About two geometry problems in IMO year 2015

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Abstract

The article resolve and gives the ideas for expanding and applications of the geometry problems in IMO year 2015.

Contents

1 Geometry problem on the first day

1.1 Introduction

The IMO exam on first day in the year 2015 [1] has interesting geometric problem as following

Problem 1.1. Let the acute triangle ABC inscribed in the circle (O) with the orthocenter H, the altitude AF and M is the midpoint of BC. The circle with the diameter HA cuts (O) at Q differently from A. The circle with the diameter HQ cuts (O) at K differently from Q. Prove that the circumcircles of the triangles KHQ and KFM touch each other.

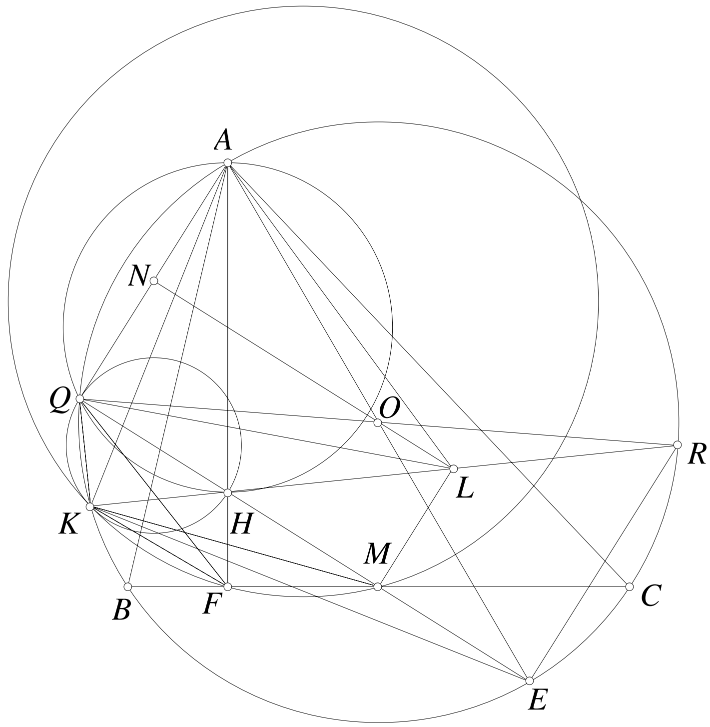

Figure 1.

First proof. Draw the diameters AE and QR of (O). L, N are the midpoints of HR, QA respectively. We have the familiar result Q, H, M, E collinear. Then deduce $ML \parallel ER \parallel QA$ v $ML = \dfrac{1}{2}ER = \dfrac{1}{2}QA = NQ$ so $NQML$ is retangular. Thence, $LN \perp QA$. Then we have $LA = LQ$. On the other hand, $ML \parallel QA$ so ML touches the circumcircles $\triangle LAQ$ at L (1). Easily seen $HA.HF = HK.HL = HQ.HM$ so the inverse with center H pow of point $\overline{HA}.\overline{HF}$ change $M \to Q, L \mapsto K, A \mapsto F, Q \mapsto M$. From (1) combine the inversation with center H deduce the circumcircles $\triangle KHQ, \triangle KFM$ touch each other. $\square$

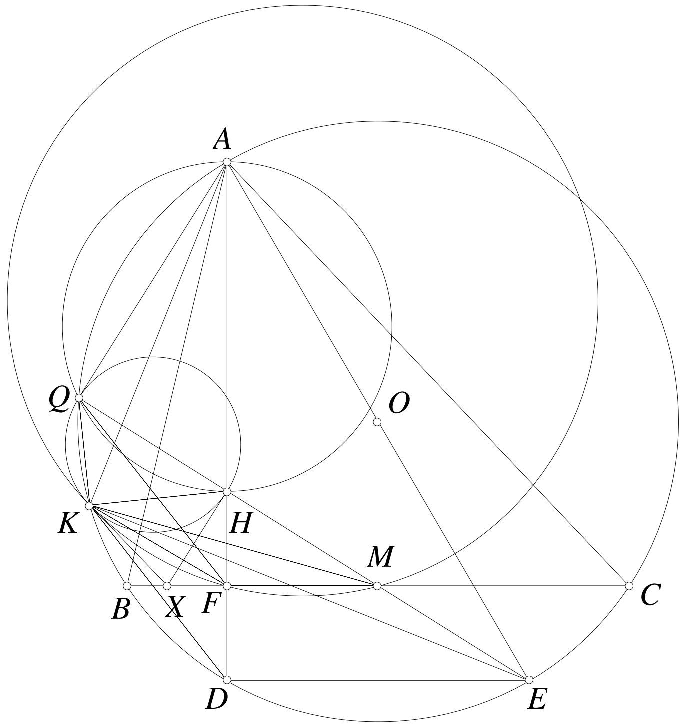

Figure 2.

Second proof. Call by AE the diameter of (O) and D is reflection of H through BC then D is on (O). Easily seen Q, H, M, E collinear. The tangents at K, H that the circumcircles of the triangles KHQ cut each other at X. We have $\angle KXH = 180^\circ - 2\angle KHX = 180^\circ - 2\angle KQH = 2(90^\circ - \angle KQH) = 2(90^\circ - \angle KAE) = 2\angle AEK = 2\angle KDH$. And have $XK = XH$, then X is the center of the circumcircles of the triangles KDH. As BC is perpendicular bisector of HD so X is on BC. Then according to trigonometry we have $XK^2 = XH^2 = XF.XM$ or XK is the common tangent of the circumcircles of the triangles KQH and KFM or two circles touch each other at K. $\square$

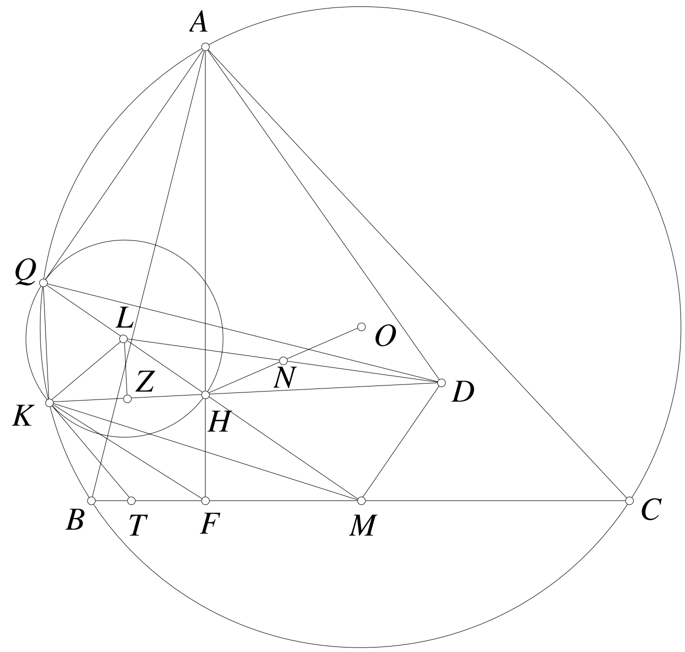

Figure 3.

Third proof. The straight line passing through M is perpendicular to QM and cuts KH at D. Call by L, Z the midpoints of HQ, HK then L, Z are on the Euler circle (N) but M is also on (N) then N is the midpoints of LD. N is also the midpoints of OH then $OD \parallel LH \perp QA$. Thence $DQ = DA$ and $HA.HF = HQ.HM = HK.HD$. Draw the tangent KT of the circumcircles of the triangles KQH we have $\angle TKF = \angle TKH - \angle HKF = \angle KQH - \angle HAD = \angle HDM - (\angle QAD - \angle QAH) = \angle HDM - \angle QDM + \angle HMF = \angle HMF - \angle QDH = \angle HMF - \angle HMK = \angle KMF$. Thence KT is also the common tangent of the circumcircles of the triangles KFM. □

Remark. This problem is third problem on first day, is supposed difficult problem. However the proof of problem with two circles touche at the available tangent point then degree of difficulty is not high so it is reasonable the third problem. The first proof is rather nature idea, because when two circles touch each other we think about inversation for proving the straight line touches the circle for reduce the quantity of the circles. This proof was suggested by **Telv Cohl** and **Trnh Huy Vu** 12A1 math special school of Natural science, Mr**Vu** helps me represent the proof as above. There are many different proofs suggested in [1]. The second proof have clever and pure geometric idea was suggested by **Jeck Lim**, nick name **oneplusone** in [1], I edit a little in building the point X and transform the angle shortly. The third proof, in fact, start at an idea of inversation by **Ho Quoc ang Hung** and I also edit more shortly, remove the present of an inversation and make its more pure geometry. In this proof, the point T is not necessary but we still build it for nice proof. The problem has many applications and expanding. In the following we would like to present some application and expanding.

1.2 Extension of IMO problem

The problem IMO has nice configuration, in which we can see many other interesting problems. The first problem was found out by chance when try to resolve the IMO problem by using similarly with median method.

Problem 1.2. Let the acute triangle ABC inscribed in the circle (O) with the orthocenter H, the altitude AF and M is the midpoint of BC. The circle with the diameter HA cuts (O) at Q differently from A. The circle with the diameter HQ cuts (O) at K differently Q. KQ cuts the circumcircle of the triangle KFM at N differently from K. Prove that MN bisects AQ.

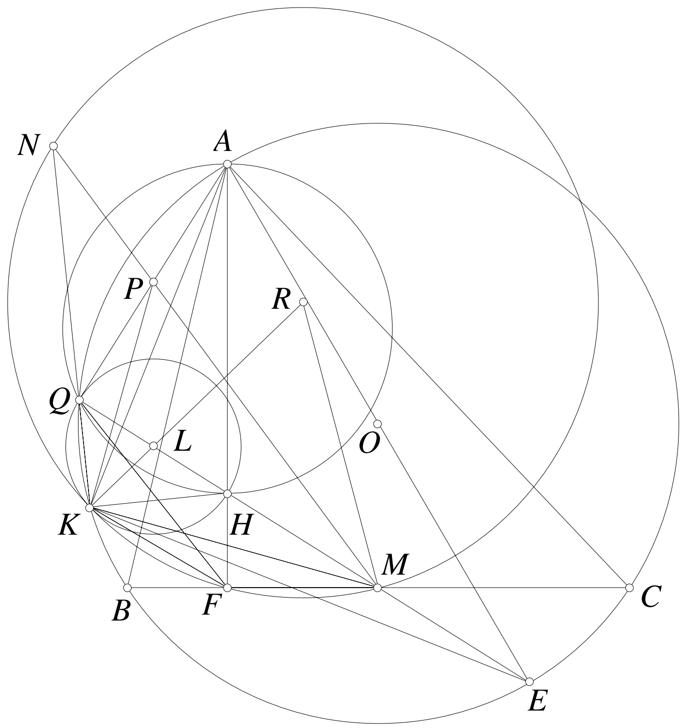

Figure 4.

Solution. Call by L, R the centers of the circumcircle of the triangles KQH and KFM then L is the midpoint of QH and according to the first problem then K, L, R are collinear. Call by P the midpoint of QA, we will prove M, N, P collinear. Indeed, call by AE the diameter of (O) then Q, H, M, E are collinear. Thence $\angle KQH = \angle KAE$ so two right triangles KQH and KAE are similar, deduce two triangles KQA and KHE are similar, their medians are KP, KM so $\angle QPK = \angle QMK$ and $\angle QKP = \angle HKM$. Then the quadrileteral $QPMK$ is cyclic. We have $\angle CMN = \angle QKF = \angle QKL + \angle LKM + \angle MKF = \angle KPM + \angle RMK + 90° - \angle RMF = 90° - \angle PMK + \angle RMK = 90° - \angle PMK + \angle RMK + 90^{c}irc - \angle RMF = 180° - \angle BMP = \angle CMP$. Then M, N, P are collinear. We are done. □

Thanks for the idea of this problem we can see the interesting when the tangent point was hidden in the origin problem

Problem 1.3. Let the acute triangle ABC with the orthocenter H, the altitude AF and M is the midpoint of BC. The circle with the diameter HA cuts HM at Q differently from A. X is on BC such that $XH \perp QM$. L, P are the midpoints of QH, QA. The straight line through Q and parallel to LX cuts MP at N. Prove that the circumcircle of the triangle NFM touches the circle with the diameter QH.

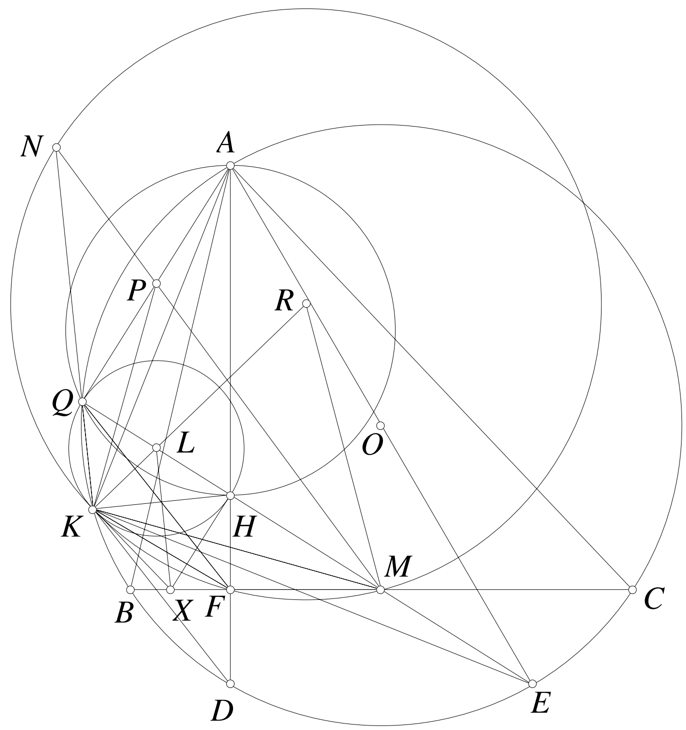

Figure 5.

The first proof. Call by (O) the circumcircle of the triangle ABC. Call by AE the diameter of (O) then Q, H, M, E are collinear. Call by D the reflection of H through BC. The circle (X, XH) touches the circle (O) at K differently D. We have $\angle XKH = \angle XHK = 90° - \angle KDH = 90° - \angle KEA = \angle KAE = \angle KQE$, then KH, KX touches the circumcircle of the triangle QKH. Moreover, we have $\angle KQH = \angle KHX = 90° - \angle KHQ$ so $\angle QKH = 90°$. K is on the circle with the diameter QH so $LX \perp KH \perp QK$ deduce $QK \parallel LX \parallel QN$ so K, Q, N are collinear. From the similar triangle KQH and the triangle KAE deduce KQA and KHE are similar, else KP, KM are their medians respectively so the triangles KQP and KHM are similar or KQH and KPM are similar. Else have $XK^2 = XH^2 = XM.XF$ deduce XK touches the circumcircle (R) of the triangle KFM. Thence K, L, R are collinear. So $\angle LKQ = \angle LQK = \angle KPM = 90° - \angle KHQ = 90° - \angle PMK$ from this easily deduce $\angle KRM = 2\angle N$. Thence N is on (R) or (R) is the circumcircle of the triangle NFM. Eevidently (R) touches the circle with the diameter QH. We are done. □

The second proof . The circle with the diameter QH cuts (O) at K differently from A and D is the reflection of H through BC. Prove analogously the origin problem then QE touches the circumcircle of the triangle KHD but $HX \perp QE$ so the center of the circumcircle of the triangle KHD is laying on HX, else have X is on the perpendicular bisector HD so the center of the circumcircle of the triangle KHD is just X so $XH = XK$. Easily seen XH touches the circumcircle of the triangle QHK so XK is the same. Then $KH \perp LX \perp QK$ so $QK \parallel LX \parallel QN$. Thence Q, K, N are collinear. We have $\angle QKF + \angle FMN = \angle QKL + \angle RKM + \angle MKF + \angle FMP = 90° - \angle KHQ + \angle RMK + 90° - \angle RMF + \angle FMP = 180°$ or the quadrileteral $NKFM$ is cyclic. Thence the circumcircle of the triangle NFM touches the circle with the diameter QH. □

We have another idea for expanding IMO problem as following

Problem 1.4. Let the acute triangle ABC inscribed in the circle (O) with the orthocenter H and the altitude AD. The circle with the diameter HA cuts (O) at G differently from A. The circle with the diameter HG cuts (O) at K differently from G. S is the reflection of D through HK. Prove that the straight line S is perpendicular to SK bisects BC.

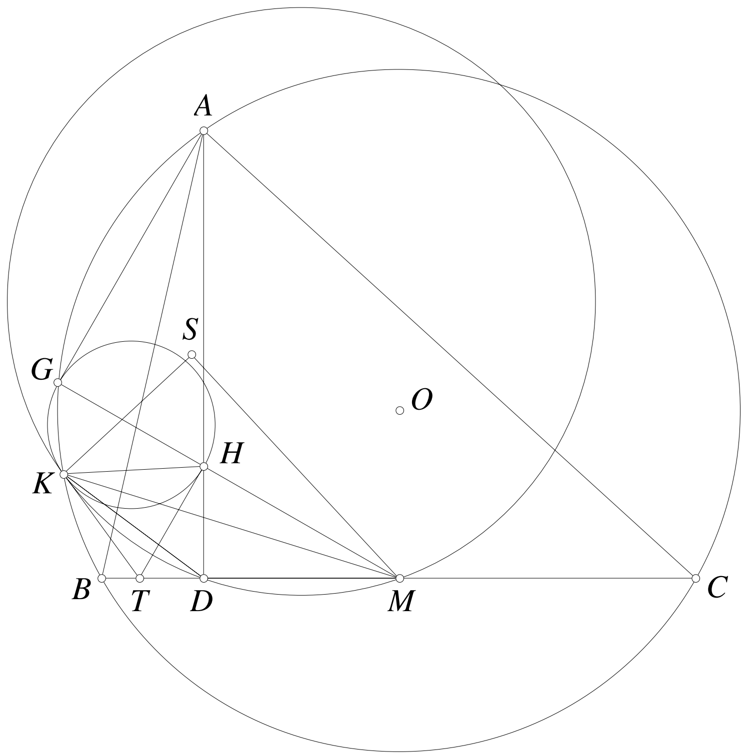

Figure 6.

Solution. Call by M the midpoint of BC, we prove that the triangle KSM is right at S. Indeed in the problem IMO, the circumcircles of the triangles KGH and KDM touch each other at K. We get T on BC such that KT is common tangent of two that circles. We have $\angle SKM = \angle SKH + \angle HKM = \angle HKD + \angle GHK - \angle GMK = \angle HKT - \angle DKT + \angle GHK - \angle GMK = \angle HGK + \angle GHK - \angle KMD - \angle GMK = 90^\circ - \angle HMD = \angle DHM$. Also according to the origin problem we also have TK and TH are the tangents of the circumcircles of the triangle KGH and two triangles TKD and TMK are similar. Then $\dfrac{KS}{KM} = \dfrac{KD}{KM} = \dfrac{TK}{TM} = \dfrac{TH}{TM} = \dfrac{HD}{HM}$. From that easily seen two triangles KSM and HDM are similar, so $\angle KSM = 90^\circ$. $\square$

From two above problems, go to the following expanding

Problem 1.5. Let the acute triangle ABC inscribed in the circle (O) with the orthocenter H and the altitude AD, the median AM. The circle with the diameter HA cuts (O) at G differently from A. The circle with the diameter HG cuts BC at K differently from G. KG cuts the circumcircle of the triangle KDM at N differently from K. KH cuts MN at Q. Prove that $QD = QM$.

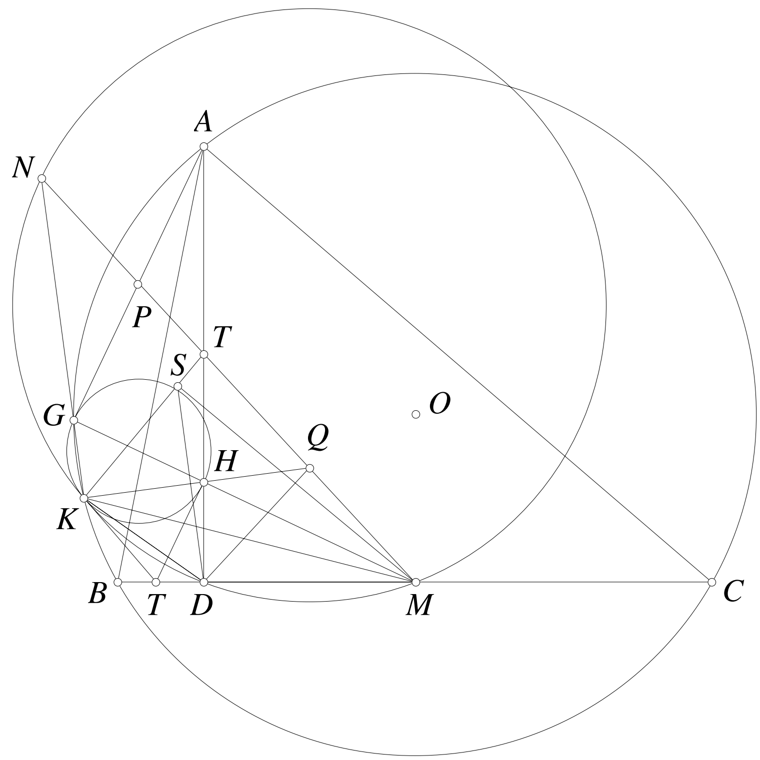

Figure 7.

The first proof based on the previous result

The first proof. Call by S the reflection of D through KH and KS cuts MM at T. According to the previous problem, MN goes through the midpoint P of GA so $\angle QHM = \angle GHK = \angle KMQ$. From this $\angle QMH = \angle QKM$. Then $\angle HKD = 90° - \angle HMD - \angle HKM = 90° - \angle QMD$. Thence $\angle KSD = 90° - \angle SKH = 90° - \angle HKD = \angle QMD$ deduce the quadrileteral $STMD$ is cyclic. Also according to the previous problem $\angle TSK = 90°$. From this deduce $\angle TDM = 90°$ or T belong to AH. And from $\angle HKD = 90° - \angle QMD = \angle MTD$ so the quadrileteral $KTQD$ is cyclic, we receive $\angle DQM = \angle TKD$ or two triangles QDM and KDS are similarly or $QD = QM$. □

The second proof was proved by **Trinh Huy Vu** the pupil from 12A1 Math, Special school of natural science.

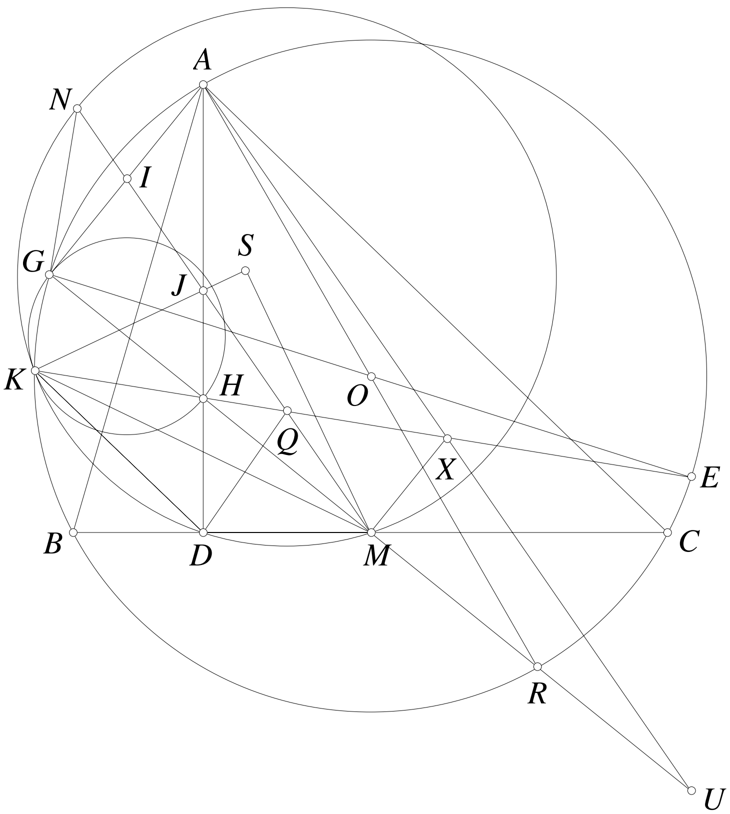

Figure 8.

proof. Draw the diameters AR, GE of (O).Call by I the midpoint of GA. From the previous problem we had I laying on MN. Call by X the midpoint of HE. We have familiar result G, H, M, R are collinear. From this deduce $MX \parallel GA$ v $MX = \dfrac{1}{2}RE = \dfrac{1}{2}.GA = IA = IG$ so $AIMX, IGMX$ is parallelogram. Thence $AX \parallel MI$ and $XI \perp GA$. From that, we receive $XA = XG$. Call by J the intersection of MI and AD. Get U symmetry of A through X. From $XA = XG$ deduce $\angle AGU = 90°$. Then U is lying on the straight line HM. So KH bisects MJ but $MJ \parallel AU$ and KH bisects AU at X. Deduce Q the midpoint of MJ, combine with $\angle JDM = 90°$, we receive $QD = QM$. □

From the result of this problem **Vu** gives the other proof for previous problem as following

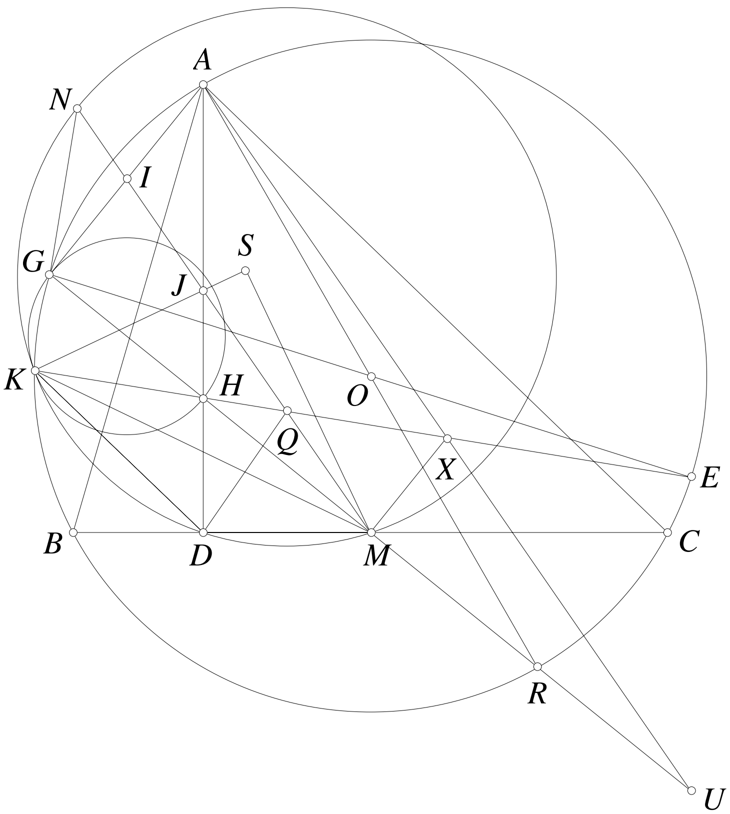

Figure 9.

Solution of the previous problem. We still use a symbol as in the second proof above. From this problem deduce Q is center of the circumcircles of the triangle DMS. Deduce $\angle DSM = \frac{1}{2}\angle DQM$. We have $HK.HX = HK.\frac{1}{2}HE = HG.\frac{1}{2}HR = HG.HM = HA.HD$. Deduce the quadrileteral $AXDK$ cyclic. So we have $\angle KSD = \angle KDS = 90° - \angle DKH = 90° - \angle DAX = 90° - \angle DJM = 90° - \frac{1}{2}.\angle DQM = 90° - \angle DSM$. So $\angle KSM = \angle KSD + \angle DSM = 90°$. □

If use the butter fly theorem we have two applications as following

Problem 1.6. Let the acute triangle ABC with the center of the circumcircles O with the orthorcenter H, the altitude AD and the median AM. G is the projector of A on HM. L is the midpoint of HG. K is reflection of G through OL. KL cuts perpendicular bisector DM at S. KG cuts BC at T. Get X belong to MK such that $TX \perp ST$. Y is symmetric of X through T. P is the midpoint of AG. Prove that KG, YD, MP concurrent.

We can present the above problem in other way, this problem also has a lot of meaning

Problem 1.7. Let the acute triangle ABC with the center of the circumcircles O with the orthocenter H, the altitude AD and the median AM. G is projector of A on HM. L is the midpoint of HG. K is the reflection of H through OL. KL cuts the perpendicular bisector DM at S. P is the midpoint of GA. N is symmetry of M through the projector of S on MP. NG cuts BC at T. Get X belong to ND such that $XT \perp ST$. Y is symmetry of X through T. Prove that MY, NG, KL concurrent.

So from the origin problem we receive some other problems, they have nice result and meaning .

1.3 Some applications

This IMO problem is nice in sense having a lot of expanding development. In [1] was given many expanding, in this article I would like to present my expanding, let go to the first expanding as following

Problem 1.8. Let the acute triangle ABC inscribed in the circle (O). P is one point in the triangle such that $\angle BPC = 180^\circ - \angle A$. PB, PC cut CA, AB at E, F. The circumcircles of the triangle AEF cuts (O) at G differently from A. The circle with the diameter PG cuts (O) at K differently from G. D is the projector of P on BC and M is the midpoint of BC. Prove that The circumcircles of the triangles KGP and KDM touche each other.

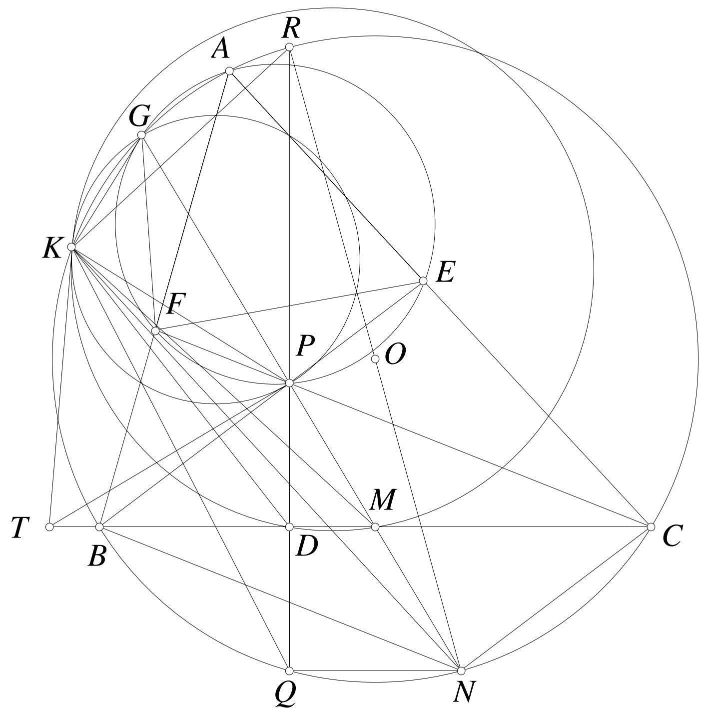

Figure 10.

Solution. Call by Q the symmetry of P through D, then Q is laying on (O). GP cuts (O) at N differently from G. We see $\angle NPC = \angle FPG = \angle FAG = \angle BNP$ deduce $BN \parallel PC$. Similarly, $CN \parallel BP$. From this M is the midpoint of PN. Call by AS, NR the diameter of (O). We easily see $\angle PQN = 90^\circ$ so P, Q, R are collinear. Thence, GN is the tangent of the circumcircle of the triangle KPQ. Call the tangent at K, P of the circumcircle of the triangle KPG cut each other at T. We have $\angle KTP = 180^\circ - 2\angle KGP = 2(90^\circ - \angle KRN) = 2\angle RNK = 2\angle KQP$ v $TK = TP$. From this T is the center of the circumcircle of the triangle KPQ but, as BC is the perpendicular bisector of PQ so T is on BC. From this, we have $TK^2 = TP^2 = TD.TM$ deduce TK is common tangent of the circumcircles of the triangles KDM and KHM or two circles touch each other at K. □

Remark. Above expand was published first time in [1] and after that it was revised shortly. When given P the orthocenter or given the angle A specially, we will receive many separate case with meaning. In other point of view, it is easier when P is the orthocenter of the triangle RBC so, we apply directly the origin problem on the triangle RBC then receive the above problem. The other expand for this problem as following

Problem 1.9. Let the triangle ABC inscribed on the circle (O). P is one point on the chord $\overset{\frown}{BC}$ not contain A. AP cuts BC at D. Q is symmetry of P through D. The circle with the diameter AQ cuts (O) at G differently from A. The circle with the diameter GQ cuts (O) at K differently from G. GQ cuts the straight line through O and parallel to AP at M. Prove that the circumcircles of the triangles KGQ and KDM touch each other.

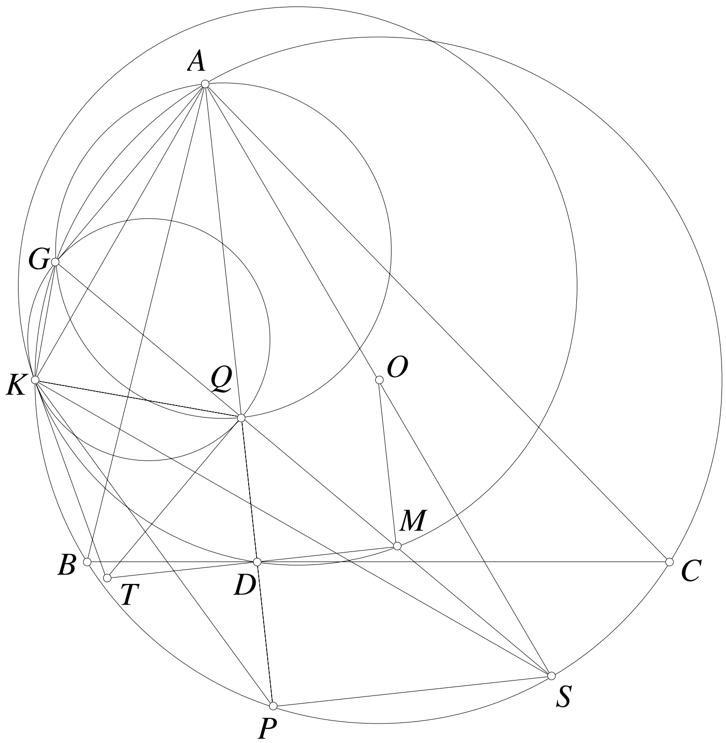

Figure 11.

Solution. Call by S the intersection again of GQ cut (O), as $\angle AGQ = 90°$ so AS is the diameter of (O). As $OM \parallel AP$ and O is the midpoint of AS so M is the midpoint of QS. Thence $DM \parallel PS \perp PA$ so DM is the perpendicular bisector of PA. Else have $\angle KQG = 90° - \angle KGQ = 90° - \angle KAS = \angle ASK = \angle QPK$. Thence GS touches the circumcircle of the triangle KQP. Call the tangent at K, Q of the circumcircle of the triangle GKQ cut each other at T. We have $\angle KTQ = 180° - 2\angle KGQ = 2\angle KQG = 2\angle KPQ$. Then T is the center of the circumcircle of the triangle KPQ. We prove that DM is the perpendicular bisector of PQ so T belongs to DM. From this, we have $TK^2 = TP^2 = TD.TM$ deduce TK is the common tangent of the circumcircles of the triangles KGQ and KDM or two that circles touch each other at K. We are done. □

Remark. This expanding is rather important because it bases on the same models as the origin problem. So the applications of the origin problem could developed on this model. However, we can see it more simple when we prolong the perpendicular bisector PQ cuts (O) at two points Y, Z then Q is the orthocenter of the triangle AYZ so apply the origin problem IMO into the triangle AYZ. We receive this problem. By the same way, you can do the following expanding problem

Problem 1.10. Let the triangle ABC and P is the point in the triangle. X, Y, Z are the reflections of P through BC, CA, AB. PX cuts the circumcircle (Q) of the triangle XYZ at T differently from X. The circle with the diameter PT cuts (Q) at G differently from T. The circle with the diameter PG cuts (Q) at K differently from G. D, M are the projectors of P, Q on BC. Prove that the circumcircles of the triangles KDM and KPG touch each other.

So, through two above problems we can see the IMO origin problem plays an important role , when apply that problem into different models then give many very interesting expanding problems .

We continue some aspect of general problem as expanding of IMO problems

Problem 1.11. Let the triangle ABC inscribed in the circle (O). P is one point on the chord $\overset{\frown}{BC}$ not contain A. AP cuts BC at D. Q is symmetry of P through D. The circle with the diameter AQ cuts (O) at G differently from A. The circle with the diameter GQ cuts (O) at K differently from G. GQ cuts the straight line through O and parallel to AP at M. KG cuts the circumcircle of the triangle KDM at N differently from K. Prove that MN bisects GA.

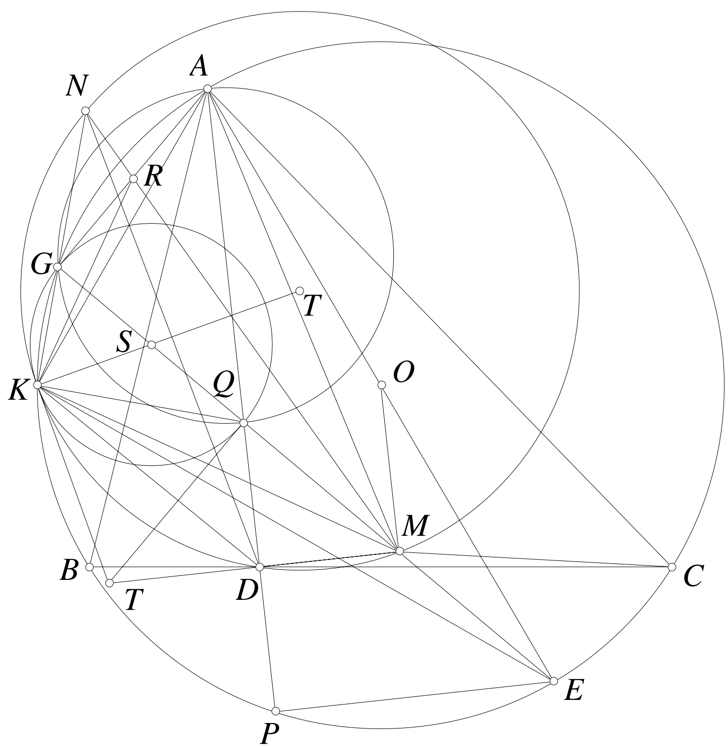

Figure 12.

Solution. Call by AE the diameter of (O). Similar proof of the problem 1.9 we have G, Q, M, E are collinear and M is the midpoint of QE. So, easily seen the right triangles KGQ and KAE are similar, deduce the triangles KGQ and KQE are similar. Call by R the midpoint of GA, so two triangles KGR and KQM are similar. So easily seen the quadrileteral $KGRM$ is cyclic. We have $\angle DMN = 180° - \angle DKN = 180° - (\angle GKS + \angle TKM + \angle MKD) = 180° - (90° - \angle KMR + \angle TMK + 90° - \angle TMD) = \angle DMR$. Thence, we have M, N, R collinear. $\square$

Problem 1.12. Let the triangle ABC inscribed in the circle (O). P is the point on the chord $\overset{\frown}{BC}$ not contain A. AP cuts BC at D. Q is the symmetry of P through D. The circle with the diameter AQ cuts (O) at G differently from A. GQ cuts the straight line through O and parallel to AP at M. The straight line through Q and perpendicular to GM cuts DM at T. S, R are the midpoints of GQ, GA. The straight line through G and parallel to ST cuts MR at N. Prove that the circumcircle of the triangle MND touches the circle with the diameter GQ.

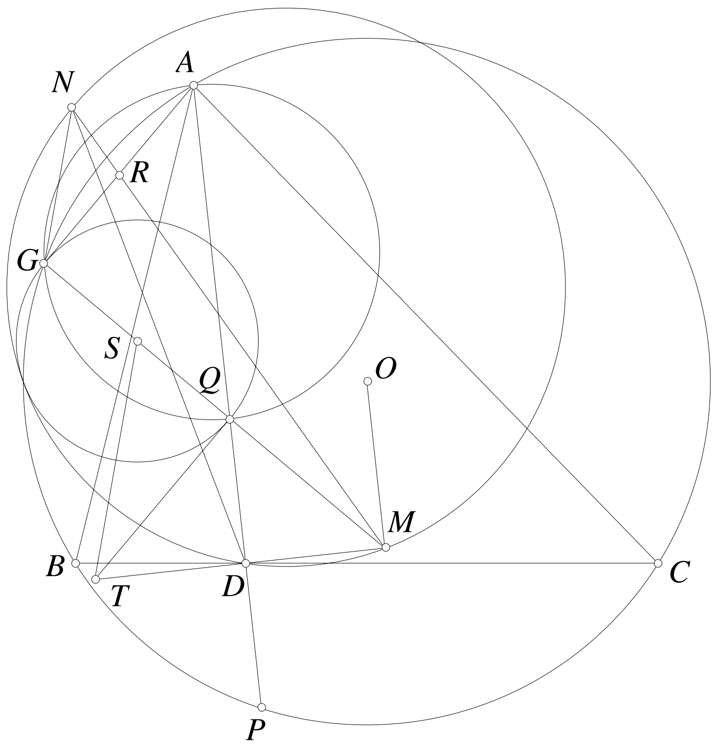

Figure 13.

We create more some other models for IMO problems as following

Problem 1.13. Let the triangle ABC right at A. P is one point on BC. The circle with the diameter BP cuts the circumcircle (K) of the triangle APC at Q differently from P. Call by M, N are the midpoints of BC, AB.

a) Prove that the circumcircle of the triangles QMN and QPB touch each other.

b) PQ cuts the circumcircle of the triangle QMN at R differently from Q. MR cuts the straight line through P and perpendicular to BC at S. Prove that $KS \parallel BC$.

c) T is the reflection of N through BQ. Prove that $\angle QTM = 90°$.

d) BQ cuts ST at L. Prove that the triangle LMN is isosceles.

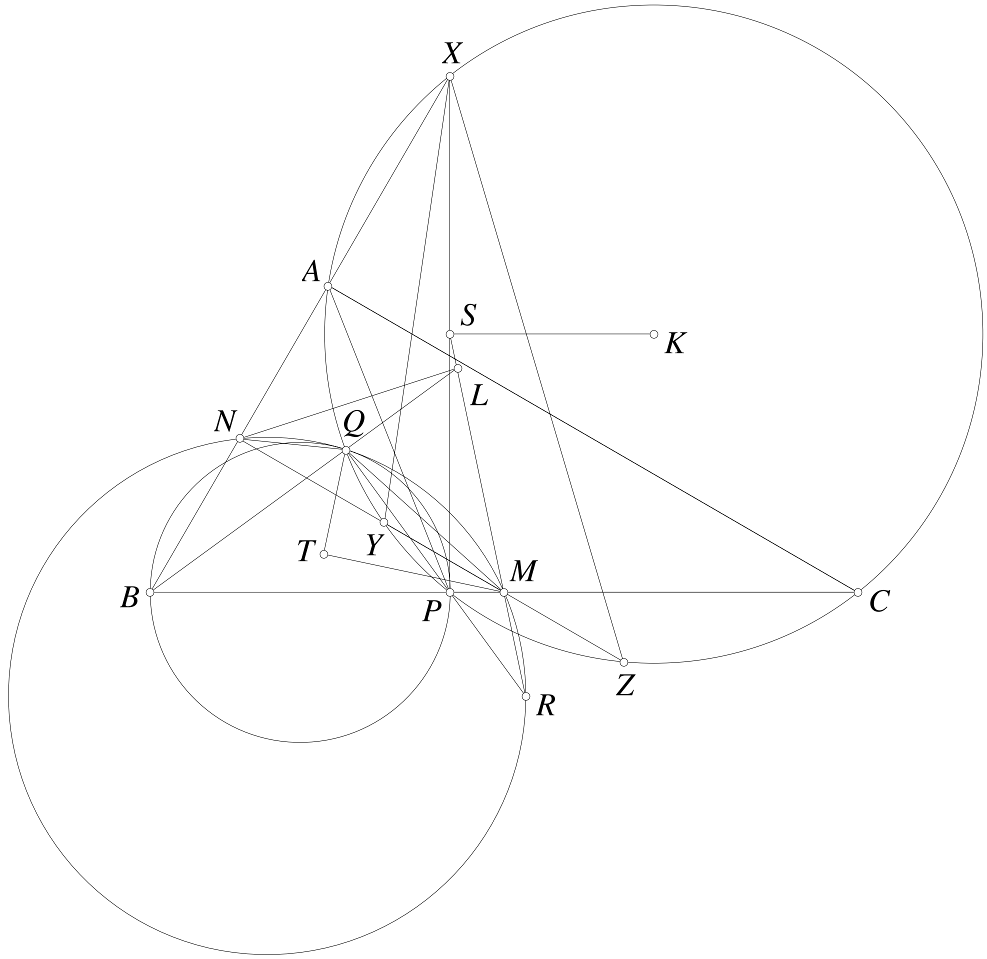

Figure 14.

Solution. Call by X the intersection again of AB cut (K) differently from A and MN cuts (K) at Y, Z. Easily seen B the orthocenter of the triangle XYZ. Apply the problems for the triangle XYZ the orthocenter B we have the thing done. □

We use once more to hide the tangent point and receive interesting problem as following

Problem 1.14. Let the triangle ABC right at A. M, N are the midpoints of BC, AB. The straight line perpendicular to BC at P cuts AB at X. S, T are the midpoints of PB, PX. Get the point L on MN such that $BL \perp BC$. Get the point R on MT such that $PR \parallel LS$. Prove that the circumcircle of the triangle RMN touch the circle with the diameter PB.

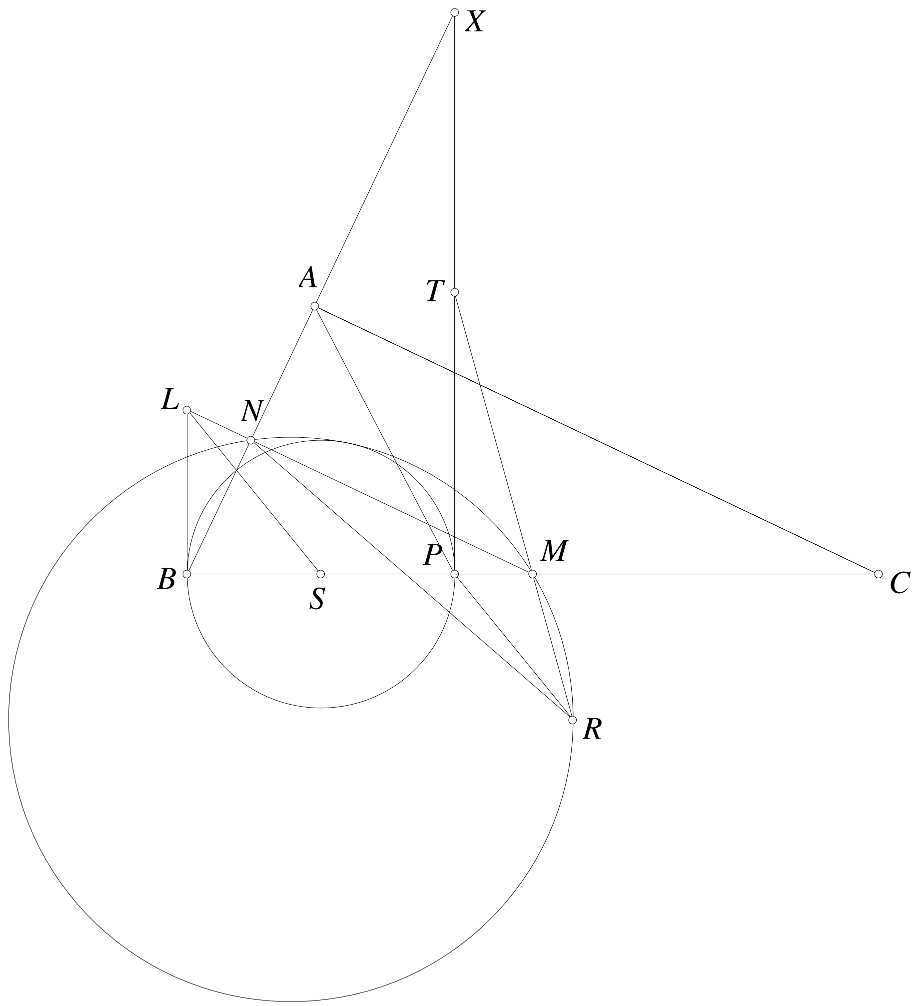

Figure 15.

In other hand, the origin problem still has many developments and expanding, do the following exercises

Problem 1.15. Let the triangle ABC inscribed in the circle (O) with the diameter AD. M is one point on BC. MD cuts (O) at G differently from D. Q is symmetry of D through M. The circle with the diameter QG cuts (O) at K differently from G. N is the projector of M on AQ.

a) Prove that the circumcircles of the triangles KMN and KQG touch each other.

b) KG cuts the circumcircle of the triangle KMN at P differently from K. Prove that MP bisects AG.

c) R is the reflection of N through QK. Prove that $\angle KRM = 90°$.

Problem 1.16. Let the triangle ABC has $\angle A = 60°$ inscribed in the circle (O). The altitudes BE, CF cut each other at H. M is the midpoint of the chord $\overset{\frown}{BC}$ contain A. MH cuts (O) at N differently from M. The circle with the diameter HN cuts (O) at K differently from N. P is the reflection of H through EF and Q is the midpoint of HM.

a) Prove that the circumcircles of the triangles KPQ and KHN touch each other.

b) KN cuts the circumcircle of the triangle KPQ at L differently from K and R is the midpoint of of the chord $\overset{\frown}{BC}$ not contain A. Prove that QL bisects KR.

c) Z is the reflection of P through KH. Prove that $\angle KZQ = 90°$.

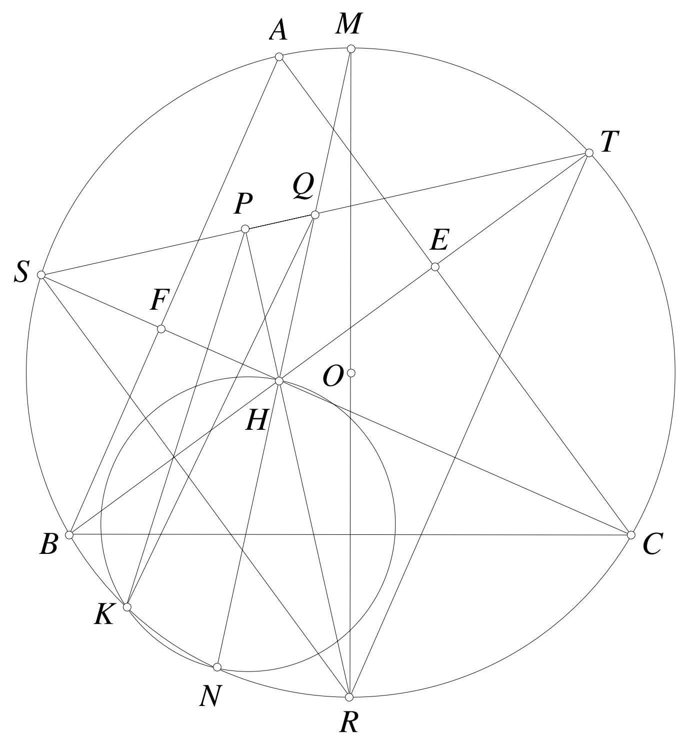

Figure 16.

Solution. Call by S, T the reflection of H through F, E and MR is the diameter of (O). From $\angle BAC = 60°$ we see H the orthocenter of the triangle RST. From that, apply the problem above for the triangle RST. We receive the thing done. □

Problem 1.17. Let the triangle ABC inscribed in the circle (O) the incircles center is I. The circle A-mixtilinear touches (O) at P. The circle with the diameter PI cuts (O) at K differently from P. N is the midpoint of AI and the perpendicular bisector AI cuts PI at M.

a) Prove that the circumcircles of the triangles KMN and KPI touch each other.

b) KP cuts the circumcircle of the triangle KMN at L differently from K. AI cuts (O) at D differently from A. Prove that ML bisects PD.

c) Q is the reflection of N through KI. Prove that $\angle KQM = 90°$.

Problem 1.18. Let the acute triangle ABC inscribed in the circle (O) has the altitudes BE, CF. K, L are the reflection of O through CA, AB. KE cuts LF at H. T belongs to the perpendicular bisector BC such that $HT \parallel OA$. M is the midpoint of AT. MO cuts the tangent through A of (O) at N. The straight line N parallel to OA cuts Euler line of the triangle ABC at P. G is the projector of T on NH. Q is the midpoint of HG. S is the reflection of G through PQ. TH cuts AN at D.

a) Prove that the circumcircles of the triangles SDN and SGH touch each other.

b) GS cuts the circumcircle of the triangle SDN at R differently from S. Prove that NR bisects TG.

c) W is the reflection of D through SH. Prove that $\angle SWN = 90°$.

In the end, one expand model in [1] was found by **Trinh Huy Vu**.

Problem 1.19. Let the triangle ABC inscribed in the circle (O). One any circle (D) passing through B, C cuts CA, AB at E, F. Draw the diameter AP of the circumcircle of the triangle AEF. K is the projector of D on AP. The circumcircle of the triangle AEF cuts (O) again at G. The circle with the diameter GP cuts (O) again at J.

a) Prove that the circumcircles of the triangles JGP and JKD touch each other.

b) JG cuts the circumcircle of the triangle JKD again at M. Prove that DM bisects GA.

c) L is the reflection of K through JP. Prove that $\angle JLD = 90°$.

2 Geometry problem on the second day

2.1 Introduction

Exam IMO second day, 2015 [2] has geometric problem interesting as following

Problem 2.1. Let the triangle ABC inscribed in the circle (O). The circle (A) with the center A cuts BC at D, E and cuts (O) at G, H such that D is between B, E and the ray AB is laying between AC, AG. The circumcircle of the triangle BDG and CEH cuts AB, AC respectively at K, L differently from B, C. Prove that GK and HL cut each other on AO.

I would like to present my proof for this problem

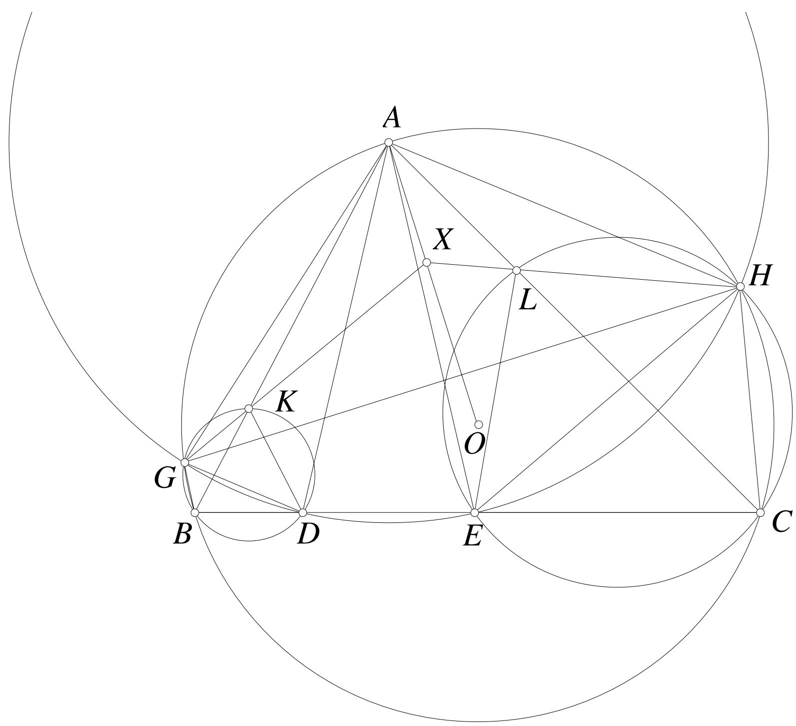

Figure 17.

Solution. Call by X the intersection of GK cuts LH we easily seen AO is the perpendicular bisector of GH. We need prove only X belong the perpendicular bisector of GH thence we are done. Indeed, we see $\angle EHC = \angle GHC - \angle GHE = 180° - \angle GBD - \angle GDB = \angle BGD$. Thence $\angle XGH = \angle XGD - \angle HGD = \angle KBD - \angle HEC = 180° - (\angle GBA + \angle BGD + \angle BDG) - \angle HEC = 180° - (\angle HCA + \angle EHC + \angle EHG) - \angle HEC = \angle ACB - \angle GHE = \angle XHE - \angle GHE = \angle XHG$. From that, the triangle XGH is isosceles, we are done. □

Remark. This problem is the forth in second day, it is supposed easily. Its proof used the way of angle added. It is nice problem, simple configuration, and has meaning for exam and development of thinking. This problem has some expanding and application, we recognize it in the next part.

2.2 Extensions and applications

At first, we can change the circle with the center A by the other with any center on the straight line AO and its proof is just the same. We see another expanding with more meaning

Problem 2.2. Let the triangle ABC inscribed in the circle (O), the altitude AD. (A) is the circle with any center A. Call by E, F two points on (A) such that E, F are the reflections through AD and the ray AE is between AB, AF. The circle (A) cuts (O) at G, H such that the ray AB is laying between two rays AG, AC. CE, BF cut the circle (A) at P, Q respectively, differently from E, F. The circumcircles of the triangles BPG and CQH cut BA, CA at K, L respectively, differently from B, C. Prove that GK and HL cut each other on AO.

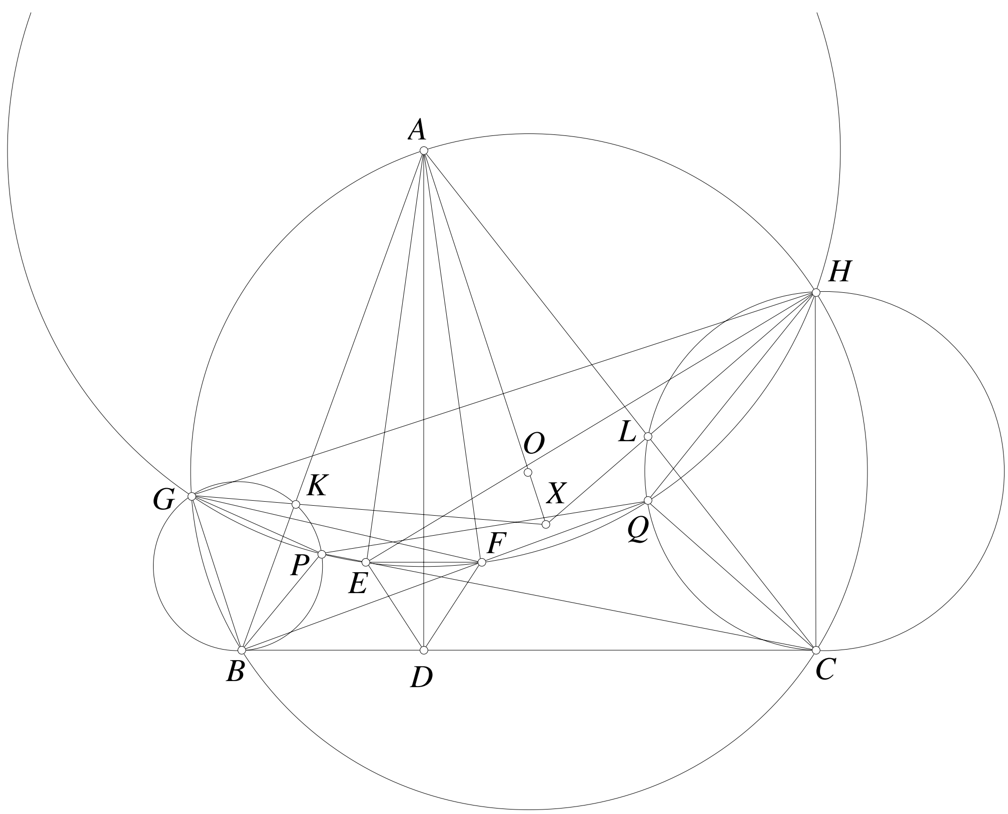

Figure 18.

Solution. At first, we have $EF \parallel BC$ so $\angle QPE = \angle EFB = \angle FBC$. Thence, the quadrileteral $PQCB$ is cyclic. Else have $\angle EHC = 180^\circ - \angle GBC - \angle GHE = 180^\circ - \angle GBP - \angle PBC - \angle GFE = \angle BGP + \angle GPB - (180^\circ - \angle BPC - \angle PCB) + 180^\circ - \angle GPE = \angle BGP + \angle FEC = \angle BGP + \angle PGF = \angle BGF$.

Thence $\angle HGX = \angle HGP - \angle PGK = \angle HEC - \angle PBK = \angle HEC - (\angle GBF - \angle GBA - \angle PBF)(1)$.

Similarly, $\angle GHX = \angle GFB - (\angle HCE - \angle HCA - \angle QCE) \quad (2)$.

Easily have $\angle GBA = \angle HCA, \angle PBF = \angle QCE$ v $\angle BGF = \angle CHE$ so $\angle GBF + \angle GFB = \angle HEC + \angle EHC$ or $\angle HEC - \angle GBF = \angle GFB - \angle EHC \quad (3)$.

From (1),(2),(3) easily deduce $\angle HGX = \angle GHX$. We are done. □

Remark. The general problem is still right when we change the circle (A) by any circle with the center belong OA and the proof with the change similar angle. To pay attention carefully in this proof as the same with the proof of the origin problem, so the change of the angle for showing $\angle EHC = \angle FGB$ that is important step.

We can see, in nature G, H are lying on (O) that is not so important, we go to the more general problem as following

Problem 2.3. Let the triangle ABC inscribed in the circle (O), the altitude AD. (A) is any circle with the center A. Call by E, F two points on (A) such that E, F are the reflection through AD and the ray AE is laying between two rays AB, AF. On the circle (A) get two points G, H such that $GH \perp OA$ in the same time, the ray AB is laying between two rays AG, AC. Call by P, Q the intersection again of CE, BF the circle (A) respectively, differently from E, F. The circumcircles of the triangles BPG and CQH cut BA, CA at K, L respectively, differently from B, C. Prove that GK and HL cut each other on AO.

On the other hand, we can see that, in the above problem we can change the circle (A) by any circle with the center belong OA. Thence, we think that, we can change the straight line OA by the perpendicular bisector of the chord of (O), we have the following problem

Problem 2.4. Let the quadrileteral $XYBC$ cyclic in the circle (O). (A) is any circle with the center A belong to the perpendicular bisector XY. D is the projector of A on BC. Call by E, F two points on (A) such that E, F are the reflection through AD and the ray AE is laying between two rays AB, AF. On the circle (A) get two points G, H such that $GH \perp OA$ in the same time, the ray AB is laying between two rays AG, AC. Call by P, Q the intersection again of CE, BF cut the circle (A) at P, Q respectively, differently from E, F. The circumcircles of the triangles BPG and CQH cut BY, CX at K, L respectively, differently from B, C. Prove that GK and HL cut each other on AO.

Hence we can exploit the problem by many ways, we present some exploiting based on the model of the problem as following

Problem 2.5. Let the triangle ABC inscribed in the circle (O). The circle (A) with the center A cuts BC at E, F and cut (O) at G, H such that E is laying between B, E and the ray AB is laying between two rays AC, AG. GH cuts the circumcircles of the triangles BEG and CFH at M, N respectively, differently from G, H. Call by P, Q the intersection of GE, HF cut BM, CN. Call by S, T the intersection of ME, GB cut NF, HC respectively. Prove that ST bisects PQ.

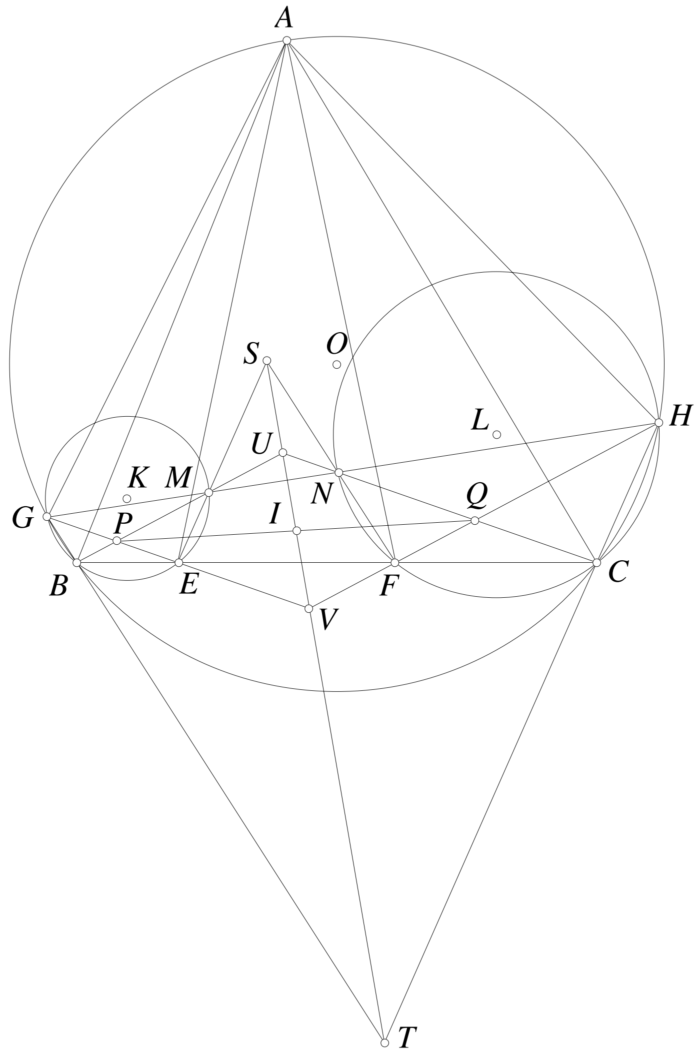

Figure 19.

Solution. According to the proof of the origin problem to show that $\angle BGE = \angle CHF$. Thence have $\angle FNH = \angle FNC + \angle CFH = \angle FHC + \angle CFH = \angle EGB + \angle EGM = \angle BGM = \angle MEF$. Then the quadrileteral $MNFE$ is cyclic. Easily have $\angle HNC = \angle HFC = \angle EGH = \angle MBE$ deduce the quadrileteral $BMNC$ is cyclic. Call by $(K), (L)$ the circumcircles of the triangles BEG and CFH then from the quadrileteral $EMNF$ and $BGHC$ cyclic, deduce ST is radical axis of (K) and (L). We easily have $\angle GEB = \angle GMB = \angle NCB$ so $GE \parallel NC$, similarly $HF \parallel MB$. Call by U, V the intersection of BM, GE cut CN, HF then $PUQV$ is the parallelogram, so UV bisects PQ. From the quadrilaterals $BMNC$ and $GEFH$ cyclic, deduce U, V also belongs to the radical axis of $(K), (L)$ is just ST. So ST bisects PQ. We are done. $\square$

By the same way, we receive the problem about the bisection of the segment on the interesting model of general problem

Problem 2.6. Let the triangle ABC inscribed in the circle (O), the altitude AD. (A) is any circle with the center A. Call by E, F two points on (A) such that E, F are the reflection through AD and the ray AE is laying between AB, AF. The circle (A) cuts (O) at G, H such that the ray AB

is laying between two rays AG, AC. CE, BF cut the circle (A) at P, Q respectively, differently from E, F. GH cuts the circumcircles of the triangles BPG and CQH at M, N respectively. MP, GB cut NQ, HC respectively at S, T. To get the points U, V on the straight lines MB, NC such that $UG \parallel NC$ and $VH \parallel MB$. Prove that ST bisects UV.

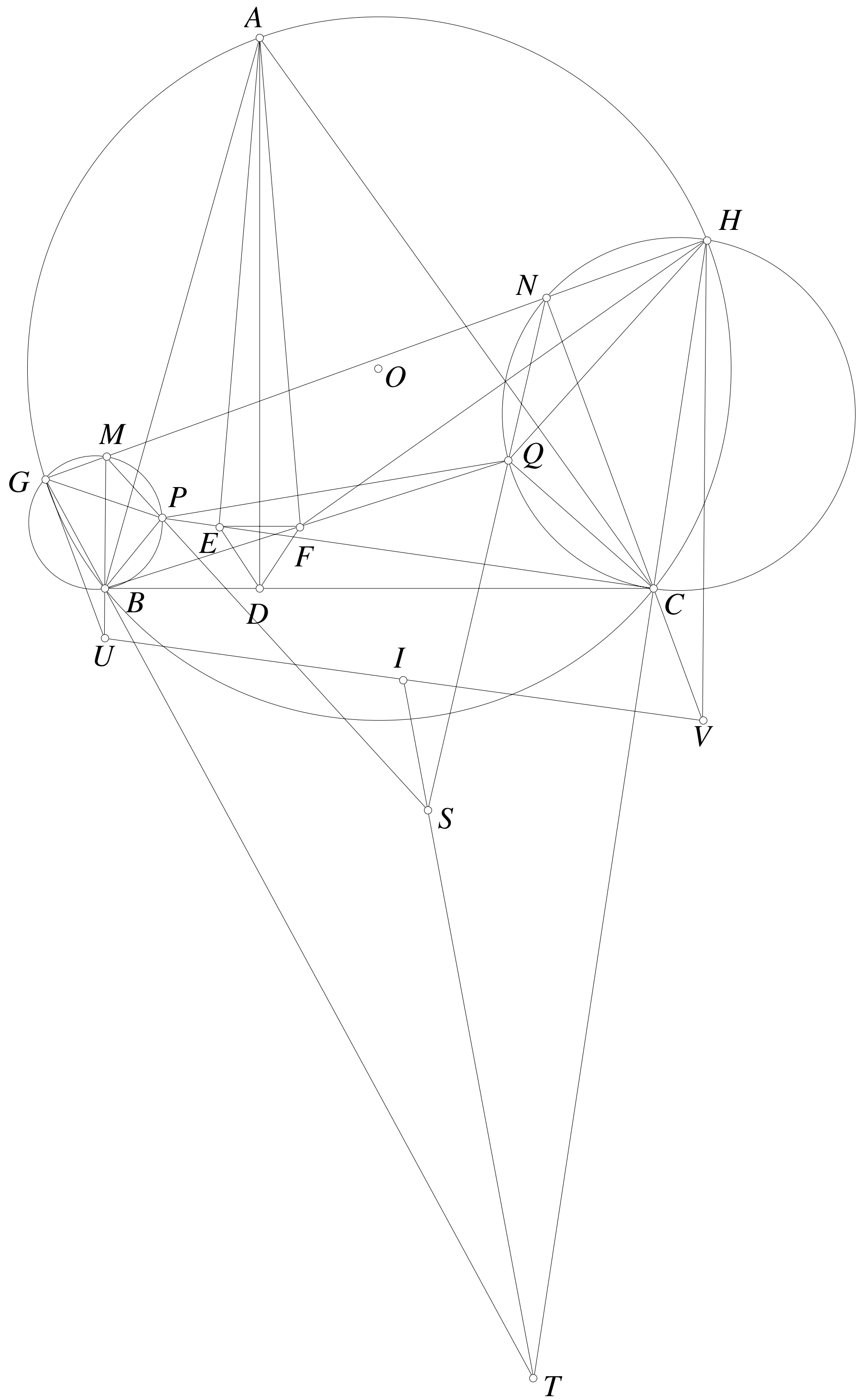

Figure 20.

If you know use the inversion you could to do the problem for exercise

Problem 2.7. Let the triangle ABC inscribed in the circle (O). The circle (A) with the center A cuts BC at E, F and cuts (O) at G, H such that E is laying between B, F and the ray AB is laying between two rays AC, AG. The circle through H, C and touches HA cuts CA at Q differently from C. The circle through G, B and touches GA cuts AB at P differently from B. The circumcircles of the triangles GPE and HQF cut AB, AC at M, N differently from P, Q. Prove that the diameter the circumcircles of the triangles AGM and AHN equally.

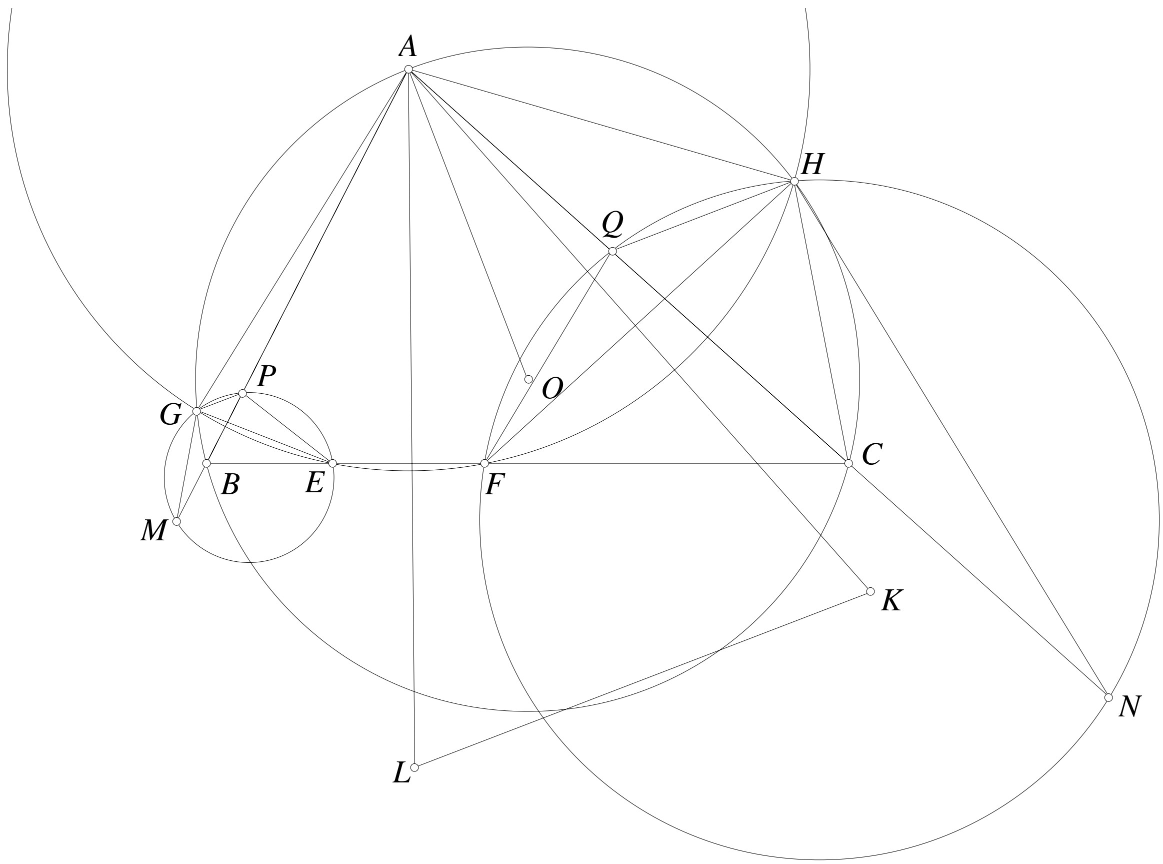

Figure 21.

3 Conclusion

The IMO exam in this year still have two geometric problem in the third and forth positions. They are interesting and have high meaning. Beside given different expanding, this article also written so nice about the problem with the bisection of the segment. And from the problem with the bisection of the segment in the first day, we receive one interesting presentation about two touching circles and thence we have the presentation of the second general problem, it gives more attraction for the problem of IMO exam. The problem with the bisection of the segment in the second day is no less interesting in comparison with the first day. That is the application of the radical axis and the parallelogram. Two geometric problem in the IMO exam this year are nice and have high suggestive and development, it is worthy to IMO exam.

In the end, I would like to thank to **Trinh Huy Vu** the pupil of 12A1 Math in my School, he is my pupil and has some contributions for this article and help me to edit it.

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[2] Topic Problem 4
http://www.artofproblemsolving.com/community/c6t48f6h1113163_problem_4

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