (Own)

Reply

skytin

#1 Jul 4, 2011, 5:20 pm

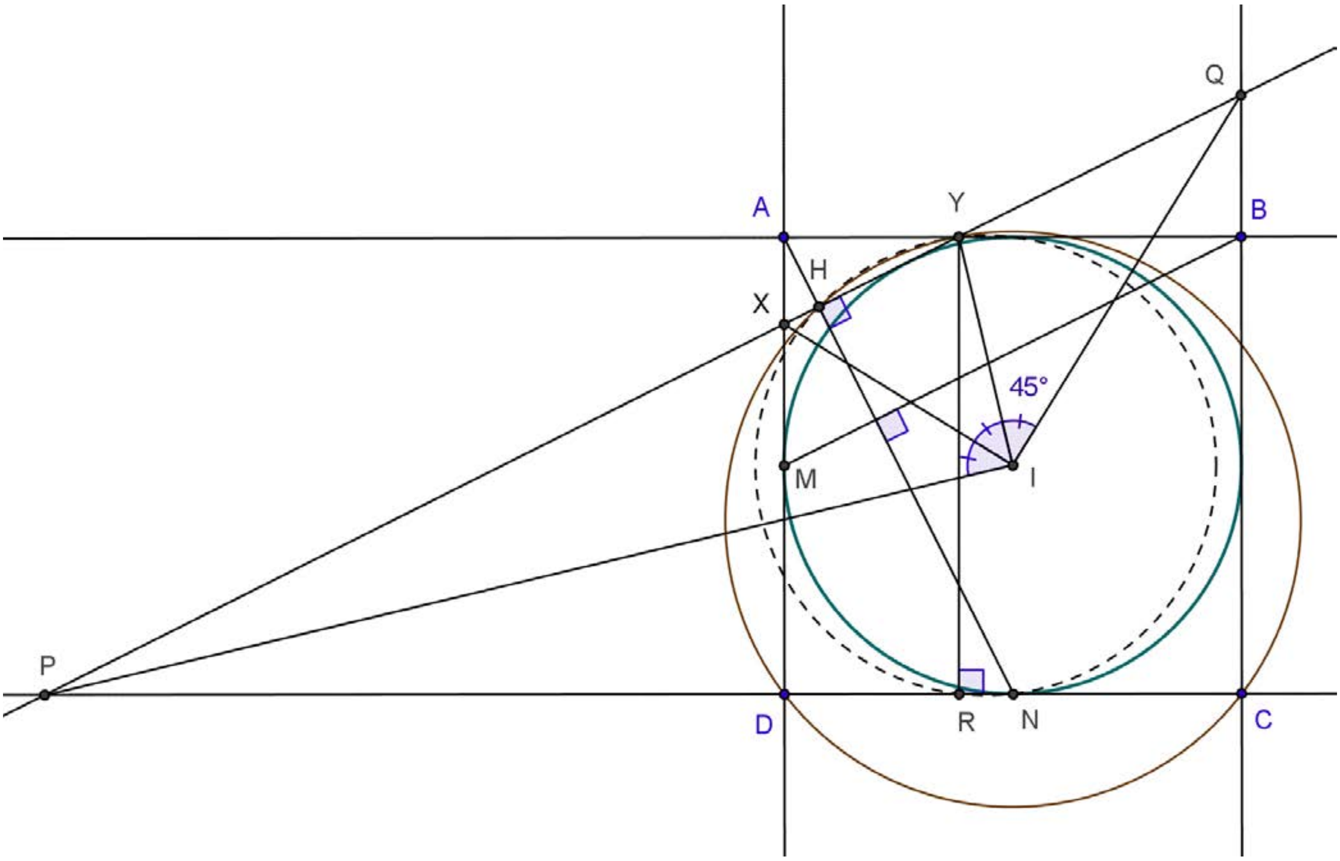
Given ABCD is square
Points Y and X are on AB and AD , such that XY is tangent to incircle of ABCD , and AX/AY = 1/2
H is foot of perpendicular from A on YX
Prove that HYCD is cyclic

Luis González

#2 Jul 5, 2011, 3:47 am

Let the incircle (I) of $ABCD$ touch DA, DC at $M, N.$ Since $AM : AB = 1 : 2,$ then $XY \parallel BM,$ i.e. $AH \perp BM.$ But $\triangle BAM \cong \triangle ADN$ (SAS) implies that $\angle MBA = \angle NAD,$ i.e. $AN \perp BM \Longrightarrow AH$ passes through $N.$ Let XY cut CD, CB at $P, Q.$ Since I becomes the P-, Q- and A-excenter of $\triangle PDX, \triangle QBY$ and $\triangle AXY,$ we deduce that $\angle PIX = \angle XIY = \angle YIQ = 45^\circ,$ i.e. IP, IY bisect $\angle QIX \Longrightarrow X, Q, Y, P$ are harmonically separeted. Thus, their orthogonal projections D, C, R, P onto the line CD are also harmonically separeted $\Longrightarrow \overline{PC} \cdot \overline{PD} = \overline{PR} \cdot \overline{PN}.$ But $HYNR$ is cyclic due to the right angles at $H, R,$ thus $\overline{PR} \cdot \overline{PN} = \overline{PY} \cdot \overline{PH} \Longrightarrow \overline{PC} \cdot \overline{PD} = \overline{PY} \cdot \overline{PH} \Longrightarrow HYCD$ is cyclic.

Attachments:



jayme

#3 Oct 8, 2011, 4:22 pm

Dear Mathlinkers,
an remastered article concerning the "'Geometric miniatures" has been put on my website
with 10 more examples.
http://perso.orange.fr/jl.ayme vol. 7 p. 72
You can use Google translator
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

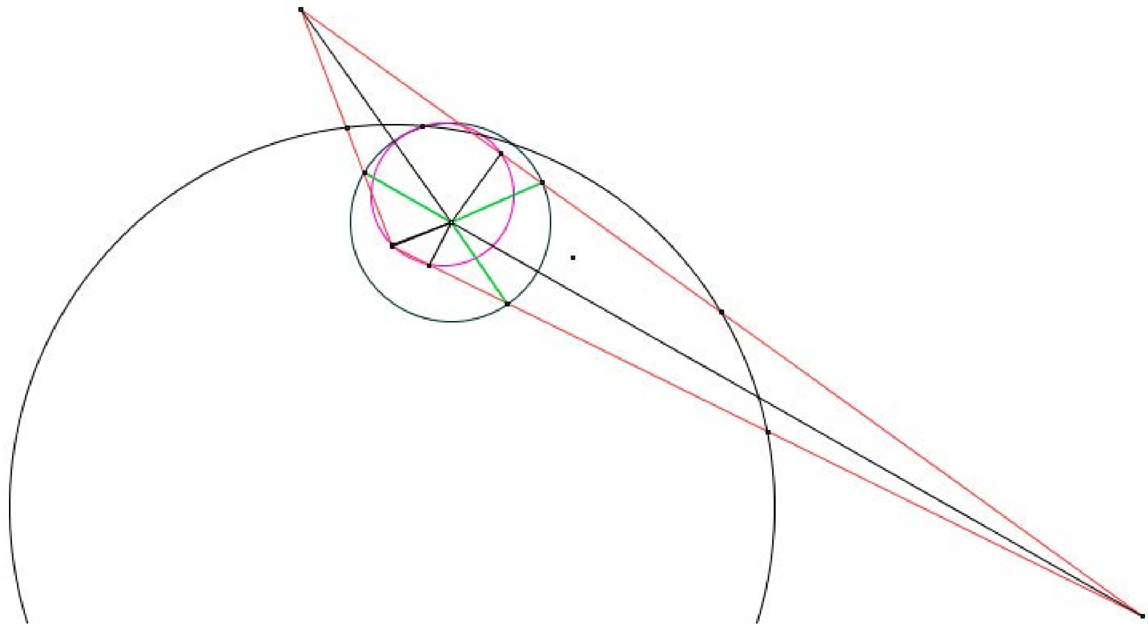
Tangent point (Own)

skytin

#1 Jul 4, 2011, 5:53 pm

Given triangle ABC and points X , Y , Z on it's sides . AX , BY , CZ intersects at same point P
a) If given that P is center of circle (XYZ) , prove that pedal circle of point P is tangent to (XYZ) at point S
b) Prove that S is on 9 point circle of ABC

Attachments:



Luis González

#2 Jul 4, 2011, 11:33 pm • 1

Let use restate the problem as follows:

$\triangle ABC$ is a triangle with circumcircle (O,R). $\triangle A'B'C'$ is the anticevian triangle of O WRT $\triangle ABC$ and D,E,F are the orthogonal projections of O onto $B'C, C'A', A'B'$. Then $\odot(DEF)$ is tangent to (O) at a point S lying on the 9-point circle of $\triangle A'B'C'$.

Proof. Let $\triangle A_0B_0C_0$ be the tangential triangle of $\triangle ABC$ and M_1, M_2, M_3 denote the midpoints of B_0C_0, C_0A_0, A_0B_0. Perpendiculars ℓ_b, ℓ_c from B, C to OM_2 and OM_3 are the polars of M_2, M_3 WRT $(O) \Longrightarrow \ell_b \cap \ell_c$ is the pole of M_2M_3 WRT (O), which lies on AO, due to $AO \perp M_2M_3$. Likewise, perpendiculars ℓ_c, ℓ_a from C, A to OM_3, OM_1 meet on BO and perpendiculars ℓ_a, ℓ_b from A, B to OM_1, OM_2 meet on $CO \Longrightarrow \triangle(\ell_a, \ell_b, \ell_c)$ coincides with the anticevian triangle of O WRT $\triangle ABC \Longrightarrow A' \equiv AO \cap \ell_b \cap \ell_c$, etc. Since $R^2 = \overline{OD} \cdot \overline{OM_1} = \overline{OE} \cdot \overline{OM_2} = \overline{OF} \cdot \overline{OM_3}$, it follows that $\odot(DEF)$ is the inverse of the 9-point circle $\odot(M_1M_2M_3)$ of $\triangle A_0B_0C_0$ WRT (O). Since (O) and $\odot(M_1M_2M_3)$ are tangent through the Feuerbach point S of $\triangle A_0B_0C_0$, then by conformity $\odot(DEF)$ is also tangent to (O) through S. Unique intersection S of the cevian and pedal circle of O WRT $\triangle A'B'C'$ is then the Poncelet point of $A'B'C'O$, thus S lies on the 9-point circle of $\triangle A'B'C'$.

skytin

#3 Jul 5, 2011, 12:30 am

My solution for (a) :
Let PQR is pedal triangle of point P wrt ABC

Make inversion of points P , Q , R wrt (XYZ) and get point P' , Q' , R'
Let lines XP' , YQ' , ZR' formed triangle LMN
Not hard to prove that P'Q' is perpendicular to PC , so P'Q' || ML
Like the same Q'R' || MN and P'R' || LN , so (P'Q'R') is 9 point circle of LMN and (XYZ) is incercle of LMN
And (P'Q'R') is tangent to (XYZ) at Feuerbach point of LMN
So after inversion we get (a)

@Luis
Nice prove for (a)
But i have question for your solution to (b) :
You wrote that

> *Quote:*
> The unique intersection S of the cevian and pedal circle of O WRT $\triangle A'B'C'$ is then the Poncelet point of $A', B', C', O,$ thus S lies on the 9-point circle of $\triangle A'B'C'$.

But in the Poncelet theorem we has :
Let A, B, C, and D be four points in the plane. The nine-point circles of triangles ABC, BCD, CDA, DAB meet at one point, the Poncelet point of the points A, B, C, and D.
So i think you have to prove that nine point circles of A'B'O , B'C'O , A'C'O are intersects at point S

Luis González
#4 Jul 5, 2011, 1:09 am

Sorry, I skipped its proof because I still don't have a complete solution. It's easy to prove by angle chasing that the pedal circle of a point P WRT ABC passes through the Poncelet point Po of ABCP. For instance, see intersection of Simson lines /K.K 6.4/2. The tough part is indeed to show that the cevian circle of P also passes through Po (see Poncelet points).

skytin
#5 Jul 5, 2011, 2:40 pm

Ok , it's correct solution
My solution based on the next problem :
Given triangle ABC , DEF is it's orthotriangle , X , Y , Z are midpoints of heights of ABC
PQR is midpoint triangle of DEF . Feuerbach point of PQR is lying on 9 point circle of XYZ

skytin
#6 Jul 7, 2011, 4:13 pm

See my solution here :
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=416422&p=2348643#p2348643

Quick Reply

High School Olympiads

The lines AX, BY, CZ are concurrent (own)

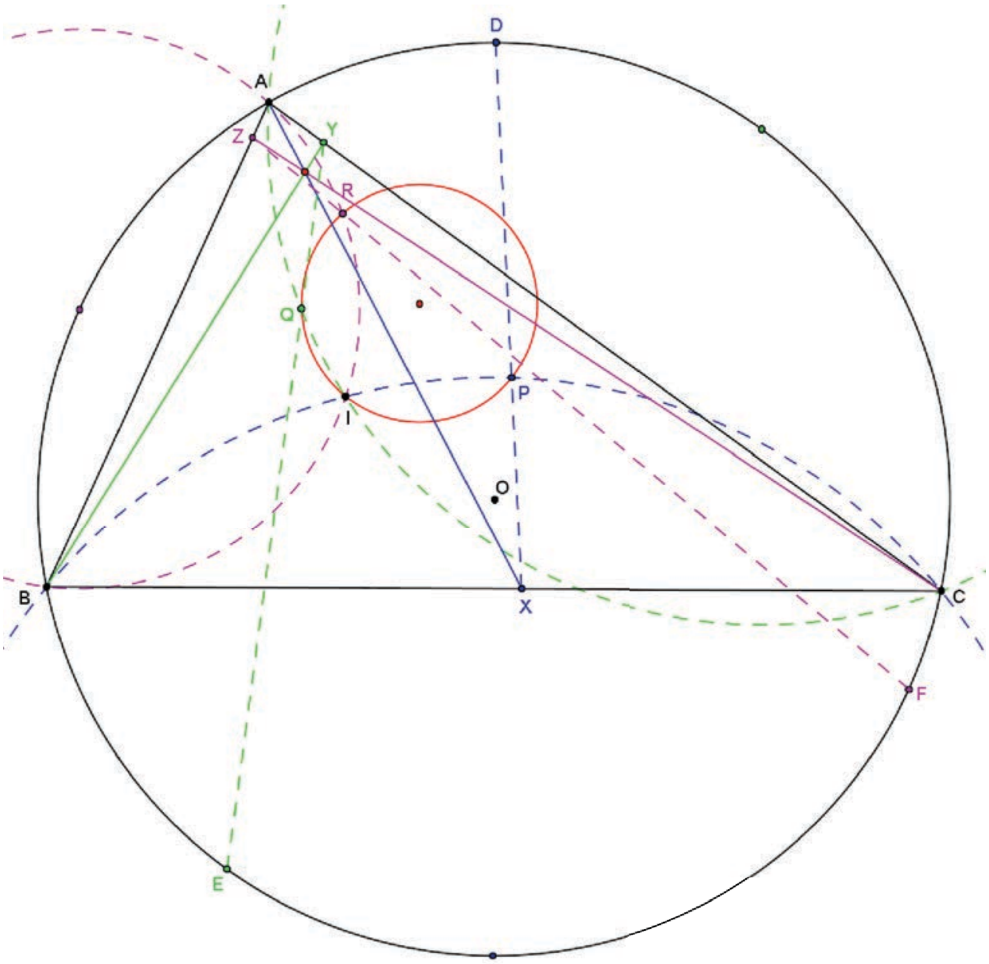
Reply

Petry

#1 Jul 3, 2011, 5:04 pm

Let ABC be a triangle, (O) is the circumcircle and I is the incenter.
Let D, E, F be the midpoints of the arcs $\widehat{CAB}, \widehat{ABC}, \widehat{BCA}$ respectively.
A circle through the point I intersects again the circumcircles of $\Delta BIC, \Delta CIA, \Delta AIB$ at P, Q, R respectively. If $\{X\} = DP \cap BC$, $\{Y\} = EQ \cap CA$, $\{Z\} = FR \cap AB$ then prove that the lines AX, BY and CZ are concurrent. :)

Attachments:



Luis González

#2 Jul 4, 2011, 3:19 am

Let $\triangle I_aI_bI_c$ be the excentral triangle of $\triangle ABC$. I_a, I_b, I_c against A, B, C. Lines IP, IQ, IR cut I_bI_c, I_cI_a, I_aI_b at P', Q', R', respectively. Since DB, DC are tangent to $\odot(IBI_aC)$, we deduce that

$$\frac{DB}{DP}=\frac{\sin\widehat{BPD}}{\sin\widehat{BI_aP}},\ \frac{DC}{DP}=\frac{\sin\widehat{CPD}}{\sin\widehat{CI_aP}} \Longrightarrow \frac{\sin\widehat{BPD}}{\sin\widehat{CPD}}=\frac{\sin\widehat{BI_aP}}{\sin\widehat{CI_aP}}=\frac{\sin\widehat{P'II_b}}{\sin\widehat{P'II_c}}$$

But $\dfrac{XB}{XC}=\dfrac{PB}{PC}\cdot\dfrac{\sin\widehat{BPD}}{\sin\widehat{CPD}}=\dfrac{\sin\widehat{BI_aP}}{\sin\widehat{CI_aP}}\cdot\dfrac{\sin\widehat{BPD}}{\sin\widehat{CPD}} \Longrightarrow$

$$\frac{XB}{XC} = \left(\frac{\sin \widehat{P'II_b}}{\sin \widehat{A'II_c}}\right)^2 = \left(\frac{II_c}{II_b} \cdot \frac{P'I_b}{P'I_c}\right)^2$$

Hence, multiplying the cyclic expressions together yields

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = \left(\frac{P'I_b}{P'I_c} \cdot \frac{Q'I_c}{Q'I_a} \cdot \frac{R'I_a}{R'I_b}\right)^2$$

Inversion with center I and power $\overline{IA} \cdot \overline{II_a}$ takes $\odot(IBC)$ into the line I_bI_c and $\odot(PQR)$ into a line $\tau \Longrightarrow P', Q', R'$ lie on the inverse τ of $\odot(PQR)$. Thus, by Menelaus' theorem for $\triangle I_aI_bI_c, \overline{P'Q'R'}$, we get

$$\frac{P'I_b}{P'I_c} \cdot \frac{Q'I_c}{Q'I_a} \cdot \frac{R'I_a}{R'I_b} = 1 \Longrightarrow \frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = 1$$

By the converse of Ceva's theorem, we conclude that AX, BY, CZ concur.

Petry

#3 Jul 4, 2011, 12:08 pm

First, let's prove the next problem.
Problem 1
Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three circles and $\{A, A'\} = \Gamma_2 \cap \Gamma_3$, $\{B, B'\} = \Gamma_3 \cap \Gamma_1$, $\{C, C'\} = \Gamma_1 \cap \Gamma_2$.
Let Γ be a circle and $\{M, N\} = \Gamma \cap \Gamma_1$, $\{P, Q\} = \Gamma \cap \Gamma_2$, $\{S, T\} = \Gamma \cap \Gamma_3$.
Prove that $\frac{MB}{MC} \cdot \frac{NB}{NC} \cdot \frac{PC}{PA} \cdot \frac{QC}{QA} \cdot \frac{SA}{SB} \cdot \frac{TA}{TB} = 1$.

Solution(see fig1):
$\{X\} = MN \cap BC$, $\{Y\} = PQ \cap CA$ and $\{Z\} = ST \cap AB$.
$\Delta XMB \sim \Delta XCN \Rightarrow \frac{XM}{XC} = \frac{MB}{CN}$ (1)
$\Delta XNB \sim \Delta XCM \Rightarrow \frac{XB}{XM} = \frac{NB}{CM}$ (2)
(1),(2)$\Rightarrow \frac{XB}{XC} = \frac{MB}{MC} \cdot \frac{NB}{NC}$
Similarly $\Rightarrow \frac{YC}{YA} = \frac{PC}{PA} \cdot \frac{QC}{QA}$ and $\frac{ZA}{ZB} = \frac{SA}{SB} \cdot \frac{TA}{TB}$.
So, $\frac{MB}{MC} \cdot \frac{NB}{NC} \cdot \frac{PC}{PA} \cdot \frac{QC}{QA} \cdot \frac{SA}{SB} \cdot \frac{TA}{TB} = \frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB}$ (3)

AA' is the radical axis of Γ_2 and Γ_3, BB' is the radical axis of Γ_3 and Γ_1 and CC' is the radical axis of Γ_1 and Γ_2.
So AA', BB', CC' are concurrent and $\{K\} = AA' \cap BB' \cap CC'$. K is the radical center of Γ_1, Γ_2 and Γ_3.
Similarly,
D is the radical center of the circles Γ, Γ_2 and Γ_3,
E is the radical center of the circles Γ, Γ_3 and Γ_1 and
F is the radical center of the circles Γ, Γ_1 and Γ_2.

$\frac{XB}{XC} = \frac{FK}{FC} \cdot \frac{EB}{EK}, \frac{YC}{YA} = \frac{DK}{DA} \cdot \frac{FC}{FK}$ and $\frac{ZA}{ZB} = \frac{EK}{EB} \cdot \frac{DA}{DK}$. So, $\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = 1$ (4)
($\Rightarrow$ the points X, Y and Z are collinear).

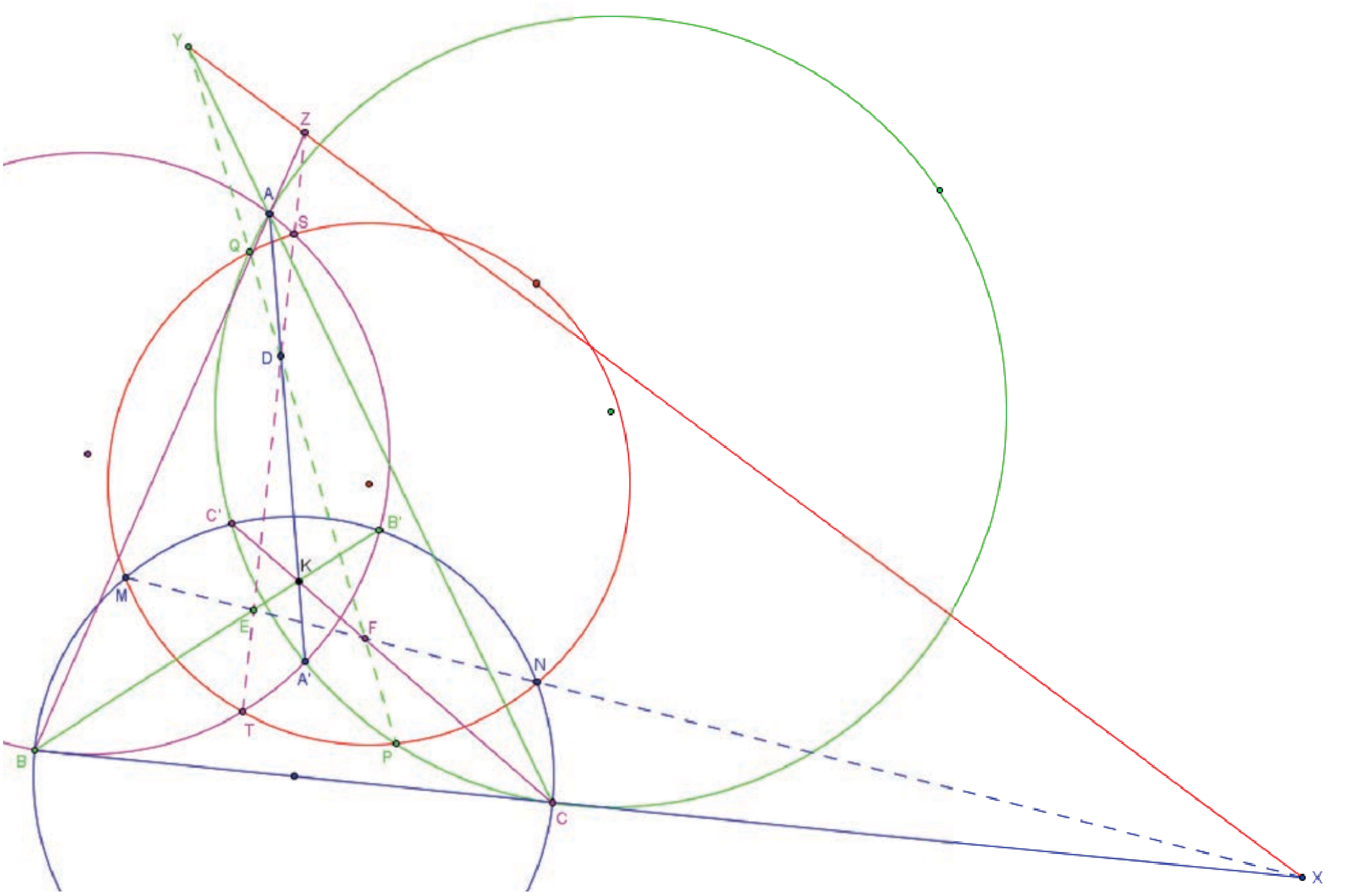
(3),(4)$\Rightarrow \frac{MB}{MC} \cdot \frac{NB}{NC} \cdot \frac{PC}{PA} \cdot \frac{QC}{QA} \cdot \frac{SA}{SB} \cdot \frac{TA}{TB} = 1$.

The solution of the proposed problem(see fig):
DB, DC are tangents to the circumcircle of $\Delta BPC \Rightarrow PX$ is symmedian in $\Delta BPC \Rightarrow \frac{XB}{XC} = \left(\frac{PB}{PC}\right)^2$.
Similarly $\Rightarrow \frac{YC}{YA} = \left(\frac{QC}{QA}\right)^2$ and $\frac{ZA}{ZB} = \left(\frac{RA}{RB}\right)^2$.
So, $\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = \left(\frac{PB}{PC} \cdot \frac{QC}{QA} \cdot \frac{RA}{RB}\right)^2$ (*)
Problem 1 $\Rightarrow \frac{IB}{} \cdot \frac{PB}{} \cdot \frac{IC}{} \cdot \frac{QC}{} \cdot \frac{IA}{} \cdot \frac{RA}{} = 1 \Rightarrow \frac{PB}{} \cdot \frac{QC}{} \cdot \frac{RA}{} = 1$ (**)

$IC \quad PC \quad IA \quad QA \quad IB \quad RB \qquad PC \quad QA \quad RB$

(*),(**)$\Rightarrow \dfrac{XB}{XC}\cdot\dfrac{YC}{YA}\cdot\dfrac{ZA}{ZB}=1 \Rightarrow$ the lines AX, BY, CZ are concurrent.

Attachments:



High School Olympiads

PB is tangent to circle (ABCD)

Reply

Source: My son & myself

sunken rock
#1 Jul 3, 2011, 7:47 pm • 1

Let $ABCD$ be a cyclic kite (i.e. BD is a perpendicular chord onto the diameter AC) and M the midpoint of AD.
The perpendicular from C onto BM intersects AD at P.
Prove that BP is tangent to the circle $\odot(ABC)$.

Best regards,
sunken rock

Luis González
#2 Jul 3, 2011, 11:25 pm • 1

Let $H \equiv AC \cap BD$ and B_∞ is the infinity point of $AD.$ Pencil $B(A,D,M,B_\infty)$ is harmonic. Therefore, pencil $C(B,H,P,D)$ formed by the perpendiculars from C to BA, BD, BM, BB_∞ is also harmonic. Then, $(Q,A,P,D)=-1,$ where $Q \equiv CB \cap AD.$ Since the pencil formed by $BA, BC \equiv BQ, BH \equiv BD$ and the tangent τ of $\odot(ABC)$ at B is clearly harmonic, then we deduce that τ passes through $P.$

sunken rock
#3 Jul 4, 2011, 7:22 pm

Another solution (my son's)
If $\{C,N\} \in CP \cap \odot(ABC) \implies \angle ANC=90^\circ \implies AN \parallel MB$, or DN is the symmedian in ΔABD, hence $ANBD$ is harmonic, consequently the tangents at D and N intersect at F, on AB. If $\{L\} \in DN \cap MB$, L is the midpoint of DN.
So, easily, the configurations B,D,L,F and B,A,N,P are mapped one over the other by a spiral similarity followed by a homothecy, both centered at B, hence $\angle PBA=\angle FBD$, done.

Best regards,
sunken rock

skytin
#4 Jul 4, 2011, 8:55 pm

Let P' is intersection point of AD and tangent to (ABC) from B
H is foot of perpendicular from P' on AC
Q is reflection of point P' wrt H
O is circumcenter of (ABC)
Angle P'QB = AP'Q = ADB = DBA = QBP' , so BP' = QP' and BP' = 2*HP'
Construct point X were HOXP' is rectangle
Easy to see that OX = HP' = BP'/2
OM = CD/2 = BC/2
Angle MOX = MP'X = P'AH = P'BC , so triangle P'BC ~ OMX
So angle MXO = MP'O = BP'C
MBOP' is cyclic and O is diameter of (BOP') , well known that if angle MP'O = BP'C then BM is perpendicular to CP' . done

Quick Reply

High School Olympiads

Prove that AO is parallel HK if IO is parallel to BC.

Source: ToT 2003 SA-4

Goutham

#1 Jul 3, 2011, 10:46 am • 1

In a triangle ABC, let H be the point of intersection of altitudes, I the center of incircle, O the center of circumcircle, K the point where the incircle touches BC. Given that IO is parallel to BC, prove that AO is parallel to HK.

Luis González

#2 Jul 3, 2011, 11:19 am • 2

Let E be the tangency point of the A-excircle (I_a) with BC. K, E are symmetrical about the midpoint M of BC and the antipode K' of K WRT (I) lies on AE, since A is the exsimilicenter of $(I) \sim (I_a)$. Thus, MI is the K-midline of $\triangle KEK'$, i.e. $MI \parallel AE$. If $T \equiv AH \cap MI$, then $ATIK'$ is a parallelogram $\Longrightarrow AT = IK' = IK$. But $IO \parallel BC$ implies then $\frac{1}{2}AH = OM = IK = AT \Longrightarrow O, T$ are the midpoints of EK', AH and the result follows.

1/0 = 42

admin25

#3 Jul 8, 2012, 6:04 am • 1

Sorry to revive, I'm a little confused with the above solution, specifically the last sentence.

> *Luis González wrote:*
> But $IO \parallel BC$ implies then $\frac{1}{2}AH = OM = IK = AT \Longrightarrow O, T$ are the midpoints of EK', AH and the result follows.

Can someone please explain this a little more in detail? Thanks.

Invader_2011

#4 Jul 8, 2012, 7:31 pm • 2

I think there are several steps skipped. $IO \parallel BC$ and $OM \parallel IK$ implies that quadrilateral $IKMO$ is a rectangle, so $IK = OM$. $\frac{1}{2}AH = OM$ is a well known fact, so the statement follows. Then since $KM = MD$, we have OM is the midline of $\Delta KEK' \Longrightarrow$ O is the midpoint of EK'. Since $ATIK'$ is a parallelogram, we have $AHKK'$ is also a parallelogram, also since O is on line AK', so $AO \parallel HK$.

Differ

#5 Sep 3, 2012, 5:26 am

Once we already know that AO goes to the tangent point of A's excircle (using collinearity), we can let P be the intersection of AH and the circumcircle. Then O, K, P are collinear, HKP is isosceles, AOP is isosceles, so angle PHK = angle PAO.

Quick Reply

High School Olympiads

from geometry unbound

Reply

eraydin

#1 Jul 1, 2011, 4:31 am

Given triangle ABC, let D,E be any points on segment BC. A circle through A cuts the lines AB, AC, AD, AE at the points P,Q,R,S. Prove that

(AP.AB - AR.AD) / (AS.AE - AQ.AC)= BD / CE

Luis González

#2 Jul 1, 2011, 6:46 am • 1

Invert the circle into a line δ through an inversion with center A and arbitrary power. δ cuts AB, AC, AD, AE at the inverses P', Q', R', S' of P, Q, R, S. Then

$$\left|\frac{AP \cdot AB - AR \cdot AD}{AS \cdot AE - AQ \cdot AC}\right| = \left|\frac{AB}{AP'} - \frac{AD}{AR'}\right| \cdot \left|\frac{AE}{AS'} - \frac{AC}{AQ'}\right|^{-1}$$

Let δ_P denote the oriented distance from P to δ. Thus, $\frac{AB}{AP'} = 1 - \frac{\delta_B}{\delta_A}$, etc.

$$\left|\frac{AB}{AP'} - \frac{AD}{AR'}\right| \cdot \left|\frac{AE}{AS'} - \frac{AC}{AQ'}\right|^{-1} = \left|1 - \frac{\delta_B}{\delta_A} - 1 + \frac{\delta_D}{\delta_A}\right| \cdot \left|1 - \frac{\delta_E}{\delta_A} - 1 + \frac{\delta_C}{\delta_A}\right|^{-1}$$

$$\left|\frac{AB}{AP'} - \frac{AD}{AR'}\right| \cdot \left|\frac{AE}{AS'} - \frac{AC}{AQ'}\right|^{-1} = \left|\frac{\delta_D - \delta_B}{\delta_C - \delta_E}\right| = \frac{BD \cdot \sin \angle(\delta, BC)}{CE \cdot \sin \angle(\delta, BC)} = \frac{BD}{CE}$$

$$\Longrightarrow \left|\frac{AP \cdot AB - AR \cdot AD}{AS \cdot AE - AQ \cdot AC}\right| = \frac{BD}{CE}$$

zero.destroyer

#3 Jul 26, 2011, 7:15 am

Is there an non-inversive proof? They posted this problem in the "Circular Reasoning" chapter before they talked about inversion.

Quick Reply

High School Olympiads

In right triangle

Reply

convexfig

#1 Jun 30, 2011, 10:58 pm

Let ABC be right triangle, $A = 90^0$. AH is altitude. HE, HF are altitudes of triangles HAB and HAC. M, N is projection of E, F on BC. P, Q is projection of E, F on AH. Prove that $\sqrt[3]{BM.PH} + \sqrt[3]{CN.QH} = \sqrt[3]{AB.AC}$.

Luis González

#2 Jul 1, 2011, 2:10 am

From $\triangle BME \sim \triangle BAC$ we deduce that

$$\frac{BM \cdot EM}{AB \cdot BC} = \frac{BM \cdot PH}{AB \cdot AC} = \frac{BE^2}{BC^2} \Longrightarrow BM \cdot PH = AB \cdot AC \cdot \frac{BE^2}{BC^2} \ (\star)$$

On the other hand, $BE = \dfrac{BH^2}{AB} = \dfrac{1}{AB}\left(\dfrac{AB^2}{BC}\right)^2 = \dfrac{AB^3}{BC^2}$

Substituting BE from this expression into $(\star)$ yields

$$BM \cdot PH = \frac{AC \cdot AB^7}{BC^6} \Longrightarrow \sqrt[3]{BM \cdot PH} = \sqrt[3]{AB \cdot AC} \cdot \frac{AB^2}{BC^2}$$

Similarly, we have $\sqrt[3]{CN \cdot QH} = \sqrt[3]{AB \cdot AC} \cdot \dfrac{AC^2}{BC^2}$. Therefore,

$$\sqrt[3]{BM \cdot PH} + \sqrt[3]{CN \cdot QH} = \sqrt[3]{AB \cdot AC}\left(\frac{AB^2 + AC^2}{BC^2}\right) = \sqrt[3]{AB \cdot AC}$$

Quick Reply

High School Olympiads

Lemoine point

Reply

Source: 0

arshakus

#1 Jun 30, 2011, 8:56 pm

Let M_a, M_b, M_c be the midpoints of sides BC, AC, AB of triangle ABC and H_a, H_b, H_c be the midpoints of altitudes from A, B, C.
i) Prove that the lines M_aH_a, M_bH_b, M_cH_c meet at one point.
ii) Prove that the point of intersection of these lines is the Lemoine point of triangle ABC.

This post has been edited 1 time. Last edited by arshakus, Jul 1, 2011, 12:45 am

Luis González

#2 Jun 30, 2011, 11:40 pm • 3

K is the symmedian point of $\triangle ABC$ and X, Y, Z are the orthogonal projections of K on BC, CA, AB. AM_a and AK are isogonals with respect to $\angle BAC \Longrightarrow AM_a \perp ZY$, i.e AM_a becomes the A-altitude of $\triangle AYZ$. Since K is the antipode of A WRT the circumcircle of $\triangle AYZ$, then its reflection E about the midpoint U of ZY is the orthocenter of $\triangle AYZ \Longrightarrow E \in AM_a$. It's well-known that K is the centroid of its pedal triangle $\triangle XYZ$, for a proof see the topic Very interesting collinear problem (post #4) or Cosmin Pohoata, A short proof of Lemoine's theorem. Consequently, X, K, E are collinear and $\overline{KX} = -2 \cdot \overline{KU} = -\overline{KE}$, i.e. K is the midpoint of XE. If D is the foot of the A-altitude, from $EX \parallel AD$, we deduce that M_aK passes through the midpoint H_a of AD. Likewise, M_bH_b and M_cH_c pass through the symmedian point K.

arshakus

#3 Jul 1, 2011, 12:49 am

> *luisgeometra wrote:*
> Since K is the antipode of A in the circumcircle of $\triangle AYZ$,

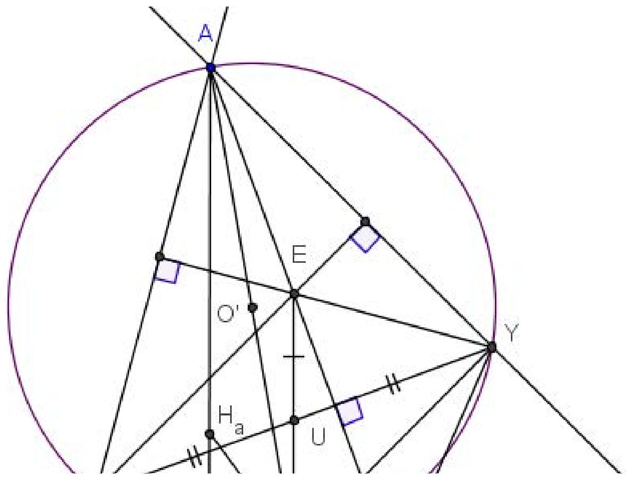
whay does it mrean?
sorry for my poor english)

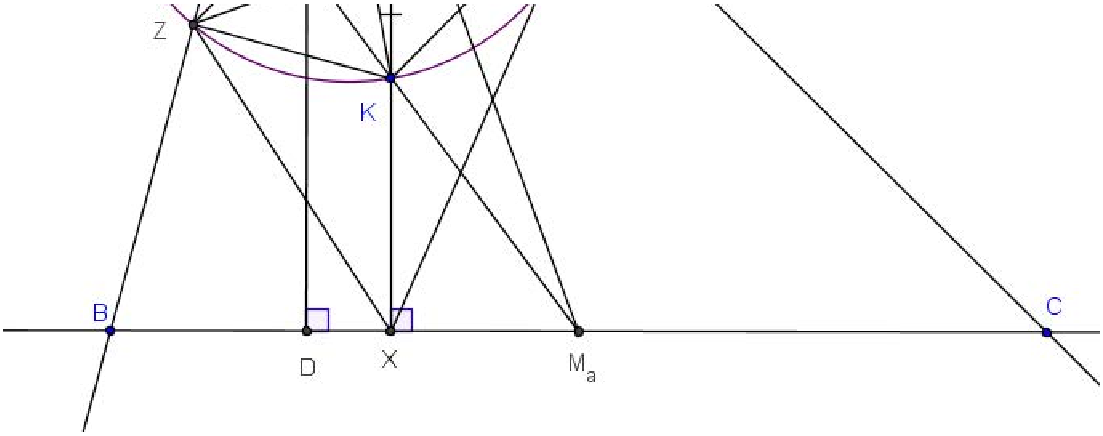
Luis González

#4 Jul 1, 2011, 1:17 am • 2

It means that K is diametrically opposed to A in the circle (O')=(AYZ)

Attachments:





jayme
#5 Jul 1, 2011, 8:22 pm • 1

Dear Mathlinkers,
this problem is the cross cevian point of G and H...
See
http://perso.orange.fr/jl.ayme vol. 3 the cross cevian point p. 20
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

ibero 1998

Reply

elegant

#1 Jun 30, 2011, 5:59 pm

The incircle of the triangle ABC touches BC, CA, AB at D, E, F respectively. AD meets the circle again at Q. Show that the line EQ passes through the midpoint of AF if AC = BC.

Luis González

#2 Jun 30, 2011, 10:45 pm

Assume that $AC = BC. \angle AEM = \angle EDA$, but since $ED \parallel AB$, we have $\angle EDA = \angle DAB \Longrightarrow \angle AEM = \angle DAB \Longrightarrow \odot(AEQ)$ is tangent to AB through A. Hence, radical axis EQ of the incircle (I) and $\odot(AEQ)$ bisects their common external tangent $\overline{AF}$. The converse is also true and we can use the same arguments.

P.S. See also http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=83932

yunxiu

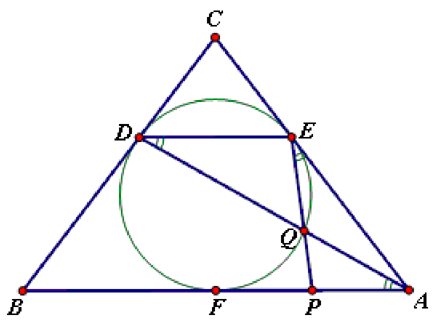
#3 Jul 1, 2011, 7:00 pm

Let $P = EQ \cap AF$, then

$$AP = PF \Leftrightarrow AP^2 = PQ \cdot PE \Leftrightarrow \Delta APQ \sim \Delta EPA$$

$$\Leftrightarrow \angle PAQ = \angle PEA = \angle QDE \Leftrightarrow DE \parallel AB \Leftrightarrow AC = BC$$

Attachments:



Quick Reply

High School Olympiads

The lines intersect on the sphere

Source: Austrian Mathematical Olympiad 1999, Part 2, D1, P2

Amir Hossein
#1 Jun 29, 2011, 2:24 am

Let ϵ be a plane and k_1, k_2, k_3 be spheres on the same side of ϵ. The spheres k_1, k_2, k_3 touch the plane at points T_1, T_2, T_3, respectively, and k_2 touches k_1 at S_1 and k_3 at S_3. Prove that the lines S_1T_1 and S_3T_3 intersect on the sphere k_2. Describe the locus of the intersection point.

Luis González
#2 Jun 29, 2011, 5:48 am • 1

O_1, O_2, O_3 are the centers of k_1, k_2, k_3. Let P be the antipode of T_2 WRT k_2. Since the radii O_1T_1 and O_2P are parallel, it follows that PT_1 passes through the exsimilicenter S_1 of $k_1 \sim k_2$. Likewise, S_3T_3 passes through P. Let the angle bisectors of $\angle T_1T_2T_3$ cut T_1T_3 at U, V. Then

$$\frac{\overline{UT_1}}{\overline{UT_3}} = -\frac{\overline{VT_1}}{\overline{VT_3}} = -\frac{T_2T_1}{T_2T_3} = -\frac{2\sqrt{r_2 \cdot r_1}}{2\sqrt{r_2 \cdot r_3}} = -\frac{\sqrt{r_1}}{\sqrt{r_3}} = \text{const.}$$

Thus, U, V are fixed $\Longrightarrow$ locus of T_2 is the circle ω with diameter $\overline{UV}$ $\Longrightarrow$ line PT_2 describes a right cylinder $\mathcal{C}$ with base ω. Since $PS_1 \cdot PT_1 = PS_3 \cdot PT_3 = PT_2{}^2$, then P has equal power WRT $k_1, k_3 \Longrightarrow P$ lies on the radical plane τ of k_1, k_3 $\Longrightarrow$ locus of P is an ellipse $\mathcal{E}$, intersection of $\mathcal{C}$ and τ.

High School Olympiads

Britain

Locked

elegant

#1 Jun 28, 2011, 5:21 am

Let and be circles meeting at the points A and B. A line through A meets at C and at D. Points M, N, K lie on the line segments CD, BC, BD respectively, with MN parallel to BD and MK parallel to BC. Let E and F be points on those arcs BC of and BD of respectively that do not contain A. Given that EN is perpendicular to BC and FK is perpendicular to BD prove that □EMF = 90 .

Luis González

#2 Jun 28, 2011, 8:15 am

This is problem 8 of IMO Shortlist 2002, which has been posted many times before, e.g.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=22201
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=17322
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=298999
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=308545

High School Olympiads

Britain

Locked

elegant

#1 Jun 28, 2011, 5:58 am

The incircle Ω of the acute-angled triangle ABC is tangent to BC at K. Let AD be an altitude of triangle ABC and let M be the midpoint of AD. If N is the other common point of Ω and KM, prove that Ω and the circumcircle of triangle BCN are tangent at N.

Luis González

#2 Jun 28, 2011, 8:05 am

This is problem 7 of IMO Shortlist 2002, which has been posted many times before, e.g.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=14741
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=17323
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=205790
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=265444

High School Olympiads

Prove concurrent

Reply

ambus

#1 Jun 28, 2011, 12:37 am

Let convex quadrilateral ABCD. The line pass A and parallel BD cut CD in F. The line pass D and parallel AC cut AB in E. M, N, P, Q are midpoint BD, AC, DE, AF. Prove: lines MN, PQ, AD concurrent

Luis González

#2 Jun 28, 2011, 5:28 am

Since PM, QN are the D- and A- midlines of $\triangle DEB$ and $\triangle AFC$, then PM, QN pass through the midpoint K of $AD.$ Let $U \equiv AC \cap BD$ and $V \equiv AF \cap DE.$ Since $AUDV$ is a parallelogram, then UV passes through $K \Longrightarrow \triangle UMN$ and $\triangle VPQ$ are perspective through $K.$ Thus, by Desargues theorem, the intersections $A \equiv UN \cap VQ, D \equiv UM \cap VP$ and $PQ \cap MN$ are collinear, i.e. lines MN, PQ, AD concur.

Quick Reply

High School Olympiads

concurrence

Reply

huyhoang

#1 Jun 27, 2011, 9:45 pm

In the circle (O), we construct the circle (O') internally tangent to circle (O) at T. Draw an arbitrary chord of circle (O), we call it BC. Let M is the midpoint of the arc BC which contains T. Construct tangent BP, CQ to (O'), and P, Q lie on the plane which does not contain T. Prove that BC, MT, PQ concur.

Luis González

#2 Jun 28, 2011, 12:19 am • 2

Already discussed on the topic Tangent circles concurrent lines. Since I will probably be asked to show the proof of the supporting lemma, then I would like to present its proof.

Theorem: Two circles $\Gamma_1(r_1)$ and $\Gamma_2(r_2)$ are internally/externally tangent to a circle $\Gamma(R)$ through A, B, respectively. The length δ_{12} of the common external tangent of Γ_1, Γ_2 is

$$\delta_{12} = \frac{AB}{R}\sqrt{(R \pm r_1)(R \pm r_2)}$$

Proof: Without loss of generality assume that $r_1 \geq r_2$ and we suppose that Γ_1 and Γ_2 are internally tangent to Γ. The remaining case will be treated analogously. A common external tangent between Γ_1 and Γ_2 touches Γ_1, Γ_2 at A_1, B_1 and A_2 is the orthogonal projection of O_2 onto O_1A_1. By Pythagorean theorem for $\triangle O_1O_2A_2$, we obtain

$\delta_{12}{}^2 = (A_1B_1)^2 = (O_1O_2)^2 - (r_1 - r_2)^2 \ \ (1)$

Let $\angle O_1OO_2 = \lambda$, thus by cosine law for $\triangle OO_1O_2$, we get

$(O_1O_2)^2 = (R - r_1)^2 + (R - r_2)^2 - 2(R - r_1)(R - r_2)\cos\lambda \ \ (2)$

By cosine law for the isosceles $\triangle OAB$, we get $AB^2 = 2R^2(1 - \cos\lambda) \ \ (3)$

Eliminating $\cos\lambda$ and O_1O_2 by combining (1), (2) and (3) yields

$\delta_{12}{}^2 = (R - r_1)^2 + (R - r_2)^2 - (r_1 - r_2)^2 - 2(R - r_1)(R - r_2)\left(1 - \frac{AB^2}{2R^2}\right)$

Subsequent simplifications give $\delta_{12} = \dfrac{AB}{R}\sqrt{(R - r_1)(R - r_2)}$.

huyhoang

#3 Jun 28, 2011, 7:56 am

really nice proof !

Quick Reply

High School Olympiads

<POB=2<PBO

Reply

jasur

#1 Jun 25, 2011, 10:28 am

The point A is on a circle with centre O. The line OA is extended to C so that OA=AC, and B is the midpoint of AC.
The point Q is on the circle such that <AOQ is obtuse. The line QO meets the perpindicular bisector of CQ at P.
Prove that <POB=2<PBO

yetti

#2 Jun 25, 2011, 9:42 pm

$OP^2 + OC^2 - 2OP \cdot OC \cdot \cos \widehat{POC} = CP^2.$ Substitute $OC = 2r, CP = QP = r + OP$ and $\angle POC \equiv \angle POB < \frac{\pi}{2} \Longrightarrow OP(1 + 2\cos \widehat{POB}) = \frac{3}{2}r = OB,$ where $\cos \widehat{POB} > 0.$
Circle (P) with radius PO cuts OB again at $S \Longrightarrow OS = OP \cdot 2\cos \widehat{POB} \Longrightarrow OP = SP = SB \Longrightarrow \triangle POS$ is P-isosceles and $\triangle SBP$ is S-isosceles $\Longrightarrow \angle POB = \angle PSO = 2\angle PBO.$

Luis González

#3 Jun 26, 2011, 2:23 am

$PC - PO = PQ - PO = R \Longrightarrow P$ lies on a hyperbola $\mathcal{H}$ with foci O, C and center $A.$ Its vertices are B and the reflection D of B about $A.$ Its directrix d with respect to the focus O cuts OC at $M.$ Then, $DM : DO = AB : AC = 1 : 2$ $\Longrightarrow M$ is the midpoints of $\overline{OB},$ i.e. d is the perpendicular bisector of $\overline{OB}.$ If d cuts PB at $U,$ from $PO = 2 \cdot \text{dist}(P, d),$ we get $\frac{\text{dist}(P,d)}{MB} = \frac{OP}{OB} = \frac{UP}{UB} \Longrightarrow OU$ bisects $\angle POB \Longrightarrow \angle POB = 2\angle PBO.$

sunken rock

#4 Jun 28, 2011, 3:30 pm

As an alternate solution to **yetti**'s:
See that $PB^2 = OP \cdot (OP + OB)$, this relation showing that, under the given conditions, $m(\widehat{POB}) = 2 \cdot m(\widehat{PBO})$.

Best regards,
sunken rock

sunken rock

#5 Jul 4, 2011, 2:59 pm

My son's solution:
Take T - midpoint of CQ, $\{B'\} \in OQ \cap BT$; easily, $2 \cdot BT = B'T\ (\ 1\ )$.
Take M - midpoint of OB', hence $CQ \parallel BM\ (*)$.
Take O' the intersection of PT with the perpendicular bisector and G - the centroid of $\triangle OBB'$.
Obviously, T is the orthocenter of $\triangle BGO'$; with (1) and G - centroid we get that $TG \parallel OB$, hence $O'B \perp OB$ and $OBO'B'$ is a kite.
Further, apply the result from http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=416180, getting that BP is tangent to the circle $\odot OBO'B'$, or $\angle TBO = \angle OBP$, done.

Best regards,
sunken rock

skytin

#6 Jul 5, 2011, 11:28 pm

Let point X is on CP and OX || QC
Easy to see that CB*CO = CX*CX , so CX is tangent to (BOX)
CBX ~ CXO , so BX/OX = 1/2 = BA/OA , so AX is anglebisector of BXO
N is midpoint of OX
Line from point P and || to OC intersects line AX at point F
XP = PF
XA is perpendicular to BN and devides this segment on two equal parts , so BF = FN = PF
So BOPF is cyclic and angle 2*PBO = BOP . done

Quick Reply

High School Olympiads

Construction of a Point Problem

Obel1x

#1 Jun 25, 2011, 3:26 am

Given a triangle ABC. Construct a point P on the side BC so that the incircles of triangles ABP and ACP have equal radii.

Luis González

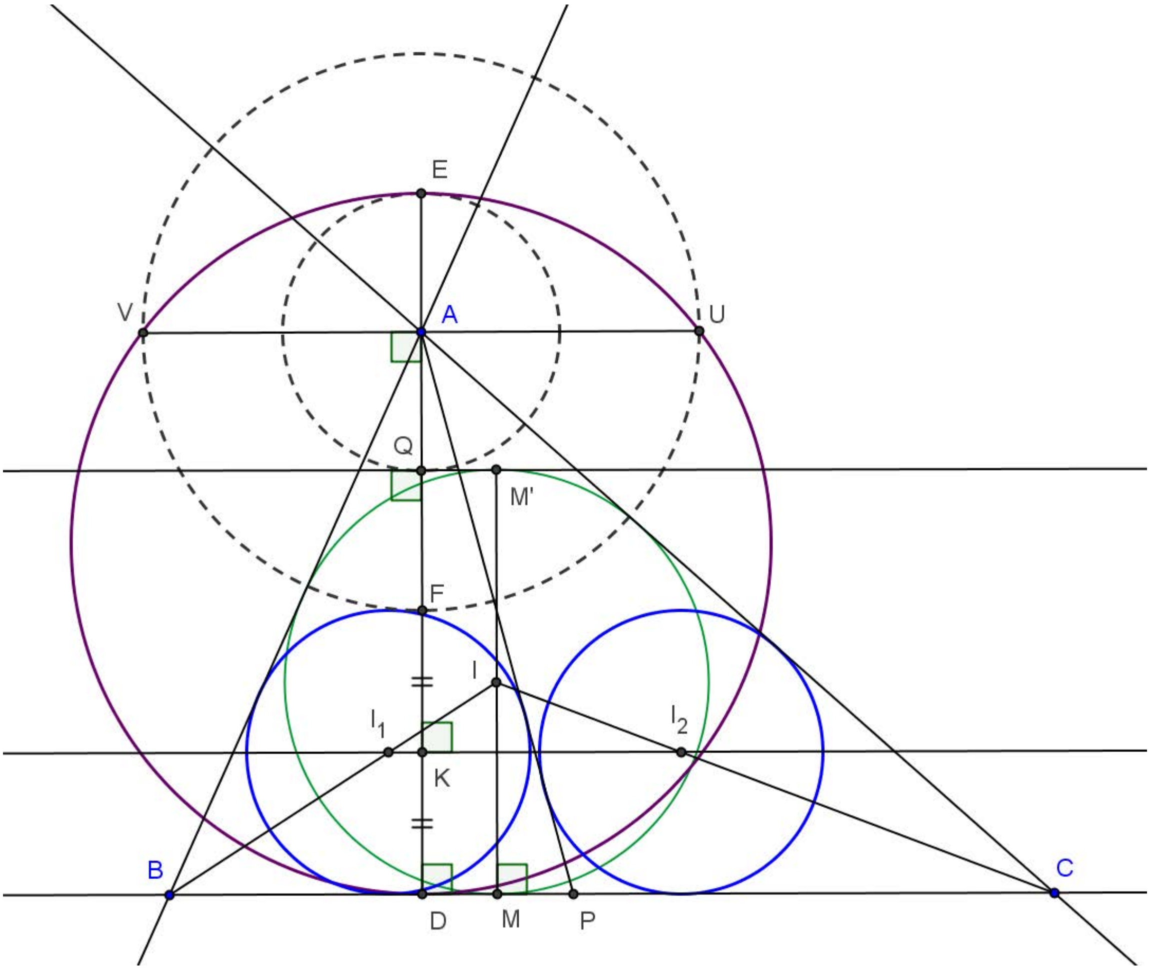
#2 Jun 25, 2011, 4:38 am • 1

Let ϱ denote the radii of the incircles $(I_1), (I_2)$ of $\triangle ABP$ and $\triangle ACP$. (I, r) is the incircle of $\triangle ABC$ and $AD = h_a$ is the A-altitude. According to The Nicest Sangaku, we have

$$1 - \frac{2r}{h_a} = \left(1 - \frac{2\varrho}{h_a}\right)^2 \Longrightarrow 2\varrho = h_a - \sqrt{h_a(h_a - 2r)}$$

Length 2ϱ is constructible with ruler and compass. Let (I) touch BC at M and let M' be the antipode of M WRT (I). Tangent of (I) at M' cuts AD at Q. E is the reflection of Q about A. Perpendicular to ED through A cuts the circle with diameter $\overline{DE}$ at U, V. Circle with center A and radius $AU = AV$ cuts the segment $\overline{AD}$ at F. Parallel to BC through the midpoint K of $\overline{DF}$ cuts BI, CI at I_1, I_2. Circles $(I_1), (I_2)$ with centers I_1, I_2 and tangent to BC are the desired incircles. The determination of P is then straightforward.

Attachments:



High School Olympiads

Triangle construction

jatin

#1 Jun 21, 2011, 3:40 pm

Construct $\triangle ABC$, given $\angle A$, length l_a of its bisector and $b+c$ where b, c are sides of $\triangle ABC$.

vanstraelen

#2 Jun 23, 2011, 1:10 am

First, some calculations.

In $\triangle ABC : a^2 = (b+c)^2 - 4bc \cdot \cos^2 \frac{\alpha}{2}$

$$4bc \cdot \cos^2 \frac{\alpha}{2} = (b+c)^2 - a^2$$

$$bc = \frac{(\frac{b+c}{2})^2 - (\frac{a}{2})^2}{\cos^2 \frac{\alpha}{2}} \qquad (1)$$

In $\triangle ABC$, the lenght l_a of the bisector: $l_a = \frac{2}{b+c}\sqrt{bcs(s-a)}$

$$(b+c)l_a = 2\sqrt{bc(\frac{b+c}{2}+\frac{a}{2})(\frac{b+c}{2}-\frac{a}{2})}$$

$$(b+c)l_a = 2\sqrt{bc\left[(\frac{b+c}{2})^2 - (\frac{a}{2})^2\right]}$$

With (1):

$$(b+c)l_a = 2\sqrt{\frac{\left[(\frac{b+c}{2})^2 - (\frac{a}{2})^2\right]^2}{\cos^2 \frac{\alpha}{2}}}$$

$$(b+c)l_a = 2\frac{\left[(\frac{b+c}{2})^2 - (\frac{a}{2})^2\right]}{\cos \frac{\alpha}{2}}$$

$$2(b+c)l_a \cos \frac{\alpha}{2} = (b+c)^2 - a^2$$

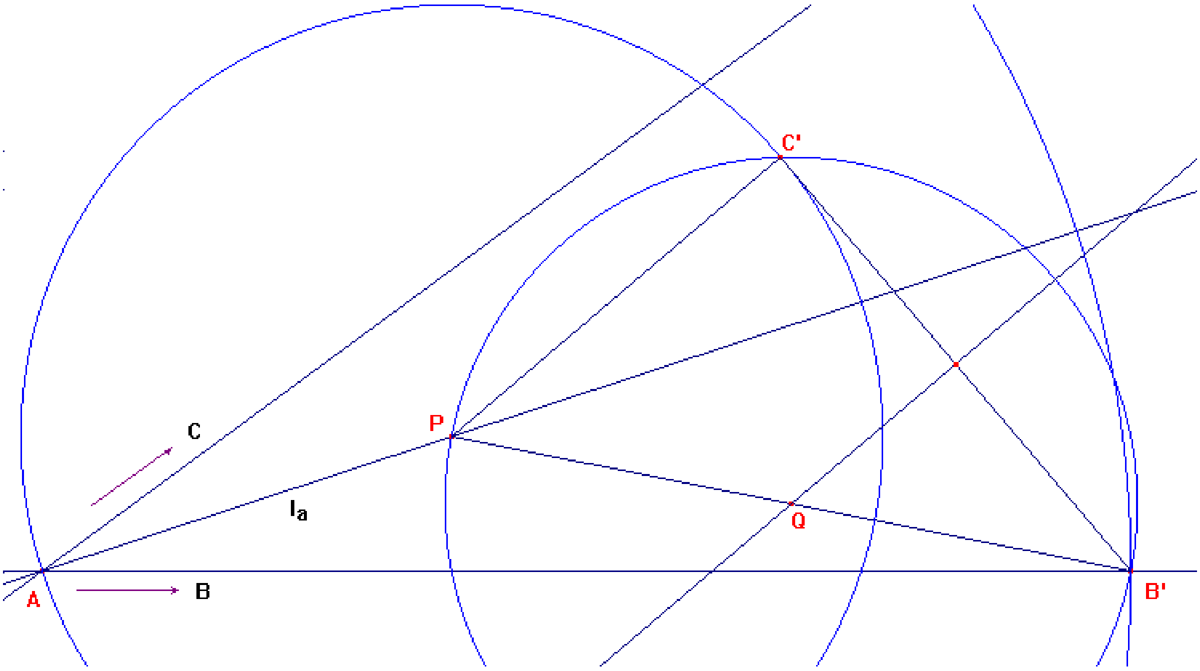
$$a^2 = (b+c)^2 - 2(b+c)l_a \cos \frac{\alpha}{2}$$

$$a^2 + (l_a)^2 = (b+c)^2 + (l_a)^2 - 2(b+c)l_a \cos \frac{\alpha}{2}$$

$$a^2 + (l_a)^2 = (b+c)^2 + (l_a)^2 - 2(b+c)l_a \cos\frac{\alpha}{2}$$

Second, the construction, see picture.
Given $\angle A, l_a = AP, b + c = AB'$
$\triangle APB' : (PB')^2 = (b+c)^2 + (l_a)^2 - 2(b+c)l_a \cos\frac{\alpha}{2}$
Hence, PB' is known, $a^2 + (l_a)^2 = (PB')^2$
Circle, midpoint Q of PB'.
Circle, midpoint P, through $A : AP = PC'$ with C' on the first circle.
Then, because $a^2 + (l_a)^2 = (PB')^2 : B'C' = a$.
The original problem becomes now: given $\angle A$, given the opposite side $a = B'C'$ and given $b + c$, construct through the point $P : BC = B'C'$. This problem is hopefully known.

Attachments:



Luis González

#3 Jun 23, 2011, 8:40 am

General problem: Given two fixed rays b, c issuing from A and a point P inside the angle $\angle(b,c)$, then find the lines through P, which intersect c, b at B, C, such that $AB + AC$ is given.

We are interested in the envelope of the lines BC, such that $AB + AC = k = \text{const}$. Construct the points X, Y on the rays c, b such that $AX = AY = \frac{1}{2}k$, thus $\overline{XB} = \overline{YC} = \frac{1}{2}|AC - AB|$. Circumcircle (O) of $\triangle ABC$ and $\odot(AXY)$ meet at A and the center D of the rotation taking the oriented segments $\overline{XB} \cong \overline{YC}$ into each other $\Longrightarrow DB = DC, DX = DY$ $\Longrightarrow$ midpoint of the arc BC of (O) coincides with the midpoint D of the arc XY of $\odot(AXY)$. Construct the reflections D_1, D_2 of D about $b, c \Longrightarrow d \equiv D_1D_2$ is the Steiner line of D WRT $\triangle ABC \Longrightarrow BC$ is always tangent to the parabola $\mathcal{P}$ with focus D and directrix d. Thus, our desired lines are the tangents from P to $\mathcal{P}$. Of course, P must lie outside the parabolic region, otherwise, there's no solution. Construct the circle with center P and radius PD that cuts the directrix d at P_1, P_2. Perpendicular bisectors of $\overline{DP_1}$ and $\overline{DP_2}$ are the desired tangents.

Quick Reply