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INTERSECTION THEOREMS AND A LEMMA OF KLEITMAN

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A lemma of Kleitman is used to derive a simple proof of an existing theorem and to confirm part of a conjecture of Katona. The lemma is extended from subsets of a set to divisors of a number, and some new results are obtained.

1. Introduction

Throughout the first three sections of this paper, S will denote a set of n elements. $\mathcal{U}$ will denote a family of subsets of S satisfying the condition

$$X \in \mathcal{U},\quad X \subseteq Y \implies Y \in \mathcal{U},$$

and $\mathcal{D}$ will similarly denote a family of subsets such that

$$X \in \mathcal{D},\ Y \subseteq X \implies Y \in \mathcal{D}.$$

We shall denote the size of the set X by $|X|$.

Kleitman [5] has proved the following elegant result.

Lemma 1.1. $|\mathcal{U} \cap \mathcal{D}| \cdot 2^n \leq |\mathcal{U}| \cdot |\mathcal{D}|$.

This has been used by Seymour [7] in a recent paper to prove that if $\mathcal{P}$ and $\mathcal{Q}$ are families of subsets of S such that no member of $\mathcal{P}$ contains or is contained in any member of $\mathcal{Q}$, then $|\mathcal{P}|^{1/2} + |\mathcal{Q}|^{1/2} \leq 2^{n/2}$. Seymour then gives a proof, based on this result, of the following theorem.

Theorem 1.2. *If $A_1, \ldots, A_r$ are subsets of S such that $A_i \cap A_j \neq \emptyset$ and $A_i \cup A_j \neq S$ for each pair i, j, then $r \leq 2^{n-2}$.*

This theorem has had other proofs, namely those by Schönheim [6] and Lovász [1]. We show that a very simple proof can be given based on Kleitman's lemma, bypassing Seymour's intermediate result. We then give an application of the lemma to a problem posed in [4] by Katona. In the remaining sections we extend Kleitman's lemma to the setting of divisors of a number (not necessarily square-free), obtaining corresponding consequences in that more general setting.

2. Proof of Theorem 1.2

Suppose that $\mathscr{P}$ is a system of subsets of S satisfying the hypotheses of the theorem. Define $\mathscr{L}$ to be the family of subsets of S consisting of the sets of $\mathscr{P}$ and all their subsets. and similarly define $\mathscr{U}$ to consist of the sets of $\mathscr{P}$ and all their supersets. Then $\mathscr{P} = \mathscr{U} \cap \mathscr{L}$, and hence, by Kleitman's lemma,

$$|\mathscr{P}| \cdot 2^n \leq |\mathscr{U}| \cdot |\mathscr{L}|.$$

Now $\mathscr{U}$ is a collection of subsets such that no two are disjoint. Thus $|\mathscr{U}| \leq 2^{n-1}$ since $\mathscr{U}$ cannot contain a subset of S and its complement. Similarly $|\mathscr{L}| \leq 2^{n-1}$, and the theorem is proved.

3. A conjecture of Katona

Katona [3] has proved that if $A_1, \ldots, A_r$ are subsets of S such that $|A_i \cap A_j| \geq k$ for each pair i, j, then $r \leq f(n, k)$ where

$$f(n,k) = \begin{cases} \displaystyle\sum_{i=(n+k)/2}^{n} \binom{n}{i} & \text{if } n+k \text{ is even,} \\[2ex] \displaystyle\binom{n-1}{(n+k-1)/2} + \sum_{i=(n+k+1)/2}^{n} \binom{n}{i} & \text{if } n+k \text{ is odd.} \end{cases}$$

He conjectured in [4] that if the further condition $A_i \cup A_j \neq S$ is added, then $r \leq f(n-1, k)$. We now prove

Theorem 3.1. *If $A_1, \ldots, A_r$ are subsets of S such that $|A_i \cap A_j| \geq k$ and $A_i \cup A_j \neq S$ for each pair i, j, then, in the above notation,*

$$r \leq \begin{cases} f(n-1,k) & \text{if } n+k \text{ is odd,} \\[2ex] f(n-1,k) + \dfrac{1}{2}\left\{\dbinom{n-2}{(n+k-4)/2} - \dbinom{n-2}{(n+k-2)/2}\right\} & \text{if } n+k \text{ is even.} \end{cases}$$

Proof. Take $\mathscr{U}$ to be the collection of sets $A_1, \ldots, A_r$ and all their supersets, and $\mathscr{L}$ to be the collection of sets $A_1, \ldots, A_r$ and all their subsets. Then by Kleitman's lemma,

$$r = |\mathscr{U} \cap \mathscr{L}| \leq 2^{-n} |\mathscr{U}| \cdot |\mathscr{L}|.$$

But $|\mathscr{L}| \leq 2^{n-1}$ as in Theorem 1.2, and $|\mathscr{U}| \leq f(n, k)$. Thus

$$r \leq \frac{1}{2} f(n, k).$$

If $n + k$ is odd, we obtain, on replacing each $\binom{n}{i}$ by $\binom{n-1}{i} + \binom{n-1}{i-1}$,

$$\frac{1}{2} f(n,k) = \frac{1}{2}\binom{n-1}{(n+k-1)/2} + \frac{1}{2}\binom{n-1}{n-1}$$
$$+ \frac{1}{2} \sum_{i=(n+k+1)/2}^{n-1} \left\{ \binom{n-1}{i} + \binom{n-1}{i-1} \right\}$$
$$= \sum_{i=(n+k-1)/2}^{n-1} \binom{n-1}{i} = f(n-1,k).$$

If $n+k$ is odd, we obtain

$$\frac{1}{2} f(n,k) = \sum_{i=(n+k)/2}^{n-1} \binom{n-1}{i} + \frac{1}{2}\binom{n-1}{(n+k-2)/2}$$

whereas

$$f(n-1,k) = \sum_{i=(n+k)/2}^{n-1} \binom{n-1}{i} + \binom{n-2}{(n+k-2)/2}.$$

The difference between these two expressions is

$$\frac{1}{2}\left\{ \binom{n-2}{(n+k-4)/2} - \binom{n-2}{(n+k-2)/2} \right\}.$$

We note that if $k=1$ this difference is zero, so that we retrieve Theorem 1.2.

4. Divisors of a number

We now extend Kleitman's lemma to the divisors of a nonsquarefree number.

Lemma 4.1. *Let $\mathcal{U}, \mathcal{L}$ be sets of divisors of $m = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ such that*

$$a \in \mathcal{U}, \quad a \mid b \implies b \in \mathcal{U},$$
$$a \in \mathcal{L}, \quad b \mid a \implies b \in \mathcal{L}.$$

Then $|\mathcal{U} \cap \mathcal{L}|\,\tau(m) \leq |\mathcal{U}| \cdot |\mathcal{L}|$, where $\tau(m)$ denotes the number of divisors of m.

Proof. We use induction on $n = \sum_{i=1}^{r} \alpha_i$. If $n = 1$ or 2 the result is trivial, so we proceed to the induction step. Writing $p_r = p$ and $\alpha_r = s$, we have $m = m'p = m''p^s$ where $(m'', p) = 1$. We partition $\mathcal{U}$ and $\mathcal{L}$ as follows:

$$\mathcal{U} = \mathcal{U}_p \cup \mathcal{U}_{\bar p}, \qquad \mathcal{L} = \mathcal{L}_p \cup \mathcal{L}_{\bar p},$$

where $\mathcal{U}_p, \mathcal{L}_p$ consist precisely of those members of $\mathcal{U}, \mathcal{L}$ respectively which are divisible by p^s. Now if $hp^s \in \mathcal{L}$, then $h, hp, \ldots, hp^{s-1}$ are all members of $\mathcal{L}_{\bar p}$, so $|\mathcal{L}_{\bar p}| \geq s\,|\mathcal{L}_p|$. Similarly, $|\mathcal{U}_{\bar p}| \leq s\,|\mathcal{U}_p|$. Thus

$$(s\,|\mathcal{U}_p| - |\mathcal{U}_{\bar p}|)(|\mathcal{L}_{\bar p}| - s\,|\mathcal{L}_p|) \geq 0,$$

whence

$$s\,|\mathcal{U}_p|\cdot|\mathcal{L}_p| + \frac{1}{s}\,|\mathcal{U}_{\bar p}|\cdot|\mathcal{L}_{\bar p}| \leq |\mathcal{U}_p|\cdot|\mathcal{L}_{\bar p}| + |\mathcal{U}_{\bar p}|\cdot|\mathcal{L}_p|. \tag{1}$$

Using the induction hypothesis for divisors of m', we have

$$|\mathcal{U}_{\bar p} \cap \mathcal{L}_{\bar p}|\,\tau(m') \leq |\mathcal{U}_{\bar p}|\cdot|\mathcal{L}_{\bar p}|. \tag{2}$$

Further, let $\mathcal{U}''_p, \mathcal{L}''_p$ denote the sets of divisors of m'' obtained by dividing each member of $\mathcal{U}_p, \mathcal{L}_p$ respectively by p'. Then, since $|\mathcal{U}_p \cap \mathcal{L}_p| = |\mathcal{U}''_p \cap \mathcal{L}''_p|$, the induction hypothesis applied to m'' gives

$$|\mathcal{U}_p \cap \mathcal{L}_p|\,\tau(m'') \leq |\mathcal{U}''_p|\cdot|\mathcal{L}''_p| = |\mathcal{U}_p|\cdot|\mathcal{L}_p|. \tag{3}$$

Using (2) and (3), and then (1),

$$\begin{aligned}
|\mathcal{U} \cap \mathcal{L}|\,\tau(m) &= |\mathcal{U}_p \cap \mathcal{L}_p|\,\tau(m) + |\mathcal{U}_{\bar p} \cap \mathcal{L}_{\bar p}|\,\tau(m) \\
&= (s+1)|\mathcal{U}_p \cap \mathcal{L}_p|\,\tau(m'') + \left(1 + \frac{1}{s}\right)|\mathcal{U}_{\bar p} \cap \mathcal{L}_{\bar p}|\,\tau(m') \\
&\leq (s+1)|\mathcal{U}_p|\cdot|\mathcal{L}_p| + \left(1 + \frac{1}{s}\right)|\mathcal{U}_{\bar p}|\cdot|\mathcal{L}_{\bar p}| \\
&= |\mathcal{U}_p|\cdot|\mathcal{L}_p| + |\mathcal{U}_{\bar p}|\cdot|\mathcal{L}_{\bar p}| + s\,|\mathcal{U}_p|\cdot|\mathcal{L}_p| + \frac{1}{s}\,|\mathcal{U}_{\bar p}|\cdot|\mathcal{L}_{\bar p}| \\
&\leq |\mathcal{U}_p|\cdot|\mathcal{L}_p| + |\mathcal{U}_{\bar p}|\cdot|\mathcal{L}_{\bar p}| + |\mathcal{U}_p|\cdot|\mathcal{L}_{\bar p}| + |\mathcal{U}_{\bar p}|\cdot|\mathcal{L}_p| \\
&= |\mathcal{U}|\cdot|\mathcal{L}|.
\end{aligned}$$

5. Applications of Lemma 4.1

We first note that the results of Seymour's paper now extend. Thus if $\mathcal{P}, \mathcal{Q}$ are two sets of divisors of m such that if $a \in \mathcal{P}$ and $b \in \mathcal{Q}$ then $a \nmid b$ and $b \nmid a$, then $|\mathcal{P}|^{1/2} + |\mathcal{Q}|^{1/2} \leq (\tau/m)^{1/2}$. Further, if $\mathcal{R}$ is a set of divisors of m and if $\mathcal{C}(\mathcal{R})$ denotes the set of all comparable divisors, i.e. those which are members of $\mathcal{R}$ or which divide or are divisible by a member of $\mathcal{R}$, then $\mathcal{C}(\mathcal{R})$ contains at least $\min\{3|\mathcal{R}|, \frac{3}{4}\tau(m)\}$ members. Simple arguments give improvements of this result: for example, if $|\mathcal{R}| > \frac{4}{5}\tau(m)$ then $|\mathcal{C}(\mathcal{R})| > \frac{99}{100}\tau(m)$.

Before giving another application of Lemma 4.1, we quote the following result of Erdös and Schönheim [2] and Woodall (unpublished).

Theorem 5.1. *Let $d_1, \ldots, d_r$ be divisors of $m = p_1^{\alpha_1} \ldots p_t^{\alpha_t}$ such that* h.c.f.$(d_i, d_j) > 1$ *for each i, j, and let $\alpha = \prod_{i=1}^{t} \alpha_i$. Then*

$$r \leq f(m) = \frac{1}{2} \sum_{I} \max\left\{ \prod_{j=1}^{w} \alpha_{i_j}, \alpha \Big/ \prod_{j=1}^{w} \alpha_{i_j} \right\}$$

where the summation is over all subsets $I = \{i_1, \ldots, i_w\}$ of $\{1, \ldots, t\}$.

We note that if $\alpha_t > \sqrt{\alpha}$ then

$$f(m) = \alpha_t(1+\alpha_1)\dots(1+\alpha_{t-1}) = \frac{\alpha_t}{\alpha_t+1}\,\tau(m).$$

Theorem 5.2. *Let* $\mathcal{P} = \{d_1, \dots, d_r\}$ *be a set of divisors of* $m = p_1^{\alpha_1}\dots p_t^{\alpha_t}$ *such that for each pair* i, j, h.c.f.$(d_i, d_j) \neq 1$ *and* l.c.m.$\{d_i, d_j\} \neq m$, *then*

$$r \leq (f(m))^2/\tau(m),$$

where $f(m)$ *is as in Theorem* 5.1.

Proof. Copying our proof of Theorem 1.2, define $\mathcal{U}, \mathcal{L}$ in analogous fashion. Then $|\mathcal{U}| \leq f(m)$, $|\mathcal{L}| \leq f(m)$ and, by Lemma 4.1 $|\mathcal{P}|\,\tau(m) \leq |\mathcal{U}| \cdot |\mathcal{L}|$.

In particular, if $\alpha_t > \sqrt{\alpha}$ we then have

$$r \leq \left(\frac{\alpha_t}{\alpha_t+1}\right)^2 \tau(m).$$

An example is the set of all divisors of m which are divisble by p_t but not by $p_t^{\alpha_t}$. Such a set $\mathcal{P}$ contains

$$\frac{\alpha_t-1}{\alpha_t+1}\,\tau(m) = \frac{\alpha_t^2-1}{(\alpha_t+1)^2}\,\tau(m)$$

divisors of m.

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