ilikemaths
50 posts

May 23, 2010, 9:35 pm • 21 PM #1

Let's start a marathon on functional equations:
When you solve a problem, you should post a new one.

Here's problem 1:
Find all functions $f:\mathbb{Q}_{>0}\to\mathbb{Q}_{>0}$ that satisfy:
$f(x+1)=f(x)+1\ \forall x\in\mathbb{Q}_{>0}$ and
$f(x^2)=f(x)^2\ \forall x\in\mathbb{Q}_{>0}$.

[moderator edit: stickied in Pre-Olympiad forum.]

Pain rinne...
1581 posts

May 23, 2010, 10:28 pm • 5 PM #2

> ilikemaths wrote:
> Let's start a marathon on functional equations:
> When you solve a problem, you should post a new one.
>
> **Problem 1:** Find all functions $f:\mathbb{Q}_+\to\mathbb{Q}_+$ that satisfy:
>
> (1).$f(x+1)=f(x)+1\ \ ,\ \forall x\in\mathbb{Q}_+$ and
>
> (2).$f(x^2)=f(x)^2\ \ ,\ \forall x\in\mathbb{Q}_+$.

I like this idea :)

Problem 1

Problem 2: Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x^3)-f(y^3)=(x^2+xy+y^2)(f(x)-f(y))\ ,\ (\forall)x,y\in\mathbb{R}$$

---

mahanmath
1356 pos...

May 23, 2010, 11:04 pm • 4 PM #3

> Quote:
> **Problem 2:** Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:
>
> $$f(x^3)-f(y^3)=(x^2+xy+y^2)(f(x)-f(y))\ ,\ (\forall)x,y\in\mathbb{R}$$

Problem 2

Problem 3 : Find all the continuous functions $f:\mathbb{R}\mapsto\mathbb{R}$ such that $\forall x,y\in\mathbb{R}$:

$(1+f(x)f(y))f(x+y)=f(x)+f(y)$

Pain rinne...
1581 posts

May 23, 2010, 11:35 pm • 2 PM #4

> mahanmath wrote:
> **Problem 3** : Find all the continuous functions $f:\mathbb{R}\mapsto\mathbb{R}$ such that $\forall x,y\in\mathbb{R}$:
>
> $(1+f(x)f(y))f(x+y)=f(x)+f(y)$

It's a too recent question: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=350104&start=0&hilit=continuous .

---

ilikemaths
50 posts

May 23, 2010, 11:39 pm • 3 PM #5

Then pose a new problem.

Pain rinne...
1581 posts

May 23, 2010, 11:43 pm • 1 PM #6

Problem 4: Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x^3+y^3)=xf(x^2)+yf(y^2)\ ,\ (\forall)x,y\in\mathbb{R}$$

---

ilikemaths
50 posts

May 23, 2010, 11:55 pm • 2 PM #7

$x=y=0$ yields $f(0)=0$.
$x=0$ yields $f(y^3)=yf(y^2)$, so the given functional equation reduces to:

$f(x^3+y^3)=f(x^3)+f(y^3)$.
Setting $a=x^3, b=y^3$ gives:
$f(a+b)=f(a)+f(b)$, which is a Cauchy-equation with solutions:
$f(x)=0$ and $f(x)=cx$ for some $c\in\mathbb{R}$.
So we have two possible functions:
$f(x)=0$ and $f(x)=cx$ for some $c\in\mathbb{R}$ and a quick check tells us that both functions satisfy.

New problem:
find all functions $f:\mathbb{R}_{>0}\to\mathbb{R}$ satisfying:
$f(x+y)-f(y)=\frac{x}{y(x+y)}$

---

mahanmath
1356 pos...

May 24, 2010, 3:32 am • 1 PM #8

Problem 5

Problem 6.Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$f\big(x+yf(x)\big)+f\big(xf(y)-y\big)=f(x)-f(y)+2xy$

---

mahanmath
1356 pos...

May 24, 2010, 5:52 am • 1 PM #9

> *ilikemaths wrote:*
> $f(a+b)=f(a)+f(b)$, which is a Cauchy-equation with solutions:
> $f(x)=0$ and $f(x)=cx$ for some $c\in\mathbb{R}$.
> So we have two possible functions:
> $f(x)=0$ and $f(x)=cx$ for some $c\in\mathbb{R}$ and a quick check tells us that both functions satisfy.
> $

I`m not sure but I think Cauchy-equation just solve continuous functions. Am I right ?

Amir Hos...
4728 pos...

May 24, 2010, 3:05 pm • 1 PM #10

> *mahanmath wrote:*
> WLOG assume $f(1)=-1$

Can we do this ?
How Without Loss Of Generality ?

Dumel
190 posts

May 24, 2010, 5:51 pm • 2 PM #11

> *mahanmath wrote:*
> **Problem 6**.Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:
> $f\big(x+yf(x)\big)+f\big(xf(y)-y\big)=f(x)-f(y)+2xy$

solution

Post a bit harder problems, please! :)
Problem 7
Find the least possible value of $f(1998)$ where $f:\mathbb{N}\to\mathbb{N}$ satisfies
$f(n^2f(m))=m(f(n))^2$

---

ilikemaths
50 posts

May 24, 2010, 7:23 pm • 1 PM #12

Dumel, your solution isn't correct:
$f(x)=x$ also satisfies!

Pain rinne...
1581 posts

May 24, 2010, 9:23 pm • 1 PM #13

> *Dumel wrote:*
> we get $f(x)=0$

$f(x)=0$ is not a solution . :)

Dumel
190 posts

May 25, 2010, 2:50 am • 1 PM #14

oh what a terrible mistake :)
At this moment I don't know how to solve this problem.

---

mahanmath
1356 pos...

May 26, 2010, 12:22 pm • 1 PM #15

> *amparvardi wrote:*
>
> > *mahanmath wrote:*
> > WLOG assume $f(1)=-1$

> Can we do this ?
> How Without Loss Of Generality ?

As I said ~~ SAME AS PROBLEM **2** ~~

---

arshakus
746 posts

May 26, 2010, 1:18 pm • 1 PM #16

problem 7
it is the problem of shortlist in1998, try to go in this link!
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=124426&sid=d54a2e73626da06fb572b3022d2bc388#p124426

---

arshakus
746 posts

May 26, 2010, 1:22 pm • 1 PM #17

problem 8 try this one....
$f: R+->R+$
$f(x+f(y))=f(x+y)+f(y)$, for every $x, y, fromR+$

---

arshakus
746 posts

May 26, 2010, 1:22 pm • 1 PM #18

find $f(x)-?$

---

mahanmath
1356 pos...

May 26, 2010, 10:38 pm • 3 PM #19

> *arshakus wrote:*
> **problem 8** try this one....
> $f: R+->R+$
> $f(x+f(y))=f(x+y)+f(y)$, for every $x, y, fromR+$

What about problem 6 ?!

BTW , I`m waiting for **pco** and his nice solutions !

---

mahanmath
1356 pos...

May 27, 2010, 7:59 pm • 3 PM #20

> *mahanmath wrote:*
> **Problem 6**.Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:
>
> $f\big(x+yf(x)\big)+f\big(xf(y)-y\big)=f(x)-f(y)+2xy$

OK , It seems hard . So I give a hint.

Mohamma...
58 posts

May 27, 2010, 8:01 pm • 2 PM #21

problem 4 :

> *ilikemaths wrote:*
> ...
> $f(a+b)=f(a)+f(b)$, which is a Cauchy-equation with solutions:
> $f(x)=0$ and $f(x)=cx$ for some $c\in\mathbb{R}$.
> ...

this is not correct !
you've only proved that $f(a)+f(b)=f(a+b)$ and $f(a^3)=af(a^2)$
now let $g(x)=f(x)/f(1)$
we have $g(x+q)=g(x)+q$ for every $q\in Q$
we have :
$g((x+q)^3)=(x+q)g((x+q)^2)=(x+q)(g(x^2)+2qg(x)+q^2)$
and also $g((x+q)^3)=g(x^3+3qx^2+3q^2x+q^3)=g(x^3)+3qg(x^2)+3q^2g(x)+q^3$
so $2g(x^2)+qg(x)=qx+2xg(x)$ or $q(g(x)-x)+2g(x^2)-2xg(x)=0$
which is true for every $q\in Q$ but it's linear and can't have more than one
zero unless $g(x)-x=0$ so $f(x)=cx$ for every $x\in R$

---

pco
12955 po...

May 29, 2010, 12:34 pm • 7 PM #22

> *mahanmath wrote:*
> **Problem 6**.Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:
>
> $f\big(x+yf(x)\big)+f\big(xf(y)-y\big)=f(x)-f(y)+2xy$

> *mahanmath wrote:*
> What about problem 6 ?!
>
> BTW , I`m waiting for **pco** and his nice solutions !

Here is a rather heavy solution :

Let $P(x,y)$ be the assertion $f(x+yf(x))+f(xf(y)-y)=f(x)-f(y)+2xy$

1) $f(x)$ is an odd function and $f(x)=0 \iff x=0$
======================================
$P(0,0) \implies f(0)=0$
$P(0,x) \implies f(-x)=-f(x)$

Suppose $f(a)=0$. Then $P(a,a) \implies 0=2a^2 \implies a=0$ and so $f(x)=0 \iff x=0$
Q.E.D

2) $f(x)$ is additive
==============

Let then $x\neq 0$ such that $f(x)\neq 0$:
$P(x,\frac{x+y}{f(x)}) \implies f(2x+y)+f(xf(\frac{x+y}{f(x)})-\frac{x+y}{f(x)})=f(x)-f(\frac{x+y}{f(x)})+2x\frac{x+y}{f(x)}$

$P(\frac{x+y}{f(x)},-x) \implies -f(xf(\frac{x+y}{f(x)})-\frac{x+y}{f(x)})-f(y)=f(\frac{x+y}{f(x)})+f(x)-2x\frac{x+y}{f(x)}$

Adding these two lines, we get : $f(2x+y)=2f(x)+f(y)$ which is obviously still true for $x=0$ and so :

New assertion $Q(x,y)$: $f(2x+y)=2f(x)+f(y)\ \forall x,y$

$Q(x,0) \implies f(2x)=2f(x)$ and so $Q(x,y)$ becomes $f(2x+y)=f(2x)+f(y)$ and so $f(x+y)=f(x)+f(y)$ and $f(x)$ is additive.
Q.E.D.

3) $f(x)$ solution implies $-f(x)$ solution and so wlog consider from now $f(1)\geq 0$
==
$P(y,x) \implies f(y+xf(y))+f(yf(x)-x)=f(y)-f(x)+2xy$
$\implies -f(-y+x(-f(y)))-f(y(-f(x))+x)=-f(x)-(-f(y))+2xy$
Q.E.D

4) $f(x)$ is bijective and $f(1)=1$
========================
Using additive property, the original assertion becomes $R(x,y)$: $f(xf(y))+f(yf(x))=2xy$

$R(x,\frac{1}{2}) \implies f(xf(\frac{1}{2})+\frac{f(x)}{2})=x$ and $f(x)$ is surjective.

So $\exists a$ such that $f(a)=1$
Then $R(a,a) \implies a^2=1$ and so $a=1$ (remember that in 3) we choosed $f(1)\geq 0$)

5) $f(x)=x$
===========
$R(x,1) \implies f(x)+f(f(x))=2x$ and so $f(x)$ is injective, and so bijective.

$R(xf(x),1) \implies f(xf(x))+f(f(xf(x)))=2xf(x)$
$R(x,x) \implies f(xf(x))=x^2$ and so $f(x^2)=f(f(xf(x)))$
Combining these two lines, we get $f(x^2)+x^2=2xf(x)$

So $f((x+y)^2)+(x+y)^2=2(x+y)f(x+y)$ and so $f(xy)+xy=xf(y)+yf(x)$

So we have the properties :
$R(x,y)$: $f(xf(y))+f(yf(x))=2xy$
$A(x,y)$: $f(xy)=xf(y)+yf(x)-xy$
$B(x)$: $f(f(x))=2x-f(x)$

So :

(a) : $R(x,x) \implies f(xf(x))=x^2$
(b) : $A(x,f(x)) \implies f(xf(x))=xf(f(x))+f(x)^2-xf(x)$
(c) : $B(x) \implies f(f(x))=2x-f(x)$

And so -(a)+(b)+x(c) : $0=x^2+f(x)^2-2xf(x)=(f(x)-x)^2$
Q.E.D.

6) synthesis of solutions
=================
Using 3) and 5), we get two solutions (it's easy to check back that these two functions indeed are solutions) :
$f(x)=x\ \forall x$
$f(x)=-x\ \forall x$

arshakus
746 posts

May 29, 2010, 1:23 pm PM #23

> *mahanmath wrote:*
>
> > *mahanmath wrote:*
> >
> > **Problem 6**.Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:

> > $f\big(x+yf(x)\big)+f\big(xf(y)-y\big)=f(x)-f(y)+2xy$
>
> OK , It seems hard . So I give a hint.

ok I do't saw that problem I will try to prove it

---

arshakus
746 posts

May 29, 2010, 4:48 pm PM #24

now problem 8
$f: R+->R+$
$f(x+f(y))=f(x+y)+f(y)$, for every x,y,from $R+$

---

ilikemaths
50 posts

May 29, 2010, 5:06 pm PM #25

Isn't that from an IMO-shortlist?

---

mahanmath
1356 pos...

May 29, 2010, 8:40 pm PM #26

> *arshakus wrote:*
> now problem 8
> $f: R+->R+$
> $f(x+f(y))=f(x+y)+f(y)$, for every x,y,from $R+$

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1165901&sid=96c6c2e3567eab5401350eb464f8fe2f#p1165901

Amir Hos...
4728 pos...

May 29, 2010, 8:53 pm PM #27

Problem 9:

Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be a function such that :

(i) For all $x,y\in\mathbb{R}$,

$$f(x)+f(y)+1\geq f(x+y)\geq f(x)+f(y)$$

(ii) For all $x\in[0,1)$, $f(0)\geq f(x)$

(iii) $-f(-1)=f(1)=1$.

Find all such functions.

---

mahanmath
1356 pos...

May 29, 2010, 9:22 pm PM #28

> *amparvardi wrote:*
> **Problem 9**:
>
> Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be a function such that :
>
> (i) For all $x,y\in\mathbb{R}$,
>
> $$f(x)+f(y)+1\geq f(x+y)\geq f(x)+f(y)$$
>
> (ii) For all $x\in[0,1)$, $f(0)\geq f(x)$
>
> (iii) $-f(-1)=f(1)=1$.
>
> Find all such functions.

Put $x=-1,y=1$ we get $f(0)\geq 0$, by plugging $(0,0)$ we get $f(0)\leq 0$ so $f(0)=0$. And then by an easy induction you can prove $f(x)=[x]$

---

mahanmath
1356 pos...

May 29, 2010, 9:22 pm PM #29

Problem 10

Find all $f:\mathbb{R}\rightarrow\mathbb{R}$ such that

$f(xy+f(x))=xf(y)+f(x)$

Pain rinne...
1581 posts

May 29, 2010, 10:32 pm PM #30

> *mahanmath wrote:*
> > *amparvardi wrote:*
> > **Problem 9**:

Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that :

(i) For all $x,y\in\mathbb{R}$,

$$f(x)+f(y)+1\geq f(x+y)\geq f(x)+f(y)$$

(ii) For all $x\in[0,1)$, $f(0)\geq f(x)$

(iii) $-f(-1)=f(1)=1$.

Find all such functions.

Put $x=-1, y=1$ we get $f(0)\geq 0$, by plugging $(0,0)$ we get $f(0)\leq 0$ so $f(0)=0$. And then by an easy induction you can prove $f(x)=[x]$

How can you use induction if $x\in\mathbb{R}$?

mahanmath
1356 pos...

May 29, 2010, 10:44 pm PM #31

OK , I should` ve explain more ! usually in $\mathbb{R}$, we induction on intervals.
(In the first step we prove $f(x)=0$ for all $x\in[0,1)$)

pco
12955 po...

May 30, 2010, 2:39 pm • 2 PM #32

mahanmath wrote:

Problem 10

Find all $f:\mathbb{R}\to\mathbb{R}$ such that

$f(xy+f(x))=xf(y)+f(x)$

My solution

I have no problem to submit. Anybody feel free to take my turn.

mahanmath
1356 pos...

May 30, 2010, 5:48 pm PM #33

Problem 11

Find all $f:\mathbb{Q}\to\mathbb{Q}$ such that :

$2(f(x))=f(2x)$ and $f(x)+f(\frac{1}{x})=1$

Farenhajt
5170 pos...

May 30, 2010, 6:26 pm PM #34

mahanmath wrote:

Problem 11

Find all $f:\mathbb{Q}\to\mathbb{Q}$ such that :

$2(f(x))=f(2x)$ and $f(x)+f(\frac{1}{x})=1$

Inductively, $f(2^nx)=2^nf(x)$ for all integer n. Since $2f(1)=1\implies f(1)=\frac{1}{2}$, we get $f(2^n)=2^{n-1}$, hence $f(2^{-n})=1-2^{n-1}$, but also $f(2^{-n})=2^{-n}f(1)=2^{-n-1}$, so we get $1-2^{n-1}=2^{-n-1}$, which obviously can't hold for all integer n. Hence there's no such function.

This post has been edited 1 time. Last edited by Farenhajt, May 30, 2010, 6:34 pm

Farenhajt
5170 pos...

May 30, 2010, 6:31 pm • 1 PM #35

Problem 12.

Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(xf(y))+f(yf(x))=\frac{1}{2}f(2x)f(2y)$ for all real values x,y.

ocha
955 posts

May 31, 2010, 12:33 pm PM #36

Farenhajt wrote:

Problem 12.

Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(xf(y))+f(yf(x))=\frac{1}{2}f(2x)f(2y)$ for all real values x,y.

May i ask where you got this problem? It seems very tough, some of the solutions are: $f(x)=kx$, $f(x)=kx+4$

May i ask where you got this problem? It seems very tough, some of the solutions are; $f(x)=kx$, $f(x)=kx+4$, and $f(x)=|kx|$, where $k\in\mathbb{R}$. But also, when $k>0$ we have solution such as

$$f(x)=\begin{cases}0 & x\le 0\\ kx & x>0\end{cases}$$

Farenhajt
5170 pos...

May 31, 2010, 12:51 pm PM #37

> *ocha wrote:*
>
> > *Farenhajt wrote:*
> > **Problem 12.**
> >
> > Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(xf(y))+f(yf(x))=\frac{1}{2}f(2x)f(2y)$ for all real values x,y.
>
> May i ask where you got this problem? It seems very tough, some of the solutions are; $f(x)=kx$, $f(x)=kx+4$, and $f(x)=|kx|$, where $k\in\mathbb{R}$. But also, when $k>0$ we have solution such as
>
> $$f(x)=\begin{cases}0 & x\le 0\\ kx & x>0\end{cases}$$

A colleague gave it to me from his private notes, with some (incomplete) outlines of the solution.

---

arshakus
746 posts

May 31, 2010, 8:58 pm PM #38

hi,
please solve this one
$f:R->R$
$f(x^5)-f(y^5)=(f(x)-f(y))(x^4+x^3y+x^2y^2+xy^3+y^4)$
$f(x)-?$

Zeus93
64 posts

May 31, 2010, 9:32 pm PM #39

The problem above is **the 13th**

Now **the 14th** :
$f:R=>R$ and $f(xf(x)+f(y))=y+f^2(x)$ for all real values x,y
find $f(x)$

---

arshakus
746 posts

May 31, 2010, 10:02 pm PM #40

at first solve the 13^{th}, then post 14^{th}

---

mahanmath
1356 pos...

May 31, 2010, 10:57 pm PM #41

> *Jumong4958 wrote:*
> The problem above is **the 13th**
>
> Now **the 14th** :
> $f:R=>R$ and $f(xf(x)+f(y))=y+f^2(x)$ for all real values x,y
> find $f(x)$

Put $y=-(f(x)^2)$. So there is a such that $f(a)=0$.
Now put $x:a$. We get $f(f(x))=x$
Put $x:f(x)$, we get $f(x)=x$ or $f(x)=-x$, but it`s easy to see the both are correct.Thus the answers are $f(x)=x$ and $f(x)=-x$

> *arshakus wrote:*
> hi,
> please solve this one
> $f:R->R$
> $f(x^5)-f(y^5)=(f(x)-f(y))(x^4+x^3y+x^2y^2+xy^3+y^4)$
> $f(x)-?$

See the **Problem 2** and **Problem 3**.

Is there any idea about **Problem 12** ?

> *Farenhajt wrote:*
> **Problem 12.**
>
> Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(xf(y))+f(yf(x))=\frac{1}{2}f(2x)f(2y)$ for all real values

Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(xf(y))+f(yf(x))=\frac{1}{2}f(2x)f(2y)$ for all real values x,y.

This post has been edited 2 times. Last edited by mahanmath, Jun 1, 2010, 10:48 pm

---

arshakus
746 posts

May 31, 2010, 11:23 pm PM #42

> *mahanmath wrote:*
>
> > *Quote:*
> > **Problem 2:** Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that:
> >
> > $$f(x^3)-f(y^3)=(x^2+xy+y^2)(f(x)-f(y))\ ,\ (\forall)x,y\in\mathbb{R}$$
>
> Problem 2
>
> **Problem 3** : Find all the continuous functions $f:\mathbb{R}\mapsto\mathbb{R}$ such that $\forall x,y\in\mathbb{R}$:
>
> $(1+f(x)f(y))f(x+y)=f(x)+f(y)$

you are not right the answer is $f(x)=kx$,

---

mahanmath
1356 pos...

May 31, 2010, 11:41 pm PM #43

> *arshakus wrote:*
> you are not right the answer is $f(x)=kx$,

No !!! Actually $k=f(1)$ and $f(x)=xf(1)+f(0)$ works . (You can check it !?!?!?!)

---

ocha
955 posts

Jun 1, 2010, 6:10 am • 8 PM #44

EDIT: The solution is now complete (it's quite long though)

> *Farenhajt wrote:*
> **Problem 12.**
>
> Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(xf(y))+f(yf(x))=\frac{1}{2}f(2x)f(2y)$ for all real values x,y.

In this proof we show that when f is not constant, it is bijective on the separate domains $(-\infty,0]$ and $[0,\infty)$, (not necessarily on $\mathbb{R}$) and then find all solutions on those domains. Then we get all functions f, by joining any two functions from the separate domains and checking they work. I mentioned some of the solutions in an earlier post.

solution

Stephen
403 posts

Jun 4, 2010, 10:15 am PM #45

Well done, Ocha!

Now what's the new problem?

---

mahanmath
1356 pos...

Jun 4, 2010, 11:33 am PM #46

Find all functions f defined on real numbers and taking real values such that $f(x)^2+2yf(x)+f(y)=f(y+f(x))$ for all real numbers x,y.

---

pco
12955 po...

Jun 4, 2010, 1:02 pm PM #47

> *mahanmath wrote:*
> Find all functions f defined on real numbers and taking real values such that $f(x)^2+2yf(x)+f(y)=f(y+f(x))$ for all real numbers x,y.

My solution

I have no problem to submit. Anybody feel free to take my turn.

Pain rinne...
1581 posts

Jun 4, 2010, 1:39 pm PM #48

Determine all the polynomial functions $f:\mathbb{R}\to\mathbb{R}$, with integer coefficients, which are bijective and satisfy the relation:

$$f^2(x)=f(x^2)-2f(x)+a\ ,\ (\forall)x\in\mathbb{R}$$

where $a\in\mathbb{R}$ is fixed.

pco
12955 po...

Jun 4, 2010, 2:18 pm PM #49

> *Pain rinnegan wrote:*
> Determine all the polynomial functions $f:\mathbb{R}\to\mathbb{R}$, with integer coefficients, which are bijective and satisfy the relation:
>
> $$f^2(x)=f(x^2)-2f(x)+a\ ,\ (\forall)x\in\mathbb{R}$$
>
> where $a\in\mathbb{R}$ is fixed.

My solution

I have no problem to submit. Anybody feel free to take my turn.

Mohamma...
58 posts

Jun 4, 2010, 7:22 pm PM #50

Problem 17:
k is a nonzero constant.
find all functions satisfying :
$f(xy)=f(x)f(y)$ and $f(x+k)=f(x)+f(k)$

Stephen
403 posts

Jun 5, 2010, 9:40 am PM #51

Solution to **Problem 17**:

$f(y)f(x)+f(y)f(k)=f(y)f(x+k)$

$f(xy)+f(ky)=f(xy+yk)$

Now we are going to prove $f(x+ky)=f(x)+f(ky)$.

If $y=0$, it's easy since $f(0)=0$.

If $y\neq 0$, then we can put $\frac{x}{y}$ in x of $f(xy)+f(ky)=f(xy+yk)$.

So $f(x+ky)=f(x)+f(ky)$.

Now, since k isn't 0, we can put $\frac{y}{k}$ in y of $f(x+ky)=f(x)+f(ky)$.

So $f(x+y)=f(x)+f(y)$.

Since is an Cauchy equation, we can know that for some constant c, that $f(q)=cq$ when q is an rational number.

But because of $f(xy)=f(x)f(y)$, c is 0 or 1.

If $c=0$, then we can easily know that $f(x)=0$ for all real number x.

If $c=1$, then $f(q)=q$.

Now let's prove $f(x)=x$.

Since $f(xy)=f(x)f(y)$, $f(x^2)=(f(x))^2$.

So if $x>0$, then $f(x)>0$ since $f(x)\neq 0$.

But $f(-x)=-f(x)$. So if $x<0$, then $f(x)<0$.

Now let a a constant that satisfies $f(a)>a$.

Then if we let $f(a)=b$, there is a rational number p that satisfies $b>p>a$.

So, $f(p-a)+f(a)=f(p)=p$.

So, $f(p-a)=p-f(a)=p-b<0$.

But, $p-a>0$. So a contradiction!

So we can know that $f(x)\leq x$.

With a similar way, we can know that $f(x)\geq x$.

So $f(x)=x$.

We can conclude that possible functions are $f(x)=0$ and $f(x)=x$.

Problem 18:

Find all continuous and strictly-decreasing function $f:R^+\to R^+$ that satisfies

$f(x+y)+f(f(x)+f(y))=f(f(x+f(y))+f(y+f(x)))$

pco
12955 po...

Jun 6, 2010, 1:46 pm PM #52

> *Stephen wrote:*
> **Problem 18:**
>
> Find all continuous and strictly-decreasing function $f:R^+\to R^+$ that satisfies
>
> $f(x+y)+f(f(x)+f(y))=f(f(x+f(y))+f(y+f(x)))$

Hello, could you confirm us that you have the full solution of this problem and could you give us a hint ?

Up to now, I proved that $f(f(x))=x$ and that $f(x)=\frac{a}{x}$ is a solution, but I'm still unable to prove if this is the only solution 🙁

Thanks for any hint.

Stephen
403 posts

Jun 6, 2010, 2:32 pm PM #53

> *pco wrote:*
>
> > *Stephen wrote:*
> > **Problem 18:**
> >
> > Find all continuous and strictly-decreasing function $f:R^+\to R^+$ that satisfies
> >
> > $f(x+y)+f(f(x)+f(y))=f(f(x+f(y))+f(y+f(x)))$
>
> Hello, could you confirm us that you have the full solution of this problem and could you give us a hint ?
>
> Up to now, I proved that $f(f(x))=x$ and that $f(x)=\frac{a}{x}$ is a solution, but I'm still unable to prove if this is the only solution 🙁
>
> Thanks for any hint.

I am really very sorry, but I don't have a solution, and I'm working on it.

To tell the truth, I just post this because I'm very curious of a solution.

But I can give you some advice.

I also did $f(f(x))=x$ and know that $f(x)=\frac{c}{x}$ is a solution.

Now, I'm trying to use k that satisfies $f(k)=k$.

Since f is continuous, we can easily prove that there exists a k that $f(k)=k$.

Good luck!

mahanmath
1356 pos...

Jun 6, 2010, 3:11 pm PM #54

Problem 18 was hard (maybe open!) and stoped the marathon, and I believe marathon should be alive !!,so I submit an easy problem.
Problem 19
Find all functions $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ of two variables satisfying for all x,y:
$f(x,x)=x$, $f(x,y)=f(y,x)$, $(x+y)f(x,y)=yf(x,x+y)$.

Farenhajt
5170 pos...

Jun 6, 2010, 8:31 pm • 2 👍 PM #55

> *mahanmath wrote:*
> Problem 18 was hard (maybe open!) and stoped the marathon, and I believe marathon should be alive !!,so I submit an easy problem.
> **Problem 19**
> Find all functions $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ of two variables satisfying for all x,y:
> $f(x,x)=x$, $f(x,y)=f(y,x)$, $(x+y)f(x,y)=yf(x,x+y)$.

Substituting $f(x,y)=\dfrac{xy}{g(x,y)}$ we get $g(x,x)=x, g(x,y)=g(y,x), g(x,y)=g(x,x+y)$. Putting $z:=x+y$, the last condition becomes $g(x,z)=g(x,z-x)$ for $z>x$. With $g(x,x)=x$ and symmetry, it is now obvious, by Euclidean algorithm, that $g(x,y)=\gcd(x,y)$, therefore $f(x,y)=[x,y]$

Someone else can post the next problem.
This post has been edited 1 time. Last edited by Farenhait, Jun 6, 2010, 10:15 pm

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Stephen
403 posts

Jun 6, 2010, 9:37 pm PM #56

Firstly I am very sorry for delaying the marathon.

Problem 20

Prove that $f(x+y+xy)=f(x)+f(y)+f(xy)$ is equivalent to $f(x+y)=f(x)+f(y)$.

---

pco
12955 po...

Jun 7, 2010, 12:58 am PM #57

> *Stephen wrote:*
> **Problem 20**
>
> Prove that $f(x+y+xy)=f(x)+f(y)+f(xy)$ is equivalent to $f(x+y)=f(x)+f(y)$.

My solution

I have no problem to submit. Anybody feel free to take my turn.

Stephen
403 posts

Jun 8, 2010, 2:21 pm PM #58

Problem 21

Find all $f:Z\to Z$ that satisfies $f(x)^3+f(y)^3+f(z)^3=f(x^3+y^3+z^3)$.

---

mahanmath
1356 pos...

Jun 8, 2010, 3:38 pm PM #59

> *Stephen wrote:*
> **Problem 21**
>
> Find all $f:Z\to Z$ that satisfies $f(x)^3+f(y)^3+f(z)^3=f(x^3+y^3+z^3)$.

#)If $x\geq 4$, x^3 can be written as a sum of five cubes such that their absolute values are less than x.
From # and induction we get the answer is $f(x)=xf(1)$ and $f(1)=1,0,-1$.

In fact , It`s a number theory problem !....

I have no problem to submit ...

---

goodar20...
1346 pos...

Jun 8, 2010, 4:52 pm • 1 PM #60

Problem 22
find all functions $f:R\longrightarrow R$ such that
$f(f(x)+y)=2x+f(f(y)-x)$
This post has been edited 1 time. Last edited by goodar2006, Aug 20, 2011, 1:57 am

---

pco
12955 po...

Jun 8, 2010, 6:39 pm • 1 PM #61

> *goodar2006 wrote:*
> Problem 22
> find all functions $f:R\longrightarrow R$ such that
> $f(f(x)+y)=2x+f(f(y)-x)$

My solution

I have no problem to submit. Anybody feel free to take my turn.

Zeus93
64 posts

Jun 9, 2010, 12:04 am PM #62

Now the 23th:
Find all $f:N\longrightarrow N$ such that
$f(f(n))+f(n+1)=n+2$
If someone see it easily, please help me in this one:
Problem 24h :Find $f:R^{+}\longrightarrow R^{+}$ such that :
$f(x)f(yf(x))=f(x+y)$ for all $x,y\in\mathbb{R}^{+}$
This post has been edited 1 time. Last edited by Zeus93, Jun 10, 2010, 6:47 pm

---

Justhalf
45 posts

Jun 10, 2010, 6:30 am PM #63

> *Jumong4958 wrote:*
> Now the 23th:
> Find $f:N\longrightarrow N$ such that
> $f(f(n))+f(n+1)=n+2$

Problem 23 is almost there...

I can get all values of $f(n)$ but I cannot find the general formula for it.
Subtitute $n=1$.
We get $f(f(1))+f(2)=3$
Since $f(n)\in\mathbb{N}$ then there are two cases:
Case 1: $f(f(1))=2$ and $f(2)=1$
Subtitute $n=2$, then we get:
$f(1)+f(3)=4$.
Now we have three cases:
Case 1.1: $f(1)=1, f(3)=3 \Rightarrow f(f(1))=1$ contradiction
Case 1.2: $f(1)=2, f(3)=2 \Rightarrow f(f(1))=1$ contradiction
Case 1.3: $f(1)=3, f(3)=1. f(f(1))=2 \Rightarrow f(3)=2$ contradiction
So case 1 fails.
So case 2 must hold.
Case 2: $f(f(1))=1, f(2)=2$
Then by induction we can get:
$f(2)=2$
$f(3)=2$
$f(4)=3$
$f(5)=4$
$f(6)=4$
$f(7)=5$
$f(8)=5$
$f(9)=6$
$f(10)=7$
$f(11)=7$
$f(12)=8$
$f(13)=9$
$f(14)=9$
$f(15)=10$
$f(16)=10$
$f(17)=11$
$f(18)=12$
$f(19)=12$
$f(20)=13$
$f(21)=13$
$f(22)=14$
$f(23)=15$
$f(24)=15$
$f(25)=16$
$f(26)=17$
$f(27)=17$
$f(28)=18$
$f(29)=18$
$f(30)=19$
$f(31)=20$

etc..
And since for all n, $1<f(n)<n$ then $f(1)=1$.
so we have the complete list of the values. But I cannot get the general formula.

---

ocha
955 posts

Jun 10, 2010, 8:39 am PM #64

> *Justhalf wrote:*
> so we have the complete list of the values. But I cannot get the general formula.

$f(n)=\left\lceil\frac{5n}{8}\right\rceil$

---

mahanmath
1356 pos...

Jun 10, 2010, 10:16 am PM #65

> *Jumong4958 wrote:*
> Problem 24h :Find $f: R^+ \longrightarrow R^+$ such that :
> $f(x)f(yf(x))=f(x+y)$ for all $x,y\in\mathbb{R}^+$

Find or Find all ? Because it has a trivial solution , $f(x)=1$!

Justhalf
45 posts

Jun 10, 2010, 10:48 am PM #66

> *ocha wrote:*
>> *Justhalf wrote:*
>> so we have the complete list of the values. But I cannot get the general formula.
>
> $f(n)=\left\lceil\frac{5n}{8}\right\rceil$

Wow, how can you find that?

ocha
955 posts

Jun 10, 2010, 11:20 am • 1

PM #67

> *Justhalf wrote:*
>
> > *ocha wrote:*
> > $f(n)=\left\lceil\frac{5n}{8}\right\rceil$
>
> Wow, how can you find that?

Just looking at the sequence. There is a pattern in the difference of consecutive terms that repeates after every 8 values of n, and causes $f(n)$ to increase 5. A rough estimate would be $f(n)=\frac{5n}{8}$. Then it isn't hard to motivate $f(n)=\left\lceil\frac{5}{8}\right\rceil$

Justhalf
45 posts

Jun 10, 2010, 11:31 am

PM #68

> *ocha wrote:*
>
> > *Justhalf wrote:*
> >
> > > *ocha wrote:*
> > > $f(n)=\left\lceil\frac{5n}{8}\right\rceil$
> >
> > Wow, how can you find that?
>
> Just looking at the sequence. There is a pattern in the difference of consecutive terms that repeates after every 8 values of n, and causes $f(n)$ to increase 5. A rough estimate would be $f(n)=\frac{5n}{8}$. Then it isn't hard to motivate $f(n)=\left\lceil\frac{5}{8}\right\rceil$

Then how we proof that it is indeed a solution?
It seems complicated to me.

ocha
955 posts

Jun 10, 2010, 12:56 pm

PM #69

> *Justhalf wrote:*
> Then how we proof that it is indeed a solution?
> It seems complicated to me.

Well observe the pattern: the differences $a_{i+1}-a_i$ are $0,1,1,0,1,0,1$ then repeat (1)

write $n=8q+r$ with $1\le r\le 7$. then $f(n)=\left\lceil\frac{5(8q+r)}{8}\right\rceil=5q+\left\lceil\frac{5r}{8}\right\rceil$.

So we have reduced the problem to showing the terms $a_r=\left\lceil\frac{5r}{8}\right\rceil$ follow the pattern (1). when $1\le r\le 6$, which is easy.

And finall if $n=8q+7$ then $a_{n+1}-a_n=5(q+1)-\left(5q+\left\lceil\frac{5\cdot 7}{8}\right\rceil\right)=0$ which is the first term in (1)

So we are done.

pco
12955 po...

Jun 10, 2010, 2:18 pm

PM #70

> *mahanmath wrote:*
>
> > *Jumong4958 wrote:*
> > Problem 24h :Find $f:R^+\longrightarrow R^+$ such that :
> > $f(x)f(yf(x))=f(x+y)$ for all $x,y\in\mathbb{R}^+$

> Find or Find all ? Because it has a trivial solution , $f(x)=1$!

A non constant solution is $f(x)=\dfrac{1}{x+1}$

All non constant solutions are continuous, strictly decreasing from $1\to 0$

But I did not succeed up to now to show that $\dfrac{1}{x+1}$ is the only non constant solution.

Justhalf
45 posts

Jun 10, 2010, 2:45 pm PM #71

> *ocha wrote:*
>
> > *Justhalf wrote:*
> > Then how we proof that it is indeed a solution?
> > It seems complicated to me.
>
> Well observe the pattern: the differences $a_{i+1}-a_i$ are $0,1,1,0,1,0,1$ then repeat (1)
>
> write $n=8q+r$ with $1\le r\le 7$. then $f(n)=\left\lceil\dfrac{5(8q+r)}{8}\right\rceil=5q+\left\lceil\dfrac{5r}{8}\right\rceil$.
>
> So we have reduced the problem to showing the terms $a_r=\left\lceil\dfrac{5r}{8}\right\rceil$ follow the pattern (1). when $1\le r\le 6$, which is easy.
>
> And finall if $n=8q+7$ then $a_{n+1}-a_n=5(q+1)-\left(5q+\left\lceil\dfrac{5\cdot 7}{8}\right\rceil\right)=0$ which is the first term in (1)
>
> So we are done.

I think you are proving that your function satisfy the pattern. But what I asked is how do you prove that the function satisfy the condition stated in the problem.

Farenhajt
5170 pos...

Jun 10, 2010, 3:19 pm • 11 PM #72

> *ocha wrote:*
> $f(n)=\left\lceil\dfrac{5n}{8}\right\rceil$

Ocha, are you sure about this?

Plugging $f(n)=\left\lceil\dfrac{5n}{8}\right\rceil$ into the initial equation, and writing $n=8q+r$ where $0\leqslant r\leqslant 7$, we find that the following must hold:

$$3q+\left\lceil\dfrac{q+5\left\lceil\frac{5r}{8}\right\rceil}{8}\right\rceil+5q+\left\lceil\dfrac{5r+5}{8}\right\rceil=8q+r+2$$

But after we cancel $8q$ from the both sides, we're left with the LHS that depends on q, hence is unbounded, and with the RHS which is bounded.

Motivated by $f(n)=\left\lceil\dfrac{an}{b}\right\rceil$, and putting the condition that q must disappear from the both sides, we arrive at $\dfrac{a^2}{b}+a=b\implies\dfrac{a}{b}=\dfrac{\sqrt5-1}{2}$. Hence we now have to prove that $f(n)=\left\lceil\dfrac{\sqrt5-1}{2}n\right\rceil=\left\lfloor\dfrac{\sqrt5-1}{2}n\right\rfloor+1$ satisfies the initial equation (the latter form of the function holds as the expression under the ceiling sign is never an integer).

(Note that $\dfrac{\sqrt5-1}{2}\approx\dfrac{5}{8}$, hence the difference between the two functions will start to show up only for a sufficiently large n, thus the first few values can definitely be misleading.)

For easier writing we'll put $\phi:=\dfrac{\sqrt5-1}{2}$ and use the designation $[\cdot]$ for the integer part function. Then we have to prove

$$[\phi([n\phi]+1)]+1+[n\phi+\phi]+1=n+2$$

$$[\phi[n\phi]+\phi]+[n\phi+\phi]=n$$

$$[n\phi^2-\phi\{n\phi\}+\phi]+[n\phi+\phi]=n$$

Since $\phi^2+\phi-1=0 \implies \phi^2=1-\phi$, we have

$[n-n\phi-\phi\{n\phi\}+\phi]+[n\phi+\phi]=n$

$[-n\phi-\phi\{n\phi\}+\phi]+[n\phi+\phi]=0$

$-[n\phi]+[-\{n\phi\}-\phi\{n\phi\}+\phi]+[n\phi]+[\phi+\{n\phi\}]=0$

$[\phi-(1+\phi)\{n\phi\}]+[\phi+\{n\phi\}]=0 \quad (*)$

First we'll note that $\{n\phi\}\neq 1-\phi$ for all natural n, since otherwise we'd have $n\phi-[n\phi]=1-\phi \iff (n+1)\phi=[n\phi]+1$, implying $\phi\in\mathbb{Q}$. Therefore we have either $0<\{n\phi\}<1-\phi$ or $1-\phi<\{n\phi\}<1$.

In the first case, $\phi-(1+\phi)(1-\phi)<\phi-(1+\phi)\{n\phi\}<\phi \iff 0<\phi-(1+\phi)\{n\phi\}<\phi$, hence $[\phi-(1+\phi)\{n\phi\}]=0$, and $\phi<\phi+\{n\phi\}<1 \implies [\phi+\{n\phi\}]=0$, therefore $(*)$ is satisfied.

In the second case, $\phi-(1+\phi)<\phi-(1+\phi)\{n\phi\}<\phi-(1+\phi)(1-\phi) \iff -1<\phi-(1+\phi)\{n\phi\}<0$, hence $[\phi-(1+\phi)\{n\phi\}]=-1$, and $1<\phi+\{n\phi\}<1+\phi \implies [\phi+\{n\phi\}]=1$, therefore $(*)$ is also satisfied.

So we finally conclude that the solution to the given equation is $f(n)=\left[\dfrac{(\sqrt{5}-1)n}{2}\right]+1$

This post has been edited 3 times. Last edited by Farenhajt, Jun 11, 2010, 2:23 pm

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Justhalf
45 posts

Jun 10, 2010, 4:31 pm PM #73

> *Farenhajt wrote:*
>
> > *ocha wrote:*
> > $f(n)=\left\lceil\dfrac{5n}{8}\right\rceil$
>
> Ocha, are you sure about this?
>
> Plugging $f(n)=\left\lceil\dfrac{5n}{8}\right\rceil$ into the initial equation, and writing $n=8q+r$ where $0\leqslant r\leqslant 7$, we find that the following must hold:
>
> $3q+\left\lceil\dfrac{q+5\left\lceil\frac{5r}{8}\right\rceil}{8}\right\rceil+5q+\left\lceil\dfrac{5r+5}{8}\right\rceil=8q+r+2$
>
> But after we cancel $8q$ from the both sides, we're left with the LHS that depends on q, hence is unbounded, and with the RHS which is bounded.
>
> Motivated by $f(n)=\left\lceil\dfrac{an}{b}\right\rceil$, and putting the condition that q must disappear from the both sides, we arrive at $\dfrac{a^2}{b}+a=b \implies \dfrac{a}{b}=\dfrac{\sqrt{5}-1}{2}$. Hence we now have to prove that $f(n)=\left\lceil\dfrac{\sqrt{5}-1}{2}n\right\rceil=\left\lfloor\dfrac{\sqrt{5}-1}{2}n\right\rfloor+1$ satisfies the initial equation (the latter form of the function holds as the expression under the ceiling sign is never an integer).
>
> (Note that $\dfrac{\sqrt{5}-1}{2}\approx\dfrac{5}{8}$, hence the difference between the two functions will start to show up only for a sufficiently large n, thus the first few values can definitely be misleading.)
>
> For easier writing we'll put $\phi:=\dfrac{\sqrt{5}-1}{2}$ and use the designation $[\cdot]$ for the integer part function. Then we have to prove
>
> $[\phi([n\phi]+1)]+1+[n\phi+\phi]+1=n+2$
>
> $[\phi[n\phi]+\phi]+[n\phi+\phi]=n$
>
> $[n\phi^2-\phi\{n\phi\}+\phi]+[n\phi+\phi]=n$
>
> Since $\phi^2+\phi-1=0 \implies \phi^2=1-\phi$, we have
>
> $[n-n\phi-\phi\{n\phi\}+\phi]+[n\phi+\phi]=n$
>
> $[-n\phi-\phi\{n\phi\}+\phi]+[n\phi+\phi]=0$
>
> $-[n\phi]+[-\{n\phi\}-\phi\{n\phi\}+\phi]+[n\phi]+[\phi+\{n\phi\}]=0$
>
> $[\phi-(1+\phi)\{n\phi\}]+[\phi+\{n\phi\}]=0 \quad (*)$

First we'll note that $\{n\phi\} \neq 1-\phi$ for all natural n, since otherwise we'd have $n\phi - [n\phi] = 1 - \phi \iff (n+1)\phi = [n\phi] + 1$, implying $\phi \in \mathbb{Q}$. Therefore we have either $0 < \{n\phi\} < 1-\phi$ or $1-\phi < \{n\phi\} < 1$.

In the first case, $\phi - (1+\phi)(1-\phi) < \phi - (1+\phi)\{n\phi\} < \phi \iff 0 < \phi - (1+\phi)\{n\phi\} < \phi$, hence $[\phi - (1+\phi)\{n\phi\}] = 0$, and $\phi < \phi + \{n\phi\} < 1 \implies [\phi + \{n\phi\}] = 0$, therefore $(*)$ is satisfied.

In the second case,
$\phi - (1+\phi) < \phi - (1+\phi)\{n\phi\} < \phi - (1+\phi)(1-\phi) \iff -1 < \phi - (1+\phi)\{n\phi\} < 0$, hence $[\phi - (1+\phi)\{n\phi\}] = -1$, and $1 < \phi + \{n\phi\} < 1+\phi \implies [\phi + \{n\phi\}] = 1$, therefore $(*)$ is also satisfied.

So we finally conclude that the solution to the given equation is $f(n) = \left[\dfrac{(\sqrt{5}-1)n}{2}\right] + 1$

Ah, this is what I call solution... =D
Thanks Farenhajt
This post has been edited 1 time. Last edited by Justhalf, Jun 10, 2010, 5:56 pm

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Farenhajt
5170 pos...

Jun 10, 2010, 5:33 pm • 3 PM #74

> *Justhalf wrote:*
> Ah, this is what I call solution... =D
> Thanks Farenhajt

Thanks to you too. There were two typos though - confer the red markings in my original post and (if you like) edit your quotation accordingly.

Zeus93
64 posts

Jun 10, 2010, 6:46 pm PM #75

> *mahanmath wrote:*
>
> > *Jumong4958 wrote:*
> > Problem 24h :Find $f: R^+ \longrightarrow R^+$ such that :
> > $f(x)f(yf(x)) = f(x+y)$ for all $x, y \in \mathbb{R}^+$
>
> Find or Find all ? Because it has a trivial solution , $f(x) = 1$!

Oh sorry, "Find all", not "Find"

---

ocha
955 posts

Jun 11, 2010, 10:55 am • 5 PM #76

Who went through this thread and marked everything as spam? Usually i wouldn't care about ratings, but even Farenhajt's nice solution got rated 1!

> *Jumong4958 wrote:*
> Problem 24h :Find $f: R^+ \longrightarrow R^+$ such that :
> $f(x)f(yf(x)) = f(x+y)$ for all $x, y \in \mathbb{R}^+4$

I have finished pco's solution
solution
This post has been edited 1 time. Last edited by ocha, Jun 11, 2010, 11:42 am

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Justhalf
45 posts

Jun 11, 2010, 11:37 am PM #77

> *ocha wrote:*
> Since it is true for all s, t we have $\dfrac{1}{s}\left(1 - \dfrac{1}{f(s)}\right) = k > 0$ is constant.

Ahhh, I missed the injectivity...
Anyway, I think for this part $k < 0$, because $1 - \dfrac{1}{f(s)} < 0$

Farenhajt
5170 pos...

Jun 11, 2010, 2:15 pm • 3 PM #78

> *ocha wrote:*
> Who went through this thread and marked everything as spam? Usually i wouldn't care about ratings, but even Farenhajt's nice solution got rated 1!

Someone's on the run - I've got over 20 of my posts spam-rated within a few hours. I contacted the administrators and they're looking into it. Anyone experiencing similar problem should do the same.

goodar20...
1346 pos...

Jun 12, 2010, 12:50 pm

PM #79

let's continue the marathon :
problem 25
find all functions $f:R\longrightarrow R$ such that
$f(xf(y)+f(x))=f(yf(x))+x$
problem 26
find all functions $f:R\longrightarrow R$ such that
$f(x^2+f(y))=y+f(x)^2$

---

jgnr
1344 pos...

Jun 12, 2010, 5:39 pm

PM #80

> goodar2006 wrote:
> let's continue the marathon :
> **problem 25**
> find all functions $f:R\longrightarrow R$ such that
> $f(xf(y)+f(x))=f(yf(x))+x$

Substitute $y=0$, f is surjective. Substitute y such that $f(y)=0$, f is injective. Substitute $x=y=0$, $f(f(0))=0$, $f(0)=0$. Substitute $y=0$, $f(f(x))=x$. Substitute $x=f(a)$, $f(f(a)f(y)+a)=f(f(a)f(y))+f(a)$. For $a\neq 0$, $f(a)f(y)$ can take any real value b, so $f(b+a)=f(b)+f(a)$ for $a\neq 0$. For $a=0$, $f(b+a)=f(b)=f(b)+f(0)$, hence $f(b+a)=f(b)+f(a)$ for all reals a,b. So $f(n)=cn$ for any rational number n. Substitute this to $f(f(x))=x$, we get $n=\pm 1$. So $f(1)=\pm 1$. If $f(1)=1$, substitute $x=1,y=f(a-1)$, $f(a)=a$. If $f(1)=-1$, note that from $f(b+a)=f(b)+f(a)$ we get f is odd, hence substituting $x=1,y=f(a+1)$ we get
$f(a)=f(-f(a+1))+1=-f(f(a+1))+1=-(a+1)+1=-a$. So the functions are $f(x)=x$ for all x and $f(x)=-x$ for all x, it is easy to check that these functions satisfy the given requirements.

Let's solve problem 26 now.

Mohamma...
58 posts

Jun 12, 2010, 7:59 pm

PM #81

I think your answer is not correct.

> Johan Gunardi wrote:
> Substitute $x=f(a)$, $f(f(a)f(y)+a)=f(f(a)f(y))+f(a)$.

how did you conclude it ?

Justhalf
45 posts

Jun 12, 2010, 11:25 pm

PM #82

> MohammadP wrote:
> I think your answer is not correct.
>
> > Johan Gunardi wrote:
> > Substitute $x=f(a)$, $f(f(a)f(y)+a)=f(f(a)f(y))+f(a)$.
>
> how did you conclude it ?

Using $f(f(a))=a$ for any $a\in\mathbb{R}$

---

pco
12955 po...

Jun 13, 2010, 11:47 am • 2

PM #83

> goodar2006 wrote:
> **problem 26**
> find all functions $f:R\longrightarrow R$ such that
> $f(x^2+f(y))=y+f(x)^2$

My solution

I have no problem to submit. Anybody feel free to take my turn.

Rijul saini
799 posts

Jun 13, 2010, 5:37 pm

PM #84

Problem 27
If $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that

$$f(x^3+y^3)=(x+y)(f(x)^2-f(x)f(y)+f(y)^2)$$

then prove that $f(1996\cdot x)=1996f(x)$

---

pco
12955 po...

Jun 14, 2010, 12:26 pm • 2

PM #85

> Rijul saini wrote:
> ***Problem 27***
> If $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that
>
> $$f(x^3+y^3)=(x+y)(f(x)^2-f(x)f(y)+f(y)^2)$$
>
> then prove that $f(1996\cdot x)=1996f(x)$

then prove that $f(1996 \cdot x) = 1996 f(x)$

My solution

I have no problem to submit. Anybody feel free to take my turn.

---

nguyenvut...
475 posts

Jun 15, 2010, 6:15 pm PM #86

Problem 28 :

Find all surjective functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that :

$f(f(x-y)) \ = \ f(x) - f(y) \ \forall \ x;y \ \in \mathbb{R}$

Problem 29 :

Find all real number k such that there exists a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that :

f is differentiable in $\mathbb{R}$, $f(1) \leq 1$

and $(f(x))^2 + (f^{'}(x))^2 \ = \ k$

Farenhajt
5170 pos...

Jun 15, 2010, 8:54 pm • 1 PM #87

> *nguyenvuthanhha wrote:*
> *Problem 28 :*
>
> *Find all surjective functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that :*
>
> $f(f(x-y)) \ = \ f(x) - f(y) \ \forall \ x;y \ \in \mathbb{R}$

Putting $y=0$ we get $f(f(x))=f(x)-f(0)$. Since f is surjective, this yields $f(x)=x-f(0)$. Plugging $x=0$ into that, we get $f(0)=0$, hence the only solution is $f(x)=x$

---

pco
12955 po...

Jun 15, 2010, 9:31 pm • 1 PM #88

> *nguyenvuthanhha wrote:*
> Problem 29 :
>
> Find all real number k such that there exists a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that :
>
> f is differentiable in $\mathbb{R}$, $f(1) \leq 1$
>
> and $(f(x))^2 + (f^{'}(x))^2 \ = \ k$

My solution

I have no problem to submit. Anybody feel free to take my turn.

---

mahanmath
1356 pos...

Jun 15, 2010, 9:46 pm PM #89

Problem 30 :
Find all $a \in \mathbb{R}$ for which there exists a non-constant function $f : (0,1] \to \mathbb{R}$ such that

$$a + f(x+y-xy) + f(x)f(y) \leq f(x) + f(y)$$

for all $x, y \in (0,1]$.

Justhalf
45 posts

Jun 16, 2010, 5:18 am • 1 PM #90

> *pco wrote:*
>
> > *nguyenvuthanhha wrote:*
> > Problem 29 :
> >
> > Find all real number k such that there exists a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that :
> >
> > f is differentiable in $\mathbb{R}$, $f(1) \leq 1$
> >
> > and $(f(x))^2 + (f^{'}(x))^2 \ = \ k$
>
> My solution
>
> I have no problem to submit. Anybody feel free to take my turn.

lol oneliner

Jun 16, 2010, 6:47 pm • 1 PM #91

pco
12955 po...

> *mahanmath wrote:*
> **Problem 30** :
> Find all $a \in \mathbb{R}$ for which there exists a non-constant function $f:(0,1] \to \mathbb{R}$ such that
> $$a + f(x+y-xy) + f(x)f(y) \leq f(x) + f(y)$$
> for all $x, y \in (0,1]$.

My solution

I have no problem to submit. Anybody feel free to take my turn.

---

gold46
593 posts

Jun 17, 2010, 2:17 pm PM #92

Find all functions $f:\mathbb{R} \to \mathbb{R}$ such that for each $n \in \mathbb{N}$,
$2n + 2009 \leq f(f(n)) + f(n) \leq 2n + 2011$.

---

pco
12955 po...

Jun 17, 2010, 2:59 pm PM #93

> *gold46 wrote:*
> Find all functions $f:\mathbb{R} \to \mathbb{R}$ such that for each $n \in \mathbb{N}$,
> $2n + 2009 \leq f(f(n)) + f(n) \leq 2n + 2011$.

Are you sure about $f:\mathbb{R} \to \mathbb{R}$?
Because then we have infinitely many solutions with very different forms ....
Maybe the real problem is $f:\mathbb{N} \to \mathbb{N}$?

---

mahanmath
1356 pos...

Jun 17, 2010, 6:26 pm PM #94

> *gold46 wrote:*
> Find all functions $f:\mathbb{R} \to \mathbb{R}$ such that for each $n \in \mathbb{N}$,
> $2n + 2009 \leq f(f(n)) + f(n) \leq 2n + 2011$.

pco is right. You can see a similar problem here.

Mr.pco your solution for **Problem30** was very nice. 😁 😉

Zeus93
64 posts

Jun 17, 2010, 10:28 pm PM #95

Problem 32:
Find all $f:\mathbb{R}^+ \to \mathbb{R}$ satisfies:
i) $f(a) = 1$with a is a given real number
ii)$f(x)f(y) + f(\frac{a}{y})f(\frac{a}{x}) = 2f(xy)$ for all x,y >0

Farenhajt
5170 pos...

Jun 18, 2010, 2:05 am PM #96

> *Jumong4958 wrote:*
> Problem 32:
> Find all $f:\mathbb{R}^+ \to \mathbb{R}$ satisfies:
> i) $f(a) = 1$with a is a given real number
> ii)$f(x)f(y) + f(\frac{a}{y})f(\frac{a}{x}) = 2f(xy)$ for all x,y >0

Putting $y = \frac{a}{x}$ we get $2f(x)f\left(\frac{a}{x}\right) = 2f(a) \iff f\left(\frac{a}{x}\right) = \frac{1}{f(x)} \quad (*)$

Putting $y = x$ we get $f^2(x) + f^2\left(\frac{a}{x}\right) = 2f(x^2) \overset{(*)}{\iff} f^2(x) + \frac{1}{f^2(x)} = 2f(x^2)$, which by AM-GM implies $f(x^2) \geqslant 1$.

Since x^2 covers $\mathbb{R}^+$ completely, we get $(\forall x \in \mathbb{R}^+) f(x) \geqslant 1$

But now $(*)$ implies $\left(\forall \frac{a}{x} \in \mathbb{R}^+\right) f\left(\frac{a}{x}\right) \leqslant 1$.

Therefore the only solution is $f(x) \equiv 1$.

Someone else can post the next problem.

---

mahanmath
1356 pos...

Jun 18, 2010, 3:47 am PM #97

Problem 33

Find all functions $f:\mathbb{Q} \mapsto \mathbb{C}$ satisfying

(i) For any $x_1, x_2, \ldots, x_{2010} \in \mathbb{Q}$, $f(x_1 + x_2 + \ldots + x_{2010}) = f(x_1)f(x_2)\ldots f(x_{2010})$.

(ii) $\overline{f(2010)}f(x) = f(2010)\overline{f(x)}$ for all $x \in \mathbb{Q}$.

pco
12955 po...

Jun 18, 2010, 1:16 pm PM #98

> *mahanmath wrote:*
> **Problem 33**
>
> Find all functions $f : \mathbb{Q} \mapsto \mathbb{C}$ satisfying
>
> (i) For any $x_1, x_2, \ldots, x_{2010} \in \mathbb{Q}$, $f(x_1 + x_2 + \ldots + x_{2010}) = f(x_1)f(x_2)\ldots f(x_{2010})$.
>
> (ii) $\overline{f(2010)}f(x) = f(2010)\overline{f(x)}$ for all $x \in \mathbb{Q}$.

My solution

I have no problem to submit. Anybody feel free to take my turn.

Pain rinne...
1581 posts

Jun 18, 2010, 4:52 pm PM #99

Problem 34: Find all the functions $f : \mathbb{Q} \to \mathbb{R}$ such that :

$$f(x+y+z) = f(x) + f(y) + f(z) + 3\sqrt[3]{f(x+y)f(y+z)f(z+x)}\ ,\ (\forall)x \in \mathbb{R}\ .$$

Mohamma...
58 posts

Jun 18, 2010, 6:15 pm PM #100

> *Pain rinnegan wrote:*
> **Problem 34:** Find all the functions $f : \mathbb{Q} \to \mathbb{R}$ such that :
>
> $$f(x+y+z) = f(x) + f(y) + f(z) + 3\sqrt[3]{f(x+y)f(y+z)f(z+x)}\ ,\ (\forall)x \in \mathbb{R}\ .$$

My Solution

I don't have any problems to submit.

Farenhajt
5170 pos...

Jun 18, 2010, 6:49 pm • 2 PM #101

> *MohammadP wrote:*
> let $(x, y, z) = (a, a, 0)$ and $(x, y, z) = (2a, -a, -a)$, we get $g(2a) = 2g(a)$

If I'm not mistaken, this actually yields $g(2a) = g(2a)$

Farenhajt
5170 pos...

Jun 18, 2010, 10:03 pm PM #102

Still on the previous problem:

Putting $(x, y, z) = (a, a, 0)$ yields $g^3(2a) - 3g(2a)g^2(a) - 2g^3(a) = 0$, which factors as $[g(2a) - 2g(a)][g(2a) + g(a)]^2 = 0$, hence either $g(2a) = 2g(a)$ or $g(2a) = -g(a)$.

Mohamma...
58 posts

Jun 18, 2010, 10:06 pm • 2 PM #103

> *Farenhajt wrote:*
>
> > *MohammadP wrote:*
> > let $(x, y, z) = (a, a, 0)$ and $(x, y, z) = (2a, -a, -a)$, we get $g(2a) = 2g(a)$
>
> If I'm not mistaken, this actually yields $g(2a) = g(2a)$

let $(x, y, z) = (a, a, 0)$, we get
$g(2a)^3 = 2g(a)^3 + 3g(2a)g(a)^2 \Rightarrow (g(2a) - 2g(a))(g(2a) + g(a))^2 = 0$
if $g(2a) \neq 2g(a)$ then $g(2a) = -g(a)$

now let $(x, y, z) = (2a, -a, -a)$ and we get
$0 = g(2a)^3 + 2g(-a)^3 + 3g(2a)g(-a)g(-a)$
and using the fact that $g(-a) = -g(a)$ and $g(2a) = -g(a)$ we get
$g(a) = g(2a) = 0 \Rightarrow g(2a) = 2g(a)$

Farenhajt
5170 pos...

Jun 18, 2010, 10:29 pm PM #104

> *MohammadP wrote:*
> let $(x, y, z) = (a, a, 0)$, we get
> $g(2a)^3 = 2g(a)^3 + 3g(2a)g(a)^2 \Rightarrow (g(2a) - 2g(a))(g(2a) + g(a))^2 = 0$
> if $g(2a) \neq 2g(a)$ then $g(2a) = -g(a)$
>
> now let $(x, y, z) = (2a, -a, -a)$ and we get
> $0 = g(2a)^3 + 2g(-a)^3 + 3g(2a)g(-a)g(-a)$
> and using the fact that $g(-a) = -g(a)$ and $g(2a) = -g(a)$ we get

$g(a) = g(2a) = 0 \Rightarrow g(2a) = 2g(a)$

Ok now. This should have been included in the original solution

Mohamma... 58 posts

Jun 19, 2010, 12:33 am • 2 — PM #105

Yes, you're right. sorry for my bad explanation.

mahanmath 1356 pos...

Jun 21, 2010, 5:27 am • 3 — PM #106

Next problem ??

ilikemaths 50 posts

Jun 21, 2010, 1:25 pm — PM #107

New problem:

Find all functions $f:\mathbb{R}\to\mathbb{R}$, satisfying:
$f(x) = max_{y\in\mathbb{R}}(2xy - f(y))$ for all $x\in\mathbb{R}$.

pco 12955 po...

Jun 21, 2010, 2:12 pm • 1 — PM #108

> ilikemaths wrote:
> New problem:
>
> Find all functions $f:\mathbb{R}\to\mathbb{R}$, satisfying:
> $f(x) = max_{y\in\mathbb{R}}(2xy - f(y))$ for all $x\in\mathbb{R}$.

My solution

I have no problem to submit. Anybody feel free to post.

Amir Hos... 4728 pos...

Jun 21, 2010, 2:24 pm • 1 — PM #109

Problem 37:
Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(f(x)+y) = f(x^2-y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$.

pco 12955 po...

Jun 21, 2010, 2:54 pm — PM #110

> amparvardi wrote:
> **Problem 37**:
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying
>
> $$f(f(x)+y) = f(x^2-y) + 4f(x)y$$
>
> for all $x, y \in \mathbb{R}$.

My solution

I have no problem to submit. Anybody feel free to take my turn.

nguyenvut... 475 posts

Jun 21, 2010, 9:41 pm — PM #111

Problem 38 :

Find all functions $f\ :\ \mathbb{R}^+ \mapsto \mathbb{R}^+$ *such that :*

$(f(x))^2 + 2yf(x) + f(y)\ =\ f(y+f(x))\ \forall y; x\ \in\mathbb{R}^+$

Farenhajt 5170 pos...

Jun 21, 2010, 10:34 pm • 2 — PM #112

> nguyenvuthanhha wrote:
> *Problem 38 :*
>
> *Find all functions* $f\ :\ \mathbb{R}^+ \mapsto \mathbb{R}^+$ *such that :*
>
> $(f(x))^2 + 2yf(x) + f(y)\ =\ f(y+f(x))\ \forall y; x\ \in\mathbb{R}^+$

The equation can be written as

$f(y+f(x)) - (y+f(x))^2 = f(y) - y^2$

Therefore $g(x) := f(x) - x^2$ is periodic, and every $f(x)$ is its period. First, $f(x)$ can be identical to zero, which satisfies the initial equation. Otherwise, $f(x)$ takes the values of the form $k\alpha$, where k is a positive integer and α the minimal (non-zero) period of g.

Hence for natural k, l such that $k > l$ we have

$f(x+k\alpha)-(x+k\alpha)^2 = f(x+l\alpha)-(x+l\alpha)^2$

$f(x+k\alpha)-f(x+l\alpha) = \alpha(k-l)(2x+(k+l)\alpha)$

By the assumption, the LHS is equal to $m\alpha$ for some natural m, which yields

$\alpha = \frac{1}{k+l}\left(\frac{m}{k-l}-2x\right)$

Therefore, the period of the function depends on x, hence any real number can be the period, hence the function is constant.

(Another line of argument: if α is rational, then the above can't be satisfied for irrational x, and vice versa, hence there can be no minimal period. Yet another: a periodic function can have both a rational and an irrational period, which is impossible if it's non-constant.)

So $f(x)-x^2 = \text{const} \iff f(x) = x^2+C, C \in \mathbb{R}$

Hence the solutions are $f(x)=0$ and $f(x)=x^2+C, C \in \mathbb{R}$

Someone else can post the next problem.

---

seifi-seifi
115 posts

Jun 21, 2010, 11:33 pm PM #113

problem39:

let $k \geq 1$ be a given integer.find all function $f: R \rightarrow R$ such that :
$f(x^k+f(y)) = y+f(x)^k$

Farenhajt
5170 pos...

Jun 23, 2010, 3:19 am • 3 PM #114

On a side note: Someone is again going through the topic and spam-rating everything (and not only on this topic). Has anyone noticed similar things elsewhere?

Justhalf
45 posts

Jun 23, 2010, 3:30 am • 3 PM #115

> *Farenhajt wrote:*
> On a side note: Someone is again going through the topic and spam-rating everything (and not only on this topic). Has anyone noticed similar things elsewhere?

You can check your foe list? lol
I didn't see this anywhere else (at least not in the combinatorial section)

Farenhajt
5170 pos...

Jun 23, 2010, 3:47 am • 4 PM #116

> *Justhalf wrote:*
> > *Farenhajt wrote:*
> > On a side note: Someone is again going through the topic and spam-rating everything (and not only on this topic). Has anyone noticed similar things elsewhere?
>
> You can check your foe list? lol

It's always been empty (and I never managed to see the point in having such lists at a board like this). Anyway, let's not go too offtopic :)

Stephen
403 posts

Jun 24, 2010, 6:46 pm • 2 PM #117

> *seifi-seifi wrote:*
> problem39:
>
> let $k \geq 1$ be a given integer.find all function $f: R \rightarrow R$ such that :
> $f(x^k+f(y)) = y+f(x)^k$

Solution of Problem 39

Let $f(0)=a$. Then if $x=0$ then $f(f(y))=y+a^k$.

If we let $y \rightarrow f(y)$, then $f(x^k+f(f(y))) = f(y)+(f(x))^k$.

So $f(x^k+y+a^k) = f(y)+(f(x))^k$.

So $f(f(x^k+y+a^k) = f((f(x))^k+f(y)) = y+(f(f(x))^k = y+(x+a^k)^k$.

But, $f(f(x^k+y+a^k)) = (x^k+y+a^k)+a^k = x^k+y+2a^k$.

So $x^k + 2a^k = (x + a^k)^k$ for all $x \in R$.

We can know easily that $a^k = 0$. So $a = 0$.

So $f(f(x)) = x$.

So $f(x+y) = f(x + f(f(y))) = f((x^{\frac{1}{k}})^k + f(f(y))) = f(y) + (f(x^{\frac{1}{k}}))^k$.

(Here, x is positive)

First, let k is even.

So, $f(x+y) = f(y) + (f(x^{\frac{1}{k}}))^k \geq f(x)$.

So, if $x_1 > x_2$, then $f(x_1) \geq f(x_2)$.

So if $x_0 < f(x_0)$ for some x_0, then

$x_0 < f(x_0) \leq f(f(x_0)) = x_0$. So a contradiction!

In a same way, we can know that there isn't some x_0 that $x_0 > f(x_0)$.

So, for every $x \in R$, $f(x) = x$.

Second, let k is odd.

Since $f(0) = 0$, letting $y = 0$ in $f(x+y) = f(y) + f(x^{\frac{1}{k}}))^k$,

we can have $(x^{\frac{1}{k}}))^k = f(x)$.

So, we can also have $f(x+y) = f(x) + f(y)$ and $f(x^k) = (f(x))^k$.

Since we produced a Cauchy equation, for every rational number q,

$f(qx) = qf(x)$, and $f(x) + f(-x) = 0$ for every $x \in R$.

So $((q+x)^k) = (f(q+x))^k = (f(q) + f(x))^k$.

So $f(\sum_{s=0}^{k} kC_s q^s x^{k-s}) = \sum_{s=0}^{k} kC_s (f(q))^s (f(x))^{k-s}$.

(Sorry, kC_s is k Combination s. I don't do latex very well.)

We can get
$f(\sum_{s=0}^{k} kC_s q^s (f(x))^{k-s}) = f(\sum_{s=0}^{k} kC_s f(q^s)(f(x))^{k-s}) = f(\sum_{s=0}^{k} kC_s q^s (f(1))(f(x))^{k-s}) = f(\sum_{s=0}^{k} kC_s q^s (f(1))^k (f(x))^{k-s})$

(because $f(x^k) = (f(x))^k$)

So for all $s \in 0, 1, 2, ..., k$, we can know that $f(x^{k-s}) = (f(1))^k (f(x))^{k-s}$.

But since $f(1) = (f(1))^k$ and $k-1$ is even, $f(1)$ is 1 or -1.

If $f(1) = 1$, then $f(x^{k-s}) = (f(x))^{k-s}$.

So $(f(x^2)) = (f(x))^2$.

So if $x > 0$, then $f(x) \geq 0$.

Since we have a Cauchy equation, we can tell f is increasing.

So, because of the Cauchy equation, $f(x) = cx$. Since $f(1) = 1$, $f(x) = x$.

If $f(1) = -1$, we can tell $f(x) = -x$ in a similar way.

To conclude, we can tell the solution is

①k is even: $f(x) = x$

②k is odd: $f(x) = x$ or $f(x) = -x$

Problem 40

Find all functions $f: R \to R$ that satisfies $f(xy) + f(x-y) \geq f(x+y)$ for all real numbers x, y.

---

peine
369 posts

Jun 27, 2010, 5:32 am *PM #118*

Solution to Problem 39:
let $y = -f(x)^k$ then there must be a number m such that $f(m) = 0$

let $y=f(m)$ then $f(x^k)=f(x)^k$ (*)
let $y=m$ then $f(x^k)=m+f(x)^k$ then $m=0$
let $x=0$ then $f(f(y))=y$.
we have then: $f(x+y)=f(x+f(f(y)))=f((x^{\frac{1}{k}})^k+f(f(y)))=f(y)+(f(x^{\frac{1}{k}}))^k=f(x)+f(y)$.
let x and two numbers such that $f(x)=f(y)$ then $f(f(x))=f(f(y))\Rightarrow x=y$ which implies that f is injective and hence it's monotonus, and we have $f(x+y)=f(x)+f(y)$ thus $f(x)=cx$ replacing in on (*) we get that $1=c^{k-1}$ if k is odd then $c=1$ or $c=-1$ if k is even then $c=1$.

For Problem 40 I think that we cannot find all the functions veryfing the conditions, actually $f(x)=ax^2+b$ is a solution for every $a>=2b>=0$

Amir Hos...
4728 pos...

Jun 29, 2010, 9:42 pm PM #119

Problem 40 didn't solve for about 4 days, so I'm posting next one.
Problem 41:
Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(1)=f(-1)$ and

$$f(m)+f(n)=f(m+2mn)+f(n-2mn)\qquad\forall m,n\in\mathbb{Z}$$

Justhalf
45 posts

Jul 9, 2010, 10:22 pm PM #120

Oh, no! This thread is dying! Are the last two problems that hard? Actually they're quite hard for me..

---

lajanugen
258 posts

Jul 13, 2010, 4:56 pm PM #121

Problem 41:
We claim that all functions that satisfy $exp_2(x)=exp_2(y)\to f(x)=f(y)$ are the solutions ($f(0)$ can be arbitrary)- this is easily seen to satisfy the equation since $exp_2(x)=exp_2(x+2xy)$ for any non-zero integer x
Plugging in $(m,n)=(n,1),(-1,n)$ and equating the expressions, we obtain that the f values of all odd numbers are equal.
Hence, for all odd n, $f(m)=f((2n+1)m)$ (Since $n-2mn$ would also be odd):
$n=-1$ gives $f(m)=f(-m)$ for all m
As n ranges through all odd values, $2n+1,-(2n+1)$ range through all odd values

waiting for problem 40 to be solved

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mahanmath
1356 pos...

Jul 29, 2010, 10:53 pm PM #122

Problem 42 : Find all functions $\mathbb{R}\to\mathbb{R}$ such that :

$f(x+f(y+f(z)))=f(x)+f(f(y))+f(f(f(z)))$

---

pco
12955 po...

Aug 2, 2010, 8:40 pm • 2 PM #123

> mahanmath wrote:
> **Problem 42** : Find all functions $\mathbb{R}\to\mathbb{R}$ such that :
>
> $f(x+f(y+f(z)))=f(x)+f(f(y))+f(f(f(z)))$

I wonder where (what course, exam,book) you found this problem.

Any solution of Cauchy's equation is obviously a solution.

But there are a lot of other strange solutions : for example, $f(x)=\lfloor 25\sin(2\pi x)\rfloor$

(in fact one of the infinitely many families of solutions is $f(x)=\lfloor h(\{x\})-h(0)\rfloor$ where $h(x)$ is any function from $\mathbb{R}\to\mathbb{R}$)

And I would be surprised if a general form existed for all these solutions.

---

mahanmath
1356 pos...

Aug 2, 2010, 9:55 pm PM #124

> pco wrote:
>> mahanmath wrote:
>> **Problem 42** : Find all functions $\mathbb{R}\to\mathbb{R}$ such that :
>>
>> $f(x+f(y+f(z)))=f(x)+f(f(y))+f(f(f(z)))$
>
> I wonder where (what course, exam,book) you found this problem.
>
> Any solution of Cauchy's equation is obviously a solution.
>
> But there are a lot of other strange solutions : for example, $f(x)=\lfloor 25\sin(2\pi x)\rfloor$
>
> (in fact one of the infinitely many families of solutions is $f(x)=\lfloor h(\{x\})-h(0)\rfloor$ where $h(x)$ is any function from $\mathbb{R}\to\mathbb{R}$)
>
> And I would be surprised if a general form existed for all these solutions.

I`m so sorry , I forgot the main condition ! The problem said find all continuous functions .

pco
12955 po...

Aug 2, 2010, 10:41 pm • 1 PM #125

> mahanmath wrote:
>
> > pco wrote:
> >
> > > mahanmath wrote:
> > > **Problem 42** : Find all functions $\mathbb{R} \to \mathbb{R}$ such that :
> > >
> > > $f(x+f(y+f(z))) = f(x)+f(f(y))+f(f(f(z)))$
> >
> > I wonder where (what course, exam,book) you found this problem.
> >
> > Any solution of Cauchy's equation is obviously a solution.
> >
> > But there are a lot of other strange solutions : for example, $f(x) = \lfloor 25\sin(2\pi x)\rfloor$
> >
> > (in fact one of the infinitely many families of solutions is $f(x) = \lfloor h(\{x\}) - h(0)\rfloor$ where $h(x)$ is any function from $\mathbb{R} \to \mathbb{R}$)
> >
> > And I would be surprised if a general form existed for all these solutions.
>
> I`m so sorry , I forgot the main condition ! The problem said find all continuous functions .

It's a pity to see how many users are quite unable to correctly copy a six words problem

With continuity, this is a trivial problem :
Let $P(x,y,z)$ be the assertion $f(x+f(y+f(z))) = f(x)+f(f(y))+f(f(f(z)))$

Subtracting $P(0,y-f(z),z)$ from $P(x,y-f(z),z)$, we get $f(x+f(y)) = f(x)+f(f(y))-f(0)$
Let $g(x) = f(x)-f(0)$ and $A = f(\mathbb{R})$

We got $g(x+y) = g(x)+g(y)\ \forall x\in\mathbb{R}, \forall y\in A$
And also $g(x-y) = g(x)-g(y)\ \forall x\in\mathbb{R}, \forall y\in A$

$g(x+y_1+y_2) = g(x+y_1)+g(y_2) = g(x)+g(y_1)+g(y_2) = g(x)+g(y_1+y_2)\ \forall x\in\mathbb{R}, \forall y_1,y_2\in A$
$g(x+y_1-y_2) = g(x+y_1)-g(y_2) = g(x)+g(y_1)-g(y_2) = g(x)+g(y_1-y_2)\ \forall x\in\mathbb{R}, \forall y_1,y_2\in A$

And, with simple induction, $g(x+y) = g(x)+g(y)\ \forall x, \forall y$ finite sums and differences of elements of A

If cardinal of A is 1, we get $f(x) = c$ and so $f(x) = 0$
If cardinal of A is not 1 and since $f(x)$ is continuous, $\exists u < v$ such that $[u,v] \subseteq A$ and any real may be represented as finite sums and differences of elements of $[u,v]$

So $g(x+y) = g(x)+g(y)\ \forall x,y$ and so, since continuous, $g(x) = ax$ and $f(x) = ax+b$

Plugging this in original equation, we get $b(a+2) = 0$

Hence the solutions :
$f(x) = ax$
$f(x) = b-2x$

Amir Hos...
4728 pos...

Aug 3, 2010, 2:37 pm PM #126

I'm posting next problem.

Problem 43 :
Let f be a real function defined on the positive half-axis for which $f(xy) = xf(y)+yf(x)$ and $f(x+1) \le f(x)$ hold for every positive x and y. Show that if $f(1/2) = 1/2$, then

$$f(x)+f(1-x) \ge -x\log_2 x-(1-x)\log_2(1-x)$$

for every $x \in (0,1)$.

Dijkschnei...
131 posts

Aug 17, 2010, 3:17 am PM #127

This marathon is interesting. Why has it stopped ?

Amir Hos...
4728 pos...

Aug 17, 2010, 12:44 pm PM #128

Yeah, it seems problem 43 is difficult.
Can I post a new problem ?

Dijkschnei...
131 posts

Aug 17, 2010, 9:19 pm PM #129

I would appreciate that.

Amir Hos...
4728 pos...

Aug 17, 2010, 9:29 pm PM #130

Problem 44 :
Let a be a real number and let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying: $f(0) = \frac{1}{2}$ and

$f(x+y) = f(x)f(a-y) + f(y)f(a-x), \forall x, y \in \mathbb{R}$.
Prove that f is constant.

pco
12955 po...

Aug 17, 2010, 9:47 pm PM #131

> *amparvardi wrote:*
> **Problem 44 :**
> Let a be a real number and let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying: $f(0) = \frac{1}{2}$ and
> $f(x+y) = f(x)f(a-y) + f(y)f(a-x), \forall x, y \in \mathbb{R}$.
> Prove that f is constant.

My solution

And I have no problem to submit. So anybody feel free to post a new problem.

Amir Hos...
4728 pos...

Aug 17, 2010, 10:36 pm PM #132

Problem 45:
Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)^3 = -\frac{x}{12} \cdot \left(x^2 + 7x \cdot f(x) + 16 \cdot f(x)^2\right), \ \forall x \in \mathbb{R}.$$

pco
12955 po...

Aug 17, 2010, 10:49 pm PM #133

> *amparvardi wrote:*
> **Problem 45**:
> Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that
>
> $$f(x)^3 = -\frac{x}{12} \cdot \left(x^2 + 7x \cdot f(x) + 16 \cdot f(x)^2\right), \ \forall x \in \mathbb{R}.$$

My solution

And I have no problem to submit. Anybody feel free to post a new one.

Amir Hos...
4728 pos...

Aug 17, 2010, 10:56 pm PM #134

So quick Mr Patrick 🙂

Problem 46:
Find all functions $f : \mathbb{R}\backslash\{0,1\} \to \mathbb{R}$ such that

$$f(x) + f\left(\frac{1}{1-x}\right) = 1 + \frac{1}{x(1-x)}.$$

pco
12955 po...

Aug 17, 2010, 11:26 pm PM #135

> *amparvardi wrote:*
> **Problem 46:**
> Find all functions $f : \mathbb{R}\backslash\{0,1\} \to \mathbb{R}$ such that
>
> $$f(x) + f\left(\frac{1}{1-x}\right) = 1 + \frac{1}{x(1-x)}.$$

My solution

And I have no problem to submit. So anybody feel free to post a new one.

Amir Hos...
4728 pos...

Aug 17, 2010, 11:32 pm PM #136

Problem 47:
Let $f(x)$ be a real-valued function defined on the positive reals such that

(1) if $x < y$, then $f(x) < f(y)$,

(2) $f\left(\frac{2xy}{x+y}\right) \geq \frac{f(x)+f(y)}{2}$ for all x.

Show that $f(x) < 0$ for some value of x.

pco
12955 po...

Aug 18, 2010, 1:47 am PM #137

> *amparvardi wrote:*
> **Problem 47**:
> Let $f(x)$ be a real-valued function defined on the positive reals such that

(1) if $x < y$, then $f(x) < f(y)$,

(2) $f\left(\frac{2xy}{x+y}\right) \geq \frac{f(x)+f(y)}{2}$ for all x.

Show that $f(x) < 0$ for some value of x.

My ugly solution

And I dont have any problem to submit. So anybody feel free to post a new one.

---

aktyw19
1315 posts

Aug 18, 2010, 11:25 am PM #138

Problem 48:
Find all continuous functions $R->R$ satisfying the equation:
$f(x+y)+f(xy)=f(x)+f(y)+f(xy+1)$

Problem 49:
Find all continuous functions $R->R$ satisfying the equation:
$f(x)+f(y)+f(z)+f(x+y+z)=f(x+y)+f(y+z)+f(z+x)+f(0)$

Dumel
190 posts

Aug 20, 2010, 3:08 am PM #139

> *amparvardi wrote:*
> **Problem 47**:
> Let $f(x)$ be a real-valued function defined on the positive reals such that
>
> (1) if $x < y$, then $f(x) < f(y)$,
>
> (2) $f\left(\frac{2xy}{x+y}\right) \geq \frac{f(x)+f(y)}{2}$ for all x.
>
> Show that $f(x) < 0$ for some value of x.

hmmm I think there is no function satisfying these conditions.
suppose that $f(x)>0$ for all $x>0$
let $a=\lim_{x\to 0} f(x)$
for $y=1$ and $x\to 0$ we get
$a \geq f(1) > a$ which is contradition.
am I wrong?

---

pco
12955 po...

Aug 20, 2010, 12:24 pm PM #140

> *Dumel wrote:*
>
> > *amparvardi wrote:*
> > **Problem 47**:
> > Let $f(x)$ be a real-valued function defined on the positive reals such that
> >
> > (1) if $x < y$, then $f(x) < f(y)$,
> >
> > (2) $f\left(\frac{2xy}{x+y}\right) \geq \frac{f(x)+f(y)}{2}$ for all x.
> >
> > Show that $f(x) < 0$ for some value of x.
>
> hmmm I think there is no function satisfying these conditions.
> suppose that $f(x)>0$ for all $x>0$
> let $a=\lim_{x\to 0} f(x)$
> for $y=1$ and $x\to 0$ we get
> $a \geq f(1) > a$ which is contradition.
> am I wrong?

Quite nice and simple ! Congrats!
Just two - very little - remarks :
a) replace your first phrase by "suppose that $f(x)\geq 0$ for all $x>0$"
b) since the rule $x<y \implies f(x)<f(y)$ is available only for $x>0$, one more line IMHO is needed to conclude $f(1)>a$ which is true

---

Amir Hos...
4728 pos...

Aug 20, 2010, 12:56 pm PM #141

But this problem is **Brazil National MO 2003** Problem 5 !

Dijkschnei...
131 posts

Aug 26, 2010, 2:10 am PM #142

> *Dumel wrote:*
>
> > *amparvardi wrote:*

amparvardi wrote:

Problem 47:
Let $f(x)$ be a real-valued function defined on the positive reals such that

(1) if $x<y$, then $f(x)<f(y)$,

(2) $f\left(\frac{2xy}{x+y}\right)\geq\frac{f(x)+f(y)}{2}$ for all x.

Show that $f(x)<0$ for some value of x.

hmmm I think there is no function satisfying these conditions.
suppose that $f(x)>0$ for all $x>0$
let $a=\lim_{x\to 0}f(x)$
for $y=1$ and $x\to 0$ we get
$a\geq f(1)>a$ which is contradition.
am I wrong?

Don't we need a continuity hypothesis to claim that f(1) > a ?
$f(1)>f(x\to 0)$, and we need continuity to have $f(1)>\lim_{x\to 0}f(x)$.

---

pco
12955 po...

Aug 26, 2010, 1:11 pm PM #143

> Dijkschneier wrote:
>
> > Dumel wrote:
> >
> > > amparvardi wrote:
> > >
> > > **Problem 47**:
> > > Let $f(x)$ be a real-valued function defined on the positive reals such that
> > >
> > > (1) if $x<y$, then $f(x)<f(y)$,
> > >
> > > (2) $f\left(\frac{2xy}{x+y}\right)\geq\frac{f(x)+f(y)}{2}$ for all x.
> > >
> > > Show that $f(x)<0$ for some value of x.
> >
> > hmmm I think there is no function satisfying these conditions.
> > suppose that $f(x)>0$ for all $x>0$
> > let $a=\lim_{x\to 0}f(x)$
> > for $y=1$ and $x\to 0$ we get
> > $a\geq f(1)>a$ which is contradition.
> > am I wrong?
>
> Don't we need a continuity hypothesis to claim that f(1) > a ?
> $f(1)>f(x\to 0)$, and we need continuity to have $f(1)>\lim_{x\to 0}f(x)$.

No, we would need continuity to write $f(1)>f(0)$

But if you have $f(1)>f(x)$ and $\lim_{x\to 0}f(x)$ exists, you can write $f(1)\geq\lim_{x\to 0}f(x)$

The fact that, here, we can write $f(1)>\lim_{x\to 0}f(x)$ instead of $f(1)\geq\lim_{x\to 0}f(x)$ is due to the fact that $f(x)$ is strictly increasing.

Dijkschnei...
131 posts

Aug 26, 2010, 11:50 pm PM #144

> pco wrote:
>
> The fact that, here, we can write $f(1)>\lim_{x\to 0}f(x)$ instead of $f(1)\geq\lim_{x\to 0}f(x)$ is due to the fact that $f(x)$ is strictly increasing.

Can you explain that, please ?

---

pco
12955 po...

Aug 27, 2010, 12:06 am PM #145

> Dijkschneier wrote:
>
> > pco wrote:
> >
> > The fact that, here, we can write $f(1)>\lim_{x\to 0}f(x)$ instead of $f(1)\geq\lim_{x\to 0}f(x)$ is due to the fact that $f(x)$ is strictly increasing.
>
> Can you explain that, please ?

$f(\frac{1}{2}) > f(x)\,\forall x < \frac{1}{2}$

So, since $\lim_{x\to 0} f(x)$ exists, we get $f(\frac{1}{2}) \ge \lim_{x\to 0} f(x)$

And since $f(1) > f(\frac{1}{2})$ we get $f(1) > \lim_{x\to 0} f(x)$

Dijkschnei... 131 posts — Aug 27, 2010, 12:40 am — PM #146

Indeed. Thanks.

---

mahanmath 1356 pos... — Sep 8, 2010, 4:41 am • 1 — PM #147

Problem 48 :

Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(f(x)+y) = f(x^2-y)+4f(x)y$$

for all $x, y \in \mathbb{R}$.

Sansa 108 posts — Sep 8, 2010, 7:46 am • 1 — PM #148

Click to reveal hidden text

Let $P(x,y)$ be the assertion of $f(f(x)+y) = f(x^2-y)+4f(x)y$.

$P(x,x^2): f(f(x)+x^2) = f(0)+4x^2f(x)$

$P(x,-f(x)): f(0) = f(f(x)+x^2)-4f(x)^2$

$\Longrightarrow f(0) = f(0)+4x^2f(X)-4f(x)^2 \Longrightarrow f(x)(f(x)-x^2)=0 \Longrightarrow f(x)=0$ or $f(x)=x^2$ or sometimes $f(x)=0$ and sometimes $f(x)=x^2, \forall x\in\mathbb{R}$

If sometimes $f(x)=0$ and sometimes $f(x)=x^2 \Longrightarrow \exists x_0, y_0$: $x_0 \neq y_0 \neq 0$ and $f(x_0)=0$ and $f(y_0)={y_0}^2$

so $P(x_0,y_0): {y_0}^2 = f({x_0}^2-y_0)$

But we know that $f({x_0}^2-y_0) = ({x_0}^2-y_0)^2$ or $f({x_0}^2-y_0)=0$ and that is not correct.

problem 49:
Find all functions f: $\mathbb{R}\to\mathbb{R}$ satisfying:

$$f(xf(y)+f(x)) = 2f(x)+xy$$

This post has been edited 2 times. Last edited by Sansa, Sep 8, 2010, 8:27 pm

---

aktyw19 1315 posts — Sep 8, 2010, 10:58 am — PM #149

unsolved problems

> *Quote:*
> **Problem 48:**
> Find all continuous functions $R->R$ satisfying the equation:
> $f(x+y)+f(xy) = f(x)+f(y)+f(xy+1)$
>
> **Problem 49:**
> Find all continuous functions $R->R$ satisfying the equation:
> $f(x)+f(y)+f(z)+f(x+y+z) = f(x+y)+f(y+z)+f(z+x)+f(0)$

---

pco 12955 po... — Sep 8, 2010, 10:50 pm • 3 — PM #150

> *Sansa wrote:*
> Let $P(x,y)$ be the assertion of $f(f(x)+y) = f(x^2-y)+4f(x)y$.
>
> $P(x,x^2): f(f(x)+x^2) = f(0)+4x^2f(x)$
>
> $P(x,-f(x)): f(0) = f(f(x)+x^2)-4f(x)^2$
>
> $\Longrightarrow f(0) = f(0)+4x^2f(X)-4f(x)^2 \Longrightarrow f(x)(f(x)-x^2)=0 \Longrightarrow f(x)=0$ or $f(x)=x^2$ or sometimes $f(x)=0$ and sometimes $f(x)=x^2, \forall x\in\mathbb{R}$

Yes. Little remark : one line for the same result : $P(x, \frac{x^2-f(x)}{2})$ 🙂

---

pco 12955 po... — Sep 9, 2010, 12:51 am • 1 — PM #151

> *Sansa wrote:*

problem 49:
Find all functions f: $\mathbb{R}\to\mathbb{R}$ satisfying:

$$f(xf(y)+f(x))=2f(x)+xy$$

Here is My solution

And anybody feel free to post the next problem.

Sansa
108 posts

Sep 12, 2010, 8:33 pm PM #152

> pco wrote:
>
> Yes. Little remark : one line for the same result : $P(x,\dfrac{x^2-f(x)}{2})$:)

It was really terrific... :)

Rijul saini
799 posts

Sep 12, 2010, 11:10 pm PM #153

Problem 50:
If the following conditions are satisfied by a function $f:\mathbb{R}\to\mathbb{R}$, then prove that it's the identity function.
1) $f(-x)=-f(x)$
2) $f(x+1)=f(x)+1$
3) $f(\frac{1}{x})=\dfrac{f(x)}{x^2}$

---

pco
12955 po...

Sep 13, 2010, 10:43 pm PM #154

> Rijul saini wrote:
>
> ***Problem 50:***
> If the following conditions are satisfied by a function $f:\mathbb{R}\to\mathbb{R}$, then prove that it's the identity function.
> 1) $f(-x)=-f(x)$
> 2) $f(x+1)=f(x)+1$
> 3) $f(\frac{1}{x})=\dfrac{f(x)}{x^2}$

Hello, could you confirm us that the function is from $\mathbb{R}\to\mathbb{R}$ (and not, for example, from $\mathbb{Q}\to\mathbb{Q}$) ?

It's easy to show that $f(x)=x\ \forall x\in\mathbb{Q}$ using, for example, induction on the length of continuous fraction of x.

But extending to $\mathbb{R}$ is not so easy. And I even think (but have not proved) that this is wrong.

---

fedja
6941 pos...

Sep 13, 2010, 11:07 pm PM #155

Let us call the operations $x\mapsto -x$, $x\mapsto 1/x$ and $x\mapsto x\pm 1$ elementary. Call two real numbers equivalent if one can be obtained from another by a chain of elementary operations (they are all invertible, so it is, indeed, an equivalence relation). Note that the values of f for the points in different equivalence classes are completely independent. All classes have just to be consistent within themselves Now, take the equivalence class C generated by $\sqrt{2}$. It is fully contained in the field $\mathbb{Q}[\sqrt{2}]$. In this field, we have an automorphism σ over $\mathbb{Q}$ that sends $\sqrt{2}$ to $-\sqrt{2}$. Obviously, $\sigma(x)$ satisfies the equation in $\mathbb{Q}[\sqrt{2}]$ and, therefore, in C. Thus, we have a non-trivial mapping in C. It can be extended to the full mapping using the trivial identity mapping in each other equivalence class. The extension is certainly not an identity.

---

arshakus
746 posts

Sep 13, 2010, 11:17 pm • 1 PM #156

> pco wrote:
>
> > Rijul saini wrote:
> >
> > ***Problem 50:***
> > If the following conditions are satisfied by a function $f:\mathbb{R}\to\mathbb{R}$, then prove that it's the identity function.
> > 1) $f(-x)=-f(x)$
> > 2) $f(x+1)=f(x)+1$
> > 3) $f(\frac{1}{x})=\dfrac{f(x)}{x^2}$
>
> Hello, could you confirm us that the function is from $\mathbb{R}\to\mathbb{R}$ (and not, for example, from $\mathbb{Q}\to\mathbb{Q}$) ?
>
> It's easy to show that $f(x)=x\ \forall x\in\mathbb{Q}$ using, for example, induction on the length of continuous fraction of x.
>
> But extending to $\mathbb{R}$ is not so easy. And I even think (but have not proved) that this is wrong.

hey everybody,
pco u r not right because it is right even that $f:R->R$.
$f(-x)=-f(x)$

$f(x+1) = f(x)+1 = f(\frac{}{x}) = f(\frac{}{x}+1)-1$
$f(\frac{1}{x}) = \frac{f(x)}{x^2} => f(\frac{x}{x+1}) = \frac{x^2}{(x+1)^2}f(\frac{x+1}{x})$
when x not equal 0 and -1=> $f(x) = x^2f(\frac{1}{x}) = x^2(f(\frac{x+1}{x})-1) = x^2(\frac{(x+1)^2}{x^2}f(\frac{x}{x+1})-1) =>$
$=> f(x) = (x+1)^2f(\frac{x}{x+1}) - x^2$
$f(\frac{x}{x+1}) = f(\frac{(x+1)-1}{x+1}) = f(1-\frac{1}{x+1}) = f(-\frac{1}{x+1})+1 = 1-f(\frac{1}{x+1}) = 1-\frac{1}{(x+1)^2}f(x+1) =>$
$=> (x+1)^2 - f(x+1) - x^2 = f(x)$
$2x+1 = f(x)+f(x+1) => f(x) = x, whenxnotequal0and-1 => whenx = 0 => f(0) = 0whenx = -1 => f(0) = f(-1)+1 =>$
$f(-1) = -1 => [url]f(x) = x[/url]$

---

fedja
6941 pos...

Sep 13, 2010, 11:33 pm PM #157

Oops, you seem to be right. I haven't noticed that the last identity uses x explicitly, not through $f()$, which invalidates my counterexample.

---

arshakus
746 posts

Sep 13, 2010, 11:37 pm PM #158

> *fedja wrote:*
> Oops, you seem to be right. I haven't noticed that the last identity uses x explicitly, not through $f()$, which invalidates my counterexample.

check it one more....

---

pco
12955 po...

Sep 13, 2010, 11:40 pm PM #159

> *arshakus wrote:*
> ...
> $f(\frac{1}{x}) = \frac{f(x)}{x^2} => f(\frac{x}{x+1}) = \frac{x^2}{(x+1)^2}f(\frac{x+1}{x})$
> when x not equal 0 and -1=> $f(x) = x^2f(\frac{1}{x}) = x^2(f(\frac{x+1}{x})-1) = x^2(\frac{(x+1)^2}{x^2}f(\frac{x}{x+1})-1) =>$
> $=> f(x) = (x+1)^2f(\frac{x}{x+1}) - x^2$
> $f(\frac{x}{x+1}) = f(\frac{(x+1)-1}{x+1}) = f(1-\frac{1}{x+1}) = f(-\frac{1}{x+1})+1 = 1-f(\frac{1}{x+1}) = 1-\frac{1}{(x+1)^2}f(x+1) =>$
> $=> (x+1)^2 - f(x+1) - x^2 = f(x)$
> $2x+1 = f(x)+f(x+1) \implies f(x) = x$,...

It seems you are right.
Nice!
And congrats.
:)

---

arshakus
746 posts

Sep 13, 2010, 11:49 pm PM #160

> *pco wrote:*
>
> > *arshakus wrote:*
> > ...
> > $f(\frac{1}{x}) = \frac{f(x)}{x^2} => f(\frac{x}{x+1}) = \frac{x^2}{(x+1)^2}f(\frac{x+1}{x})$
> > when x not equal 0 and -1=> $f(x) = x^2f(\frac{1}{x}) = x^2(f(\frac{x+1}{x})-1) = x^2(\frac{(x+1)^2}{x^2}f(\frac{x}{x+1})-1) =>$
> > $=> f(x) = (x+1)^2f(\frac{x}{x+1}) - x^2$
> > $f(\frac{x}{x+1}) = f(\frac{(x+1)-1}{x+1}) = f(1-\frac{1}{x+1}) = f(-\frac{1}{x+1})+1 = 1-f(\frac{1}{x+1}) = 1-\frac{1}{(x+1)^2}f(x+1) =>$
> > $=> (x+1)^2 - f(x+1) - x^2 = f(x)$
> > $2x+1 = f(x)+f(x+1) \implies f(x) = x$,...
>
> It seems you are right.
> Nice!
> And congrats.
> :)

thanks a lot pco

Rijul saini
799 posts

Sep 14, 2010, 3:25 am PM #161

> *pco wrote:*
> Hello, could you confirm us that the function is from $\mathbb{R}\to\mathbb{R}$ (and not, for example, from $\mathbb{Q}\to\mathbb{Q}$) ?
>
> It's easy to show that $f(x) = x$ $\forall x\in\mathbb{Q}$ using, for example, induction on the length of continuous fraction

of x.

But extending to $\mathbb{R}$ is not so easy. And I even think (but have not proved) that this is wrong.

Of course I can confirm your doubt.
It came in the INMO (Don't remember which year it was), and you can click here to check out mine (and others) proof of it.
Also, can you clarify my doubt regarding raghu.mahajan's proof in that thread?

Rijul saini
799 posts

Sep 21, 2010, 10:24 pm — PM #162

Problem 51 :
Find all one-one functions $f:\mathbb{N}\to\mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, which satisfies

$$f(f(n)) \le \frac{f(n)+n}{2}$$

This post has been edited 1 time. Last edited by Rijul saini, Sep 23, 2010, 2:15 am

---

pco
12955 po...

Sep 23, 2010, 12:21 am — PM #163

> *Rijul saini wrote:*
> ***Problem 51 :***
> Find all functions $f:\mathbb{N}\to\mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, which satisfies
> $$f(f(n)) \le \frac{f(n)+n}{2}$$

Are you sure about the problem statement ?
With your statement, there are obviously infinitely many solutions, and I'm not sure we can find a general formula for them.

For example :

$f(x)=x$
$f(x)=1$
Any $f(x)$ such that $f(f(x))=1$ (and there are infinitely many such functions)
Any $f(x)$ such that $f(1)\ge 3$ and $f(f(x))=2$ (and there are infinitely many such functions)
Any $f(x)$ such that $f(1)=f(2)=1$ and $f(f(x))\in\{1,2\}\ \forall x$
.....

---

mahanmath
1356 pos...

Sep 23, 2010, 12:30 am — PM #164

I have a short solution for injective f .

Rijul saini
799 posts

Sep 23, 2010, 2:16 am — PM #165

> *pco wrote:*
>> *Rijul saini wrote:*
>> ***Problem 51 :***
>> Find all functions $f:\mathbb{N}\to\mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, which satisfies
>> $$f(f(n)) \le \frac{f(n)+n}{2}$$
>
> Are you sure about the problem statement ?
> With your statement, there are obviously infinitely many solutions, and I'm not sure we can find a general formula for them.
>
> For example :
>
> $f(x)=x$
> $f(x)=1$
> Any $f(x)$ such that $f(f(x))=1$ (and there are infinitely many such functions)
> Any $f(x)$ such that $f(1)\ge 3$ and $f(f(x))=2$ (and there are infinitely many such functions)
> Any $f(x)$ such that $f(1)=f(2)=1$ and $f(f(x))\in\{1,2\}\ \forall x$
> .....

Sorry Patrick, missed the one-one condition

---

pco
12955 po...

Sep 23, 2010, 10:03 pm — PM #166

> *Rijul saini wrote:*
> Sorry Patrick, missed the one-one condition

It's easy to show with induction that $f^{[k]}(n)\le\frac{2f(n)+n}{3}+\frac{2}{3(-2)^k}(n-f(n))$

So, for k great enough : $f^{[k]}(n) \le \frac{2f(n)+n}{3}+1$ and so $\exists\ k_1 > k_2$ such that $f^{[k_1]}(n) = f^{[k_2]}(n)$ and, since injective :

$\forall n\ \exists p_n \ge 1$ such that $f^{[p_n]}(n) = n$

Then, setting $k = p_n$ in the above inequality, we get $n \le \frac{2f(n)+n}{3} + \frac{2}{3(-2)^{p_n}}(n - f(n))$

$\iff 0 \le (f(n)-n)(1 - \frac{1}{(-2)^{p_n}})$ and so $f(n) \ge n\ \forall n$

But $f(n) > n$ for some n and injectivity would imply $f^{[p_n]}(n) > n$ and so $f(n) = n\ \forall n$ which indeed is a solution.

Dumel
190 posts

Sep 24, 2010, 3:54 am PM #167

an alternative solution:
by strong induction we can easily prove that if $f(n) \le n$ for some n then for all natural k $f^k(n) \le n$ whence by injectivity we can simply deduce that $f(n) = n$
hence $f(n) \ge n$ for all n, so $f(n) = n$ for all n

Amir Hos...
4728 pos...

Sep 24, 2010, 10:40 am PM #168

Here is a new problem :

Problem 52 :

Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x)+y) = f(x^2-y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$.

---

pco
12955 po...

Sep 24, 2010, 12:25 pm PM #169

> *amparvardi wrote:*
> Here is a new problem :
>
> **Problem 52** :
>
> Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
>
> $$f(f(x)+y) = f(x^2-y) + 4f(x)y$$
>
> for all $x, y \in \mathbb{R}$.

Let $P(x,y)$ be the assertion $f(f(x)+y) = f(x^2-y) + 4f(x)y$

$P(x, \frac{x^2-f(x)}{2}) \implies f(x)(f(x)-x^2) = 0\ \forall x$ and so : $\forall x$, either $f(x) = 0$, either $f(x) = x^2$

Suppose now $\exists a \ne 0$ such that $f(a) = 0$ and $b \ne 0$ such that $f(b) = b^2$:

$P(a,b) \implies b^2 = f(a^2-b)$ and so $b^2 = (a^2-b)^2$ and so $b = \frac{a^2}{2}$

So, if $\exists a \ne 0$ such that $f(a) = 0$, then $f(x) = 0\ \forall x \ne \frac{a^2}{2}$ but then, choosing another a, we get that $f(\frac{a^2}{2}) = 0$ too.

Then, either $f(x) = 0\ \forall x$, either $f(x) = x^2\ \forall x$ and these two functions indeed are solutions.

Hence the only two solutions :
$f(x) = 0\ \forall x$
$f(x) = x^2\ \forall x$

Amir Hos...
4728 pos...

Sep 24, 2010, 12:37 pm PM #170

Problem 53 :

For a given natural number $k > 1$, find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,
$f[x^k + f(y)] = y + [f(x)]^k$.

---

pco
12955 po...

Sep 24, 2010, 12:44 pm PM #171

> *amparvardi wrote:*
> **Problem 53** :
>
> For a given natural number $k > 1$, find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,
> $f[x^k + f(y)] = y + [f(x)]^k$.

Are the brackets just parenthesis or integer parts ?

Amir Hos... 4728 pos... Sep 24, 2010, 1:07 pm PM #172

They are just parenthesis.

pco 12955 po... Sep 24, 2010, 8:27 pm • 1 PM #173

> amparvardi wrote:
> **Problem 53** :
>
> For a given natural number $k>1$, find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all $x,y\in\mathbb{R}$,
> $f[x^k+f(y)]=y+[f(x)]^k$.

Let $P(x,y)$ be the assertion $f(x^k+f(y))=y+f(x)^k$
Let $f(0)=a$

$P(0,y)\implies f(f(y))=y+a^k$
$P(x,0)\implies f(x^k+a)=f(x)^k$
$P(x,f(y))\implies f(x^k+y+a^k)=f(y)+f(x^k+a)$

Let then $g(x)=f(x-a^k+a)$. This last equality becomes $g(x^k+y+2a^k-a)=g(y+a^l-a)+g(x^k+a^k)$
$\iff g(x^k+a^k+y)=g(y)+g(x^k+a^k)$

And so $g(x+y)=g(x)+g(y)\ \forall x\ge a^k,\forall y$
Let then $x\ge 0$:
$g(a^k+x+y)=g(a^k+(x+y))=g(a^k)+g(x+y)$
$g(a^k+x+y)=g((a^k+x)+y)=g(a^k+x)+g(y)=g(a^k)+g(x)+g(y)$
And so $g(x+y)=g(x)+g(y)\ \forall x\ge 0,\forall y$

So $g(0)=0$ and $g(-x)=-g(x)$. Then :
$\forall x\ge 0,\forall y: -g(x-y)=-g(x)-g(-y)\implies g(-x+y)=g(-x)+g(y)$ and so $g(x+y)=g(x)+g(y)\ \forall x,y$

And so $g(px)=pg(x)\ \forall p\in\mathbb{Q},\forall x$

Then $f(x^k+a)=f(x)^k$ implies $g(x^k+a^k)=g(x+a^k-a)^k\implies g(x^k)+g(a^k)=(g(x)+g(a^k-a))^k$
Notice that $g(a^k-a)=f(0)=a$ and replace x with $x+y$ and we get :

$g((x+y)^k)+g(a^k)=(g(x)+g(y)+a)^k$

$$g(\sum_{i=0}^k\binom{k}{i}x^iy^{k-i})+g(a^k)=\sum_{i=0}^k\binom{k}{i}g(x)^i(g(y)+a)^{k-i}$$

Let then $x\in\mathbb{Q}$ and this equation becomes :

$$\sum_{i=0}^k\binom{k}{i}x^ig(y^{k-i})+g(a^k)=\sum_{i=0}^k\binom{k}{i}g(1)^ix^i(g(y)+a)^{k-i}$$

And so we have two polynomials in x (LHS and RHS) which are equal for any $x\in\mathbb{Q}$. So they are identical and all their coefficients are equal.

Since $k\ge 2$, consider the equality of coefficients of x^{k-2} :
If $k>2$, this equality is $g(y^2)=g(1)^{k-2}(g(y)+a)^2$ and $g(x)$ has a constant sign over $\mathbb{R}^+$
If $k=2$, this equality becomes $g(y^2)+g(a^2)=(g(y)+a)^2$ and $g(x)\ge -g(a^2)\ \forall x\ge 0$

In both cases, we have $g(x)$ either upper bounded, either lower-bounded on a non empty open interval, and this a classical condition to conclude to continuity and $g(x)=cx\ \forall x$

And so $f(x)=cx+d$ for some real c,d

Plugging this back in original equation, we get :

$f(x)=x\ \forall x$ which is a solution for any k
$f(x)=-x\ \forall x$ which is another solution if k is odd.

Amir Hos... 4728 pos... Sep 24, 2010, 8:30 pm PM #174

Thank you, Mr. pco :)

Problem 54 :

Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that for all $x,y\in\mathbb{Z}$:

$$f(x-y+f(y))=f(x)+f(y).$$

pco Sep 24, 2010, 9:27 pm PM #175

pco
12955 po...

> amparvardi wrote:
> **Problem 54** :
>
> Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that for all $x,y\in\mathbb{Z}$:
> $$f(x-y+f(y))=f(x)+f(y).$$

Let $P(x,y)$ be the assertion $f(x-y+f(y))=f(x)+f(y)$
Let $f(0)=a$

$P(0,0)\implies f(a)=2a$ and so $f(a)-a=a$
$P(0,a)\implies f(f(a)-a)=f(0)+f(a)$ and so $f(0)=0$

$P(0,x)\implies f(f(x)-x)=f(x)$
$P(x,f(y)-y)\implies f(x-f(y)+y+f(f(y)-y))=f(x)+f(f(y)-y)$ and so $f(x+y)=f(x)+f(y)$ and so $f(x)=xf(1)$ (remember we are in $\mathbb{Z}$)

Plugging this in original equation, we get two solutions :

$f(x)=0\ \forall x$
$f(x)=2x\ \forall x$

ArefS
297 posts

Sep 24, 2010, 9:39 pm — PM #176

My Solution

Amir Hos...
4728 pos...

Sep 24, 2010, 9:45 pm — PM #177

Problem 55 :

We denote by $\mathbb{R}^+$ the set of all positive real numbers.

Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ which have the property:
$$f(x)f(y)=2f(x+yf(x))$$

for all positive real numbers x and y.

---

pco
12955 po...

Sep 24, 2010, 11:41 pm — PM #178

> amparvardi wrote:
> **Problem 55** :
>
> We denote by $\mathbb{R}^+$ the set of all positive real numbers.
>
> Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ which have the property:
> $$f(x)f(y)=2f(x+yf(x))$$
>
> for all positive real numbers x and y.

Let $P(x,y)$ be the assertion $f(x)f(y)=2f(x+yf(x))$

Let $u,v>0$.
Let $a\in(0,u)$

Let $x=a>0$ and $y=\dfrac{u-a}{f(a)}>0$ and $z=\dfrac{2v}{f(x)f(y)}>0$

$f(x)f(y)=2f(x+yf(x))=2f(u)$ and so $f(x)f(y)f(z)=2f(u)f(z)=4f(u+zf(u))=4f(u+v)$

$f(y)f(z)=2f(y+zf(y))$ and so $f(x)f(y)f(z)=2f(x)f(y+zf(y))=4f(x+(y+zf(y))f(x))$
$=4f(x+yf(x)+zf(x)f(y))=4f(u+2v)$

And so $f(u+v)=f(u+2v)\ \forall u,v>0$ and so $f(x)=f(y)\ \forall x,y$ such that $2x>y>x>0$

And it's immediate from there to conclude $f(x)=f(y)\ \forall x,y>0$

Hence the unique solution $\boxed{f(x)=2\forall x>0}$

Amir Hos...
4728 pos...

Sep 25, 2010, 12:53 am — PM #179

Problem 56 :

Find all functions $f:\mathbb{Q}^+\mapsto\mathbb{Q}^+$ such that:

$$f(x)+f(y)+2xyf(xy)=\frac{f(xy)}{f(x+y)}.$$

It seems Problems 48 and 49 are still unsolved :

Problem 48 :
Find all continuous functions $\mathbb{R}\to\mathbb{R}$ satisfying the equation:

$$f(x+y)+f(xy)=f(x)+f(y)+f(xy+1)$$

Problem 49 :
Find all continuous functions $\mathbb{R}\to\mathbb{R}$ satisfying the equation:

$$f(x)+f(y)+f(z)+f(x+y+z)=f(x+y)+f(y+z)+f(z+x)+f(0)$$

---

pco
12955 po...

Sep 25, 2010, 12:58 am PM #180

> *amparvardi wrote:*
> **Problem 49** :
> Find all continuous functions $\mathbb{R}\to\mathbb{R}$ satisfying the equation:
> $$f(x)+f(y)+f(z)+f(x+y+z)=f(x+y)+f(y+z)+f(z+x)+f(0)$$

Let $P(x,y,z)$ be the assertion $f(x)+f(y)+f(z)+f(x+y+z)=f(x+y)+f(y+z)+f(z+x)+f(0)$

$P(x,y,y)\implies f(x+2y)-f(x+y)=f(x+y)-f(x)+(f(2y)+f(0)-2f(y))$
$P(x+y,y,y)\implies f(x+3y)-f(x+2y)=f(x+2y)-f(x+y)+(f(2y)+f(0)-2f(y))$
...
$P(x+(n-1)y,y,y)\implies$
$f(x+(n+1)y)-f(x+ny)=f(x+ny)-f(x+(n-1)y)+(f(2y)+f(0)-2f(y))$

Adding these lines gives $f(x+(n+1)y)-f(x+ny)=f(x+y)-f(x)+n(f(2y)+f(0)-2f(y))$

And so (adding this last lines for $n=0,...,k-1$) :
$f(x+ky)-f(x)=k(f(x+y)-f(x))+\frac{k(k-1)}{2}(f(2y)+f(0)-2f(y))$

Setting $x=0$ in this last equality and renaming $y\to x$ and $k\to n$, we get :

$f(nx)=\frac{f(2x)+f(0)-2f(x)}{2}n^2+\frac{4f(x)-f(2x)-3f(0)}{2}n+f(0)$

So :
$f(q\frac{p}{q})=\frac{f(2\frac{p}{q})+f(0)-2f(\frac{p}{q})}{2}q^2+\frac{4f(\frac{p}{q})-f(2\frac{p}{q})-3f(0)}{2}q+f(0)$

And since $f(q\frac{p}{q})=f(p)=\frac{f(2)+f(0)-2f(1)}{2}p^2+\frac{4f(1)-f(2)-3f(0)}{2}p+f(0)$, we get :

$(f(2)+f(0)-2f(1))p^2+(4f(1)-f(2)-3f(0))p=(f(2\frac{p}{q})+f(0)-2f(\frac{p}{q}))q^2+(4f(\frac{p}{q})-f(2\frac{p}{q})-3f(0))q$

Replacing $p\to np$ and $q\to nq$ in this equation, we get :

$(f(2)+f(0)-2f(1))p^2n^2+(4f(1)-f(2)-3f(0))pn$
$=(f(2\frac{p}{q})+f(0)-2f(\frac{p}{q}))q^2n^2+(4f(\frac{p}{q})-f(2\frac{p}{q})-3f(0))qn$ and so :

$n^2\left((f(2)+f(0)-2f(1))p^2-(f(2\frac{p}{q})+f(0)-2f(\frac{p}{q}))q^2\right)$
$+n\left((4f(1)-f(2)-3f(0))p-(4f(\frac{p}{q})-f(2\frac{p}{q})-3f(0))q\right)=0$

And since this is true for any n, we get :
$(f(2)+f(0)-2f(1))p^2-(f(2\frac{p}{q})+f(0)-2f(\frac{p}{q}))q^2=0$
$(4f(1)-f(2)-3f(0))p-(4f(\frac{p}{q})-f(2\frac{p}{q})-3f(0))q=0$

From these two lines, we get $f(\frac{p}{q})=\frac{f(2)+f(0)-2f(1)}{2}\frac{p^2}{q^2}+\frac{4f(1)-f(2)-3f(0)}{2}\frac{p}{q}+f(0)$

And so $f(x)=ax^2+bx+c$ $\forall x\in\mathbb{Q}^+$ which indeed fits whatever are a,b,c.

So $f(x)=ax^2+bx+c$ $\forall x\in\mathbb{R}^+$ (using continuity)

Let then $x>0$:
$P(-x,x,x) \implies f(-x)+3f(x)=f(2x)+3f(0)$ and, since $x\ge 0$ and $2x\ge 0$:

$f(-x)=(4ax^2+2bx+c)+3c-3(ax^2+bx+c)=ax^2-bx+c$

And so $f(x)=ax^2+bx+c\ \forall x\in\mathbb{R}$

pco
12955 po...

Sep 25, 2010, 1:09 am PM #181

> *amparvardi wrote:*
> **Problem 48** :
> Find all continuous functions $\mathbb{R}\to\mathbb{R}$ satisfying the equation:
> $$f(x+y)+f(xy)=f(x)+f(y)+f(xy+1)$$

Let $f(x)=g(x)-1$ and the equation becomes $g(x+y)+g(xy)+1=g(x)+g(y)+g(xy+1)$

Which is an old problem (see http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=336764)

pco
12955 po...

Sep 25, 2010, 1:50 am PM #182

> *amparvardi wrote:*
> **Problem 56** :
>
> Find all functions $f:\mathbb{Q}^+\mapsto\mathbb{Q}^+$ such that:
> $$f(x)+f(y)+2xyf(xy)=\frac{f(xy)}{f(x+y)}.$$

Let $P(x,y)$ be the assertion $f(x)+f(y)+2xyf(xy)=\frac{f(xy)}{f(x+y)}$
Let $f(1)=a$

$P(1,1) \implies f(2)=\frac{1}{4}$
$P(2,2) \implies f(4)=\frac{1}{16}$
$P(2,1) \implies f(3)=\frac{1}{4a+5}$
$P(3,1) \implies f(4)=\frac{1}{4a^2+5a+7}$ and so $4a^2+5a+7=16$ and so $a=1$ (remember $f(x)>0$)

$P(x,1) \implies \frac{1}{f(x+1)}=\frac{1}{f(x)}+2x+1$ and so $\frac{1}{f(x+n)}=\frac{1}{f(x)}+2nx+x^2$ and $f(n)=\frac{1}{n^2}$

$P(x,n) \implies f(nx)=\frac{f(x)+\frac{1}{n^2}}{\frac{1}{f(x)}+n^2}$

Setting $x=\frac{p}{n}$ in this last equality, we get $f(\frac{p}{n})=\frac{n^2}{p^2}$ (remember $f(x)>0$)

Hence the answer : $\boxed{f(x)=\frac{1}{x^2}}\ \forall x\in\mathbb{Q}^+$ which indeed is a solution.

Winner20...
79 posts

Sep 25, 2010, 9:55 am PM #183

Problem 57:

Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that for all $x>y>0$,

$f(x+y)-f(x-y)=4\sqrt{f(x)f(y)}$

pco
12955 po...

Sep 25, 2010, 11:52 am PM #184

> *Winner2010 wrote:*
> **Problem 57:**
>
> Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that for all $x>y>0$,
>
> $f(x+y)-f(x-y)=4\sqrt{f(x)f(y)}$

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=364296

aktyw19

Sep 25, 2010, 4:22 pm PM #185

Problem 58

1315 posts

Problem 58

Find all $f:N\to\{0,1,2,...,2000\}$
such that
$0\le n\le 2000\Rightarrow f(n)=n$
$f(f(n)+f(m))=f(m+n)$

Ramchand...
645 posts

Feb 2, 2011, 12:57 am PM #186

Why has this marathon stopped?.
Lets revive this marathon :)

Problem 59

Find all continuous $f:R\to R$ such that for reals x,y -
$f(x+f(y))=y+f(x+1)$

---

pco
12955 po...

Feb 2, 2011, 1:36 am • 1 PM #187

> *Ramchandran wrote:*
> **Problem 59**
>
> Find all continuous $f:R\to R$ such that for reals x,y -
> $f(x+f(y))=y+f(x+1)$

Let $P(x,y)$ be the assertion $f(x+f(y))=y+f(x+1)$

$P(0,y+1-f(1))\implies f(f(y+1-f(1)))=y+1$
$P(x-f(1),f(y+1-f(1)))\implies f(x-f(1)+f(f(y+1-f(1))))=f(y+1-f(1))+f(x+1-f(1))$ and so $f(x+y+1-f(1))=f(y+1-f(1))+f(x+1-f(1))$

Let then $g(x)=f(x+1-f(1))$ and we get $g(x+y)=g(x)+g(y)$ and so, since continuous, $g(x)=ax$ and $f(x)=a(x+f(1)-1)$

Plugging $f(x)=ax+b$ in original equation, we get two solutions :
$f(x)=1+x\ \forall x$
$f(x)=1-x\ \forall x$

Amir Hos...
4728 pos...

Feb 2, 2011, 3:20 pm • 1 PM #188

Problem 60.

Let $n>m>1$ be odd integers, let $f(x)=x^n+x^m+x+1$. Prove that $f(x)$ can't be expressed as the product of two polynomials having integer coefficients and positive degrees.

---

abhinavza...
418 posts

Feb 8, 2011, 1:25 pm PM #189

This Marathon **MUST NOT** Die
As Solution To 60 Is Already There.
Problem 61

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pco
12955 po...

Feb 8, 2011, 4:53 pm PM #190

> *abhinavzandubalm wrote:*
> **Problem 61** :
> $f:Z\to Z$
> $f(m+n)+f(mn-1)=f(m)f(n)+2$

Let $P(x,y)$ be the assertion $f(x+y)+f(xy-1)=f(x)f(y)+2$

$P(x,0)\implies f(x)(f(0)-1)=f(-1)-2$

If $f(0)\ne 1$, this implies $f(x)=c$ and $2c=c^2+2$ and no solution.
So $f(0)=1$ and $f(-1)=2$

Let then $f(1)=a$
$P(1,1)\implies f(2)=a^2+1$
$P(2,1)\implies f(3)=a^3+2$
$P(3,1)\implies f(4)=a^4-a^2+2a+1$
$P(2,2)\implies f(4)=a^4-a^3+2a^2+1$

And so $a^4-a^2+2a+1=a^4-a^3+2a^2+1\iff a(a-1)(a-2)=0$

If $a=0$:
Previous lines imply $f(2)=1$ and $f(3)=2$ and $f(4)=1$
$P(4,1)\implies f(5)=0$
But $P(3,2)\implies f(5)=2$ and so contradiction

If $a=1$:
Previous lines imply $f(2)=2$ and $f(3)=3$ and $f(4)=3$
$P(4,1)\implies f(5)=2$
But $P(3,2)\implies f(5)=4$ and so contradiction

If $a=2$, then $P(m+1,1) \implies f(m+2)=2f(m+1)-f(m)+2$ which is easily solved in $f(m)=m^2+1$ which indeed is a soluion.

Hence the unique solution : $\boxed{f(x)=x^2+1}$ $\forall x\in\mathbb{Z}$

abhinavza...
418 posts

Feb 8, 2011, 6:01 pm *PM #191*

Could Anyone Give The Next Problem Please.

Raja Okto...
277 posts

Feb 8, 2011, 7:39 pm *PM #192*

Problem 62. Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a function such that
$f(\sqrt{ab})=\sqrt{f(a)f(b)}$
for all $a,b\in\mathbb{R}^+$ satisfying $a^2b>2$.
Prove that the equation holds for all $a,b\in\mathbb{R}^+$.
Here, $\mathbb{R}^+$ is the set of all positive real numbers.

ocha
955 posts

Feb 10, 2011, 12:21 pm *PM #193*

> *Raja Oktovin wrote:*
> **Problem 62.** Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a function such that
> $f(\sqrt{ab})=\sqrt{f(a)f(b)}$
> for all $a,b\in\mathbb{R}^+$ satisfying $a^2b>2$.
> Prove that the equation holds for all $a,b\in\mathbb{R}^+$.
> Here, $\mathbb{R}^+$ is the set of all positive real numbers.

For some $u,v\in\mathbb{R}^+$. Take a,b sufficiently large such that such that $abuv>>2$. Then

1) $f(ab\cdot uv)^4=f(a^2b^2)^2f(u^2v^2)^2=f(a^4)f(b^4)f(u^2v^2)^2$
2) $f(au\cdot bv)^4=f(a^2u^2)^2f(b^2v^2)^2=f(a^4)f(u^4)f(b^4)f(v^4)$

Then since $f>0$ we can divide $f(a^4)f(b^4)$ and find $f(u^2v^2)^2=f(u^2)f(v^2)$. But u and v were chosen arbitrarily so $f(xy)^2=f(x^2)f(y^2)$ for all $x,y\in\mathbb{R}^+$

Problem 63
For $a,b,c\in\mathbb{N}$ suppose there exists *coprime* polynomials $P,Q,R\in\mathbb{C}[x]$ such that

$$P(x)^a+Q(x)^b=R(x)^c$$

Show that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>1$

ocha
955 posts

Feb 11, 2011, 7:51 am *PM #194*

> *amparvardi wrote:*
> **Problem 60.**
>
> Let $n>m>1$ be odd integers, let $f(x)=x^n+x^m+x+1$. Prove that $f(x)$ can't be expressed as the product of two polynomials having integer coefficients and positive degrees.

This is a scaled back proof of Ljunggren's Theorem.

Proof
Let $f^*(x)=x^{\deg(f)}f(\frac{1}{x})$ be the reciprocal polynomial of f. Suppose $f(x)=p(x)q(x)$, then we can assume $q(x)\neq q^*(x)$ because f is not reciprocal (*note:* this is where we use m,n odd). So let $g(x)=p(x)q^*(x)$ with $f\not\equiv g$ (*). Now $g\in\mathbb{Z}[x]$, and $g(x)g^*(x)=f(x)f^*(x)$

Suppose $g(x)=g_nx^n+...+g_0$. The coefficient of x^n in $f(x)f^*(x)$ is 4, and the coefficient of x^n in $g(x)g^*(x)$ is $g_n^2+g_{n-1}^2+...+g_0^2$. Since g has integer coefficients and has at least two non zero coefficients we have $g(x)=x^n\pm x^r\pm x^s\pm 1$ (wlog the coefficient of x^n in $g(x)$ is positive).

Expanding the first few terms of $f(x)f^*(x)$ and $g(x)g^*(x)$ shows that the coefficient of x^{2n-1} in $f(x)f^*(x)$ is 1, and therefore $s=1$ in $g(x)$ with x^s having a positive coefficient. Similarly looking at the coefficient of x^{n+m} in $f(x)f^*(x)$ tells us that $r=m$ in $g(x)$ and x^r has positive coefficient. Finally the x^0 coefficient in in $f(x)f^*(x)$ is 1 so $g(0)=1$.

Therefore $g\equiv f$, but this contradicts (*) so f is irreducible $\square$

ocha
955 posts

Feb 11, 2011, 2:00 pm *PM #195*

> *aktyw19 wrote:*
> **Problem 58**
> Find all $f:\mathbb{N}_0\to\{0,1,2,...,2000\}$ such that $0\leq n\leq 2000\Rightarrow f(n)=n$ and
> $f(f(n)+f(m))=f(m+n)$

Let $[x]_p$ denote the residue $x \mod p$. Suppose $f(2001) = u$, then for positive integer r such that $u + r < 2001$ we have $f(2001 + r) = f(u + r) = u + r$. Furthermore if $u + r = 2001$ then $f(2001 + r) = f(u + r) = f(2001) = u$. Hence, by induction, for $n > 2000$ we have $f(n) = u + [n - 2001]_{(2001-u)}$.

One can easily check that this works for all u.

abhinavza...
418 posts

Feb 11, 2011, 9:08 pm PM #196

Okay. As Nobody Is Giving A Problem.
Problem 63

aktyw19
1315 posts

Feb 12, 2011, 4:33 pm PM #197

problem 58

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=38&t=194959&hilit

abhinavza...
418 posts

Feb 12, 2011, 4:45 pm PM #198

Mind If I Gave Hints To People Or Would You All Like To Solve It .
Might Wait For The Reply For At Most a day.

abhinavza...
418 posts

Feb 13, 2011, 7:27 pm PM #199

Hint To Problem 63

And Solution Are

And The Next Problem 64

goodar20...
1346 pos...

Feb 13, 2011, 10:15 pm PM #200

if $degP \geq 4$ then looking at the coefficient of x^{n-1} implies $na_n = 0$ which is impossible. the rest would be easy.

SCP
1520 pos...

Feb 13, 2011, 10:51 pm PM #201

> *goodar2006 wrote:*
> if $degP \geq 4$ then looking at the coefficient of x^{n-1} implies $na_n = 0$ which is impossible. the rest would be easy.

rest: $P(1) = P(0) + 1 = P(-1)$ hence it is an even function and we find $f(x) = x^2 + c$
I let someone other post an other problem, I only complete this solution.

abhinavza...
418 posts

Feb 13, 2011, 11:06 pm PM #202

You People Must Really Hate Functional Equations. 😋
Anyway Here's a problem.
Problem 65

pco
12955 po...

Feb 14, 2011, 4:11 pm PM #203

> *abhinavzandubalm wrote:*
> Problem 65
> A Rational Function f (i.e. a function which is a quotient of two polynomials) has the property that $f(x) = f(\frac{1}{x})$
> Prove That f is a function in the variable $x + \frac{1}{x}$.

Let $f(x) = x^p(x-1)^q(x+1)^r\frac{P(x)}{Q(x)}$ where $p, q, r \in \mathbb{Z}$ and $P, Q \in \mathbb{R}[X]$ and $P(0), P(1), P(-1), Q(0), Q(1), Q(-1) \neq 0$

Let $z \in \mathbb{C}$ root of $P(x)$. $z \neq 0$ and we get $f(z) = 0$ and so $P(\frac{1}{z}) = 0$.
$z \neq \frac{1}{z}$ and so $P(x) = (x-z)(x-\frac{1}{z})P_1(x)$ and so $P(x) = x(x+\frac{1}{x}-(z+\frac{1}{z}))P_1(x)$

And so $f(x) = x^{p+1}(x-1)^q(x+1)^r(x+\frac{1}{x}-(z+\frac{1}{z}))\frac{P_1(x)}{Q(x)}$ and so $g(x) = x^{p+1}(x-1)^q(x+1)^r\frac{P_1(x)}{Q(x)}$ has the same property than $f(x)$ with degree(P_1)=degree(P) -2 and $f(x) = g(x)h(x+\frac{1}{x})$

Using this method up to degree $P = 0$ and using then the same method with $\frac{1}{f}$, we easily get :

$f(x) = x^u(x-1)^v(x+1)^w t(x+\frac{1}{x})$

Applying then $f(x) = f(\frac{1}{x})$ to this result, we get $x^{2u+v+w} = (-1)^v$ and so $v = 2k$ and $w = -2u - 2k$

And so $f(x)=x^u(x-1)^{2k}(x+1)^{-2u-2k}t(x+\frac{1}{x})=(x+\frac{1}{x}-2)^{2k+u}((x+\frac{1}{x})^2-4)^{-u-k}t(x+\frac{1}{x})$

Q.E.D.

prafullasd
25 posts

Feb 14, 2011, 6:09 pm PM #204

my solution seems very small, so pls point out the flaw if any
$f(x)=\frac{P(x)}{Q(x)}=\frac{P(\frac{1}{x})}{Q(\frac{1}{x})}=\frac{P(x)+P(\frac{1}{x})}{Q(x)+Q(\frac{1}{x})}$ .... (I)
let $R(x)=P(x)+P(\frac{1}{x})$
R(x) is a "psedo" polynomial (my notation) which is a sum of terms of form $a_n(x^n+\frac{1}{x^n})$
by an easy induction we get that R(x) is a polynomial in (x + 1/x)
similarly, the denominator S(x)
so, f(x) is a rational function of (x+1/x)
the denomianators in any of the fractions in statement (I)can become zero only a finite number of times(each of them can be expressed as a polynomial of finite deg), therefore (I) is true for infinitely many vales of x
therefore, it is true for all real values of x (as it can be expressed as a ratio of two polynomials)

pco
12955 po...

Feb 14, 2011, 6:37 pm PM #205

> *prafullasd wrote:*
> my solution seems very small, so pls point out the flaw if any
> $f(x)=\frac{P(x)}{Q(x)}=\frac{P(\frac{1}{x})}{Q(\frac{1}{x})}=\frac{P(x)+P(\frac{1}{x})}{Q(x)+Q(\frac{1}{x})}$ .... (I)
> let $R(x)=P(x)+P(\frac{1}{x})$
> R(x) is a "psedo" polynomial (my notation) which is a sum of terms of form $a_n(x^n+\frac{1}{x^n})$
> by an easy induction we get that R(x) is a polynomial in (x + 1/x)
> similarly, the denominator S(x)
> so, f(x) is a rational function of (x+1/x)
> the denomianators in any of the fractions in statement (I)can become zero only a finite number of times(each of them can be expressed as a polynomial of finite deg), therefore (I) is true for infinitely many vales of x
> therefore, it is true for all real values of x (as it can be expressed as a ratio of two polynomials)

This seems quite OK for me.
And much simpler than mine (I missed that $\frac{a}{b}=\frac{c}{d}$ implies $\frac{a}{b}=\frac{c}{d}=\frac{a+c}{b+d}$)
Congrats.

abhinavza...
418 posts

Feb 15, 2011, 3:41 pm PM #206

Sheesh.
It Doesn't Really Look Like You People Will Give Any Problems.
Problem 66

This One's Easy.
So I Hope Someone Else may Give A Problem Now.

nguyenhung
559 posts

Feb 15, 2011, 9:00 pm PM #207

> *abhinavzandubalm wrote:*
> Sheesh.
> Problem 66
> Find All Functions
> $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(x-y)=f(x+y)f(y)$

Let $P(x,y)$ be the assertion of $f\left(x-y\right)=f\left(x+y\right)f\left(y\right)$

We have

$P\left(x,0\right)\to f\left(x\right)=f\left(x\right)f\left(0\right)\Rightarrow\left[\begin{array}{l}f\left(x\right)=0\\ f\left(0\right)=0\end{array}\right.$

* Case 1: $f\left(x\right)=0$. It's easy to see that $f\left(x\right)=0$ satisfies the condition

* Case 2: $f\left(0\right)=0$ and $f\left(x\right)\neq 0$

$P\left(x,x\right)\to 0=f\left(2x\right)f\left(x\right)$, not true 'cause $f\left(x\right)\neq 0$

So the only solution is $f\left(x\right)=0$

pco
12955 po...

Feb 15, 2011, 9:01 pm • 1 PM #208

> *abhinavzandubalm wrote:*

> Problem 66 :
> Find All Functions
> $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(x-y)=f(x+y)f(y)$

Let $P(x,y)$ be the assertion $f(x-y)=f(x+y)f(y)$

$P(0,0) \implies f(0)^2=f(0)$ and so $f(0)=0$ or $f(0)=1$

If $f(0)=0$: $P(x,0) \implies f(x)=0\,\forall x$ which indeed is a solution

If $f(0)=1$:
$P(x,x) \implies f(x)f(2x)=1$ and so $f(x)\ne 0\,\forall x$
$P(\frac{2x}{3},\frac{x}{3}) \implies f(\frac{x}{3})=f(x)f(\frac{x}{3})$ and, since $f(\frac{x}{3})\ne 0$: $f(x)=1$ which indeed is a solution.

Hence the two solutions :
$f(x)=0\,\forall x$
$f(x)=1\,\forall x$

Amir Hos... 4728 pos...
Feb 15, 2011, 9:07 pm PM #209

Nice solutions, pco.

Problem 67.

Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)\cdot f(y)=f(x)+f(y)+f(xy)-2\quad \forall x,y\in\mathbb{R}.$$

---

pco 12955 po...
Feb 15, 2011, 9:22 pm PM #210

> *amparvardi wrote:*
> **Problem 67.**
>
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
>
> $$f(x)\cdot f(y)=f(x)+f(y)+f(xy)-2\quad \forall x,y\in\mathbb{R}.$$

Setting $f(x)=g(x)+1$, the equation becomes $g(xy)=g(x)g(y)$, very classical equation whose general solutions are :
$g(x)=1\,\forall x$
$g(0)=0$ and $g(x)=|x|^a\,\forall x\ne 0$ where a is any non zero real.
$g(0)=0$ and $g(x)=sign(x)|x|^a\,\forall x\ne 0$ where a is any non zero real.

Hence the three solutions of the required equation :
$f(x)=2\,\forall x$
$f(0)=1$ and $f(x)=1+|x|^a\,\forall x\ne 0$ where a is any non zero real.
$f(0)=1$ and $f(x)=1+sign(x)|x|^a\,\forall x\ne 0$ where a is any non zero real.

---

nguyenhung 559 posts
Feb 15, 2011, 9:24 pm PM #211

> *pco wrote:*
>
> > *amparvardi wrote:*
> > **Problem 67.**
> >
> > Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
> >
> > $$f(x)\cdot f(y)=f(x)+f(y)+f(xy)-2\quad \forall x,y\in\mathbb{R}.$$
>
> Seting $f(x)=g(x)+1$, the equation becomes $g(xy)=g(x)g(y)$, very classical equation whose general solutions are :
> $g(x)=1\,\forall x$
> $g(0)=0$ and $g(x)=|x|^a\,\forall x\ne 0$ where a is any non zero real.
> $g(0)=0$ and $g(x)=sign(x)|x|^a\,\forall x\ne 0$ where a is any non zero real.
>
> Hence the three solutions of the required equation :
> $f(x)=2\,\forall x$
> $f(0)=1$ and $f(x)=1+|x|^a\,\forall x\ne 0$ where a is any non zero real.
> $f(0)=1$ and $f(x)=1+sign(x)|x|^a\,\forall x\ne 0$ where a is any non zero real.

Sorry, but I remember that g must be continues before we could infer these results

---

pco 12955 po...
Feb 15, 2011, 9:27 pm PM #212

nguyenhung wrote:

> pco wrote:
> $g(xy)=g(x)g(y)$, very classical equation whose general solutions are :
> $g(x)=1\ \forall x$
> $g(0)=0$ and $g(x)=|x|^a\ \forall x\neq 0$ where a is any non zero real.
> $g(0)=0$ and $g(x)=sign(x)|x|^a\ \forall x\neq 0$ where a is any non zero real.
> ...

Sorry, but I remember that g must be continues before we could infer these results

You are perfectly right. I'm sorry (tried to answer very quickly and so missed some attention)

---

pco
12955 po...

Feb 15, 2011, 9:36 pm • 1 PM #213

And so :
...$g(xy)=g(x)g(y)$, very classical equation whose general solutions are :
$g(x)=0\ \forall x$
$g(x)=1\ \forall x$
$g(0)=0$ and $g(x)=e^{h(\ln|x|)}\ \forall x\neq 0$ where $h(x)$ is any solution of Cauchy's equation.
$g(0)=0$ and $g(x)=sign(x)e^{h(\ln|x|)}\ \forall x\neq 0$ where $h(x)$ is any solution of Cauchy's equation.

Hence the four solutions of the required equation :
$f(x)=1\ \forall x$
$f(x)=2\ \forall x$
$f(0)=1$ and $f(x)=1+e^{h(\ln|x|)}\ \forall x\neq 0$ where $h(x)$ is any solution of Cauchy's equation.
$f(0)=1$ and $f(x)=1+sign(x)e^{h(\ln|x|)}\ \forall x\neq 0$ where $h(x)$ is any solution of Cauchy's equation.

Amir Hos...
4728 pos...

Feb 15, 2011, 9:55 pm PM #214

Thanks.

Problem 68.

Find all real-valued functions f defined on $\mathbb{R}_0$, the set of all non-zero reals, such that

(a) $f(-x)=-f(x)$.

(b) $f\left(\frac{1}{x+y}\right)=f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right)+2(xy-1000)$. (for all x,y in $\mathbb{R}_0$, such that $x+y$ is in $\mathbb{R}_0$, too)

[source: 19-th Australian Mathematical Olympiad]

---

pco
12955 po...

Feb 15, 2011, 10:43 pm PM #215

> amparvardi wrote:
> Thanks.
>
> **Problem 68.**
>
> Find all real-valued functions f defined on $\mathbb{R}_0$, the set of all non-zero reals, such that
>
> **(a)** $f(-x)=-f(x)$.
>
> **(b)** $f\left(\frac{1}{x+y}\right)=f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right)+2(xy-1000)$. (for all x,y in $\mathbb{R}_0$, such that $x+y$ is in $\mathbb{R}_0$, too)
>
> *[source: 19-th Australian Mathematical Olympiad]*

Changing $x\to -x$ and $y\to -y$ in (b), using then (a) and adding to the original equation, we get $xy=1000$, impossible

So no solution.

Amir Hos...
4728 pos...

Feb 16, 2011, 8:34 pm PM #216

Problem 69.

Let $f(n)$ be defined on the set of positive integers by the rules: $f(1)=2$ and

$$f(n+1)=(f(n))^2-f(n)+1 \quad \forall n\in\mathbb{N}.$$

Prove that, for all integers $n>1$, we have

$$1-\frac{1}{2^{2^{n-1}}}<\frac{1}{f(1)}+\frac{1}{f(2)}+\frac{1}{f(3)}+\cdots+\frac{1}{f(n)}<1-\frac{1}{2^{2^n}}.$$

Note. $\mathbb{N}$ is the set of positive integers.

EDIT: Changed, thanks mahanmath. :)

nguyenhung
559 posts

Feb 17, 2011, 5:15 pm

PM #217

Solution to Problem 69

We have

$f\left(k+1\right)=\left(f\left(k\right)\right)^2-f\left(k\right)+1$

$\Leftrightarrow \frac{1}{f\left(k\right)}=\frac{1}{f\left(k\right)-1}-\frac{1}{f\left(k+1\right)-1}$

Hence

$\frac{1}{f\left(1\right)}=\frac{1}{f\left(1\right)-1}-\frac{1}{f\left(2\right)-1}$

$\frac{1}{f\left(2\right)}=\frac{1}{f\left(2\right)-1}-\frac{1}{f\left(3\right)-1}$

....

$\frac{1}{f\left(n\right)}=\frac{1}{f\left(n\right)-1}-\frac{1}{f\left(n+1\right)-1}$

$\Rightarrow S=\sum_{i=1}^{n}\frac{1}{f\left(i\right)}=\frac{1}{f\left(1\right)-1}-\frac{1}{f\left(n+1\right)-1}=1-\frac{1}{f\left(n+1\right)-1}$

We need to prove

$1-\frac{1}{2^{2^{n-1}}}<1-\frac{1}{f\left(n+1\right)-1}<1-\frac{1}{2^{2^n}}$

$\Leftrightarrow 2^{2^{n-1}}+1<f\left(n+1\right)<2^{2^n}+1$

$\Leftrightarrow 2^{2^{n-2}}+1<f\left(n\right)<2^{2^{n-1}}+1$

, which can easily prove by induction

So we've done

u2tommyf
33 posts

Feb 17, 2011, 5:45 pm

PM #218

Sorry but I don't know latex, so maybe someone will "translate"...

Problem 70

Determine all functions f defined on the set of positive integers that have:

f (x * f(y) + y) = y * f(x) + f(y), for any x,y positive integers

and f(p) prime for any p prime.

nguyenhung
559 posts

Feb 17, 2011, 5:50 pm • 2

PM #219

Latex for problem 70

Problem 70

Determine all functions f defined on the set of positive integers that have:

$f(xf(y)+y)=yf(x)+f(y)$, for any positive integers x,y

and $f(p)$ is a prime for any prime p

mousavi
222 posts

Feb 18, 2011, 12:39 am

PM #220

> *nguyenhung wrote:*
> Latex for problem 70
>
> **Problem 70**
>
> Determine all functions f defined on the set of positive integers that have:
>
> $f(xf(y)+y)=yf(x)+f(y)$, for any positive integers x,y
>
> and $f(p)$ is a prime for any prime p

$p(x,y):f(xf(y)+y)=yf(x)+f(y)$ (1)
$p(3,1):f(3f(1)+1)=f(3)+f(1)$ (2)

$p(1,1): f(f(1)+1) = 2f(1)(3)$
with (3)
$p(1, f(1)+1): f(3f(1)+1) = (f(1)+1)f(1) + 2f(1) = f(1)^2 + 3f(1)(4)$

(2),(4)$\Longrightarrow f(3)+f(1) = f(1)^2+3f(1) \Longrightarrow f(3) = f(1)(f(1)+2) \Longrightarrow f(1) = 1, f(3) = 3$

$p(x,1): f(x+1) = f(x)+1 \Longrightarrow f(x) = x$

abhinavza...
418 posts

Feb 18, 2011, 2:13 pm PM #221

Even A Lifeguard Must not be bringing the dead back to life by C.P.R. like we have to do for this marathon. 😄 🙂 😋

Problem 71

> Determine All Functions
>
> $f: \mathbb{R} - 0, 1 \rightarrow \mathbb{R}$
>
> $f(x) + f(\frac{1}{1-x}) = \frac{2(1-2x)}{x(1-x)}$

RSM
736 posts

Feb 18, 2011, 4:14 pm PM #222

> *abhinavzandubalm wrote:*
> Even A Lifeguard Must not be bringing the dead back to life by C.P.R. like we have to do for this marathon. 😄 🙂 😋
>
> Problem 71
>
> > Determine All Functions
> >
> > $f: \mathbb{R} - 0, 1 \rightarrow \mathbb{R}$
> >
> > $f(x) + f(\frac{1}{1-x}) = \frac{2(1-2x)}{x(1-x)}$

This is an INMO problem
A solution to it can be found in INMO official solution paper.

Next problem:-
Find all function f defined on real variables such that
$f(x+y) + f(x-y) = 2f(x)f(y)$
for all $x, y \in \mathbb{R}$

I could not solve this problem myself.
So I want to give one more condition that $|f(x)| \geq 1 \forall x \in \mathbb{R}$
I think it is easy now.

This post has been edited 3 times. Last edited by RSM, Feb 20, 2011, 3:23 pm

pco
12955 po...

Feb 18, 2011, 5:16 pm • 1 👍 PM #223

> *RSM wrote:*
> Find all function f defined on real variables such that
> $f(x+y) + f(x-y) = 2f(x)f(y)$
> for all $x, y \in \mathbb{R}$

Are you sure that the original problem does not contain the additional "continuity" property ?

prafullasd
25 posts

Feb 20, 2011, 10:30 am PM #224

i tried to get a discontinuous solution for the eqaution,
let $g(x)$ be any discontinuous solution of $g(x+y) = g(x) + g(y)$
let $f(x) = cos(g(x))$
then,
$f(x+y) + f(x-y) = cos(g(x+y)) + cos(g(x-y))$
$= 2cos(\frac{g(2x)}{2})cos(\frac{g(2y)}{2}) = 2cos(g(x))cos(g(y)) = 2f(x)f(y)$
as $g(x+x) = g(x) + g(x)$ therefore, $f(x)$ satisfies the equation
but, i can't show that $f(x)$ is discontinuous

magical
196 posts

Feb 20, 2011, 11:39 am PM #225

Problem 73
Find all functional $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ satisfy: $f(x^3 + 2y) + f(x+y) = g(x+2y) \; \forall x, y \in \mathbb{R}$

Feb 20, 2011, 2:10 pm PM #226

pco
12955 po...

> *RSM wrote:*
> Next problem:-
> Find all function f defined on real variables such that
> $f(x+y)+f(x-y)=2f(x)f(y)$
> for all $x, y \in \mathbb{R}$
>
> I could not solve this problem myself.
> So I want to give one more condition that $|f(x)| \geq 2 \forall x \in \mathbb{R}$
> I think it is easy now.

I think it would have been more interesting to add the continuity constraint since then this would have been the famous d'Alembert functional equation with solutions $0, \cos ax, \cosh ax$.

Just adding the constraint $|f(x)| \geq 2 \forall x \in \mathbb{R}$ is a kind of joke :
Setting $x=y=0$ in the equation implies $2f(0)=2f(0)^2$ and so $f(0)=0$ or $f(0)=1$, and so $|f(0)|<2$
So no solution.

---

pco
12955 po...

Feb 20, 2011, 2:45 pm PM #227

> *magical wrote:*
> **Problem 73**
> Find all functional $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ satisfy: $f(x^3+2y)+f(x+y)=g(x+2y)\,\forall x, y \in \mathbb{R}$

If (f, g) is a solution, so is $(f+c, g+2c)$ and so Wlog say $f(0)=0$

Setting $y=0$ in the equation gives $g(x)=f(x^3)+f(x)$
Pluging this in original equation, we get assertion $P(x,y): f(x^3+2y)+f(x+y)=f((x+2y)^3)+f(x+2y)$

Setting $x=-y$ in the equation gives $g(y)=f(2y-y^3)$ and so $g(x)=f(2x-x^3)$
Pluging this in original equation, we get assertion $Q(x,y): f(x^3+2y)+f(x+y)=f(2(x+2y)-(x+2y)^3)$

1) $f(x+\frac{1}{2})=f(x)\,\forall x$
====================
$P(1, x-\frac{1}{2}) \implies f(x+\frac{1}{2})=f((2x)^3)$
$P(0, x) \implies f(x)=f((2x)^3)$
And so $f(x+\frac{1}{2})=f(x)$
Q.E.D.

2) $f(x)=0\,\forall x\in[0,1]$
==============
Let $y\in(0,1]$
$Q(x, y-x) \implies f(x^3-2x+2y)+f(y)=f(2(2y-x)-(2y-x)^3)$
Consider now the equation $x^3-2x+2y=2(2y-x)-(2y-x)^3$
It may be written $(x-y)^2=\frac{1-y^2}{3}$ and it has always at least one solution x since $y\in(0,1]$

Choosing this value x, $f(x^3-2x+2y)+f(y)=f(2(2y-x)-(2y-x)^3)$ becomes $f(y)=0$
Q.E.D.

3) Solutions
========
2) gave $f(x)=0\,\forall x\in[0,1]$
1) gave $f(x+\frac{1}{2})=f(x)$
So $f(x)=0\,\forall x$
So $g(x)=0\,\forall x$

Hence the answer :
$\boxed{(f(x),g(x))=(c,2c)}$ for any real c

RSM
736 posts

Feb 20, 2011, 3:26 pm PM #228

I am really sorry. I made a mistake.
I have corrected the question now.

Notice:

Problem 72 is still pending.

RSM
736 posts

Feb 21, 2011, 11:45 am PM #229

Prove or disprove the statement:-
Each even function $f(x)$ can be written as $g(x)+g(-x)$
Where f and g are defined over $\mathbb{R}$.
g is not even function.

This post has been edited 1 time. Last edited by RSM, Feb 22, 2011, 5:50 am

---

pco
12955 po...

Feb 21, 2011, 1:19 pm PM #230

12955 po...

> RSM wrote:
> Prove or disprove the statement:-
> Each even function $f(x)$ can be written as $g(x)+g(-x)$
> Where f and g are defined over $\mathbb{R}$.

$f(x)=\frac{f(x)}{2}+\frac{f(-x)}{2}$

Amir Hos... 4728 pos...

Feb 21, 2011, 1:26 pm • 1 PM #231

Problem 75. (?)

For each positive integer n, let

$$f(n)=\frac{1}{\sqrt[3]{n^2+2n+1}+\sqrt[3]{n-1}+\sqrt[3]{n^2-2n+1}}.$$

Determine the value of

$$f(1)+f(3)+f(5)+\cdots+f(999997)+f(999999).$$

EDIT. Oh I am sorry. It seems it should be $\sqrt[3]{n^2-1}$ instead of $\sqrt[3]{n-1}$. Corrected. [Actually I saw this problem in one of Australian Olympiads, but it was $\sqrt[3]{n^1-1}$, which seems to be a mistake - Thanks RSM.]

This post has been edited 1 time. Last edited by Amir Hossein, Feb 22, 2011, 5:52 pm
Reason: Edited.

---

abhinavza... 418 posts

Feb 21, 2011, 6:45 pm PM #232

> amparvardi wrote:
> **Problem 75.** (?)

It's "Problem 74".

RSM 736 posts

Feb 22, 2011, 5:51 am PM #233

> pco wrote:
>> RSM wrote:
>> Prove or disprove the statement:-
>> Each even function $f(x)$ can be written as $g(x)+g(-x)$
>> Where f and g are defined over $\mathbb{R}$.
>
> $f(x)=\frac{f(x)}{2}+\frac{f(-x)}{2}$

I have changed the problem.
Otherwise it becomes trivial.

---

soulhunter 317 posts

Feb 22, 2011, 9:23 am PM #234

> RSM wrote:
>> abhinavzandubalm wrote:
>> Even A Lifeguard Must not be bringing the dead back to life by C.P.R. like we have to do for this marathon.
>>
>> Problem 71
>
> This is an INMO problem
> A solution to it can be found in INMO official solution paper.
>
> Next problem:-
> Find all function f defined on real variables such that
> $f(x+y)+f(x-y)=2f(x)f(y)$
> for all $x,y\in\mathbb{R}$
>
> I could not solve this problem myself.
> So I want to give one more condition that $|f(x)|\geq 1\forall x\in\mathbb{R}$
> I think it is easy now.

This is an famous easy problem on cauchy's equation which's solution gives us de'alembert's equation!

http://www.jstor.org/pss/2313495

or found at venkatchala's book on functional equation

pco
12955 po...

Feb 22, 2011, 1:54 pm PM #235

> *RSM wrote:*
> I have changed the problem.
> Otherwise it becomes trivial.

Sure, now it's far from being trivial

> *RSM wrote:*
> Prove or disprove the statement:-
> Each even function $f(x)$ can be written as $g(x)+g(-x)$
> Where f and g are defined over $\mathbb{R}$.
> g is not even function.

$f(x)=\frac{f(x)+x}{2}+\frac{f(-x)-x}{2}$

Amir Hos...
4728 pos...

Feb 22, 2011, 5:54 pm PM #236

I am sorry about the problem 75 I posted. There was a typo in the problem. Thanks RSM for pointing it out - edited.

prafullasd
25 posts

Feb 22, 2011, 7:43 pm PM #237

Problem 75

$f(n)=\frac{1}{\sqrt[3]{(n+1)^2}+\sqrt[3]{(n+1)(n-1)}+\sqrt[3]{(n-1)^2}}=\frac{\sqrt[3]{n+1}-\sqrt[3]{n-1}}{n+1-(n-1)}=\frac{\sqrt[3]{n+1}-\sqrt[3]{n-1}}{2}$

and ,therefore, the sum is like an telescopic sum ,

$Answer=\frac{\sqrt[3]{1000000}-\sqrt[3]{0}}{2}=50$

abhinavza...
418 posts

Feb 22, 2011, 8:01 pm PM #238

> *RSM wrote:*
> Prove or disprove the statement:-
> Each even function $f(x)$ can be written as $g(x)+g(-x)$
> Where f and g are defined over $\mathbb{R}$.
> g is not even function.

The General Solution Will Be

$f(x)=\left[\frac{f(x)}{2}+h(x)\right]+\left[\frac{f(-x)}{2}+h(-x)\right]$

Where $h(x)$ Is Any Odd Function

abhinavza...
418 posts

Feb 22, 2011, 8:08 pm PM #239

Now Lets Give The Next Problem
Yours Was Not **75** As I Pointed Out **amparvardi**
A Simple One For Your Pleasure
Problem 75

Edit. Thanks **pco** I Edited The Mistake .Its '+' Instead Of '='

This post has been edited 1 time. Last edited by abhinavzandubalm, Feb 22, 2011, 8:19 pm

pco
12955 po...

Feb 22, 2011, 8:12 pm PM #240

> *abhinavzandubalm wrote:*
> Problem 75:
> Find All The Functions Which Are Strictly Monotone Satisfying
> $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(f(x)+y)=f(x+y)=f(0)\quad\forall\ x,y\in\mathbb{R}$

None, since $f(x+y)=f(0)$ means that $f(x)$ is constant and so not strictly monotonous.

prafullasd
25 posts

Feb 22, 2011, 8:50 pm PM #241

> *abhinavzandubalm wrote:*
> Find All The Functions Which Are Strictly Monotone Satisfying
> $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(f(x)+y)=f(x+y)+f(0)\quad\forall\ x,y\in\mathbb{R}$

$y=-f(x)$ gives $f(x-f(x))=0$
as function is strictly monotone, there exists unique $a\in\mathbb{R}$ such that $f(a)=0$
therefore,
$x-f(x)=a\quad\forall\ x\in\mathbb{R}$

therefore, $f(x)=x-a \quad \forall\, x\in\mathbb{R}$, which satisfies the given functional equation

magical
196 posts

Feb 22, 2011, 9:40 pm PM #242

> *prafullasd wrote:*
> $y=-f(x)$ gives $f(x-f(x))=0$
> as function is strictly monotone, there exists unique $a\in\mathbb{R}$ such that $f(a)=0$
> therefore,
> $x-f(x)=a \quad \forall\, x\in\mathbb{R}$
> therefore, $f(x)=x-a \quad \forall\, x\in\mathbb{R}$, which satisfies the given functional equation

I think you should prove that f : is surjective before setting $y=-f(x)$

goodar20...
1346 pos...

Feb 22, 2011, 10:02 pm PM #243

> *magical wrote:*
>
> > *prafullasd wrote:*
> > $y=-f(x)$ gives $f(x-f(x))=0$
> > as function is strictly monotone, there exists unique $a\in\mathbb{R}$ such that $f(a)=0$
> > therefore,
> > $x-f(x)=a \quad \forall\, x\in\mathbb{R}$
> > therefore, $f(x)=x-a \quad \forall\, x\in\mathbb{R}$, which satisfies the given functional equation
>
> I think you should prove that f : is surjective before setting $y=-f(x)$

no, setting $y=-f(x)$ has no problem, because it's a real number, and the equation is for all $x,y\in\mathbb{R}$

magical
196 posts

Feb 23, 2011, 11:36 am PM #244

> *goodar2006 wrote:*
> no, setting $y=-f(x)$ has no problem, because it's a real number, and the equation is for all $x,y\in\mathbb{R}$

OK,thanks you

When do they can set $x=-f(x)$ into any functional equation ?

abhinavza...
418 posts

Feb 23, 2011, 12:01 pm PM #245

Now The Next Problem 76

pco
12955 po...

Feb 23, 2011, 3:57 pm PM #246

> *abhinavzandubalm wrote:*
> Problem 76 :
> Find All Functions $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)f(y)} \quad \forall\, x,y\in\mathbb{R}$

Partial result

abhinavza...
418 posts

Feb 23, 2011, 4:03 pm PM #247

> *pco wrote:*
>
> > *abhinavzandubalm wrote:*
> > Problem 76 :
> > Find All Functions $f:\mathbb{R}\to\mathbb{R}$ Such That
> > $f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)f(y)} \quad \forall\, x,y\in\mathbb{R}$
>
> Partial result

To **pco**
I Read The Question A Long Time Ago.
Hence If You Would Check Whether It Should Be Given or Not That The Function Is Continuous It Would Be Of Help?

abhinavza...
418 posts

Feb 23, 2011, 4:05 pm PM #248

And I Don't Think We Need To Put
$f(x)=tan(\pi g(x))$ Simply $f(x)=tan(g(x))$ May Also Suffice.

pco
12955 po...

Feb 23, 2011, 4:09 pm PM #249

> *abhinavzandubalm wrote:*

> And I Don't Think We Need To Put
> $f(x)=tan(\pi g(x))$ Simply $f(x)=tan(g(x))$ May Also Suffice.

It was just to get $h(x)\notin\mathbb{Q}$, which is simpler than $\frac{h(x)}{\pi}\notin\mathbb{Q}$
But this is just a detail.

My question is : could you give us a hint to prove that the results I proposed are the only one (or the contrary).
Thanks a lot.

And, btw, if you add continuity, then we obviously have only $f(x)=0$ as solution.

---

abhinavza...
418 posts

Feb 23, 2011, 4:21 pm PM #250

I Contacted Three Of My Friends And One Told Me To Check The Book
$Functional\ Equations\ -A\ Problem\ Solving\ Approach$ By $B.J.Venkatachala.$
It Was The Problem 5.21 And Continuity Was There.
The Official Solution Was To Get D'Alembert's Equation By Defining
$g(x)=\frac{1-f(x)^2}{1+f(x)^2},h(x)=\frac{2f(x)}{1+f(x)^2}$ To Get
$g(x-y)=g(x)g(y)+h(x)h(y)\implies g(x)=Cos(\alpha x),h(x)=Sin(\alpha x)$
Thus $f(x)=Tan(\alpha x)$ Is The Solution.

---

pco
12955 po...

Feb 23, 2011, 4:27 pm PM #251

> *abhinavzandubalm wrote:*
> I Contacted Three Of My Friends And One Told Me To Check The Book
> $Functional\ Equations\ -A\ Problem\ Solving\ Approach$ By $B.J.Venkatachala.$
> It Was The Problem 5.21 And Continuity Was There.
> The Official Solution Was To Get D'Alembert's Equation By Defining
> $g(x)=\frac{1-f(x)^2}{1+f(x)^2},h(x)=\frac{2f(x)}{1+f(x)^2}$ To Get
> $g(x-y)=g(x)g(y)+h(x)h(y)\implies g(x)=Cos(\alpha x),h(x)=Sin(\alpha x)$
> Thus $f(x)=Tan(\alpha x)$ Is The Solution.

You should revisit your courses : $f(x)=\tan(\alpha x)$ is not continuous if $\alpha\neq 0$ and so is NOT the solution if you add continuity.

The only continuous solution to this equation is $f(x)=0$

---

abhinavza...
418 posts

Feb 23, 2011, 5:28 pm PM #252

I Know.
That's Why I Was Having Doubts To Whether We can Use The Method Given But Disregard Continuity.

FBI__
29 posts

Feb 23, 2011, 11:41 pm PM #253

Problem 77 Find all functional $f:\mathbb{R}\to\mathbb{R}$ satisfy:$xf(x)-yf(y)=(x-y)f(x+y)$ for all $x,y\in\mathbb{R}$

---

pco
12955 po...

Feb 23, 2011, 11:56 pm PM #254

> *FBI__ wrote:*
> **Problem 77** Find all functional $f:\mathbb{R}\to\mathbb{R}$ satisfy:$xf(x)-yf(y)=(x-y)f(x+y)$ for all $x,y\in\mathbb{R}$

Let $P(x,y)$ be the assertion $xf(x)-yf(y)=(x-y)f(x+y)$

$$P(\frac{x-1}{2},\frac{1-x}{2})\implies\frac{x-1}{2}f(\frac{x-1}{2})-\frac{1-x}{2}f(\frac{1-x}{2})=(x-1)f(0)$$

$$P(\frac{1-x}{2},\frac{x+1}{2})\implies\frac{1-x}{2}f(\frac{1-x}{2})-\frac{x+1}{2}f(\frac{x+1}{2})=-xf(1)$$

$$P(\frac{x+1}{2},\frac{x-1}{2})\implies\frac{x+1}{2}f(\frac{x+1}{2})-\frac{x-1}{2}f(\frac{x-1}{2})=f(x)$$

Adding these three lines, we get $f(x)-xf(1)+(x-1)f(0)=0$ and so $f(x)=(f(1)-f(0))x-f(0)$

And so $\boxed{f(x)=ax+b}$ which indeed is a solution.

Amir Hos...
4728 pos...

Feb 24, 2011, 1:25 am PM #255

Problem 78.

For each positive integer n, let

$$f(n)=f(n)=\left[2\sqrt{n}\right]-\left[\sqrt{n-1}+\sqrt{n+1}\right].$$

Determine all values n for which $f(n)=1$.

Note. $[x]$ is the largest integer not exceeding x.

---

mousavi
222 posts

Feb 24, 2011, 3:10 am PM #256

> *amparvardi wrote:*
> **Problem 78.**
>
> For each positive integer n, let
>
> $$f(n) = f(n) = [2\sqrt{n}] - \left[\sqrt{n-1} + \sqrt{n+1}\right].$$
>
> Determine all values n for which $f(n) = 1$.
>
> **Note.** $[x]$ is the largest integer not exceeding x.

it is obvious $0 < 2\sqrt{n} - (\sqrt{n-1} + \sqrt{n+1}) < 1$ (1)

by (1) we conclude that if $n = k^2$ then $f(n) = 1$

let $n \neq k^2 \Longrightarrow [2\sqrt{n}]^2 < 4n \Longrightarrow [2\sqrt{n}] \leq \sqrt{4n-1}$

if $f(n) = 1$ then $[\sqrt{n+1} + \sqrt{n-1}] + 1 \leq \sqrt{4n-1}$

but we want to prove that $[\sqrt{n+1} + \sqrt{n-1}] + 1 > \sqrt{4n-1}$

$[\sqrt{n+1} + \sqrt{n-1}] + 1 > \sqrt{n+1} + \sqrt{n-1} > \sqrt{4n-1}$ (?)

$2n + 2\sqrt{n^2-1} > 4n - 1(?)$

$4n^2 - 4 > 4n^2 + 1 - 4n(?) \Longrightarrow 4n > 5\,(n > 1)$

---

mousavi
222 posts

Feb 25, 2011, 1:14 am PM #257

problem 79

Find all natural and odd numbers $n \geq 3$ such that the below function be injective:

$f : Q \longrightarrow Q$, $f(x) = x^n - 2x$

---

prafullasd
25 posts

Feb 25, 2011, 10:45 am PM #258

Problem 79
Click to reveal hidden text

---

prafullasd
25 posts

Feb 26, 2011, 9:04 pm PM #259

Problem 80:
Find all continuous, strictly increasing functions $f : \mathbb{R} \to \mathbb{R}\ \ such\ that$
$1)\ \ f(0) = 0$
$2)\ \ f(1) = 1$
$3)\ \ [f(x+y)] = [f(x)] + [f(y)]\ \ \forall\ \ x, y \in \mathbb{R}\ such\ that\ [x+y] = [x] + [y]$
where $[x]\ is\ largest\ integer\ less\ than\ or\ equal\ to\ x$

P.S. This is a problem which occured to me and not from any contest, and so i have no official solution, only my own.So, if u think the problem need additional constraints(or lesser!), pls tell

---

SCP
1520 pos...

Feb 26, 2011, 9:47 pm PM #260

> *prafullasd wrote:*
> **Problem 80:**
> Find all continuous, strictly increasing functions $f : \mathbb{R} \to \mathbb{R}\ \ such\ that$
> $1)\ \ f(0) = 0$
> $2)\ \ f(1) = 1$
> $3)\ \ [f(x+y)] = [f(x)] + [f(y)]\ \ \forall\ \ x, y \in \mathbb{R}\ such\ that\ [x+y] = [x] + [y]$
> where $[x]\ is\ largest\ integer\ less\ than\ or\ equal\ to\ x$
>
> P.S. This is a problem which occured to me and not from any contest, and so i have no official solution, only my own.So, if u think the problem need additional constraints(or lesser!), pls tell

I think there are infintely many:
let $f(z) = z$ for all integers and $[x] \leq f(x) \leq x$ for all x.

---

prafullasd
25 posts

Feb 26, 2011, 10:38 pm PM #261

> *SCP wrote:*
> I think there are infintely many:
> let $f(z) = z$ for all integers and $[x] \leq f(x) \leq x$ for all x.

Click to reveal hidden text

BigSams
1582 pos...

Feb 26, 2011, 11:24 pm PM #262

This is the first functional equation I ever solved, hope u like it

Problem 81.
Find all functions f mapped from $R \to R$ such that $(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^2-y^2)$.

---

mousavi
222 posts

Feb 27, 2011, 6:12 pm PM #263

> *BigSams wrote:*
> This is the first functional equation I ever solved, hope u like it
>
> **Problem 81.**
> Find all functions f mapped from $R \to R$ such that $(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^2-y^2)$.

$p(x,x) :\Longrightarrow f(0)=0$

$p(x+y,y) :\Longrightarrow xf(2x+y)-(2x+y)f(x)=4(x+y)(y)(x^2+2xy)$ (1)

$p(x,1-2x) :\Longrightarrow xf(1)-f(x)=4(1-x)(1-2x)(x^2+2x-4x^2)$ (by (1))

$\Longrightarrow f(x)=xt-4(1-x)(1-2x)(2x-3x^2)$

BigSams
1582 pos...

Feb 27, 2011, 10:30 pm PM #264

Erm no, that's not the solution, sorry.

RSM
736 posts

Feb 27, 2011, 11:05 pm PM #265

This is probably the easiest problem posted in this marathon.
First put x=y and $x \neq 0$
Then we get f(0)=0
Put x=0 to get $f(-y)=-f(y)$
Put $x=\dfrac{a+b}{2} andy=\dfrac{a-b}{2}$
and the equation turns to $bf(a)-af(b)=(a^2-b^2)ab$
$\dfrac{f(a)-a^3}{a}=\dfrac{f(b)-b^3}{b}=\text{constant}=k$
So $f(x)=x^3+kx$
k may take any real value.

---

socrates
1818 posts

Feb 28, 2011, 5:43 am PM #266

Problem 82.
Find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ such that $f(y+xf(y))=f(y)+yf(x)$ and $f(3)$ is prime.

---

mousavi
222 posts

Feb 28, 2011, 9:06 am PM #267

> *socrates wrote:*
> **Problem 82.**
> Find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ such that $f(y+xf(y))=f(y)+yf(x)$ and $f(3)$ is prime.

Problem 70

---

prafullasd
25 posts

Feb 28, 2011, 12:29 pm PM #268

No solution posted for problems 79 and 80 yet.(is my problem that difficult?)
also, i think a list should be made of the problems of the marathon which have not been solved.

RSM
736 posts

Feb 28, 2011, 3:23 pm PM #269

Solution to Problem 82:-
Suppose $f(1)=c$
Put x=1 and y=1
Put y=1
$f(1+cx)=c+f(x)$....................(1)
In this equation put x=1
$f(c+1)=2c$
Put x=1
$f(y+f(y))=f(y)+cy$
In this equation put y=c+1
$f(3c+1)=c^2+3c$.............(2)
In the equation (1) put x=3
$f(3c+1)=c+f(3)$.............(3)
Comparing (2) and (3) we have
$f(3)=c^2+2c=c(c+2)$
Since f(3) is a prime so c=1
Putting this in equation (1)
$f(c+1)=f(c)+1$
And putting x=0 in the functional equation we have $f(0)=0$
So by induction $f(x)=x\in\mathbb{N}$

Someone post the next problem,please.

RSM
736 posts

Feb 28, 2011, 5:04 pm PM #270

Solution to Problem 80:-(This problem was not solved yet)
Suppose, $x = n + a, y = n + b$ where $[x] = [y] = n$
Take two a and b such that $a + b < 1$
Then $[x] + [y] = [x + y]$
$[f(2n + a + b)] = [f(n + a)] + [f(n + b)]$
$[f(a + b)] = [f(a)] + [f(b)]$.................(1)
$[f(ka)] = k[f(a)] \forall k \in \mathbb{N}$
$[f(\frac{p}{q})] = \frac{[f(\frac{1}{p})]}{q} = \frac{[f(\frac{1}{q})]}{p}$
$[f(\frac{1}{x})] = \frac{k}{x} \forall x \in \mathbb{N}$
This implies x divides k for all x.
So k=0
$[f(\frac{1}{x})] = 0 \forall x \in \mathbb{N}$
$[f(\frac{x}{y})] = 0 \forall x, y \in \mathbb{N} and y > x$
This can be extended to irrational numbers since every irrational number can be written as a sum of rational numbers.
So $[f(a)] = 0 for all 0 < a < 1$
$[f(x)] = [f([x] + a)] = [f([x])] = [x][f(1)] = [x]$
So $[f(x)] = [x] for all x \geq 0$
In a similar manner it can be proved that $[f(x)] = [x] for all x \leq 0$
So $[f(x)] = [x]$ is the solution.

I may have made any stupid mistake.
This post has been edited 1 time. Last edited by RSM, Mar 1, 2011, 12:23 am

prafullasd
25 posts

Feb 28, 2011, 9:00 pm PM #271

why k=0?
well, the solution is wrong as there are infinite solutions
a proof (i am posting only a proof showing that the following solution satisfies the equation, not of why it is the only solution)
Click to reveal hidden text

mcrasher
1766 pos...

Feb 28, 2011, 9:11 pm PM #272

> *BigSams wrote:*
> This is the first functional equation I ever solved, hope u like it 😁
> **Problem 81.**
> Find all functions f mapped from $R \to R$ such that $(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$.

solution#81

RSM
736 posts

Feb 28, 2011, 10:32 pm PM #273

> *prafullasd wrote:*
> why k=0?
> well, the solution is wrong as there are infinite solutions
> a proof (i am posting only a proof showing that the following solution satisfies the equation, not of why it is the only solution)
> Click to reveal hidden text

I made a stupid mistake.
I have edited my proof.
This post has been edited 1 time. Last edited by RSM, Mar 1, 2011, 12:17 am

pco
12955 po...

Feb 28, 2011, 11:27 pm • 1 PM #274

> *prafullasd wrote:*
> **Problem 80:**
> Find all continuous, strictly increasing functions $f : \mathbb{R} \to \mathbb{R}$ $such\ that$
> 1) $f(0) = 0$
> 2) $f(1) = 1$
> 3) $[f(x + y)] = [f(x)] + [f(y)]$ $\forall$ $x, y \in \mathbb{R}$ $such\ that$ $[x + y] = [x] + [y]$

a) $f(x) \in (0, 1) \forall x \in (0, 1)$
Trivial using 1) 2) and increasing property

b) $[f(n)] = n \forall n \in \mathbb{Z}$
$[m + n] = [m] + [n] \forall m, n \in \mathbb{Z}$ and so $[f(m + n)] = [f(m)] + [f(n)]$ and so $[f(n)] = n[f(1)] = n$

c) $[f(x)] \geq [x] \forall x$

$x \geq [x]$ and $f(x)$ increasing implies $f(x) \geq f([x])$ and so $[f(x)] \geq [f([x])] = [x]$

d) $[f(x)] < [x] + 1\, \forall x$
If $[f(a)] \geq [a] + 1$ for some a, then :
$[f([a])] = [a]$ and so $f([a]) < [a] + 1$
Then continuity implies $\exists u \in ([a], a)$ such that $f(u) = [a] + 1$
Choosing then some $x \in ([a], u)$ and $y = a - x \in (0, 1)$ we get $[x + y] = [a] = [x] + [y]$ and so :
$[f(x + y)] = [f(x)] + [f(y)]$ which is $[f(a)] = [f(x)] + [f(y)]$ which is wrong since $[f(a)] \geq [a] + 1$ while $[f(x)] = [a]$ and $[f(y)] = 0$
So no such a

From c),d) we get $[f(x)] = [x]$ and, plugging this in original equation, we get that any strictly increasing continuous function matching 1) and 2) and $[f(x)] = [x]$ matches 3) too.

$[f(x)] = [x]$ and continuity imply $f(n) = n$

Hence the answer:
$f(x)$ solution if and only if :
$f(x) = x\, \forall x \in \mathbb{Z}$
$f(x)$ may take any values in $(n, n+1)$ when $x \in (n, n+1)$ with respect to the properties "strictly increasing and continuous"

---

abhinavza...
418 posts

Mar 2, 2011, 7:18 pm PM #275

As Solution To 81 Is Given And A Repeated Problem 82 , I assume that i have to give a New Problem 82

---

pco
12955 po...

Mar 2, 2011, 11:18 pm PM #276

> *abhinavzandubalm wrote:*
> "Problem 82"
> Find All Functions $f: \mathbb{N} \to \mathbb{N}$
> $f(m + f(n)) = n + f(m + k) \quad \forall\, m, n, k \in \mathbb{N}$ With k Being Fixed Natural Number

If $f(n) < k$ for some n, then the equation may be written $f(m + (k - f(n))) = f(m) - n\, \forall m > f(n)$
So $f(m + p(k - f(n))) = f(m) - pn$, which is impossible, since this would imply $f(x) < 0$ for some x great enough.

If $f(n) = k$ for some n, then the equation implies $n = 0$, impossible

So $f(n) > k\, \forall n$ and the equation may be written $f(m + (f(n) - k)) = n + f(m)\, \forall m > k$
And so $f(m + p(f(n) - k)) = pn + f(m)$
Choosing then $p = f(q) - k$, we get $f(m + (f(q) - k)(f(n) - k)) = (f(q) - k)n + f(m)$ and so, by symetry :
$(f(q) - k)n = (f(n) - k)q\, \forall q, n$
And so $\dfrac{f(q) - k}{q} = \dfrac{f(n) - k}{n}$ and so $f(n) = k + cn$ for some constant c

Plugging this in original equation, we get $c = 1$ and so solution $\boxed{f(n) = n + k}$

---

mousavi
222 posts

Mar 2, 2011, 11:21 pm PM #277

> *abhinavzandubalm wrote:*
> As Solution To 81 Is Given And A Repeated Problem 82 , I assume that i have to give a New Problem 82

f is injective.
$f(f(m) + f(n)) = n + f(f(m) + k) = m + f(f(n) + k)$ (1)

$p(m, f(n) + k)) : f(m + f(f(n) + k)) = f(n) + k + f(m + k)$ (2)

$p(f(m), n) : f(f(m) + f(n)) = f(m + f(f(n) + k)) = f(m) + k + f(n + k)$ (3)

by $(2), (3) \implies f(n) + f(m + k) = f(m) + f(n + k)$ (4)

by(4) $f(n) + f(2k) = f(k) + f(n + k)$ (5) and by the problem :$f(k + f(n)) = n + f(2k)$ (6)

by (5) :$f(f(n)) + f(2k) = f(k) + f(f(n) + k) = f(k) + n + f(2k) \implies f(f(n)) = f(k) + n$, (7)

by (7) and by the problem$p(m, f(k)) : f(m + k + f(k)) = f(k) + f(m + k)$

by (4) and $p(n, m + f(k)) : f(n) + f(m + f(k) + k) = f(m + f(k)) + f(n + k)$

$\implies f(n) + f(k) + f(m + k) = k + f(m + k) + f(n + k) \implies f(n) + f(k) = k + f(n + k) = f(n + f(k))$ (8)

by (8) and by the problem $f(m + f(n)) = n + f(m) + f(k) - k$ (9)

by (9) $p(m, f(n)) : f(m + n + f(k)) = f(m) + f(n) + f(k) - k$

$(8),(9): f(m+n)+f(k)=f(m)+f(n)+f(k)-k$

$\Longrightarrow f(m+n)+f(m)+f(n)-k \Longrightarrow f(m)=m+k$

Amir Hos... 4728 pos...

Mar 3, 2011, 9:28 am PM #278

Problem 83.

Let f be a function defined for all real numbers and taking real numbers as its values. Suppose that, for all real numbers x, y the function satises

$$f(2x)=f\left(\sin\left(\frac{\pi x}{2}+\frac{\pi y}{2}\right)\right)+f\left(\sin\left(\frac{\pi x}{2}-\frac{\pi y}{2}\right)\right),$$

and

$$f(x^2-y^2)=(x+y)f(x-y)+(x-y)f(x+y).$$

Show that these conditions uniquely determine

$$f\left(1990+\sqrt[2]{1990}+\sqrt[3]{1990}\right)$$

and give its value.

---

pco 12955 po...

Mar 3, 2011, 12:45 pm PM #279

> *amparvardi wrote:*
> **Problem 83.**
>
> Let f be a function defined for all real numbers and taking real numbers as its values. Suppose that, for all real numbers x, y the function satises
>
> $$f(2x)=f\left(\sin\left(\frac{\pi x}{2}+\frac{\pi y}{2}\right)\right)+f\left(\sin\left(\frac{\pi x}{2}-\frac{\pi y}{2}\right)\right),$$
>
> and
>
> $$f(x^2-y^2)=(x+y)f(x-y)+(x-y)f(x+y).$$
>
> Show that these conditions uniquely determine
>
> $$f\left(1990+\sqrt[2]{1990}+\sqrt[3]{1990}\right)$$
>
> and give its value.

Notice first that the second equation may be written $f(xy)=xf(y)+yf(x)$

Setting $x=y=0$ in the first equation, we get $f(0)=0$
Setting $x=y=1$ in the first equation, we get $f(2)=0$

Setting $y=2$ in $f(xy)=xf(y)+yf(x)$, we get $f(2x)=2f(x)$

Setting $y=x$ in the first equation, we get $2f(x)=f(\sin\pi x)$ and so :
(a) : $f(x+2)=f(x)$
(b) : $f(1-x)=f(x)$

Setting $y\to y+2$ in $f(xy)=xf(y)+yf(x)$, we get $f(xy+2x)=f(xy)+f(2x)$ and so $f(x+y)=f(x)+f(y)$

$f(2)=0$ and $f(2x)=2f(x) \Longrightarrow f(1)=0$
(b) and $f(x+y)=f(x)+f(y)$ and $f(1)=0 \Longrightarrow f(x)=f(-x)$

And so $0=f(0)=f(x+(-x))=f(x)+f(-x)=2f(x)$ and $\boxed{f(x)=0\ \forall x}$ which indeed is a solution.

---

mousavi 222 posts

Mar 3, 2011, 1:44 pm PM #280

problem 84

Find all functions $f: Z\longrightarrow Z$ such that:

$f(x^3+y^3+z^3)=f(x)^3+f(y)^3+f(z)^3$

---

mahanmath 1356 pos...

Mar 3, 2011, 6:29 pm PM #281

mousavi wrote:

problem 84

Find all functions $f:Z\longrightarrow Z$ such that:

$f(x^3+y^3+z^3)=f(x)^3+f(y)^3+f(z)^3$

mahanmath wrote:

Stephen wrote:

Problem 21

Find all $f:Z\to Z$ that satisfies $f(x)^3+f(y)^3+f(z)^3=f(x^3+y^3+z^3)$.

#)If $x\geq 4$, x^3 can be written as a sum of five cubes such that their absolute values are less than x.
From # and induction we get the answer is $f(x)=xf(1)$ and $f(1)=1,0,-1$.

In fact , It\`s a number theory problem !....

---

abhinavza...
418 posts

Mar 3, 2011, 9:16 pm PM #282

As Problem 84 Is Repeated Let Me Give A New One . Its Quite Simple.
Problem 84

---

pco
12955 po...

Mar 3, 2011, 9:28 pm PM #283

abhinavzandubalm wrote:

As Problem 84 Is Repeated Let Me Give A New One . Its Quite Simple.
Problem 84
Find All Polynomials $P(x)$ Such That
$xP(x-1)=(x-15)P(x)$

Set $x=0$ in functional equation and you get $P(0)=0$
Set $x=1$ in functional equation and you get $P(1)=0$
Set $x=2$ in functional equation and you get $P(2)=0$
...
Set $x=14$ in functional equation and you get $P(14)=0$

Then, plugging $P(x)=x(x-1)(x-2)...(x-14)Q(x)$ in equation, we get $Q(x-1)=Q(x)$

Hence the solution $\boxed{P(x)=cx(x-1)(x-2)...(x-14)}$

Babai
486 posts

Mar 4, 2011, 5:15 pm PM #284

There is a good problem-
$f:R\to R$ such that $f(x)f(yf(x)-1)=x^2f(y)-f(x)$ for real x,y.

---

pco
12955 po...

Mar 4, 2011, 5:56 pm • 1 PM #285

Babai wrote:

There is a good problem-
$f:R\to R$ such that $f(x)f(yf(x)-1)=x^2f(y)-f(x)$ for real x,y.

$f(x)=0\ \forall x$ is a solution and let us from now look for non all-zero solutions.

Let $P(x,y)$ be the assertion $f(x)f(yf(x)-1)=x^2f(y)-f(x)$
Let u such that $f(u)\neq 0$

$P(1,1)\implies f(1)f(f(1)-1)=0$ and so $\exists v$ such that $f(v)=0$
$P(v,u)\implies v^2f(u)=0$ and so $v=0$

So $f(x)=0\iff x=0$ and we got $f(1)=1$

$P(1,x)\implies f(x-1)=f(x)-1$ and so $P(x,y)$ may be written :
New assertion $Q(x,y)$: $f(x)f(yf(x))=x^2f(y)$

Let $x\neq 0$: $Q(x,x)\implies f(xf(x))=x^2$ and so any $x\geq 0$ is in $f(\mathbb{R})$

$Q(x,y)\implies f(x)f(yf(x))=x^2f(y)$
$Q(x,1)\implies f(x)f(f(x))=x^2$
$Q(x,y+1)\implies f(x)f(yf(x)+f(x))=x^2f(y)+x^2$

And so $f(x)f(yf(x)+f(x))=f(x)f(yf(x))+f(x)f(f(x))$

Choosing then $z>0$ and x such that $f(x)=z$, we get : $f(yz+z)=f(yz)+f(z)$ and so $f(x+y)=f(x)+f(y)$ $\forall x>0,\forall y$

And this immediately implies $f(x+y)=f(x)+f(y)\ \forall x,y$ ($x=0$ is obvious and using $y=-x$, we get $f(-x)=-f(x)$)

$Q(x,1)\implies f(x)f(f(x))=x^2$
$Q(x+1,1)\implies (f(x)+1)(f(f(x))+1)=x^2+2x+1$
And so $f(f(x))+f(x)=2x$

And combinaison of $f(x)f(f(x))=x^2$ and $f(f(x))+f(x)=2x$ implies $(f(x)-x)^2=0$ and so $f(x)=x\ \forall x$, which indeed is a solution

Hence the solutions :
$f(x)=0\ \forall x$
$f(x)=x\ \forall x$

---

mahanmath
1356 pos...

Mar 4, 2011, 8:49 pm PM #286

Problem 86

Prove that there is no function like $f:\mathbb{R}_+\to\mathbb{R}$ such that for all positive x,y :

$f(x+y)>y(f(x)^2)$

---

pco
12955 po...

Mar 5, 2011, 1:04 pm • 2 PM #287

> *mahanmath wrote:*
> **Problem 86**
>
> Prove that there is no function like $f:\mathbb{R}_+\to\mathbb{R}$ such that for all positive x,y :
>
> $f(x+y)>y(f(x)^2)$

Let $P(x,y)$ be the assertion $f(x+y)>yf(x)^2$

Let $x>0$: $P(\frac{x}{2},\frac{x}{2})\implies f(x)>0\ \forall x$

Let then $a>0$ and $x\in[0,a]$: $P(x,2a-x)\implies f(2a)>(2a-x)f(x)^2\ge af(x)^2$ and so $f(x)^2<\frac{f(2a)}{a}$

And so $f(x)$ is upper bounded over any interval $(0,a]$

Let then $f(1)=u>0$ and the sequence $x_0=1$ and $x_{n+1}=x_n+\frac{2}{f(x_n)}\ \forall n\ge 0$:

$P(x_n,\frac{2}{f(x_n)})\implies f(x_{n+1})>2f(x_n)$ and so $f(x_n)>2^nu\ \forall n>0$

So $x_1=1+\frac{2}{u}$ and $x_{n+1}<x_n+\frac{1}{2^{n-1}u}\ \forall n>0$

So $x_n<1+\frac{1}{u}(2+1+\frac{1}{2}+\frac{1}{4}+...\frac{1}{2^{n-1}})<1+\frac{4}{u}$

But $f(x_n)>2^nu$ and $x_n<1+\frac{4}{u}$ shows that $f(x)$ is not upper bounded over $(0,1+\frac{4}{u}]$, and so contradiction with the first sentence of this proof.

So no such function.

Amir Hos...
4728 pos...

Mar 5, 2011, 7:02 pm PM #288

Problem 87.

Let f be a function defined for positive integers with positive integral values satisfying the conditions:

(i) $f(ab)=f(a)f(b)$,

(ii) $f(a)<f(b)$ if $a<b$,

(iii) $f(3)\ge 7$.

Find the minimum value for $f(3)$.

---

pco
12955 po...

Mar 5, 2011, 7:19 pm • 1 PM #289

> *amparvardi wrote:*
> **Problem 87.**
>
> Let f be a function defined for positive integers with positive integral values satisfying the conditions:
>
> **(i)** $f(ab)=f(a)f(b)$,
>
> **(ii)** $f(a)<f(b)$ if $a<b$

(ii) $f(a) < f(b)$ if $a < b$,

(iii) $f(3) \geq 7$.

Find the minimum value for $f(3)$.

Let $m > n > 1$ two integers :

If $\frac{p}{q} < \frac{\ln m}{\ln n} < \frac{r}{s}$, with $p, q, r, s \in \mathbb{N}$, we get :

$n^p < m^q$ and so $f(n)^p < f(m)^p$ and so $\frac{p}{q} < \frac{\ln f(m)}{\ln f(n)}$

$m^s < n^r$ and so $f(m)^s < f(n)^r$ and so $\frac{\ln f(m)}{\ln f(n)} < \frac{r}{s}$

And so $\frac{\ln f(m)}{\ln f(n)} = \frac{\ln m}{\ln n}$ and $\frac{\ln f(m)}{\ln m} = \frac{\ln f(n)}{\ln n} = c$

And $f(n) = n^c$

And so $f(3) = 3^c \geq 7$

So $c = 2$ and minimum value for $f(3)$ is nine, which is reached for function $f(n) = n^2$

Amir Hos...
4728 pos...

Mar 5, 2011, 7:25 pm PM #290

So quick! :)

Problem 88.

A function $f : \mathbb{N} \to \mathbb{N}$ satisfies

(a) $f(ab) = f(a)f(b)$ whenever the greatest common divisor of a and b is 1,

(b) $f(p+q) = f(p) + f(q)$ for all prime numbers p and q.

Show that $f(2) = 2, f(3) = 3$ and $f(1999) = 1999$.

---

mousavi
222 posts

Mar 5, 2011, 8:10 pm PM #291

> *amparvardi wrote:*
> So quick! :)
>
> **Problem 88.**
>
> A function $f : \mathbb{N} \to \mathbb{N}$ satisfies
>
> **(a)** $f(ab) = f(a)f(b)$ whenever the greatest common divisor of a and b is 1,
>
> **(b)** $f(p+q) = f(p) + f(q)$ for all prime numbers p and q.
>
> Show that $f(2) = 2, f(3) = 3$ and $f(1999) = 1999$.

$f(6) = f(3)f(2)$ and $f(6) = 2f(3) \Longrightarrow f(2) = 2$, $f(4) = 2f(2) = 4$

$f(12) = f(7) + f(5) = (f(5) + f(2)) + f(5) = 2(f(3) + f(2)) + f(2) = 3f(2) + 2f(3)$ and
$f(12) = f(3)f(4) = 4f(3) \Longrightarrow f(3) = 3$

$\Longrightarrow f(2) + f(3) = f(5) \Longrightarrow f(5) = 5$

$f(7) = f(5) + f(2) = 7, f(8) = f(5) + f(3) = 8$

$f(24) = f(8)f(3) = 24, f(24) = f(7) + f(17) \Longrightarrow f(17) = 17$

$1997, 1999$ are prime.

$f(1999) = f(1997) + 2 = 4 + f(1995) = 4 + f(5)f(3)f(7)f(19) = 4 + 105(f(17) + f(2)) = 4 + 1995 = 1999$

RSM
736 posts

Mar 5, 2011, 8:40 pm PM #292

> *amparvardi wrote:*
> So quick! :)
>
> **Problem 88.**

> A function $f : \mathbb{N} \to \mathbb{N}$ satisfies
>
> **(a)** $f(ab) = f(a)f(b)$ whenever the greatest common divisor of a and b is 1,
>
> **(b)** $f(p+q) = f(p) + f(q)$ for all prime numbers p and q.
>
> Show that $f(2) = 2, f(3) = 3$ and $f(1999) = 1999$.

I found f(2) and f(3) in the same way as @mousavi.
But my solution to find f(1999) is quite different.

It has been proved that Goldbech conjecture is true in a huge range of number.
So while finding f(1999) we can apply Goldbech conjecture easily.
Suppose, $f(n) = n \forall n \leq m$
If m+1 is not prime, then if it has more than one prime divisor or a power of 2 then we are done.
Click to reveal hidden text
But if it is a prime or a power of a prime, then we can easily calculate at least one of f(m+4) or f(m+6) and so f(m+1)=f(m+4)-3.
Click to reveal hidden text

In this proof I have omitted the trivial cases, for which f(n) is easy to calculate.

socrates
1818 posts

Mar 5, 2011, 8:52 pm PM #293

Problem 89

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that:

$f(x+y) = f(x) + f(y) + f(xy)$, for all $x, y \in \mathbb{R}$.

RSM
736 posts

Mar 5, 2011, 9:25 pm PM #294

Putting x=y=2 we have f(2)=0
Putting $x = a+b, y = c$
$f(a+b+c) = f(a+b) + f(c) + f(ac+bc)$
$f(a+b+c) = f(a) + f(b) + f(c) + f(ab) + f(ac) + f(bc) + f(abc^2)$
Similarly $f(a+b+c) = f(a) + f(b) + f(c) + f(ab) + f(ac) + f(bc) + f(a^2bc)$
So we have $f(abc^2) = f(a^2bc)$
Putting $b = \frac{1}{ac}$
$f(a) = f(c) \forall a, c \in \mathbb{R}$
So f(x)=0 for all x

Amir Hos...
4728 pos...

Mar 5, 2011, 9:37 pm PM #295

Problem 90.

(a) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a^3) + f(b^3) + f(c^3) = f(3abc) \quad \forall a, b, c \in \mathbb{R}.$$

(b) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a^3)+f(b^3)+f(c^3) = a\cdot f(a^2)+b\cdot f(b^2)+c\cdot f(c^2) \quad \forall a, b, c \in \mathbb{R}.$$

pco
12955 po...

Mar 5, 2011, 9:46 pm PM #296

> *amparvardi wrote:*
> **Problem 90.**
>
> **(a)** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
>
> $$f(a^3) + f(b^3) + f(c^3) = f(3abc) \quad \forall a, b, c \in \mathbb{R}.$$

Setting $b = c = 0$, we get $f(a^3) = -f(0)$ and so $f(x)$ is constant and the only constant solution is $\boxed{f(x) = 0}$ $\forall x$

RSM
736 posts

Mar 5, 2011, 10:03 pm PM #297

> *amparvardi wrote:*
> **Problem 90.**
>
> **(a)** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
>
> $$f(a^3) + f(b^3) + f(c^3) = f(3abc) \quad \forall a, b, c \in \mathbb{R}.$$

(b) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(a^3)+f(b^3)+f(c^3)=a\cdot f(a^2)+b\cdot f(b^2)+c\cdot f(c^2)\quad \forall a,b,c\in\mathbb{R}.$$

Putting a=b=c=0 we have f(0)=0
Putting c=0 we have $f(a^3)=-f(b^3)$ for all a,b
$f(b^3)=-f(c^3)$
$f(a^3)=-f(c^3)$
And so $f(a^3)=0$
So f(x)=0 for all x.

Solution to the second part :-
Putting a=b=c=0 we have f(0)=0
Putting b=c=0 we have $f(a^3)=af(a^2)$
Define f to the set of real numbers such that all real numbers can be represented in the form $k^{2^m3^n}$ with $|m-n|=1$ with k belonging to the set and no two elements can be written in this form with same k
Now it is easy to see that $f(k^{(2^m3^n)})=k^{(2^m3^n)}g(k)$
This function satisfies the given condition.
This post has been edited 1 time. Last edited by RSM, Mar 7, 2011, 10:21 am

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pco
12955 po...

Mar 5, 2011, 10:10 pm PM #298

> *amparvardi wrote:*
> **Problem 90.**
> **(b)** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
> $$f(a^3)+f(b^3)+f(c^3)=a\cdot f(a^2)+b\cdot f(b^2)+c\cdot f(c^2)\quad \forall a,b,c\in\mathbb{R}.$$

This is equivalent to $f(x^3)=xf(x^2)$ and there are infinitely many solution.

Let $x\sim y$ the relation defined on $(1,+\infty)$ as $\frac{\ln(\ln x)-\ln(\ln y)}{\ln 3-\ln 2}\in\mathbb{Z}$

This is an equivalence relation.
Let $c(x)$ any choice function which associates to any real in $(1,+\infty)$ a representant (unique per class) of its class.
Let $n(x)=\frac{\ln(\ln x)-\ln(\ln c(x))}{\ln 3-\ln 2}\in\mathbb{Z}$
We get $x=c(x)^{\left(\frac{3}{2}\right)^{n(x)}}$ and so $f(x)=\frac{xf(c(x))}{c(x)}$

And so we can define $f(x)$ only over $c((1,+\infty))$
Let $g(x)$ any finction from $\mathbb{R}\to\mathbb{R}$

$$f(x)=\frac{xg(c(x))}{c(x)}$$

We can define in the same way $f(x)$ over $(0,1)$
We can define then $f(1)$ as any value, $f(0)$ as 0 and $f(-x)=-f(x)$

And we have got all suitable $f(x)$

Amir Hos...
4728 pos...

Mar 5, 2011, 10:17 pm PM #299

Problem 91.

Let f be a bijection from $\mathbb{N}$ into itself. Prove that one can always find three natural numbers a,b,c such that $a<b<c$ and $f(a)+f(c)=2f(b)$.

RSM
736 posts

Mar 5, 2011, 11:08 pm PM #300

> *amparvardi wrote:*
> **Problem 91.**
>
> Let f be a bijection from $\mathbb{N}$ into itself. Prove that one can always find three natural numbers a,b,c such that $a<b<c$ and $f(a)+f(c)=2f(b)$.

Suppose, $f(a)=1$
For any integer b>a we have f(b)>1
Now consider a sequence a_n such that $a_1=2$ and $a_{n+1}=2a_n-1$
Consider the sequence $f^{-1}a_n$
There are finite numbers of n for which $f^{-1}a_n<a$
So for $f^{-1}a_n>a$ If the wanted a,b,c do not exist then clearly $f^{-1}a_n>f^{-1}a_{n+1}$
So we get infinitely many $f^{-1}a_n$ less than $f^{-1}a_1$
But this is not possible.
This post has been edited 1 time. Last edited by RSM, Mar 5, 2011, 11:38 pm

pco
12955 po...

Mar 5, 2011, 11:25 pm PM #301

> *RSM wrote:*
> If the wanted a,b,c do not exist then clearly $f^{-1}a_n > f^{-1}a_{n+1}$

Not exactly. We need $f^{-1}a_n > f^{-1}a_{n+1}$ or $f^{-1}a_n < a$
Since there are at most a finite number of such $f^{-1}a_n < a$, your idea is still true with a little modification.
Congrats.

RSM
736 posts

Mar 5, 2011, 11:26 pm PM #302

> *amparvardi wrote:*
> **Problem 90.**
>
> **(a)** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
> $$f(a^3)+f(b^3)+f(c^3)=f(3abc)\quad \forall a,b,c\in\mathbb{R}.$$
>
> **(b)** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
> $$f(a^3)+f(b^3)+f(c^3)=a\cdot f(a^2)+b\cdot f(b^2)+c\cdot f(c^2)\quad \forall a,b,c\in\mathbb{R}.$$

Solution By Abhratanu(email id:- adchessman@gmail.com):-
With the condition that f is continuous at x=1

$f(a)=a^{\sum_{i=0}^{n}\frac{2^i}{3^{i+1}}}f(a^{\frac{2^{i+1}}{3^{i+1}}})$
Since f is continuous at x=0 so when n tends to infinity f(a) tends to af(1).
So we have f(a)=af(1) for all a

---

socrates
1818 posts

Mar 6, 2011, 5:37 am PM #303

Problem 92 (Baltic way)

Suppose two functions $f(x)$ and $g(x)$ are defined for all x such that $2<x<4$ and satisfy
$2<f(x)<4,$
$2<g(x)<4,$
$f(g(x))=g(f(x))=x$ and
$f(x)\cdot g(x)=x^2$, for all such values of x.

Prove that $f(3)=g(3)$.

---

mousavi
222 posts

Mar 6, 2011, 9:18 am PM #304

> *socrates wrote:*
> **Problem 92 (Baltic way)**
>
> Suppose two functions $f(x)$ and $g(x)$ are defined for all x such that $2<x<4$ and satisfy
> $2<f(x)<4,$
> $2<g(x)<4,$
> $f(g(x))=g(f(x))=x$ and
> $f(x)\cdot g(x)=x^2$, for all such values of x.
>
> Prove that $f(3)=g(3)$.

$$f(f(x))=\frac{f(x)^2}{g(f(x))}=\frac{f(x)^2}{x}$$

$$f(f(f(x)))=\frac{f(f(x))^2}{f(x)}=\frac{f(x)^3}{x^2}$$

$$\Longrightarrow f^n(x)=\frac{f(x)^n}{x^{n-1}}$$

$$2<f^n(x)=\frac{f(x)^n}{x^{n-1}}<4$$

$$\Longrightarrow f(x)=x$$

---

socrates
1818 posts

Mar 6, 2011, 11:53 am PM #305

Problem 93

Determine all monotone functions $f:\mathbb{R}\to\mathbb{Z}$ such that
$f(x)=x,\forall x\in\mathbb{Z}$ and
$f(x+y)\geq f(x)+f(y),\forall x,y\in\mathbb{R}.$

RSM
736 posts

Mar 6, 2011, 2:46 pm *PM #306*

> *socrates wrote:*
> **Problem 93**
>
> Determine all monotone functions $f:\mathbb{R}\to\mathbb{Z}$ such that
> $f(x)=x, \forall x\in\mathbb{Z}$ and
> $f(x+y)\geq f(x)+f(y), \forall x,y\in\mathbb{R}.$

I think the definition of a monotonic function is that for x>y f(x)>f(y) or f(x)<f(y)
If it is true then how f can be a monotonic function since all the f(x) for n<x<n+1(n is an integer) will have to take integer value between n and n+1.
But this is not possible.

I think $f:\mathbb{R}\to\mathbb{R}$

---

pco
12955 po...

Mar 6, 2011, 2:50 pm *PM #307*

> *socrates wrote:*
> **Problem 93**
>
> Determine all monotone functions $f:\mathbb{R}\to\mathbb{Z}$ such that
> $f(x)=x, \forall x\in\mathbb{Z}$ and
> $f(x+y)\geq f(x)+f(y), \forall x,y\in\mathbb{R}.$

Induction gives $f(qx)\ge qf(x)$ $\forall q\in\mathbb N$ and so, setting $x=\frac pq$, $f(\frac pq)\le\frac pq$.

Since $f(x)$ is non decreasing and $f(x)\in\mathbb Z$, this implies $f(x)=[x]$ $\forall x\in\mathbb Q$

Since $f(x)$ is non decreasing, this implies $f(x)=[x]$ $\forall x\in\mathbb R$

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socrates
1818 posts

Mar 6, 2011, 8:25 pm *PM #308*

Problem 94

Find all monotone functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(4x)-f(3x)=2x,$ for each $x\in\mathbb{R}.$

RSM
736 posts

Mar 6, 2011, 8:51 pm *PM #309*

> *socrates wrote:*
> **Problem 94**
>
> Find all monotone functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(4x)-f(3x)=2x,$ for each $x\in\mathbb{R}.$

Plugging $f(x)=g(x)+2x$
We get $g(4x)=g(3x)$
We can find a set of real numbers such that all real numbers can be represented in the form $k.4^m.3^n$ with $|m-n|=1$ and we say this set of $k.4^m.3^n$ the class k
m,n being integers and k belonging to that set and no two element of that set can written in this form with same k.
Define g for this set.
Easy to see that $g(k.4^m.3^n)=g(k)$
Suppose, k_1 and $k_2, k_1\geq k_2$ are two distinct elements of the set such that $g(k_1)\neq g(k_2)$
Since f is monotonic function, without loss of generality we may assume that it is non-decreasing.
$2x_{k_2}+g(k_1)\geq 2x_{k_1}+g(k_2)$ where x_{k_1} and x_{k_2} belongs to the class of k_1 and k_2 respectively.
We can infinitely increase the difference $x_{k_1}-x_{k_2}$ and this implies a contradiction since $g(k_1)$ and $g(k_2)$ are fixed.
So $g(k_1)=g(k_2)$ for all k_1, k_2
So only solution is $f(x)=2x+k$ for some constant k
This post has been edited 1 time. Last edited by RSM, Mar 7, 2011, 10:16 am

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pco
12955 po...

Mar 6, 2011, 9:01 pm *PM #310*

> *socrates wrote:*
> **Problem 94**
>
> Find all monotone functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(4x)-f(3x)=2x,$ for each $x\in\mathbb{R}.$

Forget the "monotone" constraint and the general solution of functional equation is :

$\forall x>0: f(x)=2x+h(\frac{\ln x}{\ln 4-\ln 3})$ where $h(x)$ is any function defined over $[0,1)$
$f(0)=a$
$\forall x<0: f(x)=2x+k(\frac{\ln -x}{\ln 4-\ln 3})$ where $k(x)$ is any function defined over $[0,1)$

Adding then monotone constraint and looking at $f(x)$ when $x\to 0$, we see that we must have $\sup h([0,1))=\inf h([0,1)$ and so $h(x)=c$ constant.

And then, continuity at 0 implies that $h(x)=k(x)=a$ and so $\boxed{f(x)=2x+a}$

Amir Hos...
4728 pos...

Mar 6, 2011, 9:24 pm — PM #311

a) Does it exist a function $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(f(x))=x^2-2$$

for all real numbers x ?

b) Do there exist the real coefficients a,b,c such that the following functional equation

$$f(f(x))=ax^2+bx+c$$

has at least one root?

ShahinBJK
113 posts

Mar 6, 2011, 9:58 pm — PM #312

> *amparvardi wrote:*
> **a)** Does it exist a function $f:\mathbb{R}\to\mathbb{R}$ satisfying
>
> $$f(f(x))=x^2-2$$
>
> for all real numbers x ?

see at here problem 7 http://www.imomath.com/tekstkut/funeqn_mr.pdf

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socrates
1818 posts

Mar 6, 2011, 10:39 pm — PM #313

> *amparvardi wrote:*
> **Problem 95**
> **b)** Do there exist the real coefficients a,b,c such that the following functional equation
>
> $$f(f(x))=ax^2+bx+c$$
>
> has at least one root?

Take $a=1,b=c=0$ and $f(x)=|x|^{\sqrt{2}}$.

Problem 96
Let $n\in\mathbb{N}$, such that $\sqrt{n}\notin\mathbb{N}$ and $A=\{a+b\sqrt{n}|a,b\in\mathbb{N},a^2-nb^2=1\}$.
Prove that the function $f:A\to\mathbb{N}$, such that $f(x)=[x]$ is injective but not surjective.

$(\mathbb{N}=\{1,2,...\})$

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pco
12955 po...

Mar 6, 2011, 11:01 pm — PM #314

> *socrates wrote:*
> **Problem 96**
> Let $n\in\mathbb{N}$, such that $\sqrt{n}\notin\mathbb{N}$ and $A=\{a+b\sqrt{n}|a,b\in\mathbb{N},a^2-nb^2=1\}$.
> Prove that the function $f:A\to\mathbb{N}$, such that $f(x)=[x]$ is injective but not surjective.
>
> $(\mathbb{N}=\{1,2,...\})$

If $[a+b\sqrt{n}]=p\ge 1$, then :

$p\le a+b\sqrt{n}<p+1$
$\frac{1}{p+1}<a-b\sqrt{n}<\frac{1}{p}$

Adding, we get $p+\frac{1}{p+1}<2a<p+1+\frac{1}{p}$

And since $(p+1+\frac{1}{p})-(p+\frac{1}{p+1})=1+\frac{1}{p(p+1)}<2$, this interval may contain at most one even integer.

So knowledge of $f(x)$ implies knowledge of a and so (using $a^2-nb^2=1$), knowledge of b

So $f(x)$ is injective.

Consider then $p=2$ and the equation becomes $2+\frac{1}{3}<2a<3+\frac{1}{2}$ and so $1<\frac{7}{6}<a<\frac{7}{4}<2$ and so no such a.

So $f(x)=2$ is impossible and $f(x)$ is not surjective.

ShahinBJK
113 posts

Mar 6, 2011, 11:36 pm PM #315

> *socrates wrote:*
> **Problem 96**
> Let $n\in\mathbb{N}$, such that $\sqrt{n}\notin\mathbb{N}$ and $A=\{a+b\sqrt{n}|a,b\in\mathbb{N},a^2-nb^2=1\}$.
> Prove that the function $f:A\to\mathbb{N}$, such that $f(x)=[x]$ is injective but not surjective.
>
> $(\mathbb{N}=\{1,2,...\})$

Let (a,b) and (x,y) be two solutions of $a^2-nb^2=1$ assume the contrary that let $[a+b\sqrt{n}]=[x+y\sqrt{n}]$ and $(a+b\sqrt{n})(a-b\sqrt{n})=1\Rightarrow a>b\sqrt{n}\Rightarrow 2a>[a+b\sqrt{n}]>2b\sqrt{n}$ and similarly we get $2x>[x+y\sqrt{n}]>2y\sqrt{n}$ $\Rightarrow$
$2a>[x+y\sqrt{n}]>2y\sqrt{n}$ and $2x>[a+b\sqrt{n}]>2b\sqrt{n}\Rightarrow a^2+1\geq y^2n$ and $x^2+1\geq b^2n$
$\Rightarrow 0=a^2+1-y^2n+x^2+1-b^2n\geq 0\Rightarrow a^2+1=y^2n$ which implies that $x=a$ and $y=b\Rightarrow$ contradiction

This post has been edited 1 time. Last edited by ShahinBJK, Mar 7, 2011, 1:09 am

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ShahinBJK
113 posts

Mar 6, 2011, 11:40 pm PM #316

Problem 97
Find all functions $f:N\to N$ such that $f(f(m)+f(n))=m+n$ for every two natural numbers m and n.

---

goodar20...
1346 pos...

Mar 6, 2011, 11:50 pm PM #317

plugging m,m we get $f(2f(m))=2m$, so f is injective.
suppose $f(1)=k$, so we have $f(2k)=2$ and $f(4)=4k$.
$f(f(i-1)+f(i+1))=f(2f(i))=2i$ so we have $f(i+1)-f(i)=f(i)-f(i-1)$ for all $i\in\mathbb{N}$, so for every natural number n we have $f(i+1)-f(i)=t$. so for every natural number n, we have $f(n)=k+(n-1)t$.
$f(4)=4k=k+3t$, so $t=k$ and we have $f(m)=mk$ for all natural numbers m.
plugging this in the original function, we get $k=1$ and hence the unique solution $f(m)=m$ for all natural numbers m.

---

mahanmath
1356 pos...

Mar 6, 2011, 11:54 pm PM #318

Problem 98

Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that :

$f(x^2+y^2)=f(xy)$

---

goodar20...
1346 pos...

Mar 6, 2011, 11:58 pm PM #319

problem 99
find all functions $f:\mathbb{Z}\longrightarrow\mathbb{Z}$ such that:
1) $f(1)=f(-1)$
2) $f(x)+f(y)=f(x+2xy)+f(y-2xy)\,\forall x,y\in\mathbb{Z}$.

This post has been edited 1 time. Last edited by goodar2006, Mar 7, 2011, 12:03 am

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RSM
736 posts

Mar 7, 2011, 12:00 am PM #320

> *mahanmath wrote:*
> **Problem 98**
>
> Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that :
>
> $f(x^2+y^2)=f(xy)$

Fix xy=m
Any number greater than equal to 2m can be written in the form x^2+y^2 for some x,y with xy=m
So f(x)=f(m) for all x>2m
So f(x)=c is the only solution.

---

pco
12955 po...

Mar 7, 2011, 12:05 am • 1 PM #321

> *mahanmath wrote:*
> **Problem 98**
>
> Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that :
>
> $f(x^2+y^2)=f(xy)$

The system $x^2+y^2=u$ and $xy=v$ has solution with $x,y>0$ iff $u>2v>0$

And so $f(u)=f(v)\,\forall u>2v>0$

Let then $x>y>0$:
$x>2\frac{y}{4}$ and so $f(x)=f(\frac{y}{4})$

$y>2\frac{y}{4}$ and so $f(y)=f(\frac{y}{4})$

And so $f(x)=f(y)$ and so $f(x)$ is constant.

mahanmath
1356 pos...

Mar 7, 2011, 12:08 am PM #322

> goodar2006 wrote:
> **problem 99**
> find all functions $f:\mathbb{Z}\longrightarrow\mathbb{Z}$ such that:
> **1)** $f(1)=f(-1)$
> **2)** $f(x)+f(y)=f(x+2xy)+f(y-2xy)\,\forall x,y\in\mathbb{Z}$.

> lajanugen wrote:
> Problem 41:
> We claim that all functions that satisfy $exp_2(x)=exp_2(y)\to f(x)=f(y)$ are the solutions ($f(0)$ can be arbitrary)- this is easily seen to satisfy the equation since $exp_2(x)=exp_2(x+2xy)$ for any non-zero integer x
> Plugging in $(m,n)=(n,1),(-1,n)$ and equating the expressions, we obtain that the f values of all odd numbers are equal.
> Hence, for all odd n, $f(m)=f((2n+1)m)$ (Since $n-2mn$ would also be odd):
> $n=-1$ gives $f(m)=f(-m)$ for all m
> As n ranges through all odd values, $2n+1,-(2n+1)$ range through all odd values

pco
12955 po...

Mar 7, 2011, 12:26 am PM #323

> goodar2006 wrote:
> **problem 99**
> find all functions $f:\mathbb{Z}\longrightarrow\mathbb{Z}$ such that:
> **1)** $f(1)=f(-1)$
> **2)** $f(x)+f(y)=f(x+2xy)+f(y-2xy)\,\forall x,y\in\mathbb{Z}$.

See also http://www.artofproblemsolving.com/Forum/viewtopic.php?f=56&t=386360

socrates
1818 posts

Mar 7, 2011, 1:17 am PM #324

Problem 100

Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+y)\le f(x)+f(y)$ for all $x,y\in\mathbb{R}$ and $f(x)\le e^x-1$ for each $x\in\mathbb{R}$.

pco
12955 po...

Mar 7, 2011, 2:15 am PM #325

> socrates wrote:
> **Problem 100**
>
> Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+y)\le f(x)+f(y)$ for all $x,y\in\mathbb{R}$ and $f(x)\le e^x-1$ for each $x\in\mathbb{R}$.

$f(x+0)\le f(x)+f(0)$ and so $f(0)\ge 0$ and since $f(0)\le e^0-1=0$, we get $f(0)=0$
$f(x+(-x))\le f(x)+f(-x)$ and so $f(x)+f(-x)\ge 0$

$$f(x)\le e^x-1\implies f(x)\le f(\frac{x}{2})+f(\frac{x}{2})\le 2(e^{\frac{x}{2}}-1)$$

$$f(x)\le 2(e^{\frac{x}{2}}-1)\implies f(x)\le f(\frac{x}{2})+f(\frac{x}{2})\le 4(e^{\frac{x}{4}}-1)$$

And immediate induction gives $f(x)\le 2^n(e^{\frac{x}{2^n}}-1)$

Setting $n\to+\infty$, we get $f(x)\le x$

So $f(x)+f(-x)\le x+(-x)=0$ and so, since we already got $f(x)+f(-x)\ge 0$, we get $f(x)+f(-x)=0$

Then $f(-x)\le -x\implies -f(x)\le -x\implies f(x)\ge x$

And so $\boxed{f(x)=x}$ which indeed is a solution.

socrates
1818 posts

Mar 7, 2011, 5:24 am PM #326

Problem 101

A variation on the unsolved **Problem 40**:

Let $f:R\to R$ be a function such that $f(xy)+f(x-y)\ge f(x+y)$ for all real numbers x,y.
Prove that $f(x)\ge 0$, for each $x\in\mathbb{R}$.

socrates
1818 posts

Mar 7, 2011, 5:25 am PM #327

> amparvardi wrote:

Problem 43 :
Let f be a real function defined on the positive half-axis for which $f(xy) = xf(y) + yf(x)$ and $f(x+1) \leq f(x)$ hold for every positive x and y. Show that if $f(1/2) = 1/2$, then

$$f(x) + f(1-x) \geq -x\log_2 x - (1-x)\log_2(1-x)$$

for every $x \in (0, 1)$.

Any solution?

RSM
736 posts

Mar 7, 2011, 9:09 am PM #328

> *socrates wrote:*
> **Problem 101**
>
> A variation on the unsolved **Problem 40**:
>
> Let $f : R \to R$ be a function such that $f(xy) + f(x-y) \geq f(x+y)$ for all real numbers x, y.
> Prove that $f(x) \geq 0$, for each $x \in \mathbb{R}$.

Putting $x = a+b, y = a-b$ the equation turns to
$f(a^2-b^2) \geq f(2a) - f(2b)$
Either $f(2a) \geq f(2b)$ or $f(2a) \leq f(2b)$
In the first case $f(a^2-b^2) \geq 0$ in the second case $f(b^2-a^2) \geq 0$
So we have at least one of f(x) and f(-x) is greater than 0
Putting a=b we have $f(0) \geq 0$
If $f(x) \leq 0$ for some x=2c then put b=c
and $a = \sqrt{c^2+c}$ or $a = -\sqrt{c^2+c}$ for which $f(2a) \geq 0$
The equation turns to $f(c) \geq f(2a) - f(2c) \geq 0$
So $f(x) \geq 0$ for all x

RSM
736 posts

Mar 7, 2011, 11:20 am PM #329

> *RSM wrote:*
>
> > *abhinavzandubalm wrote:*
> > Even A Lifeguard Must not be bringing the dead back to life by C.P.R. like we have to do for this marathon.
> >
> > Problem 71
>
> This is an INMO problem
> A solution to it can be found in INMO official solution paper.
>
> Next problem:-
> Find all function f defined on real variables such that
> $f(x+y) + f(x-y) = 2f(x)f(y)$
> for all $x, y \in \mathbb{R}$
>
> I could not solve this problem myself.
> So I want to give one more condition that $|f(x)| \geq 1 \forall x \in \mathbb{R}$
> I think it is easy now.

Is it so hard that no-one has posted any solution!!!!!

ShahinBJK
113 posts

Mar 7, 2011, 3:09 pm PM #330

> *abhinavzandubalm wrote:*
> **Problem 71**
> Determine All Functions
> $f : \mathbb{R} - 0, 1 \to \mathbb{R}$
> $f(x) + f(\frac{1}{1-x}) = \frac{2(1-2x)}{x(1-x)}$

Solution:
$x \to \frac{1}{1-x} \implies f(\frac{1}{1-x}) + f(\frac{x-1}{x}) = \frac{2(x+1)(1-x)}{x}$
$x \to \frac{x-1}{x} \implies f(\frac{x-1}{x}) + f(x) = \frac{2x(2-x)}{x-1}$
$\Rightarrow f(\frac{1}{1-x}) + f(\frac{x-1}{x}) = \frac{2(1-2x)}{x(1-x)} - f(x) + \frac{2x(2-x)}{x-1} - f(x) = \frac{2(x+1)(1-x)}{x}$
$\implies f(x) = \frac{x+1}{\ldots} \forall x \neq 0; 1$

$x-1$

socrates
1818 posts

Mar 7, 2011, 11:09 pm PM #331

Problem 102

Find all continuous functions $f:(0,+\infty)\to(0,+\infty)$, such that $f\left(x\right)=f\left(\sqrt{2x^2-2x+1}\right)$, for each $x>0$.

RSM
736 posts

Mar 7, 2011, 11:42 pm PM #332

> *socrates wrote:*
> **Problem 102**
>
> Find all continuous functions $f:(0,+\infty)\to(0,+\infty)$, such that $f\left(x\right)=f\left(\sqrt{2x^2-2x+1}\right)$, for each $x>0$.

The equation is equivalent to:-
$f(x)=f(\dfrac{\sqrt{2x^2-1}+1}{2})$
Consider a sequence $a_1=a>1$ and $a_{n+1}=\dfrac{\sqrt{2a_n^2-1}+1}{2}$
$f(a_n)=a$ for all a
Limit of this sequence is 1
Since f is a continuous function we have $\lim_{n\to+\infty}f(a_n)=f(\lim_{n\to+\infty}a_n)=f(1)$
So f(x)=f(1) for all x>1
Now for x<1
choose $a_1=a<1$ and $a_{n+1}=\sqrt{2a_n^2-2a_n+1}$
The limit of this sequence is also 1
So similarly we can conclude
$f(x)=c$ for all x

socrates
1818 posts

Mar 8, 2011, 6:53 am PM #333

Problem 103 (Romania 2010)

Determine all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ such that $f(a^2-b^2)=f^2(a)-f^2(b)$, for all $a,b\in\mathbb{N}_0, a\geq b$.

RSM
736 posts

Mar 8, 2011, 10:29 am PM #334

Sorry, I posted it by mistake
[Content Doleted}

See the next post.
This post has been edited 1 time. Last edited by RSM, Mar 8, 2011, 10:33 am

RSM
736 posts

Mar 8, 2011, 10:32 am PM #335

> *socrates wrote:*
> **Problem 103** (Romania 2010)
>
> Determine all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ such that $f(a^2-b^2)=f^2(a)-f^2(b)$, for all $a,b\in\mathbb{N}_0, a\geq b$.

Putting a=b we have $f(0)=0$
Putting a=1, b=0 we have $f(1)=1$ or $f(1)=0$
Putting b=0 we have $f(a^2)=f^2(a)$
$f(a^2-b^2)=(f(a)+f(b))(f(a)-f(b))$.............(1)
Suppose, S={x : f(x)=0}
T={x : $f(x)\neq 0$}
Suppose, T contains at least one element
Minimum element of T is n and maximum of S is m($\neq 0$)
My claim is m<n
Otherwise, $f(m^2-n^2)=-f^2(n)\leq 0$
But this is not possible if
So m<n
But S cannot have an upper-bound if it has any element except 1
And in this case f is a strictly increasing function(directly follows from (1))
$f(2)=c$
$f(3)=f(4)-f(1)$
$f(9)=f^2(3)$
$f(5)=f(9)-f(4)$
$f(16)=f^2(4)=f(25)-f(9)$
Solving these equations we get positive integer solutions for c are c=2 in case f(1)=1 and no solution if f(1)=0
So $f(2^{2k})=2^{2k}$
Since f is a strictly increasing function so $f(x)=x$ for all x
If T does not contain any element then $f(x)=0$ for all x
Answer:-
$f(x)=x$ for all x
$f(x)=0$ for all x

Mar 9, 2011, 11:48 am PM #336

mousavi
222 posts

problem 104

find all continues functions $f:R\longrightarrow R$ for each two real numbers x,y:

$f(x+y)=f(x+f(y))$

---

pco
12955 po...

Mar 9, 2011, 12:33 pm PM #337

> *mousavi wrote:*
> **problem 104**
>
> find all continues functions $f:R\longrightarrow R$ for each two real numbers x,y:
>
> $f(x+y)=f(x+f(y))$

If $f(x)=x\,\forall x$, we got a solution.
If $\exists a$ such that $f(a)\neq a$, then $f(x+a)=f(x+f(a))$ implies that $f(x)$ is periodic and one of its periods is $|f(a)-a|$.

Let $T=\inf\{$positive periods$\}$
If $T=0$, then $f(x)=c$ is constant and we got another solution.
if $T\neq 0$, then T is a period of $f(x)$ (since continuous) and, since any $f(y)-y$ is also a period, we get that $f(y)-y=n(y)T$ where $n(y)\in\mathbb{Z}$ but then $n(y)$ is a continuous function from $\mathbb{R}\to\mathbb{Z}$ and so is constant and $f(y)=y+kT$ which is not a periodic function.

Hence the two solutions :
$f(x)=x\,\forall x$
$f(x)=c\,\forall x$ for any $c\in\mathbb{R}$

---

socrates
1818 posts

Mar 10, 2011, 5:21 am PM #338

Problem 105

Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
• $f\left(f(x)y+x\right)=xf(y)+f(x)$, for all real numbers x,y and
• the equation $f(t)=-t$ has exactly one root.

---

pco
12955 po...

Mar 10, 2011, 1:14 pm PM #339

> *socrates wrote:*
> **Problem 105**
>
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
> • $f\left(f(x)y+x\right)=xf(y)+f(x)$, for all real numbers x,y and
> • the equation $f(t)=-t$ has exactly one root.

Let $P(x,y)$ be the assertion $f(f(x)y+x)=xf(y)+f(x)$
Let t be the unique real such that $f(t)=-t$

$f(x)=0\,\forall x$ is a solution. Let us from now look for non all-zero solutions.
Let u such that $f(u)\neq 0$

$P(1,0)\implies f(0)=0$ and so $t=0$
If $f(a)=0$, then $P(a,u)\implies af(u)=0$ and so $a=0$
So $f(x)=0\iff x=0$

If $f(1)\neq 1$, then $P(1,\frac{1}{1-f(1)})\implies f(\frac{f(1)}{1-f(1)}+1)=f(\frac{1}{1-f(1)})+f(1)$ and so $f(1)=0$, which is impossible.
So $f(1)=1$

$P(1,-1)\implies f(-1)=-1$
$P(x,-1)\implies f(x-f(x))=f(x)-x$ and so, since the only solution of $f(t)=-t$ is $t=0:f(x)=x$ which indeed is a solution.

__**Hence the two solutions**__ :
$f(x)=0\,\forall x$
$f(x)=x\,\forall x$

---

mousavi
222 posts

Mar 10, 2011, 1:47 pm PM #340

> *socrates wrote:*
> **Problem 105**
>
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
> • $f\left(f(x)y+x\right)=xf(y)+f(x)$, for all real numbers x,y and
> • the equation $f(t)=-t$ has exactly one root.

the second condition is not need.
by **pco's** way $f(1) = 0, 1$ and if $f(1) = 0$ then $f(x) = 0$(and even if $f(t) = -t$ has exactly one solution $f(x) = 0$ is possible)

let $f(1) = 1$ then:

$f(x+1) = f(x) + 1$

$p(x, y+1) : f(x + yf(x) + f(x)) = xf(y) + x + f(x) = f(x + yf(x)) + x$ let $z = x + yf(x)$ and z is surjective.

$\Longrightarrow f(z + f(x)) = f(z) + x \Longrightarrow f(f(x)) = x \Longrightarrow f(z + x) = f(z) + f(x)$

$p(f(x), y) : f(f(x) + xy) = f(x)f(y) + x \Longrightarrow f(xy) = f(x)f(y)$

$f(xy) = f(x)f(y)$ and $f(x+y) = f(x) + f(y) \Longrightarrow f(x) = x$

This post has been edited 2 times. Last edited by mousavi, Mar 10, 2011, 2:18 pm

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pco
12955 po...

Mar 10, 2011, 1:57 pm — PM #341

> *mousavi wrote:*
> $p(x, y+1) : f(x + yf(x) + f(x)) = xf(y) + x + f(x)$ let $z = x + yf(x)$ and z is surjective.
>
> $\Longrightarrow f(z + f(x)) = z + f(x) \Longrightarrow f(f(x)) = x \Longrightarrow f(z + x) = f(z) + f(x)$

I'm affraid there is an error here : in "$f(x + yf(x) + f(x)) = xf(y) + x + f(x)$", RHS contains $xf(y) + x$ and not $yf(x) + x$, so you cant substitue z

---

mousavi
222 posts

Mar 10, 2011, 2:18 pm — PM #342

> *pco wrote:*
>> *mousavi wrote:*
>> $p(x, y+1) : f(x + yf(x) + f(x)) = xf(y) + x + f(x)$ let $z = x + yf(x)$ and z is surjective.
>>
>> $\Longrightarrow f(z + f(x)) = z + f(x) \Longrightarrow f(f(x)) = x \Longrightarrow f(z + x) = f(z) + f(x)$
>
> I'm affraid there is an error here : in "$f(x + yf(x) + f(x)) = xf(y) + x + f(x)$", RHS contains $xf(y) + x$ and not $yf(x) + x$, so you cant substitue z

i edited

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socrates
1818 posts

Mar 10, 2011, 9:20 pm — PM #343

Problem 106

Find all functions $f : \mathbb{X} \to \mathbb{R}$ such that $f(x+y) + f(xy-1) = (f(x)+1)(f(y)+1)$, for all $x, y \in \mathbb{X}$, if
a) $\mathbb{X} = \mathbb{Z}$.
b) $\mathbb{X} = \mathbb{Q}$.

RSM
736 posts

Mar 11, 2011, 9:37 am • 1 — PM #344

> *socrates wrote:*
> **Problem 106**
>
> Find all functions $f : \mathbb{X} \to \mathbb{R}$ such that $f(x+y) + f(xy-1) = (f(x)+1)(f(y)+1)$, for all $x, y \in \mathbb{X}$, if
> a) $\mathbb{X} = \mathbb{Z}$.
> b) $\mathbb{X} = \mathbb{Q}$.

Lemma 1:- f(0)=0,f(-1)=1
Proof:-
Putting y=0 we have $f(x) + f(-1) = (f(x)+1)(f(0)+1)$
$f(x).f(0) = f(-1) - f(0) - 1$
If $f(0) \neq 0$ then $f(x) = constant = c$ for all x.
But putting $f(x) = c$ in the functional equation we get no real solution of c.
So $f(0) = 0$ and this implies $f(-1) = 1$

Lemma 2:- f(1)=1
Proof:-
Putting $x = y = -1$ we have $f(-2) = 4$
Putting $x = 1, y = -1$ we have $f(1) = 1$

Lemma 3:- $f(x) = x^2 \forall x \in \mathbb{Z}$
Proof:-
Putting y=1 in the functional equation we get
$f(x+1) + f(x-1) = 2f(x) + 2$
Hence by induction $f(x) = x^2 \forall x \in \mathbb{Z}$

Lemma 4:- $f(x)=x^2\forall x\in\mathbb{Q}$
Proof:-
Suppose,m be any rational number.
n is an integer such that mn is an integer.
Putting $x=m,y=n$ in the functional equation we get
$f(m+n)+(mn-1)^2=(n^2+1)(f(m)+1)$.................(1)
Putting $x=m+n,y=-n$ we get
$f(m)+(mn+n^2+1)^2=(n^2+1)(f(m+n)+1)$.......................(2)
Eliminating f(m+n) from equation (1) and (2) we get $f(m)=m^2$

Answer:-
$f(x)=x^2\forall x\in\mathbb{Q}$
This post has been edited 1 time. Last edited by RSM, Mar 11, 2011, 2:09 pm

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abhinavza...
418 posts
Mar 11, 2011, 1:56 pm *PM #345*
The final answer should be $f(x)=x^2$

RSM
736 posts
Mar 11, 2011, 2:10 pm *PM #346*

> *abhinavzandubalm wrote:*
> The final answer should be $f(x)=x^2$

Thanks, @avinab, I made a typo mistake again.
I have edited the answer.

Babai
486 posts
Mar 12, 2011, 12:32 pm *PM #347*
Find all function $f:R\to R$ such that $f(x+f(x))=2f(x)$ and $f(f(x))=f(x)$ and $f(0)=0$

---

pco
12955 po...
Mar 12, 2011, 1:44 pm • 1 *PM #348*

> *Babai wrote:*
> Find all function $f:R\to R$ such that $f(x+f(x))=2f(x)$ and $f(f(x))=f(x)$ and $f(0)=0$

I dont think this is a real olympiad exercise.
There are infinitely many very different solutions and I dont think there is a general solution.
Please, Babai, tell us from where is this problem and how are you sure there is a solution ?

Some examples of solutions :
1) $f(x)=0\ \forall x$

2) $f(x)=x\ \forall x$

3) Let $A\subseteq\mathbb{R}$ stable thru $x\to 2x$:
$f(x)=x\ \forall x\in A$
$f(x)=0\ \forall x\notin A$

4)
$f(2^n)=2^n\ \forall n\in\mathbb{N}$
$f(2^{n+1}-1)=2^{n+1}\ \forall n\in\mathbb{N}$
$f(x)=0\ \forall$ other x

and infinitely many other

Babai
486 posts
Mar 12, 2011, 1:52 pm *PM #349*
actually,It was another problem.The main problem was:$f(x+f(y))+f(f(y))=f(f(x))+2f(y)$
I solved it partially and got that $f(x+f(x))=2f(x)$ and $f(f(x))=f(x)$ while $f(0)=0$but could not solve further.So,please help.

---

pco
12955 po...
Mar 12, 2011, 2:55 pm • 2 *PM #350*

> *Babai wrote:*
> actually,It was another problem.The main problem was:$f(x+f(y))+f(f(y))=f(f(x))+2f(y)$
> I solved it partially and got that $f(x+f(x))=2f(x)$ and $f(f(x))=f(x)$ while $f(0)=0$but could not solve further.So,please help.

1) It's not very fair to transform a problem and claim that there exists a solution when your transformation is not an equivalence and so you dont know if there is such an olympiad level solution.

2) Solution of the original problem :
Let $P(x,y)$ be the assertion $f(x+f(y))+f(f(y))=f(f(x))+2f(y)$

$P(0,y)\implies f(f(y))=\frac{f(f(0))}{2}+f(y)$
$P(0,x)\implies f(f(x))=\frac{f(f(0))}{2}+f(x)$
Plugging this in $P(x,y)$, we get new assertion $Q(x,y)$: $f(x+f(y))=f(x)+f(y)$
It's immediate to see that the two assertions are equivalent.

The new assertion has been solved many times in mathlinks :

Let $A=f(\mathbb{R})$.
Using $f(x)+f(y)=f(x+f(y))$ and $f(x)-f(y)=f(x-f(y))$ (look at $Q(x-f(y),y)$), we see that A is an additive subgroup of $\mathbb{R}$

Then the relation $x\sim y\iff x-y\in A$ is an equivalence relation and let $c(x)$ any choice function which assocoates to a real x a representant (unique per class) of it's equivalence class.

Setting $g(x)=f(x)-x$, $Q(x,y)$ may be written $g(x+f(y))=g(x)$ and so $g(x)$ is constant in any equivalence class and so $f(x)-x=f(c(x))-c(x)$ and so $f(x)=h(c(x))+x-c(x)$ where $h(x)$ is a function from $\mathbb{R}\to A$

So, any solution may be written as $f(x)=x-c(x)+h(c(x))$ where :
$A\subseteq\mathbb{R}$ is an additive subgroup of $\mathbb{R}$
$c(x)$ is any choice function associating to a real x a representant (unique per class) of it's equivalence class for the equivalence relation $x-y\in A$
$h(x)$ is any function from $\mathbb{R}\to A$

Let us show now that this mandatory form is sufficient and so that we got a general solution :
Let $A\subseteq\mathbb{R}$ any additive subgroup of $\mathbb{R}$
Let $c(x)$ any choice function associating to a real x a representant (unique per class) of it's equivalence class for the equivalence relation $x-y\in A$
Let $h(x)$ any function from $\mathbb{R}\to A$
Let $f(x)=x-c(x)+h(c(x))$

$x-c(x)\in A$ and $h(c(x))\in A$ and A subgroup imply that $f(x)\in A$
So $x+f(y)\sim x$ and $c(x+f(y))=c(x)$
So $f(x+f(y))=x+f(y)-c(x+f(y))+h(c(x+f(y)))=x+f(y)-c(x)+h(c(x))=f(x)+f(y)$
Q.E.D.

And so we got a general solution.

Some examples :
1) Let $A=\mathbb{R}$ and so a unique class and $c(x)=a$ and $f(x)=x-a+h(a)$ and so the solution $\boxed{f(x)=x+b}$
(notice that $f(0)=0$ is not mandatory.

2) Let $A=\{0\}$ and so equivalence classes are $\{x\}$ and so $c(x)=x$ and $h(x)=0$ and $f(x)=x-x+0$ and so the solution $\boxed{f(x)=0}$

3) Let $A=\mathbb{Z}$ and $c(x)=x-\lfloor x\rfloor$ and $h(x)=\lfloor 2x\rfloor$
$f(x)=x-x+\lfloor x\rfloor+\lfloor 2x-2\lfloor x\rfloor\rfloor$ and so the solution $\boxed{f(x)=\lfloor 2x\rfloor-\lfloor x\rfloor}$

and infinitely many other.

---

abhinavza...
418 posts

Mar 12, 2011, 6:28 pm PM #351

Please try to number your problems
Now The Number has gone to Problem 108
This post has been edited 1 time. Last edited by abhinavzandubalm, Mar 12, 2011, 7:20 pm

---

pco
12955 po...

Mar 12, 2011, 6:55 pm • 1 PM #352

> *abhinavzandubalm wrote:*
> "Problem 107"
> Does There Exist A Function $f:\mathbb{R}\to\mathbb{R}$ Such That
> $f(1+f(x))=1-x\quad and\quad f(f(x))=x?????$

Using $x=\frac{1}{2}$, we get $f(1+f(\frac{1}{2}))=\frac{1}{2}$
Taking f or both sides, we get $f(f(1+f(\frac{1}{2})))=f(\frac{1}{2})$ and so, using $f(f(x))=x:1+f(\frac{1}{2}))=f(\frac{1}{2})$, impossible.
So no such function.

Nota : I could understand the interest of hiding solutions, but what could be the interest of hiding problems :

---

abhinavza...
418 posts

Mar 12, 2011, 7:27 pm PM #353

Just a habit formed when i started giving problems
btw it was **Problem 108**
Now to give **Problem 109**
Find all functions $f:\mathbb{R}_0\to\mathbb{R}_0$ satisfying the functional relation
$f(f(x)-x)=2x\ \forall x\in\mathbb{R}_0$

---

pco
12955 po...

Mar 12, 2011, 7:46 pm PM #354

> *abhinavzandubalm wrote:*
> Just a habit formed when i started giving problems

> btw it was **Problem 108**
> Now to give **Problem 109**
> Find all functions $f:\mathbb{R}_0\to\mathbb{R}_0$ satisfying the functional relation
> $f(f(x)-x)=2x\ \forall x\in\mathbb{R}_0$

is $\mathbb{R}_0=\mathbb{R}\setminus\{0\}$? If so, I think that, without continuity, we get infinitely many solutions.

Maybe he problem is with $\mathbb{R}^+$ (very very classical problem) ?

---

abhinavza...
418 posts

Mar 12, 2011, 8:10 pm PM #355

Its $\mathbb{R}^++0$

---

pco
12955 po...

Mar 12, 2011, 10:32 pm • 2 PM #356

> *abhinavzandubalm wrote:*
> Just a habit formed when i started giving problems
> btw it was **Problem 108**
> Now to give **Problem 109**
> Find all functions $f:\mathbb{R}_0\to\mathbb{R}_0$ satisfying the functional relation
> $f(f(x)-x)=2x\ \forall x\in\mathbb{R}_0$

Ok, so $\mathbb{R}_0$ here is the set of non negative real numbers. Then :
In order to LHS be defined, we get $f(x)\ge x\ \forall x$
So $f(f(x)-x)\ge f(x)-x\ \forall x\iff f(x)\le 3x$

So we got $x\le f(x)\le 3x$

If we consider $a_nx\le f(x)\le b_nx$, we get $a_n(f(x)-x)\le 2x\le b_n(f(x)-x)$ and so $\frac{b_n+2}{b_n}x\le f(x)\le\frac{a_n+2}{a_n}x$

And so the sequences :
$a_1=1$
$b_1=3$
$a_{n+1}=\frac{b_n+2}{b_n}$
$b_{n+1}=\frac{a_n+2}{a_n}$

And it's easy to show that :
a_n is a non decreasing sequence whose limit is 2
b_n is a non increasing sequence whose limit is 2

And so $\boxed{f(x)=2x}$ which indeed is a solution.

Babai
486 posts

Mar 13, 2011, 10:08 am PM #357

Actually in that equation I put $y=x$ and assumed that $f(0)=0$ to get a solution.

---

socrates
1818 posts

Mar 13, 2011, 7:28 pm PM #358

Problem 110 (Romania District Olympiad 2011 - Grade XI)

Find all functions $f:[0,1]\to\mathbb{R}$ for which we have:

$$|x-y|^2\le|f(x)-f(y)|\le|x-y|,$$

for all $x,y\in[0,1]$.

---

pco
12955 po...

Mar 13, 2011, 7:54 pm • 2 PM #359

> *socrates wrote:*
> **Problem 110** (Romania District Olympiad 2011 - Grade XI)
>
> Find all functions $f:[0,1]\to\mathbb{R}$ for which we have:
>
> $$|x-y|^2\le|f(x)-f(y)|\le|x-y|,$$
>
> for all $x,y\in[0,1]$.

Let $P(x,y)$ be the assertion $|x-y|^2\le|f(x)-f(y)|\le|x-y|$

Setting $y\to x$ in $P(x,y)$, we conclude that $f(x)$ is continuous.
If $f(a)=f(b)$, then $P(a,b)\implies(a-b)^2\le 0$ and so $a=b$ and $f(x)$ is injective

$f(x)$ continuous and injective implies monotonous.

$f(x)$ solution implies $f(x)+c$ and $c-f(x)$ solutions too. So Wlog say $f(0)=0$ and $f(x)$ increasing.

Then :
$P(1,0) \implies f(1)=1$ and so $f(x)\in[0,1]$
$P(x,0) \implies f(x)\le x$
$P(x,1) \implies 1-f(x)\le 1-x$

And so $f(x)=x$ which indeed is a solution.

Hence the solutions :
$f(x)=x+a$ for any real a
$f(x)=a-x$ for any real a

---

socrates
1818 posts

Mar 15, 2011, 6:39 am PM #360

Problem 111

Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x^2-f^2(y))=xf(x)-y^2$, for all real numbers x,y.

---

pco
12955 po...

Mar 15, 2011, 2:13 pm PM #361

> *socrates wrote:*
> **Problem 111**
>
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x^2-f^2(y))=xf(x)-y^2$, for all real numbers x,y.

Let $P(x,y)$ be the assertion $f(x^2-f^2(y))=xf(x)-y^2$

1) $f(x)=0 \iff x=0$
=================
Let $u=-f^2(0)$: $P(0,0) \implies f(u)=0$

$P(0,u) \implies f(0)=-u^2$ and so $u=-f^2(0)=-u^4$ and so $u\in\{-1,0\}$

If $u=-1$: $f(0)=-1$ and $P(-1,0) \implies f(0)=-f(-1)$ and so contradiction since $f(0)=-1$ while $f(-1)=f(u)=0$.
So $u=0$ and $f(0)=0$
Then $P(x,0) \implies f(x^2)=xf(x)$ and if $f(y)=0$, then $P(x,y) \implies y=0$
Q.E.D.

2) $f(x)$ is odd and surjective
==================
$P(0,x) \implies f(-f^2(x))=-x^2$ and so any non positive real may be reached
Comparing $P(x,0)$ and $P(-x,0)$, we get $xf(x)-xf(-x)$ and si $f(-x)=-f(x)\,\forall x\neq 0$, still true if $x=0$ and so $f(x)$ is odd.
So any non negative real may be reached too.
And since $f(0)=0$, $f(x)$ is surjective.
Q.E.D.

3) $f(x)=x\ \forall x$
============
$P(x,0) \implies f(x^2)=xf(x)$
$P(0,y) \implies f(-f^2(y))=-y^2$
And so $f(x^2-f^2(y))=f(x^2)+f(-f^2(y))$

And so, since surjective : $f(x+y)=f(x)+f(y)\,\forall x\ge 0, y\le 0$
And so, since odd, $f(x+y)=f(x)+f(y)\,\forall x,y$

Then from $f(x^2)=xf(x)$, we get $f((x+1)^2)=(x+1)f(x+1)$ and so
$f(x^2)+2f(x)+f(1)=xf(x)+xf(1)+f(x)+f(1)$

And so $2f(x)=xf(1)+f(x)$ and $f(x)=ax$
Plugging this back in original equation, we get $a=1$

And so the unique solution $\boxed{f(x)=x}\ \forall x$

---

socrates
1818 posts

Mar 15, 2011, 4:51 pm PM #362

Problem 112

Let $f:\mathbb{N}^*\to\mathbb{N}^*$ be a function such that
- $f(1)=1$
- $3f(n)f(2n+1)=f(2n)(1+3f(n)),$
- $f(2n)<6f(n),\ \forall n\in\mathbb{N}^*.$

Find all pairs (k,l) such that $f(k)+f(l)=293.$

This post has been edited 1 time. Last edited by socrates, Mar 16, 2011, 1:21 am

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mousavi
222 posts

Mar 15, 2011, 6:55 pm PM #363

$gcm(3f(n),1+3f(n))=1 \implies 3f(n)\mid f(2n)<6f(n) \implies 3f(n)=f(2n)$

$\Longrightarrow f(2n+1)=1+f(2n), 3f(n)=f(2n)$

$\Longrightarrow f(1)=1, f(2)=3, f(3)=4, f(4)=6, f(5)=7, f(6)=12$

for $f(n)$ we write n in base 2 and read in base 3, for example for obtaining $f(6)$: $6=(110)_2 \Longrightarrow (110)_3=12$

and easily it can be proved by induction.and $f(48)=324>293$ so some cases remain and it is easy to check.

---

socrates
1818 posts

Mar 16, 2011, 1:24 am · PM #364

Problem 113

Find all functions $f:\mathbb{N}^*\to\mathbb{N}^*$ such that $f(2x+3y)=2f(x)+3f(y)+4$, for all integers $x,y\geq 1$.

This post has been edited 1 time. Last edited by socrates, Mar 20, 2011, 9:15 pm

---

pco
12955 po...

Mar 16, 2011, 1:41 am • 1 · PM #365

> *socrates wrote:*
> **Problem 113**
>
> Find all functions $f:\mathbb{N}^*\to\mathbb{N}^*$ such that $f(2x+3y)=2f(x)+3f(y)+4$, for all integers $x,y\geq 1$.

I suppose that $N^*=\mathbb{N}$ is the set of natural numbers (positive integers)
Let $P(x,y)$ be the assertion $f(2x+3y)=2f(x)+3f(y)+4$

Subtracting $P(x+3,y)$ from $P(x,y+2)$, we get $2(f(x+3)-f(x))=3(f(y+2)-f(y))$

And so these two quantities are constant and multiple of 6 and so :
$f(x+3)=f(x)+3c$
$f(y+2)=f(y)+2c$
and (using $y=x+1$ in this last equation) : $f(x+3)=f(x+1)+2c$

and so $f(x+1)=f(x)+c$ and $f(x)=cx+d$

Plugging this in $P(x,y)$, we get $\boxed{f(x)=ax-1}$ for any real $a>1$ (the case $a=1$ must be excluded in order to have $f(1)\in\mathbb{N}$)

---

socrates
1818 posts

Mar 18, 2011, 11:18 pm · PM #366

Problem 114

Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(m+f(n))=f(m+n)+2n+1$, for all integers m,n.

This post has been edited 1 time. Last edited by socrates, Mar 20, 2011, 9:14 pm

---

pco
12955 po...

Mar 18, 2011, 11:40 pm · PM #367

> *socrates wrote:*
> **Problem 114**
>
> Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(m+f(n))=f(m+n)+2n+1$, for all integers m,n.

The equation may be written $f(m+(f(n)-n))=f(m)+2n+1$

And so $f(m+k(f(n)-n))=f(m)+k(2n+1)$
Setting $k=f(p)-p$, this becomes $f(m+(f(p)-p)(f(n)-n))=f(m)+(f(p)-p)(2n+1)$

And using symetry between n and p, we get $(f(p)-p)(2n+1)=(f(n)-n)(2p+1)$

And so $\frac{f(n)-n}{2n+1}=c$ and so $f(n)=n(2c+1)+c$ with $c=f(0)\in\mathbb{Z}$

Plugging this in original equation, we get $c=-1$ and so the solution $\boxed{f(x)=-x-1}$

---

myth_kill
5 posts

Mar 18, 2011, 11:45 pm · PM #368

> *socrates wrote:*
> **Problem 114**
>
> Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(m+f(n))=f(m+n)+2n+1$, for all integers m,n.

Click to reveal hidden text

edit: it seems i was wrong ,and pco posted the solution at same time so i did not see it.

---

socrates
1818 posts

Mar 20, 2011, 9:11 pm · PM #369

Problem 115

Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(0)=2$ and $f(x+f(x+2y))=f(2x)+f(2y)$, for all integers x,y.

pco
12955 po…

Mar 20, 2011, 10:03 pm PM #370

> *socrates wrote:*
> **Problem 115**
>
> Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(0)=2$ and $f(x+f(x+2y))=f(2x)+f(2y)$, for all integers x,y.

Let $P(x,y)$ be the assertion $f(x+f(x+2y))=f(2x)+f(2y)$

$P(0,2)\implies f(2)=4$
$P(0,1)\implies f(4)=6$
And so, using induction with $P(0,n)$, we get $f(2n)=2n+2\,\forall n\ge 0$

Let $x\ge 0: P(2x,-x)\implies f(-2x)=f(2x+2)-f(4x)=(2x+4)-(4x+2)=-2x+2$

So $f(2x)=2x+2\,\forall x\in\mathbb{Z}$ and $P(x,y)$ may be written $f(x+f(x+2y))=2x+2y+4$

If $\exists$ odd $2a+1$ such that $f(2a+1)=2b$ is even, then :
$P(2a-2b+1,b)\implies 4b=4a+6$, which is impossible modulus 4

So $f(y)$ is odd for any odd y
Let then odd $x: f(x+2y)$ is odd and so $x+f(x+2y)$ is even and so $f(x+f(x+2y))=x+f(x+2y)+2$
So $x+f(x+2y)+2=2x+2y+4$ and $f(x+2y)=x+2y+2$

And so $\boxed{f(x)=x+2}$ $\forall x\in\mathbb{Z}$, which indeed is a solution.

Dijkschnei…
131 posts

Mar 20, 2011, 10:06 pm PM #371

> *socrates wrote:*
> **Problem 115**
>
> Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(0)=2$ and $f(x+f(x+2y))=f(2x)+f(2y)$, for all integers x,y.

Let $P(x,y)$ be the assertion : $f(x+f(x+2y))=f(2x)+f(2y)$
$P(0,y)\Rightarrow f(f(2y))=f(2y)+2$, and by induction, it easily follows that for every even integer : $f(x)=x+2(1)$
Now take an odd x and suppose f(x+2y) is odd.
Then using (1) and P(x,y), it follows that f(x+2y)=(x+2y)+2, and since x+2y describes the odd numbers as long as y describes $\mathbb{Z}$, then for all integers : $f(x)=x+2$, which completes the proof.

pco
12955 po…

Mar 20, 2011, 10:08 pm PM #372

> *Dijkschneier wrote:*
>
> > *socrates wrote:*
> > **Problem 115**
> >
> > Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(0)=2$ and $f(x+f(x+2y))=f(2x)+f(2y)$, for all integers x,y.
>
> Let $P(x,y)$ be the assertion : $f(x+f(x+2y))=f(2x)+f(2y)$
> $P(0,y)\Rightarrow f(f(2y))=f(2y)+2$, and by induction, it easily follows that for every even integer : $f(x)=x+2(1)$
> Now take an odd x and suppose f(x+2y) is odd.
> Then using (1) and P(x,y), it follows that f(x+2y)=(x+2y)+2, and since x+2y describes the odd numbers as long as y describes $\mathbb{Z}$, then for all integers : $f(x)=x+2$, which completes the proof.

And what if $f(x+2y)$ is even for any x odd and any y ?

Dijkschnei…
131 posts

Mar 20, 2011, 10:16 pm PM #373

You're right pco.
I'm sorry.

myth_kill
5 posts

Mar 22, 2011, 10:13 am PM #374

Problem 116

For which integers k does there exist a function f : N → Z such that
(a) f(1995) = 1996, and
(b) f(xy) = f(x) + f(y) + kf(gcd(x, y)) for all x, y ∈ N?

pco
12955 po…

Mar 22, 2011, 1:32 pm PM #375

> *myth_kill wrote:*
> Problem 116

For which integers k does there exist a function f : N → Z such that
(a) f(1995) = 1996, and
(b) f(xy) = f(x) + f(y) + kf(gcd(x, y)) for all x, y ∈ N?

Let $P(x,y)$ be the assertion $f(xy)=f(x)+f(y)+kf(\gcd(x,y))$

$P(x,x)\implies f(x^2)=(k+2)f(x)$
$P(x^2,x)\implies f(x^3)=(2k+3)f(x)$
$P(x^3,x)\implies f(x^4)=(3k+4)f(x)$
$P(x^2,x^2)\implies f(x^4)=(k+2)^2f(x)$

So $(3k+4)f(x)=(k+2)^2f(x)$ and setting $x=1995$, we get $(k+2)^2=(3k+4)$ and so $k\in\{-1,0\}$

For $k=-1$, solutions exist. For example $f(n)=1996$ $\forall n$.

For $k=0$, solutions exist. For example $f(1)=0$ and $f(\prod_{k=1}^n p_i^{n_i})=499\sum_{k=1}^n n_i$ (where p_i are distinct primes and $n_i\in\mathbb N$).

Hence the answer : $\boxed{k\in\{-1,0\}}$

Amir Hos...
4728 pos...

Mar 22, 2011, 9:15 pm PM #376

Problem 117.

Find all surjective functions $f:\mathbb N\to\mathbb N$ such that $p\mid f(m)+f(n)\iff p\mid f(m+n)$ for all primes p.

PS. well, it's old, but very nice!

---

pco
12955 po...

Mar 23, 2011, 1:31 pm PM #377

> *amparvardi wrote:*
> **Problem 117.**
>
> Find all surjective functions $f:\mathbb N\to\mathbb N$ such that $p\mid f(m)+f(n)\iff p\mid f(m+n)$ for all primes p.
>
> PS. well, it's old, but very nice!

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=56&t=214717

Amir Hos...
4728 pos...

Mar 23, 2011, 2:48 pm PM #378

:)

Problem 118.

Find all functions $f,g:\mathbb Z\to\mathbb Z$ such that g is bijective and
$$f(g(x)+y)=g(f(y)+x).$$

---

pco
12955 po...

Mar 23, 2011, 3:04 pm PM #379

> *amparvardi wrote:*
> :)
>
> **Problem 118.**
>
> Find all functions $f,g:\mathbb Z\to\mathbb Z$ such that g is bijective and
> $$f(g(x)+y)=g(f(y)+x).$$

We just need $g(x)$ injective and we dont need the restriction $\mathbb Z\to\mathbb Z$ (it's the same result for $\mathbb R\to\mathbb R$) :

Let $P(x,y)$ be the assertion $f(g(x)+y)=g(f(y)+x)$

$P(x,g(0))\implies f(g(x)+g(0))=g(f(g(0))+x)$
$P(0,g(x))\implies f(g(0)+g(x))=g(f(g(x)))$

So $g(f(g(0))+x)=g(f(g(x)))$ and, since $g(x)$ is injective : $f(g(x))=x+f(g(0))$

$P(x,0)\implies f(g(x))=g(f(0)+x)$ and so $g(x+f(0))=x+f(g(0))$ and so $g(x)=x+a$ for some a

Then $f(g(x))=x+f(g(0))$ becomes $f(x+a)=x+f(g(0))$ and so $f(x)=x+b$ for some b

Plugging back in original equation we get that these are solutions whatever are $a,b\in\mathbb Z$

Hence the answer :
$f(x)=x+b$ $\forall x$ and for any $b\in\mathbb Z$ (or $\mathbb R$ is we move the problem in $\mathbb R$)

$g(x)=x+a\ \forall x$ and for any $a\in\mathbb{Z}$ (or $\mathbb{R}$ is we move the problem in $\mathbb{R}$)

Potla
1888 pos...

Mar 23, 2011, 3:16 pm PM #380

I hope this has not been posted already. :)
Problem 119.(Belarus 1995)
Find all $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(f(x+y))=f(x+y)+f(x)f(y)-xy\quad\forall x,y\in\mathbb{R}.$$

Amir Hos...
4728 pos...

Mar 23, 2011, 3:23 pm PM #381

Thanks for the solution, dear Patrick. :)

> *pco wrote:*
> ...$g(x+f(0))=x+f(g(0))$ and so $g(x)=x+a$ for some a

But how do you get this?

pco
12955 po...

Mar 23, 2011, 4:08 pm • 1 PM #382

> *amparvardi wrote:*
> Thanks for the solution, dear Patrick. :)
>
> > *pco wrote:*
> > ...$g(x+f(0))=x+f(g(0))$ and so $g(x)=x+a$ for some a
>
> But how do you get this?

We previously got $f(g(x))=x+f(g(0))$
Then $P(x,0)\implies f(g(x))=g(f(0)+x)$ and so $g(x+f(0))=x+f(g(0))$

From there we immediately get $g(x)=(x-f(0))+f(g(0))$ and so $g(x)=x+a$ for some $a=f(g(0))-f(0)$

pco
12955 po...

Mar 23, 2011, 5:01 pm PM #383

> *Potla wrote:*
> I hope this has not been posted already. :)
> **Problem 119.**(Belarus 1995)
> Find all $f:\mathbb{R}\to\mathbb{R}$ such that
>
> $$f(f(x+y))=f(x+y)+f(x)f(y)-xy\quad\forall x,y\in\mathbb{R}.$$

Let $P(x,y)$ be the assertion $f(f(x+y))=f(x+y)+f(x)f(y)-xy$
Let $f(0)=a$

$P(x,y)\implies f(f(x+y))=f(x+y)+f(x)f(y)-xy$
$P(x+y,0)\implies f(f(x+y))=f(x+y)+af(x+y)$
Subtracting, we get new assertion $Q(x,y):af(x+y)=f(x)f(y)-xy$

$Q(x,-x)\implies a^2=f(x)f(-x)+x^2$
$Q(x,x)\implies af(2x)=f(x)^2-x^2$
$Q(-x,2x)\implies af(x)=f(-x)f(2x)+2x^2\implies a^2f(x)=f(-x)(f(x)^2-x^2)+2ax^2$
$\implies a^2f(x)^2=f(x)f(-x)(f(x)^2-x^2)+2ax^2f(x)=(a^2-x^2)(f(x)^2-x^2)+2ax^2f(x)$

And so $x^2(f(x)-a-x)(f(x)-a+x)=0$

So : $\forall x$, either $f(x)=a+x$, either $f(x)=a-x$ (the case $x=0$ is true too)

Suppose now that $f(x)=a+x$ for some x
$P(x,0)\implies f(a+x)=(a+1)x+a(a+1)$ and so :
either $(a+1)x+a(a+1)=a+(a+x)\iff a(x+a-1)=0$
either $(a+1)x+a(a+1)=a-(a+x)\iff(a+2)x+a(a+1)=0$
And so either $a=0$, either there are at most two such $x:1-a$ and $-\dfrac{a(a+1)}{a+2}$

Suppose now that $f(x)=a-x$ for some x
$P(x,0)\implies f(a-x)=-(a+1)x+a(a+1)$ and so :
either $-(a+1)x+a(a+1)=a+(a-x)\iff a(x-a+1)=0$
either $-(a+1)x+a(a+1)=a-(a-x)\iff(a+2)x-a(a+1)=0$
And so either $a=0$, either there are at most two such $x:a-1$ and $\dfrac{a(a+1)}{a+2}$

And so $a=0$ and either $f(x)=x$, either $f(x)=-x$
If $f(1)=1$, then $Q(x,1)\implies f(x)=x\ \forall x$ which indeed is a solution
If $f(1)=-1$, then $Q(x,-1)\implies f(x)=-x\ \forall x$ which is not a solution

Hence the answer : $\boxed{f(x)=x}\ \forall x$

RSM
736 posts

Mar 24, 2011, 1:42 am PM #384

> *Potla wrote:*
> I hope this has not been posted already. 🙂
> **Problem 119.**(Belarus 1995)
> Find all $f:\mathbb{R}\to\mathbb{R}$ such that
> $$f(f(x+y))=f(x+y)+f(x)f(y)-xy\quad\forall x,y\in\mathbb{R}.$$

Putting $y=0$ we get $f(f(x))=f(x)(a+1)$..............(1) where $f(0)=a$
Putting these in the equation we get $af(x+y)=f(x)f(y)-xy$..............(2).
Suppose, $a\neq 0$
$af(x+y+z)=f(x+y)f(z)-xz-yz$
$=\frac{f(x)f(y)f(z)}{a}-xy-yz-zx+xy-\frac{xyf(z)}{a}$
So $\frac{xyf(z)}{a}-xy=\frac{yzf(x)}{a}-yz$
$\frac{f(z)-a}{z}=\frac{f(x)-x}{a}$
So the solution in this case is $f(x)=kx+a$ for some constant k.
Putting this in (2) we get $k=1$ or $k=-1$
Then putting these in (1) gives no solution.
So $f(0)=0$ which gives the following solution:-
$f(x)=x$

mousavi
222 posts

Mar 27, 2011, 11:18 am PM #385

problem 120

Find all numbers $d\in[0,1]$ such that if $f(x)$ is an arbitrary continues function with domain $[0,1]$ and $f(0)=f(1)$,there exist number $x_0\in[0,1-d]$ such that $f(x_0)=f(x_0+d)$

RSM
736 posts

Mar 27, 2011, 2:28 pm PM #386

[Content deleted]
This post has been edited 1 time. Last edited by RSM, Mar 27, 2011, 11:51 pm

pco
12955 po...

Mar 27, 2011, 4:26 pm PM #387

> *RSM wrote:*
>
> > *mousavi wrote:*
> > **problem 120**
> >
> > Find all numbers $d\in[0,1]$ such that if $f(x)$ is an arbitrary continues function with domain $[0,1]$ and $f(0)=f(1)$,there exist number $x_0\in[0,1-d]$ such that $f(x_0)=f(x_0+d)$
>
> Suppose, a is any value of x such that $f(a)$ is maximum.
> So for any $y<a$ we can find a z such that $f(z)=f(y)$.
> So if we choose $d=z-y$ then $y\in[0,1-z+y]$
> So $d\in[0,a)$

I think you misunderstood the problem : d must fit whatever is the function $f(x)$. (the only thing which depends on $f(x)$ is x_0)

I think that the result is $d\in\{0\}\cup\bigcup_{n\in\mathbb{N}}\{\frac{1}{n}\}$

pco
12955 po...

Mar 27, 2011, 10:00 pm • 1 👍 PM #388

> *mousavi wrote:*
> **problem 120**
>
> Find all numbers $d\in[0,1]$ such that if $f(x)$ is an arbitrary continues function with domain $[0,1]$ and $f(0)=f(1)$,there exist number $x_0\in[0,1-d]$ such that $f(x_0)=f(x_0+d)$

1) $d=0$ fits
=========
Just choose $x_0=0$ 🙂

2) $d=\frac{1}{n}$ fits
==========
Let $g(x)=f(x+d)=f(x+\frac{1}{n})$

Let the sequence $a_k = f(\frac{k}{n})$
$a_0 = a_n = f(0)$ and so :
either $\exists k \in [0, n-1]$ such that $a_k = a_{k+1}$ and just choose $x_0 = \frac{k}{n}$
either $a_k \neq a_{k+1}$ $\forall k \in [0, n-1]$ and then :

If $a_1 > a_0$, the sequence cannot be increasing for any k and then $\exists k \in [0, n-1]$ such that $a_k < a_{k+1}$ and $a_{k+2} < a_{k+1}$ and then :
$f(\frac{k}{n}) < g(\frac{k}{n})$ and $g(\frac{k}{n}+d) < f(\frac{k}{n}+d)$ and so $\exists x_0 \in (\frac{k}{n}, \frac{k}{n}+d)$ such that $f(x_0) = g(x_0)$ (since continuous).

If $a_1 < a_0$, the sequence cannot be decreasing for any k and then $\exists k \in [0, n-1]$ such that $a_k > a_{k+1}$ and $a_{k+2} > a_{k+1}$ and then :
$f(\frac{k}{n}) > g(\frac{k}{n})$ and $g(\frac{k}{n}+d) > f(\frac{k}{n}+d)$ and so $\exists x_0 \in (\frac{k}{n}, \frac{k}{n}+d)$ such that $f(x_0) = g(x_0)$ (since continuous).
Q.E.D

3) no other d fit
==========
Let $d \in (0,1)$ and n, r such that $1 = nd + r$ with n non negative integer and $r \in (0, d)$
Choose any $u > 0$ and any continuous $h(x)$ defined over $[0, d]$ such that :
$h(0) = 0$
$h(r) = nu$
$h(d) = -u$

And define $f(x)$ in a recursive manner :
$\forall x \in [0, d] : f(x) = h(x)$
$\forall x > d : f(x) = f(x-d) - u$

We have :
$f(x)$ continuous
$f(0) = f(1) = 0$
And the equation $f(x) = f(x+d)$ is equivalent to $f(x) = f(x) - u$ and has no solution.
Q.E.D.

Hence the result : $\boxed{d \in \{0\} \cup \left(\bigcup_{n\in\mathbb{N}} \{\frac{1}{n}\}\right)}$

Dijkschnei... *131 posts*
Mar 28, 2011, 1:48 am PM #389
I misread problem 113 and solved (well, I guess) the following problem instead :
Problem 121 :
Find all $f : \mathbb{N} \to \mathbb{N}$ such that f(2x+3y)=2f(x)+3f(y) for all $x, y \geq 1$.

---

pco *12955 po...*
Mar 28, 2011, 2:23 am PM #390

> *Dijkschneier wrote:*
> I misread problem 113 and solved (well, I guess) the following problem instead :
> **Problem 121 :**
> Find all $f : \mathbb{N} \to \mathbb{N}$ such that f(2x+3y)=2f(x)+3f(y) for all $x, y \geq 1$.

Same method as for the original problem 113 :
Let $P(x, y)$ be the assertion $f(2x+3y) = 2f(x) + 3f(y)$

Subtracting $P(x+3, y)$ from $P(x, y+2)$, we get $2(f(x+3) - f(x)) = 3(f(y+2) - f(y))$

And so these two quantities are constant and multiple of 6 and so :
$f(x+3) = f(x) + 3c$
$f(y+2) = f(y) + 2c$
and (using $y = x+1$ in this last equation) : $f(x+3) = f(x+1) + 2c$

and so $f(x+1) = f(x) + c$ and $f(x) = cx + d$

Plugging this in $P(x, y)$, we get $\boxed{f(x) = ax}$ for any $a \in \mathbb{N}$

---

mousavi *222 posts*
Mar 28, 2011, 5:53 pm PM #391
problem 122

Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$

$f(x + f(xy)) = f(x + f(x)f(y)) = f(x) + xf(y)$

Dijkschnei... *131 posts*
Mar 29, 2011, 1:33 am PM #392

> *mousavi wrote:*
> **problem 122**
>
> Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$

$f(x+f(xy))=f(x+f(x)f(y))=f(x)+xf(y)$

Please add a continuity condition, because I have a solution in this case 😁

pco
12955 po...

Mar 29, 2011, 5:12 pm PM #393

> mousavi wrote:
> **problem 122**
>
> Find all functions $f:\mathbb{R}\longrightarrow\mathbb{R}$
>
> $f(x+f(xy))=f(x+f(x)f(y))=f(x)+xf(y)$

Let $P(x,y)$ be the assertions $f(x+f(xy))=f(x+f(x)f(y))=f(x)+xf(y)$

$f(x)=0\ \forall x$ is a solution and let us from now look for non allzero solutions.
Let u such that $f(u)\neq 0$

1) $f(x)=0\iff x=0$
==================
$P(-1,-1)\implies f(-1+f(1))=f(-1+f(-1)^2)=0$ and so $\exists v$ such that $f(v)=0$
$P(v,u)\implies 0=vf(u)$ and so $v=0$
Q.E.D.

2) $f(n)=n\ \forall n\in\mathbb{N}$
==============
$P(-1,-1)\implies f(-1+f(1))=f(-1+f(-1)^2)=0$ and so, using 1) : $-1+f(1)=-1+f(-1)^2=0$
So $f(1)=1$

$P(1,x)\implies f(1+f(x))=1+f(x)$ and so from $f(1)=1$, we get $f(n)=n\ \forall n\in\mathbb{N}$
Q.E.D.

3) $f(-1)=-1$
===========
$P(-1,-1)\implies f(-1+f(1))=f(-1+f(-1)^2)=0$ and so, using 1) : $-1+f(1)=-1+f(-1)^2=0$
So $f(-1)=\pm1$
If $f(-1)=1$, then :

$P(\frac{1}{n},n)\implies f(\frac{1}{n}+1)=f(\frac{1}{n})+\frac{1}{n}f(n)$
$P(\frac{1}{n},-n)\implies f(\frac{1}{n}+1)=f(\frac{1}{n})+\frac{1}{n}f(-n)$
And so $f(-n)=f(n)=n$
Then $P(-1,2)\implies f(-1+f(-2))=f(-1+f(-1)f(2))=f(-1)-f(2)\implies 1=1=-1$, contradiction
So $f(-1)=-1$
Q.E.D.

4) $f(x)$ is injective
===========
If $f(y_1)=f(y_2)$ and $y_2=0$ then $f(y_1)=0$ and 1) gives $y_1=y_2=0$
If $f(y_1)=f(y_2)$ and $y_2\neq 0$, let $a=\frac{y_1}{y_2}$
$P(y_2,1)\implies f(y_2+f(y_2))=f(y_2)+y_2$
$P(y_2,a)\implies f(y_2+f(y_1))=f(y_2)+y_2f(a)$
And so $f(a)=1$

$P(a,1)\implies f(a+1)=a+1$

Notice that if $f(x)=x$, then :
$P(1,x)\implies f(x+1)=x+1$
$P(-1,x)\implies f(-1+f(-1)f(x))=f(-1)-f(x)\implies f(-x-1)=-x-1$

Applying this to $f(a+1)=a+1$, we get
$f(-a-2)=-a-2$ (second property)
$f(-a-1)=-a-1$ (then first property)
$f(a)=a$ (then second property)
And so $a=1$
And so $y_1=y_2$
Q.E.D.

5) $f(xy)=f(x)f(y)$
===========
This is an immediate consequence of $f(x+f(xy))=f(x+f(x)f(y))$ and $f(x)$ injective

6) $f(x)=x\ \forall x$
==========
Let $x\neq 0$
We trivially have from 5) that $f(\frac{1}{x})=\frac{1}{f(x)}$

Then $P(\frac{1}{x},x) \implies f(\frac{1}{x}+1)=\frac{1}{f(x)}+\frac{f(x)}{x}$

Then $f(x+1)=f(x(\frac{1}{x}+1))=f(x)f(\frac{1}{x}+1)=1+\frac{f(x)^2}{x}$

But $P(x,\frac{1}{x}) \implies f(x+1)=f(x)+xf(\frac{1}{x})=f(x)+\frac{x}{f(x)}$

So $1+\frac{f(x)^2}{x}=f(x)+\frac{x}{f(x)}$

$\implies xf(x)+f(x)^3=xf(x)^2+x^2$

$\implies (f(x)^2+x)(f(x)-x)=0$

And so $f(x)=x\ \forall x>0$
And since $f(-x)=f((-1)x)=f(-1)f(x)=-f(x)$, we get $f(x)=x\ \forall x$ which indeed is a solution

7) Synthesis of solutions
=================
And so we got two solutions :
$f(x)=0\ \forall x$
$f(x)=x\ \forall x$

Dijkschnei...
131 posts

Mar 29, 2011, 7:47 pm PM #394

I am very impressed by your proof of the injection.
Very good.

---

socrates
1818 posts

Mar 30, 2011, 12:01 am PM #395

Problem 123

Let $f:[0,1]\to\mathbb{R}_+^*$ be a continous function such that $f(x_1)f(x_2)...f(x_n)=e,$
for all $n\in\mathbb{N}^*$ and for all $x_1,x_2,...,x_n\in[0,1]$ with $x_1+x_2+...+x_n=1.$

Prove that $f(x)=e^x,\ x\in[0,1].$

---

pco
12955 po...

Mar 30, 2011, 12:43 am PM #396

> *socrates wrote:*
> **Problem 123**
>
> Let $f:[0,1]\to\mathbb{R}_+^*$ be a continous function such that $f(x_1)f(x_2)...f(x_n)=e,$
> for all $n\in\mathbb{N}^*$ and for all $x_1,x_2,...,x_n\in[0,1]$ with $x_1+x_2+...+x_n=1.$
>
> Prove that $f(x)=e^x,\ x\in[0,1].$

Choosing $x_i=\frac{1}{n}$, we get $f(\frac{1}{n})^n=e$ and so $f(\frac{1}{n})=e^{\frac{1}{n}}$

Let $q>p\ge 1$: choosing $n=q-p+1$ and $x_1=x_2=...=x_{n-1}=\frac{1}{q}$ and $x_n=\frac{p}{q}$, we get :
$f(\frac{1}{q})^{q-p}f(\frac{p}{q})=e$ and so $e^{\frac{q-p}{q}}f(\frac{p}{q})=e$ and so $f(\frac{p}{q})=e^{\frac{p}{q}}$

And so $f(x)=e^x\ \forall x\in\mathbb{Q}\cap(0,1)$ and continuity implies $f(x)=e^x\ \forall x\in[0,1]$ which indeed is a solution

---

aktyw19
1315 posts

Apr 2, 2011, 12:23 pm • 1 PM #397

Let $f:\mathbb{R}\to\mathbb{R}$ be a continuos function, and $a,b,c,d\in\mathbb{R}$ with $ac(a^2-1)(a^2-c^2)\neq 0$. Under these conditions, solve functioal equation of f : $af(x+b)=f(cx+d)$.

---

mousavi
222 posts

Apr 8, 2011, 11:34 pm PM #398

problem 125

Find all functions $f:R\to R$:

$f(xy)f(f(x)-f(y))=(x-y)f(x)f(y)$

---

pco
12955 po...

Apr 10, 2011, 12:33 pm PM #399

> *mousavi wrote:*
> **problem 125**
>
> Find all functions $f:R\to R$:
>
> $f(xy)f(f(x)-f(y))=(x-y)f(x)f(y)$

Where is this problem coming from ?

There are infinitely many solutions but I did not succeed up to now finding all of them.

Some solutions :

1) trivial solution $f(x)=x\ \forall x$

2) trivial solution $f(x)=0\ \forall x$

3) $f(a)=b$ and $f(x)=0\ \forall x\neq a$ where a is any nonzero real and $b\neq\pm a$

4) $f(x)=x\ \forall x\in\mathbb{Q}$ and $f(x)=0$ anywhere else

5) $f(x)=x\ \forall x\in\mathbb{Q}[\sqrt{2}]$ and $f(x)=0$ anywhere else

In fact 4) and 5) may be merged in :

$f(x)=x\ \forall x\in\mathbb{K}$ and $f(x)=0$ anywhere else where $\mathbb{K}$ is any subfield of $\mathbb{R}$

... and a lot of other.

I wonder in what contest such a problem could have been asked :

---

goodar20... 1346 pos...

Apr 14, 2011, 11:10 pm PM #400

> *pco wrote:*
>
> > *mousavi wrote:*
> > **problem 125**
> >
> > Find all functions $f:R\to R$:
> >
> > $f(xy)f(f(x)-f(y))=(x-y)f(x)f(y)$
>
> Where is this problem coming from ?
>
> There are infinitely many solutions but I did not succeed up to now finding all of them.
>
> **Some solutions** :
>
> 1) trivial solution $f(x)=x\ \forall x$
>
> 2) trivial solution $f(x)=0\ \forall x$
>
> 3) $f(a)=b$ and $f(x)=0\ \forall x\neq a$ where a is any nonzero real and $b\neq\pm a$
>
> 4) $f(x)=x\ \forall x\in\mathbb{Q}$ and $f(x)=0$ anywhere else
>
> 5) $f(x)=x\ \forall x\in\mathbb{Q}[\sqrt{2}]$ and $f(x)=0$ anywhere else
>
> In fact 4) and 5) may be merged in :
>
> $f(x)=x\ \forall x\in\mathbb{K}$ and $f(x)=0$ anywhere else where $\mathbb{K}$ is any subfield of $\mathbb{R}$
>
> ... and a lot of other.
>
> I wonder in what contest such a problem could have been asked :

dear pco,

It is ISL 2001, A4, for a solution see here. I agree with you that the solutions are not very beautiful .

---

goodar20... 1346 pos...

Apr 14, 2011, 11:28 pm PM #401

I'll post the next problem

Problem 126

find all functions f from the set $\mathbb{R}$ of real numbers into $\mathbb{R}$ which satisfy for all $x,y,z\in\mathbb{R}$ the identity

$f(f(x)+f(y)+f(z))=f(f(x)-f(y))+f(2xy+f(z))+2f(xz-yz)$

---

pco 12955 po...

Apr 14, 2011, 11:28 pm PM #402

> *goodar2006 wrote:*
> dear pco,
>
> It is ISL 2001, A4, for a solution see here. I agree with you that the solutions are not very beautiful .

Thanks for your remark, but the pointed problem is not the same than current problem Mousavi posted :

Pointed problem : $f(xy)(f(x)-f(y))=(x-y)f(x)f(y)$

Current problem : $f(xy)f(f(x)-f(y))=(x-y)f(x)f(y)$

pco
12955 po...

Apr 15, 2011, 6:52 pm PM #403

> *goodar2006 wrote:*
> I'll post the next problem
>
> **Problem 126**
>
> find all functions f from the set $\mathbb{R}$ of real numbers into $\mathbb{R}$ which satisfy for all $x,y,z\in\mathbb{R}$ the identity
>
> $f(f(x)+f(y)+f(z))=f(f(x)-f(y))+f(2xy+f(z))+2f(xz-yz)$

Still waiting for Musavi's answer about problem 125.

Solution for problem 126

socrates
1818 posts

Apr 17, 2011, 2:30 am PM #404

Problem 127 (Greek TST 2011)

Find all functions $f,g:\mathbb{Q}\to\mathbb{Q}$ such that :

$f(g(x)-g(y))=f(g(x)))-y$ and

$g(f(x)-f(y))=g(f(x)))-y$

for each $x,y\in\mathbb{Q}$.

RSM
736 posts

Apr 17, 2011, 2:51 am PM #405

> *socrates wrote:*
> **Problem 127** (Greek TST 2011)
>
> Find all functions $f,g:\mathbb{Q}\to\mathbb{Q}$ such that :
>
> $f(g(x)-g(y))=f(g(x)))-y$ and
>
> $g(f(x)-f(y))=g(f(x)))-y$
>
> for each $x,y\in\mathbb{Q}$.

f and g are bijective:-

Putting $x=y$ in the equations we get $f(g(x))=x+f(0)$ and $g(f(x))=x+g(0)$
$g(x_1)=g(x_2)\implies x_1+f(0)=x_2+f(0)\implies x_1=x_2$. So f is one-one.
Clearly f is onto. So f is bijective and similarly g is bijective.

f(0)=g(0)=0

$f(g(x)-g(y))=f(0)+x-y$
Putting $y=0$ we get $f(g(x)-g(0))=f(g(x))$
So $g(x)-g(0)=g(x)$ and so $g(0)=0$ and similarly $f(0)=0$

g(x)=$f^{-1}(x)$

because $f(g(x))=x$

f and f^{-1} are Cauchy's Function

$f(f^{-1}(x)-f^{-1}(y))=x-y$
$f^{-1}(x)-f^{-1}(y)=f^{-1}(x-y)$
So f^{-1} is Cauchy's Function. Similarly f is a Cauchy's Function.

The solution of Cauchy's Equation for domain=$\mathbb{Q}$ is $f(x)=cx$ and so
$g(x)=\frac{x}{c}$ where $c\neq 0$.

socrates
1818 posts

Apr 18, 2011, 2:00 am PM #406

> *socrates wrote:*
> **Problem 101**

A variation on the unsolved **Problem 40**:

Let $f:R\to R$ be a function such that $f(xy)+f(x-y)\geq f(x+y)$ for all real numbers x,y.
Prove that $f(x)\geq 0,$ for each $x\in\mathbb{R}$.

Also: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=38&t=160835&

Problem 128 (Greek TST 2010)

Determine all functions $f:\mathbb{R}^*\to\mathbb{R}^*$ such that $f\left(\frac{f(x)}{f(y)}\right)=\frac{1}{y}\cdot f(f(x))$, for each $x,y\in\mathbb{R}^*$ and are strictly monotonic on $(0,+\infty)$.

---

pco
12955 po...

Apr 18, 2011, 12:09 pm PM #407

> *socrates wrote:*
> **Problem 128** (Greek TST 2010)
>
> Determine all functions $f:\mathbb{R}^*\to\mathbb{R}^*$ such that $f\left(\frac{f(x)}{f(y)}\right)=\frac{1}{y}\cdot f(f(x))$, for each $x,y\in\mathbb{R}^*$ and are strictly monotonic on $(0,+\infty)$.

Still waiting for Musavi's answer about problem 125.

Solution for problem 128

---

filipbitola
124 posts

Apr 18, 2011, 5:20 pm • 1 PM #408

[moderator edit: please hide the long posts.]

Stephen wrote:

I know this is a little bit old, but from the second line onward I have a much shorter solution.
I'll upload an attachment and you will find the solution on the last page in the Note

Attachments:
Cauchy-Filip-Predrag Theorem functional equation.pdf (458kb)

---

filipbitola
124 posts

Apr 19, 2011, 1:34 am PM #409

Now that problem 128 is solved, I'd like to propose problem 129:
Problem 129: Find all functions, $f:\mathbb{R}^+\to\mathbb{R}^+$ such that:
$x^2f(f(x)+f(y))=(x+y)f(yf(x))$
for all x,y in $\mathbb{R}^+$

---

pco
12955 po...

Apr 19, 2011, 2:41 am PM #410

> *filipbitola wrote:*
> Now that problem 128 is solved, I'd like to propose problem 129:
> **Problem 129:** Find all functions, $f:\mathbb{R}^+\to\mathbb{R}^+$ such that:
> $x^2f(f(x)+f(y))=(x+y)f(yf(x))$
> for all x,y in $\mathbb{R}^+$

Still waiting for Musavi's answer about problem 125.

Solution for problem 129

Batominov...
1602 pos...

Apr 19, 2011, 8:04 am PM #411

Problem 130: Let $f:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ be a function which is bounded on the interval $[0,1]$ and obeys the inequality

$$f(x)f(y)\leq x^2f\left(\frac{y}{2}\right)+y^2f\left(\frac{x}{2}\right)$$

for each pair of nonnegative reals x and y. Prove that $f(x)\leq\frac{x^2}{2}$ for all nonnegative reals x.

Source: Adapted from Chinese 1990.

Problem 131: Let $\mathbb{N}$ denote the set of "positive" integers. Fix $c\in\mathbb{N}$. Does there exist a function $f:\mathbb{N}\to\mathbb{N}$ such that $f^{\left[f(n)\right]}(n)=n+c$ for every $n\in\mathbb{N}$? In this context, $f^{[0]}$ is the identity and $f^{[k]}:=f\circ f^{[k-1]}$ for all $k\in\mathbb{N}$.

This post has been edited 3 times. Last edited by Batominovski, Apr 20, 2011, 5:07 am

---

pco
12955 po...

Apr 19, 2011, 1:04 pm PM #412

> *Batominovski wrote:*
> **Problem 131:** Let $\mathbb{N}$ denote the set of "positive" integers. Does there exist a function $f:\mathbb{N}\to\mathbb{N}$ such that $f^{\left[f(n)\right]}(n)=n$ for every $n\in\mathbb{N}$? In this context, $f^{[0]}$ is the identity and $f^{[k]}:=f\circ f^{[k-1]}$ for all $k\in\mathbb{N}$.

Yes : $f(n)=n\,\forall n\in\mathbb{N}$

Batominov... 1602 pos...

Apr 19, 2011, 1:43 pm PM #413

> *pco wrote:*
>
> > *Batominovski wrote:*
> > **Problem 131:** Let $\mathbb{N}$ denote the set of "positive" integers. Does there exists a function $f:\mathbb{N}\to\mathbb{N}$ such that $f^{\left[f(n)\right]}(n)=n$ for every $n\in\mathbb{N}$? In this context, $f^{[0]}$ is the identity and $f^{[k]}:=f\circ f^{[k-1]}$ for all $k\in\mathbb{N}$.
>
> Yes : $f(n)=n\,\forall n\in\mathbb{N}$

Haha. This time I made a fatal mistake in my problem statement. Please take a look at it again.

---

pco 12955 po...

Apr 19, 2011, 4:52 pm PM #414

> *Batominovski wrote:*
> **Problem 130:** Let $f:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ be a function which is bounded on the interval $[0,1]$ and obeys the inequality
> $$f(x)f(y)\leq x^2 f\left(\frac{y}{2}\right)+y^2 f\left(\frac{x}{2}\right)$$
> for each pair of nonnegative reals x and y. Prove that $f(x)\leq\frac{x^2}{2}$ for all nonnegative reals x.

Setting $x=y$ in the inequality, we get $2x^2f(\frac{x}{2})\geq f(x)^2$

Setting $g(x)=\frac{2f(x)}{x^2}$ this becomes $g(\frac{x}{2})\geq g(x)^2$ and so $g(\frac{x}{2^n})\geq g(x)^{2^n}$

Suppose then that $g(u)=a>1$ for some u, then $g(\frac{u}{2^n})\geq a^{2^n}$

And so $f(\frac{u}{2^n})\geq u^2\frac{a^{2^n}}{2^{2n+1}}$

Setting $n\to+\infty$ in the above inequation, we get thet LHS is clearly unbounded, and so contradiction with the fact that $f(x)$ is bounded on $[0,1]$

So $g(x)\leq 1\,\forall x$

So $f(x)\leq\frac{x^2}{2}\,\forall x$
Q.E.D.

---

soulhunter 317 posts

Apr 20, 2011, 9:32 am PM #415

Problem no 132: Find all strictly increasing bijective function $f:R-->R$ such that $f(x)+f^{-1}(x)=2x$ for all real x.

---

pco 12955 po...

Apr 20, 2011, 1:21 pm PM #416

> *soulhunter wrote:*
> Problem no 132: Find all strictly increasing bijective function $f:R-->R$ such that $f(x)+f^{-1}(x)=2x$ for all real x.

$f(x)$ increasing bijection implies $f(x)$ continuous.
The equation may be written $f(f(x))-f(x)=f(x)-x$ and so $g(x+g(x))=g(x)$ where $g(x)=f(x)-x$ is continuous.

Let us look for continuous solutions of $g(x+g(x))=g(x)$

$g(x)=0\,\forall x$ is a solution and let us from now look for non all zero solutions.
If $g(x)$ is solution, then $-g(-x)$ is solution too and so Wlog say $g(u)=v>0$ for some u

Let $A=\{x\geq u$ such that $g(x)=g(u)=v\}$

From $g(x+g(x))=g(x)$, we get $g(x+ng(x))=g(x)$ and so $u+nv\in A\,\forall n\in\mathbb{N}\cup\{0\}$

If A is not dense in $[u,+\infty)$, let then $a,b\in A$ such that $u\leq a<b$ and $(a,b)\cap A=\emptyset$. (existence of a,b needs continuity of $g(x)$)

Let then $y\in(a,b)$. So $g(y)\neq v$
Consider then $y-a+n(g(y)-v)$ for $n\in\mathbb{N}$
Since $g(y)\neq v$, this quantity, for n great enough is out of $[-v,+v]$ and so let $m>0$ such that

$y-a+m(g(y)-v)\notin[-v,+v]$ and so such that $y+mg(y)\notin[a+(m-1)v,a+(m+1)v]$

Looking at the continuous function $h(x)=x+mg(x)$, we get :
$h(a)=a+mv\in(a+(m-1)v,a+(m+1)v)$
$h(y)=y+mg(y)\notin[a+(m-1)v,a+(m+1)v]$

So (using continuity of $h(x)$), $\exists z\in(a,y)$ such that $h(z)=a+(m-1)v$ or $h(z)=a+(m+1)v$
But then $g(h(z))=v$ and so $g(z+mg(z))=g(z)=v$, impossible since $z\in(a,b)$ and $(a,b)\cap A=\emptyset$.

So A is dense in $[u,+\infty)$

Then continuity of $g(x)$ implies $g(x)=v\ \forall x\ge u$.
Let then any $w<u$: If $g(w)>0$, then $\exists n\in\mathbb{N}$ such that $w+ng(w)>u$ and so $g(w)=v$. So $\forall x<u$: either $g(x)=v$, either $g(x)\le 0$ and continuity gives the conclusion $g(x)=v\ \forall x$

So $g(x)=c$ and $\boxed{f(x)=x+c}$ which indeed is a solution.

Btw, no longer waiting for Musavi's answer about problem 125

---

abhinavza...
418 posts

Apr 20, 2011, 6:05 pm PM #417

Problem 133
Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying
(a) $f(0)=0$
(b) $f\left(\dfrac{x^2+y^2}{2xy}\right)=\dfrac{f(x)^2+f(y)^2}{2xy}\quad\forall\ \ x,y\in\mathbb{R},x\ne 0,y\ne 0$

---

pco
12955 po...

Apr 20, 2011, 6:20 pm • 2 PM #418

> *abhinavzandubalm wrote:*
> **Problem 133**
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying
> (a) $f(0)=0$
> (b) $f\left(\dfrac{x^2+y^2}{2xy}\right)=\dfrac{f(x)^2+f(y)^2}{2xy}\quad\forall\ \ x,y\in\mathbb{R},x\ne 0,y\ne 0$

Let $P(x,y)$ be the assertion $f\left(\dfrac{x^2+y^2}{2xy}\right)=\dfrac{f(x)^2+f(y)^2}{2xy}$

$P(1,1)\implies f(1)=f(1)^2$ and so $f(1)\in\{0,1\}$

If $f(1)=0$, then $P(x,x)\implies f(x)=0\ \forall x\ne 0$ and so $f(x)=0\ \forall x$

If $f(1)=1$, then $P(x,x)\implies f(x)^2=x^2\ \forall x\ne 0$ and so $f(x)^2=x^2\ \forall x$

Then $P(x,y)$ becomes $f\left(\dfrac{x^2+y^2}{2xy}\right)=\dfrac{x^2+y^2}{2xy}$

And so $f(x)=x\ \forall x$ such that $|x|\ge 1$ and obviously $f(x)$ may be either x, either $-x$ for any other x

And so the solutions :
1) $f(x)=0\ \forall x$

2) $f(x)=e(x)x\ \forall x\in(-1,1)$ and $f(x)=x\ \forall x\in(-\infty,-1]\cup[1,+\infty)$ where $e(x)$ is any function from $(-1,1)\to\{-1,1\}$

---

filipbitola
124 posts

Apr 22, 2011, 7:19 pm PM #419

Problem 134:
This post is not that old but it's a nice function:
Find all $f:\mathbb{R}\to\mathbb{R}$ such that
$xf(y)-yf(x)=f(\dfrac{y}{x})$
for $x,y\in\mathbb{R},x\ne 0$

---

pco
12955 po...

Apr 22, 2011, 8:22 pm • 1 PM #420

> *filipbitola wrote:*
> **Problem 134:**
> This post is not that old but it's a nice function:
> Find all $f:\mathbb{R}\to\mathbb{R}$ such that
> $xf(y)-yf(x)=f(\dfrac{y}{x})$
> for $x,y\in\mathbb{R},x\ne 0$

Let $P(x,y)$ be the assertion $xf(y)-yf(x)=f(\dfrac{y}{x})$

$P(2,0)\implies f(0)=0$
$P(1,1)\implies f(1)=0$

$P(x,1) \implies f(x) = -f(\frac{1}{x}) \forall x \neq 0$

$P(\frac{1}{x},2) \implies \frac{f(2)}{x} + 2f(x) = f(2x) \forall x \neq 0$

$P(\frac{1}{2},x) \implies \frac{f(x)}{2} + xf(2) = f(2x) \forall x \neq 0$

Subtracting, we get $f(x) = \frac{2f(2)}{3}\frac{x^2-1}{x} \forall x \neq 0$

Hence the solution : $f(0) = 0$ and $f(x) = a\frac{x^2-1}{x} \forall x \neq 0$ which indeed is a solution (where a is any real)

---

nguyenvut...
475 posts

Apr 22, 2011, 8:38 pm PM #421

Problem 135 :

Find all positive real number a such that there exists a positive real number k and a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that :

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + k|x-y|^a \quad \forall x;y \ \in \mathbb{R}$$

---

pco
12955 po...

Apr 22, 2011, 8:46 pm PM #422

> *nguyenvuthanhha wrote:*
> Problem 135 :
>
> Find all positive real number a such that there exists a positive real number k and a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that :
>
> $$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + k|x-y|^a \quad \forall x;y \ \in \mathbb{R}$$

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=38&t=231740

Amir Hos...
4728 pos...

Apr 22, 2011, 9:12 pm PM #423

Problem 136. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y)+f(y+z)+f(z+x) \geq 3f(x+2y+3z), \qquad \forall x,y,z \in \mathbb{R}.$$

Have I posted this before? If yes, then

Problem 137. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ for which $f(1) = f(-1)$ and

$$f(m)+f(n) = f(m+2mn)+f(n-2mn), \qquad \forall m,n \in \mathbb{Z}.$$

Problem 138. Let $\mathbb{K}$ be the set of all real numbers x such that $0 \leq x \leq 1$. Let f be a function from K to the set of all real numbers $\mathbb{R}$ with the following properties

- $f(1) = 1$;
- $f(x) \geq 0$ for all $x \in \mathbb{K}$;
- If x, y and $x+y$ are all in $\mathbb{K}$, then

$$f(x+y) \geq f(x)+f(y).$$

Prove that $f(x) \leq 2x$, for all $x \in \mathbb{K}$.

Source of the problems: Mathematical Magazine, December, 2004 (fifth year), and Irish Olympiad.

This post has been edited 1 time. Last edited by Amir Hossein, Apr 22, 2011, 11:05 pm
Reason: edited.

---

mahanmath
1356 pos...

Apr 22, 2011, 9:34 pm PM #424

And what if you`ve already posted them ?

Problem 136 posted by user **ashkan** (with a tail of numbers) , and 137 posted at least 2 times in marathon .

Amir Hos...
4728 pos...

Apr 22, 2011, 9:46 pm PM #425

What about 138?

---

abhinavza...
418 posts

Apr 22, 2011, 10:14 pm PM #426

> *amparvardi wrote:*
> What about 138?

I think the **To Prove** condition should be $f(x) \geq x$ as $f(1) = 1$.

filipbitola
124 posts

Apr 22, 2011, 10:15 pm PM #427

> *amparvardi wrote:*
> **Problem 138.** Let $\mathbb{K}$ be the set of all real numbers x such that $0 \leq x \leq 1$. Let f be a function from K to the set of all real numbers $\mathbb{R}$ with the following properties
>
> - $f(1) = 1$;
> - $f(x) \geq 0$ for all $x \in \mathbb{K}$;
> - If x, y and $x + y$ are all in $\mathbb{K}$, then
>
> $$f(x+y) \geq f(x) + f(y).$$
>
> Prove that $f(x) \geq 2x$, for all $x \in \mathbb{K}$.
>
> *Source of the problems: Mathematical Magazine, December, 2004 (fifth year), and Irish Olympiad.*

Are you sure the problem is correct? 1 is in K, but $f(1) < 2$

abhinavza...
418 posts

Apr 22, 2011, 10:20 pm PM #428

> *abhinavzandubalm wrote:*
>> *amparvardi wrote:*
>> What about 138?
>
> I think the **To Prove** condition should be $f(x) \geq x$ as $f(1) = 1$.

I rephrase my suggestion.
If we put $x + y = 1$, then
$f(x+y) = f(1) = 1 = x + y \geq f(x) + f(y)$
So we must prove that $\boxed{f(x) \leq x}$ if we want to have a problem having some sense or else we get $f(x) = x$.
I think so.

goodar20...
1346 pos...

Apr 22, 2011, 10:30 pm PM #429

the correct statement of the problem is to prove that $f(x) \leq 2x \, \forall x \in \mathbb{K}$.

filipbitola
124 posts

Apr 22, 2011, 10:33 pm PM #430

Here is another problem:
Find all $k \in \mathbb{N}$ such that there exist exactly k functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying:
$f(x+y) = kf(x)f(y) + f(x) + f(y)$
for all x, y in $\mathbb{Q}$

pco
12955 po...

Apr 22, 2011, 10:37 pm PM #431

> *goodar2006 wrote:*
> the correct statement of the problem is to prove that $f(x) \leq 2x \, \forall x \in \mathbb{K}$.

If so, see http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=281056

pco
12955 po...

Apr 22, 2011, 10:48 pm • 1 PM #432

> *filipbitola wrote:*
> Here is another problem:
> Find all $k \in \mathbb{N}$ such that there exist exactly k functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying:
> $f(x+y) = kf(x)f(y) + f(x) + f(y)$
> for all x, y in $\mathbb{Q}$

Let $h(x) = kf(x) + 1$. The equation becomes $h(x+y) = h(x)h(y)$ and so two solutions :
$h(x) = 0 \, \forall x$
$h(x) = 1 \, \forall x$
The other solutions $h(x) = a^x$ do not fit since they are not from $\mathbb{Q} \to \mathbb{Q}$

Hence the answer $\boxed{k = 2}$

Amir Hos...
4728 pos...

Apr 22, 2011, 11:12 pm PM #433

Sorry about that typo in **Problem 138**.

Problem 140. Find all functions $f : X \to \mathbb{R}$ such that for all $x, y \in X$

$$f(x+y) + f(xy-1) = (f(x)+1)(f(y)+1),$$

a) if $X=\mathbb{Z}$;

b) if $X=\mathbb{Q}$.

---

filipbitola
124 posts

Apr 23, 2011, 12:32 am PM #434

So far I only solved part a:
Replacing $y=0$ gives:
$f(-1)=f(x)f(0)+f(0)+1$
which tells us that if $f(0)\neq 0$ then f is constant, but we can easily rule that part out.
So $f(0)=0$ and from here $f(-1)=1$
Replacing $y=-1, x=1$ in the original equation gives us:
$f(0)+f(-2)=2f(1)+2$
But, $y=-1, x=-1$, gives:
$f(-2)+f(0)=4$
Hence $f(1)=1$.
Replacing $y=1$ in the original equation we have:
$f(x+1)+f(x-1)=2f(x)+2$
and from here by induction we prove that $f(x)=x^2 \forall x\in\mathbb{Z}$, since $f(0)=0, f(1)=f(-1)=1$.
Checking in the original equation we see that $f(x)=x^2$ satisfies the given equation.

---

pco
12955 po...

Apr 23, 2011, 1:07 pm • 1 PM #435

For $b)$ (since it's now $\mathbb{Q}$ and no longer $\mathbb{R}$ 🙂) :

Using filipbitola way, one can prove that $f(x+n)=nf(x+1)-(n-1)f(x)+n^2-n$

Plugging then in original equation with $y=n$, we get : $f(nx-1)=(n^2+n)f(x)-nf(x+1)+n+1$
Using $2n$ instead of n, we get $f(2nx-1)=(4n^2+2n)f(x)-2nf(x+1)+2n+1$

Setting then $x=\frac{p}{n}$ and cancelling $f(x+1)$ between the two equations, we get $f(\frac{p}{n})=\frac{p^2}{n^2}$

And so $f(x)=x^2\ \forall x\in\mathbb{Q}$

---

filipbitola
124 posts

Apr 24, 2011, 3:28 am PM #436

Problem 141:
Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:
$f(x+y^2+z)=f(f(x))+yf(x)+f(z)\ \forall x,y,x\in\mathbb{R}$
After that problem you can try:
Problem 142:
Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:
$xf(y)-yf(x)=(x-y)f(x+y)\ \forall x,y\in\mathbb{R}$

And, thanks Patrick for doing part B on the previous problem.

---

pco
12955 po...

Apr 24, 2011, 12:27 pm PM #437

> *filipbitola wrote:*
> **Problem 141:**
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:
> $f(x+y^2+z)=f(f(x))+yf(x)+f(z)\ \forall x,y,x\in\mathbb{R}$

I suppose we must read $\forall x,y,z\in\mathbb{R}$ and not $\forall x,y,x\in\mathbb{R}$

$f(x)=0\ \forall x$ is a solution. Let us from now look for non allzero solutions.

Let $P(x,y)$ be the assertion $f(x+y^2+z)=f(f(x))+yf(x)+f(z)$
Let u such that $f(u)\neq 0$

$P(u,\frac{x-f(f(u))-f(0)}{f(u)},0)\implies f(\text{something})=x$ and so $f(x)$ is surjective.

$P(x,0,0)\implies f(x)=f(f(x))+f(0)$ and so $f(x)=x-f(0)\ \forall x\in f(\mathbb{R})$

And since $f(x)$ is surjective, we get $f(x)=x-f(0)\ \forall x\in\mathbb{R}$.

Setting then $x=0$, we get $f(0)=0$ and hence the result :

$f(x)=0\ \forall x$
$f(x)=x\ \forall x$ which indeed is a solution

---

pco
12955 po...

Apr 24, 2011, 12:30 pm PM #438

> *filipbitola wrote:*
> **Problem 142:**
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:
> $xf(y)-yf(x)=(x-y)f(x+y)\ \forall x,y\in\mathbb{R}$

Setting $y=0$ in this equation, we get $xf(0)=xf(x)$ and so $f(x)=f(0)\forall x\neq 0$ and so $f(x)=f(0)\ \forall x$, which indeed is a solution.

Hence the answer : $\boxed{f(x)=c}$ $\forall x$ and for any real c.

goodar20...
1346 pos...

Apr 24, 2011, 1:54 pm • 1 PM #439

> *pco wrote:*
>
> > *filipbitola wrote:*
> > **Problem 141:**
> > Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:
> > $f(x+y^2+z)=f(f(x))+yf(x)+f(z)\,\forall x,y,x\in\mathbb{R}$
>
> I suppose we must read $\forall x,y,z\in\mathbb{R}$ and not $\forall x,y,x\in\mathbb{R}$
>
> $f(x)=0\,\forall x$ is a solution. Let us from now look for non allzero solutions.
>
> Let $P(x,y)$ be the assertion $f(x+y^2+z)=f(f(x))+yf(x)+f(z)$
> Let u such that $f(u)\neq 0$
>
> $P(u,\frac{x-f(f(u))-f(0)}{f(u)},0)\implies f(\text{something})=x$ and so $f(x)$ is surjective.
>
> $P(x,0,0)\implies f(x)=f(f(x))+f(0)$ and so $f(x)=x-f(0)\,\forall x\in f(\mathbb{R})$
>
> And since $f(x)$ is surjective, we get $f(x)=x-f(0)\,\forall x\in\mathbb{R}$.
>
> Setting then $x=0$, we get $f(0)=0$ and hence the result :
>
> $f(x)=0\,\forall x$
> $f(x)=x\,\forall x$ which indeed is a solution

dear pco, are you sure that $f(x)=x$ is a solution?

pco
12955 po...

Apr 24, 2011, 2:16 pm • 2 PM #440

> *goodar2006 wrote:*
> dear pco, are you sure that $f(x)=x$ is a solution?

Oooooooops, not quite awaken, this morning 😁

You're quite right. $f(x)=x$ is not a solution.

Hence the unique result : $f(x)=0\,\forall x$

Thanks for the remark.

Batominov...
1602 pos...

Apr 24, 2011, 3:22 pm PM #441

> *filipbitola wrote:*
> **Problem 141 (Modified):**
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:
> $f\left(x+y^2+z\right)=f\big(f(x)\big)+yf(y)+f(z)\,\forall x,y,z\in\mathbb{R}.$

I think this version is indeed the correct one. Could you please verify filipbitola?

pco
12955 po...

Apr 24, 2011, 3:59 pm PM #442

> *Batominovski wrote:*
>
> > *filipbitola wrote:*
> > **Problem 141 (Modified):**
> > Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:
> > $f\left(x+y^2+z\right)=f\big(f(x)\big)+yf(y)+f(z)\,\forall x,y,z\in\mathbb{R}.$
>
> I think this version is indeed the correct one. Could you please verify filipbitola?

Let $P(x,y,z)$ be the assertion $f(x+y^2+z)=f(f(x))+yf(y)+f(z)$

$P(x,0,y)\implies f(x+y)=f(f(x))+f(y)$
$P(x,0,0)\implies f(x)=f(f(x))+f(0)$
Subtracting, we get $f(x+y)=f(x)+f(y)+f(0)$

So $f(x+y^2+z)=f(x)+f(y^2+z)+f(0)=f(x)+f(y^2)+f(z)+2f(0)$

And $P(x,y,z)$ becomes $f(x)+f(y^2)+f(z)+2f(0)=f(f(x))+yf(y)+f(z)$
Subtracting $f(x)=f(f(x))+f(0)$, we get :

$f(y^2)+3f(0)=yf(y)$

Setting $y=0$, we get $f(0)=0$ and so the three equations :

$f(x+y)=f(x)+f(y)$
$f(f(x))=f(x)$
$f(x^2)=xf(x)$

With these properties, we can write $f((x+1)^2)$ in two ways :
$f((x+1)^2)=(x+1)f(x+1)=xf(x)+f(x)+xf(1)+f(1)$
$f((x+1)^2)=f(x^2+2x+1)=f(x^2)+2f(x)+f(1)=xf(x)+2f(x)+f(1)$

And so $f(x)=xf(1)$

Plugging this in original equation, we get $f(1)=0$ or $f(1)=1$

And so two solutions :
$f(x)=0\ \forall x$
$f(x)=x\ \forall x$

---

filipbitola
124 posts

Apr 24, 2011, 6:19 pm PM #443

The correct version was the way I wrote it... $yf(x)$, not $yf(y)$
But nice solution btw

RSM
736 posts

Apr 25, 2011, 6:17 pm PM #444

> *filipbitola wrote:*
> **Problem 141:**
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:
> $f(x+y^2+z)=f(f(x))+yf(x)+f(z)\,\forall x,y,z\in\mathbb{R}$

Suppose, $P(x,y,z)$ is the assertion that $f(x+y^2+z)=f(f(x))+yf(x)+f(z)$
$P(x,-y,z)\implies f(x+y^2+z)=f(f(x))-yf(x)+f(z)$
Comparing the two equations we get
$2yf(x)=0$
Assume $y\neq 0$
So $f(x)=0\forall x$

Amir Hos...
4728 pos...

Apr 26, 2011, 8:19 am PM #445

Problem 143. Find all such functions $f:\mathbb{R}\to\mathbb{R}$ such that

1) for all real x,y, we have

$$f(f(x)y+x)=xf(y)+f(x),$$

2) the equation $f(t)=-t$ has exactly one root.

---

pco
12955 po...

Apr 26, 2011, 12:27 pm PM #446

> *amparvardi wrote:*
> **Problem 143.** Find all such functions $f:\mathbb{R}\to\mathbb{R}$ such that
>
> **1)** for all real x,y, we have
>
> $$f(f(x)y+x)=xf(y)+f(x),$$
>
> **2)** the equation $f(t)=-t$ has exactly one root.

$f(x)=0\ \forall x$ is a solution.
Let us from now look for non allzero solutions.

Let $P(x,y)$ be the assertion $f(f(x)y+x)=xf(y)+f(x)$
Let t such that $f(t)=-t$
Let u such that $f(u)\neq 0$

If $t\neq 0$, then $f(0)\neq 0$ and $P(0,\frac{x}{f(0)})\implies f(x)=f(0)\neq 0\,\forall x$ which is not a solution.

So $t=0$ and $f(0)=0$

$P(-1,-1)\implies f(-f(-1)-1)=0$

Then $P(-f(-1)-1,u) \implies (f(-1)+1)f(u)=0$ and so $f(-1)=-1$

$P(x,-1) \implies f(x-f(x))=f(x)-x$ and so $x-f(x)=t=0$ and $f(x)=x$ which indeed is a solution.

Hence the two solutions :
$f(x)=0\ \forall x$
$f(x)=x\ \forall x$

Amir Hos...
4728 pos...

Apr 27, 2011, 3:24 pm PM #447

Problem 12^2. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x^2-y^2)=x^2-f(y^2)$ for all reals x,y.

---

pco
12955 po...

Apr 27, 2011, 3:43 pm • 1 PM #448

> *amparvardi wrote:*
> **Problem** 12^2. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x^2-y^2)=x^2-f(y^2)$ for all reals x,y.

Let $P(x,y)$ be the assertion $f(x^2-y^2)=x^2-f(y^2)$

$P(0,0) \implies f(0)=0$

(a) : $P(\frac{x+1}{2},\frac{x-1}{2}) \implies f(x)=\frac{(x+1)^2}{4}-f(\frac{(x-1)^2}{4})$

(b) : $P(\frac{x-1}{2},\frac{x-1}{2}) \implies 0=\frac{(x-1)^2}{4}-f(\frac{(x-1)^2}{4})$

(a)-(b) : $\boxed{f(x)=x}$ which indeed is a solution

---

filipbitola
124 posts

Apr 27, 2011, 3:59 pm PM #449

> *amparvardi wrote:*
> **Problem** 12^2. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x^2-y^2)=x^2-f(y^2)$ for all reals x,y.

$x=y=0$ gives that $f(0)=0$
$y=0$ gives $f(x^2)=x^2$, hence $f(x)=x\forall x\in\mathbb{R}$
$x=0$ gives $f(-y^2)=-f(y^2)$ and so $f(-x)=-f(x)\forall x\in\mathbb{R}_0^+$.
Hence, $f(x)=x\forall x\in\mathbb{R}$

---

aktyw19
1315 posts

Apr 27, 2011, 6:47 pm PM #450

145.
Find all continuous functions $f\ :\ R_+\mapsto R_+$ such that a, b and c are the sides of any triangle, and the equality

$$\frac{f(a+b-c)+f(b+c-a)+f(c+a-b)}{3}\ =\ f\left(\ \sqrt{\frac{ab+bc+ca}{3}}\ \right)$$

---

socrates
1818 posts

Apr 28, 2011, 3:54 am PM #451

> *amparvardi wrote:*
> **Problem 143.** Find all such functions $f:\mathbb{R}\to\mathbb{R}$ such that
>
> **1)** for all real x,y, we have
> $$f(f(x)y+x)=xf(y)+f(x),$$
>
> **2)** the equation $f(t)=-t$ has exactly one root.

It's also **Problem 105**: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=350187&start=320 :)

> *amparvardi wrote:*
> **Problem 140.** Find all functions $f:X\to\mathbb{R}$ such that for all $x,y\in X$
> $$f(x+y)+f(xy-1)=(f(x)+1)(f(y)+1),$$
>
> **a)** if $X=\mathbb{Z}$;
> **b)** if $X=\mathbb{Q}$.

Also: http://www.artofproblemsolving.com/Forum/viewtopic.php?p=2201269#p2201269

BigSams
1582 pos...

May 15, 2011, 12:59 pm PM #452

> *aktyw19 wrote:*
> 145.

Find all continuous functions $f : R_+ \mapsto R_+$ such that a, b and c are the sides of any triangle, and the equality

$$\frac{f(a+b-c)+f(b+c-a)+f(c+a-b)}{3} = f\left(\sqrt{\frac{ab+bc+ca}{3}}\right)$$

It's been a while... the only idea I can think of is Click to reveal hidden text

Ravi Substitution which makes it into
$$\frac{\sum_{cyc} f(2x)}{3} = f\left(\sqrt{\frac{\sum_{cyc} x^2}{3} + \sum_{cyc} xy}\right) \forall \{x,y,z\} \in \mathbb{R}^+$$

, which makes it easier but still daunting O_o
Can you post a hint?

---

filipbitola
124 posts

May 19, 2011, 5:00 am PM #453

Problem 146:
Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that:
$xf(yz) + yf(z) + z = f(f(x)yz + f(y)z + f(z)) \, \forall x, y \in \mathbb{Q}$

---

pco
12955 po...

May 19, 2011, 3:45 pm PM #454

> *filipbitola wrote:*
> **Problem 146:**
> Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that:
> $xf(yz) + yf(z) + z = f(f(x)yz + f(y)z + f(z)) \, \forall x, y \in \mathbb{Q}$

Let $P(x, y, z)$ be the assertion $xf(yz) + yf(z) + z = f(f(x)yz + f(y)z + f(z))$

$P(x, 0, 0) \implies xf(0) = f(f(0)) \forall x$ and so $f(0) = 0$
$P(0, 0, x) \implies f(f(x)) = x$ and so $f(x)$ is an involutive bijection.

$P(-1, 1, 1) \implies 1 = f(f(-1) + 2f(1)) = f(f(1))$ and so, since injective, $f(-1) + 2f(1) = f(1)$ and so $f(1) + f(-1) = 0$
$P(0, -1, 1) \implies -f(1) + 1 = f(f(-1) + f(1)) = 0$ and so $f(1) = 1$

$P(0, x, 1) \implies x + 1 = f(f(x) + 1) = f(f(x+1))$ and so, since injective, $f(x+1) = f(x) + 1$
And so $f(x+n) = f(x) + n$ and $f(n) = n \, \forall x, \forall n \in \mathbb{Z}$

Let then $p, q \in \mathbb{Z}$ with $q \neq 0$:
$P(0, f(\frac{p}{q}), q) \implies qf(\frac{p}{q}) + q = f(p+q) = p + q$ and so $f(\frac{p}{q}) = \frac{p}{q}$

So $\boxed{f(x) = x} \, \forall x \in \mathbb{Q}$ which indeed is a solution.

USB
5 posts

May 31, 2011, 12:03 am PM #455

Problem 147: Find all such functions $f : \mathbb{R} \to \mathbb{R}$ such that:
$f(x + y + f(y)) = f(f(x)) + 2y$ for all real x, y

---

pco
12955 po...

May 31, 2011, 12:11 am PM #456

> *USB wrote:*
> **Problem 147:** Find all such functions $f : \mathbb{R} \to \mathbb{R}$ such that:
> $f(x + y + f(y)) = f(f(x)) + 2y$ for all real x, y

Let $P(x, y)$ be the assertion $f(x + y + f(y)) = f(f(x)) + 2y$

If $f(a) = f(b) = c$ for some a, b, then :
$P(a, b) \implies f(a + b + c) = f(c) + 2b$
$P(b, a) \implies f(b + a + c) = f(c) + 2a$
And so $a = b$ and $f(x)$ is injective.

Then $P(x, 0) \implies f(x + f(0)) = f(f(x))$ and so, since injective : $f(x) = x + f(0)$ which indeed is a solution whatever is $f(0)$

Hence the answer : $\boxed{f(x) = x + a} \, \forall x$ and for any real a

---

filipbitola
124 posts

May 31, 2011, 1:02 am PM #457

Problem 148:
Macedonian IMO TST 2011:
Find all functions $f : \mathbb{R} \in \mathbb{R}$
$f(x + zf(yz)) = f(x) + yf(z^2) \, \forall x, y \in \mathbb{R}$

---

mahanmath
1356 pos...

May 31, 2011, 3:19 am PM #458

> *filipbitola wrote:*

> **Problem 148:**
> Macedonian IMO TST 2011:
> Find all functions $f:\mathbb{R}\in\mathbb{R}$
> $f(x+zf(yz))=f(x)+yf(z^2)\,\forall x,y\in\mathbb{R}$

We ignore the constant function which is zero function .

If $f(0)\neq 0$ put $y=0$ to get $f(x+zf(0))=f(x)$ which is contradiction because we assumed f is non-constant .
So $f(0)=0$

If $f(z^2)=0$ for all z and $f(a)\neq 0$ then let $z=\frac{a}{y}$ then equation becomes

$f(x+\frac{a}{y}f(a))=f(x)$ so as y can run over $\mathbb{R}$ we conclude f is constant.

So there exist z_0 such that $f({z_0}^2)\neq 0$, now (x,y,z_0) tells us f is **surjective**.
(Again because y can run over $\mathbb{R}$)

Put $z=1$ we have $f(x+f(y))=f(x)+yf(1)$

because of surjectivety $f(1)\neq 0$ and equation implies f is **injective** .

Finally put $(x,y,z)=(0,1,z)$ to get $f(zf(z))=f(z^2)$. But f is injective so $zf(z)=z^2$ or $f(x)=x$.

---

iamnot
61 posts

May 31, 2011, 4:48 am PM #459

> *filipbitola wrote:*
> **Problem 148:**
> Macedonian IMO TST 2011:
> Find all functions $f:\mathbb{R}\in\mathbb{R}$
> $f(x+zf(yz))=f(x)+yf(z^2)\,\forall x,y\in\mathbb{R}$

I'll suggest a little bit easier solution than mahanmath did.

Let $P(x,y,z)$ be the assertion $f(x+zf(yz))=f(x)+yf(z^2)$.

$P(x,1,0)\implies f(0)=0.$

$P(z^2,-1,-z)\implies f(z^2-zf(z))=0.$

So, either $z^2-zf(z)=0\,\forall z\implies f(z)=z$, or there exists $a\neq 0$ such that $f(a)=0.$

$P(x,\frac{a}{z},-z)\implies \frac{a}{z}f(z^2)=0$ and so $f(z^2)=0.$

$P(x,y,z)$ now looks like $f(x+zf(yz))=f(x).$

So, either $f(x)=0\,\forall x$, or there exists b such that $f(b)\neq 0.$

$P(0,f(b),\frac{b}{f(b)})\implies f(b)=0,$ which is a contradiction.

Hence, the answer is $f(x)=x$ or $f(x)=0.$

---

iamnot
61 posts

Jun 1, 2011, 2:03 am PM #460

Problem 149: Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:
$f(x+y)f(x-y)=f^2(x)$ for all real $x,y.$

I'm not sure if there should be some additional terms like "f is continuous".

RSM
736 posts

Jun 1, 2011, 10:44 am PM #461

> *iamnot wrote:*
> **Problem 149:** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:
> $f(x+y)f(x-y)=f^2(x)$ for all real $x,y.$
>
> I'm not sure if there should be some additional terms like "f is continuous".

Consider any two reals $m,n.$
$f(m)f(n)=f^2(\frac{m+n}{2})$
So $f(m)$ and $f(n)$ have same sign $\forall m,n$
If some $f(x)$ is 0 then $f(x)f(2y-x)=f^2(y)$'
so $f(y)$ will be $0\forall y.$
Take $|f(x)|=g(x)$ and $g(x)>0\forall x$
$\ln\frac{g(x)}{g(0)}=h(x)$

So $h(x+y)+h(x-y)=2h(x)$ and $h(0)=0$
Putting $x=0$ gives $h(-x)=-h(x)$
Interchanging x,y gives $h(x-y)=h(x)-h(y)$
So h is cauchy's function.
So $f(x)=c.e^{h(x)}$ where h is cauchy's function.

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filipbitola
124 posts

Jun 8, 2011, 1:39 am PM #462

Problem 150:(yaay)
Find all functions $\mathbb{R}\to\mathbb{R}$ such that $\forall x,y\in\mathbb{R}$
$(x+y)(f(x)-f(y))=(x-y)f(x+y)$

---

socrates
1818 posts

Jun 8, 2011, 3:38 am PM #463

> *filipbitola wrote:*
> **Problem 150:**(yaay)
> Find all functions $\mathbb{R}\to\mathbb{R}$ such that $\forall x,y\in\mathbb{R}$
> $(x+y)(f(x)-f(y))=(x-y)f(x+y)$

See http://www.artofproblemsolving.com/Forum/viewtopic.php?t=39207

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r1234
462 posts

Jun 9, 2011, 3:22 pm PM #464

Problem no.151
Suppose a function $f:\mathbb{R}\to\mathbb{R}$ satisfies the conditions
$f(x+19)\le f(x)+19$ and $f(x+94)\ge f(x)+94$ for all $x\in\mathbb{R}$.Prove that $f(x+1)=f(x)+1$ for all $x\in\mathbb{R}$

---

pco
12955 po...

Jun 9, 2011, 3:48 pm PM #465

> *r1234 wrote:*
> **Problem no.151**
> Suppose a function $f:\mathbb{R}\to\mathbb{R}$ satisfies the conditions
> $f(x+19)\le f(x)+19$ and $f(x+94)\ge f(x)+94$ for all $x\in\mathbb{R}$.Prove that $f(x+1)=f(x)+1$ for all $x\in\mathbb{R}$

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=37&t=6517

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r1234
462 posts

Jun 9, 2011, 4:46 pm PM #466

My solution
First we notice that $f(x-19)\ge f(x)-19$ and $f(x)-94\le f(x)-94$ for all real x.
Now by induction we get $f(x-19n)\ge f(x)-19n$ and $f(x-94n)\le f(x)-94n$. Similarly we write $f(x+190\le f(x)+19n$ and $f(x+94n)\ge f(x)+94n$.Now we notice that $1=5.19-94$ and $1=18.94-84.19$(this can be found from the linear diophantine equation $94x\equiv\1(mod 19)$).
SO from the first representation of 1 its easy to check that $f(x+1)\le f(x)+1$ and from the second representation we will get $f(x)\ge f(x)+1$. Hence we conclude that $f(x+1)=f(x)+1$.

Problem no152
For which integers k there exists a function $f:\mathbb{N}\to\mathbb{Z}$ which satisfies
$a)$ $f(1995)=1996$
$b)$ $f(xy)=f(x)+f(y)+kf(gcd(x,y))$, for all $x,y\in\mathbb{N}$?

---

pco
12955 po...

Jun 9, 2011, 5:00 pm PM #467

> *r1234 wrote:*
> **Problem no152**
> For which integers k there exists a function $f:\mathbb{N}\to\mathbb{Z}$ which satisfies
> $a)$ $f(1995)=1996$
> $b)$ $f(xy)=f(x)+f(y)+kf(gcd(x,y))$, for all $x,y\in\mathbb{N}$?

See http://www.artofproblemsolving.com/Forum/viewtopic.php?search_id=943898879&t=277446

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socrates
1818 posts

Jun 9, 2011, 7:49 pm PM #468

Problem 153

Determine all functions $f:\mathbb{R}_{\geq 0}\rightarrow\mathbb{R}_{\geq 0}$ such that $f(1)=\frac{1}{2}$ and $f(y\cdot f(x))\cdot f(x)=f(x+y)$, for all $x,y\geq 0$.

(Germany 2008: http://www.mathematik-olympiaden.de/aufgaben/47/4/A47134a.pdf)

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pco
12955 po...

Jun 9, 2011, 9:22 pm PM #469

> *socrates wrote:*
> **Problem 153**
>
> Determine all functions $f:\mathbb{R}_{\geq 0}\rightarrow\mathbb{R}_{\geq 0}$ such that $f(1)=\frac{1}{2}$ and $f(y\cdot f(x))\cdot f(x)=f(x+y)$, for all $x,y\geq 0$.
>
> (Germany 2008: http://www.mathematik-olympiaden.de/aufgaben/47/4/A47134a.pdf)

See http://www.artofproblemsolving.com/Forum/viewtopic.php?p=2260575#p2260575

Adding the constraint $f(1)=\frac{1}{2}$, the only solution is S3 with $a=1$ and so $\boxed{f(x)=\frac{1}{x+1}}$

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arshakus
746 posts

Jun 9, 2011, 10:23 pm — PM #470

Problem 154.

$f^2(x+y)) \geq f^2(x)+f^2(y)+2f(xy)\ f(1)>0$ does there exists such a function f that for all $x \in \mathbb{R}$ $a \leq f(x) \leq b$ the equality can not to hold.

---

mahanmath
1356 pos...

Jun 9, 2011, 10:30 pm — PM #471

> *arshakus wrote:*
> Problem 154.
>
> $f^2(x+y)) \geq f^2(x)+f^2(y)+2f(xy)\ f(1)>0$ does there exists such a function f that for all $x \in \mathbb{R}$ $a \leq f(x) \leq b$ the equality can not to hold.

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=220215&sid=c2973729775dcc9ff0121fe6ef98baa4#p220215

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socrates
1818 posts

Jun 9, 2011, 10:49 pm — PM #472

Problem 155

Determine all injective functions $f:\mathbb{N}^*\to\mathbb{N}$ such that $f\left(C_n^m\right)=C_{f(n)}^{f(m)}$, for all $m,n\in\mathbb{N}^*, n\geq m$,

where $C_n^m=\binom{n}{m}$.

---

pco
12955 po...

Jun 9, 2011, 11:14 pm — PM #473

> *socrates wrote:*
> **Problem 155**
>
> Determine all injective functions $f:\mathbb{N}^*\to\mathbb{N}$ such that $f\left(C_n^m\right)=C_{f(n)}^{f(m)}$, for all $m,n\in\mathbb{N}^*, n\geq m$,
>
> where $C_n^m=\binom{n}{m}$.

If $f(1)\neq 1$, then $f(n)=f(\binom{n}{1})=\binom{f(n)}{f(1)}$ implies $f(1)=f(n)-1$ which is impossible for any n since $f(x)$ is injective.

So $f(1)=1$
Let then $n>2: f(n)=f(\binom{n}{n-1})=\binom{f(n)}{f(n-1)}$ and so either $f(n-1)=1$, impossible since injective, either $f(n-1)=f(n)-1$
So $f(n)=f(n-1)+1$ and we get $f(n)=n+c\ \forall n>1$ where $c=f(2)-2$

Using then $f(\binom{4}{2})=\binom{f(4)}{f(2)}$, we get $f(6)=\binom{c+4}{c+2}$ and so $c+6=\frac{(c+4)(c+3)}{2}$ which gives $c\in\{-5,0\}$ and so $c=0$

Hence the unique solution $\boxed{f(n)=n}$ $\forall n$, which indeed is a solution.

---

aktyw19
1315 posts

Jun 9, 2011, 11:29 pm — PM #474

Problem 156: Find all functions $f:(1,+\infty)\to R$, such that:
$f(x)-f(y)=(y-x)f(xy), \forall x,y>1$

---

socrates
1818 posts

Jun 9, 2011, 11:45 pm — PM #475

> *aktyw19 wrote:*
> **Problem 156:** Find all functions $f:(1,+\infty)\to R$, such that:
> $f(x)-f(y)=(y-x)f(xy), \forall x,y>1$

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=151&t=384559

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aktyw19
1315 posts

Jun 9, 2011, 11:59 pm — PM #476

Problem 157 Find all functions $f:R\to R$ such that:
$f(x-f(y))=f(f(y))+xf(y)+f(x)-1, \forall x,y\in R$

goodar20... 1346 pos... | Jun 10, 2011, 12:14 am | PM #477

IMO 1999 P6. here.

---

mahanmath 1356 pos... | Jun 10, 2011, 12:42 am • 1 | PM #478

Problem 158
own

Show that there are infinitely functions from $\mathbb{Q}^+$ to itself such that

$x+y+z+2=xyz$ iff $f(x)+f(y)+f(z)=1$

---

ashkan31... 69 posts | Jun 10, 2011, 4:49 pm | PM #479

> *mahanmath wrote:*
> **Problem 158**
> *own*
>
> Show that there are infinitely functions from $\mathbb{Q}^+$ to itself such that
>
> $x+y+z+2=xyz$ iff $f(x)+f(y)+f(z)=1$

doesn't it work
x=(a+b)/c y=(b+c)/a z=(a+c)/b

---

pco 12955 po... | Jun 10, 2011, 7:09 pm • 1 | PM #480

> *mahanmath wrote:*
> **Problem 158**
> *own*
>
> Show that there are infinitely functions from $\mathbb{Q}^+$ to itself such that
>
> $x+y+z+2=xyz$ iff $f(x)+f(y)+f(z)=1$

Choose for example $f(x)=\frac{ax+1-2a}{x+1}$ for any $a\in[0,\frac{1}{3})\cup(\frac{1}{3},\frac{1}{2}]$

---

jatin 547 posts | Jun 11, 2011, 2:23 pm | PM #481

Problem 159:
Find all $f:\mathbb{R}\to\mathbb{R}$, such that $\forall x,y\in\mathbb{R}$

$$|f(x)-f(y)|\le(x-y)^2$$

---

pco 12955 po... | Jun 11, 2011, 2:30 pm | PM #482

> *jatin wrote:*
> Problem 159:
> Find all $f:\mathbb{R}\to\mathbb{R}$, such that $\forall x,y\in\mathbb{R}$
>
> $$|f(x)-f(y)|\le(x-y)^2$$

$\left|\frac{f(x)-f(y)}{x-y}\right|\le|x-y|$ and so $f(x)$ is continuous and differentiable over $\mathbb{R}$ and $f'(x)=0$

Hence the solutions : $\boxed{f(x)=c}$ $\forall x\in\mathbb{R}$ and for any real c and it's immediate to check back that these functions indeed are solutions.

---

r1234 462 posts | Jun 11, 2011, 2:43 pm | PM #483

Problem No.160
Does there exist a function $f:\mathbb{R}\to\mathbb{R}$ such that $f(f(x))=x^2-2$ for all real x?

Amir Hos... 4728 pos... | Jun 11, 2011, 2:44 pm | PM #484

This may be a little harder than an Olympiad problem:

Problem 161. Let $f:[0,+\infty)\to\mathbb{R}$ be a continuous and strictly increasing function, twice differentiable in 0 with $f(0)=f'(0)=0$, $f''(0)>0$, and $\lim_{x\to\infty}f(x)=\infty$. Let $a>0$ be a real number. Show that for all positive integers n, the equation

$$\sum_{k=1}^{n}f\left(\frac{kx}{n^2}\right)=a$$

has a unique solution x_n in the interval $(0,\infty)$, and

$$\lim_{n\to\infty}\frac{x_n}{\sqrt{n}}=\sqrt{\frac{6a}{f''(0)}}.$$

EDIT. Ahh, someone else has posted 160.

---

pco
12955 po...

Jun 11, 2011, 2:49 pm PM #485

> *r1234 wrote:*
> **Problem No.160**
> Does there exist a function $f:\mathbb{R}\to\mathbb{R}$ such that $f(f(x))=x^2-2$ for all real x?

See http://www.imomath.com/tekstkut/funeqn_mr.pdf , Problem 7

---

pco
12955 po...

Jun 11, 2011, 4:13 pm PM #486

Solution to Problem #161

Amir Hos...
4728 pos...

Jun 11, 2011, 5:19 pm PM #487

This is the generalization of the problem:

Problem 162. Let $f:[0,+\infty)\to\mathbb{R}$ be a continuous and strictly increasing function, m-times differentiable in 0 with $f(0)=f'(0)=\cdots=f^{(m-1)}(0)=0$, $f(m)(0)>0$, and $\lim_{x\to\infty}f(x)=\infty$. Let $a>0$ be a real number. Show that for all positive integers n, the equation

$$\sum_{k=1}^{n}f\left(\frac{kx}{n^2}\right)=a$$

has a unique solution x_n in the interval $(0,\infty)$, and

$$\lim_{n\to\infty}\frac{x_n}{\sqrt[m]{n^{m-1}}}=\sqrt[m]{\frac{(m+1)!\cdot a}{f^{(m)}(0)}}.$$

Anyways, I don't know if this problem belongs to this marathon, so I'm posting another new problem:

Problem 163. Prove that there does not exist a rational function R with real coefficients such that

$$R(n)=\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}$$

for an infinitely of positive integers n.

---

jatin
547 posts

Jun 12, 2011, 4:39 pm PM #488

> *pco wrote:*
>
> > *jatin wrote:*
> > Problem 159:
> > Find all $f:\mathbb{R}\to\mathbb{R}$, such that $\forall x,y\in\mathbb{R}$
> > $$|f(x)-f(y)|\le(x-y)^2$$
>
> $\left|\frac{f(x)-f(y)}{x-y}\right|\le|x-y|$ and so $f(x)$ is continuous and differentiable over $\mathbb{R}$ and $f'(x)=0$
>
> Hence the solutions : $\boxed{f(x)=c}$ $\forall x\in\mathbb{R}$ and for any real c and it's immediate to check back that these functions indeed are solutions.

Can you please explain?

---

pco
12955 po...

Jun 12, 2011, 4:59 pm PM #489

> *jatin wrote:*
> Can you please explain?

What is the first step you dont understand ?

---

jatin
547 posts

Jun 12, 2011, 5:03 pm PM #490

$\left|\frac{f(x)-f(y)}{x-y}\right|\le|x-y|$

How does this imply that f is continuous and differentiable over $\mathbb{R}$?

I'm a novice at calculus

pco
12955 po...

Jun 12, 2011, 5:09 pm PM #491

> *jatin wrote:*
> $\left|\frac{f(x)-f(y)}{x-y}\right| \leq |x-y|$
>
> How does this imply that f is continuous and differentiable over $\mathbb{R}$?
>
> I'm a novice at calculus

Setting $y \to x$ in $|f(x)-f(y)| \leq |x-y|^2$, we find that $\lim_{y\to x}(f(y)-f(x))=0$ and so $f(x)$ is continuous at x

Setting $y \to x$ in $\left|\frac{f(y)-f(x)}{y-x}\right| \leq |x-y|$, we get that $\lim_{y\to x}\left|\frac{f(y)-f(x)}{y-x}\right|=0$ and so the derivative of $f(x)$ at x exists and is 0

jatin
547 posts

Jun 12, 2011, 5:12 pm PM #492

Now I understand it. Thank you very much

arshakus
746 posts

Jun 14, 2011, 4:05 pm PM #493

Find all function f.
$f: N->N$
$af(a)+bf(b)=(a+b)f(a^2+b^2)$

goodar20...
1346 pos...

Jun 14, 2011, 4:13 pm PM #494

if there are two a and b such that $f(a)<f(b)$, then $f(a)<f(a^2+b^2)<f(b)$ and so on. this is a contradiction, so the function is constant.

Amir Hos...
4728 pos...

Jun 14, 2011, 4:17 pm PM #495

> *amparvardi wrote:*
> **Problem 163.** Prove that there does not exist a rational function R with real coefficients such that
> $$R(n)=\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}$$
> for an infinitely of positive integers n.

arshakus
746 posts

Jun 14, 2011, 4:36 pm PM #496

> *goodar2006 wrote:*
> $f(a)<f(a^2+b^2)<f(b)$ and so on.

why?

goodar20...
1346 pos...

Jun 14, 2011, 4:39 pm • 1 PM #497

because of the problems condition:$af(a)+bf(b)=(a+b)f(a^2+b^2)$

arshakus
746 posts

Jun 14, 2011, 4:43 pm PM #498

> *goodar2006 wrote:*
> because of the problems condition:$af(a)+bf(b)=(a+b)f(a^2+b^2)$

ok I understand it!) but why it is contradiction?

goodar20...
1346 pos...

Jun 14, 2011, 4:53 pm • 2 PM #499

we have $f(a)<f(a^2+b^2)<f(b)$. now since $f(a)<f(a^2+b^2)$, we can do the same for it, and it implies that between $f(a)$ and $f(b)$ there are infinitely many integers, this is the contradiction.

arshakus
746 posts

Jun 14, 2011, 4:55 pm • 1 PM #500

> *goodar2006 wrote:*
> we have $f(a)<f(a^2+b^2)<f(b)$. now since $f(a)<f(a^2+b^2)$, we can do the same for it, and it implies that between $f(a)$ and $f(b)$ there are infinitely many integers, this is the contradiction.

ok thank you I understand!)
I have rated you 6!)

khaitang
168 posts

Jul 5, 2011, 8:02 am PM #501

Problem 165: Find all $f:\mathbb{R}\to\mathbb{R}$ such that:
$x^3-f(y)^3=(x-y)(f(f(x)^2)+f(x)y+yf(y))$

BigSams
1582 pos...

Jul 5, 2011, 9:52 am PM #502

> *khaitang wrote:*
> Problem 165: Find all $f:\mathbb{R}\to\mathbb{R}$ such that:
> $$x^3-f(y)^3=(x-y)(f(f(x)^2)+f(x)y+yf(y))$$

Let $x=y$.
$\implies x^3-[f(x)]^3=0$
$\implies [x-f(x)][x^2+[f(x)]^2+xf(x)]=0$
$\implies [x-f(x)][x^2+[f(x)]^2+[x+f(x)]^2]=0\,\forall x\in\mathbb{R}.$

Either $f(x)=x$ or $\{f(x)=0, x=0\}$ or both. The latter is impossible as it restricts the domain to $x=0$.
The former agrees with the given identity.
Thus, the only solution is $\boxed{f(x)=x}$. $\blacksquare$

This post has been edited 1 time. Last edited by BigSams, Jul 5, 2011, 9:54 am

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arshakus
746 posts

Jul 5, 2011, 9:54 am PM #503

> *khaitang wrote:*
> Problem 165: Find all $f:\mathbb{R}\to\mathbb{R}$ such that:
> $x^3-f(y)^3=(x-y)(f(f(x)^2)+f(x)y+yf(y))$

it is very easy problem by one step anyway here is solution)
$x=y=>x^3-f^3(x)=0=>f(x)=x$:)

---

khaitang
168 posts

Jul 5, 2011, 1:14 pm PM #504

Problem 166: Find all $f:\mathbb{R}\to\mathbb{R}$ such that:
$f(x^5-y^5)=x^2f(x^3)-y^2f(y^3)$

---

pco
12955 po...

Jul 5, 2011, 1:44 pm • 1 PM #505

> *khaitang wrote:*
> Problem 166: Find all $f:\mathbb{R}\to\mathbb{R}$ such that:
> $f(x^5-y^5)=x^2f(x^3)-y^2f(y^3)$

Let $P(x,y)$ be the assertion $f(x^5-y^5)=x^2f(x^3)-y^2f(y^3)$

$P(0,0)\implies f(0)=0$
$P(x,0)\implies f(x^5)=x^2f(x^3)$
$P(0,x)\implies f(-x^5)=-x^2f(x^3)=-f(x^5)$ and so $f(x)$ is an odd function.

So $P(x,-y)\implies f(x^5+y^5)=f(x^5)+f(y^5)$ and so $f(x+y)=f(x)+f(y)\,\forall x,y$ and so $f(qx)=qf(x)$ $\forall q\in\mathbb{Q}$

Writing then $P(x+q,0)$, we get $f(x^5+5qx^4+10q^2x^3+10q^3x^2+5q^4x+q^5)=$ $(x^2+2qx+q^2)f(x^3+3qx^2+3q^2x+q^3)$

So $f(x^5)+5qf(x^4)+10q^2f(x^3)+10q^3f(x^2)+5q^4f(x)+q^5f(1)-$ $(x^2+2qx+q^2)(f(x^3)+3qf(x^2)+3q^2f(x)+q^3f(1))=0$

This is a polynomial in q which is zero for any $q\in\mathbb{Q}$. So this is the allzero polynomial and all its coefficients are zero.

Looking at coefficient of q^4, we get then $5f(x)-3f(x)-2xf(1)=0$ and so $f(x)=xf(1)\,\forall x$

Hence the solution : $\boxed{f(x)=ax}\,\forall x$ and for any $a\in\mathbb{R}$, which indeed is a solution.

---

jatin
547 posts

Jul 5, 2011, 1:57 pm PM #506

Problem 167
Find all $f:\mathbb{R}\to\mathbb{R}$, such that:
$f(xf(y))=yf(x)$, $lim_{x\to+\infty}f(x)=0$.

---

pco
12955 po...

Jul 5, 2011, 2:17 pm PM #507

> *jatin wrote:*
> **Problem 167**
> Find all $f:\mathbb{R}\to\mathbb{R}$, such that:
> $f(xf(y))=yf(x)$, $lim_{x\to+\infty}f(x)=0$.

$f(x)=0\,\forall x$ is a solution. So let us from now look for non allzero solutions.
Let $P(x,y)$ be the assertion $f(xf(y))=yf(x)$
Let u such that $f(u)\neq 0$

$P(0,0)\implies f(0)=0$ and so $u\neq 0$

$P(0,0) \implies f(0) = 0$ and so $a \neq 0$
$P(u,x) \implies f(uf(x)) = xf(u)$ and so $f(x)$ is a bijection
$P(1,1) \implies f(f(1)) = f(1)$ and, since injective, $f(1) = 1$
$P(1,x) \implies f(f(x)) = x$
$P(-1, f(-1)) \implies 1 = f(-1)^2$ and so $f(-1) = -1$ (since injective)

$P(x, f(y)) \implies f(xy) = f(x)f(y)$
So $f(x) > 0\ \forall x > 0$
Setting then $f(x) = e^{h(\ln x)}$ for $x > 0$, we get $h(x+y) = h(x) + h(y)$ and $\lim_{x\to+\infty} h(x) = -\infty$
So $h(x)$ is a solution of Cauchy equation which is upper bounded from a given point, and so $h(x) = cx$ with $c < 0$

So $f(x) = x^c\ \forall x > 0$ and then $f(f(x)) = x$ implies $c = -1$

Hence the solutions (which indeed are solutions) :
$f(x) = 0\ \forall x$
$f(0) = 0$ and $f(x) = \frac{1}{x}\ \forall x \neq 0$

---

jatin
547 posts

Jul 5, 2011, 2:33 pm PM #508

Nice solution, Patrick :)

Problem 168
Find all $f:\mathbb{R}\to\mathbb{R}$, such that:
$f(x+y) = \frac{f(x)+f(y)}{1+f(x)f(y)}$ and f is continuous.

---

pco
12955 po...

Jul 5, 2011, 2:56 pm PM #509

> *jatin wrote:*
> Nice solution, Patrick :)

Thanks :)

> *jatin wrote:*
> **Problem 168**
> Find all $f:\mathbb{R}\to\mathbb{R}$, such that:
> $f(x+y) = \frac{f(x)+f(y)}{1+f(x)f(y)}$ and f is continuous.

Let $P(x,y)$ be the assertion $f(x+y) = \frac{f(x)+f(y)}{1+f(x)f(y)}$

$P(x,x) \implies f(2x)(1+f(x)^2) = 2f(x)$ and so :
either $f(2x) = 0$, either $f(x)^2 - \frac{2}{f(2x)}f(x) + 1 = 0$ and so the discriminant of the quadratic must be ≥ 0 :
$|f(2x)| \leq 1$

So $|f(x)| \leq 1$.

If $f(u) = +1$ for some u : $P(x-u,u) \implies f(x) = 1\ \forall x$ and we got a solution
If $f(u) = -1$ for some u : $P(x-u,u) \implies f(x) = -1\ \forall x$ and we got another solution
If $|f(x)| < 1\ \forall x$, let then $g(x) = \ln(1+f(x)) - \ln(1-f(x))$

$g(x)$ is continuous and $f(x) = \frac{e^{g(x)}-1}{e^{g(x)}+1}$

$P(x,y)$ becomes then $\frac{e^{g(x+y)}-1}{e^{g(x+y)}+1} = \frac{e^{g(x)+g(y)}-1}{e^{g(x)+g(y)}+1}$ and so $g(x+y) = g(x)+g(y)$

And since $g(x)$ is continuous, we get $g(x) = ax$

Hence the solutions (which indeed are solutions) :
$f(x) = -1\ \forall x$

$f(x) = +1\ \forall x$

$f(x) = \frac{e^{ax}-1}{e^{ax}+1}\ \forall a$ (notice that $a = 0$ gives the solution $f(x) = 0\ \forall x$

---

khaitang
168 posts

Jul 5, 2011, 4:03 pm PM #510

Problem 169
Find all $f:\mathbb{R}\to\mathbb{R}$, such that:
$xf(x) - yf(y) = (x-y)f(x+y)$

---

filipbitola
124 posts

Jul 5, 2011, 4:23 pm PM #511

Let $P(x,y)$ be the assertion $xf(x) - yf(y) = (x-y)f(x+y)$

$P(x,-y) \implies\ f(x) + f(-x) = 2f(0)$

Let $g(x) = f(x) - f(0)$

Substituting in the original equation we get that:

$xg(x) - yg(y) = (x - y)g(x + y)$

And from $f(x) + f(-x) = 2f(0)$, we have:
$g(x) + g(-x) = 0$ since $g(0) = 0$, and $g(-x) = -g(x)$

Let $Q(x, y)$ be the assertion $xg(x) - yg(y) = (x - y)g(x + y)$

$Q(x, -y) \implies xg(x) - yg(y) = (x + y)g(x - y)$

Comparing $Q(x, y)$ and $Q(x, -y)$ we get that:

$\frac{g(x+y)}{x+y} = \frac{g(x-y)}{x-y} \forall x, y \in \mathbb{R}, x^2 \neq y^2$

From here, setting $x = \frac{u+v}{2}, y = \frac{u-v}{2}$

$\frac{g(u)}{u} = \frac{v}{v} \forall u, v \in \mathbb{R}, uv \neq 0$

Hence, $g(x) = cx$, and $\boxed{f(x) = cx + b}, b = f(0)$

Checking in the original equation we see that it's a solution.

---

khaitang *168 posts*

Jul 6, 2011, 6:54 am PM #512

Problem 170 Find all function $f : \mathbb{R} \to \mathbb{R}$ such that:
$(f(x + y))^2 = f(x)f(x + 2y) + yf(y)$

---

filipbitola *124 posts*

Jul 6, 2011, 7:29 am PM #513

In my solution, $f(x)^2$ means $f(x) * f(x)$
Let $P(x, y)$ be the assertion $f(x + y)^2 = f(x)f(x + 2y) + yf(y)$
$P(x, -x) \implies f(0)^2 = f(x)f(-x) - xf(-x)$
$P(-x, x) \implies f(0)^2 = f(x)f(-x) + xf(x)$
Comparing these two lines:
$-f(x) = f(-x) \forall x \in \mathbb{R}, x \neq 0$

Now,
$P(x, -x) \implies f(0)^2 = -f(x)^2 + xf(x)$
$P(-2x, x) \implies f(-x)^2 = f(-2x)f(0) + xf(x)$ and from the previous line $f(2x)f(0) = f(0)^2$

We consider cases:

$i) f(0) \neq 0 \implies f$ is constant.
Let $f \equiv c$
Checking in the original equation we get:
$c^2 = c^2 + yc \implies yc = 0 \forall y \in \mathbb{R} \implies c = 0$, contradiction with the assumption.

$ii) f(0) = 0$
$P(0, x) \implies f(x)^2 = xf(x)$

This produces 3 cases:

$a) f \equiv 0$ which is a solution

$b) f(x) = x \forall x \in \mathbb{R}$ which is also a solution

$c) f(a) = 0, f(b) = b, a \neq b \neq 0$
$P(a, b) \implies f(a + b)^2 = b^2$
$P(b, a) \implies f(b + a)^2 = bf(2a + b)$

Note that $f(p) = q \implies p = q$ if $q \neq 0$
Comparing these lines: $f(2a + b) = b \implies 2a + b = b \implies a = 0$, contradiction.
Hence, the only solutions are $\boxed{f(x) = 0}$ and $\boxed{f(x) = x} \forall x \in \mathbb{R}$

---

khaitang *168 posts*

Jul 6, 2011, 1:08 pm PM #514

Problem 171 . Find all function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that:
$f(x + f(y)) = f(x - f(y)) + 4xf(y)$

---

pco *12955 po...*

Jul 6, 2011, 2:10 pm PM #515

> *khaitang wrote:*
> **Problem 171** . Find all function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that:
> $f(x + f(y)) = f(x - f(y)) + 4xf(y)$

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=415924

filipbitola
124 posts

Jul 6, 2011, 7:52 pm PM #516

Problem 172:
Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $\forall x,y\in\mathbb{R}$
$f(xf(y)+x)=xy+f(x)$

pco
12955 po...

Jul 6, 2011, 8:08 pm PM #517

> *filipbitola wrote:*
> **Problem 172:**
> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $\forall x,y\in\mathbb{R}$
> $f(xf(y)+x)=xy+f(x)$

Let $P(x,y)$ be the assertion $f(xf(y)+x)=xy+f(x)$

$P(1,-1-f(1))\implies f(f(-1-f(1))+1)=-1$ and let $u=f(-1-f(1))+1$ such that $f(u)=-1$

$P(x,u)\implies f(0)=ux+f(x)$ and so $f(x)=-ux+f(0)$

Plugging this in original equation, **we find two solutions** :
$f(x)=x\ \forall x$
$f(x)=-x\ \forall x$

proglote
940 posts

Jul 7, 2011, 2:12 am PM #518

Problem 173:

Show that for all integers $a,b>1$ there is a function $f:\mathbb{Z}_+^*\to\mathbb{Z}_+^*$ such that $f(a\cdot f(n))=b\cdot n$ for all positive integer n.

pco
12955 po...

Jul 7, 2011, 11:25 am PM #519

> *proglote wrote:*
> **Problem 173:**
>
> Show that for all integers $a,b>1$ there is a function $f:\mathbb{Z}_+^*\to\mathbb{Z}_+^*$ such that $f(a\cdot f(n))=b\cdot n$ for all positive integer n.

Consider the three sets :
$U_a=\mathbb{N}\setminus a\mathbb{N}$: the set of all positive integers not divisible by a
$U_b=\mathbb{N}\setminus b\mathbb{N}$: the set of all positive integers not divisible by b
$V=a\mathbb{N}\setminus ab\mathbb{N}$: the set of all positive integers divisible by a and not divisible by ab

U_a and U_b both are infinite countable (since $a,b>1$) and so $\exists$ a bijection $u(n)$ from $U_a\to U_b$

Define then $f(n)$ as :
$\forall n\in U_a$: $f(n)=u(n)$
$\forall n\in V$: $f(n)=b\times u^{-1}(\frac{n}{a})$ (notice that $n\in V\implies a|n$ and $b\not|\frac{n}{a}$)
$\forall n\notin U_a\cup V$: $f(n)=ab\times f(\frac{n}{ab})$ (notice that $n\notin U_a\cup V\implies ab|n$)

Easy to check that this function matches all requirements.

khaitang
168 posts

Jul 7, 2011, 12:18 pm PM #520

P/S: pco
When you solve a problem, you should post a new one
Thanks 🙂

khaitang
168 posts

Jul 7, 2011, 12:23 pm PM #521

Problem 174 : Find all function $f:\mathbb{R}\to\mathbb{R}$ such that: $f(xf(y)+y)=\dfrac{f(2x)f(4x)}{8}.+f(y)$

This post has been edited 2 times. Last edited by khaitang, Jul 14, 2011, 12:35 pm

pco
12955 po...

Jul 7, 2011, 12:29 pm • 3 👍 PM #522

> *khaitang wrote:*
> P/S: pco
> **When you solve a problem, you should post a new one**
> Thanks 🙂

1) I'm neither a student, neither involved in education world and so I've no book / training sheets / ... giving existing olympiad problems.
2) Creating a brand new olympiad - level problem is a quite different job than solving a problem. I have some talent in solving problems and no talent at all in creating them.

So I never propose new problems and let anybody who wants it to propose a new ploblem at my turn in marathons.
Sorry for that.

Jul 7, 2011, 1:52 pm • 1 👍 PM #523

Amir Hos...
4728 pos...

> *khaitang wrote:*
> P/S: pco
> **When you solve a problem, you should post a new one**

Let him be an exception. 🙂

I hope this problem is new:

Problem 175. Find all functions $f:\mathbb{Q}^+\to\mathbb{Q}^+$ such that

$$f(x)+f(y)+2xyf(xy)=\frac{f(xy)}{f(x+y)}.$$

Where $\mathbb{Q}^+$ is the set of positive rational numbers.

So there are two pending problems: 174 and 175.

---

pco
12955 po...

Jul 7, 2011, 4:31 pm • 4 👍 PM #524

> *amparvardi wrote:*
> **Problem 175.** Find all functions $f:\mathbb{Q}^+\to\mathbb{Q}^+$ such that
> $$f(x)+f(y)+2xyf(xy)=\frac{f(xy)}{f(x+y)}.$$
> Where $\mathbb{Q}^+$ is the set of positive rational numbers.

Let $P(x,y)$ be the assertion $f(x)+f(y)+2xyf(xy)=\dfrac{f(xy)}{f(x+y)}$

$P(1,1) \implies f(2)=\dfrac{1}{4}$
$P(2,1) \implies f(3)=\dfrac{1}{5+4f(1)}$
$P(3,1) \implies f(4)=\dfrac{f(3)}{7f(3)+1}=\dfrac{1}{12+4f(1)}$
$P(2,2) \implies f(4)=\dfrac{1}{16}$

And so $f(1)=1$ and an easy induction using $P(x,1)$: $\dfrac{1}{f(x+1)}=\dfrac{1}{f(x)}+2x+1$ gives $\dfrac{1}{f(x+n)}=2nx+n^2+\dfrac{1}{f(x)}$

And $f(n)=\dfrac{1}{n^2}$

Then $P(\frac{p}{q},q) \implies f(\frac{p}{q})+f(q)+2pf(p)=\dfrac{f(p)}{f(\frac{p}{q}+q)}$

Which becomes, using $f(p)=\dfrac{1}{p^2}$ and $f(q)=\dfrac{1}{q^2}$ and $\dfrac{1}{f(x+q)}=2qx+q^2+\dfrac{1}{f(x)}$:

$p^2f(\frac{p}{q})^2+(\frac{p^2}{q^2}-q^2)f(\frac{p}{q})-1=0$ whose unique positive root is $f(\frac{p}{q})=\dfrac{q^2}{p^2}$

Hence the answer : $\boxed{f(x)=\dfrac{1}{x^2}}$ which indeed is a solution.

---

khaitang
168 posts

Jul 12, 2011, 6:43 am PM #525

Problem 176. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that: $f(xy)(f(2x)-2f(y))=(x-y)f(x)f(2y)$.

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proglote
940 posts

Jul 12, 2011, 8:06 am PM #526

> *khaitang wrote:*
> **Problem 176.** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that: $f(xy)(f(2x)-2f(y))=(x-y)f(x)f(2y)$.

Let $P(x,y)$ be the assertion $f(xy)(f(2x)-2f(y))=(x-y)f(x)f(2y)$

$P(x,x) \implies f(x^2)(f(2x)-2f(x))=0$

Case 1) $f(x) = 0 \ \ \forall x \in \mathbb{R}_+$

$P(x,1) \implies f(x)(f(2x)) = 0$ so $f(x) = 0 \forall x \in \mathbb{R}$

Case 2) $f(2x) = 2f(x) \ \ \forall x \in \mathbb{R}$

$P(1,y) \implies f(y)(f(1) - f(y)) = (1-y)f(1)f(y)$, so $f(y) = yf(1)$ or $f(y) = 0, \ \ \forall y \in \mathbb{R}$. But the first option implies f is bijective, so these two are mutually exclusive unless $y = 0$.

So we get the following solution: $f(x) = ax \ \ \forall a, x \in \mathbb{R}$ (Notice a can be zero). Check this into the original equation and see that this is indeed the solution.

---

proglote
940 posts

Jul 12, 2011, 8:09 am PM #527

Problem 177. Find all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all nonzero reals x and y:

$$f(x)f(y) - f(xy) = \frac{x}{y} + \frac{y}{x}.$$

---

filipbitola
124 posts

Jul 12, 2011, 8:10 am PM #528

> *proglote wrote:*
>
> > *khaitang wrote:*
> > **Problem 176.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that: $f(xy)(f(2x) - 2f(y)) = (x-y)f(x)f(2y)$.
>
> Let $P(x,y)$ be the assertion $f(xy)(f(2x) - 2f(y)) = (x-y)f(x)f(2y)$
>
> $P(x,x) \implies f(x^2)(f(2x) - 2f(x)) = 0$
>
> Case 1) $f(x) = 0 \ \ \forall x \in \mathbb{R}_+$
>
> $P(x,1) \implies f(x)(f(2x)) = 0$ so $f(x) = 0 \forall x \in \mathbb{R}$
>
> Case 2) $f(2x) = 2f(x) \ \ \forall x \in \mathbb{R}$
>
> $P(1,y) \implies f(y)(f(1) - f(y)) = (1-y)f(1)f(y)$, so $f(y) = yf(1)$ or $f(y) = 0, \ \ \forall y \in \mathbb{R}$. But the first option implies f is bijective, so these two are mutually exclusive unless $y = 0$.
>
> So we get the following solution: $f(x) = ax \ \ \forall a, x \in \mathbb{R}$ (Notice a can be zero). Check this into the original equation and see that this is indeed the solution.

What about Case 3($f(x) = 0$ for some x, and $f(2x) = 2f(x)$ for all others)?

---

filipbitola
124 posts

Jul 12, 2011, 8:17 am PM #529

> *proglote wrote:*
> **Problem 177.** Find all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all nonzero reals x and y:
>
> $$f(x)f(y) - f(xy) = \frac{x}{y} + \frac{y}{x}.$$

Let $P(x,y)$ be the assertion $f(x)f(y) - f(xy) = \dfrac{x}{y} + \dfrac{y}{x}$
$P(1,1) \implies f(1) = 2$
$P(x,1) \implies f(x) = x + \dfrac{1}{x}$
Checking in the original equation we see that it satisfies the equation.
So, $\boxed{f(x) = x + \dfrac{1}{x}}$ is the only solution.

---

proglote
940 posts

Jul 12, 2011, 9:55 am PM #530

> *filipbitola wrote:*
> What about Case 3($f(x) = 0$ for some x, and $f(2x) = 2f(x)$ for all others)?

Let A and B be two sets such that $A \cap B = \{0\}$ and $A \cup B = \mathbb{R}$, $f(x) = 0 \iff x \in A$ and $2f(x) = f(2x) \iff x \in B$.

$P(2,y) \iff f(2y)(f(4) - 2f(y)) = (2-y)f(2)f(2y)$. Suppose $2y \in B, y \neq 0$. Then $f(y) = \dfrac{yf(2)}{2} - f(2) + \dfrac{f(4)}{2}$.

$P(1,y) \iff f(y)(f(2) - 2f(y)) = (1-y)f(2)f(2y)$. Suppose $y \in B, y \neq 0$.

Then$f(2)-2f(y)=2(1-y)f(1) \implies f(y)=yf(1)-f(1)+\frac{f(2)}{2}$. If $y \in B$ and $2y \in B$ then joining the two equations we get $y(\frac{f(2)}{2}-f(1))+\frac{f(4)-3f(2)}{2}+f(1)=0$ for all y such that $y \in B$ and $2y \in B$ and $y \neq 0$. The only possibility then is $\frac{f(2)}{2}=f(1)$ and $\frac{f(4)}{2}=f(2)$ so $f(y)=yf(1)$.

I'm not sure how to proceed; this is really tiring.. any help would be appreciated.

---

khaitang
168 posts

Jul 12, 2011, 4:16 pm PM #531

Problem 178 Find all function $f:\mathbb{R}\to\mathbb{R}$ such that:
$f(x+f(y))=x+f(y)+xf(y)$

Farenhajt
5170 pos...

Jul 12, 2011, 5:06 pm • 1 PM #532

> khaitang wrote:
> **Problem 178** Find all function $f:\mathbb{R}\to\mathbb{R}$ such that:
> $f(x+f(y))=x+f(y)+xf(y)$

Putting $x=t-f(0), y=0$ the equation becomes

$f(t)=(1+f(0))t-f(0)^2$

Thus $f(0)=-f(0)^2 \iff f(0)\in\{0,-1\}$

If $f(0)=0$ then $f(t)=t$, but that doesn't satisfy the initial equation.

If $f(0)=-1$ then $f(t)=-1$ and that satisfies the initial equation.

Thus $f(x)\equiv -1$ is the only solution.

I have no problem to submit, anyone's free to go.

---

filipbitola
124 posts

Jul 12, 2011, 6:13 pm PM #533

> khaitang wrote:
> **Problem 178** Find all function $f:\mathbb{R}\to\mathbb{R}$ such that:
> $f(x+f(y))=x+f(y)+xf(y)$

Alternate solution:
Let $P(x,y)$ be the assertion $f(x+f(y))=x+f(y)+xf(y)$
$P(x,0) \implies f(x+f(0))=x(f(0)+1)+f(0)$
If $f(0)=-1 \implies f(x-1)=-1 \implies f\equiv -1$ which is a solution.
Now, assume $f(0)\neq -1$.
$P(\frac{x-f(0)}{f(0)+1},0)$ implies that f is surjective.
That means that $\exists x_0\in\mathbb{R}$ such that $f(x_0)=0$
$P(x,x_0) \implies f(x)=x$ which is not a solution.
Hence, the only solution is: $\boxed{f(x)=-1}$

Problem 179: Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that both conditions are satisfied.
a) $f(x+y)=f(x)+f(y)$
b) $f(xy)=f(x)f(y)$

SCP
1520 pos...

Jul 12, 2011, 6:32 pm PM #534

> filipbitola wrote:
> Let $P(x,y)$ be the assertion $f(x)f(y)-f(xy)=\frac{x}{y}+\frac{y}{x}$
> $P(1,1) \implies f(1)=2$

In R it would give:

$f(1)=-1$ gives with $P(1,x)$ that $f(x)=-0.5(x+\frac{1}{x})$
But $P(5,5)$ isn't correct by $LH>0.25*25+0.5*25>2=RHS$
hence there is only that same solution.
This post has been edited 1 time. Last edited by SCP, Jul 12, 2011, 6:40 pm

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filipbitola
124 posts

Jul 12, 2011, 6:39 pm PM #535

> SCP wrote:
>> filipbitola wrote:
>> Let $P(x,y)$ be the assertion $f(x)f(y)-f(xy)=\frac{x}{y}+\frac{y}{x}$

> $P(1,1) \implies f(1)=2$

There are made some mistakes today,
here we have $P(1,1) \implies f(1)=2$ or $f(1)=-1$ hence that is solution which isn't correct, and 176 is pending by not correct.

To correct:

$f(1)=-1$ gives with $P(1,x)$ that $f(x)=-0.5(x+\frac{1}{x})$
But $P(5,5)$ isn't correct by $LH > 0.25*25+0.5*25>2=RHS$
hence there was indeed one solution.

$f(1)\neq -1$ because the function is from $\mathbb{R}^+$ to $\mathbb{R}^+$

---

socrates
1818 posts

Jul 12, 2011, 9:57 pm PM #536

> *filipbitola wrote:*
> **Problem 179:** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that both conditions are satisfied.
> a) $f(x+y)=f(x)+f(y)$
> b) $f(xy)=f(x)f(y)$

From $f(x^2)=f^2(x)\geq 0$ we get $f(x)\geq 0,\ \forall x\geq 0.$
So f is increasing and hence $f(x)=cx.$

Finally $c\in\{0,1\}.$

Problem 180:

Find all functions $f:[a,b]\to[a,b],\ 0<a<b,$ such that $f(x)f(f(x))=x^2,$ for each $x\in[a,b].$

Dedicated to **pco** 🙂

Djurre
389 posts

Jul 13, 2011, 2:57 am PM #537

> *socrates wrote:*
> **Problem 180:**
>
> Find all functions $f:[a,b]\to[a,b],\ 0<a<b,$ such that $f(x)f(f(x))=x^2,$ for each $x\in[a,b].$
>
> Dedicated to **pco** 🙂

Solution Problem 180

I don't have a problem right now, so feel free to post the next problem.

---

filipbitola
124 posts

Jul 13, 2011, 4:03 am PM #538

> *Djurre wrote:*
> Because f is injective, we know that:
> $f(f(x))f(f(f(x)))=f(x^2)=f(x)^2$

Can you please explain what you did here? In detail, I mean.

---

khaitang
168 posts

Jul 13, 2011, 7:24 am PM #539

Problem 181 Find all function $f:\mathbb{R}\to\mathbb{R}$ such that:
$(f(x+y))^3=f(x^3)+f(y^3)+3f(x)f(y)f(x+\frac{f(5y)}{5})$

Djurre
389 posts

Jul 13, 2011, 12:28 pm PM #540

> *filipbitola wrote:*
>> *Djurre wrote:*
>> Because f is injective, we know that:
>> $f(f(x))f(f(f(x)))=f(x^2)=f(x)^2$
>
> Can you please explain what you did here? In detail, I mean.

Yes, of course I can. I hope it's detailed

Yes, of course I can. I hope it's detailed.
What I did here is put some f signs around the expressions $f(x)$ and $f(f(x))$.
Because f is injective, we are allowed to do that *(because we proved that if $f(a)=f(b)$, then $a=b$ and so is the reverse also possible (if $a=b$, then $f(a)=f(b)$)).*
Maybe you saw that we could change x^2 in $f(x)^2$ and $f(x^2)$. We are allowed to do that, because if we remove the f's from $f(x)^2$ and $f(x^2)$, then there would be x^2, the expression were we started.
I hope you get it now.

---

filipbitola
124 posts

Jul 13, 2011, 12:39 pm — PM #541

> *Djurre wrote:*
> Maybe you saw that we could change x^2 in $f(x)^2$ and $f(x^2)$. We are allowed to do that, because if we remove the f's from $f(x)^2$ and $f(x^2)$, then there would be x^2, the expression were we started.

I don't think you are allowed to do that, even if you have injectivity.

Djurre
389 posts

Jul 13, 2011, 1:37 pm — PM #542

> *filipbitola wrote:*
>> *Djurre wrote:*
>> Maybe you saw that we could change x^2 in $f(x)^2$ and $f(x^2)$. We are allowed to do that, because if we remove the f's from $f(x)^2$ and $f(x^2)$, then there would be x^2, the expression were we started.
>
> I don't think you are allowed to do that, even if you have injectivity.

You are allowed to do that. Maybe you didn't understand that it has to be in an equation. So we have to put f's on both the LHS and the RHS.

---

filipbitola
124 posts

Jul 13, 2011, 4:40 pm — PM #543

> *Djurre wrote:*
>> *filipbitola wrote:*
>>> *Djurre wrote:*
>>> Maybe you saw that we could change x^2 in $f(x)^2$ and $f(x^2)$. We are allowed to do that, because if we remove the f's from $f(x)^2$ and $f(x^2)$, then there would be x^2, the expression were we started.
>>
>> I don't think you are allowed to do that, even if you have injectivity.
>
> You are allowed to do that. Maybe you didn't understand that it has to be in an equation. So we have to put f's on both the LHS and the RHS.

I finally think I understand what you are doing, but if that's the case, you are still wrong.
Correct me if I am wrong.
First, you substitute $x=f(x)$, to get $f(f(x))f(f(f(x)))=f(x)^2$
Then in the original equation, you take f on both sides to get that: $f(f(x)f(f(x)))=f(x^2)$
But if that is what you are doing, clearly $f(x^2)\neq f(x)^2$ since the left sides are obviously different.

SCP
1520 pos...

Jul 14, 2011, 1:17 am — PM #544

> *Djurre wrote:*
> You are allowed to do that. Maybe you didn't understand that it has to be in an equation. So we have to put f's on both the LHS and the RHS.

I agree with filip,
$f(x^2)=f(x)^2$ and let $f(x)=2x$ which is injective, then $2x^2=4x^2$ isn't true, even it is injective.

Hope it helps as example to understand you can't fill in $f(x^2)$ at the way you do.

Djurre
389 posts

Jul 14, 2011, 3:04 am — PM #545

> *SCP wrote:*
>> *Djurre wrote:*
>> You are allowed to do that. Maybe you didn't understand that it has to be in an equation. So we have to put f's on both the LHS and the RHS.

> I agree with filip,
> $f(x^2) = f(x)^2$ and let $f(x) = 2x$ which is injective, then $2x^2 = 4x^2$ isn't true, even it is injective.
>
> Hope it helps as example to understand you can't fill in $f(x^2)$ at the way you do.

The function f is injective for only the functions f who satisfying the equation (and $f(x) = 2x$ doesn't satisfy the equation ($f(x)f(f(x)) = x^2$)). Because $f(x) = 2x$ does not satisfying the equation, it seems to me, that it isn't that weird that if we put $f(x) = 2x$ in $f(x^2) = f(x)^2$ that it will not work.

---

filipbitola
124 posts

Jul 14, 2011, 3:15 am PM #546

> *Djurre wrote:*
>
> > *SCP wrote:*
> >
> > > *Djurre wrote:*
> > > You are allowed to do that. Maybe you didn't understand that it has to be in an equation. So we have to put f's on both the LHS and the RHS.
> >
> > I agree with filip,
> > $f(x^2) = f(x)^2$ and let $f(x) = 2x$ which is injective, then $2x^2 = 4x^2$ isn't true, even it is injective.
> >
> > Hope it helps as example to understand you can't fill in $f(x^2)$ at the way you do.
>
> The function f is injective for only the functions f who satisfying the equation (and $f(x) = 2x$ doesn't satisfy the equation ($f(x)f(f(x)) = x^2$)). Because $f(x) = 2x$ does not satisfying the equation, it seems to me, that it isn't that weird that if we put $f(x) = 2x$ in $f(x^2) = f(x)^2$ that it will not work.

You have no proof that $f(x^2) = f(x)^{2]}$. What you have is an assumption. What if there exists an injective function, that satisfies the original equation, but doesn't satisfy yours?

Djurre
389 posts

Jul 14, 2011, 3:42 am PM #547

> *filipbitola wrote:*
>
> > *Djurre wrote:*
> > The function f is injective for only the functions f who satisfying the equation (and $f(x) = 2x$ doesn't satisfy the equation ($f(x)f(f(x)) = x^2$)). Because $f(x) = 2x$ does not satisfying the equation, it seems to me, that it isn't that weird that if we put $f(x) = 2x$ in $f(x^2) = f(x)^2$ that it will not work.
>
> You have no proof that $f(x^2) = f(x)^{2]}$. What you have is an assumption. What if there exists an injective function, that satisfies the original equation, but doesn't satisfy yours?

I do have a proof, I proved that f was injective, so I could state that $f(f(x))f(f(f(x))) = f(x)^2 = f(x^2)$. And so it isn't an assumption. There exist no function who satisfies the original equation, but who doesn't satisfy mine. If there is such function, then show me a function, who satisfies the original equation, but who doesn't satisfy mine.

---

filipbitola
124 posts

Jul 14, 2011, 3:46 am PM #548

> *Djurre wrote:*
> I do have a proof, I proved that f was injective, so I could state that $f(f(x))f(f(f(x))) = f(x)^2 = f(x^2)$. And so it isn't an assumption. There exist no function who satisfies the original equation, but who doesn't satisfy mine. If there is such function, then show me a function, who satisfies the original equation, but who doesn't satisfy mine.

What you state has nothing to do with injectivity, and it's not a proof since I actually haven't seen the actual work that brings me to that conclusion. What you are trying to say is more or less equal to: I can't seem to spot any more functions that satisfy the equation, so I am going to say that $f(x^2) = f(x)^2$ because it satisfies the only solution I could see. But you still haven't PROVEN that.

Djurre
389 posts

Jul 14, 2011, 4:09 am PM #549

> *filipbitola wrote:*
>
> > *Djurre wrote:*
> > I do have a proof, I proved that f was injective, so I could state that $f(f(x))f(f(f(x))) = f(x)^2 = f(x^2)$. And so it isn't an assumption. There exist no function who satisfies the original equation, but who doesn't satisfy mine. If there is such function, then show me a function, who satisfies the original equation, but who doesn't satisfy mine.

> What you state has nothing to do with injectivity, and it's not a proof since I actually haven't seen the actual work that brings me to that conclusion. What you are trying to say is more or less equal to: I can't seem to spot any more functions that satisfy the equation, so I am going to say that $f(x^2)=f(x)^2$ because it satisfies the only solution I could see. But you still haven't PROVEN that.

> *Quote:*
> What you state has nothing to do with injectivity, and it's not a proof since I actually haven't seen the actual work that brings me to that conclusion.

Which proof do you mean? The proof that f is injective?

> *Quote:*
> I can't seem to spot any more functions that satisfy the equation, so I am going to say that $f(x^2)=f(x)^2$ because it satisfies the only solution I could see.

It's ridiculous, that you say that. If you want it that much, that I will not use $f(x^2)=f(x)^2$ in my solution, then fine. I can do it without that:
Solution Problem 180

> *Quote:*
> But you still haven't PROVEN that.

Do you mean, that I haven't proven that $f(x)=x$ is the only solution?

---

filipbitola
124 posts

Jul 14, 2011, 4:14 am PM #550

> *Djurre wrote:*
> It's ridiculous, that you say that. If you want it that much, that I will not use $f(x^2)=f(x)^2$ in my solution, then fine. I can do it without that:
> Solution Problem 180

And of course, you are still wrong, since these lines make no sense at all:

> *Djurre wrote:*
> We can rewrite this:
> $$x^2f\left(\frac{x^2}{f(x)}\right)=f(x)^3$$
> With injectivity we get:
> $$\frac{x^4}{f(x)}=x^3\implies f(x)=x$$

And plus, $f(x)=\frac{1}{x^2}$ is also a solution.

---

pfanni
39 posts

Jul 14, 2011, 4:16 am PM #551

> *Djurre wrote:*
> $$x^2f\left(\frac{x^2}{f(x)}\right)=f(x)^3$$
> With injectivity we get:
> $$\frac{x^4}{f(x)}=x^3\implies f(x)=x$$

Sorry. But I don't get this part.
Please explain it to me.

[edit: wrong part cited]

---

filipbitola
124 posts

Jul 14, 2011, 4:20 am PM #552

> *pfanni wrote:*
>
> > *Djurre wrote:*
> > $$x^2f\left(\frac{x^2}{f(x)}\right)=f(x)^3$$
> > With injectivity we get:
> > $$\frac{x^4}{f(x)}=x^3\implies f(x)=x$$
>
> Sorry. But I don't get this part.

Sorry. But I don't get this part.
Please explain it to me.

[edit: wrong part cited]

The problem is how to get to there.
We can rewrite $\frac{x^4}{f(x)} = x^3$ as $\frac{x}{f(x)} = 1 \Longrightarrow f(x) = x$

---

pfanni
39 posts

Jul 14, 2011, 4:23 am PM #553

> *filipbitola wrote:*
>
> > *pfanni wrote:*
> >
> > > *Djurre wrote:*
> > > $x^2 f\left(\frac{x^2}{f(x)}\right) = f(x)^3$
> > > With injectivity we get:
> > > $\frac{x^4}{f(x)} = x^3 \Longrightarrow f(x) = x$
> >
> > Sorry. But I don't get this part.
> > Please explain it to me.
> >
> > [edit: wrong part cited]
>
> The problem is how to get to there.
> We can rewrite $\frac{x^4}{f(x)} = x^3$ as $\frac{x}{f(x)} = 1 \Longrightarrow f(x) = x$

But how do you get $\frac{x^4}{f(x)} = x^3$

---

filipbitola
124 posts

Jul 14, 2011, 4:24 am • 1 PM #554

> *pfanni wrote:*
> But how do you get $\frac{x^4}{f(x)} = x^3$

You don't. It's a mistake.

Djurre
389 posts

Jul 14, 2011, 5:01 am PM #555

> *filipbitola wrote:*
> And plus, $f(x) = \frac{1}{x^2}$ is also a solution.

You're right filipbitola, that $f(x) = \frac{1}{x^2}$ is a solution. So my solution is not complete, I missed it somewhere (but I don't know where). Nevertheless I still think, that all what I did is good, except the part where I missed a step, which could lead me to $f(x) = \frac{1}{x^2}$.
Thanks for pointing it. Do you know where the error is in my problem?

Djurre
389 posts

Jul 14, 2011, 5:04 am PM #556

> *pfanni wrote:*
>
> > *filipbitola wrote:*
> > The problem is how to get to there.
> > We can rewrite $\frac{x^4}{f(x)} = x^3$ as $\frac{x}{f(x)} = 1 \Longrightarrow f(x) = x$
>
> But how do you get $\frac{x^4}{f(x)} = x^3$

Because there is a error in my solution, I will have to look first to find the error and then I will answer your question.

(Great discussion!)

filipbitola
124 posts

Jul 14, 2011, 5:12 am PM #557

> *Djurre wrote:*
>
> > *filipbitola wrote:*
> > And plus, $f(x)=\frac{1}{x^2}$ is also a solution.
>
> You're right filipbitola, that $f(x)=\frac{1}{x^2}$ is a solution. So my solution is not complete, I missed it somewhere (but I don't know where). Nevertheless I still think, that all what I did is good, except the part where I missed a step, which could lead me to $f(x)=\frac{1}{x^2}$.
> Thanks for pointing it. Do you know where the error is in my problem?

The mistake is at the part both me and pfanni cited. When you have injectivity, you can't cancel all f's on one side and all f's on the other. The only thing you can do with injectivity is say that if $f(a)=f(b) \implies a=b$

Djurre
389 posts

Jul 14, 2011, 5:28 am PM #558

I agree, that you can say with injectivity, that if $f(a)=f(b) \Longrightarrow a=b$.

> *Quote:*
> The only thing you can do with injectivity is say that if $f(a)=f(b) \implies a=b$

So you can cancel some f's.

I thought about it and maybe I do know where I made an error (but I am not sure if this is an real error):
Click to reveal hidden text

> *filipbitola wrote:*
> When you have injectivity, you can't cancel all f's on one side and all f's on the other.

I didn't cancelled all f's on one side and alle f's on the other.

filipbitola
124 posts

Jul 14, 2011, 5:30 am PM #559

> *Djurre wrote:*
> I agree, that you can say with injectivity, that if $f(a)=f(b) \Longrightarrow a=b$.
>
> > *Quote:*
> > The only thing you can do with injectivity is say that if $f(a)=f(b) \implies a=b$
>
> So you can cancel some f's.
>
> I thought about it and maybe I do know where I made an error (but I am not sure if this is an real error):
> Click to reveal hidden text
>
> > *filipbitola wrote:*
> > When you have injectivity, you can't cancel all f's on one side and all f's on the other.
>
> I didn't cancelled all f's on one side and alle f's on the other.

That is the error.

Djurre
389 posts

Jul 14, 2011, 5:32 am PM #560

You mean?:
Click to reveal hidden text

Farenhajt
5170 pos...

Jul 14, 2011, 5:37 am PM #561

> *Djurre wrote:*
> You mean?:
> Click to reveal hidden text

$f(x)=\tan x$ images $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ into $\mathbb{R}$ injectively (even bijectively, since it's strictly increasing on the interval), but you obviously can't claim $\tan a=\tan^2 b \implies a=b^2$

$f(x)=e^x$ images $\mathbb{R}$ into $(0,+\infty)$ also bijectively, but you don't have $e^a=(e^b)^2 \implies a=b^2$ - on the contrary, you have $a=2b$

etc.

Many more examples can be given to substantiate that your claim is wrong.

This post has been edited 1 time. Last edited by Farenhajt, Jul 14, 2011, 5:44 am

---

socrates
1818 posts

Jul 14, 2011, 5:42 am PM #562

Here are the last 2 problems.

Note that $f(x)=\frac{1}{x^2}$ is not a solution to problem 180.

> *socrates wrote:*
> **Problem 180:**
>
> Find all functions $f:[a,b]\to[a,b],\ 0<a<b,$ such that $f(x)f(f(x))=x^2$, for each $x\in[a,b]$.
>
> Dedicated to **pco** :)

> *khaitang wrote:*
> **Problem 181** Find all function $f:\mathbb{R}\to\mathbb{R}$ such that:
> $(f(x+y))^3=f(x^3)+f(y^3)+3f(x)f(y)f(x+\frac{f(5y)}{5})$

---

filipbitola
124 posts

Jul 14, 2011, 5:54 am PM #563

> *socrates wrote:*
> Here are the last 2 problems.
>
> Note that $f(x)=\frac{1}{x^2}$ is not a solution to problem 180.

Good point, I forgot f was bounded.

Djurre
389 posts

Jul 14, 2011, 6:01 am PM #564

> *Farenhajt wrote:*
>
> > *Djurre wrote:*
> > You mean?:
> > Click to reveal hidden text
>
> $f(x)=\tan x$ images $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ into $\mathbb{R}$ injectively (even bijectively, since it's strictly increasing on the interval), but you obviously can't claim $\tan a=\tan^2 b\implies a=b^2$
>
> $f(x)=e^x$ images $\mathbb{R}$ into $(0,+\infty)$ also bijectively, but you don't have $e^a=(e^b)^2\implies a=b^2$ - on the contrary, you have $a=2b$
>
> etc.
>
> Many more examples can be given to substantiate that your claim is wrong.

I just say that it's wrong to say if $g(a)=g(b)^2$ then $a=b^2$, with f is injective.

Djurre
389 posts

Jul 14, 2011, 6:05 am PM #565

> *filipbitola wrote:*
>
> > *socrates wrote:*
> > Here are the last 2 problems.
> >
> > Note that $f(x)=\frac{1}{x^2}$ is not a solution to problem 180.
>
> Good point, I forgot f was bounded.

Yeah, good point. I also forgot that f was bounded.

---

pfanni
39 posts

Jul 14, 2011, 12:26 pm PM #566

> *Djurre wrote:*

Djurre wrote:

> filipbitola wrote:
>
> > socrates wrote:
> > Here are the last 2 problems.
> >
> > Note that $f(x)=\frac{1}{x^2}$ is not a solution to problem 180.
>
> Good point, I forgot f was bounded.

Yeah, good point. I also forgot that f was bounded.

Why does everyone forget about the boundary conditions?
Using them, one can easily obtain $f(a)=a$ and $f(b)=b$. I think this is essential to proof the uniqueness of $f(x)=x$.

Pfanni

---

khaitang
168 posts

Jul 14, 2011, 12:35 pm • 1 PM #567

> khaitang wrote:
>
> **Problem 174** : Find all function $f:\mathbb{R}\to\mathbb{R}$ such that: $f(xf(y)+y)=\frac{f(2x)f(4x)}{8}.+f(y)$

---

filipbitola
124 posts

Jul 14, 2011, 4:21 pm • 1 PM #568

> khaitang wrote:
>
> **Problem 174** : Find all function $f:\mathbb{R}\to\mathbb{R}$ such that: $f(xf(y)+y)=\frac{f(2x)f(4x)}{8}.+f(y)$

Let $P(x,y)$ be the assertion $f(xf(y)+y)=\frac{f(2x)f(4x)}{8}+f(y)$
$P(0,y) \implies f(0)=0$
$P(x,0) \implies f(2x)f(4x)=0$
$P(x,y) \implies f(xf(y)+y)=f(y)$.
Assume $\exists y$ such that $f(y)\neq 0$.
Now, since $xf(y)$ generates all values, f is constant.
Since $f(2x)f(4x)=0 \implies f(y)=0$, contradiction. Hence, the only solution is $\boxed{f(x)=0}\,\forall x\in\mathbb{R}$

This post has been edited 1 time. Last edited by filipbitola, Jul 14, 2011, 4:59 pm

---

SCP
1520 pos...

Jul 14, 2011, 4:44 pm PM #569

> filipbitola wrote:
>
> > khaitang wrote:
> >
> > **Problem 174** : Find all function $f:\mathbb{R}\to\mathbb{R}$ such that: $f(xf(y)+y)=\frac{f(2x)f(4x)}{8}.+f(y)$
>
> Let $P(x,y)$ be the assertion $f(xf(y)+y)=\frac{f(2x)f(4x)}{8}+f(y)$
> $P(0,y) \implies f(0)=0$
> $P(x,0) \implies f(2x)f(4x)=0$
> $P(x,y) \implies f(xf(y)+y)=f(y)$ and since $xf(y)$ generates all values, f is constant.
> Since $f(2x)f(4x)=0 \implies \boxed{f(x)=0}\,\forall x\in\mathbb{R}$

$xf(y)$ generates only 0 in your solution, so you have to reword and say:
if there is a function different of the zerofunction who satisfies...,
in my opinion it is then correct.

So only pending problem is 181?

---

khaitang
168 posts

Jul 15, 2011, 2:08 pm PM #570

> filipbitola wrote:
>
> > khaitang wrote:
> >
> > **Problem 174** : Find all function $f:\mathbb{R}\to\mathbb{R}$ such that: $f(xf(y)+y)=\frac{f(2x)f(4x)}{8}.+f(y)$

> Let $P(x,y)$ be the assertion $f(xf(y)+y)=\frac{f(2x)f(4x)}{8}+f(y)$
> $P(0,y) \implies f(0)=0$
> $P(x,0) \implies f(2x)f(4x)=0$
> $P(x,y) \implies f(xf(y)+y)=f(y)$.
> Assume $\exists y$ such that $f(y)\neq 0$.
> Now, since $xf(y)$ generates all values, f is constant.
> Since $f(2x)f(4x)=0 \implies f(y)=0$, contradiction. Hence, the only solution is $\boxed{f(x)=0}\, \forall x\in\mathbb{R}$

$f(x)=x$

SCP
1520 pos...

Jul 15, 2011, 4:42 pm PM #571

> *khaitang wrote:*
>> *khaitang wrote:*
>> **Problem 174** : Find all function $f:\mathbb{R}\to\mathbb{R}$ such that: $f(xf(y)+y)=\frac{f(2x)f(4x)}{8}.+f(y)$
>
> $f(x)=x$

Wrong, it gives $xy+y=x^2+y$ which isn't correct if $x\neq y$ or you did a typo in question.

The solution of Filip is now correct in the original question formulation.

sasu1ke
34 posts

Jul 15, 2011, 5:59 pm • 1 PM #572

why does 1/x^2 not satisfy prob 180 if a<1 and b>1.

sasu1ke
34 posts

Jul 15, 2011, 6:33 pm • 1 PM #573

let f(f(...ntimes...(x))))=a(n).take the recurrence relation a(n)*a(n-1)=(a(n-2))^2 . take b(n)=log(a(n)).We have b(n)+b(n-1)=2b(n-2). Solve this equation and use f(x)>a for x>a to get f(x)=x.

sasu1ke
34 posts

Jul 15, 2011, 6:34 pm • 1 PM #574

Sorry for posting continuously but i am new to AOPS ad my two previous posts were with regard to problem f(f(x))*f(x)=x^2

khaitang
168 posts

Jul 16, 2011, 12:56 pm • 2 PM #575

@SCP: I am sorry.
Problem 182 Find all function $f:\mathbb{R}\to\mathbb{R}$ such that:
$f(xf(y)+y)=\frac{f(2x)f(4y)}{8}+f(y)$

sasu1ke
34 posts

Jul 16, 2011, 6:33 pm • 1 PM #576

Is my solution correct for problem 180 please reply.

Djurre
389 posts

Aug 1, 2011, 11:24 pm PM #577

> *khaitang wrote:*
> **Problem 182** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:
> $f(xf(y)+y)=\frac{f(2x)f(4y)}{8}+f(y)$

Let $P(x,y)$ be the assertion of $f(xf(y)+y)=\frac{f(2x)f(4y)}{8}+f(y)$.
$P(0,y)\implies f(y)=\frac{f(0)f(4y)}{8}+f(y)$
So $f(0)f(4y)=0$, thus $f(0)=0$ of $f(4y)=0$.
If we check $f(x)=0$, then we see that it satisfies the given conditions.
Now let's move on with $f(0)=0$.
$P(x,0)\implies f(0)=\frac{f(2x)f(0)}{8}+f(0)$.
So $f(2x)f(0)=0$. This is exactly the same things with $P(x,0)$.
So $f(x)=0\,\forall x\in\mathbb{R}$

proglote
940 posts

Aug 1, 2011, 11:31 pm PM #578

> *Djurre wrote:*
> $P(x,0)\implies f(0)=\frac{f(2x)f(0)}{8}+f(0)$.
> So $f(2x)f(0)=0$. This is exactly the same things with $P(x,0)$.
> So $f(x)=0\,\forall x\in\mathbb{R}$

$P(x,0)$ only implies $0=0$ in this case, nothing more.

Djurre
389 posts

Aug 1, 2011, 11:46 pm PM #579

@proglote
You're right. I will look again to it tomorrow, because I have no time at the moment.
But if this problem isn't solved within a few days, I ask khaitang to post his solution.

Djurre
389 posts

Aug 8, 2011, 6:49 am PM #580

khaitang could you give us your solution for **Problem 182**?
Because this marathon is dying!

socrates
1818 posts

Aug 8, 2011, 4:04 pm PM #581

Problem 183:
Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+y)\ge y\,f(x)+f(f(x))$, for all $x,y\in\mathbb{R}$.

abhinavza...
418 posts

Aug 9, 2011, 5:49 pm PM #582

Solution 183
Problem 184.
Find all strictly monotone functions $f:\mathbb{R}\to\mathbb{R}$,satisfying
$f(f(x)+y)=f(x+y)+f(0)\ \forall\ x,y\ \in\mathbb{R}$.

sasu1ke
34 posts

Aug 9, 2011, 6:21 pm PM #583

put y=-x,
f(f(x)-x)=2f(0),
so f(x)-x is a constant as f is montonous.
So f(x)=x+c is the only solution

socrates
1818 posts

Aug 9, 2011, 6:59 pm PM #584

> *abhinavzandubalm wrote:*
> Let $P(x,y)$ be the assertion that $f(x+y)\ge yf(x)+f(f(x))$.
> $P(x,0)\implies f(x)\ge f(f(x))$.
> $P(x,y)\implies f(x+y)\ge yf(x)+f(f(x))\ge yf(f(x))+f(f(x))=(y+1)f(f(x))$.
> Replacing y by -1.
> $f(x-1)\ge 0$.
> Hence $f(x)\in\mathbb{R}_0$
> $P(x,f(x)-x)\implies f(f(x))\ge(f(x)-x)f(x)+f(f(x))$
> $\implies f(x)\ge(f(x)-x)f(x)+f(f(x))\implies f(x)(x+1-f(x))\ge f(f(x))$.
> Now we can take x as large a negative number as we want to get the LHS as big a negative number as we want.
> But we know that RHS is positive hence we get that
> $f(x)=0\ \forall\ x\ \in\mathbb{R}$.

Your solution is not correct!

sasu1ke
34 posts

Aug 9, 2011, 7:59 pm PM #585

for question 183,
clearly
f(x)=1 is not a solution

put y=f(x)-x
=>0>=(f(x)-x)(f(x)

>eq 1
put x=0 in eq1
0>=f^2(0)
=>f(0)=0
put x=0 => f(y)>=0 for y real
if f(x) not equal to 0 for all real x,we have f(a)-a<=0 for some a for which f(a) is not equal to 0 so f(a)=0 for all negative a. if f(x) is not equal to zero for all positive real x
taking y very large we can see that f(x+y) is unbounded

>eq 2
if x+y>=0
x+y>=f(x+y)>=yf(x)+f(f(x))
=>x>=y(f(x)-1)+f(f(x)) if f(x)>1 take y=x/(f(x)-1) and the inequality doesn't hold
so f(x)<=1 but eq 2 says f is not bounded
Hence f(x)=0 is the only solution

jjax
107 posts

Aug 9, 2011, 8:14 pm PM #586

pwned for writing too slowly... nonetheless this seems a little more fleshed out, so I will post anyway.
Solution 183:
Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+y)\ge y\,f(x)+f(f(x))$, for all $x,y\in\mathbb{R}$.[/quote]

Let $P(x,y)$ be the proposition that $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+y)\ge y\,f(x)+f(f(x))$
Click to reveal hidden text

Note: This is 2011 IMO Q3, with the inequality sign flipped... Having IMO withdrawal already D:

abhinavza...

Aug 9, 2011, 10:36 pm PM #587

abhinavza...
418 posts

I apologise for the wrong answer....
Don't know what I was thinking then. 🙁

---

socrates
1818 posts

Aug 10, 2011, 4:16 pm PM #588

Problem 185 (Czech MO)
Determine all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that $f(x)f(y)=f(y)f(xf(y))+\frac{1}{xy}$, for every $x,y>0$.

$\mathbb{R}^+=(0,+\infty)$

---

goodar20...
1346 pos...

Aug 10, 2011, 4:30 pm • 1 PM #589

here we go:
Attachments:

$f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$
$\forall x,y\in\mathbb{R}^+$: $f(x)f(y)=f(y)f(xf(y))+\frac{1}{xy}$

solution:

Let (x,y) denote the assetion of functional equation $f(x)f(y)=f(y)f(xf(y))+\frac{1}{xy}$.

It's obvious that the function is injective.

let $f(1)=c$

$(1,1)\Longrightarrow f(c)=\frac{c^2-1}{c}$

$(\frac{1}{c},1)\Longrightarrow f(\frac{1}{c})=c+1$

$(\frac{1}{c+1},\frac{1}{c})\Longrightarrow f(\frac{1}{c+1})=2c$

$(\frac{1}{2c},\frac{1}{c+1})\Longrightarrow f(\frac{1}{2c})=2c+1$

$(\frac{1}{c},\frac{1}{c+1})\Longrightarrow f(2)=\frac{c+1}{2}$

$(\frac{1}{c+1},2)\Longrightarrow f(\frac{1}{2})=2c-1$

$(\frac{1}{2},\frac{1}{c+1})\Longrightarrow c=2$ so $f(1)=2$.

$(x,1)\Longrightarrow 2f(x)=2f(2x)+\frac{1}{x}$

$(x,x)\Longrightarrow f(x)^2=f(x)f(xf(x))+\frac{1}{x^2}$

$(2x,x)\Longrightarrow f(2x)f(x)=f(x)f(2xf(x))+\frac{1}{2x^2}$. using the two statements above we get $f(x)=1+\frac{1}{x}$ which is indeed a solution. so we're done.

---

socrates
1818 posts

Aug 10, 2011, 6:26 pm • 2 PM #590

Problem 186 (crazyfehmy)
Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all $x,y\in\mathbb{R}$ we have

$$f(x^2+xf(y))=[f(x)]^2+yf(x)$$

Dijkschnei...
131 posts

Aug 11, 2011, 3:13 am PM #591

Solution to problem 186 :
Click to reveal hidden text
This post has been edited 1 time. Last edited by Dijkschneier, Aug 17, 2011, 8:01 pm

---

costantin07
200 posts

Aug 11, 2011, 4:19 am • 1 PM #592

Hi! nice,it s just that : $P(0,0)$ doesn t imply $f(0)=0$
however : $P(0,y)\implies f(0)=f(0)^2+yf(0),\forall\ y\in\mathbb{R}$
so:$f(0)=0$

---

alphabeta...
176 posts

Aug 11, 2011, 9:02 pm PM #593

Problem-187 : Find all functions $f:\mathbb{R}\to\mathbb{R}$ and $x\neq 0$ satisfying the equation :

$$f\left(\frac{1}{x}\right)+x^2f(x)=0$$

This post has been edited 1 time. Last edited by alphabeta1729, Aug 12, 2011, 2:29 pm

---

filipbitola
124 posts

Aug 11, 2011, 9:44 pm PM #594

> *Dijkschneier wrote:*
> From both relations f(x²)=f(x)² and f(x+1)=f(x)+1, we infer that f is the identity function (well known fact)

Can you explain in detail(step by step)?
Thanks

Djurre
389 posts

Aug 12, 2011, 1:37 pm PM #595

> *alphabeta1729 wrote:*
> **Problem-187** : Find all functions satisfying the equation :

$$f\left(\frac{1}{x}\right)+x^2f(x)=0$$

First what's the domain and codomain?
Second which values can x take?
Third $x \neq 0$.

abhinavza...
418 posts

Aug 12, 2011, 9:51 pm PM #596

Hope that it is $\mathbb{R}^+$ or $\mathbb{R} \setminus [0]$.
Hope that it's the second one.

EDIT: OK this doesn't matter.Sorry about this.The solution is mostly independent of any differences in properties of these two domains

This post has been edited 2 times. Last edited by abhinavzandubalm, Aug 15, 2011, 6:18 pm

abhinavza...
418 posts

Aug 15, 2011, 6:13 pm PM #597

> *alphabeta1729 wrote:*
> **Problem-187** : Find all functions $f:\mathbb{R}\to\mathbb{R}$ and $x\neq 0$ satisfying the equation :
>
> $$f\left(\frac{1}{x}\right)+x^2f(x)=0$$

Solution 187

Problem 188.
Find whether or not there exists a function such that
$f:\mathbb{Q}\to\mathbb{Q}$
$f\left(\frac{p}{q}\right)=pf(p)+qf\left(\frac{p^2-q^2}{p^2+q^2}\right)$ $\forall p,q$ such that $gcd(p,q)=1, p\in\mathbb{Z}, q\in\mathbb{N}$.
And if it does find all such possible ones.

Dijkschnei...
131 posts

Aug 17, 2011, 8:00 pm PM #598

> *costantin07 wrote:*
> Hi! nice,it s just that : $P(0,0)$ doesn t imply $f(0)=0$
> however : $P(0,y) \implies f(0)=f(0)^2+yf(0), \forall\ y\in\mathbb{R}$
> so:$f(0)=0$

That's right. I'll edit my post.

> *filipbitola wrote:*
>
> > *Dijkschneier wrote:*
> > From both relations $f(x^2)=f(x)^2$ and $f(x+1)=f(x)+1$, we infer that f is the identity function (well known fact)
>
> Can you explain in detail(step by step)?
> Thanks

Consider x>1, then f(x)>1. Suppose x>f(x) so that there exists integers m,n such that $x^{2^n}>m>f(x^{2^n})$, which gives f(y)<0 with $y=x^{2^n}-m>0$, contradiction. Suppose f(x)>x, and let m be an integer such that m>m-1>f(x), then 0>f(x-m)>x-m and consequently $f((x-m)^2)=f(x-m)^2<(x-m)^2$, that is y>f(y) for $y=(x-m)^2>1$, which leads to a contradiction according to the first point. Hence f(x)=x for all x>1 and so f(x)=x for all x, that is, f is the identity over $\mathbb{R}$.

socrates
1818 posts

Aug 27, 2011, 5:13 pm PM #599

Problem 189
Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real x,y,z the inequality

$$\left(f\left(x\right)+f\left(y\right)-2f\left(xy\right)\right)\left(f\left(x\right)+f\left(z\right)-2f\left(xz\right)\right)\geq 0$$

holds.

SCP
1520 pos...

Aug 29, 2011, 12:50 am PM #600

solution 189

The soltuions are
$f(0)=a, f(x)=b$ if $x\neq 0$ for $a,b\in\mathbb{R}$.

Use the known $P(1,y,z)$ gives $f(1)$ is the minimum or maximum.
Similar $P(0,y,z)$ give $f(0)$ is a min. or max.,
then $P(x,0,1)$ gives $f(x)\in[f(0),f(1)]$

$P(x,0,\frac{1}{x})$ gives $f(x)=f(\frac{1}{x})=f(1)$ or $f(x)=g(0)$ where $x\neq 0$.
So our domain use only two values and we have only some cases.
(I have to do that yet)

---

tc1729
1221 posts

Aug 29, 2011, 12:58 am PM #601

Problem 190

Let $f(x)$ be a monic fourth degree polynomial with integer coefficients such that $f(f(1))=21$, $f(-1)=0$, and $|f(x)|\geq 85$ for all $|x|\geq 3$. Find $f(6)$.

---

littletush
761 posts

Sep 10, 2011, 11:59 am PM #602

problem 191,by OC
find all functions $R-R$ such that
$f(xf(y))=xy$
holds for any pair of (x,y).

---

filipbitola
124 posts

Sep 10, 2011, 4:50 pm PM #603

Let $P(x,y)$ be the assertion $f(xf(y))=xy$

Fixing y, we get that f is injective.
$P(1,y)\implies f(f(y))=y$
$P(x,f(y))\implies f(xy)=xf(y)$
$P(y,f(x))\implies f(yx)=yf(x)$
Comparing these 2 equations we get that:
$xf(y)=yf(x)\implies f(x)=xf(1)=cx$
Plugging in the original equation we see that the only solutions are $\boxed{f(x)=x}$ and $\boxed{f(x)=-x}$

---

socrates
1818 posts

Sep 11, 2011, 12:24 am PM #604

Problem 192
$f:\mathbb{R}\to\mathbb{R}$
$f(x^2+y^2)=f(f(x))+f(xy)+f(f(y))\,\forall x,y\in\mathbb{R}$
Find all functions f.

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1966002#p1966002

USAMOwi...
50 posts

Sep 11, 2011, 1:37 am PM #605

Problem 190 is unsolved! How can this be

FBI__
29 posts

Sep 11, 2011, 7:42 am PM #606

Problem 193
Find all functions $f:(0,+\propto)\to(0,+\propto)$ such that: $f(x^{2010})=(f(x))^{2010}\,\forall x\in(0,+\propto)$

---

anonymou...
1142 posts

Sep 11, 2011, 9:07 am PM #607

> *tc1729 wrote:*
> **Problem 190**
>
> Let $f(x)$ be a monic fourth degree polynomial with integer coefficients such that $f(f(1))=21$, $f(-1)=0$, and $|f(x)|\geq 85$ for all $|x|\geq 3$. Find $f(6)$.

$f(1)=0,-2,2.$
1. $f(1)=0$
$f(x)=x^4+ax^3+bx^2+cx+d$
so $a+b+c+d=-1, -a+b-c+d=-1$ so $a+c=0, b+d=-1$.
from $f(0)=21$ we have $d=21$ so $b=-22$.
$x^4+ax^3-ax-22x^2+21<=-85$ or $x^4+ax^3-ax-22x^2+21>=85$.
we can analyze the sign of a and see when are the minimums and maximums attained for x.

2. $f(1)=2$,$f(2)=21$, $f(-1)=0$.
so $a+b+c+d=1, -a+b-c+d=-1$ so $a+c=0, b+d=-1$ and also $16+8a+4b+2c+d=21$ so $6a+4b-1-3b=5$ so $6a+b=7$.
in the same manner we can put the conditions.
I don't have another idea to do this problem but I think this one is good. I don't finish the computations but I'll do.

---

littletush
761 posts

Sep 13, 2011, 2:15 pm • 1 PM #608

problem194
let $f:Z-R$ be a function such that
$f(x+y)f(x-y)=f(x)+f(y)$
compute $f(2011)$.

Djurre
389 posts

Sep 13, 2011, 11:46 pm PM #609

Problem 194
Let $f:Z\to R$ be a function such that
$f(x+y)f(x-y)=f(x)+f(y)$
compute $f(2011)$.

Solution

filipbitola
124 posts

Sep 13, 2011, 11:53 pm PM #610

> *Djurre wrote:*
> **Problem 194**
> Let $f:Z\to R$ be a function such that
> $f(x+y)f(x-y)=f(x)+f(y)$
> compute $f(2011)$.
>
> Solution

Not completely correct...
What if the functions are peace-wise? That's also possible

Djurre
389 posts

Sep 13, 2011, 11:56 pm PM #611

What's peace-wise?

filipbitola
124 posts

Sep 14, 2011, 12:05 am • 1 PM #612

> *Djurre wrote:*
> What's peace-wise?

A function that takes some values for certain numbers, and other values for others is called a peace-wise function.
A perfect example is $f(x)=|x|$. For positive numbers, $f(x)=x$, and for negative numbers $f(x)=-x$.

Batominov...
1602 pos...

Sep 14, 2011, 12:41 am PM #613

> *Djurre wrote:*
> What's peace-wise?

1) It's **piecewise**.
2) You have only concluded that either [a] for every x, $f(x)\in\{0,1\}$, or **for every** x, $f(x)\in\{-1,2\}$. **It is NOT equivalent to, nor can it imply the followings:** $\forall x\big[f(x)=0\big]$, $\forall x\big[f(x)=1\big]$, $\forall x\big[f(x)=-1\big]$, **or** $\forall x\big[f(x)=2\big]$.

filipbitola
124 posts

Sep 14, 2011, 12:43 am PM #614

> *Batominovski wrote:*
>
> > *Djurre wrote:*
> > What's peace-wise?
>
> It's **piecewise**.

Thanks for clearing up.

Batominov...
1602 pos...

Sep 14, 2011, 9:28 am PM #615

If there is a nicer solution to this problem, I would like to see.

> *tc1729 wrote:*
> **Problem 190**
>
> Let $f(x)$ be a monic fourth degree polynomial with integer coefficients such that $f(f(1))=21$, $f(-1)=0$, and $|f(x)|\geq 85$ for all $|x|\geq 3$. Find $f(6)$.

Write $f(x)=x^4+ax^3+bx^2+cx+d$. From $f\big(f(1)\big)=21$ and $f(-1)=0$, we see that $f(1)\in\{-2,0,+2\}$. If $f(1)=-2$, we have $f(-2)=21$, but then $3=1-(-2)\nmid(-2)-21=f(1)-f(-2)$, a contradiction. Ergo, $f(1)\in\{0,2\}$. If $f(1)=0$, then $f(0)=21$ so

$$f(x)=x^4+ax^3-22x^2-ax+21\,.$$

Now, we want $f(x)\geq 85$ whenever $|x|\geq 3$, and so we can check the inequalities at $x=-3$ and $x=+3$. It is easily seen that no values of a may satisfy the conditions. Hence, $f(1)=2$ is the only available choice. Now,

$$f(x)=x^4+ax^3-(2a-1)x^2-(a-1)x+(2a-1)\,.$$

Check the values at $x=\pm 3$ again, we can see that only $a=0$ satisfies the problem's conditions. Therefore, $f(x)=x^4+x^2+x-1$.

littletush
761 posts

Sep 14, 2011, 11:27 am PM #616

according to OC,the answer to pro 194 is
$f(2011)=0,2or-1$

$f(2011) = 0, 2$ or -1.

Batominov...
1602 pos...

Sep 15, 2011, 1:25 am PM #617

> Quote:
> **Problem 194**
> Let $f : \mathbb{Z} \to \mathbb{R}$ be a function such that
> $f(x+y)f(x-y) = f(x) + f(y)$
> compute $f(2011)$.

I shall continue from Djurre's work in Reply#608 after his arrival upon $f(0) \in \{0, 2\}$. If $f(0) = 0$, putting $y := x$, we obtain $f \equiv 0$, which may be readily checked to be a solution. From now, suppose that $f(0) = 2$. Define $A := \{x \in \mathbb{Z} | f(x) = 2\}$ and $B := \{x \in \mathbb{Z} | f(x) = -1\}$. Obviously, A and B form a partition of $\mathbb{Z}$ into two subsets. It is easy to show that A is closed under addition and all elements in A has their additive inverses in A as well. Therefore, A is a subgroup under addition of $\mathbb{Z}$. If A is indeed $\mathbb{Z}$, then $B = \emptyset$ and so $f \equiv 2$, which is a solution. Now, if $A \neq \mathbb{Z}$, then $1 \in B$. Now, iteratively, we can deduce that $2 \in B$ and $3 \in A$. Since $3 \in A$, we must then have $A = 3\mathbb{Z}$. Consequently,

$$f(x) = \{ \begin{array}{ll} 2 & \text{if } 3 \mid x\,, \\ -1 & \text{otherwise}\,. \end{array}$$

The above function is indeed a solution; therefore, $f(2011)$ may take any value within $\{-1, 0, 2\}$.

PSs:
1) I was only providing a sketch of my proof. You should fill in the blank yourselves.
2) It would be interesting if the domain of f is $\mathbb{R}$ instead of $\mathbb{Z}$. In that case, how do we construct such a function such that both$f(x) = -1$ and $f(x) = 2$ have solutions. The only info I know is: if A and B are defined similarly to their counterparts in my provided sketch, then
(a) A is still a subgroup under addition of $\mathbb{R}$,
(b) $A + B = B$,
(c) $x, y \in B$ imply $x + y \in A$ or $x - y \in A$, and
(d) all kx's such that $k \in \mathbb{Z} \setminus 3\mathbb{Z}$ belong to the same subset A or B for every real x.

littletush
761 posts

Sep 16, 2011, 9:48 am PM #618

OC's solution for problem 194:
first,$f(x) + f(-y) = f(x) + f(y)$,so $f(x)$ is even.
let $x = y$ we get $f(2x)f(0) = 2f(x)(2)$
by (2) it's easy that
$f(0) = 0$ or 2.
if $f(0) = 0$ then by (2) it's trivial that for every integer x,$f(x) = 0$.so $f(2011) = 0$.
next we assume $f(0) = 2$,by (2) we obtain
$f(2x) = f(x)$.
let $y = 0$ in the original equation then
$f^2(x) = f(x) + 2$,so
$f(x) = 2\text{or}-1$.
finally we will give examples for these two cases:
$f(x) \equiv 2$,for $f(2011) = 2$;
$f(x) = w^{2x} + w^x$,for $f(2011) = -1$,where$w = e^{\frac{2i\pi}{3}}$.
hence all possible values of $f(2011)$ are $2, 0$ and -1.

littletush
761 posts

Sep 16, 2011, 9:50 am PM #619

there's nothing to do with the so-called piecewice;after all,it's not very hard.

VHCR
90 posts

Sep 16, 2011, 7:47 pm PM #620

Problem 195:
Find all continuous solutions of $f(x-y) = f(x)f(y) + g(x)g(y)$.

littletush
761 posts

Sep 18, 2011, 11:26 am PM #621

solution195.
if f, g are constant,then every pair of (f, g) satisfying
$f = f^2 + g^2$
is the answer.
now we assume f, g are not constant.
let$x = y, y = x$,we get
$f(y-x) = f(x)f(y) + g(x)g(y) = f(x-y)$,so $f(x)$ is even.let $x = y$,then
$f(0) = f^2(x) + g^2(x)$,it's easy to obtain
$g^2(x) = g^2(-x)$.by continuousness,either
$g(x) = g(-x)$ or $g(x) = -g(-x)$
holds in a certain interval.if $g(-x) = g(x)$,then
$f(2x) = f^2(x) + g^2(x), f(2x) = f(0)$,so f is constant in an interval.
Because of continuousness,f can't be piecewice(except for $f(x_0) = 0$),so f is constant on R,a contradiction.so
$g(-x) = -g(x)$,then g is odd.then
$f(2x) = f^2(x) - g^2(x)$ and $f(0) = f^2(x) + g^2(x)$.
adding them yields
$f(2x) + f(0) = 2f^2(x)$.
let $x = 0$,them
$f(0) = 0, 1$.
if $f(0) = 0$,then
$f(x) = g(x) = 0$,a contradiction.so

$f(0) = 1, f^2 + g^2 = 1$.
let $f(x) = cos(p(x)), g(x) = sin(p(x))$,by substituting we obtain
$cos(p(x - y)) = cos(p(x) - p(y))$,so if we let
$q(x) = p(x) - 2k\pi$ for a certain integer k,we get
$q(x - y) = q(x) - q(y)$ or
$q(x - y) = q(y) - q(x)$.
the former one satisfies Cauchy's function,by continuousness it's linear,hence
$f(x) = cos(Ax), g(x) = sin(Ax)$(1)
as for the latter one,let $y = 0$ we obtain
$p(x) = \frac{p(0)}{2} = 0$
is a constant,a contradiction!
hence all non-constant solutions are expressed as (1).$\square$

littletush
761 posts
Sep 18, 2011, 11:29 am PM #622
problem196.
$f : N_+ \to N_+$ is a function that for every pair of (m, n),
$f(f(m) + f(n)) = m + n$,
compute $f(2011)$.

anonymou...
1142 posts
Sep 18, 2011, 2:37 pm PM #623
what means N_+? the natural numbers strictly greater than 0?

Kingofmat...
2216 pos...
Sep 20, 2011, 4:50 pm PM #624
Yes. That refers to the positive integers.

Djurre
389 posts
Sep 20, 2011, 8:58 pm PM #625
Problem 196
$f : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ be a function that for every pair (m, n):
$f(f(m) + f(n)) = m + n$
compute $f(2011)$.

Solution

ZetaSelb...
138 posts
Sep 21, 2011, 1:28 am PM #626
Problem 197: Find all the functions $f : \mathbb{Z} \to \mathbb{Z}$ that satisfies

1. $f(m - 1) + f(m + 1) = f(m)$ for all $m \in \mathbb{Z}$
2. $f(mn) = f(m)f(n)$ if $\gcd(m, n) = 1$
3. $f(-m) = f(m)$ for all $m \in \mathbb{Z}$

Batominov...
1602 pos...
Sep 21, 2011, 3:36 am • 1 PM #627

> *littletush wrote:*
> problem196.
> $f : \mathbb{N} \to \mathbb{N}$ is a function that for every pair of (m, n),
> $f(f(m) + f(n)) = m + n$,
> compute $f(2011)$.

Obviously, f is an injective function. Let $n \geq 2$ be an integer. Putting $m := n$ gives $f\big(2f(n)\big) = 2n$. Moreover, $m := n + 1$ and $n := n - 1$ gives $f\big(f(n + 1) + f(n - 1)\big) = 2n$. Therefore, $2f(n) = f(n + 1) + f(n - 1)$ for every $n \geq 2$. The recursion resolves to $f(n) = a + bn$, where a and b are constants. Plugging in this result, we obtain $a = 0$ and $b = 1$.

> *ZetaSelberg wrote:*
> **Problem 197**: Find all the functions $f : \mathbb{Z} \to \mathbb{Z}$ that satisfies
>
> 1. $f(m - 1) + f(m + 1) = f(m)$ for all $m \in \mathbb{Z}$
> 2. $f(mn) = f(m)f(n)$ if $\gcd(m, n) = 1$
> 3. $f(-m) = f(m)$ for all $m \in \mathbb{Z}$

Well, the general solution of the recursion in the first condition is $f(n) = A\cos\left(\frac{\pi n}{3}\right) + B\sin\left(\frac{\pi n}{3}\right)$. Since f is even (the third condition), we get $B = 0$. Now, because $f(1) = f(1 \cdot 1) = f(1) \cdot f(1)$ by the second condition, we get $f(1) \in \{0, 1\}$. This concludes that $f \equiv 0$ or $f(n) = 2\cos\left(\frac{\pi n}{3}\right)$ for every n. What remains is simply checking that the two functions satisfy the conditions.

PS: I disagree with how you deduced both sentences here:

> *Djurre wrote:*
> So f is a bijective function.
> So $f(f(f(m) + f(n))) = f(m + n) = f(m) + f(n)$

ZetaSelb...
138 posts

Sep 21, 2011, 7:07 am PM #628

Problem 198:

Find all polynomials with real coefficients that satisfies

1. $\forall x, y, z$ such that $xy + yz + zx = 0$ we have $f(x-y) + f(y-x) + f(z-x) = 2f(x+y+z)$

proglote
940 posts

Sep 21, 2011, 7:21 am PM #629

> ZetaSelberg wrote:
> **Problem 198:**
>
> Find all polynomials with real coefficients that satisfies
>
> 1. $\forall x, y, z$ such that $xy + yz + zx = 0$ we have $f(x-y) + f(y-x) + f(z-x) = 2f(x+y+z)$

Take $x = 0$ and $y = 0$. We have $2f(0) = f(z)$, i.e. $f(x) = c \ \forall x$ for some real constant c. It is easy to see that c must equal zero, therefore $f \equiv 0$.

ZetaSelb...
138 posts

Sep 21, 2011, 9:16 am • 2 PM #630

Problem 199:

Find all nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$ that satisfies

1. $f(0) = 0$
2. $f(1) = 1$
3. $\forall x, y$ such that $x < 1 < y$ we have $f(x) + f(y) = f(x)f(y) + f(x + y - xy)$

socrates
1818 posts

Sep 21, 2011, 8:08 pm • 1 PM #631

> ZetaSelberg wrote:
> **Problem 199:**
>
> Find all nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$ that satisfies
>
> 1. $f(0) = 0$
> 2. $f(1) = 1$
> 3. $\forall x, y$ such that $x < 1 < y$ we have $f(x) + f(y) = f(x)f(y) + f(x + y - xy)$

See: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=17477&

Problem 200
Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(xy + x + y) + f(xy - x - y) = 2(f(x) + f(y)), \ \forall x, y \in \mathbb{R}.$$

Find f.
This post has been edited 3 times. Last edited by socrates, Sep 24, 2011, 1:26 am

proglote
940 posts

Sep 23, 2011, 1:30 am • 1 PM #632

> socrates wrote:
> **Problem 200**
> Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that
>
> $$f(xy + x + y) + f(xy - x - y) = 2(f(x) + f(y)), \ \forall x, y \in \mathbb{R}.$$
>
> Find f.

Solution over the integers

If there is any other function satisfying the given conditions, my best guess is that it is rather not elementary.

jjax
107 posts

Sep 23, 2011, 9:46 am • 2 PM #633

> socrates wrote:
> **Problem 200**
> Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that
>
> $$f(xy + x + y) + f(xy - x - y) = 2(f(x) + f(y)), \ \forall x, y \in \mathbb{R}.$$

> Find f.

Let $P(x,y)$ be the proposition that $f(xy+x+y)+f(xy-x-y)=2(f(x)+f(y))$.

$P(0,0):f(0)=0$.
$P(x,0):f(x)=f(-x)$.
$P(x,1):f(2x+1)+f(-1)=2f(x)+2f(1)$, so if $a=2f(1)-f(-1)=f(1)$, then $f(2x+1)=2f(x)+a$.
Likewise, $P(-x,1)$ tells us that $f(2x-1)=2f(x)+a$.

$P(2x+1,y):f(2xy+2x+2y+1)+f(2xy-2x-1)=2f(2x+1)+2f(y)$.
Left hand side: $f(2(xy+x+y)+1)+f(2(xy-x)-1)=2f(xy+x+y)+2f(xy-x)+2a$
$=4f(x)+4f(y)-2f(xy-x-y)+2f(xy-x)+2a$.
Right hand side: $2f(2x+1)+2f(y)=4f(x)+2f(y)+2a$.
Comparing these gives $2f(y)+2f(xy-x)=2f(xy-x-y)$.
Since the function is even, this means that $f(xy-x-y)=f(xy-x)+f(-y)$.

For $y\neq 1$, set $x=\dfrac{y}{y-1}$ into $f(xy-x-y)=f(xy-x)+f(-y)$:
$0=f(0)=f(xy-x-y)=f(xy-x)+f(-y)=f(y)+f(-y)=2f(y)$.
Thus, for all $y\neq 1$, $f(y)=0$. Since $f(1)=f(-1)=0$, f is the zero function.
Testing, it is a solution.

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jjax
107 posts

Sep 23, 2011, 10:21 am • 1 PM #634

> *socrates wrote:*
> **Problem 192**
> $f:\mathbb{R}\to\mathbb{R}$
> $f(x^2+y^2)=f(f(x))+f(xy)+f(f(y))\,\forall x,y\in\mathbb{R}$
> Find all functions f.

A simplistic descent into brute force: Let $f(0)=a$.

Setting $x=y=0$ gives $f(f(0))=0$.
Setting $y=0$ gives $f(x^2)=f(f(x))+a$.
Thus, the original equation becomes $P(x,y):f(x^2+y^2)+2a=f(x^2)+f(xy)+f(y^2)$

Comparing $P(x,y)$ with $P(-x,y)$ shows that $f(xy)=f(-xy)$ for all xy, so the function is even.
We will subsequently only use nonnegative numbers in the substitutions.

$P(\sqrt{x},\sqrt{x}):f(2x)+2a=3f(x)$
At this point, we repeatedly use this identity to "extract" 2s:
$P(\sqrt{x},2\sqrt{x}):f(5x)+2a=f(x)+f(2x)+f(4x)$
$=f(x)+4f(2x)-2a=13f(x)-10a$.
Thus, $f(5x)+12a=13f(x)$
The same method of "extracting" prime factors from the inside of fs is used repeatedly, here for 2, and now we can use it for 5 as well. To be concise, I will skip the spammed "extractions" and jump straight to the result.

$P(\sqrt{x},5\sqrt{x}):f(13x)+60a=61f(x)$ so we can "extract" 13.
$P(3\sqrt{x},4\sqrt{x}):9f(3x)+f(9x)=88f(x)-78a$.
$P(5\sqrt{x},12\sqrt{x}):39f(3x)+27f(9x)=1184f(x)-1118a$.
Eliminating $f(9x)$ from these two equations: $204f(3x)=1192f(x)-988a$.
Reducing by a factor of 4, $51f(3x)=298f(x)-247a$. Thus, we can "extract" 3 as well.

Taking $9f(3x)+f(9x)=88f(x)-78a$, multiplying through by 51^2 and reducing $f(9x)$ to $f(3x)$ and $f(3x)$ to $f(x)$,
we get $f(x)=a$, so the function is constant for nonnegative x, and thus for all x (by evenness)

$f(x^2+y^2)=f(f(x))+f(xy)+f(f(y))$ means that the constant value is zero, so $f(x)=0$ for all x.

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socrates
1818 posts

Sep 23, 2011, 4:52 pm • 1 PM #635

Another solution to 192

Problem 201

Determine all pairs of functions $f,g:\mathbb{Q}\to\mathbb{Q}$ satisfying the following equality

$$f(x+g(y))=g(x)+2y+f(y),$$

for all $x,y\in\mathbb{Q}$.

Batominov...
1602 pos...

Sep 23, 2011, 10:59 pm • 1 PM #636

I think the correct problem is this one:

> *ZetaSelberg wrote:*
> **Problem 198:**

Find all polynomials with real coefficients that satisfies

EDIT: I was in fact mistaken. All solutions are in the form ax^2+bx^4.
This post has been edited 1 time. Last edited by Batominovski, Sep 23, 2011, 11:13 pm

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jjax
107 posts

Sep 23, 2011, 11:07 pm • 1 PM #637

Problem 198 is IMO 2004 Q2.
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=99448&sid=70ca47a9ebdd1fc5ccee2bccd54d4425#p99448

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socrates
1818 posts

Sep 24, 2011, 1:24 am • 3 PM #638

Problem 200
Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that

$$f(xy+x+y)+f(xy-x-y)=2(f(x)+f(y)),\ \forall x,y\in\mathbb{R}.$$

Find f.

Another solution to 200

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littletush
761 posts

Sep 26, 2011, 11:13 am • 1 PM #639

solution 201
let $g(0)=c$.for $x=-g(y),y=0$ we get $g(-c)=0$.letting
$x=y-g(y)$ yields $g(y-g(y))=-2y$,hence
g can equal to any number in Q.for
$y=-c$ we obtain
$g(-c)=2c$,so$2c=0,c=0$.let $y=0$,then
$f(x)=g(x)+f(0)=g(x)+d$.so by substituting this into the original functuion we know that
$g(x+g(y))=g(y)+2y$,let
$x=0$ then $g(g(y))=g(y)+2y$,so
$g(x+g(y))=g(x)+g(g(y))$.let $g(y)=z$ and z can represent any number in Q,so
$g(x+z)=g(x)+g(z)$ holds in Q.this is Cauchy's function,hence g is linear in Q and it's easy to obtain
$g(x)=2x$ or $-x$.hence all solutions are
$g(x)=2x,f(x)=2x+d$ and $g(x)=-x,f(x)=-x+d$
where $d=f(0)$ is an arbitrary rational.

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littletush
761 posts

Sep 26, 2011, 11:16 am • 3 PM #640

problem 202
$f:R_+\to R_+$ is a function such that for every $x,y>0,f(x+y)=\dfrac{f^2(x)}{x^2}+f(y)+\dfrac{2f(xy)}{xy}$,compute $f(9\sqrt{51})$.

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hungnguye...
50 posts

Sep 26, 2011, 8:06 pm PM #641

~'$f(x)^2/x^2+f(y)=f(y)^2/y^2+f(x)$ for all x,y. So $f(x)^2/x^2-f(x)=k$. And we can find $f(1)=1$(not hard). Then $f(x)=x^2$}

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filipbitola
124 posts

Sep 26, 2011, 8:10 pm • 1 PM #642

> littletush wrote:
> problem 202
> $f:R_+\to R_+$ is a function such that for every $x,y>0,f(x+y)=\dfrac{f^2(x)}{x^2}+f(y)+\dfrac{2f(xy)}{xy}$,compute $f(9\sqrt{51})$.

Another solution:

Let $g(x)=f(x)-x^2$
We rewrite the given equation as:
$g(x+y)+(x+y)^2=\dfrac{(g(x)+x^2)^2}{x^2}+g(y)+y^2+\dfrac{2g(xy)+2x^2y^2}{xy}$
$g(x+y)+(x+y)^2=\dfrac{g(x)^2}{x^2}+2g(x)+x^2+g(y)+y^2+\dfrac{2g(xy)}{xy}+2xy$
$g(x+y)=\dfrac{g(x)^2}{x^2}+2g(x)+g(y)+\dfrac{g(xy)}{xy}$
Swapping the values for x and y, and comparing, we see that $g(x)=g(y)\forall x,y\in\mathbb{R}^+$
So, $f(x)=x^2+c$
Substituting in the original equation, we see that $c=0$.
So, $f(x)=x^2$, and $f(9\sqrt{51})=81*51=4131$