High School Olympiads

Angle bisector

Source: it is a unknown lemma

Pedram-Safaei
#1 Mar 12, 2012, 8:22 pm

ABCD is a convex circumscribed quadrilateral with center I such that AD and BC intesect each other at P and circumcircles of PAB and PCD intersect each other at a point T prove that TI is angle bisector of angles BTD and CTA.

This post has been edited 2 times. Last edited by Pedram-Safaei, Mar 15, 2012, 1:56 am

Luis González
#2 Mar 14, 2012, 5:31 am

Please do not double/triple post and use meaningful subjects.

Incircle (I) of $ABCD$ touches AB, BC, CD, DA at $M, N, L, K.$ Inversion with respect to (I) takes P, A, B, C, D into the midpoints of $NK, KM, MN, NL, LK,$ respectively. Thus, the inverse images of $\odot(PAB)$ and $\odot(PCD)$ are the 9-point circles of $\triangle MNK$ and $\triangle NKL$ meeting at the midpoint of $\overline{NK}$ and the Poncelet point of $MNLK,$ i.e. Anticenter U of the cyclic $MNLK.$ So if E is the midpoint of $\overline{LK},$ then $EU \perp MN,$ i.e. $EU \parallel IB.$ Therefore, the inverse $\odot(IDT)$ of EU is tangent to IB at $I \Longrightarrow \angle TDI = \angle BIT.$ Analogously, $\angle TBI = \angle DIT \Longrightarrow \triangle TID \sim \triangle TBI \Longrightarrow \angle DTI = \angle BTI,$ i.e. TI bisects $\angle BTD.$ By similar reasoning, TI bisects $\angle CTA.$

hatchguy
#3 Mar 15, 2012, 2:02 am

My proof is wrong, I thought $ABCD$ was cyclic. It's just a coincidence that it works too. This is the proof for $ABCD$ cyclic:

Clearly PT, AB and CD concurr. Suppose their point of concurrence is K.

It is well known that $\angle ITP = 90$

(It is actually IMO 1985 problem 5 http://www.artofproblemsolving.com/Forum/viewtopic.php?p=366594&sid=cc99cea9d1a879a4b4909fde5248e61e#p366594)

So it is enough to prove $\angle PTB = \angle KTD$ and $\angle PTC = \angle KTA$.

This is easy since $\angle PTB = \angle PAB = \angle PCD = \angle KTD$.

and $\angle PTC = \angle PDC = \angle PBA = \angle KTA$ and we are done.

High School Olympiads

Acute again

Reply

Source: own

jayme

#1 Mar 13, 2012, 4:51 pm

Dear Mathlinkers,
1. ABC a triangle
2. (I) the incircle
3. DEF the contact triangle
4. Pb the parallel to AC through B
5. R, S the points of intersection of Pb wrt EF, ED

Prouve : <RIS is acute.

Sincerely
Jean-Louis

Luis González

#2 Mar 13, 2012, 11:07 pm

$\angle BSD = \angle CED = \angle BDS \Longrightarrow \triangle BDS$ is isosceles with legs $BS = BD = s - a$. Similarly $\triangle BFR$ is isosceles with legs $BR = BF = s - a \Longrightarrow B$ is midpoint of $\overline{RS}$. Since $BI > BD$, it follows that I is always outside the circle (B, BD) with diameter $\overline{RS} \Longrightarrow \angle RIS < 90^\circ$.

P.S. See also the related problem: IMO revisited and Geometry.

Quick Reply

High School Olympiads

Isogonal conjugate of the De-Longchamps point

r1234

#1 Mar 13, 2012, 12:39 am

Show that the isogonal conjugate of the De-Longchamps point of a triangle is the reflection of the Nagel Point of the tangential triangle wrt the circumcenter of the main triangle.

Luis González

#2 Mar 13, 2012, 1:38 am

Let H, O, G, L denote the orthocenter, circumcenter, centroid and De Longchamps point of the reference triangle $\triangle ABC.$ $\triangle O_AO_BO_C$ is the anticevian triangle of O WRT $\triangle ABC.$ Let D be the foot of the A-altitude, M the midpoint of $BC,$ $P \equiv AO \cap BC$ and $L_A \equiv AL \cap BC.$ Since the pencil $A(O,H,G,L)$ is harmonic, then $(P,D,M,L_A)=-1.$ But projections P,D,M,Q_A of P,A,O,O_A on BC are harmonically separeted $\Longrightarrow L_A \equiv Q_A.$ Likewise, L_B,L_C are the projections of O_B,O_C on $CA,AB.$ According to the topic A nice concurrent (see post #12), O_AL_A,O_BL_B and O_CL_C concur at the Nagel point U of the tangential triangle of $\triangle ABC.$

According to the topic Find relation (see problem a), perpendiculars to O_BO_C,O_CO_A,O_AO_B through A,B,C concur at the isogonal conjugate V of L WRT $\triangle ABC.$ Hence, U and V are isogonal conjugates WRT $\triangle O_AO_BO_C,$ being (O) their pedal circle WRT $\triangle O_AO_BO_C \Longrightarrow O$ is the midpoint of $\overline{UV}.$

r1234

#3 Mar 13, 2012, 3:14 pm

My proof:-

Let O,G,H,L,S be the circumcenter,centroid,orthocenter,De-longchamps point and Symmedian point of $\triangle ABC$.Clearly $A(O,G,H,L)=-1$.Let L^* be the Isogonal conjugate of the De-Longchamps point.Since AL,AL^* and AG,AS are also isogonal wrt $\angle OAH$ we conclude that $A(O,S,H,L^*)=-1$.... **(1)**. Now let $\triangle A_1B_1C_1$ be the tangential triangle of $\triangle ABC$.Then S is the Gergonne point of $\triangle A_1B_1C_1$.Let $\triangle A_2B_2C_2$ be the reflected triangle of $\triangle A_1B_1C_1$ wrt O.Suppose N is the Nagel Point of $\triangle A_1B_1C_1$.Then A_1N cuts $\odot ABC$ at the antipode of A, let it be A'.Then $AA_2A'A_1$ is a parallelogram which implies $AA_2 \parallel A_1N \implies AA_2$ passes through the reflection of N wrt O,let it be N'.Now $AH \parallel OA_1$.Clearly $A(A_2,O,A_1,\infty)=-1 \implies A(A_2,O,A_1,H)=-1$.Since A,A_2,N' and A,S,A_1 are collinear we conclude $A(N',O,S,H)=-1$.But from **(1)** we conclude $N' \equiv L^*$.(Since same thing holds for B,C). So done.

High School Olympiads

With the Feuerbach's point

Reply

Source: own

jayme
#1 Mar 11, 2012, 8:15 pm

Dear Mathlinkers,
1. ABC a triangle
2. B', C' the midpoints of AC, AB
3. I the incenter of ABC
4. A*the point of intersection of the perpendicular to BI through B' with the perpendicular to CI through C'
5. Fe the Feuerbach's point of ABC

Prouve : the Euler line of A*B'C' goes through Fe.

Sincerely
Jean-Louis

Luis González
#2 Mar 11, 2012, 8:49 pm

This is the same as your former post With the Feuerbach's point. If A' denotes the midpoint of BC, then A* is clearly the A-excenter of A'B'C'. Similarly B* and C* are the B' and C' excenters of A'B'C', i.e. A'B'C' becomes orthic triangle of A*B*C*. Euler lines of A*B'C', B*C'A', C*A'B' concur at X(100) of A'B'C', i.e. the Feuerbach point of ABC.

jayme
#3 Mar 11, 2012, 8:57 pm

Dear Luis and Mathlinkers,
yes you are right... I come tback to this configuration while searching another result...
Sincerely
Jean-Louis

IDMasterz
#4 Aug 29, 2015, 11:09 pm

Let DEF be the intouch triangle. Let H be the orthocentre of DEF, and H_A be the orthocentre of $AB'C'$. Let O be the circumcentre of $\odot AB'EFC'I$. D_1, D_2, D_3, K denote the reflection of I over EF, $DH \cap \odot DEF$, $DI \cap \odot DEF$, $AH_A \cap BC$ respectively.

Since $\angle F_eD_1I = \angle F_eD_2D = \angle F_eD_3I \implies F_eD_1ID_3$ are concyclic. As a result, O, F_e, D_3 are collinear. Also, notice that KDF_eH_A are concyclic, since it is the reflection of the circle $\odot AB'EFC'I$ over the A-midline of ABC (notice that $B'C'$ is the A-midline). Hence, $\angle H_AF_eD = 90 = \angle D_3F_eD \implies H_A, F_e, D_3$ are collinear. The conclusion follows.

Quick Reply

High School Olympiads

AP Geo 2

Reply

Source: own

applepi2000

#1 Mar 8, 2012, 7:40 pm

Let ABC be a triangle, and let E, F be points on AC, AB respectively of $\triangle ABC$. Let the circumcircles of triangles AFC, AEB meet again at Y, and let AY intersect the circumcircle of AEF again at X. Prove that $\frac{AF}{FB}+\frac{AE}{EC}=\frac{AX}{XY}$.

Luis González

#2 Mar 10, 2012, 9:58 pm • 1

Perform an inversion with center A and arbitrary power k^2. Label inverse points with primes. Circles $\odot(ACF), \odot(ABE)$, $\odot(AEF)$ go to the lines $C'F'$, $B'E'$, $E'F'$. Thus $Y' \equiv B'E' \cap C'F'$ and $X' \equiv AY' \cap E'F'$ are the inverse images of Y, X. By Van Aubel theorem for $\triangle AE'F'$ with concurrent cevians AX', $E'B'$, $F'C'$, we have

$$\frac{AY'}{X'Y'}=\frac{AB'}{F'B'}+\frac{AC'}{E'C'} \Longrightarrow \frac{\frac{k^2}{AY}}{k^2\left(\frac{1}{AY}-\frac{1}{AX}\right)}=\frac{\frac{k^2}{AB}}{k^2\left(\frac{1}{AB}-\frac{1}{AF}\right)}+\frac{\frac{k^2}{AC}}{k^2\left(\frac{1}{AC}-\frac{1}{AE}\right)}$$

$$\Longrightarrow \frac{AX}{AY-AX}=\frac{AX}{XY}=\frac{AF}{AB-AF}+\frac{AE}{AC-AE}=\frac{AF}{FB}+\frac{AE}{EC}.$$

Quick Reply

High School Olympiads

Two parallels

Source: own

jayme

#1 Feb 28, 2012, 8:51 pm

Dear Mathlinkers,
1. ABC a triangle
2. (O) the circumcircle of ABC
3. I the incenter of ABC
4. X the symmetric of I wrt BC
5. D the midpoint of the arch BC which doesn't contain A
6. E the second point of intersection of XD with (O)
7. P the second point of intersection of EI with (O)

Prouve : PD is parallel to AE.

Sincerely
Jean-Louis

Luis González

#2 Feb 29, 2012, 10:56 pm • 2

Power of I with respect to (O) is $\overline{IA} \cdot \overline{ID} = -2Rr \Longrightarrow \frac{AI}{AO} = \frac{IX}{ID}$. Since $\angle XID = \angle IAO$, it follows that $\triangle XID \sim \triangle IAO$ by SAS criterion. Hence $\angle AOI = \angle ADE = \angle API \Longrightarrow A, I, O, P$ are concyclic and O is midpoint of the arc AIP of their circumcircle $\Longrightarrow OI$ bisects the angle between AD and PE, i.e. cyclic quadrilateral $APDE$ is an isosceles trapezoid with $PD \parallel AE$.

buratinogigle

#3 Mar 1, 2012, 10:11 pm

Here is general problem

Let ABC be triangle with circumcircle (O, R). P, P^* are two isogonal conjugate points with respect to triangle ABC. Q is reflection of P through BC. AP, AP^* cut (O) again at D, D'. DQ cuts (O) again at E. EP^* cuts (O) again at E'. Prove that $AE \parallel D'E'$.

Sayan

#4 Mar 2, 2012, 6:09 pm

Can anyone please explain why $\angle XID = \angle IAO$ and if OI bisects why $AEPD$ is isosceles trapezium?

r1234

#5 Mar 3, 2012, 3:20 pm

From here we know that $OIDE$ is cyclic.Now invert wrt I with power $-IA.ID = -2Rr$. Now if $I' = OI \cap DE$ then from the above link its easy to get that $IO.II' = 2Rr$.Hence after inversion $\odot OIDE$ goes to the line ACI'.Hence A, P, I' are collinear.But I, A, P are the inverse of I, A, P respectively.So $OIAP$ is cyclic.Now $\angle AIP = \angle DIE \Longrightarrow \angle AOP = \angle DOE$. Hence $AP = DE$.Hence $APDE$ is an isosceles trapezium $\Longrightarrow AE \parallel PD$.

Luis González

#6 Mar 10, 2012, 5:50 am

> buratinogigle wrote:
>
> Let ABC be triangle with circumcircle (O,R). P, P^* are two isogonal conjugate points with respect to triangle ABC. Q is reflection of P through BC. AP, AP^* cut (O) again at D, D'. DQ cuts (O) again at E. EP^* cuts (O) again at E'. Prove that $AE \parallel D'E'$.

Easy angle chase reveals that $\angle CP^*D' = \angle DCP$, thus $\triangle CP^*D' \sim \triangle PCD$

$$\frac{PD}{CD'} = \frac{CP}{CP^*} \Longrightarrow \frac{PD}{2R} = \frac{CP}{CP^*} \cdot \sin\widehat{CAP'} = \frac{CP}{AP^*} \cdot \sin\widehat{BCP} = \frac{1}{2} \cdot \frac{PQ}{AP^*}$$

Hence, we have $\frac{AO}{AP^*} = \frac{PD}{PQ}$. Since $\angle OAP^* = \angle DPQ$, it follows that $\triangle OAP^*$ and $\triangle DPQ$ are similar by SAS criterion $\Longrightarrow \angle AOP^* = \angle PDE = \angle AE'E \Longrightarrow P^*$ is on perpendicular bisector of $\overline{AE} \Longrightarrow AED'E'$ is isosceles trapezoid with $AE \parallel D'E'$.

Quick Reply

High School Olympiads

angle BCM=angle DBA in a convex quad

Reply

Source: ItaMO 2010, p3

Sayan

#1 Feb 28, 2012, 9:37 pm

Let $ABCD$ be a convex quadrilateral. such that $\angle CAB = \angle CDA$ and $\angle BCA = \angle ACD$. If M be the midpoint of AB, prove that $\angle BCM = \angle DBA$.

Luis González

#2 Feb 29, 2012, 12:38 am

Let $\odot(ACD)$ cut BC again at P. Then A is the midpoint of the arc PD of $\odot(ACD) \Longrightarrow$ tangent AB of $\odot(ACD)$ at A is parallel to DP. If Q is the 2nd intersection of DB with $\odot(ACD)$, we have $\angle BCQ = \angle PDQ = \angle DBA \Longrightarrow \odot(CQB)$ is tangent to AB. Thus, radical axis CQ of $\odot(ACD)$ and $\odot(CQB)$ passes through the midpoint M of their common tangent $\overline{AB} \Longrightarrow \angle BCM \equiv \angle BCQ = \angle DBA$.

avatarofakato

#3 Mar 8, 2012, 9:12 pm

Let N be the midpoint of AD. It is easy to see that $\angle MCN = \angle BCA$ and $\angle MAN = \angle MAC + \pi - \angle ADC - \angle ACD = \pi - \angle ACD = \pi - \angle MCN$ so $MCNA$ is cyclic $\Longrightarrow$ $\angle BCM = \angle ACN = \angle AMN$ but obviously $MN \parallel BD$ hence $\angle BCM = \angle AMN = \angle ABD$. qed

Quick Reply

High School Olympiads

geometry

Reply

huyhoang
#1 Dec 23, 2010, 9:43 am

Let ABC be an acute triangle but not isoceles. Let D be the foot of the altitude form A and ω the circumcircle of the triangle. Let ω_1 be the circle tangent to AD, BD and ω. Let ω_2 be the circle tangent to AD, CD and ω . Let l be the interior common tangent to $\omega_1$and ω_2 different from CD.Prove that l passes through the midpoint of BC if and only $2BC = AB + AC$

huyhoang
#2 Dec 23, 2010, 8:16 pm

Is this problem hard ?

Luis González
#3 Feb 28, 2012, 3:49 am • 1

It's problem 4 of Romanian IMO TST 2006. Posted many times before.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=88823
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=272503
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=351270
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=399091

Quick Reply

High School Olympiads

geometry

huyhoang

#1 Aug 11, 2011, 1:04 pm

Triangle ABC is scalene with $BC > AC$. M is an arbitrary point on $[AC]$ and C', A' in segments AB, BC such that $MC' = MB, MA' = MC$. Prove that the center of the circle $A'BC'$ runs in a line.

huyhoang

#2 Sep 15, 2011, 7:53 pm

Can someone help me find the locus and I solve the rest

Luis González

#3 Feb 28, 2012, 2:58 am • 1

> huyhoang wrote:
> Triangle ABC is scalene with $BC > AC$. M is an arbitrary point on $[AC]$ and C', A' in segments AB, BC such that $MC' = MA, MA' = MC$. Prove that the center of the circle $A'BC'$ runs in a line.

Let P be the 2nd intersection of $\odot(MA'C')$ with AC. $\odot(PCA'), \odot(PAC')$ and $\odot(BA'C')$ concur at the Miquel point H of $\triangle ABC \cup A'MC'$. $\angle PHA' = 180^\circ - \angle BCA = 90^\circ + \frac{1}{2}\angle PC'A' \Longrightarrow \odot(PCA')$ passes through the incenter of $\triangle PA'C'$. Likewise, $\odot(PAC')$ passes through the incenter of $\triangle PA'C'$, i.e. H is incenter of $\triangle PA'C' \Longrightarrow$ $\angle HCB = \angle HPA' = \frac{1}{2}\angle A'MC' = 90^\circ - \angle ABC$. Similarly, $\angle HAB = 90^\circ - \angle ABC \Longrightarrow H$ is orthocenter of $\triangle ABC$. So the center of $\odot(BA'C')$ runs through the perpendicular bisector of $\overline{BH}$.

High School Olympiads

Tangential quadrilateral EDF = ADC/2

Reply

scarface

#1 Feb 26, 2012, 2:10 am

$ABCD$ is a circumscribed quadrilateral to a circle. O is the center. E is a point on OA and F is a point on OC. If $\angle EBF = \frac{\angle ABC}{2}$, the prove that $\angle EDF = \frac{\angle ADC}{2}$

Luis González

#2 Feb 26, 2012, 5:18 am • 1

Since $AB - AD = CB - CD$, then A and C lie on a hyperbola $\mathcal{H}$ with foci B, D. Internal bisectors OA, OC of $\angle BAD, \angle DCB$ are then tangents of $\mathcal{H}$ at A, C. Let the tangent from E to $\mathcal{H}$, different from OA, cut OC at F'. $P \equiv \mathcal{H} \cap EF'$. According to this problem (post #3), BE, DE, BF', DF' bisect $\angle PBA, \angle PDA, \angle PBC, \angle PDC$, respectively. Thus $\angle EDF' = \frac{1}{2}\angle ADC, \angle EBF' = \frac{1}{2}\angle ABC \Longrightarrow F \equiv F'$.

Quick Reply

High School Olympiads

Perpendicular in Quadrilateral

Locked

FoolMath

#1 Feb 25, 2012, 10:56 pm

Give a quadrilateral ABCD with AB//CD, AC meet BD at E. d is bisector of AB, d intersec CD at F. P and Q are circumcenter of triangles ADF and BCF. Prove that PQ is perpendicular to EF

Luis González

#2 Feb 26, 2012, 3:29 am

Please do a search before posting. This has been submitted many times before.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=157599
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=258337
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=282875
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=316357

High School Olympiads

four points are cyclic

Locked

Source: 2009 China Western Mathematic Olmpiad

horizon

#1 Feb 25, 2012, 9:40 pm

Let H be the orthocenter of acute triangle ABC and D the midpoint of BC. A line through H intersects AB, AC at F, E respectively, such that $AE = AF$. The ray DH intersects the circumcircle of $\triangle ABC$ at P. Prove that P, A, E, F are concyclic.

Luis González

#2 Feb 25, 2012, 11:46 pm

This is P3 (10th grade) from All-Russian olympiad 2000 and also a problem from the Swiss Imo Selection 2006. Posted many times before, so thread locked.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=89098
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=204811
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=320620
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=409570

High School Olympiads

tangency circumcircle

MBGO
#1 Feb 15, 2012, 12:01 am

In triangle ABC, P and Q are in AB and AC. Let M and N and K be the midpoint of BQ and CP and PQ, and PQ intersect BC at D. Then if the circumcircle of ABC and APQ and MKN are W1, W2 and W3, and W1 and W2 meet eachother at F and W3 tangent to line PQ, prove that W1 is tangent to FD.

Luis González
#2 Feb 15, 2012, 1:41 am

Let O be the circumcenter of $\triangle ABC$ and R the projection of O on $PQ.$ According to the topic Cyclic and the symmetry point is in Euler circle (problem a), the points M,N,K,R are concyclic. Therefore if $\odot(MKN)$ is tangent to $PQ,$ then $K \equiv R,$ i.e. OK is perpendicular bisector of $\overline{PQ}.$ By symmetry $AFPQ$ is then an isosceles trapezoid with $AF \parallel PQ.$ Since F is Miquel point of $\overline{DPQ}$ WRT $\triangle ABC,$ then F,P,B,D are concyclic $\Longrightarrow \angle DFB=\angle DPB=\angle FAP \Longrightarrow FD$ is tangent to $(O).$

hatchguy
#3 Feb 15, 2012, 2:50 am

Showing that $OP=OQ$ is problem 2 of IMO 2009. This shows $PQ \parallel AF$ and the rest is an angle chase.

I have a question Luis, I usually angle chase facts suchs as $FPBD$ being concyclic. I noticed you used Miquel's theorem for this, however I don't see how to apply the version of the Miquel's theorem I know in such configuration.

The version I know is for example, given a triangle ABC and points P,Q,R in BC,AC,AB, respectively, then the circumcircles of ARQ, BPR and CQP share a common point.

High School Olympiads

A tangent parallel to BC

Reply

Source: own ?

jayme

#1 Feb 14, 2012, 5:22 pm

Dear Mathlinkers,
1. ABC a triangle
2. I the incenter of ABC
3. B', C' the feet of the B, C-inner bissectors of ABC
4. (0) the circumcircle of ABC
5. M, N the points of intersection of B'C' with (0)
6. (1) the circumcircle of the triangle IMN
7. Ti the tangente to (1) at I.

Prouve : Ti is parallel to BC.

Sincerely
Jean-Louis

Luis González

#2 Feb 14, 2012, 10:44 pm

I can be an arbitrary point on the angle bisector of $\angle BAC.$ IB,IC cut CA,AB at B',C' and cut the circumcircle of $\triangle ABC$ again at $D,E.$ L is the midpoint of the arc BC of the circumcircle and LD,LE cut CA,AB at $U,V.$ According to Srmc (post #4), the lines $B'C',UIV,DE$ concur at P and $UIV \parallel BC.$ Therefore, $\angle EDI=\angle ECB=\angle EIP \Longrightarrow$ $PI^2=PE \cdot PD=PM \cdot PN \Longrightarrow PI \parallel BC$ is tangent to $\odot(IMN).$

Quick Reply

High School Olympiads

2012 Japan Mathematical Olympiad Finals Problem 4

Reply

Kunihiko_Chikaya

#1 Feb 11, 2012, 11:32 pm

Given two triangles PAB and PCD such that $PA = PB,\ PC = PD,\ P,\ A,\ C$ and $B,\ P,\ D$ are collinear in this order respectively.
The circle S_1 passing through $A,\ C$ intersects with the circle S_2 passing through $B,\ D$ at distinct points $X,\ Y$.
Prove that the circumcenter of the triangle PXY is the midpoint of the centers of $S_1,\ S_2$.

buratinogigle

#2 Feb 12, 2012, 12:47 am

Here is a general problem

Let ABC be triangle. M is a point on side AB. N is a point on AC such that MN is parallel to A-symmedian of triangle ABC. The circle (O_1) passing through M, B intersects circle (O_2) passing through N, C at t distinct points X, Y. Prove that circumcenter of triangle AXY is midpoint of O_1O_2.

Luis González

#3 Feb 12, 2012, 4:23 am • 1

> buratinogigle wrote:
> Let ABC be triangle. M is a point on side AB. N is a point on AC such that MN is parallel to A-symmedian of triangle ABC. The circle (O_1) passing through M, B intersects circle (O_2) passing through N, C at t distinct points X, Y. Prove that circumcenter of triangle AXY is midpoint of O_1O_2.

Let K denote the foot of the A-symmedian and $D \equiv MN \cap BC$. Then

$$\frac{\overline{AN}}{\overline{AC}} = \frac{\overline{KD}}{\overline{KC}},\ \frac{\overline{AM}}{\overline{AB}} = \frac{\overline{KD}}{\overline{KB}} \Longrightarrow \frac{\overline{AN} \cdot \overline{AC}}{\overline{AM} \cdot \overline{AB}} = \frac{\overline{KD}}{\overline{KC}} \cdot \frac{\overline{KB}}{\overline{KD}} \cdot \frac{\overline{AC}^2}{\overline{AB}^2} = -1$$

This means that the ratio of the powers of A WRT (O_1) and (O_2) equals $-1 \Longrightarrow A$ is on the circle coaxal with $(O_1), (O_2)$ whose center O divides $\overline{O_1O_2}$ in the ratio $\overline{OO_1} : \overline{OO_2} = -1$, i.e. circumcenter O of $\triangle AXY$ is midpoint of O_1O_2.

SnowEverywhere

#4 Feb 13, 2012, 9:52 pm • 4

Let O_1, O_2 be the circumcenters of S_1 and S_2, respectively. Let M be the midpoint of O_1O_2 and let $MO_1 = MO_2 = m$. Let r_1 and r_2 be the radii of S_1 and S_2, respectively. By power of a point, it follows that $PA \cdot PC = PO_1^2 - r_1^2$ and $PB \cdot PD = r_2^2 - PO_2^2$ and hence that $PO_1^2 + PO_2^2 = r_1^2 + r_2^2$. Also note that $XO_1^2 + XO_2^2 = YO_1^2 + YO_2^2 = r_1^2 + r_2^2$. By Stewart's theorem, it now follows that $2PM^2 + 2m^2 = PO_1^2 + PO_2^2 = r_1^2 + r_2^2 = 2XM^2 + 2m^2 = 2YM^2 + 2m^2$. Hence $XM = YM = PM$ and M is the circumcenter of $\triangle PXY$.

horizon

#5 Feb 28, 2012, 9:25 pm

Let K be the mipoint of segment S_1S_2,then by $PB*PD=PA*PC$,we get $R_1^2 - O_1P^2 = O_2P^2 - R_2P^2$,

then we have $PK^2 = R_1^2 + R_2^2 - \frac{1}{2}OO_2^2 = XK^2 = YK^2$

Quick Reply

High School Olympiads

Cyclic quadrialteral

Reply

buratinogigle

#1 Feb 12, 2012, 12:56 am

Let $ABCD$ be cyclic quadrilateral. d is perpendicular bisector of BD. P is a point on d. Q is reflection of P through bisector of angle $\angle BAD$. R is reflection of P through bisector of angle $\angle BCD$. Prove that AQ, CR and d are concurrent.

See more Problem 5 (Second Day) IMO 2004.

Luis González

#2 Feb 12, 2012, 3:20 am

Let d cut the circumcircle of $ABCD$ again at $M, N.$ Obviously the perpendicular lines $AN \perp AM$ bisect $\angle PAQ$ internally and externally $\Longrightarrow$ Pencil of lines $A(P,Q,N,M)$ is harmonic $\Longrightarrow$ AQ passes through the harmonic conjugate of P WRT $M,N.$ By similar reasoning, CR passes through the harmonic conjugate of P WRT $M,N,$ i.e. lines AQ,CR and d are concurrent.

genxium

#3 Feb 13, 2012, 11:27 pm

Solution from luisgeometra is perfect, I solved this problem in a similar but ugly approach.

Assume $d\cap\odot ABCD=N$ such that AN bisects $\angle BAD, AQ\cap d=X, AN\cap PQ=Y,$

$$\frac{XM}{XN}=\frac{\triangle XMA}{\triangle XNA}=\frac{AM\cdot sin\angle XAM}{AN\cdot sin\angle XAN}=\frac{AM\cdot sin(\frac{\pi}{2}+\angle XAN)}{AN\cdot sin\angle XAN}=\frac{tan\angle ANM}{tan\angle XAN}=\frac{YA}{YN}=\frac{PM}{PN}.$$

Similar process for point C. Then d, AQ, CR concurrent.

phuongtheong

#4 Feb 18, 2012, 11:18 pm

M,N is the intersection of d and $(ABCD)$. $M\in$ arc DCB, $N\in$ arc BAD.
I is the intersection of AQ and MN. We have $(MNPI)=-1$
J is the intersection of CR and MN. We have $(MNPJ)=-1$
$\Rightarrow I\equiv J$.
$\Rightarrow AQ, CR, MN$ are concurrent.

Quick Reply

High School Olympiads

Bulgaria, winter competition '99... nice circle geometry

Reply

ivanbart-15

#1 Feb 11, 2012, 6:34 pm

Let O and R be the circumcenter and the circumradius of the triangle ABC. The incircle of this triangle is tangent to BC,CA and AB at A_1, B_1 and C_1 respectively. The lines determined by the midpoints of the line segments AB_1 and AC_1,BC_1 and BA_1,CA_1 and CB_1 intersect at A_2,B_2 and C_2. Prove that the triangle $A_2B_2C_2$ has circumcenter O and circumradius $R+\frac{r}{2}$

Luis González

#2 Feb 11, 2012, 10:43 pm • 2

Let ℓ_A be the line passing through the midpoints of AB_1 and AC_1. ℓ_B and ℓ_C are defined similarly. ℓ_B is radical axis of the incircle (I) and the circle (B) with zero radius. Likewise, ℓ_C is the radical axis of (I) and $(C) \Longrightarrow A_2 \equiv \ell_B \cap \ell_C$ is the radical center of $(I), (A), (B) \Longrightarrow A_2$ is on radical axis of $(B), (C)$, i.e. A_2 is on perpendicular bisector of BC. Similarly, B_2 and C_2 lie on the perpendicular bisectors of CA, AB. Now, since $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are obviously homothetic and $OA_2 \parallel IA_1, OB_2 \parallel IB_1, OC_2 \parallel IC_2$, it follows that O is circumcenter of $\triangle A_2B_2C_2$.

Let M_a, M_b, M_c be the midpoints of BC, CA, AB and let P, Q be the midpoints of BA_1, CA_1. A-excircle (I_a, r_a) of $\triangle ABC$ touches BC at X_a. $\triangle I_aBC$ and $\triangle A_2PQ$ are obviously similar, even homothetic, with corresponding altitudes I_aX_a and A_2M_a

$$\frac{A_2M_a}{I_aX_a} = \frac{A_2M_a}{r_a} = \frac{PQ}{BC} = \frac{1}{2} \Longrightarrow A_2M_a = \tfrac{1}{2}r_a.$$

Analogously, we have $B_2M_b = \frac{1}{2}r_b$ and $C_2M_c = \frac{1}{2}r_c$. If ϱ denotes the circumradius of $\triangle A_2B_2C_2$, we have then $\varrho = OM_a + \frac{1}{2}r_a = OM_b + \frac{1}{2}r_b = OM_c + \frac{1}{2}r_c \Longrightarrow$

$$\varrho = \tfrac{1}{3}[OM_a + OM_b + OM_c + \tfrac{1}{2}(r_a + r_b + r_c)] = \tfrac{1}{3}[R + r + \tfrac{1}{2}(4R + r)] = R + \tfrac{1}{2}r$$

Quick Reply

High School Olympiads

Power of a point

Ali-mes

#1 Feb 4, 2012, 12:24 am • 1

Let $ABCD$ be a non-cyclic quadrilateral, show that:
$$\frac{1}{P_{(BCD)}(A)}+\frac{1}{P_{(ACD)}(B)}+\frac{1}{P_{(ABD)}(C)}+\frac{1}{P_{(ABC)}(D)}=0$$

$P_{(XYZ)}(M)$ denotes the power of point M wrt the circumcircle of $\triangle XYZ$.

Luis González

#2 Feb 10, 2012, 11:02 am

Let $(u:v:w)$ be the barycentric coordinates of D WRT $\triangle ABC$. Thus, barycentric equations of the circumcircles Ω_A, Ω_B, Ω_C, Ω_D of $\triangle BCD, \triangle ACD, \triangle ABD, \triangle ABC$ WRT $\triangle ABC$ are given by

$$\Omega_A \equiv a^2yz+b^2xz+c^2xy-x(x+y+z)\frac{(a^2vw+b^2wu+c^2uv)}{u(u+v+w)}=0$$

$$\Omega_B \equiv a^2yz+b^2xz+c^2xy-y(x+y+z)\frac{(a^2vw+b^2wu+c^2uv)}{v(u+v+w)}=0$$

$$\Omega_C \equiv a^2yz+b^2xz+c^2xy-z(x+y+z)\frac{(a^2vw+b^2wu+c^2uv)}{w(u+v+w)}=0$$

$$\Omega_D \equiv a^2yz+b^2xz+c^2xy=0$$

Power of A WRT Ω_A is then $p(A,\Omega_A)=\dfrac{a^2vw+b^2wu+c^2uv}{u(u+v+w)}$

Adding the reciprocals of the cyclic expressiones together yields

$$\frac{1}{p(A,\Omega_A)}+\frac{1}{p(B,\Omega_B)}+\frac{1}{p(C,\Omega_C)}=\frac{(u+v+w)^2}{a^2vw+b^2wu+c^2uv}$$

The RHS is precisely the negative reciprocal of the power of D WRT Ω_D.

$$\frac{1}{p(A,\Omega_A)}+\frac{1}{p(B,\Omega_B)}+\frac{1}{p(C,\Omega_C)}+\frac{1}{p(D,\Omega_D)}=0.$$

Ali-mes

#3 Feb 10, 2012, 7:59 pm • 1

Here's my solution without the use of barycentric coordinates:
Suppose WLOG that we have the configuration given in the figure, other cases may be treated analogically:

Let $I=(AC)\cap(BD)$, and let A' be the second intersection of line (AC) with (BCD).
thus, we have: $P_{(BCD)}(A)=AA'.AC=(AI-IA').AC=(AI-\dfrac{IB.ID}{IC}).AC$.

And by the same way we define point B' as the second intersection of line (BD) with (ACD).
$\Longrightarrow P_{(ACD)}(B)=-BB'.BD=-(IB'-IB).BD=-(\dfrac{IA.IC}{ID}-IB).BD=(IB-\dfrac{IA.IC}{ID}).BD$

Let C' be the second intersection of (AC) with (ABD).
$\Longrightarrow P_{(ABD)}(C) = CC'.AC = (IC - IC').AC = (IC - \frac{IB.ID}{IA}).AC$

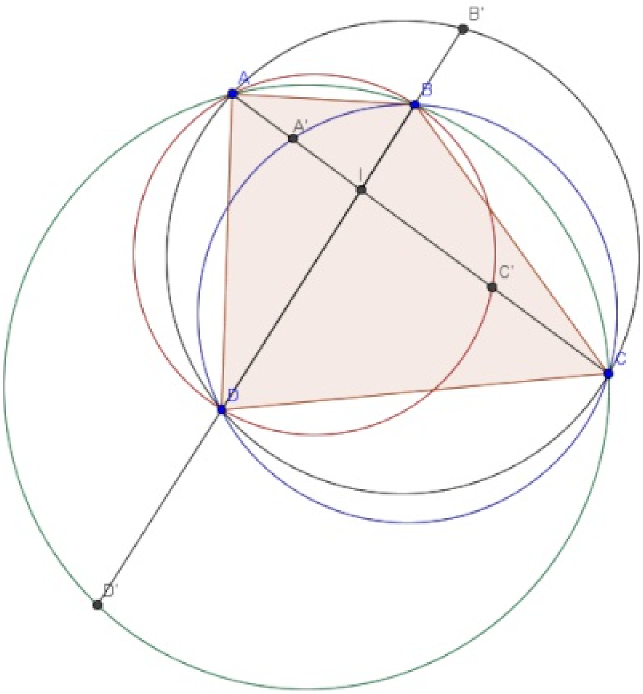
Let D' be the second intersection of (BD) with (ABC).
$\Longrightarrow P_{(ABC)}(D) = -DD'.BD = -(ID' - ID).BD = (ID - ID').BD = (ID - \frac{IA.IC}{IB}).BD$

And hence:

$$\frac{1}{P_{(BCD)}(A)} + \frac{1}{P_{(ACD)}(B)} + \frac{1}{P_{(ABD)}(C)} + \frac{1}{P_{(ABC)}(D)} = \frac{1}{(AI - \frac{IB.ID}{IC}).AC} + \frac{1}{(IB - \frac{IA.IC}{ID}).BD} + \frac{1}{(IC - \frac{IB.ID}{IA}).AC} + \frac{1}{(ID - \frac{IA.IC}{IB}).BD}$$
$$= \frac{IC}{(IA.IC - IB.ID).AC} + \frac{ID}{(IB.ID - IA.IC).BD} + \frac{IA}{(IA.IC - IB.ID).AC} + \frac{IB}{(IB.ID - IA.IC).BD}$$
$$= \frac{1}{IA.IC - IB.ID}\left(\frac{IC}{AC} - \frac{ID}{BD} + \frac{IA}{AC} - \frac{IB}{BD}\right) = \frac{1}{IA.IC - IB.ID}\left(\frac{IA + IC}{AC} - \frac{IB + ID}{BD}\right) = \frac{1}{IA.IC - IB.ID}\left(\frac{AC}{AC} - \frac{BD}{BD}\right) = 0$$

QED..

Attachments:



↩ Quick Reply

High School Olympiads

Gergonne Point

syk0526

#1 Feb 9, 2012, 9:11 am

Let $\triangle ABC$ be a scalene triangle with its incircle I. AB, BC, CA meet incircle I at D, E, F. Let $X = BC \cap EF, Y = CA \cap FD, Z = AB \cap DE, G_e = AD \cap BE \cap CF$. M is the midpoint of YZ and $N = AM \cap EF$.

(a) Prove that N, I, G_e collinear

(b) Let $AM \cap BC = T$, and F_e be a Feurbach point of $\triangle ABC$. Prove that TF_e is a tangent line of I .

(c) (My own,,, maybe wrong) AM, BE, DF concurrent $\iff N, I, G_e, Y$ collinear

Luis González

#2 Feb 10, 2012, 7:02 am

It's clear that AD, BE, CF are polars of X, Y, Z WRT $(I) \Longrightarrow G_e$ is the pole of $\overline{XYZ}$ WRT $(I) \Longrightarrow IG_e \perp \overline{XYZ}$. Thus, line ℓ_A parallel to YZ from A is the polar of $N' \equiv EF \cap IG_e$ WRT $(I) \Longrightarrow$ Pencil of lines AE, AF, AN', ℓ_A is harmonic $\Longrightarrow AN' \equiv AM \Longrightarrow N \equiv N'$, i.e. N, I, G_e are collinear.

Let P, Q, R be the midpoints of $BC, CA, AB. U \equiv RQ \cap EF$ is the A-vertex of the side triangle of $\triangle DEF$ and $\triangle PQR$ WRT $\triangle ABC \Longrightarrow AU$ passes through the intersection of the trilinear polars of the centroid G and Gergonne point G_e of $\triangle ABC \Longrightarrow AU \parallel YZ$, i.e. $AU \equiv \ell_A$. By 1st Fontené theorem, DU cuts (I) again at the Feuerbach point $F_e \equiv (I) \cap \odot(PQR)$. Since $(E, F, U, N) = -1$, then AT is the polar of U WRT (I). Thus, polar of F_e WRT (I) passes through T, i.e. TF_e is tangent to (I).

Quick Reply

High School Olympiads

projection on areal coordinates

Reply

KittyOK
#1 Jan 31, 2012, 3:46 pm

Let ABC be a triangle of reference and define the barycentric (areal) coordinates as usual. Define a point $P(x_0:y_0:z_0)$ and a line $\ell: px+qy+rz=0$. What is the coordinate of the projection of P on ℓ?

Luis González
#2 Feb 1, 2012, 9:39 am • 1

Barycentric coordinates of the projection Q of P on ℓ are actually rather complicated. Infinite point of $\perp \ell$ is $(pa^2-qS_C-rS_B : qb^2-rS_A-pS_C : rc^2-pS_B-qS_A)$. Thus equation of the perpendicular from P to ℓ is

$$\begin{bmatrix} pa^2-qS_C-rS_B & qb^2-rS_A-pS_C & rc^2-pS_B-qS_A \\ x_0 & y_0 & z_0 \\ x & y & z \end{bmatrix} = 0$$

This line cuts $\ell \equiv px+qy+rz=0$ at Q, whose first coordinate x_1 is given by

$$x_1 = pq(a^2x_0+x_0S_C)+pr(a^2y_0+x_0S_B)-qr(y_0S_B+z_0S_C-2x_0S_A)- \\ q^2(b^2x_0-y_0S_C)-r^2(c^2x_0+z_0S_B)$$

Coordinates y_1, z_1 are obtained by cyclic permutations $a\to b, b\to c, c\to a, p\to q, q\to r, r\to p, S_A\to S_B$, $S_B\to S_C, S_C\to S_A, x_0\to y_0, y_0\to z_0, z_0\to x_0$.

KittyOK
#3 Feb 1, 2012, 7:33 pm

Thanks very much, luisgeometra
Some silly questions : What is S_A? And how can you show that the infinite point of $\perp \ell$ is $(pa^2-qS_C-rS_B : qb^2-rS_A-pS_C : rc^2-pS_B-qS_A)$?

Luis González
#4 Feb 2, 2012, 9:52 am • 1

Let $\ell_1 \equiv p_1x+q_1y+r_1z=0$ and $\ell_2 \equiv p_2x+q_2y+r_2z=0$ be the barycentric equations of two lines WRT $\triangle ABC$. Vectors $\vec{\mathbf{v_1}}=(\lambda_1,\mu_1,\nu_1)$ and $\vec{\mathbf{v_2}}=(\lambda_2,\mu_2,\nu_2)$ are parallel to ℓ_1,ℓ_2, respectively.

$$\vec{\mathbf{v_1}}\cdot\vec{\mathbf{v_2}} = \left(\mu_1\overrightarrow{AB}+\nu_1\overrightarrow{AC}\right)\cdot\left(\mu_2\overrightarrow{AB}+\nu_2\overrightarrow{AC}\right) =$$

$$= c^2\mu_1\mu_2+b^2\nu_1\nu_2+bc(\mu_1\nu_2+\mu_2\nu_1)\cos A =$$

$$= (S_A+S_B)\mu_1\mu_2+(S_A+S_C)\nu_1\nu_2+S_A(\mu_1\nu_2+\mu_2\nu_1) =$$

$$= S_A(\mu_1+\nu_1)(\mu_2+\nu_2)+S_B\mu_1\mu_2+S_C\nu_1\nu_2 =$$

$$= S_A\lambda_1\lambda_2+S_B\mu_1\mu_2+S_C\nu_1\nu_2$$

Keeping in mind that the norm of $\vec{\mathbf{v}}$ is $|\mathbf{v}| = \sqrt{S_A\lambda^2+S_B\mu^2+S_C\nu^2}$, then the angle θ between $\vec{\mathbf{v_1}}$ and $\vec{\mathbf{v_2}}$ is given by $\vec{\mathbf{v_1}}\cdot\vec{\mathbf{v_2}} = |\mathbf{v_1}|\cdot|\mathbf{v_2}|\cos\theta$. Thus

$$S_A\lambda_1\lambda_2+S_B\mu_1\mu_2+S_C\nu_1\nu_2$$

$$\cos\theta = \frac{\cdots}{\sqrt{S_A{\lambda_1}^2+S_B{\mu_1}^2+S_C{\nu_1}^2}\cdot\sqrt{S_A{\lambda_2}^2+S_B{\mu_2}^2+S_C{\nu_2}^2}}$$

Hence, $\overrightarrow{\mathbf{v_1}}$ and $\overrightarrow{\mathbf{v_2}}$ are orthogonal $\Longleftrightarrow S_A\lambda_1\lambda_2+S_B\mu_1\mu_2+S_C\nu_1\nu_2=0.$ Since the components of $\overrightarrow{\mathbf{v_1}}$ and $\overrightarrow{\mathbf{v_2}}$ are proportional to the coordinates of the infinite points of $\boldsymbol{\ell_1}$ and $\boldsymbol{\ell_2},$ then it follows that $\ell_1\perp\ell_2\Longleftrightarrow$

$$S_A(q_1-r_1)(q_2-r_2)+S_B(r_1-p_1)(r_2-p_2)+S_C(p_1-q_1)(p_2-q_2)=0$$

Thus, coordinates of the infinite point P_∞ of a line perpendicular to $\ell\equiv px+qy+rz=0$ are found by solving $S_A(q-r)x+S_B(r-p)y+S_C(p-q)z=0$ and $x+y+z=0.$ Which yields

$$P_\infty(pa^2-qS_C-rS_B:qb^2-rS_A-pS_C:rc^2-pS_B-qS_A).$$

P.S. $S_A,S_B,S_C,S_\omega,S_\varphi,$ etc is Conway triangle notation.

High School Olympiads

Construction

Reply

Shravu

#1 Jan 21, 2012, 7:52 pm

Construct the triangle given
$\angle A\ AB - AC$ and the inradius r

vanstraelen

#2 Jan 26, 2012, 8:26 pm

$\triangle ACD : AC = AD = b$ (not given), see picture.
$PD = b\sin\frac{\alpha}{2}, CD = 2b\sin\frac{\alpha}{2}, \alpha = \angle A$ (given).
$\angle ADC = 90^o - \frac{\alpha}{2}, \angle CDB = 90^o + \frac{\alpha}{2}$.
$\triangle CDB : DB = c - b = t$(given).
$BC^2 = a^2 = (2b\sin\frac{\alpha}{2})^2 + (c-b)^2 - 2(2b\sin\frac{\alpha}{2})(c-b)\cos(90^o + \frac{\alpha}{2})$.
$a^2 = (c-b)^2 + 4bc\sin^2\frac{\alpha}{2} \qquad (1)$.
$\triangle AMM' : AM' = r\cot\frac{\alpha}{2}$ (r given), $BM' = c - r\cot\frac{\alpha}{2} = BQ$.
$\triangle AMN' : AN' = r\cot\frac{\alpha}{2}, N'C = b - r\cot\frac{\alpha}{2} = CQ$.
$BC = a = BQ + QC = b + c - 2r\cot\frac{\alpha}{2} \qquad (2)$.
(1) and (2), with $c - b = t$ and $c = b + t$:

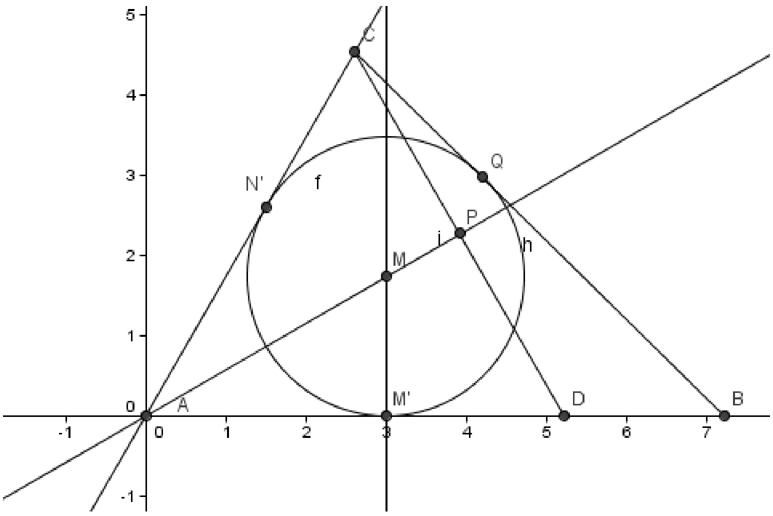
$$t^2 + 4bc\sin^2\frac{\alpha}{2} = (2b + t - 2r\cot\frac{\alpha}{2})^2$$

an equation in b; after calculating:

$$\left[b + \frac{1}{2}(t - \frac{4r}{\sin\alpha})\right]^2 = \frac{r^2}{\cos^2\frac{\alpha}{2}} + \frac{t^2}{4}$$

One solution for b, then $c = b + t$ and a.

Attachments:



Luis González

#3 Feb 1, 2012, 11:00 am

1st solution. Incircle (I) touches BC at X and M is midpoint of BC. Since $PM=\frac{1}{2}|AB-AC|$, then the right triangle $\triangle IXM$ with known catheti is constructible. Let J be the reflection of I about M, i.e $BICJ$ is a parallelogram $\Longrightarrow$ $\angle IBJ=\angle ICJ=180^\circ-\angle BIC=90^\circ-\frac{1}{2}\angle A.$ Hence, B,C are intersections of XM with the two circular arcs that see $\overline{IJ}$ under $90^\circ-\frac{1}{2}\angle A.$ Tangents from B,C to the incircle (I,IX), different from BC, meet at $A.$

2nd solution (projective). Construct two rays b,c such that $\angle(b,c)=\angle A$ and construct the incircle (I) with radius r tangent to b,c at $Y,Z.$ Pick an arbitrary C' on b (not lying on AY). Tangent from C' to (I), different from b, cuts c at B'. Then the serie C' is projective with the serie B'. We can construct a new point B'', such that $|AB''-AC'|$ equals the given difference. The serie B'' is also projective, even congruent, with the serie C'. Thus, the double points of the proyectivity between B' and B'' give the possible vertices $B.$

Shravu

#4 Apr 27, 2012, 9:58 am

Mu solution was same as your first solution.luisgeometra

Quick Reply

High School Olympiads

D,E,I collinear

r1234

#1 Jan 20, 2012, 2:56 pm • 2

In a triangle ABC, AD,BE are its altitudes from A to BC and B to CA respectively.Let I,O be the incenter and circumcenter of $\triangle ABC$.Show that if D,E,I are collinear then O,P,Q are also collinear where $P\in BC,Q\in CA$ and AP,BQ are the angle bisectors of $\angle A$ and $\angle B$.

Luis González

#2 Feb 1, 2012, 2:44 am

Posted before at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=50582
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=139275
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=219811

For a generalization see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=402242

High School Olympiads

concurrency

Reply

Babai

#1 Feb 1, 2012, 1:07 am

let X be a point.A,B,C arbitary points.A circle passing through X mee XA ,XB,XC at P,Q,R.The circle cuts circumcircles of BXC,CXA,AXB at K,L,M.Prove that PK,QL,RM are concurrent.[i tried using invetsion.but now confused what has to be proved in the new figure.so i will be happy if some one tries it using inversion.]

Luis González

#2 Feb 1, 2012, 2:10 am

Invert about X with arbitrary power. Denote inverse images with primes. Circles (BXC),(CXA) and (AXB) go to lines B'C',C'A' and A'B'. The arbitrary circle (U), passing through X, goes to an arbitrary line u cutting B'C',C'A',A'B' at K',M',L'. XA',XB',XC' cut u at P',Q',R'. Lines PK,QL,RM go to the circles (XP'K'),(XQ'L'),(XR'M'). Thus, you have to prove that these latter circles meet at another point besides X. See Geometry Problem (13).

Quick Reply

High School Olympiads

geometry

Reply

Source: zuming feng

horizon

#1 Jan 29, 2012, 10:06 pm

let ABC be an acute triangle with H as its orthocenter,circle ω passes through A and H and meets sides AB and AC at E and F,respectively,let O be the circumcenter of triangle AEF,the circumcircle of triangle OEF meet side BC at two points.prove that there must be one point D satisfy that $ED+FD=BC$

Luis González

#2 Jan 30, 2012, 1:53 am

Let OH cut BC at $P. \angle OHE=90^\circ-\angle EAH=\angle B \Longrightarrow EBPH$ is cyclic. Similarly, $FCPH$ is cyclic. Thus, $\angle EPH=\angle EBH=\angle FCH=\angle FPH \Longrightarrow PH$ bisects $\angle EPF \Longrightarrow O$ is midpoint of the arc EF of $\odot(PEF)$. Hence if $\odot(PEF)$ cuts BC again at D, we have $\angle FDC=\angle PEF=180^\circ-2\angle C \Longrightarrow \triangle CDF$ is isosceles with legs $DF=DC$. Similarly, $DE=DB$ and the conclusion follows.

Quick Reply

High School Olympiads

Circles intersecting on PA

Reply

buratinogigle

#1 Jan 22, 2012, 5:22 pm

Let ABC be triangle and P is a point. E, F are projections of P on lines CA, AB, resp. Q is isogonal conjugate of P with respect to triangle ABC. FP cuts EQ at K, EP cuts FQ at L. Prove that circumcircle (PEK) and (PFL) intersects again on line PA.

Note that, it is generalization of problem on the post circles intersecting on CH.

Luis González

#2 Jan 29, 2012, 12:13 pm • 1

This is basically a slight variation of my proof given in the referred reference.

Let $\odot(PEK)$ and $\odot(PFL)$ cut AP at U, V. If the perpendiculars to QE, QF at E, F cut AB, AC at M, N, then U and V are the second intersections of $\odot(AME)$ and $\odot(ANF)$ with AP. Let S, T denote the projections of Q on AC, AB. Since E, F, S, T lie on the pedal circle of P, Q WRT $\triangle ABC$, we have $\angle ESF = \angle ATE \Longrightarrow \angle QSF = \angle QTE \Longrightarrow \angle QNF = \angle QME$. But since P, Q are isogonal conjugates WRT $\triangle AME$ and $\triangle ANF$, then $\angle PNE = \angle PMF$, i.e. the right $\triangle PNE$ and $\triangle PMF$ are similar $\Longrightarrow \frac{PE}{PF} = \frac{EN}{FM}$.

Circles $\odot(AME)$ and $\odot(ANF)$ meet at A and the center X of the spiral similarity that takes $\overline{EN}$ into $\overline{MF}$. Hence $\frac{XE}{XM} = \frac{EN}{FM} = \frac{PE}{PF}$. As a result, $\triangle XEM$ and $\triangle PEF$ are similar by SAS $\Longrightarrow \angle PAF = \angle XEM = \angle XAM \Longrightarrow X \in AP \Longrightarrow X \equiv U \equiv V$.

panos_lo

#3 Feb 2, 2012, 1:14 am

Note that the idea of my solution in the link can also be easily applied here. I will post a complete solution of this claim later, so try using my method if you want.

Quick Reply

High School Olympiads

Mn=ae+af

Reply

Source: 17-th Iranian Mathematical Olympiad 1999/2000

Amir.S
#1 Dec 14, 2005, 6:42 pm

Circles C_1 and C_2 with centers at O_1 and O_2 respectively meet at points A and B. The radii O_1B and O_2B meet C_1 and C_2 at F and$E$. The line through B parallel to EF intersects C_1 again at M and C_2 again at N. Prove that $MN = AE + AF$.

Andreas
#2 Dec 14, 2005, 9:12 pm

$\angle BAE = 90^o = \angle BAF$.

AB is in common between ΔBAF and ΔBAE, so $A \in FE$.

Now $\angle AEB = \angle EBN$ and $\angle BNE = \angle BAE = 90^o$.

For ASA $\Delta BAE \equiv \Delta BNE \Longrightarrow AE = BN$.

Also $\Delta BAF \equiv \Delta BMF \Longrightarrow AF = BM$.

Adding this $MN = AF + AE$.

Mashimaru
#3 Feb 11, 2009, 11:07 pm

Since $\widehat{BAF} = \widehat{BAE} = \widehat{ABM} = \widehat{ABN}$ we obtain that $ABMF$ and $ABNE$ are rectangles, i.e., $AF = MB, AE = NB$. Thus: $MN = MB + BN = AE + AF$ and we are done.

Mashimaru
#4 Feb 12, 2009, 12:15 am

I think the problem has been stated wrong. It should have been $O_1B \cap (O_2) = \{B, E\}$ and $O_2B \cap (O_1) = \{B, F\}$

Amir Hossein
#5 Jan 29, 2012, 2:11 am

Please someone post a solution to this problem (to the correct version). Thanks.

Luis González
#6 Jan 29, 2012, 3:30 am • 1

Isosceles triangles $\triangle BFO_1$ and $\triangle BEO_2$ with apices O_1, O_2 are clearly similar $\Longrightarrow \angle FO_1E = \angle EO_2F \Longrightarrow EFO_1O_2$ is cyclic $\Longrightarrow \angle BFE = \angle BO_1O_2$. But $\angle BO_1O_2 = \frac{1}{2}\angle BO_1A = \angle BMA$, thus $\angle BMA = \angle BFE \Longrightarrow AM \parallel BF$, i.e. $ABFM$ is an isosceles trapezoid with congruent diagonals $AF = BM$. Similarly, $AE = BN$ and the conclusion follows.

Quick Reply

High School Olympiads

geometry- Bulgaria 96

Locked

ivanbart-15

#1 Jan 28, 2012, 6:34 pm

The circles k_1 and k_2 with respective centers O_1 and O_2 are externally tangent at the point C, while the circle k with center O is externally tangent to k_1 and k_2. Let l be the common tangent of k_1 and k_2 at the point C and let AB be the diameter of k perpendicular to l. Assume that O and A lie on the same side of l. Show that the lines AO_2, BO_1 and l have a common point.

Luis González

#2 Jan 28, 2012, 11:00 pm

Please, use the search before posting contest problems. This is IMOSL 2006 G6.

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=875026
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=278448
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=412936
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=422580

High School Olympiads

Pass through incenter and concurrent

Reply

buratinogigle

#1 Jan 27, 2012, 9:53 pm

Let ABC be a triangle and a circle (ω) passes through B,C. The circle (K) touches to segment AC,AB at E,F and externally tangent to (ω) at T. Prove that intersection other than T of circumcircle (TEC) and (TFB) is incenter of triangle ABC.

Source: the post Bisector by jayme.

More general problem

Let ABC be triangle with incenter I. A circle (ω) passes through B,C. T is a point on (ω). Circumcircle (BIT) cuts AB again at F. Circumcircle (CIT) cuts AC again at E. (K) is cirumcircle of triangle TEF.

a) Prove that K lies on AI.

b) (K) cuts AB,AC again at P,Q, resp. Prove that PQ,EF,AI are concurrent.

Luis González

#2 Jan 28, 2012, 12:58 pm • 1

T can be an arbitrary point on the plane of $\triangle ABC$, so get rid of ω. From the cyclic quadrilaterals $BITF$ and $CITE$ we obtain that $\angle FTE=\frac{1}{2}(\angle B+\angle C) \Longrightarrow \angle FKE=2\angle FTE=\angle B+\angle C \Longrightarrow A,E,F,K$ are concyclic and K is midpoint of the arc EF of $\odot(AEF)$, since $KE=KF$. Thus, AK bisects $\angle EAF$, i.e. $K\in AI$.

Proposition b) is trivial. $FPEQ$ is an isosceles trapezoid with symmetry axis AI.

Quick Reply

High School Olympiads

hard

Reply

Source: tingyan tian

horizon

#1 Jan 25, 2012, 8:24 pm

$ABCD$ is a quadrilateral,and let N_A, N_B, N_C, N_D be the center of the nine-point circle of triangle BCD, ACD, ABD, ABC respectively,then we have the angles between N_AN_C and N_BN_D is equal to the angle between AC and BD

Luis González

#2 Jan 28, 2012, 12:32 am

Let $O \equiv AC \cap BD$ and M, N, E denote the midpoints of AC, BD, CD. For the sake of ease we assume that $ABCD$ is convex. $(N_A), (N_B), (N_C), (N_D)$ concur at the Poncelet point P of $ABCD$. Since $PN \perp N_AN_C$ and $PM \perp N_BN_D$, then it suffices to prove that $\angle MPN = \angle COD$. Indeed, simple angle chase gives

$$\angle MPN = \angle EPN - \angle EPM = 180^\circ - \angle BDC - \angle ACD = \angle COD.$$

Quick Reply

High School Olympiads

Pass through midpoint and parallel

buratinogigle

#1 Jan 27, 2012, 6:58 pm

Let ABC be a triangle with circumcenter (O). P is a point on line OA. Circumcircle (PAB) cuts AC again at E. Circumcircle (PCA) cuts AB again at F. Prove that perpendicular bisector of BC passes though midpoint of EF.

Source: The post Perpendicular by 77ant.

General problem

Let ABC be a triangle with point P. Circumcircle (PAB) cuts AC again at E. Circumcircle (PCA) cuts AB again at F. M,N are midpoints of BC,EF, resp. Q is isogonal conjugate of P with respect to triangle ABC. Prove that $MN\parallel AQ$.

Luis González

#2 Jan 27, 2012, 11:05 pm • 1

Let $D\equiv BE\cap CF.$ Then MN is Newton line of the quadrangle $ACDB$ and P is its Miquel point $\Longrightarrow P\in\odot(EDC).$ If U,V denote the orthogonal projections of P on $AC,AB,$ then UV is Simson line of P WRT $\triangle ABE$ and $\triangle ECD \Longrightarrow UV$ bisects the segments connecting P with the orthocenters H_1,H_2 of $\triangle ABE,\triangle ECD \Longrightarrow UV\parallel H_1H_2.$ Since the Steiner line H_1H_2 of $ACDB$ is perpendicular to its Newton line $MN,$ then $AQ\parallel MN.$

High School Olympiads

geometry

Reply

horizon

#1 Jan 25, 2012, 7:53 pm

given triangle ABC and D be an arbitrary point on segment BC,O_1,O_2 are the circumcenters of triangles ABD and ACD respectively.DO_1 intersects circle O_2 at E,and DO_2 intersects circle O_1 at F, let the circumcenter of triangle ABC be O ,prove that
(1)A,O_1,O_2,E,F are cyclic
(2)$B,O.F$ are collinear
(3) E,O,C are collinear

Luis González

#2 Jan 27, 2012, 8:27 am • 1

Isosceles triangles $\triangle AO_1B$ and $\triangle AO_2C$ with apices O_1,O_2 are clearly spirally similar with center $A \Longrightarrow \triangle ABC$ and $\triangle AO_1O_2$ are spirally similar with center A. Since $OO_1 \perp AB$ and $OO_2 \perp AC$, then $\angle O_1OO_2 = \angle O_1AO_2 \pmod \pi$ $\Longrightarrow O \in \odot(AO_1O_2)$. Further, $\angle AED = \angle ACD = \angle AO_2O_1 \pmod \pi) \Longrightarrow E \in \odot(AO_1O_2)$. Similarly, $F \in \odot(AO_1O_2)$. Since D is reflection of A about O_1O_2, we get $\angle DAE = \angle ACD - \angle ADO_1 = \angle C - (90^\circ - \angle B) = \angle DCO \Longrightarrow O \in CE$. Similarly, $O \in BF$.

buratinogigle

#3 Jan 27, 2012, 8:17 pm

More general

Let ABC be triangle with circumcircle (O) and a point D. $(O_1),(O_2)$ are circumcircles of triangles ABD,ACD, resp. DO_1 cuts (O_2) again at E. DO_2 cuts (O_1) again at F.

a) Prove that A,E,F,O_1,O_2 lie on a circle (K).

b) DB cuts (O_2) again at M, DC cuts (O_1) again at N, EM cuts (K) again at P, FN cuts (K) again at Q. Prove that $B,P,F;C,Q,E;P,O,O_1;Q,O,O_2$ are collinear, resp.

Quick Reply

High School Olympiads

Concurrent

Reply

mathismylove
#1 Jan 10, 2012, 6:43 am

Hello.Can anyone help me this problem:
Let ABC is a non-isosceles triangle with incentre I, circumcentre O. Let x_1,y_1,z_1 be the line obtained by reflecting BC,CA,AB in the lines AI,BI,CI, respectively. Let x_2,y_2,z_2 be the tangents of (O) at A,B,C.

$$x_1 \cap y_1 = C_1, y_1 \cap z_1 = A_1, z_1 \cap x_1 = B_1$$

$$x_2 \cap y_2 = C_2, y_2 \cap z_2 = A_2, z_2 \cap x_2 = B_2$$

Prove that A_1A_2,B_1B_2,C_1C_2 concurrent.

Luis González
#2 Jan 10, 2012, 9:26 am • 1

AI, BI, CI cut BC, CA, AB at D, E, F. x_1 is the tangent of the incircle (I) passing through D. Analogously, y_1 and z_1 are tangents of (I) passing through E, F. Since x_1, y_1, z_1 are clearly antiparallel to BC, CA, AB, then $x_1 \parallel x_2, y_1 \parallel y_2$ and $z_1 \parallel z_2 \Longrightarrow \triangle A_1B_1C_1$ with incircle (I) is homothetic to $\triangle A_2B_2C_2$ with incircle $(O) \Longrightarrow A_1A_2, B_1B_2$ and C_1C_2 concur at their homothetic center, i.e. the insimilicenter X_{55} of their incircles $(I), (O)$.

Quick Reply

High School Olympiads

Kenmotu point

Reply

Source: mathworld.wolfram.com

Leon

#1 Jan 25, 2005, 3:21 am

(1) Given a triangle ABC, find a construction by ruler and compass of three equal squares such that each square touches two sides and all three squares touch at a single common point (Kenmotu point)

(2) Prove that the edge lengths of inscribed squares are

$$\ell = \frac{\sqrt{2}abc}{a^2+b^2+c^2+4\Delta}$$

where a,b,c,Δ are the sides and the area of triangle ABC.

Thanks in advance
Leon

Luis González

#2 Jan 9, 2012, 9:52 pm

Assume that $\triangle KTS, \triangle KRQ, \triangle KPU$ are congruent isosceles right triangles with common apex K. $(P,Q) \in BC$, $(R,S) \in CA, (T,U) \in AB$ (See the diagram below). Hexagon $PQRSTUV$ is obviously cyclic with circumcenter K and $UPQR$ is an isosceles trapezoid with $UR \parallel PQ \Longrightarrow TS$ is antiparallel to $UR \parallel BC$ WRT AB, AC. If M_A, M_B, M_C denote the apices of the isosceles right triangles erected outside $\triangle ABC$, then $ABM_AC \sim ASKT \Longrightarrow$ $\angle BAM_A = \angle SAK$, i.e. AM_A and AK are isogonals WRT $\angle A$. Similarly, BM_B and CM_C are the isogonals of BK and CK WRT $\angle B$ and $\angle C$. Hence, K is the isogonal conjugate of the 1st Vecten point $V \equiv AM_A \cap BM_B \cap CM_C$, i.e. the 1st Kenmotu point of $\triangle ABC$.

For the second part, we resort to calculations with barycentric coordinates with respect to $\triangle ABC$. Using Conway's notation, the coordinates of M_A, M_B and M_C are

$$M_A(-a^2 : S_C + S : S_B + S)$$

$$M_B(S_C + S : -b^2 : S_A + S)$$

$$M_C(S_B + S : S_A + S : -c^2)$$

Therefore, $K(a^2(S_A+S) : b^2(S_B+S) : c^2(S_C+S))$

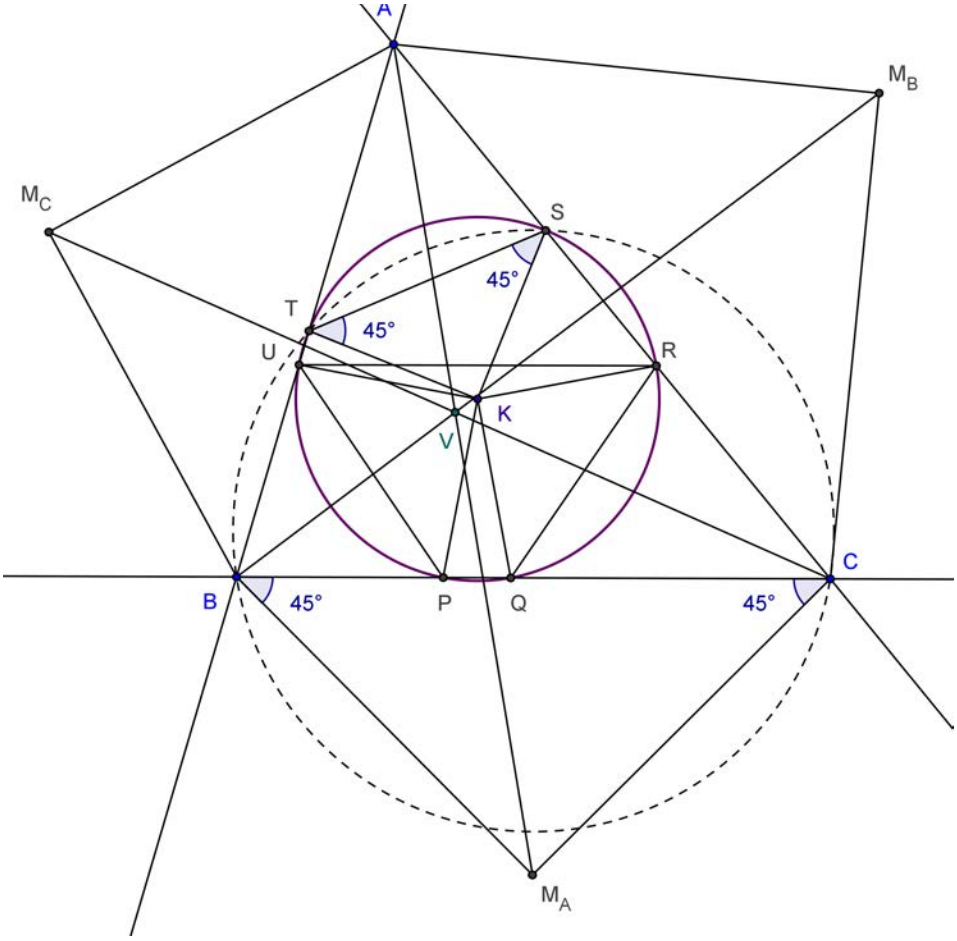
$$\frac{|\triangle KAC|}{|\triangle ABC|} = \frac{b^2}{2S^2 + S(a^2+b^2+c^2)} \Longrightarrow \text{d}(K, AC) = \frac{b(S_B+S)}{a^2+b^2+c^2}$$

$$\frac{|\triangle M_ABA|}{|\triangle ABC|} = \frac{S_B+S}{2S} \Longrightarrow \text{d}(M_A, AB) = \frac{S_B+S}{2c}$$

From $ABM_AC \sim ASKT$ we deduce that $\dfrac{\ell}{BM_A} = \dfrac{\text{d}(K,AC)}{\text{d}(M_A,AB)}$. Then

$$\ell = \frac{\sqrt{2}}{2}a \cdot \frac{2bc}{a^2+b^2+c^2+2S} = \frac{\sqrt{2}abc}{a^2+b^2+c^2+4|\triangle ABC|}.$$

Attachments:



Quick Reply

High School Olympiads

Radical axes

Reply

Source: komal

Makaveli

#1 Jun 13, 2006, 11:25 pm

Assume that a triangle ABC is not equilateral, and let $0 < t < \frac{1}{2}$ be a real number. Let points A_1 and A_2 be given on the side AB, points B_1 and B_2 on the side BC, and points C_1 and C_2 on the side CA such that
$\frac{AA_1}{AB} = \frac{A_2B}{AB} = \frac{BB_1}{BC} = \frac{B_2C}{BC} = \frac{CC_1}{CA} = \frac{C_2A}{CA} = t.$
Prove that the radical axis of the circumcircles of the triangles $A_1B_1C_1$ and $A_2B_2C_2$ depends only of the triangle ABC, but does not depend on the value of t.

Luis González

#2 Jan 9, 2011, 1:07 am

For convenience redefine the points such that A_1, A_2 lie on BC, B_1, B_2 lie on CA and C_1, C_2 lie on AB respectively. Then $\omega_1 \equiv \odot(A_1B_1C_1)$ and $\omega_2 \equiv \odot(A_2B_2C_2)$. Using barycentric coordinates with respect to $\triangle ABC$, we get

$$A_1 \ (0 : 1-t : t) \ , \ B_1 \ (t : 0 : 1-t) \ , \ C_1 \ (1-t : t : 0)$$

$$A_2 \ (0 : t : 1-t) \ , \ B_2 \ (1-t : 0 : t) \ , \ C_2 \ (t : 1-t : 0)$$

Plugging the coordinates of A_1, B_1, C_1 and A_2, B_2, C_2, respectively into the general equation of the circle $a^2yz + b^2zx + c^2xy - (x+y+z)(\alpha_i x + \beta_i y + \gamma_i z) = 0$ yields the parameters of ω_1, ω_2 as

$$\alpha_1 = \frac{b^2t^3(1-t) + c^2t(1-t)^3 - a^2t^2(1-t)^2}{t^3 + (1-t)^3}$$

$$\alpha_2 = \frac{b^2t(1-t)^3 + c^2t^3(1-t) - a^2t^2(1-t)^2}{t^3 + (1-t)^3}$$

$$\beta_1 = \frac{c^2t^3(1-t) + a^2t(1-t)^3 - b^2t^2(1-t)^2}{t^3 + (1-t)^3}$$

$$\beta_2 = \frac{c^2t(1-t)^3 + a^2t^3(1-t) - b^2t^2(1-t)^2}{t^3 + (1-t)^3}$$

$$\gamma_1 = \frac{a^2t^3(1-t) + b^2t(1-t)^3 - c^2t^2(1-t)^2}{t^3 + (1-t)^3}$$

$$\gamma_2 = \frac{a^2t(1-t)^3 + b^2t^3(1-t) - c^2t^2(1-t)^2}{t^3 + (1-t)^3}$$

$\alpha_1 - \alpha_2$, $\beta_1 - \beta_2, \gamma_1 - \gamma_2$ have common factor $\frac{t^3(1-t) - t(1-t)^3}{t^3 + (1-t)^3}$

Thus, equation of the radical axis τ of ω_1, ω_2 is indeed independent of t. Namely,

$$\tau \equiv (b^2 - c^2)x + (c^2 - a^2)y + (a^2 - b^2)z = 0$$

Which is the trilinear polar of the Steiner's point X_{99} of $\triangle ABC$. Therefore, τ passes through the centroid $G(1:1:1)$ and symmedian point $K(a^2 : b^2 : c^2)$ of $\triangle ABC$.

darij grinberg

#3 Jan 8, 2012, 10:32 pm

I have just discovered the source of this problem:

R. Tucker, *Geometrical Note*, Proceedings of the Edinburgh Mathematical Society / Volume 11 / pp 57 - 60. (With public-readable PDF.)

According to some of my notes from 4 years ago, the midpoint between the symmedian points of triangles $A_1B_1C_1$ and $A_2B_2C_2$ also seems to lie on the line joining the centroid and the symmedian point of triangle ABC. I have never checked or proved this, but maybe somebody else can come up with an idea (or a counterexample) now. (He/she would have my blessing to publish it then; there is nothing I have done for this conjecture except of conjecturing it.)

Luis González

#4 Jan 9, 2012, 11:55 am

> darij grinberg wrote:
> According to some of my notes from 4 years ago, the midpoint between the symmedian points of triangles $A_1B_1C_1$ and $A_2B_2C_2$ also seems to lie on the line joining the centroid and the symmedian point of triangle ABC.

This conjecture is true, unfortunately I don't have a synthetic proof. Again, using the barycentric coordinates of A_1, B_1, C_1 WRT $\triangle ABC$, we get that

$$|B_1C_1|^2 = S_A(1-2t)^2 + S_Bt^2 + S_C(t-1)^2$$

$$|C_1A_1|^2 = S_A(t-1)^2 + S_B(1-2t)^2 + S_Ct^2$$

$$|A_1B_1|^2 = S_At^2 + S_B(t-1)^2 + S_C(1-2t)^2$$

Thus, barycentric coordinates $(x_1 : y_1 : z_1)$ of the symmedian point K_1 of $\triangle A_1B_1C_1$ WRT $\triangle ABC$ are given by

$$K_1 \equiv |B_1C_1|^2 \cdot (0 : 1-t : t) + |C_1A_1|^2 \cdot (t : 0 : 1-t) + |A_1B_1|^2 \cdot (1-t : t : 0)$$

$$x_1 = S_A(1-t)t + S_B(3t^3 - t^2 - 2t + 1) + S_C(-3t^3 + 8t^2 - 5t + 1)$$

$$y_1 = S_A(-3t^3 + 8t^2 - 5t + 1) + S_B(1-t)t + S_C(3t^3 - t^2 - 2t + 1)$$

$$z_1 = S_A(3t^3 - t^2 - 2t + 1) + S_B(-3t^3 + 8t^2 - 5t + 1) + S_C(1-t)t$$

Similarly, barycentric coordinates $(x_2 : y_2 : z_2)$ of the symmedian point K_2 of $\triangle A_2B_2C_2$ WRT $\triangle ABC$ are given by

$$x_2 = S_A(1-t)t + S_B(-3t^3 + 8t^2 - 5t + 1) + S_C(3t^3 - t^2 - 2t + 1)$$

$$y_2 = S_A(3t^3 - t^2 - 2t + 1) + S_B(1-t)t + S_C(-3t^3 + 8t^2 - 5t + 1)$$

$$z_2 = S_A(-3t^3 + 8t^2 - 5t + 1) + S_B(3t^3 - t^2 - 2t + 1) + S_C(1-t)t$$

Clearly, $x_1 + y_1 + z_1 = x_2 + y_2 + z_2$, thus coordinates $(\overline{x} : \overline{y} : \overline{z})$ of midpoint $\overline{K}$ of K_1K_2 are found by just adding the homogeneous coordinates of K_1, K_2.

$$\overline{x} = 2S_A(1-t)t + a^2(7t^2 - 7t + 2)$$

$$\overline{y} = 2S_B(1-t)t + b^2(7t^2 - 7t + 2)$$

$$\overline{z} = 2S_C(1-t)t + c^2(7t^2 - 7t + 2)$$

Eliminating t from the coordinates of $\overline{K}$, we get that the locus of $\overline{K}$ is precisely the line through X_2 and X_6, namely $(b^2-c^2)x + (c^2-a^2)y + (a^2-b^2)z = 0$.

Quick Reply

High School Olympiads

circumscribed quadrilateral

Reply

crystallfire
#1 Jan 7, 2012, 11:47 am

Let $ABCD$ is a circumscribed quadrilateral. Let M be a point obtained by reflecting A in the line BD. Suppose $BC \cap DM = E$, $DC \cap BM = F$. Prove that $MECF$ is a circumscribed quadrilateral

unexpectedtrigbash
#2 Jan 7, 2012, 12:33 pm

Let all angles be directed.
Then
angle(EMF)=angle(DMB)=angle(BAD)(b/c it's symmetric)
=angle(BCD)(b/c ABCD is circumscribed)
=angle(ECF)
QED

notbroken
#3 Jan 7, 2012, 4:00 pm

Why "angle(BAD)=angle(BCD)"?

Petry
#4 Jan 7, 2012, 10:05 pm

Solution:

Let (I) be the circle inscribed in $ABCD$.
So, BI, CI, DI are the bisectors of the angles $\angle ABC$, $\angle BCD$, $\angle CDA$ respectively.

$\{K\} = CI \cap BD$.
CK is the bisector of $\angle BCD \Rightarrow \frac{KB}{KD} = \frac{CB}{CD} \Rightarrow \frac{BK}{BC} = \frac{DK}{DC}$ (1)

The bisector of $\angle MBC$ intersects CI at J_1.
$\angle IBJ_1 = \angle IBC + \angle CBJ_1 = \frac{\angle ABC}{2} + \frac{\angle CBM}{2} = \frac{\angle ABM}{2} =$
$= \angle ABD = \angle MBD$
$\angle IBJ_1 = \angle MBD \Rightarrow \angle IBK = \angle MBJ_1 = \angle CBJ_1 \Rightarrow \frac{IK}{IC} \cdot \frac{J_1K}{J_1C} = \left(\frac{BK}{BC}\right)^2$ (2)

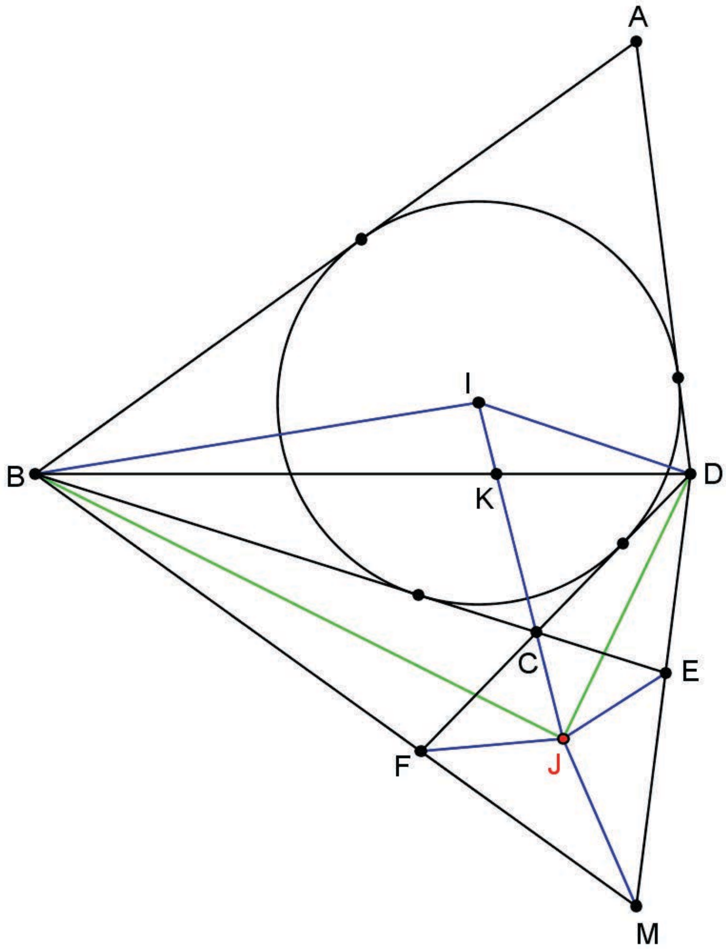
The bisector of $\angle MDC$ intersects CI at J_2. Similarly $\Rightarrow \frac{IK}{IC} \cdot \frac{J_2K}{J_2C} = \left(\frac{DK}{DC}\right)^2$ (3)

(1),(2),(3)$\Rightarrow \frac{J_1K}{J_1C} = \frac{J_2K}{J_2C} \Rightarrow J_1 = J_2 = J$. So, $J \in CI$.

CJ is the bisector of $\angle ECF$.
J is the B-excenter of $\Delta BCF \Rightarrow FJ$ is the bisector of $\angle CFM$.
J is the D-excenter of $\Delta DCE \Rightarrow EJ$ is the bisector of $\angle CEM$.
It's easy to prove that MJ is the bisector of $\angle EMF$.
So, $CEMF$ is a circumscribed quadrilateral and J is the center of the circle inscribed in $CEMF$.

Best regards,
Petrisor Neagoe

Attachments:



Luis González

#5 Jan 8, 2012, 12:37 am

(I) is incircle of $ABCD.$ Internal angle bisectors of $\angle CBM$ and $\angle CDM$ meet at $K.$
$\angle CBK=\frac{1}{2}(\angle DBC-\angle DBA)=\frac{1}{2}(\angle DBI+\angle ABI-\angle DBA)=\angle DBI.$ Similarly, we have $\angle CDK=\angle BDI \Longrightarrow BI,BK$ are isogonals WRT $\angle CBD$ and DI,DK are isogonals WRT $\angle CDB \Longrightarrow I,K$ are isogonal conjugates WRT $\triangle BDC \Longrightarrow K \in CI.$ Thus, K is common incenter of $\triangle CBF$ and $\triangle CDE \Longrightarrow EK,FK$ bisect $\angle MEC$ and $\angle MFC.$ So $CEMF$ is tangential with incircle centered at $K.$

unexpectedtrigbash

#6 Jan 9, 2012, 1:46 pm

Sorry for my mistake.
Consider a point X on line CD such that DA=DM=DX, and that C and X are on the same direction from D.
Similarly, consider a point Y on line CB such that BA=BM=BY, and that C and Y are on the same direction from B.

Now, since BA+DC=DA+BC(due to ABCD being circumscribed), or(YC=BC-BY=) BC-BA=DC-DA(=DC-DX=XC), we have YC=XC.
Also, DM=DX, and BM=BY.
Thus, the circumcenter of MXY, or the intersection of the perpendicular bisectors of MX, MY, and XY, which is the intersection of the angle bisectors of MBC, MDC, and BCD, is of equal distance from BM(or FM), BC(or EC), CD(or CF), and DM(or ME), so consider a circle centered about the circumcenter of MXY and having the radius the distance from the center to BC. That circle is circumscribed by MECF.
(Again, sorry for my messy solution)

Quick Reply

High School Olympiads

Common tangent and a simple relation.

Reply

Virgil Nicula

#1 Jan 7, 2012, 4:45 pm

PP. Let ABC be a triangle with $A \le 90^\circ$. Denote the orthocenter H of $\triangle ABC$, the midpoint M of the side $[BC]$ and the projection P of H on AM . Prove that the line BC is a common tangent to the circumcircles of APB , APC and $\frac{PB}{PC} = \frac{AB}{AC}$.

yetti

#2 Jan 7, 2012, 8:11 pm

(O) is circumcircle of $\triangle ABC$. AM cuts (O) again at Q. BH, CH cut CA, AB at E, F. EF cuts BC at R. (M) on diameter BC is circumcircle of $BCEF$.
HR is polar of A WRT $(M) \Longrightarrow HR \perp AM \Longrightarrow P \equiv HR \cap AM$ and $\overline{MP} \cdot \overline{MA} = MB^2 = MC^2$.
Power of M to (O) is $\overline{MQ} \cdot \overline{MA} = \overline{MB} \cdot \overline{MC} = -MB^2 = -MC^2 \Longrightarrow \overline{MQ} = -\overline{MP} \Longrightarrow BQCP$ is parallelogram.
$\angle QPB = \angle PQC = \angle AQC = \angle ABC \Longrightarrow BC$ is tangent of $\odot(APB)$ at B. Similarly, $\angle QPC = \angle ACB \Longrightarrow CB$ is tangent of $\odot(ACP)$ at C.
$\triangle BPQ \sim \triangle ABC \sim \triangle CQP \Longrightarrow \frac{PB}{PC} = \frac{PB}{QB} = \frac{AB}{AC}$.

Luis González

#3 Jan 7, 2012, 10:30 pm

Let D, E, F be the feet of the altitudes on BC, CA, AB. Inversion with center A and power $\overline{AH} \cdot \overline{AD}$ carries circumcircle of $HEAF$ into BC and $\odot(HBC)$ into 9-point circle $\odot(DEF)$. Thus, 2nd intersection of $\odot(HEF)$ and $\odot(HBC)$ is the inverse of $M \Longrightarrow P \in \odot(HBC)$. Since $\odot(HBC)$ is reflection of circumcircle (O) of $\triangle ABC$ about M, then AM cuts (O) again at the reflection Q of P about $M \Longrightarrow \frac{PB}{PC} = \frac{QC}{QB} = \frac{AB}{AC}$.

From parallelogram $PBQC$ we get $\angle PBM = \angle BCQ = \angle ABP \Longrightarrow \odot(PAB)$ is tangent to BC. Similarly, $\odot(PAC)$ is tangent to BC.

Petry

#4 Jan 8, 2012, 8:30 am • 1

The proposed problem can be generalized:

Let ABC be a triangle. A circle (M) through the points B, C intersects AC, AB at E, F respectively.
$\{H\} = BE \cap CF$ and $HP \perp AM$, $P \in AM$. Prove that MB is tangent to the circumcircle of ΔAPB,
MC is tangent to the circumcircle of ΔAPC and $\frac{PB}{PC} = \frac{AB}{AC}$.

Solution:

$\{V\} = EF \cap BC$
AV is the polar of H wrt $(M) \Rightarrow MH \perp AV$
AH is the polar of V wrt $(M) \Rightarrow MV \perp AH$
$\Rightarrow H$ is the orthocenter of $\Delta AVM \Rightarrow VH \perp AM$
So, the points V, H, P are collinear.
$\{K\} = MH \cap AV$
$MH \cdot MK = MP \cdot MA$ and $MH \cdot MK = MB^2 = MC^2 \Rightarrow MP \cdot MA = MB^2 = MC^2 \Rightarrow$

$\Rightarrow MB$ is tangent to the circumcircle of ΔAPB and MC is tangent to the circumcircle of ΔAPC.

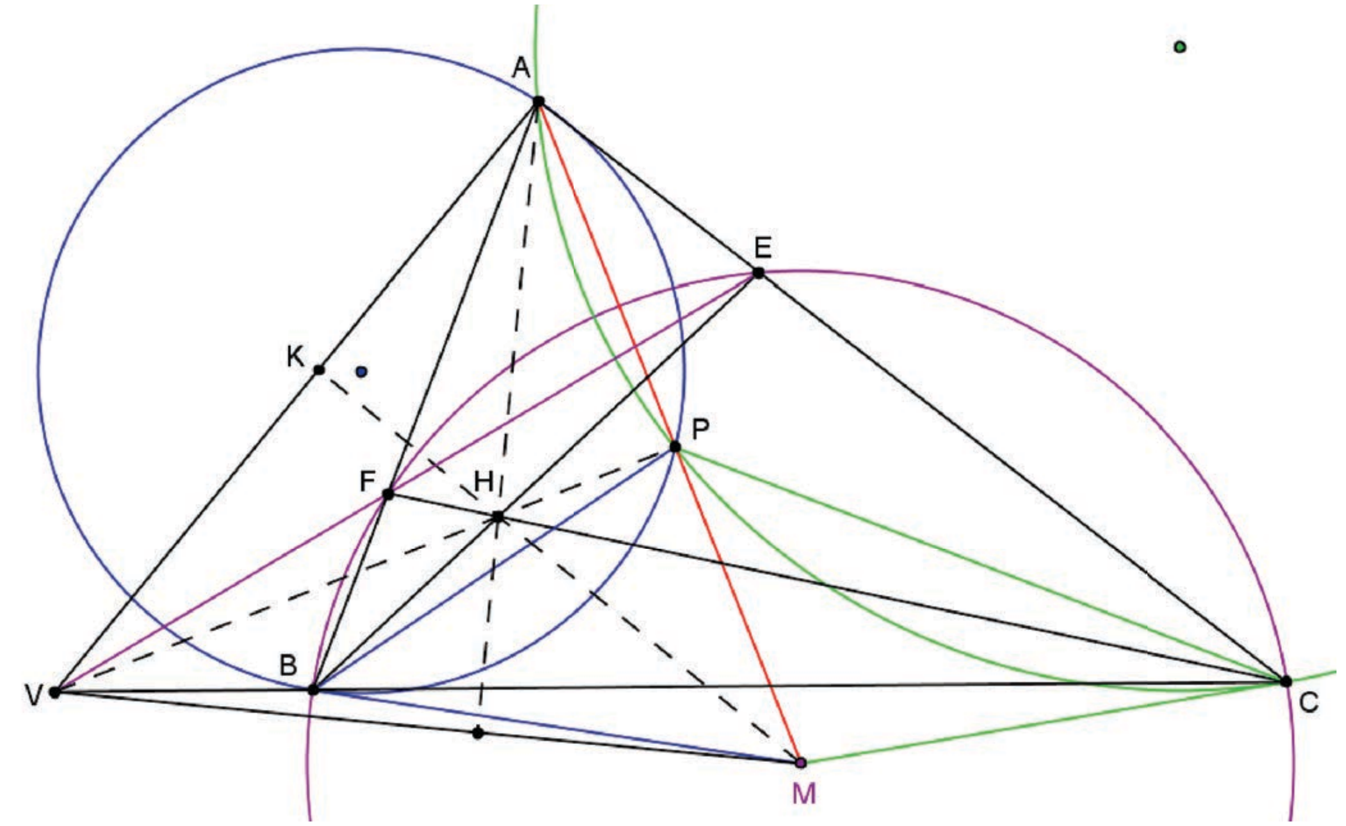
$$\frac{PB}{AB} = \frac{sin\left(\angle PAB\right)}{sin\left(\angle APB\right)} = \frac{sin\left(\angle PBM\right)}{sin\left(\angle MPB\right)} = \frac{PM}{MB} \Rightarrow \frac{PB}{AB} = \frac{PM}{MB} \text{ (*)}$$

Similarly $\Rightarrow \frac{PC}{AC} = \frac{PM}{MC}$ (**)

(*),(**) and $MB = MC \Rightarrow \frac{PB}{AB} = \frac{PC}{AC} \Rightarrow \frac{PB}{PC} = \frac{AB}{AC}$

Best regards,
Petrisor Neagoe

Attachments:



Quick Reply

High School Olympiads

Line of centroids are collinear

Reply

Goutham
#1 Jan 6, 2012, 9:47 pm

In a triangle ABC, choose points D, E, F on sides BC, CA, AB such that BD, CE and AF are equal in length. Prove that the locus of centroids of triangle DEF is a straight line.

Luis González
#2 Jan 7, 2012, 10:54 am • 1

We assume that $\triangle ABC$ is non-equilateral with centroid G. K denotes the centroid of $\triangle DEF$ and let $\overline{BD}=\overline{CE}=\overline{AF}=k$. Using barycentric coordinates with respect to $\triangle ABC,$ we have

$$D \equiv (0:a-k:k) \ , \ E \equiv (k:0:b-k) \ , \ F \equiv (c-k:k:0)$$

$$K \equiv \left(0:\frac{a-k}{a}:\frac{k}{a}\right)+\left(\frac{k}{b}:0:\frac{b-k}{b}\right)+\left(\frac{c-k}{c}:\frac{k}{c}:0\right)$$

$$K \equiv \left(\frac{k(c-b)+bc}{bc}:\frac{k(a-c)+ca}{ca}:\frac{k(b-a)+ab}{ab}\right)$$

Eliminating k from the coordinates of $K,$ we obtain that the locus of K is

$$\tau \equiv [a(b+c)-2bc]x+[b(c+a)-2ca]y+[c(a+b)-2ab]z=0$$

Which is the line τ passing through the centroid $G \equiv X_2(1:1:1)$ of $\triangle ABC$ and the isogonal conjugate $X_{513}(a(b-c):b(c-a):c(a-b))$ (at infinity) of the Feuerbach point of the antimedial triangle of $\triangle ABC.$

Goutham
#3 Jan 7, 2012, 11:06 am

Thanks, but is there a pure geometric proof for this nice result?

swaqar
#4 Jan 8, 2012, 7:54 pm • 2

Let M be the midpoint of BC and let M' be a point on AC such that MCM' is isosceles with vertex C and it can be easily seen that the locus of the midpoints of DE is the line MM'. Also note that $MD=M'E$ for all such choices and taking T to be the midpoint of DE, this gives that $\frac{AF}{MT}$ is constant. Now, consider the spiral similarity which sends the line segment AF to MT and let this similarity have center S. Now, S is also the center of spiral similarity which sends AM to FT and so, it also sends the centroid G of triangle ABC to the centroid G' of triangle DEF. Now, if there is a spiral similarity which sends A to F and G to G' with center S, then there is a spiral similarity which sends A to G and F to G' which also has S as its center. So, the segment GG' is the image of AF under the spiral similarity which has a constant center S and as spiral similarity sends lines to lines, it must send the whole line AF to the whole line GG' which makes the direction of GG' independent of the choice of F and thus, the claim is proven.

Quick Reply

High School Olympiads

Square cut by perpendicular lines

Reply

Source: Ibero-American Olympiad 1991, Problem 2

Jutaro
#1 May 3, 2007, 9:22 am • 1

A square is divided in four parts by two perpendicular lines, in such a way that three of these parts have areas equal to 1. Show that the square has area equal to 4.

Wichking
#2 Jul 2, 2009, 1:03 pm

Let the side of the square be a.
Then let the sides of one of the parts(the part that is beetwen the other two) be x, y.

We know the following

$xy = 1$
$(a-y)x = ax - xy = 1$
$(a-x)y = ay - xy = 1$

If we look at the last two (and put in the first one)

$ax - 1 = 1$
$ay - 1 = 1$

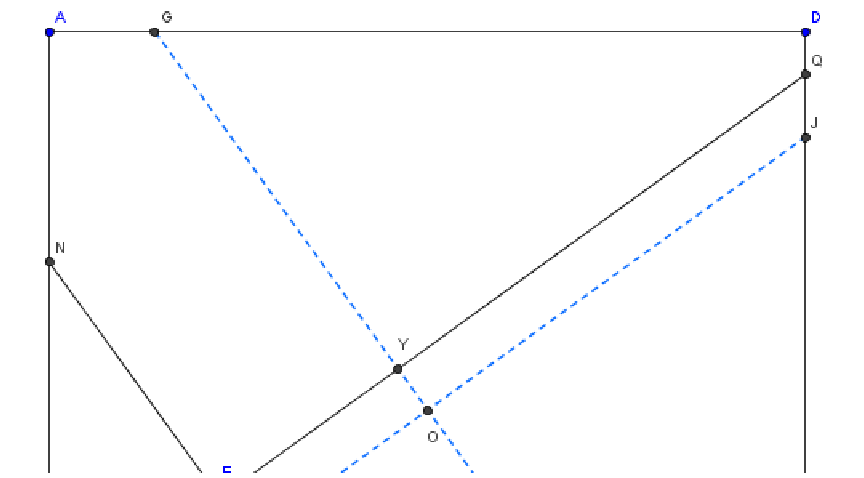
It follows that $x = y$. And from here $x = y = 1$

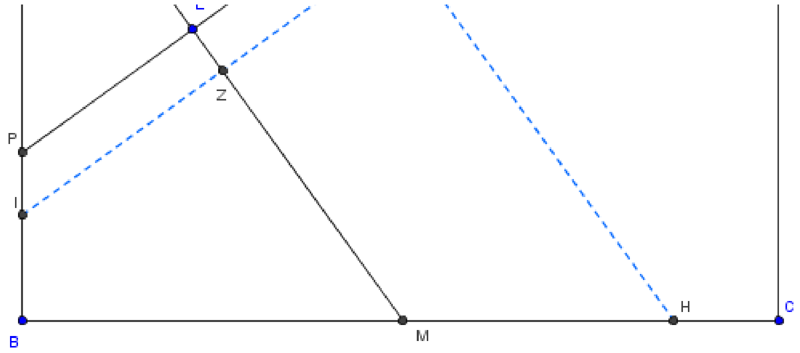
$a - 1 = 1$ so $a = 2$ so the area is indeed 4.

ith_power
#3 Jul 2, 2009, 1:33 pm

@Wichking. the perpendicular lines are not necessarily parallel to sides of square.

iTAQ
#4 Jul 2, 2009, 7:33 pm





Hmmm... Name the points as in the figure above. O is the center of the square. GH // NM and IJ // PQ. Y and Z are the lines' intercepts.
WLOG, assume that E lies in the left section of the square (in the pentagon OGAI).
Denote [polygon] be the area of the polygon.
It's easy to see that [MEPB] <= [HYPB] <= [MEQC]. Equality holds when E lies on GH. Do this similarly to the remain parts.
Then, we can show that there are 3 parts of the square having equal areas iif E is the center, which means the area of the square is 4.

iTAQ

#5 Jul 4, 2009, 11:59 pm

Is my solution correct?

Raúl

#6 Dec 6, 2009, 5:19 pm

That solution seem to lose some generality, supose that Q lie on the segment [AD]. Then it's to hard to prove the inequality [EYHM]=<[YHCQ].

It's an god solution, anyway. Those geometric problemas are very hard, it seems that is no way to prove...

Alvid

#7 Sep 11, 2010, 2:19 pm

We name the lines a and b. a and b are perpendicular, and each divides the square in two parts, one of which has area 2. This means that line b is a 90º rotation of a about the center of the square, O. We draw the other 90º rotation, c, and the 180º rotation, d. This divides the square in 9 parts.

Because each line is a rotation about O of the other lines, there are two groups of four parts that are rotations of the other ones, and therefore have the same area. Let the area of those parts be r and s, and the area of the central part be t. a and b divide the square in four parts of area r, r+s, r+s and r+2s+t. Three of them can only be equal if s=0, which means that b and c are the same line, and so are a and d, and therefore the central part has area 0. All four parts have area r=1.

Luis González

#8 Jan 4, 2012, 11:33 pm • 1

Label $P_1P_2P_3P_4$ the given square and M is a point inside it. A line through M cuts $\overline{P_1P_2}, \overline{P_3P_4}$ at P, T and another line through M perpendicular to PT cuts $\overline{P_2P_3}, \overline{P_4P_1}$ at $S, U. O$ is the center of $P_1P_2P_3P_4$. Parallel through O to PT cuts $\overline{P_1P_2}, \overline{P_3P_4}$ at E, G and parallel through O to SU cuts $\overline{P_2P_3}, \overline{P_4P_1}$ at F, H. Let $X \equiv EG \cap SU, Y \equiv FH \cap TP$. Denote $S, S_A, S_B, S_C, S_D, S_R$ the areas of $P_1P_2P_3P_4, UHYM, PEOY, SFOX, TGXM, OXMY$, respectively.

WLOG assume that O lies inside the quadrilateral MPP_2S. Clearly, the lines EG and FH divide $P_1P_2P_3P_4$ into four quadrilaterals with equal areas. Therefore, we have

$$\tfrac{1}{4}S - S_B + S_A = \tfrac{1}{4}S - S_A - S_C - S_R = \tfrac{1}{4}S - S_D + S_C = 1.$$

From these expressions we obtain $S_A + S_D = S_B + S_C \Longrightarrow$ parallelograms $PEGT$ y $SFHU$ have equal areas. But since they have equal altitudes, then it follows that $PE = FS \Longrightarrow OXMY$ is a square. Thus, $S_A = S_C$ and $S_B = S_D$. Again, using the expressions found previously, we obtain $S_A = S_B = S_C = S_D = 0 \Longrightarrow O \equiv M$ and the conclusion follows.

Quick Reply

High School Olympiads

Need euclidean prove

newbie
#1 Jan 1, 2008, 4:47 pm • 1

The diagonals AC and BD of a cyclic quadrilateral $ABCD$ meet at O. The circumcircles of triangles AOB and COD intersect again at K. Point L is such that the triangles BLC and AKD are similar and equally oriented. Prove that if the quadrilateral $BLCK$ is convex, then it is tangent [has an incircle].

newbie
#2 Jan 3, 2008, 4:53 pm

Why no one answered :(Is this to hard? Or because no sketch from me?

Vo Duc Dien
#3 Jan 4, 2012, 7:09 am

Need better translation as my graphic does not agree with the problem.

yetti
#4 Jan 4, 2012, 4:58 pm

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=105470.

Luis González
#5 Jan 4, 2012, 10:15 pm

Let $P \equiv OK \cap BC. \angle BKP = \angle BAO = \angle CDO = \angle CKP \Longrightarrow KP$ bisects $\angle BKC.$ Let the angle bisectors of $\angle KBL$ and $\angle KCL$ cut LC, LB at $U, V.$ Then

$$\angle KCL = \angle KCB + \angle ADK = \angle ACB - \angle KCO + \angle ADB + \angle ODK$$

$$\Longrightarrow \angle KCL = 2\angle BCA = \angle 2\angle KCV$$

Similarly, $\angle KBU = \angle CBD.$ Hence CA, CV and BD, BU are isogonals WRT $\angle KCB$ and $\angle KBC,$ respectively. If $I \equiv BU \cap CV,$ then O, I are isogonal conjugates WRT $\triangle KBC,$ but since KO bisects $\angle BKC,$ it follows that $I \in KP$ $\Longrightarrow$ Internal angle bisectors of the convex $BLCK$ concur at $I,$ i.e. $BLCK$ has an incircle centered at $I.$

High School Olympiads

circles intersecting on CH

Reply

paul1703

#1 Jan 2, 2012, 8:10 pm

Let H be the orthocentre and O be the circumcentre of an acute triangle ABC. Let AD and BE be the altitudes of the triangle with D on BC and E on CA. Let $K = OD \cap BE, L = OE \cap AD$. Let X be the second point of intersection of the circumcircles of triangles HKD and HLE. Prove that X lies on CH.

hatchguy

#2 Jan 3, 2012, 6:41 am

I have that the problem is equivalent to showing $(LBC) = (KAC)$..

Maybe someone has any thought on this? I am having some trouble with lines OE and OD...

paul1703

#3 Jan 3, 2012, 4:59 pm

I have also found this <=> (but no prove).
There exists a trigonometric solution (not verry long by showing that C has equal power to those 2 circle)
For sintetic solution i guess you have to intersect this 2 circles the second time with CA, CB in S,T. and find that X-S-T are collinear, i have tested in geogebra and the only importance of O is that is izogonal conjugate of H with respect to C.

Luis González

#4 Jan 3, 2012, 11:33 pm

Perpendiculars to OD, OE through D, E cut CA, CB at P, Q, respectively. KP and LQ cut CH at U, V. Since $PDKE$ and $CDHE$ are both cyclic, we have $\angle UHD = \angle PED = \angle PKD \Longrightarrow U \in \odot(HKD)$. Further, $\angle DPU = 90^\circ - \angle UHD = \angle DCU \Longrightarrow U$ is the 2nd intersection of CH with $\odot(CPD)$. Similarly, V is the 2nd intersection of CH with $\odot(CQE)$.

On the other hand, DO, DH are isogonals WRT $\angle CDP$ and CO, CH are isogonals WRT $\angle PCD \Longrightarrow O, H$ are isogonal conjugates WRT $\triangle PCD$. If N is the midpoint of AC (projection of O on PC), we have then $HP \perp DN \Longrightarrow$ $\angle DHP = \angle AHE \Longrightarrow HP$ is the reflection of HB about HD. Likewise, HQ is the reflection of HA about $HE \Longrightarrow$ right triangles $\triangle HDQ$ and $\triangle HEP$ are similar. Hence if X is the second intersection of $\odot(CPD)$ and $\odot(CQE)$ (center of the spiral similarity that takes DQ into PE), we obtain then $\frac{XD}{XP} = \frac{DQ}{PE} = \frac{HD}{HE} \Longrightarrow \triangle XDP \sim \triangle HDE$ by SAS $\Longrightarrow$ $\angle XPD = \angle XCD = \angle HED \Longrightarrow X \in CH \Longrightarrow U \equiv V \equiv X$.

panos_lo

#5 Jan 4, 2012, 9:52 pm • 2

I found a little bit different proof for this statement. It goes as follows:

Consider ponts X, Y on CB, CA such that $LY//BC, KX//CA$. Then $XKHD, LEHY$ are cyclic. It suffices to show that C belongs to the radical axis of the circumscribed circles of these quadrilaterals, or equivalently $CD * CX = CE * CY$, which is equivalent to the claim that $XYED$ is cyclic. As $ABDE$ is cyclic it is enough to show that $AB//XY$. By converse of Thales theorem, we need to show that $\frac{YA}{YC} = \frac{XB}{XC}$. By Thales theorem we need to show that $\frac{KB}{KE} = \frac{LA}{LD}$. Let S, T be the projections of B, A to OD, OE respectively and Z, W the projections of D, E to OE, OD respectively. It suffices to show that $\frac{AT}{BS} = \frac{DZ}{EW} = \frac{OD}{OE}$. This is equivalent to showing that $area(OBD) = area(OAE)$, true because (if M, N are the midpoints of CB, CA) we have $\frac{OM}{ON} = \frac{HA}{HB} = \frac{AE}{BD}$).

Quick Reply

High School Olympiads

Circumcenter of a Tetrahedron

Reply

Source: All-Russian MO 2001 Grade 11 #8

v_Enhance

#1 Jan 3, 2012, 6:39 am • 1

A sphere with center on the plane of the face ABC of a tetrahedron $SABC$ passes through A, B and C, and meets the edges SA, SB, SC again at A_1, B_1, C_1, respectively. The planes through A_1, B_1, C_1 tangent to the sphere meet at O. Prove that O is the circumcenter of the tetrahedron $SA_1B_1C_1$.

Luis González

#2 Jan 3, 2012, 11:26 am

Let $\mathcal{S}$ denote the sphere passing through A, B, C and centered on the plane ABC. Inversion with center S and power $SA \cdot SA_1 = SB \cdot SB_1 = SC \cdot SC_1$ takes $\mathcal{S}$ into itself and swaps the circumsphere $\mathcal{O}$ of $SA_1B_1C_1$ and the plane ABC. The angle between $\mathcal{S}$ and the plane ABC is right, thus by conformity the angle between $\mathcal{S}$ and $\mathcal{O}$ is also right, i.e. $\mathcal{S}$ and $\mathcal{O}$ are orthogonal $\Longrightarrow$ Tangent planes of $\mathcal{S}$ at A_1, B_1, C_1 pass through the center of $\mathcal{O}$.

Quick Reply

High School Olympiads

Concurrent 16

Reply

buratinogigle

#1 Jan 2, 2012, 11:30 pm • 1

Let ABC be a triangle and $A_1B_1C_1$ is pedal triangle of a point P with respect to triangle ABC. T is a point on circumcircle $(A_1B_1C_1)$. ℓ is line passing though T and perpendicular to PT. A_2, B_2, C_2 lie on ℓ such that $PA_2 \perp PA, PB_2 \perp PB, PC_2 \perp PC$. Prove that AA_2, BB_2, CC_2 are concurrent.

Note that, it is generalization of problem on the post concurrent.

Luis González

#2 Jan 3, 2012, 4:15 am • 3

Let $\mathcal{K}$ be the inconic with focus P and pedal circle $\odot(A_1B_1C_1)$. $\ell \perp PT$ is a tangent of $\mathcal{K}$ through M. Let $\mathcal{K}$ touch BC, CA, AB at D, E, F. According to this topic (post #3), PA bisects $\angle EPF \Longrightarrow$ pencil $P(E, F, A, A_2)$ is harmonic. Hence if p is the polar of P WRT $\mathcal{K}$ (directrix of the inconic) and $U \equiv PA \cap p$, it follows that $A_3 \equiv MU \cap EF$ is the pole of AA_2 WRT $\mathcal{K}$. Analogously, if PB, PC cut p at V, W, then $B_3 \equiv MV \cap FD$ and $C_3 \equiv MW \cap DE$ are the poles of BB_2 and CC_2 WRT $\mathcal{K}$.

Since $p \cap \mathcal{K} \equiv \emptyset$, there exists a homography that carries $\mathcal{K}$ into a circle (P^*) with center P^*, the homologous of P. The projected $\triangle A^*B^*C^*$ has incircle (P^*) and $U^*V^*W^* \equiv p^*$ is the line at infinity $\Longrightarrow A_3{}^*, B_3{}^*, C_3^*$ become orthogonal projections of M^* on the sidelines of $\triangle D^*E^*F^*$, i.e. $A_3{}^*B_3{}^*B_3{}^*$ is Simson line of M^* WRT $\triangle D^*E^*F^*$. Thus, polars $A^*A_2^*, B^*B_2^*, C^*C_2^*$ of $A_3{}^*, B_3{}^*, C_3{}^*$ WRT (P^*) concur $\Longrightarrow AA_2, BB_2, CC_2$ concur.

Quick Reply

High School Olympiads

how to draw this diagram, (1)

Source: Geometry Unbound

myceliumful

#1 Jan 2, 2012, 6:26 pm

How do you draw the diagram for this theorem?
Theorem 4.2.3
Let ΔABC be a triangle. Suppose that the lines l_1 and l_2 are perpendicular, and meet each side (or its extension) in a pair of points symmetric across the midpoint of the side. Then, the intersection of l_1 and l_2 is concyclic with the midpoints of the three sides.

Luis González

#2 Jan 2, 2012, 9:00 pm

M,N,L denote the midpoints of BC,CA,AB. $K\equiv \ell_1\cap \ell_2 \in \odot(MNL)$. Circle with center M and radius MK cuts BC at P,Q. KP cuts CA,AB at R,U and KQ cuts CA,AB a S,T. Thus, lines $\ell_1\equiv KP$ and $\ell_2\equiv KQ$ are orthogonal and verify that $\overline{MP}=-\overline{MQ},\overline{NR}=-\overline{NS},\overline{LU}=-\overline{LT}$.

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=364264.

High School Olympiads

Sawayama

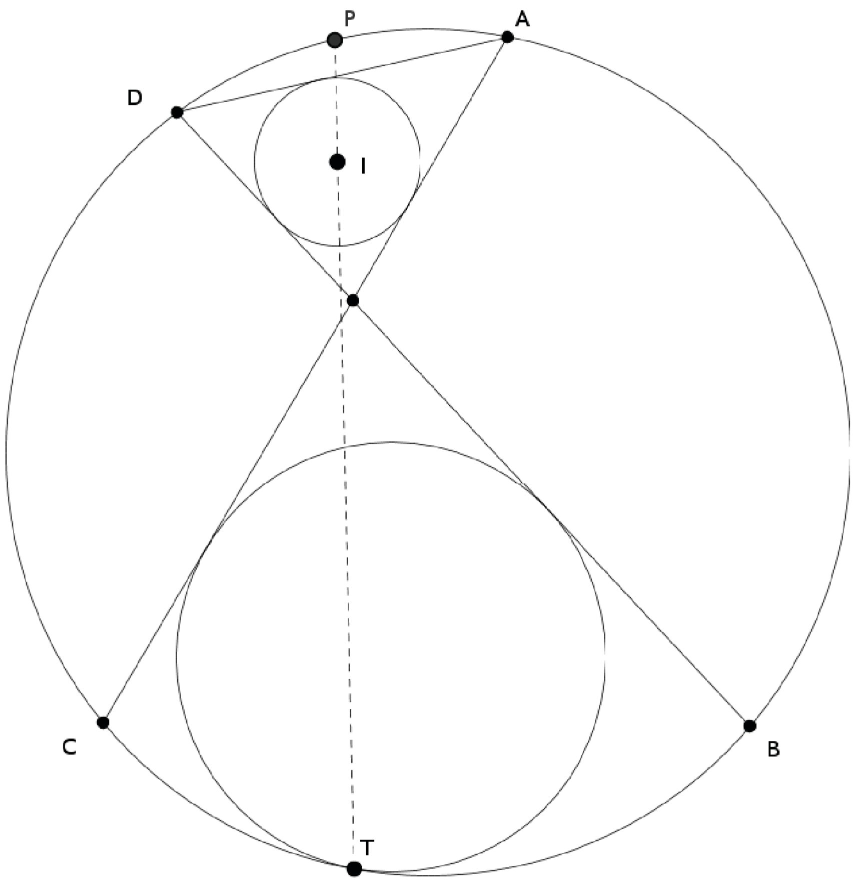
Reply

unt

#1 Jan 2, 2012, 2:05 am

Dear Mathlinkers!
Prove that points T, I, P are lie on a line.
P -- midpoint of arc DA.

Attachments:



Luis González

#2 Jan 2, 2012, 5:41 am • 2

In the given figure, let ω be the circle tangent to AC, BD and the small arc BC of $\odot(ABC)$. If X, Y denote the tangency points of ω with AC, BD, then A, I, X, T are concyclic and D, I, Y, T are concyclic, which immediatly yields $\angle ATI = \angle DTI$, i.e. T, I, P are collinear. This is basically a different fashion of an old problem. See Fairly difficult (Iran 1999), Concyclic points with triangle incenter, incenter of triangle, etc.

Quick Reply

High School Olympiads

Determine cirumradius of triangle

Reply

Goutham

#1 Dec 31, 2011, 6:04 pm

Let ABC be a triangle in which $\angle BAC = 60^{\circ}$. Let P (similarly Q) be the point of intersection of the bisector of $\angle ABC$(similarly of $\angle ACB$) and the side AC(similarly AB). Let r_1 and r_2 be the in-radii of the triangles ABC and APQ, respectively. Determine the circum-radius of APQ in terms of r_1 and r_2.

Luis González

#2 Dec 31, 2011, 11:39 pm • 1

Let $I \equiv BP \cap CQ$ be the incenter of $\triangle ABC.$ (U,R_A) is circumcircle of $\triangle APQ.$ Since $\angle PIQ=120^{\circ}$, then I is midpoint of the arc PQ of $(U) \Longrightarrow M \equiv IU \cap PQ$ is midpoint of $PQ.$ Let J be the incenter of $\triangle APQ.$ E is the projection of I on AC and N lies on $\overline{IE},$ such that $IM=IN.$ $IMEP$ is cyclic due to the right angles at $M,E.$ Thus $\angle IEM=\angle IPQ=\angle IAQ \Longrightarrow \triangle IEM \sim \triangle IAQ \Longrightarrow \frac{_{IE}}{^{IN}}=\frac{_{IA}}{^{IQ}}=\frac{_{IA}}{^{IJ}} \Longrightarrow JN \parallel AC \Longrightarrow$ $NE=r_2=IE-IN=r_1-\frac{_1}{^2}R_A.$

Quick Reply

High School Olympiads

Prove that DS bisects <BSC

Reply

Goutham
#1 Dec 31, 2011, 6:18 pm

Let I be the incentre of a triangle ABC and Γ_a be the excircle opposite A touching BC at D. If ID meets Γ_a again at S, prove that DS bisects $\angle BSC$.

Luis González
#2 Dec 31, 2011, 10:17 pm • 2

According to Geometry Problem (25), ID passes through the midpoint of the A-altitude. Thus, using the extraversion of IMO ShortList 2002, geometry problem 7 on the A-excircle Γ_a, we deduce that $\odot(SBC)$ is tangent to Γ_a at S. Let SB, SC cut Γ_a again at $B,' C'$. Since S is the exsimilicenter of $\Gamma_a \sim \odot(SBC)$, we have $BC \parallel B'C' \Longrightarrow$ arcs DB' and DC' of Γ_a are equal $\Longrightarrow SD$ bisects $\angle BSC$.

polya78
#3 Mar 15, 2013, 2:37 am • 2

Let E, F be the other points of tangency of the ex-circle. Let S' be the intersection of the circumcircles of $\triangle ICF$ and $\triangle IBE$. Then E, F are reflections in AI, so $\angle IFC = \angle IEB$, which means $S'I$ bisects $\angle BS'C$. Also $\angle IS'F = C/2$, $\angle IS'E = B/2$, so $\angle FS'E = 90 - A/2$, which means that S' lies on the excircle also. But $\angle DS'F = \angle DEF = C/2$ as well, so I, D, S' are collinear, which makes $S = S'$.

Attachments:
india postal.pdf (425kb)

Virgil Nicula
#4 Mar 16, 2013, 9:22 am • 1

See **PP5** from **here**.

MMEEvN
#5 Mar 17, 2013, 10:01 pm

Let W be the center of the ex-circle opposite A and L the midpoint of line DS. Let the incircle touch BC at E .We can easily prove that $\Delta EID \sim \Delta LDW \Rightarrow DL.DI = EI.DW$ Writing the formulas for A-ex-radius as $DW=\sqrt_ {\frac{s(s-b)(s-c)}{(s-a)}}$ and inradius as $\sqrt_ {\frac{(s-a)(s-b)(s-c)}{s}}$ gives $DL.DI = (s-b)(s-c)$ .But $DC.DB = (s-b)(s-c) \Rightarrow IBLC$ is cyclic..Let R be the midpoint of $ID \Rightarrow RBSC$ is cyclic. Let the foot of perpendicular from R on BC be J .Easy observation yields $JE = JD \Rightarrow JB = JC. \Rightarrow RB = RC \Rightarrow \angle RSB = \angle CBR = \angle RCB = \angle CSR$ as desired

Virgil Nicula
#6 Mar 19, 2013, 4:07 am

An easy extension. Let $w = C(O, r)$ be a circle what is tangent to the line d in the point $A \in d$. Let

$\{B, D, A, C\} \subset d$ in this order so that $BD = AC$. Denote the point E for which $ED \perp AB$ and

$EB \perp OB$. Define $F \in AE \cap w$. Prove that the ray $[FA$ is the bisector of the angle $\widehat{BFC}$.

Proof. Denote the diameter $[AL]$ of w and the midpoint N , M of the segments $[AE]$, $[BC]$ respectively. Therefore,

$$\left\{\begin{array}{lllll} \triangle BDE \sim \triangle OAB & \Longrightarrow & \frac{BD}{OA}=\frac{DE}{AB} & \Longrightarrow & AB\cdot AC = OA\cdot DE \\ \triangle EDA \sim \triangle AFL & \Longrightarrow & \frac{ED}{AF}=\frac{EA}{AL} & \Longrightarrow & OA\cdot DE = AN\cdot AF \end{array}\right| \Longrightarrow AB\cdot AC = AN\cdot AF \Longrightarrow$$

$BNCF$ is cyclically. Since NM is the bisector of $[BC]$ obtain that $NB=NC \implies [FA$ is the bisector of the angle $\widehat{BFC}$
.

junioragd

#7 Nov 29, 2014, 11:03 pm

Another solution:Let Ia be the excenter and let M be the midpoint of ID.Now,since it is well known that $BM=CM$ it is enough to prove that $BSCM$ is a cyclic or $DM*DS=DB*DC$,Now,since $BD*CD=r*Ra$(If E the point where the incircle touches BC we have that BIE and $BIaD$ are similar) ,since $ID=2*DM$,let N be on IaD such that $ND=r$ and let K be on $ΓA$ such that DK is the diameter,it is enough to prove that $INSK$ is cyclic,but this is easy since $<INK=<ISK=90$,so we are finished.

TelvCohl

#8 Nov 29, 2014, 11:22 pm • 1

My solution:

Let I_a be $A-$ excenter of $\triangle ABC$.
Let M be the midpoint of DS and $T=I_aM\cap BC$.

Easy to see TS is tangent to (I_a) at S .

Since $\angle IBI_a=\angle ICI_a=\angle IMI_a=90^\circ$,
so we get I, I_a, B, C, M lie on a circle with diameter II_a .

Since $TB\cdot TC=TM\cdot TI_a=TD^2=TS^2$,
so $\triangle TSB \sim \triangle \Longrightarrow TCS \angle CSD=\angle SCT-\angle SDT=\angle TSB-\angle TSD=\angle DSB$.

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Aug 21, 2015, 12:58 am

Quick Reply

High School Olympiads

An extension of Nine-point circle

Reply

Source: Own

buratinogigle

#1 Dec 29, 2011, 6:02 pm • 3

Let ABC be a triangle with circumcircle (O). (K) is a circle passing through B, C. (K) cuts CA, AB again at E, F. BE cuts CF at H_K.

a) Prove that $H_K K$ and AO intersect on (O).

b) O_K is isogonal conjugate of H_K with respect to triangle ABC. Prove that O_K lies on OK.

c) Let L, N be the points on CA, AB, resp such that $O_K L \parallel BE, O_K N \parallel CF$. Prove that $LN \parallel BC$.

d) The line passing through N parallel to BE cuts the line passing through L parallel to CF at P. Prove that P lies on AH_K.

e) Q, R lie on BE, CF, resp such that $PQ \parallel AB, PR \parallel AC$. Prove that $QR \parallel BC$.

f) Prove that NQ, LR and AH_K are concurrent.

g) D is projection of K on AH_K. Prove that DK, EF, BC are concurrent.

h) Prove that $KN \perp BE, KL \perp CF$.

i) Prove that nine points $D, E, F; P, Q, R; K, L, N$ lie on a circle (N_K).

j) Prove that N_K is midpoint of PK and KN_K is parallel to AO.

k) Prove that H_K, N_K, O are collinear.

When $K \equiv M$ midpoint of BC, we get all properities of Nine-point circle.

Luis González

#2 Dec 29, 2011, 11:51 pm • 1

a, g) Let $S \equiv EF \cap BC$. Then AS is the polar of H_K WRT (K) and AH_K is the polar of S WRT $(K) \Longrightarrow KH_K$ is perpendicular to AS through H and AH_K is perpendicular to KS through D. Hence $SH \cdot SA = SD \cdot SK = SB \cdot SC \Longrightarrow H \in (O)$. Since $\angle AHH_K = 90^\circ$, then KH_K and AO meet on (O).

b, c) $\angle O_K BC = \angle H_K BF = \angle H_K CE = \angle O_K CB \Longrightarrow O_K$ is on perpendicular bisector OK of $\overline{BC}$. $\angle BFC = \angle BNO_K = \angle BKO_K$ (mod 180) $\Longrightarrow N, B, K, O_K$ are concyclic $\Longrightarrow \angle BNK = \angle BO_K K$. But $\angle BO_K K = 90^\circ - \angle FBE = \angle FEK \Longrightarrow \angle BNK = \angle FEK$, i.e. N lies on circumcircle (N_K) of $DKEF$. Similarly, $L \in (N_K)$. Thus, LN is antiparallel to EF WRT $AE, AF \Longrightarrow LN \parallel BC$.

d, f) $\triangle PLN$ and $\triangle H_K CB$, with parallel sides, are homothetic with center $A \Longrightarrow A, P, H_K$ are collinear. Likewise, $\triangle ANL$ and $\triangle PQR$, with parallel sides, are homothetic with center $AP \cap NQ \cap LR$, i.e. AH_K, NQ, LR concur.

i) $\angle NKL = \angle NBO_K + \angle LCO_K = CBH_K + \angle BCH_K = \angle NPL$ (mod 180) $\Longrightarrow P \in (N_K)$. Further, P is the midpoint of the arc EF of (N_K), because $\angle PFK = \angle PDK = 90^\circ$, i.e. KP is perpendicular bisector of $\overline{EF}$. Now, since $\angle PQE = \angle FBE = \angle PKE$, it follows that $Q \in (N_K)$. Similarly, $R \in (N_K)$.

h, j) $D, E, F, P, Q, R, K, L, N$ lie then on a circle (N_K) with diameter KP perpendicular to EF, i.e. $KP \parallel AO$. Thus, KN is perpendicular to $PN \parallel BE$ and KL is perpendicular to $PL \parallel CF$.

e, k) $\triangle PQR \cup (N_K)$ and $\triangle ABC \cup (O)$ are homothetic with center $H_K \equiv AP \cap BQ \cap CR$, thus $QR \parallel BC$ and H_K, O, N_K are collinear.

Quick Reply

High School Olympiads

trancuong
#1 Jul 29, 2010, 11:40 am

Given convex quadrilateral ABCD with incircle (I). M, N are the midpoints of AC, BD. Prove that I, M, N are collinear

Luis González
#2 Dec 28, 2011, 11:12 pm

Locus of points P such that $[\triangle PAB]+[\triangle PCD]=\frac{1}{2}[ABCD]$ is either a line segment bounded by AB, CD or a whole line ℓ if we consider directed areas. Put $ABCD$ on a rectangular reference (x, y) and let us consider the oriented distances δ_1 and δ_2 from P to AB, CD in order not to have any issue with the area sign. Equation of the locus is obtained as $\delta_1 \cdot AB+\delta_2 \cdot CD=[ABCD]$. Since the formula of the oriented distance from $P(x, y)$ to a line in the cartesian plane is a linear fuction of x, y, it follows that $\delta_1=f(x, y)$ and $\delta_2=g(x, y)$ are linear $\Longrightarrow$ locus ℓ of $P(x, y)$ is a linear fuction of x, y.

Midpoints M, N of AC, BD lie on the locus ℓ, because

$$[\triangle MBA]+[\triangle MCD]=\frac{1}{2}[\triangle ABC+\triangle CDA]=\frac{1}{2}[ABCD]$$

$$[\triangle NBA]+[\triangle NCD]=\frac{1}{2}[\triangle BAD+\triangle DCB]=\frac{1}{2}[ABCD]$$

Incenter I of $ABCD$ lies on the locus ℓ, because

$$[\triangle IAB]+[\triangle ICD]=\frac{1}{2}r(AB+CD)=\frac{1}{2}r(BC+DA)=\frac{1}{2}[ABCD]$$

Therefore, M, N, I lie on a line ℓ whose points satisfy the referred condition.

Virgil Nicula
#3 Dec 29, 2011, 1:46 am

Lemma. Let $ABCD$ be a convex quadrilateral for which denote $O\in AB\cap CD$. Find the geometrical locus of the mobile point L which
belongs to the interior of the angle $\widehat{AOD}$ for which $\alpha\cdot [AOB]+\beta\cdot [OCD]=k$ (constant), where α and β are two given positive numbers.

Proof. Denote $\left\{\begin{array}{c} E\in (OA\ ,\ OE=\alpha\cdot AB\\\\ F\in (OD\ ,\ OF=\beta\cdot CD\end{array}\right\|$. Observe that $[LOE]=\alpha\cdot [LAB]$ and $[LOF]=\beta\cdot [LDC]$ and

$k=\alpha\cdot [AOB]+\beta\cdot [OCD]=[LOE]+[LOF]=[LEOF]=[EOF]+[ELF]\implies [ELF]=k-[EOF]$ is constant

because the triangle EOF is fixed. Since the segment $[EF]$ is fixed (with constant length) obtain that the distance of L to EF is

constant, i.e. the geometrical locus of the point L is a parallel segment with the line EF and which prop up the rays $(OA$ and $(OD$.

sunken rock
#4 Dec 20, 2014, 3:44 am

Attached, a proof by vectors, as it appeared some good years ago into Romanian Magazine "**Gazeta Matematica**".
I found it extremely interesting.

Best regards,
sunken rock

Attachments:
Newton vectorial, English.doc (44kb)

High School Olympiads

tangent

Reply

Source: 0

mathscope1995

#1 Dec 28, 2011, 4:13 pm

let ABC is a cute triangle and its circumcircle is (O).E is intersection of AO and BC.P is on (O) AP is symmedian.AD is perpencular to BC (with D on BC).prove cicumcenter of triangle PDE lies on OP

Luis González

#2 Dec 28, 2011, 9:47 pm • 2

Tangents of (O) through B and C meet on $AP \Longrightarrow ABPC$ is harmonic $\Longrightarrow$ Tangents of (O) through A and P meet a point Q lying on BC, the center of the A-Apollonius circle. Since $\angle QAE$ is right, we have $QP^2=QA^2=QD \cdot QE \Longrightarrow$ $\odot(PDE)$ is tangent to (O) at P, i.e. circumcenter of $\triangle PDE$ lies on OP.

sunken rock

#3 Jan 4, 2012, 8:52 pm

Observation:
This property is valid for each vertex of any harmonic quad $ABPC$!

Best regards,
sunken rock

Quick Reply

Spain

9th IBERO - BRAZIL 1994.

Reply

carlosbr

#1 Mar 26, 2006, 8:24 pm

9^{th} Iberoamerican Olympiad

Fortaleza - Ceara, BRAZIL. [1994]

Edited by djimenez

Carlos Bravo

Attachments:
1994.pdf (32kb)

Luis González

#2 Dec 11, 2009, 12:45 am

> *Quote:*
> **Problema 2.i)** Sea $ABCD$ un cuadrilátero cíclico y suponga que existe una circunferencia centrada en el lado BC y tangente al resto de sus lados, probar que $AB = AD + BC$.

Sea O el centro de la circunferencia centrada en AB tangente a AD, DC, CB. Los rayos OC y OD son bisectrices de $\angle BCD = \theta$ y $\angle CDA = \varphi$. Tomando un punto M en AB tal que $AD = AM$ se forma el $\triangle ADM$ isósceles del cual

$$\angle DMA = \frac{\pi - \angle DAB}{2} = \frac{\pi}{2} - \frac{\pi - \theta}{2} = \frac{\theta}{2}$$

Por tanto, como $\angle DCO = \frac{\theta}{2} \Longrightarrow$ El cuadrilátero $DCOM$ es cíclico y por consiguiente $\angle CMB = \angle ODC = \frac{\varphi}{2}$. Por otro lado en el $\triangle CMB$ se tiene:

$$\angle MCB = \pi - \frac{\varphi}{2} - (\pi - \varphi) = \frac{\varphi}{2} \Longrightarrow \angle CMB = \angle MCB = \frac{\varphi}{2}$$

$\triangle CMB$ es pues isósceles con base CM, así $AB = AM + BM = AD + BC$.

Luis González

#3 Dec 12, 2009, 6:19 am

El problema #4 puede ser generalizado de la siguiente manera:

Problema: Construir el punto P interior al $\triangle ABC$ de modo que los rayos PA, PB, PC corten los arcos BC, CA, AB de su circunferencia circunscrita en vértices de un triángulo acutángulo $\triangle XYZ$ (triángulo circunceviano de P) que sea semejante a otro dado.

Solución: Denotemos α', β', γ' los ángulos dados en X, Y, Z. Usando el Teorema generalizado de Simson, una inversión con centro en P y potencia arbitraria transformará los vértices de $\triangle ABC$ en vértices de otro triángulo semejante al triángulo pedal $\triangle A'B'C'$ de P con respecto a $\triangle ABC$. Particularmente X, Y, Z son pues los inversos de A, B, C a través de la inversión centrada en P con potencia igual a la que tiene P respecto a el circumcírculo (O) de $\triangle ABC$. Así, el problema consiste en hallar el P cuyo triángulo pedal sea semejante al dado.

Las perpendiculares a $A'B'$ y $A'C'$ desde C y B concurren el en conjugado isogonal Q de P en $\triangle ABC$, entonces de los cuadriláteros cíclicos $PA'BC'$ y $PA'CB'$ tenemos que

$$\angle QBA = \angle PBC' = \angle PA'C' \ , \ \angle QCA = \angle PCB' = \angle PA'B'$$

$$\Longrightarrow \angle B'A'C' = \angle PBC = \angle PCB = \pi - \angle BQC$$

Analogamente obtenemos $\angle CQA = \pi - \angle C'B'A'$, $\angle AQB = \pi - \angle B'C'A'$.

Así, Q es construible ya que mira a los lados BC, CA, AB bajo ángulos suplementarios a los dados α', β', γ'. Bastará pues con construir, por ejemplo, los arcos que miran a BC, CA bajo $\pi - \alpha'$ y $\pi - \beta'$, éstos se cortan en Q. Luego se sigue la construcción de P y su triángulo circunceviano buscado $\triangle XYZ$.

Luis González

#4 Dec 26, 2011, 10:35 pm

> *Quote:*
>
> **Problema 5.** Sean n, r dos enteros positivos. Se quiere formar r subconjuntos $A_1, A_2, ..., A_r$ del conjunto $\{0, 1, ..., n-1\}$, de modo que todos los subconjuntos contengan exactamente k elementos tal que, para todo entero x con $0 \le x \le n-1$ exista $x_1 \in A_1, x_2 \in A_2, ..., x_r \in A_r$ (un elemento de cada conjunto) con $x = x_1 + x_2 + ... + x_r$. Además, hallar el valor mínimo de k en función de n y r.

Para todo $x_i \in A_i$, considere la función $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + x_n$. Se quiere encontrar el valor más pequeño de k tal que la imagen de f incluya a todos los números $0, 1, 2, \dots, n-1$.

El tamaño óptimo ocurre pues cuando f is injectiva, pero inclusive asumiendo ésto, se tiene que la imagen de f es a lo más k^r. Entonces, si $k^r < n$, existen $x \in [0, n-1]$ que no pueden ser calculados. Ahora, considérese $k = \lceil n^{\frac{1}{r}} \rceil$. Si probamos que ésto es posible, entonces el problema se habrá resuelto. Entonces, consideremos los r conjuntos

$$A_1 = \{0, 1, \dots, k-1\},$$
$$A_2 = \{0, k, 2k, \dots, k(k-1)\}, ...$$
$$A_r = \{0, k^{r-1}, 2k^{r-1}, \dots, (k-1)k^{r-1}\}.$$

Finalmente, si ninguno de los elementos de A_r son mayores que $n-1$, entonces simplemente remplazelos, por decir, con 1. Es facil probar que ésto es equivalente a escribir x en base k con r dígitos y de hecho todo $x \in [0, n-1]$.

Quick Reply

High School Olympiads

Parallel (Own)

Reply

skytin

#1 Dec 17, 2011, 4:14 pm • 2

Given triangle ABC , AB = BC = AC
Let points P , Q are on (ABC) , such that PQ is diameter of (ABC)
A' , B' , C' are midpoints of cides of ABC
P' , Q' are isogonal to P , Q wrt A'B'C'
(a) Prove that P'Q' || PQ
(b) AC = 1 , find P'Q'

yetti

#2 Dec 21, 2011, 7:45 am

Incircle (O,r) of equilateral $\triangle A'B'C'$ touches $B'C'$ at its midpoint D'. Coordinate origin at O, positive x-, y-axes parallel to $\overrightarrow{OD'}, \overrightarrow{B'C'}$.
Point $P = (4r\cos\phi, 4r\sin\phi)$ is on circumcircle $(O,4r)$ of equilateral $\triangle ABC$. Foot F_0 of perpendicular from P to $B'C'$ has coordinates $F_0 = (r, 4r\sin\phi)$.
Coordinates of feet F_+, F_- of perpendiculars from P to $C'A', A'B'$ are obtained as rotations

$F_+ = \mathbf R(+\frac{_2}{^3}\pi)(r, 4r \sin (\phi -\frac{_2}{^3}\pi)) = r(-\frac{_1}{^2} + \sqrt 3 \sin \phi + 3\cos \phi, +\frac{_\sqrt 3}{^2} + \sin \phi + \sqrt 3 \cos \phi)$

$F_- = \mathbf R(-\frac{_2}{^3}\pi)(r, 4r \sin (\phi +\frac{_2}{^3}\pi)) = r(-\frac{_1}{^2} - \sqrt 3 \sin \phi + 3\cos \phi, -\frac{_\sqrt 3}{^2} + \sin \phi - \sqrt 3 \cos \phi)$

$\Longrightarrow$ perpendicular bisector of F_+F_- through its midpoint has equation $y(1+2\cos\phi) + x(2\sin\phi) = 4r\sin 2\phi\ \ (1)$.
Equation of perpendicular bisector of F_+F_0 is obtained by substituting $\phi \longrightarrow \phi + \frac{2}{3}\pi$ and $(x,y) \longrightarrow \mathbf{R}^{-1}(-\frac{2}{3}\pi)(x,y) = \mathbf{R}(+\frac{2}{3}\pi)(x,y)$:

$y(-\frac{_1}{^2} - \cos \phi + \sqrt 3 \sin \phi) + x(+\frac{_\sqrt 3}{^2} - \sqrt 3 \cos \phi - \sin \phi) =$ $-4r (\frac{_1}{^2} \sin 2 \phi + \frac{_\sqrt 3}{^2} \cos 2 \phi) \ \ (2)$

Solving equations $(1), (2)$ of the two perpendicular bisectors yields coordinates of circumcenter U of the pedal $\triangle F_-F_0F_+$ of P.
Multiplying (2) by 2, adding (1) and dividing the result by $\sqrt 3$ leads to $y(2\sin\phi) + x(1-2\cos\phi) = -4r\cos 2\phi\ \ (3)$.
Solving equations $(1), (3)$ and labeling $\varrho = \frac{4}{3}r$ leads to $x = \varrho(2\cos\phi + \cos 2\phi)$, $y = \varrho(2\sin\phi - \sin 2\phi)$,
which are parametric equations of deltoid $\mathcal D$, or hypocycloid with 3 cusps, inscribed in circle $(O,4r)$, the locus of U.
Circumcenter V of pedal triangle of Q is also on $\mathcal D$.
Its coordinates are obtained from coordinates of U by substitution $\phi \longrightarrow \phi \pm \pi$: $x = \varrho(-2\cos\phi + \cos 2\phi)$, $y = \varrho(-2\sin\phi - \sin 2\phi)$.

Slopes of PQ, UV are both equal to $\tan\phi \Longrightarrow UV \parallel PQ$. In addition, $UV = 4\varrho$.
Since isogonal conjugates P^*, Q^* of P, Q WRT $\triangle A'B'C'$ are reflections of P, Q in $U, V \Longrightarrow P^*Q^* \parallel UV \parallel PQ$.
UV is midline of trapezoid $PQQ^*P^* \Longrightarrow P^*Q^* = 2UV - PQ = 8\varrho - 6\varrho = 2\varrho$.
$\frac{P^*Q^*}{BC} = \frac{P^*Q^*}{\varrho} \cdot \frac{3\varrho}{3BC} = \frac{2}{3\sqrt 3}$.

Rotation matrix

skytin

#3 Dec 21, 2011, 11:56 pm

Very strange and interesting solution Thank you

yetti

#4 Dec 22, 2011, 3:47 pm • 2

(O,R), $(O,2R)$ are circumcircles of equilateral $\triangle A'B'C'$, $\triangle ABC$, respectively.
Rays $(OA, (OP, (OQ$ cut (O,R) at A_0, P_0, Q_0. a, p, q are Simson lines of $\triangle A'B'C'$ with the poles A_0, P_0, Q_0.
Parallels through P, Q to a cut $(O,2R)$ again at E, F. Since $p \perp q$, parallels through P, Q to p, q meet at $S \in (O,2R)$.
Since $a \parallel B'C' \Longrightarrow a \perp A'O \Longrightarrow E$ is reflection of P in $A'O$ and $A'E$ is reflection of $A'P$ in $A'O$.
$\angle EOS = 2\angle EPS = \angle P_0OA_0 = \angle POA \Longrightarrow AE \parallel PS$. Parallel to these lines through A' cuts (O,R) again at E'
$\Longrightarrow A'E' = \frac{1}{2}AE$.
Let $X \equiv AE' \cap A'E$, $K \equiv XE'A \cap PS$ and $L \equiv XA'E \cap PS$. $PK = LS$ by symmetry. Let M, N be midpoints of AE, PS. Label $\phi = \angle POA = \angle AOE = \angle EOS$.

$$\frac{KL}{PS} = \frac{KL}{AE} \cdot \frac{AE}{PS} = \frac{XN}{XM} \cdot \frac{AE}{PS} = \frac{2OM + ON}{3OM} \cdot \frac{AE}{PS} =$$
$$\frac{2\cos\frac{1}{2}\phi + \cos\frac{3}{2}\phi}{3\cos\frac{1}{2}\phi} \cdot \frac{\sin\frac{1}{2}\phi}{\sin\frac{3}{2}\phi} = \frac{2\sin\phi + (\sin 2\phi - \sin\phi)}{3(\sin 2\phi + \sin\phi)} = \frac{1}{3} \Longrightarrow PK = KL = LS = \frac{1}{3}PS.$$

Thus reflection $A'E$ of $A'P$ in $A'O$ cuts PS at L in the ratio $\frac{LS}{PS} = \frac{1}{3}$. Similarly, reflections of $B'P, C'P$ in $B'O, C'O$ cut PS in the same ratio $\Longrightarrow$

these reflections concur at L, which is therefore identical with isogonal conjugate P^* of P WRT $\triangle A'B'C' \Longrightarrow \frac{P^*S}{PS} = \frac{1}{3}$.

In exactly the same way, isogonal conjugate Q^* of Q WRT $\triangle A'B'C'$ divides QS in the ratio $\frac{Q^*S}{QS} = \frac{1}{3}$.

It follows that $P^*Q^* \parallel PQ$ and $\frac{P^*Q^*}{BC} = \frac{P^*Q^*}{PQ} \cdot \frac{PQ}{BC} = \frac{2}{3\sqrt{3}}$.

Luis González

#5 Dec 26, 2011, 9:15 pm • 2

The problem can be generalized as follows

Proposition. P is an arbitrary point on the plane of the scalene triangle $\triangle ABC$. Q is the reflection of P about the centroid G of $\triangle ABC$. P' and Q' are the isotomic conjugates of P and Q WRT $\triangle ABC$. Then $PQ \parallel P'Q'$.

Proof. Let $(u:v:w)$ be the barycentric coordinates of P WRT $\triangle ABC$. Then the coordinates of its reflection Q about $G \equiv (1:1:1)$ are $Q \equiv (2v+2w-u : 2u+2w-v : 2u+2v-w)$.

Coordinates of the isotomic conjugates P', Q' of P, Q are then

$$P'\left(\frac{1}{u} : \frac{1}{v} : \frac{1}{w}\right), \quad Q'\left(\frac{1}{2v+2w-u} : \frac{1}{2u+2w-v} : \frac{1}{2u+2v-w}\right)$$

Barycentric coordinates of line $P'Q'$ are given by

$$[u(v-w)(2v+2w-u) : v(w-u)(2w+2u-v) : w(u-v)(2u+2v-w)]$$

Infinity point of $P'Q'$ is then $P_\infty(w+v-2u : u+w-2v : u+v-2w)$, which coincides with the infinite point of line $PG \equiv (v-w)x + (w-u)y + (u-v)z = 0$.

Bonus. $R \equiv PQ' \cap QP'$ lies on the Steiner circum-ellipse of $\triangle ABC$.

Quick Reply

High School Olympiads

X Brazilian Olympic Revenge, 2011 - Problem 4

Reply

Source: X Brazilian Olympic Revenge, 2011

hvaz
#1 Dec 24, 2011, 11:34 pm • 1

Let $ABCD$ to be a quadrilateral inscribed in a circle Γ. Let r and s to be the tangents to Γ through B and C, respectively, M the intersection between the lines r and AD and N the intersection between the lines s and AD. After all, let E to be the intersection between the lines BN and CM, F the intersection between the lines AE and BC and L the midpoint of BC. Prove that the circuncircle of the triangle DLF is tangent to Γ.

Luis González
#2 Dec 25, 2011, 12:05 am

Assume that E is inside $\Gamma \Longrightarrow$ There exists a homology carrying Γ into a circle (E) with center E. Thus, $ABCD$ and $BCNM$ become isosceles trapezoids with $BC \parallel MN$. Let $P \equiv BM \cap CN$ and $G \equiv PD \cap BC$. Since EP bisects $\angle NEC$ externally, by Angle Bisector Theorem we have then $\frac{EN}{EC} = \frac{PN}{PC} = \frac{ND}{CG}$. Hence, $\triangle EDN \sim \triangle EGC$ by SAS $\Longrightarrow$ $\angle DEN = \angle GEC$, i.e. EG is the reflection of ED about $PE \Longrightarrow G \equiv F \Longrightarrow DG$ is the D-symmedian of $\triangle DBC$. Now, in the original figure, we have $\angle DLC = \angle DBC + \angle FDC \Longrightarrow \odot(DLF)$ is tangent to Γ.

See also http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=359561.

horizon
#3 Feb 5, 2012, 9:34 pm • 2

the keypoint is to prove that K, F, D is collinear

$$\frac{sin\angle BKF}{sin\angle FKC} = \frac{BF}{FC} = \frac{BE}{EC}\frac{sin\angle AEN}{sin\angle MEA} = \frac{BE}{EC}\frac{AN}{AM}\frac{ME}{EN} = \frac{ME}{EC}\frac{BE}{EN}\frac{AN}{AM} = \frac{MB}{BK}\frac{NK}{NC}\frac{MB}{MK}\frac{CK}{CN}\frac{AN}{AM} = \frac{MB^2}{NC^2}\frac{NK}{MA}\frac{NA}{MK} = \frac{MD}{MK}\frac{NK}{ND} = \frac{sin\angle MKD}{sin\angle DKN}$$

so K, D, F is collinear

Quick Reply

High School Olympiads

3 geometry problems

Reply

tkrass
#1 Feb 12, 2010, 9:16 pm

Image not found

number one: prove that AE, DG, CF meet in one point
number two: prove that AF, HE, BG meet in one point
number three: M is the midpoint of DC and AE. prove that DC and BE are perpendicular

Luis González
#2 Dec 24, 2011, 12:21 pm

1) The result is still true for arbitrary parallelograms $ABDC$ and $BFGE.$

Let $M \equiv CF \cap DG$ and $N \equiv BC \cap DG.$ From $\triangle MGF \sim \triangle MNC$ and $\triangle CND \sim \triangle ENG,$ we get

$$\frac{FM}{MC}=\frac{GF}{CN}=\frac{EB}{CN} \ , \ \frac{EG}{CD}=\frac{EN}{CN}$$

$$\Longrightarrow \frac{EG+CD}{CD}=\frac{EC}{CN} \Longrightarrow \frac{AF}{CD}=\frac{CE}{CN}$$

$$\Longrightarrow \frac{AB}{AF} \cdot \frac{FM}{MC} \cdot \frac{CE}{EB}=\frac{CD}{AF} \cdot \frac{EB}{CN} \cdot \frac{CE}{EB}=\frac{CN}{CE} \cdot \frac{CE}{CN}=1$$

By Menelaus' theorem for $\triangle BCF,$ we conclude that A,E,M are collinear, i.e. AE,DG,CF concur at $M.$

Luis González
#3 Dec 24, 2011, 12:22 pm

2) Let $P \equiv AF \cap BG.$ $\triangle CGB$ and $\triangle CAF$ are congruent by SAS, due to $\overline{CG}=\overline{CA},\overline{CF}=\overline{CB}$ and $\angle GCB=\angle ACF=90^\circ+\angle ACB.$ Therefore, $\angle CAP=\angle CGP$ and $\angle CFP=\angle CBP.$ Which implies that $CPAG$ and $CPBF$ are both cyclic, i.e circumcircles of $ACGH$ and $BCFE$ meet at $C,P.$ Since $\overline{HG}=\overline{HA},$ it follows that PH bisects $\angle GPA.$ Likewise, PE bisects $\angle FPB \Longrightarrow H,P,E$ are collinear, i.e. lines AF,HE,BG concur at $P.$

3) Obviously, $EDAC$ is a parallelogram, thus $ED \parallel CA$ and $EC \parallel AD.$ But since $AD \perp DB$ and $AC \perp CB,$ we have $EC \perp DB$ and $BC \perp ED \Longrightarrow C$ is orthocenter of $\triangle EDB \Longrightarrow DC \perp EB.$

sayantan
#4 Dec 24, 2011, 2:02 pm

In the 3rd question AB is diameter of that semicircle??
If it is so then here is the solution-
Let O be the mid point of AB i.e. center of the circle and DC intersect BE at N. Then $\angle OMN=90^\circ$ and in $\triangle ABE$, $OM \parallel EB$ so $\angle MNB=90^\circ$

This post has been edited 1 time. Last edited by sayantan, Dec 24, 2011, 9:19 pm

Virgil Nicula
#5 Dec 24, 2011, 2:04 pm

1 ▶ Let $ABCD$ and $BEGF$ be two parallelograms so that $E\in (BC)$ and BC separates A , F . Denote

$AB = a$, $AD = b$, $BF = c$ $BE = d$ and $H \in CD \cap FG$, $K \in CF \cap AE$, $L \in CF \cap DG$.
Apply the **Menelaus' theorem** to mentioned transversals and triangles :

$$\left\{\begin{array}{ccccccc} \overline{AEK}/\triangle BCF & : & \frac{AB}{AF}\cdot\frac{KF}{KC}\cdot\frac{EC}{EB}=1 & \implies & \frac{a}{a+c}\cdot\frac{KF}{KC}\cdot\frac{b-d}{d}=1 & \implies & \frac{KC}{KF}=\frac{a(b-d)}{d(a+c)} \\ \overline{DGK}/\triangle HCF & : & \frac{DC}{DH}\cdot\frac{GH}{GF}\cdot\frac{LF}{LC}=1 & \implies & \frac{a}{a+c}\cdot\frac{b-d}{d}\cdot\frac{LF}{LC}=1 & \implies & \frac{LC}{LF}=\frac{a(b-d)}{d(a+c)}\end{array}\right\|$$

$$\implies \frac{KC}{KF}=\frac{LC}{LF} \implies K\equiv L \implies AE\cap DG\cap CF\ne\emptyset\ .$$

Remark. See **PP15** from **here**.

This post has been edited 1 time. Last edited by Virgil Nicula, Dec 26, 2011, 7:38 pm

sayantan
#6 Dec 24, 2011, 9:28 pm

> *sayantan wrote:*
> In the 3rd question AB is diameter of that semicircle??
> If it is so then here is the solution-
> Let O be the mid point of AB i.e. center of the circle and DC intersect BE at N. Then $\angle OMN = 90^\circ$ and in $\triangle ABE$, $OM \parallel EB$ so $\angle MNB = 90^\circ$

my post has changed..there is some error in system..follow latex code of my post..

Quick Reply

Spain

7th IBERO - VENEZUELA 1992.

Reply

carlosbr

#1 Mar 26, 2006, 11:15 am

7^{th} Iberoamerican Olympiad
Caracas, VENEZUELA. [1992]

Edited by djimenez

Carlos Bravo

Attachments:
1992.pdf (25kb)

Luis González

#2 Oct 29, 2009, 8:52 am

> *Quote:*
> **Problema 3.** Un triángulo equilátero de lado 2 está circunscrito en una circunferencia Γ. Probar que: a) Para todo punto P sobre Γ, la suma de los cuadrados de sus distancias a A, B, C es 5. b) Para todo punto P existe un triángulo con lados PA, PB, PC y cuya área es $\frac{\sqrt{3}}{4}$.

a) En un triángulo $\triangle ABC$, P es un punto variable sobre una circunferencia centrada en su baricentro G y radio ϱ. Entonces $PA^2 + PB^2 + PC^2$ es una cantidad constante igual a $3\varrho^2 + \frac{1}{3}(a^2 + b^2 + c^2)$.

b) Si P es un punto del plano del triángulo equilátero $\triangle ABC$, entonces existe un triángulo $\triangle XYZ$ cuyos lados tienen longitudes iguales a PA, PB, PC. Este triángulo llamado de *Pompeiu* tiene varias propiedades, entre ellas:

● Cuando P yace en en la circunferencia circunscrita de $\triangle ABC$ tal triángulo se degenera.

● $\triangle XYZ$ es siempre inscriptible en $\triangle ABC$ y con esto se entiende de que siempre existe al menos una posición de $\triangle XYZ$ de modo que sus vertices X, Y, Z estén en las rectas BC, CA, AB.

● El área de $\triangle XYZ$ equivale a cuatro tercios del área del triángulo pedal de P respecto $\triangle ABC$. Lo cual implica que su área puede ser calculada usando el teorema de Euler, conociendo el circunradio de $\triangle ABC$ y la distancia de P a su circuncentro.

Luis González

#3 Nov 1, 2009, 9:40 am

> *Quote:*
> Problema 5. Sea Γ una circunferencia y h, m dos cantidades positivas tales que existe un trapecio $ABCD$ inscrito en Γ con altura h y tal que la suma de sus bases $AB + CD$ es igual a m. Construir tal trapecio.

Sean K, L, M, N los puntos medios de AB, BC, CD, DA. Es claro que como $ABCD$ es un trapecio isosceles, entonces $KLMN$ es un rombo con diagonales conocidas, la altura $KM = h$ y la base $LN = \frac{1}{2}(AB + CD) = \frac{1}{2}m$. Así la construcción del rombo $KLMN$ es inmediata y de ella se desprende la longitud KL que vale la mitad de la diagonal AC.

El triángulo $\triangle ACD$ de lado $AC = 2KL$, altura sobre DC igual a h y circuncírculo Γ es determinable.

Se fija la cuerda AC en Γ y se traza la circunferencia de diámetro AC. La circunferencia centrada en A con radio h corta a la circunferencia de diámetro AC en la proyección H de A en DC, luego el rayo CH corta a Γ en D. La paralela a CD que pasa por A corta a Γ en B completandose el trapecio.

Luis González

#4 Dec 24, 2011, 11:15 am

> Quote:
>
> **Problema 6.** En un $\triangle ABC$ se toman los puntos A_1, A_2 en las prolongaciones $\overrightarrow{BA}, \overrightarrow{CA}$, tales que $AA_1 = AA_2 = BC$. Se definen los pares de puntos B_1, B_2 y C_1, C_2 del mismo modo. Probar que el área del hexágono convexo $A_1A_2B_1B_2C_1C_2$ es mayor o igual a 13 veces el área de $\triangle ABC$.

Usando que la razón entre las áreas de dos triángulos con un par de ángulos iguales o suplementarios es igual a la razón entre el producto de los lados correspondientes que los forman, obtenemos

$$\frac{[AA_1A_2]}{[ABC]} = \frac{a^2}{bc}\ , \ \frac{[BCC_1B_2]}{[ABC]} = \frac{[AB_2C_1]}{[ABC]} - 1 = \frac{(b+c)^2}{bc} - 1 = \frac{b^2+c^2}{bc} + 1$$

Entonces, teniendo en cuenta las expresiones simétricas para el resto de las áreas, el área S del hexágono $A_1A_2B_1B_2C_1C_2$ viene dada por

$$\frac{S}{[ABC]} = \frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} + \frac{b^2+c^2}{bc} + \frac{c^2+a^2}{ca} + \frac{a^2+b^2}{ab} + 4$$

Entonces, la desiguadad pedida equivale a $(a^2+b^2+c^2)(a+b+c) \geq 9abc$, la cual se deduce por media aritmética-media geométrica aplicada sobre a, b, c y a^2, b^2, c^2, respectivamente.

Quick Reply

High School Olympiads

Bisector

Reply

mastergeo

#1 Dec 22, 2011, 8:18 pm

Let (O) and (O') are two tangent circle, and T is the tangent point. M is an arbitrary point lies in (O) and A, B lie in (O') such that A, B, M are collinear and $\overline{ABM}$ is the tangent of (O) (B lies between A, M). Denote $MT \cap (O') = N$ and C is an arbitrary point on arc NT that does not contain A. D lies in (O) such that CD is tangent to (O). $NC \cap MD = E$. Prove that AE is the internal bisector of $\angle BAC$.

Luis González

#2 Dec 23, 2011, 2:31 am • 2

We assume that (O) and (O') are externally tangent at $T.$ Lines TA, TB cut (O) again at $P, Q.$ Since T is the exsimilicenter of $(O) \sim (O'),$ then $PQ \parallel AB \Longrightarrow$ Arcs MP, MQ of (O) are congruent, i.e. M is midpoint of the arc PQ of (O) not containing $D \Longrightarrow N$ is midpoint of the arc AB of (O') containing $C \Longrightarrow CN$ bisects $\angle ACB$ externally. Now, using the extraversion of Sawayama's Lemma on the circle (O) externally tangent to $(O)'$ and tangent to AB and the C-cevian CD of $\triangle ABC,$ it follows that MD passes through the A-excenter of $\triangle ABC \Longrightarrow E \equiv NC \cap MD$ is the A-excenter of $\triangle ABC$ $\Longrightarrow AE$ bisects $\angle BAC$ internally.

Quick Reply

High School Olympiads

Moment of inertia

Reply

erkamseker
#1 Dec 22, 2011, 7:38 pm

Let ABC be a triangle and H is orhocenter, F is nine points circle, R is radius of circumcicle of ABC.
Prove that;
$FA^2+FB^2+FC^2+FH^2=3R^2$
Hint
Click to reveal hidden text

Luis González
#2 Dec 23, 2011, 12:32 am

This relation and a generalization have been discussed before

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=29235
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=29234
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=50&t=366607

Quick Reply

High School Olympiads

Equal sum

Reply

mastergeo

#1 Dec 22, 2011, 8:15 pm

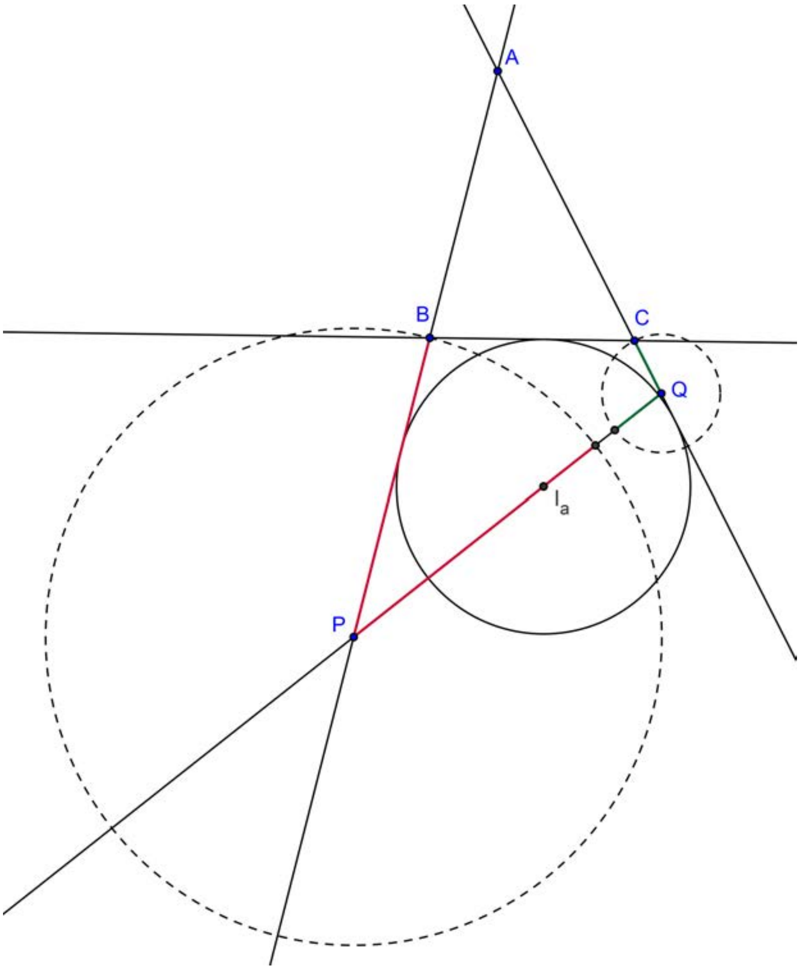
Let ABC be a triangle with excircle (I_a) with respect to A. P, Q are two points on external side AB, AC such that P, I, Q are collinear. Prove that $BP + CQ = PQ$.

Luis González

#2 Dec 22, 2011, 11:59 pm

There must be an extra condition here. My diagram reveals the result is not true.

Attachments:



Quick Reply

High School Olympiads

|SA'| |SB'| = |SN|^2

Reply

Source: Baltic Way 1994

WakeUp

#1 Dec 22, 2011, 9:55 pm

Let NS and EW be two perpendicular diameters of a circle $\mathcal{C}$. A line ℓ touches $\mathcal{C}$ at point S. Let A and B be two points on $\mathcal{C}$, symmetric with respect to the diameter EW. Denote the intersection points of ℓ with the lines NA and NB by A' and B', respectively. Show that $|SA'|\cdot|SB'|=|SN|^2$.

Luis González

#2 Dec 22, 2011, 11:26 pm

ℓ is obviously the inverse of $\mathcal{C}$ under the inversion with center N and power NS^2. Thus, A,A' and B,B' are pairs of inverse points $\Longrightarrow AA'B'B$ is cyclic $\Longrightarrow \angle NA'B'=\angle NBA$. But since $AB \parallel NS$, we have $\angle BNS=\angle NBA=\angle NA'B'$ $\Longrightarrow \odot(NA'B')$ is tangent to NS at $N \Longrightarrow SN^2=SA'\cdot SB'$.

sunken rock

#3 Dec 23, 2011, 11:04 am

$\angle NA'B'=\frac{arcNES-arcSWA}{2}=\frac{arcAN}{2}=\frac{arcBS}{2}=\angle B'NS$, implying SN tangent to the circle $\odot A'B'N$ at N, wherefrom we get the desired relation.

Best regards,
sunken rock

Quick Reply

Spain

6th IBERO - ARGENTINA 1991.

carlosbr

#1 Mar 26, 2006, 11:05 am

6^{th} Iberoamerican Olympiad

Córdoba, ARGENTINA. [1991]

Edited by djimenez

Carlos Bravo

Attachments:
1991.pdf (25kb)

Luis González

#2 Nov 13, 2010, 10:44 pm

> *Quote:*
> **Problema 1.** A cada vértice de un cubo se asigna el valor de +1 o -1, y a cada cara el producto de los valores asignados a cada vértice. ¿Qué valores puede tomar la suma de los 14 números así obtenidos?.

Al cambiar el número de un vértice se modifican 4 valores, el vértice que se cambió y las tres caras que comparten dicho vértice, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Si $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, la nueva suma S' es $S \pm 8$. Si $\lambda_1 = \lambda_2 = \lambda_3$ y $\lambda_1 = -\lambda_4$, la nueva suma S' es $S \pm 4$. Si $\lambda_1 = \lambda_2$ y $\lambda_3 = \lambda_4 = -\lambda_1$, la nueva suma S' es S.

Como toda asignación puede obtenerse a partir de una dada mediante sucesivos cambios en un vértice y $-14 \leq S \leq 14$, entonces los valores posibles de S son $14, 10, 6, 2, -2, -6, -10$ y -14.

No se puede obtener -14 pues no se puede lograr que los 14 números sean negativos, ya que si los 8 vértices son negativos las caras son positivas. No se puede obtener 10 pues para ello debe haber 12 positivos y 2 negativos. Con un vértice negativo, S = 6 y con dos vértices negativos hay además dos caras (por lo menos) negativas, que son las que tienen a uno de los vértices pero no al otro.

Luis González

#3 Dec 12, 2010, 10:48 pm

> *Quote:*
> **Problema 2.** Dos rectas perpendiculares dividen un cuadrado en cuatro partes, tres de las cuales tienen cada una área igual a 1. Demostrar que el área del cuadrado es cuatro.

Sea $P_1P_2P_3P_4$ el cuadrado dado. M es un punto interior a éste por el cual pasa una recta que corta a $\overline{P_1P_2}$ y $\overline{P_3P_4}$ en P, T y otra recta perpendicular a la anterior que corta a $\overline{P_2P_3}$ y $\overline{P_4P_1}$ en S, U. O es el centro de $P_1P_2P_3P_4$. La paralela por O a PT corta a $\overline{P_1P_2}$ y $\overline{P_3P_4}$ en E, G y la paralela a SU por O corta a $\overline{P_2P_3}$ y $\overline{P_4P_1}$ en F, H. Sean $X \equiv EG \cap SU$ y $Y \equiv FH \cap TP$. Denotemos $S, S_A, S_B, S_C, S_D, S_R$ las áreas de $P_1P_2P_3P_4, UHYM, PEOY, SFOX, TGXM, OXMY$.

Asumimos sin perdida de generalidad que O yace en el interior de MPP_2S. Es claro que las rectas EG y FH dividen $P_1P_2P_3P_4$ en cuatro cuadriláteros con igual área. Por lo tanto, se tendrá

$$\frac{1}{4}S - S_B + S_A = \frac{1}{4}S - S_A - S_C - S_R = \frac{1}{4}S - S_D + S_C = 1.$$

De éstas, se desprende que $S_A + S_D = S_B + S_C \Longrightarrow$ los paralelogramos $PEGT$ y $SFHU$ tiene igual áreas. Como además, éstos tiene alturas iguales, entonces se sigue que $PE = FS \Longrightarrow OXMY$ es un cuadrado. Por tanto $S_A = S_C$ y $S_B = S_D$. Luego, usando las expresiones conseguidas anteriormente, se llega a que $S_A = S_B = S_C = S_D = 0 \Longrightarrow O \equiv M$ y la conclusión se sigue.

Luis González

#4 Nov 19, 2011, 8:33 am

> *Quote:*
>
> Problema 4. Encontrar un número N de cinco cifras diferentes y no nulas, que sea igual a la suma de todos los números de tres cifras distintas que se pueden formar con cinco cifras de N.

El número buscado $N = abcde$ habrá de ser igual a la suma $abc + abd + ... + cde$. Esta suma tendrá $3 \cdot 4 \cdot 5 = 60$ sumandos; como primera cifra de los sumandos aparecerá igual número de veces las cinco cifras a, b, c, d y e, es decir habrá 60/12 sumandos cuya primera cifra es a, 12 sumandos cuya primera cifra es b, etc. Igual ocurrirá con las segundas y terceras cifras de los sumandos, de modo que se tendrá

$$N = (100a + 10b + c) + (100a + 10b + d) + ... + (100c + 10d + e) =$$

$$= (1200a + 120a + 12a) + (1200b + 120b + 12b) + ... + (1200e + 120e + 12e) =$$

$$= (1200 + 120 + 12)(a + b + c + d + e) = 12 \cdot 111(a + b + c + d + e)$$

Llamando $S = a + b + c + d + e$ resultará $N = 12 \cdot 111 \cdot S$.

S será máximo cuando las cifras de N sean $9, 8, 7, 6, 5$. Así, $S = 35$. S será mínimo cuando las cifras de N sean $1, 2, 3, 4, 5$. Así, $S = 15$.

Luego, S debe estar comprendido entre 15 y 35; pero la primera igualdad indica que N debe ser múltiplo de 9, ya que tanto 12 como 111 son divisibles por 3. Por lo tanto, también debe ser múltiplo de 9 la suma S de las cifras de N. Ahora bien, los únicos múltiplos de 9 comprendidos entre 15 y 35 son 18 y 27. Para S = 18 resulta N = 23976, solución no válida pues tiene S = 27≠18. Para S = 27 resulta N = 35961, que es la solución buscada pues S = 27.

Luis González

#5 Nov 19, 2011, 10:59 am

> *Quote:*
>
> **Problema 5.** Sea $P(x, y) = 2x^2 - 6xy + 5y^2$. Diremos que un número entero a es un valor de P si existen números enteros b y c tales que $a = P(b, c)$.
>
> I) Determinar cuántos elementos entre 1 y 100 son valores de P.
>
> II) Probar que el producto de valores de P es un valor de P.

Los valores de P son los enteros que pueden expresarse como suma de los cuadrados de dos enteros. Observemos en primer lugar que $P(x, y)$ puede expresarse en la forma $P(x, y) = (x - y)^2 + (x - 2y)^2$, con lo cual los valores de P son sumas de dos cuadrados. Por otro lado si $A = B^2 + C^2$, resolviendo $x - y = B \cup x - 2y = C$, resulta $P(2B - C, B - C) = A$, lo cual prueba nuestra afirmación. La parte I) se resuelve pues contando los elementos entre 1 y 100 que son sumas de dos cuadrados.

II) Suponemos que A y A' son valores de P. Con lo establecido previamente, A y A' pueden expresarse como $A = B^2 + C^2$ y $A' = B'^2 + C'^2$. Multiplicando éstos se llega a la conclusión.

Luis González

#6 Nov 19, 2011, 9:00 pm

> *Quote:*
>
> **Problema 6.** Se dan tres puntos nos alineados M, N, H. Si M, N son los puntos medios de los lados AC, AB de un triángulo $\triangle ABC$ cuyo ortocentro es H, probar que tal triángulo es construible con regla y compas.

Sea D el pie de la A-altura (desconocido) y $P = AH \cap MN$, es decir la proyección ortogonal de H en MN. Consideremos

Sea D el pie de la A-altura (desconocido) y $P = AH \cap MN$, es decir la proyección ortogonal de H en MN. Consideremos el $\triangle ABC$, en el cual A y H yacen en lados distintos de la recta MN, la otra solución posible se tratará por analogía. Es bien conocida la relación métrica $AD \cdot HD = DB \cdot DC = 4PN \cdot PM$, por consiguiente

$$HP = PD - HD = \frac{AD}{2} - HD = \frac{AD}{2} - \frac{4PN \cdot PM}{AD} \Longrightarrow$$

$$\frac{AD^2}{2} - AD \cdot HP = 4PN \cdot PM \Longrightarrow AD = HP + \sqrt{HP^2 + 8PN \cdot PM}$$

Se podrá pues construir un segmento de igual medida que la A-altura, consiguientemente determinar el vértice A y completar facilmente el $\triangle ABC$ reflejando A en M y N.

Quick Reply

High School Olympiads

Geometry

Reply

Gleek-00

#1 Nov 19, 2011, 2:25 am

Let $\triangle ABC$ be an isoceles triangle wuth $AB = AC$, let the tangents to the circumcircle of $\triangle ABC$ in A and B meet in D. Show that $\angle DCB \leq 30$

Luis González

#2 Nov 19, 2011, 4:59 am

M is the midpoint of the side $AB. CM, CD$ are the C-median and C-symmedian of $\triangle ABC \Longrightarrow \angle DCB = \angle MCA. E$ is the foot of the C-altitude and P is the projection of M on $AC.$ The right triangles $\triangle AMP$ and $\triangle ACE$ are similar with similarity coefficient $\frac{AC}{AM} = \frac{AB}{AM} = 2 \Longrightarrow CE = 2MP.$ Thus, $CM \geq CE = 2MP \Longrightarrow \angle MCP \leq 30^\circ \Longrightarrow$ $\angle DCB \leq 30^\circ$, as desired.

Quick Reply

High School Olympiads

Geometry (2)

Reply

Gleek-00

#1 Nov 19, 2011, 2:40 am

Consider two circles (C_1) and (C_2) which intersect in A and B

A tangent to (C_1) in A cuts (C_2) in D, and a tangent to (C_2) in A cuts (C_2) in C.

A ray that begins with A and found inside $\angle CAD$ cuts (C_1) in M and (C_2) in N and the circumcircle of $\triangle ACD$ in P.

Show that $AM = NP$

(Sorry for the bad wording)

[geogebra]d3a2820df2297726ae1106896adf8cdd0cac2f3d[/geogebra]

Luis González

#2 Nov 19, 2011, 3:29 am

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=187699
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=246834
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=311308
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=313905

Quick Reply

High School Olympiads

Help me a geometry problem !!!

Reply

hoangkute96
#1 Nov 18, 2011, 2:50 pm

Give quadrilateral ABCD have angle ABC = angle ADC = 90 and AB+AD=AC . The intersection point of AC and BD is K. Prove that radii of incircles of triangle ADK and triangle ABK are equal.

Luis González
#2 Nov 18, 2011, 8:48 pm • 1

Let (I,r) and (O,R) be the incircle and circumcircle of $\triangle BCD$. M,N,L are the midpoints of BD,DC,CB. Then $AC=AB+AD \Longrightarrow ON+OL=R$, but by Carnot's theorem $OM+ON+OL=R+r \Longrightarrow OM=r$. If (I,r) touches BD at X and Y is the antipode of X WRT (I), then OM and IY are congruent and parallel $\Longrightarrow OYIM$ is a parallelogram $OY \parallel MI \Longrightarrow CK$ is the C-Nagel cevian of $\triangle BCD \Longrightarrow$ Inradii of $\triangle ABK$ and $\triangle ADK$ are equal.

yetti
#3 Nov 19, 2011, 1:03 am • 1

Let $S(\triangle XYZ), p(\triangle XYZ), r(\triangle XYZ)$ be area, (semi)perimeter and inradius of $\triangle XYZ$.
(O) on diameter AC is circumcircle of $\triangle BCD$. Parallels to CD, DB, BC through B, C, D form $\triangle B'C'D'$ anticomplementary to $\triangle BCD$.
$AB+AD=AC \Longrightarrow A$ is Feuerbach point of $\triangle B'C'D'$, the tangency point of its 9-point circle (O) and incircle $(J') \Longrightarrow$ incenter J' of $\triangle B'C'D'$, identical with Nagel point of $\triangle BCD$, is on radius OA of (O)
$\Longrightarrow p(\triangle CBK)=p(\triangle CDK) \Longrightarrow$

$$\frac{r(\triangle ABK)}{r(\triangle ADK)}=\frac{S(\triangle ABK)}{S(\triangle ADK)}\cdot\frac{p(\triangle ADK)}{p(\triangle ABK)}=\frac{BK}{DK}\cdot\frac{AK}{BK}\cdot\frac{DK}{AK}\cdot\frac{p(\triangle CBK)}{p(\triangle CDK)}=1$$

skytin
#4 Nov 20, 2011, 12:04 am

Solution :
Let I is insenter of BCD , W , J are midpoints of smalest arcs BC , CD of (ABCD)
Well known that WC = WI = WB and JC = JI = JP , I is intersection point of JB , WD
Reflect point D wrt JA and get point D' , reflect B wrt WA and get point B'
B' , D' are on CA
J is center of (D'ICD) , W is center of (CB'IB)
After Reim's theorem for (D'ICD) and (ABCD) we get that ID' || WA
Like the same IB' || JA
Let O is circumcenter of (ABC) , easy to see that O is midpoint of B'D' and CA
Reflect I wrt O and get point I'
Let lines thru D , B and || to JA , AW intersects at point X
DX , BX intersects with CA at points Y , Z
Easy to see that AY = D'A = DA and AZ = AB' and YX || I'D' , B'I' || XZ , so A is midpoint of I'X
AX || I'A || CI || to angle bissector of BCD || to angle bissector of DAB
Let U is incenter of ABD , N , F are incenters of BKA , KDA
NA || BX , FA || DX , U is on BN , DF , AX , so U is homotety center of NFA and BDX
NF || BD . done

Quick Reply

High School Olympiads

A hard problem

Reply

thanhluan_LTTM
#1 Nov 16, 2011, 6:32 pm

Give circle with center O , a point M move on circle , the tangent at M cut the tangent at A at P . A circle with center B passing P ,M and BP perpendicular AP . Prove that the circle with center B exposure with a fixed circle .

tobash_co
#2 Nov 16, 2011, 7:34 pm

What is 'exposure'?

yetti
#3 Nov 16, 2011, 8:27 pm • 1

Perpendicular bisector of PM through B cuts PO at its midpoint Q. $PO \perp AM$ cuts AM at its midpoint N.

$\triangle NAO \sim \triangle APO \Longrightarrow AN = AP \cdot \dfrac{AO}{PO}$.

$\triangle PBQ \sim \triangle APM \Longrightarrow PB = AP \cdot \dfrac{PQ}{AM} = AP \cdot \dfrac{PO}{4\ AN} = \dfrac{PO^2}{4\ AO} = \dfrac{AP^2 + AO^2}{4\ AO} \Longrightarrow \overline{PB} - \dfrac{\overline{AO}}{4} = \dfrac{AP^2}{4\overline{AO}}$

Let $\overline{AV} = \dfrac{\overline{AO}}{4}$, $\overline{VF} = \overline{AO}$ and $\overline{VD} = -\overline{AO}$. Let $v, d \perp AO$ at $V, D \Longrightarrow$

B is on parabola with focus F and vertex tangent v and directrix $d \Longrightarrow BF = \text{dist}(B, d)$.

Since circle (B) is tangent to AP at P, it is also tangent to circle (F) with radius equal to $\text{dist}(P, d) = AD = \dfrac{3}{4}OA$.

Luis González
#4 Nov 16, 2011, 11:35 pm

U, V are the antipodes of A, P WRT $(O), (B)$. PU, PO cut (B) again at Q, R. Since $2\angle RVP = \angle MPA = \angle MVP$, then VR bisects $\angle OVP$, i.e. R is the midpoint of $\overline{PO}$. Let $N \equiv VQ \cap UA$. Since $VP \parallel UA$, by Reim's theorem, it follows that O, Q, R, N and U, M, Q, O are concyclic. Then

$$\angle OQU + \angle RQP = \angle OMU + \angle RVP = \angle AUM + \angle UAM = 90^\circ$$

Thus, $\angle ONR$ is right, i.e. $RN \parallel PA \Longrightarrow N$ is the midpoint of $\overline{OA}$. Hence, (B) is always tangent to the circle with diameter $UN = \frac{3}{4}UA$.

Quick Reply

Spain

5th IBERO - ESPAÑA 1990.

Reply

carlosbr

#1 Mar 26, 2006, 10:51 am

5^{th} Iberoamerican Olympiad
Valladolid, ESPAÑA. [1990]

Edited by djimenez

Carlos Bravo

Attachments:
1990.pdf (31kb)

Luis González

#2 Dec 24, 2009, 11:25 am

> *Quote:*
> **Problema 2.** En un triángulo $\triangle ABC$ sean D, E, F los puntos de tangencia de su incirculo (I) con BC, CA, AB. La recta AD corta por segunda vez a (I) en P. Si M es el punto medio de EF, probar que los puntos P, D, I, M son concíclicos o están alineados.

Como es claro que AM es mediatriz de EF, por teorema del cateto en el triángulo rectángulo $\triangle AEI$ en E se tiene $AE^2 = AM \cdot AI$. Por otro lado de la potencia de A con respecto al incirculo (I) resulta $AE^2 = AP \cdot AD$. Entonces $AE^2 = AM \cdot AI = AP \cdot AD \Longrightarrow P, D, I, M$ son concíclicos o están alineados si evidentemente $AB = AC$.

Luis González

#3 Dec 24, 2009, 11:28 pm

> *Quote:*
> **Problema 4.** Se da un circunferencia $\mathcal{C}_1$ y AB un diametro fijo de ella. t es la tangente a $\mathcal{C}_1$ en B y M es un punto en $\mathcal{C}_1$ diferente de A, B. $\mathcal{C}_2$ es la circunferencia tangente a $\mathcal{C}_1$ en M y a t en un punto P.
> a) Construir el punto de tangencia P de $\mathcal{C}_2$ con t y hallar el lugar del centro de $\mathcal{C}_2$ al variar M.
> b) Probar que existe una circunferencia ortogonal al haz $\mathcal{C}_2$.

a) La inversión positiva con respecto a la circunferencia centrada en B con radio BA transforma a t en si misma y a $\mathcal{C}_1$ en la tangente t' a $\mathcal{C}_1$ por A. Entonces, el haz $\mathcal{C}_2$ se transforma en una cadena de círculos iguales tangentes al par de paralelas t y t'. El rayo BM corta pues a t' en el inverso M' de M. La circunferencia inversa $\mathcal{C}_2'$ de $\mathcal{C}_2$ que pasa por M es pues construible ya que es tangente a t' en M' con su centro en la mediatriz ℓ de AB. Hallado el punto de tangencia P' de $\mathcal{C}_2'$ con t, que es el inverso de P, entonces bastará trazar la circunferencia que pasa por M, M' y P', esta cortará a t nuevamente en el punto buscado P. Por otro lado, para observar el lugar del centro O_2 de $\mathcal{C}_2$ basta considerar la simétrica ℓ' de la mediatriz ℓ de AB con respecto a t. Entonces si O_2P corta a ℓ' en Q, se tiene $O_2Q = O_2O_1 = r_1 + r_2 \Longrightarrow O_2$ equidista de ℓ' y O_1. Su lugar es pues la parábola con foco O_1 y directriz ℓ'.

b) Es sabido que la inversión es una transformación conforme, es decir que preserva ángulos entre curvas. Note que el lugar geométrico de los centros O_2' del haz $\mathcal{C}_2'$ inverso del haz $\mathcal{C}_2$ es al mediatriz ℓ de AB. El ángulo entre ℓ y $\mathcal{C}_2'$ es recto, por tanto el ángulo entre $\mathcal{C}_2$ y la circunferencia inversa $\mathcal{L}$ de ℓ será también recto $\Longrightarrow \mathcal{L} \perp \mathcal{C}_2$.

Luis González

#4 Nov 16, 2011, 7:04 am

> Quote:
>
> **Problema 5.** Sean A y B vértices opuestos de un tablero cuadriculado de n por n casillas $(n \geq 1)$, a cada una de las cuales se añade su diagonal de dirección AB, formándose así $2n^2$ triángulos iguales. Se mueven una ficha recorriendo un camino que va desde A hasta B formado por segmentos del tablero, y se coloca, cada vez que se recorre, una semilla en cada uno de los triángulos que admite ese segmento como lado. El camino se recorre de tal forma que no se pasa por ningún segmento más de una vez, y se observa, después de recorrido, que hay exactamente dos semillas en cada uno de los $2n^2$ triángulos del tablero. Hallar tal valor de n.

Primero observamos que como cada triángulo debe tener dos semillas necesariamente se debe pasar por sólo dos lados de cada triángulo. Además, cualquier vértice distinto de A y de B debe pertenecer a una cantidad par de aristas del camino, puesto que, por no ser puntos terminales, cada vez que se llega se tiene que salir. Por el contrario, los vértices A y B pertenecen a un número impar de aristas del camino.

Para el caso en que $n = 1$, sean P y O los otros vértices del cuadrado. Para que haya 2 semillas en $\triangle AQB$ y 2 semillas en $\triangle APB$, por lo menos un cateto de $\triangle AQB$ y un cateto de $\triangle APB$ deben pertenecer al camino. Entonces deben estar los dos catetos de los dos triángulos porque P, Q pertenecen a un número par de aristas del camino; y no puede estar la diagonal AB porque ya hay 2 semillas en cada triángulo. En consecuencia, el vértice A pertenece a dos aristas del camino. Contradicción.

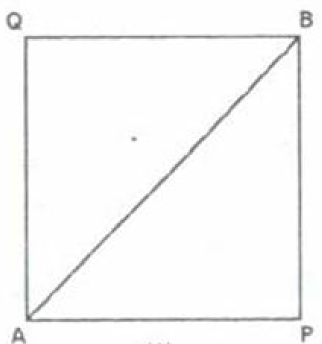
Si $n = 2$, una solución es la secuencia de aristas de 1 a 10 como se muestra en la figura 2. Para el caso $n = 3$, llamamos P y Q a los otros dos vértices extremos del tablero, numeramos las aristas como se indica en la figura 3.

Vemos lo siguiente: Como P no es un punto terminal debemos pasar por las aristas 1 y 2. Como en cada triángulo no se pueden utilizar más de dos aristas, no se usa la arista 3 y por lo tanto se deben usar la 4 y la 5. Por paridad, no se pueden utilizar ni la 10, ni la 6, lo cual obliga a usar la 11 y la 7. Por paridad, las aristas 12 y 9, o están ambas en el camino o no está ninguna. Para que haya 2 semillas en el triángulo del que son lados, deben estar ambas en el camino, lo cual obliga a que la 14 y la 8 no se utilicen. Ahora, Como 8 no se usa, estamos obligados a usar 18 y 17.

Razonando analogamente, se obliga a usar las aristas 20 y 15 y a no pasar por la 19, con lo cual el vértice B no puede ser un punto terminal, pues hay exactamente 2 aristas del camino que inciden en él. Con esto, el caso $n = 3$ no es posible.

Para $n \geq 4$, utilizando la misma notación que en el caso 3, pero llamando H a lo que era el vértice B. Como el 19 no se usó se debe utilizar el 24 (ver figura 4). Así, el vértice H no es terminal y sin embargo hay un número impar de aristas del camino que inciden en él. Con esto se ve que el caso $n \geq 4$ tampoco es posible. Así se concluye que el único caso posible es $n = 2$.

Attachments:



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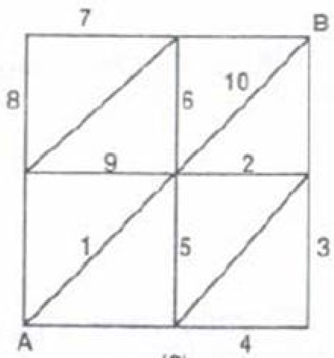
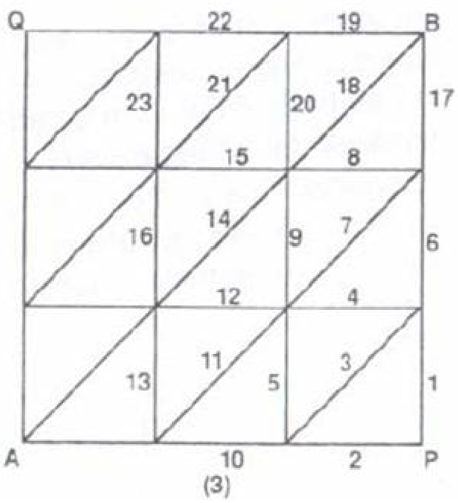
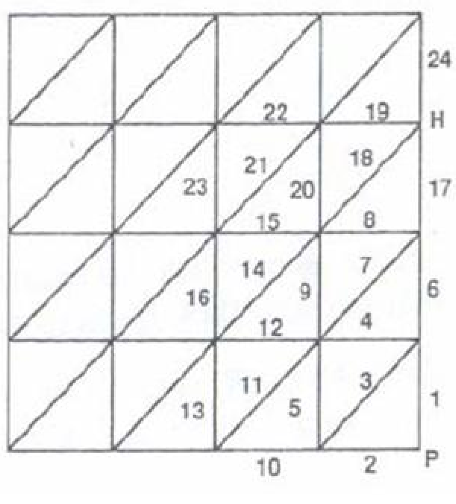


(2)



(3)



Luis González

#5 Nov 16, 2011, 9:21 am

> Quote:
>
> **Problema 6.** Sea $f(x)$ un polinomio de grado 3 con coeficientes racionales. Probar que si el gráfico de f es tangente al eje x, entonces $f(x)$ tiene sus 3 raíces racionales.

f es tangente al eje $x \Longleftrightarrow f$ tiene una raíz doble real α. Si β denota la otra raíz, entonces $f(x)=a(x-\beta)(x-\alpha)^2$. siendo a el coeficiente de x^3, entonces $a \in \mathbb{Q}$, exceptuando 0.

Así, $(x-\beta)(x-\alpha)^2=x^3-(2\alpha+\beta)x^2+(\alpha^2+2\alpha\beta)x-\alpha^2\beta$ tiene coeficientes racionales, i.e. $2\alpha+\beta, \alpha^2+2\alpha\beta$ y $\alpha^2\beta$ son racionales. Por otro lado, sabemos que en el polinomio $P(x)=x^3+bx^2+cx+d$ con b,c,d racionales, la sustitución $x \longrightarrow x-\frac{b}{3}$ es una traslación que elimina el término de 2do grado. Como $\frac{b}{3}$ es racional, las raíces racionales corresponden a raíces racionales, y las raíces irracionales corresponden a raíces irracionales. Por ende, no se pierde generalidad si consideramos $2\alpha+\beta=0$, i.e. $\beta=-2\alpha$. Así, $\alpha^2+2\alpha\beta \in \mathbb{Q} \Longrightarrow \alpha^2 \in \mathbb{Q}$ y $\alpha^2\beta \in \mathbb{Q} \Longrightarrow \alpha^3 \in \mathbb{Q}$. De ésto deducimos pues que $\alpha, \beta \in \mathbb{Q}$.

Quick Reply

Spain

4th IBERO - CUBA 1989.

Reply

carlosbr

#1 Mar 26, 2006, 10:46 am

4^{th} Iberoamerican Olympiad
La Habana , CUBA. [1989]

Edited by djimenez

Carlos Bravo

Attachments:
1989.pdf (24kb)

Luis González

#2 Dec 7, 2009, 5:36 am

> *Quote:*
> **Problema 1.** Hallar todas las ternas reales (x, y, z) que satisfacen
> $$-x-y+z=x^2-y^2+z^2=x^3-y^3-z^3=1$$

$x=z-y-1 \cup x^2-y^2+z^2=1 \Longrightarrow (z-1)(z-y)=0 \Longrightarrow z=1$ ó $y=z$.

Si $z=1$, entonces de $x=z-y-1$ se tiene que $x+y=0$. Al combinar esta última con la ecuación $x^3-y^3-z^3=1$ resulta $x=1, y=-1$. Si $y=z$, entonces $x=-1$ y $y=z=-1$. Por tanto, las soluciones son $(x, y, z)=(1,-1,1)$ ó $(-1,-1,-1)$.

Luis González

#3 Nov 7, 2011, 5:29 am

> *Quote:*
> **Problema 2.** x, y, z son números reales tales que $0<x<y<z<\frac{\pi}{2}$. Pruebe que
> $$2\sin z\cos x+2\sin x\cos y+2\sin y\cos z \geq \sin 2x+\sin 2y+\sin 2z$$

Usando las fórmulas de adición de ángulos, la desigualdad dada equivale a

$\sin(x+y)+\sin(y+z)+\sin(z+x) \geq \sin 2x+\sin 2y+\sin 2z.$

Como $(2x, 2y, 2z) \succ (x+y, x+z, y+z)$ y $\sin\theta$ es concava para $\theta \in (0, \pi)$, entonces por Desigualdad de Karamata, se deduce la desigualdad deseada.

Luis González

#4 Nov 7, 2011, 8:56 am

> *Quote:*

Problema 3. a, b, c son lados de un triángulo. Probar que $\frac{a-b}{a+b} + \frac{b-c}{b+c} + \frac{c-a}{c+a} < \frac{1}{20}$.

Denotemos $f(a, b, c)$ el lado izquierdo de la desigualdad deseada y sean A, B, C las permutaciones de a, b, c, con $A \leq B \leq C$. Si $(A, B, C) = (b, a, c), (a, c, b)$ ó (c, b, a), entonces $f(a, b, c) = \delta$, donde

$$\delta = \frac{B-A}{B+A} + \frac{C-B}{C+B} - \frac{C-A}{C+A}$$

Si $(A, B, C) = (a, b, c), (b, c, a)$ ó (c, a, b), entonces $f(a, b, c) = -\delta$. Ahora, suponemos que $B = A + h$, $C = B + k = A + h + k$, donde $h, k \geq 0$. Siendo A, B, C los lados de un triángulo, entonces tenemos $A + B > C$ ó $A > K$. Por consiguiente

$$\delta = \frac{h}{2A+h} + \frac{k}{2A+2h+k} - \frac{h+k}{2A+h+k}$$

$$\delta = \frac{h^2k + hk^2}{(2A+h)(2A+2h+k)(2A+h+k)}$$

Así, debemos probar que $20h^2k + 20hk^2 < (2k+h)(2k+2h+k)(2k+h+k)$, ya que $k < A$. Entonces, será suficiente probar que $18k^3 + 7hk^2 - 7h^2k + 2h^3 \geq 0$. Pero

$$7k^2 - 7hk + 2h^2 = 7\left(k - \frac{h}{2}\right)^2 + \frac{h^2}{4} \geq 0.$$

Así, establecemos que $\frac{1}{20} > \delta \geq 0$, que implica que $f(a, b, c) < \frac{1}{20}$.

Luis González

#5 Nov 8, 2011, 10:59 am

> *Quote:*
> **Problema 4.** El incírculo (I) de $\triangle ABC$ es tangente a AC, BC en M, N. Las bisectrices de los angulos A y B intersectan la recta MN en P y Q. Probar que $MP \cdot IA = BC \cdot IQ$.

Si D, E son los puntos medios de BC, BA, es ya sabido que DE, AI, MN y la perpendicular desde B a AI concurren en P. Por la misma razón, resulta pues que el ángulo $\angle AQI$ es recto. Como $DE \parallel CA$, es facil ver que $\triangle DNP \sim \triangle CNM$ y de ésta obtenemos la proporción:

$$\frac{NP}{MN} = \frac{ND}{CN} \Longrightarrow \frac{MP}{BC} = \frac{MN}{2CN}$$

Si $T \equiv MN \cap CI$ es claro que CT es mediatriz de MN. Entonces $\frac{MP}{BC} = \frac{TN}{CN}$

Como $\triangle CNT \sim \triangle AIQ$, ya que $\angle QAI = \frac{\pi}{2} - \frac{\pi - \gamma}{2} = \frac{\gamma}{2} = \angle TCN$

$$\Longrightarrow \frac{IQ}{IA} = \frac{TN}{CN} = \frac{MP}{BC}.$$

Luis González

#6 Nov 15, 2011, 11:03 pm

> *Quote:*
> **Problema 5.** Sea f una función definida sobre el conjunto $\{1, 2, 3, \ldots\}$, tal que $f(1) = 1$, $f(2n+1) = f(2n) + 1$ y $f(2n) = 3f(n)$. Hallar tomos los posibles valores de f.

Escribimos n en base 2 y después la leemos en base 3. La función $g(n)$ definida de esta forma, ciertamente cumple que $g(1) = 1$. Luego, $2n + 1$ tiene la misma expansión binaria que $2n$ a excepción de un 1 final, por tanto $g(2n+1) = g(2n) + 1$. Analogamente, $2n$ tiene la misma expansión binaria que n con la adición de un cero final. Así, $g(2n) = 3g(n)$. Por tanto, g es igual que f. Así, el conjunto de todo los m, tales que $m = f(n)$ para cierto n son los que se pueden escribir en base 3 sin un dígito 2.

Luis González

#7 Nov 16, 2011, 2:26 am

> Quote:
>
> **Problema 6.** Probar que hay infinidad de pares naturales que satisfacen $2x^2 - 3x = 3y^2$.

Necesitamos $3 \mid x$, así $x = 3a$ y la ecuación toma la forma $6a^2 - 3a = y^2$. Necesitamos $3 \mid y$, así $y = 3b$ y la ecuación toma la forma $2a^2 - a = 3b^2$.

Re-escribimos $16a^2 - 8a = 24b^2$, ó $(4a-1)^2 - 6(2b)^2 = 1$. Ésta es una ecuación de Pell con solución mínima $(A_0, B_0) = (5, 2)$, siendo el resto de las soluciones $(A_{n+1}, B_{n+1}) = (5A_n + 12B_n, 2A_n + 5B_n)$. En efecto, B_0 siendo par, todo B_n será pues par. Sin embargo, como $A_0 \equiv 1 \pmod 4$, se tendrá que $A_n \equiv 1 \pmod 4$. Por ende, la condición $4a - 1 = A_n$ no puede ser cumplida. La ecuación no tiene pues soluciones enteras positivas. Para tener infinitas soluciones enteras positivas debemos tomar $2x^2 + 3x = 3y^2$. Pero si permitimos cualquier solución entera, la ecuación original tendrá también infinitas soluciones.

Quick Reply

High School Olympiads

Equal areas

Locked

VHCR

#1 Nov 15, 2011, 3:39 am

Let $ABCD$ be a quadrilateral, and let H_1, H_2, H_3, H_4 be the orthocenters of the triangles DAB, ABC, BCD, CDA, respectively. Prove that the area of the quadrilateral $ABCD$ is equal to the area of the quadrilateral $H_1H_2H_3H_4$.

Luis González

#2 Nov 15, 2011, 6:00 am

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=5840
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=105181
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=222600
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=402956

High School Olympiads

Brocard axis

Source: Own

livetolove212
#1 Aug 23, 2009, 2:35 pm

Given triangle ABC with medians $AA', BB', CC'.B'C' \cap (O) = \{A_1, A_2\}$, similar for B_1, B_2, C_1, C_2. Prove that the radical center of $(A'A_1A_2), (B'B_1B_2)$ and $(C'C_1C_2)$ lies on **Brocard axis** of triangle ABC.

livetolove212
#2 Aug 23, 2009, 2:47 pm

And AA'', BB'', CC'' are the symmedians of triangle ABC, $B''C'' \cap (O) = \{A'_1, A'_2\}$, similar for B'_1, B'_2, C'_1, C'_2 then the radical center of $(A''A'_1A'_2), (B''B'_1B'_2), (C''C'_1C'_2)$ lies on **Brocard axis** of triangle ABC too

Luis González
#3 Nov 14, 2011, 3:59 am

Two nice problems forgotten for too long!.

The general barycentric coordinates of the radical center R of this triad of circles were given in the topic Radical center and line IG (see post #6). If P coincides with the centroid $X_2(1:1:1)$, then R is the Brocard midpoint (midpoint of the segment connecting the 2 Brocard points) $X_{39}(a^2(b^2+c^2) : b^2(c^2+a^2) : c^2(a^2+b^2))$, which obviously lies on the Brocard axis X_3X_6.

If P coincides with the symmedian point $X_6(a^2:b^2:c^2)$, then R is given by

$$R\ (a^2(b^2+c^2+2a^2) : b^2(c^2+a^2+2b^2) : c^2(a^2+b^2+2c^2))$$

This isn't in the current edition of ETC, but it certainly lies on the Brocard axis X_3X_6

$$\begin{bmatrix} a^2 & b^2 & c^2 \\ a^2S_A & b^2S_B & c^2S_C \\ a^2(b^2+c^2+2a^2) & b^2(c^2+a^2+2b^2) & c^2(a^2+b^2+2c^2) \end{bmatrix} = 0$$

High School Olympiads

find the angle

Reply

horizon

#1 Nov 13, 2011, 4:05 pm

given a triangle,AB is not equal to AC,the incircle of triangle ABCtangents to line AB,AC at P,Q respectively,M,N is on line BC satisfy that $MC = NB = \frac{AB + AC + BC}{2}$
(B is between M and C,and C is between B and N)if P,Q,M,N are cyclic,find all posible $\angle A$

Luis González

#2 Nov 13, 2011, 10:38 pm

Clearly, M and N are the tangency points of the C- and B- excircles with $BC.$ Thus, perpendicular bisectors of BC and MN coincide $\Longrightarrow$ circumcenter L of $PQNM$ is always the midpoint of the arc BC of $\odot(ABC),$ due to $AB \neq AC.$ Since $AP = BM, LP = LM$ and $\angle LBC = \angle LAP$, then $\triangle ALP$ and $\triangle BLM$ are pseudo-congruent $\Longrightarrow$ $\angle ALP = \angle BLM,$ i.e. $\angle ALB = \angle PLM = 2\angle PNM.$ This implies that NP is parallel to the internal angle bisector of $\angle ACB \Longrightarrow NP$ passes through the tangency point of the B-excircle with $AC.$ This latter property clearly characterizes the right triangles ABC with hypotenuse BC, i.e. $\angle A = 90^\circ.$

yetti

#3 Nov 14, 2011, 2:50 am

With the same notation: $L \in \odot(ABC)$, the intersection of perpendicular bisectors of PQ, MN, is circumcenter of $PQNM$.
Reflect A in PQ into $Z \Longrightarrow LP = LM, ZP = AP = BM$ and $\frac{\pi}{2} < \angle LZP = \pi - \frac{\angle A}{2} = \angle LBM \Longrightarrow$ $\triangle LZP \cong \triangle LBM$ by Ssa $\Longrightarrow$
$LZ = LB \Longrightarrow Z$ is incenter of $\triangle ABC \Longrightarrow$ rhombus $APZQ$ has right angles at $P, Q \Longrightarrow$ it is a square $\Longrightarrow \angle A$ is right.

Quick Reply

High School Olympiads

Conjecture with Thebault's circles and incircles

Reply

Source: own

jayme

#1 Nov 11, 2011, 11:43 am

Dear Mathlinkers,
1. ABC a triangle
2. P a point in the segment BC
3. (O) the circumcircle of ABC
4. Q the second point of intersection of AP with (O)
5. (Ob), (Oc) the Thebault's circles of ABC wrt P
5. (Ib), (Ic) the incircles of PQB, PQC.

Conjecture : (Ob) and (Oc) are equal if, and only if, (Ib) and (Ic) are equal.

Can we imagine a synthetic proof.
Sincerely
Jean-Louis

Luis González

#2 Nov 11, 2011, 12:23 pm

The conjecture is indeed true. (Ob),(Oc) are congruent iff AP is the A-Nagel cevian of ABC iff (Ib), (Ic) are congruent. See the following threads

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=201858
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=353717
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=244007

jayme

#3 Nov 11, 2011, 1:55 pm

Dear Luis and Mathlinkers,
thank you for your last reference
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=244007
where I can found the nice proof of Leonhard Euler.
I come up to this conjecture after reading again my articles on Protassov and Mixtilinear incircles I and III... where I have the differents results of Euler but not the idea to consider the Pascal theorem...
Thank again for your help.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

EF parallel to AB.

Reply

thanhnam2902
#1 Nov 10, 2011, 1:46 pm

My own!

Let (O) is the circum circle of ABC triangle. Let D is the midpoint of BC arc not contain point A, let S is the intersect of the tangents of (O) at B and C. The perpendicular bisector of AB meet AD at point E, The perpendicular bisector of AD meet AS at point F. Prove that EF parallel to AB.

Luis González
#2 Nov 11, 2011, 1:50 am

M is the midpoint of BC. AM, AS are the A-median and A-symmedian of $\triangle ABC \Longrightarrow AD$ bisects $\angle FAM$. Since $\triangle FAD$ is F-isosceles, it follows that $FD \parallel AM$. If FD cuts AB at P, we have then

$$\frac{DF}{FP}=\frac{AF}{FP}=\frac{\sin \widehat{APF}}{\sin \widehat{BAF}}=\frac{\sin \widehat{BAM}}{\sin \widehat{CAM}}=\frac{AC}{AB}$$

But $\triangle EBD \sim \triangle ABC$ yields $\frac{DE}{EB}=\frac{DE}{EA}=\frac{AC}{AB}=\frac{DF}{FP} \Longrightarrow EF \parallel AB.$

skytin
#3 Nov 17, 2011, 2:58 pm

Solution :
AD is simmedian of ABC
OS intersect BC at Y
Let N is midpoint of BD O is circumcenter of (ABC) , X is midpoint of YD
L is intersection point of OF , DA
(ONX) intersect NL at points N , N' , NL || AB , N' is on OE
Easy to see that XN' || DE , XL || DF , so O is homotety center of N'XL , EDF , so N'L || EF || AB . done

Quick Reply

High School Olympiads

Circle inscribed in a quadrangle

Locked

Stephen

#1 Nov 10, 2011, 5:56 pm

Let w a circle inscribed in $ABCD$. Let P the intersection point of AC and BC.

Prove that the incenters of triangles ABP, BCP, CPD, DPA are on a circle.

Luis González

#2 Nov 10, 2011, 9:50 pm

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=21758
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=64627
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=199861
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=228291

High School Olympiads

Hard Innercicle property

Reply

my_name_is_math

#1 Nov 10, 2011, 12:56 am

Let be ABC a triangle with circumcircle (O). Consider M the symmetric of I with respect to BC, N the midpoint of the arc BC of (O) which doesn't contain A and P the symmetric of A with respect to OI. Prove that M, N, P are collinear.

Luis González

#2 Nov 10, 2011, 6:15 am

Let NP cut OI at Q. Since QIO bisects $\angle AQN$ and $OA = ON$, we deduce that O is the midpoint of the arc AN of $\odot(AQN) \Longrightarrow \angle AOI = \angle ANP$. Power of I WRT (O) is $IA \cdot IN = 2Rr = IM \cdot OA$, i.e. $\frac{OA}{IA} = \frac{IN}{IM}$. Since $\angle MIN = \angle OIA$, then it follows that $\triangle OAI \sim \triangle NIM$ by SAS $\Longrightarrow \angle INM = \angle AOI = \angle ANP \Longrightarrow M \in PN.$

Quick Reply

High School Olympiads

Two similar triangles

Reply

Source: own

livetolove212

#1 Nov 9, 2011, 3:05 pm • 2

Given two triangles ABC and XYZ with their circumcircle (O). YZ intersects AB, AC at P, N, respectively. Let J be the projection of O on YZ. Let K, L be the midpoints of $BN, CP.A'$ be the reflection of J wrt KL. Similarly we define B', C'. Prove that two triangles XYZ and $A'B'C'$ are similar.

Luis González

#2 Nov 9, 2011, 11:50 pm • 3

XY cuts BC, CA at Q, M and XZ cuts AB, BC at $L, R.$ Tangents of $\odot(APN), \odot(BRL), \odot(CMQ)$ at A, B, C cut (O) again at $X_0, Y_0, X_0.$ $\angle X_0AZ_0 = \angle APN + \angle CQM - \angle ABC = \angle XYZ \Longrightarrow XZ = X_0Z_0.$ Similarly, $YX = Y_0X_0$ and $ZY = Z_0Y_0 \Longrightarrow \triangle XYZ \cong \triangle X_0Y_0Z_0.$ Let D, E, F be the midpoints of $BC, CA, AB.$ According to this topic (see post #2), A', B', C' lie on 9-point circle $\odot(DEF)$ and $AX_0 \parallel DA', BY_0 \parallel EB'$ and $CZ_0 \parallel FC'.$ Therefore, $\triangle ABC \cup \triangle X_0Y_0Z_0$ is similar to $\triangle DEF \cup \triangle A'B'C' \Longrightarrow \triangle XYZ$ and $\triangle A'B'C'$ are similar with similarity coefficient 2.

Quick Reply

High School Olympiads

Concurrent in bicentric hexagon

Reply

buratinogigle

#1 Nov 8, 2011, 11:09 pm

Let $ABCDEF$ be a hexagon with circumcircle (O) and incircle (I). P_1,P_2 are two points on OI. AP_1 cuts (O) again at A_1. A_1P_2 cuts (O) again at A_2. Similarly we have $B_1,B_2,C_1,C_2,D_1,D_2,E_1,E_2,F_1,F_2$. Prove that A_2D_2,B_2E_2,C_2F_2,OI are concurrent.

Note that, there are some nice other problems in the post Nice : D.

Luis González

#2 Nov 9, 2011, 12:34 pm • 1

By Poncelet porism, AD,BE,CF,OI concur at the limiting point P of (I) and (O). By Pascal theorem for AB_1BED_1D, the intersections $S \equiv AB_1 \cap ED_1, P_1 \equiv BB_1 \cap DD_1, P \equiv AD \cap BE$ are collinear. By Pascal theorem for $AE_2E_1EA_2A_1$, the intersections $Q \equiv AE_2 \cap EA_2, P_2 \equiv E_1E_2 \cap A_1A_2, P_1 \equiv AA_1 \cap EE_1$ are collinear. By Pascal theorem for $AD_2D_1EB_2B_1$, the intersections $T \equiv AD_2 \cap EB_2, P_2 \equiv D_1D_2 \cap B_1B_2, S \equiv AB_1 \cap ED_1$ are collinear $\Longrightarrow Q,T \in OI$. Now, by Pascal theorem for $AE_2B_2EA_2D_2$, the intersections $Q \equiv AE_2 \cap EA_2, U \equiv A_2D_2 \cap B_2E_2$, $T \equiv AD_2 \cap EB_2$ are collinear $\Longrightarrow A_2D_2, B_2E_2$ and OI concur at U. By similar reasoning, $U \in C_2F_2$.

Quick Reply

High School Olympiads

problem on symmedians

Reply

palashahuja

#1 Nov 8, 2011, 8:54 pm

Suppose a circle passes through the feet of the symmedians of a non-isosceles triangle ABC , and is tangent to one of the sides. Show that $a^2+b^2, b^2+c^2, c^2+a^2$ are in geometric progression when taken in some order

Luis González

#2 Nov 8, 2011, 10:28 pm

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=164755

P.S. In general, the cevian circle of a point P with respect to ABC is tangent to BC if and only if, the A-cevians of P and its cyclocevian conjugate coincide, i.e. the A-cevians of P and the isotomic conjugate of the anticomplement of the isogonal conjugate of the isotomcomplement of P coincide.

Quick Reply

High School Olympiads

Mongolia TST 2011 Test 2 #2

Reply

Source: Mongolia TST 2011 Test 2 #2

Bacteria

#1 Nov 8, 2011, 7:54 am

Let ABC be a scalene triangle. The inscribed circle of ABC touches the sides BC, CA, and AB at the points A_1, B_1, C_1 respectively. Let I be the incenter, O be the circumcenter, and lines OI and BC meet at point D. The perpendicular line from A_1 to B_1C_1 intersects AD at point E. Prove that B_1C_1 passes through the midpoint of EA_1.

Luis González

#2 Nov 8, 2011, 8:12 am

This problem and its generalizations were discussed before

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=291602
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=346956
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=420917
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=425224

Quick Reply

High School Olympiads

Mongolia TST 2011 Test 1 #3

Reply

Source: Mongolia TST 2011 Test 1 #3

Bacteria

#1 Nov 8, 2011, 4:28 am • 1

We are given an acute triangle ABC. Let (w, I) be the inscribed circle of ABC, (Ω, O) be the circumscribed circle of ABC, and A_c be the midpoint of altitude AH. w touches BC at point D. A_0D and w intersect at point P, and the perpendicular from I to A_0D intersects BC at the point M. MR and MS lines touch Ω at R and S respectively [note: I am not entirely sure of what is meant by this, but I am pretty sure it means draw the tangents to Ω from M]. Prove that the points R, P, D, S are concyclic.
(proposed by E. Enkzaya, inspired by Vietnamese olympiad problem)

Luis González

#2 Nov 8, 2011, 5:23 am • 2

According to IMO ShortList 2002, geometry problem 7, the incircle (I) and $\odot(BPC)$ are internally tangent at P. MP is then the common tangent (radical axis) of (I) and $\odot(BPC)$. Therefore, M is the radical center of (I), (O) and $\odot(BPC) \Longrightarrow$ Circle (M, MR) is orthogonal to (O) and (I), i.e. R, P, D, S lie on a circle with center M.

Quick Reply

Spain

3rd IBERO - PERU 1988.

Reply

carlosbr

#1 Mar 26, 2006, 10:26 am

3^{rd} Iberoamerican Olympiad

Lima, PERU. [1988]

Edited by djimenez

Carlos Bravo

Attachments:

1988.pdf (25kb)

Luis González

#2 May 4, 2010, 11:25 am • 1

> Quote:
>
> **Problema 1.** Los ángulos de un triángulo están en progresión aritmetica y sus alturas también. Demostrar que tal triángulo es equilátero.

Sean h_A, h_B, h_C las alturas del $\triangle ABC$ referentes a $A, B, C.$ Asumimos que tanto los ángulos A, B, C como las alturas h_A, h_B, h_C forman progresión aritmética con términos intermedios $A, h_A,$ respectivamente. Así:

$$2A = B + C \Longrightarrow \ A = 60^\circ \ \Longrightarrow a^2 = b^2 + c^2 - bc \ \ (1)$$

$$2h_A = h_B + h_C \Longrightarrow \frac{2}{a} = \frac{1}{b} + \frac{1}{c} \ \Longrightarrow a = \frac{2bc}{b+c} \ \ (2)$$

$$(1) \cup (2) \ \ \frac{4b^2c^2}{(b+c)^2} - b^2 - c^2 + bc = 0 \Longrightarrow \frac{(b-c)^2(b^2+c^2+3bc)}{(b+c)^2} = 0$$

Como $b^2 + c^2 + 3bc > 0,$ entonces $b = c \Longrightarrow \triangle ABC$ es equilátero.

Luis González

#3 Dec 14, 2010, 10:49 am

> Quote:
>
> **Problema 2.** Sean a, b, c, d, p, q números naturales no nulos positivos tales que $ad - bc = 1$ y $\frac{a}{b} > \frac{p}{q} > \frac{c}{d}.$ Demostrar que $q \geq b + d,$ y si $q = b + d,$ entonces $p = a + c.$

De $\frac{p}{q} > \frac{c}{d}$ se deduce que $pd > cq,$ entonces $pd \geq cq + 1 \Longrightarrow \frac{p}{q} \geq \frac{c}{d} + \frac{1}{qd}.$ Analogamente, de $\frac{a}{b} > \frac{p}{q}$ tenemos $\frac{a}{b} > \frac{p}{q} + \frac{1}{bq}.$ También, $\frac{a}{b} - \frac{c}{d} \geq \frac{b+d}{qbd}.$ Pero $\frac{a}{b} - \frac{c}{d} = \frac{1}{bd} \Longrightarrow q \geq b + d.$

Asumiendo que $q = b + d,$ resulta $ad - bc = 1 \leq d.$ Así, $ad + cd - d \leq bc + cd$ y de lo anterior tenemos $d(a + c - 1) \leq c(b + d).$ Entonces $p \geq a + c.$ Analogamente, $ad - bc \leq b.$ Ahora, $bc + b + ab \geq ad + ab$

$$\Longrightarrow \frac{a+c+1}{b+d} \leq \frac{a}{b} \ \wedge \ p \leq a + c \Longrightarrow p = a + c.$$

Luis González

#4 Dec 16, 2010, 11:41 pm

> Quote:
>
> **Problema 3.** Probar que entre todos los triángulos tal que la distancias de sus vértices a un punto dado P son $3, 5$ y $7,$ el que tiene mayor perímetro es aquel que tiene incentro P.

Sea $\mathcal{E}$ la elipse con focos B, C y se ubica un punto A' tal que la bisectriz de $\angle BA'C$ pasa por P. Por propiedad reflexiva de la tangente a una cónica, se sigue que AP' es la normal a la elipse $\mathcal{E}$ con focos B, C que pasa por A', i.e. el círculo ω con centro P y radio PA' es tangente internamente a $\mathcal{E}$ por A'. Así, todo punto de ω excepto A' yace en el interior de $\mathcal{E}$ y por ende se cumplirá $A'B + A'C \geq AB + AC$. Razonamiento analogo permite concluir que el $\triangle ABC$ con perímetro máximo es aquel que tiene a P por incentro.

Luis González

#5 Oct 18, 2011, 3:05 am

> Quote:
>
> **Problema 4.** Sea $\triangle ABC$ un triángulo cuyos lados son a, b, c. Se divide cada lado del triángulo en n segmentos iguales. Sea S la suma de los cuadrados de las distancias de cada vértice a cada uno de los puntos de división del lado opuesto distintos de los vértices. Demuestre que $\frac{S}{a^2+b^2+c^2}$ es un número racional.

Supongamos el lado BC dividido en n segmentos iguales mediante $n-1$ puntos $P_1, P_2, P_3, ..., P_k$. Así por el teorema de Stewart se obtiene la expresión general:

$$(AP_k)^2 = \left(\frac{k}{n}\right) b^2 + \left(1 - \frac{k}{n}\right) c^2 - \frac{k}{n}\left(1 - \frac{k}{n}\right) a^2 \ , \ \ k = 1, 2, 3, 4, 5, ..., n$$

$$(AP_k)^2 = c^2 + \left(\frac{a \cdot k}{n}\right)^2 - \frac{k}{n}(a^2 + c^2 - b^2)$$

La sumatoria S_A de los cuadrado de las las $n-1$ cevianas AP_k se plantea entonces como:

$$S_A = nc^2 + \frac{a^2}{n^2} \cdot \sum_{k=1}^{n} k^2 - \left(\frac{a^2 + c^2 - b^2}{n}\right) \cdot \sum_{k=1}^{n} k - b^2$$

Note que para $k = n,$ la ceviana AP_n se vuelve identica al lado AC. Así en la anterior sumatoria se ha sustraido b^2 puesto que no debe figurar en ésta. Desarrollando las sumatorias resultantes se obtiene la expresión:

$$S_A = nc^2 + \frac{a^2}{n^2} \cdot \left[\frac{n(n+1)(2n+1)}{6}\right] - \left(\frac{a^2 + c^2 - b^2}{n}\right) \cdot \left[\frac{n(n+1)}{2}\right] - b^2$$

$$S_A = \frac{(n-1)(b^2 + c^2)}{2} - \frac{(n^2 - 1)a^2}{6n}$$

Por permutación cíclica de letras, se obtienen analogamente las expresiones:

$$S_B = \frac{(n-1)(a^2 + c^2)}{2} - \frac{(n^2 - 1)b^2}{6n} \ , \ S_C = \frac{(n-1)(b^2 + a^2)}{2} - \frac{(n^2 - 1)c^2}{6n}$$

$$S = S_A + S_B + S_C = \left[(n-1) - \frac{(n-1)(n+1)}{6n}\right] \cdot (a^2 + b^2 + c^2)$$

$$\Longrightarrow \frac{S}{a^2 + b^2 + c^2} = \frac{(n-1)(5n-1)}{6n}.$$

Luis González

#6 Nov 6, 2011, 11:46 am

> Quote:

Problema 5. Sea $k^3 = 2$ y x, y, z racionales cualesquiera, tal que $x + yk + zk^2 > 0$. Probar que existen racionales $u, v, w,$ tales que $(x + yk + zk^2)(u + vk + wk^2) = 1$.

Primeramente, es facil ver que $x + yk + zk^2 = 0 \Longleftrightarrow x = y = z = 0$. Así,

$(x + yk + zk^2)(x + y\epsilon k + z\epsilon^2 k^2)(x + y\epsilon^2 k + z\epsilon k^2) = t \in \mathbb{Q}^*$

Siendo ϵ una raiz de $x^2 + x + 1 = 0$. Esto se debe a que $x + yk + yk^2 \neq 0 \Longrightarrow (x, y, z) \neq (0, 0, 0) \Longrightarrow$ $x + y\epsilon k + z\epsilon^2 k^2, x + y\epsilon^2 k + z\epsilon k^2 \neq 0$. Así, probamos que $1, \epsilon k, \epsilon^2 k^2$ y $1, \epsilon^2 k, \epsilon k^2$ son independendientes sobre $\mathbb{Q}$, en el mismo modo que lo probamos para $1, k, k^2$. Debido a que, en general,

$(a + b + c)(a + \epsilon b + \epsilon^2 c)(a + \epsilon^2 b + \epsilon c) = a^3 + b^3 + c^3 - 3abc.$

Por otro lado, $(x + y\epsilon k + z\epsilon^2 k^2)(x + y\epsilon^2 k + z\epsilon k^2) = a + bk + ck^2, \;\; a, b, c \in \mathbb{Q}$ y así $(u,v,w)= \left (\frac {_a}{^t},\frac{_b}{^t},\frac {_c^}{^t} \right).$

Quick Reply

Spain

1st IBERO - Colombia 1985.

carlosbr

#1 Mar 26, 2006, 10:10 am

1^{st} Iberoamerican Olympiad
Villa de Leyva, COLOMBIA. [1985]

Edited by djimenez

Carlos Bravo

Attachments:
new_1985.pdf (36kb)

Luis González

#2 Oct 25, 2009, 10:53 pm

> *Quote:*
> **Problema 1.** Hallar todas las ternas de números enteros (a, b, c), tales que:
> $$\begin{aligned} abc &= 440 \\ a+b+c &= 24 \\ a^2+b^2+c^2 &= 210 \end{aligned}$$

$24^2 - 210 = (a+b+c)^2 - a^2 - b^2 - c^2 = 2(ab+bc+ca) \Longrightarrow ab+bc+ca = 183$. Por tanto los números a, b, c son las tres raices de $x^3 - 24x^2 + 183x - 440 = 0$, que factorizando resulta $(x-5)(x-8)(x-11) = 0 \Longrightarrow$ $(a, b, c) = (5, 8, 11)$ y sus permutaciones.

Luis González

#3 Dec 18, 2010, 5:48 am

> *Quote:*
> **Problema 2.** Sea P un punto interior en un triangulo equilatero $\triangle ABC$ tal que $PA = 5, PB = 7, PC = 8$. Calcular BC.

Sea Q el homólogo de P bajo la rotación con centro C y ángulo de rotación 60° antihorario. $\triangle PQC$ es equilátero con lado 8 y $AQ = PB = 7$. Por ley del coseno en $\triangle PAQ$ resulta

$$\cos \widehat{APQ} = \frac{PA^2 + PQ^2 - AQ^2}{2 \cdot PA \cdot PQ} = \frac{5^2 + 8^2 - 7^2}{2 \cdot 5 \cdot 8} = \frac{1}{2} \Longrightarrow \widehat{APQ} = 60^\circ$$

Por lo tanto, $\widehat{APC} = 120^\circ$. Luego, por ley del coseno en $\triangle PAC$

$$AC^2 = PC^2 + PA^2 - 2 \cdot PA \cdot PC \cdot \cos 120^\circ = 5^2 + 8^2 + 5 \cdot 8 \Longrightarrow AC = \sqrt{129}$$

Luis González

#4 Oct 12, 2011, 10:33 am

> Quote:
>
> **Problema 3.** Hallar las raices r_1, r_2, r_3, r_4 de $4x^4 - ax^3 + bx^2 - cx + 5 = 0$, sabiendo que son reales y que
>
> $$\frac{r_1}{2} + \frac{r_2}{4} + \frac{r_3}{5} + \frac{r_4}{8} = 1$$

En primer lugar, el producto de tales raices es $\frac{5}{4}$. Así, usando la desigualdad media geométrica y media aritmética para $\frac{r_1}{2}$, $\frac{r_2}{4}$, $\frac{r_3}{5}$ y $\frac{r_4}{8}$, obtenemos.

$$\frac{1}{4} = \frac{\frac{r_1}{2} + \frac{r_2}{4} + \frac{r_3}{5} + \frac{r_4}{8}}{4} \geq \sqrt[4]{\frac{r_1}{2} \cdot \frac{r_2}{4} \cdot \frac{r_3}{5} \cdot \frac{r_4}{8}} = \sqrt[4]{\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{5} \cdot \frac{1}{8} \cdot \frac{5}{4}} = \frac{1}{4}$$

Es decir, la media geométrica y aritmética son iguales y sabemos que ello ocurre si y solo si los números son iguales. Así

$$\frac{r_1}{2} = \frac{r_2}{4} = \frac{r_3}{5} = \frac{r_4}{8} = \frac{1}{4} \Longrightarrow r_1 = \frac{1}{2}\ , r_2 = 1\ , r_3 = \frac{5}{4}\ , r_4 = 2.$$

Luis González

#5 Oct 22, 2011, 9:24 am

> Quote:
>
> **Problema 4.** Si $x \neq 1, y \neq 1, x \neq y \ \wedge \dfrac{yz - x^2}{1 - x} = \dfrac{xz - y^2}{1 - y}$. Probar que ambas fracciones son iguales a $x + y + z$.

$$\frac{yz - x^2}{1 - x} = \frac{xz - y^2}{1 - y} \Longrightarrow z = \frac{x + y - xy}{x + y - 1} \Longrightarrow x + y + z = \frac{x^2 + y^2 + xy}{x + y - 1}$$

Ahora, sustitutendo z en ambas fracciones, se consigue que en efecto son iguales a la expresión anterior

$$\frac{yz - x^2}{1 - x} = \frac{xz - y^2}{1 - y} = \frac{x^2 + y^2 + xy}{x + y - 1} = x + y + z.$$

Luis González

#6 Oct 23, 2011, 11:25 am

> Quote:
>
> **Problema 5.** A cada entero positivo n se asigna un entero no negativo $f(n)$ tal que $f(rs) = f(r) + f(s), f(n) = 0$ (siempre que la cifra en las unidades n sea 3) y que $f(10) = 0$. Calcular $f(1985)$.

Si $f(mn) = 0$, entonces $f(m) + f(n) = 0$. Pero como $f(m)$ y $f(n)$ son siempre positivos, se sigue que $f(m) = f(n) = 0$. Así, $f(10) = 0$ implica que $f(5) = 0$. Analogamente, se tendrá $f(3573) = 0$. Entonces $f(397) = 0$. Por consiguiente, $f(1985) = f(5) + f(397) = 0$.

Luis González

#7 Nov 6, 2011, 8:26 am

> Quote:
>
> **Problema 6.** $\triangle ABC$ es acutángulo con circuncírculo (O, R). D, E, F son puntos que yacen en BC, CA, AB tal que AD, BE, CF pasan por O. Probar que
>
> $$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{2}{R}$$

Sean P, Q, R los pies de las alturas desde A, B, C. Los rayos AP, BQ, CR cortan a (O) en M, N, L. Es sabido que M, N, L son los simétricos del ortocentro H de $\triangle ABC$ respecto a BC, CA, AB y como AD, AM son isogonales respecto a $\angle ABC$, se sigue que

$$AD \cdot AM = AB \cdot AC = 2R \cdot AP \implies \frac{1}{AD} = \frac{AM}{2R \cdot AP}$$

$$\frac{1}{AD} = \frac{1}{2R}\left(\frac{AP + HP}{AP}\right) = \frac{1}{2R}\left(1 + \frac{[\triangle BHC]}{[\triangle ABC]}\right)$$

Entonces, sumando las expresiones cíclicas obtenemos:

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{1}{2R}\left(3 + \frac{[\triangle BHC] + [\triangle CHA] + [\triangle AHB]}{[\triangle ABC]}\right)$$

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{1}{2R}\left(3 + \frac{[\triangle ABC]}{[\triangle ABC]}\right) = \frac{2}{R}$$

High School Olympiads

inversion problem

palashahuja

#1 Nov 2, 2011, 5:01 pm

How can we solve using inversion?

In a quadrilateral $ABCD$, let $\angle A + \angle C = 90^\circ$, then

$(AB \cdot CD)^2 + (BC \cdot AD)^2 = (AC \cdot BD)^2$

please solve it, I am waiting for a solution.

Luis González

#2 Nov 3, 2011, 6:34 am

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=285053.

Alternate proof: Construct points P, Q outside $ABCD$, such that $\triangle PAD \sim \triangle BCA$ and $\triangle QAB \sim \triangle DCA$. If the parallel from A to BQ cuts BD at E, we have $\angle EAB = \angle QBA = \angle CAD$, which implies that $\angle EAD = \angle CAB = \angle PDA \Longrightarrow BQ \parallel PD$. Further, $\frac{BQ}{AD} = \frac{AB}{AC} = \frac{PD}{AD} \Longrightarrow BQ = PD$. Hence, $BQPD$ is a parallelogram $\Longrightarrow BD = PQ$. Now, since $\angle BAD + \angle DCB = 90^\circ$, it follows that $\angle PAQ = 90^\circ$. Therefore, by Pythagorean theorem we get

$$PQ^2 = BD^2 = AP^2 + AQ^2 = \left(\frac{BC \cdot DA}{AC}\right)^2 + \left(\frac{AB \cdot CD}{AC}\right)^2$$

$$(AC \cdot BD)^2 = (AB \cdot CD)^2 + (BC \cdot DA)^2$$

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High School Olympiads

Prove the equality(geometry)

Reply

Source: Kazakhstan city olymp(2010)

mudok

#1 Oct 30, 2011, 8:01 pm

Given a quadrilateral which has inscribed circle with center I. Intersection point of AD and BC is R. Inscribed circle is tangent to AB and CD at points $P$and Q respectively. Perpendicular line to PR which passes through P cuts AI and BI at A_a and B_o respectively. Perpendicular line to RQ which passes through Q cuts CI and DI at C_a and D_o respectively. Prove that

$$A_oD_o = B_oC_o$$

Luis González

#2 Oct 30, 2011, 9:17 pm • 2

Let the incircle (I) touch BC, AD at $M, N.$ If $X \equiv MN \cap CD$, then RQ is the polar of X WRT $(I) \Longrightarrow IX \perp RQ.$ Since $I(C, D, Q, X)$ is harmonic, $IX \parallel C_0QD_0$ implies that Q is the midpoint of $\overline{C_0D_0}$, i.e. $\triangle RD_0C_0$ is isosceles with legs $RD_0 = RC_0.$ Since C_0 and D_0 lie on the perpendicular bisectors of $\overline{QM}$ and $\overline{QN}$, we have $C_0M = C_0Q = D_0Q = D_0N$, which yields $\triangle RMC_0 \cong \triangle RND_0$ by SSS $\Longrightarrow \angle MRN = \angle C_0RD_0.$ Similarly, $\triangle RA_0B_0$ is R-isosceles with $\angle A_0RB_0 = \angle MRN.$ As a resut, $\triangle RB_0C_0 \cong \triangle RA_0D_0$ by SAS $\Longrightarrow B_0C_0 = A_0D_0.$

mahanmath

#3 Oct 31, 2011, 9:42 am

Nice... Congrats Luis 🙂 !

Complex Bashing

Quick Reply

High School Olympiads

Concurrency of circles on a line

Reply

Source: Own and inspired by Czech-Polish-Slovak Match 2011 Day 2 P2

swaqar

#1 Oct 29, 2011, 11:40 am • 1

Let $ABCD$ be a cyclic quadrilateral and let M and N, respectively, be the midpoints of AD and BC. Suppose K and L are points on AB and CD respectively such that MK , LN and BD are concurrent. Show that the circumcircles of $\triangle BKM$ and $\triangle DML$ intersect at DB.

P.S. The problem can be generalized.

sunken rock

#2 Oct 29, 2011, 8:10 pm

My observations:
1) It's about the circles (BKN), (DML).
2) $\frac{AK}{BK}=\frac{DL}{CL}$
3) KN, ML, AC are concurrent as well hence, as per the problem, the circles $(AKM), (CLN)$ concur on AC.
4) Regarding the generalisation, most probable we may have M, N on AD, BC respectively such as $\frac{AM}{MD}=\frac{BN}{CN}$.

That's (almost) all I can do!

Best regards,
sunken rock

Luis González

#3 Oct 29, 2011, 10:36 pm • 2

Let $\odot(BKN)$ and $\odot(DML)$ cut $\overline{BD}$ again at X and Y. By Ptolemy's theorem for $BKXN$ and $DMYL$, keeping in mind that $\triangle KXN \sim \triangle ADC$ and $\triangle MYL \sim \triangle ABC$, we deduce that

$$BX \cdot AC = BK \cdot CD + BN \cdot DA \ , \ \ DY \cdot AC = DL \cdot AB + DM \cdot BC$$

$$\Longrightarrow AC(BX+DY) = BC \cdot DA + BK \cdot CD + DL \cdot AB$$

Using that $\frac{BK}{BA}=\frac{CL}{CD}$, which follows by Menelaus theorem for $\triangle ABD \cup KM$ and $\triangle CBD \cup NL$, we get

$$AC(BX+DY) = BC \cdot DA + AB \cdot CL + DL \cdot AB = BC \cdot DA + AB \cdot CD$$

$\Longrightarrow BX+DY=BD$, i.e. $X \equiv Y$ and the conclusion follows.

swaqar

#4 Oct 30, 2011, 12:17 am • 1

Proof: We generalize the problem for the case when M and N satisfy the condition $\frac{AM}{MD}=\frac{BN}{NC}$.

Lemma: The circumcircles of $\triangle CNL$, $\triangle AMK$, $\triangle DML$ and $\triangle BKN$ are concurrent.
Proof: Let AB and CD intersect in Q and AD and BC in R. Consider the circumcircles ω of $\triangle QKL$ and γ of $\triangle RMN$.
One point of intersection is obviously $\mathcal{M}$, the center of spiral symmetry which sends A to B and D to C or the Miquel Point of

$ABCD$ which lies on RQ. Let S be other intersection point of these two circles. Let SR intersect ω in T. As, QT can be seen as the inverse of γ with center of inversion being R, we have $QT \cap AD = X$ the inverse of M and so $MKTX$ is cyclic. So $\angle AKS = 180 - \angle STQ = 180 - \angle STX = 180 - \angle SMA$, proving that $AMSK$ is cyclic. $\square$

Let's prove our main result. With S as the center, apply the spiral symmetry $\mathcal{S}$ that sends the point Q to O_ω and R to O_γ. This will send the whole quadrilateral $ABCD$ to the respective centers of the four circles and so, there is a symmetry that sends the point D to B and the circumcenter of $\triangle LMD$ to the circumcenter of $\triangle BNK$ and so the intersection of the circles $ \circle LMD $ and $ \circle BNK $ other than S is collinear with DB.

Remark: If the concurrency point of the circumcirlces of $\triangle CNL$ and AMK is G and that of the other pair of circles is H and if $AC \cap BD = P$, then $SGPH$ is cyclic with the image of P under the spiral symmetry $\mathcal{S}$ is the center of this circle.

Quick Reply

High School Olympiads

Circumcenter

Reply

Source: Yufei Zhao Winter Camp Document - Spiral Similarity

yunustuncbilek

#1 Oct 29, 2011, 12:34 am

Let $ABCD$ be a quadrilateral. Let diagonals AC and BD meet at P. Let O_1 and O_2 be the circumcenters of APD and BPC. Let M, N and O be the midpoints of AC, BD and O_1O_2. Show that O is the circumcenter of MPN.

Luis González

#2 Oct 29, 2011, 3:29 am

Assume that M, N, O divide $\overline{AC}, \overline{DB}, \overline{O_1O_2}$ in the same ratio k. Thus, the ratio of the powers of M and N with respect to (O_1) and (O_2) is also k, because

$$\frac{\overline{MA} \cdot \overline{MP}}{\overline{MC} \cdot \overline{MP}} = \frac{\overline{ND} \cdot \overline{NP}}{\overline{NB} \cdot \overline{NP}} = k$$

Therefore, M and N lie on a circle coaxal with $(O_1), (O_2)$ whose center divides $\overline{O_1O_2}$ in the ratio k. In other words, O is circumcenter of $\triangle PMN$, as desired.

P.S. The proposed problem is just the case $k = -1$ discussed before in the topic Circles.

Quick Reply

High School Olympiads

Points on a chord of fixed length

sunken rock

#1 Oct 28, 2011, 9:44 pm

Let M, N be points on the chord $AB=R\sqrt{3}$ of the circle $C(O,R)$, order $A-M-N-B$ such that $\angle MON=60^\circ$.
Prove that $MN^2+AM\cdot BN=AM^2+BN^2$.

Best regards,
sunken rock

diks94

#2 Oct 28, 2011, 10:52 pm

going by basics..
drop perpendicular from O to AB let it be D.
now let $\angle MOD=x$ find all sides in terms of x and R using basic trignometry and then just prove $L.H.S=R.H.S$
well i am thinking about a solution by geometry. but angle MON making some trouble.

Luis González

#3 Oct 29, 2011, 12:29 am • 1

$\triangle OAB$ is obviously O-isosceles with $\angle AOB=120^\circ$. K is the projection of O on $\overline{AB}$ and P is the reflection of A about OM. $\angle MON=\angle AOK$ implies that $\angle ONA=\angle OAP=\angle OPA \Longrightarrow ONPA$ is cyclic. Since $OA=OP=OB$, then O is circumcenter of $\triangle PAB \Longrightarrow \angle POB=2\angle PAN=2\angle PON \Longrightarrow ON$ bisects $\angle POB$, i.e. P is reflection of B about ON. Thus, $OMPB$ is also cyclic $\Longrightarrow \angle MPN=\angle OBM+\angle OAN=60^\circ$, i.e. lengths MN, AM, BN form a triangle $\triangle PMN$ with $\angle MPN=60^\circ$. The conclusion follows.

sunken rock

#4 Oct 29, 2011, 2:50 am

Although almost similar with Luis', my solution:
Take P a random point on smaller arc $\overset{\frown}{AB}$; the angle bisectors of $\angle AOP$ and $\angle BOP$ intersect $\overline{AB}$ at M, N. $MP=AM$ and $\angle OPM=\angle OAM$ by symmetry; similarly $NP=BN$ and $\angle OPN=\angle OBN$ and, consequently $\triangle MNP$ has the sides AM, MN, NB and $\angle MPN=60^\circ$.

Best regards,
sunken rock

yetti

#5 Oct 29, 2011, 3:08 pm • 1

(O') is circumcircle of O-isosceles $\triangle ABO$. $AB=R\sqrt{3} \Longrightarrow \angle BOA=120^\circ$ and circumcenter O' is reflection of O in AB.
Perpendiculars to $O'M, O'N$ through A, B meet at C. $\angle BCA=180^\circ-\angle MO'N=180^\circ-\angle NOM=120^\circ \Longrightarrow$
$C\in(O') \Longrightarrow O'M, O'N$ are perpendicular bisectors of $AC, BC \Longrightarrow$
$\triangle AMC, \triangle BNC$ are M-, N-isosceles $\Longrightarrow AM=MC, BN=NC$ and
$\angle NCM=\angle BCA-(\angle BCN+\angle MCA)=\angle BCA-(\angle ABC+\angle CAB)=60^\circ \Longrightarrow$
$AM^2+BN^2-AM\cdot BN=MC^2+NC^2-MC\cdot NC=MN^2$.

High School Olympiads

overloading
#1 Oct 27, 2011, 3:04 pm

Give $ABCD$ is a convex quadrilateral such that $AB = BC = CD = a$. Find AD and the angel of $ABCD$ such that the area of $ABCD$ is maximum

diks94
#2 Oct 28, 2011, 1:11 am

I think i got it ..
by cosine rule $cos(\frac{\pi - C}{2}) = \frac{a^2 + BD^2 - a^2}{2.a.BD} = \frac{BD}{2a} = sin\frac{C}{2}$
also $cosC = 1 - \frac{BD^2}{2a^2}$
From these two equations you get $\angle C$ and BD
now use the formulae for area of quadrilateral $\sqrt{(s-a)(s-b)(s-c)(s-d) - abcd(cos\frac{A+C}{2})^2}$
you will get area as a function of x
rest is differentiation.
tht's one approach

Luis González
#3 Oct 28, 2011, 3:17 am

Let O, M, N be the midpoints of BC, AC, BD. Then $OM = ON = \frac{1}{2}a$. Furthermore, $[ABCD] = 4[BCNM]$. Thus, $[ABCD]$ is maximum $\Longleftrightarrow$ $[BCNM]$ is maximum. Denote $\angle BOM = \varphi_1, \angle MON = \varphi_2$ and $\angle NOC = \varphi_3$.

$[BCNM] = [BOM] + [MON] + [NOC] = \frac{1}{8}a^2(\sin\varphi_1 + \sin\varphi_2 + \sin\varphi_3)$

Obviously, $\varphi_1, \varphi_2, \varphi_3 < \pi$. Thus, by Jensen inequality for $\sin\varphi, \varphi \in (0, \pi)$, the area of $BCNM$ is maximum when $\varphi_1 = \varphi_2 = \varphi_3 = \frac{1}{3}(\varphi_1 + \varphi_2 + \varphi_3) = 60^\circ$, i.e. when $ABCD$ is isosceles trapezoid with $\angle BAD = \angle CDA = 60^\circ$, $AD = 2a$.

newsun
#4 Oct 29, 2011, 9:53 pm

Maybe here is more simpler..
Setting $\widehat{BAC} = x, \widehat{ACD} = y$ and let $BH \perp AC, H \in AC$
Then
$S_{ABCD} = S_{ABC} + S_{ACD} = \frac{1}{2}AC.BH + \frac{1}{2}AC.CD.siny \le BH.AH + AH.CD \le a.sinx.a.cos + a^2cosx$
Therefore, $S_{ABCD}^2 \le a^4cos^2x(1 + sinx)^2 = \frac{1}{3}a^4(3 - 3sinx)(1 + sinx)(1 + sinx)(1 + sinx)$
Now apply the Cauchy's inequality for 4 non-negative numbers we have $S_{ABCD} \le \frac{3a^2\sqrt{3}}{4}$.

oneplusone
#5 Nov 4, 2011, 7:27 am

An even simpler one without trigo.

Fix A, B, C and vary D. For $[ABCD]$ to be max, $[ACD]$ has to be max, so $\angle ACD = 90$. Similarly $\angle DBA = 90$. So $ABCD$ is cyclic with AD as the diameter. Since $AB = BC = CD$, we get $\angle A = \angle D = 60, \angle B = \angle C = 120$ and $AD = 2a$.

High School Olympiads

Nice locus

Reply

Omez

#1 Oct 27, 2011, 3:53 am

O is the center of inscribed circle of triangle ABC. P is a point. PA,PB and PC intersect the lines CB,AC and BA in P,Q and R. angle RPQ=90°. The symmetrics of PB and PC towards OB and OC intersect in P'. Describe the locus of P'

JSGandora

#2 Oct 27, 2011, 5:18 am

> Omez wrote:
> O is the center of inscribed circle of triangle ABC. P is a point. PA,PB and PC intersect the lines CB,AC and BA in **P**,Q and R.

Are you sure that this is correct? So you're saying that P is on CB?

Luis González

#3 Oct 27, 2011, 12:49 pm

Probably, the OP asks for the locus of the isogonal conjugate of the point P whose cevian triangle has a right angle againts A (?). Denote by $\triangle P_1P_2P_3$ the cevian triangle of P and let $(u:v:w)$ be the barycentric coordinates of P with respect to $\triangle ABC.$ Then $P_1(0:v:w), \ P_2(u:0:w), \ P_3(u:v:0)$

$$P_1P_2 \equiv vwx+uwy-uvz=0 \ , \ P_1P_3 \equiv vwx-uwy+uvz=0$$

We know that two lines ℓ_1 and ℓ_2 with infinite points $(f_1:g_1:h_1)$ and $(f_2:g_2:h_2)$ are perpendicular if and only if $S_Af_1f_2+S_Bg_1g_2+S_Ch_1h_2=0.$ Thus, the lines P_1P_2 and P_1P_3 are perpendicular if and only if

$$a^2y^2z^2-b^2x^2z^2-c^2x^2y^2-(b^2+c^2-a^2)yzx^2=0$$

Locus of P is an unicursal quartic with three double points, namely $A,B,C.$ The isogonal conjugate of this quartic is then a conic with barycentric equation

$$\mathcal{K} \equiv a^2c^2y^2+a^2b^2z^2-b^2c^2x^2+a^2(b^2+c^2-a^2)yz=0$$

Its discriminant is negative, so $\mathcal{K}$ is always a hyperbola with focal axis $S_By-S_Cz=0,$ i.e. the A-altitude of $\triangle ABC.$ The traces of $\mathcal{K}$ on the sides $y=0$ and $z=0$ are $(a:0:c), (a:0:-c)$ and $(a:b:0), (a:-b:0),$ i.e. the feet of the internal and external bisectors of $\angle ABC$ and $\angle ACB.$

Quick Reply

High School Olympiads

Calculation of Areas

Reply

silvergrasshopper

#1 Oct 27, 2011, 9:07 am

Please help me to solve the following problem. =)

Triangle $T_a = \Delta A_1A_2A_3$ is inscribed in triangle $T_b = \Delta B_1B_2B_3$ and triangle T_b is inscribed in triangle $T_c = \Delta C_1C_2C_3$ so that the sides of triangle T_a and T_c are pairwise parallel. Express the area of triangle T_b in terms of the areas of triangle T_a and T_c.

Any help will be greatly appreciated. Thank you so much =)

Luis González

#2 Oct 27, 2011, 10:45 am

This is usually known as Gergonne-Ann theorem, which states that: If a triangle T2 is inscribed in a triangle T1, and a triangle T3 is inscribed in triangle T2 such that T1 and T3 are homothetic, then the area of T2 equals the geometric mean between the areas of T1 and T3.

For a proof see the topic Area of a triangle is a GM of area of two other triangles.

Quick Reply

High School Olympiads

Nice geometry

Reply

AndreiAndronache

#1 Oct 26, 2011, 11:51 pm

Let acute triangle $\triangle ABC, M \in (\angle ABC), E = pr_{AB}M \in (AB; F = pr_{AC}M \in (AC; N \in (EF); MN \perp BC$.
Prove that $(AM$ bisect angle $\angle A$ if and only if $(AN$ bisect (BC).

Luis González

#2 Oct 27, 2011, 5:37 am

$$\frac{\mathrm{d}(N,AB)}{\mathrm{d}(N,AC)} = \frac{NE}{NF} \cdot \frac{\sin \widehat{AEF}}{\sin \widehat{AFE}} = \frac{AF}{AE} \cdot \frac{ME}{MF} \cdot \frac{\sin \widehat{EMN}}{\sin \widehat{FMN}} = \frac{ME}{MF} \cdot \frac{AF}{AE} \cdot \frac{\sin B}{\sin C}$$

$\dfrac{\mathrm{d}(N,AB)}{\mathrm{d}(N,AC)} = \dfrac{\sin B}{\sin C} \Longleftrightarrow \dfrac{ME}{AE} = \dfrac{MF}{AF}$, i.e. AN bisects $BC \Longleftrightarrow AM$ bisects $\angle A$.

Quick Reply

High School Olympiads

collinear 3 some

Reply

goodguy

#1 Oct 16, 2011, 4:31 pm

give triangle ABC inscribed a circle (O) with center O, AJ is the angle bisector of $\angle BAC$. JE, JF is perpendicular with CA, BA, at E,F. AO cut JE at Q, AO cut JF at N. CF cut BE at M, FN cut BQ at P, CN cut EQ at S. Prove that M,S,P are collinear

Luis González

#2 Oct 26, 2011, 10:43 pm

Let D be the foot of the A-altitude of $\triangle ABC.$ $AQJB$ is cyclic due to $\angle BAO=\angle CJE=90^{\circ}-\angle ACB.$ Thus, $\angle JBQ=\angle JAQ=\angle JAD \Longrightarrow BQ$ is B-altitude of $\triangle BAJ \Longrightarrow P$ is orthocenter of $\triangle BAJ,$ i.e. $P \in AD.$ By similar reasoning, $S \in AD.$ $AEJDF$ is clearly cyclic and A is the midpoint of the arc EF of its circumcircle $\Longrightarrow DA,BC$ bisect $\angle EDF \Longrightarrow$ pencil $D(E,F,A,B)$ is harmonic $\Longrightarrow M \equiv BE \cap CF \cap AD \Longrightarrow M,S,P$ lie on $AD.$

Quick Reply

High School Olympiads

About triangle centers

Reply

Stephen

#1 Oct 25, 2011, 11:32 am • 1

In triangle ABC, let A_1, B_1, C_1 the foots of perpendicular that match with A, B, C.

w, w_a, w_b, w_c are the incircle, A-excircle, B-excircle, C-excircle.

l_a is the common tangent of w, w_a that is not BC. In same way, define l_b, l_c.

A_2 is the intersection of l_b, l_c. In same way, define B_2, C_2.

(1) Prove that A_1A_2, B_1B_2, C_1C_2 is concurrent.
(Let this intersection point T_1.)

Let I, O the incenter and circumcenter. AD is the diameter of ABC's circumcircle.

D_0 is the intersection of BC and ID. In same way, define E_0, F_0.

(2) Prove that AD_0, BE_0, CF_0 is concurrent.
(Let this intersection point T_2.)

(3) Prove that T_1, T_2 is isogonal conjugate in triangle ABC.

nsato

#2 Oct 26, 2011, 6:39 am • 1

Are you sure? I find that T_1 is the Clawson Point, and that T_2 is point X_{77} in the ETC, which are not isogonal conjugates.

Luis González

#3 Oct 26, 2011, 7:01 am • 1

Dear Naoki, $\triangle A_2B_2C_2$ is not the extangents triangle of $\triangle ABC$, but its intangents triangle, which is homothetic to the orthic triangle through

$$X_{33}\left(\frac{a(S_A+bc)}{S_A}:\frac{b(S_B+ca)}{S_B}:\frac{c(S_C+ab)}{S_C}\right)$$

The perspector of $\triangle ABC$ and $\triangle A_0B_0C_0$ is indeed X_{77} (see the topic Concurrent lines). X_{33} and X_{77} are then isogonal conjugates with respect to $\triangle ABC$.

nsato

#4 Oct 26, 2011, 11:29 pm • 1

Thanks, that makes more sense.

Quick Reply

High School Olympiads

symmedian

Reply

unt

#1 Oct 25, 2011, 2:35 pm

Prove that the orthocenter H, the symmedian K and the symmedian K' of the orthic triangle are collinear.

yetti

#2 Oct 25, 2011, 5:35 pm

Let I_a, I_b, I_c be excenters of $\triangle ABC$ in $\angle A, \angle B, \angle C \Longrightarrow$ orthocenter H of $\triangle I_aI_bI_c$ is incenter of $\triangle ABC$ and $\triangle ABC$ is orthic triangle of $\triangle I_aI_bI_c$.
I_bI_cBC is cyclic $\Longrightarrow BC$ is antiparallel of I_bI_c WRT $\angle I_cI_aI_b \Longrightarrow$ symmedians I_aK, I_bK, I_cK of $\triangle I_aI_bI_c$ cut BC, CA, AB at their midpoints A', B', C'.

Barycentric equation of line I_aA' WRT $\triangle ABC$ is
$(\overrightarrow{I_a} \times \overrightarrow{A}') \cdot \overrightarrow{X} = ((-a, b, c) \times (0, 1, 1)) \cdot (x, y, z) = (b-c)x + ay - az = 0.$
Similarly, barycentric equations of lines I_bB', I_cC' WRT $\triangle ABC$ are $-bx + (c-a)y + bz = 0$ and $cx - cy + (a-b)z = 0$.
Eliminating x from the last 2 equations, $\dfrac{y}{z} = \dfrac{b(p-b)}{c(p-c)}$, where $p = \frac{1}{2}(a+b+c) \Longrightarrow$ barycentric coordinates of K WRT $\triangle ABC$ are $a(p-a) : b(p-b) : c(p-c)$.
Barycentric equation of line HK' WRT $\triangle ABC$ is $(\overrightarrow{H} \times \overrightarrow{K}') \cdot \overrightarrow{X} = \big((a, b, c) \times (a^2, b^2, c^2)\big) \cdot (x, y, z) = bc(c-b)x + ca(a-c)y + ab(b-a)z = 0$.
Barycentric coordinates of K obviously satisfy this equation.

jayme

#3 Oct 25, 2011, 5:37 pm

Dear Mathlinkers,
this is the H.H. van Aubel line. A pure synthetic proof is possible.
Sincerely
Jean-Louis

Luis González

#4 Oct 25, 2011, 8:24 pm • 2

This is equivalent to show that the incenter, Mittenpunkt and symmedian point of any $\triangle ABC$ are collinear. Denote by $O, H, I, K, M, N_a, G_e, R$ the circumcenter, orthocenter, incenter, symmedian point, Mittenpunkt, Nagel point, Gergonne point and Retrocenter of $\triangle ABC$, respectively. It's well-known that I, K, M become Nagel point, Retrocenter and Gergonne point of the medial triangle of $\triangle ABC$. So it suffices to prove that N_a, G_e, R are collinear.

H, G_e, N_a lie on a same circum-hyperbola $\mathcal{F}$ (Feuerbach hyperbola of ABC), since their isogonal conjugates, namely O and the two similitude centers of $(I) \sim (O)$ are collinear. Then the isotomic conjugation WRT $\triangle ABC$ takes N_a, G_e into each other and H into $R \Longrightarrow N_a, G_e, R$ lie on the isotomic line of $\mathcal{F}$.

P.S. The product of the isogonal and isotomic conjugation is a collineation.

unt

#5 Oct 26, 2011, 12:34 am

jayme wrote:

Dear Mathlinkers,
this is the H.H. van Aubel line. A pure synthetic proof is possible.
Sincerely
Jean-Louis

Dear, Jean-Louis where I can read it?

unt

#6 Oct 26, 2011, 12:42 am

> luisgeometra wrote:
>
> P.S. The product of the isogonal and isotomic conjugation is a collineation.

How can we prove that?

Luis González

#7 Oct 26, 2011, 1:43 am

The proof is rather simple using barycentric coordinates with respect to $\triangle ABC.$ Mappings Isogonal conjugation $\mathcal{I}$ and Isotomic conjugation $\mathcal{L}$ WRT ABC are given by

$$\mathcal{I}:(x:y:z)\mapsto\left(\frac{a^2}{x}:\frac{b^2}{y}:\frac{c^2}{z}\right)\ ,\ \mathcal{L}:(x:y:z)\mapsto\left(\frac{1}{x}:\frac{1}{y}:\frac{1}{z}\right)$$

Thus, $\mathcal{I}\circ\mathcal{L}$ carries a line ℓ with equation $px+qy+rz=0$ into a line τ with equation $a^2px+b^2qy+c^2rz=0.$ As a result, a big group of remarkable collinearities can be derived without going into calculations.

Quick Reply

High School Olympiads

Cono Sur Mathematical Olympiad 2011 P5

Reply

nunoarala

#1 Oct 25, 2011, 10:19 pm

Let ABC be a triangle and let D be a point on the side AC. If $\angle CBD - \angle ABD = 60^\circ$, $\angle BDC = 30^\circ$, and $\overline{AB} \cdot \overline{BC} = \overline{BD}^2$, find $\angle ABC$, $\angle ACB$ and $\angle BAC$.

Luis González

#2 Oct 25, 2011, 11:23 pm • 1

Let the isogonal ray of BD WRT $\angle ABC$ cut AC and the circumcircle of $\triangle ABC$ at $P, T,$ respectively. Then $\angle PBD = 60^\circ \Longrightarrow \triangle PBD$ is right at P and $BD = 2BP. \triangle BAD \sim \triangle BTC$ implies that $BD \cdot BT = AB \cdot BC.$ Thus, $AB \cdot BC = BD^2$ yields $BD = BT \Longrightarrow BP = \frac{1}{2}BT,$ i.e. T is the reflection of B about $AC \Longrightarrow \angle ABC = 90^\circ \Longrightarrow D$ is circumcenter of $\triangle ABC \Longrightarrow \angle BAC = 15^\circ, \angle BCA = 75^\circ.$

diks94

#3 Oct 26, 2011, 12:01 am

let angle ABD=x
by sin rule
$BD = \frac{cosxBC}{sin30} = \frac{sin(30-x)AB}{sin150}$
Putting in $AB.BC = BD^2$ we get
$\sqrt{3}sin2x = cos2x$
hence ur question is done.

hatchguy

#4 Oct 28, 2011, 4:55 am

Set $ABD = \angle\alpha$. From $\frac{AB}{BD} = \frac{BD}{BC}$ we have

$$\frac{\sin 150}{\sin 30 - \alpha} = \frac{cos\alpha}{\sin 30}$$

so

$$2\sin(30-\alpha)\cos\alpha = \frac{1}{2} => \sin 30 + \sin(30-2\alpha) = \frac{1}{2}$$

and $\alpha = 15$. The rest is angle chase.

sunken rock

#5 Oct 28, 2011, 11:30 am

Something similar to Luis' (reverse judgement)
Take B' the reflection of B in AC, obviously BDB' is an equilateral triangle and BB' the isogonal of BD. With $BB' = BD$, the given relation becomes $\frac{AB}{BB'} = \frac{BD}{BC}$, i.e. $\triangle ABD \sim \triangle BB'C$, or $BD = AD$.
Further, just angle chasing.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Square-square

Reply

Source: own

jayme

#1 Jul 17, 2011, 12:10 pm • 1

Dear Mathlinkers,
ABCD a square, E a point on the segment AD, AEFG a square externally to ABCD, I the midpoint of DE, K the point of intersection of BE and DG. Prove that IK, CE and AB are concurrent.
Sincerely
Jean-Louis

skytin

#2 Jul 17, 2011, 6:43 pm

BE intersect DG at Y
CE intersect perpendicular thru G on CE at X
HG is altitude of BDG
Easy to see that H is on YC
O is center of (ABCD)
YI intersect CD at J
From angle YJD = YOD = 2*YCD , so IYC = YCD = YBD = YGH = YXE = YXC , so (CYX) is tangent to IY
CE , YI , AG intersect at radical center of (BYG) , (CYX) , (GBCX) . done

jayme

#3 Jul 17, 2011, 8:18 pm

Dear skytin (A?),
thantk for your proof and tour interest...
In my proof I consider the Euler's circle of the triangle BDG and finish with the Pascal's theorem.
For me the skeletton of this problem is the geometry problem proposed in the first OIM in 1959.*Sincerely
Jean-Louis

yetti

#4 Jul 17, 2011, 11:57 pm

KI cuts AB at J. M is midpoint of GB.
$\triangle ADG \sim \triangle ABE \Longrightarrow GK \perp KB$.
$\triangle KBG \sim \triangle ABE \Longrightarrow \frac{GK}{KB} = \frac{GA}{AB} \Longrightarrow KA$ bisects $\angle BKG$.
$\triangle KBG \sim \triangle KDE \Longrightarrow KM \perp KI \Longrightarrow JKI$ is tangent to $\odot(KBG)$ and $JK = JA$.
$\Longrightarrow \triangle JKG \sim \triangle JBK \Longrightarrow \frac{JA}{JB} = \frac{JK}{JB} = \frac{GK}{KB} = \frac{EA}{CB} \Longrightarrow J \in CE$.

jayme

#5 Oct 8, 2011, 4:23 pm

Dear Mathlinkers,
an remastered article concerning the "Geometric miniatures" has been put on my website
with 10 more examples.
http://perso.orange.fr/jl.ayme vol. 7 p. 74
You can use Google translator
Sincerely

Jean-Louis

erfan_Ashorion

#6 Oct 23, 2011, 11:05 pm

oh...!
suppose CE and AB intersect each other at X we must proof I,K,X are on same line!we use Menlauos theoream!!we must proof:
$\frac{XA}{XB}\cdot\frac{KB}{KE}\cdot\frac{IE}{IA}=1$
we know that:
$\frac{XA}{AB}=\frac{AE}{ED}\to\frac{XA}{XB}=\frac{AE}{AD}$!!!
and we know that:
$\frac{KE}{KB}\cdot\frac{GB}{GA}\cdot\frac{DA}{DE}=1$
so we must proof:
$\frac{AE}{AD}\cdot\frac{GB}{GA}\cdot\frac{AD}{DE}\cdot\frac{IE}{IA}=1$
other of it is easy

Virgil Nicula

#7 Oct 25, 2011, 12:17 pm • 1

PP (An easy extension). Let $ABCD$ be a rhombus. For $E\in(AD)$ construct externally the rhombus

$AEFG$. Denote the middlepoint I of $[DE]$ and $K\in BE\cap DG$. Prove that $IK\cap CE\cap AB\ne\emptyset$.

Proof. Denote $L\in CE\cap AB$. Observe that $\frac{LA}{LB}=\frac{EA}{BC}$ (1) and $GB=GA+AB=AE+AD=2\cdot AI\implies$

$\frac{IE}{IA}=\frac{2\cdot IE}{2\cdot IA}\implies\frac{IE}{IA}=\frac{DE}{GB}$ (2) . Apply the **Menelaus**' theorem to $\overline{DKG}/\triangle AEB$: $\frac{GA}{GB}\cdot\frac{KB}{KE}\cdot\frac{DE}{DA}=1\implies$

$\frac{KB}{KE}=\frac{DA}{DE}\cdot\frac{GB}{GA}$ (3) . Therefore, $\frac{LA}{LB}\cdot\frac{KB}{KE}\cdot\frac{IE}{IA}\overset{(1)\wedge(2)\wedge(3)}{=}\frac{EA}{BC}\cdot\left(\frac{DA}{DE}\cdot\frac{GB}{GA}\right)\cdot\frac{DE}{GB}=1\implies L\in IK$.

In conclusion, the lines IK , CE and AB are concurrently . **Nice problem ! Thank you**, Jayme.

Luis González

#8 Oct 25, 2011, 9:03 pm

Since $\triangle DAG\cong\triangle BAE$ by SAS, then $\angle KDA=\angle KBA\Longrightarrow K\in\odot(ABCD)\Longrightarrow\angle BKD=90^\circ\Longrightarrow$ $K\in\odot(AEFG)$ and C,K,F are collinear. If IK cuts AB at $L,$ we have $\angle LKG=\angle KDA=KBG$ and $\angle LAK=\angle KDB=45^\circ+\angle IKD=\angle LKA\Longrightarrow LK^2=LA^2=LG\cdot LB.$ But if $N\equiv CK\cap AB,$ then $(A,N,G,B)=-1,$ since KG,KB bisect $\angle AKN.$ Therefore L is midpoint of $AN.$

On the other hand, if $M\equiv AC\cap EF,$ then $\triangle AFM$ is obviously isosceles right at $A\Longrightarrow E$ is midpoint of $MF.$ Since $MF\parallel AN,$ then C,E,L are collinear $\Longrightarrow AB,IK,CE$ concur at $L.$

Quick Reply

High School Olympiads

Symmedian point

Reply

mathetician

#1 Oct 23, 2011, 4:04 pm

Prove that the circumcentre of $\triangle ABC$ and the centroid of the anti pedal $\triangle A'B'C'$ of the symmedian point of $\triangle ABC$, coincide. Find the circumradius of $\triangle ABC$ in terms of parameters of $\triangle A'B'C'$.

Luis González

#2 Oct 24, 2011, 7:27 am

G, O, K are the centroid, circumcenter and symmedian point of $\triangle ABC. E, F$ are midpoints of AC, AB. $\angle OBA' = 90^\circ - \angle KBC + \angle OBC = \angle BEA$. Similarly, we have $\angle OCA' = \angle CFA$. So, if X, Y, Z are the projections of G on BC, CA, AB, we have then

$$\frac{\mathrm{d}(O, A'C')}{\mathrm{d}(O, A'B')} = \frac{\sin \widehat{GEY}}{\sin \widehat{GFZ}} = \frac{GY}{GZ} \cdot \frac{GF}{GE} = \frac{AB}{AC} \cdot \frac{GC}{GB} = \frac{XY}{XZ} = \frac{A'B'}{A'C'}$$

The last equality follows from the fact that $\triangle XYZ \sim \triangle A'B'C'$ are homothetic. Hence, $A'O$ is the A'-median of $\triangle A'B'C'$. Similarly, $B'O, C'O$ are the medians of $\triangle A'B'C'$ issuing from $B', C' \Longrightarrow O$ is the centroid of $\triangle A'B'C'$.

Quick Reply

High School Olympiads

triangle with bisectors and midpoints

Reply

KittyOK
#1 Oct 23, 2011, 9:54 pm

A triangle ABC has $\angle ACB = 60^\circ$. Let AA_1 and BB_1 ($AA_1 \in BC, BB_1 \in CA$) are internal angle bisectors of $\angle BAC$ and $\angle ABC$, respectively. Line A_1B_1 cuts the circumcircle of the triangle ABC at A_2 and B_2. Denote by R the midpoint of arc AB which does not contain C and P, Q midpoints of segments A_1B_1 and A_2B_2. Prove that $RP = RQ$.

Luis González
#2 Oct 23, 2011, 11:31 pm

Let I and I_a, I_b, I_c be the incenter and three excenters of $\triangle ABC. I$ and the circumcircle (O) of $\triangle ABC$ become orthocenter and 9-point circle of $\triangle I_aI_bI_c$. Thus, R is midpoint of $\overline{II_c}$ and the circumcircle (D) of $\triangle II_aI_b$ is the image of (O) under the homothety with center I_c and coefficient 2. Since $\overline{A_1B} \cdot \overline{A_1C} = \overline{A_1I_a} \cdot \overline{A_1I}$ and $\overline{B_1A} \cdot \overline{B_1C} = \overline{B_1I_b} \cdot \overline{B_1I}$, then A_1, B_1 have equal powers WRT (O) and (D), i.e. A_2B_2 is the radical axis of (O) and $(D) \Longrightarrow I_cQD \perp A_2B_2$.

Since $\angle B_1IA_1 = 120^\circ$, the quadrilateral CA_1IB_1 is cyclic and I is the midpoint of the arc A_1B_1 of its circumcircle $\Longrightarrow$ $IP \perp A_1B_1$ is the perpendicular bisector of $\overline{A_1B_1}$. So, $IPQI_c$ is a right trapezoid with $IP \parallel QI_c \Longrightarrow$ Perpendicular bisector of $\overline{PQ}$ passes through the midpoint R of $\overline{II_c} \Longrightarrow RP = RQ$.

skytin
#3 Oct 24, 2011, 12:15 pm

Solution :
Tangents to (ABC) thru A , B intersects at K
Let point X is such that X is on BC , angle BKX = B_1BC
Not hard to prove that BKX = B_1AB , KX || AA_1 and angle A_1AK = XA_1A
KA_1 = XA = B_1K
R is midpoint of KO , so use midline of trapezoid . done

Quick Reply

High School Olympiads

2 more geometry problems

votjnhtjensjnh

#1 Oct 22, 2011, 10:19 pm

1) Given an acute triangle $\triangle A_1B_1C_1$. A, B, C lie on B_1C_1, C_1A_1, A_1B_1 and satisfy the conditions: $\angle ABC = \angle A_1B_1C_1$, $\angle BAC = \angle B_1A_1C_1$, $\angle BCA = \angle B_1C_1A_1$. Let H and H_1 be the orthocenters of $\triangle ABC$ and $\triangle A_1B_1C_1$. Prove that H and H_1 are equidistant form O - the center of the circumscribed circle of $\triangle ABC$
2) (Maybe simple) $\triangle ABC$, $c \le b$,($AB = c$, $AC = b$) , $M, N \in AB, AC$ such that MN divides $\triangle ABC$ into 2 parts with the same areas. Find $minMN$

Luis González

#2 Oct 23, 2011, 12:45 am • 1

1) Posted several times before, e.g. Similar triangles, Bulgaria 1999, etc.

2) By cosine rule for $\triangle AMN$, we get $MN^2 = AM^2 + AN^2 - 2 \cdot AM \cdot AN \cos A$

$$MN^2 = AM^2 + AN^2 - 2 \cdot \frac{[ABC]}{\sin A} \cdot \cos A = AM^2 + AN^2 - 2[ABC] \cot A$$

But $AM^2 + AN^2 \ge 2 \cdot AM \cdot AN = 2[ABC] \csc A \Longrightarrow$

$$MN^2 + 2[ABC] \cot A \ge 2[ABC] \csc A \Longrightarrow MN \ge \sqrt{2[ABC] \tan \frac{A}{2}}.$$

High School Olympiads

Concurrence of angle bisectors

Source: Brazil MO #5

proglote
#1 Oct 20, 2011, 6:40 am

Let ABC be an acute triangle and H is orthocenter. Let D be the intersection of BH and AC and E be the intersection of CH and AB. The circumcircle of ADE cuts the circumcircle of ABC at $F \neq A$. Prove that the angle bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on BC.

Luis González
#2 Oct 20, 2011, 7:39 am • 2

Clearly, the reflection P of H about the midpoint M of $\overline{BC}$ is the antipode of A WRT $\odot(ABC)$. If PH cuts $\odot(ABC)$ again at F', then $\angle AF'H \equiv \angle AF'P = 90^\circ$, i.e. $F' \in \odot(ADE) \Longrightarrow F \equiv F'$. Since MD, ME are tangents of $\odot(AED)$ at D, E, then $FDHE$ is harmonic $\Longrightarrow FE \cdot HD = FD \cdot HE$, but $\triangle FBC$ and $\triangle FED$ are similar, due to $\angle EFD = \angle EAD = \angle BFC$ and $\angle FDE = \angle FAE = \angle FCB \Longrightarrow \frac{FB}{FC} = \frac{FE}{FD} = \frac{HE}{HD} = \frac{HB}{HC} \Longrightarrow$ bisectors of $\angle BFC$ and $\angle BHC$ meet on BC.

Zhero
#3 Oct 20, 2011, 8:55 am • 3

Let P_1 and P_2 be the points of intersection of the angle bisectors of $\angle BFC$ and $\angle BHC$ with BC, respectively. Since F is the Miquel point of quadrilateral $EDCB$, we have $\triangle FEB \sim \triangle FDC$, so

$$\frac{BP_1}{P_1C} = \frac{FB}{FC} = \frac{EB}{DC} = \frac{\frac{EB}{BC}}{\frac{DC}{BC}} = \frac{\sin\angle ECB}{\sin\angle DBC} = \frac{\sin\angle HCB}{\sin\angle HBC} = \frac{HB}{HC} = \frac{BP_2}{P_2C},$$

so $P_1 = P_2$, as desired.

dragon96
#4 Oct 20, 2011, 9:11 am • 1

Let H' be the reflection of H about BC and I the foot of A on BC. We start with two lemmas:

Lemma 1: AF, DE, and BC concur at a point Q.

Note that AFED, BCDE, and ACBF are all cyclic. Use the radical axis theorem.

Lemma 2: (B,C;Q,I) is a harmonic bundle.

From Ceva's Theorem with H as concurrency point, we get:

$$-\frac{IC}{IB} = -\frac{DA}{CD} \cdot \frac{EB}{AE}$$

Similarly, Menelaus' Theorem about Q yields:

$$\frac{QB}{QC} = -\frac{DA}{CD} \cdot \frac{EB}{AE}$$

$QC \qquad CD \quad AE$

Hence, we have that $\frac{QB}{QC} = -\frac{IC}{IB}$, as desired.

Proof:

Let G be the intersection point of the angle bisector of <BHC and BC. We wish to show that HC/HB = FC/FB, so the result would follow by the Angle Bisector Theorem. This is equivalent to showing that H'C/H'B = FC/FB, or that cyclic quadrilateral FBH'C is harmonic. We consider point A on its circumcircle and project the four points onto line BC. From Lemma 1, AF intersects BC at Q. Applying Lemma 2 directly gives us the desired.

hatchguy

#5 Oct 20, 2011, 10:04 am • 1

Clearly we have $\angle FBE = \angle FBA = \angle FCA = \angle FCD$ (1)

Also $\angle EFD = \angle EAD = \angle BAC = \angle BFC$ hence we have

$$\angle CFD + \angle EFC = \angle EFD = \angle BFC = \angle EFC + \angle BFE$$

and therefore $\angle BFE = \angle CFD$ (2)

From (1) and (2) we have $\triangle BFE \sim \triangle CFD => \frac{BF}{FC} = \frac{BE}{CD}$

But clearly, since $\triangle BHE \sim \triangle CHD$ we have $\frac{BE}{CD} = \frac{BH}{CH}$ and therefore

$$\frac{BF}{FC} = \frac{BE}{CD} = \frac{BH}{CH}$$

and we are done.

sunken rock

#6 Oct 21, 2011, 2:37 pm • 1

As before, H, M, P are respectively the orthocenter of $\triangle ABC$, midpoint of BC and antipode of A, it is well known that F, H, M, P are collinear and $BPCH$ is a parallelogram.
M being the midpoint of BC, it follows that the triangles $\triangle FBP, \triangle FCP$ have equal areas; as $sin\angle FBP = sin\angle FCP$, we shall get $BF \cdot BP = CF \cdot PC$; from the parallelogram property, $CP = BH, BP = CH$, hence $\frac{BF}{CF} = \frac{BH}{CH}$, done.

Best regards,
sunken rock

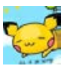
r1234

#7 Oct 21, 2011, 4:26 pm

Also there's a proof by inversion.Let $X \equiv DE \cap BC$.Now we invert this figure WRT A with power $AD.AB = k^2$.Note that F goes to X.So A, F, X are collinear.Now $BF = \frac{k^2 DX}{AD.AX}$ and

$CF = \frac{k^2 EX}{AE.AX}$. This gives $\frac{FB}{FC} = \frac{AD.XE}{AE.DX}$.Let $G \equiv AH \cap DE$.Since

$(XG; ED) = -1$, so we get $\frac{FB}{FC} = \frac{AD.EG}{AE.DG}$.Now its easy to get that $\frac{AD.EG}{AE.DG} = \frac{BH}{CH}$.Hence the result follows.



v_Enhance

#8 Oct 22, 2011, 9:16 am • 1

Note that the circumcircle of AED is simply the circle with diameter AH. Let R be the midpoint of AH. Then F is the reflection of A across RO by radical axis, where O is the circumcenter of ABC.

Now place everything on the complex plane with $o = 0$. Then $r = a + \frac{b+c}{2}$ and $f = \frac{r}{\bar{r}}\bar{a}$. We need to verify that $\left|\frac{f-b}{f-c}\right|^2 = \left|\frac{h-b}{h-c}\right|^2$ or

$$(f-b)(\bar{f}-\bar{b})(a+b)(\bar{a}+\bar{b}) = (f-c)(\bar{f}-\bar{c})(a+c)(\bar{a}+\bar{c})$$

We will now make the simplifying assumption that $r \in \mathbb{R}$; this can be done by simply rotating the triangle. Thus, $r = \bar{r}$ and $f = \bar{a}$. Now the LHS of the above is equal to

$$\begin{aligned} LHS &= (1/a-b)(a-1/b)(a+b)(1/a+1/b) \\ &= \text{kaboom} \\ &= 4-2ba-2/ba+2b/a+2a/b-b^2-1/b^2-a^2-1/a^2 \\ &= 4-a^2-1/a^2+(2/b-2b)a+(2b-2/b)/a-(b^2+1/b^2) \end{aligned}$$

Now the fact that $r = \bar{r}$ can be rearranged to

$$b-1/b = 1/c-c+(2/a-2a)$$

and also

$$b^2+1/b^2 = (c^2+1/c^2)+(4/a^2+4a^2-8)+(4/a-4a)(1/c-c) = (c^2+1/c^2)+(4/a^2+4a^2-8)-(4c-4/c)(1/a)-(4/c-4c)(a)$$

Finally,

$$\begin{aligned} LHS =& 4-a^2-1/a^2+(2/b-2b)a+(2b-2/b)/a-(b^2+1/b^2) \\ =& 4-a^2-1/a^2+(2c-2/c+4/a-4a)a+(2/c-2c+4a-4/a)/a \\ & -(c^2+1/c^2)-(4/a^2+4a^2-8)+(c-1/c)(4/a)+(c-1/c)(4a) \\ =& 4-a^2-1/a^2+((2c-2/c)+(4/c-4c))a+((2/c-2c)+(4c-4/c))/a \\ & +(4/a-4a)a+(4a-4/a)/a-(4/a^2+4a^2-8) \\ =& 4-a^2-1/a^2+(2/c-2c)a+(2c-2/c)/a-(c^2+1/c^2) \\ =& RHS \end{aligned}$$

and ~~I can now start on my English homework~~ we are done.

(For the record, the above calculations were all done by hand.)

SCP

#9 Nov 4, 2011, 12:28 am

> *dragon96 wrote:*
> Hence, we have that $\frac{QB}{QC} = -\frac{IB}{IC}$

It has to be this, you missed in the Ceva theorem.

cnyd

#10 Nov 6, 2011, 6:35 pm

Here is my solution.

From the chasing angle ,we find that ; $\angle BFH = \angle HBC$ and $\angle HFC = \angle HCB$

It means BC tangents of circumcircles of $\triangle BHF$ and $\triangle HFC$

Therefore,HF bisects BC.

If we use sinus theorem in $\triangle BHC$ we find that $\frac{BH}{} = \frac{BF}{}$

If we use sinus theorem in $\triangle BHC$,we find that $\frac{}{HC} = \frac{}{FC}$

The rest is obvious.

sayantanchakraborty

#11 Apr 7, 2014, 9:02 am

Trigo Bashing!!!

Let $\angle ABF = \theta$.Applying the sine rule in $\triangle AFE$ and $\triangle ABF$ we have $\frac{AF}{sin(C+\theta-B} = 2RcosA \Rightarrow \frac{sin\theta}{sinC+\theta-B} = cosA$
$\Rightarrow tan\theta = \frac{sin(B-C)}{2cosBcosC}$.
Now $\frac{BF}{CF} = \frac{sin(C+\theta)}{sin(B-\theta)} = \frac{sinC+cosCtan\theta}{sinB-cosBtan\theta} = \frac{cosB}{cosC}$.

jayme

#12 Apr 7, 2014, 7:17 pm

Dear Mathlinkers,
sorry, I rewrite the problem wrt the point A...

1. ABC an acute triangle
2. (O) the circumcircle of ABC
3. M the midpoint of BC
4. B', C' the feet of the B, C-altitudes of ABC
5. F the second point of intersection of (O) with the circumcircle of AB'C' (F, H and M are collinear, well known)

Prove : the F, H-inner bissector of FBC, HBC intersect on BC

An outline of my proof

1. (O') the symmetric of (O) wrt BC
2. X the midpoint of the arc BC which doesn't contai A
3. Y the second point of intersection of the H-inner bssector of HBC with (O')
4. by an angle chasing, we prove that F, H, X and Y are concyclic
5. by a converse of the three chords theorem, we are done...

Sincerely
Jean-Louis

IDMasterz

#13 Apr 8, 2014, 7:21 pm

Let the reflection of H on BC be H'. By direct application of radical axis theorem, AF, DE, BC concur at the harmonic conjugate of $AH \cap BC = T$ wrt BC, hence $BFCH'$ is an harmonic quadrilateral, and clearly the angle bisectors of $\angle BFC, \angle BH'C$ will meet at BC, but then again, the angle bisector of $\angle B'HC$ meets the angle bisector of $\angle BHC$ at BC as well due to reflexive properties.

SmartClown

#14 Apr 28, 2015, 4:14 am

By easy angle chase we get $\triangle FDB \sim \triangle FEC$ so we have $\frac{FB}{FC} = \frac{BD}{CE} = \frac{BH}{CH}$ so we are finished.

Quick Reply

High School Olympiads

3 concurrent lines

Reply

syk0526

#1 Oct 19, 2011, 1:40 pm

Let $\triangle ABC$ be an acute scalene triangle with its nine-point circle Γ . Let A_1 be the midpoint of BC, and A_2 be the foot of perpendicular from A to BC. Define B_1, C_1, B_2, C_2, respectively. It is well known that $A_1, B_1, C_1, A_2, B_2, C_2$ is on Γ. Two tangent lines from A_1, A_2 meet at point A_3. Define B_3, C_3 respectively. Prove that AA_3, BB_3, CC_3 are concurrent.

Luis González

#2 Oct 20, 2011, 12:16 am

This is a particular case of **Conway's theorem**, which states that: If $\mathcal{C}$ is a non-degenerate conic section and $\triangle ABC$ is not self-polar with respect to $\mathcal{C},$ then $\triangle ABC$ and its polar triangle with respect to $\mathcal{C}$ are in perspective. For a synthetic proof see this thread. Naturally, once we've proved the theorem for a circle $\mathcal{C},$ then it can be extented to any conic through homology. When $\mathcal{C}$ coincides with the 9-point circle (N) of $\triangle ABC,$ the perspector $U \equiv AA_3 \cap BB_3 \cap CC_3$ is the isotomic conjugate of $X_{1078}.$

General expression of the perspector

Quick Reply

High School Olympiads

Intersection of excircles and circumcircle

Source: Self-proposed

proglote

#1 Oct 19, 2011, 7:39 am

In $\triangle ABC$, let A_1 and A_2 be the intersection points of the excircle tangent to BC and the circumcircle of $\triangle ABC$, and similarly define B_1, B_2, C_1 and C_2. Let A_3 be the intersection of lines B_1B_2 and C_1C_2, and similarly define B_3 and C_3. Prove that lines AA_3, BB_3 and CC_3 concur at a single point.

Maybe it's well-known, but I have not seen it anywhere..

Luis González

#2 Oct 19, 2011, 8:23 am • 1

(O) is the circumcircle and $(I_a), (I_b), (I_c)$ are the three excircles tangent to BC, CA, AB at $D, E, F,$ respectively. EF, FD, DE cut BC, CA, AB at D', E', F'. Thus, $\ell \equiv \overline{D'E'F'}$ is the perspectrix of $\triangle ABC$ and $\triangle DEF$. Since A_1A_2 is the radical axis of (O) and (I_a), then $A_0 \equiv BC \cap A_1A_2$ has equal power WRT (O) and $(I_a) \Longrightarrow$ $A_0D^2 = \overline{A_0B} \cdot \overline{A_0B} \Longrightarrow A_0$ is the midpoint of $\overline{DD'}$, due to $(B, C, D, D') = -1$. Analogously, $B_0 \equiv CA \cap B_1B_2$ and $C_0 \equiv AB \cap C_1C_2$ are midpoints of $\overline{EE'}$ and $\overline{FF'} \Longrightarrow \overline{A_0B_0C_0}$ is Newton line of the complete quadrangle bounded by ℓ and the sidelines of $\triangle DEF \Longrightarrow \triangle A_3B_3C_3$ and $\triangle ABC$ are perspective through $\overline{A_0B_0C_0}$. Thus, by Desargues theorem, AA_3, BB_3, CC_3 concur.

jayme

#3 Oct 19, 2011, 12:11 pm

Dear Luis and Mathlinkers,
as I see the Clawson point is not easy to approach it synthetically... for the moment...
I see this result in
Yiu P., The Clawson point and excircles, Theorem 1.
The chase continues.
Sincerely
Jean-Louis

High School Olympiads

Parallels

Reply

Source: Own

jayme

#1 Oct 16, 2011, 5:46 pm

Dear Mathlinkers,
1. ABC a triangle,
2. I the incenter of ABC,
3. DEF the incentral triangle of ABC (I-cevian),
4. (O) the circumcircle of ABC,
5. Y, Z the points of intersection of EF with (O) (Y opposite to B),
6. (Ob), (Oc) the circumcircles of AIY, AIZ,
7. S the second point of intersection of YZ with (Ob).

Prouve: OcZ // ObS.

Sincerely
Jean-Louis

Luis González

#2 Oct 16, 2011, 11:41 pm

This is merely a consequence of Circle tangent to line. If I_a is the A-excenter of $\triangle ABC$, then I_aY, I_aZ are tangents of $\odot(AIY)$ and $\odot(AIZ)$ at $Y, Z \Longrightarrow$ there exists a a circle ω internally tangent to $\odot(AIY)$ and $\odot(AIZ)$ at $Y, Z.$ Thus, Y, Z are the exsimilicenters of $\omega \sim \odot(AIY)$ and $\omega \sim \odot(AIZ) \Longrightarrow YZ$ passes through the exsimilicenter of $\odot(AIY)$ and $\odot(AIZ) \Longrightarrow$ their radii O_bS and O_cZ are parallel.

Quick Reply

High School Olympiads

Prove that concurrent

Reply

Sutuxam
#1 Oct 16, 2011, 4:07 pm

Given triangle ABC, its centroid G and its symmedian point L. Let A_1, B_1, C_1 be the midpoints of BC, CA, AB, respectively.A_2, B_2, C_2 are reflections of G through B_1C_1, C_1A_1, A_1B_1, respectively. Prove that AA_2, BB_2, CC_2 concurrent at a point lies GL

This post has been edited 1 time. Last edited by Sutuxam, Oct 16, 2011, 6:40 pm

Luis González
#2 Oct 16, 2011, 10:02 pm

Let P_1, P_2, P_3 be the orthogonal projections of A_1, B_1, C_1 on B_1C_1, C_1A_1, A_1B_1. According to the topic Two perspective triangles (see post #2), AA_2, BB_2, CC_2 coincide with AP_1, BP_2, CP_3, respectively. So, in general, AA_2, BB_2, CC_2, concur at the Begonia point of G. AP_1, BP_2, CP_3 are clearly the isotomic cevians of the A-, B- and C- altitudes of $\triangle ABC$ $\Longrightarrow Q_1 \equiv BC \cap AP_1, Q_2 \equiv CA \cap BP_2$ and $Q_3 \equiv AB \cap CP_3$ are midpoints of the corresponding altitudes of the antimedial triangle $\triangle A_0B_0C_0 \Longrightarrow AA_2, BB_2, CC_2$ are Schawtt lines of $\triangle A_0B_0C_0$ concurring at its symmedian point L_0, i.e. the anticomplement of L (For a proof see the topic Lemoine point). Therefore, G, L, L_0 are collinear, such that $\overline{GL} : \overline{GL_0} = -1 : 2$.

Quick Reply

High School Olympiads

Tangents and parallel lines

Reply

Source: (mpdb)

borislav_mirchev

#1 Oct 16, 2011, 3:38 am

Let quadrilateral ABCD with intersection points of the diagonals P is inscribed in a circle k. M is the intersection point of the tangent to k through C and a line through P, parallel to BC. N is the intersection point of the tangent to k through D and a line through P, parallel to AD. Prove that CM=DN.

Luis González

#2 Oct 16, 2011, 4:00 am • 3

Let $E \equiv AD \cap BC$ and $F \equiv AB \cap DC$. EP is the polar of F with respect to $k \Longrightarrow$ Tangents of k at C, D meet at a point Q lying on EP. Thus, from $PM \parallel EC$ and $PN \parallel ED$, we deduce that $\triangle ECD$ and $\triangle PMN$ are homothetic with center $Q \Longrightarrow MN \parallel DC$. Since $\triangle QDC$ is isosceles with legs $QD = QC$, then $MNDC$ is an isosceles trapezoid with legs $CM = DN$.

borislav_mirchev

#3 Oct 16, 2011, 4:07 am

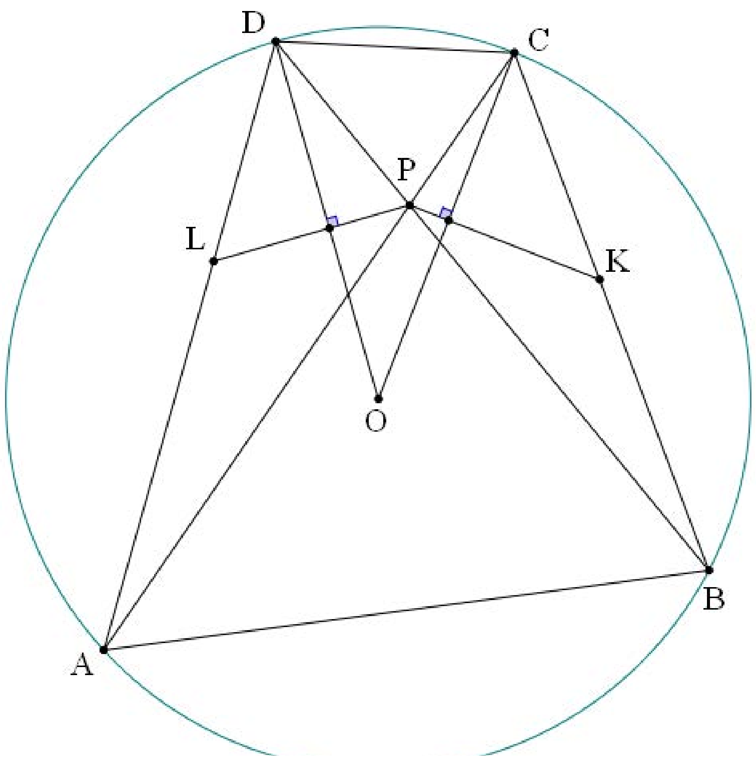
It is an excellent solution. Can the problem be solved without using polar properties?

estoyanovvd

#4 Oct 17, 2011, 9:10 pm

I think that is better one:
Prove that $PL = PK$.

Attachments:



erfan_Ashorion
#5 Oct 17, 2011, 9:35 pm • 1

> *borislav_mirchev wrote:*
> It is an excellent solution. Can the problem be solved without using polar properties?

yes!!
it is easy to show that $\triangle CPM \sim \triangle ABC$ other hand it is easy to show that $\triangle DPN \sim \triangle ABD$
so we have:
$CM/AB = PC/BC$ and $DN/AB = PD/AD$ and we know that $PC/PD = BC/AD$ ...!
so it is easy to show that $CM = DN$
so the problem is solve!
excuse me for my bad english!

erfan_Ashorion
#6 Oct 17, 2011, 10:12 pm • 1

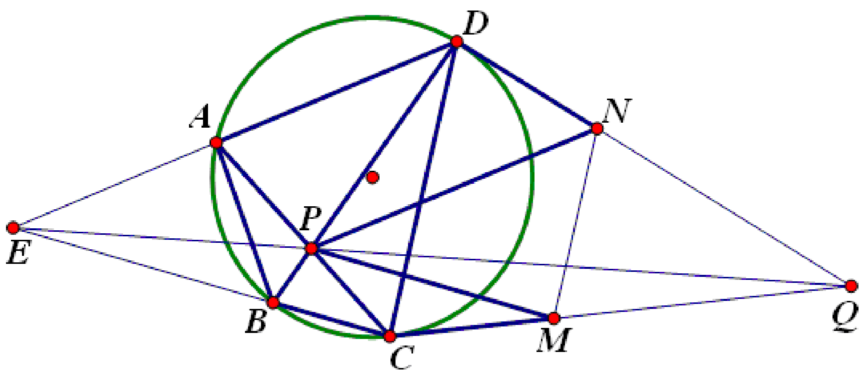
> *estoyanovvd wrote:*
> I think that is better one:
> Prove that $PL = PK$.

oh estoyanovvd you are make a problem nice!
i think that dont need to say solution but i say it!
let the intersection of tangent to circle at D and parallel line to AD meets each other at M and tangent to circle at C and a parallel line to BC meet each other at N it is easy to see that $DMPL$ and $CNPK$ are parallelogram!we know that by up solution $DM = CN$ and we know that $LP = DM$ and $PK = CN$ so $PL = PK$...!
so it is so easy problem!but nice exchange by estoyanovvd

yunxiu
#7 Oct 18, 2011, 5:59 pm

> *borislav_mirchev wrote:*
> It is an excellent solution. Can the problem be solved without using polar properties?

By Pascal theorem for $BDDACC$, $P = BD \cap AC$, $Q = DD \cap CC$, $E = DA \cap CB$ are collinear.
So $\frac{QN}{ND} = \frac{QP}{PE} = \frac{QM}{MC}$, hence $MN \parallel CD$, so $DM = CN$.
Attachments:

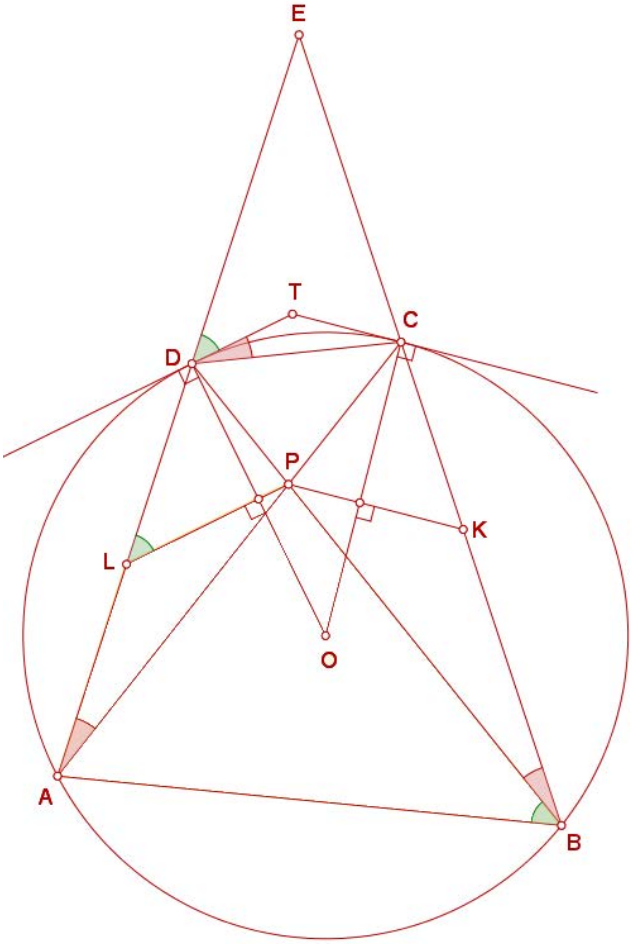


estoyanovvd
#8 Oct 19, 2011, 1:24 pm • 1

> *erfan_Ashorion wrote:*

oh estoyanovvd you are make a problem nice!

Special for you **erfan_Ashorion**!



$\angle DBC = \angle CDT$, $\angle ABC = \angle CDE \Rightarrow \angle ABD = \angle TDE$. But
$\angle TDE = \angle PLD \Rightarrow \Delta PLD \sim \Delta ABD \Rightarrow \dfrac{PL}{AB} = \dfrac{DP}{DA}$**(1)**.
Analogous $\dfrac{PK}{AB} = \dfrac{CP}{BC}$**(2)**. From **(1)** and **(2)** and $\Delta DAP \sim \Delta CBP$ we obtain $PL = PK$.
Now it is easy to prove from $TD = TC$ and $TD||PL, TC||PK$ that E, T and P lie on a same line(that is for **luisgeometra**'s solution), and even $CD||KL$

Quick Reply

High School Olympiads

Ellipse

Reply

hurricane

#1 Oct 15, 2011, 10:45 am

Let (E) be an ellipse with F_1, F_2 are two focals. (E) inscribed in a quadrilateral $ABCD$. Prove that $\angle AF_1B + \angle CF_1D = 180$

Luis González

#2 Oct 16, 2011, 12:21 am

We assume that $ABCD$ is convex. (E) touches AB, BC, CD, DA at $P, Q, R, S.$ According to this topic (post #3), the lines AF_1, BF_1, CF_1, DF_1 bisect $\angle SF_1P, \angle PF_1Q, \angle QF_1R, \angle RF_1S.$ Hence, by simple angle chase we get

$$\angle AF_1B + \angle CF_1D + \angle BF_1C + \angle AF_1D = 360^\circ \Longrightarrow$$

$$2(\angle AF_1B + \angle CF_1D) = 360^\circ \Longrightarrow \angle AF_1B + \angle CF_1D = 180^\circ.$$

Quick Reply

High School Olympiads

very nice (MOP98)

Reply

Sutuxam
#1 Oct 15, 2011, 9:01 pm

(MOP98) Let ω1 and ω2 be two circles of the same radius, intersecting at A and B. Let O be the midpoint of AB. Let CD be a chord of ω1 passing through O, and let the segment CD meet ω2 at P . Let EF be a chord of ω2 passing through O, and let the segment EF meet ω1 at Q. Prove that AB, CQ, EP are concurrent.

Luis González
#2 Oct 15, 2011, 10:55 pm • 1

Let CD and EF cut ω_2 and ω_1 again at P', Q', respectively. Since O is center of symmetry of $\omega_1 \cong \omega_2$, then O bisects $\overline{CP'}$ and $\overline{EQ'}$. Hence, $\overline{OA} \cdot \overline{OB} = \overline{OP} \cdot \overline{OP'} = -\overline{OP} \cdot \overline{OC}$ and $\overline{OA} \cdot \overline{OB} = \overline{OQ} \cdot \overline{OQ'} = -\overline{OQ} \cdot \overline{OE} \Longrightarrow$ $\overline{OP} \cdot \overline{OC} = \overline{OQ} \cdot \overline{OE} \Longrightarrow P, Q, E, C$ are concyclic. If $S \equiv EP \cap CQ$, then $\overline{SE} \cdot \overline{SP} = \overline{SC} \cdot \overline{SQ}$, i.e. S has equal power WRT ω_1 and $\omega_2 \Longrightarrow S \in AB$.

yetti
#3 Oct 16, 2011, 2:37 am • 1

C is outside and D inside of ω_2. E is outside and F inside of ω_1.
$\omega_1 \cong \omega_2 \Longrightarrow$ midpoint O of AB is also midpoint of $PD, QF \Longrightarrow PQDF$ is parallelogram $\Longrightarrow PF = QD \Longrightarrow$ $\angle QCP = \angle QCD = \angle FEP = \angle QEP \Longrightarrow CPQE$ is cyclic with circumcircle ω_3.
Pairwise radical axes AB, CQ, EP of $\omega_1, \omega_2, \omega_3$ are concurrent at their radical center.

Goutham
#4 Oct 18, 2011, 1:46 am

Let CD, EF intersect w_2, w_1 again at V, U respectively. Let CQ, EP intersect AB at S, T and let K be intersection point of UD and AB. Now, by butterfly, O is the midpoint of SK and by an half turn about O, K maps to T. So, $S \equiv T$ and we are done.

Quick Reply

High School Olympiads

MOP 1997,nice

Reply

Sutuxam

#1 Oct 15, 2011, 9:32 pm

(MOP97) Let P be a point in the plane of a triangle ABC. A circle Γ passing through P intersects the circumcircles of triangles P BC, P CA, P AB at A1, B1, C1, respectively, and lines P A, P B, P C intersect Γ at A3, B3, C3. Prove that: the lines A1A3, B1B3, C1C3 are concurrent

Luis González

#2 Oct 15, 2011, 10:07 pm • 1

> *Sutuxam wrote:*
> (MOP97) Let P be a point in the plane of a triangle ABC. A circle Γ passing through P intersects the circumcircles of triangles P BC, P CA, P AB at A1, B1, C1, respectively, and lines P A, P B, P C intersect Γ at A3, B3, C3. Prove that: the lines A1A2, B1B2, C1C2 are concurrent

Typo corrected in red color. See the topic Geometry problem (13).

Quick Reply

High School Olympiads

number of circles

Reply

hungnsl

#1 Oct 15, 2011, 8:46 pm

Given two circles (O) and (P) such that (P) is completely in the interior of (O). Suppose there exist n different circles $(C_1), (C_2), ..., (C_n)$ which are tangent to both (O) and (P), moreover, C_{i+1} is externally tangent to $C_i \forall 1 \le i \le n$, where (C_{n+1}) is (C_1). Find the minimum possible value of n

Luis González

#2 Oct 15, 2011, 9:42 pm

Inversion centered at one of the limiting points of $(O), (P)$ takes these circles into two concentric circles (U, r_1) and (U, r_2). The chain of n circles (C_i) is taken into a chain of n congruent circles (A_i) inscribed between $(U, r_1), (U, r_2)$ and tangent to each other externally $\Longrightarrow A_1, A_2, A_3, A_4, ..., A_n$ are obviously consecutive vertices of a regular n-gon with center U and circumradius $\frac{1}{2}(r_1 + r_2) \Longrightarrow n \ge 3.$

Quick Reply

High School Olympiads

MOP 98

Reply

Sutuxam

#1 Oct 15, 2011, 4:57 pm

The bisectors of angles A, B, C of triangle ABC meet its circumcircle again at the points K, L, M, respectively. Let R be an internal point on side AB. The points P and Q are defined by the conditions: RP is parallel to AK and BP is perpendicular to BL; RQ is parallel to BL and AQ is perpendicular to AK. Show that the lines KP, LQ, MR concur.

Luis González

#2 Oct 15, 2011, 8:03 pm • 1

MR cuts the circumcircle (O) again at $U. \angle AUR = \angle MBR = \angle MAR.$ But, $\angle AQR = \frac{1}{2} \angle ACB = \angle MAR \Longrightarrow \angle AQR = \angle AUR \Longrightarrow A, R, U, Q$ are concyclic. Hence, $\angle AUQ = \angle ARQ = \angle ABL = \angle AUL \Longrightarrow Q, L, U$ are collinear. Similarly, B, R, U, P are concyclic and P, K, U are collinear $\Longrightarrow$ Lines KP, LQ, MR concur at a point U lying on the circumcircle (O).

Quick Reply

High School Olympiads

midpoint

Locked

elegant

#1 Oct 14, 2011, 6:13 pm

Two circles meet at points A and B. A line through A intersects the first circle at C and the second at D. Let M and N be the midpoints of the arcs BC and BD not containing A, and let K be the midpoint of the segment CD. Prove that the angle ∠MKN is right. (It can be assumed that C and D lie on different sides of A.)

Luis González

#2 Oct 14, 2011, 11:02 pm

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=22201
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=5433
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=17322
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=105320
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=399142

High School Olympiads

Concyclic points

Reply

Source: by TianYu Wang

Manolescu

#1 Oct 14, 2011, 8:31 pm

In an acute triangle ABC, B < C
I the incenter , O the circumcenter , H the orthocenter
D the tangential point at BC on circle I
If AO is parallel to HD , OD intersects AH at E
F the midpoint of CI
Prove that: E,F,I,O are concyclic

Thanks in advance ! Luis, and can you offer me some articles about cross-ratio

Luis González

#2 Oct 14, 2011, 10:16 pm • 1

Dear Manolescu, next time use meaningful subjects. Post subjects like "Help me", "Luis González see this", etc, do not describe the purpose of the problem. So, subject edited. The problem was posted before here.

As for the cross ratio article, you may see http://www.artofproblemsolving.com/Forum/viewtopic.php?f=50&t=161310

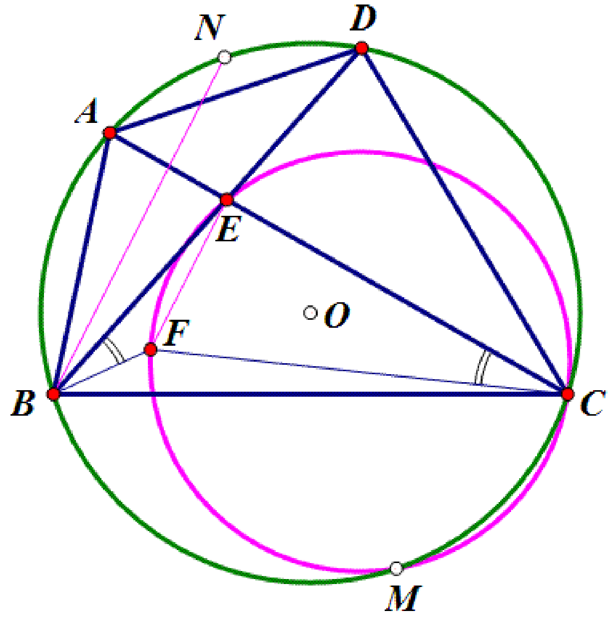
Quick Reply

High School Olympiads

lym
#1 Oct 14, 2011, 1:21 am • 1

Quadrilateral $ABCD$ is circumscribed on $\odot O$, $AC \cap BD = E$, M is midpoint of $\widehat{ABD}$ and N is the another midpoint of $\widehat{AD}$. Now give a point F is satisfied taht $\angle EBF = \angle ECF$ and $EF \parallel BN$. Then prove that C, E, F, M are concyclic

Attachments:



Luis González
#2 Oct 14, 2011, 7:51 am

Let G, H be the midpoints of $\overline{BC}, \overline{AD}$. Isogonal conjugate K of F WRT $\triangle BEC$ lies on perpendicular bisector OG of $\overline{BC}$. $\angle CEK = \angle BEF = \angle DBN = \angle ACN \Longrightarrow EK \parallel CN$, i.e. EK is perpendicular to CM at T. From $\triangle EDH \sim \triangle ECG$ and $\triangle CET \sim \triangle DMH$, we get

$$\frac{CG}{DH} = \frac{CE}{ED}, \ \frac{DH}{DM} = \frac{CT}{CE} \Longrightarrow \frac{CG}{DM} = \frac{CT}{ED}$$

Since $\angle TCG = \angle EDM$, then $\triangle TCG \sim \triangle EDM$ by SAS $\Longrightarrow \angle TGC = \angle DME$. But $TGKC$ is cyclic, due to the right angles at G and T, so $\angle CKT = \angle DME$. Hence, $\angle EFC = \angle CKT + \angle DMC = \angle EMC \Longrightarrow C, E, F, M$ are concyclic.

skytin
#3 Oct 14, 2011, 6:33 pm

Solution :
Reflect C wrt line FE and get point C'
Easy to see that F is on (BC'E) , so there exist only one point F , such that angle FBE = ECF and FE || BN
Let point X is on FE , such that MX perpendicular to EB
Easy to see that X is orthocenter of MBE
Angle MXE = 90 + BEF = 90 + NCE = 180 - ECM , so X is on (CME)
Easy to see that angle XBE = EMX = ECX , so X = F . done

High School Olympiads

Intersect on circle

Reply

buratinogigle

#1 Oct 13, 2011, 11:18 pm

Let ABC be a triangle. D is a point on BC such that $\angle BAD = \angle ACB$. E is a point on circumcircle (ACD) such that $DE \parallel AC$. P is a point on AE. PB cuts DE at F. Prove that AF and CP intersect on circumcircle (ACD).

Note that, this is generalization of a problem in the post Two perpendicular lines

Luis González

#2 Oct 14, 2011, 12:01 am • 1

$DE \parallel AC$ is unnecessary. Since $\angle BAD = \angle ACB$, then the circumcircle of $\triangle ADC$ is tangent to AB at A. Let AF cut $\odot(ADC)$ again at Q. By Pascal theorem for the degenerate cyclic hexagon $AAQCDE$, the intersections $B \equiv AA \cap DC$, $F \equiv AQ \cap DE$ and $P^* \equiv QC \cap AE$ are collinear $\Longrightarrow P$ and P^* coincide. Thus, AF and CP meet at a point Q lying on $\odot(ADC)$.

Quick Reply

High School Olympiads

Gergonne, Nagel and Incenter.

Reply

Virgil Nicula

#1 Oct 13, 2011, 1:10 am • 2

Let ABC be a triangle with incircle $w=(I)$. Denote $D\in BC\cap w$ and Γ(**Gergonne**) , N(**Nagel**)

points of $\triangle ABC$. Prov that $\boxed{\widehat{ID\Gamma}\equiv\widehat{IDN}\iff\frac{a}{b+c}=\frac{s-a}{s}}=\tan\frac{\pi}{8}$, where $2s=a+b+c$.

This post has been edited 1 time. Last edited by Virgil Nicula, Oct 14, 2011, 10:35 pm

yetti

#2 Oct 13, 2011, 5:51 pm • 1

K is foot of perpendicular from A to BC. AI cuts BC at X. Excircle (I_a) in $\angle A$ touches BC at E and $N\in AE$. ID cuts (I) again at F. M is common midpoint of BC, DE.

$(I)\sim(I_a)$ are centrally similar with center A and coefficient $\frac{s-a}{s}\Longrightarrow F\in AE$ and $\overline{NA}=-2\overline{IM}=-\overline{FE}\Longrightarrow$ $\overline{FA}=-\overline{NE}$.

Since $DF\perp DE$, DIF bisects $\angle ND\Gamma\equiv\angle NDA\Longleftrightarrow$ cross ratio $(A,N,F,E)=-1$ is harmonic $\Longleftrightarrow$

$$\frac{\overline{AF}}{\overline{AE}}=-\frac{\overline{NF}}{\overline{NE}}=\frac{\overline{AE}-2\overline{AF}}{\overline{AF}}\Longleftrightarrow\left(\frac{\overline{AF}}{\overline{AE}}\right)^2+2\frac{\overline{AF}}{\overline{AE}}-1=0\Longrightarrow 0<\frac{s-a}{s}=\frac{\overline{AF}}{\overline{AE}}=\sqrt{2}-1=\tan\frac{\pi}{8}.$$

Cross ratio $(K,X,D,E)=-1$ is always harmonic. See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=435414 for one way to show it.

Therefore, $(A,N,F,E)=-1\Longleftrightarrow NX\parallel FID\parallel AK\Longleftrightarrow\frac{s-a}{s}=\frac{\overline{AF}}{\overline{AE}}=-\frac{\overline{NF}}{\overline{NE}}=\frac{\overline{NF}}{\overline{FA}}=\frac{\overline{XI}}{\overline{IA}}=\frac{a}{b+c}$.

Luis González

#3 Oct 13, 2011, 6:43 pm • 1

Let A^* be the reflection of A about BC. Thus, DI bisects $\angle\Gamma D\mathrm{N}\Longleftrightarrow\mathrm{N},D$ and A^* are collinear. Barycentric coordinates of A^*, N and D WRT $\triangle ABC$ are given by $\mathrm{N}\ (b+c-a:c+a-b:a+b-c), D\ (0:a+b-c:c+a-b)$ and $A^*(-a^2:a^2+b^2-c^2:a^2+c^2-b^2)$. These points are collinear $\Longleftrightarrow$

$$\begin{bmatrix}-a^2 & a^2+b^2-c^2 & a^2+c^2-b^2\\ b+c-a & c+a-b & a+b-c\\ 0 & a+b-c & c+a-b\end{bmatrix}=0$$

$\Longleftrightarrow 2a(c-b)[(b+c-a)^2-2a^2]=0\Longleftrightarrow$ Either $b=c$ or $2a^2=(b+c-a)^2$

$$\Longleftrightarrow a=\sqrt{2}(s-a)\Longleftrightarrow\frac{a}{b+c}=\frac{s-a}{s}=\frac{\sqrt{2}}{\sqrt{2}+2}=\tan\frac{\pi}{8}.$$

Virgil Nicula

#4 Oct 14, 2011, 10:40 pm • 1

My proof I. Suppose w.l.o.g. that $b>c$ and denote : the orthocenter H ; $S\in BC\cap AH$; $X\in AN\cap w$; $L\in AN\cap BC$ and $K\in BC$ for which $NK\perp BC$. Show easily that $\frac{SD}{s-a}=\frac{KL}{s-a}=\frac{DK}{2a-s}=\frac{b-c}{a}$ and $NK=\frac{h_a(s-a)}{s}$. Therefore, $\widehat{ID\Gamma}\equiv\widehat{IDN}\iff\widehat{XDA}\equiv\widehat{XDN}\iff$ $\tan\widehat{XDA}=\tan\widehat{XDN}\iff\frac{SD}{SA}=\frac{KD}{KN}\iff\frac{(b-c)(s-a)}{ah_a}=\frac{(b-c)(2a-s)}{a}\cdot\frac{s}{h_a(s-a)}\iff$

$(s-a)^2 = s(2a-s) \iff [(b+c)-a]^2 = [a+(b+c)]\,[3a-(b+c)] \iff$
$a^2+2a(b+c)-(b+c)62 = 0 \iff \boxed{\frac{a}{b+c} = \sqrt{2}-1}$.

My proof II. Suppose w.l.o.g. that $b > c$. Is well-known that $X \in \overline{ANL}$. Show easily that $LS = \frac{s(b-c)}{a}$. Thus $\left\|\begin{array}{c}\widehat{XDA} \equiv \widehat{XDN} \\ \\ DX \perp DL\end{array}\right\| \iff$ the division $(A, N; X, L)$ is harmonically $\iff$ the division $(S, K; D, L)$ is harmonically
$\iff \frac{DS}{DK} = \frac{LS}{LK} \iff DS^2 = DK \cdot LS \iff (b-c)^2(s-a)^2 = (2s-a)(b-c)^2 s \iff$
$(s-a)^2 = s(2s-a) \iff \ldots \iff \frac{a}{b+c} = \sqrt{2}-1$.

Quick Reply

High School Olympiads

Distances

Source: by Dong Yi Wei

Manolescu

#1 Oct 11, 2011, 8:27 pm

Give a triangle ABC, AP is the tangent line of its circumcircle (P lies on BC), PEF intersects AC, AB at E F, respectively with the condition that <APE=<EPC , BE intersects CF at D, define that d(X,YZ) is the distance between point X and line YZ. Prove that d(D,AP)=2 d(D,BC)

Luis González

#2 Oct 11, 2011, 10:45 pm • 1

The result is true for any point P on the extensions of BC. Let AD cut BC and EF at M, Q, respectively. From the harmonic cross ratio (A, D, Q, M), we get

$$\frac{DA}{DM} \cdot \frac{QM}{QA} = \frac{DA}{DQ} \cdot \frac{QM}{AM} = 2$$

$$\frac{DA}{QA} = \frac{\text{d}(D, AP)}{\text{d}(Q, AP)}, \ \frac{QM}{DM} = \frac{\text{d}(Q, BC)}{\text{d}(D, BC)} \Longrightarrow \frac{\text{d}(D, AP)}{\text{d}(Q, AP)} \cdot \frac{\text{d}(Q, BC)}{\text{d}(D, BC)} = 2$$

Since $\text{d}(Q, AP) = \text{d}(Q, BC)$, then $\text{d}(D, AP) = 2 \cdot \text{d}(D, BC)$.

erfan_Ashorion

#3 Oct 18, 2011, 3:15 pm

nice solution luis but i think that you can make your solution so easier!
let the altitude from D to PC meet it on X and the altitude from D to PA meet it on Y..!
we know that $\frac{\sin APD}{\sin DPM} = \frac{\frac{DY}{PD}}{\frac{DX}{PD}}$ if $DX \cdot 2 = DY \to \frac{\sin APD}{\sin DPM} = 2$ →we must to proof $\frac{PA}{PM} \cdot 2 = \frac{AD}{DM}$ we know that $\frac{PA}{PM} = \frac{AQ}{QM}$ so we must to proof
$\frac{DA}{DM} \cdot \frac{QM}{QA} = 2$ oh it is really easy!
excuse me luis for my Interference

High School Olympiads

Two middles

Reply

Source: (mpdb)

borislav_mirchev
#1 Oct 11, 2011, 3:03 am

Let PA and PB are the tangents from a point P to a circle k. From A is drawn a diameter AC. M is the middle of the arc AC not containing B. D is the intersection point of BM and AC. N is the intersection point AC and MP. Prove that N is the middle of AD.

Luis González
#2 Oct 11, 2011, 3:23 am • 2

Let the tangent of k at M cut PA, PB at Q, R. k becomes the incircle of $\triangle PQR \Longrightarrow$ Lines PM, QB and RA concur at the Gergonne point of $\triangle PQR$. Thus, if $E \equiv MB \cap PQ$, then the cross ratio (P, Q, A, E) is harmonic $\Longrightarrow$ Pencil $M(N, Q, A, D)$ is harmonic. So, $MQ \parallel AD$, implies that N is the midpoint of $\overline{AD}$.

borislav_mirchev
#3 Oct 11, 2011, 3:26 am

It is a short and elegant solution but can it be solved without cross ratio?

sunken rock
#4 Oct 11, 2011, 9:00 am • 1

MP is the symmedian of $\triangle AMB$, so, if $\{K\} \in PM \cap AB$, then $\frac{AK}{BK} = \frac{AM^2}{MB^2}$. Applying Menelaos to $\triangle ABD$ with the transversal $\overline{MNK}$ it will remain to prove $AM^2 = MB \cdot MD$, which is true (inversion of pole M and power AM^2 sends the circle to its diameter AC, i.e. $B \to D$, or, even easier, see that $\triangle ABM \sim \triangle DAM$).

Best regards,
sunken rock

armpist
#5 Oct 11, 2011, 9:32 am • 1

Dear Borislav and MLs

see post #11 (and other solutions too) in
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=74386

M.T.

MariusBocanu
#6 Oct 11, 2011, 6:37 pm • 1

Denote S the midpoint of AB. We have that MS, MP are isogonal in $\triangle AMB$, so $\widehat{AMN} = \widehat{DMS}$, so is enough to prove that AS is symmedian in $\triangle AMD$, but a well-known property of symmedian tells us, that in a $\triangle ABC$, X belongs to the symmedian of A if and only if $d(X, AB).AC = d(X, AC).AB$, but in our case, denoting $\widehat{BAC} = x$, we have $\widehat{ABM} = 45$, $\widehat{ADM} = 135 - x$, so all we have to do is to prove that $SU.AM = SV.MD$, which is easy only using the law of sines, (we denote by U, V the projections of S on MD, AM).
REMARK. Indeed, the solution given by **luisgeometra** is nicer and shorter, but i tried to prove withot cross ratio.

erfan_Ashorion

#7 Oct 18, 2011, 2:03 pm • 2

oh nice problem!
lemma:
$$\frac{\sin\angle AMP}{\sin\angle BMP}=\frac{AM}{BM}$$
proof:
oh it is really easy lemma and i sure that you can proof it

proof of problem:
we want to proof that:
$$\frac{\sin\angle AMP}{\sin\angle BMP}=\frac{MD}{AM}$$
so by lemma we want to proof:
$MD\cdot MB=MA^2\longrightarrow$ we want to proof AM is tangent to circumecircle of $\triangle ABD$
we know that $\angle ABD=\angle MAC=45$ so AM is tangent to circumecircle of $\triangle ABD$

Quick Reply

High School Olympiads

Diameter and a middle

Source: (mpdb)

borislav_mirchev

#1 Oct 10, 2011, 4:19 am • 1

From point P are drawn the tangents PA and PB to a circle k. AC is a diameter of k. A line through B perpendicular to CP intersects AC and the ray PA at the points D and E respectively. Prove that D is the middle of BE.

Luis González

#2 Oct 10, 2011, 8:48 am • 2

M is the midpoint of $\overline{AB}$. CM, CP are the median and symmedian of $\triangle ABC$ issuing from $C \Longrightarrow \angle DCM = \angle BCF$. But from $BF \perp CP$ and $AB \perp BC,$ we have $\angle DBM = \angle BCF \Longrightarrow \angle DBM = \angle DCM \Longrightarrow BCDM$ is cyclic. Now, since $EAFC$ is also cyclic, due to the right angles at $A, F,$ we have $\angle AED = \angle FCD = \angle MCB = \angle MDB \Longrightarrow MD \parallel AE \Longrightarrow$ D is midpoint of BE.

estoyanovvd

#3 Oct 11, 2011, 2:54 pm • 1

It is easy to prove that $\Delta ABE \sim \Delta TAB \Rightarrow \frac{BE}{AB} = \frac{AB}{TA}$ **(1)**,

$\Delta ABD \sim \Delta TCB \Rightarrow \frac{BD}{AB} = \frac{CB}{TC}$ **(2)**.

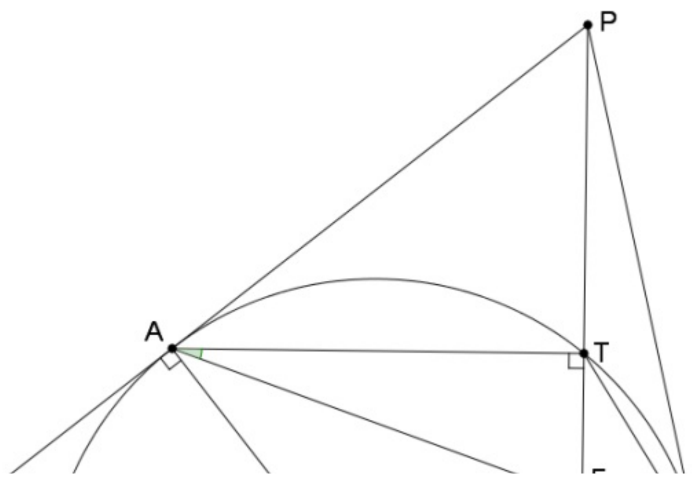
From (1) and (2) $\frac{BE}{AB} = \frac{AB.TC}{TA.CB} = \frac{TA.CB + TB.AC}{TA.CB} = 1 + \frac{TB.AC}{TA.CB}$ **(3)**. We used the theorem of Ptolemy for $ACBT$.

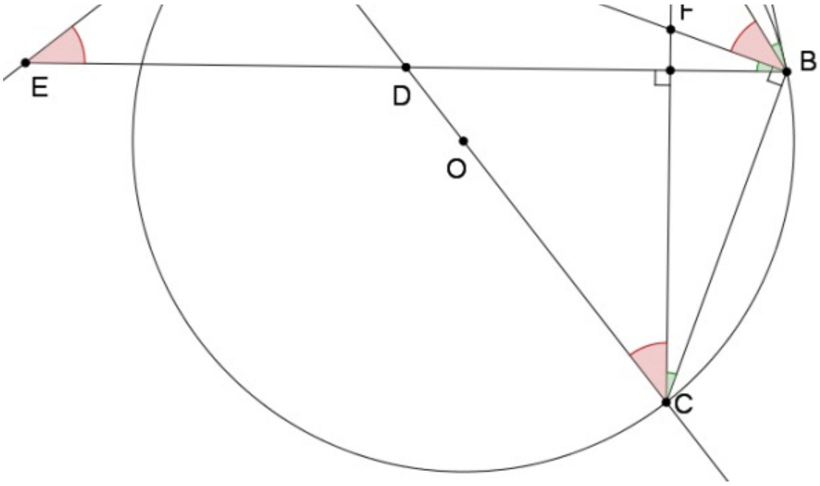
Now $\Delta ATC \sim \Delta PAC \Rightarrow \frac{TA}{AC} = \frac{AP}{PC} \Rightarrow TA = \frac{AP.AC}{PC}$**(4)**,

$\Delta CBP \sim \Delta BTP \Rightarrow \frac{CB}{BT} = \frac{PC}{PB} \Rightarrow CB = \frac{BT.PC}{PA}$**(5)**.

From **(4)** and **(5)** we obtain $TA.CB = \frac{AP.AC}{PC} . \frac{BT.PC}{PA}$ and from **(3)** $\frac{BE}{ED} = 1 + 1 = 2$. Done.

Attachments:





vittasko

#4 Oct 17, 2011, 12:55 pm

Because of AB is the polar of P, with respect to the circle (k), we have that the points $P,\ T,\ F,\ C$ are in harmonic conjugation, where $T \equiv (k) \cap PC$ and $F \equiv AB \cap PC$ and so, the pencil $A.PTFC$ is also harmonic.

Hence, because of $AT \perp PC \Longrightarrow AT \parallel BE$ we conclude that $DB = DE$, where $D \equiv AC \cap BE$ and the proof is completed.

Kostas Vittas.

skytin

#5 Oct 17, 2011, 9:09 pm

Solution :
Let O is circumcenter of (ABC)
O is midpoint of AC
Let N is midpoint of CP
Not hard to prove that POCN ~ BAFD , so D is midpoint of FB . done

estoyanovvd

#6 Oct 18, 2011, 1:22 am

> *skytin wrote:*
> Solution :
> Let O is circumcenter of (ABC)
> O is midpoint of AC
> Let N is midpoint of CP
> Not hard to prove that POCN ~ BAFD , so D is midpoint of FB . done

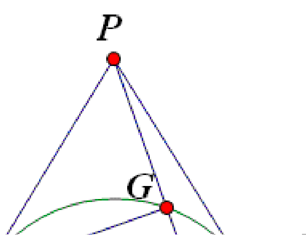
???

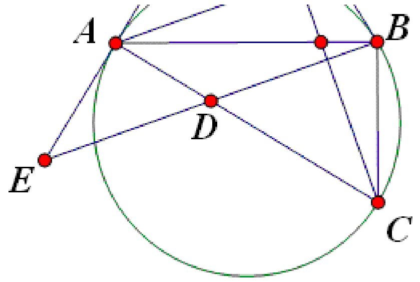
yunxiu

#7 Oct 18, 2011, 6:20 pm

Let $G = PC \cap k$, then $AG \parallel BE$. Because $ACBG$ is harmonic, so $A(EB;CG) = A(AB;CG) = -1$, hence $BD = DE$.

Attachments:





dgreenb801

#8 Nov 12, 2011, 9:10 pm

Since AC is a diameter, $AC \perp PE$.
Let O be the center of the circle.
Let X be the intersection of OP and AB.
Note that $AB \perp PO$ and $AB \perp BC$ (since AC is a diameter), so $BC \parallel XO$.
Note that $\triangle BXO \sim \triangle APO$, so
$\frac{AP}{AO} = \frac{BX}{XO}$, or
$\frac{AP}{2AO} = \frac{BX}{2XO}$, or
$\frac{AP}{AC} = \frac{BX}{BC}$
Thus, $\triangle APC \sim \triangle BXC$, so $\angle APC = \angle BXC$.
Then $\angle BDC = \angle APC = \angle BXC$.
Thus $BXDC$ is cyclic, so $\angle XDC = 90$.
Thus, $DX \parallel AE$, so since X is the midpoint of AB, D is the midpoint of BE.

armpist

#9 Nov 14, 2011, 4:40 am

Dear MLs

It turns out to be an old configuration,
a 'spiralled' out of recognition Archimedes Prop. 2
(see attachment).

This post is dedicated to Jean-Louis Ayme and Feuerbach's brother Ludwig.

Attachments:
Archi Prop. #2.doc (24kb)

estoyanovvd

#10 Jan 20, 2012, 12:02 pm

> *estoyanovvd wrote:*
>
> It is easy to prove that $\Delta ABE \sim \Delta TAB \Rightarrow \frac{BE}{AB} = \frac{AB}{TA}$ **(1)**,
>
> $\Delta ABD \sim \Delta TCB \Rightarrow \frac{BD}{AB} = \frac{CB}{TC}$ **(2)**.
>
> From (1) and (2) $\frac{BE}{BD} = \frac{AB.TC}{TA.CB} = \frac{TA.CB + TB.AC}{TA.CB} = 1 + \frac{TB.AC}{TA.CB}$ **(3)**. We used the theorem of Ptolemy for $ACBT$.
>
> Now $\Delta ATC \sim \Delta PAC \Rightarrow \frac{TA}{AC} = \frac{AP}{PC} \Rightarrow TA = \frac{AP.AC}{PC}$**(4)**,
>
> $\Delta CBP \sim \Delta BTP \Rightarrow \frac{CB}{BT} = \frac{PC}{PB} \Rightarrow CB = \frac{BT.PC}{PA}$**(5)**.
>
> From **(4)** and **(5)** we obtain $TA.CB = \frac{AP.AC}{PC} . \frac{BT.PC}{PA}$ and from **(3)** $\frac{BE}{BD} = 1 + 1 = 2$. Done.

Quick Reply

High School Olympiads

Coaxal circles

buratinogigle

#1 Oct 9, 2011, 11:58 pm • 1

Let ABC be a triangle and point P. $A'B'C'$ is pedal triangle of P with respect to triangle ABC. O is circumcenter of triangles ABC, (O') is circumcircle of triangle $A'B'C'$. PA', PB', PC' intersects (O') again at A_1, B_1, C_1, respectively. Assume that P, O, O' are collinear. Prove that circumcirles $(PAA_1), (PBB_1), (PCC_1)$ have a common point other than P.

This is generalization of a problem in the post A nice problem.

Luis González

#2 Oct 10, 2011, 5:11 am • 1

Let PA, PB, PC cut B_1C_1, C_1A_1, A_1B_1 at X, Y, Z, respectively. From the cyclic quadrilaterals $B'C'B_1C_1$ and $PB'AC'$ we have $\angle PB_1X = \angle PC'B'$ and $\angle XPB_1 = \angle AC'B'$. Thus, $\angle PB_1X + \angle XPB_1 = \angle PC'A = 90^\circ$, i.e. $PX \perp B_1C_1$. Likewise, $PY \perp C_1A_1$ and $PZ \perp A_1B_1$, i.e. $\triangle XYZ$ is pedal triangle of P WRT $\triangle A_1B_1C_1$. Inversion with center P and power equal to the power of P WRT (O') carries A_1, B_1, C_1 into A', B', C' and the circles $\odot(AB'C')$, $\odot(BC'A'), \odot(CA'B')$ into the lines B_1C_1, C_1A_1, A_1B_1, respectively. Therefore, X, Y, Z are the inverses of $A, B, C \Longrightarrow$ $\odot(PAA_1), \odot(PBB_1), \odot(PCC_1)$ are taken into the lines XA', YB', ZC', respectively $\Longrightarrow$ Circumcircle (U) of $\triangle XYZ$ is the inverse of (O) under the referred inversion $\Longrightarrow P, U, O, O'$ are collinear $\Longrightarrow$ Pedal triangle $\triangle XYZ$ and circumcevian triangle $\triangle A'B'C'$ of P WRT $\triangle A_1B_1C_1$ are perspective, even homothetic $\Longrightarrow XA', YB', ZC', UO'$ concur at the homothetic center of $\triangle XYZ \sim \triangle A'B'C'$. Hence, their inverses $\odot(PAA_1), \odot(PBB_1), \odot(PCC_1)$ are coaxal with common radical axis the line OP.

P.S. Locus of points P satisfying the coaxiality is then the M'Cay cubic of ABC.

Quick Reply

High School Olympiads

Isotomcomplement of K

Reply

jayme

#1 Aug 8, 2008, 8:05 pm

Dear Mathlinkers,
let ABC be a triangle, Q, Q' the first and second Brocard's points of ABC, K the Lemoine's point of ABC and K# the isotomcomplément of K wrt ABC.
Prove that K# is the midpoint of QQ'.

Sincerely
Jean-Louis

Luis González

#2 Oct 9, 2011, 10:55 pm

We'll use barycentric coordinates WRT $\triangle ABC$. Let δ_{AB} be the circle passing through A, B tangent to BC. Its equation satisfies $A(1:0:0)$ and $B(0:1:0) \Longrightarrow \delta_{AB} \equiv a^2yz + b^2zx + c^2xy - rz(x+y+z) = 0$. Since δ_{AB} is tangent to $x = 0$ at $B(0:1:0)$, then $z(a^2y - ry - rz) = 0$ has double root equal to zero $\Longrightarrow r = a^2$.

$$\Longrightarrow \ \delta_{AB} \equiv a^2yz + b^2zx + c^2xy - a^2z(x+y+z) = 0$$

Similarly, we get the equation of the circle δ_{BC} passing through B, C tangent to CA and the circle δ_{CA} passing through C, A tangent to AB.

$$\delta_{BC} \equiv a^2yz + b^2zx + c^2xy - b^2x(x+y+z) = 0$$

$$\delta_{CA} \equiv a^2yz + b^2zx + c^2xy - c^2y(x+y+z) = 0$$

Thus, circles $\delta_{AB}, \delta_{BC}, \delta_{CA}$ concur at the 1st Brocard point $\Omega_1\left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right)$. Analogously, the another triad of circles δ_{BA}, δ_{CB}, δ_{AC} concur at the 2nd Brocard point $\Omega_2\left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}\right)$.

Midpoint of $\overline{\Omega_1\Omega_2}$ is then $\left(\frac{1}{b^2} + \frac{1}{c^2} : \frac{1}{c^2} + \frac{1}{a^2} : \frac{1}{a^2} + \frac{1}{b^2}\right)$, which indeed coincide with the coordinates of the isotomcomplement of the symmedian point $K(a^2 : b^2 : c^2)$.

lym

#3 Oct 10, 2011, 2:34 am

Let K' be the isotomic conjugate of K with relation to $\triangle ABC$, $\triangle A_1B_1C_1$ the First brocard triangle of $\triangle ABC$. M, N, L respectively the midpoints of BC, AC, AB, M', N', L' respectively the midpoints of B_1C_1, A_1C_1, A_1B_1. Then AA_1 pass through K' and easy to prove the midpoint of QQ', $M\ M'$ are collinear, it means that MM', NN', LL' are concurrent at the midpoint of QQ' and use congruent we can get $M'N = M'L$. $\angle NM'L = \angle BA_1C$, so $MM' \parallel AA_1$ i.e. $MM' \parallel AK'$, so the midpoint of QQ' is the isotomcomplément of K.

PS. Let O and H respectively be the circumcenter and orthocenter of $\triangle ABC$, then $OK \parallel HK'$. I let Fang-jh posted my proof is here: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=182853

TelvCohl

#4 Oct 22, 2014, 10:05 pm • 1

Notice that the isotomcomplement of P is the center of the conic which is tangent to BC, CA, AB at $AP \cap BC, BP \cap CA$, $CP \cap AB$. Since the Brocard ellipse is an conic tangent to BC, CA, AB at the vertices of the cevian triangle of the Symmedian point, so the isotomcomplement of the Symmedian point is the center of the Brocard ellipse which is the midpoint of the first Brocard point and the second Brocard point .

This post has been edited 1 time. Last edited by TelvCohl, Aug 21, 2015, 2:40 am

Quick Reply

High School Olympiads

problem

Reply

robinson123

#1 Oct 9, 2011, 9:04 pm

Given quadrilateral $ABCD$ inscribed (O). AB meets CD at P. AD and BC meets at Q. Prove that $\overrightarrow{OP}\cdot\overrightarrow{OQ}=R^2$.

Luis González

#2 Oct 9, 2011, 10:49 pm

Let $E\equiv AC\cap BD$. Since the pencil $Q(A,B,E,P)$ is harmonic, it follows that QE is the polar of P with respect to (O,R) $\Longrightarrow QE$ is perpendicular to OP at S. If $M,N\equiv QE\cap(O)$, then PM,PN are tangents of $(O)\Longrightarrow$

$$OM^2=R^2=|OS|\cdot|OP|=|OP|\cdot|OQ|\cdot\cos\angle POQ=\overrightarrow{OP}\cdot\overrightarrow{OQ}.$$

High School Olympiads

Common midpoint

Source: me

yetti

#1 Aug 18, 2011, 5:47 pm • 2

G is centroid, (I) is incircle and (N) 9-point circle of $\triangle ABC$. GI cuts BC at P. Radical axis of $(I),(N)$ cuts BC at Q. Show that midpoint of BC is also midpoint of PQ.

jayme

#2 Aug 18, 2011, 6:07 pm • 2

Dear Mathlinkers,
nice problem...
we have to consider the Nagel point Na....
Sincerely
Jean-Louis

Luis González

#3 Aug 19, 2011, 12:09 am • 4

Very nice problem Vladimir, alas, I don't have a synthetic approach yet. We use barycentric coordinates with respect to ABC. Equations of (I) and (N) are given by

$$(I) \equiv a^2yz+b^2zx+c^2xy-(x+y+z)((s-a)^2x+(s-b)^2y+(s-c)^2z)=0$$

$$(N) \equiv a^2yz+b^2zx+c^2xy-\tfrac{1}{2}(x+y+z)(S_Ax+S_By+S_Cz)=0$$

We obtain the equation of the radical axis τ of $(I),(N)$ by subtracting their equations

$$\tau \equiv [S_A-2(s-a)^2]x+[S_B-2(s-b)^2]y+[S_C-2(s-c)^2]z=0 \Longrightarrow$$

$$\tau \equiv \frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}=0 \Longrightarrow Q \equiv BC \cap \tau \equiv \left(0:\frac{1}{a-b}:\frac{1}{a-c}\right).$$

$$IG \equiv (b-c)x+(c-a)y+(a-b)z=0 \Longrightarrow P \equiv IG \cap BC \equiv (0:a-b:a-c)$$

Coordinates of P,Q reveal that they are isotomic points in the sideline BC, i.e. midpoints of $\overline{BC}$ and $\overline{PQ}$ coincide.

yetti

#4 Aug 19, 2011, 3:29 am

AI cuts BC at K and circumcircle (O) of $\triangle ABC$ again at X. OX is perpendicular bisector of BC, cutting it at its midpoint A'.
Incircle (I) and A-excircle (I_a) touch BC at D, D_a. I is Nagel point of the complementary triangle $\Longrightarrow L \equiv IG \cap AD_a$ is Nagel point of $\triangle ABC$. By van Aubel theorem, $\dfrac{AL}{LD_a}=\dfrac{p-b}{p-a}+\dfrac{p-c}{p-a}=\dfrac{a}{p-a}$. In addition, $AL=2IA'$ and $AL \parallel IA'$ $\Longrightarrow \dfrac{PD_a}{PA'}=\dfrac{LD_a}{IA'}=\dfrac{b+c-a}{a}$.

J is reflection of I in BC. Radical axis of $(I),(N)$ is their single common tangent at the Feuerbach point. By the problem http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=215210, X, J, Q are collinear. $XB=XC=XI \Longrightarrow$ by Ptolemy for cyclic $ABXC$, $\dfrac{XA}{\ldots}=\dfrac{b+c}{\ldots}$. By inversion with center X and power XI^2,

$$\frac{QD}{QA'} = \frac{JD}{XA'} = \frac{DI}{XA'} = \frac{KI}{XK} = \frac{XI}{XK} - 1 = \frac{XA}{XI} - 1 = \frac{b+c-a}{a}.$$

As a result, $\frac{D_aA'}{PA'} = 1 - \frac{PD_a}{PA'} = 1 - \frac{QD}{QA'} = \frac{DA'}{QA'}$. Since midpoint A' of BC is also midpoint of DD_a, it follows that it is also midpoint of PQ .

skytin

#5 Aug 19, 2011, 6:01 pm

You can find another solution with using first Fontene theorem

lym

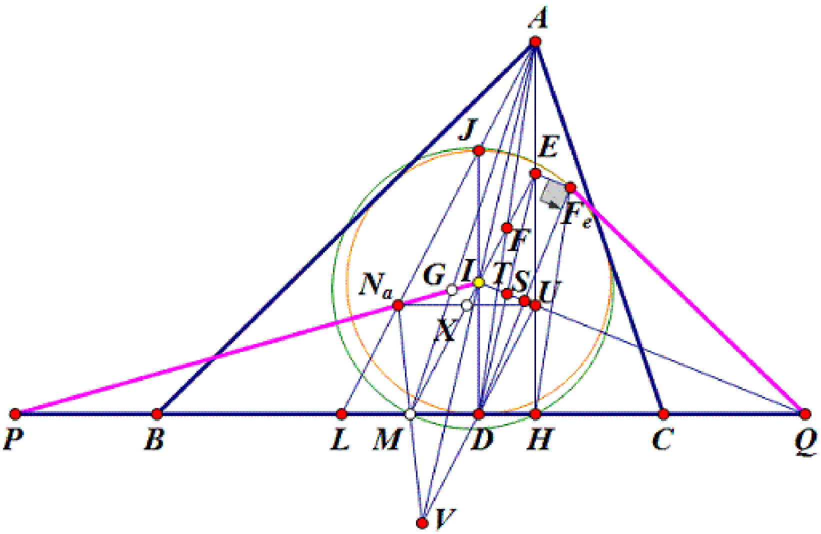
#6 Aug 22, 2011, 1:55 am

Let M be the midpoint of $BC, ID \perp BC, AH \perp BC, MI$ cut AH, AD at $E, F, DI \cap (I) = J, AJ$ cut BC, IG at L, N_a. F_e is the feuerbach point of $\triangle ABC, IQ$ cut AH, DF_e, DE at U, S, T, DU intersect AI at V, UN_a cut IM at X.

Solution

As well known, $MD = ML$, so $IM \parallel JL$, so F is the midpoint of AD, so $AIDE$ is a parallelogram. So $\angle DEH = \angle DF_eH = \frac{\angle C - \angle B}{2}$, hence DEF_eH is a circle, so $DF_e \perp EF_e$,but $IQ \perp DF_e$. So $IU \parallel EF_e$ and S is the midpoint DF_e, so T is the midpoint DE. Then $IEDU$ is a parallelogram, so $DU = DV, AI = IV$. So G is the centroid of $\triangle AN_aV, V, M, N_a$, are collinear $MV = MN_a$. But $LN_a \parallel IE \parallel UV$, So $LN_a = DV = DU$, so $LDUN_a$ is a parallelogram, so $UN_a \parallel PQ$. Cuz M is the midpoint of DL, so X is the midpoint of UN_a, so M is the midpoint of PQ. Done .

Attachments:



Luis González

#7 Oct 6, 2011, 10:06 pm • 3

Okay, I found a proof without barycentric coordinates.

Incircle (I) and A-excircle touch BC at D, E, respectively, M is the midpoint of BC and $V \equiv AI \cap BC$ is the foot of the internal bisector of $\angle BAC$. $\triangle A_0B_0C_0$ is the anticomplementary triangle of $\triangle ABC$ with incenter N_a (Nagel point of ABC), i.e. $N_a \in GI$, such that $\overline{GI} : \overline{GN_a} = -1 : 2$. $A_0N_a \parallel AI$ and $AI = \frac{1}{2}A_0N_a \Longrightarrow MN_a$ passes through the reflection A^* of A about I. Thus if $U \equiv A_0N_a \cap BC$, then $N_a(A, A^*, I, U)$ is harmonic $\Longrightarrow (E, M, P, U) = -1$.

On the other hand, radical axis of $(I), (N)$ is their common tangent through the Feuerbach point F_e. Tangent from V to (I), different from BC, touches (I) at F. If $X \equiv QF_e \cap VF$, then $\triangle DFF_e$ becomes the X-extouch triangle of $\triangle XQV$. Since $M \in FF_e$ (well-known), it follows that $(D, M, Q, V) = -1$, so the reflections E, M, Q^*, U of D, M, Q, V about M are also harmonically separeted, i.e. $(E, M, Q^*, U) = -1 \Longrightarrow P \equiv Q^*$.

Quick Reply

High School Olympiads

perpendiculars

elegant

#1 Oct 4, 2011, 6:55 am

In a circle with radii R is inscribed a quadrilateral with perpendicular diagonals. From the intersecting point of the diagonals are drawn perpendiculars to the sides of the quadrilateral. (a) Prove that the feet of these perpendiculars P1, P2, P3, P4 are vertices of the quadrilateral that is inscribed and circumscribed. (b) Prove the inequalities: 2r1 ≤ R1√2 ≤ R, where R1 and r1 are radiuces respectively of the circumcircle and inscircle to the quadrilateral P1P2P3P4. When does equalities holds?

Luis González

#2 Oct 5, 2011, 6:51 am

Label the given quadrilateral $ABCD$ with circumcircle $(O,R).$ Diagonals AC and BD are perpendicular at P and P_1,P_2,P_3,P_4 are the projections of P on $AB,BC,CD,DA.$ Quadrilaterals $PP_1AP_4, PP_2BP_1, PP_3CP_2, PP_4DP_3$ are clearly cyclic. Thus, $\angle PP_1P_2=\angle PBC=\angle PAD=\angle PP_1P_4 \Longrightarrow PP_1$ bisects $\angle P_4P_1P_2.$ Similarly PP_2, PP_3 bisect $\angle P_1P_2P_3, \angle P_2P_3P_4 \Longrightarrow P_1P_2P_3P_4$ has incircle $(P).$

Further, $\angle P_4P_1P_2+\angle P_2P_3P_4=\angle PAD+\angle PBC+\angle PCB+\angle PDA=180^\circ \Longrightarrow P_1P_2P_3P_4$ is also cyclic with circumcircle $(U).$ If $M \equiv PP_1 \cap CD,$ then $\angle CP_3P_2=\angle CBD=\angle PP_1P_2$ implies that $M \in \odot(P_1P_2P_3)$ and from $\angle BPP_1=\angle PAB=\angle PDC,$ we get that $\triangle MPD$ is isosceles with apex $M,$ i.e. M is the midpoint of $CD.$ Similarly, (U) cuts AB,BC,DA again at their midpoints $K,L,N.$ Thus, (U) is circumcircle of the rectangle $KLMN \Longrightarrow$

$$4{R_1}^2=KN^2+KL^2=\tfrac{1}{4}(AC^2+BD^2) \le 2R^2 \Longrightarrow \sqrt{2}R_1 \le R.$$

On the other hand, by Fuss theorem for the bicentric quadrilateral $P_1P_2P_3P_4,$ we get

$$\frac{1}{{r_1}^2}=\frac{1}{(R_1+UP)^2}+\frac{1}{(R_1-UP)^2} \ge \frac{2}{{R_1}^2} \Longrightarrow 2r_1 \le \sqrt{2}R_1.$$

Obviously, the equalities hold if and only if $P \equiv U \equiv O,$ i.e. $ABCD$ is a square.

Quick Reply

High School Olympiads

squares

elegant

#1 Oct 4, 2011, 4:44 am

On the sides BC, CA, AB of acute angled triangle ABC externally are constructed squares which centers are denoted by M, N, P. Find k : [MNP] ≥ k [ABC].

Luis González

#2 Oct 4, 2011, 8:06 am

According to the topic Triangle, we have $[MNP]=\frac{1}{8}(a^2+b^2+c^2)+[ABC]$

Thus, from $a^2+b^2+c^2 \ge 4\sqrt{3}[ABC]$ (Weitzenbock) we get $\frac{[MNP]}{[ABC]} \ge \frac{\sqrt{3}}{2}+1.$

High School Olympiads

common point

Reply

elegant

#1 Oct 4, 2011, 5:45 am

The opposite sides AB and CD of inscribed in the circle k quadrilateral ABCD intersects at a point M. Tangent MN (N belongs to k) is parallel to the diagonal AC. NB intersects AC at the point P. Prove that the lines AN, DB and PM intersects at a common point.

Luis González

#2 Oct 4, 2011, 7:02 am

The condition $MN \parallel AC$ is unnecessary. By Pascal theorem for the degenerate cyclic hexagon NNBDCA, the intersections $M \equiv NN \cap DC, P \equiv NB \cap CA$ and $Q \equiv BD \cap AN$ are collinear, in other words, AN, DB and PM concur at Q, as desired.

dragon96

#3 Oct 4, 2011, 7:21 am

Big Diagram

Alternative solution that uses MN||AC:

Solution

skytin

#4 Oct 4, 2011, 5:46 pm

Hint :
Let (MDN) intersect DB at points D , X , MBNX is parallelogram

Quick Reply

High School Olympiads

CP=2AP

Source: (mpdb)

borislav_mirchev
#1 Oct 3, 2011, 3:33 am

Triangle ABC is inscribed in a circle k. The height from C intersects AB and k at the points D and E respectively. F is the middle of the segment BD. A line l through D, perpendicular to EF intersects the segment AC at the point P. Prove that CP=2AP.

Luis González
#2 Oct 3, 2011, 9:34 am • 1

$$\frac{\overline{PC}}{\overline{PA}}=-\frac{DC}{DA}\cdot\frac{\sin\widehat{PDC}}{\sin\widehat{PDA}}=-\frac{DC}{DA}\cdot\frac{\cos\widehat{DEF}}{\cos\widehat{DFE}}=-\frac{DC}{DA}\cdot\frac{DE}{DF}=-2.$$

The last equality follows from $p(D,k)=DC\cdot DE=DB\cdot DA=2\cdot DF\cdot DA.$

armpist
#3 Oct 3, 2011, 4:55 pm • 2

Dear Borislav and MLs

This problem is a play on a parallelogram property
(see attachment).

Attachments:
435505.doc (24kb)

borislav_mirchev
#4 Oct 4, 2011, 12:23 am

This problem is definitely not for this section - it is too easy but I think it is useful to see different approaches to solve it. You can see more ways.
http://www.math10.com/f/viewtopic.php?f=10&t=7341
http://dxdy.ru/topic49705.html
The problem can be solved also using trigonometry.
Hope you like the problem.

sunken rock
#5 Oct 7, 2011, 3:09 am

I was not able to open the suggested links (by the way, if a problem is known to have solution(s), why is it posted under 'Open Questions' Section??) hence I come with this idea:
Take Q the midpoint of CD, see that, obvious, $\triangle ADQ\sim\triangle EDF$ and it is well known that PD passes through R, the midpoint of AQ. Next, apply Menelaos to $\triangle QAC$ with the transversal PRD and get the required relation.

Best regards,
sunken rock

borislav_mirchev
#6 Oct 7, 2011, 3:34 am

Because when I posted it I didn't know a solution. Indeed I'm not good in geometry

Because when I posted it I didn't know a solution. Indeed I'm not good in geometry.

estoyanovvd

#7 Oct 7, 2011, 3:14 pm

> *armpist wrote:*
>
> ...This problem is a play on a parallelogram property .

They are really beautiful similar rectangles!

Quick Reply

High School Olympiads

An easy Geometry

Locked

undefeatedturk

#1 Oct 2, 2011, 7:29 pm

ABCD is cyclic quadrilateral. M and N are midpoints of AC and BD respectively. AC is bisector of angle BMD. Prove that; BD is bisector of angle ANC.

Luis González

#2 Oct 2, 2011, 9:14 pm

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=50002
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=6557
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=204810
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=328964

High School Olympiads

Concurrent 1

Reply

dazai

#1 Oct 2, 2011, 2:06 pm

Let ABC be a triangle inscribed the circle of center O with the altitudes AH, BK, CL. Denote by

A_0, B_0, C_0 respectively, the midoints of AH, BK, CL. The circle of center I inscribed in triangle

ABC, touches the sides BC, CA, AB at D, E, F respectively. Prove that the four lines

A_0D, B_0E, C_0F, OI are concurrent

yetti

#2 Oct 2, 2011, 3:51 pm

> dazai wrote:
> Let ABC be a triangle inscribed the circle of center O with the altitudes AH, BK, CL. Denote by
>
> A_0, B_0, C_0 respectively, the midoints of AH, BK, CL. The circle of center I inscribed in triangle
>
> ABC, touches the sides BC, CA, AB at D, E, F respectively. Prove that the four lines
>
> A_0D, B_0E, C_0F, OI are concurrent

A', B', C' are midpoints of BC, CA, AB. $(I_A), (I_B), (I_C)$ are excircles in $\angle A, \angle B, \angle C$. Excircle (I_A) touches BC at D'. AI cuts BC at X.
$AX, A'B', FD$ concur at P. $B'P = B'A = B'C$ and $A'P = A'D = A'D' \Longrightarrow \angle APC, \angle DPD'$ are right $\Longrightarrow$ $\angle DPX = \frac{1}{2}\angle C$ and $\angle HPX = \frac{\pi}{2} - \angle CPH = \frac{\pi}{2} - \angle CAH = \angle C \Longrightarrow PD, PD'$ bisect $\angle HPX$ internally/externally $\Longrightarrow$ cross ratio $(D, D', X, H) = -1$ is harmonic.
Line pencil $I_A(D', D, X, H) \equiv I_A(D', D, A, H)$ is harmonic and $AH \parallel I_AD' \Longrightarrow I_AD$ cuts AH at its midpoint A_0.
Likewise, I_BE, I_CF cut BK, CL at their midpoints B_0, C_0.
I is orthocenter and (O) 9-point circle of $\triangle I_AI_BI_C \Longrightarrow$ reflection J of I in O is its circumcenter.
$\triangle I_AI_BI_C \sim \triangle DEF$ with their circumcircles $(J), (I)$ are centrally similar $\Longrightarrow I_ADA_0, I_BEB_0, I_CFC_0, JOI$ concur at their similarity center.

Luis González

#3 Oct 2, 2011, 8:18 pm

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=331144
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=265276
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=230481
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=48453
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=42412

Quick Reply

High School Olympiads

Prove perpendicular

Reply

shorin

#1 Oct 2, 2011, 9:12 am

Let M, N be two points interior circle (O) such that O is the midpoint of MN. Let S is an arbitrary point on (O).Denote by E, F, G the intersection of line SM, SN, SO with (O). EF intersect MN at K such that K lie on rays NM, FE. SO intersect (O) at G. Prove that KG, GS are perpendicular.

Moderator edit: Do not change the problem once a solution is given. If you want to discuss other problems then submit a new thread. The original proposition was

> *shorin wrote:*
> Let ABC be a triangle and H be a orthocenter, O be a center of circumcircle of its. Take the point I lie on BC. The circle with diameter AI intersect (O) at K and KH intersect (O) at M. The line through K and perpendicular OI intersect (O) at T. The line through O and perpendicular TM intersect BC at N. Prove that AM, MN are perpendicular.

This post has been edited 2 times. Last edited by shorin, Oct 2, 2011, 1:55 pm

Luis González

#2 Oct 2, 2011, 12:10 pm • 1

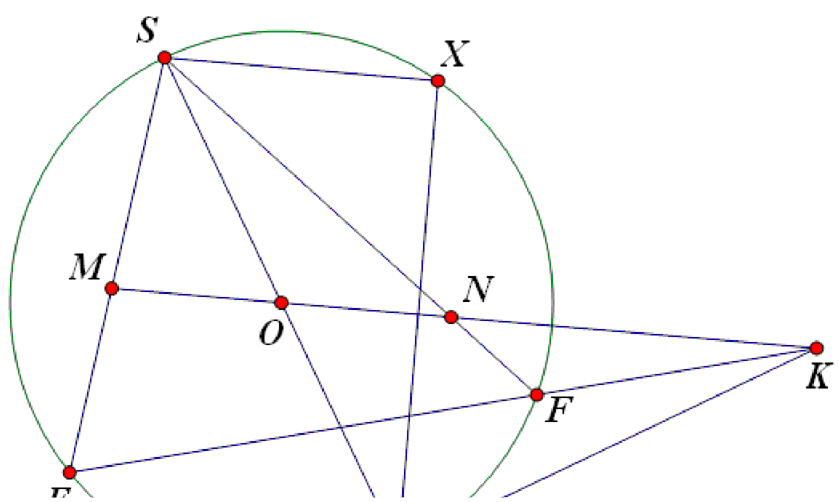
D is the foot of the A-altitude and AD cuts (O) again at the reflection P of H about D. AO cuts (O) again at S and SI obviously cuts (O) again at K. Let $R \equiv MK \cap BC$ and PR cuts (O) again at T^*. Since AO, AH are isogonals WRT $\angle BAC$, then $BC \parallel PS \Longrightarrow \angle KIR = \angle KSP = \angle KT^*R \Longrightarrow K, T^*, I, R$ are concyclic and I is the midpoint of the arc KRT^* since BC bisects $\angle PRH$. Consequently, OI is the perpendicular bisector of $\overline{KT^*} \Longrightarrow T \equiv T^*$. Hence, BC and the perpendicular bisector of $\overline{MT}$ meet at the midpoint N of the arc MT of $\odot(RMT) \Longrightarrow$ $\angle AMN = \angle TMN + \angle TMA = \angle PRD + \angle RPD = 90^\circ$.

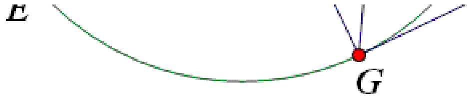
yunxiu

#3 Oct 2, 2011, 6:21 pm • 1

Let X on the circle and $XG \perp MN$, than $\angle SXG = 90^\circ$, so $SX \parallel MN$. As $MO = NO$, $S(XG; FE) = S(XO; NM) = -1$, so $XFGE$ is Harmonic.
Let PG, PX tangent to the circle, then P is on EF, and by the symmetry of G, X, P is also on MN, so $P = K$, hence $KG \perp SG$.

Attachments:





mahanmath

#4 Oct 2, 2011, 8:38 pm • 1

Dear **yunxiu** , here is what I said in PM

I worked backward and prove that :

If b, c lie on unit circle and bc intersects the line tangent to unit circle at 1 at p and op intersect lines passing trough $-1, b$ and $-1, c$ at m, n then m, n are symmetric wrt o .

First we calculate p :

It lies on $bc \implies p + \bar{p}bc = b + c$

It lies on tangent line to unit circle at $1 \implies p + \bar{p} = 2$

$$\implies p = \frac{b+c-2bc}{1-bc}.$$

Now we calculate m :

It lies on the line passing trough $-1, b \implies m - \bar{m}b = b - 1$ (#)

It lies on op so has same argument as $p \implies \frac{m}{\bar{m}} = \frac{p}{\bar{p}} = \frac{\frac{b+c-2bc}{1-bc}}{\frac{b+c-2}{bc-1}} = \frac{b+c-2bc}{2-b-c}$

So from (#) we have $\bar{m}(\frac{b+c-2bc}{2-b-c} - b) = b - 1$ but $\frac{b+c-2bc}{2-b-c} - b = \frac{(b-1)(b-c)}{2-b-c}$ and it show the symmetry that we wanted !

Quick Reply

High School Olympiads

PA^2+PB^2=4PT^2

Source: Pan African Olympiad 2008

WakeUp

#1 Oct 1, 2011, 10:02 pm

Let C_1 be a circle with centre O, and let AB be a chord of the circle that is not a diameter. M is the midpoint of AB. Consider a point T on the circle C_2 with diameter OM. The tangent to C_2 at the point T intersects C_1 at two points. Let P be one of these points. Show that $PA^2+PB^2=4PT^2$.

Luis González

#2 Oct 1, 2011, 11:21 pm • 1

Label $C_1 \equiv (O,R)$ and $C_2 \equiv (U,r)$. By Apollonius theorem, we have

$$PA^2+PB^2=2(PM^2+MA^2)\ ,\ R^2+PM^2=2(PU^2+r^2) \Longrightarrow$$

$$PA^2+PB^2=4PU^2+4r^2-2R^2+2MA^2$$

But by Pythagoren theorem in $\triangle OAM$, we obtain $AM^2=R^2-4r^2$. Hence

$$PA^2+PB^2=4PU^2+4r^2-2R^2+2R^2-8r^2=4(PU^2-r^2)=4PT^2.$$

High School Olympiads

If DEF is equilateral then AD, BE and CR are concurrent

Reply

Source: Pan African Olympiad 2009

WakeUp
#1 Oct 1, 2011, 9:32 pm

Point P lies inside a triangle ABC. Let D, E and F be reflections of the point P in the lines BC, CA and AB, respectively. Prove that if the triangle DEF is equilateral, then the lines AD, BE and CF intersect in a common point.

Luis González
#2 Oct 1, 2011, 10:17 pm

Clearly, if DEF is equilateral, then the pedal triangle of P WRT ABC is also equilateral, i.e. P is an isodynamic point of ABC. Thus, according to the problem http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=304719, AD,BE,CF concur at the isogonal conjugate of P, namely a Fermat point of ABC.

P.S. For a generalization, see Reflections of a point M.

r1234
#3 May 4, 2012, 11:01 pm

Since $\triangle DEF$ is equilateral we conclude that P is the first Isodyanamic point of $\triangle ABC$.We will prove that AD, BE, CF concur at the Fermat point of $\triangle ABC$ i.e at P', the isogonal conjugate of P wrt $\triangle ABC$.For this note that D is isogonal conjugate of A wrt $\triangle P'BC$.We know that AP' bisects $\angle BP'C$.Hence $A, P'D$ are collinear. So done.

jred
#4 Mar 13, 2015, 1:32 pm

Wow, I don't know what it is " *isodynamic point* ", however this problem can be solved by some basic knowledge.
Easily we see $AE = AP = AF$, and also $DE = DF$, thus $\triangle ADE$ is congruent to $\triangle ADF$, so we have DA is the bisector of $\angle EDF$, similarly we get BE, CF are bisectors of $\angle DEF, \angle DFE$ respectively. hence we know lines AD, BE and CF are concurrent at the incenter of $\triangle DEF$.

This post has been edited 1 time. Last edited by jred, Mar 13, 2015, 1:33 pm
Reason: minor typo

Quick Reply

High School Olympiads

Turkey NMO 2001 Problem 1

Reply

Source: Turkey NMO 2001 Problem 1

mestavk

#1 Sep 30, 2011, 2:11 pm

Let $ABCD$ be a convex quadrilateral. The perpendicular bisectors of the sides $[AD]$ and $[BC]$ intersect at a point P inside the quadrilateral and the perpendicular bisectors of the sides $[AB]$ and $[CD]$ also intersect at a point Q inside the quadrilateral. Show that, if $\angle APD = \angle BPC$ then $\angle AQB = \angle CQD$

yetti

#2 Oct 1, 2011, 9:56 am

P-isosceles $\triangle DPA \sim \triangle BPC$ are similar. Let points $B_0 \in (PB, C_0 \in (PC$ be such that $\triangle DPA \cong \triangle B_0PC_0$ are congruent $\Longrightarrow B_0C_0 \parallel BC$. P is circumcenter of AB_0C_0D with $DA = B_0C_0 \Longrightarrow$ it is isosceles trapezoid with $AB_0 \parallel DC_0$. Let $Q_0 \equiv AC_0 \cap B_0D$ be its diagonal intersection. Let S be circumcenter of $\triangle DPA$. Undefine Q. S-isosceles $\triangle ASP \cong \triangle PSD$ and Q_0-isosceles $\triangle AQ_0B_0 \sim \triangle C_0Q_0D$ are all similar $\Longrightarrow$ segment $|SQ_0|$ is spirally similar to segments $|PB_0|, |PC_0|$ with centers A, D and the same coefficient $\frac{AQ_0}{AB_0} = \frac{AS}{AP} = k = \frac{DS}{DP} = \frac{DQ_0}{DC_0}$. Since $\frac{\overline{BP}}{\overline{BB_0}} = \frac{\overline{CP}}{\overline{CC_0}}$, these spiral similarities take $B \in PB_0, C \in PC_0$ to the same point $Q \in SQ_0$, such that $\angle QAB = \angle SAP = \angle PDS = \angle QDC$ and $\frac{AQ}{AB} = k = \frac{DQ}{DC} \Longrightarrow \triangle AQB \sim \triangle CQD$ are Q-isosceles and similar $\Longrightarrow Q$ is intersection of perpendicular bisectors of $|AB|, |CD|$ and $\angle AQB = \angle CQD$.

Luis González

#3 Oct 1, 2011, 10:33 am

If $\angle APD = \angle BPC$, then $\angle APC = \angle DPB \Longrightarrow \triangle APC$ and $\triangle DPB$ are congruent by SAS criterion, since $PA = PD$ and $PC = PB$. Therefore, $AC = BD$, which implies that $\triangle AQC$ and $\triangle BQD$ are congruent by SSS criterion, due to $QA = QB$ and $QC = QD \Longrightarrow \angle AQC = \angle BQD \Longrightarrow \angle AQB = \angle CQD$.

Quick Reply

High School Olympiads

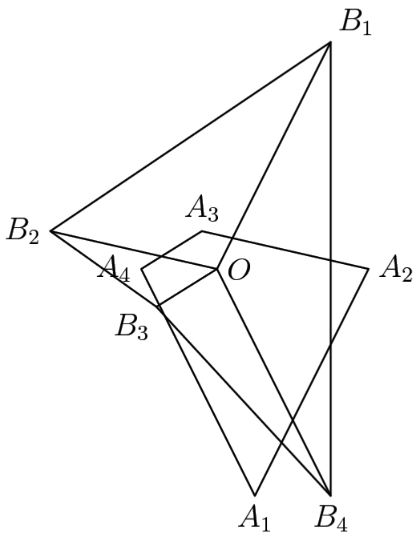
Show that the area of A'B'C'D' is twice that of ABCD

Source: Canada National Mathematical Olympiad 1982 - Problem 1

Amir Hossein

#1 Sep 30, 2011, 3:13 pm

In the diagram, OB_i is parallel and equal in length to A_iA_{i+1} for $i = 1$, 2, 3, and 4 ($A_5 = A_1$). Show that the area of $B_1B_2B_3B_4$ is twice that of $A_1A_2A_3A_4$.



Luis González

#2 Oct 1, 2011, 6:53 am

$\overline{AB} \cong \overline{PA'}, \overline{BC} \cong \overline{PB'}$ and $\measuredangle A'PB' = \measuredangle ABC \pmod \pi$ yield $[PA'B'] = [ABC]$. Similarly, $[PB'C'] = [BCD]$, $[PC'D'] = [CDA]$ and $[PD'A'] = [DAB]$. Thus

$$[A'B'C'D'] = [PA'B'] + [PB'C'] + [PC'D'] + [PD'A'] =$$

$$= [ABC] + [BCD] + [CDA] + [DAB] = [ABCD] + [ABCD] = 2[ABCD].$$

High School Olympiads

Turkey NMO 2001 Problem 5, locus of point P_G

Reply

Source: Turkey NMO 2001 Problem 5

mestavk
#1 Sep 30, 2011, 2:17 pm

Two nonperpendicular lines throught the point A and a point F on one of these lines different from A are given. Let P_G be the intersection point of tangent lines at G and F to the circle through the point A, F and G where G is a point on the given line different from the line FA. What is the locus of P_G as G varies.

Luis González
#2 Sep 30, 2011, 9:22 pm

Let the perpendicular bisector of $\overline{AF}$ cut AG at $E.$ The isosceles triangles $\triangle EAF$ and $\triangle P_GGF$ are similar due to $\angle EFA=\angle EAF=\angle P_GGF=\angle P_GFG.$ Therefore, $\angle AEF=\angle GP_GF \Longrightarrow$ The points E,F,G,P_G are concyclic $\Longrightarrow \angle FEP_G=\angle FGP_G=\angle EFA \Longrightarrow EP_G \parallel AF,$ i.e. Locus of P_G is the parallel line to AF through $E.$

Number1
#3 Oct 2, 2011, 10:16 pm

Let M be midpoint of GF. Then as G varies, M descirbes line ℓ parallel to AF, since: $d(M,AF)=\frac{1}{2}d(G,AF)=const.$

Now $\angle GFP_G=\angle(FA,GA)=\alpha$ is also constant, since lines FA and GA are fixed.

Finally, map $M\mapsto P_G$ is spiral similarity with center at F, and since M describes line ℓ, then P_G describes line ℓ' which we get with rotating ℓ by angle α around F. So ℓ' is parallel to AG.

xeroxia
#4 Jan 12, 2013, 3:34 am

Trigonometric Approach:

$\angle FAG=\angle P_GFG=\angle P_GGF.$

$\frac{AF}{FG}=\frac{\sin\angle AGF}{\sin\angle FAG}.$

$\frac{FG}{P_GF}=\frac{\sin\angle FP_GG}{\sin\angle P_GFG}=\frac{\sin 2\cdot\angle FAG}{\sin\angle FAG}=2\cos\angle FAG$

$\frac{AF}{FG}\cdot\frac{FG}{P_GF}=\frac{AF}{P_GF}=2\cot\angle FAG\cdot\sin\angle AGF.$

$\angle(FA,FP_G)=\angle AGF$ or $\angle(FA,FP_G)=180^\circ-\angle AGF.$
So the distance of P_G to AF is
$d(P_G,AF)=P_GF\cdot\sin\angle AGF=\frac{AF\cdot\tan\angle FAG}{2}$ which is constant. So locus of P_G is a a line parallel to AF with distance $\frac{AF\cdot\tan\angle FAG}{2}.$ $\blacksquare$

Now, we will show for every point on the line parallel to AF with distance $\frac{AF\cdot\tan\angle FAG}{2}$, we can find G on the given line.

Let the line perpendicular to AF at F meet the line AG (Here G is not defined, but the line AG is constant) at Q. Let H be the foot of perpendicular from P_G to QF.
Since $HF = \dfrac{AF \cdot \tan \angle FAG}{2}$ and $QF = AF \tan \alpha$, H is the midpoint of QF.
So $P_G Q = P_G F$. Let the circle with center P_G and radius $P_G F = P_G Q$ cut AQ at G.
Since $\angle GP_G F = 2 \cdot \angle FQA = 180^\circ - 2 \cdot \angle FAG$, and $P_G F = P_G G$, then $\angle P_G FG = \angle P_G GF = \angle FAG$. So $P_G F$ and $P_G G$ are tangent to the circumcircle of $\triangle AFG$.

Quick Reply

High School Olympiads

Cyclic Quads

Locked

mastergeo

#1 Sep 29, 2011, 8:52 pm

Let $ABCD$ be a convex quadrilateral. P, Q are two points lie inside $ABCD$ such that $ABQP$ and $CDPQ$ are concyclic. Assume that there exists $E \in [PQ]$ such that $\angle PAE = \angle QBE, \angle PDE = \angle QCE$. Prove that $ABCD$ is concyclic.

Luis González

#2 Sep 29, 2011, 9:48 pm

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=356615
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=287865
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=275065

High School Olympiads

Turkey NMO 1997 Problem 5, PQ=m

Reply

Source: Turkey NMO 1997 Problem 5

mestavk

#1 Sep 28, 2011, 3:43 pm

In a triangle ABC, the inner and outer bisectors of the $\angle A$ meet the line BC at D and E, respectively. Let d be a common tangent of the circumcircle (O) of $\triangle ABC$ and the circle with diameter DE and center F. The projections of the tangency points onto FO are denoted by P and Q, and the length of their common chord is denoted by m. Prove that $PQ = m$

Luis González

#2 Sep 29, 2011, 11:06 am

Let d touch $(O), (F)$ at U, V. R is the second intersection of $(O), (F)$ and $M \equiv AR \cap OF$ is the midpoint of $\overline{PQ}$. K and $J \equiv UV \cap OF$ are the insimilicenter and exsimilicenter of $(O) \sim (F)$, respectively. Since $(B, C, D, E) = -1$, then $(O) \perp (F)$, thus if UF cuts (O) and (F) again at T and X, Y, we have $(U, T, X, Y) = -1 \Longrightarrow VT$ is the polar of U WRT $(F) \Longrightarrow VT \perp FU \Longrightarrow VT$ goes through the antipode U^* of U WRT $(O) \Longrightarrow K \in VT$. Since $U(J, K, O, F) = -1$, then $(V, K, U^*, T) = -1 \Longrightarrow VP \perp OF$ is the polar of K WRT $(O) \Longrightarrow OA^2 = OK \cdot OP \Longrightarrow \angle APK = \angle OAK$. But $\frac{OA}{FA} = \frac{KO}{KF}$ implies that AK bisects $\angle OAF$, i.e. $\angle OAK = 45^{\circ}$, so $\angle APK = 45^{\circ} \Longrightarrow$ the right $\triangle MPA$ is isosceles with $MA = MP$, i.e. $PQ = m$.

yetti

#3 Sep 29, 2011, 12:10 pm • 1

WLOG $AB > AC$. $\angle FAC = \angle FAD - \angle CAD = \angle ADF - \angle DAB = \angle ABC \Longrightarrow AF$ is tangent of $(O) \Longrightarrow AO \perp AF$.

The common tangent touches $(O), (F)$ at U, V. $PQ = \frac{UV^2}{OF} = \frac{OF^2 - (AO - AF)^2}{OF} = \frac{2 \cdot AO \cdot AF}{OF} = m$.

Quick Reply

High School Olympiads

Ineq-G134 - Geometry

Reply

Ligouras

#1 May 15, 2011, 9:10 pm

Suppose AA_1, BB_1 and CC_1 are the altitudes of an acute-angled triangle ABC, AA_1 meets B_1C_1 in a point K.
The circumcircles of triangles A_1KC_1 and A_1KB_1 intersect the lines AB and AC for the second time at points N and L respectively.
Prove that

$$\frac{BB_1}{NA_1}+\frac{CC_1}{LA_1}\geq 4$$

Luis González

#2 Sep 28, 2011, 12:16 am • 1

From $\triangle ABC \sim \triangle AB_1C_1$ and $\triangle AKC_1 \sim \triangle ANA_1$, we obtain

$$\frac{AC}{AC_1}=\frac{BC}{B_1C_1}, \ \frac{NA_1}{KC_1}=\frac{AA_1}{AC_1} \Longrightarrow \frac{NA_1}{BB_1}=\frac{KC_1}{AC_1}\cdot\frac{AA_1}{BB_1}=\frac{KC_1}{AC_1}\cdot\frac{AC}{BC}=\frac{KC_1}{B_1C_1}$$

Similarly, we have $\frac{LA_1}{CC_1}=\frac{KB_1}{B_1C_1}$. Therefore, using AM-HM we get

$$\frac{BB_1}{NA_1}+\frac{CC_1}{LA_1}\geq\frac{4}{\frac{KC_1}{B_1C_1}+\frac{KB_1}{B_1C_1}}=\frac{4\cdot B_1C_1}{KC_1+KB_1}=4.$$

Quick Reply

High School Olympiads

Turkey TST 2005 Problem 5, BD is tangent to circle of ADZ

Reply

Source: Turkey TST 2005 Problem 5

mestavk

#1 Sep 27, 2011, 9:36 pm

Let ABC be a triangle such that $\angle A = 90$ and $\angle B < \angle C$. The tangent at A to its circumcircle Γ meets the line BC at D. Let E be the reflection of A across BC, X the foot of the perpendicular from A to BE, and Y be the midpoint of AX. Let the line BY meet Γ again at Z. Prove that the line BD is tangent to circumcircle of triangle ADZ .

Luis González

#2 Sep 27, 2011, 10:19 pm

Please, try searching before posting contest problems

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=18215
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=198944

Quick Reply

High School Olympiads

Ayme's theorem or the four points theorem

Reply

Source: own

jayme

#1 Sep 26, 2011, 6:37 pm

Dear Mathlinkers,
I present a new theorem which I have proved synthetically (next on my site) and lead to new centers
1. ABC a triangle
2. (O) the circumcircle of ABC
3. P, Q, R three points
4. P1P2P3 the P-cevian triangle of ABC
5. Q1Q2Q3 the Q-cevian triangle of ABC
6. R1R2R3 the R-circumcevian triangle of ABC
7. (O1) the circumcircle of P1Q1R1 and circularly
8. S1 the second point of intersection of (O1) with (O) and circularly.

Conclusion : AS1, BS2 and CS3 are concurrent at a point S.

Other proofs or ideas are welcome
Thank to Francisco Javier Garcia Capitan for the coordinate of S and the remark that
P=X(1), Q=X(3) and R=X(5) lead to a new center
P=X(1), Q=X(4) and R=X(6) lead to X(19) the Clawson point.

We have constructed a map from a triangle PQR to a point S. Is this map known?
Sincerely
Jean-Louis

Luis González

#2 Sep 26, 2011, 10:49 pm

AS_1, AR_1 cut BC at A_1, A_2 and R_1S_1 cut BC at A_3. From the problem Three collinear points, we get that

$$\frac{\overline{A_3B}}{\overline{A_3C}} = \frac{\overline{BP_1}}{\overline{P_1C}} \cdot \frac{\overline{BQ_1}}{\overline{Q_1C}} = \frac{\overline{S_1B}}{\overline{S_1C}} \cdot \frac{\overline{R_1B}}{\overline{R_1C}} = \frac{AC^2}{AB^2} \cdot \frac{\overline{BA_1}}{\overline{A_1C}} \cdot \frac{\overline{BA_2}}{\overline{A_2C}} \Longrightarrow$$

$$\frac{\overline{BA_1}}{\overline{A_1C}} = \frac{AB^2}{AC^2} \cdot \frac{\overline{BP_1}}{\overline{P_1C}} \cdot \frac{\overline{BQ_1}}{\overline{Q_1C}} \cdot \frac{\overline{A_2C}}{\overline{BA_2}}$$

Thus, if $(u_1 : v_1 : w_1), (u_2 : v_2 : w_2), (u_3 : v_3 : w_3)$ are the barycentric coordinates of P, Q, R with respect to $\triangle ABC$, then AS_1, BS_2, CS_3 concur at the barycentric product of $P \cdot Q$ and the isogonal conjugate of R. Namely

$$S\left(\frac{a^2u_1u_2}{u_3} : \frac{b^2v_1v_2}{v_3} : \frac{c^2w_1w_2}{w_3}\right)$$

jayme

#3 Nov 24, 2011, 4:56 pm

Dear Mathlinkers,
an article concerning the “Ayme’s theorem” has been put on my website.
http://perso.orange.fr/jl.ayme vol. 20
You can use Google translator
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Two perspective triangles

Source: (own) Dedicated to Luis and Borislav

jayme

#1 Sep 25, 2011, 8:36 pm

Dear Mathlinkers,
ABC a triangle,
H the orthocenter of ABC,
A'B'C' the orthic triangle of ABC,
1a, 1b, 1c the resp. circle with center A', B', C' and passing through H
A+, B+, C+ the second point of intersection of 1b and 1c, 1c and 1a, 1a and 1b.
Prouve that ABC and A+B+C+ are perspective at perhaps a new center.
Sincerely
Jean-Louis

Luis González

#2 Sep 25, 2011, 9:35 pm

Thanks for the dedication dear Jean Louis. We can generalized the problem as follows: $\triangle A'B'C'$ is the cevian triangle of P WRT $\triangle ABC.$ Circles $(B', B'P), (C', C'P)$ meet again at $A^+.$ We define B^+ and C^+ cyclically. Then $\triangle ABC$ and $\triangle A^+B^+C^+$ are perspective.

Proof: Let $\triangle A_1B_1C_1$ be the orthic triangle of $\triangle A'B'C'.$ Since A^+ is clearly the reflection of P about the line $B'C',$ then we have $A_1(P, A^+, C'A') = -1,$ but $A_1(P, A, C'A') = -1 \Longrightarrow A^+ \in AA_1.$ Similarly, $B^+ \in BB_1$ and $C^+ \in CC_1.$ Thus, by Cevian Nest theorem AA^+, BB^+, CC^+ concur at a point $B_e,$ which is usually known as Begonia Point of P WRT $\triangle ABC.$ When P coincides with the orthocenter of $\triangle ABC,$ as your problem states, then B_e is X_{24} of $\triangle ABC.$

High School Olympiads

Collinear

Reply

dazai

#1 Sep 25, 2011, 6:25 am

Let ABC be a triangle and BD, CE are altitudes of its. Denote bt M, I, K be the midpoints of segments BC, MD, ME. Denote by d be the line through A and parallel BC. The intersection of d and the perpendicular bisector of AM is N. Prove that N, I, K are collinear

This post has been edited 1 time. Last edited by dazai, Sep 25, 2011, 7:19 am

Luis González

#2 Sep 25, 2011, 9:31 am • 3

$H \equiv BD \cap CE$ and P is the midpoint of $\overline{AH}$. Since P, M are the centers of the circles $\odot(ADHE)$ and $\odot(BCD)$, then $\angle ADP = \angle DAH = \angle DBM = \angle BDM \Longrightarrow \angle MDP = \angle BDA = 90^\circ$, i.e. MD is tangent to $\odot(ADHE)$. Likewise, ME is tangent to $\odot(ADHE)$. Thus, IK is the radical axis of $\odot(ADHE)$ and the circle (M) with zero radius $\Longrightarrow N' \equiv IK \cap d$ has equal power WRT (M) and $\odot(ADHE)$, but d is obviously tangent to $\odot(ADHE)$ at A, so $N'M = N'A \Longrightarrow N \equiv N'$.

sunken rock

#3 Sep 25, 2011, 2:02 pm • 1

w.l.o.g. suppose $AB < AC$; call P the midpoint of DE. Then, if $m(\angle AMB) = \alpha$, the following yelds: $\angle AMB = \angle MAN = \angle AMN = \angle APD = \alpha$ and, subsequently $\angle ANM = 180^\circ - 2\alpha$ and $\angle APM = 90^\circ + \alpha$, or N is the circumcenter of $\triangle APM$ and, as KI is the perpendicular bisector of the chord MP, it passes through the circumcenter N, done.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Dedicated to Luis-A Locus problem

Reply

Goutham

#1 Sep 24, 2011, 11:11 pm • 1

In triangle ABC, let $A'B'C'$ be the circumcevian triangle of a point P, and $A''B''C''$ be the triangle formed by the orthogonal projections of $A'B'C'$ on the sides of ABC. Find the locus of P so that $A''B''C''$ is a Cevian triangle.

Luis González

#2 Sep 25, 2011, 6:30 am • 2

First of all, thanks for this dedication dear Goutham!

P^* is the isogonal conjugate of P, $\triangle P_1P_2P_3$ is the pedal triangle of P WRT $\triangle ABC$ and $\triangle A_1B_1C_1$ is the pedal triangle of P WRT $\triangle P_1P_2P_3$. Lines AP^*, BP^*, CP^* cut P_2P_3, P_3P_1, P_1P_2 at A_2, B_2, C_2. $\triangle PP_2P_3$ and $\triangle A'CB$ are similar with corresponding altitudes $PA_1, A'A'' \Longrightarrow \frac{BA''}{A''C} = \frac{P_3A_1}{A_1P_2} = \frac{P_2A_2}{A_2P_3}$. So, multiplying the symmetric ratios together and using the converse of Ceva's theorem, we conclude that if AA'', BB'', CC'' concur, then P_1A_2, P_2B_2, P_3C_2 concur. Thus, by Cevian nest theorem, AP_1, BP_2, CP_3 concur, i.e. $\triangle ABC$ and the pedal triangle of P are perspective $\Longrightarrow$ Locus of P is the Darboux cubic of the scalene $\triangle ABC$.

Quick Reply

High School Olympiads

A Problem of Ramanujan

tc1729

#1 Sep 23, 2011, 3:48 am • 1

Let AB be a diameter and BC be a chord of a circle ABC. Bisect the minor arc BC at M; and draw a chord BN equal to half of the chord BC. Join AM. Describe two circles with A and B as centers and AM and BN as radii, cutting each other at S and S', and cutting the given circle again at the points M' and N' respectively. Join AN and BM intersecting at R, and also join AN' and BM' intersecting at R'. Through B draw a tangent to the given circle, meeting AM and AM' produced at Q and Q' respectively. Produce AN and $M'B$ to meet at P, and also produce AN' and MB to meet at P'. Show that the eight points $P, Q, R, S, S', R', Q', P'$ are cyclic, and that the circle passing through these eight points is orthogonal to the given circle ABC.

Luis González

#2 Sep 25, 2011, 3:45 am

O is the center of the circle with diameter $\overline{AB}$, D is the midpoint of $\overline{BC}$ and F is the projection of M on AB. Clearly, $D \in OM$, thus $MBFD$ is an isosceles trapezoid with $DF \parallel MB$. Hence if $V \equiv DF \cap QB$, then $\angle BDF = \angle BMF = \angle BVF \Longrightarrow \triangle BDV$ is isosceles with legs $BD = DV$, i.e. V is on the circle (B) with center B and radius $BD = BN$. If $T \equiv DF \cap MA$, then $\overline{AT} \cdot \overline{AQ} = \overline{AF} \cdot \overline{AB} = AM^2 \Longrightarrow \overline{FDT} \perp \overline{MA}$ is the polar of Q WRT the circle (A) with center A and radius $AM \Longrightarrow p(T,(A)) = \overline{TA} \cdot \overline{TQ}$. Thus, $\frac{p(T,(A))}{p(T,(B))} = \frac{\overline{TA} \cdot \overline{TQ}}{\overline{TD} \cdot \overline{TV}}$. But on the other hand, we have

$$\frac{\overline{TQ}}{\overline{TV}} = \frac{\overline{MQ}}{\overline{MB}}, \ \frac{\overline{FA}}{\overline{TA}} = \frac{\overline{MQ}}{\overline{FB}}, \ \frac{\overline{TD}}{\overline{FB}} = \frac{\overline{FB}}{\overline{MB}} \Longrightarrow$$

$$\frac{\overline{TA} \cdot \overline{TQ}}{\overline{TD} \cdot \overline{TV}} = \frac{\overline{FA}}{\overline{FB}} \Longrightarrow \frac{p(T,(A))}{p(T,(B))} = \frac{\overline{FA}}{\overline{FB}}$$

Which means that the circle (F) with center F and radius FT is coaxal with (A) and $(B) \Longrightarrow S, S' \in (F)$. Inversion with center A and radius $AM = AS$ carries Q into T and S, S' into themselves $\Longrightarrow (F)$ is the inverse of $\odot(QSS') \equiv \omega$. Since (O) is taken into the line MF perpendicular to (F), then by conformity $(O) \perp \omega$.

If QV cuts (B) again at U, then clearly $MU \parallel AB$. Thus

$$\frac{p(Q,(A))}{p(Q,(B))} = \frac{\overline{QT} \cdot \overline{QA}}{\overline{QU} \cdot \overline{QV}} = \frac{\overline{QB} \cdot \overline{QV}}{\overline{QU} \cdot \overline{QV}} = \frac{\overline{QB}}{\overline{QU}} = \frac{\overline{QA}}{\overline{QM}} = \frac{\overline{AB}}{\overline{FB}}$$

$$\frac{p(R,(A))}{p(R,(B))} = \frac{RM^2}{RN^2} = \frac{MA^2}{NB^2} = \frac{MA^2}{MF^2} = \frac{\overline{AF} \cdot \overline{AB}}{\overline{AF} \cdot \overline{FB}} = \frac{\overline{AB}}{\overline{FB}}$$

$$\Longrightarrow \frac{p(Q,(A))}{p(Q,(B))} = \frac{p(R,(A))}{p(R,(B))}$$

Which implies that P, Q lie on a same circle coaxal with (A) and $(B) \Longrightarrow R \in \omega$. Let AR cut ω again at P^*. Since ω is the inverse of (F) under the inversion (A, AM^2), then ω is tangent to AM at $Q \Longrightarrow QB \perp AB$ is the polar of A WRT $\omega \Longrightarrow$ $B(R, P^*, Q, A) = -1$. But by obvious symmetry, BQ, BA bisect $\angle MBP$, so $B(R, P, Q, A) = -1 \Longrightarrow P \equiv P^*$, i.e. $P \in \omega$. Now, P', R', Q' are obviously the reflections of P, Q, R across AB, thus, we conclude that P, Q, R, S, P', Q', R', S' lie on a circle ω orthogonal to (O).

skytin

#3 Sep 25, 2011, 9:33 pm

Solution :
Let a is circle wich is tangent to AQ' , AQ at points Q' , Q
O is center of a , OB*OA = OQ*OQ , so a is orthogonal to (ABC)
O' is center of (ABC)
I is incenter of Q'AQ , easy to see that I is on AB , a
a is Apollonius circle of segment AB and point I on it
O'M intersect BC at point N
MH is a altitude of MOB
HM = BN , BN'/AM = MH/AM = BM/AB = BQ/AQ , so intersections of circles with centers A , B and radii MA , BN are placed on a
S , S' are on a
RB/RA = NB/MA = BQ/AQ , so R , R' are on a
PB/AP = BN'/MA = BQ/AQ = IB/AI , so P , P' are on a . done

Quick Reply

High School Olympiads

Fixed circle

Reply

buratinogigle

#1 Sep 23, 2011, 12:23 am

Let ABC be a triangle. Incircle (I) touches BC, CA, AB at D, E, F. M is a point on circle center A which passes though E, F.

a) Prove that pedal triangle XYZ of M with respect to triangle DEF is right triangle.

b) DM cuts IA at K. MI cuts EF at T. Prove that K lies on circumcircle (DEF) if only if T lies on circumcircle (XYZ).

c) M^* is isogonal conjugate of M with respect to triangle DEF. Prove that M^* always lies on fixed circle.

Luis González

#2 Sep 23, 2011, 12:27 pm • 1

It's clear that $\angle EDF = 90^\circ - \frac{1}{2}\angle A$ and $\angle EMF = 180^\circ - \frac{1}{2}\angle A$, but by properties of isogonal conjugates we have $\angle EMF + \angle EM^*F = 180^\circ + \angle EDF \Longrightarrow \angle EM^*F = 90^\circ \Longrightarrow M^*$ lies on the circle with diameter $\overline{EF}$ and $XZ \perp EM^*, XY \perp FM^*$ implies that $\angle YXZ$ is also right, i.e. $\triangle XYZ$ is right at X. Now, assume that $K \in (I)$, thus DK bisects $\angle EDF$ internally and $M^* \in DK$. If T^* is the projection of M^* onto EF, then

$$\sqrt{\frac{ET^*}{FT^*}} = \frac{EM^*}{FM^*} = \frac{\sin \widehat{DFM^*}}{\sin \widehat{DEM^*}} \cdot \frac{\mathrm{d}(M^*, DE)}{\mathrm{d}(M^*, DF)} = \frac{\sin \widehat{DFM^*}}{\sin \widehat{DEM^*}} = \frac{EM}{FM}$$

Which means that T^* coincides with the foot of the M-symmedian MI of $\triangle MEF \Longrightarrow T \equiv T^* \Longrightarrow T \in \odot(XYZ)$. The converse is proved with the same arguments.

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