AN ANOTHER UNLIKELY CONCURRENCE [1]

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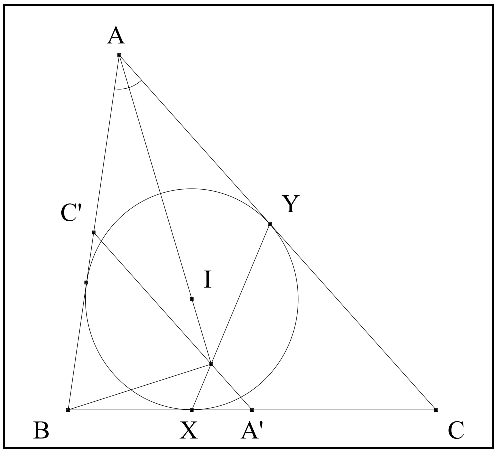
Abstract: the result in plane geometry presented in this article may be seen as a twin-theorem.

A point of history.



In his book, Honsberger[2] attracs the lecture's attention on *An Unlikely concurrence*: "let I be the incenter of a triangle ABC, X and Y be the points of contact of the incircle on BC and CA, then the lines AI, XY and the perpendicular from B to AI are concurrent". This result has already been proposed as an excercise by Altshiller-Court[3] and studied only for a right triangle by Papelier[4]. The german historian Simon[5] pointed out that a former student of Gerono, Lascases[6] from the city of Lorient (France) has initially situated in 1859 the point of intersection of AI and the perpendicular on A'C' where A' and C' are the midpoints of BC and AB respectively.

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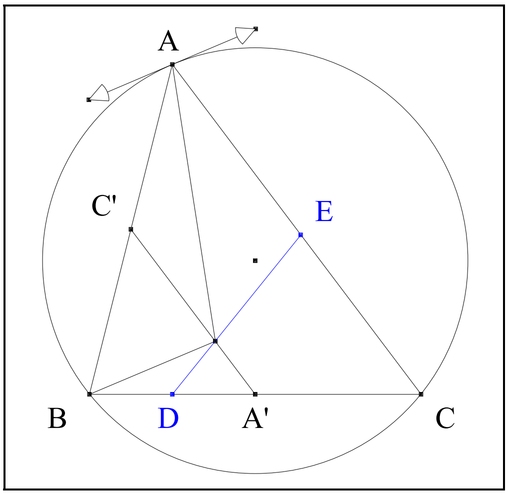
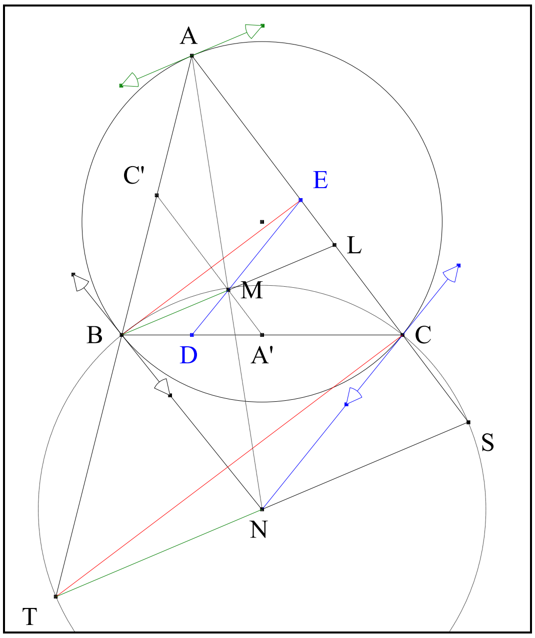
1 Ayme J.-L;, *Crux Mathematicorum*, (Canada) 8 (2003) 511-513.
2 Honsberger R., *Episodes in Nineteeth and Twentieth Century Euclidean Geometry*, MAA (1995) 31.
3 Altshiller-Court N., exercise 43, *College Geometry*, Barnes & Noble, Inc., (1952) 118.
4 Papelier G., *Exercices de géométrie Modernes*, Pôles et polaires (1927) 19.
5 Simon M., *Über die Entwicklung der Elementar-Geometrie im XIX-Jahrhundert* (1906) 127, 133.
6 Lascases or Lescaze Arth., *Nouvelles Annales* 18 (1859) 171.

The twin-theorem.



Let ABC be a triangle and Γ its circumcircle. Let D and E be the feet of the altitudes from A, B to BC, CA respectively and A', C' the midpoints of BC and AB respectively. Let Δa be the symmedian issued from A. Let Db be the parallel passing through B to the tangent at the point A to Γ.
Then the four lines Δa, Db, DE and A'C' are concurrent.

Proof.

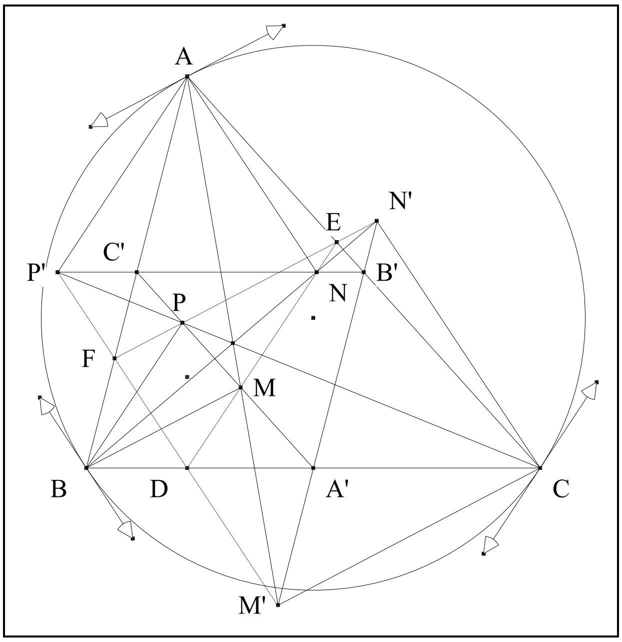


• Notations :	let	Ta, Tb, Tc	be the tangents at the points A, B, C to Γ respectively,
		L	the intersection of the lines Db and AC,
		M	the intersection of the lines Db and DE,
		N	the intersection of Tb and Tc,
		Δn	the parallel passing through N to Ta
	and	S, T	the intersections of Δ_N respectively with AC and AB.

- According to a Lemoine's result[7], Δa = AN.
- According to Boutin's theorem[8], the four points B, C, S and T lie on a circle having ST for diameter and N for center;
  we have, CT ⊥ ACS ;
  but since, ACS ⊥ BE
  we have, CT // BE.
- According to Carnot' s theorem[9], Tc // DE.
- Since, Db // Ta and Ta // Δn, we get : Db // Δn.
- According to Desargues's theorem[10] applied to the homothetic triangles BME and TNC, the three points M, N and A are collinear ;
  so M is the midpoint of BL[11] and the three points A', M et C' are collinear.
- Thus the four lines Δa = AN, Db, DE and A'C' are concurrent in M.

Remarks. **(1)** a second concurrence: mutatis mutandis[12], we would show that
the four lines Δa, Dc, FD and B'A' are concurrent in M'
whereas Dc is the parallel passing through C to the tangent at the point A to Γ,
F the feet of the altitude from C to AB and B' the midpoint of CA.

(2) The complete figure :



7 The symmedian issued from a vertex of a triangle passes through the point of intersection of the tangents to the circumcircle at the other two vertices of the triangle.
8 Boutin M. A., (1858-?), *Journal de Mathématiques Elémentaires* (1890) 113: if three circles are externally tangent in pair and we join the point of contact of the first two circles to each of the other points of contact, then these two lines meet the third circle at the extremity of a same diameter, and this diameter is parallel to the line center of the two first circles.
9 The sides of the orthic triangle are parallel to the tangents to the circumcircle at the vertices.
10 If two triangles have an axis of perpective (here the line at infinity), they have a center of perspective.
11 A line antiparallel to a side of a triangle is bissect by the symmedian from the corresponding vertex.
12 i.e. changing what has to be changed.