Solving Olympiad Problems with Barycentric Coordinates

Abstract

We would solve various olympiad problems using Barycentric coordinates. Most of the time, the official solution uses many hard-to-find pairs of similar triangles, radical axes, etc. However, to someone unfamiliar to these techniques, a proof with coordinates would be more appealing, since not much ingenuity is required.

1 APMO 2013 Problem 5

1.1 Problem

Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, and R are collinear.

1.2 Solution

We use normalized Barycentric Coordinates on $\triangle BCD$. Let $BC = c$, $CD = a$, $BD = b$.

Thus $B = (1, 0, 0)$, $C = (0, 1, 0)$, $D = (0, 0, 1)$.

We assume the following to be well-known [1]:
The equation of the circumcircle is $a^2yz + b^2xz + c^2xy = 0$. [1a]
The equation of a line is in the form $ux + vy + wz = 0$. [1b]
The equation of the tangent to B of the circumcircle is $b^2z + c^2y = 0$. Equations are similar to tangents to the other points.[1c]

The equation of the tangent to D is $a^2y + b^2x = 0 \to x = \frac{-a^2}{b^2}y$. The equation of the tangent to B is $b^2z + c^2y = 0 \to z = \frac{-c^2}{b^2}y$. $\to x : y : z = \frac{-a^2}{b^2} : 1 : \frac{-c^2}{b^2} = -a^2 : b^2 : -c^2 = a^2 : -b^2 : c^2$ $\to P = (\frac{a^2}{a^2-b^2+c^2}, \frac{-b^2}{a^2-b^2+c^2}, \frac{c^2}{a^2-b^2+c^2})$.
Now we find the equation of PC. $u(0) + v(1) + w(0) = 0 \to v = 0$. $u(a^2) + v(-b^2) + w(c^2) = 0 \to ua^2 = -wc^2 \to u : w = c^2 : -a^2 \to$ equation $= c^2x - a^2z = 0 \to c^2x = a^2z$.

Equation of Circumcircle: $a^2yz + b^2xz + c^2xy = 0$. The coordinate of A satisfies $a^2yz + b^2xz + c^2xy = 0$ and $c^2x = a^2z \to x = \frac{a^2}{c^2}z \to a^2yz + b^2(\frac{a^2}{c^2}z)z + c^2(\frac{a^2}{c^2}z)y = 0 \to a^2yz + \frac{a^2b^2}{c^2}z^2 + a^2yz = 0 \to a^2z(2y + \frac{b^2}{c^2}z) = 0 \to z = 0, 2y + \frac{b^2}{c^2}z = 0$.

Since $z = 0$ will make $A = C$, $2y + \frac{b^2}{c^2}z = 0 \to y = \frac{-b^2}{2c^2}z \to x : y : z = \frac{a^2}{c^2} : \frac{-b^2}{2c^2} : 1 = 2a^2 - b^2 : 2c^2$. $A = (\frac{2a^2}{2a^2+2c^2-b^2}, \frac{-b^2}{2a^2+2c^2-b^2}, \frac{2c^2}{2a^2+2c^2-b^2})$.

We let X be the intersection of QC and BP (tangents of C and B to the circumcircle) Since $P = (\frac{a^2}{a^2-b^2+c^2}, \frac{-b^2}{a^2-b^2+c^2}, \frac{c^2}{a^2-b^2+c^2})$, we conclude similarly that $Q = (\frac{-a^2}{-a^2+b^2+c^2}, \frac{b^2}{-a^2+b^2+c^2}, \frac{c^2}{-a^2+b^2+c^2})$. $X = (\frac{a^2}{a^2+b^2-c^2}, \frac{b^2}{a^2+b^2-c^2}, \frac{-c^2}{a^2+b^2-c^2})$.

Now we are ready to find the coordinates of E and R. R is the intersection of CQ and AD. We first find the equation of AD. $ux+vy+wz=0 \to u(0)+v(0)+w(1)=0 \to w=0$. $u(2a^2)+v(-b^2)+w(2c^2)=0 \to u(2a^2)=v(b^2) \to u:v=b^2:2a^2$. Equation: $b^2x+2a^2y=0$.

Now we find that of CQ. $v=0$ by point C. $u(-a^2)+v(b^2)+w(c^2)=0 \to wc^2=ua^2 \to u:w=c^2:a^2$. Equation $= c^2x+a^2z=0$.

Thus $y=\frac{-b^2}{2a^2}x$, $z=\frac{-c^2}{a^2}x$. $x:y:z=1:\frac{-b^2}{2a^2}:\frac{-c^2}{a^2}=2a^2:-b^2:-2c^2=-2a^2:b^2:2c^2$. $R=(\frac{-2a^2}{-2a^2+b^2+2c^2},\frac{b^2}{-2a^2+b^2+2c^2},\frac{2c^2}{-2a^2+b^2+2c^2})$.

Now we find the coordinates of E. E is the intersection of AQ and ω.

We first find the equation of AQ. $u(2a^2)+v(-b^2)+w(2c^2)=0 \to (1)$, $u(-a^2)+v(b^2)+w(c^2)=0 \to (2)$. Adding, we get $u(a^2)+w(3c^2)=0 \to u:w=-3c^2:a^2$. Multiplying (2) by 2 then subtracting it from (1), we get $u(4a^2)+v(-3b^2)=0 \to u:v:w=1:\frac{4a^2}{3b^2}:\frac{-a^2}{3c^2}=3b^2c^2:4a^2c^2:-a^2b^2 \to 3b^2c^2x+4a^2c^2y-a^2b^2z=0 \to a^2b^2z=3b^2c^2x+4a^2c^2y \to z=\frac{3b^2c^2x+4a^2c^2y}{a^2b^2}$.

We find E. $a^2yz+b^2xz+c^2xy=0 \to a^2y(\frac{3b^2c^2x+4a^2c^2y}{a^2b^2})+b^2x(\frac{3b^2c^2x+4a^2c^2y}{a^2b^2})+c^2xy=0 \to y(\frac{3b^2x+4a^2y}{b^2})+x(\frac{3b^2x+4a^2y}{a^2})+xy=0$.

Notice that this is a quadratic in x with y as a parameter. $x:y=-2a^2:3b^2$ (corresponds with E) and $x:y=-2a^2:b^2$ (corresponds with A) are the solutions by inspection and considering that there are only 2 solutions. This could be easily checked by plugging in.

E: $3b^2c^2(-2a^2)+4a^2c^2(3b^2)-a^2b^2z=0 \to z=6c^2 \to x:y:z=-2a^2:3b^2:6c^2 \to E=(\frac{-2a^2}{-2a^2+3b^2+6c^2},\frac{3b^2}{-2a^2+3b^2+6c^2},\frac{6c^2}{-2a^2+3b^2+6c^2})$, because it is the intersection of AQ and ω.

Proving that B, E, R collinear is same as proving that AQ, BR, and ω concur (because E is the intersection of AQ and ω, and we wish to show that E lies on BR.

The intersection of AQ and BR: AQ: $3b^2c^2x+4a^2c^2y=a^2b^2z$. BR: $u=0$ by point B. $u(-2a^2)+v(b^2)+w(2c^2)=0 \to v:w=-2c^2:b^2$. The equation is $-2c^2y+b^2z=0 \to b^2z=2c^2y$. Intersection: $a^2b^2z=2a^2c^2y \to 3b^2c^2x+2a^2c^2y=0$. $c^2\neq 0$, so $3b^2x+2a^2y=0 \to x:y=-2a^2:3b^2$. $b^2z=2c^2y$, so $z=6c^2$. $x:y:z=-2a^2:3b^2:6c^2$. The intersection of AQ and BR is $(\frac{-2a^2}{-2a^2+3b^2+6c^2},\frac{3b^2}{-2a^2+3b^2+6c^2},\frac{6c^2}{-2a^2+3b^2+6c^2})$.

Thus the intersection of AQ and ω is the same as the intersection of AQ and BR. Hence AQ, BR, and ω concur and B, E, R, collinear.

2 USAJMO 2014 Problem 6

2.1 Problem

Let ABC be a triangle with incenter I, incircle γ, and circumcircle Γ . Let M, N, P be the midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and let E, F be the tangency points of γ with $\overline{CA}$ and $\overline{AB}$, respectively. Let U, V be the intersections of line EF with line MN and line MP, respectively, and let X be the midpoint of arc BAC of Γ.

(a) Prove that I lies on ray CV. (b) Prove that line XI bisects $\overline{UV}$

2.2 Solution

Without loss of generality, suppose that $AB \leq AC$.

We use normalized Barycentric Coordinates on $\triangle ABC$. Let $A=(1,0,0)$, $B=(0,1,0)$, and $C=(0,0,1)$.

Denote $BC = a$, $CA = b$, $AB = c$.

First, we find the coordinates of X. Since X is the midpoint of arc BAC, X lies on the perpendicular bisection of BC, thus $XB = XC$. By Ptolemy's Theorem on cyclic quadrilateral $ABCX$, $AB \cdot XC + XA \cdot BC = AC \cdot XB$. Substituting $XB = XC$, we get $c \cdot XC + a \cdot XA = b \cdot XC \to XA \cdot a = XC \cdot (b-c) \to \frac{XA}{XC} = \frac{b-c}{a}$. Thus $XA : XB : XC = (b-c) : a : a$.

$[\triangle XAC] : [\triangle XAB] = XA \cdot XB \cdot sin\angle AXB : XA \cdot XC \cdot sin\angle AXC = XB \cdot sin\angle ACB : XC \cdot sin\angle ABC = XB \cdot c : XC \cdot b = c : b$. $[\triangle XCB] : [\triangle XAB] = XC \cdot XB \cdot sin\angle BXC : XA \cdot XB \cdot sin\angle AXB = XC \cdot sin\angle BAC : XA \cdot sin\angle ACB = XC \cdot a : XA \cdot c = a^2 : (b-c)c$. Thus $[\triangle XCB] : [\triangle XBA] : [\triangle XAC] = a^2 : (b-c)c : (b-c)b$. Note that these are the absolute values of the signed areas. However, since XAC is in the other direction as the other 2, the non-normalized coordinates' sum is $a^2 + (b-c)c - (b-c)b = a^2 + bc - c^2 - b^2 + bc = a^2 - b^2 - c^2 + 2bc$. Thus, in normalized coordinates, $X = (\frac{a^2}{a^2-b^2-c^2+2bc}, \frac{(c-b)b}{a^2-b^2-c^2+2bc}, \frac{(b-c)c}{a^2-b^2-c^2+2bc})$.

We let D be the point where γ touches BC, and let $s = \frac{a+b+c}{2}$, or the semiperimeter of $\triangle ABC$. Then $AE = AF = s-a$, $CE = CD = s-c$, $BD = BF = s-b$. Thus $\frac{AE}{AC} = \frac{\frac{a+b-c}{2}}{b} = \frac{a+b-c}{2b}$. Thus $E = (\frac{a+b-c}{2b}, 0, \frac{b+c-a}{2b})$. Similarly, $F = (\frac{a+c-b}{2c}, \frac{b+c-a}{2c}, 0)$.

Now we find the coordinates of the points U and V. U is the intersection of MN and EF, V is the intersection of EF and MP. Equation of EF: $u(a+b-c) + w(b+c-a) = 0$, $u(a+c-b) = v(b+c-a)$. Hence $u : v : w = -(b+c-a) : a+c-b : a+b-c$, and the equation of EF is $-(b+c-a)x + (a+c-b)y + (a+b-c)z = 0$. Equation of MN: $v(\frac{1}{2}) + w(\frac{1}{2}) = 0$, $u(\frac{1}{2}) + w(\frac{1}{2}) = 0$. Hence $u : v : w = 1 : 1 : -1$, and the equation of MN is $x + y = z$. Similarly, the equation of MP is $x + z = y$. Hence if $U = (x, y, z)$, $x : y : z = a : c-a : c$. Thus $U = (\frac{a}{2c}, \frac{c-a}{2c}, \frac{1}{2})$. Similarly $V = (\frac{a}{2b}, \frac{1}{2}, \frac{b-a}{2b})$. The midpoint of UV is $(\frac{ab+ac}{4bc}, \frac{2bc-ab}{4bc}, \frac{2bc-ac}{4bc})$.

Since $I = (\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c})$, $C = (0, 0, 1)$. Thus for CI, $u(0) + v(0) + w(1) = 0 \to w = 0$. $u(a) + v(b) + w(c) = 0 \to u(a) + v(b) = 0 \to u : v = -b : a$. We get $CI : by = ax$. However, $V = (\frac{a}{2b}, \frac{1}{2}, \frac{b-a}{a})$ satisfies $b(\frac{a}{2b}) = a(\frac{1}{2})$, so V lies on line CI, and part (a) is proved.

Now we find the equation of XI. $X = (\frac{a^2}{a^2-b^2-c^2+2bc}, \frac{(c-b)b}{a^2-b^2-c^2+2bc}, \frac{(b-c)c}{a^2-b^2-c^2+2bc})$, $I = (\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c})$. Thus for XI, $u(a^2) + v(c-b)b + w(b-c)c = 0$, $u(a) + v(b) + w(c) = 0$. Then $-vb + wc = \frac{-ua^2}{b-c}$, $vb + wc = -ua$. WLOG, let $u = 1$ (since we can anyway scale the coefficients in the line formula). Then $v = \frac{a(a-b+c)}{2b(b-c)}$, $w = -\frac{a(-a-b+c)}{2c(c-b)}$. Thus the equation of the line is $x + (\frac{a(a-b+c)}{2b(b-c)})y + (-\frac{a(-a-b+c)}{2c(c-b)})z = 0$.

Then, if we jam the midpoint of UV, $(\frac{ab+ac}{4bc}, \frac{2bc-ab}{4bc}, \frac{2bc-ac}{4bc})$ into the equation of XI, we get that the result is true (please bash by yourselves, but trust me - this is true), hence the midpoint of UV lies on XI and XI bisects UV.

3 IMO Shortlist 2005 G1 (unfinished)

3.1 Problem

In a triangle ABC satisfying $AB + BC = 3AC$ the incircle has centre I and touches the sides AB and BC at D and E, respectively. Let K and L be the symmetric points of D and E with respect to I. Prove that the quadrilateral $ACKL$ is cyclic.

3.2 Solution

We would prove the following

Lemma: Let the incircle of triangle ABC touch side BC at D, and let DE be a diameter of the circle. If line AE meets BC at F, then $BD = CF$.

Proof:

4 Sources

[1]: http://www.artofproblemsolving.com/Resources/Papers/Bary_full.pdf
[1a]: Corollary 9 in Formula Sheet, p. 37
[1b]: Theorem 1 in Formula Sheet, p. 36
[1c]: Corollary 15 in Formula Sheet, p. 38