From Baltic Way to Feuerbach - a geometrical excursion

Darij Grinberg

1. Introduction

Though mathematical competitions highly contribute to the popularization of elementary geometry among the mathematicians of tomorrow, the geometry one gets confronted with in olympiads is rather a geometry of problems and tricks than a coherent theory. One solves proposed problems, using a number of more or less known techniques, sometimes generalizing, but generally one is seldomly interested in systematics. Yet, Euclidean geometry is one of the most interconnected fields in olympiad mathematics. An example of such interconnections will be shown in this note. Starting at a relatively easy competition problem, we set out for a trip through the world of elementary geometry. Through a deeper exploration of the configuration, we obtain some additional remarkable results, which lead us to a milestone of triangle geometry - the Feuerbach theorem about the tangency of the nine-point circle of a triangle with its incircle. Finally, we will establish two rather elaborate properties of the point of tangency using the results we obtained during our journey.

Readers wishing to improve their problem-solving skills are invited to consider some of the theorems we will meet below as exercises to prove. Most of them, in fact, allow for various different approaches, and the proofs presented in this note are by far not the only possible ones.

Prerequisites of the journey are some interest in geometry and knowledge not far above the high school level - besides the standard properties of cyclic quadrilaterals, the four basic triangle centers and fundamental properties of similitude transformations, the Euler line and the nine-point circle (also called Euler circle or Feuerbach circle) of a triangle will be used (see [7], §1.7-§1.8, or [8], or lots of other sources the reader could easily find). Furthermore, directed angles modulo 180° will be used throughout the article - this kind of angles is introduced in [4], §1.7 (as *directed angles*), [5] and [6].

Finally, a terminological convention: In the following, when a line g_1, a circle k_1 and a point P_1 will be given, and both the line g_1 and the circle k_1 pass through the point P_1, we will often speak of the "point of intersection of the line g_1 with the circle k_1 different from the point P_1". What this means is clear if the line g_1 and the circle k_1 indeed have two different points of intersection. However, if the line g_1 touches the circle k_1, then this formulation will simply mean the point P_1.

2. A problem from the Baltic Way 1995

We start our journey with a property of triangles which was given as problem 18 at the Baltic Way team contest 1995 ([1], [2], [3] i)):

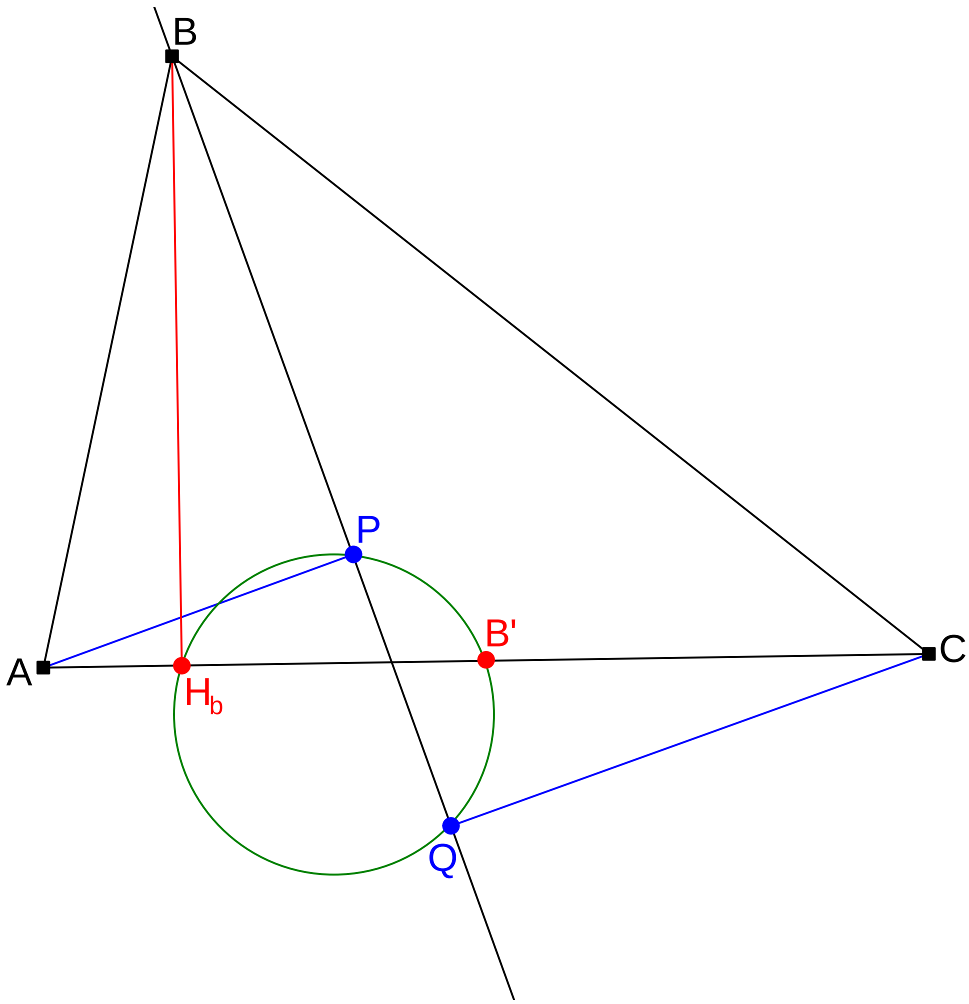

Fig. 1

Theorem 1. Let ABC be a triangle, and B' the midpoint of its side CA. Denote by H_b the foot of the B-altitude of triangle ABC, and by P and Q the orthogonal projections of the points A and C on the bisector of angle ABC. Then, the points H_b, B', P, Q lie on one circle. (See Fig. 1.)

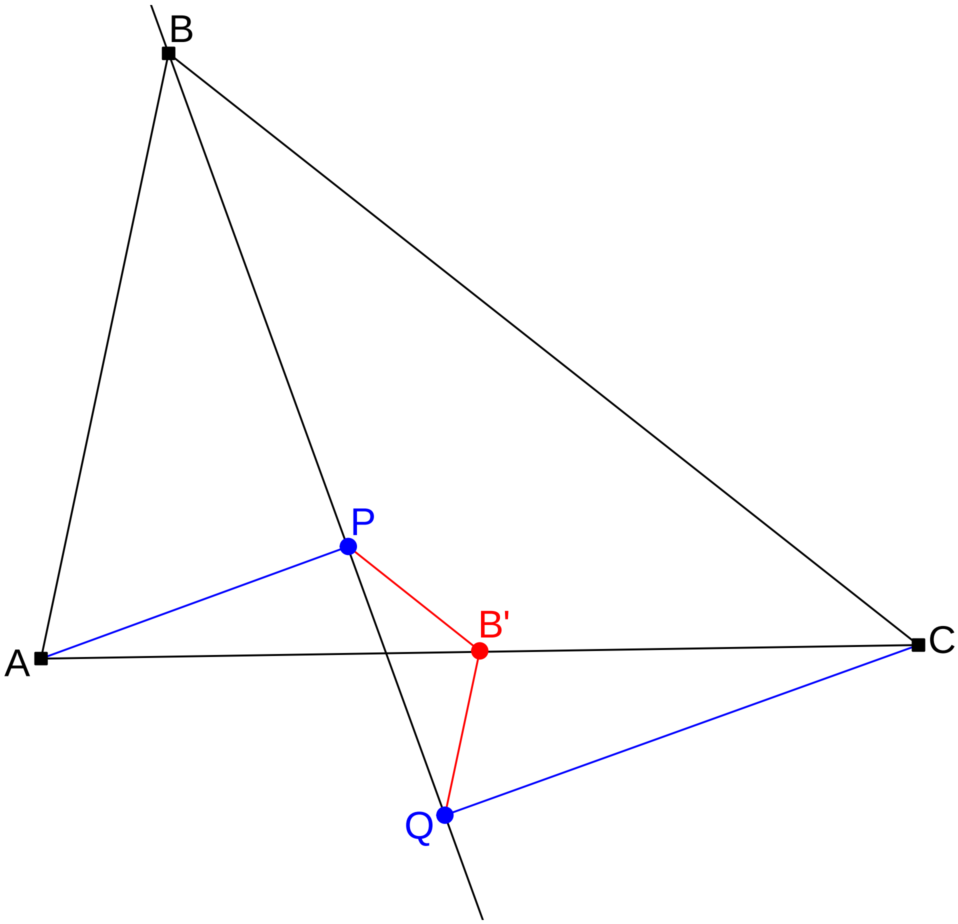

Fig. 2

This is not a tough problem, and different proofs are possible. One of them starts off with a simple, but not uninteresting lemma:

Theorem 2. We have $B'P \parallel BC$ and $B'Q \parallel AB$. (See Fig. 2.)

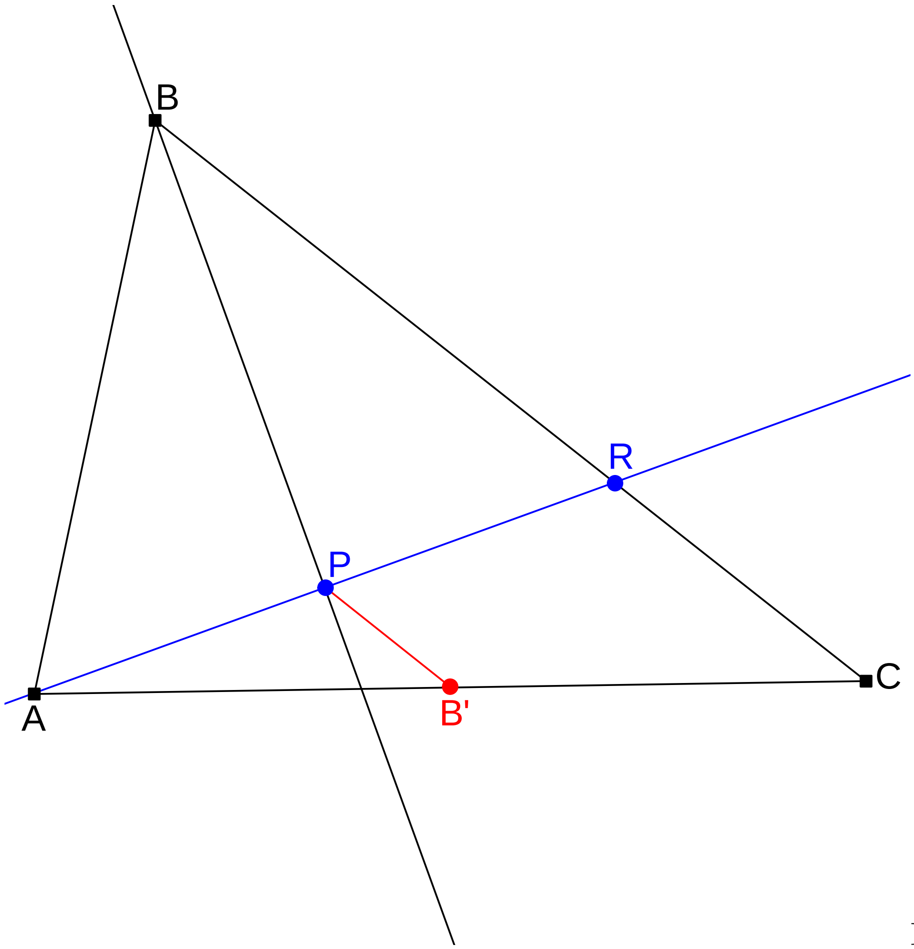

Fig. 3

Proof of Theorem 2. (See Fig. 3.) Let the line AP meet the line BC at a point R. Since the point P lies on the bisector of angle ABC, we have $\measuredangle ABP = -\measuredangle RBP$. Furthermore, $\measuredangle APB = -\measuredangle RPB$, since $\measuredangle APB = 90^\circ$ and $\measuredangle RPB = 90^\circ$, and as we are operating with directed angles modulo 180°, we have $90^\circ = -90^\circ$. From $\measuredangle ABP = -\measuredangle RBP$, $\measuredangle APB = -\measuredangle RPB$ and $BP = BP$, it follows that triangles ABP and RBP are inversely congruent, and thus $AP = PR$. In other words, the point P is the midpoint of the segment AR. On the other hand, the point B' is the midpoint of the segment CA. Thus, the line $B'P$ passes through the midpoints of two sides of triangle CAR, so that $B'P \parallel CR$; equivalently, $B'P \parallel BC$. Similarly, $B'Q \parallel AB$. Theorem 2 is proven.

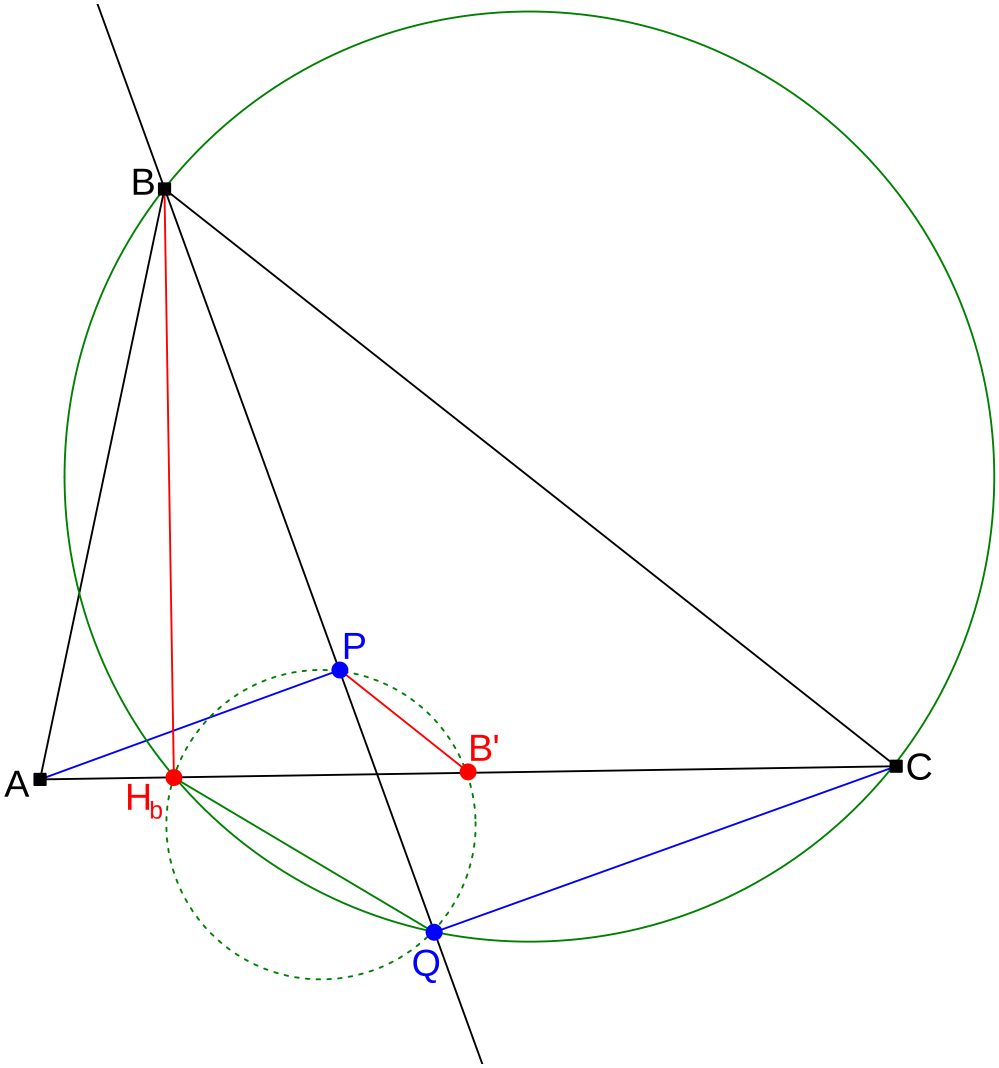

Fig. 4

Now, the *proof of Theorem 1* becomes really easy: (See Fig. 4.)

Since $\measuredangle BH_bC = 90^\circ$ and $\measuredangle BQC = 90^\circ$, the points H_b and Q lie on the circle with diameter BC. Thus, $\measuredangle BQH_b = \measuredangle BCH_b$, what becomes $\measuredangle PQH_b = \measuredangle BCA$. But as Theorem 2 yields $B'P \parallel BC$, we have $\measuredangle (BC;\ CA) = \measuredangle (B'P;\ CA)$, hence $\measuredangle BCA = \measuredangle PB'H_b$. Therefore, $\measuredangle PQH_b = \measuredangle BCA = \measuredangle PB'H_b$, and this shows that the points H_b, B', P, Q are concyclic. This completes the proof of Theorem 1.

Playing around with the configuration, one can come up with two more simple, but notable properties:

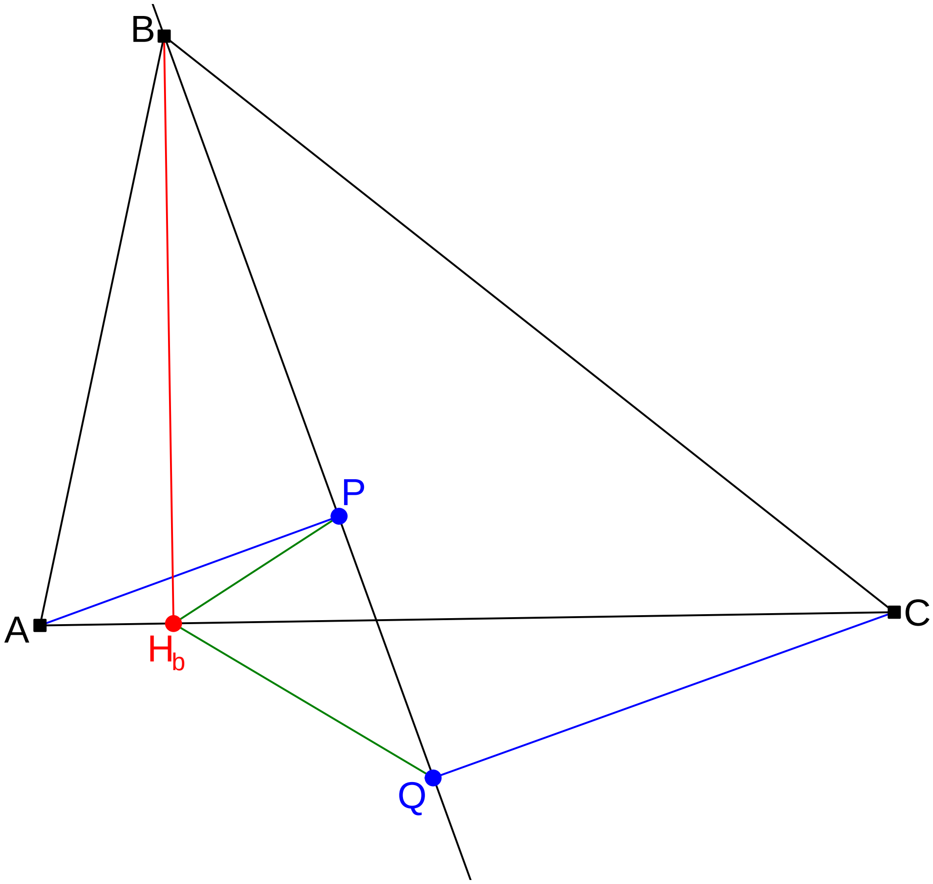

Fig. 5

Theorem 3. The triangles PH_bQ and ABC are inversely similar. (See Fig. 5.)

Proof. As shown in the proof of Theorem 1, we have $\measuredangle PQH_b = \measuredangle BCA$, thus $\measuredangle PQH_b = -\measuredangle ACB$. Similarly, $\measuredangle QPH_b = -\measuredangle CAB$. Thus, the triangles PH_bQ and ABC are inversely similar, what proves Theorem 3.

Theorem 4. We have $B'P = B'Q$. (See Fig. 2.)

Proof. According to Theorem 2, we have $B'P \parallel BC$, so that $\measuredangle (B'P;\ PQ) = \measuredangle (BC;\ PQ)$, and $B'Q \parallel AB$, so that $\measuredangle (PQ;\ B'Q) = \measuredangle (PQ;\ AB)$. But since the line PQ is the bisector of angle ABC, we have $\measuredangle (BC;\ PQ) = \measuredangle (PQ;\ AB)$. Thus, $\measuredangle (B'P;\ PQ) = \measuredangle (PQ;\ B'Q)$. Equivalently, $\measuredangle B'PQ = \measuredangle PQB'$. This shows that the triangle $PB'Q$ is isosceles with $B'P = B'Q$, and Theorem 4 is proven.

Now, we broaden our configuration by adding a new point - the center of the circle through the points H_b, B', P, Q. The following theorem (partly contained in [3], ii)) identifies this center:

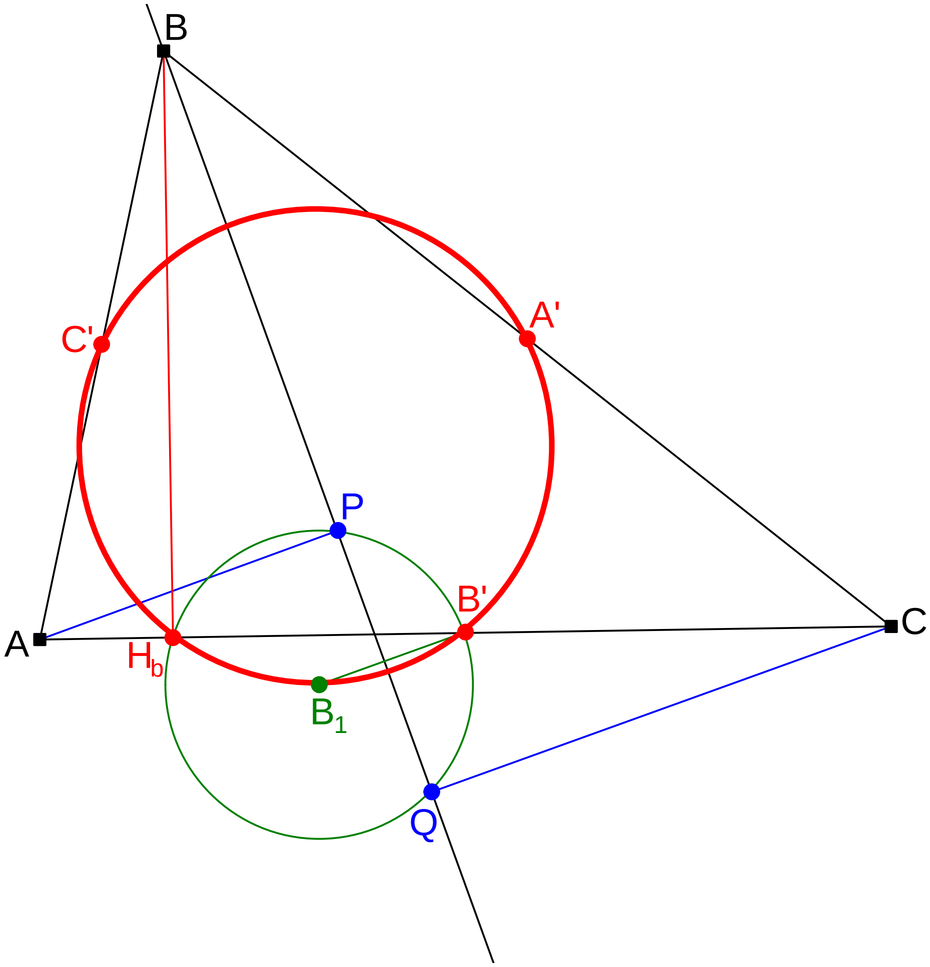

Fig. 6

Theorem 5. The center B_1 of the circle through the points H_b, B', P, Q lies on the nine-point circle of triangle ABC, and the line $B'B_1$ is perpendicular to the bisector of angle ABC.

In other words, the point B_1 is the point of intersection of the nine-point circle of triangle ABC with the perpendicular to the bisector of angle ABC through the point B' different from B'. (See Fig. 6.)

Proof. The point B_1 is the center of the circle through the points H_b, B', P, Q, hence the circumcenter of triangle $PB'Q$; consequently, it lies on the perpendicular bisector of the side PQ of this triangle. On the other hand, the point B' lies on the perpendicular bisector of this segment PQ, since $B'P = B'Q$ by Theorem 4. Thus, the line $B'B_1$ is the perpendicular bisector of the segment PQ; thus, it is perpendicular to the line PQ, i. e. to the bisector of angle ABC.

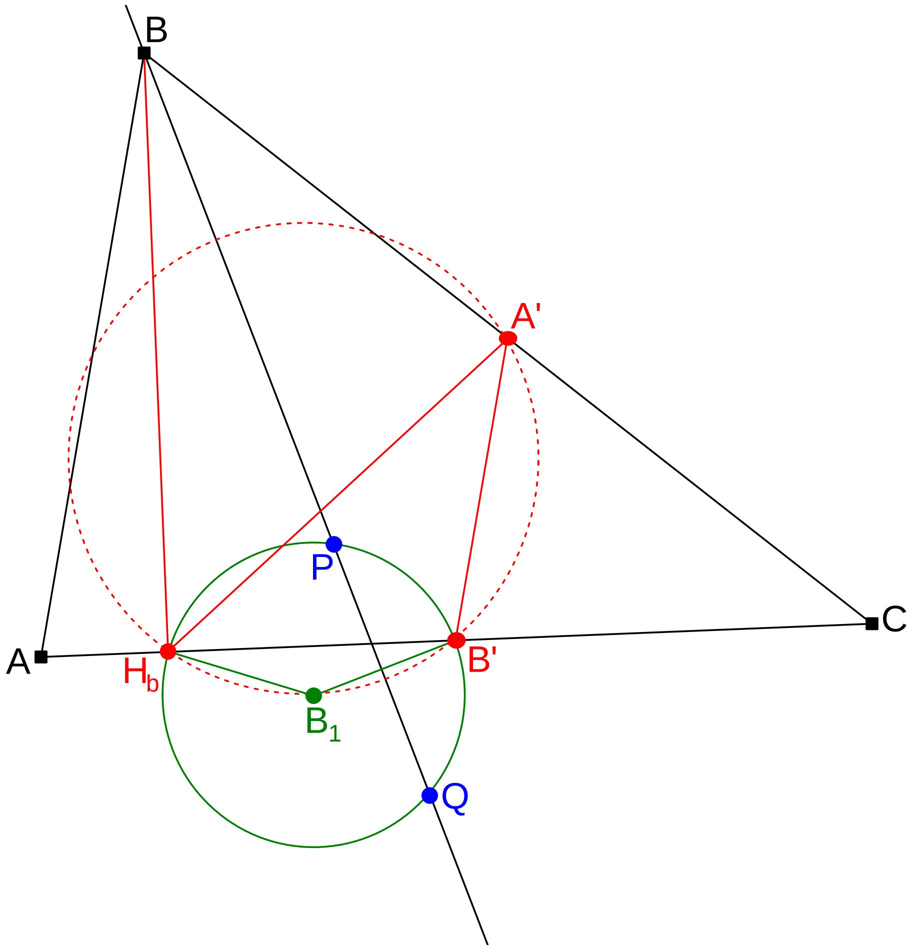

Fig. 7

It remains to prove that the point B_1 lies on the nine-point circle of triangle ABC. This can, e. g., be shown by chasing angles (Fig. 7): In addition to the midpoint B' of the side CA of triangle ABC, we introduce the midpoint A' of the side BC. The nine-point circle of triangle ABC is known to pass through the midpoints A' and B' of the sides BC and CA and through the foot H_b of the B-altitude of triangle ABC.

Since $\measuredangle BH_bC = 90^\circ$, the point H_b lies on the circle with diameter BC. The center of this circle is the midpoint A' of the segment BC. Thus, $A'H_b = A'C$. Consequently, triangle $H_bA'C$ is isosceles, so that $\measuredangle A'H_bC = \measuredangle H_bCA'$. On the other hand, the point B_1 is the center of the circle through the points H_b, B', P, Q; this yields $B_1H_b = B_1B'$, so that the triangle H_bB_1B' is isosceles, and $\measuredangle B'H_bB_1 = \measuredangle B_1B'H_b$. Finally, from $B'B_1 \perp PQ$ we conclude that $\measuredangle(B'B_1;\ PQ) = 90^\circ = -90^\circ$. Thus,

$$\begin{aligned}\measuredangle A'H_bB_1 &= \measuredangle A'H_bC + \measuredangle B'H_bB_1 = \measuredangle H_bCA' + \measuredangle B_1B'H_b = \measuredangle(CA;\ BC) + \measuredangle(B'B_1;\ CA)\\ &= \measuredangle(B'B_1;\ BC) = \measuredangle(B'B_1;\ PQ) + \measuredangle(PQ;\ BC) = 90^\circ + \measuredangle(PQ;\ BC).\end{aligned}$$

Since the line PQ is the bisector of angle ABC, we have $\measuredangle(PQ;\ BC) = \measuredangle(AB;\ PQ)$, and this becomes $\measuredangle A'H_bB_1 = 90^\circ + \measuredangle(AB;\ PQ)$.

Since the points A' and B' are the midpoints of the sides BC and CA of triangle ABC, the line $A'B'$ is parallel to the line AB. Consequently, $\measuredangle(A'B';\ B'B_1) = \measuredangle(AB;\ B'B_1)$, and

$$\begin{aligned}\measuredangle A'B'B_1 &= \measuredangle(A'B';\ B'B_1) = \measuredangle(AB;\ B'B_1) = \measuredangle(AB;\ PQ) - \measuredangle(B'B_1;\ PQ)\\ &= \measuredangle(AB;\ PQ) - (-90^\circ) = 90^\circ + \measuredangle(AB;\ PQ) = \measuredangle A'H_bB_1.\end{aligned}$$

Thus, the points A', B', H_b and B_1 are concyclic. In other words, the point B_1 lies on the circle through the points A', B', H_b, i. e. on the nine-point circle of triangle ABC. This completes our proof of Theorem 5.

3. The incenter

Now we move to deeper waters and enhance the configuration by a new point; we start with a fact first noted by Grobber in [2]:

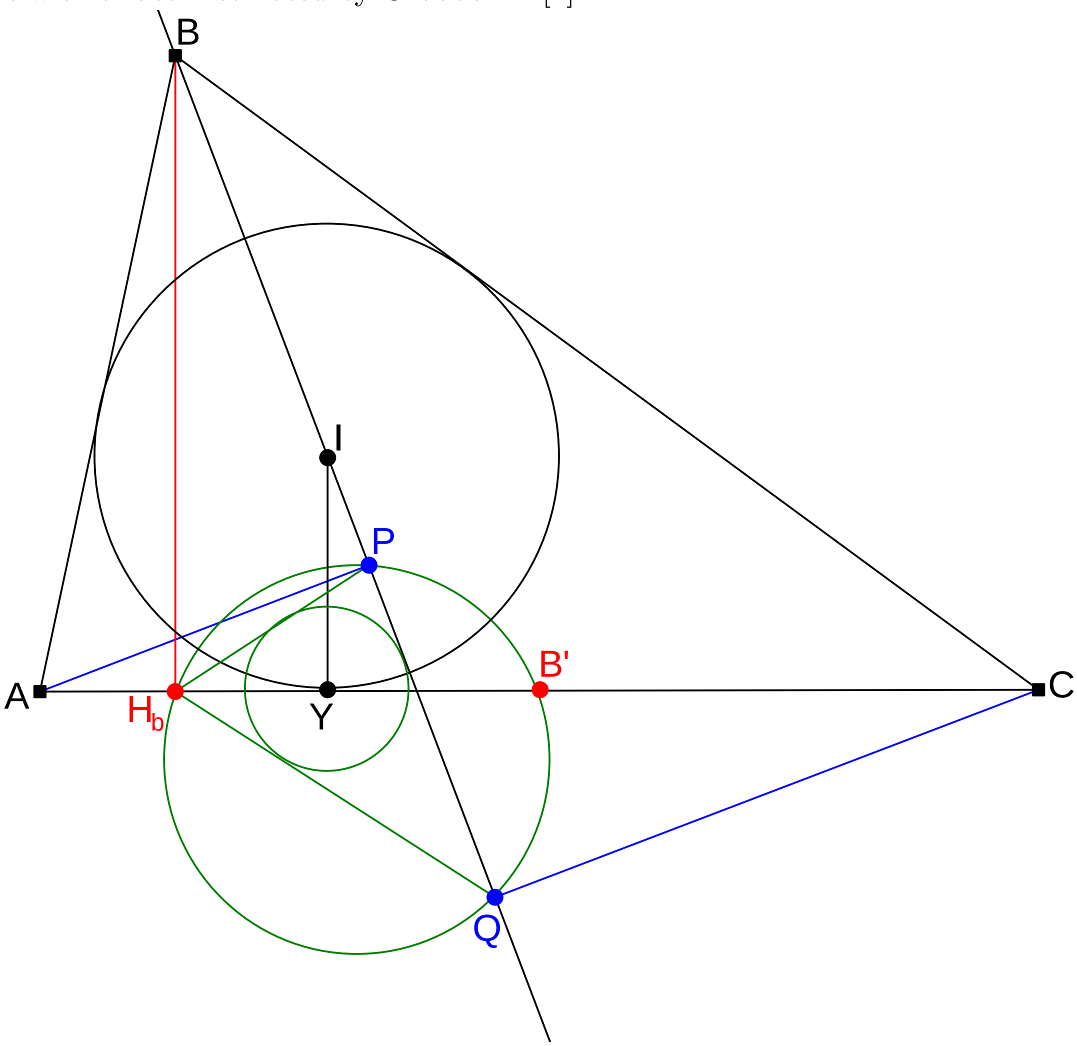

Fig. 8

Theorem 6. Let the incircle of triangle ABC touch its side CA at a point Y. Then, this point Y is the incenter of triangle PH_bQ. (See Fig. 8.)

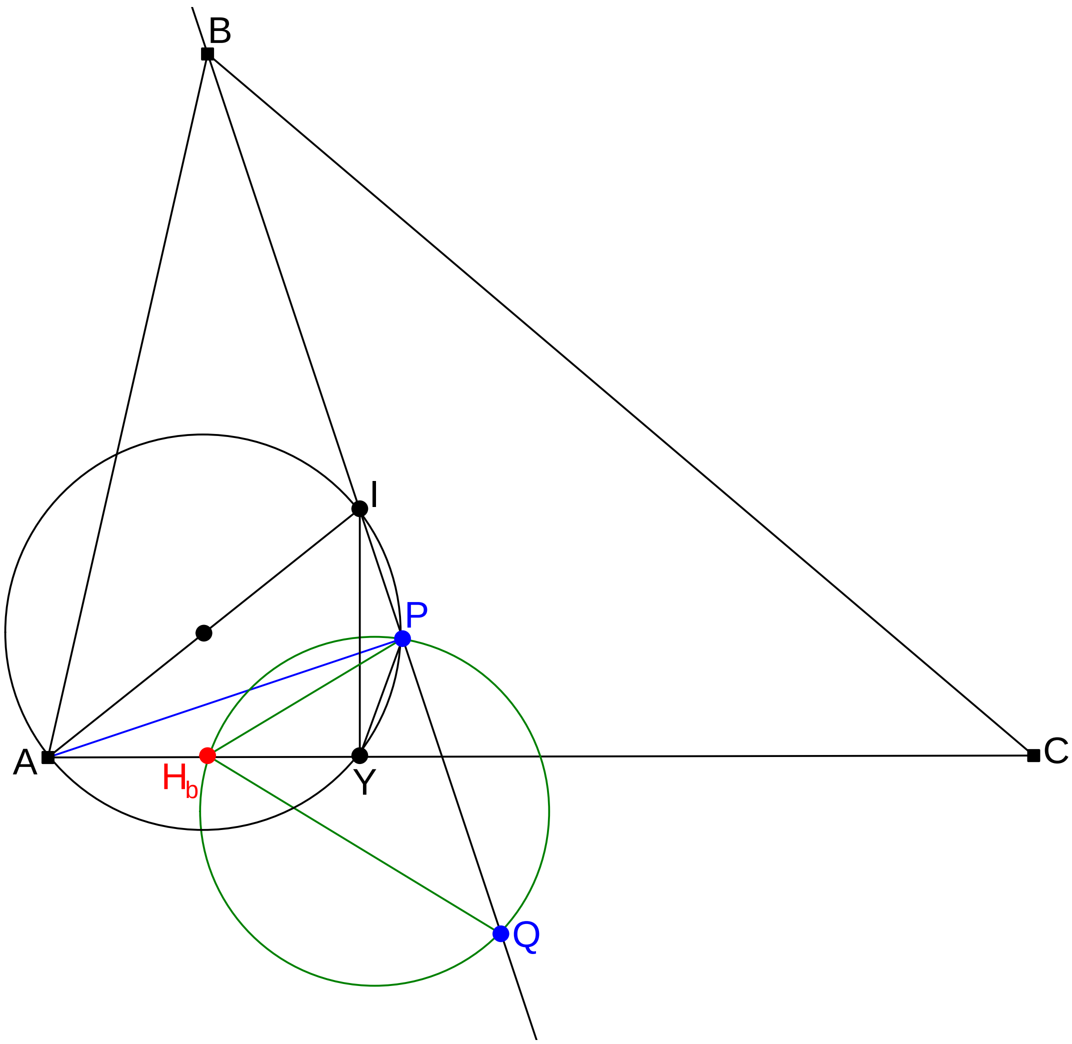

Fig. 9

Proof. Let I' be the incenter of triangle PH_bQ. Theorem 6 asserts that $Y = I'$.

(See Fig. 9.) Let I be the incenter of triangle ABC. This incenter I must obviously lie on the bisector of angle ABC, this means, on the line through the points B, P, Q.

Since the incircle of triangle ABC has the center I and touches the side CA at the point Y, we have $IY \perp CA$, so that $\measuredangle AYI = 90°$. Together with $\measuredangle API = 90°$, this shows that the points Y and P lie on the circle with diameter AI. Hence, $\measuredangle YPI = \measuredangle YAI$. In other words, $\measuredangle YPQ = -\measuredangle IAC$.

Now, according to Theorem 3, the triangles PH_bQ and ABC are inversely similar; i. e., there exists an indirect similitude which maps triangle ABC to triangle PH_bQ. This similitude, of course, must also map the incenter I of triangle ABC to the incenter I' of triangle PH_bQ, and since directed angles change their sign under an indirect similitude, this point I' satisfies $\measuredangle I'PQ = -\measuredangle IAC$. Comparison with $\measuredangle YPQ = -\measuredangle IAC$ yields $\measuredangle I'PQ = \measuredangle YPQ$; thus, the point Y lies on the line $I'P$. Similarly, the point Y lies on the line $I'Q$. But the lines $I'P$ and $I'Q$ have only one point in common, namely I'; thus, $Y = I'$, what proves Theorem 6.

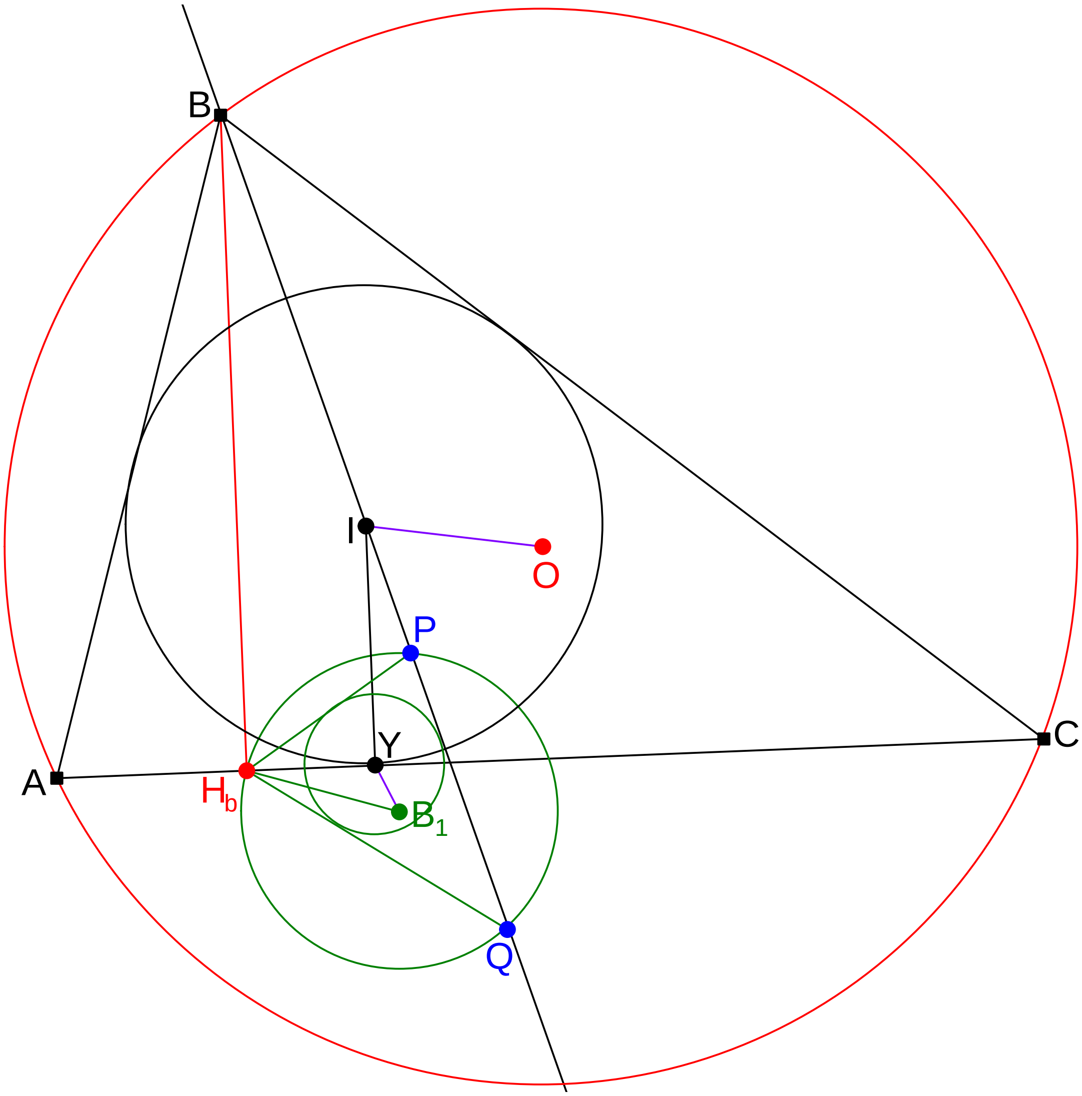

Fig. 10

(See Fig. 10.) Now, we denote by O the circumcenter of triangle ABC. Furthermore, we note that the point B_1 is the circumcenter of triangle PH_bQ (being the center of the circle through the points H_b, B', P, Q). The indirect similitude which maps triangle ABC to triangle PH_bQ must map the incenter I of triangle ABC to the incenter Y of triangle PH_bQ and the circumcenter O of triangle ABC to the circumcenter B_1 of triangle PH_bQ. Since directed angles change their sign under indirect similitudes, we thus have $\measuredangle(YB_1;\ QP) = -\measuredangle(IO;\ CA)$. As the line QP coincides with the line BI, this becomes $\measuredangle(YB_1;\ BI) = -\measuredangle(IO;\ CA)$, or, equivalently, $\measuredangle(YB_1;\ BI) = \measuredangle(CA;\ IO)$.

Using the alternative description of the point B_1 which was given in Theorem 5, the result just obtained can be stated as follows:

Theorem 7. Let ABC be a triangle, I its incenter, O its circumcenter, and B' the midpoint of its side CA. Further, let Y be the point of tangency of the incircle of triangle ABC with its side CA, and let B_1 be the point of intersection of the nine-point circle of triangle ABC with the perpendicular to the bisector of angle ABC through the point B' different from B'. Then, $\measuredangle(YB_1;\ BI) = \measuredangle(CA;\ IO)$.

4. Nine-point circle and incircle

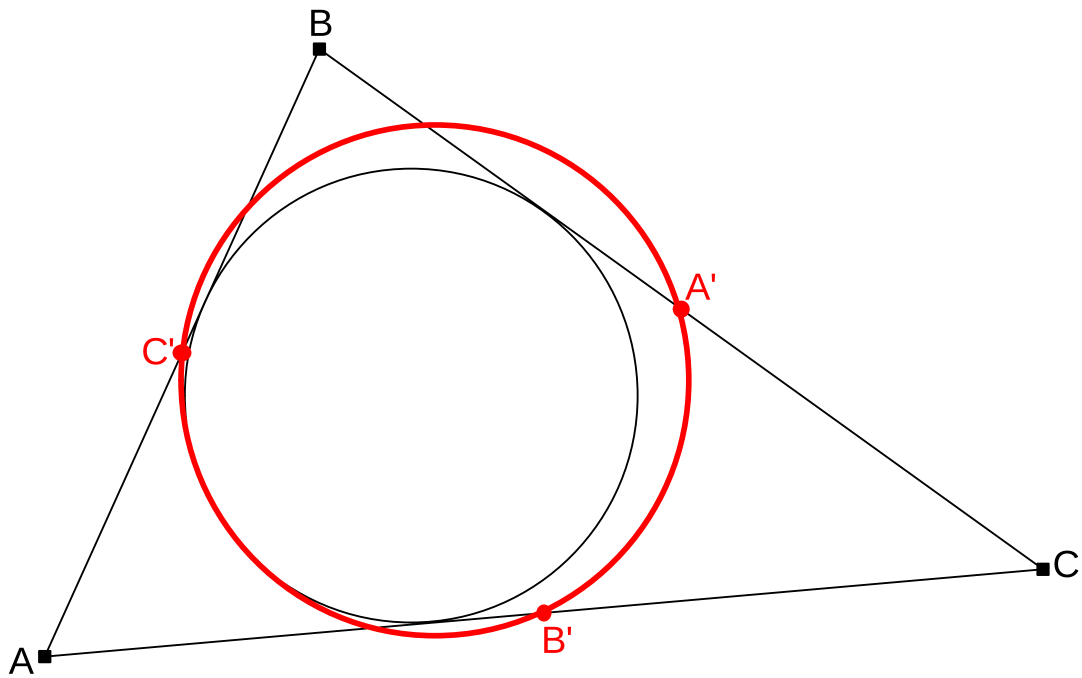

Fig. 11

Theorem 7 differs in some respect from the results before: The points H_b, P and Q don't occur anymore; it's mainly a property of the incircle and the nine-point circle of the triangle ABC. This suggests a connection to a famous theorem of triangle geometry, the **Feuerbach theorem** (see, e. g., [7], §5.6, theorem 5.61):

Theorem 8. The nine-point circle of any triangle ABC touches the incircle of triangle ABC. (See Fig. 11.)

In its more general form, the Feuerbach theorem states that the nine-point circle of triangle ABC touches its incircle and its three excircles; however, we will only prove the tangency with the incircle (i. e. the assertion of Theorem 8) below, as the proof of the tangency with the excircles can be done in a completely analogous way - our observations are, thanks to the use of directed angles modulo 180°, independent of the arrangement, and transferring them from the incircle to an excircle comes down to just replacing some internal angle bisectors by external angle bisectors.

Now, we are going to prove Theorem 8 with the help of our Theorem 7.

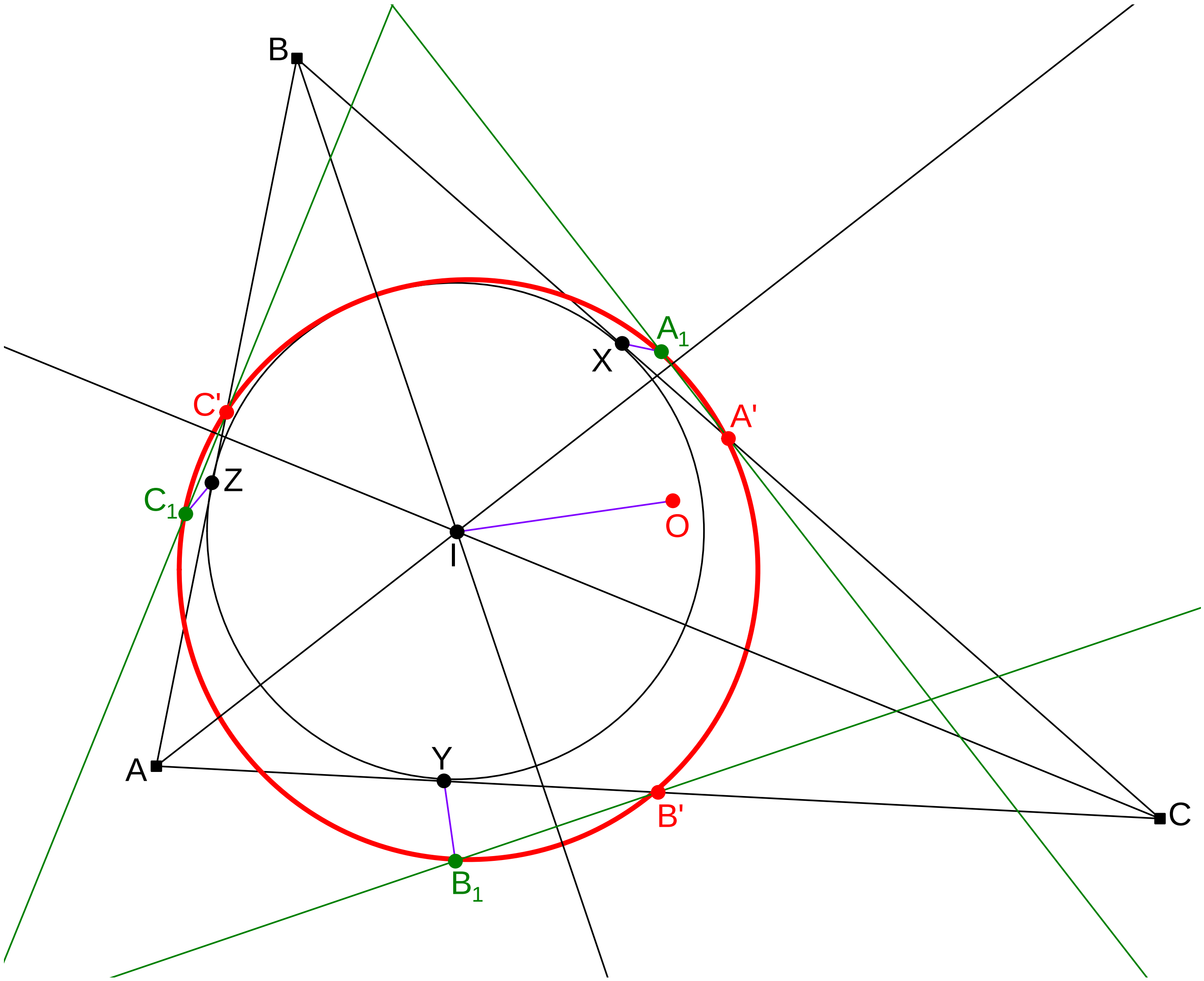

Fig. 12

(See Fig. 12.) First, we extend our configuration to a symmetric one:

Let A', B', C' be the midpoints of the sides BC, CA, AB of triangle ABC. Then, the lines $B'C'$, $C'A'$, $A'B'$ are parallel to the sidelines BC, CA, AB, respectively. Furthermore, the points A', B', C' lie on the nine-point circle of triangle ABC.

(See Fig. 13.) Let X, Y, Z be the points of tangency of the incircle of triangle ABC with its sides BC, CA, AB. Then, the points Y and Z are symmetric to each other with respect to the bisector of angle CAB; thus, the line YZ is perpendicular to the bisector of the angle CAB, i. e. to the line AI (since the point I is the incenter of triangle ABC). Similarly, the lines ZX and XY are perpendicular to the bisectors of the angles ABC and BCA, i. e. to the lines BI and CI, respectively.

(See Fig. 12 again.) Let A_1, B_1, C_1 be the points of intersection of the nine-point circle of triangle ABC with the perpendiculars to the bisectors of angles CAB, ABC, BCA through the points A', B', C' different from A', B', C', respectively.

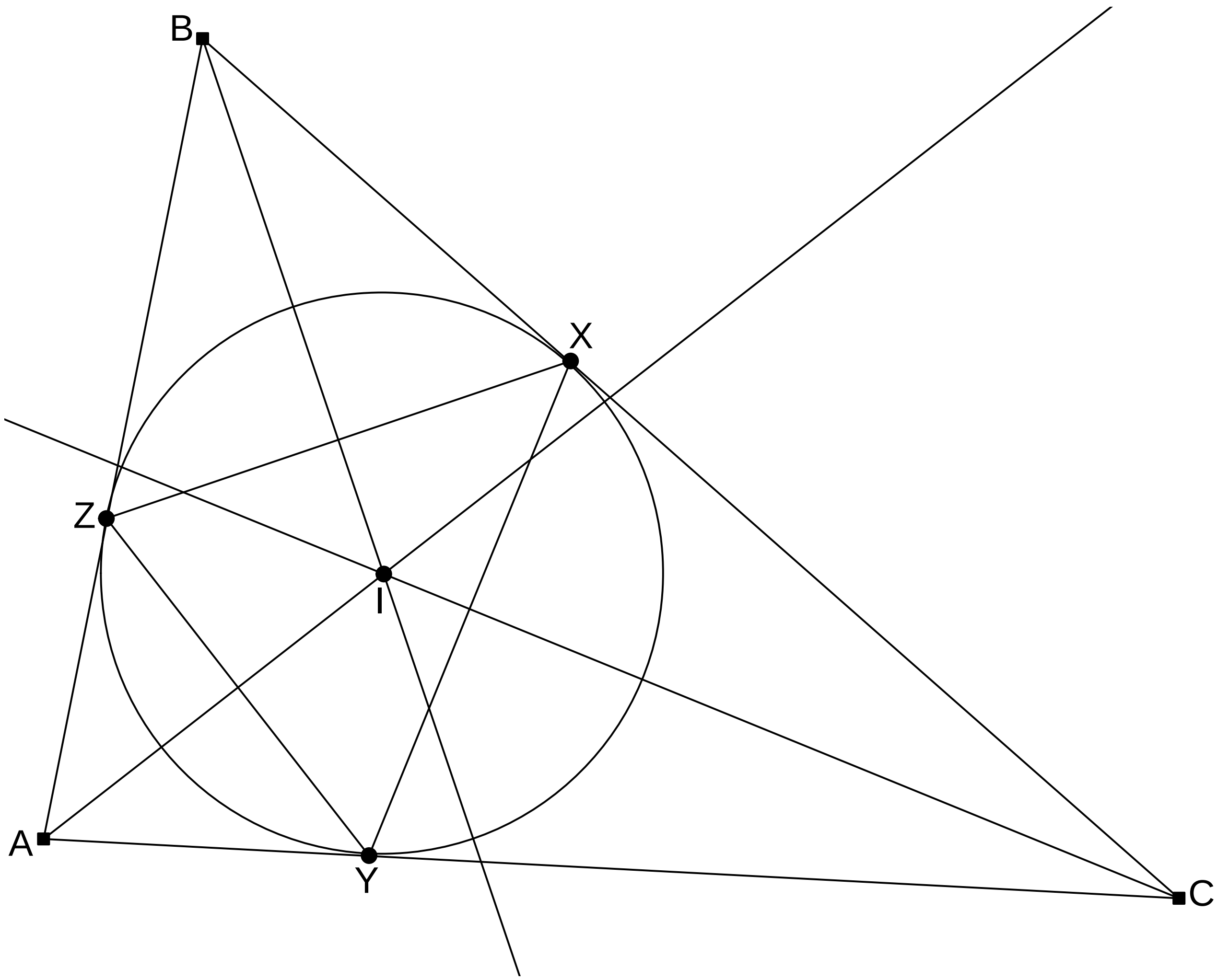

Fig. 13

Theorem 7 yields $\measuredangle(YB_1;\ BI) = \measuredangle(CA;\ IO)$. Similarly, we can get $\measuredangle(ZC_1;\ CI) = \measuredangle(AB;\ IO)$ and $\measuredangle(XA_1;\ AI) = \measuredangle(BC;\ IO)$. This entails

$$\begin{aligned}\measuredangle(YB_1;\ ZC_1) &= \measuredangle(YB_1;\ BI) + \measuredangle(BI;\ CI) - \measuredangle(ZC_1;\ CI)\\ &= \measuredangle(CA;\ IO) + \measuredangle(BI;\ CI) - \measuredangle(AB;\ IO)\\ &= (\measuredangle(CA;\ IO) - \measuredangle(AB;\ IO)) + \measuredangle(BI;\ CI) = \measuredangle(CA;\ AB) + \measuredangle(BI;\ CI)\\ &= (\measuredangle(CA;\ BI) + \measuredangle(BI;\ AB)) + \measuredangle(BI;\ CI)\\ &= (\measuredangle(CA;\ BI) + \measuredangle(BI;\ CI)) + \measuredangle(BI;\ AB) = \measuredangle(CA;\ CI) + \measuredangle(BI;\ AB).\end{aligned}$$

Now, as the lines CI and BI are the bisectors of angles BCA and ABC, we have $\measuredangle(CA;\ CI) = \measuredangle(CI;\ BC)$ and $\measuredangle(BI;\ AB) = \measuredangle(BC;\ BI)$, so that

$$\measuredangle(YB_1;\ ZC_1) = \measuredangle(CA;\ CI) + \measuredangle(BI;\ AB) = \measuredangle(CI;\ BC) + \measuredangle(BC;\ BI) = \measuredangle(CI;\ BI).$$

(See Fig. 13.) From $ZX \perp BI$, we conclude that $\measuredangle(BI;\ ZX) = 90^\circ$, and from $XY \perp CI$, we get $\measuredangle(CI;\ XY) = 90^\circ$. Hence,

$$\begin{aligned}\measuredangle(YB_1;\ ZC_1) &= \measuredangle(CI;\ BI) = \measuredangle(CI;\ XY) + \measuredangle(XY;\ ZX) - \measuredangle(BI;\ ZX)\\ &= 90^\circ + \measuredangle(XY;\ ZX) - 90^\circ = \measuredangle(XY;\ ZX) = \measuredangle YXZ.\end{aligned}$$

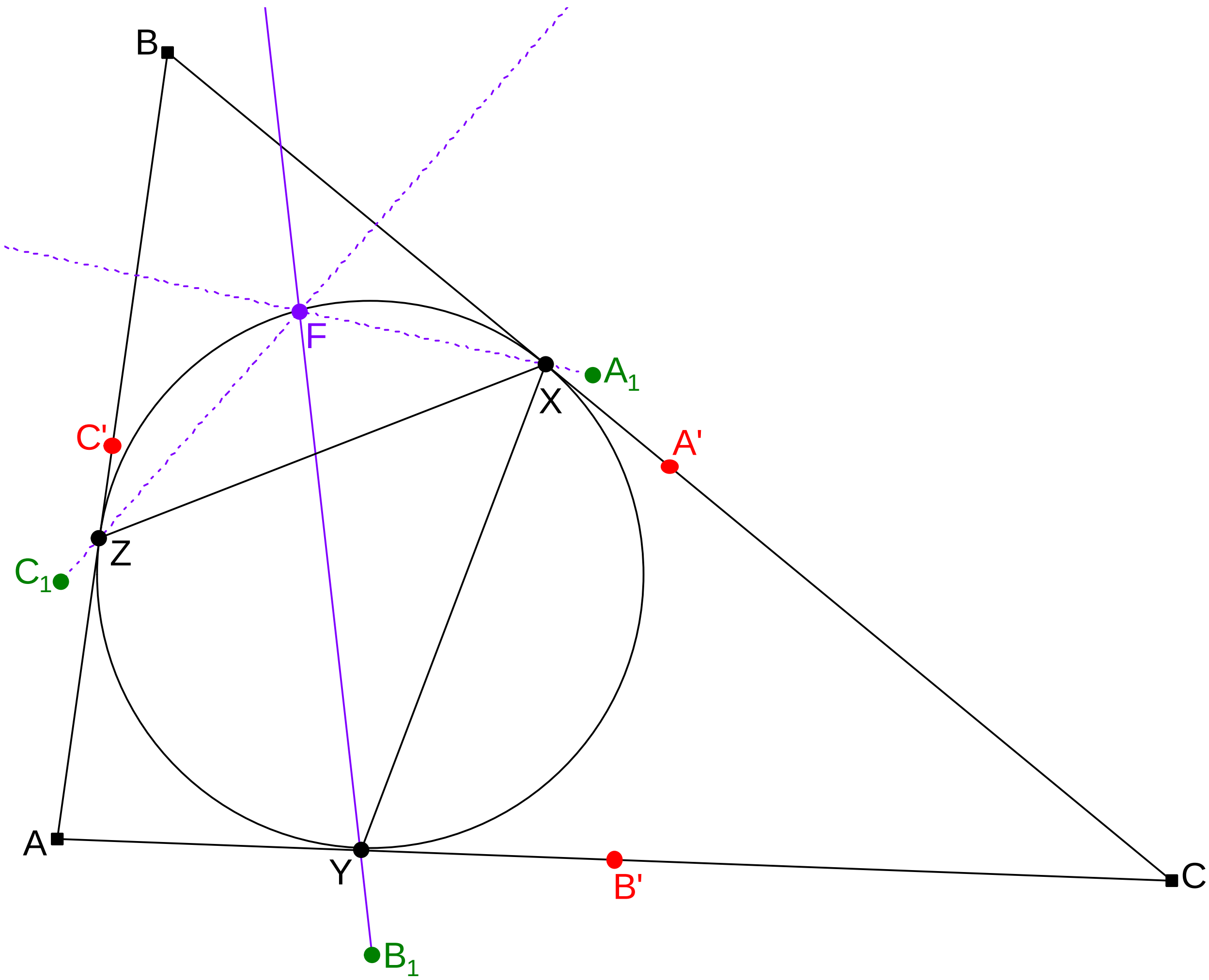

Fig. 14

(See Fig. 14.) Now, let F be the point of intersection of the line B_1Y with the incircle of triangle ABC different from Y. Then, $\measuredangle YFZ = \measuredangle YXZ$. But, as we showed above, $\measuredangle(YB_1;\ ZC_1) = \measuredangle YXZ$. Thus, $\measuredangle YFZ = \measuredangle(YB_1;\ ZC_1)$, or, equivalently, $\measuredangle(YB_1;\ FZ) = \measuredangle(YB_1;\ ZC_1)$. This yields that the lines FZ and ZC_1 are parallel. Since they have a common point (the point Z), they must therefore coincide; i. e., the point F lies on the line C_1Z. Similarly, we show that the point F lies on the line A_1X.

We note for our further reasoning that the point F lies on the incircle of triangle ABC and is the point of intersection of the three lines A_1X, B_1Y, C_1Z.

Now, we are going to show a simple lemma:

Theorem 9. The triangles $A_1B_1C_1$ and XYZ are homothetic.

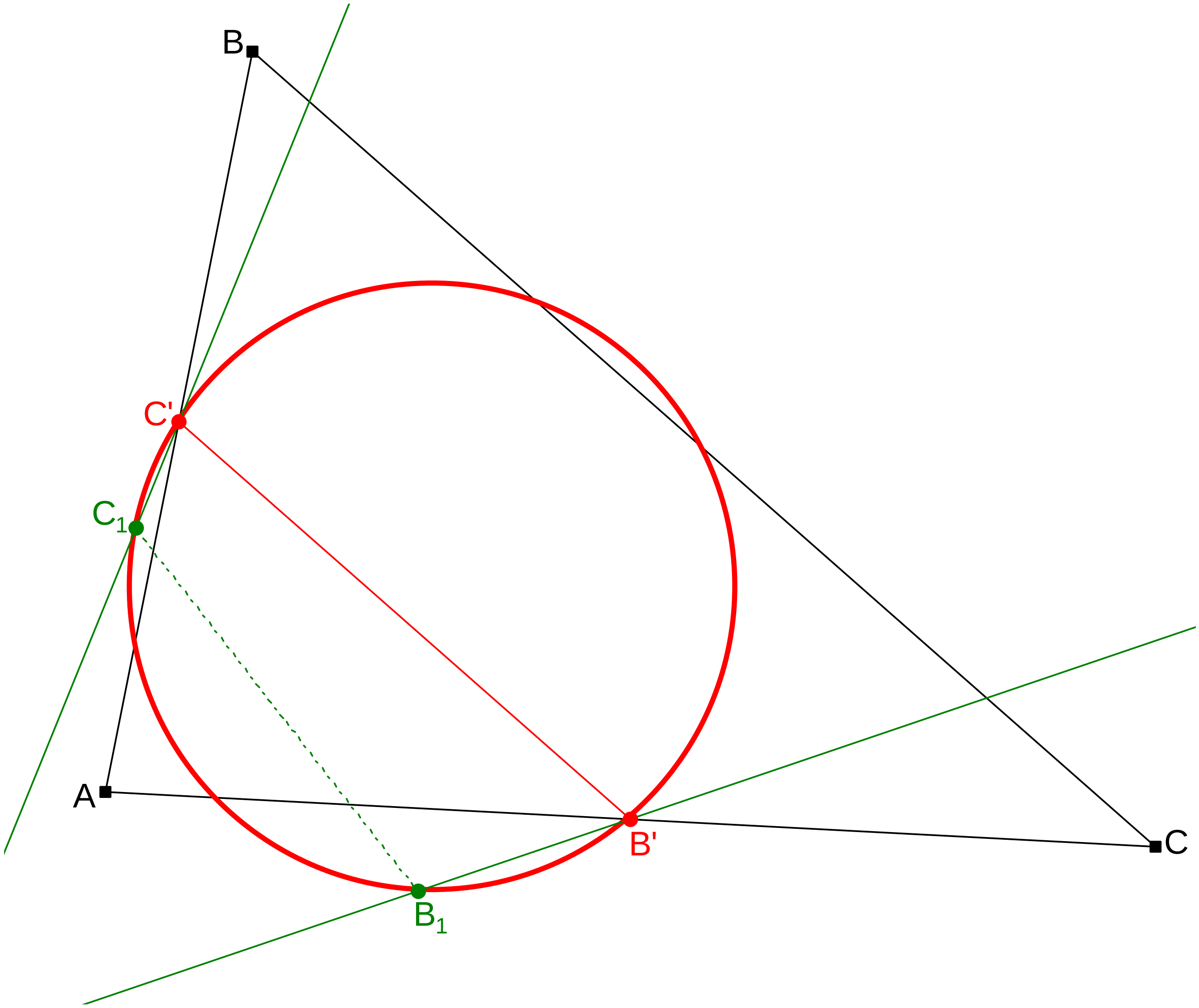

Fig. 15

Proof of Theorem 9. (See Fig. 15.) Since the points B', C', B_1, C_1 all lie on the nine-point circle of triangle ABC, we have $\measuredangle B_1C_1C' = \measuredangle B_1B'C'$, thus $\measuredangle(B_1C_1;\ C'C_1) = \measuredangle(B'B_1;\ B'C')$. Now, the lines $B'B_1$ and ZX are both perpendicular to the bisector of angle ABC; thus, $B'B_1 \parallel ZX$. Similarly, $C'C_1 \parallel XY$.

From $C'C_1 \parallel XY$, it follows that $\measuredangle(B_1C_1;\ C'C_1) = \measuredangle(B_1C_1;\ XY)$, while $B'B_1 \parallel ZX$ and $B'C' \parallel BC$ yield $\measuredangle(B'B_1;\ B'C') = \measuredangle(ZX;\ BC)$. Thus, the equation $\measuredangle(B_1C_1;\ C'C_1) = \measuredangle(B'B_1;\ B'C')$ becomes $\measuredangle(B_1C_1;\ XY) = \measuredangle(ZX;\ BC)$. Now, the angle $\measuredangle(ZX;\ BC)$ is the angle between the chord ZX of the incircle of triangle ABC and the tangent BC to this incircle at the point X, and thus, according to the tangent-chordal angle theorem, equals to the angle $\measuredangle ZYX$ subtended by the chord ZX. Hence, we have $\measuredangle(B_1C_1;\ XY) = \measuredangle ZYX$, or, equivalently, $\measuredangle(B_1C_1;\ XY) = \measuredangle(YZ;\ XY)$. Thus, $B_1C_1 \parallel YZ$. Similarly, $C_1A_1 \parallel ZX$ and $A_1B_1 \parallel XY$. Hence, the triangles $A_1B_1C_1$ and XYZ are homothetic, and Theorem 9 is proven.

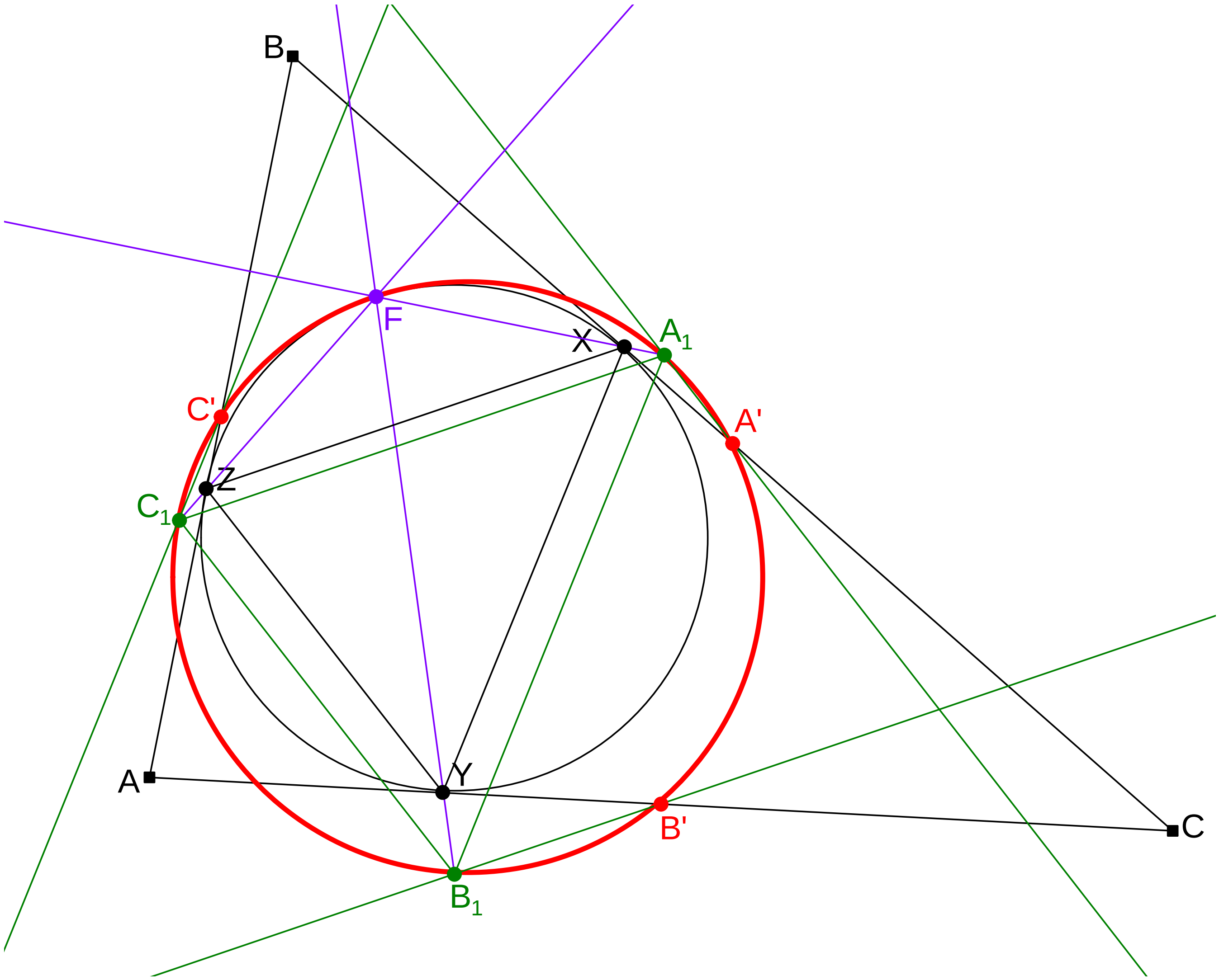

Fig. 16

(See Fig. 16.) Consider the two homothetic triangles $A_1B_1C_1$ and XYZ. Their homothetic center must be the point of intersection of the three lines A_1X, B_1Y, C_1Z, so it is the point F. Thus, there exists a homothety with center F which maps the triangle XYZ to the triangle $A_1B_1C_1$. Of course, this homothety must then map the circumcircle of triangle XYZ to the circumcircle of triangle $A_1B_1C_1$. Since the circumcircle of triangle XYZ is the incircle of triangle ABC, and the circumcircle of triangle $A_1B_1C_1$ is the nine-point circle of triangle ABC, we thus conclude that our homothety with center F transforms the incircle of triangle ABC into the nine-point circle of triangle ABC. On the other hand, the homothety fixes the point F (since it is the center of the homothety). Thus, as the point F lies on the incircle of triangle ABC, its image under the homothety - i. e. the point F again - must lie on the image of the incircle - i. e. on the nine-point circle of triangle ABC. Therefore, the point F is a common point of the incircle and the nine-point circle of triangle ABC. Moreover, since our homothety with center F maps the incircle to the nine-point circle and leaves the point F fixed, it must map the tangent to the incircle at the point F to the tangent to the nine-point circle at the point F; on the other hand, the tangent to the incircle at the point F is a line through the center F of our homothety and thus must remain fixed under the homothety. Hence, the tangent to the incircle at the point F must coincide with the tangent to the nine-point circle at the point F. In other words, the incircle and the nine-point circle of triangle ABC have a common tangent at their common point F. Therefore, they must touch each other at the point F. This not only establishes Theorem 8, but also yields a characterization of the point of tangency F:

Theorem 10. The point of tangency F of the incircle and the nine-point circle of triangle ABC is the homothetic center of the homothetic triangles $A_1B_1C_1$ and XYZ. (See Fig. 16.)

This point of tangency F is usually referred to as the **Feuerbach point** of triangle ABC. [1]

5. The Feuerbach point as an Anti-Steiner point

The main theorem is proven, but geometry doesn't stop here. In fact, the Feuerbach point F is one of the richest in properties geometrical objects and was subject to numerous publications. Here, we are going to show two characteristics of F which are, in my opinion, much less popular than they deserve.

We will formulate these characteristics using the notion of *Anti-Steiner points*. This notion is based on the following fact:

[1] A little bit of care is necessary when consulting literature, as some authors use the term "Feuerbach point" for the center of the nine-point circle; this is, however, pretty seldom.

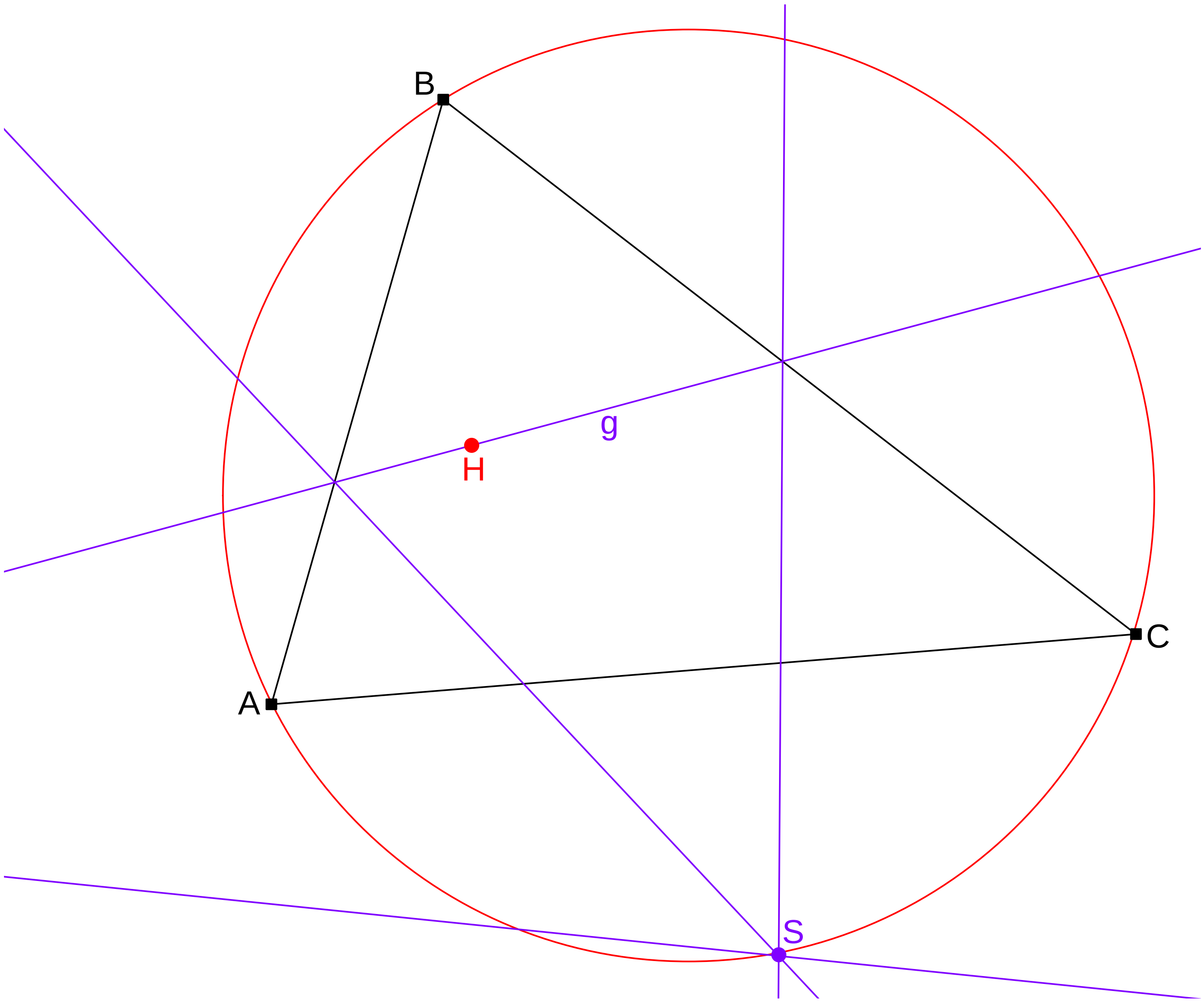

Fig. 17

Theorem 11. Let ABC be a triangle and H its orthocenter. Let g be a line through the point H. Then, the reflections of the line g in the lines BC, CA, AB concur at a point S, and this point S lies on the circumcircle of triangle ABC. Furthermore, $\measuredangle BCS = \measuredangle BAS = 90^\circ - \measuredangle (CA;\ g)$.

We will call the point S the **Anti-Steiner point** of the line g with respect to triangle ABC. [2] (See Fig. 17.)

[2]The denotation "Anti-Steiner point" was choosen by me for the following reason:
If a point lies on the circumcircle of triangle ABC, then the reflections of this point in the sidelines BC, CA, AB lie on one line, the so-called **Steiner line** of this point with respect to triangle ABC. Now, the point S is called the Anti-Steiner point of the line g since the Steiner line of the point S is the line g (as one can easily see). The name "Steiner point" may be more appropriate, but unfortunately, it is already used for at least three different triangle centers!

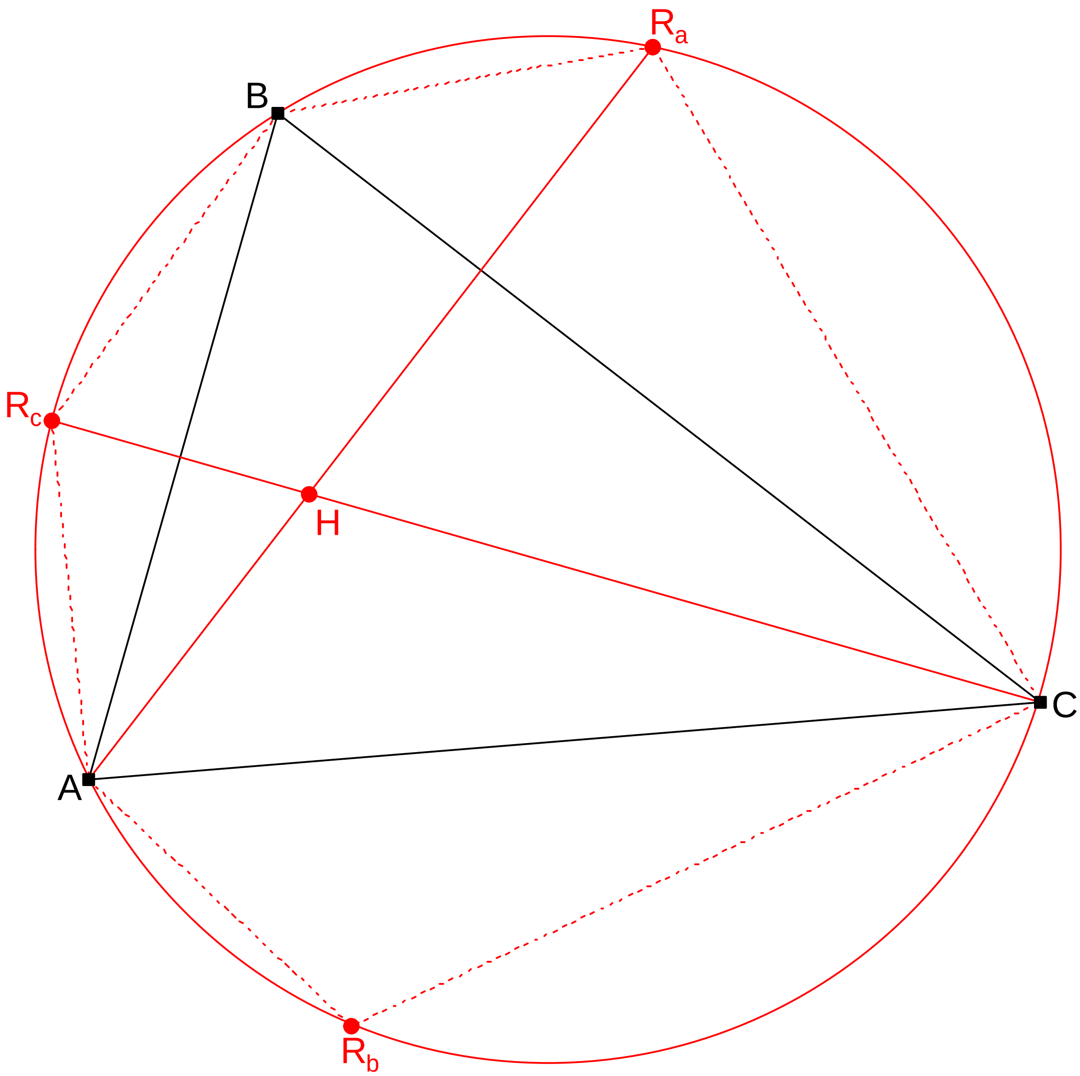

Fig. 18

The *proof of Theorem 11* relies on a very well-known fact:

Theorem 12. Let ABC be an arbitrary triangle and H its orthocenter. Then, the reflections R_a, R_b, R_c of the point H in the lines BC, CA, AB lie on the circumcircle of triangle ABC. (See Fig. 18.)

Proof of Theorem 12. Since H is the orthocenter of triangle ABC, we have $BH \perp CA$ and $CH \perp AB$, thus $\measuredangle (BH;\ CA) = 90^\circ$ und $\measuredangle (CH;\ AB) = 90^\circ$. Consequently,

$$\begin{aligned}\measuredangle BHC &= \measuredangle (BH;\ CH) = \measuredangle (BH;\ CA) + \measuredangle (CA;\ AB) - \measuredangle (CH;\ AB) \\ &= 90^\circ + \measuredangle (CA;\ AB) - 90^\circ = \measuredangle (CA;\ AB) = \measuredangle CAB.\end{aligned}$$

Since R_a is the reflection of the point H in the line BC, we have $\measuredangle BR_aC = -\measuredangle BHC$, thus $\measuredangle BR_aC = -\measuredangle CAB = \measuredangle BAC$. Hence, the point R_a lies on the circumcircle of triangle ABC. Similarly, we can show that the points R_b and R_c lie on this circumcircle, and Theorem 12 is proven.

Now, we come to the actual proof of Theorem 11:

Let g_a, g_b, g_c be the reflections of the line g in the lines BC, CA, AB. We have to show that these reflections g_a, g_b, g_c concur at one point S, and that this point S lies on the circumcircle of triangle ABC and satisfies $\measuredangle BCS = \measuredangle BAS = 90^\circ - \measuredangle (CA;\ g)$.

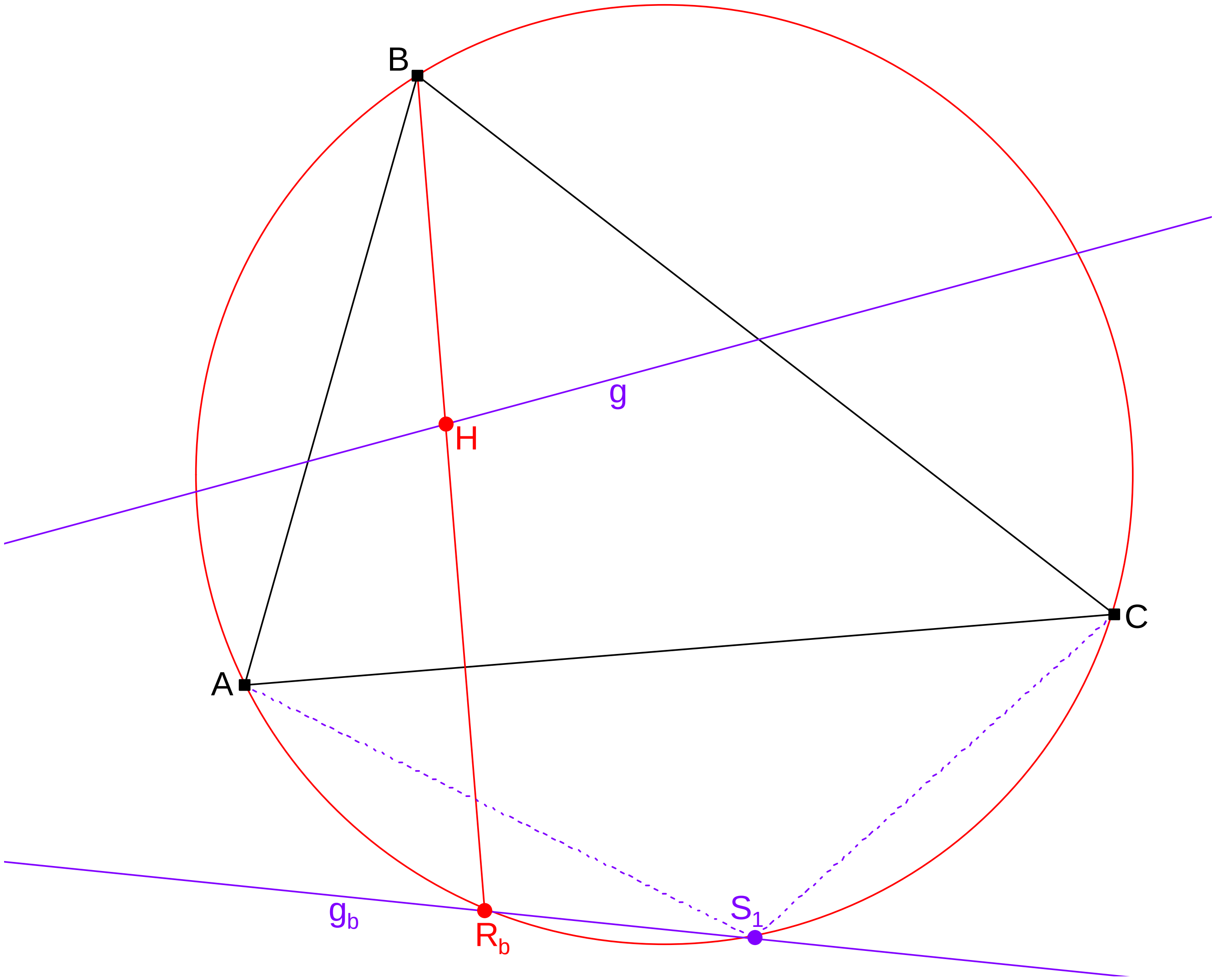

Fig. 19

(See Fig. 19.) We have $BH \perp CA$ (since H is the orthocenter of triangle ABC) and $HR_b \perp CA$ (since R_b is the reflection of the point H in the line CA). Thus, the points B, H, R_b lie on one line perpendicular to the line CA.

Since the line g passes through the point H, the reflection g_b of the line g in the line CA passes through the reflection R_b of the point H in the line CA. Since, according to Theorem 12, the point R_b lies on the circumcircle of triangle ABC, it is therefore a point of intersection of the line g_b and the circumcircle of triangle ABC. Let S_1 be the point of intersection of the line g_b and the circumcircle of triangle ABC different from R_b. Then, $\measuredangle BAS_1 = \measuredangle BR_bS_1$. On the other hand, $BH \perp CA$ yields $\measuredangle (BH;\ CA) = 90^\circ$. Finally, $\measuredangle (CA;\ g_b) = -\measuredangle (CA;\ g)$ because the line g_b is the reflection of the line g in the line CA. Hence,

$$\begin{aligned}\measuredangle BAS_1 = \measuredangle BR_bS_1 &= \measuredangle (BH;\ g_b) = \measuredangle (BH;\ CA) + \measuredangle (CA;\ g_b) \\ &= 90^\circ + (-\measuredangle (CA;\ g)) = 90^\circ - \measuredangle (CA;\ g)\,.\end{aligned}$$

Similarly, $\measuredangle BCS_1 = 90^\circ - \measuredangle (CA;\ g)$. Thus, we get $\measuredangle BCS_1 = \measuredangle BAS_1 = 90^\circ - \measuredangle (CA;\ g)$.

Consequently,

$$\begin{aligned}\measuredangle CAS_1 &= \measuredangle CAB + \measuredangle BAS_1 = \measuredangle\left(CA;\ AB\right) + \left(90^\circ - \measuredangle\left(CA;\ g\right)\right) \\ &= 90^\circ - \left(\measuredangle\left(CA;\ g\right) - \measuredangle\left(CA;\ AB\right)\right) = 90^\circ - \measuredangle\left(AB;\ g\right).\end{aligned}$$

Now, we have defined the point S_1 as the point of intersection of the line g_b with the circumcircle of triangle ABC different from R_b. Similarly, we can denote by S_2 the point of intersection of the line g_c with the circumcircle of triangle ABC different from R_c, and in the same way as we showed $\measuredangle BCS_1 = 90^\circ - \measuredangle\left(CA;\ g\right)$ we can then show $\measuredangle CAS_2 = 90^\circ - \measuredangle\left(AB;\ g\right)$ for our point S_2. Comparing this with $\measuredangle CAS_1 = 90^\circ - \measuredangle\left(AB;\ g\right)$, we see that $\measuredangle CAS_2 = \measuredangle CAS_1$. Hence, the point S_2 lies on the line AS_1. Since the point S_2 also lies on the circumcircle of triangle ABC, this point S_2 must therefore be the point of intersection of the line AS_1 with the circumcircle of triangle ABC different from A. Thus, the point S_2 coincides with the point S_1; since the point S_2 lies on the line g_c, we have herewith shown that the point S_1 lies on the line g_c. Similarly, we can see that the point S_1 lies on the line g_a.

Altogether, we now know that the point S_1 lies on the lines g_a, g_b, g_c; in other words, the lines g_a, g_b, g_c concur at the point S_1; also, we know that this point S_1 lies on the circumcircle of triangle ABC and satisfies $\measuredangle BCS_1 = \measuredangle BAS_1 = 90^\circ - \measuredangle\left(CA;\ g\right)$. Thus, Theorem 11 is proven, and it is clear that our point S_1 coincides with the point S from Theorem 11.

Now, we can easily show the first characteristic of F:

Theorem 13. The Feuerbach point F of triangle ABC is the Anti-Steiner point of the line IO with respect to triangle $A'B'C'$. (See Fig. 20.)

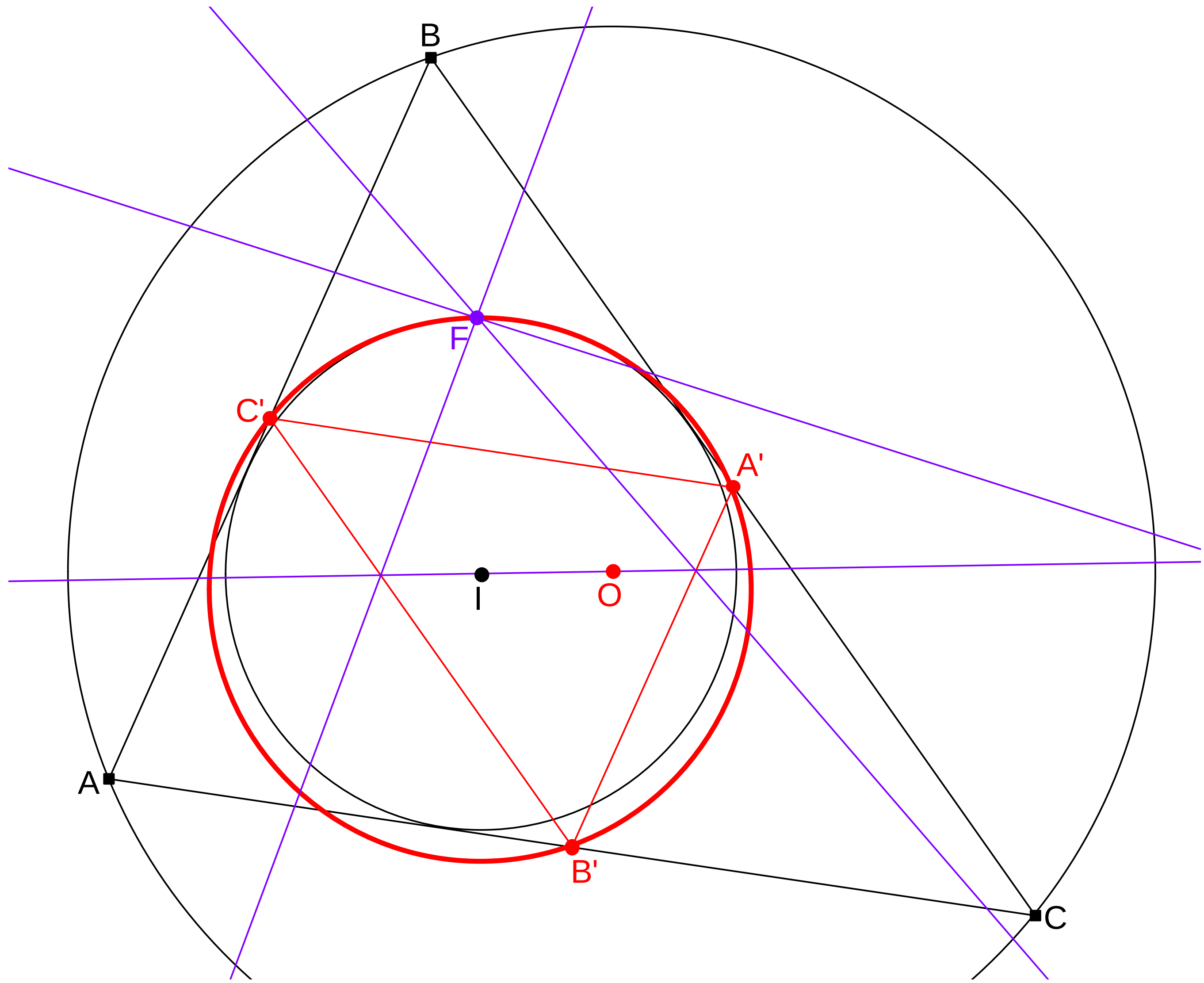

Fig. 20

Proof of Theorem 13. (See Fig. 21.) First, we have to show that the Anti-Steiner point of the line IO with respect to triangle $A'B'C'$ is defined at all; this requires showing that the line IO passes through the orthocenter of triangle $A'B'C'$. This, however, is clear because the point O is the orthocenter of triangle $A'B'C'$ (in fact, since the point O is the circumcenter of triangle ABC, it lies on the perpendicular bisector of its side BC; this yields $OA' \perp BC$, since A' is the midpoint of this side BC; using $B'C' \parallel BC$, this transforms into $OA' \perp B'C'$, and similarly we get $OB' \perp C'A'$ and $OC' \perp A'B'$, what shows that the point O is the orthocenter of triangle $A'B'C'$).

Now, as we have shown that the line IO passes through the orthocenter of triangle $A'B'C'$, Theorem 11 yields that there exists an Anti-Steiner point F_1 of this line IO with respect to triangle $A'B'C'$, and that this point F_1 satisfies the equation $\measuredangle B'C'F_1 = 90^\circ - \measuredangle (C'A';\ IO)$.

On the other hand, Theorem 7 yields $\measuredangle (YB_1;\ BI) = \measuredangle (CA;\ IO)$. Since $C'A' \parallel CA$, we have $\measuredangle (CA;\ IO) = \measuredangle (C'A';\ IO)$, so this becomes $\measuredangle (YB_1;\ BI) = \measuredangle (C'A';\ IO)$.

After Theorem 5, the line $B'B_1$ is perpendicular to the bisector of the angle ABC, i. e. to the line BI; this entails $\measuredangle (B'B_1;\ BI) = 90^\circ$. Since the points B', F, C', B_1 all lie on the nine-point circle of triangle ABC, we have

$$\measuredangle B'C'F = \measuredangle B'B_1F = \measuredangle\left(B'B_1;\ YB_1\right) = \measuredangle\left(B'B_1;\ BI\right) - \measuredangle\left(YB_1;\ BI\right) = 90^\circ - \measuredangle\left(C'A';\ IO\right).$$

Comparing this with $\measuredangle B'C'F_1 = 90^\circ - \measuredangle\left(C'A';\ IO\right)$, we get $\measuredangle B'C'F = \measuredangle B'C'F_1$. Therefore, the point F lies on the line $C'F_1$. Similarly, it can be shown that the point F lies on the lines $A'F_1$ and $B'F_1$. But the lines $A'F_1$, $B'F_1$, $C'F_1$ have only one point in common, namely the point F_1. Thus, $F = F_1$; in other words, the point F coincides with the Anti-Steiner point F_1 of the line IO with respect to triangle $A'B'C'$. This proves Theorem 13.

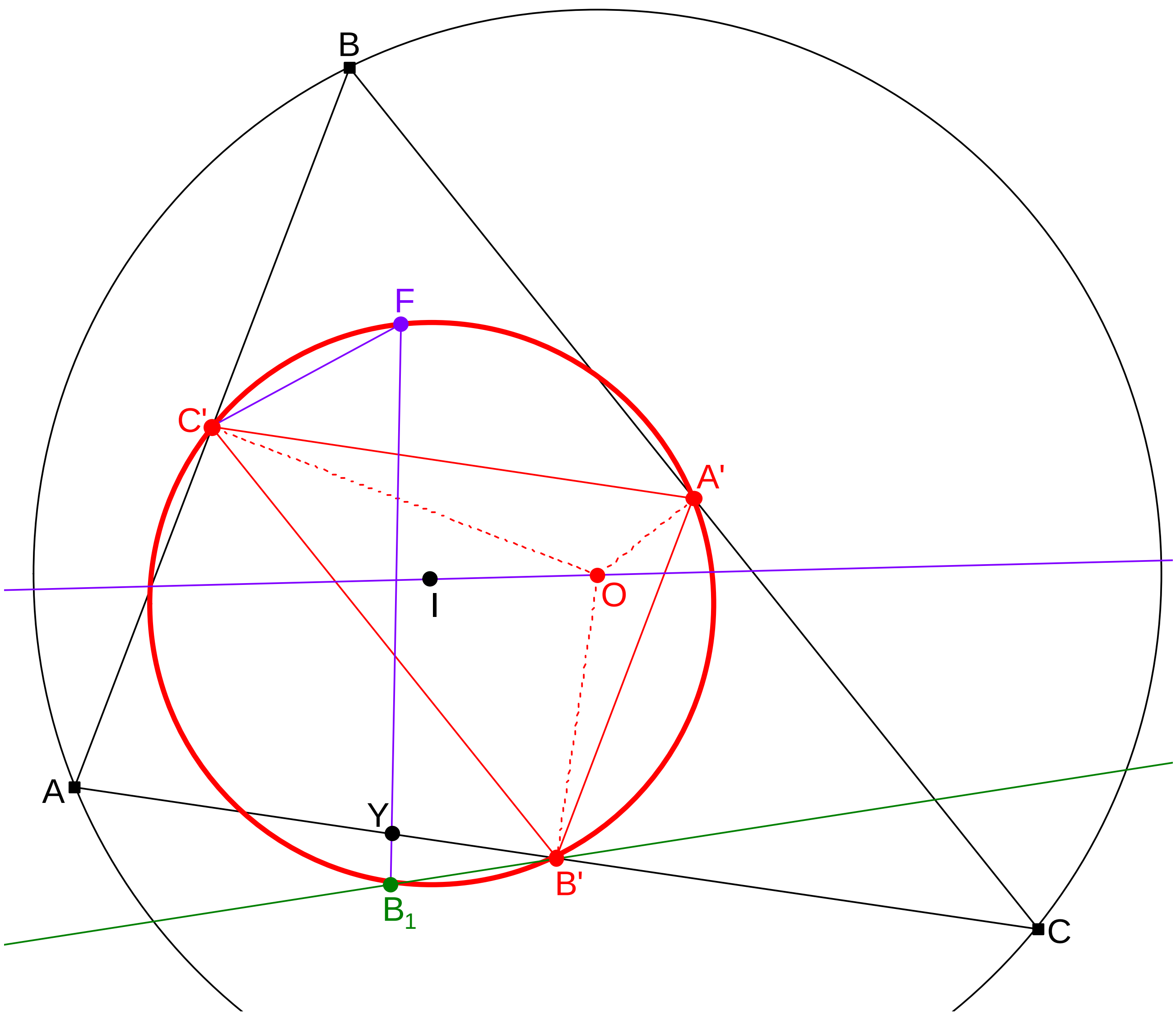

Fig. 21

The second characteristic of the Feuerbach point is going to be similar to the first one, though harder to prove:

Theorem 14. The Feuerbach point F of triangle ABC is the Anti-Steiner point of the line IO with respect to triangle XYZ. (See Fig. 22.)

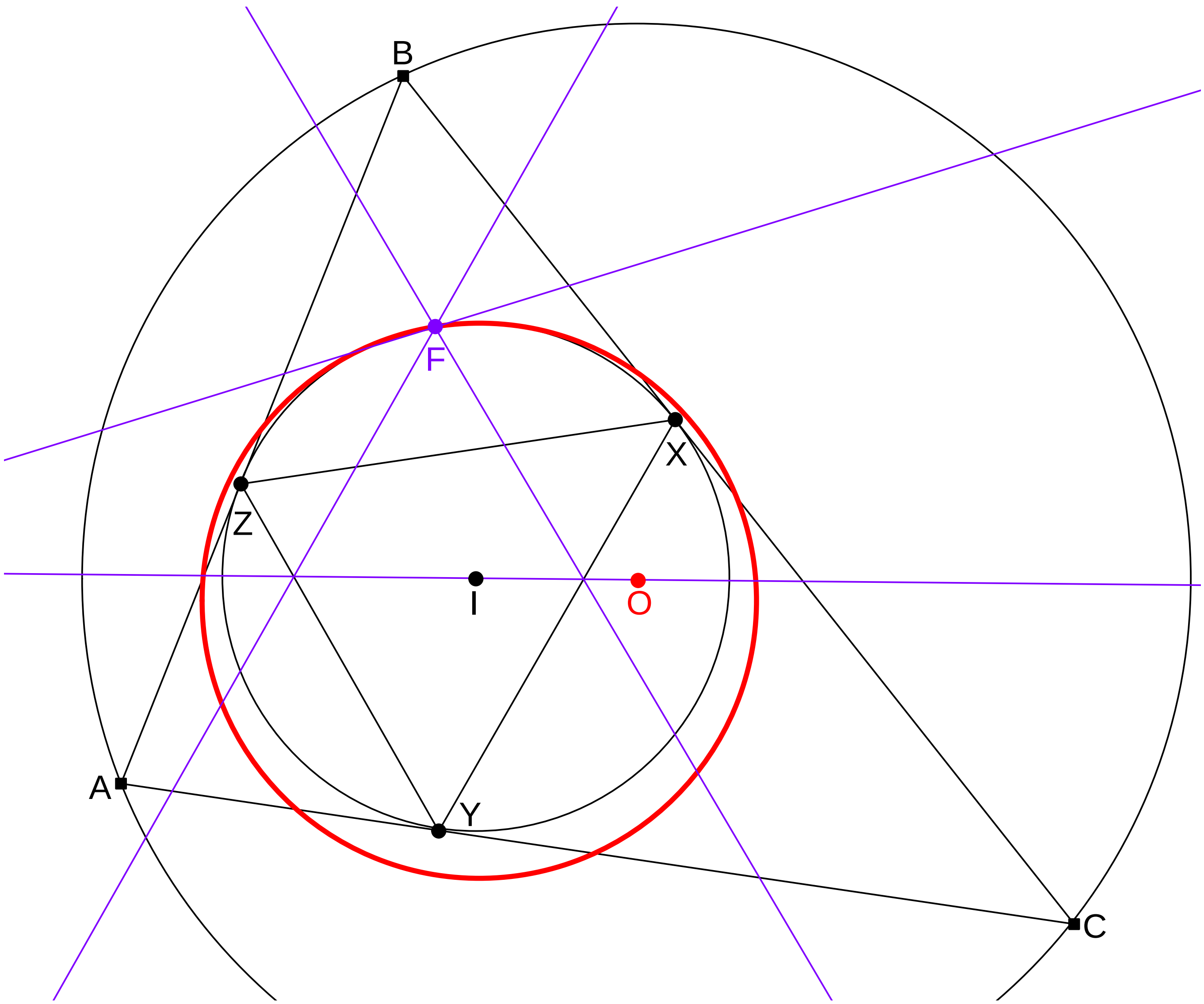

Fig. 22

Proof of Theorem 14. Again, we first have to show that the line IO passes through the orthocenter of triangle XYZ. This is an old result, and a number of proofs can be found at [9]; here, for the sake of completeness, we give a self-contained proof:

Theorem 15. The orthocenter of triangle XYZ lies on the line IO. Equivalently: The line IO is the Euler line of triangle XYZ.

Proof of Theorem 15. (See Fig. 23.) Let X', Y', Z' be the points of intersection of the X-altitude, Y-altitude, Z-altitude of triangle XYZ with the incircle of triangle ABC different from X, Y, Z, respectively.[3]

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[3] Theorem 12 would now almost immediately yield that these points X', Y', Z' are the reflections of the orthocenter of triangle XYZ in its sides YZ, ZX, XY, but this won't be of use in our proof.

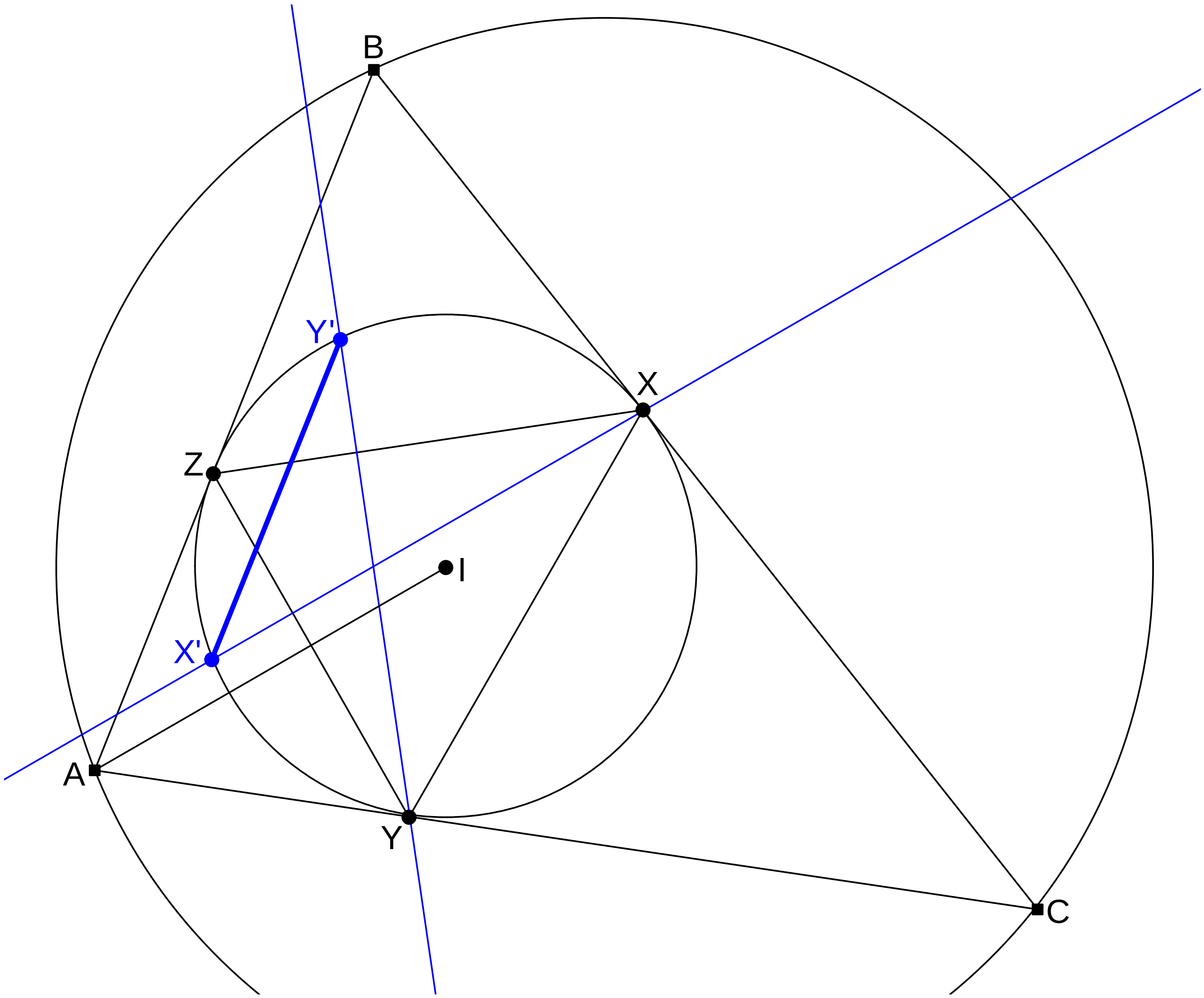

Fig. 23

As the points X, Y, X', Y' all lie on the incircle of triangle ABC, we have $\measuredangle XX'Y' = \measuredangle XYY'$. The line YY', being an altitude in triangle XYZ, is perpendicular to its side ZX; thus, $\measuredangle\left(ZX;\ YY'\right) = 90^\circ$. Therefore,

$$\begin{aligned}\measuredangle XX'Y' &= \measuredangle XYY' = \measuredangle\left(XY;\ YY'\right) = \measuredangle\left(XY;\ ZX\right) + \measuredangle\left(ZX;\ YY'\right) \\ &= \measuredangle\left(XY;\ ZX\right) + 90^\circ = \measuredangle YXZ + 90^\circ.\end{aligned}$$

From $YZ \perp AI$, it follows that $\measuredangle\left(AI;\ YZ\right) = 90^\circ$. On the other hand, the tangent-chordal theorem yields $\measuredangle YXZ = \measuredangle\left(YZ;\ AB\right)$, since $\measuredangle YXZ$ is the angle subtended by the chord YZ in the incircle of triangle ABC and AB is the tangent to this incircle at the point Z. Thus,

$$\measuredangle XX'Y' = \measuredangle YXZ + 90^\circ = \measuredangle\left(YZ;\ AB\right) + \measuredangle\left(AI;\ YZ\right) = \measuredangle\left(AI;\ AB\right) = \measuredangle IAB.$$

In other words, $\measuredangle\left(XX';\ X'Y'\right) = \measuredangle\left(AI;\ AB\right)$.

Now, the line XX', being an altitude in triangle XYZ, is perpendicular to its side YZ; together with $YZ \perp AI$, this results in $XX' \parallel AI$, thus $\measuredangle\left(XX';\ X'Y'\right) = \measuredangle\left(AI;\ X'Y'\right)$. Consequently, the equation $\measuredangle\left(XX';\ X'Y'\right) = \measuredangle\left(AI;\ AB\right)$ becomes

$\measuredangle(AI;\ X'Y') = \measuredangle(AI;\ AB)$, so that $X'Y' \parallel AB$. Similarly, $Y'Z' \parallel BC$ and $Z'X' \parallel CA$. This shows that triangles $X'Y'Z'$ and ABC are homothetic; i. e., there exists a homothety which maps the triangle ABC to the triangle $X'Y'Z'$. Denote by T the center of this homothety.

Now, this homothety, as it maps the triangle ABC to the triangle $X'Y'Z'$, must also take the circumcenter of triangle ABC to the circumcenter of triangle $X'Y'Z'$; since the circumcenter of triangle ABC is the point O, while the circumcenter of triangle $X'Y'Z'$ is the point I (in fact, the circumcircle of triangle $X'Y'Z'$ is the incircle of triangle ABC and thus centered at I), this means that our homothety maps the point O to the point I; since the center of the homothety is T, we thus conclude that the points O, I, T are collinear.

(See Fig. 24.) Now, let H' be the image of the point I under our homothety with center T which maps the triangle ABC to the triangle $X'Y'Z'$. Then, the points I, H', T are collinear; i. e., the point H' lies on the line IT. Since the points O, I, T are collinear, this line IT coincides with the line IO; thus, we see that the point H' lies on the line IO.

Now, as the point H' is the image of the point I under a homothety which maps the triangle ABC to the triangle $X'Y'Z'$, we must have $\measuredangle H'X'Y' = \measuredangle IAB$ (homotheties leave directed angles invariant). Comparing this with the equality $\measuredangle XX'Y' = \measuredangle IAB$ we got above, we obtain $\measuredangle H'X'Y' = \measuredangle XX'Y'$; thus, the point H' lies on the line XX', i. e. on the X-altitude of triangle XYZ. Similarly, the point H' lies on the other two altitudes of triangle XYZ; thus, the point H' is the orthocenter of triangle XYZ. Since we already know that the point H' lies on the line IO, we have thus proven that the orthocenter of triangle XYZ lies on the line IO.

On the other hand, the circumcenter of triangle XYZ is the point I (in fact, the circumcircle of triangle XYZ is the incircle of triangle ABC and has the center I); this point I, trivially, also lies on the line IO.

Thus, the line IO passes through both the orthocenter and the circumcenter of triangle XYZ; this means that the line IO is the Euler line of triangle XYZ. This completes the proof of Theorem 15.

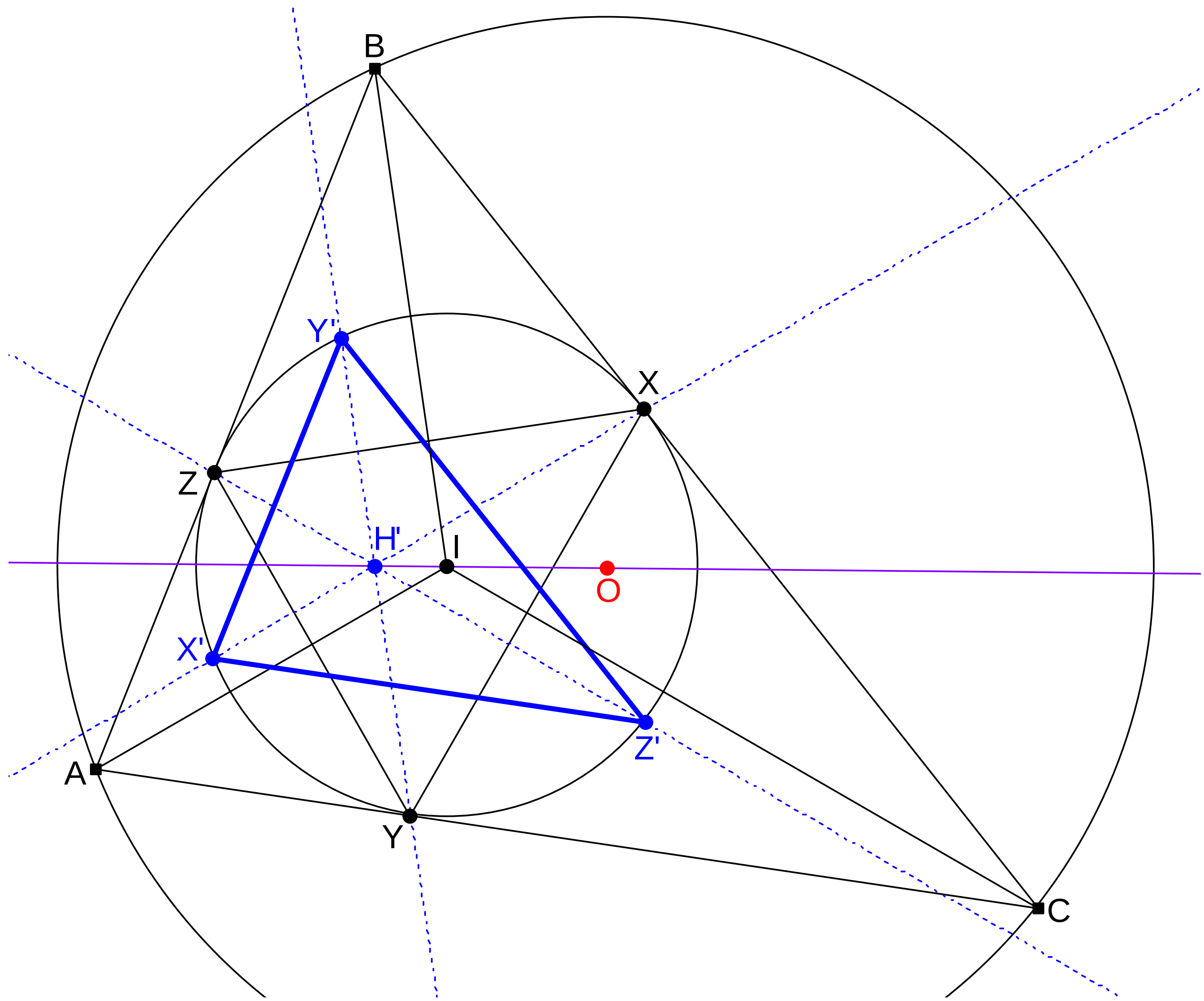

Fig. 24

As we now have shown that the line IO passes through the orthocenter of triangle XYZ, according to Theorem 11, there exists an Anti-Steiner point F_2 of this line IO with respect to triangle XYZ, and this point F_2 satisfies $\measuredangle YZF_2 = 90° - \measuredangle(ZX;\ IO)$.

(See Fig. 25.) Now, $\measuredangle YZF = \measuredangle(CA;\ YF)$ by the tangent-chordal angle theorem, since $\measuredangle YZF$ is the angle subtended by the chord YF in the incircle of triangle ABC and CA is the tangent to this incircle at the point Y. Furthermore, $\measuredangle(YB_1;\ BI) = \measuredangle(CA;\ IO)$ by Theorem 7. Finally, $ZX \perp BI$ leads to $\measuredangle(ZX;\ BI) = 90°$. This all yields

$$\begin{aligned}\measuredangle YZF &= \measuredangle(CA;\ YF) = \measuredangle(CA;\ YB_1) = \measuredangle(CA;\ BI) - \measuredangle(YB_1;\ BI) = \measuredangle(CA;\ BI) - \measuredangle(CA;\ IO) \\ &= \measuredangle(IO;\ BI) = \measuredangle(ZX;\ BI) - \measuredangle(ZX;\ IO) = 90° - \measuredangle(ZX;\ IO),\end{aligned}$$

so that $\measuredangle YZF = \measuredangle YZF_2$. Hence, the point F lies on the line ZF_2. Similarly, the point F also lies on the lines XF_2 and YF_2. But the lines XF_2, YF_2, ZF_2 have only one point in common, namely the point F_2; thus, the point F must coincide with the point F_2. In other words, the point F is the Anti-Steiner point F_2 of the line IO with respect to triangle XYZ. This completes the proof of Theorem 14.

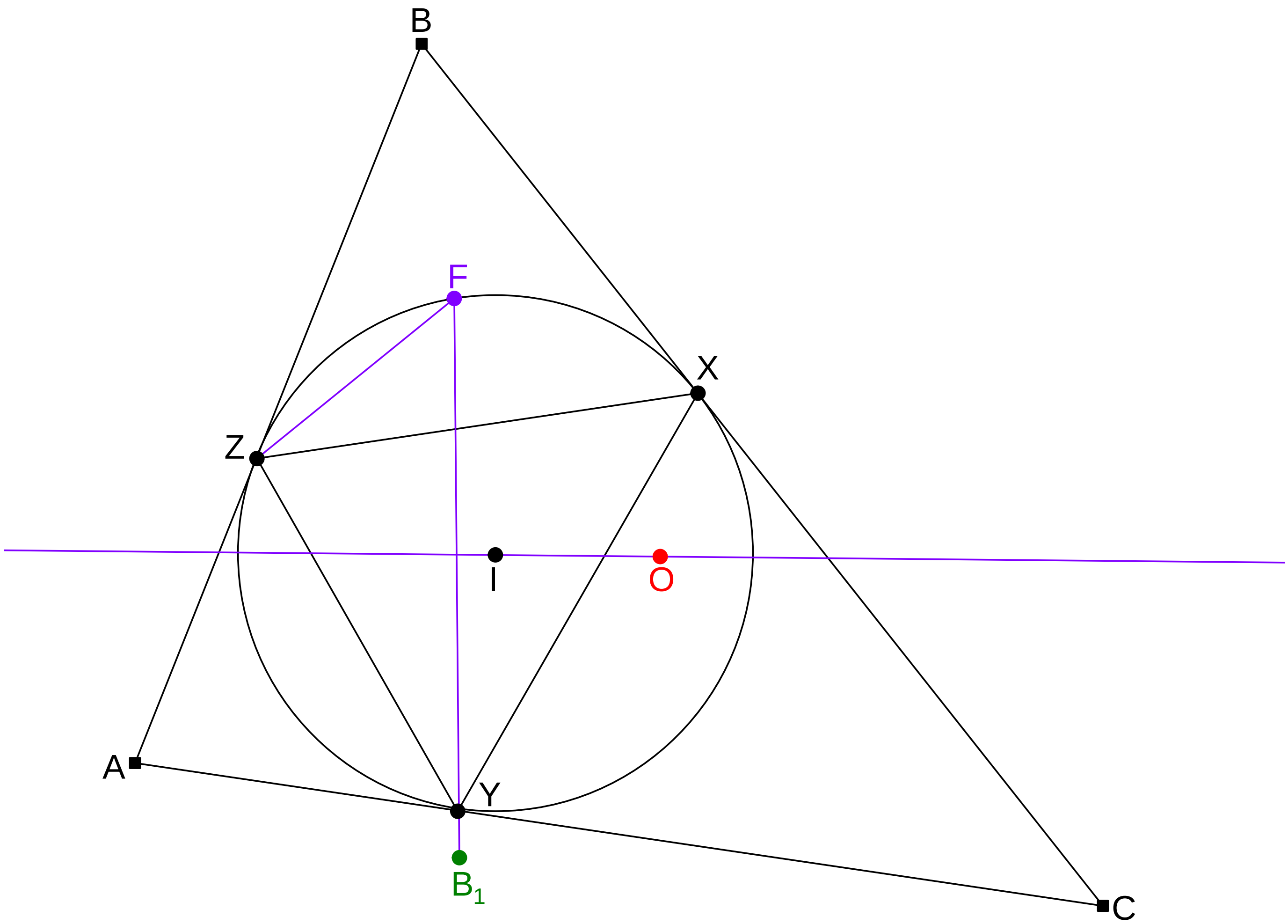

Fig. 25

Theorems 14 and 15 allow a simple characterization of the Feuerbach point F of triangle ABC from the viewpoint of triangle XYZ: The Feuerbach point F of triangle ABC is the Anti-Steiner point of the Euler line IO of triangle XYZ with respect to triangle XYZ.

References

[1] Baltic Way, *Problems of the Baltic Way Competition 1995.*
`http://www.hh.schule.de/ifl/mathematik/baltic-way.htm`

[2] MathLinks topic, *concyclic points in a triangle*.
`http://www.mathlinks.ro/Forum/viewtopic.php?t=13998`

[3] MathLinks topic, *Geometry , gem 4 [a circle centered on nine-point circle]*.
`http://www.mathlinks.ro/Forum/viewtopic.php?t=52070`

[4] Kiran Kedlaya, *Geometry Unbound*, version of January 18, 2006.
`http://www-math.mit.edu/~kedlaya/geometryunbound`

[5] R. A. Johnson, *Directed Angles in Elementary Geometry*, American Mathematical Monthly 24 (1917) nr. 3, pp. 101-105.

[6] R. A. Johnson, *Advanced Euclidean Geometry*, Dover reprint 1925.

[7] H. S. M. Coxeter, Samuel L. Greitzer: *Geometry Revisited*, MAA 1967.

[8] Alexander Bogomolny, *The Euler Line and the 9-Point Circle*.
http://www.cut-the-knot.org/triangle/EulerLine.shtml

[9] MathLinks topic, *i find it difficult [Euler line of intouch triangle]*.
http://www.mathlinks.ro/Forum/viewtopic.php?t=6228