Lester's Circle

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Introduction

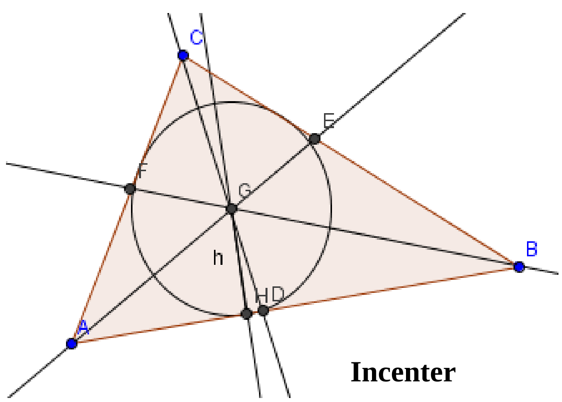
Euclid is credited with most of the theorems in geometry textbooks today. Around 300 B.C., Euclid produced a thirteen volume publication called *The Elements*. These volumes include much information others had studied. Throughout history, many great mathematicians including Pasch, Hilbert, and Birkhoff have studied and tried to improve Euclidean geometry. Groups such as University of Chicago School Mathematics Project have made improvements on the Euclid's axiomatic system.

In 1996, June Lester discovered a new theorem. Lester's Theorem states in a scalene triangle the circumcenter, the nine-point center, and the first and second Fermat points of a triangle lie on a circle. Definitions of these centers will be provided later. It is an amazing feat to discover a new theorem in Euclidean Geometry.

History

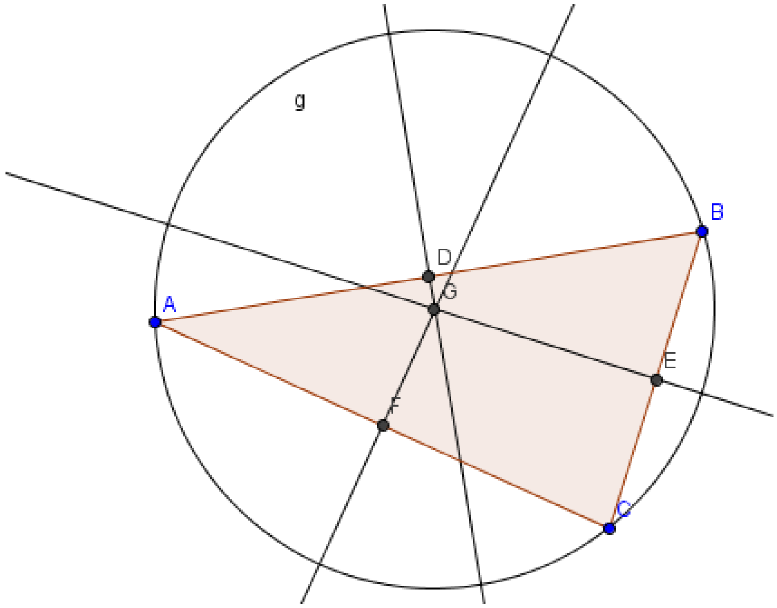
Many people have studied the centers of triangles prior to June Lester. These centers include the incenter, circumcenter, orthocenter and the centroid. Pierre de Fermat was a mathematician in the 1600's. He contributed to the study of triangle centers. Fermat actually discovered two triangle centers which are now appropriately called Fermat Points. In 1820, French mathematicians Charles Brianchon and Jean Vicory Poncelet published a paper that contained the proof of the following statement: The circle which passes through the feet of the perpendiculars of any triangle, passes also through the midpoints of these sides as well as through the midpoints of the segments which join the vertices to the point of concurrency of the perpendiculars. This circle is known as the Nine-Point Circle. Recall that three points determine a circle. Given this, it is phenomenal that they proved that nine points, which are seemingly unrelated, always lie on the same circle. Discussion will follow on each of these centers and how they relate to Lester's Circle.

The *incenter* is the point of concurrency of the angle bisectors of the triangle. This is the center of the circle inscribed in the triangle. The radius of the inscribed circle is the segment from the incenter, perpendicular to a side of the triangle.



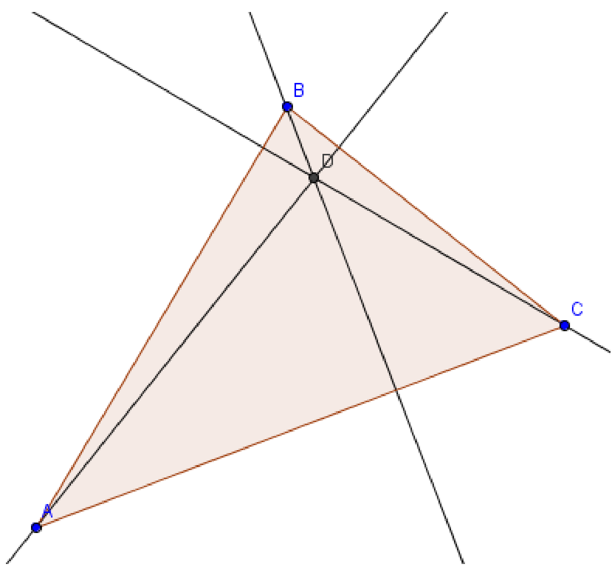
Incenter

The *circumcenter* of a triangle is the point of concurrency of the perpendicular bisectors of the triangle. This is the center of the circle circumscribed about the triangle. The radius of this circle is the segment from the circumcenter to a vertex of the triangle.



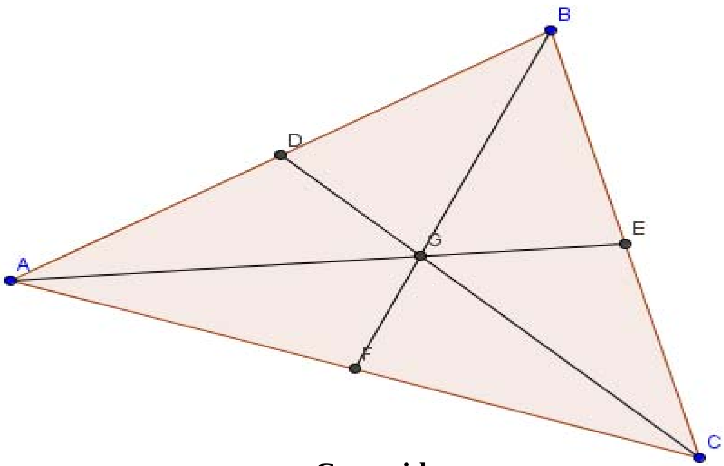
Circumcenter

The *orthocenter* of a triangle is the point of intersections of the altitudes.



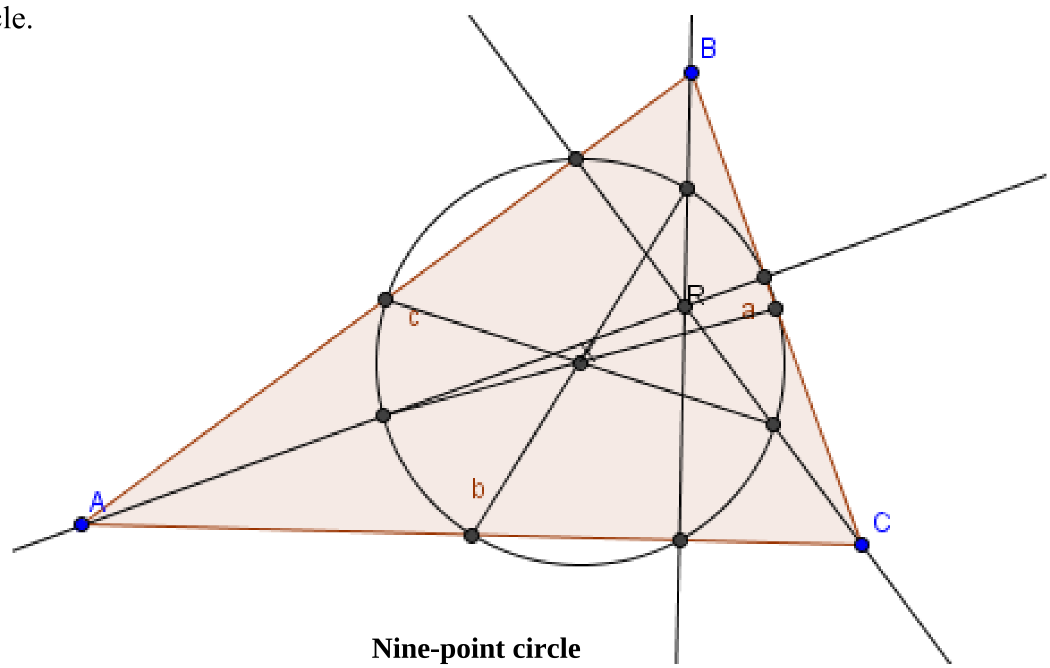
Orthocenter

The *centroid* is the point of concurrency of the medians of a triangle. It is also called the center of mass of the triangle.



Centroid

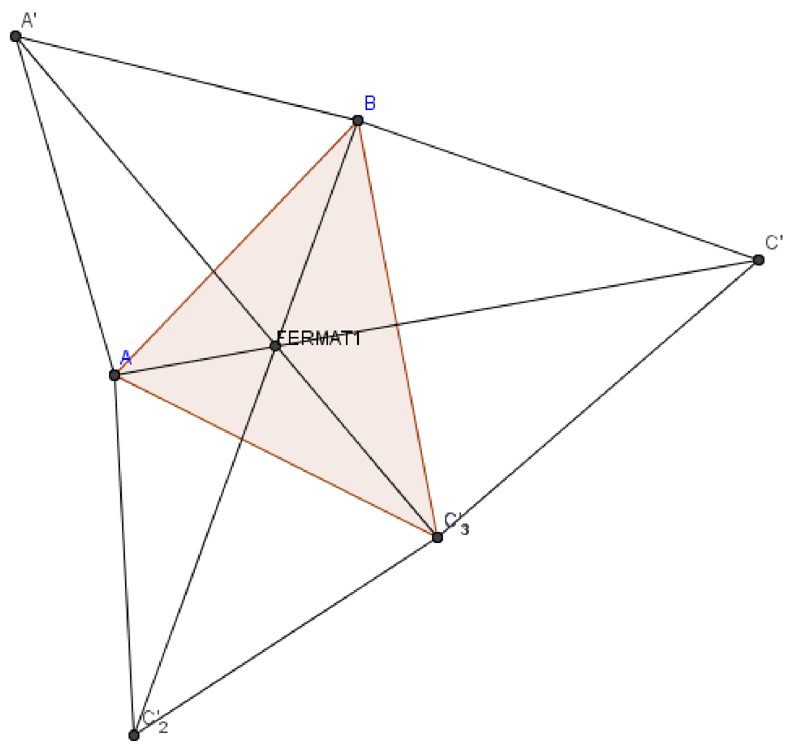
The *nine-point circle* includes some of the above constructions. To construct the nine-point circle, one must find the orthocenter, find the midpoint of each side of the triangle, find the midpoints of the segments from each vertex to the orthocenter, and construct the segments from the midpoint of the sides to the midpoint of the segment connecting the vertex of the triangle and orthocenter. The point of concurrency of the last three segments is the center of the nine-point circle.

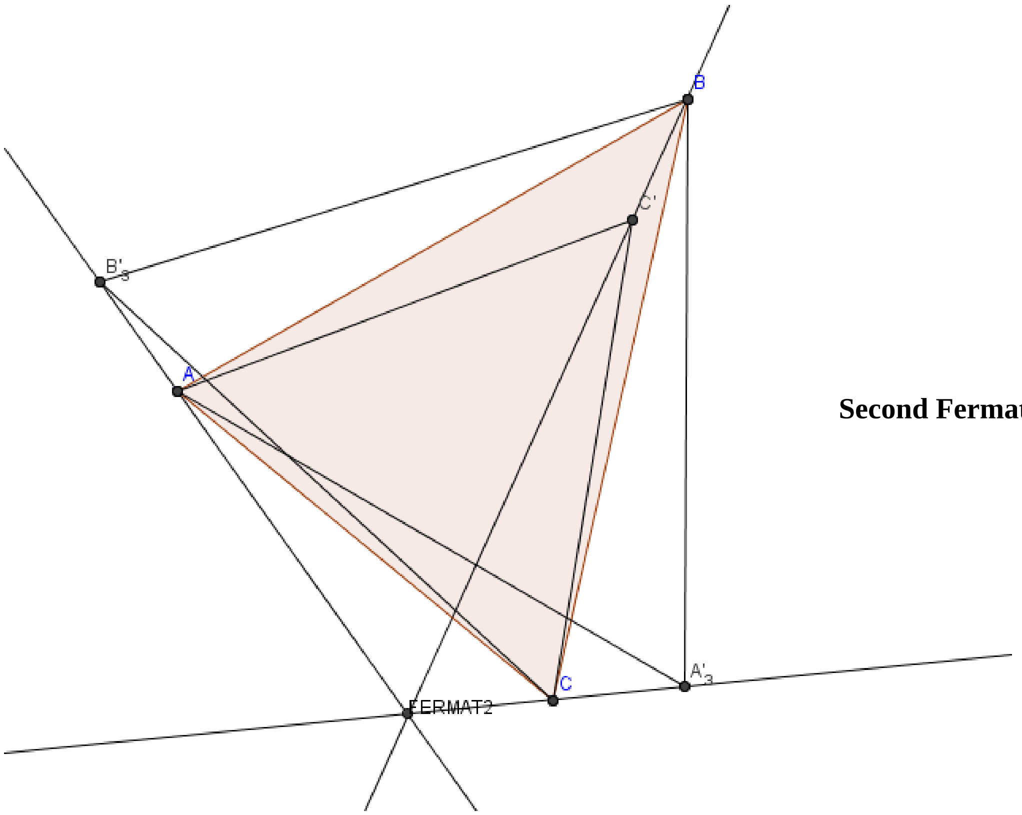


Nine-point circle

Pierre de Fermat was a French mathematician who lived in the 1600's. He is credited with many numerical contributions, but is also responsible for two different centers of triangles. The *First Fermat Point* is found by constructing an equilateral triangle on the outside of each side of a triangle and drawing the segment from the opposite vertex of the original triangle to the vertex of the equilateral triangle. The *First Fermat Point* is the intersection of these segments. The *Second Fermat Point* is found by constructing equilateral triangles on each side of a triangle toward the interior of the triangle. Again the segments from the opposite vertices of the original triangle to the vertices of the equilateral triangle are constructed and the *Second Fermat Point* is the intersection of these segments.

First Fermat Point

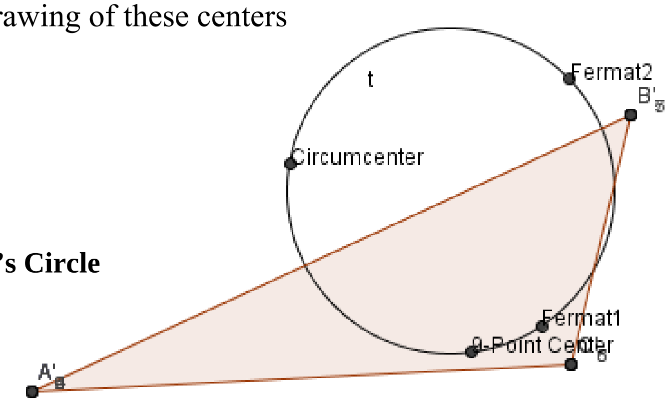




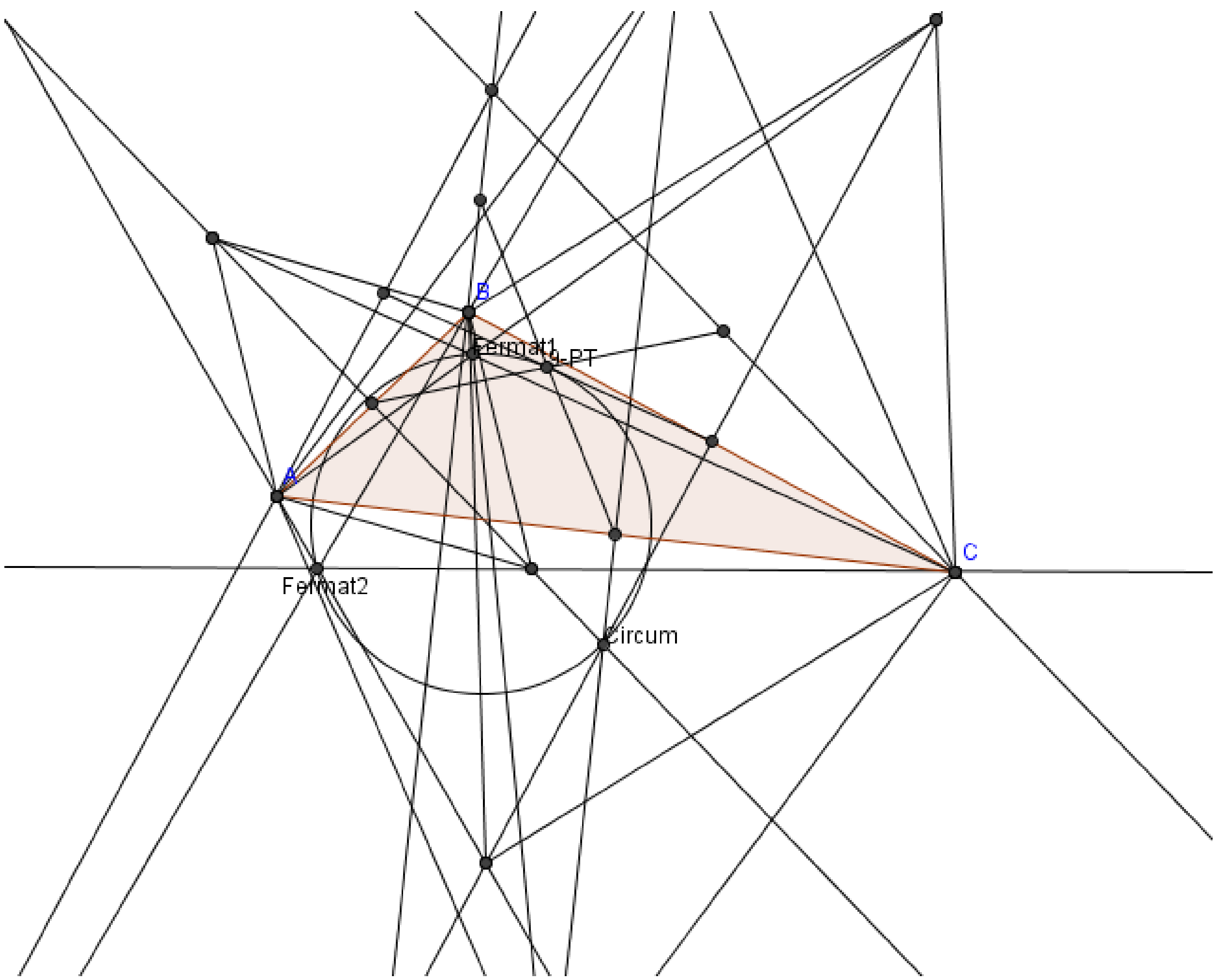
Second Fermat point

June Lester

June Lester discovered Lester's Theorem in 1996. In Euclidean plane geometry, Lester's theorem states that in any scalene triangle, the two Fermat points, the nine-point center, and the circumcenter are concyclic. She also states that the order of the points is always the same, either clockwise or counterclockwise. The points occur in the following order: circumcenter, nine-point center, first Fermat point and second Fermat point. The direction around the circle is dependent on the type of triangle. If the triangle becomes isosceles, the circle becomes a line and if the three vertices of the triangle are collinear, some of the points do not exist. Lester discovered this theorem while exploring ways of using complex numbers to study Euclidean geometry. She has written three articles about triangles in the study of this theorem. Lester was a professor of mathematics at the University of New Brunswick in Fredericton, New Brunswick, Canada at the time she discovered Lester's Theorem. Since that time Lester has begun studying technology and its role in mathematics. She is currently a graduate student working toward a doctorate degree in "onscreen mathematics communication and design" at Simon Fraser University's School of Interactive Arts and Technology. A drawing of these centers and the *Lester's Circle* are shown.



Lester's Circle



Lester's Circle with construction lines

The diagram shown above gives the lines of construction for the four parts of Lester's theorem. With so many lines, it is hard to see the circle. However, this is a good representation of how much information is included in Lester's Theorem.

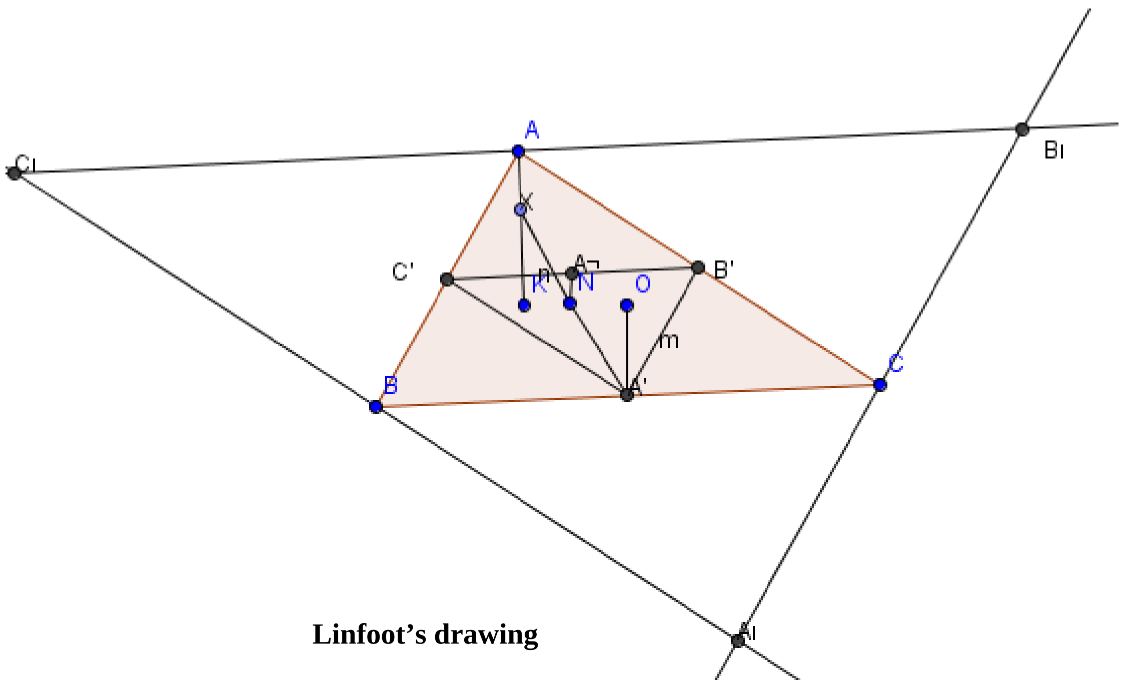
Even though the constructions completed with technology, such as the one above, give one the power to explore theorems in general, they are not a replacement for an actual proof. A proof shows the mathematics behind the constructions. Throughout history many people have completed proofs for the previously mentioned centers. Some of the proofs are very extensive and others rather simple and straight forward. The following pages include proofs of some of the centers of triangle.

Proofs

One can prove the circumcenter is the same distance from all three vertices of any triangle because any point that lies on a perpendicular bisector of a segment is the same distance from the endpoints of the segment. Thus, if one has triangle XYZ and the perpendicular bisectors intersect at K, the lengths of segments KX, KY, and KZ are all equal because K is the same distance from the endpoints of each side of the triangle.

The following is a proof of the nine-point circle completed by Joyce Linfoot in 1957.

> "Consider ΔABC, together with the ΔA'B'C' formed by joining the midpoints of its sides and the $\Delta A_1B_1C_1$ formed by drawing lines through its vertices parallel to its sides. The Δs ABC, A'B'C' and $A_1B_1C_1$ are similar and corresponding lengths are in the ratios of 2: 1: ½. Now the circumcenter of $\Delta A_1B_1C_1$ is the orthocenter K of ΔABC. Let O be the circumcenter of ΔABC and N be the circumcenter of ΔA'B'C'. Let A" be the midpoint of B'C'. Then AK:A'O:A"N=2: 1: ½ and A, A' and A" are collinear. Let AA' meet NO at G. Then from the similar Δs A"NG and A'OG, A"G=2GA, whence A'G=1/3 AA' and G is the centroid of ΔABC and also of Δs $A_1B_1C_1$ and A'B'C'. Let A'N meet AK at X. From the Δs A'AX, A'A"N, AX=2A"N= 1/2AK. Also XN=NA', So the circumcircle of ΔA'B'C' passes through X, the midpoint of AK, and, since A'X is a diameter, also through the foot of the perpendicular from A to BC. With these guiding lines the proof of any required nine-point circle property can be fairly easily reconstructed." (The Mathematical Gazette, Vol. 141, No. 388, Dec. 1957, p. 295)



Linfoot's drawing

The following is a proof completed in 1946 about Fermat points. It uses the many properties of equilateral and 30-60-90 triangles.

THE FERMAT AND HESSIAN POINTS OF A TRIANGLE

H. E. FETTIS, Dayton, Ohio

1. Introduction. The properties of the triangle which are presented here are proved analytically in the Morley *Inversive Geometry.** The properties seem so noteworthy as to demand straightforward proofs by the methods of modern geometry, since, to the author's knowledge, the relations are neither mentioned nor proved in any of the standard works on the subject. The paper will confine itself to the discussion of those properties which are less familiar, it being assumed that the reader is acquainted with certain terms and theorems which are standard topics in modern geometry.

2. The Fermat points. The first points to be considered are those known as the Fermat points.† These may be defined in the following manner:

Consider a triangle $A_1A_2A_3$ with equilateral triangles $A_2A_3P_1$, $A_3A_1P_2$, $A_1A_2P_3$ described externally on its sides. Then it is known that the lines A_1P_1, A_2P_2, A_3P_3 are concurrent at a point F, which is known as a Fermat point of the triangle. Similarly, if equilateral triangles $A_2A_3P_1'$, $A_3A_1P_2'$, $A_1A_2P_3'$ are constructed internally on the sides of $A_1A_2A_3$ ($A_1A_2A_3$ being, in this case, non-equilateral) the lines A_1P_1', A_2P_2', A_3P_3' are concurrent at a second Fermat point, F'.

The Fermat points are characterized by the fact that the segments A_iP_i are equal and intersect at angles of 60°, and the segments A_iP_i' are equal and intersect at angles of 60°. The points are named from the fact that F is (for triangles having no angle exceeding 120°) the solution to Fermat's problem: to find a point the sum of whose distances from the vertices of a triangle is a minimum.

Let M_1 be the midpoint of A_2A_3 and let N_1 and N_1' be the centers of the equilateral triangles $A_2A_3P_1$ and $A_2A_3P_1'$ respectively. The centroid, G, of $A_1A_2A_3$ lies on A_1M_1, and $A_1G=2GM_1$. Then, since M_1 is the midpoint of P_1P_1', it is clear that G is also the centroid of $A_1P_1P_1'$, and therefore divides the median of this triangle drawn from P_1 in the ratio 2:1. Thus since $P_1N_1'=2N_1'P_1'$, GN_1' is parallel to A_1P_1'. Similarly it may be shown that GN_1 is parallel to A_1P_1.

Let N_1G and $N_1'G$ be produced to intersect A_1P_1' and A_1P_1 at R_1 and R_1' respectively. Then, since G is the centroid of $A_1P_1P_1'$, N_1R_1 and $N_1'R_1'$ are bisected at G, and circles may be described with centers at G and diameters N_1R_1 and $N_1'R_1'$. Let these circles respectively intersect A_1P_1' and A_1P_1 again in points E' and E. Now, since $N_1'E$ is perpendicular to A_1P_1 and N_1E' is perpendicular to A_1P_1', it follows that the circles circumscribing triangles $A_2A_3P_1$ and $A_2A_3P_1'$ pass, respectively, through E and E'. From previous considerations, therefore, we see that E and E' coincide with the Fermat points F and F' of $A_1A_2A_3$. In addition, since GN_1 and GN_1' are parallel to A_1F and A_1F', it follows that

$$\measuredangle FA_1F' = \measuredangle GFA_1 = \measuredangle A_1F'G = \tfrac{1}{3}\measuredangle FGF', \pmod{\pi}.$$

Summarizing we have: *GN_i and GN_i' are parallel to A_iP_i and A_iP_i' respectively*; *the points N_i and N_i' are the vertices of two equilateral triangles* (*with sides perpendicular to A_iF and A_iF'*) *inscribed in the circles with centers at G and radii equal to GF' and GF respectively*; *and*

$$\measuredangle FA_iF' = \measuredangle GFA_i = \measuredangle A_iF'G = \tfrac{1}{3}\measuredangle FGF', \pmod{\pi}.$$

(Fettis, 1946).

The following are the words of June Lester according to her website. It includes her explanation of the math she addressed while discovering Lester's Theorem.

The mathematics behind the theorem

The material below is a brief outline of the mathematics used to discover and prove that the four points lie on a circle. It's by no means a complete description, but is meant only to give a small taste of the "flavour" of the mathematics. Full details can be found in the original papers, listed in the bibliography.

You'll need to know what a complex number is to understand the definitions and some of the fundamental facts and ideas behind the proof.

Background

The idea behind using complex numbers in Euclidean geometry is to choose some rectangular coordinate system in the plane and then make every point (x, y) into a complex number:

$(x,y)\rightarrow \mathbf{z}=x+iy$.

Instead of two coordinates for every point, a single complex number now represents every point.

Two basic tools help with calculations involving these points.

The first tool is a special combination of numbers (points): the cross ratio of any four distinct points a, b, c and d in the plane is the complex number

$$[\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d}] = \frac{(\mathbf{a}-\mathbf{c})(\mathbf{b}-\mathbf{d})}{(\mathbf{a}-\mathbf{d})(\mathbf{b}-\mathbf{c})}.$$

The order of the points is important - rearranging the points may give a different cross ratio - but given the cross ratio for one order, the cross ratio for any rearrangement is easy to find. Cross ratios are extremely useful in geometric proofs; for example, the cross ratio of four points is real if and only if the points lie on a line or on a circle. The order of the points along that line or circle can then be determined from this real number.

The second tool is a "pretend" point added to the plane; the point at infinity, or antipode. Think of it as being infinitely far away from any other point - what happens to z when x and y get infinite. To see how the point at infinity works in cross ratios, suppose the first number a in the cross ratio definition is very big. Divide the first term top and bottom by a:

$$\frac{(\mathbf{a}-\mathbf{c})(\mathbf{b}-\mathbf{d})}{(\mathbf{a}-\mathbf{d})(\mathbf{b}-\mathbf{c})} = \frac{(1-\mathbf{c}/\mathbf{a})(\mathbf{b}-\mathbf{d})}{(1-\mathbf{d}/\mathbf{a})(\mathbf{b}-\mathbf{c})}.$$

The larger a is, the smaller c/a and d/a are and the closer those two terms are to 1. So for "infinite" a, it makes sense to define

$$[\infty,\mathbf{b};\mathbf{c},\mathbf{d}] = \frac{(\mathbf{b}-\mathbf{d})}{(\mathbf{b}-\mathbf{c})}.$$

This version of the cross ratio is very useful too: it turns out that three points lie on a line if and only if their cross ratio with the point at infinity is real. Another useful way of looking at it: think of lines as being circles through the point at infinity.

(You can learn more about cross ratios and the point at infinity from various books on complex numbers, for example

I. M. Yaglom, Complex Numbers in Geometry, Academic Press, New York, 1968.

H. Schwerdtfeger, Geometry of Complex Numbers, University of Toronto Press, Toronto, 1962.)

Shapes

Three distinct points a, b and c in the plane determine a triangle (it will be degenerate if the points lie on a line). The shape of this triangle is the complex number

$$\Delta_{\mathbf{abc}} = [\infty,\mathbf{a};\mathbf{b},\mathbf{c}].$$

The important thing about shapes is that two triangles are similar if and only if they have the same shape. This means that proofs of theorems about similar triangles can sometimes be reduced to calculating their shapes. It also means that the numerical values of the angles of a triangle and ratios of the lengths of its sides can all be extracted from its shape.

Triangle coordinates

To look at the geometry of points relative to a given, fixed triangle, it helps to have a coordinate system adapted to that triangle. The triangle coordinate of any point p with respect to the triangle with vertices a, b and c is the number

$$\mathbf{p}_\Delta = [\mathbf{p},\mathbf{a};\mathbf{b},\mathbf{c}].$$

Triangle coordinates of points related to the geometry of the triangle tend to be simpler than arbitrary complex coordinates in the plane. What makes triangle coordinates most useful is that the cross ratio of any four points equals the cross ratio of their triangle coordinates:

$$[\mathbf{p},\mathbf{q};\mathbf{r},\mathbf{s}] = [\mathbf{p}_\Delta,\mathbf{q}_\Delta;\mathbf{r}_\Delta,\mathbf{s}_\Delta]$$

for any four points p, q, r and s. This means that any geometry we can describe through cross ratios of points can also be described through cross ratios of their triangle coordinates.

Triangle functions

"Special" points of a triangle are points like the circumcentre, the nine-point centre, etc. - basically, points that are defined in terms of the geometry of the triangle. Triangle functions relate the triangle coordinate of any special point to the shape of the triangle: every special point s of a triangle has a corresponding triangle function S such that the triangle coordinate of s with respect to any triangle of shape Δ is

$$\mathbf{s}_\Delta = \mathcal{S}(\Delta).$$

For example, the triangle function of the nine-point centre is

$$N(\mathbf{z}) = \frac{\mathbf{z}^2 - 2\mathbf{z} + \overline{\mathbf{z}}}{-\mathbf{z}^2 - \mathbf{z} + 2\mathbf{z}\overline{\mathbf{z}}}\mathbf{z}$$

(overbars denote conjugates).
Essentially, a triangle function"embeds" the definition/construction of the special point inside a single function. Since triangle functions are the triangle coordinates of special points, it follows that to prove that four special points always lie on a circle or on a line, it is enough to prove that the cross ratio of their triangle functions is identically real.

How the theorem was discovered

I discovered the theorem by searching through a large number of special triangle points for quadruples of points which lie on a circle.

First, I needed a database of special points and their coordinates; I got it from Clark Kimberling's list of triangle centres (now expanded and online). Clark uses trilinear coordinates, so I had to figure out how to convert trilinear coordinates into complex triangle coordinates.

I next input everything - points, conversion formulas, cross ratio formulas - into an easy-to-use computer math program, Theorist (which has since evolved into LiveMath).

Then I input a single numerical shape and set a search going for quadruples of special points with a real cross ratio. The computer was a then state-of-the-art Mac IIcx; the computation was nevertheless time-consuming (several hours), and had to be repeated multiple times to be sure that those real cross ratios found were not a coincidence.

Along the way, I discovered other quadruples of special points apparently on circles. These involve more obscure triangle centres than the four on the Lester circle, so I noted them in Triangles III, but didn't look for a proof.

How the theorem was proved

The idea of the proof is to show algebraically that the cross ratio of the triangle functions of the four points is real. Finding the functions themselves was a non-trivial process; calculating their cross ratio was horrendous. As motivation for such drudgery, it helps to have solid experimental evidence that what you're trying to prove is actually true. More practically, it also helps to have ways of simplifying cross ratio calculations, such as rearranging the order of the functions. Combined with the hindsight gleaned from doing the calculation in several different non-optimal ways, I eventually pared it down to the form that appears in Triangles III. (Lester)

Conclusion

June Lester is a twentieth century mathematician who made an amazing discovery. She discovered Lester's Theorem in 1996. It is phenomenal that there are new things to be discovered in Euclidean Geometry, since Euclidean Geometry has been around since 300 B. C. June Lester has shared her knowledge with others through speaking engagements, publications, and through her teaching. She is currently studying how to enhance mathematics through technology; I am sure she has reached many people through this medium also.

OUTLINE FOR TEACHING LESTER'S THEOREM

Lester's Theorem states that the circumcenter, nine-point center and the first and second Fermat points all lie on a circle.

A. Circumcenter
    a. Definition
    b. How to construct perpendicular bisectors
B. Nine-point center
    a. Definition
    b. How to construct the orthocenter
    c. How to construct midpoint of sides
    d. How do I draw appropriate segments
    e. How to construct midpoint of segments from vertices to orthocenter
C. First Fermat Point
    a. Definition
    b. How to construct equilateral triangles outside of original triangle
    c. Rotation-counterclockwise and clockwise
D. Second Fermat Point
    a. Definition
    b. How to construct equilateral triangles inside of original triangle
E. Lester's Circle
    a. How to draw a circle through four points

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