Algebra: Quadratic Polynomials

A **quadratic polynomial** is a polynomial of the form $ax^2 + bx + c$ with $a \neq 0$.

The roots (solutions) of a quadratic polynomial are $\boxed{x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

There are many problems about roots of quadratic polynomials, however it is not always the best idea to use the above formula. Some other techniques include:

1. <u>Vieta's Theorem</u>: $ax^2 + bx + c = a(x - x_1)(x - x_2)$ where x_1, x_2 are the roots. Therefore:

$$x_1 + x_2 = -\frac{b}{a}, \quad x_1 x_2 = \frac{c}{a}$$

2. <u>Discriminant</u>: The discriminant of an equation $ax^2 + bx + c$ is $D = b^2 - 4ac$. The equation has two real roots if $D > 0$, one real root if $D = 0$, and no real roots if $D < 0$.

3. <u>Graph</u>: It is sometimes useful to consider the graph of a function. The graph of $f(x) = ax^2 + bx + c$ is a parabola that opens up if $a > 0$ and a parabola that opens down if $a < 0$. The graph will cross the x-axis if there are two real roots, touch it if there is one real root, and not intersect it at all if there are no real roots.
   The graph is symmetric with respect to the vertical line $x = -\frac{b}{2a}$.

4. <u>Manipulation</u>: Algebraic manipulations are very important in solving many mathematical problems, not just ones about quadratic polynomials. Seeing and knowing how to manipulate a given expression to get something useful is a handy skill that comes with practice. Factoring is always a good thing to try.

Problems

1. Find the sum of the roots of equation $(x+1)(x+2) + (x+3)(x+4) + \ldots + (x+99)(x+100) = 0$.

2. Solve $(x^2 + 3x + 2)(x^2 + 7x + 12) + (x^2 + 5x + 6) = 0$.

3. Let k be a non-zero real number. Find, in terms of k, the roots of $x^2 + (k - \frac{1}{k})x - 1 = 0$.

4. Let the roots of $x^2 - 17x + 13 = 0$ be r and s. What is the value of:
   (a) $r^2 s^2$ (b) $r^2 + s^2$ (c) $r^2 s + s^2 r$ (d) $r^3 + s^3$
   (Hint: you are not supposed to use the quadratic formula)

5. Do there exist positive integers a, b, c such that each of the equations $ax^2 + bx + c = 0, ax^2 - bx + c = 0, ax^2 + bx - c = 0, ax^2 - bx - c = 0$, has two integer roots?

6. The absolute value of the difference of the roots of the quadratic equation $ax^2 + bx + c$ is 2.
   (a) What is the absolute value of the difference of the roots of the equation $ax^2 + 6bx + 36c$?
   (b) Prove the absolute value of the difference of the roots of the equation $ax^2 + 10bx + 27c$ is at least $2\sqrt{27}$.
   (Hint: think about how you can use Vieta's formula)

Some Problems from Competitions
Recall that $a^2 - b^2 = (a - b)(a + b)$; this may help for some of the problems below.

1. (Euclid 2010) Determine all real values of x such that $(x + 8)^4 = (2x + 16)^2$.

2. (Euclid 2008) Determine all $k \neq 0$ such that the equation $y = kx^2 + (5k + 3)x + (6k + 5)$ touches the $x-$axis.

3. (Fermat 2008) For how many integers k do the graphs of $y = -\frac{1}{8}x^2 + 4$ and $y = x^2 - k$ intersect on or above the $x-$axis?

4. (COMC 2008) Determine all real x such that
   (a) $(x + 3)(x - 6) = -14$
   (b) $2^{2x} - 3(2^x) - 4 = 0$
   (c) $(x^2 - 3x)^2 = 4 - 3(3x - x^2)$
   (Hint: for (b), (c), use substitution).

5. (COMC 2004) (a) Determine the two values of x such that $x^2 - 4x - 12 = 0$.
   (b) Determine the one value of x such that $x - \sqrt{4x + 12} = 0$.
   (c) (Hard) Determine all real c such that $x^2 - 4x - c - \sqrt{8x^2 - 32x - 8c} = 0$ has exactly two real solutions for x.

Harder Problems

1. It is known that $(a + b + c)c < 0$. Prove that $b^2 > 4ac$.
   (Hint: $b^2 - 4ac$ is a discriminant of a certain equation...)

2. $f(x)$ and $g(x)$ are quadratic polynomials. The polynomial $f(x) + g(x)$ has two distinct roots, each of which is a root of $f(x) - (g(x))^3 = 0$. Prove that $f(x) = Cg(x)$ for some constant C.
   (Hint: you have $f(x) + g(x) = 0$ and $f(x) - (g(x))^3 = 0$ when x is a root of $f(x) + g(x) = 0$.)

3. Let $f(x) = x^2 + bx + c$. Prove that at least one of the numbers $|f(-1)|, |f(0)|, |f(1)|$ is greater than $\frac{1}{2}$.

4. There are several quadratic polynomials, such that each of them has two real roots, and the difference of any two of them has no real roots. Prove that the sum of all of the polynomials has at least one real root. (Hint: look at the graphs of the polynomials.)

5. A quadratic polynomial $P(x)$ satisfies $-1 \leq P(x) \leq 1$ for $0 \leq x \leq 1$. Prove that $P(-\frac{1}{2}) \leq 7$.