High School Olympiads

Intersection of two Simson lines lie on a fixed line.

Source: HGSG VMO team training test 2013

buratinogigle

#1 Jan 19, 2015, 11:59 am

Let ABC be a triangle inscribed circle (O). M,N move on (O) such that MN is always parallel to a fixed line. Prove that intersection of two Simson lines of M,N with respect to triangle ABC lie on a fixed line when M,N move.

Luis González

#2 Jan 19, 2015, 12:12 pm • 1

Let Y,Z be the projections of B,C on $MN.$ It's known that the Simson lines of M and N meet at the orthopole T of MN WRT $\triangle ABC$ (for a proof see the lemma at Six orthopoles lie on a circle). Therefore all lines TY and TZ remain perpendicular to $AC,AB \Longrightarrow$ all $\triangle TYZ$ are homothetic $\Longrightarrow T$ moves on a fixed line parallel to $BY \parallel CZ,$ i.e. perpendicular to $MN.$

In general, when all lines MN go through a fixed point, not necessarily at infinity, then the locus is the orthopolar ellipse of this point WRT ABC. See Orthopole lies on ellipse.

buratinogigle

#3 Jan 19, 2015, 12:41 pm

My solution

Denote by s_X is Simson line of point X lie on circumcircle of triangle ABC with respect to triangle ABC. and $\mathcal{R}_d$ is reflection through axis d.

Lemma 1. Triangle ABC is inscribed circle (O) and M,N lie on (O) then $(s_M,s_N)=\frac{1}{2}(\overrightarrow{ON},\overrightarrow{OM})(\bmod \pi)$.

Lemma 2. Triangle ABC is inscribed circle (O) and orthocenter H and P lies on (O) then s_P passes through midpoint of PH.

Lemma 3. Triangle ABC is inscribed circle (O) and M,N lie on (O). P is a point on circle diameter MN. m,n are two line passing through M,N and parallel to s_M,s_N, resp. Then reflection of m,n through PM,PN, reps, intersect on (O).

Proof. Let m',n' be reflection of m,n through PM,PN, resp. We have

$(m',n')=(m',PM)+(PM,PN)+(PN,n')(\bmod \pi)$

$=(PM,m)+(PN,PM)+(n,PN)(\bmod \pi)$ (Because $PM\perp PN$ thus $(PM,PN)=\frac{\pi}{2}=-\frac{\pi}{2}=(PN,PM)(\bmod \pi)$)

$=(n,m)(\bmod \pi)$

$=\frac{1}{2}(\overrightarrow{OM},\overrightarrow{ON})(\bmod \pi)$.

Note that, m' and n' pass through M,N therefore m',n' intersect on (O). We are done.

Lemma 4. Triangle ABC is inscribed circle (O), orthocenter H. MN and PQ are perpendicular chords of (O) and they intersect at S. s_M and s_N intersect at K. s_P and s_Q intersect at L. Then K,L are symmetric through midpoint T of HS.

Proof. Let K',L' be images of K,L thourgh the dilation center H ratio 2. Because S is image of T through the dilation center

H ratio 2, so we need to prove S is midpoint of $K'L'$ then we are done. Indeed, from lemma 2 we see the dilation center H ratio 2 maps s_M, s_N to $K'M, K'N$, resp. From this $(K'M, K'N) = (s_M, s_N) = \frac{1}{2}(\overrightarrow{ON}, \overrightarrow{OM})(\bmod \pi)$.

Let R_1 be reflection of K' through MN then $(R_1M, R_1N) = -(K'M, K'N) = \frac{1}{2}(\overrightarrow{OM}, \overrightarrow{ON})(\bmod \pi)$ thus R_1 lies on (O).

Similarly R_2 is reflection of L' through PQ then R_2 lies on (O).

We easily seen R_1N, R_2P are reflection of NK', PL' through SN, SP, reps. S lie on circle diameter NP from lemma 3 they intersect on (O). Thus $R_1 \equiv R_2 \equiv R \in (O)$.

Because MN, PQ are perpendicular $L' = \mathcal{R}_{PQ}(R) = \mathcal{R}_{PQ}[\mathcal{R}_{MN}(K')] = \mathcal{R}_{PQ} \circ \mathcal{R}_{MN}(K') = \mathcal{S}_S(K')$ thus K' and L' are symmetry through S.

Proof of problem. Let PQ is diameter perpendicular to MN then P, Q are fixed points. PQ cuts MN at U. H is orthocenter of ABC. s_P, s_Q intersect at S then S is fixed point. s_M, s_N intersect at T. We will prove that T lie on a fixed line. Indeed, from lemma 4 we have S, T are symmetric through midpoint V of HU. H is fixed and U moves on PQ so V move on image of PQ through dilation center H ratio $\frac{1}{2}$. Because T is image of V throgh dilation center S ratio 2. From this, T lie on fixed line is image of PQ through product of two dilations $\mathcal{D}_S^2 \circ \mathcal{D}_H^{\frac{1}{2}}$. We are done.

TelvCohl

#4 Jan 19, 2015, 12:52 pm • 1

Remark:

This fixed line is the Simson line of $\triangle ABC$ with direction $\perp MN$. In general, the locus of the orthopole of a pencil with center P is an ellipse $\mathcal{E}_P$ (orthopolar ellipse), it passes through the projection of P on BC, CA, AB, resp. and the center of $\mathcal{E}_P$ is the midpoint of PH where H is the orthocenter of $\triangle ABC$.

Moreover, the length of major axis and minor axis of $\mathcal{E}_P$ is $\mathrm{R} + OP, \mathrm{R} - OP$, respectively ($O$ is the circumcenter of $\triangle ABC$ and R is the radius of $\odot(O)$) and the axis of $\mathcal{E}_P$ is parallel to the asymptote of the isogonal conjugate of OP WRT $\triangle ABC$.

This post has been edited 1 time. Last edited by TelvCohl, Jan 16, 2016, 7:43 pm

Quick Reply

High School Olympiads

Perpendicular line

Reply

Source: Vietnam IMO team training 2014 (Own)

buratinogigle

#1 Jan 17, 2015, 2:39 pm • 1

Let ABC be a triangle inscribed circle (O), orthocenter H. E, F lie on (O) such that $EF \parallel BC$. D is midpoint of HE. The line passing though O and parallel to AF cuts AB at G. Prove that $DG \perp DC$.

TelvCohl

#2 Jan 17, 2015, 4:54 pm • 3

My solution:

Let $\mathcal{C}$ be a circumconic of $\triangle ABC$ passing through E and H .

Since E is the isogonal conjugate of the infinity point on OG WRT $\triangle ABC$,
so $\mathcal{C}$ is the isogonal conjugate of OG WRT $\triangle ABC$ which is a rectangle hyperbola ,
hence we get E is the forth intersection of $\odot(ABC)$ and $\mathcal{C} \Longrightarrow D$ is the center of $\mathcal{C}$.
Since the pedal circle of G WRT $\triangle ABC$ pass through the center D of $\mathcal{C}$ (well-known) ,
so we get $\angle CDG = 90^\circ$.

Q.E.D

buratinogigle

#3 Jan 17, 2015, 5:24 pm • 1

Very nice and short proof dear Telv Cohl. Here is my solution

Let M be midpoint of BC and AP is diameter of (O) thus M is midpoint of HP. CG cuts (O) again at Q. T is reflection of A through OG then T lies on (O). $AT \perp OG \parallel AF$ so FT is diameter of (O) hence $AT = PF$. We see $\angle PEF = \angle BMD$ by parallel lines. Therefore $\angle AQT = 180^\circ - \angle PEF = 180^\circ - \angle BMD = \angle DMC \quad (1)$.

S is projection of G on CT. Note that, $AF = PT$, we have

$\angle EAP = \angle EAC - \angle PAC = \angle BAF - \angle PAC = \angle BCF - \angle PAC = \angle ACF - \angle ACB - \angle PAC = \angle TAP - \angle PAC - \angle ACB = \angle TAC - (\angle GAT - \angle TCA) = 180^\circ - \angle GTA - \angle ATC = \angle GTS.$

From this $\triangle GTS \sim \triangle PAE$ deduce $\frac{GT}{AP} = \frac{GS}{PE} \quad (2)$.

Easily seen $\angle GCS = \angle QTF$ thus $\triangle TQF \sim \triangle GSC$ deduce $\frac{QT}{GS} = \frac{TF}{CG} \quad (3)$.

From (2),(3) deduce $\frac{QT}{2DM} = \frac{QT}{PE} = \frac{TF}{CG} \cdot \frac{GT}{AP} = \frac{GT}{GC} = \frac{GA}{GC} = \frac{AQ}{BC} = \frac{AQ}{2BM}$ suy ra $\frac{QT}{DM} = \frac{AQ}{BM} \quad (4)$.

From (1),(4) deduce $\triangle DMC \sim \triangle TQA$ thus $\angle MDC = \angle QTA = \angle QCA$. Hence if N is midpoint of AB then

$\angle NMD = \angle NMB + \angle BMD = \angle ACB + \angle MCD + \angle MDC = \angle ACD + \angle QTA = \angle ACD + \angle QAC = \angle QCD.$

Let CK be altitude, easily seen $KDMN$ is cyclic deduce $\angle NKD = 180^\circ - \angle MND = 180^\circ - \angle QCD$. Thus, $GKDC$ is cyclic $\angle GDC = \angle GKC = 90^\circ$. We are done.

Attachments:
Figure2380.pdf (13kb)

PROF65
#4 Jan 17, 2015, 9:17 pm • 1

Just a a little remark F must be chosen s.t. AF isn't parallel to the AB-bisector else $D=G$

TelvCohl
#5 Jan 17, 2015, 11:42 pm • 1

I found a solution without using conic

My solution:

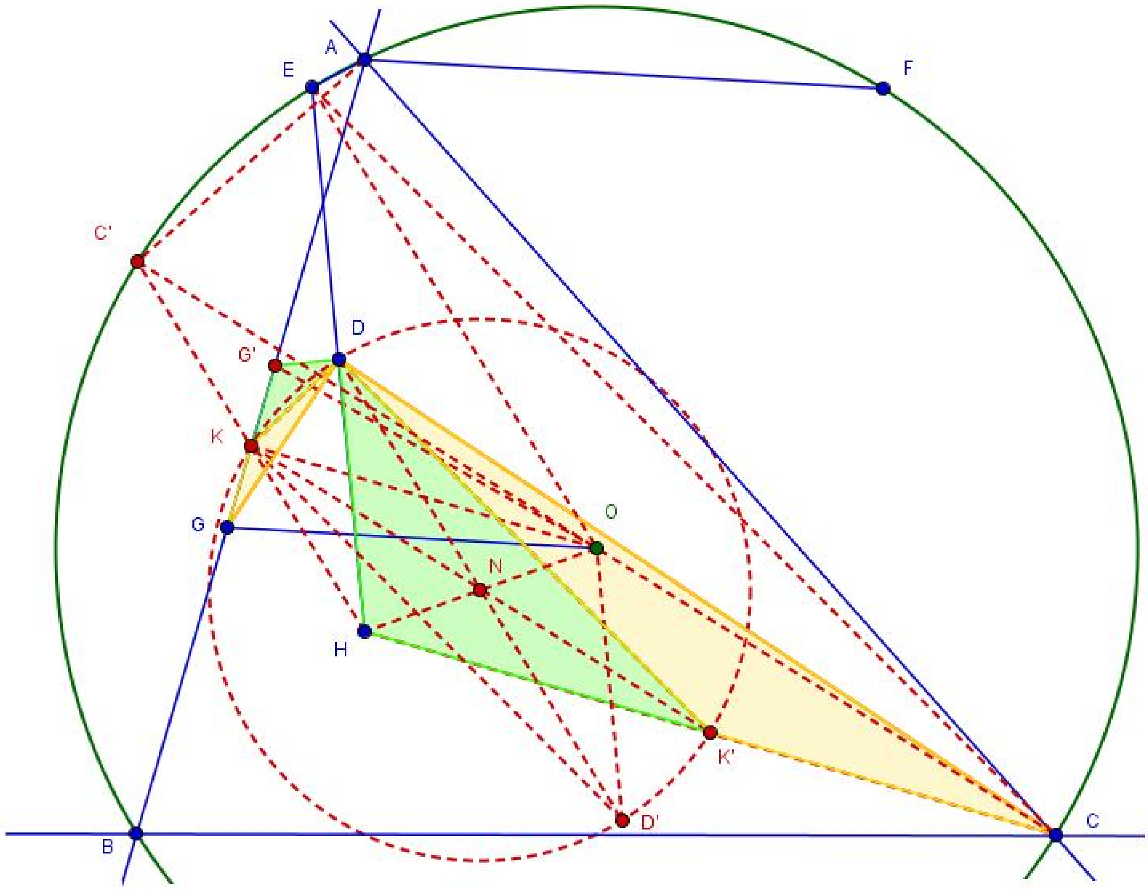
Let N be the nine point center of $\triangle ABC$.
Let K be the midpoint of AB and K' be the reflection of K in N .
Let C', D', G' be the reflection of C, D, G in O, N, K, respectively .

Easy to see AE, AF are isogonal conjugate of $\angle BAC$.

Since $C'A \perp CA$,
so $\angle G'OK = \angle KOG = \angle EAC' = \angle ECC' = \angle DK'K$,
hence we get $Rt\triangle KOG' \sim Rt\triangle KK'D \Longrightarrow \triangle KG'D \sim \triangle KOD' \sim \triangle K'HD$. $(\star)$
Since G, C is the reflection of G', H in K, K', respectively ,
so combine with $(\star)$ we get $\triangle DKG \sim \triangle DK'C \Longrightarrow \angle GDC = \angle KDK' = 90^\circ$.

Q.E.D

Attachments:



Luis González
#6 Jan 19, 2015, 11:22 am • 2

Let X, Y, Z be the projections of A, B, C on BC, CA, AB and let U be the midpoint of $AH. GO$ cuts AC at Q and DU cuts AB, AC at R, S. Clearly D is on 9-point circle $\odot(XYZ)$ of $\triangle ABC$.

Since $\angle AGO = \angle BAF = \angle CAE = \angle ASR \Longrightarrow RQSG$ is cyclic $\Longrightarrow \angle OQC = \angle URZ.$ Thus since $\triangle ABC \sim \triangle AYZ$ with corresponding circumcenters $O, U \Longrightarrow AQ : AC = AR : AZ \Longrightarrow RQ \parallel CZ.$ Hence $\angle ACZ = \angle AQR = \angle AGS$ and $\angle ZDU = \angle ZXA = \angle ZCA \Longrightarrow SCDZG$ is cyclic $\Longrightarrow \angle CDG = \angle CZG = 90^\circ.$

Quick Reply

High School Olympiads

Perpendiculars and Cyclic Quad

Reply

Source: SMO(O) 2014 #1

Konigsberg

#1 Jan 18, 2015, 10:43 am

The quadrilateral ABCD is inscribed in a circle which has diameter BD. Points A' and B' are symmetric to A and B with respect to the line BD and AC respectively. If the lines A'C, BD intersect at P and AC, B'D intersect at Q, prove that PQ is perpendicular to AC.

TelvCohl

#2 Jan 18, 2015, 11:12 am • 1

My solution:

Let $X = AC \cap BD$.
Let O be the center of $\odot(ABCD)$.

Since $\angle XOA' = \angle ACA' = \angle XCA'$,
so we get A', C, O, X are concyclic and $PB \cdot PD = PX \cdot PO$,
hence from $BO = OD \Longrightarrow (P, X; B, D) = -1 \Longrightarrow \frac{PB}{PD} = \frac{XB}{XD}$ . ... (1)
Since XQ is the external bisector of $\angle DXB'$,
so we get $\frac{QB'}{QD} = \frac{XB'}{XD} = \frac{XB}{XD}$ . ... (2)

From $(1), (2) \Longrightarrow PQ \parallel BB' \Longrightarrow PQ \perp AC$.

Q.E.D

Luis González

#3 Jan 19, 2015, 8:43 am

Let $E \equiv AC \cap BD$. By obvious symmetry the tangents of $\odot(ABCD)$ at A and A' intersect on $BD \Longrightarrow ABA'D$ is harmonic $\Longrightarrow$ pencil $C(A, A', B, D) = -1$ is harmonic $\Longrightarrow (E, P, B, D) = -1$. But since QE bisects $\angle BQB' \equiv \angle BQD \Longrightarrow QE, QP$ bisect $\angle BQD \Longrightarrow PQ \perp AC.$

Quick Reply

High School Olympiads

Isosceles triangle

Locked

Source: SMO(O) 2007 #3

Konigsberg

#1 Jan 18, 2015, 10:22 pm

Let A_1, B_1 be two points on the base AB of an isosceles triangle ABC, with $\angle C > 60^\circ$, such that $\angle A_1CB_1 = \angle ABC$. A circle externally tangent to the circumcircle of $\triangle A_1B_1C$ is tangent to the rays CA and CB at points A_2 and B_2, respectively. Prove that $A_2B_2 = 2AB$.

Luis González

#2 Jan 19, 2015, 5:33 am

Posted many times before as SAMO 2004 (Q4), so for further discussions use any of the links below. Thread locked.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=23118
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=203188
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=481364
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=484873

High School Olympiads

surface ineq

Reply

Source: nice !!!

saif

#1 Sep 11, 2009, 11:29 pm

Consider triangle ABC, and three squares $BCDE$, $CAFG$ and $ABHI$ constructed on its sides, outside the triangle. Let XYZ the triangle en-closed by the lines DE, HI and FG
prove that $S_{XYZ} \geq (13+4\sqrt{3})S_{ABC}$

Luis González

#2 Jan 17, 2015, 9:29 am

Denote $X \equiv GF \cap IH, Y \equiv IH \cap ED, Z \equiv ED \cap GF.$ It's well known that AX, BY, CZ are the symmedians of $\triangle ABC \Longrightarrow \triangle ABC$ and $\triangle XYZ$ are homothetic with homothetic center the symmedian point K of $\triangle ABC.$ Hence the homothety coefficent k equals to the ratio of the distances from K to YZ and $BC.$ Thus if d_a denotes the distance from K to $BC,$ we have

$$k=\frac{d_a+a}{d_a}=\frac{a \cdot d_a+a^2}{a \cdot d_a} \Longrightarrow k=1+\frac{a^2}{2 \cdot [KBC]}.$$

But $\dfrac{[KBC]}{[ABC]}=\dfrac{a^2}{a^2+b^2+c^2} \Longrightarrow k=1+\dfrac{a^2+b^2+c^2}{2 \cdot [ABC]} \Longrightarrow$

$$\frac{[XYZ]}{[ABC]}=k^2=(1+\frac{a^2+b^2+c^2}{2 \cdot [ABC]})^2.$$

Combined with Weitzenbock's inequality $a^2+b^2+c^2 \geq 4\sqrt{3}[ABC],$ we get $\dfrac{[XYZ]}{[ABC]} \geq (\dfrac{4\sqrt{3}}{2}+1)^2=13+4\sqrt{3}.$

Quick Reply

High School Olympiads

Small problem about Feuerbach point

Reply

Source: Own

Luis González

#1 Jan 17, 2015, 9:07 am • 2

$\triangle ABC$ is non-equilateral and let D,E,F be the midpoints of BC,CA,AB. $Fe \in \odot(DEF)$ is the Feuerbach point of $\triangle ABC$ and $\triangle XYZ$ is the tangential triangle of $\triangle DEF$. Show that the sum of the distances from Fe to the sides of $\triangle XYZ$ is twice the sum of the signed distances from Fe to the sides of $\triangle DEF$.

P.S. The signed distance from a point X to a side of $\triangle DEF$, say EF, is positive if X and D are on a same side of the line EF, otherwise negative.

Quick Reply

High School Olympiads

Concurrent lines with angle bisector

Reply

Source: Own

buratinogigle

#1 Jan 16, 2015, 3:34 pm

Let ABC be a triangle and bisector AD, BE, CF. Perpendicular bisector of AD, BE, CF cut EF, FD, DE at X, Y, Z, resp. Prove that AX, BY, CZ are concurrent.

Luis González

#2 Jan 16, 2015, 11:30 pm • 1

This configuration was discussed before. The said lines concur at X_{100}, i.e. the Feuerbach point of the antimedial triangle of $\triangle ABC.$ Letting U, V be the reflections of D on $BE, CF,$ then it's known that UV, EF and the perpendicular bisector of AD concur at $X.$ Now see the topic three concurrent lines (straightforward proof by RSM).

See also Two perspective triangles (own) for a short proof with barycentric coordinates.

Quick Reply

High School Olympiads

Geometry Problem

Reply

Jul

#1 Jan 15, 2015, 4:22 pm

Let ABC be a triangle and I, J similary be incenter and A excenter. (I) touchs BC at D. The line thourgh pass I and parallel to BC cuts AB, AC at E, F similary. Let X, Y, Z be the intersection points of DF and AB, DE and AC, AD and XY similary.
a) Prove that ZJ parallel to BC.
b) Prove that $ZB = ZC$.

sasanineq

#2 Jan 15, 2015, 5:55 pm

my proof is great jul,but you can use sin! theorem!

TelvCohl

#3 Jan 15, 2015, 9:19 pm

My solution:

Let M be the midpoint of arc BC in $\odot(ABC)$.
Let ℓ be a line passing through A and parallel to BC .
Let $D' = AI \cap BC, E' = DE \cap \ell, F' = DF \cap \ell$.

Redefine Z as the projection of J on the perpendicular bisector of BC .

From $\triangle IDD'$ and $\triangle MZJ$ are homothetic $\Longrightarrow Z \in AD$,

so it's suffices to prove $Z \in XY \Longleftrightarrow \frac{EA}{XE} + \frac{FA}{YF} = \frac{AD}{ZD}$ (Van Aubel theorem) .

Easy to see $\frac{AD}{ZD} = \frac{AD'}{JD'}$ . ... (1)

From $(A, D'; I, J) = -1$ we get $\frac{ID'}{JD'} = \frac{AI}{AJ} = \frac{AI - ID'}{AD'}$ . ... (2)

Since $\frac{EA}{XE} + \frac{FA}{YF} = \frac{AF' - EF}{EF} + \frac{AE' - EF}{EF} = \frac{E'F' - 2EF}{EF} = \frac{AI - ID'}{ID'}$,

so combine with (1) and (2) we get $\frac{EA}{XE} + \frac{FA}{YF} = \frac{AI - ID'}{ID'} = \frac{AD'}{JD'} = \frac{AD}{ZD}$.

Q.E.D

Luis González

#4 Jan 15, 2015, 10:24 pm • 1

A-excircle (J) touches BC at U and let V be its antipode WRT (J). Since A is the exsimilicenter of $(I) \sim (J)$, then A, D, V are collinear. Since the midpoint M of BC is also midpoint of DU, then the perpendicular bisector of BC is D-midline of $\triangle DUV$, cutting DV at its midpoint $Z^* \Longrightarrow JZ^* \parallel DU \equiv BC$. Thus if $K \equiv AJ \cap BC$ and $L \equiv AD \cap EF$, from $IL \parallel KD \parallel JZ^*$, we get $(A, D, L, Z^*) = (A, K, I, J) = -1$. But from the complete quadrilateral $EFYX$, we have $(A, D, L, Z) = -1 \Longrightarrow Z \equiv Z^* \Longrightarrow ZB = ZC$ and $JZ \parallel BC$.

Quick Reply

High School Olympiads

double angle, parallel

Reply

ricarlos

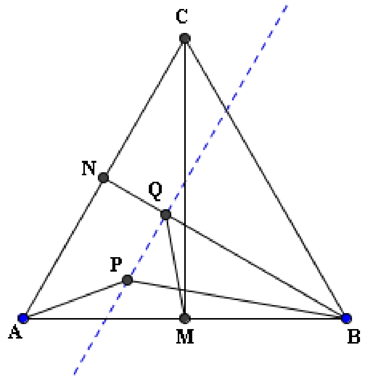
#1 Jan 14, 2015, 4:31 am

Let ABC be an equilateral triangle and M,N midpoints of AB and AC, respectively.
Let P be a point in the same semiplane that C w.r.t. the line AB and Q a point on line BN so that

$$\angle PBA = \angle QMC = \frac{\angle PAB}{2}$$

Prove that $PQ \parallel AC$.

Attachments:



Luis González

#2 Jan 15, 2015, 8:32 am • 1

Let $S \equiv PB \cap CM$ and let X be the projection of P on CM. $\angle SAB = \angle PBA = \frac{1}{2}\angle PAB \Longrightarrow AS$ bisects $\angle PAB$ $\Longrightarrow \frac{PA}{AB} = \frac{SP}{SB} = \frac{PX}{BM} \Longrightarrow \frac{PA}{PX} = 2 \Longrightarrow P$ is on hyperbola $\mathcal{H}$ with focus A, directrix CM and whose eccentricity equals 2 $\Longrightarrow$ it has vertex $D \in AM$, such that $DA = 2 \cdot DM$ and center the reflection of D on $M \Longrightarrow B$ is the other vertex. When S coincides with the reflection of C on M, then P coincides with the point at infinity B_∞ of AC, i.e. $\mathcal{H}$ has an asymptote parallel to AC.

If we redefine Q as the intersection of the perpendicular from M to AS with the parallel from P to AC, then we need to show that $Q \in BN$. Since the pencils $MQ \mapsto AS \mapsto BP \mapsto B_\infty P$ are projective, then it suffices to prove that $Q \in BN$ holds for 3 positions of S and this is trivial when S coincides with M, the center of $\triangle ABC$ and C.

Quick Reply

High School Olympiads

Two parallels

Reply

Source: own?

jayme

#1 Jan 14, 2015, 5:12 pm

Dear Mathlinkers,

1. ABC a triangle
2. I the incenter
3. A', A" the vertex of the orthic, Nagel triangle on BC
4. A* the second point of intersection of AI with the circumcircle of ABC.

Prove : A'I is parallel to A"A*.

Sincerely
Jean-Louis

TelvCohl

#2 Jan 14, 2015, 5:48 pm

My solution:

Let M be the midpoint of BC .
Let P, Q be the tangent point of $\odot(I)$ with BC, AB, respectively .

Since $Rt\triangle AIQ \sim Rt\triangle BA^*M$,

so $\frac{A'P}{A''M} = \frac{A'P}{PM} = \frac{AI}{IA^*} = \frac{AI}{BA^*} = \frac{IQ}{A^*M} = \frac{IP}{A^*M}$

$\Longrightarrow Rt\triangle IPA' \sim Rt\triangle A^*MA'' \Longrightarrow \angle A''A'I = \angle A'A''A^* \Longrightarrow A'I \parallel A''A^*$.

Q.E.D

jayme

#3 Jan 14, 2015, 6:31 pm

Dear Mathlinkers,
thank for your proof...
This is just an application of a generalization of the Reim's theorem (obtained with calculation also) that you can see on

http://jl.ayme.pagesperso-orange.fr/Docs/Le%20cercle%20de%20Hagge.pdf p. 52

Sincerely
Jean-Louis

Luis González

#4 Jan 15, 2015, 12:44 am

Let AI cut BC at J and let E be the A-excenter of $\triangle ABC$ (reflection of I on A^*). Since $AA' \parallel EA''$, then it suffices to show that $\frac{AI}{EA^*} = \frac{AA'}{EA''}$.

$(A, J, I, E) = -1 \Longrightarrow \frac{JA}{JE} = 2 \cdot \frac{IA}{IE} = \frac{IA}{EA^*} \Longrightarrow \frac{AI}{EA^*} = \frac{JA}{JE} = \frac{AA'}{EA''}$.

Quick Reply

High School Olympiads

Fixed point(s) for a family of circles

Source: unknown

mavropnevma

#1 Jan 14, 2015, 2:56 am • 1

I have been asked to post this question.

Let M be any point inside the acute-angled triangle ABC, such that $\angle ABC + \angle AMC = \pi$. Let AM meet BC at E and let CM meet AB at F.
Prove that the circumcircle of $\triangle BEF$ passes through a second fixed point (other than B).

Further question: if so, is that other fixed point the foot of the perpendicular from H onto the line OB? where H, respectively O, are the orthocentre, respectively circumcentre, of $\triangle ABC$.

TelvCohl

#2 Jan 14, 2015, 5:31 am

My solution:

Let $X = \odot(CME) \cap \odot(AMF)$.

From $\angle FBE + \angle EMF = \pi \Longrightarrow M \in \odot(BEF)$,
so $\angle CXA = \angle CXM + \angle MXA = \angle BEM + \angle MFB = \pi \Longrightarrow X \in AC$.

Let Y be the second intersection of $\odot(EFX)$ and AC.

Since $\angle FEY = \angle FMA = \pi - \angle ABC = \angle CME = \angle YFE$,
so all $\triangle YEF$ are similar when M varies on $\odot(AHC) \Longrightarrow \odot(BEF)$ pass through a fixed point.

Consider the case $M \equiv A$ and the case $M \equiv C$
$\Longrightarrow$ the fixed point is the projection of H onto the B-median of $\triangle ABC$.

Q.E.D

Luis González

#3 Jan 14, 2015, 6:00 am

Posted before. It's Turkey 2010 TST second day Q2 and also ELMO Shortlist 2013 (G3).

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=341591
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=545085

ibp

#4 Jan 14, 2015, 6:11 pm

Please explain this part:

> *TelvCohl wrote:*
> so all $\triangle YEF$ are similar when M varies on $\odot(AHC) \Longrightarrow \odot(BEF)$ pass through a fixed point.

High School Olympiads

Area is Constant

Source: Own

TelvCohl

#1 Jan 11, 2015, 12:51 pm • 2

Given a fixed point P and a circle Ω .
Let Q be a point varies on Ω and R is a point satisfy all $\triangle PQR$ are directly similar .

Prove that there exist a point S such that $[QRS] = \text{const}$.

61plus

#2 Jan 11, 2015, 10:42 pm • 2

Cool qn!!

We first find the point S. Let centre of circle be O.
First we draw these few points: the reference point Q defining R, extending PQ to meet circle again at q defining r (s.t. $\triangle PQR \sim \triangle Pqr$), reflecting Q across PO to Q' defining R', and extending PQ' to meet circle again at q' defining r'. Now reflect P across perpendicular bisector of Qq,$Q'q'$,Rr,$R'r'$ to X, X', Y, Y'.

Clearly, PRr,$PR'r'$ are straight lines, and $QR||qr$,$Q'R'||q'r'$. It is easy to check that O is circumcentre of PXX', and $\triangle PXY$ $\triangle PX'Y'$ $\triangle PQR$, hence S', intersection of XY and $X'Y'$, lies on circle with radius OP and centre O, and also arc PS' has fixed length since $\angle PXY = \angle PQR$ is fixed. Thus, S' is fixed no matter where we shift the reference point Q.
Now we show that it has properties of S.

$S_{\triangle QRS'} = \frac{1}{2} \cdot QR \cdot d(S', QR) = \frac{1}{2} \cdot QR \cdot d(X, QR) = \frac{1}{2} \cdot QR \cdot d(P, qr) = \frac{Pq}{PQ} \cdot S_{\triangle PQR}$. We fix a reference $\triangle PAB$ similar to $\triangle PQR$ and satisfying its properties with area U. Thus $S_{\triangle QRS'} = \frac{Pq}{PQ} \cdot S_{\triangle PAB} \cdot \frac{PQ^2}{PA^2} = \frac{PQ \cdot Pq}{PA^2} \cdot S_{\triangle PAB}$ which is fixed from power of point, hence S' is the desired S.

Luis González

#3 Jan 13, 2015, 3:40 am • 1

R moves on circle $\Omega' \equiv (O')$ image of $\Omega \equiv (O)$ under spiral similarity with center P, rotational angle $\angle QPR$ and coefficient $\tfrac{PR}{PQ}$. QR cuts $(O), (O')$ again at U, V, resp and $M \equiv OQ \cap O'R, N \equiv OU \cap O'V$.

Since P is center of the spiral similarity that takes OQ into $O'R$, it follows that $P \in \odot(MOO')$. Since $\angle NUV = \angle MQU$ and $\angle NVU = \angle MRV$, then $\angle QMR = \angle UNV \Longrightarrow \odot(ONO')$ is reflection of $\odot(OMM')$ on $OO' \Longrightarrow \odot(NOO')$ passes through the reflection S on P on OO' and since $\tfrac{SO}{SO'} = \tfrac{PO}{PO'} = \tfrac{OU}{O'V}$, then S is center of spiral similarity that takes OU into $O'V \Longrightarrow$

$$\frac{\text{dist}(S, UV)}{\text{dist}(S, OO')} = \frac{UV}{OO'} \Longrightarrow \frac{2 \cdot [SQR]}{\text{dist}(S, OO')} = \frac{UV \cdot QR}{OO'} \ (\star).$$

If U' is the reflection of U on OO' and PQ cuts (O) again at X, we have $\angle UU'X = \angle UQP = \angle POO' \Longrightarrow PO \perp XU'$ $\Longrightarrow U'$ is reflection of X on PO

$$\Longrightarrow \frac{UV}{OO'} = \frac{SU}{SO} = \frac{PU'}{PO} = \frac{PX}{PO}, \text{but since } \frac{QR}{OO'} = \frac{PQ}{PO} \Longrightarrow$$

$$\frac{UV \cdot QR}{OO'^2} = \frac{PQ \cdot PX}{PO^2} \Longrightarrow UV \cdot QR = \text{const}.$$

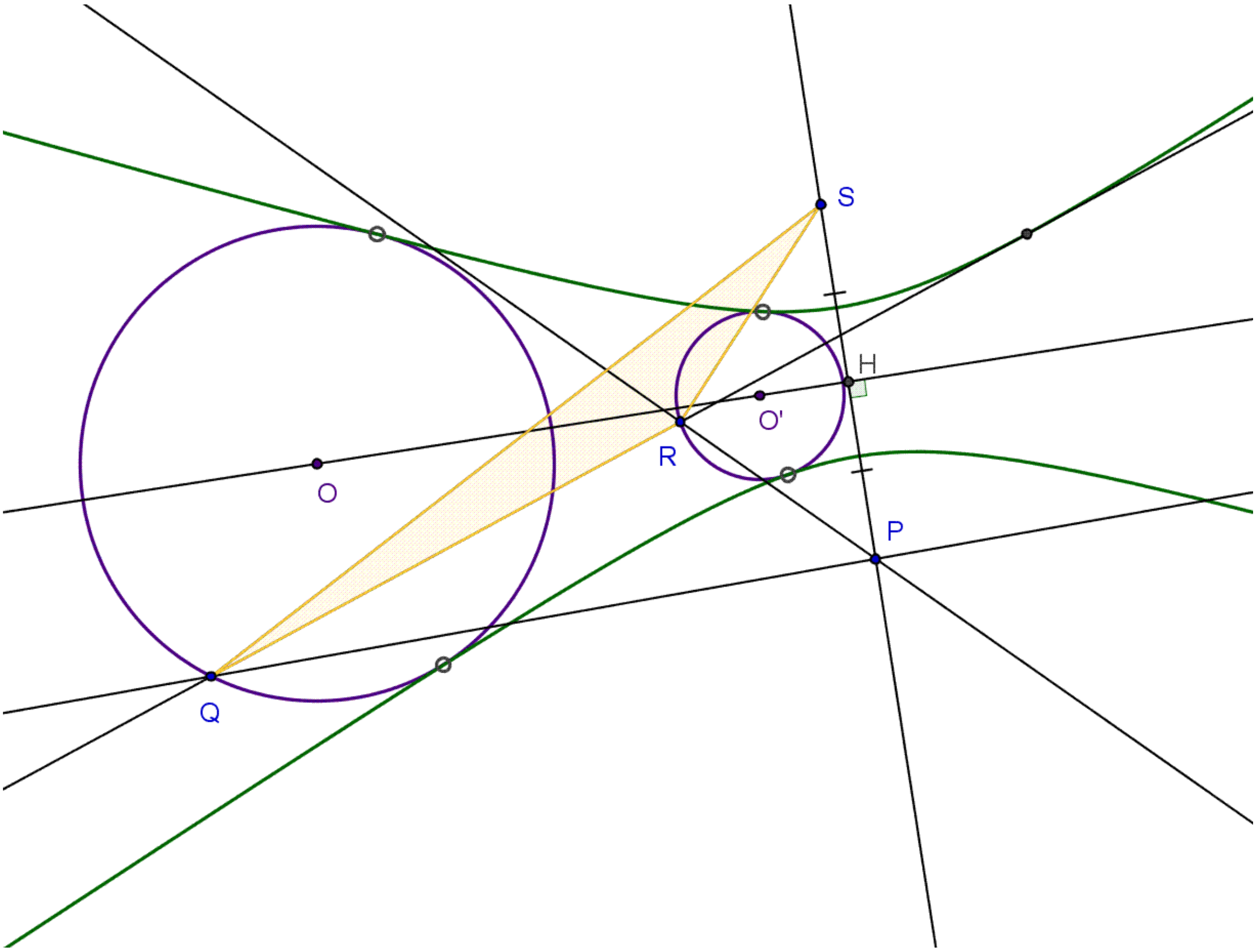
Combined with $(\star)$, it follows that $[SQR]$ is constant, as desired.

Luis González

#4 Jan 13, 2015, 4:50 am • 1

A nice remark from this configuration: If P is outside of $(O) \equiv \Omega$, then the envelope of the lines QR is a hyperbola $\mathcal{H}$ with foci P, S and bitangent to Ω, Ω'. I have a synthetic approach.

Attachments:



TelvCohl

#5 Jan 13, 2015, 6:37 pm • 1

Thanks **61plus** and **Luis** for your interest.

Here is my solution for original problem and the remark by **Luis** :

(Original problem)

Let O be the center of Ω and ℓ be a line through O satisfy $\angle(\ell, OP) = \angle RQP$.
Let S be the reflection of P in ℓ and V be the second intersection of PQ and $\odot(O, OP)$.

Let's prove $[QRS] = \text{const}$ when Q varies on Ω .

Since $\angle SVP = \frac{1}{2}\angle SOP = \angle RQP \Longrightarrow PQ \parallel SV$,

so $[QRS] = [QRV] = \frac{1}{2} \cdot QV \cdot QR \cdot \sin \angle RQP = \frac{1}{2} \cdot QV \cdot QP \cdot \frac{QR}{QP} \cdot \sin \angle RQP$ . ... $(\star)$

Since $QV \cdot QP$ is the power of Q WRT $\odot(O, OP)$ which is fixed when Q movess on Ω ,
so combine with $(\star)$ we get $[QRS]$ is constant when Q varies on Ω .

(Remark by Luis)

Lemma :

Let $\mathcal{H}$ be a hyperbola with focus $\{F_1, F_2\}$.
Let O_1 be the center of $\mathcal{H}$ and O_2 be a point on the perpendicular bisector of F_1F_2 .
Let $\odot(O_1), \odot(O_2)$ be a circle with center O_1, O_2 and tangent to $\mathcal{H}$, respectively .

Then $R_{\odot(O_1)} : R_{\odot(O_2)} = F_1O_1 : F_1O_2$ ($R_\odot$ is the radius of $\odot$) .

Proof of the lemma :

Let T be one of the tangent points of $\odot(O_2)$ and $\mathcal{H}$.

Since TO_2 is the external bisector of $\angle F_1TF_2$,
so O_2 is the midpoint of arc F_1TF_2 in $\odot(TF_1F_2)$.
From Ptolemy theorem $\Longrightarrow R_{\odot(O_2)} \cdot F_1F_2 = |TF_2 - TF_1| \cdot F_1O_2 \Longrightarrow \dfrac{F_1O_2}{F_1O_1} = \dfrac{R_{\odot(O_2)}}{R_{\odot(O_1)}}$

Back to the main problem :

Let P', S' be the reflection of P, S in QR, respectively .
Let X, Y be the midpoint of PS, PP', respectively .

Since $Rt\triangle POX \sim Rt\triangle PQY$ $(\star)$,
so Y moves on a circle $\odot(X)$ with center X when Q varies on Ω ,
hence the locus of P' is a circle $\odot(S)$ with center S when Q varies on Ω ,
so from symmetry we get the locus of S' is a circle $\odot(P)$ with center P .
(The radius of $\odot(P), \odot(S)$ is equal to the length of diameter of $\odot(X)$)

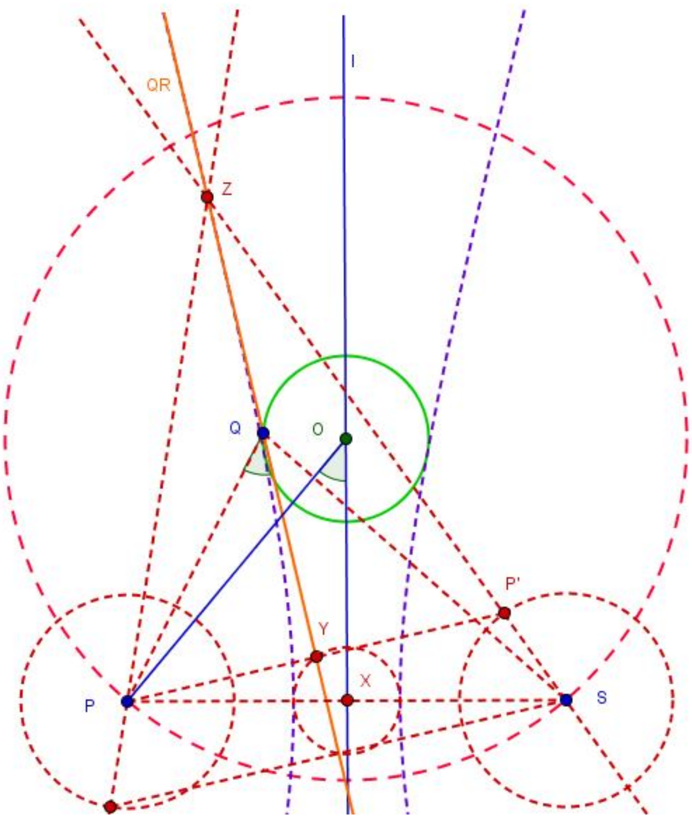
Since $|SZ - PZ| = PS' = \text{const}$ $(Z = PS' \cap QR)$,
so the envelope of QR is a hyperbola $\mathcal{H}$ with focus $\{P, S\}$.

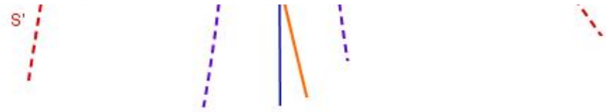
Since the length of the diameter of $\odot(X)$ is equal to PS' ,
so we get $\odot(X)$ is tangent to $\mathcal{H}$.
From $(\star)$ we get $R_\Omega : R_{\odot(X)} = PO : PX$,
so combine with the lemma we get Ω is tangent to $\mathcal{H}$.
(Notice there is only one circle with center O and tangent to $\mathcal{H}$)

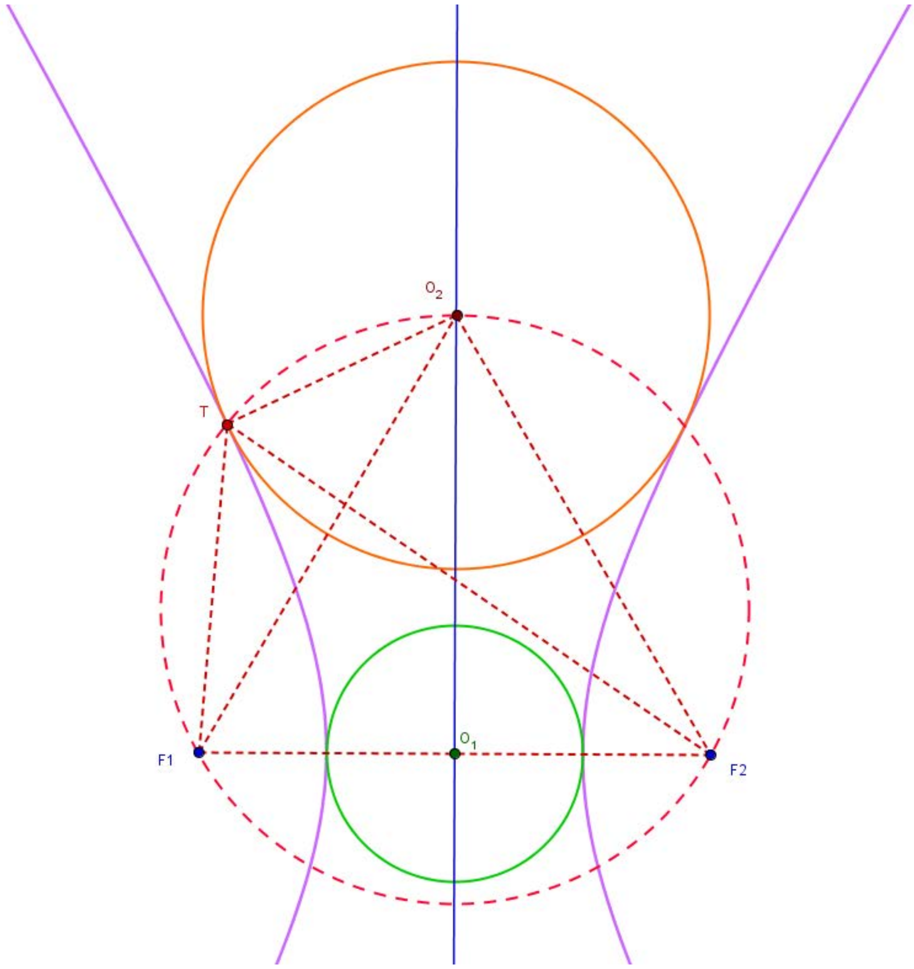
Similarly, we can prove Ω' is tangent to $\mathcal{H}$.

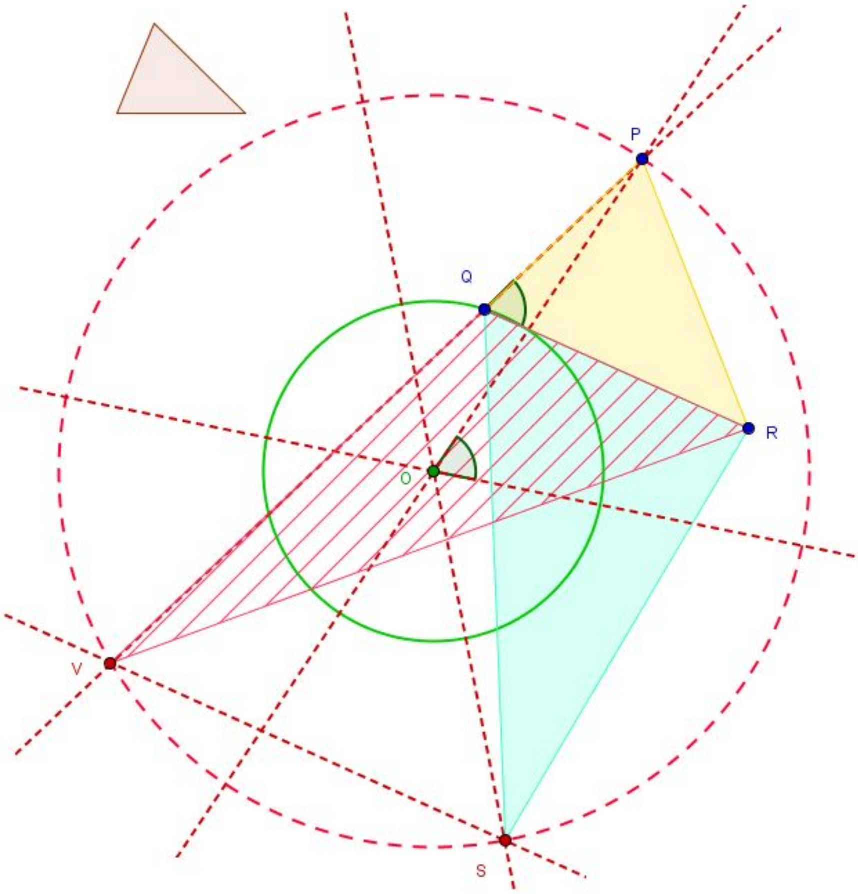
Q.E.D

Attachments:









This post has been edited 1 time. Last edited by TelvCohl, Aug 7, 2015, 1:27 pm

PROF65

#6 Jan 13, 2015, 10:32 pm • 1

Let us,first ,prove two lemmas:
lemma 1:
BC and DE two segments parallel identically directed. The locus of point X s.t. XBC and XED have the same area is the two parallel lines to BC that pass resp. through S, T the symmetrics of their ex and in-similicenters S, T in the mid-line of BC and ED.
proof:

S,T are the homothecy centers then $\frac{SH_1}{SH_2} = \frac{BC}{DE}$ where SH_1, SH_2 are the altitudes of SBC, SDE thus
$SH_1.DE = SH_2.BC \implies S'H_2.DE = S'H_1.BC \implies S'BC$ and $S'DE$ have the same area
identically for T'
lemma 2:
let $(O'), l'$ the images of $(O), l$ by the similitude of center K, l is line that pass through K and intersect (O) at M, N; l' intersect (O') at M', N' ;I, I' midpoints of MN, $M'N'$; $OO' \cap II' = H$ then
the K-altitude of KOO', OO' and II' are concurrent
proof:

$\widehat{KIH} = \widehat{KOH}$ then $KIOH$ cylic then $\widehat{KHO} = \widehat{KIO} = \frac{\pi}{2}$
back to the problem
let PQ, PR intersect Ω, Ω' (the image of Ω by s the similarity that maps $P \to P, Q \to R$) at Q', R' then $PQR, PQ'R'$ are homothetic thus by applying lemma 1 ,the line d_Q parallel to QR passing through $\overline{P}$, the symmetric of P in the midline II' of $QR, Q'R'$ is the locus of isoarea points wrt QR and $Q'R'$
we'll prove that P' the symmetric of P in OO' lie on d_Q indeed the lemma 2 assert that H the midpoint of PP' is also on II' then $P'\overline{P} \parallel II'$
it remain to prove that the area is fixed:
$\mathbb{S} = S_{P'QR} = S_{P'Q'R'} \implies PH_2 \cdot QR = PH_1 \cdot Q'R' \implies \mathbb{S}^2 = PH_2 \cdot QR \cdot PH_1 \cdot Q'R' = S_{PQR} \cdot S_{PQ'R'} = PQ \cdot PR \cdot \sin\alpha \cdot PQ' \cdot PR' \cdot \sin\alpha$
where α is the angle of the similarity s thus $\mathbb{S}^2 = P_\Omega \cdot P_{\Omega'} \cdot \sin\alpha$ which is independent from Q ; P_Ω is the power of P wrt Ω.

Luis González

#7 Jan 14, 2015, 12:00 am • 1

Thanks Telv for your proof to the remark. This is how I got it:

We work with the same notations of my previous post. $Q \mapsto U$ is a homography on Ω, as it is an involution with pole P followed by rotation $(O, 2\angle PQR)$ counterclockwise. Thus, $QU \equiv QR$ envelopes a conic $\mathcal{H}$ bitangent to Ω through the fixed points of the homography and they do exist; they are the intersections of Ω with the circle that sees $\overline{PO}$ under $90^\circ - \angle PQR$ (mod 180º). In addition, $\mathcal{H}$ is hyperbola as QR clearly passes through O twice. By similar reasoning $\mathcal{H}$ is bitangent to Ω'.

Let H, T be the projections P on OO', QR. Then $\triangle POQ \sim \triangle PHT \Longrightarrow \frac{HT}{OQ} = \frac{PH}{PO} \Longrightarrow HT$ is constant $\Longrightarrow$ $\odot(H, HT)$ is pedal circle of $\mathcal{H} \Longrightarrow \mathcal{H}$ has center H and foci P and its reflection S on H.

IDMasterz

#8 Jan 15, 2015, 11:41 pm

Area problem: Let QR meet the circles again at Q', R'. Let the centre of spiral similarity between the series $Q' \mapsto R'$ be S. The ratio of the spiral similarity S is the inverse of the ratio of spiral similarity w.r.t. P, so done. (This is because Q' and R' are on opposite sides).

With Luis' theorem for this homography on conic, then one can see that if the tangency points with two circles are $\{X, Y\}, \{X', Y'\}$, then X is the point when $Q = Q'$. If one lets $XR \cap OO' = T$, where O, O' are the centres of the original circles, then note $XYOF_1F_2$ are concyclic where F_1, F_2 are the focii, so F_1, F_2 are on the circle with diametre OT. Define T' similarly for O', then we uniquely define F_1, F_2 as P, S by the property discussed in previous

Quick Reply

High School Olympiads

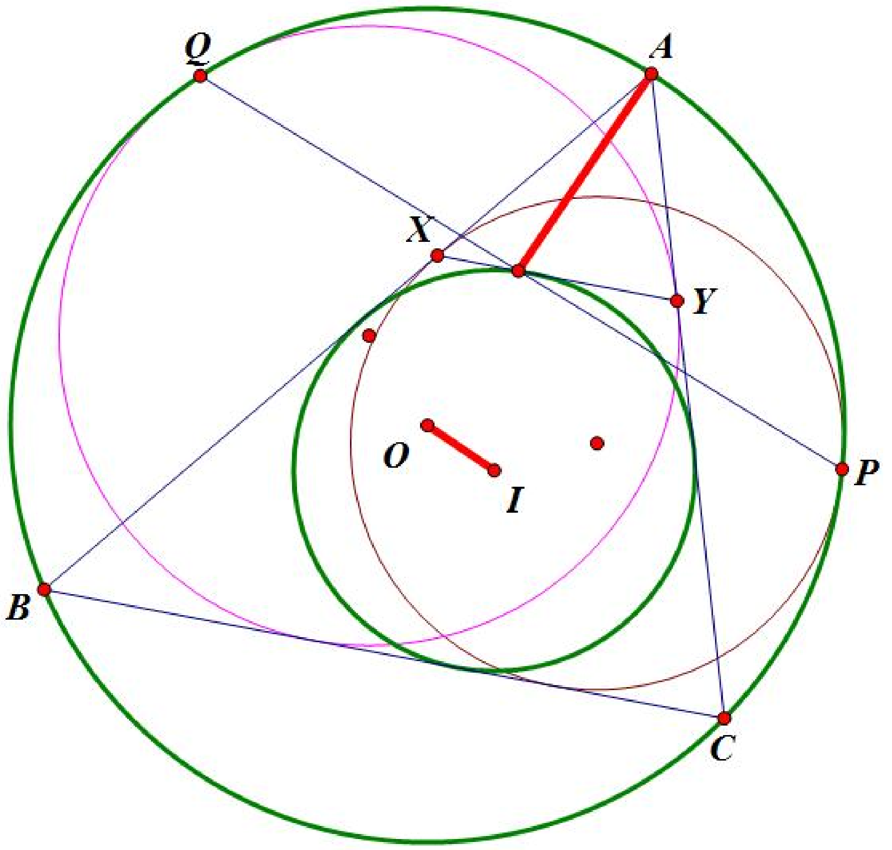
An interesting problem related to Mixtilinear incircle

Reply

Butterfly
#1 Apr 28, 2012, 1:22 pm

Let (O) and (I) be the circumcircle and incircle of the triangle ABC. The **B-mixtilinear incircle** contacts AB and (O) at X and P respectively, and similarly the **C-mixtilinear incircle** contacts AC and (O) at Y and Q in order. Suppose that XY intersects PQ at M□prove that the two lines AM and OI are perpendicular to each other.

Attachments:



Butterfly
#2 Apr 29, 2012, 3:38 pm

This is a solid problem. I have racked my brains but gained nothing. Who can solve it?

sunken rock
#3 Apr 29, 2012, 9:31 pm

Additional remark (open):
If $\{Z\} \in PX \cap QY$, then O, I, Z are collinear.

Best regards,
sunken rock

RSM

#4 Apr 30, 2012, 4:35 pm

I am sorry that my proof is not completely synthetic. At first for the remark of sunken rock, note that, if CC', CC_1 are the internal and external angle-bisector of $\angle ACB$ where C', C_1 lie on $\odot ABC$, C_1, I, Q are collinear and P, X, C' are collinear. Similarly define B', B_1. So applying Pascal's theorem on $C_1QB'B_1PC'$ we get that PX, QY intersect on OI.
Now note that, $XY \parallel BC$ and I is the A-excenter of ΔAXY. We will prove that, if $U = B'C' \cap XY$, then AU and AM are isogonal conjugates wrt $\angle BAC$. $V = B'Q \cap C'P$. By Menelaus theorem, $\frac{XM}{YM} = \frac{XQ}{VQ} \cdot \frac{VP}{YP}$ and $\frac{XU}{YU} = \frac{XB'}{B'V} \cdot \frac{VC'}{C'Y}$. But we have $\frac{VC'}{VB'} = \frac{VQ}{VP}$ and $XB'.XQ = XA.XB$ and $YC'.YP = YA.YC$. So $\frac{XM.XU}{YM.YU} = \frac{AX.XB}{AY.YC} = \frac{AX^2}{AY^2}$. So AM, AU are isogonal conjugates. But from here we have that the isogonal conjugate of AU is perpendicular to OI. So done.

TelvCohl

#5 Jan 11, 2015, 7:51 pm

My solution:

Let $B' = QY \cap \odot(ABC), C' = PX \cap \odot(ABC)$.
Let $U = PQ \cap BC, V = C'Q \cap AB, Z = PQ \cap AB$.

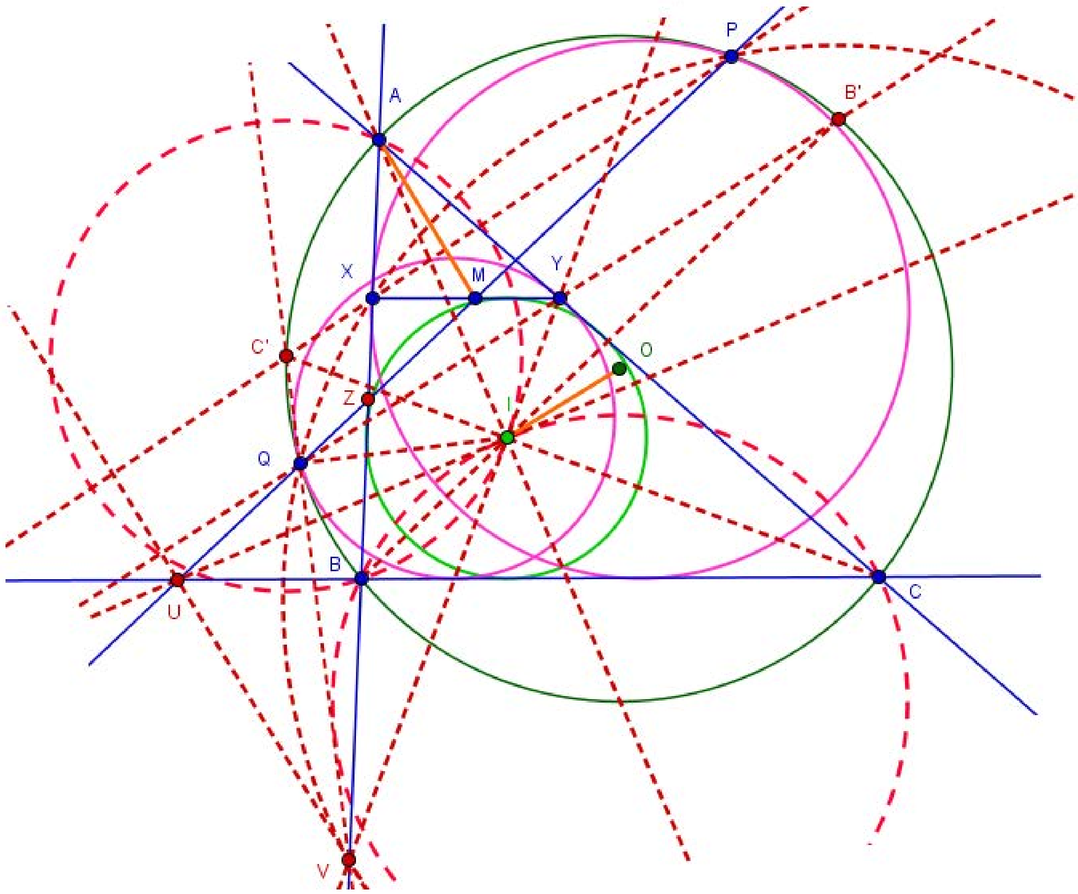
From homothety we get B', C' is the midpoint of CA, AB, respectively .

Since UI is tangent to $\odot(IBC)$ at I (well-known) ,
so we get $UI^2 = UB \cdot UC \Longrightarrow U$ lie on the radical axis of $\odot(O)$ and degenerate circle I .
Similarly, VI the tangent of $\odot(IAB) \Longrightarrow V$ lie on the radical axis of $\odot(O)$ and degenerate circle I ,
so we get UV is the radical axis of $\odot(O)$ and degenerate circle $I \Longrightarrow UV \perp OI$. $(\star)$

Since $\angle IQC' = 90^\circ$ (well-known) ,
so from $C'Q \cdot C'V = C'I^2 = C'A^2 = C'X \cdot C'P$ we get P, Q, V, X are concyclic ,
hence $ZX \cdot ZV = ZP \cdot ZQ = ZA \cdot ZB \Longrightarrow \frac{ZM}{ZU} = \frac{ZX}{ZB} = \frac{ZA}{ZV} \Longrightarrow AM \parallel UV$,
so combine with $(\star)$ we get $AM \perp OI$.

Q.E.D

Attachments:



Luis González

#6 Jan 12, 2015, 4:28 am

Let the A-mixtilinear incircle touch (O) at R and let the B- and C- mixtilinear incircles touch BC at $X', Y'.$ Since I is midpoint of $XX', YY',$ it follows that $XY \parallel BC.$ If PQ, QR cut BC, AC at $D, E,$ then DE is the polar of $AR \cap BP \cap CQ$ (the exsimilicenter of (O)~(I)) WRT $(O) \Longrightarrow DE \perp OI.$

On the other hand, if L is the midpoint of the arc AC of $(O),$ it's known that D, Y, L are collinear and that $E \equiv AC \cap QR \cap PL.$ Hence $\angle PEC = \frac{1}{2}|\angle PAC - \angle PCA| = \angle PQL \Longrightarrow PEQY$ is cyclic. Thus if $K \equiv AC \cap PQ,$ we have $KA \cdot KC = KP \cdot KQ = KE \cdot KY \Longrightarrow \frac{KA}{KE} = \frac{KY}{KC} = \frac{KM}{KD} \Longrightarrow (AM \parallel DE) \perp OI.$

IDMasterz

#7 Jan 14, 2015, 9:25 pm

Let R be the A-mixtillinear touch point. Note that $PQ \cap BC = T$ is the pole of DH w.r.t. the incircle of ABC, where D is the foot of I onto B and H is the orthocnetre of the intouch triangle of ABC. Hence, if $QR \cap AC = U$, then $TU \perp OI$. Let M be the midpoint of arc AC. Then, under inversion about the circle through IAC, we have $P \mapsto U, Y \mapsto Q \implies PQTY$ are concyclic. Hence, PQ meets AC at V s.t. $\frac{VA}{VU} = \frac{VY}{VC}$, so V is the centre of homothety mapping $AYM \mapsto UCT \implies AM \parallel TU \perp OI$.

Note: My motivation actually came from inverting about the incircle :)

Quick Reply

High School Olympiads

Well known concurrency problem

Locked

dothef1

#1 Jan 10, 2015, 11:09 pm

This lemma is well known , and i'm looking for various proofs if possible .
Let ABC be a triangle with incenter I .
E and F are the points of tangency of the incircle with sides AB and AC respectively .
M and N are midpoints of BC and AC .
Prove that (MN) , (EF) and (BI) are concurrent .

Luis González

#2 Jan 10, 2015, 11:20 pm

Use the search for old or well-known problems. It has been posted many times, e.g.
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=23677
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=217827
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=132338
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=302750

For a generalization see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=363070

High School Olympiads

Concurrency related to G

Reply

Source: Own

TheOverlord
#1 Jan 9, 2015, 9:44 pm

P is a point in triangle ABC. G is the centroid of this triangle. Line parallel to AP from G intersects BC at P_a. Q_a is reflection of P_a through G. Q_b and Q_c are defined similarly. Prove that AQ_a, BQ_b and CQ_c are concurrent.

Edited: typo fixed

Luis González
#2 Jan 10, 2015, 12:09 am • 1

Let A' be the reflection of A on the midpoint M of $BC.$ Let U be the midpoint of AG and AQ_a cuts BC at $X.$ Since G is also midpoint of $MU,$ then $UQ_a \parallel BC \Longrightarrow AQ_a : AX = AU : AM = AG : AA' = 1:3 \Longrightarrow A'X \parallel GQ_a \parallel AP.$ Hence if $Y \equiv AP \cap BC,$ then $AXA'Y$ is parallelogram $\Longrightarrow AP$ and AQ_a are isotomics WRT $\triangle ABC.$ Hence, we conclude that AQ_a, BQ_b, CQ_c concur at the isotomic conjugate of $P.$

TelvCohl
#3 Jan 10, 2015, 12:28 am • 1

My solution:

Let ℓ be a line passing through A and parallel to BC .
Let P^* be the isotomic conjugate of P WRT $\triangle ABC$.

Easy to see the reflection of G in Q_a lie on ℓ ,
so from $(AP, AQ_a; AG, \ell) = -1$ we get AQ_a pass through P^* .
Similarly, we can prove $P^* \in BQ_b, P^* \in CQ_c \Longrightarrow AQ_a, BQ_b, CQ_c$ are concurrent at P^* .

Q.E.D

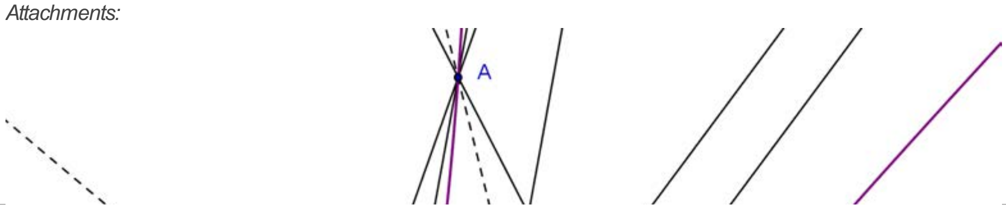
This post has been edited 1 time. Last edited by TelvCohl, Aug 2, 2015, 12:10 am

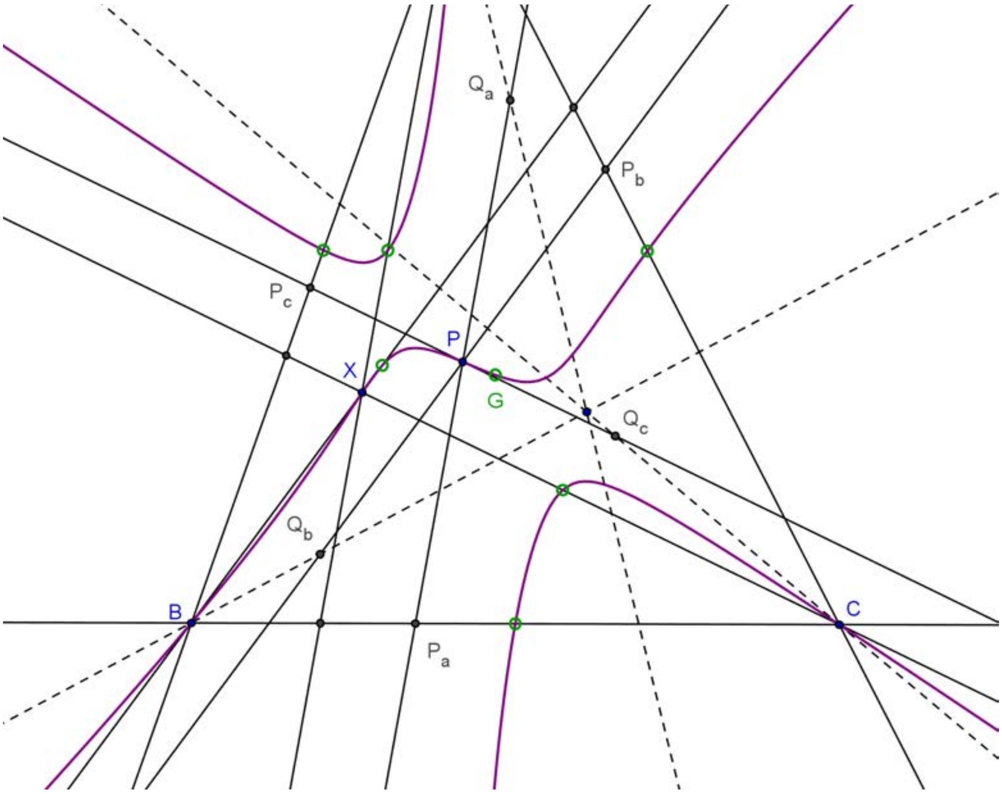
Luis González
#4 Jan 10, 2015, 4:15 am • 1

More general: Let X be a fixed point inside of $\triangle ABC,$ while P is a variable point. Parallel from P to AX cuts BC at P_a and Q_a is the reflection of P_a on $P.$ Q_b and Q_c are defined cyclically. Then AQ_a, BQ_b, CQ_c concur if and only if P is on unique cubic $\mathcal{Q}$ through A,B,C the midpoints of BC,CA,AB and the midpoints of the cevians of $X.$ Furthermore, $\mathcal{Q}$ goes through X and the centroid G of $\triangle ABC.$

Parallel project $\triangle ABC \cup X$ into an acute $\triangle ABC$ with orthocenter $X.$ $\triangle P_aP_bP_c$ becomes pedal triangle of P WRT $\triangle ABC$ and $\triangle Q_bQ_cQ_b$ is its reflection on $P.$ Hence AQ_a, BQ_b, CQ_c concur $\Longleftrightarrow P$ is on Thomson cubic K002 of $\triangle ABC.$ Thus, back in the primitive figure, locus of P is the afffine equivalence of K002, i.e. the unique cubic $\mathcal{Q}$ through A,B,C the midpoints of BC,CA,AB and the midpoints of the cevians of $X.$ In addition, it contains X and the centroid G of $\triangle ABC.$

Attachments:





Quick Reply

High School Olympiads

O is the orthocenter of MNP

Reply

Source: own

THVSH
#1 Jan 9, 2015, 9:13 pm

Let ABC be a triangle with O is the circumcenter. D,E,F are midpoints of the arc BAC,ABC,ACB of (ABC), respectively. EF,FD,DE intersect BC,CA,AB at M,N,P, respectively. Prove that O is the orthocenter of the triangle MNP.

Luis González
#2 Jan 9, 2015, 9:47 pm

More general: In a triangle $\triangle ABC,$ let D,E,F be the midpoints of $BC,CA,AB.$ P is an arbitrary point and $\triangle P_AP_BP_C$ is the pedal triangle of P WRT $\triangle ABC.$ $X \equiv EF \cap P_BP_C,$ $Y \equiv P_CP_A \cap FD,$ $Z \equiv P_AP_B \cap DE.$ Then the orthocenter of $\triangle XYZ$ is the circumcenter of $\triangle P_AP_BP_C.$

From the first Fontené theorem, the lines P_AX,P_BY,P_CZ concur on $\odot(P_AP_BP_C),$ i.e. $\triangle XYZ$ is a cevian triangle of $\triangle P_AP_BP_C$ WRT a point on its circumcircle. By Brokart's theorem, its orthocenter is the center of $\odot(P_AP_BP_C).$

TelvCohl
#3 Jan 9, 2015, 10:54 pm

Another solution:

Let I_a,I_b,I_c be the excenters of $\triangle ABC$.
Let $I \equiv I_aA \cap I_bB \cap I_cC$ be the incenter of $\triangle ABC$.
Let $G \equiv I_aD \cap I_bE \cap I_cF$ be the centroid of $\triangle I_aI_bI_c$.
Let $\mathcal{H}$ be the conic passing through I_a,I_b,I_c,I,G and $T=BN \cap CP$.

Since BC,EF is the polar of A,D WRT $\mathcal{H}$, respectively ,
so $M \equiv BC \cap EF$ is the pole of $AD \equiv I_bI_c$ WRT $\mathcal{H}$,
hence we get MI_b,MI_c are the tangent of $\mathcal{H}$.
Similarly, we can prove NI_c,NI_a,PI_a,PI_b are the tangent of $\mathcal{H}$,
so we get $I_a \in NP,I_b \in PM,I_c \in MN$.

From Pascal theorem (for $BTCFDE$) we get $T \equiv BN \cap CP \in \odot(O)$.
Similarly, we can prove $CP \cap AM \in \odot(O)$ and $AM \cap BN \in \odot(O)$,
so we get AM,BN,CP are concurrent at a point T lie on $\odot(O)$,
hence from Brokart's theorem we get O is the orthocenter of $\triangle MNP$.

Q.E.D

LeVietAn
#4 Jun 17, 2015, 9:48 pm

This problem also has the following property:
Let G,G' be the centroids of the triangles DEF,MNP, reps. Prove that $\angle OGG'=90^\circ$. :)

TelvCohl
#5 Jun 17, 2015, 10:43 pm

LeVietAn wrote:

> This problem also has the following property:
> Let G, G' be the centroids of the triangles DEF, MNP, reps. Prove that $\angle OGG' = 90^\circ$. :)

My solution **:**

Let M_A be the midpoint of NP .
Let I_a, I_b, I_c be the A-excenter, B-excenter, C-excenter of $\triangle ABC$, respectively .

From Side triangle (2) $\Longrightarrow I_aM \parallel I_bN \parallel I_cP$ (all $\perp OG$) ,
so combine $DI_b = DI_c \Longrightarrow DM_A \perp OG \Longrightarrow GG' \perp OG$ ($\because \frac{MG'}{M_AG'} = \frac{I_aG}{DG} = 2$) .

Q.E.D

Quick Reply

High School Olympiads

Nice result

Reply

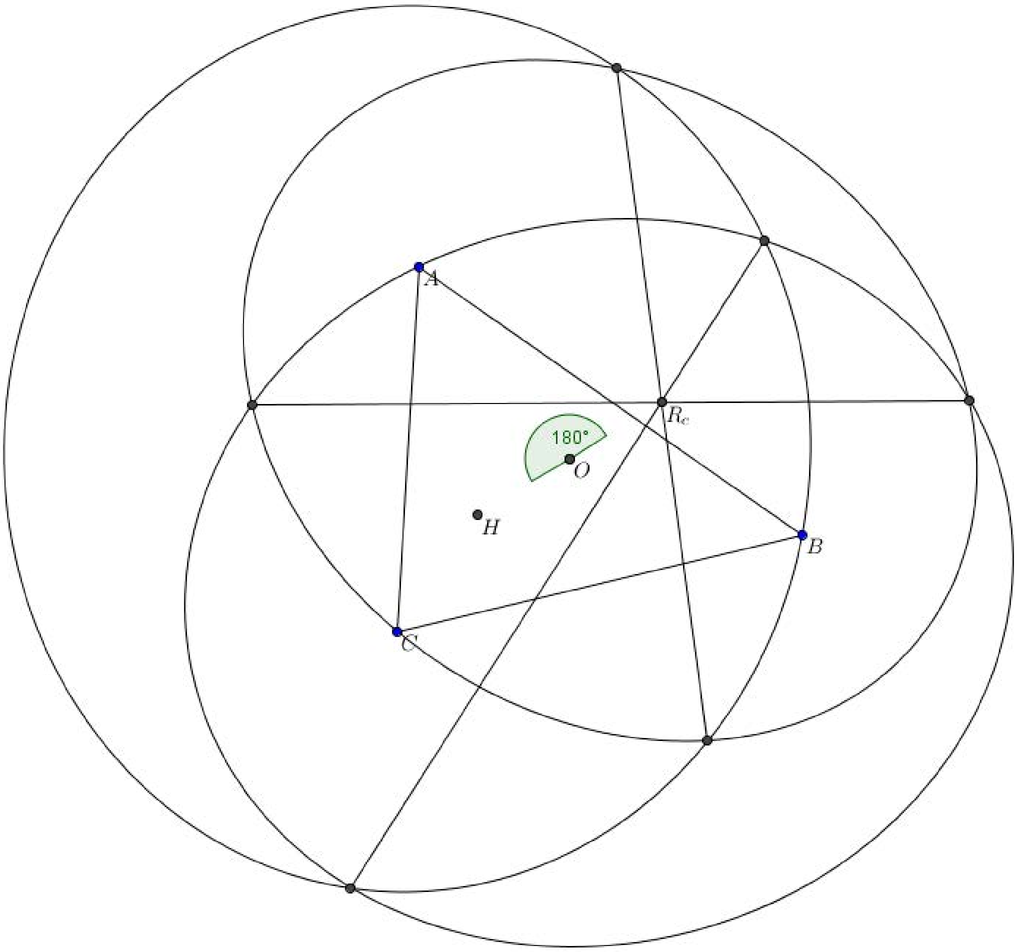
Source: own

eziz

#1 Jan 9, 2015, 5:12 pm

Let ABC acute triangle.T_1 ellipse by focies B and C and point A, similarly T_2, T_3.Prove that:
1) Radical center of T_1, T_2, T_3 is R_c and it lie on Euler line of the triangle ABC
2) R_c is symmetry of H by O.

Attachments:



TelvCohl

#2 Jan 9, 2015, 5:39 pm

From the proof by **skytin** at conics we know R_c is the radical center of $\{\odot(A,a), \odot(B,b), \odot(C,c)\}$ which is the orthocenter of the antimedial triangle of $\triangle ABC$ (a, b, c is the length of BC, CA, AB, respectively .) . $\Longrightarrow$ R_c is the reflection of H in O

Luis González

#3 Jan 9, 2015, 9:16 pm

See also http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=604217 (post #5). Check solutions (1) and (2); the common chords of these three ellipses is the De Longchamps point of ABC, reflection of H on O.

Quick Reply

High School Olympiads

Concurrent Simson lines

Reply

Source: Own

buratinogigle
#1 Jan 9, 2015, 9:15 am

Let ABC be a triangle with altitudes AA_1,BB_1,CC_1 are concurrent at H and Euler circle is (N). A_2,B_2,C_2 are symmetric of A_1,B_1,C_1 through N. A_3,B_3,C_3 are symmetric of H through A_2,B_2,C_2 then A_3,B_3,C_3 lie on circumcirle (O) of ABC. Prove that Simson lines of A_3,B_3,C_3 with respect to triangle ABC are concurrent.

TelvCohl
#2 Jan 9, 2015, 9:52 am • 1

My solution:

Let A' be the antipode of A in $\odot(ABC)$.
Let $\mathcal{S}_A,\mathcal{S}_B,\mathcal{S}_C$ be the Simson line of A_3,B_3,C_3 WRT $\triangle ABC$, respectively .

Since H is the exsimilicenter of $(N) \sim (O)$,
so we get $AA_3 \parallel BC, BB_3 \parallel CA, CC_3 \parallel AB$.

Since the $A'A_3 \perp BC$,
so we get $\mathcal{S}_A \parallel AO$,
From Steiner theorem we get $A_2 \in \mathcal{S}_A$,
so $\mathcal{S}_A$ is A_2- altitude of $\triangle A_2B_2C_2$.

Similarly, $\mathcal{S}_B,\mathcal{S}_C$ is B_2- altitude, C_2- altitude of $\triangle A_2B_2C_2$, respectively ,
so we get $\mathcal{S}_A,\mathcal{S}_B,\mathcal{S}_C$ are concurrent at the orthocenter of $\triangle A_2B_2C_2$.

Q.E.D

buratinogigle
#3 Jan 9, 2015, 10:01 am

Thank you Telv Cohl for quick solution, I see the general problem as a property of McKay cubic,

Let ABC be a triangle with circumcircle (O) and P is a point such that P,O and isogonal conjugate of P are collinear. A',B',C' lie on (O) such that AA',BB',CC' are perpendicular to PA,PB,PC, reps. Prove that Simson lines of A',B',C' with respect to triangle ABC are concurrent.

Luis González
#4 Jan 9, 2015, 10:42 am • 1

Let $\triangle XYZ$ be the antimedial triangle of $\triangle ABC.$ Clearly $\triangle A_3B_3C_3$ is the orthic triangle of $\triangle XYZ.$ Let D be the projection of A_3 on BC (midpoint of XA_3) and let U,V be the projections of A_3 on $XY,XZ.$ Obviously D is circumcenter of $\triangle A_3UV.$

Since the Simson line ℓ_{A_3} of A_3 WRT $\triangle ABC$ bisects $\overline{A_3H},$ it goes through D parallel to $XH;$ X-circumdiameter of $\triangle XYZ.$ But UV is antiparallel to YZ WRT $XY,XZ,$ thus $\ell_{A_3} \perp UV \Longrightarrow \ell_{A_3}$ is perpendicular bisector of $\overline{UV} \Longrightarrow \ell_{A_3}$ goes through the center of the Taylor circle T of $\triangle XYZ.$ Likewise ℓ_{B_3} and ℓ_{C_3} go through $T.$

IDMasterz
#5 Jan 9, 2015, 4:05 pm • 1

The general problem:

Let $A^*B^*C^*$ be the antipedal triangle of P w.r.t. ABC. Let the orthocentre of ABC be H, the orthocentre of $A^*B^*C^*$ be H^* and the orthocentre of $A'B'C'$ be H'. Let the isogonal conjugate of P w.r.t. $A^*B^*C^*$ be P^* and let the centres of the P^*-Carnot circles w.r.t. $A^*B^*C^*$ be $O_AO_BO_C$. Suppose P' is the isogonal conjugate of P w.r.t. ABC and let ℓ_A, ℓ_B, ℓ_C be the point at infinity of the A', B', C' simson lines w.r.t. ABC. It suffice to show $A'\ell_A, B'\ell_B, C'\ell_C$ concur.

Note that by the Liang-Zelich theorem (A theorem that was discovered by XML(Evan Liang) and I) we have PP^* passes through the circumcentre of $A^*B^*C^*$. Note $\ell_A \parallel AP'$, which is parallel to P^*O_A. We have the parallels through A^*, B^*, C^* to P^*O_A, P^*O_B, P^*O_C respectively concur at say X. Since $X = HP \cap OP^*$, it follows $X = P \implies \ell_A \parallel AP$, hence $A'\ell_A, B'\ell_B, C'\ell_C$ concur at H' :) Evidently, if the midpoints of HA', HB', HC' are A_1, B_1, C_1, then the aforementioned simson lines concur at the orthocentre of $A_1B_1C_1$.

XmL

#6 Feb 1, 2015, 4:30 am • 1

> *IDMasterz wrote:*
> The general problem:
>
> Let $A^*B^*C^*$ be the antipedal triangle of P w.r.t. ABC. Let the orthocentre of ABC be H, the orthocentre of $A^*B^*C^*$ be H^* and the orthocentre of $A'B'C'$ be H'. Let the isogonal conjugate of P w.r.t. $A^*B^*C^*$ be P^* and let the centres of the P^*-Carnot circles w.r.t. $A^*B^*C^*$ be $O_AO_BO_C$. Suppose P' is the isogonal conjugate of P w.r.t. ABC and let ℓ_A, ℓ_B, ℓ_C be the point at infinity of the A', B', C' simson lines w.r.t. ABC. It suffice to show $A'\ell_A, B'\ell_B, C'\ell_C$ concur.
>
> Note that by the Liang-Zelich theorem (A theorem that was discovered by XML(Evan Liang) and I) we have PP^* passes through the circumcentre of $A^*B^*C^*$. Note $\ell_A \parallel AP'$, which is parallel to P^*O_A. We have the parallels through A^*, B^*, C^* to P^*O_A, P^*O_B, P^*O_C respectively concur at say X. Since $X = HP \cap OP^*$, it follows $X = P \implies \ell_A \parallel AP$, hence $A'\ell_A, B'\ell_B, C'\ell_C$ concur at H' :) Evidently, if the midpoints of HA', HB', HC' are A_1, B_1, C_1, then the aforementioned simson lines concur at the orthocentre of $A_1B_1C_1$.

Using our theorem is actually unecessary. If we let O'_A be circumcenter of BPC; then it's well-known (property of the McCay cubic, easily proven by a property discussed in our paper) that $AP' \parallel PO'_A \parallel PA^* \implies A'\ell_A \parallel PA^* \perp B'C' \implies A'\ell_A, B'\ell_B, C'\ell_C$ concur at the orthocenter of $A'B'C'$.

Quick Reply

High School Olympiads

Orthotransversal and antigonal conjugate points

Reply

Source: Own

buratinogigle

#1 Sep 16, 2012, 12:35 am

Prove that two orthotransversals of two antigonal conjugate points with respect to a triangle are parallel.

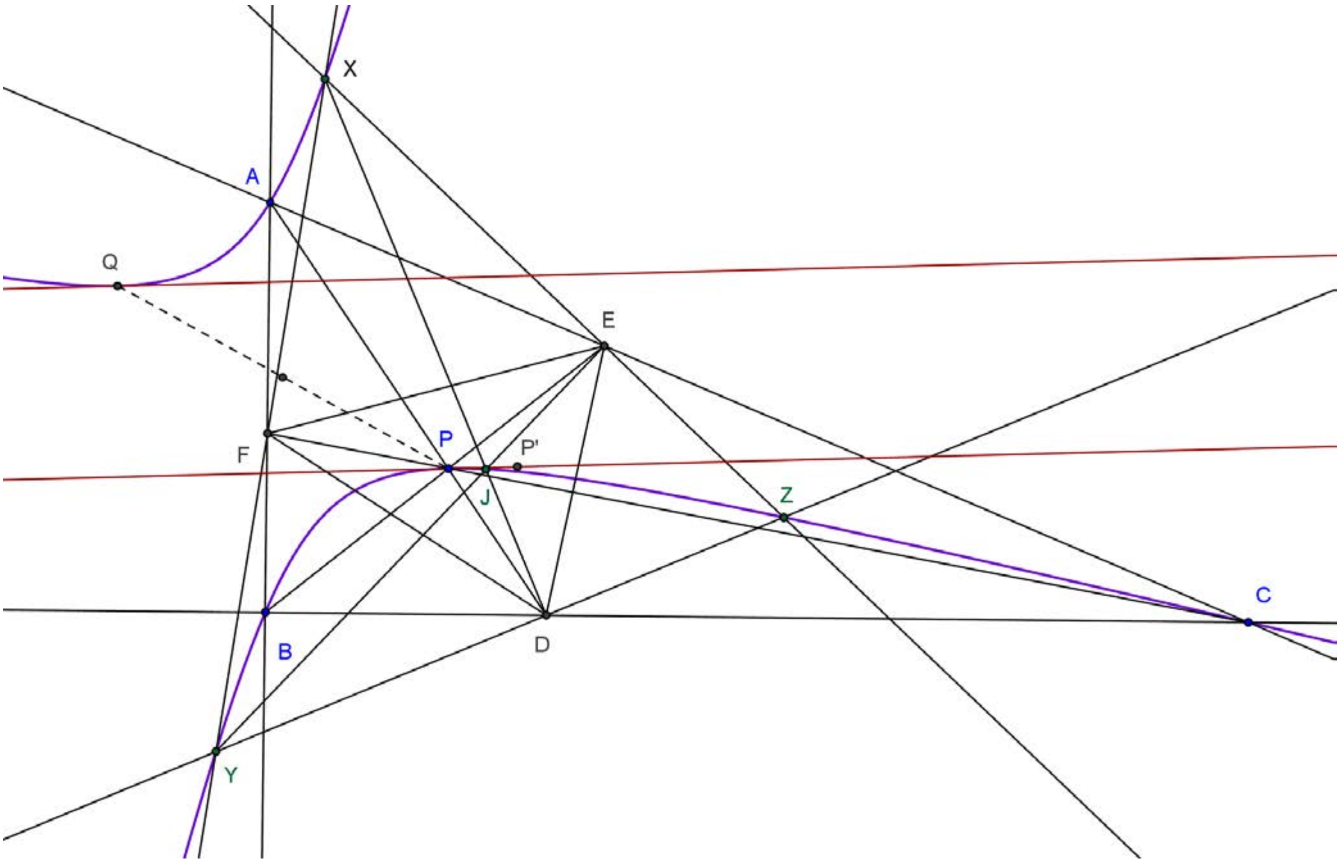
Luis González

#2 Sep 14, 2013, 2:25 am • 2

Let P be a point on the plane of $\triangle ABC.$ $\mathcal{H}$ is the rectangular hyperbola through $A,B,C,P.$ Let Q be the antigonal conjugate of P WRT $\triangle ABC,$ i.e. the reflection of P on the center of $\mathcal{H}.$ $\triangle DEF$ is the cevian triangle of P WRT $\triangle ABC$ with incenter J and excenters X,Y,Z against $D,E,F.$ These J,X,Y,Z lie on $\mathcal{H},$ for a proof see the thread Poncelet points (post #6).

It's known that the polars of a fixed P WRT the pencil of conics through 4 points X,Y,Z,J go through a fixed point. For the degenerate conic $DX \cup YZ,$ the polar of P is clearly the reflection of DP on $DJ,$ i.e. the isogonal of DP WRT $\angle EDF,$ passing through the isogonal conjugate P' of P WRT $\triangle DEF.$ Similarly, polar of P WRT the degenerate conic $EY \cup ZX$ is $EP' \Longrightarrow P'$ is the fixed point in the referred conjugation $\Longrightarrow PP' \equiv p$ is the polar of P WRT $\mathcal{H} \Longrightarrow p$ is tangent of $\mathcal{H}.$ Now, according to Isogonal conjugate and perpendicularity, p is perpendicular to the orthotransversal τ_P of P WRT $\triangle ABC.$ Analogously, the tangent q of $\mathcal{H}$ at Q is perpendicular to the orthotransversal τ_Q of Q WRT $\triangle ABC.$ Since P,Q are symmetric about the center of $\mathcal{H},$ then $p \parallel q \Longrightarrow \tau_P \parallel \tau_Q.$

Attachments:



Luis González

#3 Jan 9, 2015, 8:04 am • 3

Using the result discussed in the thread Collinearity with Symmedian Point (see post #16 and the subsequent replies) for $P \equiv Q$, we get a much simpler proof to this problem. We immediately get that the orthotransversal τ_P of P is perpendicular to the tangent p of $\mathcal{H}$ through P and similarly for Q. Since $p \parallel q \Longrightarrow \tau_P \parallel \tau_Q$, as desired.

buratinogigle

#4 Aug 10, 2015, 4:56 pm • 1

Thank you so much dear Luis for your solution, I have seen this problem in the post http://artofproblemsolving.com/community/c6t48f6h1127393_orthotransversal_of_ferma_point and I have an idea

If P, Q are two antigonal conjugate with respect to triangle ABC. Let P^* and Q^* are isogonal conjugate of P, Q then P, Q, O are collinear with O is circumcenter of ABC. Choose point R^* on that line such that $(OR^*, P^*Q^*) = -1$ and R is isogonal conjugate of R^*. H is orthocenter of triangle ABC. Prove that two orthotransversals of P, Q are perpendicular to RH.

TelvCohl

#5 Aug 10, 2015, 5:29 pm • 1

> *buratinogigle wrote:*
>
> If P, Q are two antigonal conjugate with respect to triangle ABC. Let P^* and Q^* are isogonal conjugate of P, Q then P^*, Q^*, O are collinear with O is circumcenter of ABC. Choose point R^* on that line such that $(OR^*, P^*Q^*) = -1$ and R is isogonal conjugate of R^*. H is orthocenter of triangle ABC. Prove that two orthotransversals of P, Q are perpendicular to RH.

Let $\mathcal{H}$ be the isogonal conjugate of P^*Q^* WRT $\triangle ABC$ ($\mathcal{H}$ passes through the isogonal conjugate H, P, Q, R of O, P^*, Q^*, R^* WRT $\triangle ABC$, resp). From $(H, R; P, Q) = (O, R^*; P^*, Q^*) = -1 \Longrightarrow$ the tangents τ_P, τ_Q of $\mathcal{H}$ passing through P, Q, resp and RH are concurrent, so from $\tau_P \parallel \tau_Q$ ($\because P, Q$ are symmetry WRT the center of $\mathcal{H}$) we get $RH \parallel \tau_P \parallel \tau_Q$, hence RH is perpendicular to the orthotransversals of P, Q WRT $\triangle ABC$ (see the solution by Luis at post #3).

buratinogigle

#6 Aug 10, 2015, 5:41 pm

Thank you so much dear Telv Cohl, follow this idea, I have an idea to see Lester circle in general case

If P, Q are two antigonal conjugate with respect to triangle ABC. Let P^* and Q^* are isogonal conjugate of P, Q then P^*, Q^*, O are collinear with O is circumcenter of ABC. Choose point R^* on that line such that $(OR^*, P^*Q^*) = -1$ and R is isogonal conjugate of R^*. H is orthocenter of triangle ABC.

a) Prove that PQ pass through midpoint G of RH.

b) Let N be midpoint of GR and K is symmetric of H though N. Prove that P, Q, N, K are concyclic.

When P, Q are two Fermat point we have Lester's circle.

TelvCohl

#7 Aug 10, 2015, 6:14 pm

> *buratinogigle wrote:*
>
> If P, Q are two antigonal conjugate with respect to triangle ABC. Let P^* and Q^* are isogonal conjugate of P, Q then P^*, Q^*, O are collinear with O is circumcenter of ABC. Choose point R^* on that line such that $(OR^*, P^*Q^*) = -1$ and R is isogonal conjugate of R^*. H is orthocenter of triangle ABC.
>
> a) Prove that PQ pass through midpoint G of RH.
>
> b) Let N be midpoint of GR and K is symmetric of H though N. Prove that P, Q, N, K are concyclic.
>
> When P, Q are two Fermat point we have Lester's circle.

Let $\mathcal{H}$ be the isogonal conjugate of P^*Q^* WRT $\triangle ABC$. From my proof at post #5 $\Longrightarrow RH$ is parallel to the tangents of $\mathcal{H}$ passing through P and Q, so the pole of RH WRT $\mathcal{H}$ lie on PQ and PQ passes through the midpoint G of RH.

From the theorem mentioned at post #2 in Rectangular circumhyperbola and circle (see the proof at post #2 and post #4) $\Longrightarrow$ P, Q is the image of each other under the inversion $\mathbf{I}(\odot(RH))$ so $GN \cdot GK = GP \cdot GQ \Longrightarrow P, Q, N, K$ are concyclic.

P', Q' is the image of each other under the inversion [illegible], so [illegible] are concyclic.

Quick Reply

High School Olympiads

Concurrent, collinear and perpendicular

Source: Inspire from VMO 2015

buratinogigle
#1 Jan 8, 2015, 9:31 pm

Let ABC be acute triangle with altitudes AD,BE,CF are concurrent at H. Circles passing through E,F tangent to BC at X_1,X_2 such that X_1 lie on segment BC. Similarly we have Y_1,Y_2,Z_1,Z_2.

a) Prove that AX_1,BY_1,CZ_1 are concurrent at P.

b) Prove that X_2,Y_2,Z_2 lie on a line d.

c) Prove that PH and d are perpendicular.

TelvCohl
#2 Jan 8, 2015, 11:23 pm • 1

My solution:

Let $B' = BE \cap \odot(EFX_1), C' = CF \cap \odot(EFX_1)$.
Let $X = EF \cap BC, Y = FD \cap CA, Z = DE \cap AB$.
Let the tangent of $\odot(AH)$ through X tangent to $\odot(AH)$ at X_1', X_2' .
(X_1' is in $\triangle ABC$ and X_2' is out of $\triangle ABC$)

From Reim theorem we get $BC \parallel B'C'$,

so $\frac{{BX_1}^2}{{CX_1}^2} = \frac{BB' \cdot BE}{CC' \cdot CF} = \frac{BH \cdot BE}{CH \cdot CF}$.

Similarly, we can get $\frac{{CY_1}^2}{{AY_1}^2} = \frac{CH \cdot CF}{AH \cdot AD}$ and $\frac{{AZ_1}^2}{{BZ_1}^2} = \frac{AH \cdot AD}{BH \cdot BE}$,

so multiple these three equality we get $\frac{BX_1}{CX_1} \cdot \frac{CY_1}{AY_1} \cdot \frac{AZ_1}{BZ_1} = 1$.

i.e. AX_1, BY_1, CZ_1 are concurrent at P

Since $XE \cdot XF = {XX_1}^2 = {XX_2}^2 = XB \cdot XC$,
so we get $(X_2, X_1; B, C) = -1$.
Similarly, we can prove $(Y_2, Y_1; C, A) = -1$ and $(Z_2, Z_1; A, B) = -1$,
so from **(a)** we get X_2, Y_2, Z_2 are collinear at the trilinear polar d of P .

Let H^* be the projection of H on d .

Since ${XX_1'}^2 = XE \cdot XF = XB \cdot XC$,
so XX_1' is also the tangent of $\odot(BCX_1')$,
hence $\odot(AH)$ and $\odot(BCX_1')$ are external tangent at X_1' .

Invert with center A and factor $AF \cdot AB = AE \cdot AC$.

Since $BC \longleftrightarrow \odot(AH)$,
so X_1' is the image of X_1 ,

hence we get A, X_1, X_1' are collinear .
Similarly, we can prove X_2' is the image of X_2 and $X_2' \in AX_2$.

Invert with center H which swap $\odot(ABC)$ and $\odot(DEF)$.

Since $BC \longleftrightarrow \odot(AH)$,
so $X_1 \longleftrightarrow X_2', X_2 \longleftrightarrow X_1'$,
hence we get $X_1' \in HX_2$ and $X_2' \in HX_1$.
Since $HA \cdot HD = HX_1 \cdot HX_2' = HX_2 \cdot HX_1'$,
so we get H is the orthocenter of $\triangle AX_1X_2$.

Since X_2H is the diameter of $\odot(HX_2X_2')$
so the image $\odot(HX_2X_2')$ of AX_1 pass through H^* .
Similarly, we can prove the image of BY_1, CZ_1 pass through H^* ,
so we get H^* is the image of P and $PH \perp d$.

Q.E.D

buratinogigle
#3 Jan 9, 2015, 12:35 am

My solution:

We have the general problem as the lemma

Lemma. Let ABC be a triangle. Circle (K) passing through B, C cuts CA, AB again at E, F. BE cuts CF at H. AH cuts BC at D. Circle passing through E, F touches BC tại S, T (T lies on segment BC.) Prove that $\frac{TB^2}{TC^2} = \frac{DB}{DC}$.

Proof. EF cuts BC at G then G is midpoint of ST. By power of point we have $GS^2 = GT^2 = \overline{GE}.\overline{GF} = \overline{GB}.\overline{GC}$ deduce $(ST, BC) = -1$. Easily seen $(GD, BC) = -1$. From this, $\overline{CB}.\overline{CG} = \overline{CT}.\overline{CS}$ and $\overline{BC}.\overline{BG} = \overline{BT}.\overline{BS}$ deduce $\frac{\overline{DB}}{\overline{DC}} = -\frac{\overline{BG}}{\overline{CG}} = \frac{\overline{BG}.\overline{BC}}{\overline{CG}.\overline{CB}} = \frac{\overline{BT}.\overline{BS}}{\overline{CT}.\overline{CS}} = \frac{\overline{BT}}{\overline{CT}}.\frac{\overline{BS}}{\overline{CS}} = -\frac{TB^2}{TC^2}$. We are done.

Now when H is orthocenter then $\frac{X_1B}{X_1C} = \sqrt{\frac{DB}{DC}} = \sqrt{\frac{S_B}{S_C}}$. So AX_1, BY_1, CZ_1 are concurrent at point $P(\frac{1}{\sqrt{S_A}}, \frac{1}{\sqrt{S_B}}, \frac{1}{\sqrt{S_C}})$ and X_2, Y_2, Z_2 are collinear on line d is tripolar of P.

$d: \sqrt{S_A}x + \sqrt{S_B}y + \sqrt{S_C}z = 0$ has infinite point $(\sqrt{S_B} - \sqrt{S_C}, \sqrt{S_B} - \sqrt{S_C}, \sqrt{S_B} - \sqrt{S_C})$.

$H: (\frac{1}{S_A}, \frac{1}{S_B}, \frac{1}{S_C})$ so

$$PH: (\frac{1}{S_B\sqrt{S_C}} - \frac{1}{S_C\sqrt{S_B}})x + (\frac{1}{S_C\sqrt{S_A}} - \frac{1}{S_A\sqrt{S_C}})y + (\frac{1}{S_A\sqrt{S_B}} - \frac{1}{S_B\sqrt{S_A}})z = 0$$

has infinite point

$$(\frac{1}{S_C\sqrt{S_A}} - \frac{1}{S_A\sqrt{S_C}} - \frac{1}{S_A\sqrt{S_B}} + \frac{1}{S_B\sqrt{S_A}},,)$$

We can see $\sum_{cyc} S_A(\sqrt{S_B} - \sqrt{S_C})(\frac{1}{S_C\sqrt{S_A}} - \frac{1}{S_A\sqrt{S_C}} - \frac{1}{S_A\sqrt{S_B}} + \frac{1}{S_B\sqrt{S_A}}) = 0$.

Thus $PH \perp d$. We are done.

Luis González
#4 Jan 9, 2015, 3:58 am • 1

Problems a) and b) are particular cases of the configuration discussed at Double elements of three involutions when the object

Problems a) and b) are particular cases of the configuration discussed at Double elements of three involutions when the object conic is the 9-point circle of ABC. Moreover, from the relation found there, we deduce that P none other than the barycentric square root of $H.$

c) If $U \equiv EF \cap BC,$ then $UX_1{}^2 = UX_2{}^2 = UE \cdot UF \Longrightarrow$ circle with diameter $\overline{X_1X_2}$ is orthogonal to the circle with diameter $\overline{AH},$ thus it follows that H is orthocenter of $\triangle AX_1X_2$ (this is in fact valid for all conjugate pairs $\{X_1, X_2\} \in BC$ WRT the circle with diameter $\overline{AH}$). Hence if S is the projection of H on $APX_1,$ we have $HX_2 \cdot HS = HA \cdot HD = -k^2$ $\Longrightarrow X_2$ is on the inverse of the circle with diameter $\overline{HP}$ under inversion $(H, -k^2)$ and similarly Y_2 and $Z_2 \Longrightarrow$ $\overline{X_2Y_2Z_2} \perp PH.$

High School Olympiads

Concurrent line on incircle

Source: Own

buratinogigle

#1 Jan 8, 2015, 8:40 am

Let ABC be a triangle. P is a point on its incircle (I). PA cuts (I) again at D. Tangent at D of (I) cuts BC at X. Similarly we have Y,Z. Prove that AX,BY,CZ are concurrent.

Especially, $P=F$ Feuerbach point then $AX\parallel BY\parallel CZ$.

Luis González

#2 Jan 8, 2015, 9:23 am • 1

Let (I) touch BC,CA,AB at U,V,W and let PA,PB,PC cut BC,CA,AB at P_a,P_b,P_c. By dual of Desargues involution theorem for the fourline BC,BC,XD,XD, it follows that AX is fixed under the involution $AB\mapsto AC$, $AD\mapsto AU\Longrightarrow (X,B,C,P_a)=(X,C,B,U)\Longrightarrow$

$$\frac{\overline{XB}}{\overline{XC}}\cdot\frac{\overline{P_aC}}{\overline{P_aB}}=\frac{\overline{XC}}{\overline{XB}}\cdot\frac{\overline{UB}}{\overline{UC}}\Longrightarrow\frac{\overline{XB}^2}{\overline{XC}^2}=\frac{\overline{UB}}{\overline{UC}}\cdot\frac{\overline{P_aB}}{\overline{P_aC}}.$$

Multiplying the cyclic expressions together, we conclude by Ceva's theorem that AX,BY,CZ concur.

TelvCohl

#3 Jan 8, 2015, 9:36 am • 1

My solution:

Let A',B',C' be the tangent point of $\odot(I)$ with BC,CA,AB, respectively .
Let $D'=A'D\cap B'C', E'=B'E\cap C'A', F'=C'F\cap A'B'$.

Since $B'C'$ is the polar of A WRT $\odot(I)$,
so the intersection of the tangent of $\odot(I)$ through P,D lie on $B'C'$,
hence we get $PB'DC'$ is a harmonic quadrilateral and $A'(P,D';B',C')=-1$.
Similarly, we can prove $B'(P,E';C',A')=-1$ and $C'(P,F';A',B')=-1$,
so D',E',F' are collinear at the trilinear polar $\mathcal{T}$ of P WRT $\triangle A'B'C'$.

On the other hand, D',E',F' is the pole of AX,BY,CZ WRT $\odot(I)$, respectively ,
so we get AX,BY,CZ are concurrent at the pole of $\mathcal{T}$ WRT $\odot(I)$.

Q.E.D

Luis González

#4 Jan 8, 2015, 10:06 am • 1

Now if P is the Feuerbach point Fe, from the relation found in my previous post we get

$$\frac{\overline{XB}^2}{\overline{XC}^2}=\frac{s-b}{s-c}\cdot\frac{[FeAB]}{[FeAC]}=\frac{s-b}{s-c}\cdot\frac{(s-c)(a-b)^2}{(s-b)(c-a)^2}\Longrightarrow\frac{\overline{XB}}{\overline{XC}}=\frac{a-b}{c-a},$$

which means that AX is the A-cevian of $X_{514}\equiv(b-c:c-a:a-b)$ at infinity. Thus $AX\parallel BY\parallel CZ$ and this direction is X_{514} of $\triangle ABC$.

TelvCohl

#5 Jan 8, 2015, 1:01 pm • 1

Since the Feuerbach point F_e is the Kiepert focus of $\triangle A'B'C'$,
so the trilinear polar $\mathcal{T}$ of F_e WRT $\triangle A'B'C'$ is the Brocard axis of $\triangle A'B'C'$,
hence from $I \in \mathcal{T}$ we get the pole $\mathcal{P}$ of $\mathcal{T}$ WRT $\odot(I)$ is at infinity . i.e. $AX \parallel BY \parallel CZ$
(the direction of $\mathcal{P}$ is perpendicular to the Soddy line of $\triangle ABC$(Brocard axis of $\triangle A'B'C'$))

Quick Reply

High School Olympiads

Three lines converge.

Reply

HHT23

#1 Jan 7, 2015, 4:33 pm

Let ABC be a triangle with two points M, M' inside it. Points X, Y, Z lie in BC, CA, AB respectively, such that $M'X, M'Y, M'Z$ are parallel to MA, MB, MC. Points X', Y', Z lie in BC, CA, AB respectively, such that MX', MY', MZ' are parallel to $M'A, M'B, M'C$. Prove that AX, BY, CZ converge if and only if AX', BY', CZ' converge.

TelvCohl

#2 Jan 7, 2015, 8:53 pm • 1

My solution:

From the condition :
$\angle BM'X = \angle Y'MA, \angle XM'C = \angle AMZ'$ ... (1)
$\angle CM'Y = \angle Z'MB, \angle YM'A = \angle BMX'$ ... (2)
$\angle AM'Z = \angle X'MC, \angle ZM'B = \angle CMY'$ ... (3)

From Ceva's theorem :

$$\frac{\sin \angle MCB}{\sin \angle CBM} \cdot \frac{\sin \angle MAC}{\sin \angle ACM} \cdot \frac{\sin \angle MBA}{\sin \angle BAM} = 1 \text{ ... } (4)$$

$$\frac{\sin \angle M'CB}{\sin \angle CBM'} \cdot \frac{\sin \angle M'AC}{\sin \angle ACM'} \cdot \frac{\sin \angle M'BA}{\sin \angle BAM'} = 1 \text{ ... } (5)$$

From $(1), (2), (3), (4), (5)$ we get

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB}$$

$$= \frac{\sin \angle BM'X}{\sin \angle XM'C} \cdot \frac{\sin \angle M'CB}{\sin \angle CBM'} \cdot \frac{\sin \angle CM'Y}{\sin \angle YM'A} \cdot \frac{\sin \angle M'AC}{\sin \angle ACM'} \cdot \frac{\sin \angle AM'Z}{\sin \angle ZM'B} \cdot \frac{\sin \angle M'BA}{\sin \angle BAM'}$$

$$= \frac{\sin \angle BM'X}{\sin \angle XM'C} \cdot \frac{\sin \angle CM'Y}{\sin \angle YM'A} \cdot \frac{\sin \angle AM'Z}{\sin \angle ZM'B}$$

$$= \frac{\sin \angle Y'MA}{\sin \angle AMZ'} \cdot \frac{\sin \angle Z'MB}{\sin \angle BMX'} \cdot \frac{\sin \angle X'MC}{\sin \angle CMY'}$$

$$= \frac{\sin \angle X'MC}{\sin \angle BMX'} \cdot \frac{\sin \angle CBM}{\sin \angle MCB} \cdot \frac{\sin \angle Y'MA}{\sin \angle CMY'} \cdot \frac{\sin \angle ACM}{\sin \angle MAC} \cdot \frac{\sin \angle Z'MB}{\sin \angle AMZ'} \cdot \frac{\sin \angle BAM}{\sin \angle MBA}$$

$$= \frac{X'C}{BX'} \cdot \frac{Y'A}{CY'} \cdot \frac{Z'B}{AZ'}$$

i.e. AX, BY, CZ are concurrent $\Longleftrightarrow$ AX', BY', CZ' are concurrent

Q.E.D

Luis González

#3 Jan 7, 2015, 11:44 pm • 1

There exists an affine homology taking $\triangle ABC \cup M$ into an acute $\triangle ABC$ with orthocenter M, thus it suffices to show the problem for $M \equiv H$ the orthocenter of $\triangle ABC$. $\triangle H_AH_BH_C$ is the orthic triangle of $\triangle ABC$, $\triangle A_0B_0C_0$ is the antipedal triangle of M' WRT $\triangle ABC$ and B_0C_0, C_0A_0, A_0B_0 cut BC, CA, AB at A_1, B_1, C_1. H_1, H_2, H_3 are the projections of H on $B_0C_0, C_0, A_0, A_0B_0 \Longrightarrow X', Y', Z'$ are intersections of HH_1, HH_2, HH_3 with BC, CA, AB.

Since $HX' \cdot HH_1 = HA \cdot HH_A = k^2$, then AX' is the inverse of the circle (O_A) with diameter $\overline{HA_1}$ under inversion (H, k^2). Likewise, BY', CZ' are the inverses of the circles $(O_B), (O_C)$ with diameters $\overline{HB_1}, \overline{HC_1}$ under the referred inversion. Hence AX', BY', CZ' concur $\Longleftrightarrow (O_A), (O_B), (O_C)$ are coaxal $\Longleftrightarrow O_A, O_B, O_C$ are collinear $\Longleftrightarrow$ A_1, B_1, C_1 are collinear $\Longleftrightarrow \triangle ABC$ and $\triangle A_0B_0C_0$ are perspective $\Longleftrightarrow \triangle ABC$ and $\triangle XYZ$ are perspective.

Quick Reply

High School Olympiads

Concurrent from reflection

Source: Own

buratinogigle

#1 Jan 6, 2015, 10:14 pm

Let ABC be a triangle with altitude AD, BE, CF. A_b, A_c are reflection of D through AC, AB. BA_b cuts CA_c at A_a. Similarly we have B_b, C_c. Prove that AA_a, BB_b, CC_c are concurrent.

Luis González

#2 Jan 7, 2015, 3:40 am • 1

Let AA_a, BB_b, CC_c cut BC, CA, AB at X, Y, Z. If BA_b, CA_c cut AC, AB at M, N, we have

$$\frac{CM}{MA}=\frac{[BCA_b]}{[BAA_b]}=\frac{BC\cdot CD\cdot\sin 2\widehat{C}}{AB\cdot AD\cdot\sin(\widehat{A}+90^\circ-\widehat{C})}=\frac{BC\cdot CD\cdot\sin 2\widehat{C}}{AB\cdot AD\cdot\cos(\widehat{C}-\widehat{A})}$$

Similarly, we get $\frac{BN}{NA}=\frac{BC\cdot BD\cdot\sin 2\widehat{B}}{AC\cdot AD\cdot\cos(\widehat{A}-\widehat{B})}$. Thus by Ceva's theorem, we have

$$\frac{BX}{XC}=\frac{BN}{NA}\cdot\frac{MA}{CM}=\frac{AB}{AC}\cdot\frac{BD}{CD}\cdot\frac{\sin 2\widehat{B}}{\sin 2\widehat{C}}\cdot\frac{\cos(\widehat{C}-\widehat{A})}{\cos(\widehat{A}-\widehat{B})}.$$

Multiplying $\frac{BX}{XC}$ and cyclic expressions for $\frac{CY}{YA}, \frac{AZ}{ZB}$ together, we conclude by Ceva's theorem that $AX\equiv AA_a, BY\equiv BB_b$ and $CZ\equiv CC_c$ concur. This concurrency point is not in the current edition of ETC.

TelvCohl

#3 Jan 7, 2015, 10:04 am • 1

My solution:

Let A^* be the reflection of A in BC .
Let O be the circumcenter of $\triangle ABC$.
Let $A'=BA_c\cap CA_b, B'=CB_a\cap AB_c, C'=AC_b\cap BC_a$.

Since A' is the isogonal conjugate of A^* WRT $\triangle ABC$.
so AA' is the isogonal conjugate of AD WRT $\angle BAC$. i.e. $O\in AA'$
Similarly, we can prove $O\in BB'$ and $O\in CC'$.

From Dual Desargue theorem we get $A(B,C;A_a,A_c)=A(C,B;A',A_b)$

$$\Longrightarrow \frac{\sin\angle BAA_a}{\sin\angle CAA_a}=\frac{\sin\angle CAO}{\sin\angle BAO}\cdot\frac{\sin\angle BAA_b}{\sin\angle CAA_b}\cdot\frac{\sin\angle BAA_c}{\sin\angle CAA_c}\text{ ... }(1)$$

Similarly, we can get following two expressions :

$$\frac{\sin\angle CBB_b}{\sin\angle ABB_b}=\frac{\sin\angle ABO}{\sin\angle CBO}\cdot\frac{\sin\angle CBB_c}{\sin\angle ABB_c}\cdot\frac{\sin\angle CBB_a}{\sin\angle ABB_a}\text{ ... }(2)$$

$$\frac{\sin\angle ACC_c}{\sin\angle BCC_c}=\frac{\sin\angle BCO}{\sin\angle ACO}\cdot\frac{\sin\angle ACC_a}{\sin\angle BCC_a}\cdot\frac{\sin\angle ACC_b}{\sin\angle BCC_b}\text{ ... }(3)$$

From $(1), (2), (3)$ we get

$$\frac{\sin\angle BAA_a}{\sin\angle CAA_a}\cdot\frac{\sin\angle CBB_b}{\sin\angle ABB_b}\cdot\frac{\sin\angle ACC_c}{\sin\angle BCC_c}=\frac{\sin\angle BAA_b}{\sin\angle CAA_c}\cdot\frac{\sin\angle CBB_c}{\sin\angle ABB_a}\cdot\frac{\sin\angle ACC_a}{\sin\angle BCC_b}$$

$$=\frac{|\cos(\angle BCA-\angle CAB)|}{|\cos(\angle ABC-\angle CAB)|}\cdot\frac{|\cos(\angle CAB-\angle ABC)|}{|\cos(\angle BCA-\angle ABC)|}\cdot\frac{|\cos(\angle ABC-\angle BCA)|}{|\cos(\angle CAB-\angle BCA)|}$$

$$=1$$

so from Ceva's theorem we get AA_a, BB_b, CC_c are concurrent .

Q.E.D

buratinogigle

#4 Jan 7, 2015, 1:32 pm

Thank you so much dear **Luis** and **Telv** for nice solution. I see the general problem is a property of Neuberg cubic as following

Let ABC be a triangle. P is a point such that Euler lines of triangles PBC, PCA, PAB are concurrent. AP, BP, CP cut BC, CA, AB at D, E, F, resp. A_b, A_c are reflection of D through AC, AB. BA_b cuts CA_c at A_a. Similarly we have B_b, C_c. Prove that AA_a, BB_b, CC_c are concurrent.

Quick Reply

High School Olympiads

Find the square - Nice construction

Reply

Seventh

#1 Jan 7, 2015, 1:18 am

Are given four points A, B, C, D on a same plane, and is known that A, B, C, D are on the sides XY, YW, WZ, ZX of a square $XYWZ$. Find $XYWZ$, knowing the points A, B, C, D.

Luis González

#2 Jan 7, 2015, 1:48 am • 1

In general, we can circumscribe a quadrilateral $XYWZ$ similar to another given quadrilateral.

As the angles $\angle XYW$ and $\angle WZY$ are known, then Y, Z lie on the circular arcs ω_1 and ω_2 (constructible) that see $\overline{AB}$ and $\overline{CD}$ under the referred angles. Now, since the angles $\angle XYZ \equiv AYZ$ and $\angle XZY \equiv DZY$ are known, then YZ cuts ω_1, ω_2 again at known points M, N. This line MN will cut ω_1, ω_2 at Y, Z and the rest is straightforward.

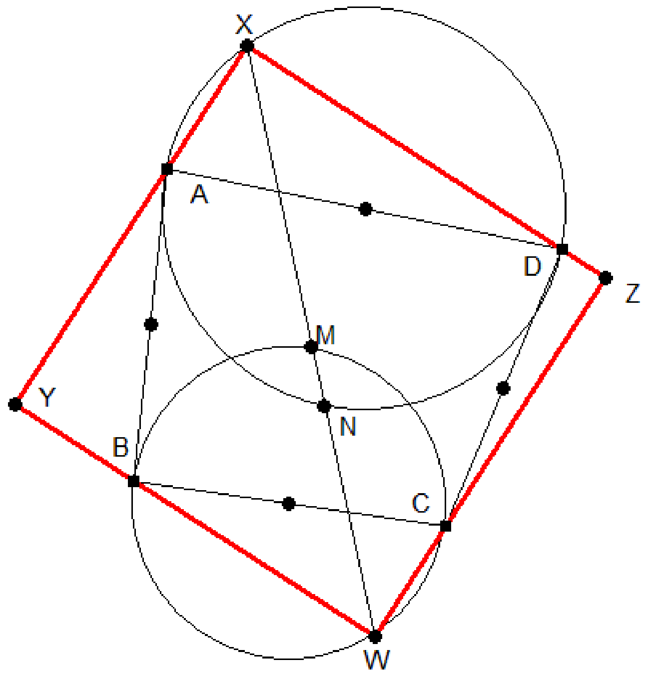
sunken rock

#3 Jan 7, 2015, 8:52 am • 1

Draw the circles of diameters BC, AD respectively and note M, N the midpoints of the inner semicircles, MN will 2nd time intersect the circles at X, W respectively. See the drawing.

Best regards,
sunken rock

Attachments:



Quick Reply

High School Olympiads

geometry with equation

Reply

junior2001

#1 Jan 7, 2015, 12:43 am

Let P be arbitrary point inside the triangle ABC, w_A, w_B and w_C be the circumcircles of triangles BPC, APC and APB respectively. Denote by X, Y, Z the intersection points of straight lines AP, BP, CP with circles w_A, w_B, w_C respectively.
Prove that $\frac{AP}{AX}+\frac{BP}{BY}+\frac{CP}{CZ}=1.$

Luis González

#2 Jan 7, 2015, 1:14 am

Invert with center P and arbitrary positive power; label inverse points with primes. Circles w_a, w_b, w_c go to lines $B'C', C'A', A'B'$ and X, Y, Z go to $X' \equiv PA' \cap B'C', Y' \equiv PB' \cap C'A', Z' \equiv PC' \cap A'B'$. Since $\tfrac{AP}{AX}=\tfrac{X'P}{X'A'}$ and cyclic expressions $\Longrightarrow$

$$\frac{AP}{AX}+\frac{BP}{BY}+\frac{CP}{CZ}=\frac{PX'}{A'X'}+\frac{PY'}{B'Y'}+\frac{PZ'}{C'Z'}=$$

$$=\frac{[PB'C']}{[A'B'C']}+\frac{[PC'A']}{[A'B'C']}+\frac{[PA'B']}{[A'B'C']}=\frac{[A'B'C']}{[A'B'C']}=1.$$

Quick Reply

High School Olympiads

Point M is on Euler 's circle (Feuerbach or nine-point)

Reply

tohoproirac

#1 Jan 7, 2015, 12:00 am

Let ABC be acute triangle inscribed (O). Line d passes O intersect $BC and AC$. L, K denote the feet of perpendicular from A, B to d respectively. d_1(passes L and $d_1 \perp BC$) intersect d_2 (passes K and $d_2 \perp AC$)
at M Prove that : Point M is on Euler 's circle (Feuerbach or nine-point)

Luis González

#2 Jan 7, 2015, 12:32 am • 1

This is well-known. If a line d passes through the circumcenter O of $\triangle ABC,$ its orthopole M lies on the 9-point circle of $\triangle ABC.$ Alternatively, you can prove that M is in fact the intersection of the Simson lines of $d \cap (O).$

For a more general problem see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=498545

tohoproirac

#3 Jan 7, 2015, 7:43 am

> Luis González wrote:
> This is well-known. If a line d passes through the circumcenter O of $\triangle ABC,$ its orthopole M lies on the 9-point circle of $\triangle ABC.$ Alternatively, you can prove that M is in fact the intersection of the Simson lines of $d \cap (O).$
>
> For a more general problem see
> http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=498545

This isn't well-known for me :p
Could u explain it clearlier ? Tks

TelvCohl

#4 Jan 7, 2015, 8:30 am • 1

Dear **tohoproirac**
You can see Darij Grinberg, Generalization of the Feuerbach point (available at Darij Grinberg's website) .
Hope it will be helpful for you !

This post has been edited 1 time. Last edited by TelvCohl, Jul 3, 2015, 1:40 am

Quick Reply

High School Olympiads

Excircle

Reply

Source: Sample USAMO

KudouShinichi

#1 Jan 5, 2015, 10:26 pm

14. Let □ABC be a triangle, and let the excircle opposite A be tangent to BC, CA, and AB at D,E, and F, respectively. Let AD meet EF at P and intersect the excircle again at G. Let M be the midpoint of DG, and let Q be the second intersection point of the circumcircles of □MFB and □MEC. Prove that PQ is perpendicular to BC.

Luis González

#2 Jan 6, 2015, 1:38 am

Redefine Q as the projection of P on $BC.$ As $DEGF$ is harmonic, BC,EF and the tangent to the A-excircle at G concur at $S,$ thus from cyclic $PMSQ,$ we have $\widehat{PMQ}=\widehat{FSB}.$ Since FE and FM are the F-symmedian and F-median of $\triangle FDG,$ they are isogonals WRT $\widehat{DFG},$ thus it follows that $\triangle FMG \sim \triangle FDE \Longrightarrow \widehat{FMD}=\pi-\widehat{FDE}=\widehat{BFS}.$ Hence $\widehat{FMQ}=\widehat{PMQ}+\widehat{FMD}=\widehat{FSB}+\widehat{BFS}=\widehat{ABC} \Longrightarrow Q \in \odot(MFB)$ and similarly $Q \in \odot(MEC).$

P.S. See also Mock USAMO by DBR, et al. (Day 1 Problem #3)

TelvCohl

#3 Jan 6, 2015, 4:04 am

My solution:

Let I_a be $A-$ excenter of $\triangle ABC$.

Since $\angle BQM+\angle MQC=(180^\circ-\angle MFB)+(180^\circ-\angle CEM)=180^\circ$,
so we get Q lie on BC .
Since EF is the polar of A WRT $\odot(I_a)$,
so BC, EF, I_aM are concurrent at the pole X of AD WRT $\odot(I_a)$.
Since $\angle MQX=90^\circ-\angle I_aEM=90^\circ-\angle I_aAM=\angle MI_aA=\angle MPX$,
so we get M, P, Q, X are concyclic and $\angle PQX=90^\circ$. i.e. $PQ \perp BC$

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Jan 6, 2015, 7:08 am

PROF65

#4 Jan 6, 2015, 5:04 am

dear **telvcohl**
I think you have taken the colinearity of B, C and Q for granted ,which is easy to prove

PROF65

#5 Jan 6, 2015, 7:56 pm

My solution is based on the fact that the circles (MPF) and (MPE) are tangent with the sides AB and AC resp. which is deduced from $(A,P;D,D')=-1$ i.e $AM.AP=AD.AD'=AF^2=AE^2$
then $\widehat{MQB}=\pi-\widehat{MFB}=\widehat{MPF}$ and $\widehat{MQC}=\pi-\widehat{MEC}=\widehat{MPE}$ therfore
$\widehat{MQB}+\widehat{MQC}=\widehat{MPF}+\widehat{MPE}=\pi$ i.e B,Q and C are colinear . Further let T be the intersection of BC and EF:
$\widehat{MQT}=\widehat{MQB}=\widehat{MPE}=\widehat{MPT}$ i.e $MPQT$ cyclic besides AD is the polar of T thus $PM \perp MT$ hence
$PQ \perp QT$ i.e $PQ \perp BC$

Quick Reply

High School Olympiads

A, D, E, F lie on a circle

Reply

Source: own

THVSH

#1 Jan 4, 2015, 10:00 pm

Let ABC be a triangle. P is an arbitrary point. AP intersects (ABC) at D. Q is an arbitrary point lying on (ABC). M is midpoint of AQ. QP intersects (ABC) again at K. A circle ω passes through P, K and tangent to AP. AK, MP intersect ω again at E, F, respectively. Prove that A, D, E, F lie on a circle.

Luis González

#2 Jan 4, 2015, 10:31 pm • 2

Invert with center P and power $PA \cdot PD = PK \cdot PQ$. ω goes to a parallel from Q to AD cutting PE and PM at the inverses E', F' of E, F. Since M is the midpoint of $\overline{AQ}$, it follows that $PAF'Q$ is parallelogram $\Longrightarrow AF' \parallel KQ$. Since P, Q, D, E' are concyclic on the inverse of AK, we have $\angle(F'E', F'A) = \angle(QE', QP) = \angle(DE', DA) \Longrightarrow$ A, D, E', F' are concyclic $\Longrightarrow A, D, E, F$ are also concyclic.

TelvCohl

#3 Jan 5, 2015, 9:00 am

My solution:

Let P' be the reflection of P in M and $R = EP \cap QP'$.

Since $APQP'$ is a parallelogram ,
so $\angle P'AE = \angle PKE = 180^\circ - \angle EFP'$,
hence we get A, E, F, P' are concyclic . ... (1)

Since AP is tangent to ω at P ,
so from Reim theorem (for $R - P - E$ and $P' - P - F$) we get E, F, P', R are concyclic . ... (2)

Since $\angle RPD = \angle AKQ = \angle PDQ$,
so $PRQD$ is an isosceles trapezoid ,
hence from $AP' = PQ = DR$ we get $AP'RD$ is an isosceles trapezoid ,
so A, P', R, D are concyclic . ... (3)

From $(1), (2), (3)$ we get A, D, E, F, P', R are concyclic .

Q.E.D

Quick Reply

High School Olympiads

Good problem!

Reply

AdithyaBhaskar

#1 Jan 4, 2015, 4:27 pm

Let $ABCD$ be a quadrilateral. Let diagonals AC and BD meet at P. Let O_1 and O_2 be the circumcenters of APD and BPC, respectively. Let M, N and O be the midpoints of AC, BD and O_1O_2, respectively. Show that O is the circumcenter of MPN.

TelvCohl

#2 Jan 4, 2015, 4:37 pm • 1

My solution:

Let $\mathcal{P}(Z, \odot)$ be the power of Z WRT $\odot$.

Since $\frac{\mathcal{P}(M, \odot(O_1))}{\mathcal{P}(M, \odot(O_2))} = \frac{MA}{MC} = -1, \frac{\mathcal{P}(N, \odot(O_1))}{\mathcal{P}(N, \odot(O_2))} = \frac{ND}{NB} = -1$,

so M, N both lie on the midcircle of $\{\odot(O_1), \odot(O_2)\}$ .ie. O is the center of $\odot(PMN)$

Q.E.D

Luis González

#3 Jan 4, 2015, 9:09 pm

Posted several times before. It still holds for points M,N and O dividing AC,BD and O1O2 in the same ratio.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=441718
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=514278
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=580130

PROF65

#4 Jan 4, 2015, 9:20 pm • 1

MY SOLUTION:
Let K the intersection of O_1 and O_2 (other than P if they have two common points). K is center of similitude that maps $A \to C, B \to D, O_1 \to O_2$ thus $BKD \sim AKC \sim O_1KO_2 \implies BKN \sim AKM \sim O_1KO$ .Let ϕ the similitude of center K that maps $A \to M$ then it maps $B \to C, O_1 \to O$ hence M, N and K are in the circle of center O.besides $\widehat{MKN} = \widehat{AKB} = \widehat{MPN}$ (Oriented angles) and the result follow

Quick Reply

High School Olympiads

Isosceles Trapezoid

Reply

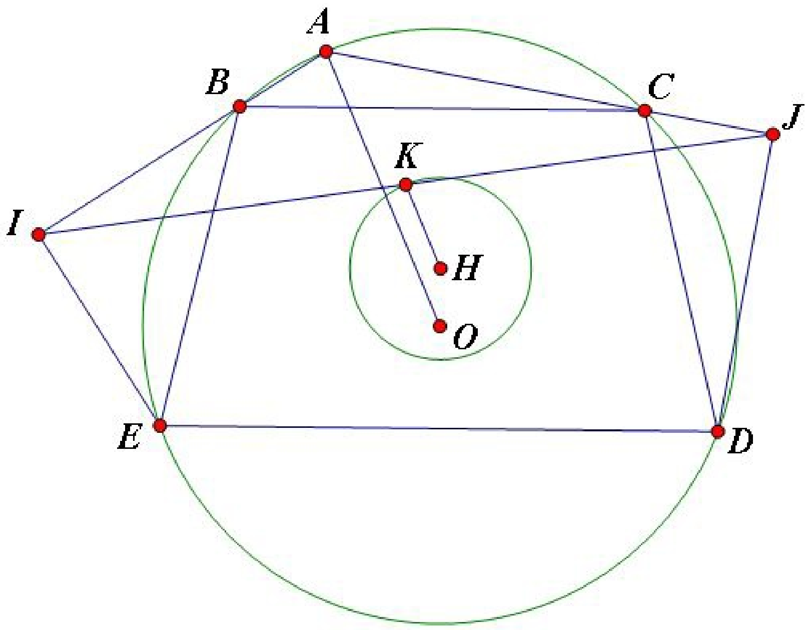
Source: Own

BeyondTheBoundary

#1 Jan 4, 2015, 10:08 am

Let $BCDE$ be an isosceles trapezoid inscribed in (O) with $CD = BE$. A is a point moving on (O). I, J are projections of E, D on line AB, AC, respectively. K is the midpoint of IJ. Prove that:
1. Whenever A moves, K always lies on a fixed circle with center H
2. $AO \parallel KH$

Attachments:



TelvCohl

#2 Jan 4, 2015, 12:14 pm

My solution:

Let X, Y, H be the midpoint of BE, CD, XY, respectively .
Let I', J' be two points satisfy $I'I \parallel JJ' \parallel XY, HI' \parallel XI, HJ' \parallel YJ$.

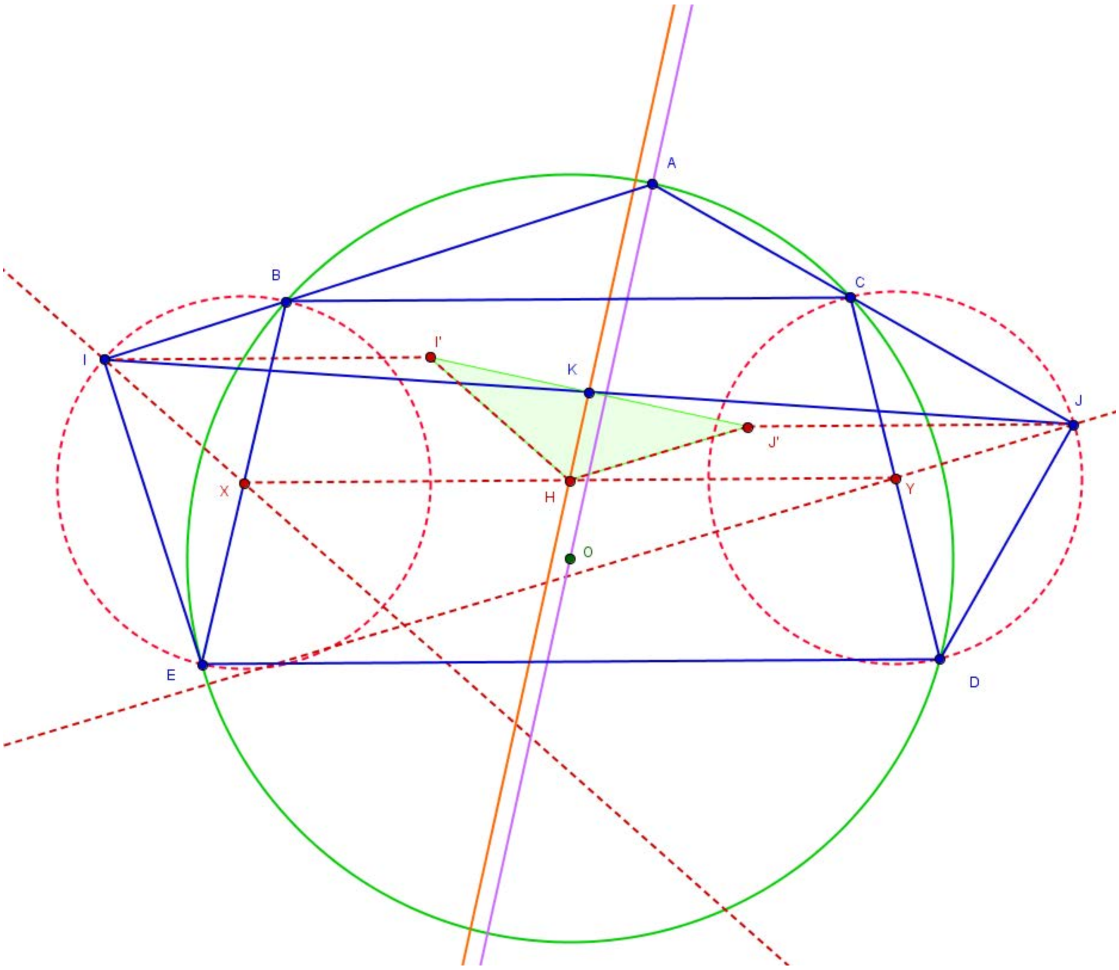
Since $\angle J'HI' = 360^\circ - \angle BAC - \angle XIB - \angle CJY$
$= 360^\circ - \angle BAC - (180^\circ - \angle EBA) - (180^\circ - \angle ACD)$
$= \angle EBA + \angle ACD - \angle BAC = 180^\circ + 2\angle EBC - 2\angle BAC = \text{const.}$,
so combine with $HI' = XI = \frac{1}{2}BE = \frac{1}{2}CD = YJ = HJ'$ we get all $\triangle HI'J'$ are congruent

Since $II'JJ'$ is a parallelogram ,
so K is also the midpoint of $I'J'$,
hence the distance between H and K is fixed when A moves .
ie. the locus of K is a circle with center H when A varies on (O)

Since $\angle(OA, XI) = 180^\circ - \angle XIB - \angle BAO$
$= \angle EBA - (90^\circ - \angle ACB) = \angle EBC + 90^\circ - \angle BAC = \angle KHI'$,
so from $I'H \parallel XI$ we get $AO \parallel KH$.

Q.E.D

Attachments:



![avatar]

Luis González

#3 Jan 4, 2015, 12:24 pm

It's a particular case of the configuration discussed at Vietnam NMO 2000_2 and All Russian olympiad 1961.

As A moves on $(O), I, J$ move on circles $(U), (V)$ with diameters $\overline{BE}, \overline{CD}$, all with the same speed and direction. Thus, the midpoint K of $\overline{IJ}$ moves on a circle centered at the midpoint H of $\overline{UV}$ with the same speed and direction. To prove that $OA \parallel HK$, it suffices to show that they are either parallel or coincide at one time and this is obvious when A coincides with the midpoint of the arc BC of (O), i.e. O, H, K, A are collinear on the symmetry axis of the trapezoid.

Quick Reply

High School Olympiads

A well-known geometry problem?

Reply

Tung-CHL

#1 Jan 4, 2015, 10:20 am

For triangle $\triangle ABC$, altitudes AD, BE, CF. $(X), (Y), (Z)$ are incircle of triangle AEF, BDF, CED. Let d_1 be common external tangent of $(Y), (Z), d_1$ isn't BC. d_2, d_3 are defined cyclically. Prove that d_1, d_2, d_3 concur.

Moderator says: Problem LaTeXed and edited to make it clearer

Luis González

#2 Jan 4, 2015, 10:40 am

Tung-CHL, next time please give your posts meaningful subjects and try to use $\LaTeX$ to make your post more comprehensive. Just type equations sandwiched between dollar signs.

As for the problem, it was posted before, see the thread tangent and orthocenter. These common tangents concur at the orthocenter of the intouch triangle of ABC.

Quick Reply

High School Olympiads

It's Nice result

Reply

math1200

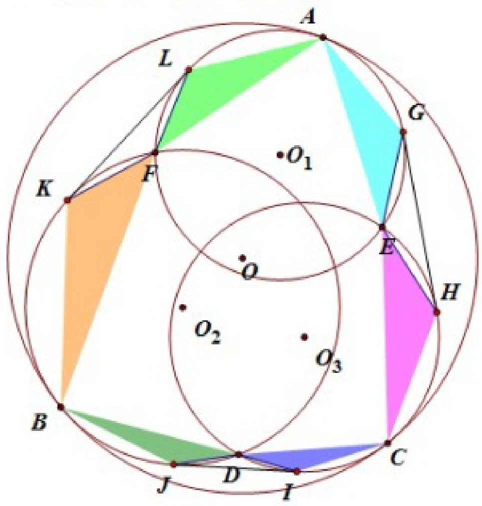
#1 Jan 3, 2015, 8:09 pm

Let $\odot O_1, \odot O_2$ and $\odot O_3$ internally tangent to $\odot O$ at A, B and C respectively and mutually intersecting at D, E, F respectively. (As shown in Figure)

Assume that GH, IJ, KL is external common tangenst of $\odot O_1, \odot O_2$ and $\odot O_3$ respectively. Show that

$$\frac{S_{FLA}}{S_{AGE}} \cdot \frac{S_{EHC}}{S_{CID}} \cdot \frac{S_{DJB}}{S_{BKF}} = 1$$

Attachments:



Luis González

#2 Jan 4, 2015, 12:15 am

Since A and B are the exsimilicenters of $(O_1) \sim (O)$ and $(O_2) \sim (O)$, then by Monge & d'Alembert theorem, $U \equiv AB \cap KL$ is the exsimilicenter of $(O_1) \sim (O_2)$, which is also their center of direct inversion $\Longrightarrow AL, BK$ are antiparallel WRT UA, UL and UF touches $\odot(FKL)$ and $\odot(FAB)$. This yields

$$\frac{[FLA]}{[FKB]} = \frac{AL \cdot AF \cdot \sin \widehat{KLF}}{BK \cdot BF \cdot \sin \widehat{LKF}} = \frac{AL}{BK} \cdot \frac{AF}{BF} \cdot \frac{KF}{LF} =$$

$$= \frac{UL}{UB} \cdot \frac{UF}{UB} \cdot \frac{UK}{UF} = \frac{UL \cdot UK}{UB^2}.$$

Similarly, we get $\frac{[FKB]}{[FLA]} = \frac{UL \cdot UK}{UA^2} \Longrightarrow \frac{[FLA]}{[FKB]} = \frac{UA}{UB}$.

Let $V \in BC$ and $W \in CA$ denote the exsimilicenters of $(O_2) \sim (O_3)$ and $(O_3) \sim (O_1)$. Multiplying the cyclic expressions together gives

$$\frac{[FLA]}{[FKB]} \cdot \frac{[DJB]}{[DIC]} \cdot \frac{[EHC]}{[EGA]} = \frac{UA}{UB} \cdot \frac{VB}{VC} \cdot \frac{WC}{WA}.$$

$\overline{U, V, W}$ are collinear on the positive homothety axis of $(O_1), (O_2), (O_3)$, thus by Menelaus' theorem for $\triangle ABC$, cut by $\overline{UVW}$, the RHS of the latter expression equals 1, as desired.

Quick Reply

High School Olympiads

Collinearity with Symmedian Point

Source: ELMO 2014 Shortlist G1, by Sammy Luo

v_Enhance

#1 Jul 24, 2014, 7:34 pm • 1

Let ABC be a triangle with symmedian point K. Select a point A_1 on line BC such that the lines AB, AC, A_1K and BC are the sides of a cyclic quadrilateral. Define B_1 and C_1 similarly. Prove that A_1, B_1, and C_1 are collinear.

Proposed by Sammy Luo

thkim1011

#2 Jul 24, 2014, 9:21 pm

Proof?

v_Enhance

#3 Jul 24, 2014, 9:45 pm • 1

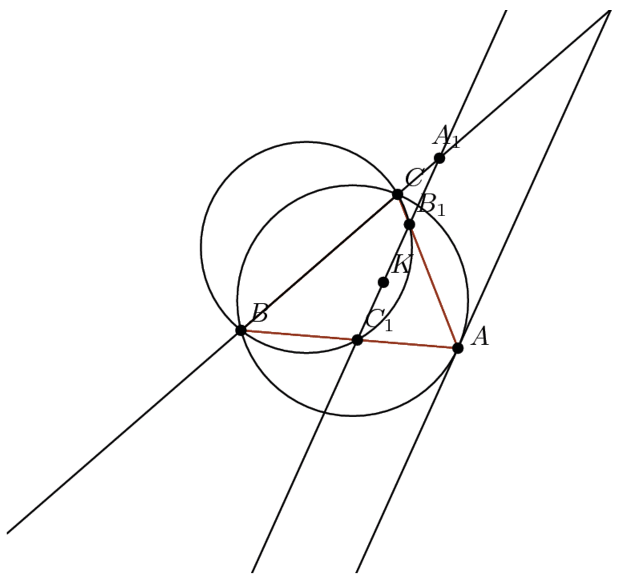
> *thkim1011 wrote:*
> It's pretty easy to see that B_1C_1BC is cyclic.

This seems false?

In particular it's not possible for A_1, B_1, C_1 to lie on a line parallel to the tangent at A, because then the same would be true of the tangents at B and C.

thkim1011

#4 Jul 24, 2014, 10:06 pm



It is true for B and C too (doing the same thing with B and C gives you 6 more points). It has to be cyclic from the alternate segment theorem segment theorem.

v_Enhance

#5 Jul 25, 2014, 1:14 am • 1

I think we're not doing the same problem. B_1 is defined as the point on line CA such that BA, BC, B_1K, CA are the sides of a cyclic quadrilateral.

thkim1011

#6 Jul 25, 2014, 3:07 am

I'm still not understanding. I'm pretty sure the sides mentioned form a cyclic quadrilateral. Can you post a diagram?

XmL

#7 Jul 25, 2014, 9:02 am

The construction of point X_1 is the intersection of its opposite side with the line through K perpendicular to XO where $X = A, B, C$ and O is the circumcenter of ABC. Let $A_1K \cap AB, AC = A_C, A_B$, similarly define B_A, B_C, C_A, C_B. It's well known that these points lie on the Lemoine circle with center K.

Now A_1, B_1, C_1 are collinear $\iff$ their polars wrt (K) concur. Let $A_BB_A \cap A_CC_A = A_2$, similarly define B_2, C_2, where by the obvious parallels $A_2B_2C_2$ is homothetic(congruent as well) with ABC. Moreover, the construction of the polar of X wrt (K) is the line through X_2 perpendicular to X_1K or parallel to XO. By the correspondence of the homothety the polars concur at the circumcenter of $A_2B_2C_2$ and we done.

v_Enhance

#8 Jul 26, 2014, 10:12 pm • 1

> *thkim1011 wrote:*
> I'm still not understanding. I'm pretty sure the sides mentioned form a cyclic quadrilateral. Can you post a diagram?

Oof yeah, I managed to break the problem while editing it. The original wording was something like

> *Quote:*
> The anti-parallel of BC through K wrt AB and AC meets BC again at A_1. Then define B_1 and C_1 similarly.

Sorry about that.

thkim1011

#9 Jul 27, 2014, 1:00 am

Construct the tangents to the circumcircle of ABC at A, B, C and let it intersect at the three points A_3, B_3, C_3 where each point is opposite of their corresponding point. Apply pascals on degenerate hexagon $AABBCC$. This creates a new set of 3 collinear points A_2, B_2, C_2. We construct the antiparallels by constructing the parallels to the three tangent. We note that there must be a homothety from B mapping two antiparallels to two tangents (this is true because K lies on the line connecting B and the intersection of the tangents at A and C. Thus A_1C_1 is parallel to A_2C_2. Similarly, we can find that A_1B_1 is parallel to A_2B_2 and B_1C_1 is parallel to B_2C_2. Thus A_1, B_1, and C_1 are collinear.

61plus

#10 Jul 27, 2014, 2:17 am • 1

Another finish from xml's lemma that 6 points lying on the lemoine circle centered at K
note that A_1, B_1, C_1 have same power to the lemoine circle and circumcircle of ABC, so they must be on the radical axis of the two circles, thus collinear

thecmd999

#11 Oct 23, 2014, 11:50 am

Solution

IDMasterz

#12 Jan 1, 2015, 5:17 pm • 1

Let AO meet the line through K at A_2. Easy to see that under inversion about the second lemoine circle and a flip accros the lemoine point that $A_1 \mapsto A_2$. Hence, the circles with diametres AA_1 have common radical axis HK. So, $A_1B_1C_1$ are collinear. But, furthermore, we have HK is perpendicular to the isotomic line of $A_1B_1C_1$, and even more $A_1B_1C_1$ is parallel to the polar line of O wrt K. With all this, we appear to have some sort of mapping between isotomic lines; interesting

IDMasterz

#13 Jan 1, 2015, 10:48 pm

Here is a generalisation found with Telv

jlammy

#14 Jan 1, 2015, 10:50 pm

The requested blog entry does not exist

nsato

#15 Jan 2, 2015, 8:52 pm

What points other than K have the given property? Experimentation suggests that the locus is the Jerabek hyperbola.

IDMasterz

#16 Jan 3, 2015, 12:01 am • 2

Oops! Yes it is. But more generally, here is the result found by Telv and me;

Let ABC be a triangle. Let P, Q be two points on a rectangular circum-hyperbola $\mathcal{H}$. Let the perpendicular from P to AQ meet BC at A_1 and define B_1, C_1 similarly. Then $A_1B_1C_1$ is a line.

Found with Telv Cohl

Proof: Suppose BQ meets the perpendicular to CQ from P at Y and define X similarly for point on CQ. Of course, $XY \perp PQ$. Now, let the orthocentre of BQC be H^*, and note $H^* \in \mathcal{H}$. We have

$$(\infty, P; X, B_1) = (H^*, P; Q, A) = (\infty, P; Y, C_1) \implies XY \parallel B_1C_1$$

$$\implies B_1C_1 \perp PQ.$$

So, $A_1B_1C_1 \perp PQ$.

Luis González

#17 Jan 3, 2015, 11:39 am • 3

Another proof to the previous generalization:

As $\mathcal{H}$ is rectangular, the orthocenters H_B and H_C of $\triangle BPQ$ and $\triangle CPQ$ are the second intersections of $\mathcal{H}$ with PB_1 and PC_1, respectively. By Pascal theorem for PH_CCABH_B, the intersections $C_1 \equiv PH_C \cap AB, B_1 \equiv PH_B \cap AC$ and $CH_C \cap BH_B$ (point at infinity of $\perp PQ$) are collinear $\Longrightarrow (B_1C_1 \parallel BH_B \parallel CH_C) \perp PQ$. Hence, we conclude that A_1, B_1, C_1 lie on a perpendicular to PQ.

rodinos

#18 Jan 4, 2015, 12:40 am

Variation of the generalization:

Let ABC be a triangle and P,P* two isogonal conjugate points.

A1 = (Perpendicular to AP* through P) /\ BC
Similarly B1, C1.

Which is the locus of P such that A1,B1,C1 are collinear?

The locus is the McCay cubic + Circumcircle + line at infinity

How about a synthetic proof? ie

Let P,P* be two isogonal conjugate points collinear with O.
Let A1 = (Perpendicular to AP* through P) /\ BC
Similarly B1, C1.
To prove: A1,B1,C1 are collinear.

See also another locus related to circunormal triangle by Cesar Lozada:
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/22977

TelvCohl
#19 Jan 4, 2015, 12:57 am • 1

> *rodinos wrote:*
> Let P,P* be two isogonal conjugate points collinear with O.
> Let A1 = (Perpendicular to AP* through P) /\ BC
> Similarly B1, C1.
> To prove: A1,B1,C1 are collinear.

Since P, P^*, O are collinear ,
so the rectangular circum-hyperbola $\mathcal{H}$ passing through P also contain P^* ,
hence from the generalization above we get A_1, B_1, C_1 are collinear .

rodinos
#20 Jan 4, 2015, 2:40 am

Well... that's the problem: if P,P*,O are collinear then P* lies on the r. c/hyperbola through P :)

The isogonal conjugate of the line PP*O contains the points P*,(P*)*, O* = P*, P, H : on the r. c/hyperbola through P

daothanhoai
#21 May 12, 2015, 1:18 pm

Please see:

https://groups.yahoo.com/neo/groups/AdvancedPlaneGeometry/conversations/messages/1327

Quick Reply

High School Olympiads

the line pass through midpoint of segment

phuong

#1 Jan 1, 2015, 3:01 pm

Let ABC be a triangle and AH is the altitude. Circle with diameter AD cuts AB, AC at X, Y respectively. The tangents at E, F of the circle with diameter AD cut BC at E, F respectively. BY intersects CX at P, EY cuts FX at Q. Prove that the line PQ pass through midpoint of BC

TelvCohl

#2 Jan 1, 2015, 3:53 pm

> phuong wrote:
> Let ABC be a triangle and AD is the altitude. Circle with diameter AD cuts AB, AC at X, Y respectively. The tangents at X, Y of the circle with diameter AD cut BC at E, F respectively. BY intersects CX at P, EY cuts FX at Q. Prove that the line PQ pass through midpoint of BC

Typo corrected :)

My solution:

Let $X' = FX \cap (AD), Y' = EY \cap (AD)$ and M be the midpoint of BC .

Since $\angle BXD = 90^\circ$,
so from $EX = ED$ we get E is the midpoint of BD .
Similarly, we can prove F is the midpoint of CD .
Since $FC^2 = FX \cdot FX', EB^2 = EY \cdot EY'$,
so BC is the common tangent of $\odot(BYY')$ and $\odot(CXX')$.

Since $MB = MC$,
so M lie on the radical axis of $\{\odot(BYY'), \odot(CXX')\}$ . ... (1)
Since $XQ \cdot QX' = YQ \cdot QY'$,
so Q lie on the radical axis of $\{\odot(BYY'), \odot(CXX')\}$ . ... (2)
Since $\angle AYX = \angle 90^\circ - \angle XDB = \angle CBA$,
so we get B, C, X, Y are concyclic and $BP \cdot PY = CP \cdot PX$,
hence P lie on the radical axis of $\{\odot(BYY'), \odot(CXX')\}$ . ... (3)

From $(1), (2), (3)$ we get P, Q, M are collinear .

Q.E.D

P.S

We can also use following property of quadrilateral (well-known) to prove this problem:

Let $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ be two quadrilateral .
Let $U = A_1B_1 \cap A_2B_2, V = B_1C_1 \cap B_2C_2, W = C_1D_1 \cap C_2D_2$.
Let $X = D_1A_1 \cap D_2A_2, Y = A_1C_1 \cap A_2C_2, Z = B_1D_1 \cap B_2D_2$.

If U, V, W, X, Y all lie on line ℓ , then $Z \in \ell$.

From this property (for $PXQY$ and $AYZX$ $(Z = XE \cap YF$)) we can get the conclusion :) .

Luis González

#3 Jan 2, 2015, 12:18 pm

Denote $M \equiv PQ \cap BC.$ Under the homology that sends $P \mapsto Q,$ fixing the line XY and the line pencil through $M,$ we have $B \mapsto E$ and $C \mapsto F.$ Therefore $A \equiv BX \cap CY$ goes to $R \equiv EX \cap FY \Longrightarrow M \in AR.$ But since XY is antiparallel to BC WRT $AB, AC,$ then it follows that the A-symmedian AR of $\triangle AXY$ becomes A-median of $\triangle ABC \Longrightarrow M$ is the midpoint of $\overline{BC}.$

Quick Reply

High School Olympiads

A short probem but very nice

Reply

Source: Test 2015

kingmathvn

#1 Jan 2, 2015, 9:57 am

Let O be the center of (ABCD). E be a point change on AB. DE meets BC at F. DE meets (O) again at P. BP meets AF at Q. Prove that QE always passes through a fixed point.

Luis González

#2 Jan 2, 2015, 10:45 am • 1

As E varies, the pencils $DP \equiv DF$ and BP are proyective $\Longrightarrow$ pencils BP and AF are projective $\Longrightarrow$ Q describes a conic $\mathcal{K}$ that passes through $A, B.$ It's easy to check that it is tangent to $AD,$ since A goes to $AD \cap BC,$ and it passes through C for $E \equiv AB \cap CD.$ Now, since $E \mapsto Q$ has fixed points at A and $B,$ then it is a stereographic projection of $\mathcal{K}$ onto $AB \Longrightarrow$ all lines QE go through the 2nd intersection V of $\mathcal{K}$ with $CD.$

P.S. The problem was posted before at V is a fixed point (own), where you can see more elementary solutions.

Quick Reply

High School Olympiads

MN goes through fixed point

Source: Iran TST 2014, second exam, day 1, problem 2

TheOverlord
#1 Jan 2, 2015, 2:24 am

Point D is an arbitary point on side BC of triangle ABC. I,I_1 andI_2 are the incenters of triangles ABC,ABD and ACD respectively. $M\neq A$ and $N\neq A$ are the intersections of circumcircle of triangle ABC and circumcircles of triangles IAI_1 and IAI_2 respectively. Prove that regardless of point D, line MN goes through a fixed point.

Luis González
#2 Jan 2, 2015, 2:57 am

Let O_1 and O_2 denote the centers of $\odot(AII_1)$ and $\odot(AII_2)$. Then $\angle AI_1I=\angle ADI_2=90^\circ-\frac{1}{2}\angle ADB$ and similarly $\angle AI_2I=\angle ADI_1$. Hence $\angle IAO_1=90^\circ-\angle AI_1I=90^\circ-\angle ADI_2=\angle AI_2I \Longrightarrow AO_1$ is tangent to $\odot(AII_2) \Longrightarrow$ $\odot(AII_1)$ and $\odot(AII_2)$ are orthogonal. Now according to the problem Exsimilar center, all lines MN go through the exsimilicenter of the circumcircle (O) and incircle (I) of $\triangle ABC,$ obviously fixed.

TelvCohl
#3 Jan 2, 2015, 3:49 am

My solution:

Let I, O be the incenter, circumcenter of $\triangle ABC$, respectively .

From incenter of triangle we get M, N are the tangent point of the Thebault circles of the cevian AD ,
so from Mabey Thebault have oveloоked we get MN pass through the exsimilicenter S of $(I) \sim (O)$.
ie. MN pass through a fixed point S

Q.E.D

andria
#4 Jun 19, 2015, 3:16 pm

My solution:

Note that $\angle AI_2I+\angle AI_1I=(\frac{\angle B}{2}+\frac{\angle BAD}{2})+(\frac{\angle C}{2}+\frac{\angle CAD}{2})=\frac{A+B+C}{2}=90$ so $\odot(\triangle AI_1I),\odot(\triangle AI_2I)$ are orthogonal now apply an inversion with center A and radius $\sqrt{AB.AC}$ and then reflection throw the bisector of $\angle A$ note that $I\longleftrightarrow I_A,B\longleftrightarrow C$ under the inversion and $\odot(\triangle AI_1I),\odot(\triangle AI_2I)$ are taken to two perpendicular lines throw I_A and M,N are taken to intersections of these two perpendicular lines with BC(call them M',N') let X projection of I_A on BC and note that $XM'.XN'=I_AX^2$ is fixed this means that power of X WRT $\odot(\triangle AM'N')$ is fixed so all of the circles $\odot(AM'N')$ are coaxal so MN passes throw the fixed point
DONE

This post has been edited 2 times. Last edited by andria, Jun 20, 2015, 2:13 am

High School Olympiads

First Pappus circle

Reply

jayme

#1 Dec 31, 2014, 5:47 pm

Dear Mathlinkers,

propose a construction without inversion of the first Pappus circle which is tangent tangent to the three circles of an Arbelos.

Sincerely
Happy New Year
Jean-Louis

TelvCohl

#2 Dec 31, 2014, 6:22 pm • 1

My solution:

Assume $\odot(O_2)$ and $\odot(O_3)$ are external tangent at P and internal tangent to $\odot(O_1)$ at A, B .

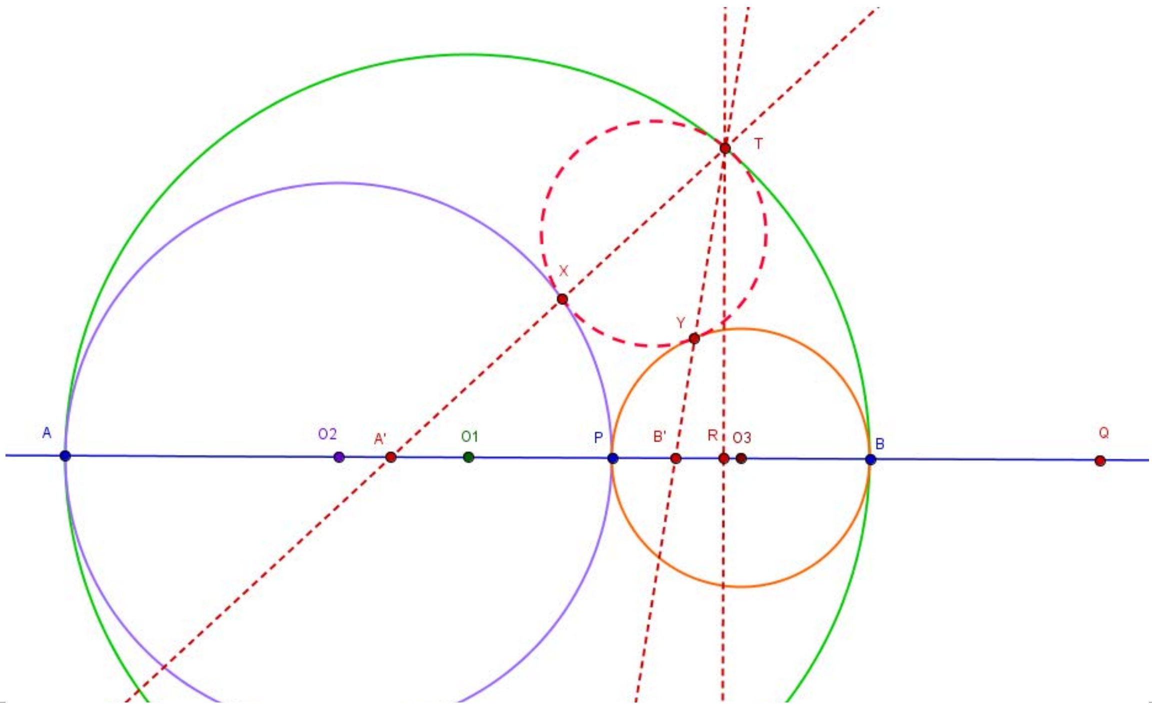
Construction:

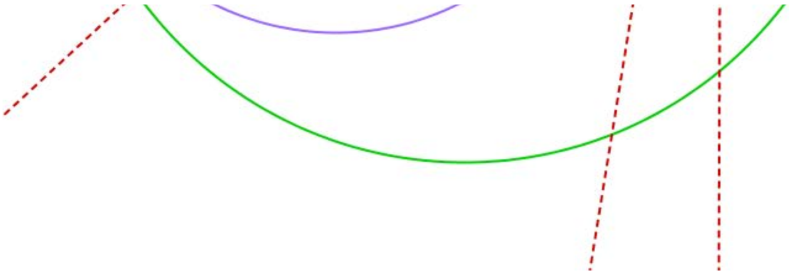
(1) Construct $Q \in \overline{O_1O_2O_3}$ satisfy $(O_2, O_3; P, Q) = -1$
(2) Construct $R \in \overline{O_1O_2O_3}$ satisfy $(A, B; R, Q) = -1$
(3) Construct $T \in \odot(O_1)$ satisfy $TR \perp \overline{O_1O_2O_3}$
(4) Construct $A' \in \overline{O_1O_2O_3}$ satisfy $(A, A'; O_2, O_1) = -1$
(5) Construct $B' \in \overline{O_1O_2O_3}$ satisfy $(B, B'; O_3, O_1) = -1$
(6) Construct $X = TA' \cap \odot(O_2)$ and $Y = TB' \cap \odot(O_3)$

Then $\odot(TXY)$ is the first Pappus circle

Q.E.D

Attachments:





jayme
#3 Dec 31, 2014, 6:29 pm

Dear Mathlinkers,
thank for this first proof...

and now without harmonic division... for beginners...

Sincerely
Jean-Louis

jlammy
#4 Dec 31, 2014, 7:28 pm

Unless I am wrong, is this not the Problem of Appolonius, as considered by Soddy?

jayme
#5 Dec 31, 2014, 8:02 pm

Dear jlammy and Mathlinkers,
yes this is also a point of view... after my lecture Archimedes must have being considering this circle in lemme 6 if I am not wrong...
By considering the chain of circle initiated by Pappus, it has this name...

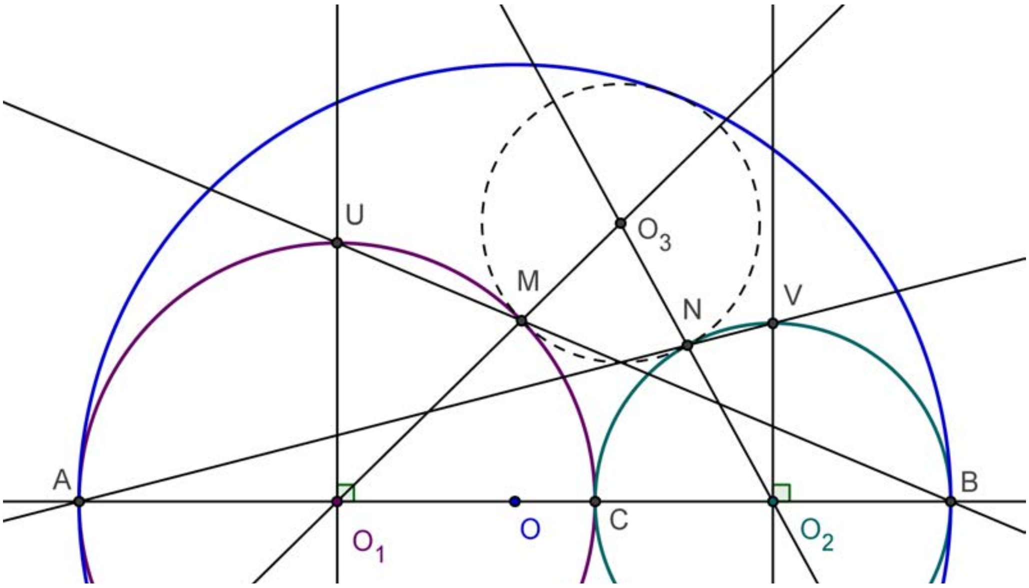
Point of view...

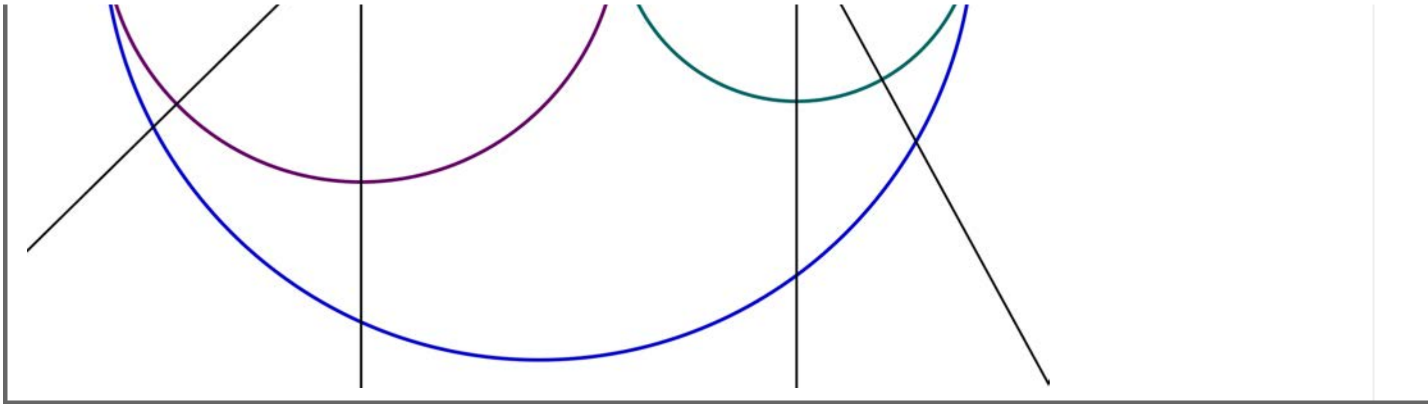
Sincerely
Jean-Louis

Luis González
#6 Jan 1, 2015, 12:56 am

More straightforward: Let $C \in \overline{AB}$ and label $(O), (O_1), (O_2)$ the circles with diameters $\overline{AB}, \overline{AC}, \overline{BC}.$ Construct midpoints U, V of the arcs AC, BC of $(O_1), (O_2). BU$ cuts (O_1) again at M and AV cuts (O_2) again $N.$ Then $O_3 \equiv MO_1 \cap NO_2$ $\Longrightarrow \odot(O_3, O_3M)$ touches $(O), (O_1), (O_2).$

Attachments:





Quick Reply

High School Olympiads

Concurrent on (O)

Source: HSGS TST 2014

buratinogigle
#1 Dec 29, 2014, 10:18 pm

Let ABC be a triangle isncribed (O), incenter I, median AM. E,F lie on CA,AB such that $ME\perp IC,MF\perp IB$. (MEF) cuts BC again at D. S is midpoint of arc BC contain A of (O). The line passing through S parrallel to OI cuts the line passing through D perpendicular to BC at T. K,L are symmetric of T through E,F. Prove that CK,BL and ST are concurrent on (O).

TelvCohl
#2 Dec 30, 2014, 10:35 am • 2

My solution:

Let O' be the reflection of O in I .
Let B' be the reflection of B in F .
Let C' be the reflection of C in E .
Let $X_1=TS \cap BL , X_2=TS \cap CK$.

Since $B'B=BC=CC'$,
so we get $S \in \odot (AB'C')$ (see hard problem (#post 11)) .
Since IB, IC is the perpendicular bisector of MF, ME, respectively ,
so we get I is the center of $\odot (MEF)$ and DT is the reflection of MS in I ,
hence the reflection O' of O in I lie on DT .

From hard problem (#post 9) we get $OI \perp B'C'$ and the length of OI is equal to the radius of $\odot (AB'C')$,
so combine with $OO' \parallel ST$ we get T is the midpoint of arc $B'C'$ in $\odot (AB'C')$ and ST is the diameter of $\odot (AB'C')$.

Easy to see $TB' \parallel LB$ and $TC' \parallel KC$.
Since $\angle BX_1S=180^{\circ}-\angle STB'=180^{\circ}-\angle SCB$,
so we get $X_1 \equiv TS \cap BL \in (O)$.
Similarly, we can prove $X_2 \equiv TS \cap CK \in (O)$,
so we get $X_1 \equiv X_2$ and TS, BL, CK are concurrent on (O) .

Q.E.D

buratinogigle
#3 Dec 30, 2014, 11:16 am • 1

Nice solution dear Telv Cohl, here is my solution

AI cuts (O) again at N. SN is diameter of (O). G is midpoint of DM and J is midpoint of ST. IG and JG are perpendicular to DM so I,J,G are collinear. This, $IJ\parallel SO$ and $JS\parallel IO$ deduce $IJSO$ is parallelogram. Therefore $ITJO$ is parallelogram, too, so $TI\parallel JO$. Easily seen $TN\parallel JO$. Therefore T,I,N are collinear or T lies on AI.

Let P is symmetric of B through F. We have $BP=2BF=2BM=BC$. From OBN and SBC are similar. We have $\frac{BP}{BS}=\frac{BC}{BS}=\frac{BN}{BO}=\frac{NI}{NO}=\frac{NT}{NS}$. Easily seen $\angle PBS=\angle TNS$ thus PBS and TNS are similar. Deduce $\angle BPS=\angle NTS$ thus $ATPS$ is cyclic. ST cuts (O) again at R. We have angle charsing $\angle ABL=\angle TPF=\angle TSA=\angle ABR$. Thus BL passe through R. Similarly CK passes through R. We are done.

Attachments:
Figure2588.pdf (13kb)

Luis González

#4 Dec 30, 2014, 12:11 pm • 1

Let Y, Z be the reflections of C, B on E, F and let R be the midpoint of the arc BC of (O). Clearly $BZ = BC = CY$ and S is the center of the rotation taking CY into BZ, thus $A \in \odot(SYZ) \Longrightarrow AR$ cuts $\odot(SYZ)$ again at the midpoint T' of its arc YZ. Hence, the spiral similarity with center S, taking YZ into CB, takes T' into $R \Longrightarrow \triangle SRT' \sim \triangle SCY \Longrightarrow \frac{RS}{RT'} = \frac{CS}{CY} = \frac{CS}{BC} = \frac{OR}{RB} = \frac{OR}{RI} \Longrightarrow ST' \parallel OI \Longrightarrow I$ is midpoint of RT'. But since I is circumcenter of $\triangle MEF$, then M, D are symmetric across the perpendicular from I to $DM \Longrightarrow (DT' \parallel MR) \perp BC \Longrightarrow T \equiv T'$. Now if ST cuts (O) again at J, we have $\angle ACJ = \angle AST = \angle AYT \Longrightarrow YT \parallel CJ$, i.e. $J \in CK$ and similarly $J \in BL$.

toto1234567890

#5 Dec 31, 2014, 7:22 pm

Nice problem And by the way @buratinogigle what is HSGS TST 2014? Is HSGS a nation?

dibyo_99

#6 Jan 1, 2015, 12:42 pm • 1

Let B', C' on AB, AC such that $BB' = BC = CC'$. We know that S is the center of the spiral similarity taking BB' to CC'. Consequently, $ASB'C'$ is cyclic.

Now, since $IE = IM$ and $IM = IF$, note that I is the center of (MEF). Therefore, $IM = ID$, implying that I lies on the perpendicular bisector of MD. Since $STDM$ is a trapezium, the perpendicular bisector l of DM cuts ST at its midpoint. Let $P = SO \cap TI$. Then, lSO, meaning I is the midpoint of TP. Since $IOST$, O must be the midpoint of SP. Therefore, P is the midpoint of arc BC of (ABC).

Let $ST \cap (ABC) = X$. Note that $LBTB'$ is a parallelogram. Then,

$$\angle ABX = \angle ASX = \angle AST = \angle AB'T = 180^\circ - \angle ABL$$

So, L, B, X are collinear. Similarly, C, K, X are also collinear. Therefore, CK, BL, ST are all concurrent on (O).

Quick Reply

High School Olympiads

PQ bisects EF

Seventh
#1 Dec 30, 2014, 2:37 am

Let $\triangle ABC$ and its incircle ω touches BC, AC and AB at D, E and F, respectively. Let H be the orthocenter of $\triangle DEF$, and let P and Q be the projections of H in BC and AD, respectively. Prove that PQ bisects EF.

Luis González
#2 Dec 30, 2014, 4:44 am

Let Y,Z be the projections of H on $DE,DF,$ these are the 2nd intersections of DE,DF with the circle $\odot(HPDQ)$ with diameter $\overline{DH}.$ It's well-known that the tangents of this circle at Y,Z meet at the midpoint M of $\overline{EF}.$ Now, since $Q(P,Q,Y,Z)=H(P,Q,Y,Z)=D(P,Q,Y,Z) \equiv D(B,A,E,F)=-1,$ it follows that $QYPZ$ is harmonic $\Longrightarrow$ P,Q and M are collinear.

TelvCohl
#3 Dec 30, 2014, 4:47 am • 1

My solution:

Let $T=EH \cap FD, S=FH \cap DE$.
Let O,M be the midpoint of DH,EF, respectively .

Easy to see $D,H,P,Q,T,S \in (O)$.

Since AD is $D-$ symmedian of $\triangle DEF$,
so we get $H(P,Q;T,S)=D(D,A;F,E)=-1$.
ie. $PTQS$ is a harmonic quadrilateral ... $(\star)$

Since $Rt\triangle TEF \cap M \sim Rt\triangle TDH \cap O$,
so we get $\angle OTM=\angle DTE=90^{\circ}$.
ie. MT is tangent to (O) at T
Similarly, we can prove MS is tangent to (O) at S ,
so combine with $(\star)$ we get M,Q,P are collinear .
ie. PQ bisects EF

Q.E.D

armpist
#4 Dec 30, 2014, 6:22 am

Dear MLs,

There is an old problem that states:

line connecting perpendiculars from orthocenter onto internal
and external bisectors of A passes thru midpoint of BC.

In the problem above we have perpendiculars onto internal
and external A-symmedians with the same midpoint of BC property.

It is clear that it should work for all internal-external pairs.

And this knowledge will make it easier to approach the "Hard to approach it"

And this knowledge will make it easier to approach the "Hard to approach it" problem from Swiss test 2006:

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=89098

Happy New Year!

M.T.

TelvCohl
#5 Dec 30, 2014, 7:18 am • 1

> armpist wrote:
> It is clear that it should work for all internal-external pairs.

Yes :)
We can generalize the original problem as following:

Let $\triangle DEF$ be the cevian triangle of R WRT $\triangle ABC$.
Let H be the orthocenter of $\triangle DEF$.
Let P, Q be the projection of H on BC, AD, respectively .

Prove that PQ bisect EF

My proof is almost same as what I did above:

Let M be the midpoint of EF .
Let $T = EH \cap DF, S = FH \cap DE$.

Easy to see D, H, P, Q, T, S are concyclic (denote as $\mathcal{C}$) .

Since $H(P, Q; T, S) = D(P, A; F, E) = -1$,
so we get $PTQS$ is a harmonic quadrilateral ,
hence combine with MT, MS are the tangent of $\mathcal{C}$ we get $M \in PQ$.
ie. PQ bisect EF

Q.E.D

Quick Reply

High School Olympiads

Synthetic Geometry

Locked

AdithyaBhaskar
#1 Dec 29, 2014, 12:12 pm

Can anyone help me solve this problem?
The point E is the midpoint of the segment connecting the orthocenter of the scalene triangle ABC and the point A. The incircle of triangle ABC is tangent to AB and AC at points C0 and B0, respectively. Prove that point F, the point symmetric to point E with respect to line B0C0, lies on the line that passes through both the circumcenter and the incenter of triangle ABC.

[Mod: Do not double post: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=619235.]

Luis González
#2 Dec 30, 2014, 2:14 am

Posted before at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=481938
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=506190

For a generalization see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=506193

High School Olympiads

The Kosnita point lies on AU

Reply

THVSH

#1 Dec 29, 2014, 10:04 pm

Let ABC be a triangle. D is the feet of the altitude through A. Let E, F be reflection of D with respect to CA, AB. The circumcircles of the triangles ABE and ACF intersect at a point U. Prove that the Kosnita point of the triangle ABC lies on AU.

Luis González

#2 Dec 29, 2014, 11:10 pm

Let Y, Z be the projections of D on $AC, AB.$ Inversion WRT $\odot(A, AD)$ takes $\odot(BAE)$ and $\odot(CAF)$ into the lines EZ and $FY,$ resp $\Longrightarrow G \equiv FY \cap EZ$ is the inverse of $U \Longrightarrow G \in AU.$ But G is centroid of $\triangle DEF \Longrightarrow AGU$ is Euler line of $\triangle DEF$ passing through its orthocenter $L.$

Since $EL \parallel AB, FL \parallel AC$ and EF is antiparallel to BC WRT $AB, AC,$ then it follows that $\triangle ABC \sim \triangle LFE$ are inversely similar. LA is clearly L-cevian of the 9-point center of $\triangle LFE,$ thus if N is the 9-point center of $\triangle ABC,$ we get $\angle BAN = \angle FLA = \angle CAL \Longrightarrow AL \equiv AU$ is the isogonal of AN WRT $\angle BAC.$

IDMasterz

#3 Dec 29, 2014, 11:16 pm

invert about A and flip over angle bisector s.t. $B \mapsto C$. After some paralleograms, it becomes to prove the centroid of HBC lies on AN. Then, just use $A(H, G; N, O) = -1$.

TelvCohl

#4 Dec 30, 2014, 2:54 pm • 1

Here is another solution without inversion:

Let $T = BF \cap CE$ and O be the center of $\odot(ABC)$.
Let B', C' be the antipode of A in $\odot(ACF), \odot(ABE)$, respectively .

From symmetry we get BF, CE are the tangent of $\odot(A, AD)$,
so $\odot(A, AD)$ is the $T-$ excenter of $\triangle TBC$ . ... (α)
Since T is the isogonal conjugate of the reflection of A in BC WRT $\triangle ABC$,
so combine with (α) we get A, O, T are collinear witch is the bisector of $\angle CTB$.

Since $\angle TOC = 180^\circ - 2\angle CBA = \angle TBC$,
so we get O, T, B, C are concyclic . ... (β)

Since $\angle ACB' = \angle ABC' = 90^\circ$,
so BC', CB' is the bisector of $\angle TBC, \angle TCB$, respectively ,
hence from $AU \perp B'C', (\alpha), (\beta)$ we get AU pass through the center of $\odot(OBC)$ (see **(1)**).
ie. AU is $A-$ Kosnita line of $\triangle ABC$

Q.E.D

P.S I use following fact at **(1)** :

Let O, I, I_a be the circumcenter, incenter, $A-$ excenter of $\triangle ABC$, respectively .
Let E, F be the intersection of BI, CI, with AC, AB, respectively .

Then $OI_a \perp EF$

Quick Reply

High School Olympiads

Maximal area construction

Reply

Source: IMO 1967, Day 2, Problem 4

DPopov

#1 Oct 15, 2005, 1:45 am

Let $A_0B_0C_0$ and $A_1B_1C_1$ be any two acute-angled triangles. Consider all triangle ABC that are similar to $\triangle A_1B_1C_1$ (so that vertices A_1, B_1, C_1 correspond to vertices A, B, C, respectively) and circumscribed about triangle $A_0B_0C_0$ (where A_0 lieson BC, B_0 lies on CA, and C_0 lies on AB). Of all such possible triangles, determine the one with the largest area and construct it.

Luis González

#2 Dec 29, 2014, 2:44 am • 1

Since the angles $\angle B_0AC_0 = \angle B_1A_1C_1, \angle C_0BA_0 = \angle C_1B_1A_1$ and $\angle A_0CB_0 = \angle A_1C_1B_1$ are constant, then the circles $(X) \equiv \odot(B_0C_0A), (Y) \equiv \odot(C_0A_0B), (Z) \equiv \odot(A_0B_0C)$ are fixed. Their construction is straightforward.

As $\triangle ABC$ are all similar, then the area $[ABC]$ will attain its maximum when BC attains its maximal length. If N, L are the projections of Y, Z on BC, then from the right trapezoid $YZLN$, we get $YZ \ge NL = (NA_0 + LA_0) = \frac{1}{2}BC \Longrightarrow BC \le 2 \cdot YZ$ with equality iff $YZLN$ is rectangle, i.e. $YZ \parallel NL$. Thus, the desired triangle $\triangle ABC$ is that having its sides parallel to $\triangle XYZ$.

Quick Reply

High School Olympiads

a triangle ABC

Reply

calhanSPheiro2

#1 Dec 27, 2014, 3:41 am • 1

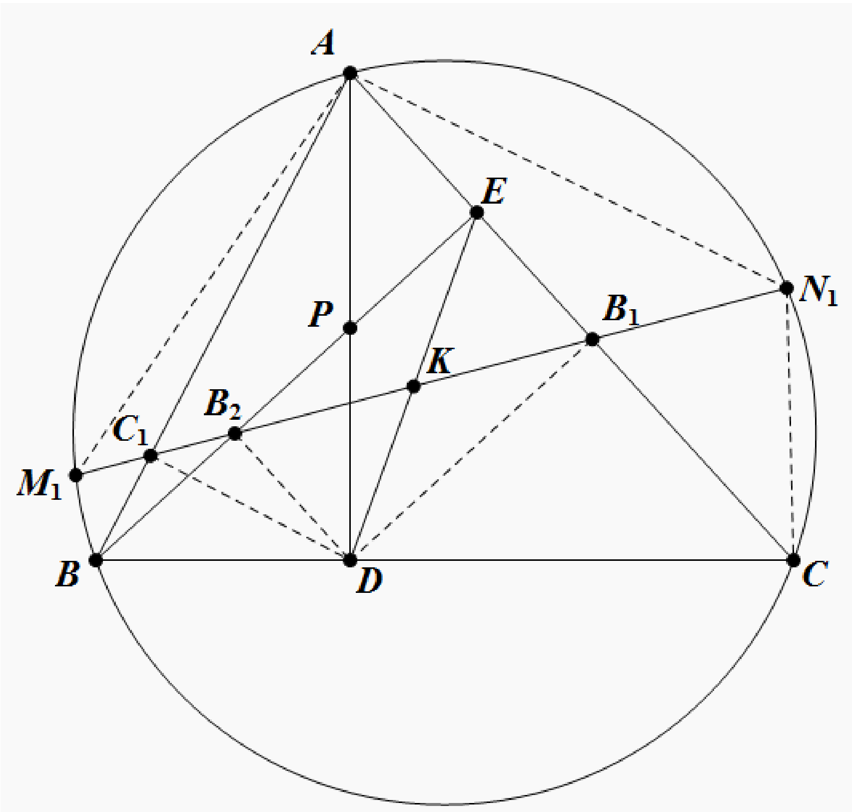
A triangle ABC (AB<AC) is inscribed (O) and AD,BE,CF is three altitudes of ABC. Let's (O') is a circle that pass through D and intersect (O) at M,N. MN intersects DE,DF at K,H. Prove K,H is midpoint of DE,DF.

Arab

#2 Dec 27, 2014, 7:31 am • 5

I think it should be 'circle O' is centered at A and passes through D'.

Denote by B_1, C_1 the perpendicular projections of D on CA, AB respectively. Let B_1C_1 intersect ω, the circumcircle of $\triangle ABC$, at M_1, N_1, with C_1 in between M_1, B_1. Since $DB_1 \perp CA, DC_1 \perp AB, AD \perp BC$, we have $AD^2 = AB_1 \cdot AC = AC_1 \cdot AB$, so B, C, B_1, C_1 are concyclic. Meanwhile, $\angle AB_1N_1 = 180^\circ - \angle AB_1C_1 = 180^\circ - \angle ABC = \angle AN_1C$, then we have $AN_1^2 = AB_1 \cdot AC = AD^2$, and hence $AN_1 = AD$. Similarly, $AM_1 = AD$, therefore $M_1 \equiv M, N_1 \equiv N$. Let BE intersect MN, AD at B_2, P respectively, then we obtain that $AD \perp BC, BE \perp CA \implies C, D, P, E$ are concyclic, so $\angle APE = \angle ACB = \angle AC_1B_1$, and hence A, P, B_2, C_1 are concyclic, therefore $\angle BB_2C_1 = \angle BAP = \angle BDC_1$, meaning that B, D, B_2, C_1 are concyclic, therefore $DB_2 \perp BE$. Since $DB_1 \perp CA, BE \perp CA$, we have $DB_1 \parallel BE$, and hence $DB_1 \perp DB_2$, then quadrilateral DB_1EB_2 is a rectangle, so B_1B_2 bisects DE. Consequently, MN bisects DE. Similarly, MN bisects DF, and we are done. $\blacksquare$



TelvCohl

#3 Dec 27, 2014, 2:35 pm • 2

My solution:

My solution:

Let O be the center of $\odot(ABC)$.
Let $T = AD \cap EF$ and X be the midpoint of DT.
Let $D' = AD \cap \odot(ABC)$ and D'' be the antipode of D in $\odot(A, AD)$.

Since $(D, T; H, A) = -1$,
so $XA \cdot HD = XA \cdot (XH + XD) = XD^2 + XA \cdot XD = XD \cdot AD$,
hence $XA \cdot XD' = XA \cdot HD + XD \cdot XA = XD \cdot AD + XD \cdot XA = XD \cdot XD''$.
ie. X lie on the radical axis MN of $\{\odot(A, AD), \odot(ABC)\}$ ... $(\star)$

Since MN is the radical axis of $\{\odot(A, AD), \odot(ABC)\}$,
so we get $MN \perp AO$ (ie. $MN \parallel EF$),
hence combine with $(\star)$ we get MN is $D-$ midline of $\triangle DEF$.

Q.E.D

sunken rock

#4 Dec 27, 2014, 6:31 pm • 1

Remark: Excellent solution, **Arab** now we can see that $C_1 - B_2 - B_1$ is **Simson line** of D w.r.t. $\triangle ABE$ and it passes through the midpoint of segment joining the point D and the orthocenter (E) of the triangle.

Best regards,
sunken rock

wiseman

#5 Dec 27, 2014, 9:42 pm

__Lemma__: Let $\mathcal{W}_1(O_1, R_1)$ and $\mathcal{W}_2(O_2, R_2)$ be two arbitrary circles with radical axis d. Let K be an arbitrary point in the plane and let $KH \perp d$ where $H \in d$. Prove that $|P^K_{\mathcal{W}_1} - P^K_{\mathcal{W}_2}| = 2.KH.O_1O_2$ where $P^K_{\mathcal{W}}$ is the power of K with respect to $\mathcal{W}(O, R)$.
__Proof of the lemma__: The proof is easy, just notice that $P^K_{\mathcal{W}} = |KO^2 - R^2|$, then a one or two lines of calculation and you'll get the result.

$\rightarrow$ For easier calculations, I assume that $\triangle ABC$ is acute. You can apply the same idea for the obtuse ones.
$\rightarrow$ L =The orthocenter of $\triangle ABC$.
$\rightarrow$ $BD.CD = AD.DL = AD^2 - AE^2 - (AE.CE)$.
$\Rightarrow \dfrac{|P^D_{\odot(MDN)} - P^D_{\odot(ABC)}|}{2AO} = \dfrac{|P^E_{\odot(MDN)} - P^E_{\odot(ABC)}|}{2AO}$.
$\Longrightarrow$ By the lemma we get that $D \text{and} E$ have equal distances to $MN \Longrightarrow EK = DK \blacksquare$.

Luis González

#6 Dec 28, 2014, 12:39 pm • 4

Let Y, Z be the projections of D on AC, AB. Inversion with center A and power $AD^2 = AY \cdot AC = AZ \cdot AB$ swaps (O) and the line $MN \Longrightarrow \{Y, Z\} \in MN$ and $MN \parallel EF$ (both perpendicular to AO). If K' denotes the midpoint of DE, we have then $\angle EK'Y = 2 \cdot \angle EDY = 2 \cdot \angle DEB = \angle DEF \Longrightarrow YK' \parallel EF \Longrightarrow K \equiv K'$. Analogously we get that H is midpoint of DF.

PROF65

#7 Dec 28, 2014, 7:14 pm • 1

consider the inversion wrt the circle (A,AD) .I,J intersection of MN with AB ,AC resp. it maps MM to the circle(ABC) then I to B and J to C thus IJ is antiparallel to BC besides EF is also antiparallel to BC then MN=IJ || EF
take D' the symmetric of D in AB .it s clear that D' is on EF and I is the midpoint of DD'(AD tangent to(DIB) ie <AID=<ADB=90°)
therfore NM intersect DE and DF at their midpoints

Quick Reply

High School Olympiads

Find BC

SymbatK
#1 Dec 27, 2014, 1:07 pm

The circle C1 with radius r1 and the circle C2 with radius r2 are externally tangent to each other at A. The circle C3 which is externally tangent to C1 and C2 has a radius with length r3. The common tangent C1 and C2 which passes through A meets C3 at B and C. Find BC

Luis González
#2 Dec 28, 2014, 11:12 am

Let O_1, O_2, O_3 denote the centers of the given circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$. Let PQ be a common external tangent of $\mathcal{C}_1, \mathcal{C}_2$, with $P \in \mathcal{C}_1$ and $Q \in \mathcal{C}_2$, and assume that B is inside of O_1O_2QP. From the problem Three tangent circles, we have $O_3B \perp PQ$, hence if $H \equiv PQ \cap O_1O_2$ and M is the projection of O_3 on BC, the right triangles $\triangle BMO_3$ and $\triangle HPO_1$ are similar $\Longrightarrow$

$$\frac{MB}{BO_3} = \frac{\frac{1}{2}BC}{r_3} = \frac{HP}{HO_1} = \frac{PQ}{O_1O_2} = \frac{2\sqrt{r_1 \cdot r_2}}{r_1 + r_2} \Longrightarrow BC = \frac{4r_3\sqrt{r_1 \cdot r_2}}{r_1 + r_2}.$$

High School Olympiads

A,B', C'; A',B, C'; A',B', C are collinear (IMO SL 1987-P12)

Reply

Amir Hossein

#1 Aug 19, 2010, 4:27 pm • 1

Given a nonequilateral triangle ABC, the vertices listed counterclockwise, find the locus of the centroids of the equilateral triangles $A'B'C'$ (the vertices listed counterclockwise) for which the triples of points A, B', C'; A', B, C'; and A', B', C are collinear.

Proposed by Poland.

Luis González

#2 Aug 21, 2010, 4:22 am • 1

Fixed circumcircles $(X), (Y), (Z)$ of $\triangle A'BC, \triangle C'AB, \triangle B'CA$ meet at the Miquel point of $\triangle A'B'C' \cup ABC$, i.e. the 1st Fermat point F of $\triangle ABC$. Let T be the midpoint of $B'C'$, running on midcircle (K) of $(Y), (Z)$, and let L be the 2nd intersection of $A'T$ with (X); midpoint of its arc BFC. Since AF bisects $\angle BFC$, it follows that FL is the external bisector of $\angle BFC \Longrightarrow \angle AFL = 90^\circ \Longrightarrow L \in (K)$.

Let G and U be the centroids of $\triangle ABC$ and $\triangle A'B'C'$. Since G is also the centroid of $\triangle XYZ$, then $\frac{\overline{GX}}{\overline{GK}} = -2 \ (*)$. But $\frac{\overline{UA'} \cdot \overline{UL}}{\overline{UL} \cdot \overline{UT}} = -2 \Longrightarrow$ powers of U WRT circles $(X), (K)$ are in the same ratio $\Longrightarrow U$ is on circle Ω coaxal with $(X), (K)$, but from $(*)$ we deduce that G is the center of Ω. Therefore, locus of U is the circle Ω centered at the centroid G of $\triangle ABC$ and passing through its 1st Fermat point F.

skytin

#3 Nov 6, 2011, 1:12 pm

Hint :
take midpoints of arc's AB , BC , CA of (ABC') , (BCA') , (CAB')

Luis González

#4 Dec 26, 2014, 5:10 am • 1

Another proof, following skytin's idea:

Circumcircles $(X), (Y), (Z)$ of $\triangle A'BC, \triangle B'CA$ and $\triangle C'AB$ concur at the Fermat point F of $\triangle ABC$. Since U is the incenter of $\triangle A'B'C'$, then UA', UB', UC' cut $(X), (Y), (Z)$ again at the midpoints A_0, B_0, C_0 of its arcs FBC, FCA, FAB, these are none other than the reflections of X, Y, Z on BC, CA, AB. Thus $\triangle A_0B_0C_0$ is the inner equilateral Napoleon triangle of $\triangle ABC$, whose center is the centroid G of $\triangle ABC$. Its circumcircle passes through F and U, since $\angle(FC_0, FB_0) = 60^\circ$ and $\angle(UC_0, UB_0) = 60^\circ \pmod \pi$. The conclusion follows.

wiseman

#5 Dec 26, 2014, 5:59 pm

A more generalized decision: Let K be an arbitrary point in the plane.Draw circles $\odot(BKC), \odot(CKA), \odot(AKB)$. We'll prove that the locus of the centroid G of triangles like $\triangle MNP$ such that $P \in \odot(BKC), N \in \odot(CKA), M \in \odot(AKB)$ and MN, MP, NP passes through A, B, C respectively, is a circle centered at the centroid L of triangle $\triangle A'B'C'$ passing through K where A', B', C' are the circumcircles of $\odot(BKC), \odot(CKA), \odot(AKB)$ respectively.

$\rightarrow \vec{LG} = \frac{1}{3}.(\vec{MC'} + \vec{NB'} + \vec{PA'})$.

$\rightarrow$ Now let M traverses a distance equal to α on the perimeter of $\odot(AKB)$ and makes the point M' ($\widehat{M'C'M} = 2\alpha$).

WLOG suppose the movement is clockwise; Then if we call the intersection points of $M'A'$ and $M'B'$ with $\odot(CKA)$ and $\odot(BKC)$ as N', P' respectively, we obviously have $\widehat{M'C'M} = \widehat{N'B'N} = \widehat{P'A'P} = 2\alpha \Rightarrow$ If G' be the centroid of $\triangle M'N'P'$ then $\vec{GG'}$ is the rotatated vector of $\vec{LG}$ with angle 2α(Note that N' and P' are also traversing clockwise). So the length of $\vec{LG}$ is constant $\Longrightarrow$ The locus of G is a circle centered at the centroid L of triangle $\triangle A'B'C'$ passing through $K\blacksquare$.

TelvCohl

#6 Dec 26, 2014, 6:44 pm • 2

> *wiseman wrote:*
>
> **A more generalized decision**: Let K be an arbitrary point in the plane.Draw circles $\odot(BKC), \odot(CKA), \odot(AKB)$. We'll prove that the locus of the centroid G of triangles like $\triangle MNP$ such that $P \in \odot(BKC), N \in \odot(CKA), M \in \odot(AKB)$ and MN, MP, NP passes through A, B, C respectively, is a circle centered at the centroid L of triangle $\triangle A'B'C'$ passing through K where A', B', C' are the circumcircles of $\odot(BKC), \odot(CKA), \odot(AKB)$ respectively.

Another proof of **wiseman**'s generalization:

Let P', N', M' be the midpoint of NM, MP, PN, respectively .
Let $\triangle A_1B_1C_1$ be the anti-pedal triangle of K WRT $\triangle ABC$.
Let G_1 be the centroid of $\triangle A_1B_1C_1$.

Easy to see A', B', C' is the midpoint of KA_1, KB_1, KC_1, respectively .

Since all $\triangle PNM$ are similar,
so PP' pass through a fixed point $P_1 \in \odot(KBC)$.
Similarly, NN', MM' pass through a fixed point $N_1 \in \odot(KCA), M_1 \in \odot(KAB)$, respectively .

Since PP', NN', MM' are concurrent at G,
so we get G, K, P_1, N_1, M_1 are concyclic .
ie. the locus of G is a circle $\mathcal{C}$ passing through K

Consider the case $P \equiv A_1, N \equiv B_1, M \equiv C_1$.

Since KA_1 is the diameter of $\odot(KBC)$,
so we get P_1 is the projection of K on A_1G_1 .
Similarly, N_1, M_1 is the projection of K on B_1G_1, C_1G_1, respectively ,
so we get KG_1 is the diameter of $\mathcal{C}$. ie. L is the center of $\mathcal{C}$

Q.E.D

Luis González

#7 Dec 26, 2014, 9:09 pm • 2

Even more general, all points U that verify $\triangle MNP \cup U \sim \triangle A'B'C' \cup U'$ orbit on a circle with center U' that passes through K. My proof is exactly the same what I did in my 1st post.

If MU cuts PN at D and cuts $\odot(KBC)$ again at M', we have $\angle KM'M = \angle KCM = \angle KAN \Longrightarrow M' \in \odot(KAD)$. Powers of D WRT $\odot(KCA), \odot(KAB)$ are in constant ratio $\overline{DN} : \overline{DP} \Longrightarrow$ center of $\odot(KAD)$ is the intersection $D' \equiv A'U' \cap B'C'$ that verifies $\overline{D'B'} : \overline{D'C'} = \overline{DN} : \overline{DP}$. Now, powers of U WRT $\odot(KBC)$ and $\odot(KAD)$ are in constant ratio $\overline{UM} : \overline{UD} \Longrightarrow$ locus of U is a circle centered at $U' \in A'D'$, because $\overline{U'A'} : \overline{U'D'} = \overline{UM} : \overline{UD}$, and passing through K, M'.

P.S. Telv's method also works in this general configuration.

Quick Reply

High School Olympiads

tangent circles

junior2001

#1 Dec 26, 2014, 3:47 pm

Let I be the incentre of triangle ABC. A circle containing the points B and C meets the segments BI and CI at points P and Q respectively. It is known that $BP \cdot CQ = PI \cdot QI$. Prove that the circumcircle of the triangle PQI is tangent to the circumcircle of ABC.

Luis González

#2 Dec 26, 2014, 8:13 pm • 2

Let X,Y,Z be the tangency points of (I) with BC,CA,AB. Inversion WRT (I) takes A,B,C into the midpoints A',B',C' of YZ,ZX,XY and takes P,Q into P',Q' such that B',C',Q',P' are concyclic. Since I becomes similitude center of $\odot(BCQP)$ and its inverse $\odot(B'C'Q'P') \Longrightarrow BCQP \sim P'Q'C'B' \Longrightarrow \frac{P'B'}{B'I} = \frac{IC'}{C'Q'} \Longrightarrow$ parallels from B',C' to IC,IB intersect at $D \in P'Q'$. This is the reflection of I on $B'C'$ lying on the 9-point circle $\odot(A'B'C')$ of $\triangle XYZ$ and since $P'Q'$ is antiparallel to YZ WRT XY,XZ, we deduce it is tangent to $\odot(A'B'C')$ at D. Thus, by conformity $\odot(IPQ)$ is tangent to $\odot(ABC)$.

TelvCohl

#3 Dec 26, 2014, 8:35 pm • 2

My solution:

Let T be a point satisfy $TQ \parallel IB, TP \parallel IC$.
Let T' be the reflection of T in PQ and ℓ be the reflection of BC in PQ .

From the condition $BP \cdot CQ = PI \cdot QI \Longrightarrow \frac{BP}{PI} = \frac{IQ}{QC} \Longrightarrow T \in BC$.

Since $\angle PTB = \angle ICB = \angle QPI = \angle PQT$,
so $\odot(PQT)$ is tangent to BC at T ,
hence we get $\odot(IPQ)$ is tangent to ℓ at T' ,
so from my problem8 !! we get $\odot(IPQ)$ is tangent to $\odot(ABC)$.

Q.E.D

61plus

#4 Jan 6, 2015, 12:30 pm • 2

Let R,S be midpoint of minor arcs AB,AC. It is easy to see that $RS||PQ$. Now $\triangle BRI$ and $\triangle ISC$ are similar isosceles triangles, Hence from $\frac{BP}{PI} = \frac{IQ}{QC}$, $\triangle BRP \sim \triangle ISQ$. Let RP intersect SQ at T, hence T lies on $\odot(ABC)$. Also from angle chase we can get $IPTQ$ concyclic. From homothety we deduce that $\odot(IPQ)$ tangent to $\odot(ABC)$.

shinichiman

#5 Mar 1, 2015, 9:10 am

Lemma. Let TF be a tangent of $\odot(O)$. A line passing through T intersects (O) at B,C. We will have $\frac{TB}{TC} = \left(\frac{FB}{FC}\right)^2$.

Back to the problem, let PQ cut BC at T. Let TF be the tangent of $\odot(ABC)$ such that F and A are in the different side wrt

line BC. Applying Meneleaus theorem with $\overline{T,P,Q}$ and $\triangle IBC$ we have $\frac{TB}{TC}=\frac{QI}{QC}\cdot\frac{PB}{PI}=\left(\frac{QI}{QC}\right)^2=\left(\frac{PB}{PI}\right)^2$.
Applying the lemma, we also have $\frac{TB}{TC}=\left(\frac{FB}{FC}\right)^2$. This follows $\frac{QI}{QC}=\frac{PB}{PI}=\frac{FB}{FC}$.

The bisector of $\angle BFC$ cuts BC at E. Hence, $\frac{FB}{FC}=\frac{EB}{EC}=\frac{PB}{PI}=\frac{QI}{QC}$. Therefore, $EP\parallel IC, EQ\parallel IB$. From here, we get $\angle BPE=\angle BIC=\angle EQC=90^\circ+\frac{1}{2}\angle A$. We also have $\angle BFE=\angle EFC=\frac{1}{2}BFC=90^\circ-\frac{1}{2}\angle A$. This follows $BPEF, CQEF$ are cyclic. We get $\angle PFE=\angle PBE=\frac{1}{2}ABC$ and $\angle QFE=\angle QCE=\frac{1}{2}ACB$. Thus, $\angle PFQ=90^\circ-\frac{1}{2}\angle A=180^\circ-\angle PIQ$. Hence, $IPFQ$ are cyclic or $I\in\odot(FPQ)$.
On the other hand, we have $PQCB$ are cyclic so $TB\cdot TC=TP\cdot TQ=TF^2$. This means TF is the tangent of $\odot(FPQ)$ and $\odot(ABC)$ or we can say TF is the tangent of $\odot(IPQ)$ and $\odot(ABC)$.
Thus, $\odot(IPQ)$ is tangent to $\odot(ABC)$.

sunken rock

#6 Mar 2, 2015, 3:01 am • 1

Remark: From the proof of **61plus** we easily see that T, the tangency point, coincides with the contact point of the $A-$mixtilear incircle with the circle $\odot(ABC)$.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Locus with equilateral triangles

Reply

Source: Romania TST 2014 Day 1 Problem 3

ComplexPhi

#1 Dec 26, 2014, 2:56 am

Let $A_0A_1A_2$ be a scalene triangle. Find the locus of the centres of the equilateral triangles $X_0X_1X_2$, such that A_k lies on the line $X_{k+1}X_{k+2}$ for each $k=0,1,2$ (with indices taken modulo 3).

This post has been edited 1 time. Last edited by ComplexPhi, Jan 22, 2015, 12:22 am

Luis González

#2 Dec 26, 2014, 3:22 am

Posted before, it is actually problem #12 from IMO shortlist 1987. The locus of the centers of all the equilateral triangles circumscribed to A0A1A2 is the circle centered at its centroid G and passing through its first Fermat point F. See the topic A,B', C'; A',B, C'; A',B', C are collinear (IMO SL 1987-P12).

Quick Reply

High School Olympiads

hardest

Reply

junior2001
#1 Dec 24, 2014, 12:14 pm

O is a point inside triangle ABC such that $OA = OB + OC$. Suppose B', C' be midpoints of arcs $\overset{\frown}{AOC}$ and AOB. Prove that circumcircles COB' and BOC' are tangent to each other

TelvCohl
#2 Dec 24, 2014, 1:08 pm • 3

My solution:

Let $Y, Z \in AO$ satisfy $OY = OB, OZ = OC$.

Since $AY = AO - YO = CO, AZ = AO - ZO = BO$,
so from $AB' = CB'$ and $\angle B'AY = \angle B'CO$ we get $\triangle AYB' \cong \triangle COB'$,
hence combine with $OZ = OC$ we get $\triangle B'YO \sim \triangle B'AC \sim \triangle OZC$.
Similarly, we can prove $\triangle AZC' \cong \triangle BOC'$ and $\triangle C'ZO \sim \triangle C'AB \sim \triangle OYB$.

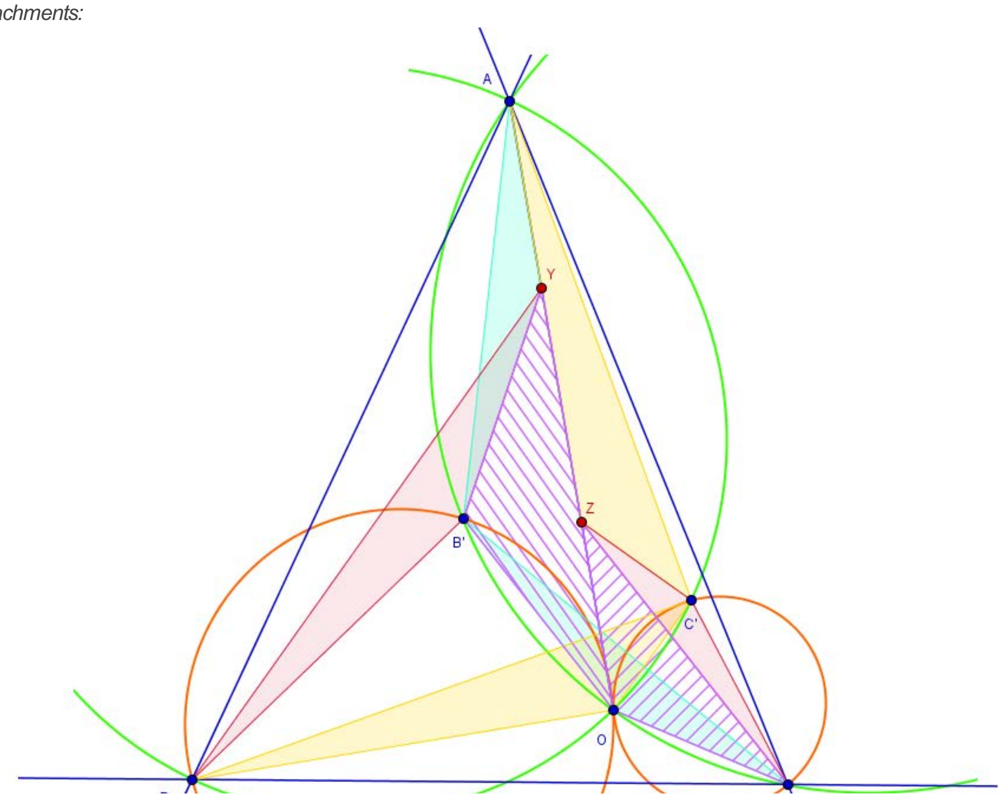
Since $\angle BYB' = \angle BYO - \angle B'YO = \angle OZC' - \angle OZC = \angle CZC'$,

so combine with $\frac{YB}{YB'} = \frac{ZO \cdot \frac{YO}{ZC'}}{ZO \cdot \frac{YO}{ZC}} = \frac{ZC}{ZC'}$ we get $\triangle YBB' \sim \triangle ZCC'$ and $\angle B'BY = \angle C'CZ$,

hence $\angle OBB' + \angle C'CO = \angle OBY + \angle ZCO = \angle C'OZ + \angle YOB' = \angle C'OB'$.
ie. (OBB') and (OCC') are tangent at O

Q.E.D

Attachments:



Luis González

#3 Dec 24, 2014, 1:17 pm • 3

Inverting the figure with center O and arbitrary positive power, we get the equivalent problem: O is inside $\triangle ABC$, such that $\frac{1}{OA}=\frac{1}{OB}+\frac{1}{OC}$. External bisectors of $\angle AOB$ and $\angle AOC$ cut AB, AC at B', C', respectively. Then $BC' \parallel CB'$.

By angle bisector theorem, we get $\dfrac{B'A}{B'B}=\dfrac{OA}{OB}$ and $\dfrac{CA}{CC'}=\dfrac{OC-OA}{OC}$.

But $\dfrac{OA}{OB}=\dfrac{OC-OA}{OC} \Longrightarrow \dfrac{B'A}{B'B}=\dfrac{CA}{CC'} \Longrightarrow BC' \parallel CB'$.

drmzjoseph

#4 Feb 4, 2015, 12:56 pm

Iran Team Selection Test 2007 - Day 3 - Problem 3

Quick Reply

High School Olympiads

Post modern Sangaku

Source: Miguel Ochoa Sanchez

leonardg
#1 Dec 23, 2014, 4:01 pm

Here you can see the draw :
https://www.facebook.com/photo.php?fbid=528776990593074&set=gm.555342234602222&type=1&theater

Attachments:

Miguel

Let ABC be an equilateral triangle and w be its incircle. Consider $P \in w$ and denote with x, y, z the distances from P to BC, CA and AB respectively. Prove that $\dfrac{x^2+y^2+z^2}{xy+yz+zx}=2$.

TelvCohl
#2 Dec 23, 2014, 6:08 pm • 2

My solution:

Let $BC=CA=AB=t$

From Viviani theorem we get $x+y+z=\frac{\sqrt{3}}{2}t$ . ... (1)

From Gergonne theorem we get $xy+yz+zx=\frac{3}{16}t^2$ . ... (2)

Combine with (1) and (2) we get $\frac{x^2+y^2+z^2}{xy+yz+zx}=\frac{(x+y+z)^2-2(xy+yz+zx)}{xy+yz+zx}=\frac{\frac{3}{4}t^2-\frac{3}{8}t^2}{\frac{3}{16}t^2}=2$.

Q.E.D

leonardg
#3 Dec 23, 2014, 6:20 pm • 1

Demonstracion :

Attachments:

Miguel

We choose incircle $x^2+y^2=1$, $A(0,2)$, $B(-\sqrt{3},-1)$ *and* $C(\sqrt{3},-1)$. *Also* $P(\cos t, \sin t)$, $t \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

So $BC: y=-1$, $AB: y-2=x\sqrt{3}$ *and* $AC: y-2=-x\sqrt{3}$. $PD=1+\sin t \Longrightarrow \sqrt{PD}=\sin\frac{t}{2}+\cos\frac{t}{2}$.

$PE=\frac{2-\sqrt{3}\cos t-\sin t}{2}=\frac{2-2\sin\left(t+\frac{\pi}{3}\right)}{2} \Longrightarrow \sqrt{PE}=\sin\left(\frac{t}{2}+\frac{\pi}{6}\right)-\cos\left(\frac{t}{2}+\frac{\pi}{6}\right)$. *Analog,*

$$PF = \frac{2+\sqrt{3}\cos t - \sin t}{2} = \frac{2+2\sin\left(t+\frac{2\pi}{3}\right)}{2} \Longrightarrow \sqrt{PF} = \sin\left(\frac{t}{2}+\frac{\pi}{3}\right) + \cos\left(\frac{t}{2}+\frac{\pi}{3}\right)$$. *In conclusion ,*

$\sqrt{PD} = \sqrt{PE} + \sqrt{PF}$. *Done* !!!

Leo

Luis González

#4 Dec 23, 2014, 8:19 pm • 1

More general, for any scalene triangle $\triangle ABC,$ all points P on its inellipse $\mathcal{E}$ tangent to BC, CA, AB through the feet of the internal bisectors verify the same relation. Working with trilinear coordinates $P \equiv (\alpha : \beta : \gamma)$ WRT $\triangle ABC,$ this simply follows from the trilinear equation of $\mathcal{E}$ WRT $\triangle ABC,$ namely $\mathcal{E} \equiv \alpha^2+\beta^2+\gamma^2-2(\alpha\cdot\beta+\beta\cdot\gamma+\gamma\cdot\alpha)=0.$ In other words, we have $\sqrt{\alpha}=\sqrt{\beta}+\sqrt{\gamma}$ or cyclic permutations.

Quick Reply

High School Olympiads

X, Y lie onto the C-median

Reply

Source: Own, maybe

sunken rock

#1 Dec 21, 2014, 6:57 pm • 2

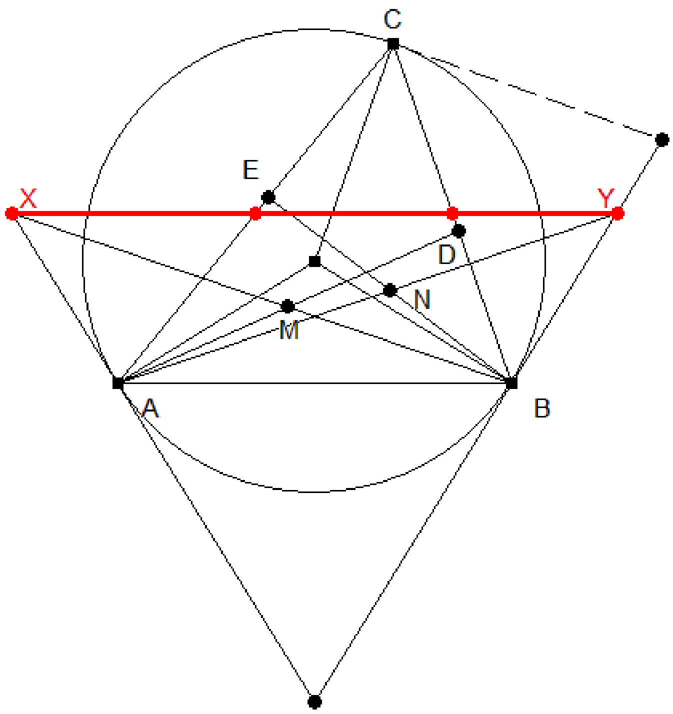
Let M, N be the midpoints of the symmedians AD, BE respectively of the triangle $\triangle ABC$ (D, E lie onto the sides BC, CA). BM intersects the tangent at A to the circumcircle of $\triangle ABC$ at X, and AN intersects the tangent at B to the same circle at Y.
Prove that XY is the $C-$median of $\triangle ABC$.

In connection with http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=275380

[Edit]: added a drawing, for a better understanding

Best regards,
sunken rock

Attachments:



TelvCohl

#2 Dec 21, 2014, 10:30 pm • 4

My solution:

Let P, Q be the midpoint of CB, CA, respectively .
Let $P' = AP \cap (ABC), D' = AD \cap (ABC), T = BM \cap P'Q$.

From Pascal theorem (for $ACBTP'D'$) we get $T \in (ABC)$.
From Pascal theorem (for $AACBTP'$) we get $X \in PQ$.

Similarly, we can prove $Y \in PQ$

Similarly, we can prove $Y \in PQ$,
so we get XY is $C-$ midline of $\triangle ABC$.

Q.E.D

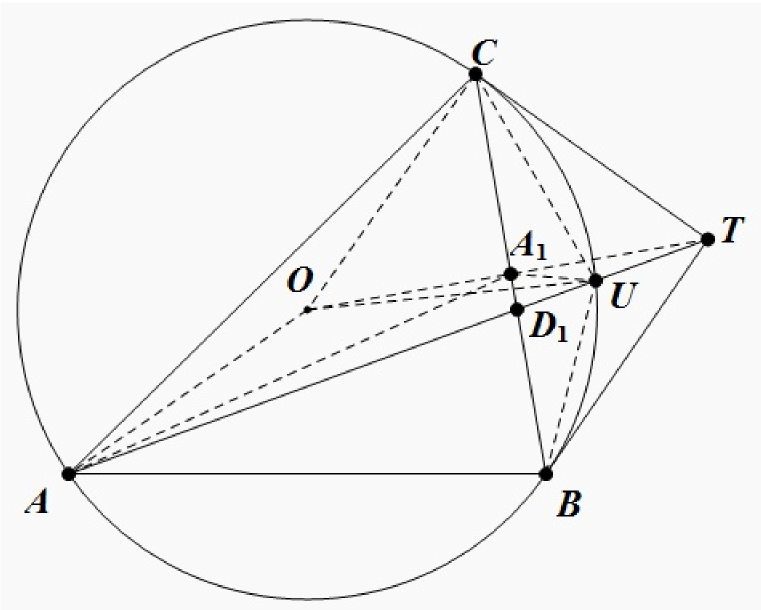

Luis González
#3 Dec 21, 2014, 10:49 pm • 2

Let $\triangle A'B'C'$ be the tangential triangle of $\triangle ABC.$ Parallel from C to BEB' cuts $AB, A'C'$ at P, Y', respectively. Since $B(C,A,B',A') \equiv B(C,P,\infty,Y') = -1 \Longrightarrow Y'$ is midpoint of $CP \Longrightarrow Y'$ is on C-midline of $\triangle ABC$ and A, N, Y' are collinear $\Longrightarrow Y' \equiv Y.$ Likewise, X is on C-midline of $\triangle ABC.$


Arab
#4 Dec 22, 2014, 2:09 am • 2

Let the lines of tangency of the circumcircle of $\triangle ABC$ at B, C intersect at T, and O be the circumcenter of $\triangle ABC$.

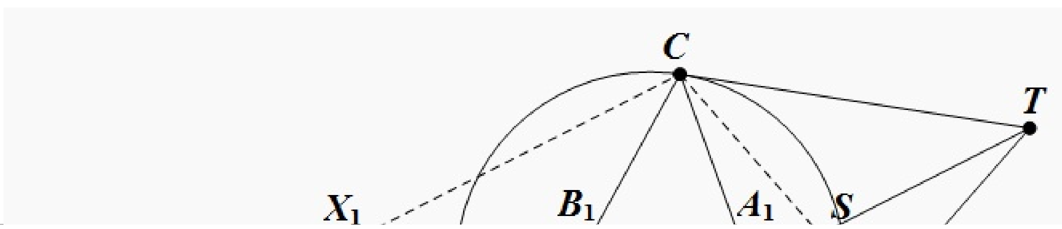
Lemma (well-known)
A, D, T are collinear.

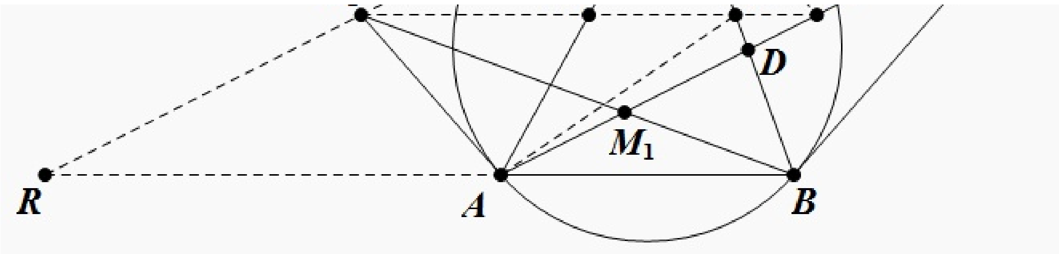


Proof of lemma
Let U be the second point of intersection of AT and the circumcircle of $\triangle ABC$, and A_1 be the midpoint of CB. Without loss of generality, we may assume that $D_1 = AT \cap CB$ lies on A_1B, then we obtain that $OT \perp CB, TC \perp OC$, so $TA_1 \cdot TO = TC^2 = TU \cdot TA$, and hence A_1, O, A, U are concyclic, then $\angle AOU = \angle AA_1U$. Note that CB bisects $\angle AA_1U$ since quadrilateral $ABUC$ is harmonic, then we have $\angle AA_1C = 180^\circ - \angle AA_1D_1 = 180^\circ - \dfrac{\angle AA_1U}{2} = 180^\circ - \dfrac{\angle AOU}{2} = 180^\circ - ACU = \angle ABU$, meaning that $\angle CAA_1 = \angle UAB$ (because $\angle ACB = \angle AUB$), so $D_1 \equiv D$. $\blacksquare$

Proof of the original problem

Denote by B_1 the midpoint of CA, and let A_1B_1 intersect the line of tangency of $\triangle ABC$ at A and the line AT at X_1, S respectively, and let M_1 be the point of intersection of BX_1, AT. Note that $\angle ACA_1 = 180^\circ - \angle X_1AB = \angle AX_1A_1$, so A, X, C, A_1 are concylic, and hence $AB_1^2 = AB_1 \cdot B_1C = X_1B_1 \cdot B_1A_1$. Meanwhile, since $\angle B_1AA_1 = \angle BAT = \angle ASB_1$, we have $AB_1^2 = B_1A_1 \cdot B_1S$, and therefore $X_1B_1 = B_1S$, meaning that quadrilateral AX_1CS is a parallelogram, so $CX_1 \parallel AS$. Let $R = CX_1 \cap AB$, then since $RA \parallel X_1S$, we obtain that quadrilateral X_1RAS is a parallelogram, so $X_1R = AS = X_1C$, and hence $AM_1 = M_1D$, therefore $M_1 \equiv M$, which implies that $X_1 \equiv X$, as desired. $\blacksquare$





Remark
From above we obtain, though not the most elegant one, a solution to this problem (because $XY \parallel AB$, quadrilateral $AXYB$ is an isosceles trapezoid, which follows the angles being equal).

wiseman
#5 Dec 22, 2014, 7:47 pm

We change the decision; Let $\triangle A'B'C'$ be the tangential triangle of $\triangle ABC$.Let P, Q be the midpoints of BC, AC respectively. Denote by X, Y the intersection points of PQ with $B'C'$ and $A'C'$ respectively. We'll prove that if BX intersects with AD at M, then $AM = MD$.

$\rightarrow$ First of all, it's well-known that $BD = \dfrac{ac^2}{b^2+c^2}$. (a)
$\rightarrow XY \parallel BY \Rightarrow AX = BY \Rightarrow \widehat{ABX} = \widehat{BXY}$. (b)
$\rightarrow \widehat{XAQ} = \widehat{ABC}, \widehat{AQX} = \widehat{BAC} \Rightarrow \triangle AQX \sim \triangle BAC \Rightarrow QX = \dfrac{b^2}{2c}$
$\Rightarrow PX = \dfrac{b^2+c^2}{2c}$. (c)
$\rightarrow$ Combining (a),(b),(c) yields that: $\dfrac{\sin(\widehat{ABX})}{\sin(\widehat{XBC})} = \dfrac{\sin(\widehat{BXY})}{\sin(\widehat{XBP})} = \dfrac{a}{2PX} = \dfrac{ac}{b^2+c^2} = \dfrac{BD}{AB} \Rightarrow AM = MD\blacksquare$.

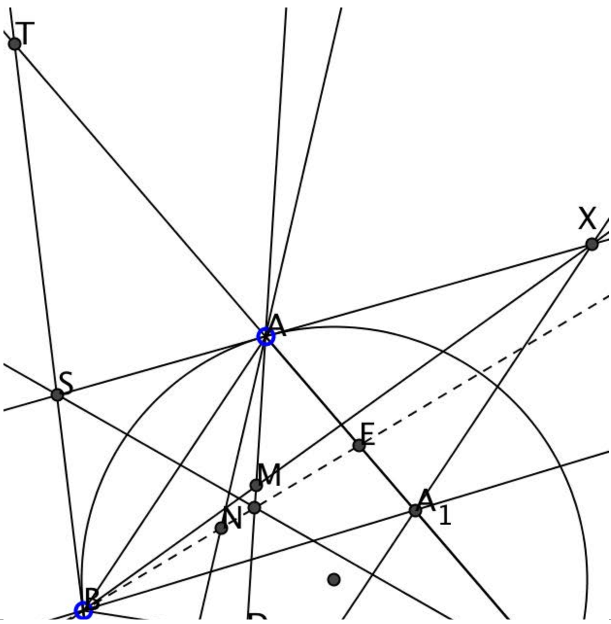
shinichiman
#6 Dec 25, 2014, 2:29 pm • 1

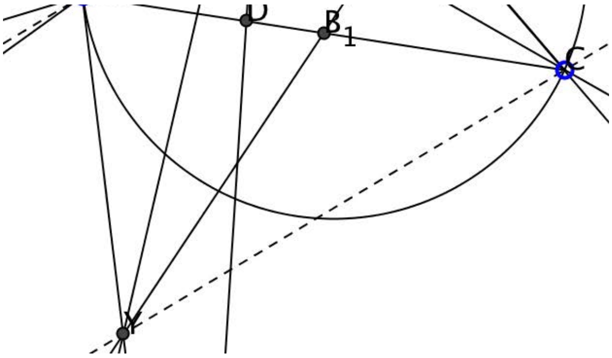
Let A_1, B_1 be the midpoint of CA, CB,respectively. S is the intersection between tangents from A and B, $T = BS \cap AC, BS \cap A_1B_1 = Y'$. We can easily see that $BY'CA_1$ are cyclic so

$$\angle BCY' = \angle BA_1A = \angle BCA + \angle CBA_1 = \angle SBA + \angle NBA = \angle NBS.$$

Therefore, $BE \parallel Y'C$. We get $\dfrac{\overline{Y'B}}{\overline{Y'T}} = \dfrac{\overline{CE}}{\overline{CT}}$. Also note that $(TE, AC) = -1$ so $\dfrac{\overline{Y'B}}{\overline{Y'T}} = -\dfrac{\overline{AE}}{\overline{AT}}$. From here we get $\dfrac{\overline{Y'B}}{\overline{Y'T}} \cdot \dfrac{\overline{AT}}{\overline{AE}} \cdot \dfrac{\overline{NE}}{\overline{NB}} = 1$. By Menelaus theorem we get Y', A, N are collinear or $Y \equiv Y'$. We obtain Y, A_1, B_1 are collinear. Similarly, we also get X, A_1, B_1 are collinear.

Attachments:





Quick Reply

High School Olympiads

Perpendicular

Reply

ntn0797

#1 Dec 21, 2014, 8:46 pm

Given triangle ABC with (I) is incircle and (I) touchs BC at D. Let H and K be orthocenter of $\triangle IAB$ and $\triangle IAC$ respectively. $E = DI \cap (I)$. Prove that
1. H, D, K are collinear
2. AE is perpendicular to HK

Luis González

#2 Dec 21, 2014, 10:01 pm • 1

If M, N, L are the midpoints of $BC, CA, AB,$ then H, K are the poles of MN, ML WRT (I) (this is a particular case of the problem Pole of midline) $\Longrightarrow H, D, K$ are collinear on the polar of M WRT (I). Thus, if HK cuts (I) again at S, then MS is tangent to (I). If the A-excircle of $\triangle ABC$ touches BC at D', we have that A, E, D' are collinear and $MD = MD' = MS \Longrightarrow \angle DSD' = 90^\circ \Longrightarrow S \in AD' \Longrightarrow AE \perp DS \equiv HK$.

TelvCohl

#3 Dec 22, 2014, 11:17 am • 1

Another solution for 2. :

Let I_a, I_b, I_c be three excenters of $\triangle ABC$.
Let F be the tangent point of (I_a) with BC .
Let T be a point s.t. $BHKT$ is a parallelogram .

Easy to see A, E, F are collinear and $T \in CK$.

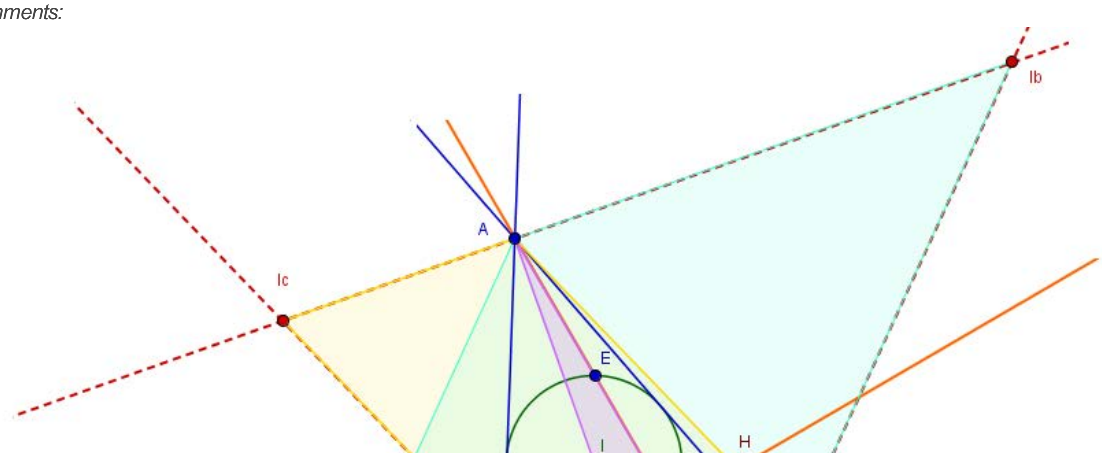
Since $AHBI_c$ and $AKCI_b$ are parallelogram ,
so $CT = CK + KT = CK + HB = I_bI_c$.

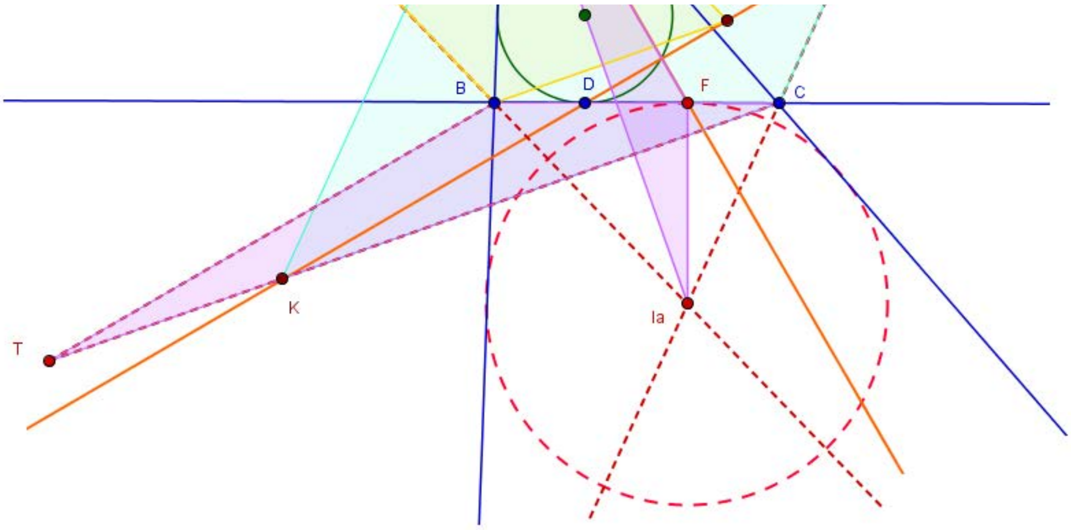
From $\triangle I_aBC \sim \triangle I_aI_bI_c \Longrightarrow \frac{CB}{CT} = \frac{CB}{I_bI_c} = \frac{I_aF}{I_aA}$ . ... $(\star)$

From $I_aF \perp CB$ and $I_aA \perp CT \Longrightarrow \angle FI_aA = \angle BCT$,
so combine with $(\star)$ we get $\triangle I_aAF \sim \triangle CTB$ and $TB \perp AF$.
i.e. $AE \perp HK$

Q.E.D

Attachments:





This post has been edited 1 time. Last edited by TelvCohl, Jun 23, 2015, 6:21 pm

Arab

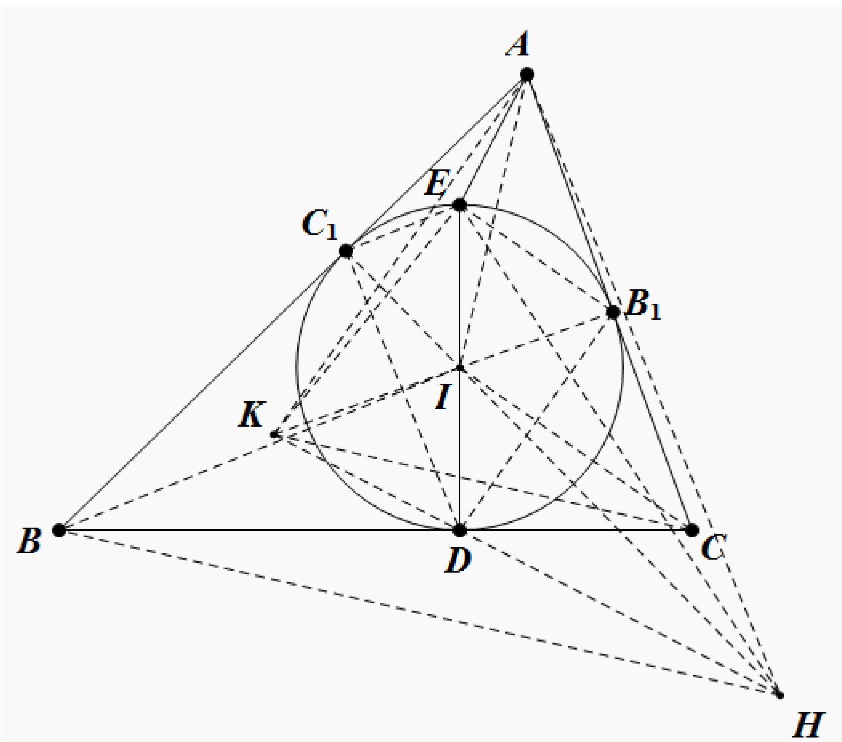
#4 Dec 22, 2014, 1:56 pm

1. Without loss of generality, we may assume that the locations of all elements are as illustrated. Let the incircle of $\triangle ABC$ touch CA, AB at B_1, C_1 respectively, then $B_1 \in IK, C_1 \in IH$. Note that $\frac{BH}{CK} = \frac{BH}{AI} \cdot \frac{AI}{CK} = \frac{BC_1}{IC_1} \cdot \frac{IB_1}{CB_1} = \frac{BD}{ID} \cdot \frac{ID}{CD} = \frac{BD}{CD}$, therefore $D \in HK$ (because $BH \perp AI, CK \perp AI \implies BH \parallel CK \implies \angle DBH = \angle DCK \implies \triangle BDH \sim \triangle CDK \implies \angle BDH = \angle CDK$).

2. Since $EC_1 \perp C_1D, AH \perp BI, C_1D \perp BI$, we have $EC_1 \perp AH$, and similarly, $EB_1 \perp AK$, and hence we obtain that

$$\begin{aligned} AH^2 - AK^2 &= (AC_1^2 + HC_1^2) - (AB_1^2 + KB_1^2) \\ &= HC_1^2 - KB_1^2 \\ &= (AC_1^2 + EH^2 - EA^2) - (AB_1^2 + EK^2 - EA^2) \\ &= EH^2 - EK^2 \end{aligned}$$

meaning that $AE \perp HK$, as desired. $\blacksquare$



shinichiman

#5 Dec 25, 2014, 4:37 pm

Since H, K is the othocenter of $\triangle IAB, \triangle IAC$ so $\sin \frac{\angle BAC}{2} = \frac{BD}{BH} = \frac{CD}{CK}$ or $\frac{DB}{DC} = \frac{BH}{CK}$. Hence, K, H, D are collinear. (I) touches AB at F, AI cuts BH at L. We have $\triangle AIL \sim \triangle HBL$ (A.A) so $\frac{BH}{AI} = \frac{BD}{ID}$. Since $BDLI$ is cyclic so $\angle DBH = \angle DIL = \angle EIA$. From here we get $\triangle HBD \sim \triangle AIE$ (S.A.S). Thus, $\angle HDC= \180^{\circ}- \angle IEA=\angle IEx$. Note that $ED \perp BC$, we obtain $AE \perp HK$.

jayme

#6 Jun 23, 2015, 6:05 pm

Dear Mathlinkers,
this reference
http://www.artofproblemsolving.com/community/c6t48f6h1104088_beautiful_geometry
help to have a nice proof.

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

A well-known geometry problem?

Reply

Tung-CHL

#1 Dec 21, 2014, 8:38 pm

For a non-isosceles triangle ABC inscribed (O) circle. AD, BE, CF are hightways, they converge at H. Points A1, A2 lies on (O) so that (A1EF) and (A2EF) contact with (O). Similar to B1, B2, C1, C2. Prove that A1A2, B1B2, C1C2, OH are converge. (Sorry for my bad English :(

This post has been edited 1 time. Last edited by Tung-CHL, Dec 29, 2014, 9:37 pm

Luis González

#2 Dec 21, 2014, 9:15 pm

Lines EF, BC and the common tangent of $(O), \odot(EFA_1)$ are pairwise radical axes of $(O), \odot(EFA_1)$ and $\odot(BCEF)$ concurring at their radical center $X.$ Similarly the common tangent of $(O), \odot(A_2EF)$ goes through $X \Longrightarrow A_1A_2$ is the polar of X WRT $(O).$ Likewise, B_1B_2 and C_1C_2 are the polars of $Y \equiv FD \cap CA$ and $Z \equiv DE \cap AB$ WRT $(O) \Longrightarrow$ A_1A_2, B_1B_2, C_1C_2 concur at the pole of the orthic axis $\overline{XYZ}$ WRT (O) (Gob's point of ABC). This lies on $OH,$ because $OH \perp \overline{XYZ}$ (orthic axis is radical axis of (O) and the 9-point circle).

Quick Reply

High School Olympiads

Not so easy

bethebest
#1 Dec 20, 2014, 9:23 pm

Tangents at A and B of the $\triangle ABC$ intersect the tangent at C at points U and T respectively. AT intesects BC at P and BU intersects AC at R. If Q is the midpoint of AP and S is the midpoint of BR prove that $\angle ABQ = \angle BAS$.

TelvCohl
#2 Dec 20, 2014, 9:57 pm • 1

See
IMO Shortlist 2000 G5
Romania TST 2001 Day 4 Problem 3
China Team Training 2003
angle with tangential circle

Luis González
#3 Dec 20, 2014, 11:42 pm • 1

More general: P is a point on the plane of $\triangle ABC.$ PB, PC cut AC, AB at $Y, Z,$ respectively and N, L denote the midpoints of $BY, CZ,$ respectively. Then $\angle BCN = \angle CBL$ if and only if P is on the line passing through the midpoint of BC and the symmedian point K of $\triangle ABC.$

Proof: Let D, E, F be the midpoints of BC, CA, AB and let $U \in EF$ be the midpoint of the A-altitude of $\triangle ABC.$ It's known that D, K, U are collinear (Schawtt line). It suffices to find the locus of P that verifies $\angle BCN = \angle CBL,$ i.e. $Q \equiv BL \cap CN$ on the perpendicular bisector ℓ of $BC.$

Since $BL \mapsto CN$ induces a perspectivity between $DE, DF,$ then $BN \mapsto CL$ is also a perspectivity, as $BN \equiv CL$ when $Q \equiv D \Longrightarrow P$ is on a fixed line through $D.$ When $Q \equiv \ell \cap EF,$ then AQ, CL cut $DE, EF,$ resp, at the reflections L', Q' of L, Q on $E,$ thus $QLQ'L'$ becomes parallelogram $\Longrightarrow CQ'L \parallel AQL' \parallel DU$ and analogously we get $BN \parallel DU \Longrightarrow P$ becomes point at infinity of $DKU.$ Consequently, all points P lie on the line $DK,$ as desired.

TelvCohl
#4 Dec 21, 2014, 1:32 am • 2

Generalization:

Let M be the midpoint of BC and K' be a point lie on AM .
Let K be the isogonal conjugate of K' WRT $\triangle ABC$ and P be a point on MK .
Let $B' = BK' \cap AC, C' = CK' \cap AB, E = BP \cap AC, F = CP \cap AB$.
Let $Y \in BE, Z \in CF$ satisfy $BY : YE = CZ : ZF = BC' : C'A = CB' : B'A$.

Then $\angle YCB = \angle ZBC$

Proof:

Let P' be the isogonal conjugate of P WRT $\triangle ABC$.
Let $E' = BK' \cap (ABC), F' = CK' \cap (ABC), U = BP' \cap (ABC), V = CP' \cap (ABC)$.
Let $R = E'V \cap F'U, T = B'V \cap (ABC), S = C'U \cap (ABC)$.

Easy to see $\triangle KBC \sim \triangle AF'E'$.

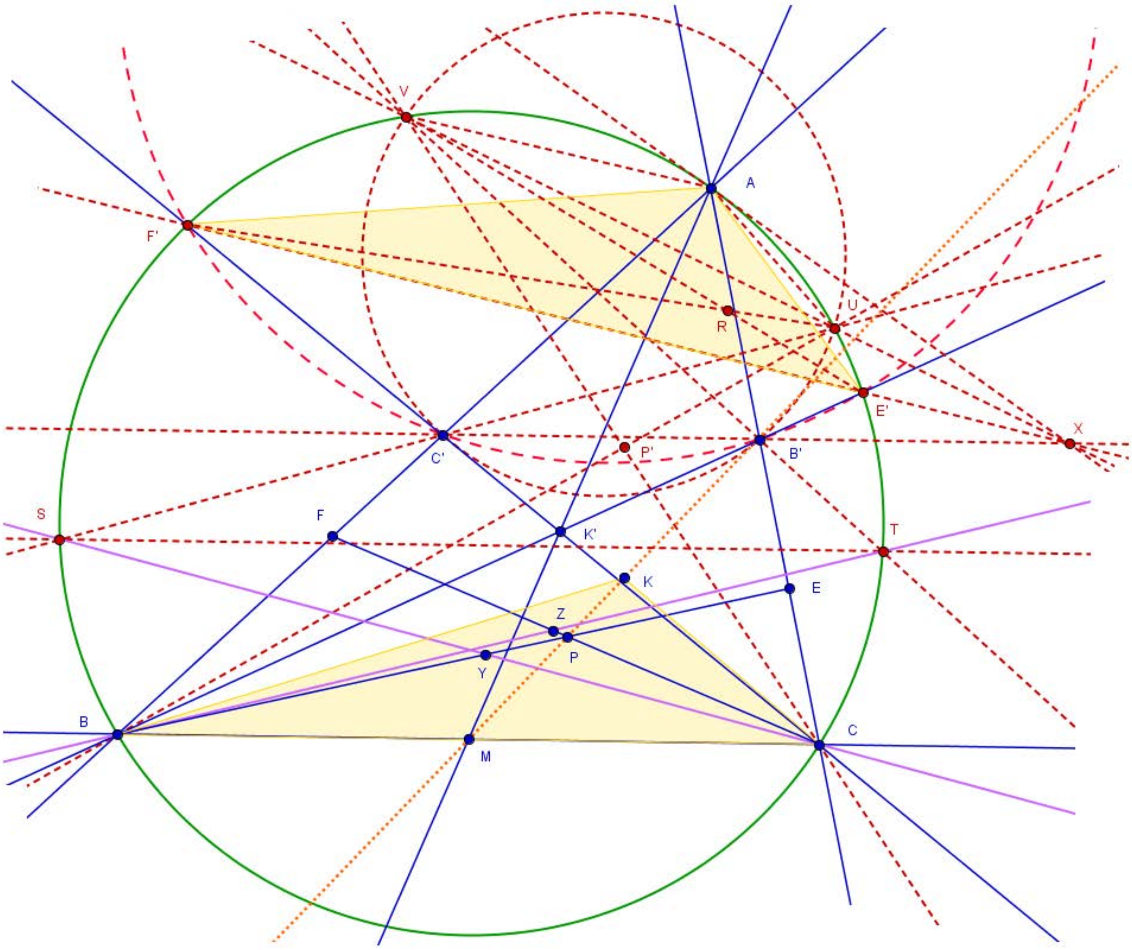
Since $B'C' \parallel BC$,
so from Reim theorem we get B', C', E', F' are concyclic .
Since $\angle AE'R = \angle PCB, \angle AF'R = \angle PBC$,
so from $P \in KM$ we get AR is $A-$ symmedian line of $\triangle AF'E'$,
hence the tangent at A and $UV, E'F'$ are concurrent at X .

From pascal theorem (for $AABE'F'C$) we get $X \in B'C'$.

Since $XB' \cdot XC' = XE' \cdot XF' = XU \cdot XV$,
so we get U, V, B', C' are concyclic ,
hence from Reim theorem we get $TS \parallel BC$ and $\angle SUB = \angle TVC$ . ... $(\star)$
Since $\triangle BCE \cap Y \sim \triangle BUA \cap C', \triangle CBF \cap Z \sim \triangle CVA \cap B'$,
so combine with $(\star)$ we get $\angle YCB = \angle C'UB = \angle B'VC = \angle ZBC$.

Q.E.D

Attachments:



This post has been edited 1 time. Last edited by TelvCohl, Dec 21, 2014, 7:09 am

Luis González

#5 Dec 21, 2014, 5:12 am

Another proof to Telv's generalization:

It suffices to find the locus of $P,$ as $Q \equiv BZ \cap CY$ runs on perpendicular bisector ℓ of $BC.$ As before, the pencils BY, CZ are perspective since $BY \equiv CZ$ for $P \equiv M \Longrightarrow P$ is on a fixed line through $M.$ Hence, it is enough to prove that P becomes the point at infinity of MK when $Q \equiv B'C' \cap \ell.$

For this, animate K' on AM and let V be the intersection of the parallel to AB through B' with the parallel to MK through $C.$ Since $K \barwedge K' \barwedge B' \barwedge Q,$ the pencils $CV, B'V$ are homographic $\Longrightarrow V$ is on a fixed conic $\mathcal{C}$ that passes through C, B and $\ell \cap AC$ (here Q=V). Now since $Q \barwedge V$ and bearing in mind that B, Q, V are collinear when K' coincides with M (trivial) and when K coincides with the centroid (already proved in my first post), then it follows that $Q \mapsto V$ is a stereographic projection of $\mathcal{C}$ onto $\ell \Longrightarrow B, Q, V$ are collinear for all K'. Therefore, $CZ \parallel BY \parallel MK$ for $Q \equiv B'C' \cap \ell,$ as desired.

buratinogigle

#6 Feb 13, 2016, 4:34 pm

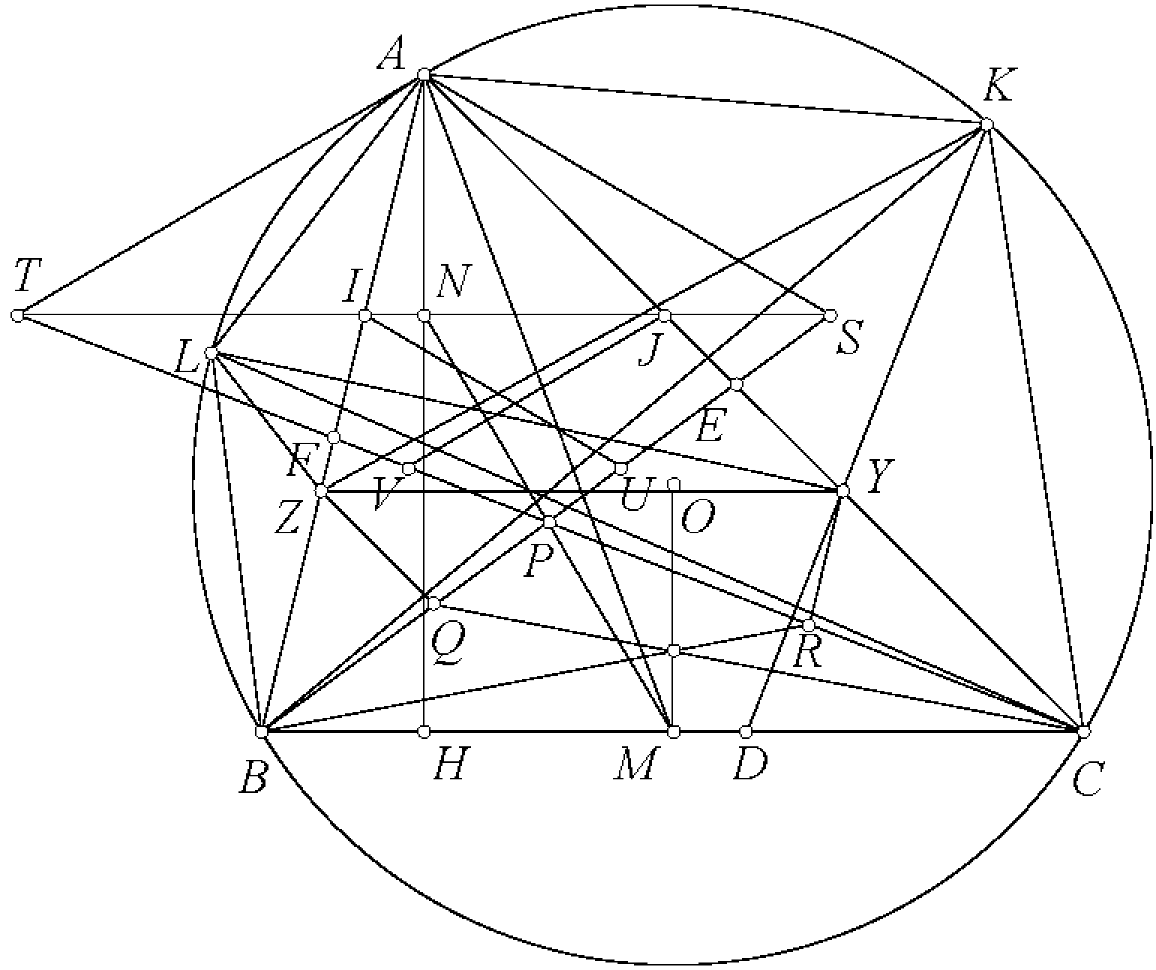
Write Telv's problem as following

Problem. Let ABC be a triangle with median AM and altitude AH. P is an any point. PB, PC cut CA, AB at E, F, reps. PM cuts AH at N. Q, R lie on BE, CF, reps such that $\frac{QB}{QE} = \frac{RC}{RF} = \frac{NA}{NH}$. Prove that $\angle RBC = \angle QCB$.

Here is the synthetic solution which I write base on idea of my pupil **Trinh Huy Vu**.

Proof. Let the line passes through N and is parallel to BC which cut CA, AB at J, I. Let PB, PC cut JI at S, T. Let K, L be on (ABC) such that $\angle KBA = \angle PCB$ and $\angle LCA = \angle PBC$. Let U, V be on BK, CL such that $IU \parallel AS$ and $JV \parallel AT$. Y, Z lie on CA, AB such that $CY = AJ$ and $BZ = AI$. We easily seen triangles CJT and ALB are similar with Z, V devide AB, CT in the same ratio. Similarly, triangles BIS and AKC are similar with Y, U devide AC, BS in the same ratio. We also have M is midpoint of BC so N is midpoint of ST, thus AST is isoceles triangle. Hence, $\angle BLZ = \angle VJT = \angle ATS = \angle AST = \angle UIS = \angle CKY$. Let KY cut BC at D, we have $\angle ZLK = \angle BLK - \angle BLZ = \angle 180^\circ - \angle BCK - \angle CKY = \angle CDK = \angle DYZ$. Therefore $KLZY$ is cyclic. From here $\angle ZKB = \angle ZKY - \angle YKB = \angle ZKY - (\angle CKB - \angle CKY) = \angle ZLY - (\angle CLB - \angle BLZ) = \angle YLC$. Note that triangles CBE and KBA are similar with Q, Z devide BE, BA in the same ratio. Similarly, triangles CBF and LCA are similar and R, Y devide CF, CA in the same ration. We deduce $\angle QCB = \angle ZKB = \angle YLC = \angle RBC$. We are done.

Attachments:



Lin_yangyuan

#7 Mar 26, 2016, 7:28 am

Discussed before here (#10 posted by lym) http://bbs.cnool.net/m/cthread-56219342.html

Quick Reply

High School Olympiads

Orthologic pedals

Reply

Source: Barry Wolk (Hyacinthos #22916)

rodinos

#1 Dec 21, 2014, 3:59 am

The pedal triangles of any 2 points on the Euler line are orthologic.

Conversely, let Q be any point on the Euler line except the circumcenter
and the 2 points where the Euler line meets the circumcircle. For any P,
if the pedal triangles of P and Q are orthologic, then P is also on the Euler line.

(Barry Wolk)

Synthetic proof?

Luis González

#2 Dec 21, 2014, 4:27 am • 1

Actually the direct proposition can be more general: If P and Q are two points collinear with the circumcenter O of ABC, then the pedal triangles of P and Q WRT ABC are orthologic. See the thread http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=571620 (post #6)

rodinos

#3 Dec 22, 2014, 12:21 am

.... and so, as lemma, can be applied to find which is the locus of P such that the pedal triangles of P, P* are orthologic where P* is the isogonal conjugate of P

aph

Quick Reply

High School Olympiads

Segment ratio

Reply

Fang-jh
#1 Dec 23, 2008, 8:15 pm • 1

Let $ABCD$ be a quadrilateral (convex or concave). Let the internal common tangent (different from BD) of the incircles of triangles ABD and BCD cut the diagonal AC at M. Prove that: $\frac{AM}{MC}=\frac{tan\frac{C}{2}}{tan\frac{A}{2}}$.

Zhang Fangyu
#2 Feb 9, 2009, 7:27 am

any solution?

Zhang Fangyu
#3 May 5, 2009, 8:30 pm

nobody can solve it?

Ponclete
#4 Jul 21, 2009, 11:25 am • 3

Let P,Q,R,S be the intersection of second internal common tangent to AB,BC,BD,CD, and N,M be intersection of diagonal AR,DQ and, BS,CR.And E,F,G,H are tangency between incircle of ABD,BCD and $AB,QR.BC,BR$.
Suppose that P lies on expanded line BC, then by menelaus's theorem on triangle BCD by cut PRS , $\frac{BP}{CP}\cdot\frac{DR}{BR}\cdot\frac{CS}{DS}=1$,
and on triangle BDS by cut CMR , $\frac{BR}{BD}\cdot\frac{CM}{RM}\cdot\frac{DS}{CS}=1$.
In short form, $\frac{BP}{CP}\cdot\frac{CM}{RM}\cdot\frac{DR}{BD}=1$.
And simmilar way, $\frac{DR}{BD}\cdot\frac{AN}{RN}\cdot\frac{BQ}{AQ}=1$.
Now, Let T be the intersection of AC and the second internal common tangent.By menelaus's theorem on ABC by cut PQT, $\frac{BP}{CP}\cdot\frac{AQ}{BQ}\cdot\frac{BT}{AT}=1$.
Therefore, $\frac{BT}{AT}=\frac{RN}{AN}\cdot\frac{CM}{RM}=\frac{RG}{AE}\cdot\frac{CF}{RH}=\frac{r}{AE}\cdot\frac{CF}{r'}=\frac{tan\frac{A}{2}}{tan\frac{C}{2}}$.

Tiks
#5 Jun 26, 2010, 10:00 pm • 1

Fang-jh, is your solution the same as Ponclete's?

> *Fang-jh wrote:*
> Let $ABCD$ be a quadrilateral (convex or concave). Let the internal common tangent (different from BD) of the incircles of triangles ABD and BCD cut the diagonal AC at M. Prove that: $\frac{AM}{MC}=\frac{tan\frac{C}{2}}{tan\frac{A}{2}}$.

TelvCohl

#6 Dec 20, 2014, 9:45 pm • 1

My solution:

Let I_1, I_2 be the incenter of $\triangle ABD, \triangle CBD$, respectively .
Let ℓ be the internal common tangent of (I_1) and (I_2) (different from BD)
Let $X = \ell \cap BD, Y = \ell \cap DA, Z = \ell \cap BC$ and r_1, r_2 be the radius of $(I_1), (I_2)$, respectively .

Since $\angle DI_1B = 90^\circ + \angle YAI_1, \angle BI_2D = 90^\circ + \angle ZCI_2$,

so $\dfrac{AM}{MC} = \dfrac{AY \cdot \sin \angle MYA}{CZ \cdot \sin \angle MZC}$

$= \dfrac{r_1 \cdot (\cot \angle YAI_1 + \cot \angle I_1YA) \cdot \sin \angle MYA}{r_2 \cdot (\cot \angle ZCI_2 + \cot \angle I_2ZC) \cdot \sin \angle MZC}$

$= \dfrac{r_1 \cdot (\tan \angle DI_1I_2 - \tan \angle DI_1B) \cdot \sin \angle MYA}{r_2 \cdot (\tan \angle I_1I_2D - \tan \angle BI_2D) \cdot \sin \angle MZC}$

$= \dfrac{\mathrm{S}[\triangle I_1BD] \cdot (\tan \angle DI_1I_2 - \tan \angle DI_1B) \cdot \sin \angle MYA}{\mathrm{S}[\triangle I_2BD] \cdot (\tan \angle I_1I_2D - \tan \angle BI_2D) \cdot \sin \angle MZC}$

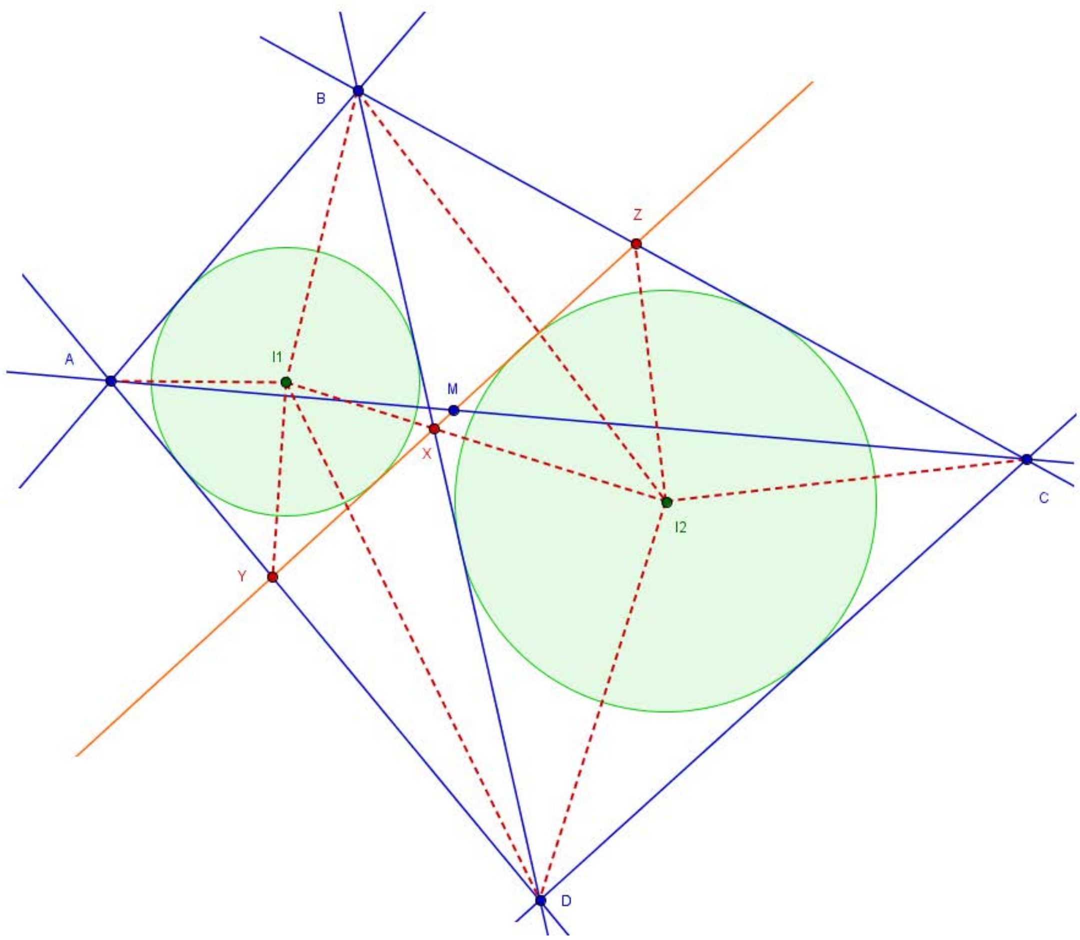
$= \dfrac{(I_1B \cdot I_1D \cdot \sin \angle DI_1B) \cdot \sin DI_1I_2 \cdot \sin \angle I_2I_1B \cdot \cos \angle BI_2D}{(I_2B \cdot I_2D \cdot \sin \angle BI_2D) \cdot \sin I_1I_2D \cdot \sin \angle BI_2I_1 \cdot \cos \angle DI_1B}$

$= \dfrac{(I_1B \cdot I_1D \cdot \sin \angle DI_1B) \cdot I_2D \cdot I_2B \cdot \cos \angle BI_2D}{(I_2B \cdot I_2D \cdot \sin \angle BI_2D) \cdot I_1D \cdot I_1B \cdot \cos \angle DI_1B}$

$= \dfrac{\cot \angle BI_2D}{\cot \angle DI_1B} = \dfrac{\tan \frac{1}{2}\angle BCD}{\tan \frac{1}{2}\angle DAB}$

Q.E.D

Attachments:



Luis González

#7 Dec 20, 2014, 10:12 pm

This was also discussed in the following threads:

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=487427 (Lemma)
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=354536 (Lemma at post #2)

Quick Reply

High School Olympiads

Pass through a Fixed point

Reply

Source: Own (Inspired from 2005 ISL G6)

TelvCohl

#1 Dec 7, 2014, 12:51 pm • 1

Let P be a fixed point on BC .
Let (I) be the incircle of $\triangle ABC$.
Let Q and Q' lie on BC satisfy $(B, C; P, Q) = (B, C; Q', P)$.
Let X, Y be the intersection of AQ and (I) $(AX < AY)$.
Let X', Y' be the intersection of AQ' and (I) ($AX' < AY'$) .

Prove that XY' pass through a fixed point when Q varies on BC

Luis González

#2 Dec 19, 2014, 11:02 pm • 2

Let (I) touch AC, AB at E, F. AP cuts EF at M and the arc EXF of (I) at R. XM cuts (I) again at Z'. Tangents of (I) at X, Z', R cut EF at $U, V, N,$ resp; these are the poles of AX, AZ', AP WRT (I).

By Desargues involution theorem for $XXZ'Z'$ cut by EF, if follows that M, N are the fixed points of the involution $\{U \mapsto V, F \mapsto E\}$, because $(E, F, M, N) = -1$. Thus AM is fixed under involution $\{AX \mapsto AZ', AF \mapsto AE\}$. This coincides then with the described involution $\{AB \mapsto AC, AQ \mapsto AQ'\}$ that fixes $AP \Longrightarrow AQ' \equiv AZ' \Longrightarrow Z' \equiv Y' \Longrightarrow$ all lines XY' go through the fixed point M.

Quick Reply

High School Olympiads

Nice geometry problem

Reply

Sardor
#1 Dec 18, 2014, 11:19 pm

Let I be the incenter of a triangle ABC and let IA_1, IB_1, IC_1 be symmedians of triangles BIC, CIA, AIB, respectively. Prove that AA_1, BB_1, CC_1 are concurrent at some point P. Prove that P lies on the line GT, where G is centroid and T is Gergonne point of triangle ABC

Luis González
#2 Dec 19, 2014, 12:10 am • 2

The concurrency point P is X_{57} of $\triangle ABC$; perspector of $\triangle ABC$ and the orthic triangle of $\triangle DEF$. This lies on the line connecting the centroid and Gergonne point.

Tangent of $\odot(IBC)$ at I (perpendicular to AI at I) cuts BC at A_2 and AA_2 cuts EF at X. A_3 is the projection of D on EF. Since EF is the polar of A WRT (I) and DA_3 is the polar of A_2 WRT (I), it follows that AA_2 is the polar of A_3 WRT $(I) \Longrightarrow A(E,F,A_3,X) = A(B,C,A_3,A_2) = -1$. But $(B,C,A_1,A_2) = -1 \Longrightarrow A_3 \in AA_1$. Likewise, BB_1, CC_1 pass through the projections of E, F on $FD, DE \Longrightarrow X_{57} \equiv AA_1 \cap BB_1 \cap CC_1$.

TelvCohl
#3 Dec 19, 2014, 10:34 am • 2

I think you forgot to mention $A_1 \in BC, B_1 \in CA, C_1 \in AB$:)

My solution:

I'll prove AA_1, BB_1, CC_1 are concurrent at X_{57} of $\triangle ABC$.

Let D, E, F be the tangent point of (I) with BC, CA, AB, respectively .
Let $X = AI \cap BC, Y = AI \cap EF$ and Z be the projection of D on EF .

Since $Z(Y,D;B,C) = -1$,
so DZ is the bisector of $\angle BZC$,
hence we get $\triangle BFZ \sim \triangle CEZ$.

Since $(E,F;Y,Z) = \frac{FZ}{EZ} = \frac{BZ}{CZ} = \frac{BD}{CD} = \frac{\cot\frac{\angle ABC}{2}}{\cot\frac{\angle BCA}{2}}$

$$= \frac{\sin^2\frac{\angle BCA}{2}}{\sin^2\frac{\angle ABC}{2}} \cdot \frac{\sin\angle ABC}{\sin\angle BCA} = \frac{IB^2}{IC^2} \cdot \frac{AC}{AB} = (C,B;X,A_1)$$

so we get A, Z, A_1 are collinear . ie. $X_{57} \in AA_1$
Similarly, we can prove $X_{57} \in BB_1$ and $X_{57} \in CC_1$,
so we get AA_1, BB_1, CC_1 are concurrent at X_{57} of $\triangle ABC$.

Q.E.D

PS. Notice that X_{57} is the homothetic center of intouch triangle and excentral triangle,
so from $\text{intouch}\triangle \cap X_7 \sim \text{excentral}\triangle \cap X_9$ and X_9 is the complement of X_7 we get $X_{57} \in X_2X_7$:) .

(By the way, this problem implies that X_{57} is the Orthocorrespondent of X_1 :D .)

rodinos

#4 Dec 19, 2014, 11:02 am

Let ABC be a triangle and P a point. The symmedians of the triangles PBC, PCA, PAB corresponding to sides BC, CA, AB, respectively, intersect these sides at A', B', C' respectively. Then the lines AA', BB', CC' are concurrent.

A discussion in Hyacinthos (January 2000):
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/topics/219

Luis González

#5 Dec 19, 2014, 11:08 am • 1

Antreas, it simply follows from Ceva's theorem:

$$\frac{A'B}{A'C}\cdot\frac{B'C}{B'A}\cdot\frac{C'A}{C'B}=\frac{PB^2}{PC^2}\cdot\frac{PC^2}{PA^2}\cdot\frac{PA^2}{PB^2}=1\Longrightarrow AA',BB',CC' \text{ concur.}$$

In general, for $P(x:y:z)$, the barycentric coordinates of $X\equiv AA'\cap BB'\cap CC'$ are

$$X\equiv(b^2z^2+c^2y^2+2S_Ayz:c^2x^2+a^2z^2+2S_Bzx:a^2y^2+b^2x^2+2S_Cxy).$$

Quick Reply

High School Olympiads

hard geo 2

Reply

junior2001
#1 Dec 18, 2014, 11:47 pm

Let ABC be a triangle and let $D; E; F$ be the tangency points of the incircle with $BC; CA; AB$, respectively. Let EF meet the circumcircle (O) of ABC at X and Y . Furthermore, let T be the second intersection of the circumcircle of DXY with the incircle. Prove that AT passes through the tangency point T_a of the A—mixtilinear incircle with (O).

TelvCohl
#2 Dec 19, 2014, 12:41 am • 4

My solution:

Let I be the incenter of $\triangle ABC$.

Invert WRT (I) and denote P' as the image of P .

Since it's well known T_aI pass through the midpoint of arc BAC ,
so we get (AIT_a) is orthogonal to (ABC) and T'_a is the antipode of A' in $(A'B'C')$.

Now we get the new problem:

Let A', B', C' be the midpoint of EF, FD, DE, respectively .
Let T'_a be the antipode of A' and I be the center of (DEF) .
Let $\{X', Y'\} = (IEF) \cap (A'B'C')$ and $T' = (DX'Y') \cap (I)$.

Prove that T', T'_a, A', I are concyclic

Proof of the new problem:

Let H be the orthocenter of $\triangle DEF$ and $R = DT' \cap EF$.

Since $DT', X'Y', EF$ are concurrent at the radical center of $\{(I), (DX'Y'), (IEF)\}$,
so we get R, X', Y' are collinear and $RX' \cdot RY' = RE \cdot RF$.
ie. R lie on the radical axis of (DEF) and $(A'B'C')$

Since $T' \in (DH)$ and $T' \in HA'$ (well known) ,
so we get $T'T'_a = HT'_a = A'I$ and $T'A' \parallel T'_aI$,
hence $T'T'_aIA'$ is an isosceles trapezoid ,
so we get T', T'_a, A', I are concyclic .

Q.E.D

Luis González
#3 Dec 19, 2014, 3:47 am • 3

Let M be the midpoint of EF and let Q be the projection of D on EF. DA, DQ cut $(I) \equiv \odot(DEF)$ again at P, S, respectively and SP cuts EF at R. It suffices to show that A, S, T are collinear, since AS goes through the exsimilicenter of $(O) \sim (I)$.

Since $DEPF$ is harmonic, it follows that $\triangle PME \sim \triangle PFD$, i.e. $\angle PME = \angle PFD = \angle PSD \Longrightarrow PSQM$ is cyclic $\Longrightarrow R$ is radical center of $\odot(PSQM), (I)$ and the 9-point circle ω of $\triangle DEF \Longrightarrow R$ is on radical axis of the pencil $(I), (O), \omega \Longrightarrow RX \cdot RY = RE \cdot RF \Longrightarrow R \in TD$. Now, since R is on the polar EF of A WRT (I), the 2nd intersections S, T of RP, RD with (I) are collinear with A (valid for any R on EF actually).

Dukejukem

#4 Apr 5, 2015, 4:24 am

> Luis González wrote:
>
> Let M be the midpoint of EF and let Q be the projection of D on EF. DA, DQ cut $(I) \equiv \odot(DEF)$ again at $P, S,$ respectively and SP cuts EF at R. It suffices to show that A, S, T are collinear, since AS goes through the exsimilicenter of $(O) \sim (I)$.
>
> Since $DEPF$ is harmonic, it follows that $\triangle PME \sim \triangle PFD$, i.e. $\angle PME = \angle PFD = \angle PSD \Longrightarrow PSQM$ is cyclic $\Longrightarrow R$ is radical center of $\odot(PSQM), (I)$ and the 9-point circle ω of $\triangle DEF \Longrightarrow R$ is on radical axis of the pencil $(I), (O), \omega \Longrightarrow RX \cdot RY = RE \cdot RF \Longrightarrow R \in TD$. Now, since R is on the polar EF of A WRT (I), the 2nd intersections S, T of RP, RD with (I) are collinear with A (valid for any R on EF actually).

I have two questions: First of all, how do we know that AS goes through the exsimilicenter of $(O) \sim (I)$? Second, how did we go from R is the radical center of $\odot(PSQM), (I), \omega$ to R is also the radical center of $(I), (O), \omega$?

Quick Reply

High School Olympiads

Mini Sangaku

Reply

Source: Ruben Dario

leonardg

#1 Dec 15, 2014, 10:32 am

An interesting relation :

Attachments:

Dario

Let ABC be a triangle and G be its centroid . An arbitrary line through G intersects the segments (BC) and (AC) in P and respectively and same line intersects the line BA in M . Prove that

$$\frac{1}{GP}=\frac{1}{GN}+\frac{1}{GM} \ .$$

leonardg

#2 Dec 15, 2014, 10:34 am

On 2-nd row , after P it follows N .

Luis González

#3 Dec 18, 2014, 12:11 pm • 1

Let $\triangle A'B'C'$ be the antimedial triangle of $\triangle ABC$. Let $L \equiv PG \cap B'C'$ and $D \equiv AG \cap BC, F \equiv AG \cap AB$ are the midpoints of BC, AB. From $AL \parallel DP$, we get $\frac{GL}{GP}=\frac{GA}{GD}=2$ and since $A(C,F,G,C')=-1 \Longrightarrow$ $(N,M,G,L)=-1 \Longrightarrow \frac{1}{GN}+\frac{1}{GM}=\frac{2}{GL}=\frac{1}{GP}$.

leonardg

#4 Dec 20, 2014, 2:17 am

Beautiful.

PROF65

#5 Dec 20, 2014, 11:02 pm

K is the intersection of the parallel to AB through C and MN

$(CA, CB; CG, CK)$ is harmonic then $\frac{2}{GK}=\frac{1}{GN}+\frac{1}{GP}$

besides $\frac{GK}{GM}=\frac{GC}{GI}=-2$ thus $\frac{1}{GN}+\frac{1}{GP}+\frac{1}{GM}=0$

if P is on segment BC , N is on segment AC and M is on ray BA then GN and GM has the same direction unlike GP therfore with distances the equality occur

Quick Reply

High School Olympiads

Either PQE or PQF is right

Source: USA December TST for the 56th IMO, by Evan Chen

v_Enhance

#1 Dec 18, 2014, 3:21 am • 7

Let ABC be a non-isosceles triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Let P be the point inside the triangle on line AI for which $MD = MP$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$.

Proposed by Evan Chen

pi37

#2 Dec 18, 2014, 3:37 am • 1

Let D', X be the reflections of D across M, I respectively. It is well known that D' is the point at which the excircle touches BC, and also that A, X, D' are collinear. If Q' is the second intersection of this line and the incircle, then $\angle AQ'D = \angle XQ'D = 90^\circ$. So $Q' = Q$, and Q lies on AXD'.

Let ω be the incircle and ω_1 be the circle with diameter DD'. Note that Q lies on ω_1, which is also the circle centered at M with radius MD. Thus P is one of the two intersections of AI and ω_1.

Note that Q is the spiral similarity center mapping DD' to XD. The same spiral similarity then maps ω_1 to ω. Thus it suffices to show that this spiral similarity maps AI to EF. This is because if this is true, P maps to either E or F, so if it WLOG maps to E, $\angle PQE = \angle D'QD = 90^\circ$.

So note that the spiral similarity has angle 90°, and that $AI \perp EF$. Thus it suffices to show that there is a point on AI which maps to EF. Specifically, we show that I maps to a point I' on EF. Note that since the similarity maps M to I, it then maps I to the point I' on MQ such that $\angle I'IM = 90^\circ$ (since $\angle IQM = 90^\circ$). But MQ is tangent to ω since $MQ = MD$, so the polar of I' is the line through Q parallel to IM. But by homothety centered at D $IM \parallel AXQD'$, so said line is A. And thus the polar of A, EF, passes through I'.

Ifetahu

#3 Dec 18, 2014, 3:42 am

I wonder whether there is any solution that does not use spiral similarity.

Luis González

#4 Dec 18, 2014, 4:12 am • 5

WLOG assume that $AC > AB.$ A-excircle (I_a) touches BC at D' and U is the antipode of D WRT $(I).$ As A is the exsimilicenter of $(I) \sim (I_a),$ it follows that A,U,D' are collinear. Thus if AUD' cuts (I) again at $Q',$ we have $\angle UQ'D \equiv \angle AQ'D = 90^\circ \Longrightarrow Q \equiv Q'.$ Since M is also midpoint of $DD',$ then $DPQD'$ is cyclic with circumcenter $M.$

Note that P is intersection of AI with the B-midline of $\triangle ABC,$ thus it is on DE (well-known). Hence $\angle PQE = \angle UQE + \angle UQP = \angle UDE + \angle EDC = 90^\circ,$ as desired.

superpi83

#5 Dec 18, 2014, 4:16 am • 1

Assume WLOG $AB < AC$. We will show $\angle PQE = 90^\circ$.

Lemma I: AI, DE, and the line through M parallel to BC are concurrent. Furthermore their intersection is P.

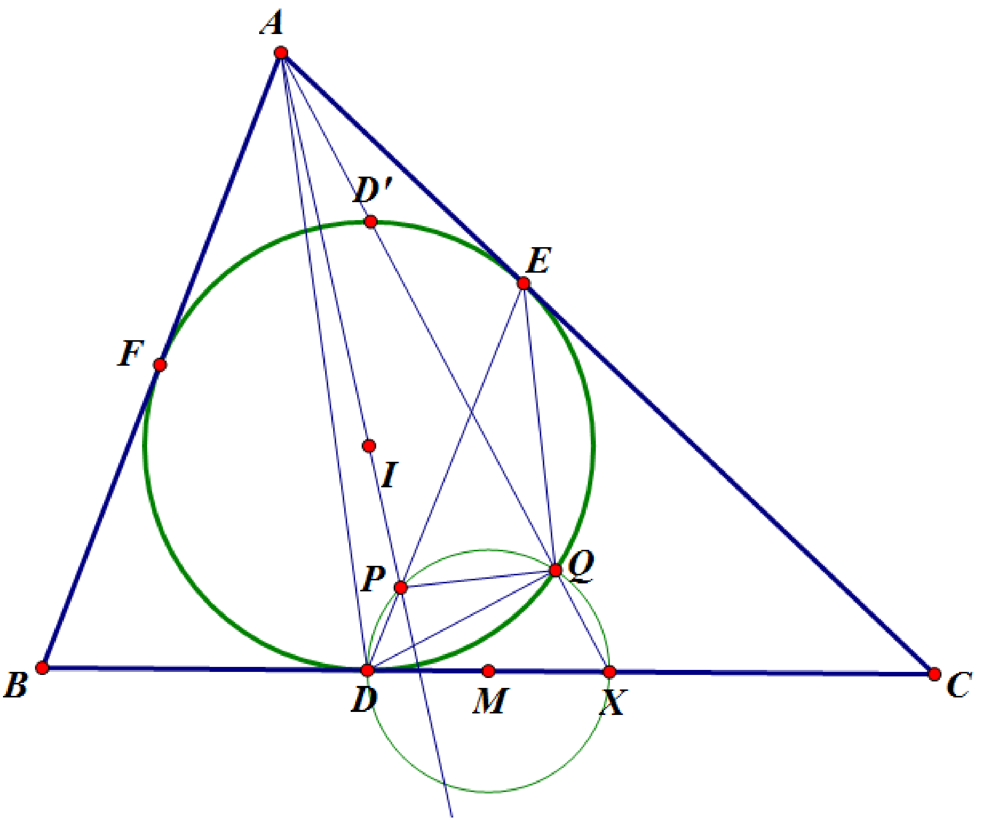
Proof: The lines are concurrent by the Midline Concurrency Lemma (see e.g. here). Say this intersection is P'. Then $\angle DP'M = \angle DEC = \angle EDC = \angle P'DM$. So, $P'M = DM$ and P' lies on AI. Hence $P' = P$.

Since $\angle AQD = 90^\circ$, AQ passes through D', the point diametrically across D on the incircle. By a well-known fact of D', AQ passes through the tangency point of the A-excircle on BC, which we call X.

Since $BD = CX$, M is the midpoint of DX. Since $AQ \perp QX$, Q lies on the circle with diameter AX. It follows that A, P, Q, X are all on the circle with center M and radius DM.

Since we have $\angle DQD' = 90^\circ$, it suffices to show $\angle DQP = \angle D'QE$. Since DD' is tangent to the circle with center M and radius DM, we have $\angle DQP = \angle D'DP$. We also have $\angle D'QE = \angle D'DE$. By collinearity of D, P, E (Lemma I), $\angle D'DE = \angle D'DP$ and we are done.

Attachments:



djmathman

#6 Dec 18, 2014, 4:25 am • 1

Let ω be the incircle of $\triangle ABC$, and WLOG let $AB > AC$. We will show that $\angle PQF = 90^\circ$ in this case; if this holds true then trivially $\angle PQE = 90^\circ$ if $AC > AB$, completing the proof.

First, we show that M is the circumcenter of $\triangle DPQ$. Let $N = AQ \cap BC$ and $D' = AQ \cap \omega$. Then since $AQ \perp QD$ we have D' and D antipodal, so a homothety centered at A sending ω to the A-excircle of $\triangle ABC$ sends D' to the tangency point of the A-excircle of $\triangle ABC$ with BC, which is therefore N. It is well-known that $BN = CD$, and since $BM = MC$ trivially we have $NM = MD$. From the previously-mentioned perpendicularity M is the midpoint of the hypotenuse $\overline{ND}$ of right triangle NQD, so $MQ = MP = MD$.

Next, we show that F, P, and D are collinear. Let FD intersect AI at a point P'. Angle chasing gives

$$\angle AP'F = \angle BFD - \angle FAP' = \frac{\pi - \angle B}{2} - \frac{\angle A}{2} = \frac{\angle C}{2}.$$

Therefore $\angle IP'F \equiv \angle AP'F = \angle IP'D$, so $ICDP'$ is a cyclic quadrilateral. Hence $\angle IP'C = \angle IDC = 90^\circ$, so P' is the foot of the perpendicular from C to the bisector from $\angle A$. From here it is well-known that $P'M \parallel AB$, so more angle chasing gives

$$\angle MP'D = \angle PAB + \angle ICD = \frac{1}{2}\angle A + \frac{1}{2}\angle C = \frac{\pi - \angle B}{2} = \angle FDM \equiv \angle P'DM.$$

Hence P' lies on the circle centered at M with radius $MD \implies P \equiv P'$.

To finish, note that $D'D \perp BC$ from the fact that D' and D are antipodes. This makes $\angle PDN$ and $\angle D'DF$ complementary, so $\angle PND = \angle D'DF = \angle D'QF$. Therefore

$$\angle FQP = \angle FQD' + \angle D'QP = \angle D'QP + \angle PQD = 90^\circ,$$

which is what we wanted.

v_Enhance

#7 Dec 18, 2014, 5:05 am • 1

> *lfetahu wrote:*
> I wonder whether there is any solution that does not use spiral similarity.

The solution I had in mind is the same as Luis's -- I was not actually aware of the spiral similarity solution until after the fact.

This problem was created by just messing around in GeoGebra (I had a lot of time in April). I was just curious what would come up once I constructed the circle with center M and radius D. I noticed the collinearity of D, P, E (in $AB < AC$ case) after about 20 minutes, and I remember I was initially very confused about how this was not symmetric. The observation about $\angle PQE$ came about during my half-asleep attempts to prove the collinearity.

TelvCohl

#8 Dec 18, 2014, 5:07 am • 5

My solution:

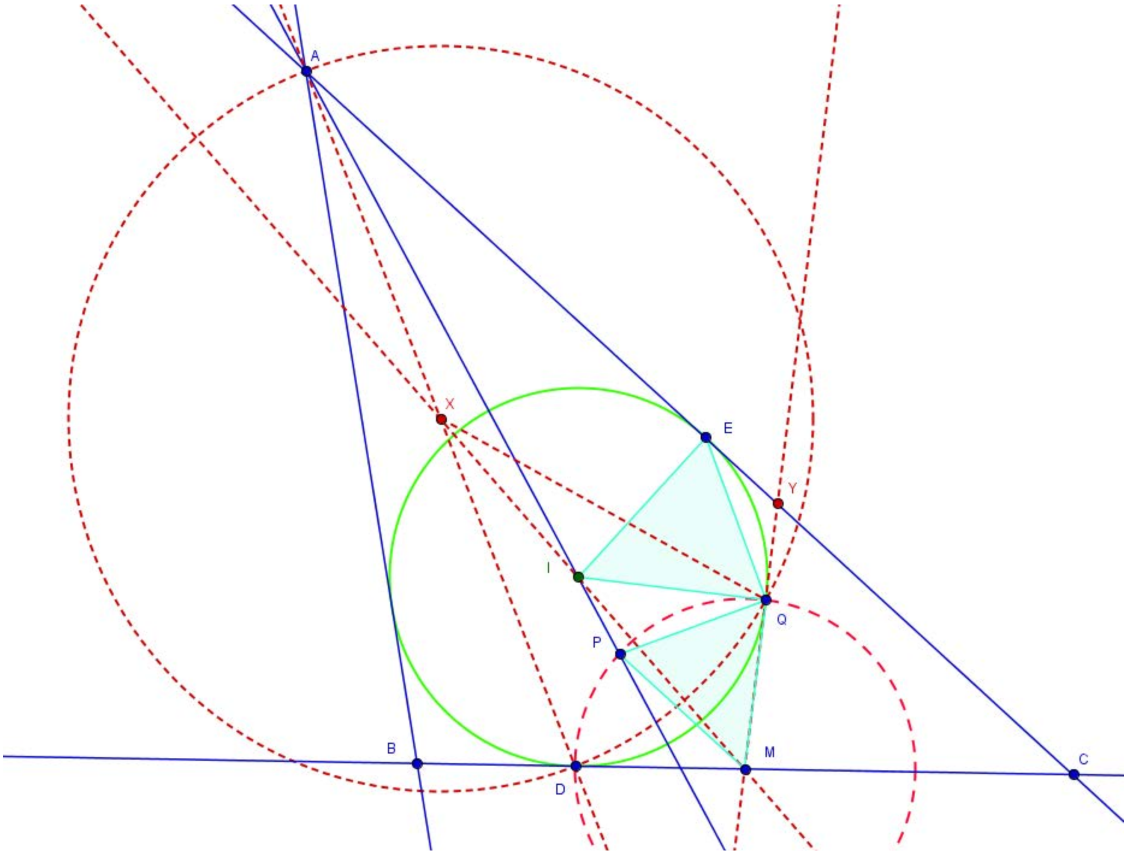
Let X be the midpoint of AD and $Y = MQ \cap AC$.

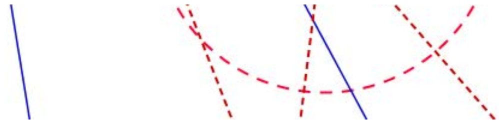
Since $Q \equiv (I) \cap (AD)$,
so we get XI is the perpendicular bisector of DQ .
Since $X \in IM$ (well known) ,
so we get MQ is tangent to (I) at Q .

Since $MP \parallel AC$ (well known) ,
so $\angle QMP = \angle MYC = \angle QIE$,
hence we get $\triangle QMP \sim \triangle QIE$ and $\angle EQP = \angle IQM = 90^\circ$.

Q.E.D

Attachments:





buratinogigle

#9 Dec 18, 2014, 8:50 am

Some consequence from this nice configuration

The circle (M, MQ) intersect AI at P, R (P is inside triangle) and intersect BC again at S.

- Note that Spiral similarity center Q transform triangle DEF to triangle SPR. So triangle SPR and DEF are similar.

- Tangents of (M) from P, R intersect at T then $AT \perp BC$.

- Tangents of (M) from P, S intersect at U and tangents of (M) from R, S intersect at V then $BU \parallel CV$.

Konigsberg

#10 Dec 18, 2014, 9:41 am • 1

We will bary you!

buratinogigle

#11 Dec 18, 2014, 10:08 am • 2

General problem

Let ABC be a triangle and DEF is pedal triangle of P. (DEF) cut BC again at G. The line passes through P and perpendicular to EF cuts DE, DF at M, N. Circle (DMN) cuts (DEF) again at Q. T is a point such that $TM \perp AC, TN \perp AB$. AT cuts BC at S. Prove that four points A, Q, G, S are concyclic.

TelvCohl

#12 Dec 18, 2014, 11:08 am • 4

> *buratinogigle wrote:*
>
> General problem
>
> Let ABC be a triangle and DEF is pedal triangle of P. (DEF) cut BC again at G. The line passes through P and perpendicular to EF cuts DE, DF at M, N. Circle (DMN) cuts (DEF) again at Q. T is a point such that $TM \perp AC, TN \perp AB$. AT cuts BC at S. Prove that four points A, Q, G, S are concyclic.

My solution:

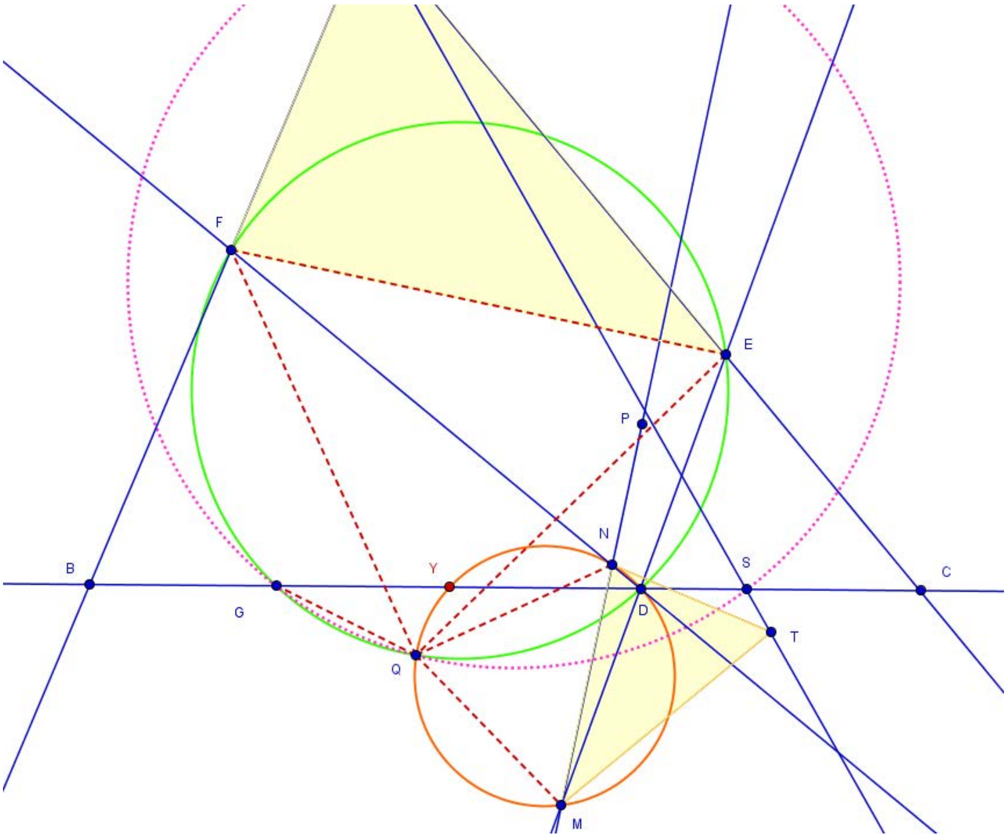
Let $Y = (DMN) \cap BC$.

Since $MN \perp EF, TM \perp AE, TN \perp AF$,
so we get $\triangle TMN \sim \triangle AEF$.
Since $\angle MQN = \angle EDF = \angle EQF$,
so Q is the spiral center of $\triangle TMN \mapsto \triangle AEF$ . ... $(\star)$

Since $\angle YNQ = \angle GFQ$,
so $\triangle TMN \cap Y \sim \triangle AEF \cap G$,
hence combine with $(\star)$ we get $\angle QGS = \angle QGY = \angle QAT = \angle QAS$.
ie. A, Q, G, S are concyclic

Q.E.D

Attachments:



Luis González

#13 Dec 18, 2014, 11:41 am • 3

> *buratinogigle wrote:*
>
> General problem
>
> Let ABC be a triangle and DEF is pedal triangle of $P.$ (DEF) cut BC again at $G.$ The line passes through P and perpendicular to EF cuts DE, DF at $M, N.$ Circle (DMN) cuts (DEF) again at $Q.$ T is a point such that $TM \perp AC, TN \perp AB.$ AT cuts BC at $S.$ Prove that four points A, Q, G, S are concyclic.

Clearly Q is center of spiral similarity that carries $\triangle QMN$ into $\triangle QEF$ and it also carries T into A as $\triangle TMN$ and $\triangle AEF$ are directly similar. Since MN, EF form right angle, it follows that the rotational angle is right $\Longrightarrow$ $\angle MQE = \angle TQA = 90^\circ.$ Hence if $U \equiv TM \cap AC,$ then $MQUE$ and $TQUA$ are cyclic $\Longrightarrow$ $\angle TAQ = \angle MUQ = \angle DEQ = \angle CGQ \Longrightarrow AQGS$ is cyclic.

dibyo_99

#14 Dec 18, 2014, 2:48 pm

WLOG, $AB > AC$. Let $AQ \cap (I) = T$. Since, $\angle TQD = 90^\circ$, so T is the antipode of D w.r.t (I). Therefore, AQ intersects BC at the A-extouch point D'. Since $\angle DQD' = 90^\circ$ and M is the midpoint of the hypotenuse of $\triangle DQD'$, M is the circumcentre of $\triangle DD'Q$. This also means that $DD'PQ$ is cyclic with centre M.

Let $DF \cap AI = P'$. Then, it is well known that MP' is the C-midline and that I, P', C, D, E are concyclic. Let $MP' \cap (IPD) = T$. Then

$$\angle TCD = \angle MP'D = \angle BFD = \angle BFD = \angle BDF$$

$$\angle MDF = \angle MDP' \implies MD = MP'$$

Therefore, $P = P'$, giving $MPAB$. Some more angle chasing gives

$$\angle DQF = 180^\circ - \frac{\angle FID}{2} = 90^\circ + B/2 = 90^\circ + \frac{\angle DMP}{2}$$

$$= 90^\circ + \angle DQP \implies \angle PQF = 90^\circ$$

shinichiman
#15 Dec 18, 2014, 3:39 pm • 1

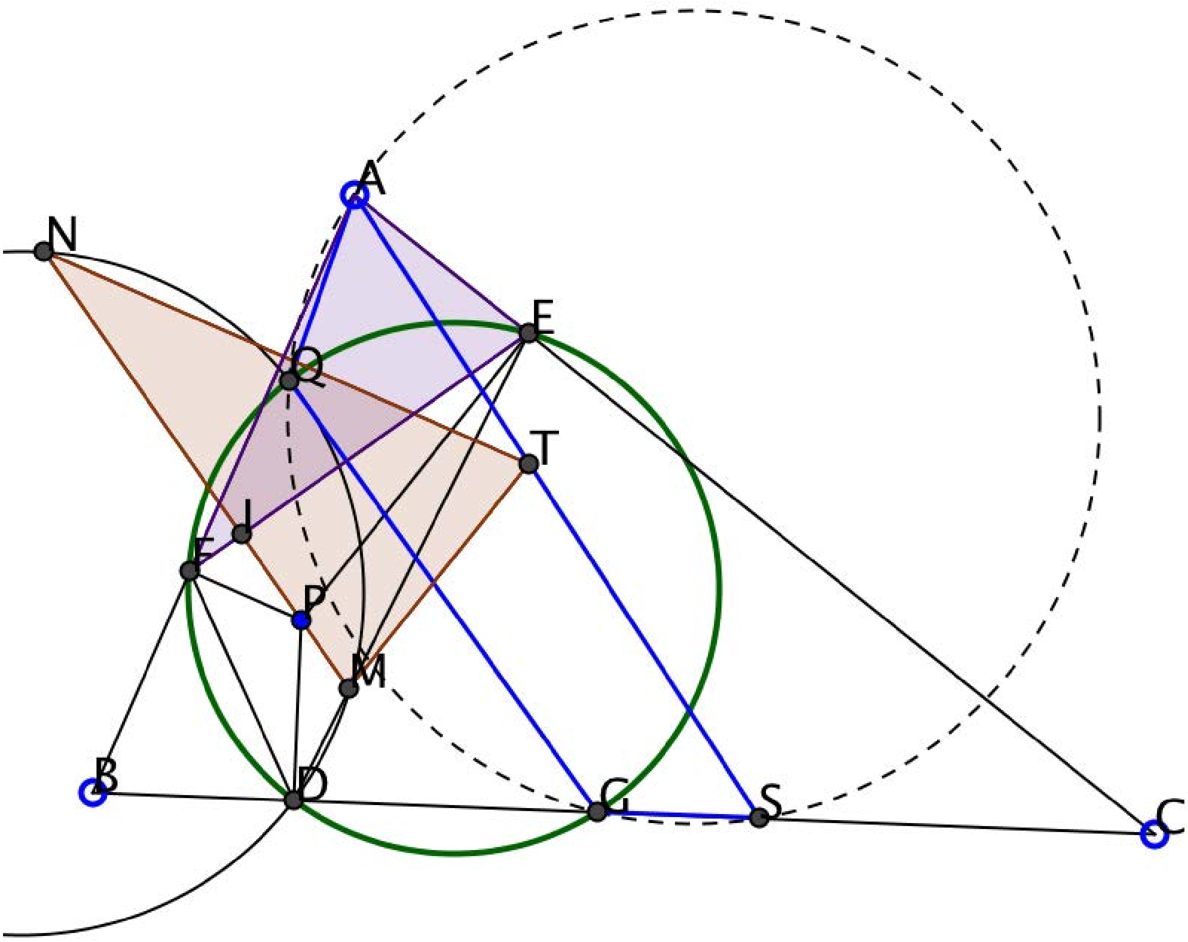
> *buratinogigle wrote:*
> General problem
>
> Let ABC be a triangle and DEF is pedal triangle of P. (DEF) cut BC again at G. The line passes through P and perpendicular to EF cuts DE, DF at M, N. Circle (DMN) cuts (DEF) again at Q. T is a point such that $TM \perp AC, TN \perp AB$. AT cuts BC at S. Prove that four points A, Q, G, S are concyclic.

MN intersects EF at I. We have

$$\begin{aligned}(QF,QN) &\equiv (QF,QD)+(QD,QN) \pmod{\pi}\\ &\equiv (EF,ED)+(MD,MN) \pmod{\pi}\\ &\equiv (EI,EM)+(ME,MI)\equiv 90^\circ \pmod{\pi}.\end{aligned}$$

This follows $QE \perp QM$. Similarly, we obtain $QF \perp QN$. Since $(FD,FQ) \equiv (ED,EQ) \pmod{\pi}$ so $\triangle EQM \sim \triangle FQN$ (A.A). From here, we can easily get $\triangle QMN \sim \triangle QEF$ (S.A.S). On the other hand, we can easily prove that $\triangle TMN \sim \triangle AEF$ (A.A). Therefore, from these two couple of similar triangle, we obtain $\triangle QEA \sim \triangle QMT$ (S.A.S). This follows $\angle AQT = \angle MQE = 90^\circ$ and $\frac{AQ}{TQ} = \frac{EQ}{QM}$. Hence, $\triangle AQT \sim \triangle EQM$ (S.A.S). We get $(AQ,AS) \equiv (EQ,ED) \equiv (GQ,GS) \pmod{\pi}$ or Q, A, G, S are concyclic. ■

Attachments:



jayme
#16 Dec 18, 2014, 5:36 pm

Dear Mathlinkers,
for the first problem

1. D, P and E are collinear (well known see : http://jl.ayme.pages perso-orange.fr/ vol. 4, Un unlikely concurrence…p. 4-5)
2. AQ goes thrhrough the antipole of D wrt (I)
3. the circle centered at M and passing through D is orthogonal to (I) : it passes also by Q (according to a developpment with the

Reim's theorem)
4. then QPE is Q-rectangular…

Sincerely
Jean-Louis

AnonymousBunny
#17 Dec 18, 2014, 11:41 pm

Solution

sayantanchakraborty
#18 Dec 19, 2014, 2:58 pm

I also have a solution.

Without loss of generality let $AB \leq AC$ then Q lies on the minor arc DE of the incircle.Let $X = AQ \cap BC$ and $Y = AQ \cap (I)$.It is well known that the homothety with center A sending Y to X sends I to $I_a \implies X$ is the point of contact of the excircle with $BC \implies BD = CX$.Thus M is the midpoint of DX,and as a consequence $DPQX$ is cyclic with center M.Let $P' = DE \cap AJ$ where AJ is the angle bisector of A.A very quick bash shows that $\frac{JP'}{JI} = \frac{JX}{JC} = \frac{(b-c)(b+c-a)}{2ab}$ so $XP' \parallel CI$.This means that $XP' \perp P'D$ or in other words $MP' = MD \implies P' \equiv P$.Thus we have $\angle PQE = \angle PDI = \angle EQY$ or $\angle PQE = 90^\circ$ as desired.

A good geometry by v_Enhance.Thank you.

Konigsberg
#19 Dec 21, 2014, 10:36 am • 1

I reduced the problem to showing that Q is on the circle with radius MD, but I can't seem to do that. Any help?

AnonymousBunny
#20 Dec 21, 2014, 11:20 am • 1

^Well, that follows trivially -- (as in the other solutions...) if AQ intersects BC at T, then T is the point where the A excircle touches BC, so the circle with radius MD is actually the circle with diameter DT. Now the result follows by noting that $DQ \perp QT$.

djmathman
#21 Dec 21, 2014, 7:53 pm

Eh, I wouldn't say that follows *trivially*... if you happen to recognize the configuration then of course extending AD down to BC is natural, but it doesn't follow immediately.

Wolstenholme
#22 Jan 7, 2015, 11:39 am

I got a 7/7 on this problem, but my solution was rather unique - it was a mix of synthetic observations and complex numbers.

Assume WLOG that $AC > AB$. Let ω denote the incircle of $\triangle ABC$ and let X be the antipode of D with respect to ω. It's clear that line AQ passes through X since DX is a diameter of ω.

Now let $D' = AQ \cap BC$. It is well-known (and easily proved by homothety) that M is the midpoint of DD'. Now let Ω denote the circle centered at M with radius MD. Since $MD = MP$ and since $\angle DQD' = 90$ we have that points D, D', P, Q all lie on this circle. Let E' be the second intersection of line EQ with Ω. By the standard Yufei configuration, it is clear that D is the center of the spiral similarity taking EE' to XD'. Therefore $\angle EDE' = \angle XDD' = 90$.

Now, I will prove that $E.P, D$ are collinear with complex numbers. Let $P' = AI \cap ED$. Let the lowercase letter denote the complex coordinate of the point denoting its corresponding uppercase letter. Assume WLOG that ω is the unit circle. We clearly have $a = \frac{2ef}{e+f}$ and $b = \frac{2df}{d+f}$ and $c = \frac{2de}{d+e}$. Therefore $m = \frac{de}{d+e} + \frac{df}{d+f}$.

Now the equation of line AI is $z = ef\overline{z}$ and the equation of line DE is $z = e + d - ed\overline{z}$. Solving, we find that

$p' = \dfrac{f(e+d)}{d+f}$. Now to prove that $P = P'$ it suffices to show that $MP' = MD$ which in complex numbers translates to showing that $(m-p')(\overline{m}-\overline{p'}) = (m-d)(\overline{m}-\overline{d})$ which is relatively easy.

Therefore E, P, D are collinear as desired. Therefore $\angle PQE = 180 - \angle PQE' = \angle PDE' = \angle EDE' = 90$ so we are done.

Motivation:

Seeing the definition of Q, I immediately noticed the construction of line AX since this is a pretty standard configuration. After constructing point D' I tried to use the definition of P and found an easy way to do so by drawing Ω since this seemed WAY more natural than $MD = MP$. Now that I had two intersecting circles in the diagram I immediately thought about the spiral similarity and the rest of the steps followed naturally from there.

ATimo

#23 Jan 22, 2015, 7:06 pm • 1

USA TST 2015.

supercomputer

#24 Feb 8, 2015, 7:18 am

Let X be the antipode of D on the incircle of ABC and $AQ \cup BC = D'$. Note that once you show that D, P, E are collinear and M is the center of $DPQD'$, the solution can be finished as follows:

We find that $\triangle DPM \sim \triangle DEC$ so that $\angle PDD' = \angle PDM = 90 - \dfrac{C}{2}$. Then, $\angle PQD' = 180 - \angle PDD' = 90 + \dfrac{C}{2}$. Thus, $\angle XQP = 180 - \angle PQD' = 90 - \dfrac{C}{2}$.

Then, simply note that $\angle DIE = 180 - C$ so that $\angle XIE = 180 - \angle DIE = C$. Then, $\angle XQE = \dfrac{\angle XIE}{2} = \dfrac{C}{2}$.

Thus, $\angle PQE = \angle PQX + \angle XQE = 90 - \dfrac{C}{2} + \dfrac{C}{2} = 90$ as desired.

So all we really needed to do was find the circle DPQ with center M (follows from angle chase+the excircle tangency point) and show that DE and AI intersect on this circle (we can use complex numbers or synthetic or trig bash). The rest follows from an angle chase. In particular, no spiral similarity or very many extra constructions (besides X and D') are necessary.

Radar

#25 Feb 22, 2015, 5:35 am

WLOG $AC > AB$. Let D' and E' be the points on the incircle of $\triangle ABC$ opposite to D and E. Let P' be the intersection of line segment $E'Q$ and circle ω with center M and radius MD. It's well-known that Q lies on ω. Obviously $D'D$ is tangent to ω and obviously arcs ED' and $E'D$ are equally long. Therefore $\angle P'DD' = \angle P'QD = \angle E'QD = \angle E'ED = \angle EDD'$, so P' lies on ED. From Pascal theorem applied on $DEEE'QD'$ we get, that A, I and P' are collinear, so $P' = P$. But then $\angle PQE = \angle E'QE = 90^\circ$, so we're done.

Q.E.D.

EulerMacaroni

#26 Aug 6, 2015, 4:42 am • 1

Let G be the antipode of D with respect to the incircle and H be the tangency point of the A-excircle on BC. Remark that by homothety, A, G and H are collinear. It is clear by reflection that $MD = MH$, so M is the circumcenter of $\triangle DPH$, and this circle has diameter $\overline{DH}$, so by Thales' theorem, Q lies on the circle as well, since $\angle AQD = \angle DQH = 90^\circ$.

Now, we claim that $MP \parallel AC$.

Proof

We see that by the homothety that sends $\triangle DMP$ to $\triangle DCE$, D, P and E are collinear, whence

$$\angle PQE = \angle PQG + \angle GQE = 180^\circ - \angle PQH + \angle GDE = \angle PDH + \angle GDP = 90^\circ$$

as desired.

This post has been edited 1 time. Last edited by EulerMacaroni, Aug 6, 2015, 4:45 am

Reason: angle chase error

rkm0959

#27 Oct 30, 2015, 10:06 am

WLOG $AB < AC$. We show that $\angle PQE = 90$.
Let the antipode of D be D'. It is clear that A, D', Q are colinear. Also, let $AD' \cap BC = X$.
It is well known that $BD = CX \implies DM = MX$. Therefore,
$MD = MP = MX \implies \angle DPX = \angle DQX = 90 \implies D, P, Q, X$ are cyclic.
Now let $P' = AI \cap DE$. I claim that $P' = P$. We use barycentric coordinates.
It can be easily calculated that $P'(\frac{b-c}{2b} : \frac{1}{2} : \frac{c}{2b})$, so P' lie on the B-midline. Therefore $P'M \parallel AC$.
So $\frac{DM}{MP'} = \frac{DC}{EC} = 1$, so $MD = MP'$, as desired.
Now $\angle PQE = \angle PQD' + \angle D'QE = \angle PDM + \angle D'DE = \angle EDM + \angle D'DE = \angle DD'E + \angle D'DE = 90$
$\blacksquare$.

This post has been edited 2 times. Last edited by rkm0959, Oct 30, 2015, 10:07 am

AstrapiGnosis

#28 Dec 10, 2015, 4:33 am

using barycentric coordinates to bash out the coordinates of P (with no synthetic observations) works as well, but on an actual olympiad would we have to show every step of solving whatever quadratic pops up and grouping terms and whatever to show the grader we aren't just bs'ing??

This post has been edited 1 time. Last edited by AstrapiGnosis, Dec 10, 2015, 4:34 am
Reason: no synthetic!!!

K6160

#29 Apr 5, 2016, 1:52 pm

WLOG let $AB < AC$. Let K be the antipode of D w.r.t. the incircle and D' be the point of tangency of the A excircle with BC. Then Q is just the second intersection of AD' with the incircle. Also, P is the intersection of AI with the circumcircle of $\triangle PDD'$ since D' is the reflection of D about M. Let P' be the intersection of AI and DE. Then we know that $P'M$ is parallel to EC and this is sufficient to show that $P = P'$. Therefore, $P, D,$ and E are collinear. We have that $\angle KDQ = \angle QD'D = \angle EPQ$ and $\angle DKQ = \angle PEQ$, so $\angle PQE = 90$. $\square$

Quick Reply

High School Olympiads

Exsimilar center

Reply

Source: Unknown

shinny98NT

#1 Dec 14, 2014, 11:26 pm

Given triangle ABC, I is the incenter and O is the circumcenter of it. Two circle passing through AI and orthogonal to each other cut the circumference of triangle ABC at M and N. Prove that the exsimilar center (center of positive homothety) of (O) and (I) lies on MN.

TelvCohl

#2 Dec 15, 2014, 2:33 am • 1

My solution:

Let $X = AI \cap (ABC)$, $Y = (AI) \cap (ABC)$ ($X, Y \neq A$).
Let D, E, F be the tangent point of (I) with BC, CA, AB, respectively .

Invert WRT (I) and denote P' as the image of P .

Easy to see A', B', C' is the midpoint of EF, FD, DE, respectively .

Since the image of $(AIM), (AIN)$ are two lines passing through A' and perpendicular to each other ,
so we get $M'N'$ is the diameter of $(A'B'C')$ and the center T of $(A'B'C')$ is the midpoint of $M'N'$,
hence $(IM'N')$ form a coaxial system with radical axis $TI \equiv OI$ when M, N move on (ABC) .
ie. MN pass through a fixed point on OI ... $(\star)$

Consider the case $M \equiv X, N \equiv Y$.

Since it's well known that X, D, Y are collinear (see incenter I and touches BC side with D) ,
so $XY \equiv XD$ pass through the exsimilicenter of $(O) \sim (I)$ ($\because (O) \cap X \sim (I) \cap D$) ,
hence combine with $(\star)$ we get MN pass through the exsimilicenter of $(O) \sim (I)$.

Q.E.D

Luis González

#3 Dec 18, 2014, 12:12 am • 1

Inversion WRT $\odot(A, AI)$ takes $\odot(AIM) \equiv (O_1), \odot(AIN) \equiv (O_2)$ into 2 perpendicular lines through I, cutting the inverse line of (O) at the inverses M', N' of M, N. As $IM' \mapsto IN'$ is an involution, then $AM' \equiv AM \mapsto AN' \equiv AN$ induces an involution on $(O) \Longrightarrow MN$ goes through a fixed point.

Take $M \equiv B$. If Y is the tangency point of the B-mixtilinear incircle ω_B with AB, we get
$\widehat{IYB} = 90^\circ - \frac{1}{2}\widehat{ABI} = \frac{1}{2}\widehat{AO_2I} = \widehat{ANI} \Longrightarrow ANIY$ is cyclic $\Longrightarrow N$ is the tangency point of ω_B with $(O) \Longrightarrow$
$MN \equiv BN$ goes to the exsimilicenter X_{56} of $(I) \sim (O)$ and the same happens when $N \equiv C$. So X_{56} is the fixed point.

TelvCohl

#4 Dec 18, 2014, 6:40 am • 2

Another solution:

Lemma:

Let P be a point on BC .

Let I be the incenter of $\triangle ABC$.
Let ω_B, ω_C be the Thebault circles of cevian AP .
Let Y, Z be the tangent point of (ABC) with ω_B, ω_C, respectively .

Then (AIY) and (AIZ) are orthogonal

Proof of the lemma:

Let I_b, I_c be the incenter of $\triangle ABP, \triangle ACP$, respectively .

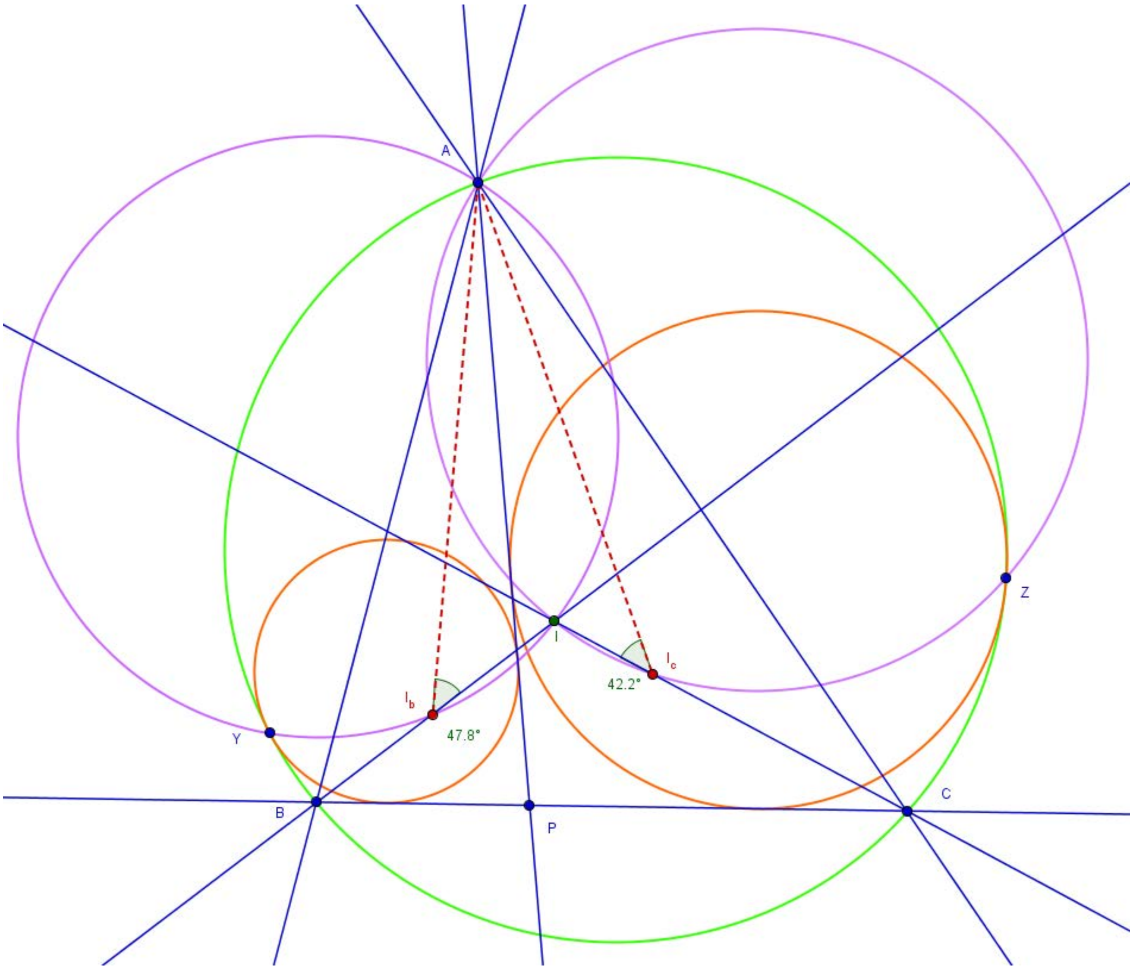
It's well known that $I_b \in (AIY)$ and $I_c \in (AIZ)$,
so from $\angle AI_bI + \angle AI_cI = (90^\circ - \frac{1}{2}\angle APB) + (90^\circ - \frac{1}{2}\angle APC) = 90^\circ$ we get $(AIY) \perp (AIZ)$.

Back to the main problem:

By the lemma we know we can find $P \in BC$ satisfy M, N are the tangent point of Thebault circles of AP with (ABC) ,
so from the problem Mabey Thebault have ovelooked we get $X_{56} \in MN$.

Q.E.D

Attachments:



High School Olympiads

Homography on a Conic

Reply

TelvCohl

#1 Dec 3, 2014, 3:04 pm

$P \mapsto Q$ is a homography on conic $\mathcal{C}$.

Prove that the envelope of PQ is a conic

Luis González

#2 Dec 17, 2014, 8:42 am • 2

Let $A,B \in \mathcal{C}$ be homologous points under the referred homography, different from P,Q. a,b,p,q denote the tangents of $\mathcal{C}$ at $A,B,P,Q,$ respectively. $M \equiv p \cap q, N \equiv q \cap b, L \equiv b \cap a$ and $O \equiv a \cap p$. Clearly, we have $X \equiv PB \cap QA \cap ML \cap NO$ and $Y \equiv PA \cap QB \cap ML.$

X moves on axis ℓ of the homography $P \mapsto Q$ that fixes $\mathcal{C}$ (invariant line of the extension of $P \mapsto Q$ on the plane). As the pencils AP, BQ are homographic, Y is on a fixed conic through $A,B.$ But NO is the polar of Y WRT $\mathcal{C} \Longrightarrow$ $N(B,Q,X,Y) = -1 \Longrightarrow (L,M,X,Y) = -1 \Longrightarrow X \mapsto Y$ is a homology fixing ℓ and all lines through $L.$ Consequently, M is on a fixed conic $\mathcal{M} \Longrightarrow PQ$ envelopes the dual conic of $\mathcal{M}$ WRT $\mathcal{C}.$

Quick Reply

High School Olympiads

???

Reply

Source: TelvCohl

toto1234567890
#1 Dec 14, 2014, 9:10 pm

Let M, M' be isogonal conjugate of $\triangle ABC$.
Let P be a point on MM' and P' be the isogonal conjugate of P .

Show that $P', M, (P/M)$ are collinear :)
Is there a synthetic proof?
(P/M): perspector of cevian triangle of P and anti-cevian triangle of M.

IDMasterz
#2 Dec 16, 2014, 10:54 am

First time I have attempted to prove this property - yay :)

First note that (P/M), as P moves, moves on a conic passing through $M_AM_BM_C$ (anticevian triangle) and P. Furthermore, the map $P' \mapsto P \mapsto (P/M)$ is clearly preserves cross-ratio. For the points when P are on the sidelines of ABC we note (P/M) coincides with M_A, M_B, M_C, P' concides with A, B, C respectively, so $(P/M), P', M'$ are collinear for these three points, so by cross ratio the result holds for all P on MM'.

Question is: Is there a better way to use $P \in MM'$ for practical purposes?

Luis González
#3 Dec 17, 2014, 7:25 am • 1

To the original poster, please give your posts meaningful subjects and correct sources. This is an old property posted a while back on TTW; so it was problably discussed first at Hyacinthos messages or somewhere else; see r1824. Similarly, the property holds for a pair of isotomic points; see r1825 (the proof is exactly the same).

For a proof see Property of Cevian quotient (post #3).

toto1234567890
#4 Dec 17, 2014, 6:09 pm

I didn't know Mr. Luis Gonzalez. :) And do you have a proof without using conics?
Is there a solution to properly use $P \in \overline{MM'}$?

Quick Reply

High School Olympiads

O, P, H are collinear

Source: Own

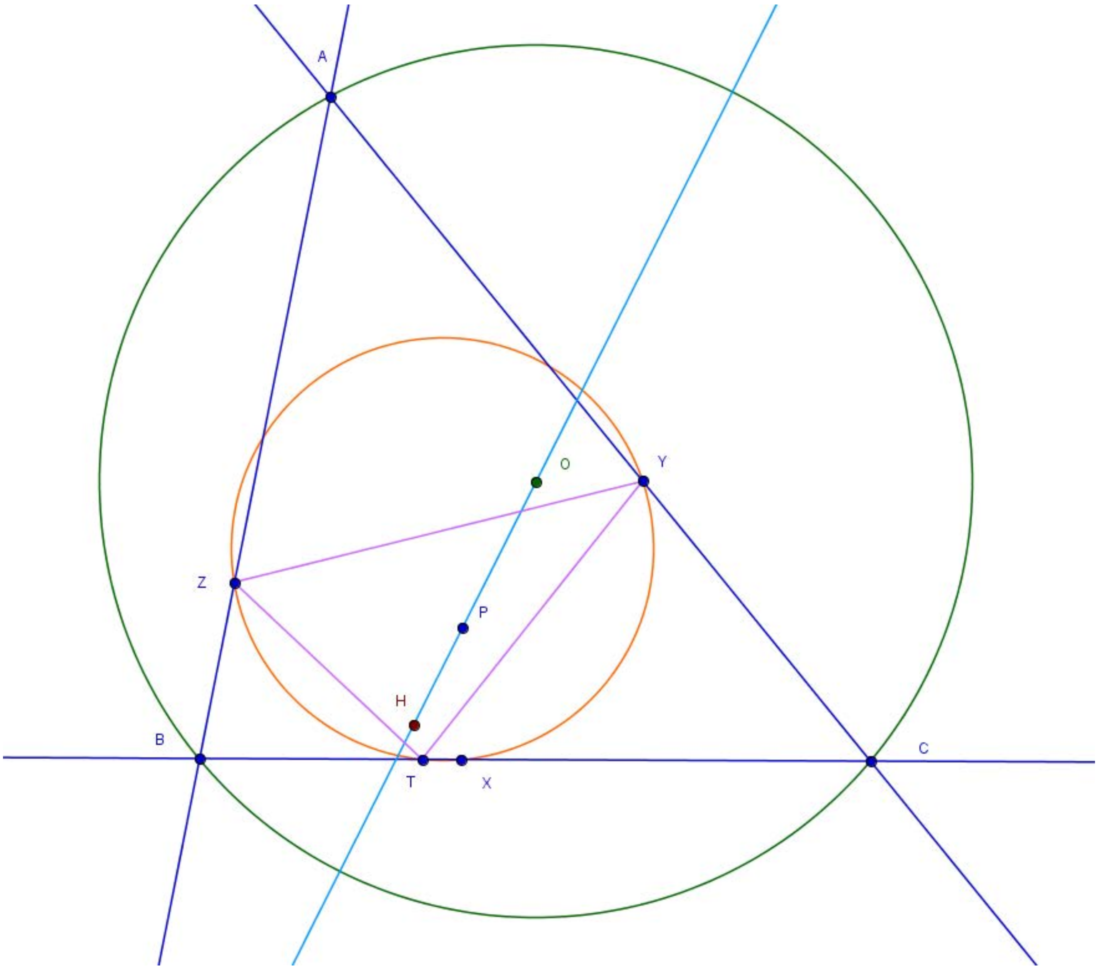
TelvCohl

#1 Nov 30, 2014, 8:32 pm • 1

Let O be the circumcenter of $\triangle ABC$.
Let P be a point on the bisector of $\angle BAC$.
Let X, Y, Z be the projection of P on BC, CA, AB, respectively .
Let T be the other intersection of (XYZ) and BC .
Let H be the orthocenter of $\triangle TYZ$.

Prove that O, P, H are collinear

Attachments:



Luis González

#2 Dec 2, 2014, 6:05 am • 1

Let Q be the isogonal conjugate of P WRT $\triangle ABC$, i.e. intersection of AP with the perpendicular to BC at T. Let M be the midpoint of YZ and let R be the reflection of P on M. If K is the midpoint of PQ (circumcenter of $\triangle TYZ$), we have $TH = 2 \cdot KM = KR - KP = KR - KQ = QR \Longrightarrow THRQ$ is parallelogram $\Longrightarrow (RH \parallel QT) \perp BC$.

As P varies, all $AYPZ \cup R$ are similar $\Longrightarrow R \mapsto P$ is a homothety with center A and clearly $P \barwedge Q \barwedge T \Longrightarrow R \barwedge T$. Since the directions $RH \perp BC$ and $TH \parallel AP$ are fixed, H describes a hyperbola $\mathcal{H}$ through A with asymptotes parallel to

AP and $\perp BC$. Hence $H \mapsto P$ is a stereographic projection of $\mathcal{H}$ onto AP, as it fixes A and the point at infinity of AP. So it suffices to check that PH goes through O for two positions of P. Indeed, when P coincides with the incenter or the A-excenter, PH becomes Euler line of the intouch, or A-extouch triangle, both passing through O. Hence, O, P, H are collinear for any P.

TelvCohl

#3 Dec 2, 2014, 10:20 pm • 3

Thank you for your nice solution

My solution:

Let I be the incenter of $\triangle ABC$.
Let M, N be the midpoint of $\overset{\frown}{\mathrm{BC}}, \overset{\frown}{\mathrm{BAC}}$, respectively .
Let Q be the isogonal conjugate of P WRT $\triangle ABC$ and Q' be the reflection of Q in BC .

Since $\triangle APZ \sim \triangle NMB$,

so we get $\frac{2OA}{MB} = \frac{MN}{MB} = \frac{AP}{PZ}$ . ... (1)

Since BI bisect $\angle PBQ$ and $\triangle BQT \sim \triangle BPZ$,

so we get $\frac{MQ}{MB} = \frac{BQ}{BP} = \frac{QQ'}{2PZ}$ . ... (2)

From (1) and (2) we get $\frac{OA}{AP} = \frac{QM}{QQ'}$,

so combine with $\angle PAO = \angle Q'QM$ we get $\triangle OAP \sim \triangle MQQ'$.

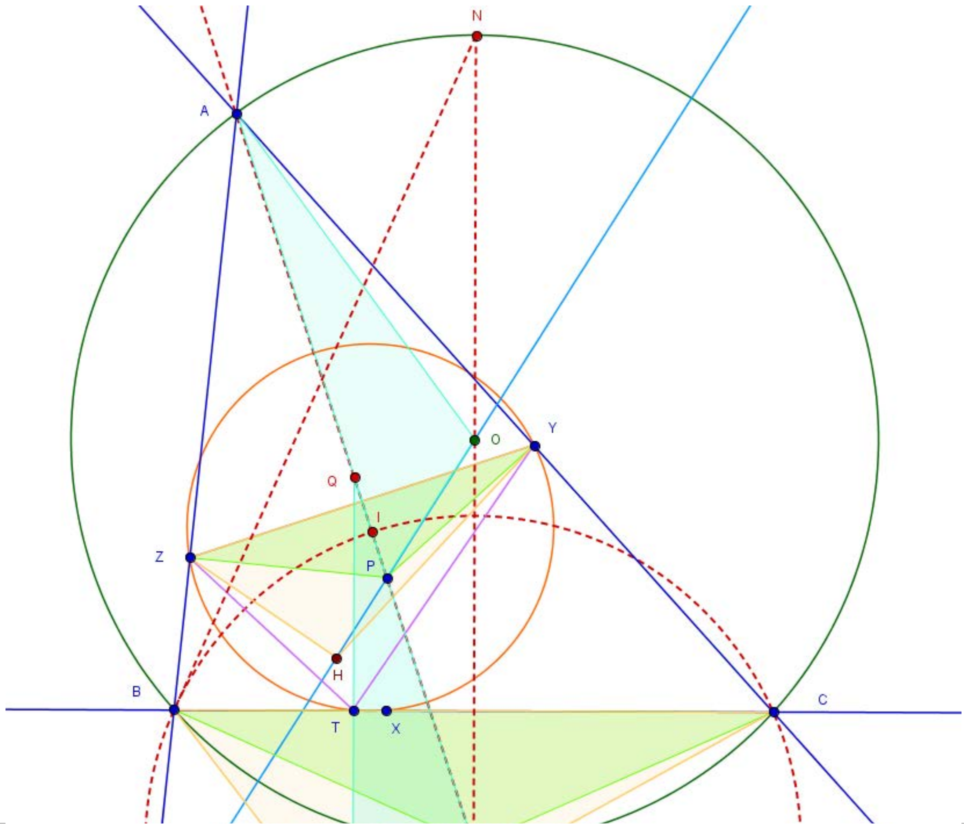
Since
$\angle HYZ = 90^\circ - \angle TZY = 90^\circ - \angle CXY = 90^\circ - \angle CPY = \angle Q'CB$
$\angle HZY = 90^\circ - \angle TYZ = 90^\circ - \angle BXZ = 90^\circ - \angle BPZ = \angle Q'BC$,

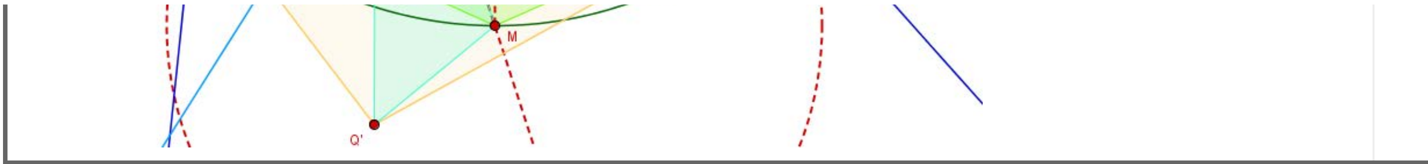
so we get $\triangle PYZ \cap H \sim \triangle MCB \cap Q'$,

hence $\angle HPA + \angle APO = \angle Q'MO + \angle APO = 180^\circ$. ie. H, P, O are collinear

Q.E.D

Attachments:





Quick Reply

High School Olympiads

Contemporan Sangaku

Source: Own

leonardg

#1 Dec 1, 2014, 12:47 pm

I hope that it's new .
Beautiful is for sure !

Attachments:

Own ?

Let ABC be a triangle with its angles less than 120° .Denote T its Torricelli point and w its circumcircle .The lines AT ,BT and CT intersect second time w in D , E and F respectivelly .

Prove that for any point M from triangle's plane the following inequality holds :

$$MD+ME+MF \geq TA+TB+TC\,.$$

Author : Leo Giugiuc , Romania

Luis González

#2 Dec 1, 2014, 10:56 pm

Antipedal triangles $\triangle PQR$ and $\triangle P_1Q_1R_1$ of T WRT $\triangle ABC$ and $\triangle DEF$ are clearly equilateral with parallel sides. Since QR, RP, PQ cut $\omega \equiv (O)$ again at the antipodes of $D, E\ F$, we deduce that $\triangle PQR$ and $\triangle P_1Q_1R_1$ are symmetric about O, i.e. $\triangle PQR \cong \triangle P_1Q_1R_1$. Thus, by Viviani's theorem we get

$$TA+TB+TC=\frac{\sqrt{3}}{2}PQ=\frac{\sqrt{3}}{2}P_1Q_1=TD+TE+TF.$$

T is also Torricelli point of $\triangle DEF$, because $\widehat{ETF}=\widehat{FTD}=\widehat{DTE}=120^\circ$. Hence, any point M on its plane verifies $MD+ME+MF \geq TD+TE+TF=TA+TB+TC$, as desired.

leonardg

#3 Dec 1, 2014, 10:58 pm

Great !!!!!

High School Olympiads

The Schröder's point

Reply

jayme
#1 Nov 30, 2014, 2:00 pm

Dear Mathlinkers,

1. ABC a triangle
2. I the incenter
3. DEF the contact triangle of ABC

Prove without inversion : the circles (AID), (BIE) and (CIF) are coaxal.

Sincerely
Jean-Louis

TelvCohl
#2 Nov 30, 2014, 2:10 pm

We can prove the generalization of this problem without Inversion (see Coaxal conjecture) :)

Luis González
#3 Nov 30, 2014, 9:21 pm • 1

In general, for a PC-point P and its pedal triangle DEF, circles (APD), (BPE) and (CPF) are coaxal, i.e. they intersect at P and the inverse of Q* in the circumcircle, where Q* is the isogonal conjugate of the Ceva point Q of P. For a proof see Concurrent 2 (PC point again) (post #4 and the remark between bold lines). When P=I, they intersect at I and the inverse of the isogonal conjugate of the Gergonne point in the circumcircle, i.e. the Schröder's point of ABC.

P.S. A more general coaxiality was discussed at Coaxal circles.

Quick Reply

High School Olympiads

Calculation of length

Reply

Source: India Postal Coaching 2014 Set 3 Problem 2

hajimbrak

#1 Nov 29, 2014, 6:03 pm

Let O be the centre of the square $ABCD$. Let P,Q,R be respectively on the segments OA,OB,OC such that $OP=3,OQ=5,OR=4$. Suppose S is on OD such that $X=AB\cap PQ,Y=BC\cap QR$ and $Z=CD\cap RS$ are collinear. Find OS.

Luis González

#2 Nov 30, 2014, 4:11 am

From problem condition, $ABCD$ and $PQRS$ are perspective. Thus $O,M\equiv PQ\cap RS$ and $AB\cap CD$ are collinear and $O,N\equiv QR\cap PS$ and $BC\cap DA$ are collinear $\Longrightarrow OM\parallel AB$ and $ON\parallel BC$, i.e. OM,ON bisect $\angle POQ$. Hence, by angle bisector theorem and Van Aubel's theorem for the cevian triangle $\triangle OMN$ of S WRT $\triangle PQR$, we obtain

$$\frac{OQ}{OP}+\frac{OQ}{OR}=\frac{MQ}{MP}+\frac{NQ}{NR}=\frac{QS}{OS}=\frac{OQ+OS}{OS}\Longrightarrow\frac{1}{OP}+\frac{1}{OR}=\frac{1}{OQ}+\frac{1}{OS}.$$

$$OS=\frac{1}{\frac{1}{OP}+\frac{1}{OR}-\frac{1}{OQ}}=\frac{1}{\frac{1}{3}+\frac{1}{4}-\frac{1}{5}}=\frac{60}{23}.$$

sunken rock

#3 Nov 30, 2014, 7:25 pm

Remark: $ABCD$ could be a rectangle, and we get the same result:
Let's lift O above the plan $ABCD$ to V so that $VO\perp ABCD$. $ABCD$ and $PQRS$ being perspective, P,Q,R,S remain coplanar points, i.e. PR,QS intersect the common angle bisector of $\angle PVR,\angle QVS$ at the same point O'. Now let's rotate the triangle $\triangle VAC$ over triangle VBD, such as $A\equiv B,C\equiv D$ and note P',Q',R',S' the new positions of P,Q,R,S respectively. $P'R'$ intersects $Q'S'$ at O' onto the bisector of angle $\angle AVC$ and let's note $O"=VO'\cap Q'R'$. The segments VP', a.s.o. are proportional to OP, a.s.o. and we may apply Ceva: $\frac{VP'}{P'Q'}\cdot\frac{Q'O"}{O"R'}\cdot\frac{R'S'}{S'V"}=1$, with $\frac{Q'O"}{O"R'}=\frac{5}{4}$ and $\frac{VS'}{S'R'}=\frac{x}{4-x}$. We shall get $60=23x$.

Best regards,
sunken rock

Quick Reply

High School Olympiads

nice problem

Locked

DonaldLove

#1 Nov 29, 2014, 10:12 pm

Given circles (O_1), (O_2) with different radiuses. T_1T_2 and T_3T_4 are the circles' inside common tangent lines, with T_1, T_3 on (O_1) and T_2, T_4 on (O_2). T_1T_2 intersects T_3T_4 at P. Consider F on PT_1 and F' on PT_4. x is the line through F, different from T_1T_2 and touches (O_1). y is the line through F, different from T_3T_4 and touches (O_2). z is the line through F', different from T_1T_2 and touches (O_2). d is the line through F', different from T_3T_4 and touches (O_1). y intersects z at A. x intersects d at B. PRove that AB goes through a fixed point when F.F' changes

Luis González

#2 Nov 29, 2014, 11:00 pm • 1

First off, there are some typos: y should be the line through F', different from T_3T_4, touching (O_2) and z should be the line through F, different from T_1T_2, touching (O_2). In this configuration, AB always goes through the exsimilicenter of $(O_1) \sim (O_2)$. You posted this problem before at Fix point involving tangent lines, so what's the object of posting the same problem again?. Topic locked.

High School Olympiads

Parallel lines and circumscribed quadrilaterals

Reply

Source: India Postal Coaching 2014 Set 2 Problem 2 & Sharygin 2014

hajimbrak
#1 Nov 29, 2014, 5:59 pm

Let $ABCD$ be a circumscribed quadrilateral. Its incircle ω touches the sides BC and DA at points E and F respectively. It is known that lines AB, FE and CD concur. The circumcircles of triangles AED and BFC meet ω for the second time at points E_1 and F_1. Prove that EF is parallel to E_1F_1.

Luis González
#2 Nov 29, 2014, 10:27 pm • 2

Let ω touch CD, BA at $G, H.$ If AB, CD, EF concur, $EGFH$ is harmonic $\Longrightarrow AD, BC, GH$ concur at K and by Newton's theorem AC, BD, EF, GH concur at $P.$ If EE_1 cuts AD at $U,$ we have $UA \cdot UD = UE \cdot UE_1 = UF^2,$ but from the complete $ABCD,$ we have $(A, D, F, K) = -1 \Longrightarrow U$ is midpoint of KF and similarly FF_1 goes through the midpoint V of $KE.$ Now, by the symmetry of the K-isosceles $\triangle KEF,$ we obtain $EF \parallel E_1F_1.$

TelvCohl
#3 Nov 30, 2014, 1:03 am • 2

My solution:

Let I be the incenter of $ABCD$.
Let X, Y be the tangent point of (I) with $AB, CD,$ respectively .
Let P, Q, R, S be the midpoint of EX, EY, FX, FY, respectively .
Let $T = AD \cap BC \cap XY, U = PQ \cap BC, V = RS \cap AD$.

Since $EP \cdot PX = BP \cdot PI, EQ \cdot QY = CQ \cdot QI$,
so we get PQ is the radical axis of $\{(I), (BCI)\}$,
hence the midpoint U of TE is the radical center of $\{(I), (BCI), (BCF)\}$,
so U lie on the radical axis FF_1 of $\{(I), (BCF)\}$.
Similarly, we can prove the midpoint V of TF lie on EE_1 .
Since $\{E, F\}, \{U, V\}$ are symmetry WRT TI ,
so we get $\{E_1 \equiv EV \cap (I), F_1 \equiv FU \cap (I)\}$ are symmetry WRT TI .
ie. $EF \parallel E_1F_1$

Q.E.D

Quick Reply

High School Olympiads

Three tangent circles

Reply

Source: India Postal Coaching 2014 Set 4 Problem 3 & Komal

hajimbrak

#1 Nov 29, 2014, 6:09 pm

The circles $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$ are pairwise externally tangent to each other; the point of tangency betwwen $\mathcal{K}_1$ and $\mathcal{K}_2$ is T. One of the external common tangents of $\mathcal{K}_1$ and $\mathcal{K}_2$ meets $\mathcal{K}_3$ at points P and Q. Prove that the internal common tangent of $\mathcal{K}_1$ and $\mathcal{K}_2$ bisects the arc PQ of $\mathcal{K}_3$ which is closer to T.

Luis González

#2 Nov 29, 2014, 9:09 pm • 1

Label O_1, O_2, O_3 the centers of the referred circles. Internal common tangent of $\mathcal{K}_1, \mathcal{K}_2$ cuts $\mathcal{K}_3$ at M, N such that T, N are on the same side of the line PQ. Inversion WRT $\odot(M, MT)$ fixes $\mathcal{K}_1, \mathcal{K}_2$ and carries $\mathcal{K}_3$ into the other external common tangent τ of $\mathcal{K}_1, \mathcal{K}_2 \Longrightarrow MO_3 \perp \tau$. Hence, if $H \equiv O_1O_2 \cap PQ \cap \tau$, we have
$\angle O_3NM = \angle O_3MN = \angle(\tau, O_1O_2) = \angle PHT \Longrightarrow NO_3 \perp PQ \Longrightarrow NO_3$ is perpendicular bisector of $\overline{PQ}$, i.e. N is midpoint of the arc PQ of $\mathcal{K}_3$.

randomusername

#3 Nov 29, 2014, 9:23 pm

http://www.komal.hu/verseny/feladat.cgi?a=feladat&f=A607&l=en

TelvCohl

#4 Nov 29, 2014, 9:44 pm • 2

My solution:

Let M be the midpoint of arc PQ .
Let R, S be the tangent point of $\mathcal{K}_3$ with $\mathcal{K}_1, \mathcal{K}_2$, respectively .
Let X, Y be the tangent point of PQ with $\mathcal{K}_1, \mathcal{K}_2$, respectively .

From homothety with center R send $\mathcal{K}_1$ to $\mathcal{K}_3$ we get M, R, X are collinear .
From homothety with center S send $\mathcal{K}_2$ to $\mathcal{K}_3$ we get M, S, Y are collinear .

Since $\angle YXM = \angle MSR$,
so we get R, S, X, Y are concyclic,
hence we get M is the radical center of $\{\mathcal{K}_1, \mathcal{K}_2, (RSXY)\}$,
so M lie on the radical axis of $\{\mathcal{K}_1, \mathcal{K}_2\}$ which is the internal common tangent of $\mathcal{K}_1, \mathcal{K}_2$.

Q.E.D

sunken rock

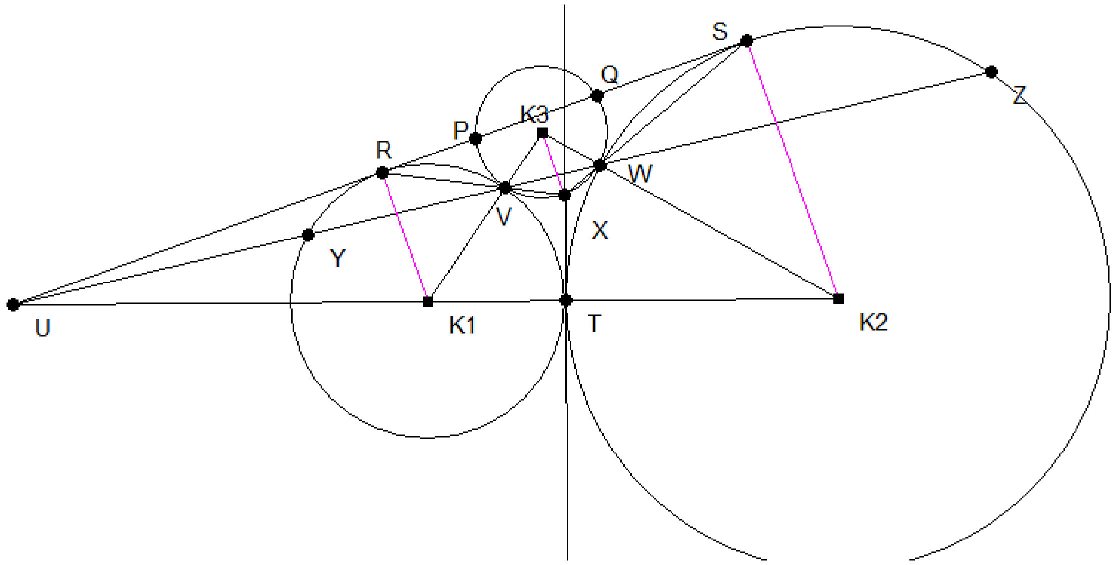
#5 Nov 30, 2014, 2:34 am • 1

See the drawing for notations; $VW \cap K_1K_2 = U$ which is, from Menelaos in $\triangle K_1K_2K_3$, the ex-similicenter of the circles $\mathcal{K}_1, \mathcal{K}_2$, meaning that PQ passes through U and $\angle QSW = \angle RVY$, or $RVWS$ is cyclic and $X = RV \cap SW$ belongs to the radical axis of the 2 circles, from $XV \cdot XR = XW \cdot XS$.
On the other way V, W are the insimilicenters of $\mathcal{K}_1$ and $\mathcal{K}_3$, respectively $\mathcal{K}_2$ and $\mathcal{K}_3$, so RV and SW intersect the circle $\mathcal{K}_3$ the second time at the same point, and this is X, with $XK_3 \parallel RK_1$, or $XK_3 \perp PQ$, done.

Best regards,

sunken rock

Attachments:



High School Olympiads

Nice problem related to four triangle centers

Reply

Source: Own

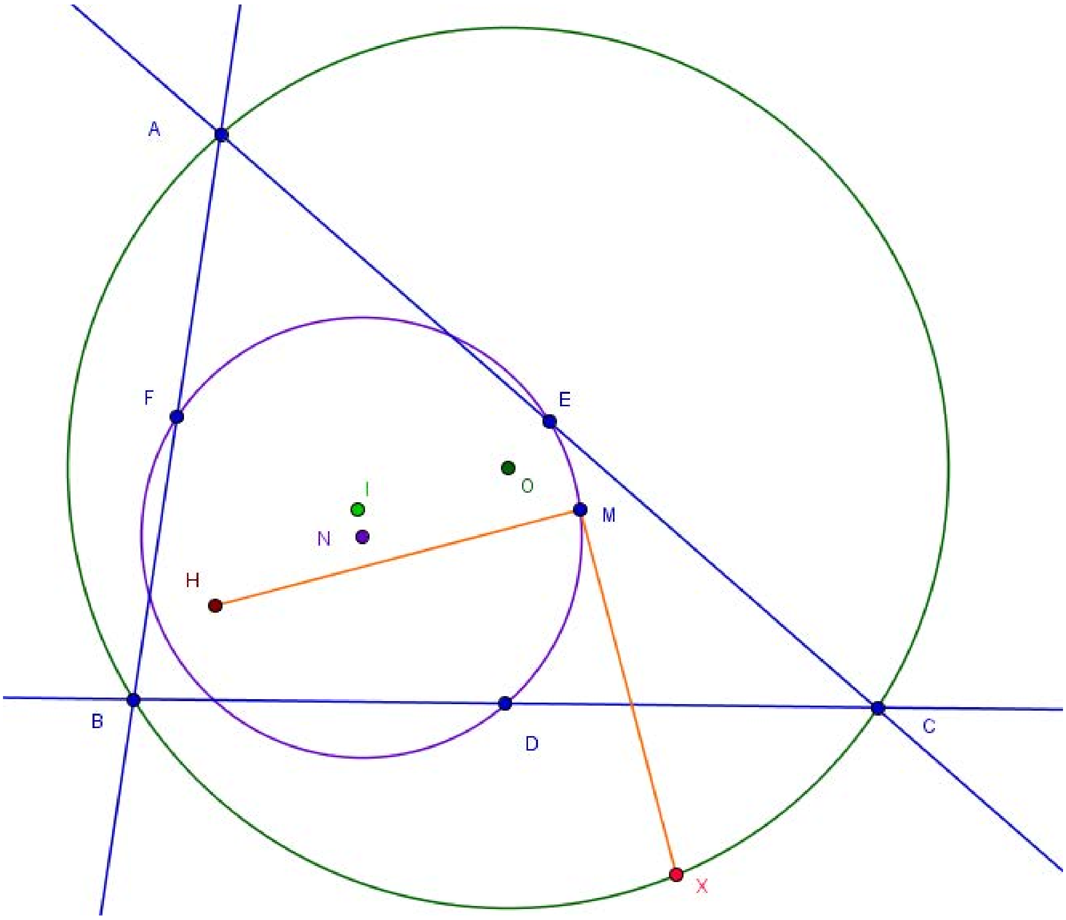
TelvCohl

#1 Oct 11, 2014, 6:34 pm • 2

D, E, F is the midpoint of BC, CA, AB , respectively.
I, O, H, N is the incenter, circumcenter, orthocenter, nine point center of ABC , respectively.
M is the Miquel point of complete quadrilateral formed by triangle DEF and line OI .
X is the pole of the Simson line WRT $\triangle ABC$ which is perpendicular to IN .

Prove that $MH = MX$

Attachments:



Luis González

#2 Nov 29, 2014, 5:26 am • 3

Lemma: Let ℓ be a line through the orthocenter H of $\triangle ABC$. M and S are the Miquel point and anti-Steiner point of ℓ WRT $\triangle ABC$. If MH cuts the circumcircle (O) again at L, then $SL \parallel \ell$.

Proof: ℓ cuts AC, AB at U, V, respectively and BH cuts (O) again at the reflection Y of H on $AC \Longrightarrow S \in YU$. Angle chase (mod 180°) gives $\angle YMU = \angle AMU - \angle AMY = \angle AVU - \angle ABH = \angle YHU \Longrightarrow Y, M, H, U$ are concyclic $\Longrightarrow \angle(HL, HU) = \angle(YM, YU) = \angle(LH, LS) \Longrightarrow SL \parallel HU \equiv \ell$.

Back to the main problem. Ray NI cuts 9-point circle $\odot(DEF) \equiv (N)$ at the Feuerbach point Fe, which is the anti-Steiner

point of OI WRT $\triangle DEF$ (well-known), and MO cuts (N) again at $R.$ From the previous lemma, we have $F_eR \parallel OI,$ hence if the parallel through D to IN cuts (N) again at $T,$ we have
$\angle(RT, RFe) = \angle(DT, DFe) = \angle(FeN, FeD) = \angle(DN, DFe).$ As Fe is isogonal conjugate of the point at infinite of $\perp OI$ WRT $\triangle DEF,$ it follows that the angle between DN, DFe equals the angle between DO and the perpendicular from D to $OI,$ precisely the angle between OI and $EF.$ Therefore $\angle(RT, RFe) = \angle(EF, OI) = \angle(EF, FeR) \Longrightarrow$ $RT \parallel EF \Longrightarrow DR, DT$ are isogonals WRT $\angle EDF \Longrightarrow R$ is the isogonal conjugate of the point at infinite of IN WRT $\triangle DEF,$ i.e. unique pole of Simson line perpendicular to $IN.$

As a result, X is the image of R under the indirect homothety, with center the centroid, that takes (N) into the circumcircle (O) of $\triangle ABC.$ Thus $NR,$ being parallel to $OX,$ cuts HX at its midpoint $Z,$ lying on $(N),$ because (N) and (O) are homothetic under homothety with center H and coefficient $\frac{1}{2}.$ Together with $ZM \perp (MOR \parallel HX),$ it follows that MZ is perpendicular bisector of $\overline{HX}$ or $MH = MX.$

Lyub4o

#3 Nov 29, 2014, 4:51 pm

> *Luis González wrote:*
> the Feuerbach point $Fe,$ which is the anti-Steiner point of OI WRT $\triangle DEF$ (well-known)

Where can I find proof of this fact? Thanks in advance!

Quick Reply

High School Olympiads

coaxal circles formed by excenters

Reply

math4evernever
#1 Nov 26, 2014, 9:27 pm

Let ABC be a triangle with I_A, I_B, I_C the corresponding excenters. Let A' be the intersection between the A-excircle and side BC, B' and C' are defined analogously. Prove that the circles $(AA'I_A), (BB'I_B), (CC'I_C)$ are coaxal.

Luis González
#2 Nov 26, 2014, 9:38 pm

Posted before. These circles are coaxal with common radical axis the line connecting the incenter I and the Mittenpunkt X_9 of $\triangle ABC$; the symmedian point of its excentral triangle $\triangle I_AI_BI_C$.

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=569311.

Quick Reply

High School Olympiads

Geometry Locus Problem

Reply

Source: The Greek Math. Magazine EUCLID, May 1971.

rodinos
#1 Nov 25, 2014, 9:20 am

Let ABC be a triangle and L a line passing through A (not perpendicular
to BC). If B, C are fixed and A moves on the L, which is the locus
of the centers of the squares inscribed in ABC with base on the line BC ?
(ie two vertices of the square are on the line BC)

Source: The Greek Math. Magazine EUCLID, May 1971.

Variation:
Which is the same locus if A moves on the circumcircle of ABC?
(ie the angle BAC is fixed)?

APH

Luis González
#2 Nov 25, 2014, 11:57 am • 1

Label $PQRS$ the vertices of the inscribed square with center $O,$ such that $R\in AB$ and $S\in AC.$ If $P'Q'BC$ is a square erected outside of $\triangle ABC,$ then A is the center of the homothety that takes $PQRS$ into $P'Q'BC \Longrightarrow P\in AP'$ and $Q\in AQ'.$ If $P'S$ cuts BC at $D,$ then clearly AD passes through the midpoints M,N of CP',SP and $(A,D,N,M)=-1 \Longrightarrow A\mapsto N$ is an involutive homology fixing $BC.$ But since $N\mapsto O$ is an affine homology fixing $BC,$ because all $\triangle PQS$ are homothetic with homothety centers falling on $PQ\equiv BC,$ then it follows that the application $A\mapsto O$ is a homology fixing $BC.$

Hence, in conclusion, if A moves on a line $\ell,$ then O moves on another line passing through $\ell\cap BC$ and if A moves on a conic $\mathcal{C},$ then O moves on another conic passing through $BC\cap\mathcal{C}.$

Quick Reply

High School Olympiads

Geometry Problem

Reply

Source: Greek Math. Magazine EUCLID, April 1971

rodinos

#1 Nov 25, 2014, 6:27 am

Let (O, R) be a circle and S a point outside the circle.
Draw a line through S intersecting the circle at A,B such that:
If A', B' are the orthogonal projections of A,B on SO, then A'B' = k (given)

APH

Luis González

#2 Nov 25, 2014, 8:06 am • 1

Let C and D be the reflections of B and A on $SO,$ lying on $(O).$ $ABCD$ is an isosceles trapezoid with bases $BC \parallel AD$ and diagonal intersection $P \equiv AC \cap BD \cap SO$ the pole of the normal to SO at S WRT $(O),$ hence constructible. From $(A',B',P,S)=-1,$ it follows that the circle (J) with diameter $\overline{SP}$ and the circle (M) with diameter $\overline{A'B'}$ are orthogonal, hence M is found intersecting SO (inside of (O)) with the circle with center J and radius $\sqrt{JS^2+\frac{1}{4}k^2}.$ Once M is determined, the construction of A',B' and A,B is straightforward.

Quick Reply

High School Olympiads

Isogonal conjugate of Tangential quadrilateral

Reply

Source: Own

TelvCohl
#1 Nov 22, 2014, 1:41 pm • 1

$ABCD$ is a tangential quadrilateral with incircle (I) .
Let $\{P, P'\}$ be the isogonal conjugate of $ABCD$.
Let M be the Miquel point of $ABCD$ and ℓ be the perpendicular bisector of IP .

Prove that the reflection of M in ℓ lie on the Newton line of $ABCD$

toto1234567890
#2 Nov 23, 2014, 1:35 pm

Do you know if P' exists?

IDMasterz
#3 Nov 24, 2014, 11:05 pm

Yes, they could be focii of the inellipses and in general there are conditions for P for their to be P'.

Luis González
#4 Nov 25, 2014, 4:40 am • 1

We consider the configuration where $\{P, P'\}$ are inside of $ABCD.$ The remaining configurations are treated analogously. Denote Φ the composition of the symmetry about MI followed by inversion WRT $\odot(M, MI).$ According to Circumscribed quadrilateral with inversion (see post #2) we have $\Phi : A \mapsto C, B \mapsto D \Longrightarrow$ line AD goes to $\odot(BCXM) \equiv (L),$ where $X \equiv AB \cap CD.$

If $(U), (V)$ are the circumcircles of $\triangle PAD, \triangle P'BC,$ we have $\widehat{CVL} = \widehat{LCB} - \widehat{VCB} = 90^\circ - \widehat{BXC} - (\widehat{BP'C} - 90^\circ) \Longrightarrow$ $180^\circ - \widehat{CVL} = \widehat{BXC} + \widehat{BP'C} = 180^\circ - \widehat{BPC} = \widehat{APD},$ which means that the angle between (U) and AD is equal to the angle between (V) and $(L).$ Hence, by conformity $\Phi : (U) \mapsto (V)$ and similarly it carries the circumcircle of $\triangle PAB$ into the circumcircle of $\triangle P'CD.$ As a result, $\Phi : P \mapsto P' \Longrightarrow MP$ and MP' are symmetric WRT MI and $MI^2 = MP \cdot MP' \Longrightarrow \triangle MPI \sim \triangle MIP' \Longrightarrow \odot(MIP)$ touches IP' and $\odot(MIP')$ touches $IP.$ Thus any inversion with center I takes $\odot(IMP)$ to a paralell to IP' through the inverse S of P and $\odot(IMP')$ to a parallel to IP through the inverse T of P'. These parallels meet at the inverse R of M forming the parallelogram $ISRT \Longrightarrow IMR$ bisects antiparallel ST of PP' WRT $IP, IP' \Longrightarrow IM$ is I-symmedian of $\triangle IPP'$.

Since $\{P, P'\}$ are foci of a conic inscribed in $ABCD,$ its center N (midpoint of PP') is on Newton line of $ABCD \Longrightarrow IN$ is Newton line of $ABCD.$ Now, $\widehat{PIN} = \widehat{MIP'} = \widehat{MPI} \Longrightarrow MP$ and IN are symmetric WRT the perpendicular bisector of IP and the result follows.

Quick Reply

High School Olympiads

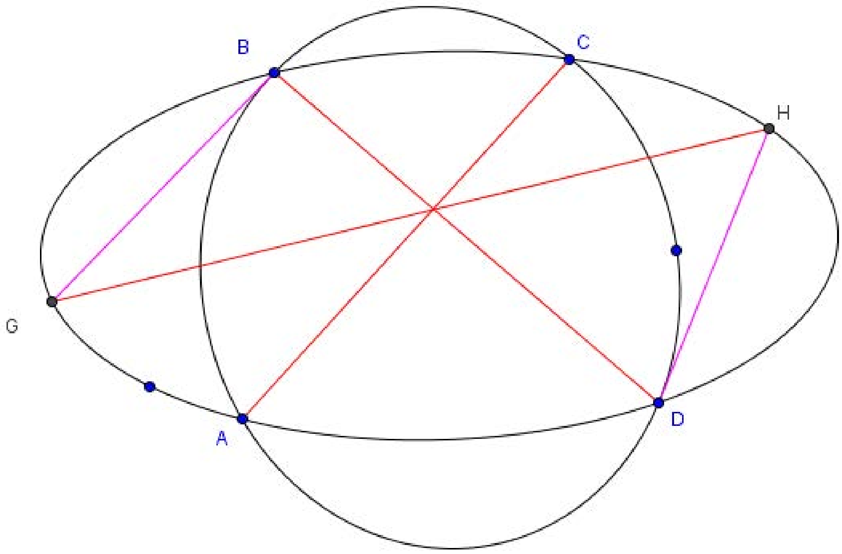
Two conics problem

daothanhoai

#1 Nov 5, 2014, 8:48 pm

Two conic problem: Let two conics through four common points A, B, C, D. Tangent line of the first conic at B, D meets the second conic at G, H respectively. Show that AC, BD, GH are concurrent

Attachments:



TelvCohl

#2 Nov 5, 2014, 9:34 pm • 2

My solution:

We call these two conics ω_1 and ω_2 (G, H lie on ω_2).

Since

$$(H, B; C, A) = D(H, B; C, A) = (D, B; C, A) = (B, D; A, C) = B(G, D; A, C) = (G, D; A, C),$$

so $G \mapsto H, A \mapsto C, B \mapsto D$ is an involution $\Longrightarrow GH, AC, BD$ are concurrent at the pole of this involution. .

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Mar 31, 2016, 3:15 am

Luis González

#3 Nov 24, 2014, 9:36 pm • 1

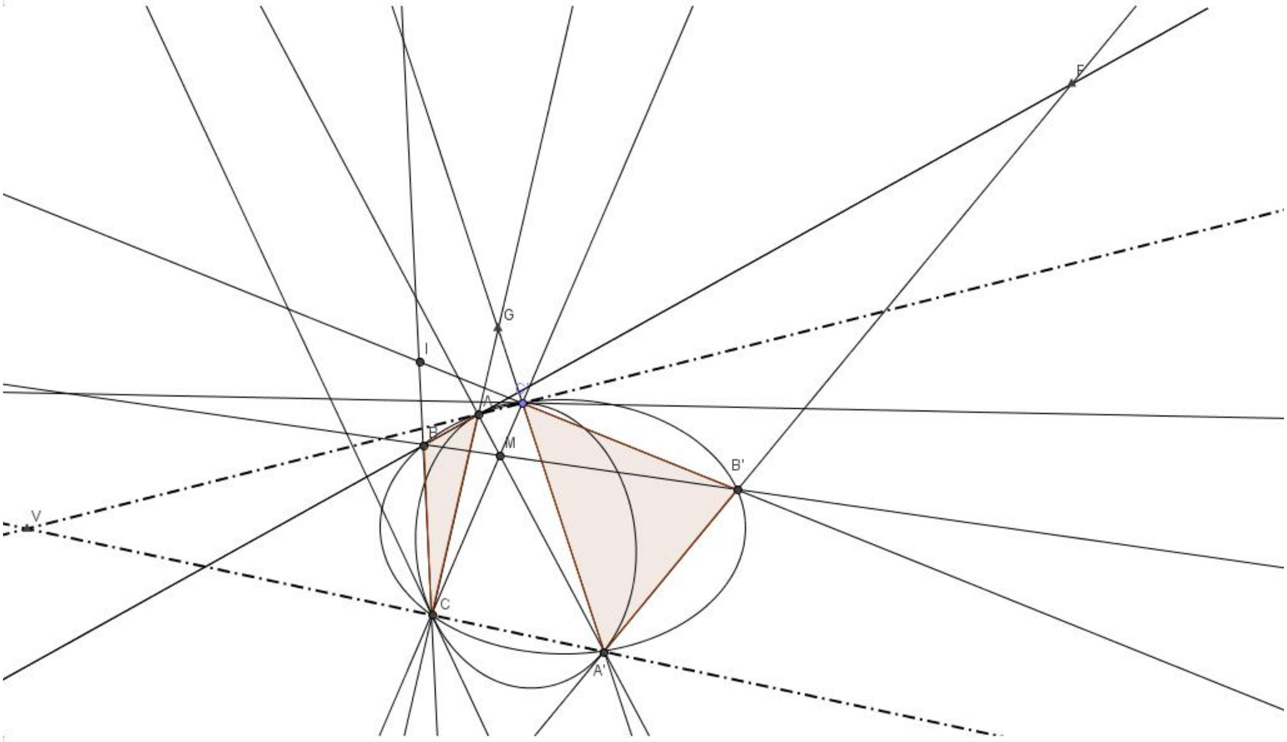
Label $\mathcal{C}_1$ the conic through A, B, C, D and $\mathcal{C}_2$ the conic through A, B, C, D, G, H. Consider a homology transforming $ABCD$ into a rectangle $A'B'C'D'$. By symmetry $O' \equiv A'C' \cap B'D'$ is common center of $\mathcal{C}_1{}', \mathcal{C}_2{}'$ and tangents of $\mathcal{C}_1{}'$ at B', D' are parallel. Again by symmetry, G' and H' are diametrically opposed in $\mathcal{C}_2{}' \Longrightarrow O' \equiv A'C' \cap B'D' \cap G'H' \Longrightarrow$ primitive lines AC, BD, GH also concur.

wiseman

#4 Nov 25, 2014, 12:43 am • 1

I'm using the figure below :
$\rightarrow \mathcal{W} =$ The conic passing through B.
$\rightarrow \mathcal{W}' =$ The conic which is not passing through B.
$\rightarrow AA' \cap CC' = M$
$\rightarrow AC' \cap A'C = V$
$\rightarrow BC \cap B'C' = I$
$\rightarrow AC \cap A'C' = G$
$\rightarrow AB \cap A'B' = F$
$\rightarrow$ Using the pascal theorem for $ABCA'B'C'A$ we get that V, I, F are concurrent. (a)
$\rightarrow$ Obviously, V, G, F lie on the polar of M WRT $\mathcal{W}'$. (b)
Combining (a) and (b) yields that F, G, I are concurrent $\Rightarrow$ by the Desargues' theorem for $\triangle ABC$ and $\triangle A'B'C'$ we get that AA', BB', CC' are concurrent.$\blacksquare$.

Attachments:



Quick Reply

High School Olympiads

the center of homothety of two circles is on PQ

Reply

DonaldLove

#1 Nov 6, 2014, 9:57 pm

Given (O_1) and (O_2) outside of each other and have different radius. AB,CD are two outside common tangent lines (A,D are on (O_1) and B,C are on (O_2)). MN is the inside common tangent line (M is on (O_1) and N is on (O_2)). AM intersects CN at P and BN intersects DM at Q. prove that the outside center of homothety of these two circles is on PQ.

TelvCohl

#2 Nov 6, 2014, 10:46 pm • 1

My solution:

Let $X = AB \cap CD, Y = AB \cap MN, Z = CD \cap MN.$

WLOG (O_1) is smaller than (O_2) ,
then $(O_1), (O_2)$ become the incircle, X-excenter of $\triangle XYZ$, respectively.
Easy to see XM, YD, ZA are concurrent at T and XN, YC, ZB are concurrent at S ,
so we get A, B, C, D, M, N lie on the bicevian conic of $\{T, S\}$ WRT $\triangle XYZ$,
hence from **Pascal theorem** (for $CNBAMD$) we get P, Q, X are collinear .
ie. PQ pass through the exsimilicenter X of $(O_1) \sim (O_2)$

Q.E.D

DonaldLove

#3 Nov 7, 2014, 6:11 am

Thanks for your solution. Can we find another one using simpler method, not bicevian conic ?

Luis González

#4 Nov 24, 2014, 2:18 am

Let $H \equiv AB \cap CD \cap O_1O_2$ and $J \equiv MN \cap O_1O_2$. MN cuts AB, CD at U, V. WLOG (O_1) and (O_2) become the incircle and H-excircle of $\triangle HUV$. Let X and Y be the projections of V and U on O_1O_2.

From cyclic O_1DVMX, we get $\angle XMU = \angle VO_1J = 90^\circ - \frac{1}{2}\widehat{HUV} = \widehat{UMA} \Longrightarrow X \in AM$. By extraversion for the excircle, we have $X \in BN$ and analogously $Y \equiv DM \cap CN$. Thus if $V' \equiv VX \cap AB$ and $U' \equiv UY \cap CD$, then $UU'VV'$ is isosceles trapezoid with $UU' \parallel VV'$ and diagonal intersection $J \Longrightarrow (X, Y, J, H) = -1$. But from the complete $XYPQ$, the line PQ hits XY at the harmonic conjugate of J WRT X, Y, i.e. $H \in PQ$.

TelvCohl

#5 Nov 24, 2014, 10:36 am • 1

My solution:

Let $X = AB \cap CD, Y = MN \cap CD, Z = MN \cap AB$.
Let T, S be the projection of Y, Z on O_1O_2, respectively .

It's well known that $T \in AM, T \in BN, S \in CN, S \in DM$.

Since $ABCD$ is a isosceles trapezoid ,
so $AC \cap BD \in O_1O_2 \equiv TS$,
hence we get $\triangle ABT$ and $\triangle CDS$ are perspective .
From **Desargue theorem** (for $\triangle ABT$ and $\triangle CDS$)we get P, Q, X are collinear .
ie. The exsimilicenter X of $(O_1) \sim (O_2)$ lie on PQ

Q.E.D

Quick Reply

High School Olympiads

Geometry Problem Anopolis #2075

Reply

Source: Anopolis #2075

rodinos

#1 Nov 23, 2014, 11:06 pm

Let ABC be a triangle and A'B'C', A"B"C" the orthic, medial triangles resp.

Denote:

Ab, Ac = the reflections of A' in BH,CH, resp.

Na = the NPC center of A"AbAc. Similarly Nb,Nc.

A'B'C', NaNbNc are orthologic.
(The orthologic center (NaNbNc, A'B'C') is the midpoint of HN.)

Synthetic Proof?

Luis González

#2 Nov 23, 2014, 11:34 pm • 1

Clearly H becomes circumcenter of $\triangle AA_bA_c \Longrightarrow$ perpendicular h_a from H to $A_bA_c \parallel B'C'$ is perpendicular bisector of $\overline{A_bA_c}$. Since $A''N$ is perpendicular bisector of $\overline{B'C'}$, then it becomes A"- altitude of $\triangle A''A_bA_c$. Thus N_a is on the midparallel of $A''N$ and h_a, cutting then $\overline{HN}$ at its midpoint. Likewise, perpendiculars from N_b and N_c to $C'A'$ and $A'B'$ concur at the midpoint of $\overline{HN}$ and the conclusion follows.

rodinos

#3 Nov 23, 2014, 11:37 pm

Thanks, Luis!

The other question is not synthetic ! 🙂

Which point is the other orthologic center ?

Quick Reply

High School Olympiads

Miquel circle of Tangential quadrilateral

Reply

Source: Own

TelvCohl

#1 Nov 10, 2014, 10:04 am • 2

$ABCD$ is a tangential quadrilateral with incircle (I) .
Let M be the Miquel point of $ABCD$ and ℓ be the perpendicular bisector of IM .

Prove that the reflection of the center of the Miquel circle of $ABCD$ in ℓ lie on the Newton line of $ABCD$

Luis González

#2 Nov 22, 2014, 11:17 am • 2

Let $P \equiv AD \cap BC.$ Circles $\odot(PAB) \equiv (K)$ and $\odot(PCD) \equiv (O)$ meet again at the Miquel point M of $ABCD$ (WLOG we assume that M is on the arc PC of (O) not containing D). $(J) \equiv \odot(MKO)$ is Miquel circle of $ABCD.$ According to Circumscribed quadrilateral with inversion (see post #2), we have $\triangle MBI \sim \triangle MID.$ Thus angle chasing, keeping in mind that $\triangle MAB \sim \triangle MDC,$ gives:

$$\widehat{JMI} = \widehat{JMB} + \tfrac{1}{2}\widehat{BMD} = \widehat{JMO} - \widehat{BMO} + \tfrac{1}{2}\widehat{BMD} =$$

$$= \widehat{PMK} - \widehat{BMO} + \tfrac{1}{2}\widehat{BMD} = \widehat{PMK} - (\widehat{BMD} - \widehat{OMD}) + \tfrac{1}{2}\widehat{BMD} =$$

$$= \widehat{PMK} + \widehat{KMA} - \tfrac{1}{2}\widehat{BMD} = \widehat{APM} - \widehat{PBM} - \tfrac{1}{2}\widehat{BMD} \ \ (1).$$

Let the Newton line τ of $ABCD,$ passing through $I,$ cut MJ at $V.$ Since the Miquel point of any $ABCD$ is isogonal conjugate of the point at infinity of its Newton line, it follows that $\widehat{PIV} = \widehat{MPI} \Longrightarrow \widehat{MIV} = \widehat{MPI} - \widehat{PIM} \ \ (2).$ Now, reflection of J on perpendicular bisector of $\overline{MI}$ is on τ if and only if $\triangle VMI$ is isosceles, in other words, iff the expressions (1) and (2) are equal. Indeed, equating them leads to a trivial identity

$$\widehat{MPI} - \widehat{PIM} = \widehat{APM} - \widehat{PBM} - \tfrac{1}{2}\widehat{BMD}$$

$$\widehat{BMI} - \widehat{PIM} = \tfrac{1}{2}\widehat{APB} - \widehat{PBM}$$

$$\widehat{BMI} + \widehat{MDI} - \widehat{PIB} = \tfrac{1}{2}\widehat{APB} - \widehat{PBM}$$

$$180^\circ - \widehat{MBI} - \tfrac{1}{2}\widehat{PAB} = \tfrac{1}{2}\widehat{APB} - \widehat{PBM}$$

$$180^\circ - (\widehat{MBA} + 90^\circ - \tfrac{1}{2}\widehat{PBA}) - \tfrac{1}{2}\widehat{PAB} = \tfrac{1}{2}\widehat{APB} - \widehat{PBM}$$

$$90^\circ - \widehat{MBA} + \tfrac{1}{2}\widehat{PBA} - \tfrac{1}{2}\widehat{PAB} = \tfrac{1}{2}\widehat{APB} - \widehat{PBM}$$

$$90^\circ = \tfrac{1}{2}\widehat{APB} + \widehat{PBA} - \tfrac{1}{2}\widehat{PBA} + \tfrac{1}{2}\widehat{PAB} = 90^\circ.$$

TelvCohl

#3 Nov 22, 2014, 1:17 pm • 1

Nice solution ! Thank you very much :)

Here is my solution:

Invert WRT $\odot(M, MI)$ and then reflect in MI

Invert WRT $\odot(M, MT)$ and then reflect in MT .
Denote T' as the image of T under this transformation .

From Circumscribed quadrilateral with inversion we get $A' = C, C' = A, B' = D, D' = B$,
so the image of the Miquel circle $\odot(J)$ of $ABCD$ is the Steiner line $\mathcal{S}$ of $ABCD \Longrightarrow J'$ is the reflection of M in $\mathcal{S}$.

Since the Newton line is perpendicular to $\mathcal{S}$,
so MJ' is parallel to the Newton line $\mathcal{N}$ of $ABCD \Longrightarrow$ the reflection of MJ in ℓ is parallel $\mathcal{N}$ . ... $(\star)$
From $I \in \mathcal{N}$ and $(\star) \Longrightarrow$ the reflection of MJ in ℓ coincide with $\mathcal{N} \Longrightarrow$ the reflection of J in ℓ lie on the $\mathcal{N}$.

Q.E.D

High School Olympiads

nice geometry

Locked

DonaldLove

#1 Nov 21, 2014, 9:39 pm

AD,BE,CF are altitudes of acute triangle ABC. Assume that A_1, B_1, C_1 are on AD,BE,CF such that the circle $(A_1B_1C_1)$ is inside triangle ABC. prove that orthocenter H of triangle ABC is on $(A_1B_1C_1)$ <=> $S_{A_1BC} + S_{B_1CA} + S_{C_1AB} = S_{ABC}$

Luis González

#2 Nov 21, 2014, 10:23 pm

Posted several times before. The converse is proved with exactly the same arguments.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=302249
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=313868
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=452583

High School Olympiads

Pole of Midline

Reply

Source: Own

TelvCohl
#1 Oct 30, 2014, 10:24 pm • 5

P, P^* are isogonal conjugate of $\triangle ABC$.
Let ω be the pedal circle of $\{P, P^*\}$ WRT $\triangle ABC$.
Let H, H^* be the orthocenter of $\triangle PBC, \triangle P^*BC$, respectively .

Prove that the pole of A- midline WRT ω is the midpoint of HH^*

Luis González
#2 Nov 4, 2014, 10:19 am • 4

Very nice problem. It generalizes, for example, the result r276 of TTW.

Let $\triangle DEF$ and $\triangle D^*E^*F^*$ be the pedal triangles of P and P^* WRT $\triangle ABC$. Let K be the midpoint of $\overline{PP^*}$; center of ω and let Z, Z^* be the midpoints of $\overline{CP}, \overline{CP^*}$; centers of the circumcircles of $PDCE, P^*D^*CE^*$. Thus DE and D^*E^* are pairwise radical axes of $\omega, (Z), (Z^*)$ meeting at their radical center $U \Longrightarrow UC$ is radical axis of $(Z), (Z^*) \Longrightarrow$ $UC \perp (ZZ^* \parallel PP^*)$. Similarly, if $V \equiv DF \cap D^*F^*$, we have $BV \perp PP^*$. Now, by Pappus theorem for B, D^*, C and the infinite points of $(BH \parallel D^*E^*) \perp CP$, $(CH \parallel D^*F^*) \perp BP$ and $(CU \parallel BV) \perp \underline{PP^*}$, we get that $H \in UV$ and analogously $H^* \in UV$. Perpendicular to BC through K cuts then UV at the midpoint J of $\overline{HH^*}$.

Since CU is the polar of $DE^* \cap D^*E$ WRT ω, then the tangents of ω at E, D^* and CU concur at the pole L of ED^* WRT ω. If $S \equiv KJ \cap DE$, we get $\angle DSK = \angle KED = \angle ELK \Longrightarrow S$ is the 2nd intersection of $\odot(LEKD^*)$ with $KJ \Longrightarrow$ $LS \perp KJ$. Hence $\triangle JSL$ and $\triangle HDC$ are homothetic with center $U \Longrightarrow JL \parallel CH \parallel D^*F^* \Longrightarrow JL$ is the polar of the intersection X of ED^* with the perpendicular bisector of $\overline{D^*F^*}$ WRT ω. But according to Prove LY=LZ, X is on the A-midline ℓ_A of $\triangle ABC$. Thus, $\ell_A \perp KJ$ is the polar of J WRT ω.

buratinogigle
#3 Dec 27, 2014, 9:15 am

Very nice problem. If we call K_a be pole of HH^* with respect to pedal circle then K_a lies on A-midline. Similalry we have K_b, K_c then AK_a, BK_b, CK_c are perpendicular to PP^*, thus easy show $2[K_aK_bK_c] = [ABC]$. And let $HH^* = \ell_a$ similarly we have ℓ_b, ℓ_c then ℓ_a, ℓ_b, ℓ_c bound a triangle that is perspective to triangle $K_aK_bK_c$ and the perspectrix is PP^*. If M_a, M_b, M_c are midpoint of BC, CA, AB then triangle $K_aK_bK_c$ and triangle $M_aM_bM_c$ are perspective, also.

TelvCohl
#4 Apr 14, 2016, 7:55 pm • 1

After proceeding as the first paragraph of the proof at post #2, we can finish the proof as following :

Let T be the point on the A-midline of $\triangle ABC$ such that $AT \parallel BV \parallel CU$. Obviously, the reflection of T in the midpoint of CA lies on CU, so $C(U, T; A, B) = -1 \Longrightarrow CT$ is the polar of U WRT ω. Similarly, we can prove T lies on the polar of V WRT ω, so UV is the polar of T WRT ω. Combining $JK \perp BC$ we get J is the pole of the A-midline of $\triangle ABC$ WRT ω.

Quick Reply

High School Olympiads

Prove LY=LZ

Reply

livetolove212
#1 Aug 21, 2010, 9:19 am • 2

Problem (own): Given an acute triangle ABC. Let P be an arbitrary point inside ΔABC. Let XYZ be the pedal triangle of P wrt ΔABC. (XYZ) intersects BC at X and T. Let M, N be the midpoints of BC, AB. MN intersects YT at L. Prove that $LY = LZ$.

oneplusone
#2 Aug 22, 2010, 6:48 pm • 2

Let D be on the extension of AC such that $\angle PDY = \angle PBZ$. Let L' be the midpoint of BD. Then L' lies on MN. Since $\triangle PZB \sim \triangle PYD$, it is well known that $L'Z = L'Y$ and $\angle LYZ = \angle BPZ$. Thus $\angle ZYL' = \angle ZPB = \angle ZXB$, thus $L'Y$ passes through T, so $L \equiv L'$, and we are done.

Luis González
#3 Nov 4, 2014, 10:19 am • 2

Redefine L as the intersection of TY with the perpendicular bisector of $\overline{YZ}$. We prove that L is then on the B-midline of $\triangle ABC$. Since $\widehat{ZYT} = \widehat{ZXB} = \widehat{ZPB}$ and $\widehat{AYT} = \widehat{APB} \pmod \pi$, we get

$$\text{dist}(L, AC) = LY \cdot \sin \widehat{AYT} = LY \cdot \sin \widehat{APB} = \frac{YZ}{2 \cdot \cos \widehat{ZYT}} \cdot \sin \widehat{APB} =$$

$$= \frac{YZ}{2} \cdot \frac{\sin \widehat{APB}}{\sin \widehat{PBA}} = \frac{AB}{2} \cdot \frac{YZ}{PA} = \frac{AB \cdot \sin \widehat{A}}{2} = \frac{\text{dist}(B, AC)}{2},$$

which means that L is on the B-midline of $\triangle ABC$, as desired.

TelvCohl
#4 Nov 26, 2014, 2:52 pm • 2

My solution is almost same as **oneplusone**'s solution

Let me write down it with more detail

Let D be the point on AC satisfy $\triangle PYD \sim \triangle PZB$.
Let L' be the midpoint of BD and S be the projection of P on S .
Let P' be the isogonal conjugate of P WRT $\triangle ABD$.

Since $\angle DBP' = \angle PBA = \angle ADP = \angle P'DB$,
so L' is the projection of P' on BD . ie. S, L, Y, Z are concyclic
Since $\angle ZYL' = \angle ZSL' = \angle ZPB = \angle DPY = \angle DSY = \angle L'ZY$,
so we get $\triangle L'YZ$ is an isosceles triangle . i.e. $L'Y = L'Z$ ... (1)

Since $\angle ZYL' = \angle ZPB = \angle ZXT = \angle ZYT$,
so we get Y, L', T are collinear . ... (2)

From $(1), (2)$ we get $L \equiv L'$ and $LY = LZ$.

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Aug 4, 2015, 10:07 pm

shinichiman

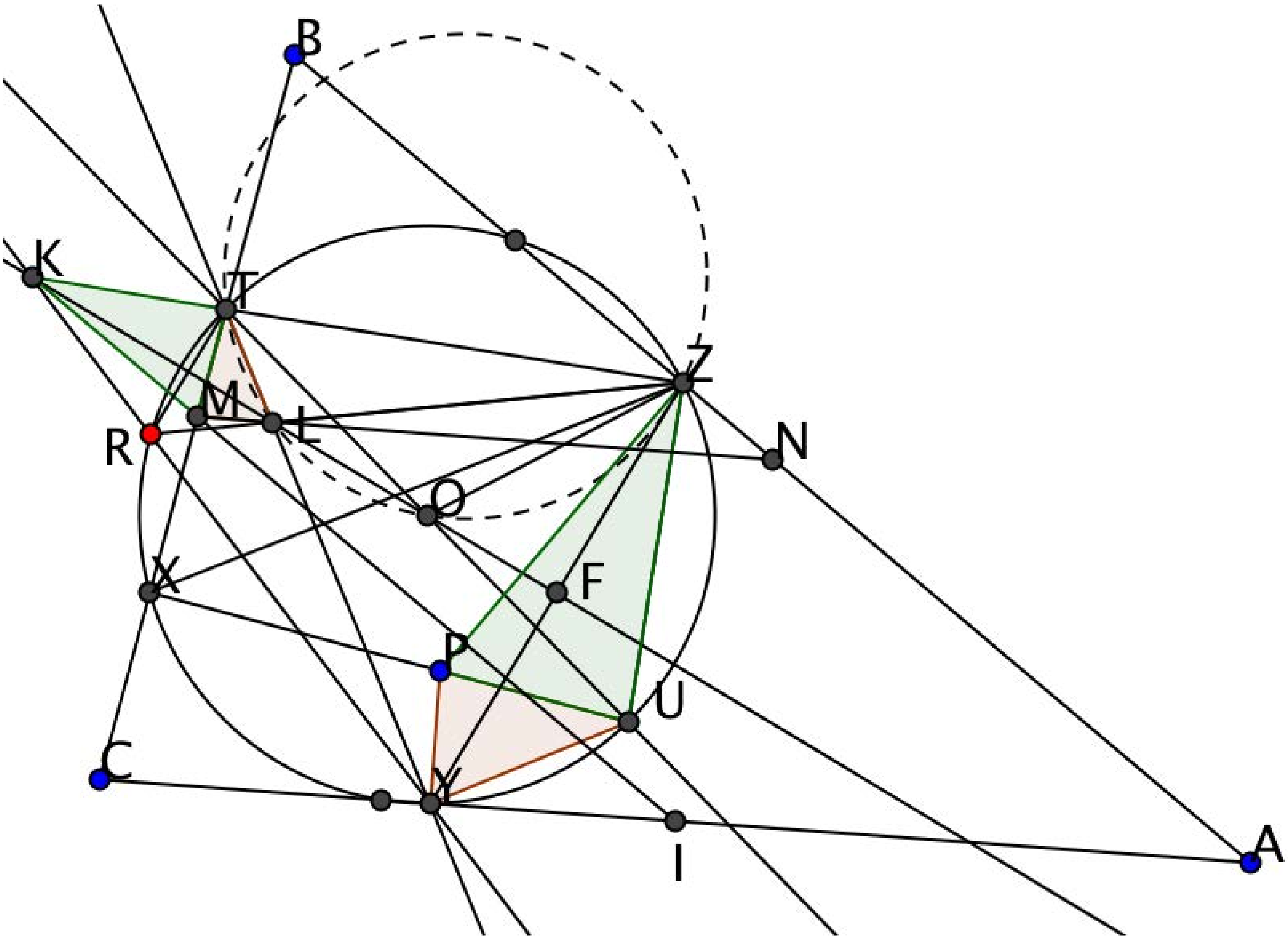
#5 Jan 1, 2015, 7:15 pm • 1

Let I be the midpoint of CA. $MI \cap ZT = K$. Let (O) be the circumcenter of $\triangle XYZ$. $TO \cap XP = U$. Since $XT \perp XP$ so $U \in (O)$. Since $PY \perp AC, UY \perp TY$ so $\angle PYU = \angle TYC = \angle TLM$. We also have $\angle PUI = \angle LTM$ so $\triangle PUY \sim \triangle MTL$ (A.A). Similarly, we also get $\triangle PUZ \sim \triangle MTK$ (A.A). From here we obtain $\triangle KTL \sim \triangle ZUY$ (S.A.S). Hence, $\angle LKT = \angle YZU$. Also note that $KZ \perp ZU$, we obtain $KL \perp ZY \qquad (1)$.

Since $KM \parallel AB$ and $ML \parallel CA$ so $\frac{KT}{KZ} = \frac{LT}{LY}$. This follows that $RT \parallel YZ$ with $R = KY \cap ZL$. $KL \cap YZ = F$. Applying Ceva's theorem we have $\frac{RK}{RY} \cdot \frac{FY}{FZ} \cdot \frac{TZ}{TK} = 1$. Since $RT \parallel YZ$ so $FZ = FY \qquad (2)$.

From (1) and (2) we have that KT is the perpendicular bisector of YZ. Therefore, $LY = LZ$.

Attachments:



Quick Reply

High School Olympiads

Tangent circles and collinearity

Reply

Source: Problem 6, Brazilian MO 2014

cyshine
#1 Nov 4, 2014, 2:52 am

Let ABC be a triangle with incenter I and incircle ω. Circle ω_A is externally tangent to ω and tangent to sides AB and AC at A_1 and A_2, respectively. Let r_A be the line A_1A_2. Define r_B and r_C in a similar fashion. Lines r_A, r_B and r_C determine a triangle XYZ. Prove that the incenter of XYZ, the circumcenter of XYZ and I are collinear.

Luis González
#2 Nov 4, 2014, 4:26 am • 2

Let ω touch BC, CA, AB at $D, E, F.$ Internal common tangents ℓ_A, ℓ_B, ℓ_C of ω with $\omega_A, \omega_B, \omega_C$ obviously bound a triangle $\triangle D'E'F'$ with incircle ω homothetic to $\triangle DEF.$ $J \equiv DD' \cap EE' \cap FF'$ is the homothetic center and this homothety carries $\triangle DEF$ with circumcenter I to $\triangle D'E'F'$ with circumcenter $K \Longrightarrow J, I, K$ are collinear. Hence, the circumcenter and incenter of any triangle homothetic to $\triangle D'E'F'$ under a homothety with center J will lie on the fixed line $JIK.$ Indeed, $\triangle XYZ$ is homothethic to $\triangle D'E'F'$ through J (as r_A, r_B, r_C are the reflections of EF, FD, DE across $\ell_A, \ell_B, \ell_C,$ then X, Y, Z are the reflections of D, E, F on D', E', F', respectively).

TelvCohl
#3 Nov 4, 2014, 5:06 am

My solution:

Let I', O' be the incenter, circumcenter of $\triangle XYZ$, respectively.
Let l_a, l_b, l_c be the internal common tangent of $(\omega, \omega_a), (\omega, \omega_b), (\omega, \omega_c)$, respectively.
Let $\triangle X'Y'Z'$ be the triangle formed by l_a, l_b, l_c .
Let D, E, F be the tangent point of (I) with BC, CA, AB, respectively .

Since $\triangle DEF$ and $\triangle XYZ$ are homothetic ,
so DX, EY, FZ are concurrent at a point H .
Since $\triangle DEF \cap I \sim \triangle XYZ \cap O'$,
so we get I, O', H are collinear . ... (1)
Since l_b pass through the midpoint of DB_1, FB_2 and l_c pass through the midpoint of DC_2, EC_1,
so $X' = l_b \cap l_c$ is the midpoint of DX .
Similarly, we can prove Y', Z' is the midpoint of EY, FZ, respectively .
Since $\triangle X'Y'Z' \cap I \sim \triangle XYZ \cap I'$,
so we get I, I', H are collinear. ... (2)
From (1) and (2) we get I, I', O' are collinear.

Q.E.D

StanleyST
#4 Nov 10, 2014, 12:04 am

Remark: Actually, it is true that DD' is perpendicular on EF, and from there it is easy because it is a well-known result that in any $\triangle ABC$ with $\triangle DEF$, its tangency triangle, the circumcenter of $\triangle ABC$, the incenter of $\triangle ABC$ and the orthocenter of $\triangle DEF$ are collinear.

If i will have enough time, i will post a full solution...

JuanOrtiz

JuanOrtiz

#5 Jan 23, 2015, 12:52 am

Homothety! That is enough!

shinichiman

#6 Jan 24, 2015, 8:23 am • 1

Let O, I' be circumcenter and incenter of $\triangle XYZ$, respectively. ω touches $BC, CA, AB, \omega_A, \omega_B, \omega_C$ at D, E, F, M, N, P, respectively.
Since $\triangle DEF$ and $\triangle XYZ$ are two non-congruent triangles with parallel corresponding sides, therefore there exists unique homothety with center H that maps $\triangle DEF$ to $\triangle XYZ$. Hence, this homothety sends circumcenter of $\triangle DEF$ to circumcenter of $\triangle XYZ$ (or H, I, O are collinear) and sends incenter I_1 of $\triangle DEF$ to incenter I' of $\triangle XYZ$ (or H, I_1, I' are collinear).

Now we will prove I_1, I, I' are collinear.

First, we will prove XM is bisector of $\angle YXZ$. Indeed, denote by t' the line tangent to ω parallel to B_2C_1 such that ω lies between t' and B_2C_1. Denote by L the point where t' is tangent to ω. Homothety with center P that maps ω_C to ω sends B_2C_1 to t' and hence B_2 to L implying that B_2, P, L are collinear. Similarly, we get L, N, C_1 are collinear. In other words, B_2P, C_1N intersect at $L \in \omega$. On the other hand, we also have B_2P, C_1N are the bisector or $\angle XB_2C_1, \angle B_2C_1X$, respectively. This follows L is the incenter of $\triangle XB_2C_1 \qquad (1)$.

We have

$$\begin{aligned} \angle B_2LD = 90^\circ - \angle LB_2D &= 90^\circ - \tfrac{1}{2}\angle B_1B_2C \\ &= 90^\circ - \tfrac{1}{2}\left(90^\circ - \tfrac{1}{2}\angle ACB\right) \\ &= \tfrac{1}{2}\left(90^\circ + \tfrac{1}{2}\angle ACB\right) \\ &= \tfrac{1}{2}\angle AIB = \tfrac{1}{2}\angle MIN = \angle MLN. \end{aligned}$$

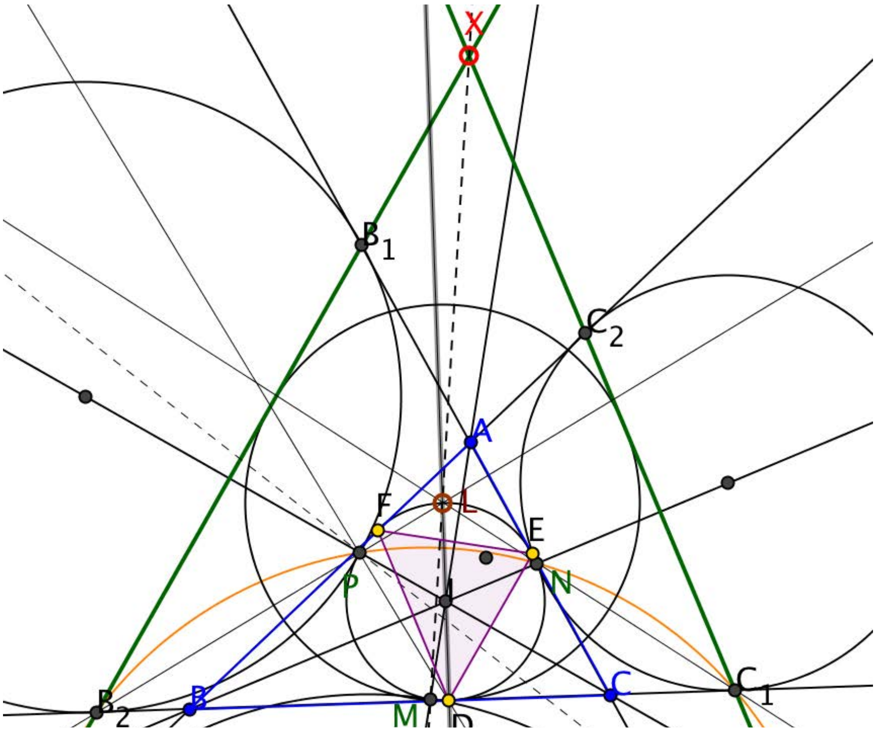
This follows LM, LD are isogonal conjugates wrt $\angle B_2LC_1 \qquad (2)$.

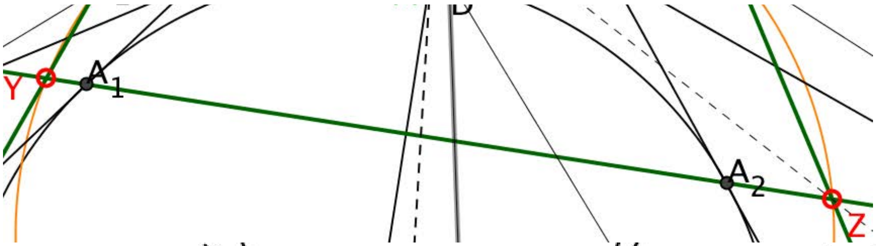
From (1) and (2) we get X, M, L are collinear or LM is the bisector of $\angle B_2XC_1$. Therefore, the problem about collinearity of I_1, I, I' can be shown as similar problem: *Let O, I be circumcenter and incenter of $\triangle ABC$, respectively. AI, BI, CI cut (O) at D, E, F, respectively. $DD', EE', FF', AA', BB', CC'$ are diameter of (O). Prove that $A'D', B'E', C'F'$ are concurrent at K and I, O, K are collinear.*
To prove this, note that $AD \parallel D'A'$, take K' such that O is midpoint of IK' then $K' \in D'A'$. Similarly, we have $K' \in B'E', C'F'$.

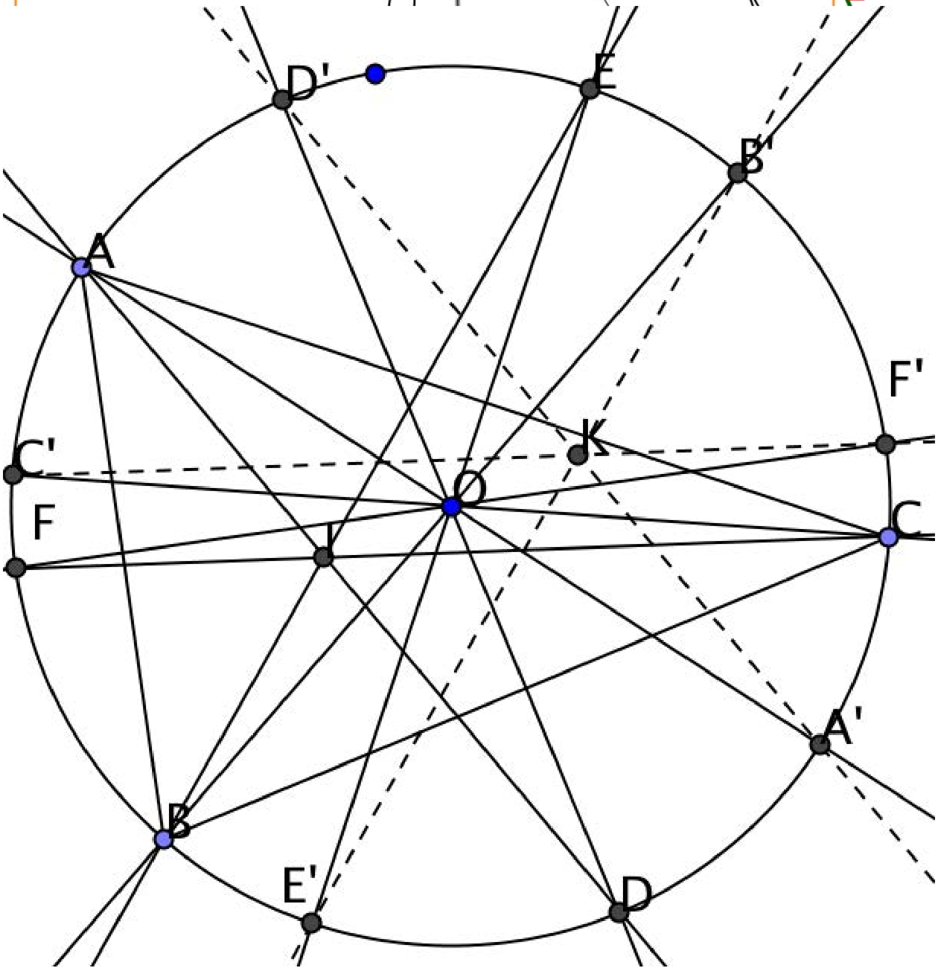
Back to the main problem, by applying above problem to $\triangle DEF$, we can easily have I, I_1, I' are collinear.

Thus, we have $\overline{H, I, O}, \overline{H, I', I_1}$ and $\overline{I, I', I_1}$ so I, O, I' are collinear.

Attachments:







Quick Reply

High School Olympiads

Dilation and Concurrent

Reply

Source: Own

TelvCohl

#1 Nov 3, 2014, 9:49 pm • 1

Let I be the incenter of $\triangle ABC$.
Let $X = AI \cap (ABC), Y = BI \cap (ABC), Z = CI \cap (ABC)$.
Let D, E, F be the midpoint of BC, CA, AB, respectively .
Let D', E', F' be a point on ID, IE, IF satisfy $\frac{ID'}{ID} = \frac{IE'}{IE} = \frac{IF'}{IF}$.

Prove that $D'X, E'Y, F'Z$ are concurrent

Click to reveal hidden text

rodinos

#2 Nov 3, 2014, 10:09 pm

Also, if ID'/ID = IE'/IE = IF'/IF = t, then which is the locus of the point of concurrence as t varies?

aph

TelvCohl

#3 Nov 3, 2014, 10:18 pm

> rodinos wrote:
>
> Also, if ID'/ID = IE'/IE = IF'/IF = t, then which is the locus of the point of concurrence as t varies?
>
> aph

The locus of the point of concurrent is **Jerabek Hyperbola** of $\triangle XYZ$:)

Luis González

#4 Nov 4, 2014, 12:00 am • 1

As D', E', F' run on ID, IE, IF, they induce an afine homography beween these lines $\Longrightarrow YE' \mapsto ZF'$ is a homography $\Longrightarrow P \equiv YE' \cap ZF'$ is on a fixed conic $\mathcal{J}$ through Y, Z. When $E' \equiv F' \equiv I$, then $P \equiv I$ and when $E' \equiv E, F' \equiv F$, then $P \equiv YE \cap ZF \equiv O$ is the circumcenter of $\triangle ABC$.

Let I_a, I_b, I_c be the excenters of $\triangle ABC$ againts A, B, C. Now assume the case when E' and F' coincide with the reflections of I on E, F. These are clearly the orthocenters of $\triangle I_bCA$ and $\triangle I_cAB \Longrightarrow YE'$ and ZF' become Euler lines of $\triangle I_bCA$ and $\triangle I_cAB$. Since $\triangle XYZ \cup OI \sim \triangle I_bAC \cup YE' \Longrightarrow \angle(YX, YE') = \angle(I_bC, YE') = \angle(OI, ZY)$, thus parallel from Y to OI is the isogonal of YE' WRT $\triangle XYZ$. Similarly parallel from Z to OI is the isogonal of ZF' WRT $\triangle XYZ \Longrightarrow P$ is isogonal conjugate of the point at infinity of OI WRT $\triangle XYZ$; 4th intersection of (O) with the isogonal of OI WRT $\triangle XYZ$, i.e. the Jerabek hyperbola of $\triangle XYZ \Longrightarrow \mathcal{J}$ is the Jerabek hyperbola of $\triangle XYZ$. Similarly, intersection $P^* \equiv ZF' \cap XD'$ is on $\mathcal{J} \Longrightarrow P \equiv P^* \Longrightarrow XD', YE', ZF'$ concur at a point P lying on Jerabek hyperbola of $\triangle XYZ$.

buratinogigle

#5 Nov 8, 2014, 7:53 am • 1

This problem is true for all P lies on McKay Cubic as following

Let ABC be a triangle and P is a point such that if XYZ is circumcevian triangle of P and D, E, F are projection of X, Y, Z on BC, CA, AB then DX, EY, FZ are concurrent. Let U, V, W lie on PD, PE, PF such that $\frac{PU}{PD} = \frac{PV}{PE} = \frac{PW}{PF}$. Prove that XU, YV, ZW are concurrent.

Quick Reply

High School Olympiads

A conics theorem

Reply

Source: A Greek Book

rodinos

#1 Nov 2, 2014, 1:03 am

Let (C1), (C2) be two conics and A,B,C,D their intersection points.

The circumcircle of ABC intersects the two conics again at M1,M2.

Denote: <ABC> = the angle inscribed in the arc (M1M2) [see figure]
Similarly <BCD>, <CDA>, <DAB>

The four angles are equal.

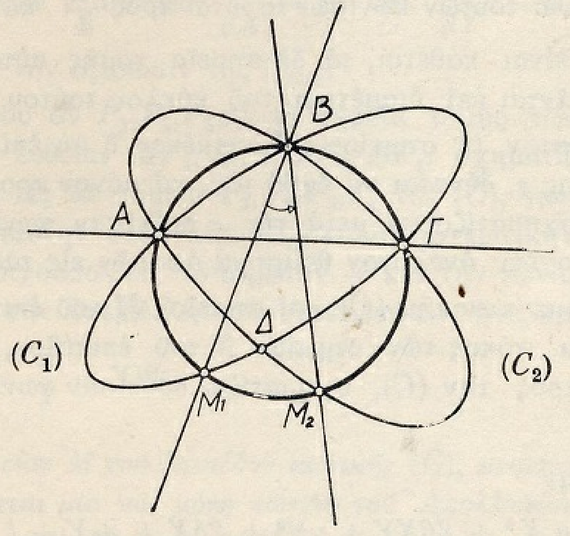
Reference: P. Ladopoulos, Projective Geometry [in Greek], vol. II, p. 269.

The author proves it by the notion of angles of curves, which is his own invention (*).

How to prove it "classically" ie without that notion?

(*) See an old post of mine
http://mathforum.org/kb/thread.jspa?forumID=125&threadID=349129&messageID=1070072

Attachments:



Σχ. 149

τῶν δύο κωνικῶν $(C_1), (C_2)$ ἰσοῦται μὲ τὴν γωνίαν τῶν ἐκφυλισμένων κωνικῶν $(A\Gamma, BM_1), (A\Gamma, BM_2)$, δηλαδὴ μὲ τὴν γωνίαν (BM_1, BM_2), τὴν ἐγγεγραμμένην εἰς τὸ τόξον $M_1 M_2$ τοῦ κύκλου $AB\Gamma$, ὅθεν:

Αἱ ἐγγεγραμμέναι γωνίαι εἰς τὰ τόξα τὰ ὁριζόμενα ἐπὶ ἑκάστου τῶν τεσσάρων κύκλων, τῶν διερχομένων διὰ τριῶν ἐκ τῶν τεσσάρων σημείων τομῆς δύο κωνικῶν, ὑπὸ τῶν κωνικῶν τούτων, εἶναι ἴσαι.

Luis González

#2 Nov 2, 2014, 3:00 am • 1

If $\mathcal{C}_1, \mathcal{C}_2$ cut the circle $\odot(ABD)$ again at $N_1, N_2,$ respectively, then it suffices to show that $\angle M_1AM_2 = \angle N_1BN_2.$

Arbitrary circle through A, B meets $\mathcal{C}_1$ again at two points and the direction of this line is independent of the circle (see the solution of Problem of conic and three circles!). Hence $CM_2 \parallel DN_1$ and similarly $CM_1 \parallel DN_2 \Longrightarrow \angle M_1CM_2$ and $\angle N_1DN_2$ having parallel sides are equal $\Longrightarrow \angle M_1AM_2 = \angle M_1CM_2 = \angle N_1DN_2 = \angle N_1BN_2.$

TelvCohl

#3 Nov 2, 2014, 4:58 am • 1

My solution:

This problem can be deduced from the following lemma:

Lemma:

Let Ω be a conic and $A, B \in \Omega$.
Let C, D be two points on Ω satisfy A, B, C, D are concyclic .

Then AB and CD are symmetry WRT the axis of Ω

Proof of the lemma:

Let another circle through A, B cut Ω at C_1, D_1 .
Let $X = AC \cap (ABCD), Y = BD \cap (ABCD)$.
Let A', B' be the reflection of A, B WRT the axis of Ω , respectively.

From **Reim theorem** we get $CD \parallel XY$ . ... (1)
From **Pascal theorem** (for $DCAC_1D_1B$)
we get $DC \cap C_1D_1, CA \cap D_1B, AC_1 \cap BD$ are collinear. ... (2)
From **Pascal theorem** (for $YXAC_1D_1B$)
we get $YX \cap C_1D_1, CA \equiv XA \cap D_1B, AC_1 \cap BY \equiv BD$ are collinear . ... (3)

From $(1), (2), (3)$ we get $CD \parallel C_1D_1$,
so CD has same direction when C and D varies on Ω (satisfy A, B, C, D are concyclic).
Since it's easy to see A, B, A', B' are concyclic and $A', B' \in \Omega$,
so AB and CD are symmetry WRT the axis of Ω .

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Aug 21, 2015, 2:05 am

IDMasterz

#4 Nov 2, 2014, 12:43 pm

This is a direct corollary of the three conics theorem.

Quick Reply

High School Olympiads

Transformation a quadrilateral to a rectangle.

Reply

Source: A book

CTK9CQT

#1 Sep 19, 2014, 11:10 pm

Let $ABCD$ be a quadrilateral. $I = AD \cap BC, K = AB \cap CD$. d is an arbitrary line parallel to IK. $AB \cap d = X, AD \cap d = Y$. A' be an arbitrary point on the circle with diameter XY. AA' cuts the line through K and parallel to XA' at O. $B' = OB \cap XA', D' = OD \cap YA'$. Prove that the line through B' and perpendicular to XA' cuts the line through D' and perpendicular to YA' at a point on OC.

Luis González

#2 Oct 5, 2014, 9:00 pm • 1

Consider the homology fixing the pencil O and the line d. It carries the figure $\{A, B, D\}$ into the figure $\{A', B', D'\}$. The infinite point of the line $A'B'$ goes then to the intersection K of AB with the parallel through O to $A'B' \Longrightarrow$ parallel KI from K to d is therefore the limiting line of the figure $\{A, B, D\}$. As a result, C goes to the point C' where the parallels from B' and D' to $A'D'$ and $A'B'$ intersect, in other words the intersection of the perpendicular to XA' at B' and the perpendicular to YA' at D' lies on OC.

CTK9CQT

#3 Oct 31, 2014, 2:16 pm

Can you explain to me some concepts: homology, infinite point, limiting line? Thank you in advance

Luis González

#4 Nov 1, 2014, 10:57 pm • 1

The concept of point at infinity is exactly the same thing as direction in the plane, e.g. a set of parallel lines (having the same direction) is a set of lines passing through a point at infinity. All points at infinity lie on the line at infinity. The Euclidean plane with this line added is known as projective plane. So here, any two lines always intersect, parallels are no exception.

A bijection between two figures $\mathcal{F} \mapsto \mathcal{F}'$, such that a point and an indicent line go to a point a an indicent line, is called homography. The limiting line is simply the image of the line at infinity under a homographic transformation. Hence, in general there are 2 limiting lines ℓ, ℓ' in a bijection, one for each figure $\mathcal{F}, \mathcal{F}'$. A homology is a homography that fixes a line.

Quick Reply

High School Olympiads

Angle is Constant

Reply

Source: Own

TelvCohl

#1 Oct 27, 2014, 6:08 am

Let E, E' be two points on AC and F, F' be two points on AB satisfy $EF \parallel E'F'$.
Let $X = BE \cap CF, X' = BE' \cap CF', Y = (AEF) \cap (ABC), Y' = (AE'F') \cap (ABC)$.

Prove that $\angle AYX = \angle AY'X'$.

This post has been edited 1 time. Last edited by TelvCohl, Aug 21, 2015, 2:09 am

wiseman

#2 Oct 30, 2014, 11:00 pm • 1

Restating the problem in other words, if E,F be points on AC and AB respectively such that EF be perpendicular to a fixed line ℓ, then we are to prove that $\widehat{AYX}$ is constant.
$\rightarrow$ R =The circumcenter of $\triangle ABC$.
$\rightarrow$ $D = (EF \cap BC)$
$\rightarrow$ Obviously, Y is Michel point (If we call the concurrency point of the circumcircles of the four triangles formed by four distinct lines as the Michel point of the quadrilateral made by the mentioned lines) of $BFEC$.
$\rightarrow$ $\widehat{AYC} = \widehat{ABC}$. (a)
$\rightarrow$ $\widehat{FYC} = \widehat{FYE} - \widehat{EYC} = \widehat{BAC} - \widehat{EDC} = \text{constant}$.(b)

$$\rightarrow \frac{\sin(\widehat{CYX})}{\sin(\widehat{XYF})} = \frac{CX}{XF}.\frac{YF}{YC} = \frac{\sin(\widehat{EBC})}{\sin(\widehat{EBA})}.\frac{BC}{BF}.\frac{YF}{YC} = \frac{AY}{2R\sin(\widehat{EDC})}.\frac{EC.AB}{AE.YC}$$

$$= \frac{AB}{2R\sin(\widehat{AFE})} = \text{constant}. \text{ (c)}$$

$\Rightarrow$ Combining (b) and (c) yields that $\widehat{CYX}$ is constant. (d)
$\Longrightarrow$ Combining (a) and (d) yields that $\widehat{AYX}$ is constant $\blacksquare$.

This post has been edited 2 times. Last edited by wiseman, Nov 16, 2014, 2:54 am

Luis González

#3 Oct 31, 2014, 9:26 am • 1

E, F move on AC, AB such that all lines EF are parallel. By Desargues involution theorem for the quadrangle $AFXE$, it follows that $YA \mapsto YX, YB \mapsto YC, YF \mapsto YE$ is an involution. But since the oriented angles $\angle(YA, YB) = \angle(CA, CB), \angle(YA, YC) = \angle(BA, BC), \angle(YA, YF) = \angle(EA, EF)$ and $\angle(YA, YE) = \angle(FA, FE)$ are all constant, then it follows that the pencils $Y(B, C, E, F, A, X)$ are all similar, in other words $\angle(YA, YX)$ is also constant.

TelvCohl

#4 Oct 31, 2014, 2:15 pm

> *Luis González wrote:*
> By Desargues involution theorem for the quadrangle $AFXE$

That's the idea how I created this problem :)

My solution is just a bit different from the proof of Luis González :

We only have to prove XY pass through a fixed point on (ABC) when EF varies (with same direction)

Let $E_1 = YE \cap (ABC), F_1 = YF \cap (ABC), E_1F_1 \cap BC = Z, W = AZ \cap (ABC)$.
Let O, O' be the circumcenter of $\triangle ABC, \triangle AEF$, respectively .

From homothety we get the locus of O' is a line passing through A when EF varies .

Since $\angle YCE = \angle O'OA, \angle CEY = \angle AO'O$,
so $\angle E_1YC = \angle OAO' = \text{const.}$ when EF varies ,
hence E_1 is a fixed point on (ABC). Similarly, F_1 is a fixed point on (ABC).

Since E_1, F_1 are fixed point, so we get W, Z are fixed point.
From **Desargue involution theorem** we get $(YE_1, YF_1), (YB, YC), (YA, YX)$ are in the same involution .

Since $Z \equiv E_1F_1 \cap BC$ is the pole of this involution,
so YX pass through $W \equiv AZ \cap (ABC)$ which is a fixed point on (ABC).

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Aug 21, 2015, 2:11 am

Quick Reply

High School Olympiads

Easy but Nice 4

Reply

Source: Own

TelvCohl

#1 Oct 30, 2014, 1:48 am

Let I be the incenter of $\triangle ABC$ ($AC > AB$) .
Let $\triangle DEF$ be the intouch triangle of $\triangle ABC$.
Let $X = DE \cap AB, Y = DF \cap AC$.
l is a line passing through I and perpendicular to XY .
Let $P = l \cap DF, R = l \cap EF, Q = AR \cap DE$.

Prove that B, P, Q are collinear

Luis González

#2 Oct 30, 2014, 7:41 am

Since BE and CF are the polars of Y and X WRT (I), then $G \equiv AD \cap BE \cap CF$ is the pole of XY WRT $(I) \Longrightarrow IG \perp XY \Longrightarrow G \in \ell$. Now, forget about the incircle (I), the collinearity of B, P, Q holds for any cevian triangle $\triangle DEF$ of a point G and any line ℓ through G.

Let BP cut DE, EF at Q', M and let CM cut DE, DF at L, S. By Pappus theorem for $BEDFCM$, we deduce that P, G, L are collinear. By Pappus theorem for $BFDECM$, we deduce that S, A, Q' are collinear. If DM cuts PL, AQ' at U, V and AG cuts EF at J, then from the complete quadrilaterals $AEGF$ and $LPSQ'$, we get $(D, J, G, A) = (D, M, U, V) = -1 \Longrightarrow UG \equiv \ell, MJ \equiv EF$ and $AV \equiv AQ'$ concur at $R \Longrightarrow Q \equiv Q' \Longrightarrow B, P, Q$ are collinear.

Quick Reply

High School Olympiads

Tangential trapezoid concurrency

Reply

Source: Own

El_Ectric

#1 Aug 13, 2014, 4:00 am

Let $ABCD$ be a tangential trapezoid with $AB \parallel DC$ and incircle ω. ω is tangent to AD at F and to BC at E. Let FB and EA intersect ω at G and H, respectively. Let the intersections of DE and CF with ω be J and I respectively. Show that AD, BC, HJ and GI are all concurrent if and only if $AD = BC$.

TelvCohl

#2 Oct 30, 2014, 1:37 am

My solution:

$AD = BC \implies AD, BC, HJ, GI$ are concurrent :

Let $X = AD \cap BC, Y = AC \cap BD, M = \omega \cap AB, N = \omega \cap CD$.

From Newton theorem we get A, Y, C are collinear and B, Y, D are collinear .

Since $AB \parallel CD \parallel EF$,
so $(E, F; H, J) = E(E, F; H, J) = (X, F; A, D) = (X, Y; M, N) = -1$,
i.e. $EHFJ$ is a harmonic quadrilateral
hence we get X, H, J are collinear .
Similarly, we can prove X, G, I are collinear ,
so we get AD, BC, HJ, GI are concurrent .

AD, BC, HJ, GI are concurrent $\implies AD = BC$:

Let $X = AD \cap BC \cap HJ \cap GI$.

Since $HFEJ, EGFI$ are harmonic quadrilateral ,
so $(X, F; A, D) = E(E, F; H, J) = (E, F; H, J) = -1 = (F, E; G, I) = F(F, E; G, I) = (X, E; B, C)$.
i.e. EF, AB, CD are concurrent ... $(*)$

Since $AB \parallel CD$,
so from $(*)$ we get $AB \parallel EF \parallel CD$.
i.e. $ABCD$ is an isosceles trapezoid and $AD = BC$

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Mar 31, 2016, 3:26 am

Luis González

#3 Oct 30, 2014, 3:43 am

Let $X \mapsto X'$ be the homology sending the parallel through $P \equiv AD \cap BC$ to $AB \parallel CD$ to infinity and taking ω into another circle ω'. $A'B'C'D'$ goes to a rhombus with incircle ω' and by symmetry $H'J' \parallel G'I'$. Hence $A'D', B'C', H'J', G'I'$ concur $\Longleftrightarrow H'J' \parallel A'D'$, but by symmetry, this clearly happens $\Longleftrightarrow A'B'C'D'$ is a square $\Longleftrightarrow \omega'$ touches $C'D'$ at its midpoint. As $CD \mapsto C'D'$ is affine, then AD, BC, HJ, GI concur $\Longleftrightarrow \omega$ touches CD at its midpoint $\Longleftrightarrow ABCD$ is an isosceles trapezoid with $AD = BC$ (oblique-angled rhombi excluded).

Quick Reply

High School Olympiads

Easy but Nice 3

Reply

Source: Own

TelvCohl

#1 Oct 28, 2014, 11:42 pm • 1

Let H be the orthocenter of $\triangle ABC$.
Let $D = AH \cap BC, E = BH \cap CA, F = CH \cap AB$.
Let X, Y, Z be the midpoint of BC, CA, AB , respectively .
Let (N) be the nine point circle of $\triangle ABC$.
Let $P = BY \cap (N), Q = CZ \cap (N), R = XQ \cap EF$.

Prove that $\angle ADP = \angle RDA$

Luis González

#2 Oct 29, 2014, 8:45 am • 2

By Pascal theorem for the hexagon $FEYXQZ,$ the intersections $R \equiv FE \cap XQ, C \equiv EY \cap QZ$ and $YX \cap ZF$ (at infinity) are collinear $\Longrightarrow CR \parallel AB.$ Thus perpendicular bisector XY of CF meets FR at its midpoint L and CL cuts AB at the reflection M of C on $L.$ From L-isosceles $\triangle LFC,$ we get $\angle LCF = \angle LFC = \angle DFC \Longrightarrow MC \parallel FD \Longrightarrow BL$ passes through the midpoint S of FD and $BSL \parallel DR.$ But since BS is B-symmedian of $\triangle ABC \Longrightarrow$ $\angle RDC = \angle SBC = \angle PBA = \angle BYX = \angle BDP \Longrightarrow \angle ADP = \angle ADR.$

aopsermath

#3 Oct 30, 2014, 1:41 am

How is BS the B-symmedian of ABC?

Quick Reply

High School Olympiads

Two cevian points & 3 circles

Reply

Source: Mathematical Reflections 4

prowler

#1 Nov 4, 2006, 12:58 pm

This is a small beautiful problem, which was created by treegoner and me a few monthes ago:

Consider a triangle ABC and two points P, Q in it's plane. Let A_1, B_1, C_1 and A_2, B_2, C_2 be cevians in our triangle (intersections of AP, BP, CP, AQ, BQ, CQ with opposite sides). Denote U, V, W second intersections of circles $(AA_1A_2), (BB_1B_2), (CC_1C_2)$ with circumcircle (ABC). Let X be point of the intersection of AU with BC, similarly Y for BV and AC and Z for CW and AB. Prove that X, Y, Z are collinear.

treegoner

#2 Nov 6, 2006, 7:23 am

A short note to this problem.

Actually this problem is still true if you replace the condition $A_1, A_2, B_1, B_2, C_1, C_2$ are the traces of P and Q by the codition 6 points $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a conic.

TelvCohl

#3 Oct 23, 2014, 6:05 am • 2

I'll prove the generalization of this problem ($A_1, A_2, B_1, B_2, C_1, C_2$ lie on a conic)

My solution:

Lemma:

Let D, D' be two point on BC .
Let (w) be the circle pass through B and C .
Let (k) be the circle pass through D and D' .
Let M be the intersection of BC and the radical axis of $\{(w), (k)\}$.

Then $\frac{MB}{MC} = \frac{BD}{DC} \cdot \frac{BD'}{D'C}$

Proof of the lemma:

Since $MB \cdot (MB + BD + DC) = MB \cdot MC = MD \cdot MD' = (MB + BD) \cdot (MB + BD')$,

so we get $MB = \frac{BD \cdot BD'}{DC - BD'}$,

hence

$$\frac{MB}{MC} = \frac{MB}{MB + BD + DC} = \frac{BD \cdot BD'}{BD \cdot BD' + (BD + DC)(DC - BD')}$$

$$= \frac{BD \cdot BD'}{DC \cdot (BD + DC - BD')} = \frac{BD \cdot BD'}{DC \cdot (BC - BD')} = \frac{BD}{DC} \cdot \frac{BD'}{D'C} .$$

Back to the main problem

Since $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a conic ,

so from Carnot theorem we get $\frac{A_1C}{BA_1} \cdot \frac{A_2C}{BA_2} \cdot \frac{B_1A}{CB_1} \cdot \frac{B_2A}{CB_2} \cdot \frac{C_1B}{AC_1} \cdot \frac{C_2B}{AC_2} = 1$ . ... (1)

From the lemma we get

$\frac{XB}{XC} = \frac{BA_1}{A_1C} \cdot \frac{BA_2}{A_2C}$ ... (2)

$\frac{YC}{YA} = \frac{CB_1}{B_1A} \cdot \frac{CB_2}{B_2A}$ ... (3)

$\frac{ZA}{ZB} = \frac{AC_1}{C_1B} \cdot \frac{AC_2}{C_2B}$ ... (4)

so combine $(1), (2), (3), (4)$ we get X, Y, Z are collinear .

Q.E.D

Remark:

For the original problem (AA_1, BB_1, CC_1 are concurrent at P, AA_2, BB_2, CC_2 are concurrent at Q):
From the proof above we get X, Y, Z lie on the trilinear polar of the barycentric product of $P \cdot Q$.

This post has been edited 2 times. Last edited by TelvCohl, Feb 13, 2015, 9:56 pm

TelvCohl

#4 Oct 27, 2014, 9:11 pm • 1

Another proof of the lemma :

Since M is the center of the involution determined by $\{(B, C), (D, D')\}$,

so we get $(M, D; B, C) = (\infty, D'; C, B) \Longrightarrow \frac{MB}{MC} = (\frac{BD}{DC})(\frac{BD'}{D'C})$.

This post has been edited 1 time. Last edited by TelvCohl, Aug 21, 2015, 2:26 am

Luis González

#5 Oct 28, 2014, 11:37 pm • 2

More general: A conic $\mathcal{C}$ cuts BC, CA, AB at the pairs of points $\{A_1, A_2\}, \{B_1, B_2\}, \{C_1, C_2\}$. J, K are two arbitrary points on the plane and $\mathcal{K}$ is the conic through A, B, C, J, K. The conic $\mathcal{K}_A$ through A, A_1, A_2, J, K cuts $\mathcal{K}$ again at U and AU cuts BC at X. The points $Y \in CA$ and $Z \in AB$ are defined similarly. Then X, Y, Z are collinear.

Let JK cut BC, CA, AB at D, E, F. By Desargues theorem, the conics $\mathcal{K}, \mathcal{K}_A$ and $AU \cup JK$ of the pencil $\{A, J, K, U\}$ form an involution on the line $BC \Longrightarrow (D, A_1, B, C) = (X, A_2, C, B) \Longrightarrow$

$\frac{DB}{DC} \cdot \frac{A_1C}{A_1B} = \frac{XC}{XB} \cdot \frac{A_2B}{A_2C} \Longrightarrow \frac{XC}{XB} = \frac{DB}{DC} \cdot \frac{A_1C}{A_1B} \cdot \frac{A_2C}{A_2B}$.

Multiplying the cyclic expressions together gives the desired result.

P.S. Another proof, though not synthetic, is working on $\mathbb{C}^2$ and considering any projective transformation taking J, K into the umbilics. This will produce the configuration of the original problem, then proceed as TelvCohl did.

Quick Reply

High School Olympiads

Perpendicular bisectors create concyclic points

jlammy
#1 Oct 4, 2014, 6:26 pm

Let ABC be an acute-angled triangle. Let the perpendicular bisector of AB intersect BC, CA at P, Q respectively, and let the perpendicular bisector of CA intersect BC, AB at R, S respectively. Prove that P, Q, R, S are concyclic.

Sardor
#2 Oct 4, 2014, 7:12 pm

Let $SR \cap PQ = O$.We need to prove that $OR \cdot OS = OQ \cdot OP$.It's easy by sine law.

jayme
#3 Oct 4, 2014, 8:13 pm • 1

Dear Mathlinkers,

1. Y, Z the midpoints of AC, AB
2. Y, Z, Q and S are on the circle with diameter QS
3. YZ// PR
4/ according to a converse of the Reim's theorem, we are done...

Sincerely
Jean-Louis

mavropnevma
#4 Oct 4, 2014, 8:43 pm

First problem in the Oxford camp Test 2014; is it from there you have it?

jlammy
#5 Oct 4, 2014, 8:55 pm • 1

I created this problem by playing with Reim's theorem in the context of the triangle (hence my intended solution was that of **jayme**).

mavropnevma
#6 Oct 4, 2014, 9:01 pm

The Oxford camp 2014 was a math training camp in Oxford, UK at the end of August, ending with a test where this was Problem 1, in this exact wording!

TelvCohl
#7 Oct 27, 2014, 12:31 am

My solution:

Let O be the circumcenter of $\triangle ABC$.

Since $\angle OBR = 90^\circ - \angle BAC = \angle OSB$,
so OB is tangent to (RSB) at B , i.e. $OR \cdot OS = OB^2$ ... (1)

so OB is tangent to (BSR) at B . i.e. $OR \cdot OS = OB^2$ ... (1)
Similarly, we can prove $OP \cdot OQ = OC^2$ . ... (2)
From (1) and (2) we get $OR \cdot OS = OP \cdot OQ$. i.e. P, Q, R, S are concyclic

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Feb 23, 2015, 9:38 pm

rodinos

#8 Oct 27, 2014, 6:19 am

Rephrasing it:

Let ABC be a triangle.

Denote:
Ab = AC /\ (perpendicular bisector of BC)
Ac = AB /\ (perpendicular bisector of BC)

Similarly Bc, Ba and Ca, Cb.

Ba, Ca, Cb, Bc are concyclic.
Let Oa be the center of the circle (Ba, Ca, Cb, Bc)
Simiarly Ob, Oc.

The triangles ABC, OaObOc are perspective and orthologic. (*)

Synthetic proofs?

(*) See
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/22683

TelvCohl

#9 Oct 27, 2014, 6:56 am

> *rodinos wrote:*
> Rephrasing it:
>
> Let ABC be a triangle.
>
> Denote:
> Ab = AC /\ (perpendicular bisector of BC)
> Ac = AB /\ (perpendicular bisector of BC)
>
> Similarly Bc, Ba and Ca, Cb.
>
> Ba, Ca, Cb, Bc are concyclic.
> Let Oa be the center of the circle (Ba, Ca, Cb, Bc)
> Simiarly Ob, Oc.
>
> The triangles ABC, OaObOc are perspective and orthologic. (*)
>
> Synthetic proofs?
>
> (*) See
> https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/22683

My solution:

Since $A_bA_c = (O_b) \cap (O_c)$,
so $O_bO_c \perp A_bA_c$. ie. $O_bO_c \parallel BC$
Similarly, we can prove $O_cO_a \parallel CA$ and $O_aO_b \parallel AB$,
so $\triangle O_aO_bO_c$ and $\triangle ABC$ are homothety ,
hence $\triangle O_aO_bO_c$ and $\triangle ABC$ are perspective and orthologic.

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Feb 23, 2015, 9:39 pm

Luis González

#10 Oct 27, 2014, 9:24 am • 1

Antreas, two homothetic triangles are trivially perspective through their homothetic center and orthologic through their orthocenters. What is interesting here is that the homothetic center is X(184); isogonal conjugate of the isotomic conjugate of O. The orthocenter of OaObOc is not a known Kimberling center.

Since $OB^2=OC^2=OB_a\cdot OB_c=OC_a\cdot OC_b,$ then circumcircles (O) and (O_a) of $\triangle ABC$ and $C_aC_bB_aB_c$ are orthogonal. B_cC_b is clearly antiparalell to BC WRT $AB,AC,$ thus inversion with center A and power $AB\cdot AB_c=AC\cdot AC_b$ takes (O) into $B_cC_b.$ By conformity (O_a) goes to the circle (K_a) through B,C orthogonal to B_cC_b $\Longrightarrow K_a$ is the intersection of the perpendicular bisector of $\overline{BC}$ with $B_cC_b.$

Let U be the circumcenter of $\triangle AB_cC_b$ lying on the A-altitude of $\triangle ABC.$ AU cuts B_cC_b at D and V is the reflection of U on $B_cC_b.$ As O is orthocenter of $\triangle AB_cC_b,$ then $AUVO$ is parallelogram $\Longrightarrow (OV \parallel AU) \perp BC \Longrightarrow K_a \equiv OV \cap B_cC_b$ and $UDVK_a$ is a rhombus $\Longrightarrow K_a$ is reflection of D on the midpoint of $\overline{B_cC_b},$ i.e. $AK_a \equiv AO_a$ is the isotomic of AU WRT $\triangle AB_cC_b.$ Since $\triangle ABC \sim \triangle AC_bB_c,$ then we deduce that AO_a is then the isogonal of the isotomic conjugate of AO WRT $\triangle ABC \Longrightarrow AO_a$ goes through the isogonal conjugate of the isotomic conjugate of $O;$ $X_{184}.$ Likewise, BO_b and CO_c go through $X_{184}.$

TelvCohl

#11 Oct 27, 2014, 11:51 am • 1

My solution for the homothetic center is X_{184} :

Lemma:

Let O be the circumcenetr of $\triangle ABC$.
Let H be the orthocenter of $\triangle ABC$.
Let D,E,F be the projection of H on BC,CA,AB .
Let the line passing through H and perpendicular to EF cut BC at X and $Y=AO \cap BC$.

Then X,Y are symmetry WRT the midpoint of BC .

Proof of the lemma :

Let M be the midpoint of BC .

Easy to see $AO \perp EF$.
Since A,H is the D -excenetr, incenter of $\triangle DEF$,respectively ,
so HX and AO are are symmetry WRT the perpendicular bisector of EF ,
hence we get M is the midpoint of XY . i.e. X,Y are symmetry WRT the midpoint of BC

Back to the main problem:

Let $D=AB_a \cap (BB_cB_a), E=AC_a \cap (CC_aC_b)$.
Let O' be the circumcenter of $\triangle AB_cC_b$.

Since $OA^2=OB_c\cdot OB_a=OC_a\cdot OC_b$,
so OA is the common tangent of (AB_cB_a) and (AC_aC_b) .
ie. (AB_cB_a) and (AC_aC_b) are tangent at A ... $(\star)$
Since B_c,B_a,C_a,C_b are concyclic ,
so $\angle C_bB_cA=\angle C_bB_cO+\angle OB_cA=(90^\circ-\angle CBA)+(90^\circ-\angle BAC)=\angle ACB$,
hence we get B_c,C_b,B,C are concyclic .

Invert with center A and factor $AD\cdot AB_a=AB\cdot AB_c=AC\cdot AC_b=AE\cdot AC_a$,
then $(O_a) \longleftrightarrow (BCDE) \equiv (T)$.
From $(\star)$ we get $BCDE$ is a isosceles trapezoid ,
so T is the intersection of B_cC_b and the perpendicular bisector of BC .
Notice that O is the orthocenter of $\triangle AB_cC_b$,
so from the lemma we get $AO' \cap B_cC_b$ and T are symmetry WRT the midpoint of B_cC_b .
Since BC and B_cC_b are antiparallel with respect to $\angle BAC$,
so $AT \equiv AO_a$ is the isogonal of the isotomic conjugate of AO WRT $\triangle ABC$
ie. AO_a pass through X_{184}

Similarly, we can prove X_{184} lie on BO_b and CO_c ,
so we get the homothetic center of $\triangle ABC$ and $\triangle O_aO_bO_c$ is X_{184} .

Q.E.D

This post has been edited 2 times. Last edited by TelvCohl, Jan 8, 2016, 6:33 am

Quick Reply

High School Olympiads

Like a G6

Reply

Source: ELMO 2014 Shortlist G11, by Yang Liu

v_Enhance

#1 Jul 24, 2014, 7:49 pm

Let ABC be a triangle with circumcenter O. Let P be a point inside ABC, so let the points D,E,F be on BC,AC,AB respectively so that the Miquel point of DEF with respect to ABC is P. Let the reflections of D,E,F over the midpoints of the sides that they lie on be R,S,T. Let the Miquel point of RST with respect to the triangle ABC be Q. Show that $OP=OQ$.

Proposed by Yang Liu

XmL

#2 Jul 25, 2014, 12:00 pm • 1

Same problem as: http://www.artofproblemsolving.com/Forum/viewtopic.php?p=3338548&sid=18791f5c99978f09fcf6cef32e379ef1#p3338548

leminscate

#3 Jul 25, 2014, 12:03 pm

This is just a generalisation of 2012 G6! Couldn't be any more like a G6!

AnonymousBunny

#4 Jul 26, 2014, 9:51 am

I see someone posted this generalization in the 2012 G6 thread. Also, as XmL pointed out, see Proofathon December 2013 geometry contest.

sayantanchakraborty

#5 Oct 27, 2014, 1:33 am

Yeah,its like ISL 2012 G6.

I drew two diagrams for this one,one for P as Miquel point and the other for Q as miquel point.Let BI meet the circle BDF at Y,circle BTR at Y' and circle ABC at X,where I is the incenter of $\triangle ABC$.Then applying Ptolemy's theorem in $BDYF$ we get $BY=\frac{(BD+BF)\cdot DY}{FD}$.Also note that $\triangle DYF\sim\triangle CXA$ so $\frac{DY}{FD}=\frac{CX}{AC}$.Combining we get $BY=\frac{(BD+BF)\cdot CX}{AC}$.Analogously we get $BY'=\frac{(BR+BT)\cdot CX}{AC}$.Adding the two relations we get $BY+BY'=\frac{(AB+BC)\cdot CX}{AC}$.(Here we have used the reflection property)Also by applying Ptolemy's theorem in $ABCX$ we get $BX=\frac{(AB+BC)\cdot CX}{AC}$.So $BY+BY'=BX\implies BY=Y'X$.This means that $\triangle OBY\cong\triangle OXY'$ so $OY=OY'$.Analogously if $CI\cap CDE=Z,CI\cap CRS=Z'$ we get $OZ=OZ'$ and if $AI\cap AEF=X,AI\cap ATS=X'$ we get $OX=OX'$.

Next note that $\angle ZPY=\angle ZCB+\angle YBC=\angle IBC+\angle ICB=\angle YIZ$ so Y,Z,P,I lie on a circle.By combination of the analogous facts we get Y,Z,X,P,I lie on a circle.Similarly Y',Z',X',Q,I lie on a circle.

Its now clear that $OP=OQ$.

TelvCohl

#6 Oct 27, 2014, 5:51 am • 1

Generalization:

Let D, E, F be the point on BC, CA, AB , respectively .
Let K be a point and X, Y, Z be the projection of K on BC, CA, AB, respectively .
Let R, S, T be the reflection of D, E, F in X, Y, Z , respectively .
Let P be the Miquel point of $\{D, E, F\}$ and Q be the Miquel point of $\{R, S, T\}$.

Then $KP = KQ$.

Proof:

Let P_x, P_y, P_z be the reflection of P in KX, KY, KZ , respectively .
Let $Q_x = P_xR \cap \odot(P_xP_yP_z), Q_y = P_yS \cap \odot(P_xP_yP_z), Q_z = P_zT \cap \odot(P_xP_yP_z)$.

From $\angle Q_xP_xP = \angle RDP = \angle SEP = \angle Q_yP_yP \Longrightarrow Q_x \equiv Q_y$.
Similarly, we can prove $Q_x \equiv Q_y \Longrightarrow P_xR, P_yS, P_zT$ are concurrent at $Q' \in \odot(K, KP)$.

From $\angle TAS = \angle TQ'S \Longrightarrow A, T, S, Q'$ are concyclic .
Similarly, we can prove $Q' \in \odot(BRT)$ and $Q' \in \odot(CRS)$,
so $Q' \equiv Q$ which is the Miquel point of $\{R, S, T\}$ and lie on $\odot(K, KP)$.
i.e. $KP = KQ$

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, May 14, 2015, 6:08 pm

Luis González

#7 Oct 27, 2014, 6:29 am

The referred generalization was posted before:

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=298400
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=363074 (Lemma)

Quick Reply

High School Olympiads

Concyclic points from mixtilinear incircles

Reply

Source: Own

buratinogigle

#1 Oct 26, 2014, 11:25 pm

Let ABC be a triangle and D is a point on BC. Let $(K),(L)$ be D-mixtilinear incircles of triangles DAB,DAC, respectively. Common external tangent other than BC touch $(K),(L)$ at E,F, respectively. Incircle (I) of ABC touches BC at P. Prove that D,E,F,P are concyclic.

TelvCohl

#2 Oct 27, 2014, 12:06 am • 1

My solution:

Let $(K),(L)$ touch BC at E',F', respectively.
Let $(K),(L)$ touch (ABC) at X,Y, respectively.
Let $S=EF \cap BC$.

From D'Alembert theorem we get X,Y,S are collinear .
From my post at here we get X,Y,P,D are concyclic .

Easy to see X,Y,E',F' are concyclic,
so we get $SE \cdot SF=SE' \cdot SF'=SX \cdot SY=SP \cdot SD$.
i.e. D,E,F,P are concyclic.

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Aug 21, 2015, 2:31 am

Luis González

#3 Oct 27, 2014, 1:23 am • 2

Dear TelvCohl, you missundertood the problem, (K) and (L) are the D-mixtilinear incircles of DAB and DAC, not the Thebault circles of the cevian AD, i.e. (K) touches segments DA,DB and the circle (ADB) and similarly (L).

Let $(K),(L)$ touch BC at U,V and let I_1,I_2 be the incenters of $\triangle DAB,\triangle DAC,$ resp. Then $UI_1 \parallel DI_2L$ and $VI_2 \parallel DI_1K.$ By Pappus theorem for U,D,V and the infinite points of $UJ,VJ,UK,$ we get $J \equiv KL \cap UI_1 \cap VI_2.$

On the other hand, it's known that I_1,I_2,D,P are concyclic (see problem 1 at two problems about cyclic quadrilateral or incenters and cyclic). Hence J is the antipode of D on $\odot(DI_1I_2),$ i.e. $J \in IP.$ Now, if M is the projection of D on KL (2nd intersection of $\odot(DI_1I_2)$ with KL), we get $\angle MUV=\angle MKD=\angle LJV \Longrightarrow JMUV$ is cyclic. Thus if $S \equiv EF \cap BC,$ we have $SE \cdot SF=SU \cdot SV=SM \cdot SJ=SP \cdot SD \Longrightarrow D,E,F,P$ are concyclic.

Quick Reply

High School Olympiads

I lies on EF iff

Reply

Source: Own

buratinogigle
#1 Oct 25, 2014, 9:49 pm

Let ABC be a triangle with incenter I. Excircles touches BC, CA, AB at D, E, F, respectively. Prove that I lies on EF iff (DEF) passes though A.

Luis González
#2 Oct 26, 2014, 5:58 am • 2

Let I_a, I_b, I_c denote the excenters of $\triangle ABC$ against A, B, C. Then I_aD, I_bE and I_cF concur at the circumcentre P of $\triangle I_aI_bI_c$ (Bevan point of ABC). If $AEDF$ is cyclic, then we necessarily have $P \equiv D$, as $AEPF$ is cyclic due to the right angles at E and F. Now, by Pappus' theorem for hexagon I_cDI_bBAC, intersections $E \equiv I_bD \cap CA, F \equiv DI_c \cap AB$ and $I \equiv BI_b \cap CI_c$ are collinear. Conversely, if I, E, F are collinear, then by Pappus' theorem for hexagon I_bII_cFAE, we get that B, C, P are collinear $\Longrightarrow P \equiv D \Longrightarrow AEDF$ is cyclic due to the right angles at E and F.

Quick Reply

High School Olympiads

about euler line and excenters

Reply

Stephen
#1 Feb 8, 2014, 4:33 pm • 1

ABC is a triangle that is not equilateral. Let I_a,I_b,I_c be its excenters.

A line that passes A and parallel to euler line meets BC at X. Similarly, we define Y, Z.

Prove that I_aX,I_bY,I_cZ meet at the circumcircle of $I_aI_bI_c$.

willcai
#2 Feb 8, 2014, 7:15 pm

first prove the three lines meet at the same point . we can use ceva. it's easy.
then prove the point and Ia,Ib,Ic on a circle.
make IaX and the circumcircle meet at G. connect IcG and IbG. BA and IcG meet at Y'.
PROVE Y' IS Y. Use some similar triangles ,that's easy.
the same with Z.

Stephen
#3 Feb 8, 2014, 8:23 pm

That doesn't seem very easy to me. Can you show your detailed proof?

willcai
#4 Feb 8, 2014, 10:24 pm

Sorry, give you tomorrow. Today I have something else to do. Very sorry.

Luis González
#5 Feb 8, 2014, 11:03 pm • 4

It has anything to do with the Euler line itself; any three parallel lines through A,B,C do the work. In particular, when this direction coincides with Euler's infinite point of $\triangle ABC$ (as the proposed problem) then the concurrency point $I_aX \cap I_bY \cap I_cZ$ is X_{2383} of the excentral triangle $\triangle I_aI_bI_c$, but we will not need it.

Denote P_{∞} the direction $AX \parallel BY \parallel CZ$. By Cevian Nest theorem I_aX,I_bY,I_cZ concur at K, the cevian quotient P_{∞}/I where $I \equiv AI_a \cap BI_b \cap CI_c$ is the incenter. As P_{∞} varies, the series X,Y,Z are clearly projective, even similar $\Longrightarrow$ they induce a homography between the pencils $I_bY,I_cZ \Longrightarrow K$ moves on a conic $\mathcal{C}$ through I_b,I_c.

When P_{∞} coincides with the infinite point of BC, then $K \equiv I_a$. When P_{∞} coincides with the infinite point of BI, then Z becomes the reflection of C on $I_aI_c \Longrightarrow I_cZ$ intersects I_bYB at the reflection J_b of I on B, lying on $\odot(I_aI_bI_c)$, since I is orthocenter $\triangle I_aI_bI_c$. Similarly when P_{∞} coincides with the infinite point of CI, then K coincides with the reflection J_c of I on C, lying on $\odot(I_aI_bI_c)$. As a result, K is always on unique conic $\mathcal{C} \equiv \odot(I_aI_bI_c)$ through I_a,I_b,I_c,J_b,J_c.

mathuz
#6 Feb 9, 2014, 2:08 am • 1

let A',B',C' are symmetry point of I respect to the points I_a,I_b,I_c.
I think I_aX,I_bY,I_cZ are intersect at one point which lies on the Euler circle of $A'B'C'$. :)

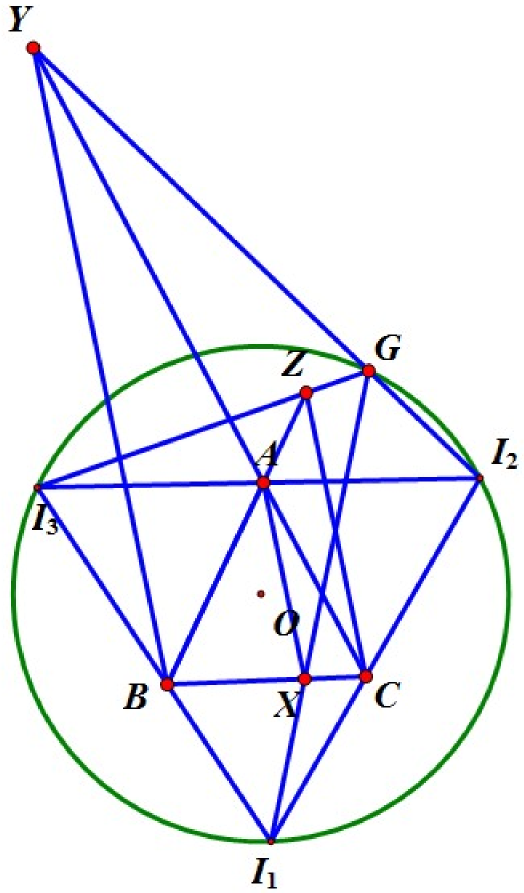
willcai

#7 Feb 9, 2014, 7:49 am • 1

yeah, the ruler line doesn't make sense for the problem.
using ceva can easily get the the three lines meet a point G.
just to prove sin□I3I1G*sin□GI2I1*sin□GI3I2=sin□GI1I2*sin□GI2I3*sin□GI3I1
This is easy. you can change it to BA/AZ,CA/AY, BX/XC and I3B/BI1,I1C/CI2,I2A/AI3

but in fact use the second way is easier and can get G,I1,I2,I3 are on a circle
let I1X and the circumcircle meet at G. connect I3G,and I2G
BA and I3G meet at Z', CA and I2G meet at Y'
□□I3B=□CBI1 and □ZI3B= BXI1
□ZI3B□□I1XB
□BZ=BI1*BI3/BX
we are to prove BZ'/BA=BC/BX□so you can get AX□Z'C)
□we are to prove BA*BC=BI3*BI1
□□BI3C=□BAI1 AND □I3BC=□ABI1
□□I3BC□□ABI1
□ BA*BC=BI3*BI1
so Z' is z
the same with Y.

Attachments:



TelvCohl

#8 Oct 24, 2014, 11:07 pm • 1

Generalization:

Let $\triangle A'B'C'$ be the anti-cevian triangle of P and Q be a point on L .
Then (Q/P) move on the conic passing through A', B', C' with perspector (p_L/P) when Q move on L
(p_L is the trilinear pole of L with respect to $\triangle ABC$)

When P be the incenter of $\triangle ABC$ and L be the infinity line we get the original problem .

Luis González

#9 Oct 25, 2014, 1:44 am • 2

TelvCohl, regarding your generalization, you forgot to mention that the perspector is in reference to the anticevian triangle A'B'C' and not ABC, the latter is self polar WRT the referred conic (no perspector here).

Consider the homography leaving $\{A, B, C\}$ in place and sending P to the centroid G of $\triangle ABC.$ If Q' is the anticomplement of Q WRT $\triangle ABC,$ then $K \equiv Q/G$ is none other than the isotomic conjugate of Q' WRT the antimedial $\triangle A'B'C' \Longrightarrow K$ runs on isotomic conic $\mathcal{K}$ of $\ell \equiv GQ$ WRT $\triangle A'B'C'$ tangent to ℓ at the fixed point G of $Q' \mapsto K.$

Now let ℓ cut BC, CA, AB at $D, E, F.$ Since BC is the polar of A WRT $\mathcal{K},$ then $A'D$ is tangent of $\mathcal{K}$ at A' and similarly $B'E$ and $C'F$ are tangents of $\mathcal{K}$ at $B', C' \Longrightarrow U \equiv BC \cap B'E \cap C'F, V \equiv CA \cap C'F \cap A'D$ and $W \equiv AB \cap A'D \cap B'E$ are then the vertices of the polar triangle of $\mathcal{K}$ WRT $\triangle A'B'C'.$ Moreover, from the complete $EFWV,$ we have $(B, C, U, D) = -1 \Longrightarrow A'U$ passes through p_L/G and similarly $B'V, C'W$ go through $p_L/G.$ So back to the primitive figure, Q/P runs a conic through A', B', C', P with perspector p_L/P WRT $\triangle A'B'C'.$

Quick Reply

High School Olympiads

Concurrent from mixtiinear incircles

Source: Own

buratinogigle
#1 Oct 23, 2014, 10:27 pm

Let ABC be a triangle with circumcircle (O). A-mixtilinear circle (O_a) cuts BC at A_1,A_2. AO_a cuts (O) again at A_3. Circumcircle of triangle $A_1A_2A_3$ cuts (O) again at A_4. Similarly we have B_4,C_4. Prove that AA_4,BB_4,CC_4 are concurrent.

Luis González
#2 Oct 24, 2014, 5:58 am • 1

Let P,Q,R be the tangency points of $(O_a),(O_b),(O_c)$ with (O). BC,A_3A_4 and the common tangent of $(O),(O_a)$ are pairwise radical axes of $(O),(O_a),\odot(A_1A_2A_3)$ concurring at their radical center $X,$ thus $\frac{XB}{XC}=\frac{PB^2}{PC^2}$. But as A_4A_3 bisects $\angle BA_4C$ externally, we have $\frac{XB}{XC}=\frac{A_4B}{A_4C} \Longrightarrow \frac{A_4B}{A_4C}=\frac{PB^2}{PC^2}$. Multiplying the cyclic expressions together gives

$$\frac{A_4B}{A_4C}\cdot\frac{B_4C}{B_4A}\cdot\frac{C_4A}{C_4B}=\left(\frac{PB}{PC}\cdot\frac{QC}{QA}\cdot\frac{RA}{RB}\right)^2.$$

Since AP,BQ,CR concur at the exsimilicenter of (O) and the incircle of $\triangle ABC$ (well-known), the RSH of the latter expressions equals 1, which yields that AA_4,BB_4,CC_4 concur.

TelvCohl
#3 Oct 25, 2014, 2:45 pm • 1

Remark:

From the proof by Luis González and **Jerabek theorem** we get
A_3,B_3,C_3 can be replaced by three vertices of any circumcevian triangle .
(ie. A_3,B_3,C_3 can be any points on (ABC) satisfy AA_3,BB_3,CC_3 are concurrent)

buratinogigle
#4 Oct 25, 2014, 9:26 pm

Yes, you are right, thank you much !

High School Olympiads

Property of Cevian quotient

Reply

Source: Own

TelvCohl

#1 Oct 22, 2014, 10:14 pm • 3

Let I be the incenter of $\triangle ABC$.
Let P' be the isogonal conjugate of P with respect to $\triangle ABC$.

Prove that $I , P' , (P/I)$ are collinear

v_Enhance

#2 Oct 23, 2014, 8:54 pm • 1

□□□□□□□ ><

Barycentric coordinates on $\triangle ABC$.
Let $P = (u : v : w)$, so $P' = \left(\frac{a^2}{u} : \frac{b^2}{v} : \frac{c^2}{w}\right)$.
By http://forumgeom.fau.edu/FG2013volume13/FG201324.pdf we have

$$P/I = \left(a\left(-\frac{a}{u}+\frac{b}{v}+\frac{c}{w}\right) : b\left(\frac{a}{u}-\frac{b}{v}+\frac{c}{w}\right) : c\left(\frac{a}{u}+\frac{b}{v}-\frac{c}{w}\right)\right).$$

So we want to verify

$$0 = \det \begin{pmatrix} a & b & c \\ \frac{a^2}{u} & \frac{b^2}{v} & \frac{c^2}{w} \\ a\left(-\frac{a}{u}+\frac{b}{v}+\frac{c}{w}\right) & b\left(\frac{a}{u}-\frac{b}{v}+\frac{c}{w}\right) & c\left(\frac{a}{u}+\frac{b}{v}-\frac{c}{w}\right) \end{pmatrix}$$

but the third row is the sum of $\frac{a}{u}+\frac{b}{v}+\frac{c}{w}$ times the first row and -2 times the second row.

Luis González

#3 Oct 23, 2014, 9:58 pm • 2

More general: M and M^* are isogonal conjugates WRT $\triangle ABC$ and P is an arbitrary point on MM^*. Then M, the isogonal conjugate P^* of P and the cevian quotient $Q \equiv P/M$ are collinear.

As P runs on MM^*, its isogonal conjugate P^* WRT $\triangle ABC$ runs on the isogonal circum-conic $\mathcal{C}$ through A, B, C, M, M^* and the cevian quotient $Q \equiv P/M$ WRT $\triangle ABC$ runs on another conic $\mathcal{K}$ through A', B', C', M, where A', B', C' are the vertices of the anticevian triangle of M WRT $\triangle ABC$. Clearly $P \barwedge P^*$ and since $AP \mapsto A'Q$ is a perspectivity, it follows that $P^* \barwedge Q$. Thus, all we need to prove is that these radiations on $\mathcal{C}$ and $\mathcal{K}$ coincide.

When $P \equiv MM^* \cap BC$, then P^* goes to A and Q goes to $A' \Longrightarrow M \in QP^*$. The same happens when P coincides with the traces of MM^* on CA, AB. Therefore, the aforementioned radiations coincide $\Longrightarrow M, Q$ and P^* are collinear for any P.

TelvCohl

#4 Apr 17, 2016, 1:31 pm • 1

> Luis González wrote:

More general: M and M^* are isogonal conjugates WRT $\triangle ABC$ and P is an arbitrary point on MM^*. Then M, the isogonal conjugate P^* of P and the cevian quotient $Q \equiv P/M$ are collinear.

We can prove the converse is also true, let me rephrase the problem as following **:**

Given a $\triangle ABC$ and a point U. Then U lies on the Pivotal Isogonal cubic with pivot P if and only if U, the isogonal conjugate P^* of P WRT $\triangle ABC$, the cevian quotient P/U WRT $\triangle ABC$ are collinear.

Proof :

Lemma : Given a $\triangle ABC$ and a point P. Let T be a point varies on a fixed line ℓ passing through P. Then the Cevapoint R of P and T WRT $\triangle ABC$ moves on a circumconic $\mathcal{C}$ of $\triangle ABC$. Furthermore, ℓ is the tangent to $\mathcal{C}$ at P.

Proof : Consider a homography taking P into the centroid of $\triangle ABC$. Then R is the isotomic conjugate (WRT $\triangle ABC$) of the complement of T WRT $\triangle ABC$, so R lies on the isotomic conjugate $\mathcal{C}$ of ℓ WRT $\triangle ABC$ and ℓ is tangent to $\mathcal{C}$. So back in the primitive figure we conclude that R varies on a circumconic of $\triangle ABC$ which is tangent to ℓ at P.

Back to the main problem :

Let U^* be the isogonal conjugate of U WRT $\triangle ABC$. Let $\mathcal{C}$ be the circumconic of $\triangle ABC$ passing through P, U and let $\mathcal{C}^*$ be the circumconic of $\triangle ABC$ passing through P^*, U. From **Lemma** we get the tangent of $\mathcal{C}$ through U passes through P/U, so $U, P^*, P/U$ are collinear $\Longleftrightarrow (U, A; B, C)$ (cross ratio on $\mathcal{C}$) $= (P^*, A; B, C)$ (cross ratio on $\mathcal{C}^*$).

Since $\mathcal{C}^*$ is the isogonal conjugate of PU^* WRT $\triangle ABC$, so if PU^* meets BC at K, then the isogonal conjugate of AK WRT $\angle A$ is tangent to $\mathcal{C}^* \Longrightarrow (P^*, A; B, C)$ (cross ratio on $\mathcal{C}^*$) $= A(P, K; C, B) = P(A, U^*; C, B) = P(U^*, A; B, C)$, hence $P \in UU^* \Longleftrightarrow (P^*, A; B, C)$ (cross ratio on $\mathcal{C}^*$) $= P(U, A; B, C) \Longleftrightarrow P/U \in P^*U$.

Quick Reply

High School Olympiads

Nice concurrent problem

Reply

Source: Own

TelvCohl

#1 Oct 13, 2014, 3:35 pm • 3

Given a triangle ABC and a point P
D, E, F is the projection of P on BC, CA, AB, respectively
L is the trilinear polar of P and L' is the orthotransversal of P

Prove that the intersection of L and L' lie on the polar of P with respect to (DEF)

Click to reveal hidden text

rodinos

#2 Oct 13, 2014, 5:41 pm

Hi Telv,

I asked for the barycentric coordinates of the point of concurrence of T1,T2,T3.

Peter Moses replied:

I think that for some point P{p, q, r}, T1, T2, T3 are concurrent at:
p (a^2 q r - p (p SA - q SB - r SC)) (p (c^2 q^2 - b^2 r^2) + 2 q r (q SB - r SC))::

https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/22633

APH

Luis González

#3 Oct 23, 2014, 7:28 pm • 4

Let $\triangle A'B'C'$ be the anticevian triangle of P WRT $\triangle ABC$. Trilinear polar of P WRT $\triangle ABC$ is the line through $B_0 \equiv AC \cap A'C'$ and $C_0 \equiv AB \cap A'B'$. Perpendicular from P to PB cuts $AC, A'C'$ at B_1, B_2 and the perpendicular form P to PC cuts $AB, A'B'$ at C_1, C_2, thus B_1C_1 and B_2C_2 are the orthopolars of P WRT $\triangle ABC$ and $\triangle A'B'C'$, respectively.

Since $P \equiv C_1C_2 \cap B_1B_2, A' \equiv C_2C_0 \cap B_2B_0$ and $A \equiv C_0C_1 \cap B_0B_1$ are collinear, then $\triangle B_0B_1B_2$ and $\triangle C_0C_1C_2$ are perspective. By Desargues theorem, we deduce that lines B_0C_0, B_1C_1 and B_2C_2 concur, but from problem Isogonal conjugate and perpendicularity, B_2C_2 is presicely the polar of P WRT $\odot(DEF)$. Hence, intersection of the trilinear polar and orthopolar of P WRT $\triangle ABC$ lies on the polar of P WRT $\odot(DEF)$.

Quick Reply

High School Olympiads

similar triangles in cyc hexagon

Reply

Source: Geometry Unbound

Samiur

#1 Oct 21, 2014, 10:34 pm

Let $ABCDEF$ be a cyclic hexagon with $AB = CD = EF$.
Prove that the intersectons of AC with BD, of CE with DF and of EA with FB form a triangle similar to ΔBDF.

Luis González

#2 Oct 23, 2014, 7:38 am • 3

Label $M \equiv CE \cap DF, N \equiv EA \cap FB$ and $L \equiv BD \cap AC$. $AB = CD = EF \Longrightarrow ABCD, CDEF, EFAB$ are isosceles trapezoids whose pairs of diagonals form the same angle, i.e. $\angle EMF = \angle ANB = \angle DLC \Longrightarrow MEFN$, $NABL$ and $LCDM$ are cyclic $\Longrightarrow \angle LMC = \angle BDC = \angle BAC$ and $\angle NMF = \angle AEF = \angle ABF \Longrightarrow$ $\angle NML = 180^\circ - \angle DLC - \angle BAC - \angle ABF = \angle FBD$. Analogously, we get $\angle LNM = \angle ECA$ and $\angle MLN = \angle BFD \Longrightarrow \triangle MNL \sim \triangle BDF$.

Quick Reply

High School Olympiads

Common chords of circumcircle and excircles

Reply

bcp123

#1 Oct 11, 2014, 2:00 am

ABC is a triangle with circumcircle ω and excircles $\omega_A, \omega_B, \omega_C$. ℓ_A is the common chord of ω,ω_A and ℓ_B,ℓ_C are defined similarly. Prove that ℓ_A,ℓ_B and ℓ_C form a triangle perspective with $\triangle ABC$.

TelvCohl

#2 Oct 11, 2014, 6:09 am • 1

My solution:

Let A' be the intersection of A -excircle and BC.
Let A'' be the intersection of BC and the polar of A with respect to A -excircle.
Let X, Y, Z be the intersection of BC, CA, AB and ℓ_A , ℓ_B , ℓ_C , respectively.
Define B' , B'' , C' , C'' Similarly.

Since $(B, C; A', A'') = (C, A; B', B'') = (A, B, C', C'') = -1$,
so A'' , B'' , C'' all lie on the trilinear polar of the Nagel point of $\triangle ABC$.
Since X has equal power with respect to ω and ω_A ,
so X is the midpoint of $A'A''$.
We can prove Y, Z is the midpoint of $B'B''$, $C'C''$ similarly,
so X, Y, Z all lie on the Newton line of complete quadrilateral formed by $\triangle A'B'C'$ and line $A''B''C''$,
hence by Desargue theorem we get ℓ_A , ℓ_B , ℓ_C form a triangle perspective with $\triangle ABC$.

Q.E.D

This post has been edited 2 times. Last edited by TelvCohl, Oct 23, 2014, 6:43 am

jayme

#3 Oct 11, 2014, 2:02 pm

Dear Mathlinkers,
you can see and more

http://jl.ayme.pagesperso-orange.fr/ vol. 11 Ayme-Moses perspector...

Sincerely
Jean-Louis

Luis González

#4 Oct 23, 2014, 6:32 am

Posted several times before. The perspectrix is the trilinear polar of the barycentric square of the Nagel point and the perspector is the Clawson point.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=385591
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=439325

buratinogigle

#5 Oct 23, 2014, 6:55 am

Dear friends, we can replace ω by a circle concentric with ω.

Quick Reply

High School Olympiads

Concurrent on OI line

Reply

Source: Own

buratinogigle

#1 Oct 22, 2014, 11:09 pm • 1

Let ABC be a triangle with bisector AD, BE, CF and incircle (I). M, N, P are midpoints of BC, CA, AB. (K_a) is the circle passing through M, D and touches (I) at X. Similarly we have Y, Z. AI, BI, CI cut circumcircle (O) of ABC again at U, V, W, repsectively. Prove that UX, VY, WZ are concurrent at a point on OI.

Luis González

#2 Oct 23, 2014, 6:06 am • 2

Let T be the tangency point of (I) with BC and let L be the antipode of T WRT (I). LU cuts BC at S. As UI is U-median of $\triangle UTL$ and $UM \parallel TL$, then $U(T, L, I, M) = -1 \Longrightarrow (T, S, D, M) = -1$. But if XM, XD cut (I) again at M', D', then $M'D' \parallel DM$, as X is insimilicenter of $(I) \sim (K_a) \Longrightarrow$ arcs TD' and TM' of (I) are equal $\Longrightarrow XT$ and XL bisect $\angle M'XD' \equiv \angle MXD \Longrightarrow X(T, L, D, M) = -1$, forcing $X \in UL$. Now, from parallel radii $IL \parallel OU$, we deduce that UXL goes through the insimilicenter X_{55} of $(I) \sim (O)$ lying on OI. Likewise, lines VY and WZ pass through X_{55}.

TelvCohl

#3 Oct 23, 2014, 8:33 am • 2

My solution :

Lemma:

Let I be the incenter of $\triangle ABC$ and D be the tangency point of $\odot(I)$ with BC .
Let H be the projection of A on BC and the tangent pass through H tangent to $\odot(I)$ at T .
Let M, K be the midpoint of BC , arc BC and $T' = MT \cap \odot(I)$.
Let D' be the reflection of D in I .

Then D', T', K are collinear .

Proof :

Let I_a be A -excenter of $\triangle ABC$ and $X = KD \cap \odot(I)$.
Let Y be the tangency point of $\odot(I_a)$ with BC and Y' be the reflection of Y in I_a
Let $h_a = AH$ and r, r_a be the radius of $\odot(I), \odot(I_a)$.

Since $(A, D'; I, I_a) = -1$
so $r \cdot r_a = h_a \cdot KM$ . ... (1)

From homothety we get A, D, Y' are collinear ,
so $\frac{DH}{DY} = \frac{h_a}{2r_a}$. i.e. $\frac{DH}{DM} = \frac{h_a}{r_a}$ ... (2)

Combine with $(1), (2)$ we get $\triangle IHD \sim \triangle KDM$. i.e. $XT \parallel DD'$
From Pascal theorem (for $TT'D'DDX$) we get D', T', K are collinear .

Back to the main problem:

Let I_a, I_b, I_c be the projection of I on BC, CA, AB .
Let I'_a, I'_b, I'_c be the reflection of I_a, I_b, I_c in I .
Let H_a, H_b, H_c be the projection of A, B, C on BC, CA, AB .
Let the tangent pass through H_a, H_b, H_c tangent to $\odot(I)$ at X', Y', Z'

Let the tangent pass through I_a, I_b, I_c tangent to $\odot(I)$ at X', Y', Z'.

Since $\odot(I)$ is fixed under the inversion with respect to $\odot(M, MI_a)$,
so the inversion with respect to $\odot(M, MI_a)$ swaps X and X'. i.e. X, X', M are collinear
From the lemma we get UX pass through I_a'. i.e. UX pass through the insimilicenter of $\odot(I) \sim \odot(O)$.
Similarly, we can prove VY, WZ passes through the insimilicenter of $\odot(I) \sim \odot(O)$,
so we get UX, VY, WZ are concurrent at the insimilicenter of $\odot(I) \sim \odot(O)$ (X_{55} in ETC)

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Aug 20, 2015, 10:39 pm

Quick Reply

High School Olympiads

Beautiful locus 8

wiseman

#1 Aug 28, 2014, 3:44 pm

$\triangle ABC$ is an acute triangle. AD,BE and CF are perpendicular to sides BC,AC and AB respectively. (Note that $D \in BC$,$E \in AC$,and $F \in AB$). Let $\mathcal{W}_a$ be an ellipse with foci E and F which passes through D. Ellipses $\mathcal{W}_b$ and $\mathcal{W}_c$ are defined respectively. Let A' and A'' be the intersection points of $\mathcal{W}_b$ and $\mathcal{W}_c$. Points B',B'',C' and C'' are defined respectively.

1) Prove that Points A, A' and A'' are collinear.

2) Let l_a be the line passing through A,A' and A''. Lines l_b and l_c are defined respectively. Prove that lines l_a,l_b and l_c are concurrent. Name the concurrency point as $P_{\triangle ABC}$.

3) Let B,C and circle $\mathcal{W}(O,R)$ be three fixed objects such that B,$C \in \mathcal{W}$. Let A be an arbitrary point lying on $\mathcal{W}$. Find the locus of $P_{\triangle ABC}$. (Note that $P_{\triangle ABC}$ is defined if and only if $\triangle ABC$ is acute).

yetti

#2 Oct 6, 2014, 5:03 pm • 2

(1) If reference $\triangle ABC$ is acute, then A, B, C are D-, E-, F-excenters, resp., of its orthic $\triangle DEF$. Let $[EF] = d, [FD] = e, [DE] = f$ be sides of orthic triangle.
Let $\mathcal{K}, \mathcal{L}, \mathcal{M}$ be ellipses with foci $(E, F), (F, D), (D, E)$ and passing through D, E, F, resp. Their major axes are then $p = e + f, q = f + d, r = d + e$
and their excentricities $\kappa = \frac{d}{p}, \lambda = \frac{e}{q}, \mu = \frac{f}{r}$. Let $(A', A'') \in \mathcal{L} \cap \mathcal{M}, (B', B'') \in \mathcal{M} \cap \mathcal{K}, (C', C'') \in \mathcal{K} \cap \mathcal{L}$ be their pairwise intersections.
Let $(K_E, K_F), (L_F, L_D), (M_D, M_E)$ be corresponding vertices of $\mathcal{K}, \mathcal{L}, \mathcal{M}$. Let $(k_e \parallel k_f), (l_f \parallel l_d), (m_d \parallel m_e)$ be their corresponding directrices.

Since they are polars of foci WRT their pedal circles $\Longrightarrow$ their distances are $\delta(k_e, k_f) = \frac{p^2}{d} = \frac{p}{\kappa}, \delta(l_f, l_d) = \frac{q^2}{e} = \frac{q}{\lambda},$
$\delta(m_d, m_e) = \frac{r^2}{f} = \frac{r}{\mu}.$

Let $X \equiv l_d \cap m_d, Y \equiv m_e \cap k_e, Z \equiv k_f \cap l_f$. Since $\frac{\delta(A', l_d)}{\delta(A', m_d)} = \frac{\delta(A', l_d)}{[AA']} \cdot \frac{[AA']}{\delta(A', m_d)} = \frac{\lambda}{\mu}$ and similarly,
$\frac{\delta(A'', l_d)}{\delta(A'', m_d)} = \frac{\lambda}{\mu} \Longrightarrow X \in A'A''$. Likewise, $Y \in B'B''$ and $Z \in C'C''$.
Let $U \equiv l_d \cap m_e, V \equiv l_f \cap m_d, W \equiv l_f \cap m_e$, so that $XUWV$, formed by directrix pairs $(l_f \parallel l_d), (m_d \parallel m_e)$, is parallelogram.
Let common chord $XA'A''$ of ellipses $\mathcal{L}, \mathcal{M}$ cut parallelogram diagonal UV at P and directrices l_f, m_e at Q, R $\Longrightarrow$
$\frac{\overline{PU}}{\overline{PV}} = -\frac{[XUP]}{[XVP]} = -\frac{\delta(P, l_d)}{\delta(P, m_d)} \cdot \frac{[XU]}{[XV]} = -\frac{\mu}{\lambda} \cdot \frac{\delta(m_d, m_e)}{\delta(l_f, l_d)} = -\frac{r}{q}.$
$\overline{VQ} \parallel \overline{XU}$ and $\overline{UR} \parallel \overline{XV} \Longrightarrow \frac{\overline{PX}}{\overline{PQ}} = \frac{\overline{PU}}{\overline{PV}} = -\frac{r}{q}$ and $\frac{\overline{PX}}{\overline{PR}} = \frac{\overline{PV}}{\overline{PU}} = -\frac{q}{r} \Longrightarrow$
$\overline{XQ} = \overline{PQ} - \overline{PX} = -\frac{q+r}{r} \cdot \overline{PX}$ and $\overline{XR} = \overline{PR} - \overline{PX} = -\frac{q+r}{q} \cdot \overline{PX} \Longrightarrow \frac{\overline{XQ}}{\overline{XR}} = \frac{q}{r}.$
Since $[DL_F] = [DM_E] = \frac{1}{2}(d + e + f) = s$, semiperimeter of $\triangle DEF \Longrightarrow$ ellipse vertices L_F, M_E are tangency points of its D-excircle (A) with sidelines $DF, DE \Longrightarrow$
ellipse vertex tangents $v_{lf} \perp DF, v_{me} \perp DE$ at L_F, M_E, resp., intersect at D-excenter A. Let common chord $XA'A''$ of ellipses $\mathcal{L}, \mathcal{M}$ meet their vertex tangents v_{lf}, v_{me} at A_L, A_M, resp. $\Longrightarrow$
$\frac{\overline{XA_L}}{\overline{XQ}} = \frac{\delta(l_d, v_{lf})}{\delta(l_d, l_f)} = \frac{\frac{1}{2}(\delta(l_d, l_f) + q)}{\delta(l_d, l_f)} = \frac{1}{2}(1 + \lambda) = \frac{\frac{1}{2}(q + e)}{q} = \frac{s}{q}$ and likewise,
$\frac{\overline{XA_M}}{\overline{XR}} = \frac{\delta(m_d, v_{me})}{\delta(\ldots)} = \frac{\frac{1}{2}(\delta(m_d, m_e) + r)}{\delta(\ldots)} = \frac{1}{2}(1 + \mu) = \frac{\frac{1}{2}(r + f)}{\ldots} = \frac{s}{\ldots} \Longrightarrow$

$\dfrac{XA_L}{XA_M} = \dfrac{s}{q} \cdot \dfrac{r}{s} \cdot \dfrac{XQ}{XR} = 1 \Longrightarrow$ points $A_L \equiv A_M \equiv A$ are identical and ellipse common chord $XA'A''$ passes through A. In the same way, we can show that $B \in YB'B''$ and $C \in ZC'C''$.

(2) For a more general problem of ellipses $\mathcal{K}, \mathcal{L}, \mathcal{M}$ with foci $(B, C), (C, A), (A, B)$, resp., at vertices of $\triangle ABC$ and witrh arbitrary major axes, see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=29201 (no solution),
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=348789,
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=366631 (link to book).

(3) Locus of $P_{\triangle ABC}$ appears to be ellipse centered at midpoint of $[BC]$, with major axis line BC. No proof yet.

Pirkuliyev Rovsen
#3 Oct 6, 2014, 7:32 pm

Glad to see you here yetti

wiseman
#4 Oct 7, 2014, 1:29 am

Thank you really much dear yetti! I'm happy that someone posted an answer finally. I hope you can solve the last one!

Luis González
#5 Oct 7, 2014, 8:52 am • 3

(1) D-excircle (A) of $\triangle DEF$ touches EF, FD, DE at X, Y, Z. Ellipse $\mathcal{W}_B$ is the locus of the centers of all circles externally tangent to $(F) \equiv \odot(F, FX)$ and internally tangent to $(D) \equiv \odot(D, DY)$. Similarly, ellipse $\mathcal{W}_C$ is the locus of the centers of all circles externally tangent to $(E) \equiv \odot(E, EX)$ and internally tangent to $(D) \Longrightarrow \{A', A''\} \equiv \mathcal{W}_B \cap \mathcal{W}_C$ are the centers of the circles (A') and (A'') internally tangent to (D) and externally tangent to both (E) and (F). Since (A) is orthogonal to $(D), (E), (F)$, then the inversion WRT (A) leaves them invariant and swaps (A') and (A'') by conformity $\Longrightarrow$ their centers A' and A'' are collinear with the inversion center A. Analogously $B \in B'B''$ and $C \in C'C''$.

(2) $\ell_a \equiv AA'A''$ is the D- Soddy line of $\triangle DEF$ passing through its De Longchamps point and similarly $\ell_b \equiv BB'B''$ and $\ell_c \equiv CC'C''$ are Soddy lines of $\triangle DEF$ passing through $L \Longrightarrow \ell_a, \ell_b, \ell_c$ concur at the De Longchamps point $P_{\triangle ABC} \equiv L$ of $\triangle DEF$.

(3) Let T and N be the orthocenter and circumcenter of $\triangle DEF$ (9-point center of ABC) $\Longrightarrow L$ is the reflection of T on N. M is the midpoint of BC and U is the projection of L on BC. MN cuts LU at J and parallel DT from D to NM passes through the reflection S of N on BC and the reflection K of L on J. $\triangle NJS$ is medial triangle of $\triangle KLT$.

$$NJ = TS = TD + NM = 2 \cdot NM \cos \widehat{EDF} + NM \Longrightarrow$$

$$MJ = NJ - NM = 2 \cdot NM \cos \widehat{EDF} = |\cos 2\widehat{A}|R = \text{const}.$$

$\dfrac{UJ}{\frac{1}{2}NS} = \dfrac{UJ}{\frac{1}{2}LJ} = \dfrac{MJ}{MN} = |\cos 2\widehat{A}| \Longrightarrow \dfrac{UL}{UJ} = 1 - \dfrac{1}{|\cos 2\widehat{A}|} \Longrightarrow (J, L, A_\infty, U) = 1 - \dfrac{1}{|\cos 2\widehat{A}|} = \text{const}$, where A_∞ is the infinite point of $\perp BC \Longrightarrow L \mapsto J$ is an affine homology fixing BC with affine direction $A_\infty \Longrightarrow$ locus of L is the ellipse image of the circle $\odot(M, |\cos 2\widehat{A}|R)$. Thus, it has pedal circle $\odot(M, |\cos 2\widehat{A}|R)$ and an axis along BC.

Quick Reply

High School Olympiads

elipse 3

Reply

EdsonBR

#1 Oct 6, 2014, 6:45 am

Consider a elipse of foci F and F' and variable focal rays FM and F'M' parallels and same direction. Determinate The locus of The intersection of The tangents to The elipse drawn from M and M' .

Luis González

#2 Oct 7, 2014, 12:54 am

Generalization: $\mathcal{C}$ is a fixed conic with center O and F, F' are two fixed points on its plane, such that O is midpoint of $\overline{FF'}$. Points M and M' vary on $\mathcal{C}$, such that FM and $F'M'$ are parallel and have the same direction. Tangents of $\mathcal{C}$ at M and M' meet at P. Then the locus of P is another conic $\mathcal{F}$ bitangent to $\mathcal{C}$ through $FF' \cap \mathcal{C}$, thus its center is also O.

Since $\mathcal{C}$ is symmetric WRT O, it follows that $F'M'$ cuts $\mathcal{C}$ again at the reflection N of M on O. The application $\mathbb{H} : M \mapsto M'$ being the composition of the symmetry about O followed by involution with pole F' is homographic fixing the conic $\mathcal{C}$. If we fix two homologous points $\mathbb{H} : X \mapsto X' \in \mathcal{E}$, then $U \equiv XM' \cap X'M$ is on homography axis FF'.

Let $\ell_M, \ell_{M'}, \ell_X, \ell_{X'}$ be the tangents of $\mathcal{C}$ at M, M', X, X', resp. $Q \equiv \ell_{M'} \cap \ell_{X'}, R \equiv \ell_{X'} \cap \ell_X$ and $S \equiv \ell_X \cap \ell_M$. Then $U \equiv QS \cap PR$ and $V \equiv XM \cap X'M' \cap PR$ is the pole of QS WRT $\mathcal{C} \Longrightarrow Q(M', X', U, V) = -1 \Longrightarrow$ $(P, R, U, V) = -1$. But since $XM \mapsto X'M'$ is a homography, then V moves on a conic $\mathcal{K}$ through $X, X' \Longrightarrow P$ moves on the image $\mathcal{F}$ of $\mathcal{K}$ under the involutive homology that fixes the pencil R and the line FF'. It touches $\mathcal{C}$ at its intersections with the homography axis $FF' \Longrightarrow$ it also has center O. $\blacksquare$

When F, F' coincide with the focus of $\mathcal{C}$, as the original problem, then $\mathcal{F}$ becomes the pedal circle of $\mathcal{C}$.

Quick Reply

High School Olympiads

ellipse 4

Reply

EdsonBR

#1 Oct 6, 2014, 6:49 am

It's given an ellipse . It's chosen a chord RS which passes through the center and take the tangents that passes through R and S , also take the tangents parallel to RS . Show that the area of the formed parallelogram is independent of the choice of RS

Luis González

#2 Oct 6, 2014, 10:49 am • 1

Label $ABCD$ the referred parallelogram. An affine homography, taking the given ellipse with center O into a circle with center O', takes $ABCD$ into a square $A'B'C'D'$ with incircle (O'). So if ϱ denotes the radius of (O'), we have $[A'B'C'D'] = 4\varrho^2 = \text{const}.$ As affine homographies can be obtained by orthogonal projection followed by a similarity, it follows that the ratio of the areas of two figures is preserved, thus the area of the parallelogram $ABCD$ is also constant.

EdsonBR

#3 Oct 6, 2014, 11:47 pm

Really cool ! Is there any solution using only classical geometry?

Quick Reply

High School Olympiads

ellipse 1

Reply

EdsonBR
#1 Oct 6, 2014, 4:47 am

Using ruler and compass determinate the center of a given ellipse.

Luis González
#2 Oct 6, 2014, 10:00 am

This depends on how the given conic $\mathcal{C}$ is defined. If the curve is given then it is possible two construct two parallel chords, the line passing through their midpoints contains the center of the ellipse. Repeating the construction for other pair of parallel chords yields the center of the conic.

If the conic is defined by, for example, five points, four points and an incident tangent, three points and two incident tangents, etc, then the previous construction falls apart. All of these cases can be reduced to the case where four points A, B, C, D and the tangent d at D are known, by resorting to Pascal-Brianchon theorems. Now constructing an arbitrary circle ω tangent to d at D, there is a homology fixing the pencil D that transforms $\mathcal{C}$ into ω. Now constructing the images A', B', C' of A, B, C and the limiting line ℓ' of the figure ω in the referred homology, the center of $\mathcal{C}$ is just the image of the pole of ℓ' WRT ω.

Quick Reply

High School Olympiads

elipse 2

EdsonBR

#1 Oct 6, 2014, 4:53 am

Determinate The locus of The midpoints of The pararell cords (same given direction) of a ellipse.

Luis González

#2 Oct 6, 2014, 7:22 am

In general, given a fixed ellipse $\mathcal{E}$ with center O and a fixed point P on its plane, the midpoints of the chords $\overline{AB}$ passing through P lie on an ellipse $\mathcal{F}$ homothetic to $\mathcal{E}$ that has a diameter $\overline{PO}$.

There is an affine homography (fixing the line at infinity) that takes $\mathcal{E}$ into a circle (O'). Midpoint M of $\overline{AB}$ goes to the midpoint M' of $\overline{A'B'}$. In this figure, we obviously have $O'M' \perp A'B' \Longrightarrow M'$ lies on fixed circle $\mathcal{F}'$ with diameter $\overline{P'O'} \Longrightarrow M$ lies on a fixed ellipse $\mathcal{F}$ that has a diameter $\overline{PO}$. Now, as any homothetic application remains invariant under affine homographies, we deduce that that $\mathcal{E}$ and $\mathcal{F}$ are homothetic.

When $\mathcal{E}$ is instead a hyperbola, the locus of M is still a conic through O, P but no longer homothetic to $\mathcal{E}$. When P is at infinite, as the problem states, then M lies on a fixed line through O.

High School Olympiads

fIND THE LOCUS

Reply

Submathematics

#1 Oct 3, 2014, 7:30 pm

In a variable triangle ABC the basis BC is fixed and the magnitude $\angle ABC - \angle ACB = k$, where k is constant. Find the equation of the locus of the vertex A, preferably by Cartesian coordinates.

Luis González

#2 Oct 4, 2014, 11:21 am

For convenience rename $A \equiv P$, so we'll find the locus of the points P satisfying $\widehat{PBC} - \widehat{PCB} = k = \text{const}.$ If we fix a point A, such that $\widehat{ABC} - \widehat{ACB} = k$, then we get $\widehat{ABC} - \widehat{PBC} = \widehat{ACB} - \widehat{PCB} \Longrightarrow \widehat{ABP} = \widehat{ACP}.$ Hence, if Q is the isogonal conjugate of P WRT $\triangle ABC$, we have $\widehat{QBC} = \widehat{PBA} = \widehat{PCA} = \widehat{QCB} \Longrightarrow QB = QC$, i.e. Q is on perpendicular bisector ℓ of $\overline{BC} \Longrightarrow$ locus of P is the isogonal conjugate of ℓ WRT $\triangle ABC$; a rectangular hyperbola $\mathcal{H}$ through $A, B, C.$

The isogonal conjugate of the point D forming the parallelogram $ABDC$ is the point $E \in \ell$ where the tangents of $\odot(ABC)$ at B, C intersect, and the isogonal conjugates of the midpoints of the arcs BC and BAC of $\odot(ABC)$ are the infinity points of the angle bisectors of $\angle BAC.$ Therefore, we conclude that $\mathcal{H}$ is centered at the midpoint of $\overline{BC}$ and its asymptotes form angles of $\frac{k}{2}$ and $90^\circ + \frac{k}{2}$ with $BC.$

Quick Reply

High School Olympiads

long solution using trigometry

Reply

EdsonBR

#1 Oct 4, 2014, 3:41 am

In a triangle ABC, the points M and N lie on the sides AB and AC, respectively, and such that MB=BC=CN. Let R and r be circunradius and inradius of ABC. Give MN/BC in function of R and r.

Luis González

#2 Oct 4, 2014, 9:43 am

Discussed before. If I and O denote the incenter and circumcenter of $\triangle ABC,$ we have $\frac{MN}{BC}=\frac{OI}{R}=\frac{\sqrt{R(R-2r)}}{R}$.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=20406
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=30955
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=321293

Quick Reply

High School Olympiads

Three collinear points

Reply

jayme
#1 Oct 2, 2014, 8:00 pm

Dear Mathlinkers,

Let ABC be a triangle inscribed in a circle, and let P, Q be two inverse points with respect to the circle.
Take L, M, N, the images (symmetric points) of P in BC, CA, AB,
and join QL, QM, QN cutting BC, CA, AB in X, Y, Z respectively.
Then the points X, Y, Z are collinear.

Sincerely
Jean-Louis

Luis González
#2 Oct 4, 2014, 2:24 am • 1

There is more to say in this configuration. X,Y,Z are collinear and this line also passes through the orthopole Or of PQ WRT $\triangle ABC,$ center of the rectangular hyperbola $\mathcal{H}$ isogonal of $PQ.$

Fix the line $\ell \equiv PQ$ passing through the circumcenter O of $\triangle ABC.$ Let P^*,Q^* be the isogonal conjugates of P,Q WRT $\triangle ABC$ lying on $\mathcal{H}.$ Since $P \mapsto Q$ is an involution and $P \barwedge P^*,$ it follows that $P^* \mapsto Q^*$ is consequently an involutive homography fixing $\mathcal{H}.$ When P coincides with the intersections of ℓ with $(O),$ then P^* goes to infinite in the direction of the asymptotes of $\mathcal{H},$ hence the pole of this involution is none other than the center Or of $\mathcal{H},$ in other words P^* and Q^* are symmetric WRT $Or.$

As Or is Poncelet point of $ABCP^*$ and $ABCQ^*,$ then the pedal circles of P^* and Q^* WRT $\triangle ABC$ intersect at $Or \Longrightarrow$ perpendicular τ to P^*Q^* at Or touches the inconics with foci $\{P,P^*\}$ and $\{Q,Q^*\},$ resp $\Longrightarrow P,P^*$ and Q,Q^* are isogonal conjugates WRT the complete quadrangle $BC,CA,AB,\tau.$ Hence if τ cuts BC at $U,$ we have $\angle(UC,UP)=\angle(UP^*,\tau)=\angle(\tau,UQ^*)=\angle(UQ,UC),$ which means that UQ and UP are symmetric WRT $BC,$ i.e. $L \in QU \Longrightarrow U \equiv X \equiv \tau \cap BC.$ Likewise, τ hits CA and AB at Y and $Z \Longrightarrow X,Y,Z$ lie on a line τ passing through the orthopole Or of PQ WRT $\triangle ABC.$

Quick Reply

High School Olympiads

Concurrent Circles

Reply

Source: me?

mohohoho

#1 Oct 1, 2014, 8:50 pm • 1

Let ABC be a triangle.
Let P be a point on the plane of ABC.
Let a, b, c, are the sides opposite vertices A, B, C.
Let P_a be the reflection of P around the perpendicular
bisector of a. Define P_b, P_c cyclically.

Then, circles (AP_bP_c), (BP_aP_c), (CP_aP_b) are concurrent at
the circumcircle of ABC.

Anyone has reference for this problem?

Regards,
Emmanuel.

Luis González

#2 Oct 1, 2014, 10:41 pm • 2

Let Q be the second intersection of $\odot(AP_bP_c)$ with the circumcircle (O) of $\triangle ABC$. Since O is on perpendicular bisector of $\overline{AQ}$ and $OP = OP_b = OP_c$, by obvious symmetry it follows that the cyclic AQP_bP_c is an isosceles trapezoid with $AQ \parallel P_bP_c$. Now let ℓ_a be the parallel to P_bP_c through P and let τ_a be the tangent of $\odot(PP_bP_c)$ at P. Then $\angle(PP_b, \ell_a) = \angle(P_bP, P_bP_c) = \angle(\tau_a, PP_c) \Longrightarrow \ell_a$ and τ_a are isogonals WRT $\angle P_bPP_c$. But since $\ell_a \parallel P_bP_c \parallel AQ$, $PP_b \parallel AC$ and $PP_c \parallel AB$, then it follows that AQ is the isogonal of the perpendicular from A to OP WRT $\angle BAC \Longrightarrow Q$ is therefore the isogonal conjugate of the infinite point of $\perp OP$ WRT $\triangle ABC$. Similarly, circles $\odot(BP_cP_a)$ and $\odot(CP_aP_b)$ hit (O) again at Q.

mohohoho

#3 Oct 1, 2014, 10:57 pm

Thank you, Luis.

buratinogigle

#4 Oct 15, 2014, 9:45 pm

Variation of this problem

Let ABC be a triangle and three Apollonius circles with respect to A, B, C are $(K_a), (K_b), (K_c)$. P is a point on plane. Let D, E, F are inversions of P through $(K_a), (K_b), (K_c)$, respectively.

a) Prove that P, D, E, F lie on a circle with center lie on Brocard Axis of ABC.

b) Prove that circumcircles of triangle AEF, BFD, CDE are concurrent.

TelvCohl

#5 Oct 16, 2014, 1:03 am • 2

> buratinogigle wrote:

Variation of this problem

Let ABC be a triangle and three Apollonius circles with respect to A, B, C are $(K_a), (K_b), (K_c)$. P is a point on plane. Let D, E, F are inversions of P through $(K_a), (K_b), (K_c)$, respectively.

a) Prove that P, D, E, F lie on a circle with center lie on Brocard Axis of ABC.

b) Prove that circumcircles of triangle AEF, BFD, CDE are concurrent.

My solution:

(a)

It's well known that $(K_a), (K_b), (K_c)$ concur at isodynamic points S, S' of triangle ABC .
Let (T) be the circle pass through P and orthogonal to $(K_a), (K_b), (K_c)$.
Let P' be the reflection of P with respect to T .

Since (T) is orthogonal to (K_a) ,
so the intersection of PK_a and (T) is the inversion of P through (K_a) . ie. D lie on (T)
Similarly, we can prove E, F lie on (T) .

Since (T) is orthogonal to $(K_a), (K_b), (K_c)$,
so we get T lie on the radical axis of $(K_a), (K_b), (K_c)$ and P, D, E, F lie on (T) .
ie. P, D, E, F lie on a circle with center lie on Brocard Axis of ABC

(b)

Invert with respect to S and denote Z' as the image of Z under this inversion.

Since $SA * BC = SB * CA = SC * AB$ and (ABC) is orthogonal to $(K_a), (K_b), (K_c)$,
so $A'B'C'$ is an equilateral triangle and $(K_a)', (K_b)', (K_c)'$ is the perpendicular bisector of $B'C'$, $C'A'$, $A'B'$,
so D' , E' , F' is the reflection of P' with respect to the perpendicular bisector of $B'C'$, $C'A'$, $A'B'$.

Since the midpoint of $P'D'$, $P'E'$, $P'F'$ lie on $(O'P')$ (O' is the center of $A'B'C'$)
so we get P' , D' , E' , F' are concyclic with center O' and $D'E'F'$ is also an equilateral triangle .
hence the line pass through A' , B' , C' and parallel to $E'F'$, $F'D'$, $D'E'$ concur at a point G' on $(A'B'C')$
By symmetry it's easy to see $A'G'F'E'$ is a isosceles trapezoid,
so we get A' , G' , E' , F' are concyclic .
Similarly, we can prove B' , G' , F' , D' are concyclic and C' , G' , D' , E' are concyclic,
so we get $(A'B'C'), (A'E'F'), (B'F'D'), (C'D'E')$ are concurrent.
ie. $(ABC), (AEF), (BFD), (CDE)$ are concurrent

Q.E.D

rodinos

#6 Oct 16, 2014, 5:04 am • 1

Emmanuelle,

If A', B', C' are three points on the plane of triangle ABC such that
the circumcircles of A'BC, B'CA, C'AB are concurent, then the circumcircles
of AB'C', BC'A',CA'B' are also concurrent. (it is an old theorem)
In your problem the circumcircles of PaBC, PbCA, PcAB concur at P, therefore
the circumcircles of APbPc, BPcPa, CPaPb are concurrent, and as Luis proved, the
point of concurrence is lying on the circumcircle of ABC.

APH

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