Problem set 2

Combi Geo

1) Prove that in every polygon there is a diagonal that cuts off a triangle and lies within the polygon

2) In an isosceles right-angled triangle shaped billiards table , a ball starts moving from one of the vertices adjacent to hypotenuse. When it reaches to one side then it will reflect its path. Prove that if we reach to a vertex then it is not the vertex at initial position

3) Prove that there exists a set S of $n-2$ points inside a convex polygon P with n sides, such that any triangle determined by 3 vertices of P contains exactly one point from S inside or on the boundaries

Poly

1) Let P, Q be two monic polynomials with complex coefficients such that $P(P(x)) = Q(Q(x))$ for all x. Prove that $P = Q$.

2) Let $f(x)$ be a real function such that for each positive real c there exist a polynomial $P(x)$ (maybe dependent on c) such that $|f(x) - P(x)| \leq c \cdot x^{1998}$ for all real x. Prove that f is a real polynomial.

3) Find all values of the positive integer m such that there exists polynomials $P(x), Q(x), R(x, y)$ with real coefficient satisfying the condition: For every real numbers a, b which satisfying $a^m - b^2 = 0$, we always have that $P(R(a, b)) = a$ and $Q(R(a, b)) = b$

NT

1) Show that: a) There are infinitely many positive integers n such that there exists a square equal to the sum of the squares of n consecutive positive integers (for instance, 2 is one such number as $5^2 = 3^2 + 4^2$). b) If n is a positive integer which is not a perfect square, and if x is an integer number such that $x^2 + (x+1)^2 + ... + (x+n-1)^2$ is a perfect square, then there are infinitely many positive integers y such that $y^2 + (y+1)^2 + ... + (y+n-1)^2$ is a perfect square.

2) Let p be an odd prime and $a_1, a_2, ..., a_p$ be integers. Prove that the following two conditions are equivalent:

a) There exists a polynomial $P(x)$ with degree $\leq \frac{p-1}{2}$ such that $P(i) \equiv a_i \pmod{p}$ for all $1 \leq i \leq p$

b) For any natural $d \leq \frac{p-1}{2}$,

$$\sum_{i=1}^{p} (a_{i+d} - a_i)^2 \equiv 0 \pmod{p}$$

where indices are taken $\pmod{p}$

3) Let $m,\ n \geq 3$ be positive odd integers. Prove that $2^m - 1$ doesn't divide $3^n - 1$.

Geo

1) In an acute $\triangle ABC$ $(AB \neq AC)$ with angle α at the vertex A, point E is the nine-point center, and P a point on the segment AE. If $\angle ABP = \angle ACP = x$, prove that $x = 90\ - 2\alpha$

2) Let P and Q be points inside triangle ABC satisfying $\angle PAC = \angle QAB$ and $\angle PBC = \angle QBA$.

a) Prove that feet of perpendiculars from P and Q on the sides of triangle ABC are concyclic.

b) Let D and E be feet of perpendiculars from P on the lines BC and AC and F foot of perpendicular from Q on AB. Let M be intersection point of DE and AB. Prove that $MP \perp CF$.