At least $|S|^{k-1} \cdot (|S|-1)$ frontiers: a graph theory problem

Darij Grinberg

October 18, 2015 (modified version of a note from 2009)

1. Problem

Let S be a finite set. Let k be a positive integer. Let A be a subset of S^k satisfying $|A| = |S|^{k-1}$. Let $B = S^k \setminus A$.

For every $v \in S^k$ and every $i \in \{1, 2, \ldots, k\}$, we denote by v_i the i-th component of the k-tuple v (remember that v is an element of S^k, that is, a k-tuple of elements of S). Then, every $v \in S^k$ satisfies $v = (v_1, v_2, \ldots, v_k)$.

Let F be the set of all pairs $(a, b) \in A \times B$ for which there exists an $i \in \{1, 2, \ldots, k\}$ satisfying $(a_j = b_j \text{ for all } j \neq i)$ [1]. (Speaking less formally, let F be the set of all pairs $(a, b) \in A \times B$ for which the k-tuples a and b differ in at most one position.)

Prove that $|F| \geq |S|^{k-1} \cdot (|S| - 1)$.

2. Remark

In the case $|S| = 2$, this is an old problem (which appeared, for example, in a Moscow MO 1962 preparation booklet, and which is a particular case of Cheeger's inequality for the hypercube).

I use to call the elements of F "frontiers" between the sets A and B.

3. Solution

Since $A \subseteq S^k$ and $B = S^k \setminus A$, we have $A \cap B = \varnothing$ and $A \cup B = S^k$. Hence, $S^k \setminus B = A$.

Define a map $\phi : A \times B \to F$ as follows:

[1] Of course, "for all $j \neq i$" means "for all $j \in \{1, 2, \ldots, k\}$ satisfying $j \neq i$" here.

Let $(u,v) \in A \times B$ be a pair. Then, $u \in A$ and $v \in B$, so that $u \notin B$ and $v \notin A$ (since $A \cap B = \varnothing$).

We define a subset T of $\{0,1,2,\ldots,k\}$ by

$$T = \{i \in \{0,1,2,\ldots,k\} \mid (v_1, v_2, \ldots, v_i, u_{i+1}, u_{i+2}, \ldots, u_k) \in B\}$$

[2]. Then, $0 \notin T$ (since $(u_1,u_2,\ldots,u_k) = u \notin B$) and $k \in T$ (since $(v_1,v_2,\ldots,v_k) = v \in B$). In particular, $k \in T$ yields $T \neq \varnothing$. Thus, the set T has a minimal element (since T is a finite set). Let α be this minimal element. Then, $\alpha \in T$ and $\alpha - 1 \notin T$. We have $\alpha \neq 0$ (since $\alpha \in T$ but $0 \notin T$). Thus, $\alpha - 1 \in \{0,1,2,\ldots,k\}$ (since $\alpha \in T \subseteq \{0,1,2,\ldots,k\}$).

Now, $\alpha \in T$ yields $(v_1,v_2,\ldots,v_\alpha,u_{\alpha+1},u_{\alpha+2},\ldots,u_k) \in B$, while $\alpha - 1 \notin T$ yields $(v_1,v_2,\ldots,v_{\alpha-1},u_\alpha,u_{\alpha+1},\ldots,u_k) \notin B$, so that $(v_1,v_2,\ldots,v_{\alpha-1},u_\alpha,u_{\alpha+1},\ldots,u_k) \in S^k \setminus B = A$. Set $a = (v_1,v_2,\ldots,v_{\alpha-1},u_\alpha,u_{\alpha+1},\ldots,u_k)$ and $b = (v_1,v_2,\ldots,v_\alpha,u_{\alpha+1},u_{\alpha+2},\ldots,u_k)$. Then, $a = (v_1,v_2,\ldots,v_{\alpha-1},u_\alpha,u_{\alpha+1},\ldots,u_k) \in A$ and $b = (v_1,v_2,\ldots,v_\alpha,u_{\alpha+1},u_{\alpha+2},\ldots,u_k) \in B$, so that $(a,b) \in A \times B$. Besides, there exists an $i \in \{1,2,\ldots,k\}$ satisfying $\left(a_j = b_j \text{ for all } j \neq i\right)$ (namely, $i = \alpha$ [3]). Hence, $(a,b) \in F$ (by the definition of F).

Now set $\phi(u,v) = (a,b)$. Thus we have defined a map $\phi : A \times B \to F$.

Next, we will prove that $\left|\phi^{-1}(\{(a,b)\})\right| \leq |S|^{k-1}$ for every $(a,b) \in F$. In fact, let $(a,b) \in F$. Since $(a,b) \in F$, we have $(a,b) \in A \times B$, so that $a \in A$ and $b \in B$, so that $a \neq b$ (since $A \cap B = \varnothing$). But since $(a,b) \in F$,

$$\text{there exists an } i \in \{1,2,\ldots,k\} \text{ satisfying } \left(a_j = b_j \text{ for all } j \neq i\right). \tag{1}$$

Consider this i.

We must have $a_i \neq b_i$ [4].

Now, consider some $(u,v) \in \phi^{-1}(\{(a,b)\})$. Then, $\phi(u,v) = (a,b)$. Thus, by the definition of ϕ, there exists an $\alpha \in \{0,1,2,\ldots,k\}$ such that

$$a = (v_1,v_2,\ldots,v_{\alpha-1},u_\alpha,u_{\alpha+1},\ldots,u_k) \qquad \text{and} \qquad b = (v_1,v_2,\ldots,v_\alpha,u_{\alpha+1},u_{\alpha+2},\ldots,u_k). \tag{2}$$

Consider this α.

[2] For $i = 0$, the notation $(v_1,v_2,\ldots,v_i,u_{i+1},u_{i+2},\ldots,u_k)$ means $(u_1,u_2,\ldots,u_k)$.
For $i = k$, the notation $(v_1,v_2,\ldots,v_i,u_{i+1},u_{i+2},\ldots,u_k)$ means $(v_1,v_2,\ldots,v_k)$.

[3] In fact, $a_j = b_j$ for all $j \neq \alpha$ (in fact, for any j, we have $a_j = \begin{cases} v_j, & \text{if } j < \alpha; \\ u_j, & \text{if } j \geq \alpha \end{cases}$ and $b_j = \begin{cases} v_j, & \text{if } j \leq \alpha; \\ u_j, & \text{if } j > \alpha \end{cases}$; thus, if $j \neq \alpha$, this simplifies to $a_j = \begin{cases} v_j, & \text{if } j < \alpha; \\ u_j, & \text{if } j > \alpha \end{cases}$ and $b_j = \begin{cases} v_j, & \text{if } j < \alpha; \\ u_j, & \text{if } j > \alpha \end{cases}$, so that $a_j = b_j$ for all $j \neq \alpha$).

[4] In fact, otherwise, we would have $a_i = b_i$, what, combined with $a_j = b_j$ for all $j \neq i$, would yield $a_j = b_j$ for all $j \in \{1,2,\ldots,k\}$, so that $a = b$, contradicting $a \neq b$.

We must have $u_\alpha \neq v_\alpha$ [5]. Since $a_\alpha = u_\alpha$ and $b_\alpha = v_\alpha$, this yields $a_\alpha \neq b_\alpha$. Hence, $\alpha = i$ (since otherwise, we would have $\alpha \neq i$, so that $a_\alpha = b_\alpha$ by (1), contradicting $a_\alpha \neq b_\alpha$). Thus, (2) becomes

$$a = (v_1, v_2, \ldots, v_{i-1}, u_i, u_{i+1}, \ldots, u_k) \qquad \text{and} \qquad b = (v_1, v_2, \ldots, v_i, u_{i+1}, u_{i+2}, \ldots, u_k).$$

Now, $a = (v_1, v_2, \ldots, v_{i-1}, u_i, u_{i+1}, \ldots, u_k)$ yields $a_j = u_j$ for all $j \geq i$. Hence,

$$\begin{aligned} u &= (u_1, u_2, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_k) = (u_1, u_2, \ldots, u_{i-1}, a_i, a_{i+1}, \ldots, a_k) \\ &\in S^{i-1} \times \{a_i\} \times \{a_{i+1}\} \times \cdots \times \{a_k\}. \end{aligned} \tag{3}$$

[6]

Also, $b = (v_1, v_2, \ldots, v_i, u_{i+1}, u_{i+2}, \ldots, u_k)$ yields $b_j = v_j$ for all $j \leq i$. Hence,

$$\begin{aligned} v &= (v_1, v_2, \ldots, v_i, v_{i+1}, v_{i+2}, \ldots, v_k) = (b_1, b_2, \ldots, b_i, v_{i+1}, v_{i+2}, \ldots, v_k) \\ &\in \{b_1\} \times \{b_2\} \times \cdots \times \{b_i\} \times S^{k-i}. \end{aligned} \tag{4}$$

[7]

By (3) and (4), we have

$$(u, v) \in \left(S^{i-1} \times \{a_i\} \times \{a_{i+1}\} \times \cdots \times \{a_k\}\right) \times \left(\{b_1\} \times \{b_2\} \times \cdots \times \{b_i\} \times S^{k-i}\right)$$

for every $(u, v) \in \phi^{-1}(\{(a, b)\})$. Thus,

$$\begin{aligned} &\left|\phi^{-1}(\{(a, b)\})\right| \\ &\leq \left|\left(S^{i-1} \times \{a_i\} \times \{a_{i+1}\} \times \cdots \times \{a_k\}\right) \times \left(\{b_1\} \times \{b_2\} \times \cdots \times \{b_i\} \times S^{k-i}\right)\right| \\ &= \left|S^{i-1} \times \{a_i\} \times \{a_{i+1}\} \times \cdots \times \{a_k\}\right| \cdot \left|\{b_1\} \times \{b_2\} \times \cdots \times \{b_i\} \times S^{k-i}\right| \\ &= \left(\underbrace{\left|S^{i-1}\right|}_{=|S|^{i-1}} \cdot \underbrace{|\{a_i\}|}_{=1} \cdot \underbrace{|\{a_{i+1}\}|}_{=1} \cdot \cdots \cdot \underbrace{|\{a_k\}|}_{=1}\right) \cdot \left(\underbrace{|\{b_1\}|}_{=1} \cdot \underbrace{|\{b_2\}|}_{=1} \cdot \cdots \cdot \underbrace{|\{b_i\}|}_{=1} \cdot \underbrace{\left|S^{k-i}\right|}_{=|S|^{k-i}}\right) \\ &= |S|^{i-1} \cdot |S|^{k-i} = |S|^{k-1} \end{aligned} \tag{5}$$

---

[5]because otherwise, we would have $u_\alpha = v_\alpha$ and thus

$$a = \left(v_1, v_2, \ldots, v_{\alpha-1}, \underbrace{u_\alpha}_{=v_\alpha}, u_{\alpha+1}, \ldots, u_k\right) = (v_1, v_2, \ldots, v_\alpha, u_{\alpha+1}, u_{\alpha+2}, \ldots, u_k) = b,$$

contradicting $a \neq b$.

[6]By abuse of notation, we are writing $S^{i-1} \times \{a_i\} \times \{a_{i+1}\} \times \cdots \times \{a_k\}$ for $\underbrace{S \times S \times \cdots \times S}_{i-1 \text{ factors}} \times \{a_i\} \times \{a_{i+1}\} \times \cdots \times \{a_k\}$ here.

[7]By abuse of notation, we are writing $\{b_1\} \times \{b_2\} \times \cdots \times \{b_i\} \times S^{k-i}$ for $\{b_1\} \times \{b_2\} \times \cdots \times \{b_i\} \times \underbrace{S \times S \times \cdots \times S}_{k-i \text{ factors}}$ here.

for every $(a,b) \in F$.

Thus,

$$\begin{aligned}
|A \times B| &= \sum_{(a,b)\in F} |\{(u,v) \in A \times B \mid \phi(u,v) = (a,b)\}| \\
&= \sum_{(a,b)\in F} \underbrace{\left|\phi^{-1}(\{(a,b)\})\right|}_{\substack{\leq |S|^{k-1} \\ \text{(by (5))}}} \leq \sum_{(a,b)\in F} |S|^{k-1} \\
&= |F| \cdot |S|^{k-1}.
\end{aligned}$$

But

$$\begin{aligned}
|A \times B| &= |A| \cdot |B| = |A| \cdot \left|S^k \setminus A\right| = |A| \cdot \left(\left|S^k\right| - |A|\right) \\
&= |S|^{k-1} \cdot \left(\left|S^k\right| - |S|^{k-1}\right) = |S|^{k-1} \cdot \left(|S|^k - |S|^{k-1}\right),
\end{aligned}$$

so this becomes

$$|S|^{k-1} \cdot \left(|S|^k - |S|^{k-1}\right) \leq |F| \cdot |S|^{k-1},$$

thus $|S|^k - |S|^{k-1} \leq |F|$, so that

$$|F| \geq |S|^k - |S|^{k-1} = |S|^{k-1} \cdot (|S|-1),$$

qed.