Practice Problems in Geometry

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Abstract

The problems here are not sorted in order of difficulty because sometimes after seeing the source of the problem, people get intimidated. The best way is to look at the problem first and then look at the source after solving it.[1]

Invert the diagram about any point circle in the plane. All concurrencies become obvious.

- A solution which gets a 7/7.

1 Symmedians

1. Telv Cohl: Given a $\triangle ABC$ inscribed in $\odot(O)$ and a point P. Let $\triangle DEF$ be the tangential triangle of $\triangle ABC$ and let K_A, K_B, K_C be the symmedian point of $\triangle BPC$, $\triangle CPA$, $\triangle APB$, respectively. Prove that OP, DK_A, EK_B, FK_C are concurrent.

2. Russia 2009: Let be given a triangle ABC and its internal angle bisector BD ($D \in BC$). The line BD intersects the circumcircle Ω of triangle ABC at B and E. Circle ω with diameter DE cuts Ω again at F. Prove that BF is the symmedian line of triangle ABC.

3. Romania TST 2014: Let $\triangle ABC$ be an acute triangle of circumcentre O. Let the tangents to the circumcircle of $\triangle ABC$ in points B and C meet at point P. The circle of centre P and radius $PB = PC$ meets the internal angle bisector of $\angle BAC$ inside $\triangle ABC$ at point S, and $OS \cap BC = D$. The projections of S on AC and AB respectively are E and F. Prove that AD, BE and CF are concurrent.

4. andria (AoPS):let ABC be a triangle. let A_0, B_0, C_0 be the midpoints of BC, CA, AB respectively. let Ω be the nine point circle of $\triangle ABC$. let $AA_0 \cap \Omega = A_1, BB_0 \cap \Omega = B_1, CC_0 \cap \Omega = C_1$. let A_2, B_2, C_2 be the reflections of A, B, C WRT A_1, B_1, C_1 respectively. let A_3, B_3, C_3 be the reflections of A_2, B_2, C_2 WRT BC, CA, AB respectively. prove that AA_3, BB_3, CC_3 are concurrent.

5. andria: In triangle ABC with circumcenter O points M, N are midpoints of AB, AC the tangents from B, C intersect at T let $\odot(\triangle BON) \cap \odot(\triangle COM) = S, AT \cap \odot(\triangle ABC) = R$ prove that A, O, S, R lie on the circle.

6. Tran Quang Hung: Let ABC be a triangle with external bisector AD, incenter I and circumcenter O. Perpendicular bisector of AI cuts OA at J. K, L are circumcenter of triangle ABD, ACD. S, T are reflection of A through JK, JL. AE, AF are symmedian of triangle AIC, AIB with E, F lie on IC, IB. Prove that SF, TE and BC are concurrent.

[1]Some of the problems here which are supposedly high-numbered problems from the ISLs etc are actually not hard. In the words of v_Enhance, try each problem on each contest religiously for at least half an hour to get a feel for its difficulty. Some problems turn out to be really foolish.

7. ELMO 2014: Let ABC be a triangle with circumcenter O and orthocenter H. Let ω_1 and ω_2 denote the circumcircles of triangles BOC and BHC, respectively. Suppose the circle with diameter AO intersects ω_1 again at M, and line AM intersects ω_1 again at X. Similarly, suppose the circle with diameter AH intersects ω_2 again at N, and line AN intersects ω_2 again at Y . Prove that lines MN and XY are parallel.

8. See Yufei Zhao's handout "Lemmas in Geometry" for more.

2 Orthocenters

1. Own, easy: Let H be the orthocenter of a triangle ABC. Let M be the midpoint of BC, and let E, F be the feet of the B and the C altitudes onto the opposite sides. Let X be the intersection of ray MA with the circumcircle of BHC. Prove that HX, EF and BC concur at a point and also show that the line joining that point and A is perpendicular to the line HM.

2. USA TSTST 2011 P4: Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $OA \perp RA$.

3. IMO 2008 P1: Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1, and C_2 are concyclic.

4. IMO 2015 P3:[2] Let ABC be an acute triangle with $AB \gneq AC$. Let Γ be its cirumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = \pi/2$ and let K be the point on Γ such that $\angle HKQ = \pi/2$. Assume that the points A, B, C, K, and Q are all different and lie on Γ in this order. Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

5. EGMO 2012 P7: Let ABC be an acute-angled triangle with circumcircle Γ and orthocenter H. Let K be a point of Γ on the other side of BC from A. Let L be the reflection of K across AB, and let M be the reflection of K across BC. Let E be the second point of intersection of Γ with the circumcircle of triangle BLM. Show that the lines KH, EM, and BC are concurrent.

6. APMO 2012 P4: Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC, by M the midpoint of BC, and by H the orthocenter of ABC. Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the ray MH, and F be the point of intersection (other than E) of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{BC}$ must hold.

7. USA January TST 2014: Let $ABCD$ be a cyclic quadrilateral, and let E, F, G, and H be the midpoints of AB, BC, CD, DA respectively. Let W, X, Y, and Z be the orthocenters of triangles AHE, BEF, CFG,and DGH, respectively. Prove that quadrilaterals $ABCD$ and $WXYZ$ have the same area.[3]

8. APMO 2010 P4: Let ABC be an acute angled triangle satisfying the conditions $AB > BC$ and $AC > BC$. Denote by O and H the circumcentre and orthocentre, respectively, of the triangle ABC. Suppose that the circumcircle of the triangle AHC intersects the line AB at M different from A, and the circumcircle of the triangle AHB intersects the line AC at N different from A. Prove that the circumcentre of the triangle MNH lies on the line OH.

[2]Don't get intimidated by the presence of IMO P3's and P6's, as some of them quite often turn out to be somewhat misplaced IMO problems, and this problem actually has a very short solution.

[3]This problem is probably most easily solved using complex numbers but finding a synthetic solution provides a lot of insight.

9. NIMO 2014: Let ABC be a triangle and let Q be a point such that $AB \perp QB$ and $AC \perp QC$. A circle with center I is inscribed in $\triangle ABC$ which touches BC, CA, AB at D, E, F respectively. If ray QI intersects EF at P, prove that $DP \perp EF$.

10. USA TST 2009 P2: Let ABC be an acute triangle. Point D lies on side BC. Let O_B, O_C be the circumcenters of triangles ABD and ACD, respectively. Suppose that the points B, C, O_B, O_C lies on a circle centered at X. Let H be the orthocenter of triangle ABC. Prove that $\angle DAX = \angle DAH$.

11. ISL 2005 G5: Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocentre of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.

12. ISL 1998: Let ABC be a triangle, H its orthocentre, O its circumcentre, and R its circumradius. Let D be the reflection of the point A across the line BC, let E be the reflection of the point B across the line CA, and let F be the reflection of the point C across the line AB. Prove that the points D, E and F are collinear if and only if $OH = 2R$.

13. USA TSTST 2016 P2: Let ABC be a scalene triangle with orthocenter H and circumcenter O. Denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line AN at a point $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect at a point T on $\overline{PN}$.

3 In/Excenters

1. Sharygin: Let triangle $A_1B_1C_1$ be symmetric to ABC w.r.t. the incentre of its medial triangle. Prove that the orthocentre of $A_1B_1C_1$ coincides with the circumcentre of the triangle formed by the excentres of ABC.

2. Sharygin: The incircle of triangle ABC touches BC, CA, AB at points A_1, B_1, C_1, respectively. The perpendicular from the incentre I to the median from vertex C meets the line A_1B_1 in point K. Prove that CK is parallel to AB.

3. Russia 2006: Let K and L be two points on the arcs AB and BC of the circumcircle of a triangle ABC, respectively, such that $KL \parallel AC$. Show that the incentres of triangles ABK and CBL are equidistant from the midpoint of the arc ABC of the circumcircle of triangle ABC.

4. Russia 2012:The point E is the midpoint of the segment connecting the orthocentre of the scalene triangle ABC and the point A. The incircle of triangle ABC incircle is tangent to AB and AC at points C' and B' respectively. Prove that point F, the point symmetric to point E with respect to line $B'C'$, lies on the line that passes through both the circumcentre and the incentre of triangle ABC.

5. Russia 2012: The points A_1, B_1, C_1 lie on the sides sides BC, AC and AB of the triangle ABC respectively. Suppose that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let I_A, I_B, I_C be the incentres of triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the circumcentre of triangle $I_AI_BI_C$ is the incentre of triangle ABC.

6. Vietnam TST 2003: Given a triangle ABC. Let O be the circumcentre of this triangle ABC. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by A_0, B_0, C_0 the midpoints of these altitudes AH, BK, CL, respectively. The incircle of triangle ABC has centre I and touches the sides BC, CA, AB at the points D, E, F, respectively. Prove that the four lines A_0D, B_0E, C_0F and OI are concurrent. (When the point O coincides with I, we consider the line OI as an arbitrary line passing through O.)

7. ISL 2011 G6: Let ABC be a triangle with $AB = AC$ and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B and C at the point E inside the triangle ABC. The line BD intersects the circle through A, E and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incentre of triangle KAB.

8. RMM 2012: Let ABC be a triangle and let I and O denote its incentre and circumcentre respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO.

9. ISL 2012: Let ABC be a triangle with circumcentre O and incentre I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

10. Own: Let ABC be a triangle with all angles$> 45°$. D, E, F are the feet of the altitudes from A, B, C. G is the centroid. Intersection of DG and (ABC) is A'. Define B' and C' analogously.$A'B'$ intersects with AC at L_{ac} and $A'B'$ with BC at L_{bc}. Define the L's with other subscripts similarly. Let (O_a) be the circle passing through A and L_{ac} and tangent to $A'B'$. Let (O'_a) be the circle through A and L_{ab} and tangent to $C'A'$. Here we denote by (X) a circle with center as X. Define the circles with the other subscripts in the same manner. Let the circumcentre of $AO_aO'_a$ be O''_a. Define other centres similarly, using a cyclic shift of variable names. Extend $A'O''_a$, $B'O''_b$ and $C'O''_c$ to form a triangle $A_1B_1C_1$. Let the mixtilinear incircle touch points with its circumcircle be X, Y, and Z. Prove that the cevians A_1X etc concur at the point P such that the circumcentre of the cevian triangle of the isotomic conjugate of isogonal conjugate of P with respect to $A_1B_1C_1$ is the circumcentre of ABC.[4]

11. Let I be the incenter of an acute-angled triangle ABC. Line AI cuts the circumcircle of BIC again at E. Let D be the foot of the altitude from A to BC, and let J be the reflection of I across BC. Show D, J and E are collinear.

12. Russia 1999: A triangle ABC is inscribed in a circle S. Let A_0 and C_0 be the midpoints of the arcs BC and AB on S, not containing the opposite vertex, respectively. The circle S_1 centered at A_0 is tangent to BC, and the circle S_2 centered at C_0 is tangent to AB. Prove that the incenter I of $\triangle ABC$ lies on a common tangent to S_1 and S_2.

13. Taiwan 2014 TST Quiz: Let ABC be a triangle with incenter I and circumcenter O. A straight line L is parallel to BC and tangent to the incircle. Suppose L intersects IO at X, and select Y on L such that YI is perpendicular to IO. Prove that A, X, O, Y are concyclic.

14. Let I be the incenter of a triangle ABC and let the A, B and C mixtilinear incircles touch the circumcircle of ABC at T_A, T_B and T_C respectively. Let IT_A, IT_B, IT_C cut BC, CA, AB at X, Y and Z respectively. Prove that AX, BY and CZ concur at a point joining the centroid of ABC to the Gergonne point of ABC.

15. Russia 2014: Let ABC be a triangle with $AB > BC$ and circumcircle Γ. Points M, N lie on the sides AB, BC respectively, such that $AM = CN$. Lines MN and AC meet at K. Let P be the incenter of the triangle AMK, and let Q be the K-excenter of the triangle CNK. If R is midpoint of arc ABC of Γ, then prove that $RP = RQ$.

[4]Another problem: Draw its diagram. It has a surprisingly simple solution, even though the notation is quite intimidating.

4 Altitudes and Midpoints

1. ISL 2011 G4[5]: Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

2. Russia: An acute-angled ABC ($AB < AC$) is inscribed into a circle ω. Let M be the centroid of ABC, and let AH be an altitude of this triangle. A ray MH meets ω at A'. Prove that the circumcircle of the triangle $A'HB$ is tangent to AB.

3. ThE-dArK-lOrD (AoPS):Given $\triangle ABC$ with altitude AH_A, BH_B, CH_C where $H_A, H_B, H_C \in BC, CA, AB$ respectively Let D, E, F are points in segment AH_A, BH_B, CH_C such that $\angle BDC = \angle CEA = \angle AFB = 90°$ And let $X \neq D, Y \neq E, Z \neq F$ lie on AH_A, BH_B, CH_C such that $\angle BXC = \angle CYA = \angle AZB = 90°$ Let M is circumcenter of $\triangle DEF$ and N is circumcenter of $\triangle XYZ$

   1) Prove that H, M, N collinear where H is orthocenter of $\triangle ABC$

   2) Prove that Z, D, Y collinear if and only if Z, N, Y collinear.

4. CentroAmerican Olympiad: Let ABC be an acute-angled triangle, Γ its circumcircle and M the midpoint of BC. Let N be a point in the arc BC of Γ not containing A such that $\angle NAC = \angle BAM$. Let R be the midpoint of AM, S the midpoint of AN and T the foot of the altitude through A. Prove that R, S and T are collinear.

5. Tran Quang Hung: Let ABC be a triangle with orthocenter H, circumcenter O and nine point circle (N). Construct parallelogram $ABDC$. Let P be a point on radical axis of (N) and circle diameter OD such that $OP \parallel BC$. K lies on OH such that $PK = PO$. Prove that radical axis of (N) and circle diameter HK passes through O.

6. Own: Let the perimeter of a triangle ABC be 2 and let BC be the smallest side. Let P and Q be on AC and AB such that $AP + PB = AQ + QC = 1$. A line parallel to the internal angle bisector of B through P meets the perpendicular bisector of BC at T. BP intersects QC at W. Prove that A, W, T are collinear iff $AB = AC$.[6]

5 Parallelograms

1. USAMO 2003: Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB.MD = MC^2$.

2. Let ABC be a triangle with circumcenter O and orthocenter H, and let M and N be the midpoints of AB and AC. Rays MO and NO meet line BC at Y and X, respectively. Lines MX and NY meet at P. Prove that OP bisects AH.

3. Let $ABCD$ be a parallelogram. Let E and F be points on AB and AD such that $BE = DF$. Prove that DE, BF and the angle bisector of $\angle BCD$ concur.

4. Let $ABCD$ be a parallelogram. P is a point such that $\angle PDC = \angle PBC$. Prove that $\angle PAB = \angle PCB$.

5. Let $ABCD$ be a parallelogram in which the angle bisector of $\angle ADC$ meets AB and BC at X and Y respectively. Prove that the circumcenter of BXY lies on the circumcircle of ABC.

[5]A solution using a $\sqrt{bc}$ inversion is also possible. Try finding it after a solution using one of our lemmas.

[6]This may look weird, but the idea of the solution is quite natural at heart.

6. ELMO 2012: Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $PD \parallel AE$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.

7. Tran Quang Hung: Let $ABCD$ be a parallelogram. (O) is the circumcircle of triangle ABC. P is a point on BC. K is circumcenter of triangle PAB. L is on AB such that $KL \perp BC$. CL cuts (O) again at M. Prove that M, P, C, D are concyclic.

8. ISL 2013 G5: Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals AD, BE, and CF are concurrent.[7]

9. ELMO 2016: Oscar is drawing diagrams with trash can lids and sticks. He draws a triangle ABC and a point D such that DB and DC are tangent to the circumcircle of ABC. Let B' be the reflection of B over AC and C' be the reflection of C over AB. If O is the circumcenter of $DB'C'$, help Oscar prove that AO is perpendicular to BC.

6 Miscellaneous Problems

1. Romania TST: Let ABC be a triangle, D be a point on side BC, and let $\mathcal{O}$ be the circumcircle of triangle ABC. Show that the circles tangent to $\mathcal{O}, AD, BD$ and to $\mathcal{O}, AD, DC$ are tangent to each other if and only if $\angle BAD = \angle CAD$.

2. Two circles intersect at two points A and B. A line ℓ which passes through the point A meets the two circles again at the points C and D, respectively. Let M and N be the midpoints of the arcs BC and BD (which do not contain the point A) on the respective circles. Let K be the midpoint of the segment CD. Prove that $\measuredangle MKN = 90°$.

3. The tangents at A and B to the circumcircle of the acute triangle ABC intersect the tangent at C at the points D and E, respectively. The line AE intersects BC at P and the line BD intersects AC at R. Let Q and S be the midpoints of the segments AP and BR respectively. Prove that $\angle ABQ = \angle BAS$.

4. Russia: Two circles intersect at A and B. A line through A meets the first circle again at C and the second circle again at D. Let M and N be the midpoints of the arcs BC and BD not containing A, and let K be the midpoint of the segment CD. Show that $\angle MKN = \pi/2$. (You may assume that C and D lie on opposite sides of A.)

5. A circle centred at O and inscribed in triangle ABC meets sides AC;AB;BC at K;M;N, respectively. The median BB_1 of the triangle meets MN at D. Show that O;D;K are collinear.

6. In scalene $\triangle ABC$, the tangent from the foot of the bisector of $\angle A$ to the incircle of $\triangle ABC$, other than the line BC, meets the incircle at point K_a. Points K_b and K_c are analogously defined. Prove that the lines connecting K_a, K_b, K_c with the midpoints of BC, CA, AB, respectively, have a common point on the incircle.

7. In triangle ABC, a circle passes through A and B and is tangent to BC. Also, a circle that passes through B and C is tangent to AB. These two circles intersect at a point K other than B. If O is the circumcentre of ABC, prove that $\angle BKO = 90°$.

8. Sharygin: Quadrilateral $ABCD$ is circumscribed around a circle with centre I. Prove that the projections of points B and D to the lines IA and IC lie on a single circle.

[7] Yes, this problem comes under the section parallelograms.

9. ISL 2015 G5: Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

10. ISL 2015 G7: Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segment PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

11. ISL 2014 G7: Let ABC be a triangle with circumcircle Ω and incentre I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V , respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y . Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points I, X, and Y are collinear, then the points I, W, and Z are also collinear.

12. ISL 2014 G6: Let ABC be a fixed acute-angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of EF . Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T , respectively. We call the pair (E, F) *interesting*, if the quadrilateral $KSAT$ is cyclic. Suppose that the pairs (E_1, F_1) and (E_2, F_2) are interesting. Prove that $\dfrac{E_1E_2}{AB} = \dfrac{F_1F_2}{AC}$

13. ISL 2014 G4: Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0, 1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.[8]

14. IMO 2013: Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

15. ISL 2015 G3: Let ABC be a triangle with $\angle C = 90°$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

16. China TST: A circle Γ through A meets AB, AC at E, F respectively and (ABC) at P. Prove that the reflection of P across EF lies on BC if and only if Γ passes through O.

17. Let the circumcenter of ABC be O. Let the projection of A onto BC be H_A. AO meets (BOC) again at A'. Projections of A' onto AB, AC are D, E. Let H_ADE have the circumcenter O_A. Define H_B, H_C, O_B, O_C respectively. Prove that O_AH_A, O_BH_C and O_CH_C concur at a point.

18. Diagonals of a cyclic quadrilateral $ABCD$ meet at P and $\exists$ a circle Γ tangent to the extensions of AB, BC, AD, DC at X, Y, Z, T respectively. Circle Ω passes through A, B, and is tangent to Γ at S. Prove that $SP \perp ST$.

19. Let D, E be the feet of the B and the C altitudes in an acute $\triangle ABC$. Let the reflection of E with respect to AC, BC be S, T. The circumcircle of $\triangle CST$ meets AC again at X. Denote the circumcenter of $\triangle CST$ as O. Prove that $XO \perp DE$.

[8] After a suitable transformation, this problem can be finished even by coordinates.

20. USA TST 2016: Let ABC be a scalene triangle with circumcircle Ω and suppose that the incircle of ABC touches BC at D. Let the angle bisector of BC and Ω at E and F. The circumcircle of DEF meets the A-excircle at S_1 and S_2 and Ω at $T \neq F$. Prove that AT passes through either S_1 or S_2.

21. USA TST 2015: Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively. Let S be a point lying on the Euler line. Denote by X, Y, Z the second intersections of M_aS, M_bS, M_cS with the nine-point circle. Prove that AX, BY, CZ are concurrent.

22. In acute ABC, $\angle A$ is the smallest. P is a variable point on BC and D, E lie on AB, AC respectively such that $BP = PD$ and $CP = PE$. Prove that as P varies on BC, circumcircle of ADE passes through a fixed point.

23. Let ABC be an acute scalene triangle inscribed in circle Ω. A circle ω centered at O meets AB, AC again at E, D respectively. Point P lies on arc BAC of Ω. Prove that OP, BD, CE are concurrent if and only if the incircles of PBD and PCE are concentric.

24. The incircle ω of a quadrilateral $ABCD$ touches AB, BC, CD, DA at E, F, G, H respectively. X is an arbitrary point on segment AC inside ω .The segments XB, XD meet ω at I, J. Prove that FJ, IG and AC concur at a point.

25. Let the incircle (I) of $\triangle ABC$ touch BC, CA, AB at D, E, F respectively and let their midpoints be P, Q, R. The reflections of D, E, F over AI, BI, CI are D', E', F'. Prove that PD', QE', RF' concur at a point.

7 Further solving

1. Yufei Zhao's notes
2. Evan Chen's articles
3. Sharygin Geometrical Olympiad
4. TSTs of various countries
5. Forum Geometricorum (Good for reading)
6. Akopyan, Geometry in figures
7. Roger Johnson, Advanced Euclidean Geometry
8. I.F. Sharygin, Problems in Plane Geometry
9. Aref, Wernick, Problems and Solutions in Euclidean Geometry