High School Olympiads

triangle, excircles

Reply

Source: Kürschak 2012, problem 1

randomusername

#1 Sep 12, 2013, 3:27 am

The centers of the excircles of triangle ABC opposite to A and B are J_A, J_B, resp. Let PQ be a chord of circle ABC parallel to AB that intersects segments AC and BC. Denote $R = AB \cap CP$. Prove that $\angle J_AQJ_B + \angle J_ARJ_B = \pi$.

Luis González

#2 Sep 12, 2013, 10:07 am

From $\triangle BCJ_A \sim \triangle J_BCA$ we obtain $CJ_A \cdot CJ_B = CA \cdot CB.$ Hence if $\odot(QJ_AJ_B)$ cuts CQ again at $S,$ we get $CQ \cdot CS = CJ_A \cdot CJ_B = CA \cdot CB.$ But $PQ \parallel AB \Longrightarrow \angle BCQ = \angle ACP$ and since $\angle CQB = \angle CAR$, then $\triangle CQB \sim \triangle CAR,$ which yields $CA \cdot CB = CR \cdot CQ \Longrightarrow CR = CS.$ Since CR and $CQ \equiv CS$ are symmetric about $J_AJ_B,$ then it follows that S is the reflection of R on $J_AJ_B \Longrightarrow \angle J_AQJ_B = \pi - \angle J_ASJ_B = \pi - \angle J_ARJ_B.$

Quick Reply

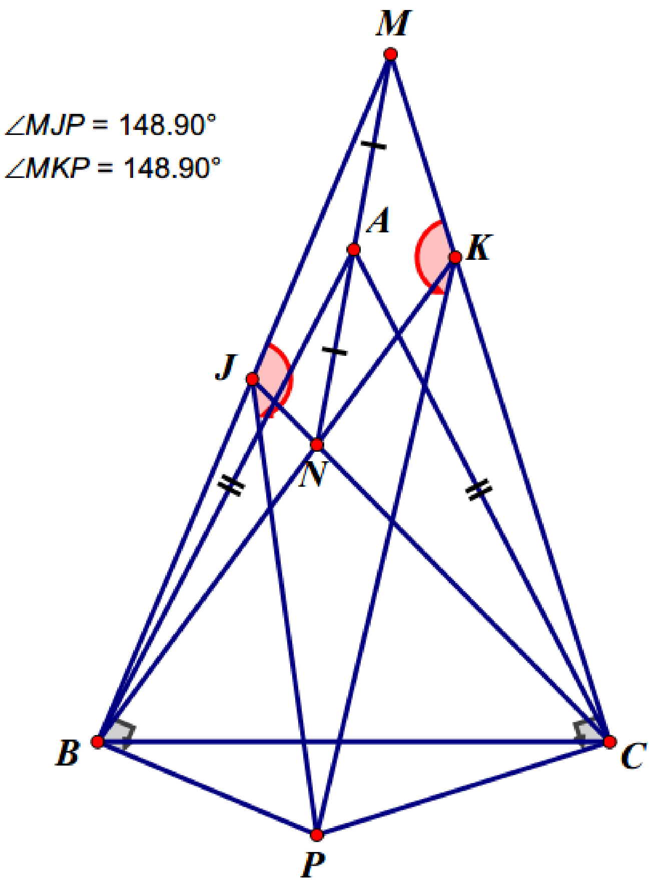
High School Olympiads

An elegant geometry problem

David_Forest
#1 Jul 3, 2015, 7:53 pm

In triangle ABC, $AB = AC$, N is a point inside the triangle and M is a point outside such that A is exactly the midpoint of MN. $BP \perp BM, CP \perp CM$ and $BP \cap CP = P$. Suppose $CN \cap BM = J$ and $BN \cap CM = K$. Prove that $\angle MJP = \angle MKP$.

Attachments:



David_Forest
#2 Jul 3, 2015, 8:23 pm

Has anyone got an idea?

David_Forest
#3 Jul 3, 2015, 11:52 pm

Well, I've got a solution.

My solution:
Lemma: Let AQ be an altitude of ΔABC, K be the midpoint of BC. P lies on BC such that $PK = QK$. D is a point on the line perpendicular to BC which passes through P. $CD \cap AB = E$ and $BD \cap AC = F$. If L is any point on AQ then $S_{\Delta BEL} = S_{\Delta FCL}$.

Proof of Lemma: Suppose KM is the perpendicular bisector of BC which passes through AD. By Newton's theorem easily we get MK bisects EF at N. Therefore, if $EX \perp BC$ at X, $FY \perp BC$ at Y, we have $KX = KY$. Then suppose $DP \cap AB = Z$. We have

$$S_{\Delta BLZ} = S_{\Delta ALB} \cdot \frac{BZ}{AB} = S_{\Delta ALB} \cdot \frac{BP}{BQ} = S_{\Delta ALB} \cdot \frac{CQ}{BQ} = S_{\Delta ALC}$$

Then we only need $\frac{BZ}{BE} = \frac{AC}{CF}$. Note that

$$\frac{BZ}{BE} = \frac{BP}{BX} = \frac{CQ}{CY} = \frac{AC}{CF}$$

Then the lemma is proved.

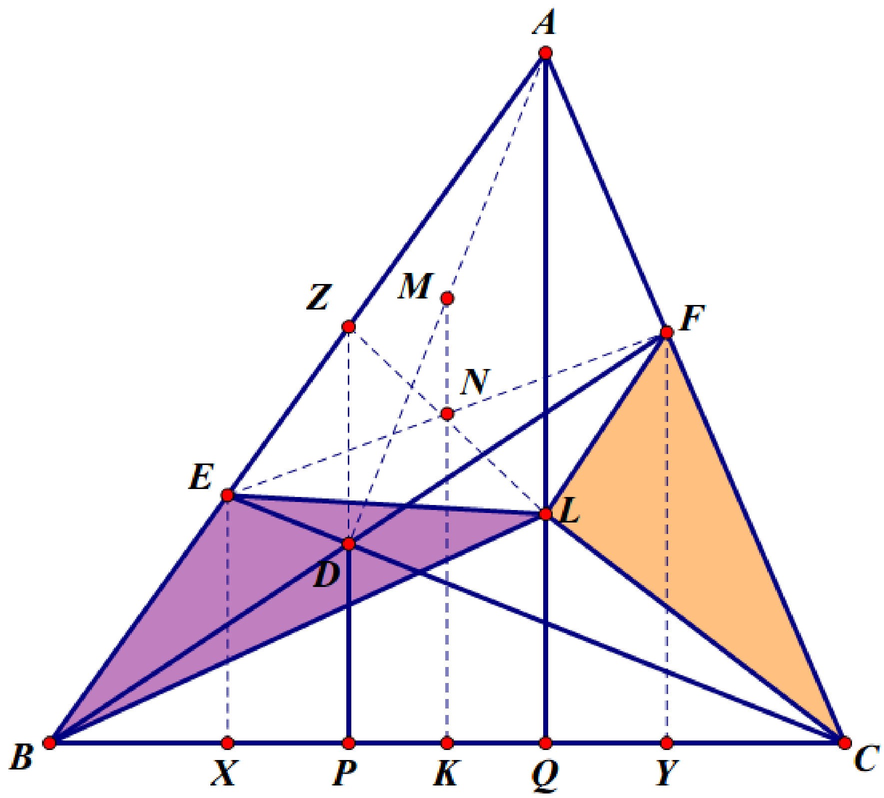
Back to the main problem.
Suppose P' is symmetrical to P about the midpoint of BC, then $P'BPC$ is a parallelogram.
So $CP' \perp BM$ and $BP' \perp CM$, therefore P' is the orthocenter of ΔBCM.
By our lemma we have $S_{\Delta BJP'} = S_{\Delta CKP'}$, since $\angle JBP' = \angle KCP'$ we have

$$BJ \cdot BP' = CK \cdot CP' \iff \frac{CP'}{BJ} = \frac{BP'}{CK}$$

$$\iff \frac{BP}{BJ} = \frac{CP}{CK}$$

Therefore $\Delta JBP \sim \Delta KCP$, so the problem is proved.

Attachments:



This post has been edited 1 time. Last edited by David_Forest, Jul 3, 2015, 11:53 pm
Reason: Wrong

David_Forest
#4 Jul 4, 2015, 12:10 am

Still looking for simpler proof.

farfadet29
#5 Jul 4, 2015, 1:55 am

when i see such a problem, this is what come through my mind:

is every condition projective: yes
can you fix all points except a set of points and describe nicely what happend when those points move on their locus: yes

then it is easy, you just solve the problem in 3 cases.

now rewrite the problem to make it look easy
let B,C and M be fixed points
P is the point such that $MCP = PBM = 90$ (angle, degree)
let (D) be the line image of the mediator of BC from the homothetie of center M and ratio 2

N is a variable point on (D)
$CN \cap BM = J$ and $BN \cap CM = K$

Prove that $\angle MJP = \angle MKP$.
(remark: this is not true in general (the cases when N not Inside ABC), but $(MJ; JP) = (PK; KM)$ so it doesnt matter)

all is projective so solve only three cases

for N on BC it is obvious since $\angle MJP = \angle MKP = 90$
the case N on BM and the case N on CM are similars to your prove but with less points

i hope it was clear

best regard

Luis González
#6 Jul 4, 2015, 2:43 am

Let D be the point forming the parallelogram $MBDC$. Clearly P is orthocenter of $\triangle DBC$ and $N \in DP$. If $F \equiv DP \cap BC, U \equiv BN \cap DC$ and $V \equiv CN \cap DB$, then by Ceva's theorem for $\triangle DBC$ and the cevian triangle $\triangle UVF$ of N, we get

$$\frac{FC}{FB} = \frac{UC}{UD} \cdot \frac{DV}{BV} = \frac{CK}{DB} \cdot \frac{DC}{BJ} \Longrightarrow \frac{CK}{BJ} = \frac{FC}{FB} \cdot \frac{DB}{DC} = \frac{[PCD]}{[PBD]} \cdot \frac{DB}{DC} = \frac{PC \cdot DC}{PB \cdot DB} \cdot \frac{DB}{DC} = \frac{PC}{PB},$$

which means that $\triangle PBJ$ and $\triangle PCK$ are similar by SAS. Thus we deduce that $\angle MKP = \angle MJP$, as desired.

David_Forest
#7 Jul 6, 2015, 6:16 am

> *farfadet29 wrote:*
> when i see such a problem, this is what come through my mind:
>
> is every condition projective: yes
> can you fix all points except a set of points and describe nicely what happend when those points move on their locus: yes
>
> then it is easy, you just solve the problem in 3 cases.
>
> now rewrite the problem to make it look easy
> let B,C and M be fixed points
> P is the point such that $MCP = PBM = 90$ (angle, degree)
> let (D) be the line image of the mediator of BC from the homothetie of center M and ratio 2
>
> N is a variable point on (D)
> $CN \cap BM = J$ and $BN \cap CM = K$
>
> Prove that $\angle MJP = \angle MKP$.
> (remark: this is not true in general (the cases when N not Inside ABC), but $(MJ; JP) = (PK; KM)$ so it doesnt matter)
>
> all is projective so solve only three cases

all is projective so solve only three cases

for N on BC it is obvious since $\angle MJP = \angle MKP = 90$
the case N on BM and the case N on CM are similars to your prove but with less points

i hope it was clear

best regard

I'm still unsure about how you extend this from 3 special cases into general.

David_Forest

#8 Jul 6, 2015, 6:21 am

> *Luis González wrote:*
>
> Let D be the point forming the parallelogram $MBDC$. Clearly P is orthocenter of $\triangle DBC$ and $N \in DP$. If $F \equiv DP \cap BC, U \equiv BN \cap DC$ and $V \equiv CN \cap DB$, then by Ceva's theorem for $\triangle DBC$ and the cevian triangle $\triangle UVF$ of N, we get
>
> $$\frac{FC}{FB} = \frac{UC}{UD} \cdot \frac{DV}{BV} = \frac{CK}{DB} \cdot \frac{DC}{BJ} \Longrightarrow \frac{CK}{BJ} = \frac{FC}{FB} \cdot \frac{DB}{DC} = \frac{[PCD]}{[PBD]} \cdot \frac{DB}{DC} = \frac{PC \cdot DC}{PB \cdot DB} \cdot \frac{DB}{DC} = \frac{PC}{PB},$$
>
> which means that $\triangle PBJ$ and $\triangle PCK$ are similar by SAS. Thus we deduce that $\angle MKP = \angle MJP$, as desired.

Good method!

David_Forest

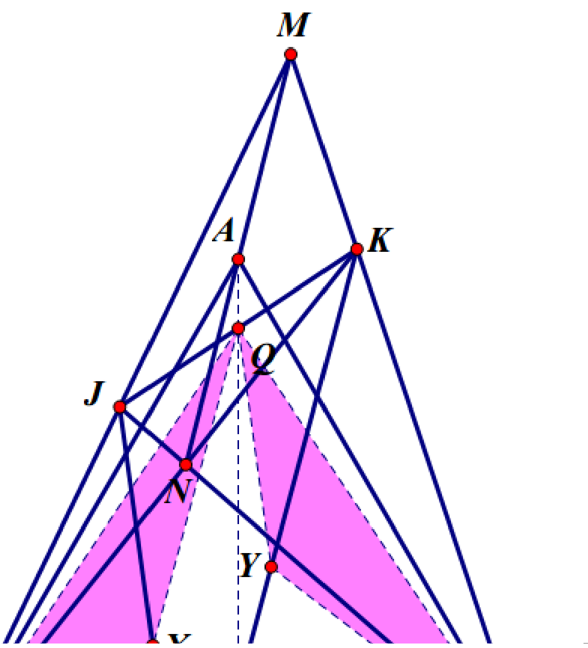
#9 Jul 6, 2015, 6:51 am

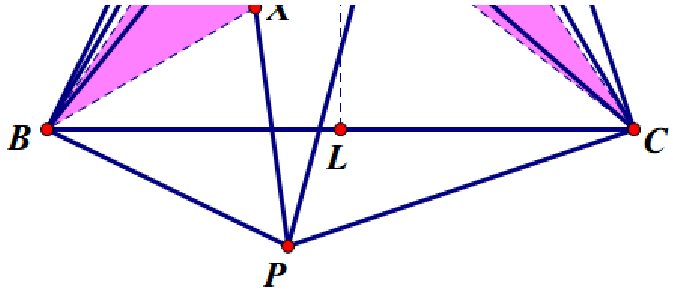
One of my friends has got a really nice solution.

His Solution:
Let X and Y be the midpoints of JP and PK respectively and suppose $AL \perp BC$ at L. Let $AL \cap JK = Q$. Then by Newton's theorem we have Q is the midpoint of JK and since AL is the perpendicular bisector, we also have $BQ = CQ$. Now since Q, X, Y are all midpoints, we get $QY = \frac{1}{2}JP = BX$ and similarly $QX = CY$. Then we can deduce that $\Delta QXB \cong \Delta QYC$ by SSS. Now since $\angle KYQ = \angle KPJ = \angle QXJ$ and $\angle QXB = \angle QYC$, by simple subtraction we have $\angle JXB = \angle KYC$ which means their halfs $\angle JPB$ and $\angle KPC$ are equal. Therefore $\angle MJP = \angle MKP (= \angle JPB + 90°)$.

Q.E.D.

Attachments:





This post has been edited 1 time. Last edited by David_Forest, Jul 6, 2015, 6:52 am
Reason: typo

Quick Reply

High School Olympiads

angle NAK=angle OAN^*

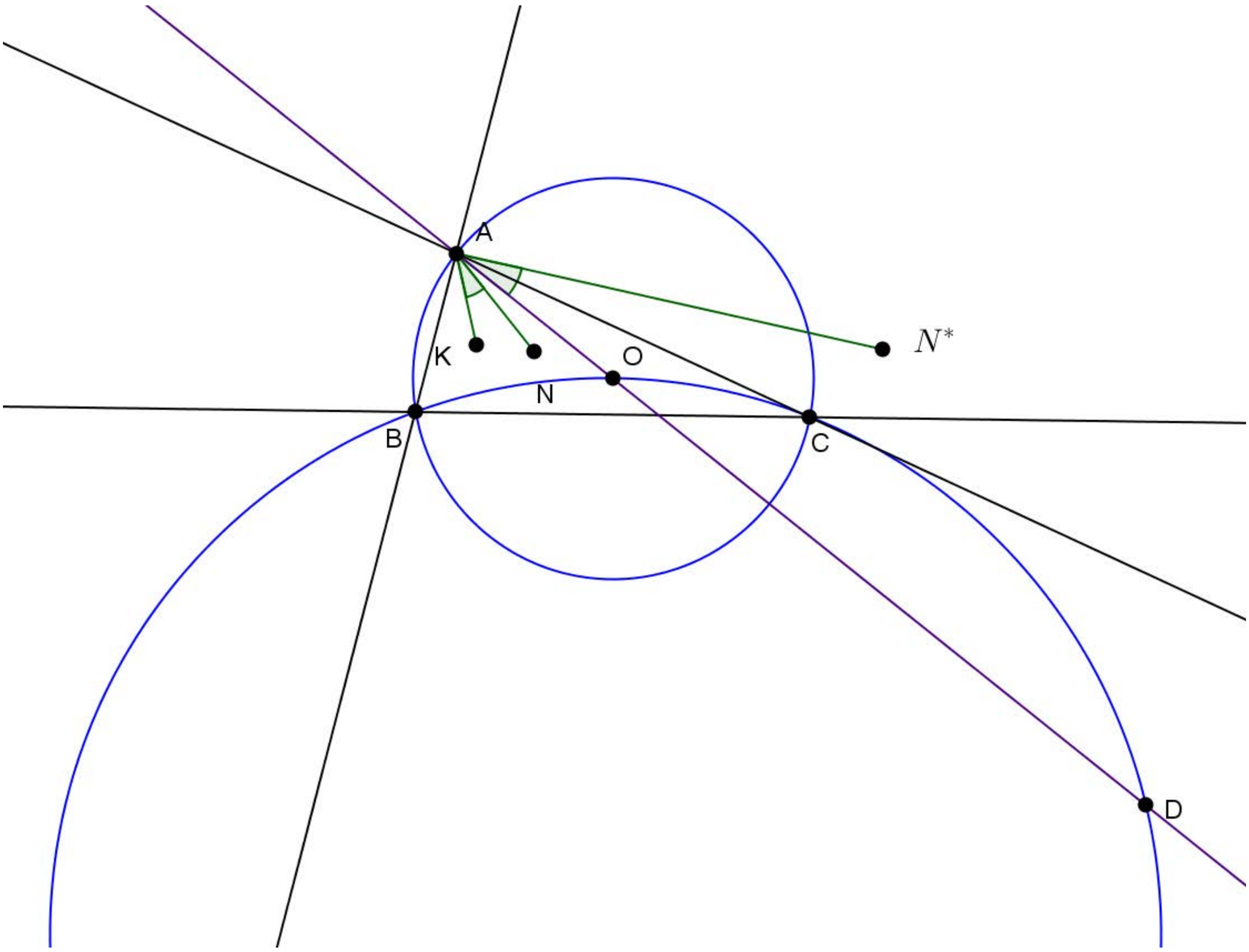
Reply

THVSH

#1 Jul 3, 2015, 9:36 pm • 1

Let ABC be a triangle with circumcenter O, nine point center N and Kosnita point K. $AO \cap \odot(OBC) = D \neq O$. N^* is the nine point center of $\triangle DBC$.
Prove that $\angle NAK = \angle OAN^*$

Attachments:



TelvCohl

#2 Jul 3, 2015, 11:34 pm

My solution :

Let M, X be the midpoint of AD, BC, respectively .
Let H, T be the projection of A, D on BC, respectively .
Let O^* be the reflection of O in BC (It's well-known $O^* \in AN$) .
Let Z be the reflection of H in A and A^* be the reflection of A in Z .

Since $\angle DBC = \angle DOC = 180^\circ - 2\angle CBA, \angle DCB = \angle DOB = 180^\circ - 2\angle ACB$,
so BA, CA is the external bisector of $\angle DBC, \angle DCB$, respectively $\Longrightarrow$ A is D-excenter of $\triangle DBC$,
hence D-excircle $\odot(A)$ of $\triangle DBC$ and the 9-point circle $\odot(N^*)$ of $\triangle DBC$ are tangent at R (Feuerbach theorem) .

From Intersect on circle (post #4) $\Longrightarrow$ $R \in MZ$

From intersect on circle (post #4) $\Longrightarrow R \in MZ$.
Since AX is parallel to the D-Gergonne line DZ of $\triangle DBC$ (well-known) ,
so from $D(T, Z; A^*, A) = -1 = A(H, X; O^*, O) \Longrightarrow DA^* \parallel AO^* \Longrightarrow RZ \parallel AO^*$,
hence $AN \perp HR \Longrightarrow$ combine with $AH = AR$ we get AN is the bisector of $\angle HAN^*$,
so $\angle KAN = \angle HAN - \angle HAK = \angle NAN^* - \angle NAO = \angle OAN^*$.

Q.E.D

Luis González

#3 Jul 4, 2015, 12:12 am • 1

It also follows from the problem Reflections (N) in the altitudes (see the solution at post #2).

Quick Reply

High School Olympiads

Reflections (N) in the altitudes

Reply

Source: Hyacinthos #22491

rodinos
#1 Jul 3, 2014, 5:14 pm

Let ABC be a triangle, H the orthocenter and (N) the Nine Point circle.
Denote:
(N1) = the reflection of (N) in AH
(Na) = the reflection of (N1) in BC
(Nna) = the reflection of (Na) in NA.

The circles (Na) and (Nna) are tangent at a point Ma.
Similarly Mb,Mc.

The triangles ABC, MaMbMc are perspective,
(ie AMa, BMb, CMc are concurrent)

aph

Luis González
#2 Jul 4, 2014, 2:08 am • 2

Let D,E,F denote the feet of the altitudes on $BC,CA,AB.$ It's clear that the composition of the symmetries accross perpendicular axes AH and BC is a symmetry about $D \equiv AH \cap BC \Longrightarrow (N)$ and (N_a) are tangent at $D,$ hence M_a is just the reflection of D on $AN.$

Let the reflections of BC towards AB,AC intersect at $J.$ Hence A becomes incenter or J-excenter of $\triangle JBC$ (WLOG we assume the former case), thus M_a is the 2nd intersection of its J-excircle (A,AD) and the 9-point circle (N) of $\triangle ABC,$ i.e. the Poncelent point of $JBCA \Longrightarrow M_a$ is the J-Feuerbach point of $\triangle JBC,$ i.e. tangency point of (A,AD) with its 9-point circle $(X) \Longrightarrow AM_a$ goes through 9-point center X of $\triangle JBC.$ But if U is the 9-point center of $\triangle DEF,$ we have $\angle CAX=\angle BAU$ because of $AEDF \sim ABJC \Longrightarrow AM_a$ goes through the isogonal conjugate of U WRT $\triangle ABC$ and similarly BM_b and $CM_c \Longrightarrow \triangle ABC$ and $\triangle M_aM_bM_c$ are perspective through the isogonal conjugate of U WRT $\triangle ABC;$ X_{252} on ETC.

rodinos
#3 Jul 4, 2014, 2:28 am

Yes, Luis, Yes ... Just the reflection of the foot af the A-altitude to AN.... 🙂
And I noticed that later....

https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/22492

Thanks ! 🙂

Quick Reply

High School Olympiads

Construction Pole of Pascal line

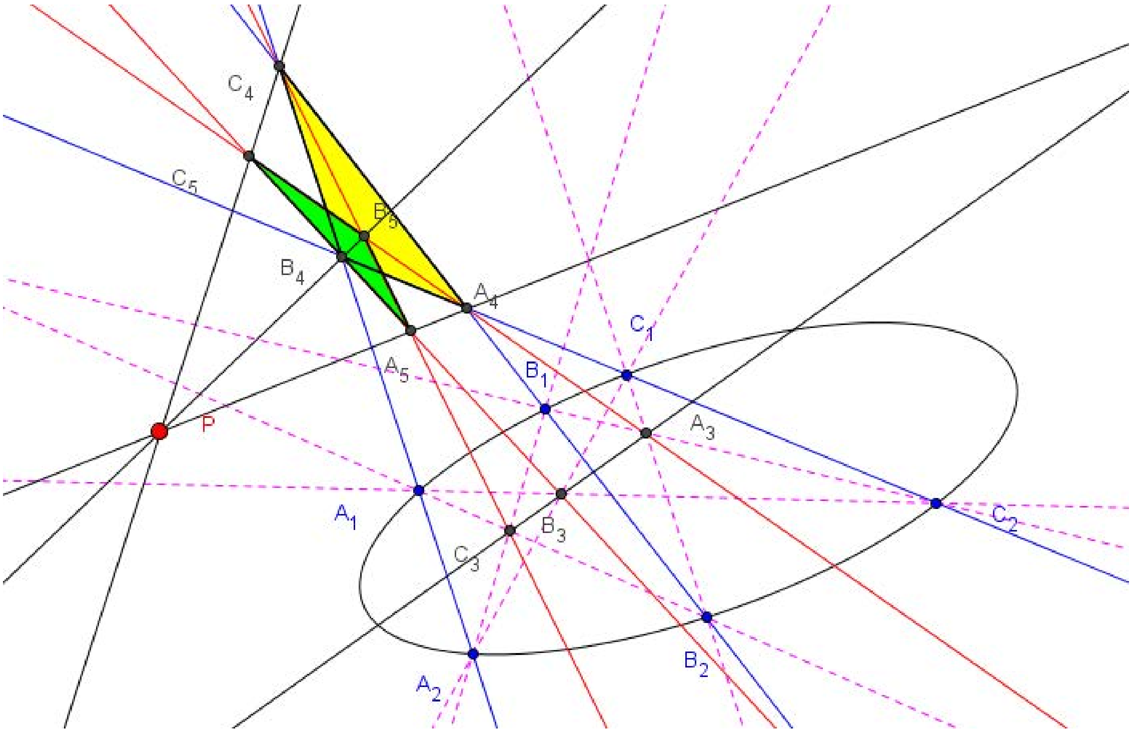
Reply

daothanhoai

#1 Jul 3, 2015, 6:50 pm

Let $A_1, B_1, C_1, C_2, B_2, A_2$ lie on a conic. Let $A_3 = B_1C_2 \cap B_2C_1$, $B_3 = A_1C_2 \cap A_2C_1$, $C_3 = A_1B_2 \cap A_2B_1$. Let $A_4B_4C_4$ be a triangle bounded by A_1A_2, B_1B_2, C_1C_2. Let $A_5B_5C_5$ be a triangle bound by A_3A_4, B_3B_4, C_3C_4 then show that two triangle $A_4B_4C_4$ and $A_5B_5C_5$ are perpective, the perpector ispole of Pascal line $\overline{A_3B_3C_3}$.

Attachments:



Luis González

#2 Jul 3, 2015, 9:25 pm

By Pascal theorem for $A_2C_2B_1A_1C_1B_2$, the intersections $U \equiv A_1B_1 \cap A_2B_2, V \equiv A_2C_2 \cap A_1C_1$ and $A_3 \equiv C_2B_1 \cap C_1B_2$ are collinear. From the complete quadrilaterals $A_1A_2B_2B_1$ and $A_1A_2C_2C_1$, it follows that U and V are the poles of C_3C_4 and B_3B_4 WRT the conic $\mathcal{C} \Longrightarrow \overline{UVA_3}$ is the polar of $A_5 \equiv C_3C_4 \cap B_3B_4$ WRT $\mathcal{C}$. But from the complete quadrilateral $B_1B_2C_2C_1$, it follows that A_4 is on the polar of A_3 WRT $\mathcal{C} \Longrightarrow A_4A_5$ is the polar of A_3 WRT $\mathcal{C}$ and similarly B_4B_5 and C_4C_5 are the polars of B_3 and C_3 WRT $\mathcal{C} \Longrightarrow P \equiv A_4A_5 \cap B_4B_5 \cap C_4C_5$ is the pole of $\overline{A_3B_3C_3}$ WRT $\mathcal{C}$.

Quick Reply

High School Olympiads

Line passes through intersection of two diagonals

Reply

Source: Own

buratinogigle

#1 Jul 3, 2015, 11:34 am

Let $ABCD$ be cyclic quadrilateral with circumcenter O. AC cuts BD at E. We known that orthopole of line OE with respect to triangles ABC, BCD, CDA, DAB are collinear on line ℓ. Prove that ℓ passes through E.

Luis González

#2 Jul 3, 2015, 1:00 pm • 1

Let M,N,L be the midpoints of BD,DA,AB and let P be the projection of A on $OE.$ Orthopole X of OE WRT $\triangle ABD$ is on its 9-point circle $\odot(MNL),$ thus it's on the reflection $\odot(ALOPN)$ of $\odot(MNL)$ on $NL \Longrightarrow X$ is reflection of P on $NL \Longrightarrow X$ is on circle with diameter $\overline{AE} \Longrightarrow \angle AEX=\angle APX=\angle PED \Longrightarrow EX,EO$ are isogonals WRT $\angle(AC,BD).$ Similarly, if Y is the orthopole of OE WRT $\triangle CBD,$ then EY,EO are isogonals WRT $\angle(AC,BD) \Longrightarrow$ $E \in XY \equiv \ell.$

TelvCohl

#4 Jul 3, 2015, 3:50 pm • 1

Another solution :

Let $F \equiv AB \cap CD, G \equiv DA \cap BC$.
Let B', D' be the projection of B, D on OE, respectively .
Let $\mathcal{O}_A, \mathcal{O}_C$ be the orthopole of OE WRT $\triangle CBD, \triangle ABD$, respectively .

Since $\mathcal{O}_AB' \perp CD, \mathcal{O}_AD' \perp CB, \mathcal{O}_CB' \perp AD, \mathcal{O}_CD' \perp AB, OE \perp FG$,
so $\mathcal{O}_A, B', \mathcal{O}_C, D'$ are concyclic and $\triangle\mathcal{O}_AB'D' \sim \triangle CFG, \triangle\mathcal{O}_CD'B' \sim \triangle AFG$.

From $\triangle FBC \sim FDA, \triangle GDC \sim \triangle GBA \Longrightarrow \frac{AB}{AG}=\frac{CD}{CG}, \frac{CB}{CF}=\frac{AD}{AF} \Longrightarrow \frac{AB \cdot CB}{AD \cdot CD}=\frac{CF \cdot AG}{CG \cdot AF}$,

so $\frac{B'E}{D'E}=\frac{BE}{DE}=\frac{[ABC]}{[CDA]}=\frac{AB \cdot CB}{AD \cdot CD}=\frac{CF \cdot AG}{CG \cdot AF}=\frac{\mathcal{O}_AB' \cdot \mathcal{O}_CB'}{\mathcal{O}_AD' \cdot \mathcal{O}_CD'}=\frac{[\mathcal{O}_AB'\mathcal{O}_C]}{[\mathcal{O}_CD'\mathcal{O}_A]} \Longrightarrow E \in \mathcal{O}_A\mathcal{O}_C$.

Q.E.D

buratinogigle

#5 Jul 3, 2015, 10:51 pm

Thank you very much dear friends for your interest, I don't have solution but I have seen Euler-Poncelet point of $ABCD$ lies on ℓ, too.

TelvCohl

#6 Jul 3, 2015, 10:57 pm • 1

> buratinogigle wrote:
> I don't have solution but I have seen Euler-Poncelet point of $ABCD$ lies on ℓ, too.

See A generalization of the Simson line theorem (Problem 2) :)

Quick Reply

High School Olympiads

cyclic quadrilateral

Reply

Source: Own

andria

#1 Jun 19, 2015, 2:23 pm • 1

In $\triangle ABC$ assume that R, r_a are radius of circumcircle and A-excircle let the A-mixtilinear incircle touch AB, AC at F, E let $BE \cap CF = S$ prove that $AESF$ is cyclic if and only $R = r_a$.

Luis González

#2 Jul 3, 2015, 11:09 am

Let X, Y, Z be the tangency points of the A-excircle (I_a) with $BC, CA, AB.$ For any $\triangle ABC, J \equiv AX \cap BY \cap CZ$ and its isogonal conjugate J^* WRT $\triangle ABC$ is the exsimilicenter of (I_a) and the circumcircle (O) (this is an extraversion of the incircle case). It's also known that $AE = AF = \frac{bc}{s} \Longrightarrow \frac{AE}{AC} = \frac{c}{s} = \frac{AB}{AZ} \Longrightarrow BE \parallel CZ$ and likewise $CF \parallel BY.$

From the above claims we conclude that $AESF$ is cyclic $\Longleftrightarrow ABJC$ is cyclic $\Longleftrightarrow J \in (O) \Longleftrightarrow J^*$ is at infinity $\Longleftrightarrow (O) \cong (I_a) \Longleftrightarrow R = r_a.$

Quick Reply

High School Olympiads

Two lines are parallel again

Reply

Source: Own

buratinogigle
#1 Jul 3, 2015, 2:45 am • 1

Let $ABCD$ be cyclic quadrilateral. AB cuts CD at E and AD cuts BC at F. We known that orthopoles of lines EF with respec to triangles ABC,BCD,CDA,DAB are collinear on line ℓ. Prove that ℓ is parallel to Newton line of $ABCD$.

Luis González
#2 Jul 3, 2015, 6:43 am • 4

Let B',D' be the projections of B,D on $EF.$ Perpendiculars from D',B' to $AB,AD,$ respectively, meet at the orthopole X of EF WRT $\triangle ABD$ and the perpendiculars from D',B' to $CB,CD,$ respectively, meet at the orthopole Y of EF WRT $\triangle CBD \Longrightarrow \ell \equiv XY.$ Since $\angle D'XB'=\angle D'YB=\angle BAD=\angle BCD$ (mod 180º), then X,Y,B',D' are concyclic, i.e. ℓ is antiparallel to EF WRT $D'X,B'Y.$ Thus, it suffices to prove that the Newton line n of $ABCD$ is antiparallel to EF WRT $D'X,B'Y,$ which in turn is equivalent to prove that $\angle (n,CD)=\angle (AB,EF).$

Let $M \in EF$ be the Miquel point of $ABCD;$ which is the isogonal conjugate of the point at infinity of n WRT $ABCD.$ Hence $\angle (AB,EF)=\angle ADM=\angle FCM=\angle (n,CD),$ as desired.

TelvCohl
#3 Aug 9, 2015, 3:11 pm • 3

My solution **:**

Lemma :

Given a cyclic quadrilateral $ABCD$ and a line ℓ. Let P be the point on $\odot (ABCD)$ such that $PD \perp \ell$. Then the orthopolar line of ℓ WRT $ABCD$ is parallel to the Steiner line of P WRT $\triangle ABC$.

Proof :

Let Y,Z be the projection of B,C on ℓ, respectively. Let $\mathcal{O}_A,\mathcal{O}_D$ be the orthopole of ℓ WRT $\triangle BCD,\triangle ABC$, respectively. Let H be the orthocenter of $\triangle ABC$ and Q be the reflection of P in BC (QH is the Steiner line of P WRT $\triangle ABC$). From $Y\mathcal{O}_A \perp DC,Y\mathcal{O}_D \perp AC,Z\mathcal{O}_A \perp DB,Z\mathcal{O}_D \perp AB \Longrightarrow \measuredangle Y\mathcal{O}_AZ=\measuredangle CDB=\measuredangle CAB=\measuredangle Y\mathcal{O}_DZ$, so $Y,Z,\mathcal{O}_A,$ $\mathcal{O}_D$ are concyclic. Notice Q lie on the reflection $\odot (BHC)$ of $\odot (ABCD)$ in BC we get $\measuredangle Z\mathcal{O}_D\mathcal{O}_A=\measuredangle ZY\mathcal{O}_A=\measuredangle PDC$ ($\because DP \perp YZ,DC \perp Y\mathcal{O}_A$) $=\measuredangle PBC=\measuredangle CBQ=\measuredangle CHQ$, so combine $Z\mathcal{O}_D \parallel CH$ ($\perp AB$) $\Longrightarrow \mathcal{O}_A\mathcal{O}_D \parallel QH$.

Back to the main problem :

Let $V \equiv AC \cap BD$ and O be the center of $\odot (ABCD)$. Let M,N be the midpoint of AC,BD, respectively (MN is the Newton line of $ABCD$). Let X be the point on $\odot (O)$ such that $DX \perp EF$ and Y be the point on $\odot (O)$ such that $XY \perp CA$. Since BY is parallel to the Steiner line of X WRT $\triangle ABC$ (well-known), so from the lemma $\Longrightarrow BY \parallel \ell$ $(\star)$. Since $O,$ V,M,N lie on a circle (with diameter OV), so $\measuredangle NMO=\measuredangle NVO=\measuredangle BDX$ ($\because OV \perp EF \Longrightarrow OV \parallel XD$) $=\measuredangle BYX,$ hence combine $OM \parallel XY$ ($\perp AC$) we get $MN \parallel BY \Longrightarrow MN \parallel \ell$ (from $(\star)$).

Quick Reply

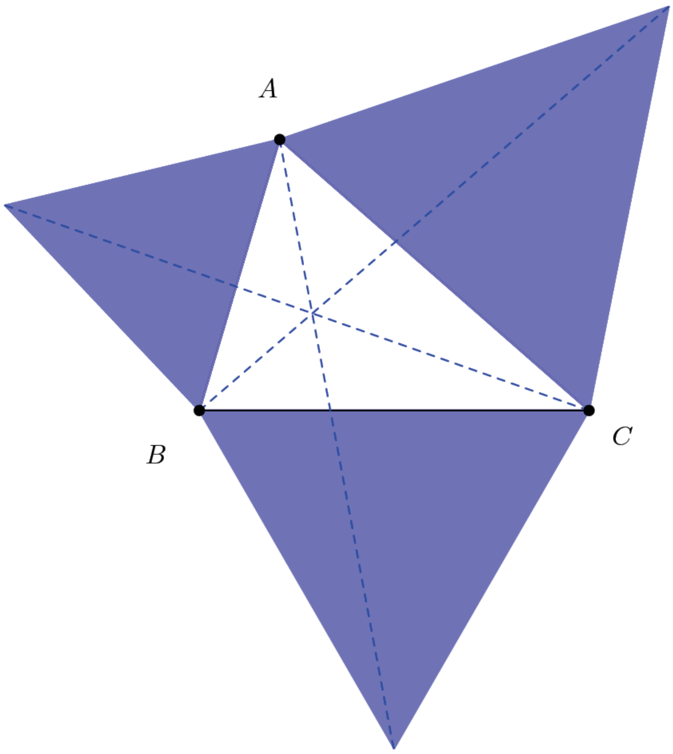
High School Olympiads

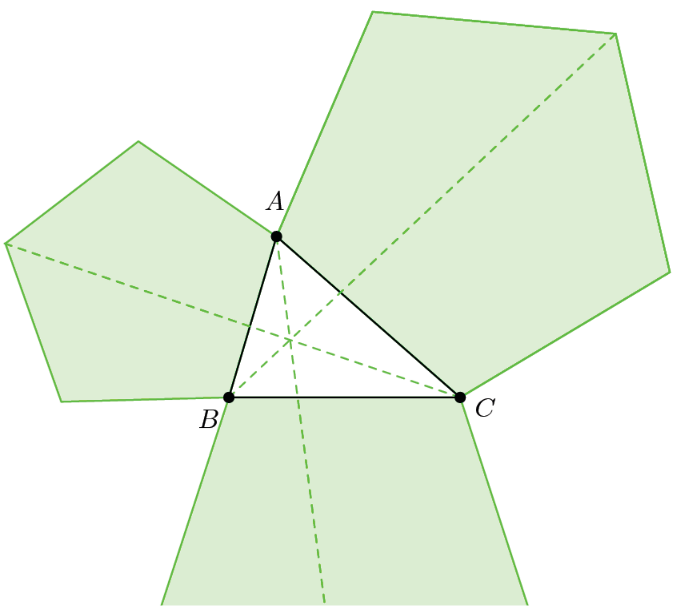
Always Concurrent???

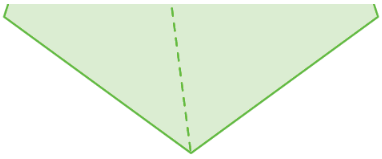
tkhalid

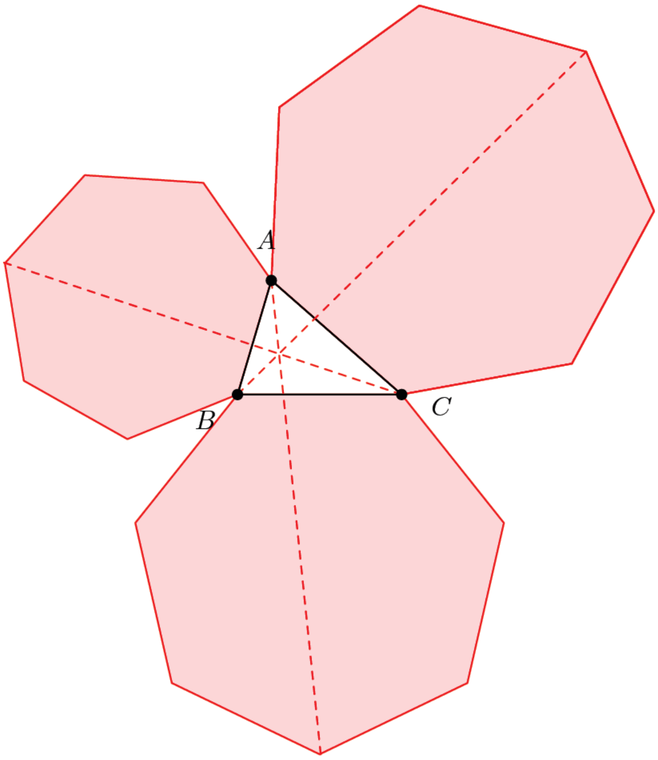
#1 Jul 3, 2015, 5:43 am

I noticed that when we construct a 3, 5, or 7 sided regular polygon on the sides of a triangle, then the lines connecting a vertex of the triangle to the vertex of the regular polygon farthest opposite it are concurrent. Is this true for all regular polygons with an odd number of sides? And if so, how do we prove it?









Luis González

#2 Jul 3, 2015, 5:50 am • 1

Posted before at http://www.artofproblemsolving.com/community/c6h440633. Obviously it follows from Kiepert's theorem.

tkhalid

#3 Jul 3, 2015, 5:55 am

Oops, that was kind of obvious. Thanks Luis González

Quick Reply

High School Olympiads

Regular polygons external to a triangle

Reply

Source: self-proposed

proglote

#1 Oct 25, 2011, 5:13 am

Given $\triangle ABC$, construct the regular $2n+1$- gon $AC_1...C_n..C_{2n-2}C_{2n-1}B$ externally to side AB, and similarly define A_n and B_n. Prove or disprove : $\forall n \in \mathbb{N}^*$, the lines CC_n, AA_n and BB_n are concurrent.

$n=1$ is well-known result. An example for $n=2$ is on the Applet.

[geogebra]df8915b72104cbab63fea76eb20b1f739ebde5d6[/geogebra]

Maybe it's an old result, but I have not seen anywhere..

Luis González

#2 Oct 25, 2011, 9:17 pm • 1

Note that An,Bn,Cn are apices of similar isosceles triangles erected outside ABC. So this is equivalent to Kiepert's theorem, which is a particular case of Jacobi's theorem.

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=186725
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=50&t=154396
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=404816

Concyclicboy

#3 Nov 1, 2011, 3:12 am

Trigonometric Ceva for each isosceles similar external triangles: BCA', CAB', ABC'

Let be X = <CBA'=<BCA'=ABC'=BAC'=ACB'=CAB' and k= BA'/BC=CA'/BC= AB'/AC=CA'/AC=AC'/AB=BC'/AB

sin BAA'/sin CAA' = [(BA') * sin(B+X) / AA'] / [(CA') * sin(C+X)/AA'] = [(BC) *(K)*sin(B+X)/AA'] / [(BC) *(K)*sin(C+X)/AA'] = sin(B+X)/sin(C+X) and finish

Quick Reply

High School Olympiads

Midpoint

Headhunter

#1 Feb 18, 2011, 3:21 am

Hello.

For a triangle ABC with its circumcircle (O), let K_1 be the circle which are tangent to AB, AC, and (O) internally.
Let P_1 be the circle which is tangent to AB, AC, and (O) externally. Likewise, define K_2, P_2 and K_3, P_3
K is the radical center of K_1, K_2, K_3 and P is the radical center of P_1, P_2, P_3
Show that O is the midpoint of KP

Luis González

#2 Feb 18, 2011, 7:20 am

The notation is somewhat inconvenient, thus I will denote the A-, B-,C- mixtilinear incircles as $\omega_a, \omega_b, \omega_c$ and the A-,B-,C- mixtilinear excircles as $\Omega_a, \Omega_b, \Omega_c$, respectively.

Let $\triangle I_aI_bI_c$ be the excentral triangle of $\triangle ABC.$ Incircle (I) and A-excircle (I_a) touch BC at $X, Y.$ B- and C- mixtilinear incircles ω_b and ω_c touch BC at D, E and touch circumcircle (O) of $\triangle ABC$ at $B_0, C_0.$ Thus, B_0D and C_0E bisect $\angle BB_0C$ and $\angle CC_0B \Longrightarrow B_0D$ and C_0E pass through the midpoint L of the arc BC of $(O).$ Since BC is the image of (O) under the inversion with center L and radius $LB = LC,$ it follows that $LB^2 = LD \cdot LB_0 = LE \cdot LC_0 \Longrightarrow L$ has equal power to ω_b and ω_c, thus radical axis τ_a of ω_b, ω_c goes through L and the midpoint U of $DE.$

On the other hand, it's well-known that $DI \parallel BI_a$ and $EI \parallel CI_a \Longrightarrow \triangle DEI$ and $\triangle BCI_a$ are homothetic. Therefore, if M denotes the midpoint of $BC,$ we deduce that $I_aM \parallel IU \ (\star).$ Let $R \equiv I_aM \cap IX.$ Since $\overline{MX} = -\overline{MY} \Longrightarrow XRYI_a$ is a parallelogram $\Longrightarrow M$ is the midpoint of $RI_a.$ Together with $(\star),$ it follows that $IU \cap I_aX$ is the reflection of I about $U \Longrightarrow UL \equiv \tau_a$ is the I-midline of $\triangle IXI_a.$ Now, since I_aX cuts IO at the exsimilicenter X_{57} of (I) and $\odot(I_aI_bI_c),$ then τ_a passes through the midpoint X_{999} of $IX_{57}.$ Analogously, radical axes ℓ_b, ℓ_c of mixtilinear incircles ω_c, ω_a and ω_a, ω_b pass through $X_{999} \Longrightarrow X_{999}$ is radical center of the three mixtilinear incircles $\omega_a, \omega_b, \omega_c.$

Ω_b and Ω_c touch the sideline BC at $D', E'.$ Using analogous arguments as the mixtilinear incircles, we get that the radical axis ℓ'_a of Ω_b and Ω_c is the line connecting the midpoint U' of $D'E'$ and the midpoint of the arc BAC of $(O).$ Likewise, we'll have that the reflection of $V \equiv I_bD' \cap I_cE'$ about U' lies on I_aX and since L' is the midpoint of I_aV (due to the parallelogram $I_aI_bVI_c$), it follows that $\ell'_a \parallel I_aX \Longrightarrow \ell'_a \parallel LX_{999}.$ Thus, $\overline{OL} = -\overline{OL'}$ implies that ℓ'_a cuts IO at the reflection of X_{999} about $O.$ Radical center of $\Omega_a, \Omega_b, \Omega_c$ is the reflection of X_{999} about $O.$

High School Olympiads

Feuerbach Points and Concurrency

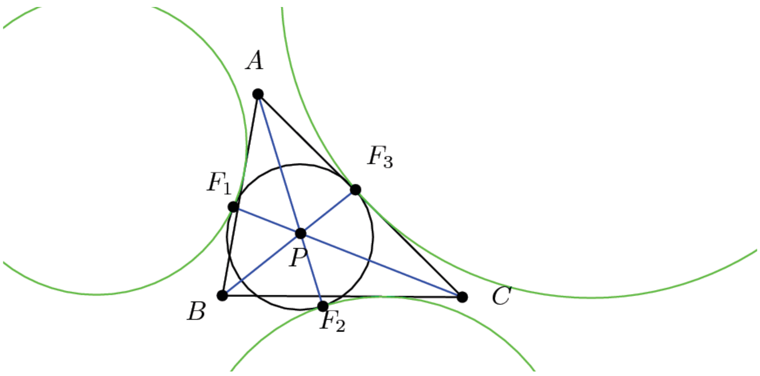
Reply

tkhalid

#1 Jun 25, 2015, 12:33 pm

Does anyone know a nice proof for the following problem?

Let F_1, F_2, and F_3 be the tangency points of the C, A, and B excircles with the Nine Point circle of $\triangle ABC$ respectively. Prove that lines CF_1, AF_2, and BF_3 are concurrent.



Luis González

#2 Jul 3, 2015, 12:23 am • 1

This is an old problem likely posted before somewhere. Monge & d'Alembert kill it.

Label (I) the incircle of $\triangle ABC$ and $(I_a),(I_b),(I_c)$ its excircles against $A,B,C.$ A is the exsimilicenter of $(I) \sim (I_a)$ and F_2 is the insimilicenter of the 9-point circle (N) and $(I_a).$ Thus by Monge & d'Alembert theorem for $(I),(N),(I_a),$ it follows that A,F_2 and the insimilicenter X_{12} of $(I) \sim (N)$ are collinear. Similarly BF_3 and CF_1 go through $X_{12}.$

tkhalid

#3 Jul 3, 2015, 12:30 am

Thanks Luis and good solution :). TelvCohl pm-ed me a while ago with the solution too.

Gibby

#4 Jul 3, 2015, 5:37 am

There is also an interesting solution to this involving inversion, try drawing the incircle and the circle with the tangency point of the incircle and tangency of the excircle as endpoints of diameter. Surely you could look this up and read more nice proofs as this is a well-known theorem. Additional note: the proof I have mentioned is not mine.

Quick Reply

High School Olympiads

Interesting problem with I-inscribed circle center

Reply

DanDumitrescu
#1 Jul 2, 2015, 7:21 pm

Prove that the I-inscribed circle center of the trinagle ABC there is in the interior of the median triangle of ABC.

ComplexPhi
#2 Jul 2, 2015, 7:52 pm • 1

It suffices to show that $\frac{AI}{ID} > 1$ and the analogues (where D is the intersection between AI and BC) .
Using the angle-bisector theorem we can easily get $\frac{AI}{DI} = \frac{b+c}{a} > 1$ because of the triangle inequality . So it follows that the conclusion holds

Luis González
#3 Jul 2, 2015, 11:51 pm

More general: The centers of all the ellipses $\mathcal{E}$ inscribed inside of the triangle $\triangle ABC$ lie inside its medial triangle $\triangle DEF.$

Proof: Parallel project $\mathcal{E}$ with center J into a circle (J^*), i.e. $\triangle ABC$ goes to a $\triangle A^*B^*C^*$ with incenter J^*. It's well known for any $\triangle A^*B^*C^*$ that J^* is the Nagel point of its medial $\triangle D^*E^*F^*$, thus being the Nagel cevians interior rays of $\widehat{D^*}, \widehat{E^*}, \widehat{F^*}$, it follows that J^* is inside of $\triangle D^*E^*F^* \Longrightarrow J$ is inside of $\triangle DEF$, as desired.

Quick Reply

High School Olympiads

bisector

Reply

Source: Balkan MO 2014 G-6

dizzy

#1 Jun 10, 2015, 8:42 pm

In $\triangle ABC$ with $AB = AC$,M is the midpoint of BC,H is the projection of M onto AB and D is arbitrary point on the side AC.Let E be the intersection point of the parallel line through B to HD with the parallel line through C to AB.Prove that DM is the bisector of $\angle ADE$.

Luis González

#2 Jun 17, 2015, 2:43 am • 1

Let X be the reflection of A on BC. Thus $ABXC$ is a rhombus with incircle $\omega \equiv \odot(M, MH)$ and let the tangent from E to ω, other than CX, cut AC, AB at D', F. Tangents $D'F, XB, XC$ of ω induce a proyectivity between AB, AC, thus if B_∞, C_∞ denote the points at infinity of AB, AC, we get $(H, B, F, B_\infty) = (A, C_\infty, D', C) \Longrightarrow \frac{HB}{HF} = \frac{D'C}{D'A} = \frac{D'E}{D'F} \Longrightarrow HD' \parallel BE \Longrightarrow D' \equiv D \Longrightarrow DM$ bisects $\angle CDF \equiv \angle ADE$.

Octophi

#3 Jun 18, 2015, 9:49 pm

Is there a proof without projective geometry?

navredras

#4 Aug 23, 2015, 3:09 am

Any other solutions?

Quick Reply

High School Olympiads

I need to find an old link

Reply

A-B-C

#1 Jun 16, 2015, 8:04 pm

Dear Mathlinkers,
I have seen a problem on AoPS, which was solved by Luiz Gonzales, Telv Cohl,...
That problem is (may be in the link, the name of some points are quite different) :
Given $\triangle ABC$ and a point P. PA, PB, PC intersects BC, CA, AB at D, E, F.
$(ABE) \cap (ACF) = X$
$(BCF) \cap (BAD) = Y$
$(CAD) \cap (CBE) = Z$
Prove that: AX, BY, CZ are concurrent.
I also remember that Luis Gonzales has a synthetic proof and a proof using barycentric coordnates, Telv Cohl has a synthetic proof.
I really need to find that link.
Thank you
A-B-C

Luis González

#3 Jun 16, 2015, 9:33 pm • 1

I believe you mean the topic http://www.artofproblemsolving.com/community/c6h282732. The concurrency point is the isogonal conjugate of the complement of P.

Quick Reply

High School Olympiads

Three concurrent radical axes

Reply

Source: 0

Luis González

#1 Jun 14, 2009, 12:03 am • 1

P is an arbitrary point on the plane of $\triangle ABC$ and let $\triangle A'B'C'$ be the cevian triangle of P WRT $\triangle ABC$. The circles $\odot(ABB')$ and $\odot(ACC')$ meet at A, X. Similarly, define the points Y and Z WRT B and C. Prove that the lines AX, BY, CZ concur at the isogonal conjugate of the complement of P WRT $\triangle ABC$.

jayme

#2 Jun 14, 2009, 11:10 am

Dear Luis,
nice result.
See for historical note
Stevanovic M. R., Symmedian as radical axis, Message Hyacinthos:
http://tech.groups.yahoo.com/group/Hyacinthos/message/10931
http://tech.groups.yahoo.com/group/Hyacinthos/message/10905
http://tech.groups.yahoo.com/group/Hyacinthos/message/10904
Sincerely
Jean-Louis

Luis González

#3 Jun 14, 2009, 10:47 pm

Thanks for the links Jean-Louis, I found this result very interesting and with lots of applications. There is also a couple of additional things that I would like to highlight.

Let $P\ (u:v:w)\ ,\ A'\ (0:v:w)\ ,\ B'(u:0:w)\ ,\ C'(u:v:0)$

Barycentric equations of circles $\odot(ABB')$ and $\odot(ACC')$ are then:

$$a^2yz+b^2xz+c^2xy-\frac{ub^2z}{u+w}(x+y+z)=0$$

$$a^2yz+b^2xz+c^2xy-\frac{uc^2y}{u+v}(x+y+z)=0$$

Thus, their radical axis AX has equation $AX\equiv\frac{c^2y}{u+v}-\frac{b^2z}{u+w}=0$

Similarly, equations of BY, CZ are given by:

$$BY\equiv\frac{c^2x}{u+v}-\frac{a^2z}{w+v}=0\ ,\ CZ\equiv\frac{a^2y}{v+w}-\frac{b^2x}{u+w}=0$$

Therefore, AX, BY, CZ concur at $J\left(\frac{a^2}{v+w}:\frac{b^2}{u+w}:\frac{c^2}{u+v}\right)$,

which is the isogonal conjugate of the complement of P

Additionally, if we consider the three centers of the spiral similarities that takes the vectors $\overrightarrow{BP},\overrightarrow{CP},\overrightarrow{AP}$ into the vectors $\overrightarrow{CA},\overrightarrow{AB},\overrightarrow{BC}$, then the triangle formed by these three centers is perspective with $\triangle ABC$ and the perspector is J.

Further, the incenter of this triangle is the first Stevanovic point X_{1130} of $\triangle ABC$.

jayme

#4 Jun 15, 2009, 7:01 pm

Dear Luis,
sorry, I have enconter no name for that point.
Sincerely
Jean-Louis

TelvCohl

#5 Feb 19, 2015, 9:54 pm

My solution:

Let ℓ_A be the bisector of $\angle BAC$.
Let $Q \in AC, R \in AB$ such that $BQ \parallel CP, CR \parallel BP$ and $T = BQ \cap CR$.
Let Ψ be the composition of inversion $\mathbf{I}(A, \sqrt{AB \cdot AC})$ and reflection $\mathbf{R}(\ell_A)$.

Since $AB \cdot AC = AB' \cdot AR$,
so R is the image of B' under Ψ .
Similarly we can prove Q is the image of C' under Ψ ,
so BQ, CR is the image of $\odot(ACC'), \odot(ABB')$ under Ψ, respectively ,
hence $T \equiv BQ \cap CR$ is the image of $X \equiv \odot(ACC') \cap \odot(ABB')$ under Ψ . ... $(\star)$

Since AT pass through the complement P^* of P WRT $\triangle ABC$,
so from $(\star)$ we get AX passes through the isogonal conjugate V of P^* WRT $\triangle ABC$.

Similarly we can prove BY, CZ pass through V ,
so we conclude that AX, BY, CZ are concurrent at V .

Q.E.D

This post has been edited 3 times. Last edited by TelvCohl, Apr 29, 2016, 12:17 am

Luis González

#7 Apr 17, 2015, 2:44 am

Here is my proof without using barycentric coordinates:

Let D be the point forming the parallelogram $BPCD \Longrightarrow AD$ is the A-cevian of the complement of P. Since $[DCB'] = [DCP] = [DBP] = [DBC'] \Longrightarrow \text{dist}(D, AC) : \text{dist}(D, AB) = BC' : CB'$. But since X is center of the spiral similarity that swaps $\overline{CB'}$ and $\overline{C'B}$, we have $\text{dist}(X, AB) : \text{dist}(X, AC) = BC' : CB' \Longrightarrow$ $\text{dist}(D, AC) : \text{dist}(D, AB) = \text{dist}(X, AB) : \text{dist}(X, AC) \Longrightarrow AD$ and AX are isogonals WRT $\angle BAC \Longrightarrow AX$ goes through the isogonal of the complement of P and similarly BY and CZ.

TelvCohl

#8 Apr 17, 2015, 10:15 am

Another proof:

Let P_A, P_B, P_C be the reflection of P in the midpoint of BC, CA, AB, respectively .

Since X is the center of spiral similarity that maps $CP \mapsto AB$,
so $\triangle XCP \sim \triangle XAB \Longrightarrow XC : BP_A = XC : PC = XA : AB$ . ... $(\star)$
Since $\angle P_ABA = \angle PCB + \angle CBP + \angle PBA = \angle B'XA + \angle CXB' = \angle CXA$,
so combine with $(\star)$ we get $\triangle CXA \sim \triangle P_ABA \Longrightarrow \angle BAP_A = \angle XAC$.

Similarly we can prove $\angle CBP_B = \angle YBA, \angle ACP_C = \angle ZCB$,
so AX, BY, CZ are concurrent at the isogonal conjugate of the complement of P WRT $\triangle ABC$.

Q.E.D

andria

#11 Apr 17, 2015, 12:56 pm

Another solution:let S reflection of P in the midpoint of BC then note that $\angle PXB = \angle CC'A = \angle CXA, \angle XBP = \angle XC'B = \angle XCA \longrightarrow \triangle PXB \sim \triangle CXA$ so $\frac{AC}{PB} = \frac{AX}{BX}$ $\longrightarrow \frac{AC}{CS} = \frac{AX}{BX}$ also $\angle ACS = \angle AB'B = \angle AXB$ so $\triangle ACS \sim \triangle AXB \longrightarrow \angle BAX = \angle CAS$ DONE

pi37

#12 Apr 28, 2016, 11:58 pm

Let D be the reflection of P across M, the midpoint of BC, and let E be the intersection of CX and BD. Since $PC \parallel BD$,

$$\angle XEB = \angle XCP = \angle XB'P = \angle XAB$$

so $AB'XBE$ is cyclic. Now $B'B \parallel DC$, so by Reim's theorem $ACDE$ is cyclic. Then

$$\angle XAB = \angle XEB = \angle CED = \angle CAD$$

as desired.

Quick Reply

High School Olympiads

mixtilinear excircles

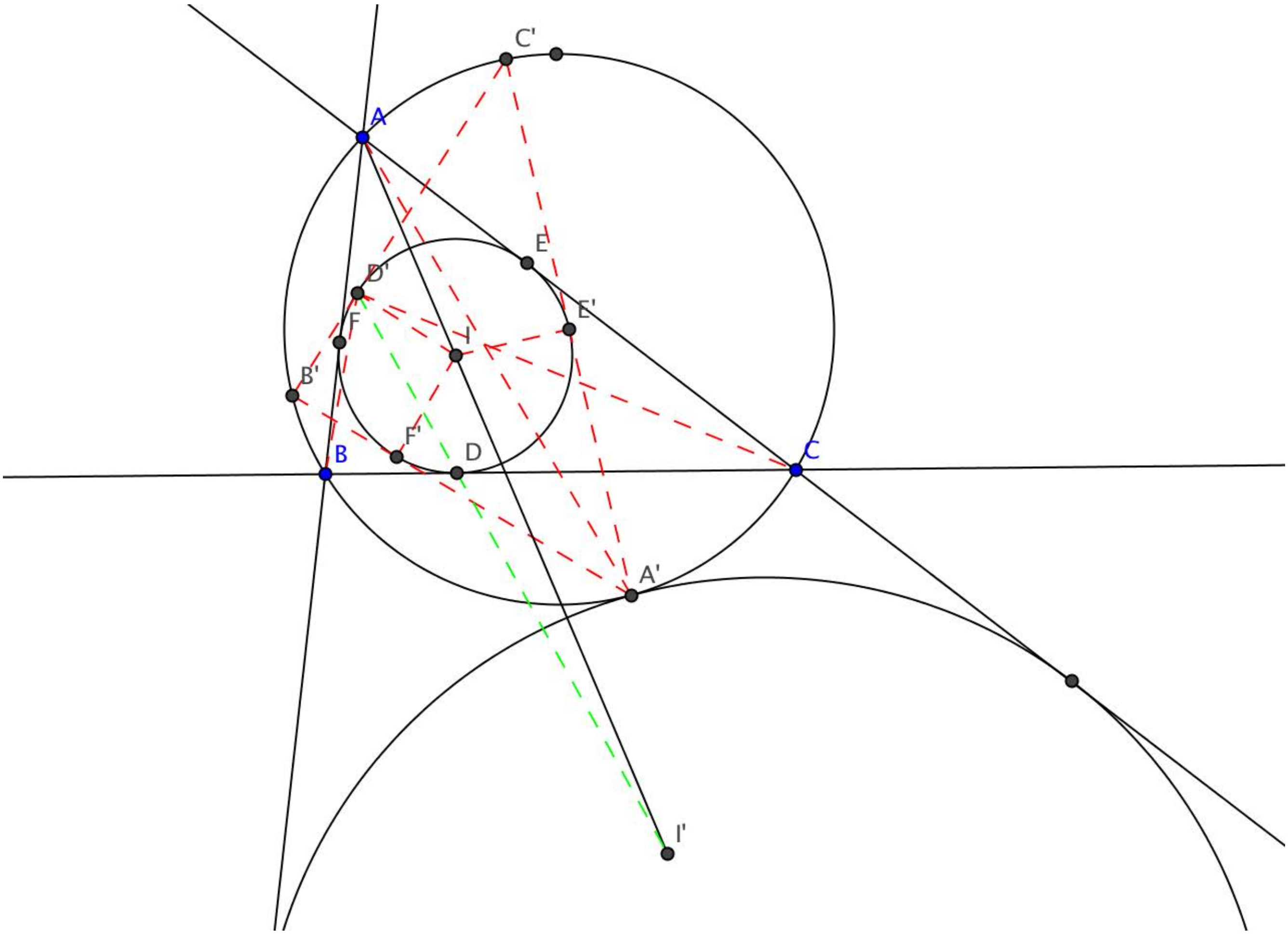
Reply

mineiraojose

#1 Jun 16, 2015, 4:04 pm • 1

Let ABC be a triangle and A' the A - mixtilinear excircle tangency point with the circumcircle of ABC. Let B', C' be points on circumcircle of ABC such that ABC and $A'B'C'$ have the same incircle γ (B', C' exist from Poncelet Theorem). Let D and D' be the tangency point of γ with BC and $B'C'$. Prove that A is the A' - mixtilinear excircle tangency point for triangle $A'B'C'$ and DD' pass through the A - excenter of ABC.

Attachments:



TelvCohl

#3 Jun 16, 2015, 5:09 pm • 2

My solution :

Let $X \equiv A'D \cap \odot(ABC), Y \equiv AD \cap \odot(ABC), Z \equiv AD' \cap \odot(ABC)$.

Since $AY \equiv AD, AA'$ are isogonal conjugate WRT $\angle BAC$ (well-known) ,
so we get $A'Y \parallel BC \Longrightarrow \odot(ADX)$ is tangent to BC at D (Reim theorem) .
From the dual of Desargue involution theorem (for ABDC) $\Longrightarrow A'(B', B, A) \longleftrightarrow A'(C', C, D)$ is an involution ,
so $BC, B'C', AX$ are concurrent at the pole P of this involution $\Longrightarrow PD'^2 = PD^2 = PA \cdot PX = PB \cdot PC$,
hence $\odot(BD'C)$ is tangent to γ at $D' \Longrightarrow DD'$ passes through A-excenter of $\triangle ABC$ (see 2002 ISL G7 and here) .

Similarly, we can prove A', Z, P are collinear $\Longrightarrow PA' \cdot PZ = PX \cdot PA = PD'^2$,
so $\odot(A'D'Z)$ is tangent to $B'C'$ at $D' \Longrightarrow A'D', A'A$ are isogonal conjugate WRT $\angle B'A'C'$,
hence we get A is the tangency point of A'-mixtilinear excircle of $\triangle A'B'C'$ with $\odot(A'B'C')$ (well-known) .

Q.E.D

Luis González

#4 Jun 16, 2015, 9:23 pm • 1

Also discussed at http://www.artofproblemsolving.com/community/c6h591004.

Quick Reply

High School Olympiads

An important lemma for another problem

Source: Own

jayme

#1 May 24, 2014, 12:29 pm

Dear Mathlinkers,

1. ABC a triangle
2. (O), (I) the circumcircle, incircle of ABC
3. DEF the contact triangle of ABC
4. A' the midpoint of the A-altitude of ABC
5. M the second point of intersection of DA' and (I)
6. X the center of homothetie between (O) and (I)
7. Ia the A-excenter of ABC
8. A+ the midpoint of the arc BAC
9. A* the second point of intersection of A+Ia and (O).

Prove : A*I and XM intersect on (O).

Sincerely
Jean-Louis

jayme

#2 May 24, 2014, 4:47 pm

Dear Mathlinkers,
any ideas?
Sincerely
Jean-Louis

Luis González

#3 Jun 8, 2014, 8:56 am • 2

Clearly A^* is the tangency point of the A-mixtilinear excircle with $(O) \Longrightarrow AD, AA^*$ are isogonals WRT $\widehat{BAC}$. Thus if A^*D cuts (O) again at Q, we have $\widehat{QAD} = \widehat{QAB} + \widehat{BAD} = \widehat{BA^*D} + \widehat{CAA^*} = \widehat{BA^*D} + \widehat{CBA^*} = \widehat{QDB} \Longrightarrow \odot(AQD)$ is tangent to BC at D $(\star)$.

Let the tangents from A^* to (I) cut (O) again at U, V. By Poncelet porism, UV touches (I) at N and by dual of Desargues theorem for the degenerate quadrilateral $ABDC$ circumscribed to (I), the pencils $A^*(A, U, B) \mapsto A^*(D, V, C)$ are involutive $\Longrightarrow B \mapsto C, U \mapsto V, A \mapsto Q$ is an involution on $(O) \Longrightarrow P \equiv BC \cap UV \cap AQ$. Using $(\star)$, we get $PB \cdot PC = PA \cdot PQ = PD^2 = PN^2 \Longrightarrow \odot(BNC)$ is tangent to (I) at N. But according to IMO Shortlist 2002 G7, the circle $\odot(MBC)$ is tangent to (I) at $M \Longrightarrow M \equiv N$. Since A^*I cuts (O) again at the midpoint L of the arc UV, then $OL \parallel IM$ (both perpendicular to UV) $\Longrightarrow ML$ goes through the exsimilicenter X of $(I) \sim (O)$, i.e. XM and A^*I intersect at a point $L \in (O)$, as desired.

Quick Reply

High School Olympiads

nice theorem related of Desargues theorem theorem

Reply

daothanhoai

#1 Jun 16, 2015, 11:38 am

Let ABC and $A_0B_0C_0$ perpective, the perpector is D. Let A_1, B_1, C_1 lie on B_0C_0, C_0A_0, A_0B_0 respectively. DA_1, DB_1, DC_1 meets BC, CA, AB at A_2, B_2, C_2. Then show that A_1, B_1, C_1 collinear if only A_2, B_2, C_2 are collinear.

http://tube.geogebra.org/material/show/id/1322479

This post has been edited 1 time. Last edited by daothanhoai, Jun 16, 2015, 11:38 am

Luis González

#2 Jun 16, 2015, 11:55 am

$\triangle ABC$ and $\triangle A_0B_0C_0$ are homologic being $\{A_1, A_2\}, \{B_1, B_2\}, \{C_1, C_2\}$ pairs of homologous points in this homology. Thus A_1, B_1, C_1 are collinear $\Longleftrightarrow A_2, B_2, C_2$ are collinear.

tranquanghuy7198

#3 Jun 16, 2015, 3:10 pm

An elementary solution with Menelaus theorem:

Lemma. In the figure below: $\frac{\overline{DK}}{\overline{KE}} = \frac{\overline{DA}}{\overline{AB}} \cdot \frac{\overline{BL}}{\overline{LC}} \cdot \frac{\overline{CA}}{\overline{AE}}$

Proof. Apply Menelaus theorem

Back to our main problem.

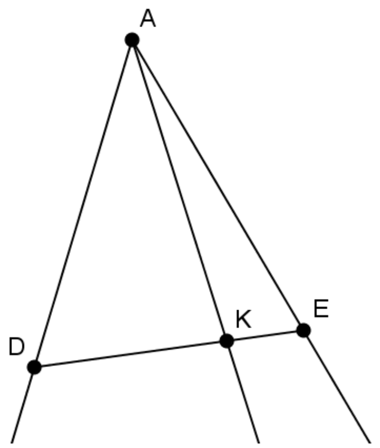
We have: $\frac{\overline{B_0A_1}}{\overline{A_1C_0}} = \frac{\overline{B_0D}}{\overline{DB}} \cdot \frac{\overline{BA_2}}{\overline{A_2C}} \cdot \frac{\overline{CD}}{\overline{DC_0}}$

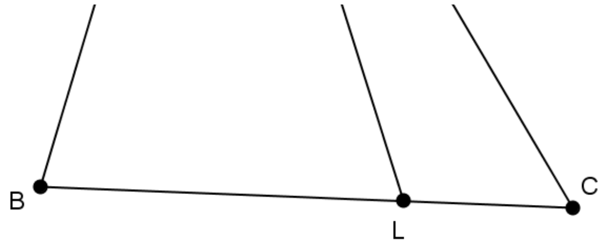
$\Rightarrow \prod \frac{\overline{B_0A_1}}{\overline{A_1C_0}} = \prod \frac{\overline{B_0D}}{\overline{DB}} \cdot \prod \frac{\overline{BA_2}}{\overline{A_2C}} \cdot \prod \frac{\overline{CD}}{\overline{DC_0}}$

$\Rightarrow \prod \frac{\overline{B_0A_1}}{\overline{A_1C_0}} = \prod \frac{\overline{BA_2}}{\overline{A_2C}}$ and the conclusion follows.

Q.E.D

Attachments:





Quick Reply

High School Olympiads

Anti-steiner point, Feuerbach point, Perpendicular

Reply

Source: Own

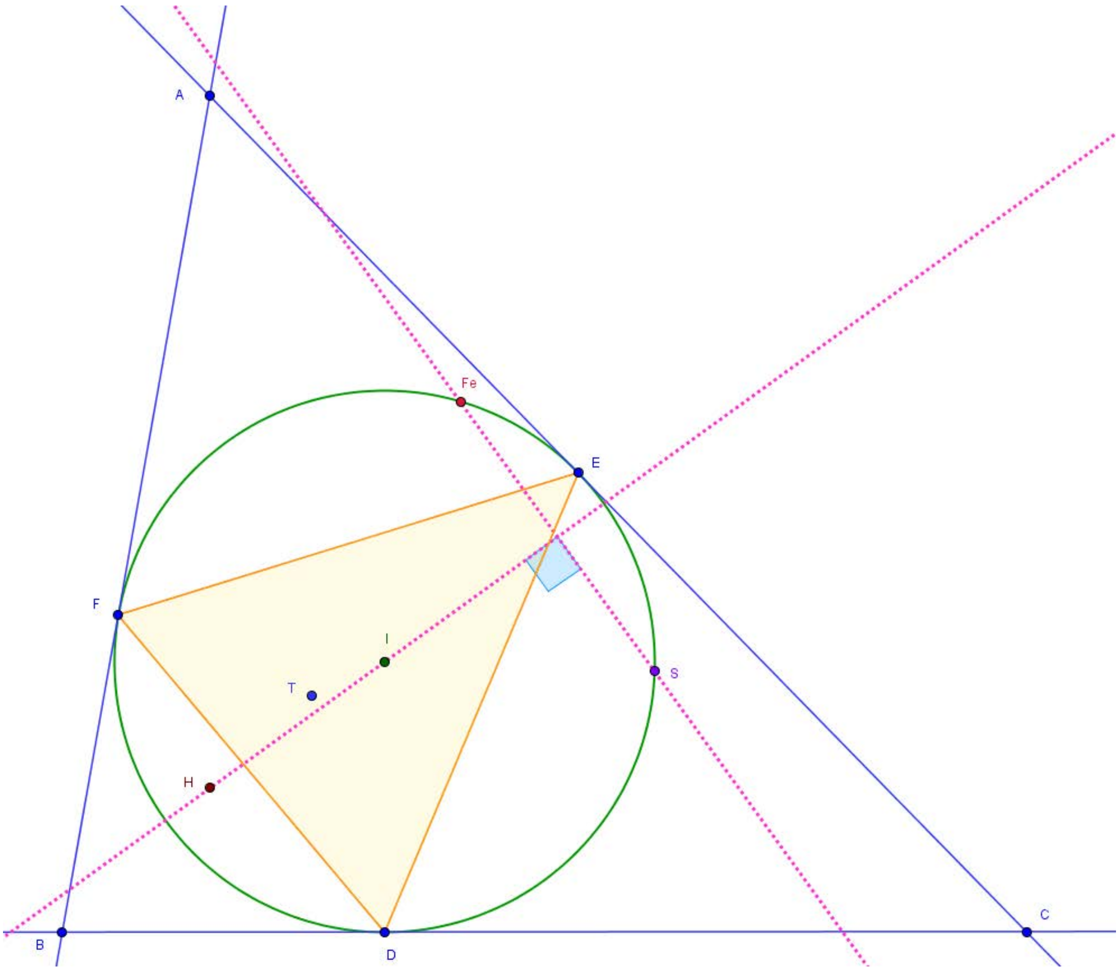
TelvCohl

#1 May 22, 2015, 9:46 pm • 4

Let I, H be the incenter, orthocenter of $\triangle ABC$, respectively .
Let $\triangle DEF$ be the intouch triangle of $\triangle ABC$ and T be the orthocenter of $\triangle DEF$.
Let Fe be the Feuerbach point of $\triangle ABC$ and S be the Anti-steiner point of TFe WRT $\triangle DEF$.

Prove that $IH \perp SFe$

Attachments:



SalaF

#2 May 23, 2015, 9:59 pm • 3

I will give a sketch of a proof; it is indeed quite long but easily manageable for those who have a good knowledge of this kind of geometry. This being said, I'll leave most of the details to the readers.

If we have a triangle ABC inscribed into (O) and ℓ is a line through O, then its orthopole K lies on the Euler circle of ABC; moreover, if we take $D = \ell \cap BC$ and respectively E, F we get that the circles with diameter AD, BE, CF concur at K and at $L \in (O)$. Clearly ℓ is the orthotransversal of L wrt ABC.
If A_0, B_0, C_0 are the midpoints of BC, CA, AB then the Steiner line of K wrt $A_0B_0C_0$ is ℓ; having called H the orthocenter of ABC the midpoints of AH, BH, CH are antipodal to A_0, B_0, C_0 in the Euler circle, so that the Steiner line of K wrt that

triangle is perpendicular to ℓ. Finally, if K_1 is the symmetric point of H in K then the Steiner line of K_1 wrt ABC is perpendicular to ℓ (just consider the homothety of center H and factor 2). So we have this

Property: If a line passing through two points $X, Y \in \odot(ABC)$ contains the orthocenter of ABC then the orthotransversal of X wrt ABC is perpendicular to the Steiner line of Y wrt ABC.

Let us take X as the Euler reflection point of ABC and let $\mathcal{P}$ be the isogonal conjugate of the orthotransversal of X; the **Property** implies that $Y \in \mathcal{P}$. We recall that the orthotransversal of a point P is perpendicular to the line through P which is tangent to the rectangular hyperbola $ABCHP$; this implies that the line through Y tangent to $\mathcal{P}$ (and so the one through H, as they're symmetric wrt the center of the hyperbola) is parallel to OH (by the **Property** with X and Y switched): in particular, the latter coincides with OH. Then the orthotransversal of X is tangent to the Jarebek hyperbola at O, and it is perpendicular to the orthotransversal of O wrt ABC.

Let Z be the anti-Steiner point of HX wrt ABC: from here (round 2-2011, problem 3) we can easily derive that the perpendicular bisector of XZ is perpendicular to the Steiner line of Y wrt ABC, and so it's parallel to the orthotransversal of O wrt ABC.

Then we can turn to the main problem: clearly Fe is the Euler reflection point of DEF so it is sufficient to show that IH is perpendicular to t, that is the orthotransversal of I (wrt DEF): but this is obvious as the inverse of H through the incircle is the projection of I on t (quite known, I think, and easy to show).

This post has been edited 1 time. Last edited by SalaF, May 23, 2015, 10:05 pm
Reason: typo fixed

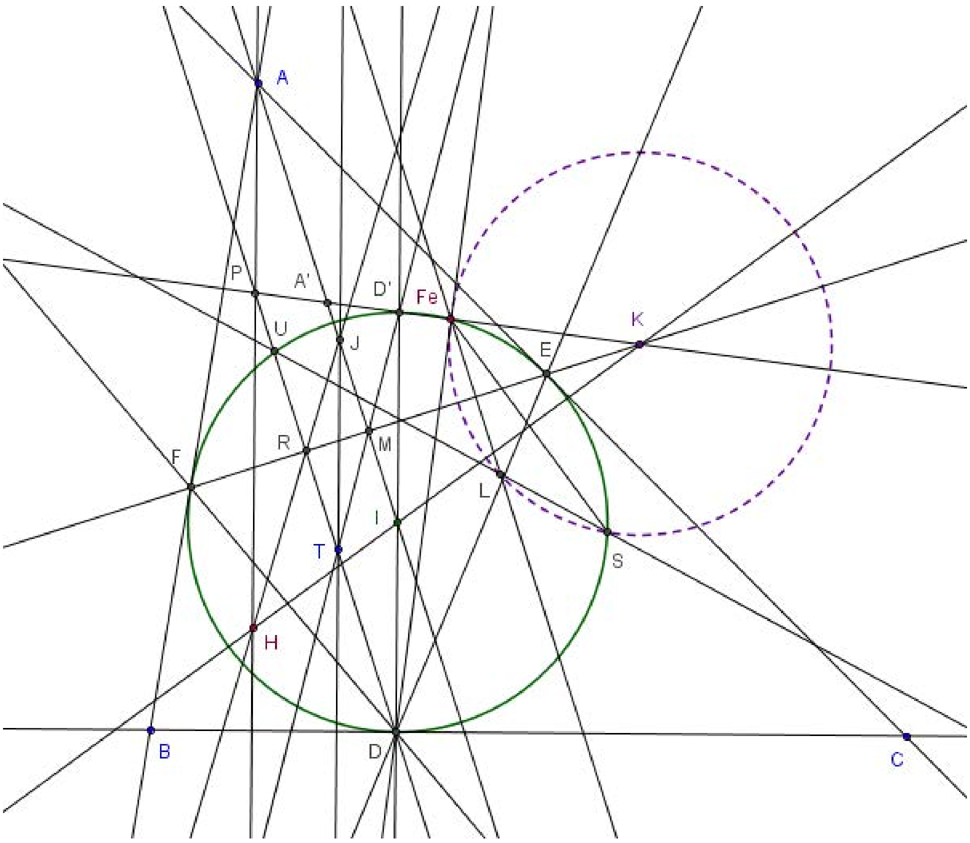
Luis González

#3 Jun 16, 2015, 3:53 am • 2

Let D' be the antipode of D on (I) and let R be the foot of the D-altitude of $\triangle DEF$. DR cuts (I) again at U; reflection of T on EF. If $L \in SU$ is the reflection of Fe on EF, we have $\angle FeSL = \angle FeDU = \angle DFeL \Longrightarrow DFe$ is tangent of $\odot(SLFe)$, hence EF and the perpendicular FeD' from Fe to DFe meet at the center K of $\odot(SLFe)$.

According to problem intersect on circle, Fe, D' and the midpoint A' of $\overline{AI}$ are collinear. Hence if $A'D'$ cuts AH at P, then $APID'$ is a parallelogram $\Longrightarrow AIDP$ is a parallelogram $\Longrightarrow (DP \parallel AI) \perp EF \Longrightarrow DP$ is D-altitude of $\triangle DEF$. If M is the midpoint of $\overline{EF}$ and J is the reflection of I on EF, then H, R, J are collinear (for a proof see post #8 at the topic Orthotransversal, precisely 1st paragraph of the proof). As a result, $\triangle HIJ$ and $\triangle PD'T$ are perspective with perspector $PH \parallel TJ \parallel ID'$ at infinity. Thus by Desargues theorem, we have $K \equiv PD' \cap RM \equiv EF \cap IH \Longrightarrow \overline{HIK}$ is perpendicular bisector of $\overline{SFe}$ or $HI \perp SFe$, as desired.

Attachments:



IDMasterz

#4 Jun 28, 2015, 6:59 pm • 1

Let D' be the antipode of D w.r.t. (I). According to lemma 1, we have the tangent to the circumrectangular hyperbola of DEF through Fe is SFe. According to here, we get that SFe is perpendicular to the orthotransversal t of Fe w.r.t. DEF.

Since Fe is the isogonal conjugate of the point at infinity perpendicular to OI, we have
$D'(D, Fe; E, F) = I(A, O; B, C) = (AIBC) \text{ (w.r.t. the hyperbola } ABCHI) = H(A, I; B, C) = I(D, H; E, F) \implies D'Fe \cap IH \in EF$
We conclude that $t \equiv IH \perp SFe$, as desired.

This post has been edited 3 times. Last edited by IDMasterz, Jun 28, 2015, 7:01 pm
Reason: clarity

Quick Reply

High School Olympiads

Orthotransversal

Reply

tranquanghuy7198
#1 May 17, 2015, 4:57 pm • 1

Let the incircle (I) of $\triangle ABC$ touch BC, CA, AB at D, E, F, resp. The orthotransversal of I WRT $\triangle ABC$ intersects BC, CA, AB at X, Y, Z, resp. Prove that the orthotransversal of I WRT $\triangle DEF$ bisects AX, BY, CZ

TelvCohl
#2 May 17, 2015, 7:27 pm • 1

My solution :

Let H be the orthocenter of $\triangle ABC$.
Let $\triangle D'E'F'$ be the orthic triangle of $\triangle DEF$.
Let $\overline{PQR}$ be the orthotransversal of I WRT $\triangle DEF$.
Let A', B', C' be the reflection of I in EF, FD, DE, respectively .
Let V^* be the image of V under the Inversion $\mathbf{I}(\odot(I))$ (for any point V) .

Since P^* is the intersection of IP with $\odot(IEF)$,
so P^* is the projection of A on IP . i.e. $P^* \equiv IP \cap AH$
Similarly, we can prove $Q^* \equiv IQ \cap BH, R^* \equiv IR \cap CH$,
so I, H, P^*, Q^*, R^* are concyclic at a circle with diameter IH .

Since the polar of X WRT $\odot(I)$ is D-altitude of $\triangle DEF$,
so X^* is the projection of I on D-altitude of $\triangle DEF \Longrightarrow A^*X^*$ is I-midline of $\triangle IA'D'$,
hence from $H \in A'D'$ (see 2014 HSGS TST (lemma)) $\Longrightarrow A^*X^*$ pass through the midpoint of IH .

Similarly, we can prove B^*Y^* and C^*Z^* pass through the midpoint of IH ,
so the perpendicular bisectors of IP^*, IQ^*, IR^* and A^*X^*, B^*Y^*, C^*Z^* are concurrent ,
hence we get $\odot(P, PI), \odot(Q, QI), \odot(R, RI), \odot(AIX), \odot(BIY), \odot(CIZ)$ are coaxial ,
so $\overline{PQR}$ pass through the center of $\odot(AIX), \odot(BIY), \odot(CIZ)$ (midpoint of AX, BY, CZ) .

Q.E.D

buratinogigle
#3 May 17, 2015, 8:05 pm • 2

Nice problem :D, I have an idea

Let P, Q be two isognal conjugate points on A-bisector of triangle ABC. E, F are projection of P on CA, AB and D is projection of Q on BC. X lies on BC such that $PX \perp PA$. Prove that orthotransversal of P wrt triangle DEF bisects AX.

THVSH
#4 May 17, 2015, 8:46 pm • 2

On the same way with **buratinogigle**, we can generalize the lemma in 2014 HSGS TST

Let P, Q be two isogonal conjugate points on A-bisector of triangle ABC. E, F are projection of P on CA, AB and D is projection of Q on BC. G is the projection of D on EF. H is the orthocenter of $\triangle ABC$. Prove that GD is the bisector of $\angle PGH$.

buratinogigle
#5 May 17, 2015, 9:09 pm

Yes nice dear THVSH, I known it before, I try to find that problem with two arbitrary isogonal conjgate point.

THVSH
#6 May 17, 2015, 9:11 pm • 1

> THVSH wrote:
> On the same way with **buratinogigle**, we can generalize the lemma in 2014 HSGS TST
>
> Let P,Q be two isogonal conjugate points on A-bisector of triangle ABC. E,F are projection of P on CA,AB and D is projection of Q on BC. G is the projection of D on EF. H is the orthocenter of $\triangle ABC$. Prove that GD is the bisector of $\angle PGH$.

We need to prove that AD,BE,CF are concurrent, which is easily got by Ceva's theorem.
And then, **TelvCohl**'s proof in post #4 in 2014 HSGS TST still works with this problem!

tranquanghuy7198
#7 May 17, 2015, 9:12 pm • 1

Thanks Telv for your solution, which is the same as mine. Thanks Mr.Buratino and THVSH for great ideas. And I'm very happy that the solution for the original problem still works (but there are a bit difference, and all of you have found)

This post has been edited 1 time. Last edited by tranquanghuy7198, May 17, 2015, 9:39 pm

Luis González
#8 May 18, 2015, 2:16 am

> tranquanghuy7198 wrote:
> Let the incircle (I) of $\triangle ABC$ touch BC,CA,AB at D,E,F, resp. The orthotransversal of I WRT $\triangle ABC$ intersects BC,CA,AB at X,Y,Z, resp. Prove that the orthotransversal of I WRT $\triangle DEF$ bisects AX,BY,CZ

Let H,T be the orthocenters of $\triangle ABC,\triangle DEF$. DT cuts EF at U and cuts (I) again at S. If M is the midpoint of AH and (N) is the 9-point circle of $\triangle ABC$, we clearly have $NM \parallel IS \Longrightarrow MS$ goes through the exsimilicenter Fe of $(I) \sim (N)$; Feuerbach point of $\triangle ABC$ and it's known that Fe is the anti-Steiner point of IT WRT $\triangle DEF \Longrightarrow$ reflection V of I of EF is on MS. Since $VT \parallel ID \parallel AH$, we have $V(A,H,M,T)=-1$, but $V(A,U,M,T)=-1 \Longrightarrow V,U,H$ are collinear. Hence if J is the projection of I on DU and D' is the midpoint of EF, then $D'J$, parallel to VU, cuts IH at its midpoint P.

Let the perpendicular to ID at I cut EF at X' and Y',Z' are defined similarly $\Longrightarrow \overline{X'Y'Z'}$ is orthotransversal of I WRT $\triangle DEF$. Inversion WRT (I) takes $\odot(AIX)$ into $D'J$ and X',Y',Z' into the projections X'',Y'',Z'' of I on $AH,BH,CH \Longrightarrow \overline{X'Y'Z'}$ goes to circle (P) with diameter $\overline{IH}$. Since $D'J$ goes through its center P, then by conformity $\overline{X'Y'Z'}$ goes through the center of $\odot(AIX)$, i.e. the midpoint of $\overline{AX}$.

drmzjoseph
#9 May 18, 2015, 3:49 am

> tranquanghuy7198 wrote:
> Let the incircle (I) of $\triangle ABC$ touch BC,CA,AB at D,E,F, resp. The orthotransversal of I WRT $\triangle ABC$ intersects BC,CA,AB at X,Y,Z, resp. Prove that the orthotransversal of I WRT $\triangle DEF$ bisects AX,BY,CZ

$N\equiv EF\cap AX$.Let $M\in EF$ be such that $MD\perp EF \Rightarrow MD$ is the polar of X WRT $\odot(I) \Rightarrow XA$ is the polar of M WRT $\odot(I) \Rightarrow (E,F,M,N)=-1$ Let $\mathcal{L}$ the perpendicular bisector of $AI \Rightarrow$ the midpoint A^* of AX belongs $\mathcal{L},B_1\equiv \mathcal{L}\cap AE, C_1\equiv \mathcal{L}\cap AF, Q\equiv C_1I\cap FD, R\equiv B_1I\cap ED$ Since $B_1I\parallel AF$ we get RQ is thee orthotransversal of I WRT $\triangle DEF$

$\dfrac{A^*C_1}{A^*B_1}=\dfrac{NF}{NE}=\dfrac{MF}{ME}$, and in the $\triangle IB_1E$ we get $\dfrac{B_1R}{IR}=\dfrac{B_1E}{IE}\times\cot\angle IED=\dfrac{B_1E}{IE}\times\cot\angle FDM$ analogously

$$\frac{IQ}{C_1Q}=\frac{IF}{FC_1}\times\tan\angle IFD=\frac{IF}{FC_1}\times\tan\angle MDE\Rightarrow\frac{A^*C_1}{A^*B_1}\times\frac{B_1R}{IR}\times\frac{IQ}{C_1Q}=\frac{MF}{ME}\times\cot\angle FDM\times\tan\angle MDE=1$$

By Menalus' theorem in $\triangle C_1B_1I-R,Q,A^*$ we get $A^*\in RQ$ as desired.

This post has been edited 1 time. Last edited by dmmzjoseph, May 18, 2015, 3:50 am

Quick Reply

High School Olympiads

Two circles & angle bisectors

Source: Germany 2013, Grade 11/12, Round 3 - #5

Kezer

#1 Jun 16, 2015, 12:31 am

Let two circles k_1 and k_2 intersect in two distinct points $P \neq Q$. A line intersects k_1 in A, C and k_2 in B, D, such that B is between A and C and C is between B and D.
Prove: If the segment PC is an internal angle bisector of triangle DPB, then segment QA is an external angle bisector of triangle BQD.

Luis González

#2 Jun 16, 2015, 1:26 am • 1

PC cuts k_2 again at the midpoint M of its arc BD. Since $\angle(DB, DM) = \angle(PM, PD)$, then $\angle(CD, CP) = \angle(DM, DP) = \angle(QM, QP)$. Thus since $APCQ$ is cyclic, it follows that A, Q, M are collinear $\Longrightarrow$ QA is external bisector of $\angle BQD$.

MillenniumFalcon

#3 Jun 16, 2015, 10:37 am

Can I just add my proof for the last sentence: Draw N diametrically opposite to M on the circle k2. Since M is the midpoint of arc BD, N should also be the midpoint of the arc (but on the other side), so NQ bisects BQD. Also angle MQN=90 degree because it's a diameter, so AQ is perpendicular to the internal angle bisector, so it is the external angle bisector

High School Olympiads

A metrical relation in a trapezoid.

Reply

Virgil Nicula

#1 Jun 15, 2015, 10:59 pm

PP4. A trapezoid $ABCD\ ,\ AD\parallel BC$ with $I\in AC\cap BD\ ;\ P\in (AB)\ ,\ R\in (CD)$ so that $I\in PR\ ;\ S\in AR\cap BD$. Prove that $CS\parallel AB\iff \frac {PB}{PA}+\frac {BC}{AD}=1$.

Luis González

#2 Jun 16, 2015, 12:02 am

Assume that $CS \parallel AB$. CS cuts AD at E and BE cuts AC,PR at $M,G,$ resp. $AECB$ is parallelogram with diagonal intersection $M \Longrightarrow M$ is midpoint of $\overline{AC}$ and from the complete $CRSI,$ it follows that $(M,B,G,E)=I(C,S,R,E)=-1 \Longrightarrow \overline{GM}:\overline{GB}=-\overline{EM}:\overline{EB}=-1:2 \Longrightarrow G$ is centroid of $\triangle ABC \Longrightarrow$ $\tfrac{PB}{PA}+\tfrac{IC}{IA}=1$ (Cristea's theorem) $\Longrightarrow \tfrac{PB}{PA}+\tfrac{BC}{AD}=1.$

The converse is taken for granted by the uniqueness of $S \in BD$ and $P \in AB.$

drmzjoseph

#3 Jun 16, 2015, 12:22 am

$X\equiv AB\cap CD, CS\parallel AB\iff(\infty,B,A,X)=(S,B,I,D)=R(A,B,P,X)$
$\iff \frac{BX}{AB}=\frac{AP}{BP}\times \frac{BX}{AX}\iff \frac{BP}{AP}=\frac{AB}{AX}=1-\frac{BC}{AD}$

Virgil Nicula

#4 Jun 16, 2015, 10:01 pm

See **PP4** (with my proof) from **here**.

This post has been edited 1 time. Last edited by Virgil Nicula, Jun 16, 2015, 10:03 pm

Quick Reply

High School Olympiads

Collinearity and points on the circumcenter

Reply

premiumsound
#1 Jul 27, 2011, 12:33 pm

Let M be a point on the cicumcircle of $\triangle ABC$. Given a point R, denote by A_1, B_1, C_1 the intersection points of the lines AR, BR, CR with the circumcircle. Prove that the intersection points of the lines MA_1 and BC, MB_1 and CA, MC_1 and AB are collinear.

Luis González
#2 Jul 27, 2011, 8:44 pm

This is a particular case of the following configuration (discussed before):

Proposition. P, Q are two arbitrary points in the plane of $\triangle ABC$ with circumcircle (O). PA, PB, PC cut (O) again at P_1, P_2, P_3 and QA, QB, QC cut (O) again at Q_1, Q_2, Q_3. P_1Q_1, P_2Q_2, P_3Q_3 cut BC, CA, AB at A_1, B_1, C_1. Then A_1, B_1, C_1 are collinear.

$$\frac{A_1B}{A_1C} = \frac{BP_1}{CP_1} \cdot \frac{\sin \widehat{A_1P_1B}}{\sin \widehat{A_1P_1C}} = \frac{\sin \widehat{BAP_1}}{\sin \widehat{CAP_1}} \cdot \frac{\sin \widehat{BAQ_1}}{\sin \widehat{CAQ_1}}$$

Multiplying cyclic expressions together, keeping in mind that

$$\frac{\sin \widehat{BAP_1}}{\sin \widehat{CAP_1}} \cdot \frac{\sin \widehat{CBP_2}}{\sin \widehat{ABP_2}} \cdot \frac{\sin \widehat{ACP_3}}{\sin \widehat{BCP_3}} = 1 \ , \ \frac{\sin \widehat{BAQ_1}}{\sin \widehat{CAQ_1}} \cdot \frac{\sin \widehat{CBQ_2}}{\sin \widehat{ABQ_2}} \cdot \frac{\sin \widehat{ACQ_3}}{\sin \widehat{BCQ_3}} = 1$$

We get $\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1 \Longrightarrow A_1, B_1, C_1$ are collinear.

Bigwood
#3 Jul 1, 2012, 10:04 am

I am not sure if it is correct, as I managed to come up with extremely simple solution.
Let the intersections A_2, B_2, C_2. Pascal's theorem wrt B_1AC_1BMC implies B_2, C_2, P collinear. Similarly A_2, C_2, P collinear. Thus we get the conclusion.

jayme
#4 Jul 1, 2012, 1:16 pm

Dear Mathlinkers,
for the proof and more, see also

http://perso.orange.fr/jl.ayme vol. 3 La P-transversale de Q

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Prove JM=JN

Reply

Source: VMF

kaito_shinichi

#1 Jun 15, 2015, 1:19 am

Given triangle ABC inscribed (O). J is center of A- excircle. (J) touches BC at D. $AJ \cap (O) \equiv E$. A line passes through J and perp with OJ intersect AO, DE at M, N, respectively. Prove that $JM = JN$

This post has been edited 1 time. Last edited by kaito_shinichi, Jun 15, 2015, 2:19 am

Luis González

#2 Jun 15, 2015, 2:49 am • 2

If ED cuts (O) again at X, then $\angle AXJ = 90^\circ$ (this is an extraversion of the problem incenter I and touches BC side with D) $\Longrightarrow JX$ passes through the antipode Y of A on (O). Thus, by Butterfly theorem for the complete cylic $AEYX$, we deduce that J is midpoint of MN.

Quick Reply

High School Olympiads

geometry

Reply

JimMoriarty

#1 Jun 14, 2015, 11:18 pm

In ABC AD is height,I is incenter ofABC and the incircle of ABC cuts AD at X such that X,I and P are collinear. M,P is midpoint of minor arc BC and BC resptively.BK,CN are heights of BIM,CIM respectively.Prove $IK=MN$

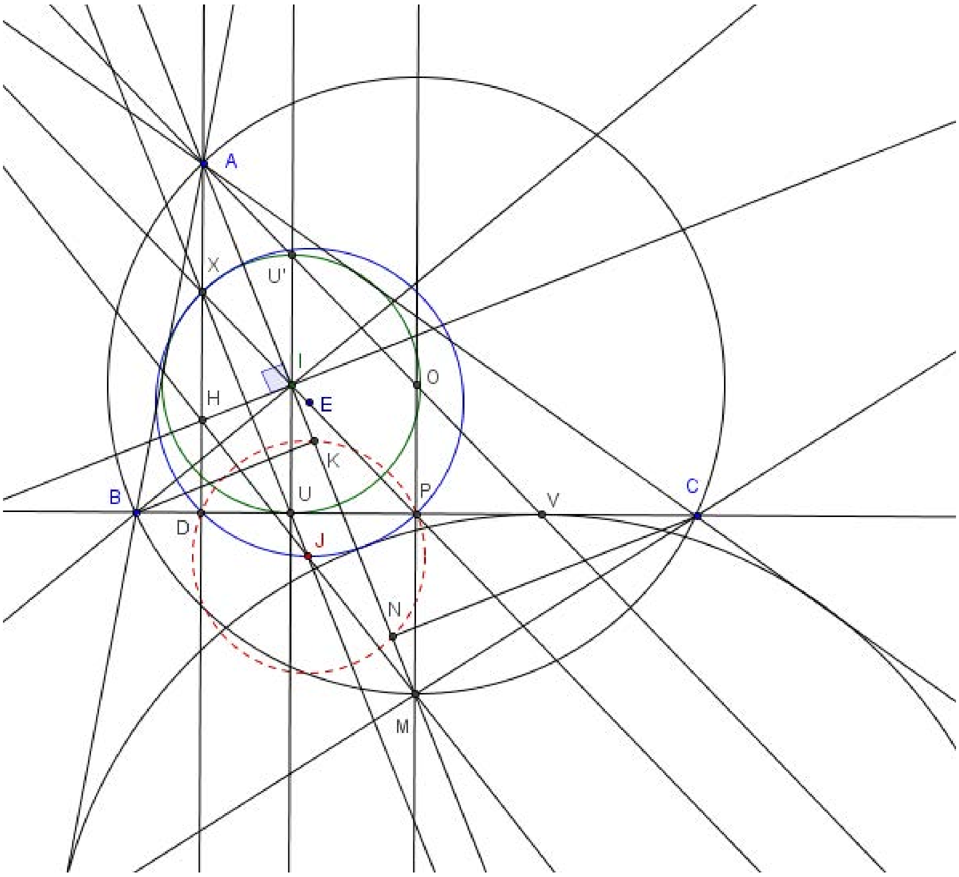
Luis González

#2 Jun 15, 2015, 12:48 am

Let O,H be the circumcenter and orthocenter of $\triangle ABC.$ Incircle (I) and A-excircle touch BC at $U,V,$ resp and U' is the antipode of U on $(I).$ Since A,V,U' are collinear and P is also midpoint of $UV,$ then $AU'IX$ is a rhombus whose sides equal the iradius $IU=r \Longrightarrow AU'$ is the reflection of $AX \equiv AH$ on $AI \Longrightarrow O \in AU'V$ and $OPXA$ is then a parallelogram $\Longrightarrow OP=AX=\tfrac{1}{2}AH \Longrightarrow XH=r=IU \Longrightarrow IXHU$ is a rhombus. Since $AIUX$ is parallelogram, then $(XU \parallel AI) \perp IH.$

Since $PIX \parallel AO,$ then PX is diameter of the 9-point circle $(E).$ Thus XU is internal bisector of $\angle DXP$ cutting the arc DP of (E) at its midpoint $J,$ which is circumcenter of $\triangle DKN$ (see 9-points circle of ABC, concyclic points in a triangle and elsewhere). Since (E) is image of (O) under homothethy $(H,\tfrac{1}{2}),$ then J is midpoint of $HM \Longrightarrow J$ is circumcenter of the right $\triangle MIH,$ hence we have $JN=JK$ and $JM=JI \Longrightarrow K,N$ are isotomic points WRT $IM,$ or $IK=MN.$

Attachments:



TelvCohl

#3 Jun 15, 2015, 2:59 pm

My solution :

Let $B^* \equiv BK \cap CA, C^* \equiv CN \cap AB$.
Let $T \equiv \odot(I) \cap BC$ and T^* be the antipode of T in $\odot(I)$.

From $BK = B^*K, CN = C^*N \Longrightarrow PK \parallel CA, PN \parallel AB$,
so $\angle NKP = \angle MAC = \angle MAB = \angle PNK \Longrightarrow PK = PN$ . ... (1)

Since A-Nagel line AT^* passes through the reflection S of T in P ,
so IP is T-midline of $\triangle TT^*S \Longrightarrow XI \parallel AT^* \Longrightarrow AXIT^*$ is a parallelogram ,
hence $AX = T^*I = XI \Longrightarrow \triangle AXI$ is an isosceles triangle $\Longrightarrow PI = PM$ . ... (2)

From (1) and (2) $\Longrightarrow IK = MN$.

Q.E.D

Quick Reply

High School Olympiads

9-points circle of ABC

Reply

Amir Hossein

#1 Aug 27, 2010, 2:17 pm

in $\triangle ABC$ let AH be the altitude from A to side BC and let M, N be images of B, C on the bisector of A, respectively. Prove that circumcenter of $\triangle HMN$ lies on 9-points circle of $\triangle ABC$.

mahanmath

#2 Aug 27, 2010, 3:08 pm

@amparvardi

Amir Hossein

#3 Aug 27, 2010, 3:10 pm

> mahanmath wrote:
> @amparvardi

@mahanmath

jayme

#4 Aug 27, 2010, 3:45 pm

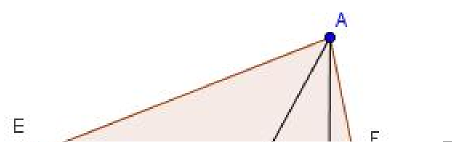
Dear Mathlinkers,
this is the circle of Calabre...
A proof can be seen on
http://perso.orange.fr/jl.ayme vol. 4 Symétrique de (OI)... p. 6-7 in French
or
Ayme J.-L., Revistaoim (Espagne) 26 (2006) ; /http://www.campus-oei.org/oim/revistaoim/ in Spain
Sincerely
Jean-Louis

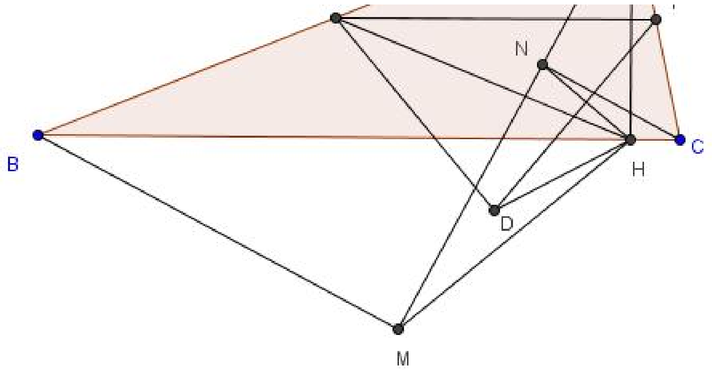
tuanh208

#5 Aug 27, 2010, 4:20 pm

If $AB = AC$ then $D \equiv H$ and the problem is proved
If $AB \neq AC$ then suppose that $AB > AC$
Let E, F are midpoints of AB and AC respectively. We just need to prove that $EFHD$ is cyclic.
We have $\widehat{ANC} = \widehat{AHC} = 90^0$ so $FN = FH$ but $DN = DH$ hence $FD \perp HN$
So $\widehat{FDH} = \frac{1}{2}\widehat{NDH} = \widehat{AMH} = \widehat{ABH}(AMBH$ is cyclic$) = \widehat{FEA}$
But we have $EA = EH, FA = FH$ so $\widehat{FEA} = \widehat{FEH}$
Thus $\widehat{FDH} = \widehat{FEH}$ and $EFHD$ is cyclic.

Attachments:





This post has been edited 1 time. Last edited by tuanh208, Aug 27, 2010, 4:24 pm

Amir Hossein

#6 Aug 27, 2010, 4:22 pm

Thank you :)

Quick Reply

High School Olympiads

concyclic points in a triangle

Reply

Source: BW 95

Maverick

#1 Jul 11, 2004, 2:00 pm

Let M be the midpoint of the side AC of a triangle ABC and let H be the footpoint of the altitude from B. Let P and Q be orthogonal projections of A and C on the bisector of the angle B. Prove that the four points H,P,M and Q lie on the same circle

grobber

#2 Jul 11, 2004, 4:01 pm • 1

I don't think the solution will have all the necessary details, but here are main ideas:

Let I^* be the projection of I (the incenter) on AC. Try to show that HQP is similar to BCA (I don't know if this is necessary, bu it gives some insight), and that I^* is the incenter of HQP. Show that M is equidistant from $Q,P \Rightarrow M$ is the point where the angle bisector of $\angle QHP$ cuts the circumcircle of $HQP \Rightarrow MI^*=MQ(=MP)$. If B^* is the point where the bisector of $\angle B$ cuts AC, then the fact that $MQHP$ is cyclic is equivalent to $MI^*^2=MB^*\cdot MH$, and this really is well-known (it's also easy to prove).

darij grinberg

#3 Jul 12, 2004, 9:04 pm

I haven't got the time to read your solution right now, but I think I have a simpler one.

If the line CQ meets AB at R, then the triangles RQB and CQB are congruent (since BQ = BQ, < RBQ = < CBQ, < RQB = < CQB = 90°); hence, RQ = CQ, and Q is the midpoint of the segment CR. On the other hand, M is the midpoint of the segment CA. Hence MQ || RA, or, in other words, MQ || AB. Thus, if the angle bisector of the angle ABC meets the line CA at X, we have XM / XQ = XA / XB. But on the other hand, since < AHB = < APB = 90°, the points A, B, H and P lie on one circle, and thus by the intersecting chords theorem we have $XH\cdot XA=XP\cdot XB$, thus XA / XB = XP / XH. Together with the former equality XM / XQ = XA / XB, this entails XM / XQ = XP / XH, so that $XM\cdot XH=XP\cdot XQ$. Now, by the converse of the intersecting chords theorem, the points M, H, P and Q are concyclic. OED (maybe you know I'm in Greece right now).

darij

fuzzylogic

#4 Jul 13, 2004, 2:26 am

Nice Proof! This can be done by just chasing angles.

> darij grinberg wrote:
> ... in other words, MQ || AB. ...

After noting that MQ || AB, we could go like this:

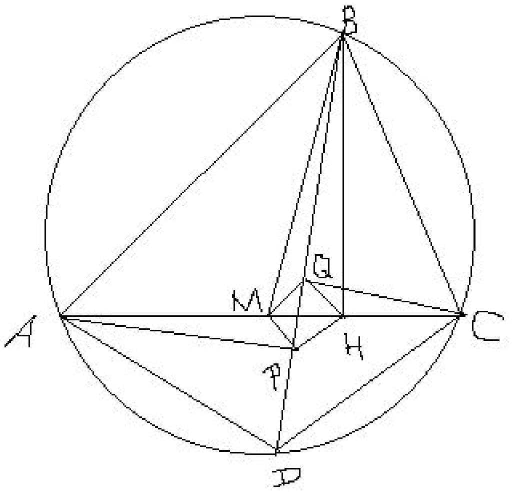
:ang: MQP = :ang: ABP = :ang: AHP (:ang: MHP) since A,B,H,P are concyclic. Hence M,Q,H,P are concyclic.

It's interesting to note that:

The triangle PMQ is isosceles.

The quadrilateral PHQM is similar to the quadrilateral ABCD where D is the intersection of the bisector of :ang: ABC and the circumcircle of ABC.

Attachments:



fuzzylogic

#5 Jul 13, 2004, 7:48 am

Another Proof:

Refering to the diagram above. MD is the perpendicular bisector of AC (An easy fact being used for #1 of IMO2004, what a coincidence 🙂)

:ang: AMD= :ang: APD=90 :^o: ==> A,M,P,D are concyclic ==> :ang: PMH= :ang: ADB

:ang: BQC= :ang: BHC=90 :^o: ==> B,Q,H,C are concyclic ==> :ang: PQH= :ang: ACB

Hence :ang: PMH= :ang: PQH since :ang: ADB= :ang: ACB.

Therefore P,M,Q,H are concyclic.

QED.

No Reason

#6 Feb 7, 2009, 8:13 pm

Let $CQ \cap AB = R$.Then BRC is isosceles triangle.From this we have Q is midpoint of CR.It follows that QM || AB.After that is just angle-chasing.

jayme

#7 Feb 8, 2009, 4:40 pm

Dear Mathlinkers,
this nice circle is from Calabre.
A proof can be see on my website : http://perso.orange.fr/jl.ayme vol. 1 Le théorème de Feuerbach Exemple 1. (in French)
or in spain : http://www.oei.es/oim/revistaoim/numero26.htm el theorema de Feuerbach
Sincerely
Jean-Louis

Luis González

#8 Feb 16, 2009, 8:54 am

> *Maverick wrote:*
>
> Let M be the midpoint of the side AC of a triangle ABC and let H be the footpoint of the altitude from B. Let P and Q be orthogonal projections of A and C on the bisector of the angle B. Prove that the four points H,P,M and Q lie on the same circle

Furthermore, the center of this circle lies on the 9 point circle of ABC.

dgreenb801

#9 Aug 24, 2009, 1:28 am

Assume WLOG H is between A and M.
Extend AP to meet BC at R, extend CQ to meet AB at S. Then $\triangle ARB$ and $\triangle SBC$ are isoceles, so $AS = RC = a - c$ and since M is the midpoint of AC, P the midpoint of AR, $PM = \frac{a-c}{2}$, similarly $QM = \frac{a-c}{2}$. So $\triangle$ PMQ is isoceles, so PHQM is cyclic if $\angle PHM = \angle PQM$, or $\angle PHM = \angle XPM$, or $\triangle MPX \sim \triangle MPH$, or $MX \cdot MH = MP^2$. But $MH = MA - HA = \frac{b}{2} - c \cdot cosA = \frac{b}{2} - \frac{b^2+c^2-a^2}{2b}$. Also $MX = MA - XA = \frac{b}{2} - \frac{bc}{a+c}$. So we have to show
$(\frac{b}{2} - \frac{b^2+c^2-a^2}{2b})(\frac{b}{2} - \frac{bc}{a+c}) = (\frac{a-c}{2})^2$
Which is easy to verify.

Quick Reply

High School Olympiads

concurrent lines

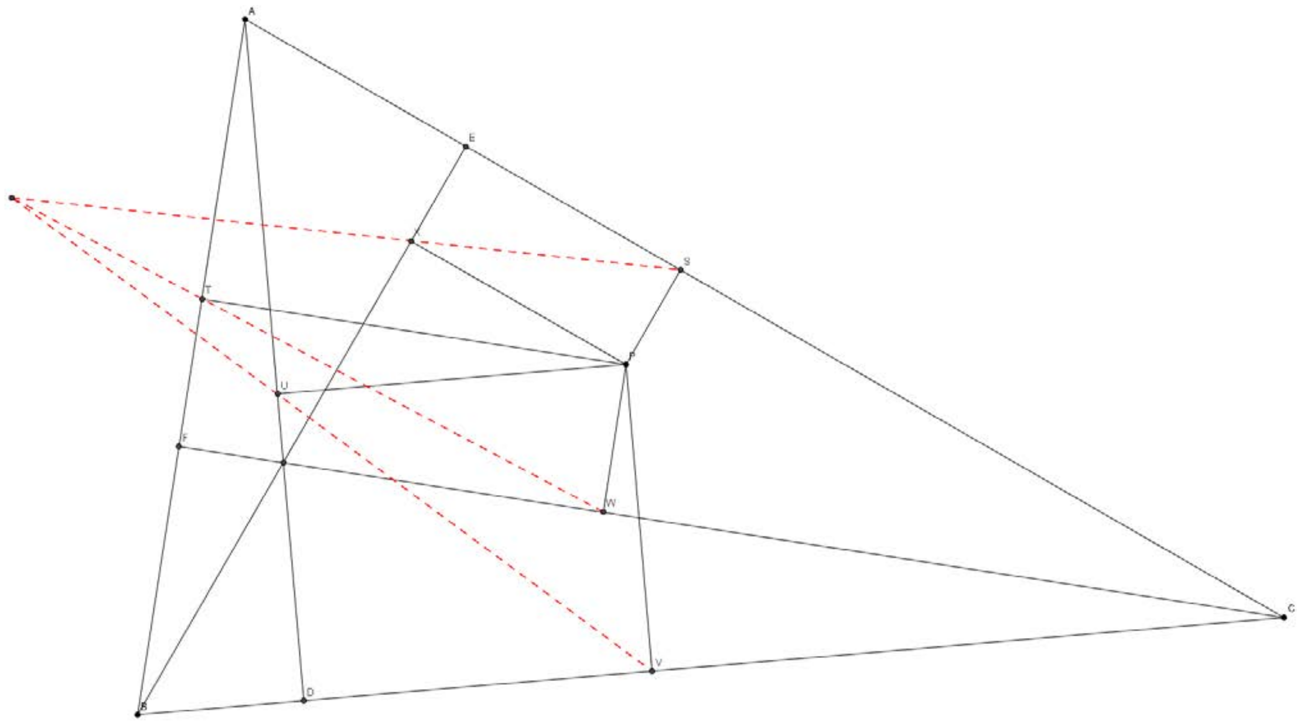
Reply

andria

#1 Jun 14, 2015, 11:11 pm

In $\triangle ABC$ with altitudes AD, BE, CF,P is an arbitrary point in a plain of $\triangle ABC$ let S, T, X, W, U, V are projections of P on AC, AB, BE, CF, AD, BC prove that UV, WT, SX are concurrent.

Attachments:



Luis González

#2 Jun 14, 2015, 11:17 pm

Posted before at http://www.artofproblemsolving.com/community/c6h378378.

Quick Reply

High School Olympiads

mamangava12345678
#1 Jun 14, 2015, 2:07 am

A scalene triangle ABC is inscribed within circumference w. Tangent to the circumference at point C intersects line AB at point D. Let I be the center of the circumference inscribed within triangle ABC. Lines AI and BI intersect the bisector of angle CDB in points Q and P, respectively. Let M be the midpoint of QP. Prove that MI passes through the middle of arc ACB of circumference w.

Luis González
#2 Jun 14, 2015, 3:31 am

WLOG we assume that $CB > CA$. We have $\widehat{QDA} = \frac{1}{2}\widehat{CDA} = \frac{1}{2}(\widehat{A} - \widehat{B}) \Longrightarrow$ $\widehat{DQA} = \widehat{IAB} - \widehat{QDA} = \frac{1}{2}\widehat{A} - \frac{1}{2}(\widehat{A} - \widehat{B}) = \frac{1}{2}\widehat{B} = \widehat{PBA} \Longrightarrow APQB$ is cyclic, i.e. PQ is antiparallel to AB WRT $IA, IB \Longrightarrow$ I-median IM of $\triangle IPQ$ is the I-symmedian of $\triangle IAB$. But if L is the midpoint of the arc ACB, then IL is precisely the I-symmedian of $\triangle IAB$, because LA, LB are tangents of $\odot(IAB)$, due to $\widehat{LBA} = \widehat{LAB} = \widehat{PIA}$. Hence I, M, L are collinear.

andria
#3 Jun 14, 2015, 4:37 am

See http://www.artofproblemsolving.com/community/q3h1093525p4881175

TelvCohl
#5 Jun 15, 2015, 2:47 pm • 1

My solution :

Let I_a, I_b be the A-excenter, B-excenter of $\triangle ABC$, respectively .
Let $R \equiv CI \cap AB$ and X be the midpoint of arc ACB in $\odot(ABC)$.

From $\angle DCR = \angle CRD \Longrightarrow DC = DR \Longrightarrow PQ \perp CI \Longrightarrow PQ \parallel I_bI_a$,
so the I-median IX of $\triangle II_aI_b$ passes through the midpoint of PQ . i.e. I, M, X are collinear

Q.E.D

PROF65
#6 Jun 16, 2015, 4:49 am

An other solutution !

Q is the D-excenter wrt DAC then $\widehat{PQA} = \frac{\widehat{ACD}}{2} = \frac{\widehat{CBA}}{2} = \widehat{B'A'A}$ where A', B' midpoints of minor arcs $\widehat{BC}, \widehat{CA}$ resp. thus $PQ \parallel A'B'$ but we know that LI pass through the midpoint of XY and so through the midpoint of $A'B'$ where L, X, Y touch-points of C-mixtlinear with the circle and sides resp.
WCP

sunken rock
#8 Jun 18, 2015, 1:48 am

Remark (easy): **The circles $\odot(ABC)$ and $\odot(PIQ)$ are tangent.**

Best regards,
sunken rock

Quick Reply

High School Olympiads

Two mixtilinear incircles : two parallels again

Reply

jayme
#1 Jun 13, 2015, 6:42 pm

Dear Mathlinkers,

1. ABC a triangle
2. (O) the circumcircle of ABC
3. 1b the B-mixtilinear incircle of ABC
3. D, E, F the points of contact of 1b wrt AB, BC, (O)
4. 1c the C-mixtilinear incircle of ABC
5. P, Q, R the points of contact of 1c wrt BC, CA, (O)
6. U, V the feet of the F, R-inner bisectors wrt triangles FPQ, RDE.

Prove : UV is parallel to BC.

Sincerely
Jean-Louis

Luis González
#2 Jun 13, 2015, 11:00 pm

This is a consequence of the problem http://www.artofproblemsolving.com/community/c6h479272. By angle bisector theorem, we get $\frac{PU}{UQ}=\frac{FP}{FQ}=\frac{RE}{RD}=\frac{EV}{VD}$. But since $DQ \parallel BC$ ($PEQD$ is parallelogram with center the incenter of ABC), then $UV \parallel BC.$

jayme
#3 Jun 14, 2015, 12:30 pm

Dear Luis and Mathlinkers,
thank for your answer...
I was coming to this problem after reading your proof in the above link... I was just waiting for a new approach without trigonometry
Sincerely
Jean-Louis

This post has been edited 1 time. Last edited by jayme, Jun 14, 2015, 12:31 pm
Reason: more precision

Quick Reply

High School Olympiads

The two ratios are equal

Reply

Fang-jh

#1 May 12, 2012, 6:30 pm • 1

Given a triangle ABC. Let its B-mixtilinear incircle touch AB, BC, and its circumcircle at D, E, and F, respectively. Let C-mixtilinear incircle touch BC, CA, and the circumcircle at P, Q, and R, respectively. Prove that $\frac{PF}{RE} = \frac{QF}{RD}$.

Luis González

#2 Jun 7, 2012, 2:45 am • 1

Let the A-mixtilinear incircle touch the circumcircle (O) of $\triangle ABC$ at T. Then $\triangle TFR$ is circumcevian triangle of the isogonal conjugate of the Nagel point N of $\triangle ABC$. In addition, FQ, FT are isogonals WRT $\angle AFC$ and RD, RT are isogonals WRT $\angle ARB$ (For a proof see On mixtilinear incircles 2). Therefore, we have

$$FQ \cdot FT = FA \cdot FC \ , \ RD \cdot RT = RA \cdot RB \Longrightarrow$$

$$\frac{FQ}{RD} = \frac{FC}{RB} \cdot \frac{FA}{RA} \cdot \frac{RT}{FT} = \frac{FC}{RB} \cdot \frac{\sin \widehat{NBC}}{\sin \widehat{NCB}} \cdot \frac{\sin \widehat{NCA}}{\sin \widehat{NBA}} \cdot \frac{b}{c} =$$

$$= \frac{FC}{RB} \cdot \frac{\sin \widehat{NBC}}{\sin \widehat{NBA}} \cdot \frac{\sin \widehat{NCA}}{\sin \widehat{NCB}} \cdot \frac{b}{c} = \frac{FC}{RB} \cdot \frac{\sin \widehat{NAC}}{\sin \widehat{NAB}} \cdot \frac{b}{c} = \frac{FC}{RB} \cdot \frac{s-b}{s-c} \ (1)$$

On the other hand, from the internal tangency of (O) with $\odot(DEF)$ and $\odot(PQR)$, we deduce that FE and RP bisect $\angle BFC$ and $\angle BRC$ internally, i.e. $M \equiv EF \cap PR$ is the midpoint of the arc BC of (O). Since $EFRP$ is cyclic, due to $MB^2 = MC^2 = ME \cdot MF = MP \cdot MR$, then $\triangle FPM \sim \triangle REM \Longrightarrow$

$$\frac{FP}{RE} = \frac{MP}{ME} = \frac{\sin \widehat{FEC}}{\sin \widehat{RPB}} = \frac{FC}{RB} \cdot \frac{PB}{EC} \cdot \frac{\sin \widehat{MFC}}{\sin \widehat{MRB}} = \frac{FC}{RB} \cdot \frac{PB}{EC}$$

But $\frac{PB}{a} = \frac{s-b}{s} \ , \ \frac{EC}{a} = \frac{s-c}{s} \Longrightarrow \frac{FP}{RE} = \frac{FC}{RB} \cdot \frac{s-b}{s-c} \ (2)$

From (1) and (2) we get then $\frac{FP}{RE} = \frac{FQ}{RD}$.

Quick Reply

High School Olympiads

Three coaxial circles

Reply

Source: Own

THVSH

#1 Jun 12, 2015, 11:33 pm • 1

Let ABC be a triangle with incircle $\odot(I)$ and the Feuerbach point F_e. (I) touches BC, CA, AB at D, E, F, respectively. E', F' are symmetric of E, F wrt the midpoint of CA, AB, respectively. X is the orthogonal projection of A on BC. Prove that $\odot(F_eDX); \odot(F_eEE'); \odot(F_eFF')$ are coaxial.

P.S

Luis González

#2 Jun 13, 2015, 11:08 am • 1

Clearly E', F' are the tangency points of the B- and C- excircle with AC, AB. If U is the tangency point of the B-excicle with BC, then it's known that UE' cuts $\overline{AX}$ at a point J verifying $AJ = ID = r$ (see for instance Inradius and altitude). If S is the antipode of D WRT (I), then $AJIS$ is parallelogram $\Longrightarrow JS$ passes through the midpoint A' of AI, but A', S, Fe are collinear (see intersect on circle) $\Longrightarrow J, S, Fe$ are collinear $\Longrightarrow \angle AE'J = \angle CE'U = \angle SDE = \angle JFeE \Longrightarrow J \in \odot(FeEE')$ and similarly $J \in \odot(FeFF')$. But $\angle SFeD = \angle JXD = 90^\circ \Longrightarrow J \in \odot(FeDX) \Longrightarrow \odot(FeDX)$, $\odot(FeEE')$ and $\odot(FeFF')$ are coaxal with common radical axis FeJ.

Quick Reply

High School Olympiads

Inradius and altitude

Source: Mine

juancarlos

#1 Jan 8, 2006, 9:15 pm

Let AH be altitude of triangle ABC, the excircle (I_c) touch BC, AB at P, Q respectively, so PQ meet AH at J. Prove that:
1) $AJ = r$
2) $[I_bJI_c] = [JBC]$
Where: r=inradius of ABC and I_b, I_c are excenters of ABC

Virgil Nicula

#2 Jan 10, 2006, 10:26 pm

$\blacksquare$ $1^\circ.\ 2s = a+b+c;\ QA = s-b;\ PB = QB = s-a;\ m(\widehat{HPQ}) = \frac{B}{2}.$

Apply the Menelaus' theorem to the transversal $\overline{PQJ}$ and the triangle ABH :

$$\frac{PB}{PH}\cdot\frac{JH}{JA}\cdot\frac{QA}{QB} = 1 \Longrightarrow JA = QA\cdot\frac{JH}{PH} \Longrightarrow JA = (s-b)\tan\frac{B}{2} \Longrightarrow \boxed{JA = r}$$

$\blacksquare$ $2^\circ.\ [JBC] = \frac{1}{2}BC\cdot JH = \frac{1}{2}a(h_a - r) = \frac{2S - ar}{2} \Longrightarrow$

$[JBC] = \frac{r(2s-a)}{2} = \frac{r(b+c)}{2} \Longrightarrow [JBC] = \frac{r(b+c)}{2}\quad (1)$

$I_bI_c = AI_b + AI_c = \frac{s-c}{\sin\frac{A}{2}} + \frac{s-b}{\sin\frac{A}{2}} = \frac{a}{\sin\frac{A}{2}} = 4R\cos\frac{A}{2} \Longrightarrow \boxed{I_bI_c = 4R\cos\frac{A}{2}}$

$m(\widehat{JAI_c}) = m(\widehat{JAB}) + m(\widehat{BAI_c}) = (90^\circ - B) + \left(90^\circ - \frac{A}{2}\right) = C + \frac{A}{2} \Longrightarrow$

$[I_bJI_c] = \frac{1}{2}\cdot I_bI_c\cdot AJ\cdot\sin\widehat{JAI_c} = 2Rr\cos\frac{A}{2}\sin\left(C + \frac{A}{2}\right) \Longrightarrow$

$Rr(\sin B + \sin C) = \frac{r(b+c)}{2} \Longrightarrow [I_bJI_c] = \frac{r(b+c)}{2}\quad (2)$

From the relations (1) and (2) results $\boxed{[I_bJI_c] = [JBC]}$

sprmnt21

#3 Jan 19, 2006, 4:07 pm

Let M be the ,nearest to C , intersection of CQ with the incircle (I). It is easy to see that CMI ~ CQI_c. Let F be the intersection between I_cA and PQ, by angle chasing one can prove that <CFI_c = 90° then CFI_c ~ IAI_c. From those relation we have that r_c/r = I_cC/IC = I_cF/AF. From the last, since AH//I_cP it holds that I_cF/AF = I_cP/AJ = r_c/AJ then the thesis.

High School Olympiads

A property of Fuhrman triangle

Reply

THVSH

#1 Jun 12, 2015, 11:41 pm

Prove that the **Fuhrman** triangle and the **excentral** triangle are both perspective and orthologic.

TelvCohl

#2 Jun 13, 2015, 12:18 am

My solution :

Let I be the incenter of $\triangle ABC$.
Let D, E, F be the midpoint of BC, CA, AB, respectively .
Let I_a, I_b, I_c be the A-excenter, B-excenter, C-excenter of $\triangle ABC$, respectively .
Let M_a, M_b, M_c be the midpoint of arc BC, arc CA, arc AB in $\odot(ABC)$, respectively .
Let $Mt \equiv I_aD \cap I_bE \cap I_cF, Be$ be the symmedian point, circumcenter of $\triangle I_aI_bI_c$, respectively .
Let T_a, T_b, T_c be the reflection of M_a, M_b, M_c in BC, CA, AB, respectively ($\triangle T_aT_bT_c$ is Fuhrmann triangle of $\triangle ABC$) .

Since I_a, I_b, I_c, I, Mt, Be lie on a conic $\mathcal{J}$ (Jerabek hyperbola of $\triangle I_aI_bI_c$) ,
so from $I_a(M_a, T_a; D, Be) = -1 \Longrightarrow (I, P \equiv I_aT_a \cap \mathcal{J}; Mt, Be) = -1$.
Similarly, we can prove $P \in I_bT_b$ and $P \in I_cT_c \Longrightarrow \triangle I_aI_bI_c$ and $\triangle T_aT_bT_c$ are perspective .

Since $\triangle T_aT_bT_c \sim \triangle M_aM_bM_c \sim \triangle I_aI_bI_c$ (see Some properties of Hagge circle (theorem 4)) ,
so the perpendicular from T_a, T_b, T_c to I_bI_c, I_cI_a, I_aI_b, respectively are concurrent on $\odot(T_aT_bT_c)$.
i.e. $\triangle I_aI_bI_c$ and $\triangle T_aT_bT_c$ are orthologic

Q.E.D

P.S. From the proof above we get P is the Kosnita point of $\triangle I_aI_bI_c$:)

Luis González

#3 Jun 13, 2015, 9:27 am

Let $\triangle I_aI_bI_c$ be the excentral triangle of $\triangle ABC. D, E, F$ are the midpoints of the arcs BC, CA, AB of its circumcircle (O) and D', E', F' are the reflections of D, E, F on $BC, CA, AB. \triangle D'E'F'$ is Fuhrmann triangle of $\triangle ABC.$

If U is the circumcenter of $\triangle I_aI_bI_c$ and X is the reflection of U on $I_bI_c,$ we have $\angle BD'C = \angle BDC = 2\angle BI_aC = \angle I_bUI_c = \angle I_bXI_c \Longrightarrow$ isosceles $\triangle D'BC$ and $\triangle XI_bI_c$ are similar $\Longrightarrow$ $I_aBD'C \sim I_aI_bXI_c \Longrightarrow I_aD'$ and I_aX are isogonals WRT $\angle I_bI_aI_c \Longrightarrow I_aD'$ is the isogonal of the I_a-cevian of the 9-point center of $\triangle I_aI_bI_c \Longrightarrow I_aD'$ goes through the Kosnita point K of $\triangle I_aI_bI_c$ and similarly I_bE', I_cF' go through $K.$

Let M, N, L be the midpoints of BC, CA, AB and let S be the incenter of $\triangle MNL.$ Perpendicular from D' to I_bI_c is parallel to $DI \parallel MS,$ cutting IS at the reflection of I on $S;$ the Nagel point Na of $\triangle ABC.$ Likewise perpendiculars from E' and F' to I_cI_a, I_aI_b go through $Na \Longrightarrow \triangle I_aI_bI_c$ and $\triangle D'E'F'$ are orthologic with an orthology center $Na.$

Quick Reply

High School Olympiads

tangent circles

Reply

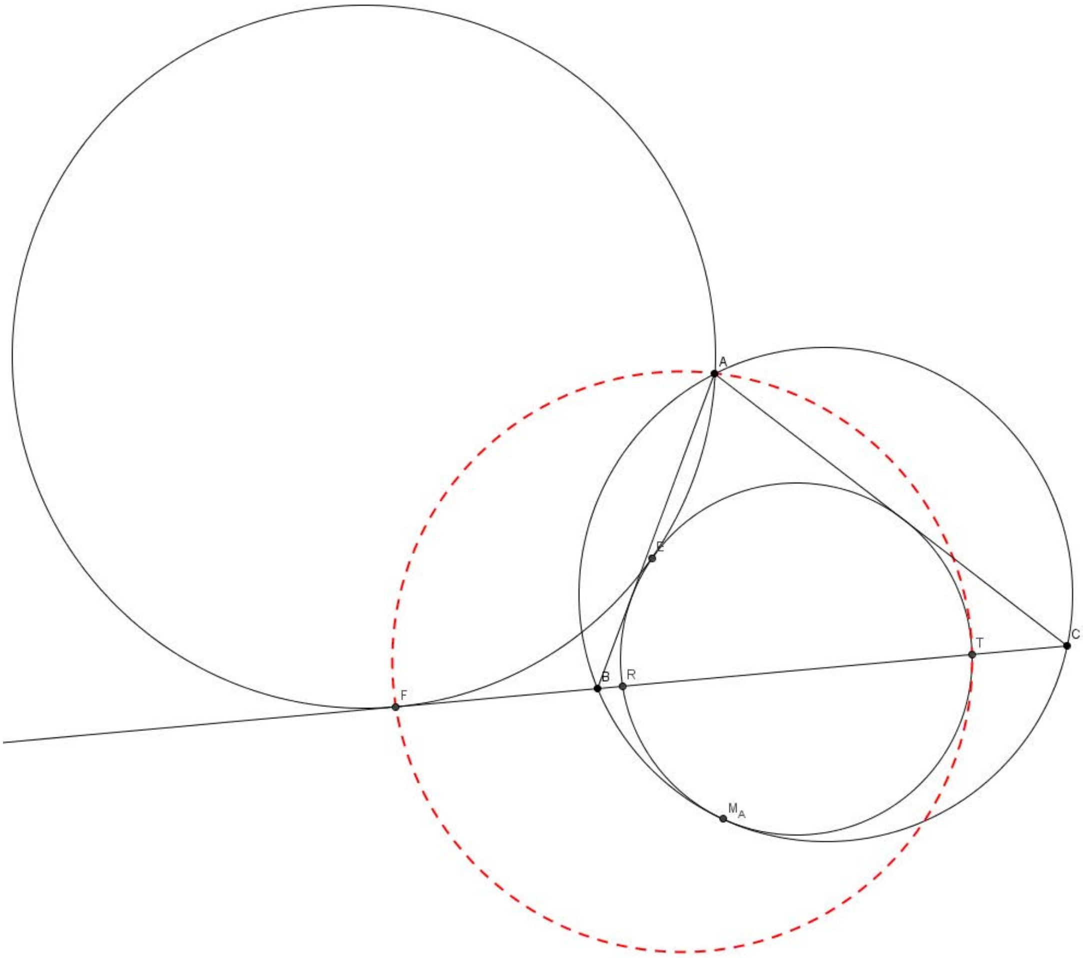
Source: Own

andria

#1 Jun 12, 2015, 2:20 pm

In $\triangle ABC$ let the A_mixtilinear incircle cut BC at R,T(R between B,T) ω is a circle that it passes throw A and it is tangent to A_mixtilinear incircle and BC at E,F prove that $\odot(\triangle AFT)$ is tangent to the A_mixtilinear incircle at T.

Attachments:



Luis González

#2 Jun 13, 2015, 8:12 am • 1

Lemma: The circumcircle (O) of $\triangle ABC$ and its A-excircle (I_a) meet at X,Y (B,X,Y,C lie on the arc BC in that order). External common tangent of $(O),(I_a)$ touches the arc AB of (O) at U. Then UY is tangent of (I_a).

Let P be a variable point on (O), not lying on its arc XY, and the tangents from P to (I_a) cut (O) again at X',Y'. By Poncelet porism for $\{\triangle ABC \cup (O),(I_a)\}$, it follows that $X'Y'$ touches (I_a). When $P \equiv U$, then $Y \equiv Y'$ and PX' becomes external common tangent of $(O),(I_a)$, forcing $X' \equiv U \Longrightarrow UY$ touches (I_a). $\blacksquare$

Back to the problem. Consider the inversion with center A taking the A-mixtilinear incircle ω_A into the incircle (I). Labeling inverse points with primes, (O) goes to a tangent $B'C'$ of (I) leaving A,I on different sides and BC goes to $\odot(AB'C') \Longrightarrow T' \equiv (I) \cap \odot(AB'C')$ closer to C'. $\odot(AFT)$ goes then to the common external tangent of $\odot(AB'C'),(I)$ touching the arc AB' of $\odot(AB'C')$ at F'. Using the previous lemma for $\triangle AB'C'$, it follows that $F'T'$ is tangent to (I) at $T' \Longrightarrow \odot(AFT)$ is tangent to ω_A at T.

Quick Reply

High School Olympiads

geometry

Reply

andria

#1 Jun 10, 2015, 2:05 pm

k is a constant positive real number.point P is an arbitrary point Inside triangle ABC let the parallel line from P to BC intersect AB,AC at A_1,A_2 we define B_1,B_2,C_1,C_2 similarly find the locus of point P such that $PA_1PA_2+PB_1PB_2+PC_1PC_2=k$ is constant.

andria

#2 Jun 11, 2015, 4:08 pm

Hint: the locus is a circle with center O where O is circumcenter of $\triangle ABC$ before proving this show that $PA_1PA_2+PB_1PB_2+PC_1PC_2$ is power of P WRT $\odot(\triangle ABC)$.

Luis González

#3 Jun 11, 2015, 10:10 pm

See the general problem at http://www.artofproblemsolving.com/community/c6h384789 (posts #2 and #4).

Quick Reply

High School Olympiads

Equality with circumradius [ILL 1974]

Amir Hossein

#1 Jan 3, 2011, 5:44 am

Through the circumcenter O of an arbitrary acute-angled triangle, chords A_1A_2, B_1B_2, C_1C_2 are drawn parallel to the sides BC, CA, AB of the triangle respectively. If R is the radius of the circumcircle, prove that

$$A_1O \cdot OA_2 + B_1O \cdot OB_2 + C_1O \cdot OC_2 = R^2.$$

Luis González

#2 Jan 3, 2011, 9:19 am • 5

The proposed problem can be generalized for any point on the plane of $\triangle ABC$.

Proposition. Let P be a point on the plane of $\triangle ABC$ with circumcircle (O, R). Parallel through P to BC cuts AB, AC at A_1, A_2 and define similarly the points B_1, B_2 and C_1, C_2. Then

$$\overline{PA_1} \cdot \overline{PA_2} + \overline{PB_1} \cdot \overline{PB_2} + \overline{PC_1} \cdot \overline{PC_2} = |PO^2 - R^2|$$

Proof. Let X, Y, Z be the orthogonal projections of P onto BC, CA, AB. R_1 denotes the circumradius of $\triangle AA_1A_2$. By Euler theorem for P and its pedal triangle $\triangle PYZ$ WRT $\triangle AA_1A_2$, we get:

$\dfrac{|\triangle PYZ|}{|\triangle AA_1A_2|} = \dfrac{\overline{PA_1} \cdot \overline{PA_2}}{4{R_1}^2}$. But $\triangle ABC \sim \triangle AA_1A_1$ gives $\dfrac{|\triangle ABC|}{|\triangle A_1B_1C_1|} = \dfrac{R^2}{{R_1}^2}$

$$\Longrightarrow \frac{|\triangle PYZ|}{|\triangle ABC|} = \frac{\overline{PA_1} \cdot \overline{PA_2}}{4R^2}$$

By cyclic exchange of elements we obtain the expressions:

$$\frac{|\triangle PZX|}{|\triangle ABC|} = \frac{\overline{PB_1} \cdot \overline{PB_2}}{4R^2}, \ \frac{|\triangle PXY|}{|\triangle ABC|} = \frac{\overline{PC_1} \cdot \overline{PC_2}}{4R^2} \Longrightarrow$$

$$\frac{\overline{PA_1} \cdot \overline{PA_2} + \overline{PB_1} \cdot \overline{PB_2} + \overline{PC_1} \cdot \overline{PC_2}}{4R^2} = \frac{|\triangle PYZ| + |\triangle PZX| + |\triangle PXY|}{|\triangle ABC|}$$

$$\overline{PA_1} \cdot \overline{PA_2} + \overline{PB_1} \cdot \overline{PB_2} + \overline{PC_1} \cdot \overline{PC_2} = 4R^2 \cdot \frac{|\triangle XYZ|}{|\triangle ABC|} = 4R^2 \cdot \frac{|PO^2 - R^2|}{4R^2}$$

$$\overline{PA_1} \cdot \overline{PA_2} + \overline{PB_1} \cdot \overline{PB_2} + \overline{PC_1} \cdot \overline{PC_2} = |PO^2 - R^2|.$$

Virgil Nicula

#3 May 26, 2011, 9:39 am • 1

$$\left\{\begin{array}{ccccc} \triangle OAA_1 & : & m\left(\widehat{OAA_1}\right) = 90^\circ - C & \Longrightarrow & \frac{OA_1}{\cos C} = \frac{R}{\sin B} \\ \triangle OAA_2 & : & m\left(\widehat{OAA_2}\right) = 90^\circ - B & \Longrightarrow & \frac{OA_2}{\cos B} = \frac{R}{\sin C} \end{array}\right\| \Longrightarrow OA_1 \cdot OA_2 = R^2 \cot B \cot C \Longrightarrow$$

$\sum OA_1 \cdot OA_2 = R^2$ because $\sum \cot B \cot C = 1$. **Very nice** Luis's **extension**. **Thank you**. Amparvardi & Luis

$\sum OH_1 \cdot OH_2 = R^2$ because $\sum \cot B \cot C = 1$. **Very nice** Luis's **extension**. **Thank you**, Amparvardi & Luis.

This post has been edited 1 time. Last edited by Luis González, Jun 11, 2015, 10:07 pm

yetti

#4 May 26, 2011, 11:58 am

The general problem was posted before at http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=118621.

Quick Reply

High School Olympiads

easy geometry

Source: Own

andria

#1 Jun 11, 2015, 4:24 pm

In triangle ABC with incenter I let the incircle and A_excircle touch BC at A', A'' let $\odot(A'A'') \cap AI = \{A_1, A_2\}$ point A_3 lies on BC such that $AA_3 \perp BC$ let O_A, I_A circumcenter and incenter of $\triangle A_1A_2A_3$ we define O_B, I_B, O_C, I_C similarly Prove that O_AI_A, O_BI_B, O_CI_C are concurrent.

Luis González

#2 Jun 11, 2015, 9:18 pm • 1

From the problem Prove D is the incenter, we have that $A' \equiv I_A$ is the incenter of $\triangle A_1A_2A_3.$ Furthermore, if D is the midpoint of $BC,$ then O_A is the midpoint of the arc A_3D of the 9-point circle (N) of $\triangle ABC$ (see for instance 9-points circle of ABC, concyclic points in a triangle and elsewhere). Now since the Feuerbach point Fe is the exsimilicenter of $(I) \sim (N),$ it follows that FeI_A bisects $\angle DFeA_3 \Longrightarrow O_A \in FeI_A,$ i.e. O_AI_A passes through Fe and similarly O_BI_B and O_CI_C go through $Fe.$

tranquanghuy7198

#3 Jun 12, 2015, 3:55 pm • 1

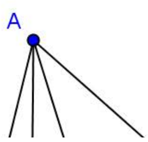
My solution:
M is the midpoint of BC and also the center of $(A'A")$
I_a is the A-excenter, $AI \cap BC = D$
From here we get: $MA_1 \parallel AC, MA_2 \parallel AB$
Notice that: $(ADII_a) = -1 \Rightarrow (A_3DA'A") = -1$
$\Rightarrow MA'^2 = MD.MA_3$
$\Rightarrow \angle MA_3A_1 = \angle MA_1D = \frac{1}{2}\angle BAC$
$\Rightarrow BA_1 \perp AD$ and similarly we have: $CA_2 \perp AD$
By simple angle chasing we get: $\triangle A_3A_1A_2 \cup D \sim \triangle ABC \cup D$
$\Rightarrow$ If l_a is the bisector of (BC, AD) then:

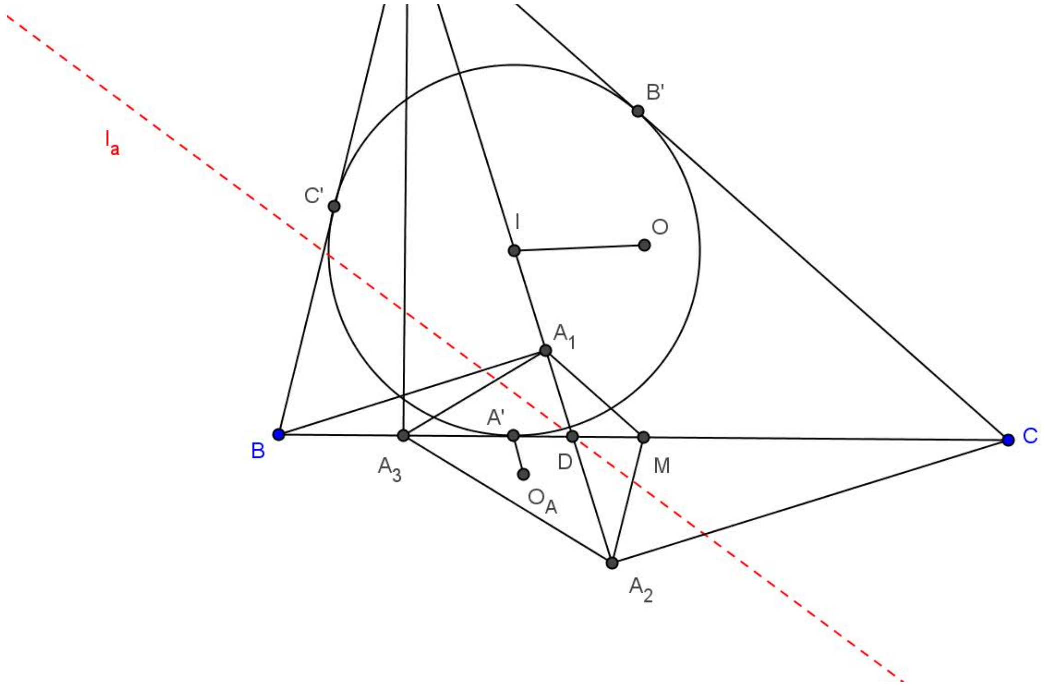
$$R_{l_a}.H_D : A_3 \mapsto A, A_1 \mapsto B, A_2 \mapsto C$$

$\Rightarrow OI \parallel (R_{l_a}(O_AI_A))$ $(I_A \equiv A')$
Similarly: l_b is the bisector of (CA, BI), $OI \parallel (R_{l_b}(O_BB'))$

Now, because of all the above, by angle chasing, we have: $(A'O_A, B'O_B) = (C'A', C'B')$ (mod π)
$\Rightarrow (A'O_A \cap B'O_B) \in (I)$
Similarly, we have: $(C'O_C \cap B'O_B) \in (I)$
$\Rightarrow A'O_A, B'O_B, C'O_C$ concur (on (I))
Q.E.D

Attachments:





andria

#4 Jun 12, 2015, 4:45 pm

Thank you for your solutions
Remark: this problem shows another way to prove that incircle and nine point circle are tangent to each other.

Quick Reply

High School Olympiads

Prove D is the incenter

Reply

maths_lover5
#1 Nov 27, 2014, 1:22 am

Given triangle ABC, $AB < AC$. The inscribed circle with center I touches BC, AC, AB in D, E, F Points respectively. The line AI Intersects DE, DF in points X, Y respectively. Let AZ be perpendicular to BC. Prove that D is the incenter of triangle XYZ

Luis González
#2 Nov 27, 2014, 2:35 am • 1

Since $\angle EDY = 90^\circ + \frac{1}{2}\angle BAC = \angle FIY$, then the quadrilateral $FIXD$ is cyclic, i.e. X is on circle with diameter $\overline{BI}$ $\Longrightarrow X$ is the projection of B on AI and similarly Y is the projection of C on AI. Thus, from cyclic quadrilaterals $ABZX$ and $IBDX$, we get $\angle YXZ = \angle ABC$ and $\angle YXD = \angle IBD = \frac{1}{2}\angle ABC \Longrightarrow XD$ bisects $\angle YXZ$ and similarly YD bisects $\angle XYZ \Longrightarrow D$ is the incenter of $\triangle XYZ$.

jayme
#3 Nov 27, 2014, 4:45 pm

Dear Mathlinkers,
you can see

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=607046

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Prove that IK parallels with EF

Reply

tdl

#1 Jun 7, 2015, 12:28 am

Let a triangle ABC with incircle (I) tangents AC, AB at E, F. $J \in EF$ so that BJ parallels with AC. Let $CJ \cap AB = K$. Prove that IK parallels with EF.

---

v_Enhance

#2 Jun 7, 2015, 12:59 am • 1

Using $\triangle AEF \sim \triangle BJF$ and $\triangle JBK \sim \triangle CAK$ we derive

$$\frac{JK}{KC} = \frac{JB}{AC} = \frac{BF}{AC} = \frac{s-b}{b}.$$

Now let $L = \overline{EF} \cap \overline{CI}$. By Sine Law on $\triangle ECL$ we deduce

$$\frac{LI}{IC} = \frac{\frac{\sin\frac{1}{2}\angle A}{\sin\frac{1}{2}\angle B}}{\frac{\sin 90^\circ}{\sin\frac{1}{2}\angle C}} = \frac{\sin\frac{1}{2}\angle A \sin\frac{1}{2}\angle C}{\sin\frac{1}{2}\angle B}.$$

We may conclude from this using some routine calculations that $\dfrac{JK}{KC} = \dfrac{LI}{IC}$ as desired.

Luis González

#3 Jun 7, 2015, 2:59 am

Let P be the pole of BJ WRT (I), lying on perpendicular IE from I to BJ. Since P, J are conjugate points WRT (I), then PJ and DE are conjugate lines WRT $(I) \Longrightarrow C, P, J$ are collinear. If $Q \equiv DE \cap BJ$, then CJ is the polar of Q WRT (I), because P, Q are conjugate points WRT (I) and Q is on the polar DE of C WRT (I). Thus if L is the 2nd intersection of IE with (I), from the complete cyclic $EFLD$, it follows that $Q \in FL$ and KL is tangent of $(I) \Longrightarrow FL$ is the polar of K WRT $(I) \Longrightarrow IK \perp FL$, but $FL \perp EF \Longrightarrow IK \parallel EF$.

---

tranquanghuy7198

#4 Jun 7, 2015, 9:09 am

My solution:
We have: $AE = AF \Rightarrow BJ = BF$

$\Rightarrow \dfrac{KA}{KB} = \dfrac{AC}{BJ} = \dfrac{AC}{BF} = \dfrac{b}{s-b}$

$\Rightarrow \dfrac{KA}{c} = \dfrac{KA}{KA+KB} = \dfrac{b}{b+(s-b)} = \dfrac{b}{s}$

$\Rightarrow KA = \dfrac{bc}{s}$

$\Rightarrow AK.AF = \dfrac{bc(s-a)}{s} = AI^2$

$\Rightarrow IK \perp IA$ (because $FI \perp FA$)

$\Rightarrow IK \parallel EF$

Q.E.D

EulerMacaroni

#5 Jun 8, 2015, 4:58 am

This is bashable extremely quickly.

Solution

This post has been edited 1 time. Last edited by EulerMacaroni, Jun 8, 2015, 6:18 pm
Reason: Det calculation error

nguyenvanthien63

#6 Jun 8, 2015, 7:56 am

> *Luis González wrote:*
> Let P be the pole of BJ WRT (I), lying on perpendicular IE from I to BJ. Since P, J are conjugate points WRT (I), then PJ and DE are conjugate lines WRT $(I) \Longrightarrow C, P, J$ are collinear. If $Q \equiv DE \cap BJ$, then CJ is the polar of Q WRT (I), because P, Q are conjugate points WRT (I) and Q is on the polar DE of C WRT (I). Thus if L is the 2nd intersection of IE with (I), from the complete cyclic $EFLD$, it follows that $Q \in FL$ and KL is tangent of $(I) \Longrightarrow$ FL is the polar of K WRT $(I) \Longrightarrow IK \perp FL$, but $FL \perp EF \Longrightarrow IK \parallel EF$.

where is D?

Luis González

#7 Jun 8, 2015, 8:37 am • 1

Sorry I forgot to mention that D is the tangency point of (I) with BC. By the way, there is a generalization of this problem:

P is abritrary point on the plane of $\triangle ABC$. BP, CP cut AC, AB at E, F and $J \in EF$, such that $BJ \parallel AC$. CJ cuts AB at K. If P^* is the isotomcomplement of P WRT $\triangle ABC$, then $P^*K \parallel EF$.

Bearing in mind that P^* is the center of the inconic with perspector P, the projective proof in my previous post #3 still holds for this generalization.

Quick Reply

High School Olympiads

AB+AC=3BC

Reply

Source: Own

LeVietAn

#1 May 29, 2015, 9:45 am • 1

Dear Mathlinkers,
Given triangle ABC with $AB+AC=3BC$ and (I) be the incircle. The circumcircle o triangle IBC and (I) intersect at the point X and Y. The circle (I) touches AB, BC, CA at D, E, F, respectively. Prove that XY passes through the orthocenter of the triangle DEF.

This post has been edited 1 time. Last edited by LeVietAn, May 29, 2015, 9:46 am

tranquanghuy7198

#2 May 29, 2015, 10:32 am • 2

My solution:
Denote by $d(A,d)$ the distance from the point A to the line d
Denote by $\angle D, \angle E, \angle F$ the angles of $\triangle DEF$
Denote by r the inradius of $\triangle ABC$
Let DX, FY be the diameters of (I), H is the orthocenter of $\triangle DEF$
Because $AB+AC=3BC$, it is well-known that $X, Y \in (BIC)$ (proof uses Ptolemy theorem), hence XY is the radical axis of (BIC) and (I)
Now: $H \in XY$
$\Longleftrightarrow d(H,DF)=2.d(I,DF)$ (because $DFXY$ is the rectangle)
$\Longleftrightarrow d(H,DF)=EH$
$\Longleftrightarrow 2r.cosD.cosF=2r.cosE$
$\Longleftrightarrow cosD.cosF=cosE$
$\Longleftrightarrow sin\angle C/2.sin\angle B/2=sin\angle A/2$
$\Longleftrightarrow \sqrt{\dfrac{(s-a)(s-b)}{ab}}.\sqrt{\dfrac{(s-a)(s-c)}{ac}}=\sqrt{\dfrac{(s-c)(s-b)}{cb}}$
$\Longleftrightarrow s-a=a$
$\Longleftrightarrow b+c-a=2a$ (right)
Q.E.D

TelvCohl

#3 May 29, 2015, 1:43 pm

My solution :

Let H be the orthocenter of $\triangle DEF$.
Let $M \equiv AI \cap \odot(ABC)$ and $T \equiv EI \cap \odot(I)$.
Let $K \equiv EI \cap DF$ and N be the midpoint of BC .

It's well-known that A, K, N are collinear .

From $AB+AC=3BC \Longrightarrow AD=AF=BC$,
so combine $\triangle BMN \sim \triangle AID \Longrightarrow AI=2BM=2IM \Longrightarrow IK=\frac{2}{3}\cdot MN=\frac{1}{3}r$,

hence we get $\dfrac{\text{dist}(H,DF)}{\text{dist}(E,DF)}=\dfrac{\text{dist}(T,DF)}{\text{dist}(E,DF)}=\dfrac{TK}{EK}=\dfrac{r-IK}{r+IK}=\dfrac{1}{2}$ . ... $(\star)$

Since $\odot(IBC)$ is the image of E-midline of $\triangle DEF$ under the inversion $\mathbf{I}(\odot(I))$,
so XY is the E-midline of $\triangle DEF \Longrightarrow$ combine $(\star)$ we get H lie on the E-midline XY of $\triangle DEF$.

Q.E.D

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, May 29, 2015, 2:08 pm

Luis González

#4 Jun 7, 2015, 7:00 am • 1

Let T be the orthocenter of $\triangle DEF$ and let $\triangle I_aI_bI_c$ be the excentral triangle of $\triangle ABC. U, V$ denote the midpoints of I_aI_b, I_aI_c. Since $AI_a : AI = s : s - a$, it follows that $I \in UV \Longleftrightarrow AI_a = 2 \cdot AI \Longleftrightarrow s = 2(s-a) \Longleftrightarrow b + c = 3a$.

If B', C' are the midpoints of DF, DE, then $B'D \cdot B'F = B'B \cdot B'I \Longrightarrow B'$ has equal power WRT $(I), \odot(BIC)$ and so does $C' \Longrightarrow XY \equiv B'C'$. Now since $\triangle DEF \cup T \sim \triangle I_aI_bI_c \cup I$. it follows that $T \in XY \Longleftrightarrow I \in UV \Longleftrightarrow$ $b + c = 3a$.

buratinogigle

#6 Jun 7, 2015, 10:07 am • 1

It was posted before http://www.artofproblemsolving.com/community/c6h526036

Quick Reply

High School Olympiads

geometry

Reply

Source: Own?

andria

#1 Jun 6, 2015, 11:29 pm

In $\triangle ABC, AD, BE, CF$ are altitudes and O is circumcenter of $\triangle ABC$ let $AO \cap EF = T$ and R foot of A on DT prove that $\angle ERT = \angle FRT$.

Luis González

#2 Jun 7, 2015, 12:10 am

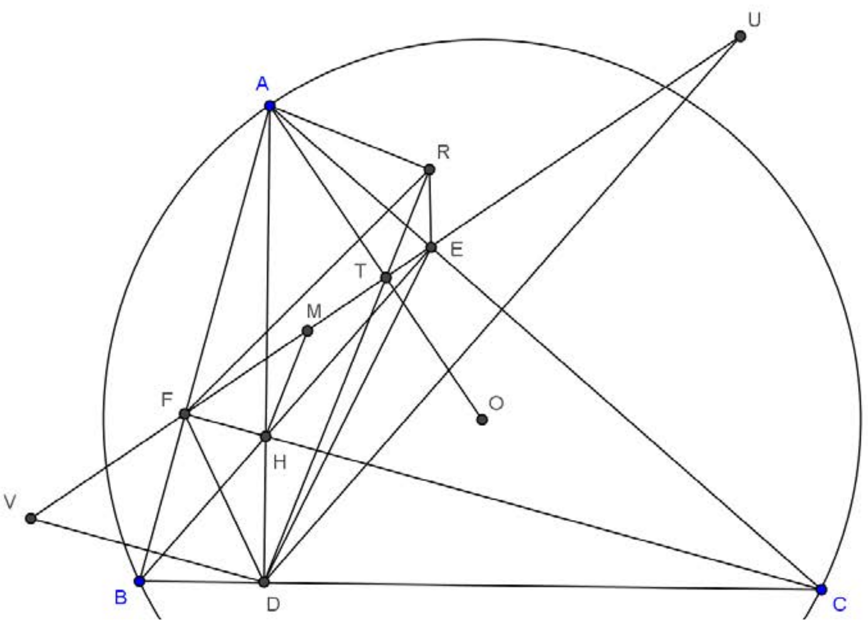
Since A is D-excenter of $\triangle DEF$ and $AT \perp EF$, then $\odot(A, AT)$ is the A-excircle of $\triangle DEF$. Hence $U \equiv AR \cap EF$ is the pole of DT WRT $\odot(A, AT) \Longrightarrow (E, F, T, U) = -1$. Together with $RT \perp RA$, it follows that RT, RA bisect $\angle ERF$, or $\angle ERT = \angle FRT$.

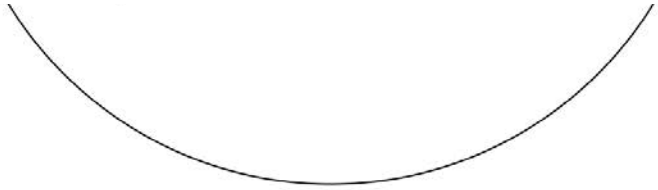
tranquanghuy7198

#3 Jun 7, 2015, 9:40 am

My solution:
$U, V \in EF : DU \perp AC, DV \perp AB$, x is the line passing through D which is parallel to EF
H is the orthocenter of $\triangle ABC$, M is the midpoint of EF
Consider $\triangle DEF$ with incenter H and T is the tangent point of EF and D-excircle
$\Rightarrow HM \parallel DT (\because DT$ passes through the Nagel point of $\triangle DEF)$
Notice that: $DU, DV, DT \parallel HE, HF, HM$
$\Rightarrow TU = TV$ (because $ME = MF$)
$\Rightarrow D(UVTx) = -1$
$\Rightarrow A(EFRT) = -1$ (because $AE \perp DU, AF \perp DV, \ldots$)
$\Rightarrow R(EFAT) = -1$
$\Rightarrow RT$ bisects $\angle ERF$ (because $RT \perp RA$)
Q.E.D

Attachments:





hayoola

#4 Sep 24, 2015, 1:41 am

let P be the interestion between AR, FE and let k be the interestion betwwen FE, BC so we must prove that $(EFTP) = -1$ so we have to prove that $A(EFTP) = -1$ by reflecting AE, AF, AT, AP about the internal angle bisector of angle BAC we must prove that $(BCDK) = -1$ at it is khown

Quick Reply

High School Olympiads

Line passes through circumcenter

Reply

Source: Own

LeVietAn
#1 Jun 5, 2015, 12:11 pm

Dear Mathlinkers,
Let P and Q be on segment BC of a triangle ABC such that $\measuredangle BAP = \measuredangle CAQ = 90^0$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN; H is the intersection of BM and CN. Prove that AH passes through the circumcenter of the triangle ABC.

rkm0959
#2 Jun 5, 2015, 12:41 pm

Barycentric Coordinates kill this problem!
Set $A(1,0,0), B(0,1,0), C(0,0,1)$
Then $P(0, \frac{a*cosB-c}{a*cosB}, \frac{c}{a*cosB}), Q(0, \frac{b}{a*cosC}, \frac{a*cosC-b}{a*cosC})$
$M(-1, \frac{2(a*cosB-c)}{a*cosB}, \frac{2c}{a*cosB}) N(-1, \frac{2b}{a*cosC}, \frac{2(a*cosC-b)}{a*cosC})$
Line BM : $\frac{2c}{a*cosB}x+z=0$
Line CN: $\frac{2b}{a*cosC}x+y=0$
Now calculate H and standard formula for the circumcenter and we are done.

This post has been edited 3 times. Last edited by rkm0959, Jun 7, 2015, 12:53 am

TelvCohl
#3 Jun 5, 2015, 12:59 pm • 1

My solution :

Let Ψ be the composition of inversion $\mathbf{I}(A, \sqrt{AB \cdot AC})$ and reflection $\mathbf{R}(\ell)$ (ℓ is the bisector of $\angle BAC$) .

From $\angle BAP = \angle CAQ = 90^\circ \Longrightarrow \Psi(Q), \Psi(P)$ is the antipode of B, C in $\odot(ABC)$, respectively ,
so $\Psi(M), \Psi(N)$ is the midpoint of $A\Psi(P), A\Psi(Q)$, respectively and $\Psi(H) \equiv \odot(AB\Psi(N)) \cap \odot(AC\Psi(M))$.

Since the center of $\odot(AB\Psi(N)), \odot(AC\Psi(M))$ is the midpoint of $B\Psi(N), C\Psi(M)$, respectively ,
so from $\Psi(N)\Psi(M) \parallel BC \Longrightarrow$ the radical axis $A\Psi(H)$ of $\odot(AB\Psi(N)), \odot(AC\Psi(M))$ is A-altitude of $\triangle ABC$,
hence AH is the isogonal conjugate of A-altitude WRT $\angle BAC \Longrightarrow AH$ passes through the circumcenter of $\triangle ABC$.

Q.E.D

TelvCohl
#4 Jun 5, 2015, 1:08 pm • 1

Another solution :

Let O be the circumcenter of $\triangle ABC$.
Let B', C' be the antipode of B, C in $\odot(ABC)$, respectively .

From $PQ \equiv BC \parallel B'C' \parallel MN \Longrightarrow (A, M; P, B') = (A, N; Q, C')$,
so $B(A, M; P, B') = C(A, N; Q, C') \Longrightarrow A, H \equiv BM \cap CN, O \equiv BB' \cap CC'$ are collinear .

Q.E.D

Luis González

#5 Jun 5, 2015, 1:09 pm • 2

Let MN cut AC, AB at $U, V.$ Perpendiculars to AU, AV at C, B are their perpendicular bisectors meeting at the circumcenter K of $\triangle AUV.$ Since $BC \parallel MN, KC \parallel AN, KB \parallel AM \Longrightarrow \triangle AMN$ and $\triangle KBC$ are homothetic with center $H \equiv AK \cap BM \cap CN,$ but AK is also the A-cevian of the circumcenter of $\triangle ABC,$ due to $BC \parallel UV \Longrightarrow AH$ passes through the circumcenter of $\triangle ABC.$

v_Enhance

#6 Jun 7, 2015, 12:40 am

The midpoint condition that $AP = PM$ and $AQ = QN$ is a red herring: it is sufficient that $\frac{AP}{PM} = \frac{AQ}{QN}$ (with appropriately directed ratios).

Quick Reply

High School Olympiads

Locus of P

Reply

trunglqd91

#1 Jun 5, 2015, 10:31 am

Let a circle (O) with diameter BC. B,C are fixed points. A is an arbitrary point in (O). The altitude AH. H lie on BC. M is the midpoint of AH. BM cut (O) at N again. The tangent from N of (O) cut AC at P. Find the locus of P when A is portable.

This post has been edited 1 time. Last edited by trunglqd91, Jun 5, 2015, 12:04 pm
Reason: A mistake make the problem is wrong

Luis González

#2 Jun 5, 2015, 11:49 am

Let the fixed tangent of (O) at B cut AC at D. We have $\frac{PA}{PC}=\frac{NA^2}{NC^2}=\frac{NA^2}{AC^2}$ and $\frac{DC}{DA}=\frac{BC}{BH} \Longrightarrow$

$(P,A,C,D)=\frac{NA^2}{AC^2} \cdot \frac{BC}{BH}=\frac{4 \cdot BH \cdot CH}{CH \cdot BC} \cdot \frac{BC}{BH}=4.$

Thus $P \mapsto A$ is a homography transforming a certain conic $\mathcal{H}$ into its pedal circle (O). Clearly P is at infinity when $\triangle ACN$ is equilateral, so $\mathcal{H}$ is a hyperbola with pedal circle (O) and whose asymptotes form 60°.

trunglqd91

#4 Jun 5, 2015, 12:09 pm

Sorry **Luis González** very much.

I have just edited my problem.
BM cut (O) not AM.

Luis González

#5 Jun 5, 2015, 12:45 pm

Surprisingly the other problem turned out to be more interesting than the corrected version. Just notice that M is the symmedian point of $\triangle ABC \Longrightarrow BN$ is the B-symmedian $\Longrightarrow BANC$ is harmonic $\Longrightarrow P$ moves on the tangent of (O) at B.

phuocdinh_vn99

#6 Jun 5, 2015, 2:15 pm

> *Luis González wrote:*
>
> Let the fixed tangent of (O) at B cut AC at D. We have $\frac{PA}{PC}=\frac{NA^2}{NC^2}=\frac{NA^2}{AC^2}$ and $\frac{DC}{DA}=\frac{BC}{BH} \Longrightarrow$
>
> $(P,A,C,D)=\frac{NA^2}{AC^2} \cdot \frac{BC}{BH}=\frac{4 \cdot BH \cdot CH}{CH \cdot BC} \cdot \frac{BC}{BH}=4.$
>
> Thus $P \mapsto A$ is a homography transforming a certain conic $\mathcal{H}$ into its pedal circle (O). Clearly P is at infinity when $\triangle ACN$ is equilateral, so $\mathcal{H}$ is a hyperbola with pedal circle (O) and whose asymptotes form 60°.

For the orignal problem: If we use O-coordinate with O is midpoint of BC then the locus is the hyperbola: $x^2-3y^2=1$

This post has been edited 1 time. Last edited by phuocdinh_vn99, Jun 5, 2015, 2:15 pm
Reason: 1

Quick Reply

High School Olympiads

geometry

Reply

Source: Own

andria

#1 May 29, 2015, 2:14 pm

Points B, C lie on the circle ω points M, N, S are midpoints of BC, CA, AB and P, Q, R are midpoints of the altitudes $AD, BE, CF(D, E, F$ lie on $BC, CA, AB)$ let J radical center of three circles $\odot(\triangle APM), \odot(\triangle BQN), \odot(\triangle CRS)$ find the locus of point J when A varies on ω.

Luis González

#2 Jun 5, 2015, 5:03 am

According to http://www.artofproblemsolving.com/community/c6h194794, the radical center J of $\odot(APM), \odot(BQN)$ and $\odot(CRS)$ is the 9-point center of $\triangle ABC$. Therefore, the locus of J is the circle with center M and radius half the radius of ω.

Quick Reply

High School Olympiads

hard to approach; N is radical centre of three circles

Reply

Source: Math Reflections O78, Cosmin Pohoata

epitomy01

#1 Mar 19, 2008, 2:44 pm

Let ABC be a triangle and let M, N, P be the midpoints of sides BC, CA, AB respectively. Denote by X, Y, Z the midpoints of the altitudes emerging from vertices A, B, C respectively. Prove that the radical centre of the circles AMX, BNY, CPZ is the centre of the nine-point circle of triangle ABC.

This is from Math Reflections 2008 Issue 1, and since deadline is over for those problems I think I can post them here.

yetti

#2 Mar 20, 2008, 3:22 pm • 1

Let G be the centroid and O the circumcenter of the $\triangle ABC.$ G is also the centroid of the medial $\triangle MNP$ and O is its orthocenter. Let U, V, W be midpoints of $NP, PM, MN.$ Let $\mathcal Q_m, \mathcal Q_n, \mathcal Q_p$ be circles internally tangent to the circles $\odot(AMX), \odot(BNY), \odot(CPZ)$ at M, N, P and with similarity coefficients $1/3.$

Let D, E, F be the altitude feet of the medial $\triangle MNP.$ Since the circles $\odot(AMX), \odot(BNY), \odot(CPZ)$ pass through the reflections A', B', C' of M, N, P in the perpendicular bisectors of AX, BY, CZ and $\overline{MA'} = 3\ \overline{MD}, \overline{NB'} = 3\ \overline{NE},$ $\overline{PC'} = 3\ \overline{PF},$ the circles $\mathcal Q_m, \mathcal Q_n, \mathcal Q_p$ pass through the altitude feet D, E, F of the medial $\triangle MNP$ and they are centered on the sidelines VW, WU, UV of its medial $\triangle UVW.$ Thus the powers of the orthocenter O of the medial $\triangle MNP$ to the circles $\mathcal Q_m, \mathcal Q_n, \mathcal Q_p$ are all equal to half the power of orthocenter O to the circumcircle (K) of this medial triangle, identical with the 9-point circle of the $\triangle ABC$:

$$\overline{OD} \cdot \overline{OM} = \overline{OE} \cdot \overline{ON} = \overline{OF} \cdot \overline{OP} = \tfrac{1}{2}\ p(O, (K)).$$

Since the circles $\odot(AMX), \odot(BNY), \odot(CPZ)$ pass through the reflections A, B, C of M, N, P in U, V, W and $\overline{MA} = 3\ \overline{MG}, \overline{NB} = 3\ \overline{NG}, \overline{PC} = 3\ \overline{PG},$ the circles $\mathcal Q_m, \mathcal Q_n, \mathcal Q_p$ all pass through the centroid $G.$ Thus the powers of the centroid G of the medial $\triangle MNP$ to the circles $\mathcal Q_m, \mathcal Q_n, \mathcal Q_p$ are all equal to zero.

Consequently, the circles $\mathcal Q_m, \mathcal Q_n, \mathcal Q_p$ are coaxal and the common Euler line GO of the $\triangle MNP, \triangle ABC$ is their common radical axis. The circumcenter K of the medial $\triangle MNP,$ identical with the 9-point center of the $\triangle ABC,$ then lies on their common radical axis. Let the lines MK, NK cut the circles $\mathcal Q_m, \mathcal Q_n$ again at $M_q, N_q.$ Since $\overline{KM} = \overline{KN}$ and $\overline{KM} \cdot \overline{KM_q} = \overline{KN} \cdot \overline{KN_q},$ it follows that $\overline{KM_q} = \overline{KN_q}$ and $\overline{MM_q} = \overline{NN_q}.$

Let the lines KM, KN cut the circles $\odot(AMX), \odot(BNY)$ again at $M_0, N_0.$ Since M, N are similarity centers of $\mathcal Q_m \sim \odot(AMX), \mathcal Q_n \sim \odot(BNY),$ we get $\overline{MM_0} = 3\ \overline{MM_q} = 3\ \overline{NN_q} = \overline{NN_0}.$ Using $\overline{KM} = \overline{KN}$ again, this yields $\overline{KM_0} = \overline{KN_0}$ and $\overline{KM_0} \cdot \overline{KM} = \overline{KN_0} \cdot \overline{KM}.$ In conclusion, K is on the radical axis of the circles $\odot(AMX), \odot(BNY)$ and similarly, K is on the pairwise radical axes of $\odot(BNY), \odot(CPZ)$ and $\odot(CPZ), \odot(AMX).$ As a result, K is (at least) the radical center of these 3 circles.

Luis González

#3 Dec 9, 2009, 12:02 pm • 1

Let D be the foot of the A-altitude and H, N the orthocenter and the 9-point center of $\triangle ABC.$ Let L denote the midpoint of AH (orthocenter of APN) and let the ray ML cut the circle $\mathcal{G}_a \equiv \odot(AMX)$ at $T.$ Since LM is a diameter of the 9-point circle $(N),$ we have that $p(N, \mathcal{G}_a) = NM \cdot NT$ and from the power of L WRT $\mathcal{G}_a$ we get $R \cdot TL = AL \cdot LX$ since $\triangle ABC \sim \triangle APN$ with similarity coefficient $2.$ Then the power $AL \cdot LX$ equals a quarter of the power $AH \cdot HD.$

$$p(N, \mathcal{G}_a) = \frac{R}{2}\left(\frac{R}{2} + TL\right) = \frac{R}{2}\left(\frac{R}{2} + \frac{AH \cdot HD}{4R}\right)$$

$$p(N, \mathcal{G}_a) = \frac{R^2}{4} + \frac{AH \cdot HD}{8} = \frac{R^2}{4} + \frac{k^2}{8}$$

$AH \cdot HD$ equals the power of inversion k^2 that takes the circumcircle (O) of $\triangle ABC$ into its 9-point circle $(N),$ thus we conclude that $p(N, \mathcal{G}_a) = p(N, \mathcal{G}_b) = p(N, \mathcal{G}_c) \Longrightarrow N$ is the radical center of $\mathcal{G}_a, \mathcal{G}_b, \mathcal{G}_c.$

Quick Reply

High School Olympiads

Three points are conlinear

Reply

Jul

#1 May 25, 2015, 6:20 pm • 1

Suppose that $ABCD$ is a quadrilateral inscribed in the circle (O). Let E, F, G be the intersection of the pair of the lines $(AB, CD), (AD, BC), (AC, BD)$ respectively. Let H be the point which lies on (O) and M, N be the intersection of HE, HF with (O) respectively. Prove that M, N, G are conlinear.

TelvCohl

#2 May 25, 2015, 6:48 pm • 1

If P is a point varies on $\odot(O)$ and $R \equiv PE \cap \odot(O), S \equiv PF \cap \odot(O)$, then $ES \cap FR \in \odot(O)$ due to E, F are the conjugate points WRT $\odot(O)$ ... $(\star)$. Since $P \mapsto R$ is an Involution with pole E on $\odot(O)$ and $P \mapsto S$ is an Involution with pole F on $\odot(O)$, so $R \mapsto S$ is a homography Ψ on $\odot(O)$. Notice that if Ψ maps X to Y, then $Y \mapsto X$ under Ψ (from $(\star)$) $\Longrightarrow$ Ψ is an Involution on $\odot(O) \Longrightarrow RS$ pass through a fixed point when P varies on $\odot(O)$ (pole of the involution) .

In the original problem, the fixed point is G ($\because A \mapsto C, B \mapsto D$ under Ψ), so $G \in MN$.

shinichiman

#3 May 25, 2015, 7:49 pm • 1

We have a lemma:
Lemma 1. Let X, Y, Z, T, U, V are six points lying on a circle. Hence, XY, ZT, UV are concurrent if and only if $\frac{ZX}{ZY} \cdot \frac{TX}{TY} = \frac{UX}{UY} \cdot \frac{VX}{VY}$.
<u>*Proof.*</u> We using another lemma to prove this lemma.
Lemma 2. Let $ABCD$ is a cyclic quadrilateral. $BC \cap AD = E$ then $\frac{EB}{EC} = \frac{AB}{AC} \cdot \frac{DB}{DC}$.

The problem can be seen as: Given A, B, C, D, M, H, N in a circle. AB, CD, MH are concurrent; NH, AD, BC are concurrent. Prove that AC, MN, BD are concurrent.

From the concurrence of AB, CD, MH, we have

$$\frac{CA}{CB} \cdot \frac{DA}{DB} = \frac{MA}{MB} \cdot \frac{HA}{HB}$$
$$\frac{AC}{AD} \cdot \frac{BC}{CD} = \frac{MC}{MD} \cdot \frac{HC}{HD}$$

By timing these two identities we get

$$\left(\frac{AC}{BD}\right)^2 = \frac{MA}{MB} \cdot \frac{MC}{MD} \cdot \frac{HA}{HB} \cdot \frac{HC}{HD}.$$

Similarly to NH, AD, BC, we have

$$\left(\frac{AC}{BD}\right)^2 = \frac{HA}{HB} \cdot \frac{HC}{HD} \cdot \frac{NA}{NB} \cdot \frac{NC}{ND}.$$

Therefore, we have

$$\frac{MA}{MB} \cdot \frac{MC}{MD} = \frac{NA}{NB} \cdot \frac{NC}{ND}$$

or

$$\frac{AN}{AM} \cdot \frac{CN}{CM} = \frac{BN}{BM} \cdot \frac{DN}{DM}.$$

From here we obtain that AC, MN, BD are concurrent or M, N, G are collinear.

tranquanghuy7198

#4 May 25, 2015, 8:41 pm

My solution:
$MN \cap AB, CD = X, Y$
Now we have:
$\overline{M, G, N}$
$\iff \overline{X, G, Y}$
$\iff (EAXB) = (ECYD)$
$\iff M(EAXB) = M(ECYD)$
$\iff M(HANB) = M(HCND)$
$\iff D(HANB) = B(HCND)$
$\iff D(HFNB) = B(HFND)$
$\iff \overline{H, F, N}$ (right)
Q.E.D

mnguyen99

#5 May 25, 2015, 8:59 pm

> *tranquanghuy7198 wrote:*
> My solution:
> $MN \cap AB, CD = X, Y$
> Now we have:
> $\overline{M, G, N}$
> $\iff \overline{X, G, Y}$
> $\iff (EAXB) = (ECYD)$
> $\iff M(EAXB) = M(ECYD)$
> $\iff M(HANB) = M(HCND)$
> $\iff D(HANB) = B(HCND)$
> $\iff D(HFNB) = B(HFND)$
> $\iff \overline{H, F, N}$ (right)
> Q.E.D

Why you have it

Luis González

#6 Jun 5, 2015, 4:43 am

Posted before at http://www.artofproblemsolving.com/community/c6h573041.

Quick Reply

High School Olympiads

Concurrent(well-known?)

Reply

Iambosama
#1 Jan 27, 2014, 2:14 am

Give the cyclic quadrilateral ABCD. Let $AB \cap CD \equiv E, AD \cap BC \equiv F$. M is a point lie on $(ABCD)$. P, Q are the intersection of ME, MF with $(ABCD)$. Prove that: AC, BD, PQ are concurrent

IMOwnik
#2 Jan 27, 2014, 12:25 pm

It's well known that AC, BD, PQ are concurrent if and only if $AB \cdot PC \cdot DQ = BP \cdot CD \cdot QA$, since points A, B, P, C, D, Q lie on one circle .(Try to prove by yourself...). Which is easy follows)))

ThirdTimeLucky
#3 Jan 28, 2014, 12:21 am • 1

Another approach: Suppose $DQ \cap BP = K, BQ \cap DP = L, AC \cap BD = G, PQ \cap BD = G'$. By Pascal's Theorem on $PMQDCB$, we get $PM \cap CD = E, MQ \cap CB = F, QD \cap BP = K$ collinear. Using it again on $QMPDAB$, $QM \cap DA = F, MP \cap AB = E, PD \cap BQ = L$ are collinear. Therefore, $\overline{EF} = \overline{KL}$, i.e the polar of G is the same as the polar of G', so $G = G'$ i.e, AC, BD, PQ are concurrent

jlammy
#4 Jan 28, 2014, 12:42 am

Very short, slick solution on http://mathsolympian.wordpress.com/geometry-notes.

Check it out.

Edit: I've realised I misread the question, and it is now on the website with slick solution using Ceva.

Luis González
#5 Jan 28, 2014, 2:54 am

As M runs on the circumcircle ω, $M \mapsto P$ and $M \mapsto Q$ are homographic involutions with poles E, F, respectively $\Longrightarrow$ $\mathbf{H} : P \mapsto Q$ is a homography on ω. When $M \equiv C$, we have $P \equiv D, Q \equiv B$ and when $M \equiv A$, we have $P \equiv B, Q \equiv D$, thus $\mathbf{H}$ is an involution $\Longrightarrow$ all lines PQ go through the fixed pole of the involution. Making $M \equiv B$, we deduce that the fixed pole is then $AC \cap BD$.

IDMasterz
#6 Jan 29, 2014, 3:36 pm

Take EF to infinity under a homology, P, Q go to antipodal points so done.

jayme
#7 Jan 29, 2014, 7:08 pm

Dear Mathlinkers,

1. X the point of intersection of AC and BD
2. EF is the polar of X
3. Y the point of intersection of AP and CQ

3. Y the point of intersection of AP and CQ
4. EFY is the Pascal's line of PMQCDAP
Conclusion : PQ goes through X.

Sincerely
Jean-Louis

Luis González
#8 Jan 29, 2014, 11:21 pm

> *IDMasterz wrote:*
> Take EF to infinity under a homology, P, Q go to antipodal points so done.

This argument only works if the line EF does not cut the circle (conic). The concurrency holds for any quadrilateral, i.e. any 4 concyclic points A,B,C,D, hence the line EF can intersect the circle and such a homology does not exist.

Mosquitall
#9 Jan 29, 2014, 11:31 pm

What about complex plane (add complex points and make complex transformations)?
P.S. On complex plane every line intersect circle but transformation exists.

Mosquitall
#10 Jan 29, 2014, 11:43 pm

My previous post is only $conjecture$ about projective transformations on plane with complex points.

IDMasterz
#11 Jan 30, 2014, 3:52 pm

> *Luis González wrote:*
> This argument only works if the line EF does not cut the circle (conic). The concurrency holds for any quadrilateral, i.e. any 4 concyclic points A,B,C,D, hence the line EF can intersect the circle and such a homology does not exist.

Yes, true. Though there might be some bad analytical way to patch it up, it probably isn't pretty.

In general, I believe we take diagonal of complete $ABCD$ that is outside the circle to infinity. If it happens to be in the case you say, I believe we can do this:

If $AB \cap CD$ is inside $\odot ABCD$, then $AC \cap BD = X$, and take XF to infinity. Then, we get a rectangle as described before. Also, $AB \cap CD$ goes to the centre O of $\odot ABCD$. Since $\angle MQP = 90$ and MQ is the line parallel to AD etc... we have $PQ \parallel AC, BD$ are the result follows. (I am doing this in my head, but I think this works).

Quick Reply

High School Olympiads

Triangles of equal perimeters !!

Reply

Source: Romania TST 2015 Day4 Problem 1

ComplexPhi
#1 Jun 4, 2015, 10:45 pm

Let ABC and ABD be coplanar triangles with equal perimeters. The lines of support of the internal bisectrices of the angles CAD and CBD meet at P. Show that the angles APC and BPD are congruent.

aditya21
#2 Jun 4, 2015, 11:00 pm

> *ComplexPhi wrote:*
> The lines of support of the internal bisectrices of the angles CAD and CBD meet at P

what do you mean by this? please elaborate?
do you mean angle bisectors??

This post has been edited 1 time. Last edited by aditya21, Jun 4, 2015, 11:00 pm
Reason: e

ComplexPhi
#3 Jun 4, 2015, 11:06 pm

Yes, I mean the lines of support of the internal angle bisectors of CAD and CBD meet at P .

Luis González
#4 Jun 5, 2015, 4:29 am

As $\triangle ABC$ and $\triangle ABD$ have equal perimeter, then $CA+CB=DA+DB \Longrightarrow D$ is on the ellipse $\mathcal{E}$ with foci A,B that passes through C. If the tangents of $\mathcal{E}$ at C,D meet at P^*, then by conic tangent property, AP^* and BP^* bisect $\angle CAD$ and $\angle CBD \Longrightarrow P \equiv P^* \Longrightarrow PA,PB$ are isogonals WRT $\angle CPD$, i.e. $\angle APC=\angle BPD$.

Quick Reply

High School Olympiads

THVSH

#1 May 17, 2015, 10:12 pm • 1

Let ABC be a triangle with circumcenter O. D, E, F is the centers of $\odot(OBC)$; $\odot(OCA)$; $\odot(OAB)$, respectively. X, Y, Z is reflection of D, E, F in BC, CA, AB, respectively. Prove that the nine point center of $\triangle XYZ$ lies on the Euler line of $\triangle ABC$.

This post has been edited 1 time. Last edited by THVSH, May 17, 2015, 10:13 pm

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TelvCohl

#3 May 17, 2015, 11:49 pm • 3

My solution :

Lemma :

Let P, Q be the isogonal conjugate of $\triangle ABC$ and $\triangle DEF$ be the pedal triangle of Q .
Let N, N_a, N_b, N_c be the 9-point center of $\triangle ABC, \triangle PBC, \triangle PCA, \triangle PAB$, respectively .

Then $\triangle DEF \cup Q \sim \triangle N_aN_bN_c \cup N$

Proof :

Let A', B', C' be the midpoint of BC, CA, AB, respectively .
Let X', Y', Z' be the midpoint of PA, PB, PC, respectively .

It's well-known that $\odot(N), \odot(N_a), \odot(N_b), \odot(N_c)$ are concurrent at R .

Since $\angle NN_bN_c = \angle B'RX' = \angle PAC = \angle QAB = \angle QEF$ (similarly, $\angle NN_cN_b = \angle QFE$) ,
so $\triangle NN_bN_c \sim \triangle QEF$ (similarly, $\triangle NN_cN_a \sim \triangle QFD$) $\Longrightarrow \triangle DEF \cup Q \sim \triangle N_aN_bN_c \cup N$.

Back to the main problem :

Let T be the 9-point center of $\triangle XYZ$.
Let $\triangle PQR$ be the orthic triangle of $\triangle ABC$.
Let $N, H,$ be the 9-point center, orthocenter of $\triangle ABC$, respectively .
Let X', Y', Z' be the 9-point center of $\triangle OBC, \triangle OCA, \triangle OAB$, respectively .
(It's well-known that X', Y', Z' is the midpoint of OX, OY, OZ, respectively)

From lemma $\Longrightarrow \triangle PQR \cup H \sim \triangle X'Y'Z' \cup N$,
so $\triangle PQR \cup H \sim \triangle XYZ \cup H \Longrightarrow H$ is the incenter of $\triangle XYZ$.

From $\measuredangle ZOY + \measuredangle RHQ = 180^\circ \Longrightarrow \measuredangle ZOY + \measuredangle ZHY = 180^\circ$,
so we get $\odot(OYZ)$ and $\odot(HYZ)$ are symmetry with respect to YZ .
Similarly, $\odot(OZX)$ and $\odot(HZX)$ are symmetry with respect to ZX ,
so O is the antigonal conjugate of H WRT $\triangle XYZ$ (X_{80} of $\triangle XYZ$) ,
hence we get N is the Feuerbach point of $\triangle XYZ$ and $T \in NH$ (Euler line of $\triangle ABC$) .

Q.E.D

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buratinogigle

#5 May 18, 2015, 9:58 am • 2

This problem is true for all P on Neuberg cubic. Now I repropose problem

Let ABC be a triangle and P is a point such that Euler line of triangle PBC, PCA, PAB are concurrent. Let N_a, N_b, N_c, N_p be ninepoint center of triangles $PBC, PCA, PAB, N_aN_bN_c$, reps. Prove that symmetric of P through N_p lies on Euler line of triangle ABC.

I think we can use nice lemma of Telv !

High School Olympiads

About 9-point centers

Stephen

#1 Oct 24, 2011, 5:28 am

In triangle ABC, I is a incenter. Let N_a, N_b, N_c the 9-point centers of triangles BIC, CIA, AIB.

Prove that I is the orthocenter of triangle $N_aN_bN_c$.

Luis González

#2 Oct 24, 2011, 9:45 am • 1

Incircle (I) touches BC at P and X is the midpoint of IA. 9-point circles $(N), (N_a), (N_b), (N_c)$ of $\triangle ABC, \triangle IBC$, $\triangle ICA, \triangle IAB$ concur at the Feuerbach point F_e of $\triangle ABC$ (Poncelet point of ABCI). So, XF_e is radical axis of (N_b) and $(N_c) \Longrightarrow N_bN_c \perp XF_e$. PF_e is radical axis of (I) and $(N_a) \Longrightarrow IN_a \perp PF_e$. But XF_e cuts (I) again at the antipode of P WRT (I), i.e. $XF_e \perp PF_e$. Hence, $IN_a \perp N_bN_c$. Similarly, $IN_b \perp N_cN_a$ and $IN_c \perp N_aN_b \Longrightarrow I$ is the orthocenter of $\triangle N_aN_bN_c$.

Reference: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=325489.

DC93

#3 Oct 30, 2011, 7:01 am

> *luisgeometra wrote:*
> But XF_e cuts (I) again at the antipode of P WRT (I)

Could you tell me why?, Thank you!

simplependulum

#4 Oct 30, 2011, 12:16 pm

Let A_1 be the intersection point of the bisector of $\angle A$ and the circumcircle of ABC, A_2 be the reflection of A_1 in BC. Define B_2, C_2 similarly.

We know that N_a, N_b, N_c is the midpoint of IA_2, IB_2, IC_2, respectively, but at the same time Fuhrmann theorem states that I is the orthocentre of triangle $\Delta N_aN_bN_c$. Considering the dilation $I(\frac{1}{2})$, we get the result.

Luis González

#5 Oct 30, 2011, 9:51 pm

@DC93. See the post #4 in the given reference. When P coincides with the incenter I of $\triangle ABC$, P_1 becomes the antipode of P_A WRT (I) and U is the unique intersection of (I) and the 9-point circle of $\triangle ABC$.

Additional remark: 9-point center N of $\triangle ABC$ is circumcenter of $\triangle N_aN_bN_c$.

High School Olympiads

Concentric circles !

Source: Romania TST 2015 Day 2 Problem 2

ComplexPhi
#1 Jun 4, 2015, 9:09 pm

Let ABC be a triangle . Let A' be the center of the circle through the midpoint of the side BC and the orthogonal projections of B and C on the lines of support of the internal bisectrices of the angles ACB and ABC , respectively ; the points B' and C' are defined similarly . Prove that the nine-point circle of the triangle ABC and the circumcircle of $A'B'C'$ are concentric.

This post has been edited 1 time. Last edited by ComplexPhi, Jun 4, 2015, 10:24 pm

Luis González
#2 Jun 4, 2015, 9:57 pm

Let D,E,F be the midpoints of $BC,CA,AB.$ Circle (A') through D and the projections of B,C on internal bisectors of $\angle ACB, \angle ABC$ is 9-point circle of $\triangle IBC,$ where I is incenter of $\triangle ABC$ and similarly $(B'),(C')$ are 9-point circles of $\triangle ICA, \triangle IAB.$ They concur then at the Poncelet point Fe of $ABCI;$ Feuerbach point of $\triangle ABC.$

Let X,Y,Z be the midpoints of IA,IB,IC and let N be the 9-point center of $\triangle ABC.$ We have $NB' \perp FeE$ and $B'C' \perp FeX \Longrightarrow \angle NB'C' = \angle XFeE$ and similarly $\angle NC'B' = \angle XFeF.$ But since FeX bisects $\angle EFeF$ (well-known) then $\angle NB'C' = \angle NC'B' \Longrightarrow NB' = NC'$ and similarly we'll get $NC' = NA' \Longrightarrow N$ is circumcenter of $\triangle A'B'C'.$

Luis González
#3 Jun 4, 2015, 10:15 pm

For a more general configuration see Nine point center lies on the Euler line (lemma at post #3). Additionally, the incenter I of ABC is orthocenter of A'B'C'. For a proof See About 9-point centers.

tranquanghuy7198
#4 Jun 6, 2015, 10:42 pm

> *ComplexPhi wrote:*
> Let ABC be a triangle . Let A' be the center of the circle through the midpoint of the side BC and the orthogonal projections of B and C on the lines of support of the internal bisectrices of the angles ACB and ABC , respectively ; the points B' and C' are defined similarly . Prove that the nine-point circle of the triangle ABC and the circumcircle of $A'B'C'$ are concentric.

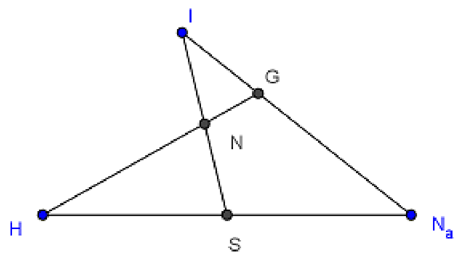
My solution.

Lemma.
Given $\triangle ABC$ and O,N are its circumcenter and nine-point center. $M = R_{BC}(O)$ then N is the midpoint of AM
Proof. Too trivial

Back to our main problem.
Let I be the incenter of $\triangle ABC$
D,E,F are the midpoints of $\overset{\frown}{BC}, \overset{\frown}{CA}, \overset{\frown}{AB}$
X,Y,Z are the reflections of D,E,F WRT BC,CA,AB
It's clear that D is the circumcenter of $\triangle IBC$, $X = R_{BC}(D)$, A' is the nine-point center of $\triangle IBC$
$\Rightarrow$ A' is the midpoint of IX (lemma)
Analogously, we have: $H_I^2 : A' \mapsto X, B' \mapsto Y, C' \mapsto Z$ (1)
$\Rightarrow$ Center of $(A'B'C') \mapsto$ center of (XYZ)

$\Rightarrow$ Center of $(A'B'C') \mapsto$ center of (XYZ)
But where is the center of (XYZ)? We must remember that (XYZ) is the Fuhrmann circle of $\triangle ABC$, and HN_a is its diameter, i.e the center of (XYZ) is the midpoint S of HN_a!
Notice that the centroid G of $\triangle ABC$ lies on IN_a s.t. $\dfrac{GI}{GN_a} = \dfrac{-1}{2}$. Moreover, the nine-point center N of $\triangle ABC$ lies on HG s.t. $\dfrac{NH}{NG} = -3$ (as you can see in the figure)
Because of all that, N must be the midpoint of IS, then due to the homothety (1), N is the center of $(A'B'C')$.
Q.E.D

Attachments:



anantmudgal09

#5 Sep 12, 2015, 4:47 am

Use the right angles on the in-touch chord lemma(see Yufei Zhao) to observe that A' is the nine point center of $\triangle BIC$. Now just complex bash it(pretty straight forward.)

P.S.- The complex bash is similar to the solution given below.

This post has been edited 1 time. Last edited by anantmudgal09, Oct 14, 2015, 12:02 am

Dukejukem

#6 Sep 12, 2015, 7:46 am

We begin with a lemma.

Back to the problem at hand, note that A' is just the nine-point center of $\triangle BIC$, where I is the incenter. Now, let M, N, P be the midpoints of arcs $\widehat{BC}, \widehat{CA}, \widehat{AB}$ on $\odot(ABC)$. It is well-known that M is the circumcenter of $\triangle BIC$. Thus, if M' is the reflection of M in BC, it follows from the lemma that A' is the midpoint of $\overline{IM'}$. Now, we use complex numbers.

WLOG set $\odot(ABC)$ to be the unit circle. It is well-known that there exist complex numbers x, y, z such that

$$a = x^2, b = y^2, c = z^2, m = -yz, n = -zx, p = -xy, i = -xy - yz - zx.$$

Thus, we compute

$$a' = \frac{i + m'}{2} = \frac{i + (b + c - bc\overline{m})}{2} = \frac{y^2 + z^2 - x(y+z)}{2}.$$

Then if Q is the nine-point center of $\triangle ABC$, we have $q = \frac{x^2+y^2+z^2}{2}$, implying that

$$|q - a'| = \left|\frac{x(x+y+z)}{2}\right| = \frac{1}{2}|x+y+z|,$$

which is symmetric in x, y, z. Therefore, $|q - a'| = |q - b'| = |q - c'|$, implying that Q is the circumcenter of $\triangle A'B'C'$, as desired. $\square$

This post has been edited 1 time. Last edited by Dukejukem, Sep 25, 2015, 7:16 am

utkarshgupta
#7 Oct 13, 2015, 9:46 pm

> *ComplexPhi wrote:*
> Let ABC be a triangle . Let A' be the center of the circle through the midpoint of the side BC and the orthogonal projections of B and C on the lines of support of the internal bisectrices of the angles ACB and ABC , respectively ; the points B' and C' are defined similarly . Prove that the nine-point circle of the triangle ABC and the circumcircle of $A'B'C'$ are concentric.

Just use homothety centred at G with ratio -1/2.
Will add details soon.

Dukejukem
#8 Dec 21, 2015, 5:59 am

Let I, G, O, N be the incenter, centroid, circumcenter, nine-point center of $\triangle ABC.$ Let G' be the centroid of $\triangle BIC$, and note that A' is just the nine-point center of $\triangle BIC.$ Let M and D be the midpoints $\overline{BC}$ and arc $\widehat{BC}$ on $\odot(ABC).$

The homothety $\mathbf{H}(G, -2)$ sends $\overline{MN} \mapsto \overline{AO}.$ Similarly, since D is the circumcenter of $\triangle BIC$ (well-known), the homothety $\mathbf{H}(G', -2)$ sends $\overline{MA'} \mapsto \overline{ID}.$ Thus, if J is the reflection of I in the midpoint of $\overline{AD}$, we have $AJ = ID = 2 \cdot MA'.$ Moreover, $AO = 2 \cdot MN$ and $\angle(AO, AJ) = \angle(MN, MA')$ since homothety preserves angles. This is enough to imply that $\triangle AOJ \sim \triangle MNA'$ with similarity coefficient $2.$ Therefore, $NA' = \frac{1}{2} \cdot OJ = \frac{1}{2} \cdot OI$, where the last step follows from symmetry in the perpendicular bisector of $\overline{AD}.$ Writing analagous relations, we obtain $NA' = NB' = NC'$, as desired. $\square$

This post has been edited 1 time. Last edited by Dukejukem Dec 21, 2015, 6:09 am

Quick Reply

High School Olympiads

Line is tangent to circle

Reply

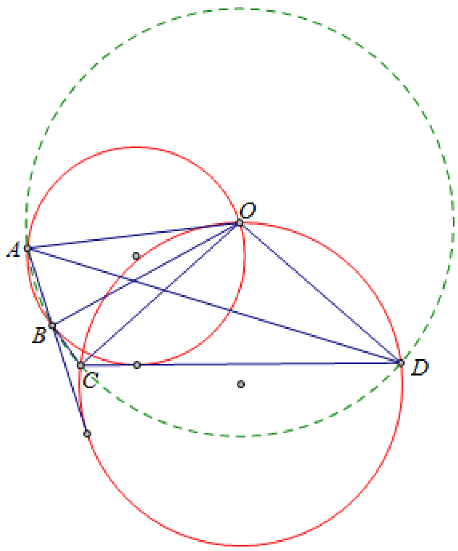
Source: Own

LeVietAn

#1 Jun 4, 2015, 7:52 am

Dear Mathlinkers,
Let $ABCD$ be a cyclic quadrilateral with center O. Prove that AB is tangent to $\odot(OCD) \Leftrightarrow CD$ is tangent to $\odot(OAB)$.

Attachments:



This post has been edited 1 time. Last edited by LeVietAn, Jun 4, 2015, 7:59 am

TelvCohl

#2 Jun 4, 2015, 8:05 am • 1

My solution :

Since $AB \longleftrightarrow \odot(OAB), CD \longleftrightarrow \odot(OCD)$ under $\mathbf{I}(\odot(O))$,
so AB is tangent to $\odot(OCD)$ if and only if CD is tangent to $\odot(OAB)$.

Q.E.D

LeVietAn

#3 Jun 4, 2015, 11:12 am

The problem is a case of the following problem:
Two circles (O) and (O') intersect at A and B. A line t varies, which does not contain O and is tangent to the circle (O') such that t intersects the circle (O) at E and F. Prove that the circumcircle of the triangle OEF alway is tangent to either a fixed line or a fixed circle.

Luis González

#4 Jun 4, 2015, 1:25 pm

> *LeVietAn wrote:*
>
> The problem is a case of the following problem:
> Two circles (O) and (O') intersect at A and B. A line t varies, which does not contain O and is tangent to the circle (O') such that t intersects the circle (O) at E and F. Prove that the circumcircle of the triangle OEF alway is tangent to either a fixed line or a fixed circle.

Let M be the midpoint of $\overline{EF}$. The antipode J of O WRT $(K) \equiv \odot(OEF)$ is clearly the inverse of M under inversion WRT (O). Since M runs on the pedal curve of O WRT (O') (a limaçon $\mathcal{L}$ with cusp O), then J moves a conic $\mathcal{C}$ with focus O and focal axis along OO'. Thus by homothety with center O and coefficient $\frac{1}{2}$, it follows that K moves on a central conic with foci O and $O'' \Longrightarrow$ either $KO + KO'' = \text{const} = \varrho$ or $|KO - KO''| = \text{const} = \varrho$. Hence, in both cases, $\odot(OEF)$ touches the circle (O'', ϱ).

From the above reasoning, we also conclude that $\odot(OEF)$ touches a line $\Longleftrightarrow O \in (O')$.

TelvCohl

#5 Jun 4, 2015, 2:28 pm

> *LeVietAn wrote:*
>
> The problem is a case of the following problem:
> Two circles (O) and (O') intersect at A and B. A line t varies, which does not contain O and is tangent to the circle (O') such that t intersects the circle (O) at E and F. Prove that the circumcircle of the triangle OEF alway is tangent to either a fixed line or a fixed circle.

My solution **:**

Let ℓ be the polar of O' WRT $\odot(O)$ and P be the pole of EF WRT $\odot(O)$.

From 1st Salmon theorem $\Longrightarrow \frac{OP}{OO'} = \frac{\text{dist}(P, \ell)}{\text{dist}(O', EF)} \Longrightarrow \frac{OP}{\text{dist}(P, \ell)} = \text{constant}$,

so P moves on a fixed conic $\mathcal{C}$ (with focus O (and directrix ℓ) and T) $\Longrightarrow \odot(P, PO)$ is tangent to a fixed circle $\odot(T)$,

hence we get $\odot(OEF) \equiv \odot(OP)$ is tangent to a fixed cirlce $\odot(T')$ (image of $\odot(T)$ under the homothety $\mathbf{H}(O, \frac{1}{2})$) .

Q.E.D

---

tranquanghuy7198

#6 Jun 4, 2015, 9:28 pm

> *LeVietAn wrote:*
>
> The problem is a case of the following problem:
> Two circles (O) and (O') intersect at A and B. A line t varies, which does not contain O and is tangent to the circle (O') such that t intersects the circle (O) at E and F. Prove that the circumcircle of the triangle OEF alway is tangent to either a fixed line or a fixed circle.

My solution:
Let $\omega = I_{(O)}(\odot(O'))$ is a fixed circle.
Now: $I_{(O)} : E \mapsto E, F \mapsto F, (OEF) \mapsto EF, \omega \mapsto (O')$
Since EF is tangent to (O') then (OEF) is tangent to ω :)
Q.E.D

LeVietAn

#7 Jun 4, 2015, 10:23 pm

My Solution:

Let the opposite ray of the ray OO' intersects (O') at C, r is radius of (O); $D \in OC$ such that $\overline{OD}.\overline{OC} = r^2$

Let the opposite ray of the ray OO' intersects (O') at C, r is radius of (O); $D \in OC$ such that $OD.OC = r^2$,
$t \cap (O') = T$ and $OT \cap (O) = \{O, M\}$.
$\mathbf{I}(O, r^2)$: $\overline{EFT} \leftrightarrow \odot(EFM), \odot(ABC) \leftrightarrow (ABD)$
Thence, because $\overline{EFT}$ is tangent $\odot(ABC)$ deduced $\odot(OEF) \equiv (EFM)$ is tangent to (ABD).
Q.E.D

This post has been edited 1 time. Last edited by LeVietAn, Jun 4, 2015, 10:23 pm

Quick Reply

High School Olympiads

nice geometry

Reply

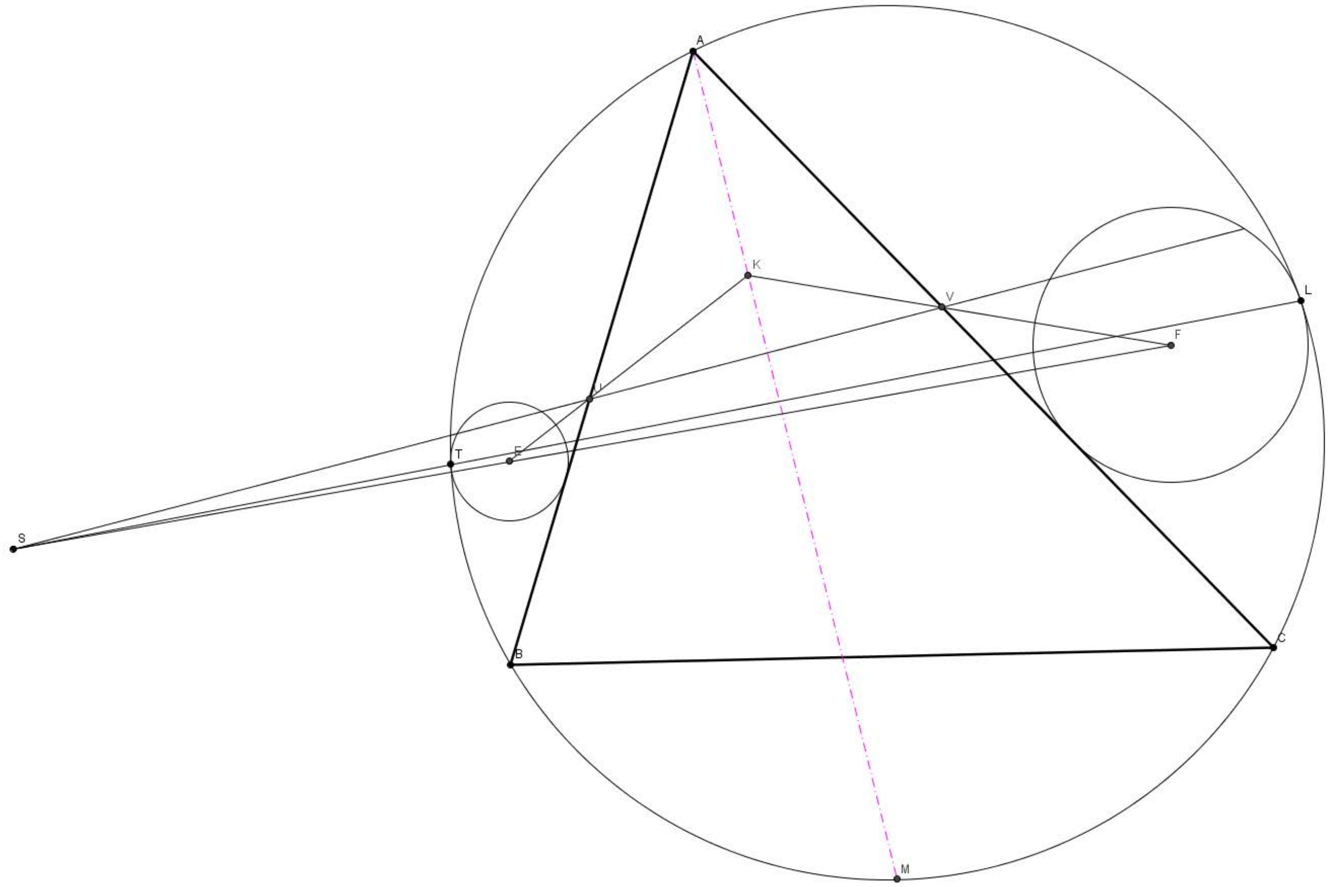
Source: Own

andria

#1 Jun 2, 2015, 4:23 pm

In triangle ABC M is midpoint of arc BC(doesn't contain A),ω_C is an arbitrary circle with center E that it is tangent to AB and $\odot(\triangle ABC)$ at T similarly ω_B is an arbitrary circle with center F that it is tangent to AC and $\odot(\triangle ABC)$ at L. let $LT\cap EF=S$, ℓ is an arbitrary line passing throw S that it intersect AB, AC at U, V let $EU\cap FV=K$. prove that A, K, M are collinear.

Attachments:



TelvCohl

#2 Jun 2, 2015, 5:22 pm • 1

My solution :

Let τ_1, τ_2 be the common external tangents of ω_B and ω_C .
Let $W\equiv \tau_1\cap AB, X\equiv \tau_1\cap AC, Y\equiv \tau_2\cap AB, Z\equiv \tau_2\cap AB$.

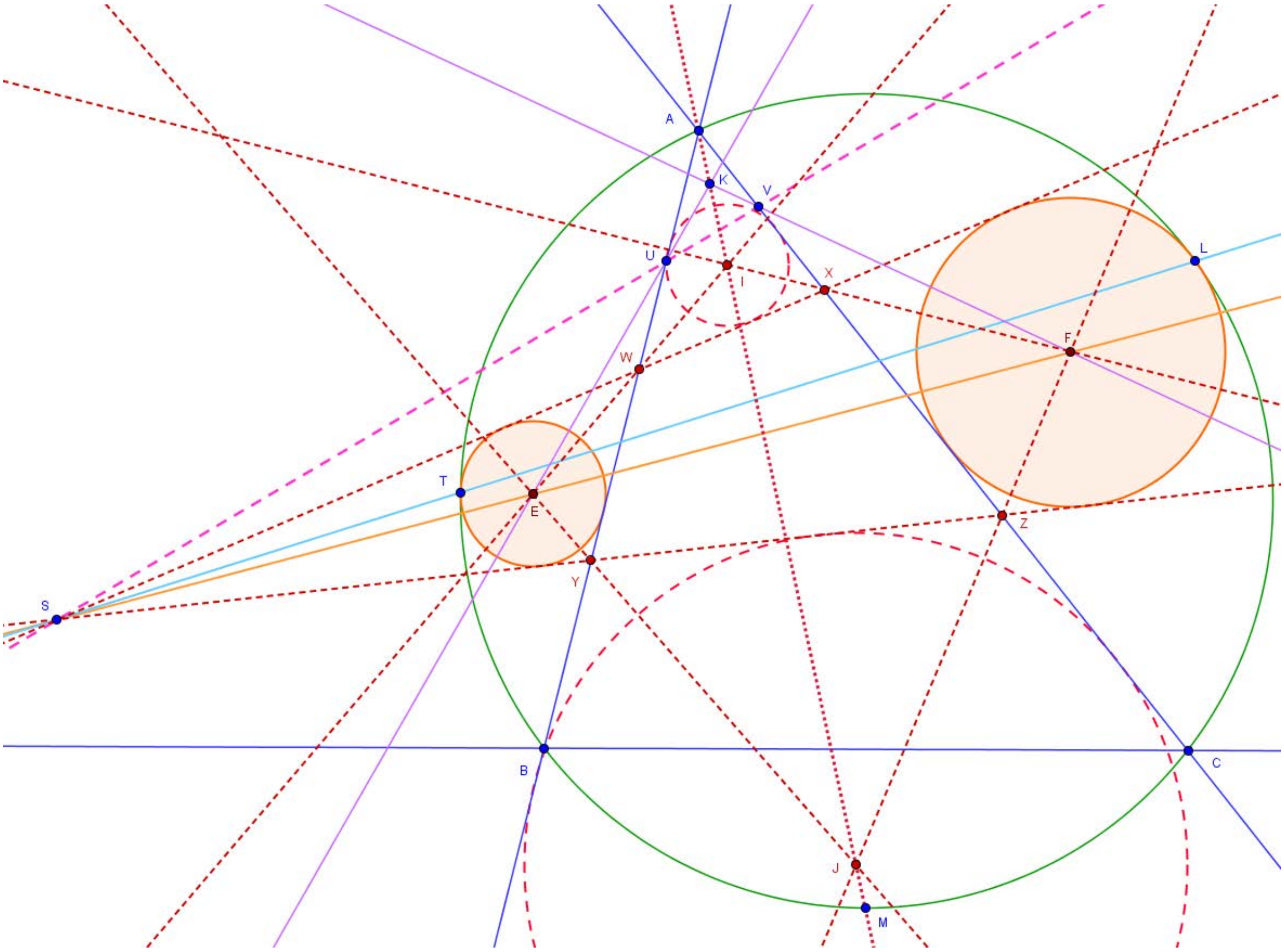
From D'Alembert theorem (for $\omega_B, \omega_C, \odot(ABC)$) $\Longrightarrow$ S is the exsimilicenter of $\omega_B\sim\omega_C \Longrightarrow S\equiv \tau_1\cap\tau_2$.

From symmetry $\Longrightarrow$ EW, FX is the internal bisector of $\angle W, \angle X$ of $\triangle AWX$, respectively ,
so $I\equiv EW\cap FX$ is the incenter of $\triangle AWX \Longrightarrow I$ lie on the internal bisector AM of $\angle BAC$.
Similarly,we can prove $J\equiv EY\cap FZ$ is the A-excenter of $\triangle AYZ$ and $J\in AM \Longrightarrow A, I, J, M$ are collinear . ... $(\star)$

Since $E(A, U; W, Y) = (A, U; W, Y) = (A, V; X, Z) = F(A, V; X, Z)$, so from $(\star)$ we get $K \equiv EU \cap FV$ lie on $\overline{AIJ} \Longrightarrow A, K, M, I, J$ are collinear .

Q.E.D

Attachments:



Luis González

#3 Jun 4, 2015, 11:24 am

Since T and L are the exsimilicenters of $\omega_C \sim \odot(ABC)$ and $\omega_B \sim \odot(ABC)$, by Monge's & d'Alembert theorem it follows that S is the exsimilicenter of $\omega_C \sim \omega_B$. Thus when ℓ varies, $U \mapsto V$ is a perspectivity between $AB, AC \Longrightarrow EU \mapsto FV$ is a perspectivity as well, because $EU \equiv FV$ when $U \in EF \cap AB \Longrightarrow K$ moves on a fixed line. When ℓ coincides with the common external tangents of ω_B and ω_C, then K becomes either incenter or A-excenter of $\triangle AUV \Longrightarrow AK$ is internal bisector of $\angle BAC$ and the conclusion follows.

Quick Reply

High School Math

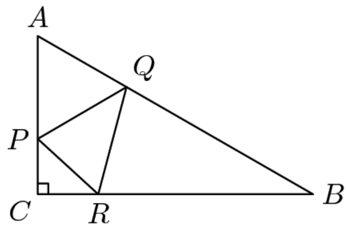
Side Lengths of Inscribed Triangle

Reply

djmathman

#1 May 22, 2015, 4:43 am

In right triangle ABC, $\overline{PQ}$ and $\overline{QR}$ are drawn so that $AQ = AP$ and $BQ = BR$. If $PQ = 4\sqrt{3}$ and $QR = 3\sqrt{6}$, what is PR?



Luis González

#2 May 22, 2015, 5:06 am

From isosceles $\triangle APQ$ and $\triangle BRQ$, we get $\widehat{AQP} = 90^\circ - \frac{1}{2}\widehat{A}$ and $\widehat{BQR} = 90^\circ - \frac{1}{2}\widehat{B} \Longrightarrow$ $\widehat{AQP} + \widehat{BQR} = 180^\circ - \frac{1}{2}(\widehat{A} + \widehat{B}) = 135^\circ \Longrightarrow \widehat{PQR} = 45^\circ$. By cosine rule we get then

$$PR^2 = (4\sqrt{3})^2 + (3\sqrt{6})^2 - 2 \cdot 4\sqrt{3} \cdot 3\sqrt{6} \cdot \cos 45^\circ = 30 \Longrightarrow PR = \sqrt{30}.$$

Quick Reply

High School Olympiads

Source: vankhea

vankhea

#1 May 20, 2015, 10:30 pm

Let D, E, F be points on the sides BC, CA, AB such that AD, BE, CF concurrent at O. Let T be point on BC. The rays TE and TF cut AD at P and Q.

Prove that: $\frac{DO}{OQ}\cdot\frac{QD}{DA}\cdot\frac{AP}{PD}=1$

THVSH

#2 May 20, 2015, 10:50 pm

My solution:

By Menelaus and Ceva theorem, we have 5 equations as following:

$(1)\frac{OD}{OQ}.\frac{CT}{CD}.\frac{FQ}{FT}=1$

$(2)\frac{DQ}{DA}.\frac{BA}{BF}.\frac{TF}{TQ}=1$

$(3)\frac{PA}{PD}.\frac{EC}{EA}.\frac{TD}{TC}=1$

$(4)\frac{EA}{EC}.\frac{DC}{DB}.\frac{FB}{FA}=1$

$(5)\frac{TQ}{FQ}.\frac{AF}{AB}.\frac{DB}{DT}=1$

From all of them, we get : $\frac{DO}{OQ}\cdot\frac{QD}{DA}\cdot\frac{AP}{PD}=1$ Q.E.D

This post has been edited 1 time. Last edited by THVSH, May 20, 2015, 10:52 pm

Luis González

#3 May 21, 2015, 3:26 am

The relation is equivalent to $(P, A, D, Q) = (Q, O, A, D)$, thus it is invariant under a homology sending AT to the line at infinity. In this figure, we have $FB \parallel EC \parallel OD$ and $EP \parallel FQ \parallel BC \Longrightarrow$

$$\frac{DO}{OQ}=\frac{OC}{OF}=\frac{CE}{BF}=\frac{PD}{QD}\Longrightarrow\frac{DO}{OQ}=\frac{PD}{QD}\Longrightarrow\frac{DO}{OQ}\cdot\frac{QD}{PD}\cdot\frac{AP}{AD}=1.$$

tranquanghuy7198

#4 May 21, 2015, 8:36 am

My solution bases on Menelaus theorem, too. But I only use it 3 times:

$EF\cap AD=K$. We have:

$\frac{AP}{PD}.\frac{DQ}{QO}=(\frac{AE}{EC}.\frac{CT}{TD}).(\frac{DT}{TC}.\frac{CF}{FO})=\frac{AE}{EC}.\frac{CF}{FO}=\frac{AK}{KO}=\frac{AD}{DO}$ and the conclusion follows.

vankhea

#5 May 21, 2015, 7:07 pm

Thanks you THVSH and Luis González and tranquanghuy71 for all your short solution.

Now i have one more problem that it is a generalization of above problem.

Let E and F be points on the side CA and AB such that BE cut CF at O. Let l be line pass through O cut BC at S. Let T be point on BC. The rays TE, TF, TA cuts the line l at P, Q, R.
Prove that $\frac{SO}{OQ} \cdot \frac{QS}{SR} \cdot \frac{RP}{PS} = 1$

vankhea

#6 May 21, 2015, 7:55 pm

i got this problem with my proving in this equality $\frac{1}{SO} + \frac{1}{SR} = \frac{1}{SP} + \frac{1}{SQ}$

Luis González

#7 May 22, 2015, 4:42 am

Here is a proof to the generalized relations posted above:

By dual of Desargues involution theorem for the complete quadrangle $AFOE,$ it follows that $TA \mapsto TO, TE \mapsto TF,$ $TB \mapsto TC$ is an involution $\Longrightarrow$ S is fixed in the involution $P \mapsto Q, O \mapsto R$ on the line $l \Longrightarrow$

$$(S, O, P, R) = (S, R, Q, O) \Longrightarrow (S, O, P, R) \cdot (R, O, S, Q) = 1 \Longrightarrow \frac{SO}{OQ} \cdot \frac{QS}{SR} \cdot \frac{RP}{PS} = 1.$$

We also have $(P, R, Q, S) = (Q, O, P, S) = (O, Q, S, P) \Longrightarrow$

$$\frac{OQ}{OS} \cdot \frac{PS}{PQ} = \frac{PR}{PQ} \cdot \frac{SQ}{SR} \Longrightarrow \frac{OQ}{SO \cdot SQ} = \frac{PR}{SP \cdot SR} \Longrightarrow$$

$$\frac{SQ - SO}{SO \cdot SQ} = \frac{SR - SP}{SP \cdot SR} \Longrightarrow \frac{1}{SO} + \frac{1}{SR} = \frac{1}{SP} + \frac{1}{SQ}.$$

Quick Reply

High School Math

Geometry

Reply

IstekOlympiadTeam

#1 May 21, 2015, 9:00 pm

ABC be triangle and let M be the midpoint of the side BC. Triangles ABM and ACM are inscribed in circles w_1 and w_2.Points P and Q are the midpoints of arcs AB and AC. Prove that PQ is perpendicular to AM

tranquanghuy7198

#2 May 21, 2015, 9:34 pm • 1

My solution:
Let O_1, O_2 be the centers of $\omega_1, \omega_2 \Rightarrow O_1O_2 \perp AM$; S is the midpoint of O_1O_2, U, V are the projections of P, Q on O_1O_2
We have: $\angle PO_1U = 180 - \angle AO_1P - \angle AO_1S = 180 - \angle ASO_1 - \angle AO_1S = \angle SAO_1$
Analogously, we have: $\angle QO_2V = \angle SAO_2$
Because: $\frac{sin\angle SAO_1}{sin\angle SAO_2} = \frac{AO_2}{AO_1} \Rightarrow \frac{PO_1}{sin\angle PO_1U} = \frac{QO_2}{sin\angle QO_1V} \Rightarrow PU = QV \Rightarrow PQ \parallel O_1O_2 \Rightarrow PQ \perp AM$
Q.E.D

Luis González

#3 May 22, 2015, 1:58 am

Inverting the figure with center M and arbitrary power, the problem becomes: In a $\triangle ABC, M$ is the midpoint of BC and the internal bisectors of $\angle AMB$ and $\angle AMC$ cut AB and AC at P and Q. Then the circumcenter of $\triangle MPQ$ lies on AM.

By angle bisector theorem we have $\tfrac{AP}{PB} = \tfrac{AM}{MB} = \tfrac{AM}{MC} = \tfrac{AQ}{QC} \Longrightarrow PQ \parallel BC$. Thus if $O \equiv AM \cap PQ$, then $\angle OMP = \angle BMP = \angle OPM \Longrightarrow OP = OM$ and similarly $OQ = OM \Longrightarrow OP = OQ = OM \Longrightarrow O$ is circumcenter of $\triangle MPQ$, lying on AM, as desired.

Quick Reply

High School Olympiads

BMO1 2006/07 Question 4 Geometry Problem

Locked

Source: BMO1 2006/07

MadChickenMan

#1 May 21, 2015, 10:57 pm

$4.$ Two touching circles S and T share a common tangent which meets S at A and T at B. Let AP be a diameter of S and let the tangent from P to T touch it at Q. Show that $AP = PQ$.

I'm aware that this problem has been posted before on this site, but I couldn't really understand the proof. If someone could give a simple proof, preferably with a visual aid, that would be great.

Luis González

#2 May 21, 2015, 11:11 pm

Please do not repost contest problems. You can bump BrMO R1 2006/07 P4 to ask for more solutions or clarifications anytime.

High School Olympiads

Tangents of circles

Reply

Source: BrMO R1 2006/07 P4

utkarshgupta
#1 Jun 2, 2014, 11:57 am

Two touching circles S and T share a common tangents which meets S at A and T at B. Let AP be a diameter of S and let the tangent from P to T touch it at Q. Show that $AP = PQ$.

BBAI
#2 Jun 2, 2014, 4:02 pm

Let the common tangency point be X. Let the centre of S be U and centre of T be V.
As $UA||VB$ So by angle chasing we see that $\angle AXB = 90^0$ So when BX is produced to meet at S at a point R So $\angle AXR = 90^0$(as $\angle AXB = 90^0$). So AR is a diameter
So $R = P$.
and X, P, B are collinear.
Also by angle chasing we see that $\angle XBA = \angle XAP \implies AP$ is tangent to $\odot AXB \implies AP^2 = PX.PB$
So PQ is tangent to $T \implies PQ^2 = PX.PB$
So $AP = PQ$
Hnece proved

inthedarkness
#3 Jun 2, 2014, 4:39 pm

We consider, $S = (O_1, r_1)$ and $T = (O_2, r_2)$ and we assume $r_1 > r_2$. Now draw, $O_2S \perp AP$, clearly, $(O_2S)^2 = 4r_1r_2$ and $SP^2 = (2r_1 - r_2)^2$,
so, we have, $PQ^2 = (PO_2)^2 - (r_2)^2 = (SO_2)^2 + SP^2 - (r_2)^2 = 4r_1r_2 + (2r_1 - r_2)^2 - (r_2)^2 = (4r_1)^2 = AP^2$

utkarshgupta
#4 Jun 11, 2014, 4:32 pm

What do you denote by S?
First you assign them as circles and then you use SP^2

Am I missing something? (I sure am)

inthedarkness
#5 Jun 11, 2014, 5:45 pm

Omg! Just change the name of first circle instead of S

utkarshgupta
#6 Jun 11, 2014, 5:54 pm

Then how do you define S & T ?

inthedarkness
#7 Jun 11, 2014, 6:07 pm

For example SO_2 here, S is a point, if you change the name of the first circle M instead of S then the solution becomes clear!

utkarshgupta
#8 Jun 11, 2014, 6:21 pm

That's what I am asking!
Which point is S?

inthedarkness
#9 Jun 11, 2014, 6:28 pm

Here S is the foot of perp. To AP from O_2 , actually I should select another name for this point because S is already using as the name of the first circle, Got it?

jlammy
#10 Jul 4, 2014, 10:05 pm • 1

Here is a solution using inversion.

Consider the circle U with centre P, and which passes through A. Invert with respect to U. The line AB exchanges with circle S. The line PQ inverts to itself. The circle T is tangent to the circle S, the line AB and the line PQ, so T inverts to itself. The point Q inverts to a point on the line PQ and also on the circle T, so Q inverts to Q. Therefore, $PQ = AP$, as required.

Quick Reply

High School Olympiads

Circles with a common point

Reply

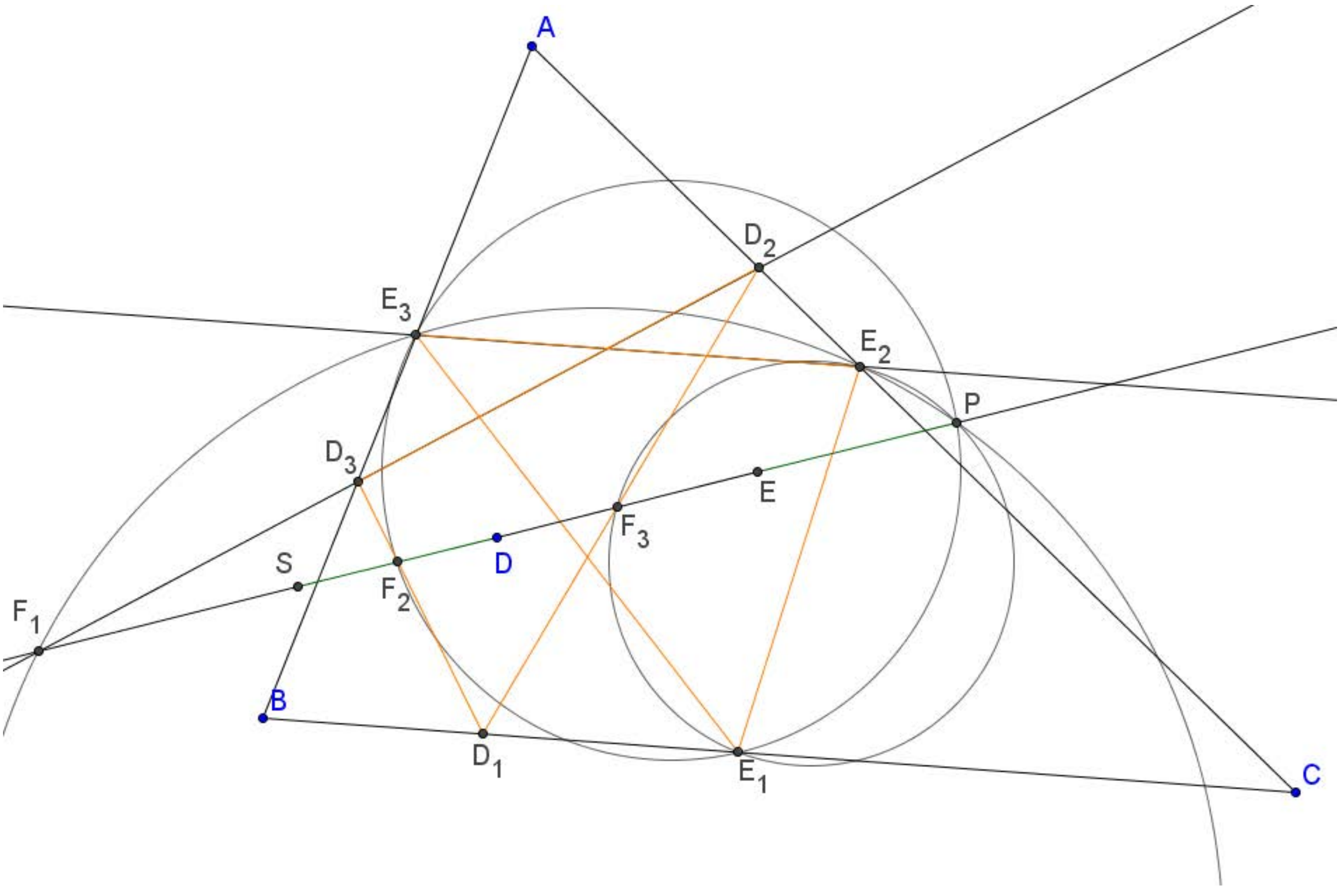
Source: A generalization of a problem from journal of classical geometry.

Cezar

#1 May 20, 2015, 12:42 am • 1

Let E be the isogonal conjugate of D in $\triangle ABC$.
Let $\triangle D_1D_2D_3$ and $\triangle E_1E_2E_3$ be the pedal triangles of points D and E.
Let $D_2D_3 \cap DE = F_1$, $D_1D_3 \cap DE = F_2$, $D_1D_2 \cap DE = F_3$.
1)Prove that $\odot F_3E_2E_1 \cap \odot F_2E_1E_3 \cap \odot F_1E_2E_3 = P$ and that P belongs to DE.
2)Let S be the point which has the same construction as P, for S we take the intersections of DE and $\triangle E_1E_2E_3$. Prove $SD = EP$.

Attachments:



Luis González

#2 May 20, 2015, 5:25 am • 2

It's well-known that $\triangle D_1D_2D_3$ and $\triangle E_1E_2E_3$ have common circumcircle ω centered at the midpoint of DE and it's also known that $O \equiv D_2D_3 \cap E_2D_3$ is on DE, being the pole of AM WRT ω, where M is the projection of A on DE (for a proof see post #1 at http://www.artofproblemsolving.com/community/c6h438841).

Let $J \equiv E_2E_3 \cap DE$ and let P denote the 2nd intersection of $\odot(F_1E_2E_3)$ with DE. Then J is the center of the involution that swaps $\{U, V\}, \{E, M\}$ and $\{F_1, P\} \Longrightarrow$

$$(P, U, V, \infty) = (F_1, V, U, J) \Longrightarrow \frac{PU}{PV} = \frac{F_1V}{F_1U} \cdot \frac{JU}{JV} \quad (1).$$

But by Desargues involution theorem for $E_2D_2E_3D_3$ cut by DE, we get

$$(F_1,V,U,O)=(J,U,V,O)\implies \frac{F_1V}{F_1U}=\frac{JU}{JV}\cdot\frac{OV^2}{OU^2}\quad (2).$$

Combining (1) and (2), keeping in mind that $(U,V,M,O)=-1$ and $(M,V,U,\infty)=(E,U,V,J)$, we obtain

$$\frac{PU}{PV}=\left(\frac{JU}{JV}\cdot\frac{OV}{OU}\right)^2=\left(\frac{JU}{JV}\cdot\frac{MV}{MU}\right)^2=\left(\frac{JU}{JV}\cdot\frac{EU}{EV}\cdot\frac{JV}{JU}\right)^2=\frac{EU^2}{EV^2}.$$

This reveals that P is determined by U,V,E only, which implies that P is also on circles $\odot(F_2E_1E_3)$ and $\odot(F_3E_2E_1)$. By similar reasoning, the point S verifies $\frac{SV}{SU}=\frac{DV^2}{DU^2}$. Since D,E are isotomic points WRT UV, then it follows that P,S are also isotomic points WRT U,V, i.e. $SD=EP$.

TelvCohl

#3 May 20, 2015, 9:36 am • 1

My solution :

Let Ω be the common circumcircle of $\triangle D_1D_2D_3$ and $\triangle E_1E_2E_3$.
Let $E_1'=AE\cap D_2D_3, E_2'=BE\cap D_3D_1, E_3'=CE\cap D_1D_2, X=D_2D_3\cap E_2E_3, A'=AX\cap DE$.

Since D_2D_3, E_2E_3 is the radical axis of $\{\Omega,\odot(AD_2D_3)\},\{\Omega,\odot(AE_2E_3)\}$, respectively ,
so X is the radical center of $\{\Omega,\odot(AD_2D_3)\equiv\odot(AD),\odot(AE_2E_3)\equiv\odot(AE)\}\implies AA'\perp DE$.

From $\angle AA'F_1=\angle AE_1'F_1=90^\circ\implies A,A',E_1',F_1$ are concyclic ,
so $E_1'X\cdot F_1X=AX\cdot A'X=E_2X\cdot E_3X\implies E_1'\in\odot(F_1E_2E_3)$.
Similarly, we can prove $E_2'\in\odot(F_2E_3E_1)$ and $E_3'\in\odot(F_3E_1E_2)$.

Let $P=\odot(F_1E_2E_3)\cap DE$.

Since E,E_1',E_2',D_3,E_3 are concyclic at the circle with diameter ED_3 ,
so from $\angle E_3E_2'F_2=180^\circ-\angle D_3E_2'E_3=180^\circ-\angle D_3E_1'E_3=180^\circ-\angle F_2PE_3\implies P\in\odot(F_2E_3E_1)$.
Similarly, we can prove $P\in\odot(F_3E_1E_2)\implies\odot(F_1E_2E_3),\odot(F_2E_3E_1),\odot(F_3E_1E_2)$ are concurrent at $P\in DE$.

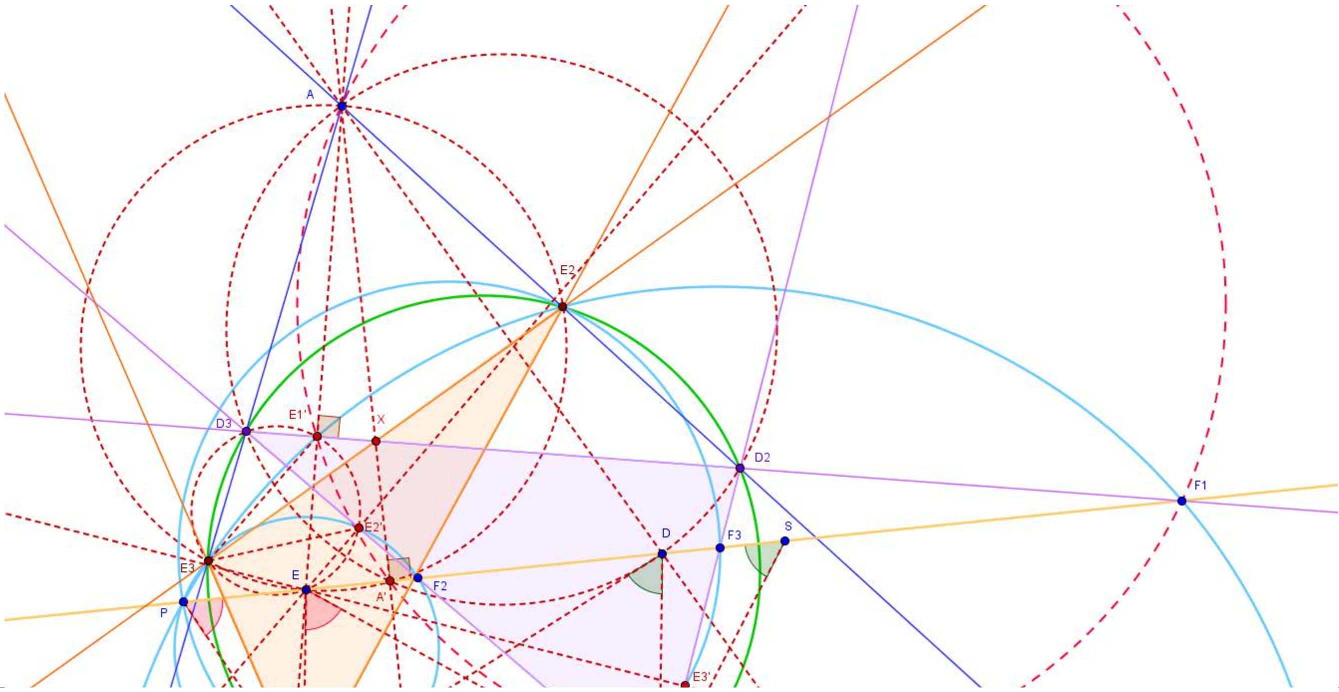
Similarly, we can prove the existence of S .

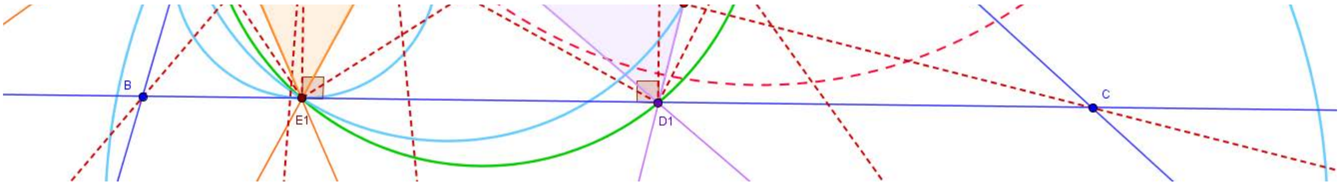
Since $\angle E_1PE=\angle E_1E_2'D_1=\angle E_1ED_1$ (similarly, $\angle D_1SD=\angle D_1DE_1$) ,

so $ES:DP=\dfrac{E_1D_1}{\sin\angle EPE_1}\cdot\dfrac{\sin\angle SD_1E}{\sin\angle DSD_1}:\dfrac{D_1E_1}{\sin\angle DSD_1}\cdot\dfrac{\sin\angle PE_1D}{\sin\angle EPE_1}=1:1\implies SD=EP$.

Q.E.D

Attachments:





bobthesmartypants

#4 May 20, 2015, 11:18 am • 1

@TelvCohl Every time you write a solution I look at the diagram and it's way too overwhelming. How do you do anything on that?

Luis González

#5 May 20, 2015, 12:15 pm • 1

Just an observation. In my solution I forgot to mention that $\{U, V\}$ are the intersections of ω with DE and it should be $O \equiv D_2E_3 \cap E_2D_3.$

@bobthesmartypants, just bear in mind that Geogebra allows one to hide everything that is unnecessary at the moment. So the submitted diagram, with colors and stuffs, is usually busier.

mineiraojose

#6 Jul 1, 2015, 1:25 am

Additional remark :) : $F_1E_3 \cap F_3E_1$, $F_2E_3 \cap F_3E_2$, $F_2E_1 \cap F_1E_2$ lie on the common circumcircle of $D_1D_2D_3$ and $E_1E_2E_3$

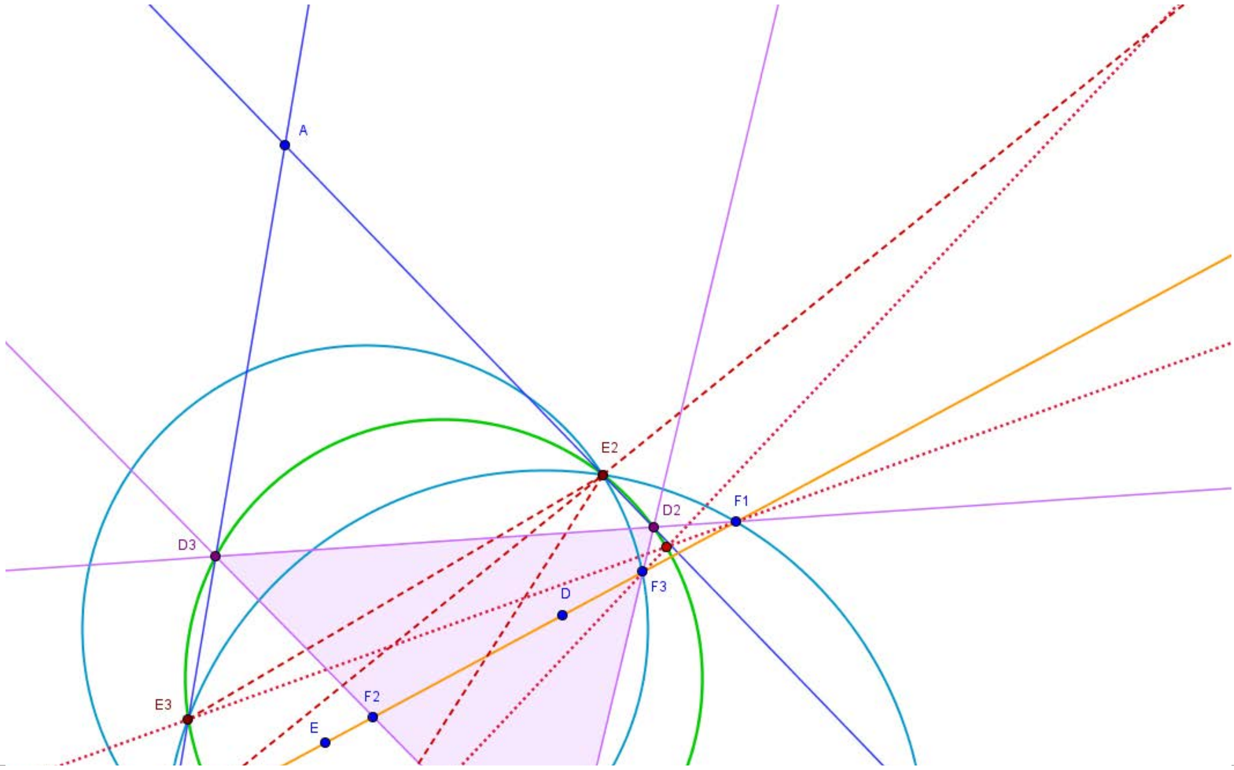
TelvCohl

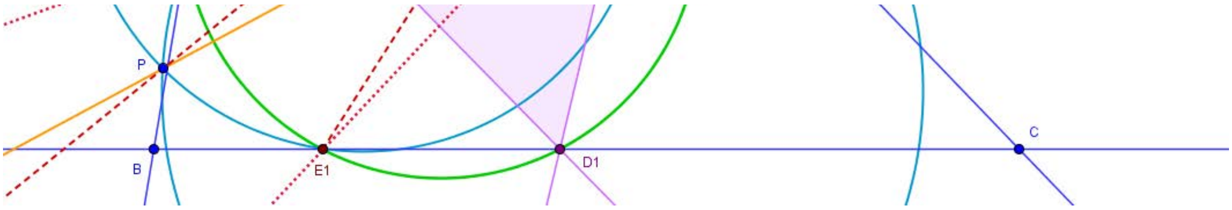
#7 Jul 1, 2015, 11:39 am

> *mineiraojose wrote:*
> Additional remark :) : $F_1E_3 \cap F_3E_1$, $F_2E_3 \cap F_3E_2$, $F_2E_1 \cap F_1E_2$ lie on the common circumcircle of $D_1D_2D_3$ and $E_1E_2E_3$

Since $\angle(E_3F_1, PE_2) = \angle E_3E_2P + \angle F_1PE_2, \angle(PE_2, E_1F_3) = \angle PE_2E_1 - \angle F_3PE_2$,
so $\angle(E_3F_1, E_1F_3) = \angle E_3E_2P + \angle PE_2E_1 = \angle E_3E_2E_1$. i.e. $E_3F_1 \cap E_1F_3 \in \odot(E_1E_2E_3)$

Attachments:





Quick Reply

High School Olympiads

Tangent to the Circumcircle

Locked

Seventh

#1 May 20, 2015, 7:26 am

Let $\triangle ABC$ be a triangle, I its incenter, Γ its circumcircle and γ its incircle. The circumcircle of $\triangle BIC$ meets γ at X and Y. The common external tangents to the circumcircle of $\triangle BIC$ and γ meets at Z. Prove that the circumcircle of $\triangle XYZ$ is tangent to Γ.

Luis González

#2 May 20, 2015, 7:37 am

It's ARMO 2013 Grade 11 P4. See Circumcircle of XYZ is tangent to circumcircle of ABC.

High School Olympiads

Circumcircle of XYZ is tangent to circumcircle of ABC

Reply

Source: ARMO 2013 Grade 11 Day 2 P4

mathuz
#1 May 22, 2013, 9:39 pm • 4

Let ω be the incircle of the triangle ABC and with centre I. Let Γ be the circumcircle of the triangle AIB. Circles ω and Γ intersect at the point X and Y. Let Z be the intersection of the common tangents of the circles ω and Γ. Show that the circumcircle of the triangle XYZ is tangent to the circumcircle of the triangle ABC.

hal9v4ik
#2 May 22, 2013, 9:58 pm • 1

This one was proposed by Saken Ilyas (Kazakhstan).

mathuz
#3 May 22, 2013, 10:55 pm • 1

you are right!
Very nice problem by Saken Ilyasov.

Luis González
#4 May 22, 2013, 11:01 pm • 4

Let F be the tangency point of (I) with AB. It's well known that the center J of Γ is the midpoint of the arc AB of the circumcircle $\odot(ABC) \equiv (O)$. Let JF cut (O) again at P. Inversion WRT Γ swaps (O) and AB and X, Y are double $\Longrightarrow$ (I) goes to the circle $\odot(PXY)$ tangent to (O) due to conformity. $\odot(PXY)$ cuts CI at U, V (U is between I,J and V is between C,I). It suffices to show that $V \equiv Z$.

Let D be the midpoint of the arc ACB of (O). Since $JI^2 = JF \cdot JP$, we obtain $\angle JPI = \angle FIJ = \angle CJD = \angle CPD$ $\Longrightarrow \angle CPI = \angle JPD = 90^\circ$. Since $\angle UPV$ is also right, then $\angle CPV = \angle UPI$. From the tangency of (O) and $\odot(PUV) \equiv \odot(PXY)$, it follows that PU, PV are isogonals WRT $\angle CPJ$, hence $\angle CPV = \angle JPU = \angle IPU \Longrightarrow$ PU, PV bisect $\angle IPJ \Longrightarrow \odot(PXY)$ becomes Apollonius circle of $\overline{IJ}$ referent to the ratio $\frac{PI}{PJ} = \frac{XI}{XJ} \Longrightarrow \frac{VI}{VJ} = \frac{XI}{XJ} \Longrightarrow$ $V \equiv Z$ is the exsimilicenter of $(I) \sim \Gamma$, as desired.

mathuz
#5 May 25, 2013, 11:02 am • 2

Hi;
It's beautiful problem, but don't very hard.

My solution:
Hint: after the Poncelet's theorem, $\Rightarrow$ mixtilinear circle.

Let $CI \cap \Gamma = D(.)$, a circle $k(T, r')$ - circumcircle of the triangle XYZ and R' - radius of the circle Γ, r - radius of the circle ω.
We have that

$$\frac{ZI}{ZD} = \frac{r}{R'}$$

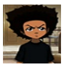
and

$$ZI = \frac{R' \cdot r}{R' - r}.$$

So, we easily get that

$$\frac{DI}{DT} = \frac{r}{r'}$$

and ω and k are gomothetic similar (1), which D is center of the gomothety.
Let E and F are intersection point of the Γ and DM, DN, respectively. (here, DM, DN are tangents to the circle k, $M, N \in k$.) Since (1), DM and DN are tangents to the circle ω too.
Finally,
since the Poncelet's theorem $\rightarrow \omega$ is incircle of the triangle DEF and MN passes throught the point $I \rightarrow k$ is mixtilinear circle of the triangle $DEF \Rightarrow k$ is tangent to Γ.



thecmd999

#6 Sep 25, 2014, 11:20 pm

Solution

JuanOrtiz

#7 May 22, 2015, 9:44 am

An amusing result indeed!

Let M be the midpoint of AB, center of Γ, and let D be the tangency of incircle with AB. Let $P = MD \cap$ circumcircle of ABC. It is easy to prove using power of a point from $XY \cap AB$ that circumcircle of XYP is tangent to circumcircle of ABC. So what is left is to prove $PZXY$ is cyclic. Let Z' be intersection of incircle and MI, then what is left is to prove $MZ * MZ' = MI^2$, since then we are done by inversion at M.

However notice that if r, R are radii of incircle and Γ, then $R = IM = ZM - ZI = ZM\left(\frac{R-r}{R}\right)$ and from this $ZM = R^2/(R-r)$ but notice $MZ' = MI - IZ' = R - r$ so $MZ' * MZ = R^2$ indeed, done!

anantmudgal09

#8 Jan 21, 2016, 2:07 am

Here is a very nice solution which seems kind of strange so please point out to me if this is incorrect.

We shall assume to the contrary and let (XYZ) meet (ABC) at two distinct points P, Q respectively. We shall now prove that infact we must have $P = Q = T$ where T is a point on (ABC) such that $\angle ATI = 90$.

Consider the following

Lemma

Given circles ω_1, ω_2 with intersection points A, B, the locus of a point V such that $\frac{p(V, \omega_1)}{p(V, \omega_2)} = k$ where k is a constant is a circle co-axial to both ω_1, ω_2.

Proof:- This follows by some power of point and length chasing combined with Reim's Theorem. $\blacksquare$

Now, let $\Omega = (AIB)$ and $\omega = (DEF)$ where D, E, F are the touch-points of the incircle on BC, CA, AB respectively.
Now, let $\Gamma = (XYZ)$.

We thus, know that Γ, Ω, ω are co-axial and so by our *Lemma*, we have that

$$\frac{PI^2 - r^2}{PM^2 - MI^2} = \frac{QI^2 - r^2}{QM^2 - MI^2} = \frac{ZI^2 - r^2}{ZM^2 - MI^2}$$

wherein the equalities follow since I is the center of ω and M is the center of Ω. Now, point Z is the ex-similicenter of Ω, ω and so $Z \in IM$. Moreover, by this property of Z, we have that

$$\frac{r^2}{MI^2} = \frac{ZI^2}{ZM^2} = \frac{ZI^2 - r^2}{ZM^2 - MI^2} = \frac{PI^2 - r^2}{PM^2 - MI^2} = \frac{QI^2 - r^2}{QM^2 - MI^2}$$

Now, we see that when we invert about I, both P, Q map to the mid-point of the arc BAC of the triangle $B'A'C'$ and so we are done with that $P = Q$.

Now, it is just a re-application of the above steps to show that if T is a point on the circumcircle such that $\angle ATI = 90$, then X, Y, Z, T are concyclic.

These two ideas thus, complete our proof. ■

livetolove212

#9 Jan 21, 2016, 10:37 am • 1

> *anantmudgal09 wrote:*
> Here is a very nice solution which seems kind of strange so please point out to me if this is incorrect.
>
> We shall assume to the contrary and let (XYZ) meet (ABC) at two distinct points P, Q respectively. We shall now prove that infact we must have $P = Q = T$ where T is a point on (ABC) such that $\angle ATI = 90$.

You have to prove that (XYZ) meets (O). So I recommend that first we prove $XYZT$ are concyclic by using ratio of power lemma, then assume the contrary. By the way, nice idea :)

This post has been edited 1 time. Last edited by livetolove212, Jan 21, 2016, 10:43 am

anantmudgal09

#10 Jan 21, 2016, 2:27 pm

Um, okay, I would edit that soon. At that time I thought that just reversing the argument would do, I.e., applying the Ratio Lemma for that part as well. I guess I would have to be more precise with that.

Thanks for the suggestion. :)

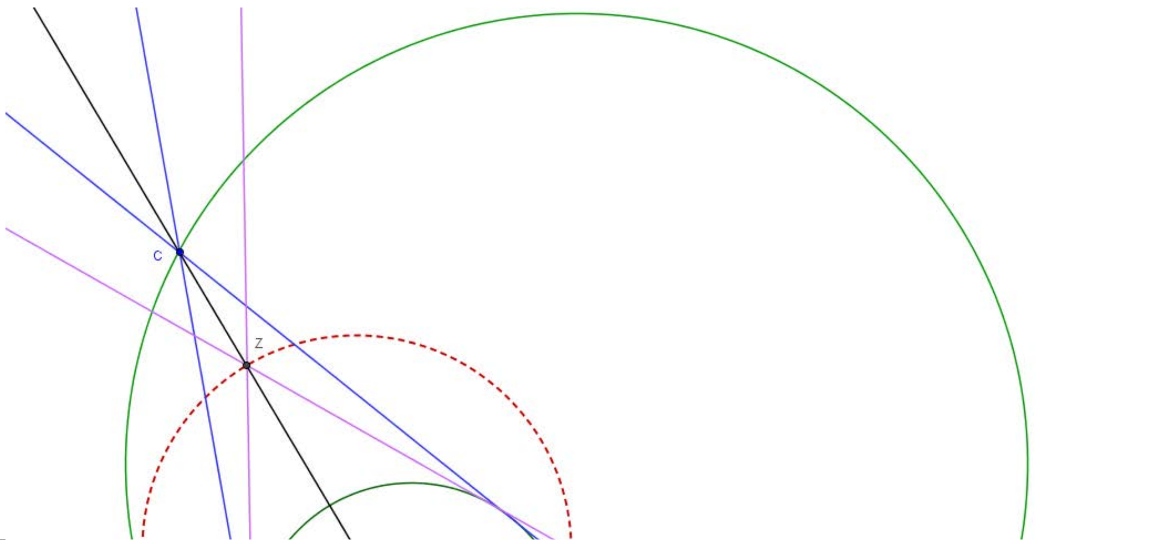
TelvCohl

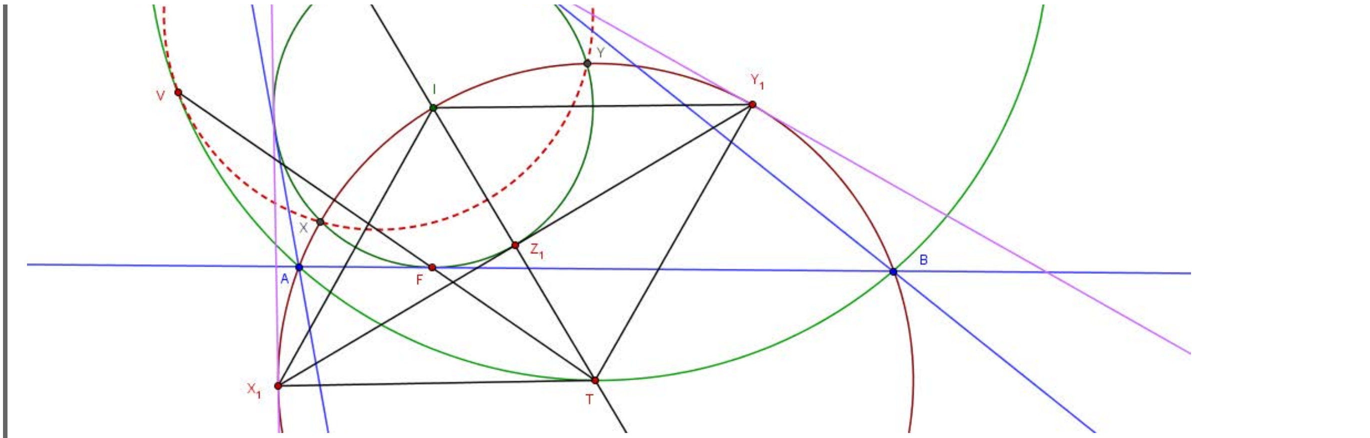
#11 Jan 21, 2016, 3:15 pm • 1

Let T be the midpoint of arc AB in $\odot(ABC)$ (center of Γ) and let $F \equiv \odot(I) \cap AB, V \equiv FT \cap \odot(ABC)$. Let the common tangent of ω and Γ touches Γ at X_1, Y_1, respectively. Since $\measuredangle IX_1Z = \measuredangle IY_1X_1 = \measuredangle Y_1X_1I$ and similarly we get $\measuredangle IY_1Z = \measuredangle X_1Y_1I$, so I is the incenter of $\triangle X_1Y_1Z \Longrightarrow \omega$ is the incircle of $\triangle X_1Y_1Z$, hence notice $ZX_1 = ZY_1$ we get the projection Z_1 of Z on X_1Y_1 lies on $\odot(I)$. i.e. the image of Z under the inversion $\mathbf{I}(\Gamma)$ lies on ω

Since F, V is the image of each other under $\mathbf{I}(\Gamma)$ and $\odot(I)$ is tangent to AB at F, so after performing $\mathbf{I}(\Gamma)$ we conclude that V, X, Y, Z lie on a circle which is tangent to $\odot(ABC)$ (image of AB under $\mathbf{I}(\Gamma)$) at V.

Attachments:





Quick Reply

High School Olympiads

vov

Reply

Source: it is brillant

vatoz51

#1 Oct 17, 2011, 2:39 pm

Let P and Q be isogonal conjugate points with respect to $\triangle ABC$ and assume that $\triangle P_1P_2P_3$ and $\triangle Q_1Q_2Q_3$ are their pedal triangles, respectively. Let $X_1 = P_2Q_3 \cap P_3Q_2, X_2 = P_1Q_3 \cap P_3Q_1$ and $X_3 = P_1Q_2 \cap P_2Q_1$. Prove that the points X_1, X_2, X_3 belong to the line PQ.

Luis González

#2 Oct 17, 2011, 9:32 pm • 1

It's well known that $P_1, P_2, P_3, Q_1, Q_2, Q_3$ lie on a same circle, whose center U is the midpoint of $\overline{PQ}$. Let M be the orthogonal projection of A on PQ, i.e. 2nd intersection of the circumcircles of PP_2AP_3 and $QQ_2AQ_3 \Longrightarrow P_2P_3, Q_2Q_3$, AM are pairwise radical axes of $(U), \odot(AP_2P_3), \odot(AQ_2Q_3)$ concurring at their radical center D. Since AD is the polar of X_1 WRT (U), then $AD \perp \overline{PUQ}$ implies that $X_1 \in PQ$. Similarly, X_2 and X_3 lie on PQ.

yetti

#3 Oct 17, 2011, 10:05 pm • 1

$P_1Q_1P_2Q_2P_3Q_3$ is cyclic with circumcenter U at the midpoint of PQ. PP_1, PP_2 cut its circumcircle (U) again at R_1, R_2.
Due to right angles $\angle Q_1P_1R_1, \angle Q_2P_2R_2 \Longrightarrow Q_1R_1, Q_2R_2$ are diameters of (U).
By Pascal theorem for $P_1Q_1R_2R_1Q_2P_2$, intersections $P \equiv P_1R_1 \cap P_2R_2, U \equiv Q_1R_1 \cap Q_2R_2, X_3 \equiv P_1Q_2 \cap P_2Q_1$ are collinear $\Longrightarrow X_3 \in PUQ$. Similarly, $X_1, X_2 \in PUQ$.

jayme

#4 May 20, 2015, 8:01 pm

Dear Mathlinkers,
you can have a look at

http://jl.ayme.pagesperso-orange.fr/Docs/Les%20deux%20points%20de%20Schroeter.pdf p. 1-4

which proof can be generalized with two isogonal points.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Line Tangent (continue)

Reply

Source: Own

LeVietAn
#1 May 20, 2015, 12:38 am

Dear Mathlinkers,
Suppose that ABC is a triangle inscribed in the circle (O) and M is the midpoint of side BC. On the line segment BC, choose T such that $\angle TAB = \angle MAC$. Points E and F lie on sides CA and AB, respectively, such that $AETF$ is a parallelogram. The circle (O) meets EF at S and R. Prove that line BC is tangent to the circumcircle of triangle STR.

Luis González
#2 May 20, 2015, 3:26 am • 1

Clearly any point T on the A-symmedian verifies $\triangle ABC \sim \triangle TFE$ are inversely similar $\Longrightarrow \widehat{BTF} = \widehat{C} = \widehat{TEF} \Longrightarrow$ BC is tangent of $\odot(TEF)$ and $\widehat{BFE} = \widehat{B} + \widehat{A} = 180^\circ - \widehat{BCE} \Longrightarrow BFEC$ is cyclic. Hence if $P \equiv EF \cap BC$, we have $PT^2 = PE \cdot PF = PB \cdot PC = PS \cdot PR \Longrightarrow BC$ is tangent of $\odot(TSR)$.

tranquanghuy7198
#3 May 20, 2015, 9:03 am • 1

In my solution, the problem is still right if T is an arbitrary point on BC
$EF \cap BC = K$
We have: $\frac{KB}{KT} = \frac{KF}{KE} = \frac{KT}{KC}$
$\Rightarrow KB.KC = KT^2$
$\Rightarrow KS.KR = KT^2$
$\Rightarrow (TRS)$ is tangent to BC as desired

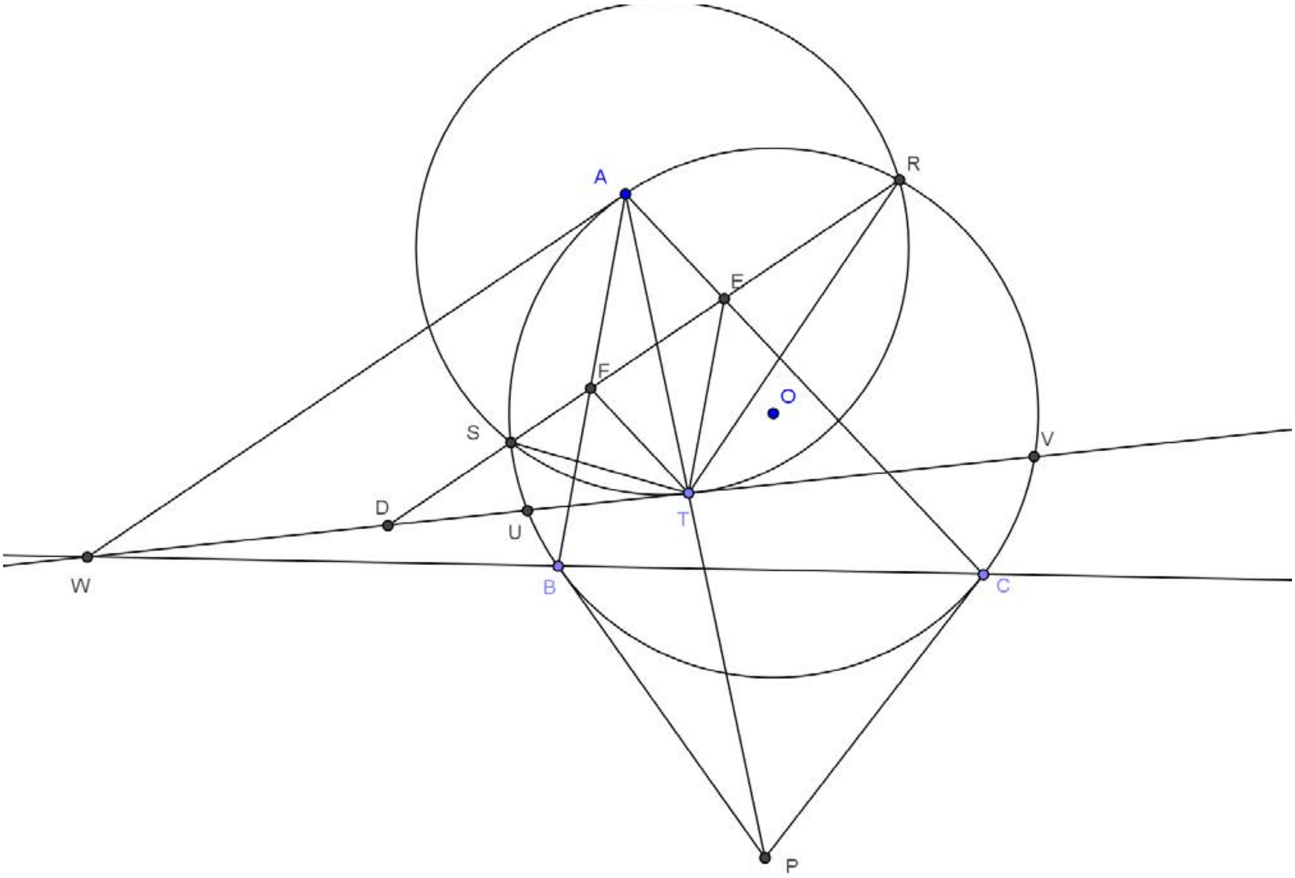
LeVietAn
#4 May 20, 2015, 6:12 pm • 1

Thanks Luis González and tranquanghuy7198. And this is apply:
Let ABC is a non-isosceles traingle inscribed in the circle (O). The tangent lines at B, C of (O) intersect at P. A point T varies arbitrarily on the line segment AP. Points E and F lie on sides CA and AB, respectively, such that $TE \parallel AB, TF \parallel AC$. The circle (O) meets EF at S and R.The circle (O) meets EF at S and R. Prove that the line is tangent to the circumcircle of triangle STR at T always gose through a fixed point.

tranquanghuy7198
#5 May 20, 2015, 8:14 pm

My solution:
$AA \cap BC = W, SR \cap WT = D, WT \cap (O) = U, V$
We will prove that WT is tangent to (TRS). Indeed:
$AETF$ is the parallelogram $\Rightarrow EF$ bisects AT
$\Rightarrow EF$ bisects WT (because $EF \parallel AW$) $\Rightarrow DW = DT$
On the other hand: $(WTUV) = -1$ (basic) $\Rightarrow DT^2 = DU.DV$ (Newton) $\Rightarrow DT^2 = DR.DS \Rightarrow DT$ is tangent to (TRS)
Q.E.D

Attachments:



Quick Reply

High School Olympiads

Concurrency in Equiangular Hexagon

Reply

Eray

#1 May 18, 2015, 2:19 am

Let $ABCDEF$ be an equiangular hexagon. Let P, Q, R, S, T, U be the midpoints of the segments AB, BC, CD, DE, EF, FA, respectively. Prove that the lines PS, QT, RU are concurrent.

Luis González

#2 May 18, 2015, 2:55 am

Clearly all its interior angles equal 120°. Hence if $X \equiv AB \cap FE$ and $Y \equiv AB \cap CD$, then $\triangle XAF$ and $\triangle YBC$ are equilateral $\Longrightarrow \angle XAF = \angle XYC = 60^\circ \Longrightarrow FA \parallel CD$ and likewise $AB \parallel DE$ and $BC \parallel EF$. Thus by the converse of Pascal theorem, it follows that $ABCDEF$ is inscribed in a conic $\mathcal{C}$ and PS, QT, RU are the polars of the infinity points of $AB \parallel DE, BC \parallel EF, FA \parallel CD \Longrightarrow PS, QT, RU$ pass through the center O of $\mathcal{C}$.

Eray

#3 May 19, 2015, 7:01 pm

Any other solutions excluding projective geometry?

Luis González

#4 May 19, 2015, 10:07 pm

The key is to show that if a hexagon has parallel opposite sides, then the 3 lines connecting their midpoints concur. The above proof is one way, for other ideas see Hexagon [lines joining side midpoints are concurrent] or Hexagon with parallel sides.

Quick Reply

High School Olympiads

Hexagon [lines joining side midpoints are concurrent]

Reply

Source: unknow

Bajoukx

#1 Jan 4, 2005, 7:05 am

Let $A_0A_1A_2A_3A_4A_5$ be a hexagon with parallel oposite sides.
Let M_i be the midpoint of segments A_iA_{i+1} and s_j the line defined by M_jM_{j+3}, $j \in \{1,2,3\}$.

Prove that s_1, s_2, s_3 intersect at one single point.

Eduardo

Myth

#2 Jan 4, 2005, 2:31 pm • 1

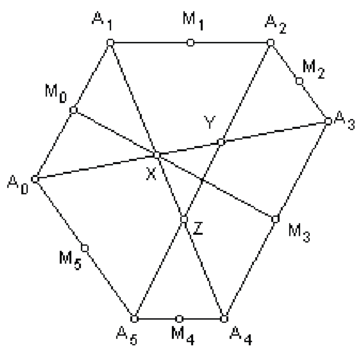
Let $X = A_0A_3 \cap A_1A_4$. Since $A_0A_1A_3A_4$ is a trapezium, we conclude that M_0, X and M_3 are collinear. So M_0M_3, M_1M_4 and M_2M_5 are cevians in XYZ. We will apply Ceva for XYZ in sine form. So first of all we need to calculate $\frac{\sin ZXM_3}{\sin YXM_3}$. Indeed, using sine theorem to triangles A_4XM_3 and A_3XM_3 we obtain:
<center>$\frac{\sin A_4XM_3}{A_4M_3} = \frac{\sin A_4M_3X}{A_4X}$ and $\frac{\sin A_3XM_3}{A_3M_3} = \frac{\sin A_3M_3X}{A_3X}$.</center>
Taking into consideration $A_3M_3 = A_4M_3$ and $\sin A_4M_3X = \sin A_3M_3X$, we obtain

$$\frac{\sin ZXM_3}{\sin YXM_3} = \frac{\sin A_4XM_3}{\sin A_3M_3X} = \frac{XA_3}{XA_4} = \frac{A_0A_3}{A_1A_4}$$

(last equality is due to similarity of triangles XA_0A_1 and XA_3A_4).
In the same way we can write ratios for cevians M_1M_4 and M_2M_5. Multiplying together these ratios we obviously obtain 1.

That's all.

Attachments:



sprmnt21

#3 Jan 4, 2005, 11:22 pm

The claim is a corollary of the following lemma RL04012005.

Let A_0A_1A_2A_3A_4A_5 be a hexagon with parallel oposite sides.

Let B_i = A_iA_(i+5)^A_(i+1)A_(i+2) where (i+k)'s are numbers module 6.

Let P_0 a generic point if P_1, P_2, ..., P_5 are the points "similarly" disposed wrt to the correspondent triangles (*) as P_0 is wrt

Let P_0 a generic point if P_1, P_2, ..., P_5 are the points "similarly" disposed wrt to the correspondent triangles (*) as P_0 is wrt A_0B_0A_1, then P_0P_3, P_1P_4 and P_2P_5 are concurrent.

(*) a point P_i is correspondent to the triangle A_iB_iA_(i+1).

<<"similarly" disposed >> means that P_(i+1) is the correspondent of P_i with the homotetie of center A_i and ratio B_(i+1)A_(i+1)/A_(i+1)A_i.

Or perhaps, but i'm not sure because i don't know these matters, the P_i's should have the same trilinear coordinates wrt the omologue triangles.

Quick Reply

High School Olympiads

Hexagon with parallel sides

Reply

Source: maybe well-known

treegoner
#1 Jan 22, 2006, 11:53 am

Let $ABCDEF$ be a convex hexagon such that $AB//DE, BC//EF, CD//FA$. If M, N, P, Q, R, S are the midpoints of AB, BC, CD, DE, EF, FA, then show that MQ, NR, PS are concurrent.

James Potter
#2 Jan 22, 2006, 3:27 pm

Yes this is a well-known problems.
Solution 1
Solution 2

Quick Reply

High School Olympiads

Concurrent on cyclic pentagon

Reply

Source: Own

buratinogigle

#1 May 19, 2015, 3:51 pm

Let $ABCDE$ be cyclic pentagon inscribed circle (O). Tangents of (O) at A,E intersect at M. Tangents of (O) at B,C intersect at N. DM,DN cut CE at P,Q, resp. AP,BQ cut (O) again at S,T. Prove that ST,AB and tangent of (O) at D are concurrent.

TelvCohl

#2 May 19, 2015, 7:50 pm • 1

My solution :

Let $Y=DM \cap \odot(O), Z=DN \cap \odot(O)$.

Apply inversion with center D we get the new problem as following
(notice that $DAYE$ and $DBZC$ are harmonic quadrilateral)

Problem :

Let C,B,A,E be four points lie on a line ℓ (in order) .
Let Y,Z be the midpoint of AE,CB, respectively and $P=DY \cap \odot(DCE), Q=DZ \cap \odot(DCE)$.
Let $T=\ell \cap \odot(DBQ), S=\ell \cap \odot(DAP)$ ($T \neq B, S \neq A$) and τ be the line passing through D and parallel to ℓ .

Prove that $\tau, \odot(DAB), \odot(DST)$ have a common point other than D .

Proof :

Let X be the second intersection of τ with $\odot(DAB)$.

Easy to see X is the reflection of D in the perpendicular bisector of AB .

From $TZ \cdot BZ=QZ \cdot DZ=CZ \cdot EZ \Longrightarrow TZ=EZ \Longrightarrow TB=CE$.
Similarly, we can prove $SA=CE \Longrightarrow XTSD$ is an Isosceles trapezoid $\Longrightarrow X \in \odot(DST)$.

Q.E.D

Luis González

#3 May 19, 2015, 9:27 pm • 1

Tangents of (O) at A,B meet at X, tangents of (O) at C,S meet at Y, tangents of (O) at E,T meet at Z and $R \equiv YS \cap ZT.$ By Newton's theorem for the tangential quadrangle bounded by $AM,SY,YC,ZM,$ it follows that M,P,Y are collinear and similarly N,Q,Z are collinear. Thus by Brianchon's theorem for $XNYRZM,$ it follows that $D \equiv XR \cap MY \cap NZ,$ i.e. R,X,D are collinear $\Longrightarrow$ their polars WRT (O) concur, namely ST,AB and the tangent of (O) at $D.$

Quick Reply

High School Olympiads

Geometry in a 45-60-75 triangle

Reply

socrates

#1 May 19, 2015, 4:27 am

From the foot D of the height CD in the triangle ABC, perpendiculars to BC and AC are drawn, which they intersect at points M and N. Let $\angle CAB = 60^\circ$, $\angle CBA = 45^\circ$, and H be the orthocentre of MNC. If O is the midpoint of CD, find $\angle COH$.

Luis González

#2 May 19, 2015, 10:31 am • 1

Since CD is diameter of $\odot(CMN)$, then O is circumcenter of $\triangle CMN$. Further $\angle CMN = \angle CDN = \angle BAC = 60^\circ$ $\Longrightarrow \angle CON = \angle CHN = 120^\circ \Longrightarrow CHON$ is cyclic $\Longrightarrow \angle COH = \angle CNH = 90^\circ - 75^\circ = 15^\circ$.

professordad

#3 May 19, 2015, 7:15 pm

$\overline{NO}$ is a median of 30-60-90 $\triangle CND$, so $\angle CNO = 30^\circ$. $\angle CNM = \angle CDM = 45^\circ$. Thus $\angle CNH = \angle HNO = \angle MNO = 15^\circ$. $\angle HCM = \angle HNM = 30^\circ$, so $\angle OCH = 15^\circ$. So $CHON$ is cyclic and $\angle COH = 15^\circ$.

Quick Reply

Site Support

Post number glitch?

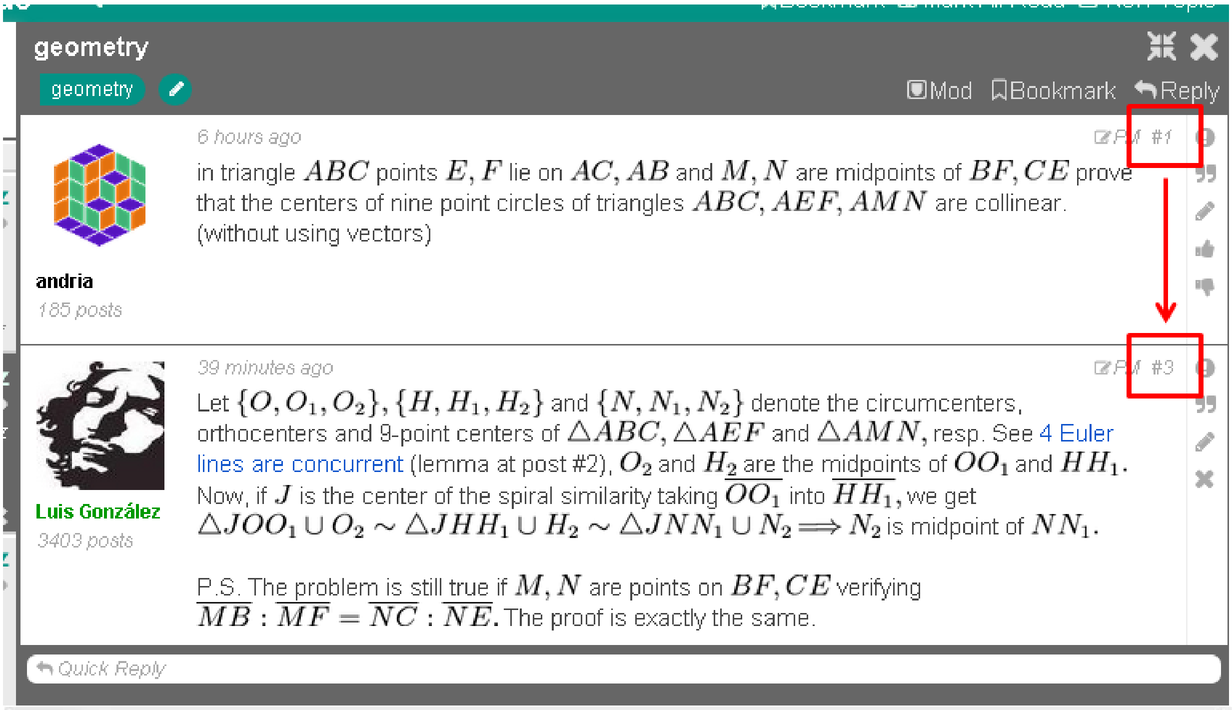
Reply

Luis González

#1 May 19, 2015, 8:22 am

I just noticed when a message (reply) is deleted by its poster or a moderator, its post number is not replaced by the post number of the subsequent message. See for example the screen captures below. Is this a glitch?; wouldn't it be better to enumerate the existent posts only?.

Attachments:



geometry

geometry | Mod | Bookmark | Reply

6 hours ago — PM #1

in triangle ABC points E, F lie on AC, AB and M, N are midpoints of BF, CE prove that the centers of nine point circles of triangles ABC, AEF, AMN are collinear. (without using vectors)

andria

185 posts

39 minutes ago — PM #3

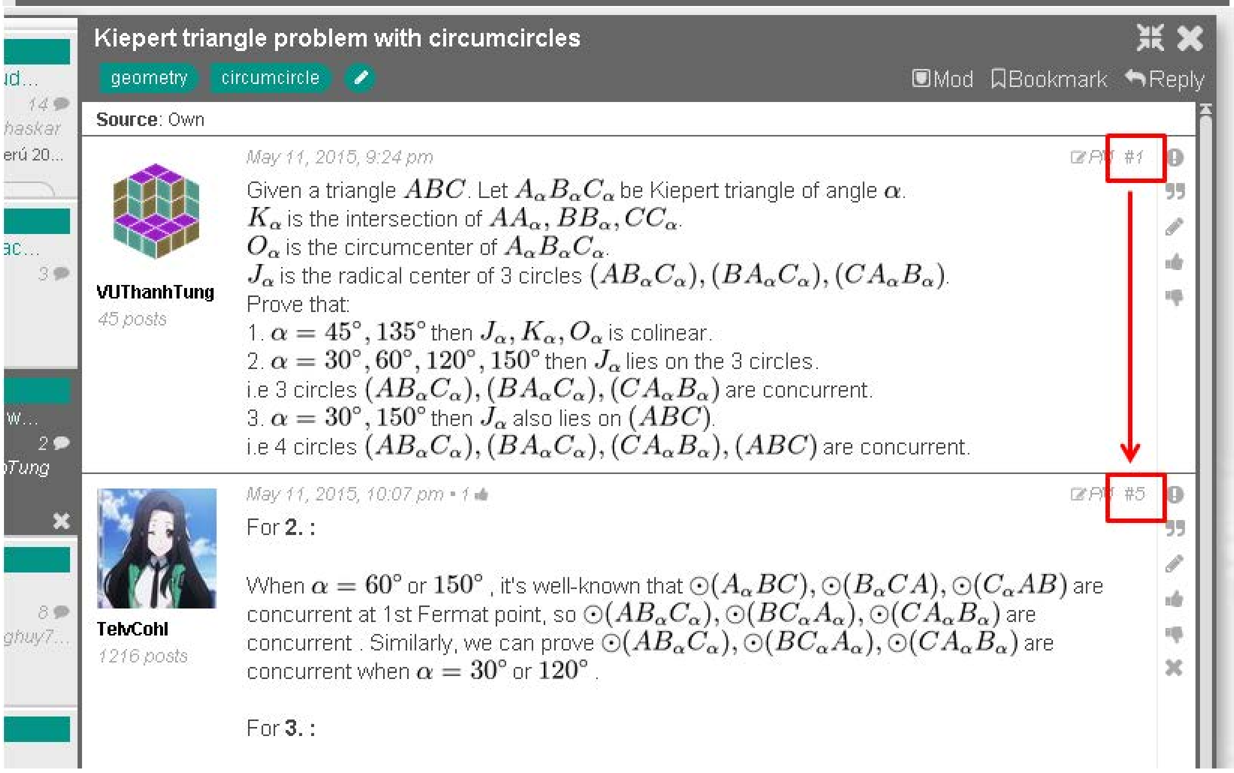
Let $\{O, O_1, O_2\}, \{H, H_1, H_2\}$ and $\{N, N_1, N_2\}$ denote the circumcenters, orthocenters and 9-point centers of $\triangle ABC, \triangle AEF$ and $\triangle AMN$, resp. See 4 Euler lines are concurrent (lemma at post #2), O_2 and H_2 are the midpoints of OO_1 and HH_1. Now, if J is the center of the spiral similarity taking $\overline{OO_1}$ into $\overline{HH_1}$, we get $\triangle JOO_1 \cup O_2 \sim \triangle JHH_1 \cup H_2 \sim \triangle JNN_1 \cup N_2 \Longrightarrow N_2$ is midpoint of NN_1.

P.S. The problem is still true if M, N are points on BF, CE verifying $\overline{MB} : \overline{MF} = \overline{NC} : \overline{NE}$. The proof is exactly the same.

Luis González

3403 posts

Quick Reply



Kiepert triangle problem with circumcircles

geometry | circumcircle | Mod | Bookmark | Reply

Source: Own

May 11, 2015, 9:24 pm — PM #1

Given a triangle ABC. Let $A_\alpha B_\alpha C_\alpha$ be Kiepert triangle of angle α.
K_α is the intersection of $AA_\alpha, BB_\alpha, CC_\alpha$.
O_α is the circumcenter of $A_\alpha B_\alpha C_\alpha$.
J_α is the radical center of 3 circles $(AB_\alpha C_\alpha), (BA_\alpha C_\alpha), (CA_\alpha B_\alpha)$.
Prove that:
1. $\alpha = 45^\circ, 135^\circ$ then $J_\alpha, K_\alpha, O_\alpha$ is colinear.
2. $\alpha = 30^\circ, 60^\circ, 120^\circ, 150^\circ$ then J_α lies on the 3 circles.
i.e 3 circles $(AB_\alpha C_\alpha), (BA_\alpha C_\alpha), (CA_\alpha B_\alpha)$ are concurrent.
3. $\alpha = 30^\circ, 150^\circ$ then J_α also lies on (ABC).
i.e 4 circles $(AB_\alpha C_\alpha), (BA_\alpha C_\alpha), (CA_\alpha B_\alpha), (ABC)$ are concurrent.

VUThanhTung

45 posts

May 11, 2015, 10:07 pm • 1 — PM #5

For **2.** :

When $\alpha = 60^\circ$ or 150° , it's well-known that $\odot(A_\alpha BC), \odot(B_\alpha CA), \odot(C_\alpha AB)$ are concurrent at 1st Fermat point, so $\odot(AB_\alpha C_\alpha), \odot(BC_\alpha A_\alpha), \odot(CA_\alpha B_\alpha)$ are concurrent . Similarly, we can prove $\odot(AB_\alpha C_\alpha), \odot(BC_\alpha A_\alpha), \odot(CA_\alpha B_\alpha)$ are concurrent when $\alpha = 30^\circ$ or 120° .

For **3.** :

TelvCohl

1216 posts

$\odot(AB_\alpha C_\alpha), \odot(BC_\alpha A_\alpha), \odot(CA_\alpha B_\alpha)$ are concurrent at Euler reflection point (X_{110} in ETC) of $\triangle ABC$
(see Cosmin Pohoata, On the Euler reflection point, Forum Geometricorum, 10 (2010) 157--163.)

May 11, 2015, 10:18 pm PM #6

Thank you Do you know if the the points of concurrence in 2. for $\alpha = 60^\circ, 120^\circ$ are in

nosaj

#2 May 19, 2015, 8:41 am • 2

This is not a glitch; it is by design. If someone references another post like

> *Quote:*
> See the hint in post #5

and all of the post numbers change when a post is deleted, then such references will become incorrect.

CaptainFlint

#3 May 19, 2015, 8:49 am • 1

> *levans wrote:*
> This is intentional so that you can reference post numbers. Say I want to direct someone to post number 3 in this thread, I would say, "See post number 3 in this thread." If numbers adjusted when posts were deleted, post number 3 would no longer make sense.

Quick Reply

High School Olympiads

Intersection on a A-gergonian

Reply

Source: own

jayme
#1 May 18, 2015, 1:53 pm

Dear Mathlinkers,

1. ABC a triangle
2. DEF the contact triangle of ABC
3. PQR the A-excontact triangle of ABC
4. U the point of intersection of the parallel to EF with DF through B.

Prove : RU, QP and AD are concurrent.

Sincerely
Jean-Louis

Luis González
#2 May 19, 2015, 6:47 am

Let I be the incenter of $\triangle ABC$ and AI cuts DE at S. It's well-known (easy to show by simple angle chase) that $S \in PQ$ and $\angle ASB = 90^\circ$, i.e. $BS \parallel EF \Longrightarrow U \in BS$. Now, $\triangle DSU$ and $\triangle AQR$ are perspectrix, EF being their perspectrix, thus by Desargues theorem $RU, QP \equiv QS$ and AD concur.

aiyer12
#3 May 19, 2015, 7:02 am

> *jayme wrote:*
> Dear Mathlinkers,
>
> 1. ABC a triangle
> 2. DEF the contact triangle of ABC
> 3. PQR the A-excontact triangle of ABC
> 4. U the point of intersection of the parallel to EF with DF through B.
>
> Prove : RU, QP and AD are concurrent.
>
> Sincerely
> Jean-Louis

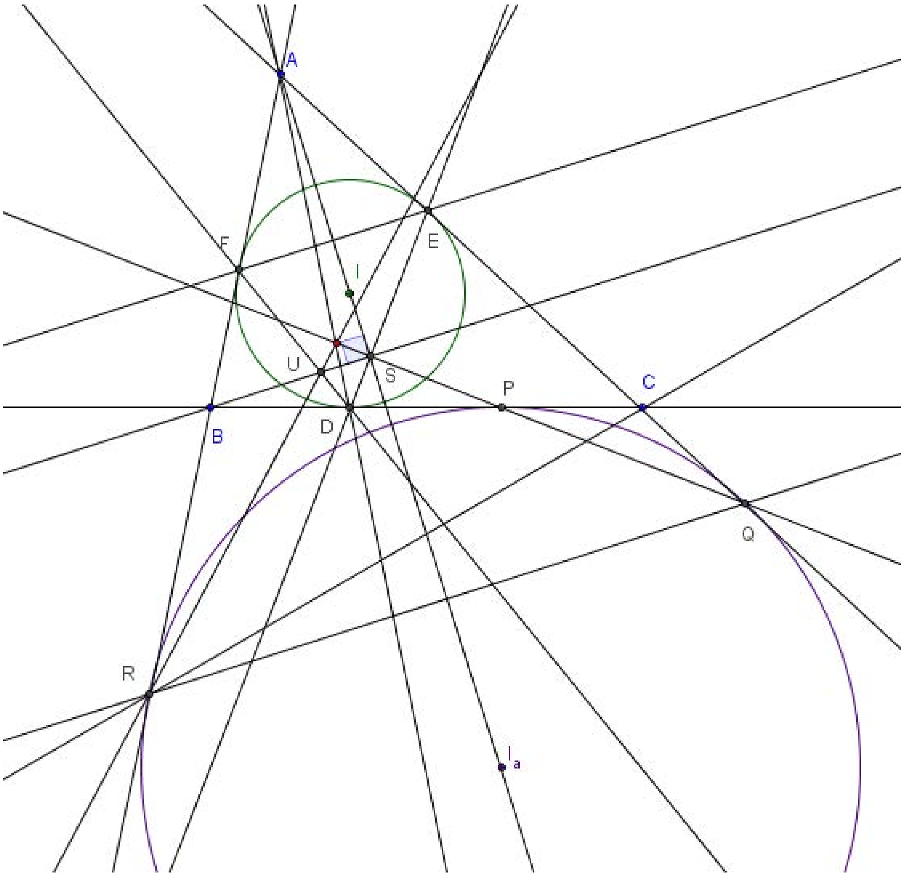
Hi,
what is the excontact triangle of a triangle? I think the contact triangle is the triangle formed by the points of tangency of the incircle in a triangle correct?

Thanks!

Luis González
#4 May 19, 2015, 7:29 am • 1

D,E,F are the tangency points of the incircle with BC,CA,AB, while P,Q,R are the tangency points of the A-excircle with BC,CA,AB I trust.

Attachments:



High School Olympiads

Geometry in a square

Reply

socrates

#1 May 19, 2015, 4:17 am

Points M and N lie on the sides BC and CD of the square $ABCD,$ respectively, and $\angle MAN=45^\circ$. The circle through A,B,C,D intersects AM and AN again at P and Q, respectively. Prove that $MN||PQ.$

Luis González

#2 May 19, 2015, 4:46 am • 1

Since $\angle PAQ=\angle BAC=45^\circ \Longrightarrow PQ=BC=CD \Longrightarrow BPCQ$ is an isosceles trapezoid $\Longrightarrow PB=QC$ and similarly $PC=QD \Longrightarrow \triangle PBC \cong \triangle QCD \Longrightarrow PBDC \cong QCAD.$ Hence if $L \equiv PD \cap BC,$ it follows that $QN:QA=PL:LD=PM:MA \Longrightarrow MN \parallel PQ.$

tranquanghuy7198

#4 May 19, 2015, 10:45 am

My solution:
S is the projection of A on $MN \Rightarrow AB=AS=AD$ (familiar property)
It's easy to see that: $\triangle APC \sim \triangle ASN, \triangle AQC \sim \triangle ASM$

$$\Rightarrow \frac{AP}{AC}=\frac{AS}{AN}, \frac{AC}{AQ}=\frac{AM}{AS}$$

$\Rightarrow \dfrac{AP}{AQ}=\dfrac{AM}{AN}$ and the conclusion follows

sunken rock

#5 May 19, 2015, 11:40 am

Let O be the centre of the square, $K=AM\cap BD, L=AN\cap BD$; $\angle MAL=\angle BML=45^\circ \implies ML\perp AN.$ Similarly $NK\perp AM$, hence $MNLK$ is cyclic, wherefrom $\angle AMN=\angle ALK\ (\ 1\ )$.
$PC\perp AP\implies PCOK$ cyclic, so $AK\cdot AP=AO\cdot AC\ (\ 2\ )$. Similarly $LOCQ$ is cyclic, therefore $AL\cdot AQ=AO\cdot AC\ (\ 3\ )$. From (2) and (3) $PQLK$ is cyclic, so $\angle KPQ=\angle ALK\ (\ 4\ )$; with (1) and (4) we are done.
However **Luis's** solution is really original and effective!

Best regards,
sunken rock

jayme

#6 May 19, 2015, 4:43 pm

Dear Mathlinkers,

1. MLN is the Pascal's line of APDCBQA
2. PCQD is a trapeze and PD perpendicular to AQ
3. PBQC is a trapeze and BQ perpendicular to AP
4. L is the orthocenter of APQ
5. X the symmetric of L wrt PQ
6. ML is the Pascal's line of XCBQPAX and ML // PQ

and we are done without any calculation…

Sincerely
Jean-Louis

This post has been edited 1 time. Last edited by jayme, May 19, 2015, 4:44 pm
Reason: typo

Quick Reply

High School Olympiads

Symmetric orthotransversal

Reply

Source: Own

buratinogigle
#1 May 19, 2015, 12:55 am

Prove that orthotransversal of incenter wrt reference triangle and wrt incentral triangle are symmetric through such this incenter.

Luis González
#2 May 19, 2015, 3:38 am • 1

I is incenter of $\triangle ABC$ and AI, BI, CI cut BC, CA, AB at D, E, F. Perpendicular to AI at I cuts BC, EF at X, X^*, respectively. Points $\{Y, Y^*\}$ and $\{Z, Z^*\}$ are defined cyclically. $\overline{XYZ}$ and $\overline{X^*Y^*Z^*}$ are the orthotransversals of I WRT $\triangle ABC$ and $\triangle DEF$.

If XX^* cuts AC, AB at U, V, then by Desargues involution theorem for $BFEC$, it follows that $X \mapsto X^*, U \mapsto V$ is an involution where I is double and U, V are symmetric WRT $I \Longrightarrow$ it coincides with the central symmetry WRT $I \Longrightarrow X, X^*$ are symmetric WRT I and similarly Y, Y^* and Z, Z^* are symmetric WRT $I \Longrightarrow \overline{XYZ}$ and $\overline{X^*Y^*Z^*}$ are symmetryc WRT I.

TelvCohl
#3 May 19, 2015, 5:44 am • 1

My solution :

Let I be the incenter of $\triangle ABC$ and $\triangle DEF$ be its cevian triangle WRT $\triangle ABC$.
Let the orthotransversal of I WRT $\triangle ABC$ cuts BC, CA, AB at X, Y, Z, respectively .
Let the orthotransversal of I WRT $\triangle DEF$ cuts EF, FD, DE at X^*, Y^*, Z^*, respectively .
Let T be the intersection of BC with A-external bisctor of $\triangle ABC$ (it's well-known $T \in EF$) .

From $T(A, I; X^*, X) = -1$ and $XX^* \parallel TA$ (both $\perp$ AI) $\Longrightarrow XI = X^*I$.
Similarly, we can prove $YI = Y^*I$ and $ZI = Z^*I \Longrightarrow \overline{XYZ}$ and $\overline{X^*Y^*Z^*}$ are symmetry WRT I .

Q.E.D

buratinogigle
#4 May 19, 2015, 7:20 am

Thank you so much, actually I have found solution the same as Telv base on idea of Luis but Telv posted. I only note that. Then line passes through I cuts BC, EF at G, H then AG, AH are isogonal conjugate wrt $\angle BAC$.

tranquanghuy7198
#5 Jun 2, 2015, 10:49 pm • 1

> *Mr.Buratino wrote:*
> Prove that orthotransversal of incenter wrt reference triangle and wrt incentral triangle are symmetric through such this incenter (post #1).

My solution:

Lemma 1. Let X, Y be 2 points on the side BC of $\triangle ABC$. We have: $\odot(AXY)$ is tangent to $\odot(ABC) \Longleftrightarrow AX, AY$ are isogonal in $\angle BAC$

isogonal in $\angle BAC$
Proof. Simple angle chasing.

Lemma 2. Given $\triangle ABC, E = R_{AC}(B), F = R_{AB}(C)$. The line passing through A which is tangent to $\odot(AEF)$ intersects BC, EF at X, Y. Prove that: $AX = AY$

Back to our main problem:
Consider $\triangle ABC$ and the bisectors AD, BE, CF concur at I
The orthotransversal of I WRT $\triangle ABC$ intersects BC, CA, AB at X, Y, Z
The orthotransversal of I WRT $\triangle DEF$ intersects EF, FD, DE at M, N, P
In order to prove $\overline{X, Y, Z} = S_I(\overline{M, N, P})$, we only need to indicate that $IX = IM$
$K = R_{IF}(E), L = R_{IE}(F) \Rightarrow K, L \in BC$
Notice that $\angle CIK = \angle CIE = \angle BIF = \angle BIL$
$\Rightarrow (IKL), (IBC)$ are tangent to each other at I (lemma 1)
$\Rightarrow (IKL)$ is tangent to $\overline{X, I, M}$
Now apply the lemma 2 for $\triangle IEF$ we receive: $IX = IM$ and our proof is completed!
Q.E.D

This post has been edited 1 time. Last edited by tranquanghuy7198, Jun 2, 2015, 10:50 pm

jayme
#6 Jun 3, 2015, 2:11 pm

Dear Mathlinkers,

a proof based on anharmonic pencils can be very nice...

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

geometry

Reply

Source: Own

andria

#1 May 18, 2015, 11:44 pm

In triangle ABC, M_A is A_mixtilinear point.let the bisector of $\angle BM_AC$ cut BC at A' we define B', C' similarly.
a) Prove that AA', BB', CC' are collinear at S.
b) Prove that J, G, S are collinear where G, J are centroid and gergonne point of $\triangle ABC$.

Luis González

#2 May 19, 2015, 3:09 am • 3

a) Let I be the incenter of $\triangle ABC$ and AI cuts $\odot(ABC)$ again at D; midpoint of the arc BC of $\odot(ABC)$. If the A-mixtilinear incircle touches AC, AB at Y, Z, then YZ, BC, DM_A concur at N_A (see Internally tangent circles and lines and concurrency and elsewhere).Thus if N_B, N_C are defined simialrly, then $\tau \equiv \overline{N_AN_BN_C}$ is the orthopolar of I WRT $\triangle ABC$ and since $(B, C, A', N_A) = -1$, etc $\Longrightarrow AA', BB', CC'$ concur at the tripole of τ, i.e. the ortocorrespondent $S \equiv X_{57}$ of I, which is the isogonal of the Mittenpunkt M_T.

b) It's known that X_{57} is the homothetic center of the intouch triangle $\triangle A_0B_0C_0$ and excentral triangle $\triangle I_aI_bI_c$ of $\triangle ABC$ $\Longrightarrow X_{57}$ is collinear with the symmedian points J and M_T of $\triangle A_0B_0C_0$ and $\triangle I_aI_bI_c$, but M_T is the complement of J, so $G \in M_TJ \Longrightarrow X_{57}, G, J$ are collinear.

tranquanghuy7198

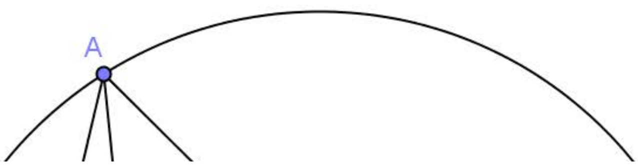
#3 May 19, 2015, 2:58 pm • 1

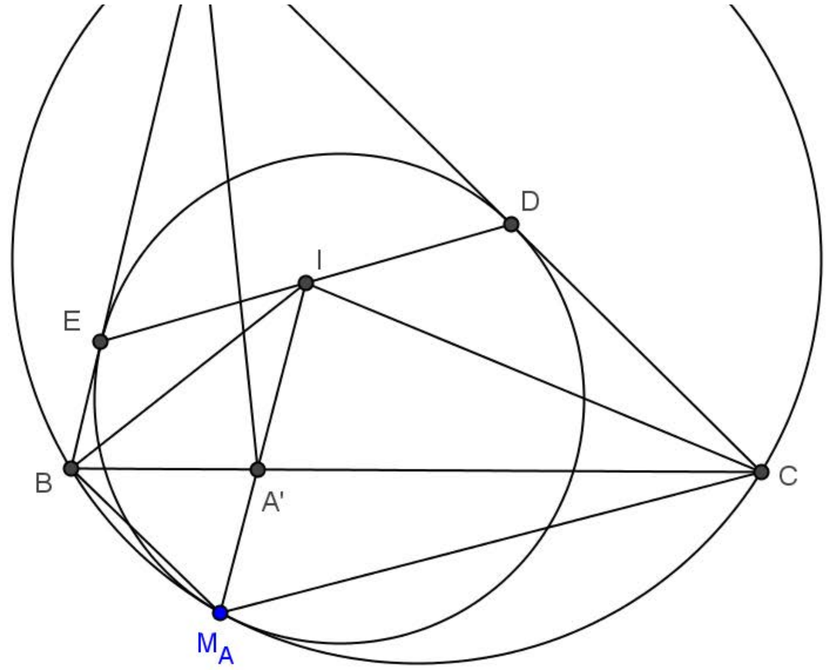
My solution:

Lemma. Let I be the incenter of $\triangle ABC$ then we have $IA = \sqrt{\frac{bc(s-a)}{s}}$ and analogously, we can compute the length of IB, IC

Back to our main problem.
I is the incenter of $\triangle ABC$.
It is well-known that $\overline{M_A, A', I}$ and $\triangle M_ABI \sim \triangle M_AIC$
We have: $\frac{BA'}{A'C} = \frac{BM_A}{M_AC} = \frac{BM_A}{M_AI} . \frac{IM_A}{M_AC} = (\frac{BI}{IC})^2 = \frac{(\frac{c}{s-c})}{(\frac{b}{s-b})}$ (apply the lemma)
Now let S be the point such that:
$S = \frac{\Sigma\frac{a}{s-a}A}{\Sigma\frac{a}{s-a}}$
$\Rightarrow \overline{A, S, A'}$ and $\overline{B, S, B'}$, $\overline{C, S, C'}$ (analogously)
Moreover:
$(\Sigma\frac{a}{s-a})S = \Sigma\frac{a}{s-a}A = s\Sigma\frac{1}{s-a}A$-$\Sigma A = s.(\Sigma\frac{1}{s-a})J - 3G$
$\Rightarrow \overline{S, G, J}$
Q.E.D

Attachments:





drmzjoseph

#4 May 20, 2015, 6:05 am

> *tranquanghuy7198 wrote:*
> My solution:
>
> Lemma. Let I be the incenter of $\triangle ABC$ then we have $IA = \sqrt{\frac{bc(s-a)}{s}}$ and analogously, we can compute the length of IB, IC
>
> Back to our main problem.
> I is the incenter of $\triangle ABC$.
> It is well-known that $\overline{M_A, A', I}$ and $\triangle M_ABI \sim \triangle M_AIC$
> We have: $\frac{BA'}{A'C} = \frac{BM_A}{M_AC} = \frac{BM_A}{M_AI}.\frac{IM_A}{M_AC} = (\frac{BI}{IC})^2 = \frac{(\frac{c}{s-c})}{(\frac{b}{s-b})}$ (apply the lemma)
> Now let S be the point such that:
> $S = \frac{\Sigma\frac{a}{s-a}A}{\Sigma\frac{a}{s-a}}$
> $\Rightarrow \overline{A, S, A'}$ and $\overline{B, S, B'}$, $\overline{C, S, C'}$ (analogously)
> Moreover:
> $(\Sigma\frac{a}{s-a})S = \Sigma\frac{a}{s-a}A = s\Sigma\frac{1}{s-a}A\text{-}\Sigma A = s.(\Sigma\frac{1}{s-a})J - 3G$
> $\Rightarrow \overline{S, G, J}$
> Q.E.D

Nice solution, I want add

You obtain $S \equiv a(s-a)^{-1} : b(s-b)^{-1} : c(s-c)^{-1}$
If the A-Mixtilinear excircle touch $\odot(ABC)$ at A_1, and A_1A_2 is the angle bisector of $\angle BA_1C$ then we get, with $A_2 \in BC$, analogously is defined B_2 and C_2 then
$AA_2 \cap BB_2 \cap CC_2 \equiv a(s-a) : b(s-b) : c(s-c)$ (same argument)
This is the Mittepunkt point that say **Luis González**, and is the conjugate isogonal of $S \equiv X_{57}$

JuanOrtiz

#5 Jun 2, 2015, 6:52 am

Let P be any point in ABC and define A', B', C' as the intersections of P with BC, CA, AB and define A'', B'', C'' as the intersection of BC with the angle bisector of $BA'C$. Then it's easy to see AA'', BB'', CC'' concur, define this point to be $f(P)$.

Then f maps lines to lines. From this we finish easily. It is worth noting $f(O)$ also lies on the line JGS. :)

This post has been edited 1 time. Last edited by JuanOrtiz, Jun 2, 2015, 7:04 am

drmzjoseph

#6 Jun 2, 2015, 7:41 am • 1

> *JuanOrtiz wrote:*
>
> Let P be any point in ABC and define A', B', C' as the intersections of P with BC, CA, AB and define A'', B'', C'' as the intersection of BC with the angle bisector of $BA'C$. Then it's easy to see AA'', BB'', CC'' concur, define this point to be $f(P)$.
>
> Then f maps lines to lines. From this we finish easily. It is worth noting $f(O)$ also lies on the line JGS. :)

You have typo:

Let $\triangle A'B'C'$ be the **circumcevian** triangle of P with respect to $\triangle ABC$, define A'' as the intersection of BC with the angle bisector of $BA'C$ and define analogously B'' and C''. Then it's easy to see AA'', BB'', CC'' concur, define this point to be $f(P)$.

$f(P)$ is a **Affine Transformation,** that maps lines a lines, that preserves the ratio of distances.

Saludos amigo, de México.

JuanOrtiz

#7 Jun 2, 2015, 10:09 am

Yes I meant to say that. Also, SIO are collinear :)

drmzjoseph

#8 Jun 2, 2015, 11:42 am

> *JuanOrtiz wrote:*
>
> Yes I meant to say that. Also, SIO are collinear :)

$\ell \equiv \overline{IO}$, since $f(I) = G$ and $f(X_{56}) = S$
we get $f(\ell) = GS$ this explain that $f(O) \in \overline{JGS}$
Remember that $X_{56} \in AM_A$ and this is the exsimilicenter of the incircle and circumcircle, and too the isogonal of the Nagel Point.

Moreover AA' cut again $\odot(ABC)$ at Lyou can prove that if the incircle of $\triangle ABC$ touch BC at A_1, obtain that M_AA' cut to A_1L at $\odot(ABC)$. $\star$
$\star$ This is useful for the prove that X_{57} (Like the isogonal of Mittepunkt point) belongs AA'
So $f(S) = J \Rightarrow S \in \ell$

This post has been edited 3 times. Last edited by drmzjoseph, Jun 2, 2015, 11:55 am

Quick Reply

High School Olympiads

geometry

Reply

andria

#1 May 18, 2015, 4:02 pm

in triangle ABC points E, F lie on AC, AB and M, N are midpoints of BF, CE prove that the centers of nine point circles of triangles ABC, AEF, AMN are collinear.(without using vectors)

Luis González

#3 May 18, 2015, 9:13 pm • 1

Let $\{O, O_1, O_2\}, \{H, H_1, H_2\}$ and $\{N, N_1, N_2\}$ denote the circumcenters, orthocenters and 9-point centers of $\triangle ABC$, $\triangle AEF$ and $\triangle AMN$, resp. See 4 Euler lines are concurrent (lemma at post #2), O_2 and H_2 are the midpoints of OO_1 and HH_1. Now, if J is the center of the spiral similarity taking $\overline{OO_1}$ into $\overline{HH_1}$, we get $\triangle JOO_1 \cup O_2 \sim \triangle JHH_1 \cup H_2 \sim \triangle JNN_1 \cup N_2 \Longrightarrow N_2$ is midpoint of NN_1.

P.S. The problem is still true if M, N are points on BF, CE verifying $\overline{MB} : \overline{MF} = \overline{NC} : \overline{NE}$. The proof is exactly the same.

Quick Reply

High School Olympiads

4 Euler lines are concurrent

Source: Vietnam IMO training 2015 test- Own

livetolove212
#1 May 12, 2015, 12:39 am

Given triangle ABC with its Euler center E. Let XYZ be the pedal triangle of E wrt triangle ABC. Prove that Euler lines of 4 triangles AYZ, BXZ, CXY, ABC are concurrent.

Luis González
#2 May 12, 2015, 5:25 am • 7

Lemma: In a triangle $\triangle ABC$, let B_1, C_1 be arbitrary points on AB, AC and let B_2, C_2 be the midpoints of BB_1, CC_1. Then the orthocenter H_2 of $\triangle AB_2C_2$ is the midpoint between the orthocenters H, H_1 of $\triangle ABC$ and $\triangle AB_1C_1$.

Proof: Let $(O), (O_1), (O_2)$ be the circumcircles of $\triangle ABC, \triangle AB_1C_1, \triangle AB_2C_2$. (O_2) is midcircle of $(O), (O_1)$, thus O_2 is midpoint OO_1. Since AO, AO_1, AO_2 are the isogonals of AH, AH_1, AH_2 WRT $\angle BAC$ and $\frac{AH}{AO} = \frac{AH_1}{AO_1} = \frac{AH_2}{AO_2} = 2\cos A$, then it follows that $\triangle AOO_1 \cup O_2 \sim \triangle AHH_1 \cup H_2 \Longrightarrow H_2$ is midpoint of HH_1.

Back to the proposed problem. Let O, H be the circumcenter and orthocenter of $\triangle ABC$. Let R, Q be the feet of the altitudes on AC, AB and N, L the midpoints of AC, AB.

Since Y, Z are the midpoints of RN, QL, then from the previous lemma, it follows that the orthocenter T of $\triangle AYZ$ is the midpoint between the orthocenters S, K of $\triangle ANL$ and $\triangle ARQ$. The line joining T and the midpoint of AE (circumcenter of $\triangle AYZ$) is then median of the trapezoid $ASEO$, cutting OE at its midpoint, i.e. Euler line UT of $\triangle AYZ$ goes through the midpoint of OE and similarly Euler lines of $\triangle BXZ$ and $\triangle CXY$ go through the midpoint of OE.

Luis González
#3 May 12, 2015, 5:36 am

On a side note, the locus of the points E, such that the Euler lines of $\triangle AYZ, \triangle BXZ, \triangle CXY$ concur is the Napoleon-Feuerbach cubic of $\triangle ABC$. But their concurrency point is not, in general, on the Euler line of $\triangle ABC$.

buratinogigle
#4 May 12, 2015, 7:20 am • 1

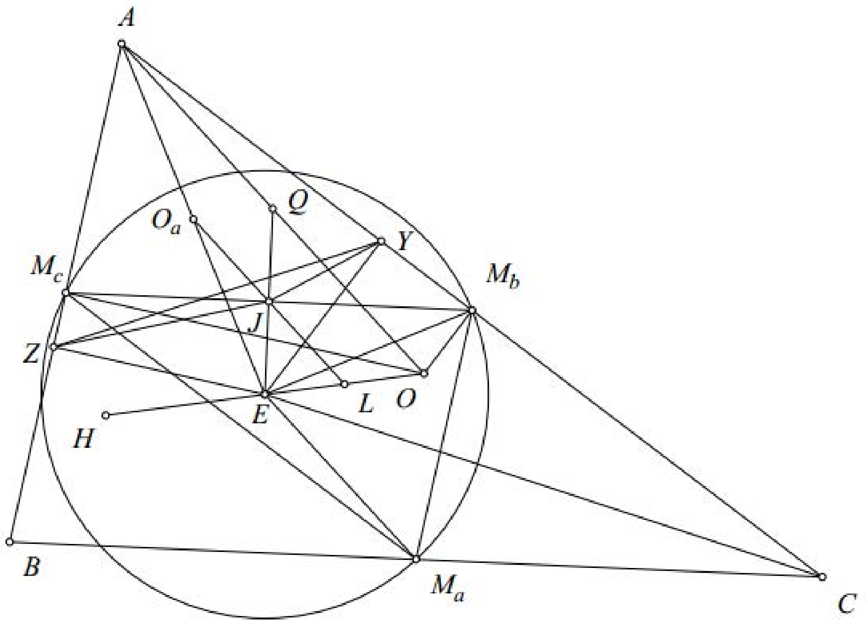
It was posted before http://www.artofproblemsolving.com/community/c6h209318

livetolove212
#5 May 12, 2015, 8:08 am

My solution.
Lemma: Given triangle ABC. Let D be a point such that $\angle DBA = \angle BAC = \angle DCA$. Then D lies on Euler line of triangle ABC.
This lemma is well-known. We can prove it by using Pappus theorem.
Back to our problem.
Let O be the circumcenter of triangle ABC, L be the midpoint of EO. We will prove Euler lines of triangles AYZ, BXZ, CXY pass through L.
Let O_a be the circumcenter of AYZ, J be the midpoint of O_aL. EJ cuts AO at Q.
We have O_a is the midpoint of AE then $O_aL \parallel AO$. We get Q is the midpoint of AO.
Let M_a, M_b, M_c be the midpoints of BC, CA, AB. Since Q and E are the circumcenters of AM_bM_c and $M_aM_bM_c$ hence Q and E are symmetric wrt M_bM_c. This means J is the midpoint of M_bM_c and $EJ \perp M_bM_c$.

Therfore $EJYM_b$ is concyclic, we get $\angle JYA = \angle JEM_b = \angle M_bM_aM_c = \angle BAC$.
Similarly, $\angle AYJ = \angle AZJ = \angle YAZ$. Using the lemma above, J lies on Euler line of triangle AYZ. Then O_aL is Euler line of triangle AYZ. We are done.

Attachments:



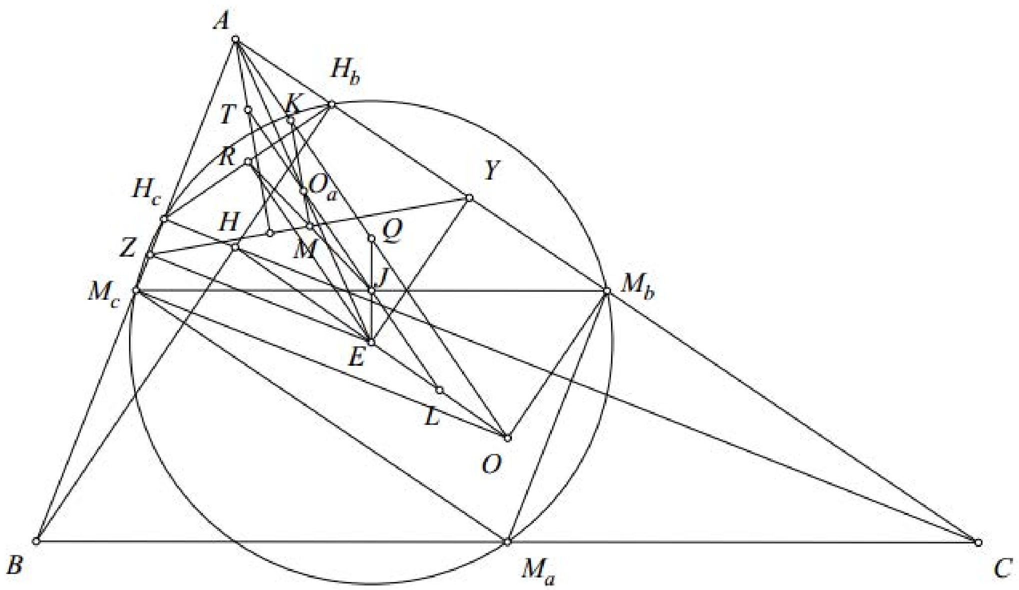
livetolove212

#6 May 12, 2015, 8:15 am

Another proof.

Let Q, Q_a, J, L be the midpoints of AO, AE, EQ, EO; M_a, M_b, M_c be the midpoints of BC, CA, AB. We have $(M_aM_bM_c)$ and (AM_bM_c) are symmetric through M_bM_c then the circumcenters E and Q are symmetric through M_bM_c. This means J is the midpoint of M_bM_c. Obviously, J is the midpoint of O_aL.
Denote H_b, H_c the projection of B, C on AC, AB, R, M the midpoints of H_bH_c, XY.
Since $\dfrac{H_bY}{YM_b} = \dfrac{H_cZ}{ZM_c} = 1$ and R, M, J are the midpoints of H_bH_c, YZ, M_bM_c then applying ERIQ theorem, M is the midpoint of RJ.
We have $ER \perp H_bH_c, AO \perp H_bH_c, O_aL \parallel AO$ hence $ER \parallel O_aL \parallel AO$.
Let K be the intersection of MO_a and AO.
Denote $d_{A/l}$ the distance from A to line l. Since $EL = LO$ then $d_{R/AO} = 2d_{J/AO}$. M is the midpoint of RJ then $d_{M/AO} = \dfrac{3}{2}d_{J/AO}$. Therefore $O_aK = 2O_aM$.
Let T be the orthocenter of AYZ we get $AT \parallel= 2O_aM$ or $AT \parallel= O_aK$, then $TO_a \parallel AO$. Hence T, O_a, L are collinear or Euler line of triangle AYZ passes through the midpoint of EO. Similarly we are done.

Attachments:



This post has been edited 1 time. Last edited by livetolove212, May 12, 2015, 8:16 am

buratinogigle

#11 May 12, 2015, 8:52 am • 1

Change the notation

Let ABC be a triangle with nine point center N and DEF is pedal triangle of N. Prove that Euler lines of triangle AEF, BFD, CDE, ABC are concurrent.

Solution of my student Phan Minh Nghia

O_a is midpoint of AN and H_a is orthocenter of AEF. Let BY, CZ are altitudes of ABC and M_a is midpoint of BC. Let G_a be centroid of M_aYZ. P, Q are midpoint of CA, AB and H_aENF is parallelogram so
$2\overrightarrow{O_aH_a} = \overrightarrow{H_aA} + \overrightarrow{H_aE} + \overrightarrow{H_aF} = \overrightarrow{H_aN} + \overrightarrow{NA} + \overrightarrow{H_aE} + \overrightarrow{H_aF} = 2\overrightarrow{H_aE} + 2\overrightarrow{H_aF} + \overrightarrow{NA} = \overrightarrow{HY} + \overrightarrow{OP} + \overrightarrow{HZ} + \overrightarrow{OQ} + \overrightarrow{NH} + \overrightarrow{HA} = 3\overrightarrow{HG_a} + 3\overrightarrow{NH} = 3\overrightarrow{NG_a}$
so $O_aH_a \parallel NG_a \perp YZ \perp OA$. Thus $O_aH_a \parallel OA$ deduce O_aH_a passes through midpoint of ON.

An other way to generalize this problem

Let ABC be a triangle with centroid G and P is a point. Q is symmetric of P through midpoint of PG. R is midpoint of PQ. D, E, F lie on BC, CA, AB such that $RD \parallel PA, RE \parallel PB, RF \parallel PC$. ℓ_a is the line passes through midpoint of RA and centroid of triangle AEF. Similarly, we have ℓ_b, ℓ_c. Prove that $\ell_a.\ell_b, \ell_c$ are concurrent.

Luis González

#12 May 12, 2015, 9:17 am • 1

> *buratinogigle wrote:*
> An other way to generalize this problem
>
> Let ABC be a triangle with centroid G and P is a point. Q is symmetric of P through midpoint of PG. R is midpoint of PQ. D, E, F lie on BC, CA, AB such that $RD \parallel PA, RE \parallel PB, RF \parallel PC$. ℓ_a is the line passes through midpoint of RA and centroid of triangle AEF. Similarly, we have ℓ_b, ℓ_c. Prove that $\ell_a.\ell_b, \ell_c$ are concurrent.

If P is inside of $\triangle ABC$, there exists an affine homology carring $\triangle ABC \cup P$ into an acute $\triangle ABC$ with orthocenter P. Then it becomes the original problem for the Euler center.

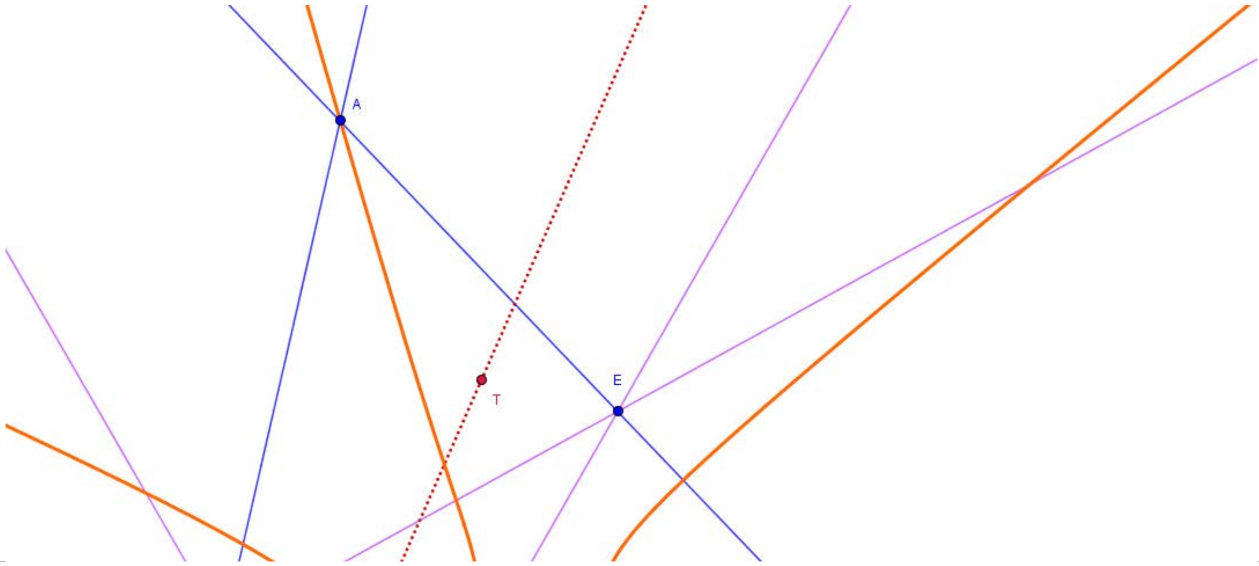
TelvCohl

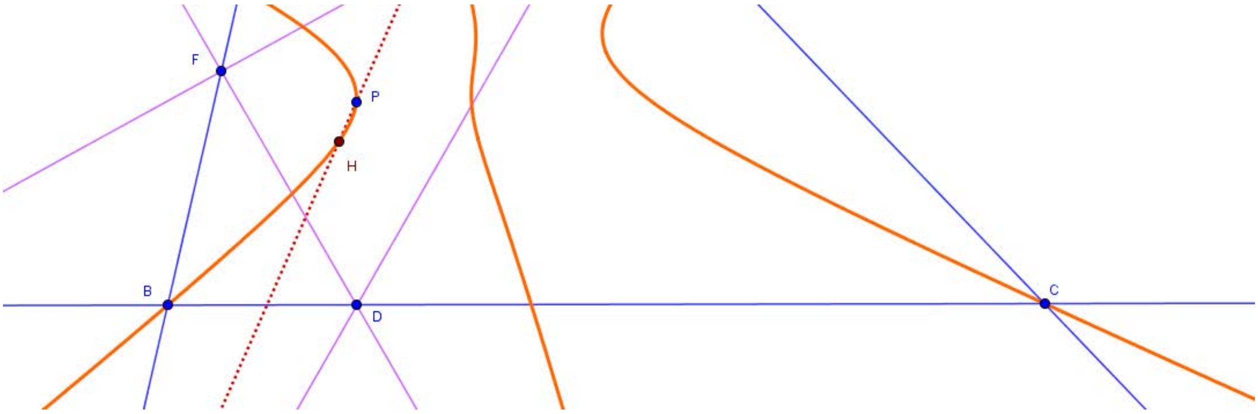
#13 May 12, 2015, 5:34 pm • 8

I found a generalization of the original problem , but I don't have the proof yet .

Generalization : Let P be a point on Napoleon-Feuerbach cubic of $\triangle ABC$ and $\triangle DEF$ be its pedal triangle . T is the intersection of the Euler lines of $\triangle AEF, \triangle BFD$ and $\triangle CDE$. Then PT pass through the orthocenter H of $\triangle ABC$.

Attachments:





SalaF

#14 May 17, 2015, 12:18 am • 3

I have not thought about this generalization yet, however I made a computer drawing. Here is what I found: the symmetric point of P wrt T belongs to the line OP', where O is the circumcenter of ABC and P' is the isogonal conjugate of P wrt ABC. This seems interesting, as the point $HP \cap OP'$ usually has a lot of nice properties when we consider the isopivotal cubics with pivot on the Euler line (Neuberg, McCay, Darboux,...).
I hope that someone might find this useful.

SalaF

#15 May 17, 2015, 3:13 pm • 6

As it often happens, the properties of an isopivotal cubic can be completely generalized. Here is the

Complete generalization:

Let ABC be a triangle and P any point in the same plane. Let P_1 be the isogonal conjugate of P wrt ABC and let us call O, H the circumcenter and the orthocenter of ABC.
Let $X = PP_1 \cap OH$ and let us denote $k = \frac{XO}{XH}$ (the lenghts are considered to be oriented). Let P_A, P_B, P_C be the reflections of P wrt BC, CA, AB and let P'_A, P'_B, P'_C be the image of $P_AP_BP_C$ after a homothety of center P and factor $\frac{k}{2k+1}$. Let Q_A be a point such that $PP'_BQ_AP'_C$ is a parallelogram, and let Q_B, Q_C be similarly defined.
Then the five lines $AQ_A, BQ_B, CQ_C, HP, OP_1$ are concurrent.

When $k = -1$ we obtain the particular case found by Telv Cohl.

This post has been edited 1 time. Last edited by SalaF, May 17, 2015, 4:20 pm

AdithyaBhaskar

#16 May 17, 2015, 3:14 pm • 1

What is the euler center here? Is it the nine-point center?

TelvCohl

#17 Feb 26, 2016, 7:22 pm • 1

> *TelvCohl wrote:*
>
> **Generalization :**
>
> Let P be a point on Napoleon-Feuerbach cubic of $\triangle ABC$ and $\triangle DEF$ be its pedal triangle . T is the intersection of the Euler lines of $\triangle AEF, \triangle BFD$ and $\triangle CDE$. Then PT pass through the orthocenter H of $\triangle ABC$.

We can prove it by using some results in ★ this paper by Ivan Zelich and Xuming Liang.

Let H_d, H_e, H_f be the orthocenter of $\triangle PEF, \triangle PFD, \triangle PDE$, respectively. Let O_a be the circumcenter of $\triangle AEF$ and let O_A be the reflection of O_a in P (define O_b, O_B, O_c, O_C similarly). Since P lies on Napoleon-Feuerbach cubic of $\triangle ABC$, so from **Theorem 0.1** in ★ $\Longrightarrow$ P lies on Napoleon-Feuerbach cubic of $\triangle DEF \Longrightarrow \triangle DEF, \triangle O_AO_BO_C$ are perspective (**Theorem 0.2** in ★) and orthologic, hence notice the orthocenter of $\triangle O_AO_BO_C$ lies on PH (which is the image of H under the homothety $(P, \frac{-1}{2})$) we get $PH, O_AH_d, O_BH_e, O_CH_f$ are concurrent at S (**Theorem 2.5** in ★).

Since the orthocenter H_a of $\triangle AEF$ is the reflection of P in the midpoint M of EF, so from $\frac{MO_a}{PH_d}=\frac{1}{2}$ we get the reflection of H_d in P lies on $O_aH_a \Longrightarrow$ the Euler line of $\triangle AEF$ is the reflection of O_AH_d in P. Analogously, we can prove the Euler line of $\triangle BFD, \triangle CDE$ is the reflection of O_BH_e, O_CH_f in P, respectively, so from the discussion above we conclude that the Euler line of $\triangle AEF, \triangle BFD, \triangle CDE$ and PH are concurrent at the reflection T of S in P.

Remark : Let O be the circumcenter of $\triangle ABC$ and let Q be the isogonal conjugate of P WRT $\triangle ABC$. Then we can prove the reflection R of P in T lies on OQ as following **:**

Since $\triangle O_AO_BO_C, \triangle H_dH_eH_f$ are orthologic (**Theorem 2.5** in ★), so the circum-rectangular hyperbola of $\triangle ABC$ passing through Q is the locus of V such that the parallel from H_d, H_e, H_f to AV, BV, CV, resp. are concurrent (**Lemma 1.3** in ★), hence notice $H_dS \parallel AR, H_eS \parallel BR, H_fS \parallel CR$ we get H, Q, R lie on a circumconic of $\triangle ABC$. Since R lies on PH and the circumconic of $\triangle ABC$ passing through H and Q, so R also lies on OQ (**Theorem 1.5** in ★).

This post has been edited 1 time. Last edited by TelvCohl, Feb 27, 2016, 1:08 pm

TelvCohl

#18 May 22, 2016, 6:21 am

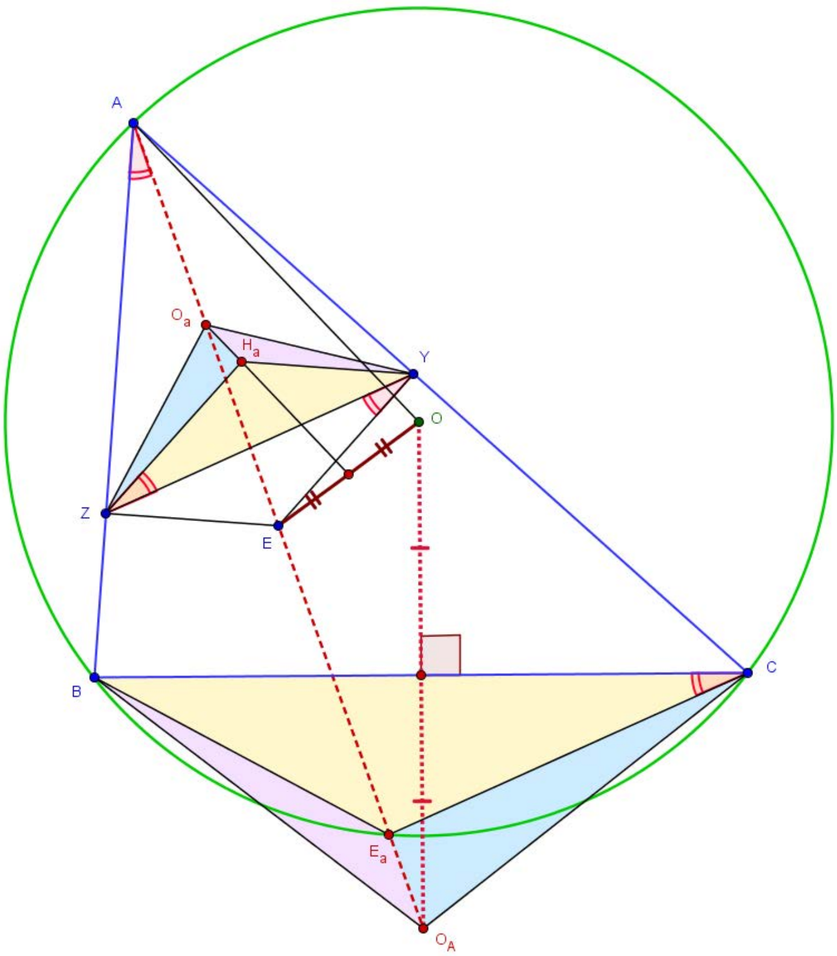
Another proof to the original problem :

Let O be the circumcenter of $\triangle ABC$ and let O_a, H_a be the circumcenter, orthocenter of $\triangle AYZ$, respectively. Let O_A be the reflection of O in BC and let $AO_A \equiv AE$ cuts $\odot(O)$ again at E_a. Notice EYH_aZ is a parallelogram, so

$$\left\{\begin{array}{l} \angle H_aZY = \angle EYZ = \angle EAZ = \angle E_aCB \\ \\ \angle H_aYZ = \angle EZY = \angle EAY = \angle E_aBC \end{array}\right\| \Longrightarrow \triangle O_aYZ \cup H_a \overset{+}{\sim} \triangle O_ABC \cup E_a,$$

hence $\angle(O_aH_a, AE) = \angle(O_aH_a, O_AE_a) = \angle(YZ, BC) = \angle(\perp YZ, \perp BC) = \angle(AO, AE) \Longrightarrow AO \parallel O_aH_a \Longrightarrow$ the Euler line O_aH_a of $\triangle AYZ$ passes through the midpoint of OE.

Attachments:



Quick Reply

High School Olympiads

PF is the bisector of angle BPC

Reply

Source: Own

THVSH
#1 May 13, 2015, 7:16 pm

Let ABC be a triangle. P lies inside $\triangle ABC$ such that $\angle PBA = \angle PCA$. $AP \cap \odot(ABC) = Q$. The bisector of $\angle BAC$ intersects $\odot(ABC)$ at D. DE is the diameter of $\odot(ABC)$. $EQ \cap BC = F$. Prove that PF is the bisector of $\angle BPC$

Hint

TelvCohl
#3 May 13, 2015, 8:50 pm

My solution :

Let $B' = BP \cap AC, C' = CP \cap AB$.

From $\angle B'BC' = \angle B'CC' \Longrightarrow B, C, B', C'$ are concyclic ,
so $\triangle PBC' \sim \triangle PCB' \Longrightarrow \sin \angle BAP : \sin \angle CAP = \text{dist}(P, AB) : \text{dist}(P, AC) = BP : CP$ . ... $(\star)$

Since QF is the bisector of $\angle CQB$,
so $BF : CF = BQ : CQ = \sin \angle BAP : \sin \angle CAP$,
hence combine with $(\star)$ we get $BF : CF = BP : CP$. i.e. PF is the bisector of $\angle BPC$

Q.E.D

THVSH
#4 May 13, 2015, 9:02 pm

Thank you, Telv :)
This is my solution:

We have $\frac{S_{APB}}{S_{AQB}} = \frac{AP}{AQ} = \frac{S_{APC}}{S_{AQC}} \Longrightarrow \frac{S_{APB}}{S_{APC}} = \frac{S_{AQB}}{S_{AQC}}.(1)$
On the other hand, since, $\angle PBA = \angle PCA; \angle QBA + \angle QCA = 180^\circ \Longrightarrow \frac{S_{APB}}{S_{APC}} = \frac{AB.BP}{AC.CP}; \frac{S_{AQB}}{S_{AQC}} = \frac{AB.BQ}{AC.CQ}$
(2)
From $(1), (2)$, we get $\frac{BP}{CP} = \frac{BQ}{CQ} = \frac{BF}{CF}$ (since QF is the bisector of $\angle BQC$)
$\Longrightarrow PF$ is the bisector of $\angle BPC$ Q.E.D

andria
#5 May 13, 2015, 9:18 pm

Too trivial problem:
Let $\angle ABP = \angle ACP = x$ note that $\frac{AP}{\sin x} = \frac{BP}{\sin \angle BAQ} = \frac{CP}{\sin \angle CAQ} \longrightarrow \frac{\sin \angle BAQ}{\sin \angle CAQ} = \frac{PB}{PC} = \frac{BQ}{QC}$(1) since QE is bisector of $\angle BQC$ we have $\frac{BQ}{QC} = \frac{BF}{FC}$ so from (1) we get that $\frac{PB}{PC} = \frac{BF}{FC}$
DONE

LeVietAn

#6 May 13, 2015, 9:28 pm

My solution:
Let BP cut (ABC) again at K, CP cut (ABC) at again L.
the tangent line at Q to (ABC) meet BC at S.
From $\angle PBA = \angle PCA \Rightarrow AK = AL \Rightarrow SQ = SP$ (see IMO 2010 Problem 4:

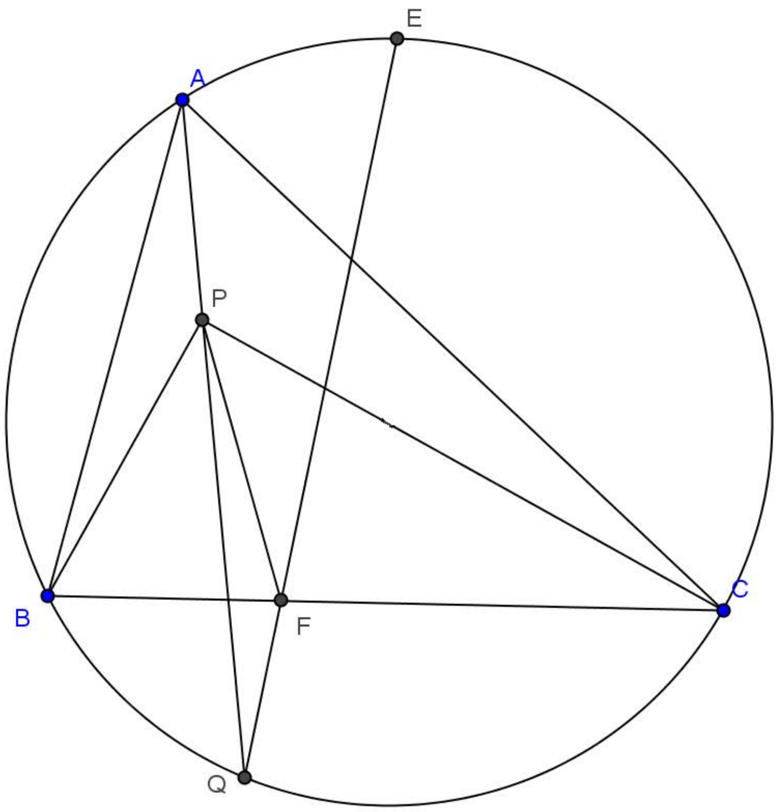
http://artofproblemsolving.com/community/c6h356195p1936916)
Since $SP^2 = SQ^2 = SB.SC \Rightarrow SP$ is tangent to (BPC).
So $\Delta SBP \sim \Delta SPC$ and $\Delta SBQ \sim \Delta SQC \Rightarrow \frac{PB}{CP} = \frac{SP}{SC} = \frac{SQ}{SC} = \frac{QB}{CQ}$. i.e.
QED

tranquanghuy7198

#7 May 14, 2015, 11:12 pm

My solution:
Notice that: QF is the bisector of $\angle BQC$, then:
PF is the bisector of $\angle BPC$
$\Longleftrightarrow \frac{BP}{PC} = \frac{BF}{FC}$
$\Longleftrightarrow \frac{BP}{PC} = \frac{BQ}{QC}$ (1)
Now we prove (1) by the inversion centered at A and using the length formula for (1): Given triangle ABC and a point P such that $\angle APB = \angle APC$. $AP \cap BC = Q$. Prove that $\frac{BP}{PC} = \frac{BQ}{QC}$.
This is obviously right due to the property of the angle bisector. Q.E.D

Attachments:



Luis González

#8 May 18, 2015, 4:53 am

Let the tangents of $\odot(ABC)$ and $\odot(PBC)$ at Q and P intersect at X. We have
$\widehat{XPQ} = \widehat{PAB} + \widehat{PBA} + \angle PCB = \widehat{QCB} + \widehat{PCA} + \widehat{PCB} = \widehat{QCA} = \widehat{XQP} \Longrightarrow XQ = XP \Longrightarrow X$ is on

radical axis BC of $\odot(ABC), \odot(PBC) \Longrightarrow P$ is on the Q-Apollonius circle $\odot(X, XQ)$ of $\triangle QBC \Longrightarrow PF$ bisects $\widehat{BPC}.$

Quick Reply

High School Olympiads

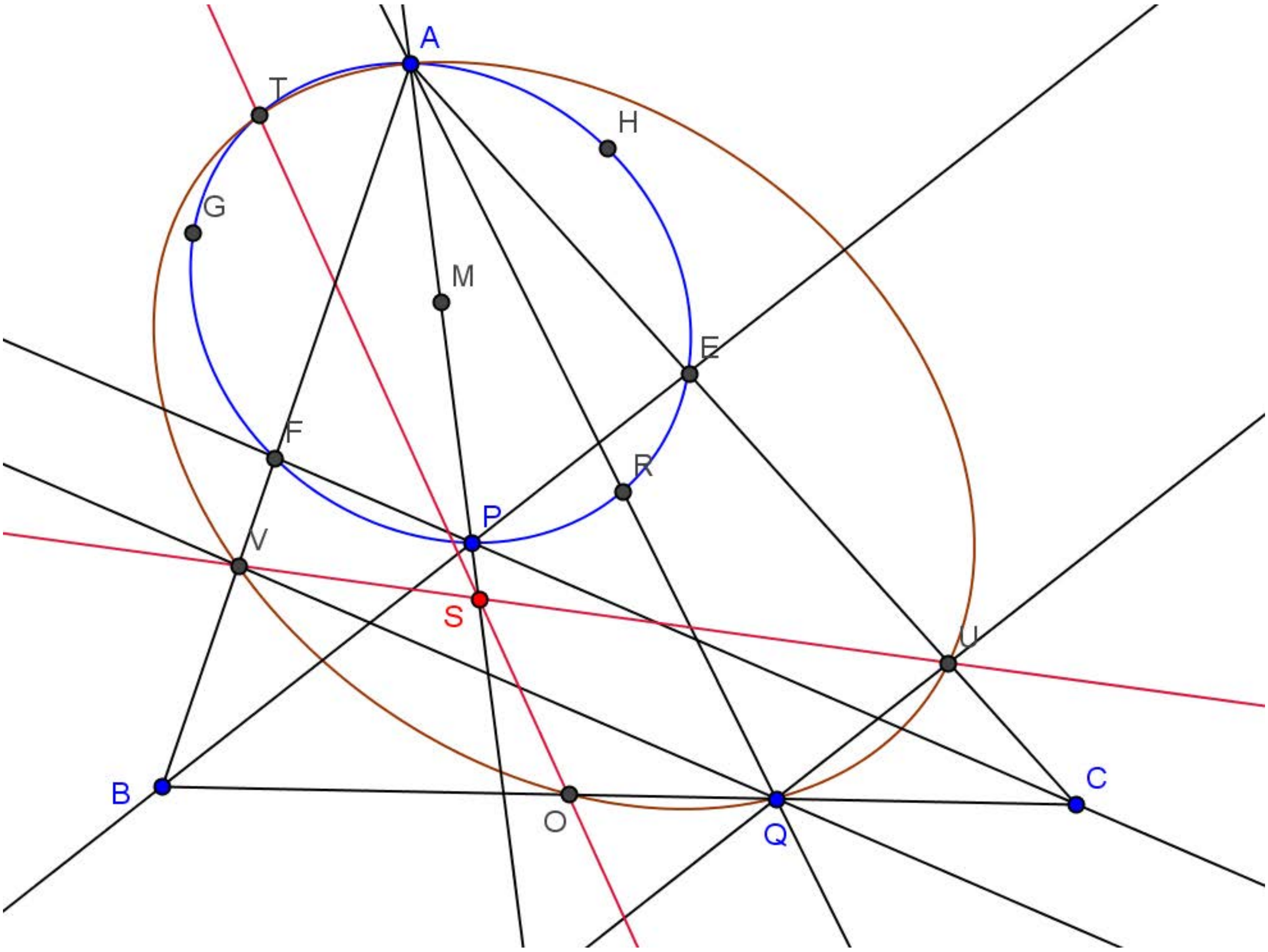
A,P,S are collinear

THVSH

#1 May 16, 2015, 10:16 pm • 1

Let ABC be a triangle. Arbitrary points $P, Q(Q \in BC)$. $BP \cap CA = E, CP \cap AB = F$. M is the midpoint of PA. G, H are symmetric of E, F wrt M, respectively. Then A, E, F, G, H, P lie on a conic with center M, denote it by $\mathcal{C}$. $AQ \cap \mathcal{C} = \{A, R\}$. T is symmetric of R wrt M. $U \in CA, V \in AB$ such that $QU \parallel BP; QV \parallel CP$. A conic $\mathcal{C}_1$ passes through A, T, Q, U, V. $\mathcal{C}_1 \cap BC = \{Q, O\}$. $TO \cap UV = S$. Prove that A, P, S are collinear

Attachments:



Luis González

#2 May 17, 2015, 5:42 am • 1

Denote $D \equiv AP \cap BC$. Since AP passes though the center M of $\mathcal{C}$, then from the complete $AEPF$, we deduce that the tangent of $\mathcal{C}$ at P is parallel to BC. Thus since $QA \parallel PT, PE \parallel QU, QV \parallel PF \Longrightarrow$ $Q(A, D, V, U) = P(T, P, F, E) = A(T, P, F, E) \equiv A(T, D, V, U) \Longrightarrow AT$ is tangent of the conic through $AVDQU$. Hence, by Pascal theorem for $AAUVQD$, we get that $C, X \equiv AT \cap VQ$ and $S^* \equiv UV \cap AD$ are collinear. Thus if $O^* \equiv TS^* \cap BC$, by the converse of Pascal theorem, the hexagon $TAUVQO^*$ is inscribed in the conic $\mathcal{C}_1$, i.e. $O^* \equiv O$ $\Longrightarrow S \equiv S^* \Longrightarrow A, P, S$ are collinear.

High School Olympiads

Prove that the line is perpendicular !

Reply

arberig

#1 May 17, 2015, 12:18 am

Let I be the incenter of $\triangle ABC$ and let H_a, H_b and H_c be the orthocenters of $\triangle BIC, \triangle CIA$ and $\triangle AIB$, respectively. The line H_aH_b meets AB at $X$and the line H_aH_c meets AC at Y. If the midpoint T of the median AM of $\triangle ABC$ lies on XY. Prove that the line H_aT is perpendicular to BC.

Luis González

#2 May 17, 2015, 12:40 am

Since H_a, H_b, H_c are the poles of the A-,B- and C- midline of $\triangle ABC$ WRT its incircle (I), it follows that X, Y are nothing but the tangency points of (I) with AB, AC. It's also known for any $\triangle ABC$, that AM, XY and the perpendicular from I to BC concur at a point S. Therefore $T \equiv S \Longleftrightarrow TH_a \perp BC$.

Quick Reply

High School Olympiads

Lots of circles and lines

Reply

Source: Bosnia and Herzegovina TST 2015 day 1 problem 2

gobathegreat
#1 May 16, 2015, 11:26 pm • 1

Let D be an arbitrary point on side AB of triangle ABC. Circumcircles of triangles BCD and ACD intersect sides AC and BC at points E and F, respectively. Perpendicular bisector of EF cuts AB at point M, and line perpendicular to AB at D at point N. Lines AB and EF intersect at point T, and the second point of intersection of circumcircle of triangle CMD and line TC is U. Prove that $NC = NU$

Luis González
#2 May 16, 2015, 11:55 pm

Since $\angle ADE = \angle ACB = \angle BDF \Longrightarrow AB$ bisects $\angle EDF$ externally $\Longrightarrow ND$ bisects $\angle EDF$ internally $\Longrightarrow M, N$ are midpoints of the arcs EDF and EF of $\odot(DEF) \Longrightarrow NE = NF$ and $\angle ENF = 180^\circ - \angle EDF = 180^\circ - (180^\circ - 2\angle ACB) = 2\angle ACB \Longrightarrow N$ is circumcenter of $\triangle CEF$. Now, since $TE \cdot TF = TD \cdot TM = TU \cdot TC \Longrightarrow U \in \odot(CEF)$ or $NC = NU$.

samithayohan
#3 Jun 5, 2015, 11:46 pm • 1

Its realy "Lots and lots of circles". My obvious fist attempt was inversion ,without any succes.So here goes an elementary proof by angle chasing.
First we are going to show $FDME$ are cyclic.Assume that M' is the second intersection of $\odot FDE$ withAB.
$\angle M'EF$=180-$\angle M'FE$-$\angle EM'F$=180-$\angle FDE$-$\angle M'DE$($FDM'E$ cyclic)=180-(180-$\angle M'DE$-$\angle ADE$)-$\angle M'DE$.
$ADFC$ cyclic and $BDEC$ cyclic $\Longrightarrow \angle ADE$=$\angle BDF$=$\angle ECF$
Hence$\angle M'EF$=$\angle ECF$.On the other hand we have that $\angle M'FE$=$\angle ECF$ $\Longrightarrow \angle M'EF$=$\angle M'FE$ $\Longrightarrow M'E$=$M'F$
Hence M' must lie on the perpendicular bisectosr of EF $\Longrightarrow M'$=M $\Longrightarrow EMDF$ cyclic.
Another easy angle chasing will tell that N lies on $\odot EMDF$
The rest of my proof is very similar to **Luis Gonzalezs** solution.

> Quote:
> $\angle ENF = 180^\circ - \angle EDF = 180^\circ - (180^\circ - 2\angle ACB) = 2\angle ACB \Longrightarrow N$ is circumcenter of $\triangle CEF$.
> Now, since $TE \cdot TF = TD \cdot TM = TU \cdot TC \Longrightarrow U \in \odot(CEF)$ or $NC = NU$.

.
Hence we are done.

Quick Reply

High School Olympiads

Metric relation in a right triangle

Reply

socrates

#1 May 16, 2015, 10:01 pm

Let ABC be a right triangle with the hypotenuse BC. Let BE be the bisector of the angle $\angle ABC.$ The circumcircle of the triangle BCE cuts the segment AB again at F. Let K be the projection of A on $BC.$ The point L lies on the segment AB such that $BL=BK.$ Prove that

$$\frac{AL}{AF}=\sqrt{\frac{BK}{BC}}.$$

Luis González

#2 May 16, 2015, 11:34 pm

Since $BCEF$ is cyclic $\Longrightarrow AF \cdot AB=AE \cdot AC$ and since $BK=BL \Longrightarrow KL \perp BE \Longrightarrow \angle LKA=\angle EBC$ and $\angle LAK=\angle ECB \Longrightarrow \triangle ALK \sim \triangle CEB \Longrightarrow \frac{AL}{CE}=\frac{AK}{BC} \Longrightarrow$

$$\frac{AL}{AF}=\frac{CE}{AE} \cdot \frac{AB}{BC} \cdot \frac{AK}{AC}=\frac{AK}{AC}=\frac{\sqrt{BK \cdot KC}}{\sqrt{KC \cdot BC}}=\sqrt{\frac{BK}{BC}}.$$

sunken rock

#3 May 17, 2015, 9:24 am

You may see it at https://www.facebook.com/photo.php?fbid=10205234557896577&set=gm.747055895407306&type=1&theater as well.

Best regards,
sunken rock

Quick Reply

High School Math

Find the lenght

Reply

drmzjoseph

#1 May 16, 2015, 7:11 am

Given a acute triangle PA_1B_1 is inscribed in the circle Γ with radius 1. for all integers $n \geq 1$ are defined:
C_n the foot of the perpendicular from P to A_nB_n
O_n is the center of $\odot(PA_nB_n)$
A_{n+1} is the foot of the perpendicular from C_n to PA_n
$B_{n+1} \equiv PB_n \cap O_nA_{n+1}$

If $PC_1 = \sqrt{2}$, find the length of PO_{2015}

Source

This post has been edited 1 time. Last edited by drmzjoseph, May 16, 2015, 7:47 am

drmzjoseph

#2 May 16, 2015, 8:52 am

Solution

Luis González

#3 May 16, 2015, 9:04 am

Let the perpendicular to PO_1 at O_1 cut PA_1, PB_1 at A_2', B_2', respectively and PO_1 cuts $\odot(PA_1B_1)$ again at Q_1. Since $|PC_1|^2 = 2 = |PO_1| \cdot |PQ_1|$, then O_1A_2' is the inverse of $\odot(PA_1B_1)$ under inversion WRT $\odot(P, PC_1)$. A_2', B_2' are the inverse images of $A_1, B_1 \Longrightarrow |PC_1|^2 = |PA_2'| \cdot |PA_1| = |PB_2'| \cdot |PB_1| \Longrightarrow A_2', B_2'$ coincide with the projections A_2, B_2 of C_1 on $PA_1, PB_1 \Longrightarrow PC_1$ is diameter of the circumcircle of $\triangle PA_2B_2$, i.e. O_2 is midpoint of $PC_1 \Longrightarrow O_n$ is the midpoint of $PC_{n-1} \Longrightarrow |PO_n|$ forms a geometric progression with ratio $\frac{\sqrt{2}}{2} \Longrightarrow |PO_n| = |PO_1| \cdot \left(\frac{\sqrt{2}}{2}\right)^{n-1} \Longrightarrow$ $|PO_{2015}| = \left(\frac{\sqrt{2}}{2}\right)^{2014}$.

Quick Reply

High School Olympiads

andria

#1 May 16, 2015, 2:22 am

In triangle ABC, BE, CF, AD are altitudes (E, F, D on AC, AB, BC) O is circumcenter of $\odot(\triangle ABC)$ let $AO \cap \odot(\triangle AEF) = R$ point M is miquel point of quadrilateral $AERF$ prove that $\angle AMD = 90$.

drmzjoseph

#2 May 16, 2015, 4:15 am

Is sufficient, by angle-chasing, prove that $M \in \odot(EFD)$, let $AR \cap EF \equiv Z$
Let N be the midpoint of BC, $\Rightarrow N$ is the pole of EF WRT $\odot(AEF) \Rightarrow Z$ and N are conjugate points.
Since YX is the polar of Z WRT $\odot(AEF) \Rightarrow Y, X$ and N are collinear.
$FR \cap AE \equiv X, ER \cap AF \equiv Y \Rightarrow M \in XY$ because $AFRE$ is cyclical.
$\angle BAC = \angle EDC = \angle EMY \Rightarrow M \in \odot(EDN) \Rightarrow M \in \odot(EFD)$

Remark

This post has been edited 1 time. Last edited by drmzjoseph, May 16, 2015, 4:26 am

Luis González

#3 May 16, 2015, 5:01 am

Inverting with center A and power $AE \cdot AC = AF \cdot AB$, the problem becomes: H is orthocenter and AO cuts BC at R. $\odot(ABR)$ cuts AC again at S and BS cuts $\odot(CRS)$ again at M. Then $MH \perp AH$.

Since $\angle BMC = \angle BRS = 180^\circ - \angle BAC = \angle BHC \Longrightarrow BCMH$ is cyclic. But $\angle BCM = \angle BSR = \angle BAO = \angle CBH \Longrightarrow BCMH$ is an isosceles trapezoid with bases $\overline{HM} \parallel \overline{BC} \Longrightarrow MH \perp AH.$

drmzjoseph

#4 May 16, 2015, 5:03 am

Another proof

Lemma (trivial)
Let X a point variable on $\odot(ABC)$, then the locus of the miquel point of $ABRC$ is a circle that passes through B and C

Let H the orthocenter of $\triangle ABC$, then the miquel point of $AFHE$ is D, then the miquel point of $AERF$, by lemma, belongs $\odot(EFD)$

This post has been edited 1 time. Last edited by drmzjoseph, May 16, 2015, 5:04 am

tranquanghuy7198

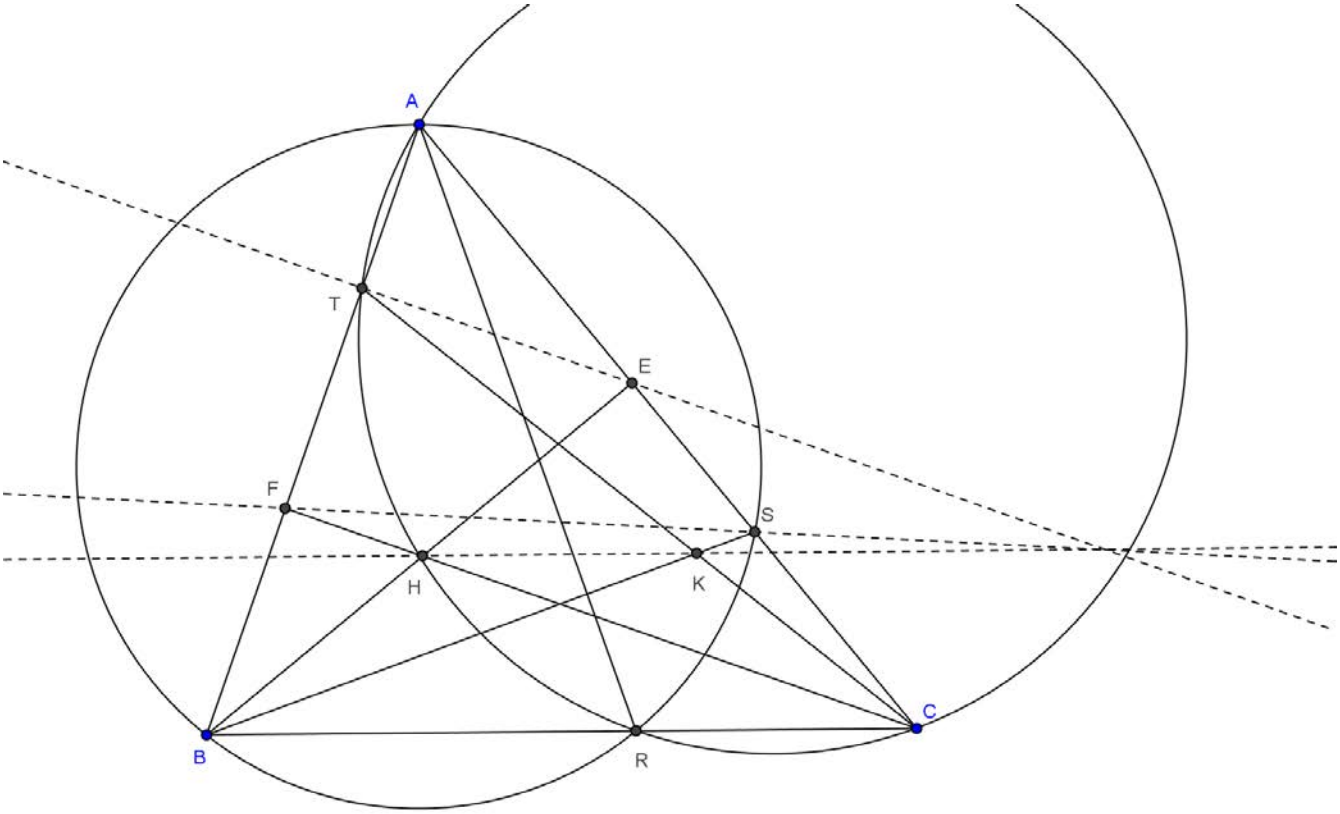
#5 May 16, 2015, 10:43 am

My solution bases on inversion, too.
Invert with center A and the power $AB.AF = AC.AE$ to get a new problem: Given $\triangle ABC$, altitudes $BE \cap CF = H$. $R \in BC$ such that $\angle HAB = \angle RAC$. $(ABR) \cap AC = S$, $(ACR) \cap AB = T$. Prove that: ET, FS and the line passing through H which is parallel to BC are concurrent.

Proof.
$BS \cap CT = K$
Notice that: $\angle KBC = \angle SBR = \angle SAR = \angle HCB$
Analogously: $\angle KCB = \angle HBC$

Analogously: $\angle KCB = \angle HBC$
$\Rightarrow HK \parallel BC$
Apply the Pappus theorem for (TFBSEC) to get the conclusion.

Attachments:



andria

#7 May 16, 2015, 3:38 pm

thank you for your solutions I found this problem with inversion with center A in this problem.

Quick Reply

High School Olympiads

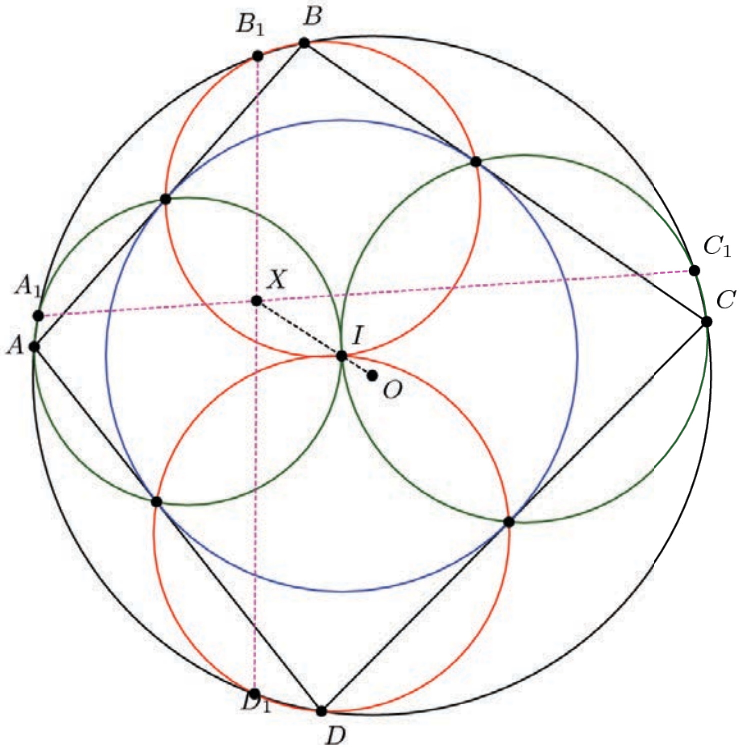
Quadrilateral bicentric

Source: own invention

drmzjoseph

#1 May 15, 2015, 9:22 am • 1

Let $ABCD$ a quadrilateral bicentric, with $\odot(I)$ incircle, and $\odot(O)$ circumcircle, if $(AI), (BI), (CI)$ and (DI) cut again to $\odot(O)$ at A_1, B_1, C_1 and D_1 respectively. $X \equiv A_1C_1 \cap B_1D_1$. Prove that X, I and O are collinear.



THVSH

#2 May 15, 2015, 2:06 pm • 2

My solution:

Let $\odot(I)$ intersect AB, BC, CD, DA at A', B', C', D'. M, N, P, Q is the midpoint of $A'B', B'C', C'D', D'A'$
We have: $AC, BD, A'C', B'D', IO$ are concurrent at a point Y (well-known)
Now we only need to prove that X, Y, I are collinear.
Invert WRT $\odot(I)$
$A \longrightarrow Q; B \longrightarrow M; C \longrightarrow N; D \longrightarrow P$
$\odot(AI) \longrightarrow A'D'; \odot(BI) \longrightarrow A'B'; \odot(CI) \longrightarrow B'C'; \odot(DI) \longrightarrow C'D'.$
$A_1 \longrightarrow A_1' = A'D' \cap \odot(MNPQ); B_1 \longrightarrow B_1' = A'B' \cap \odot(MNPQ)$
$C_1 \longrightarrow C_1' = B'C' \cap \odot(MNPQ); D_1 \longrightarrow D_1' = C'D' \cap \odot(MNPQ)$
$X \longrightarrow X' = \odot(IA_1'C_1') \cap \odot(IB_1'D_1')$

Let $A_1'C_1' \cap B_1'D_1' = X_1$

By easy angle chasing, we have $Y \in MD_1; NA_1; PB_1; QC_1$
$A'_1C'_1$ is the radical axis of $\odot(IA'_1C'_1)$ and $\odot(MNPQ)$; $B'_1D'_1$ is the radical axis of $\odot(MNPQ)$ and $\odot(IB'_1D'_1)$
$\Longrightarrow X_1$ is the radical center of $\odot(IA'_1C'_1); \odot(IB'_1D'_1); \odot(MNPQ) \Longrightarrow X_1 \in IX'$

Now we only need to prove that I, X_1, Y are collinear. If that, we get I, X', Y are collinear and then from the inversion I, X, Y are collinear
It is easy to see that $MNPQ$ is a rectangle, so the center of $(MNPQ)$ is the common midpoint R of MP, NQ.
Since $O \in IY \Longrightarrow R \in IY$. (1)
From Pascal's theorem for $A'_1B'_1QD'_1C'_1P$, we get X_1Y, A'_1P, D'_1Q are concurrent
From Pascal's theorem for $A'_1QMD'_1PN$, we get RY, A'_1P, D'_1Q are concurrent
$\Longrightarrow X_1, Y, R$ are collinear. From (1), we get I, X_1, Y are collinear. Q.E.D

TelvCohl

#5 May 15, 2015, 5:20 pm • 3

My solution :

Let $T = AC \cap BD, Y = BB_1 \cap DD_1, Z = AA_1 \cap CC_1$ (it's well-known $T \in OI$) .

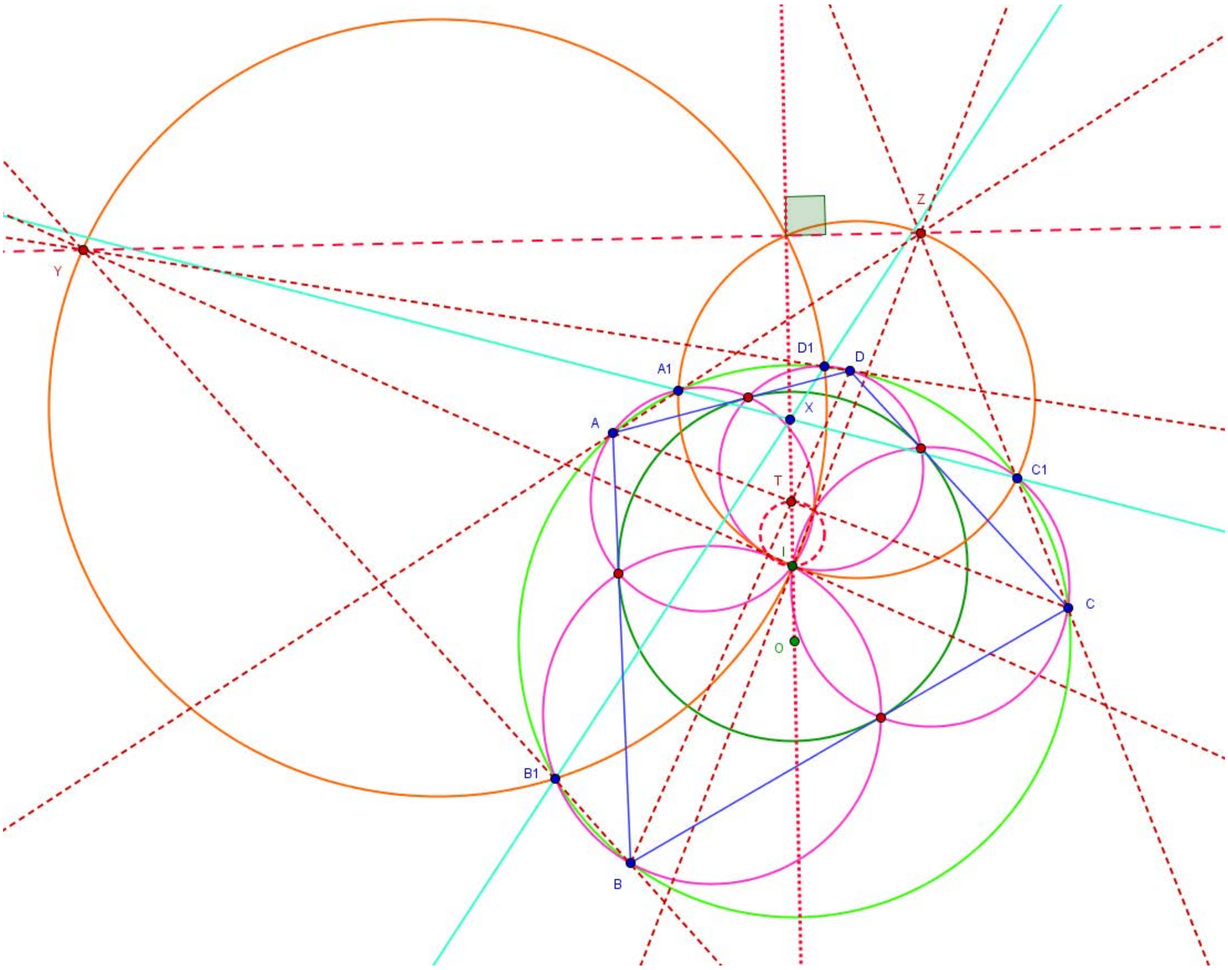
Easy to see $\odot(IT), \odot(IBB_1), \odot(IDD_1)$ are coaxial and $\odot(IT), \odot(IAA_1), \odot(ICC_1)$ are coaxial .

From $A_1X \cdot C_1X = B_1X \cdot D_1X \Longrightarrow X$ lie on the radical axis of $\{\odot(IB_1D_1), \odot(IA_1C_1)\}$,
so IX is the radical axis of $\{\odot(IB_1D_1) \equiv \odot(IY), \odot(IA_1C_1) \equiv \odot(IZ)\} \Longrightarrow IX \perp YZ$.

Since BB_1, DD_1 is the radical axis of $\{\odot(O), \odot(IBB_1)\}, \{\odot(O), \odot(IDD_1)\}$, respectively ,
so $Y = BB_1 \cap DD_1$ is the radical center of $\{\odot(O), \odot(IBB_1), \odot(IDD_1), \odot(IT)\}$.
Similarly, we can prove Z is the radical center of $\{\odot(O), \odot(IAA_1), \odot(ICC_1), \odot(IT)\}$,
so YZ is the radial axis of $\{\odot(O), \odot(IT)\} \Longrightarrow IT \perp YZ \Longrightarrow I, O, X, T$ are collinear .

Q.E.D

Attachments:



drmzjoseph

#6 May 15, 2015, 6:03 pm • 1

My solution:

$\odot(I)$ touch AB, BC, CD and DA at M, N, P and Q respectively. By Brianchon's theorem in the degenerate hexagon $AMBCDQ$ we get BQ, DM and CA are concurrent. If $Y \equiv AC \cap BD \Rightarrow \frac{BM}{DQ} = \frac{BY}{YD}$ (Ceva's theorem). Also A_1 is spiral center $MB \to QD \Rightarrow \frac{A_1B}{A_1D} = \frac{BM}{DQ} = \frac{BY}{YD} \Rightarrow \angle BA_1Y = \angle YA_1D$.
Let A_2, B_2, C_2 and D_2 are the second points of intersection of A_1Y, B_1Y, C_1Y and D_1Y at $\odot(O)$.So $O \equiv A_2C_2 \cap B_2D_2$

$Z \equiv A_1D_2 \cap B_1C_2 \Rightarrow X, Y, Z$ are collinear (By Pascal's Theorem $C_1, D_1, A_1, C_2, D_2, B_1$)
From Pascal's Theorem $B_2, A_1, C_2, D_2, B_1, A_2$ we obtain Y, O, Z are collinear.
Since I, O, Y are collinear (well-known) $\Rightarrow X, Y, O, I$ are collinear.

drmzjoseph
#7 May 15, 2015, 6:40 pm

Extension:

$AC \cap BD \equiv Y$
IO cut agian to (IA) at A^*, Prove that $\odot(A_1A^*Y)$ is tangent to $\odot(O)$

buratinogigle
#8 May 15, 2015, 7:41 pm

Nice problem dear drmzjoseph, here is an extension I found

Let $ABCD$ be cyclic quadrilateral inscribed circle (O). AC cuts BD at E. P is a point on OE. Circles diameter PA, PB, PC, PD cut (O) again at X, Y, Z, T. Prove that XZ, YT and OE are concurrent.

THVSH
#9 May 15, 2015, 7:51 pm • 1

> *drmzjoseph wrote:*
> Extension:
>
> $AC \cap BD \equiv Y$
> IO cut agian to (IA) at A^*, Prove that $\odot(A_1A^*Y)$ is tangent to $\odot(O)$

My solution:
Let J be the center of $\odot(A_1A^*Y)$. AA' is the diameter of (O), then A_1, I, A' are collinear.
Since $\angle BA_1Y = \angle YA_1D$ (see in **drmzjoseph**'s proof), we get A_1Y, AI intersect at point K which is the midpoint of arc BD(not contain A) of (O).
Then $\angle JA_1Y = \angle A_1A^*I - 90^\circ = 90^\circ - \angle A_1AI = \angle AIA_1 = \angle IA_1K + \angle IKA_1$
$= \angle IA_1K + \angle OA'A_1 = \angle IA_1K + \angle OA_1I = \angle OA_1Y$
$\Longrightarrow O, J, A_1$ are collinear. It means $\odot(A_1A^*Y)$ is tangent to $\odot(O)$. Q.E.D

TelvCohl
#10 May 15, 2015, 9:24 pm • 2

> *buratinogigle wrote:*
> here is an extension I found
>
> Let $ABCD$ be cyclic quadrilateral inscribed circle (O). AC cuts BD at E. P is a point on OE. Circles diameter PA, PB, PC, PD cut (O) again at X, Y, Z, T. Prove that XZ, YT and OE are concurrent.

My proof in post #5 still works for this extension, but I rewrite it in your notation :

Let $Q = BY \cap DT, R = AX \cap CZ, S = XZ \cap YT$.

Easy to see $\odot(PE), \odot(PAX), \odot(PCZ)$ are coaxial and $\odot(PE), \odot(PBY), \odot(PDT)$ are coaxial .

From $XS \cdot ZS = YS \cdot TS \Longrightarrow S$ lie on the radical axis of $\{\odot(PXZ), \odot(PYT)\}$,
so PS is the radical axis of $\{\odot(PXZ) \equiv \odot(PR), \odot(PYT) \equiv \odot(PQ)\} \Longrightarrow PS \perp QR$.

Since BY, DT is the radical axis of $\{\odot(O), \odot(PBY)\}, \{\odot(O), \odot(PDT)\}$, respectively,
so $Q = BY \cap DT$ is the radical center of $\{\odot(O), \odot(PBY), \odot(PDT), \odot(PE)\}$.
Similarly, we can prove R is the radical center of $\{\odot(O), \odot(PAX), \odot(PCZ), \odot(PE)\}$,
so QR is the radical axis of $\{\odot(O), \odot(PE)\} \Longrightarrow PE \perp QR \Longrightarrow O, P, E, S$ are collinear.
i.e. OE, XZ, YT are concurrent

Q.E.D

buratinogigle

#11 May 16, 2015, 2:26 am

Thank TelvCohl for your very nice solutions, here is the others extension

Let $ABCDEF$ be cyclic hexagon with AD, BE, CF are concurrent at S. P is a point on line OS. Circles diameter PA, PB, PC, PD, PE, PF cut (O) again at X, Y, Z, T, U, V. Prove that XT, YU, ZV and OS are concurrent.

I think that this problems is also true for cyclic $2n$-gon with the principle diagonals are concurrent.

Luis González

#12 May 16, 2015, 3:33 am • 1

> *buratinogigle wrote:*
> Let $ABCDEF$ be cyclic hexagon with AD, BE, CF are concurrent at $S. P$ is a point on line $OS.$ Circles diameter PA, PB, PC, PD, PE, PF cut (O) again at $X, Y, Z, T, U, V.$ Prove that XT, YU, ZV and OS are concurrent.
>
> I think that this problems is also true for cyclic $2n$-gon with the principle diagonals are concurrent.

Yes, the problem can be extended to a cyclic 2n-gon $P_1P_2P_3P_4 \cdots P_n$ with circumcircle (O) whose main diagonals concur at a point $S.$ P is a point on OS and the circle with diameter PP_1 cuts (O) again at X_1 and $X_2, X_3, X_4, \cdots, X_n$ are defined similarly. Then $X_1X_2X_3X_4 \cdots X_n$ is a 2n-gon with main diagonals concurring on $OS.$

Proof: Let OS cut (O) at $U, V.$ Clearly $PX_1, PX_2, PX_3, PX_4, \cdots PX_n$ cut (O) again at the antipodes $Q_1, Q_2, Q_3, Q_4, \cdots, Q_n$ of $P_1, P_2, P_3, P_4, \cdots, P_n.$ From rectangle $P_1P_{\frac{n}{2}+1}Q_1Q_{\frac{n}{2}+1}$, it follows that $Q_1Q_{\frac{n}{2}+1}$ passes through the reflection M of S on O and similarly all lines $Q_2Q_{\frac{n}{2}+2}, Q_3Q_{\frac{n}{2}+3}, \cdots, Q_iQ_{\frac{n}{2}+i}$ go through $M.$ Now the homography $X_i \mapsto X_{\frac{n}{2}+1}$ is the composition of involutions with poles P, M, P that fixes (O) and interchanges $U, V \Longrightarrow$ it is an involution on (O) that interchanges $U, V \Longrightarrow$ all lines $X_iX_{\frac{n}{2}+1}$ go through the pole of this involution lying on $UV \equiv OS.$

Quick Reply

High School Olympiads

nice and easy geometry

Locked

Source: Own

andria

#1 May 16, 2015, 1:41 am

In triangle ABC, D is a variable point on BC let $\odot(\triangle ABD) \cap AC = E, \odot(\triangle ACD) \cap AB = F$ prove that circumcircle of triangle AEF passes throw the fixed point as D varies on BC.

Luis González

#2 May 16, 2015, 1:50 am

Same as ELMO SL G3. The fixed point is the projection of the orthocenter of ABC on its A-median. See (AEF) passes through a fixed point on A-median.

High School Olympiads

(AEF) passes through a fixed point on A-median

Reply

Source: ELMO Shortlist 2013: Problem G3, by Allen Liu

v_Enhance

#1 Jul 23, 2013, 7:31 am

In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

Proposed by Allen Liu

Luis González

#2 Jul 23, 2013, 10:28 am • 1

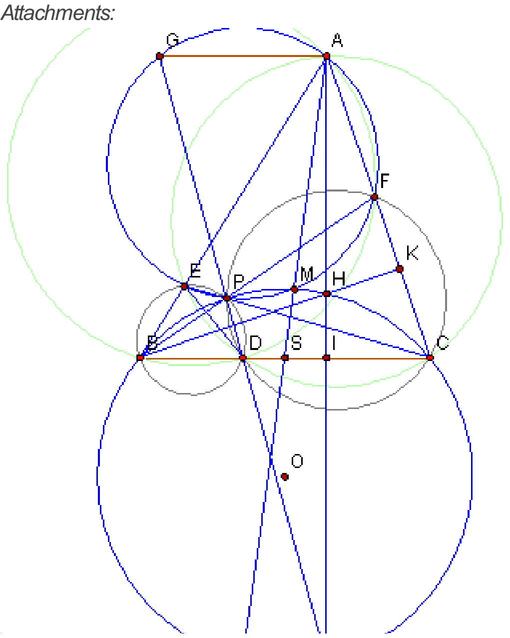
Let $P \equiv BF \cap CE. \angle(FA, FB) = \angle(DA, DB) = \angle(EA, EC) \Longrightarrow A, E, P, F$ are concyclic. From the complete cyclic $AEPF$, it follows that the polar of B WRT $\odot(AEF)$ goes through C and vice versa $\Longrightarrow B, C$ are conjugate points WRT $\odot(AEF) \Longrightarrow \odot(AEF)$ is orthogonal to the circle (M) with diameter $\overline{BC}$. Now all circles through a fixed point A and orthogonal to a fixed circle (M) forms a pencil, the second common point of this pencil is the intersection of AM with the polar of A WRT (M), which in fact is the projection of the orthocenter of $\triangle ABC$ on the median AM.

vslmat

#3 Jul 31, 2013, 7:14 pm

We know that if BF cuts CE at P then P lies on the circle (AEF).
Let H be the orthocenter of ΔABC with altitudes AI and BK, then B, P, H, C are concyclic. Easy to see that the reflection of A over BC lies on this circle (BHC).
Notice that both $BDPE$ and $DCFP$ are cyclic. Let (AEF) cuts (BHC) at another point M and the line through A parallel to BC cuts (AEF) at G. As $\angle GPF = 180^\circ - \angle GAF = \angle BAC = \angle BED = \angle BPD$, G, P, D are collinear.
Let GD, AM cut (BHC) at N, Q, resp., then $QN \parallel GA \parallel BC$. As $\angle BCN = \angle BPD = \angle ACB$, N is the reflection of A over BC and N lies on the altitude AI.
Let AQ meets BC at S; I is the midpoint of AN, S is the midpoint of AQ, buts as $BQ = CN = AC$, S is the midpoint of BC. As M also lies on (BHC), M is a fixed point on the A-median of ΔABC.
(Note: M also lies on the circle (AHK)).

Attachments:



thecmd999

#4 Apr 24, 2014, 3:27 am

Solution

leader

#5 Apr 24, 2014, 5:14 am

Here is a solution that uses 0 creativity xD $b = AC, c = AB, a = BC$

Let K be on AC such that $\angle KBC = \angle BAC$ and let X be on circle ABK on arc BK without A such that $\frac{BX}{XK} = \frac{b}{c}$.

Now by power of point $c \cdot BE = BD \cdot a, CK \cdot b = a^2, CF \cdot b = CD \cdot a$ now $BE = \frac{BD \cdot a}{c}$ and $KF = CK - CF = \frac{BD \cdot a}{b}$ so we get $\frac{BE}{KF} = \frac{b}{c} = \frac{BX}{XK}$ now along with $\angle XBE = \angle XKF$ we have that $BXE \sim XKE$ yeilding $\angle BEX = \angle XFA$ means $AEXF$ is cyclic.

The only thing you need to do is notice K when $D = B$ and just check that $\frac{BE}{KF}$ is fixed.

leminscate

#6 Jun 23, 2014, 9:30 am

Barycentrics WRT $\triangle ABC$ gives a 5 minute solution. The equation of (AEF) is $-a^2yz - b^2zx - c^2xy + (x+y+z)(ux+vy+wz) = 0$ where u, v, w are the powers of A, B, C WRT (AEF), respectively. So $u = 0, v = aBD, w = aCD$, by Power of a Point and the cyclic quadrilaterals $AEDB, AFDC$. Now let $y = z = 1$ in the equation, which gives $-a^2 - b^2x - c^2x + (x+2)a^2 = 0$ since $a(BD) + a(CD) = a^2$. We get a fixed value for x so we're done.

Wolstenholme

#7 Jul 28, 2014, 10:41 pm

I will also provide a barycentric coordinate solution.

Let $A = (1,0,0), B = (0,1,0), C = (0,0,1)$ and let $a = BC, b = CA, c = AB$. Let $D = (0, r, 1-r)$ for some real r.

We find that the equation of the circumcircle of $\triangle ABD$ is given by: $a^2yz + b^2xz + c^2xy = (x+y+z)a^2rz$ and it is easy to see that this intersects line AC at the point $E = (a^2r : 0 : b^2 - a^2r)$. Similarly we find that $F = (a^2(1-r) : c^2 - a^2(1-r) : 0)$.

Now it is easy to find that the circumcircle of $\triangle AEF$ has equation $a^2yz + b^2xz + c^2xy = (x+y+z)(a^2(1-r)y + a^2rz)$. It is easy to compute that regardless of the value of r, the point $(a^2 : b^2 + c^2 - a^2 : b^2 + c^2 - a^2)$ lies on this circumcircle, and moreover lies on the A-median of $\triangle ABC$ so we are done.

The motivation for using barycentric coordinates is that despite the large number of circles in this diagram, all are defined by vertices or points on sides of $\triangle ABC$ and so their equations are easy to deal with. Moreover, the condition that our fixed point lies on the relevant median implies that its barycentric coordinates will be nice.

XmL

#8 Jul 29, 2014, 12:21 am

Let $DE, DF \cap (AEF) = E', F'$ again, since $\angle E'DC = \angle A = \angle EF'D$, therefore $EE' \parallel BC$ and $EE' \parallel FF' \parallel BC$ analogously. Let $BE' \cap (AEF) = G$, since $BE * AB = BG * BE' = BD * BC, G, E', C, D$ are concyclic $\Rightarrow \angle E'GC = \angle E'DC = \angle 180 - \angle F'GE' \Rightarrow G \in CF'$. Now let $AG \cap BC = M$, since $\angle GCM = \angle FF'C = \angle FAG, \angle GBC = \angle BAG$, therefore by similarity $CM^2 = GM * AM = BM^2 \Rightarrow$ M is the midpoint of BC. The length of GM dictates that G is a fixed point and we are done.

andria

#9 Apr 22, 2015, 10:59 pm

My solution: apply an inversion with center A with radius $\sqrt{AB \cdot AC}$ and reflection WRT bisector of $\angle BAC$(assume that X' is inverse of X) then $B \longleftrightarrow C$ under the inversion and D' lies on the arc BC and (C, D', E') , (B, D', F') are collinear. let $AD' \cap BC = S$ and the tangents from B, C to $\odot(\triangle ABC)$ meet at R because $E'F'$ is a polar of S ;$E'F'$ passes throw point R which is fixed. So $\odot(\triangle AEF)$ passes throw the inverse of R which lies on the median M_a DONE

Dukejukem

#10 May 16, 2015, 5:10 am

Let $\mathcal{I} : X \mapsto X'$ be the composition of an inversion with center A and radius $\sqrt{bc}$, combined with a reflection in the A-angle bisector. It is easy to see that $B' \equiv C$ and $C' \equiv B.$ Hence, $\mathcal{I}$ takes line BC to $\omega \equiv \odot(ABC).$ It follows that $D' \in \omega$, and from basic inversive properties (circles through the center of inversion are sent to lines not passing through the center of inversion), we find that $E' \equiv A'B' \cap C'D'$ and $F' \equiv A'C' \cap B'D'.$

Now, note that the image of $\odot(AEF)$ is the line $E'F'$, and the image of the A-median is the A-symmedian. Therefore, to prove that $\odot(AEF)$ passes through a fixed point on the A-median, it suffices to prove that $E'F'$ passes through a fixed point on the A-symmedian. To see this, let the tangents to ω at X (note that X is a fixed point, regardless of where D' lies on ω). We will prove that X is the desired point that we seek. First, recall that X lies on the A-symmedian (Lemma 1), as needed. Second, by applying Pascal's Theorem to cyclic "hexagon" $AC'C'D'B'B'$, it follows that F', X, E' are collinear, i.e. $E'F'$ passes through $X.$ This completes the proof. $\square$

Quick Reply

High School Olympiads

Intersecting circles

Reply

eager

#1 May 15, 2015, 9:17 pm • 1

Let A be one of the intersection points of circles Ω_1 , Ω_2 with centers O_1,O_2.The line ℓ is tangent to Ω_1,Ω_2 at B,C respectively.Let O_3 be the circumcenter of $triangleABC$.Let D be a point such that A is the midpoint of O_3D.Let M be the midpoint of O_1O_2.Prove that $\angle O_1DM$=$\angle O_2DA$.

TelvCohl

#2 May 15, 2015, 10:11 pm

My solution :

Let $X = \Omega_1 \cap \Omega_2, T = AX \cap \odot(O_3)$ ($X \neq A$) .

Since $\angle XCB = \angle XAC = \angle TBC, \angle XBC = \angle XAB = \angle TCB$,
so $XB \parallel TC, XC \parallel TB \Longrightarrow XBTC$ is a parallelogram $\Longrightarrow AX$ pass through the midpoint V of BC ,
hence from $1 : 1 = BV : CV = [ABT] : [ACT] = AB \cdot BT : AC \cdot CT \Longrightarrow AB : AC = TC : TB$.

From $\triangle AO_1O_3 \sim \triangle ABC \Longrightarrow AO_1 : AD = AO_1 : AO_3 = AB : AC = TC : TB$,
so combine $\angle DAO_1 = 180^\circ - \angle O_1AO_3 = 180^\circ - \angle BAC = \angle BTC \Longrightarrow \triangle AO_1D \sim \triangle TCB$.
Similarly, we can prove $\triangle AO_2D \sim \triangle TBC \Longrightarrow \triangle AO_1D \sim \triangle ADO_2 \Longrightarrow DA$ is D-symmedian of $\triangle DO_1O_2$.
i.e. $\angle O_1DM = \angle O_2DA$

Q.E.D

Luis González

#3 May 15, 2015, 10:36 pm

Posted before at http://www.artofproblemsolving.com/community/c6h348427.

buratinogigle

#5 May 16, 2015, 1:34 pm

A little extension

Let ABC be a triangle and E lies on segment BC and F lie on BC such that B is between E,F and $\angle EAC=\angle FAB$. Let K,L,N be circumcenters of triangles AEC,AFB,AEF, resp. Then K,L,O,N lie on circle ω and let T be intersection of bisector $\angle NAO$ with minnor arc ON of ω and S is symmetric of T through A. Prove that ST is symmedian of triangle SKL.

TelvCohl

#6 May 16, 2015, 2:26 pm • 1

> buratinogigle wrote:
>
> A little extension
>
> Let ABC be a triangle and E lies on segment BC and F lie on BC such that B is between E,F and $\angle EAC=\angle FAB$. Let K,L,N be circumcenters of triangles AEC,AFB,AEF, resp. Then K,L,O,N lie on circle ω and let T be intersection of bisector $\angle NAO$ with minnor arc ON of ω and S is symmetric of T through A. Prove that ST is symmedian of triangle SKL.

My solution :

Let $X = KL \cap BC, Y = KL \cap AT$.

Since $\triangle LAN \sim \triangle BAE \sim \triangle OAK$,
so $\angle LAN = \angle OAK \Longrightarrow AT$ is the bisector of $\angle LAK$.

From $\angle FAB = \angle EAC \Longrightarrow BC$ pass through the exsimilicenter of $\odot(L) \sim \odot(K)$,
so X is the exsimilicenter of $\odot(L) \sim \odot(K) \Longrightarrow AX$ is the external bisector of $\angle LAK$,
hence we get AX is the perpendicular bisector of ST and $(X, Y; L, K) = -1 \Longrightarrow AY \equiv ST$ is the polar of X WRT ω ,
so XS and XT are the tangents of $\omega \Longrightarrow SLTK$ is a harmonic quadrilateral $\Longrightarrow ST$ is the S-symmedian of $\triangle SKL$.

Q.E.D

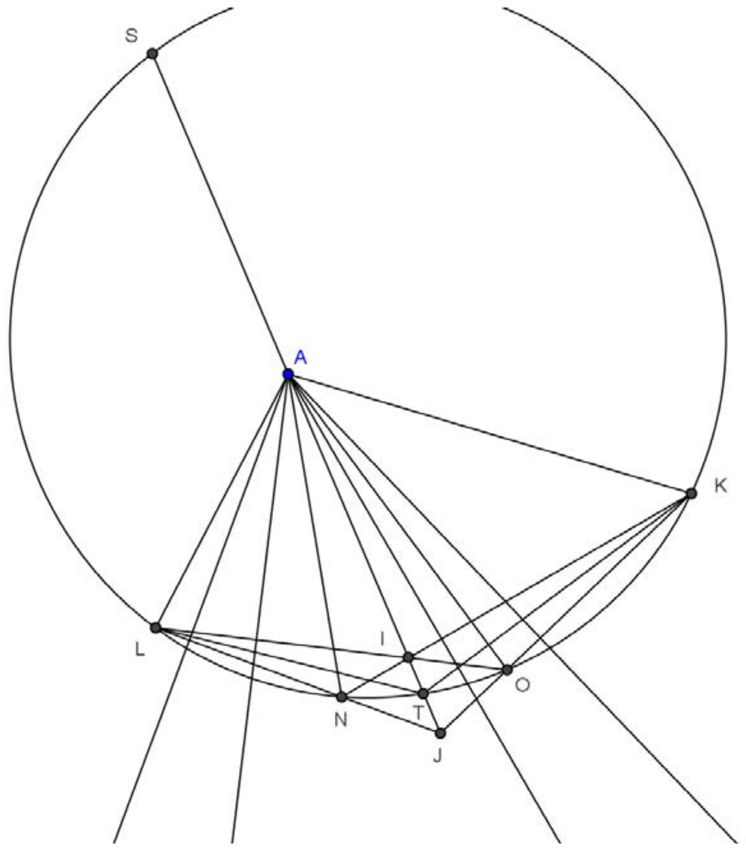
tranquanghuy7198

#7 May 16, 2015, 8:54 pm

My solution for the extension uses pure angle chasing.
Lemma.
Let D be an arbitrary point on the line BC of $\triangle ABC$. M, N are the circumcenters of $\triangle ADB, \triangle ADC$, then we have:
$\triangle AMN \sim \triangle ABC$ (prove by angle chasing)

Back to our main problem.
$OL \cap KN = I, OK \cap NL = J$
Apply the lemma to receive: $\triangle ALN \sim \triangle AOK (\sim \triangle ABE)$
$\Rightarrow \triangle ALO \sim \triangle ANK$
$\Rightarrow \angle ALO = \angle ANK$
$\Rightarrow A, L, N, I$ are concyclic
Analogously, we have:
A, K, O, I are concyclic, A, L, J, O are concyclic, A, K, J, N are concyclic
$\Rightarrow \angle LAI = 180 - \angle LNI = 180 - \angle KOI = \angle KAI \Rightarrow AI$ is the angle bisector of $\angle KAL$, and so is AJ (prove analogously)
$\Rightarrow \overline{A, I, T, J}$
Moreover: $\angle ATL = \angle AOL + \angle IAO - \angle OLT = \angle AKN + \angle IKO - \angle OKT = \angle AKT$
$\Rightarrow \triangle ATL \sim \triangle AKT$
$\Rightarrow \triangle ASL \sim \triangle AKS$ (because $AT = AS$)
$\Rightarrow SA$ is the symmedian of $\triangle SKL$. Q.E.D

Attachments:



Quick Reply

High School Olympiads

Problem

Reply

gold46

#1 May 8, 2010, 3:00 pm

Let w_1 and w_2 circles on a plane.Let A one of intersections of circles.l line tangents w_1, w_2 at B, C , respectively.
Let O_3 circumcentre of ABC. Let D on line 0_3A such that A is midpoint of O_3D.Let M be midpoint of O_1O_2.
Prove that $\angle O_1DM = \angle O_2DA$.

Luis González

#2 May 9, 2010, 12:01 am

Let $P \equiv BC \cap O_1O_2$ be the exsimilicenter of ω_1 and ω_1, which is also center of the positive inversion which swaps ω_1, ω_2. Since B, C are inverse points and A is double, it follows that $\odot(ABC)$ is double $\Longrightarrow PA \perp AD \Longrightarrow AD$ bisects $\angle O_1AO_2$, hence AD passes through their insimilicenter Q. Let N be the midpoint of BC. Then $AN \perp O_1O_2$ at R. Clearly A, N, O_3, P are concyclic. Therefore $\angle RAQ = \angle NPO_3 = \angle APQ$, this is $\angle APO_3 = \angle APD = \angle NPM \Longrightarrow$ M, P, D, O_3 are concyclic. Consequently, from the harmonic cross ratio (O_1, O_2, Q, P), we have

$QO_1 \cdot QO_2 = QM \cdot QP = QD \cdot QO_3 \Longrightarrow O_1, O_2, O_3, D$ are concyclic.

Then AQ is the polar of P with respect to $\odot(O_1O_2O_3) \Longrightarrow$ tangents of $\odot(O_1O_2O_3)$ at O_3, D pass through P, which implies that $O_1DO_2O_3$ is harmonic $\Longrightarrow DO_3 \equiv DA$ is the D-symmedian of $\triangle DO_1O_2 \Longrightarrow \angle O_1DM = \angle O_2DA$.

oneplusone

#4 May 10, 2010, 1:37 pm

My solution is about the same as Luis(at least the last part is)

$\angle AO_2C = 2\angle ACB = \angle AO_3B$. Similarly $\angle BO_1A = \angle AO_3C$ and thus the kites BO_1AO_3 and CO_3AO_2 are similar and so the triangles O_1AO_3 and O_3AO_2 are similar. Now let E be such that DO_1O_3E is a paralellogram. So A is the midpoint of O_1E and $\triangle O_1O_3E$ is similar to $\triangle O_3O_2D$. So now we get $\angle DO_2O_3 + \angle DO_1O_3 = \angle O_1O_3E + \angle DO_1O_3 = 180$ as DO_1O_3E is a parallelogram. Thus $DO_1O_3O_2$ is cyclic and furthermore, $\dfrac{DO_2}{O_3O_2} = \dfrac{EO_3}{O_1O_3} = \dfrac{DO_1}{O_3O_1}$. Thus $DO_1O_3O_2$ is harmonic and the result follows by properties of harmonic cyclic quads.

This post has been edited 1 time. Last edited by Luis González, May 15, 2015, 10:31 pm
Reason: Fixing typo

Quick Reply

High School Olympiads

locus of

Reply

yabi
#1 May 15, 2015, 3:35 am

We have a curve y=x^2
We draw two tangent lines to this curve so that these two tangents are perpendicular.
Lets name the intersection point of these two lines with C.
What is the locus of point C

Luis González
#2 May 15, 2015, 4:04 am

This is well-known. The locus of the points that see a parabola $\mathcal{P}$ at a right angle is its directrix. Moreover if X,Y are points where these tangents touch $\mathcal{P},$ then it focus F is the projection of C on XY (however this holds for any focal chord of a conic).

By reflective property, the reflections U and V of its focus F across the tangents CX,CY lie on its directrix. But C becomes then circumcenter of the right triangle $\triangle FUV$ at $F \Longrightarrow C \in UV.$

Quick Reply

High School Olympiads

Two circles and tangents

Reply

gobathegreat

#1 May 15, 2015, 2:14 am

Let k_1 be a circle witch center O and diameter AB. Let C be a point on k_1 such that $90 < \angle AOC < 180$. Let K be point on OC such that C lies between K and O. Furthermore, let k_2 be a circle with center K passing through C. If E is second intersection between KB and k_1, and S and T two points on k_2 such that ES and ET are tangents to k_2, prove that lines AC, EK and ST are concurrent

This post has been edited 1 time. Last edited by gobathegreat, May 15, 2015, 2:15 am

Luis González

#2 May 15, 2015, 3:28 am

If AC cuts EK at F, we have $\angle CEK = \angle CAB = \angle ACO \equiv \angle FCK \Longrightarrow KC$ is tangent to $\odot(CEF) \Longrightarrow KC^2 = KE \cdot KF$. Thus if $\{U, V\} \equiv k_2 \cap KE$, then $KU^2 = KV^2 = KE \cdot KF \Longrightarrow (U, V, E, F) = -1 \Longrightarrow F$ lies on the polar ST of E WRT $k_2 \Longrightarrow F \equiv AC \cap EK \cap ST$.

Quick Reply

High School Olympiads

Tangent circles

Reply

tranquanghuy7198

#1 May 14, 2015, 3:57 pm

Let $ABCD$ be the isoceles trapezium ($AB \parallel CD$) inscribed in the circle ω. $J \in \omega$ such that the tangent to ω at J is perpendicular to AB, CD. $M \in JC$ such that $AM \perp AC$. $N \in JB$ such that $DN \perp DB$. Prove that $\odot(JMN)$ is tangent to ω

THVSH

#2 May 14, 2015, 4:38 pm

My solution:

Let $AM \cap DN = E$. BI is diameter of ω. O, O_1 are the center of $\omega, \odot(JMN)$, respectively.
Then A, G, D, E are concyclic and E, D, N, I are collinear.
We have $\angle AGD = 2.\angle ABG = \angle AOD \implies A, G, O, D, E$ are concyclic.
$\implies OE \parallel AB \parallel CD$ ($\perp OG$)
Since the tangent of J is perpendicular to $AB, CD \implies OJ \parallel AB \parallel CD \implies J \in OE$
From Menelaus's theorem, we get $\frac{NJ}{NB} = \frac{EJ}{EO}.\frac{IO}{IB} = \frac{EJ}{2EO}$. Similarly, $\frac{MJ}{MC} = \frac{EJ}{2EO}$
$\implies \frac{NJ}{NB} = \frac{MJ}{MC} \implies MN \parallel BC \implies \triangle JBC \sim \triangle JNM$
$\implies O, J, O_1$ are collinear. It means $\odot JMN$ is tangent to ω. Q.E.D

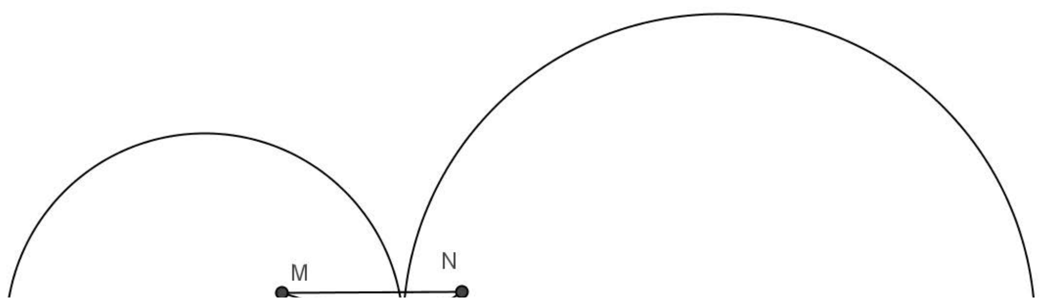
tranquanghuy7198

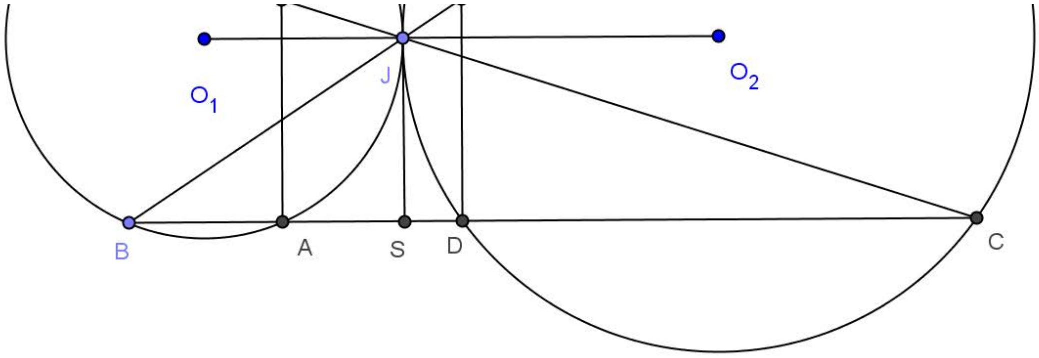
#3 May 14, 2015, 10:36 pm

Thank you, THVSH, for your perfect solution. And this is mine:
Consider the inversion centered at J, we have a new problem: Given 2 circles (O_1) and (O_2) externally tangent to each other at J. An arbitrary line which is parallel to O_1O_2 intersects 2 circles at A, B, C, D. $M \in JC$ such that $AM \perp AC$. $N \in JB$ such that $DN \perp DB$. Prove that $MN \parallel ABCD$.

Proof.
Construct $S \in ABCD$ such that $JS \perp ABCD$
$\Rightarrow S$ has the same power WRT (O_1) and (O_2)
$\Rightarrow SA.SB = SC.SD$
$\Rightarrow \frac{SA}{SC} = \frac{SD}{SB}$
$\Rightarrow \frac{JM}{JC} = \frac{JN}{JB}$
$\Rightarrow MN \parallel ABCD$.

Attachments:





buratinogigle

#4 May 15, 2015, 12:35 am

Nice problem and solution, here is an easy extension we can see

Let $ABCD$ be cyclic quadrilateral with circumcircle (O). AB cuts CD at S. P is a point on (O) such that P lies on OS. Let Q be on CP such that $AQ \perp AC$ and R be on BP such that $DR \perp DB$. Prove that (PQR) is tangent to (O).

Solution by inversion is available.

Luis González

#5 May 15, 2015, 2:38 am • 1

> buratinogigle wrote:
> Let $ABCD$ be cyclic quadrilateral with circumcircle (O). AB cuts CD at S. P is a point on (O) such that P lies on OS. Let Q be on CP such that $AQ \perp AC$ and R be on BP such that $DR \perp DB$. Prove that (PQR) is tangent to (O).

Let X, Y be the projections of P on AC, BD. According to nice lemma on cyclic quadrilateral, we have $\frac{CX}{XA} = \frac{BY}{YD}$ (valid for any P on OS). Hence $\frac{BP}{PR} = \frac{BY}{YD} = \frac{CX}{XA} = \frac{CP}{PQ} \Longrightarrow RQ \parallel BC \Longrightarrow \odot(PQR)$ is tangent to $\odot(PBC) \equiv (O)$.

THVSH

#6 May 15, 2015, 8:43 pm • 1

> buratinogigle wrote:
> Let $ABCD$ be cyclic quadrilateral with circumcircle (O). AB cuts CD at S. P is a point on (O) such that P lies on OS. Let Q be on CP such that $AQ \perp AC$ and R be on BP such that $DR \perp DB$. Prove that (PQR) is tangent to (O).

My solution:
Let $AQ \cap DR = T$, then ET is the diameter of $\odot(EAD)$
Let $AC \cap BD = E$; $AD \cap BC = F$. $\odot(EAD) \cap \odot(EBC) = G$
EG, AD, BC are radical axis of $\{\odot(EAD), \odot(EBC)\}; \{\odot(EAD), \odot(O)\}; \{\odot(O), \odot(EBC)\}$, so they are concurrent.
From Brokart's theorem, we get $EG \equiv EF \perp OS$

Let I, J be the center of $\odot(EAD), \odot(EBC)$. Then $OIEJ$ is a parallelogram, so IJ passes through the midpoint of OE. (1)
On the other hand, $IJ \parallel OS (\perp EG)$. (2)
From $(1), (2), \Longrightarrow$ the symmetric of E wrt I lies on OS. In the other word, $T \in OS$

Construct the diameter BB' of (O), then T, D, R, B' are collinear.
From Menelaus's theorem, $\frac{RP}{RB} = \frac{TP}{TO}.\frac{B'O}{B'B} = \frac{TP}{2TO}$. Similarly, $\frac{QP}{QC} = \frac{TP}{2TO}$
$\Longrightarrow \frac{RP}{RB} = \frac{QP}{QC} \Longrightarrow RQ \parallel BC \Longrightarrow (PQR)$ is tangent to (O). Q.E.D

$\implies \frac{}{RB} = \frac{}{QC} \implies RQ \parallel BC \implies (PQR)$ is tangent to (O). Q.E.D

This post has been edited 1 time. Last edited by THVSH, May 15, 2015, 8:49 pm

Quick Reply

High School Olympiads

Points collinear

Reply

Source: Maybe own - easy problem

drmzjoseph

#1 May 14, 2015, 1:21 pm • 1

Given a triangle ABC with $AB = BC$, let P and Q be two points on AC, such that $BQ = QP$, let M and N be the midpoints of AP and PC respectively. R and S lies on BM and BN respectively, such that $AR = RB$ and $BS = SC$. Prove that Q, R and S are collinear.

Luis González

#2 May 14, 2015, 11:50 pm • 2

Invert the figure WRT the circle $\odot(B, BA)$. P, M, N go to points P', M', N' on $(O) \equiv \odot(ABC)$ and BM', BN' become B-symmedians of $\triangle BAP', \triangle BCP' \Longrightarrow$ tangents of (O) at A, P' meet at $X \in BM'$ and the tangents of (O) at C, P' meet at $Y \in BN'$. $\odot(Q, QB)$ goes to $P'O$ and $\odot(R, RB), \odot(S, SB)$ go to the perpendiculars τ_A, τ_C from A, C to BX, BY.

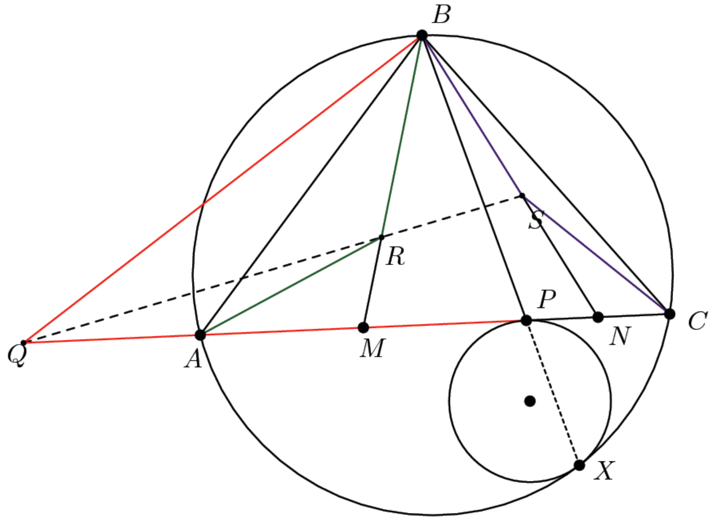
Perpendiculars from B, X, Y to CA, AP', CP' concur at $O \Longrightarrow \triangle BXY$ and $\triangle P'CA$ are orthologic $\Longrightarrow$ perpendiculars OP', τ_A, τ_C from P', A, C to XY, BX, BY concur. Consequently, their inverses $\odot(Q, QB), \odot(R, RB)$ and $\odot(S, SB)$ are coaxal $\Longrightarrow$ their centers Q, R, S are collinear.

drmzjoseph

#3 May 15, 2015, 10:25 am • 1

My solution:

$X \equiv \odot(ABC) \cap BP, B \neq X$, Is well-known that exist a circle (ω) tangent to AC at P and $\odot(ABC)$ at X, such that B, P, X are collinear. $AB^2 = XB.PB \Rightarrow B$ belongs to radical axis between A (radius 0) and ω.
Then MB is the radical axis between A and $\omega \Rightarrow \mathcal{P}(R) = AR^2 = BR^2$ where $\mathcal{P}(X)$ is the power of P with respect to ω
So R belongs to radical axis between B (radius 0) and ω, analogously S belongs, too. Is trivial that Q belongs to radical axis, too. Then $\overline{QRS}$ is the radical axis between B and ω



High School Olympiads

Problem about locus

Reply

tranquanghuy7198
#1 May 14, 2015, 3:59 pm

Given the angle $\angle xOy$ and the circle ω touches the rays Ox, Oy. A, B vary on the rays Ox, Oy, resp. such that AB always touches ω. M, N are 2 fixed points on the plane. $AM \cap BN = S$.
a) Find the locus of S when A, B vary.
b) Find the necessary and enough condition of M, N such that the locus above is a straight line.

Luis González
#2 May 14, 2015, 10:49 pm

When the tangent AB of $(K) \equiv \omega$ varies, then $A \mapsto B$ is a proyectivity between Ox, Oy, due to $\angle(KB, KA) = 90^\circ - \frac{1}{2}\angle(Ox, Oy) = \text{const}$. Thus the pencils MA and NB are proyective $\Longrightarrow S \equiv AM \cap BN$ moves on a conic $\mathcal{C}$ passing through M, N.

S describes a line $\Longleftrightarrow$ pencils MA and NB are perspective $\Longleftrightarrow MA \equiv NB$ for $S \equiv M \Longleftrightarrow A \in MN \Longleftrightarrow MN$ is tangent of ω.

Quick Reply

High School Olympiads

A beautiful theorem

Reply

Source: vankhea

vankhea

#1 May 14, 2015, 11:24 am

Let O be point inside of triangle ΔABC. The rays AO, BO, CO cuts BC, CA, AB at D, E, F respectively.
Let (l) be a line cuts AO, BO, CO at X, Y, Z respectively.
Prove that:

$$\frac{AO}{OX}\cdot\frac{XD}{DA}+\frac{BO}{OY}\cdot\frac{YE}{EB}+\frac{CO}{OZ}\cdot\frac{ZF}{FC}=2$$

This post has been edited 1 time. Last edited by vankhea, May 20, 2015, 9:23 am

Luis González

#2 May 14, 2015, 11:35 am

Considering a homology sending (l) to the line at infinity (of course cross ratios are invariant), the problem reduces to prove $\frac{AO}{AD}+\frac{BO}{BE}+\frac{CO}{CF}=2$ or equivalently $\frac{OD}{AD}+\frac{OE}{BE}+\frac{OF}{CF}=1$, which is the well-known Gergonne-Euler theorem.

Quick Reply

High School Olympiads

Trilinear pole, Polar of Incenter, Centroid, Symmedian point

Reply

Source: Own

TelvCohl

#1 May 14, 2015, 1:37 am • 3

Let I, G, K be the Incenter, Centroid, Symmedian point of $\triangle ABC$, respectively .
Let τ be the polar of I WRT $\odot(ABC)$ and T be the trilinear pole of τ WRT $\triangle ABC$.

Prove that T, G, K are collinear

Luis González

#2 May 14, 2015, 6:39 am • 4

Very nice problem!, but I found a really hardcore solution. Fortunately this is yielding some other interesting properties in this configuration.

(O) is the circumcircle of $\triangle ABC$ and J is its Euler's reflection point (Feuerbach point of its tangential triangle $\triangle XYZ$). Let $\mathcal{F}$ be the Feuerbach hyperbola of $\triangle XYZ,$ passing through $O,$ the Gergonne point of $\triangle XYZ$ and the incenter I of its cevian triangle $\triangle ABC$ (see Poncelet points posts #6~#8). Moreover, $\mathcal{F}$ is tangent to the Euler line e of $\triangle ABC$ at O (fixed point of the isogonal conjugation WRT $\triangle XYZ$)

The dual of $\mathcal{F}$ WRT (O) is a conic $\mathcal{P}$ tangent to BC, CA, AB and the line at infinity as $O \in \mathcal{F} \Longrightarrow \mathcal{P}$ is a parabola whose axis is perpendicular to the tangent e of $\mathcal{F}$ at $O \Longrightarrow$ its focus is then the isogonal conjugate J of $\perp e$ WRT $\triangle ABC,$ i.e. $\mathcal{P}$ is the Kiepert parabola of $\triangle ABC \Longrightarrow$ polar τ of I WRT (O) touches $\mathcal{P}.$

It's known that the tripoles of all the tangents to a fixed inconic of $\triangle ABC$ fall on a fixed line. When this inconic is the Kiepert parabola $\mathcal{P},$ the fixed line passes through G (tripole of the line at infinity) and the tripole K of the Lemoine axis $\ell,$ since ℓ is tangent to $\mathcal{P},$ i.e. J is Miquel point of $\{BC, CA, AB, \ell\}$ (for a proof see A nice problem with Feuerbach point). Therefore the tripole T of τ is on $GK.$

buratinogigle

#3 May 14, 2015, 8:16 am • 2

Nice problem, I have seen the generalization

Let ABC be a triangle and incircle (I) touches BC, CA, AB at D, E, F. P is a point on OI line of ABC and Q is isogonal conjugate of P wrt ABC. ℓ is polar of Q wrt (I) and R is tripole of ℓ wrt DEF. Prove that R lies on line connecting Lemoine point and centroid of triangle DEF.

Luis González

#4 May 14, 2015, 8:54 am • 2

> *buratinogigle wrote:*
> Let ABC be a triangle and incircle (I) touches BC, CA, AB at D, E, F. P is a point on OI line of ABC and Q is isogonal conjugate of P wrt ABC. ℓ is polar of Q wrt (I) and R is tripole of ℓ wrt DEF. Prove that R lies on line connecting Lemoine point and centroid of triangle DEF.

My previous solution still works for this generalization. Isogonal conjugate Q of P WRT $\triangle ABC$ is on its Feuerbach hyperbola; dual of the Kiepert parabola $\mathcal{P}$ of $\triangle DEF$ WRT $(I) \Longrightarrow$ polar ℓ or Q WRT (I) touches $\mathcal{P} \Longrightarrow$ its tripole WRT $\triangle DEF$ is on the line joining the centroid and symmedian point of $\triangle DEF.$

buratinogigle

#5 May 14, 2015, 9:05 am

Thank you dear Luis, I see that, when P lies on a circumconic of ABC then pole wrt DEF of polar of P wrt (O) lies on a fixed line, too !

Quick Reply

High School Olympiads

Poncelet points

Source: posted by Darij long time ago; no proofs whatsoever

pohoatza
#1 Jul 3, 2011, 9:58 pm • 1

Let ABC be a triangle and let P be a point in its plane. Let $A_1B_1C_1$ be its cevian triangle and $A_2B_2C_2$ be its pedal triangle (both with respect to ABC, of course). Prove that the circumcircles of triangles $A_1B_1C_1$ and $A_2B_2C_2$ meet at a point on the nine-point circle of ABC.

A synthetic proof would be great. Any ideas?

Mohammadi
#2 Jun 23, 2012, 3:56 pm

Can anybody solve this?

r1234
#3 Jun 23, 2012, 9:29 pm

see here (by RSM),"The cevian circle problem".

MBGO
#4 Jul 5, 2012, 3:45 pm • 1

dear all

this three circles with 14 other circles meet at a one point. Noticed by "Chris Van Tienhoven".

jayme
#5 Jul 7, 2012, 7:03 pm • 1

Dear Mathlinkers,
I come back with this nice problem.

1. By considering the quadrilateral ABPC, the nine-point circle of ABC and the P-pédal circle (A2B2C2) intersect in two points, one being the Euler-Poncelet point of ABPC.
See for example
http://perso.orange.fr/jl.ayme vol. 8 Le point d'Euler-Poncelet p. 23-27.

2. The circle (A2B2C2) passes also through this last point ; this is the first Brianchon-Poncelet circle
http://perso.orange.fr/jl.ayme vol. 8 Le point d'Euler-Poncelet p. 27-28.
I notice in my text that I was unable to find a synthetic proof (this research is open). The proof of Brianchon ans Poncelet appears at the end of the text p. 21 and uses a rectangular hyperbol and polar

2. At p. 80 you will discover a lot of circle going through this point following an idea of Quang Tuan Bui on Hyacinthos site.

Now, it will be interesting to find a synthetic proof without conic for the point 2.

Sincerely
Jean-Louis

Luis González

#6 Jul 15, 2013, 11:42 pm • 1

Pedal circle $\odot(A_2B_2C_2)$ of P and 9-point circle (N) meet at the poncelet point P_0 of A,B,C,P (see intersection of Simson lines /K.K 6.4 /2). This is the center of the equilateral hyperbola $\mathcal{H}$ through $A,B,C,P,$ so it remains to show that $\odot(A_1B_1C_1)$ goes through the center of $\mathcal{H}.$

Let J be the incenter of $\triangle A_1B_1C_1$ and X,Y,Z its excenters referent to $A_1,B_1,C_1,$ respectively. Consider $\triangle XYZ$ just as the anticevian triangle of J WRT $\triangle A_1B_1C_1.$ Consider the homology that transforms the quadrangle $ABCP$ into a square. A_1 and C_1 go to infinity in perpendicular directions and the quadrilateral $XYZJ$ becomes a rectangle whose sides are parallel to the sides of the square $ABCP.$ The image of B_1 is then the common center of $ABCP$ and $XYZJ.$ Now, by symmetry the conic passing through A,B,C,P and one of the vertices of the rectangle $XYZJ,$ also passes through its remaining three vertices. Thereby, in the original figure, A,B,C,P,J,X,Y,Z are on a same conic. Since J is orthocenter of $\triangle XYZ,$ then this conic is the rectangular hyperbola $\mathcal{H},$ whose center is then on the 9-point circle $\odot(A_1B_1C_1)$ of $\triangle XYZ.$

TelvCohl

#7 Oct 29, 2014, 8:14 am

My solution **:**

We only have to prove the Poncelet point of $ABCP$ lie on $\odot(A_1B_1C_1)$.

Let I be the incenter of $\triangle A_1B_1C_1$ and I_a,I_b,I_c be the excenters of $\triangle A_1B_1C_1$.
Let $X \equiv B_1C_1 \cap A_1I_a, Y \equiv AB \cap A_1I_a, Y' \equiv CP \cap A_1I_a, Z \equiv AC \cap A_1I_a, Z' \equiv BP \cap A_1I_a$.

Consider the involution on A_1I_a defined by the pencil of conic passing through $\{A,B,C,P\}$

From $(Y,Y';A_1,X)=(Z,Z';A_1,X)=-1 \Longrightarrow A_1,X$ are the fixed points of this involution ,
so from $(I,I_a;A_1,X)=-1 \Longrightarrow I_a$ lie on the conic Ω passing through $\{A,B,C,P,I\}$

Similarly, we can prove I_b and I_c lie on Ω ,
so A,B,C,P,I,I_a,I_b,I_c lie on a conic Ω .

Since I is the orthocenter of $\triangle I_aI_bI_c$,
so Ω is a rectangular hyperbola .
Since $\odot(A_1B_1C_1)$ is the nine point circle of $\triangle I_aI_bI_c$,
so $\odot(A_1B_1C_1)$ pass through the center of Ω which is the Poncelet point of $ABCP$.

Q.E.D

Remark:

(1) From this proof we get the interesting property **:**

Let $\triangle DEF$ be the cevain triangle of P .
Let I be the incenter of $\triangle DEF$ and O be the circumcenter of $\triangle ABC$.
Let P' be the isogonal conjugate of P with respect to $\triangle ABC$.

Then the isogonal conjugate of I with respect to $\triangle ABC$ lie on OP' .

(2) For another proof you can see here (post #7 **(1)**)

This post has been edited 2 times. Last edited by TelvCohl, Sep 2, 2015, 10:42 pm

IDMasterz

#8 Oct 29, 2014, 8:45 pm

> *TelvCohl wrote:*
> Consider the involution on A_1I_a defined by the pencil of conic passing through $\{A,B,C,P\}$

This is exactly my idea, except I think I have done it a bit differently:

Consider fixed points $ABCP$. Let ℓ be a fixed line through A_1 and D be a variable point on ℓ. As D moves, the pencil of conics through $ABCPD$ intersect ℓ again at D' such that $\{D,D'\}$ are pairs of involutions. When $D=A_1$, then the conic degenerates into two lines and hence A_1 is a double point. Also, notice that if $\ell \cap B_1C_1 = E$ we have:

$$E(A_1,P,B,C)=P(A_1,E,B,C)=A(P,E,B,C)=A(E,P,B,C)$$

$\implies EA_1 \equiv \ell$ is tangent to the conic when $D=E$, hence E is also a double point. Hence, $(A_1,E,D,D')=-1$ so we conclude for a point D on a conic through $ABCP$, the anticevian triangle of D wrt $A_1B_1C_1$ lies on the conic.

Consider the conic through $ABCPI$ where I is the incentre of $A_1B_1C_1$. Then $I, I_A, I_B, I_C, A, B, C, P$ lie a conic, but since I is the orthocentre of $I_AI_BI_C$, it is an equilateral hyperbola and hence passes through H.

Quick Reply

High School Olympiads

A nice problem with Feuerbach point

Reply

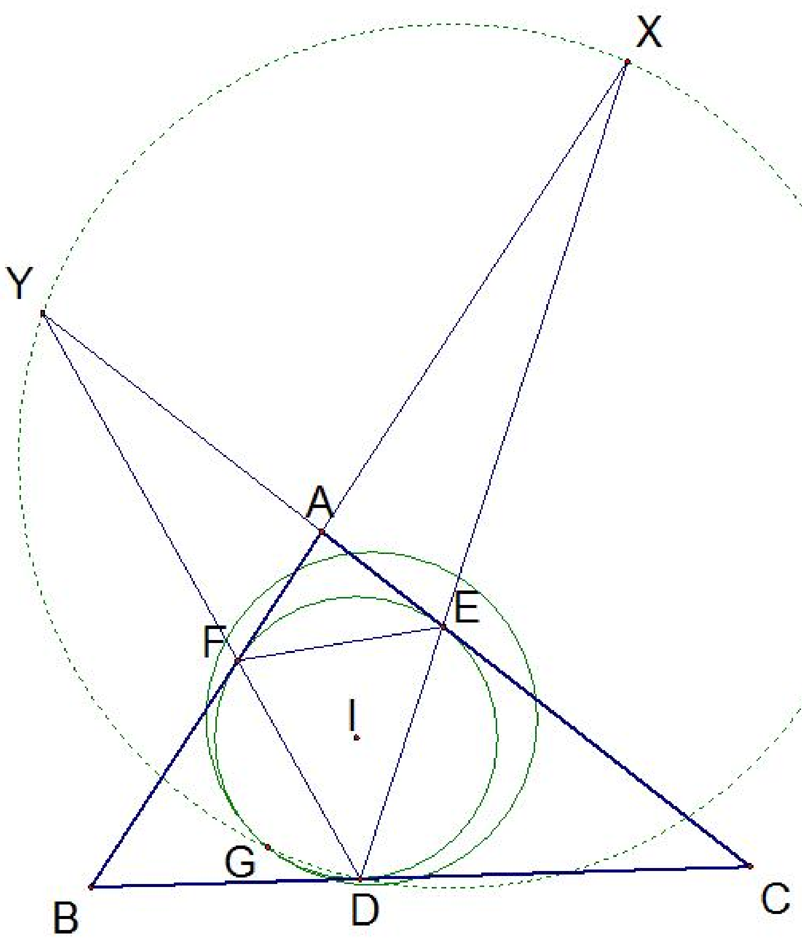
Isogonics

#1 Jan 24, 2013, 5:26 pm

Let incircle of $\triangle ABC$ meets with BC, CA, AB at D, E, F, respectively.
Let DE and AB meets at X and DF and CA meets at Y.
Let G the Feuerbach point of $\triangle ABC$.

Prove that G, D, X, Y are concyclic.

Attachments:



jayme

#2 Jan 25, 2013, 6:37 pm

Dear Mathlinkers,
you can see

http://www.artofproblemsolving.com/Forum/viewtopic.php?search_id=64018334&t=158254

Sincerely
Jean-Louis

Luis González

#3 Jan 26, 2013, 3:27 am

Denote by (I) the incircle of $\triangle ABC.$ D' is the antipode of D WRT (I) and FD', ED' cut DE, DF at $P, Q,$ respectively. D' becomes then orthocenter of $\triangle DPQ \Longrightarrow (I)$ is orthogonal to the circle with diameter $\overline{PQ} \Longrightarrow A$ is midpoint of $\overline{PQ}.$ Midpoint U of $\overline{IA}$ is then 9-point center of $\triangle DPQ \Longrightarrow UD'$ is Euler line of $\triangle DPQ.$

From the topic Intersect on circle (see the proposition in the post #4), we deduce that UD' cuts (I) again at the Feuerbach point G of $\triangle ABC.$ Then using the result of the problem Concyclic Quadrilateral for $\triangle DPQ,$ we get that G, D, X, Y are concyclic.

Quick Reply

High School Olympiads

U lies on (Oa) and AU passes through the Gergonne point

Reply

THVSH
#1 May 12, 2015, 11:07 pm • 1

Let ABC be a triangle with circumcircle (O). (O_a) is the $A-$mixtilinear incircle of $\triangle ABC$.
$(O_a)\cap(O)=D; (O_a)\cap CA=E; (O_a)\cap AB=F$.
$DB\cap(O_a)=\{D,G\}; DC\cap(O_a)=\{D,H\}.GE\cap BC=M; HF\cap BC=N.FM\cap EN=U$.
Prove that $U\in(O_a)$ and AU passes through the Gergonne point of $\triangle ABC$.

This post has been edited 1 time. Last edited by THVSH, May 12, 2015, 11:10 pm

Luis González
#2 May 13, 2015, 2:06 am • 1

Incircle (I) touches BC at A' and Y is the midpoint of the arc AC of (O). It's well known that I is the midpoint of EF.

Since D is the exsimilicenter of $(O)\sim(O_a)$, it follows that D,E,Y are collinear and $GE \parallel BY$, i.e. $EM\parallel IB$ and similarly $FN\parallel IC$. Thus $\angle EMC=\angle IBC=\angle EIC\Longrightarrow CEIM$ is cyclic $\Longrightarrow IE=IM\Longrightarrow\angle EMF=90^\circ$ and similarly $\angle ENF=90^\circ$. Thus if $J\equiv EM\cap FN$, we get $\angle EUF=\angle NJM=\angle CIB=\angle AEF$ (mod 180º) $\Longrightarrow U\in(O_a)$.

Since MN is antiparallel to EF WRT UE,UF, then $UO_a\perp MN\Longrightarrow O_aU$ and IA' are parallel radii of (O_a) and $(I)\Longrightarrow UA'$ passes through their exsimilicenter A, in other words, AU is the A-cevian of the Gergonne point of $\triangle ABC$.

tranquanghuy7198
#3 May 13, 2015, 4:54 pm

Luis González wrote:
Since MN is antiparallel to EF WRT UE,UF, then $UO_a\perp MN\Longrightarrow O_aU$ and IA' are parallel radii of (O_a) and $(I)\Longrightarrow UA'$ passes through their exsimilicenter A, in other words, AU is the A-cevian of the Gergonne point of $\triangle ABC$.

That is perfect solution. We can say in other word:
MN is antiparallel to EF WRT UE,UF. Moreover, the tangents of (O_a) at E,F intersect at A, so UA is the symmedian of $\triangle UEF$. Because of all that, UA passes through the midpoint of MN, which is A'.

THVSH
#4 May 13, 2015, 5:29 pm • 1

Thank you for your interest
This is my solution:

Since the problem in All-Russian Olympiad 2006 finals, problem 9.4, we get $BM=BF$ and $CN=CE$.
Then $\angle UMN=\angle FMB=90^\circ-\frac{1}{2}.\angle ABC$. Similarly, $\angle UNM=90^\circ-\frac{1}{2}.\angle ACB$
$\Longrightarrow\angle EUF=180^\circ-\angle UMN-\angle UNM=90^\circ-\frac{1}{2}.\angle BAC=\frac{1}{2}.\angle BDC=\angle EDF$
$\Longrightarrow U\in(O_a)$.

Let the incircle (I) intersect BC,CA,AB at X,Y,Z, respectively.
We have $YZ\parallel EF(\perp AI); XZ\parallel UF(\perp BI); XY\parallel UE(\perp CI)\Longrightarrow UX,EY,FZ$ are concurrent.
It means $A\in UX\Longrightarrow A,U,X$ are collinear. In the other word, AU passes through the Gergonne point of $\triangle ABC$. Q.E.D

buratinogigle
#5 May 13, 2015, 5:34 pm • 1

#5 May 13, 2015, 5:54 pm

Nice dear THVSH and Luis, here is an easily seen extension

Let $ABCD$ be quadrialateral inscribed circle (O). AC cuts BD at S. (K) is a circle touches SC, SD at M, N and touches (O) internally at P. PC, PD cut (K) again at E, F. EN, FM cut CD at G, H. Incircle of triangle SCD touches CD at R. Prove that MG, NH and SR are concurrent on (K).

THVSH

#6 May 13, 2015, 6:19 pm

> *buratinogigle wrote:*
>
> Nice dear THVSH and Luis, here is an easily seen extension
>
> Let $ABCD$ be quadrialateral inscribed circle (O). AC cuts BD at S. (K) is a circle touches SC, SD at M, N and touches (O) internally at P. PC, PD cut (K) again at E, F. EN, FM cut CD at G, H. Incircle of triangle SCD touches CD at R. Prove that MG, NH and SR are concurrent on (K).

My solution still works with this extension

TelvCohl

#10 May 13, 2015, 6:57 pm

> *buratinogigle wrote:*
>
> Let $ABCD$ be quadrialateral inscribed circle (O). AC cuts BD at S. (K) is a circle touches SC, SD at M, N and touches (O) internally at P. PC, PD cut (K) again at E, F. EN, FM cut CD at G, H. Incircle of triangle SCD touches CD at R. Prove that MG, NH and SR are concurrent on (K).

My solution :

Let $U = MG \cap NH$.

From homothety (with center P that maps $\odot(K) \mapsto \odot(O)$) we get $EF \parallel CD$.
From All-Russian Olympiad 2006 finals, problem 9.4 we get $CM = CG, DN = DH$.

From Reim theorem $\Longrightarrow H, G, M, N$ are concyclic ,
so $\angle UMC = \angle CGM = \angle UNM, \angle UND = \angle DHN = \angle UMN$,
hence $\odot(UMN)$ is tangent to SC, SD at M, N, respectively $\Longrightarrow U \in \odot(K)$.

From $\angle FMU = \angle ENU \Longrightarrow U$ is the midpoint of arc EF in $\odot(K)$,
so from homothety with center S that maps the incircle of $\triangle SCD$ to $\odot(K)$ we get $U \in SR$.

Q.E.D

Luis González

#11 May 14, 2015, 1:15 am

> *buratinogigle wrote:*
>
> Let $ABCD$ be quadrialateral inscribed circle (O). AC cuts BD at S. (K) is a circle touches SC, SD at M, N and touches (O) internally at P. PC, PD cut (K) again at E, F. EN, FM cut CD at G, H. Incircle of triangle SCD touches CD at R. Prove that MG, NH and SR are concurrent on (K).

Let Y be the midpoint of the arc ABC of (O). Since P is the exsimilicenter of $(K) \sim (O)$, then P, M, Y are collinear and $MF \parallel DY, EF \parallel CD \Longrightarrow \angle MHC = \angle YDC = \angle MPC \Longrightarrow CMHP$ is cyclic and similarly $DNGP$ is cyclic and $\angle HMN = \angle FEN = \angle HGN \Longrightarrow MNHG$ is cyclic.

According to problem incenter of triangle, the incenter J of $\triangle SCD$ is the 2nd intersection of $\odot(CMP)$ and $\odot(DNP) \Longrightarrow JM = JH$ and $JN = JG \Longrightarrow J$ is the center of $\odot(MNHG) \Longrightarrow JM = JG \Longrightarrow G$ is reflection of M across CJ $\Longrightarrow MG \perp JC$ and similarly $NH \perp JD$. So if $U \equiv MG \cap NH$, then $\angle MUN = \angle(JC, JD) = \angle SMN \Longrightarrow U \in (K)$. Since HG is antiparallel to MN WRT UM, UN, then $KU \perp CD$. Hence by the direct homothety with center S sending (J) to (K), it follows that S, R, U are collinear.

sending (J) to (K), it follows that D, K, C are collinear.

THVSH

#12 May 15, 2015, 10:15 pm • 1

Another extension:

Let ABC be a triangle. An arbitrary circle $\odot(I)$ tangent to CA, AB at E, F, respectively. D lies on $\odot(I). DB\cap\odot(I)=\{D,G\}; DC\cap\odot(I)=\{D,H\}. EG\cap BC=M; FH\cap BC=N.$
Prove that EN, FM intersect on (I).

TelvCohl

#13 May 15, 2015, 10:48 pm

> THVSH wrote:
>
> Another extension:
>
> Let ABC be a triangle. An arbitrary circle $\odot(I)$ tangent to CA, AB at E, F, respectively. D lies on $\odot(I). DB\cap\odot(I)=\{D,G\}; DC\cap\odot(I)=\{D,H\}. EG\cap BC=M; FH\cap BC=N.$
> Prove that EN, FM intersect on (I).

My solution :

Let $J=EF\cap BC$ and $K=DJ\cap\odot(I)$ ($K\neq D$) .

From Pascal theorem (for $FFEGDK$) we get $K\in FM$.
Similarly, we can prove $K\in EN \Longrightarrow EN\cap FM\in\odot(I)$.

Q.E.D

Quick Reply

High School Olympiads

A, M, N, P are concyclic iff d passes through I- [VMO 2011]

Reply

Potla

#1 Jan 12, 2011, 11:33 am • 1

Let $\triangle ABC$ be a triangle such that $\angle C$ and $\angle B$ are acute. Let D be a variable point on BC such that $D \neq B, C$ and AD is not perpendicular to BC. Let d be the line passing through D and perpendicular to BC. Assume $d \cap AB = E, d \cap AC = F$. If M, N, P are the incentres of $\triangle AEF, \triangle BDE, \triangle CDF$. Prove that A, M, N, P are concyclic if and only if d passes through the incentre of $\triangle ABC$.

Love_Math1994

#2 Jan 12, 2011, 4:20 pm

Solution of me in Vietnamese

Attachments:

newfile8.pdf (121kb)

vulalach

#3 Jan 14, 2011, 10:13 am

my solution in Vietnamese
http://www.mediafire.com/file/r8jxz0zzzjpr6lm/loi%20giai%20bai%206.pdf

My solution in English
http://www.mediafire.com/file/8oom6cbu55ycvuc/loi%20giai%20bai%206%20%20inenglish.docx

This post has been edited 1 time. Last edited by vulalach, Jan 14, 2011, 1:00 pm

TheIronChancellor

#4 Jan 14, 2011, 11:59 am

Can we get a solution in english ?

sankha012

#5 Jan 14, 2011, 1:30 pm

Construct the incenter I of $\triangle ABC$.Let $DE \cap CP$ be K and $EN \cap AI$ be J.Also,$AI \cap ED$ is S.

We have $\angle FKP = \pi - \frac{C}{2} - \left(\frac{\pi}{2} - C\right) = \frac{\pi}{2} + C = \angle JIN$.

$\angle JMF = \angle MEF + \angle MFE = \frac{\pi}{4} - \frac{B}{2} + \frac{\pi}{4} - \frac{C}{2} = \frac{A}{2} = \angle JAF$.So A, M, F, J are concyclic and $\angle NJI = \angle AJM = \angle AFM = \angle KFP$.Thus $\triangle NIJ$ and $\triangle PFK$ are similar implying that $\frac{IJ}{FK} = \frac{JN}{FP}$(*)

Now A, M, P, N are concyclic $\Leftrightarrow \angle ANJ = \angle APF$.Note that this can only happen when $\triangle AJN$ and AFP are similar as we already have $\angle AJN = \pi - \angle NJI = \pi - \angle AFM = \angle AFP$.So A, M, P, N are concyclic iff $\frac{AF}{AJ} = \frac{FP}{JN}$.Also as FJ is an angle bisector or $\angle AFD$ we have $\frac{AF}{AJ} = \frac{FS}{JS} \Rightarrow \frac{FS}{JS} = \frac{FP}{JN} = \frac{FK}{IJ}$ from (*).This implies that $IK||FJ$ which is impossible if I is not on d..QED
this is a mere translation..
CAN'T U VIETNAMESE WRITE IN ENGLISH?

sankha012

#6 Jan 14, 2011, 6:21 pm

@Vulalach
We Mathlinkers are accustomed to Vietnamese English so please post your solutions in English even if they look bad.

Quick Reply

High School Olympiads

Incenters and point A are concyclic

Reply

SMOJ

#1 May 13, 2015, 8:32 am

Let the incircle with centre I of triangle ABC touch BC at D. Let AB, AC meet DI at X, Y respectively. Prove that A and the incentres of triangles AXY, BXD and CYD are concyclic.

This post has been edited 1 time. Last edited by SMOJ, May 13, 2015, 8:32 am

FabrizioFelen

#2 May 13, 2015, 10:23 am • 1

My solution :
To solve the problem we use the following lemma :
Lemma
Let M be the incenter of $\triangle BXD$
Let N be the incenter of $\triangle CDY$
Let P be he incenter of $\triangle AXY$
$\Longrightarrow \angle BAM = \angle IAN$ and $\angle MAI = \angle CAN$
Proof : It is easy to see that the law of sines in the triangle ABI and ACI :
$\frac{sen\angle BID}{sen\angle IBD} = \frac{BM}{MI} = \frac{sen\angle AIM.sen\angle BAM}{sen\angle ABI.sen\angle MAI} \Longrightarrow \frac{sen\angle BAM}{sen\angle MAI} = \frac{sen\angle AIB}{sen\angle BID}$ similary
$\frac{sen\angle IAN}{sen\angle NAC} = \frac{sen\angle DIC}{sen\angle AIC} \Longrightarrow \frac{sen\angle IAN}{sen\angle NAC} = \frac{sen\angle BAM}{sen\angle MAI}$
also $\angle IAN + \angle NAC = \angle BAM + \angle IAM$ then by the Cuya's lemma $\angle BAM = \angle IAN$ and $\angle MAI = \angle CAN$
then $2\angle MAN = \angle ABC$. Using the Lemma on the problem : $2\angle MAN = 2\angle MPN = \angle BAC$ then $APMN$ is cyclic.

This post has been edited 1 time. Last edited by FabrizioFelen, Oct 7, 2015, 11:38 am
Reason: error in the post

tranquanghuy7198

#3 May 13, 2015, 10:35 am

See VMO 2011 problem 6

Luis González

#4 May 13, 2015, 10:44 am

Denote by J, K, L the incenters of $\triangle BXD, \triangle CDY, \triangle AXY$, respectively.

$\angle BID = 90^\circ - \frac{1}{2}\angle ABC = \angle(IA, IC) \Longrightarrow ID, IA$ are isogonals WRT $\angle JIK$ and since
$\angle JDK + \angle JAK = 90^\circ + \frac{1}{2}\angle BAC = \angle JIK$, then we deduce that A and D are isogonal conjugates WRT $\triangle IJK$
$\Longrightarrow \angle AJK = \angle IJD = 45^\circ + \frac{1}{2}\angle ABC$. Now since
$\angle ALY = 90^\circ + \frac{1}{2}\angle AXY = 90^\circ + \frac{1}{2}(90^\circ - \angle ABC) = 135^\circ - \frac{1}{2}\angle ABC \Longrightarrow \angle AJK + \angle ALY = 180^\circ \Longrightarrow$
$AJKL$ is cyclic.

Luis González

#5 May 13, 2015, 10:53 am

Thanks tranquanghuy7198 for the reference. VMO 2011 P6 is actually harder as it asks for the converse. See A, M, N, P are concyclic iff d passes through I - [VMO 2011] for other solutions.

TelvCohl

TelvCohl

#7 May 13, 2015, 11:58 am

Another solution **:**

Let the tangent from A to the incircle of $\triangle BXD$ cut ID at G .
Let I_1, I_2, I_3 be the incenter of $\triangle AXY, \triangle BXD, \triangle CYD$, respectively .

From quadrilateral $AGDB$ has incircle $\Longrightarrow AG + BD = AB + GD$,
so combine $AB + CD = AC + BD$ we get $AG + CD = AC + GD$,
hence quadrilateral $AGDC$ has incircle $\Longrightarrow$ the common internal tangent of $\odot(I_2)$ and $\odot(I_3)$ pass through A .

Since the reflection of A in I_2I_3, I_3I_1, I_1I_2 all lie on ID ,
so A is the anti-steiner point of ID WRT $\triangle I_1I_2I_3 \Longrightarrow A \in \odot(I_1I_2I_3)$.

Q.E.D

Remark : From my proof above, we can generalize the original problem as following **:**

Let I be the incenter of $\triangle ABC$ and $\odot(I)$ touch BC at D .
Let ℓ be a line through D cut AB, AC at X, Y, resepctively .
Let I_1, I_2, I_3 be the incenter of $\triangle AXY, \triangle BXD, \triangle CYD$, respectively .

Then A, I_1, I_2, I_3 are concyclic

Quick Reply

High School Olympiads

XY is perpendicular to BC

Reply

Source: Own

THVSH

#1 May 12, 2015, 9:26 pm

Let ABC be a triangle. M, N is the midpoints of CA, AB. A fixed line d intersect the segment BC at D. The circles $(K), (L)$ change, (K) passes through B, D, (L) passes through C, D. BE, CF are the diameters of $(K), (L)$, respectively. $(K) \cap d = I; (L) \cap d = J$. $NI \cap (K) = G; MJ \cap (L) = H$.
a)When $(K), (L)$ change, prove that EG passes through a fixed point X and FH passes through a fixed point Y.
b)Prove that $XY \perp d$

tranquanghuy7198

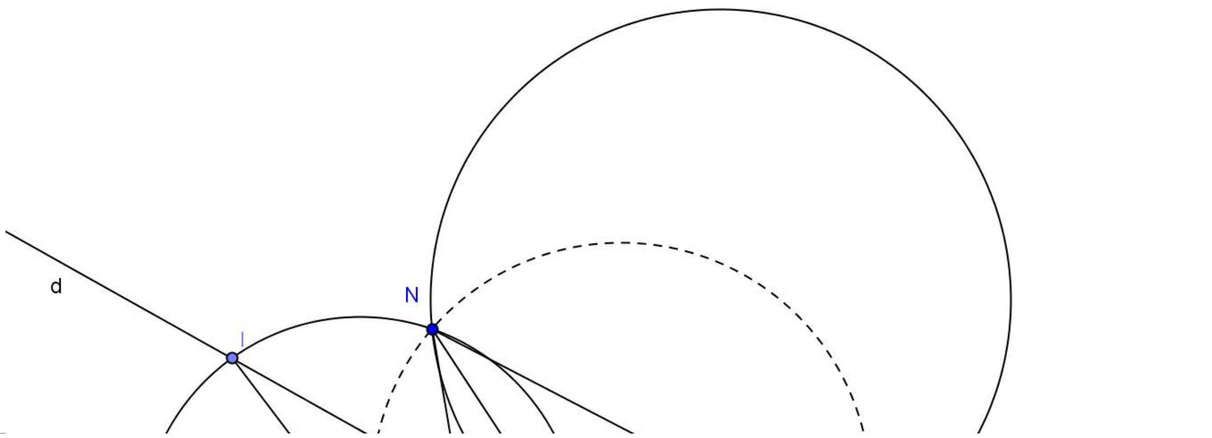
#2 May 12, 2015, 11:52 pm

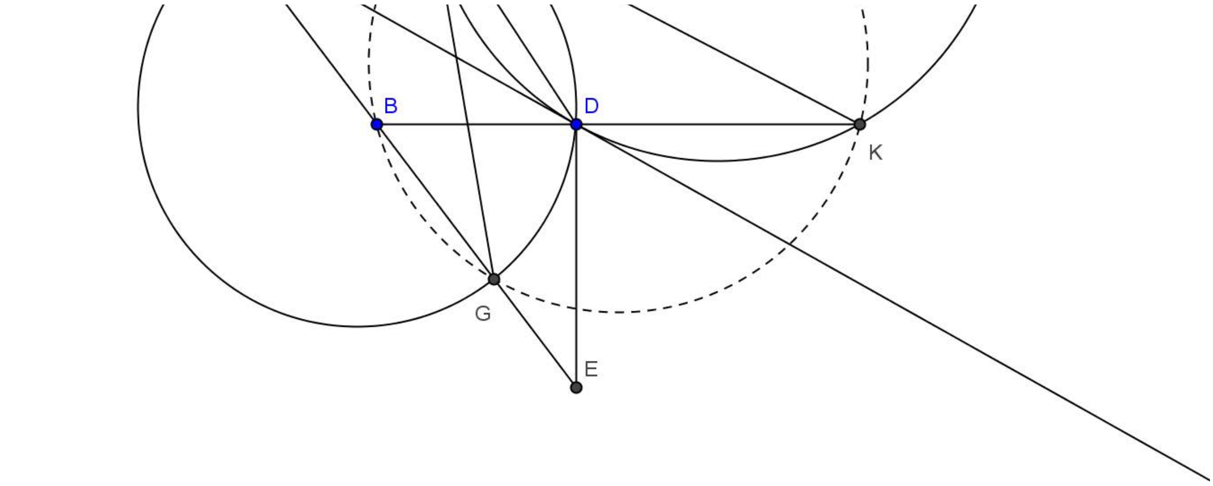
My solution:
a) Firstly, we'll find the fixed points X, Y
With the notice that $\angle BGE$=90, finding the fixed point X means that finding the locus of G (easy to check). Now we'll find the locus of G by inversion. Indeed:
(FIG. 1)
The problem is restated: Given 3 fixed points D, N, B and d is the fixed line passing through D. I varies on d.
NI∩(BID)≡G. Find the locus of G.

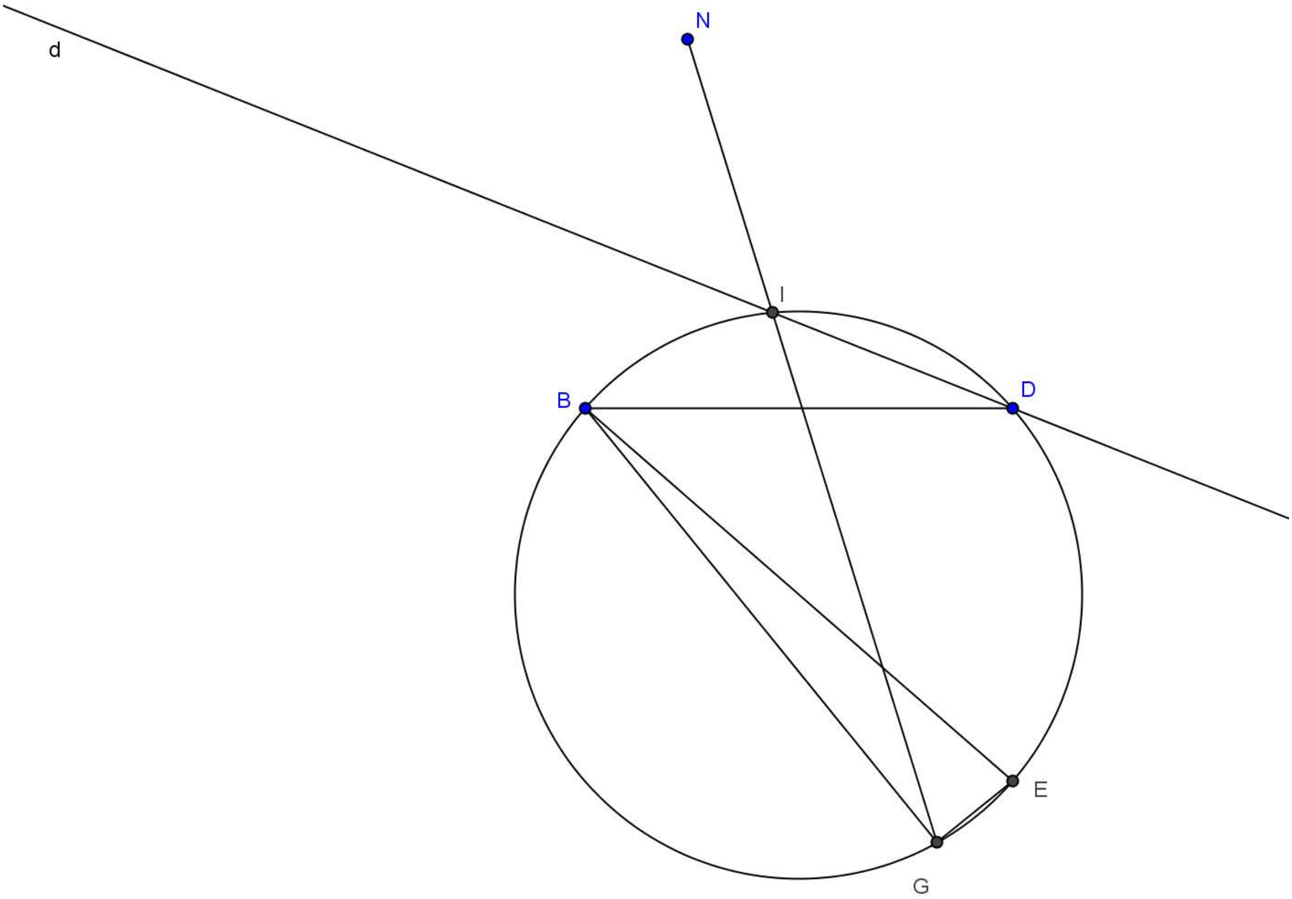
Solution.
Consider the inversion centered at D with an arbitrary power, we receive the new problem: (FIG. 2) Given 3 fixed points D, N, B and d is the fixed line passing through D. I varies on d. (DNI)∩BI≡G. Find the locus of G.
Let K be the point on BD such that (DNK) is tangent to d, so K is the fixed point.
We have:
$\angle BGN$=$\angle IGN$=$\angle IDN$=$\angle DKN$=$\angle BKN$
⇒ G is on (BNK)
⇒ the locus of G is the fixed circle (BNK)

Back to our main problem
Let K, L be the points on BC such that $NK \parallel ML \parallel d$
According to the inversion above, G is on (BNK). However, if the arbitrary circle passing through B, D mentioned in THVSH's problem passes through N (it means that we are considering the special position of that circle), GE turns out to be the perpendicular bisector of AB (easy to check). Because of all that, X, which is the diametrical opposite point of B WRT (BNK), is the intersection of the perpendicular bisector of AB and the line which passes through K and are perpendicular to BC. That is the fixed point we are trying to determine. So is Y. Q.E.D.

Attachments:





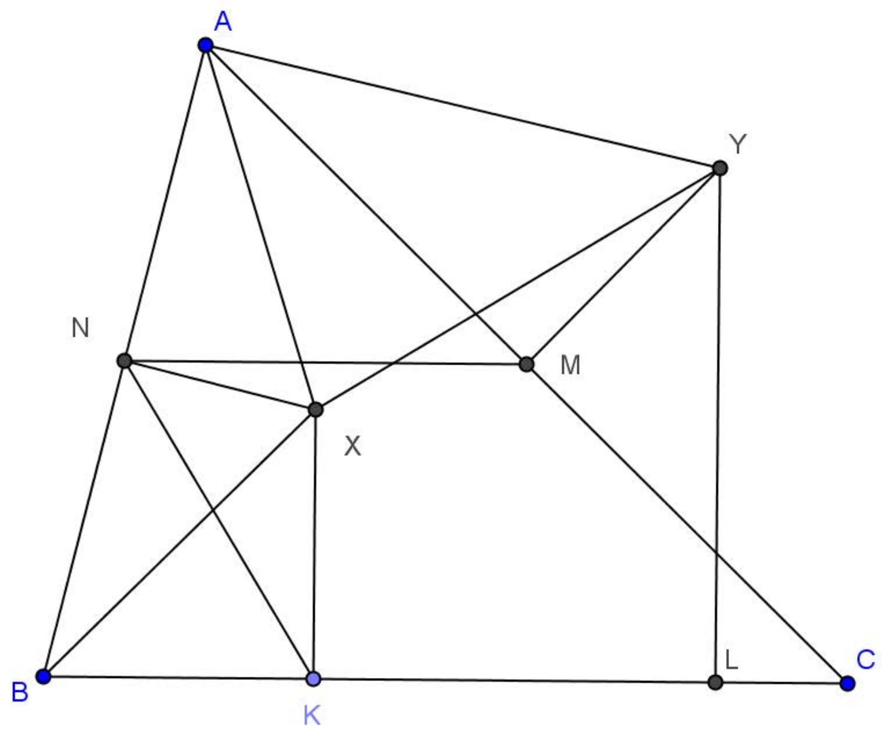


tranquanghuy7198

#3 May 12, 2015, 11:55 pm

b) (FIG. 3)According to the result above, the problem reduces to: Given $\triangle ABC$ and M, N are the midpoints of CA, AB, resp. K, L are on BC such that $NK \parallel ML$. X, Y are the points such that:
$XK \perp BC$, $XN \perp AB$, $YL \perp BC$, $YM \perp CA$.
Prove that $XY \perp NK, ML$.
We have:
$$\frac{AN}{NX}=\frac{BN}{NX}=\frac{sin\angle BKN}{sin\angle NKX}=\frac{sin\angle CLM}{sin\angle MLY}=\frac{CM}{MY}=\frac{AM}{MY}$$
$\Rightarrow \triangle ANX \sim \triangle AMY$
$\Rightarrow \triangle ANM \sim \triangle AXY$
$\Rightarrow$
$$(XY, NK)=(XY, NM)+(NM, NK)=(AX, AN)+(NM, NK)=(BN, BX)+(NM, NK)=(KN, KX)+(NM, NK$$
Q.E.D.

Attachments:



tranquanghuy7198

#4 May 12, 2015, 11:57 pm

I think that combining 2 problems like that isn't a good way of creating new problem.

Luis González

#6 May 13, 2015, 12:36 am

a) E moves on perpendicular τ to BC at D. Hence since $\angle(GN, GB) = \angle(d, BC)$ and $\angle(GE, GI) = \angle(\tau, d)$ are constant, it follows that the circle $\omega_B \equiv \odot(BGN)$ is fixed and EG cuts its arc BN at a fixed point X and similarly Y is a fixed point on the circle $\omega_C \equiv \odot(CFM)$.

b) If ω_B, ω_C cut BC again at U, V, we have $\angle NUB = \angle BGI = \angle(BC, d) \Longrightarrow NU \parallel d$ and similarly $MV \parallel d$. Moreover $\angle BUX = \angle BGE = 90^\circ \Longrightarrow XU \perp BC$ and similarly $YV \perp BC$. If P is the midpoint of BC, then from parallelogram $MNUV$, it follows that $\triangle PNU$ is image of $\triangle CMV$ under the translation $\overrightarrow{MN} \Longrightarrow$ projection S of Y on XU is on $\odot(PNU)$. Now since $\angle NXS = \angle ABC$ and $\angle NSX = \angle BPN = \angle BPN = \angle BCA \Longrightarrow$ $\triangle ABC \sim \triangle NXS \Longrightarrow \frac{SN}{SX} = \frac{CA}{CB} = \frac{PN}{MN} = \frac{PN}{YS} \Longrightarrow$ right triangles $\triangle XYS$ and $\triangle SPN$ are similar $\Longrightarrow$ $\angle SYX = \angle SPN = \angle SUN \Longrightarrow XY \perp NU$ or $XY \parallel d$.

THVSH

#7 May 13, 2015, 12:20 pm • 2

Dear **Luis González** and **trangquanghuy7198**,
Thank you for your interest.
Your solution is very good, **tranquanghuy7198**, but that 's not the way I created this problem!

This post has been edited 1 time. Last edited by THVSH, May 13, 2015, 12:20 pm

Quick Reply

High School Olympiads

Hard geometry

Reply

Source: Iran TST 2015 ,exam 1, day 2 problem 3

TheOverlord

#1 May 11, 2015, 8:09 pm • 1

$ABCD$ is a circumscribed and inscribed quadrilateral. O is the circumcenter of the quadrilateral. E,F and S are the intersections of AB,CD , AD,BC and AC,BD respectively. E' and F' are points on AD and AB such that $A\hat{E}E'=E'\hat{E}D$ and $A\hat{F}F'=F'\hat{F}B$. X and Y are points on OE' and OF' such that $\frac{XA}{XD}=\frac{EA}{ED}$ and $\frac{YA}{YB}=\frac{FA}{FB}$.
M is the midpoint of arc BD of (O) which contains A.
Prove that the circumcircles of triangles OXY and OAM are coaxal with the circle with diameter OS.

This post has been edited 1 time. Last edited by TheOverlord, May 11, 2015, 8:10 pm

Luis González

#2 May 12, 2015, 11:58 am

If I is the incenter of $ABCD$, then $E'\equiv IE\cap AD$ and $F'\equiv IF\cap AB$. Condition $\frac{XA}{XD}=\frac{EA}{ED}$ means that X is on the E-Apollonius circle of $\triangle EAD$, which is orthogonal to any circle through A,D, thus particularly orthogonal to the circumcircle (O,R) of $ABCD \Longrightarrow R^2=OE'\cdot OX$ and similarly $R^2=OF'\cdot OY$.

Inversion WRT (O,R) takes X,Y into E',F', takes the circle with diameter $\overline{OS}$ into EF (as EF is the polar of S WRT (O)) and fixes A,M. If $E'F',IA$ cut EF at T,K, then from the complete quadrilateral $EE'F'F$, it follows that $(E,F,K,T)=-1$. But since AM,AI bisect $\angle BAD$, we have $A(E,F,K,M)=-1 \Longrightarrow M\in AT \Longrightarrow E'F',AM$ and EF concur at $T \Longrightarrow$ their inverses $\odot(OXY),\odot(OAM)$ and the circle with diameter $\overline{OS}$ intersect at a second point, i.e. they are coaxal.

Quick Reply

High School Olympiads

Possible pascal's?

Reply

SMOJ

#1 May 12, 2015, 9:50 am

Let ABC be an acute triangle with circumcircle ω. Let D be the midpoint of minor arc BC. Let M,N be points on AB,AC respectively such that MN is parallel to BC. Lines DM,DN intersect ω again at P,Q. Let PQ intersect MN at R. Prove that RA is tangent to ω.

I think Pascal can be used here but I do not get the result I want.

Luis González

#2 May 12, 2015, 10:33 am • 1

Let MN cut ω at $U,V.$ Obviously D is also midpoint of the arc UV of $\omega \Longrightarrow DU^2=DM \cdot DP=DN \cdot DQ \Longrightarrow$ $MNQP$ is cyclic. Therefore, MN,PQ and the tangent of ω at A are pairwise radical axes of $\omega,\odot(AMN)$ and $\odot(MNQP)$ concurring at their radical center $R \Longrightarrow RA$ is tangent to $\omega.$

TelvCohl

#3 May 12, 2015, 11:09 am • 1

Here is the solution by using Pascal theorem :

From Pascal theorem (for $ABQDPC$) $\Longrightarrow$ BQ,CP,MN are concurrent at X .

From Pascal theorem (for $ABCPDD$) $\Longrightarrow$ A,D,X are collinear .

From Pascal theorem (for $AABQPD$) $\Longrightarrow$ RA is the tangent to $\odot(ABC)$ at A .

Q.E.D

FabrizioFelen

#4 May 12, 2015, 11:55 am

My solution in the Theorem Pascal's :
Let $S=BQ\cap CP$. By Pascal's theorem in $(PAQCDB)$: $S=BQ\cap CP, M=PD\cap BA, N=DQ\cap CA$ are collinear. By Ceva's trigonometric theorem in $\triangle PBS$: $sen\angle PBM.sen\angle MSB=sen\angle MBS.sen\angle MSP \Longrightarrow$ $\frac{sen\angle PCN}{sen\angle MSP}=\frac{sen\angle MBS}{sen\angle MSB}=\frac{NS}{NC}=\frac{MS}{MB}$ then it is easy too see that $\frac{AM}{MS}=\frac{AN}{NS} \Longrightarrow AS$ is bisector of $\angle ABC$ $\Longrightarrow A,S,D$ are collinear. Let $U=AB\cap CP$ and $V=AC\cap BQ$
The $\triangle PUM$ and $\triangle QVN$ with the same perspective since $D=PM\cap QN, A=MU\cap VN, S=PU\cap VQ$ are collinear then by Desargues theorem : PQ,UV,MN are concurrent in R. By Pascal´s theorem $(AABQPD)$: By tangent for A,PQ,UV are concurrent in R
$\Longrightarrow RA$ is the tangent to $\odot(ABC)$

tranquanghuy7198

#5 May 12, 2015, 1:09 pm • 1

Here's my solution.
$AD\cap MN\equiv S$
Because: $DM.DP=DN.DQ=DS.DA$
$\Rightarrow A,S,N,Q$ are cyclic
$\Rightarrow \angle SQN=\angle SAN=\angle DAC=\angle DAB$

$\Rightarrow B, S, Q$ are collinear
Apply Pascal for $(AQDPAB) we have the conclusion.

jayme
#6 May 12, 2015, 3:25 pm

Dear Mathlinkers,

my answer isn't with what it was asking... but

1. U, V the points of intersection of BC with resp. DP, DQ
2. consider (O) circumcircle of ABC, (X) circumcircle of AMN and (Y) the circumcircle of PQMN
which can be obtain for the two last with the Reim's theorem
3. By the three cords theorem we are done.

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Incenter and excenter

Reply

tranquanghuy7198

#1 May 11, 2015, 9:42 pm

Given $\triangle ABC$ and S varies on the line BC. I, J are the incenter OR excenter of $\triangle ASB$ and $\triangle ASC$, respectively, SUCH THAT S, I, J are collinear. Prove that the intersection of the lines which pass through I, J and are perpendicular to CJ, BI,resp., is on BC.

TelvCohl

#2 May 11, 2015, 10:21 pm

My solution :

(I'll only prove the case I is the incenter of $\triangle ASB$ and J is the A-excenter of $\triangle ASC$)

Let K be the incenter of $\triangle ABC$.
Let L, T be the A-excenter, B-excenter of $\triangle ABC$, respectively .
Let $I^*, J^* \in BC$ be the points such that $II^* \perp CJ, JJ^* \perp BI$.
Let $\mathcal{P}(P, \odot)$ be the power of a point P with respect to a circle $\odot$.

Since $\angle TIA = 90^\circ - \frac{1}{2}\angle ASB = \angle TJA$,
so A, I, J, T are concyclic $\Longrightarrow \odot(TCK), \odot(TIJ), \odot(TLB)$ are coaxial (with common radical axis AT) ,
hence $\mathcal{P}(I, \odot(TCK)) : \mathcal{P}(I, \odot(TLB)) = \mathcal{P}(J, \odot(TCK)) : \mathcal{P}(J, \odot(TLB)) \Longrightarrow BI : IK = LJ : JC$,
so we get $BI^* : I^*C = BI : IK = LJ : JC = BJ^* : J^*C \Longrightarrow I^* \equiv J^*$.

Q.E.D

THVSH

#3 May 11, 2015, 10:42 pm

My solution:

In my proof, I is the incenter of $\triangle ASB$, J is $A-$excenter of $\triangle ASC$.
Let G be the incenter of $\triangle ABC$.
Let I_b be the $B-$excenter of $\triangle ABC$, then C, J, I_b are collinear.
Let the line passing through I and perpendicular to CJ intersect BC at T. We only need to prove that $JT \perp BI$
Since $\triangle BAI_b \sim \triangle BGC$, we get $BG.BI_b = BA.BC$.
$IT \parallel CG \Longrightarrow \dfrac{BI}{BG} = \dfrac{BT}{BC} \Longrightarrow BI.BI_b = BT.BA$
$\Longrightarrow \triangle BIA \sim \triangle BTI_b \Longrightarrow \angle BI_bT = \angle BAI = 90^\circ - \angle SII_b$
$\Longrightarrow I_bT \perp IS \equiv IJ \Longrightarrow T$ is the orthocenter of ΔIJI_b
Then $JT \perp BI$. Q.E.D

This post has been edited 1 time. Last edited by THVSH, May 11, 2015, 10:43 pm

Luis González

#4 May 12, 2015, 7:49 am

Assume that I is the A-excenter of $\triangle ABS$ and J is the incenter of $\triangle ACS$. If K is the incenter of $\triangle ABC$ and I_a is its excenter againts A, then clearly it's enough to show that $CJ : JK = I_aI : IB$.

$\angle CAJ = \frac{1}{2}\angle CAS = \frac{1}{2}(\angle BAC - \angle BAS) = \angle BAK - \angle BAI = \angle I_aAI$, which means that ΔKAC and $\triangle BAI_a$ are similar with corresponding cevians $AJ, AI \Longrightarrow CJ : JK = I_aI : IB$, as desired.

tranquanghuy7198

#5 May 12, 2015, 10:21 am

Thank you, TelvCohl, THVSH, Luis Gonzalez, those are perfect solutions.

Quick Reply

High School Olympiads

Circles and Diameters

Reply

KudouShinichi

#1 May 12, 2015, 1:34 am

Among the points A, B, C, D no three are collinear. The lines AB and CD intersect at E, and BC and DA intersect at F. Prove that either the circles with diameters AC, BD, EF pass through a common point, or no two of them have any common point.

Luis González

#2 May 12, 2015, 3:20 am

Rather old problem; it certainly has been posted before. It follows from the fact that the aforementioned circles are always coaxal, so they either have common points (elliptic or parabolic pencil) or not (hyperbolic pencil). e.g. see Gauss + Miquel + Aubert [a. k. a. Steiner] and elsewhere.

Quick Reply

High School Olympiads

Gauss + Miquel + Aubert [a. k. a. Steiner]

grobber
#1 Sep 8, 2003, 5:51 am • 3

We have a quadrilateral $ABCD$. Let E be the point of intersection of the lines AB and CD. Let F be the point of intersection of the lines AD and BC. Prove the following facts:

a) The midpoints of the segments AC, BD, EF are collinear.
The line through these midpoints is called the *Gauss line* of the quadrilateral $ABCD$.

b) The orthocenters of the triangles ABF, CDF, BCE, ADE are collinear on a line perpendicular to the Gauss line.
The line joining these orthocenters is known as the *Aubert line*, or, also, as the *Steiner line* of the quadrilateral $ABCD$.

c) The circumcircles of the four triangles ABF, CDF, BCE, ADE intersect at a point M.
This point M is called the *Miquel point* of the quadrilateral $ABCD$.
The circumcenters of these four triangles ABF, CDF, BCE, ADE lie on a circle which also passes through M.

Lagrangia
#2 Sep 8, 2003, 4:18 pm

a) you use Menelau's theorem

Lagrangia
#3 Sep 8, 2003, 4:34 pm • 1

For the Aubert line:

You use the following:

Lemma :

Let ABC be a triangle and A' the projection of A on BC, B' the projection of B on CA and C' the projection of C on AB. Then we have HA*HA'=HB*HB'=HC*HC', where H is the orthocenter.

Proof :
It is easy to see that (HB'A with HA'B) and (HA'C and HC'A) (all are triangles) are proportional (this is not the right word .. I am sure.. can't find the right word, but you will figure out what I am trying to say) we have:

HA*HA'=HB*HB' and HA*HA'=HC*HC' so lemma proved.

Proof of Aubert line :
We know from Gauss that the middle points of the diagonals AC, BD , EF are on Gauss line, which we denote d (lets consider the points : O1, O2,O3).

We consider the circles C1,C2,C3 of diameters AC, BD and EF. We denote H as the orthocenter of ADE and let A' be the projection of A on DE , D' the projection of D on AE and E' the projection of E on AD. It is easy to see that A' is on C1, D' on C2 and E' on C3. Using the previous lemma we get that: HA*HA'=HD*HD'=HE*HE'.
These relations show that H has equal point powers on C1,C2 and C3.

Now we consider dij the radical axe of circles Ci,Cj (i,j belong to {1,2,3} and i<>j). From HA*HA'=HD*HD' we have that H is on d12, from HD*HD'=HE*HE' we have that H is on d23. because d12 is perpendicular on d and d23 is perp on d, and from H we can only draw one perpendicular on d, we have that d12 and d23 are one and the same line,
Then d12=d23=d13=d' which is perpendicular on d.

We showed that H is on d'. You do the same with the ortocenters of ABF, DCF and BEC.

So theorem proved!
cheers!

Lagrangia
#4 Sep 8, 2003, 4:45 pm

Miquel point:

Lets consider M the 2nd point of meeting of the circumcircles of triangles:
BCE and DFC.

Because we have m(CMF)=m(CDA) and m(CBA)=m(CME) we have that
m(BAF)+m(BMF)=m(CMF)+m(BMC)+m(BAF)=m(ADE)+m(EAD)+m(AED)=180. So we get that ABMF can be inscriebed. you do the same with AEMD. So you get what we call Miquel point!

cheers!

grobber
#5 Sep 8, 2003, 4:59 pm

Cool! That's prwtty much what I did too.

galois
#6 Sep 8, 2003, 9:19 pm

my proofs are same xcept for the 1st one where i used vectors and came up with a pretty neat solution .neway,here's a related problem on complete quadrangles
each diagonal of a complete quadrangle is cut harmonically by the other two.try this.it's nice
best regards

sprmnt21
#7 Nov 3, 2004, 4:11 am

> *Lagrangia wrote:*
> a) you use Menelau's theorem

a different proof here

jayme
#8 Nov 7, 2008, 5:31 pm

Dear Mathlinkers,
a synthetic proof of the orthogonality of the Gauss and Steiner's line can be found on my website
http://perso.orange.fr/jl.ayme vol. 4 La droite de Gauss et la droite de Steiner
Sincerely
Jean-Louis

April
#9 Nov 8, 2008, 7:08 am

Related paper: http://forumgeom.fau.edu/FG2004volume4/FG200405.pdf

balan razvan
#10 Apr 22, 2009, 1:54 am

see also miquel's star that uses his point
http://www.mathlinks.ro/viewtopic.php?search_id=1105712782&t=188934

http://www.mathlinks.ro/viewtopic.php?p=1132671#1132671

Quick Reply

High School Olympiads

Given midpoint and circumcircle conditions, prove angle same

SMOJ

#1 May 11, 2015, 9:17 am

Let D, E, F be the midpoint of BC, AC, AB.BE intersects the circumcircle of BCF again at P, AD intersects the circumcircle of ABE again at Q. Prove that $\angle AQF = \angle BPD$

TelvCohl

#3 May 11, 2015, 11:00 am

My solution :

Lemma :

Let D, E, F be the midpoint of BC, CA, AB, respectively .
Let G be the centroid of $\triangle ABC$ and $E' = CF \cap \odot(BCE), F' = BE \cap \odot(BCF)$.

Then $\angle DE'G = \angle DF'G$

Proof :

Let Y be the reflection of C in E' and Z be the reflection of B in F' .

Since
$BG \cdot GZ = BG(GF' + BF') = BG^2 + 2BG \cdot GF' = BG^2 + 2CG \cdot GF = BG^2 + CG^2$
$CG \cdot GY = CG(GE' + CE') = CG^2 + 2CG \cdot GE' = CG^2 + 2BG \cdot GE = CG^2 + BG^2$,
so $BG \cdot GZ = CG \cdot GY \Longrightarrow B, C, Y, Z$ are concyclic $\Longrightarrow \angle DE'G = \angle BYG = \angle CZG = \angle DF'G$.

Back to the main problem :

Let G be the centroid of $\triangle ABC$ and $R = CF \cap \odot(BCE)$.

Since $AG \cdot GQ = BG \cdot GE = CG \cdot GR$,
so A, C, Q, R are concyclic $\Longrightarrow \angle QDF = \angle DAC = \angle QRC$,
hence D, F, Q, R are concyclic $\Longrightarrow \angle AQF = \angle CRD = \angle BPD$ (from lemma) .

Q.E.D

Luis González

#4 May 11, 2015, 11:31 am • 2

Moreover $\angle AQF = \angle BPD = \omega$ equals the Brocard angle of $\triangle ABC$.

Let the A-symmedian cut BE at S. Inversion with center A, power $AB \cdot AE = AC \cdot AF$ followed by reflection on the angle bisector of $\angle BAC$ swaps BE and $\odot(ABE) \Longrightarrow$ it swaps S and $Q \Longrightarrow AQ \cdot AS = AC \cdot AF$ or $\frac{AC}{AS} = \frac{AQ}{AF} \Longrightarrow \triangle AQF \sim \triangle ACS \Longrightarrow \angle AQF = \angle ACS$. But it's known that the A-symmedian, the B-median and the C-cevian of the 1st Brocard point concur (for a proof you can see the topic symmedian post #2). Hence $\angle AQF = \angle ACS = \omega$ is the Brocard angle and in the same way we have $\angle BPD = \omega$.

saturzo

#5 May 12, 2015, 12:03 am • 1

Let P' be the point on the same side of AB as G [the centroid of $\triangle ABC$], s.t., $\triangle P'AB \sim \triangle PBC$. Let $P'B \cap CF = R$