Homogeneity in Inequalities

I was reading this proof of 2001 IMO Problem #2, proving for $a, b, c \in \mathbb{R}^+$ that $\sum \frac{a}{\sqrt{a^2+8bc}} \geq 1$. In a proof using Jensen's inequality, it says

> This inequality is homogeneous so we can assume without loss of generality $a+b+c=1$.

I've read multiple proofs saying we can assume whatever because the terms are homogeneous, but what can you assume without loss of generality from homogeneity?

#Proofs #Advice #Math

Note by **Cody Johnson**
4 years, 8 months ago

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In this post I consider only inequalities in three variables, but it extends to any number of variables. Expression $f(a,b,c)$ is said to be homogeneous of degree k if and only if there exists real k such that for every $t>0$ we have

$$t^k \cdot f(a,b,c) = f(ta,tb,tc)$$

For instance in your example we have

$$f(a,b,c) = \sum_{\text{cyc}} \frac{a}{\sqrt{a^2+8bc}} - 1$$

and this expression f is homogeneous of degree 0, i.e.

$$f(a,b,c) = f(ta,tb,tc)$$

Okay, now finally why can the assumption $a+b+c=1$ be made? Assume that

$$a+b+c=m$$

for $m>0$ i.e.

$$\frac{a}{m}+\frac{b}{m}+\frac{c}{m}=1$$

Let $a'=\frac{a}{m}, b'=\frac{b}{m}, c'=\frac{c}{m}$. Then

$$a'+b'+c'=1$$

But the homegeneity of degree 0 tells us that

$$f(a',b',c') = f\left(\frac{a}{m}+\frac{b}{m}+\frac{c}{m}\right) = f(a,b,c)$$

(in case it's not clear, we used $\frac{1}{m}=t$, remember that t can be arbitrary positive real number). Hence proving $f(a',b',c') \geq 0$ is equivalent to proving $f(a,b,c) \geq 0$ and we have the nice condition that

$$a'+b'+c'=1$$

You can assume many other things (but only one assumption at a time), like

$$a=1$$
$$b=1$$
$$c=1$$
$$abc=1$$
$$ab+bc+ca=1$$
$$a^2+b^2+c^2=1$$

etc. also the number on right-hand side of these assumptions doesn't need to be 1.

Jan J. - 4 years, 8 months ago

15 0

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http://www.artofproblemsolving.com/Forum/viewtopic.php?f=52&t=386799&p=2148037#p2148037 in here

Truong Nguyen Ngoc - 4 years, 8 months ago

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