FROM

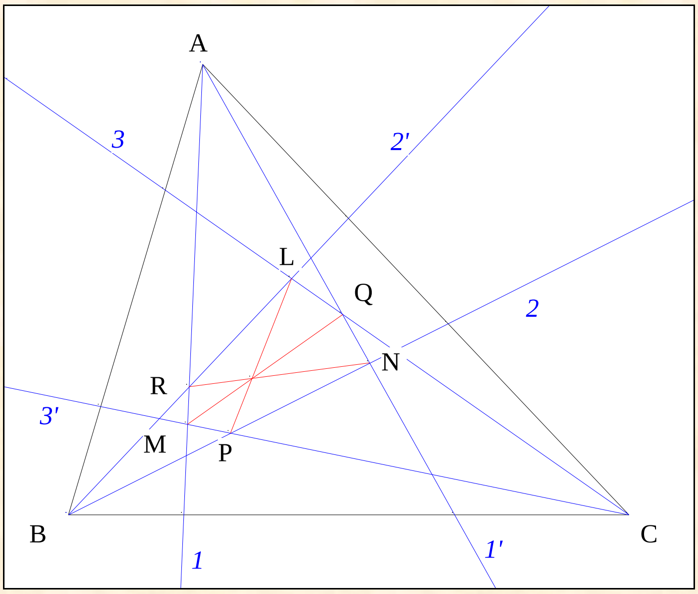
ROBSON TECHNIQUE

TO

MORLEY's TRISECTOR THEOREM

†

Jean - Louis AYME



Abstract. The author presents the forgotten synthetic proof of Alan Robson concerning the Morley's trisector theorem. Short biographies, the digital journal *Revistaoim* and two archives are given.
The figures are all in general position and all the theorems quoted can be proved synthetically.

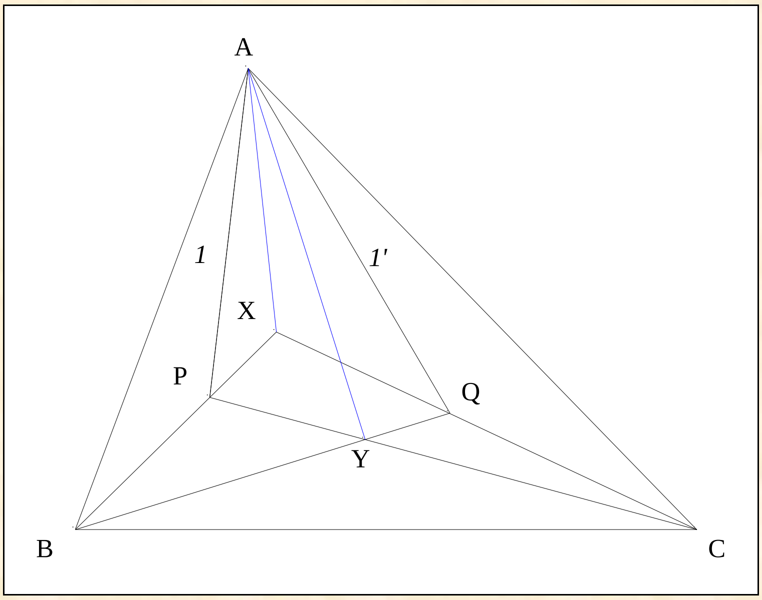
Summary

A. ROBSON TECHNIQUE

1. Two isogonal lines from Darij Grinberg

VISION

Figure :

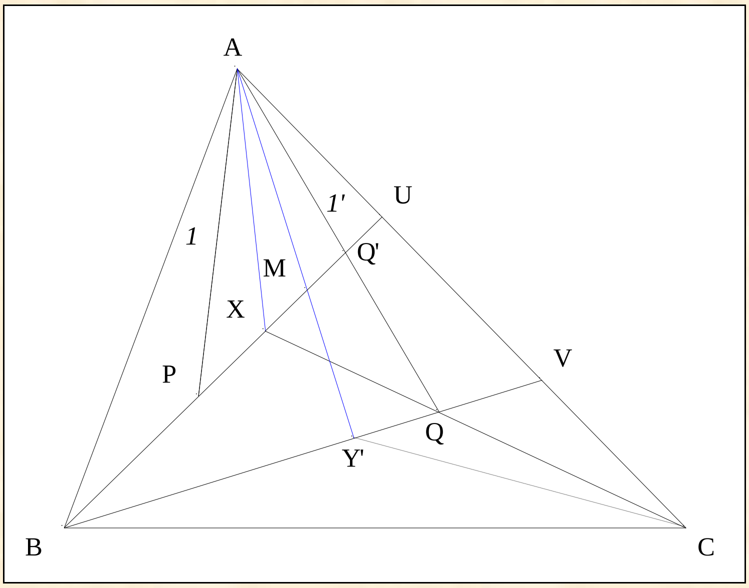


Features : ABC a triangle,
1, 1' two A-isogonal lines of ABC,

	P, Q	two points resp. on *1*, *1'*
and	X, Y	the points of intersection resp. of BP and CQ, BQ and CP.

Given : AX and AY are two A-isogonal lines of ABC.[1]

VISUALIZATION



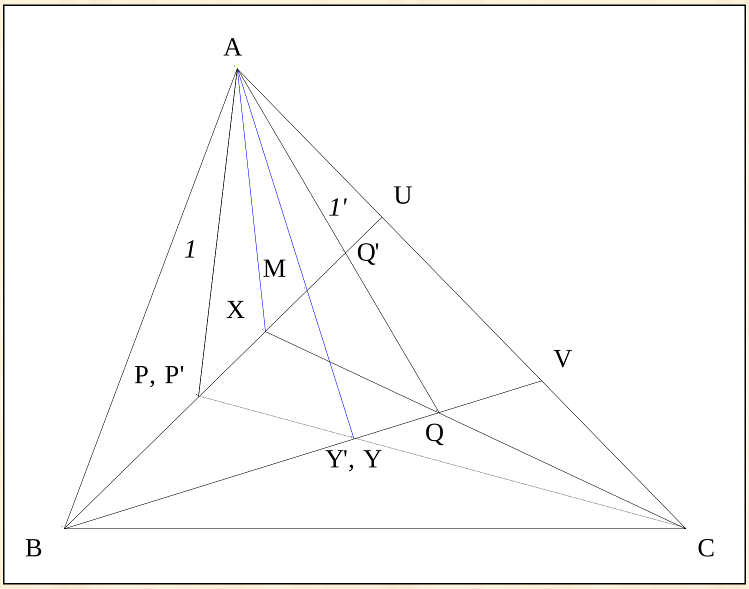
- Note Q', U, V the points of intersection resp. of AQ and BP, BP and AC, BQ and AC,
  M the point on BP so that AX and AM are two A-isogonal lines of ABC
  and Y' the point of intersection of BQ and AM.

- We consider the cross-ration (BXPU) ;
  by symmetry wrt the A-bissector of ABC, (BXPU) = (UMQ'B) ;
  by A-perpectivity, (UMQ'B) = (VY'QB).

- By definition of the equality of two pencils, (C ; VY'QB) = (A ; VY'QB) ;
  or by A-perpectivity, (A ; VY'QB) = (A ; UMQ'B) ;

  by symmetry wrt the A-bissector of ABC, (A ; UMQ'B) = (A ; BXPU) ;
  by permutation (Cf. Appendix)[2], (A ; BXPU) = (A ; XBUP) ;
  by permutation (Cf. Appendix)[3], (A ; XBUP) = (A ; UPXB) ;
  by transitivity of the relation =, (C ; VY'QB) = (A ; UPXB).

[1] Isogonal conjugates - an exercise, Message *Hyacinthos* # 9835 du 01/06/2004 ; http://tech.groups.yahoo.com/group/Hyacinthos/.
[2] The quaterne does not change if it swaps at the same time the first with the second point, the third with the fourth.
[3] The quaterne does not change if you swap the first two points with the last two.



- Note P' the point of intersection of CY' and AP.

- The two last pencils having the ray CV (= AU) in common,
  the points of intersection of CY' and AP, CQ and AX, CB and AB are collinear
  i.e. P', X and B are collinear ;

  consequently, **(1)** P' and P are identic
  (2) Y' and Y are identic.

- **Conclusion :** AX and AY are two A-isogonal lines of ABC.

Historic note : a particular case of this problem where P is the orthocenter and Q the center of the circumcircle of ABC has been proposed in 2004

by Arend Bayer in the IMO training at

Mathematical Research Institute of Oberwolfach (Bade-Württemberg, Germany) in May 2004.

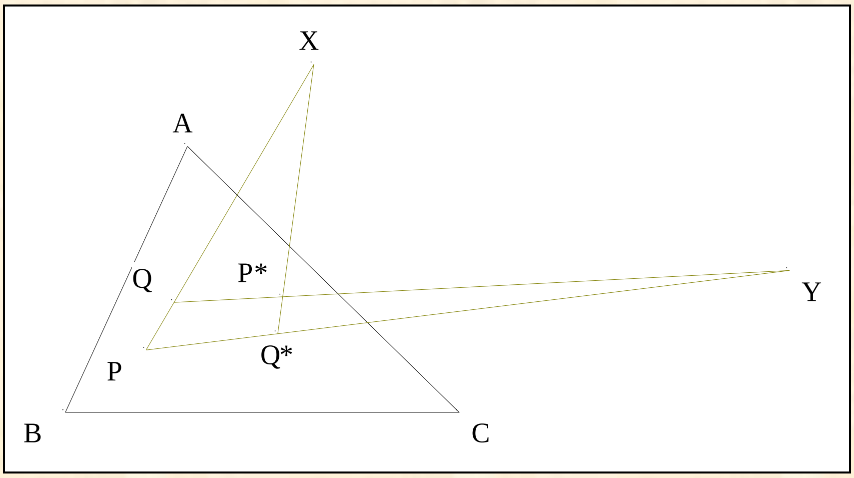
Darij Grinberg who has participated to this seminar, proposed two proofs for this nice problem, one metric and the other based on the Desargues involution theorem. The generalization presented above is from Darij Griberg [4].

2. Hesse's theorem

VISION

Figure :

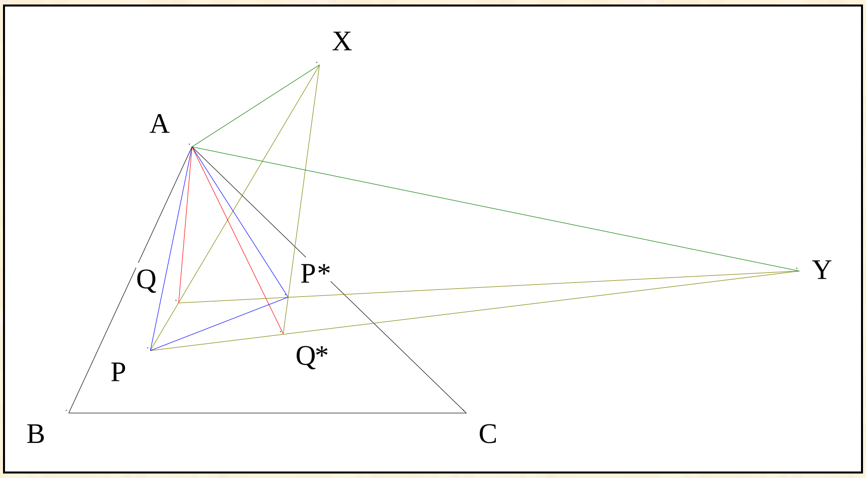
[4] Isogonal conjugates - an exercise, Message *Hyacinthos* # 9835 du 01/06/2004 ; http://tech.groups.yahoo.com/group/Hyacinthos/.



Features : ABC a triangle,
P a point,
P* the isogonal of P wrt ABC,
Q a point,
Q* the isogonal of Q wrt ABC
and X, Y the points of intersection resp. of PQ and P*Q*, PQ* and P*Q.

Given : X and Y are two isogonal points of ABC.[5]

VISUALIZATION



- **Remarks :** **(1)** AP and AP* are two A-isogonal lines of ABC
  (2) AQ and AQ* are two A-isogonal lines of ABC.

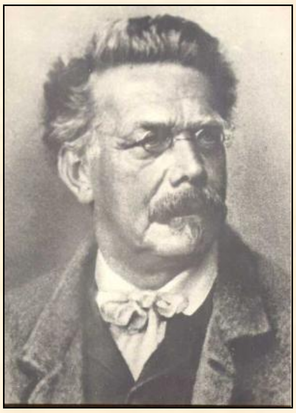
- According to **A. 1.** Two isogonal lines from Darij Grinberg,
  applied to the triangle APP* with the points Q and Q*, AX and AY are two A-isogonal lines of APP* ;
  consequently, AX and AY are two A-isogonal lines of ABC.

- Mutatis mutandis, we would prove that BX and BY are two B-isogonal lines of ABC
  CX and CY are two A-isogonal lines of ABC.

- **Conclusion :** X and Y are two isogonal points of ABC.

[5] Hesse L. O., *Journal de Crelle* vol. **20** (1840) ; http://gdz.sub.uni-goettingen.de/no_cache/dms/load/toc/?IDDOC=238618.

3. A short biography of Ludwig Otto Hesse

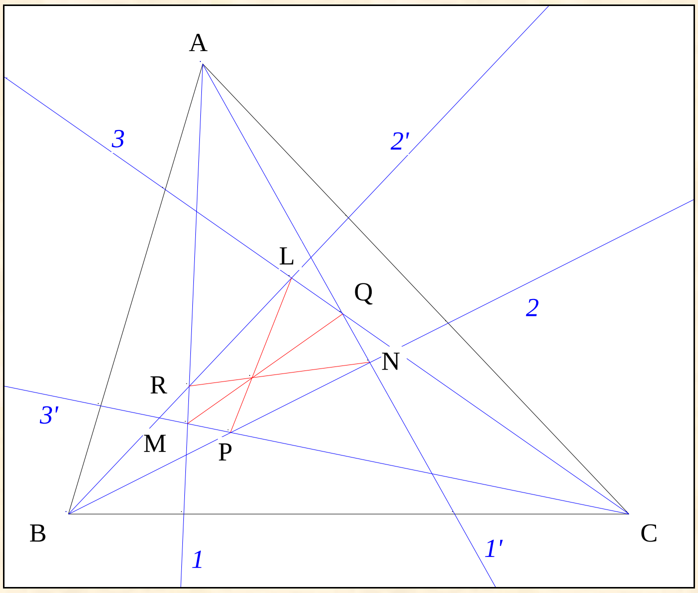


Ludwig Otto Hesse was born in Konigsberg (Prussia, now Kaliningrad, Russia), April 22, 1811. Student, then teacher in a school of Konigsberg, he became professor at the University of that city in 1845. From 1856 to 1868, he professes at the University of Heidelberg before teaching at Polytechnic School in Munich. From 1868, he became member of the Bavarian Academy of sciences.
He died 4 August 1874 in Munich (Germany).

3. Robson technique to prove that three diagonals are concurrent

VISION

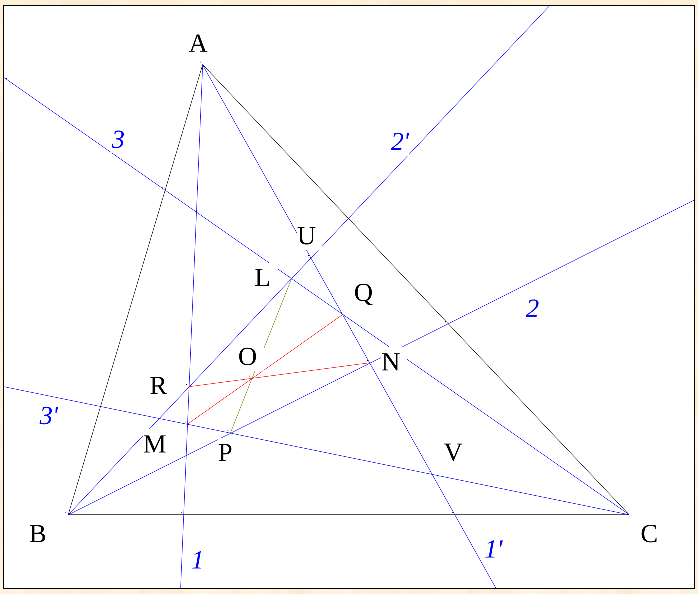
Figure :



Features :	ABC	a triangle,
	1, *1'*	two A-isogonal lines of ABC,
	2, *2'*	two B-isogonal lines of ABC,
	3, *3'*	two C-isogonal lines of ABC,
	P, Q, R	the points of intersection resp. of *2* and *3'*, *3* and *1'*, *1* and *2'*,
and	L, M, N	the points of intersection resp. of *2'* and *3*, *3'* and *1*, *1'* and *2*.

Given : PL, QM and RN are concurrent.[6]

VISUALIZATION



- Note U, V, O the points of intersection resp. of BR and AQ, AQ and CP, QM and RN.

- **Remarks :** **(1)** BP and BR are two B-isogonal lines of ABC
  (2) CP and CQ are two C-isogonal lines of ABC.

- According to **A. 2.** Hesse's theorem, AP and AL are two A-isogonal lines of ABC.

- By symmetry wrt the A-bissector of ABC, (A ; BLRU) = (A ; CPVM).

- By change of origin, (N ; BLRU) = (Q ; CPVM) ;
  by swapping the first two points, and the last two points, (Q ; CPVM) = (Q ; PCMV) ;
  by transitivity of the relation =, (N ; BLRU) = (Q ; PCMV).

- **Remark :** these three pencils have the ray NUQV in common.

- According "A ray in common" (Cf. Appendix), P, L and O are collinear.

- **Conclusion :** PL, QM and RN are concurrent.

Remark : LRMPNQL is "a Robson's hexagon".

Historic note : this technique above has been initiated by Alan Robson to demonstrate the Morley's trisectors theorem.
The author met this technique in the digital Journal *Revista Escolar Olimpiada Iberoamericana de Matematica*[7] (REOIM) headed by Professor Francisco Bellot Rosado by reading the article by Professor Juan Manuel Code (Alicante, Spain) regarding "el Morley theorema".

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6 Robson A., *The Mathematical Gazette* **11** (1922-23) 310-311.
7 Code J. M., *Revistaoim* **14** (Julio-Agosto) ; http://www.oei.es/oim/revistaoim/index.html.

3. Revistaoim

[8]

This magazine school mathematics digital is promoted by the Professor Francisco Bellot Rosado.


[9]

The idea of a school journal in Spanish date 1987 during the regional days of Castile - Leon of Didactics of mathematics where Professor Bellot presents a paper entitled "A necessity: a journal of mathematics".
Professor Bellot noticed that most countries with strong tradition and good results in International Math Olympics have excellent Journals School of mathematics. It notes also that in the Iberoamerican area, the situation is different with the exception of Brazil with his Eureka magazine that is available on paper or on the Internet, the Argentina and the Mexico.
Ten years later, he became editor of SIPROMA, a paper and ephemeral review for the advancement of mathematics, published under the auspices of the Organization of the Iberoamerican States (O.E.I.). In April 2002, he sent a draft to the O.I.E. concerning a publication exclusively digital and free, written in Portuguese and Spanish (the two official languages of the O.E.I.).
The first issue of the REIOM hosted by the site of O.E.I. appears in May 2002.
Today, this magazine whose logo was chosen by O.I.E. has more than 30000 readers.

4. A very short biography of Alan Robson

[8] http://www.oei.es/oim/revistaoim/index.html.
[9] Bellot Rosado F., Congrès THALES, Cordoue (Espagne) 2010.

We know as little about the renowned mathematics Professor Alan Robson taught at Marlborough college (Wiltshire, England) founded in 1843.
In 1939 he published a book titled *Advanced Trigonometry* in collaboration with his former pupil Clément Vavasor Durell (06/06/1882-12/10/1968).
In 1939, he is President of *The Mathematical Gazette Magazine* to devote an obituary note written by Durell.
He die in 1956.

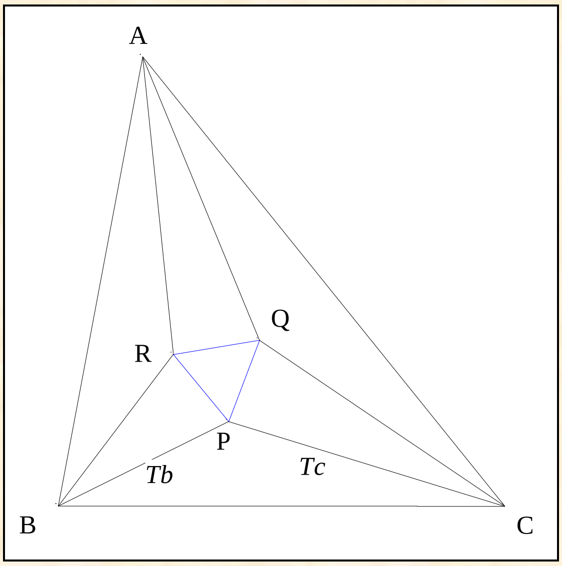
B. TWO NICE APPLICATIONS

I. MORLEY's TRISECTOR THEOREM

1. The problem

VISION

Figure :

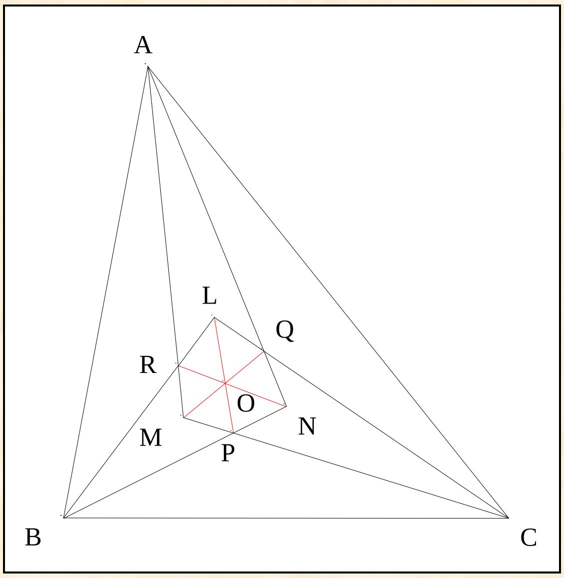


Features :	ABC	a triangle,
	Tb, *Tc*	the adjacent trisectors of <B, <C,
	P	the point of intersection of *Tb* and *Tc*,
and	Q, R	the corresponding points obtained in an analogous manner.

Given : the triangle PQR is equilateral.

VISUALIZATION [10]

[10] Robson A., *The Mathematical Gazette* **11** (1922-23) 310-311.

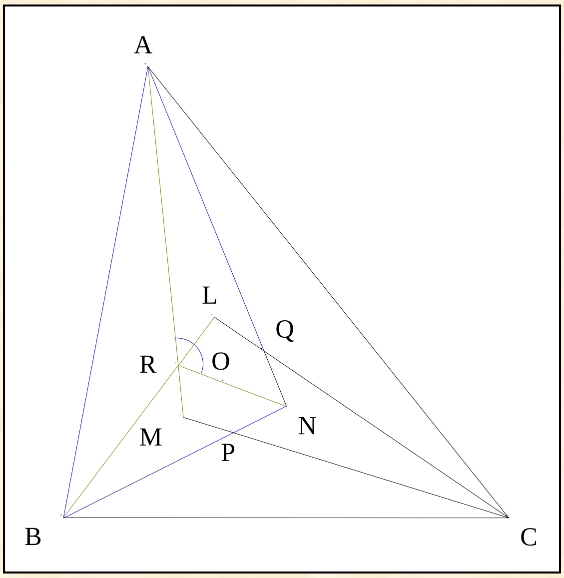


- Start again with the Robson's technique.

- Note L, N, M the points of intersection of BR and CQ, AQ and BP, CP and AR.

- **Partial conclusion :** according to **A. 3.** Robson technique to prove that three diagonals are concurrent, PL, QM and RN are concurrent.

- Note O this point of concurs.

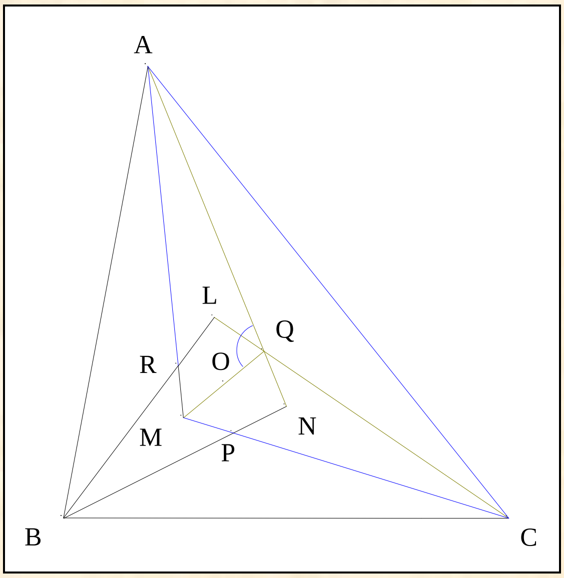


- **Comment :** now begin a geometric angle chasing.

- **Remark :** R is the incenter of the triangle ANB.

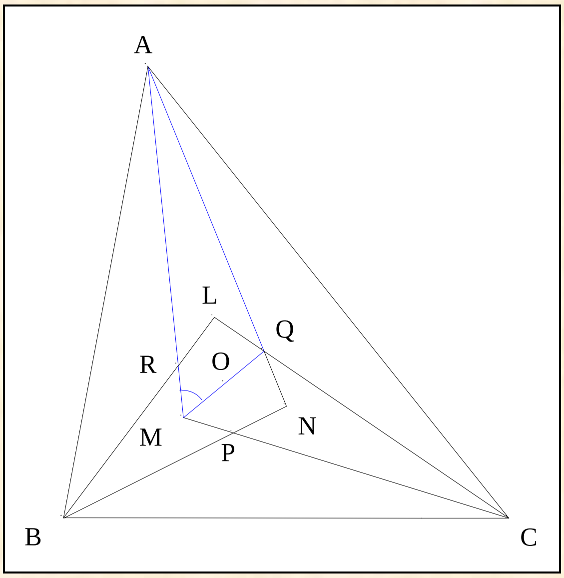
- According to "The angle I" (Cf. Annex 1) wrt B, <ARN = 90° + 1/2.<ABN ;
  by hypothesis, <ABN= 2/3.<B ;
  by substitution, <ARN = 90° + 1/3.<B .

- **Partial conclusion :** <ARO = 90° + 1/3.<B .



- **Remark :** Q is the incenter of the triangle AMC.

- According to "The angle I" (Cf. Annex 1) wrt C,
  we would prove <AQM = 90° + 1/2.<C .



- According to the theorem "Sum of the angles of a triangle"
  applied to the triangle AMQ

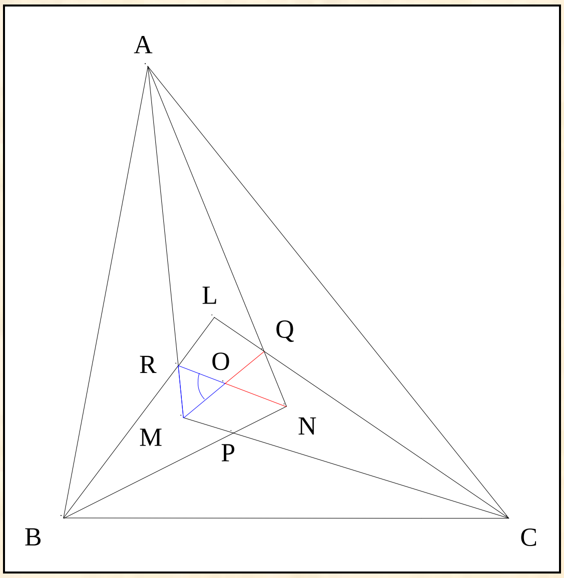
  <QMA = 180° - 1/3.<A - <AQM ;

  by substitution, <QMA = 180° - 1/3.<A - (90°+1/3.<C) ;

  i.e. <QMA = 90° - 1/3.<A – 1/3.<C .

- **Remark :** <QMA = <OMR.

- **Partial conclusion :** <OMR = 90° - 1/3.<A – 1/3.<C .

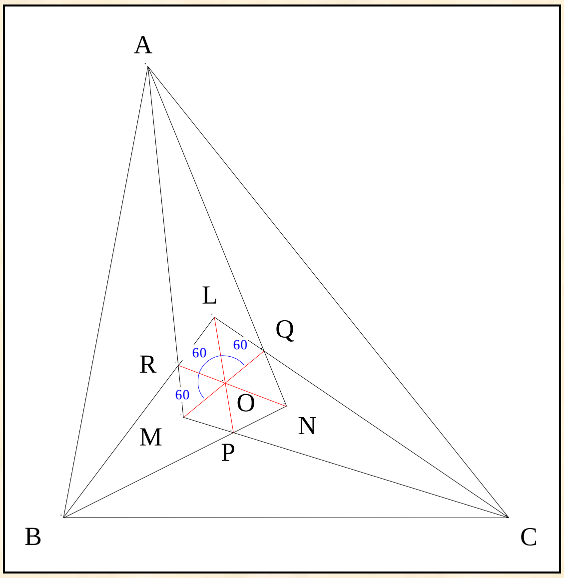


- According to the theorem "Sum of the angles of a triangle" applied to the triangle OMR

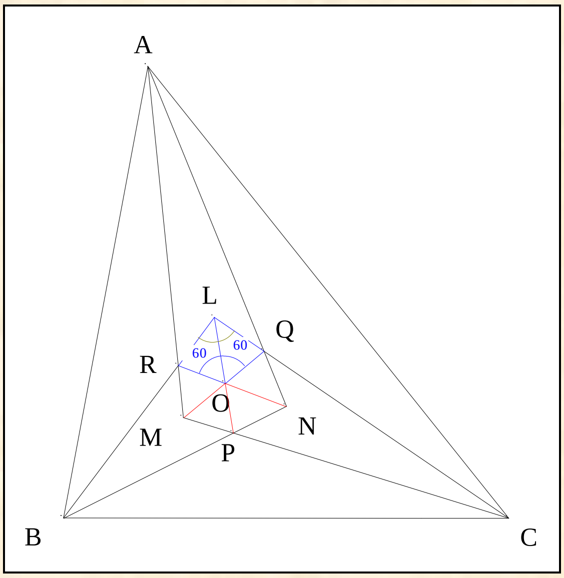
  <ROM = <ARO - <OMR ;

  by substitution, <ROM = [90° + 1/3.<B] – [90° - 1/3.<A – 1/3.<C]

  by reduction, <ROM = 60°.



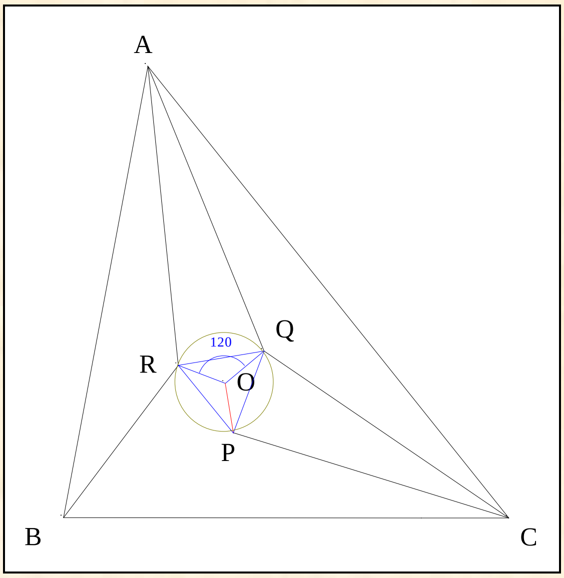
- Mutatis mutandis, we would prove <LOR = 60° and <LOQ = 60°.



- **Remarks :** **(1)** P is the incenter of the triangle BLC
  (2) <PLB = <PLC or <OLR = OLQ.

- According to "a.s.a. theorem"[11] applied to the triangles LOQ and LOR, OQ = OR.

- Mutatis mutandis, we would prove OR = OP.



- Consequently, O is the center of the circumcircle of PQR.

- According to the "Chordal angle theorem", <QPR = 60°.

- Mutatis mutandis, we would prove <RQP = 60° and <PRQ = 60°.

- **Conclusion :** the triangle PQR is equilateral.

Theorem : the points of intersection of the adjacent trisectors of the angles of any triangle
are
the vertices of an equilateral triangle.

[11] a.s.a. means angle-side-angle.

Remarks : **(1)** PQR is "the Morley's triangle of ABC"

(2) Pierre Laurent Wantzel[12] demonstrated in 1836 that a trissector cannot be constructed using a ruler and a compass.

Historic note :

This ingenious theorem that today bears his name has been found by chance after many figures in 1904 when Frank Morley worked from 1899 on the centers of tangent cardioids on three sides of a triangle. Frank Morley who had not demonstrated this result spoke at Richmond of Cambridge and Wittaker of Edinburgh which spread this discovery in 1904 in the world as a research topic. It is E. J. Ebden who first formatted and spread it in 1908 in the *Educational Times* without reference to Morley. Two solutions were sent the year after; the first trigonometric of Satyanarayana[13] and the second by Delahaye and Lez in *Mathesis*[14]. The next year, the Indian M.T. Naraniengar[15] proposed a geometric solution which will be followed by a dozen other later including W. E. Philip in 1914, Raoul Bricard[16] in 1922, and J. M. Child[17] in the same year. Frank Morley[18] published finally in 1924 in Japan his heavy proof which involved a cardioid, then an article in 1929 in the *American Journal of Mathematics*[19].
Remember that the Naraniengar solution was added a translation of a card sent to Morley at the Japanese Professor T. Hayasi asking him to publish his result as well as a commentary by Leon Bankoff concerning already known proofs.

Comment : Alan Robson's proof is the shortest among all those who have been proposed.
It should be noted that these of Henri Lebesgue[20] approximates that of Robson.
Remember that a small book of André Viricel entitled "Morley theorem" published in 1993 by the *Association for the development of Culture Science* (A.D.C.S.), presents

* the elementary proofs of E. Ehrhart[21] with its triaxes, R. Sasportès, Niewenglowski, Raoul Bricard[22], Claude Frasnay with the concept of tripod, F. Glanville, F. G. Taylor and W. L. Muir[23], Mirimanoff[24]

* the analytic proof of André Viricel

* the trigonometric proof of Jacques Bouteloup, Commeau, D. J. Newman en 1996

* the complex proof of J. Hoffmann.

Finally, remind these of A. H. Holmes, B. Gambier and Alain Connes en 1998[25].

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12 Wantzel P. L. (Paris 05/06/1814-Paris 21/05/1848),
Recherche sur les moyens de reconnaître si un problème de géométrie peut se résoudre avec la règle et le compas,
Journal de Mathématiques Pures et Appliquées **1** (2) (1837) 366-372.

13 Satyanarayana M., Solution to problem n° 16381, *The Educational Times* New Series, **61** (July 1908) 308.

14 Delahaye T., Lez H., *Mathesis*, problem n° 1655, 3-ième Série, **8** (1908) 138-139.

15 Naraniengar M. T., *Mathematical Questions and Solutions* from *The Educational Times*, New Series **15** (1909) 47.
It will be rediscovered by J. M. Child en 1922.

16 Bricard R. (1870-1944), has been engineer at Dijon, then Repeater at École Polytechnique, and editor of the *Nouvelles Annales* in 1903 ; he signed a few articles by R. B..

17 Child J. M., A proof of Morley's theorem, *Mathematical Gazette* (1922) 171.

18 Morley F., *Mathematical Association of Japan for Secondary Mathematics*, vol. **6** (December 1924) 260-262.

19 Morley F., *American Journal of Mathematics*, **51** (1929) 465-472.

20 Lebesgues H., *L'Enseignement Mathématique* **38** (1940) 29.

21 Ehrhart E., Le triangle Orienté, *Mathesis* (fév.-avr. 1951).

22 BricarD R., *Nouvelles Annales de Mathématiques* 5ième Série I(1922), 5-ième Série II (1923).

23 Glainville F., Taylor F. G., The relation of Morley'theoremto the Hessian axis and the circumcentre, *Proc. Edinburgh Math. Soc.*, **32**, (1913-1914) 132-135.
Mathematical Association of Japan for Secondary Mathematics vol. **6,** (décembre 1924).

24 Marchand J., *L'Enseignement Mathématique* (1931) 29.

25 Connes A., A new proof of Morley's theorem ; http://www.alainconnes.org/docs/morley.pdf.

2. A short biography of Frank Morley

Frank Morley was born September 9, 1860, in Woodbridge Suffolk (England).
Son of Elizabeth Muskett and Joseph Roberts Morley, a Quaker who kept a Chinese shop, Frank Morley emigrates in 1887 after his studies at King's college, Cambridge, to Pennsylvania (United States) where he taught until 1900 at Haverford College, a suburb of Philadelphia (Pennsylvania, United States).

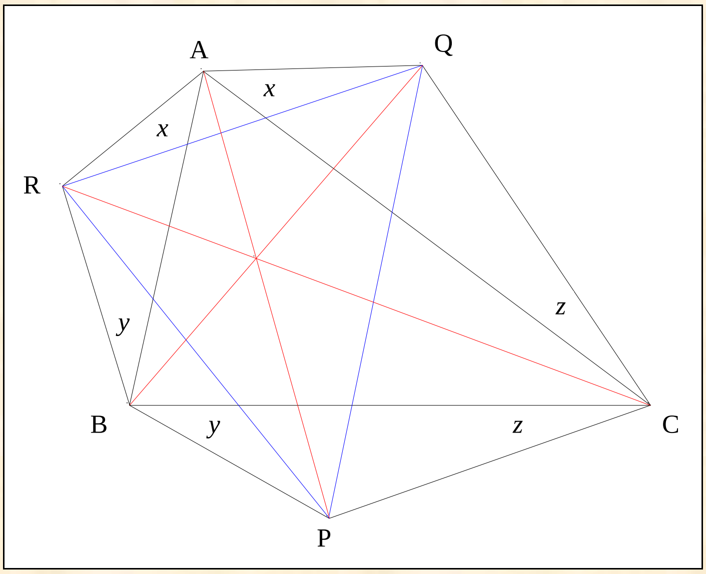
Then he becomes Professor at Johns Hopkins University.
Editor of the *American Journal of Mathematics*, he was elected to the year 1919-20, President of the *American Journal of Mathematics*.
He has three sons, Christopher who will write the new *Thunder on the Left*, Felix who won the Pullitzer's price and Frank Vigor with whom he wrote in 1933 the "stimulating volume" *Inversive Geometry*.
Frank Morley so excelled at Chess that he defeated once a world champion title, Emmanuel Lasker.
Remind us that he managed 50 PhD.
He died on October 17, 1937 at Baltimore (Maryland, United States) without ever having renounced his British citizenship.

II. JACOBI's THEOREM

1. The problem

VISION

Figure :



Features :	ABC	a triangle,
	PCB, QAC, RBA	three triangles outside (or inside) ABC such that
		<PBC = <ABR (= y), <QCA = <BCP (= z), <RAB = <CAQ (= x).

Given : AP, BQ and CR are concurrent.[26]

Definition : PQR is "a Jacobi's triangle of ABC".

Remarks : **(1)** PQR is perspective to ABC
(2) the Morley's triangle of ABC is perspective to ABC.

2. Comment :

the proof based on the Robson technique can be seen in an article by the author.[27]

[26] Jacobi C. F. A., *De triangulorum rectilineorum proprietatibus quibusdam nondum satis cognitis*, Naumburg (1825).
[27] Ayme J.-L., Le théorème de Jacobi, G.G.G. vol. **5** ; http://perso.orange.fr/jl.ayme.

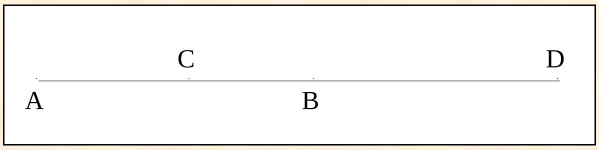
D. APPENDIX

I. ANHARMONIC RATIO

OF

FOUR COLLINEAR POINTS

1. Definition



Given a system of four collinear points A, B, C, D,

we call "anharmonic ratio of these four points in the order ACBD", the quantity $\dfrac{\overline{CA}}{\overline{CB}} : \dfrac{\overline{DA}}{\overline{DB}}$

that we will write (ABCD).

Note that this quantity is independent of the origin, the direction and the unit chosen on this line.

2 . Two remarkable results

We call "permutation of four letters ABCD" the different way to write these four letters on the same line in all possible orders.

2. 1. The quaterne does not change if we swap the first two points with the last two.

For example : (ABCD) = (CDAB).

2. 2. The quaterne does not change
if we swap at the same time the first with the second point, the third with the fourth.

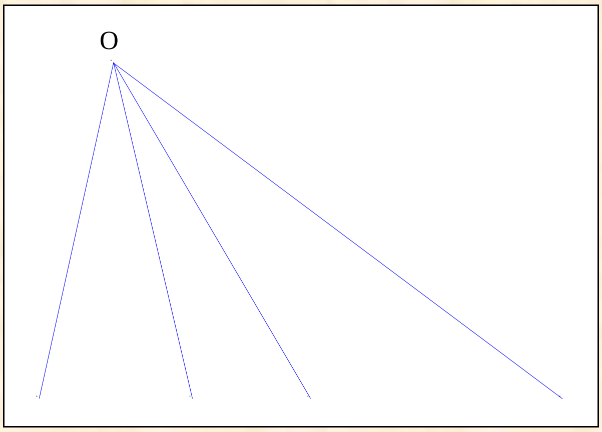
For example : (ABCD) = (BADC).

II. ANHARMONIC RATIO

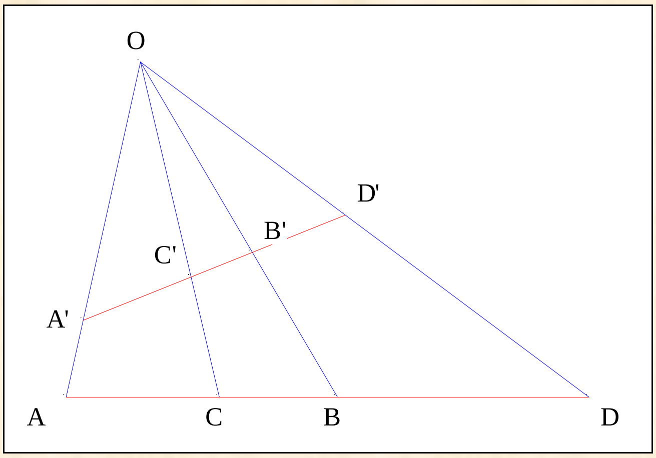
OF

FOUR CONCURRENT LINES

1. Definition



We call "pencil of four lines" a set of four lines passing through a point.
These lines are called "rays of the pencil" and their meeting point, the "summit of the pencil".



If a pencil of four rays is cut by two any transversals ACBD, A'C'B'D' in this order,
then (ABCD) = (A'B'C'D').

The anharmonic ratio of the four points on any transversal cutting a pencil being constant,
it is called "the anharmonic ratio of the pencil" and noted (O ; ABCD).

2 . A remarkable result

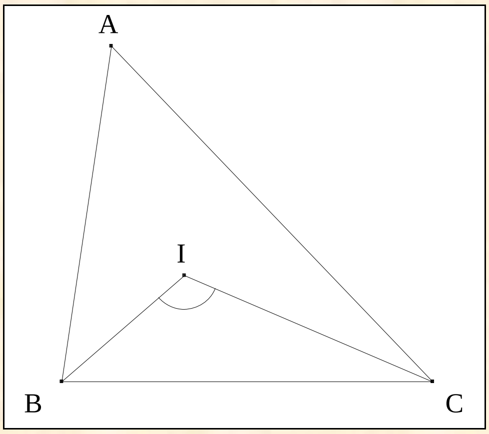
If two equal pencils have a common ray,
then the intersections of the remaining three homologous pairs of rays are collinear.

E. ANNEX

1. Angle I

VISION

Figure :



Features : ABC a triangle
and I the incenter of ABC.

Given : <BIC = 90° + 1/2.<BAC.

E. ARCHIVES

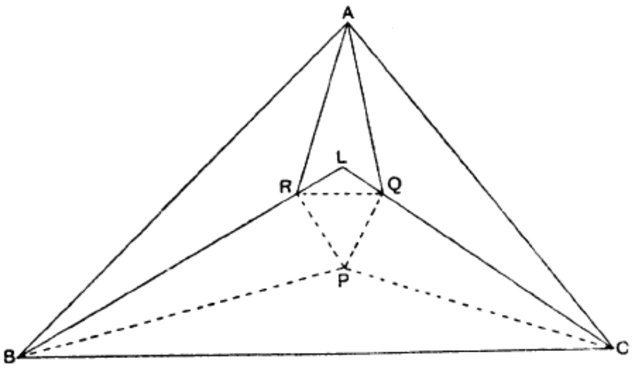
MATHEMATICAL NOTES. 85

621. [$\mathbf{K}^1$. 1. c.] *Morley's Theorem.*

ABC being any triangle with all its angles trisected: if the two trisectors of angle BAC intersect the adjacent trisectors of angles ABC, BCA in R and Q respectively, and if BR, CQ be produced to intersect in L, then

$$RL = QL.$$

For, if—in accompanying figure—P be the third point of intersection, obviously, in the triangle BLC, PL will bisect the angle BLC, and the triangles



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660. [$\mathbf{K}^1$. 1. c.] *Morley's Theorem* (*v.* Note 621).

In the figure, *Gazette*, vol. xi. p. 85, let BRL cut AQ in U; AQ produced cuts BP in N and CP in V; CP cuts AR in M; QM cuts RN in O.

Then BP, BL are isogonal, and so are CP, CL;

$$\therefore\ AP,\ AL \text{ are also isogonal};$$

$$\therefore\ A(BRLU) = A(CVPM);$$

$$\therefore\ N(BRLU) = Q(CVPM) = Q(PMCV),$$

and these pencils have a common ray; $\therefore$ their corresponding rays have collinear intersections, *i.e.* P, O, L are collinear.

As R is the in-centre of ANB, $A\hat{R}N = 90° + \frac{1}{3}B$.

As Q is the in-centre of AMC, $R\hat{M}Q = 90° - \frac{1}{3}A - \frac{1}{3}C$;

$$\therefore \text{ the difference, viz. } R\hat{O}M = 60°.$$

Similarly the other angles at O are $60°$; since they have a common base and equal angles at each of its extremities, the triangles ORL, OQL are congruent, and so are the triangles PRL, PQL.

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[28] C. H. Chepmell C. H., Morley's Theorem, *The Mathematical Gazette*, Vol. **11**, No. 158 (May, 1922) 85

[29] A. Robson, Morley's Theorem (v. Note 621), *The Mathematical Gazette*, Vol. **11**, No. 164 (May, 1923) 310-311.