Gauss's line and some little appications

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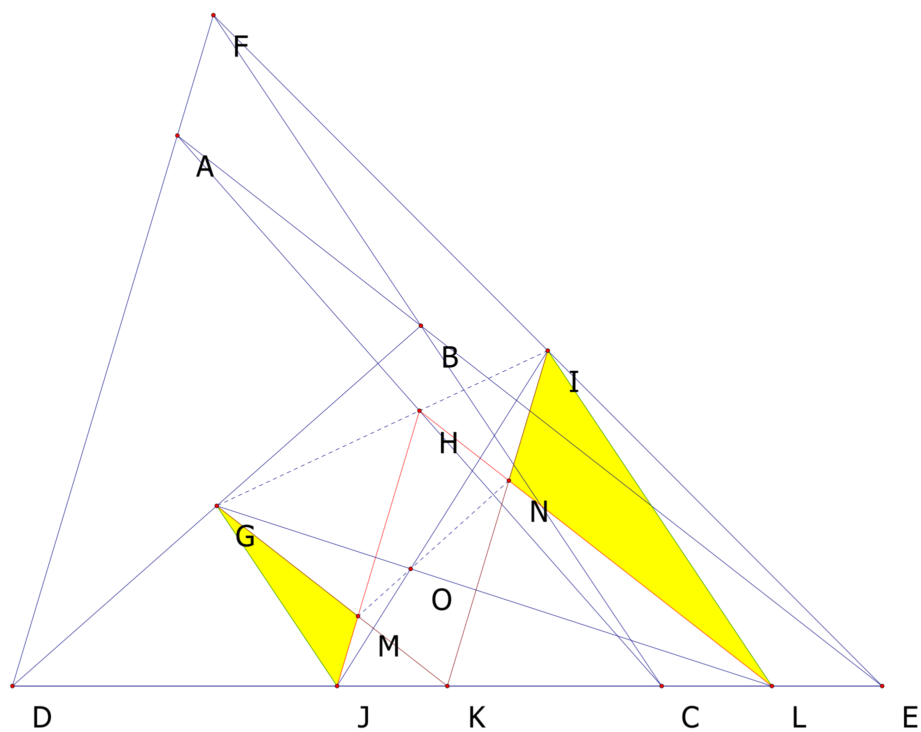
A/Introduce

Gauss's line: Let $ABCDEF$ be a complete quadrilateral.Then the midpoint of $AC; BD; EF$ respectively are collinear

There are many solution of this problem .I'll introduce a interesting solution

Proof



Let $J; L; K$ be the midpoint of $DC; CE; DE$ respectively

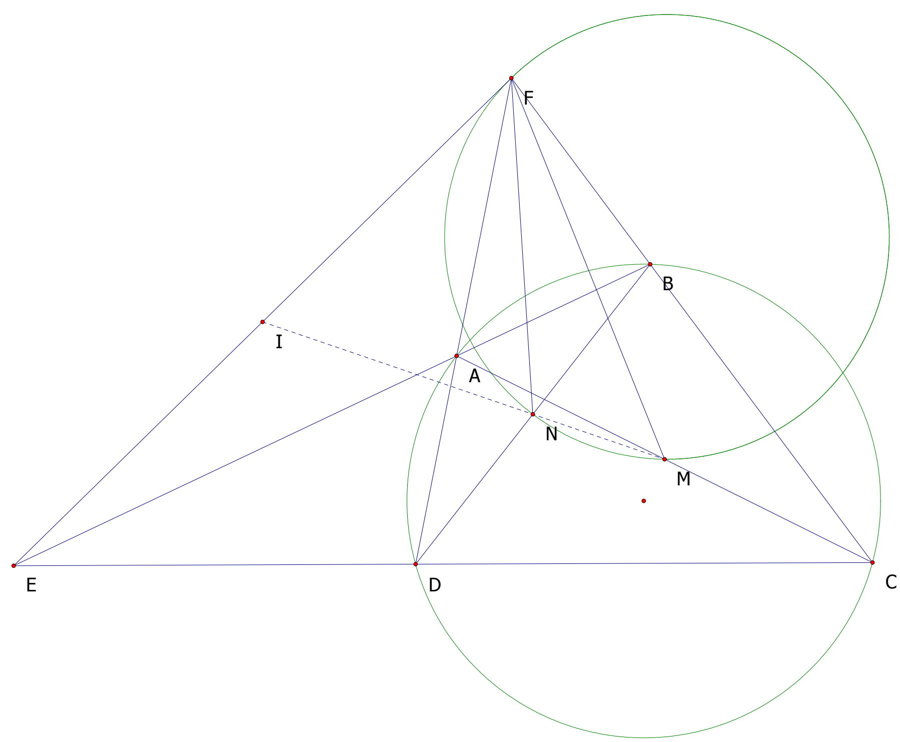
It's easy to see that $GM//NL; GJ//IL; MJ//IN$ which implies that $\triangle GMJ$ and $\triangle INL$ are homothetic

So $M;N;O$ are collinear.Applying Pappus's theorem.We'll get $G;H;I$ are collinear (QED)

B/ Some little appication

Problem 1:Let $ABCD$ be a convex quadrilateral and it's circumcircle (O).$E \equiv AB \cap CD; F \equiv AD \cap BC$.$M;N$ be the midpoint of $AC;BD$ respectively.Prove that EF is the tangency of (MNF)

Proof



The tangency at A meet the tangency at C at G.The tangency at B meet the tangency D at H (The tangency of (O)).It's easy to see that the pencil $(EFHG) = -1$

Let I be the midpoint of $EF \Rightarrow \overline{INM}$ is the Gauss's line of complete quadrilateral $ABCDEF$

According to Newton's theorem $IF^2 = IG.IH(1)$Morever ,$GHNM$ is cyclic so $IN.IM = IG.IH(2)$

From $(1);(2) \Rightarrow IF^2 = IM.IN$ which implies that IF is the tangency of (MNF).Our proof is completed then

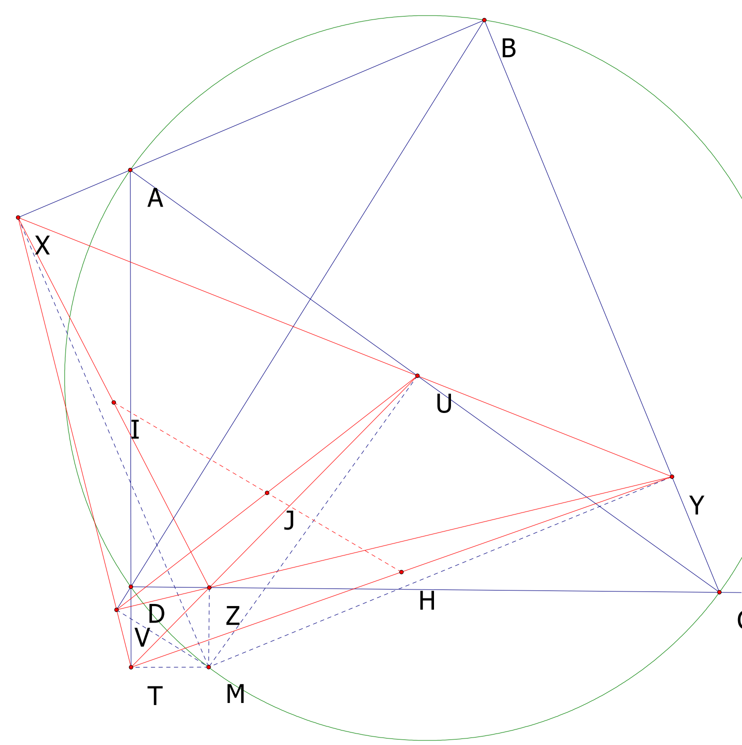
Problem 2:Let $ABCD$ incribed (O) and a point so-called M.Call $X;Y;Z;T;U;V$ are the projection of M onto $AB;BC;CD;DA;CA;BD$ respectively.Call $I;J;H$ are the midpoint of $XZ;UV;YT$ respectively.Prove that $\overline{N;P;Q}$

Proof

There are three case for consideration

+Case 1: $M \equiv O$.This case make the problem become trivial

+Case 2:M lies on (O).According to the Simson's line then $XYZTUV$ become a complete quadrilateral and we can conclude that $\overline{IJH}$ is the Gauss's line of $XYZTUV$ (QED)

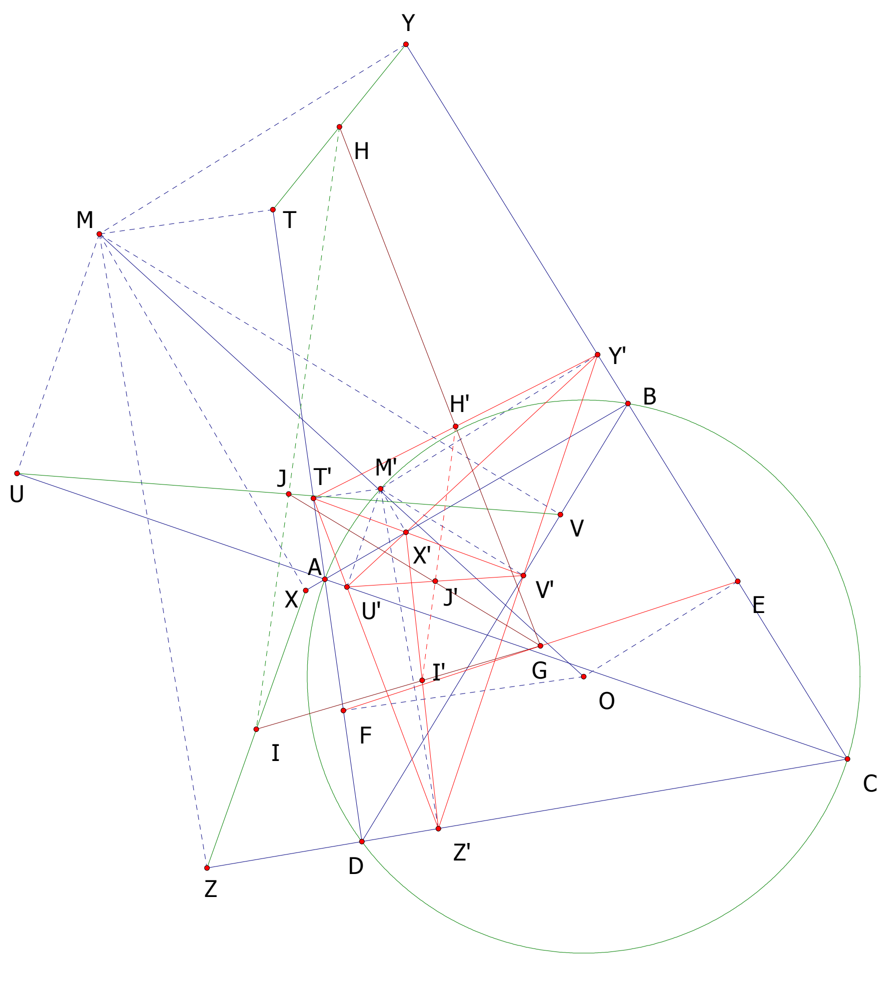


+ Case 3: M not coincide O and not lies on (O).First of all,We need a lemma

Lemma: Given 2 line $\triangle_1; \triangle_2 . A_1, B_1, C_1 \epsilon \triangle_1; A_2, B_2, C_2 \epsilon \triangle_2, A_3 \epsilon A_1A_2; B_3 \epsilon B_1B_2; C_3 \epsilon C_1C_2$ such that: $\dfrac{A_1B_1}{A_1C_1} = \dfrac{A_2B_2}{A_2C_2}$ and $\dfrac{A_3A_1}{A_3A_2} = \dfrac{B_3B_1}{B_3B_2} = \dfrac{C_3C_1}{C_3C_2}$

Then $\overline{A_3, B_3, C_3}$ and $\dfrac{A_3B_3}{A_3C_3} = \dfrac{A_2B_2}{A_2C_2} = \dfrac{A_1B_1}{A_1C_1}$ (I use directed length)

BACK TO OUR PROBLEM



+Let OM meet (O) at M'.Call X', Y', Z', T', U', V' are the projections of M' onto AB, BC, CD, DA, AC, BD.For the same reason at Case 2,We'll have I', J', H' are collinear (With I', J', H' are the midpoint of $X'Z', U'V', Y'T'$ respectively

+Let $E; F$ be the midpoint of $BC; AD$ respectively and G be the centroid of quadrilateral $ABCD \Rightarrow G$ is the midpoint of EF.We'll have :

+$\frac{YY'}{YE} = \frac{MM'}{MO} = \frac{TT'}{TF}$.Applying Lemma above we'll get $\overline{H, H', G}$ and $\frac{GH'}{GH} = \frac{EY'}{EY} = \frac{OM'}{OM} = k$

+Anagolously, We'll get $\overline{I, I', G}; \overline{J, J', G}$ and $\frac{GI'}{GI} = \frac{GJ'}{GJ} = \frac{GH'}{GH} = k(i)$.Morever, $\overline{I', J', H'}(ii)$

+From $(i); (ii) \Rightarrow \overline{I, J, H}$(QED)

LAST FAREWELL