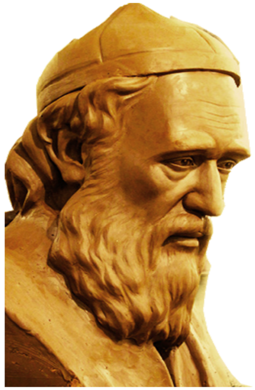
Omar Khayyám



Ghiyās od-Dīn Abol-Fath Omār ibn Ebrāhīm Khayyām Neyshābūri (18th May 1048 - 4th December 1131) was a Persian poet, mathematician, philosopher and astronomer Who lived in Iran. He is best known for his poetry, and outside Iran, for the quatrains (rubaiyaas) in Rubaiyat of Omar Khayyam, popularized through Edward Fitzgerald's re-created translation. Khayyám is the author of one of the most important treatises on algebra written before modern times, the Treatise on Demonstration of Problems of Algebra, which includes a geometric method for solving cubic equations by intersecting a hyperbola with a circle. In the Treatise he wrote on the triangular array of binomial coefficients known as Pascal's triangle. In 1077, Khayyam wrote "Sharhe Ma Ashkala Men Mosadarat Ketab Uqlidos" (Explanations of the Difficulties in the Postulates of Euclid) published in English as "On the Difficulties of Euclid's Definitions". An important part of the book is concerned with Euclid's famous parallel postulate. Omar Khayyám had notable works in geometry, specifically on the theory of proportions. He also contributed to calendar reform and may have proposed a heliocentric theory well before Copernicus.

29th Iranian Mathematical Olympiad

2011-2012
Young Scholars Club
Ministry of Education, I.R.Iran
www.ysc.ac.ir

This booklet is dedicated to

our science and technology martyrs;

the scientists who sacrificed their lives

for the progress of our country.

Majid Shahriari

Born: 1966

Assassinated: 2010

Masoud Alimohammadi

Born: 1959

Assassinated: 2010

Mostafa Ahmadi-Roshan

Born: 1980

Assassinated: 2012

Daryoush Rezaeinejad

Born: 1976

Assassinated: 2011

Iranian Team Members in 53th IMO (Mar Del Plata-Argentina) (from left to right):

- Mina Dalirrooyfard
- Goodarz Mehr
- Alek Bedroya
- Pedram Safaei
- Alireza Fallah
- Amir Ali Moinfar

This booklet is prepared by Ali Khezeli, Hesam Rajabzadeh,
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With special thanks to Javad Abedi, Shayan Aziznejad, Ali Babaei, Atieh Khoshnevis, Omid Naghshineh Arjmand and Sina Rezaie.

29^{th}

Iranian Mathematical Olympiad

2011-2012

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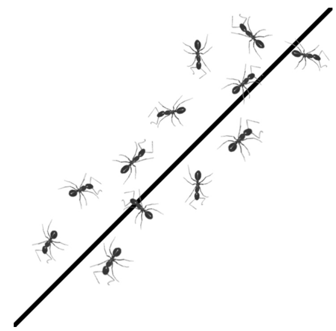
Problems

First Round

1. (Mohsen Jamali) 1390 ants are near a straight line such that the distance between the head of each ant and the line is less than 1 centimeter and the distance between the head of each two ants is more than 2 centimeters. Prove that there exist two ants which their heads are at least 10 meters far. (Assume that the head of each ant is a point!) (→p. 18)

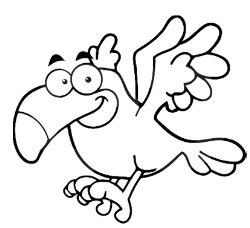


2. (MirSaleh BahavarNia) In triangle ABC we have $\angle BAC = 60°$. The perpendicular line to AB at B intersects the bisector of $\angle BAC$ at D and the perpendicular line to BC at C meets the bisector of $\angle ABC$ at E. Prove that $\angle BED \leq 30°$. (→p. 18)

3. (Mohsen Jamali) Determine all increasing sequence $a_1, a_2, a_3, \dots$ of positive integers such that for every $i, j \in \mathbb{N}$, the number of positive divisors of $i + j$ and $a_i + a_j$ are equal (A sequence $a_1, a_2, a_3, \dots$ is increasing if $i \leq j$ implies $a_i \leq a_j$). (→p. 19)

4. (Mohammad Mansouri, Shayan Dashmiz) Find the smallest positive integer n such that there exist n real numbers in the interval $(-1,1)$ such that their sum is zero and the sum of their squares equals 20. (→p. 19)

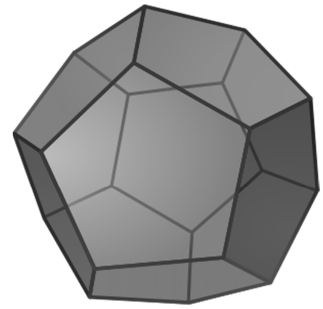
5. (Mohsen Jamali) A beautiful rare bird called n*-colored Rainbow* can be seen in one of n different colors each day and its color is always different from the previous day. Recently, scientists have discovered a new fact about this bird's life: "there does not exist four days like i, j, k, l in its life such that $i < j < k < l$ with colors namely a, b, c, d respectively, such that $a = c \neq b = d$". Find the maximum possible age of an n-colored Rainbow as a function of n. (→p. 20)



6. (Ali Khezeli) We have extended the sides AB and AC of triangle ABC from B and C respectively to intersects a given line l at D and E respectively. Suppose the reflection of l with respect to perpendicular bisector of BC intersects mentioned extensions at D' and E' respectively. If $BD + CE = DE$, show that $BD' + CE' = D'E'$. (→p. 20)

Second Round

1. (Kasra Alishahi) A regular dodecahedron is a convex polyhedron such that its faces are regular pentagons. It has 20 vertices and 3 edges connected to each vertex. (As you see in the picture.)



   Suppose that we have marked 10 vertices of a regular dodecahedron.
   a) Prove that we can rotate the dodecahedron in such a way that the dodecahedron is mapped to itself and at most four marked vertices are mapped to a marked vertex.
   b) Prove that number 4 cannot be replaced with number 3 in the previous part.
   (→p. 22)

2. (Mohammad Mansouri) Prove that for every positive integers k and n there exist k monic polynomials $P_1(x), P_2(x), \ldots, P_k(x)$ of degree n with integer coefficient such that each two of them have no common factor and the sum of each arbitrary number of them has all its roots real. (→p. 22)

3. (Erfan Salavati) Four metal pieces are joined to each other to form a quadrilateral in the space. The angle between them can vary freely. In a case that the quadrilateral is not planar, we mark one point of each piece such that the points lie in a plane. Prove that these four points are always coplanar as the quadrilateral varies.

   (→p. 23)

4. (Mohammad Ghiasi) The escalator of "Champion Butcher" metro station has this property that if m persons are on it, its speed is $m^{-\alpha}$ where α is a positive constant number. Suppose that n persons want to go upstairs by the escalator. If the length of the escalator is l, what is the least time required for these persons to go to upstairs? (Suppose the persons can use the escalator simultaneously at any time).

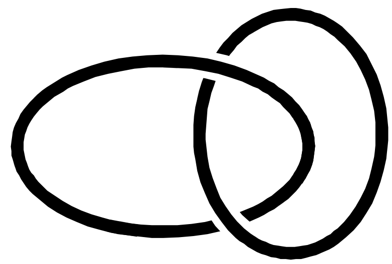
   (→p. 23)

5. (Mahyar Sefidgaran, Mostafa Eynollahzadeh) Let α be a real number and $a_1, a_2, a_3, \ldots$ a strictly increasing sequence of positive integers such that for every $n \in \mathbb{N}$, $a_n \le n^{\alpha}$. A prime number q is called *golden* if there is a positive integer m such that $q \mid a_m$. Suppose that $q_1 < q_2 < q_3 < \cdots$ are all *golden* prime numbers.

   a) Prove that if $\alpha = 1.5$, then $q_n \le 1390^n$.

   b) Prove that if $\alpha = 2.4$, then $q_n \le 1390^{2n}$. (→p. 24)

6. (Ali Khezeli) Two circles in the space are called *linking* if they intersect at two points or they are interlocked. Find a necessary and sufficient condition for four distinct points A, B, A', B' in the space such that every two different circles passing through A, B and the other passing through A', B' respectively are *linking.* (→p. 25)



7. (Sepehr GhaziNezami) For a function $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ and a subset $A \subseteq \mathbb{N}$, we say f is A*-predictor* if the set $\{x \in \mathbb{N} \mid x \notin A, f(A \cup \{x\}) \neq x\}$ is finite. Prove that there exists a function that for every subset A of natural numbers is A*-predictor.* (→p. 26)

8. (Ali Khezeli) A sequence $d_1, \ldots, d_n$ of not necessarily distinct natural numbers is called a *covering sequence* if there exist arithmetic progressions of the form $\{a_i + kd_i : k = 0,1,2,\ldots\}$ such that every natural number comes in at least one of them. We call this sequence *minimal* if we cannot delete any of $d_1, \ldots, d_n$ such that the resulting sequence is still *covering.*

   a) Suppose $d_1, \ldots, d_n$ is a *minimal covering sequence* and suppose we've covered all the natural numbers with arithmetic progressions $\{a_i + kd_i : k = 0,1,2,\ldots\}$. Suppose that p is a prime number that divides $d_1, \ldots, d_k$ but does not divide $d_{k+1}, \ldots, d_n$. Prove that the remainders of $a_1, \ldots, a_k$ modulo p contain all the numbers $0,1,\ldots,p-1$.

   b) Write anything you can about *covering sequences* and *minimal covering sequences* in the case that each of $d_1, \ldots, d_n$ has only one prime divisor. (→p. 26)

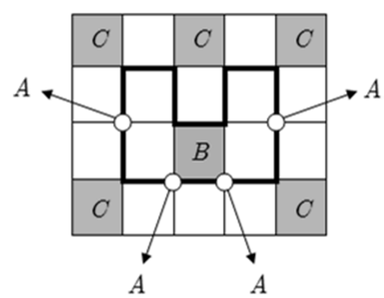
Third Round

1. (Mahdi E'tesamiFard) Find all natural numbers $n \geq 2$ such that for every pair of integers $i, j \in [0, n]$, $i + j$ and $\binom{n}{i} + \binom{n}{j}$ have the same parity. (→. 29)

2. (Mahdi E'tesamiFard) Let ω be the circumcircle of an acute triangle ABC. Let D be the midpoint of arc $\overset{\frown}{BAC}$ in ω and I be the incenter of triangle ABC. Suppose DI intersects BC at E and ω at F for the second time. Suppose the parallel line to AI from E meets AF at P. Prove that PE is the bisector of $\angle BPC$. (→p. 30)

3. (Erfan Salavati) Let n be a natural number. A subset S of points in the plane has following properties:
   i) There are not n lines in the plane such that each element of S lies on at least one of them.
   ii) For every $X \in S$ there exist n lines in the plane such that each point of $S - \{X\}$ lies on at least one of them.
   Find the maximum possible number of points in S. (→p. 30)

4. (Ali Khezeli) There are $m+1$ horizontal and $n+1$ vertical lines $(m, n \geq 4)$ that make a $m \times n$ table. Consider a closed path that does not intersect itself and passes through all $(m-1)(n-1)$ interior vertices and none of the outer vertices. (Each vertex is the intersection point of two lines!) A is the number of interior vertices that the path passes through them straightforward, B is the number of squares in the table that only two opposite sides of them are used in the path and C is the number of squares that none of its sides are used in the path. Prove that $A = B - C + m + n - 1$. (→p. 31)



5. (Masoud Shafaei) Let $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be a function such that for all $a, b \in \mathbb{R}^{\geq 0}$:
   i) $f(a) = 0 \Leftrightarrow a = 0$.
   ii) $f(ab) = f(a)f(b)$.
   iii) $f(a+b) \leq 2\max\{f(a), f(b)\}$.

Prove that for every $a,b \in \mathbb{R}^{\geq 0}$, $f(a+b) \leq f(a) + f(b)$. (→p. 33)

6. (Morteza Saghafian, Ali Khezeli) Let $ABCDE$ be a cyclic pentagon with circumcircle ω. Suppose that $\omega_a, \omega_b, \omega_c, \omega_d, \omega_e$ are the reflections of ω with respect to AB, BC, CD, DE and EA respectively. A' is the second intersection of ω_a and ω_e. B', C', D', E' are defined similarly. Prove that
$$2 \leq \frac{S(A'B'C'D'E')}{S(ABCDE)} \leq 3,$$
where $S(X)$ denotes the area of figure X. (→p. 34)

7. (Morteza Saghafian) Is it possible to write $\binom{n}{2}$ consecutive natural numbers on the edges of a complete graph with n vertices such that for every path (or cycle) of length 3 with edges a,b,c (b lies between a,c) the greatest common divisor of the numbers of edges a and c divides the number of edge b? (→p. 35)

8. (Mohammad Ja'fari) Let g be a polynomial of degree at least 2 with nonnegative coefficients. Find all functions $f: \mathbb{R}^+ \to \mathbb{R}^+$ such that for every $x,y \in \mathbb{R}^+$
$$f(f(x) + g(x) + 2y) = f(x) + g(x) + 2f(y).$$
(→p. 36)

9. (Ali Khezeli) Let $ABCD$ be a parallelogram. Consider circles ω_1 and ω_2 such that ω_1 is tangent to segments AB, AD and ω_2 is tangent to segments BC, CD. Suppose that there exist a circle tangent to lines AD, DC and externally tangent to ω_1, ω_2. Prove that there exists a circle tangent to lines AB and BC and externally tangent to ω_1, ω_2. (→p. 37)

10. (Morteza Saghafian) Let a,b,c be positive real numbers such that $ab+bc+ca=1$. Show that
$$\sqrt{3}(\sqrt{a}+\sqrt{b}+\sqrt{c}) \leq \frac{a\sqrt{a}}{bc} + \frac{b\sqrt{b}}{ca} + \frac{c\sqrt{c}}{ab}.$$
(→p. 37)

11. (Mahdi E'tesamiFard, Ali Khezeli) Let A,B be two different points on a circle ω with center O such that $60^\circ \leq \angle AOB \leq 120^\circ$. Let C be the circumcenter of $\triangle AOB$. l is a line passing through C such that the angle between l and OC is 60°. Tangent lines to ω at A,B meets l at M,N respectively. Suppose the circumcircle of

triangles CAM and CBN intersect ω for the second time at Q and R respectively and meets each other in P (different from C). Prove that $OP \perp QR$. (→p. 38)

12. (Javad Abedi) A subset B of natural numbers is called *loyal* if there exist positive integers $i \le j$ such that $B = \{i, i+1, \ldots, j\}$. Q is the collection of *loyal* subsets of natural numbers. For every subset $A = \{a_1 < a_2 < \cdots < a_k\}$ of $\{1,2,\ldots,n\}$ we define:

$$g(A) = \max_{B \subset A, B \in Q} \left|B\right| \text{ and } f(A) = \max_{1 \le i \le k-1} a_{i+1} - a_i.$$

Also, Define

$$G(n) = \sum_{A \subseteq \{1,2,\ldots,n\}} g(A) \text{ and } F(n) = \sum_{A \subseteq \{1,2,\ldots,n\}} f(A).$$

Prove there exists $m \in \mathbb{N}$ such that for every positive integer n greater than m we have $F(n) > G(n)$. ($\left|A\right|$ denotes the number of elements of a set A and if $\left|A\right| \le 1$ we set $f(A) = 0$). (→p. 39)

13. (Hesam Rajabzadeh) Consider a regular 2^k-gon with center O in the plane and let $l_1, l_2, \ldots, l_{2^k}$ be its sides with the clockwise order. Reflect O with respect to l_1, reflect the resulting point with respect to l_2 and continue this process until the last side. Prove that the distance between final point and O is less than the perimeter of the mentioned 2^k-gon. (→p. 40)

14. (Morteza Saghafian) Are there 2000 real numbers (not necessarily distinct), not all zero, such that if we put any 1000 of these numbers as roots of a monic polynomial of degree 1000, its coefficients (except the coefficient of x^{1000}) are a permutation of the 1000 remaining numbers? (→p. 42)

15. (Mahyar Sefidgaran) Determine all integers x, y satisfying the equation

$$(y^3 + xy - 1)(x^2 + x - y) = (x^3 - xy + 1)(y^2 + x - y).$$ (→p. 43)

16. (Mohyeddin Motevassel) Let p be an odd prime number. We say a polynomial

$$f(x) = \sum_{j=0}^{n} a_j x^j \text{ is } i\text{-}residue \text{ if } \sum_{p-1 | j, j>0} a_j \equiv i \pmod p.$$

Show that $\{f(1), \ldots, f(p-1)\}$ is a complete residue system modulo p if and only if polynomials $f(x), \ldots, f(x)^{p-2}$ are 0-*residue* and $(f(x))^{p-1}$ is 1-*residue*. (→p. 44)

17. (Ali Khezeli) n is a positive integer. Let A, B be two sets of n points in the plane such that no three points of them are collinear. Denote by $T(A)$ the number of non-self-intersecting broken lines containing $n-1$ segments such that its vertices in A. Define $T(B)$ similarly. If the elements of B are the vertices of a convex n-gon but the elements of A are not, prove that $T(B) < T(A)$. (→p. 45)

18. (Mahdi E'tesamiFard) Let O be the circumcenter of triangle ABC. Points A', B', C' lie on the segments BC, CA, AB respectively such that the cicumcircles of triangles $AB'C'$, $BC'A'$ and $CA'B'$ pass through O. Denote by l_a the radical axis of the circle with center B' and radius $B'C$ and circle with center C' and radius $C'B$. Define l_b and l_c similarly. Prove that the lines l_a, l_b, l_c form a triangle such that its orthocenter coincides with the orthocenter of triangle ABC. (→p. 46)

Solutions

First Round

1. Consider the line in the problem as the $x-$axis in the coordinate plane and assume that the $y-$axis is an arbitrary line perpendicular to the $x-$axis.

   Let $A_i = (x_i, y_i)$, $1 \le i \le 1390$ be the coordinates of the head of ants. We can assume that $x_1 \le x_2 \le \cdots \le x_{1390}$.

   Denote by C_i, $1 \le i \le 1390$ the circle with center A_i and radius 1. According to the problem's condition, we have $A_iA_j > 2$ for all $1 \le i < j \le 1390$. So the circles are pairwise disjoint and all of them are in the rectangle

   $$A = \{(x,y) : x_1 - 1 < x < x_{1390} + 1,\ -2 < y < 2;\ 1 \le i \le 1390\}.$$

   Thus the sum of the areas of the circles is less than the area of the rectangle. Therefore

   $$1390 \times \pi \le 4(x_{1390} - x_1 + 2).$$

   So

   $$\begin{aligned}&\Rightarrow 1390 \times 3 < 4(x_n - x_1 + 2) \Rightarrow 1042.5 < x_n - x_1 + 2\\ &\Rightarrow x_n - x_1 > 1000(cm) = 10m.\end{aligned}$$

   Finally by Pythagorean Theorem we have $A_nA_1 > x_n - x_1 > 10m$. $\square$

2. Denote by I the intersection point of AD and BE, so I is the incenter of triangle ABC. Suppose that $\alpha = \dfrac{\angle BAC}{2}$. We have

   $$\begin{aligned}\angle IBD &= 90^\circ - \angle IBA = 90^\circ - \frac{\angle CBA}{2} = 90^\circ - 30^\circ = 60^\circ,\\ \angle CEB &= 90^\circ - \angle CBE = 90^\circ - \frac{\angle CBA}{2} = 90^\circ - 30^\circ = 60^\circ.\end{aligned}$$

   Thus in triangles BEC and BED we have $\angle CEB = \angle EBD = 60^\circ$ and BE is side of both of them. Since $\angle BEC = 30^\circ$ it suffices to prove $BD < CE$ but we have $\dfrac{BD}{AB} = \tan\alpha$ and $\dfrac{EC}{BC} = \tan 30^\circ$, So

   $$\begin{aligned}&BD \le CE \Leftrightarrow AB\tan\alpha \le BC\tan 30^\circ \Leftrightarrow \frac{AB}{BC}.\tan\alpha \le \tan 30^\circ\\ &\Leftrightarrow \tan\alpha\frac{\sin(120^\circ - 2\alpha)}{\sin(2\alpha)} \le \tan 30^\circ \quad \text{(Law of Sines)}\\ &\Leftrightarrow \frac{\cancel{\sin}\,\alpha}{\cos\alpha}\frac{\sin(120^\circ - 2\alpha)}{2\cancel{\sin}\,\alpha.\cos\alpha} \le \tan 30^\circ\\ &\Leftrightarrow \sin(120^\circ - 2\alpha) \le 2\cos^2\alpha.\tan 30^\circ\end{aligned}$$

$$\Leftrightarrow \sin 120^\circ.\cos(2\alpha) - \sin 120^\circ.\sin(2\alpha) \le (1+\cos(2\alpha))\tan 30^\circ$$
$$\Leftrightarrow (\sin 120^\circ - \tan 30^\circ).\cos(2\alpha) - \cos 120^\circ.\sin(2\alpha) \le \tan 30^\circ.$$

We know by Cauchy-Schwarz inequality that

$$(LHS)^2 \le ((\sin 120^\circ - \tan 30^\circ)^2 + (\cos 120^\circ)^2)(\cos^2(2\alpha) + \sin^2(2\alpha))$$
$$= (\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{3})^2 + (\frac{-1}{2})^2 = \frac{3}{36} + \frac{1}{4} = \frac{1}{3} = (\tan 30^\circ)^2 = (RHS)^2. \qquad \square$$

3. First it is easy to show that the sequence is strictly increasing. Assume that $a_i = a_{i+1}$ for some i. Let $j = p - i$ for a large prime p. Now $i + j$ is a prime number. So $a_i + a_j$ is prime too. But from $a_i = a_{i+1}$ we get $a_{i+1} + a_j$ and $i + 1 + j$ are prime numbers. So $i + j$ and $i + j + 1$ are two consecutive large prime numbers, contradiction.

Now put $i = j = 2^{p-2}$ for a large prime p. So the number of divisors of $2a_i = 2^{p-1}$ equals p. Hence $2a_i$ is of the form q^{p-1} for a prime q. Obviously q should be 2 and therefore we have $a_{2^{p-2}} = 2^{p-2}$.

Now we have a strictly increasing sequence of integers with infinitely many fixed points. So, for each n, $a_n = n$. $\square$

4. Suppose that $a_1, a_2, \ldots, a_n$ satisfies the conditions. First, we have

$$20 = a_1^2 + a_2^2 + \cdots + a_n^2 < \underbrace{1 + 1 + \cdots + 1}_{n} = n.$$

So $21 \le n$. We want to show that $n = 22$ is the answer. So we prove that there are not 21 numbers $a_1, a_2, \ldots, a_{21}$ in the interval $(-1,1)$ such that $a_1 + a_2 + \ldots + a_{21} = 0$ and $a_1^2 + a_2^2 + \cdots + a_{21}^2 = 20$. Assume that sequence a_i is in increasing order so $a_1 \le \frac{a_1 + a_2 + \cdots + a_{21}}{21} \le a_{21}$. Thus $a_1 \le 0 \le a_{21}$. But because of minimality of number 21, we have $a_i \ne 0$ for $1 \le i \le 21$. So there exist a unique number $1 \le k < 21$ such that

$$-1 < a_1 \le a_2 \le \cdots \le a_k < 0 < a_{k+1} \le \cdots \le a_{21} < 1.$$

We know that numbers $-a_1, -a_2, \ldots, -a_{21}$ satisfies the problem condition, too.

Thereby we can assume that $k \le \frac{21}{2}$ and since $k \in \mathbb{Z}$ we have $k \le 10$.

Now for every $k + 1 \le i \le 21$. We have $0 < a_i < 1$, so $0 < a_i^2 < a_i$.

$$\begin{aligned} 20 &= a_1^2 + a_2^2 + \cdots + {a_{21}}^2 = (a_1^2 + \cdots + a_k^2) + (a_{k+1}^2 + \cdots + a_{21}^2) \\ &< (a_1^2 + \cdots + a_k^2) + (a_{k+1} + \cdots + a_{21}) \\ &< (a_1^2 + \cdots + a_k^2) + (-a_1 - a_2 - \cdots - a_k) \\ &< 2k \le 20. \end{aligned}$$

This contradiction shows that $n \geq 22$. The following numbers are an example for $n = 22$ and so the answer is 22.

$$a_i = \sqrt{\frac{11}{10}} \quad (1 \leq i \leq 11) \quad \text{and} \quad a_i = -\sqrt{\frac{11}{10}} \quad (12 \leq i \leq 22). \qquad \square$$

5. We prove by induction that the maximum number of days it can live is $2n-1$. The construction is easy, like $1,2,\ldots,n-1,n,n-1,\ldots,2,1$ (These are the labels of colors). For $n = 1$ the result is evident. Suppose that it is true for numbers smaller than n. Let its color in the first day be R, and this colors appear k times, at days $R_1, R_2, \ldots, R_k$.
Consider the interval of days $(R_1,R_2),(R_2,R_3),\ldots,(R_{k-1},R_k),(R_k,\ldots)$. If some color appears in two intervals it contradicts the problem statement. So each color appears in exactly one interval. Suppose that there are C_i colors in the interval (R_i, R_{i+1}), so $\sum_{i=1}^{k} C_i = n-1$. By the induction hypothesis, the interval (R_i, R_{i+1}) consists of at most $2C_i - 1$ days. Only the last interval can be empty. If it is empty then the number of days is at most $k + \sum_{i=1}^{k-1}(2C_i - 1) = 2n-1$. Otherwise, the number of days is at most $k + \sum_{i=1}^{k}(2C_i - 1) = 2n-2$ and we're done. $\square$

6. Suppose that M is the midpoint of side BC and a its perpendicular bisector. X is the intersection point of lines a and l. Put $\angle MXE = \alpha, \angle ABC = \beta$ and $\angle ACB = \gamma$. Let Y be a point on segment DE such that $EY = CE$ and Z the reflection of Y with respect to a, So Z lies on l'. It is obvious that $BCYZ$ is a trapezoid and hence cyclic. Denote by K, the intersection point of l' and circumcircle of $BCYZ$ (The solution is similar when the circle is tangent to l'). Now we want to prove that $D'B = D'K$.

$$\begin{aligned}\angle CED &= 360^\circ - (\angle MCE + \angle CMX + \angle MXE) \\ &= 360^\circ - (180^\circ - \gamma + 90^\circ + \alpha) = 90^\circ + \gamma - \alpha\end{aligned}$$

Obviously,

$$\Rightarrow \quad \angle CYE = \frac{180^\circ - \angle CED}{2} = \frac{180^\circ - 90^\circ - \gamma + \alpha}{2} = 45^\circ - \frac{\gamma}{2} + \frac{\alpha}{2}. \quad (1)$$

triangle ZXY is isosceles, so $\angle XYZ = 90^\circ - \angle MXE = 90^\circ - \alpha$. (2)

$$(1),(2) \quad \Rightarrow \quad \angle CYZ = 180^\circ - (90^\circ - \alpha) - (45^\circ - \frac{\gamma}{2} - \frac{\alpha}{2}) = 45^\circ + \frac{\gamma}{2} + \frac{\alpha}{2}.$$

$CYKZ$ is cyclic, so $\angle CKZ = \angle CYZ = 45^\circ + \dfrac{\gamma}{2} + \dfrac{\alpha}{2}$. Therefore

$$\angle CKE' = 180^\circ - \angle CKZ = 135^\circ - \frac{\gamma}{2} - \frac{\alpha}{2}. \quad (3)$$

$$\begin{aligned}\angle CE'K &= 360^\circ - (\angle MCE' + \angle XMC + \angle MXE') \\ &= 360^\circ - (180^\circ - \gamma + 90^\circ + 180^\circ - \alpha) = \alpha + \gamma - 90^\circ. \quad (4)\end{aligned}$$

So

$$\begin{aligned}&\angle KCE' = 180^\circ - (\angle CE'K + \angle CKE') \\ &\overset{(3),(4)}{\Rightarrow} \angle KCE' = 180^\circ - (\alpha + \gamma - 90^\circ + 135^\circ - \frac{\alpha}{2} - \frac{\gamma}{2}) \\ &\qquad = 135^\circ - \frac{\alpha}{2} - \frac{\gamma}{2} = \angle CKE'.\end{aligned}$$

Thereby triangle $CE'K$ is isosceles and hence $CE' = KE'$. Similarly, we have $BD' = KD'$ and so $D'E' = D'K + KE' = BD' + CE'$ and this is desired assertion. $\square$

Second Round

1. a) First we count the number of rotations of a dodecahedron. In 12 ways we can change a face of the dodecahedron with the down face. Although we can put this face in 5 distinct ways but one of this $12\times 5=60$ states is the original state, so we have 59 states for rotating the dodecahedron.

   Now consider one of the marked vertices named A. We have 10 marked vertices and each of them can lie on its locations in 3 distinct ways, so in $3\times 10-1=29$ rotations (other than the original state) a special vertex lie on the location of A in original state, therefore totally $29\times 10=290$ times a marked vertex lie on a marked vertex of the original state.

   Now by the *Pigeonhole principle* less than $\dfrac{290}{59}<5$ of this coincidences is in one of the states and so we're done. $\square$

   b) It suffices to give an example that in each rotation at least 4 marked vertices lie on a marked vertex of original state.

   Mark the vertices of two opposite faces. Suppose that there exists a rotation with at most 3 coincidences. Consider this state is ξ.

   So one of the marked faces in ξ has at most one marked vertex of the original state, but each face has at least two marked vertices, contradiction. $\square$

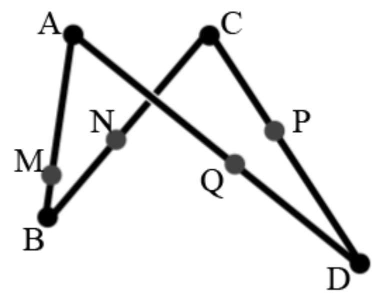
2. For each $1\leq i\leq k$ we define
$$P_i(x)=(x-i)(x-(k+i))\cdots(x-((n-1)k+i)).$$
   We claim that these polynomials satisfy the problem condition.

   For each $1\leq i\leq k$ and each $0\leq j\leq n-1$, $P_i(x)$ has exactly one simple root in the interval $(jk+\frac{1}{2},(j+1)k+\frac{1}{2})$ so invoking the *mean value theorem* we deduce that $P_i(jk+\frac{1}{2})$ and $P_i((j+1)k+\frac{1}{2})$ have different signs. Note that $P_i(nk+\frac{1}{2})>0$ because P_i is monic and so is positive for large positive values and does not have any root greater than n. Thus for each $1\leq i\leq k$, $P_i(jk+\frac{1}{2})>0$ if $j\equiv n\,(\mathrm{mod}\,2)$ and $P_i(jk+\frac{1}{2})<0$ if $j\not\equiv n\,(\mathrm{mod}\,2)$.

Now let $Q(x) = P_{i_1}(x) + P_{i_2}(x) + \cdots + P_{i_t}(x)$ where $i_1, i_2, \ldots, i_t \in \{1,2,\ldots,k\}$ are distinct. Obviously $Q(x) \in \mathbb{Z}[x]$ is a polynomial of degree n.

For each $0 \le j \le n-1$, numbers $Q(jk + \frac{1}{2})$ and $Q((j+1)k + \frac{1}{2})$ have different signs because $P_{i_1}, P_{i_2}, \ldots, P_{i_t}$ have this property. So again according to mean value theorem we deduce that Q has a root in the interval $(jk + \frac{1}{2}, (j+1)k + \frac{1}{2})$ and $Q(x)$ has at most n real roots so all its roots are real, hence the claim is proved. $\square$

3. Let quadrilateral that this four pieces form in the space be $ABCD$ and we marked points M, N, P and Q on sides AB, BC, CD and DA respectively, So that the marked points are on a plane named π. Let a, b, c and d be the distances from A, B, C and D to π respectively. So we have



$$\frac{AM}{MB} = \frac{a}{b}, \frac{BN}{NC} = \frac{b}{c}, \frac{CP}{PD} = \frac{c}{d}, \frac{DQ}{QA} = \frac{d}{a} \Rightarrow \frac{AM}{MB} \times \frac{BN}{NC} \times \frac{CP}{PD} \times \frac{DQ}{QA} = 1.$$

Now $ABCD$ varies, let π' be the plane passing through M, N, P. Suppose that a', b', c' and d' are the distances of A, B, C and D to π' respectively. Thus

$$\frac{AM}{MB} = \frac{a'}{b'}, \frac{BN}{NC} = \frac{b'}{c'}, \frac{CP}{PD} = \frac{c'}{d'} \Rightarrow \frac{DQ}{QA} = \frac{d'}{a'}.$$

So Q must be on π' too. $\square$

4. In every moment consider the number of persons that are on the escalator at that time. Now consider the intervals such that in every time of such intervals the number of persons on the escalator is equal to a fixed integer. Suppose that we have k intervals $I_1, I_2, \ldots, I_k$ and for $1 \le i \le k$, a_i and l_i denotes the length and number of persons on the escalator in every moment of I_i. We claim that $\sum_{i=1}^{k} a_i^{1-\alpha} t_i = nl$.

Since every person have moved a distance equal l so the sum of travelled distance of people is nl. On the other hand travelled distance of a person who is on the escalator on the interval I_i equals $a_i^{-\alpha} t_i$. So the sum of travelled distance in the interval I_i equals $a_i . a_i^{-\alpha} t_i = a_i^{1-\alpha} t_i$, hence the total travelled distance is $\sum_{i=1}^{k} a_i^{1-\alpha} t_i$.

So the claim is proved.

Now consider the following cases.

Case 1. $\alpha \geq 1$. If $a_i \geq 1$ since $\alpha \geq 1$ we have $a_i^{\alpha-1} \geq 1$, hence $a_i^{1-\alpha} \leq 1$. If $a_i = 0$ also $a_i^{1-\alpha} = 0 \leq 1$. So $nl = \sum_{i=1}^{k} a_i^{1-\alpha} t_i \leq \sum_{i=1}^{k} t_i$. So the required time is at least nl. If each person goes on the escalator when the previous one reached the top of the escalator the required time equals nl. Hence in this case the required time is at least nl.

Case 2. $\alpha < 1$. Since $a_i < n$ and $1-\alpha > 0$ we have $a_i^{1-\alpha} < n^{1-\alpha}$ so

$$nl = \sum_{i=1}^{k} a_i^{1-\alpha} t_i \leq \sum_{i=1}^{k} n^{1-\alpha} t_i = n^{1-\alpha} \sum_{i=1}^{k} t_i \quad \Rightarrow \quad n^{\alpha} l \leq \sum_{i=1}^{k} t_i$$

So the required time is at least $n^{\alpha} l$. If all n people go on the escalator together the velocity equals $n^{-\alpha}$ and so the required time equals $\frac{l}{n^{-\alpha}} = n^{\alpha} l$. Hence in this case the required time is at least $n^{\alpha} l$. $\square$

5. a) Denote by t the number of *golden* prime numbers less than or equal to 1390^n. We want to show that $t \geq n$. Suppose that S is collection of all natural numbers less than or equal to 1390^n with prime factors from the set $\{q_1, q_2, \ldots, q_t\}$. Obviously each element of S can be written in the form $a^2 b$ where $a, b \in \mathbb{N}$ and b is out of square. So $a \leq \sqrt{1390^n} = 1390^{\frac{n}{2}}$ and $b = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_t^{\alpha_t}$ such that $\alpha_i \in \{0, 1\}$. Therefore a and b have $1390^{\frac{n}{2}}$ and 2^t states respectively, and so $|S| \leq 2^t \times 1390^{\frac{n}{2}}$.
On the other hand for each integer $1 \leq i \leq k = 1390^{\frac{2}{3}n}$ we have

$$a_i \leq i^{1.5} \leq k^{1.5} = 1390^{\frac{2}{3}n \times 1.5} = 1390^n.$$

And all prime divisors of a_i are in the set $\{q_1, q_2, \ldots, q_t\}$, so $a_i \in S \, (1 \leq i \leq 1390^{\frac{2}{3}n})$.
Therefore S has at least $\lfloor k \rfloor$ elements. So $1390^{\frac{2}{3}n} - 1 < \lfloor k \rfloor \leq |S| \leq 2^t \times 1390^{\frac{n}{2}}$, But it is easy to check that $2 \times 1390^{\frac{1}{2}} \leq 1390^{\frac{2}{3}} - 1$ and this implies $t \geq n$, because

$$2^n \times 1390^{\frac{n}{2}} = (2 \times 1390^{\frac{1}{2}})^n \leq (1390^{\frac{2}{3}} - 1)^n \leq 1390^{\frac{2}{3}n} - 1 < 2^t \times 1390^{\frac{n}{2}}. \qquad \square$$

b) The proof of this part is very similar to part a. Denote by t the number of *golden* prime numbers less than or equal to 1390^{2n}. We want to show that $t \geq n$. Suppose that S is collection of all natural numbers less than or equal to 1390^{2n} with prime factors from the set $\{q_1, q_2, \ldots, q_t\}$. Then every element of S can be written in the form a^4b^2c where $a, b, c \in \mathbb{N}$ and b, c are out of square. (In part a writing a as x^2y where $x, y \in \mathbb{N}$ and y is out of square implies this claim.). Now $a \le \sqrt[4]{1390^{2n}} = 1390^{\frac{n}{2}}$, $b = q_1^{\alpha_1}q_2^{\alpha_2}\ldots q_t^{\alpha_t}$ and $c = q_1^{\beta_1}q_2^{\beta_2}\ldots q_t^{\beta_t}$ such that $\alpha_i, \beta_i \in \{0,1\}$. Thus we have $1390^{\frac{n}{2}}, 2^t$ and 2^t states for a, b and c respectively and so $|S| \le 2^{2t} \times 1390^{\frac{n}{2}}$.

In this case if $1 \le i \le k = 1390^{\frac{5}{6}n}$ $(i \in \mathbb{N})$ then $a_i \le i^{2.4} \le k^{2.4} = 1390^{\frac{5}{6}n \times 2.4} = 1390^{2n}$. Although prime divisors of $a_i\,(1 \le i \le k)$ are in the set $\{q_1, q_2, \ldots, q_t\}$ so $a_i \in S$, hence $1390^{\frac{5}{6}n} - 1 < \lfloor k \rfloor \le |S| \le 2^{2t} \times 1390^{\frac{n}{2}}$. On the other hand $4 \times 1390^{\frac{1}{2}} \le 1390^{\frac{5}{6}} - 1$ and so

$$2^{2n} \times 1390^{\frac{n}{2}} = (4 \times 1390^{\frac{1}{2}})^n \le (1390^{\frac{5}{6}} - 1)^n \le 1390^{\frac{5}{6}n} - 1 < 2^{2t} \times 1390^{\frac{n}{2}},$$

which implies $t \ge n$. $\square$

6. The points should be on a circle (or line) and A, B should separate A', B' on it. To prove necessity, first suppose that the points are not coplanar. Then there exist two parallel planes passing through A, B and A', B' respectively. Any two circles in these planes are not linking. So the points should be coplanar.
Now suppose B' is not on the circumcircle of ABA' (which can be a line). So we can slightly change the circle to find a circle passing through A, B such that A', B' are both outside or both inside it. Now, this circle is not linking with the circle with diameter $A'B'$ orthogonal to the plane containing the points.
So the points should be on a circle (or line). Now, suppose A, B do not separate A', B' on the circle. If we change the circle slightly, still passing through A, B, then A', B' will be both inside or both outside the new circle and we arrive to a contradiction like the previous case. So, the necessity of the condition is proved.
To prove sufficiency, Let C, C' be two different circles passing through A, B and A', B' respectively. Let P, P' be the planes containing C, C' respectively. If the points are collinear, then $C' \cap P$ is consisted of a point inside C and a point outside C. So C, C' are interlocked. So, suppose the points are on a circle. Let M be the intersection of the segments AB and $A'B'$. We have $MA.MB = MA'.MB'$. Let

$l = P \cap P'$ which passes through M. M is inside C, so l intersects C at two points like X, Y and M is between X, Y. Similarly, l intersects C' at X', Y' namely, and M is between X', Y'. Suppose X, X' are in one side of M. We have

$$MX.MY = MA.MB = MA'.MB' = MX'.MY'.$$

So if $MX \le MX'$, then $MY \ge MY'$ and vice versa. So the points of $C' \cap P = \{X', Y'\}$ are in different sides of C or both are on C. So C, C' are *linking* and sufficiency of the condition is proved. □

7. Define $f(A) = \max(A)$ when A is finite. Evidently, f is A-*predicator* when A is finite. We extend f to all subsets of $\mathbb{N}$. We say two subsets A, B are *equivalent* if B is derived from A by adding and deleting a finite number of elements; i.e. $A \Delta B$ is finite. This is an equivalence relation. By the Axiom of Choice, we can select an element from each class of equivalency. For an arbitrary proper subset A, let S_A be the selected element from the class of A. So, $A \Delta S_A$ is finite. Define $f(A) = \max(A \Delta S_A)$ when $A \ne S_A$ and define $f(S_A)$ arbitrarily. We claim this function is A-*predicator* for all subsets $A \subseteq \mathbb{N}$.

   Let x be a natural number such that $x \notin A$. A and $A \cup \{x\}$ are equivalent, so $S_A = S_{A \cup \{x\}}$. So

   $$f(A \cup \{x\}) = f(A \Delta \{x\}) = \max((A \Delta \{x\}) \Delta S_A) = \max((A \Delta S_A) \Delta \{x\}).$$

   For $x > \max(A \Delta S_A)$ we have $(A \Delta S_A) \Delta \{x\} = (A \Delta S_A) \cup \{x\}$. therefore $f(A \cup \{x\}) = x$ and the claim is proved. □

8. a) Let's name the arithmetic progressions as $S_1, \dots, S_n$. Suppose the remainders of $a_1, \dots, a_k$ modulo p don't contain r. So no member of $S_1, \dots, S_k$ is $r \pmod p$. Consider the arithmetic progression $S = \{r + ip : i = 0, 1, 2, \dots\}$ which has empty intersection with $S_1, \dots, S_k$. So S is covered by $S_{k+1}, \dots, S_n$.

   Lemma. *If d, d' are coprime natural numbers, then intersection of the arithmetic progressions $\{a + kd : k = 0, 1, 2, \dots\}$ and $\{a' + kd' : k = 0, 1, 2, \dots\}$ is an arithmetic progression whose common difference is dd'.*

   Proof. The sequences have a nonempty intersection by the Chinese remainder theorem. The difference of two consecutive elements of the intersection is the least common multiple of d, d' which is dd'. □

   According to the lemma, the sets $S \cap S_{k+1}, \dots, S \cap S_n$ are arithmetic progressions with common differences $pd_{k+1}, \dots, pd_n$. Consider the map $f : S \to \{0, 1, 2, \dots\}$ with formula $f(x)=\frac{x-r}{p}$. So, the sets $f(S\cap S_{k+1}),\ldots,f(S\cap S_n)$ are arithmetic progressions with common differences $d_{k+1},\ldots,d_n$. These sets cover the natural numbers, because S is covered by $S_{k+1},\ldots,S_n$. So $d_{k+1},\ldots,d_n$ is also a *covering sequence* which contradicts the minimality of $d_1,\ldots,d_n$. $\square$

b) Let $p_1,\ldots,p_k$ be the prime factors of $d_1,\ldots,d_n$ and let $I_i=\left\{j : p_i \mid d_j\right\}$ (which can contain multiplicities). Suppose the natural numbers are covered by arithmetic progressions $S_i=\left\{a_i+kd_i : k=0,1,2,\ldots\right\}$. We claim that at least one of the sets of sequences $\mathfrak{S}_i=\left\{S_j : j\in I_i\right\}$ covers the natural numbers. As a result, one of the sequences $\left\{d_j : p_i \mid d_j\right\}$ is a *covering sequence*.

Suppose for each i the sequences in $\mathfrak{S}_i$ doesn't cover a number r_i. Let $D_i=\prod_{p_i\mid d_j} d_j$.

By the Chinese remainder theorem, there is a natural number r such that $r\equiv r_i$ mod D_i for each i. So r is not covered by any of the arithmetic progressions, a contradiction. So the claim is proved.

Now, it is enough to suppose $d_i=p^{r_i}$. We claim it is a *covering sequence* if and only if $\sum_i \frac{1}{d_i}\geq 1$ and it is *minimal* if and only if $\sum_i \frac{1}{d_i}=1$.

i) First, suppose we have covered the natural numbers by progressions S_i as above. For any natural number N, S_i covers at most $\frac{N+1}{d_i}$ members of $\left\{1,\ldots,N\right\}$. So we have $\sum_i \frac{N+1}{d_i}\geq N$. So $\sum_i \frac{1}{d_i}\geq \frac{N}{N+1}$. So we should have $\sum_i \frac{1}{d_i}\geq 1$.

ii) Second, suppose $\sum_i \frac{1}{d_i}\geq 1$. We use induction on n to prove that this is a *covering sequence*. If $n=1$ it is evident. If not, let n_i be the number of p^i's in $d_1,\ldots,d_n$. We can suppose $n_0=0$. There should exist i such that $n_i\geq p$, or else $\sum_i \frac{1}{d_i}<(p-1)(\frac{1}{p}+\frac{1}{p^2}+\cdots)=1$. Remove p ones of p^i's and add a p^{i-1} to $d_1,\ldots,d_n$. The sum doesn't change. So, cover the natural numbers by progressions according to the induction hypothesis. Now, split a progression with common difference p^{i-1} into p progressions with common difference p^i. The induction claim is proved.

iii) Moreover, suppose $\sum_i \frac{1}{d_i} > 1$. Suppose $d_1 \leq \cdots \leq d_n$. We have $\sum_i \frac{d_n}{d_i} > d_n$ and the summands are natural numbers. So $\sum_i \frac{1}{d_i} \geq 1 + \frac{1}{d_n}$. So, we can remove d_n and the sequence is not *minimal*. $\square$

Third Round

1. **Lemma.** *Suppose* $n \geq 2$ *is an integer, then all of the numbers* $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ *are odd if and only if* $n = 2^k - 1$ *for an integer* $k \geq 2$.

Proof. For an integer t, let $v(t)$ be the greatest integer u such that $2^u \mid n$. We know that

$$\text{“ If } p, q \in \mathbb{N} \text{ and } 0 \leq p \leq 2^q \text{ then } v(p) = v(2^q + p) = v(2^q - p)\text{”} \qquad (1)$$

Because if $p = 2^a b$ with $a, b \in \mathbb{N}$ and b an odd number then $a < q$ and $2^q \pm p = 2^a(2^{q-a} \pm b)$ where $2^{q-a} \pm b$ are odd numbers.

Now there is one and only one $m \in \mathbb{N}$ such that $2^m \leq n < 2^{m+1}$. Let $n = 2^m + s$ with $0 \leq s < 2^m$. Now consider the number $\binom{n}{2^m - 1}$. We have

$$\binom{n}{2^m - 1} = \binom{2^m + s}{2^m - 1} = \binom{2^m + s}{s + 1} = \frac{(2^m + s)(2^m + s - 1)\cdots(2^m + 1)(2^m)}{s(s-1)\cdots(1)(s+1)}.$$

By (1) we have $v(2^m + i) = v(i)$ where $1 \leq i \leq s$, therefore

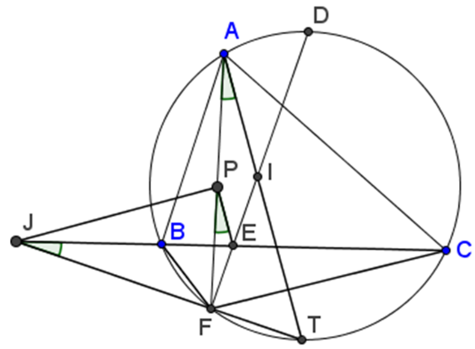
$$v((2^m + s)\cdots(2^m + 1)) = v(s!),$$

and by assumption $\binom{n}{2^m - 1}$ is odd therefore $v(2^m) = v(s+1)$ and consequently $2^m \mid s+1$ and $s + 1 \geq 1$. Hence we have $2^m - 1 \leq s$ and therefore $s = 2^m - 1$ and $n = 2^m + s = 2^{m+1} - 1$.

Now if $n = 2^k - 1$ for some natural number k, for each $1 \leq c \leq n$ we have $\binom{2^k - 1}{c} = \frac{(2^k - 1)(2^k - 2)\cdots(2^k - c)}{(1)(2)\cdots(c)}$ and by (1) we know for $1 \leq l \leq c$ $v(2^k - l) = v(l)$, so $\binom{2^k - 1}{c}$ is odd for $0 \leq c \leq n$. $\square$

Now we return to main problem:

Positive integer n has the property of the problem if and only if all the numbers $\binom{n}{i} - i$ $(0 \leq i \leq n)$ have the same parity. It means that for every $0 \leq i \leq n - 1$, $\binom{n}{i}, \binom{n}{i+1}$ have different parities. So $\binom{n+1}{i+1} = \binom{n}{i} + \binom{n}{i+1}$ $(1 \leq i \leq n - 1)$ is odd and as we know $\binom{n+1}{0} = 1$ is also odd. Therefore by the lemma this is equivalence to $n+1 = 2^k - 1$ for some integer $k \geq 2$ so $n = 2^k - 2$ where $k \geq 2$ is an integer. $\square$



2. Let T be the point of intersection of perpendicular bisector of BC and circle ω, so TD is a diagonal of ω and $\angle DFT = 90°$. Since D is the midpoint of arc $\overset{\frown}{BAC}$, FD is the angle bisector of $\angle BFC$. Therefore $(JEBC) = -1$ where J is the intersection point of line TF and extension of BC.

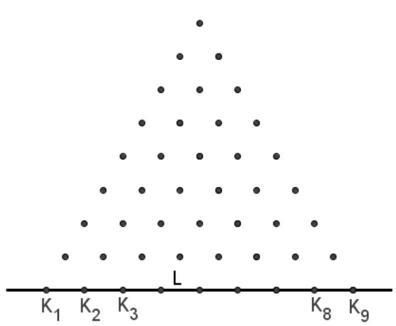
$$\angle EJF = \frac{1}{2}(\overset{\frown}{CT} - \overset{\frown}{BF}) = \frac{1}{2}(\overset{\frown}{BT} - \overset{\frown}{BF}) = \frac{1}{2}\overset{\frown}{FT} = \angle TAF = \angle EPF$$

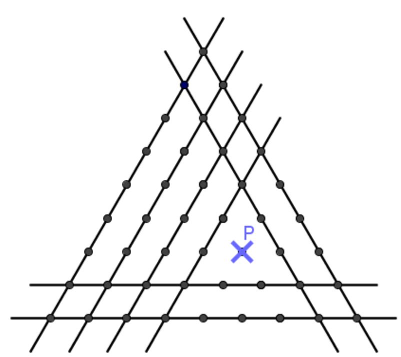
So quadrilateral $PEFJ$ is cyclic and consequently $\angle JPE = 90°$. This fact and $(JEBC) = -1$ implies that PE is the angle bisector of $\angle BPC$ and this is what we wanted to prove. $\square$

3. For each $p = (x_p, y_p) \in S$ suppose that $L_{i,p} : a_{i,p}x + b_{i,p}y + c_{i,p} = 0$ $(1 \leq i \leq n)$ are n lines that cover $S - \{p\}$. We know that $p \not\in L_{1,p} \cup L_{2,p} \cup \cdots \cup L_{n,p}$. Let

$$P_p(x,y) = \prod_{i=1}^{n} \left(\frac{a_{i,p}x + b_{i,p}y + c_{i,p}}{a_{i,p}x_p + b_{i,p}y_p + c_{i,p}}\right) \quad (p \not\in L_{i,p} \Rightarrow a_{i,p}x_p + b_{i,p}y_p + c_{i,p} \neq 0)$$

P_p is a polynomial of degree n in two variables and we have $P_p|_{S-\{p\}} \equiv 0$ and $P_p(p) = 1$ so $\{P_p\}_{p \in S}$ are linearly independent in F_n, where F_n denotes the vector space of polynomials in two variables of degree less than or equal to n, therefore $|S| \leq dimF_n = \binom{n+2}{2}$.





Now we give an example of S with $\binom{n+2}{2}$ points.

Consider U_n be $\binom{n+2}{2}$ points forming a triangular lattice (The left figure above shows U_8.) Then $|U_n| = 1 + 2 + \cdots + (n+1) = \binom{n+2}{2}$. For every point $p \in U_n$ we can cover $U_n - \{p\}$ with n lines (Right figure above shows an example for $n = 8$). Now we use induction on n to prove that we cannot cover U_n with n lines. The base $n = 1$ is obvious. Assume the statement to be true for $n = k + 1$ and suppose U_n is covered with $k + 1$ lines. Now consider line L which points $A_1, A_2, \ldots, A_{k+2}$ lie on it. We must have L in our $k + 1$ lines because if we do not have it then we must cover $A_1, A_2, \ldots, A_{k+2}$ with at least $k + 2$ distinct lines. Now $U_k = U_{k+1} - L$ and according to the induction hypothesis U_k cannot be covered with k lines so the statement was proved. $\square$

Comment. The minimum possible number of elements of S is $2n + 1$. Obviously we have $|S| \geq 2n + 1$. Also any $2n + 1$ points such that no three of them are collinear is an example.

4. **Solution 1.** Let N_i for $0 \leq i \leq 3$ be the set of squares with i sides in the cycle and let n_i be the number of such squares. There are a total number of mn squares, so

$$\sum_i n_i = mn \qquad (1)$$

Each vertex that is not on the boundary is adjacent to 2 edges and there are $(m-1)(n-1)$ such vertices. So

$$\sum_i i n_i = 2(m-1)(n-1) \qquad (2)$$

For each 90 degree turn in the cycle, consider the square that contains its two adjacent edges. This square has at least two adjacent edges in the cycle. If we let N_2' be the set of squares with only two opposite sides in the cycle, then each square in $N_2 \setminus N_2'$ is counted once, each square in N_3 is counted twice and no other squares is counted. So

$$A' = (n_2 - B) + 2n_3$$

Where A' is the number of 90 degree turns. By subtracting equation (1) from equation (2) we get

$$A' = n_0 - B + 2(m-1)(n-1) - mn$$

and so

$$\begin{aligned} A &= (m-1)(n-1) - A' \\ &= B - n_0 + mn - (m-1)(n-1) \\ &= B - C + m + n - 1. \end{aligned}$$

□

Solution 2.

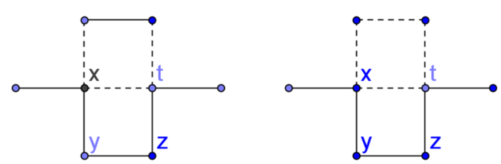
Lemma 1. *Let P be an arbitrary cycle in the graph with length more than 4, such that no vertex is inside P. Then the number of vertices of P with no 90 degree turns (A) is $2B - 2C + 2$, where B is the number of squares inside P with only two opposite sides in P and C is the number of squares inside P with no edge in P.*

Proof. We use induction on k, the number of squares inside P, which is more than one by assumption. For $k = 2$ the assertion is trivial. Suppose $k > 2$. Consider a new graph whose nodes are the centers of the squares inside P and two nodes are adjacent if and only if their corresponding squares have a common edge (not necessarily in P). This graph is trivially connected and has no cycles, because if it has a cycle, then there is a vertex inside it which is also inside P. So the graph is a tree and has a leaf. Consequently, there is a square Q inside P with three sides in P. Consider the induction hypothesis for the cycle obtained by deleting these three edges and adding the remaining side of Q:

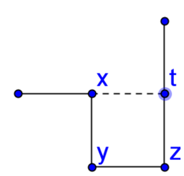
$$A' = 2B' - 2C' + 2.$$

Let $xyzt$ be the path in P with the vertices of Q. Consider the following cases. The claims can be easily proved.

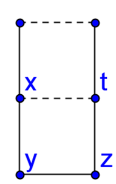
- There are 90 degree turns at both x and t. In this case we have $A = A' - 2$. Also $B = B' - 1, C = C'$ or $B = B', C = C' + 1$ depending on Q's adjacent square inside P.



- There is a 90 degree turn at only one of x, t. In this case we have $A = A'$, $B = B'$ and $C = C'$.



- There is no 90 degree turn at x and t. In this case we have $A = A' + 2$, $B = B' + 1$ and $C = C'$.



The induction claim follows trivially in each case. □

Using this lemma, we get

$$A = 2B' - 2C' + 2, \qquad (3)$$

where B',C' are the number of squares inside P with the same properties as B,C. Let R be the inner $(m-2)\times(n-2)$ rectangles and draw its boundary together with the edges of P. These edges partition the square into some regions. It is easily seen that the boundary of each region, other than the inside of P and the out-most region, is a cycle containing one edge in the boundary of R and a path in P. Use the lemma for each region to get

$$A_i = 2B_i - 2C_i + 2, \qquad 1 \le i \le t, \quad (4)$$

where t is the number of mentioned regions. It is easily seen that

$$B = B' + \sum_i B_i + 1,$$
$$C = C' + \sum_i C_i + t + 4,$$
$$A = \frac{1}{2}(A' + \sum_i A_i + A''),$$

where A'' is the number of vertices without 90 degree turns that on the boundary of R. Now the assertion follows easily by substituting equation (3) and (4) in $A - B + C$. □

5. We claim that for every $k \in \mathbb{N}$ and real numbers $a_1, a_2, \dots, a_{2^k}$:

$$f(a_1 + a_2 + \dots + a_{2^k}) \le 2^k \max\left\{f(a_1), f(a_2), \dots, f(a_{2^k})\right\}$$

Proof is done by induction on k .Basis is obviously the condition (iii). Suppose the claim is true for k. For $k+1$ we have:

$$\begin{aligned}
&f(a_1 + a_2 + \dots + a_{2^{k+1}}) = f((a_1 + a_2 + \dots + a_{2^k}) + (a_{2^k+1} + \dots + a_{2^{k+1}}))\\
&\le 2\max\{f(a_1 + a_2 + \dots + a_{2^k}) + f(a_{2^k+1} + \dots + a_{2^{k+1}})\}\\
&\le 2\max\left\{2^k\max\{f(a_1), f(a_2), \dots, f(a_{2^k})\}, 2^k\max\{f(a_{2^k+1}), \dots, f(a_{2^{k+1}})\}\right\}\\
&\le 2^{k+1}\max\{f(a_1), f(a_2), \dots, f(a_{2^{k+1}})\}
\end{aligned}$$

Now suppose that $2^{k-1} < n \le 2^k$:

$$\begin{aligned}
&f(a_1 + \dots + a_n) = f(a_1 + \dots + a_n + \underbrace{0 + 0 + \dots + 0}_{2^k - n})\\
&\le 2^k \max\{f(a_1), \dots, f(a_n), f(0), \dots, f(0)\}\\
&= 2^k \max\{f(a_1), \dots, f(a_n)\} \le 2n \max\{f(a_1), \dots, f(a_n)\}
\end{aligned}$$

$2^{k-1} < n$ So $2^k < 2n$ and this implies the last inequality.

If we put $a_1 = a_2 = \cdots = a_n = 1$ then $f(n) = f(\underbrace{1+1+\cdots+1}_{n}) \le 2nf(1)$. Therefore

$$\begin{aligned}(f(a+b))^n &= f((a+b)^n) = f(\sum_{i=0}^{n}\binom{n}{i}a^ib^{n-i}) \le 2(n+1)\max_{0\le i\le n}\{f(\binom{n}{i}a^ib^{n-i})\}\\ &\le 2(n+1)\sum_{i=0}^{n}f(\binom{n}{i})f(a^i)f(b^{n-i}) \le 4(n+1)f(1)\sum_{i=0}^{n}\binom{n}{i}f(a)^if(b)^{n-i}\\ &= 4(n+1)f(1)(f(a)+f(b))^n \Rightarrow f(a+b) \le \sqrt[n]{4(n+1)f(1)}(f(a)+f(b))\end{aligned}$$

If n tends to infinity, we have $\sqrt[n]{4(n+1)f(1)} \to 1$ and this implies the desired result.

□

6. $\omega, \omega_a, \omega_e$ are three equal circles and A', B, E are other intersection points of these circles. By some angle chasing it is easy to prove that A, A', B, E form an orthocentric system of points.(each one is the orthocenter of others.) so A' is orthocenter of triangle EAB. Similarly B', C', D' and E' are orthocenter of triangles ABC, BCD, CDE and DEA respectively. Note that $A'E$ and $B'C$ are both perpendicular to AB so they are parallel. On the other hand $AA' \perp BE$ so

$$\angle A'AE = 90^\circ + \angle AEB = 90^\circ + \frac{1}{2}\overset{\frown}{AB},$$

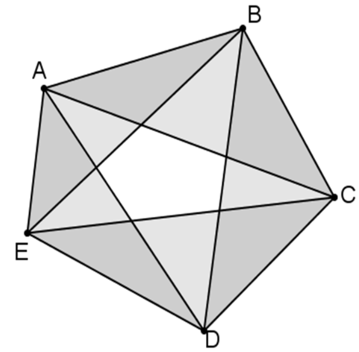
Similarly $\angle B'BC = 90^\circ + \angle BCA = 90^\circ + \frac{1}{2}\overset{\frown}{AB}$. Therefore $\angle A'AE = \angle B'BC$. Since ω_b, ω_e have equal radius, $A'E = B'C$ are of equal length and parallel, consequently quadrilateral $A'B'CE$ is parallelogram. Thereby segment $A'B'$ is parallel to the diagonal CE of pentagon and they have equal length. Similarly pairs of segments $(B'C', AD), (C'D', BE), (D'E', CA)$ and $(E'A', DB)$ are parallel and have equal length. So the sides of pentagon $A'B'C'D'E'$ and diagonals of pentagon $ABCDE$ are of equal length and angle between sides equals to angle between corresponding diagonals.

Now consider pentagon $A_1B_1C_1D_1E_1$, similar to $ABCDE$ such that $A_1B_1 \parallel AB$ and $A_1B_1 = 2AB$ (and similar relations for other sides.). Let A_2, B_2, C_2, D_2, E_2 be midpoints of sides $C_1D_1, D_1E_1, E_1A_1, A_1B_1$ and B_1C_1 respectively. By *Thales'* theorem, $A_2B_2 \parallel C_1E_1$ and $A_2B_2 = \frac{1}{2}C_1E_1$. Other sides have similar situation, so pentagons $ABCDE$ and $A_2B_2C_2D_2E_2$ are equal. Therefore we have

$$\begin{aligned}&S(ABCDE) = S(A_2B_2C_2D_2E_2) =\\ &= S(A_1B_1C_1D_1E_1) - S(A_2B_2D_1) - S(B_2C_2E_1) - S(C_2D_2A_1) - S(D_2E_2B_1) - S(E_2A_2C_1) =\\ &= 4S(ABCDE) - S(CDE) - S(DEA) - S(EAB) - S(ABC) - S(BCD)\end{aligned}$$

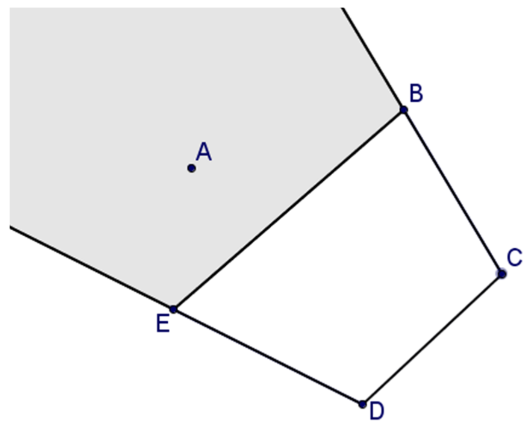
Let S be area of pentagon $ABCDE$, S_1 area of pentagon $A_1B_1C_1D_1E_1$ and by S' sum of area of triangles ABC, BCD, CDE, DEA, EAB so the assertion is equivalent to

$$S \le S' \le 2S. \quad (*)$$



The right hand inequality in (*) is easy, because if we color triangles ABC, BCD, CDE, DEA and EAB each region in pentagon $ABCDE$ is colored at most two times.

We claim that in every convex pentagon the sum of area of corner triangles is more than or equal to area of pentagon.



Consider a convex pentagon $ABCDE$. Obviously, There exist two adjacent angles with some more than $180°$. Suppose that these two angles are $\angle D$ and $\angle E$. Now fix points B, C, D and E in the plane and move point A. Since $ABCDE$ is convex, place of point A must be in the shaded region (Look at the figure) which is convex itself.

Let $f(A) = S(ABC) + S(BCD) + S(CDE) + S(DEA) - S(BCDE)$. We must prove $f(A) \ge 0$ for every point A in mentioned convex region. f is a linear function of A defined in a convex region, so f takes its minimum at corner points B, E or ∞.

$$f(\infty) = +\infty > 0,$$
$$f(B) = 0 + S(BCD) + S(CDE) + S(DEB) - S(BCDE) \ge 0,$$
$$f(E) = S(EBC) + S(BCD) + S(CDE) + 0 - S(BCDE) \ge 0.$$

So f is nonnegative in this region and the assertion is proved. $\square$

7. First we claim that for every positive integer k, the edges which their numbers are divisible by k form a cluster. Indeed, suppose on the contrary that S is the largest cluster such that the number of its edges is divisible by k and $v \notin S$ is another such edge. Applying problem statement on edge v and edges of cluster we get there exists a larger cluster with number on edges divisible by k.

Now suppose that $p \,\Big|\, \binom{n}{2}$ where p is a prime number. Therefore the edges with numbers divisible by $\frac{1}{p}\binom{n}{2}$ form a cluster and the number of such edges is p. So

$$p = \binom{t}{2} \Rightarrow 2p = t(t-1)$$

$$\Rightarrow \begin{cases} p \mid t \;\Rightarrow\; t-1 \mid 2 \;\Rightarrow\; t=2 \text{ or } t=3 \;\Rightarrow\; t=3, p=3 \\ p \mid t-1 \;\Rightarrow\; t \mid 2 \;\Rightarrow\; t=1 \text{ or } t=2 \;\Rightarrow\; \textit{Contradiction}! \end{cases}$$

Thereby the only prime divisor of $\binom{n}{2}$ is three, so $\binom{n}{2} = 3^a$ where $a \in \mathbb{N}$. If $a = 1$ and so $n = 3$ and numbers 1,2,3 satisfies problem statement. If $a > 1$ there exist 9 multiple of $\frac{1}{9}\binom{n}{2}$ among numbers, but 9 cannot be written in the form $\binom{t}{2}$ so for $n > 3$ such numbers do not exist. □

8. Let $h(x) = f(x) - x$, So we have

$$h(h(x) + g(x) + x + 2y) = 2h(y). \quad (1)$$

Hence For every $x, y, z \in \mathbb{R}^+$ we have

$$h(h(x) + g(x) + x + 2y) = 2h(y) = h(h(z) + g(z) + z + 2y). \quad (2)$$

There exist some $x, z \in \mathbb{R}^+$ such that $T = h(x) + g(x) + x - h(z) - g(z) - z$ is positive. Otherwise $h(x) + g(x) + x$ must be constant. So

$$h(x) + g(x) + x = h(1) + g(1) + 1 \Rightarrow h(x) = -g(x) - x + C,$$

where $C = h(1) + g(1) + 1$ (constant). By definition of $h(x)$ we get

$$f(x) = -g(x) + C.$$

But coefficients of $g(x)$ all are positive, therefore $g(x) \to +\infty$ as $x \to +\infty$ so $g(x) > C$ for large values x, Which contradicts the assumption that $f(x) \in \mathbb{R}^+$ for every $x \in \mathbb{R}^+$. Therefore function $h(x)$ is periodic because according to equation (2) for each $x, z \in \mathbb{R}^+$, $T = h(x) + g(x) + x - h(z) - g(z) - z$ is a period for h. (We can choose $x, z \in \mathbb{R}^+$ such that $T > 0$.) Therefore

$$S(x) = h(x+T) + g(x+T) + (x+T) - h(x) - g(x) - x = g(x+T) - g(x) - T.$$

$S(x)$ is a period for h for every $x \in \mathbb{R}^+$. Since $g(x)$ is a polynomial of degree at least two, $S(x)$ is not constant and is a function of x such that its image contains all numbers greater than a fixed positive real number A and this implies that $h(x)$ is constant for every $x \in \mathbb{R}^+$ and so $h(x) = K$ for some constant K. Now we replace function h by K in (1)

$$h(h(x) + g(x) + x + 2y) = 2h(y) \Rightarrow K = 2K \Rightarrow K = 0$$

$$\Rightarrow h(x) = 0 \;\; \forall x \in \mathbb{R}^+ \Rightarrow f(x) = x \;\; \forall x \in \mathbb{R}^+.$$

□

9. Suppose ω_1 is tangent to AB, AD at P_1, Q_1 respectively and ω_2 is tangent to CD, BC at P_2, Q_2 respectively.

Lemma. *Let a circle ω be tangent to the half-lines BA, BC at P, Q respectively. ω is tangent to ω_1 if and only if $\sqrt{BP} = \sqrt{AB} \pm \sqrt{AP_1}$ for some choice of the sign.*

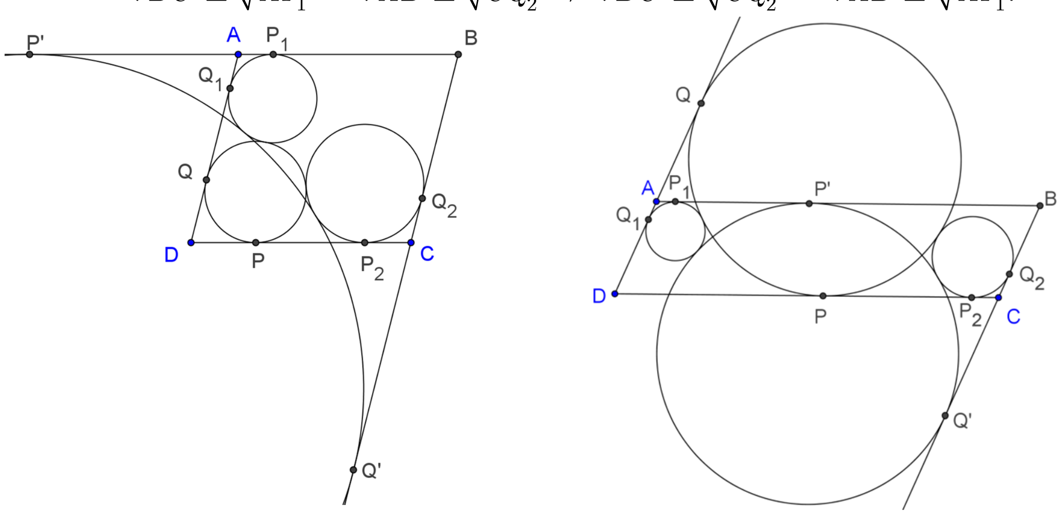
Proof. PP_1 is the common external tangent of ω and ω_1. So ω, ω_1 are tangent if and only if $PP_1 = 2\sqrt{rr_1}$, where r, r_1 are the radii of ω, ω_1 respectively. If we let $\angle ABC = 2\alpha$, then $r = BP.\tan\alpha$ and $r_1 = AP.\cot\alpha$. So $2\sqrt{rr_1} = 2\sqrt{BP.AP_1}$. So ω, ω_1 are tangent if and only if $AB - AP_1 - BP = \pm 2\sqrt{BP.AP_1}$ and the claim follows. □

By assumption, there is a circle ω tangent to the lines DA, DC at Q, P respectively and externally tangent to ω_1, ω_2. It is easily seen that ω should be inside the angel $\angle ADC$. So, by the lemma we have:

$$\sqrt{DQ} = \sqrt{AD} \pm \sqrt{AQ_1}$$
$$\sqrt{DP} = \sqrt{CD} \pm \sqrt{CP_2}$$

for some choice of the signs. We have $DP = DQ$, $AQ_1 = AP_1$, $CP_2 = CQ_2$, $AD = BC$ and $CD = AB$. So by the above equations we get

$$\sqrt{BC} \pm \sqrt{AP_1} = \sqrt{AB} \pm \sqrt{CQ_2} \Rightarrow \sqrt{BC} \pm \sqrt{CQ_2} = \sqrt{AB} \pm \sqrt{AP_1}.$$



Let ω' be a circle tangent to the half-lines BA, BC at P', Q' respectively, such that both sides of this equation are equal to $\sqrt{BP'}$ (note that the value is positive). So ω' is tangent to ω_1, ω_2 by the lemma and the assertion is proved. □

10. **Solution 1.** By Holder Inequality we have

$$(\sum_{cyc}\frac{a\sqrt{a}}{bc})(\sum_{cyc}bc)(\sum_{cyc}1) \geq (\sum_{cyc}\sqrt{a})^3 \Rightarrow \sum_{cyc}\frac{a\sqrt{a}}{bc} \geq \frac{1}{3}(\sum_{cyc}\sqrt{a})^3.$$

So it suffices to have

$$\frac{1}{3}(\sum_{cyc}\sqrt{a})^3 \geq \sqrt{3}(\sum_{cyc}\sqrt{a}) \Leftrightarrow (\sum_{cyc}\sqrt{a})^2 \geq 3\sqrt{3} \Leftrightarrow \sum_{cyc}\sqrt{a} \geq \sqrt[4]{27}$$

Now suppose that $\sum_{cyc}\sqrt{a} < \sqrt[4]{27}$. It suffices to prove that $\sum_{cyc}\frac{a\sqrt{a}}{bc} \geq \sqrt{3}.\sqrt[4]{27}$, but we have $\sum_{cyc}\frac{a\sqrt{a}}{bc} \geq 3\sqrt[3]{\prod\frac{a\sqrt{a}}{bc}} = 3(abc)^{-\frac{1}{6}}$ $\quad(*)$ by AM-GM inequality

$$1 = \sum_{cyc}ab \geq 3(abc)^{\frac{2}{3}} \Rightarrow (abc)^{-\frac{2}{3}} \geq 3 \Rightarrow (abc)^{-\frac{1}{6}} \geq \sqrt[4]{3}$$

$$\overset{(*)}{\Rightarrow}\sum_{cyc}\frac{a\sqrt{a}}{bc} \geq 3(abc)^{-\frac{1}{6}} \geq 3\sqrt[4]{3} = 3^{\frac{5}{4}} = 3^{\frac{1}{2}}.3^{\frac{3}{4}} = \sqrt{3}.\sqrt[4]{27}.$$

$\square$

Solution 2. First by Cauchy-Schwarz inequality we have

$$(\sum_{cyc}\frac{a^2}{bc\sqrt{a}})(\sum_{cyc}bc\sqrt{a}) \geq (\sum_{cyc}a)^2.$$

So it suffices to prove that

$$(\sum_{cyc}a)^2\sqrt{\sum_{cyc}ab} \geq \sqrt{3} \times \sqrt{abc}(\sum_{cyc}\sqrt{bc})(\sum_{cyc}\sqrt{a}).$$

Since $\sum_{cyc}ab = 1$.

But by AM-GM and Cauchy-Schwarz we have

$$\sqrt{\sum_{cyc}ab} \geq \frac{1}{\sqrt{3}}\sum_{cyc}\sqrt{ab} \quad \text{(Cauchy–Schwarz)} \quad (1)$$

$$\sum_{cyc}a \geq \frac{1}{3}(\sum_{cyc}\sqrt{a})^2 \quad \text{(Cauchy–Schwarz)} \quad (2)$$

$$\sum_{cyc}\sqrt{a} \geq 3.(abc)^{\frac{1}{6}} \quad (AM-GM) \quad (3)$$

$$\sum_{cyc}a \geq 3.(abc)^{\frac{1}{3}} \quad (AM-GM) \quad (4)$$

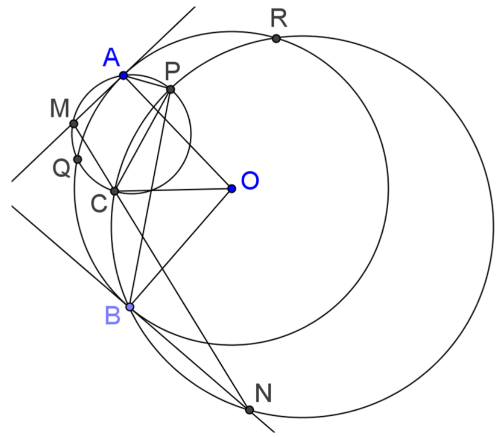
Multiplying (1),(2),(3) and (4) finishes the proof. $\square$

11. We can suppose the 60° angle between l and half-line CO is in the same side of CO as B. We take all angles with a plus or minus sign according to their

orientations and consider them modulo 180°. It can be seen that points X, Y, Z, T are on a circle (or line) if and only if $\angle XYZ = \angle XTZ$.

Let $\angle AOC = \angle COB = \alpha$. We have

$$\angle CMA = 360^\circ - (90^\circ + \alpha + 120^\circ) = 150^\circ - \alpha \Rightarrow \angle APC = \alpha + 30^\circ$$
$$\angle CNB = (60^\circ + \alpha) - 90^\circ = \alpha - 30^\circ \Rightarrow \angle CPB = \alpha - 30^\circ$$
$$\Rightarrow \angle APB = 2\alpha = \angle AOB.$$



Thus, point P is on the circumcircle of AOB and so $CP = CA = CO$.

So $\angle CAP = \angle APC = \alpha + 30^\circ$. Since $\angle CAO = \angle AOC = \alpha$ then $\angle OAP = 30^\circ$

Now $\angle OCP = \widehat{OP} = 2\angle OAP = 60^\circ$. So, triangle OCP is equilateral and so $OC = OP$. So, ω and the circumcircles of triangles CAM and CBN are symmetric with respect to the perpendicular bisector of CP. By considering their intersections, we get $PQ = CA$ and $PR = CB$. So, $PQ = PR$ and OP is the perpendicular bisector of QR (If D is the intersection of AM and CN, the condition on $A\hat{O}B$ only ensures that B is between D, N to have similar figures). $\square$

12. We divide subgroups into three groups
    1) Subset ϕ such that $f(\phi) = g(\phi) = 0$.
    2) One element subsets. Of course we know that for an arbitrary element of this group like X, $f(X) = 0$ and $g(X) = 1$.
       So for one element subsets we have $\sum_{|A|=1} (f(A) - g(A)) = -n$.
    3) Subsets with at least two elements.

For subsets in group 3. For integers $1 \le i \le j \le n$ Let A_{ij} be the collection of such binary strings of length n that the first 1 lies in the i's place and the last 1 lies in the j's place. Hence, if $i < j$ and $X = a_1 a_2 \cdots a_n \in A$ then $a_{i+1}, \ldots, a_j$ can be 0 or 1 so $|A_{ij}| = 2^{j-i-1}$. For $a \in \{0,1\}$ let $\bar{a} = 1$ if $a = 0$ and $\bar{a} = 0$ if $a = 1$.

For every $X = a_1 a_2 \cdots a_n \in A_{ij}$ we define $\bar{X} = a_1 \cdots a_i \bar{a}_{i+1} \cdots \bar{a}_{j-1} a_j \cdots a_n$. (We change the numbers between first and last 1.) Now we compare $f(\bar{X})$ and $g(X)$. (There is bijection between binary strings of length n and subsets of $\{1,2,\ldots,n\}$ so we can define f and g on binary strings.)

For the largest block of 1's in X we have three cases:

Case 1. The largest block contains places i and j. So $a_i = a_{i+1} = \cdots = a_j = 1$, hence $f(\bar{X}) = j - i$ and $g(X) = j - i + 1$. Thus $f(\bar{X}) - g(X) = -1$.

Case 2. The largest block contains exactly one of places i or j. So for example for some k $(i \le k < j)$, $a_i = a_{i+1} = \cdots = a_k = 1$ and $a_{k+1} = 0$. Therefore we deduce that $g(X) = k - i + 1$ and $f(\bar{X}) \ge k - i + 1$. Hence $f(\bar{X}) - g(X) \ge 0$

Case 3. The largest block contains none of i and j. So for example for some k and l $i < k \le l < j$ we have $a_k = 0, a_{k+1} = \cdots = a_{l-1} = 1$ and $a_l = 0$. Thus $g(X) = l - k - 1$ and $f(\bar{X}) \ge l - k$. Hence $f(\bar{X}) - g(X) \ge 1$.

Note that for each $1 \le i < j \le n$ one of elements of A_{ij} satisfies the condition of *case1* and there are $2^{n-4} - 1$ binary strings $X = a_1 a_2 \cdots a_n$ such that $a_1 = a_n = 0$ and $a_2 = a_{n-1} = 0$ also they have another 1.

$$\sum_{\substack{X \in A_{ij} \\ 1 \le i < j \le n}} (f(\bar{X}) - g(X)) \ge (-1)\binom{n}{2} + (+1)(2^{n-4} - 1).$$

Our conclusions are summed up in the following result.

$$\sum_{X} (f(X) - g(X)) \ge 2^{n-4} - 1 - \binom{n}{2} - n.$$

But it is very easy to prove by induction that for $n \ge 10$ we have

$2^{n-4} > \binom{n}{2} + n + 1$ and this implies the assertion. □

13. **Lemma 1.** *Suppose that l_1 and l_2 are two lines which intersect at O and the angle between them is α. Then the composition of reflections with respect to l_1 and l_2 is rotation with center O and angle 2α.* □

Lemma 2. *Suppose that R_1 and R_2 are two rotations with centers O_1 and O_2 and directed angles α and β respectively. Then if $\alpha + \beta \ne 2k\pi$ $(k \in \mathbb{Z})$ the composition of R_2 and R_1 is a rotation with center O and directed angle $\alpha + \beta$ where O is a point such that $\angle OO_1O_2 = \frac{1}{2}\alpha$ and $\angle OO_2O_1 = \frac{1}{2}\beta$ else if $\alpha + \beta = 2k\pi$ for some integer k then this composition is a translation.* □

Lemma 3. *Consider a 2^k-gon with center O and radius of circumcircle r. Let $A_1, A_2, \ldots, A_{2^k}$ be its vertices clockwise and R_i be the rotation with center A_i and angle*

$\frac{2\pi}{2^k}$ *clockwise. Then* $O'O = 2^{k+1} r\sin(\frac{\pi}{2^k})$ *where* $O' = R_{2^k} \circ \mathrm{R}_{2^k-1} \circ \cdots \circ R_1(O)$. *(Note that* $2^{k+1} r\sin(\frac{\pi}{2^k})$ *is the perimeter of* 2^k *-gon.)*

Proof. Proof is by induction on k. For the base case $k=2$ it is easy to verify that $O'O = 4r\sqrt{2} = 2^{2+1} r\sin(\frac{\pi}{4})$. Now suppose that $A_1A_2\ldots A_{2^{k+1}}$ is a 2^{k+1} -gon with radius of circumcircle r and let R_i be the rotation with center A_i and angle $\frac{2\pi}{2^{k+1}}$ clockwise ($1 \le i \le 2^k$). According to the lemma 2 composition $S_i = R_{2i} \circ \mathrm{R}_{2i-1}$ is a rotation of angle $2 \times \frac{2\pi}{2^{k+1}} = \frac{2\pi}{2^k}$ clockwise and center B_i. Where B_i is a point such that $\angle B_iA_{2i}A_{2i-1} = \angle B_iA_{2i-1}A_{2i} = \frac{2\pi}{2^{k+2}}$. By definition of B_i we have

$$\angle B_iA_{2i}O = \angle B_iA_{2i-1}O = \frac{\pi}{2} \Rightarrow B_iO = \frac{A_{2i}O}{\cos(\frac{2\pi}{2^{k+2}})} = \frac{r}{\cos(\frac{2\pi}{2^{k+2}})}.$$

And also

$$\angle B_iOB_{i+1} = \angle B_iOA_{2i} + \angle A_{2i}OA_{2i+1} + \angle A_{2i+1}OB_{i+1} = \frac{2\pi}{2^{k+2}} + \frac{2\pi}{2^{k+1}} + \frac{2\pi}{2^{k+2}} = \frac{2\pi}{2^k}.$$

So $B_1B_2\cdots B_{2^k}$ is a regular 2^k -gon with center O and radius of circumcircle $\frac{r}{\cos(\frac{2\pi}{2^{k+2}})}$. Therefore according to the induction hypothesis we have

$$OO' = 2^{k+1}\frac{r}{\cos(\frac{\pi}{2^{k+1}})}\sin(\frac{\pi}{2^k}) = 2^{k+2} r\sin(\frac{\pi}{2^{k+1}}).$$

Where $O' = S_{2^k} \circ S_{2^k-1} \circ \cdots \circ S_1(O)$, but by definition $S_i = R_{2i} \circ \mathrm{R}_{2i-1}$ and so
$O' = S_{2^k} \circ S_{2^k-1} \circ \cdots \circ S_1(O) = R_{2^{k+1}} \circ R_{2^{k+1}-1} \circ \cdots \circ R_1(O)$
This completes the proof of the lemma. □

We proceed the proof in two different ways.

Solution 1. Let $A_1, A_2, \ldots, A_{2^k}$ be the vertices of 2^k-gon such that $l_i = A_iA_{i+1}$ where $A_{2^k+1} = A_1$ and L_i be the reflection with respect to l_i ($1 \le i \le 2^k$). Invoking the lemma1 the composition $R_i = L_{2i} \circ L_{2i-1}$ ($1 \le i \le 2^{k-1}$) is a rotation with center A_{2i} and angle $\frac{2\pi}{2^{k-1}}$. $A_2A_4\cdots \mathrm{A}_{2^k}$ are vertices of a 2^{k-1}-gon with center O. So by lemma3 if $O' = R_{2^{k-1}} \circ R_{2^{k-1}-1} \circ \cdots \circ R_1(O)$, We have $O'O = 2^k r\sin(\frac{\pi}{2^{k-1}})$. Since

$R_i = L_{2i} \circ L_{2i-1}$ then $O' = L_{2^k} \circ L_{2^k-1} \circ \cdots \circ L_1(O)$ so it suffices to prove that $2^k r \sin(\frac{\pi}{2^{k-1}})$ is less than the perimeter of 2^k-gon which equals $2^{k+1} r \sin(\frac{\pi}{2^k})$.

$$\begin{aligned}
&\cos(\frac{\pi}{2^k}) < 1 \Rightarrow 2\sin(\frac{\pi}{2^k})\cos(\frac{\pi}{2^k}) = \sin(\frac{\pi}{2^{k-1}}) < 2\sin(\frac{\pi}{2^k}) \\
&\Rightarrow 2^k r \sin(\frac{\pi}{2^{k-1}}) < 2^{k+1} r \sin(\frac{\pi}{2^k})
\end{aligned}$$

And this finishes our proof. $\square$

Solution 2. We prove the assertion of problem for regular polygon with even number of sides. Denote by $A_1, A_2, \ldots, A_{2k}$ the vertices of this polygon clockwise and let $l_i = A_i A_{i+1}$ for $1 \le i \le 2k$ where $A_{2k+1} = A_1$.

Now suppose that L_i is the reflection with respect to l_i. Invoking the lemma 1 we deduce $R_i = L_{2i} \circ L_{2i-1}$ $(1 \le i \le k)$ is a rotation with center A_{2i} and angle $\frac{2\pi(k-1)}{k}$ in triangular direction, but we have $k \times \frac{2\pi(k-1)}{k} = 2\pi(k-1)$ so the composition $R_k \circ R_{k-1} \circ \cdots \circ R_1$ is a translation by lemma2. So it suffices to prove that the length of the vector of translation is not greater than the perimeter of $2k$ – gon.

Let $B_1 = R_1(A_1), B_2 = R_2(B_1), \ldots, B_k = R_k(B_{k-1})$ and l be the length of sides of polygon. We must prove that $B_k A_1 \le 2kl$. We have

$$\begin{aligned}
A_2A_1 = l &\Rightarrow A_2B_1 = l \ (R_1 \text{ is a rotation with center } A_2) \\
&\Rightarrow A_4B_1 \le A_4A_3 + A_3A_2 + A_2B_1 \le 3l \\
&\Rightarrow A_4B_2 \le 3l \ (R_2 \text{ is a rotation with center } A_4) \\
&\Rightarrow A_6B_2 \le A_6A_5 + A_5A_4 + A_4B_2 \le 5l \\
&\qquad \vdots \\
&\Rightarrow A_{2k}B_k \le (2k-1)l \Rightarrow A_1B_k \le A_1A_{2k} + A_{2k}B_k \le l + (2k-1)l = 2kl
\end{aligned}$$

So it finishes the proof. $\square$

Comment. Another proof can be given by complex numbers and think of rotations as multiplying by a complex number.

14. First suppose the case that none of the numbers are zero. Note that there exist at least 1000 positive numbers or at least 1000 negative numbers among these 2000 numbers. If there exist at least 1000 negative numbers and we put these 1000 numbers as roots of a degree 1000 polynomial all its coefficients are positive. So in every case we have 1000 positive numbers.

Now take 1000 positive numbers and put the 1000 remaining numbers as roots of a polynomial, all of its coefficients are positive, so the numbers must be negative. Therefore there is 1000 positive and 1000 negative numbers. If we put 1000 positive numbers as roots of a polynomial its coefficients are alternating positive and negative. This is a contradiction because remaining 1000 numbers all are negative. The contradiction shows that there at least one number equal to zero among numbers.

Denote by k the number of zeros among numbers so $k > 0$. If $k < 1000$, then put these k numbers zero and arbitrary $1000 - k$ numbers among others as roots of a polynomial. The product of roots is zero so there exists another number equal to zero and this contradicts the definition of k, thereby $k > 1000$. Now put 1000 numbers equal to zero as roots of a polynomial this polynomial is x^{1000} , so other numbers are equal to zero. So there do not exist 2000 numbers with mentioned property. □

15. We have

$$\frac{x^3 - xy + 1}{x^2 + x - y} = \frac{y^3 + xy - 1}{y^2 + x - y} \Rightarrow x - \frac{x^2 - 1}{x^2 + x - y} = y + \frac{y^2 - 1}{y^2 + x - y}$$
$$\Rightarrow x - y = \frac{x^2 - 1}{x^2 + x - y} + \frac{y^2 - 1}{y^2 + x - y}. \quad (*)$$

Now we have three cases:

Case1.

$$x = y \Rightarrow 2\frac{x^2 - 1}{x} = 0 \Rightarrow x^2 = 1 \text{ or } x = 0 \Rightarrow (x,y) = (0,0),(1,1),(-1,-1)$$

Case2. $x > y$. Let $k = x - y > 0$ so by $(*)$ we have $k = \dfrac{x^2 - 1}{x^2 + k} + \dfrac{y^2 - 1}{y^2 + k}$. If $\left|x\right| \le 1$ or $\left|y\right| \le 1$ by using main equation we get $(x,y) = (-1,0),(1,2),(0,1),(-2,-1)$. Now we can assume $\left|x\right|,\left|y\right| > 1$

$x^2 - 1 < x^2 + k \Rightarrow 0 < \dfrac{x^2 - 1}{x^2 + k} < 1$ and similarly $0 < \dfrac{y^2 - 1}{y^2 + k} < 1$, therefore

$$0 < k = \frac{x^2 - 1}{x^2 + k} + \frac{y^2 - 1}{y^2 + k} < 2 \Rightarrow k = 1 \Rightarrow 1 = \frac{x^2 - 1}{x^2 + 1} + \frac{y^2 - 1}{y^2 + 1}. \quad (**)$$

On the other hand $\left|x\right| > 1$ so $x^2 > 3$ and $2x^2 - 2 > x^2 + 1$ consequently $\dfrac{x^2 - 1}{x^2 + 1} > \dfrac{1}{2}$, similarly $\dfrac{y^2 - 1}{y^2 + 1} > \dfrac{1}{2}$, therefore $\dfrac{x^2 - 1}{x^2 + 1} + \dfrac{y^2 - 1}{y^2 + 1} > 1$. Contradicts $(**)$

Case3. $x < y$. Let $t = y - x > 0$ so by $(*)$ we have $t = \dfrac{x^2-1}{t-x^2} + \dfrac{y^2-1}{t-y^2}$ and by adding $2 = \dfrac{t-x^2}{t-x^2} + \dfrac{t-y^2}{t-y^2}$ to both sides we get $t+2 = \dfrac{t-1}{t-x^2} + \dfrac{t-1}{t-y^2}$.

We claim that $t-x^2$ and $t-y^2$ are not positive, simultaneously. Assume by contrary $t-x^2 > 0, t-y^2 > 0$ then

$$t > x^2, t > y^2 \Rightarrow 2(y-x) = 2t > x^2 + y^2 \Rightarrow (x+1)^2 + (y-1)^2 < 2\,.$$

This contradicts because $|x|,|y| > 1$. Therefore at least one of fractions $\dfrac{t-1}{t-x^2}$ and $\dfrac{t-1}{t-y^2}$ is less than or equal to 0 and the other less than or equal to $t-1$. So

$$t+2 = \frac{t-1}{t-x^2} + \frac{t-1}{t-y^2} \le 0 + (t-1) < t.$$

This contradiction shows that *case3* does not have a new solution and these are all the solutions: $(0,0),(1,1),(-1,-1),(-1,0),(1,2),(0,1),(-2,-1)$. $\square$

16. **Lemma.** *Let p be an odd prime number and k a positive integer then we have*

$$\sum_{i=0}^{p-1} i^k \equiv \begin{cases} 0 & p-1 \nmid k \\ -1 & p-1 \mid k \end{cases} \pmod p.$$

Proof. If $p-1 \mid k$ the statement is obvious by *Fermat Little Theorem.* For $p-1 \nmid k$ consider g a primitive root modulo p then

$$\sum_{i=0}^{p-1} i^k \equiv \sum_{i=1}^{p-1} i^k \equiv \sum_{i=1}^{p-1} g^{ik} \equiv \frac{g^{k(p-1)}-1}{g^k-1} \equiv 0 \pmod p \quad (p-1 \nmid k \Rightarrow g^k \not\equiv 1).$$

$\square$

Now for every polynomial $f \in \mathbb{Z}[x]$ with $f(x) = \sum_{n=0}^{m} a_n x^n$, we have

$$\sum_{i=0}^{p-1} f(i) \equiv \sum_{i=0}^{p-1}\sum_{n=0}^{m} a_n i^n \equiv \sum_{n=0}^{m} a_n \sum_{i=0}^{p-1} i^n \equiv -\sum_{p-1 \mid n, n>0} a_n \pmod p.$$

Therefore if $f(0), f(1), \dots, f(p-1)$ are a complete system of residues modulo p then we have $\sum_{i=0}^{p-1} f(i)^s \equiv 0 \pmod p$ for $1 \le s \le p-2$ so $f(x), (f(x))^2, \dots, (f(x))^{p-2}$ are all 0-*residue* and $-\sum_{i=0}^{p-1} f(i)^{p-1} \equiv 1 \pmod p$ so $(f(x))^{p-1}$ is 1-*residue.*

Now for reverse by (1) it suffices to prove that if for p numbers $a_0, a_1, \dots, a_{p-1} \in \mathbb{Z}$ we have

$$
\begin{aligned}
&a_0^i + a_1^i + \cdots + a_{p-1}^i \equiv 0 \qquad (\bmod p) \quad (1 \le i \le p-2) \\
&a_0^{p-1} + a_1^{p-1} + \cdots + a_{p-1}^{p-1} \equiv -1 \quad (\bmod p)
\end{aligned}
$$

Then $\{a_0, a_1, \ldots, a_{p-1}\}$ is a complete system of residues. Now Suppose that

$$g(x) = (x-a_0)(x-a_1)\cdots(x-a_{p-1}) = x^p + b_1 x^{p-1} + \cdots + b_{p-1}x + b_p$$

and $S_i = a_0^i + a_1^i + \cdots + a_{p-1}^i$ for $i \in \mathbb{N}$.

If for all $0 \le i \le p-1$ we have $a_i \not\equiv 0 \,(\bmod p)$ then invoking the Fermat little theorem we have $1 \equiv a_0^{p-1} + a_0^{p-1} + \cdots + a_0^{p-1} \equiv \underbrace{1+1+\cdots+1}_{p\text{ times}} \equiv 0 \,(\bmod p)$

Contradiction. So there exists a $0 \le j \le p-1$ such that $a_j \equiv 0 \,(\bmod p)$ and therefore $b_p \equiv 0 \,(\bmod p)$. Now by Newton identities we have

$$
\begin{cases}
S_1 + b_1 = 0 \\
S_2 + b_1 S_1 + 2b_2 = 0 \\
S_3 + b_1 S_2 + b_2 S_1 + 3b_3 = 0 \\
\quad \vdots \\
S_{p-1} + b_1 S_{p-2} + \cdots + \mathrm{b}_{p-2} S_1 + (p-1) b_{p-1} = 0
\end{cases}
$$

Hence

$$
\begin{aligned}
&S_1 + b_1 = 0, S_1 \equiv 0 \,(\bmod p) \Rightarrow b_1 \equiv 0 \,(\bmod p) \\
&S_2 + b_1 S_1 + 2b_2 = 0,\ S_1 \equiv S_2 \equiv 0 \,(\bmod p) \Rightarrow 2b_2 \equiv 0 \,(\bmod p) \Rightarrow b_2 \equiv 0 \,(\bmod p) \\
&\quad \vdots \\
&S_{p-2} + b_1 S_{p-3} + \cdots + \mathrm{b}_{p-3} S_1 + (p-2) b_{p-2} = 0,\ S_1 \equiv S_2 \equiv \cdots \equiv S_{p-2} \equiv 0 \,(\bmod p) \\
&\qquad \Rightarrow (p-2) b_{p-2} \equiv 0 \,(\bmod p) \Rightarrow b_{p-2} \equiv 0 \,(\bmod p)
\end{aligned}
$$

and

$$
\left.
\begin{aligned}
S_{p-1} + b_1 S_{p-2} + \cdots + \mathrm{b}_{p-2} S_1 + (p-1) b_{p-1} = 0 \\
S_1 \equiv S_2 \equiv \cdots \equiv S_{p-2} \equiv 0, S_{p-1} \equiv -1 \,(\bmod p)
\end{aligned}
\right\}
$$
$$\Rightarrow (p-1) b_{p-1} \equiv 1 \,(\bmod p) \Rightarrow b_{p-1} \equiv -1 \,(\bmod p)$$

So in $\mathbb{Z}_p[x]$, $g(x)$ is $x^p - x = x(x-1)\cdots(x-(p-1))$ and $\mathbb{Z}_p$ is a field so $\mathbb{Z}_p[x]$ is UFD and therefore $\{a_0, a_1, \ldots, a_{p-1}\}$ is a complete system of residues and this finishes our proof. $\square$

17. We call such a broken line a *good path.*

Lemma. *Let C be a set of $n \ge 2$ points in the plane, no three of which are collinear and let x_0 be a vertex of the convex hull of C. The number of good paths with vertices of C starting at x_0 is at least 2^{n-2}. Equality holds only when C is convex.*

Proof. We use induction on n. For $n = 2$ it is trivial. Suppose the claim is true for $n-1$. Let $x_0 y$ be a line such that $C - \{x_0\}$ is entirely in one side of it. Sort the vertices of $C - \{x_0\}$ as $x_1, x_2, \ldots, x_{n-1}$ such that the angles $\angle x_i x_0 y$ are increasing. So $C - \{x_0, x_1\}$ is entirely in one side of $x_0 x_1$ and $C - \{x_0, x_{n-1}\}$ is entirely in one side of $x_0 x_{n-1}$. There are at least $2 \times 2^{n-3}$ good paths with vertices of $C - \{x_0\}$ starting at either x_1 or x_{n-1}. By joining the segments $x_0 x_1$ or $x_0 x_{n-1}$ we obtain at least 2^{n-2} good paths with the vertices of C starting at x_0.

If C is convex, then in any good path starting at x_0, x_0 should be joined to x_1 or x_{n-1}, because in other cases the vertices will be in both sides of the first segment and the path will intersect the first segment. So equality for convex sets follows by the induction hypothesis.

Now, suppose C is not convex. Let z be a vertex of C not on the convex hull. Either z is in triangle $x_0 x_1 x_{n-1}$ or is inside the convex hull of $C - \{x_0\}$ (depending on which side of $x_1 x_{n-1}$ that z is in). In the first case, if z' is the farthest vertex from line in triangle $x_0 x_1 x_{n-1}$ (other than x_0), then the segment $x_0 z'$ doesn't intersect the convex hull of $C - \{x_0\}$ and there is a good path starting with $x_0 z'$ by the induction hypothesis. In the second case, $C - \{x_0\}$ is not convex and the number of good paths starting with $x_0 x_1$ is more than 2^{n-3} by the induction hypothesis. So the lemma is proved. □

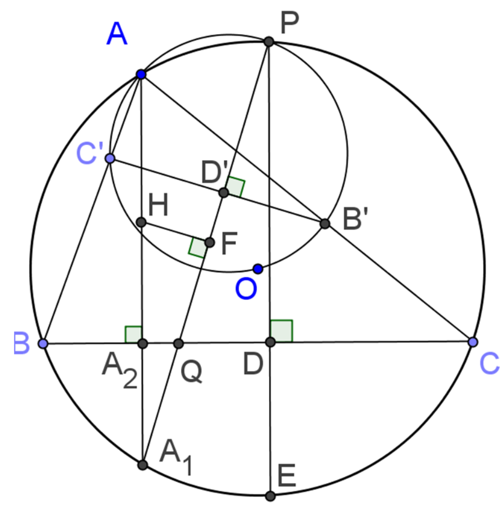
According to the lemma, we have $T(B) = n2^{n-3}$. We prove $T(A) > n2^{n-3}$. Let x_0 be a vertex on the convex hull of A and sort the other vertices of A as described in the lemma. For any $1 \le i \le n-2$, by joining any two good paths starting at x_0 with vertices of $\{x_0, x_1, \ldots, x_i\}$ and $\{x_0, x_{i+1}, \ldots, x_{n-1}\}$, we get a good path with the vertices of A, because the two sets can be divided by a line through x_0. This way we get $\sum_{i=1}^{n-2} 2^{i-1} \times 2^{n-i-2} = (n-2)2^{n-3}$ good vertices not starting at x_0. There are more than 2^{n-2} good vertices starting at x_0 and so $T(A) > n2^{n-3}$. So, the assertion is proved.

□

18. **Lemma 1.** *Triangle $A'B'C'$ is similar to triangle ABC and O is its orthocenter. Moreover, the corresponding sides make the same angle.*

Proof. We have $\angle OB'C' = \angle OAC' = 90° - \angle C$ and $\angle OB'A' = \angle OCA' = 90° - \angle A$. So $\angle C'B'A' = \angle B$. Compute the other angles similarly. This way it is proved that the triangles are similar and O is the orthocenter of triangle $A'B'C'$. Now, the altitudes of triangle $A'B'C'$ make the same angle with the corresponding sides of triangle ABC (angles $\angle OB'A$ and so on).So, the corresponding sides of the triangles also make the same angle. □

Lemma 2. *Let B',C' be points on AC,AB respectively such that $AB'OC'$ is cyclic. The circle w_1 with center B' passing through C intersects the circle ω_2 with center C' passing through B at points namely P,Q such that P is on the circumcircle of ABC and Q is on BC. Moreover, PQ and the altitude of A intersect on the circumcircle of ABC.*



Proof. Let P be the second intersection of the circumcircles of triangles ABC and $AB'C'$. We claim P is on both ω_1,ω_2. We have

$$\angle PBA = \angle PCA = \frac{1}{2}\overset{\frown}{PA} = \frac{1}{2}\angle POA,$$
$$\angle PC'A = \angle PB'A = \angle POA.$$

So, triangles $C'PB$ and $B'PC$ are isosceles and P is on both ω_1,ω_2. Also, these triangles are similar. Let $\alpha = \frac{1}{2}\angle POA$.

Triangles PBC and $PC'B'$ are similar (Consider the angles of P and the ratio of the sides incident with P in these triangles). The ratio of similarity is $2\cos\alpha$ and the corresponding sides make angles equal to α. So, if D,D' are the orthogonal projections of P on BC and $C'B'$ respectively, then $\frac{PD}{PD'} = 2\cos\alpha$ and $\angle D'PD = \alpha$. So triangles $D'PD$ and $C'PB$ are similar. So, $D'P = D'D$. Thus, if PD' intersects BC at Q, then D' is the midpoint of the hypotenuse in the right angled triangle PDQ. So, $B'C'$ is the perpendicular bisector of PQ and Q is the second intersection of ω_1,ω_2.

Suppose PD and AH intersect the circumcircle of triangle ABC for the second time at E and A_1 respectively. We have

$$\angle QPE = \angle D'PD = \alpha = \frac{1}{2}\overset{\frown}{PA} = \frac{1}{2}\overset{\frown}{A_1E} = \angle A_1PE.$$

So, PQ passes through A_1 and the lemma is proved. □

According to the lemma 2, the angle between PQ and AH is equal to the angle between $B'C'$ and BC. So, by lemma 1 the radical axis's in the problem are obtained by rotating the altitudes of triangle ABC with a fixed angle (but with different centers). The orientations of the rotations are the same, because all the equations in the proof remain valid by considering orientations. So, the triangle formed by the radical axis's is similar to triangle ABC. Therefore, it suffices to prove that the ratios of distances of H from the radical axis's is the same as the ratios of distances of H from sides of triangle ABC.

Let F, A_2 be the orthogonal projections of H on PQ and BC as in the lemma. We have $\frac{HF}{HA_2} = 2\frac{HF}{HA_1} = 2\sin\alpha$ which is the same as the other ratios of distances of H from the sides of the mentioned triangles. So the assertion is proved. $\square$