On cyclic quadrilaterals and the butterfly theorem

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1. The butterfly theorem(s)

Being one of the most well-known geometric facts beyound the usual school curriculum, the butterfly theorem has received attention of various authors. In [1], 14 proofs of this theorem and a number of generalizations are presented. In this note, we are going to show two new approaches to the butterfly theorem and incidentally prove an important fact on cyclic quadrilaterals.

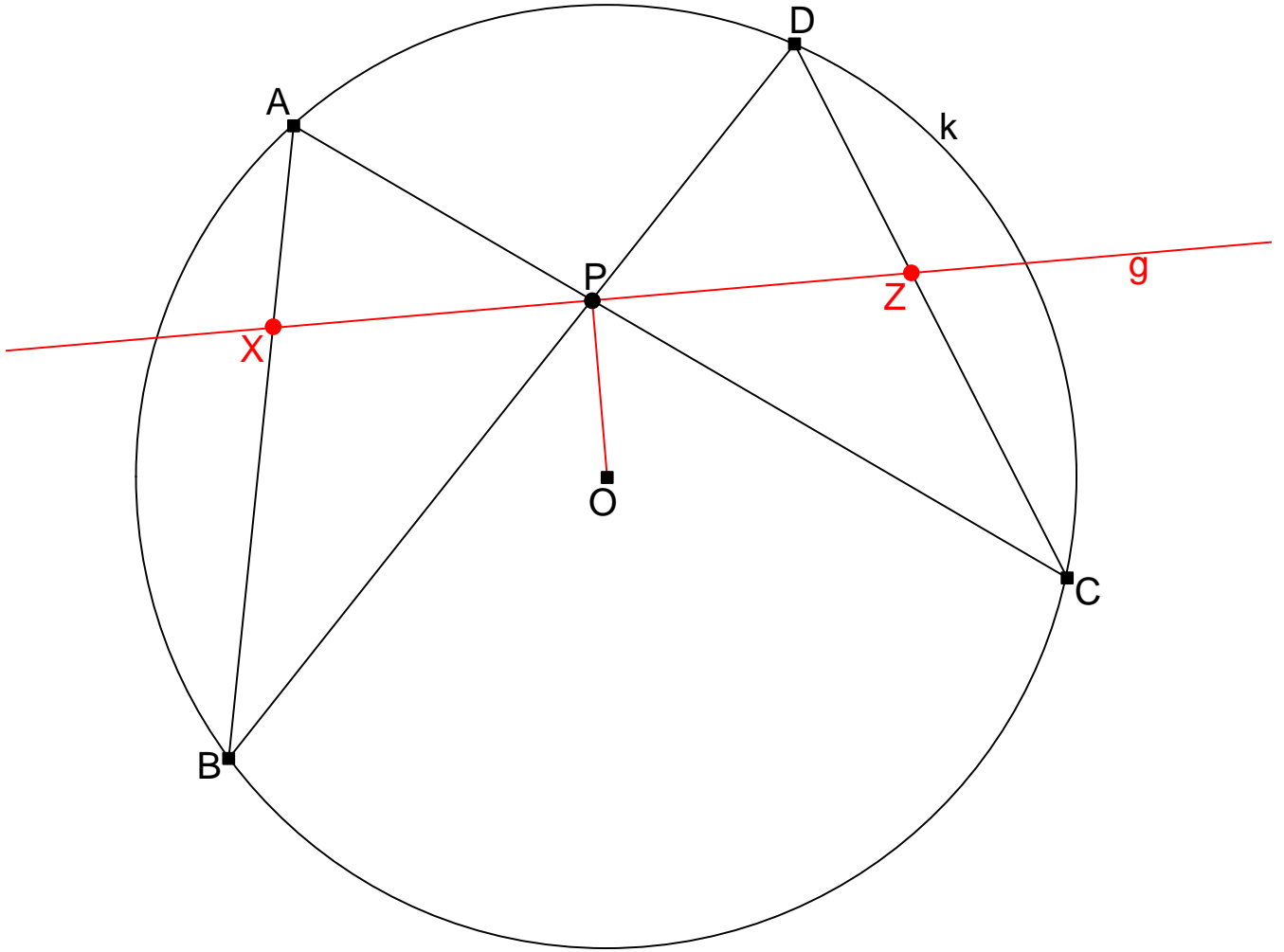The butterfly theorem is known in two versions, a "strong" and a "weak" one:[1]

Fig. 1

Theorem 1, the strong butterfly theorem. Let k be a circle with center O, and let A, B, C, D be four points on this circle k. Let $P = AC \cap BD$. Let g be a line through the point P such that the point P is the orthogonal projection of the point O on this line g. [2] Let $X = g \cap AB$ and $Z = g \cap CD$. Then, the point P is the midpoint

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[1] In the following, the point of intersection of two lines u and v will be denoted by $u \cap v$.

[2] In the case when $P \neq O$, this simply means that the line g is the perpendicular to the line OP at the point P. In the case $P = O$, however, g can be any arbitrary line through the point P.

Note that, while we have thus taken care of the case $P = O$ in the formulation of Theorem 1, we

of the segment XZ. (See Fig. 1.)

Theorem 2, the weak butterfly theorem. Let k be a circle with center O, and let T_1, T_2, A, B, C, D be six points on this circle k. Let $P = AC \cap BD$. Assume that the point P is the midpoint of the segment T_1T_2. Let $X = T_1T_2 \cap AB$ and $Z = T_1T_2 \cap CD$. Then, the point P is the midpoint of the segment XZ. (See Fig. 2.)
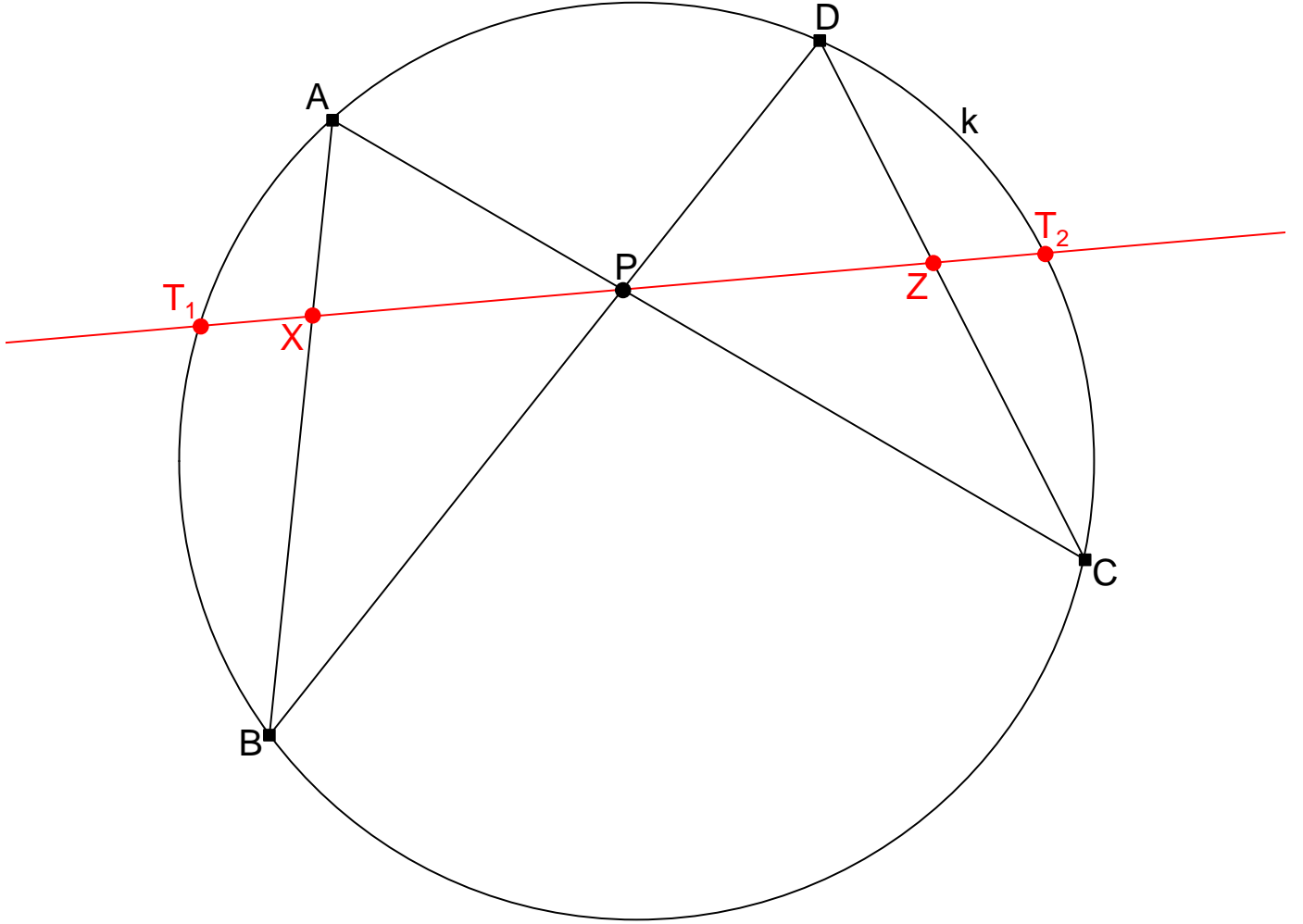

Fig. 2

First a *remark* on degenerate cases: In the formulations of both theorems, we didn't exclude the case when some of the points A, B, C, D coincide. In such cases, the following convention is to be applied: If two points P_1 and P_2, which have been defined as two points on a circle k, coincide, then the line P_1P_2 has to be understood as the tangent to the circle k at the point P_1 (or, what is the same, at the point P_2).

Theorems 1 and 2 are called strong and weak butterfly theorem for the reason that Theorem 2 readily follows from Theorem 1, but not conversely (it is possible to infer Theorem 1 from Theorem 2 using an algebraic argument, but this is not quite trivial).

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won't always take care of such particular cases in the proofs of our theorems.
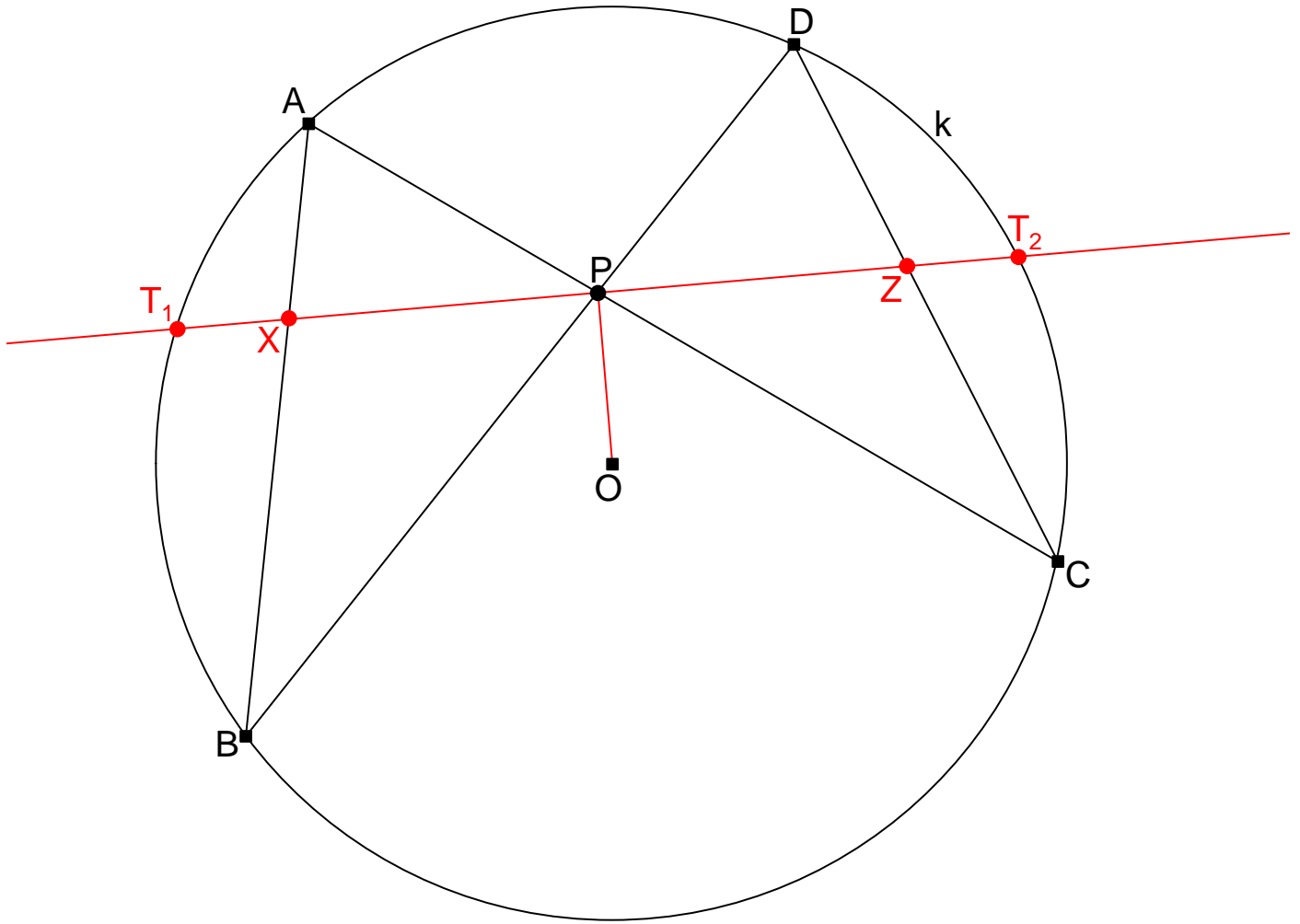
Fig. 3

The *proof of Theorem 2* using Theorem 1 works out as follows: (See Fig. 3.) Since T_1T_2 is a chord of the circle k, while O is the center of this circle k, the point O lies on the perpendicular bisector of T_1T_2 (since the perpendicular bisector of a chord of a circle passes through the center of the circle). Since P is the midpoint of T_1T_2, this yields that P is the orthogonal projection of the point O on the line T_1T_2. Thus, Theorem 2 follows from Theorem 1 (we just have to apply Theorem 1 to the line T_1T_2 in the role of the line g).

Most of the literature considers the weak Theorem 2 as "the" butterfly theorem and disregards Theorem 1 - a pity, for Theorem 1 has numerous applications in geometry which are harder (if possible at all) to handle with Theorem 2. We are going to forget about Theorem 2 for the rest of this note and work with Theorem 1 only.

Note that proofs of Theorem 1 are also featured in [14] and [15], and [16] indicates a proof of Theorem 2 which can be easily extended to a proof of the stronger Theorem 1.

We are going to give two proofs of Theorem 1.

2. The first proof
(requirements: Pascal theorem)

First proof of Theorem 1. We show a generalization of Theorem 1:

Theorem 3, the strong Klamkin butterfly theorem. Let k be a circle with center O, and let A, B, C, D be four points on this circle k. Let g be an arbitrary line, and let P be the orthogonal projection of the point O on this line g. The line g intersects the lines AB, BC, CD, DA, AC, BD at the points X, Y, Z, W, U, V. Then, the following three assertions are pairwisely equivalent:

Assertion 1: The point P is the midpoint of the segment XZ.

Assertion 2: The point P is the midpoint of the segment YW.

Assertion 3: The point P is the midpoint of the segment UV.

(See Fig. 4.)

A few words about the name of Theorem 3: Murray Klamkin found a weaker version of Theorem 3 which generalizes the weak butterfly theorem (Theorem 2) in the same way as Theorem 3 generalizes the strong butterfly theorem (Theorem 1). See [1] (Remark to Proof 5') for details on Klamkin's result. In the form given here, Theorem 3 has been proven by Virgil Nicula in [11], post #2 (P.B.2).
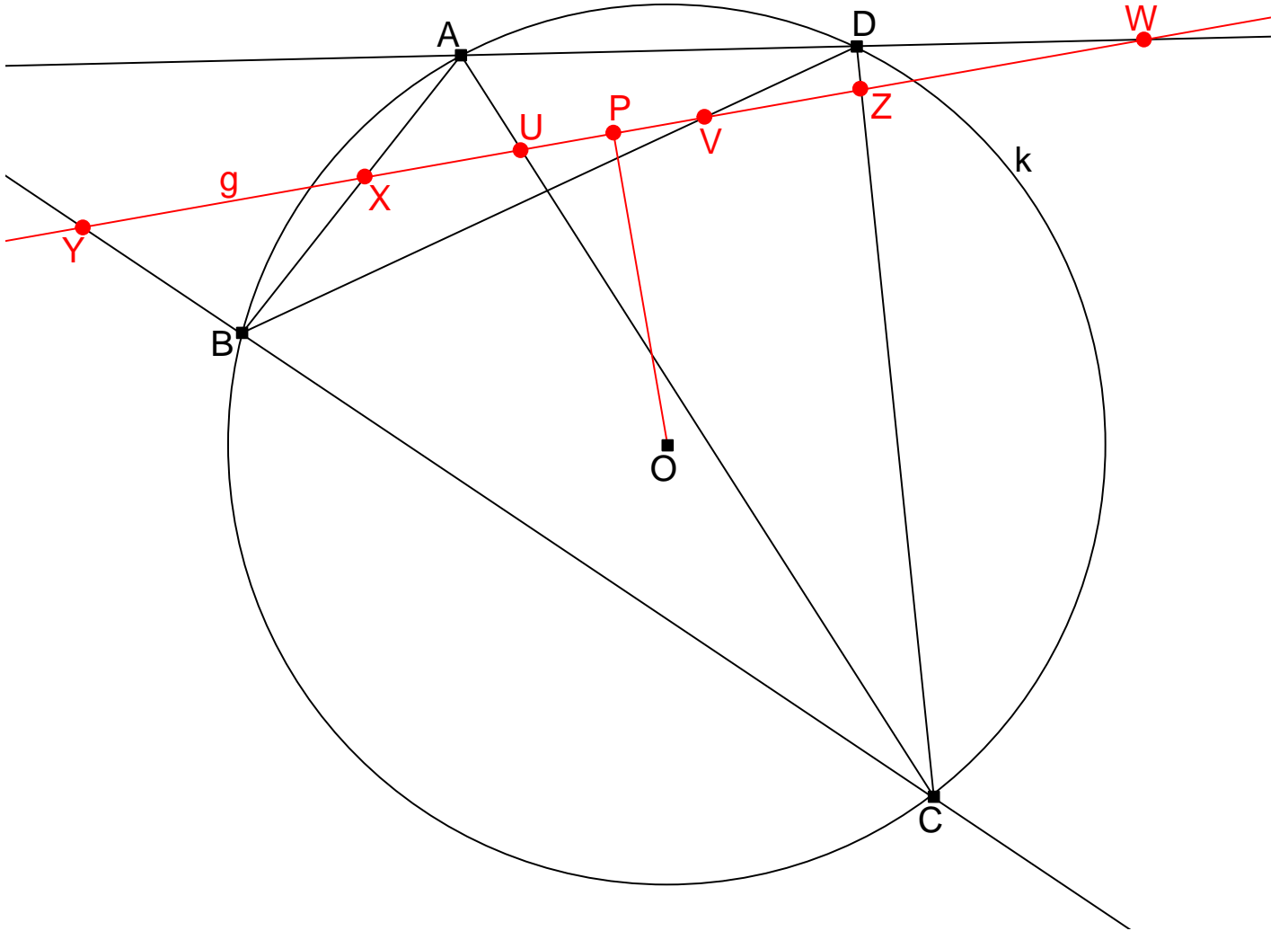

Fig. 4

The following *proof of Theorem 3* is apparently new: We start with a property of

triangles which, at the first sight, seems unrelated to our subject, but turns out to be equivalent to Theorem 3. This property was formulated by a MathLinks user with the nickname "Bismarck" in [2], post #4:

Theorem 4. Let ABC be a triangle, and g an arbitrary line. The line g intersects the lines BC, CA, AB at the points X, Y, Z.

Let O be the circumcenter of triangle ABC, and let P be the orthogonal projection of the point O on the line g. Let X', Y', Z' be the reflections of the points X, Y, Z in the point P.

Then, the lines AX', BY', CZ' concur at one point, and this point lies on the circumcircle of triangle ABC. (See Fig. 5.)

Theorem 4 is related to the Blaikie theorem, which states that the lines AX', BY', CZ' concur for *any* point P on the line g - not just for P being the orthogonal projection of the point O on the line g. The point of concurrence is called the <mark>*Blaikie point*</mark> of the line g and the point P with respect to triangle ABC. In the context of this result, Theorem 4 shows that, if the point P is the orthogonal projection of the point O on the line g, then the Blaikie point of the line g and the point P with respect to triangle ABC lies on the circumcircle of triangle ABC.
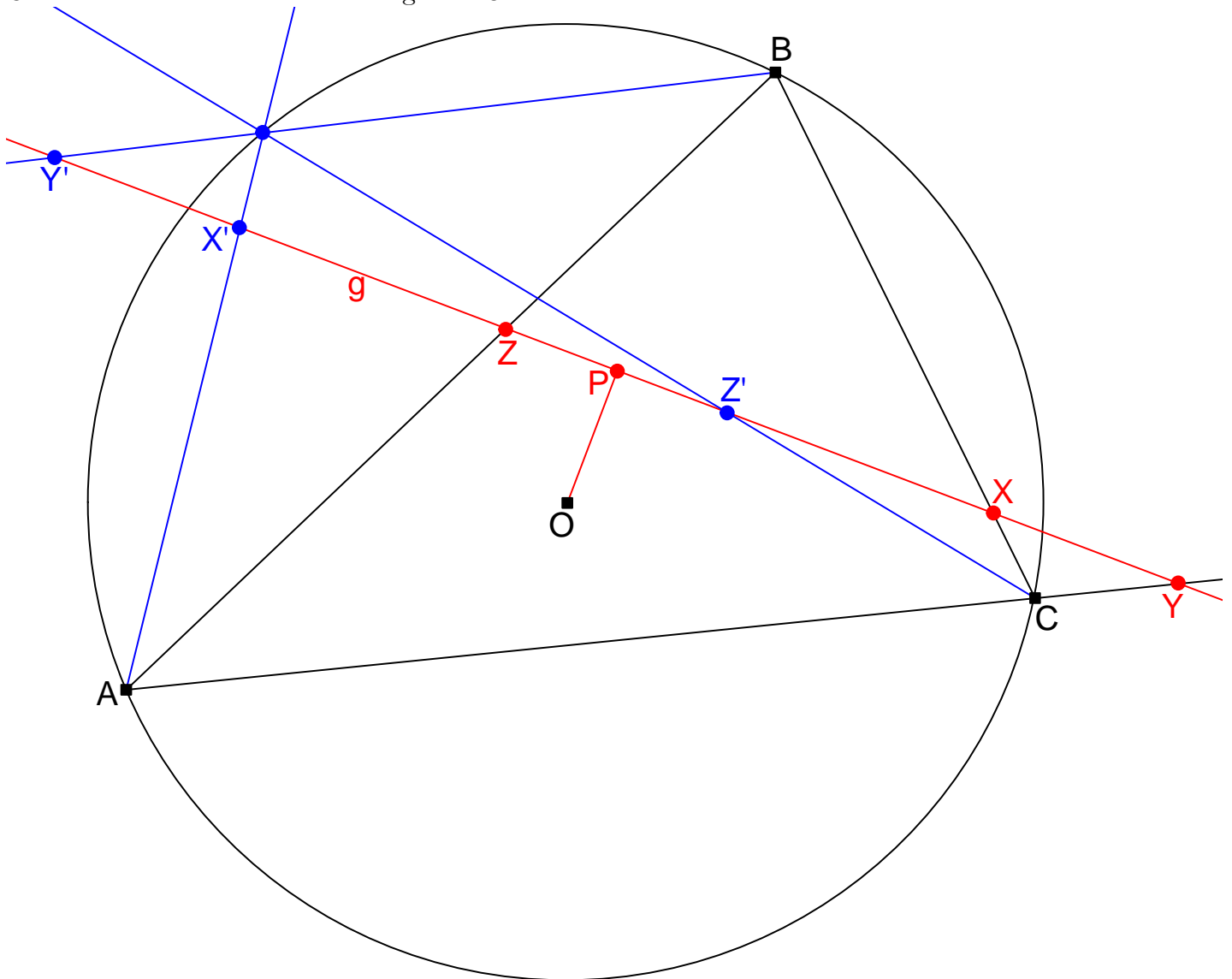
Fig. 5

Proof of Theorem 4. (See Fig. 6.) (In the following, we will often speak of the line OP. In the case when $O = P$, the line OP will mean the perpendicular to the line g at the point O.)

Since X' is the reflection of X in the point P, the point P is the midpoint of the segment XX'. Also, $g \perp OP$. Thus, the line OP is perpendicular to the segment XX' and passes through the midpoint P of this segment. Hence, the line OP is the perpendicular bisector of the segment XX'. Thus, X' is the reflection of X in the line OP.

Let A', B', C' be the reflections of the points A, B, C in the line OP. Then, $AA' \perp OP$ and $BB' \perp OP$. On the other hand, $g \perp OP$. Thus, the lines AA', BB' and g are parallel to each other; therefore, they concur at one point - namely, at an infinite point. In other words: The (infinite) point of intersection $AA' \cap BB'$ lies on the line g.

The reflection in the line OP maps the circumcircle of triangle ABC to itself (since the line OP passes through the center O of this circumcircle). Since the points A, B, C lie on the circumcircle of triangle ABC, it thus follows that their reflections in the line OP, i. e. the points A', B', C', also lie on the circumcircle of triangle ABC.

Now, let S be the point of intersection of the line $A'X$ with the circumcircle of triangle ABC different from A'. Then, the hexagon $A'SB'BCA$ has a circumcircle (namely, the circumcircle of triangle ABC). Therefore, by the Pascal theorem, the points $A'S \cap BC$, $SB' \cap CA$ and $B'B \cap AA'$ lie on one line. This line must coincide with the line g, since two points on this line (namely, the point $A'S \cap BC = X$ and the point $B'B \cap AA' = AA' \cap BB'$) lie on the line g. Hence, the point $SB' \cap CA$ lies on the line g. Thus, $SB' \cap CA = g \cap CA$. But $g \cap CA = Y$. Hence, $SB' \cap CA = Y$. This signifies that the point S lies on the line $B'Y$. Similarly, the point S lies on the line $C'Z$.

Altogether, the point S lies on the circumcircle of triangle ABC and on the lines $A'X$, $B'Y$, $C'Z$.

Let S' be the reflection of the point S in the line OP. Since the point S lies on the circumcircle of triangle ABC, its reflection S' in the line OP also lies on the circumcircle of triangle ABC (since the reflection in the line OP maps this circumcircle into itself).

Since A' is the reflection of A in the line OP, the point A is the reflection of A' in the line OP. On the other hand, X' is the reflection of X in the line OP. Thus, the line AX' is the reflection of the line $A'X$ in the line OP. Similarly, the lines BY' and CZ' are the reflections of the lines $B'Y$ and $C'Z$ in the line OP. Since the point S lies on the lines $A'X$, $B'Y$, $C'Z$, its reflection S' in the line OP must lie on the reflections of the lines $A'X$, $B'Y$, $C'Z$ in the line OP, i. e. on the lines AX', BY', CZ'.

Altogether, the point S' lies on the lines AX', BY', CZ' and on the circumcircle of triangle ABC. This proves Theorem 4.
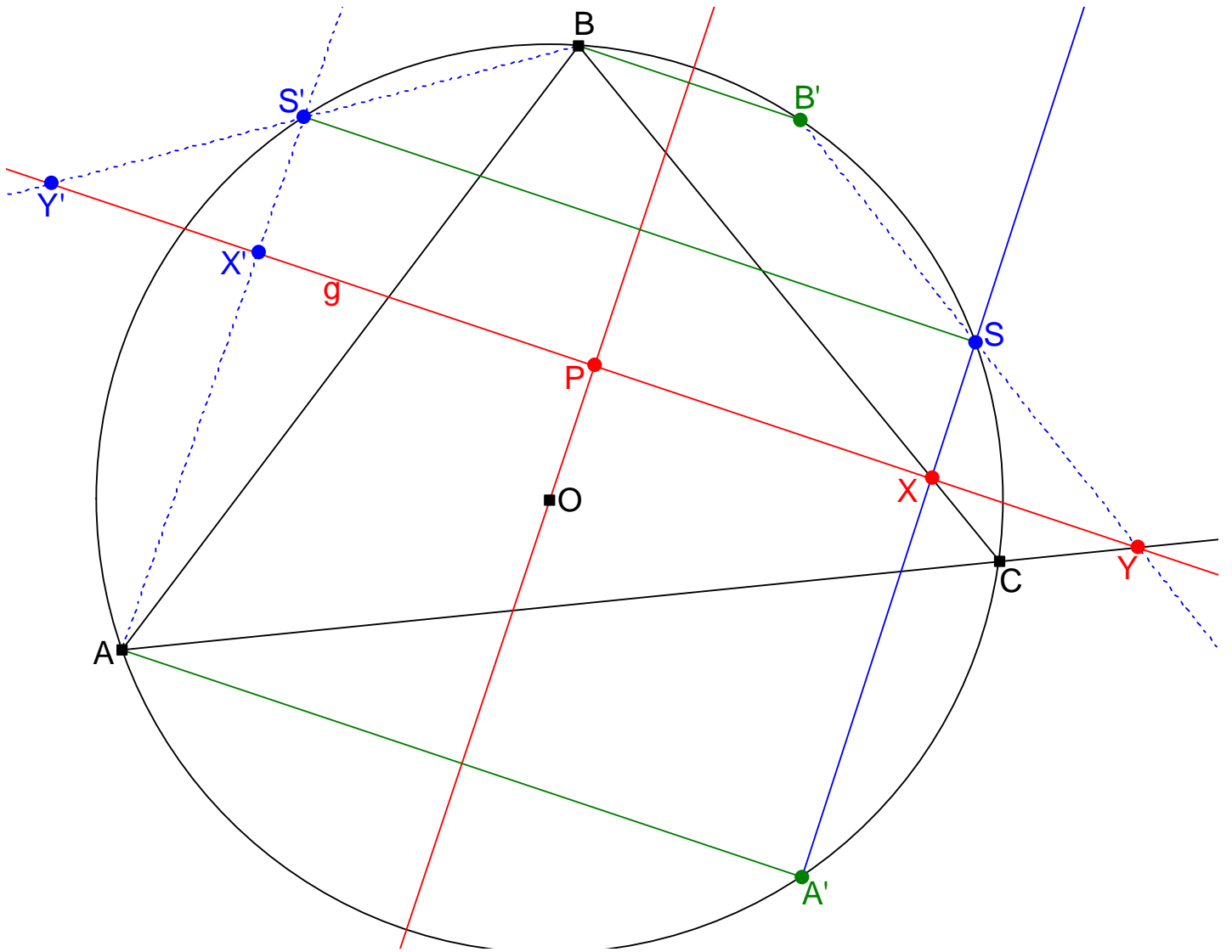
Fig. 6

Now we are going to derive Theorem 3 from Theorem 4:

(See Fig. 7.) First, we are going to show that Assertion 1 yields Assertion 2. In order to prove this, we assume Assertion 1 to hold, i. e. we assume that P is the midpoint of XZ. Then, Z is the reflection of X in the point P. Let V' and W' be the reflections of U and Y in the point P. Then, we have the following configuration:

A triangle ABC is given. The line g intersects the lines BC, CA, AB at the points Y, U, X. The point O is the circumcenter of triangle ABC (in fact, the circumcircle of triangle ABC is the circle k and thus has center O), and the point P is the orthogonal projection of the point O on the line g. Finally, W', V', Z are the reflections of the points Y, U, X in the point P.

Hence, Theorem 4 yields that the lines AW', BV', CZ concur at one point, and this point lies on the circumcircle of triangle ABC. We denote this point by D'. Thus, this point D' is the point of intersection of the line CZ with the circumcircle of triangle ABC different from C. But the point of intersection of the line CZ with the circumcircle of triangle ABC different from C is the point D (in fact, the circumcircle of triangle ABC is the circle k and intersects the line CZ at the points C and D). Hence, $D' = D$. As we know that D' lies on AW', we thus conclude that D lies on AW'. In other words,

W' lies on DA. On the other hand, W' lies on the line g (since W' is the reflection of Y in P, and both points Y and P lie on the line g). Hence, $W' = DA \cap g$. But $DA \cap g = W$. Thus, $W' = W$. Since we have introduced the point W' as the reflection of Y in the point P, it thus follows that W is the reflection of Y in the point P. Thus, P is the midpoint of YW. Hence, Assertion 2 holds.

Thus we have shown that Assertion 1 yields Assertion 2. Similarly we can show (or conclude from the already shown result by permutation of the points A, B, C, D) that Assertion 2 yields Assertion 3 and that Assertion 3 yields Assertion 1. Hence, the Assertions 1, 2 and 3 are pairwisely equivalent. This completes the proof of Theorem 3.
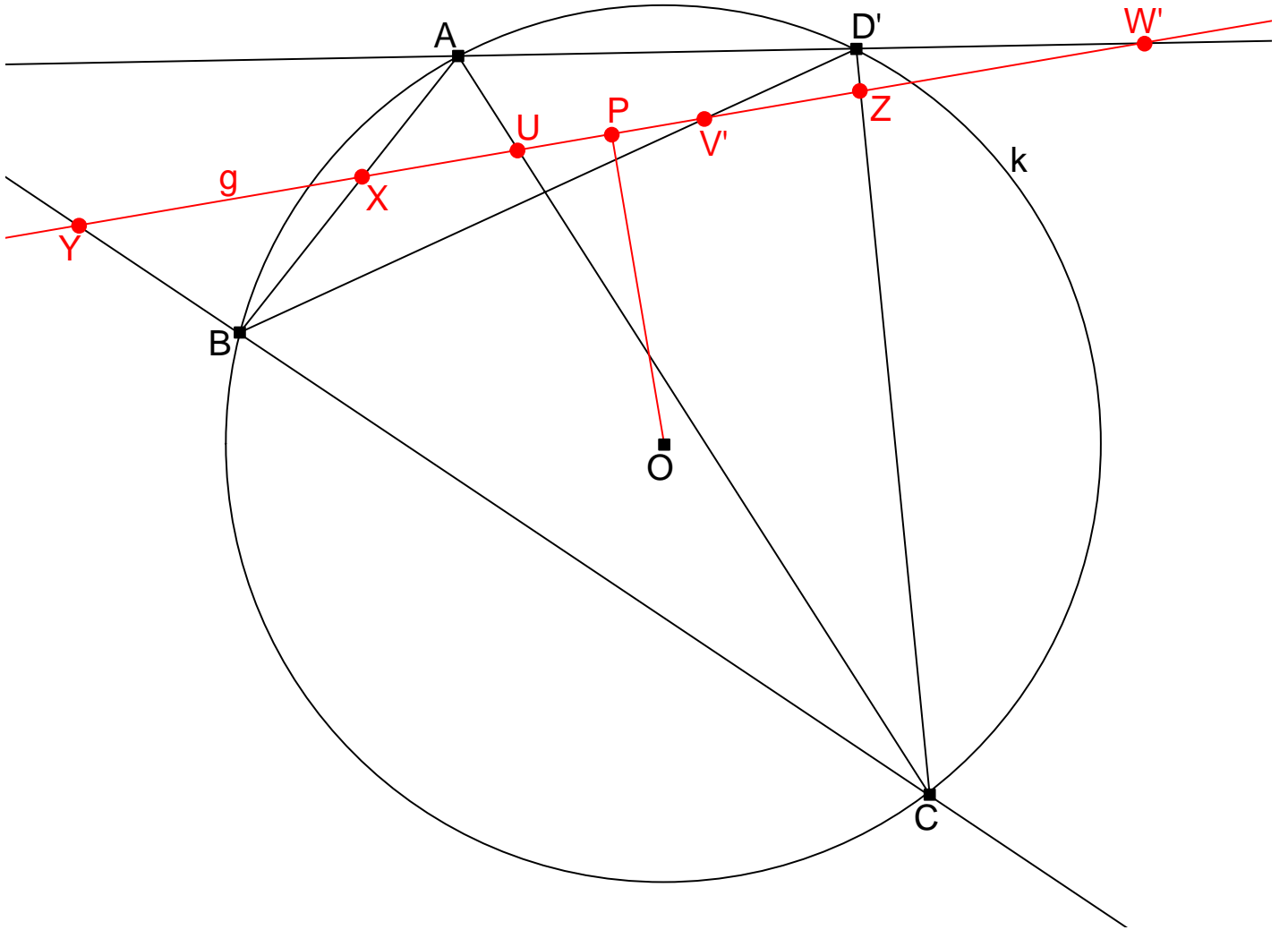

Fig. 7

Now, after Theorem 3 is verified, we can finally establish Theorem 1:

Consider the particular case of Theorem 3 when the point P happens to coincide with the point $AC \cap BD$. Then, $U = g \cap AC = P$ and $V = g \cap BD = P$. Hence, Assertion 3 of Theorem 3 is equivalent to the point P being the midpoint of the segment PP. Of course, this assertion is trivially valid. Since, according to Theorem 3, the Assertions 1, 2 and 3 are pairwisely equivalent, this entails that Assertion 1 of Theorem 3 is also

valid, i. e. the point P is the midpoint of the segment XZ. Thus we have proven the following: If the point P in Theorem 3 coincides with the point $AC \cap BD$, then the point P is the midpoint of the segment XZ. But this is exactly what Theorem 1 asserts. Hence, Theorem 1 is proven.

3. The second proof

(requirements: Ceva AND (Desargues OR (invariance of cross-ratio AND Menelaos)) AND (polarity with respect to circles OR inversion OR radical axes))

Second proof of Theorem 1. This second proof of Theorem 1 is more or less a variation of Proof 12 in [1] - the idea is exactly that of Proof 12, but the advanced concepts used will be reduced to a significantly lower amount, and a number of useful facts will be gathered on the way.

Our first lemma is an affine theorem which has been proposed independently as a problem in [3]:

Theorem 5. Let A, B, C, D be four points in the plane. Let $P = AC \cap BD$, $Q = AB \cap CD$ and $R = BC \cap DA$. The parallel to the line QR through the point P intersects the lines AB and CD at the points X and Z. Then, the point P is the midpoint of the segment XZ. (See Fig. 8.)
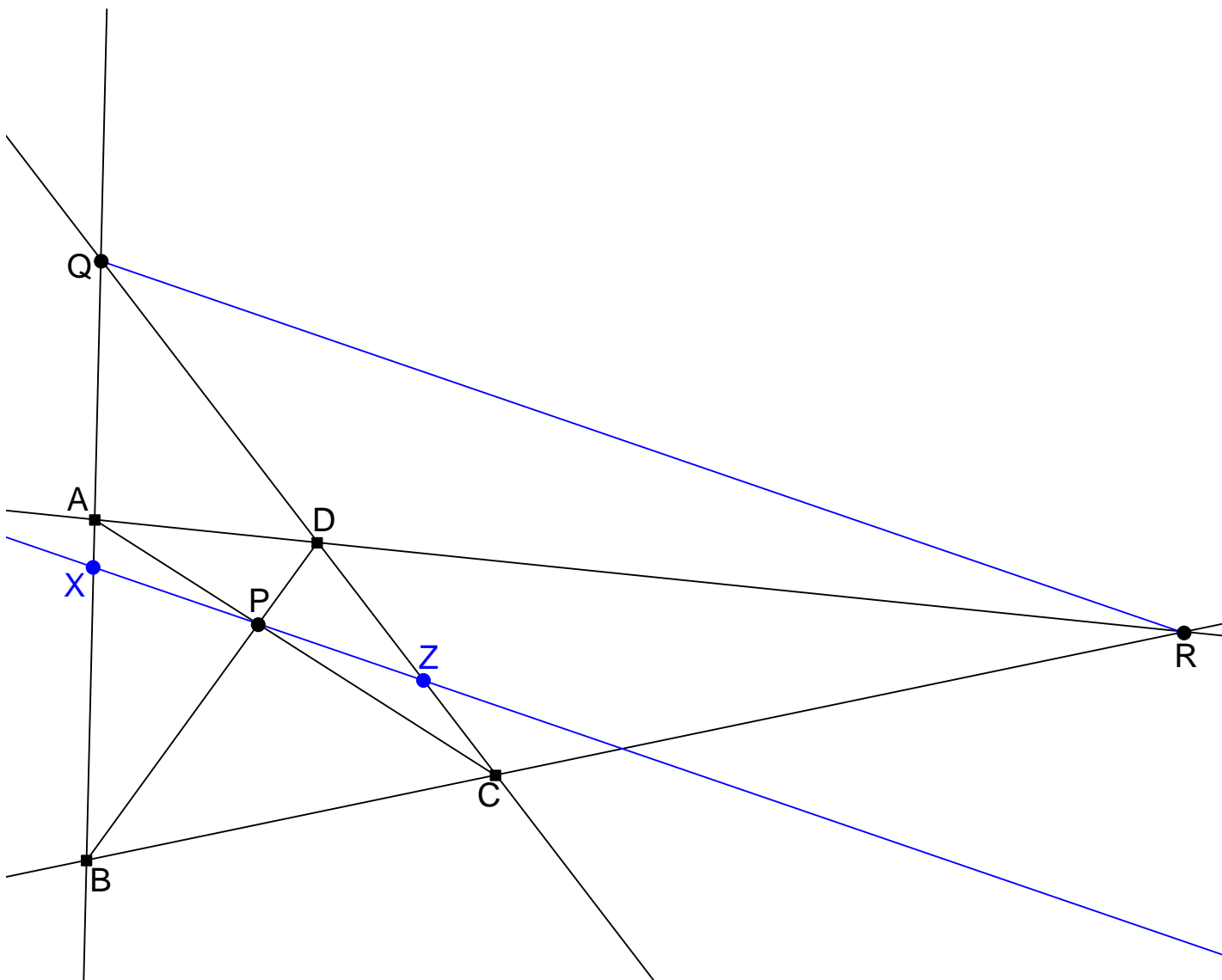
Fig. 8

We give two proofs of this fact - one using some basic projective geometry and one using just the Ceva and Desargues theorems.

First proof of Theorem 5. The following proof of Theorem 5 uses some projective geometry - namely the basic properties of infinite points and the invariance of the cross-ratio.

We work with directed segments. (See Fig. 9.) Let $Q' = QP \cap BC$. Since the lines QQ', BD, CA concur (in fact, they concur at the point P), the Ceva theorem (applied to triangle QBC and the points Q', D, A on its sides BC, CQ, QB) yields

$$\frac{BQ'}{Q'C} \cdot \frac{CD}{DQ} \cdot \frac{QA}{AB} = 1.$$

Since the points R, D, A are collinear, the Menelaos theorem (applied to triangle QBC and the points R, D, A on its sides BC, CQ, QB) yields

$$\frac{BR}{RC} \cdot \frac{CD}{DQ} \cdot \frac{QA}{AB} = -1.$$

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Comparison of these two equations yields $\dfrac{BR}{RC} = -\dfrac{BQ'}{Q'C}$, so that $\dfrac{BR}{RC} : \dfrac{BQ'}{Q'C} = -1$.

Since $XZ \parallel QR$, the lines XZ and QR intersect at an infinite point. Denote this point by T. Then, the four points B, C, R, Q' lie on the line BC, and the points X, Z, T, P are the projections of these four points from the point Q onto the line XZ. Thus, by the invariance of the cross-ratio under central projection, we have $\dfrac{XT}{TZ} : \dfrac{XP}{PZ} = \dfrac{BR}{RC} : \dfrac{BQ'}{Q'C}$. Since $\dfrac{BR}{RC} : \dfrac{BQ'}{Q'C} = -1$, this becomes $\dfrac{XT}{TZ} : \dfrac{XP}{PZ} = -1$, so that $\dfrac{XT}{TZ} = -\dfrac{XP}{PZ}$. But since T is the infinite point of the line XZ, we have $\dfrac{XT}{TZ} = -1$. Thus, $-1 = -\dfrac{XP}{PZ}$, so that $1 = \dfrac{XP}{PZ}$. Hence, $XP = PZ$, so that the point P is the midpoint of XZ. This proves Theorem 5.
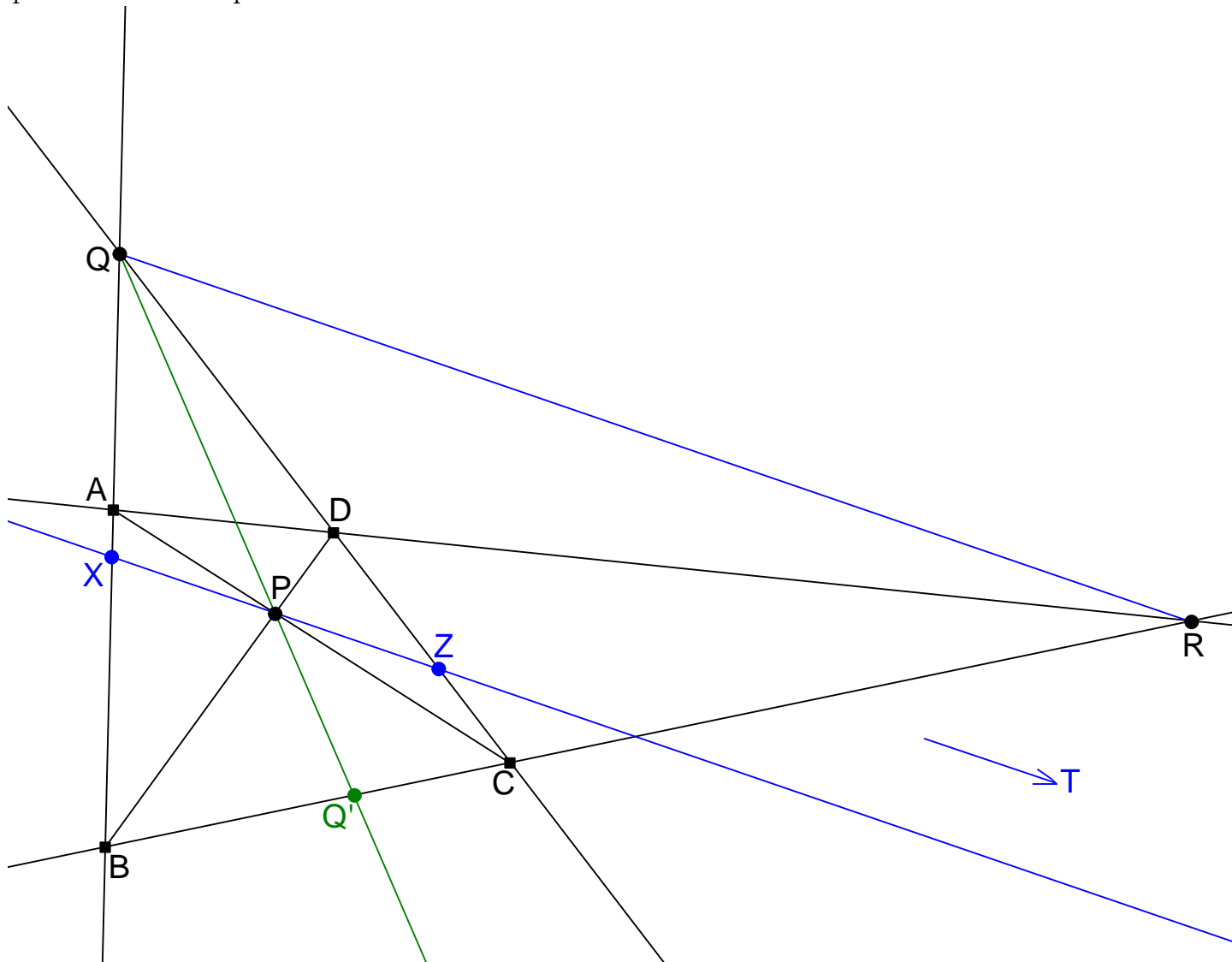

Fig. 9

Second proof of Theorem 5. We will use directed segments. Hereby, the parallel lines XZ and QR are assumed to be directed equally. (See Fig. 10.) Let $B' = BD \cap QR$ and $C' = AC \cap QR$. Since $XZ \parallel QR$, Thales yields $\dfrac{XP}{QB'} = \dfrac{BP}{BB'}$ and $\dfrac{C'Q}{PZ} = \dfrac{CC'}{CP}$, and

therefore

$$\frac{XP}{PZ} = \frac{XP}{QB'} \cdot \frac{QB'}{C'Q} \cdot \frac{C'Q}{PZ} = \frac{BP}{BB'} \cdot \frac{QB'}{C'Q} \cdot \frac{CC'}{CP} = \left(-\frac{PB}{BB'}\right) \cdot \frac{B'Q}{QC'} \cdot \left(-\frac{C'C}{CP}\right)$$
$$= \frac{PB}{BB'} \cdot \frac{B'Q}{QC'} \cdot \frac{C'C}{CP}.$$

Now, the points $B'C' \cap CB = R$, $C'P \cap BQ = A$ and $PB' \cap QC = D$ are collinear. According to the Desargues theorem (applied to the triangles $PB'C'$ and QCB), this yields that the lines PQ, $B'C$ and $C'B$ concur. Hence, according to the Ceva theorem (applied to the triangle $PB'C'$ and the points Q, C, B on its sidelines $B'C'$, $C'P$, PB'), we have $\frac{PB}{BB'} \cdot \frac{B'Q}{QC'} \cdot \frac{C'C}{CP} = 1$. Thus, $\frac{XP}{PZ} = \frac{PB}{BB'} \cdot \frac{B'Q}{QC'} \cdot \frac{C'C}{CP} = 1$, so that $XP = PZ$. Consequently, P is the midpoint of XZ. This proves Theorem 5.
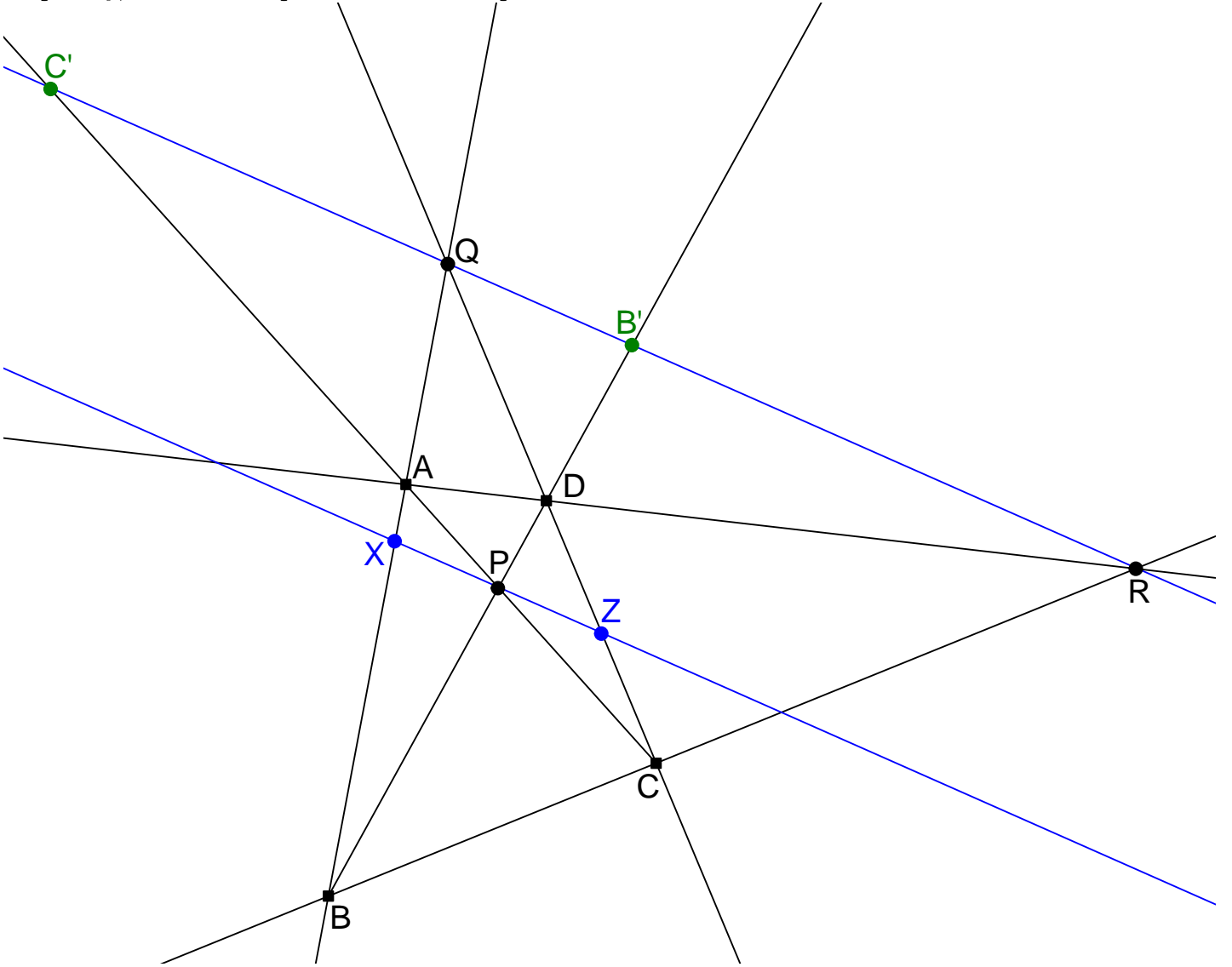

Fig. 10

Our second lemma is an important and known fact from the geometry of cyclic quadrilaterals. In an equivalent version, it has been discussed, e. g., at [4]. We will use it in the following form:

Theorem 6. Let k be a circle with center O, and let A, B, C, D be four points on this circle k. Let $P = AC \cap BD$, $Q = AB \cap CD$ and $R = BC \cap DA$. Then, the point O is the orthocenter of triangle PQR. (See Fig. 11.)
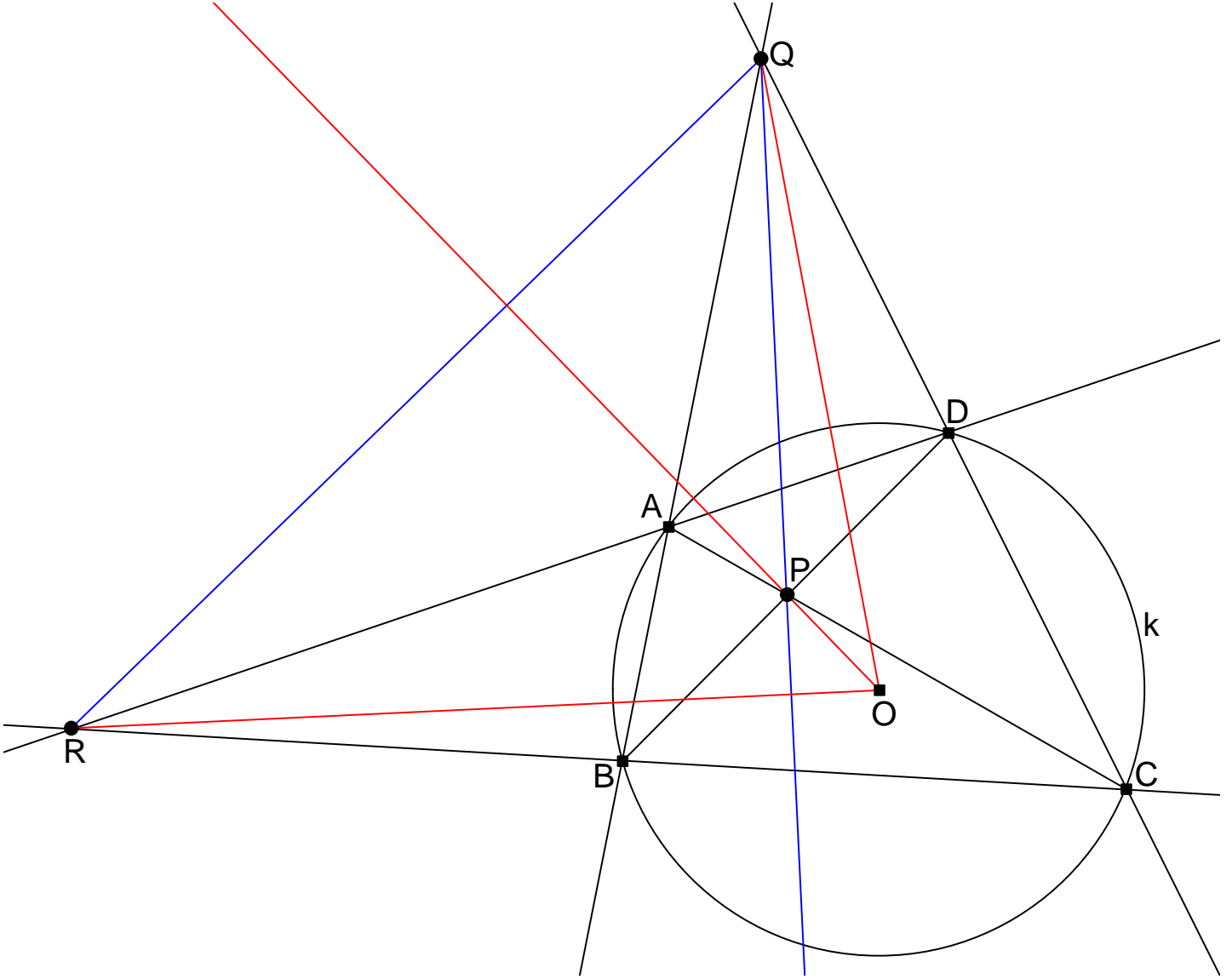

Fig. 11

We will give two proofs of Theorem 6. The first one deduces it from a result in the theory of poles and polars with respect to a circle, while the second one uses no theory beyound radical axes *or* inversion - either of these is enough! - and directed angles modulo 180°.

First proof of Theorem 6. This proof relies on a fact from the theory of poles and polars with respect to a circle:

Theorem 7. Let X, Y, Z, W be four points on a circle k_1. Then, the point $XY \cap ZW$ lies on the polar of the point $XZ \cap YW$ with respect to the circle k_1.

This fact appears as Theorem 1 in [5], where it is proven using the Pascal theorem. Below we will give a different proof of this fact based on our second proof of Theorem 6.

Assuming Theorem 7 as given, Theorem 6 is easy to verify:

Applying Theorem 7 to the points $X = A$, $Y = B$, $Z = C$, $W = D$ on the circle

$k_1 = k$, we see that the point $AB \cap CD$ lies on the polar of the point $AC \cap BD$ with respect to the circle k. Since $AC \cap BD = P$ and $AB \cap CD = Q$, this means that Q lies on the polar of P with respect to k. Applying Theorem 7 to the points $X = B$, $Y = C$, $Z = D$, $W = A$ on the circle $k_1 = k$, we obtain that the point $BC \cap DA$ lies on the polar of the point $BD \cap CA$ with respect to the circle k. Since $BC \cap DA = R$ and $BD \cap CA = P$, this means that R lies on the polar of P with respect to k.

Since the two points Q and R both lie on the polar of P with respect to k, this yields that the line QR is the polar of P with respect to k. Now, since O is the center of k, and since the polar of a point with respect to a circle is always perpendicular to the line joining this point with center of the circle, we thus obtain $OP \perp QR$. Similarly, $OQ \perp RP$ and $OR \perp PQ$. Thus, the lines OP, OQ, OR are the altitudes of triangle PQR. Hence, the orthocenter of triangle PQR is the point of intersection of these lines OP, OQ, OR, so it must be the point O. This proves Theorem 6.

Second proof of Theorem 6. In the following proof, we are going to use directed angles modulo 180°.

We commence with a classical result about four lines in a plane:

Theorem 8, the Miquel fourline theorem. Let a, b, c, d be four lines in the plane. Denote $A = b \cap c$, $B = c \cap a$, $C = a \cap b$, $D = a \cap d$, $E = b \cap d$, $F = c \cap d$. Then, the circumcircles of triangles EAF, BDF, EDC, BAC have a common point.

This point is called the *Miquel point* of the four lines a, b, c, d. (See Fig. 12, where the Miquel point is marked red.)
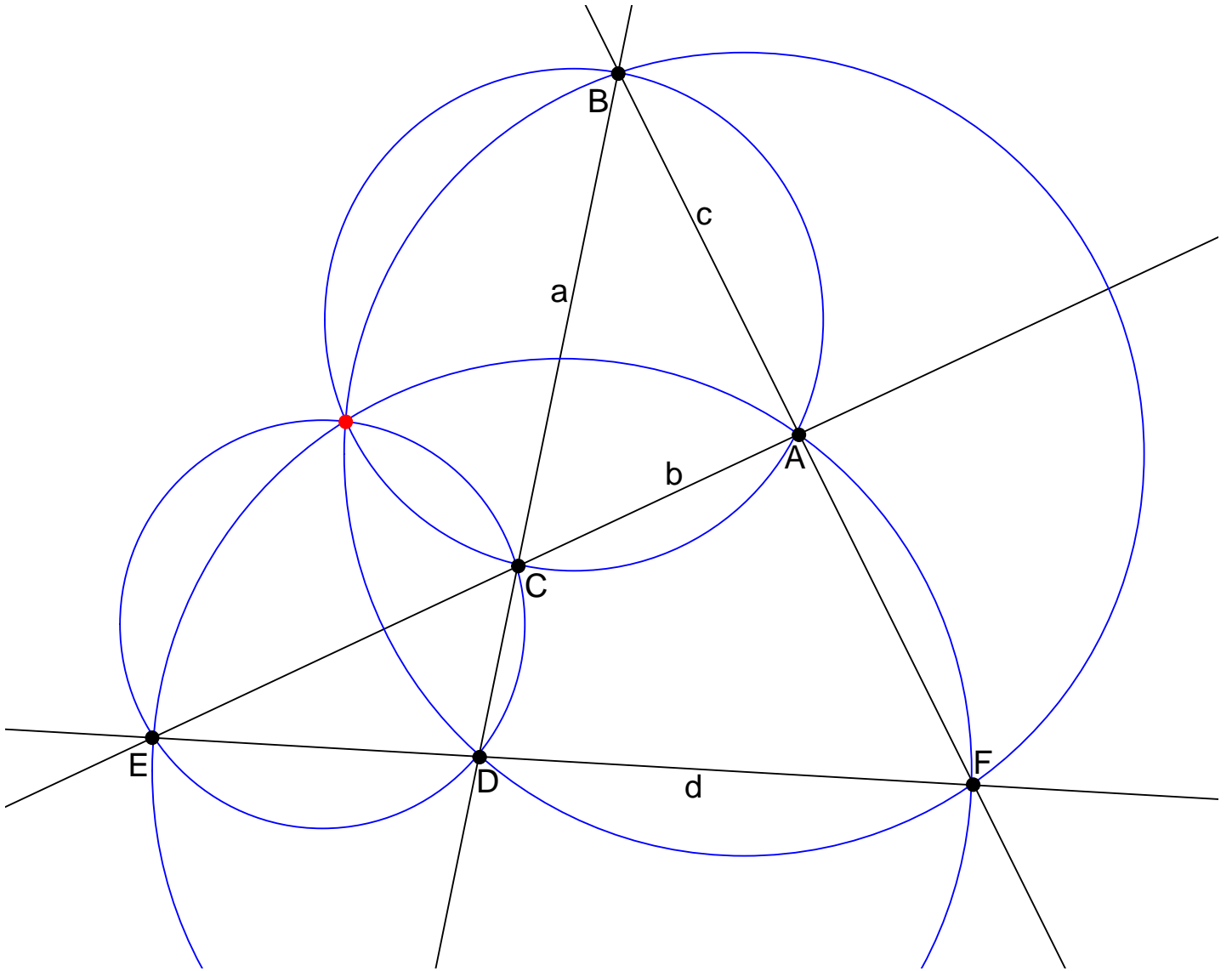
Fig. 12

Proof of Theorem 8. (See Fig. 13.) Let M be the point of intersection of the circumcircles of triangles EAF and EDC distinct from E. Then, $\measuredangle MCD = \measuredangle MED$ (since M lies on the circumcircle of triangle EDC) and $\measuredangle MEF = \measuredangle MAF$ (since M lies on the circumcircle of triangle EAF). Hence, $\measuredangle MCB = \measuredangle MCD = \measuredangle MED = \measuredangle MEF = \measuredangle MAF = \measuredangle MAB$. Thus, the points M, B, C, A lie on one circle. Equivalently, M lies on the circumcircle of triangle BAC. Similarly, M lies on the circumcircle of triangle BDF. Thus, the circumcircles of triangles EAF, BDF, EDC, BAC have a common point - namely, the point M. This proves Theorem 8.
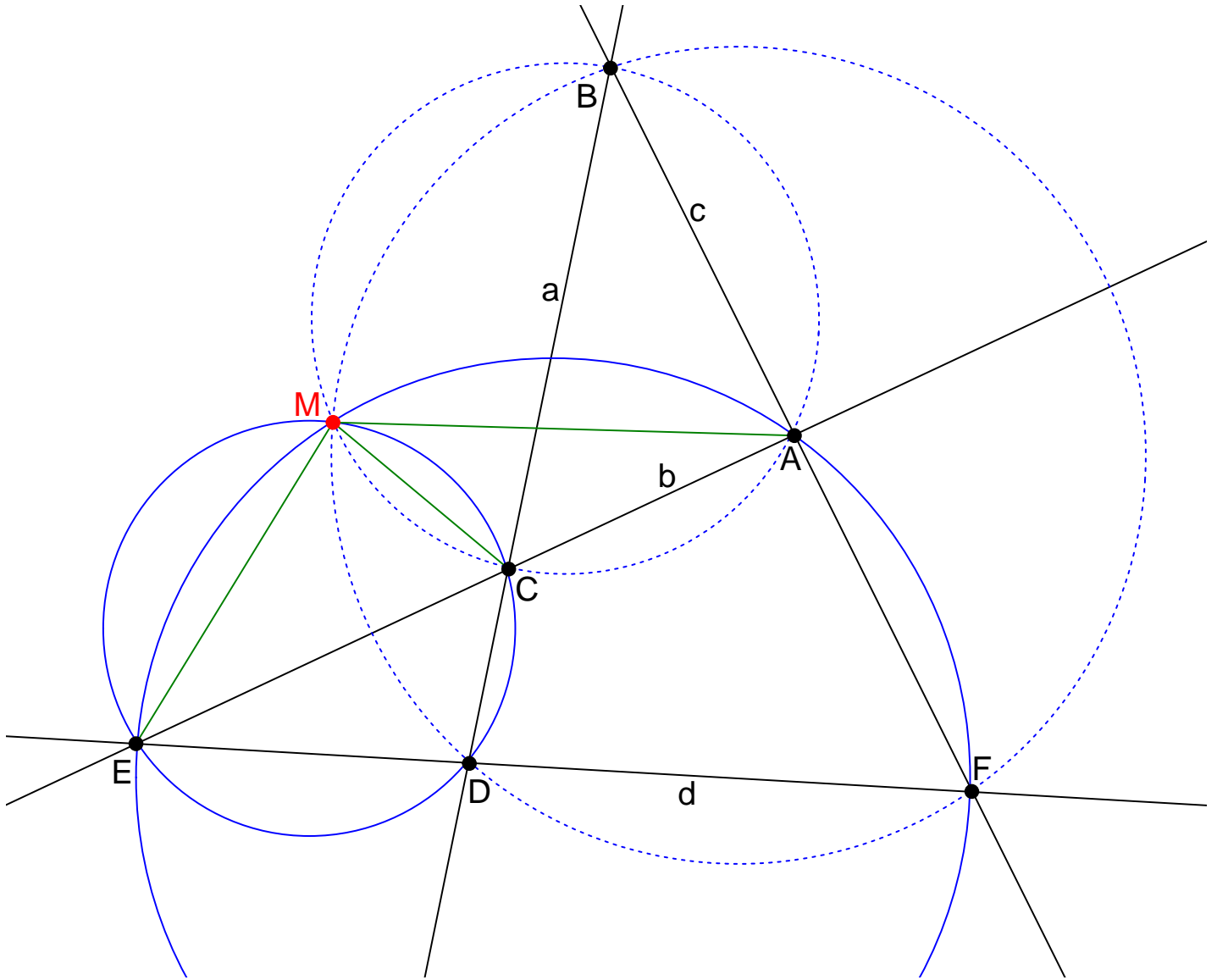
Fig. 13

Throughout mathematics it can be observed that proving more is usually easier than proving less. According to this, and also for the sake of completeness, with the next theorem we are going to prove a whole catalogue of properties of a configuration, despite the fact that the only one that we will need in our proof of Theorem 6 is Theorem 9 **h)**. [Thus, the reader not familiar with inversion can skip Theorem 9 **f)**, and the reader not familiar with polarity can skip **i)**.] Note that the other properties are of interest, too: as an exercise, the reader can kill three olympiad problems - [6], [7], [8] - using Theorem 9. Theorem 9 **d)** has also been discussed in [9], while Theorem 9 **b)**, **d)** and **f)** yield the result of [10].

Fig. 14

Theorem 9. Let k be a circle with center O, and let A, B, C, D be four points on this circle k. Let M be the Miquel point of the four lines AB, BC, CD, DA. Let $P = AC \cap BD$, $Q = AB \cap CD$ and $R = BC \cap DA$.
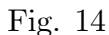
a) The point M lies on the circumcircles of triangles RCD, QAD, RAB, QCB.

b) The point M lies on the line QR. (See Fig. 14.)

c) The point M lies on the circumcircles of triangles AOC and BOD. (See Fig. 15.)

d) The point M lies on the line OP.

e) The line OP bisects the angle AMC and bisects the angle BMD.

f) The point M is the image of the point P under the inversion with respect to the circle k.

g) The point M is the orthogonal projection of the point P on the line QR. (See Fig. 16.)

h) We have $OP \perp QR$.

i) The line QR is the polar of the point P with respect to the circle k.
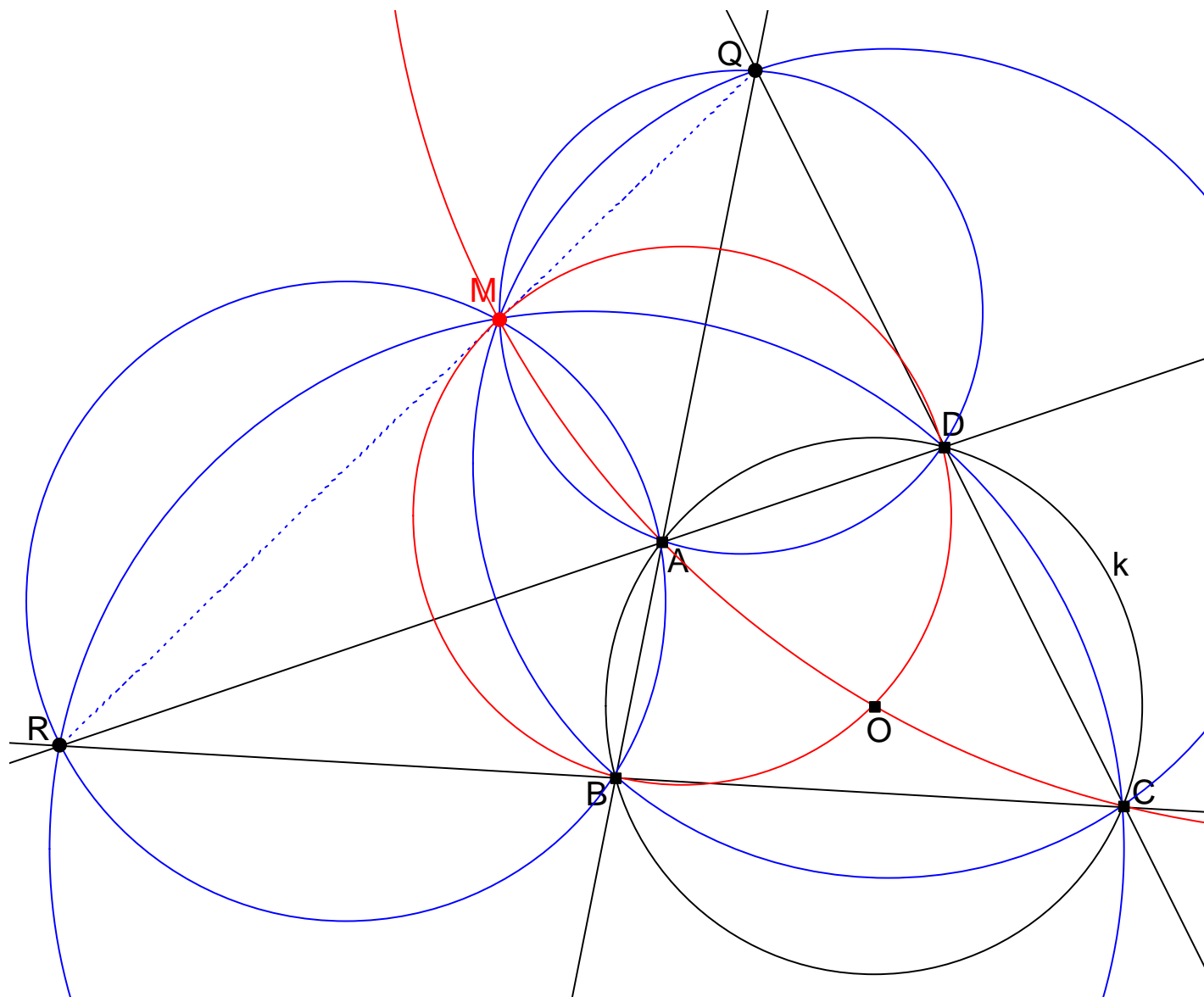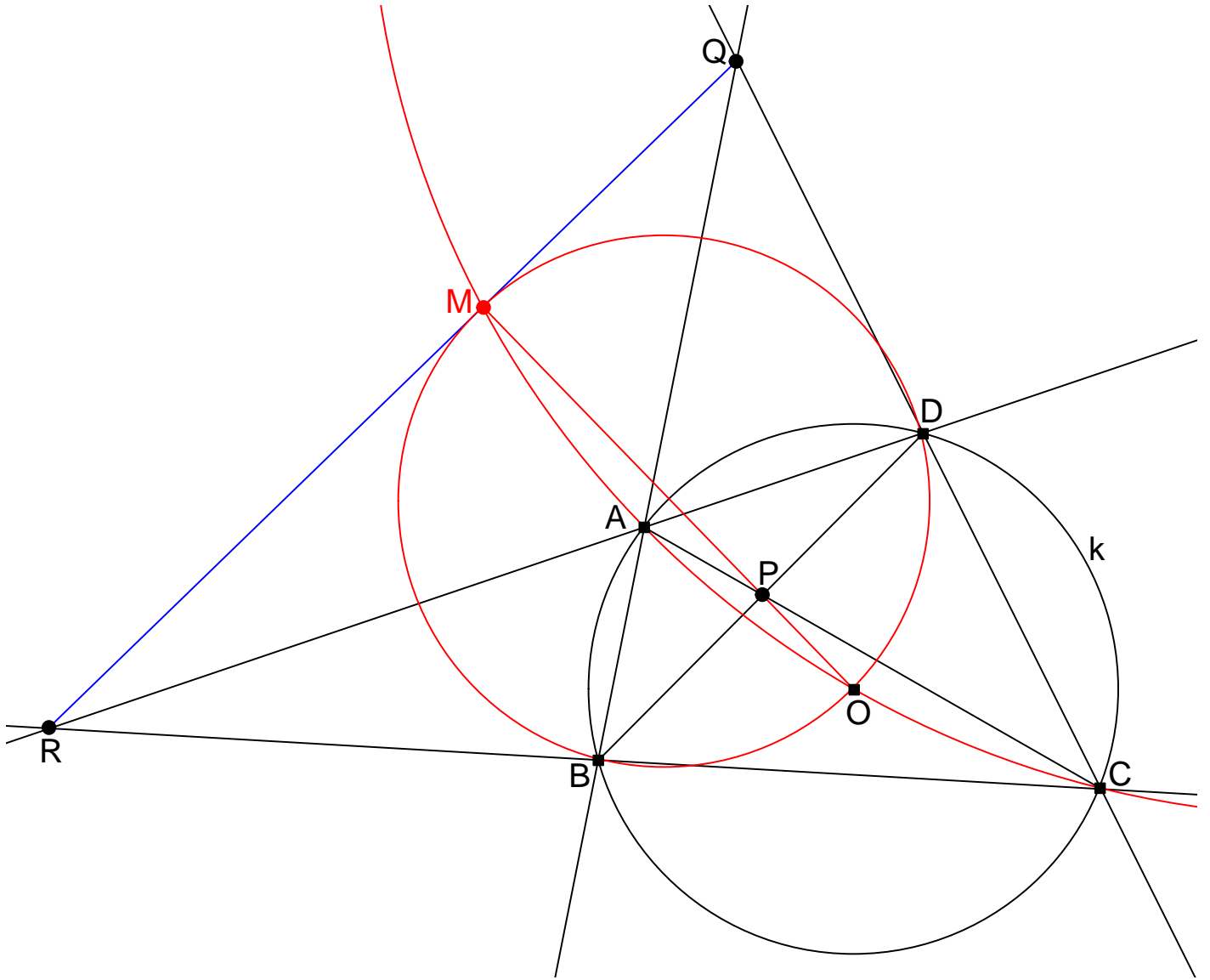
Fig. 15

Fig. 16

Proof of Theorem 9. We have $BC \cap CD = C$, $CD \cap AB = Q$, $AB \cap BC = B$, $AB \cap DA = A$, $BC \cap DA = R$, $CD \cap DA = D$. Thus, due to its definition, the Miquel point M of the four lines AB, BC, CD, DA is the common point of the circumcircles of triangles RCD, QAD, RAB, QCB. This proves Theorem 9 **a)**.

(See Fig. 17.) Since M lies on the circumcircle of triangle RAB, we have $\angle RMB = \angle RAB$. Since the points A, B, C, D lie on one circle (namely, on the circle k), we have $\angle DAB = \angle DCB$. Since M lies on the circumcircle of triangle QCB, we have $\angle QCB = \angle QMB$. Hence, $\angle RMB = \angle RAB = \angle DAB = \angle DCB = \angle QCB = \angle QMB$. Thus, the points M, Q, R are collinear, i. e. the point M lies on the line QR. This proves Theorem 9 **b)**.

Since O is the center of the circle k, while the points B, C, D lie on this circle, the central angle theorem yields $\angle BOD = 2 \cdot \angle BCD$. Now we know that $\angle RMB = \angle RAB$, and similarly we can get $\angle RMD = \angle RCD$. With the aid of the relation

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$\sphericalangle DAB = \sphericalangle DCB$ shown above, we thus get

$$\sphericalangle BMD = \sphericalangle RMD - \sphericalangle RMB = \sphericalangle RCD - \sphericalangle RAB = \sphericalangle BCD - \sphericalangle DAB = \sphericalangle BCD - \sphericalangle DCB$$
$$= \sphericalangle BCD - (-\sphericalangle BCD) = 2 \cdot \sphericalangle BCD = \sphericalangle BOD.$$

Thus, the points B, D, M, O lie on one circle, i. e. the point M lies on the circumcircle of triangle BOD. Similarly, M also lies on the circumcircle of triangle AOC. This proves Theorem 9 **c)**.
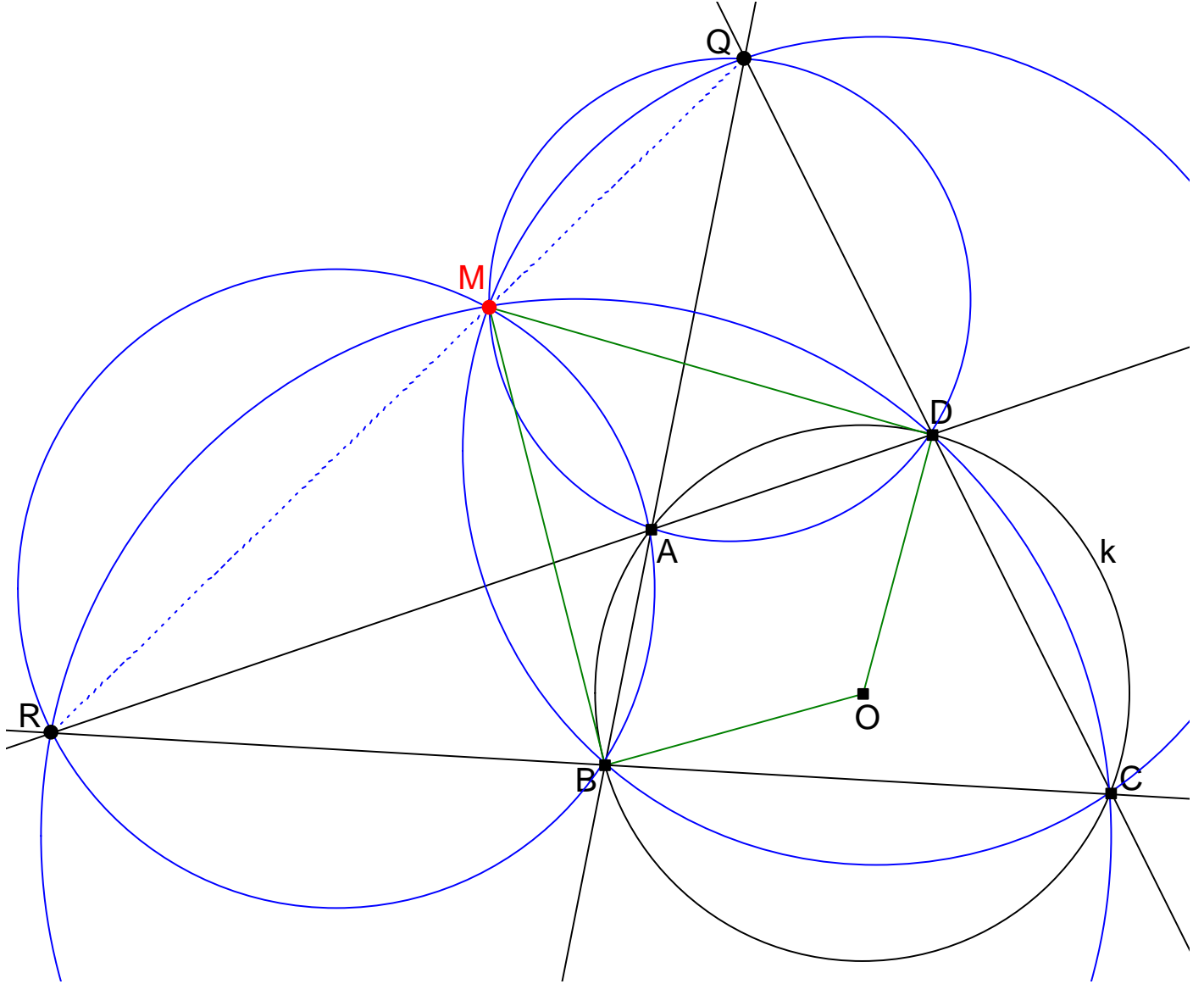
Fig. 17

Now we are going to prove Theorem 9 **d)** and **f)** in two different ways - a first one using inversion, and a second one using radical axes:

*First proof of Theorem 9 **d)** and **f)**.* (See Fig. 16.) The inversion with respect to the circle k maps the points A and C to themselves (since these points lie on k). On the other hand, the inversion with respect to the circle k maps circles through the point O to lines (since O is the center of the circle k, hence the center of our inversion, and any inversion maps circles through the center of inversion to lines). Hence, the image of the circumcircle of triangle AOC under this inversion is a line (since the circumcircle of

triangle AOC is a circle through the point O), and this line passes through the points A and C (in fact, this line is the image of the circumcircle of triangle AOC under our inversion, and thus must pass through the images of the points A and C under this inversion - but these images are these points A and C themselves). Thus, the image of the circumcircle of triangle AOC under our inversion is the line AC. Similarly, the image of the circumcircle of triangle BOD under our inversion is the line BD.

Now, since the point M lies on the circumcircles of triangles AOC and BOD, the image of this point M under the inversion with respect to the circle k must lie on the images of these circumcircles, thus on the lines AC and BD. Hence, the image of the point M under the inversion with respect to k is the point of intersection of the lines AC and BD. But the point of intersection of the lines AC and BD is the point P. Hence, the image of the point M under the inversion with respect to k is the point P. Thus, in turn, the point M is the image of the point P under the inversion with respect to k. Theorem 9 **f)** is hence proven.[3]

The image of a point under the inversion with respect to a circle always lies on the line through the center of the circle and the point. Since M is the image of the point P under the inversion with respect to the circle k, we can thus conclude that the point M lies on the line OP (since O is the center of the circle k). This proves Theorem 9 **d)**.

Second proof of Theorem 9 **d)** *and* **f)**. If two circles intersect, then the line joining the two points of intersection is the radical axis of the two circles. This yields that:

- The line OM is the radical axis of the circumcircles of triangles AOC and BOD (since the two points of intersection of these circles are O and M).

- The line AC is the radical axis of the circle k and the circumcircle of triangle AOC (since the two points of intersection of these circles are A and C).

- The line BD is the radical axis of the circle k and the circumcircle of triangle BOD (since the two points of intersection of these circles are B and D).

Now, the pairwise radical axes of three circles always concur. Applied to the circle k and the circumcircles of triangles AOC and BOD, this yields that the lines OM, AC and BD concur. In other words, the line OM passes through the point $AC \cap BD$. Since $AC \cap BD = P$, this is equivalent to saying that the line OM passes through P. In other words, M lies on the line OP. Hence, Theorem 9 **d)** is proven.

(See Fig. 18.) Since O is the center of the circle k, while the points B and D lie on this circle, we have $OB = OD$. Thus, triangle BOD is isosceles, so that $\angle OBD = \angle BDO$. Since M lies on the circumcircle of triangle BOD, we have $\angle OBD = \angle OMD$ and $\angle BDO = \angle BMO$. Hence, $\angle OBD = \angle BDO$ becomes $\angle OMD = \angle BMO$. This equation shows that the line OP bisects the angle BMD. Similarly, the line OP bisects the angle AMC. Thus we have shown Theorem 9 **e)**.

From $\angle OBD = \angle BDO$ and $\angle BDO = \angle BMO$ we can conclude that $\angle OBD = \angle BMO$, so that $\angle OBP = -\angle OMB$. Further, obviously $\angle BOP = -\angle MOB$. Thus,

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[3]This proof was not particularly watertight. In fact, as we are working in the inversive plane, the lines AC and BD have not just one, but two points of intersection: the usual point of intersection P and the infinite point of the inversive plane. But it is readily seen that the image of the point M under the inversion with respect to k is the "right" point of intersection, i. e. the point P.

the triangles OBP and OMB are oppositely similar. This entails $OP : OB = OB : OM$, or $OP \cdot OM = OB^2$. A more accurate reasoning shows that this equation $OP \cdot OM = OB^2$ holds even if the segments are considered directed. Since O is the center and OB is the radius of the circle k, and since the point M lies on the line OP, this equation yields that the point M is the image of the point P under the inversion with respect to the circle k. Thus, Theorem 9 **f)** is established.

Fig. 18

Now, we have shown Theorem 9 **a)**, **b)** and **c)**, then proved Theorem 9 **d)** and **f)** in two ways, additionally showing Theorem 9 **e)** in the second proof of Theorem 9 **d)** and **f)**. What remains now is to verify Theorem 9 **g)**, **h)** and **i)**:

Since M lies on the circumcircle of triangle RAB, we have $\angle RMB = \angle RAB$. Since M lies on the circumcircle of triangle BOD, we have $\angle BMO = \angle BDO$. Since O is the center of the circle k, while the points B, D, A lie on this circle k, the central angle theorem yields $\angle BDO = 90° - \angle DAB$. Thus,

$$\angle RMO = \angle RMB + \angle BMO = \angle RAB + \angle BDO = \angle DAB + (90° - \angle DAB) = 90°,$$

hence $OM \perp QR$. Since the point M lies on OP, this becomes $OP \perp QR$. Thus,

Theorem 9 **h)** is proven. The relation $OP \perp QR$, together with the fact that the point M lies on the lines OP and QR, yields that the point M is the orthogonal projection of the point P on the line QR. Hence, Theorem 9 **g)** is proven as well.

The polar of the point P with respect to the circle k is defined as the perpendicular to the line OP through the image of the point P under the inversion with respect to the circle k (since O is the center of k). Now, the image of the point P under the inversion with respect to the circle k is the point M. Hence, the polar of the point P with respect to the circle k is the perpendicular to the line OP through M. This perpendicular is obviously the line QR (since the line QR passes through M and is perpendicular to OP). Hence, the polar of the point P with respect to the circle k is the line QR. Hence, Theorem 9 **i)** is proven, what concludes our proof of Theorem 9.

Now, Theorem 9 swiftly implies Theorem 6: Applying Theorem 9 **h)** directly to the points A, B, C, D on the circle k, we get $OP \perp QR$. But applying Theorem 9 **h)** to the points C, B, D, A (in this order) on the circle k, we get $OQ \perp RP$, and applying Theorem 9 **h)** to the points B, D, C, A (in this order) on the circle k, we obtain $OR \perp PQ$.

Since $OP \perp QR$, $OQ \perp RP$ and $OR \perp PQ$, the lines OP, OQ, OR must be the altitudes of the triangle PQR. Thus, the point O, being the point of intersection of these lines OP, OQ, OR, must be the point of intersection of the altitudes of triangle PQR, i. e. the orthocenter of triangle PQR. Thus, Theorem 6 is proven.

As we promised, we can also immediately conclude Theorem 7 from Theorem 9: In the configuration of Theorem 9, according to Theorem 9 **i)**, the line QR is the polar of the point P with respect to the circle k. Now, the point Q lies on the line QR; hence, the point Q lies on the polar of the point P with respect to the circle k. In other words: The point $AB \cap CD$ lies on the polar of the point $AC \cap BD$ with respect to the circle k. Renaming the points A, B, C, D into X, Y, Z, W and the circle k into k_1 in this assertion, we get Theorem 7.

After Theorem 6 has been proved in two ways, we finally deduce Theorem 1 from Theorems 5 and 6:

(See Fig. 19.) Consider the configuration of Theorem 1. Assume that $O \neq P$ (in fact, in the case $O = P$, the lines AC and BD are diameters of the circle k, so that the quadrilateral $ABCD$ is symmetric with respect to O, and thus Theorem 1 becomes trivial from symmetry). Then, we can speak of the line OP. Define two points $Q = AB \cap CD$ and $R = BC \cap DA$. Then, according to Theorem 6, the point O is the orthocenter of triangle PQR. Hence, $OP \perp QR$. Together with $OP \perp g$, this yields $g \parallel QR$. Thus, the line g is the parallel to the line QR through the point P. Since this line g intersects the lines AB and CD at the points X and Z, Theorem 5 now yields that the point P is the midpoint of the segment XZ. Thus, Theorem 1 is proven.

Fig. 19

4. An application of the first proof
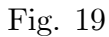
We have proven Theorem 1 in two different ways now. We conclude this note with an application of Theorem 3 noticed by Virgil Nicula in [12].

We start with a trivial particular case of Theorem 3:

Theorem 10. Let ABC be a triangle with the circumcenter O. Let g be an arbitrary line, and let P be the orthogonal projection of the point O on this line g. The line g intersects the lines BC, CA, AB at some points X, Y, Z. Let W be the point of intersection of the line g with the tangent to the circumcircle of triangle ABC at the point A. Then, the point P is the midpoint of the segment YZ if and only if the point P is the midpoint of the segment XW. (See Fig. 20.)
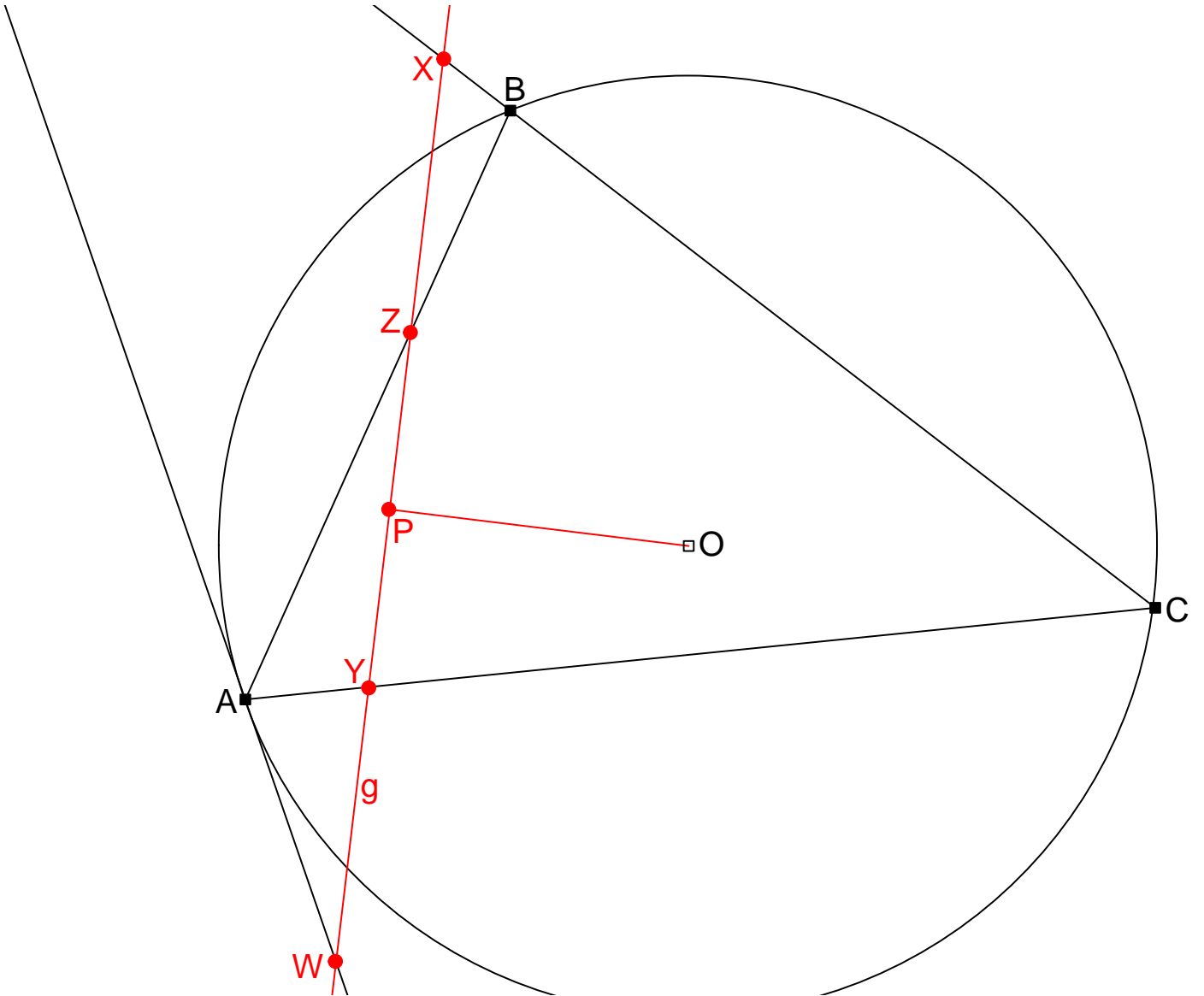
Fig. 20

Proof of Theorem 10. We consider our configuration from a slightly different viewpoint:

The circumcircle of triangle ABC has the center O, and A, C, B, A are four points on this circumcircle. The point P is the orthogonal projection of the point O on the line g. The line g intersects the lines AC, CB, BA, AA, AB, CA at the points Y, X, Z, W, Z, Y (hereby, the line AA is considered to mean the tangent to the circumcircle of triangle ABC at the point A).

Hence we can apply Theorem 3 to the circumcircle of triangle ABC (in the role of the circle k), the four points A, C, B, A on this circumcircle (in the role of the points A, B, C, D) and the line g (in the role of the line g), and we conclude that the following three assertions are pairwisely equivalent:

Assertion 1: The point P is the midpoint of the segment YZ.

Assertion 2: The point P is the midpoint of the segment XW.

Assertion 3: The point P is the midpoint of the segment ZY.

The equivalence of Assertions 1 and 2 is exactly the statement of Theorem 10. Thus, Theorem 10 is proven.

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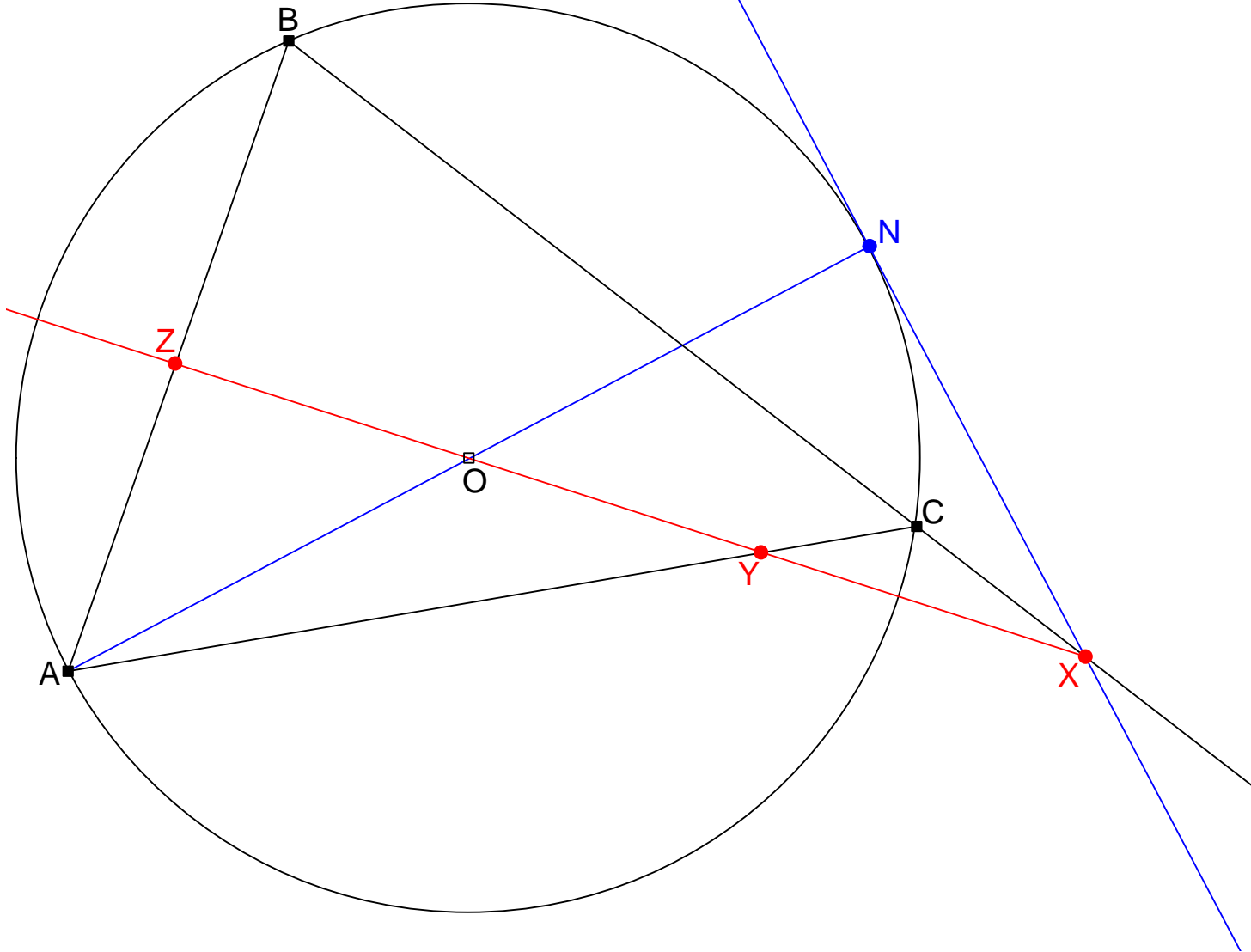An application of Theorem 10 is the following fact from triangle geometry:

Fig. 21

Theorem 11. Let O be the circumcenter of a triangle ABC. Let N be the reflection of the point A in the point O, or, equivalently, the point diametrically opposite to the point A on the circumcircle of triangle ABC. The tangent to the circumcircle of triangle ABC at the point N intersects the line BC at a point X. The line OX intersects the lines CA and AB at the points Y and Z. Then, the point O is the midpoint of the segment YZ. (See Fig. 21.)

This theorem has been discussed in [12] and [13] and allows for different approaches. It has been given in a slightly more complicated form as problem 6 in the selection round of the St. Petersburg Mathematical Olympiad 2002 (SPbMO) for the 9th grade. Here we show two proofs of Theorem 11 - one by Virgil Nicula using Theorem 3 and one being a slight variation of the proposed solution of the SPbMO problem.
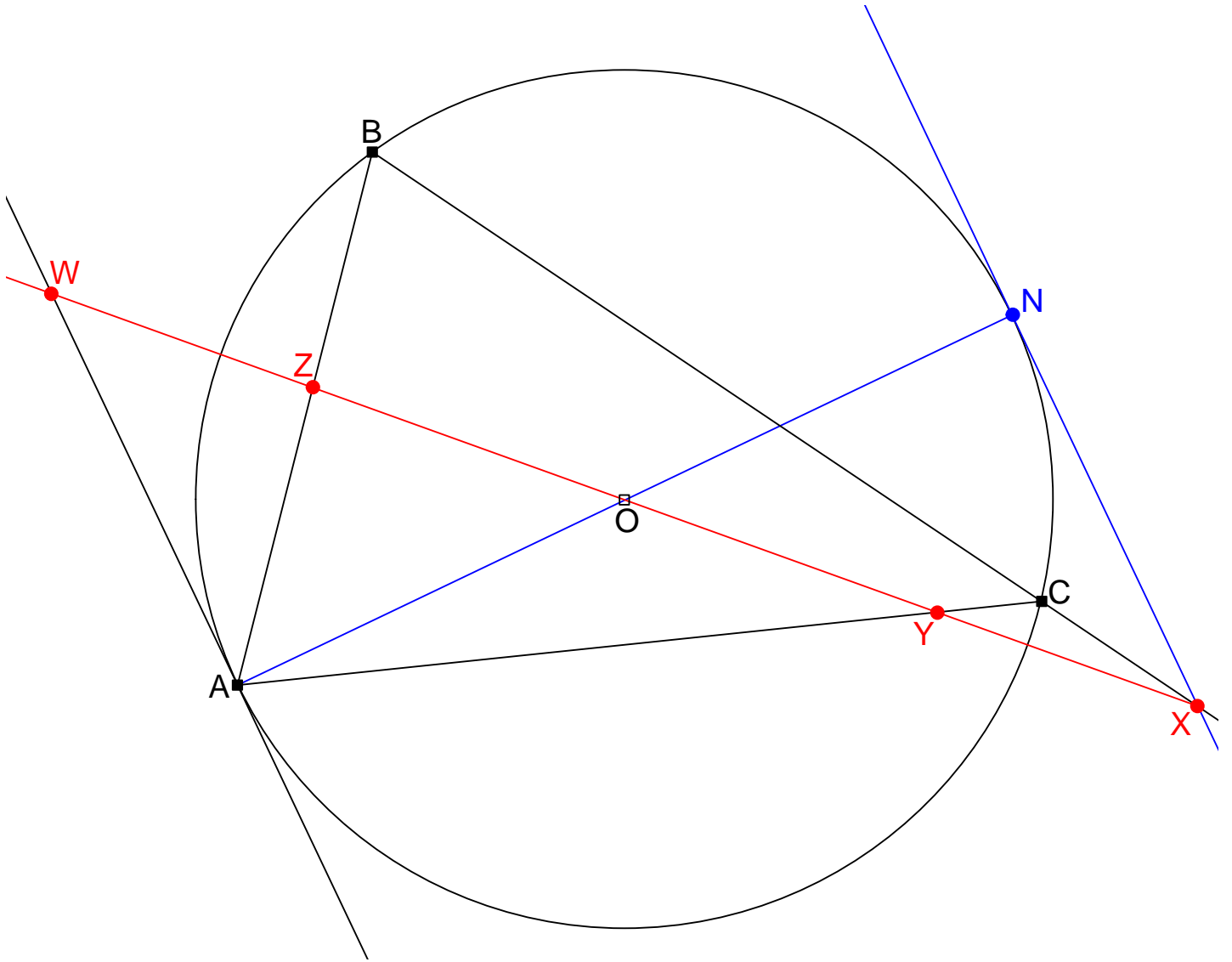
Fig. 22

First proof of Theorem 11 (by Virgil Nicula). (See Fig. 22.) Let W be the point of intersection of the line OX with the tangent to the circumcircle of triangle ABC at the point A. We consider our configuration as follows:

The triangle ABC has the circumcenter O. The point O is the orthogonal projection of the point O on the line OX (obviously, since it lies on this line). The line OX intersects the lines BC, CA, AB at the points X, Y, Z. The point W is the point of intersection of the line OX with the tangent to the circumcircle of triangle ABC at the point A.

Hence, according to Theorem 10, the point O is the midpoint of the segment YZ if and only if the point O is the midpoint of the segment XW. Hence, in order to prove Theorem 11 (which states that the point O is the midpoint of the segment YZ), it is enough to prove that O is the midpoint of the segment XW.

This is rather obvious: Since AW is the tangent to the circumcircle of triangle ABC at A, while O is the center of this circumcircle, we have $AW \perp AO$. Since NX is the tangent to the circumcircle of triangle ABC at N, while O is the center of this circumcircle, we have $NX \perp NO$. This rewrites as $NX \perp AO$. Together with

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$AW \perp AO$, this yields $NX \parallel AW$. Hence, after the Thales theorem, $\dfrac{XO}{OW} = \dfrac{NO}{OA}$. Now, $NO = OA$ (since N is the reflection of A in the point O), and thus $\dfrac{XO}{OW} = \dfrac{NO}{OA} = \dfrac{OA}{OA} = 1$, so that $XO = OW$. Thus, the point O is the midpoint of the segment XW. As we have said, this proves Theorem 11.

Second proof of Theorem 11. As a contrast, here comes a completely elementary proof of Theorem 11 - actually, more or less a restatement of the proposed solution of the SPbMO problem. We will use directed angles modulo 180°, but we will use non-directed segments. (See Fig. 23.) Let M be the midpoint of the segment BC. Being the circumcenter of triangle ABC, the point O must lie on the perpendicular bisector of its side BC. Thus, $OM \perp BC$. In other words, $\angle OMX = 90°$. On the other hand, NX is the tangent to the circumcircle of triangle ABC at N, and thus $NX \perp NO$ (since O is the center of this circumcircle). This yields $\angle ONX = 90°$.

Since $\angle OMX = 90°$ and $\angle ONX = 90°$, the points M and N lie on the circle with diameter OX. Thus, $\angle XMN = \angle XON$. On the other hand, $\angle BCN = \angle BAN$ since the point N lies on the circumcircle of triangle ABC.

Thus, $\angle CMN = \angle XMN = \angle XON = -\angle AOZ$ and $\angle MCN = \angle BCN = \angle BAN = -\angle OAZ$. Hence, the triangles CMN and AOZ are oppositely similar. This entails $\dfrac{OZ}{AO} = \dfrac{MN}{CM}$. Similarly, $\dfrac{OY}{AO} = \dfrac{MN}{BM}$. Since $CM = BM$ (what is because the point M is the midpoint of BC), this leads to $\dfrac{OZ}{AO} = \dfrac{MN}{CM} = \dfrac{MN}{BM} = \dfrac{OY}{AO}$, and thus $OZ = OY$. Hence, O is the midpoint of YZ, and Theorem 11 is proven once again.
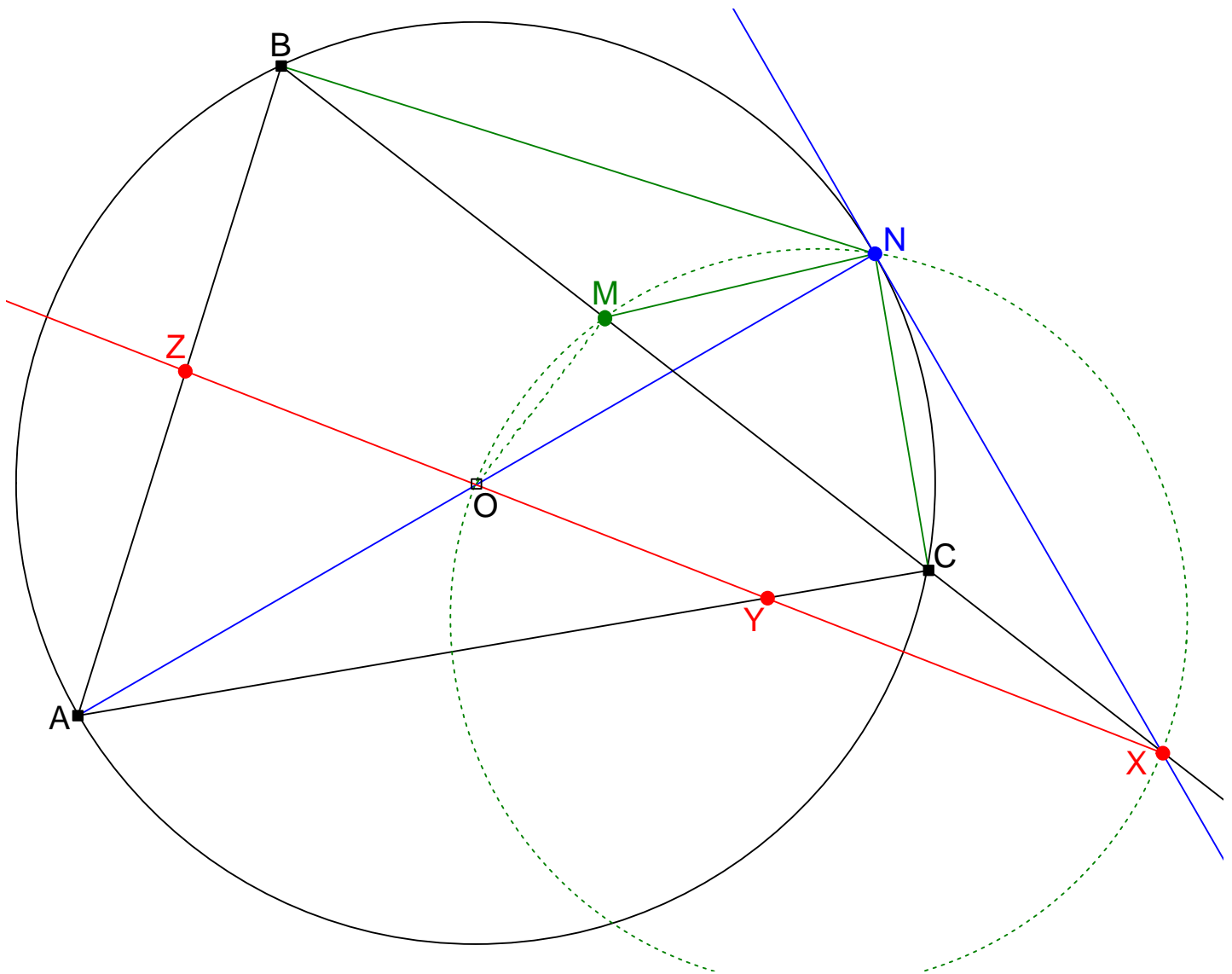
Fig. 23

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