Specialized Discrete Continuity

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Abstract

In this paper we talk about **peaks** in the graph of a discrete function, and how to use it to solve complex problems. This is inspired by the widely used **discrete continuity**, and hence the name. First we formally define the notion of a peak. After that, we prove some lemmas regarding the existence of a peak. Then we solve two problems from the IMO shortlist using the lemmas. Information about the peak is often vital because it gives us insight on the behaviour of the function.

1 Introduction

Most olympiad mathematics problems are quite easy to understand. The solutions also seem very easy when seen. But rarely are these problems easy to solve. One reason of this is because an olympiad problem often have various ways to meddle with. Most of the approaches are fruitless. A veteran in olympiad problem solving will tell you that they rely on certain techniques, certain ways that are often fruitful. We discuss one such technique in this paper. Which is looking at peaks in the graph of a discrete function.

For a function f where the domain is $\mathbb{N}$, we say n is a peak of the function, if

1. $\Delta f(n-1) > 0$
2. $\Delta f(n) \leq 0$

Where $\Delta f(n) = f(n+1) - f(n)$.

Information about a peak is sometimes vital in problem solving as peaks give many unique properties. In the following sections, we will prove some lemmas and use the lemmas in problems from IMO and IMOSL.

2 Lemmas

We will prove some lemmas here which involve the existence of a peak.

Lemma 1: For a function $f : \mathbb{N} \to \mathbb{R}$, if there exists $a, b \in \mathbb{N}$ so that $a < b$, $\Delta f(a) > 0$ and $\Delta f(b) < 0$, there exists a peak in the set $S = \{x | x \in \mathbb{N}, a < x \leq b\}$.

Lemma 2: For a function $f : \mathbb{N} \to \mathbb{R}$, if there exists $a, b \in \mathbb{N}$ so that $a < b$ and $f(a) < f(b)$, then there exists $c \in S$ so that $\Delta f(c) > 0$ where $S = \{x | x \in \mathbb{N}, a \leq x < b\}$.

Note that if we had $f(a) > f(b)$ in the statement, then we would've gotten a $c \in S$ so that $\Delta f(c) < 0$.

Lemma 3: For a function $f : \mathbb{N} \to \mathbb{R}$, if there are exists infinite sets $A, B \subset \mathbb{N}$ where $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$, so that for all $i \in \mathbb{N}$, $\Delta f(a_i) > 0$ and $\Delta f(b_i) < 0$, then there are infinitely many peaks.

Note that **Lemma 3** is the infinite version of **Lemma 1**.
The proofs of these lemmas are left to the readers.

3 Problems

Now we show some applications of the lemmas in some problems from mathematical Olympiads.

The key insight to these problems is directly intertwined with our discussed technique.

3.1 2013 IMO Shortlist N3 [1]

The Statement: Prove that there exist infinitely many positive integers n such that the largest prime divisor of n^4+n^2+1 is equal to the largest prime divisor of $(n+1)^4+(n+1)^2+1$.

At first glance, the problem doesn't seem to bear any clue to our discussed lemmas. So we venture a bit. We notice that we're dealing with prime numbers here. This gives us the intuition that factorizing might give us a better insight into the problem. And the expression $n^4 + n^2 + 1$ factorizes nicely.

Let us define $f(x) = x(x+1)+1$. We observe that $n^4+n^2+1 = f(n^2) = f(n)f(n-1)$. One can easily show that $f(n-1)$ and $f(n)$ is coprime. So, the largest prime divisor of n^4+n^2+1 must be the largest prime divisor of either $f(n) = n(n + 1) + 1$ or $f(n - 1) = n(n - 1) + 1$. In more technical terms,

$$g(n^4 + n^2 + 1) = g(f(n^2)) = \max\{g(f(n)), g(f(n - 1))\} \tag{1}$$

where $g(x)$ is the greatest prime divisor of x. Note that for $n \in \mathbb{N}$, $f(n) > 1$ and $f(n) \in \mathbb{N}$, so $g(f(n))$ is defined.

Now, we need to prove that there exists infinitely many n such that

$$\max\{g(f(n)), g(f(n - 1))\} = \max\{g(f(n + 1)), g(f(n))\}$$

Note that since $f(n)$ and $f(n + 1)$ are always co-prime and greater than 1, $\Delta g(f(n)) \neq 0$. We see that the if we can prove that there are infinitely many 'peaks' in the graph of $g(f(x))$, we immediately solve the problem since $\max\{g(f(n)), g(f(n-1))\} = \max\{g(f(n+1)), g(f(n))\} = g(f(n))$ for a peak n.

From **Schur's theorem**[2], which states that "For every polynomial $P(x)$ with integer coefficients, there are infinitely many primes that divide some member of the set $\{P(n) \neq 0 : n \in \mathbb{N}\}$", $f(n)$ has infinitely many prime divisors. So $g(f(n))$ is unbounded above. So $\Delta g(f(n)) > 0$ for infinitely many n.

From (1) we can say that $g(f(n^2)) = g(f(n))$ or $g(f(n^2)) = g(f(n-1))$ depending on which one is larger. Let the larger one be $k = n$ if $g(f(n))$ is larger and $k = n-1$ otherwise. Now, we know that when $n > 1$ we have $k + 1 < n^2$. Also, $g(f(k)) = g(f(n^2))$. Now, we let $q = k + 1$. Since $f(a)$ and $f(a + 1)$ are co-prime, $g(f(q)) \neq g(f(k))$. If $g(f(q)) < g(f(k))$, Let $c = q$. If $g(f(q)) > g(f(k)) = g(f(n^2))$, since $q < n^2$, By using **Lemma 2**, we get that there is an integer m so that $q \leq m < n^2$ so that $\Delta g(f(m)) < 0$. We let $c = m$ in this case. So for every n, We get a $c \in \mathbb{N}$ So that $\Delta g(f(c)) < 0$ and $n - 2 < c < n^2$. So $\Delta g(f(n)) < 0$ for infinitely many n.

By using **Lemma 3**, We see that there are infinitely many peaks. Hence the problem is solved.

3.2 Based on 2006 IMO Shortlist N3 [3]

The Statement: We define a sequence $(f(1), f(2), f(3), \ldots)$ by

$$f(n) = \frac{1}{n}\left(\left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor\right),$$

where $\lfloor x \rfloor$ denotes the integer part of x.
Prove that there are infinitely many n so that $f(n+1) \geq f(n)$ and $f(n-1) \geq f(n)$.

Note that, the statement is essentially a weaker version of a peak. So, we just need to find infinitely many peaks.

First we do some experimentation. One pattern immediately jumps out here and that is $f(2n) > f(n)$. So we try to prove this.

$$\begin{aligned} f(2n) &= \frac{1}{2n}\left(\left\lfloor \frac{2n}{1} \right\rfloor + \left\lfloor \frac{2n}{2} \right\rfloor + \cdots + \left\lfloor \frac{2n}{n} \right\rfloor + \left\lfloor \frac{2n}{n+1} \right\rfloor + \cdots + \left\lfloor \frac{2n}{2n} \right\rfloor\right) \\ &= \frac{1}{2n}\left(\left\lfloor \frac{2n}{1} \right\rfloor + \left\lfloor \frac{2n}{2} \right\rfloor + \cdots + \left\lfloor \frac{2n}{n} \right\rfloor + n\right) \end{aligned}$$

Now we prove the following lemma: $\left\lfloor \frac{2x}{y} \right\rfloor \geq 2\left\lfloor \frac{x}{y} \right\rfloor$

It is well known that[4] $\lfloor a+b \rfloor \geq \lfloor a \rfloor + \lfloor b \rfloor$. So, we have $\left\lfloor \frac{2x}{y} \right\rfloor \geq 2\left\lfloor \frac{x}{y} \right\rfloor$

Using this lemma, we get,

$$f(2n) = \frac{1}{2n}\left(\left\lfloor \frac{2n}{1} \right\rfloor + \left\lfloor \frac{2n}{2} \right\rfloor + \cdots + \left\lfloor \frac{2n}{n} \right\rfloor + n\right) \geq \frac{1}{2n}\left(2\left\lfloor \frac{n}{1} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor + \cdots + 2\left\lfloor \frac{n}{n} \right\rfloor + n\right)$$

$$= \frac{1}{n}\left(\left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor\right) + \frac{1}{2} = f(n) + \frac{1}{2} > f(n)$$

So we have proved that $f(2n) > f(n)$ for all $n \in \mathbb{N}$. By using **Lemma 2** we get that for each $n \geq 2$ there exists $n \leq x < 2n$ such that $\Delta f(x) > 0$. This implies that there are infinitely many x so that $\Delta f(x) > 0$.

Another observation shows that $f(p) < f(p-1)$ where p is a prime number greater than 5. We will try to prove this observation now.

Note that $\left\lfloor \frac{p}{x} \right\rfloor = \left\lfloor \frac{p-1}{x} \right\rfloor$ for $x \in \{2, 3, 4, \ldots, p-1\}$. So,

$$\begin{aligned} f(p) &= \frac{1}{p}\left(\left\lfloor \frac{p}{1} \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor + \cdots + \left\lfloor \frac{p}{p} \right\rfloor\right) \\ &= \frac{p+1}{p} + \frac{1}{p}\left(\left\lfloor \frac{p-1}{2} \right\rfloor + \left\lfloor \frac{p-1}{3} \right\rfloor + \cdots + \left\lfloor \frac{p-1}{p-1} \right\rfloor\right) \\ &= \frac{p+1}{p} + \frac{1}{p}\left(\left\lfloor \frac{p-1}{1} \right\rfloor + \left\lfloor \frac{p-1}{2} \right\rfloor + \left\lfloor \frac{p-1}{3} \right\rfloor + \cdots + \left\lfloor \frac{p-1}{p-1} \right\rfloor - \left\lfloor \frac{p-1}{1} \right\rfloor\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{p+1}{p} + \frac{1}{p}\left((p-1)f(p-1) - (p-1)\right) \\
&= \frac{2+(p-1)f(p-1)}{p}
\end{aligned}
$$

So we have $f(p) = \dfrac{2+(p-1)f(p-1)}{p}$. Let's assume for contradiction that $f(p) \geq f(p-1)$.

$$
\begin{aligned}
&\implies \frac{2+(p-1)f(p-1)}{p} \geq f(p-1) \\
&\implies 2+(p-1)f(p-1) \geq pf(p-1) \\
&\implies 2 \geq f(p-1) \\
&\implies 2 \geq \frac{1}{p-1}\left(\left\lfloor \frac{p-1}{1} \right\rfloor + \left\lfloor \frac{p-1}{2} \right\rfloor + \cdots + \left\lfloor \frac{p-1}{p-1} \right\rfloor\right) \\
&\implies 2p-2 \geq \left\lfloor \frac{p-1}{1} \right\rfloor + \left\lfloor \frac{p-1}{2} \right\rfloor + \cdots + \left\lfloor \frac{p-1}{p-1} \right\rfloor \\
&\implies p-1 \geq \left\lfloor \frac{p-1}{2} \right\rfloor + \cdots + \left\lfloor \frac{p-1}{p-1} \right\rfloor
\end{aligned}
$$

The right hand side is the sum of $p-2$ positive integers. So if at least two of the integers is greater than 1, Then the inequality would be false. Note that if $p > 5$, $\left\lfloor \frac{p-1}{2} \right\rfloor \geq 2$ and $\left\lfloor \frac{p-1}{3} \right\rfloor \geq 2$. Which means the assertion is false for $p > 5$. So, we have $\Delta f(p-1) < 0$ by contradiction for every prime more than 5.

Finally, we end our proof with **Lemma 3**. We have proved the existence of infinitely many integers such that Δf is positive. And we also have proved the existence of infinitely many integers such that Δf is negative. So, by **Lemma 3**, we can say there exists infinitely many peaks.

4 Conclusion

As evident from the examples, the main power of the technique lies in **Lemma 3**, by which we can find infinitely many peaks. After the problem has been suitably restructured to a peak finding problem, one just has to find infinitely many integers so that $\Delta f(a) > 0$ and infinitely many integers b so that $\Delta f(b) < 0$, possibly by using **Theorem 2**, and his work would be done. Note that these techniques also work for troughs, in fact in **Problem 2** we can also find infinitely many troughs by using an alternate version of **Lemma 3**.

The paper has principally been focused on the use of discrete continuity outside of its usual roles in combinatorial structures and discrete geometry. For further reading about discrete continuity, the authors would like to suggest **Olympiad Combinatorics**.[olycombi]

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