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25^{th}
Iranian Mathematical Olympiad
2007-2008

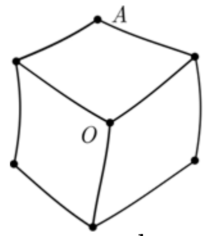
Contents

Problems

First Round

1. Show that for each natural number n, there exist n distinct natural numbers, whose sum is the square of a natural number, and whose product is the cube of a natural number.

2. Consider a $1 \times 1 \times 1$ cube. Let O and A be two of its vertices such that OA is the diagonal of a face of the cube. Which one is larger: The number of paths of length 1386 beginning at O and ending at O, or the number of paths of length 1386 beginning at O and ending at A. (A path of length n on the cube is a sequence of $n+1$ vertices, such that the distance between each two consecutive vertices is 1.)



3. Circles C_1 and C_2 are externally tangent to each other at the point P. From the arbitrary point A on C_1 draw the two tangents AM and AM' to C_2 (M and M' are the points of tangency). Let the second intersections of AM and AM' with C_1 be N and N' respectively. Show that

$$PN \times M'N' = PN' \times MN$$

4. Three Ping Pong teams, each having 10 members, compete in a trilateral tournament. In each game two members of two different teams play against each other; and no two members play against each other more than once. Show that after the 201^{st} game there exist three persons, such that each two of them have had played against each other.

5. In the triangle ABC, $\angle A$ is right. M is the midpoint of BC. Choose D on AC such that $AD = AM$. Let the second intersection of the circumcircles of AMC and BDC be P. Prove that CP bisects the angle $\angle ACB$.

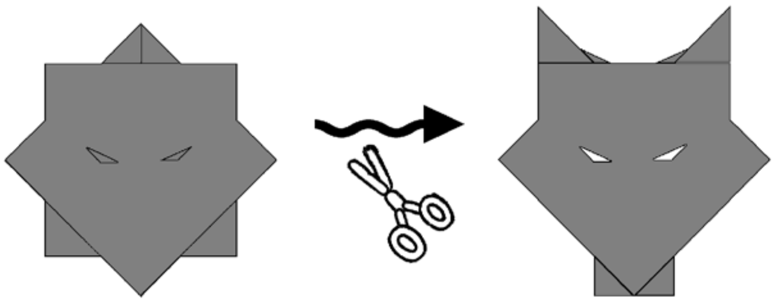
6. There are some buildings, located in a city. We say that a building overlooks a shorter one, if the line connecting the top of the two buildings makes an angle of more than 45° with the ground. Assume that in our city no building overlooks another. We want to construct a new building in a given location. Show that we can do this in such a way that, even after the construction, no building overlooks another.

   (Consider the city to be the xy plane in the 3D space, and the buildings to be line segments parallel to the z axis.)

Second Round

In this round, eight problems are given to the contestants one by one; each problem has its own time duration, and the contestants are asked to hand in their solutions to each problem before the next problem is handed out. Time durations are mentioned after each problem's statement.

1. Consider two polygons P and Q in the plane. We want to cut P along straight lines, dividing it to smaller pieces, move the pieces (without rotating or reflecting them) and glue the pieces together, such that the resulting shape is Q. No piece can overshadow another in the final configuration. Show that:



   (a) Each two rectangles with the same area are convertible to each other.

   (b) Two triangles are convertible to each other iff one is a translated version of the other.

90 minutes

2. A map $\Delta : \mathbb{Z}\backslash\{0\} \to \mathbb{Z}$ is said to be a degree map, iff for each $a, b \in \mathbb{Z}$ where $b \neq 0$ and $b \nmid a$, $\exists r, s \in \mathbb{Z}$ such that $a = br + s$, and $\Delta(s) < \Delta(b)$.

   (a) Show that the following map is a degree map:

$$\delta(n) = \text{ Number of digits of } |n| \text{ in base 2}$$

   (b) Show that there exists a degree map Δ_0 which is smaller than any other degree map. That is for every degree map Δ and $n \neq 0$, $\Delta_0(n) \leq \Delta(n)$.

   (c) Prove that $\Delta_0 = \delta$.

90 minutes

3. Two arrangements of some non-intersecting circles in the plane are called equivalent if we can start with one arrangement; move the circles, enlarge them or shrink them (such that the circles do not cross each other in between the moves) and arrive at the other arrangement.

   The number of nonequivalent arrangements of n circles is shown by a_n.

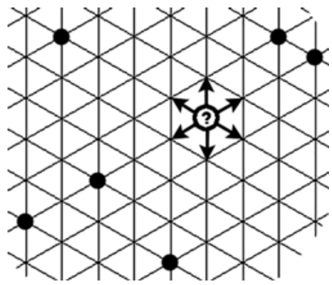
   The first terms of the sequence $\{a_n\}$ are $1, 2, 4, 9, \ldots$.

   If positive numbers A, B, a, b are found such that $Aa^n \leq a_n \leq Bb^n$ we say that the growth rate of $\{a_n\}$ is larger than or equal to a and smaller than or equal to b.

   Prove that the growth rate of $\{a_n\}$ is bigger than or equal to 2 and smaller than or equal to 4.

90 minutes

4. In the following triangular network, the distance between two vertices is the length of the smallest network-path between them.
   Let $A_1, \ldots, A_n$ be some fixed vertices of the network. We want to find a vertex whose sum of distances to $A_1, \ldots, A_n$ is minimum.



   For this, we use the following algorithm: We start with an arbitrary vertex and in each stage, we move to one of the neighbors of this vertex, whose sum of distances is smaller than this vertex. If there are many, we choose an arbitrary one. We do this until there is no neighbor with a smaller sum of distances.

   It is obvious that this algorithm always ends.

   (a) Prove that the final point is the point we are looking for.

   (b) Does this algorithm work correctly, if we replace the triangular network with an arbitrary connected graph?

60 minutes

5. (a) Consider a process like the following one: First we write two fractions $\frac{0}{1}$ and $\frac{1}{0}$. At each stage in the process we write $\frac{a+c}{b+d}$ between every two consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$.

$$\frac{0}{1} \qquad \frac{1}{0}$$

$$\frac{0}{1} \qquad \frac{1}{1} \qquad \frac{1}{0}$$

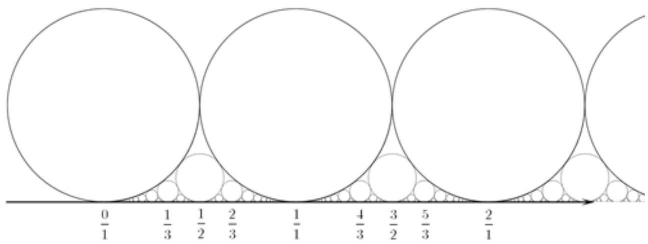
$$\frac{0}{1} \quad \frac{1}{2} \quad \frac{1}{1} \quad \frac{2}{1} \quad \frac{1}{0}$$

$$\frac{0}{1} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{1} \quad \frac{3}{2} \quad \frac{2}{1} \quad \frac{3}{1} \quad \frac{1}{0}$$

$$\frac{0}{1} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{1}{1} \quad \frac{4}{3} \quad \frac{3}{2} \quad \frac{5}{3} \quad \frac{2}{1} \quad \frac{5}{2} \quad \frac{3}{1} \quad \frac{4}{1} \quad \frac{1}{0}$$

Show that each fraction which appears in this process is simplified. (Hint: Prove that if $\frac{a}{b}$ and $\frac{c}{d}$ are two consecutive terms in some stage of the process, then $bc - ad = 1$.)

(b) On the real line, we have placed a circle of diameter 1, tangent to the line, above each of the numbers $0, 1, \ldots$. At each stage, for each two consecutive (that is their tangency points with the real line are consecutive) circles, we draw a circle tangent to them and the real line, between them.



Show that the numbers that appear as the tangency points of the circles are exactly those that appear in (a). (Except $\frac{1}{0}$ which is not a number)

(c) Prove that in (a) and (b), every rational number appears in some stage of the process.

120 minutes

6. Scientists, with the help of powerful microscopes, have found some new numbers between the real numbers! After extensive research, they have extended the real numbers system to a larger system of numbers. In this system we can compare each two numbers, and we can do normal arithmetical operations ($+ - \times \div$) on these numbers; and luckily all the algebraical and ordinal relations of the real numbers system, hold.

(a) Prove that there exists a new number, which is smaller than any positive real number, but greater than zero.

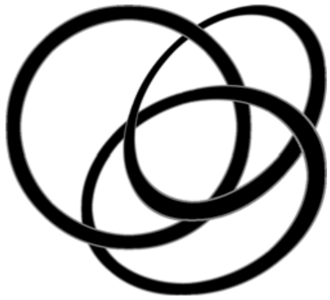
(b) Prove that no new number is a root of a polynomial with real coefficients.

60 minutes

7. A ring is the area between two co-planar circles with the same center. The width of the ring is the difference between the circles' radii.

(a) Can we find an uncountable number of disjoint rings with the width 1 in the space, such that each two of them are tied to each other?

(b) What is the answer when the width of 1 is replaced by 0 (rings become circles)?



90 minutes

8. In this question, you are to label an anlog clock. You have to construct each of the numbers $1, 2, \ldots, 12$ by using exactly three 2 and mathematical symbols which do not contain any alphabetical letter.

20 minutes

Third Round

First Exam

1. I_a is the excenter of the triangle ABC with respect to A, and AI_a intersects the circumcircle of ABC at T. Let X be a point on TI_a such that $XI_a^2 = XA.XT$. Draw a perpendicular line from X to BC so that it intersects BC in A'. Define B' and C' in the same way. Prove that AA', BB' and CC' are concurrent.

2. Find all polynomials P with integer coefficients such that, for every $a, b \in \mathbb{N}$ if $a + b$ is a perfect square, then $P(a) + P(b)$ is also a perfect square.

3. Let $a, b, c > 0$ be such that $ab + bc + ca = 1$. Prove that

$$\sqrt{a^3 + a} + \sqrt{b^3 + b} + \sqrt{c^3 + c} \geq 2\sqrt{a + b + c}$$

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the following identity for each $x, y \in \mathbb{R}$:

$$f(xf(y)) + y + f(x) = f(x + f(y)) + yf(x)$$

5. Suppose that I is the incircle of the triangle ABC, and l' is a tangent to this circle. l is another line which intersects the lines BC,CA,AB in A',B',C' respectively. Draw a tangent from A' to I, other than BC, and name its intersection with l', A_1. B_1 and C_1 are defined in the same way. Prove that AA_1, BB_1 and CC_1 are concurrent.

6. Let T be a tree with k edges. Show that one can partition the edges of the k dimensional cube into subgraphs isomorphic to T.

Second Exam

1. In the triangle ABC, $\angle B$ is greater than $\angle C$. T is the midpoint of the arc BAC from the circumcircle of ABC and I is the incenter of ABC. E is a point such that $\angle AEI = 90^\circ$ and $AE \parallel BC$. TE intersects the circumcircle of ABC for the second time in P. If $\angle B = \angle IPB$, find the angle $\angle A$.

2. Show that in a tournament with 799 teams, there exist 14 teams, such that, each of the first seven teams have won the game against each of the last seven teams.

3. Let P_1, P_2, P_3, P_4 be arbitrary points on the unit sphere. Show that $\sum_{i \neq j} \frac{1}{|P_i - P_j|}$ has its minimum value iff P_1, P_2, P_3, P_4 are the vertices of a regular tetrahedron.

4. Suppose that S is a set with size n and F is a family of subsets of S with 2^{n-1} such subsets, for which each three of them have nonempty intersection.

   (a) Show that the intersection of all elements of F is nonempty.

   (b) If we replace 2^{n-1} by $2^{n-1} - 1$ does the answer change? Why?

5. k is a given natural number. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for each $m, n \in \mathbb{N}$ the following holds:

$$f(m) + f(n) \mid (m+n)^k$$

6. In the acute-angled triangle ABC, D is the intersection of the altitude passing through A with BC and I_a is the excenter of the triangle with respect to A. K is a point on the extension of AB from B, for which $\angle AKI_a = 90^\circ + \frac{3}{4}\angle C$. I_aK intersects the extension of AD at L. Prove that DI_a bisects the angle $\angle AI_aB$ iff $AL = 2R$. (R is the circumradius of ABC.)

Solutions

First Round

1. The numbers $1^3, 2^3, \ldots, n^3$ are n distinct numbers. Since each one is a perfect cube, their product is also a perfect cube.

   Next we will show that their sum is $\left(\frac{n(n+1)}{2}\right)^2$. We use induction on n. For $n = 1$ it is obvious.

   Now we have

$$\left(\frac{n(n-1)}{2}\right)^2 + n^3 = \frac{n^4 - 2n^3 + n^2}{4} + n^3 = \frac{n^4 + 2n^3 + n^2}{4} = \left(\frac{n(n+1)}{2}\right)^2$$

   which completes the induction.

2. Let a_n be the number of paths of length n from O to itself, and b_n be the number of paths of the same length from O to A.

   Consider a path from O to itself. Consider the $n-1^{th}$ vertex of the path. A careful examination shows that it should have been either O or the other endpoint of a diagonal of a face, which contains O as one endpoint. The number of paths of length 2 from O to itself is 3. The number of paths of length 2 from A to O is 2. So we have the relation:

$$a_n = 2(b_{n-2} + b_{n-2} + b_{n-2}) + 3a_{n-2}$$

   One can similarly show that (considering all the possibilities for the $n-1^{th}$ vertex of the path)

$$b_n = 2(a_{n-2} + b_{n-2} + b_{n-2}) + 3b_{n-2}$$

   Now, taking the difference of the two relations, we arrive at

$$a_n - b_n = a_{n-2} - b_{n-2}$$

   So we have

$$a_{1386} - b_{1386} = a_{1384} - b_{1384} = \cdots = a_0 - b_0 = 1$$

   So $a_{1386} > b_{1386}$.

3. A homothety H with P as the center and suitable coefficient, would map C_1 onto C_2. The coefficient is $k = -\frac{R_2}{R_1}$ where R_i is the radius of C_i. Now let Q be $H(N)$ and Q' be $H(N')$.

Now consider the power of the point N with respect to C_2. Since NM is tangent to C_2, this power is equal to NM^2. But it is also equal to $NP.NQ$. We have $PQ = -kNP$ (because of the homothety). So $NQ = NP + PQ = (1-k)NP$ and hence the power is equal to $(1-k)NP^2$. So we have

$$(1-k)NP^2 = NM^2$$

from which we have $\frac{NM}{NP} = \sqrt{1-k}$.

We can do the same for N' and get $\frac{N'M'}{N'P} = \sqrt{1-k} = \frac{NM}{NP}$ which proves the claim.

4. Consider the graph of the games till the end of the 201^{st} game. For each member there is one vertex, and for each game there is an edge between the persons who participated in that game.

   The graph is obviously tripartite, with each team as a part. Name the parts of the graph as A, B, C. Let N be the number of pairs of vertices from different parts, that are not connected by an edge. Since the number of pairs of vertices from different parts is $3 \times 10^2 = 300$, and the graph has 201 edges, $N = 300 - 201 = 99$.

   Now assume that the claim of the problem is false. So, for each $a \in A, b \in B, c \in C$, at least one of the pairs between a, b, c are not connected by an edge. There are $10^3 = 1000$ such triples. But each pair of vertices from different parts, that are not connected by an edge, is counted in 10 different triples (the number of ways to choose the third vertex).

   So $N \geq \frac{1000}{10} = 100$, which is a contradiction. So the claim should have been true.

5. Since B, P, D, C are on a circle, $\angle PBC = 180° - \angle CDP = \angle ADP$. Since A, P, M, C are on a circle, $\angle CAP = 180° - \angle PMC = \angle PMB$. Since ABC is a right-angled triangle $AM = MB$; but $AM = AD$, so $AD = MB$.

   Therefor the triangles PMB and PAD are congruent. So the altitudes from P in these two triangles are of equal length, which shows that P is on the bisector of $\angle C$.

6. Let's denote the height of the building located in a point P (on the ground) by h_P. Then the condition that, the buildings located in points A, B do not overlook each other is easily translated to $|h_A - h_B| \leq |A - B|$.

Assume that the new building is to be constructed on the point P and is going to have a height of h. Then for the condition of the problem to hold; for each other building located in a point like A, we should have $|h - h_A| \leq |P - A|$ which specifies an interval for h. We need to show that all these intervals have a common point greater than zero. Let h be equal to the minimum of all the upper bounds of these intervals. Since each upper bound is something like $h_A + |P - A|$ which is greater than zero, h is greater than zero. Since it is a minimum over a finite set, there exists a building A for which $h = h_A + |P - A|$.

Now we want to show that h is inside each of those intervals. Assume the contrary, and let B be the building for which h is not inside the interval specified by B. h is obviously less than or equal to the upper bound of this interval. So the only possibility that remains is for h to be less than the lower bound. That is $h < h_B - |P - B|$. But then

$$h_A + |P - A| = h < h_B - |P - B|$$

which shows that

$$h_B - h_A > |P - A| + |P - B| \geq |B - A|$$

which shows that the building on the point B overlooks the building on the point A. A contradiction!

Second Round

1. For each polygon in the plane, direct its perimeter in the clockwise direction, so that each of its edges becomes a directed line segment (or simply a vector).

   Now consider the vectors parallel to a given line l. A cut along a line, not parallel to l, produces no new vectors of this kind. A cut along a line parallel to l, produces two vectors of this kind in opposite directions.

   Therefor if we consider the sum of such vectors, it doesn't change by any cut.

   Obviously moving the polygons doesn't change the sum either.

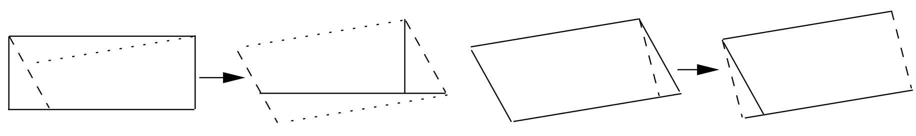
   Gluing along edges, not parallel to l does not change the vectors; and gluing along an edge parallel to l, is equivalent to removing two vectors, and putting their sum instead.

   So this sum is invariant under the moves specified in the problem.

   For two triangles ABC and $A'B'C'$ to be convertible, the invariant we introduced should be the same for each line l. Putting l as the lines passing through vertices of ABC, shows that the sides of ABC and $A'B'C'$ should be equal. (when treated as vectors) So $A'B'C'$ should be a translated version of ABC.

   For the first part of the problem, the figure below shows one way to convert a rectangle to a parallelogram and then to another rectangle. One side of the parallelogram is arbitrary (the dotted line is arbitrary in both direction and length). This side is exactly one of the sides of the resulting rectangle. So the second rectangle is arbitrary (among the rectangles with the same area).

   Of course the dotted side should be somehow close to the top edge of the first rectangle for this conversion to work, but note that you can use several conversions in series, and the fact that being convertible is an equivalence relation.

2. For (a), let $a, b \in \mathbb{Z}$ be such that $b \neq 0$ and $b \nmid a$. Now divide a by b and write $a = br' + s'$ where $0 < s' < |b|$. If $s' < \frac{|b|}{2}$, then put $r = r', s = s'$; and if $s' \geq \frac{|b|}{2}$ then put $r = r' \pm 1, s' = s - |b|$ (the sign $\pm$ is the same as b's sign).

One can easily see that $a = br + s$ and $|s| \leq \frac{|b|}{2}$ which shows that $\delta(s) \leq \delta(b) - 1$.

For (b), let $\Delta_0(n)$ be the minimum of $\Delta(n)$ for every degree map Δ. It suffices to show that Δ_0 is a degree map. Given a, b, there exists a degree map Δ for which $\Delta_0(b) = \Delta(b)$. Now take the r, s given by Δ. So $a = br + s$ and $\Delta(s) < \Delta(b)$. But then $\Delta_0(s) \leq \Delta(s) < \Delta(b) = \Delta_0(b)$ which shows that this r, s work for Δ_0 too.

Now for (c): For $n = \pm 1$, $\delta(n) \geq \Delta_0(n) \geq 1 = \delta(n)$ which shows that $\delta(n) = \Delta_0(n)$. Let n be a number such that $\Delta_0(n) < \delta(n)$. If there are many, choose one with the minimum $\Delta_0(n)$. $n \neq \pm 1$ because of the previous result. Now take $a = \lfloor \frac{n}{2} \rfloor$ and $b = n$. Then $b \nmid a$ because $a \neq \pm 1$.

Let r, s be such that $\Delta_0(s) < \Delta_0(n)$ and $a = br + s$. Since $\Delta_0(s) < \Delta_0(n)$, we have $\Delta_0(s) = \delta(s)$. But it can be easily shown that whatever the value of s is, $|s| \geq \lfloor \frac{|n|}{2} \rfloor$ which is at least $\delta(n) - 1$. So $\delta(n) \geq \Delta_0(n) > \delta(n) - 1$ which shows that $\Delta_0(n) = \delta(n)$. A contradiction!

3. For the upper bound, to each arrangement assign a string of parentheses. First arrange the circles in such a way that each circle's center is on the x axis, and then cut the tops and bottoms of every circle.

   The remaining of each circle consists of two arcs shaped either like (or like). Assign this string to this arrangement. (Of course there may be several ways to assign a string to a single arrangement)

   This string is correctly parenthesized, because each (is matched to a corresponding), and the substring between these two, corresponds to a subarrangement with fewer circles (that is those circles contained in one of the circles in the original arrangement)

   It is straightforward to construct back the arrangement from the string of parentheses. (Complete the circles by joining each (with its matching).)

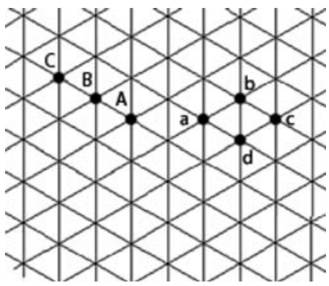
   Therefor a_n is smaller than or equal to the Catalan number of order n which is $\frac{1}{n+1}\binom{2n}{n} \leq 2^{2n} = 4^n$.

   For the lower bound, select a string of 0's and 1's of length n and assign an arrangement to it. Reading the string from left to right, replace each 0 by a circle containing no other circles, and each 1 by a circle containing all the other previous circles.

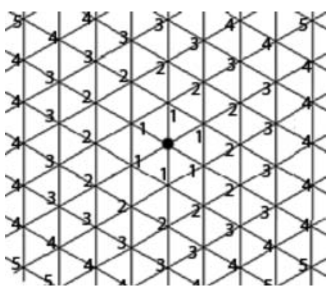
Of course some arrangements can not be achieved by this method. But if an arrangement can be achieved, it can be achieved from at most one string. This is because, we can easily reverse the procedure. first consider the outermost circles; at most one can contain other circles. Put as many 0's as there are empty circles at the end of the string. If no circle is left, then we are done. If there is one left, put a 1 before these zeros, and then put the string representing the contents of this circle at the beginning of our string.

This shows that $a_n \geq 2^n$.

4. A real valued function defined on the vertices of this network is called convex iff for each three vertices positioned like A, B, C and for each four vertices positioned like a, b, c, d in the following figure, the relations $f(A) + f(C) \geq f(B) + f(B)$ and $f(a) + f(c) \geq f(b) + f(d)$ always hold.



It is obvious that the sum of two convex functions is a convex function. The distance from a single point is a convex function, as it can be verified by examining different cases in this figure:



Next, let's show that if we replace the sum of distances to some given points, by a convex function f with a minimum, the algorithm works. (if the algorithm stops, of course)

Assume that the final point given by our algorithm is P, and Q is a point with $f(Q)$ being minimum. There is a path from P to Q with at most one corner. (A corner with angle $120°$) Assume that the points on this path are $P = P_1, P_2, \ldots, P_n = Q$. We can assume W.L.O.G. that $f(P_{n-1}) > f(Q)$, or else we could replace Q by P_{n-1} and still P_{n-1} would have been a minimum. (Again if $f(P_{n-2}) = f(P_{n-1})$ we could have moved to P_{n-2}.)

Now note that except for the P_i at the corner of the path, we have $f(P_{i-1}) + f(P_{i+1}) \geq 2f(P_i)$. If $f(P_i) > f(P_{i+1})$, from the last inequality we would get $f(P_{i-1}) > f(P_i)$. So using backward induction, we can show that $f(P_n) < f(P_{n-1}) < f(P_{n-2}) < \cdots < f(P_i)$, where P_i is the point at the corner of the path. If $f(P_i) < f(P_{i-1})$ we can continue, and deduce that $f(P_2) < f(P)$ which contradicts the assumption that P was the final point of our algorithm. So assume that $f(P_i) \geq f(P_{i-1})$

Now the second relation of convexity becomes handy. Name the reflection of P_i with respect to the line $P_{i-1}P_{i+1}$ as R. Then we would have $f(P_{i-1}) + f(P_{i+1}) \geq f(P_i) + f(R)$. But from the fact that $f(P_i) > f(P_{i+1})$ we can find out that $f(P_{i-1}) > f(R)$. Now replace P_i by R, and we still have a path. Again if $f(P_{i-2}) > f(P_{i-1})$ we get $f(P_2) < f(P)$ and we are done. Otherwise we can do the same things to P_{i-1} as we did to P_i and get an R'.

Continuing this way we will finally find that either $f(P_2) < f(P)$ or $f(S) < f(P)$ where S is the reflected version of P_2 with respect to PP_3. But S and P_2 are both neighbors of P, and this contradicts the assumption that P was our algorithm's final point.

5. For part (a) we use induction. If $bc - ad = 1$ then $b(a+c) - a(b+d) = ba + bc - ab - ad = bc - ad = 1$ and $(b+d)c - (a+d)c = bc + dc - ad - dc = bc - ad = 1$ which proves the induction step. It is obvious that it is true at the beginning of the process.

   Now if $bc - ad = 1$ then $(a, b) = 1$. So every fraction appearing in this process is simplified.

   We will show that if x is a number generated in one of the processes, then $\frac{1}{x}, x + 1, x - 1$ are also produced.

   For the first operation: If the i^{th} number, counting from the beginning of the sequence, is $\frac{a}{b}$, then the i^{th} number, counting from the end of the sequence, would be $\frac{b}{a}$. This is true at the beginning. $(\frac{1}{0}, \frac{0}{1})$ And using induction, it is no hard to prove it.

Now note that the sequence generated by $a+1, b+1$ as the first and the last fraction is exactly the sequence generated by a, b as the first and the last fraction, plus 1. That is a 1 is added to each term of the sequence) Again this is easy to prove using induction. At the beginning it is true. If $\frac{a}{b}, \frac{c}{d}$ are generated somewhere in the second sequence, then $\frac{a+b}{b}, \frac{c+d}{d}$ are generated in the first sequence. The next fraction appearing between them is $\frac{a+c+b+d}{c+d}$ which is one unit more than $\frac{a+c}{b+d}$ (the one appearing in the second sequence)

Now note that, in the 1^{st} step of the process, the number appearing just before $\frac{1}{0}$ is $\frac{1}{1}$. Therefor the numbers appearing between $\frac{1}{1}, \frac{1}{0}$ are exactly one unit more than the numbers appearing between $\frac{0}{1}, \frac{1}{0}$. (Which are all the numbers appearing in the process) So if x appears in the process, $x+1$ does, and if $x+1$ does, then so does x.

Now for the second process: It is obvious that if x appears, then so does $x+1$, because the first circles are one unit apart from each other. A unit inversion from the 0 of the x axis, maps circles to circles, and tangent ones to tangent ones. The circle above 0 is mapped to the line $y=1$. So the circles tangent to it, are mapped to the circles tangent to $y=0$ and $y=1$. Each two consecutive circles among these ones are mapped to tangent circles. Therefor these circles are mapped to circles above natural numbers, with diameter 1. (the first circles of the process) Since the inverse of inversion is itself, the circles above natural numbers are mapped to circles appearing in the process and tangent to the circle above 0. This inversion maps the point x of the real line to $\frac{1}{x}$. Therefor if x appears in the process, $\frac{1}{x}$ appears in the image of the process under the inversion which also appears in the first sequence itself.

So in both processes, if x appears, then so does $\frac{1}{x}$ and $x+1$ and $x-1$. (if it's not negative)

Now consider the Euclidean algorithm for a pair of relatively prime numbers (a, b). In each step if our pair is (a, b) consider the fraction $\frac{a}{b}$. Then a step consisting of swapping the numbers is the same as replacing our fraction x by $\frac{1}{x}$. A step consisting of replacing (a, b) by $(a-b, b)$ is the same as replacing x by $x-1$. Therefor after some number of moves of the form $x \to \frac{1}{x}$ and $x \to x-1$, x becomes $\frac{0}{1}$. (Because (a, b) were relatively prime)

Using the inverse operations, which are $x \to \frac{1}{x}$ and $x \to x+1$, we see that we can reach each rational number from $\frac{0}{1}$ which is present in both processes.

So all the rational numbers appear in both processes. What remains to prove, is to show that only rational numbers appear in the second process. But it is a straightforward induction. Assume that two tangent circles C_1, C_2 appeared in the process and we know that their tangency points with the x axis are rational numbers.

Using some unit translations and inversion from 0, we can map C_1 to the circle above 0. We have already shown that these operations and their inverse maps, map circles appearing in the process to circles appearing in the process. So now we have two circles appearing in the process, and one of them is the one above 0. We know that the circles tangent to this one are the circles above $\frac{1}{n}$'s. So the circle between C_1, C_2 is mapped to a circle above $\frac{1}{n}$ for some n which has a rational tangency point. Since our operations map rational numbers to rational numbers, the original tangency point should have been a rational number; which completes the proof.

6. Let α be a new number. If $\alpha < 0$ one can consider $-\alpha$ which is greater than 0. So assume that $\alpha > 0$.

   If for each positive real number r, $\alpha > r$ then, $\frac{1}{\alpha}$ is the number we are after, because for each positive number r, $\alpha > \frac{1}{r}$ which gives $\frac{1}{\alpha} < r$. But $\alpha > 0$ which shows that $\frac{1}{\alpha} > 0$.

   So assume that there is a positive real number R such that $\alpha < R$. Let r be the superimum of all the real numbers less than α. (This exists because $\alpha < R$ and 0 is less than α)

   Let $\epsilon = \alpha - r$. Then $\epsilon \neq 0$ because α is not a real number.

   If $\epsilon > 0$, there is no positive real number a, less than ϵ. Because if a is such a number, $r + a < \alpha$ which contradicts r being the superimum.

   But if $\epsilon < 0$ then there is no negative real number a greater than ϵ. Because if a is such a number, then $\alpha < r + a$ which shows that the superimum should have been less than or equal to $r + a$. So in this case, $-\epsilon$ is the required number.

   For the second part of the problem, assume P is a polynomial with real coefficients. P can be factored into linear and quadratic terms, each with real coefficients. So for α to be a root of this polynomial, it has to be a root of one of the terms. (Because the new system of numbers is a field) If $ax + b$ is a linear term ($a \neq 0$), then $a\alpha + b = 0$ gives $\alpha = \frac{-b}{a}$ which shows that α is a real number.

If $ax^2 + bx + c$ is a quadratic term, ($a \neq 0$ and $b^2 < 4ac$ so that this term cannot be factored into linear ones) then $a\alpha^2 + b\alpha + c = a\left((\alpha + \frac{b}{2a})^2 + \frac{4ac-b^2}{4a^2}\right)$ But each term of the last expression is positive, because if c is a number in the new system, either $c \leq 0$ which shows that $c \times c \geq 0 \times 0 = 0$, or $c \geq 0$ which shows that $c \times c \geq 0 \times 0 = 0$. So $c^2 \geq 0$. So $(\alpha + \frac{b}{2a})^2 \geq 0$. And $4ac - b^2 > 0$ which gives $\frac{4ac-b^2}{4a^2} > 0$.

Hence α can not be a root of this term, which shows that α is not a root of the original polynomial.

7. The answer for (a) is no. Assume that two rings A, A' with centers O, O' are tied together. We will show that $|OO'| \geq 1$.

   Let R, R' denote the inner radii of the rings. Assume that $R' \geq R$. The plane of the ring A' intersects the ring A, in two intervals, one inside the ring, and the other outside. Let U be a point on the outer border of A' that is on the plane and inside the ring A. Then $R' + 1 = |O'U| \leq |O'O| + |OU| \leq |OO'| + R \leq |OO'| + R'$ which shows that $|OO'| \geq 1$.

   Now for each ring with O as the center, consider the sphere with radius $\frac{1}{3}$ around O. This sphere contains a point with rational coordinates. So there is an injective mapping from the rings to the rational points. (It is injective because no two such spheres intersect) So the number of rings must be countable.

   The answer for (b) is yes. Consider a circle with center $(0, 0, 0)$ lying in the xy plane and having radius 1. For each $0 \leq \theta \leq 0.1$, move the circle by θ along the positive direction of the x axis, and rotate it along this axis by θ radians.

   It is obvious that the rings constructed in this way are pairwise tied to each other. They are in one-to-one correspondence to $[0, 0.1]$ which is uncountable.

8. Some possible answers are shown in the following table

$1 = 2^{2-2}$	$2 = 2 + 2 - 2$	$3 = 2 + \frac{2}{2}$
$4 = \sqrt{2 \times 2}^2$	$5 = 2 \times 2 + \lfloor\sqrt{2}\rfloor$	$6 = 2 + 2 + 2$
$7 = \lceil\sqrt{(2+2)!}\rceil + 2$	$8 = 2 \times 2 \times 2$	$9 = (2 + \lfloor\sqrt{2}\rfloor)^2$
$10 = 2 \times \lceil\sqrt{(2+2)!}\rceil$	$11 = \frac{22}{2}$	$12 = \frac{(2+2)!}{2}$

Third Round

First Exam

1. This proof tries to use Ceva's theorem for concurrency. For this we need to know BA' and CA'.

   AI_a is the bisector of the angle $\angle A$. So T is the midpoint of the arc BC. Hence the projection of T on BC is the midpoint of BC, which we name M. Let the projection of I_a on BC be the point P, and the projection of A on BC, which is the foot of the altitude drawn from A, be H. Since projection from a line onto another line is a linear map, we have $A'P^2 = A'M.A'H$. Let $m = BM, h = BH, x = BA', u = BP$. Then the relation $A'P^2 = A'M.A'H$ is easily translated to $(x-u)^2 = (x-h)(x-m)$ which after simplification yields $x = \frac{u^2-hm}{2u-h-m}$.

   Now let A, B, C denote the angles $\angle A, \angle B, \angle C$ and a, b, c denote the lengths BC, CA, AB. Let p denote half the perimeter of the triangle.

   Writing the previous relations, changing the roles of B, C, gives

   $$(a-x) = \frac{(a-u)^2-(a-h)(a-m)}{2(a-u)-(a-h)-(a-m)} = \frac{(a-u)^2-(a-h)(a-m)}{h+m-2u}$$

   So dividing x by $a-x$ (as is needed in Ceva's theorem)

   $$\frac{x}{a-x} = \frac{u^2-hm}{(a-h)(a-m)-(a-u)^2}$$

   Now, note that $u = p-b$, $h = c\cos(B)$, $m = \frac{a}{2}$ and $a-h = b\cos(C)$, $a-m = \frac{a}{2}$, $a-u = p-c$. Together all these give

   $$\frac{x}{a-x} = \frac{(p-b)^2 - \frac{ac\cos(B)}{2}}{\frac{ab\cos(C)}{2} - (p-c)^2}$$

   Multiplying the above for A', B', C' results in numerators and denominators simplifying each other.

   Hence Ceva's theorem shows that AA', BB', CC' are concurrent.

2. We will need a lemma for this problem. Since it is famous enough, the proof is avoided here; but it can be found in some number theory textbooks.

Lemma. If f is a polynomial with integer coefficients, and $f(n)$ is a perfect square for every natural number n, then there is a polynomial g with integer coefficients such that $f = g^2$.

Now let P be the polynomial from the problem statement. If $a+b$ is a perfect square, then so is $ax^2 + bx^2$. Therefor $P(ax^2) + P(bx^2)$ gives a perfect square for each natural number x, which shows that it is the square of an integer polynomial.

The leading coefficient of the square of an integer polynomial, is a perfect square. So is the leading coefficient of $P(ax^2) + P(bx^2)$. If the leading term of P is Ax^n, then the leading coefficient of $P(ax^2) + P(bx^2)$ would be $A(a^n + b^n)$. For the moment, assume that $n > 0$.

Consider a very big prime p which doesn't divide A. Let $a = 1, b = p^2 + 2p$. Then $a + b = (p+1)^2$, so $A(a^n + b^n)$ should be a perfect square, and so it must be a quadratic residue modulo p. But modulo p it is equal to A.

Now let $a = 1, b = p^2 - 1$. Then $a + b = p^2$. So $A(a^n + b^n)$ should be a perfect square and hence a quadratic residue modulo p. But if n is even, then this is equal to $2A$ modulo p. This shows that 2 must be a quadratic residue, too. But by choosing p as a prime of the form $8k + 3$, then this cannot happen.

So let's analyze the case where n is odd. Let $a = 2, b = 2$. Then $2^{n+1}A$ is a perfect square. But since n is odd, A must be a perfect square too. Since A is a perfect square, if $A(a^n + b^n)$ is a perfect square, then so is $a^n + b^n$. Now put $a = 1, b = 3$ to get that $3^n + 1$ is a perfect square. Let $3^n + 1 = m^2$ which shows that $3^n = (m-1)(m+1)$. But $\gcd(m-1, m+1)$ is at most 2; so one of them must be 1, which shows that $m - 1 = 1$ or $m = 2$ or equivalently $n = 1$.

So the degree of P is at most 1. Therefor $P(x) = cx + d$. From the previous section, we know that c is a perfect square. If $c = 0$, then $P(a) + P(b) = 2d$. Which shows that $c = 2k^2$ for some integer k. The polynomial $P(x) = 2k^2$ satisfies the problem conditions.

So assume that $c \neq 0$ and $c = k^2$ for some integer k. Then $P(a) + P(b) = k^2(a + b) + 2d$. $k^2(a + b)$ is a perfect square, since $(a + b)$ is. But since the distance between consecutive perfect squares goes larger and larger, as they grow larger, $|2d|$ must be either 0 or bigger than any given number; which is not possible. So $d = 0$, and $P(x) = k^2x$ which satisfies the problem conditions.

3. Let's homogenize the terms:

$$\sqrt{a^3+a} = \sqrt{a(a^2+1)} = \sqrt{a(a^2+ab+bc+ca)} = \sqrt{a(a+b)(a+c)}$$

Replace all the terms in the l.h.s. by their homogenized versions, and then take the square of each side. After cleaning up the terms, we arrive at (each cyclic sum has three terms generated by cyclically permuting a, b, c)

$$\sum_{\text{cyclic}} a^3 + 2\sum_{\text{cyclic}} \sqrt{ab(a+b)^2(a+c)(b+c)} \geq 3\sum_{\text{cyclic}} (a^2b+ab^2) + 9abc$$

From the Cauchy's inequality we have

$$\sqrt{(a+c)(b+c)} \geq \sqrt{ab}+c$$

So multiplying by $\sqrt{ab(a+b)^2}$ we get

$$\sqrt{ab(a+b)^2(a+c)(b+c)} \geq \sqrt{ab(a+b)^2}\left(\sqrt{ab}+c\right) = ab(a+b)+c\sqrt{ab}(a+b)$$

Substituting a, b, c by x^2, y^2, z^2, and using the above inequality we arrive at

$$\sum_{\text{cyclic}} x^6 + 2\sum_{\text{cyclic}} (x^4y^2+x^2y^4) + 2\sum_{\text{cyclic}} (x^3y^2z+x^3yz^2) \geq 3\sum_{cyclic} (x^4y^2+x^2y^4) + 9x^2y^2z^2$$

and after simplification

$$\sum_{\text{cyclic}} x^6 + 2\sum_{\text{cyclic}} (x^3y^2z+x^3yz^2) \geq \sum_{cyclic} (x^4y^2+x^2y^4) + 9x^2y^2z^2$$

But from AM-GM inequality we have

$$\sum_{\text{cyclic}} (x^3y^2z+x^3yz^2) \geq 6x^2y^2z^2$$

and from Schur's inequality we have

$$\sum_{\text{cyclic}} x^6 + 3x^2y^2z^2 \geq \sum_{\text{cyclic}} (x^4y^2+x^2y^4)$$

Adding two times the first inequality to the second, we get the inequality we were to prove.

4.

$$f(xf(y)) + y + f(x) = f(x + f(y)) + yf(x) \tag{1}$$

In 1, put $x = 0$ to get

$$f(f(y)) = y(1 - f(0)) + 2f(0) \tag{2}$$

Setting $y = 2$ in 2 results in $f(f(2)) = 2$. Now put $y = 1$ and substitute x by $x - f(1)$ in 1 to get

$$f((x - f(1))f(1)) = f(x) - 1 \tag{3}$$

This shows that the image of f is closed under the operator $x \to x - 1$. We already know that 2 is in the image of f. So is 0.

Let a be a number such that $f(a) = 0$. Set $x = a$ in 1 to get

$$f(0) + a = af(x)$$

If $a \neq 0$, the above shows that f is constant. But substituting a constant c as f in 1 shows

$$c + y = cy$$

which shows that $c = 0$ and $c = 1$ which cannot hold simultaneously. So f is not a constant, and hence $a = 0$. That is $f(0) = 0$.

Now 2 becomes

$$f(f(y)) = y \tag{4}$$

This shows that f is both injective and surjective. (injective because if $f(a) = f(b)$, then $a = f(f(a)) = f(f(b)) = b$)

Now replacing y by $f(y)$ in 1 and using the fact that $f(f(y)) = y$, yields

$$f(xy) + f(x) + f(y) = f(x)f(y) + f(x + y) \tag{5}$$

Setting $x = y = 2$ in 5 gives $2f(2) = f(2)^2$. Since f is injective and $f(0) = 0$, $f(2) \neq 0$, and hence $f(2) = 2$.

Let $f(1) = a$, then from 4 we have $f(a) = 1$. Set $x = y = 1$ in 5 to get $3a = a^2 + 2$ which shows that either $a = 1$ or $a = 2$. But $f(2) = 2$ which excludes $a = 2$ from the list. Hence $a = 1$, and we have $f(1) = 1$.

Put $y = 1$ in 5 to get

$$f(x + 1) = f(x) + 1 \tag{6}$$

Now replace x by $x+1$ in 5 and use 6 to get

$$f(xy+y)+f(x)+f(y)=f(x)f(y)+f(y)+f(x+y) \tag{7}$$

Subtracting 5 from 7 yields

$$f(xy+y)=f(xy)+f(y)$$

For $a,b\neq 0$, one can find x,y such that $xy=a, y=b$ which gives $f(a+b)=f(a)+f(b)$. But for $a=0$ or $b=0$ the last equation is obvious. Hence $f(a+b)=f(a)+f(b)$ for every a,b.

Using this fact and simplifying 5 yields $f(xy)=f(x)f(y)$ which for $x=y$ shows that $f(x^2)=f(x)^2$. This shows that f maps nonnegative numbers to nonnegative numbers. This along with the Cauchy's equation $f(a+b)=f(a)+f(b)$ shows that $f(x)=cx$ for a suitable c. But $f(1)=1$, so $c=1$ and $f(x)=x$ which satisfies the problem statement's conditions.

5. Whenever the word pole or polar is used in this proof, it implicitly means with respect to I.

   Let D,E,F be the tangency points of BC,CA,AB with I. Let Q be the pole of l, and P be the pole of l'. (which is also its tangency point) Since A' is on l and BC, its polar is the line joining poles of l and BC which is the line DQ. If the tangency point of $A'A_1$ with I is D' then D' is the pole of the line $A'A_1$ and hence it is on the line DQ. (polar of A') So DD' passes through Q. Similarly define E',F', and we would have that EE',FF' pass through Q. Pole of the line $A'A_1$ is D' and pole of the line l' is P. So PD' is the polar of A_1. Polar of A is EF, because poles of the lines AB and AC are E,F.

   So pole of the line AA_1 is $PD'\cap EF$. So we need to show that $PD'\cap EF, PE'\cap FD, PF'\cap DE$ are collinear. (poles are collinear iff polars are concurrent)

   We will show that they are collinear with Q. Use the Pappus's theorem for circles (similar to Pascal's theorem) for the triples (P,D,E) and (F,E',D'), which shows that $PE'\cap DF, PD'\cap EF, DD'\cap EE'=Q$ are collinear. And this is exactly what we wanted to show. Using similar arguments $PF'\cap DE$ is also collinear with these three.

6. Let Q_k denote the k dimensional cube. Each vertex of Q_k has degree k, and there are 2^k vertices in total. So Q_k has $k2^{k-1}$ edges. Therefor if we are to partition Q_k by T, we will exactly need 2^{k-1} copies.

Label the vertices of Q_k by sequences of length k consisting of $0, 1$ as their coordinates in the k dimensional space. Name a vertex an odd vertex if the sum of its coordinates is odd, and even if the sum is even. From each two adjacent vertices, one should be odd and one should be even.

Now we will prove a stronger statement than the one in the problem statement.

We can partition Q_k into copies of T in such a way that for each vertex v of T, the v's in the copies of T are either all the even vertices of Q_k or all the odd vertices. (Note that because the number of copies is 2^{k-1} as is the number of odd or even vertices, this shows that no two copies of T share the same vertex as their v's)

For $k = 1$, the only tree is K_2 and $Q_1 = K_2$. The only copy of T needed here is Q_1 itself. This partitioning satisfies all the extra conditions put on the partitioning.

We will proceed using induction on k. Let T be a given tree with k edges. Let u be a leaf of T. Remove u to obtain T'. Now partition Q_{k-1} into copies of T' with all the conditions already mentioned.

Q_k is obtained from Q_{k-1} by putting two copies of Q_{k-1} side by side, and drawing an edge between each two dual vertices from the two copies. (dual vertices are the vertices that were the same, when being copied) Partition these two Q_{k-1}'s by copies of T' using the following method. Partition the first copy as the one partitioning you already have. For the second one, partition it like the first one, and then reflect the vertices. (that is swap the place of each two vertices having the same coordinates, except their first coordinate)

Now if v is a vertex of T' and the v's in the copies of T' were all of the even vertices in Q_{k-1}, v's in the new partitioning are all the even vertices too, because the even vertices of the first Q_{k-1} are even in Q_k and odd vertices of the second Q_{k-1} are even in Q_k. (reflection, as we did to the second Q_{k-1}, swaps the place of even vertices and odd vertices)

If u's parent in T was w, then w's in the current partitioning are either all the even vertices or all the odd vertices. Let's assume w's are placed at the even vertices. Odd vertices will be the new u's. Simply connect each odd vertex to its dual. Since the dual of each odd vertex is even, it will be a w for exactly one copy of T'. Therefor the new partitioning has all the conditions we were after, and the theorem is proved.

Second Exam

1. If AH is the altitude drawn from A, then $IE \parallel AH$, so

$$\angle AIE = \angle IAH = \angle IAB - \angle IAH = \frac{\angle A}{2} - (90^\circ - \angle B) = \frac{\angle B - \angle C}{2}$$

But $\angle APE$ is equal to half the arc AT which is equal to $\frac{\angle B - \angle C}{2}$.

So $\angle APE = \angle AIE$ which shows that $APEI$ is a circumscribed quadrangle. But since $\angle AEI = 90^\circ$, we have $\angle API = 90^\circ$. From the assumptions, $\angle IPB = \angle B$, which shows that $\angle APB = 90^\circ + \angle B$. But it is also equal to $180^\circ - \angle C = \angle A + \angle B$, which shows that $\angle A = 90^\circ$.

2. Throughout this solution, a^+ denotes $\max\{a, 0\}$.

Consider the tournament as a directed graph, where an edge from v to u shows that v has won its game against u. Let's count the number of directed 7-paw's. (An n-paw is a subgraph consisting of n edges coming out from a vertex.)

If the out degree of vertices are denoted by d_i then this number is obviously equal to $\sum_i \binom{d_i}{7}$.

Assuming that the claim of the problem is not satisfied, for each 7 vertices there are at most 6 vertices having edges coming out to each of those 7. Hence the number of 7-paw's is at most $6\binom{n}{7}$ where n is 799, the number of vertices.

The remaining parts of the proof try to show that $\sum_i \binom{d_i}{7} > 6\binom{n}{7}$.

Let $a_i = (d_i - 6)^+$. Then $\binom{d_i}{7} = \frac{d_i(d_i-1)\ldots(d_i-6)}{7!} = \frac{a_i(a_i+1)\ldots(a_i+6)}{7!}$. This is because if $d_i > 6$, then $a_i = d_i - 6$ and this identity obviously holds. And if $d_i \leq 6$ then both sides are 0 and the identity still holds.

Now let $f(x) = \frac{x(x+1)\ldots(x+6)}{7!} = \sum_i c_i x^i$. Of course c_i's are all non-negative.

$$\sum_i \binom{d_i}{7} = \sum_i f(a_i) = \sum_i \sum_j c_j a_i^j = \sum_j c_j \sum_i a_i^j$$

Note that $c_0 = 0$. For $j \geq 1$, we have the following inequality between j^{th} exponents mean and arithmetic mean.

$$\left(\frac{\sum_i a_i^j}{n}\right)^{\frac{1}{j}} \geq \frac{\sum_i a_i}{n}$$

But since our graph is a tournament $\sum_i d_i = \binom{n}{2}$. Hence $\sum_i a_i \geq \binom{n}{2} - 6n$. This shows that $\frac{\sum_i a_i}{n} = \frac{n-1}{2} - 6 = \frac{n-13}{2}$. Name this constant as A. Substituting in our inequality gives

$$\sum_i a_i^j \geq nA^j$$

Now substituting in our previous identities, we get

$$\sum_i \binom{d_i}{7} \geq \sum_j c_j n A^j = n \sum_j c_j A^j = nf(A)$$

But f is defined so that $f(x) = \binom{x+6}{7}$. So $nf(A) = n\binom{\frac{n-1}{2}}{7}$. A straightforward calculation reveals that this number is 242071561222413021 while $6\binom{n}{7}$ is 241047478096826154, a slightly smaller number. This completes the proof.

3. From the QM-HM inequality for the six numbers $|P_i - P_j|$ we have

$$\sum_{i \neq j} \frac{1}{|P_i - P_j|} \geq C \left(\sum_{i \neq j} |P_i - P_j|^2 \right)^{-\frac{1}{2}}$$

where C is a constant. The equality holds iff $|P_i - P_j|$'s are all equal, which is satisfied when P_i's are the vertices of a regular tetrahedron.

We now try to find an upper bound for $\sum_{i \neq j} |P_i - P_j|^2$. Considering P_i's as vectors in 3D space, then this sum becomes $3\sum_i (P_i.P_i) - 2\sum_{i \neq j}(P_i.P_j)$. Adding and subtracting $\sum_i (P_i.P_i)$ we get $4\sum_i (P_i.P_i) - (\sum_i P_i).(\sum_i P_i)$. But $P_i.P_i$ is a constant. Therefor the expression is maximized when $\sum_i P_i = 0$ for which it is equal to 4. This condition is again satisfied when P_i's are the vertices of a regular tetrahedron.

Therefore, the condition of minimality of $\sum_{i \neq j} \frac{1}{|P_i - P_j|}$ becomes that, $|P_i - P_j|$'s should be equal for different i, j's and that $\sum_i P_i = 0$. But if $|P_i - P_j|$ has a constant value, then every face of the tetrahedron is an equilateral triangle, which itself shows that the tetrahedron is regular.

4. For (a): Let X^c denote the complement of X. From X, X^c only one can be in F, because their intersection is empty. But since F has 2^{n-1} members, for each X, from X, X^c at least one must be a member. This shows that if $X \notin F$ then $X^c \in F$.

Now let $A, B \in F$. If $(A \cap B)^c \in F$ then $A, B, (A \cap B)^c$ are three members of F having empty intersection. So $(A \cap B)^c \notin F$ which shows that $A \cap B \in F$.

So F is closed under intersection. Therefor the intersection of all its members is a member of F. But the empty set cannot be in F. So this intersection is non-empty.

If we replace 2^{n-1} by $2^{n-1}-1$ the answer doesn't change. From previous arguments it is obvious that there is a subset C, for which neither C nor C^c are in F, but for every other X, X^c, at least one of them is in F. If X is in F and X does not contain neither C nor C^c, then for every other $Y \in F$, $X \cap Y$ doesn't contain C or C^c, so $(X \cap Y)^c$ is not C or C^c; this shows that (using similar arguments as for the first part) $X \cap Y \in F$. But $X \cap Y \subset X$, so it doesn't contain C or C^c, too. Proceeding like this, we can show that the intersection of all members of F is nonempty.

So assume that each member of F either contains C or C^c. C or C^c can not be empty because if it so, then adding the whole set to F yields a family with 2^{n-1} members, each three having nonempty intersection; and from the previous part we can see that the intersection of all members of F is nonempty.

Now count the number of sets containing either C or C^c. There are $2^{|C^c|}$ sets containing C and $2^{|C|}$ sets containing C^c. But the whole set is counted twice. So the number is $2^{|C|}+2^{|C^c|}-1$. But C, C^c themselves should not be counted as they are not members of F. Hence $2^{n-1}-1 = |F| \leq 2^{|C|}+2^{|C^c|}-3$. Let $\max\{|C|, |C^c|\} = t$. Then $2^{n-1}-1 \leq 2^t+2^t-3 = 2^{t+1}-3$. This shows that $t \geq n-1$ and since neither C nor C^c are the empty set, $t = n-1$. For simplicity assume that $|C| = 1$ and $|C^c| = n-1$. But now each set containing C^c is either C^c or the whole set. Since C^c is not in F, the only set containing C^c is the whole set which also contains C. Therefor each member of F contains C, which proves that the intersection of all members of F is not empty.

5. First we will show that f is injective. If $a \neq b$ but $f(a) = f(b)$, then for each n we have $f(a)+f(n) \mid (a+n)^k$ and $f(a)+f(n) = f(b)+f(n) \mid (b+n)^k$. So $f(a)+f(n)$ is a common divisor of $(a+n)^k$ and $(b+n)^k$. If n satisfies the condition that $\gcd(a+n, b+n) = 1$ then this can not happen. But $\gcd(a+n, b+n) = \gcd(a+n, b-a)$ and if $b-a \neq 0$ there is number $a+n$ that is relatively prime to it. (for example a very big prime)

   Now let b be a natural number. For every n we have $f(n)+f(b) \mid (n+b)^k$ and $f(n)+f(b+1) \mid (n+b+1)^k$. But $(n+b)$ and $(n+b+1)$ are relatively prime to each other. So $1 = \gcd(f(n)+f(b), f(n)+f(b+1)) = \gcd(f(n)+f(b), f(b+1)-f(b))$.

We want to show that $f(b+1) - f(b) = \pm 1$. If it is not equal to ± 1, then there is a prime p which divides it. Let a be such that $p^a > b$. Now put $n = p^a - b$. We have $f(n) + f(b) \mid (n+b)^k = p^{ak}$, so $p \mid f(n) + f(b)$. We had $p \mid f(b+1) - f(b)$, so $p \mid \gcd(f(n) + f(b), f(b+1) - f(b))$ which contradicts the previous arguments.

So $f(b+1) - f(b) = \pm 1$ for every number b. But since f is injective, it is either always equal to 1 or always equal to -1. (Because for two consecutive b's it cannot change sign)

But it cannot always be equal to -1, because f takes only natural values. So $f(b+1) - f(b) = 1$ for every number b. Hence there is a number c such that $f(n) = n + c$. c is non-negative because $f(1) = 1 + c$ is positive. If c is positive, then take a prime p greater than $2c$. Now $f(1) + f(p-1) \mid p^k$ which shows that $p \mid f(1) + f(p-1) = p + 2c$. But it's a contradiction because $2c < p$.

So $c = 0$ and the function is $f(n) = n$ which obviously satisfies the conditions of the problem statement.

6. Let Q be the point on the extension of AD (from D) such that $AQ = 2R$. Let I_aQ intersect the extension of AB at P. Then it is no hard to see that AK being equal to $2R$ is equivalent to L being the same as Q which is itself equivalent to P being the same as K which may be said in another language as $API_a = 90° + \frac{3}{4}\angle C$. So from now on we forget about L, K and think only about P, Q.

   A key fact in this problem is to realize that $\angle QI_aA = \angle QAI_a$. This can be proved by showing that the triangles QI_aD and QAI_a are similar. They have a common angle, so we just need to prove that $\frac{QI_a}{QD} = \frac{QA}{QI_a}$ or equivalently $QI_a^2 = QD.QA$.

   This requires a bit of calculations. Throughout S denotes the area, p half the perimeter, a, b, c length of the sides BC, CA, AB, r_a radius of the excircle with respect to A, h_a length of the altitude drawn from A.

   We have $QA = 2R$ and so $QD = 2R - h_a$. So $QA.QD = 2R(2R - h_a) = 4R^2 - 2Rh_a$.

   If H is the the intersection of the line drawn from I_a perpendicular to AC, with AC itself, then it is well known that $AH = p$. $I_aH = r_a$ and using Pythagoras we have $AI_a^2 = p^2 + r_a^2$. Let U be on AQ such that $I_aU \perp AU$.

Then it's no hard to see that $AU = r_a + h_a$. Therefor $I_aU^2 = AI_a^2 - AU^2 = r_a^2 + p^2 - (r_a + h_a)^2$. But we are after I_aQ^2. $UQ = AU - AQ = r_a + h_a - 2R$. So $I_aQ^2 = r_a^2 + p^2 - (r_a + h_a)^2 + (r_a + h_a - 2R)^2 = r_a^2 + p^2 + 4R^2 - 4R(r_a + h_a)$.

To prove that it is equal to $4R^2 - 2Rh_a$ we need to show that (after simplifications) $r_a^2 + p^2 = 4Rr_a + 2Rh_a$.

This is the point that well-known relations about triangle constants, become useful. We have $r_a = \frac{S}{p-a}$ and $4RS = abc$, which together give $4Rr_a = \frac{abc}{p-a}$. Also it is obvious that $h_a = \frac{2S}{a}$ which gives $2Rh_a = bc$. So one side of the equality we are trying to prove, is $bc + \frac{abc}{p-a} = \frac{pbc}{p-a}$.

For the other side, note that $r_a^2 = \frac{S^2}{(p-a)^2}$ and from Heron's relation we have $S^2 = p(p-a)(p-b)(p-c)$ which gives $r_a^2 = \frac{p(p-b)(p-c)}{p-a}$. Adding p^2 gives $r_a^2 + p^2 = \frac{p(p-b)(p-c)+p^2(p-a)}{p-a}$ which after simplification gives $\frac{pbc}{p-a}$, which proves the equality we were after.

$$\angle I_aAD = \angle I_aAB - \angle DAB = \frac{\angle A}{2} - (90^\circ - \angle B) = \frac{\angle B - \angle C}{2}$$

So we have $\angle QI_aD = \frac{\angle B - \angle C}{2}$. Let's name the unknown angle $\angle DI_aA$ as α. Then $\angle QI_aA = \frac{\angle B - \angle C}{2} + \alpha$. Writing the sum of the angles in the triangle PAI_a, and simplifying a little bit, we get $\angle I_aPA = 90^\circ + \angle C - \alpha$.

Now note that $\angle BAI_a + \angle BI_aA = \angle PBI_a = \frac{180^\circ - \angle B}{2} = \frac{\angle A + \angle C}{2}$. This shows that $\angle BI_aA = \frac{\angle C}{2}$. So DI_a bisecting $\angle BI_aA$ is equivalent to α being $\frac{\angle C}{4}$ which in turn is equivalent to $\angle I_aPA$ being $90^\circ + \frac{3}{4}\angle C$. This finishes the proof.