The Lamoen circle

Darij Grinberg

Let ΔABC be an arbitrary triangle, M_a, M_b, M_c the midpoints of its sides BC, CA, AB, and S its centroid, i. e. the intersection of the lines AM_a, BM_b and CM_c (Fig. 1). We get six triangles: AM_bS, CM_bS, CM_aS, BM_aS, BM_cS and AM_cS. These triangles have some interesting properties. At first, their areas are equal. The area of each one of these triangles will be denoted by k.

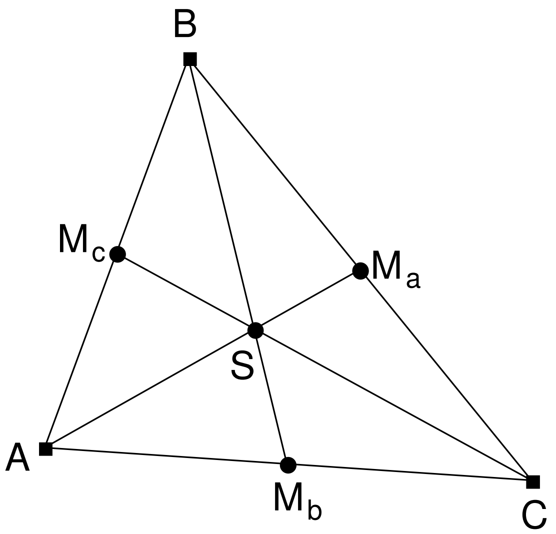

Fig. 1

Another interesting property, which turned out to be a theorem of Floor van Lamoen, is that the circumcenters of these six triangles are concyclic (Fig. 2). More precisely:

Theorem 1: Let A_b, C_b, C_a, B_a, B_c, A_c be the circumcenters of triangles AM_bS, CM_bS, CM_aS, BM_aS, BM_cS, AM_cS. Then, A_b, C_b, C_a, B_a, B_c, A_c lie on one circle (Fig. 2).

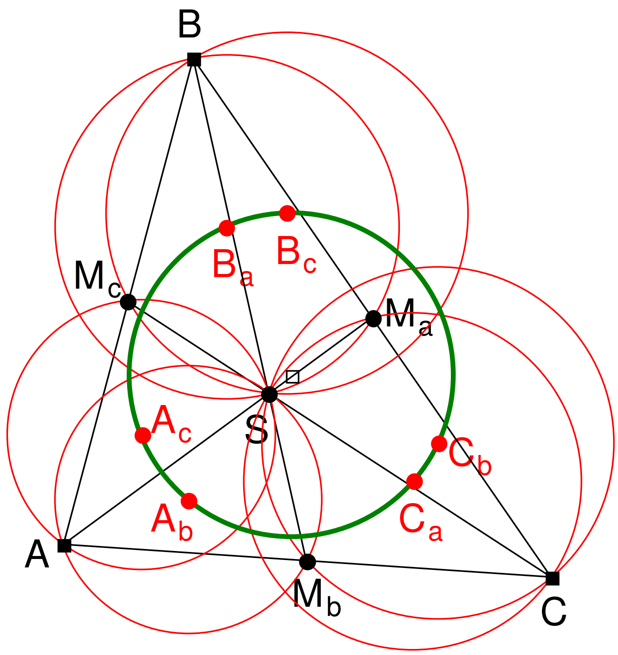

Fig. 2

After his discoverer, I call this circle the **Lamoen circle** of ΔABC.

Here is a half-synthetical *proof* of Theorem 1 (Fig. 3). Regard the circumcenters B_a and

B_c; they both lie on the perpendicular bisector of the segment BS. Hence, $B_aB_c \perp BS$. On the other hand, the circumcenters A_b and C_b both lie on the perpendicular bisector of the segment SM_b, hence, $A_bC_b \perp SM_b$. For BS and SM_b are the same line, we have $B_aB_c \parallel A_bC_b$. Analogously, we show that $A_cA_b \parallel C_aB_a$ and $C_bC_a \parallel B_cA_c$. Therefore, the opposite sides of the hexagon $A_bA_cB_cB_aC_aC_b$ are respectively parallel.

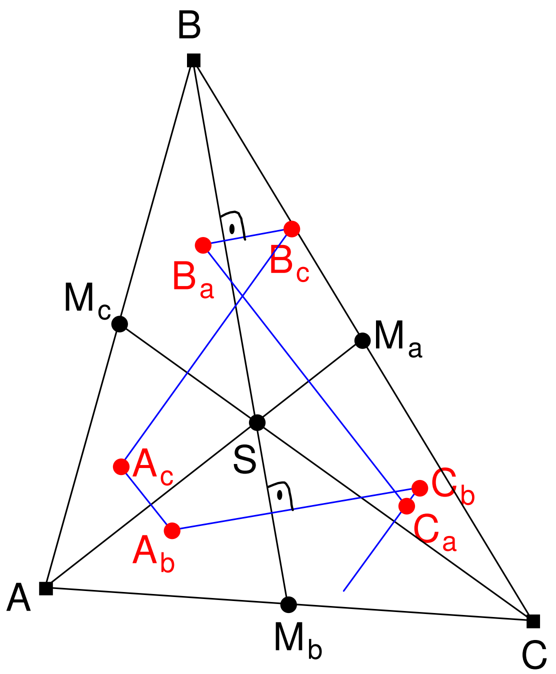

Fig. 3

Now we have the following theorem ([1] Aufgabe 34; [4] problem 109; [5] problem 131):

Theorem 2: A hexagon, whose opposite sides are respectively parallel, and whose main diagonals are of equal length, has a circumcircle.

Thus, in order to show that the hexagon $A_bA_cB_cB_aC_aC_b$ has a circumcircle, we must prove:

$$A_bB_a = A_cC_a = B_cC_b.$$

We will calculate A_cC_a after the Cosine Law in triangle ΔA_cSC_a; but for this aim we must know the two other sides and the opposite angle. The side A_cS is the circumradius of ΔAM_cS; so we have

$$k = \frac{AS \bullet SM_c \bullet M_cA}{4 \bullet A_cS} = \frac{AS \bullet \frac{1}{2}CS \bullet \frac{1}{2}c}{4 \bullet A_cS} = \frac{AS \bullet CS \bullet c}{16 \bullet A_cS},$$

hence

$$A_cS = \frac{AS \bullet CS \bullet c}{16 \bullet k}.$$

Analogously,

$$C_aS = \frac{AS \bullet CS \bullet a}{16 \bullet k}.$$

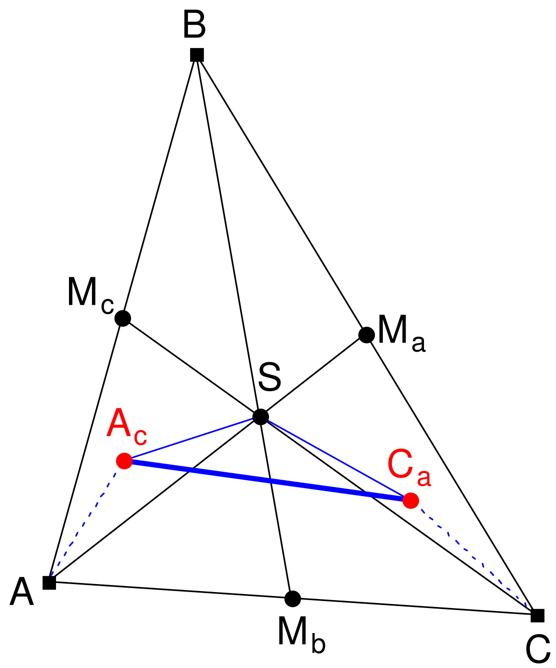

Fig. 4

Now we will calculate $\measuredangle A_cSC_a$. (Our arguments depend on the arrangement of points on Fig. 4, but can be done analogously for other positions.) In the isosceles ΔAA_cS, we have

$$\begin{aligned}\measuredangle A_cSA &= 90° - \frac{1}{2}\measuredangle AA_cS\\ &= 90° - \measuredangle AM_cS \qquad \text{(central angle)},\end{aligned}$$

and similarly $\measuredangle C_aSC = 90° - \measuredangle SM_aC$. Thus,

$$\begin{aligned}\measuredangle A_cSC_a &= \measuredangle A_cSA + \measuredangle ASC + \measuredangle C_aSC\\ &= (90° - \measuredangle AM_cS) + \measuredangle ASC + (90° - \measuredangle SM_aC)\\ &= (180° - \measuredangle AM_cS - \measuredangle SM_aC) + \measuredangle ASC\\ &= (180° - \measuredangle AM_cS - \measuredangle SM_aC) + (180° - \measuredangle M_cSA)\\ &= (180° - \measuredangle AM_cS - \measuredangle M_cSA) + (180° - \measuredangle SM_aC)\\ &= \measuredangle M_cAS + (180° - \measuredangle SM_aC)\\ &= \measuredangle BAM_a + \measuredangle SM_aB\\ &= \measuredangle BAM_a + \measuredangle AM_aB = 180° - \beta.\end{aligned}$$

Now, we can apply the Cosine Law to ΔA_cSC_a:

$$\begin{aligned}
A_cC_a^{\ 2} &= A_cS^{\ 2} + C_aS^{\ 2} - 2 \bullet A_cS \bullet C_aS \bullet \cos\triangle A_cSC_a \\
&= \left(\frac{AS \bullet CS \bullet c}{16 \bullet k}\right)^2 + \left(\frac{AS \bullet CS \bullet a}{16 \bullet k}\right)^2 \\
&\quad - 2 \bullet \frac{AS \bullet CS \bullet c}{16 \bullet k} \bullet \frac{AS \bullet CS \bullet a}{16 \bullet k} \bullet \cos(180° - \beta) \\
&= \left(\frac{AS \bullet CS}{16 \bullet k}\right)^2 \bullet (c^2 + a^2 - 2ca \bullet \cos(180° - \beta)) \\
&= \left(\frac{AS \bullet CS}{16 \bullet k}\right)^2 \bullet (c^2 + a^2 + 2ca\cos\beta) \\
&= \left(\frac{AS \bullet CS}{16 \bullet k}\right)^2 \bullet (2 \bullet BM_b)^2 \qquad \text{(after a formula for a triangle median)} \\
&= \left(\frac{AS \bullet CS}{16 \bullet k}\right)^2 \bullet \left(2 \bullet \frac{3}{2} \bullet BS\right)^2 \\
&= \left(\frac{AS \bullet CS}{16 \bullet k}\right)^2 \bullet (3 \bullet BS)^2 = \left(\frac{3}{16} \bullet \frac{AS \bullet BS \bullet CS}{k}\right)^2,
\end{aligned}$$

therefore

$$A_cC_a = \frac{3}{16} \bullet \frac{AS \bullet BS \bullet CS}{k}.$$

Analogously, one gets the same expression for A_bB_a and B_cC_b, and the equation $A_bB_a = A_cC_a = B_cC_b$ is proven!

References

[1] H. Dörrie: *Mathematische Miniaturen*, Wiesbaden 1969.

[2] D. O. Shkljarskij, N. N. Chenzov, I. M. Jaglom: *Izbrannye zadachi i teoremy elementarnoj matematiki*: *Chastj* 2 (*Planimetrija*), Moscow 1952.

[3] D. O. Shkljarskij, N. N. Chenzov, I. M. Jaglom: *Izbrannye zadachi i teoremy planimetrii*, Moscow 1967.