Generalization of the Feuerbach point

Darij Grinberg

In this note, we are going to use directed angles modulo 180°, also called crosses. See [2] for three references.

We will abbreviate the circle through three given points P_1, P_2, P_3 to "circle $P_1P_2P_3$".

Consider a triangle ABC. The midpoints A', B', C' of its sides BC, CA, AB form a triangle $A'B'C'$ called the **medial triangle** of triangle ABC. The circumcircle of this medial triangle is the nine-point circle of triangle ABC. Let U be the circumcenter of triangle ABC, and P an arbitrary point different from U.

The circumcenter U of ΔABC is the meet of the perpendicular bisectors of the sides BC, CA, AB; hence, $UA' \perp BC$, $UB' \perp CA$, $UC' \perp AB$. Since $B'C' \parallel BC$, $C'A' \parallel CA$, $A'B' \parallel AB$, we also have $UA' \perp B'C'$, $UB' \perp C'A'$, $UC' \perp A'B'$, and hence U lies on the three altitudes of $\Delta A'B'C'$. Consequently, U is the orthocenter of triangle $A'B'C'$.

According to [1] and [2], if a line that passes through the orthocenter of a triangle is reflected in the sidelines, the three reflections meet at one point lying on the circumcircle of the triangle. This point is called the **Anti-Steiner point** of the line with respect to the triangle. Applying this to the line PU passing through the orthocenter U of triangle $A'B'C'$, we infer that the reflections of PU in the sidelines $B'C'$, $C'A'$, $A'B'$ of triangle $A'B'C'$ meet at one point lying on the circumcircle of $\Delta A'B'C'$; this point is the Anti-Steiner point of PU with respect to $\Delta A'B'C'$.

Now, the circumcircle of $\Delta A'B'C'$ is the nine-point circle of ΔABC; hence we may state:

Theorem 1.1: The reflections x, y, z of the line PU in the sidelines $B'C'$, $C'A'$, $A'B'$ of the medial triangle $A'B'C'$ meet at one point L lying on the nine-point circle of triangle ABC. This L is the Anti-Steiner point of the line PU with respect to triangle $A'B'C'$.

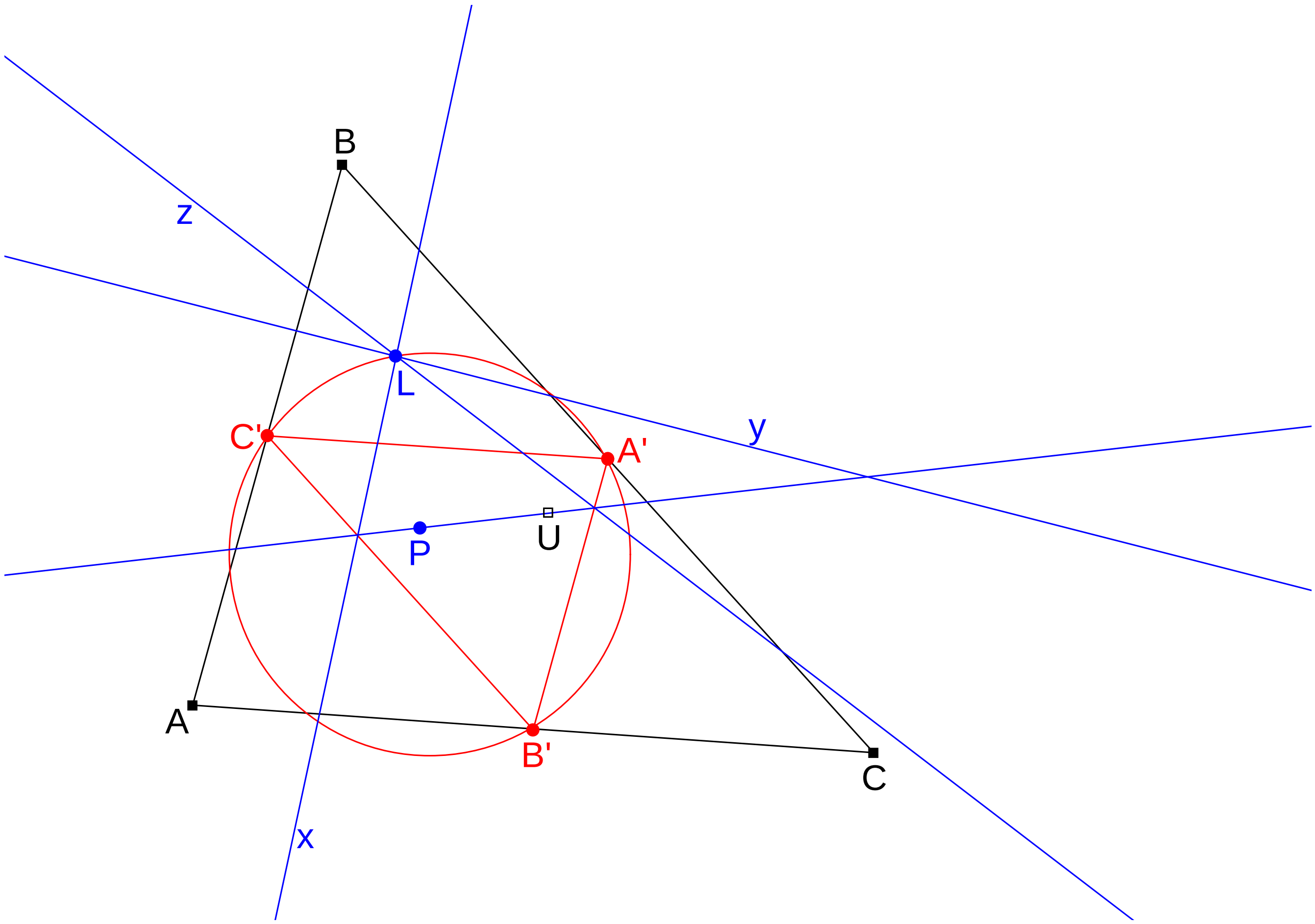

Fig. 1

The position of this point L depends only on the direction of the line PU, not of the actual position of the point P on this line.

A first property of L will be (Fig. 2):

Theorem 1.2: The reflections X', Y', Z' of L in the sidelines $B'C'$, $C'A'$, $A'B'$ of triangle $A'B'C'$ are the feet of the perpendiculars from A, B, C to the line PU.

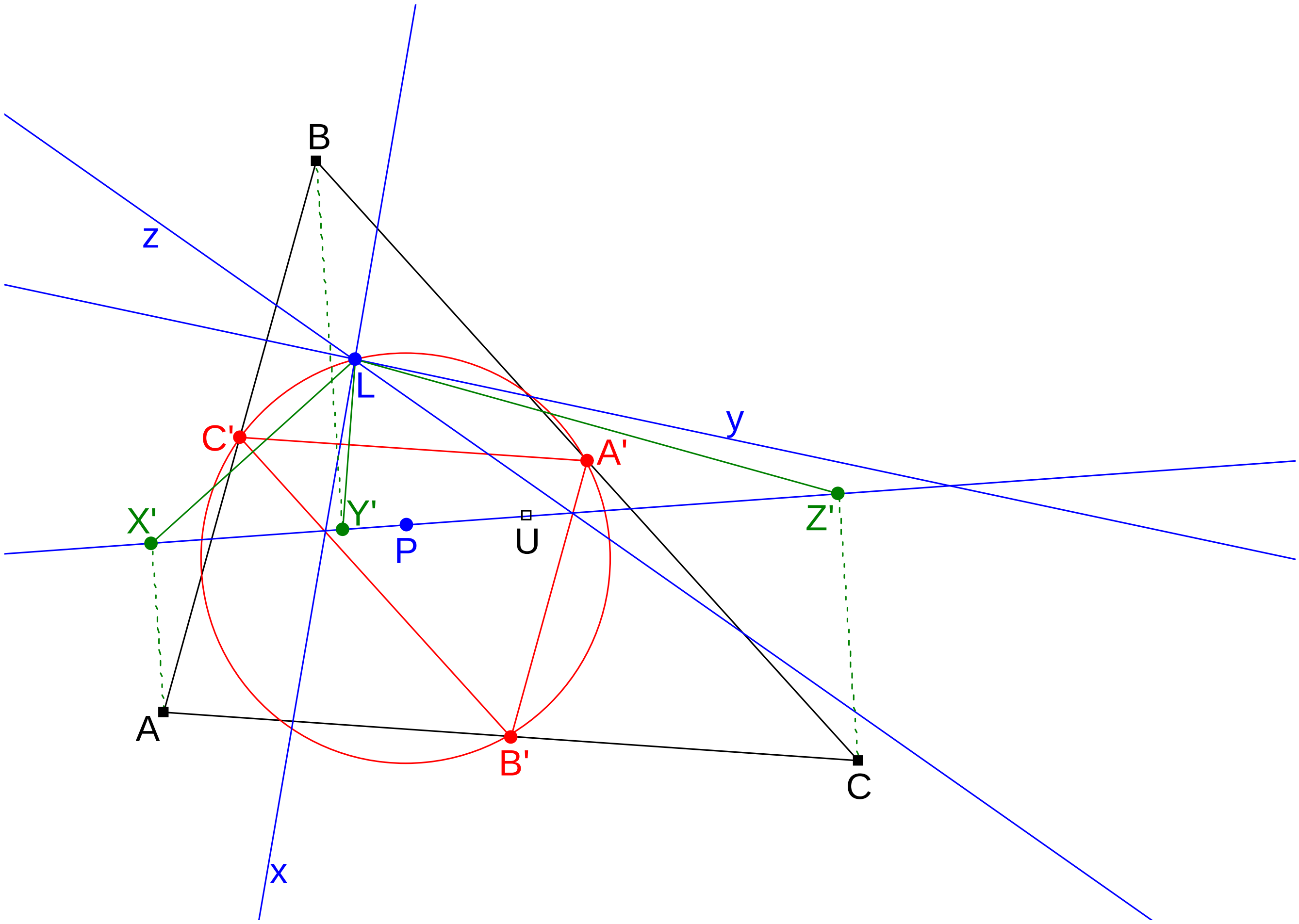

Fig. 2

Proof (Fig. 3). The lines PU and x are symmetrically placed with respect to the line $B'C'$. Therefore, as L lies on x, its reflection X' in $B'C'$ must lie on PU.

For $\measuredangle AB'U = 90^\circ$ and $\measuredangle AC'U = 90^\circ$, the points B' and C' lie on the circle with diameter AU. In other words, the circle $AB'C'$ is the circle with diameter AU. The circles $A'B'C'$ and $AB'C'$ are congruent (being the circumcircles of the congruent triangles $A'B'C'$ and $AB'C'$); hence, these circles are symmetrically placed with respect to the line $B'C'$. Since L lies on the nine-point circle of ΔABC, i. e. on the circle $A'B'C'$, its reflection X' in $B'C'$ must therefore lie on the circle $AB'C'$, i. e. on the circle with diameter AU. Hence, $\measuredangle AX'U = 90^\circ$ and $AX' \perp PU$. Thus, X' is the foot of the perpendicular from A to PU. Parallel reasoning establishes the same for Y' and Z', and Theorem 1.2 is proven.

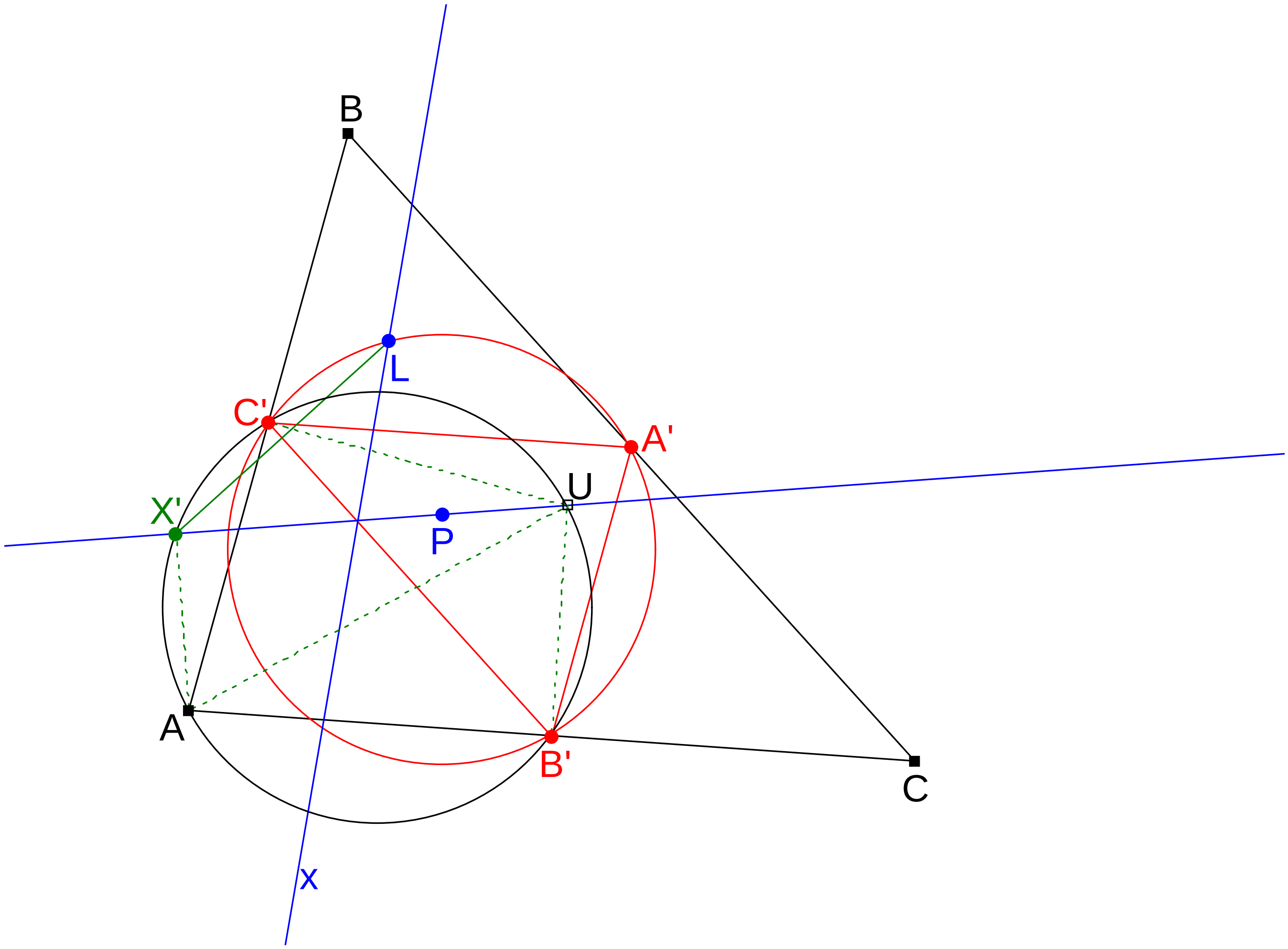

Fig. 3

As a corollary, we find (Fig. 4):

Theorem 1.3: The point L is the orthopole of the line PU with respect to triangle ABC.

Proof. The orthopole of a line with respect to a triangle is defined as follows: From the vertices of the triangle, perpendiculars are dropped to the line, and from the feet of these perpendiculars, we drop perpendiculars to the corresponding sidelines of the triangle. Then, these new perpendiculars meet at one point, the so-called **orthopole** of the line with respect to the triangle.

Now, considering our triangle ABC and the line PU, the points X', Y', Z' are the feet of the perpendiculars from the vertices A, B, C to the line PU. But on the other hand, X' is the reflection of L in $B'C'$, hence $X'L \perp B'C'$, and $X'L \perp BC$ (since $B'C' \parallel BC$). This indicates that L lies on the perpendicular from X' to BC. Similarly, L lies on th perpendiculars from Y' to CA and from Z' to AB, and thus L is the orthopole of PU with respect to triangle ABC. This proves Theorem 1.3.

Note. As a consequence of Theorem 1.3, we find a well-known result:

Theorem 1.4: The orthopole of a line passing through the circumcenter of a triangle always lies on the nine-point circle of the triangle.

In fact, our line PU passing through the circumcenter U of ΔABC has its orthopole L on the nine-point circle of ΔABC.

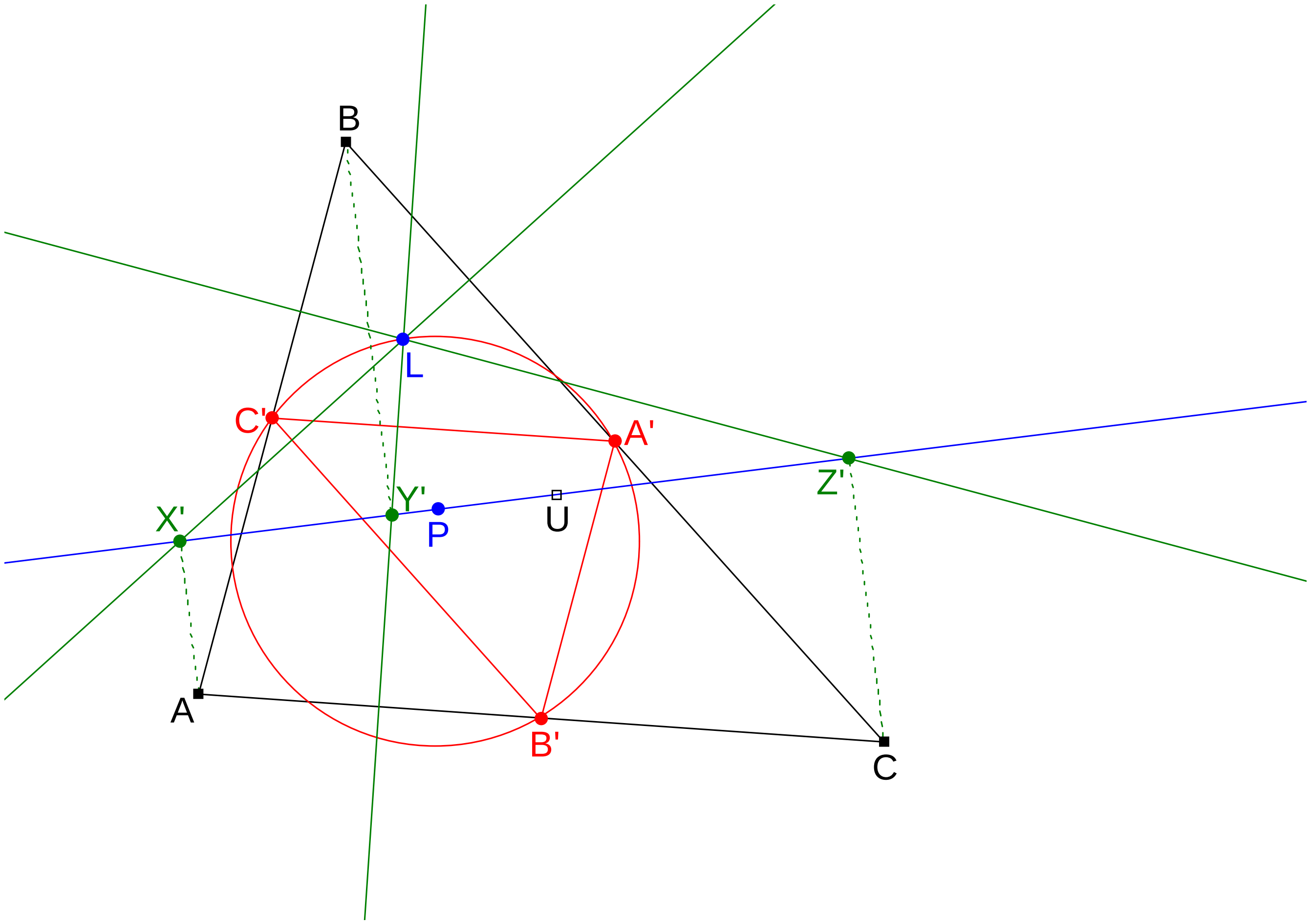

Fig. 4

Let H_a, H_b, H_c be the feet of the altitudes of triangle ABC from A, B, C. These points H_a, H_b, H_c are known to lie on the nine-point circle of ΔABC; this can be proven as follows: As we know, the circles $AB'C'$ and $A'B'C'$ are symmetrically placed with respect to the line $B'C'$. Since A lies on the circle $AB'C'$, its reflection H_{a1} in $B'C'$ must lie on the circle $A'B'C'$, i. e. on the nine-point circle. But since H_{a1} is the reflection of A in $B'C'$, the segment AH_{a1} is perpendicular to $B'C'$ and twice as long as the distance from A to $B'C'$. Hence, H_{a1} is the foot of the altitude of triangle ABC from A. [1] Hence, $H_{a1} = H_a$, and consequently, H_a lies on the nine-point circle. Similarly, H_b and H_c lie on the nine-point circle, qed..

Incidentally, we have just shown that H_a is the reflection of A in the line $B'C'$. On the other hand, X' is the reflection of L in this line, i. e. L is the reflection of X'. Hence, $H_aL = AX'$, and similarly $H_bL = BY'$ and $H_cL = CZ'$. We record this:

Theorem 1.5: The distances from the feet H_a, H_b, H_c of the altitudes to the point L are equal to the distances from the points A, B, C to the line PU. I. e., $H_aL = AX'$, $H_bL = BY'$, $H_cL = CZ'$. (See Fig. 5.)

[1] In fact, $AH_{a1} \perp B'C'$ yields $AH_{a1} \perp BC$ (since $B'C' \parallel BC$), and the segment AH_{a1} is twice as long as the distance from A to $B'C'$, i. e. just as long as the distance from A to BC.

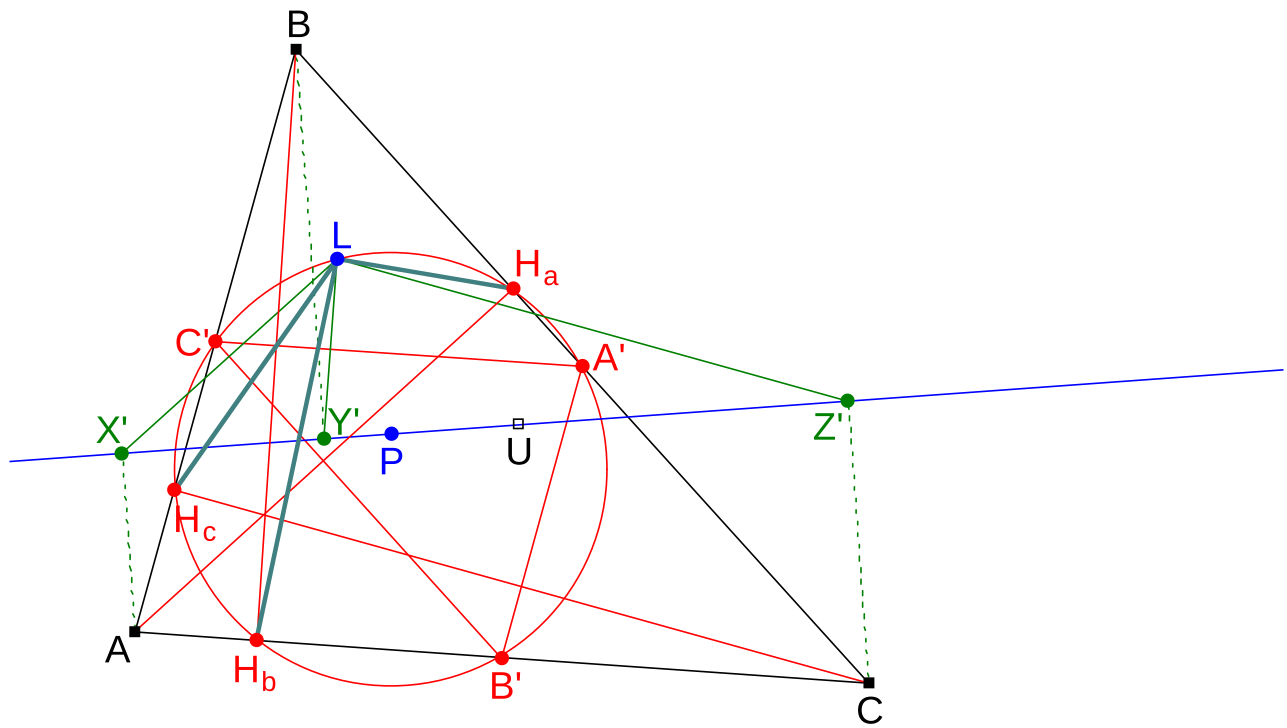

Fig. 5

Now, let X, Y, Z be the feet of the perpendiculars from P to the lines BC, CA, AB. The triangle XYZ is called the **pedal triangle** of P with respect to triangle ABC. If X'' is the reflection of X in $B'C'$, then obviously $XX'' \perp B'C'$ and therefore $XX'' \perp BC$ (since $B'C' \parallel BC$), so that the points P, X, X'' lie on one line perpendicular to BC. (See Fig. 6.)

The point H_a is the reflection of A in $B'C'$; hence, A is the reflection of H_a in $B'C'$. The point X'' is the reflection of X in $B'C'$. Hence, the line AX'' is the reflection of the line H_aX in $B'C'$. But since the line H_aX (i. e., the line BC) is parallel to $B'C'$, its image AX'' is parallel to $B'C'$, too. Hence, also $AX'' \parallel BC$.

Now, since $AX'' \parallel BC$ and $PXX'' \perp BC$, it follows that $\measuredangle AX''P = 90^\circ$; thus, the point X'' lies on the circle with diameter AP. This circle also contains the points Y and Z (since $\measuredangle AYP = 90^\circ$ and $\measuredangle AZP = 90^\circ$) and the point X' (since $AX' \perp PU$ entails $\measuredangle AX'P = 90^\circ$).

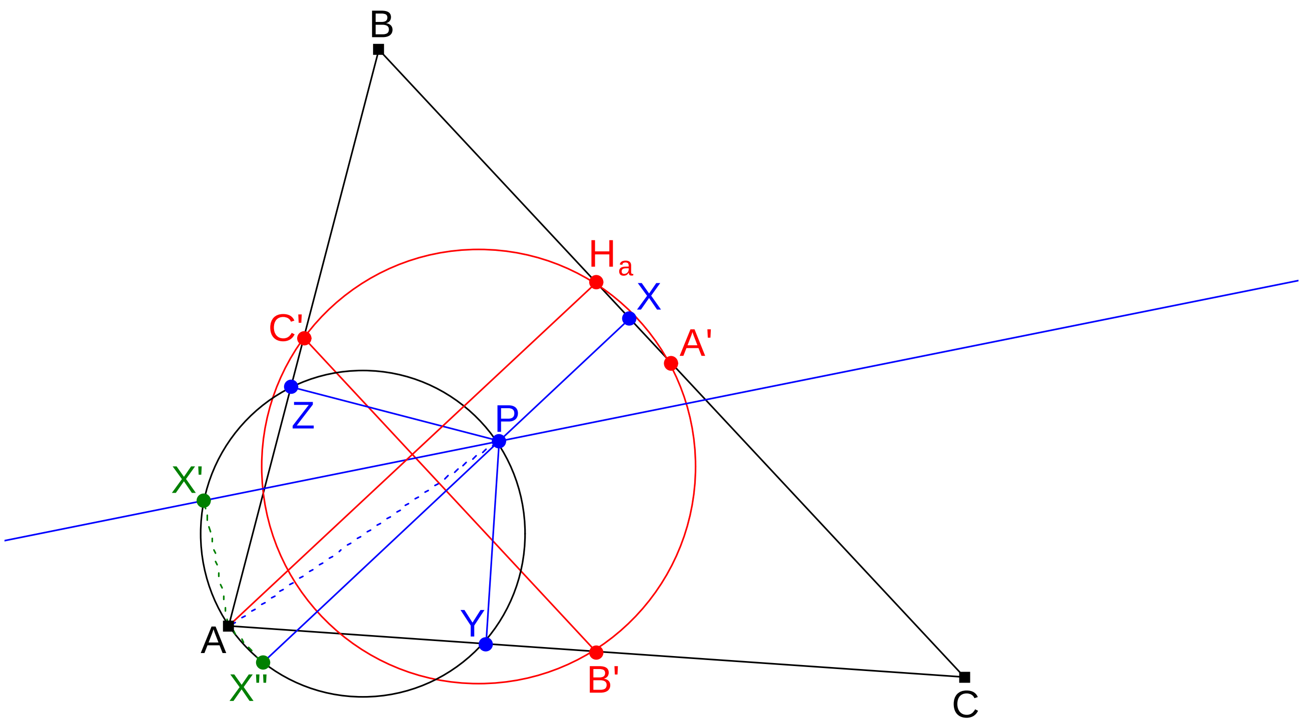

Fig. 6
We define the intersections

$$A'' = B'C' \cap YZ; \qquad B'' = C'A' \cap ZX; \qquad C'' = A'B' \cap XY.$$

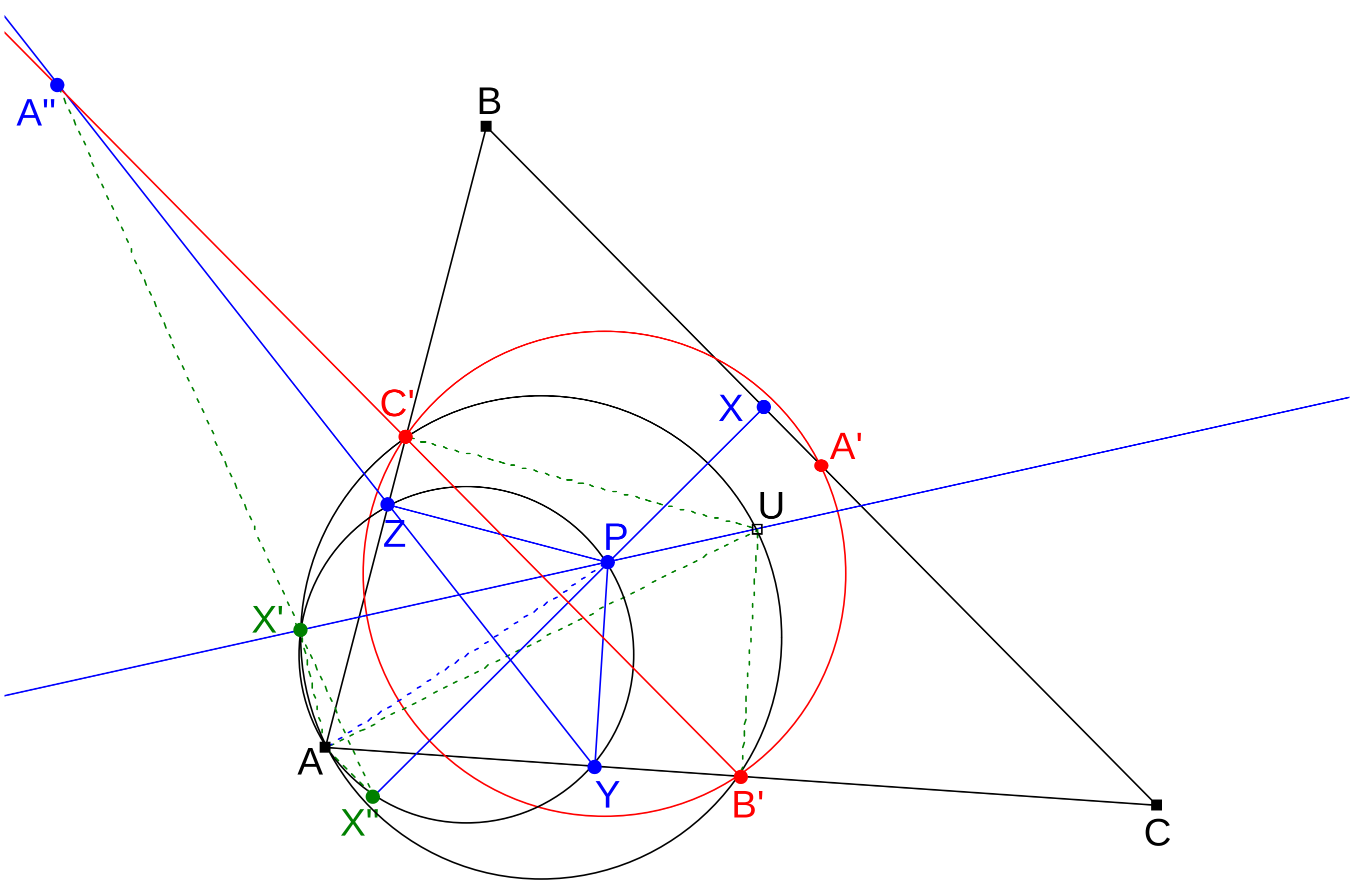

Fig. 7

As we know, B', C' and X' lie on the circle with diameter AU, and Y, Z, X' and X'' lie on the circle with diameter AP. Hence, $\measuredangle AX'C' = \measuredangle AB'C'$ and $\measuredangle ZX'A = \measuredangle ZYA$, and consequently

$$\begin{aligned}
\measuredangle ZX'C' &= \measuredangle ZX'A + \measuredangle AX'C' = \measuredangle ZYA + \measuredangle AB'C' \\
&= \measuredangle A''YB' + \measuredangle YB'A'' = (\measuredangle A''YB' + \measuredangle YB'A'' + \measuredangle B'A''Y) - \measuredangle B'A''Y \\
&= 0^\circ - \measuredangle B'A''Y \\
&= -\measuredangle B'A''Y = \measuredangle YA''B' = \measuredangle ZA''C'.
\end{aligned}$$

Therefore, X' lies on the circle $ZA''C'$. This entails $\measuredangle A''X'Z = \measuredangle A''C'Z$. Furthermore, $\measuredangle ZX'X'' = \measuredangle ZPX''$ follows from the circle with diameter AP. Hence,

$$\begin{aligned}
\measuredangle A''X'X'' &= \measuredangle A''X'Z + \measuredangle ZX'X'' = \measuredangle A''C'Z + \measuredangle ZPX'' \\
&= \measuredangle (B'C';\ AB) + \measuredangle (PZ;\ PX) \\
&= \measuredangle (B'C';\ AB) + \measuredangle (PZ;\ AB) + \measuredangle (AB;\ BC) + \measuredangle (BC;\ PX) \\
&= \measuredangle (B'C';\ AB) + 90^\circ + \measuredangle (AB;\ BC) + 90^\circ \\
&= \measuredangle (B'C';\ AB) + \measuredangle (AB;\ BC) \\
&= \measuredangle (BC;\ AB) + \measuredangle (AB;\ BC) \qquad \text{(for } B'C' \parallel BC\text{)} \\
&= 0^\circ.
\end{aligned}$$

We conclude that the points A'', X' and X'' are collinear. But X' is the reflection of L in $B'C'$, X'' is the reflection of X, and A'' is its own reflection (since A'' lies on $B'C'$). Since the points A'', X' and X'' are collinear, their preimages A'', L and X are collinear, too, i. e. L lies on the line XA''. Similarly, L lies on YB'' and ZC''. Summarizing:

Theorem 1.6: The point L lies on the lines XA'', YB'', ZC''. (See Fig. 9.)

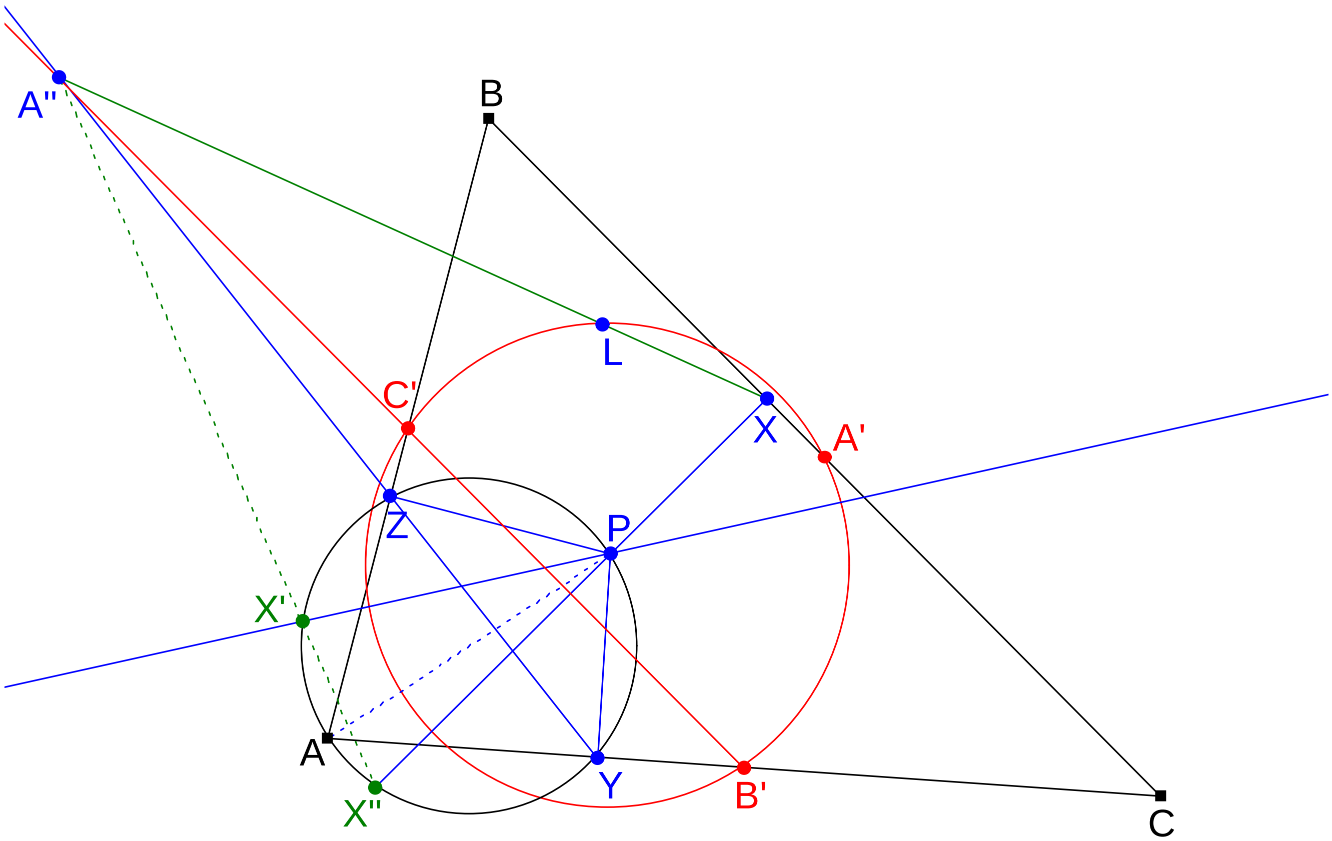

Fig. 8

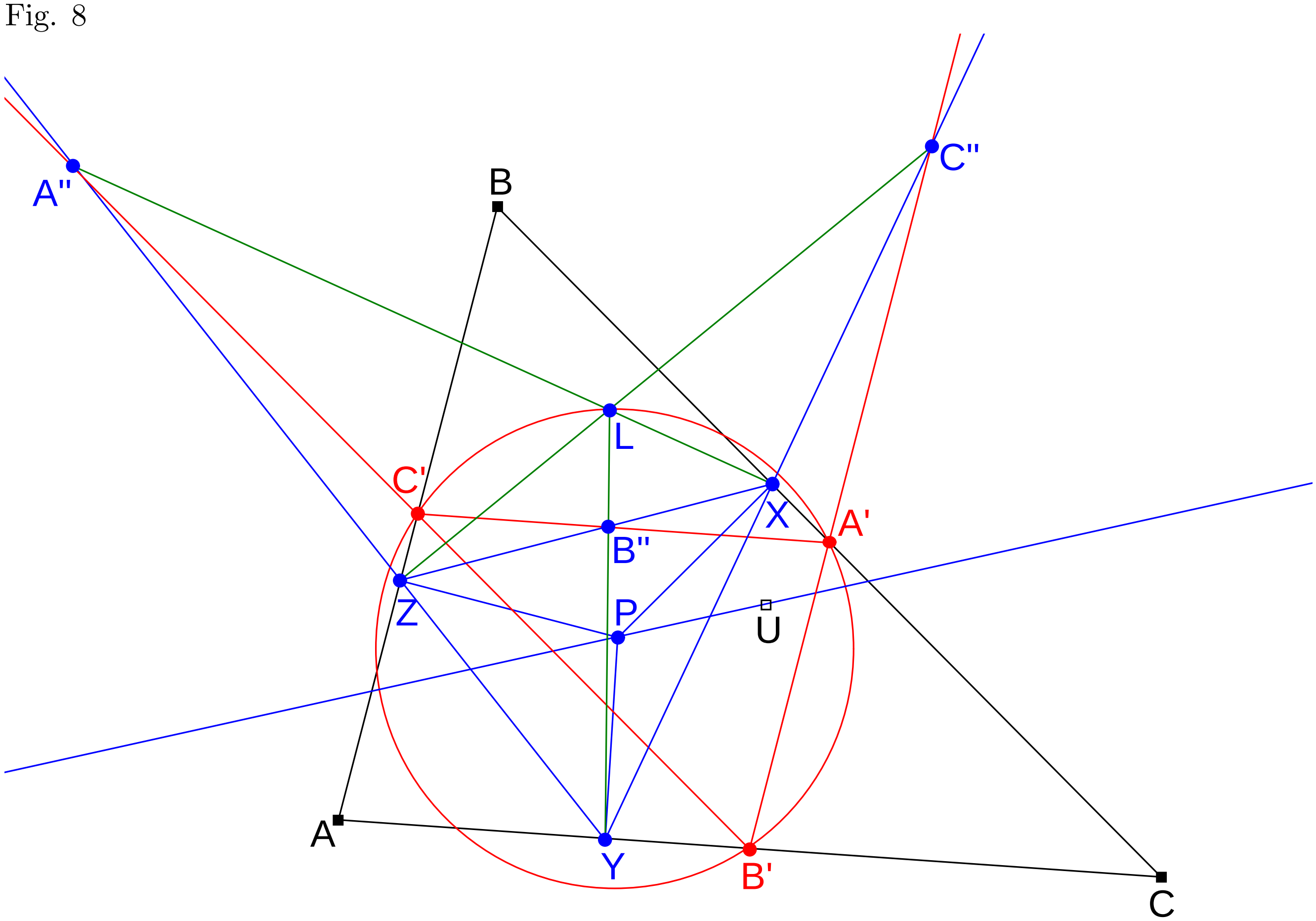

Fig. 9

The points Y, Z, X', X'' being concyclic (they lie on the circle with diameter AP), we get $A''X' \cdot A''X'' = A''Y \cdot A''Z$. Since X', X'' and A'' are the reflections of L, X, A'' in $B'C'$, we have $A''X' = A''L$ and $A''X'' = A''X$, thus $A''L \cdot A''X = A''X' \cdot A''X''$. Hence, $A''L \cdot A''X = A''Y \cdot A''Z$, and the points L, X, Y, Z are concyclic, i. e. the point L lies on the circle XYZ. This circle is called the **pedal circle** of P with respect to triangle ABC. We record this fact:

Theorem 1.7: The point L lies on the pedal circle XYZ of P with respect to triangle ABC. (See Fig. 10.)

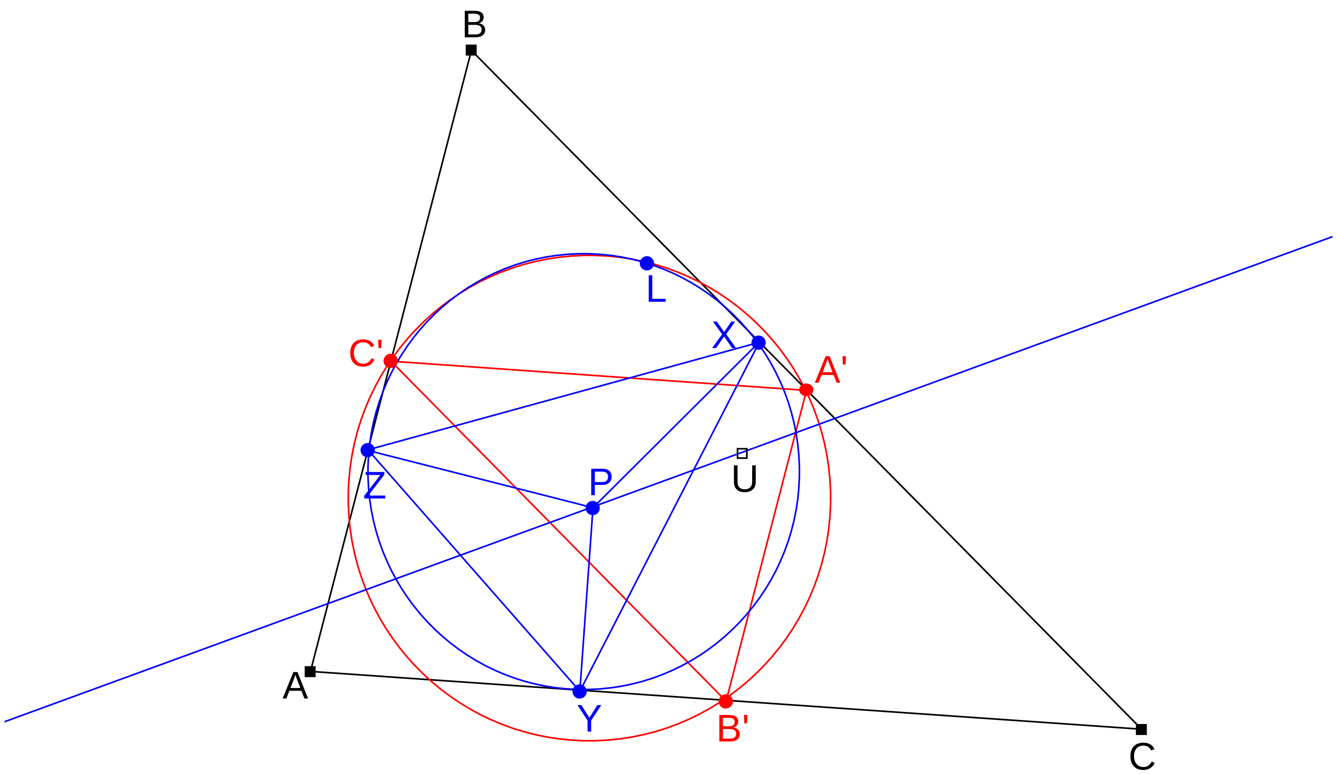

Fig. 10

Herewith, we have proven the First and the Second Fontene theorem; however, we won't stop here, for there are many more properties to discover.

We begin with a reminder. If Δ_1 and Δ_2 are two similar triangles, there is a similitude transformation ϕ mapping Δ_1 to Δ_2. This similitude equally maps any notable point of triangle Δ_1 to the *corresponding* point of Δ_2. Hereby, the term "corresponding" makes sense only if the notable point of triangle Δ_1 is defined by a certain chain of construction steps (applied to triangle Δ_1); in this case, we may simply apply this chain to triangle Δ_2 and get the corresponding point. But if we arbitrarily pick a point P_1 in the plane of triangle Δ_1, we cannot immediately say where the "corresponding" point of triangle Δ_2 is. Yet, it suggests itself that we regard the image P_2 of P_1 in the similitude ϕ as the "corresponding" point of Δ_2. In the following, we will make use of this definition of corresponding points; similarly, corresponding lines, or segments, or angles, or any kinds of figures can be defined. We will often say "corresponding point of the point P_1 in triangle Δ_2" instead of "the point of Δ_2 corresponding to the point P_1", and similarly for lines.

(Note that this notion makes sense for similar triangles Δ_1 and Δ_2 only. If triangles

Δ_1 and Δ_2 are not similar, I would advise against using the term "corresponding point" at all!)

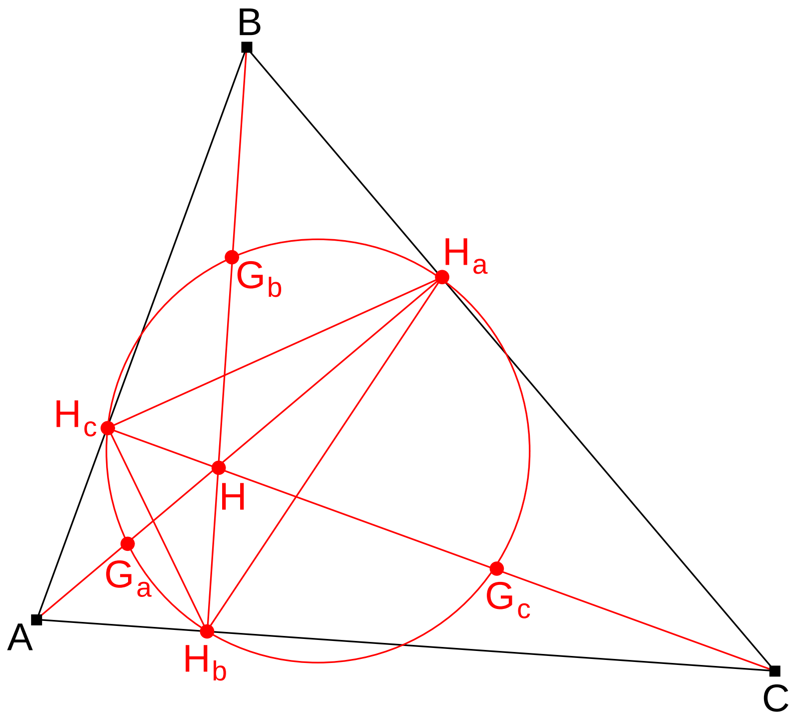

Fig. 11

Now, we continue considering the feet H_a, H_b, H_c of the altitudes of ΔABC. The triangle $H_aH_bH_c$ is called **orthic triangle** of triangle ABC.

Since $\measuredangle BH_bC = 90°$ and $\measuredangle BH_cC = 90°$, the points H_b and H_c lie on the circle with diameter BC, and thus $\measuredangle CH_bH_c = \measuredangle CBH_c$, i. e. $\measuredangle AH_bH_c = \measuredangle CBA = -\measuredangle ABC$. Analogously, $\measuredangle AH_cH_b = -\measuredangle ACB$, hence $\measuredangle H_bH_cA = -\measuredangle AH_cH_b = \measuredangle ACB = -\measuredangle BCA$. Therefore, triangles AH_bH_c and ABC are inversely similar.

Likewise, triangles H_aBH_c and ABC are inversely similar, and triangles H_aH_bC and ABC are inversely similar. Let x', y', z' be the corresponding lines of the line PU in the triangles AH_bH_c, H_aBH_c, H_aH_bC. Then, we have:

Theorem 1.8: The lines x', y', z' pass through L. (See Fig. 12.)

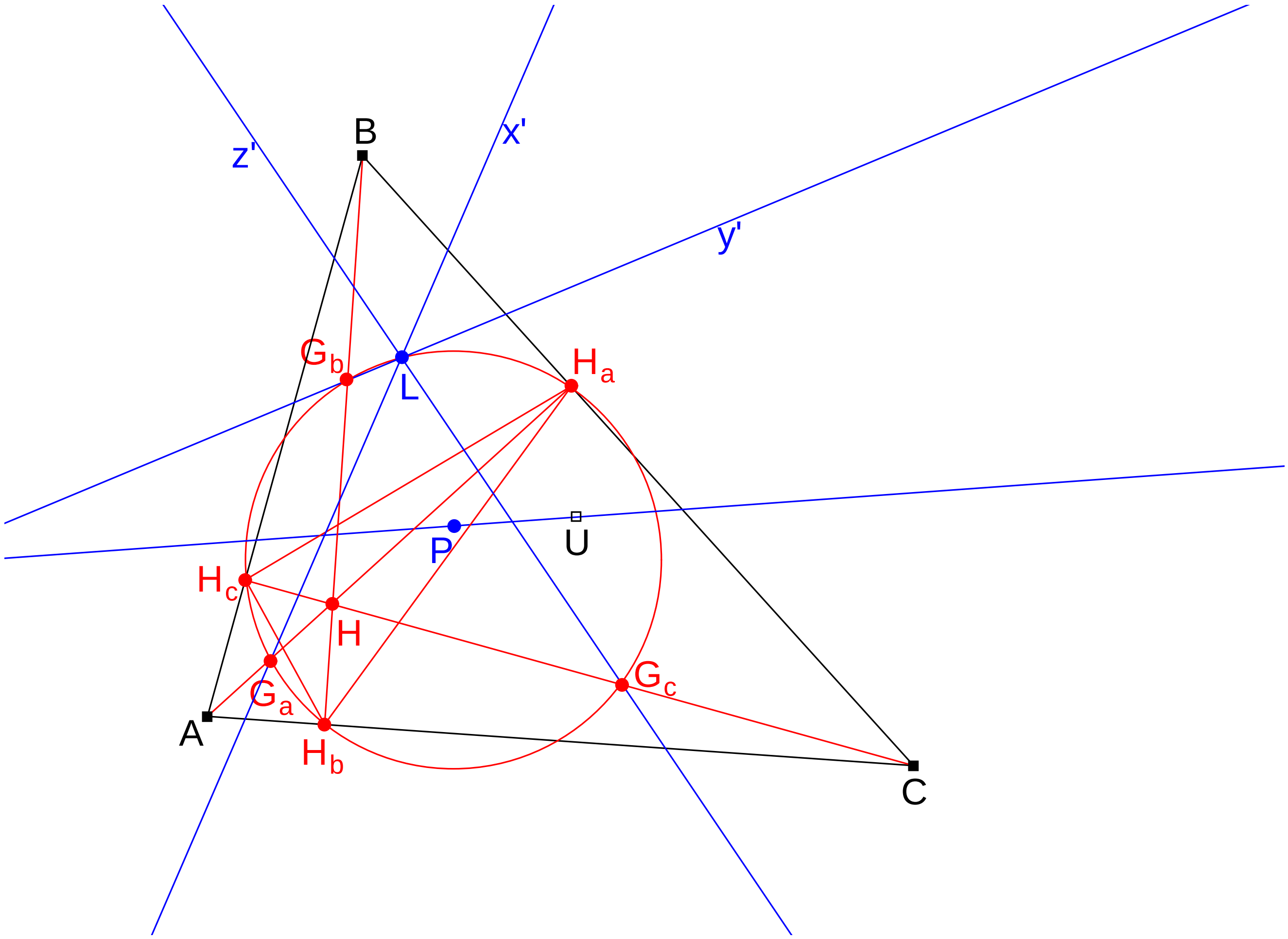

Fig. 12

Proof. If H is the orthocenter of triangle ABC, the midpoints G_a, G_b, G_c of the segments AH, BH, CH lie on the nine-point circle of triangle ABC.

As $\measuredangle AH_bH = 90°$ and $\measuredangle AH_cH = 90°$, the points H_b and H_c lie on the circle with diameter AH. In other words, the circumcircle of triangle AH_bH_c is the circle with diameter AH; thus, the circumcenter of triangle AH_bH_c is the midpoint G_a of AH.

Since the line PU passes through the circumcenter U of ΔABC, its corresponding line x' in triangle AH_bH_c passes through the circumcenter G_a of ΔAH_bH_c. Similarly, the lines y' and z' pass through G_b and G_c.

(See Fig. 13.) Since B' and C' are the midpoints of CA and AB, the triangle $AC'B'$ is the image of triangle ABC in the homothety with center A and factor $\frac{1}{2}$. Hence, the orthocenter of triangle $AC'B'$ is the image of the orthocenter H of ΔABC in this homothety, i. e. the midpoint G_a of the segment AH.

After [2], Lemma 1, the reflections of the orthocenter of a triangle in the sidelines lie on the circumcircle of the triangle. Thus, the reflection U_a of the orthocenter G_a of triangle $AC'B'$ in the sideline $B'C'$ lies on the circumcircle of triangle $AC'B'$. But as the points B' and C' lie on the circle with diameter AU, this circumcircle is just the circle with diameter AU. Consequently, U_a lies on the circle with diameter AU. As we know, X' lies on this circle, too. Hence, $\measuredangle U_aX'U = \measuredangle U_aAU$. We have

$$\begin{aligned}
\measuredangle\left(U_aX';\ BC\right) &= \measuredangle\left(U_aX';\ PU\right) + \measuredangle\left(PU;\ BC\right) = \measuredangle U_aX'U + \measuredangle\left(PU;\ BC\right) \\
&= \measuredangle U_aAU + \measuredangle\left(PU;\ BC\right) = \measuredangle\left(AH_a;\ AU\right) + \measuredangle\left(PU;\ BC\right) \\
&= \measuredangle\left(AH_a;\ BC\right) + \measuredangle\left(BC;\ CA\right) + \measuredangle\left(CA;\ AU\right) + \measuredangle\left(PU;\ BC\right) \\
&= 90^\circ + \measuredangle BCA + \measuredangle CAU + \measuredangle\left(PU;\ BC\right).
\end{aligned}$$

Since U is the center of the circle ABC, we have $\measuredangle CAU = 90^\circ - \measuredangle ABC$, thus

$$\begin{aligned}
\measuredangle\left(U_aX';\ BC\right) &= 90^\circ + \measuredangle BCA + \left(90^\circ - \measuredangle ABC\right) + \measuredangle\left(PU;\ BC\right) \\
&= 180^\circ + \left(\measuredangle BCA - \measuredangle ABC\right) + \measuredangle\left(PU;\ BC\right) \\
&= \measuredangle BCA - \measuredangle ABC + \measuredangle\left(PU;\ BC\right) \\
&= \measuredangle BCA - \measuredangle\left(AB;\ BC\right) + \measuredangle\left(PU;\ BC\right) \\
&= \measuredangle BCA + \measuredangle\left(PU;\ AB\right).
\end{aligned}$$

Now, U_a and X' are the reflections of G_a and L in $B'C'$; therefore, $\measuredangle\left(G_aL;\ B'C'\right) = -\measuredangle\left(U_aX';\ B'C'\right)$, and

$$\begin{aligned}
\measuredangle\left(G_aL;\ CA\right) &= \measuredangle\left(G_aL;\ B'C'\right) + \measuredangle\left(B'C';\ CA\right) \\
&= -\measuredangle\left(U_aX';\ B'C'\right) + \measuredangle\left(B'C';\ CA\right) \\
&= -\measuredangle\left(U_aX';\ BC\right) + \measuredangle\left(BC;\ CA\right) \qquad \text{(since } B'C' \parallel BC\text{)} \\
&= -\left(\measuredangle BCA + \measuredangle\left(PU;\ AB\right)\right) + \measuredangle BCA = -\measuredangle\left(PU;\ AB\right).
\end{aligned}$$

On the other hand, x' is the corresponding line of PU in triangle AH_bH_c. Hence, the angle between the line x' and the sideline AH_b of triangle AH_bH_c is oppositely equal[2] to the angle between the line PU and the sideline AB of triangle ABC. This means:

$$\measuredangle\left(x';\ AH_b\right) = -\measuredangle\left(PU;\ AB\right),$$

thus $\measuredangle\left(x';\ CA\right) = -\measuredangle\left(PU;\ AB\right) = \measuredangle\left(G_aL;\ CA\right)$. Therefore, the lines x' and G_aL are parallel, and, as both of them pass through G_a, they must coincide, so that L lies on x'. Similarly, L lies on y' and z', proving Theorem 1.8.

[2] Oppositely equal because triangles AH_bH_c and ABC are inversely similar.

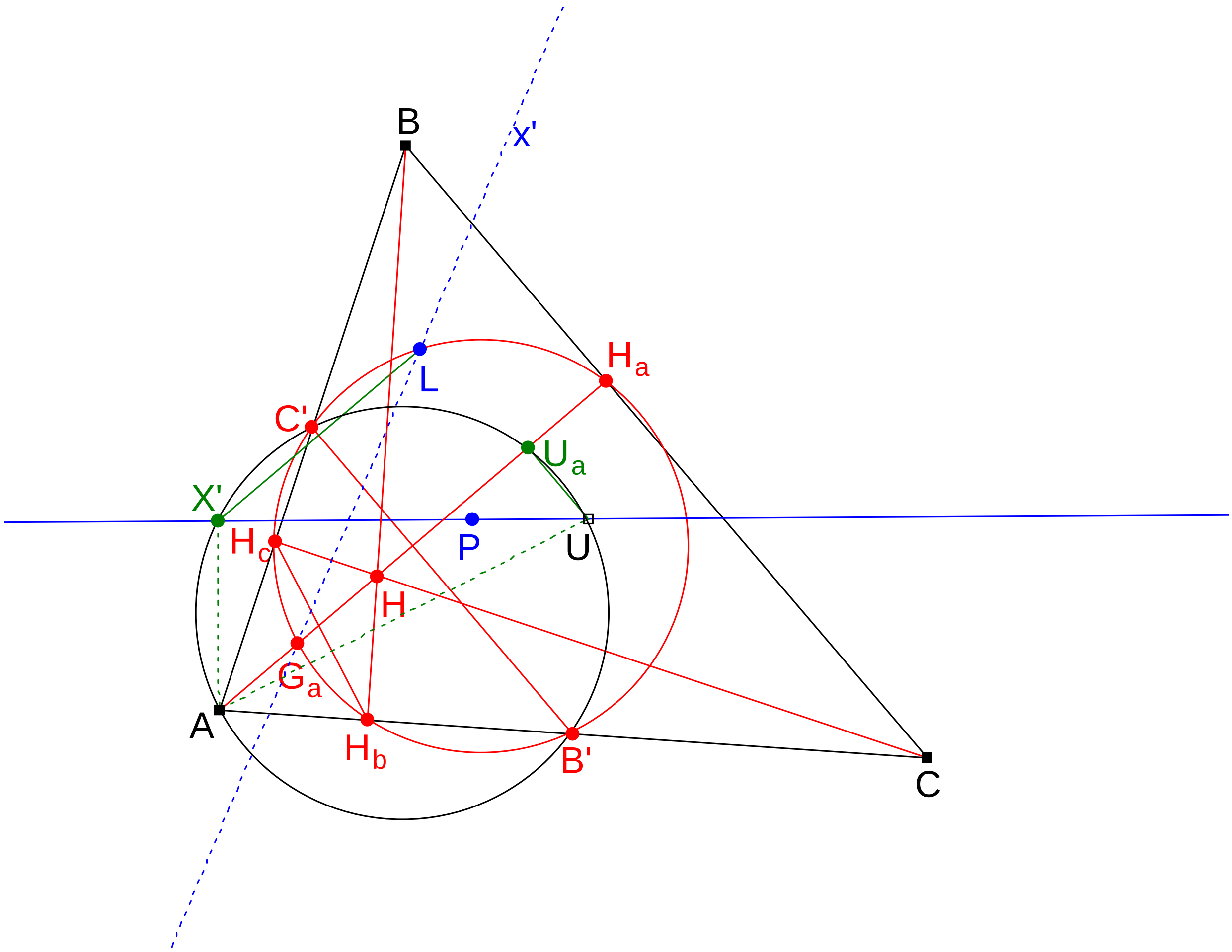

Fig. 13

Theorem 1.9: The orthocenters D, E, F of triangles AYZ, BZX, CXY are simultaneously the corresponding points of P in triangles AH_bH_c, H_aBH_c, H_aH_bC.

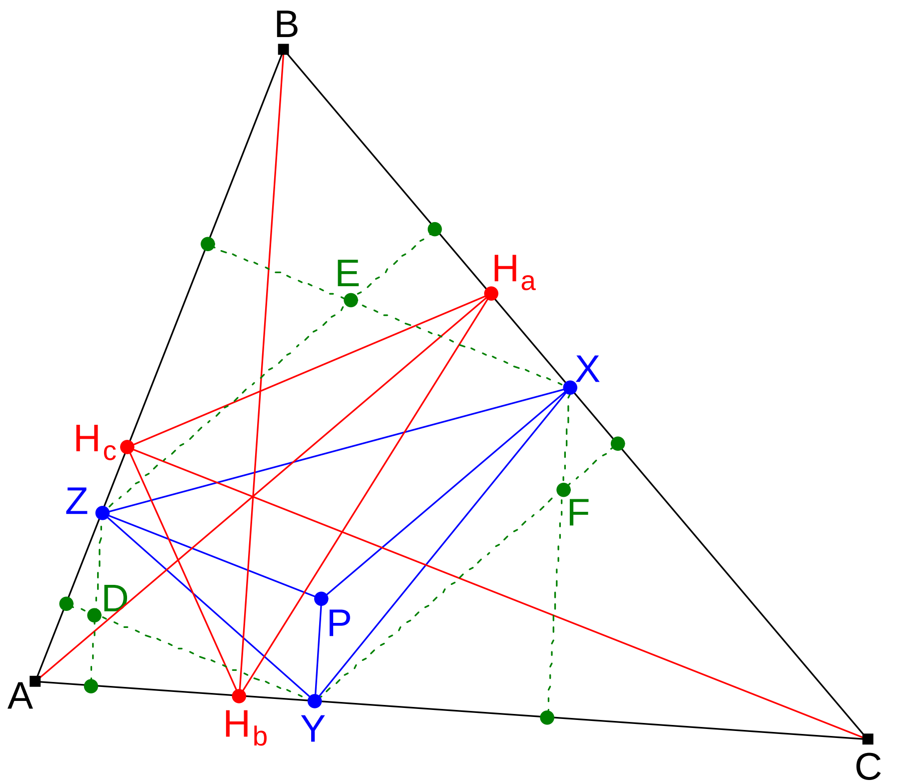

Fig. 14

Proof (Fig. 15). If Y_c is the foot of the altitude of triangle AYZ from Y, then the orthocenter D of ΔAYZ lies on YY_c.

The segments YY_c and CH_c are parallel (since both are perpendicular to AB); hence, $AH_c : H_cY_c = AC : CY$.

Being an altitude in triangle AYZ, AD is perpendicular to YZ; hence

$$\begin{aligned}\measuredangle DAY_c &= \measuredangle(AD;\ AZ) = \measuredangle(AD;\ YZ) + \measuredangle(YZ;\ AZ) = 90^\circ + \measuredangle(YZ;\ AZ) \\ &= 90^\circ + \measuredangle YZA.\end{aligned}$$

The points Y and Z lie on the circle with diameter AP, entailing $\measuredangle YZA = \measuredangle YPA$, and thus $\measuredangle DAY_c = 90^\circ + \measuredangle YPA = \measuredangle AYP + \measuredangle YPA$. Now, since $\measuredangle AYP + \measuredangle YPA = -\measuredangle PAY$, we have $\measuredangle DAY_c = -\measuredangle PAY$. Furthermore, obviously $\measuredangle AY_cD = 90^\circ = -90^\circ = -\measuredangle AYP$. Hence, the triangles DAY_c and PAY are inversely similar. And since $AH_c : H_cY_c = AC : CY$, the points H_c and C are *corresponding* points on their sidelines AY_c and AY, respectively. Corresponding points in similar triangles produce equal angles; hence, the angles $\measuredangle AH_cD$ and $\measuredangle ACP$ are oppositely equal[3]. We have thus shown $\measuredangle AH_cD = -\measuredangle ACP$; likewise, $\measuredangle AH_bD = -\measuredangle ABP$.

[3] Oppositely because triangles DAY_c and PAY are inversely similar.

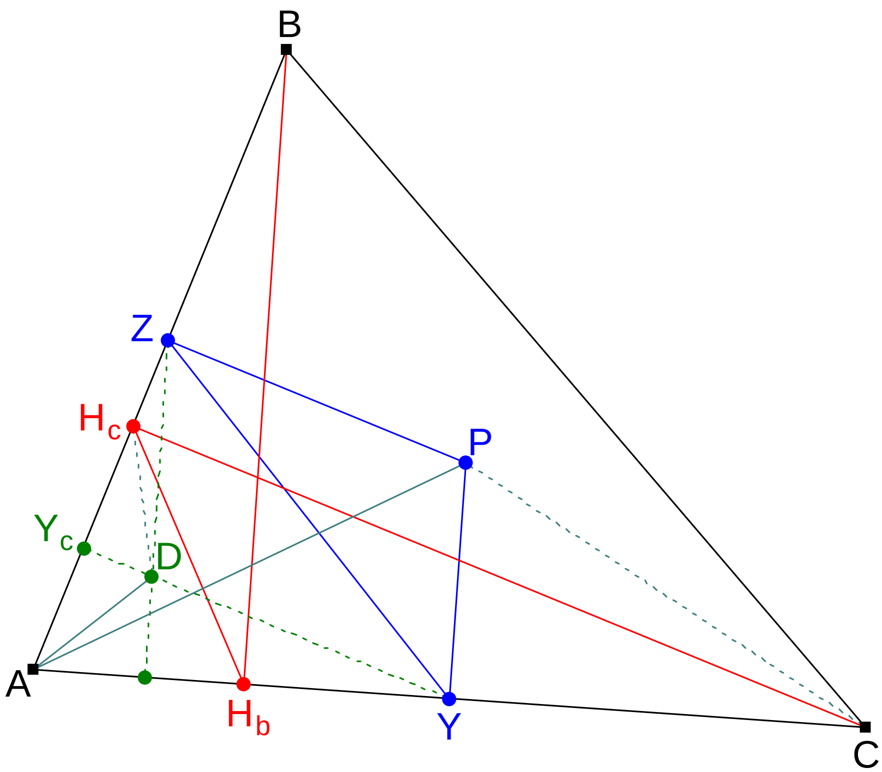

Fig. 15

On the other hand, if D_1 is the corresponding point of P in triangle AH_bH_c, we have $\measuredangle AH_cD_1 = -\measuredangle ACP$ and $\measuredangle AH_bD_1 = -\measuredangle ABP$, since corresponding points in similar triangles produce equal angles. Hence, $\measuredangle AH_cD = \measuredangle AH_cD_1$ and $\measuredangle AH_bD = \measuredangle AH_bD_1$. The point D_1 must therefore lie on H_cD and H_bD, what shows that $D_1 = D$, i. e. that D is the corresponding point of P in triangle AH_bH_c. Analogous reasoning shows the same for E and F, and Theorem 1.9 is established.

(See Fig. 16.) Theorem 1.9 entails that the points D, E, F lie on the lines x', y', z'. In fact, since P lies on PU, the corresponding point D of P in triangle AH_bH_c lies on the corresponding line x' of PU in this triangle, and equally E lies on y' and F on z'. Hence,

$$\measuredangle ELF = \measuredangle(y';\ z') = \measuredangle(y';\ BC) - \measuredangle(z';\ BC)\,.$$

But just as we have shown $\measuredangle(x';\ CA) = -\measuredangle(PU;\ AB)$ previously, we may find $\measuredangle(y';\ BC) = -\measuredangle(PU;\ AB)$ and $\measuredangle(z';\ BC) = -\measuredangle(PU;\ CA)$; it follows that

$$\begin{aligned}\measuredangle ELF &= (-\measuredangle(PU;\ AB)) - (-\measuredangle(PU;\ CA)) = \measuredangle(PU;\ CA) - \measuredangle(PU;\ AB) \\ &= \measuredangle(AB;\ CA)\,.\end{aligned}$$

On the other hand, since E and F are the orthocenters of ΔBZX and ΔCXY, we get $XE \perp AB$ and $XF \perp CA$, thus

$$\begin{aligned}\measuredangle EXF &= \measuredangle(XE;\ XF) = \measuredangle(XE;\ AB) + \measuredangle(AB;\ CA) + \measuredangle(CA;\ XF) \\ &= 90^\circ + \measuredangle(AB;\ CA) + 90^\circ = 180^\circ + \measuredangle(AB;\ CA) = \measuredangle(AB;\ CA) = \measuredangle ELF.\end{aligned}$$

Hence, L lies on the circle EXF.

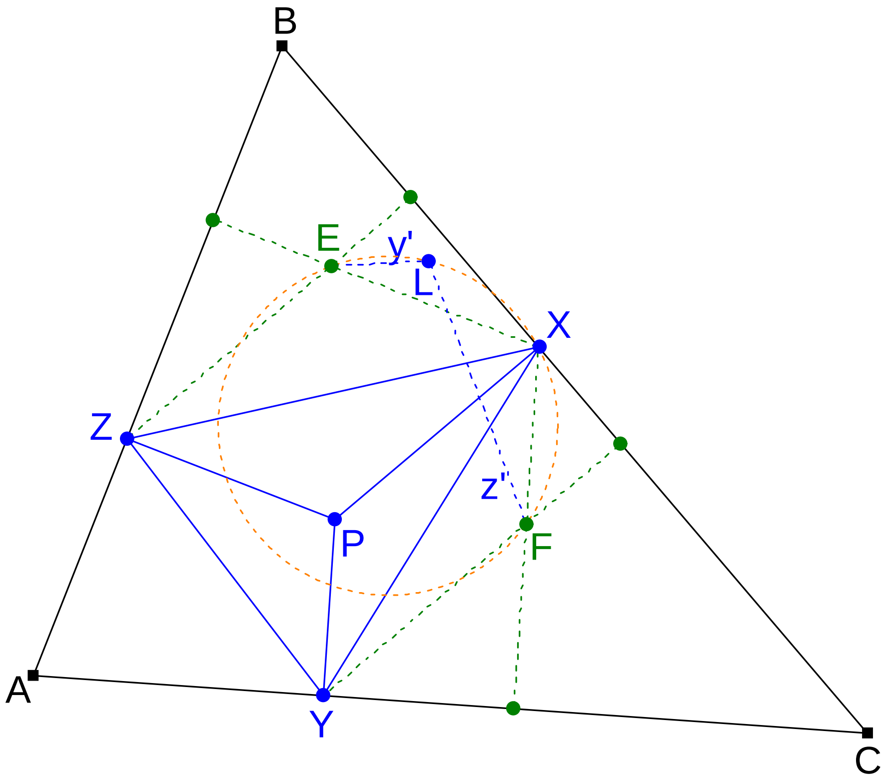

Fig. 16

(See Fig. 17.) For the orthocenter E of triangle BZX, we have $XE \perp AB$; on the other hand, $PZ \perp AB$. Hence, $XE \parallel PZ$, and similarly $ZE \parallel PX$, proving the quadrilateral $ZEXP$ a parallelogram. We conclude $\overrightarrow{ZE} = \overrightarrow{PX}$. Similarly, $\overrightarrow{YF} = \overrightarrow{PX}$. If R_a is the foot of the perpendicular from P to AH_a, then the quadrilateral PR_aH_aX is a rectangle, and $\overrightarrow{R_aH_a} = \overrightarrow{PX}$. In other words, the translation by the vector $\overrightarrow{PX}$ maps the points P, Y, Z, R_a to X, F, E, H_a, respectively. Since the points P, Y, Z, R_a all lie on the circle with diameter AP (R_a does, since $\measuredangle AR_aP = 90°$), their images X, F, E, H_a lie on the image of this circle in the translation. But we know that L lies on the circle EXF; combining, we obtain:

Theorem 1.10: The points X, E, F, H_a and L lie on one circle, and this circle is the image of the circle with diameter AP in the translation by the vector $\overrightarrow{PX}$.

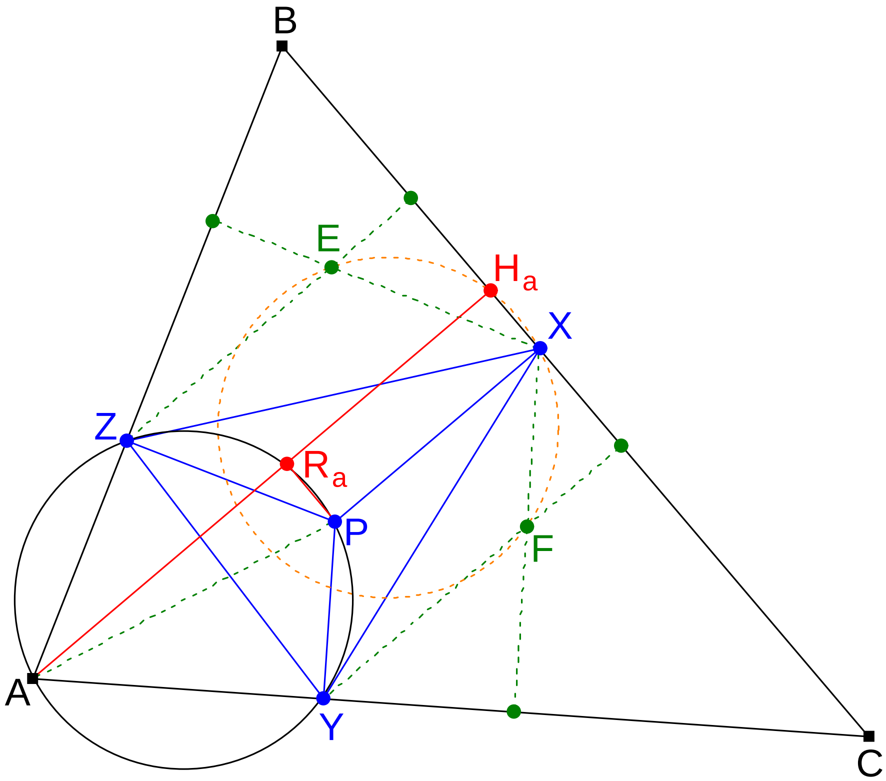

Fig. 17

In addition, we have (Fig. 18):

Theorem 1.11: This circle through X, E, F, H_a, L is as well the reflection of the circle with diameter AP in the line $B'C'$.

Proof. The points X'' and X' are the reflections of X and L in $B'C'$; conversely, X and L are the reflections of X'' and X' in $B'C'$. Moreover, H_a is the reflection of A in $B'C'$. Since the points X'', X', A lie on the circle with diameter AP, their reflections X, L, H_a are placed on the reflection of this circle in $B'C'$. I. e., the circle XLH_a is the reflection of this circle with diameter AP in $B'C'$; but this is just the statement of Theorem 1.11.

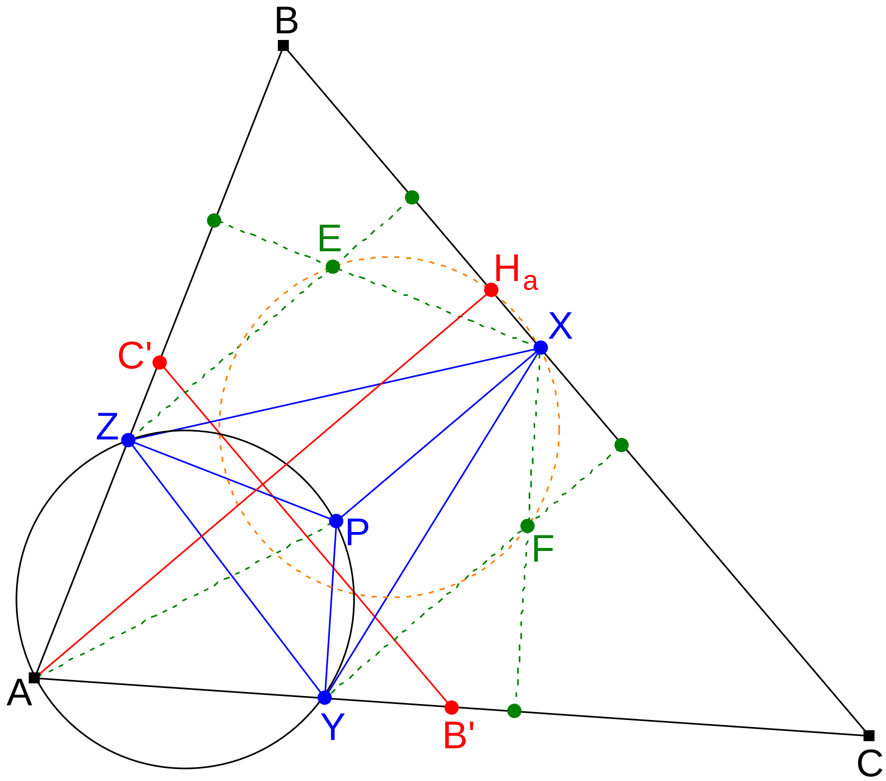

Fig. 18

Now we leave the unending series of results concerning pedal circles for considering two prominent special cases (we will get back to the general case at the end of the paper):

<u>**Case 1: P is the orthocenter of ΔABC**</u>

We first consider the case where P is the orthocenter H of triangle ABC. A special feature of this case is that the feet X, Y, Z of the perpendiculars from $P = H$ to the sidelines BC, CA, AB coincide with the feet H_a, H_b, H_c of the altitudes. The line $PU = HU$ is the well-known **Euler line** of triangle ABC.

Theorem 1.1 yields (Fig. 19):

Theorem 2.1: The reflections x, y, z of the Euler line HU of triangle ABC in the sidelines $B'C'$, $C'A'$, $A'B'$ of the medial triangle $A'B'C'$ concur at one point L lying on the nine-point circle of triangle ABC. This L is the Anti-Steiner point of the Euler line HU of triangle ABC with respect to triangle $A'B'C'$.

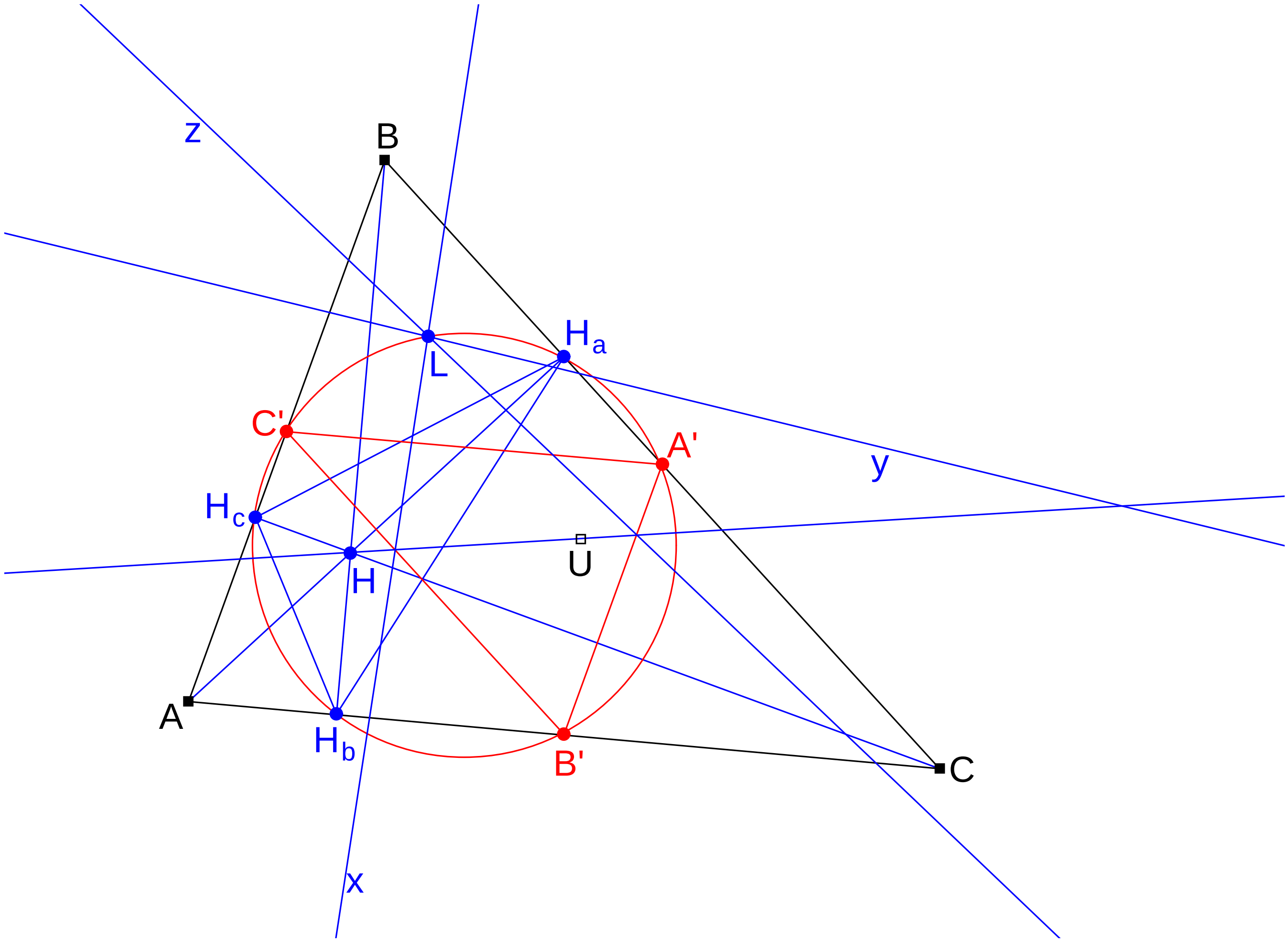

Fig. 19

It is interesting to note that HU is the Euler line of triangle $A'B'C'$, too.[4] Thus, the lines x, y, z are the reflections of the Euler line of $\Delta A'B'C'$ in the sidelines $B'C'$, $C'A'$, $A'B'$ of $\Delta A'B'C'$. According to [2], Note 3, their intersection L is the Euler reflection point of triangle $A'B'C'$. We record this fact:

Theorem 2.2: The point L is the Euler reflection point of the medial triangle $A'B'C'$.

Since for any point defined in triangle ABC, the corresponding point of the medial triangle $A'B'C'$ is called the **complement** of this point, we can rewrite Theorem 2.2 as follows:

Theorem 2.3: The point L is the complement of the Euler reflection point of triangle ABC.

[4] For the sake of completeness, I give a *proof.* Triangle $A'B'C'$ is the medial triangle of triangle ABC; hence, it has the same centroid as ΔABC, and the circumcenter of ΔABC is the orthocenter of $\Delta A'B'C'$. The Euler line of triangle $A'B'C'$ passes through the orthocenter and the centroid of $\Delta A'B'C'$, i. e. through the circumcenter and the centroid of ΔABC, and thus coincides with the Euler line HU of ΔABC.

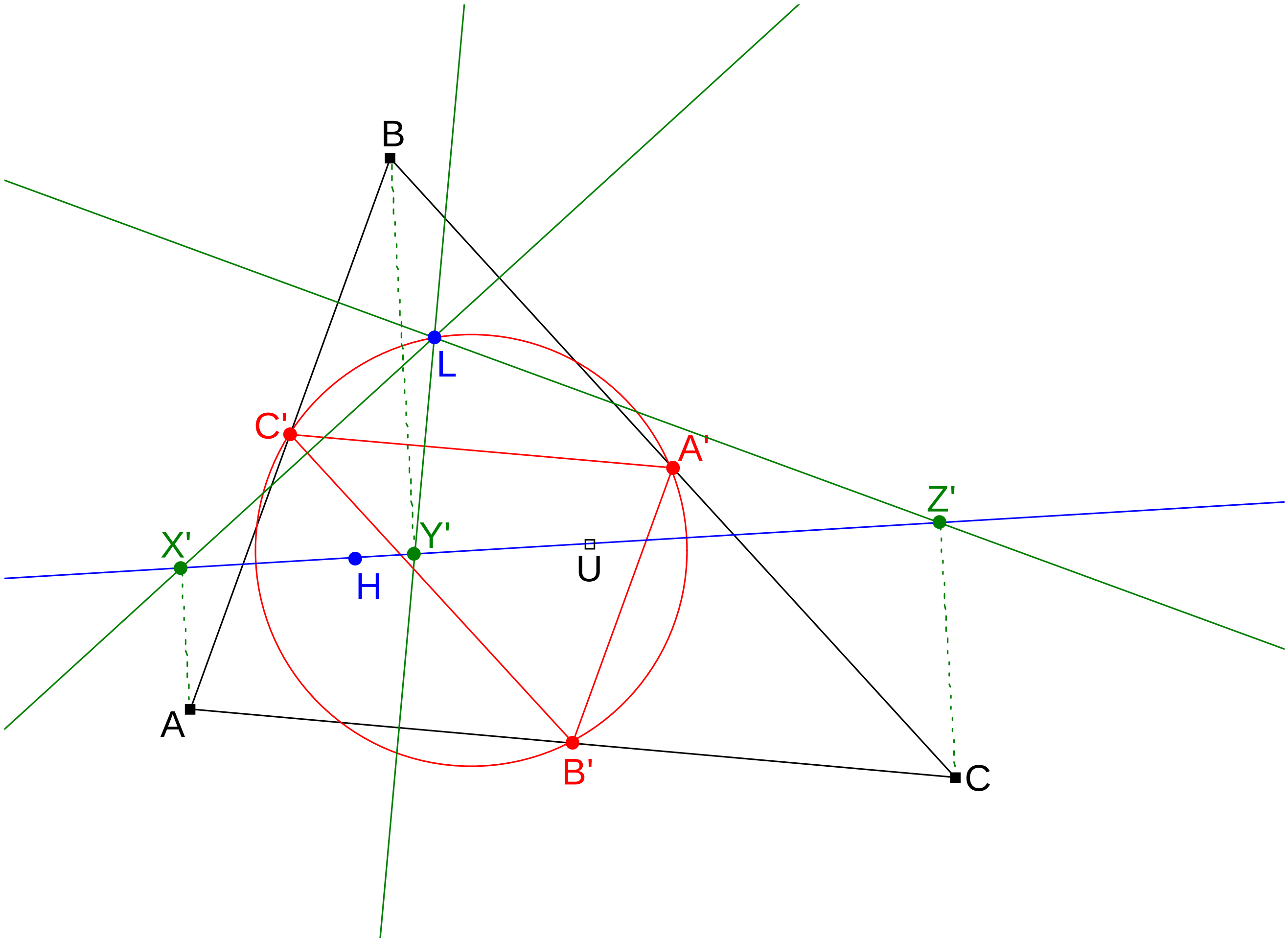

Fig. 20

Theorem 1.3 yields:

Theorem 2.4: The point L is the orthopole of the Euler line HU of triangle ABC. (See Fig. 20.)

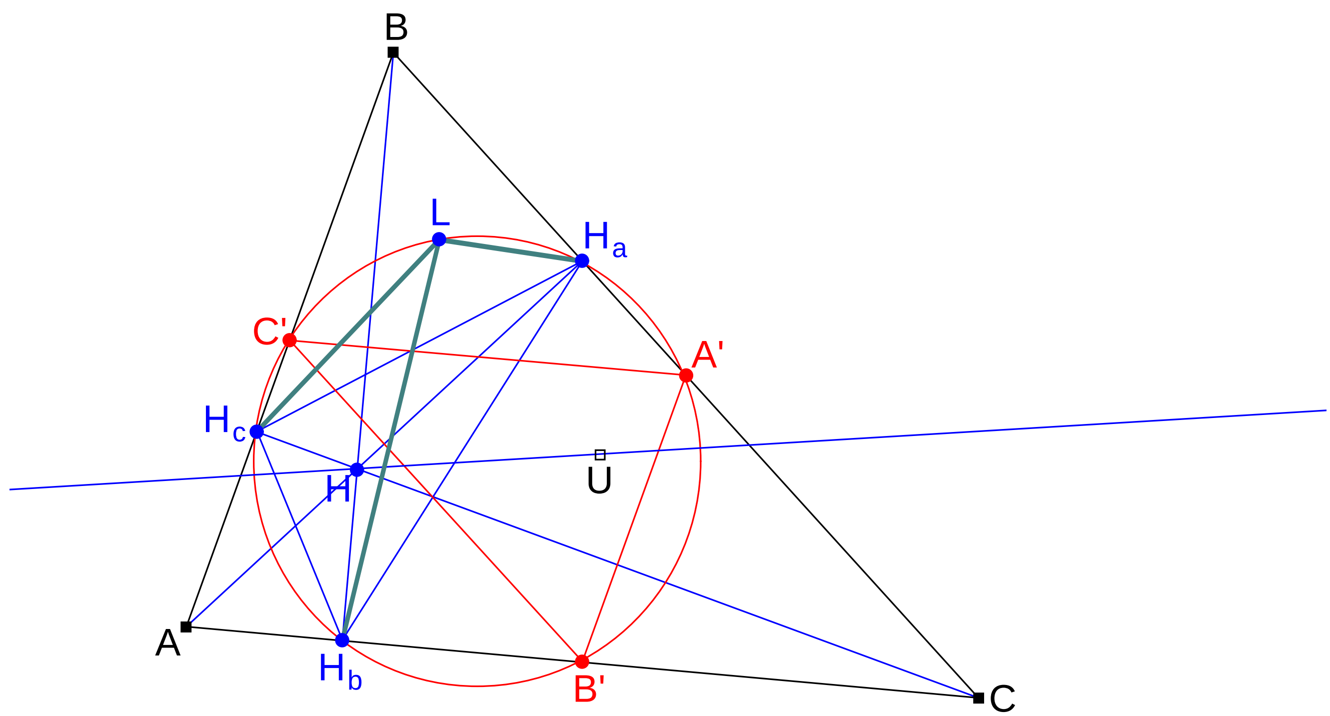

Fig. 21

Now we will apply Theorem 1.5. The segments AX', BY', CZ' are the distances from the points A, B, C to the Euler line HU. The Euler line HU passes through the centroid of ΔABC.

A theorem states that if a line g passes through the centroid of a triangle ABC, and the points A and C lie on one side of g and the point B on the other one, then $d(A;\ g)+d(C;\ g)=d(B;\ g)$, where the abbreviation $d(P_1;\ g_1)$ is used for the distance of a point P_1 to a line g_1. [5] Applying this to the Euler line $g = HU$, we obtain $d(A;\ HU)+d(C;\ HU) = d(B;\ HU)$, i. e. $AX' + CZ' = BY'$. Of course, this holds only for A and C lying on one side of HU and B on the other side. Else, $AX' + BY' = CZ'$ or $BY' + CZ' = AX'$. Altogether, we can say that the longest of the three segments AX', BY', CZ' equals the sum of the other two.

After Theorem 1.5, $H_aL = AX'$, $H_bL = BY'$, $H_cL = CZ'$. Hence we get:

Theorem 2.5: The longest of the three segments H_aL, H_bL, H_cL equals the sum of the other two. (See Fig. 21.)

[5] *Proof.* If S is the centroid of triangle ABC, and X', Y', Z' are the feet of the perpendiculars from A, B, C to g, and M_y is the midpoint of $Z'X'$, then the segment $B'M_y$ is a midparallel in the trapezium $AX'Z'C$, hence $B'M_y \parallel AX'$ and $B'M_y = \frac{1}{2}(AX' + CZ')$. But $B'M_y \parallel AX' \parallel BY'$ and $BS : SB' = 2$ imply $BY' : B'M_y = 2$, hence $BY' = 2 \cdot B'M_y = AX' + CZ'$, and $d(A;\ g) + d(C;\ g) = d(B;\ g)$.

See, e. g., [3] for references.

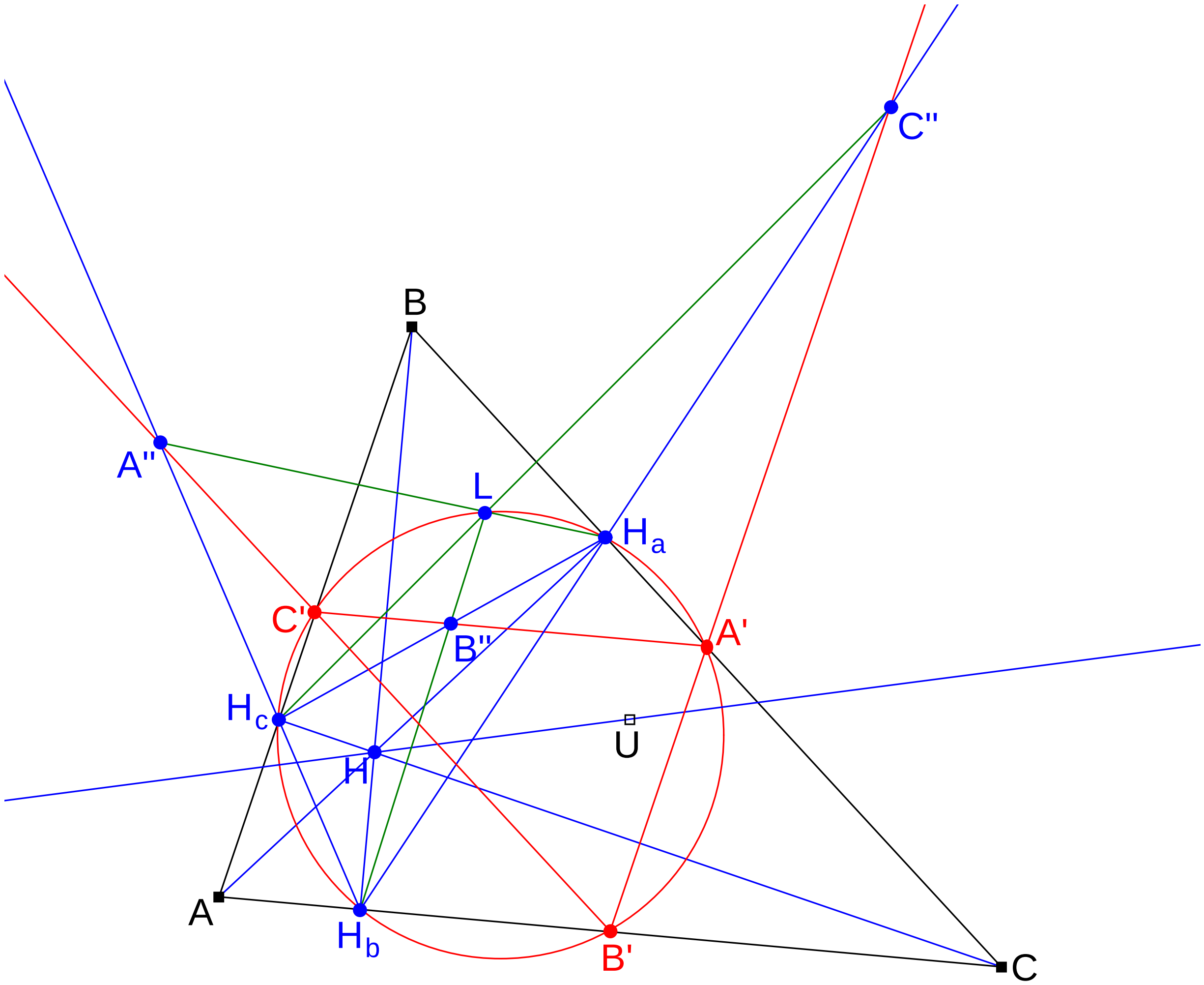

Fig. 22

According to Theorem 1.6, we have:

Theorem 2.6: The point L lies on the lines H_aA'', H_bB'', H_cC'', where $A'' = B'C' \cap H_bH_c$, $B'' = C'A' \cap H_cH_a$, $C'' = A'B' \cap H_aH_b$. (See Fig. 22.)

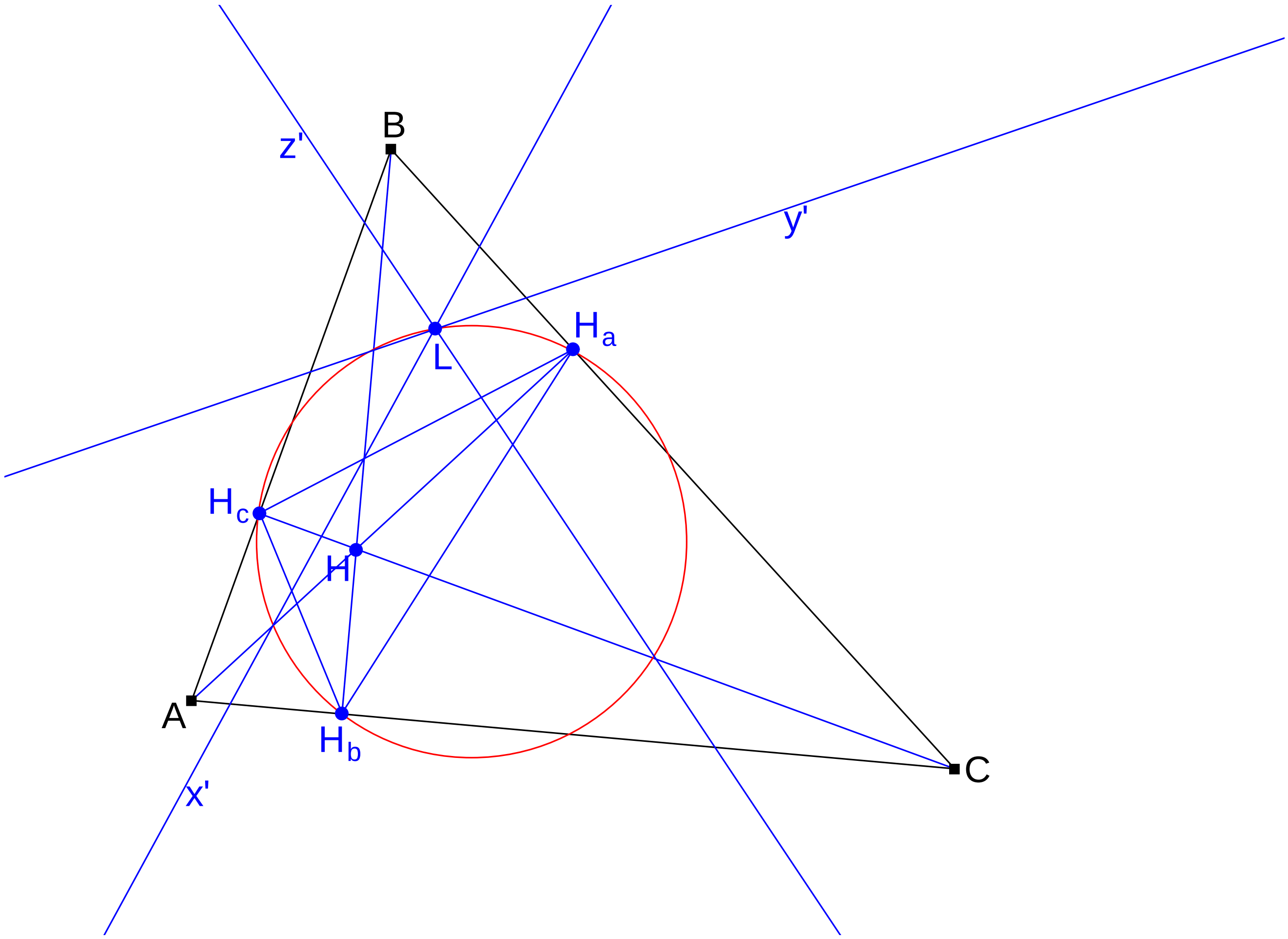

Fig. 23

The lines x', y', z' are the corresponding lines of the Euler line HU in the triangles AH_bH_c, H_aBH_c, H_aH_bC, i. e. simply the Euler lines of these triangles. Hence, Theorem 1.8 yields:

Theorem 2.7: The Euler lines of the triangles AH_bH_c, H_aBH_c, H_aH_bC pass through L. (See Fig. 23.)

Note that Theorem 2.7, together with Theorems 2.5 and 2.1, provides a solution of the following problem [4] by Victor Thebault: Show that the Euler lines of the triangles AH_bH_c, H_aBH_c, H_aH_bC meet at a point L lying on the nine-point circle of ΔABC, such that the longest of the three segments H_aL, H_bL, H_cL equals the sum of the other two.

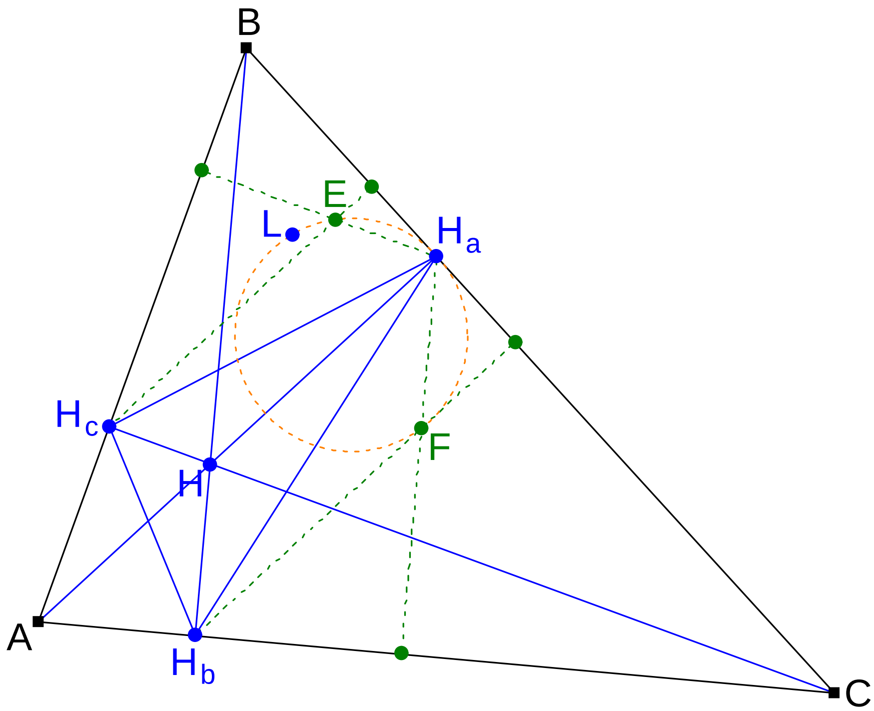

Fig. 24

The points D, E, F are the orthocenters of triangles AH_bH_c, H_aBH_c, H_aH_bC (since $H_a = X$, $H_b = Y$, $H_c = Z$). In accordance with Theorem 1.10, the points X, E, F, H_a, L lie on one circle; but since X and H_a coincide, this circle touches BC. We summarize:

Theorem 2.8: The points E, F, H_a, L lie on one circle touching the line BC.

Note that this can be proven in a simpler way.

Case 2: P is the incenter of ΔABC

Now we are going to consider another special case, namely let P be the incenter O of triangle ABC. The feet X, Y, Z of the perpendiculars from $P = O$ to the lines BC, CA, AB are the points where the incircle of ΔABC touches the sides BC, CA, AB. The triangle XYZ is called **Gergonne triangle** of triangle ABC. The pedal circle of O is the circle XYZ, the incircle of triangle ABC.

The line $PU = OU$ will be called the **diacentral line** of triangle ABC. (It is better known as the "OI line", but the term "diacentral line" has at least two notable advantages: it is, at first, independent of the notations; also, it is constructed in analogy to the corresponding line of a bicentric quadrilateral.)

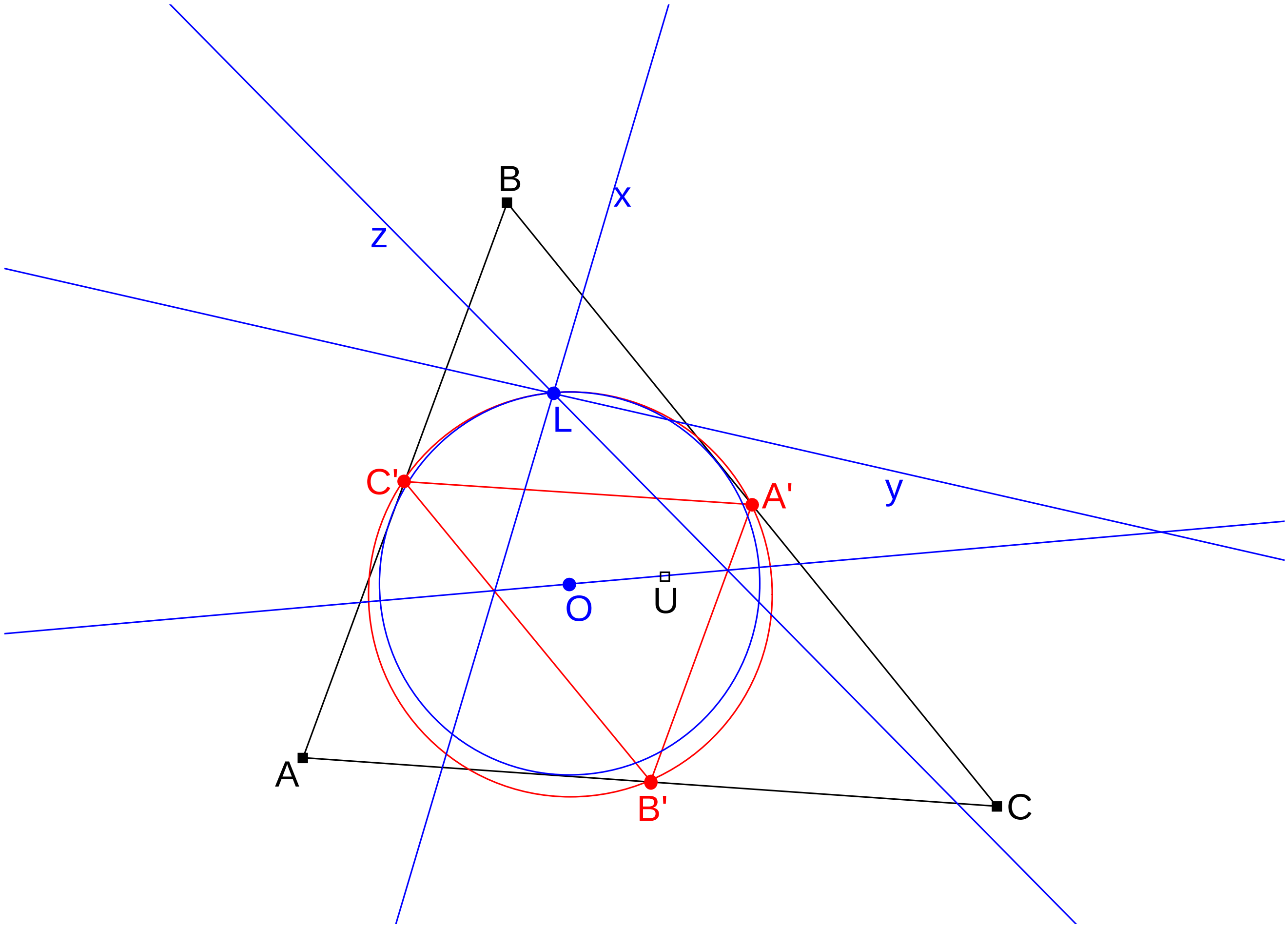

Fig. 25

Theorem 1.1 yields (Fig. 25):

Theorem 3.1: The reflections x, y, z of the diacentral line OU of triangle ABC in the sidelines $B'C'$, $C'A'$, $A'B'$ of the medial triangle $A'B'C'$ meet at a point L lying on the nine-point circle of triangle ABC. This L is the Anti-Steiner point of the diacentral line OU of triangle ABC with respect to triangle $A'B'C'$.

As a consequence of Theorem 1.7, L lies on the circle XYZ, i. e. on the incircle of ΔABC. In fact, more is true:

Theorem 3.2: The incircle and the nine-point circle of triangle ABC touch each other internally in L.

We will prove this later. The point L is called **Feuerbach point** or **Feuerbach tangency point** of triangle ABC. Thus, we can restate Theorem 3.1 as follows:

The Anti-Steiner point of the diacentral line of a triangle with respect to the medial triangle is the Feuerbach point of the original triangle.

We will study some properties of L before proving Theorem 3.2.

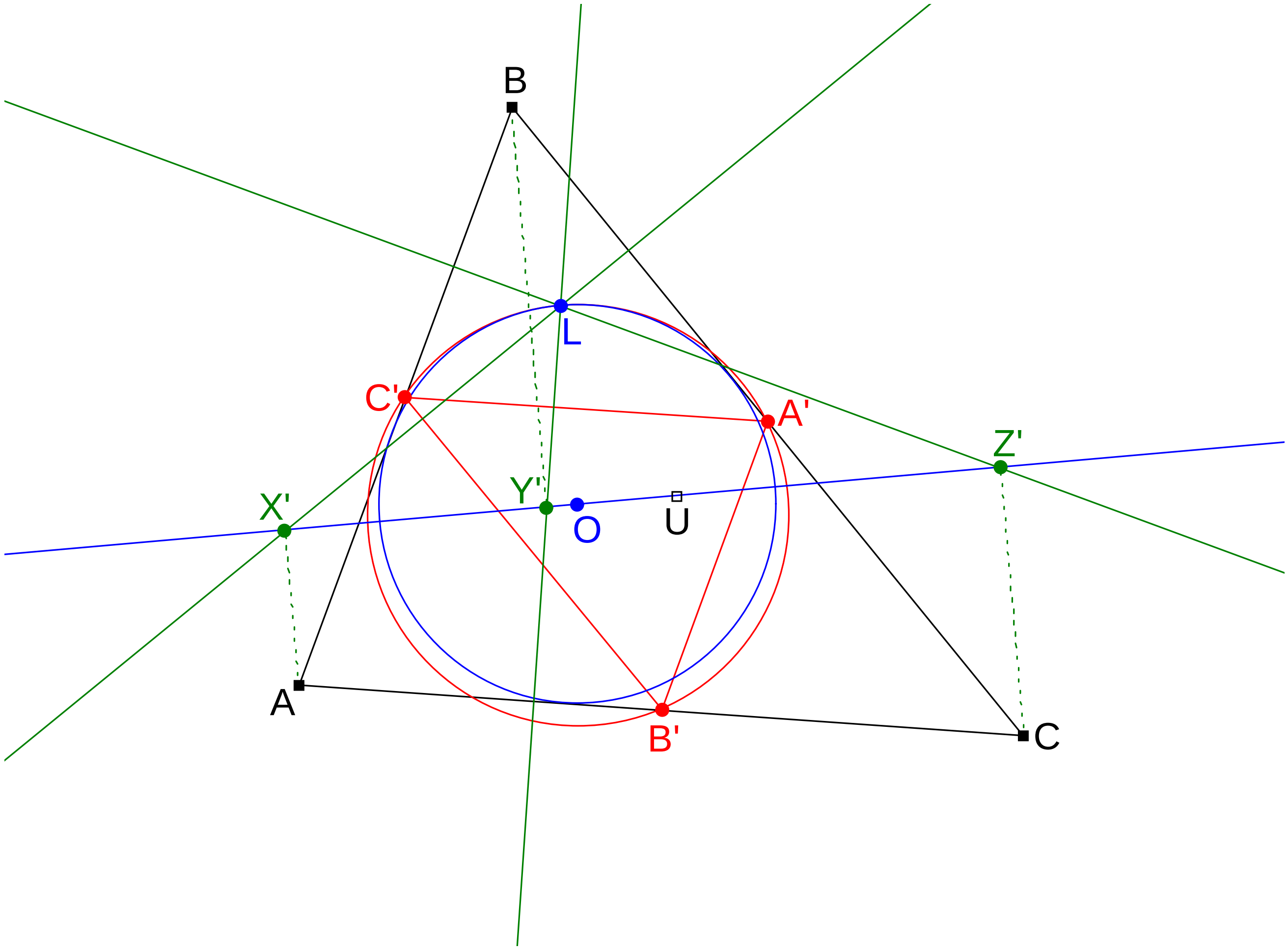

Fig. 26

Application of Theorem 1.3 yields:

Theorem 3.3: The Feuerbach point L is the orthopole of the diacentral line OU of triangle ABC. (See Fig. 26.)

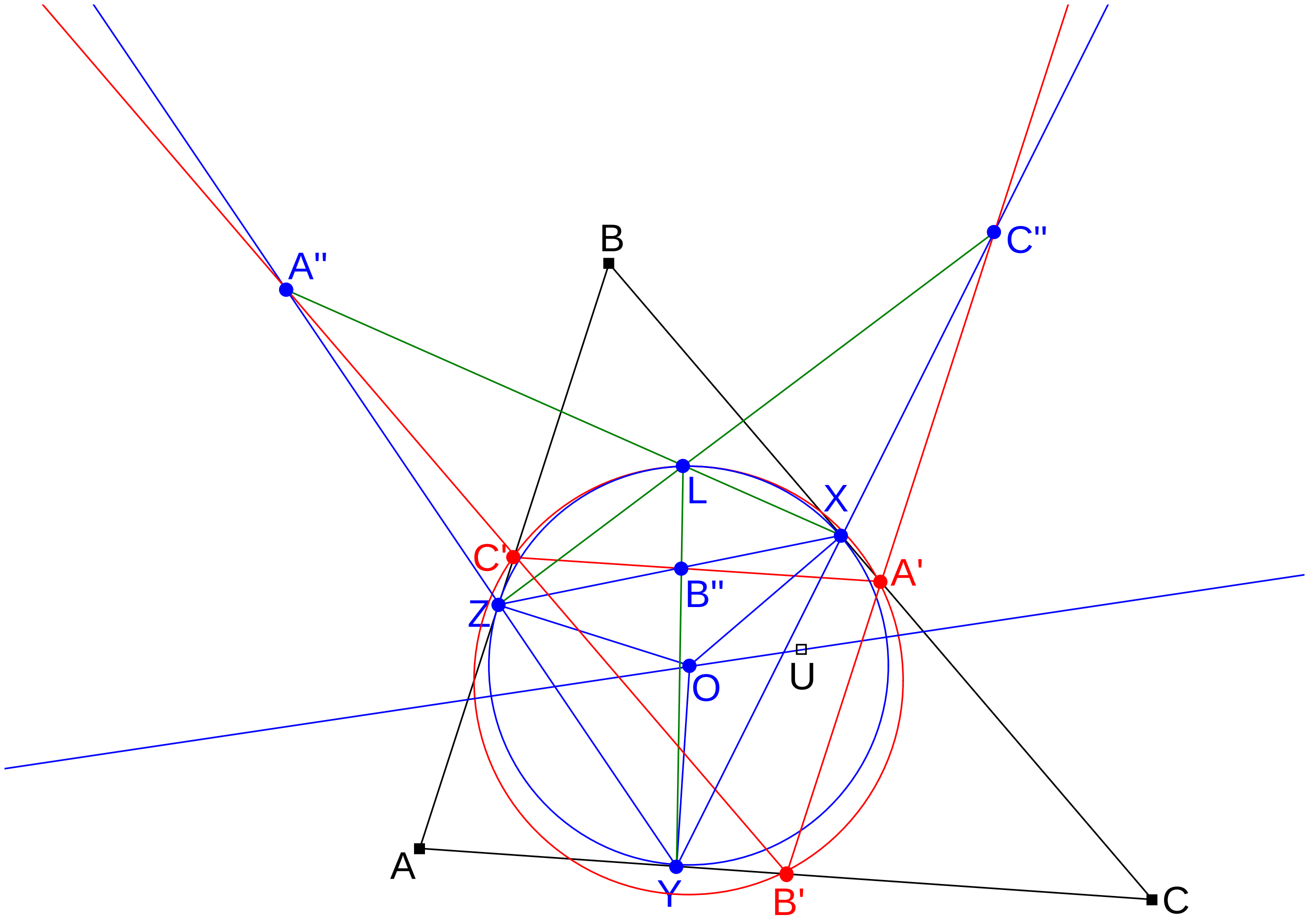

Fig. 27

Theorem 1.6 gives:

Theorem 3.4: The Feuerbach point L lies on the lines XA'', YB'', ZC'', where $A'' = B'C' \cap YZ$, $B'' = C'A' \cap ZX$, $C'' = A'B' \cap XY$. (See Fig. 27.)

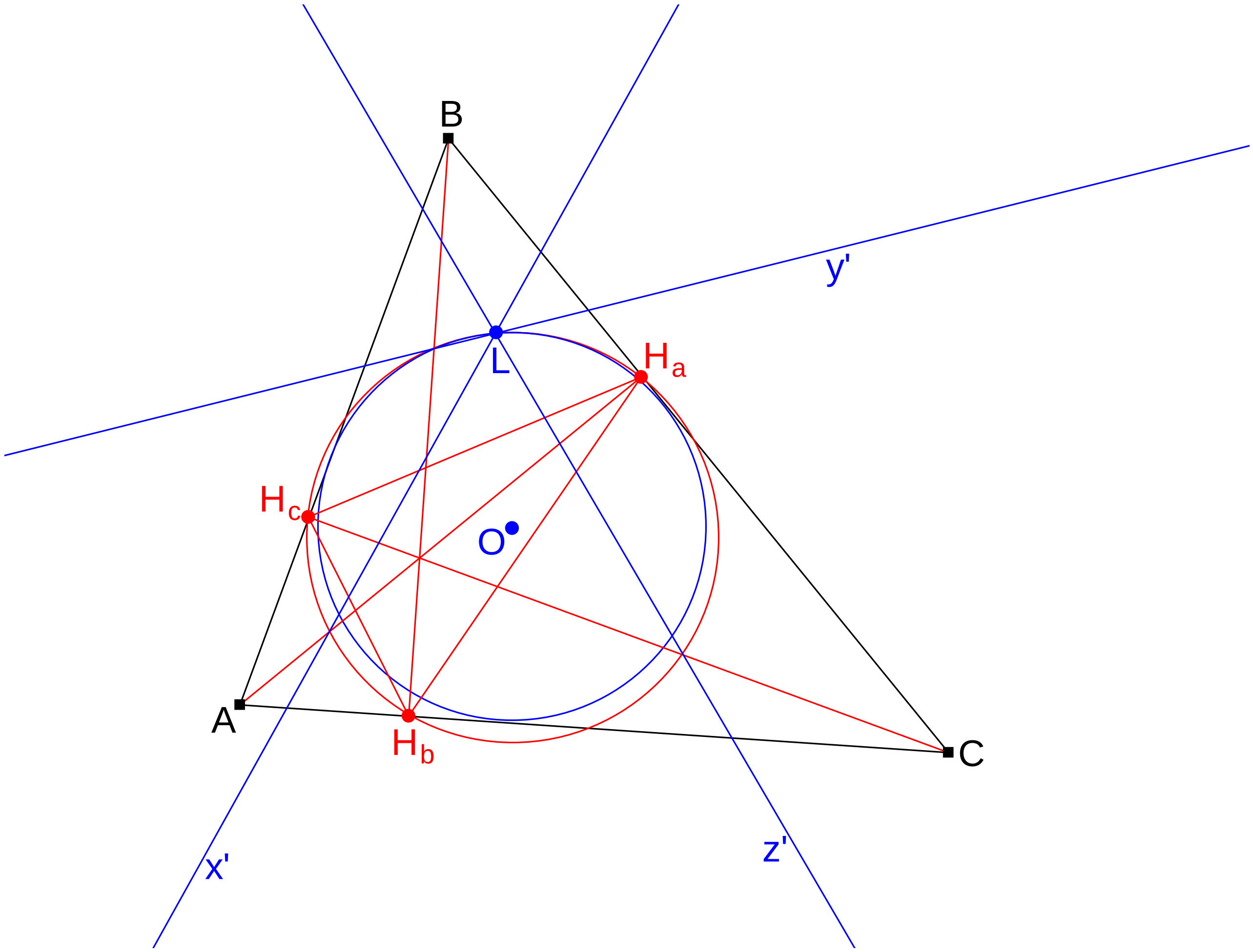

Fig. 28

The lines x', y', z' are the corresponding lines of the diacentral line OU in the triangles AH_bH_c, H_aBH_c, H_aH_bC, thus simply the diacentral lines of these triangles. Hence, as a consequence of Theorem 1.8, we get:

Theorem 3.5: The diacentral lines of the triangles AH_bH_c, H_aBH_c, H_aH_bC pass through the Feuerbach point L of triangle ABC. (See Fig. 28.)

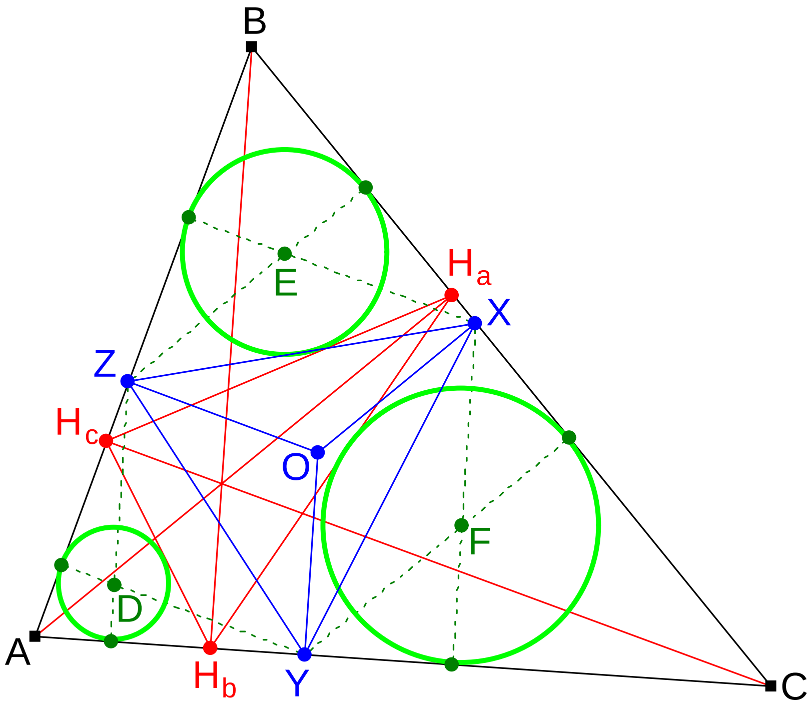

Fig. 29

The points D, E, F are the orthocenters of triangles AYZ, BZX, CXY. The point P is the incenter of ΔABC; hence, the corresponding points of P in triangles AH_bH_c, H_aBH_c, H_aH_bC are the incenters of these triangles. Hence, Theorem 1.9 yields:

Theorem 3.6: The orthocenters D, E, F of triangles AYZ, BZX, CXY are simultaneously the incenters of triangles AH_bH_c, H_aBH_c, H_aH_bC. (See Fig. 29.)

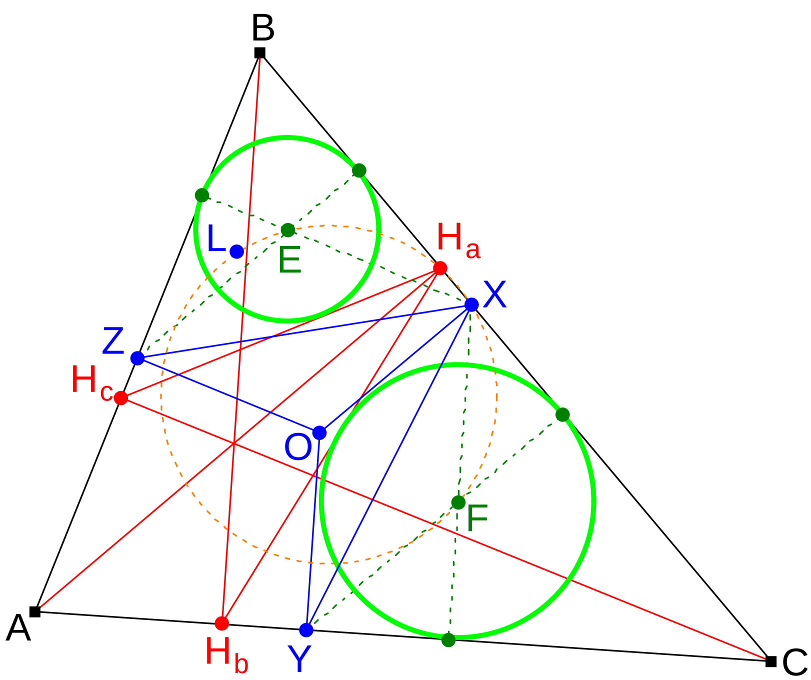

Fig. 30

Finally, we apply Theorem 1.10:

Theorem 3.7: The points X, E, F, H_a and the Feuerbach point L lie on one circle. (See Fig. 30.)

This circle is the image of the circle with diameter AO in the translation by the vector $\overrightarrow{OX}$. If A_n is the center of the circle through X, E, F, H_a, L, and A_m is the center of the circle with diameter AO, then we conclude that A_n is the image of A_m in the translation by the vector $\overrightarrow{OX}$. Thus, $A_mA_n \parallel OX$ and $A_mA_n = OX$, implying that the quadrilateral A_mA_nXO is a parallelogram. Since the diagonals of a parallelogram bisect each other, the midpoint of XA_m is simultaneously the midpoint of OA_n. Denote this midpoint by A_q.

(See Fig. 30a.) Now consider the incircle of ΔABC with center O, the circle through X, E, F, H_a, L with center A_n, and the circle with diameter XA_m with center A_q (remember that A_q is the midpoint of XA_m). All three circles pass through X. The first two of these circles also pass through L; we suspect that the third circle passes through L, too.

In order to prove this, we remember that the common points of two circles are symmetrically placed with respect to the line joining the centers. Hence, the common points X and L of the incircle of ΔABC and the circle through X, E, F, H_a, L are symmetrically placed with respect to the line OA_n; i. e., the point L is the reflection of X in OA_n. But OA_n contains the center A_q of the circle with diameter XA_m (because A_q is the midpoint of OA_n). Since a circle is symmetric with respect to any line through its center, the circle with diameter XA_m must be symmetric with respect to OA_n. Hence, since X lies on this circle, ist reflection L in the line OA_n must lie on this circle, too.

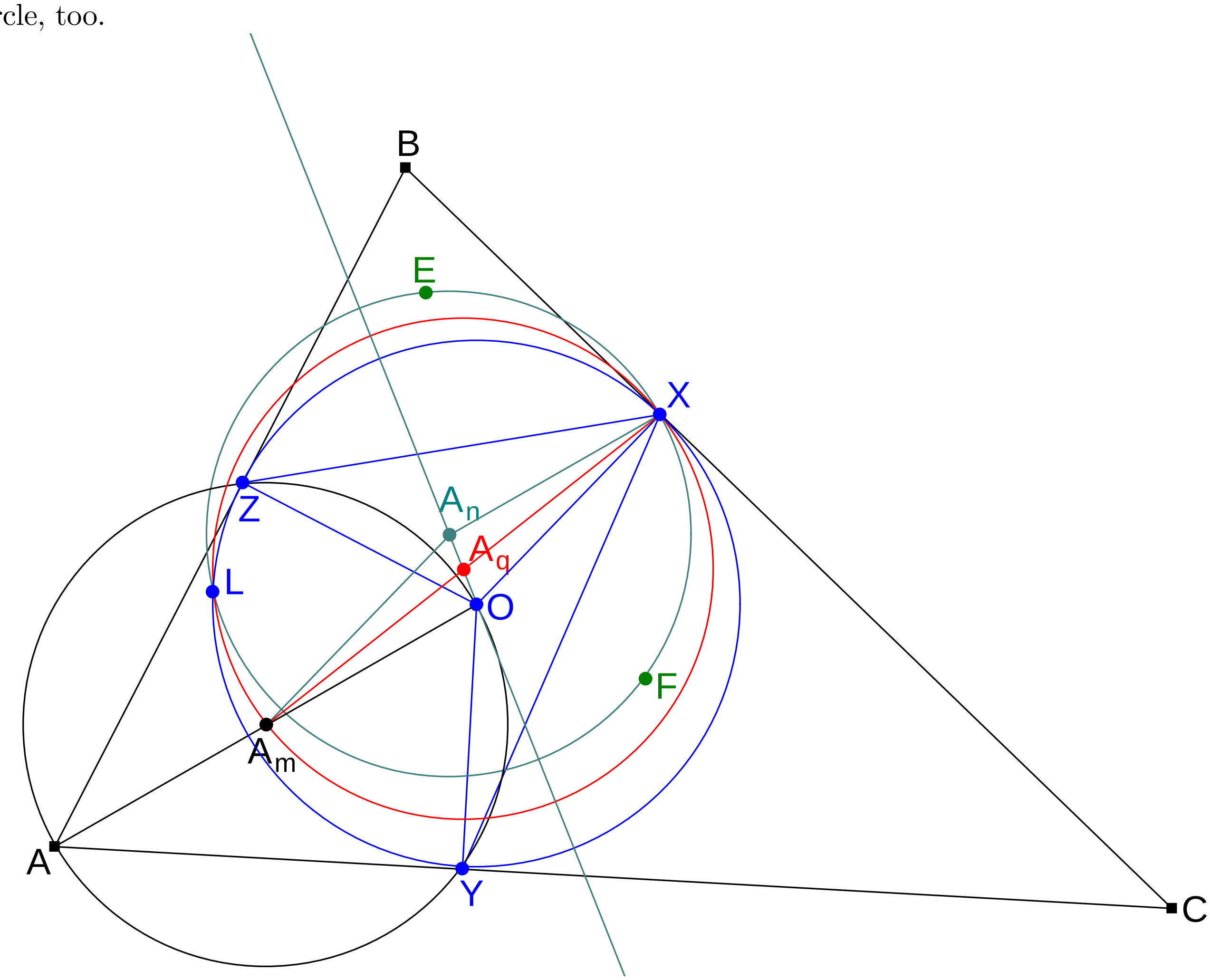

Fig. 30a

We have just shown that L lies on the circle with diameter XA_m. Hereby, A_m is the center of circle with diameter AO, i. e. the midpoint of the segment AO and the circumcenter of triangle OYZ (because the circumcircle of ΔOYZ is the circle with diameter AO, as $\measuredangle AYO = 90^\circ$ and $\measuredangle AZO = 90^\circ$ makes the points Y and Z lie on the circle with diameter AO).

Similarly, we can introduce the midpoints B_m, C_m of the segments BO, CO and show that they are the circumcenters of triangles OZX, OXY, and that L lies on the circles with diameters YB_m and ZC_m. We sum up:

Theorem 3.7a: The midpoints A_m, B_m, C_m of the segments AO, BO, CO are the circumcenters of triangles OYZ, OZX, OXY. The Feuerbach point L of triangle ABC lies on the circles with diameters XA_m, YB_m, ZC_m. (See Fig. 30b.)

This result was communicated to me by Michel Garitte in a somewhat different form.

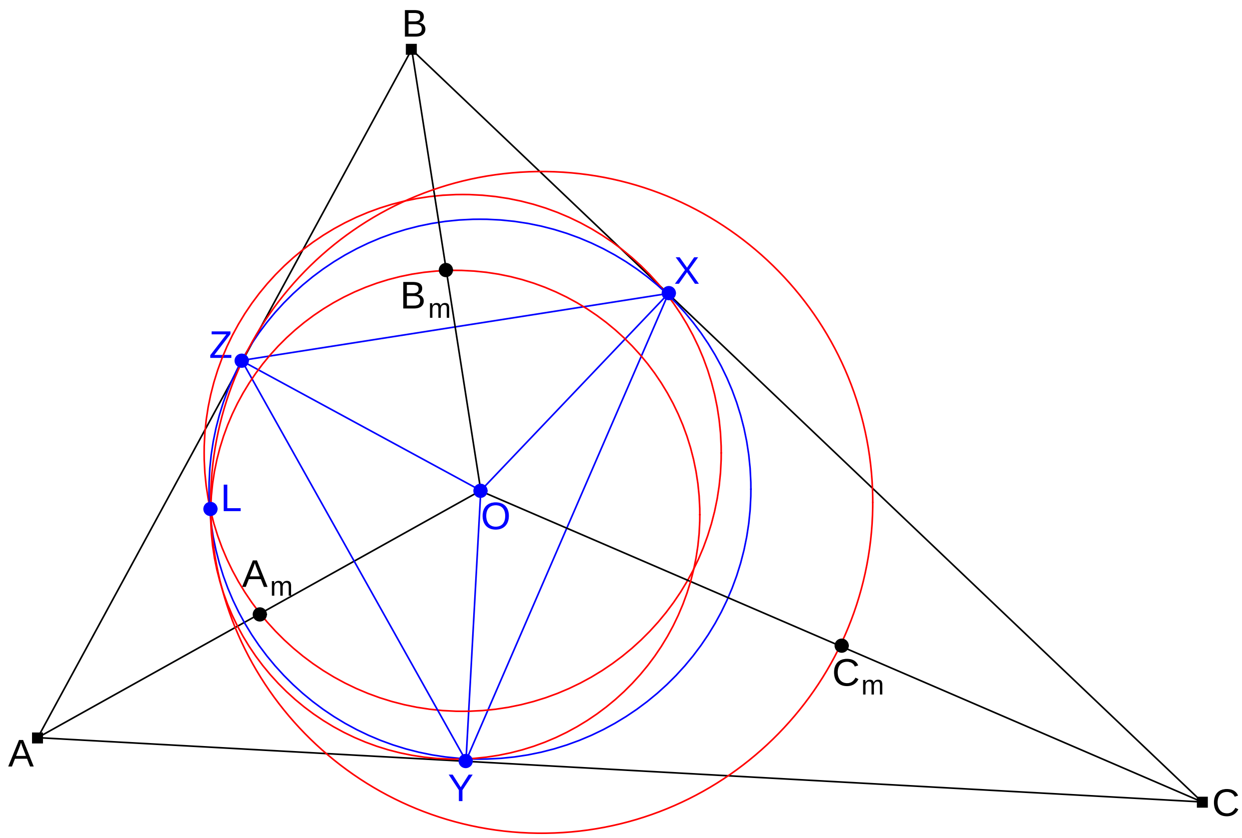

Fig. 30b

(See Fig. 30c.) The point L lying on the circles with diameters XA_m and YB_m, we get $\measuredangle XLA_m = 90^\circ$ and $\measuredangle YLB_m = 90^\circ$, and thus

$$\begin{aligned}\measuredangle A_m L B_m &= \measuredangle A_m LX + \measuredangle XLY + \measuredangle YLB_m = -\measuredangle XLA_m + \measuredangle XLY + \measuredangle YLB_m \\ &= -90^\circ + \measuredangle XLY + 90^\circ = \measuredangle XLY.\end{aligned}$$

Since L lies on the circle XYZ, we have $\measuredangle XLY = \measuredangle XZY$, and $\measuredangle A_m L B_m = \measuredangle XZY = \measuredangle (ZX;\ YZ)$. But the points Y and Z are symmetrically placed with respect to the angle bisector AO of the angle CAB; thus, $YZ \perp AO$, and likewise, $ZX \perp BO$.

Herewith,

$$\begin{aligned}\measuredangle A_m L B_m &= \measuredangle(ZX;\ YZ) = \measuredangle(ZX;\ BO) + \measuredangle(BO;\ AO) + \measuredangle(AO;\ YZ) \\ &= 90^\circ + \measuredangle BOA + 90^\circ = 180^\circ + \measuredangle BOA = \measuredangle BOA.\end{aligned}$$

The triangle $B_m C' A_m$ is formed by the midpoints of the sides of ΔAOB and is therefore the medial triangle of ΔAOB. Since any triangle is directly similar to its medial triangle, triangles AOB and $B_m C' A_m$ are directly similar, and $\measuredangle BOA = \measuredangle A_m C' B_m$, so that the equation above becomes $\measuredangle A_m L B_m = \measuredangle A_m C' B_m$. Hence, the point L lies on the circle $A_m B_m C'$. But for A_m, B_m, C' are the midpoints of the sides of ΔAOB, the circle $A_m B_m C'$ is the nine-point circle of ΔAOB; hence, L lies on the nine-point circle of ΔAOB. Likewise, L lies on the nine-point circles of ΔBOC and ΔCOA.

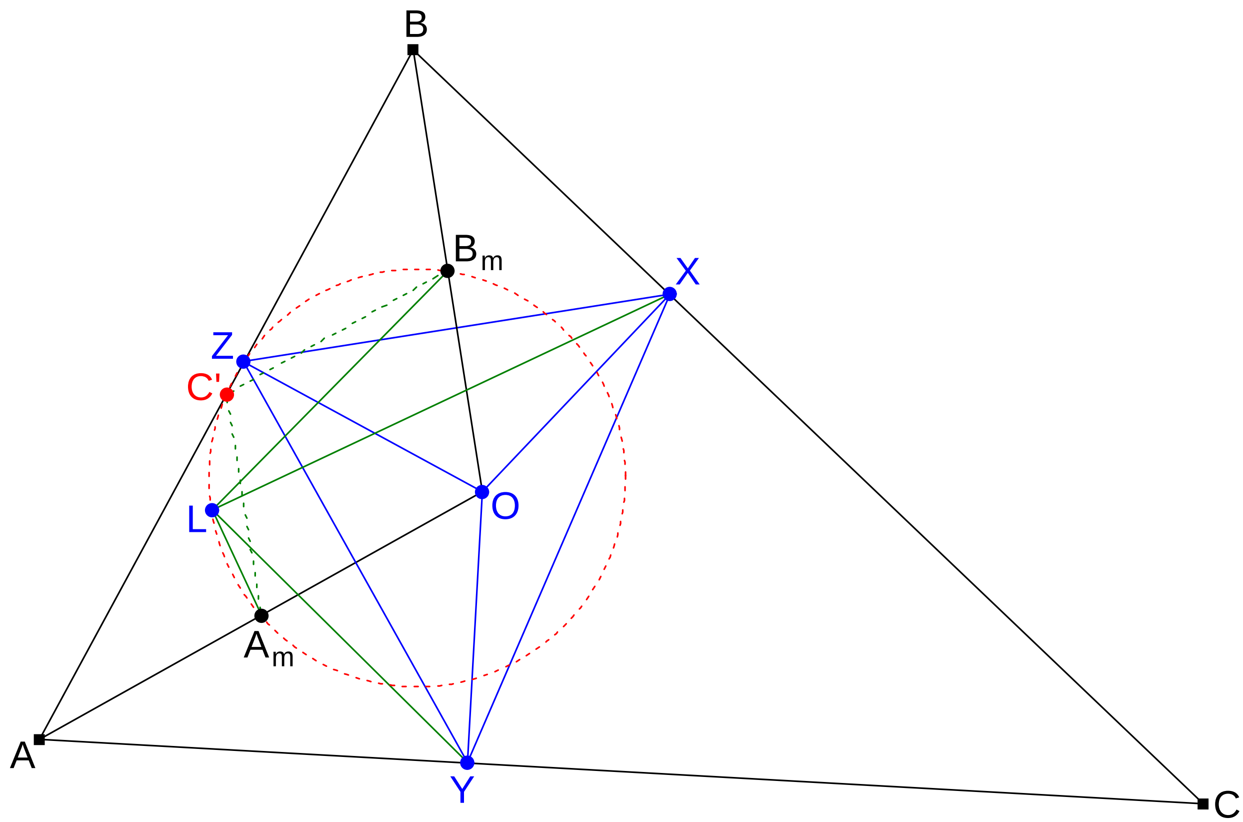

Fig. 30c

We record this result:

Theorem 3.7b: The Feuerbach point L of a triangle ABC lies on the nine-point circles of triangles BOC, COA, AOB, where O is the incenter of triangle ABC. (See Fig. 30d.)

We note in passing that there are simpler ways to establish this theorem.

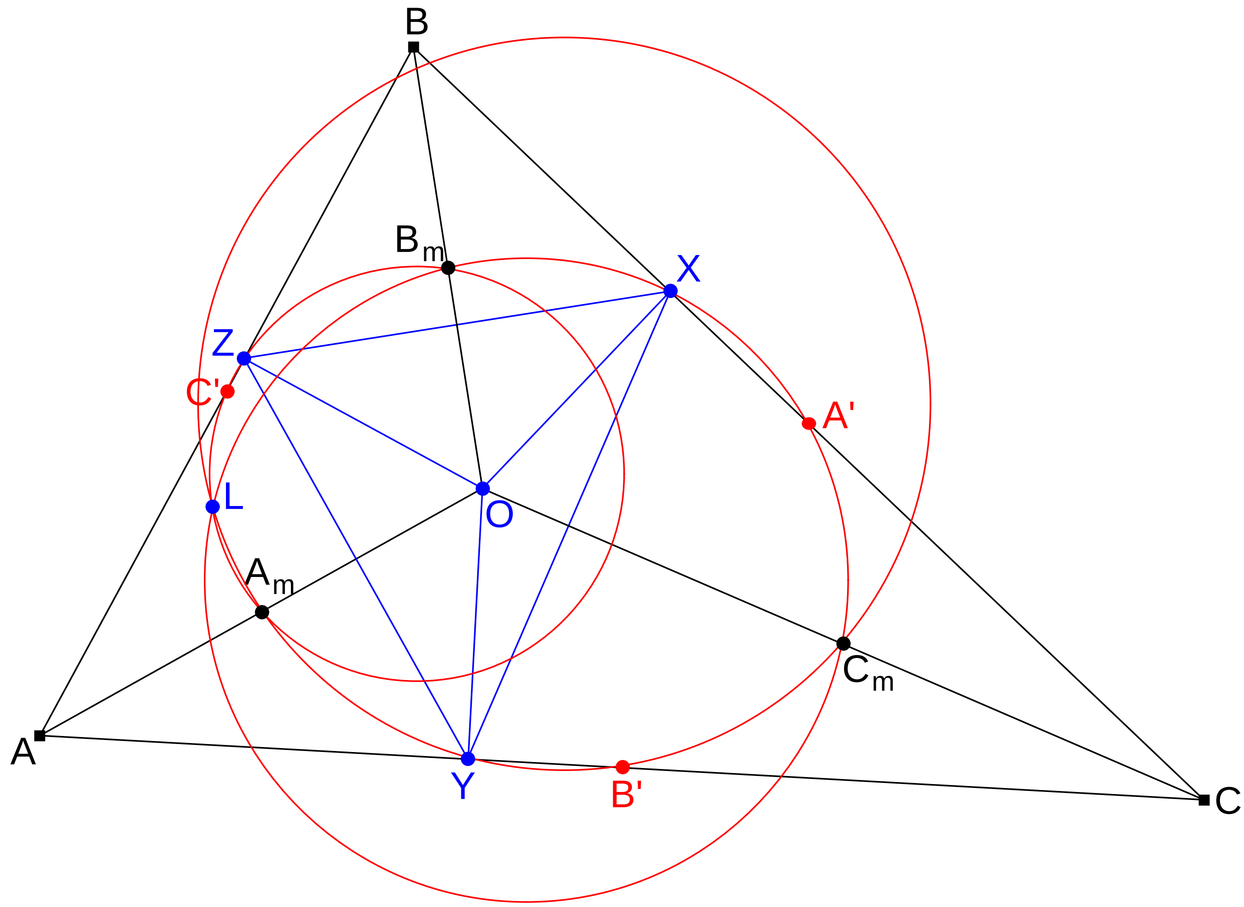

Fig. 30d

We continue with the *Proof of Theorem 3.2:* As D is the orthocenter of ΔAYZ, $YD \perp AB$. On the other hand, $OZ \perp AB$. Thus, $YD \parallel OZ$, and similarly $ZD \parallel OY$, proving the quadrilateral $OYDZ$ a parallelogram. It is even a rhombus (as $OY = OZ$); thus, D is the reflection of O in YZ. Let x_1' be the reflection of the line OU in YZ; then, x_1' passes through D, since OU passes through O.

We also have $\measuredangle (x_1';\ YZ) = -\measuredangle (OU;\ YZ)$ by the definition of x_1'.

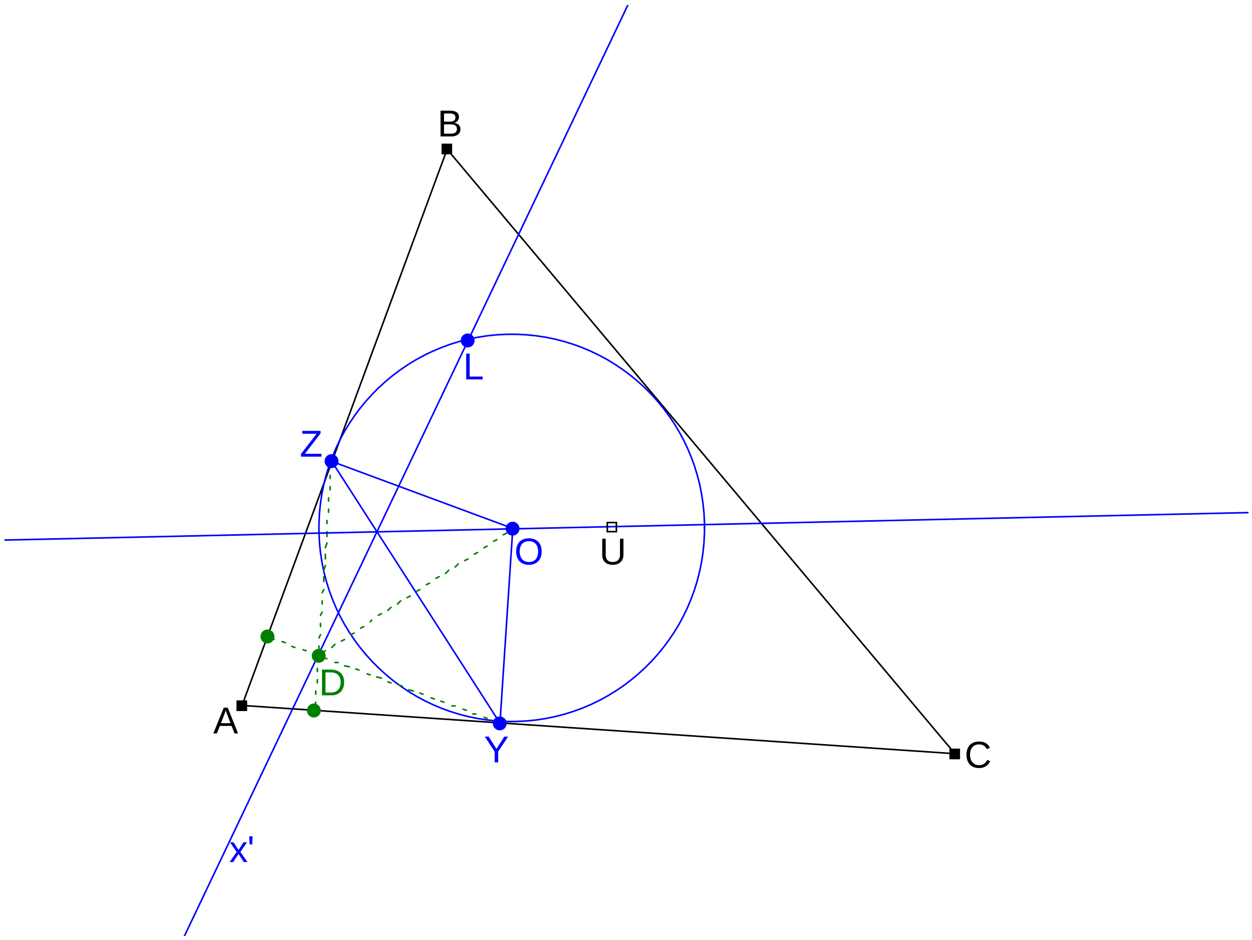

Fig. 31

In the proof of Theorem 1.8, we have shown $\measuredangle(x';\ CA) = -\measuredangle(PU;\ AB)$; with $P = O$, this becomes $\measuredangle(x';\ CA) = -\measuredangle(OU;\ AB)$, and hence

$$\measuredangle(x';\ YZ) = \measuredangle(x';\ CA) + \measuredangle(CA;\ YZ) = -\measuredangle(OU;\ AB) + \measuredangle(CA;\ YZ).$$

By symmetry, $YZ \perp AO$, and since AO is the angle bisector of the angle CAB, we have $\measuredangle(CA;\ AO) = \measuredangle(AO;\ AB)$; thus,

$$\begin{aligned}
\measuredangle(x';\ YZ) &= -\measuredangle(OU;\ AB) + \measuredangle(CA;\ YZ) \\
&= -\measuredangle(OU;\ AB) + \measuredangle(CA;\ AO) + \measuredangle(AO;\ YZ) \\
&= -\measuredangle(OU;\ AB) + \measuredangle(CA;\ AO) + 90^\circ \\
&= -\measuredangle(OU;\ AB) + \measuredangle(AO;\ OB) + 90^\circ \\
&= -\measuredangle(OU;\ AB) + \measuredangle(AO;\ AB) + \measuredangle(YZ;\ AO) \\
&= -\measuredangle(OU;\ YZ) = \measuredangle(x_1';\ YZ).
\end{aligned}$$

Hence, the lines x' and x_1' are parallel. But as they both pass through D [6], they coincide. I. e., the line x' is the reflection of OU in YZ. Similarly, y' and z' are the reflections of OU in ZX and XY. We summarize:

Theorem 3.8: The diacentral lines x', y', z' of the triangles AH_bH_c, H_aBH_c, H_aH_bC are the reflections of the diacentral line OU of ΔABC in the sidelines YZ, ZX, XY of the Gergonne triangle XYZ. (See Fig. 32.)

[6] Being the corresponding point of P in triangle AH_bH_c, D lies on x'.

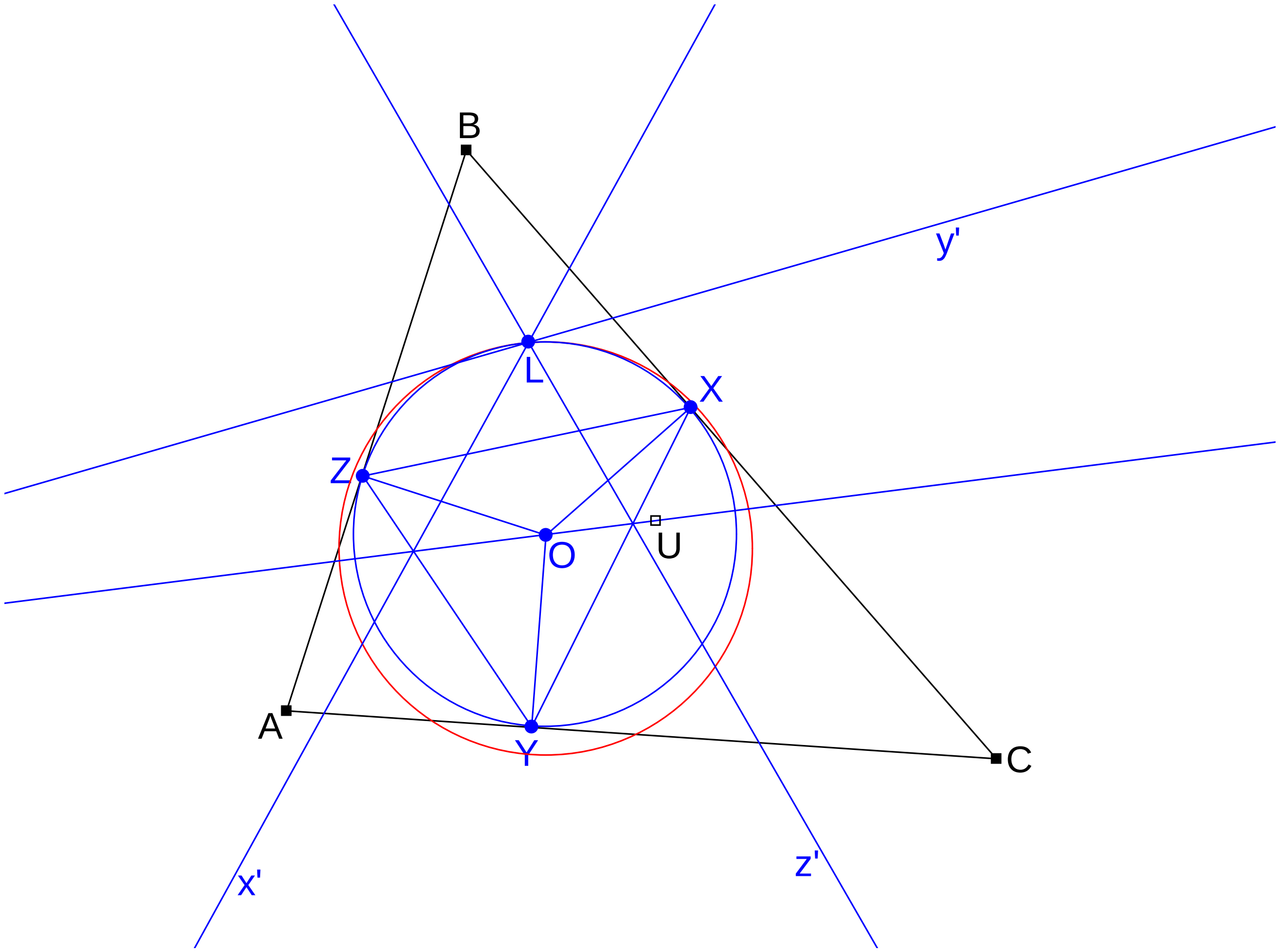

Fig. 32

The triangle ABC is the tangential triangle of ΔXYZ. It is well-known that the Euler line of a triangle passes through the circumcenter of the tangential triangle; hence, the Euler line of ΔXYZ passes through the circumcenter U of ΔABC. On the other hand, this Euler line passes through the circumcenter O of ΔXYZ and hence coincides with the line OU. Thus, we have shown:

Theorem 3.9: The diacentral line OU of triangle ABC is the Euler line of the Gergonne triangle XYZ.

After Theorem 3.8, the lines x', y', z' are the reflections of the Euler line of triangle XYZ in its sidelines YZ, ZX, XY; their meet L is therefore the Euler reflection point of ΔXYZ. We emphasize this result:

Theorem 3.10: The Feuerbach point L of triangle ABC is the Euler reflection point of the Gergonne triangle XYZ.

But we still haven't established Theorem 3.2. Regard the Gergonne triangle XYZ (Fig. 33): The orthocenter H' of ΔXYZ lies on the Euler line of ΔXYZ, i. e. on OU. Consequently, the reflections Q_a, Q_b, Q_c of H' in YZ, ZX, XY lie on the reflections x', y', z' of OU in YZ, ZX, XY. On the other hand, these reflections Q_a, Q_b, Q_c lie on the incircle of ΔABC, since the reflections of the orthocenter of a triangle in its sidelines lie on the circumcircle of the triangle ([2], Lemma 1), and the circumcircle of ΔXYZ is the incircle of ΔABC.

Now, the feet T_a, T_b, T_c of the altitudes of ΔXYZ from X, Y, Z are the midpoints of the segments $H'Q_a$, $H'Q_b$, $H'Q_c$. As a midparallel in $\Delta Q_bH'Q_c$, the line T_bT_c is parallel to Q_bQ_c.

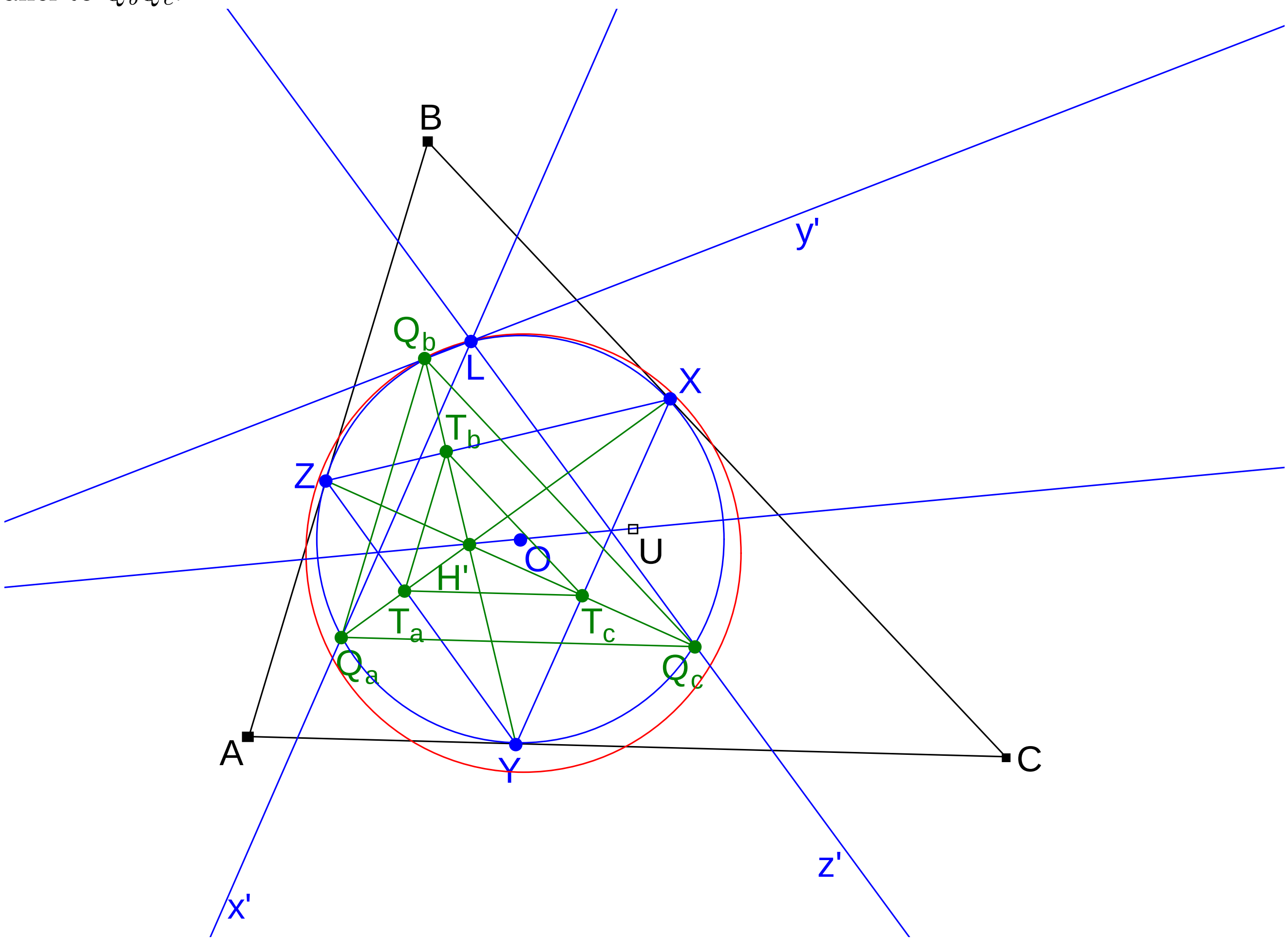

Fig. 33

Note that ΔABC is the tangential triangle of ΔXYZ, and $\Delta T_aT_bT_c$ is the orthic triangle of ΔXYZ. It may be readily shown that the orthic triangle and the tangential triangle of a triangle are homothetic; hence, $\Delta T_aT_bT_c$ and ΔABC are homothetic, and $T_bT_c \parallel BC$. Remembering that $T_bT_c \parallel Q_bQ_c$, we conclude $Q_bQ_c \parallel BC$.

We know that the lines x', y', z' also pass through the midpoints G_a, G_b, G_c of AH, BH, CH. The circumcircle of $\Delta G_aG_bG_c$ is the nine-point circle of ΔABC. As a midparallel in ΔBHC, the line G_bG_c is parallel to BC; together with $Q_bQ_c \parallel BC$, this entails $G_bG_c \parallel Q_bQ_c$. The same reasoning shows $G_cG_a \parallel Q_cQ_a$ and $G_aG_b \parallel Q_aQ_b$. Thus, triangles $G_aG_bG_c$ and $Q_aQ_bQ_c$ are homothetic, and the homothetic center lies on the lines G_aQ_a, G_bQ_b, G_cQ_c. But these lines are simply x', y', z' and intersect at L, making L the homothetic center of $\Delta G_aG_bG_c$ and $\Delta Q_aQ_bQ_c$. The homothety with center L transforming $\Delta G_aG_bG_c$ to $\Delta Q_aQ_bQ_c$ (the factor of this homothety is positive, as the figure shows), maps the circumcircle of $\Delta G_aG_bG_c$ to the circumcircle of $\Delta Q_aQ_bQ_c$, i. e. the nine-point circle of ΔABC to the incircle of ΔABC. Hence, the homothetic center L is the external center of similitude of the nine-point circle and the incircle, and therefore lies on the line joining the centers of these two circles. But L also lies on the two circles themselves; hence, the two circles must touch each other at L. The tangency is internal, since L is the external center of similitude. Theorem 3.2 is proven.

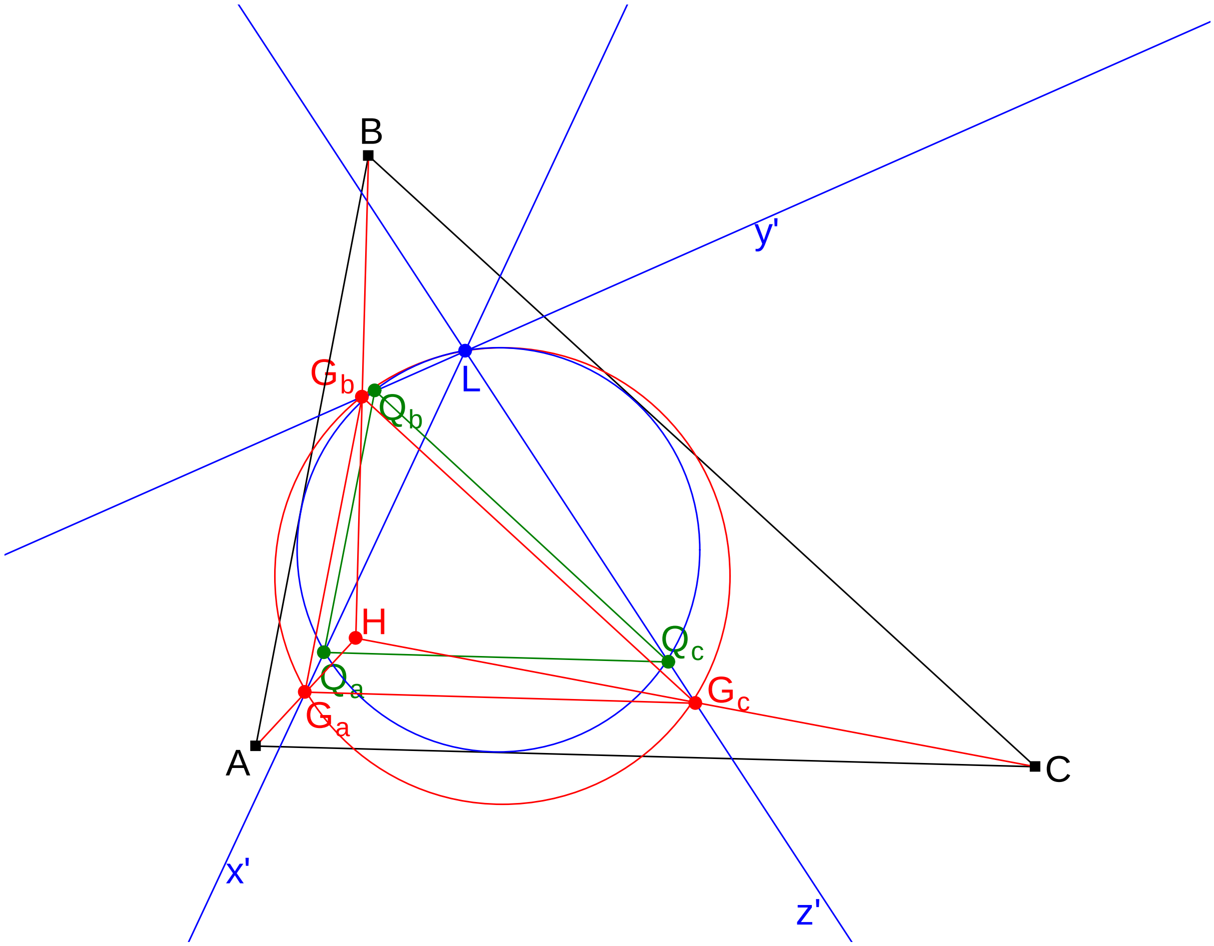

Fig. 34

Mutatis mutandis, any of the three excircles of ΔABC touches the nine-point circle, this time externally. We summarize:

Theorem 3.11: The nine-point circle of triangle ABC touches the incircle internally and any of the three excircles externally.

This result is the **Feuerbach theorem**; a lot of proofs are known, but this one seems to be new.

<u>**Generalization of the Feuerbach theorem**</u>

In this last paragraph, we will establish a generalization of the Feuerbach theorem attributed to V. Ramaswamy Aiyer, the **Aiyer theorem**. Again it deals with an *arbitrary* point P in the plane of ΔABC, the pedal triangle XYZ of P and the pedal circle XYZ.

Before stating the theorem I make a little remark: If we speak of an angle between two circles, we mean the angle between the tangents to the two circles at one of their common points. This angle is defined *except for its sign*, since the angles at the two common points are oppositely equal to each other: If two circles k and k' intersect at P and P', then the (directed) angle between the tangents to k and k' at P is oppositely equal to the (directed) angle between the tangents to k and k' at P'. But we can also speak of the angle between the circles k and k' *in the common point* P; this angle is uniquely defined as the directed angle between the tangents to k and k' at P.

Now we can state the **Aiyer theorem**:

Theorem 4.1: The angle between the nine-point circle of triangle ABC and the pedal circle of the point P in their common point L is $90° - \measuredangle PBC - \measuredangle PCA - \measuredangle PAB$.

Remark. This expression is obviously asymmetric, meaning that we can show as well that the angle between the nine-point circle and the pedal circle of P in their common point L is $90° - \measuredangle PCB - \measuredangle PAC - \measuredangle PBA$. However, it can be easily verified that $90° - \measuredangle PBC - \measuredangle PCA - \measuredangle PAB = 90° - \measuredangle PCB - \measuredangle PAC - \measuredangle PBA$. (See Fig. 35.)

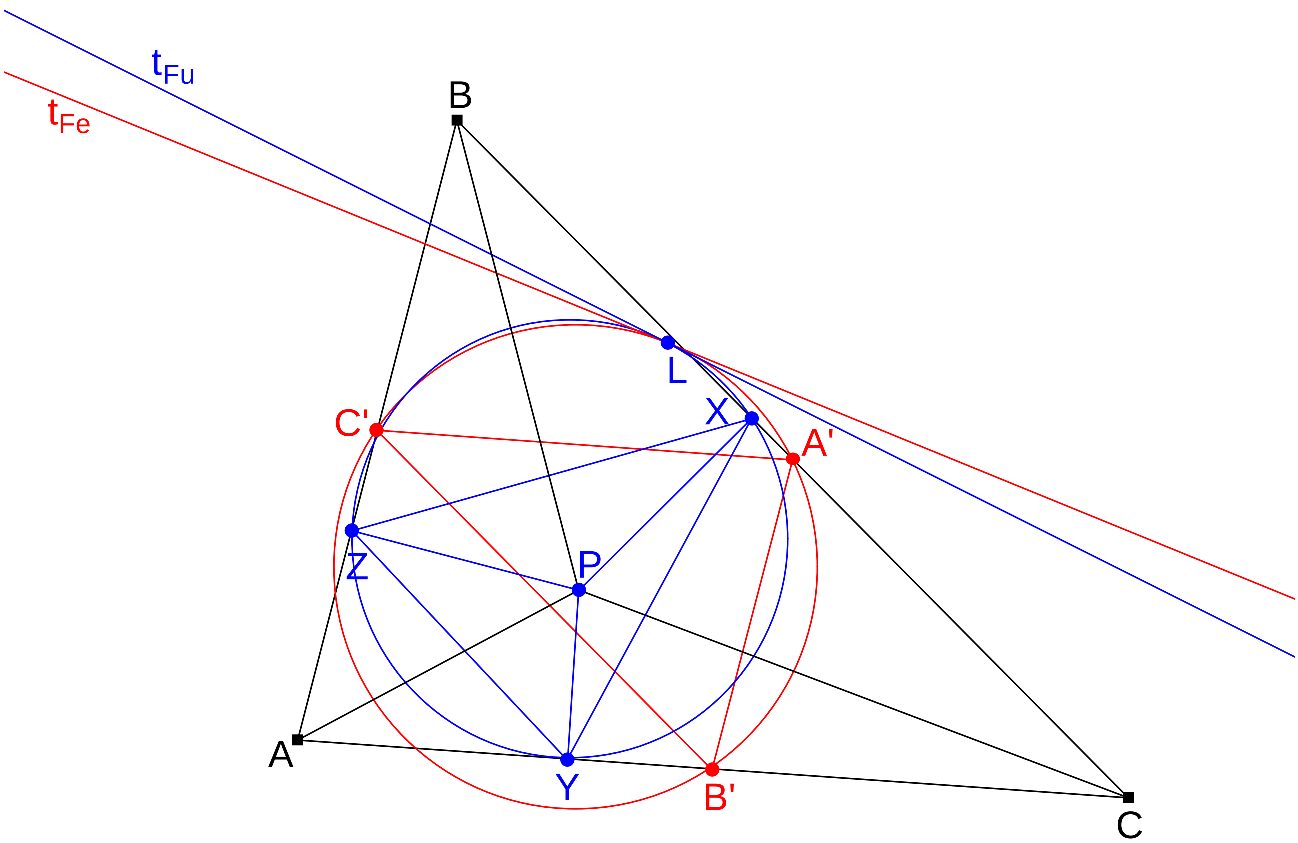

Fig. 35

Proof of Theorem 4.1. Let t_{Fe} and t_{Fu} be the tangents to the nine-point circle and to the pedal circle of P in the point L, respectively. We have to prove that

$$\measuredangle (t_{Fe};\ t_{Fu}) = 90° - \measuredangle PBC - \measuredangle PCA - \measuredangle PAB.$$

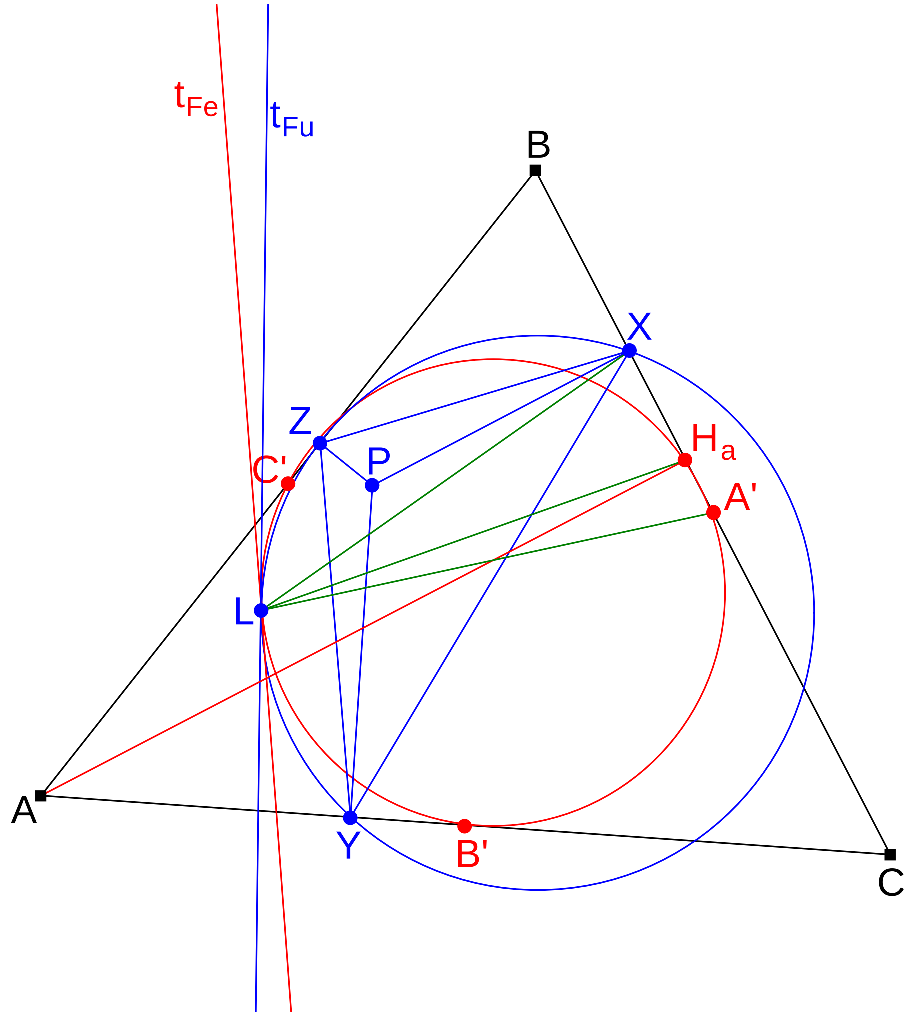

Fig. 36

(See Fig. 36.) We have obviously

$$\measuredangle(t_{Fe};\ t_{Fu}) = \measuredangle(t_{Fe};\ A'L) + \measuredangle(A'L;\ XL) + \measuredangle(XL;\ t_{Fu}).$$

According to the theorem about the tangent-chordal angle, we have $\measuredangle(t_{Fe};\ A'L) = \measuredangle(LH_a;\ H_aA')$ in the nine-point circle, and $\measuredangle(XL;\ t_{Fu}) = \measuredangle(XZ;\ ZL)$ in the pedal circle. Therefore,

$$\begin{aligned}
\measuredangle(t_{Fe};\ t_{Fu}) &= \measuredangle(LH_a;\ H_aA') + \measuredangle(A'L;\ XL) + \measuredangle(XZ;\ ZL)\\
&= \measuredangle(LH_a;\ BC) + \measuredangle(A'L;\ XL) + \measuredangle(ZX;\ ZL)\\
&= \measuredangle(LH_a;\ BC) + (\measuredangle(A'L;\ LH_a) + \measuredangle(LH_a;\ XL))\\
&\quad + (\measuredangle(ZX;\ BC) + \measuredangle(BC;\ XL) + \measuredangle(XL;\ ZL))\\
&= \measuredangle(LH_a;\ BC) + \measuredangle A'LH_a + \measuredangle(LH_a;\ XL)\\
&\quad + \measuredangle ZXB + \measuredangle(BC;\ XL) + \measuredangle XLZ\\
&= (\measuredangle(LH_a;\ BC) + \measuredangle(BC;\ XL)) + \measuredangle(LH_a;\ XL)\\
&\quad + \measuredangle A'LH_a + \measuredangle ZXB + \measuredangle XLZ\\
&= 2\cdot\measuredangle(LH_a;\ XL) + \measuredangle A'LH_a + \measuredangle ZXB + \measuredangle XLZ.
\end{aligned}$$

Since L lies on the circle XYZ, $\measuredangle XLZ = \measuredangle XYZ$, and since L lies on the nine-point circle of ΔABC, $\measuredangle A'LH_a = \measuredangle A'C'H_a$. Herewith,

$$\measuredangle(t_{Fe};\ t_{Fu}) = 2\cdot\measuredangle(LH_a;\ XL) + \measuredangle A'C'H_a + \measuredangle ZXB + \measuredangle XYZ. \tag{1}$$

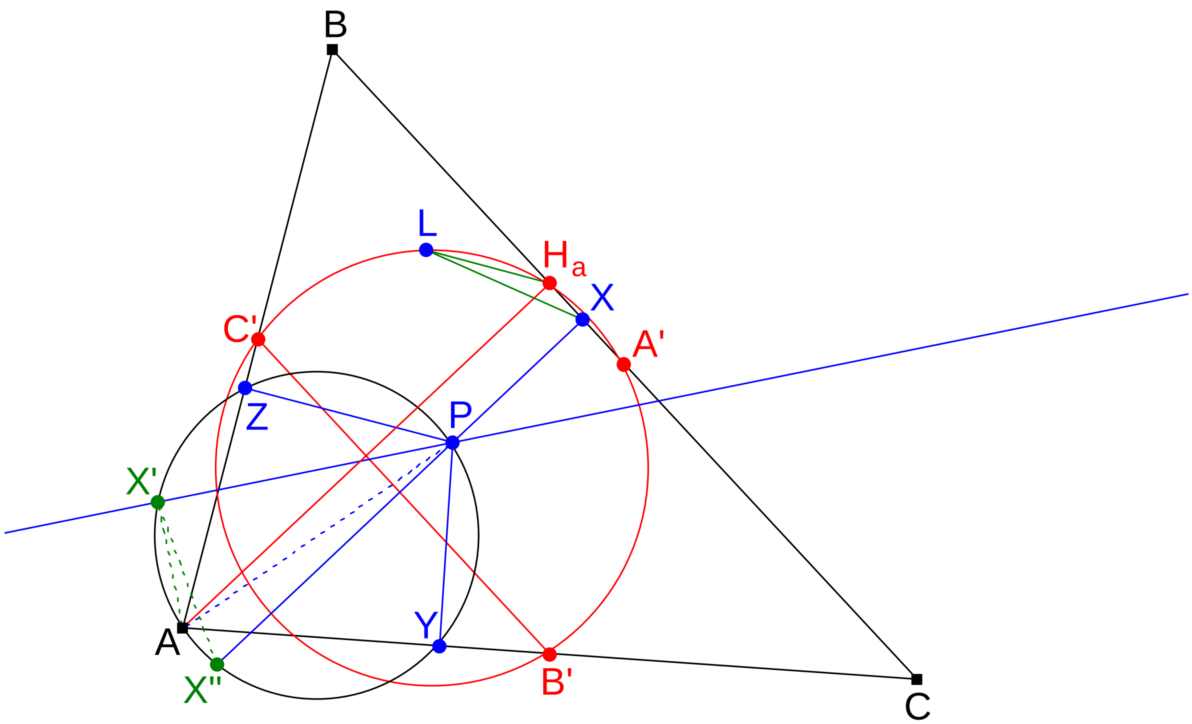

Fig. 37

(See Fig. 37.) The point H_a is the reflection of A in $B'C'$; i. e., A is the reflection of H_a in $B'C'$. We also know that X' and X'' are the reflections of L and X in $B'C'$. Hence, the lines $X'A$ and $X''X'$ are the reflections of LH_a and XL in $B'C'$, what yields $\measuredangle(X'A;\ B'C') = -\measuredangle(LH_a;\ B'C')$ and $\measuredangle(X''X';\ B'C') = -\measuredangle(XL;\ B'C')$, therefore

$$\begin{aligned}
\measuredangle(X'A;\ X''X') &= \measuredangle(X'A;\ B'C') - \measuredangle(X''X';\ B'C') \\
&= (-\measuredangle(LH_a;\ B'C')) - (-\measuredangle(XL;\ B'C')) \\
&= \measuredangle(B'C';\ LH_a) + \measuredangle(XL;\ B'C') = -\measuredangle(LH_a;\ XL),
\end{aligned}$$

hence

$$\measuredangle(LH_a;\ XL) = -\measuredangle(X'A;\ X''X') = \measuredangle X''X'A.$$

For the points X' and X'' lying on the circle with diameter AP, we get $\measuredangle X''X'A = \measuredangle X''PA$, and

$$\begin{aligned}
\measuredangle(LH_a;\ XL) &= \measuredangle X''PA = \measuredangle(PX;\ AP) = \measuredangle(PX;\ BC) + \measuredangle(BC;\ AP) \\
&= 90^\circ + \measuredangle(BC;\ AP), \qquad \text{thus} \\
2\cdot\measuredangle(LH_a;\ XL) &= 2\cdot(90^\circ + \measuredangle(BC;\ AP)) = 180^\circ + 2\cdot\measuredangle(BC;\ AP) = 2\cdot\measuredangle(BC;\ AP).
\end{aligned}$$

Hence, (1) becomes

$$\measuredangle(t_{Fe};\ t_{Fu}) = 2\cdot\measuredangle(BC;\ AP) + \measuredangle A'C'H_a + \measuredangle ZXB + \measuredangle XYZ. \qquad (2)$$

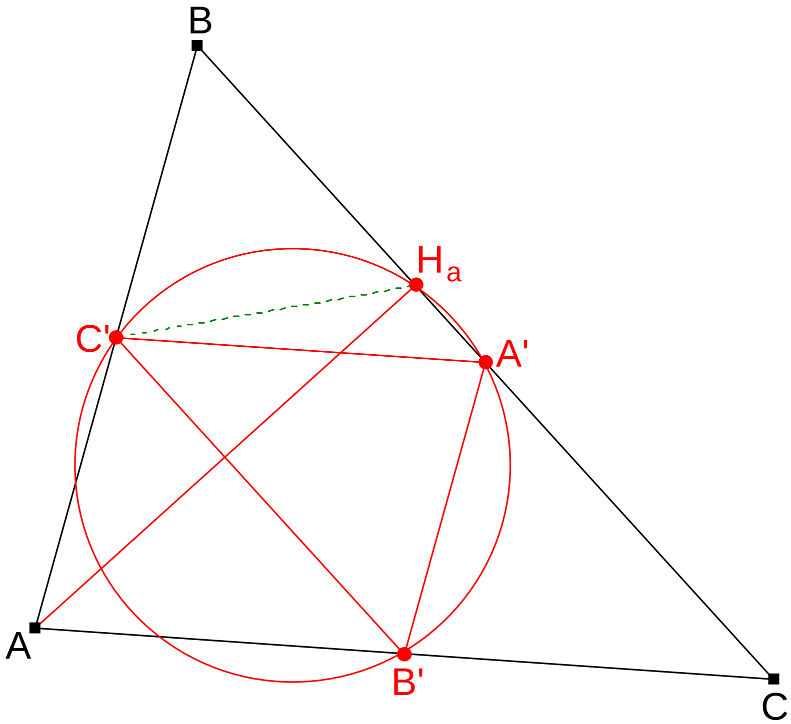

Fig. 38

It remains to calculate $\measuredangle A'C'H_a$ (Fig. 38). Since the circumcenter of a right-angled triangle is the midpoint of the hypotenuse, the midpoint C' of AB is the circumcenter of the right-angled ΔAH_aB; thus, $C'B = C'H_a$, and the isosceles triangle $BC'H_a$ yields $\measuredangle C'H_aB = \measuredangle H_aBC'$. On the other hand, $\measuredangle BC'H_a + \measuredangle C'H_aB + \measuredangle H_aBC' = 0°$, and we obtain $\measuredangle BC'H_a = -\measuredangle C'H_aB - \measuredangle H_aBC' = -2 \cdot \measuredangle H_aBC' = -2 \cdot \measuredangle CBA = 2 \cdot \measuredangle ABC$.

Since $C'A' \parallel CA$, we have $\measuredangle A'C'B = \measuredangle CAB$. Therewith,

$$\begin{aligned} \measuredangle A'C'H_a &= \measuredangle A'C'B + \measuredangle BC'H_a = \measuredangle CAB + 2 \cdot \measuredangle ABC \\ &= (\measuredangle CAB + \measuredangle ABC + \measuredangle BCA) + \measuredangle ABC - \measuredangle BCA \\ &= 0° + \measuredangle ABC - \measuredangle BCA = \measuredangle ABC - \measuredangle BCA. \end{aligned}$$

Substituting in (2),

$$\measuredangle (t_{Fe};\ t_{Fu}) = 2 \cdot \measuredangle (BC;\ AP) + (\measuredangle ABC - \measuredangle BCA) + \measuredangle ZXB + \measuredangle XYZ.$$

Due to $\measuredangle BZP = 90°$ and $\measuredangle BXP = 90°$, the points Z and X lie on the circle with diameter BP; hence,

$$\measuredangle ZXB = \measuredangle ZPB = \measuredangle (PZ;\ BP) = \measuredangle (PZ;\ AB) + \measuredangle (AB;\ BP) = 90° + \measuredangle (AB;\ BP).$$

Analogously, $\measuredangle XYC = 90° + \measuredangle (BC;\ CP)$ and $\measuredangle ZYA = 90° + \measuredangle (BA;\ AP)$, leading to

$$\begin{aligned} \measuredangle XYZ &= \measuredangle XYC + \measuredangle CYZ = \measuredangle XYC - \measuredangle ZYA \\ &= (90° + \measuredangle (BC;\ CP)) - (90° + \measuredangle (BA;\ AP)) \\ &= \measuredangle (BC;\ CP) - \measuredangle (BA;\ AP). \end{aligned}$$

Thus,

$$\begin{aligned}
\measuredangle\left(t_{Fe};\ t_{Fu}\right) &= 2\cdot\measuredangle\left(BC;\ AP\right)+\left(\measuredangle ABC-\measuredangle BCA\right)+\measuredangle ZXB+\measuredangle XYZ \\
&= 2\cdot\measuredangle\left(BC;\ AP\right)+\left(\measuredangle ABC-\measuredangle BCA\right) \\
&\quad +\left(90^\circ+\measuredangle\left(AB;\ BP\right)\right)+\left(\measuredangle\left(BC;\ CP\right)-\measuredangle\left(BA;\ AP\right)\right) \\
&= 2\cdot\measuredangle\left(BC;\ AP\right)+\measuredangle\left(AB;\ BC\right)-\measuredangle\left(BC;\ CA\right) \\
&\quad +90^\circ+\measuredangle\left(AB;\ BP\right)+\measuredangle\left(BC;\ CP\right)-\measuredangle\left(AB;\ AP\right) \\
&= \left(\measuredangle\left(BC;\ AP\right)+\measuredangle\left(AB;\ BC\right)\right)+\measuredangle\left(BC;\ AP\right) \\
&\quad +\left(\measuredangle\left(BC;\ CP\right)-\measuredangle\left(BC;\ CA\right)\right) \\
&\quad +90^\circ+\left(\measuredangle\left(AB;\ BP\right)-\measuredangle\left(AB;\ AP\right)\right) \\
&= \measuredangle\left(AB;\ AP\right)+\measuredangle\left(BC;\ AP\right)+\measuredangle\left(CA;\ CP\right)+90^\circ+\measuredangle\left(AP;\ BP\right) \\
&= \measuredangle\left(AB;\ AP\right)+\left(\measuredangle\left(BC;\ AP\right)+\measuredangle\left(AP;\ BP\right)\right)+\measuredangle\left(CA;\ CP\right)+90^\circ \\
&= \measuredangle\left(AB;\ AP\right)+\measuredangle\left(BC;\ BP\right)+\measuredangle\left(CA;\ CP\right)+90^\circ \\
&= \measuredangle BAP+\measuredangle CBP+\measuredangle ACP+90^\circ \\
&= 90^\circ-\measuredangle PBC-\measuredangle PCA-\measuredangle PAB,
\end{aligned}$$

qed..

Two direct corollaries of Theorem 4.1 should be mentioned. First, the nine-point circle and the pedal circle are orthogonal if and only if the angle between them is 90°, i. e. $90^\circ-\measuredangle PBC-\measuredangle PCA-\measuredangle PAB=90^\circ$, i. e. $\measuredangle PBC+\measuredangle PCA+\measuredangle PAB=0^\circ$. We record this result:

Theorem 4.2: The nine-point circle and the pedal circle of P are orthogonal if and only if $\measuredangle PBC+\measuredangle PCA+\measuredangle PAB=0^\circ$.

The nine-point circle touches the pedal circle of P if and only if the angle between the two circles is 0°, i. e. $90^\circ-\measuredangle PBC-\measuredangle PCA-\measuredangle PAB=0^\circ$, i. e. $\measuredangle PBC+\measuredangle PCA+\measuredangle PAB=90^\circ$. We record this, too:

Theorem 4.3: The nine-point circle touches the pedal circle of P if and only if $\measuredangle PBC+\measuredangle PCA+\measuredangle PAB=90^\circ$.

This is a generalization of the Feuerbach theorem. In fact, for the incenter O of triangle ABC, we have $\measuredangle OBC+\measuredangle OCA+\measuredangle OAB=90^\circ$, what can be easily proved:

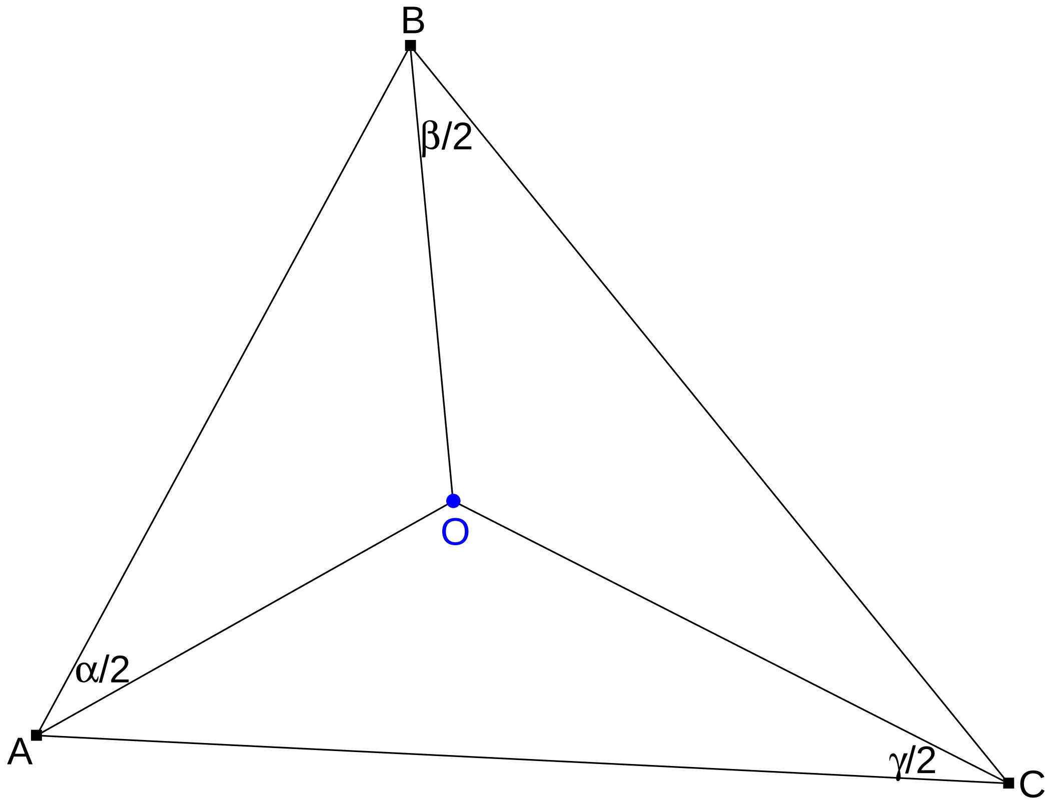

Fig. 39

If α, β, γ are the Euclidean (non-directed) angles of ΔABC, then $\measuredangle OBC = \frac{\beta}{2}$, $\measuredangle OCA = \frac{\gamma}{2}$, $\measuredangle OAB = \frac{\alpha}{2}$, thus $\measuredangle OBC + \measuredangle OCA + \measuredangle OAB = \frac{\alpha + \beta + \gamma}{2} = \frac{180^\circ}{2} = 90^\circ$ with Euclidean angles. Now, the arrangement of points (Fig. 39) makes clear that this equation $\measuredangle OBC + \measuredangle OCA + \measuredangle OAB = 90^\circ$ holds for directed angles modulo 180°, too.[7] According to Theorem 4.3, this indicates that the nine-point circle touches the pedal circle of O, i. e. the incircle of ΔABC. Similar reasoning proves the same for the excircles. This proves Theorem 3.11 again.

References

[1] S. N. Collings: *Reflections on a triangle 1*, Mathematical Gazette 1973, pages 291-293.

[2] Darij Grinberg: *Anti-Steiner points with respect to a triangle.*

[3] Milorad Stevanovic: *Hyacinthos message #6563.*

[4] Victor Thebault, (somebody else?): *Solution of Problem 4328*, American Mathematical Monthly 1951, page 45.

[7] This equation - signifying that the sum of the angles between the three internal angle bisectors of the triangle and adjacent triangle sides in cyclic order is 90° - plays a crucial role in the geometry of directed angles modulo 180°. This equation cannot be shown without the use of Euclidean angles; in fact, directed angles modulo 180° don't allow distinguishing between internal and external angle bisectors, but if we replace one of the internal angle bisectors by an external one, the sum of the angles will be 0°.