ilikemaths
50 posts

May 23, 2010, 9:35 pm • 21 PM #1

Let's start a marathon on functional equations:
When you solve a problem, you should post a new one.

Here's problem 1:
Find all functions $f:\mathbb{Q}_{>0}\to\mathbb{Q}_{>0}$ that satisfy:
$f(x+1)=f(x)+1\forall x\in\mathbb{Q}_{>0}$ and
$f(x^2)=f(x)^2\forall x\in\mathbb{Q}_{>0}$.

[moderator edit: stickied in Pre-Olympiad forum.]

Pain rinnegan
1581 posts

May 23, 2010, 10:28 pm • 5 PM #2

> *ilikemaths wrote:*
> Let's start a marathon on functional equations:
> When you solve a problem, you should post a new one.
>
> **Problem 1:** Find all functions $f:\mathbb{Q}_+\to\mathbb{Q}_+$ that satisfy:
>
> (1).$f(x+1)=f(x)+1\ ,\ \forall x\in\mathbb{Q}_+$ and
>
> (2).$f(x^2)=f(x)^2\ ,\ \forall x\in\mathbb{Q}_+$.

I like this idea

Problem 1

From (1), we can easily find by induction that $f(x+n)=f(x)+n\ ,\ (\forall)x\in\mathbb{Q}_+\ ,\ (\forall)n\in\mathbb{N}$. Therefore by (2) we have:

$$f((x+n)^2)=f^2(x+n)\Leftrightarrow f(x^2+2nx+n^2)=(f(x)+n)^2\Leftrightarrow$$

$$f(x^2+2nx)+n^2=f^2(x)+2nf(x)+n^2\Leftrightarrow f(x^2+2nx)=f^2(x)+2nf(x)$$

Now let's put $x=\frac{p}{q}\ ,\ p,q\in\mathbb{N}^*$ and let $n\to q$

$$\Rightarrow f\left(\frac{p^2}{q^2}+2p\right)=f^2\left(\frac{p}{q}\right)+2qf\left(\frac{p}{q}\right)\Leftrightarrow$$

$$f\left(\frac{p^2}{q^2}\right)+2p=f\left(\frac{p^2}{q^2}\right)+2qf\left(\frac{p}{q}\right)$$

So $f\left(\frac{p}{q}\right)=\frac{p}{q}$. So $f(x)=x\ ,\ (\forall)x\in\mathbb{Q}_+$ which verifies the initial equation.

Problem 2: Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x^3)-f(y^3)=(x^2+xy+y^2)(f(x)-f(y))\ ,\ (\forall)x,y\in\mathbb{R}$$

---

mahanmath
1355 posts

May 23, 2010, 11:04 pm • 4 PM #3

> *Quote:*
> **Problem 2:** Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:
>
> $$f(x^3)-f(y^3)=(x^2+xy+y^2)(f(x)-f(y))\ ,\ (\forall)x,y\in\mathbb{R}$$

Problem 2

WLOG assume $f(0)=0$ (otherwise let $F(x)=f(x)-f(0)$ then you can easily see that it works in equation!) .

Now put $y=0$, we get $f(x^3)=(x^2)f(x)$. Substitute in the main equation we get $f(x)=xf(1)$.

So the answer is $f(x)=xf(1)+f(0)$

Problem 3 : Find all the continuous functions $f:\mathbb{R}\mapsto\mathbb{R}$ such that $\forall x,y\in\mathbb{R}$:

$(1+f(x)f(y))f(x+y)=f(x)+f(y)$

Pain rinnegan
1581 posts

May 23, 2010, 11:35 pm • 2 PM #4

> *mahanmath wrote:*
> **Problem 3** : Find all the continuous functions $f:\mathbb{R}\mapsto\mathbb{R}$ such that $\forall x,y\in\mathbb{R}$:
>
> $(1+f(x)f(y))f(x+y)=f(x)+f(y)$

It's a too recent question: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=350104&start=0&hilit=continuous .

ilikemaths
50 posts

May 23, 2010, 11:39 pm • 3 PM #5

Then pose a new problem.

Pain rinnegan
1581 posts

May 23, 2010, 11:43 pm • 1 PM #6

Problem 4: Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x^3+y^3)=xf(x^2)+yf(y^2)\ ,\ (\forall)x,y\in\mathbb{R}$$

ilikemaths
50 posts

May 23, 2010, 11:55 pm • 2 PM #7

$x=y=0$ yields $f(0)=0$.
$x=0$ yields $f(y^3)=yf(y^2)$, so the given functional equation reduces to:
$f(x^3+y^3)=f(x^3)+f(y^3)$.
Setting $a=x^3, b=y^3$ gives:
$f(a+b)=f(a)+f(b)$, which is a Cauchy-equation with solutions:
$f(x)=0$ and $f(x)=cx$ for some $c\in\mathbb{R}$.
So we have two possible functions:
$f(x)=0$ and $f(x)=cx$ for some $c\in\mathbb{R}$ and a quick check tells us that both functions satisfy.

New problem:
find all functions $f:\mathbb{R}_{>0}\to\mathbb{R}$ satisfying:
$f(x+y)-f(y)=\dfrac{x}{y(x+y)}$

mahanmath
1355 posts

May 24, 2010, 3:32 am • 1 PM #8

Problem 5

Same as problem 2 ! WLOG assume $f(1)=-1$, We claim that $f(x)=\dfrac{-1}{x}$.To see this fact just put $y=1$. Hence the answer is $f(x)=\dfrac{-1}{x}+c$ for some real c.

Problem 6. Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f\big(x+yf(x)\big)+f\big(xf(y)-y\big)=f(x)-f(y)+2xy$$

mahanmath
1355 posts

May 24, 2010, 5:52 am • 1 PM #9

> *ilikemaths wrote:*
> $f(a+b)=f(a)+f(b)$, which is a Cauchy-equation with solutions:
> $f(x)=0$ and $f(x)=cx$ for some $c\in\mathbb{R}$.
> So we have two possible functions:
> $f(x)=0$ and $f(x)=cx$ for some $c\in\mathbb{R}$ and a quick check tells us that both functions satisfy.
> $

I`m not sure but I think Cauchy-equation just solve continuous functions. Am I right ?

Amir Hossein
4719 posts

May 24, 2010, 3:05 pm • 1 PM #10

> *mahanmath wrote:*
> WLOG assume $f(1)=-1$

Can we do this ?
How Without Loss Of Generality ?

Dumel
190 posts

May 24, 2010, 5:51 pm • 2 PM #11

> *mahanmath wrote:*
> **Problem 6.**Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:
> $f\big(x+yf(x)\big)+f\big(xf(y)-y\big)=f(x)-f(y)+2xy$

solution

$P(0,0)$ gives $f(0)=0$. now $P(0,x)\rightarrow\ f(-x)=-f(x)$
Finally adding $P(x,y)$ and $P(-y,x)$ we get $f(x)=0$

Post a bit harder problems, please! 🙂
Problem 7
Find the least possible value of $f(1998)$ where $f:\mathbb{N}\to\mathbb{N}$ satisfies
$f(n^2f(m))=m(f(n))^2$

ilikemaths
50 posts

May 24, 2010, 7:23 pm • 1 PM #12

Dumel, your solution isn't correct:
$f(x)=x$ also satisfies!

Pain rinnegan
1581 posts

May 24, 2010, 9:23 pm • 1 PM #13

> *Dumel wrote:*
> we get $f(x)=0$

$f(x)=0$ is not a solution . 🙁

Dumel
190 posts

May 25, 2010, 2:50 am • 1 PM #14

oh what a terrible mistake 😲
At this moment I don't know how to solve this problem.

mahanmath
1355 posts

May 26, 2010, 12:22 pm • 1 PM #15

> *amparvardi wrote:*
>
> > *mahanmath wrote:*
> > WLOG assume $f(1)=-1$

> WLOG assume $f(1) = -1$
>
> Can we do this ?
> How Without Loss Of Generality ?

As I said ~~ SAME AS PROBLEM **2** ~~

---

arshakus
748 posts

May 26, 2010, 1:18 pm • 1 PM #16

problem 7
it is the problem of shortlist in1998, try to go in this link!
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=124426&sid=d54a2e73626da06fb572b3022d2bc388#p124426

---

arshakus
748 posts

May 26, 2010, 1:22 pm • 1 PM #17

problem 8 try this one....
$f:R+->R+$
$f(x+f(y))=f(x+y)+f(y)$, for every $x,y, fromR+$

---

arshakus
748 posts

May 26, 2010, 1:22 pm • 1 PM #18

find $f(x)-?$

---

mahanmath
1355 posts

May 26, 2010, 10:38 pm • 3 PM #19

> *arshakus wrote:*
> **problem 8** try this one....
> $f:R+->R+$
> $f(x+f(y))=f(x+y)+f(y)$, for every $x,y, fromR+$

What about problem 6 ?!

BTW , I`m waiting for **pco** and his nice solutions !

---

mahanmath
1355 posts

May 27, 2010, 7:59 pm • 3 PM #20

> *mahanmath wrote:*
> **Problem 6.**Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:
>
> $f\Big(x+yf(x)\Big)+f\Big(xf(y)-y\Big)=f(x)-f(y)+2xy$

OK , It seems hard . So I give a hint

> prove $f(x+y)=f(x)+f(y)$

.

MohammadP
58 posts

May 27, 2010, 8:01 pm • 2 PM #21

problem 4 :

> *ilikemaths wrote:*
> ...
> $f(a+b)=f(a)+f(b)$, which is a Cauchy-equation with solutions:
> $f(x)=0$ and $f(x)=cx$ for some $c\in\mathbb{R}$.
> ...

this is not correct !
you've only proved that $f(a)+f(b)=f(a+b)$ and $f(a^3)=af(a^2)$
now let $g(x)=f(x)/f(1)$
we have $g(x+q)=g(x)+q$ for every $q \in Q$
we have :
$g((x+q)^3)=(x+q)g((x+q)^2)=(x+q)(g(x^2)+2qg(x)+q^2)$
and also $g((x+q)^3)=g(x^3+3qx^2+3q^2x+q^3)=g(x^3)+3qg(x^2)+3q^2g(x)+q^3$
so $2g(x^2)+qg(x)=qx+2xg(x)$ or $q(g(x)-x)+2g(x^2)-2xg(x)=0$
which is true for every $q \in Q$ but it's linear and can't have more than one
zero unless $g(x)-x=0$ so $f(x)=cx$ for every $x \in R$

---

pco
14052 posts

May 29, 2010, 12:34 pm • 7 PM #22

> *mahanmath wrote:*
> **Problem 6.**Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:
>
> $f\Big(x+yf(x)\Big)+f\Big(xf(y)-y\Big)=f(x)-f(y)+2xy$

> *mahanmath wrote:*
> What about problem 6 ?!
>
> BTW , I`m waiting for **pco** and his nice solutions !

Here is a rather heavy solution :

Let $P(x,y)$ be the assertion $f(x+yf(x))+f(xf(y)-y)=f(x)-f(y)+2xy$

1) $f(x)$ is an odd function and $f(x)=0\iff x=0$
=====================================
$P(0,0)\implies f(0)=0$
$P(0,x)\implies f(-x)=-f(x)$

Suppose $f(a)=0$. Then $P(a,a)\implies 0=2a^2\implies a=0$ and so $f(x)=0\iff x=0$
Q.E.D

2) $f(x)$ is additive
==============

Let then $x\neq 0$ such that $f(x)\neq 0$:
$P(x,\frac{x+y}{f(x)})\implies f(2x+y)+f(xf(\frac{x+y}{f(x)})-\frac{x+y}{f(x)})=f(x)-f(\frac{x+y}{f(x)})+2x\frac{x+y}{f(x)}$

$P(\frac{\quad}{f(x)}, -x) \implies -f(x)f(\frac{\quad}{f(x)}) - \frac{\quad}{f(x)}) - f(y) = f(\frac{\quad}{f(x)}) + f(x) - 2x\frac{\quad}{f(x)}$

Adding these two lines, we get : $f(2x+y) = 2f(x) + f(y)$ which is obviously still true for $x = 0$ and so :

New assertion $Q(x,y)$: $f(2x+y) = 2f(x) + f(y) \forall x, y$

$Q(x,0) \implies f(2x) = 2f(x)$ and so $Q(x,y)$ becomes $f(2x+y) = f(2x) + f(y)$ and so $f(x+y) = f(x) + f(y)$ and $f(x)$ is additive.
Q.E.D.

3) $f(x)$ solution implies $-f(x)$ solution and so wlog consider from now $f(1) \geq 0$
==
$P(y,x) \implies f(y + xf(y)) + f(yf(x) - x) = f(y) - f(x) + 2xy$
$\implies -f(-y + x(-f(y))) - f(y(-f(x)) + x) = -f(x) - (-f(y)) + 2xy$
Q.E.D

4) $f(x)$ is bijective and $f(1) = 1$
=========================
Using additive property, the original assertion becomes $R(x,y)$: $f(xf(y)) + f(yf(x)) = 2xy$

$R(x, \frac{1}{2}) \implies f(xf(\frac{1}{2}) + \frac{f(x)}{2}) = x$ and $f(x)$ is surjective.

So $\exists a$ such that $f(a) = 1$
Then $R(a,a) \implies a^2 = 1$ and so $a = 1$ (remember that in 3) we choosed $f(1) \geq 0$)

5) $f(x) = x$
===========
$R(x,1) \implies f(x) + f(f(x)) = 2x$ and so $f(x)$ is injective, and so bijective.

$R(xf(x),1) \implies f(xf(x)) + f(f(xf(x))) = 2xf(x)$
$R(x,x) \implies f(xf(x)) = x^2$ and so $f(x^2) = f(f(xf(x)))$
Combining these two lines, we get $f(x^2) + x^2 = 2xf(x)$

So $f((x+y)^2) + (x+y)^2 = 2(x+y)f(x+y)$ and so $f(xy) + xy = xf(y) + yf(x)$

So we have the properties :
$R(x,y)$: $f(xf(y)) + f(yf(x)) = 2xy$
$A(x,y)$: $f(xy) = xf(y) + yf(x) - xy$
$B(x)$: $f(f(x)) = 2x - f(x)$

So :

(a) : $R(x,x) \implies f(xf(x)) = x^2$
(b) : $A(x, f(x)) \implies f(xf(x)) = xf(f(x)) + f(x)^2 - xf(x)$
(c) : $B(x) \implies f(f(x)) = 2x - f(x)$

And so -(a)+(b)+x(c) : $0 = x^2 + f(x)^2 - 2xf(x) = (f(x) - x)^2$
Q.E.D.

6) synthesis of solutions
=================
Using 3) and 5), we get two solutions (it's easy to check back that these two functions indeed are solutions) :
$f(x) = x \forall x$
$f(x) = -x \forall x$

---

arshakus
748 posts

May 29, 2010, 1:23 pm PM #23

> *mahanmath wrote:*
>
> > *mahanmath wrote:*
> > **Problem 6.**Determine all the functions $f : \mathbb{R} \to \mathbb{R}$ such that:
> >
> > $f\big(x + yf(x)\big) + f\big(xf(y) - y\big) = f(x) - f(y) + 2xy$
>
> OK , It seems hard . So I give a hint.

ok I do't saw that problem I will try to prove it

---

arshakus
748 posts

May 29, 2010, 4:48 pm PM #24

now problem 8
$f : R+ -> R+$
$f(x + f(y)) = f(x + y) + f(y)$, for every x, y,from $R+$

---

ilikemaths
50 posts

May 29, 2010, 5:06 pm PM #25

Isn't that from an IMO-shortlist?

---

mahanmath
1355 posts

May 29, 2010, 8:40 pm PM #26

> *arshakus wrote:*
> now problem 8
> $f : R+ -> R+$
> $f(x + f(y)) = f(x + y) + f(y)$, for every x, y,from $R+$

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1165901&sid=96c6c2e3567eab5401350eb464f8fe2f#p1165901

Amir Hossein
4719 posts

May 29, 2010, 8:53 pm PM #27

Problem 9:

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that :

(i) For all $x, y \in \mathbb{R}$,

$$f(x) + f(y) + 1 \geq f(x+y) \geq f(x) + f(y)$$

(ii) For all $x \in [0,1)$, $f(0) \geq f(x)$

$(iii) - f(-1) = f(1) = 1$

$(iii)-f(-1)=f(1)=1.$

Find all such functions.

---

mahanmath
1355 posts

May 29, 2010, 9:22 pm PM #28

> amparvardi wrote:
> **Problem 9:**
>
> Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that :
>
> (i) For all $x,y\in\mathbb{R}$,
> $$f(x)+f(y)+1\geq f(x+y)\geq f(x)+f(y)$$
>
> (ii) For all $x\in[0,1)$, $f(0)\geq f(x)$
>
> $(iii)-f(-1)=f(1)=1.$
>
> Find all such functions.

Put $x=-1,y=1$ we get $f(0)\geq 0$, by plugging $(0,0)$ we get $f(0)\leq 0$ so $f(0)=0$. And then by an easy induction you can prove $f(x)=[x]$

---

mahanmath
1355 posts

May 29, 2010, 9:22 pm PM #29

Problem 10

Find all $f:\mathbb{R}\to\mathbb{R}$ such that

$f(xy+f(x))=xf(y)+f(x)$

Pain rinnegan
1581 posts

May 29, 2010, 10:32 pm PM #30

> mahanmath wrote:
>> amparvardi wrote:
>> **Problem 9:**
>>
>> Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that :
>>
>> (i) For all $x,y\in\mathbb{R}$,
>> $$f(x)+f(y)+1\geq f(x+y)\geq f(x)+f(y)$$
>>
>> (ii) For all $x\in[0,1)$, $f(0)\geq f(x)$
>>
>> $(iii)-f(-1)=f(1)=1.$
>>
>> Find all such functions.
>
> Put $x=-1,y=1$ we get $f(0)\geq 0$, by plugging $(0,0)$ we get $f(0)\leq 0$ so $f(0)=0$. And then by an easy induction you can prove $f(x)=[x]$

How can you use induction if $x\in\mathbb{R}$?

---

mahanmath
1355 posts

May 29, 2010, 10:44 pm PM #31

OK , I should\`ve explain more ! usually in $\mathbb{R}$, we induction on intervals.
(In the first step we prove $f(x)=0$ for all $x\in[0,1)$)

---

pco
14052 posts

May 30, 2010, 2:39 pm • 2 PM #32

> mahanmath wrote:
> **Problem 10**
>
> Find all $f:\mathbb{R}\to\mathbb{R}$ such that
>
> $f(xy+f(x))=xf(y)+f(x)$

My solution

Let $P(x,y)$ be the assertion $f(xy+f(x))=xf(y)+f(x)$

$f(x)=0\forall x$ is a solution and we'll consider from now that $\exists a$ such that $f(a)\neq 0$

1) $f(0)=0$ and $f(f(x))=f(x)$
==============================
Suppose $f(0)\neq 0$. Then $P(x,0)\implies f(f(x))=xf(0)+f(x)$ and so $f(x_1)=f(x_2)\implies x_1=x_2$ and $f(x)$ is injective.
Then $P(0,0)\implies f(f(0))=f(0)$ and, since $f(x)$ is injective, $f(0)=0$, so contradiction.
So $f(0)=0$ and $P(x,0)\implies f(f(x))=f(x)$
Q.E.D.

2) $f(-1)=-1$
==========
$P(f(a),-1)\implies 0=f(a)(f(-1)+1)$ and so $f(-1)=-1$
Q.E.D.

3) $f(x)=x\,\forall x$
================
Let $g(x)=f(x)-x$

Suppose now $\exists b$ such that $f(b)\neq b$

$$P(\frac{x}{f(b)-b},b)\implies f(b\frac{x}{f(b)-b}+f(\frac{x}{f(b)-b}))=\frac{x}{f(b)-b}f(b)+f(\frac{x}{f(b)-b})$$

and so $f(b\frac{}{f(b)-b}+f(\frac{}{f(b)-b}))-(b\frac{}{f(b)-b}+f(\frac{}{f(b)-b}))=x$

and so $g(b\frac{x}{f(b)-b}+f(\frac{x}{f(b)-b}))=x$ and $g(\mathbb{R})=\mathbb{R}$

but $P(x,-1)\implies f(f(x)-x)=f(x)-x$ and so $f(x)=x\,\forall x\in g(\mathbb{R})$
And it's immediate to see that this indeed is a solution
Q.E.D.

4) Synthesis of results
================
We got two solutions :
$f(x)=0\,\forall x$
$f(x)=x\,\forall x$

I have no problem to submit. Anybody feel free to take my turn.

---

mahanmath
1355 posts

May 30, 2010, 5:48 pm — PM #33

Problem 11

Find all $f:\mathbb{Q}\to\mathbb{Q}$ such that :

$2(f(x))=f(2x)$ and $f(x)+f(\frac{1}{x})=1$

Farenhajt
5170 posts

May 30, 2010, 6:26 pm — PM #34

> *mahanmath wrote:*
> **Problem 11**
>
> Find all $f:\mathbb{Q}\to\mathbb{Q}$ such that :
>
> $2(f(x))=f(2x)$ and $f(x)+f(\frac{1}{x})=1$

Inductively, $f(2^nx)=2^nf(x)$ for all integer n. Since $2f(1)=1\implies f(1)=\frac{1}{2}$, we get $f(2^n)=2^{n-1}$, hence $f(2^{-n})=1-2^{n-1}$, but also $f(2^{-n})=2^{-n}f(1)=2^{-n-1}$, so we get $1-2^{n-1}=2^{-n-1}$, which obviously can't hold for all integer n. Hence there's no such function.

This post has been edited 1 time. Last edited by Farenhajt, May 30, 2010, 6:34 pm

---

Farenhajt
5170 posts

May 30, 2010, 6:31 pm • 1 — PM #35

Problem 12.

Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(xf(y))+f(yf(x))=\frac{1}{2}f(2x)f(2y)$ for all real values x,y.

---

ocha
955 posts

May 31, 2010, 12:33 pm — PM #36

> *Farenhajt wrote:*
> **Problem 12.**
>
> Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(xf(y))+f(yf(x))=\frac{1}{2}f(2x)f(2y)$ for all real values x,y.

May i ask where you got this problem? It seems very tough, some of the solutions are; $f(x)=kx$, $f(x)=kx+4$, and $f(x)=|kx|$, where $k\in\mathbb{R}$. But also, when $k>0$ we have solution such as

$$f(x)=\left\{\begin{array}{cc}0 & x\le 0\\ kx & x>0\end{array}\right.$$

Farenhajt
5170 posts

May 31, 2010, 12:51 pm — PM #37

> *ocha wrote:*
>
> > *Farenhajt wrote:*
> > **Problem 12.**
> >
> > Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(xf(y))+f(yf(x))=\frac{1}{2}f(2x)f(2y)$ for all real values x,y.
>
> May i ask where you got this problem? It seems very tough, some of the solutions are; $f(x)=kx$, $f(x)=kx+4$, and $f(x)=|kx|$, where $k\in\mathbb{R}$. But also, when $k>0$ we have solution such as
>
> $$f(x)=\left\{\begin{array}{cc}0 & x\le 0\\ kx & x>0\end{array}\right.$$

A colleague gave it to me from his private notes, with some (incomplete) outlines of the solution.

---

arshakus
748 posts

May 31, 2010, 8:58 pm — PM #38

hi,
please solve this one
$f:R->R$
$f(x^5)-f(y^5)=(f(x)-f(y))(x^4+x^3y+x^2y^2+xy^3+y^4)$
$f(x)-?$

Zeus93
64 posts

May 31, 2010, 9:32 pm — PM #39

The problem above is the 13th

Now the 14th :
$f:R=>R$ and $f(xf(x)+f(y))=y+f^2(x)$ for all real values x,y
find $f(x)$

---

arshakus

May 31, 2010, 10:02 pm — PM #40

arshakus
748 posts

at first solve the 13^{th}, then post 14^{th}

mahanmath
1355 posts

May 31, 2010, 10:57 pm

PM #41

> Jumong4958 wrote:
>
> The problem above is the 13th
>
> Now the 14th :
> $f : R => R$ and $f(xf(x) + f(y)) = y + f^2(x)$ for all real values x,y
> find $f(x)$

Put $y = -(f(x)^2)$. So there is a such that $f(a) = 0$.
Now put $x : a$. We get $f(f(x)) = x$
Put $x : f(x)$, we get $f(x) = x$ or $f(x) = -x$, but it`s easy to see the both are correct.Thus the answers are $f(x) = x$ and $f(x) = -x$

> arshakus wrote:
>
> hi,
> please solve this one
> $f : R- > R$
> $f(x^5) - f(y^5) = (f(x) - f(y))(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$
> $f(x)-?$

See the **Problem 2** and **Problem 3**.

Is there any idea about **Problem 12** ?

> Farenhajt wrote:
>
> **Problem 12.**
>
> Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xf(y)) + f(yf(x)) = \frac{1}{2}f(2x)f(2y)$ for all real values x, y.

This post has been edited 2 times. Last edited by mahanmath, Jun 1, 2010, 10:48 pm

arshakus
748 posts

May 31, 2010, 11:23 pm

PM #42

> mahanmath wrote:
>
> > Quote:
> >
> > **Problem 2:** Determine all the functions $f : \mathbb{R} \to \mathbb{R}$ such that:
> >
> > $$f(x^3) - f(y^3) = (x^2 + xy + y^2)(f(x) - f(y))\ ,\ (\forall)x, y \in \mathbb{R}$$
>
> Problem 2
>
> **Problem 3** : Find all the continuous functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that $\forall x, y \in \mathbb{R}$:
>
> $(1 + f(x)f(y))f(x + y) = f(x) + f(y)$

you are not right the answer is $f(x) = kx$,

mahanmath
1355 posts

May 31, 2010, 11:41 pm

PM #43

> arshakus wrote:
>
> you are not right the answer is $f(x) = kx$,

No !!! Actually $k = f(1)$ and $f(x) = xf(1) + f(0)$ works . (You can check it !?!?!?!)

ocha
955 posts

Jun 1, 2010, 6:10 am • 8

PM #44

EDIT: The solution is now complete (it's quite long though)

> Farenhajt wrote:
>
> **Problem 12.**
>
> Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xf(y)) + f(yf(x)) = \frac{1}{2}f(2x)f(2y)$ for all real values x, y.

In this proof we show that when f is not constant, it is bijective on the separate domains $(-\infty, 0]$ and $[0, \infty)$, (not necessarily on $\mathbb{R}$) and then find all solutions on those domains. Then we get all functions f, by joining any two functions from the separate domains and checking they work. I mentioned some of the solutions in an earlier post.

solution

Assume f is not constant and let $P(x, y)$ be the assertion $f(xf(y)) + f(yf(x)) = \frac{1}{2}f(2x)f(2y)$.

$P(0, 0) :\quad 4f(0) = f(0)^2 \implies f(0) = 0$ or $4 \qquad (1)$.

Injectivity
As $f(x) = |x|$ is a solution, we cannot prove that f is injective on $\mathbb{R}$, instead we show it is injective on the domains $(-\infty, 0]$ and $[0, \infty)$. So suppose there were two reals $a \neq b$ such that $f(a) = f(b)$, then we have

$$\frac{1}{4}f(2a)^2 + \frac{1}{4}f(2b)^2 = f(af(a)) + f(bf(b)) = f(af(b)) + f(bf(a)) = \frac{1}{2}f(2a)f(2b)$$

Which implies $\frac{1}{4}[f(2a) - f(2b)]^2 = 0 \implies f(2a) = f(2b)$

Moreover, $f(af(x)) + f(xf(a)) = \frac{1}{2}f(2a)f(2x) = \frac{1}{2}f(2b)f(2x) = f(bf(x)) + f(xf(b))$

This then implies $f(af(x)) = f(bf(x))$ for all $x \in \mathbb{R}$ $(\star)$.

Case 1: f(0)=0

First we will show that f is injective on $[0, \infty)$. So for the sake of contradiction assume there existed $a > b > 0$ such that $f(a) = f(b)$.
Since $f(x)$ is continuous and not constant when $x > 0$, there must be some interval $[0, c_1]$ or $[-c_1, 0]$ such that f is surjective onto that interval. wlog that interval is $[0, c_1]$. So, motivated by $(\star)$ we define a strictly decreasing sequence $u_0 \in [0, c_1]$, $u_{n+1} = \frac{b}{a} u_n$. We find that $u_n \in [0, c_1]$ for all n and therefore $f(au_0) = f(bu_0) = f(au_1) = \cdots = f(au_n)$.

Now $\lim_{n\to\infty} u_n \to 0$, so by the continuity of f we have $\lim_{n\to\infty} f(au_n) = f\left(\lim_{n\to\infty} au_n\right) = f(0) = 0$. This implies that $f(au_0) = 0$ for all $u_0 \in [0, c_1]$, and therefore $f(x) = 0$ when $x \in [0, ac_1]$.

But then for any $x \in [0, ac_1]$ we have $P(x, x): \ 0 = f(xf(x)) = \frac{1}{4} f(2x)^2$, hence $f(2x) = 0$. Inductively we find that $f(x) = 0$ for all $x \in \mathbb{R}^+$. Contradicting the assumption that f was not constant on that domain. Hence f is injective on the domain $[0, \infty)$.

As for the domain $(-\infty, 0]$, simply alter the original assumption to $a < b < 0$ such that $f(a) = f(b)$ and the same proof applies. Hence f is injective on $(-\infty, 0]$ and $[0, \infty)$

Case 2: f(0)=4

Again we will consider the case $x \in [0, \infty)$. Assume there exists $a > b > 0$ such that $f(a) = f(b)$

$P\left(\frac{x}{2}, 0\right): \ f(2x) + 4 = 2f(x) \iff f(2x) - 4 = 2[f(x) - 4]$, and inductively $f(2^n x) - 4 = 2^n [f(x) - 4]$. So assuming there exists atleast one value such that $f(x) - 4 \neq 0$, we will have $f(2^n) \to \pm\infty$. And since f is continuous, f will also be surjective onto at least one of: $[4, \infty)$ or $(-\infty, 4]$. wlog, we will assume it $[4, \infty)$

Similar to the previous case we define the increasing sequence $u_0 \in [4, \frac{a}{b} 4]$ and $u_{n+1} = \frac{a}{b} u_n$. Again $u_n \in [4, \infty)$ and therefore $f(bu_0) = f(au_0) = f(bu_1) = \cdots = f(bu_n)$.

Now for any $y \in [4, \infty)$ there must exists a $u_0 \in [4, \frac{a}{b} 4]$, such that $y = bu_n = b \frac{a^n}{b^n} u_0$ for some n. Hence for any value, v in the range of f, there exists some value in $x \in [4b, 4a]$ such that $f(x) = v$.

But f is continuous on the domain $[4b, 4a]$ therefore achieves a (finite) maximum. This contradicts the fact that f is surjective on $[4, \infty)$, hence our assumption is false and $f(x)$ is injective on the domain $[0, \infty)$.

We handle the negative domain $(\infty, 0]$ by changing the assumption to $a < b < 0$ and $f(a) = f(b)$. Therefore $f(x)$ is injective on both domains $x \in (-\infty, 0]$ and $[0, \infty)$. (in fact, it is bijective)

Surjectivity
We already know that $f(x)$ is surjective on either $(-\infty, 4]$ or $[4, \infty)$ when $f(0) = 4$, so consider, $f(0) = 0$. We know that there exists some interval $[-c_1, 0]$ or $[0, c_1]$ such that f is surjective onto that range and f is monotonic increasing/decreasing (following from f being injective and continuous), so we consider two cases.

Case 1: f is surjective on [0,c_1]

Suppose f is bounded above, let $\lim_{x\to\infty} f(x) \to L_1$. Then when $f(y) > 00$ we have

$$P(\infty, y): \ L_1 + f(L_1 y) = \frac{L_1}{2} f(2y).$$

So let $y = u_0 > 0$, and $u_{n+1} = \frac{u_n}{L}$, and as we send $n \to \infty$, by the continuity of f we have:
$L_1 + f(0) = \frac{L_1}{2} f(0) \implies L_1 = 0.$

But this implies f is constant, and contradicts that f is surjective on $[0, c_1]$, hence f is not bounded above, and must be surjective onto $[0, \infty)$.

Case 2: f is surjective on [-c_1,0]

Suppose f is bounded below, let $\lim_{n\to\infty} f(x) \to L_2$, then when $f(y) < 0$ we have

$P(\infty, y): \ L_2 + f(L_1 y) = \frac{L_1}{2} f(2y)$. By a similar argument to case 1, we find $L_2 = 0$, contradicting that f is not constant. Hence $f(x)$ has no lower bound and must be surjective onto $[0, -\infty)$

Conclusion
functions when f(0)=0

When $f(0) = 0$, we know that there exists $2c \in \mathbb{R}$ such that $f(2c) = 4$, hence

$f(cf(c)) = \frac{1}{4} f(2c)^2 = 4 = f(2c)$ So by the fact that f is injective $cf(c) = 2c \Rightarrow f(c) = 2$.

$$P(x, c): \ f(2x) + f(cf(x)) = \frac{1}{2} f(2c) f(2x) = 2f(2x), \implies f(cf(x)) = f(2x)$$
$$\implies f(x) = \frac{2}{c} x$$

Since c can be any real value, let $\frac{2}{c} = k$ we have $f(x) = kx$ $(\star\star)$.

functions when f(0)=4

When $f(0)=4$ the above doesn't work because $c=0$. But we do know that $f(2^nx)=4+2^n[f(x)-4]$. So let $f(x)=g(x)+4$ so that $g(2^nx)=2^ng(x) \qquad (2)$.

Now $P(x,x):\quad f(xf(x))=\frac{1}{4}f(2x)^2=(f(x)-2)^2 \iff g(xg(x)+4x)=g(x)^2+4g(x)$.

Applying (2) gives $g(2^nxg(x)+x)=2^ng(x)^2+g(x)$, which holds for all $n\in\mathbb{Z}, x\in\mathbb{R}^+$

Now there must exist $c\in\mathbb{R}$ such that $f(c)=1$, so, letting $x=c$ gives: $g(2^nc+c)=2^n+1$ and applying (2) gives

$f(2^{n+m}c+2^m)=2^{n+m}+2^m \qquad (3)$ which also holds for all $n,m\in\mathbb{Z}$ and $x\in\mathbb{R}$.

So now we will define a sequence that has a limit at any positive real number we choose, let that limit be $a\in\mathbb{R}^+$, and show that $g(ac)=a$, it will follow that $g(cx)=x$ for all $x\in\mathbb{R}^+$.

So pick two integers $k,\ell\in\mathbb{Z}$ such that $2^k+2^\ell<a$, and let $u_0=2^k+2^\ell$.

Now the next term in the sequence is defined by $u_{n+1}=2^{k_{n+1}}u_n^2+u_n$, where k_{n+1} is the largest possible integer such that $u_{n+1}<a$. Then the limit of this sequence as $n\to\infty$ is a.

But from (3) we have $g(cu_n)=u_n$ for all $n\in\mathbb{N}$, so by the continuity of g,

$\limits_{n\to \infty} g(cu_n) = g\left(\limits_{n\to \infty} cu_n\right) = g(ca) = a$.

This is true for all real $a\in\mathbb{R}^+$, so we have $g(x)=\frac{x}{c}$ or $f(x)=\frac{x}{c}+4$, for some $c\neq 0$. so let $\frac{1}{c}=k$ and $f(x)=kx+4 \qquad (\star\star\star)$

Here are all the solution of f

All functions

$f(x)=kx \qquad k\in\mathbb{R}$

$f(x)=kx+4 \qquad k\in\mathbb{R}$

And when when $k_1\le 0,\ k_2\ge 0$, we also have

$$f(x)=\begin{cases} k_1x & x<0 \\ k_2x & x\ge 0\end{cases}$$

$$f(x)=\begin{cases} k_1x+4 & x<0 \\ k_2x+4 & x\ge 0\end{cases}$$

Stephen
403 posts

Jun 4, 2010, 10:15 am PM #45

Well done, Ocha!

Now what's the new problem?

mahanmath
1355 posts

Jun 4, 2010, 11:33 am PM #46

Find all functions f defined on real numbers and taking real values such that
$f(x)^2+2yf(x)+f(y)=f(y+f(x))$ for all real numbers x,y.

pco
14052 posts

Jun 4, 2010, 1:02 pm PM #47

> *mahanmath wrote:*
> Find all functions f defined on real numbers and taking real values such that
> $f(x)^2+2yf(x)+f(y)=f(y+f(x))$ for all real numbers x,y.

My solution

Let $P(x,y)$ be the assertion $f(x)^2+2yf(x)+f(y)=f(y+f(x))$

$f(x)=0\forall x$ is a solution.
So we'll look from now for non all-zero solutions.

Let $f(a)\neq 0$: $P(a,\frac{u-f(a)^2}{2f(a)}) \implies u=f(\text{something})-f(\text{something else})$ and so any real may be written as a difference $f(v)-f(w)$

$P(w,-f(w)) \implies -f(w)^2+f(-f(w))=f(0)$
$P(v,-f(w)) \implies f(v)^2-2f(v)f(w)+f(-f(w))=f(f(v)-f(w))$

Subtracting the first from the second implies $f(v)^2-2f(v)f(w)+f(w)^2=f(f(v)-f(w))-f(0)$ and so $f(f(v)-f(w))=(f(v)-f(w))^2+f(0)$

And so $f(x)=x^2+f(0)\forall x\in\mathbb{R}$ which indeed is a solution.

Hence the two solutions :
$f(x)=0\forall x$
$f(x)=x^2+a\,\forall x$

I have no problem to submit. Anybody feel free to take my turn.

Pain rinnegan
1581 posts

Jun 4, 2010, 1:39 pm PM #48

Determine all the polynomial functions $f:\mathbb{R}\to\mathbb{R}$, with integer coefficients, which are bijective and satisfy the relation:

$$f^2(x)=f(x^2)-2f(x)+a\ ,\ (\forall)x\in\mathbb{R}$$

where $a \in \mathbb{R}$ is fixed.

pco
14052 posts

Jun 4, 2010, 2:18 pm PM #49

> *Pain rinnegan wrote:*
> Determine all the polynomial functions $f : \mathbb{R} \to \mathbb{R}$, with integer coefficients, which are bijective and satisfy the relation:
>
> $$f^2(x) = f(x^2) - 2f(x) + a\ ,\ (\forall)x \in \mathbb{R}$$
>
> where $a \in \mathbb{R}$ is fixed.

My solution

Let $g(x) = f(x) + 1$. The equation may be written $g(x)^2 = g(x^2) + a$

So $g(x)^2 = g(-x)^2$ and two possibilities :

1) $g(x)$ is odd
So $g(0) = 0$ and so $a = 0$ and we get $g(x)^2 = g(x^2)$
It's easy to see that if $\rho e^{i\theta}$ is root of $g(x)$, then so is $\sqrt{\rho} e^{i\frac{\theta}{2}}$ and so the only roots may be 0 and 1 and, since 1 does not fit, the only odd polynomials matching $g(x)^2 = g(x^2)$ are 0 and x^{2n+1}

2) $g(x)$ is even
Then :
Either $g(x) = c$ and $c^2 - c = a$
Either $g(x) = h(x^2)$ and the equation becomes $h(x^2)^2 = h(x^4) + a$ and so $h(x)^2 = h(x^2) + a$ (remember these are polynomials).
We then just have to repeat the process and the conclusion is :

The only solutions for $g(x)$ are $g(x) = c$ and $g(x) = x^n$

Hence the only solutions for $f(x)$:

If $\nexists c \in \mathbb{Z}$ such that $a = c^2 - c$, then no solution.
If $a = c^2 - c$ for some integer c, we have a solution $f(x) = c - 1$
If $a = 0$, we also have $f(x) = x^n - 1$

I have no problem to submit. Anybody feel free to take my turn.

MohammadP
58 posts

Jun 4, 2010, 7:22 pm PM #50

Problem 17:
k is a nonzero constant.
find all functions satisfying :
$f(xy) = f(x)f(y)$ and $f(x + k) = f(x) + f(k)$

Stephen
403 posts

Jun 5, 2010, 9:40 am PM #51

Solution to **Problem 17:**

$f(y)f(x) + f(y)f(k) = f(y)f(x + k)$

$f(xy) + f(ky) = f(xy + yk)$

Now we are going to prove $f(x + ky) = f(x) + f(ky)$.

If $y = 0$, it's easy since $f(0) = 0$.

If $y \neq 0$, then we can put $\frac{x}{y}$ in x of $f(xy) + f(ky) = f(xy + yk)$.

So $f(x + ky) = f(x) + f(ky)$.

Now, since k isn't 0, we can put $\frac{y}{k}$ in y of $f(x + ky) = f(x) + f(ky)$.

So $f(x + y) = f(x) + f(y)$.

Since is an Cauchy equation, we can know that for some constant c, that $f(q) = cq$ when q is an rational number.

But because of $f(xy) = f(x)f(y)$, c is 0 or 1.

If $c = 0$, then we can easily know that $f(x) = 0$ for all real number x.

If $c = 1$, then $f(q) = q$.

Now let's prove $f(x) = x$.

Since $f(xy) = f(x)f(y)$, $f(x^2) = (f(x))^2$.

So if $x > 0$, then $f(x) > 0$ since $f(x) \neq 0$.

But $f(-x) = -f(x)$. So if $x < 0$, then $f(x) < 0$.

Now let a a constant that satisfies $f(a) > a$.

Then if we let $f(a) = b$, there is a rational number p that satisfies $b > p > a$.

So, $f(p - a) + f(a) = f(p) = p$.

So, $f(p - a) = p - f(a) = p - b < 0$.

But, $p - a > 0$. So a contradiction!

So we can know that $f(x) \leq x$.

With a similar way, we can know that $f(x) \geq x$.

So $f(x) = x$.

We can conclude that possible functions are $f(x) = 0$ and $f(x) = x$.

Problem 18:

Find all continuous and strictly-decreasing function $f : R^+ \to R^+$ that satisfies

$f(x+y) + f(f(x) + f(y)) = f(f(x + f(y)) + f(y + f(x)))$

pco
14052 posts

Jun 6, 2010, 1:46 pm PM #52

> *Stephen wrote:*
> **Problem 18:**
>
> Find all continuous and strictly-decreasing function $f : R^+ \to R^+$ that satisfies
>
> $f(x+y) + f(f(x) + f(y)) = f(f(x + f(y)) + f(y + f(x)))$

Hello, could you confirm us that you have the full solution of this problem and could you give us a hint ?

Up to now, I proved that $f(f(x)) = x$ and that $f(x) = \frac{a}{x}$ is a solution, but I'm still unable to prove if this is the only solution

Thanks for any hint.

Stephen
403 posts

Jun 6, 2010, 2:32 pm PM #53

> *pco wrote:*
>
> > *Stephen wrote:*
> > **Problem 18:**
> >
> > Find all continuous and strictly-decreasing function $f : R^+ \to R^+$ that satisfies
> >
> > $f(x+y) + f(f(x) + f(y)) = f(f(x + f(y)) + f(y + f(x)))$
>
> Hello, could you confirm us that you have the full solution of this problem and could you give us a hint ?
>
> Up to now, I proved that $f(f(x)) = x$ and that $f(x) = \frac{a}{x}$ is a solution, but I'm still unable to prove if this is the only solution
>
> Thanks for any hint.

I am really very sorry, but I don't have a solution, and I'm working on it.

To tell the truth, I just post this because I'm very curious of a solution.

But I can give you some advice.

I also did $f(f(x)) = x$ and know that $f(x) = \frac{c}{x}$ is a solution.

Now, I'm trying to use k that satisfies $f(k) = k$.

Since f is continuous, we can easily prove that there exists a k that $f(k) = k$.

Good luck!

mahanmath
1355 posts

Jun 6, 2010, 3:11 pm PM #54

Problem 18 was hard (maybe open!) and stoped the marathon, and I believe marathon should be alive !!,so I submit an easy problem.
Problem 19
Find all functions $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ of two variables satisfying for all x, y:
$f(x,x) = x, f(x,y) = f(y,x), (x+y)f(x,y) = yf(x,x+y)$.

Farenhajt
5170 posts

Jun 6, 2010, 8:31 pm • 2 PM #55

> *mahanmath wrote:*
> Problem 18 was hard (maybe open!) and stoped the marathon, and I believe marathon should be alive !!,so I submit an easy problem.
> **Problem 19**
> Find all functions $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ of two variables satisfying for all x, y:
> $f(x,x) = x, f(x,y) = f(y,x), (x+y)f(x,y) = yf(x,x+y)$.

Substituting $f(x,y) = \frac{xy}{g(x,y)}$ we get $g(x,x) = x, g(x,y) = g(y,x), g(x,y) = g(x,x+y)$. Putting $z := x + y$, the last condition becomes $g(x,z) = g(x, z-x)$ for $z > x$. With $g(x,x) = x$ and symmetry, it is now obvious, by Euclidean algorithm, that $g(x,y) = \gcd(x,y)$, therefore $f(x,y) = [x,y]$

Someone else can post the next problem.
This post has been edited 1 time. Last edited by Farenhajt, Jun 6, 2010, 10:15 pm

Stephen
403 posts

Jun 6, 2010, 9:37 pm PM #56

Firstly I am very sorry for delaying the marathon.

Problem 20

Prove that $f(x+y+xy) = f(x) + f(y) + f(xy)$ is equivalent to $f(x+y) = f(x) + f(y)$.

pco
14052 posts

Jun 7, 2010, 12:58 am PM #57

> *Stephen wrote:*
> **Problem 20**
>
> Prove that $f(x+y+xy) = f(x) + f(y) + f(xy)$ is equivalent to $f(x+y) = f(x) + f(y)$.

My solution

Let $P(x,y)$ be the assertion $f(x+y+xy)=f(x)+f(y)+f(xy)$

1) $f(x+y)=f(x)+f(y)\implies P(x,y)$
===========================
Trivial.

2) $P(x,y)\implies f(x+y)=f(x)+f(y)\forall x,y$
=====================================
$P(x,0)\implies f(0)=0$
$P(x,-1)\implies f(-x)=-f(x)$

2.1) new assertion $R(x,y)$: $f(x+y)=f(x)+f(y)\forall x,y$ such that $x+y\neq -2$

Let x,y such that $x+y\neq -2$:
$P(\frac{x+y}{2},\frac{x-y}{x+y-2})\implies f(x)=f(\frac{x+y}{2})+f(\frac{x-y}{x+y-2})+f(\frac{x^2-y^2}{x+y-2})$

$P(\frac{x+y}{2},\frac{y-x}{x+y-2})\implies f(y)=f(\frac{x+y}{2})-f(\frac{x-y}{x+y-2})-f(\frac{x^2-y^2}{x+y-2})$

Adding these two lines gives new assertion $Q(x,y)$: $f(x)+f(y)=2f(\frac{x+y}{2})\forall x,y$ such that $x+y\neq -2$
$Q(x+y,0)\implies f(x+y)=2f(\frac{x+y}{2})$
and so $f(x+y)=f(x)+f(y)$
Q.E.D.

2.2) $f(x+y)=f(x)+f(y)\forall x,y$ such that $x+y=-2$

If $x=-2$, then $y=0$ and $f(x+y)=f(x)+f(y)$
If $x\neq -2$, then $(x+2)+(-2)\neq -2$ and then $R(x+2,-2)\implies f(x)=f(x+2)+f(-2)$ and so
$f(x)+f(-2-x)=f(-2)$ and so $f(x)+f(y)=f(x+y)$

Q.E.D.

I have no problem to submit. Anybody feel free to take my turn.

Stephen
403 posts

Jun 8, 2010, 2:21 pm PM #58

Problem 21

Find all $f:Z\to Z$ that satisfies $f(x)^3+f(y)^3+f(z)^3=f(x^3+y^3+z^3)$.

---

mahanmath
1355 posts

Jun 8, 2010, 3:38 pm PM #59

> Stephen wrote:
> **Problem 21**
>
> Find all $f:Z\to Z$ that satisfies $f(x)^3+f(y)^3+f(z)^3=f(x^3+y^3+z^3)$.

#)If $x\geq 4$, x^3 can be written as a sum of five cubes such that their absolute values are less than x.
From # and induction we get the answer is $f(x)=xf(1)$ and $f(1)=1,0,-1$.

In fact , It`s a number theory problem !....

I have no problem to submit ...

---

goodar2006
1344 posts

Jun 8, 2010, 4:52 pm • 1 PM #60

Problem 22
find all functions $f:R\longrightarrow R$ such that
$f(f(x)+y)=2x+f(f(y)-x)$

This post has been edited 1 time. Last edited by goodar2006, Aug 20, 2011, 1:57 am

---

pco
14052 posts

Jun 8, 2010, 6:39 pm • 1 PM #61

> goodar2006 wrote:
> Problem 22
> find all functions $f:R\longrightarrow R$ such that
> $f(f(x)+y)=2x+f(f(y)-x)$

My solution

Let $P(x,y)$ be the assertion $f(f(x)+y)=2x+f(f(y)-x)$

$P(\frac{f(0)-x}{2},-f(\frac{f(0)-x}{2}))\implies x=f(f(-f(\frac{f(0)-x}{2}))-\frac{f(0)-x}{2})$ and so $f(x)$ is surjective.

So :
$\exists u$ such that $f(u)=0$
$\exists v$ such that $f(v)=x+u$

And then $P(u,v)\implies f(x)=x-u$ which indeed ,is a solution

Hence the answer : $\boxed{f(x)=x+c}$

I have no problem to submit. Anybody feel free to take my turn.

Zeus93
64 posts

Jun 9, 2010, 12:04 am PM #62

Now the 23th:
Find all $f:N\longrightarrow N$ such that
$f(f(n))+f(n+1)=n+2$

If someone see it easily, please help me in this one:
Problem 24h :Find $f : R^+ \longrightarrow R^+$ such that :
$f(x)f(yf(x)) = f(x+y)$ for all $x, y \in \mathbb{R}^+$
This post has been edited 1 time. Last edited by Zeus93, Jun 10, 2010, 6:47 pm

Justhalf
45 posts
Jun 10, 2010, 6:30 am PM #63

> *Jumong4958 wrote:*
> Now the 23th:
> Find $f : N \longrightarrow N$ such that
> $f(f(n)) + f(n+1) = n+2$

Problem 23 is almost there...
I can get all values of $f(n)$ but I cannot find the general formula for it.
Subtitute $n = 1$.
We get $f(f(1)) + f(2) = 3$
Since $f(n) \in \mathbb{N}$ then there are two cases:
Case 1: $f(f(1)) = 2$ and $f(2) = 1$
Subtitute $n = 2$, then we get:
$f(1) + f(3) = 4$.
Now we have three cases:
Case 1.1: $f(1) = 1, f(3) = 3 \Rightarrow f(f(1)) = 1$ contradiction
Case 1.2: $f(1) = 2, f(3) = 2 \Rightarrow f(f(1)) = 1$ contradiction
Case 1.3: $f(1) = 3, f(3) = 1. f(f(1)) = 2 \Rightarrow f(3) = 2$ contradiction
So case 1 fails.
So case 2 must hold.
Case 2: $f(f(1)) = 1, f(2) = 2$
Then by induction we can get:
$f(2) = 2$
$f(3) = 2$
$f(4) = 3$
$f(5) = 4$
$f(6) = 4$
$f(7) = 5$
$f(8) = 5$
$f(9) = 6$
$f(10) = 7$
$f(11) = 7$
$f(12) = 8$
$f(13) = 9$
$f(14) = 9$
$f(15) = 10$
$f(16) = 10$
$f(17) = 11$
$f(18) = 12$
$f(19) = 12$
$f(20) = 13$
$f(21) = 13$
$f(22) = 14$
$f(23) = 15$
$f(24) = 15$
$f(25) = 16$
$f(26) = 17$
$f(27) = 17$
$f(28) = 18$
$f(29) = 18$
$f(30) = 19$
$f(31) = 20$

etc..
And since for all n, $1 < f(n) < n$ then $f(1) = 1$.
so we have the complete list of the values. But I cannot get the general formula.

ocha
955 posts
Jun 10, 2010, 8:39 am PM #64

> *Justhalf wrote:*
> so we have the complete list of the values. But I cannot get the general formula.

$$f(n) = \left\lceil \frac{5n}{8} \right\rceil$$

mahanmath
1355 posts
Jun 10, 2010, 10:16 am PM #65

> *Jumong4958 wrote:*
> Problem 24h :Find $f : R^+ \longrightarrow R^+$ such that :
> $f(x)f(yf(x)) = f(x+y)$ for all $x, y \in \mathbb{R}^+$

Find or Find all ? Because it has a trivial solution , $f(x) = 1$!

Justhalf
45 posts
Jun 10, 2010, 10:48 am PM #66

> *ocha wrote:*
>
> > *Justhalf wrote:*
> > so we have the complete list of the values. But I cannot get the general formula.
>
> $$f(n) = \left\lceil \frac{5n}{8} \right\rceil$$

Wow, how can you find that?

ocha
955 posts
Jun 10, 2010, 11:20 am • 1 PM #67

> *Justhalf wrote:*

> ocha wrote:
> $f(n) = \left\lceil \frac{5n}{8} \right\rceil$
>
> Wow, how can you find that?

Just looking at the sequence. There is a pattern in the difference of consecutive terms that repeates after every 8 values of n, and causes $f(n)$ to increase 5. A rough estimate would be $f(n) = \frac{5n}{8}$. Then it isn't hard to motivate $f(n) = \left\lceil \frac{5}{8} \right\rceil$

Justhalf
45 posts

Jun 10, 2010, 11:31 am PM #68

> ocha wrote:
>> Justhalf wrote:
>>> ocha wrote:
>>> $f(n) = \left\lceil \frac{5n}{8} \right\rceil$
>>
>> Wow, how can you find that?
>
> Just looking at the sequence. There is a pattern in the difference of consecutive terms that repeates after every 8 values of n, and causes $f(n)$ to increase 5. A rough estimate would be $f(n) = \frac{5n}{8}$. Then it isn't hard to motivate $f(n) = \left\lceil \frac{5}{8} \right\rceil$

Then how we proof that it is indeed a solution?
It seems complicated to me.

---

ocha
955 posts

Jun 10, 2010, 12:56 pm PM #69

> Justhalf wrote:
> Then how we proof that it is indeed a solution?
> It seems complicated to me.

Well observe the pattern: the differences $a_{i+1} - a_i$ are $0, 1, 1, 0, 1, 0, 1$ then repeat (1)

write $n = 8q + r$ with $1 \le r \le 7$. then $f(n) = \left\lceil \frac{5(8q+r)}{8} \right\rceil = 5q + \left\lceil \frac{5r}{8} \right\rceil$.

So we have reduced the problem to showing the terms $a_r = \left\lceil \frac{5r}{8} \right\rceil$ follow the pattern (1). when $1 \le r \le 6$, which is easy.

And finall if $n = 8q + 7$ then $a_{n+1} - a_n = 5(q+1) - \left(5q + \left\lceil \frac{5 \cdot 7}{8} \right\rceil\right) = 0$ which is the first term in (1)

So we are done.

---

pco
14052 posts

Jun 10, 2010, 2:18 pm PM #70

> mahanmath wrote:
>> Jumong4958 wrote:
>> Problem 24h :Find $f : R^+ \longrightarrow R^+$ such that :
>> $f(x)f(yf(x)) = f(x+y)$ for all $x, y \in \mathbb{R}^+$
>
> Find or Find all ? Because it has a trivial solution , $f(x) = 1$!

A non constant solution is $f(x) = \frac{1}{x+1}$

All non constant solutions are continuous, strictly decreasing from $1 \to 0$

But I did not succeed up to now to show that $\frac{1}{x+1}$ is the only non constant solution.

Justhalf
45 posts

Jun 10, 2010, 2:45 pm PM #71

> ocha wrote:
>> Justhalf wrote:
>> Then how we proof that it is indeed a solution?
>> It seems complicated to me.
>
> Well observe the pattern: the differences $a_{i+1} - a_i$ are $0, 1, 1, 0, 1, 0, 1$ then repeat (1)
>
> write $n = 8q + r$ with $1 \le r \le 7$. then $f(n) = \left\lceil \frac{5(8q+r)}{8} \right\rceil = 5q + \left\lceil \frac{5r}{8} \right\rceil$.
>
> So we have reduced the problem to showing the terms $a_r = \left\lceil \frac{5r}{8} \right\rceil$ follow the pattern (1). when $1 \le r \le 6$, which is easy.
>
> And finall if $n = 8q + 7$ then $a_{n+1} - a_n = 5(q+1) - \left(5q + \left\lceil \frac{5 \cdot 7}{8} \right\rceil\right) = 0$ which is the first term in (1)
>
> So we are done.

I think you are proving that your function satisfy the pattern. But what I asked is how do you prove that the function satisfy the condition stated in the problem.

Farenhajt
5170 posts

Jun 10, 2010, 3:19 pm • 11 PM #72

> ocha wrote:
> $f(n)=\left\lceil\frac{5n}{8}\right\rceil$

Ocha, are you sure about this?

Plugging $f(n)=\left\lceil\frac{5n}{8}\right\rceil$ into the initial equation, and writing $n=8q+r$ where $0\leqslant r\leqslant 7$, we find that the following must hold:

$$3q+\left\lceil\frac{q+5\left\lceil\frac{5r}{8}\right\rceil}{8}\right\rceil+5q+\left\lceil\frac{5r+5}{8}\right\rceil=8q+r+2$$

But after we cancel $8q$ from the both sides, we're left with the LHS that depends on q, hence is unbounded, and with the RHS which is bounded.

Motivated by $f(n)=\left\lceil\frac{an}{b}\right\rceil$, and putting the condition that q must disappear from the both sides, we arrive at $\frac{a^2}{b}+a=b\implies\frac{a}{b}=\frac{\sqrt5-1}{2}$. Hence we now have to prove that $f(n)=\left\lceil\frac{\sqrt5-1}{2}n\right\rceil=\left\lfloor\frac{\sqrt5-1}{2}n\right\rfloor+1$ satisfies the initial equation (the latter form of the function holds as the expression under the ceiling sign is never an integer).

(Note that $\frac{\sqrt5-1}{2}\approx\frac{5}{8}$, hence the difference between the two functions will start to show up only for a sufficiently large n, thus the first few values can definitely be misleading.)

For easier writing we'll put $\phi:=\frac{\sqrt5-1}{2}$ and use the designation $[\cdot]$ for the integer part function. Then we have to prove

$$[\phi([n\phi]+1)]+1+[n\phi+\phi]+1=n+2$$

$$[\phi[n\phi]+\phi]+[n\phi+\phi]=n$$

$$[n\phi^2-\phi\{n\phi\}+\phi]+[n\phi+\phi]=n$$

Since $\phi^2+\phi-1=0\implies\phi^2=1-\phi$, we have

$$[n-n\phi-\phi\{n\phi\}+\phi]+[n\phi+\phi]=n$$

$$[-n\phi-\phi\{n\phi\}+\phi]+[n\phi+\phi]=0$$

$$-[n\phi]+[-\{n\phi\}-\phi\{n\phi\}+\phi]+[n\phi]+[\phi+\{n\phi\}]=0$$

$$[\phi-(1+\phi)\{n\phi\}]+[\phi+\{n\phi\}]=0\quad(*)$$

First we'll note that $\{n\phi\}\neq 1-\phi$ for all natural n, since otherwise we'd have $n\phi-[n\phi]=1-\phi\iff(n+1)\phi=[n\phi]+1$, implying $\phi\in\mathbb{Q}$. Therefore we have either $0<\{n\phi\}<1-\phi$ or $1-\phi<\{n\phi\}<1$.

In the first case, $\phi-(1+\phi)(1-\phi)<\phi-(1+\phi)\{n\phi\}<\phi\iff 0<\phi-(1+\phi)\{n\phi\}<\phi$, hence $[\phi-(1+\phi)\{n\phi\}]=0$, and $\phi<\phi+\{n\phi\}<1\implies[\phi+\{n\phi\}]=0$, therefore $(*)$ is satisfied.

In the second case, $\phi-(1+\phi)<\phi-(1+\phi)\{n\phi\}<\phi-(1+\phi)(1-\phi)\iff -1<\phi-(1+\phi)\{n\phi\}<0$, hence $[\phi-(1+\phi)\{n\phi\}]=-1$, and $1<\phi+\{n\phi\}<1+\phi\implies[\phi+\{n\phi\}]=1$, therefore $(*)$ is also satisfied.

So we finally conclude that the solution to the given equation is $f(n)=\left\lfloor\frac{(\sqrt5-1)n}{2}\right\rfloor+1$

This post has been edited 3 times. Last edited by Farenhajt, Jun 11, 2010, 2:23 pm

Justhalf
45 posts

Jun 10, 2010, 4:31 pm PM #73

> Farenhajt wrote:
>> ocha wrote:
>> $f(n)=\left\lceil\frac{5n}{8}\right\rceil$
>
> Ocha, are you sure about this?
>
> Plugging $f(n)=\left\lceil\frac{5n}{8}\right\rceil$ into the initial equation, and writing $n=8q+r$ where $0\leqslant r\leqslant 7$, we find that the following must hold:
>
> $$3q+\left\lceil\frac{q+5\left\lceil\frac{5r}{8}\right\rceil}{8}\right\rceil+5q+\left\lceil\frac{5r+5}{8}\right\rceil=8q+r+2$$
>
> But after we cancel $8q$ from the both sides, we're left with the LHS that depends on q, hence is unbounded, and with the RHS which is bounded.
>
> Motivated by $f(n)=\left\lceil\frac{an}{b}\right\rceil$, and putting the condition that q must disappear from the both sides, we arrive at $\frac{a^2}{b}+a=b\implies\frac{a}{b}=\frac{\sqrt5-1}{2}$. Hence we now have to prove that $f(n)=\left\lceil\frac{\sqrt5-1}{2}n\right\rceil=\left\lfloor\frac{\sqrt5-1}{2}n\right\rfloor+1$ satisfies the initial equation (the latter form of the function holds as the expression under the ceiling sign is never an integer).
>
> (Note that $\frac{\sqrt5-1}{2}\approx\frac{5}{8}$, hence the difference between the two functions will start to show up only for a sufficiently large n, thus the first few values can definitely be misleading.)
>
> For easier writing we'll put $\phi:=\frac{\sqrt5-1}{2}$ and use the designation $[\cdot]$ for the integer part function. Then we have to prove

$$[\phi([n\phi]+1)]+1+[n\phi+\phi]+1=n+2$$

$$[\phi[n\phi]+\phi]+[n\phi+\phi]=n$$

$$[n\phi^2-\phi\{n\phi\}+\phi]+[n\phi+\phi]=n$$

Since $\phi^2+\phi-1=0 \implies \phi^2=1-\phi$, we have

$$[n-n\phi-\phi\{n\phi\}+\phi]+[n\phi+\phi]=n$$

$$[-n\phi-\phi\{n\phi\}+\phi]+[n\phi+\phi]=0$$

$$-[n\phi]+[-\{n\phi\}-\phi\{n\phi\}+\phi]+[n\phi]+[\phi+\{n\phi\}]=0$$

$$[\phi-(1+\phi)\{n\phi\}]+[\phi+\{n\phi\}]=0 \quad (*)$$

First we'll note that $\{n\phi\}\neq 1-\phi$ for all natural n, since otherwise we'd have $n\phi-[n\phi]=1-\phi \iff (n+1)\phi=[n\phi]+1$, implying $\phi\in\mathbb{Q}$. Therefore we have either $0<\{n\phi\}<1-\phi$ or $1-\phi<\{n\phi\}<1$.

In the first case, $\phi-(1+\phi)(1-\phi)<\phi-(1+\phi)\{n\phi\}<\phi \iff 0<\phi-(1+\phi)\{n\phi\}<\phi$, hence $[\phi-(1+\phi)\{n\phi\}]=0$, and $\phi<\phi+\{n\phi\}<1 \implies [\phi+\{n\phi\}]=0$, therefore $(*)$ is satisfied.

In the second case, $\phi-(1+\phi)<\phi-(1+\phi)\{n\phi\}<\phi-(1+\phi)(1-\phi) \iff -1<\phi-(1+\phi)\{n\phi\}<0$, hence $[\phi-(1+\phi)\{n\phi\}]=-1$, and $1<\phi+\{n\phi\}<1+\phi \implies [\phi+\{n\phi\}]=1$, therefore $(*)$ is also satisfied.

So we finally conclude that the solution to the given equation is $f(n)=\left[\dfrac{(\sqrt{5}-1)n}{2}\right]+1$

Ah, this is what I call solution... =D
Thanks Farenhajt

This post has been edited 1 time. Last edited by Justhalf, Jun 10, 2010, 5:56 pm

---

Farenhajt
5170 posts

Jun 10, 2010, 5:33 pm • 3 PM #74

> *Justhalf wrote:*
> Ah, this is what I call solution... =D
> Thanks Farenhajt

Thanks to you too. There were two typos though - confer the red markings in my original post and (if you like) edit your quotation accordingly.

Zeus93
64 posts

Jun 10, 2010, 6:46 pm PM #75

> *mahanmath wrote:*
>
> > *Jumong4958 wrote:*
> > Problem 24h :Find $f:R^+\longrightarrow R^+$ such that :
> > $f(x)f(yf(x))=f(x+y)$ for all $x,y\in\mathbb{R}^+$
>
> Find or Find all ? Because it has a trivial solution , $f(x)=1$!

Oh sorry, "Find all", not "Find"

---

ocha
955 posts

Jun 11, 2010, 10:55 am • 5 PM #76

Who went through this thread and marked everything as spam? Usually i wouldn't care about ratings, but even Farenhajt's nice solution got rated 1!

> *Jumong4958 wrote:*
> Problem 24h :Find $f:R^+\longrightarrow R^+$ such that :
> $f(x)f(yf(x))=f(x+y)$ for all $x,y\in\mathbb{R}^+4$

I have finished pco's solution
solution

There is a trivial solution $f(x)\equiv 1$, now suppose f is not constant. Let $P(x,y)$ be the assertion $f(x)f(yf(x))=f(x+y)$. We show that f is strictly decreasing $(\star)$.

For the sake of contradiction assume there existed $a\in\mathbb{R}^+$ such that $f(a)>1$. Note that $f(x)>0\forall x\in\mathbb{R}^+$, therefore

$P(a,\dfrac{a}{f(a)-1})\implies f(a)f\left(\dfrac{af(a)}{f(a)+1}\right)=f\left(\dfrac{af(a)}{f(a)+1}\right)\implies f(a)=1$ which contradicts our original assumption. Therefore $f(x)\leq 1$ for all x

So for any $\epsilon>0$, $P(x,\epsilon)\Rightarrow f(x+\epsilon)=f(x)f(\text{something})\leq f(x)$, hence f is monotonic decreasing $\qquad (1)$.

Now suppose there existed $x_0\in\mathbb{R}^+$ such that $f(x_0)=1$, then $P(x_0,x_0)\Rightarrow f(x_0)^2=f(2x_0)$, so $f(2x_0)=1$. By induction $f(2^nx_0)=1$ for all $n\in\mathbb{N}$, but since f is monotonic decreasing and $f\leq 1$, we must have f constant, contradicting our assumption. Therefore $\nexists x_0\in\mathbb{R}$ such that $f(x_0)=1$. Now using that result in (1), it follows that f is strictly decreasing $(\star)$.

Now for some $s,t\in\mathbb{R}^+$

$$P(s,\frac{t}{f(s)})\Rightarrow f(s)f(t)=f(s+\frac{t}{f(s)}) \qquad (2)$$

Also, $P(t,\dfrac{s}{f(t)})\Rightarrow f(t)f(s)=f(t+\dfrac{s}{f(t)}) \qquad (3)$

Since f is strictly decreasing, it is injective. So from (2) and (3),

$$s+\frac{t}{f(s)}=t+\frac{s}{f(t)}\Leftrightarrow \frac{1}{s}\left(\frac{1}{f(s)}-1\right)=\frac{1}{t}\left(\frac{1}{f(t)}-1\right).$$

Since it is true for all s, t we have $\frac{1}{s}\left(\frac{1}{f(s)} - 1\right) = k > 0$ is constant.

$\therefore f(x) = \frac{1}{kx+1}$ for any $k > 0$. Checking back into the equation shows it is indeed a solution.

This post has been edited 1 time. Last edited by ocha, Jun 11, 2010, 11:42 am

---

Justhalf
45 posts

Jun 11, 2010, 11:37 am PM #77

> *ocha wrote:*
> Since it is true for all s, t we have $\frac{1}{s}\left(1 - \frac{1}{f(s)}\right) = k > 0$ is constant.

Ahhh, I missed the injectivity...
Anyway, I think for this part $k < 0$, because $1 - \frac{1}{f(s)} < 0$

Farenhajt
5170 posts

Jun 11, 2010, 2:15 pm • 3 PM #78

> *ocha wrote:*
> Who went through this thread and marked everything as spam? Usually i wouldn't care about ratings, but even Farenhajt's nice solution got rated 1!

Someone's on the run - I've got over 20 of my posts spam-rated within a few hours. I contacted the administrators and they're looking into it. Anyone experiencing similar problem should do the same.

---

goodar2006
1344 posts

Jun 12, 2010, 12:50 pm PM #79

let's continue the marathon :
problem 25
find all functions $f : R \longrightarrow R$ such that
$f(xf(y) + f(x)) = f(yf(x)) + x$
problem 26
find all functions $f : R \longrightarrow R$ such that
$f(x^2 + f(y)) = y + f(x)^2$

---

jgnr
1344 posts

Jun 12, 2010, 5:39 pm PM #80

> *goodar2006 wrote:*
> let's continue the marathon :
> **problem 25**
> find all functions $f : R \longrightarrow R$ such that
> $f(xf(y) + f(x)) = f(yf(x)) + x$

Substitute $y = 0$, f is surjective. Substitute y such that $f(y) = 0$, f is injective. Substitute $x = y = 0$, $f(f(0)) = 0$, $f(0) = 0$. Substitute $y = 0$, $f(f(x)) = x$. Substitute $x = f(a)$, $f(f(a)f(y) + a) = f(f(a)f(y)) + f(a)$. For $a \neq 0$, $f(a)f(y)$ can take any real value b, so $f(b + a) = f(b) + f(a)$ for $a \neq 0$. For $a = 0$, $f(b + a) = f(b) = f(b) + f(0)$, hence $f(b + a) = f(b) + f(a)$ for all reals a, b. So $f(n) = cn$ for any rational number n. Substitute this to $f(f(x)) = x$, we get $n = \pm 1$. So $f(1) = \pm 1$. If $f(1) = 1$, substitute $x = 1, y = f(a - 1)$, $f(a) = a$. If $f(1) = -1$, note that from $f(b + a) = f(b) + f(a)$ we get f is odd, hence substituting $x = 1, y = f(a + 1)$ we get $f(a) = f(-f(a + 1)) + 1 = -f(f(a + 1)) + 1 = -(a + 1) + 1 = -a$. So the functions are $f(x) = x$ for all x and $f(x) = -x$ for all x, it is easy to check that these functions satisfy the given requirements.

Let's solve problem 26 now.

MohammadP
58 posts

Jun 12, 2010, 7:59 pm PM #81

I think your answer is not correct.

> *Johan Gunardi wrote:*
> Substitute $x = f(a)$, $f(f(a)f(y) + a) = f(f(a)f(y)) + f(a)$.

how did you conclude it ?

Justhalf
45 posts

Jun 12, 2010, 11:25 pm PM #82

> *MohammadP wrote:*
> I think your answer is not correct.
>
> > *Johan Gunardi wrote:*
> > Substitute $x = f(a)$, $f(f(a)f(y) + a) = f(f(a)f(y)) + f(a)$.
>
> how did you conclude it ?

Using $f(f(a)) = a$ for any $a \in \mathbb{R}$

---

pco
14052 posts

Jun 13, 2010, 11:47 am • 2 PM #83

> *goodar2006 wrote:*
> **problem 26**
> find all functions $f : R \longrightarrow R$ such that
> $f(x^2 + f(y)) = y + f(x)^2$

My solution

Let $P(x, y)$ be the assertion $f(x^2 + f(y)) = y + f(x)^2$

$P(0, y) \implies f(f(y)) = y + f(0)^2$ and then : $P(x, f(y - f(0)^2)) \implies f(x^2 + y) = f(y - f(0)^2) + f(x)^2$
Setting $x = 0$ in this last equality, we get $f(y) = f(y - f(0)^2) + f(0)^2$ and so $f(x^2 + y) = f(y) + f(x)^2 - f(0)^2$
Setting $y = 0$ in this last equality, we get $f(x^2) = f(0) + f(x)^2 - f(0)^2$ and so $f(x^2 + y) = f(y) + f(x^2) - f(0)$

Let then $g(x) = f(x) - f(0)$. We got $g(x + y) = g(x) + g(y) \forall x \geq 0, \forall y$
It's immediate to establish $g(0) = 0$ and $g(-x) = -g(x)$ and so $g(x + y) = g(x) + g(y) \forall x, y$

$P(x, 0) \implies f(x^2 + f(0)) = f(x)^2 \implies f(x^2 + f(0)) - f(0) = f(x)^2 - f(0)$ and so $g(x) \geq -f(0) \forall x \geq f(0)$

So $g(x)$ is a solution of Cauchy equation with a lower bound on some non empty open interval.

> So $g(x)=ax$ and $f(x)=ax+b$
>
> Plugging this back in original equation, we get $a=1$ and $b=0$ and the unique solution $\boxed{f(x)=x}$

I have no problem to submit. Anybody feel free to take my turn.

Rijul saini
799 posts

Jun 13, 2010, 5:37 pm PM #84

Problem 27
If $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that

$$f(x^3+y^3)=(x+y)(f(x)^2-f(x)f(y)+f(y)^2)$$

then prove that $f(1996\cdot x)=1996f(x)$

---

pco
14052 posts

Jun 14, 2010, 12:26 pm • 2 PM #85

> *Rijul saini wrote:*
> **Problem 27**
> If $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that
>
> $$f(x^3+y^3)=(x+y)(f(x)^2-f(x)f(y)+f(y)^2)$$
>
> then prove that $f(1996\cdot x)=1996f(x)$

My solution

> Let $P(x,y)$ be the assertion $f(x^3+y^3)=(x+y)(f(x)^2-f(x)f(y)+f(y)^2)$
> Let $A=\{a\in\mathbb{R}$ such that $f(ax)=af(x)\forall x\in\mathbb{R}\}$
> $1\in A$
> $P(0,0)\implies f(0)=0$ and so $0\in A$
>
> $P(\sqrt[3]{x},0)\implies f(x)=\sqrt[3]{x}f(\sqrt[3]{x})^2$ and so $f(x)=0$ or $f(x)$ and x have same sign.
>
> Let $a\neq 0\in A$ and let $x\neq 0$ Then :
> $P(\sqrt[3]{ax},0)\implies f(ax^3)=\sqrt[3]{ax}f(\sqrt[3]{ax})^2$ and, since $f(ax^3)=af(x^3)$: $(\sqrt[3]{a})^2f(x^3)=xf(\sqrt[3]{ax})^2$
> $P(x,0)\implies f(x^3)=xf(x)^2$
> So $f(\sqrt[3]{ax})^2=(\sqrt[3]{a})^2f(x)^2$
> And, since $f(x)=0$ or $f(x)$ and x have same sign : $f(\sqrt[3]{ax})=\sqrt[3]{a}f(x)$ and this is still true if $a=0$ or $x=0$
> So (1) $a\in A\implies\sqrt[3]{a}\in A$
>
> Let $a\in A$. Then :
> $P(x,0)\implies f(x^3)=xf(x)^2$
> $P(ax,x)\implies f((a^3+1)x^3)=(a+1)x(a^2f(x)^2-af(x)^2+f(x)^2)=(a^3+1)xf(x)^2=(a^3+1)f(x^3)$
> So (2) $a\in A\implies a^3+1\in A$
>
> So $a\in A\implies$(using (1)) $\sqrt[3]{a}\in A\implies$(using (2)) $a+1\in A$
>
> And since $1\in A$, we get with immediate induction $1996\in A$

I have no problem to submit. Anybody feel free to take my turn.

---

nguyenvut...
475 posts

Jun 15, 2010, 6:15 pm PM #86

Problem 28 :

Find all surjective functions $f:\mathbb{R}\mapsto\mathbb{R}$ such that :

$f(f(x-y))\ =\ f(x)-f(y)\ \forall\ x;y\ \in\mathbb{R}$

Problem 29 :

Find all real number k such that there exists a function $f:\mathbb{R}\mapsto\mathbb{R}$ such that :

f *is differentiable in* $\mathbb{R}$, $f(1)\le 1$

and $(f(x))^2+(f^{'}(x))^2\ =\ k$

Farenhajt
5170 posts

Jun 15, 2010, 8:54 pm • 1 PM #87

> *nguyenvuthanhha wrote:*
> Problem 28 :
>
> *Find all surjective functions $f:\mathbb{R}\mapsto\mathbb{R}$ such that :*
>
> $f(f(x-y))\ =\ f(x)-f(y)\ \forall\ x;y\ \in\mathbb{R}$

Putting $y=0$ we get $f(f(x))=f(x)-f(0)$. Since f is surjective, this yields $f(x)=x-f(0)$. Plugging $x=0$ into that, we get $f(0)=0$, hence the only solution is $f(x)=x$

---

pco
14052 posts

Jun 15, 2010, 9:31 pm • 1 PM #88

> *nguyenvuthanhha wrote:*
> Problem 29 :
>
> Find all real number k such that there exists a function $f:\mathbb{R}\mapsto\mathbb{R}$ such that :
>
> f is differentiable in $\mathbb{R}$, $f(1)\le 1$
>
> and $(f(x))^2+(f^{'}(x))^2\ =\ k$

My solution

> Obviously $k\geq 0$
> And any $k\geq 0$ matches the requirement : just consider $f_k(x)=\sqrt{k}\sin(x-1)$

I have no problem to submit. Anybody feel free to take my turn.

mahanmath
1355 posts

Jun 15, 2010, 9:46 pm PM #89

Problem 30 :
Find all $a \in \mathbb{R}$ for which there exists a non-constant function $f : (0,1] \to \mathbb{R}$ such that

$$a + f(x + y - xy) + f(x)f(y) \le f(x) + f(y)$$

for all $x, y \in (0,1]$.

Justhalf
45 posts

Jun 16, 2010, 5:18 am • 1 PM #90

> pco wrote:
>
> > nguyenvuthanhha wrote:
> > Problem 29 :
> >
> > Find all real number k such that there exists a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that :
> >
> > f is differentiable in $\mathbb{R}$, $f(1) \le 1$
> >
> > and $(f(x))^2 + (f'(x))^2 \ = \ k$
>
> My solution
>
> I have no problem to submit. Anybody feel free to take my turn.

lol oneliner

pco
14052 posts

Jun 16, 2010, 6:47 pm • 1 PM #91

> mahanmath wrote:
> **Problem 30 :**
> Find all $a \in \mathbb{R}$ for which there exists a non-constant function $f : (0,1] \to \mathbb{R}$ such that
>
> $$a + f(x + y - xy) + f(x)f(y) \le f(x) + f(y)$$
>
> for all $x, y \in (0,1]$.

My solution

Let $g(x)$ from $[0,1) \to \mathbb{R}$ such that $g(x) = f(1-x) - 1$
$a + f(x + y - xy) + f(x)f(y) \le f(x) + f(y) \iff g((1-x)(1-y)) + g(1-x)g(1-y) \le -a \iff$
$g(xy) + g(x)g(y) \le -a\ \forall x, y \in [0,1)$

Let $P(x,y)$ be the assertion $g(xy) + g(x)g(y) \le -a$

$P(0,0) \implies g(0) + g(0)^2 \le -a \iff a \le \frac{1}{4} - (g(0) + \frac{1}{2})^2$ and so $a \le \frac{1}{4}$

If $a < \frac{1}{4}$:

Let us consider $g(x) = -\frac{1}{2} \forall x \in (0,1)$ and $g(0) = -\frac{1}{2} - \sqrt{\frac{1}{4} - a} \ne -\frac{1}{2}$ (so that $g(x)$ is not constant) :
If $x = y = 0$: $g(xy) + g(x)g(y) = -a \le -a$
If $x = 0$ and $y \ne 0$: $g(xy) + g(x)g(y) = -\frac{1}{4} - \frac{1}{2}\sqrt{\frac{1}{4} - a} < -\frac{1}{4} < -a$
If $x, y \ne 0$: $g(xy) + g(x)g(y) = -\frac{1}{4} < -a$

If $a = \frac{1}{4}$:
$P(0,0) \implies g(0) + g(0)^2 \le -\frac{1}{4}$ and so $g(0) = -\frac{1}{2}$
$P(x,0) \implies g(x) \ge -\frac{1}{2}$
$P(\sqrt{x}, \sqrt{x}) \implies g(x) + g(\sqrt{x})^2 \le -\frac{1}{4} \implies g(x) \le -\frac{1}{4}$
Let then the sequence u_n defined as :
$u_0 = -\frac{1}{4}$
$u_{n+1} = -\frac{1}{4} - a_n^2$
It's easy to show with induction that $-\frac{1}{2} \le g(x) \le a_n < 0 \forall x \in [0,1)$
It's then easy to show that a_n is a decreasing sequence whose limit is $-\frac{1}{2}$
And so the unique solution for $a = \frac{1}{4}$ is $g(x) = -\frac{1}{2}$ which is not a solution (since constant).

Hence the answer : $\boxed{a \in (-\infty, \frac{1}{4})}$

I have no problem to submit. Anybody feel free to take my turn.

gold46
593 posts

Jun 17, 2010, 2:17 pm PM #92

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for each $n \in \mathbb{N}$,
$2n + 2009 \le f(f(n)) + f(n) \le 2n + 2011.$

pco
14052 posts

Jun 17, 2010, 2:59 pm PM #93

> gold46 wrote:
> Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for each $n \in \mathbb{N}$,
> $2n + 2009 \le f(f(n)) + f(n) \le 2n + 2011.$

Are you sure about $f : \mathbb{R} \to \mathbb{R}$?
Because then we have infinitely many solutions with very different forms

Because then we have infinitely many solutions with very different forms ....
Maybe the real problem is $f:\mathbb{N}\to\mathbb{N}$?

mahanmath
1355 posts

Jun 17, 2010, 6:26 pm PM #94

> *gold46 wrote:*
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for each $n\in\mathbb{N}$,
> $2n+2009\le f(f(n))+f(n)\le 2n+2011$.

pco is right. You can see a similar problem here

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=76767&start=0

.

Mr.pco your solution for **Problem30** was very nice.

Zeus93
64 posts

Jun 17, 2010, 10:28 pm PM #95

Problem 32:
Find all $f:\mathbb{R}^+\to\mathbb{R}$ satisfies:
i) $f(a)=1$with a is a given real number
ii)$f(x)f(y)+f(\frac{a}{y})f(\frac{a}{x})=2f(xy)$ for all x,y >0

Farenhajt
5170 posts

Jun 18, 2010, 2:05 am PM #96

> *Jumong4958 wrote:*
> Problem 32:
> Find all $f:\mathbb{R}^+\to\mathbb{R}$ satisfies:
> i) $f(a)=1$with a is a given real number
> ii)$f(x)f(y)+f(\frac{a}{y})f(\frac{a}{x})=2f(xy)$ for all x,y >0

Putting $y=\frac{a}{x}$ we get $2f(x)f\left(\frac{a}{x}\right)=2f(a)\iff f\left(\frac{a}{x}\right)=\frac{1}{f(x)}\quad(*)$

Putting $y=x$ we get $f^2(x)+f^2\left(\frac{a}{x}\right)=2f(x^2)\overset{(*)}{\iff} f^2(x)+\frac{1}{f^2(x)}=2f(x^2)$, which by AM-GM implies $f(x^2)\geqslant 1$.

Since x^2 covers $\mathbb{R}^+$ completely, we get $(\forall x\in\mathbb{R}^+)f(x)\geqslant 1$

But now $(*)$implies $\left(\forall\frac{a}{x}\in\mathbb{R}^+\right)f\left(\frac{a}{x}\right)\leqslant 1$.

Therefore the only solution is $f(x)\equiv 1$.

Someone else can post the next problem.

mahanmath
1355 posts

Jun 18, 2010, 3:47 am PM #97

Problem 33

Find all functions $f:\mathbb{Q}\mapsto\mathbb{C}$ satisfying

(i) For any $x_1,x_2,\ldots,x_{2010}\in\mathbb{Q}$, $f(x_1+x_2+\ldots+x_{2010})=f(x_1)f(x_2)\ldots f(x_{2010})$.

(ii) $\overline{f(2010)}f(x)=f(2010)\overline{f(x)}$ for all $x\in\mathbb{Q}$.

pco
14052 posts

Jun 18, 2010, 1:16 pm PM #98

> *mahanmath wrote:*
> **Problem 33**
>
> Find all functions $f:\mathbb{Q}\mapsto\mathbb{C}$ satisfying
>
> (i) For any $x_1,x_2,\ldots,x_{2010}\in\mathbb{Q}$, $f(x_1+x_2+\ldots+x_{2010})=f(x_1)f(x_2)\ldots f(x_{2010})$.
>
> (ii) $\overline{f(2010)}f(x)=f(2010)\overline{f(x)}$ for all $x\in\mathbb{Q}$.

My solution

Let $a=f(0)$

Using $x_1=x_2=...=x_p=x$ and $x_{p+1}=...=x_{2010}=0$, (i) $\Longrightarrow f(px)=a^{2010-p}f(x)^p$
$\forall x\in\mathbb{Q},\forall 0\le p\le 2010\in\mathbb{Z}$

Setting $x=0$ in the above equation, we get $a=a^{2010}$ and so :
Either $a=0$ and so $f(x)=0\forall x$, which indeed is a solution.
Either $a^{2009}=1$ and we get $f(px)=a^{1-p}f(x)^p$

Let then $g(x)=\frac{f(x)}{a}$ and we got $g(px)=g(x)^p\forall 0\le p\le 2010\in\mathbb{Z}$
A simple induction using (i) shows that $g(px)=g(x)^p\,\forall p\in\mathbb{N}\cup\{0\}$

And it's then immediate to get $g(\frac{x}{p})=g(x)^{\frac{1}{p}}$ and so $g(x)=c^x\,\forall x\in\mathbb{Q}$

So $f(x)=a\cdot c^x$
(ii) implies then $c=\bar{c}$ and so $c\in\mathbb{R}$

Hence the solutions :
$f(x)=0\forall x$

$f(x)=e^{i\frac{2k\pi}{2009}}c^x$ with $k\in\mathbb{Z}$ and $c\in\mathbb{R}$ (according to me, better to say $c\in\mathbb{R}^+$)

I have no problem to submit. Anybody feel free to take my turn.

Pain rinnegan
1581 posts

Jun 18, 2010, 4:52 pm PM #99

Problem 34: Find all the functions $f:\mathbb{Q}\to\mathbb{R}$ such that :

$$f(x+y+z)=f(x)+f(y)+f(z)+3\sqrt[3]{f(x+y)f(y+z)f(z+x)}\ ,\ (\forall)x\in\mathbb{R}.$$

MohammadP
58 posts

Jun 18, 2010, 6:15 pm PM #100

> *Pain rinnegan wrote:*
> **Problem 34:** Find all the functions $f:\mathbb{Q}\to\mathbb{R}$ such that :
>
> $$f(x+y+z)=f(x)+f(y)+f(z)+3\sqrt[3]{f(x+y)f(y+z)f(z+x)}\ ,\ (\forall)x\in\mathbb{R}.$$

My Solution

let $g(x)=\sqrt[3]{f(x)}$
so $g(x+y+z)^3=g(x)^3+g(y)^3+g(z)^3+3g(x+y)g(y+z)g(z+x)$
let $(x,y,z)=(a,-a,0)$ so we get $g(-a)=-g(a)$
let $(x,y,z)=(a,a,0)$ and $(x,y,z)=(2a,-a,-a)$, we get $g(2a)=2g(a)$
and now using induction it's easy to show that $g(na)=ng(a)$ for every $n\in\mathbb{N}$
(just let $(x,y,z)=((n-1)a,a,0)$)

now we have
$mg(\frac{n}{m})=g(n)=ng(1)\Rightarrow g(\frac{n}{m})=\frac{n}{m}g(1)$
$\Rightarrow g(x)=kx$ with $k\in\mathbb{R}$
so the answer is $f(x)=cx^3$ with $c\in\mathbb{R}$

I don't have any problems to submit.

Farenhajt
5170 posts

Jun 18, 2010, 6:49 pm • 2 PM #101

> *MohammadP wrote:*
> let $(x,y,z)=(a,a,0)$ and $(x,y,z)=(2a,-a,-a)$, we get $g(2a)=2g(a)$

If I'm not mistaken, this actually yields $g(2a)=g(2a)$

Farenhajt
5170 posts

Jun 18, 2010, 10:03 pm PM #102

Still on the previous problem:

Putting $(x,y,z)=(a,a,0)$ yields $g^3(2a)-3g(2a)g^2(a)-2g^3(a)=0$, which factors as $[g(2a)-2g(a)][g(2a)+g(a)]^2=0$, hence either $g(2a)=2g(a)$ or $g(2a)=-g(a)$.

MohammadP
58 posts

Jun 18, 2010, 10:06 pm • 2 PM #103

> *Farenhajt wrote:*
>
> > *MohammadP wrote:*
> > let $(x,y,z)=(a,a,0)$ and $(x,y,z)=(2a,-a,-a)$, we get $g(2a)=2g(a)$
>
> If I'm not mistaken, this actually yields $g(2a)=g(2a)$

let $(x,y,z)=(a,a,0)$, we get
$g(2a)^3=2g(a)^3+3g(2a)g(a)^2\Rightarrow(g(2a)-2g(a))(g(2a)+g(a))^2=0$
if $g(2a)\neq 2g(a)$ then $g(2a)=-g(a)$

now let $(x,y,z)=(2a,-a,-a)$ and we get
$0=g(2a)^3+2g(-a)^3+3g(2a)g(-a)g(-a)$
and using the fact that $g(-a)=-g(a)$ and $g(2a)=-g(a)$ we get
$g(a)=g(2a)=0\Rightarrow g(2a)=2g(a)$

Farenhajt
5170 posts

Jun 18, 2010, 10:29 pm PM #104

> *MohammadP wrote:*
> let $(x,y,z)=(a,a,0)$, we get
> $g(2a)^3=2g(a)^3+3g(2a)g(a)^2\Rightarrow(g(2a)-2g(a))(g(2a)+g(a))^2=0$
> if $g(2a)\neq 2g(a)$ then $g(2a)=-g(a)$
>
> now let $(x,y,z)=(2a,-a,-a)$ and we get
> $0=g(2a)^3+2g(-a)^3+3g(2a)g(-a)g(-a)$
> and using the fact that $g(-a)=-g(a)$ and $g(2a)=-g(a)$ we get
> $g(a)=g(2a)=0\Rightarrow g(2a)=2g(a)$

Ok now. This should have been included in the original solution

MohammadP
58 posts

Jun 19, 2010, 12:33 am • 2 PM #105

Yes, you're right. sorry for my bad explanation.

mahanmath
1355 posts

Jun 21, 2010, 5:27 am • 3 PM #106

Next problem ??

ilikemaths
50 posts

Jun 21, 2010, 1:25 pm PM #107

New problem:

Find all functions $f:\mathbb{R}\to\mathbb{R}$, satisfying:
$f(x)=max_{y\in\mathbb{R}}(2xy-f(y))$ for all $x\in\mathbb{R}$.

pco
14052 posts

Jun 21, 2010, 2:12 pm • 1 PM #108

> *ilikemaths wrote:*
> New problem:
>
> Find all functions $f:\mathbb{R}\to\mathbb{R}$, satisfying:
> $f(x)=max_{y\in\mathbb{R}}(2xy-f(y))$ for all $x\in\mathbb{R}$.

My solution

1) $f(x)\ge x^2\forall x$
==================
$f(x)\ge 2xy-f(y)\forall x,y$. Choosing $y=x$, we get $f(x)\ge x^2$
Q.E.D

2) $f(x)\le x^2\forall x$
==============
Let $x\in\mathbb R$
Since $f(x)=\max_{y\in\mathbb R}(2xy-f(y))$, $\exists$ a sequence y_n such that $\lim_{n\to+\infty}(2xy_n-f(y_n))=f(x)$

So $\lim_{n\to+\infty}(f(y_n)-y_n^2+(x-y_n)^2)=x^2-f(x)$
And since we know that $f(y_n)-y_n^2\ge 0$, then $LHS\ge 0$ and so $RHS\ge 0$
Q.E.D

So $\boxed{f(x)=x^2}$ which indeed is a solution

I have no problem to submit. Anybody feel free to post.

Amir Hossein
4719 posts

Jun 21, 2010, 2:24 pm • 1 PM #109

Problem 37:
Find all functions $f:\mathbb R\to\mathbb R$ satisfying

$$f(f(x)+y)=f(x^2-y)+4f(x)y$$

for all $x,y\in\mathbb R$.

---

pco
14052 posts

Jun 21, 2010, 2:54 pm PM #110

> *amparvardi wrote:*
> **Problem 37:**
> Find all functions $f:\mathbb R\to\mathbb R$ satisfying
>
> $$f(f(x)+y)=f(x^2-y)+4f(x)y$$
>
> for all $x,y\in\mathbb R$.

My solution

Let $P(x,y)$ be the assertion $f(f(x)+y)=f(x^2-y)+4f(x)y$

$P(x,\frac{x^2-f(x)}{2})\implies f(x)(f(x)-x^2)=0$ and so : $\forall x$, either $f(x)=0$, either $f(x)=x^2$

$f(x)=0\forall x$ is a solution
$f(x)=x^2\forall x$ is also a solution.

Suppose now that $\exists a\ne 0$ such that $f(a)=0$
Then if $\exists b\ne 0$ such that $f(b)\ne 0$:
$f(b)=b^2$ and $P(a,b)\implies b^2=f(a^2-b)$ and so $b^2=(a^2-b)^2$ and so $b=\frac{a^2}{2}$
So there is a unique such b (equal to $\frac{a^2}{2}$)
But then there at at most two such a (a and $-a$)
And it is is impossible to have at most one $x\ne 0$ such that $f(x)=x^2$ and at most two $x\ne 0$ such that $f(x)=0$

So we have only two solutions :
$f(x)=0\forall x$
$f(x)=x^2\forall x$

I have no problem to submit. Anybody feel free to take my turn.

---

nguyenvut...
475 posts

Jun 21, 2010, 9:41 pm PM #111

Problem 38 :

Find all functions $f\ :\ \mathbb R^+\mapsto\mathbb R^+$ *such that :*

$(f(x))^2+2yf(x)+f(y)\ =\ f(y+f(x))\ \forall y;x\ \in\mathbb R^+$

Farenhajt
5170 posts

Jun 21, 2010, 10:34 pm • 2 PM #112

> *nguyenvuthanhha wrote:*
> Problem 38 :
>
> *Find all functions* $f\ :\ \mathbb R^+\mapsto\mathbb R^+$ *such that :*
>
> $(f(x))^2+2yf(x)+f(y)\ =\ f(y+f(x))\ \forall y;x\ \in\mathbb R^+$

The equation can be written as

$f(y+f(x))-(y+f(x))^2=f(y)-y^2$

Therefore $g(x):=f(x)-x^2$ is periodic, and every $f(x)$ is its period. First, $f(x)$ can be identical to zero, which satisfies the initial equation. Otherwise, $f(x)$ takes the values of the form $k\alpha$, where k is a positive integer and α the minimal (non-zero) period of g.

Hence for natural k,l such that $k>l$ we have

$f(x+k\alpha)-(x+k\alpha)^2=f(x+l\alpha)-(x+l\alpha)^2$

$f(x+k\alpha)-f(x+l\alpha)=\alpha(k-l)(2x+(k+l)\alpha)$

By the assumption, the LHS is equal to ... for some natural ... which yields

By the assumption, the LHS is equal to $m\alpha$ for some natural m, which yields

$$\alpha = \frac{1}{k+l}\left(\frac{m}{k-l} - 2x\right)$$

Therefore, the period of the function depends on x, hence any real number can be the period, hence the function is constant.

(Another line of argument: if α is rational, then the above can't be satisfied for irrational x, and vice versa, hence there can be no minimal period. Yet another: a periodic function can have both a rational and an irrational period, which is impossible if it's non-constant.)

So $f(x) - x^2 = \text{const} \iff f(x) = x^2 + C, C \in \mathbb{R}$

Hence the solutions are $f(x) = 0$ and $f(x) = x^2 + C, C \in \mathbb{R}$

Someone else can post the next problem.

---

seifi-seifi
115 posts

Jun 21, 2010, 11:33 pm PM #113

problem39:

let $k \geq 1$ be a given integer.find all function $f: R \to R$ such that :
$f(x^k + f(y)) = y + f(x)^k$

Farenhajt
5170 posts

Jun 23, 2010, 3:19 am • 3 PM #114

On a side note: Someone is again going through the topic and spam-rating everything (and not only on this topic). Has anyone noticed similar things elsewhere?

Justhalf
45 posts

Jun 23, 2010, 3:30 am • 3 PM #115

> *Farenhajt wrote:*
> On a side note: Someone is again going through the topic and spam-rating everything (and not only on this topic). Has anyone noticed similar things elsewhere?

You can check your foe list? lol
I didn't see this anywhere else (at least not in the combinatorial section)

Farenhajt
5170 posts

Jun 23, 2010, 3:47 am • 4 PM #116

> *Justhalf wrote:*
>
> > *Farenhajt wrote:*
> > On a side note: Someone is again going through the topic and spam-rating everything (and not only on this topic). Has anyone noticed similar things elsewhere?
>
> You can check your foe list? lol

It's always been empty (and I never managed to see the point in having such lists at a board like this). Anyway, let's not go too offtopic 🙂

Stephen
403 posts

Jun 24, 2010, 6:46 pm • 2 PM #117

> *seifi-seifi wrote:*
> problem39:
>
> let $k \geq 1$ be a given integer.find all function $f: R \to R$ such that :
> $f(x^k + f(y)) = y + f(x)^k$

Solution of Problem 39

Let $f(0) = a$. Then if $x = 0$ then $f(f(y)) = y + a^k$.

If we let $y \to f(y)$, then $f(x^k + f(f(y))) = f(y) + (f(x))^k$.

So $f(x^k + y + a^k) = f(y) + (f(x))^k$.

So $f(f(x^k + y + a^k) = f((f(x))^k + f(y)) = y + (f(f(x))^k = y + (x + a^k)^k$.

But, $f(f(x^k + y + a^k)) = (x^k + y + a^k) + a^k = x^k + y + 2a^k$.

So $x^k + 2a^k = (x + a^k)^k$ for all $x \in R$.

We can know easily that $a^k = 0$. So $a = 0$.

So $f(f(x)) = x$.

So $f(x + y) = f(x + f(f(y))) = f((x^{\frac{1}{k}})^k + f(f(y))) = f(y) + (f(x^{\frac{1}{k}}))^k$.

(Here, x is positive)

First, let k is even.

So, $f(x + y) = f(y) + (f(x^{\frac{1}{k}}))^k \geq f(x)$.

So, if $x_1 > x_2$, then $f(x_1) \geq f(x_2)$.

So if $x_0 < f(x_0)$ for some x_0, then

$x_0 < f(x_0) \leq f(f(x_0)) = x_0$. So a contradiction!

In a same way, we can know that there isn't some x_0 that $x_0 > f(x_0)$.

So, for every $x \in R$, $f(x) = x$.

Second, let k is odd.

Since $f(0) = 0$, letting $y = 0$ in $f(x + y) = f(y) + f(x^{\frac{1}{k}}))^k$, we can have $(x^{\frac{1}{k}}))^k = f(x)$.

So, we can also have $f(x+y) = f(x) + f(y)$ and $f(x^k) = (f(x))^k$.

Since we produced a Cauchy equation, for every rational number q,

$f(qx) = qf(x)$, and $f(x) + f(-x) = 0$ for every $x \in R$.

So $((q+x)^k) = (f(q+x))^k = (f(q) + f(x))^k$.

So $f(\sum_{s=0}^{k} kC_s q^s x^{k-s}) = \sum_{s=0}^{k} kC_s (f(q))^s (f(x))^{k-s}$.

(Sorry, kC_s is k Combination s. I don't do latex very well.)

We can get
$f(\sum_{s=0}^{k} kC_s q^s (f(x))^{k-s}) = f(\sum_{s=0}^{k} kC_s f(q^s)(f(x))^{k-s}) = f(\sum_{s=0}^{k} kC_s q^s (f(1))(f(x))^{k-s}) = f(\sum_{s=0}^{k} kC_s q^s (f(1))^k (f(x))^{k-s})$

(because $f(x^k) = (f(x))^k$)

So for all $s \in 0, 1, 2, ..., k$, we can know that $f(x^{k-s}) = (f(1))^k (f(x))^{k-s}$.

But since $f(1) = (f(1))^k$ and $k-1$ is even, $f(1)$ is 1 or -1.

If $f(1) = 1$, then $f(x^{k-s}) = (f(x))^{k-s}$.

So $(f(x^2)) = (f(x))^2$.

So if $x > 0$, then $f(x) \geq 0$.

Since we have a Cauchy equation, we can tell f is increasing.

So, because of the Cauchy equation, $f(x) = cx$. Since $f(1) = 1$, $f(x) = x$.

If $f(1) = -1$, we can tell $f(x) = -x$ in a similar way.

To conclude, we can tell the solution is

①k is even: $f(x) = x$

②k is odd: $f(x) = x$ or $f(x) = -x$

Problem 40

Find all functions $f : R \to R$ that satisfies $f(xy) + f(x-y) \geq f(x+y)$ for all real numbers x, y.

---

peine
369 posts

Jun 27, 2010, 5:32 am PM #118

Solution to Problem 39:
let $y = -f(x)^k$ then there must be a number m such that $f(m) = 0$
let $y = f(m)$ then $f(x^k) = f(x)^k$ (*)
let $y = m$ then $f(x^k) = m + f(x)^k$ then $m = 0$
let $x = 0$ then $f(f(y)) = y$.
we have then: $f(x+y) = f(x + f(f(y))) = f((x^{\frac{1}{k}})^k + f(f(y))) = f(y) + (f(x^{\frac{1}{k}}))^k = f(x) + f(y)$.
let x and two numbers such that $f(x) = f(y)$ then $f(f(x)) = f(f(y)) \Rightarrow x = y$ which implies that f is injective and hence it's monotonus, and we have $f(x+y) = f(x) + f(y)$ thus $f(x) = cx$ replacing in on (*) we get that $1 = c^{k-1}$ if k is odd then $c = 1$ or $c = -1$ if k is even then $c = 1$.

For Problem 40 I think that we cannot find all the functions veryfing the conditions, actually $f(x) = ax^2 + b$ is a solution for every $a >= 2b >= 0$:)

Amir Hossein
4719 posts

Jun 29, 2010, 9:42 pm PM #119

Problem 40 didn't solve for about 4 days, so I'm posting next one.
Problem 41:
Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(1) = f(-1)$ and

$$f(m) + f(n) = f(m+2mn) + f(n-2mn) \qquad \forall m, n \in \mathbb{Z}$$

Justhalf
45 posts

Jul 9, 2010, 10:22 pm PM #120

Oh, no! This thread is dying! Are the last two problems that hard? Actually they're quite hard for me..

---

lajanugen
258 posts

Jul 13, 2010, 4:56 pm PM #121

Problem 41:
We claim that all functions that satisfy $exp_2(x) = exp_2(y) \to f(x) = f(y)$ are the solutions ($f(0)$ can be arbitrary)- this is easily seen to satisfy the equation since $exp_2(x) = exp_2(x + 2xy)$ for any non-zero integer x
Plugging in $(m, n) = (n, 1), (-1, n)$ and equating the expressions, we obtain that the f values of all odd numbers are equal.
Hence, for all odd n, $f(m) = f((2n+1)m)$(Since $n - 2mn$ would also be odd):
$n = -1$ gives $f(m) = f(-m)$ for all m
As n ranges through all odd values, $2n+1, -(2n+1)$ range through all odd values

waiting for problem 40 to be solved

---

mahanmath
1355 posts

Jul 29, 2010, 10:53 pm PM #122

Problem 42 : Find all functions $\mathbb{R} \to \mathbb{R}$ such that :

$f(x + f(y + f(z))) = f(x) + f(f(y)) + f(f(f(z)))$

---

pco
14052 posts

Aug 2, 2010, 8:40 pm • 2 PM #123

> mahanmath wrote:
> **Problem 42** : Find all functions $\mathbb{R} \to \mathbb{R}$ such that :
>
> $f(x + f(y + f(z))) = f(x) + f(f(y)) + f(f(f(z)))$

I wonder where (what course, exam,book) you found this problem.

Any solution of Cauchy's equation is obviously a solution.

But there are a lot of other strange solutions : for example, $f(x)=\lfloor 25\sin(2\pi x)\rfloor$

(in fact one of the infinitely many families of solutions is $f(x)=\lfloor h(\{x\})-h(0)\rfloor$ where $h(x)$ is any function from $\mathbb{R}\to\mathbb{R}$)

And I would be surprised if a general form existed for all these solutions.

mahanmath
1355 posts

Aug 2, 2010, 9:55 pm PM #124

> *pco wrote:*
>
> > *mahanmath wrote:*
> > **Problem 42** : Find all functions $\mathbb{R}\to\mathbb{R}$ such that :
> >
> > $f(x+f(y+f(z)))=f(x)+f(f(y))+f(f(f(z)))$
>
> I wonder where (what course, exam,book) you found this problem.
>
> Any solution of Cauchy's equation is obviously a solution.
>
> But there are a lot of other strange solutions : for example, $f(x)=\lfloor 25\sin(2\pi x)\rfloor$
>
> (in fact one of the infinitely many families of solutions is $f(x)=\lfloor h(\{x\})-h(0)\rfloor$ where $h(x)$ is any function from $\mathbb{R}\to\mathbb{R}$)
>
> And I would be surprised if a general form existed for all these solutions.

I`m so sorry , I forgot the main condition ! The problem said find all continuous functions .

pco
14052 posts

Aug 2, 2010, 10:41 pm • 1 PM #125

> *mahanmath wrote:*
>
> > *pco wrote:*
> >
> > > *mahanmath wrote:*
> > > **Problem 42** : Find all functions $\mathbb{R}\to\mathbb{R}$ such that :
> > >
> > > $f(x+f(y+f(z)))=f(x)+f(f(y))+f(f(f(z)))$
> >
> > I wonder where (what course, exam,book) you found this problem.
> >
> > Any solution of Cauchy's equation is obviously a solution.
> >
> > But there are a lot of other strange solutions : for example, $f(x)=\lfloor 25\sin(2\pi x)\rfloor$
> >
> > (in fact one of the infinitely many families of solutions is $f(x)=\lfloor h(\{x\})-h(0)\rfloor$ where $h(x)$ is any function from $\mathbb{R}\to\mathbb{R}$)
> >
> > And I would be surprised if a general form existed for all these solutions.
>
> I`m so sorry , I forgot the main condition ! The problem said find all continuous functions .

It's a pity to see how many users are quite unable to correctly copy a six words problem

With continuity, this is a trivial problem :
Let $P(x,y,z)$ be the assertion $f(x+f(y+f(z)))=f(x)+f(f(y))+f(f(f(z)))$

Subtracting $P(0,y-f(z),z)$ from $P(x,y-f(z),z)$, we get $f(x+f(y))=f(x)+f(f(y))-f(0)$
Let $g(x)=f(x)-f(0)$ and $A=f(\mathbb{R})$

We got $g(x+y)=g(x)+g(y)\forall x\in\mathbb{R},\forall y\in A$
And also $g(x-y)=g(x)-g(y)\forall x\in\mathbb{R},\forall y\in A$

$g(x+y_1+y_2)=g(x+y_1)+g(y_2)=g(x)+g(y_1)+g(y_2)=g(x)+g(y_1+y_2)\forall x\in\mathbb{R},\forall y_1,y_2\in A$
$g(x+y_1-y_2)=g(x+y_1)-g(y_2)=g(x)+g(y_1)-g(y_2)=g(x)+g(y_1-y_2)\forall x\in\mathbb{R},\forall y_1,y_2\in A$

And, with simple induction, $g(x+y)=g(x)+g(y)\forall x,\forall y$ finite sums and differences of elements of A

If cardinal of A is 1, we get $f(x)=c$ and so $f(x)=0$
If cardinal of A is not 1 and since $f(x)$ is continuous, $\exists u<v$ such that $[u,v]\subseteq A$ and any real may be represented as finite sums and differences of elements of $[u,v]$

So $g(x+y)=g(x)+g(y)\forall x,y$ and so, since continuous, $g(x)=ax$ and $f(x)=ax+b$

Plugging this in original equation, we get $b(a+2)=0$

Hence the solutions :
$f(x)=ax$
$f(x)=b-2x$

Amir Hossein
4719 posts

Aug 3, 2010, 2:37 pm PM #126

I'm posting next problem.

Problem 43 :
Let f be a real function defined on the positive half-axis for which $f(xy)=xf(y)+yf(x)$ and $f(x+1)\leq f(x)$ hold for every positive x and y. Show that if $f(1/2)=1/2$, then

$$f(x)+f(1-x)\geq -x\log_2 x-(1-x)\log_2(1-x)$$

for every $x\in(0,1)$.

Dijkschneier
131 posts

Aug 17, 2010, 3:17 am PM #127

This marathon is interesting. Why has it stopped ?

Amir Hossein
4719 posts

Aug 17, 2010, 12:44 pm PM #128

Yeah, it seems problem 43 is difficult.
Can I post a new problem ?

Dijkschneier
131 posts

Aug 17, 2010, 9:19 pm PM #129

I would appreciate that.

Amir Hossein
4719 posts

Aug 17, 2010, 9:29 pm PM #130

Problem 44 :

Let a be a real number and let $f:\mathbb{R}\to\mathbb{R}$ be a function satisfying: $f(0)=\frac{1}{2}$ and $f(x+y)=f(x)f(a-y)+f(y)f(a-x), \forall x,y\in\mathbb{R}.$
Prove that f is constant.

---

pco
14052 posts

Aug 17, 2010, 9:47 pm PM #131

> amparvardi wrote:
> **Problem 44 :**
>
> Let a be a real number and let $f:\mathbb{R}\to\mathbb{R}$ be a function satisfying: $f(0)=\frac{1}{2}$ and $f(x+y)=f(x)f(a-y)+f(y)f(a-x), \forall x,y\in\mathbb{R}.$
> Prove that f is constant.

My solution

Let $P(x,y)$ be the assertion $f(x+y)=f(x)f(a-y)+f(y)f(a-x)$

$P(0,0)\implies f(a)=\frac{1}{2}$
$P(x,0)\implies f(x)=f(a-x)$ and so $P(x,y)$ may also be written $Q(x,y)$: $f(x+y)=2f(x)f(y)$

$Q(a,-x)\implies f(a-x)=f(-x)$ and so $f(x)=f(-x)$

Then, comparing $Q(x,y)$ and $Q(x,-y)$, we get $f(x+y)=f(x-y)$ and choosing $x=\frac{u+v}{2}$ and $y=\frac{u-v}{2}$, we get $f(u)=f(v)$
Q.E.D

And I have no problem to submit. So anybody feel free to post a new problem.

Amir Hossein
4719 posts

Aug 17, 2010, 10:36 pm PM #132

Problem 45:
Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)^3=-\frac{x}{12}\cdot\left(x^2+7x\cdot f(x)+16\cdot f(x)^2\right),\ \forall x\in\mathbb{R}.$$

---

pco
14052 posts

Aug 17, 2010, 10:49 pm PM #133

> amparvardi wrote:
> **Problem 45**:
> Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that
>
> $$f(x)^3=-\frac{x}{12}\cdot\left(x^2+7x\cdot f(x)+16\cdot f(x)^2\right),\ \forall x\in\mathbb{R}.$$

My solution

This equation may be written $(f(x)+\frac{x}{2})^2(f(x)+\frac{x}{3})=0$ and so 4 solutions :

S1 : $f(x)=-\frac{x}{2}\forall x$

S2 : $f(x)=-\frac{x}{3}\forall x$

S3 : $f(x)=-\frac{x}{2}\forall x<0$ and $f(x)=-\frac{x}{3}\forall x\ge 0$

S4 : $f(x)=-\frac{x}{2}\forall x>0$ and $f(x)=-\frac{x}{3}\forall x\le 0$

And I have no problem to submit. Anybody feel free to post a new one.

Amir Hossein
4719 posts

Aug 17, 2010, 10:56 pm PM #134

So quick Mr Patrick :)

Problem 46:
Find all functions $f:\mathbb{R}\backslash\{0,1\}\to\mathbb{R}$ such that

$$f(x)+f\left(\frac{1}{1-x}\right)=1+\frac{1}{x(1-x)}.$$

---

pco
14052 posts

Aug 17, 2010, 11:26 pm PM #135

> amparvardi wrote:
> **Problem 46:**
> Find all functions $f:\mathbb{R}\backslash\{0,1\}\to\mathbb{R}$ such that
>
> $$f(x)+f\left(\frac{1}{1-x}\right)=1+\frac{1}{x(1-x)}.$$

My solution

$(E1)$: $f(x)+f(\frac{1}{1-x})=1+\frac{1}{x(1-x)}\forall x\notin\{0,1\}$

Setting $x\to\frac{1}{1-x}$ and since $x\notin\{0,1\}$ implies $\frac{1}{1-x}\notin\{0,1\}$, we get :

$(E2)$: $f($ ...

$(E2): f(\frac{1}{1-x}) + f(1-\frac{1}{x}) = 3 - x - \frac{1}{x}$

Setting in $(E1) x \to 1-\frac{1}{x}$ and since $x \notin \{0,1\}$ implies $1-\frac{1}{x} \notin \{0,1\}$, we get :

$(E3): f(1-\frac{1}{x}) + f(x) = 1 + \frac{x^2}{x-1}$

$(E1)+(E3)-(E2) \implies f(x) = x + \frac{1}{x} \forall x \notin \{0,1\}$ which indeed is solution

And I have no problem to submit. So anybody feel free to post a new one.

Amir Hossein
4719 posts

Aug 17, 2010, 11:32 pm PM #136

Problem 47:
Let $f(x)$ be a real-valued function defined on the positive reals such that

(1) if $x < y$, then $f(x) < f(y)$,

(2) $f\left(\frac{2xy}{x+y}\right) \geq \frac{f(x)+f(y)}{2}$ for all x.

Show that $f(x) < 0$ for some value of x.

---

pco
14052 posts

Aug 18, 2010, 1:47 am PM #137

> *amparvardi wrote:*
> **Problem 47**:
> Let $f(x)$ be a real-valued function defined on the positive reals such that
>
> (1) if $x < y$, then $f(x) < f(y)$,
>
> (2) $f\left(\frac{2xy}{x+y}\right) \geq \frac{f(x)+f(y)}{2}$ for all x.
>
> Show that $f(x) < 0$ for some value of x.

My ugly solution

Here is a rather ugly proof :

1) $f(x)$ is concave.
====================
If $x < y$: $\frac{x+y}{2} > \frac{2xy}{x+y}$ and so $f(\frac{x+y}{2}) > \frac{f(x)+f(y)}{2}$
Using this plus the fact that $f(x)$ is stricly increasing, we get immediately the result.

2) $\frac{f(x)-f(\frac{x}{2})}{\frac{x}{2}} \geq 2\frac{f(2x)-f(x)}{x}$
==
Let $a > 1$. From the original inequality, using $y = ax$, we get $f(\frac{2a}{a+1}x) \geq \frac{f(x)+f(ax)}{2}$

$\implies f(\frac{2a}{a+1}x) - f(x) \geq \frac{f(ax)-f(x)}{2}$

$\implies \frac{f(\frac{2a}{a+1}x)-f(x)}{\frac{2a}{a+1}x - x} \geq \frac{a+1}{2}\frac{f(ax)-f(x)}{ax-x}$

Let then the sequence a_n defined as $a_1 = 2$ and $a_{n+1} = \frac{2a_n}{a_n+1}$. We got :

$\frac{f(a_{n+1}x)-f(x)}{a_{n+1}x-x} \geq \frac{a_n+1}{2}\frac{f(a_nx)-f(x)}{a_nx-x}$

And, since $f(x)$ is concave, we get also $\frac{f(x)-f(\frac{x}{2})}{\frac{x}{2}} \geq \frac{f(a_nx)-f(x)}{a_nx-x}$

And so $\frac{f(x)-f(\frac{x}{2})}{\frac{x}{2}} \geq \left(\prod_{k=1}^{n}\frac{a_k+1}{2}\right)\frac{f(2x)-f(x)}{x}$

And since $\prod_{k=1}^{+\infty}\frac{a_k+1}{2} = 2$, we got the required result in title of paragraph 2.
(just write $\frac{a_k+1}{2} = \frac{a_k}{a_{k+1}}$).

3) Final result
===============
From 2), we got $f(x) - f(\frac{x}{2}) \geq f(2x) - f(x)$

And so $f(\frac{x}{2}) - f(\frac{x}{4}) \geq f(x) - f(\frac{x}{2}) \geq f(2x) - f(x)$
...
$f(\frac{x}{2^{n-1}}) - f(\frac{x}{2^n}) \geq f(2x) - f(x)$

... and so (summing these lines) : $f(x) - f(\frac{x}{2^n}) \geq n(f(2x) - f(x))$

Which may be written $f(\frac{x}{2^n}) \leq f(x) - n(f(2x) - f(x))$

And, since $f(2x) > f(x)$, and choosing n great enough, we get $f(\frac{x}{2^n}) < 0$
Q.E.D.

And I dont have any problem to submit. So anybody feel free to post a new one.

aktyw19
1315 posts

Aug 18, 2010, 11:25 am PM #138

Problem 48:
Find all continuous functions $R->R$ satisfying the equation:
$f(x+y)+f(xy)=f(x)+f(y)+f(xy+1)$

Problem 49:
Find all continuous functions $R->R$ satisfying the equation:
$f(x)+f(y)+f(z)+f(x+y+z)=f(x+y)+f(y+z)+f(z+x)+f(0)$

Dumel
190 posts

Aug 20, 2010, 3:08 am PM #139

> *amparvardi wrote:*
> **Problem 47**:
> Let $f(x)$ be a real-valued function defined on the positive reals such that
>
> (1) if $x<y$, then $f(x)<f(y)$,
>
> (2) $f\left(\frac{2xy}{x+y}\right)\geq\frac{f(x)+f(y)}{2}$ for all x.
>
> Show that $f(x)<0$ for some value of x.

hmmm I think there is no function satisfying these conditions.
suppose that $f(x)>0$ for all $x>0$
let $a=\lim_{x\to 0}f(x)$
for $y=1$ and $x\to 0$ we get
$a\geq f(1)>a$ which is contradition.
am I wrong?

pco
14052 posts

Aug 20, 2010, 12:24 pm PM #140

> *Dumel wrote:*
>> *amparvardi wrote:*
>> **Problem 47**:
>> Let $f(x)$ be a real-valued function defined on the positive reals such that
>>
>> (1) if $x<y$, then $f(x)<f(y)$,
>>
>> (2) $f\left(\frac{2xy}{x+y}\right)\geq\frac{f(x)+f(y)}{2}$ for all x.
>>
>> Show that $f(x)<0$ for some value of x.
>
> hmmm I think there is no function satisfying these conditions.
> suppose that $f(x)>0$ for all $x>0$
> let $a=\lim_{x\to 0}f(x)$
> for $y=1$ and $x\to 0$ we get
> $a\geq f(1)>a$ which is contradition.
> am I wrong?

Quite nice and simple ! Congrats!
Just two - very little - remarks :
a) replace your first phrase by "suppose that $f(x)\geq 0$ for all $x>0$"
b) since the rule $x<y\implies f(x)<f(y)$ is available only for $x>0$, one more line IMHO is needed to conclude $f(1)>a$ which is true

Amir Hossein
4719 posts

Aug 20, 2010, 12:56 pm PM #141

But this problem is **Brazil National MO 2003** Problem 5 !

Dijkschneier
131 posts

Aug 26, 2010, 2:10 am PM #142

> *Dumel wrote:*
>> *amparvardi wrote:*
>> **Problem 47**:
>> Let $f(x)$ be a real-valued function defined on the positive reals such that
>>
>> (1) if $x<y$, then $f(x)<f(y)$,
>>
>> (2) $f\left(\frac{2xy}{x+y}\right)\geq\frac{f(x)+f(y)}{2}$ for all x.
>>
>> Show that $f(x)<0$ for some value of x.
>
> hmmm I think there is no function satisfying these conditions.
> suppose that $f(x)>0$ for all $x>0$
> let $a=\lim_{x\to 0}f(x)$
> for $y=1$ and $x\to 0$ we get
> $a\geq f(1)>a$ which is contradition.
> am I wrong?

Don't we need a continuity hypothesis to claim that f(1) > a ?
$f(1)>f(x\to 0)$, and we need continuity to have $f(1)>\lim_{x\to 0}f(x)$.

pco
14052 posts

Aug 26, 2010, 1:11 pm PM #143

> *Dijkschneier wrote:*
>> *Dumel wrote:*
>>> *amparvardi wrote:*
>>> **Problem 47**:
>>> Let $f(x)$ be a real-valued function defined on the positive reals such that

(1) if $x < y$, then $f(x) < f(y)$,

(2) $f\left(\frac{2xy}{x+y}\right) \geq \frac{f(x)+f(y)}{2}$ for all x.

Show that $f(x) < 0$ for some value of x.

hmmm I think there is no function satisfying these conditions.
suppose that $f(x) > 0$ for all $x > 0$
let $a = \lim_{x\to 0} f(x)$
for $y = 1$ and $x \to 0$ we get
$a \geq f(1) > a$ which is contradition.
am I wrong?

Don't we need a continuity hypothesis to claim that f(1) > a ?
$f(1) > f(x \to 0)$, and we need continuity to have $f(1) > \lim_{x\to 0} f(x)$.

No, we would need continuity to write $f(1) > f(0)$

But if you have $f(1) > f(x)$ and $\lim_{x\to 0} f(x)$ exists, you can write $f(1) \geq \lim_{x\to 0} f(x)$

The fact that, here, we can write $f(1) > \lim_{x\to 0} f(x)$ instead of $f(1) \geq \lim_{x\to 0} f(x)$ is due to the fact that $f(x)$ is strictly increasing.

Dijkschneier
131 posts

Aug 26, 2010, 11:50 pm PM #144

> *pco wrote:*
> The fact that, here, we can write $f(1) > \lim_{x\to 0} f(x)$ instead of $f(1) \geq \lim_{x\to 0} f(x)$ is due to the fact that $f(x)$ is strictly increasing.

Can you explain that, please ?

pco
14052 posts

Aug 27, 2010, 12:06 am PM #145

> *Dijkschneier wrote:*
>> *pco wrote:*
>> The fact that, here, we can write $f(1) > \lim_{x\to 0} f(x)$ instead of $f(1) \geq \lim_{x\to 0} f(x)$ is due to the fact that $f(x)$ is strictly increasing.
>
> Can you explain that, please ?

$f(\frac{1}{2}) > f(x) \forall x < \frac{1}{2}$

So, since $\lim_{x\to 0} f(x)$ exists, we get $f(\frac{1}{2}) \geq \lim_{x\to 0} f(x)$

And since $f(1) > f(\frac{1}{2})$ we get $f(1) > \lim_{x\to 0} f(x)$

Dijkschneier
131 posts

Aug 27, 2010, 12:40 am PM #146

Indeed. Thanks.

mahanmath
1355 posts

Sep 8, 2010, 4:41 am • 1 PM #147

Problem 48 :

Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(f(x)+y) = f(x^2-y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$.

Sansa
108 posts

Sep 8, 2010, 7:46 am • 1 PM #148

Click to reveal hidden text

Let $P(x,y)$ be the assertion of $f(f(x)+y) = f(x^2-y)+4f(x)y$.

$P(x, x^2)$: $f(f(x)+x^2) = f(0) + 4x^2 f(x)$

$P(x, -f(x))$: $f(0) = f(f(x)+x^2) - 4f(x)^2$

$\Longrightarrow f(0) = f(0) + 4x^2 f(X) - 4f(x)^2 \Longrightarrow f(x)(f(x)-x^2) = 0 \Longrightarrow f(x) = 0$ or $f(x) = x^2$ or sometimes $f(x) = 0$ and sometimes $f(x) = x^2, \forall x \in \mathbb{R}$

If sometimes $f(x) = 0$ and sometimes $f(x) = x^2 \Longrightarrow \exists x_0, y_0$: $x_0 \neq y_0 \neq 0$ and $f(x_0) = 0$ and $f(y_0) = {y_0}^2$

so $P(x_0, y_0)$: ${y_0}^2 = f({x_0}^2 - y_0)$

But we know that $f({x_0}^2 - y_0) = ({x_0}^2 - y_0)^2$ or $f({x_0}^2 - y_0) = 0$ and that is not correct.

problem 49:
Find all functions f: $\mathbb{R}\to\mathbb{R}$ satisfying:

$$f(xf(y)+f(x)) = 2f(x) + xy$$

This post has been edited 2 times. Last edited by Sansa, Sep 8, 2010, 8:27 pm

aktyw19
1315 posts

Sep 8, 2010, 10:58 am PM #149

unsolved problems

> *Quote:*
> **Problem 48:**
> Find all continuous functions $R->R$ satisfying the equation:
> $f(x+y)+f(xy) = f(x)+f(y)+f(xy+1)$

Problem 49:
Find all continuous functions $R->R$ satisfying the equation:
$f(x)+f(y)+f(z)+f(x+y+z)=f(x+y)+f(y+z)+f(z+x)+f(0)$

pco
14052 posts

Sep 8, 2010, 10:50 pm • 3

PM #150

> *Sansa wrote:*
> Let $P(x,y)$ be the assertion of $f(f(x)+y)=f(x^2-y)+4f(x)y$.
>
> $P(x,x^2)$: $f(f(x)+x^2)=f(0)+4x^2f(x)$
>
> $P(x,-f(x))$: $f(0)=f(f(x)+x^2)-4f(x)^2$
>
> $\Longrightarrow f(0)=f(0)+4x^2f(X)-4f(x)^2 \Longrightarrow f(x)(f(x)-x^2)=0 \Longrightarrow f(x)=0$ or $f(x)=x^2$ or sometimes $f(x)=0$ and sometimes $f(x)=x^2, \forall x\in\mathbb{R}$

Yes. Little remark : one line for the same result : $P(x,\frac{x^2-f(x)}{2})$:)

pco
14052 posts

Sep 9, 2010, 12:51 am • 1

PM #151

> *Sansa wrote:*
> problem 49:
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying:
>
> $$f(xf(y)+f(x))=2f(x)+xy$$

Here is My solution

Let $P(x,y)$ be the assertion $f(xf(y)+f(x))=2f(x)+xy$

$P(1,x-2f(1)) \implies f(\text{something})=x$ and $f(x)$ is surjective.
If $f(a)=f(b)$, subtracting $P(1,a)$ from $P(1,b)$ implies $a=b$ and $f(x)$ is injective, and so bijective.

Let $f(0)=a$ and u such that $f(u)=0$

$P(u,0) \implies f(au)=0=f(u)$ and so, since injective, $au=u$

If $u=0$, then $a=0$ and $P(x,0) \implies f(f(x))=2f(x)$ and so, since surjective, $f(x)=2x$ which is not a solution.

So $u\neq 0$ and $a=1$. Then $P(u,u) \implies 1=u^2$ and so $u=\pm 1$
If $u=1$, $P(0,-1) \implies 0=2$, impossible.

So $a=0$ and $u=-1$: $f(-1)=0$ and $f(0)=1$ and $P(0,-1) \implies f(1)=2$

$P(-1,x) \implies f(-f(x))=-x$
$P(x,-f(1)) \implies f(f(x)-x)=2(f(x)-x)$

Let then $x\in\mathbb{R}$ and z such $f(z)=f(x)-x$ which exists since $f(x)$ is surjective.
Using last equation, we get $f(f(z))=2f(z)$
$P(z,-1) \implies f(f(z))=2f(z)-z$

And so $z=0$ and $f(z)=1$ and $f(x)=x+1$, which indeed is a solution.

Hence the answer : $\boxed{f(x)=x+1}$

And anybody feel free to post the next problem.

Sansa
108 posts

Sep 12, 2010, 8:33 pm

PM #152

> *pco wrote:*
> Yes. Little remark : one line for the same result : $P(x,\frac{x^2-f(x)}{2})$:)

It was really terrific... :)

Rijul saini
799 posts

Sep 12, 2010, 11:10 pm

PM #153

Problem 50:
If the following conditions are satisfied by a function $f:\mathbb{R}\to\mathbb{R}$, then prove that it's the identity function.
1) $f(-x)=-f(x)$
2) $f(x+1)=f(x)+1$
3) $f(\frac{1}{x})=\frac{f(x)}{x^2}$

pco
14052 posts

Sep 13, 2010, 10:43 pm

PM #154

> *Rijul saini wrote:*
> **Problem 50:**
> If the following conditions are satisfied by a function $f:\mathbb{R}\to\mathbb{R}$, then prove that it's the identity function.
> 1) $f(-x)=-f(x)$
> 2) $f(x+1)=f(x)+1$
> 3) $f(\frac{1}{x})=\frac{f(x)}{x^2}$

Hello, could you confirm us that the function is from $\mathbb{R}\to\mathbb{R}$ (and not, for example, from $\mathbb{Q}\to\mathbb{Q}$) ?

It's easy to show that $f(x)=x\ \forall x\in\mathbb{Q}$ using, for example, induction on the length of continuous fraction of x.

But extending to $\mathbb{R}$ is not so easy. And I even think (but have not proved) that this is wrong.

fedja
6941 posts

Sep 13, 2010, 11:07 pm

PM #155

Let us call the operations $x\mapsto -x$, $x\mapsto 1/x$ and $x\mapsto x\pm 1$ elementary. Call two real numbers equivalent if one can be obtained from another by a chain of elementary operations (they are all invertible, so it is, indeed, an equivalence relation). Note that the values of f for the points in different equivalence classes are completely independent. All classes have just to be consistent within themselves

Now, take the equivalence class C generated by $\sqrt{2}$. It is fully contained in the field $\mathbb{Q}[\sqrt{2}]$. In this field, we have an automorphism σ over $\mathbb{Q}$ that sends $\sqrt{2}$ to $-\sqrt{2}$. Obviously, $\sigma(x)$ satisfies the equation in $\mathbb{Q}[\sqrt{2}]$ and, therefore, in C. Thus, we have a non-trivial mapping in C. It can be extended to the full mapping using the trivial identity mapping in each other equivalence class. The extension is certainly not an identity.

arshakus
748 posts

Sep 13, 2010, 11:17 pm • 1 PM #156

> *pco wrote:*
>
> > *Rijul saini wrote:*
> > **Problem 50:**
> > If the following conditions are satisfied by a function $f:\mathbb{R}\to\mathbb{R}$, then prove that it's the identity function.
> > 1) $f(-x)=-f(x)$
> > 2) $f(x+1)=f(x)+1$
> > 3) $f(\frac{1}{x})=\frac{f(x)}{x^2}$
>
> Hello, could you confirm us that the function is from $\mathbb{R}\to\mathbb{R}$ (and not, for example, from $\mathbb{Q}\to\mathbb{Q}$) ?
>
> It's easy to show that $f(x)=x\ \forall x\in\mathbb{Q}$ using, for example, induction on the length of continuous fraction of x.
>
> But extending to $\mathbb{R}$ is not so easy. And I even think (but have not proved) that this is wrong.

hey everybody,
pco u r not right because it is right even that $f:R->R.$
$f(-x)=-f(x)$
$f(x+1)=f(x)+1=f(\frac{1}{x})=f(\frac{1}{x}+1)-1$
$f(\frac{1}{x})=\frac{f(x)}{x^2}=>f(\frac{x}{x+1})=\frac{x^2}{(x+1)^2}f(\frac{x+1}{x})$
when x not equal 0 and -1=> $f(x)=x^2f(\frac{1}{x})=x^2(f(\frac{x+1}{x})-1)=x^2(\frac{(x+1)^2}{x^2}f(\frac{x}{x+1})-1)=>$
$=>f(x)=(x+1)^2f(\frac{x}{x+1})-x^2$
$f(\frac{x}{x+1})=f(\frac{(x+1)-1}{x+1})=f(1-\frac{1}{x+1})=f(-\frac{1}{x+1})+1=1-f(\frac{1}{x+1})=1-\frac{1}{(x+1)^2}f(x+1)=>$
$=>(x+1)^2-f(x+1)-x^2=f(x)$
$2x+1=f(x)+f(x+1)=>f(x)=x,whenxnotequal0and-1=>whenx=0=>f(0)=0whenx=-1=>f(0)=f(-1)+1=>$
$f(-1)=-1=>[url]f(x)=x[/url]$

fedja
6941 posts

Sep 13, 2010, 11:33 pm PM #157

Oops, you seem to be right. I haven't noticed that the last identity uses x explicitly, not through $f()$, which invalidates my counterexample.

arshakus
748 posts

Sep 13, 2010, 11:37 pm PM #158

> *fedja wrote:*
> Oops, you seem to be right. I haven't noticed that the last identity uses x explicitly, not through $f()$, which invalidates my counterexample.

check it one more....

pco
14052 posts

Sep 13, 2010, 11:40 pm PM #159

> *arshakus wrote:*
> ...
> $f(\frac{1}{x})=\frac{f(x)}{x^2}=>f(\frac{x}{x+1})=\frac{x^2}{(x+1)^2}f(\frac{x+1}{x})$
> when x not equal 0 and -1=> $f(x)=x^2f(\frac{1}{x})=x^2(f(\frac{x+1}{x})-1)=x^2(\frac{(x+1)^2}{x^2}f(\frac{x}{x+1})-1)=>$
> $=>f(x)=(x+1)^2f(\frac{x}{x+1})-x^2$
> $f(\frac{x}{x+1})=f(\frac{(x+1)-1}{x+1})=f(1-\frac{1}{x+1})=f(-\frac{1}{x+1})+1=1-f(\frac{1}{x+1})=1-\frac{1}{(x+1)^2}f(x+1)=>$
> $=>(x+1)^2-f(x+1)-x^2=f(x)$
> $2x+1=f(x)+f(x+1)\Longrightarrow f(x)=x$,...

It seems you are right.
Nice!
And congrats.
:)

arshakus
748 posts

Sep 13, 2010, 11:49 pm PM #160

> *pco wrote:*
>
> > *arshakus wrote:*
> > ...
> > $f(\frac{1}{x})=\frac{f(x)}{x^2}=>f(\frac{x}{x+1})=\frac{x^2}{(x+1)^2}f(\frac{x+1}{x})$
> > when x not equal 0 and -1=> $f(x)=x^2f(\frac{1}{x})=x^2(f(\frac{x+1}{x})-1)=x^2(\frac{(x+1)^2}{x^2}f(\frac{x}{x+1})-1)=>$
> > $=>f(x)=(x+1)^2f(\frac{x}{x+1})-x^2$
> > $f(\frac{x}{x+1})=f(\frac{(x+1)-1}{x+1})=f(1-\frac{1}{x+1})=f(-\frac{1}{x+1})+1=1-f(\frac{1}{x+1})=1-\frac{1}{(x+1)^2}f(x+1)=>$
> > $=>(x+1)^2-f(x+1)-x^2=f(x)$
> > $2x+1=f(x)+f(x+1)\Longrightarrow f(x)=x$,...
>
> It seems you are right.
> Nice!
> And congrats.
> :)

thanks a lot pco

Sep 14, 2010, 3:25 am PM #161

Rijul saini
799 posts

> *pco wrote:*
> Hello, could you confirm us that the function is from $\mathbb{R} \to \mathbb{R}$ (and not, for example, from $\mathbb{Q} \to \mathbb{Q}$) ?
>
> It's easy to show that $f(x) = x\,\forall x \in \mathbb{Q}$ using, for example, induction on the length of continuous fraction of x.
>
> But extending to $\mathbb{R}$ is not so easy. And I even think (but have not proved) that this is wrong.

Of course I can confirm your doubt.
It came in the INMO (Don't remember which year it was), and you can click here to check out mine (and others) proof of it.
Also, can you clarify my doubt regarding raghu.mahajan's proof in that thread?

Rijul saini
799 posts

Sep 21, 2010, 10:24 pm PM #162

Problem 51:
Find all one-one functions $f : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, which satisfies

$$f(f(n)) \leq \frac{f(n)+n}{2}$$

This post has been edited 1 time. Last edited by Rijul saini, Sep 23, 2010, 2:15 am

---

pco
14052 posts

Sep 23, 2010, 12:21 am PM #163

> *Rijul saini wrote:*
> **Problem 51:**
> Find all functions $f : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, which satisfies
>
> $$f(f(n)) \leq \frac{f(n)+n}{2}$$

Are you sure about the problem statement ?
With your statement, there are obviously infinitely many solutions, and I'm not sure we can find a general formula for them.

For example :

$f(x) = x$
$f(x) = 1$
Any $f(x)$ such that $f(f(x)) = 1$ (and there are infinitely many such functions)
Any $f(x)$ such that $f(1) \geq 3$ and $f(f(x)) = 2$ (and there are infinitely many such functions)
Any $f(x)$ such that $f(1) = f(2) = 1$ and $f(f(x)) \in \{1, 2\}\,\forall x$
.....

---

mahanmath
1355 posts

Sep 23, 2010, 12:30 am PM #164

I have a short solution for injective f .

Rijul saini
799 posts

Sep 23, 2010, 2:16 am PM #165

> *pco wrote:*
>
> > *Rijul saini wrote:*
> > **Problem 51:**
> > Find all functions $f : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, which satisfies
> >
> > $$f(f(n)) \leq \frac{f(n)+n}{2}$$
>
> Are you sure about the problem statement ?
> With your statement, there are obviously infinitely many solutions, and I'm not sure we can find a general formula for them.
>
> For example :
>
> $f(x) = x$
> $f(x) = 1$
> Any $f(x)$ such that $f(f(x)) = 1$ (and there are infinitely many such functions)
> Any $f(x)$ such that $f(1) \geq 3$ and $f(f(x)) = 2$ (and there are infinitely many such functions)
> Any $f(x)$ such that $f(1) = f(2) = 1$ and $f(f(x)) \in \{1, 2\}\,\forall x$
> .....

Sorry Patrick, missed the one-one condition

---

pco
14052 posts

Sep 23, 2010, 10:03 pm PM #166

> *Rijul saini wrote:*
> Sorry Patrick, missed the one-one condition

It's easy to show with induction that $f^{[k]}(n) \leq \frac{2f(n)+n}{3} + \frac{2}{3(-2)^k}(n - f(n))$

So, for k great enough : $f^{[k]}(n) \leq \frac{2f(n)+n}{3} + 1$ and so $\exists\, k_1 > k_2$ such that $f^{[k_1]}(n) = f^{[k_2]}(n)$ and, since injective :

$\forall n\, \exists p_n \geq 1$ such that $f^{[p_n]}(n) = n$

Then, setting $k = p_n$ in the above inequality, we get $n \leq \frac{2f(n)+n}{3} + \frac{2}{3(-2)^{p_n}}(n - f(n))$

$\iff 0 \leq (f(n) - n)(1 - \frac{1}{(-2)^{p_n}})$ and so $f(n) \geq n\,\forall n$

But $f(n) > n$ for some n and injectivity would imply $f^{[p_n]}(n) > n$ and so $f(n) = n\,\forall n$ which indeed is a solution.

Dumel
190 posts

Sep 24, 2010, 3:54 am PM #167

an alternative solution:
by strong induction we can easily prove that if $f(n) \leq n$ for some n then for all natural k $f^k(n) \leq n$ whence by injectivity we can simply deduce that $f(n) = n$
hence $f(n) \geq n$ for all n, so $f(n) = n$ for all n

Amir Hossein
4719 posts

Sep 24, 2010, 10:40 am PM #168

Here is a new problem :

Problem 52 :

Find all functions $f:\mathbb R\to\mathbb R$ satisfying

$$f(f(x)+y)=f(x^2-y)+4f(x)y$$

for all $x,y\in\mathbb R$.

---

pco
14052 posts

Sep 24, 2010, 12:25 pm PM #169

> *amparvardi wrote:*
> Here is a new problem :
>
> **Problem 52** :
>
> Find all functions $f:\mathbb R\to\mathbb R$ satisfying
>
> $$f(f(x)+y)=f(x^2-y)+4f(x)y$$
>
> for all $x,y\in\mathbb R$.

Let $P(x,y)$ be the assertion $f(f(x)+y)=f(x^2-y)+4f(x)y$

$P(x,\frac{x^2-f(x)}2)$ $\implies f(x)(f(x)-x^2)=0$ $\forall x$ and so : $\forall x$, either $f(x)=0$, either $f(x)=x^2$

Suppose now $\exists a\neq 0$ such that $f(a)=0$ and $b\neq 0$ such that $f(b)=b^2$:

$P(a,b)$ $\implies b^2=f(a^2-b)$ and so $b^2=(a^2-b)^2$ and so $b=\frac{a^2}2$

So, if $\exists a\neq 0$ such that $f(a)=0$, then $f(x)=0$ $\forall x\neq\frac{a^2}2$ but then, choosing another a, we get that $f(\frac{a^2}2)=0$ too.

Then, either $f(x)=0$ $\forall x$, either $f(x)=x^2$ $\forall x$ and these two functions indeed are solutions.

Hence the only two solutions :
$f(x)=0$ $\forall x$
$f(x)=x^2\forall x$

Amir Hossein
4719 posts

Sep 24, 2010, 12:37 pm PM #170

Problem 53 :

For a given natural number $k>1$, find all functions $f:\mathbb R\to\mathbb R$ such that for all $x,y\in\mathbb R$, $f[x^k+f(y)]=y+[f(x)]^k$.

---

pco
14052 posts

Sep 24, 2010, 12:44 pm PM #171

> *amparvardi wrote:*
> **Problem 53** :
>
> For a given natural number $k>1$, find all functions $f:\mathbb R\to\mathbb R$ such that for all $x,y\in\mathbb R$, $f[x^k+f(y)]=y+[f(x)]^k$.

Are the brackets just parenthesis or integer parts ?

Amir Hossein
4719 posts

Sep 24, 2010, 1:07 pm PM #172

They are just parenthesis.

---

pco
14052 posts

Sep 24, 2010, 8:27 pm • 1 PM #173

> *amparvardi wrote:*
> **Problem 53** :
>
> For a given natural number $k>1$, find all functions $f:\mathbb R\to\mathbb R$ such that for all $x,y\in\mathbb R$, $f[x^k+f(y)]=y+[f(x)]^k$.

Let $P(x,y)$ be the assertion $f(x^k+f(y))=y+f(x)^k$
Let $f(0)=a$

$P(0,y)$ $\implies f(f(y))=y+a^k$
$P(x,0)$ $\implies f(x^k+a)=f(x)^k$
$P(x,f(y))$ $\implies f(x^k+y+a^k)=f(y)+f(x^k+a)$

Let then $g(x)=f(x-a^k+a)$. This last equality becomes $g(x^k+y+2a^k-a)=g(y+a^l-a)+g(x^k+a^k)$ $\iff$
$g(x^k+a^k+y)=g(y)+g(x^k+a^k)$

And so $g(x+y)=g(x)+g(y)\forall x\ge a^k,\forall y$
Let then $x\ge 0$:
$g(a^k+x+y)=g(a^k+(x+y))=g(a^k)+g(x+y)$
$g(a^k+x+y)=g((a^k+x)+y)=g(a^k+x)+g(y)=g(a^k)+g(x)+g(y)$
And so $g(x+y)=g(x)+g(y)\forall x\ge 0,\forall y$

So $g(0)=0$ and $g(-x)=-g(x)$. Then :
$\forall x\ge 0,\forall y$: $-g(x-y)=-g(x)-g(-y)$ $\implies g(-x+y)=g(-x)+g(y)$ and so $g(x+y)=g(x)+g(y)\forall x,y$

And so $g(px)=pg(x)\forall p\in\mathbb Q,\forall x$

Then $f(x^k+a)=f(x)^k$ implies $g(x^k+a^k)=g(x+a^k-a)^k$ $\implies g(x^k)+g(a^k)=(g(x)+g(a^k-a))^k$
Notice that $g(a^k-a)=f(0)=a$ and replace x with $x+y$ and we get :

$$g((x+y)^k)+g(a^k)=(g(x)+g(y)+a)^k$$

$$g(\sum_{i=0}^k\binom ki x^iy^{k-i})+g(a^k)=\sum_{i=0}^k\binom ki g(x)^i(g(y)+a)^{k-i}$$

Let then $x\in\mathbb Q$ and this equation becomes :

$$\sum_{i=0}^{k}\binom{k}{i}x^i g(y^{k-i})+g(a^k)=\sum_{i=0}^{k}\binom{k}{i}g(1)^i x^i(g(y)+a)^{k-i}$$

And so we have two polynomials in x (LHS and RHS) which are equal for any $x\in\mathbb{Q}$. So they are identical and all their coefficients are equal.

Since $k\ge 2$, consider the equality of coefficients of x^{k-2}:
If $k>2$, this equality is $g(y^2)=g(1)^{k-2}(g(y)+a)^2$ and $g(x)$ has a constant sign over $\mathbb{R}^+$
If $k=2$, this equality becomes $g(y^2)+g(a^2)=(g(y)+a)^2$ and $g(x)\ge -g(a^2)\forall x\ge 0$

In both cases, we have $g(x)$ either upper bounded, either lower-bounded on a non empty open interval, and this a classical condition to conclude to continuity and $g(x)=cx\,\forall x$

And so $f(x)=cx+d$ for some real c,d

Plugging this back in original equation, we get :

$f(x)=x\,\forall x$ which is a solution for any k
$f(x)=-x\,\forall x$ which is another solution if k is odd.

Amir Hossein
4719 posts

Sep 24, 2010, 8:30 pm PM #174

Thank you, Mr. pco 🙂

Problem 54 :

Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that for all $x,y\in\mathbb{Z}$:

$$f(x-y+f(y))=f(x)+f(y).$$

---

pco
14052 posts

Sep 24, 2010, 9:27 pm PM #175

> amparvardi wrote:
> **Problem 54** :
>
> Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that for all $x,y\in\mathbb{Z}$:
>
> $$f(x-y+f(y))=f(x)+f(y).$$

Let $P(x,y)$ be the assertion $f(x-y+f(y))=f(x)+f(y)$
Let $f(0)=a$

$P(0,0)\implies f(a)=2a$ and so $f(a)-a=a$
$P(0,a)\implies f(f(a)-a)=f(0)+f(a)$ and so $f(0)=0$

$P(0,x)\implies f(f(x)-x)=f(x)$
$P(x,f(y)-y)\implies f(x-f(y)+y+f(f(y)-y))=f(x)+f(f(y)-y)$ and so $f(x+y)=f(x)+f(y)$ and so $f(x)=xf(1)$ (remember we are in $\mathbb{Z}$)

Plugging this in original equation, we get two solutions :

$f(x)=0\,\forall x$
$f(x)=2x\,\forall x$

ArefS
297 posts

Sep 24, 2010, 9:39 pm PM #176

My Solution

let $P(x,y)$ be the assertion: $f(x-y+f(y))=f(x)+f(y)$
$P(-f(y),y)\to f(-y)=f(-f(y))+f(y); (y=0)\Rightarrow f(-f(0))=0$
$P(x,-f(0))\to f(x+f(0))=f(x); (x=-f(0))\Rightarrow f(0)=f(-f(0))=0$
Define the function $g:\mathbb{Z}\to\mathbb{Z}; g(x)=f(x)-x$
rewriting the functional equation as tems of g, we get that:
$g(x+g(y))=g(x)+y$, and from $f(0)=0$ we get that also $g(0)=0$.
let $Q(x,y)$ be the assertion: $g(x+g(y))=g(x)+y$.
$Q(0,y)\to g(g(y))=y$ substituting $z=g(y)$; by:$y\to Z$ we get that: $g(x+z)=g(x)+g(z)$; Hence we have got two solution for g and therefore f:
$g(x)=x$ and $g(x)=-x$

Amir Hossein
4719 posts

Sep 24, 2010, 9:45 pm PM #177

Problem 55 :

We denote by $\mathbb{R}^+$ the set of all positive real numbers.

Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ which have the property:

$$f(x)f(y)=2f(x+yf(x))$$

for all positive real numbers x and y.

---

pco
14052 posts

Sep 24, 2010, 11:41 pm PM #178

> amparvardi wrote:
> **Problem 55** :
>
> We denote by $\mathbb{R}^+$ the set of all positive real numbers.
>
> Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ which have the property:
>
> $$f(x)f(y)=2f(x+yf(x))$$
>
> for all positive real numbers x and y.

Let $P(x,y)$ be the assertion $f(x)f(y)=2f(x+yf(x))$

Let $u,v>0$.
Let $a\in(0,u)$

Let $x=a>0$ and $y=\frac{u-a}{f(a)}>0$ and $z=\frac{2v}{f(x)f(y)}>0$

$f(x)f(y)=2f(x+yf(x))=2f(u)$ and so $f(x)f(y)f(z)=2f(u)f(z)=4f(u+zf(u))=4f(u+v)$

$f(y)f(z)=2f(y+zf(y))$ and so $f(x)f(y)f(z)=2f(x)f(y+zf(y))$$=4f(x+(y+zf(y))f(x))$
$=4f(x+yf(x)+zf(x)f(y))$$=4f(u+2v)$

And so $f(u+v)=f(u+2v)\forall u,v>0$ and so $f(x)=f(y)\forall x,y$ such that $2x>y>x>0$

And it's immediate from there to conclude $f(x)=f(y)\forall x,y>0$

Hence the unique solution $\boxed{f(x)=2\forall x>0}$

Amir Hossein
4719 posts

Sep 25, 2010, 12:53 am PM #179

Problem 56 :

Find all functions $f:\mathbb{Q}^+\mapsto\mathbb{Q}^+$ such that:

$$f(x)+f(y)+2xyf(xy)=\frac{f(xy)}{f(x+y)}.$$

It seems Problems 48 and 49 are still unsolved :

Problem 48 :
Find all continuous functions $\mathbb{R}\to\mathbb{R}$ satisfying the equation:

$$f(x+y)+f(xy)=f(x)+f(y)+f(xy+1)$$

Problem 49 :
Find all continuous functions $\mathbb{R}\to\mathbb{R}$ satisfying the equation:

$$f(x)+f(y)+f(z)+f(x+y+z)=f(x+y)+f(y+z)+f(z+x)+f(0)$$

---

pco
14052 posts

Sep 25, 2010, 12:58 am PM #180

> *amparvardi wrote:*
> **Problem 49** :
> Find all continuous functions $\mathbb{R}\to\mathbb{R}$ satisfying the equation:
>
> $$f(x)+f(y)+f(z)+f(x+y+z)=f(x+y)+f(y+z)+f(z+x)+f(0)$$

Let $P(x,y,z)$ be the assertion $f(x)+f(y)+f(z)+f(x+y+z)=f(x+y)+f(y+z)+f(z+x)+f(0)$

$P(x,y,y)\implies f(x+2y)-f(x+y)=f(x+y)-f(x)+(f(2y)+f(0)-2f(y))$
$P(x+y,y,y)\implies f(x+3y)-f(x+2y)=f(x+2y)-f(x+y)+(f(2y)+f(0)-2f(y))$
...
$P(x+(n-1)y,y,y)\implies$
$f(x+(n+1)y)-f(x+ny)=f(x+ny)-f(x+(n-1)y)+(f(2y)+f(0)-2f(y))$

Adding these lines gives $f(x+(n+1)y)-f(x+ny)=f(x+y)-f(x)+n(f(2y)+f(0)-2f(y))$

And so (adding this last lines for $n=0,...,k-1$) :
$f(x+ky)-f(x)=k(f(x+y)-f(x))+\frac{k(k-1)}{2}(f(2y)+f(0)-2f(y))$

Setting $x=0$ in this last equality and renaming $y\to x$ and $k\to n$, we get :

$$f(nx)=\frac{f(2x)+f(0)-2f(x)}{2}n^2+\frac{4f(x)-f(2x)-3f(0)}{2}n+f(0)$$

So :
$$f(q\frac{p}{q})=\frac{f(2\frac{p}{q})+f(0)-2f(\frac{p}{q})}{2}q^2+\frac{4f(\frac{p}{q})-f(2\frac{p}{q})-3f(0)}{2}q+f(0)$$

And since $f(q\frac{p}{q})=f(p)=\frac{f(2)+f(0)-2f(1)}{2}p^2+\frac{4f(1)-f(2)-3f(0)}{2}p+f(0)$, we get :

$$(f(2)+f(0)-2f(1))p^2+(4f(1)-f(2)-3f(0))p=(f(2\frac{p}{q})+f(0)-2f(\frac{p}{q}))q^2+(4f(\frac{p}{q})-f(2\frac{p}{q})-3f(0))q$$

Replacing $p\to np$ and $q\to nq$ in this equation, we get :

$(f(2)+f(0)-2f(1))p^2n^2+(4f(1)-f(2)-3f(0))pn$
$=(f(2\frac{p}{q})+f(0)-2f(\frac{p}{q}))q^2n^2+(4f(\frac{p}{q})-f(2\frac{p}{q})-3f(0))qn$ and so :

$$n^2\left((f(2)+f(0)-2f(1))p^2-(f(2\frac{p}{q})+f(0)-2f(\frac{p}{q}))q^2\right)$$
$$+n\left((4f(1)-f(2)-3f(0))p-(4f(\frac{p}{q})-f(2\frac{p}{q})-3f(0))q\right)=0$$

And since this is true for any n, we get :
$(f(2)+f(0)-2f(1))p^2-(f(2\frac{p}{q})+f(0)-2f(\frac{p}{q}))q^2=0$
$(4f(1)-f(2)-3f(0))p-(4f(\frac{p}{q})-f(2\frac{p}{q})-3f(0))q=0$

From these two lines, we get $f(\frac{p}{q})=\frac{f(2)+f(0)-2f(1)}{2}\frac{p^2}{q^2}+\frac{4f(1)-f(2)-3f(0)}{2}\frac{p}{q}+f(0)$

And so $f(x)=ax^2+bx+c\forall x\in\mathbb{Q}^+$ which indeed fits whatever are a,b,c.

So $f(x)=ax^2+bx+c\forall x\in\mathbb{R}^+$(using continuity)

Let then $x>0$:
$P(-x,x,x) \implies f(-x)+3f(x)=f(2x)+3f(0)$ and, since $x\ge 0$ and $2x\ge 0$:

$f(-x)=(4ax^2+2bx+c)+3c-3(ax^2+bx+c)=ax^2-bx+c$

And so $f(x)=ax^2+bx+c\,\forall x\in\mathbb{R}$

pco
14052 posts

Sep 25, 2010, 1:09 am — PM #181

> amparvardi wrote:
> Problem 48 :
> Find all continuous functions $\mathbb{R}\to\mathbb{R}$ satisfying the equation:
> $$f(x+y)+f(xy)=f(x)+f(y)+f(xy+1)$$

Let $f(x)=g(x)-1$ and the equation becomes $g(x+y)+g(xy)+1=g(x)+g(y)+g(xy+1)$

Which is an old problem (see http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=336764)

pco
14052 posts

Sep 25, 2010, 1:50 am — PM #182

> amparvardi wrote:
> Problem 56 :
>
> Find all functions $f:\mathbb{Q}^+\mapsto\mathbb{Q}^+$ such that:
> $$f(x)+f(y)+2xyf(xy)=\frac{f(xy)}{f(x+y)}.$$

Let $P(x,y)$ be the assertion $f(x)+f(y)+2xyf(xy)=\dfrac{f(xy)}{f(x+y)}$
Let $f(1)=a$

$P(1,1)\implies f(2)=\frac{1}{4}$
$P(2,2)\implies f(4)=\frac{1}{16}$
$P(2,1)\implies f(3)=\frac{1}{4a+5}$
$P(3,1)\implies f(4)=\frac{1}{4a^2+5a+7}$ and so $4a^2+5a+7=16$ and so $a=1$ (remember $f(x)>0$)

$P(x,1)\implies\frac{1}{f(x+1)}=\frac{1}{f(x)}+2x+1$ and so $\frac{1}{f(x+n)}=\frac{1}{f(x)}+2nx+x^2$ and $f(n)=\frac{1}{n^2}$

$P(x,n)\implies f(nx)=\dfrac{f(x)+\frac{1}{n^2}}{\frac{1}{f(x)}+n^2}$

Setting $x=\frac{p}{n}$ in this last equality, we get $f(\frac{p}{n})=\frac{n^2}{p^2}$ (remember $f(x)>0$)

Hence the answer : $\boxed{f(x)=\frac{1}{x^2}}\,\forall x\in\mathbb{Q}^+$ which indeed is a solution.

Winner2010
79 posts

Sep 25, 2010, 9:55 am — PM #183

Problem 57:

Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that for all $x>y>0$,

$f(x+y)-f(x-y)=4\sqrt{f(x)f(y)}$

pco
14052 posts

Sep 25, 2010, 11:52 am — PM #184

> Winner2010 wrote:
> **Problem 57:**
>
> Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that for all $x>y>0$,
>
> $f(x+y)-f(x-y)=4\sqrt{f(x)f(y)}$

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=364296

aktyw19
1315 posts

Sep 25, 2010, 4:22 pm — PM #185

Problem 58
Find all $f:N\to\{0,1,2,...,2000\}$
such that
$0\le n\le 2000\Rightarrow f(n)=n$
$f(f(n)+f(m))=f(m+n)$

Ramchandran
645 posts

Feb 2, 2011, 12:57 am — PM #186

Why has this marathon stopped?.
Lets revive this marathon :)

Problem 59

Find all continuous $f:R\to R$ such that for reals x,y -
$f(x+f(y))=y+f(x+1)$

pco
14052 posts

Feb 2, 2011, 1:36 am • 1 👍 — PM #187

> Ramchandran wrote:
> **Problem 59**
>
> Find all continuous $f:R\to R$ such that for reals x,y -
> $f(x+f(y))=y+f(x+1)$

Let $P(x,y)$ be the assertion $f(x+f(y))=y+f(x+1)$

$P(0,y+1-f(1)) \implies f(f(y+1-f(1)))=y+1$
$P(x-f(1),f(y+1-f(1))) \implies f(x-f(1)+f(f(y+1-f(1))))=f(y+1-f(1))+f(x+1-f(1))$ and so $f(x+y+1-f(1))=f(y+1-f(1))+f(x+1-f(1))$

Let then $g(x)=f(x+1-f(1))$ and we get $g(x+y)=g(x)+g(y)$ and so, since continuous, $g(x)=ax$ and $f(x)=a(x+f(1)-1)$

Plugging $f(x)=ax+b$ in original equation, we get two solutions :
$f(x)=1+x\forall x$
$f(x)=1-x\forall x$

Amir Hossein
4719 posts

Feb 2, 2011, 3:20 pm • 1 PM #188

Problem 60.

Let $n>m>1$ be odd integers, let $f(x)=x^n+x^m+x+1$. Prove that $f(x)$ can't be expressed as the product of two polynomials having integer coefficients and positive degrees.

---

abhinavzan...
418 posts

Feb 8, 2011, 1:25 pm PM #189

This Marathon **MUST NOT** Die
As Solution To 60 Is Already There.
Problem 61

---

pco
14052 posts

Feb 8, 2011, 4:53 pm PM #190

> abhinavzandubalm wrote:
> Problem 61 :
> $f:Z\to Z$
> $f(m+n)+f(mn-1)=f(m)f(n)+2$

Let $P(x,y)$ be the assertion $f(x+y)+f(xy-1)=f(x)f(y)+2$

$P(x,0) \implies f(x)(f(0)-1)=f(-1)-2$

If $f(0)\ne 1$, this implies $f(x)=c$ and $2c=c^2+2$ and no solution.
So $f(0)=1$ and $f(-1)=2$

Let then $f(1)=a$
$P(1,1) \implies f(2)=a^2+1$
$P(2,1) \implies f(3)=a^3+2$
$P(3,1) \implies f(4)=a^4-a^2+2a+1$
$P(2,2) \implies f(4)=a^4-a^3+2a^2+1$

And so $a^4-a^2+2a+1=a^4-a^3+2a^2+1 \iff a(a-1)(a-2)=0$

If $a=0$:
Previous lines imply $f(2)=1$ and $f(3)=2$ and $f(4)=1$
$P(4,1) \implies f(5)=0$
But $P(3,2) \implies f(5)=2$ and so contradiction

If $a=1$:
Previous lines imply $f(2)=2$ and $f(3)=3$ and $f(4)=3$
$P(4,1) \implies f(5)=2$
But $P(3,2) \implies f(5)=4$ and so contradiction

If $a=2$, then $P(m+1,1) \implies f(m+2)=2f(m+1)-f(m)+2$ which is easily solved in $f(m)=m^2+1$ which indeed is a soluion.

Hence the unique solution : $\boxed{f(x)=x^2+1}\ \forall x\in\mathbb{Z}$

---

abhinavzan...
418 posts

Feb 8, 2011, 6:01 pm PM #191

Could Anyone Give The Next Problem Please.

Raja Oktovin
277 posts

Feb 8, 2011, 7:39 pm PM #192

Problem 62. Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a function such that
$f(\sqrt{ab})=\sqrt{f(a)f(b)}$
for all $a,b\in\mathbb{R}^+$ satisfying $a^2b>2$.
Prove that the equation holds for all $a,b\in\mathbb{R}^+$.
Here, $\mathbb{R}^+$ is the set of all positive real numbers.

---

ocha
955 posts

Feb 10, 2011, 12:21 pm PM #193

> Raja Oktovin wrote:
> **Problem 62.** Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a function such that
> $f(\sqrt{ab})=\sqrt{f(a)f(b)}$
> for all $a,b\in\mathbb{R}^+$ satisfying $a^2b>2$.
> Prove that the equation holds for all $a,b\in\mathbb{R}^+$.
> Here, $\mathbb{R}^+$ is the set of all positive real numbers.

For some $u,v\in\mathbb{R}^+$. Take a,b sufficiently large such that such that $abuv>>2$. Then

1) $f(ab\cdot uv)^4=f(a^2b^2)^2f(u^2v^2)^2=f(a^4)f(b^4)f(u^2v^2)^2$
2) $f(au\cdot bv)^4=f(a^2u^2)^2f(b^2v^2)^2=f(a^4)f(u^4)f(b^4)f(v^4)$

Then since $f>0$ we can divide $f(a^4)f(b^4)$ and find $f(u^2v^2)^2=f(u^2)f(v^2)$. But u and v were chosen arbitrarily so $f(xy)^2=f(x^2)f(y^2)$ for all $x,y\in\mathbb{R}^+$

Problem 63
For $a,b,c\in\mathbb{N}$ suppose there exists *coprime* polynomials $P,Q,R\in\mathbb{C}[x]$ such that

$$P(x)^a+Q(x)^b=R(x)^c$$

Show that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>1$

$a \quad b \quad c$

ocha
955 posts

Feb 11, 2011, 7:51 am PM #194

> amparvardi wrote:
>
> **Problem 60.**
>
> Let $n > m > 1$ be odd integers, let $f(x) = x^n + x^m + x + 1$. Prove that $f(x)$ can't be expressed as the product of two polynomials having integer coefficients and positive degrees.

This is a scaled back proof of Ljunggren's Theorem.

Proof
Let $f^*(x) = x^{\deg(f)} f(\frac{1}{x})$ be the reciprocal polynomial of f. Suppose $f(x) = p(x)q(x)$, then we can assume $q(x) \neq q^*(x)$ because f is not reciprocal (*note*: this is where we use m, n odd). So let $g(x) = p(x)q^*(x)$ with $f \not\equiv g$ (*). Now $g \in \mathbb{Z}[x]$, and $g(x)g^*(x) = f(x)f^*(x)$

Suppose $g(x) = g_n x^n + ... + g_0$. The coefficient of x^n in $f(x)f^*(x)$ is 4, and the coefficient of x^n in $g(x)g^*(x)$ is $g_n^2 + g_{n-1}^2 + ... + g_0^2$. Since g has integer coefficients and has at least two non zero coefficients we have $g(x) = x^n \pm x^r \pm x^s \pm 1$ (wlog the coefficient of x^n in $g(x)$ is positive).

Expanding the first few terms of $f(x)f^*(x)$ and $g(x)g^*(x)$ shows that the coefficient of x^{2n-1} in $f(x)f^*(x)$ is 1, and therefore $s = 1$ in $g(x)$ with x^s having a positive coefficient. Similarly looking at the coefficient of x^{n+m} in $f(x)f^*(x)$ tells us that $r = m$ in $g(x)$ and x^r has positive coefficient. Finally the x^0 coefficient in in $f(x)f^*(x)$ is 1 so $g(0) = 1$.

Therefore $g \equiv f$, but this contradicts (*) so f is irreducible $\square$

ocha
955 posts

Feb 11, 2011, 2:00 pm PM #195

> aktyw19 wrote:
>
> **Problem 58**
> Find all $f : \mathbb{N}_0 \to \{0, 1, 2, ..., 2000\}$ such that $0 \leq n \leq 2000 \Rightarrow f(n) = n$ and
> $f(f(n) + f(m)) = f(m + n)$

Let $[x]_p$ denote the residue $x \mod p$. Suppose $f(2001) = u$, then for positive integer r such that $u + r < 2001$ we have $f(2001 + r) = f(u + r) = u + r$. Furthermore if $u + r = 2001$ then $f(2001 + r) = f(u + r) = f(2001) = u$. Hence, by induction, for $n > 2000$ we have $f(n) = u + [n - 2001]_{(2001-u)}$.

One can easily check that this works for all u.

abhinavzan...
418 posts

Feb 11, 2011, 9:08 pm PM #196

Okay. As Nobody Is Giving A Problem.
Problem 63

> Find All Functions
> $f : Z \to Z$ Such That
> $[f(m) + f(n)]f(m - n) = [f(m) - f(n)]f(m + n) \quad \forall \quad m, n \quad \in Z$

aktyw19
1315 posts

Feb 12, 2011, 4:33 pm PM #197

problem 58

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=38&t=194959&hilit

abhinavzan...
418 posts

Feb 12, 2011, 4:45 pm PM #198

Mind If I Gave Hints To People Or Would You All Like To Solve It .
Might Wait For The Reply For At Most a day.

abhinavzan...
418 posts

Feb 13, 2011, 7:27 pm PM #199

Hint To Problem 63

> Try To Use
> $\mod 3$
> $\mod 4$
> And Some Others.

And Solution Are

> $f(0) = 0$
> $f(n) + f(-n) = 0$
> Let $m = 2, n = 1$
> Therefore By Rearranging The Terms
> $[f(3) - 1][f(2) - 1] = 2$
> **Case 1**
> $f(2) = 2, f(3) = 3$
> By Induction We Get $f(n) = n$
>
> **Case 2**
> $f(2) = 3, f(3) = 2$
> This Gives $f(4) = 9 \implies 8f(5) = 20$
> Hence a Contradiction.
>
> **Case 3**
> $f(2) = 0, f(3) = -1$
> Use Induction To Prove That
> $f(2n) = 0, f(4n + 1) = 1, f(4n - 1) = -1$
>
> **Case 4**
> $f(2) = -1, f(3) = 0$
> Use Induction To Prove That
> $f(3n) = 0, f(3n + 1) = 1, f(3n - 1) = -1$
>
> Hence we Can Find The Solutions

And The Next Problem 64

Find All Polynomials
$P(x+1)=P(x)+2x+1$

goodar2006
1344 posts
Feb 13, 2011, 10:15 pm PM #200
if $degP \geq 4$ then looking at the coefficient of x^{n-1} implies $na_n = 0$ which is impossible. the rest would be easy.

SCP
1520 posts
Feb 13, 2011, 10:51 pm PM #201

> *goodar2006 wrote:*
> if $degP \geq 4$ then looking at the coefficient of x^{n-1} implies $na_n = 0$ which is impossible. the rest would be easy.

rest: $P(1)=P(0)+1=P(-1)$ hence it is an even function and we find $f(x)=x^2+c$
I let someone other post an other problem, I only complete this solution.

abhinavzan...
418 posts
Feb 13, 2011, 11:06 pm PM #202
You People Must Really Hate Functional Equations.
Anyway Here's a problem.
Problem 65

> A Rational Function f (i.e. a function which is a quotient of two polynomials) has the property that
> $f(x)=f(\frac{1}{x})$
> Prove That f is a function in the variable $x+\frac{1}{x}$.

pco
14052 posts
Feb 14, 2011, 4:11 pm PM #203

> *abhinavzandubalm wrote:*
> Problem 65
> A Rational Function f (i.e. a function which is a quotient of two polynomials) has the property that
> $f(x)=f(\frac{1}{x})$
> Prove That f is a function in the variable $x+\frac{1}{x}$.

Let $f(x)=x^p(x-1)^q(x+1)^r\frac{P(x)}{Q(x)}$ where $p,q,r\in\mathbb{Z}$ and $P,Q\in\mathbb{R}[X]$ and
$P(0),P(1),P(-1),Q(0),Q(1),Q(-1)\neq 0$

Let $z\in\mathbb{C}$ root of $P(x)$. $z\neq 0$ and we get $f(z)=0$ and so $P(\frac{1}{z})=0$.
$z\neq\frac{1}{z}$ and so $P(x)=(x-z)(x-\frac{1}{z})P_1(x)$ and so $P(x)=x(x+\frac{1}{x}-(z+\frac{1}{z}))P_1(x)$

And so $f(x)=x^{p+1}(x-1)^q(x+1)^r(x+\frac{1}{x}-(z+\frac{1}{z}))\frac{P_1(x)}{Q(x)}$ and so $g(x)=x^{p+1}(x-1)^q(x+1)^r\frac{P_1(x)}{Q(x)}$ has the same property than $f(x)$ with degree(P_1)=degree(P) -2 and $f(x)=g(x)h(x+\frac{1}{x})$

Using this method up to degree $P=0$ and using then the same method with $\frac{1}{f}$, we easily get :

$f(x)=x^u(x-1)^v(x+1)^w t(x+\frac{1}{x})$

Applying then $f(x)=f(\frac{1}{x})$ to this result, we get $x^{2u+v+w}=(-1)^v$ and so $v=2k$ and $w=-2u-2k$

And so $f(x)=x^u(x-1)^{2k}(x+1)^{-2u-2k}t(x+\frac{1}{x})=(x+\frac{1}{x}-2)^{2k+u}((x+\frac{1}{x})^2-4)^{-u-k}t(x+\frac{1}{x})$

Q.E.D.

prafullasd
25 posts
Feb 14, 2011, 6:09 pm PM #204
my solution seems very small, so pls point out the flaw if any
$f(x)=\frac{P(x)}{Q(x)}=\frac{P(\frac{1}{x})}{Q(\frac{1}{x})}=\frac{P(x)+P(\frac{1}{x})}{Q(x)+Q(\frac{1}{x})}$ .... (I)
let $R(x)=P(x)+P(\frac{1}{x})$
R(x) is a "psedo" polynomial (my notation) which is a sum of terms of form $a_n(x^n+\frac{1}{x^n})$
by an easy induction we get that R(x) is a polynomial in (x + 1/x)
similarly, the denominator S(x)
so, f(x) is a rational function of (x+1/x)
the denomianators in any of the fractions in statement (I)can become zero only a finite number of times(each of them can be expressed as a polynomial of finite deg), therefore (I) is true for infinitely many vales of x
therefore, it is true for all real values of x (as it can be expressed as a ratio of two polynomials)

pco
14052 posts
Feb 14, 2011, 6:37 pm PM #205

> *prafullasd wrote:*
> my solution seems very small, so pls point out the flaw if any
> $f(x)=\frac{P(x)}{Q(x)}=\frac{P(\frac{1}{x})}{Q(\frac{1}{x})}=\frac{P(x)+P(\frac{1}{x})}{Q(x)+Q(\frac{1}{x})}$ .... (I)
> let $R(x)=P(x)+P(\frac{1}{x})$
> R(x) is a "psedo" polynomial (my notation) which is a sum of terms of form $a_n(x^n+\frac{1}{x^n})$
> by an easy induction we get that R(x) is a polynomial in (x + 1/x)
> similarly, the denominator S(x)
> so, f(x) is a rational function of (x+1/x)
> the denomianators in any of the fractions in statement (I)can become zero only a finite number of times(each of them can be expressed as a polynomial of finite deg), therefore (I) is true for infinitely many vales of x
> therefore, it is true for all real values of x (as it can be expressed as a ratio of two polynomials)

This seems quite OK for me.
And much simpler than mine (I missed that $\frac{a}{b}=\frac{c}{d}$ implies $\frac{a}{b}=\frac{c}{d}=\frac{a+c}{b+d}$)

Congrats.

abhinavzan...
418 posts

Feb 15, 2011, 3:41 pm PM #206

Sheesh.
It Doesn't Really Look Like You People Will Give Any Problems.
Problem 66

> Find All Functions
> $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(x-y)=f(x+y)f(y)$

This One's Easy.
So I Hope Someone Else may Give A Problem Now.

nguyenhung
559 posts

Feb 15, 2011, 9:00 pm PM #207

> *abhinavzandubalm wrote:*
> Sheesh.
> Problem 66
> Find All Functions
> $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(x-y)=f(x+y)f(y)$

Let $P(x,y)$ be the assertion of $f\left(x-y\right)=f\left(x+y\right)f\left(y\right)$

We have

$$P\left(x,0\right)\to f\left(x\right)=f\left(x\right)f\left(0\right)\Rightarrow\left[\begin{array}{l}f\left(x\right)=0\\ f\left(0\right)=0\end{array}\right.$$

* Case 1: $f\left(x\right)=0$. It's easy to see that $f\left(x\right)=0$ satisfies the condition

* Case 2: $f\left(0\right)=0$ and $f\left(x\right)\neq 0$

$P\left(x,x\right)\to 0=f\left(2x\right)f\left(x\right)$, not true 'cause $f\left(x\right)\neq 0$

So the only solution is $f\left(x\right)=0$

pco
14052 posts

Feb 15, 2011, 9:01 pm • 1 PM #208

> *abhinavzandubalm wrote:*
> Problem 66 :
> Find All Functions
> $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(x-y)=f(x+y)f(y)$

Let $P(x,y)$ be the assertion $f(x-y)=f(x+y)f(y)$

$P(0,0)\implies f(0)^2=f(0)$ and so $f(0)=0$ or $f(0)=1$

If $f(0)=0$: $P(x,0)\implies f(x)=0$ $\forall x$ which indeed is a solution

If $f(0)=1$:
$P(x,x)\implies f(x)f(2x)=1$ and so $f(x)\neq 0$ $\forall x$
$P(\frac{2x}{3},\frac{x}{3})\implies f(\frac{x}{3})=f(x)f(\frac{x}{3})$ and, since $f(\frac{x}{3})\neq 0$: $f(x)=1$ which indeed is a solution.

Hence the two solutions :
$f(x)=0$ $\forall x$
$f(x)=1$ $\forall x$

Amir Hossein
4719 posts

Feb 15, 2011, 9:07 pm PM #209

Nice solutions, pco.

Problem 67.

Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)\cdot f(y)=f(x)+f(y)+f(xy)-2\quad\forall x,y\in\mathbb{R}.$$

pco
14052 posts

Feb 15, 2011, 9:22 pm PM #210

> *amparvardi wrote:*
> **Problem 67.**
>
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
>
> $$f(x)\cdot f(y)=f(x)+f(y)+f(xy)-2\quad\forall x,y\in\mathbb{R}.$$

Setting $f(x)=g(x)+1$, the equation becomes $g(xy)=g(x)g(y)$, very classical equation whose general solutions are :
$g(x)=1$ $\forall x$
$g(0)=0$ and $g(x)=|x|^a$ $\forall x\neq 0$ where a is any non zero real.
$g(0)=0$ and $g(x)=sign(x)|x|^a$ $\forall x\neq 0$ where a is any non zero real.

Hence the three solutions of the required equation :
$f(x)=2$ $\forall x$
$f(0)=1$ and $f(x)=1+|x|^a$ $\forall x\neq 0$ where a is any non zero real.
$f(0)=1$ and $f(x)=1+sign(x)|x|^a$ $\forall x\neq 0$ where a is any non zero real.

nguyenhung
559 posts

Feb 15, 2011, 9:24 pm PM #211

> *pco wrote:*
>
> > *amparvardi wrote:*
> > **Problem 67.**
> >
> > Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x) \cdot f(y) = f(x) + f(y) + f(xy) - 2 \quad \forall x, y \in \mathbb{R}.$$

Seting $f(x) = g(x) + 1$, the equation becomes $g(xy) = g(x)g(y)$, very classical equation whose general solutions are :
$g(x) = 1 \forall x$
$g(0) = 0$ and $g(x) = |x|^a \forall x \neq 0$ where a is any non zero real.
$g(0) = 0$ and $g(x) = sign(x)|x|^a \forall x \neq 0$ where a is any non zero real.

Hence the three solutions of the required equation :
$f(x) = 2 \forall x$
$f(0) = 1$ and $f(x) = 1 + |x|^a \forall x \neq 0$ where a is any non zero real.
$f(0) = 1$ and $f(x) = 1 + sign(x)|x|^a \forall x \neq 0$ where a is any non zero real.

Sorry, but I remember that g must be continues before we could infer these results

pco
14052 posts

Feb 15, 2011, 9:27 pm PM #212

> *nguyenhung wrote:*
>
> > *pco wrote:*
> > $g(xy) = g(x)g(y)$, very classical equation whose general solutions are :
> > $g(x) = 1 \forall x$
> > $g(0) = 0$ and $g(x) = |x|^a \forall x \neq 0$ where a is any non zero real.
> > $g(0) = 0$ and $g(x) = sign(x)|x|^a \forall x \neq 0$ where a is any non zero real.
> > ...
>
> Sorry, but I remember that g must be continues before we could infer these results

You are perfectly right. I'm sorry (tried to answer very quickly and so missed some attention)

pco
14052 posts

Feb 15, 2011, 9:36 pm • 1 PM #213

And so :
...$g(xy) = g(x)g(y)$, very classical equation whose general solutions are :
$g(x) = 0 \forall x$
$g(x) = 1 \forall x$
$g(0) = 0$ and $g(x) = e^{h(\ln |x|)} \forall x \neq 0$ where $h(x)$ is any solution of Cauchy's equation.
$g(0) = 0$ and $g(x) = sign(x)e^{h(\ln |x|)} \forall x \neq 0$ where $h(x)$ is any solution of Cauchy's equation.

Hence the four solutions of the required equation :
$f(x) = 1 \forall x$
$f(x) = 2 \forall x$
$f(0) = 1$ and $f(x) = 1 + e^{h(\ln |x|)} \forall x \neq 0$ where $h(x)$ is any solution of Cauchy's equation.
$f(0) = 1$ and $f(x) = 1 + sign(x)e^{h(\ln |x|)} \forall x \neq 0$ where $h(x)$ is any solution of Cauchy's equation.

Amir Hossein
4719 posts

Feb 15, 2011, 9:55 pm PM #214

Thanks.

Problem 68.

Find all real-valued functions f defined on $\mathbb{R}_0$, the set of all non-zero reals, such that

(a) $f(-x) = -f(x)$.

(b) $f\left(\frac{1}{x+y}\right) = f\left(\frac{1}{x}\right) + f\left(\frac{1}{y}\right) + 2(xy - 1000)$. (for all x, y in $\mathbb{R}_0$, such that $x + y$ is in $\mathbb{R}_0$, too)

[source: 19-th Australian Mathematical Olympiad]

pco
14052 posts

Feb 15, 2011, 10:43 pm PM #215

> *amparvardi wrote:*
> Thanks.
>
> **Problem 68.**
>
> Find all real-valued functions f defined on $\mathbb{R}_0$, the set of all non-zero reals, such that
>
> (a) $f(-x) = -f(x)$.
>
> (b) $f\left(\frac{1}{x+y}\right) = f\left(\frac{1}{x}\right) + f\left(\frac{1}{y}\right) + 2(xy - 1000)$. (for all x, y in $\mathbb{R}_0$, such that $x + y$ is in $\mathbb{R}_0$, too)
>
> *[source: 19-th Australian Mathematical Olympiad]*

Changing $x \to -x$ and $y \to -y$ in (b), using then (a) and adding to the original equation, we get $xy = 1000$, impossible

So no solution.

Amir Hossein
4719 posts

Feb 16, 2011, 8:34 pm PM #216

Problem 69.

Let $f(n)$ be defined on the set of positive integers by the rules: $f(1) = 2$ and

$$f(n+1) = (f(n))^2 - f(n) + 1 \quad \forall n \in \mathbb{N}.$$

Prove that, for all integers $n > 1$, we have

$$1 - \frac{1}{2^{2^{n-1}}} < \frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \cdots + \frac{1}{f(n)} < 1 - \frac{1}{2^{2^n}}.$$

Note. $\mathbb{N}$ is the set of positive integers.

EDIT: Changed, thanks mahanmath.

nguyenhung
559 posts

Feb 17, 2011, 5:15 pm PM #217

Solution to Problem 69

We have

$$f(k+1) = (f(k))^2 - f(k) + 1$$

$$\Leftrightarrow \frac{1}{f(k)} = \frac{1}{f(k)-1} - \frac{1}{f(k+1)-1}$$

Hence

$$\frac{1}{f(1)} = \frac{1}{f(1)-1} - \frac{1}{f(2)-1}$$

$$\frac{1}{f(2)} = \frac{1}{f(2)-1} - \frac{1}{f(3)-1}$$

....

$$\frac{1}{f(n)} = \frac{1}{f(n)-1} - \frac{1}{f(n+1)-1}$$

$$\Rightarrow S = \sum_{i=1}^{n} \frac{1}{f(i)} = \frac{1}{f(1)-1} - \frac{1}{f(n+1)-1} = 1 - \frac{1}{f(n+1)-1}$$

We need to prove

$$1 - \frac{1}{2^{2^{n-1}}} < 1 - \frac{1}{f(n+1)-1} < 1 - \frac{1}{2^{2^n}}$$

$$\Leftrightarrow 2^{2^{n-1}} + 1 < f(n+1) < 2^{2^n} + 1$$

$$\Leftrightarrow 2^{2^{n-2}} + 1 < f(n) < 2^{2^{n-1}} + 1$$

, which can easily prove by induction

So we've done

---

u2tommyf
33 posts

Feb 17, 2011, 5:45 pm PM #218

Sorry but I don't know latex, so maybe someone will "translate"...

Problem 70

Determine all functions f defined on the set of positive integers that have:

f (x * f(y) + y) = y * f(x) + f(y), for any x,y positive integers

and f(p) prime for any p prime.

---

nguyenhung
559 posts

Feb 17, 2011, 5:50 pm • 2 PM #219

Latex for problem 70

Problem 70

Determine all functions f defined on the set of positive integers that have:

$f(xf(y)+y) = yf(x)+f(y)$, for any positive integers x, y

and $f(p)$ is a prime for any prime p

---

mousavi
222 posts

Feb 18, 2011, 12:39 am PM #220

> *nguyenhung wrote:*
> Latex for problem 70
>
> **Problem 70**
>
> Determine all functions f defined on the set of positive integers that have:
>
> $f(xf(y)+y) = yf(x)+f(y)$, for any positive integers x, y
>
> and $f(p)$ is a prime for any prime p

$p(x,y): f(xf(y)+y) = yf(x)+f(y)$(1)
$p(3,1): f(3f(1)+1) = f(3)+f(1)$(2)
$p(1,1): f(f(1)+1) = 2f(1)$(3)
with (3)
$p(1,f(1)+1): f(3f(1)+1) = (f(1)+1)f(1)+2f(1) = f(1)^2+3f(1)$(4)

(2),(4)$\Longrightarrow f(3)+f(1) = f(1)^2+3f(1) \Longrightarrow f(3) = f(1)(f(1)+2) \Longrightarrow f(1) = 1, f(3) = 3$

$p(x,1): f(x+1) = f(x)+1 \Longrightarrow f(x) = x$

---

abhinavzan...
418 posts

Feb 18, 2011, 2:13 pm PM #221

Even A Lifeguard Must not be bringing the dead back to life by C.P.R. like we have to do for this marathon.

Problem 71

> Determine All Functions
>
> $f : \mathbb{R} - 0, 1 \to \mathbb{R}$
>
> $$f(x) + f(\frac{1}{1-x}) = \frac{2(1-2x)}{x(1-x)}$$

RSM
736 posts

Feb 18, 2011, 4:14 pm PM #222

> *abhinavzandubalm wrote:*

Even A Lifeguard Must not be bringing the dead back to life by C.P.R. like we have to do for this marathon. 😁 🙂 😋

Problem 71

This is an INMO problem
A solution to it can be found in INMO official solution paper.

Next problem:-
Find all function f defined on real variables such that
$f(x+y)+f(x-y)=2f(x)f(y)$
for all $x,y\in\mathbb{R}$

I could not solve this problem myself.
So I want to give one more condition that $|f(x)|\geq 1\forall x\in\mathbb{R}$
I think it is easy now.
This post has been edited 3 times. Last edited by RSM, Feb 20, 2011, 3:23 pm

---

pco
14052 posts

Feb 18, 2011, 5:16 pm • 1👍 PM #223

> *RSM wrote:*
> Find all function f defined on real variables such that
> $f(x+y)+f(x-y)=2f(x)f(y)$
> for all $x,y\in\mathbb{R}$

Are you sure that the original problem does not contain the additional "continuity" property ?

---

prafullasd
25 posts

Feb 20, 2011, 10:30 am PM #224

i tried to get a discontinuous solution for the eqaution,
let $g(x)$ be any discontinuous solution of $g(x+y)=g(x)+g(y)$
let $f(x)=cos(g(x))$
then,
$f(x+y)+f(x-y)=cos(g(x+y))+cos(g(x-y))$
$=2cos(\frac{g(2x)}{2})cos(\frac{g(2y)}{2})=2cos(g(x))cos(g(y))=2f(x)f(y)$
as $g(x+x)=g(x)+g(x)$ therefore, $f(x)$ satisfies the equation
but, i can't show that $f(x)$ is discontinuous

---

magical
196 posts

Feb 20, 2011, 11:39 am PM #225

Problem 73
Find all functional $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ satisfy: $f(x^3+2y)+f(x+y)=g(x+2y)\forall x,y\in\mathbb{R}$

---

pco
14052 posts

Feb 20, 2011, 2:10 pm PM #226

> *RSM wrote:*
> Next problem:-
> Find all function f defined on real variables such that
> $f(x+y)+f(x-y)=2f(x)f(y)$
> for all $x,y\in\mathbb{R}$
>
> I could not solve this problem myself.
> So I want to give one more condition that $|f(x)|\geq 2\forall x\in\mathbb{R}$
> I think it is easy now.

I think it would have been more interesting to add the continuity constraint since then this would have been the famous d'Alembert functional equation with solutions $0,\cos ax,\cosh ax$.

Just adding the constraint $|f(x)|\geq 2\forall x\in\mathbb{R}$ is a kind of joke :
Setting $x=y=0$ in the equation implies $2f(0)=2f(0)^2$ and so $f(0)=0$ or $f(0)=1$, and so $|f(0)|<2$
So no solution.

---

pco
14052 posts

Feb 20, 2011, 2:45 pm PM #227

> *magical wrote:*
> **Problem 73**
> Find all functional $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ satisfy: $f(x^3+2y)+f(x+y)=g(x+2y)\forall x,y\in\mathbb{R}$

If (f,g) is a solution, so is $(f+c,g+2c)$ and so Wlog say $f(0)=0$

Setting $y=0$ in the equation gives $g(x)=f(x^3)+f(x)$
Pluging this in original equation, we get assertion $P(x,y)$: $f(x^3+2y)+f(x+y)=f((x+2y)^3)+f(x+2y)$

Setting $x=-y$ in the equation gives $g(y)=f(2y-y^3)$ and so $g(x)=f(2x-x^3)$
Pluging this in original equation, we get assertion $Q(x,y)$: $f(x^3+2y)+f(x+y)=f(2(x+2y)-(x+2y)^3)$

1) $f(x+\frac{1}{2})=f(x)\forall x$
====================
$P(1,x-\frac{1}{2})\implies f(x+\frac{1}{2})=f((2x)^3)$
$P(0,x)\implies f(x)=f((2x)^3)$
And so $f(x+\frac{1}{2})=f(x)$
Q.E.D.

2) $f(x)=0\,\forall x\in[0,1]$
=============
Let $y\in(0,1]$
$Q(x,y-x)\implies f(x^3-2x+2y)+f(y)=f(2(2y-x)-(2y-x)^3)$
Consider now the equation $x^3-2x+2y=2(2y-x)-(2y-x)^3$
It may be written $(x-y)^2=\frac{1-y^2}{3}$ and it has always at least one solution x since $y\in(0,1]$

Choosing this value x, $f(x^3-2x+2y)+f(y)=f(2(2y-x)-(2y-x)^3)$ becomes $f(y)=0$
Q.E.D.

3) Solutions
========
2) gave $f(x)=0\forall x\in[0,1]$
1) gave $f(x+\frac{1}{2})=f(x)$

So $f(x) = 0 \forall x$
So $g(x) = 0 \forall x$

Hence the answer :
$\boxed{(f(x), g(x)) = (c, 2c)}$ for any real c

RSM
736 posts

Feb 20, 2011, 3:26 pm PM #228

I am really sorry. I made a mistake.
I have corrected the question now.

Notice:

Problem 72 is still pending.

RSM
736 posts

Feb 21, 2011, 11:45 am PM #229

Prove or disprove the statement:-
Each even function $f(x)$ can be written as $g(x) + g(-x)$
Where f and g are defined over $\mathbb{R}$.
g is not even function.

This post has been edited 1 time. Last edited by RSM, Feb 22, 2011, 5:50 am

---

pco
14052 posts

Feb 21, 2011, 1:19 pm PM #230

> *RSM wrote:*
>
> Prove or disprove the statement:-
> Each even function $f(x)$ can be written as $g(x) + g(-x)$
> Where f and g are defined over $\mathbb{R}$.

$f(x) = \frac{f(x)}{2} + \frac{f(-x)}{2}$

Amir Hossein
4719 posts

Feb 21, 2011, 1:26 pm • 1 PM #231

Problem 75. (?)

For each positive integer n, let

$$f(n) = \frac{1}{\sqrt[3]{n^2 + 2n + 1} + \sqrt[3]{n - 1} + \sqrt[3]{n^2 - 2n + 1}}.$$

Determine the value of

$$f(1) + f(3) + f(5) + \cdots + f(999997) + f(999999).$$

EDIT. Oh I am sorry. It seems it should be $\sqrt[3]{n^2 - 1}$ instead of $\sqrt[3]{n - 1}$. Corrected. [Actually I saw this problem in one of Australian Olympiads, but it was $\sqrt[3]{n^1 - 1}$, which seems to be a mistake - Thanks RSM.]

This post has been edited 1 time. Last edited by Amir Hossein, Feb 22, 2011, 5:52 pm
Reason: Edited.

---

abhinavzan...
418 posts

Feb 21, 2011, 6:45 pm PM #232

> *amparvardi wrote:*
>
> **Problem 75.** (?)

It's "Problem 74".

RSM
736 posts

Feb 22, 2011, 5:51 am PM #233

> *pco wrote:*
>
> > *RSM wrote:*
> >
> > Prove or disprove the statement:-
> > Each even function $f(x)$ can be written as $g(x) + g(-x)$
> > Where f and g are defined over $\mathbb{R}$.
>
> $f(x) = \frac{f(x)}{2} + \frac{f(-x)}{2}$

I have changed the problem.
Otherwise it becomes trivial.

---

soulhunter
317 posts

Feb 22, 2011, 9:23 am PM #234

> *RSM wrote:*
>
> > *abhinavzandubalm wrote:*
> >
> > Even A Lifeguard Must not be bringing the dead back to life by C.P.R. like we have to do for this marathon.
> >
> > Problem 71
> >
> > Determine All Functions
> >
> > $f : \mathbb{R} - 0, 1 \to \mathbb{R}$
> >
> > $f(x) + f(\frac{1}{1 - x}) = \frac{2(1 - 2x)}{x(1 - x)}$
>
> This is an INMO problem
> A solution to it can be found in INMO official solution paper.
>
> Next problem:-
> Find all function f defined on real variables such that
> $f(x + y) + f(x - y) = 2f(x)f(y)$
> for all $x, y \in \mathbb{R}$
>
> I could not solve this problem myself.
> So I want to give one more condition that $|f(x)| > 1 \forall x \in \mathbb{R}$

I think it is easy now.

This is an famous easy problem on cauchy's equation which's solution gives us de'alembert's equation!

http://www.jstor.org/pss/2313495

or found at venkatchala's book on functional equation

pco
14052 posts

Feb 22, 2011, 1:54 pm PM #235

> RSM wrote:
> I have changed the problem.
> Otherwise it becomes trivial.

Sure, now it's far from being trivial

> RSM wrote:
> Prove or disprove the statement:-
> Each even function $f(x)$ can be written as $g(x)+g(-x)$
> Where f and g are defined over $\mathbb{R}$.
> g is not even function.

$$f(x)=\frac{f(x)+x}{2}+\frac{f(-x)-x}{2}$$

Amir Hossein
4719 posts

Feb 22, 2011, 5:54 pm PM #236

I am sorry about the problem 75 I posted. There was a typo in the problem. Thanks RSM for pointing it out - edited.

prafullasd
25 posts

Feb 22, 2011, 7:43 pm PM #237

Problem 75

$$f(n)=\frac{1}{\sqrt[3]{(n+1)^2}+\sqrt[3]{(n+1)(n-1)}+\sqrt[3]{(n-1)^2}}=\frac{\sqrt[3]{n+1}-\sqrt[3]{n-1}}{n+1-(n-1)}=\frac{\sqrt[3]{n+1}-\sqrt[3]{n-1}}{2}$$

and ,therefore, the sum is like an telescopic sum ,

$$Answer=\frac{\sqrt[3]{1000000}-\sqrt[3]{0}}{2}=50$$

abhinavzan...
418 posts

Feb 22, 2011, 8:01 pm PM #238

> RSM wrote:
> Prove or disprove the statement:-
> Each even function $f(x)$ can be written as $g(x)+g(-x)$
> Where f and g are defined over $\mathbb{R}$.
> g is not even function.

The General Solution Will Be

$$f(x)=\left[\frac{f(x)}{2}+h(x)\right]+\left[\frac{f(-x)}{2}+h(-x)\right]$$

Where $h(x)$ Is Any Odd Function

abhinavzan...
418 posts

Feb 22, 2011, 8:08 pm PM #239

Now Lets Give The Next Problem
Yours Was Not **75** As I Pointed Out **amparvardi**
A Simple One For Your Pleasure
Problem 75

> Find All The Functions Which Are Strictly Monotone Satisfying
> $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(f(x)+y)=f(x+y)+f(0)\quad\forall\, x,y\in\mathbb{R}$

Edit. Thanks **pco** I Edited The Mistake .Its '+' Instead Of '='

This post has been edited 1 time. Last edited by abhinavzandubalm, Feb 22, 2011, 8:19 pm

pco
14052 posts

Feb 22, 2011, 8:12 pm PM #240

> *abhinavzandubalm wrote:*
> Problem 75:
> Find All The Functions Which Are Strictly Monotone Satisfying
> $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(f(x)+y)=f(x+y)=f(0)\quad\forall\, x,y\in\mathbb{R}$

None, since $f(x+y)=f(0)$ means that $f(x)$ is constant and so not strictly monotonous.

prafullasd
25 posts

Feb 22, 2011, 8:50 pm PM #241

> *abhinavzandubalm wrote:*
> Find All The Functions Which Are Strictly Monotone Satisfying
> $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(f(x)+y)=f(x+y)+f(0)\quad\forall\, x,y\in\mathbb{R}$

$y=-f(x)$ gives $f(x-f(x))=0$
as function is strictly monotone, there exists unique $a\in\mathbb{R}$ such that $f(a)=0$
therefore,
$x-f(x)=a\quad\forall\, x\in\mathbb{R}$
therefore, $f(x)=x-a\quad\forall\, x\in\mathbb{R}$, which satisfies the given functional equation

magical
196 posts

Feb 22, 2011, 9:40 pm PM #242

> prafullasd wrote:
> $y=-f(x)$ gives $f(x-f(x))=0$
> as function is strictly monotone, there exists unique $a\in\mathbb{R}$ such that $f(a)=0$
> therefore,
> $x-f(x)=a\quad\forall\, x\in\mathbb{R}$
> therefore, $f(x)=x-a\quad\forall\, x\in\mathbb{R}$, which satisfies the given functional equation

I think you should prove that f :is surjective before setting $y=-f(x)$

goodar2006
1344 posts

Feb 22, 2011, 10:02 pm — PM #243

> magical wrote:
>
> > prafullasd wrote:
> > $y=-f(x)$ gives $f(x-f(x))=0$
> > as function is strictly monotone, there exists unique $a\in\mathbb{R}$ such that $f(a)=0$
> > therefore,
> > $x-f(x)=a \quad \forall\ x\in\mathbb{R}$
> > therefore, $f(x)=x-a \quad \forall\ x\in\mathbb{R}$, which satisfies the given functional equation
>
> I think you should prove that f :is surjective before setting $y=-f(x)$

no, setting $y=-f(x)$ has no problem, because it's a real number, and the equation is for all $x,y\in\mathbb{R}$

magical
196 posts

Feb 23, 2011, 11:36 am — PM #244

> goodar2006 wrote:
> no, setting $y=-f(x)$ has no problem, because it's a real number, and the equation is for all $x,y\in\mathbb{R}$

OK,thanks you

When do they can set $x=-f(x)$ into any functional equation ?

abhinavzan...
418 posts

Feb 23, 2011, 12:01 pm — PM #245

Now The Next Problem 76

> Find All Functions $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)f(y)} \quad \forall\ x,y\in\mathbb{R}$

pco
14052 posts

Feb 23, 2011, 3:57 pm — PM #246

> abhinavzandubalm wrote:
> Problem 76 :
> Find All Functions $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)f(y)} \quad \forall\ x,y\in\mathbb{R}$

Partial result

> Obviously $f(x)\notin\{-1,1\}$
> Setting $f(x)=\tan(\pi g(x))$, with $g(x)\in(-\frac{1}{2},+\frac{1}{2})\setminus\{-\frac{1}{4},+\frac{1}{4}\}$, we get :
> $\tan(\pi g(x+y))$=$\tan(\pi(g(x)+g(y)))$
>
> And so $g(x+y)=g(x)+g(y)+n(x,y)$ where $n(x,y)\in\mathbb{Z}$
>
> So we trivially have at least the solutions $f(x)=\tan(\pi h(x))$ where $h(x)$ is any solution of Cauchy equation such that $h(x)\notin\mathbb{Q}\forall x\neq 0$
>
> But I'm stucked here : are these the only solutions?
>
> Any hint for this remaining part, abhinavzandubalm, please ?

abhinavzan...
418 posts

Feb 23, 2011, 4:03 pm — PM #247

> pco wrote:
>
> > abhinavzandubalm wrote:
> > Problem 76 :
> > Find All Functions $f:\mathbb{R}\to\mathbb{R}$ Such That
> > $f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)f(y)} \quad \forall\ x,y\in\mathbb{R}$
>
> Partial result

To **pco**
I Read The Question A Long Time Ago.
Hence If You Would Check Whether It Should Be Given or Not That The Function Is Continuous It Would Be Of Help?

abhinavzan...
418 posts

Feb 23, 2011, 4:05 pm — PM #248

And I Don't Think We Need To Put
$f(x)=tan(\pi g(x))$ Simply $f(x)=tan(g(x))$ May Also Suffice.

pco
14052 posts

Feb 23, 2011, 4:09 pm — PM #249

> abhinavzandubalm wrote:
> And I Don't Think We Need To Put
> $f(x)=tan(\pi g(x))$ Simply $f(x)=tan(g(x))$ May Also Suffice.

It was just to get $h(x)\notin\mathbb{Q}$, which is simpler than $\dfrac{h(x)}{\pi}\notin\mathbb{Q}$
But this is just a detail.

My question is : could you give us a hint to prove that the results I proposed are the only one (or the contrary). Thanks a lot.

And, btw, if you add continuity, then we obviously have only $f(x)=0$ as solution.

abhinavzan...
418 posts

Feb 23, 2011, 4:21 pm — PM #250

I Contacted Three Of My Friends And One Told Me To Check The Book
$Functional\ Equations\ -A\ Problem\ Solving\ Approach$ By $B.J.Venkatachala.$
It Was The Problem 5.21 And Continuity Was There

It Was The Problem 5.21 And Continuity Was There.
The Official Solution Was To Get D'Alembert's Equation By Defining
$g(x) = \frac{1-f(x)^2}{1+f(x)^2}, h(x) = \frac{2f(x)}{1+f(x)^2}$ To Get
$g(x-y) = g(x)g(y) + h(x)h(y) \implies g(x) = Cos(\alpha x), h(x) = Sin(\alpha x)$
Thus $f(x) = Tan(\alpha x)$ Is The Solution.

---

pco
14052 posts

Feb 23, 2011, 4:27 pm PM #251

> *abhinavzandubalm wrote:*
> I Contacted Three Of My Friends And One Told Me To Check The Book
> $Functional\ Equations\ -A\ Problem\ Solving\ Approach$ By $B.J.Venkatachala$.
> It Was The Problem 5.21 And Continuity Was There.
> The Official Solution Was To Get D'Alembert's Equation By Defining
> $g(x) = \frac{1-f(x)^2}{1+f(x)^2}, h(x) = \frac{2f(x)}{1+f(x)^2}$ To Get
> $g(x-y) = g(x)g(y) + h(x)h(y) \implies g(x) = Cos(\alpha x), h(x) = Sin(\alpha x)$
> Thus $f(x) = Tan(\alpha x)$ Is The Solution.

You should revisit your courses : $f(x) = \tan(\alpha x)$ is not continuous if $\alpha \neq 0$ and so is NOT the solution if you add continuity.

The only continuous solution to this equation is $f(x) = 0$

---

abhinavzan...
418 posts

Feb 23, 2011, 5:28 pm PM #252

I Know.
That's Why I Was Having Doubts To Whether We can Use The Method Given But Disregard Continuity.

FBI__
29 posts

Feb 23, 2011, 11:41 pm PM #253

Problem 77 Find all functional $f:\mathbb{R}\to\mathbb{R}$ satisfy: $xf(x) - yf(y) = (x-y)f(x+y)$ for all $x, y \in \mathbb{R}$

---

pco
14052 posts

Feb 23, 2011, 11:56 pm PM #254

> *FBI__ wrote:*
> **Problem 77** Find all functional $f:\mathbb{R}\to\mathbb{R}$ satisfy: $xf(x) - yf(y) = (x-y)f(x+y)$ for all $x, y \in \mathbb{R}$

Let $P(x,y)$ be the assertion $xf(x) - yf(y) = (x-y)f(x+y)$

$P(\frac{x-1}{2}, \frac{1-x}{2}) \implies \frac{x-1}{2}f(\frac{x-1}{2}) - \frac{1-x}{2}f(\frac{1-x}{2}) = (x-1)f(0)$

$P(\frac{1-x}{2}, \frac{x+1}{2}) \implies \frac{1-x}{2}f(\frac{1-x}{2}) - \frac{x+1}{2}f(\frac{x+1}{2}) = -xf(1)$

$P(\frac{x+1}{2}, \frac{x-1}{2}) \implies \frac{x+1}{2}f(\frac{x+1}{2}) - \frac{x-1}{2}f(\frac{x-1}{2}) = f(x)$

Adding these three lines, we get $f(x) - xf(1) + (x-1)f(0) = 0$ and so $f(x) = (f(1) - f(0))x - f(0)$

And so $\boxed{f(x) = ax+b}$ which indeed is a solution.

Amir Hossein
4719 posts

Feb 24, 2011, 1:25 am PM #255

Problem 78.

For each positive integer n, let

$$f(n) = f(n) = \left[2\sqrt{n}\right] - \left[\sqrt{n-1} + \sqrt{n+1}\right].$$

Determine all values n for which $f(n) = 1.$

Note. $[x]$ is the largest integer not exceeding $x.$

---

mousavi
222 posts

Feb 24, 2011, 3:10 am PM #256

> *amparvardi wrote:*
> **Problem 78.**
>
> For each positive integer n, let
>
> $$f(n) = f(n) = \left[2\sqrt{n}\right] - \left[\sqrt{n-1} + \sqrt{n+1}\right].$$
>
> Determine all values n for which $f(n) = 1.$
>
> **Note.** $[x]$ is the largest integer not exceeding $x.$

it is obvious $0 < 2\sqrt{n} - (\sqrt{n-1} + \sqrt{n+1}) < 1$ (1)

by (1) we conclude that if $n = k^2$ then $f(n) = 1$

let $n \neq k^2 \Longrightarrow [2\sqrt{n}]^2 < 4n \Longrightarrow [2\sqrt{n}] \leq \sqrt{4n-1}$

if $f(n) = 1$ then $[\sqrt{n+1} + \sqrt{n-1}] + 1 \leq \sqrt{4n-1}$

but we want to prove that $[\sqrt{n+1} + \sqrt{n-1}] + 1 > \sqrt{4n-1}$

$[\sqrt{n+1} + \sqrt{n-1}] + 1 > \sqrt{n+1} + \sqrt{n-1} > \sqrt{4n-1}$ (?)

$2n + 2\sqrt{n^2-1} > 4n - 1$(?)

$4n^2 - 4 > 4n^2 + 1 - 4n$(?)$\Longrightarrow 4n > 5$ $(n > 1)$

---

mousavi
222 posts

Feb 25, 2011, 1:14 am PM #257

problem 79

Find all natural and odd numbers $n \geq 3$ such that the below function be injective:

$f : Q \longrightarrow Q, f(x) = x^{n} - 2x$

prafullasd
25 posts

Feb 25, 2011, 10:45 am PM #258

Problem 79

Click to reveal hidden text

@mousavi
i tried to find all n for which it is not true, but ended up getting a contradiction,using a parity argument, alongwith a little number theory
i am not so good at latex, so i cant type the solution
so , it is injective for all natural and odd numbers $n \geq 3$??

prafullasd
25 posts

Feb 26, 2011, 9:04 pm PM #259

Problem 80:
Find all continuous, strictly increasing functions $f : \mathbb{R} \to \mathbb{R} \ \ such\ that$
$1)\ \ f(0) = 0$
$2)\ \ f(1) = 1$
$3)\ \ [f(x+y)] = [f(x)] + [f(y)] \ \ \forall\ \ x, y \in \mathbb{R}\ such\ that\ [x+y] = [x] + [y]$
where $[x]\ is\ largest\ integer\ less\ than\ or\ equal\ to\ x$

P.S. This is a problem which occured to me and not from any contest, and so i have no official solution, only my own.So, if u think the problem need additional constraints(or lesser!), pls tell

SCP
1520 posts

Feb 26, 2011, 9:47 pm PM #260

> prafullasd wrote:
> **Problem 80:**
> Find all continuous, strictly increasing functions $f : \mathbb{R} \to \mathbb{R} \ \ such\ that$
> $1)\ \ f(0) = 0$
> $2)\ \ f(1) = 1$
> $3)\ \ [f(x+y)] = [f(x)] + [f(y)] \ \ \forall\ \ x, y \in \mathbb{R}\ such\ that\ [x+y] = [x] + [y]$
> where $[x]\ is\ largest\ integer\ less\ than\ or\ equal\ to\ x$
>
> P.S. This is a problem which occured to me and not from any contest, and so i have no official solution, only my own.So, if u think the problem need additional constraints(or lesser!), pls tell

I think there are infintely many:
let $f(z) = z$ for all integers and $[x] \leq f(x) \leq x$ for all x.

prafullasd
25 posts

Feb 26, 2011, 10:38 pm PM #261

> SCP wrote:
> I think there are infintely many:
> let $f(z) = z$ for all integers and $[x] \leq f(x) \leq x$ for all x.

Click to reveal hidden text

yep, i know but you need to find all (ie the general solution)
the general solution is Click to reveal hidden text

$f(z) = z$ for all integers. In between any two integers, ie $z < x < z+1$, f(x) can be any strictly increasing continuous function such that $z < f(x) < z+1$

BigSams
1581 posts

Feb 26, 2011, 11:24 pm PM #262

This is the first functional equation I ever solved, hope u like it 😄

Problem 81.
Find all functions f mapped from $R \to R$ such that $(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$.

mousavi
222 posts

Feb 27, 2011, 6:12 pm PM #263

> BigSams wrote:
> This is the first functional equation I ever solved, hope u like it 😄
>
> **Problem 81.**
> Find all functions f mapped from $R \to R$ such that $(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$.

$p(x, x) :\Longrightarrow f(0) = 0$

$p(x+y, y) :\Longrightarrow xf(2x+y) - (2x+y)f(x) = 4(x+y)(y)(x^2 + 2xy)$ (1)

$p(x, 1-2x) :\Longrightarrow xf(1) - f(x) = 4(1-x)(1-2x)(x^2 + 2x - 4x^2)$ (by (1))

$\Longrightarrow f(x) = xt - 4(1-x)(1-2x)(2x - 3x^2)$

BigSams
1581 posts

Feb 27, 2011, 10:30 pm PM #264

Erm no, that's not the solution, sorry.

RSM
736 posts

Feb 27, 2011, 11:05 pm PM #265

This is probably the easiest problem posted in this marathon.
First put x=y and $x \neq 0$
Then we get f(0)=0
Put x=0 to get $f(-y) = -f(y)$
Put $x = \frac{a+b}{2} and y = \frac{a-b}{2}$
and the equation turns to $bf(a) - af(b) = (a^2 - b^2)ab$
$\frac{f(a) - a^3}{a} = \frac{f(b) - b^3}{b} = \text{constant} = k$
So $f(x) = x^3 + kx$
k may take any real value.

socrates
1872 posts

Feb 28, 2011, 5:43 am PM #266

Problem 82.
Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(y + xf(y)) = f(y) + yf(x)$ and $f(3)$ is prime.

mousavi
222 posts

Feb 28, 2011, 9:06 am PM #267

> socrates wrote:
> **Problem 82.**
> Find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ such that $f(y+xf(y))=f(y)+yf(x)$ and $f(3)$ is prime.

Problem 70

---

prafullasd
25 posts

Feb 28, 2011, 12:29 pm — PM #268

No solution posted for problems 79 and 80 yet.(is my problem that difficult?)
also, i think a list should be made of the problems of the marathon which have not been solved.

RSM
736 posts

Feb 28, 2011, 3:23 pm — PM #269

Solution to Problem 82:-
Suppose $f(1)=c$
Put x=1 and y=1
Put y=1
$f(1+cx)=c+f(x)$....................(1)
In this equation put x=1
$f(c+1)=2c$
Put x=1
$f(y+f(y))=f(y)+cy$
In this equation put y=c+1
$f(3c+1)=c^2+3c$.............(2)
In the equation (1) put x=3
$f(3c+1)=c+f(3)$..............(3)
Comparing (2) and (3) we have
$f(3)=c^2+2c=c(c+2)$
Since f(3) is a prime so c=1
Putting this in equation (1)
$f(c+1)=f(c)+1$
And putting x=0 in the functional equation we have $f(0)=0$
So by induction $f(x)=x\in\mathbb{N}$

Someone post the next problem,please.

RSM
736 posts

Feb 28, 2011, 5:04 pm — PM #270

Solution to Problem 80:-(This problem was not solved yet)
Suppose, $x=n+a, y=n+b$ where $[x]=[y]=n$
Take two a and b such that $a+b<1$
Then $[x]+[y]=[x+y]$
$[f(2n+a+b)]=[f(n+a)]+[f(n+b)]$
$[f(a+b)]=[f(a)]+[f(b)]$................(1)
$[f(ka)]=k[f(a)]\forall k\in\mathbb{N}$
$[f(\frac{p}{q})]=\frac{[f(\frac{1}{p})]}{q}=\frac{[f(\frac{1}{q})]}{p}$
$[f(\frac{1}{x})]=\frac{k}{x}\forall x\in\mathbb{N}$
This implies x divides k for all x.
So k=0
$[f(\frac{1}{x})]=0\forall x\in\mathbb{N}$
$[f(\frac{x}{y})]=0\forall x,y\in\mathbb{N}\text{and}y>x$
This can be extended to irrational numbers since every irrational number can be written as a sum of rational numbers.
So $[f(a)]=0 forall 0<a<1$
$[f(x)]=[f([x]+a)]=[f([x])]=[x][f(1)]=[x]$
So $[f(x)]=[x]forallx\geq 0$
In a similar manner it can be proved that $[f(x)]=[x]forallx\leq 0$
So $[f(x)]=[x]$ is the solution.

I may have made any stupid mistake.

This post has been edited 1 time. Last edited by RSM, Mar 1, 2011, 12:23 am

---

prafullasd
25 posts

Feb 28, 2011, 9:00 pm — PM #271

why k=0?
well, the solution is wrong as there are infinite solutions
a proof (i am posting only a proof showing that the following solution satisfies the equation, not of why it is the only solution)
Click to reveal hidden text

---

mcrasher
1766 posts

Feb 28, 2011, 9:11 pm — PM #272

> BigSams wrote:
> This is the first functional equation I ever solved, hope u like it
> **Problem 81.**
> Find all functions f mapped from $R\to R$ such that $(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^2-y^2)$.

solution#81

RSM
736 posts

Feb 28, 2011, 10:32 pm — PM #273

> prafullasd wrote:
> why k=0?
> well, the solution is wrong as there are infinite solutions
> a proof (i am posting only a proof showing that the following solution satisfies the equation, not of why it is the only solution)
> Click to reveal hidden text
>
> > My general solution:
> > f(x) is any continuous strictly increasing function for all real x with f(z)=z for all integers z (in between two integers, the only thing it needs to satisfy is that f(x) should remain continuous and strictly increasing
> > to show that this satisfies the equation:
> > In between any two integers, ie $z<x<z+1$, f(x) satisfies $z<f(x)<z+1$
> > therefore, $[f(x)]=[x]$(both equal to z)$\forall\ \ x\in\mathbb{R}$
> > plugging this in the equation, it is clearly satisfied

I made a stupid mistake.
I have edited my proof.

This post has been edited 1 time. Last edited by RSM, Mar 1, 2011, 12:17 am

---

pco
14052 posts

Feb 28, 2011, 11:27 pm • 1 — PM #274

> prafullasd wrote:

Problem 80:
Find all continuous, strictly increasing functions $f:\mathbb{R}\to\mathbb{R}\ \ such\ that$
$1)\ \ f(0)=0$
$2)\ \ f(1)=1$
$3)\ \ [f(x+y)]=[f(x)]+[f(y)]\ \ \forall\ \ x,y\in\mathbb{R}\ such\ that\ [x+y]=[x]+[y]$

a) $f(x)\in(0,1)\forall x\in(0,1)$
Trivial using 1) 2) and increasing property

b) $[f(n)]=n\forall n\in\mathbb{Z}$
$[m+n]=[m]+[n]\forall m,n\in\mathbb{Z}$ and so $[f(m+n)]=[f(m)]+[f(n)]$ and so $[f(n)]=n[f(1)]=n$

c) $[f(x)]\ge[x]\forall x$
$x\ge[x]$ and $f(x)$ increasing implies $f(x)\ge f([x])$ and so $[f(x)]\ge[f([x])]=[x]$

d) $[f(x)]<[x]+1\forall x$
If $[f(a)]\ge[a]+1$ for some a, then :
$[f([a])]=[a]$ and so $f([a])<[a]+1$
Then continuity implies $\exists u\in([a],a)$ such that $f(u)=[a]+1$
Choosing then some $x\in([a],u)$ and $y=a-x\in(0,1)$ we get $[x+y]=[a]=[x]+[y]$ and so :
$[f(x+y)]=[f(x)]+[f(y)]$ which is $[f(a)]=[f(x)]+[f(y)]$ which is wrong since $[f(a)]\ge[a]+1$ while $[f(x)]=[a]$ and $[f(y)]=0$
So no such a

From c),d) we get $[f(x)]=[x]$ and, plugging this in original equation, we get that any strictly increasing continuous function matching 1) and 2) and $[f(x)]=[x]$ matches 3) too.

$[f(x)]=[x]$ and continuity imply $f(n)=n$

<u>Hence the answer</u>:
$f(x)$ solution if and only if :
$f(x)=x\forall x\in\mathbb{Z}$
$f(x)$ may take any values in $(n,n+1)$ when $x\in(n,n+1)$ with respect to the properties "strictly increasing and continuous"

---

abhinavzan...
418 posts

Mar 2, 2011, 7:18 pm PM #275

As Solution To 81 Is Given And A Repeated Problem 82 , I assume that i have to give a New
Problem 82

Find All Functions $f:\mathbb{N}\to\mathbb{N}$
$f(m+f(n))=n+f(m+k)\ \ \forall\ m,n,k\in\mathbb{N}$ With k Being Fixed Natural Number.

---

pco
14052 posts

Mar 2, 2011, 11:18 pm PM #276

> *abhinavzandubalm wrote:*
> "Problem 82"
> Find All Functions $f:\mathbb{N}\to\mathbb{N}$
> $f(m+f(n))=n+f(m+k)\ \ \forall\ m,n,k\in\mathbb{N}$ With k Being Fixed Natural Number

If $f(n)<k$ for some n, then the equation may be written $f(m+(k-f(n)))=f(m)-n\forall m>f(n)$
So $f(m+p(k-f(n)))=f(m)-pn$, which is impossible, since this would imply $f(x)<0$ for some x great enough.

If $f(n)=k$ for some n, then the equation implies $n=0$, impossible

So $f(n)>k\forall n$ and the equation may be written $f(m+(f(n)-k))=n+f(m)\forall m>k$
And so $f(m+p(f(n)-k))=pn+f(m)$
Choosing then $p=f(q)-k$, we get $f(m+(f(q)-k)(f(n)-k))=(f(q)-k)n+f(m)$ and so, by symetry :
$(f(q)-k)n=(f(n)-k)q\forall q,n$
And so $\frac{f(q)-k}{q}=\frac{f(n)-k}{n}$ and so $f(n)=k+cn$ for some constant c

Plugging this in original equation, we get $c=1$ and so solution $\boxed{f(n)=n+k}$

---

mousavi
222 posts

Mar 2, 2011, 11:21 pm PM #277

> *abhinavzandubalm wrote:*
> As Solution To 81 Is Given And A Repeated Problem 82 , I assume that i have to give a New
> Problem 82

f is injective.
$f(f(m)+f(n))=n+f(f(m)+k)=m+f(f(n)+k)(1)$

$p(m,f(n)+k)):f(m+f(f(n)+k))=f(n)+k+f(m+k)(2)$

$p(f(m),n):f(f(m)+f(n))=f(m+f(f(n)+k))=f(m)+k+f(n+k)(3)$

by $(2),(3)\implies f(n)+f(m+k)=f(m)+f(n+k)(4)$

by(4) $f(n)+f(2k)=f(k)+f(n+k)(5)$ and by the problem :$f(k+f(n))=n+f(2k)(6)$

by (5):$f(f(n))+f(2k)=f(k)+f(f(n)+k)=f(k)+n+f(2k)\implies f(f(n))=f(k)+n,(7)$

by (7) and by the problem$p(m,f(k)):f(m+k+f(k))=f(k)+f(m+k)$

by (4) and $p(n,m+f(k)):f(n)+f(m+f(k)+k)=f(m+f(k))+f(n+k)$

$\implies f(n)+f(k)+f(m+k)=k+f(m+k)+f(n+k)\implies f(n)+f(k)=k+f(n+k)=f(n+f(k))(8)$

by (8) and by the problem $f(m+f(n))=n+f(m)+f(k)-k(9)$

by (9) $p(m,f(n)):f(m+n+f(k))=f(m)+f(n)+f(k)-k$

$(8),(9)$:$f(m+n)+f(k)=f(m)+f(n)+f(k)-k$

$\implies f(m+n)+f(m)+f(n)-k\implies f(m)=m+k$

Amir Hossein
4719 posts

Mar 3, 2011, 9:28 am PM #278

Problem 83.

Let f be a function defined for all real numbers and taking real numbers as its values. Suppose that, for all real numbers x, y the function satises

$$f(2x) = f\left(\sin\left(\frac{\pi x}{2} + \frac{\pi y}{2}\right)\right) + f\left(\sin\left(\frac{\pi x}{2} - \frac{\pi y}{2}\right)\right),$$

and

$$f(x^2 - y^2) = (x+y)f(x-y) + (x-y)f(x+y).$$

Show that these conditions uniquely determine

$$f\left(1990 + \sqrt[2]{1990} + \sqrt[3]{1990}\right)$$

and give its value.

pco
14052 posts

Mar 3, 2011, 12:45 pm PM #279

> *amparvardi wrote:*
> **Problem 83.**
>
> Let f be a function defined for all real numbers and taking real numbers as its values. Suppose that, for all real numbers x, y the function satises
>
> $$f(2x) = f\left(\sin\left(\frac{\pi x}{2} + \frac{\pi y}{2}\right)\right) + f\left(\sin\left(\frac{\pi x}{2} - \frac{\pi y}{2}\right)\right),$$
>
> and
>
> $$f(x^2 - y^2) = (x+y)f(x-y) + (x-y)f(x+y).$$
>
> Show that these conditions uniquely determine
>
> $$f\left(1990 + \sqrt[2]{1990} + \sqrt[3]{1990}\right)$$
>
> and give its value.

Notice first that the second equation may be written $f(xy) = xf(y) + yf(x)$

Setting $x = y = 0$ in the first equation, we get $f(0) = 0$
Setting $x = y = 1$ in the first equation, we get $f(2) = 0$

Setting $y = 2$ in $f(xy) = xf(y) + yf(x)$, we get $f(2x) = 2f(x)$

Setting $y = x$ in the first equation, we get $2f(x) = f(\sin \pi x)$ and so :
(a) : $f(x+2) = f(x)$
(b) : $f(1-x) = f(x)$

Setting $y \to y+2$ in $f(xy) = xf(y) + yf(x)$, we get $f(xy+2x) = f(xy) + f(2x)$ and so $f(x+y) = f(x) + f(y)$

$f(2) = 0$ and $f(2x) = 2f(x) \implies f(1) = 0$
(b) and $f(x+y) = f(x) + f(y)$ and $f(1) = 0 \implies f(x) = f(-x)$

And so $0 = f(0) = f(x + (-x)) = f(x) + f(-x) = 2f(x)$ and $\boxed{f(x) = 0\ \forall x}$ which indeed is a solution.

mousavi
222 posts

Mar 3, 2011, 1:44 pm PM #280

problem 84

Find all functions $f : Z \longrightarrow Z$ such that:

$f(x^3 + y^3 + z^3) = f(x)^3 + f(y)^3 + f(z)^3$

mahanmath
1355 posts

Mar 3, 2011, 6:29 pm PM #281

> *mousavi wrote:*
> **problem 84**
>
> Find all functions $f : Z \longrightarrow Z$ such that:
>
> $f(x^3 + y^3 + z^3) = f(x)^3 + f(y)^3 + f(z)^3$

> *mahanmath wrote:*
>
> > *Stephen wrote:*
> > **Problem 21**
> >
> > Find all $f : Z \to Z$ that satisfies $f(x)^3 + f(y)^3 + f(z)^3 = f(x^3 + y^3 + z^3)$.
>
> #)If $x \geq 4$, x^3 can be written as a sum of five cubes such that their absolute values are less than x.
> From # and induction we get the answer is $f(x) = xf(1)$ and $f(1) = 1, 0, -1$.
>
> In fact , It`s a number theory problem !....

abhinavzan...
418 posts

Mar 3, 2011, 9:16 pm PM #282

As Problem 84 Is Repeated Let Me Give A New One . Its Quite Simple.
Problem 84

> Find All Polynomials $P(x)$ Such That
> $xP(x-1) = (x-15)P(x)$

pco

Mar 3, 2011, 9:28 pm PM #283

14052 posts

> abhinavzandubalm wrote:
>
> As Problem 84 Is Repeated Let Me Give A New One . Its Quite Simple.
> Problem 84
> Find All Polynomials $P(x)$ Such That
> $xP(x-1)=(x-15)P(x)$

Set $x=0$ in functional equation and you get $P(0)=0$
Set $x=1$ in functional equation and you get $P(1)=0$
Set $x=2$ in functional equation and you get $P(2)=0$
...
Set $x=14$ in functional equation and you get $P(14)=0$

Then, plugging $P(x)=x(x-1)(x-2)...(x-14)Q(x)$ in equation, we get $Q(x-1)=Q(x)$

Hence the solution $\boxed{P(x)=cx(x-1)(x-2)...(x-14)}$

Babai
487 posts

Mar 4, 2011, 5:15 pm PM #284

There is a good problem-
$f:R\to R$ such that $f(x)f(yf(x)-1)=x^2f(y)-f(x)$ for real x,y.

---

pco
14052 posts

Mar 4, 2011, 5:56 pm • 1 PM #285

> Babai wrote:
>
> There is a good problem-
> $f:R\to R$ such that $f(x)f(yf(x)-1)=x^2f(y)-f(x)$ for real x,y.

$f(x)=0\forall x$ is a solution and let us from now look for non all-zero solutions.

Let $P(x,y)$ be the assertion $f(x)f(yf(x)-1)=x^2f(y)-f(x)$
Let u such that $f(u)\neq 0$

$P(1,1)\implies f(1)f(f(1)-1)=0$ and so $\exists v$ such that $f(v)=0$
$P(v,u)\implies v^2f(u)=0$ and so $v=0$

So $f(x)=0\iff x=0$ and we got $f(1)=1$

$P(1,x)\implies f(x-1)=f(x)-1$ and so $P(x,y)$ may be written :
New assertion $Q(x,y)$: $f(x)f(yf(x))=x^2f(y)$

Let $x\neq 0$: $Q(x,x)\implies f(xf(x))=x^2$ and so any $x\geq 0$ is in $f(\mathbb{R})$

$Q(x,y)\implies f(x)f(yf(x))=x^2f(y)$
$Q(x,1)\implies f(x)f(f(x))=x^2$
$Q(x,y+1)\implies f(x)f(yf(x)+f(x))=x^2f(y)+x^2$

And so $f(x)f(yf(x)+f(x))=f(x)f(yf(x))+f(x)f(f(x))$

Choosing then $z>0$ and x such that $f(x)=z$, we get : $f(yz+z)=f(yz)+f(z)$ and so $f(x+y)=f(x)+f(y)\forall x>0$, $\forall y$

And this immediately implies $f(x+y)=f(x)+f(y)\forall x,y$ ($x=0$ is obvious and using $y=-x$, we get $f(-x)=-f(x)$)

$Q(x,1)\implies f(x)f(f(x))=x^2$
$Q(x+1,1)\implies (f(x)+1)(f(f(x))+1)=x^2+2x+1$
And so $f(f(x))+f(x)=2x$

And combinaison of $f(x)f(f(x))=x^2$ and $f(f(x))+f(x)=2x$ implies $(f(x)-x)^2=0$ and so $f(x)=x\forall x$, which indeed is a solution

Hence the solutions :
$f(x)=0\forall x$
$f(x)=x\forall x$

---

mahanmath
1355 posts

Mar 4, 2011, 8:49 pm PM #286

Problem 86

Prove that there is no function like $f:\mathbb{R}_+\to\mathbb{R}$ such that for all positive x,y :

$f(x+y)>y(f(x)^2)$

---

pco
14052 posts

Mar 5, 2011, 1:04 pm • 2 PM #287

> mahanmath wrote:
>
> **Problem 86**
>
> Prove that there is no function like $f:\mathbb{R}_+\to\mathbb{R}$ such that for all positive x,y :
>
> $f(x+y)>y(f(x)^2)$

Let $P(x,y)$ be the assertion $f(x+y)>yf(x)^2$

Let $x>0$: $P(\frac{x}{2},\frac{x}{2})\implies f(x)>0\,\forall x$

Let then $a>0$ and $x\in[0,a]$: $P(x,2a-x)\implies f(2a)>(2a-x)f(x)^2\geq af(x)^2$ and so $f(x)^2<\frac{f(2a)}{a}$

And so $f(x)$ is upper bounded over any interval $(0,a]$

Let then $f(1)=u>0$ and the sequence $x_0=1$ and $x_{n+1}=x_n+\frac{2}{f(x_n)}\forall n\geq 0$:

$P(x_n,\frac{2}{f(x_n)})\implies f(x_{n+1})>2f(x_n)$ and so $f(x_n)>2^nu\,\forall n>0$

So $x_1=1+\frac{2}{u}$ and $x_{n+1}<x_n+\frac{1}{2^{n-1}u}\forall n>0$

So $x_n<1+\frac{1}{u}(2+1+\frac{1}{2}+\frac{1}{4}+...\frac{1}{2^{n-1}})<1+\frac{4}{u}$

But $f(x_n) > 2^n u$ and $x_n < 1 + \frac{4}{u}$ shows that $f(x)$ is not upper bounded over $(0, 1 + \frac{4}{u}]$, and so contradiction with the first sentence of this proof.

So no such function.

Amir Hossein
4719 posts

Mar 5, 2011, 7:02 pm PM #288

Problem 87.

Let f be a function defined for positive integers with positive integral values satisfying the conditions:

(i) $f(ab) = f(a)f(b),$

(ii) $f(a) < f(b)$ if $a < b,$

(iii) $f(3) \geq 7.$

Find the minimum value for $f(3).$

---

pco
14052 posts

Mar 5, 2011, 7:19 pm • 1 PM #289

> *amparvardi wrote:*
> **Problem 87.**
>
> Let f be a function defined for positive integers with positive integral values satisfying the conditions:
>
> (i) $f(ab) = f(a)f(b),$
>
> (ii) $f(a) < f(b)$ if $a < b,$
>
> (iii) $f(3) \geq 7.$
>
> Find the minimum value for $f(3).$

Let $m > n > 1$ two integers :

If $\frac{p}{q} < \frac{\ln m}{\ln n} < \frac{r}{s}$, with $p, q, r, s \in \mathbb{N}$, we get :

$n^p < m^q$ and so $f(n)^p < f(m)^p$ and so $\frac{p}{q} < \frac{\ln f(m)}{\ln f(n)}$

$m^s < n^r$ and so $f(m)^s < f(n)^r$ and so $\frac{\ln f(m)}{\ln f(n)} < \frac{r}{s}$

And so $\frac{\ln f(m)}{\ln f(n)} = \frac{\ln m}{\ln n}$ and $\frac{\ln f(m)}{\ln m} = \frac{\ln f(n)}{\ln n} = c$

And $f(n) = n^c$

And so $f(3) = 3^c \geq 7$

So $c = 2$ and minimum value for $f(3)$ is nine, which is reached for function $f(n) = n^2$

Amir Hossein
4719 posts

Mar 5, 2011, 7:25 pm PM #290

So quick! 🙂

Problem 88.

A function $f : \mathbb{N} \to \mathbb{N}$ satisfies

(a) $f(ab) = f(a)f(b)$ whenever the greatest common divisor of a and b is $1,$

(b) $f(p + q) = f(p) + f(q)$ for all prime numbers p and $q.$

Show that $f(2) = 2, f(3) = 3$ and $f(1999) = 1999.$

---

mousavi
222 posts

Mar 5, 2011, 8:10 pm PM #291

> *amparvardi wrote:*
> So quick! 🙂
>
> **Problem 88.**
>
> A function $f : \mathbb{N} \to \mathbb{N}$ satisfies
>
> (a) $f(ab) = f(a)f(b)$ whenever the greatest common divisor of a and b is $1,$
>
> (b) $f(p + q) = f(p) + f(q)$ for all prime numbers p and $q.$
>
> Show that $f(2) = 2, f(3) = 3$ and $f(1999) = 1999.$

$f(6) = f(3)f(2)$ and $f(6) = 2f(3) \Longrightarrow f(2) = 2$, $f(4) = 2f(2) = 4$

$f(12) = f(7) + f(5) = (f(5) + f(2)) + f(5) = 2(f(3) + f(2)) + f(2) = 3f(2) + 2f(3)$ and
$f(12) = f(3)f(4) = 4f(3) \Longrightarrow f(3) = 3$

$\Longrightarrow f(2) + f(3) = f(5) \Longrightarrow f(5) = 5$

$f(7) = f(5) + f(2) = 7, f(8) = f(5) + f(3) = 8$

*** $f(24) = f(8)f(3) = 24, f(24) = f(7) + f(17) \Longrightarrow f(17) = 17$ ***

$1997, 1999$ are prime.

$f(1999) = f(1997) + 2 = 4 + f(1995) = 4 + f(5)f(3)f(7)f(19) = 4 + 105(f(17) + f(2)) = 4 + 1995 = 1999$

RSM
736 posts

Mar 5, 2011, 8:40 pm PM #292

> *amparvardi wrote:*
> So quick!
>
> **Problem 88.**
>
> A function $f:\mathbb{N}\to\mathbb{N}$ satisfies
>
> (a) $f(ab)=f(a)f(b)$ whenever the greatest common divisor of a and b is 1,
>
> (b) $f(p+q)=f(p)+f(q)$ for all prime numbers p and q.
>
> Show that $f(2)=2, f(3)=3$ and $f(1999)=1999.$

I found f(2) and f(3) in the same way as @mousavi.
But my solution to find f(1999) is quite different.

It has been proved that Goldbech conjecture is true in a huge range of number.
So while finding f(1999) we can apply Goldbech conjecture easily.
Suppose, $f(n)=n\forall n\leq m$
If m+1 is not prime, then if it has more than one prime divisor or a power of 2 then we are done.
Click to reveal hidden text

> For we can write m+1 as a product of two co-prime integers, if m+1 has more than one prime divisors. If m+1 is a power of 2, then it can be written as a sum of two primes and so done.

But if it is a prime or a power of a prime, then we can easily calculate at least one of f(m+4) or f(m+6) and so f(m+1)=f(m+4)-3.
Click to reveal hidden text

> Since none of m+4 and m+6 cannot be a power of a prime(except 2) and both of them cannot be power of 2 if m+1 is a prime or a power of prime, so we can write at least one of them as a product of two co-prime integers and so done.

In this proof I have omitted the trivial cases, for which f(n) is easy to calculate.

---

socrates
1872 posts

Mar 5, 2011, 8:52 pm PM #293

Problem 89

Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$f(x+y)=f(x)+f(y)+f(xy)$, for all $x,y\in\mathbb{R}.$

RSM
736 posts

Mar 5, 2011, 9:25 pm PM #294

Putting x=y=2 we have f(2)=0
Putting $x=a+b, y=c$
$f(a+b+c)=f(a+b)+f(c)+f(ac+bc)$
$f(a+b+c)=f(a)+f(b)+f(c)+f(ab)+f(ac)+f(bc)+f(abc^2)$
Similarly $f(a+b+c)=f(a)+f(b)+f(c)+f(ab)+f(ac)+f(bc)+f(a^2bc)$
So we have $f(abc^2)=f(a^2bc)$
Putting $b=\frac{1}{ac}$
$f(a)=f(c)\forall a,c\in\mathbb{R}$
So f(x)=0 for all x

Amir Hossein
4719 posts

Mar 5, 2011, 9:37 pm PM #295

Problem 90.

(a) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(a^3)+f(b^3)+f(c^3)=f(3abc)\quad\forall a,b,c\in\mathbb{R}.$$

(b) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(a^3)+f(b^3)+f(c^3)=a\cdot f(a^2)+b\cdot f(b^2)+c\cdot f(c^2)\quad\forall a,b,c\in\mathbb{R}.$$

---

pco
14052 posts

Mar 5, 2011, 9:46 pm PM #296

> *amparvardi wrote:*
> **Problem 90.**
>
> **(a)** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
>
> $$f(a^3)+f(b^3)+f(c^3)=f(3abc)\quad\forall a,b,c\in\mathbb{R}.$$

Setting $b=c=0$, we get $f(a^3)=-f(0)$ and so $f(x)$ is constant and the only constant solution is $\boxed{f(x)=0}\ \forall x$

RSM
736 posts

Mar 5, 2011, 10:03 pm PM #297

> *amparvardi wrote:*
> **Problem 90.**
>
> **(a)** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
>
> $$f(a^3)+f(b^3)+f(c^3)=f(3abc)\quad\forall a,b,c\in\mathbb{R}.$$
>
> **(b)** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
>
> $$f(a^3)+f(b^3)+f(c^3)=a\cdot f(a^2)+b\cdot f(b^2)+c\cdot f(c^2)\quad\forall a,b,c\in\mathbb{R}.$$

Putting a=b=c=0 we have f(0)=0

Putting c=0 we have $f(a^3) = -f(b^3)$ for all a,b
$f(b^3) = -f(c^3)$
$f(a^3) = -f(c^3)$
And so $f(a^3) = 0$
So f(x)=0 for all x.

Solution to the second part :-
Putting a=b=c=0 we have f(0)=0
Putting b=c=0 we have $f(a^3) = af(a^2)$
Define f to the set of real numbers such that all real numbers can be represented in the form $k^{2^m3^n}$ with $|m-n| = 1$ with k belonging to the set and no two elements can be written in this form with same k
Now it is easy to see that $f(k^{(2^m3^n)}) = k^{(2^m3^n)}g(k)$
This function satisfies the given condition.

This post has been edited 1 time. Last edited by RSM, Mar 7, 2011, 10:21 am

---

pco
14052 posts

Mar 5, 2011, 10:10 pm PM #298

> *amparvardi wrote:*
> **Problem 90.**
> **(b)** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
> $$f(a^3)+f(b^3)+f(c^3) = a\cdot f(a^2)+b\cdot f(b^2)+c\cdot f(c^2) \quad \forall a,b,c\in\mathbb{R}.$$

This is equivalent to $f(x^3) = xf(x^2)$ and there are infinitely many solution.

Let $x\sim y$ the relation defined on $(1,+\infty)$ as $\frac{\ln(\ln x)-\ln(\ln y)}{\ln 3-\ln 2}\in\mathbb{Z}$

This is an equivalence relation.
Let $c(x)$ any choice function which associates to any real in $(1,+\infty)$ a representant (unique per class) of its class.
Let $n(x) = \frac{\ln(\ln x)-\ln(\ln c(x))}{\ln 3-\ln 2}\in\mathbb{Z}$
We get $x = c(x)^{(\frac32)^{n(x)}}$ and so $f(x) = \frac{xf(c(x))}{c(x)}$

And so we can define $f(x)$ only over $c((1,+\infty))$
Let $g(x)$ any finction from $\mathbb{R}\to\mathbb{R}$

$$f(x) = \frac{xg(c(x))}{c(x)}$$

We can define in the same way $f(x)$ over $(0,1)$
We can define then $f(1)$ as any value, $f(0)$ as 0 and $f(-x) = -f(x)$

And we have got all suitable $f(x)$

Amir Hossein
4719 posts

Mar 5, 2011, 10:17 pm PM #299

Problem 91.

Let f be a bijection from $\mathbb{N}$ into itself. Prove that one can always find three natural numbers a,b,c such that $a<b<c$ and $f(a)+f(c) = 2f(b)$.

RSM
736 posts

Mar 5, 2011, 11:08 pm PM #300

> *amparvardi wrote:*
> **Problem 91.**
>
> Let f be a bijection from $\mathbb{N}$ into itself. Prove that one can always find three natural numbers a,b,c such that $a<b<c$ and $f(a)+f(c) = 2f(b)$.

Suppose, $f(a) = 1$
For any integer b>a we have f(b)>1
Now consider a sequence a_n such that $a_1 = 2$ and $a_{n+1} = 2a_n - 1$
Consider the sequence $f^{-1}a_n$
There are finite numbers of n for which $f^{-1}a_n < a$
So for $f^{-1}a_n > a$ If the wanted a,b,c do not exist then clearly $f^{-1}a_n > f^{-1}a_{n+1}$
So we get infinitely many $f^{-1}a_n$ less than $f^{-1}a_1$
But this is not possible.

This post has been edited 1 time. Last edited by RSM, Mar 5, 2011, 11:38 pm

---

pco
14052 posts

Mar 5, 2011, 11:25 pm PM #301

> *RSM wrote:*
> If the wanted a,b,c do not exist then clearly $f^{-1}a_n > f^{-1}a_{n+1}$

Not exactly. We need $f^{-1}a_n > f^{-1}a_{n+1}$ or $f^{-1}a_n < a$
Since there are at most a finite number of such $f^{-1}a_n < a$, your idea is still true with a little modification.
Congrats.

RSM
736 posts

Mar 5, 2011, 11:26 pm PM #302

> *amparvardi wrote:*
> **Problem 90.**
>
> **(a)** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
> $$f(a^3)+f(b^3)+f(c^3) = f(3abc) \quad \forall a,b,c\in\mathbb{R}.$$
>
> **(b)** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
> $$f(a^3)+f(b^3)+f(c^3) = a\cdot f(a^2)+b\cdot f(b^2)+c\cdot f(c^2) \quad \forall a,b,c\in\mathbb{R}.$$

Solution By Abhratanu(email id:- adchessman@gmail.com):-
With the condition that f is continuous at x=1

$f(a) = a^{\sum_{i=0}^{n} \frac{2^i}{3^{i+1}}} f(a^{\frac{2^{i+1}}{3^{i+1}}})$
Since f is continuous at x=0 so when n tends to infinity f(a) tends to af(1).
So we have f(a)=af(1) for all a

---

socrates
1872 posts

Mar 6, 2011, 5:37 am *PM #303*

Problem 92 (Baltic way)

Suppose two functions $f(x)$ and $g(x)$ are defined for all x such that $2 < x < 4$ and satisfy
$2 < f(x) < 4,$
$2 < g(x) < 4,$
$f(g(x)) = g(f(x)) = x$ and
$f(x) \cdot g(x) = x^2,$ for all such values of $x.$

Prove that $f(3) = g(3).$

---

mousavi
222 posts

Mar 6, 2011, 9:18 am *PM #304*

> *socrates wrote:*
> **Problem 92 (Baltic way)**
>
> Suppose two functions $f(x)$ and $g(x)$ are defined for all x such that $2 < x < 4$ and satisfy
> $2 < f(x) < 4,$
> $2 < g(x) < 4,$
> $f(g(x)) = g(f(x)) = x$ and
> $f(x) \cdot g(x) = x^2,$ for all such values of $x.$
>
> Prove that $f(3) = g(3).$

$$f(f(x)) = \frac{f(x)^2}{g(f(x))} = \frac{f(x)^2}{x}$$

$$f(f(f(x))) = \frac{f(f(x))^2}{f(x)} = \frac{f(x)^3}{x^2}$$

$$\Longrightarrow f^n(x) = \frac{f(x)^n}{x^{n-1}}$$

$$2 < f^n(x) = \frac{f(x)^n}{x^{n-1}} < 4$$

$$\Longrightarrow f(x) = x$$

---

socrates
1872 posts

Mar 6, 2011, 11:53 am *PM #305*

Problem 93

Determine all monotone functions $f : \mathbb{R} \to \mathbb{Z}$ such that
$f(x) = x, \forall x \in \mathbb{Z}$ and
$f(x+y) \geq f(x) + f(y), \forall x, y \in \mathbb{R}.$

RSM
736 posts

Mar 6, 2011, 2:46 pm *PM #306*

> *socrates wrote:*
> **Problem 93**
>
> Determine all monotone functions $f : \mathbb{R} \to \mathbb{Z}$ such that
> $f(x) = x, \forall x \in \mathbb{Z}$ and
> $f(x+y) \geq f(x) + f(y), \forall x, y \in \mathbb{R}.$

I think the definition of a monotonic function is that for x>y f(x)>f(y) or f(x)<f(y)
If it is true then how f can be a monotonic function since all the f(x) for n<x<n+1(n is an integer) will have to take integer value between n and n+1.
But this is not possible.

I think $f : \mathbb{R} \to \mathbb{R}$

---

pco
14052 posts

Mar 6, 2011, 2:50 pm *PM #307*

> *socrates wrote:*
> **Problem 93**
>
> Determine all monotone functions $f : \mathbb{R} \to \mathbb{Z}$ such that
> $f(x) = x, \forall x \in \mathbb{Z}$ and
> $f(x+y) \geq f(x) + f(y), \forall x, y \in \mathbb{R}.$

Induction gives $f(qx) \geq q f(x) \forall q \in \mathbb{N}$ and so, setting $x = \frac{p}{q}$, $f(\frac{p}{q}) \leq \frac{p}{q}$.

Since $f(x)$ is non decreasing and $f(x) \in \mathbb{Z}$, this implies $f(x) = [x] \forall x \in \mathbb{Q}$

Since $f(x)$ is non decreasing, this implies $f(x) = [x] \forall x \in \mathbb{R}$

---

socrates
1872 posts

Mar 6, 2011, 8:25 pm *PM #308*

Problem 94

Find all monotone functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(4x) - f(3x) = 2x,$ for each $x \in \mathbb{R}.$

RSM
736 posts

Mar 6, 2011, 8:51 pm *PM #309*

> *socrates wrote:*
> **Problem 94**
>
> Find all monotone functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(4x) - f(3x) = 2x,$ for each $x \in \mathbb{R}.$

Plugging $f(x) = g(x) + 2x$
We get $g(4x) = g(3x)$
We can find a set of real numbers such that all real numbers can be represented in the form $k.4^m.3^n$ with $|m-n| = 1$ and we say this set of $k.4^m.3^n$ the class k

m,n being integers and k belonging to that set and no two element of that set can written in this form with same k.
Define g for this set.
Easy to see that $g(k.4^m.3^n) = g(k)$
Suppose, k_1 and $k_2, k_1 \geq k_2$ are two distinct elements of the set such that $g(k_1) \neq g(k_2)$
Since f is monotonic function, without loss of generality we may assume that it is non-decreasing.
$2x_{k_2} + g(k_1) \geq 2x_{k_1} + g(k_2)$ where x_{k_1} and x_{k_2} belongs to the class of k_1 and k_2 respectively.
We can infinitely increase the difference $x_{k_1} - x_{k_2}$ and this implies a contradiction since $g(k_1)$ and $g(k_2)$ are fixed.
So $g(k_1) = g(k_2)$ for all k_1, k_2
So only solution is $f(x) = 2x + k$ for some constant k

This post has been edited 1 time. Last edited by RSM, Mar 7, 2011, 10:16 am

---

pco
14052 posts

Mar 6, 2011, 9:01 pm PM #310

> *socrates wrote:*
> **Problem 94**
>
> Find all monotone functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(4x)-f(3x)=2x$, for each $x\in\mathbb{R}$.

Forget the "monotone" constraint and the general solution of functional equation is :

$\forall x>0$: $f(x)=2x+h(\frac{\ln x}{\ln 4-\ln 3})$ where $h(x)$ is any function defined over $[0,1)$
$f(0)=a$
$\forall x<0$: $f(x)=2x+k(\frac{\ln -x}{\ln 4-\ln 3})$ where $k(x)$ is any function defined over $[0,1)$

Adding then monotone constraint and looking at $f(x)$ when $x\to 0$, we see that we must have $\sup h([0,1))=\inf h([0,1)$ and so $h(x)=c$ constant.

And then, continuity at 0 implies that $h(x)=k(x)=a$ and so $\boxed{f(x)=2x+a}$

Amir Hossein
4719 posts

Mar 6, 2011, 9:24 pm PM #311

a) Does it exist a function $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(f(x))=x^2-2$$

for all real numbers x ?

b) Do there exist the real coefficients a,b,c such that the following functional equation

$$f(f(x))=ax^2+bx+c$$

has at least one root?

ShahinBJK
113 posts

Mar 6, 2011, 9:58 pm PM #312

> *amparvardi wrote:*
> **a)** Does it exist a function $f:\mathbb{R}\to\mathbb{R}$ satisfying
>
> $$f(f(x))=x^2-2$$
>
> for all real numbers x ?

see at here problem 7 http://www.imomath.com/tekstkut/funeqn_mr.pdf

---

socrates
1872 posts

Mar 6, 2011, 10:39 pm PM #313

> *amparvardi wrote:*
> **Problem 95**
> **b)** Do there exist the real coefficients a,b,c such that the following functional equation
>
> $$f(f(x))=ax^2+bx+c$$
>
> has at least one root?

Take $a=1, b=c=0$ and $f(x)=|x|^{\sqrt{2}}$.

Problem 96
Let $n\in\mathbb{N}$, such that $\sqrt{n}\notin\mathbb{N}$ and $A=\{a+b\sqrt{n}|a,b\in\mathbb{N}, a^2-nb^2=1\}$.
Prove that the function $f:A\to\mathbb{N}$, such that $f(x)=[x]$ is injective but not surjective.

$(\mathbb{N}=\{1,2,...\})$

---

pco
14052 posts

Mar 6, 2011, 11:01 pm PM #314

> *socrates wrote:*
> **Problem 96**
> Let $n\in\mathbb{N}$, such that $\sqrt{n}\notin\mathbb{N}$ and $A=\{a+b\sqrt{n}|a,b\in\mathbb{N}, a^2-nb^2=1\}$.
> Prove that the function $f:A\to\mathbb{N}$, such that $f(x)=[x]$ is injective but not surjective.
>
> $(\mathbb{N}=\{1,2,...\})$

If $[a+b\sqrt{n}]=p\geq 1$, then :

$p\leq a+b\sqrt{n}<p+1$
$\frac{1}{p+1}<a-b\sqrt{n}<\frac{1}{p}$

Adding, we get $p+\frac{1}{p+1}<2a<p+1+\frac{1}{p}$

And since $(p+1+\frac{1}{p})-(p+\frac{1}{p+1})=1+\frac{1}{p(p+1)}<2$, this interval may contain at most one even integer.

So knowledge of $f(x)$ implies knowledge of a and so (using $a^2-nb^2=1$), knowledge of b

So $f(x)$ is injective.

Consider then $p=2$ and the equation becomes $2+\frac{1}{3}<2a<3+\frac{1}{2}$ and so $1<\frac{7}{6}<a<\frac{7}{4}<2$ and so no such a.
So $f(x)=2$ is impossible and $f(x)$ is not surjective.

ShahinBJK
113 posts

Mar 6, 2011, 11:36 pm PM #315

> *socrates wrote:*
> **Problem 96**
> Let $n\in\mathbb{N}$, such that $\sqrt{n}\notin\mathbb{N}$ and $A=\{a+b\sqrt{n}|a,b\in\mathbb{N},a^2-nb^2=1\}$.
> Prove that the function $f:A\to\mathbb{N}$, such that $f(x)=[x]$ is injective but not surjective.
>
> $(\mathbb{N}=\{1,2,...\})$

Let (a,b) and (x,y) be two solutions of $a^2-nb^2=1$ assume the contrary that let $[a+b\sqrt{n}]=[x+y\sqrt{n}]$ and
$(a+b\sqrt{n})(a-b\sqrt{n})=1\Rightarrow a>b\sqrt{n}\Rightarrow 2a>[a+b\sqrt{n}]>2b\sqrt{n}$ and similarly we get $2x>[x+y\sqrt{n}]>2y\sqrt{n}\Rightarrow$
$2a>[x+y\sqrt{n}]>2y\sqrt{n}$ and $2x>[a+b\sqrt{n}]>2b\sqrt{n}\Rightarrow a^2+1\geq y^2n$ and $x^2+1\geq b^2n$
$\Rightarrow 0=a^2+1-y^2n+x^2+1-b^2n\geq 0\Rightarrow a^2+1=y^2n$ which implies that $x=a$ and $y=b\Rightarrow$ contradiction

This post has been edited 1 time. Last edited by ShahinBJK, Mar 7, 2011, 1:09 am

---

ShahinBJK
113 posts

Mar 6, 2011, 11:40 pm PM #316

Problem 97
Find all functions $f:N\to N$ such that $f(f(m)+f(n))=m+n$ for every two natural numbers m and n.

---

goodar2006
1344 posts

Mar 6, 2011, 11:50 pm PM #317

plugging m,m we get $f(2f(m))=2m$, so f is injective.
suppose $f(1)=k$, so we have $f(2k)=2$ and $f(4)=4k$.
$f(f(i-1)+f(i+1))=f(2f(i))=2i$ so we have $f(i+1)-f(i)=f(i)-f(i-1)$ for all $i\in\mathbb{N}$, so for every natural number n we have $f(i+1)-f(i)=t$. so for every natural number n, we have $f(n)=k+(n-1)t$.
$f(4)=4k=k+3t$, so $t=k$ and we have $f(m)=mk$ for all natural numbers m.
plugging this in the original function, we get $k=1$ and hence the unique solution $f(m)=m$ for all natural numbers m.

---

mahanmath
1355 posts

Mar 6, 2011, 11:54 pm PM #318

Problem 98

Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that :

$f(x^2+y^2)=f(xy)$

---

goodar2006
1344 posts

Mar 6, 2011, 11:58 pm PM #319

problem 99
find all functions $f:\mathbb{Z}\longrightarrow\mathbb{Z}$ such that:
1) $f(1)=f(-1)$
2) $f(x)+f(y)=f(x+2xy)+f(y-2xy)\forall x,y\in\mathbb{Z}$.

This post has been edited 1 time. Last edited by goodar2006, Mar 7, 2011, 12:03 am

---

RSM
736 posts

Mar 7, 2011, 12:00 am PM #320

> *mahanmath wrote:*
> **Problem 98**
>
> Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that :
>
> $f(x^2+y^2)=f(xy)$

Fix xy=m
Any number greater than equal to 2m can be written in the form x^2+y^2 for some x,y with xy=m
So f(x)=f(m) for all x>2m
So f(x)=c is the only solution.

---

pco
14052 posts

Mar 7, 2011, 12:05 am • 1 PM #321

> *mahanmath wrote:*
> **Problem 98**
>
> Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that :
>
> $f(x^2+y^2)=f(xy)$

The system $x^2+y^2=u$ and $xy=v$ has solution with $x,y>0$ iff $u>2v>0$

And so $f(u)=f(v)\forall u>2v>0$

Let then $x>y>0$:
$x>2\frac{y}{4}$ and so $f(x)=f(\frac{y}{4})$

$y>2\frac{y}{4}$ and so $f(y)=f(\frac{y}{4})$

And so $f(x)=f(y)$ and so $f(x)$ is constant.

---

mahanmath
1355 posts

Mar 7, 2011, 12:08 am PM #322

> *goodar2006 wrote:*
> **problem 99**
> find all functions $f:\mathbb{Z}\longrightarrow\mathbb{Z}$ such that:
> 1) $f(1)=f(-1)$
> 2) $f(x)+f(y)=f(x+2xy)+f(y-2xy)\forall x,y\in\mathbb{Z}$.

> *lajanugen wrote:*
> Problem 41:
> We claim that all functions that satisfy $exp_2(x)=exp_2(y)\to f(x)=f(y)$ are the solutions ($f(0)$ can be arbitrary)- this is easily seen to satisfy the equation since $exp_2(x)=exp_2(x+2xy)$ for any non-zero integer x
> Plugging in $(m,n)=(n,1),(-1,n)$ and equating the expressions, we obtain that the f values of all odd numbers are equal.
> Hence, for all odd n, $f(m)=f((2n+1)m)$ (Since $n-2mn$ would also be odd):
> $n=-1$ gives $f(m)=f(-m)$ for all m
> As n ranges through all odd values, $2n+1,-(2n+1)$ range through all odd values

pco
14052 posts

Mar 7, 2011, 12:26 am PM #323

> *goodar2006 wrote:*
> **problem 99**
> find all functions $f:\mathbb{Z}\longrightarrow\mathbb{Z}$ such that:
> 1) $f(1)=f(-1)$
> 2) $f(x)+f(y)=f(x+2xy)+f(y-2xy)\forall x,y\in\mathbb{Z}$.

See also http://www.artofproblemsolving.com/Forum/viewtopic.php?f=56&t=386360

---

socrates
1872 posts

Mar 7, 2011, 1:17 am PM #324

Problem 100 😁

Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+y)\le f(x)+f(y)$ for all $x,y\in\mathbb{R}$ and $f(x)\le e^x-1$ for each $x\in\mathbb{R}$.

---

pco
14052 posts

Mar 7, 2011, 2:15 am PM #325

> *socrates wrote:*
> **Problem 100** 😁
>
> Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+y)\le f(x)+f(y)$ for all $x,y\in\mathbb{R}$ and $f(x)\le e^x-1$ for each $x\in\mathbb{R}$.

$f(x+0)\le f(x)+f(0)$ and so $f(0)\ge 0$ and since $f(0)\le e^0-1=0$, we get $f(0)=0$
$f(x+(-x))\le f(x)+f(-x)$ and so $f(x)+f(-x)\ge 0$

$f(x)\le e^x-1\implies f(x)\le f(\frac x2)+f(\frac x2)\le 2(e^{\frac x2}-1)$

$f(x)\le 2(e^{\frac x2}-1)\implies f(x)\le f(\frac x2)+f(\frac x2)\le 4(e^{\frac x4}-1)$

And immediate induction gives $f(x)\le 2^n(e^{\frac x{2^n}}-1)$

Setting $n\to+\infty$, we get $f(x)\le x$

So $f(x)+f(-x)\le x+(-x)=0$ and so, since we already got $f(x)+f(-x)\ge 0$, we get $f(x)+f(-x)=0$

Then $f(-x)\le -x\implies -f(x)\le -x\implies f(x)\ge x$

And so $\boxed{f(x)=x}$ which indeed is a solution.

---

socrates
1872 posts

Mar 7, 2011, 5:24 am PM #326

Problem 101

A variation on the unsolved **Problem 40**:

Let $f:R\to R$ be a function such that $f(xy)+f(x-y)\ge f(x+y)$ for all real numbers x,y.
Prove that $f(x)\ge 0,$ for each $x\in\mathbb{R}$.

---

socrates
1872 posts

Mar 7, 2011, 5:25 am PM #327

> *amparvardi wrote:*
> **Problem 43** :
> Let f be a real function defined on the positive half-axis for which $f(xy)=xf(y)+yf(x)$ and $f(x+1)\le f(x)$ hold for every positive x and y. Show that if $f(1/2)=1/2$, then
>
> $$f(x)+f(1-x)\ge -x\log_2 x-(1-x)\log_2(1-x)$$
>
> for every $x\in(0,1)$.

Any solution?

RSM
736 posts

Mar 7, 2011, 9:09 am PM #328

> *socrates wrote:*
> **Problem 101**
>
> A variation on the unsolved **Problem 40**:
>
> Let $f:R\to R$ be a function such that $f(xy)+f(x-y)\ge f(x+y)$ for all real numbers x,y.
> Prove that $f(x)\ge 0,$ for each $x\in\mathbb{R}$.

Putting $x=a+b,y=a-b$ the equation turns to
$f(a^2-b^2)\ge f(2a)-f(2b)$
Either $f(2a)\ge f(2b)$ or $f(2a)\le f(2b)$
In the first case $f(a^2-b^2)\ge 0$ in the second case $f(b^2-a^2)\ge 0$
So we have at least one of f(x) and f(-x) is greater than 0
Putting a=b we have $f(0)\ge 0$
If $f(x)\le 0$ for some x=2c then put b=c
and $a=\sqrt{c^2+c}$ or $a=-\sqrt{c^2+c}$ for which $f(2a)\ge 0$
The equation turns to $f(c)\ge f(2a)-f(2c)\ge 0$
So $f(x)\ge 0$ for all x

RSM
736 posts

Mar 7, 2011, 11:20 am PM #329

> *RSM wrote:*
>
> > *abhinavzandubalm wrote:*
> > Even A Lifeguard Must not be bringing the dead back to life by C.P.R. like we have to do for this marathon. 😁 🙂 😋
> >
> > Problem 71
> >
> > > Determine All Functions
> > >
> > > $f:\mathbb{R}-0,1\to\mathbb{R}$

$$f(x)+f(\frac{1}{1-x})=\frac{2(1-2x)}{x(1-x)}$$

This is an INMO problem
A solution to it can be found in INMO official solution paper.

Next problem:-
Find all function f defined on real variables such that
$f(x+y)+f(x-y)=2f(x)f(y)$
for all $x,y\in\mathbb{R}$

I could not solve this problem myself.
So I want to give one more condition that $|f(x)|\geq 1\forall x\in\mathbb{R}$
I think it is easy now.

Is it so hard that no-one has posted any solution!!!!!

ShahinBJK
113 posts

Mar 7, 2011, 3:09 pm PM #330

> *abhinavzandubalm wrote:*
> **Problem 71**
> Determine All Functions
> $f:\mathbb{R}-0,1\to\mathbb{R}$
> $f(x)+f(\frac{1}{1-x})=\frac{2(1-2x)}{x(1-x)}$

__Solution:__
$$x\to\frac{1}{1-x}\implies f(\frac{1}{1-x})+f(\frac{x-1}{x})=\frac{2(x+1)(1-x)}{x}$$
$$x\to\frac{x-1}{x}\implies f(\frac{x-1}{x})+f(x)=\frac{2x(2-x)}{x-1}$$
$$\Rightarrow f(\frac{1}{1-x})+f(\frac{x-1}{x})=\frac{2(1-2x)}{x(1-x)}-f(x)+\frac{2x(2-x)}{x-1}-f(x)=\frac{2(x+1)(1-x)}{x}$$
$$\implies f(x)=\frac{x+1}{x-1}\forall x\neq 0;1$$

socrates
1872 posts

Mar 7, 2011, 11:09 pm PM #331
Problem 102

Find all continuous functions $f:(0,+\infty)\to(0,+\infty)$, such that $f\left(x\right)=f\left(\sqrt{2x^2-2x+1}\right)$, for each $x>0$.

RSM
736 posts

Mar 7, 2011, 11:42 pm PM #332

> *socrates wrote:*
> **Problem 102**
>
> Find all continuous functions $f:(0,+\infty)\to(0,+\infty)$, such that $f\left(x\right)=f\left(\sqrt{2x^2-2x+1}\right)$, for each $x>0$.

The equation is equivalent to:-
$f(x)=f(\frac{\sqrt{2x^2-1}+1}{2})$
Consider a sequence $a_1=a>1$ and $a_{n+1}=\frac{\sqrt{2a_n^2-1}+1}{2}$
$f(a_n)=a$ for all a
Limit of this sequence is 1
Since f is a continuous function we have $\lim_{n\to+\infty}f(a_n)=f(\lim_{n\to+\infty}a_n)=f(1)$
So f(x)=f(1) for all x>1
Now for x<1
choose $a_1=a<1$ and $a_{n+1}=\sqrt{2a_n^2-2a_n+1}$
The limit of this sequence is also 1
So similarly we can conclude
$f(x)=c$ for all x

socrates
1872 posts

Mar 8, 2011, 6:53 am PM #333
Problem 103 (Romania 2010)

Determine all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ such that $f(a^2-b^2)=f^2(a)-f^2(b)$, for all $a,b\in\mathbb{N}_0, a\geq b$.

RSM
736 posts

Mar 8, 2011, 10:29 am PM #334
Sorry, I posted it by mistake
[Content Doleted}

See the next post.
This post has been edited 1 time. Last edited by RSM, Mar 8, 2011, 10:33 am

RSM
736 posts

Mar 8, 2011, 10:32 am PM #335

> *socrates wrote:*
> **Problem 103** (Romania 2010)
>
> Determine all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ such that $f(a^2-b^2)=f^2(a)-f^2(b)$, for all $a,b\in\mathbb{N}_0, a\geq b$.

Putting a=b we have $f(0)=0$
Putting a=1, b=0 we have $f(1)=1$ or $f(1)=0$
Putting b=0 we have $f(a^2)=f^2(a)$
$f(a^2-b^2)=(f(a)+f(b))(f(a)-f(b))$............(1)
Suppose, S={x : f(x)=0}
T={x : $f(x)\neq 0$}
Suppose, T contains at least one element
Minimum element of T is n and maximum of S is m($\neq 0$)
My claim is m<n
Otherwise, $f(m^2-n^2)=-f^2(n)\leq 0$
But this is not possible if

So m<n
But S cannot have an upper-bound if it has any element except 1
And in this case f is a strictly increasing function(directly follows from (1))
$f(2) = c$
$f(3) = f(4) - f(1)$
$f(9) = f^2(3)$
$f(5) = f(9) - f(4)$
$f(16) = f^2(4) = f(25) - f(9)$
Solving these equations we get positive integer solutions for c are c=2 in case f(1)=1 and no solution if f(1)=0
So $f(2^{2k}) = 2^{2k}$
Since f is a strictly increasing function so $f(x) = x$ for all x
If T does not contain any element then $f(x) = 0$ for all x
Answer:-
$f(x) = x$ for all x
$f(x) = 0$ for all x

---

mousavi
222 posts

Mar 9, 2011, 11:48 am PM #336

problem 104

find all continues functions $f : R \longrightarrow R$ for each two real numbers x, y:

$f(x + y) = f(x + f(y))$

---

pco
14052 posts

Mar 9, 2011, 12:33 pm PM #337

> *mousavi wrote:*
> **problem 104**
>
> find all continues functions $f : R \longrightarrow R$ for each two real numbers x, y:
>
> $f(x + y) = f(x + f(y))$

If $f(x) = x \, \forall x$, we got a solution.
If $\exists a$ such that $f(a) \neq a$, then $f(x + a) = f(x + f(a))$ implies that $f(x)$ is periodic and one of its periods is $|f(a) - a|$.

Let $T = \inf\{$positive periods$\}$
If $T = 0$, then $f(x) = c$ is constant and we got another solution.
if $T \neq 0$, then T is a period of $f(x)$ (since continuous) and, since any $f(y) - y$ is also a period, we get that $f(y) - y = n(y)T$ where $n(y) \in \mathbb{Z}$ but then $n(y)$ is a continuous function from $\mathbb{R} \to \mathbb{Z}$ and so is constant and $f(y) = y + kT$ which is not a periodic function.

Hence the two solutions :
$f(x) = x \, \forall x$
$f(x) = c \, \forall x$ for any $c \in \mathbb{R}$

---

socrates
1872 posts

Mar 10, 2011, 5:21 am PM #338

Problem 105

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
• $f\left(f(x)y + x\right) = xf(y) + f(x)$, for all real numbers x, y and
• the equation $f(t) = -t$ has exactly one root.

---

pco
14052 posts

Mar 10, 2011, 1:14 pm PM #339

> *socrates wrote:*
> **Problem 105**
>
> Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
> • $f\left(f(x)y + x\right) = xf(y) + f(x)$, for all real numbers x, y and
> • the equation $f(t) = -t$ has exactly one root.

Let $P(x, y)$ be the assertion $f(f(x)y + x) = xf(y) + f(x)$
Let t be the unique real such that $f(t) = -t$

$f(x) = 0 \, \forall x$ is a solution. Let us from now look for non all-zero solutions.
Let u such that $f(u) \neq 0$

$P(1, 0) \implies f(0) = 0$ and so $t = 0$
If $f(a) = 0$, then $P(a, u) \implies af(u) = 0$ and so $a = 0$
So $f(x) = 0 \iff x = 0$

If $f(1) \neq 1$, then $P(1, \frac{1}{1 - f(1)}) \implies f(\frac{f(1)}{1 - f(1)} + 1) = f(\frac{1}{1 - f(1)}) + f(1)$ and so $f(1) = 0$, which is impossible.
So $f(1) = 1$

$P(1, -1) \implies f(-1) = -1$
$P(x, -1) \implies f(x - f(x)) = f(x) - x$ and so, since the only solution of $f(t) = -t$ is $t = 0$: $f(x) = x$ which indeed is a solution.

<u>**Hence the two solutions**</u> :
$f(x) = 0 \, \forall x$
$f(x) = x \, \forall x$

---

mousavi
222 posts

Mar 10, 2011, 1:47 pm PM #340

> *socrates wrote:*
> **Problem 105**
>
> Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
> • $f\left(f(x)y + x\right) = xf(y) + f(x)$, for all real numbers x, y and
> • the equation $f(t) = -t$ has exactly one root.

the second condition is not need.
by **pco's** way $f(1) = 0, 1$ and if $f(1) = 0$ then $f(x) = 0$(and even if $f(t) = -t$ has exactly one solution $f(x) = 0$ is possible)

let $f(1) = 1$ then:

$f(x + 1) = f(x) + 1$

$p(x, y + 1) : f(x + yf(x) + f(x)) = xf(y) + x + f(x) = f(x + yf(x)) + x$ let $z = x + yf(x)$ and z is surjective.

$\Longrightarrow f(z+f(x))=f(z)+x \Longrightarrow f(f(x))=x \Longrightarrow f(z+x)=f(z)+f(x)$

$p(f(x),y): f(f(x)+xy)=f(x)f(y)+x \Longrightarrow f(xy)=f(x)f(y)$

$f(xy)=f(x)f(y)$ and $f(x+y)=f(x)+f(y) \Longrightarrow f(x)=x$

This post has been edited 2 times. Last edited by mousavi, Mar 10, 2011, 2:18 pm

---

pco
14052 posts

Mar 10, 2011, 1:57 pm PM #341

> *mousavi wrote:*
> $p(x,y+1): f(x+yf(x)+f(x))=xf(y)+x+f(x)$ let $z=x+yf(x)$ and z is surjective.
>
> $\Longrightarrow f(z+f(x))=z+f(x) \Longrightarrow f(f(x))=x \Longrightarrow f(z+x)=f(z)+f(x)$

I'm affraid there is an error here : in "$f(x+yf(x)+f(x))=xf(y)+x+f(x)$", RHS contains $xf(y)+x$ and not $yf(x)+x$, so you cant substitue z

---

mousavi
222 posts

Mar 10, 2011, 2:18 pm PM #342

> *pco wrote:*
>
> > *mousavi wrote:*
> > $p(x,y+1): f(x+yf(x)+f(x))=xf(y)+x+f(x)$ let $z=x+yf(x)$ and z is surjective.
> >
> > $\Longrightarrow f(z+f(x))=z+f(x) \Longrightarrow f(f(x))=x \Longrightarrow f(z+x)=f(z)+f(x)$
>
> I'm affraid there is an error here : in "$f(x+yf(x)+f(x))=xf(y)+x+f(x)$", RHS contains $xf(y)+x$ and not $yf(x)+x$, so you cant substitue z

i edited

---

socrates
1872 posts

Mar 10, 2011, 9:20 pm PM #343

Problem 106

Find all functions $f:\mathbb{X}\to\mathbb{R}$ such that $f(x+y)+f(xy-1)=(f(x)+1)(f(y)+1)$, for all $x,y\in\mathbb{X}$, if
$\mathrm{a)}\mathbb{X}=\mathbb{Z}.$
$\mathrm{b)}\mathbb{X}=\mathbb{Q}.$

RSM
736 posts

Mar 11, 2011, 9:37 am • 1 PM #344

> *socrates wrote:*
> **Problem 106**
>
> Find all functions $f:\mathbb{X}\to\mathbb{R}$ such that $f(x+y)+f(xy-1)=(f(x)+1)(f(y)+1)$, for all $x,y\in\mathbb{X}$, if
> $\mathrm{a)}\mathbb{X}=\mathbb{Z}.$
> $\mathrm{b)}\mathbb{X}=\mathbb{Q}.$

Lemma 1:- f(0)=0,f(-1)=1
Proof:-
Putting y=0 we have $f(x)+f(-1)=(f(x)+1)(f(0)+1)$
$f(x).f(0)=f(-1)-f(0)-1$
If $f(0)\neq 0$ then $f(x)=constant=c$ for all x.
But putting $f(x)=c$ in the functional equation we get no real solution of c.
So $f(0)=0$ and this implies $f(-1)=1$

Lemma 2:- f(1)=1
Proof:-
Putting $x=y=-1$ we have $f(-2)=4$
Putting $x=1,y=-1$ we have $f(1)=1$

Lemma 3:- $f(x)=x^2\forall x\in\mathbb{Z}$
Proof:-
Putting y=1 in the functional equation we get
$f(x+1)+f(x-1)=2f(x)+2$
Hence by induction $f(x)=x^2\forall x\in\mathbb{Z}$

Lemma 4:- $f(x)=x^2\forall x\in\mathbb{Q}$
Proof:-
Suppose,m be any rational number.
n is an integer such that mn is an integer.
Putting $x=m,y=n$ in the functional equation we get
$f(m+n)+(mn-1)^2=(n^2+1)(f(m)+1)$.................(1)
Putting $x=m+n,y=-n$ we get
$f(m)+(mn+n^2+1)^2=(n^2+1)(f(m+n)+1)$.......................(2)
Eliminating f(m+n) from equation (1) and (2) we get $f(m)=m^2$

Answer:-
$f(x)=x^2\forall x\in\mathbb{Q}$

This post has been edited 1 time. Last edited by RSM, Mar 11, 2011, 2:09 pm

---

abhinavzan...
418 posts

Mar 11, 2011, 1:56 pm PM #345

The final answer should be $f(x)=x^2$

RSM
736 posts

Mar 11, 2011, 2:10 pm PM #346

> *abhinavzandubalm wrote:*
> The final answer should be $f(x)=x^2$

Thanks, @avinab, I made a typo mistake again.
I have edited the answer.

Babai
487 posts

Mar 12, 2011, 12:32 pm PM #347

Find all function $f:R\to R$ such that $f(x+f(x))=2f(x)$ and $f(f(x))=f(x)$ and $f(0)=0$

---

pco
14052 posts

Mar 12, 2011, 1:44 pm • 1 PM #348

> *Babai wrote:*

Find all function $f: R \rightarrow R$ such that $f(x+f(x)) = 2f(x)$ and $f(f(x)) = f(x)$ and $f(0) = 0$

I dont think this is a real olympiad exercise.
There are infinitely many very different solutions and I dont think there is a general solution.
Please, Babai, tell us from where is this problem and how are you sure there is a solution ?

Some examples of solutions :
1) $f(x) = 0 \forall x$

2) $f(x) = x \forall x$

3) Let $A \subseteq \mathbb{R}$ stable thru $x \rightarrow 2x$:
$f(x) = x \forall x \in A$
$f(x) = 0 \forall x \notin A$

4)
$f(2^n) = 2^n \forall n \in \mathbb{N}$
$f(2^{n+1} - 1) = 2^{n+1} \forall n \in \mathbb{N}$
$f(x) = 0 \forall$ other x

and infinitely many other

Babai
487 posts

Mar 12, 2011, 1:52 pm PM #349

actually,It was another problem.The main problem was:$f(x+f(y)) + f(f(y)) = f(f(x)) + 2f(y)$
I solved it partially and got that $f(x+f(x)) = 2f(x)$ and $f(f(x)) = f(x)$ while $f(0) = 0$ but could not solve further.So,please help.

pco
14052 posts

Mar 12, 2011, 2:55 pm • 2 PM #350

> *Babai wrote:*
> actually,It was another problem.The main problem was:$f(x+f(y)) + f(f(y)) = f(f(x)) + 2f(y)$
> I solved it partially and got that $f(x+f(x)) = 2f(x)$ and $f(f(x)) = f(x)$ while $f(0) = 0$ but could not solve further.So,please help.

1) It's not very fair to transform a problem and claim that there exists a solution when your transformation is not an equivalence and so you dont know if there is such an olympiad level solution.

2) Solution of the original problem :
Let $P(x, y)$ be the assertion $f(x+f(y)) + f(f(y)) = f(f(x)) + 2f(y)$

$P(0, y) \implies f(f(y)) = \frac{f(f(0))}{2} + f(y)$
$P(0, x) \implies f(f(x)) = \frac{f(f(0))}{2} + f(x)$
Plugging this in $P(x, y)$, we get new assertion $Q(x, y)$: $f(x+f(y)) = f(x) + f(y)$
It's immediate to see that the two assertions are equivalent.

The new assertion has been solved many times in mathlinks :

Let $A = f(\mathbb{R})$.
Using $f(x) + f(y) = f(x+f(y))$ and $f(x) - f(y) = f(x-f(y))$ (look at $Q(x-f(y), y)$), we see that A is an additive subgroup of $\mathbb{R}$

Then the relation $x \sim y \iff x - y \in A$ is an equivalence relation and let $c(x)$ any choice function which assocoates to a real x a representant (unique per class) of it's equivalence class.

Setting $g(x) = f(x) - x$, $Q(x, y)$ may be written $g(x+f(y)) = g(x)$ and so $g(x)$ is constant in any equivalence class and so $f(x) - x = f(c(x)) - c(x)$ and so $f(x) = h(c(x)) + x - c(x)$ where $h(x)$ is a function from $\mathbb{R} \rightarrow A$

<u>**So, any solution may be written as**</u> $f(x) = x - c(x) + h(c(x))$ where :
$A \subseteq \mathbb{R}$ is an additive subgroup of $\mathbb{R}$
$c(x)$ is any choice function associating to a real x a representant (unique per class) of it's equivalence class for the equivalence relation $x - y \in A$
$h(x)$ is any function from $\mathbb{R} \rightarrow A$

<u>**Let us show now that this mandatory form is sufficient and so that we got a general solution**</u> :
Let $A \subseteq \mathbb{R}$ any additive subgroup of $\mathbb{R}$
Let $c(x)$ any choice function associating to a real x a representant (unique per class) of it's equivalence class for the equivalence relation $x - y \in A$
Let $h(x)$ any function from $\mathbb{R} \rightarrow A$
Let $f(x) = x - c(x) + h(c(x))$

$x - c(x) \in A$ and $h(c(x)) \in A$ and A subgroup imply that $f(x) \in A$
So $x + f(y) \sim x$ and $c(x+f(y)) = c(x)$
So $f(x+f(y)) = x + f(y) - c(x+f(y)) + h(c(x+f(y)))$ $= x + f(y) - c(x) + h(c(x))$ $= f(x) + f(y)$
Q.E.D.

And so we got a general solution.

<u>**Some examples**</u> :
1) Let $A = \mathbb{R}$ and so a unique class and $c(x) = a$ and $f(x) = x - a + h(a)$ and so the solution $\boxed{f(x) = x + b}$ (notice that $f(0) = 0$ is not mandatory.

2) Let $A = \{0\}$ and so equivalence classes are $\{x\}$ and so $c(x) = x$ and $h(x) = 0$ and $f(x) = x - x + 0$ and so the solution $\boxed{f(x) = 0}$

3) Let $A = \mathbb{Z}$ and $c(x) = x - \lfloor x \rfloor$ and $h(x) = \lfloor 2x \rfloor$
$f(x) = x - x + \lfloor x \rfloor + \lfloor 2x - 2\lfloor x \rfloor \rfloor$ and so the solution $\boxed{f(x) = \lfloor 2x \rfloor - \lfloor x \rfloor}$

and infinitely many other.

abhinavzan...
418 posts

Mar 12, 2011, 6:28 pm PM #351

Please try to number your problems
Now The Number has gone to Problem 108

This post has been edited 1 time. Last edited by abhinavzandubalm, Mar 12, 2011, 7:20 pm

pco
14052 posts

Mar 12, 2011, 6:55 pm • 1 PM #352

> *abhinavzandubalm wrote:*

> "Problem 107"
> Does There Exist A Function $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(1+f(x))=1-x \quad and \quad f(f(x))=x$?????

Using $x=\frac{1}{2}$, we get $f(1+f(\frac{1}{2}))=\frac{1}{2}$
Taking f or both sides, we get $f(f(1+f(\frac{1}{2})))=f(\frac{1}{2})$ and so, using $f(f(x))=x$: $1+f(\frac{1}{2}))=f(\frac{1}{2})$, impossible.
So no such function.

Nota : I could understand the interest of hiding solutions, but what could be the interest of hiding problems :

---

abhinavzan...
418 posts

Mar 12, 2011, 7:27 pm PM #353

Just a habit formed when i started giving problems
btw it was **Problem 108**
Now to give **Problem 109**
Find all functions $f:\mathbb{R}_0\to\mathbb{R}_0$ satisfying the functional relation
$f(f(x)-x)=2x\ \forall x\in\mathbb{R}_0$

---

pco
14052 posts

Mar 12, 2011, 7:46 pm PM #354

> *abhinavzandubalm wrote:*
> Just a habit formed when i started giving problems
> btw it was **Problem 108**
> Now to give **Problem 109**
> Find all functions $f:\mathbb{R}_0\to\mathbb{R}_0$ satisfying the functional relation
> $f(f(x)-x)=2x\ \forall x\in\mathbb{R}_0$

is $\mathbb{R}_0=\mathbb{R}\setminus\{0\}$? If so, I think that, without continuity, we get infinitely many solutions.

Maybe he problem is with $\mathbb{R}^+$ (very very classical problem) ?

---

abhinavzan...
418 posts

Mar 12, 2011, 8:10 pm PM #355

Its $\mathbb{R}^++0$

---

pco
14052 posts

Mar 12, 2011, 10:32 pm • 2 PM #356

> *abhinavzandubalm wrote:*
> Just a habit formed when i started giving problems
> btw it was **Problem 108**
> Now to give **Problem 109**
> Find all functions $f:\mathbb{R}_0\to\mathbb{R}_0$ satisfying the functional relation
> $f(f(x)-x)=2x\ \forall x\in\mathbb{R}_0$

Ok, so $\mathbb{R}_0$ here is the set of non negative real numbers. Then :
In order to LHS be defined, we get $f(x)\ge x\,\forall x$
So $f(f(x)-x)\ge f(x)-x\,\forall x\iff f(x)\le 3x$

So we got $x\le f(x)\le 3x$

If we consider $a_nx\le f(x)\le b_nx$, we get $a_n(f(x)-x)\le 2x\le b_n(f(x)-x)$ and so $\frac{b_n+2}{b_n}x\le f(x)\le\frac{a_n+2}{a_n}x$

And so the sequences :
$a_1=1$
$b_1=3$
$a_{n+1}=\frac{b_n+2}{b_n}$
$b_{n+1}=\frac{a_n+2}{a_n}$

And it's easy to show that :
a_n is a non decreasing sequence whose limit is 2
b_n is a non increasing sequence whose limit is 2

And so $\boxed{f(x)=2x}$ which indeed is a solution.

Babai
487 posts

Mar 13, 2011, 10:08 am PM #357

Actually in that equation I put $y=x$ and assumed that $f(0)=0$ to get a solution.

---

socrates
1872 posts

Mar 13, 2011, 7:28 pm PM #358

Problem 110 (Romania District Olympiad 2011 - Grade XI)

Find all functions $f:[0,1]\to\mathbb{R}$ for which we have:

$$|x-y|^2\le|f(x)-f(y)|\le|x-y|,$$

for all $x,y\in[0,1]$.

---

pco
14052 posts

Mar 13, 2011, 7:54 pm • 2 PM #359

> *socrates wrote:*
> **Problem 110** (Romania District Olympiad 2011 - Grade XI)
>
> Find all functions $f:[0,1]\to\mathbb{R}$ for which we have:
>
> $$|x-y|^2\le|f(x)-f(y)|\le|x-y|,$$
>
> for all $x,y\in[0,1]$.

Let $P(x,y)$ be the assertion $|x-y|^2\le|f(x)-f(y)|\le|x-y|$

Setting $y\to x$ in $P(x,y)$, we conclude that $f(x)$ is continuous.
If $f(a)=f(b)$, then $P(a,b)\implies(a-b)^2\le 0$ and so $a=b$ and $f(x)$ is injective

$f(x)$ continuous and injective implies monotonous.
$f(x)$ solution implies $f(x)+c$ and $c-f(x)$ solutions too. So Wlog say $f(0)=0$ and $f(x)$ increasing.

Then :
$P(1,0) \implies f(1)=1$ and so $f(x)\in[0,1]$
$P(x,0) \implies f(x)\le x$
$P(x,1) \implies 1-f(x)\le 1-x$

And so $f(x)=x$ which indeed is a solution.

<u>**Hence the solutions**</u> :
$f(x)=x+a$ for any real a
$f(x)=a-x$ for any real a

---

socrates
1872 posts

Mar 15, 2011, 6:39 am PM #360

Problem 111

Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x^2-f^2(y))=xf(x)-y^2$, for all real numbers x,y.

---

pco
14052 posts

Mar 15, 2011, 2:13 pm PM #361

> *socrates wrote:*
> **Problem 111**
>
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x^2-f^2(y))=xf(x)-y^2$, for all real numbers x,y.

Let $P(x,y)$ be the assertion $f(x^2-f^2(y))=xf(x)-y^2$

1) $f(x)=0 \iff x=0$
=================
Let $u=-f^2(0)$: $P(0,0) \implies f(u)=0$

$P(0,u) \implies f(0)=-u^2$ and so $u=-f^2(0)=-u^4$ and so $u\in\{-1,0\}$

If $u=-1$: $f(0)=-1$ and $P(-1,0) \implies f(0)=-f(-1)$ and so contradiction since $f(0)=-1$ while $f(-1)=f(u)=0$.
So $u=0$ and $f(0)=0$
Then $P(x,0) \implies f(x^2)=xf(x)$ and if $f(y)=0$, then $P(x,y) \implies y=0$
Q.E.D.

2) $f(x)$ is odd and surjective
==================
$P(0,x) \implies f(-f^2(x))=-x^2$ and so any non positive real may be reached
Comparing $P(x,0)$ and $P(-x,0)$, we get $xf(x)-xf(-x)$ and si $f(-x)=-f(x)\forall x\ne 0$, still true if $x=0$ and so $f(x)$ is odd.
So any non negative real may be reached too.
And since $f(0)=0$, $f(x)$ is surjective.
Q.E.D.

3) $f(x)=x\forall x$
============
$P(x,0) \implies f(x^2)=xf(x)$
$P(0,y) \implies f(-f^2(y))=-y^2$
And so $f(x^2-f^2(y))=f(x^2)+f(-f^2(y))$

And so, since surjective : $f(x+y)=f(x)+f(y)\forall x\ge 0, y\le 0$
And so, since odd, $f(x+y)=f(x)+f(y)\forall x,y$

Then from $f(x^2)=xf(x)$, we get $f((x+1)^2)=(x+1)f(x+1)$ and so
$f(x^2)+2f(x)+f(1)=xf(x)+xf(1)+f(x)+f(1)$

And so $2f(x)=xf(1)+f(x)$ and $f(x)=ax$
Plugging this back in original equation, we get $a=1$

And so the unique solution $\boxed{f(x)=x}\ \forall x$

---

socrates
1872 posts

Mar 15, 2011, 4:51 pm PM #362

Problem 112

Let $f:\mathbb{N}^*\to\mathbb{N}^*$ be a function such that
$\bullet\, f(1)=1$
$\bullet\, 3f(n)f(2n+1)=f(2n)(1+3f(n)),$
$\bullet\, f(2n)<6f(n),\ \ \forall n\in\mathbb{N}^*.$

Find all pairs (k,l) such that $f(k)+f(l)=293.$

This post has been edited 1 time. Last edited by socrates, Mar 16, 2011, 1:21 am

---

mousavi
222 posts

Mar 15, 2011, 6:55 pm PM #363

$gcm(3f(n),1+3f(n))=1 \Longrightarrow 3f(n)\mid f(2n)<6f(n) \Longrightarrow 3f(n)=f(2n)$

$\Longrightarrow f(2n+1)=1+f(2n), 3f(n)=f(2n)$

$\Longrightarrow f(1)=1, f(2)=3, f(3)=4, f(4)=6, f(5)=7, f(6)=12$

for $f(n)$ we write n in base 2 and read in base 3, for example for obtaining $f(6)$: $6=(110)_2 \Longrightarrow (110)_3=12$

and easily it can be proved by induction.and $f(48)=324>293$ so some cases remain and it is easy to check.

---

socrates
1872 posts

Mar 16, 2011, 1:24 am PM #364

Problem 113

Find all functions $f:\mathbb{N}^*\to\mathbb{N}^*$ such that $f(2x+3y)=2f(x)+3f(y)+4$, for all integers $x,y\ge 1.$

This post has been edited 1 time. Last edited by socrates, Mar 20, 2011, 9:15 pm

---

pco
14052 posts

Mar 16, 2011, 1:41 am • 1 PM #365

> *socrates wrote:*
> **Problem 113**
>
> Find all functions $f:\mathbb{N}^*\to\mathbb{N}^*$ such that $f(2x+3y)=2f(x)+3f(y)+4$, for all integers $x,y\ge 1.$

I suppose that $N^*=\mathbb{N}$ is the set of natural numbers (positive integers)

Let $P(x,y)$ be the assertion $f(2x+3y)=2f(x)+3f(y)+4$

Subtracting $P(x+3,y)$ from $P(x,y+2)$, we get $2(f(x+3)-f(x))=3(f(y+2)-f(y))$

And so these two quantities are constant and multiple of 6 and so :
$f(x+3)=f(x)+3c$
$f(y+2)=f(y)+2c$
and (using $y=x+1$ in this last equation) : $f(x+3)=f(x+1)+2c$

and so $f(x+1)=f(x)+c$ and $f(x)=cx+d$

Plugging this in $P(x,y)$, we get $\boxed{f(x)=ax-1}$ for any real $a>1$ (the case $a=1$ must be excluded in order to have $f(1)\in\mathbb{N}$)

socrates
1872 posts

Mar 18, 2011, 11:18 pm PM #366

Problem 114

Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(m+f(n))=f(m+n)+2n+1,$ for all integers $m,n.$

This post has been edited 1 time. Last edited by socrates, Mar 20, 2011, 9:14 pm

pco
14052 posts

Mar 18, 2011, 11:40 pm PM #367

> *socrates wrote:*
> **Problem 114**
>
> Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(m+f(n))=f(m+n)+2n+1,$ for all integers $m,n.$

The equation may be written $f(m+(f(n)-n))=f(m)+2n+1$

And so $f(m+k(f(n)-n))=f(m)+k(2n+1)$
Setting $k=f(p)-p$, this becomes $f(m+(f(p)-p)(f(n)-n))=f(m)+(f(p)-p)(2n+1)$

And using symetry between n and p, we get $(f(p)-p)(2n+1)=(f(n)-n)(2p+1)$

And so $\frac{f(n)-n}{2n+1}=c$ and so $f(n)=n(2c+1)+c$ with $c=f(0)\in\mathbb{Z}$

Plugging this in original equation, we get $c=-1$ and so the solution $\boxed{f(x)=-x-1}$

myth_kill
5 posts

Mar 18, 2011, 11:45 pm PM #368

> *socrates wrote:*
> **Problem 114**
>
> Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(m+f(n))=f(m+n)+2n+1,$ for all integers $m,n.$

Click to reveal hidden text

> I am not sure if this is right approach but on solving this further, i am getting that this f(x) should satisfy
>
> f(x)-f(y)=x-y+2 .... (1)
>
> and no such function exists .. if thats the answer i will post my working on how i geot to the relation (1)
>
> someone please verify

edit: it seems i was wrong ,and pco posted the solution at same time so i did not see it.

socrates
1872 posts

Mar 20, 2011, 9:11 pm PM #369

Problem 115

Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(0)=2$ and $f(x+f(x+2y))=f(2x)+f(2y),$ for all integers $x,y.$

pco
14052 posts

Mar 20, 2011, 10:03 pm PM #370

> *socrates wrote:*
> **Problem 115**
>
> Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(0)=2$ and $f(x+f(x+2y))=f(2x)+f(2y),$ for all integers $x,y.$

Let $P(x,y)$ be the assertion $f(x+f(x+2y))=f(2x)+f(2y)$

$P(0,2)\implies f(2)=4$
$P(0,1)\implies f(4)=6$
And so, using induction with $P(0,n)$, we get $f(2n)=2n+2\forall n\ge 0$

Let $x\ge 0$: $P(2x,-x)\implies f(-2x)=f(2x+2)-f(4x)=(2x+4)-(4x+2)=-2x+2$

So $f(2x)=2x+2\forall x\in\mathbb{Z}$ and $P(x,y)$ may be written $f(x+f(x+2y))=2x+2y+4$

If $\exists$ odd $2a+1$ such that $f(2a+1)=2b$ is even, then :
$P(2a-2b+1,b)\implies 4b=4a+6$, which is impossible modulus 4

So $f(y)$ is odd for any odd y
Let then odd x : $f(x+2y)$ is odd and so $x+f(x+2y)$ is even and so $f(x+f(x+2y))=x+f(x+2y)+2$
So $x+f(x+2y)+2=2x+2y+4$ and $f(x+2y)=x+2y+2$

And so $\boxed{f(x)=x+2}$ $\forall x\in\mathbb{Z}$, which indeed is a solution.

Dijkschneier
131 posts

Mar 20, 2011, 10:06 pm PM #371

> *socrates wrote:*
> **Problem 115**
>
> Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(0)=2$ and $f(x+f(x+2y))=f(2x)+f(2y),$ for all integers $x,y.$

Let $P(x,y)$ be the assertion : $f(x+f(x+2y))=f(2x)+f(2y)$
$P(0,y)\Rightarrow f(f(2y))=f(2y)+2$, and by induction, it easily follows that for every even integer : $f(x)=x+2(1)$
Now take an odd x and suppose f(x+2y) is odd.
Then using (1) and P(x,y), it follows that f(x+2y)=(x+2y)+2, and since x+2y describes the odd numbers as long as y describes $\mathbb{Z}$, then for all integers : $f(x)=x+2$, which completes the proof.

---

pco
14052 posts

Mar 20, 2011, 10:08 pm PM #372

> *Dijkschneier wrote:*
>
> > *socrates wrote:*
> > **Problem 115**
> >
> > Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(0)=2$ and $f(x+f(x+2y))=f(2x)+f(2y)$, for all integers x,y.
>
> Let $P(x,y)$ be the assertion : $f(x+f(x+2y))=f(2x)+f(2y)$
> $P(0,y)\Rightarrow f(f(2y))=f(2y)+2$, and by induction, it easily follows that for every even integer : $f(x)=x+2(1)$
> Now take an odd x and suppose f(x+2y) is odd.
> Then using (1) and P(x,y), it follows that f(x+2y)=(x+2y)+2, and since x+2y describes the odd numbers as long as y describes $\mathbb{Z}$, then for all integers : $f(x)=x+2$, which completes the proof.

And what if $f(x+2y)$ is even for any x odd and any y ?

Dijkschneier
131 posts

Mar 20, 2011, 10:16 pm PM #373

You're right pco.
I'm sorry.

---

myth_kill
5 posts

Mar 22, 2011, 10:13 am PM #374

Problem 116

For which integers k does there exist a function f : N → Z such that
(a) f(1995) = 1996, and
(b) f(xy) = f(x) + f(y) + kf(gcd(x, y)) for all x, y ∈ N?

---

pco
14052 posts

Mar 22, 2011, 1:32 pm PM #375

> *myth_kill wrote:*
> Problem 116
>
> For which integers k does there exist a function f : N → Z such that
> (a) f(1995) = 1996, and
> (b) f(xy) = f(x) + f(y) + kf(gcd(x, y)) for all x, y ∈ N?

Let $P(x,y)$ be the assertion $f(xy)=f(x)+f(y)+kf(\gcd(x,y))$

$P(x,x)\implies f(x^2)=(k+2)f(x)$
$P(x^2,x)\implies f(x^3)=(2k+3)f(x)$
$P(x^3,x)\implies f(x^4)=(3k+4)f(x)$
$P(x^2,x^2)\implies f(x^4)=(k+2)^2f(x)$

So $(3k+4)f(x)=(k+2)^2f(x)$ and setting $x=1995$, we get $(k+2)^2=(3k+4)$ and so $k\in\{-1,0\}$

For $k=-1$, solutions exist. For example $f(n)=1996\,\forall n$.

For $k=0$, solutions exist. For example $f(1)=0$ and $f(\prod_{k=1}^n p_i^{n_i})=499\sum_{k=1}^n n_i$ (where p_i are distinct primes and $n_i\in\mathbb{N}$).

Hence the answer : $\boxed{k\in\{-1,0\}}$

Amir Hossein
4719 posts

Mar 22, 2011, 9:15 pm PM #376

Problem 117.

Find all surjective functions $f:\mathbb{N}\to\mathbb{N}$ such that $p\mid f(m)+f(n)\iff p\mid f(m+n)$ for all primes p.

PS. well, it's old, but very nice!

---

pco
14052 posts

Mar 23, 2011, 1:31 pm PM #377

> *amparvardi wrote:*
> **Problem 117.**
>
> Find all surjective functions $f:\mathbb{N}\to\mathbb{N}$ such that $p\mid f(m)+f(n)\iff p\mid f(m+n)$ for all primes p.
>
> PS. well, it's old, but very nice!

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=56&t=214717

Amir Hossein
4719 posts

Mar 23, 2011, 2:48 pm PM #378

:)

Problem 118.

Find all functions $f,g:\mathbb{Z}\to\mathbb{Z}$ such that g is bijective and

$$f(g(x)+y)=g(f(y)+x).$$

---

pco
14052 posts

Mar 23, 2011, 3:04 pm PM #379

> *amparvardi wrote:*
> :)
>
> **Problem 118.**
>
> Find all functions $f,g:\mathbb{Z}\to\mathbb{Z}$ such that g is bijective and
>
> $$f(g(x)+y)=g(f(y)+x).$$

We just need $g(x)$ injective and we dont need the restriction $\mathbb{Z}\to\mathbb{Z}$ (it's the same result for $\mathbb{R}\to\mathbb{R}$) :

Let $P(x,y)$ be the assertion $f(g(x)+y)=g(f(y)+x)$

$P(x,g(0))\implies f(g(x)+g(0))=g(f(g(0))+x)$
$P(0,g(x))\implies f(g(0)+g(x))=g(f(g(x)))$

So $g(f(g(0))+x)=g(f(g(x)))$ and, since $g(x)$ is injective : $f(g(x))=x+f(g(0))$

$P(x,0)\implies f(g(x))=g(f(0)+x)$ and so $g(x+f(0))=x+f(g(0))$ and so $g(x)=x+a$ for some a

Then $f(g(x))=x+f(g(0))$ becomes $f(x+a)=x+f(g(0))$ and so $f(x)=x+b$ for some b

Plugging back in original equation we get that these are solutions whatever are $a,b\in\mathbb{Z}$

Hence the answer :
$f(x)=x+b\,\forall x$ and for any $b\in\mathbb{Z}$ (or $\mathbb{R}$ is we move the problem in $\mathbb{R}$)
$g(x)=x+a\,\forall x$ and for any $a\in\mathbb{Z}$ (or $\mathbb{R}$ is we move the problem in $\mathbb{R}$)

Potla
1888 posts

Mar 23, 2011, 3:16 pm PM #380

I hope this has not been posted already. :)
Problem 119.(Belarus 1995)
Find all $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(f(x+y))=f(x+y)+f(x)f(y)-xy\quad\forall x,y\in\mathbb{R}.$$

Amir Hossein
4719 posts

Mar 23, 2011, 3:23 pm PM #381

Thanks for the solution, dear Patrick. :)

> *pco wrote:*
> ...$g(x+f(0))=x+f(g(0))$ and so $g(x)=x+a$ for some a

But how do you get this?

---

pco
14052 posts

Mar 23, 2011, 4:08 pm • 1 PM #382

> *amparvardi wrote:*
> Thanks for the solution, dear Patrick. :)
>
> > *pco wrote:*
> > ...$g(x+f(0))=x+f(g(0))$ and so $g(x)=x+a$ for some a
>
> But how do you get this?

We previously got $f(g(x))=x+f(g(0))$
Then $P(x,0)\implies f(g(x))=g(f(0)+x)$ and so $g(x+f(0))=x+f(g(0))$

From there we immediately get $g(x)=(x-f(0))+f(g(0))$ and so $g(x)=x+a$ for some $a=f(g(0))-f(0)$

---

pco
14052 posts

Mar 23, 2011, 5:01 pm PM #383

> *Potla wrote:*
> I hope this has not been posted already. :)
> **Problem 119.**(Belarus 1995)
> Find all $f:\mathbb{R}\to\mathbb{R}$ such that
>
> $$f(f(x+y))=f(x+y)+f(x)f(y)-xy\quad\forall x,y\in\mathbb{R}.$$

Let $P(x,y)$ be the assertion $f(f(x+y))=f(x+y)+f(x)f(y)-xy$
Let $f(0)=a$

$P(x,y)\implies f(f(x+y))=f(x+y)+f(x)f(y)-xy$
$P(x+y,0)\implies f(f(x+y))=f(x+y)+af(x+y)$
Subtracting, we get new assertion $Q(x,y)$: $af(x+y)=f(x)f(y)-xy$

$Q(x,-x)\implies a^2=f(x)f(-x)+x^2$
$Q(x,x)\implies af(2x)=f(x)^2-x^2$
$Q(-x,2x)\implies af(x)=f(-x)f(2x)+2x^2\implies a^2f(x)=f(-x)(f(x)^2-x^2)+2ax^2$
$\implies a^2f(x)^2=f(x)f(-x)(f(x)^2-x^2)+2ax^2f(x)=(a^2-x^2)(f(x)^2-x^2)+2ax^2f(x)$

And so $x^2(f(x)-a-x)(f(x)-a+x)=0$

So : $\forall x$, either $f(x)=a+x$, either $f(x)=a-x$ (the case $x=0$ is true too)

Suppose now that $f(x)=a+x$ for some x
$P(x,0)\implies f(a+x)=(a+1)x+a(a+1)$ and so :
either $(a+1)x+a(a+1)=a+(a+x)\iff a(x+a-1)=0$
either $(a+1)x+a(a+1)=a-(a+x)\iff (a+2)x+a(a+1)=0$
And so either $a=0$, either there are at most two such x : $1-a$ and $-\dfrac{a(a+1)}{a+2}$

Suppose now that $f(x)=a-x$ for some x
$P(x,0)\implies f(a-x)=-(a+1)x+a(a+1)$ and so :
either $-(a+1)x+a(a+1)=a+(a-x)\iff a(x-a+1)=0$
either $-(a+1)x+a(a+1)=a-(a-x)\iff (a+2)x-a(a+1)=0$
And so either $a=0$, either there are at most two such x : $a-1$ and $\dfrac{a(a+1)}{a+2}$

And so $a=0$ and either $f(x)=x$, either $f(x)=-x$
If $f(1)=1$, then $Q(x,1)\implies f(x)=x\,\forall x$ which indeed is a solution
If $f(1)=-1$, then $Q(x,-1)\implies f(x)=-x\,\forall x$ which is not a solution

Hence the answer : $\boxed{f(x)=x}\ \forall x$

RSM
736 posts

Mar 24, 2011, 1:42 am PM #384

> *Potla wrote:*
> I hope this has not been posted already. :)
> **Problem 119.**(Belarus 1995)
> Find all $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(f(x+y)) = f(x+y) + f(x)f(y) - xy \quad \forall x, y \in \mathbb{R}.$$

Putting $y=0$ we get $f(f(x)) = f(x)(a+1)$..............(1) where $f(0)=a$
Putting these in the equation we get $af(x+y) = f(x)f(y) - xy$.............(2).
Suppose, $a \neq 0$
$af(x+y+z) = f(x+y)f(z) - xz - yz$
$= \frac{f(x)f(y)f(z)}{a} - xy - yz - zx + xy - \frac{xyf(z)}{a}$
So $\frac{xyf(z)}{a} - xy = \frac{yzf(x)}{a} - yz$
$\frac{f(z)-a}{z} = \frac{f(x)-x}{a}$
So the solution in this case is $f(x) = kx + a$ for some constant k.
Putting this in (2) we get $k=1$ or $k=-1$
Then putting these in (1) gives no solution.
So $f(0)=0$ which gives the following solution:-
$f(x)=x$

---

mousavi
222 posts

Mar 27, 2011, 11:18 am PM #385

problem 120

Find all numbers $d \in [0,1]$ such that if $f(x)$ is an arbitrary continues function with domain $[0,1]$ and $f(0)=f(1)$, there exist number $x_0 \in [0, 1-d]$ such that $f(x_0) = f(x_0+d)$

RSM
736 posts

Mar 27, 2011, 2:28 pm PM #386

[Content deleted]

This post has been edited 1 time. Last edited by RSM, Mar 27, 2011, 11:51 pm

---

pco
14052 posts

Mar 27, 2011, 4:26 pm PM #387

> *RSM wrote:*
>
> > *mousavi wrote:*
> > **problem 120**
> >
> > Find all numbers $d \in [0,1]$ such that if $f(x)$ is an arbitrary continues function with domain $[0,1]$ and $f(0)=f(1)$, there exist number $x_0 \in [0,1-d]$ such that $f(x_0) = f(x_0+d)$
>
> Suppose, a is any value of x such that $f(a)$ is maximum.
> So for any $y < a$ we can find a z such that $f(z) = f(y)$.
> So if we choose $d = z - y$ then $y \in [0, 1-z+y]$
> So $d \in [0, a)$

I think you misunderstood the problem : d must fit whatever is the function $f(x)$. (the only thing which depends on $f(x)$ is x_0)

I think that the result is $d \in \{0\} \cup \bigcup_{n\in\mathbb{N}} \{\frac{1}{n}\}$

---

pco
14052 posts

Mar 27, 2011, 10:00 pm • 1 PM #388

> *mousavi wrote:*
> **problem 120**
>
> Find all numbers $d \in [0,1]$ such that if $f(x)$ is an arbitrary continues function with domain $[0,1]$ and $f(0)=f(1)$, there exist number $x_0 \in [0,1-d]$ such that $f(x_0) = f(x_0+d)$

1) $d=0$ fits
=========
Just choose $x_0 = 0$:)

2) $d = \frac{1}{n}$ fits
==========
Let $g(x) = f(x+d) = f(x+\frac{1}{n})$
Let the sequence $a_k = f(\frac{k}{n})$
$a_0 = a_n = f(0)$ and so :
either $\exists k \in [0, n-1]$ such that $a_k = a_{k+1}$ and just choose $x_0 = \frac{k}{n}$
either $a_k \neq a_{k+1} \forall k \in [0, n-1]$ and then :

If $a_1 > a_0$, the sequence cannot be increasing for any k and then $\exists k \in [0, n-1]$ such that $a_k < a_{k+1}$ and $a_{k+2} < a_{k+1}$ and then :
$f(\frac{k}{n}) < g(\frac{k}{n})$ and $g(\frac{k}{n}+d) < f(\frac{k}{n}+d)$ and so $\exists x_0 \in (\frac{k}{n}, \frac{k}{n}+d)$ such that $f(x_0) = g(x_0)$ (since continuous).

If $a_1 < a_0$, the sequence cannot be decreasing for any k and then $\exists k \in [0, n-1]$ such that $a_k > a_{k+1}$ and $a_{k+2} > a_{k+1}$ and then :
$f(\frac{k}{n}) > g(\frac{k}{n})$ and $g(\frac{k}{n}+d) > f(\frac{k}{n}+d)$ and so $\exists x_0 \in (\frac{k}{n}, \frac{k}{n}+d)$ such that $f(x_0) = g(x_0)$ (since continuous).
Q.E.D

3) no other d fit
==========
Let $d \in (0,1)$ and n, r such that $1 = nd + r$ with n non negative integer and $r \in (0,d)$
Choose any $u > 0$ and any continuous $h(x)$ defined over $[0,d]$ such that :
$h(0)=0$
$h(r)=nu$
$h(d)=-u$

And define $f(x)$ in a recursive manner :
$\forall x \in [0,d]$: $f(x) = h(x)$
$\forall x > d$: $f(x) = f(x-d) - u$

We have :
$f(x)$ continuous
$f(0)=f(1)=0$
And the equation $f(x) = f(x+d)$ is equivalent to $f(x) = f(x) - u$ and has no solution.

Q.E.D.

Hence the result : $\boxed{d \in \{0\} \cup \left(\bigcup_{n\in\mathbb{N}}\{\frac{1}{n}\}\right)}$

Dijkschneier
131 posts

Mar 28, 2011, 1:48 am PM #389

I misread problem 113 and solved (well, I guess) the following problem instead :
Problem 121 :
Find all $f:\mathbb{N}\to\mathbb{N}$ such that f(2x+3y)=2f(x)+3f(y) for all $x,y\ge 1$.

pco
14052 posts

Mar 28, 2011, 2:23 am PM #390

> Dijkschneier wrote:
> I misread problem 113 and solved (well, I guess) the following problem instead :
> **Problem 121 :**
> Find all $f:\mathbb{N}\to\mathbb{N}$ such that f(2x+3y)=2f(x)+3f(y) for all $x,y\ge 1$.

Same method as for the original problem 113 :
Let $P(x,y)$ be the assertion $f(2x+3y)=2f(x)+3f(y)$

Subtracting $P(x+3,y)$ from $P(x,y+2)$, we get $2(f(x+3)-f(x))=3(f(y+2)-f(y))$

And so these two quantities are constant and multiple of 6 and so :
$f(x+3)=f(x)+3c$
$f(y+2)=f(y)+2c$
and (using $y=x+1$ in this last equation) : $f(x+3)=f(x+1)+2c$

and so $f(x+1)=f(x)+c$ and $f(x)=cx+d$

Plugging this in $P(x,y)$, we get $\boxed{f(x)=ax}$ for any $a\in\mathbb{N}$

mousavi
222 posts

Mar 28, 2011, 5:53 pm PM #391

problem 122

Find all functions $f:\mathbb{R}\longrightarrow\mathbb{R}$

$f(x+f(xy))=f(x+f(x)f(y))=f(x)+xf(y)$

Dijkschneier
131 posts

Mar 29, 2011, 1:33 am PM #392

> mousavi wrote:
> **problem 122**
>
> Find all functions $f:\mathbb{R}\longrightarrow\mathbb{R}$
>
> $f(x+f(xy))=f(x+f(x)f(y))=f(x)+xf(y)$

Please add a continuity condition, because I have a solution in this case

pco
14052 posts

Mar 29, 2011, 5:12 pm PM #393

> mousavi wrote:
> **problem 122**
>
> Find all functions $f:\mathbb{R}\longrightarrow\mathbb{R}$
>
> $f(x+f(xy))=f(x+f(x)f(y))=f(x)+xf(y)$

Let $P(x,y)$ be the assertions $f(x+f(xy))=f(x+f(x)f(y))=f(x)+xf(y)$

$f(x)=0\forall x$ is a solution and let us from now look for non allzero solutions.
Let u such that $f(u)\ne 0$

1) $f(x)=0\iff x=0$
==================
$P(-1,-1)\implies f(-1+f(1))=f(-1+f(-1)^2)=0$ and so $\exists v$ such that $f(v)=0$
$P(v,u)\implies 0=vf(u)$ and so $v=0$
Q.E.D.

2) $f(n)=n\forall n\in\mathbb{N}$
==============
$P(-1,-1)\implies f(-1+f(1))=f(-1+f(-1)^2)=0$ and so, using 1) : $-1+f(1)=-1+f(-1)^2=0$
So $f(1)=1$

$P(1,x)\implies f(1+f(x))=1+f(x)$ and so from $f(1)=1$, we get $f(n)=n\forall n\in\mathbb{N}$
Q.E.D.

3) $f(-1)=-1$
===========
$P(-1,-1)\implies f(-1+f(1))=f(-1+f(-1)^2)=0$ and so, using 1) : $-1+f(1)=-1+f(-1)^2=0$
So $f(-1)=\pm 1$
If $f(-1)=1$, then :

$P(\frac{1}{n},n)\implies f(\frac{1}{n}+1)=f(\frac{1}{n})+\frac{1}{n}f(n)$
$P(\frac{1}{n},-n)\implies f(\frac{1}{n}+1)=f(\frac{1}{n})+\frac{1}{n}f(-n)$
And so $f(-n)=f(n)=n$
Then $P(-1,2)\implies f(-1+f(-2))=f(-1+f(-1)f(2))=f(-1)-f(2)\implies 1=1=-1$, contradiction
So $f(-1)=-1$
Q.E.D.

4) $f(x)$ is injective
===========
If $f(y_1)=f(y_2)$ and $y_2=0$ then $f(y_1)=0$ and 1) gives $y_1=y_2=0$
If $f(y_1)=f(y_2)$ and $y_2\ne 0$, let $a=\frac{y_1}{y_2}$
$P(y_2,1)\implies f(y_2+f(y_2))=f(y_2)+y_2$
$P(y_2,a)\implies f(y_2+f(y_1))=f(y_2)+y_2f(a)$
And so $f(a)=1$

$P(a,1) \implies f(a+1)=a+1$

Notice that if $f(x)=x$, then :
$P(1,x) \implies f(x+1)=x+1$
$P(-1,x) \implies f(-1+f(-1)f(x))=f(-1)-f(x) \implies f(-x-1)=-x-1$

Applying this to $f(a+1)=a+1$, we get
$f(-a-2)=-a-2$ (second property)
$f(-a-1)=-a-1$ (then first property)
$f(a)=a$ (then second property)
And so $a=1$
And so $y_1=y_2$
Q.E.D.

5) $f(xy)=f(x)f(y)$
===========
This is an immediate consequence of $f(x+f(xy))=f(x+f(x)f(y))$ and $f(x)$ injective

6) $f(x)=x\forall x$
==========
Let $x\neq 0$
We trivially have from 5) that $f(\frac{1}{x})=\frac{1}{f(x)}$

Then $P(\frac{1}{x},x) \implies f(\frac{1}{x}+1)=\frac{1}{f(x)}+\frac{f(x)}{x}$

Then $f(x+1)=f(x(\frac{1}{x}+1))=f(x)f(\frac{1}{x}+1)=1+\frac{f(x)^2}{x}$

But $P(x,\frac{1}{x}) \implies f(x+1)=f(x)+xf(\frac{1}{x})=f(x)+\frac{x}{f(x)}$

So $1+\frac{f(x)^2}{x}=f(x)+\frac{x}{f(x)}$

$\implies xf(x)+f(x)^3=xf(x)^2+x^2$

$\implies (f(x)^2+x)(f(x)-x)=0$

And so $f(x)=x\forall x>0$
And since $f(-x)=f((-1)x)=f(-1)f(x)=-f(x)$, we get $f(x)=x\forall x$ which indeed is a solution

7) Synthesis of solutions
=================
And so we got two solutions :
$f(x)=0\forall x$
$f(x)=x\forall x$

Dijkschneier
131 posts

Mar 29, 2011, 7:47 pm PM #394

I am very impressed by your proof of the injection.
Very good.

---

socrates
1872 posts

Mar 30, 2011, 12:01 am PM #395

Problem 123

Let $f:[0,1]\to\mathbb{R}_+^*$ be a continous function such that $f(x_1)f(x_2)...f(x_n)=e$,
for all $n\in\mathbb{N}^*$ and for all $x_1,x_2,...,x_n\in[0,1]$ with $x_1+x_2+...+x_n=1.$

Prove that $f(x)=e^x,\ x\in[0,1].$

---

pco
14052 posts

Mar 30, 2011, 12:43 am PM #396

> *socrates wrote:*
> **Problem 123**
>
> Let $f:[0,1]\to\mathbb{R}_+^*$ be a continous function such that $f(x_1)f(x_2)...f(x_n)=e$,
> for all $n\in\mathbb{N}^*$ and for all $x_1,x_2,...,x_n\in[0,1]$ with $x_1+x_2+...+x_n=1.$
>
> Prove that $f(x)=e^x,\ x\in[0,1].$

Choosing $x_i=\frac{1}{n}$, we get $f(\frac{1}{n})^n=e$ and so $f(\frac{1}{n})=e^{\frac{1}{n}}$

Let $q>p\geq 1$: choosing $n=q-p+1$ and $x_1=x_2=...=x_{n-1}=\frac{1}{q}$ and $x_n=\frac{p}{q}$, we get :
$f(\frac{1}{q})^{q-p}f(\frac{p}{q})=e$ and so $e^{\frac{q-p}{q}}f(\frac{p}{q})=e$ and so $f(\frac{p}{q})=e^{\frac{p}{q}}$

And so $f(x)=e^x\forall x\in\mathbb{Q}\cap(0,1)$ and continuity implies $f(x)=e^x\forall x\in[0,1]$ which indeed is a solution

---

aktyw19
1315 posts

Apr 2, 2011, 12:23 pm • 1 PM #397

Let $f:\mathbb{R}\to\mathbb{R}$ be a continuos function, and $a,b,c,d\in\mathbb{R}$ with $ac(a^2-1)(a^2-c^2)\neq 0$. Under these conditions, solve functioal equation of $f:af(x+b)=f(cx+d)$.

---

mousavi
222 posts

Apr 8, 2011, 11:34 pm PM #398

problem 125

Find all functions $f:R\to R$:

$f(xy)f(f(x)-f(y))=(x-y)f(x)f(y)$

---

pco
14052 posts

Apr 10, 2011, 12:33 pm PM #399

> *mousavi wrote:*
> **problem 125**
>
> Find all functions $f:R\to R$:
>
> $f(xy)f(f(x)-f(y))=(x-y)f(x)f(y)$

Where is this problem coming from ?

There are infinitely many solutions but I did not succeed up to now finding all of them.

Some solutions :

1) trivial solution $f(x) = x \forall x$

2) trivial solution $f(x) = 0 \forall x$

3) $f(a) = b$ and $f(x) = 0 \forall x \neq a$ where a is any nonzero real and $b \neq \pm a$

4) $f(x) = x \forall x \in \mathbb{Q}$ and $f(x) = 0$ anywhere else

5) $f(x) = x \forall x \in \mathbb{Q}[\sqrt{2}]$ and $f(x) = 0$ anywhere else

In fact 4) and 5) may be merged in :

$f(x) = x \forall x \in \mathbb{K}$ and $f(x) = 0$ anywhere else where $\mathbb{K}$ is any subfield of $\mathbb{R}$

... and a lot of other.

I wonder in what contest such a problem could have been asked :

---

goodar2006
1344 posts

Apr 14, 2011, 11:10 pm PM #400

> *pco wrote:*
>
> > *mousavi wrote:*
> > **problem 125**
> >
> > Find all functions $f: R \to R$:
> >
> > $f(xy)f(f(x) - f(y)) = (x - y)f(x)f(y)$
>

> Where is this problem coming from ?
>
> There are infinitely many solutions but I did not succeed up to now finding all of them.
>
> **Some solutions :**
>
> 1) trivial solution $f(x) = x \forall x$
>
> 2) trivial solution $f(x) = 0 \forall x$
>
> 3) $f(a) = b$ and $f(x) = 0 \forall x \neq a$ where a is any nonzero real and $b \neq \pm a$
>
> 4) $f(x) = x \forall x \in \mathbb{Q}$ and $f(x) = 0$ anywhere else
>
> 5) $f(x) = x \forall x \in \mathbb{Q}[\sqrt{2}]$ and $f(x) = 0$ anywhere else
>
> In fact 4) and 5) may be merged in :
>
> $f(x) = x \forall x \in \mathbb{K}$ and $f(x) = 0$ anywhere else where $\mathbb{K}$ is any subfield of $\mathbb{R}$
>
> ... and a lot of other.
>
> I wonder in what contest such a problem could have been asked :

dear pco,

It is ISL 2001, A4, for a solution see here. I agree with you that the solutions are not very beautiful .

---

goodar2006
1344 posts

Apr 14, 2011, 11:28 pm PM #401

I'll post the next problem

Problem 126

find all functions f from the set $\mathbb{R}$ of real numbers into $\mathbb{R}$ which satisfy for all $x, y, z \in \mathbb{R}$ the identity

$f(f(x) + f(y) + f(z)) = f(f(x) - f(y)) + f(2xy + f(z)) + 2f(xz - yz)$

---

pco
14052 posts

Apr 14, 2011, 11:28 pm PM #402

> *goodar2006 wrote:*
> dear pco,
>
> It is ISL 2001, A4, for a solution see here. I agree with you that the solutions are not very beautiful .

Thanks for your remark, but the pointed problem is not the same than current problem Mousavi posted :

Pointed problem : $f(xy)(f(x) - f(y)) = (x - y)f(x)f(y)$

Current problem : $f(xy)f(f(x) - f(y)) = (x - y)f(x)f(y)$

---

pco
14052 posts

Apr 15, 2011, 6:52 pm PM #403

> *goodar2006 wrote:*
> I'll post the next problem
>
> **Problem 126**
>
> find all functions f from the set $\mathbb{R}$ of real numbers into $\mathbb{R}$ which satisfy for all $x, y, z \in \mathbb{R}$ the identity
>
> $f(f(x) + f(y) + f(z)) = f(f(x) - f(y)) + f(2xy + f(z)) + 2f(xz - yz)$

Still waiting for Musavi's answer about problem 125.

Solution for problem 126

---

socrates
1872 posts

Apr 17, 2011, 2:30 am PM #404

Problem 127 (Greek TST 2011)

Find all functions $f, g : \mathbb{Q} \to \mathbb{Q}$ such that :

$f(g(x) - g(y)) = f(g(x)) - y$ and

$g(f(x) - f(y)) = g(f(x)) - y$

for each $x, y \in \mathbb{Q}$.

RSM
736 posts

Apr 17, 2011, 2:51 am PM #405

> socrates wrote:
> **Problem 127** (Greek TST 2011)
>
> Find all functions $f, g : \mathbb{Q} \to \mathbb{Q}$ such that :
>
> $f(g(x) - g(y)) = f(g(x)) - y$ and
>
> $g(f(x) - f(y)) = g(f(x)) - y$
>
> for each $x, y \in \mathbb{Q}$.

f and g are bijective:-

Putting $x = y$ in the equations we get $f(g(x)) = x + f(0)$ and $g(f(x)) = x + g(0)$
$g(x_1) = g(x_2) \implies x_1 + f(0) = x_2 + f(0) \implies x_1 = x_2$. So f is one-one.
Clearly f is onto. So f is bijective and similarly g is bijective.

f(0)=g(0)=0

$f(g(x) - g(y)) = f(0) + x - y$
Putting $y = 0$ we get $f(g(x) - g(0)) = f(g(x))$
So $g(x) - g(0) = g(x)$ and so $g(0) = 0$ and similarly $f(0) = 0$

g(x)=$f^{-1}(x)$

because $f(g(x)) = x$

f and f^{-1} are Cauchy's Function

$f(f^{-1}(x) - f^{-1}(y)) = x - y$
$f^{-1}(x) - f^{-1}(y) = f^{-1}(x - y)$
So f^{-1} is Cauchy's Function. Similarly f is a Cauchy's Function.

The solution of Cauchy's Equation for domain=$\mathbb{Q}$ is $f(x) = cx$ and so
$g(x) = \frac{x}{c}$ where $c \neq 0$.

---

socrates
1872 posts

Apr 18, 2011, 2:00 am PM #406

> socrates wrote:
> **Problem 101**
>
> A variation on the unsolved **Problem 40**:
>
> Let $f : R \to R$ be a function such that $f(xy) + f(x - y) \geq f(x + y)$ for all real numbers x, y.
> Prove that $f(x) \geq 0$, for each $x \in \mathbb{R}$.

Also: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=38&t=160835&

Problem 128 (Greek TST 2010)

Determine all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ such that $f\left(\frac{f(x)}{f(y)}\right) = \frac{1}{y} \cdot f(f(x))$, for each $x, y \in \mathbb{R}^*$
and are strictly monotonic on $(0, +\infty)$.

---

pco
14052 posts

Apr 18, 2011, 12:09 pm PM #407

> socrates wrote:
> **Problem 128** (Greek TST 2010)
>
> Determine all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ such that $f\left(\frac{f(x)}{f(y)}\right) = \frac{1}{y} \cdot f(f(x))$, for each $x, y \in \mathbb{R}^*$
> and are strictly monotonic on $(0, +\infty)$.

Still waiting for Musavi's answer about problem 125.

Solution for problem 128

---

filipbitola
124 posts

Apr 18, 2011, 5:20 pm • 1 PM #408

[moderator edit: please hide the long posts.]

Stephen wrote:

I know this is a little bit old, but from the second line onward I have a much shorter solution.
I'll upload an attachment and you will find the solution on the last page in the Note

Attachments:
Cauchy-Filip-Predrag Theorem functional equation.pdf (458kb)

---

filipbitola
124 posts

Apr 19, 2011, 1:34 am PM #409

Now that problem 128 is solved, I'd like to propose problem 129:
Problem 129: Find all functions, $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that:
$x^2 f(f(x) + f(y)) = (x + y) f(y f(x))$
for all x,y in $\mathbb{R}^+$

pco
14052 posts

Apr 19, 2011, 2:41 am PM #410

> *filipbitola wrote:*
> Now that problem 128 is solved, I'd like to propose problem 129:
> **Problem 129:** Find all functions, $f:\mathbb{R}^+\to\mathbb{R}^+$ such that:
> $x^2f(f(x)+f(y))=(x+y)f(yf(x))$
> for all x,y in $\mathbb{R}^+$

Still waiting for Musavi's answer about problem 125.

Solution for problem 129

for each pair of nonnegative reals x and y. Prove that $f(x)\le\frac{x^2}{2}$ for all nonnegative reals x.

Source: Adapted from Chinese 1990.

Problem 131: Let $\mathbb{N}$ denote the set of "positive" integers. Fix $c\in\mathbb{N}$. Does there exists a function $f:\mathbb{N}\to\mathbb{N}$ such that $f^{\left[f(n)\right]}(n)=n+c$ for every $n\in\mathbb{N}$? In this context, $f^{[0]}$ is the identity and $f^{[k]}:=f\circ f^{[k-1]}$ for all $k\in\mathbb{N}$.

This post has been edited 3 times. Last edited by Batominovski, Apr 20, 2011, 5:07 am

pco
14052 posts

Apr 19, 2011, 1:04 pm PM #412

> *Batominovski wrote:*
> **Problem 131:** Let $\mathbb{N}$ denote the set of "positive" integers. Does there exists a function $f:\mathbb{N}\to\mathbb{N}$ such that $f^{\left[f(n)\right]}(n)=n$ for every $n\in\mathbb{N}$? In this context, $f^{[0]}$ is the identity and $f^{[k]}:=f\circ f^{[k-1]}$ for all $k\in\mathbb{N}$.

Yes : $f(n)=n\forall n\in\mathbb{N}$

Batominovski
1602 posts

Apr 19, 2011, 1:43 pm PM #413

> *pco wrote:*
>
> > *Batominovski wrote:*
> > **Problem 131:** Let $\mathbb{N}$ denote the set of "positive" integers. Does there exists a function $f:\mathbb{N}\to\mathbb{N}$ such that $f^{\left[f(n)\right]}(n)=n$ for every $n\in\mathbb{N}$? In this context, $f^{[0]}$ is the identity and $f^{[k]}:=f\circ f^{[k-1]}$ for all $k\in\mathbb{N}$.
>
> Yes : $f(n)=n\forall n\in\mathbb{N}$

Haha. This time I made a fatal mistake in my problem statement. Please take a look at it again.

pco
14052 posts

Apr 19, 2011, 4:52 pm PM #414

> *Batominovski wrote:*
> **Problem 130:** Let $f:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ be a function which is bounded on the interval $[0,1]$ and obeys the inequality
> $$f(x)f(y)\le x^2f\left(\frac{y}{2}\right)+y^2f\left(\frac{x}{2}\right)$$
> for each pair of nonnegative reals x and y. Prove that $f(x)\le\frac{x^2}{2}$ for all nonnegative reals x.

Setting $x=y$ in the inequality, we get $2x^2f(\frac{x}{2})\ge f(x)^2$

Setting $g(x)=\frac{2f(x)}{x^2}$ this becomes $g(\frac{x}{2})\ge g(x)^2$ and so $g(\frac{x}{2^n})\ge g(x)^{2^n}$

Suppose then that $g(u)=a>1$ for some u, then $g(\frac{u}{2^n})\ge a^{2^n}$

And so $f(\frac{u}{2^n})\ge u^2\frac{a^{2^n}}{2^{2n+1}}$

Setting $n\to+\infty$ in the above inequation, we get thet LHS is clearly unbounded, and so contradiction with the fact that $f(x)$ is bounded on $[0,1]$

So $g(x)\le 1\forall x$

So $f(x)\le\frac{x^2}{2}\forall x$
Q.E.D.