On the Taylor center of a triangle / Darij Grinberg

Let AH_a, BH_b and CH_c be the altitudes of a triangle ABC. From the foot H_a we construct the perpendiculars H_aB_a to CA and H_aC_a to AB; so we get the feet B_a and C_a. Analogously, the points C_b, A_b and A_c, B_c are defined.

The following result was proven in [1] and [2]: The points B_a, C_a, C_b, A_b, A_c and B_c lie on one circle. This circle is called **Taylor circle** or **Catalan circle** of triangle ABC. Its center T lies on the Brocard axis of ΔABC (for the Taylor circle is a Tucker circle: [1]); T is called the **Taylor center** of triangle ABC. We shall show a possibly new theorem on this point T:

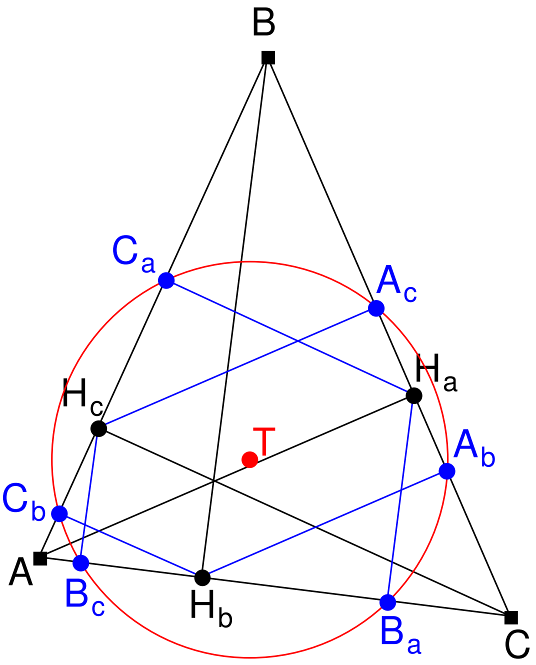

Fig. 1

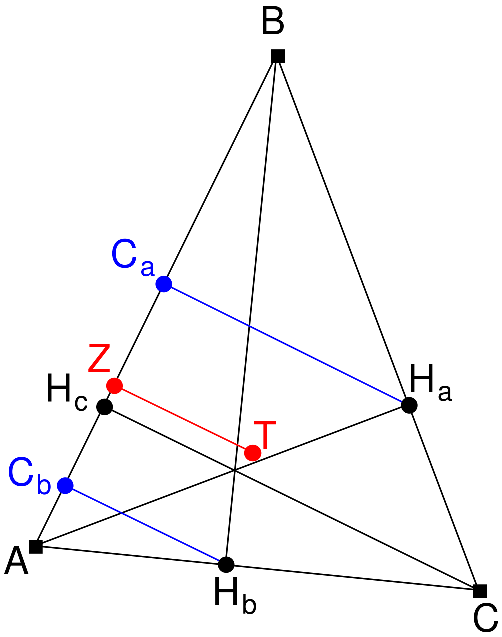

Fig. 2

Theorem 1. The distances of T to the vertices of ΔABC satisfy the equations

$$AT^2 - h_a^2 = BT^2 - h_b^2 = CT^2 - h_c^2,$$

where $h_a = AH_a$, $h_b = BH_b$ and $h_c = CH_c$ are the altitudes of ΔABC.

Proof. Let us use directed edges, where the sideline AB is oriented in direction $A \to B$, i. e. the length of the segment AB is positive, and the length of the segment BA is negative.

Let Z be the midpoint of segment C_aC_b. For the point T, being the center of the Taylor circle, lies on the perpendicular bisector of its chord C_aC_b, this perpendicular bisector is ZT; thus $ZT \perp AB$. Now we have

$$\begin{aligned} AT^2 - BT^2 &= (AZ^2 + ZT^2) - (BZ^2 + ZT^2) \qquad \text{(Pythagoras)} \\ &= AZ^2 - BZ^2 = AZ^2 - ZB^2 \qquad \text{(since } BZ^2 = (-ZB)^2 = ZB^2\text{)} \\ &= (AZ - ZB)(AZ + ZB) = (AZ - ZB) \bullet AB = c \bullet (AZ - ZB). \end{aligned}$$

Since Z is the midpoint of C_aC_b, we get:

$$\begin{aligned} AZ &= \frac{1}{2}(AC_a + AC_b) = \frac{1}{2}(AB + BC_a + AC_b) \\ &= \frac{1}{2}(c - C_aB + AC_b) \\ &= \frac{1}{2}(c - BH_a \bullet \cos\beta + AH_b \bullet \cos\alpha) \\ &= \frac{1}{2}(c - c\cos\beta \bullet \cos\beta + c\cos\alpha \bullet \cos\alpha) \\ &= \frac{1}{2}c \bullet (1 - \cos^2\beta + \cos^2\alpha), \end{aligned}$$

and analogously

$$ZB = \frac{1}{2}c \bullet (1 - \cos^2\alpha + \cos^2\beta),$$

and thus

$$\begin{aligned} AT^2 - BT^2 &= c \bullet (AZ - ZB) \\ &= c \bullet \left(\frac{1}{2}c \bullet (1 - \cos^2\beta + \cos^2\alpha) - \frac{1}{2}c \bullet (1 - \cos^2\alpha + \cos^2\beta)\right) \\ &= \frac{1}{2}c^2((1 - \cos^2\beta + \cos^2\alpha) - (1 - \cos^2\alpha + \cos^2\beta)) \\ &= \frac{1}{2}c^2(2\cos^2\alpha - 2\cos^2\beta) = c^2 \bullet (\cos^2\alpha - \cos^2\beta) \\ &= c^2 \bullet \left(\left(1 - \sin^2\alpha\right) - \left(1 - \sin^2\beta\right)\right) \\ &= c^2 \bullet \left(\sin^2\beta - \sin^2\alpha\right) = (c\sin\beta)^2 - (c\sin\alpha)^2 = h_a^2 - h_b^2. \end{aligned}$$

This formula $AT^2 - BT^2 = h_a^2 - h_b^2$ yields $AT^2 - h_a^2 = BT^2 - h_b^2$; analogously we can prove

$$BT^2 - h_b^2 = CT^2 - h_c^2$$

and Theorem 1 is established.

Theorem 1 can be paraphrased in a geometric disguise:

The circle centered at A and having the radius h_a passes through H_a and touches the sideline BC (since the tangent to this circle at the point H_a is orthogonal to the radius AH_a, i. e. it is the line BC itself). Further, we construct the circle centered at B and having the radius h_b and the circle centered at C and having the radius h_c.

Theorem 2. The radical center of these three circles is the Taylor center T.

Proof. The power of T with respect to the circle centered at A and having the radius h_a is $AT^2 - h_a^2$, and similarly, $BT^2 - h_b^2$ and $CT^2 - h_c^2$ are the powers of T with respect to the two other circles. After Theorem 1, they are equal, i. e. T is in fact the radical center of the three circles.

This result is likely known. In fact, I have noticed that in [3], where some points on the Brocard axis are listed, one of them is ”the radical center of the circles centered at A, B, C, which touch the opposite sidelines”.

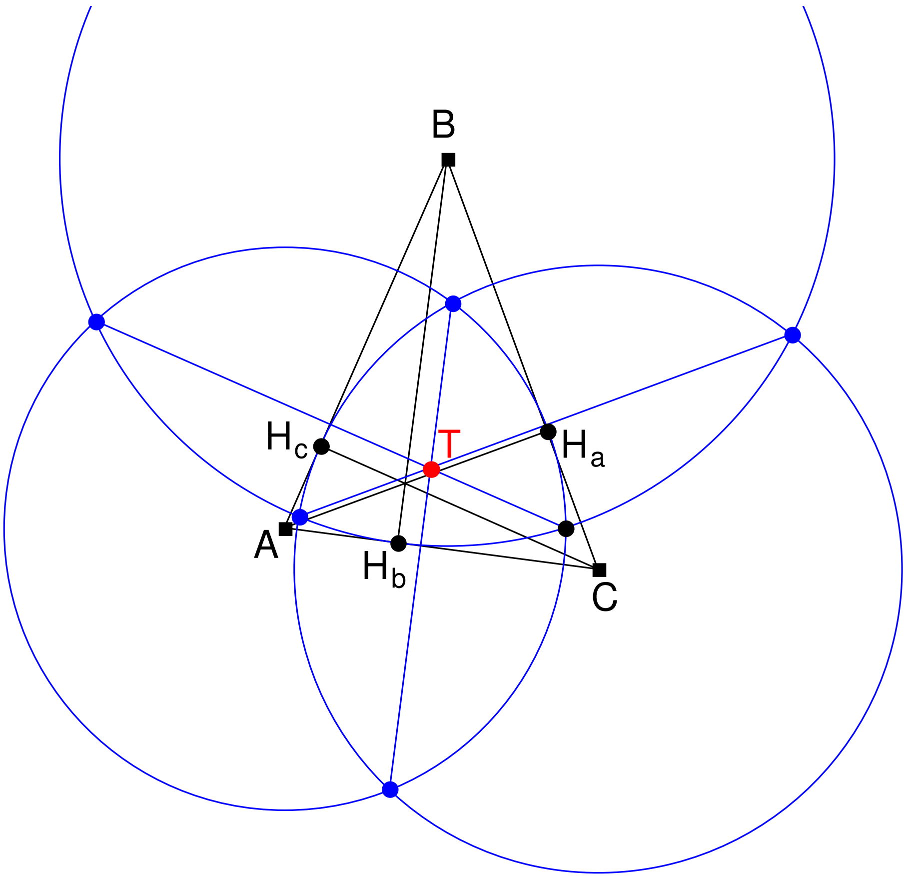

Fig. 3

References

[1] R. Honsberger: *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, USA 1995.

[2] A. Bogomolny: `http://cut-the-knot.com/triangle/Taylor.shtml`

[3] R. Stärk: *Ein Verfahren*, *Punkte der Tuckergeraden eines Dreiecks zu konstruieren*, Praxis der Mathematik 5/1992 pages 213-215.