Schröder Points Database / Darij Grinberg

This page contains some results and questions about the Schröder points of a triangle and configuration related to these. The dates are the dates of last changes.

The idea of this page was to collect all known results about the Schröder points of a triangle from different sources (Hyacinthos messages, MathLinks discussions). Feel free to mail me (see main site for address) if you have something to add.

[Schroeder2] The Schröder points
Darij Grinberg, 4 Jul 2004

This message contains all properties and generalizations of the Schröder points known hitherto.

1. **The Schröder point by inversion**

In Hyacinthos message #6319, I defined the Schröder point as follows:

Let ABC be a triangle with the incenter I. The incircle of triangle ABC touches BC, CA, AB at X, Y, Z. The triangle XYZ is called the Gergonne triangle (= intouch triangle = contact triangle = pedal triangle of I) of triangle ABC. Then the circles AIX, BIY and CIZ are coaxal and concur at two points. The first point of concurrence is obviously I; the second one is a point Sc which I call **Schröder point** of triangle ABC. My naming refers to the article

Heinz Schröder: *Die Inversion und ihre Anwendung im Unterricht der*

Oberstufe, Der Mathematikunterricht 1/1957, pages 59 - 80.

In this article, the concurrence is proven using inversion.

One can verify that the vertices of the tangential triangle of an arbitrary triangle ABC are the inverses of the midpoints of ABC's sides in the circumcircle. Applying this fact to triangle XYZ, we easily see that the circles AIX, BIY and CIZ are the inverses of the medians of triangle XYZ in the circumcircle of triangle XYZ, i. e. in the incircle of triangle ABC. Hence, the Schröder point Sc is the inverse of the centroid of XYZ in the incircle of ABC. In other words, Sc is the Far-out point of the Gergonne triangle XYZ.

From this we get that Sc lies on the Euler line of XYZ, i. e. on the line OI, where O is the circumcenter and I is the incenter of ABC. (This follows from the theorem that the Euler line of XYZ is the line OI, which can be shown using the homothety of triangle XYZ with the excentral triangle of ABC.)

The common chord of the circles AIX, BIY and CIZ is the line OI.

2. Trilinears of the Schröder point

Homogeneous trilinears for the Schröder point Sc found by Jean-Pierre Ehrmann are

$$((b-c)^2 + a(b+c-2a) : (c-a)^2 + b(c+a-2b) : (a-b)^2 + c(a+b-2c)).$$

Later I proved another trilinear representation:

$$(\cos B + \cos C - 2 \cos A : \cos C + \cos A - 2 \cos B : \cos A + \cos B - 2 \cos C).$$

Clark Kimberling has included the Schröder point Sc as X(1155) in the ETC (Encyclopedia of Triangle Centers).

I also propose to call Sc the **Gergonne-Schröder point** of triangle ABC, since we will later find some analogues related to the Nagel and Bevan points.

Clark Kimberling introduces the Schröder point as follows:

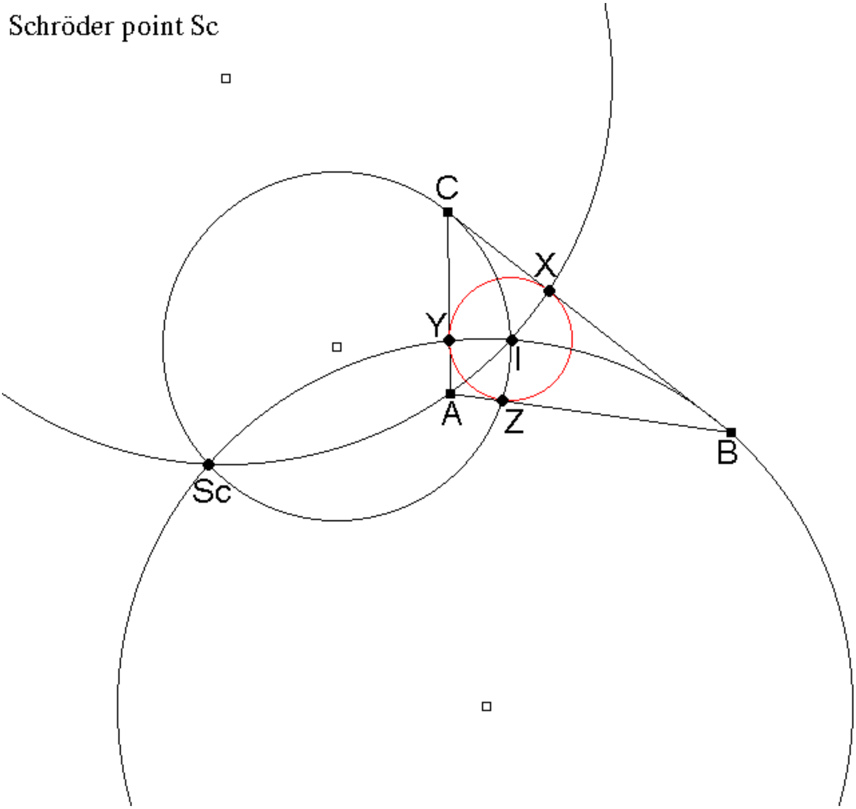
X(1155) = SCHRÖDER POINT

Let XYZ be the intouch triangle of ABC; i.e., the pedal triangle of the incenter, I. The circles AIX, BIY, CIZ concur in two points. One of them is I; the other is X(1155). This result is obtain by inversion in

Heinz Schröder, "Die Inversion und ihre Anwendung im Unterricht der Oberstufe," *Der Mathematikunterricht* 1 (1957) 59-80.

Each vertex of the tangential triangle of any triangle T is the inverse-in-the-circumcircle-of-T of the midpoints of the sides of T. Applying this to triangle XYZ shows that X(1155) is the inverse-in-the-incircle of the centroid of XYZ; i.e., X(1155) is X(23)-of-the-intouch-triangle. (Darij Grinberg, #6319, 1/11/03; coordinates by Jean-Pierre Ehrmann, #6320, 1/11/03)

In Clark Kimberling's ETC, the isogonal conjugate of the Schröder point Sc is the point X(1156); this point lies on the line GF, where G is the Gergonne point and F the Feuerbach point of triangle ABC.



3. Jean-Pierre Ehrmann's approach

The key to the trilinear coordinates of Sc is another proof of the fact that the circles AIX, BIY and CIZ concur at two points. In fact, if the external bisectors of A, B, C in triangle ABC meet the opposite sides in A", B", C", then the points A", B", C" lie on one line h, the so-called tripolar of the incenter I of triangle ABC.

Since A", B", C" lie on one line h, the midpoints of segments IA", IB", IC" lie on the image of this line in the homothety with center I and factor 1/2. Call this image h'.

Now, the circle AIX is the circle with diameter IA", because the angles IAA" and IXA" are 90°. Analogously, the circles BIY and CIZ are the circles with diameters IB" and IC". Hence, the centers of the circles AIX, BIY and CIZ lie on the line h'. Consequently, they pass through the reflection of I in h'. This proves again that they have two common points, but this also gives more: In fact, this reflection is easily seen to be the pedal of I on h. This yields that the Schröder point lies on h and that ISc is orthogonal to h. But we know that the Schröder point Sc lies on OI. Thus, we get the following result:

The tripolar h of the incenter of a triangle ABC is orthogonal to the line joining its circumcenter O with its incenter I, and these two lines meet at the Schröder point Sc of triangle ABC.

From this result, we easily get the trilinears of Sc given above.

These nice observations are due to Jean-Pierre Ehrmann (Hyacinthos messages #6326 and #6327) - many thanks!

4. The Schröder point as inversive image of X(55)

The Schröder point Sc of triangle ABC is the inversive image of the centroid S' of the Gergonne triangle XYZ in the incircle of ABC. Note that S' is X(354) in Kimberling's ETC, known as the **Weill point** of triangle ABC. On the other hand, in Kimberling's ETC I read that the Schröder point Sc is the inversive image of X(55) in the circumcircle of ABC. The point X(55) is the internal center of similitude of the circumcircle and the incircle of triangle ABC, also known as the isogonal conjugate of the Gergonne point.

I have found a proof that Sc is the inverse of X(55), using some ratios between points on the line OI. This proof is not easy enough to describe it here, but it is pretty straightforward, using the relation

$$\frac{IS'}{OI} = \frac{r}{3R},$$

where r is the inradius and R is the circumradius of triangle ABC, and the Euler formula $OI^2 = R^2 - 2Rr$.

5. The Nagel-Schröder point

In Hyacinthos message #6544, I have defined three analogues of the Schröder point. While the Schröder point is the second intersection of the circles AIX, BIY and CIZ, where I is the incenter and X, Y, Z are the vertices of the Gergonne triangle, other points can be obtained by replacing the incenter by the excenters or the Gergonne triangle by the Nagel triangle, for example.

Draw the points X', Y', Z' where the sides BC, CA, AB of triangle ABC meet the respective excircles. The triangle X'Y'Z' is called the Nagel triangle (= extouch triangle = pedal triangle of X(40)) of triangle ABC.

Now I conjectured that the circles AIX', BIY' and CIZ' are coaxal and concur at two points. The first point is I, and the second point, denoted by Sn, is called the **Nagel-Schröder point** of triangle ABC.

Unlike the Schröder point Sc, for whose existence we have two rather simple synthetic proofs, the first elementary proof for the existence of Sn was found more than a year after the discovery of the point itself, and this proof is very long.

The homogeneous trilinears of Sn are also much more difficult:

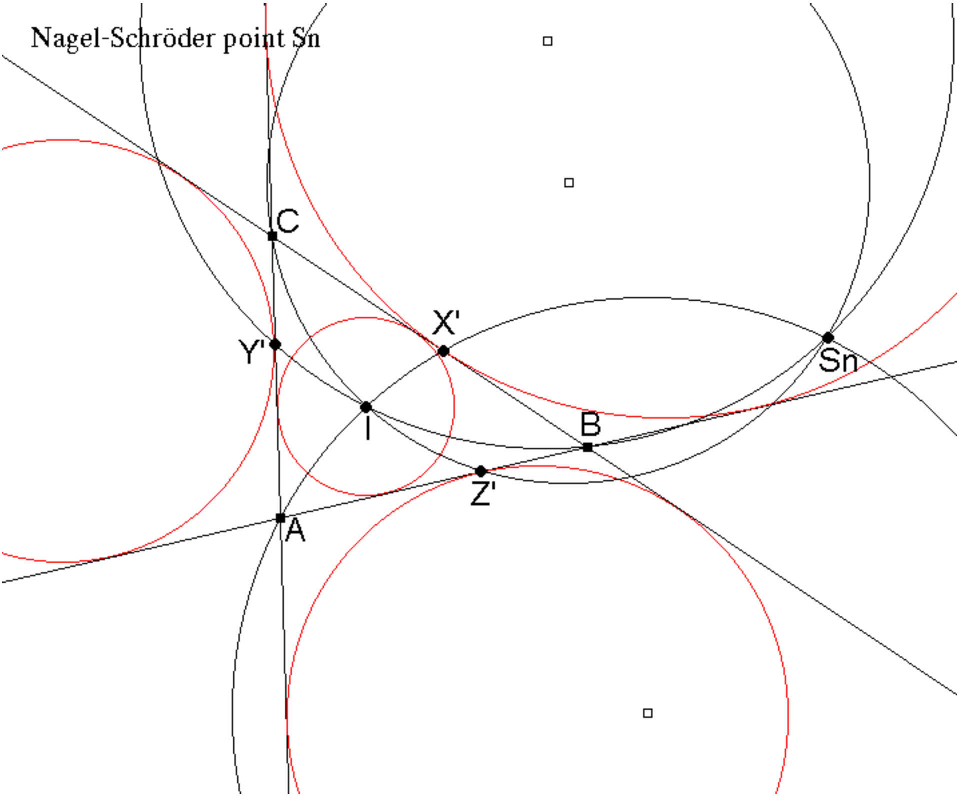
$$\left(\frac{(b+c-2a)(2bc(b+c-a)-(a+b+c)S_A)}{b+c-3a} : \dots \right)$$

(the two other coordinates follow by symmetry), where

$$S_A = \frac{b^2+c^2-a^2}{2},$$ and analogously S_B and S_C.

These trilinears were found by Jean-Pierre Ehrmann (and I have verified them with a dynamic sketch).

In Clark Kimberling's ETC, the Nagel-Schröder point is X(1339).



6. The Stutensee point

One of my attempts to prove the existence of the Nagel-Schröder point Sn synthetically was an inversion argument like that I used for Sc. In fact, the inverses of the circles AIX', BIY' and CIZ' in the incircle of triangle ABC are the lines M_xX'', M_yY'' and M_zZ'', where M_x, M_y and M_z are the midpoints of the sides of triangle XYZ, and X", Y" and Z" are the inverses of X', Y' and Z' in the incircle of ABC. By inversion, the concurrence of the lines M_xX'', M_yY'' and M_zZ'' is equivalent to the coaxality of the circles AIX', BIY' and CIZ'.

I had no success with this method but it produced a new triangle center. Namely, the intersection of the lines M_xX'', M_yY'' and M_zZ'', or equivalently the inverse of the Nagel-Schröder point Sn in the incircle of triangle ABC, is a triangle center not in Kimberling's ETC; I call it the **Stutensee point** of triangle ABC. [My naming is voluntary and has no reasons.]

Thanks to Edward Brisse for trilinears of the Stutensee point.

Unfortunately, these trilinears are too long to be shown here.

7. The Mitten-Schröder points

A further variation of the Schröder point is not only replacing the Gergonne triangle by the Nagel triangle, but also the incenter I by the excenters I_a, I_b, I_c. In fact, in Hyacinthos message #6544, I showed:

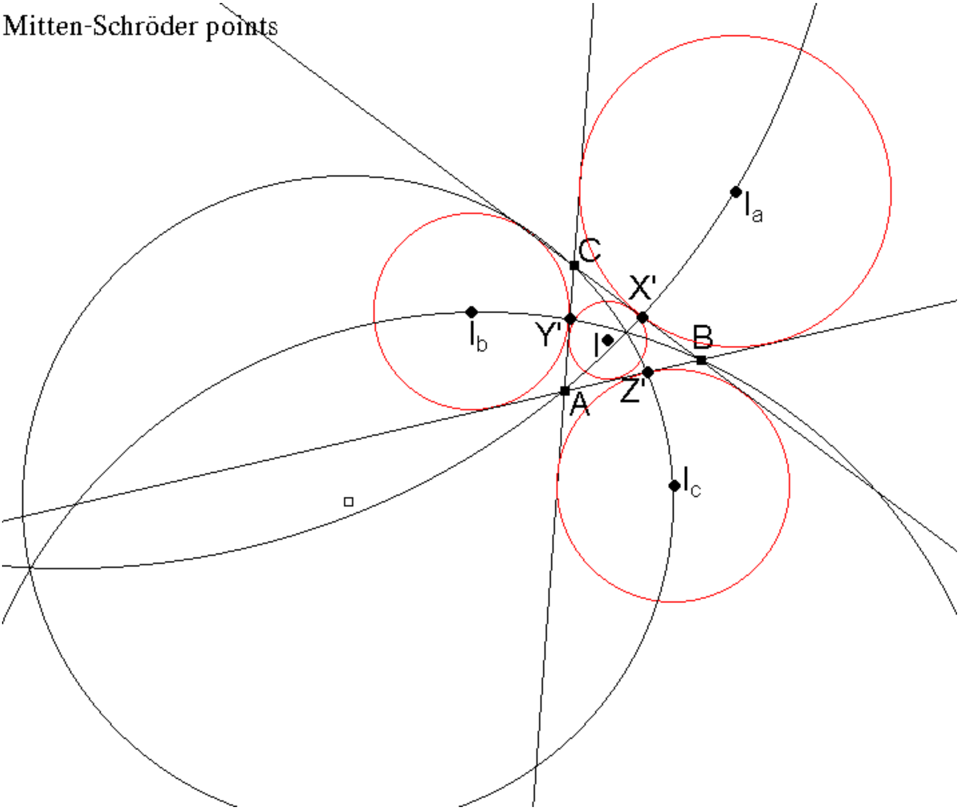
The circles AI_aX', BI_bY', CI_cZ' are coaxal. They intersect at two points. The common chord passes through I.

My proof was similar to Jean-Pierre Ehrmann's proof for the Schröder point Sc. The circle AI_aX' has diameter I_aA'', where A" is the intersection of the external angle bisector of A with BC, since the angles $I_aX'A''$ and I_aAA'' are both 90°. So we have to prove that the circles having diameters I_aA'', I_bB'', I_cC'' are coaxal, and that I lies on the radical axis. But from the Bodenmiller and Steiner theorems [see Hyacinthos #6124, §4, or the equivalent geometry-college messages], applied to the complete quadrilateral by the lines I_bI_cA'', I_cI_aB'', I_aI_bC'' and A"B"C", this circles are coaxal, and the orthocenter of triangle $I_aI_bI_c$, i. e. the incenter I of ABC, lies on the radical axis. Now it remains to prove that in our case, the circles have common points (in fact, there could be also the case that they don't have common points). But this is easy: the incenter I lies on the chords AI_a, BI_b, CI_c of each of these circles; therefore, it is an inner point of all three circles, and they must have common points.

One could think that this completely closes the problem. But Jean-Pierre Ehrmann added that the common chord of the three circles also passes through the symmedian point K of triangle ABC. I. e., the common chord is the line IK, and after a well-known theorem it also passes through the Mitten point of triangle ABC.

The two points at which the circles AI_aX', BI_bY', CI_cZ' intersect can be called the **Mitten-Schröder points** of triangle ABC. I don't know their trilinears.

I am also very interested in a synthetic proof that the common chord of the circles AI_aX', BI_bY', CI_cZ' is the line IK.



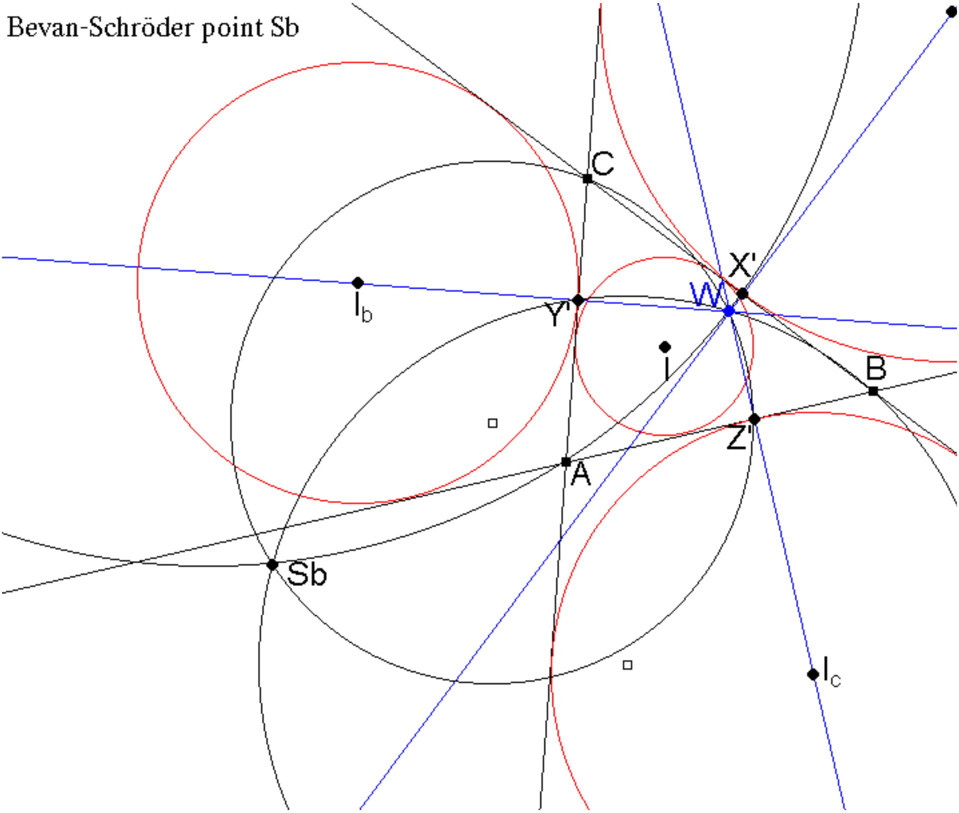
8. The Bevan-Schröder point

Now we are going to consider the last and most fruitful analogon of the Schröder point. The **Bevan point** W of triangle ABC is defined as the circumcenter of the excentral triangle $I_aI_bI_c$; it is known to be the reflection of the incenter I of ABC in the circumcenter O of ABC, and it is also the intersection of the lines I_aX', I_bY', I_cZ'. In fact, X', Y' and Z' are the pedals of the Bevan point W on BC, CA and AB.

The Bevan point is X(40) in Clark Kimberling's ETC.

Now I found that the circles AWX', BWY' and CWZ' are coaxal and intersect at two points. Their first intersection is W; the other intersection is a point which I call **Bevan-Schröder point** of triangle ABC and denote by Sb.

The common chord of the circles AWX', BWY' and CWZ' is the line OI; the Bevan-Schröder point Sb lies on OI.



9. Trilinears of the Bevan-Schröder point

Homogeneous trilinears for the Bevan-Schröder point Sb were found by Jean-Pierre Ehrmann:

$$\left(\frac{b+c-2a}{b+c-a} : \frac{c+a-2b}{c+a-b} : \frac{a+b-2c}{a+b-c} \right).$$

In Clark Kimberling's ETC, the Bevan-Schröder point Sb occurs as X(1319).

The isogonal conjugate of the Bevan-Schröder point Sb is the point X(1320); this point lies on the line NF, where N is the Nagel point and F the Feuerbach point of triangle ABC.

10. Some synthetic conjectures about the Bevan-Schröder point

Again, I am missing a synthetic proof of the existence of the Bevan-Schröder point. This time, we have two results about Sb which remind on similar results for Sc.

In fact, while the Schröder point Sc is the inverse of the centroid of triangle XYZ in the incircle of ABC, the Bevan-Schröder point Sb is the inverse of the orthocenter of triangle XYZ in the incircle of ABC. And while the Schröder point Sc is the inverse of X(55), the internal center of similtude of circumcircle and incircle, in the circumcircle of ABC, the Bevan-Schröder point Sb is the inverse of X(56), the external center of similtude of circumcircle and incircle, in the circumcircle of ABC.

The centroid of triangle XYZ is X(354) in Kimberling's ETC, while the orthocenter of triangle XYZ is X(65) in Kimberling's ETC. So we can state the two results on Sb as follows:

The Bevan-Schröder point Sb is the inverse of X(65) in the incircle of triangle ABC and the inverse of X(56) in the circumcircle of triangle ABC.

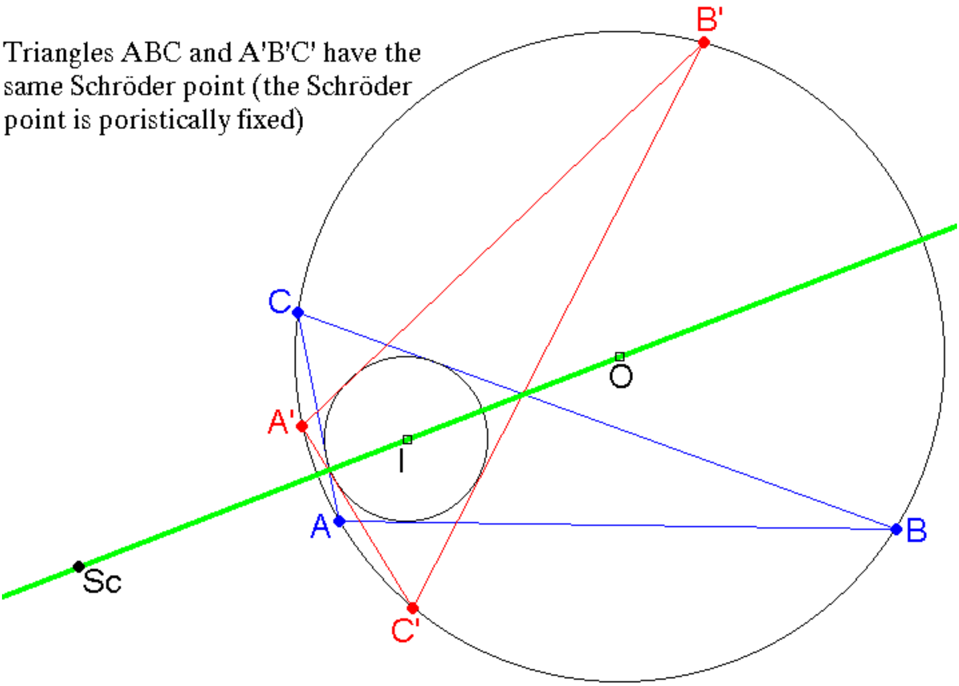
I can't prove either of these two facts. However, it is not hard to show that the inverse of X(65) in the incircle coincides with the inverse of X(56) in the circumcircle, i. e. it is sufficient to establish one of the two results.

Remark that X(56) is the isogonal conjugate of the Nagel point.

11. Poristically fixed points

A triangle center of a triangle is called **poristically fixed** if every triangle having the same circumcircle and the same incircle has the same corresponding triangle center. For example, the circumcenter is obviously poristically fixed. The incenter is also poristically fixed. The midpoint of OI is poristically fixed (where O is the circumcenter and I is the incenter). It can be also shown that the isogonal Mitten point (i. e. the isogonal conjugate of the Mitten point) is poristically fixed, the orthocenter X(65) of the Gergonne triangle XYZ is poristically fixed, and the centroid X(354) of the Gergonne triangle XYZ is poristically fixed.

The internal and external centers of similtude of the circumcircle and the incircle are poristically fixed. By inversion in the circumcircle, this yields that the Schröder point and the Bevan-Schröder point are poristically fixed (what means that all triangles which share the same circumcircle and the same incircle have the same Schröder point and the same Bevan-Schröder point).



The Nagel-Schröder point is not poristically fixed. The examples suggest the following theorem:

Fundamental Poristic Theorem.
All poristically fixed triangle centers lie on the line OI.

NOTE. This theorem must be understood with a slight restriction: For example, constructing equilateral triangles OX_1I and OX_2I on the segment OI yields two points X_1 and X_2, which are no triangle centers yet, but can be made triangle centers by defining X_1 as the point for which triangle OX_1I has the same orientation as the original triangle ABC and X_2 as the point for which triangle OX_2I has opposite orientation to triangle ABC. But the orientation of triangle ABC is not poristically fixed, i. e. the points X_1 and X_2 can be interchanged if we get over to another triangle with the same circumcircle and the same incircle.

A proof of the Fundamental Poristic Theorem was given by Barry Wolk (I cite Hyacinthos message #6895):

> If you reflect ABC about a line through O, the image triangle has the same circumcircle as ABC. Similarly, if you reflect ABC about a line through I, the image has the same incircle

as ABC. So the reflection of ABC about its OI line is poristic with ABC. This easily shows that any poristically fixed triangle center must lie on the line OI.

12. A generalization of the Schröder and Bevan-Schröder points leading to the Darboux cubic

Now we are going to treat two generalizations of the Schröder points and their variations.

The Schröder point is the second intersection of the circles AIX, BIY and CIZ, where I is the incenter of triangle ABC and XYZ is the pedal triangle of I.

The Bevan-Schröder point is the second intersection of the circles AWX', BWY' and CWZ', where W is the Bevan point of triangle ABC and X'Y'Z' is the pedal triangle of W.

This challenges the following generalizing question (Floor van Lamoen, Hyacinthos message #6321):

For which points P with pedal triangle $P_aP_bP_c$ do the circles APP_a, BPP_b and CPP_c have two common points?

In Hyacinthos message #6329, Floor van Lamoen proved that they have two common points if and only if P lies on the Darboux cubic of triangle ABC. I cite Floor van Lamoen (notations changed):

> The perpendicular bisectors of AP, BP, CP, P_aP, P_bP and P_cP are the homothetics through P with factor 1/2 of the sidelines of ABC and the prepedal triangle of P. The circumcenters are the intersections of the corresponding sides of these smaller triangles. So the circumcenters are collinear iff ABC and prepedal of P are lineperspective iff P lies on Darboux. Of course the second point of intersection of the three circles lies on the perspectrix of ABC and the prepedal triangle of P.

Two <u>notes</u>: The prepedal triangle of a point P (also called antipedal triangle) is the triangle whose the sides are the perpendiculars to AP, BP, CP at A, B, C. The idea of Floor van Lamoen is a generalization of the proof that Jean-Pierre Ehrmann gave for the Schröder point (see 3.).

Since the Darboux cubic is also the locus of all P whose pedal triangle $P_aP_bP_c$ is perspective with ABC, we can rewrite our result as follows:

The circles APP_a, BPP_b and CPP_c have two common points if and only if the lines AP_a, BP_b and CP_c concur.

Here are some points P on the Darboux cubic and the corresponding second intersections Q of the circles APP_a, BPP_b and CPP_c:

- If P is the incenter X(1) of triangle ABC, then Q is the Schröder point X(1155) and has trilinears

  $$((b-c)^2 + a(b+c-2a) : (c-a)^2 + b(c+a-2b) : (a-b)^2 + c(a+b-2c)).$$

- If P is the circumcenter X(3) of triangle ABC, then Q is the triangle center X(187), defined as the inversive image of the symmedian point in the circumcircle, and known as the **Schoute point** of triangle ABC. This point has trilinears

  $$(a(b^2+c^2-2a^2) : b(c^2+a^2-2b^2) : c(a^2+b^2-2c^2)),$$

  and lies on the Brocard axis and on the Lemoine axis of ABC.

- If P is the orthocenter X(4) of triangle ABC, then Q is the infinite point of the inversive plane.
  The circles APP_a, BPP_b and CPP_c degenerate to lines.

- If P is the Longchamps point X(20) of triangle ABC, then Q is X(468), having trilinears

  $$\left(\frac{b^2+c^2-2a^2}{b^2+c^2-a^2} : \frac{c^2+a^2-2b^2}{c^2+a^2-b^2} : \frac{a^2+b^2-2c^2}{a^2+b^2-c^2} \right).$$

  Isn't it a good idea to call X(468) the **Longchamps-Schröder point** of triangle ABC ?

- If P is the Bevan point X(40) of triangle ABC, then Q is the Bevan-Schröder point X(1319) and has trilinears

  $$\left(\frac{b+c-2a}{b+c-a} : \frac{c+a-2b}{c+a-b} : \frac{a+b-2c}{a+b-c} \right).$$

The trilinears of X(468) were found by Floor van Lamoen in Hyacinthos message #6352.

In Hyacinthos messages #6385 and #6390, Floor van Lamoen and Bernard Gibert discussed a variation of this generalized Schröder points.

13. A generalization of the Schröder and Nagel-Schröder points and the Feuerbach hyperbola

In Hyacinthos message #6764, I suggested another generalization of the Schröder point. While the previous generalization covered the Schröder and Bevan-Schröder point, this one contains the Schröder and Nagel-Schröder points. Here is how I introduced the latter generalization:

If ABC is a triangle with incenter I, and XYZ is the cevian triangle of the Gergonne point (called Gergonne triangle, intouch triangle, contact triangle), then the circles AIX, BIY and CIZ are coaxal. Their second point of concurrence is called the Schröder point or the 1st Schröder point of triangle ABC.

If X'Y'Z' is the cevian triangle of the Nagel point (called Nagel triangle, extouch triangle), then the circles AIX', BIY' and CIZ' are also coaxal. Their second point of concurrence is called the Nagel-Schröder point of triangle ABC.

Generalization: If P is a point with cevian triangle A'B'C', then I conjecture that the circles AIA', BIB', CIC' are coaxal if and only if P lies on the Feuerbach hyperbola of triangle ABC. By the way, what curve is drawn by the second points of concurrence of the circles?

In Hyacinthos message #6785, Barry Wolk verified my conjecture with slight adjustment: The circles are coaxal if and only if P lies on the the Feuerbach hyperbola or on the line at infinity.

Barry Wolk also generalized the generalization (I cite Hyacinthos message #6785, with little changes):

> I tried generalizing, from using the incenter I to using an arbitrary point Q. Given Q, find all P such that the circles AQA', BQB' and CQC' have collinear centers, where A'B'C' is the cevian triangle of P. The answer in general is a cubic, with a complicated equation. That cubic factors into (line at infinity)·(a conic), only when Q=I or Q=an excenter. And when Q=I the conic is indeed the Feuerbach hyperbola.
>
> A few other choices for Q didn't give anything interesting. When Q=G, the cubic is isotomic, with pivot ($a^4 + b^2c^2 - b^4 - c^4$: :) in barycentrics.

The latter pivot has a name: It is the so-called **Droussent pivot** of triangle ABC, the point X(316) in Kimberling's ETC.

Another nice choice for Q is the circumcenter of triangle ABC. Then the locus of P is the union of the Euler line and the circumcircle of triangle ABC, as Paul Yiu and Jean-Pierre Ehrmann found in some later Hyacinthos messages.

Darij Grinberg

[Schroeder3] Incentral Triangle Question
Darij Grinberg, 27 Feb 2004

I post this problem here, since it has to do with Schröder points.

Here are two of my Hyacinthos messages (slightly edited):

Hyacinthos message #6542

Subject: Incentral Triangle question
From: Darij Grinberg

Dear friends,

Let the internal bisectors of a triangle ABC intersect the opposite sides at A', B', C'. To prove that

the distance of I to B'C' is rR / OI_a and
B'C' orthogonal OI_a,

where O is the circumcenter of ABC, I is the incenter, I_a is the a-excenter, r is the inradius and R is the circumradius.

(The orthogonality relation I have proven, but how to manage the equation?)

Note that if we also take the points A", B", C" where the external bisectors intersect the opposite sides, we get the collinear triples of points A"B'C', A'B"C', A'B'C" and A"B"C", and then the lines A"B'C', A'B"C', A'B'C", A"B"C" are orthogonal to OI_a, OI_b, OI_c, OI respectively, where I_a, I_b, I_c are the excenters of triangle ABC and I is the incenter.

Darij Grinberg

Hyacinthos message #6627

Subject: Re: Incentral Triangle question
From: Darij Grinberg

In message #6542, I wrote:

>> Let the internal bisectors of a triangle ABC
>> intersect the opposite sides at A', B', C'.
>> To prove that
>>
>> the distance of I to B'C' is rR / OI_a and
>> B'C' orthogonal OI_a,
>>

>> where O is the circumcenter of ABC, I is the
>> incenter, I_a is the a-excenter, r is the
>> inradius and R is the circumradius.
>>
>> (The orthogonality relation I have proven, but
>> how to manage the equation?)
>>
>> Note that if we also take the points A", B", C"
>> where the external bisectors intersect the
>> opposite sides, we get the collinear triples
>> of points A"B'C', A'B"C', A'B'C" and A"B"C",
>> and then the lines A"B'C', A'B"C', A'B'C",
>> A"B"C" are orthogonal to OI_a, OI_b, OI_c, OI
>> respectively, where I_a, I_b, I_c are the
>> excenters of triangle ABC and I is the
>> incenter.

Here is a little idea: Call M the midpoint of I_bI_c, then OM = R. Let L_a be the pedal of I on B'C', and X be the pedal of I on BC, then IX = r. To prove that the distance of I to B'C' is rR / OI_a, we have to show that triangles L_aIX and MOI_a are similar. I have tested this by computer drawing, but still don't have a glimmer how to prove it. (Angles L_aIX and MOI_a are equal, but we need more.)

Darij Grinberg

Eric Danneels has found a trigonometric proof - see [Schroeder4].
Later, I found a synthetic proof - see [Schroeder6].

Darij Grinberg

[Schroeder4] Re: Incentral Triangle Question
Eric Danneels, 3 May 2003 (edited by Darij Grinberg)

Dear Darij,

in Forum Geometricorum Volume I pages 121-124 Lev Emelyanov and Tatiana Emelyanova proved that

$B'C' = abc\cdot\mathrm{sqrt}(R\cdot(R + 2\cdot r_a)) / [R\cdot(a + b)\cdot(a + c)]$

Application of the sinus law in triangles AIB' and AIC' leads to

$IB' = b\cdot c\cdot\sin(A/2) / [(a + c)\cdot\cos(C/2)]$
and

IC' = b·c·sin(A/2) / [(a + b)·cos(B/2)]

So the surface of triangle IB'C' becomes 1/2·IB'·IC'·cos(A/2)
with $\sin^2(A/2)$ = (s-b)·(s-c)/bc and $\cos^2(A/2)$ = s·(s-a)/bc etc...
this becomes S(IB'C') = a·b·c·r / [2·(a + b)·(a + c)]
and therefore IL_a = 2·S(IB'C') / B'C' = R·r / sqrt(R·(R + 2·r_a))
since sqrt(R·(R + 2·r_a)) = OI_a (Euler) we have

IL_a = R·r / OI_a

I hope this can be of some help

Kind regards

Eric Danneels

[Schroeder5] Poristically fixed points
Darij Grinberg, 25 Mar 2004

In [Schroeder2], 11. I have discussed "poristically fixed" points. With the theoretical assistance of Barry Wolk and computational help of Paul Yiu - many thanks - I have made a list of all triangle centers on the line OI showing which of them are poristically fixed.

See also Hyacinthos messages #6873, #6895, #6897 and #6900.

The following list contains all points on the line OI from X(1) up to X(2445).

Triangle centers X(i) in Clark Kimberling's ETC lying on OI, where I = X(1) and O = X(3).

X(1) = incenter; PORISTICALLY FIXED
X(3) = circumcenter; PORISTICALLY FIXED
X(35) = harmonical conjugate of X(36) with respect to OI;
 PORISTICALLY FIXED
X(36) = inverse of the incenter in the circumcircle;
 PORISTICALLY FIXED
X(40) = Bevan point = reflection of I in O; PORISTICALLY
 FIXED
X(46) = reflection of I in X(56); PORISTICALLY FIXED
X(55) = internal center of similtude of circumcircle and
 incircle; PORISTICALLY FIXED
X(56) = external center of similtude of circumcircle and
 incircle; PORISTICALLY FIXED
X(57) = isogonal Mitten point; PORISTICALLY FIXED

X(65) = orthocenter of intouch triangle; PORISTICALLY FIXED
X(165) = centroid of excentral triangle; PORISTICALLY FIXED
X(171) = isogonal conjugate of 1st Sharygin point;
NOT FIXED
X(241) = intersection of OI line and Gergonne axis;
NOT FIXED
X(260) = isogonal conjugate of 1st mid-arc point;
NOT FIXED
X(354) = Weill point; PORISTICALLY FIXED
X(484) = (1st) Evans perspector = reflection of I in X(36);
PORISTICALLY FIXED
X(517) = intersection of OI and line at infinity;
PORISTICALLY FIXED
X(559) ; a fissile and quartile (--> octile) point;
NOT FIXED
X(940) = intersection of OI and GK, where G centroid and
K symmedian point; NOT FIXED
X(942) = midpoint of I and X(65); PORISTICALLY FIXED
X(980) ; NOT FIXED
X(982) ; NOT FIXED
X(986) ; NOT FIXED
X(988) ; NOT FIXED
X(999) = midpoint of I and X(57); PORISTICALLY FIXED
X(1038) ; NOT FIXED
X(1040) ; NOT FIXED
X(1060) ; NOT FIXED
X(1062) ; NOT FIXED
X(1082) ; a fissile and quartile (--> octile) point;
NOT FIXED
X(1155) = Schröder point; PORISTICALLY FIXED
X(1159) = Greenhill point; PORISTICALLY FIXED
X(1214) ; NOT FIXED
X(1319) = Bevan-Schröder point; PORISTICALLY FIXED
X(1381) = 1st intercept of line OI and circumcircle;
PORISTICALLY FIXED
X(1382) = 2nd intercept of line OI and circumcircle;
PORISTICALLY FIXED
X(1385) = midpoint of OI; PORISTICALLY FIXED
X(1388) = midpoint of I and X(36); PORISTICALLY FIXED
X(1402) ; NOT FIXED
X(1403) ; NOT FIXED
X(1420) ; PORISTICALLY FIXED
X(1429) ; NOT FIXED
X(1454) ; PORISTICALLY FIXED
X(1460) ; NOT FIXED
X(1466) ; PORISTICALLY FIXED
X(1467) ; PORISTICALLY FIXED
X(1470) ; PORISTICALLY FIXED
X(1482) = reflection of O in I; PORISTICALLY FIXED
X(1617) ; PORISTICALLY FIXED
X(1622) ; NOT FIXED

X(1697) = internal center of similtude of incircle of
triangle ABC and circumcircle of excentral
triangle; PORISTICALLY FIXED
X(1715) ; NOT FIXED
X(1735) ; NOT FIXED
X(1754) ; NOT FIXED
X(1758) ; NOT FIXED
X(1771) ; NOT FIXED
X(1936) ; NOT FIXED
X(2061) ; NOT FIXED
X(2077) = reflection of X(36) in O; PORISTICALLY FIXED
X(2078) = inverse of X(57) in the circumcircle;
PORISTICALLY FIXED
X(2093) = reflection of I in X(57); PORISTICALLY FIXED
X(2095) = reflection of O in X(57); PORISTICALLY FIXED
X(2098) = reflection of X(56) in I; PORISTICALLY FIXED
X(2099) = reflection of X(55) in I; PORISTICALLY FIXED
X(2223) ; NOT FIXED
X(2283) ; NOT FIXED
X(2352) ; NOT FIXED

PS. Many thanks to Edward Brisse and to Paul Yiu for determining the poristic behaviour of many ETC centers.

Darij Grinberg

[Schroeder6] Re: Incentral Triangle Question
Darij Grinberg, 27 Feb 2004

In [Schroeder3], I wrote:

>> Let the internal bisectors of a triangle ABC intersect the opposite
>> sides at A', B', C'. To prove that
>>
>> the distance of I to B'C' is rR / OI_a and
>> B'C' orthogonal OI_a,
>>
>> where O is the circumcenter of ABC, I is the incenter, I_a is the
>> a-excenter, r is the inradius and R is the circumradius.

Here is a *synthetic proof*.

I am going to use directed angles modulo 180°. I will write "<" for "angle".

We will use the following lemmata:

Lemma 1. Let H be the orthocenter of a triangle ABC, and X, Y, Z be the feet of the altitudes from A, B, C. The sidelines YZ, ZX, XY of triangle XYZ intersect the sidelines BC, CA, AB of triangle ABC at the points O, P, Q, respectively. Then the points O, P, Q lie on one line - the so-called **orthic axis** of triangle ABC -, and this line is perpendicular to the Euler line of triangle ABC.

Proof. Let **o** be the circumcircle and **n** the nine-point circle of triangle ABC. The circumcircle **o** passes through A, B, C, and the nine-point circle **n** passes through X, Y, Z.

Since the points Z and X lie on the circle with diameter CA, we have PZ * PX = PC * PA, and hence the point P has equal powers with respect to the circles **n** and **o** (in fact, PZ * PX is the power of P with respect to **n**, and PC * PA is the power of P with respect to **o**). Hence, P lies on the radical axis of **n** and **o**. Similarly, O and Q also lie on this radical axis; altogether, the points O, P, Q lie on one line, namely the radical axis of **n** and **o**. Moreover, this line is perpendicular to the Euler line of triangle ABC, since the radical axis of two circles is perpendicular to the central line, while the centers of **n** and **o** lie on the Euler line of triangle ABC. Lemma 1 is proven.

Lemma 2. Let H be the orthocenter of a triangle ABC, let X and Y be the feet of the altitudes from A and B, and C' the midpoint of the side AB. Finally, call Q the intersection of the lines XY and AB. Then, the line HQ is perpendicular to the median CC'.

Proof (from: M. Volchkevich, *Reshenie zadachi M1724*, Kvant). Let N be the foot of the perpendicular from H to CQ. The points X, Y and N lying on the circle with diameter CH, we have < CNX = < CYX. The points X and Y lying on the circle with diameter AB, we get < ABX = < AYX, i. e. < ABX = < CYX = < CNX. In other words, < QBX = < QNX. Consequently, the points Q, N, X and B lie on a circle, and < NBQ = < NXQ. But, again, the circle with diameter CH passing through X, Y and N, we obtain < NXY = < NCY, and thus < NBA = < NBQ = < NXQ = < NCY = < NCA. Hence, the points N, B, A, C lie on one circle, i. e., the point N lies on the circumcircle of triangle ABC. Let the line NH meet the circumcircle again at the point R (apart from N). Since < RNC = 90°, the segment RC is a diameter of the circumcircle, so that < RAC = 90°, and RA is perpendicular to CA. Therefore, RA is parallel to the altitude BH. Similarly, RB || AH. Hence, the quadrilateral AHBR is a parallelogram, and the diagonal HR passes through the midpoint C' of the diagonal AB. In other words, the line C'H is perpendicular to CQ. On the other hand, the line CH is perpendicular to the line C'Q (since CH is perpendicular to AB). Hence, the point H lies on two altitudes of triangle CC'Q; consequently, it also lies on the third altitude, i. e. the line HQ is perpendicular to CC', qed..

Okay, now we go on to the proof.

Let the external angle bisector of A meet the sideline BC at A".

The excenters of triangle ABC will be called I_a, I_b, I_c. The points A, B, C are the feet of the altitudes of triangle $I_aI_bI_c$, and the point I is the orthocenter of this triangle. On the other hand, we may consider the triangle II_cI_b; the feet of the altitudes of this triangle are A, B, C, again, and the orthocenter of this triangle is I_a. Now, the sidelines BC, CA, AB of triangle ABC intersect the sidelines I_cI_b, I_bI, II_c of triangle II_cI_b in the points A", B', C', respectively; hence, after Lemma 1, the points A", B', C' lie on one line perpendicular to the Euler line of triangle II_cI_b. However, the Euler line of this triangle is the line OI_a (because O is the nine-point center and I_a is the orthocenter of this triangle); hence, the points A", B', C' lie on one line perpendicular to OI_a. This proves the second part of our problem.

Now to the first part. Let X be the orthogonal projection of I on the line BC, i. e. the point where the incircle of triangle ABC touches BC. Then, IX = r.

Since A, B, C are the feet of the altitudes of triangle $I_aI_bI_c$, the circumcircle of triangle ABC is the nine-point circle of triangle $I_aI_bI_c$; hence, it also passes through the midpoints of the sides of triangle $I_aI_bI_c$. For instance, it passes through the midpoint M of the side I_bI_c. Hence, OM = R.

As the points B and C lie on the circle with diameter I_bI_c (remember $\angle I_bBI_c = 90°$ and $\angle I_bCI_c = 90°$), the perpendicular bisector of BC passes through the center of this circle, i. e. through the midpoint M of I_bI_c. On the other hand, this perpendicular bisector obviously passes through the circumcenter O of triangle ABC. Consequently, MO is perpendicular to BC. Together with IX perpendicular to BC (this is trivial), we obtain MO || IX.

We have shown before that the line A"B'C' is perpendicular to OI_a. On the other hand, if L_a is the orthogonal projection of I on the line A"B'C', the line IL_a is perpendicular to the line A"B'C'. Hence, the lines IL_a and OI_a are parallel.

Since MO || IX and OI_a || IL_a, we have

$$\angle(IX; IL_a) = \angle(MO; OI_a) = -\angle(OI_a; MO),$$

i. e. $\angle XIL_a = -\angle I_aOM$.

Now apply Lemma 2 to the triangle $I_aI_bI_c$ with I as orthocenter, A and B as feet of the altitudes, and A" as intersection of BC with I_bI_c.

This yields that the line IA" is perpendicular to the median I_aM of triangle $I_aI_bI_c$.

The points X and L_a lie on the circle with diameter IA"; thus, $\angle IXL_a = \angle IA''L_a$. But $\angle IA''L_a = \angle (IA''; B'C')$. Now, on one hand, IA" is perpendicular to I_aM; on the other hand, B'C' is perpendicular to OI_a. Hence,

$$\begin{aligned} \angle IA''L_a &= \angle (IA''; B'C') = \angle (IA''; I_aM) + \angle (I_aM; OI_a) + \angle (OI_a; B'C') \\ &= 90° + \angle MI_aO + 90° = 180° + \angle MI_aO = \angle MI_aO = - \angle OI_aM. \end{aligned}$$

Therefore, $\angle IXL_a = - \angle OI_aM$. On the other hand, $\angle XIL_a = - \angle I_aOM$, as we have seen before. Hence, triangles IXL_a and OI_aM are oppositely similar, and

$$\frac{IL_a}{IX} = \frac{OM}{OI_a},$$

thus

$$IL_a = \frac{IX * OM}{OI_a} = \frac{rR}{OI_a}.$$

In other words, the distance of I to the line B'C' is rR / OI_a, proving the second assertion.

Sometimes geometry is nontrivial...

Darij Grinberg

[Schroeder7] Some newer results from MathLinks
Darij Grinberg, 4 Jul 2004

1. Treegoner's Theorem

In [Schroeder2], I wrote:

>> Let ABC be a triangle with the incenter I. [...]
>> Draw the points X', Y', Z' where the sides BC, CA, AB of triangle
>> ABC meet the respective excircles. [...] Now I conjectured that
>> the circles AIX', BIY' and CIZ' are coaxal and concur at two
>> points. The first point is I, and the second point, denoted by
>> Sn, is called the **Nagel-Schröder point** of triangle ABC.

Lately, the existence of the Nagel-Schröder point was proven elementarily by Treegoner on the MathLinks forum. See the thread

http://www.mathlinks.ro/viewtopic.php?p=22210

started by Treegoner. We begin with the following theorem he has discovered:

Theorem 1. Let ABC be a triangle, let A', B', C' be the midpoints of its sides BC, CA, AB, and let X, Y, Z be the feet of its altitudes. These altitudes AX, BY, CZ concur at the orthocenter H of triangle ABC. Finally, let (W) be the nine-point circle of triangle ABC, passing through the points A', B', C', X, Y, Z; and let X', Y', Z' be the second intersections of the lines A'H, B'H, C'H with (W). Then,

(a) The lines XX', YY', ZZ' meet at one point P, which lies on the Euler line of triangle ABC.
(b) The lines AX', BY', CZ' meet at one point Q.

Here are my *proofs* of **(a)** and **(b)**. Another proof of **(a)**, using inversion with respect to the polar circle of triangle ABC, was given by Grobber on MathLinks.

Let W be the center of the nine-point circle (W). Also, the midpoints A_1, B_1, C_1 of the segments AH, BH, CH lie on the nine-point circle.

The segments A_1A', B_1B', C_1C' are diameters of the nine-point circle (this is clear from $\measuredangle A_1XA' = 90°$, $\measuredangle B_1YB' = 90°$, $\measuredangle C_1ZC' = 90°$); hence, the center W of the nine-point circle must lie on these segments and bisect them, and the distances WA_1, WA', WB_1, WB', WC_1, WC' must all be equal to the radius of the nine-point circle. Hence, $WA' * WA_1 = WB' * WB_1 = WC' * WC_1$. In other words, the powers of the point W with respect to the circles HA_1A', HB_1B', HC_1C' are equal. But the point H must also have equal powers with respect to these three circles (since it actually lies on these circles). Hence, we have found two points - H and W - with equal powers with respect to the circles HA_1A', HB_1B', HC_1C'. Therefore, these circles are coaxal, and their common radical axis is the line HW, i. e. the Euler line of triangle ABC. Since our circles meet at H, they must also meet at another point R on the Euler line. We state this fact:

(c) The circles HA_1A', HB_1B', HC_1C' meet at H and at another point R on the Euler line of triangle ABC.

Another proof of **(c)** is obtained quickly by defining the point R as the point on the Euler line of triangle ABC satisfying the equation $WH * WR = -t^2$, where t is the radius of the nine-point circle of triangle ABC. Then, since the segments WA' and WA_1 both equal t but with opposite signs, we have $WH \cdot WR = -t^2 = WA' \cdot WA_1$, so that the point R lies on the circle HA_1A', and similarly R lies on the circles HB_1B' and HC_1C'. Thus we have not only proved **(c)** again, but also shown that

(d) We have $WH \cdot WR = -t^2$, where t is the radius of the nine-point circle of triangle ABC.

(In other words, if we invert the point H with respect to the nine-point circle of triangle ABC, and then reflect the resulting inversive image in W, then we get the point R.)

Now let the line ZZ' meet the Euler line at a point P. Then

$$\angle PZH = \angle Z'ZC_1 = \angle Z'C'C_1 = \angle HC'C_1 = \angle HRC_1 = \angle C_1RH.$$

Also, evidently, $\angle PHZ = \angle C_1HR$. Hence, the triangles PZH and C_1RH are similar, so that $HP : HZ = HC_1 : HR$, and $HP \cdot HR = HZ \cdot HC_1$. But $HZ \cdot HC_1 = p$, where p is the power of the point H with respect to the nine-point circle of triangle ABC. Hence, $HP \cdot HR = p$. This is a symmetric term, so the point P does not really depend on the choice of the line ZZ', but could be obtained for the lines XX' and YY' as well. Hence, the point P lies on the lines XX', YY', ZZ' and the Euler line of triangle ABC. This proves Theorem 1 **(a)**.

Now, in an arbitrary circle, the ratio of two chords equals the ratio of the sines of their chordal angles. Hence, in the nine-point circle,

$$\frac{XZ'}{Z'Y} = \frac{\sin \angle XZZ'}{\sin \angle Z'ZY}.$$

Similarly, we can find $YX' / X'Z$ and $ZY' / Y'X$, and see that

$$\frac{XZ'}{Z'Y} \cdot \frac{YX'}{X'Z} \cdot \frac{ZY'}{Y'X} = \frac{\sin \angle XZZ'}{\sin \angle Z'ZY} \cdot \frac{\sin \angle YXX'}{\sin \angle X'XZ} \cdot \frac{\sin \angle ZYY'}{\sin \angle Y'YX} = 1,$$

the latter following from the trigonometric version of Ceva's theorem, applied to the concurrent lines XX', YY', ZZ' in triangle XYZ. Now, using the sine law in triangles XCZ' and YCZ', we have

$$\frac{\sin \angle BCZ'}{\sin \angle Z'CA} = \frac{\sin \angle XCZ'}{\sin \angle Z'CY} = \frac{XZ' \cdot \sin \angle CXZ' : CZ'}{Z'Y \cdot \sin \angle CYZ' : CZ'}$$

$$= \frac{XZ'}{} \cdot \frac{\sin \angle CXZ'}{} = \frac{XZ'}{} \cdot \frac{\sin \angle A'XZ'}{} = \frac{XZ'}{} \cdot \frac{\sin \angle A'C'Z'}{}$$

Z'Y   sin < CYZ'   Z'Y   sin < B'YZ'   Z'Y   sin < B'C'Z'

$$= \frac{XZ'}{Z'Y} \cdot \frac{\sin \angle A'C'H}{\sin \angle B'C'H},$$

and similarly

$$\frac{\sin \angle CAX'}{\sin \angle X'AB} = \frac{YX'}{X'Z} \cdot \frac{\sin \angle B'A'H}{\sin \angle C'A'H};$$

$$\frac{\sin \angle ABY'}{\sin \angle Y'BC} = \frac{ZY'}{Y'X} \cdot \frac{\sin \angle C'B'H}{\sin \angle A'B'H}.$$

Hence,

$$\frac{\sin \angle BCZ'}{\sin \angle Z'CA} \cdot \frac{\sin \angle CAX'}{\sin \angle X'AB} \cdot \frac{\sin \angle ABY'}{\sin \angle Y'BC}$$

$$= \left(\frac{XZ'}{Z'Y} \cdot \frac{\sin \angle A'C'H}{\sin \angle B'C'H} \right) \cdot \left(\frac{YX'}{X'Z} \cdot \frac{\sin \angle B'A'H}{\sin \angle C'A'H} \right) \cdot \left(\frac{ZY'}{Y'X} \cdot \frac{\sin \angle C'B'H}{\sin \angle A'B'H} \right)$$

$$= \left(\frac{XZ'}{Z'Y} \cdot \frac{YX'}{X'Z} \cdot \frac{ZY'}{Y'X} \right) \cdot \left(\frac{\sin \angle A'C'H}{\sin \angle B'C'H} \cdot \frac{\sin \angle B'A'H}{\sin \angle C'A'H} \cdot \frac{\sin \angle C'B'H}{\sin \angle A'B'H} \right)$$

$$= \frac{\sin \angle A'C'H}{\sin \angle B'C'H} \cdot \frac{\sin \angle B'A'H}{\sin \angle C'A'H} \cdot \frac{\sin \angle C'B'H}{\sin \angle A'B'H}$$

$$\left(\text{since } \frac{XZ'}{Z'Y} \cdot \frac{YX'}{X'Z} \cdot \frac{ZY'}{Y'X} = 1 \right)$$

$$= 1;$$

this time, the last step was a consequence of the trigonometric version of Ceva's theorem applied to the concurrent lines A'H, B'H, C'H in triangle A'B'C'. Hence, the trigonometric version of Ceva's theorem, applied to triangle ABC, yields that the lines AX', BY', CZ' are concurrent, proving Theorem 1 **(b)**.

2. More circles

In the following, the **antipode** of a point P on a circle k will mean the point lying diametrically opposite to the point P on k.

Now consider the reflections A_0, B_0, C_0 of the point H in the points A', B', C'. Since A_0 is the reflection of H in A', the point A' is the midpoint of the segment HA_0. But remember that we have defined the point A' as the midpoint of the segment BC. Hence, the midpoints of the segments HA_0 and BC coincide, and the quadrilateral $BHCA_0$ is a parallelogram. Therefore, BA_0 || CH, or, in other words, BA_0 is perpendicular to AB, so that < ABA_0 = 90°. Similarly, < ACA_0 = 90°. Therefore, the points B and C lie on the circle with diameter AA_0. This is equivalent to saying that the point A_0 is the antipode of A on the circumcircle of triangle ABC. Similar facts will hold for B_0 and C_0, and thus we can summarize that the points A_0, B_0, C_0 are the antipodes of the points A, B, C on the circumcircle of triangle ABC.

A nearly trivial observation is that the circles AHA_0, BHB_0, CHC_0 concur at one point L (apart from H); this point L is the reflection of the inverse of H with respect to the circumcircle of triangle ABC in the circumcenter of triangle ABC.

Now we are going to prove:

Theorem 2.

(a) The circles HXA_0, HYB_0, HZC_0 concur at one point on the line HQ (apart from the point H).

(b) These circles pass through the orthogonal projections A_2, B_2, C_2 of the points A, B, C onto the lines YZ, ZX, XY, respectively.

Proof. Let S be the point on the line HQ satisfying the equation HQ * HS = 2p, where p is the power of the point H with respect to the nine-point circle of triangle ABC. (Of course, the segments are directed.)

We have p = HA_1 * HX, so that 2p = 2 HA_1 * HX = HA * HX (remember that A_1 is the midpoint of AH). Also, since the point A_0 is the reflection of H in A', we have HA_0 = 2 HA', so that 2p = 2 HX' * HA' = HX' * (2 HA') = HX' * HA_0. Altogether we see that

2p = HQ * HS = HA * HX = HX' * HA_0.

From HQ * HS = HX' * HA_0, we conclude HQ / HX' = HA_0 / HS; this, together with < QHX' = < A_0HS, shows that the triangles QHX' and A_0HS are similar. Therefore, < HQX' = < HA_0S.

On the other hand, from HQ * HS = HA * HX, it follows that HQ / HA = HX / HS, and this, combined with < QHA = < XHS, leads to the similarity of the triangles QHA and XHS. Thus, < HQA = < HXS. Therefore,

$\angle HA_0S = \angle HQX' = \angle HQA = \angle HXS$.

It follows that the point S lie on the circle HXA_0. Similarly, the same point S lies on the circles HYB_0 and HZC_0. This proves Theorem 2 **(a)**.

Now, we have to show that the orthogonal projection A_2 of the point A onto the line YZ lies on the circle HXA_0. Here is a proof of this:

We will work with directed angles modulo 180°.

The triangles AYZ and ABC are indirectly similar. In the triangle AYZ, the point A_2 is the foot of the A-altitude, and the point H is the antipode of A on the circumcircle of triangle AYZ. In the triangle ABC, the point X is the foot of the A-altitude, and the point A_0 is the antipode of A on the circumcircle of triangle ABC. Hence, the points A_2 and X are corresponding points in the triangles ABC and AYZ, and so are H and A_0. Since corresponding points in indirectly similar triangles form oppositely equal angles, we get $\angle A_2AH = -\angle XAA_0$ and $\angle AHA_2 = -\angle AA_0X$. At first, from $\angle A_2AH = -\angle XAA_0$, it follows that $\angle A_2AH = -\angle XAA_0 = \angle A_0AH$, so that the point A_2 lies on the line AA_0. Then, from $\angle AHA_2 = -\angle AA_0X$, we imply that $\angle XHA_2 = \angle AHA_2 = -\angle AA_0X = \angle XA_0A_2$, so that the points H, X, A_2 and A_0 lie on one circle, i. e. the point A_2 lies on the circle HXA_0.

Similarly, we can show that the points B_2 and C_2 lie on the circles HYB_0 and HZC_0, respectively. This finally proves Theorem 2 **(b)**.

3. The Nagel-Schröder point

Theorem 2 **(a)** states that the circles HXA_0, HYB_0, HZC_0 concur at one point (apart from H). But after Theorem 2 **(b)**, these circles coincide with the circles HXA_2, HYB_2, HZC_2, respectively. Therefore, we can state:

Corollary 3. The circles HXA_2, HYB_2, HZC_2 concur at one point, apart from H.

If we now consider a triangle ABC with its incenter I and its excenters I_a, I_b, I_c, then it is well-known that the points A, B, C are the feet of the altitudes of triangle $I_aI_bI_c$, and the point I is the orthocenter of triangle $I_aI_bI_c$. The orthogonal projections of the points I_a, I_b, I_c onto the lines BC, CA, AB are the points X', Y', Z' where the A-, B-, C-excircles of triangle ABC touch the sides BC, CA, AB. Hence, from Corollary 3, the circles IAX', IBY', ICZ' concur at one point, apart from H. The existence of the Nagel-Schröder point is proven!

4. ETC reference

Here are the numbers under which the points defined above occur in Clark Kimberling's ETC:

P = triangle center X(235) = the midpoint between the orthocenter H = X(4) and the triangle center X(24).
X(24) = a very interesting triangle center. Here are three constructions of this point:

- If you take the orthic triangle XYZ of triangle ABC, and the orthic triangle $X_1Y_1Z_1$ of triangle XYZ, then the lines AX_1, BY_1, CZ_1 concur at X(24).
- If you take the inverses X_v, Y_v, Z_v, of the points X, Y, Z in the circumcircle of triangle ABC, then the lines AX_v, BY_v, CZ_v concur at X(24).
- If you call O the circumcenter of triangle ABC, and X_2, Y_2, Z_2 the circumcenters of triangles BOC, COA, AOB, then the lines XX_2, YY_2, ZZ_2 concur at X(24).

Q : not in the ETC.
R : not in the ETC, but:
R = the reflection of X(403) in X(5).
X(403) = the inverse of the orthocenter H in the nine-point circle of triangle ABC. Two properties of X(403):

- If O is the circumcenter of triangle ABC, then the circles AOX, BOY, COZ pass through X(403).
- The point X(403) is the inverse of X(24) in the circumcircle of triangle ABC.

L : not in the ETC, but:
L = the reflection of X(186) in X(3).
X(186) = "threefold angle point" = the inverse of H with respect to the circumcircle of triangle ABC.
S : not in the ETC.

5. Alternative proof of Theorem 1 (a) by Treegoner

At first, we define the notion of *circumcevian triangles*:

If ABC is a triangle, and P is a point, then the **circumcevian triangle** of the point P with respect to the triangle ABC is the triangle A'B'C', where A', B', C' are the points of intersection of the lines AP, BP, CP with the cirucmcircle of triangle ABC (different from A, B, C, of course).

An alternative proof of Theorem 1 **(a)** found by Treegoner used the following lemma:

Lemma 4. Let M and N be two points in the plane of a triangle ABC. Let $A_1B_1C_1$ be the circumcevian triangle of the point M with respect to triangle ABC. Let $A_3B_3C_3$ be the circumcevian triangle of the point N with respect to triangle $A_1B_1C_1$. Let $A_4B_4C_4$ be the circumcevian triangle of the point M with respect to triangle $A_3B_3C_3$. Then the lines AA_4, BB_4, CC_4 are concurrent at a point which lies on the line MN.

Now apply this lemma to triangle XYZ with its M = H and N = W, and you get a new proof of Theorem 1 **(a)**.

The *proof of Lemma 4* is very simple:

In the following, I will denote by g /\ h the point of intersection of two given lines g and h.

After Pascal's theorem, applied to the cyclic hexagon $A_1A_3BB_1B_3A$, the points A_1A_3 /\ B_1B_3 (= N), A_3B /\ B_3A and BB_1 /\ AA_1 (= M) lie on one line. Hence, the point A_3B /\ B_3A lies on the line MN. On the other hand, after Pascal's theorem, applied to the cyclic hexagon $A_4A_3BB_4B_3A$, the points A_4A_3 /\ B_4B_3 (= M), A_3B /\ B_3A (lying on MN) and BB_4 /\ AA_4 are collinear. Hence, the point BB_4 /\ AA_4 lies on the line MN, i. e. the lines AA_4, BB_4 and MN concur. Similarly, the lines BB_4, CC_4 and MN concur. Hence, the four lines AA_4, BB_4 and CC_4 concur at one point, and Lemma 4 is proven.

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