Functional Equations

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Contents

1 Basic Methods For Solving Functional Equations

- Substituting the values for variables. The most common first attempt is with some constants (eg. 0 or 1), after that (if possible) some expressions which will make some part of the equation to become constant. For example if $f(x+y)$ appears in the equations and if we have found $f(0)$ then we plug $y=-x$. Substitutions become less obvious as the difficulty of the problems increase.

- Mathematical induction. This method relies on using the value $f(1)$ to find all $f(n)$ for n integer. After that we find $f\left(\frac{1}{n}\right)$ and $f(r)$ for rational r. This method is used in problems where the function is defined on $\mathbb{Q}$ and is very useful, especially with easier problems.

- Investigating for injectivity or surjectivity of functions involved in the equaiton. In many of the problems these facts are not difficult to establish but can be of great importance.

- Finding the fixed points or zeroes of functions. The number of problems using this method is considerably smaller than the number of problems using some of the previous three methods. This method is mostly encountered in more difficult problems.

- Using the Cauchy's equation and equation of its type.

- Investigating the monotonicity and continuity of a function. Continuity is usually given as additional condition and as the monotonicity it usually serves for reducing the problem to Cauchy's equation. If this is not the case, the problem is on the other side of difficulty line.

- Assuming that the function at some point is greater or smaller then the value of the function for which we want to prove that is the solution. Most often it is used as continuation of the method of mathematical induction and in the problems in which the range is bounded from either side.

- Making recurrent relations. This method is usually used with the equations in which the range is bounded and in the case when we are able to find a relashionship between $f(f(n))$, $f(n)$, and n.

- Analyzing the set of values for which the function is equal to the assumed solution. The goal is to prove that the described set is precisely the domain of the function.

- Substituting the function. This method is often used to simplify the given equation and is seldom of crucial importance.

- Expressing functions as sums of odd and even. Namely each function can be represented as a sum of one even and one odd function and this can be very handy in treating "linear" functional equations involving many functions.

- Treating numbers in a system with basis different than 10. Of course, this can be used only if the domain is $\mathbb{N}$.

- For the end let us emphasize that it is very important to guess the solution at the beginning. This can help a lot in finding the appropriate substitutions. Also, at the end of the solution, DON'T FORGET to verify that your solution satisfies the given condition.

2 Cauchy Equation and Equations of the Cauchy type

The equation $f(x+y) = f(x) + f(y)$ is called the Cauchy equation. If its domain is $\mathbb{Q}$, it is well-known that the solution is given by $f(x) = xf(1)$. That fact is easy to prove using mathematical induction. The next problem is simply the extention of the domain from $\mathbb{Q}$ to $\mathbb{R}$. With a relatively easy counter-example we can show that the solution to the Cauchy equation in this case doesn't have to be $f(x) = xf(1)$. However there are many additional assumptions that forces the general solution to be of the described form. Namely if a function f satisfies any of the conditions:

- monotonicity on some interval of the real line;

- continuity;

- boundedness on some interval;

- positivity on the ray $x \geq 0$;

then the general solution to the Cauchy equation $f : \mathbb{R} \to S$ has to be $f(x) = xf(1)$.

The following equations can be easily reduced to the Cauchy equation.

- All continuous functions $f : \mathbb{R} \to (0, +\infty)$ satisfying $f(x+y) = f(x)f(y)$ are of the form $f(x) = a^x$. Namely the function $g(x) = \log f(x)$ is continuous and satisfies the Cauchy equation.

- All continuous functions $f : (0, +\infty) \to \mathbb{R}$ satisfying $f(xy) = f(x) + f(y)$ are of the form $f(x) = \log_a x$. Now the function $g(x) = f(a^x)$ is continuous and satisfies the Cauchy equation.

- All continuous functions $f : (0, +\infty) \to (0, +\infty)$ satisfying $f(xy) = f(x)f(y)$ are $f(x) = x^t$, where $t = \log_a b$ and $f(a) = b$. Indeed the function $g(x) = \log f(a^x)$ is continuous and satisfies the Cauchy equation.

3 Problems with Solutions

The following examples should illustrate the previously outlined methods.

Problem 1. *Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(1) = 2$ and $f(xy) = f(x)f(y) - f(x+y) + 1$.*

Solution. This is a classical example of a problem that can be solved using mathematical induction. Notice that if we set $x=1$ and $y=n$ in the original equation we get $f(n+1)=f(n)+1$, and since $f(1)=2$ we have $f(n)=n+1$ for every natural number n. Similarly for $x=0$ and $y=n$ we get $f(0)n=f(n)-1=n$, i.e. $f(0)$. Now our goal is to find $f(z)$ for each $z\in\mathbb{Z}$. Substituting $x=-1$ and $y=1$ in the original equation gives us $f(-1)=0$, and setting $x=-1$ and $y=n$ gives $f(-n)=-f(n-1)+1=-n+1$. Hence $f(z)=z+1$ for each $z\in\mathbb{Z}$. Now we have to determine $f\left(\frac{1}{n}\right)$. Plugging $x=n$ and $y=\frac{1}{n}$ we get

$$f(1)=(n+1)f\left(\frac{1}{n}\right)-f\left(n+\frac{1}{n}\right)+1. \tag{1}$$

Furthermore for $x=1$ and $y=m+\frac{1}{n}$ we get $f\left(m+1+\frac{1}{n}\right)=f\left(m+\frac{1}{n}\right)+1$, hence by the mathematical induction $f\left(m+\frac{1}{n}\right)=m+f\left(\frac{1}{n}\right)$. Iz (1) we now have

$$f\left(\frac{1}{n}\right)=\frac{1}{n}+1,$$

for every natural number n. Furthermore for $x=m$ and $y=\frac{1}{n}$ we get $f\left(\frac{m}{n}\right)=\frac{m}{n}+1$, i.e. $f(r)=r+1$, for every positive rational number r. Setting $x=-1$ and $y=r$ we get $f(-r)=-f(r-1)+1=-r+1$ as well hence $f(x)=x+1$, for each $x\in\mathbb{Q}$.
Verification: Since $xy+1=(x+1)(y+1)-(x+y+1)+1$, for all $x,y\in\mathbb{Q}$, f is the solution to our equation. $\triangle$

Problem 2. *(Belarus 1997) Find all functions $g:\mathbb{R}\to\mathbb{R}$ such that for arbitrary real numbers x and y:*

$$g(x+y)+g(x)g(y)=g(xy)+g(x)+g(y).$$

Solution. Notice that $g(x)=0$ and $g(x)=2$ are obviously solutions to the given equation. Using mathematical induction it is not difficult to prove that if g is not equal to one of these two functions then $g(x)=x$ for all $x\in\mathbb{Q}$. It is also easy to prove that $g(r+x)=r+g(x)$ and $g(rx)=rg(x)$, where r is rational and x real number. Particularly from the second equation for $r=-1$ we get $g(-x)=-g(x)$, hence setting $y=-x$ in the initial equation gives $g(x)^2=g(x^2)$. This means that $g(x)\geq 0$ for $x\geq 0$. Now we use the standard method of extending to $\mathbb{R}$. Assume that $g(x)<x$. Choose $r\in\mathbb{Q}$ such that $g(x)<r<x$. Then

$$r>g(x)=g(x-r)+r\geq r,$$

which is clearly a contradiction. Similarly from $g(x)>x$ we get another contradiction. Thus we must have $g(x)=x$ for every $x\in\mathbb{R}$. It is easy to verify that all three functions satisfy the given functional equation. $\triangle$

Problem 3. *The function $f:\mathbb{R}\to\mathbb{R}$ satisfies $x+f(x)=f(f(x))$ for every $x\in\mathbb{R}$. Find all solutions of the equation $f(f(x))=0$.*

Solution. The domain of this function is $\mathbb{R}$, so there isn't much hope that this can be solved using mathematical induction. Notice that $f(f(x))-f(x)=x$ and if $f(x)=f(y)$ then clearly $x=y$. This means that the function is injective. Since $f(f(0))=f(0)+0=f(0)$, because of injectivity we must have $f(0)=0$, implying $f(f(0))=0$. If there were another x such that $f(f(x))=0=f(f(0))$, injectivity would imply $f(x)=f(0)$ and $x=0$. $\triangle$

Problem 4. *Find all injective functions $f:\mathbb{N}\to\mathbb{R}$ that satisfy:*

$$(a)\ f(f(m)+f(n))=f(f(m))+f(n),\quad (b)\ f(1)=2,\ f(2)=4.$$

Solution. Setting $m = 1$ and n first, and $m = n$, $n = 1$ afterwards we get

$$f(f(1)+f(n)) = f(f(1))+f(n), \quad f(f(n)+f(1)) = f(f(n))+f(1).$$

Let us emphasize that this is one standard idea if the expression on one side is symmetric with respect to the variables while the expression on the other side is not. Now we have $f(f(n)) = f(n)-f(1)+f(f(1)) = f(n)-2+f(2) = f(n)+2$. From here we conclude that $f(n) = m$ implies $f(m) = m+2$ and now the induction gives $f(m+2k) = m+2k+2$, for every $k \geq 0$. Specially if $f(1) = 2$ then $f(2n) = 2n+2$ for all positive integers n. The injectivity of f gives that at odd numbers (except 1) the function has to take odd values. Let p be the smallest natural number such that for some k $f(k) = 2p+1$. We have $f(2p+2s+1) = 2p+2s+3$ for $s \geq 0$. Therefore the numbers $3,5,\dots,2p-1$ are mapped into $1,3,\dots,2p+1$. If $f(t) = 1$ for some t, then for $m = n = t$ $4 = f(2) = f(f(t)+f(t)) = f(f(t))+f(t) = 3$, which is a contradiction. If for some t such that $f(t) = 3$ then $f(3+2k) = 5+2k$, which is a contradiction to the existence of such t. It follows that the numbers $3,5,\dots,2p-1$ are mapped into $5,7,\dots,2p+1$. Hence $f(3+2k) = 5+2k$. Thus the solution is $f(1) = 2$ and $f(n) = n+2$, for $n \geq 2$.
It is easy to verify that the function satisfies the given conditions. $\triangle$

Problem 5. *(BMO 1997, 2000) Solve the functional equation*

$$f(xf(x)+f(y)) = y+f(x)^2, \; x,y \in \mathbb{R}.$$

Solution. In probelms of this type it is usually easy to prove that the functions are injective or surjective, if the functions are injective/surjective. In this case for $x = 0$ we get $f(f(y)) = y+f(0)^2$. Since the function on the right-hand side is surjective the same must hold for the function on the left-hand side. This implies the surjectivity of f. Injectivity is also easy to establish. Now there exists t such that $f(t) = 0$ and substitution $x = 0$ and $y = t$ yields $f(0) = t+f(0)^2$. For $x = t$ we get $f(f(y)) = y$. Therefore $t = f(f(t)) = f(0) = t+f(0)^2$, i.e. $f(0) = 0$. Replacing x with $f(x)$ gives

$$f(f(x)x+f(y)) = x^2+y,$$

hence $f(x)^2 = x^2$ for every real number x. Consider now the two cases:
First case $f(1) = 1$. Plugging $x = 1$ gives $f(1+f(y)) = 1+y$, and after taking squares $(1+y)^2 = f(1+f(y))^2 = (1+f(y))^2 = 1+2f(y)+f(y)^2 = 1+2f(y)+y^2$. Clearly in this case we have $f(y) = y$ for every real y.
Second case $f(1) = -1$. Plugging $x = -1$ gives $f(-1+f(y)) = 1+y$, and after taking squares $(1+y)^2 = f(-1+f(y))^2 = (-1+f(y))^2 = 1-2f(y)+f(y)^2 = 1-2f(y)+y^2$. Now we conclude $f(y) = -y$ for every real number y.
It is easy to verify that $f(x) = x$ and $f(x) = -x$ are indeed the solutions. $\triangle$

Problem 6. *(IMO 1979, shortlist) Given a function $f : \mathbb{R} \to \mathbb{R}$, if for every two real numbers x and y the equality $f(xy+x+y) = f(xy)+f(x)+f(y)$ holds, prove that $f(x+y) = f(x)+f(y)$ for every two real numbers x and y.*

Solution. This is a clasical example of the equation that solution is based on a careful choice of values that are plugged in a functional equation. Plugging in $x = y = 0$ we get $f(0) = 0$. Plugging in $y = -1$ we get $f(x) = -f(-x)$. Plugging in $y = 1$ we get $f(2x+1) = 2f(x)+f(1)$ and hence $f(2(u+v+uv)+1) = 2f(u+v+uv)+f(1) = 2f(uv)+2f(u)+2f(v)+f(1)$ for all real u and v. On the other hand, plugging in $x = u$ and $y = 2v+1$ we get $f(2(u+v+uv)+1) = f(u+(2v+1)+u(2v+1)) = f(u)+2f(v)+f(1)+f(2uv+u)$. Hence it follows that $2f(uv)+2f(u)+2f(v)+f(1) = f(u)+2f(v)+f(1)+f(2uv+u)$, i.e.,

$$f(2uv+u) = 2f(uv)+f(u). \tag{1}$$

Plugging in $v = -1/2$ we get $0 = 2f(-u/2)+f(u) = -2f(u/2)+f(u)$. Hence, $f(u) = 2f(u/2)$ and consequently $f(2x) = 2f(x)$ for all reals. Now (1) reduces to $f(2uv+u) = f(2uv)+f(u)$. Plugging in $u = y$ and $x = 2uv$, we obtain $f(x)+f(y) = f(x+y)$ for all nonzero reals x and y. Since $f(0) = 0$, it trivially holds that $f(x+y) = f(x)+f(y)$ when one of x and y is 0. $\triangle$

Problem 7. *Does there exist a function $f:\mathbb{R}\to\mathbb{R}$ such that $f(f(x))=x^2-2$ for every real number x?*

Solution. After some attempts we can see that none of the first three methods leads to a progress. Notice that the function g of the right-hand side has exactly 2 fixed points and that the function $g\circ g$ has exactly 4 fixed points. Now we will prove that there is no function f such that $f\circ f=g$. Assume the contrary. Let a,b be the fixed points of g, and a,b,c,d the fixed points of $g\circ g$. Assume that $g(c)=y$. Then $c=g(g(c))=g(y)$, hence $g(g(y))=g(c)=y$ and y has to be on of the fixed points of $g\circ g$. If $y=a$ then from $a=g(a)=g(y)=c$ we get a contradiction. Similarly $y\neq b$, and since $y\neq c$ we get $y=d$. Thus $g(c)=d$ and $g(d)=c$. Furthermore we have $g(f(x))=f(f(f(x)))=f(g(x))$. Let $x_0\in\{a,b\}$. We immediately have $f(x_0)=f(g(x_0))=g(f(x_0))$, hence $f(x_0)\in\{a,b\}$. Similarly if $x_1\in\{a,b,c,d\}$ we get $f(x_1)\in\{a,b,c,d\}$, and now we will prove that this is not possible. Take first $f(c)=a$. Then $f(a)=f(f(c))=g(c)=d$ which is clearly impossible. Similarly $f(c)\neq b$ and $f(c)\neq c$ (for otherwise $g(c)=c$) hence $f(c)=d$. However we then have $f(d)=f(f(c))=g(c)=d$, which is a contradiction, again. This proves that the required f doesn't exist. $\triangle$

Problem 8. *Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that $f(x)f(yf(x))=f(x+y)$ for every two positive real numbers x,y.*

Solution. Obviously $f(x)\equiv 1$ is one solution to the problem. The idea is to find y such that $yf(x)=x+y$ and use this to determine $f(x)$. For every x such that $\dfrac{x}{f(x)-1}\geq 0$ we can find such y and from the given condition we get $f(x)=1$. However this is a contradiction since we got that $f(x)>1$ implies $f(x)=1$. One of the consequences is that $f(x)\leq 1$. Assume that $f(x)<1$ for some x. From the given equation we conclude that f is non-increasing (because $f(yf(x))\leq 1$). Let us prove that f is decreasing. In order to do that it is enough to prove that $f(x)<1$, for each x. Assume that $f(x)=1$ for every $x\in(0,a)$ $(a>0)$. Substituting $x=y=\dfrac{2a}{3}$ in the given equation we get the obvious contradiction. This means that the function is decreasing and hence it is injective. Again everything will revolve around the idea of getting rid of $f(yf(x))$. Notice that $x+y>yf(x)$, therefore

$$f(x)f(yf(x))=f(x+y)=f(yf(x)+x+y-yf(x))=f(yf(x))f\Big(f\Big(yf(x)\Big)(x+y-yf(x))\Big),$$

i.e. $f(x)=f\Big(f\Big(yf(x)\Big)(x+y-yf(x))\Big)$. The injectivity of f implies that $x=f\Big(yf(x)\Big)(x+y-yf(x))$. If we plug $f(x)=a$ we get

$$f(y)=\frac{1}{1+\alpha z},$$

where $\alpha=\dfrac{1-f(a)}{af(a)}$, and according to our assumption $\alpha>0$.

It is easy to verify that $f(x)=\dfrac{1}{1+\alpha x}$, for $\alpha\in\mathbb{R}^+$, and $f(x)\equiv 1$ satisfy the equation. $\triangle$

Problem 9. *(IMO 2000, shortlist) Find all pairs of functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ such that for every two real numbers x,y the following relation holds:*

$$f(x+g(y))=xf(y)-yf(x)+g(x).$$

Solution. Let us first solve the problem under the assumption that $g(\alpha)=0$ for some α.

Setting $y=\alpha$ in the given equation yields $g(x)=(\alpha+1)f(x)-xf(\alpha)$. Then the given equation becomes $f(x+g(y))=(\alpha+1-y)f(x)+(f(y)-f(\alpha))x$, so setting $y=\alpha+1$ we get $f(x+n)=mx$, where $n=g(\alpha+1)$ and $m=f(\alpha+1)-f(\alpha)$. Hence f is a linear function, and consequently g is also linear. If we now substitute $f(x)=ax+b$ and $g(x)=cx+d$ in the given equation and compare the coefficients, we easily find that

$$f(x)=\frac{cx-c^2}{1+c}\quad\text{and}\quad g(x)=cx-c^2,\qquad c\in\mathbb{R}\setminus\{-1\}.$$

Now we prove the existence of α such that $g(\alpha)=0$. If $f(0)=0$ then putting $y=0$ in the given equation we obtain $f(x+g(0))=g(x)$, so we can take $\alpha=-g(0)$.

Now assume that $f(0)=b\neq 0$. By replacing x by $g(x)$ in the given equation we obtain $f(g(x)+g(y))=g(x)f(y)-yf(g(x))+g(g(x))$ and, analogously, $f(g(x)+g(y))=g(y)f(x)-xf(g(y))+g(g(y))$. The given functional equation for $x=0$ gives $f(g(y))=a-by$, where $a=g(0)$. In particular, g is injective and f is surjective, so there exists $c\in\mathbb{R}$ such that $f(c)=0$. Now the above two relations yield

$$g(x)f(y)-ay+g(g(x))=g(y)f(x)-ax+g(g(y)). \tag{1}$$

Plugging $y=c$ in (1) we get $g(g(x))=g(c)f(x)-ax+g(g(c))+ac=kf(x)-ax+d$. Now (1) becomes $g(x)f(y)+kf(x)=g(y)f(x)+kf(y)$. For $y=0$ we have $g(x)b+kf(x)=af(x)+kb$, whence

$$g(x)=\frac{a-k}{b}f(x)+k.$$

Note that $g(0)=a\neq k=g(c)$, since g is injective. From the surjectivity of f it follows that g is surjective as well, so it takes the value 0. $\triangle$

Problem 10. *(IMO 1992, shortlist) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ which satisfy*

$$f(f(x))+af(x)=b(a+b)x.$$

Solution. This is a typical example of a problem that is solved using recurrent equations. Let us define x_n inductively as $x_n=f(x_{n-1})$, where $x_0\geq 0$ is a fixed real number. It follows from the given equation in f that $x_{n+2}=-ax_{n+1}+b(a+b)x_n$. The general solution to this equation is of the form

$$x_n=\lambda_1 b^n+\lambda_2(-a-b)^n,$$

where $\lambda_1,\lambda_2\in\mathbb{R}$ satisfy $x_0=\lambda_1+\lambda_2$ and $x_1=\lambda_1 b-\lambda_2(a+b)$. In order to have $x_n\geq 0$ for all n we must have $\lambda_2=0$. Hence $x_0=\lambda_1$ and $f(x_0)=x_1=\lambda_1 b=bx_0$. Since x_0 was arbitrary, we conclude that $f(x)=bx$ is the only possible solution of the functional equation. It is easily verified that this is indeed a solution. $\triangle$

Problem 11. *(Vietnam 2003) Let F be the set of all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ which satisfy the inequality $f(3x)\geq f(f(2x))+x$, for every positive real number x. Find the largest real number α such that for all functions $f\in F$: $f(x)\geq\alpha\cdot x$.*

Solution. We clearly have that $\frac{x}{2}\in F$, hence $\alpha\leq\frac{1}{2}$. Furthermore for every function $f\in F$ we have $f(x)\geq\frac{x}{3}$. The idea is the following: Denote $\frac{1}{3}=\alpha_1$ and form a sequence $\{\alpha_n\}$ for which $f(x)\geq\alpha_n x$ and which will (hopefully) tend to $\frac{1}{2}$. This would imply that $\alpha\geq\frac{1}{2}$, and hence $\alpha=\frac{1}{2}$. Let us constract a recurrent relation for α_k. Assume that $f(x)\geq\alpha_k x$, for every $x\in\mathbb{R}^+$. From the given inequality we have

$$f(3x)\geq f(f(2x))+x\geq\alpha_k f(2x)+x\geq\alpha_k\cdot\alpha_k\cdot 2x+x=\alpha_{k+1}\cdot 3x.$$

This means that $\alpha_{n+1}=\frac{2\alpha_n^2+1}{3}$. Let us prove that $\lim_{n\to+\infty}\alpha_n=\frac{1}{2}$. This is a standard problem. It is easy to prove that the sequence α_k is increasing and bounded above by $\frac{1}{2}$. Hence it converges and its limit α satisfies $\alpha=\frac{2\alpha^2+1}{3}$, i.e. $\alpha=\frac{1}{2}$ (since $\alpha<1$). $\triangle$

Problem 12. *Find all functions $f,g,h:\mathbb{R}\to\mathbb{R}$ that satisfy*

$$f(x+y)+g(x-y)=2h(x)+2h(y).$$

Solution. Our first goal is to express f and g using h and get the equation involving h only. First taking $y=x$ and substituting $g(0)=a$ we get $f(2x)=4h(x)-a$. Furthermore by putting $y=0$ we get $g(x)=2h(x)+2b-4h\left(\frac{x}{2}\right)+a$, where $h(0)=b$. Now the original equation can be written as

$$2\left[h\left(\frac{x+y}{2}\right)+h\left(\frac{x-y}{2}\right)\right]+h(x-y)+b=h(x)+h(y). \tag{2}$$

Let $H(x)=h(x)-b$. These "longer" linear expressions can be easily handled if we express functions in form of the sum of an even and odd function, i.e. $H(x)=H_e(x)+H_o(x)$. Substituting this into (2) and writing the same expressions for $(-x,y)$ and $(x,-y)$ we can add them together and get:

$$2\left[H_e\left(\frac{x-y}{2}\right)+H_e\left(\frac{x+y}{2}\right)\right]+H_e(x-y)=H_e(x)+H_e(y). \tag{3}$$

If we set $-y$ in this expression and add to (3) we get (using $H_e(y)=H_e(-y)$)

$$H_e(x+y)-H_e(x-y)=2H_e(x)+2H_e(y).$$

The last equation is not very difficult. Mathematical induction yields $H_e(r)=\alpha r^2$, for every rational number r. From the continuity we get $H_e(x)=\alpha x^2$. Similar method gives the simple relation for H_o

$$H_o(x+y)+H_o(x-y)=2H_o(x).$$

This is a Cauchy equation hence $H_o(x)=\beta x$. Thus $h(x)=\alpha x^2+\beta x+b$ and substituting for f and g we get:

$$f(x)=\alpha x^2+2\beta x+4b-a, \quad g(x)=\alpha x^2+a.$$

It is easy to verify that these functions satisfy the given conditions.

Problem 13. *Find all functions* $f:\mathbb{Q}\to\mathbb{Q}$ *for which*

$$f(xy)=f(x)f(y)-f(x+y)+1.$$

Solve the same problem for the case $f:\mathbb{R}\to\mathbb{R}$.

Solution. It is not hard to see that for $x=y=0$ we get $(f(0)-1)^2=0$, i.e. $f(0)=1$. Furthermore, setting $x=1$ and $y=-1$ gives $f(-1)=f(1)f(-1)$, hence $f(-1)=0$ or $f(1)=1$. We will separate this into two cases:

1° Let $f(-1)=0$. In this innocent-looking problems that are resistent to usual ideas it is sometimes successful to increase the number of variables, i.e. to set yz instead of y:

$$f(xyz)=f(x)f(yz)-f(x+yz)+1=f(x)(f(y)f(z)-f(y+z)+1)-f(x+yz)+1.$$

Although it seems that the situation is worse and running out of control, that is not the case. Namely the expression on the left-hand side is symmetric, while the one on the right-hand side is not. Writing the same expression for x and equating gives

$$f(x)f(y+z)-f(x)+f(x+yz)=f(z)f(x+y)-f(z)+f(xy+z). \tag{4}$$

Setting $z=-1$ (we couldn't do that at the beginning, since $z=1$ was fixed) we get $f(x)f(z-1)-f(x)+f(x-y)=f(xy-1)$, and setting $x=1$ in this equality gives

$$f(y-1)(1-f(1))=f(1-y)-f(1). \tag{5}$$

Setting $y=2$ gives $f(1)(2-f(1))=0$, i.e. $f(1)=0$ or $f(1)=2$. This means that we have two cases here as well:

1.1° If $f(1)=0$, then from (5) plugging $y+1$ instead of y we get $f(y)=f(-y)$. Setting $-y$ instead of y in the initial equality gives $f(xy)=f(x)f(y)-f(x-y)+1$, hence $f(x+y)=f(x-y)$, for every two rational numbers x and y. Specially for $x=y$ we get $f(2x)=f(0)=1$, for all $x\in\mathbb{Q}$. However this is a contradiction with $f(1)=0$. In this case we don't have a solution.

1.2° If $f(1)=2$, setting $y+1$ instead of y in (5) gives $1-f(y)=f(-y)-1$. It is clear that we should do the substitution $g(x)=1-f(x)$ because the previous equality gives $g(-x)=-g(x)$, i.e. g is odd. Furthermore substituting g into the original equality gives

$$g(xy)=g(x)+g(y)-g(x)g(y)-g(x+y). \tag{6}$$

Setting $-y$ instead of y we get $-g(xy)=g(x)-g(y)+g(x)g(y)-g(x-y)$, and adding with (6) yields $g(x+y)+g(x-y)=2g(x)$. For $x=y$ we have $g(2x)=2g(x)$ therefore we get $g(x+y)+g(x-y)=g(2x)$. This is a the Cauchy equation and since the domain is $\mathbb{Q}$ we get $g(x)=rx$ for some rational number r. Plugging this back to (6) we obtain $r=-1$, and easy verification shows that $f(x)=1+x$ satisfies the conditions of the problem.

2° Let $f(1)=1$. Setting $z=1$ in (4) we get

$$f(xy+1)-f(x)f(y+1)+f(x)=1,$$

hence for $y=-1$ we get $f(1-x)=1$, for every rational x. This means that $f(x)\equiv 1$ and this function satisfies the given equation.

Now let us solve the problem where $f:\mathbb{R}\to\mathbb{R}$. Notice that we haven't used that the range is $\mathbb{Q}$, hence we conclude that for all rational numbers q $f(q)=q+1$, or $f(q)\equiv 1$. If $f(q)=1$ for all rational numbers q, it can be easily shown that $f(x)\equiv 1$. Assume that $f(q)\not\equiv 1$. From the above we have that $g(x)+g(y)=g(x+y)$, hence it is enough to prove monotonicity. Substitute $x=y$ in (6) and use $g(2x)=2g(x)$ to get $g(x^2)=-g(x)^2$. Therefore for every positive r the value $g(r)$ is non-positive. Hence if $y>x$, i.e. $y=x+r^2$ we have $g(y)=g(x)+g(r^2)\le g(x)$, and the function is decreasing. This means that $f(x)=1+\alpha x$ and after some calculation we get $f(x)=1+x$. It is easy to verify that so obtained functions satisfy the given functional equation. △

Problem 14. *(IMO 2003, shortlist) Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ that satisfy the following conditions:*

(i) $f(xyz)+f(x)+f(y)+f(z)=f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$

(ii) $f(x)<f(y)$ *for all* $1\le x<y$.

Solution. First notice that the solution of this functional equation is not one of the common solutions that we are used to work with. Namely one of the solutions is $f(x)=x+\dfrac{1}{x}$ which tells us that this equality is unlikely to be shown reducing to the Cauchy equation. First, setting $x=y=z=1$ we get $f(1)=2$ (since $f(1)>0$). One of the properties of the solution suggested above is $f(x)=f(1/x)$, and proving this equality will be our next step. Putting $x=ts$, $y=\frac{t}{s}$, $z=\frac{s}{t}$ in (i) gives

$$f(t)f(s)=f(ts)+f(t/s). \tag{7}$$

In particular, for $s=1$ the last equality yields $f(t)=f(1/t)$; hence $f(t)\ge f(1)=2$ for each t. It follows that there exists $g(t)\ge 1$ such that $f(t)=g(t)+\frac{1}{g(t)}$. Now it follows by induction from (7) that $g(t^n)=g(t)^n$ for every integer n, and therefore $g(t^q)=g(t)^q$ for every rational q. Consequently, if $t>1$ is fixed, we have $f(t^q)=a^q+a^{-q}$, where $a=g(t)$. But since the set of a^q ($q\in\mathbb{Q}$) is dense in $\mathbb{R}^+$ and f is monotone on $(0,1]$ and $[1,\infty)$, it follows that $f(t^r)=a^r+a^{-r}$ for every real r. Therefore, if k is such that $t^k=a$, we have

$$f(x)=x^k+x^{-k}\qquad\text{for every } x\in\mathbb{R}.\ \triangle$$

Problem 15. *Find all functions* $f : [1,\infty) \to [1,\infty)$ *that satisfy:*

(i) $f(x) \le 2(1+x)$ *for every* $x \in [1,\infty)$;

(ii) $xf(x+1) = f(x)^2 - 1$ *for every* $x \in [1,\infty)$.

Solution. It is not hard to see that $f(x) = x+1$ is a solution. Let us prove that this is the only solution. Using the given conditions we get

$$f(x)^2 = xf(x+1)+1 \le x(2(x+1))+1 < 2(1+x)^2,$$

i.e. $f(x) \le \sqrt{2}(1+x)$. With this we have found the upper bound for $f(x)$. Since our goal is to prove $f(x) = x+1$ we will use the same method for lowering the upper bound. Similarly we get

$$f(x)^2 = xf(x+1)+1 \le x(\sqrt{2}(x+1))+1 < 2^{1/4}(1+x)^2.$$

Now it is clear that we should use induction to prove

$$f(x) < 2^{1/2^k}(1+x),$$

for every k. However this is shown in the same way as the previous two inequalities. Since $2^{1/2^k} \to 1$ as $k \to +\infty$, hence for fixed x we can't have $f(x) > x+1$. This implies $f(x) \le x+1$ for every real number $x \ge 1$. It remains to show that $f(x) \ge x+1$, for $x \ge 1$. We will use the similar argument. From the fact that the range is $[1,+\infty)$ we get $\dfrac{f(x)^2-1}{x} = f(x+1) \ge 1$, i.e. $f(x) \ge \sqrt{x+1} > x^{1/2}$. We further have $f(x)^2 = 1+xf(x+1) > 1+x\sqrt{x+2} > x^{3/2}$ and similarly by induction

$$f(x) > x^{1-1/2^k}.$$

Passing to the limit we further have $f(x) \ge x$. Now again from the given equality we get $f(x)^2 = 1+xf(x+1) \ge (x+1/2)^2$, i.el $f(x) \ge x+1/2$. Using the induction we get $f(x) \ge x+1-\dfrac{1}{2^k}$, and passing to the limit we get the required inequality $f(x) \ge x+1$. △

Problem 16. *(IMO 1999, probelm 6) Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x-f(y)) = f(f(y))+xf(y)+f(x)-1.$$

Solution. Let $A = \{f(x) \,|\, x \in \mathbb{R}\}$, i.e. $A = f(\mathbb{R})$. We will determine the value of the function on A. Let $x = f(y) \in A$, for some y. From the given equality we have $f(0) = f(x)+x^2+f(x)-1$, i.e.

$$f(x) = \frac{c+1}{2} - \frac{x^2}{2},$$

where $f(0) = c$. Now it is clear that we have to analyze set A further. Setting $x = y = 0$ in the original equation we get $f(-c) = f(c)+c-1$, hence $c \ne 0$. Furthermore, plugging $y = 0$ in the original equation we get $f(x-c)-f(x) = cx+f(c)-1$. Since the range of the function (on x) on the right-hand side is entire $\mathbb{R}$, we get $\{f(x-c)-f(x) \,|\, x \in \mathbb{R}\} = \mathbb{R}$, i.e. $A-A = \mathbb{R}$. Hence for every real number x there are real numbers $y_1, y_2 \in A$ such that $x = y_1 - y_2$. Now we have

$$\begin{aligned} f(x) &= f(y_1-y_2) = f(y_1-f(z)) = f(f(z))+y_1f(z)+f(y_1)-1 \\ &= f(y_1)+f(y_2)+y_1y_2-1 = c-\frac{x^2}{2}. \end{aligned}$$

From the original equation we easily get $c = 1$. It is easy to show that the function $f(x) = 1-\dfrac{x^2}{2}$ satisfies the given equation. △

Problem 17. *Given an integer n, let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying $f(0)=0$, $f(1)=1$, and $f^{(n)}(x)=x$, for every $x\in[0,1]$. Prove that $f(x)=x$ for each $x\in[0,1]$.*

Solution. First from $f(x)=f(y)$ we have $f^{(n)}(x)=f^{(n)}(y)$, hence f is injective. The idea for what follows is clear once we look at the graphical representation. Namely from the picture it can be easily deduced that the function has to be strictly increasing. Let us prove that formally. Assume the contrary, that for some two real numbers $x_1<x_2$ we have $f(x_1)\geq f(x_2)$. The continuity on $[0,x_1]$ implies that there is some c such that $f(c)=f(x_2)$, which contradicts the injectivity of f. Now if $x<f(x)$, we get $f(x)<f(f(x))$ etc. $x<f(n)(x)=x$. Similarly we get a contradiction if we assume that $x>f(x)$. Hence for each $x\in[0,1]$ we must have $f(x)=x$. △

Problem 18. *Find all functions $f:(0,+\infty)\to(0,+\infty)$ that satisfy $f(f(x)+y)=xf(1+xy)$ for all $x,y\in(0,+\infty)$.*

Solution. Clearly $f(x)=\dfrac{1}{x}$ is one solution to the functional equation. Let us prove that the function is non-increasing. Assume the contrary that for some $0<x<y$ we have $0<f(x)<f(y)$. We will consider the expression of the form $z=\dfrac{yf(y)-xf(x)}{y-x}$ since it is positive and bigger then $f(y)$. We first plug $(x,z-f(y))$ instead of (x,y) in the original equation, then we plug $z-f(x)$ instead of y, we get $x=y$, which is a contradiction. Hence the function is non-decreasing.

Let us prove that $f(1)=1$. Let $f(1)\neq 1$. Substituting $x=1$ we get $f(f(1)+y)=f(1+y)$, hence $f(u+|f(1)-1|)=f(u)$ for $u>1$. Therefore the function is periodic on the interval $(1,+\infty)$, and since it is monotone it is constant. However we then conclude that the left-hand side of the original equation constant and the right-hand side is not. Thus we must have $f(1)=1$. Let us prove that $f(x)=\dfrac{1}{x}$ for $x>1$. Indeed for $y=1-\dfrac{1}{x}$ the given equality gives $f\Big(f(x)-\dfrac{1}{x}\Big)=xf(x)$. If $f(x)>\dfrac{1}{x}$ we have $f\Big(f(x)-\dfrac{1}{x}+1\Big)\leq f(1)=1$ and $xf(x)>1$. If $f(x)<\dfrac{1}{x}$ we have $f\Big(f(x)-\dfrac{1}{x}+1\Big)\geq f(1)=1$, and $xf(x)<1$. Hence $f(x)=\dfrac{1}{x}$. If $x<1$, plugging $y=\dfrac{1}{x}$ we get

$$f\Big(f(x)+\frac{1}{x}\Big)=xf(2)=\frac{x}{2},$$

and since $\dfrac{1}{x}\geq 1$, we get $f(x)+\dfrac{1}{x}=\dfrac{2}{x}$, i.e. $f(x)=\dfrac{1}{x}$ in this case, too. This means that $f(x)=\dfrac{1}{x}$ for all positive real numbers x. △

Problem 19. *(Bulgaria 1998) Prove that there is no function $f:\mathbb{R}^+\to\mathbb{R}^+$ such that $f(x)^2\geq f(x+y)(f(x)+y)$ for every two positive real numbers x and y.*

Solution. The common idea for the problems of this type is to prove that $f(y)<0$ for some $y>0$ which will lead us to the obvious contradiction. We can also see that it is sufficient to prove that $f(x)-f(x+1)\geq c>0$, for every x because the simple addition gives $f(x)-f(x+m)\geq mc$. For sufficiently large m this implies $f(x+m)<0$. Hence our goal is finding c such that $f(x)-f(x+1)\geq c$, for every x. Assume that such function exists. From the given inequality we get $f(x)-f(x+y)\geq \dfrac{f(x+y)y}{f(x)}$ and the function is obviously decreasing. Also from the given equality we can conclude that

$$f(x)-f(x+y)\geq\frac{f(x)y}{f(x)+y}.$$

Let n be a natural number such that $f(x+1)n\geq 1$ (such number clearly exists). Notice that for $0\leq k\leq n-1$ the following inequality holds

$$f\Big(x+\frac{k}{n}\Big)-f\Big(x+\frac{k+1}{n}\Big)\geq\frac{f\left(x+\frac{k}{n}\right)\frac{1}{n}}{f\left(x+\frac{k}{n}\right)+\frac{1}{n}}\geq\frac{1}{2n},$$

and adding similar realitions for all described k yields $f(x)-f(x+1)\geq \frac{1}{2}$ which is a contradiction. $\triangle$

Problem 20. *Let $f:\mathbb{N}\to\mathbb{N}$ be a function satisfying*

$$f(1)=2,\quad f(2)=1,\quad f(3n)=3f(n),\quad f(3n+1)=3f(n)+2,\quad f(3n+2)=3f(n)+1.$$

Find the number of integers $n\leq 2006$ for which $f(n)=2n$.

Solution. This is a typical problem in which the numbers should be considered in some base different than 10. For this situation the base 3 is doing the job. Let us calculate $f(n)$ for $n\leq 8$ in an attempt to guess the solution. Clearly the given equation can have only one solution.

$$f((1)_3)=(2)_3,\ f((2)_3)=(1)_3,\ f((10)_3)=6=(20)_3,\ f((11)_3)=8=(22)_3,$$

$$f((12)_3)=7=(21)_3,\ f((20)_3)=3=(10)_3,\ f((21)_3)=5=(12)_3,\ f((22)_3)=4=(11)_3.$$

Now we see that $f(n)$ is obtained from n by changing each digit 2 by 1, and conversely. This can be now easily shown by induction. It is clear that $f(n)=2n$ if and only if in the system with base 3 n doesn't contain any digit 1 (because this would imply $f(n)<2n$). Now it is easy to count the number of such n's. The answer is 127. $\triangle$

Problem 21. *(BMO 2003, shortlist) Find all possible values for $f\left(\frac{2004}{2003}\right)$ if $f:\mathbb{Q}\to[0,+\infty)$ is the function satisfying the conditions:*

(i) $f(xy)=f(x)f(y)$ *for all* $x,y\in\mathbb{Q}$*;*

(ii) $f(x)\leq 1\Rightarrow f(x+1)\leq 1$ *for all* $x\in\mathbb{Q}$*;*

(iii) $f\left(\frac{2003}{2002}\right)=2.$

Solution. Notice that from (i) and (ii) we conclude that $f(x)>0$, for every rational x. Now (i) implies that for $x=y=1$ we get $f(1)=0$ and similarly for $x=y=-1$ we get $f(-1)=1$. By induction $f(x)\leq 1$ for every integer x. For $f(x)\leq f(y)$ from $f\left(\frac{y}{x}\right)f(y)=f(x)$ we have that $f\left(\frac{y}{x}\right)\leq 1$, and according to (ii) $f\left(\frac{y}{x}+1\right)\leq 1$. This implies

$$f(x+y)=f\left(\frac{y}{x}+1\right)f(x)\leq f(x),$$

hence $f(x+y)\leq\max\{f(x),f(y)\}$, for every $x,y\in\mathbb{Q}$. Now you might wonder how did we get this idea. There is one often neglected fact that for every two relatively prime numbers u and v, there are integers a and b such that $au+bv=1$. What is all of this good for? We got that $f(1)=1$, and we know that $f(x)\leq 1$ for all $x\in\mathbb{Z}$ and since 1 is the maximum of the function on $\mathbb{Z}$ and since we have the previous inequality our goal is to show that the value of the function is 1 for a bigger class of integers. We will do this for prime numbers. If for every prime p we have $f(p)=1$ then $f(x)=1$ for every integer implying $f(x)\equiv 1$ which contradicts (iii). Assume therefore that $f(p)\neq 1$ for some $p\in\mathbb{P}$. There are a and b such that $ap+bq=1$ implying $f(1)=f(ap+bq)\leq\max\{f(ap),f(bq)\}$. Now we must have $f(bq)=1$ implying that $f(q)=1$ for every other prime number q. From (iii) we have

$$f\left(\frac{2003}{2002}\right)=\frac{f(2003)}{f(2)f(7)f(11)f(13)}=2,$$

hence only one of the numbers $f(2),f(7),f(11),f(13)$ is equal to $1/2$. Thus $f(3)=f(167)=f(2003)$ giving:

$$f\left(\frac{2004}{2003}\right)=\frac{f(2)^2f(3)f(167)}{f(2003)}=f(2)^2.$$

If $f(2)=1/2$ then $f\left(\frac{2003}{2002}\right)=\frac{1}{4}$, otherwise it is 1.

It remains to construct one function for each of the given values. For the first value it is the multiplicative function taking the value $1/2$ at the point 2, and 1 for all other prime numbers; in the second case it is a the multiplicative function that takes the value $1/2$ at, for example, 7 and takes 1 at all other prime numbers. For these functions we only need to verify the condition (ii), but that is also very easy to verify. △

Problem 22. *Let $I=[0,1]$, $G=I\times I$ and $k\in\mathbb{N}$. Find all $f:G\to I$ such that for all $x,y,z\in I$ the following statements hold:*

(i) $f(f(x,y),z)=f(x,f(y,z))$;

(ii) $f(x,1)=x$, $f(x,y)=f(y,x)$;

(ii) $f(zx,zy)=z^kf(x,y)$ *for every $x,y,z\in I$, where k is a fixed real number.*

Solution. The function of several variables appears in this problem. In most cases we use the same methods as in the case of a single-variable functions. From the condition (ii) we get $f(1,0)=f(0,1)=0$, and from (iii) we get $f(0,x)=f(x,0)=x^kf(1,0)=0$. This means that f is entirely defined on the edge of the region G. Assume therefore that $0<x\leq y<1$. Notice that the condition (ii) gives the value for one class of pairs from G and that each pair in G can be reduced to one of the members of the class. This implies

$$f(x,y)=f(y,x)=y^kf\left(1,\frac{x}{y}\right)=y^{k-1}x.$$

This can be written as $f(x,y)=\min(x,y)(\max(u,v))^{k-1}$ for all $0<x,y<1$. Let us find all possible values for k. Let $0<x\leq\frac{1}{2}\leq y<1$. From the condition (i), and the already obtained results we get

$$f\left(f\left(x,\frac{1}{2}\right),y\right)=f\left(x\left(\frac{1}{2}\right)^{k-1},y\right)=f\left(x,f\left(\frac{1}{2}\right)\right)=f\left(x,\frac{1}{2}y^{k-1}\right).$$

Let us now consider $x\leq 2^{k-1}y$ in order to simplify the expression to the form $f\left(x,\frac{1}{2}y^{k-1}\right)=x\left(\frac{y}{2}\right)^{k-1}$, and if we take x for which $2x\leq y^{k-1}$ we get $k-1=(k-1)^2$, i.e. $k=1$ or $k=2$. For $k=1$ the solution is $f(x,y)=\min(x,y)$, and for $k=2$ the solution is $f(x,y)=xy$. It is easy to verify that both solutions satisfy the given conditions. △

Problem 23. *(APMO 1989) Find all strictly increasing functions $f:\mathbb{R}\to\mathbb{R}$ such that*

$$f(x)+g(x)=2x,$$

where g is the inverse of f.

Solution. Clearly every function of the form $x+d$ is the solution of the given equation. Another useful idea appears in this problem. Namely denote by S_d the the set of all numbers x for which $f(x)=x+d$. Our goal is to prove that $S_d=\mathbb{R}$. Assume that S_d is non-empty. Let us prove that for $x\in S_d$ we have $x+d\in S_d$ as well. Since $f(x)=x+d$, according to the definition of the inverse function we have $g(x+d)=x$, and the given equation implies $f(x+d)=x+2d$, i.e. $x+d\in S_d$. Let us prove that the sets $S_{d'}$ are empty, where $d'<d$. From the above we have that each of those sets is infinite, i.e. if x belongs to some of them, then each $x+kd$ belongs to it as well. Let us use this to get the contradiction. More precisely we want to prove that if $x\in S_d$ and $x\leq y\leq x+(d-d')$, then $y\notin S_{d'}$. Assume the contrary. From the monotonicity we have $y+d'=f(y)\geq f(x)=x+d$, which is a contradiction to our assumption. By further induction we prove that every y satisfying

$$x+k(d-d')\leq y<x+(k+1)(d-d'),$$

can't be a member of $S_{d'}$. However this is a contradiction with the previously established properties of the sets S_d and $S_{d'}$. Similarly if $d' > d$ switching the roles of d and d' gives a contradiction.

Simple verification shows that each $f(x) = x + d$ satisfies the given functional equation. $\triangle$

Problem 24. *Find all functions $h : \mathbb{N} \to \mathbb{N}$ that satisfy*

$$h(h(n)) + h(n+1) = n + 2.$$

Solution. Notice that we have both $h(h(n))$ and $h(n+1)$, hence it is not possible to form a recurrent equation. We have to use another approach to this problem. Let us first calculate $h(1)$ and $h(2)$. Setting $n = 1$ gives $h(h(1)) + h(2) = 3$, therefore $h(h(1)) \le 2$ and $h(2) \le 2$. Let us consider the two cases:

1° $h(2) = 1$. Then $h(h(1)) = 2$. Plugging $n = 2$ in the given equality gives $4 = h(h(2)) + h(3) = h(1) + h(3)$. Let $h(1) = k$. It is clear that $k \neq 1$ and $k \neq 2$, and that $k \le 3$. This means that $k = 3$, hence $h(3) = 1$. However from $2 = h(h(1)) = h(3) = 1$ we get a contradiction. This means that there are no solutions in this case.

2° $h(2) = 2$. Then $h(h(1)) = 1$. From the equation for $n = 2$ we get $h(3) = 2$. Setting $n = 3,4,5$ we get $h(4) = 3, h(5) = 4, h(6) = 4$, and by induction we easily prove that $h(n) \ge 2$, for $n \ge 2$. This means that $h(1) = 1$. Clearly there is at most one function satisfying the given equality. Hence it is enough to guess some function and prove that it indeed solves the equation (induction or something similar sounds fine). The solution is

$$h(n) = \lfloor n\alpha \rfloor + 1,$$

where $\alpha = \dfrac{-1+\sqrt{5}}{2}$ (this constant can be easily found $\alpha^2 + \alpha = 1$). Proof that this is a solution uses some properties of the integer part (although it is not completely trivial). $\triangle$

Problem 25. *(IMO 2004, shortlist) Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equality*

$$f(x^2 + y^2 + 2f(xy)) = f(x+y)^2.$$

Solution. Let us make the substitution $z = x + y$, $t = xy$. Given $z, t \in \mathbb{R}$, x, y are real if and only if $4t \le z^2$. Define $g(x) = 2(f(x) - x)$. Now the given functional equation transforms into

$$f\left(z^2 + g(t)\right) = (f(z))^2 \ \text{ for all } \ t, z \in \mathbb{R} \ \text{ with } \ z^2 \ge 4t. \tag{8}$$

Let us set $c = g(0) = 2f(0)$. Substituting $t = 0$ into (8) gives us

$$f(z^2 + c) = (f(z))^2 \ \text{ for all } \ z \in \mathbb{R}. \tag{9}$$

If $c < 0$, then taking z such that $z^2 + c = 0$, we obtain from (9) that $f(z)^2 = c/2$, which is impossible; hence $c \ge 0$. We also observe that

$$x > c \quad \text{implies} \quad f(x) \ge 0. \tag{10}$$

If g is a constant function, we easily find that $c = 0$ and therefore $f(x) = x$, which is indeed a solution.

Suppose g is nonconstant, and let $a, b \in \mathbb{R}$ be such that $g(a) - g(b) = d > 0$. For some sufficiently large K and each $u, v \ge K$ with $v^2 - u^2 = d$ the equality $u^2 + g(a) = v^2 + g(b)$ by (8) and (10) implies $f(u) = f(v)$. This further leads to $g(u) - g(v) = 2(v - u) = \frac{d}{u+\sqrt{u^2+d}}$. Therefore every value from some suitably chosen segment $[\delta, 2\delta]$ can be expressed as $g(u) - g(v)$, with u and v bounded from above by some M.

Consider any x,y with $y > x \geq 2\sqrt{M}$ and $\delta < y^2 - x^2 < 2\delta$. By the above considerations, there exist $u,v \leq M$ such that $g(u) - g(v) = y^2 - x^2$, i.e., $x^2 + g(u) = y^2 + g(v)$. Since $x^2 \geq 4u$ and $y^2 \geq 4v$, (8) leads to $f(x)^2 = f(y)^2$. Moreover, if we assume w.l.o.g. that $4M \geq c^2$, we conclude from (10) that $f(x) = f(y)$. Since this holds for any $x,y \geq 2\sqrt{M}$ with $y^2 - x^2 \in [\delta, 2\delta]$, it follows that $f(x)$ is eventually constant, say $f(x) = k$ for $x \geq N = 2\sqrt{M}$. Setting $x > N$ in (9) we obtain $k^2 = k$, so $k = 0$ or $k = 1$.

By (9) we have $f(-z) = \pm f(z)$, and thus $|f(z)| \leq 1$ for all $z \leq -N$. Hence $g(u) = 2f(u) - 2u \geq -2 - 2u$ for $u \leq -N$, which implies that g is unbounded. Hence for each z there exists t such that $z^2 + g(t) > N$, and consequently $f(z)^2 = f(z^2 + g(t)) = k = k^2$. Therefore $f(z) = \pm k$ for each z.

If $k = 0$, then $f(x) \equiv 0$, which is clearly a solution. Assume $k = 1$. Then $c = 2f(0) = 2$ (because $c \geq 0$), which together with (10) implies $f(x) = 1$ for all $x \geq 2$. Suppose that $f(t) = -1$ for some $t < 2$. Then $t - g(t) = 3t + 2 > 4t$. If also $t - g(t) \geq 0$, then for some $z \in \mathbb{R}$ we have $z^2 = t - g(t) > 4t$, which by (8) leads to $f(z)^2 = f(z^2 + g(t)) = f(t) = -1$, which is impossible. Hence $t - g(t) < 0$, giving us $t < -2/3$. On the other hand, if X is any subset of $(-\infty, -2/3)$, the function f defined by $f(x) = -1$ for $x \in X$ and $f(x) = 1$ satisfies the requirements of the problem.

To sum up, the solutions are $f(x) = x$, $f(x) = 0$ and all functions of the form

$$f(x) = \begin{cases} 1, & x \notin X, \\ -1, & x \in X, \end{cases}$$

where $X \subset (-\infty, -2/3)$. $\triangle$

4 Problems for Independent Study

Most of the ideas for solving the problems below are already mentioned in the introduction or in the section with solved problems. The difficulty of the problems vary as well as the range of ideas used to solve them. Before solving the problems we highly encourage you to first solve (or look at the solutions) the problems from the previous section. Some of the problems are quite difficult.

1. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ that satisfy $f(x+y) = f(x) + f(y) + xy$.

2. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ for which we have $f(0) = 1$ and $f(f(n)) = f(f(n+2)+2) = n$, for every natural number n.

3. Find all functions $f : \mathbb{N} \to \mathbb{N}$ for which $f(n)$ is a square of an integer for all $n \in \mathbb{N}$, and that satisfy $f(m+n) = f(m) + f(n) + 2mn$ for all $m,n \in \mathbb{N}$.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f((x-y)^2) = f(x)^2 - 2xf(y) + y^2$.

5. Let $n \in \mathbb{N}$. Find all monotone functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x + f(y)) = f(x) + y^n.$$

6. (USA 2002) Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equality $f(x^2 - y^2) = xf(x) - yf(y)$.

7. (Mathematical High Schol, Belgrade 2004) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(m) + f(n)) = m + n$ for every two natural numbers m and n.

8. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xy) = xf(y) + yf(x)$.

9. (IMO 1983, problem 1) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

   (i) $f(xf(y)) = yf(x)$, for all $x,y \in \mathbb{R}$;

   (ii) $f(x) \to 0$ as $x \to +\infty$.

10. Let $f:\mathbb{N}\to\mathbb{N}$ be strictly increasing function that satisfies $f(f(n))=3n$ for every natural number n. Determine $f(2006)$.

11. (IMO 1989, shortlist) Let $0<a<1$ be a real number and f continuous function on $[0,1]$ which satisfies $f(0)=0$, $f(1)=1$, and
$$f\left(\frac{x+y}{2}\right)=(1-a)f(x)+af(y),$$
for every two real numbers $x,y\in[0,1]$ such that $x\le y$. Determine $f\left(\frac{1}{7}\right)$.

12. (IMO 1996, shortlist) Let $f:\mathbb{R}\to\mathbb{R}$ be the function such that $|f(x)|\le 1$ and
$$f\left(x+\frac{13}{42}\right)+f(x)=f\left(x+\frac{1}{6}\right)+f\left(x+\frac{1}{7}\right).$$
Prove that f is periodic.

13. (BMO 2003, problem 3) Find all functions $f:\mathbb{Q}\to\mathbb{R}$ that satisfy:
    (i) $f(x+y)-yf(x)-xf(y)=f(x)f(y)-x-y+xy$ for every $x,y\in\mathbb{Q}$;
    (ii) $f(x)=2f(x+1)+2+x$, for every $x\in\mathbb{Q}$;
    (iii) $f(1)+1>0$.

14. (IMO 1990, problem 4) Determine the function $f:\mathbb{Q}^+\to\mathbb{Q}^+$ such that
$$f(xf(y))=\frac{f(x)}{y},\ \text{for all } x,y\in\mathbb{Q}^+.$$

15. (IMO 2002, shortlist) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
$$f(f(x)+y)=2x+f(f(y)-x).$$

16. (Iran 1997) Let $f:\mathbb{R}\to\mathbb{R}$ be an increasing function such that for all positive real numbers x and y:
$$f(x+y)+f(f(x)+f(y))=f(f(x+f(y))+f(y+f(x))).$$
Prove that $f(f(x))=x$.

17. (IMO 1992, problem 2) Find all functions $f:\mathbb{R}\to\mathbb{R}$, such that $f(x^2+f(y))=y+f(x)^2$ for all $x,y\in\mathbb{R}$.

18. (IMO 1994, problem 5) Let S be the set of all real numbers strictly greater than -1. Find all functions $f:S\to S$ that satisfy the following two conditions:
    (i) $f(x+f(y)+xf(y))=y+f(x)+yf(x)$ for all $x,y\in S$;
    (ii) $\dfrac{f(x)}{x}$ is strictly increasing on each of the intervals $-1<x<0$ and $0<x$.

19. (IMO 1994, shortlist) Find all functions $f:\mathbb{R}^+\to\mathbb{R}$ such that
$$f(x)f(y)=y^\alpha f(x/2)+x^\beta f(y/2),\ \text{ for all } x,y\in\mathbb{R}^+.$$

20. (IMO 2002, problem 5) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
$$(f(x)+f(z))(f(y)+f(t))=f(xy-zt)+f(xt+yz).$$

21. (Vietnam 2005) Find all values for a real parameter α for which there exists exactly one function $f:\mathbb{R}\to\mathbb{R}$ satisfying
$$f(x^2+y+f(y)) = f(x)^2+\alpha\cdot y.$$

22. (IMO 1998, problem 3) Find the least possible value for $f(1998)$ where $f:\mathbb{N}\to\mathbb{N}$ is a function that satisfies
$$f(n^2f(m)) = mf(n)^2.$$

23. Does there exist a function $f:\mathbb{N}\to\mathbb{N}$ such that
$$f(f(n-1)) = f(n+1)-f(n)$$
for each natural number n?

24. (IMO 1987, problem 4) Does there exist a function $f:\mathbb{N}_0\to\mathbb{N}_0$ such that $f(f(n)) = n+1987$?

25. Assume that the function $f:\mathbb{N}\to\mathbb{N}$ satisfies $f(n+1) > f(f(n))$, for every $n\in\mathbb{N}$. Prove that $f(n) = n$ for every n.

26. Find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$, that satisfy:
    (i) $2f(m^2+n^2) = f(m)^2+f(n)^2$, for every two natural numbers m and n;
    (ii) If $m\geq n$ then $f(m^2)\geq f(n^2)$.

27. Find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ that satisfy:
    (i) $f(2) = 2$;
    (ii) $f(mn) = f(m)f(n)$ for every two relatively prime natural numbers m and n;
    (iii) $f(m) < f(n)$ whenever $m < n$.

28. Find all functions $f:\mathbb{N}\to[1,\infty)$ that satisfy conditions (i) and (ii9) of the previous problem and the condition (ii) is modified to require the equality for every two natural numbers m and n.

29. Given a natural number k, find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ for which
$$f(f(n))+f(n) = 2n+3k,$$
for every $n\in\mathbb{N}_0$.

30. (Vijetnam 2005) Find all functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy $f(f(x-y)) = f(x)f(y)-f(x)+f(y)-xy$.

31. (China 1996) The function $f:\mathbb{R}\to\mathbb{R}$ satisfy $f(x^3+y^3) = (x+y)\Big(f(x)^2-f(x)f(y)+f(y)^2\Big)$, for all real numbers x and y. Prove that $f(1996x) = 1996f(x)$ for every $x\in\mathbb{R}$.

32. Find all functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy:
    (i) $f(x+y) = f(x)+f(y)$ for every two real numbers x and y;
    (ii) $f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}$ for $x\neq 0$.

33. (IMO 1989, shortlist) A function $f:\mathbb{Q}\to\mathbb{R}$ satisfy the following conditions:
    (i) $f(0) = 0$, $f(\alpha) > 0$ za $\alpha\neq 0$;

(ii) $f(\alpha\beta) = f(\alpha)f(\beta)$ i $f(\alpha+\beta) \le f(\alpha) + f(\beta)$, for all $\alpha, \beta \in \mathbb{Q}$;
(iii) $f(m) \le 1989$ za $m \in \mathbb{Z}$.

Prove that $f(\alpha+\beta) = \max\{f(\alpha), f(\beta)\}$ whenever $f(\alpha) \neq f(\beta)$.

34. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for every two real numbers $x \neq y$ the equality

$$f\left(\frac{x+y}{x-y}\right) = \frac{f(x)+f(y)}{f(x)-f(y)}$$

is satisfied.

35. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying:

(i) $f(x+1) = f(x)+1$ for all $x \in \mathbb{Q}^+$;
(ii) $f(x^3) = f(x)^3$ for all $x \in \mathbb{Q}^+$.

36. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equality

$$f(x+y) + f(xy) = f(x) + f(y) + f(xy+1).$$

37. Find all continuous functions $f, g, h, k : \mathbb{R} \to \mathbb{R}$ that satisfy the equality

$$f(x+y) + g(x-y) = 2h(x) + 2k(y).$$

38. (IMO 1996, shortlist) Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f(m+f(n)) = f(f(m)) + f(n).$$

39. (IMO 1995, shortlist) Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the conditions:

(i) There exists a positive real number M such that $-M \le f(x) \le M$ for all $x \in \mathbb{R}$;
(ii) $f(1) = 1$;
(iii) If $x \neq 0$ then $f\left(x + \frac{1}{x^2}\right) = f(x) + \left[f\left(\frac{1}{x}\right)\right]^2$?

40. (Belarus) Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(f(x)) = f(x) + 2x.$$

41. Prove that if the function $f : \mathbb{R}^+ \to \mathbb{R}$ satisfy the equality

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{2xy}{x+y}\right) = f(x) + f(y),$$

the it satisfy the equality $2f(\sqrt{xy}) = f(x) + f(y)$ as well.

42. Find all continuous functions $f : (0,\infty) \to (0,\infty)$ that satisfy

$$f(x)f(y) = f(xy) + f(x/y).$$

43. Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ that satisfy the inequality $f(y) > (y-x)f(x)^2$, for every two real numbers x and y.

44. (IMC 2001) Prove that there doesn't exist a function $f:\mathbb{R}\to\mathbb{R}$ for which $f(0)>0$ and
$$f(x+y)\geq f(x)+yf(f(x)).$$

45. (Romania 1998) Find all functions $u:\mathbb{R}\to\mathbb{R}$ for which there exists a strictly monotone function $f:\mathbb{R}\to\mathbb{R}$ such that
$$f(x+y)=f(x)u(y)+f(y),\quad \forall x,y\in\mathbb{R}.$$

46. (Iran 1999) Find all functions $f:\mathbb{R}\to\mathbb{R}$ for which
$$f(f(x)+y)=f(x^2-y)+4f(x)y.$$

47. (IMO 1988, problem 3) A function $f:\mathbb{N}\to\mathbb{N}$ satisfies:
   (i) $f(1)=1,\ f(3)=3;$
   (ii) $f(2n)=f(n);$
   (iii) $f(4n+1)=2f(2n+1)-f(n)$ and $f(4n+3)=3f(2n+1)-2f(n),$

   for every natural number $n\in\mathbb{N}$. Find all natural numbers $n\leq 1998$ such that $f(n)=n$.

48. (IMO 2000, shortlist) Given a function $F:\mathbb{N}_0\to\mathbb{N}_0$, assume that for $n\geq 0$ the following relations hold:
   (i) $F(4n)=F(2n)+F(n);$
   (ii) $F(4n+2)=F(4n)+1;$
   (iii) $F(2n+1)=F(2n)+1.$

   Prove that for every natural number m, the number of positive integers n such that $0\leq n<2^m$ and $F(4n)=F(3n)$ is equal to $F(2^{m+1})$.

49. Let $f:\mathbb{Q}\times\mathbb{Q}\to\mathbb{Q}^+$ be a function satisfying
$$f(xy,z)=f(x,z)f(y,z),\quad f(z,xy)=f(z,x)f(z,y),\quad f(x,1-x)=1,$$
   for all rational numbers x,y,z. Prove that $f(x,x)=1$, $f(x,-x)=1$, and $f(x,y)f(y,x)=1$.

50. Find all functions $f:\mathbb{N}\times\mathbb{N}\to\mathbb{R}$ that satisfy
$$f(x,x)=x,\quad f(x,y)=f(y,x),\quad (x+y)f(x,y)=yf(x,x+y).$$