TWO CIRCLES, EXTERNALLY TANGENT

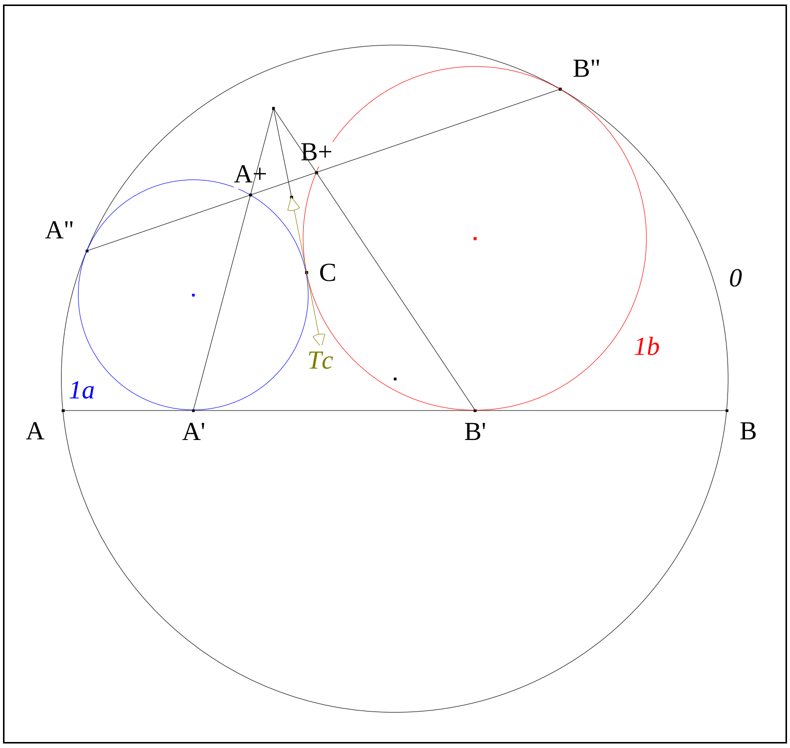
AND

A THIRD CIRCLE COMES IN

AN AESTHETIC PROOF [1]

†

Jean - Louis AYME



Abstract. The author present an aesthetic proof concerning three circles in a special position. The figures are all in general position and all the theorems quoted can be proved synthetically.

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[1] 24/09/2010.

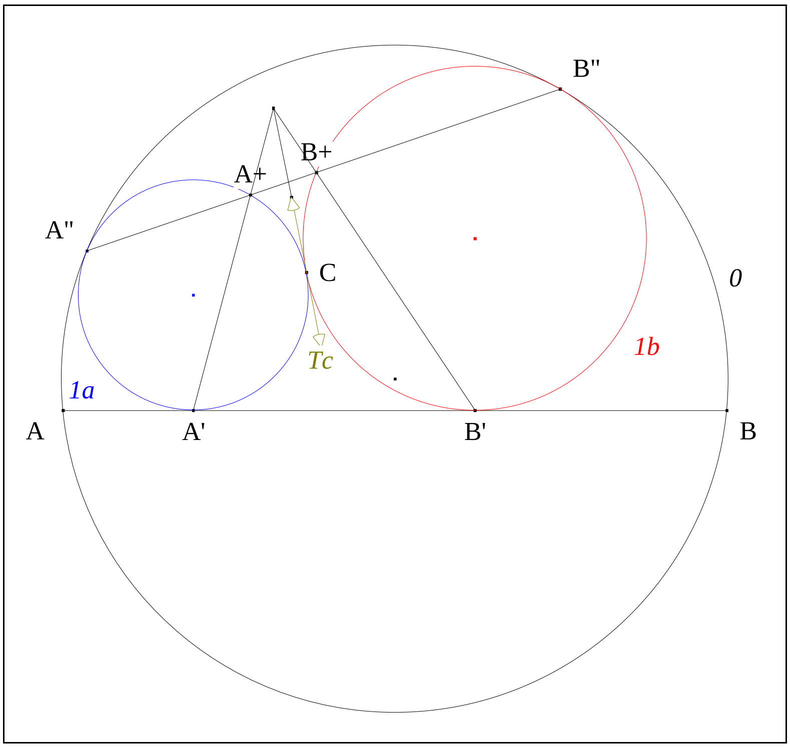
Summary

A. The problem 2
B. Three lemmas 3
1. A problem from the *Leybourn's Mathematical repository*
2. The Monge's theorem or the three circles theorem
3. The five circles theorem
4. A short biography of Henri-Léon Lebesgue
C. An aesthetic proof 11
D. Annexe 12
1. A converse of the pivot theorem
2. Radical axis of two secant circles
3. The Desargues' week theorem

A. THE PROBLEM

VISION

Figure :



Features :	*0*	a circle,
	AB	a chord
	1a	a circle tangent to AB and internally to *0*,
	A', A"	the points of contact of *1a* wrt. AB, *0*,
	1b	the circle tangent to AB, internally to *0*, externally to *1a*,
	A', A", C	the points of contact of *1b* wrt. AB, *0*, *1a*
	Tc	the common tangent to *1a* and *1b* at C
and	A+, B+	the second points of intersection of A"B" wrt. *1a*, *1b*.

Given : A'A+, B'B+ and *Tc* are concurrent.[2]

Historical note : this problem coming from a Greek mathematical competition has been presented in 2006 by Valentin Vornicu, Administrator of the well known site *Art of Problem Solving*.
This problem has been solved in different ways…

B. THREE LEMMAS

1. A problem from the *Leybourn's Mathematical repository*

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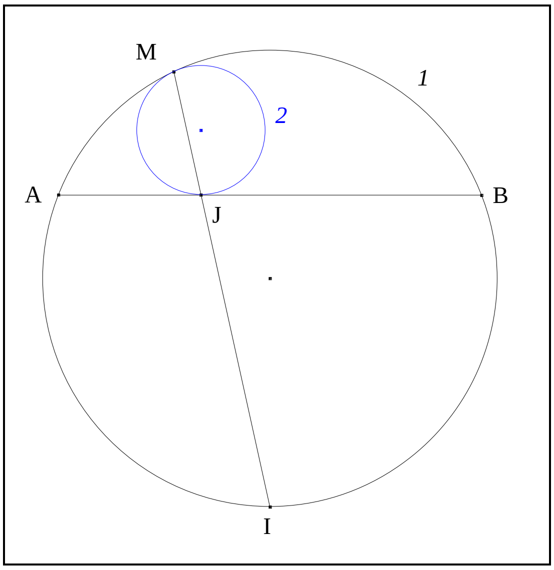

Features :

	1	a circle,
	AB	a chord of *1*,
	2	a circle tangent to AB and internally to *1*,
	J, M	the points of contact of *2* wrt. AB, *1*
and	I	the midpoint of the arch AB which doesn't contains M.

Given : I, J and M are collinear. [3]

VISUALIZATION

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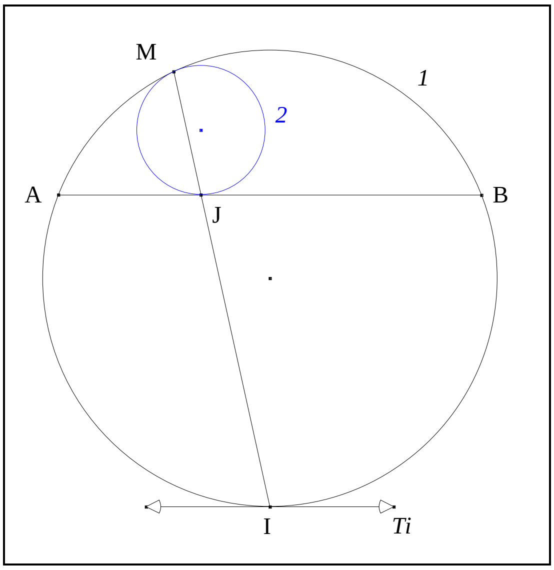
[2] Greece, Two circles, externally tangent and a third circle comes in, *Art of Problem Solving* (24/04/2006) ; http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=85161.

[3] *Leybourn's Mathematical repository* (Nouvelle série) 6 tome I (1811) 209.

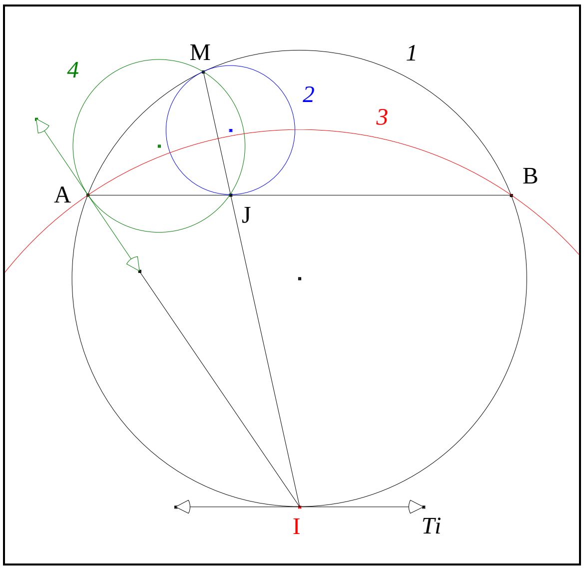


- Note *Ti* the tangent to *1* at I.

- **Conclusion :** the tangent circles *1* et *2*, the basic point M, the monian IMJ, the parallels *Ti* and AB, lead to the Reim's theorem **8'** ; consequently, I, M and J are collinear.

Historical note : this kind of San Gaku[4] can be also found at Aichi prefecture (Japon) on a wooden Tablet dating from 1843 which has vanished to day.

Remarks : **(1)** an orthogonal circle



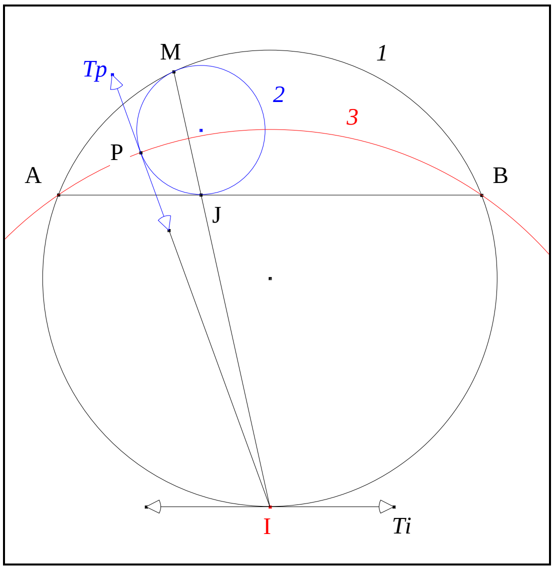
- Note *3* the circle of center I, passing through A
  and *4* the circle passing through A, J and M.

- According to a converse of "The pivot theorem" (Cf. Annexe 1) applied to triangle IJA with the three concurrent circles *1*, *2* and *4*, *4* is tangent to IA at A ;
  in consequence, *3* is orthogonal to *4* .

[4] Fukagawa H., Pedoe D., 1. 6., *Japanese Temple Geometry Problems*, Charles Babbage Research Centre (1989) 14, 88-89.

- **Conclusion :** according to Gaultier "Radical axis of two secant circles" (Cf. Annexe 2), I being on the radical axis MJ of *2* and *4*, *2* is orthogonal to *3*.

(2) A tangent [5]



- Note P one of the two points of intersection of *2* and *3*,
  and *Tp* the tangent to *2* at P.

- **Conclusion :** *2* and *3* being orthogonal, *Tp* passes through I.

2. The Monge's theorem or the three circles theorem

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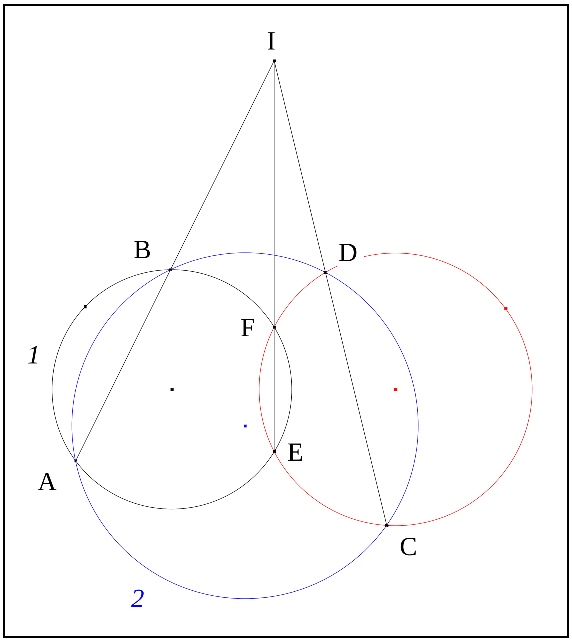
Figure :

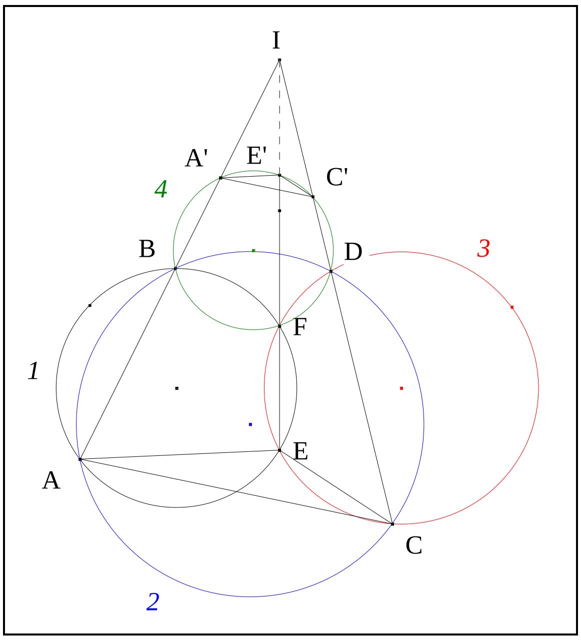
5 Nordic Math Contest (avril 2005) problème 4.



Features :	*1*, *2*	two secant circles,
	A, B	the points of intersection of *1* and *2*,
	C, D	two points on *2*,
	E, F	two points on *1*
et	I	the point of intersection of AB and CD.

Given : C, D, E et F are concyclic *if, and oly if,* EF passes through I.

NECESSARY VISUALIZATION



- Note

	3	the circle passing through C, D, E, F,
	4	the circle passing through B, F, D
et	A', C', E'	the resp. second points of of AB, CD, EF with *4*.

- The circles *2* and *4*, the basic points B and D, the monians ABA' and CDC', lead to the Reim's theorem **0** ; consequently, AC // A'C'.

- The circles *3* and *4*, the basic points D and F, the monians CDC' and EFE', lead to the Reim's theorem **0** ; consequently, CE // C'E'.

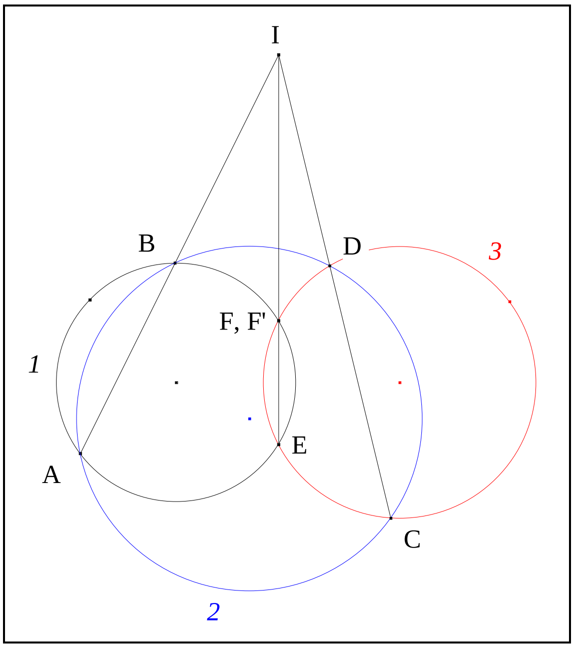
- The circles *1* and *4*, the basic points B and F, the monians ABA' and EFE', lead to the Reim's theorem **0** ; consequently, EA // E'A'.

- According to "The Desargues's week theorem" (Cf. Annexe 3) applied to the homothetic triangles EAC and E'A'C', EE' passes through I.

- **Conclusion :** EF passes through I.

Monge's theorem : *if three circles are secant in pair, then the three intersection lines are concurrent.*

SUFFICIENT VISUALIZATION



- Note *3* the circle passing through C, D, E
  and F' the second point of intersection of *3* and *1*.

- According to the necessary visualization, EF' passes through I.

- F' being on the same time on EI and on *1*, coincides with F.

- **Conclusion :** C, D, E et F are concyclic**.**

Remark : this result holds in case of tangency.

3. The five circles theorem [6]

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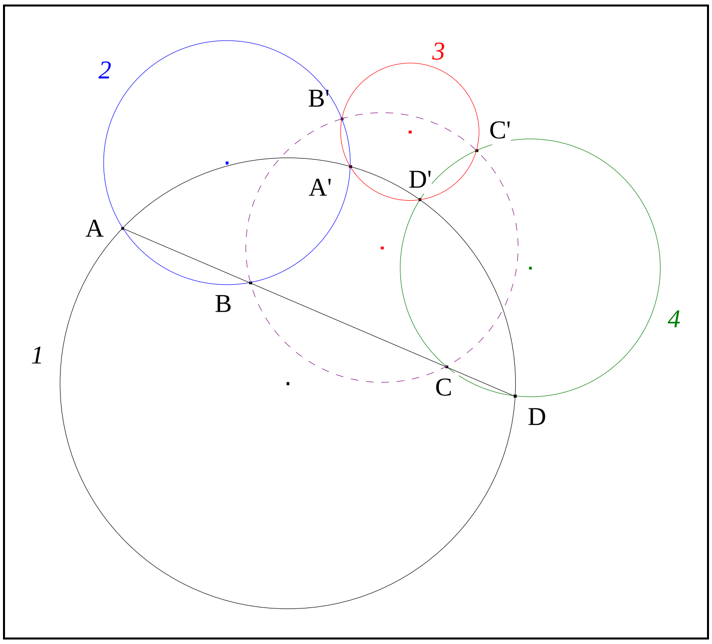
[6] Lebesgue H. L., Sur deux théorèmes de Miquel et de Clifford, *Nouvelles Annales de Mathématiques* (1916).

VISION

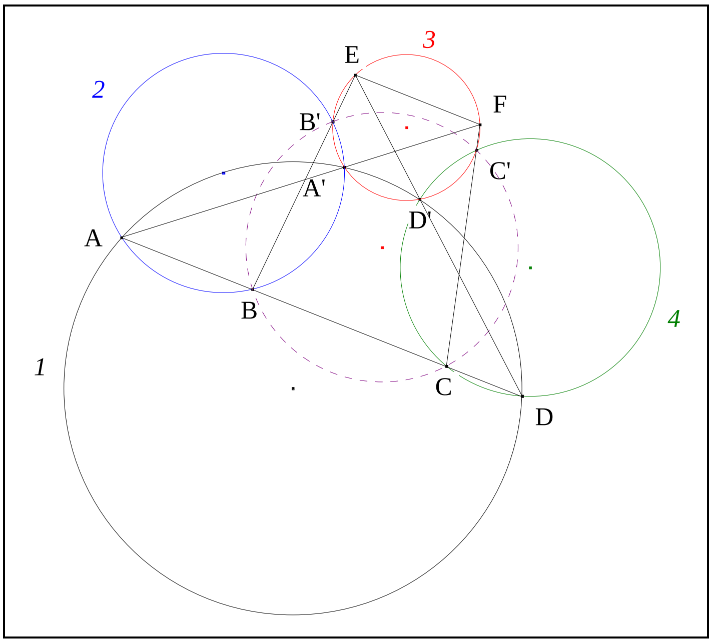
Figure :



Features :	*1*	a circle,
	A, B', D', D	four points in this order on *1*,
	2	a circle passing through A and A',
	B	the second point of intersection of AD with *2*,
	3	a circle passing through A' and D',
	B'	the second point of intersection of *3* and *2*,
	4	a circle passing through D' and D,
	C'	the second point of intersection of *4* and *3*,
and	C	the second point of intersection of AD with *4*.

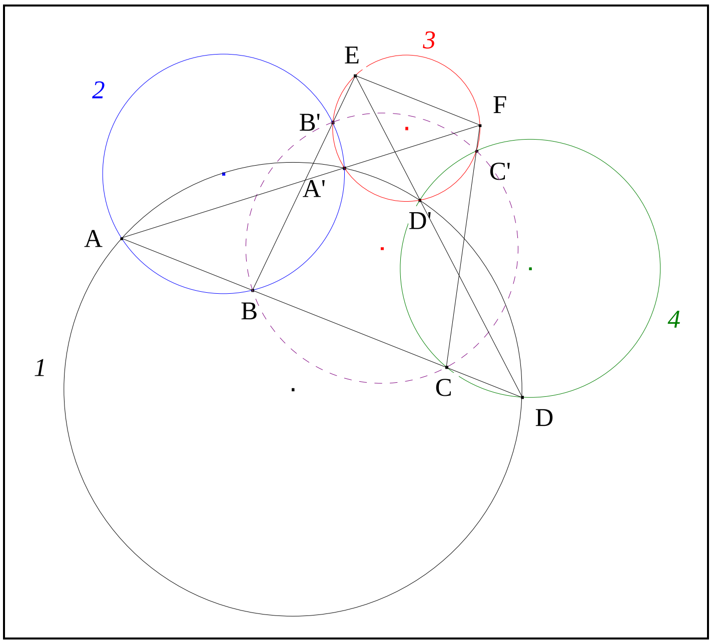
Given : B, C, B' et C' are concyclic.

VISUALIZATION

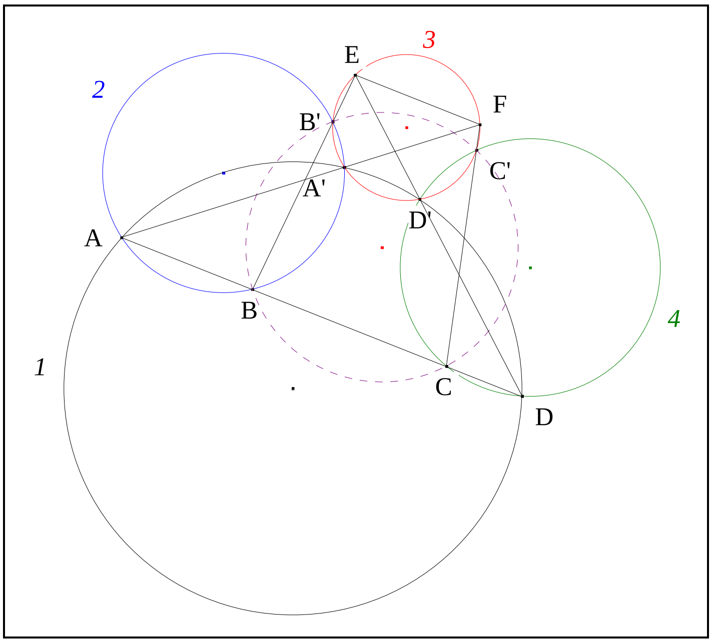


- Note E, F the resp. second points of intersection of BB', CC' with *3*.

- According to a converse of the "The pivot theorem" (Cf. Annexe 1)
  applied to the triangle DBE with the three concurrent circles *1*, *2* and *3*, E, D' and D are collinear.



- According to a converse of the "The pivot theorem" (Cf. Annexe 1)
  applied to the triangle ACF with the three concurrent circles *1*, *3* and *4*, F, A' and A are collinear.



- The circle *1* and *3*, the basic points A' and D', the monians AA'F and DD'E, lead to the Reim's theorem **0** ; consequently, AD is parallel to FE.

- **Conclusion :** the circle *3*, the basic points B' and C', the borning monians EB'B and FC'C, the parallels EF and BC, lead to the Reim's theorem **0"** ; consequently, B', C', B and C are concyclic.

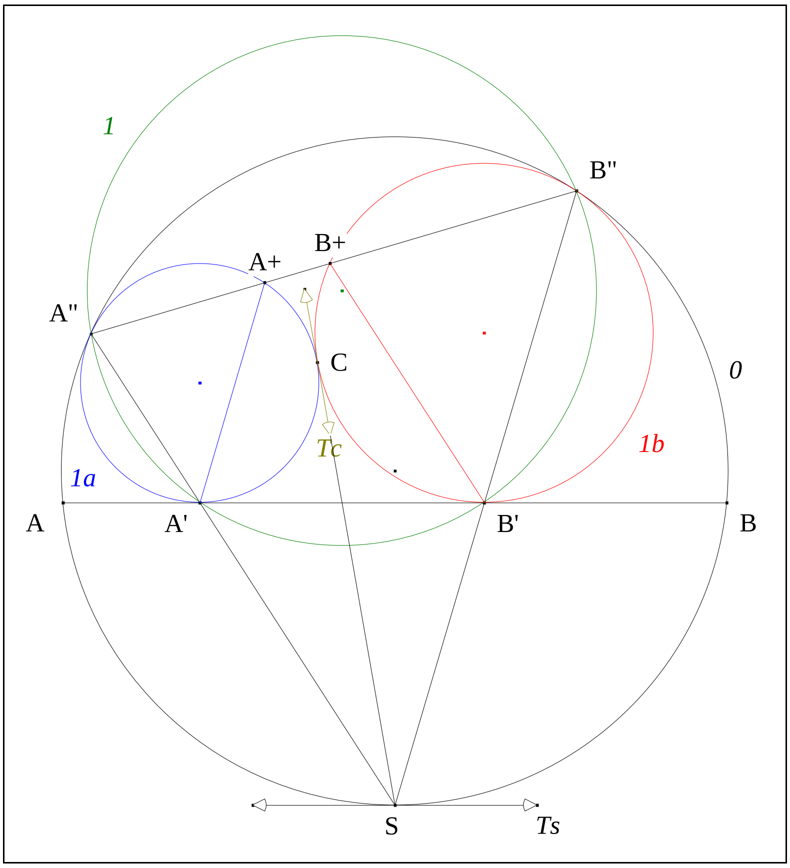
Remark : this result holds in case of tangency.

4. A short biography of Henri-Léon Lebesgue

Henri Lebesgue is born at Beauvais (France), the 28 of June 1875.
At the age of 19 years old, he enters at the Ecole Normale Supérieure. Three years after, he teaches at the central College of Nancy. From 1902 to 1906, he teaches at the University of Rennes, later at Poitiers University which he left in 1910 for the Sorbonne where he became "maître de conference", then professor in 1919. Selected for being professor at the College de France in 1921, he is elected the year after at the Académie des Sciences.
His remarkable works concerning the integration theory and the algebraic topology, did not make him forget his interest in Geometry where he publishes an article entitled *Sur deux théorèmes de Miquel et de Clifford*, in 1916

in the *Nouvelles Annales de Mathématiques*. In this paper, he is the first geometer to make a link between the Miquel's circle and the Clifford's point.
He died in Paris, the 26 of July 1941.

C. AN AESTHETIC PROOF



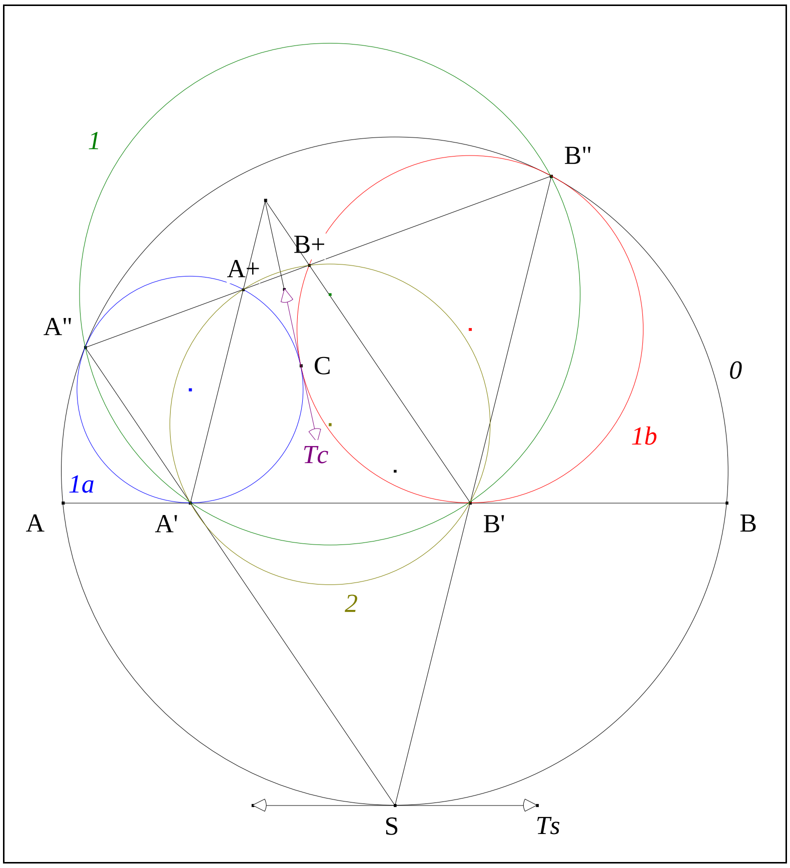
- Note S
  and *Ts* the tangent to *0* en S.

- **Remarks : (2)** *Tc* goes through I
  (1) *Ts* //A'B'.

- According to B. 1. A problem from…, A'A" and B'B" go through S.

- According to B. 2. The Monge's theorem, A", B", A' and B' are concyclic.

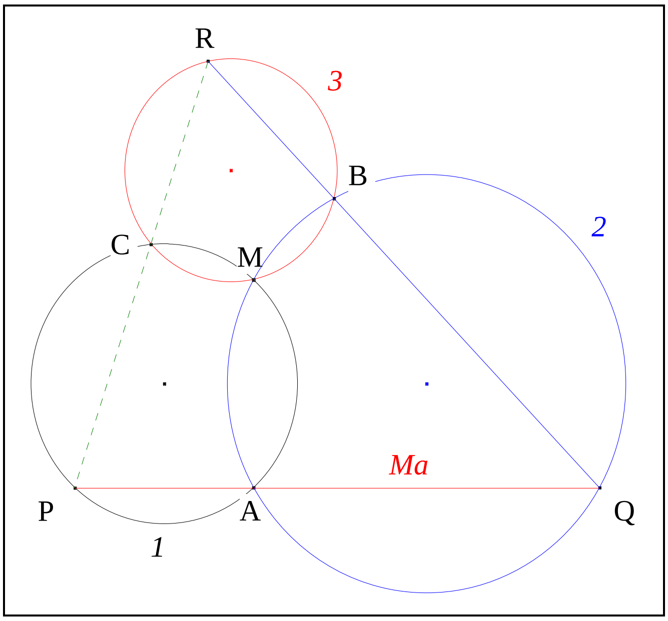
- Note *1* this circle.



- According to B. 3. The five circles theorem, A', B', A+, B+ are concyclic.
- Note 2 this circle.
- **Conclusion :** according to B. 2. The Monge's theorem, A'A+, B'B+ and *Tc* are concurrent.

D. ANNEXE

1. A converse of the pivot theorem

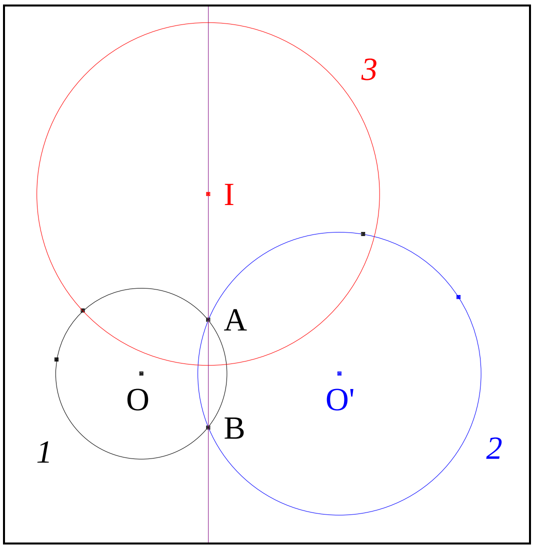


Features : *1*, *2*, *3* three concurrent circles,
M the point of concurs of *1*, *2*, *3*,
A the second point of intersection of *1* and *2*,
Ma a A-monian of *1* and *2*,
P, Q the second points of intersection of *Ma* wrt. *1*, *2*,
B, C the second points of intersection of *3* wrt. *2*, *1*
et R a point of *3*.

Given : QBR is a monian of *2* and *3* *if, and only if,* PCR is a C-monian of *1* and *3*.

Remark : this result holds in case of tangency.

2. Radical axis of two secant circles [7]



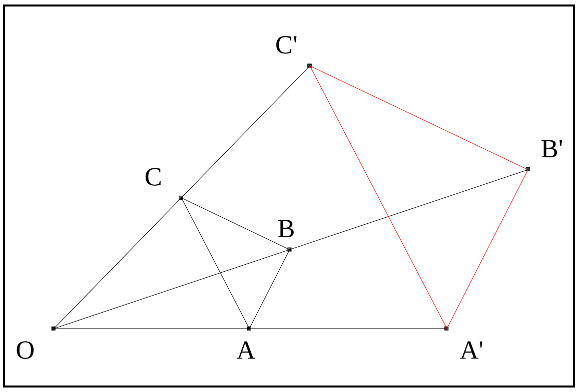
Features : *1*, *2* two secant circles,
O, O' the resp. centers of *1*, *2*,

[7] Gaultier (de Tours) Louis, Les contacts des cercles, *Journal de l'École Polytechnique, Cahier* **16** (1813) 124-214.

	A, B	the points of intersection of *1* et *2*,
	3	an orthogonal circle to *2*
et	I	the center of *3*.

Given : I is on AB *if, and only if,* *3* is orthogonal to *2*.

3. The Desargues's week theorem



Features : ABC a triangle,
and A'B'C' a triangle so that
(1) AA' and BB' are concurrent at O
(2) AB is parallel to A'B'
(3) BC is parallel to B'C'.

Given : CC' goes through O *if, and only if,* AC is parallel to A'C'.