31^{st}

Iranian Mathematical Olympiad

Selected Problems with Solutions

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This booklet is prepared by Goodarz Mehr, Hesameddin Rajabzadeh and Morteza Saghafian.

With special thanks to Roohallah Mahkam and Erfan Salavati.

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Iranian Team Members in the 55^{th} IMO
(Cape Town - South Africa)

From left to right:

- Yeganeh Alimohammadi
- Shayan Gholami
- Seyed Mohammad Hosseion Seyedsalehi
- Arian HosseinGholizadeh
- Armin Behnamnia
- Ehsan Mokhtarian

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Preface

The 31^{st} Iranian National Mathematical Olympiad consisted of four rounds. The First Round was held on 14 February 2013 all over the country. The exam consisted of 15 short-answer questions and 10 multiple-choice questions to be solved in 3 hours. In total more than 25000 students participated in the exam and 1699 of them were validated for the next round.
The Second Round was held on 2 and 3 May 2013. In each day, participants were given 3 problems to be solved in 4.5 hours. In this round, from a total of 1699 participants, 43 of them were chosen to participate in the Third Round.
The examination of the Third Round consisted of five separate exams, and a final exam containing 8 questions, each having it's specified time to be solved in. In the end, 9 people were awarded Bronze Medal, 20 people were awarded Silver Medal, and 13 people were awarded Gold Medal, which the following list represents the names of the Gold Medalists:

Yeganeh Alimohammadi
Armin Behnamnia
Ali Daei Nabi
Mohammadreza Daneshvar Amoli
Khashayar Gatmiri
Shayan Gholami
Saman Hadian
Arian HosseinGholizadeh
Ehsan Mokhtarian
Pantea Naderian
Abolfazl Najafian
Seyyed Mohammadhossein Seyyedsalehi
Ali Zoelm

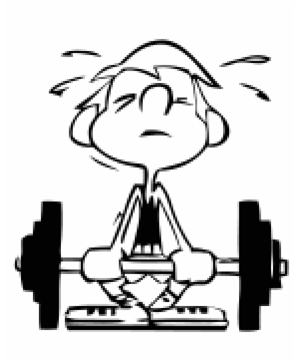
The Team Selection Test was held on 6 days, each day having 3 problems to be solved in 4.5 hours. In the end, the top 6 participants were chosen to participate in the 55^{th} IMO as members of the Iranian Team.
In this booklet, we present 6 problems of the Second Round, 8 problems of the final exam of the Third Round, and 18 problems of the Team Selection Test, together with their solutions.
It's a pleasure for the authors to offer their grateful appreciation to all the people who have contributed to the conduction of the 31^{st} Iranian Mathematical Olympiad, including the National Committee of Mathematics Olympiad, problem proposals, problem selection teams, exam preparation teams, coordinators, editors, instructors and all who have shared their knowledge and effort to increase the Mathematics enthusiasm in our country, and assisted in various ways to the conduction of this scientific event.

Problems

Second Round

1 . (Ali Khezeli) Find all coprime natural numbers a and b for which $b.a = \frac{a}{b}$ (Note that if $a = 92$ and $b = 13$, then $b.a = 13.92$). ($\rightarrow$ p.26)

2. (Pedram Safaei) Suppose that natural numbers $\omega_1, \omega_2, ..., \omega_n$ are the weights of n weights. We call this set of weights **complete** if for each natural number W less than $\omega_1 + \omega_2 + ... + \omega_n$, W can be represented as the sum of weights of some of these weights. Prove that if we omit the heaviest weight from this set, the resulting set of weights is still complete.



($\rightarrow$ p.26)

3 . (Mehdi E'tesami Fard) Let ABC be a triangle with circumcircle ω and circumcenter O. Denote by M the midpoint of that arc BC of ω which does not contain vertex A. Lines passing through O parallel to MB and MC intersect sides AB and AC at points K and L, respectively. If the perpendicular from vertex A to side BC intersects ω at point N, show that $NK = NL$. ($\rightarrow$ p.26)

4 . (Ali Zamani) Let C be a circle and P a point outside of it. PA and PB are the two tangent lines to this circle and point K is chosen arbitrarily on the segment AB. The circumcircle of triangle PBK intersects circle C for the second time at T. Let P' be the reflection of P with respect to A. Show that $\angle PBT = \angle P'KA$. ($\rightarrow$ p.28)

5 . (Omid Naghshineh Arjmand) In each unit cell of a $n \times m$ table an integer number is written. An **diagonal** is formed by those cells of the table for which the difference between their column number and their row number is a constant number. We want to make all the numbers in the table 0 during a finite number of steps. In each step, we can select a row, a column or a diagonal and add 1 or -1 to the number in every cell of it. Prove that if we can make all the numbers in each 3×3 subtable 0 without altering the rest of the table, then we can make all the numbers in the table 0. For example, in the 5×9 table below, cells of a diagonal and also cells of a 3×3 subtable are shown. Note that the rightmost upper cell (row 1, column 9) is also considered a diagonal.

	1	2	3	4	5	6	7	8	9
1						♣			
2		◊	◊	◊			♣		
3		◊	◊	◊				♣	
4		◊	◊	◊					♣
5									

($\rightarrow$ p.28)

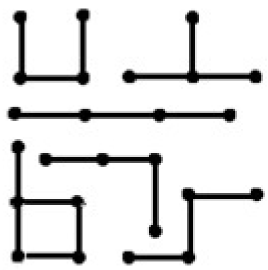
6 . (Morteza Saghafian) The sequence $\{a_n\}_{n=1}^{\infty}$ of natural numbers satisfies the following relation:

$$a_{n+2} = \lfloor \tfrac{2a_{n+1}}{a_n} \rfloor + \lfloor \tfrac{2a_n}{a_{n+1}} \rfloor,$$

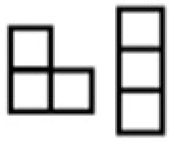
for which by $\lfloor x \rfloor$ we mean the integer part of x. Prove that there exists natural number m such that $a_m = 4$ and $a_{m+1} \in \{3, 4\}$. ($\rightarrow$ p.29)

Third Round

1 . (Morteza Saghafian) **Polystick!**
A n-**stick** is a connected figure made by n unity horizontal or vertical sticks. If two polysticks can be obtained from one another by rotation, translation or reflection, they are considered the same. For example, in the right figure all 3-sticks and one 5-stick can be seen.



A n-**mino** is a figure made by attaching n unity squares to one another from edges in such a way that for every two unity squares, there is a path of unity squares between them (i.e. the figure is connected).
Suppose that S_n is the number of n-sticks and M_n is the number of n-minoes. For example, from the two above figures it can be seen that $S_3 = 5$ and $M_3 = 2$.



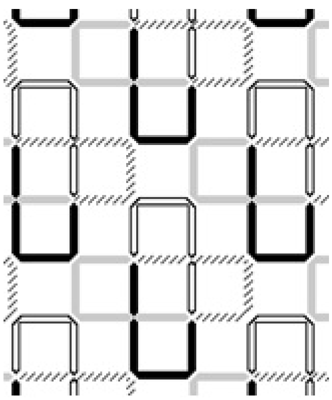
a) Prove that for every natural number n,

$$S_n \geq M_{n+1}.$$

b) Prove that for sufficiently large natural number n,

$$2.4^n \leq S_n \leq 16^n.$$

A **lattice edge** is a segment of unit length in the cartesian plane that the coordinates of its vertices are integer numbers. We call a polystick **wise** if it can be used to tile the lattice edges of the cartesian plane (using translation, rotation and reflection). Otherwise it is called **stupid**. For example, the figure below shows that the 4-stick is wise. Also, it can be seen that the 5-stick is stupid.



c) Prove that at least 2^{n-6} stupid n-sticks exist.
d) Prove that the polystick having the shape of a path that alternatively goes up or right is wise.
e) (Extra points) Show that for sufficiently large $n \in \mathbb{N}$,

$$3^n \leq S_n \leq 12^n.$$

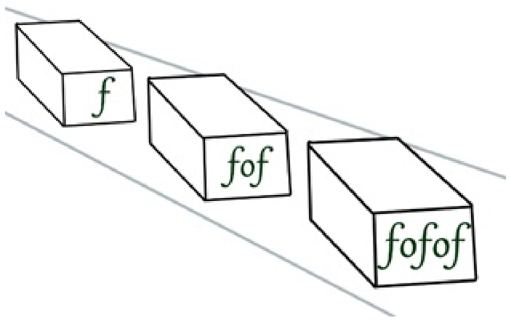
120 minutes (→ p.31)

2 . (Ja'far Namdar) **Distance Between Circles!**
The **distance** between two circles ω and ω' is defind as the length of their common external tangent and is represented as $d(\omega, \omega')$. If two circles don't have a common external tangent, the distance between them is not defined. Also, note that a point is a circle with zero radius and that the distance between two circles can be zero.
a) **Centroid**. Circles $\omega_1, \omega_2, ..., \omega_n$ are given in the plane where n is a natural number. Prove that a unique circle $\bar{\omega}$ exists such that for an arbitrary circle ω in that plane, the difference between the square of the distance between ω and $\bar{\omega}$ and the average of the squares of the distances between ω and ω_i ($1 \leq i \leq n$) is constant (for those circles ω that all these distances are defined). That is,

$$\forall\omega : d(\omega, \omega')^2 - \tfrac{1}{n} \textstyle\sum_{i=1}^{n} d(\omega, \omega_i)^2 = \text{constant}.$$

$\bar{\omega}$ is called the **centroid** of these circles, for it is similar to the centroid of n points in the plane.
b) **Perpendicular bisector**. Suppose that circle ω is equidistant from circles ω_1 and ω_2. Let ω_3 be an arbitrary circle whose center is on the centerline of circles ω_1 and ω_2 and is tangent to the common external tangent of circles ω_1 and ω_2. Prove that "the distance between ω and the centroid of circles ω_1 and ω_2" is not more than "the distance between ω and ω_3" (for the case that all these distances are defined).
c) **Circumcenter**. Let $\mathcal{C}$ be the set of all circles in the plane that each one of them is equidistant from three fixed circles ω_1, ω_2 and ω_3. Prove that a fixed point exists in the plane that is the direct homothetic center of each two circles in $\mathcal{C}$.
d) **Regular tetrahedron**. Do there exist four circles in the plane for which the distance between any two of them is unity? 150 minutes ($\rightarrow$ p.34)

3 . (Morteza Saghafian) **Function Generation!**



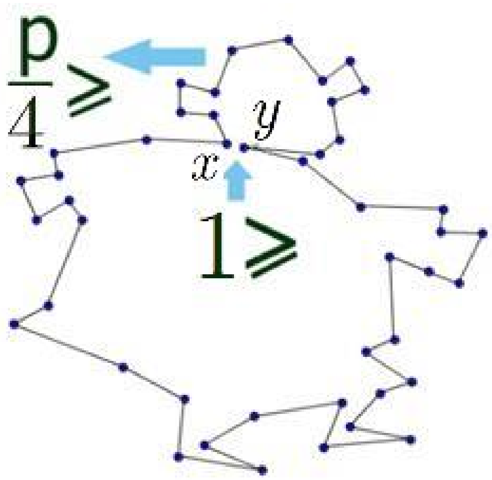
Function f is said to **generate** function g (and denote it by $f \rightarrow g$), if g can be written as the composition of function f with itself for several times; i.e., natural number k exists for which $f \circ f \circ \cdots \circ f = g$ (k times).
We want to explore some properties of this relation. For example, it can easily be shown that if $f \rightarrow g$ and $g \rightarrow h$, then $f \rightarrow h$. (transitivity)
a) Show that two functions $f \neq g : \mathbb{R} \rightarrow \mathbb{R}$ exist for which $f \rightarrow g$ and $g \rightarrow f$.
b) Prove that for each function $f : \mathbb{R} \rightarrow \mathbb{R}$ there are a finite number of functions $g : \mathbb{R} \rightarrow \mathbb{R}$ for which $f \rightarrow g$ and $g \rightarrow f$.
c) Does a function $g : \mathbb{R} \rightarrow \mathbb{R}$ exist such that the only function that generates it is itself?
d) Does there exist a function that generates functions x^3 and x^5?

e) Prove that if there exists a function that generates two linear polynomials P and Q, then there exists a linear polynomial that generates both P and Q.

75 minutes ($\rightarrow$ p.39)

4 . (Erfan Salavati) **Rotund Polygon!**

Simple polygon A with perimeter p is called a **rotund polygon**, if for each two points x and y on the perimeter of A that have a distance of at most 1 in the plane, their distance on A (i.e. the smaller part of the perimeter of A that lies between them) is at most $\frac{p}{4}$. We want to prove that a circle with radius $\frac{1}{4}$ can be drawn entirely inside a rotund polygon.



Intellectuals of the Earth and researchers of the planet **Hot Dog**! have devised two completely different approaches to solve the problem. In both approaches a **chord** is a line segment with its end points lying on the perimeter of the polygon. A **diameter** is a chord with its endpoints being vertices of the polygon. An **inner chord** is a chord that lies entirely inside or on the perimeter of the polygon. The **distance on the perimeter** between two points on the polygon is defined as the length of the smaller part of the perimeter between them.

The Earth approach: the maximum chord

We know as a fact that for each polygon, an inner chord xy with a length less than or equal to unity can be found such that for each inner chord $x'y'$ with a length less than or equal to unity, the distance on the perimeter of x and y is greater than or equal to the distance on the perimeter of x' and y'. This chord is called **the maximum chord**.

In a rotund polygon A_o there are two possibilities for the maximum chord:

a) **First possibility**: The length of the maximum chord equals unity. Prove that in this case a semicircle with the maximum chord as its diameter can fit completely inside the polygon A_o; therefore, a circle with radius $\frac{1}{4}$ can be drawn entirely inside the polygon.

b) **Second possibility**: The length of the maximum chord is less than unity. Prove that still in this case one can find a circle with radius $\frac{1}{4}$ that fits completely inside the polygon A_o.

Intellectuals of the Earth thought for many times that they have tackled the problem, but each time they noticed a small flaw in their proof, until they could finally solve it.

The Hot Dog approach: triangulation

Cosider the two statements below:

First statement: "For each arbitrary polygon with edges of at most unity length for which there are no circles of radius $\frac{1}{4}$ that could be fit entirely inside it, it's possible to triangulate it with diameters of at most unit length."
Second statement: "For each arbitrary polygon that a circle of radius $\frac{1}{4}$ cannot be fit entirely inside it, it's possible to triangulate it with chords of at most unit length."
Researchers of Hot Dog could prove that if the second statement is true, then one can draw a circle of radius $\frac{1}{4}$ completely inside a rotund polygon.
c) Prove that if the second statement is true, then one can draw a circle of radius $\frac{1}{4}$ completely inside a rotund polygon.
They could easily deduce that if the first statement is true, then the second one is also true. So they announced that a house full of Hot Dogs would be the reward of anyone who could prove the first statement! After a while, a young barber called **J.N** who considerd himself to be from the Earth, succeeded in contradicting the first statement and was rewarded the house, as promised.
d) Construct a 1392-gon that contradicts the first statement.
Still, researchers of Hot Dog have the hope to prove the second statement directly.
e) Write anything that you think might be true about the second statement.

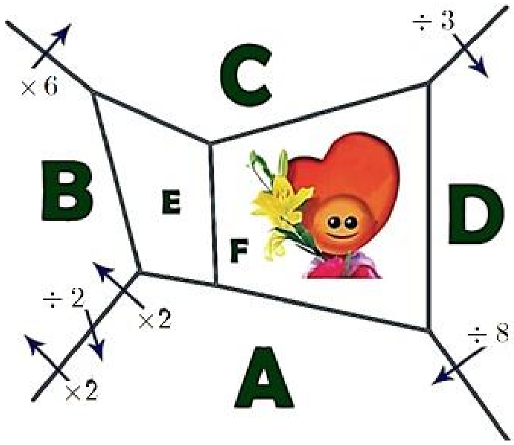
150 minutes (→ p.39)

5 . (Morteza Saghafian, Alireza Fallah, Pooya Honaryar) **In The Search Of Lost Numbers!**
A **partial sum** of n real numbers $a_1, a_2, ..., a_n$ is the sum of some of them; that is, $\epsilon_1 a_1 + \epsilon_2 a_2 + ... + \epsilon_n a_n$, where for each $1 \leq i \leq n$, ϵ_i is either 0 or 1 and at least one of them is nonzero. Now, having these partial sums, we want to find the numbers.
Years ago a valuable list containing n real (not necessarily distinct) numbers and all their $2^n - 1$ partial sums was shown to the public in a museum. Some strange creatures from the planet Hot Dog (after being defeated in solving the Rotund Polygon problem!) have stolen our original n numbers and the only thing that is left are those $2^n - 1$ partial sums.
a) Prove that if all the partial sums are positive, all the stolen numbers can be determined uniquely.
b) Suppose that some of the partial sums are positive and some of them negative, but none of them zero. Prove that still all the stolen numbers can be determined uniquely.
c) Prove that for $n = 1392$, an example can be constructed to show it's not possible to determine all of the stolen numbers uniquely, by only having their $2^n - 1$ partial sums.

75 minutes (→ p.42)

6 . (Amir Ali Moinfar) **Travelers Of The Planet Hot Dog!**
During interplanetary battles, skilled astronomers of the Earth discovered the planet Hot Dog in the Sandwich Galaxy. This planet has the shape of a convex n-hedral. By analyzing the spectrum of the waves emitted by this planet, scientists could discover some interesting facts about the strange way of life there. For example, each face of this planet is a country, each country has its own currency, and along the border between two countries, there is a **constant conversion factor** to chang currencies. All the people that pass through a border between two countries must change all their money into the currency of their destination and there is no other way for changing currencies. Surprisingly, astronomers observed that it's possible for a passenger to travel to different countries and without spending any money, upon reaching home own an amount of money different from the one owned when they started to travel.

Philosophers think that this is a result of the difference between those constant conversion factors and relative values of the currencies. For example, in the figure below, if someone starts from country A and travels to countries B, A, B, C and D and then travels back home, the amount of money owned by that person will be half of the amount owned upon starting the travel. But, if someone travels to a neighbouring country and returns, the amount of money owned won't change (because the product of the two conversion factors along a border is 1).



During a research project, a group of travelers were discovered that started to travel from the same country and owned the same amount of money. Each of them traveled a path in the form of a closed broken line and returned home! At most how many of these travelers will have mutually different amounts of money when they are back home?

Note 1.: The only parameter is the number of countries, n. All other things like the conversion factors and the arrangement of the countries are variables, so your answer must be only in terms of n.

Note 2.: None of the travelers spend money during the journey!

90 minutes (→ p.42)

7 . (Mohyeddin Motevassel) **Interesting Properties Of Interesting Equations!**

The equation $P(x) = Q(y)$ is called **interesting** if both P and Q are polynomials with integer coefficients and degree at least one and the equation has infinitely many solutions in natural numbers. We say that equation $F(x) = G(y)$ **results** from equation $P(x) = Q(y)$, if polynomial R with rational coefficients exists such that $F(x) = R(P(x))$ and $G(y) = R(Q(y))$.

a) Suppose that S is an infinite subset of the set $\mathbb{N} \times \mathbb{N}$. We say that S **satisfies** the interesting equation $P(x) = Q(y)$, if each element of it satistfies this equation. Prove that an interesting equation $P_o(x) = Q_o(y)$ exists such that every equation that is satisfied by S (if any), has resulted from the equation $P_o(x) = Q_o(y)$.

b) The degree of an interesting equation is defined as the largest degree between the degrees of both P and Q. We call an interesting equation **primitive** if it is not resulted from any other interesting equation with a lesser degree. Prove that if $P(x) = Q(y)$ is a primitive interesting equation and both P and Q are monadic polynomials, then the degrees of P and Q are coprime natural numbers. 120 minutes (→ p.44)

8 . (Mostafa Eynollahzadeh) **Rational Pentagon!**
Suppose that $A_1A_2A_3A_4A_5$ is a convex pentagon in the cartesian plane with all its vertices having rational coordinates. For each $1 \leq i \leq 5$, define B_i as the intersection point of the extensions of the sides $A_{i+1}A_{i+2}$ and $A_{i+3}A_{i+4}$ (the vertices of the pentagon are numbered in a cyclic manner, i.e for each $1 \leq i \leq 5$, $A_i = A_{i+5}$). Prove that at most three of the lines A_iB_i, $1 \leq i \leq 5$, are concurrent. 75 minutes (→ p.46)

Team Selection Test

1 . (Mehdi E'tesami Fard) Point O is the center of the circumcircle ω of the acute-angled triangle ABC. A circle centered at O tangent to side BC of the triangle is drawn. Let X and Y be the intersection points of tangents from A to this circle with side BC in such a way that points X and B are on one side of the line AO. A line parallel to side AC is drawn from X to intersect the tangent at point B to the circle ω at T. Similarly, a line parallel to side AB is drawn from Y to intersect the tangent at point C to the circle ω at S. Prove that line TS is tangent to ω. ($\rightarrow$ p.47)

2 . (Mohyeddin Motevassel) Find all Polynomials with integer coefficients like P such that the set $P(\mathbb{Z}) = \{P(a) : a \in \mathbb{Z}\}$ contains an infinite geometric progression.
($\rightarrow$ p.47)

3 . (Morteza Saghafian) Determine all $n \times n$ tables of nonnegative integers with rows and columns labeled by $0, 1, \ldots, n-1$ such that for each $0 \leq i, j \leq n-1$ the number in cell (i, j) be the number of i's of row j. One such table can be seen below.

	Column 0	Column 1	Column 2	Column 3	Column 4
Row 0	1	0	3	3	4
Row 1	1	3	2	1	1
Row 2	0	1	0	1	0
Row 3	2	1	0	0	0
Row 4	1	0	0	0	0

($\rightarrow$ p.48)

4 . (Mahyar Sefidgaran) Find the maximum number of permutations of the set $\{1, 2, ..., 2014\}$ such that for each two distinct numbers a and b of this set, one could find at most one permutation in which b has appeared exactly after a. ($\rightarrow$ p.51)

5 . (Morteza Saghafian) Let n be a natural number. For positive real numbers $x_1, x_2, ..., x_{n+1}$ with unit product prove that

$$\sqrt[x_1]{n} + \sqrt[x_2]{n} + ... + \sqrt[x_{n+1}]{n} \geq n^{\sqrt[n]{x_1}} + n^{\sqrt[n]{x_2}} + ... + n^{\sqrt[n]{x_{n+1}}}.$$

($\rightarrow$ p.52)

6 . (Ali Zamani) Suppose that I is the incenter of triangle ABC. The perpendicular to line AI from point I intersects sides AC and AB in points B' and C', respectively. Points B_1 and C_1 are placed on half-lines BC and CB, respectively, in such a way that $AB = BB_1$ and $AC = CC_1$. If T is the second intersection point of the circumcircles of triangles AB_1C' and AC_1B', prove that the circumcenter of triangle ATI lies on the line BC. ($\rightarrow$ p.52)

7 . (Mostafa Eynollahzadeh) The vertices of an n-vertex tree are labled by numbers $1, 2, \ldots, n$. At each stage one can choose an edge that has not been selected before, and switch the lables of it's two ends, until all of the edges have been selected. Show that the final permutation of the lables is a complete cycle of n letters. ($\rightarrow$ p.53)

8 . (Mehdi E'tesami Fard) Let D be a variable point on side BC of triangle ABC. The incenters of triangles ABC, ABD and ACD are I, I_1 and I_2, respectively. Points M and N are the second intersection points of circumcircles of triangles IAI_1 and IAI_2 with the circumcircle of triangle ABC, respectively. Prove that as D varies on side BC, line MN passes through a fixed point in the plane. ($\rightarrow$ p.53)

9 . (Mohyeddin Motevassel) Prove that for each natural number $k > 1$, only a finite number of k-tuples of consecutive natural numbers exist such that their product square.

($\rightarrow$ p.53)

10 . (Mohsen Jamali) Let n be a natural number. Permutation $a_1, a_2, ..., a_n$ of numbers $1, 2, ..., n$ is called square (cubic), if for each natural number $1 \leq i \leq n-1$, $a_i a_{i+1} + 1$ is a perfect square (cube).
a) Prove that for infinitely many natural numbers n there exists at least one square permutation of numbers $1, 2, ..., n$.
b) Prove that for no natural number n there exists a cubic permutation of numbers $1, 2, ..., n$. ($\rightarrow$ p.54)

11 . (Jafar Namdar) Suppose that x, y and z are positive real numbers and $x^2 + y^2 + z^2 = x^2y^2 + y^2z^2 + z^2x^2$. Prove that

$$(x-y)^2(y-z)^2(z-x)^2 \leq (x^2-y^2)^2 + (y^2-z^2)^2 + (z^2-x^2)^2.$$

($\rightarrow$ p.54)

12 . (Morteza Saghafian, Ali Khezeli, Pooya Honaryar) There are n line segments in the plane such that no two of them intersect with each other and among the $2n$ endpoints of these line segments no three are collinear. Is it always possible to find a simple $2n$-gon such that its vertices are the endpoints of these line segments and also each of these line segments is completely inside or on the perimeter of this $2n$-gon?

($\rightarrow$ p.55)

13 . (Meysam Aghighi) Points A_1 and A_2 are the intersection points of the incircle and the angle bisector of vertex A with side BC of triangle ABC, respectively. Points B_1, B_2, C_1 and C_2 are defined similarly. Suppose that the perpendicular from A_1 to line B_2C_2 intersects the angle bisector of vertex A in A'. Points B' and C' are defined similarly. Prove that the two triangles $A_1B_1C_1$ and $A'B'C'$ are congruent. ($\rightarrow$ p.55)

14 . (Morteza Saghafian) Does there exist a non-identity function $f : \mathbb{N} \rightarrow \mathbb{N}$ that:

The number of divisors of m is $f(n)$, if and only if the number of divisors of $f(m)$ is n, for each two natural numbers m and n. ($\rightarrow$ p. 56)

15 . (Mohammad Ja'fari) m and n are two natural numbers and $p(x)$, $q(x)$ and $h(x)$ are polynomials with real coefficients such that $p(x)$ is a decreasing polynomial. Also, for each real number x

$$p(q(nx+m) + h(x)) = n(p(q(x)) + h(x)) + m.$$

Prove that function $g : \mathbb{R} \rightarrow \mathbb{R}$ does not exist such that

$$g(q(p(x)) + h(x)) = (g(x))^2 + 1.$$

($\rightarrow$ p.56)

16 . (Goodarz Mehr) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all positive real numbers x and y,

$$f(\tfrac{y}{f(x+1)}) + f(\tfrac{x+1}{xf(y)}) = f(y).$$

(→ p.56)

17 . (Mahyar Sefidgaran) Set $X = \{x_1 < x_2 < ... < x_n\}$ of natural numbers has the property that for each $1 < i \leq n$, $1 \leq x_i - x_{i-1} \leq 2$. A real number a is called **good** if there exists a $1 \leq j \leq n$ such that $|x_j - a| \leq \frac{1}{2}$. Also, a subset of X is called **dense** if the average of its members is a good number. Show that at least 2^{n-3} subsets of the set X are dense. (→ p.57)

18 . (Ali Zamani) Let I be the incenter of triangle ABC. X is a point on arc BC of the circumcircle of triangle ABC such that if E and F are the feet of the perpendiculars from X to BI and CI, respectively, and M is the midpoint of EF, then $MB = MC$. If D is the foot of the perpendicular from I to BC, show that $\angle BAD = \angle CAX$.

(→ p.58)

Solutions

Second Round

1 . Let k be the number of digits of a. Therefore, $10^{k-1} \leq a < 10^k$. We have $\frac{a}{b} = b.a = b + \frac{a}{10^k}$ and consequently $\frac{a-b^2}{b} = \frac{a}{10^k}$. Since $(a,b) = 1$, $(a-b^2, b) = 1$; therefore, $\frac{a-b^2}{b}$ is the irreducible form of $\frac{a}{10^k}$. Thus, for some natural number s, $sb = 10^k$ and $s(a-b^2) = a$.

Now, $a - b^2 | a$ and $(a-b^2, a) = 1$. Hence, $a - b^2 = \pm 1$, but $\frac{a-b^2}{b} = \frac{a}{10^k} > 0$, so $a - b^2 = 1$. Therefore, $a = b^2 + 1$ and $\frac{1}{b} = \frac{a}{10^k}$.

Since $a \geq 10^{k-1}$, we have $\frac{1}{b} = \frac{a}{10^k} \geq \frac{1}{10}$ which implies $b \leq 10$. On the other hand, $ab = b(b^2+1) = 10^k$, so prime factors of b and b^2+1 are only 2 or 5. Since $b \leq 10$, we have $b \in \{1, 2, 4, 5, 8, 10\}$. A simple calculation shows that only for $b = 2$, $b(b^2+1)$ is a power of 10. For this case, $a = 5$ and $k = 1$. Therefore, $\frac{5}{2} = 2.5$, this is the only solution.

2 . Throughout the solution by $\{\omega_1 \leq \cdots \leq \omega_n\} \to a$, we mean that $a \in \mathbb{N}$ can be represented as a sum of some of the ω_i's and say that the set $\{\omega_1 \leq \cdots \leq \omega_n\}$ generates a.

We claim that natural numbers $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n$ form a complete set, if and only if

$$\omega_1 = 1, \quad \omega_i \leq \omega_1 + \cdots + \omega_{i-1} + 1 \text{ (for every } 2 \leq i \leq n). \tag{1}$$

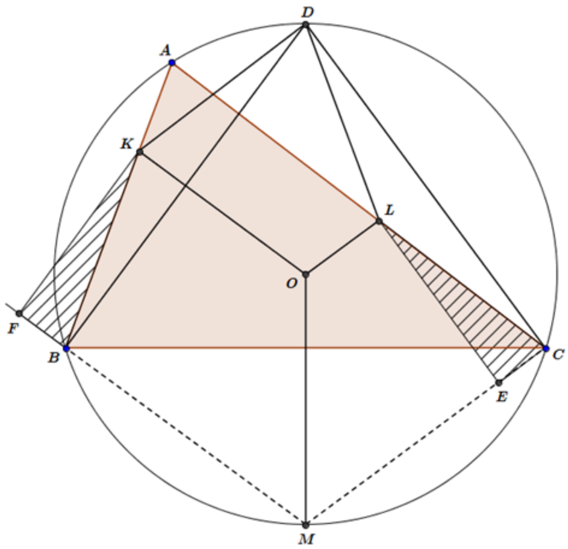
One part of the claim is trivial, because if $\omega_1 > 1$ or $\omega_i > \sum_{j=1}^{i-1} \omega_j + 1$ for some $2 \leq i \leq n$, then 1 or $\sum_{j=1}^{i-1} \omega_j + 1$ cannot be generated by the set $\{\omega_1 \leq \cdots \leq \omega_n\}$.

For the other part, suppose that numbers $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n$ satisfy (1). Our goal is to show that they form a complete set. This is proved by induction on n. The base case $n = 1$ is trivial. Now, suppose that the assertion is true for $n-1$. Since $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n$ satisfies (1), $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_{n-1}$ satisfies (1) too. Thus, by the induction hypothesis $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_{n-1}$ is a complete set. This means that for every natural number $W \leq \sum_{j=1}^{n-1} \omega_j$, $\{\omega_1 \leq \cdots \leq \omega_n\} \to W$. For every $\sum_{i=1}^{n-1} \omega_i < W \leq \sum_{i=1}^{n} \omega_i$ we have

$$-1 \leq \sum_{i=1}^{n-1} \omega_i - \omega_n < W - \omega_n \leq \sum_{i=1}^{n-1} \omega_i.$$

Now, if $W = \omega_n$, then obviously $\{\omega_1 \leq \cdots \leq \omega_n\} \to W$. If $1 \leq W - \omega_n \leq \sum_{i=1}^{n-1} \omega_i$, then by the induction hypothesis $\{\omega_1 \leq \cdots \leq \omega_{n-1}\} \to W - \omega_n$. Adding ω_n to this set, we get $\{\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n\} \to W$. This shows that $\{\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n\}$ is a complete set and the proof is complete.

3 . First, we prove that $KB = LC$.



Let F and E be the feet of perpendiculars from K and L to MB and MC, respectively. We have

$$OK = MB \quad \Rightarrow \quad KF = \text{The distance from } O \text{ to } MB.$$
$$OL = MC \quad \Rightarrow \quad LE = \text{The distance from } O \text{ to } MC.$$

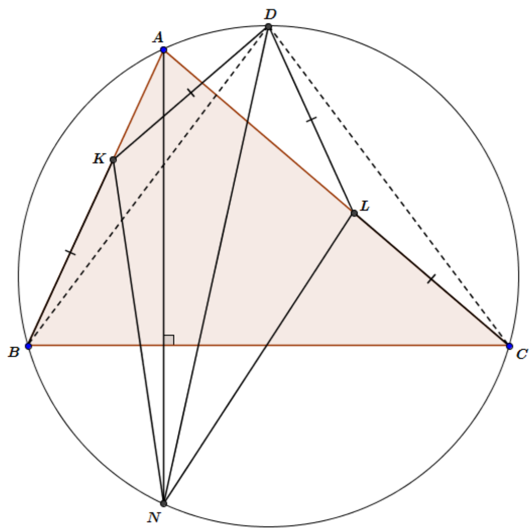
Since $MB = MC$, O (circumcenter of triangle MBC) is equidistant from these equal chords, and therefore $KF = LE$ (1). On the other hand, since $ABMC$ is cyclic, $\angle KBF = \angle LCE$ (2).
(1) and (2) imply that the two right-angled triangles KBF and LCE are congruent, so their hypotenuses have the same length ($KB = LC$).
Let D be the midpoint of arc $B\overset{\frown}{A}C$. We have

$$\left.\begin{array}{c} DB = DC \\ \angle DBA = \angle DCA \\ KB = LC \end{array}\right\} \Rightarrow D\overset{\triangle}{B}K \equiv D\overset{\triangle}{C}L \Rightarrow \begin{cases} \angle KDL = \angle BDC = \angle BAC \\ KD = LD \end{cases}$$

Since $\angle KOL = \angle BMC$, we deduce that $AKOLD$ is cyclic. Therefore,

$$\left.\begin{array}{c} \angle AKD = \angle ALD = \angle AOD = \angle B - \angle C \\ \angle KBD = \angle LCD = \frac{\angle B - \angle C}{2} \end{array}\right\} \Rightarrow \angle KDB = \angle LDC = \frac{\angle B - \angle C}{2}$$

$$\Rightarrow KB = KD = LD = LC \text{ (3)}.$$

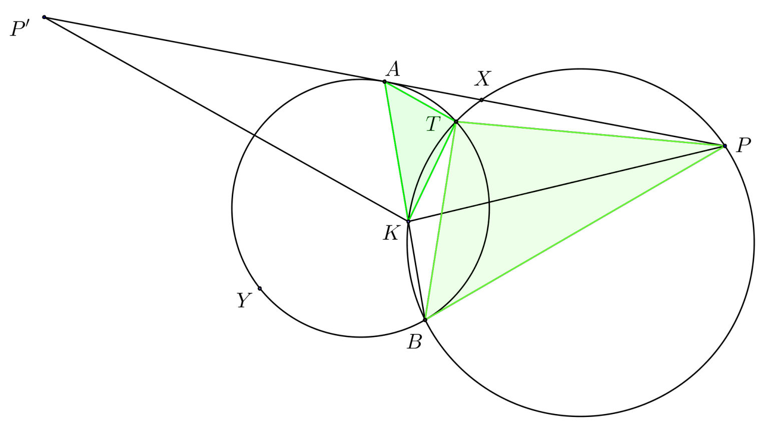


$$\left.\begin{array}{l}\angle NDK = \angle NDB + \angle BDK = (90^\circ - \angle B) + (\frac{\angle B - \angle C}{2}) = \frac{\angle A}{2} \\ \angle NDL = \angle NDC - \angle CDL = (90^\circ - \angle C) - (\frac{\angle B - \angle C}{2}) = \frac{\angle A}{2}\end{array}\right\}$$

$$\Rightarrow \angle NDK = \angle NDL \quad (4).$$

(3) and (4) imply that two triangles NDK and NDL are congruent. Therefore, we have $NK = NL$.

4 . In this solution, all of the arcs considered are from circle C.



Since quadrilatral $KTPB$ is cyclic, $\angle AKT = \angle BPT$.

$$\left.\begin{array}{c}\angle TAK = \frac{\overset{\frown}{TB}}{2} = \angle TBP \\ \angle AKT = \angle BPT\end{array}\right\} \Rightarrow T\overset{\triangle}{A}K \sim T\overset{\triangle}{B}P.$$

Therefore,

$$\frac{TA}{TB} = \frac{AK}{BP} = \frac{AK}{AP'} \Rightarrow \frac{AP'}{TB} = \frac{AK}{TA} \quad (1).$$

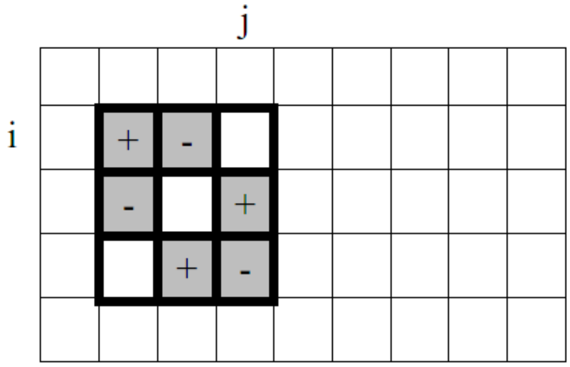
On the other hand,

$$\left.\begin{array}{c}\angle P'AK = \frac{\overset{\frown}{AYB}}{2} = \angle BTA \\ (1)\end{array}\right\} \Rightarrow P'\overset{\triangle}{A}K \sim B\overset{\triangle}{T}A.$$

So, $\angle P'KA = \angle BAT = \angle PBT$, and the assertion is proved.

5 . Let $A_{i,j}$ denote the number in the cell located in the i-th row and j-th column of the table, and $S(i,j)$ the 3×3 subtable whose rightmost upper number is $A_{i,j}$
First, assume that $m, n \geq 3$ (If $m \leq 2$ or $n \leq 2$, there is no 3×3 subtable). For $S(i,j)$, consider the following value

$$M(i,j) = A_{i+1,j} - A_{i+2,j} + A_{i+2,j-1} - A_{i+1,j-2} + A_{i,j-2} - A_{i,j-1}.$$



Note that if we choose a row, a column or a diagonal of the table and add 1 or -1 to the number in every cell of it, this value stays invariant.
Since we can make all the numbers in each subtable zero, we have

$$M(i,j) = 0 \quad \forall 1 \le i \le n-2, 3 \le j \le m.$$

The assertion is proved by induction on $m+n$. If $m+n=6$, then $m=n=3$ and the assertion is trivial. If $m+n>6$, either $m>3$ or $n>3$. Without loss of generality we can assume that $m>3$. By omitting the last column, we get a $n \times (m-1)$ table. By the induction hypothesis, we can make the numbers in the cells of this $n \times (m-1)$ subtable zero, using allowable operations (except for the last column and the diagonal $A_{1,m}$ of the initial table).
Now, the numbers in the last column of the table may be nonzero. Considering the invariant values $M(i,m)$ $(1 \le i \le n-2)$ implies that $A_{i+1,m} = A_{i+2,m}$ for $1 \le i \le n-2$. So $A_{2,m} = A_{3,m} = \cdots = A_{n,m}$. By using the last column several times all the numbers in the table expect $A_{1,m}$ may become zero. But $A_{1,m}$ is itself a diagonal, hence we can also make this number zero.
If $m+n<6$, where either $m \le 2$ or $n \le 2$, the proof is very easy and is left to the reader.

6 . First we prove some lemmas.

Lemma 1. *$a_n \ge 3$ for all natural numbers $n \ge 3$.*

Proof. Suppose that $a_n < 3$ for some natural nnumber $n \ge 3$. If $a_{n-1} \ge a_{n-2}$, then

$$\frac{2a_{n-1}}{a_{n-2}} \ge 2 \Rightarrow \left[\frac{2a_{n-1}}{a_{n-2}}\right] \ge 2 \Rightarrow \left[\frac{2a_{n-2}}{a_{n-1}}\right] = 0 \Rightarrow a_{n-1} > 2a_{n-2} \Rightarrow \frac{2a_{n-1}}{a_{n-2}} > 4.$$

This is a contradiction since $a_n < 3$. If $a_{n-2} \ge a_{n-1}$, the proof is similar to the former case because the recursive relation for a_n is symmetric with respect to both a_{n-1} and a_{n-2}. $\square$

Lemma 2. *For each natural number $n \ge 3$,*

$$a_{n+1} = a_n, \text{ or } a_{n+2} < \max\{a_n, a_{n+1}\}.$$

Proof. Suppose that $a_{n+1} \ne a_n$ for some $n \in \mathbb{N}$. If $a_n = \max\{a_n, a_{n+1}\}$, $\frac{2a_{n+1}}{a_n} < 2$. On the other hand,

$$a_{n+1}, a_n \ge 3 \Rightarrow \frac{2a_n}{a_{n+1}} \le \frac{2a_n}{3} \Rightarrow a_{n+2} = \left[\frac{2a_{n+1}}{a_n}\right] + \left[\frac{2a_n}{a_{n+1}}\right] \le 1 + \frac{2a_n}{3} \le \frac{a_n}{3} + \frac{2a_n}{3} = a_n.$$

Now, if $a_{n+2} \geq \max\{a_n, a_{n+1}\} = a_n$, the above inequalities will become equalities; therefore, $a_n = a_{n+1} = 3$ which contradicts our assumption $a_n \neq a_{n+1}$. The case $a_{n+1} = \max\{a_n, a_{n+1}\}$ is proved similarly. □

Lemma 3. *There exists some natural number k such that $a_k = a_{k+1}$.*

Proof. Assume to the contrary that the equality never holds, hence by lemma 2

$$a_{n+2} < \max\{a_n, a_{n+1}\},$$

and

$$a_{n+3} < \max\{a_{n+1}, a_{n+2}\} < \max\{a_n, a_{n+1}\}.$$

Therefore,

$$\max\{a_{n+2}, a_{n+3}\} < \max\{a_n, a_{n+1}\}.$$

Thus $\max\{a_{2k}, a_{2k+1}\}$ is a strictly decreasing sequence of natural numbers. But there is no such sequence; therefore, our initial assupmtion is not true and the lemma is proved.

□

By lemma 3, there exists some natural number $k \geq 1$ such that $a_k = a_{k+1}$. This implies that $a_{k+2} = 4$. Now, if $a_k = a_{k+1} \in \{3, 4\}$, then $a_{k+3} \in \{3, 4\}$ and there is nothing left to prove, but if $a_k = a_{k+1} > 4$, there exists some natural number m such that $a_{k+m} = a_{k+m+1}$. Using the proof of lemma 3, we have:

- If m is odd,

$$a_{k+1} = \max\{a_{k+1}, a_{k+2} = 4\} > \max\{a_{k+3}, a_{k+4}\} > \dots$$

$$> \max\{a_{k+m}, a_{k+m+1}\} = a_{k+m}.$$

- If m is even,

$$a_{k+1} = \max\{a_{k+1}, a_{k+2} = 4\} > \max\{a_{k+3}, a_{k+4}\} > \dots$$

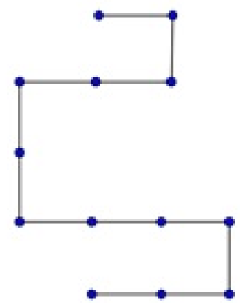
$$> \max\{a_{k+m-1}, a_{k+m}\} \geq a_{k+m}.$$

Therefore, if two equal consecutive terms in the sequence are greater than 4, there is another pair of equal consecutive terms in the sequence with a smaller value. We can continue this until we find two equal consecutive terms less than or equal 4. This completes the proof because the terms after that pair are $4, 3$ or $4, 4$.

Third Round

1 . a) We build a graph by drawing line segments joining the centers of adjacent squares of each $(n+1)$-mino. Obviously this graph is connected and hence has at least n edges. Now, consider a subtree of this graph with n edges. It is a $n-$stick. On the other hand, the $(n+1)$-mino corresponding to each $n-$stick is uniquely determined (if such an $(n+1)$-mino exists). So

$$S_n \geq \text{The number of } n\text{-sticks without any cycles} \geq M_{n+1}.$$



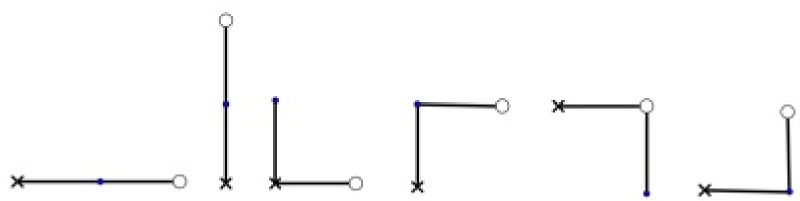
b) Consider the lattice paths of length n which alternatively go up, right or left. There is an obvious bijection between these paths and strings of length n with letters from the set $\{\rightarrow, \leftarrow, \uparrow\}$, provided that there are no two adjacent $\rightarrow$ and $\leftarrow$ letters. Let A_n be the number of such strings. We know that $8S_n \geq A_n$, because we can put each n-stick in the plane in at most 8 different ways. It is easy to see that the sequence A_n satisfies the following relations:

$$\begin{aligned} &A_1 = 3, A_2 = 7 \\ &A_n = 2A_{n-1} + A_{n-2} \quad n \geq 2. \end{aligned}$$

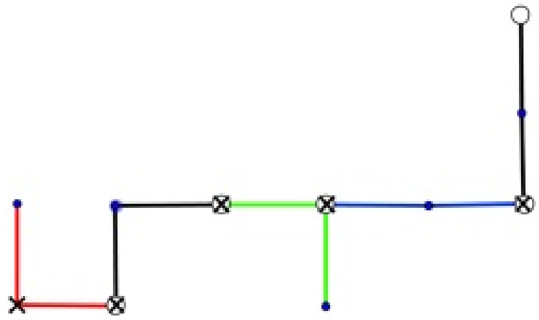
Therefore, the explicit form for A_n is:

$$A_n = \frac{1}{2}\left((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right),$$

but $1+\sqrt{2} > 2.41$ and $|1-\sqrt{2}| < 1$. So for sufficiently large values of n, we have $S_n \geq \frac{A_n}{8} \geq (2.41)^n$.
As yet another approach, consider all 2-sticks that none of them can be mapped to another one using only translations. There are only 6 such 2-sticks. In each of them, we mark the topmost vertex among the rightmost vertices by $\circ$, and the bottommost vertex among the leftmost vertices by $\times$. The following figure can help visualize the procedure.



For sufficiently large values of n, we choose $\left[\frac{n}{2}\right]$ 2-sticks and put the $\times$ vertex of each one on the $\circ$ vertex of the previous. For odd values of n we add an edge to the $\circ$ vertex of the last $2-$stick.



We have 6 ways of choosing each 2-stick and hence there are at least $6^{\left[\frac{n}{2}\right]}$ n-sticks that do not map to each other using only translations. Now, since each n-stick can be seen at most 8 times among these polysticks we get:

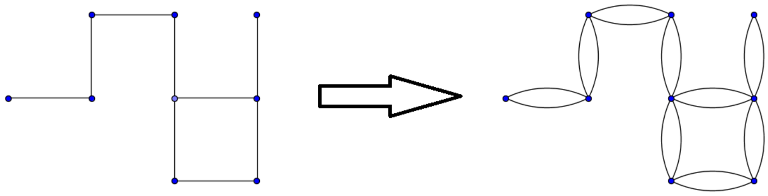
$$8S_n \geq 6^{\left[\frac{n}{2}\right]} \geq \sqrt{6}^{n-1}.$$

But $\sqrt{6} > 2.4$, so $S_n \geq (2.4)^n$.
To prove the upper bound we need the following lemma:

Lemma 1. *In a connected graph with even degree vertices there is a cycle passing through each edge exactly once. This cycle is called an Eulerian tour.*

Proof. It is a well-known theorem and the proof is very easy. $\square$

Unfortunately, we cannot use this lemma here, because the degree of vertices in an arbitrary n-stick are not necesserily even. For this reason, we can draw each edge of the polystick twice.



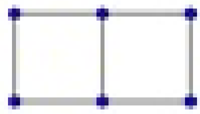
Now, each vertex in this new graph has an even degree and hence by the previous lemma there is a cycle passing through all the edges. Note that this new graph has $2n$ edges, therefore an upper bound for S_n is the number of lattice cycles with $2n$ edges. Hence we will estimate the number of such cycles.
The number of cycles of length $2n$ is at most as large as the number of strings consisting of letters $\leftarrow, \rightarrow, \uparrow, \downarrow$ such that

$$\begin{aligned} \text{Number of } \rightarrow\text{'s} &= \text{Number of } \leftarrow\text{'s} \\ \text{Number of } \uparrow\text{'s} &= \text{Number of } \downarrow\text{'s}. \end{aligned}$$

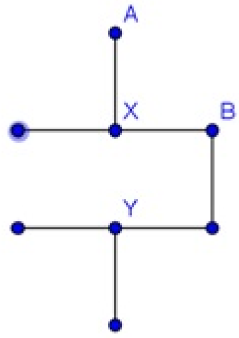
By a combinatorial arguments it is easy to see that the number of such strings is

$$\sum_{k=1}^{n} \binom{2n}{k}\binom{2n-k}{k}\binom{2n-k}{n-k} = \binom{2n}{n}^2 < 16^n.$$

Therefore, $S_n \leq 16^n$.
c) The following figure shows a stupid 7-stick.



For $n > 7$, if we start from A or B in the former 7-stick and add $n-7$ horizontal or vertical edges, the final polystick is obviously stupid, because the segment xy in the polystick which covers it does not have any adjacent edges.



d) Consider an infinite path (say L) in the plane that alternatively goes up or right.

Lemma 2. *Let L_n be the translation of L by vector $(-n, n)$. Then for integers $i < j$, L_i and L_j do not have any common vertices.*

Proof. Let (x, y) be a common vertex of both L_i and L_j. Then

$$(x, y) \in L_i \Rightarrow (x + i, y - i) \in L$$

$$(x, y) \in L_j \Rightarrow (x + j, y - j) \in L.$$

This is a contradiction, since L goes alternatively up or right. □

Let L' be the reflection of L with respect to the line $y = x$.

Lemma 3. *L and L' do not have a common edge.*

Proof. Assume to the contrary that L and L' have some edge e in common. So e and its reflection with respect to the line $y = x$ are both edges of L which is a contradiction, because L goes alternatively up or right. □

Therefor, L and L' and all their translations by vectors of the form $(-n, n)$ do not have an edge in common. Note that these paths cover all the lattice edges, because for every edge e we can find an integer n such that translation of e by vector $(-n, n)$ intersects L. Now, if they intersect in an edge, e lies in some translation of L and if they intersect in a vertex, e lies in some translation of L'. Finally, note that L can be built by the head to tail attachment of an infinte number of polysticks that go alternatively up or right.
e) If we use 4-sticks instead of 2-sticks in the second solution of part b, we get

$$S_n \geq (\sqrt[4]{88})^n \simeq (3.06)^n.$$

To prove the upper bound, we introduce at most 16×8^n connected polysticks. We claim that each n-sticks has appeared among them at least once. First, mark one vertex in the plane. We have $16 = 2^4$ ways for choosing its adjacent edges. Therefore, we have 16 polysticks. Now, in each step mark the topmost vertex among the rightmost unmarked vertices of each polystick. We can draw edges adjacent to this vertex in at most 8 ways. Obviously, we can reach any n-stick by n times repeating this process. On the other hand, note that the number of polysticks that are made in the i-th step of the process is 16×8^i. So

$$S_n \leq 16 + 16 \times 8 + \cdots + 16 \times 8^{n-1} \leq 16 \times 8^n.$$

Obviously, for large values of n, $16 \times 8^n < 12^n$.

2 . Denote by $C(O, R)$ the circle with center O and radius R. If $\omega_1 = C(O_1, R_1)$ and $\omega_2 = C(O_2, R_2)$, then

$$d(\omega_1, \omega_2)^2 = O_1O_2{}^2 - (R_1 - R_2)^2,$$

and $d(\omega_1, \omega_2)$ is defined if and only if the right hand side is nonnegative. This is equivalent to the existence of the common external tangents of these circles.
a) Let $\omega = C(O, R)$ and $\omega_i = C(O_i, R_i)$ for $1 \leq i \leq n$. We have

$$\frac{1}{n}\sum_{i=1}^{n} d(\omega, \omega_i)^2 = \frac{1}{n}\sum_{i=1}^{n} OO_i^2 + \frac{1}{n}\sum_{i=1}^{n}(R - R_i)^2.$$

Suppose that $\bar{R} = \frac{1}{n}(R_1 + R_2 + \cdots + R_n)$. Then

$$\frac{1}{n}\sum_{i=1}^{n}(R - R_i)^2 = R^2 - 2R\bar{R} + \frac{1}{n}\sum_{i=1}^{n} R_i^2 = (R - \bar{R})^2 + \frac{1}{n}\sum_{i=1}^{n}(\bar{R} - R_i)^2.$$

Similarly, let $\bar{O}$ be the centroid of all the O_i's ($1 \leq i \leq n$). Then

$$\frac{1}{n}\sum_{i=1}^{n} OO_i{}^2 = O\bar{O}^2 + \frac{1}{n}\sum_{i=1}^{n}\bar{O}O_i^2.$$

Therefore, the circle $C(\bar{O}, \bar{R})$ satisfies the problems condition. Now, if two circles $\bar{\omega_1} = C(P_1, r_1)$ and $\bar{\omega_2} = C(P_2, r_2)$ both satisfy the problems condition, for each circle ω we can write (C is a constant value here):

$$\begin{aligned} &d(\omega, \bar{\omega}_1)^2 - d(\omega, \bar{\omega}_2)^2 = \mathrm{C} \\ &\Rightarrow OP_1^2 - OP_2^2 + (R - r_1)^2 - (R - r_2)^2 = \mathrm{C} \\ &\Rightarrow OP_1^2 - OP_2^2 - 2R(r_1 - r_2) = \mathrm{C} \end{aligned}$$

By fixing O, we have $r_1 = r_2$ and hence $OP_1^2 - OP_2^2$ is a constant value. Thus $P_1 = P_2$ and $\bar{\omega_1} = \bar{\omega_2}$. Therefore, $\bar{\omega} = C(\bar{O}, \bar{R})$ is the unique circle satisfying the problems condition.
b) Let $\omega_i = C(O_i, R_i)$ for $1 \leq i \leq n$, and $\omega = C(O, R)$. Furthermore, without loss of generality we can assume that $O_i = (x_i, 0)$ and $O = (x, y)$.
We can write $x_3 = (1 - \alpha)x_1 + \alpha x_2$ for some $\alpha \in \mathbb{R}$. Since the distances of O_1 and O_2 from the common tangent are R_1 and R_2, respectively, we conclude that the distance

of O_3 from this line is $|(1-\alpha)R_1+\alpha R_2|$ (If the number in the absolute value sign is negative, it means that O_1 and O_2 are in the two sides of that line). So

$$R_3 = |(1-\alpha)R_1+\alpha R_2|.$$

Furthermore, the centroid of ω_1 and ω_2 is $\bar{\omega} = C(\frac{x_1+x_2}{2}, \frac{R_1+R_2}{2})$. Therefore, putting $\alpha = \frac{1}{2}$ in the definition of ω_3 implies

$$\begin{aligned}
&d(\omega,\omega_1) = d(\omega,\omega_2)\\
&\Rightarrow OO_1^2 - (R-R_1)^2 = OO_2^2 - (R-R_2)^2\\
&\Rightarrow OO_1^2 - OO_2^2 = {R_1}^2 - R_2^2 - 2R(R_1-R_2)\\
&\Rightarrow (x-x_1)^2 - (x-x_2)^2 = {R_1}^2 - R_2^2 - 2R(R_1-R_2)\\
&\Rightarrow 2x(x_1-x_2) - 2R(R_1-R_2) = x_1^2 - x_2^2 - {R_1}^2 + R_2^2.
\end{aligned}$$

We have

$$\begin{aligned}
&d(\omega,\omega_3)^2 = (x-x_3)^2 + y^2 - (R-R_3)^2\\
&= (x-((1-\alpha)x_1+\alpha x_2))^2 + y^2 - (R-((1-\alpha)R_1+\alpha R_2))^2\\
&= ((x-x_1)+\alpha(x_1-x_2))^2 + y^2 - ((R-R_1)+\alpha(R_1-R_2))^2.
\end{aligned}$$

Suppose that circles ω, ω_1 and ω_2 are fixed and α varies. We can write $d(\omega,\omega_3)^2$ as a polynomial of α such as $p_2\alpha^2+p_1\alpha+p_0$. We know $p_2 = (x_1-x_2)^2 - (R_1-R_2)^2$ is nonnegative because $d(\omega_1,\omega_2)$ is defined. Furthermore, the coefficient of α is

$$p_1 = 2(x_1-x_2)(x-x_1) - 2(R_1-R_2)(R-R_1).$$

Because of the relation between X and R,

$$x_1^2 - R_1^2 - x_2^2 + R_2^2 - 2x_1(x_1-x_2) + 2R_1(R_1-R_2) = -(x_1-x_2)^2 + (R_1-R_2)^2.$$

If $p_1 = p_2 = 0$, $d(\omega,\omega_3)$ is independent of α. Otherwise, the minimum value of $d(\omega,\omega_3)$ is for $\alpha = \frac{1}{2}$ and the assertion is proved.

c) First, note that the radical center of three circle ω_1, ω_2 and ω_3, as a circle with radius zero is equidistant from these circles (if this point is outside of all the circles). Therefore, we have

Lemma 1. *If two circles C_1 and C_2 are both equidistant from ω_1 and ω_2, then the direct homothetic center of C_1 and C_2 lies on the radical axis of ω_1 and ω_2. If two circles C_1 and C_2 have equal radius then the direct homothetic center of them is not defined. In this case, the line passing through the centers of C_1 and C_2 is parallel to the radical axis of ω_1 and ω_2.*

Let $\omega_i = C(O_i,R_i)$ and $C_i = C(P_i,r_i)$. We can assume that $O_i = (x_i,0)$ and $P_i = (a_i,b_i)$. If $(x,0)$ lies on the radical axis of ω_1 and ω_2, then

$$(x-x_1)^2 - R_1^2 = (x-x_2)^2 - R_2^2 \Rightarrow x = \frac{x_1^2 - x_2^2 - R_1^2 + R_2^2}{2(x_1-x_2)}.$$

Denote this value by c. We proved in part b that

$$2a_i(x_1 - x_2) - 2r_i(R_1 - R_2) = x_1^2 - {R_1}^2 - x_2^2 + R_2^2 \Rightarrow a_i = r_i \frac{R_1 - R_2}{x_1 - x_2} + c.$$

The direct homothetic center of C_1 and C_2 lies on P_1P_2 and hence it can be shown that

$$S = \frac{r_2}{r_2 - r_1} P_1 - \frac{r_1}{r_2 - r_1} P_2 \,(\text{whenever}\, r_1 \neq r_2),$$

so the first coordinate of S is

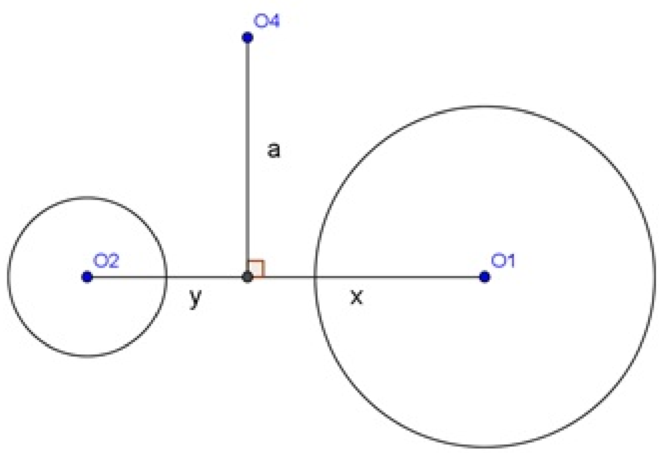
$$\frac{r_2a_1 - r_1a_2}{r_2 - r_1} = \frac{1}{r_2 - r_1}\left[r_2\left(r_1\frac{R_1 - R_2}{x_1 - x_2} + c\right) - r_1\left(r_2\frac{R_1 - R_2}{x_1 - x_2} + c\right)\right] = c.$$

Hence S lies on the radical axis of ω_1 and ω_2.
Otherwise, if $r_1 = r_2$, by the above equations we get $a_1 = a_2$ and thus the line passing through the centers of C_1 and C_2 is parallel to the radical axis of ω_1 and ω_2.
d) The answer is no. Let $\omega_i = C(O_i, R_i)$ and $d_{ij} = |O_i - O_j|$. We assume that $R_1 \geq R_2 \geq R_3 \geq R_4$. According to the problems assumption we must have

$$d_{ij}^2 - (R_i - R_j)^2 = 1.$$

Note that these equations are invariant under addition of a constant number to R_i's $(1 \leq i \leq n)$. Therefore, we can suppose that $R_4 = 0$ and O_4 lies on the radical axis of ω_1 and ω_2. According to the following figure we have:

$$\begin{aligned} a^2 + x^2 - R_1^2 &= 1, \\ a^2 + y^2 - R_2^2 &= 1, \\ (x+y)^2 - (R_1 - R_2)^2 &= 1. \end{aligned}$$



Substracting the sum of the first two equations from the third one implies:

$$2xy + 2R_1R_2 + 2 - 2a^2 = 1 \Rightarrow a = \sqrt{\frac{1}{2} + xy + R_1R_2}.$$

On the other hand, by substraction of the first equation frome the second one we have:

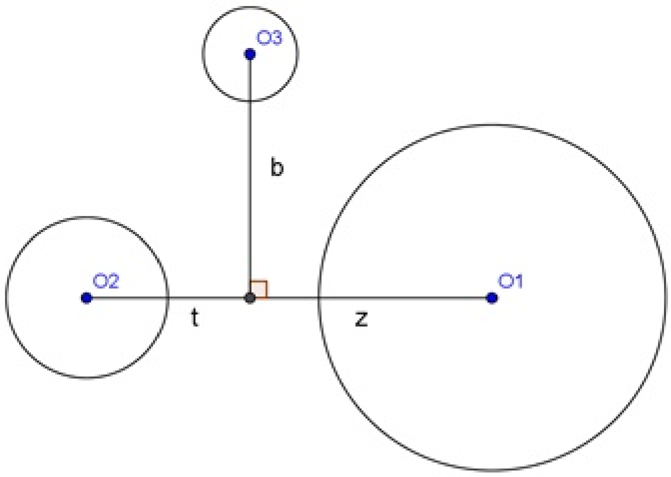
$$\begin{cases} x^2 - y^2 = R_1^2 - R_2^2 \\ x + y = d_{12} \end{cases} \Rightarrow x - y = \frac{R_1^2 - R_2^2}{d_{12}}$$
$$\Rightarrow \{x, y\} = \frac{d_{12}}{2} \pm \frac{R_1^2 - R_2^2}{2d_{12}}$$
$$\Rightarrow xy = \frac{d_{12}^2}{4} - (\frac{R_1^2 - R_2^2}{d_{12}})^2.$$

Hence

$$a = \sqrt{\frac{1}{2} + \frac{d_{12}^2}{4} - \frac{(R_1^2 - R_2^2)^2}{4d_{12}^2} + R_1R_2}.$$

Now, replacing R_1, R_2 and R_3 by $R_1 - R_3$, $R_2 - R_3$ and 0 yields:

$$b = \sqrt{\frac{1}{2} + \frac{d_{12}^2}{4} - \frac{((R_1 - R_3)^2 - (R_2 - R_3)^2)^2}{4d_{12}^2} + (R_1 - R_3)(R_2 - R_3)}.$$



There are two possiblities.
Case 1. O_3 and O_4 are not on the same side of O_1O_2. In this case we have

$$O_3O_4^2 - R_3^2 = 1 \Rightarrow 1 + R_3^2 = O_3O_4^2 \geq (a + b)^2 \geq a^2 + b^2.$$

On the other hand,

$$a^2 > \frac{1}{2} + R_1R_2,$$

$$b^2 > \frac{1}{2} + (R_1 - R_3)(R_2 - R_3),$$

so

$$R_3^2 > R_1R_2 + (R_1 - R_3)(R_2 - R_3) \Rightarrow (R_1 + R_2)R_3 > 2R_1R_2.$$

But this is in contradiction with the assumption that $R_1 \geq R_2 \geq R_3$.

Case 2. O_3 and O_4 are on the same side of O_1O_2. In this case we have

$$\begin{aligned}
&1+R_3^2=(a-b)^2+(x-z)^2=(a^2+x^2)+(b^2+z^2)-2ab-2xz\\
&=1+{R_1}^2+1+(R_1-R_3)^2-2ab-2xz\\
&\Rightarrow 0=1+2R_1^2-2R_1R_3-2ab-2xz\\
&\Rightarrow 4ab+4xz=2+4R_1(R_1-R_3).
\end{aligned}$$

Substituting $d_{12}^2=1+(R_1-R_2)^2$ implies

$$a=\sqrt{\frac{1}{2}+\frac{d_{12}^2}{4}-\frac{(R_1^2-R_2^2)^2}{4d_{12}^2}+R_1R_2}=\sqrt{\frac{3}{4}+\frac{(R_1+R_2)^2}{4}-\frac{(R_1-R_2)^2(R_1+R_2)^2}{4(1+(R_1-R_2)^2)}}.$$

Similarly,

$$b=\sqrt{\frac{3}{4}+\frac{(R_1+R_2-2R_3)^2}{4}-\frac{(R_1-R_2)^2(R_1+R_2-2R_3)^2}{4(1+(R_1-R_2)^2)}}.$$

Defining

$$\begin{aligned}
t&:=R_1-R_2,\\
s&:=R_1+R_2-2R_3,\\
r&:=R_1+R_2,
\end{aligned}$$

we have

$$\begin{aligned}
a&=\sqrt{\frac{3}{4}+\frac{r^2}{4}-\frac{r^2t^2}{4(1+t^2)}}=\sqrt{\frac{3}{4}+\frac{r^2}{4(1+t^2)}},\\
b&=\sqrt{\frac{3}{4}+\frac{s^2}{4(1+t^2)}}.
\end{aligned}$$

Also,

$$xz=\left(\frac{d_{12}}{2}+\frac{R_1^2-R_2^2}{2d_{12}}\right)\left(\frac{d_{12}}{2}+\frac{(R_1-R_3)^2-(R_2-R_3)^2}{2d_{12}}\right)=\frac{1+t^2}{4}+\frac{rt}{4}+\frac{st}{4}+\frac{rst^2}{4(1+t^2)}.$$

Now, we have:

$$\begin{aligned}
&4ab+4xz=2+4R_1(R_1-R_3)\\
&\Rightarrow\sqrt{\left(3+\frac{r^2}{1+t^2}\right)\left(3+\frac{s^2}{1+t^2}\right)}+1+t^2+rt+st+\frac{rst^2}{1+t^2}=2+(r+t)(s+t).
\end{aligned}$$

Multiplication by $1+t^2$ and some computations yields:

$$\sqrt{(3+3t^2+r^2)\,(3+3t^2+s^2)}=1+t^2+rs,$$

which is in contradiction with the Cauchy-Schwarz inequality.

3 . a) Let

$$f(x) = \begin{cases} 2 & x = 1 \\ 3 & x = 2 \\ 1 & x = 3 \\ x & x \neq \{1,2,3\} \end{cases} \qquad g(x) = \begin{cases} 3 & x = 1 \\ 1 & x = 2 \\ 2 & x = 3 \\ x & x \neq \{1,2,3\} \end{cases}$$

It is easy to check that $g^2 = f$ and $f^2 = g$.
b) Suppose g is a function that $f \to g$ and $g \to f$. It means that there are integers m and n such that $f^m = g$ and $g^n = f$. So $f^{mn} = f$ and f generates at most $mn - 1$ functions. Therefore, there are only a finite number of functions g.
c) Define $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} x+1 & x \in \mathbb{Z} \\ x & x \notin \mathbb{Z} \end{cases}$$

Let f be a real function such that $f \to g$ ($f^k = g$). Note that g is bijective, and hence f is bijective too. If $f(z_0) \notin \mathbb{Z}$ for some $z_0 \in \mathbb{Z}$, we have:

$$f(z_0) = g(f(z_0)) = f^{k+1}(z_0) = f(g(z_0)) = f(z_0 + 1),$$

which leads to a contradiction since f is injective. Hence $f(\mathbb{Z}) \subseteq \mathbb{Z}$. Now, for each integer number z we have:

$$f(z+1) = f(g(z)) = f^{k+1}(z) = g(f(z)) = f(z) + 1.$$

Therefore, $f(z) = z + t$ for some integer number t and every integer number z. Next, for every integer number z we have:

$$z + kt = f^k(z) = g(z) = z + 1 \Rightarrow kt = 1 \Rightarrow k = t = 1.$$

Therefore, f must be equal to g.
d) If $f^m(x) = x^3$ and $f^n(x) = x^5$ for some $m, n \in \mathbb{N}$, then $x^{3^n} = f^{mn}(x) = x^{5^m}$. Therefore, $3^n = 5^m$. But this equation does not have any solutions in natural numbers.
e) First, we prove a lemma.

Lemma 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $a > b$ two integer numbers such that f^a and f^b are both polynomials of degree 1. Then f^{a-b} is also a polynomial of degree 1.*

Proof. $f^{a-b}(x) = f^a \circ ((f^b)^{-1}(x))$, and the inverse of every polynomial with degree 1 is again a polynomial of degree 1. □

Now, assume that $f^m = P$, $f^n = Q$ and $(m, n) = d$ for some $m, n, d \in \mathbb{N}$. Referring to the lemma and using Euclidean algorithm, f^d is a polymial of degree 1 that generates both P and Q.

4 . Throughout the solution the following notations will be used:
$C_1(a, b)$: The smaller part of the perimeter of the polygon which lies between a and b.
$C_2(a, b)$: The other part of the perimeter of the polygon joining a and b.
$d(a, b)$: The length of $C_1(a, b)$.
$[a, b]$: A chord of the polygon whose end points are a and b.

a) Let $[x, y]$ be the maximum chord of A_o with unit length. Our aim is to show that in this case the semicircle with diameter $[x, y]$, which is on the same side of $[x, y]$ as $C_2(x, y)$, lies entirely inside A_o.

Lemma 1. $C_2(x,y) \cap [x,y] = \{x,y\}$.

Proof. Assume that $z \in C_2(x,y) \cap ([x,y] - \{x,y\})$. Obviously $[x,z]$ and $[y,z]$ are both inner chords of A_o. Note that if $y \in C_1(x,z)$, then $d(x,z) > d(x,y)$ which is a contradiction chord $[x,y]$ is maximal, hence $y \notin C_1(x,z)$. Similarly, $x \notin C_1(y,z)$. Therefore, $C_1(x,y)$, $C_1(y,z)$ and $C_1(z,x)$ form a partition of the perimeter of A_o. This leads to a contradiction because A_o is rotund and $d(x,y)$, $d(y,z)$ and $d(z,x)$ are all less than $\frac{p}{4}$ and consequently their sum is less than p. □

Lemma 2. *The common part of $C_2(x,y)$ and edges of A_o that contain x lies outside the semicircle mentioned above. A similar assertion holds for y.*

Proof. Assume to the contrary that z is a point inside that semicircle, and it also lies on the common part of $C_2(x,y)$ and the edge containing x such that $d(x,z)$ is very small. By lemma 1, we can choose z such that $[y,z]$ intersects the perimeter of A_o only at y and z. Since $d(x,y) \leq \frac{p}{4}$ and z is near to x, $yxz = C_1(y,z)$ which is a contradiction with the maximality of $[x,y]$. □

Lemma 3. *x and y are the only intersection points of $C_2(x,y)$ and the semicircle.*

Proof. Assume to the contrary that the intersection set is not empty. Let z be a point in this set which has the minimum distance to the segment $[x,y]$. Note that by the above lemmas we know that this minimum is positive. We claim that $[x,z]$ and $[y,z]$ are both inner chords of A_o, because if not, there is another point of the polygon inside triangle xyz, which contradicts the minimality of z.
Rest of the proof is similar to that of lemma 1. $y \notin C_1(x,z)$ and $x \notin C_1(y,z)$, so $C_1(x,y)$, $C_1(y,z)$ and $C_1(z,x)$ form a partition of the perimeter of A_o, but we know that the length of each one is at most $\frac{p}{4}$. □

Using lemma 3, $C_2(x,y)$ does not intersect the semicircle (except at x and y). Therefore, $C_1(x,y)$ does not intersect the semicircle because the polygon is not self-intersecting. Hence the semicircle fits completely inside A_o.
b) Let $[x,y]$ be the maximum chord. Draw semicircles of radius 1 and centers at x and y such that they are on the same side of line xy as $C_2(x,y)$. We claim that the common part of these two semicircles (say S) fits completely inside A_o.

Lemma 4. $C_2(x,y) \cap [x,y] = \{x,y\}$.

Proof. The proof is the same as that one given in part a. □

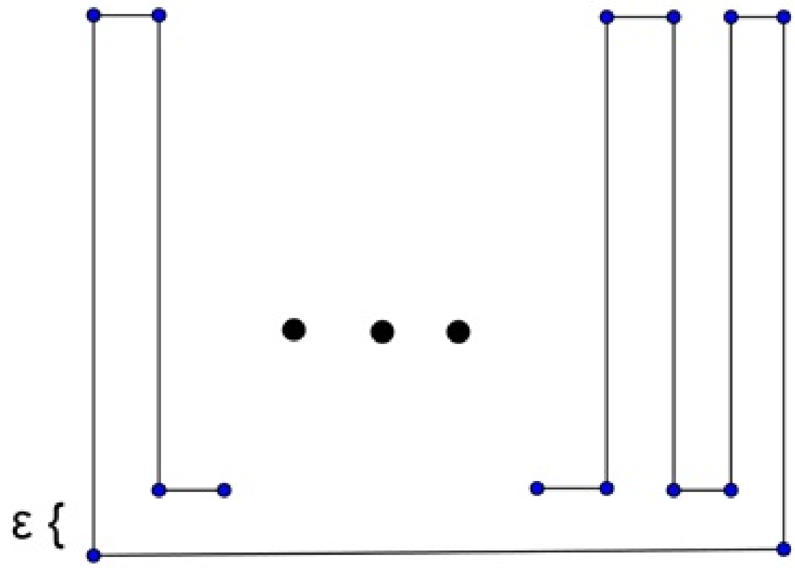
Lemma 5. *The common part of $C_2(x,y)$ and edges of A_o which contain x lies outside of S. A similar assertion holds for y.*

Proof. The proof is similar to the proof of lemma 2.
Assume to the contrary that $z \in S$ is a point in the common part of $C_2(x,y)$ and the edge containing x such that $d(x,z)$ is very small. By lemma 4, we can choose z such that $[y,z]$ intersects the perimeter of A_o exactly at y and z. Since $d(x,y) \leq \frac{p}{4}$ and z is near x, $yxz = C_1(y,z)$ which contradicts the maximality of $[x,y]$. □

Lemma 6. $C_2 \cap S = \{x,y\}$.

Proof. Assume to the contrary that there are points other than x and y in this intersection. Let z be a point in this intersection which has the minimum distance from the segment $[x, y]$. By the above lemmas, we know that this minimum is positive. We claim that $[x, z]$ and $[y, z]$ are both inner chords of A_o, because if not, there exists another point of the polygon inside triangle xyz which contradicts the minimality of z. So $[x, z]$ and $[y, z]$ are both inner chords of A_o. But z lies in S, therefore the lengths of $[x, z]$ and $[y, z]$ are less than 1. This implies that $d(x, z)$ and $d(y, z)$ are both less than $\frac{p}{4}$.
Rest of the proof is similar to lemma 1. $y \notin C_1(x, z)$ and $x \notin C_1(y, z)$, so $C_1(x, y)$, $C_1(y, z)$ and $C_1(z, x)$ form a partition of the perimeter of A_o, but we know that the length of each one is at most $\frac{p}{4}$. □

By lemma 6, $C_2(x, y)$ does not intersect the region S (except at x and y). Therefore, $C_1(x, y)$ does not intersect S because the polygon is not self-intersecting. Hence the region S fits completely inside A_o. It is easy to see that a circle with radius $\frac{1}{4}$ can be drawn entirely inside S.
c) Suppose that there exists a rotund polygon A such that no circle of radius $\frac{1}{4}$ completely fits inside it. By the second statement, there is a triangulation of the polygon with chords of at most unit length. Let xy be the chord of the triangulation that has the maximum $d(x, y)$. There are two possibilities:
Case 1. There is no vetex of A in $C_2(x, y)$ other than x and y. Therefore, $[x, y]$ must be an edge of A. Hence $C_2(x, y) = [x, y]$ and the length of $C_2(x, y)$ is less than $C_1(x, y)$ because of the triangle inequality which is a contradiction.
Case 2. There is another vertex of A in $C_2(x, y)$ other than x and y. Therefore, there is some vertex $z \in C_2(x, y)$ such that xyz forms one of the triangles of the triangultion. Note that if $y \in C_1(x, z)$, then $d(x, z) > d(x, y)$ and this contradicts the maximality of $[x, y]$. Thus $y \in C_2(x, z)$ and similarly $z \in C_2(x, y)$. Therefore, $C_1(x, y)$, $C_2(x, y)$ and $C_3(x, y)$ form a partition of the perimeter of A. This is a contradiction because the lengths of all these parts are less than $\frac{p}{4}$.
d) If ϵ is sufficiently small, the following polygon is a counterexample for the first statement.



5 . a) Let $a_1 \le a_2 \le \cdots \le a_n$ be the stolen numbers. Since all the partial sums are positive, all the stolen numbers must be positive. Obviously a_1 is the smallest number among the partial sums. Suppose that numbers $a_1, a_2, \ldots, a_i$ have been determined. Omit all the partial sums of $a_1, a_2, \ldots, a_i$ from the partial sums of $a_1, a_2, \ldots, a_n$. The smallest number among the remaining numbers, must be a_{i+1}, so all the numbers will be determined uniquely. b) Suppose that $a_1 \le \cdots \le a_k < 0 \le a_{k+1} \le \cdots \le a_n$ are the stolen numbers and $s_1 \le s_2 \le \ldots \le s_{2^n-1}$ are the partial sums. Note that if $s_1 > 0$, the problem is already solved in part (a). Therefore, we can assume that $s_1 < 0$. We have

$$(1+x^{a_1})(1+x^{a_2})\cdots(1+x^{a_n}) = 1 + x^{s_1} + x^{s_2} + \cdots + x^{s_{2^n-1}}.$$

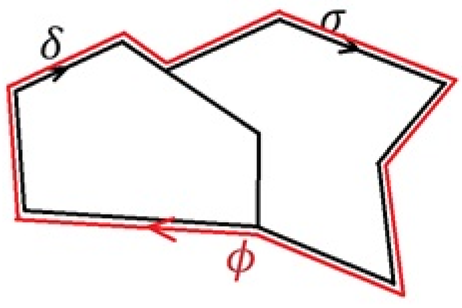
Obviously, s_1 is the sum of negative numbers $a_1, a_2, \ldots, a_k$. Multiplying the above equation by x^{-s_1} implies

$$(1+x^{-a_1})\cdots(1+x^{-a_k})(1+x^{a_{k+1}})\cdots(1+x^{a_n}) = x^{-s_1} + x^{s_1-s_1} + x^{s_2-s_1} + \cdots + x^{s_{2^n-1}-s_1}.$$

Hence we can say that we have the partial sums of stolen numbers $|a_1|, |a_2|, \ldots, |a_n|$, and by part a, it is possible to find them uniquely.
Finally, we must show that x^{s_1} can be uniquely written as a product of x^{a_i}'s. If there were two ways to do this, we would obtain a partial sum of a_i's equal to 0, which leads to a contradiction.
c) The partial sums of two sets $\{1, 2, -3\}$ and $\{-1, -2, 3\}$ are the same, so adding 1389 0's to these sets will not change the partial sums.

6 . For each directed closed broken line σ in the plane, let κ_σ be the product of conversion factors along σ.

Lemma 1. *Let σ and δ be two non-self-intersecting directed closed broken lines with opposite directions and $\phi = (\sigma \cup \delta) - (\sigma \cap \delta)$. Obviously, ϕ is a directed closed broken line with the same direction as σ and δ. We have $\kappa_\phi = \kappa_\sigma \kappa_\delta$.*



Proof. Note that each part of ϕ is a part of exactly one of σ and δ. Furthermore, each part of $\sigma \cup \delta - \phi$ is a part of both σ and δ and has opposite directions in these two directed closed broken lines. Hence the factors corresponding to the common parts of σ and δ cancel each other and therefore $\kappa_\phi = \kappa_\sigma \kappa_\delta$. □

Using a stereographic projection from a vertex of the polyhedral, we can map the polyhedral on a plane. Note that the faces adjacent to the center of projection map to unbounded regions in the plane. Other faces and travel paths are sent to polygons in the plane.
For each vertex v of this graph in the plane we define c_v to be the product of conversion factors of its adjacent edges while traveling clockwise. We claim that for each closed

broken line σ, κ_σ is a function of the factors that have been assigned to the vertices enclosed by σ.

Lemma 2. *For each directed closed broken line σ, we have:*

$$\begin{aligned} \kappa_\sigma &= \prod_{v \text{ inside } \sigma} c_v \quad \text{If } \sigma \text{ is clockwise,} \\ \kappa_\sigma &= \prod_{v \text{ inside } \sigma} c_v^{-1} \quad \text{If } \sigma \text{ is counterclockwise.} \end{aligned}$$

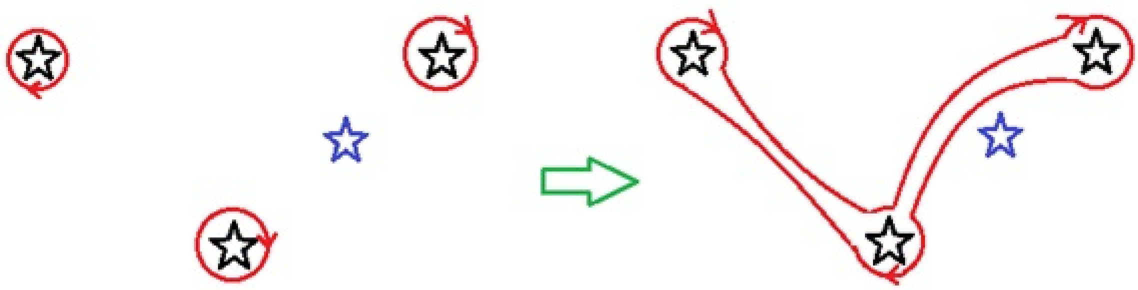
Proof. Obviously, it is sufficient to prove the lemma for the first case. If there are no vertices inside σ, the proof is easy. For each edge of the graph, σ intersects that edge a same number of times from both directions and hence $\kappa_\sigma = 1$. Now, if some clockwise path σ, that is not self-intersecting, encloses some vertices (say $v_1, v_2, \ldots, v_k$), consider small counterclockwise paths σ_i with $\kappa_{\sigma_i} = c_{v_i}^{-1}$ such that σ_i encloses only v_i. We can write σ as a union of $\sigma_1, \sigma_2 \cdots, \sigma_k$ and some paths that do not enclose any vertices and thus by lemma 1,

$$\kappa_\sigma = \kappa_{\sigma_1}^{-1}\kappa_{\sigma_2}^{-1}\cdots\kappa_{\sigma_k}^{-1} = c_{v_1}c_{v_2}\cdots c_{v_k}.$$

□

Now, according to the former lemma, for each closed boken line σ that is not self-intersecting, $\kappa_\sigma = \prod_{v\in A} c_v$ or $\kappa_\sigma = \prod_{v\in A} c_v^{-1}$, where A is a subset of the vertices (except the one that maps to infinity). The number of such subsets is 2^{v-1} where v is the number of the vertices of the polyhedral. Note that an empty set gives the $\kappa = 1$ in both of the equations and hence the number of mutually different amounts of money is at most $2 \times (2^{v-1}) - 1 = 2^v - 1$.

Now, we claim that for each nonempty subset A of the vertices in the plane and each arbitrary direction there is a non-self-intersecting path in the form of a closed broken line with vertices of A as its inner points. To prove it, draw small curves around points of A with the given direction. We can combine these curves using arcs joining the points of A to get the desired path.



We claim that by assigning different prime numbers to different edges of the polyhedral as conversion factors, two paths with different κ have different inner points or different directions. Assume that for two paths σ_1 and σ_2, $\kappa_{\sigma_1} = \kappa_{\sigma_2}$. If they have the same direction, we may combine them to get a path with $\kappa = \kappa_{\sigma_1}^2$, and if they have different directions, we first change the direction of σ_2 to get another path σ_2' and then combine σ_1 and σ_2' to get a path with $\kappa = 1$. Let this new path be γ. Note that in each case, the power of each prime number p in κ_γ is even (0, 2 or -2). It means that every edge of the graph has an even number of adjacent vertices among vertices inside γ. But

this is a contradition since there is always an edge that is adjacent to only one vertex among a given set of vertices (note that there are also some unbound edges).
For this reason, the maximum number of κ's is $2^v - 1$. Now, our goal is to find an upper bound for v in terms of n. Since the projected graph is planar, we have

$$v = 2 - n + e \leq 2 - n + 3n - 6 = 2n - 4,$$

and equality holds when all the the vertices of the polyhedral have degree 3. Obviously, a $(n-2)$-gonal prism is an example of such polyhedral. Hence we have proved that the desired number is $2^{2n-4} - 1$.

7 . a) First, note that for each $x \in \mathbb{N}$ there are at most a fininte number of elements of S with first coordinate x. Therefore, for each positive number R, there are at most a finite number of elements of S with a first coordinate less than R. A same asseretion is true for the second coordinate.
Suppose that the set of interesting equations that S satisfies in not empty. Let $P_0(x) = Q_0(y)$ be the interesting equation with minimum degree among the interesting equtions that are satisfied by an infinite subset of S (by the degree of the equation $P(x) = Q(y)$, we mean $\deg(P(x) - Q(y))$ as a polynomial with two variables). Denote by S_0 the subset of S that satisfies this equation.
Suppose that $P(x) = Q(y)$ is an arbitrary equation that S satisfies. Referring to the division algorithm in polynomials with rational coefficients, there exist rational polynomials $A(x)$, $B(x)$, $C(x)$ and $D(x)$ such that

$$P(x) = A(x)P_0(x) + B(x), \deg B < \deg P_0.$$
$$Q(y) = C(y)Q_0(y) + D(y), \deg D < \deg Q_0.$$

Denote by N the least common multiple of the denominators of the coefficients in the polynomials of the above equations. By multiplying these equations by N, we get

$$\begin{array}{r} NP(x) = A'(x)P_0(x) + B'(x) \\ NQ(y) = C'(y)Q_0(y) + D'(y) \end{array} \quad (*)$$

where A', B', C' and D' are polynomials with integer coefficients.
If $(x_0, y_0) \in S_0$, there are some integers a and b such that $P(x_0) = Q(y_0) = a$ and $P_0(x_0) = Q_0(y_0) = b$. By substracting the two equalities in $(*)$ we obtain

$$(A'(x_0) - C'(y_0))b = B'(x_0) - D'(y_0). \quad (**)$$

On the other hand, we know that $\deg B' = \deg B < \deg P_0$ and $\deg D' = \deg D < \deg Q_0$. Therefore, there is a real number $R > 0$ such that whenever $|x|, |y| > R$,

$$|B'(x)| < \frac{1}{2}|P_0(x)|, \quad |D'(y)| < \frac{1}{2}|Q_0(y)|.$$

Using $(**)$ and the triangle inequality, if $|x_0|, |y_0| > R$

$$|A'(x_0) - C'(y_0)||b| = |B'(x_0) - D'(y_0)| \leq |B'(x_0)| + |D'(y_0)| < \frac{1}{2}|b| + \frac{1}{2}|b| = |b|.$$

This implies that $A'(x_0) = C'(y_0)$ and $B'(x_0) = D'(y_0)$. We know that except for a finite number of elements of S_0, both coordinates of the elements are greater than R.

Therefore, the equation $B'(x) = D'(y)$ has infinitely many solutions in S. On the other hand, the degree of this equation is less than the degree of the equation $P_0(x) = Q_0(y)$. Hence B' and D' must be constant polynomials. It means that there exists some $c \in \mathbb{Z}$ such that $B'(x) = D'(y) = c$. Hence

$$A'(x) = \frac{NP(x) - c}{P_0(x)}, \quad C'(y) = \frac{NQ(y) - c}{Q_0(y)}.$$

Now, if the equation $A'(x) = C'(y)$ is a new interesting equation, we can repeat the above steps for this new equation.
Continuing this process, we see that there is some polynomial F with rational coefficients such that $P(x) = F(P_0(x))$ and $Q(x) = F(Q_0(x))$. This means that the interesting equation $P(x) = Q(y)$ has resulted from the interesting equation $P(x) = Q(y)$.
b) First, we prove a useful lemma.

Lemma 1. *Let $P(x) \in \mathbb{Z}[x]$ be a monic polynomial and d be a positive integer such that $d | \deg P$. Show that there exist a natural number N and some polynomials $T(x)$ and $R(x)$ with integer coefficients such that*

$$NP(x) = (T(x))^d + R(x),$$

and for sufficiently large values of x,

$$(T(x))^d \leq NP(x) \leq (T(x) + 1)^d$$

Proof. Suppose that $\deg P = dm$ for some $m \in \mathbb{N}$. First, we will find a polynomial $T_1(x) \in \mathbb{Q}[x]$ such that $\deg\left(P(x) - (T_1(x))^d\right) < (m-1)d$. To do this, assume that

$$P(x) = x^{md} + a_{md-1}x^{md-1} + \cdots + a_1x + a_0.$$

We must find rational numbers $b_0, b_1, \ldots, b_{m-1}$ such that for each natural number $m(d-1) \leq i \leq md$, the coefficient of x^i in $(x^m + b_{m-1}x^{m-1} + \cdots + b_1 + b_0)^d$ be a_i. We can find b_i's recursively as rational functions of a_i's. Now, let n be the least common multiple of denominators of the coefficients of T_1. Define $T_2(x) = nT_1(x)$, $N = n^d$ and $S_2(x) = NP(x) - (T_2(x))^d$. Obviously, $S_2(x), T_2(x) \in \mathbb{Z}[x]$ and $\deg S_2 > (m-1)d$.
If the leading coefficient of S_2 is positive, set $T(x) = T_2(x)$ and $R(x) = S_2(x)$. Otherwise, if the leading coefficient of S_2 is negative, set $T(x) = T_2(x) - 1$ and $S(x) = NP(x) - (T(x))^d$. It is easy to check that these polynomials have the desired properties. □

Suppose that $d = (\deg P, \deg Q)$ for some natural number d. Using the lemma, there exist some natural number N and polynomials U, V, T and R with integer cooefficients such that

$$NP(x) = (T(x))^d + R(x), \quad NQ(y) = (U(y))^d + V(y),$$

and for sufficiently large values of x and y,

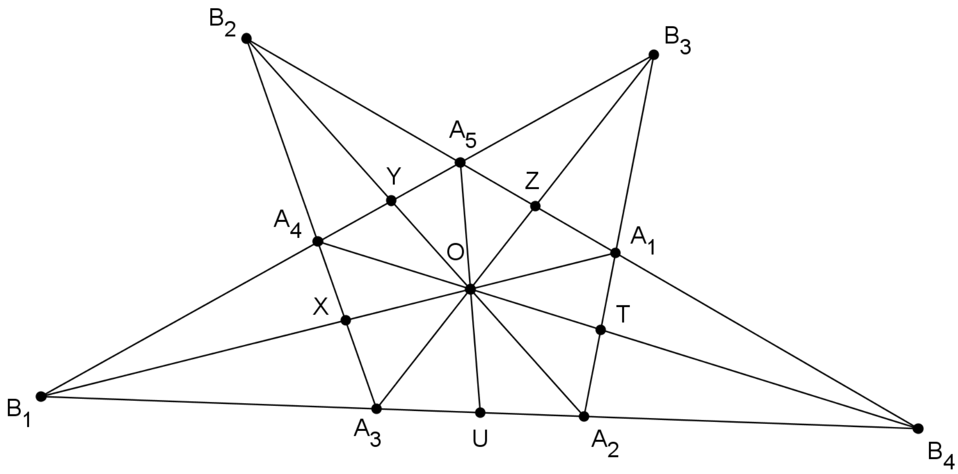
$$(T(x))^d \leq NP(x) < (T(x) + 1)^d, \quad (U(y))^d \leq NQ(y) < (U(y) + 1)^d.$$

Note that if $d > 1$, the degree of the equation $T(x) = U(y)$ is less than the degree of the equation $P(x) = Q(y)$. Furthermore, from the previous inequalities we can deduce that every solution of $P(x) = Q(y)$ is a solution of $T(x) = U(y)$ and by part a these interesting equations are both resulted from a common interesting equation with a degree less than the degree of $P(x) = Q(y)$. This contradicts the assumption of $P(x) = Q(y)$ being primitive. Hence $(\deg P, \deg Q) = d = 1$.

8 . First, note that if (x_1, y_1) and (x_2, y_2) are two points with rational coordinates, then the equation of the line passing through them in the cartesian coordinates is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Therefore, the equation of this line can be written using only rational numbers. We will call such lines a rational line. It is easy to see that the intersection point of two rational lines has rational coordinates. Furhtermore, the ratio of the lengths of two line segments with rational end points is a rational number. Consequently, every four collinear points with rational coordinates have a rational cross ratio.
Assume to the contrary that $A_1A_2A_3A_4A_5$ is a rational pentagon that at least four lines from the lines A_iB_i are concurrent. Without loss of generality, we can assume these four lines are A_iB_i for $1 \leq i \leq 4$. We wall use the notation given in the following figure.



Let $\lambda = (B_1A_3, A_2B_4)$. According to the figure we have

$$\begin{aligned}
\lambda &= (B_1A_3, A_2B_4) &= (B_1B_4, A_2U) \quad &\text{(Projection from } O\text{)} \\
\lambda &= (B_1A_3, A_2B_4) &= (A_5Z, A_1B_4) \quad &\text{(Projection from } B_3\text{)} \\
& &= (UA_3, B_1B_4) \quad &\text{(Projection from } O\text{)} \\
& &= (B_1B_4, UA_3) \quad &
\end{aligned}$$

Using the properties of cross ratio we get

$$\lambda^2 = (B_1B_4, A_2U)(B_1B_4, UA_3) = (B_1B_4, A_2A_3) = \frac{\lambda}{\lambda - 1}$$

Hence λ is a root of the equation $x^2 = x + 1$ that has no rational root. We assumed that points A_i, $1 \leq i \leq 5$, have rational coordinates. Therefore, all the lines and points in the figure are rational. Furthermore, every four collinear points with rational coordinates have a rational cross ratio. Consequently, $\lambda = (B_1A_3, A_2B_4)$ must be rational which is a contradiction since $\lambda^2 = \lambda + 1$. Therefore, at most three of the lines A_iB_i are concurrent.

Team Selection Test

1 . Let M and N be the feet of the perpendicular lines from O to BY and AY, respectively. Since O is a point on the perpendicular bisector of AB, we have $OA = OB$ and hence

$$AN^2 = AO^2 - ON^2 = BO^2 - OM^2 = BM^2.$$

So $BY = BM + MY = AN + NY = AY$ and hence

$$\angle AYS = \angle BAY = \angle ABY = \angle ACS.$$

Therefore, the quadrilateral $ACYS$ is cyclic and thus $\angle SAC = 180^\circ - \angle BYS = \angle ABC$. This implies that SA is tangent to ω. By a similar arguments we can prove that TA is tangent to ω.

2 . Suppose that the image of integer numbers under the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ contains an infinite geometric progression with common ratio $a \in \mathbb{Z} - \{0\}$. For each $b \in \mathbb{Z}$ we have

$$P(ax + b) = a_n a^n x^n + (na_n a^{n-1} b + a_{n-1}a^{n-1})x^{n-1} + \cdots ,$$

$$a^n P(x) = a_n a^n x^n + a_{n-1}a^n x^{n-1} + \cdots + a_0 a^n.$$

We can find some $b_1, b_2 \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that for every $x \geq N$,

$$P(ax + b_2) < a^n P(x) < P(ax + b_1),$$

and for every $x \leq -N$,

$$P(ax + b_2) < a^n P(x) < P(ax + b_1),$$
$$\text{or } P(ax + b_1) < a^n P(x) < P(ax + b_2).$$

For each $P(x)$ in the geometric progression, $aP(x)$,$a^2P(x)$,...,$a^nP(x)$,... are all in $P(\mathbb{Z})$, hence there exists some $y \in \mathbb{Z}$ such that $a^n P(x) = P(y)$. If $|x|$ is sufficiently large (there are infinitely many values of $P(x)$ in the geometric progression), using the above inequalities, $ax + b_1 < y < ax + b_2$. Therefore, $y - ax$ is a constant value between b_1 and b_2. Hence there exists some constant number c such that the equation $a^n P(x) = P(ax + c)$ has infinitely many solutions and consequently $P(ax + c)$ and $a^n P(x)$ are two equal polynomials.
Now, our goal is to find all polynomials $P(x) \in \mathbb{Z}[x]$ satisfying the equation $a^n P(x) = P(ax + c)$. Let $Q(x) = ax + c$. If α is a root of $P(x)$, setting $x = \alpha$ in the equation implies that $P(Q(\alpha)) = 0$ and hence $Q(\alpha), Q^2(\alpha), \ldots$ are all roots of P. Since $P(x)$ has a finite number of roots, there are some natural numbers $m_1 > m_2$ such that $Q^{m_1}(\alpha) = Q^{m_2}(\alpha)$. This implies $Q^{m_1 - m_2}(\alpha) = \alpha$, since Q is injective. Note that if $\beta = \frac{c}{1-a}$, then $Q(\beta) = \beta$ and hence $Q^{m_1-m_2}(\beta) = \beta$. On the other hand, $Q^{m_1-m_2}(x) - x$ is a linear polynomial. Therefore, it has at most one root, hence $\alpha = \frac{c}{1-a}$ is the only root of $P(x)$. Consequently, $P(x)$ has the form $r(x - \frac{p}{q})^n = \frac{r}{q^n}(qx - p)^n$, where p, q and r are three integers such that $(p, q) = 1$.
$P(x) \in \mathbb{Z}[x]$, hence

$$\frac{r}{q^n}p^n \in \mathbb{Z} \Rightarrow q^n | rp^n \overset{(q,p)=1}{\Rightarrow} q^n | r \Rightarrow \exists s \in \mathbb{Z};\ r = q^n s.$$

Therefore, we have $P(x) = s(qx - p)^n$, for some $p, q, s \in \mathbb{Z}$ that $(p, q) = 1$.
We claim that each polynomial of this form satisfies the problems condition. It is enough to show that the polynomial $qx - p$ satisfies the property. This is equivalent to showing the existence of a geometric progression whose elements are all congruent to $-p$ modulo q. Obviously, $\{-p(q+1)^m\}_{m\geq 0}$ is an example of such progression and this completes our proof.

3 . The solution contains 10 steps. Furthermore, throughout the solution denote by $a_{i,j}$ $(0 \leq i, j \leq n-1)$ the entry in cell (i, j) of the table and by c_i and r_i $(0 \leq i \leq n-1)$ the sum of the entries of row i and column i of the table, respectively.

Step 1. $a_{0,0} > 0$.

Proof. If $a_{0,0} = 0$, then the number of 0's of row zero must be zero, but $a_{0,0}$ is itself zero, which is a contradiction. □

Step 2. For each $0 \leq i \leq n-1$, $c_i \leq n$.

Proof.

$$c_i = \sum_{j=0}^{n-1} a_{j,i} = \sum_{j=0}^{n-1} (\text{number of } j'\text{s of row } i) \leq n.$$

□

Corollary 1. Let S be the sum of the entries of the table. We have $S \leq n^2$.

Step 3. There is no $0 \leq i, j \leq n-1$ such that $a_{i,j} \geq n$.

Proof. By definition, $a_{i,j} \leq n$ for each $0 \leq i, j \leq n-1$ (Since $a_{i,j}$ is the number of i's of row j). Assume that there exists an entry equal to n (let it be $a_{i,j}$). So all of the entries of row j must be i, Hence $a_{j,j} = i$ and $n \geq c_j \geq a_{i,j} + a_{j,j} = n + i$, so $i = 0$ and $a_{0,j} = n$. Since $c_j \leq n$ and $a_{1,j} = a_{2,j} = \cdots = a_{n-1,j} = 0$, for each $1 \leq k \leq n-1$, row k contains at least one 0 entry. Hence $a_{0,i} > 0$ for each $1 \leq i \leq n-1$. By step 1, $a_{0,0} > 0$ and so all the entries of row 0 are positive, which is a contradiction. □

Step 4. $c_i = n$ for each $0 \leq i \leq n-1$ and consequently, $S = n^2$.

Proof. Since all the entries of the table are natural numbers between 0 and $n-1$, we have

$$c_i = \sum_{j=0}^{n-1} a_{j,i} = \sum_{j=0}^{n-1} (\text{number of } j'\text{s of row } i) = \text{all the entries of row } i = n$$

□

Step 5. $a_{0,0} = 1$.

Proof. Assume to the contrary that $a_{0,0} \geq 2$. So row 0 contains at least two 0 entries (say $a_{0,i_1} = a_{0,i_2} = 0$). So rows i_1 and i_2 do not have any 0 entry and all their entries are positive, thus for each $1 \leq j \leq n-1$, $a_{i_1,j} \geq 1$ and $a_{i_2,j} \geq 1$. It means that row j contains entries i_1 and i_2, so for each $0 \leq j \leq n-1$, $r_j \geq i_1 + i_2$. Because r_j is equal to the number of j's in the table, $S \geq \sum_{j=0}^{n-1}(i_1+i_2)j$. Hence

$$n^2 = S \geq \sum_{j=0}^{n-1}(i_1+i_2)j \geq (1+2)\sum_{j=0}^{n-1} j = 3\frac{n(n-1)}{2} \Rightarrow n \leq 3.$$

If $n = 2$, then number 2 cannot appear in the table (by step 3) and if $n = 3$, then we must complete table 1 with entries equal to 1 or 2. So row 0 must contain at least a 1 entry, which is a contradiction. □

2	0	0

Table 1

Step 6. $a_{0,1} = 0$ and $a_{0,i} > 0$.

Proof. If $i \geq 3$, a method like that of step 5 leads to a contradiction (if $n = 3$, we do not have $a_{0,3}$ in the table).
If $a_{0,2} = 0$, then all the entries of row 2 are positive. So each row contains at least one entry equal to 2 and so each number $0 \leq j \leq n-1$ has appeared in the table at least 2 times. Since row 2 contains at least one entry equal to 2, we deduce that $r_2 \geq n+1$. Therefore, 2 has appeared in the table at least $n+1$ times. Hence

$$n^2 = S \geq 2(\sum_{j \neq 2} j) + 2(n+1) = n^2 + n - 2 \Rightarrow n \leq 2.$$

But 2×2 and 1×1 tables do not contain entry $a_{0,2}$. So we have proved that $a_{0,2} > 0$ and $a_{0,1} = 0$.

□

Step 7. $r_1 \geq 2n - 2$.

Proof. By step 6, all the entries of row 1 are positive. Suppose that $a_{1,1} = t$. If $t = 1$, then other entries of row 1 must be at least 2 and so $r_1 \geq 2n - 1$. If $t > 1$, then row 1 contains at least one entry equal to t, t entries equal to 1 and other entries are at least 2, so $r_1 \geq 1 \times t + t \times 1 + 2 \times (n - t - 1) = 2n - 2$. □

Step 8. $r_0 + (r_2 + \cdots + r_{n-1}) \geq n^2 - 2n + 1$.

Proof. r_0 is the number of 0's of the table. Since row zero contains exactly one entry equal to 0 and all of the entries of row 1 are positive, the number of 0's in rows $2, 3, \ldots, n-1$ is equal to $r_0 - 1$ and thus $n(n-2) - (r_0 - 1)$ entries of these rows are positive. Hence

$$r_2 + \cdots + r_{n-1} \geq n(n-2) - (r_0 - 1) \Rightarrow r_0 + (r_2 + \cdots + r_{n-1}) \geq n^2 - 2n + 1.$$

□

Summing the expressions obtained in steps 7 and 8, we get

$$n^2 = S = r_0 + r_1 + \cdots + r_{n-1} \geq n^2 - 1$$

Corollary 2. $r_1 = 2n - 1$ or $2n - 2$.

Corollary 3. Rows $2, 3, \ldots, n-1$ do not contain any 3's and at most one of the entries of these rows can be 2. Therefore, $a_{i,j} = 0$ for each $i \geq 3, j \geq 2$ and $a_{2,2} + a_{2,3} + \cdots + a_{2,n-1} \leq 1$.

Corollary 4. By corollary 3, there are at least $n - 2$ entries equal to 0 in row j. So $a_{0,j} = n - 2$ or $n - 1$.

Step 9. $n \leq 5$.

Proof. By corollary 4, the number of entries in row 0 equal to $n-1$ or $n-2$ is at least $n - 3$, so $a_{n-2,0} + a_{n-1,0} \geq n - 3$. On the other hand, by corollary 3, at most one of $a_{n-1,0}$ and $a_{n-2,0}$ is 2. Therefore, $a_{n-2,0} + a_{n-1,0} \leq 3$, thus $n - 3 \leq 3$ and so $n \leq 6$. But if $n = 6$ by corollary 3, $a_{2,2} + a_{2,3} + a_{2,4} + a_{2,5} \leq 1$, hence at most one of them can be nonzero and so $a_{0,2} \geq 3$. Therefore, row 0 does not have any 2's and so $a_{2,0} = 0$. Consequently, $a_{0,2} \geq 4$. On the other hand, $a_{0,3}, a_{0,4}, a_{0,5} \geq 4$. So four entries of row 0 are equal to 4 or5, hence $a_{4,0} + a_{5,0} = 4$ which leads to a contradiction with corollary 3. Therefore, $n \leq 5$. □

Step 10. Finding the tables!

Proof. (i) If $n = 2$, $a_{0,0} = 1$ and $a_{0,1} = 0$, so $a_{1,1} = 2$ which contradicts step 4.
(ii) If $n = 3$, $r_1 = 4$ or 5 (corollary 2), but $S = 9$ and all other entries are even. So $r_1 = 5$ and $r_0 = r_2 = 2$. Since $r_1 = 5$, by the proof of step 7 we get table 2. By step 4, $a_{2,0} = a_{2,2} = 0$, contradicting the value of $a_{0,2}$.

1	0	1
2	1	2

Table 2

(iii) If $n = 4$, then $r_1 = 6$ or 7.
Case 1. $r_1 = 6$. Since $S = 16$ is even, the number of 3's in the table must be even and so $r_3 = 2$ ($r_3 = 0$ implies $a_{0,3} = 4$ which is a contradiction!). So $r_2 = 2$ and $r_0 = 6$. On the other hand by corollary 3, there exists exactly one entry equal to 2 in the union of rows 2 and 3, so row 1 contains at most one entry equal to 2. Since $r_1 = 6$ and all of the entries of row 1 are positive, row 1 does not have any 2's. Therefore, $a_{1,0} = a_{1,2} = a_{1,3} = 1$ and $a_{1,1} = 3$. On the other hand, $\{a_{2,3}, a_{2,2}\} = \{1, 0\}$. According to the values of $c_2 = c_3 = 4$, one of $a_{0,3}$ and $a_{0,2}$ is 2 and the other one is 3. Therefore, $a_{2,0} = a_{3,0} = 1$. Now, we do not have any 2's in rows 2 and 3, so $S = 15$ which is a contradiction.
Case 2. $r_1 = 7$. Since $S = 16$ is even, the number of 3's in the table must be odd, so $r_3 = 1$ ($r_3 = 0$ implies $r_2 = 0$ which is a contradiction!). On the other hand, since $r_1 = 7$, row 1 contains one entry equal to 2, one entry equal to 3 and two entries equal to 1 (if there exist 3 entries equal to 2, $a_{2,1} = 3$ which contradicts corollary 3).

Therefore, $a_{3,1} = 1$. Now, since $r_3 = 1$, we get $a_{3,0} = a_{3,2} = a_{3,3} = 0$ and consequently $a_{0,3} = 3$, contradicting the value of $a_{3,0}$.
(iv) If $n = 5$, then $r_1 = 8$ or $r_1 = 9$.
Case 1. $r_1 = 9$. By corollary 3, rows $2, 3, 4$ and 5 do not contain 2. So for each $j \geq 2$, $a_{2,j} = 0$ $(\star)$ and thus $a_{0,2}, a_{0,3}, a_{0,4} \geq 3$. Therefore, two of them are 3 and the others are 4. So one of $a_{3,0}$ and $a_{4,0}$ is at least 4, contradicting $(\star)$.
Case 2. $r_1 = 8$. By the proof of step 7 we have $a_{1,1} > 1$.
(a) $a_{1,1} = 2$. In this case row 1 contains two 1's and three 2's, so $a_{2,1} = 3$, contradicting corollary 3.
(b) $a_{1,1} = 3$. In this case row 1 contains three entries equal to 1, one entry equal to 2 and one entry equal to 3. So $a_{2,1} = a_{3,1} = 1$ and $a_{4,1} = 0$. Since $a_{1,4} > 0$, by corollary 3 we get $a_{4,0} = 1$ and consequently $a_{0,4} = 4$ and $a_{2,4} = a_{3,4} = 0$. On the other hand, $a_{0,3} \geq 3$, but row 0 contains one entry 4 ($a_{0,4}$), so $a_{0,3} = 3$. Again by corollary 3, we get $a_{2,2} + a_{2,3} + a_{2,4} = 1$. Therefore, $\{a_{1,2}, a_{2,2}\} = \{0, 1\}$. Then $a_{1,2} = 2$ and $a_{1,0} = a_{1,3} = 1$. Consequently, $a_{2,3} = 1$, $a_{2,2} = 0$, $a_{0,2} = 3$ and $a_{0,3} = 2$. (See Table 3)
(c) $a_{1,1} = 4$. This implies $a_{1,j} = 1$ for $j \neq 1$, so $a_{2,1} = a_{3,1} = 0$ and $a_{4,1} = 1$. Therefore, $a_{0,3} = 4$, $a_{2,3} = 0$ and $a_{3,0} = 1$ (since $a_{1,3} = 1$). We know that $a_{4,0} \geq 1$ and $a_{4,1} = a_{1,4} = 1$. So $a_{4,0} = 2$ and hence $a_{2,4} = 1$ and $a_{2,0} = 0$. These imply $a_{0,4} = 3$, $a_{0,2} = 4$ and $a_{2,2} = 0$. (See table 4)

1	0	3	3	4
1	3	2	1	1
0	1	0	1	0
2	1	0	0	0
1	0	0	0	0

Table 3

1	0	4	4	3
1	4	1	1	1
0	0	0	0	1
1	0	0	0	0
2	1	0	0	0

Table 4

$\square$

By these 10 steps, we deduce that the two tables 3 and 4 are the only ones satisfying the problems condition.

4 . There are 2014×2013 ordered pairs (a, b) such that $1 \leq a \neq b \leq 2014$. On the other hand, there are 2013 consequtive pairs in each permutation. Therefore, the number of such permuations is at most 2014.
Consider the 2014 permutations of the form

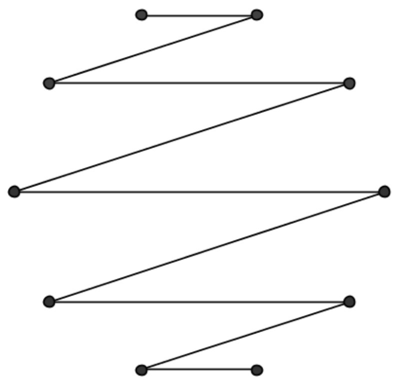
$$\sigma_i = (2014 + i, 1 + i, 2013 + i, 2 + i, \cdots, 1007 + i, 1006 + i), \quad 1 \leq i \leq 2014,$$

where all the numbers are considered modulo 2014 (2014 itself being an exception). We claim that for each $a \neq b$ there is exactly one index i such that b has appeared after a in σ_i. To prove it, note that the differences of consequtive terms in all these permutations are

$$(2013, 2, 2011, 4, 2009, 6, \ldots, 2012, 1),$$

and all the numbers are modulo 2014. So the position of b and a is uniquely determined by the value of $b - a$ and the index of permutation is uniquely determined by the value of b.

Comment 1. If we assign a directed hamiltonian path in a complete graph with 2014 veritces labled with numbers $1, 2, ..., 2014$ to each permutation, the equivalent problem is to find a partition of the complete graph with 2104 vertices to directed hamiltonian paths. So our proof implies the existence of such a partition. Consider the following path and its rotations.



Comment 2. A similar proof will work for every even number instead of 2014.

5 . Using AM-GM inequality two times, we get

$$\begin{aligned} n\sum_{i=1}^{n+1} \sqrt[x_i]{n} &= \sum_{j=1}^{n+1}\left(\sum_{i\neq j} n^{\frac{1}{x_i}}\right) \geq \sum_{j=1}^{n+1}\left(nn^{\frac{1}{n}\sum_{i\neq j}\frac{1}{x_i}}\right) \\ &\geq \sum_{j=1}^{n+1} nn^{\sqrt[n]{\prod_{i\neq j}\frac{1}{x_i}}} = n\sum_{j=1}^{n+1} n^{\sqrt[n]{x_j}}. \end{aligned}$$

Hence the assetion is proved.

6 . Throughout the solution, let ω and ω_a be the circumcirle of triangle ABC and the excircle of ABC tangent to side BC, respectively. Furthermore, O and I_a are the centers of ω and ω_a, respectively.
Under an inversion with center A and power $AB \times AC$, and then a reflection with respect to the bisector of $\angle BAC$, B' is sent to some point of line AB, and C' to some point of line AC. Denote by B'' and C'' these two points, respectively. Note that

$$AB'.AC'' = AC'.AB'' = AB.AC = AI.AI_a.$$

Therefore, the quadrilaterals $II_aB''C'$ and $II_aC''B'$ are cyclic. Since $\angle I_aIB' = 90°$, we get $I_aC'' \perp AC$ and hence C'' is the tangency point of ω_a and AC. Similarly, B'' is the tangency point of ω_a and AB.
Now, we are going to find the image of B_1 and C_1 under the transformation mentioned above. Denote by M and N the midpoints of arcs ACB and ABC of circle ω, repectively. Then $\angle BAC_1 = \angle CAM$. We know that under the transformation, line BC transforms to circle ω. Hence C_1 is sent to M. Analogously, B_1 is sent to N. So T is sent to the intersection point of lines NC'' and MB'', say T'. Note that the tangent to ω at M is parallel to the tangent to ω_a at C''. Thus the direct homothetic center of ω and ω_a, lies on line NC''. Denote this point by T''. By a similar argument, T'' lies on MB'' and hence $T'' = T'$ lies on line OI_a. This implies that $T'I_a$ is perpendicular to ω. Since the transformation above preserves angles, we deduce that the circumcircle of ATI (the transformation of $T'I_a$) is perpendicular to BC (the transformation of ω) and hence the center of this circle lies on the side BC.

7 . We will prove the statement by induction on n. Let e be the last edge, with i, j as the lables of it's vertices. Deleting e divides the tree into two trees with vertices $I = \{i = i_1, i_2, \ldots, i_k\}, J = \{j = j_1, \ldots, j_l\}$ (we denote every vertex by it's initial lable). By the induction hypothesis, before the last switch the permutation of lables is the product of two cyclic permutations on I and J, which after renumbering the indices can be written in the form

$$(i_1 i_2 \cdots i_k)(j_1 j_2 \ldots j_l).$$

So the endpoint lables of e before the last operation are i_2, j_2, and after this operation the permutation of lables becomes

$$(i_2 j_2)(i_1 \cdots i_k)(j_1 \cdots j_l) = (i_1 j_2 j_3 \cdots j_l j_1 i_2 i_3 \cdots i_k),$$

which is a cycle of order n.

8 . First, note that the circumcircles of triangles AII_1 and AII_2 are penpendicular, because

$$\angle AI_1I + \angle AI_2I = \frac{\angle B + \angle BAD}{2} + \frac{\angle C + \angle CAD}{2} = 90^\circ$$

Under an inversion with center A and power $AB \times AC$ and then a reflection with respect to the bisector of $\angle BAC$, pairs (B, C) and (I, I_a) are mapped to each other. Let M' and N' be the image of M and N under this transformation, respectively. Circumcirles of AI_1I and AI_2I (which pass through the center of inversion) are mapped to lines I_aM' and I_aN'. It is obvious that in this transformation, the circumcircle of triangle ABC and the line BC are mapped to each other, so M' and N' are on the line BC. Now, the assertion becomes equivalent to proving that the circumcircle of $AM'N'$ passes through some constant point other than A as D varies.
Since inversion and reflection both preserve the angles, we have $I_aM' \perp I_aN'$. If X is the foot of the perpendicular line from I_a to BC, then the power of X with respect to this circle is $M'X.N'X = I_aX^2$ which is constant as D varies. Therefore, the power of X with respect to the circumcircle of $AM'N'$ is constant. Hence X lies on the radical axis of these circles and another point on the line AX is the common point of all these circles.

9 . Using the lemma proved in the solution of the 7th problem of the Third Round, we can say that if k is even, there exist an integer a and some polynomial $Q(x) \in \mathbb{Z}[x]$ such that

$$(Q(x))^2 \leq a^2(x+1)(x+2)\cdots(x+k) < (Q(x)+1)^2, \quad \text{for large values of } x.$$

Since $Q(x)$ is a perfect square for infinitely many values of x, we get

$$(Q(x))^2 = a^2(x+1)(x+2)\cdots(x+k).$$

This is a contradiction since $a^2(x+1)(x+2)\cdots(x+k)$ does not have any multiple roots.
For odd values of k, we claim that if $\prod_{i=1}^{k}(a+i)$ is a perfect square, there is some nonempty proper subset A of $\{1, 2, \ldots, k\}$ such that $\prod_{i \in A}(a+i)$ is a perfect square. To prove the claim, consider all of the subsets of $S = \{a+1, a+2, \ldots, a+k\}$. For

any subset A of S, let $F(A)$ be the product of elements of A ($F(\emptyset)=1$) and let $g(A)$ be the square free part of $F(A)$.
Note that all of the prime divisors of $g(\{a+1\}), g(\{a+2\}), \ldots, g(\{a+k\})$ are less than k, because if some prime number $p \geq k$ divides two of the numbers $g(\{a+1\}), g(\{a+2\}), \ldots, g(\{a+k\})$, for example $p|a+i$ and $p|a+j$, we obtain $p|j-i$ and hence $p<k$. If for some i, $1 \leq i \leq k$, p divides only $g(\{a+i\})$, this leads to a contradiction because $g(S)=1$ is a perfect square, and the power of p in the factorization of $\prod_{i=1}^{k} a+i$ should be even.
For any $A \subseteq S$, $g(A)|g(\{a+1\})g(\{a+2\})\cdots g(\{a+k\})$ and so the prime divisors of all $g(A)$'s are less than k. This implies that for each $A \subseteq S$, $g(A)$ is a divisor of $2 \times 3 \times \cdots \times p_{\pi(k-1)}$. So we have $2^{\pi(k-1)}$ cases for $g(A)$.
Since $k-1 > \pi(k-1)$, $2^k > 2 \times 2^{\pi(k-1)}$. This implies that there are two subsets $A, B \subseteq S$ such that $A \neq B, S-B$ and $g(A)=g(B)$ (note that $g(A)=g(S-A)$ because $g(S)=1$). Now, $g(A)=g(B)$, therefore $F(A)F(B)$ is a perfect squre. We have

$$F(A)F(B)=F(A\cap B)^2F(A\Delta B).$$

Hence $F(A\Delta B)$ is a perfect square. But $A\Delta B \neq \emptyset, S$ since $A \neq B, S-B$, and this is the desired subset.
By replacing $A\Delta B$ with $S-A\Delta B$ if necessery, we will get a nonempty proper subset X of $\{1,2,\ldots,k\}$ with an even number of elements such that $\prod_{i\in X}(x+i)$ is a perfect square. Using the lemma in the solution of the 7th problem of the Third Round again, this product can be a perfect square for at most a finite number of values of x. Thus if k is odd, then $(x+1)(x+2)\cdots(x+k)$ is a perfect square for at most a finite number of values of x.

10 . a) Let $a_1=2$, $a_2=4, \ldots, a_k=2k, a_{k+1}=1$, $a_{k+2}=3, \ldots, a_{2k}=2k-1$. We can easily check that $a_ia_{i+1}+1$ is a perfect square for $1 \leq i \leq 2k$ except $i=k$, which can be repaired if $2k+1$ is a perfect square which is possible for infinitely many values of k.
b) Let $a_1, a_2, \ldots, a_n$ be a cubic permutation. Let 2^k be the largest power of 2 less than or equal to n. By the definition of the cubic permutation we know that $2^ku+1=x^3$, where u is an element of the permutation. So we have $2^ku=(x-1)(x^2+x+1)$. Hence we conclude that $2^k|(x-1)$. Because of the way that k is chosen, we have $n<2^{k+1}$. So we have $2^k \leq x-1 \leq n^{\frac{2}{3}} < 2^{\frac{2}{3}(k+1)}$ which is a contradiction. Hence no such permutation exists.

11 . Because of the problems assumption, it is enough to prove that

$$(\prod(x-y))^2(\sum x^2) \leq \sum(x^2-y^2)^2(\sum(xy)^2).$$

By Cauchy-Schwarz inequality we have

$$(\sum xy(x^2-y^2))^2 \leq \sum(x^2-y^2)^2(\sum(xy)^2). \quad (1)$$

On the other hand, an easy calculation shows that

$$(\prod(x-y))^2(\sum x)^2 = (\sum xy(x^2-y^2))^2.$$

Finally, we have

$$(\prod(x-y))^2(\sum x^2) \leq (\prod(x-y))^2(\sum x)^2 = (\sum xy(x^2-y^2))^2. \quad (2)$$

(1) and (2) imply the desired assertion.

13 . We start by proving a simple lemma.

Lemma 1. *Let X' be the reflection of the circumcenter of triangle XYZ with respect to side YZ. Then A' and X are symmetric with respect to the center of the nine-point circle of triangle XYZ.*

Proof. Let O and H be the circumcenter and orthocenter of triangle XYZ, respectively. It is well-known that the center of the nine-point circle is the midpoint of HO. On the other hand, it is easy to see that $OX' = XH$ and $OX' \parallel XH$. So the quadrilateral $XHX'O$ is a parallelogram and the proof is complete. □

Using the lemma, it is enough to show that A' and I are symmetric with respect to B_1C_1. To prove it, let A'' be the reflection of I with respect to B_1C_1. We claim that $A'' = A'$. We will show that $A_1A'' \perp B_2C_2$. By cosine formula in triangles A_1IB_2, A_1IC_2, $A''IB_1$ and $A''IC_1$, we get:

$$\begin{aligned}
A_1B_2^2 &= r^2 + IB_2^2 - 2r.IB_2\cos(90^\circ + \tfrac{\angle B}{2}), &(1)\\
A_1C_2^2 &= r^2 + IC_2^2 - 2r.IC_2\cos(90^\circ + \tfrac{\angle C}{2}), &(2)\\
A''B_2^2 &= r^2 + B_1B_2^2 - 2r.B_1B_2\cos(90^\circ - \angle A), &(3)\\
A''C_2^2 &= r^2 + C_1C_2^2 - 2r.C_1C_2\cos(90^\circ - \angle A), &(4)
\end{aligned}$$

where r indicates the radius of the incircle of triangle ABC. Substracting equations (1) from (2) and (3) from (4) imply

$$A_1B_2^2 - A_1C_2^2 = IB_2^2 - IC_2^2 - 2r(IB_2\cos(90^\circ + \tfrac{\angle B}{2}) - IC_2\cos(90^\circ + \tfrac{\angle C}{2})),$$

$$A''B_2^2 - A''C_2^2 = B_1B_2^2 - C_1C_2^2 - 2r(B_1B_2\cos(90^\circ - \angle A) - C_1C_2\cos(90^\circ - \angle A)).$$

Note that

$$B_2C_1^2 - B_1C_2^2 = r^2 + B_1B_2^2 - r^2 - C_1C_2^2 = B_1B_2^2 - C_1C_2^2.$$

Therefore, we must show

$$IB_2\cos(90^\circ + \tfrac{\angle B}{2}) - IC_2\cos(90^\circ + \tfrac{\angle C}{2}) = (B_1B_2 - C_1C_2)\cos(90^\circ - \angle A).$$

Let M and N be the perpendicular projections of B_2 and C_2 on AC and AB, respectively. It is easy to see that $\angle IB_2M = 90^\circ - \frac{\angle B}{2}$ and $\angle IC_2M = 90^\circ - \frac{\angle C}{2}$. So

$$\begin{aligned}
IB_2\cos(90^\circ + \tfrac{\angle B}{2}) - IC_2\cos(90^\circ + \tfrac{\angle C}{2}) &= IC_2\cos(90^\circ - \tfrac{\angle C}{2}) - IB_2\cos(90^\circ - \tfrac{\angle B}{2})\\
&= (C_2N - r) - (B_2M - r)\\
&= C_2N - B_2M.
\end{aligned}$$

On the other hand, $B_1B_2 - C_1C_2 = AC_2 - AB_2$ and hence

$$(B_1B_2 - C_1C_2)\cos(90^\circ - \angle A) = (AC_2 - AB_2)\sin(\angle A) = C_2N - B_2M$$

This completes the proof.

14 . The answer is Yes!
Let $d(n)$ be the number of divisors of natural number n. We want to construct f such that for any positive integer m, $f(d(f(m)) = d(m)$. Let $A_k = \{n \in \mathbb{N} | d(n) = k\}$. For example, $A_1 = \{1\}$ and A_2 is the set of prime numbers. Note that A_k has an infinite number of elements for every $k > 1$, because $p^{k-1} \in A_k$ for every prime number p.
To define f, set $f(1) = 1, f(2) = 2, f(3) = 5$ and $f(5) = 3$. For each $n \geq 4$, suppose that $f(k)$ is defined for $1 \leq k \leq n-1$. If $f(n)$ is not defined then let $j = f(d(n))$. j is well defined because $d(n) < n$. Let t be the least element of A_j that f has not been defined on it yet, so we have $d(t) = j$. Define $f(n) = t$ and $f(t) = n$.
Therefore, for each natural number n, these properties are gained inductively:

$$f(d(n)) = j = d(t) = d(f(n)),$$

$$f(f(n)) = n,\ f(f(t)) = t,$$

$$f(d(t)) = f(j) = f(f(d(n))) = d(n) = d(f(t)).$$

Hence for every $m \in \mathbb{N}$, we have $f(d(f(m))) = f(f(d(m))) = d(m)$.
Comment 1. Indeed, f has the property that for each natural number n, f serves as a bijection between A_n and $A_{f(n)}$.

15 . Since $p(x)$ is a decreasing function and $nx + m$ is strictly increasing, there exists a unique real number α such that $p(\alpha) = n\alpha + m$. Setting $x = \alpha$ in the problems statement implies

$$p\big(q(\underbrace{n\alpha + m}_{p(\alpha)} + h(\alpha)\big) = n\big(q(p(\alpha) + h(\alpha)\big) + m$$

This means that $q(p(\alpha)) + h(\alpha)$ is a solution of $p(x) = nx + m$. But we know that the root of this equation is unique, so $q(p(\alpha)) + h(\alpha) = \alpha$. Now, for every function g, $g\big(q(p(\alpha)) + h(\alpha)\big) = g(\alpha) < g(\alpha)^2 + 1$. Therefore, no such function g exists.

16 . Let $P(x, y)$ be the assertion $f(\frac{y}{f(x+1)}) + f(\frac{x+1}{xf(y)}) = f(y)$.
Rewriting the equation, we get $f(\frac{y}{f(x)}) + f(\frac{x}{(x-1)f(y)}) = f(y)$ which is valid for each $x > 1, y \in \mathbb{R}^+$.
If there exists $a \in \mathbb{R}^+$ for which $f(a) > \frac{1}{a}$, we get

$$P(\frac{af(a)}{af(a)-1}, a) \Rightarrow f(\frac{a}{f(\frac{af(a)}{af(a)-1})}) = 0,$$

which is a contradiction. Therefore, for each $x \in \mathbb{R}^+$, $f(x) \leq \frac{1}{x}$.
Now, we can write

$$P(x > 1, y) \Rightarrow f(y) = f(\frac{y}{f(x)}) + f(\frac{x}{(x-1)f(y)}) \leq \frac{f(x)}{y} + (1 - \frac{1}{x})f(y) \Rightarrow yf(y) \leq xf(x).$$

So for all $x, y > 1 \in \mathbb{R}^+$, $xf(x) = yf(y)$. Hence for each $x > 1 \in \mathbb{R}^+$, $f(x) = \frac{C}{x}$ where C is a constant number. If we choose real numbers x, y greater than 1 such that y is also greater than C, substituting these values for x and y in the equation shows that $C = 1$. Therefore, $\forall x > 1 \in \mathbb{R}^+$, $f(x) = \frac{1}{x}$.

Since $f(1) \leq 1$, we have

$$P(x > 1, 1) \Rightarrow f(1) = f(\frac{1}{f(x)}) + f(\frac{x}{(x-1)f(1)}) = f(x) + \frac{x-1}{x}f(1)$$

$$\Rightarrow \frac{1}{x}f(1) = \frac{1}{x} \Rightarrow f(1) = 1.$$

Now, for $\frac{1}{2} \leq x < 1$, we can write

$$P(2, x) \Rightarrow f(\frac{x}{f(2)}) + f(\frac{2}{f(x)}) = f(x) \Rightarrow f(2x) + f(\frac{2}{f(x)}) = f(x)$$

$$\Rightarrow \frac{1}{2x} + \frac{f(x)}{2} = f(x) \Rightarrow f(x) = \frac{1}{x}.$$

In a similar way, by using induction on n one can prove that for each positive real x in the interval $\frac{1}{2^n} \leq x < \frac{1}{2^{n-1}}$, $f(x) = \frac{1}{x}$. Therefore, for each $x \in \mathbb{R}^+$, $f(x) = \frac{1}{x}$. It's easy to verify that this solution is indeed an answer to the functional equation.

17 . First, note that the statement of the problem is invariant under translations of the set X. Therefore, without loss of generality we may assume that $X = \{x_1 = 0 < x_2 < \cdots < x_n\}$. Next, we shall prove some lemmas.

Lemma 1. *Let $0 \leq a < b \leq x_n$ be two real numbers such that $b - a = 1$. Then either a or b is good.*

Proof. The proof is very easy. □

Lemma 2. *Under the assumptions of the problem, for a given index $1 < i \leq n$, there are at least $\binom{n-2}{x_i-1}$ dense subsets of X with x_i elements.*

Proof. Let A be an arbitrary subset of $X - \{x_1, x_i\}$ such that $|A| = x_i - 1$ (there are $\binom{n-2}{x_i-1}$ such subsets). We claim that either $A \cup \{x_1\}$ or $A \cup \{x_i\}$ is dense. Note that the average of the members of these two sets are $\frac{x_1+\sum_{x\in A} x}{x_i} = \frac{\sum_{x\in A} x}{x_i}$ and $\frac{x_i+\sum_{x\in A} x}{x_i} = \frac{\sum_{x\in A} x}{x_i} + 1$, respectively. So by lemma 1, at least one of these two averages is good and the claim is proved. Finally, note that different subsets of $X - \{x_1, x_i\}$ lead to different dense subsets during this process and thus we have at least $\binom{n-2}{x_i-1}$ dense subsets of size x_i. □

Lemma 2 implies that there are at least $\sum\limits_{i=1}^{n} \binom{n-2}{x_i-1}$ dense subsets. Now, we shall show that this number is not less than 2^{n-3}. The following lemma will be useful here.

Lemma 3. *Let $\{a_i\}_{i=0}^{\infty}$ be a strictly increasing sequence of integers such that $a_1 = -1$ and $a_i - a_{i-1} \in \{1, 2\}$ for each $i > 1$. Then $\sum\limits_{i=1}^{\infty} \binom{n}{a_i} \geq 2^{n-1}$.*

Proof. We use induction on n. The base cases in trivial. For the induction step, the Pascal's rule will help us as follows:

$$\sum_{i=1}^{\infty} \binom{n}{a_i} = \sum_{i=1}^{\infty} \binom{n-1}{a_i} + \sum_{i=1}^{\infty} \binom{n-1}{a_i - 1} \geq 2^{n-2} + 2^{n-2} = 2^{n-1}$$

□

Lemma 3 obviousely yields the assertion.

18 . First, we prove two lemmas.

Lemma 1. *Let l_1 and l_2 be two lines and A, B and C three points in the plane. Denote by A_1 and A_2 the perpendicular projections of A on l_1 and l_2, resepectively. Furthermore, let A' be the midpoint of A_1A_2. B' and C' are defined similarly. A, B and C are collinear if and only if A', B' and C' are collinear.*

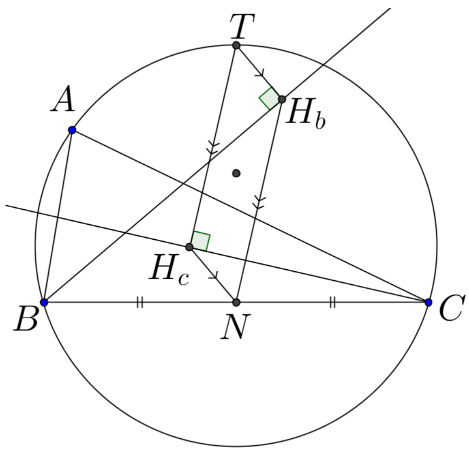
Proof. The key observation is that $\overrightarrow{A'B'} = \frac{1}{2}(\overrightarrow{A_1B_1} + \overrightarrow{A_2B_2})$ and $\overrightarrow{B'C'} = \frac{1}{2}(\overrightarrow{B_1C_1} + \overrightarrow{B_2C_2})$. Using these observations and the Thales' theorem, it is easy to check that the assertion is equivalent to the equality $\frac{A_1B_1}{A_2B_2} = \frac{B_1C_1}{B_2C_2}$. □

Lemma 2. *Let T be the midpoint of arc BAC of the circumcircle of triangle ABC. Furthermore, let H_b and H_c be the feet of the perpendicular lines from T to the internal bisectors of $\angle B$ and $\angle C$, respectively. If T' is the midpoint of H_bH_c, then T' lies on the perpendicular bisector of side BC.*

Proof. Let N be the midpoint of side BC. Note that $TH_c \perp CH_c$ and $TN \perp BC$, hence the quadrilateral TH_cNC is cyclic and

$$\angle H_cNB = \angle H_cTC = 90^\circ - \angle TCH_c = \angle H_cCB + \angle NTC = 90^\circ - \frac{\angle B}{2}.$$

It means that H_cN is parallel to the external bisector of $\angle BAC$. On the other hand, TH_b is also parallel to the external bisector of $\angle ABC$ and thus $TH_b \parallel H_cN$. Similarly, $TH_c \parallel H_bN$. So TH_cNH_b is a parallelogram and hence T', the midpoint of H_bH_c, lies on TN. But TN is the perpendicular bisector of side BC, which is what we desired to prove.



□

We will keep using the notations in lemma 2. We will now apply lemma 1, with IC, IB, I_a, X and M playing the roles of l_1, l_2, A, B and C, respectively. Since N, M and T' are collinear, we obtain that I_a, X and T are collinear, too.
Now, by an inversion with center A and power $AB \times AC$ and then a reflection with respect to the internal bisector of angle $\angle BAC$, B is sent to C, C is sent to B, I_a is sent to I and M is sent to the foot of the external bisector of $\angle A$, say M'. If X' is the image of X under this transformation, then points A, I, X' and M' are on a common circle. So $\angle IX'M' = \angle IAM' = 90^\circ$. On the other hand, since the line BC is the

image of the circumcircle of triangle ABC under this transformation, X' lies on the line BC. This implies that $X' = D$ and thus $\angle BAD = \angle CAX$.