Solutions for *The Art and Craft of Problem Solving*

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Chapter 2: Strategies for Investigating Problems

2.2: Strategies for Getting Started

2.2.9 Let's examine the first few iterations:

$$\begin{aligned} f(x) &= \frac{1}{1-x} \\ f^2(x) &= \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{-x} \\ f^3(x) &= \frac{1}{1+\frac{1-x}{x}} = x \end{aligned}$$

Now we see that the function is periodic with period 3. Then $f^{1999}(2000) = f(2000) = \frac{1}{1-2000} = \boxed{-\frac{1}{1999}}$.

2.2.13 When we add the nth line, it appears as if the number of regions is increased by n. The first line divides the plane into two regions, so the total should be $\boxed{\frac{n(n+1)}{2}+1}$.

2.2.14

2.2.15 We clear denominators:

$$mx + my = xy$$

Now factor:

$$m^2 = (x-m)(y-m)$$

Thus, all x and y such that $x - m | m^2$ and $y - m | m^2$ work except the case $x = y = 2m$.

2.2.16

2.2.18 We claim that the only such n are of the form $k^2 - 1$, where k is an integer. First, we show that any such number satisfies $q(n) > q(n+1)$.

$$\begin{aligned} q(k^2-1) &= \left\lfloor \frac{k^2-1}{\lfloor\sqrt{k^2-1}\rfloor} \right\rfloor \\ &= \left\lfloor \frac{k^2-1}{k-1} \right\rfloor \\ &= k+1 \\ q(k^2) &= \left\lfloor \frac{k^2}{\lfloor\sqrt{k^2}\rfloor} \right\rfloor \\ &= \left\lfloor \frac{k^2}{k} \right\rfloor \\ &= k \end{aligned}$$

Now suppose that $n = k^2 - j$, where $j > 1$.

$$\begin{aligned} q(k^2 - j) &= \left\lfloor \frac{k^2 - j}{\lfloor \sqrt{k^2 - j} \rfloor} \right\rfloor \\ &= \left\lfloor \frac{k^2 - j}{k - 1} \right\rfloor \\ q(k^2 - j + 1) &= \left\lfloor \frac{k^2 - j + 1}{\lfloor \sqrt{k^2 - j + 1} \rfloor} \right\rfloor \\ &= \left\lfloor \frac{k^2 - j + 1}{k - 1} \right\rfloor \end{aligned}$$

Clearly, $\left\lfloor \frac{k^2-j}{k-1} \right\rfloor \leq \left\lfloor \frac{k^2-j+1}{k-1} \right\rfloor$. Therefore, we conclude that only number of the form $k^2 - 1$ satisfy $q(n) > q(n+1)$.

2.2.21 Experimentation shows that $s(1) = 1$, $s(2) = 2$, $s(3) = 4$, and $s(4) = 8$. It looks like 2^{n-1}. For a proof, consider partitioning a number n into n ones, with spaces between adjacent ones. Then the number of ways to partition n into an ordered sum of k elements is the number of ways to insert dividers in the $n - 1$ spaces between the ones, which is $\binom{n-1}{k}$. Therefore, the total number of ordered sums is

$$\sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$$

2.2.22 One such solution is

$$1^2 + 2^2 + 2^2 = 3^2$$

If we multiply this by a scalar, the equality will preserved. Thus, we can generate infinitely many solutions.

2.2.23 We note that

$$n^2 - (n+1)^2 - (n+2)^2 + (n+3)^2 = (2n+5) - (2n+1) = 4$$

Therefore, if we append $n^2 - (n+1)^2 - (n+2)^2 + (n+3)^2$ to a string of sums, we increase the overall sum by 4. Therefore, if we can find 4 consecutive numbers of the form $\pm 1^2 \pm 2^2$..., we can express all numbers thereafter as $\pm 1^2 \pm 2^2$....

$$\begin{aligned} 1 &= 1^2 \\ 2 &= -1^2 - 2^2 - 3^2 + 4^2 \\ 3 &= -1^2 + 2^2 \\ 4 &= 1^2 - 2^2 - 3^2 + 4^2 \end{aligned}$$

Therefore, all positive integers can be obtained.

2.2.24 Let's write out the first few terms of this product:

$$(1 + x^3)(1 + 2x^9)(1 + 3x^{27})...$$

For now, ignore the coefficient of the x terms and focus on the exponents. In base three, they look like

$$(1 + x^{10})(1 + x^{100})(1 + x^{1000})...$$

Now we examine the possible exponents of x in the product. Each is formed by a string of 0's and 1's in base 3. Furthermore, each exponent is achieved by some unique combination of factors as dictated by its base 3 representation. The last digit must be a 0, since 10_3 is the smallest positive exponent we have. The first digit must be a 1. All of the other digits can be 0 or 1. Thus, there are 2^{n-2} different possibilities for an n-digit exponent.

We wish to find the 1996th exponent. Summing $1 + 2 + 4 + 8 + ... + 512 = 1023$, we know that k_{1996} is the 973^{rd} 12-digit base 3 number consisting of only 0's and 1's and with a last digit of 0.

We can find this number by converting to binary. For the moment, ignore the last 0 and the first 1. Then the number we seek is the 972^{nd} binary number, with a 1 appended to the beginning and a 0 appended to the end.

$$962 = 1111001100_2$$

Therefore, $k_{1996} = 111110011000_3$. This implies that a_{1996} is the product of the coefficients of $x^{3^{11}}, x^{3^{10}}, x^{3^9}$, $x^{3^8}, x^{3^7}, x^{3^4}, x^{3^3}$, which is $11 * 10 * 9 * 8 * 7 * 4 * 3 = \boxed{665280}$.

2.2.25

2.2.26

2.2.27 Take all terms in base 2. Every row that is a power of 2 looks like

$$1\ 0\ 0\ 0\ ...\ 0\ 1$$

Therefore, the triangle "repeats" after every power of 2 from the ones on the end. Consider the base 2 representation of n. The number of ones in the nth row is two times the number of ones in the $n - 2^k$th row, where 2^k is the largest power of two less than n. Therefore, the number of ones is doubled for each 1 in the base two representation of n.

2.2.28 We claim that only the perfect squares are eventually constant. If A is a perfect square, then $S(A) = 0$, so the sequence is clearly constant. Suppose that $A = m^2 + r$, with $r \leq 2m$. Then $S(A) = r$, so $m^2 < A + S(A) = m^2 + 2r < (m + 2)^2$. Also, $A + S(A) \neq (m + 1)^2$ because $2r \neq 2m + 1$. Therefore, the sequence $(a_k)_{k=0}^{\infty}$ never encounters a perfect square, so $S(a_k)$ is positive for all k, implying that the sequence is never constant.

2.2.29 Denote a person holding cards $a, b, a < b$, as $\frac{a}{b}$. Then the arrangement of cards is

$$\frac{a_1}{a_2}\ \frac{a_3}{a_4}\ \dots$$

With the top row passed along.

We first claim that, after some number of passes, all the cards on the bottom are greater than or equal to all the cards in the top row. Suppose to the contrary that there is always some number in the top row that is greater than some number in the bottom row. But then, after a finite number of passes the number in the top row is either switched to the bottom row, or it reaches the same hand as the smaller number, at which point it is also switched to the bottom row.

Once we reach the point where all the numbers in the bottom row are greater than or equal to the numbers in the bottom row, then the top row merely rotates while the bottom row is static. Since there are an odd number of slots in each row, and two of each card, there must be some number that appears both in the top row and in the bottom. Then, after a finite number of rotations, these numbers appear in the same hand, and we are done.

2.2.30 Let us re-examine the false conjecture: If m has remainder b, then $f^2(m)$ has remainder $b-1$. We know that it isn't quite true, but perhaps some investigation will shed light on something similar. Let $m = n^2 + b$, where $b \leq 2n$. Then

$$\begin{aligned} f(n) &= (n^2+b) + \lfloor\sqrt{n^2+b}\rfloor \\ &= n^2+b+n \\ f^2(n) &= (n^2+n+b) + \lfloor\sqrt{n^2+n+b}\rfloor \end{aligned}$$

This is a little bit trickier. If $b + n \leq 2n$, then

$$\begin{aligned} f^2(n) &= n^2+2n+b \\ &= (n+1)^2 + (b-1) \end{aligned}$$

as desired. However, if $b + n \geq 2n + 1$, then

$$\begin{aligned} f^2(n) &= n^2+2n+b+1 \\ &= (n+1)^2 + b \end{aligned}$$

This contradicts the desired conjecture, but perhaps we can manipulate this result further to reach the desired result. After each double-iteration of f, the remainder either decreases by 1 if $b \leq n$ or stays the same if $b > n$. However, after each double-iteration the base n increases by 1. Suppose that sequence never reaches a perfect square; that is, b never decreases to 0. Since b is always non-increasing, it must eventually stop decreasing, which implies that $b > n$. However, as shown above, n increases by one after every double-iteration, so after some finite number of steps, we must have $b < n$. Contradiction. Therefore, we are forced to conclude that the sequence always includes a perfect square.

2.2.31 It suffices to prove that $f(m)$ is not a perfect square for any positive integer m. Suppose that $n^2 \leq k \leq (n+1)^2$. Then $n^2 < n^2 + n \leq f(k) \leq n^2 + 3n + 2 \leq (n+2)^2$. Thus, $f(k)$ can be a perfect square iff it is equal to $(n+1)^2$. We consider to cases:

Case 1: $k \geq n^2+n+1$. Then $\sqrt{k} > \sqrt{n^2+n+\frac{1}{4}} = n+\frac{1}{2}$. Therefore, $\{\sqrt{k}\} = n+1$ and $f(k) = n^2+2n+2$. Thus, if $k \geq n^2+n+1$, then $k \neq (n+1)^2$.

Case 2: $k \leq n^2+n$. Then $\sqrt{k} < \sqrt{n^2+n+\frac{1}{4}} = n+\frac{1}{2}$. Therefore, $\{\sqrt{k}\} = n$ and $f(k) = n^2+2n$. Thus, if $k \leq n^2+n$, then $k \neq (n+1)^2$.

We conclude that the sequence $f(m), f(f(m)), ...$ never contains any perfect squares.

2.3: Methods of Argument

2.3.11 If any one of a, b, c is even, then the product abc is even. Suppose that abc is odd. Then a^2, b^2, and c^2 are odd. Since

$$a^2 + b^2 = c^2$$

Then we have the sum of two odds, which is even, is equal to an odd number, which is impossible.

2.3.12 (a) Suppose that $\sqrt{3}$ can be expressed as a quotient $\frac{a}{b}$ in lowest terms. Then $3b^2 = a^2$. This implies that $3|a^2$, so $9|a^2$. Similarly we obtain $3|b^2$, so $3|b$. The contradicts the assumption that a and b are in lowest terms. We conclude that $\sqrt{3}$ is irrational.

(b) Again, suppose that $\sqrt{6}$ can be expressed as $\frac{a}{b}$ in lowest terms. Then $6b^2 = a^2$. Therefore, $a = 2 * 3 * a'$. Then $6b^2 = 36a'^2$. Now $b^2 = 6a^2$. Similarly, $2|b$ and $3|b$. This contradicts our assumption that a and b are in lowest terms. We conclude that $\sqrt{6}$ is irrational.

(c) As in the previous cases, we reach $49b^2 = a^2$. Then $a = 7a'$, so $b^2 = a'^2$. At this point, we cannot assert anything further.

2.3.13 Suppose that there exists a smallest real number a. Consider $\frac{a}{2}$. It both positive and smaller than a, contradicting our assumption. We conclude that there is no smallest real number.

2.3.14 Suppose that $\log_1 02 = \frac{a}{b}$, with a and b integers. Then $2 = 10^{\frac{a}{b}}$, so $2^b = 10^a$. However, the right side is divisible by 5 and the left side cannot be, so there exist no such a and b.

2.3.15 Suppose that $\sqrt{2} + \sqrt{3} = \frac{a}{b}$. Then $5 + 2\sqrt{6} = \frac{a^2}{b^2}$. As proved earlier, $\sqrt{6}$ is irrational. Thus, the left side is irrational and the right side is rational, which is impossible. We conclude that $\sqrt{2} + \sqrt{3}$ is irrational.

2.3.17 Suppose that $a + b$ can be expressed as $\frac{c}{d}$. Since a is irrational, it can be expressed as $\frac{x}{y}$. Then

$$b = \frac{c}{d} - \frac{x}{y} = \frac{cy - xd}{dy}$$

This shows that b can be expressed as a quotient of integers, implying that b is rational, which contradicts our assumption. We conclude that the sum of an irrational and a rational number is irrational.

2.3.18 False. Consider the number $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$. As proved before, $\sqrt{2}$ is irrational. If $\sqrt{2}^{\sqrt{2}}$ is rational, then let $a = b = \sqrt{2}$ and we have a counterexample. Otherwise, let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$, so $a^b = \sqrt{2}^{(\sqrt{2})^2} = \sqrt{2}^2 = 2$, which is rational.

2.3.20 Consider the m numbers

$$x, 2x, 3x, 4x, ...mx$$

We claim that, of these, no two have the same residue mod m. Suppose otherwise; then for some a and b,

$$ax \equiv bx \pmod{m}$$

But since $\gcd(x, m) = 1$, this implies that $a = b$. We conclude that all m numbers are distinct modulo m, so x has a unique multiplicative inverse.

2.3.21 By hypothesis, Q is composite and must be divisible by at least one of $p_1, p_2, ...p_n$. But this is impossible because Q is one more than a multiple of each of $p_1, p_2, ...p_n$. We are forced to conclude that there are infinitely many primes.

2.3.22 Suppose to the contrary that neither T nor U are closed under multiplication. Then $\exists a, b \in T$ such that $ab \in U$ and $\exists c, d \in U$ such that $cd \in T$. But by the given information, $a * b * cd \in T$ and $c * d * ab \in U$, which is impossible because they are disjoint. We conclude that at least one of T, U is closed over multiplication.

2.3.23 Suppose that we do have such a configuration $a_1, a_2, ...a_{1995}$. WLOG, $a_2 = p_1 a_1$. Then

$$\frac{a_2}{a_3} = p_2, \frac{1}{p_2}$$

$$\frac{a_3}{a_4} = p_3, \frac{1}{p_3}$$

$$...$$

$$\frac{a_{1995}}{a_1} = p_{1995}, \frac{1}{p_{1995}}$$

Multiplying these all together,

$$\frac{a_2}{a_1} = \frac{\prod p_k}{\prod p_j} = p_1$$

Where $\prod p_k$ is some product of the elements in a subset of $\{p_2, p_3, ...p_{1995}\}$. We clear denominators to get

$$p_1 \prod p_j = \prod p_k$$

Now, by unique prime factorization, there must be some prime in the set of p_k that is equal to p_1. We can divide these out from both sides. Again, there must be two primes common to the sets of p_j and p_k. We can continue to divide these out two at a time, but since there are a total of 1995 numbers, there will always be one prime left over. Therefore, it is impossible that the two are equal. Contradiction.

2.3.24 Our hypothesis is that $\frac{1}{1*2} + \frac{1}{2*3} + ... + \frac{1}{n*(n+1)} = \frac{n}{n+1}$. For the base case,

$$\frac{1}{1*2} = \frac{1}{2}$$

Now suppose that the hypothesis is true for n. Then

$$\begin{aligned} \frac{1}{1*2} + \frac{1}{2*3} + ... + \frac{1}{n*(n+1)} + \frac{1}{(n+1)*(n+2)} &= \frac{n}{n+1} + \frac{1}{(n+1)*(n+2)} \\ &= \frac{n(n+2)}{(n+1)*(n+2)} + \frac{1}{(n+1)*(n+2)} \\ &= \frac{n^2+2n+1}{(n+1)*(n+2)} \\ &= \frac{(n+1)^2}{(n+1)*(n+2)} \\ &= \frac{n+1}{n+2} \end{aligned}$$

2.3.25 For the base case, 0 lines divides the plane into 1 region. Now assume that, for n lines, the plane is divided int $\frac{n^2+n+2}{2}$ regions. When we add the $n+1$st line, it intersects n times. Since each intersection occurs between two regions, the line passes through $n+1$ regions, dividing each one into two. Therefore the number of regions becomes

$$\frac{n^2+n+2}{2}+\frac{2n+2}{2}=\frac{2n^2+3n+4}{2}=\frac{(n+1)^2+(n+1)+2}{2}$$

Which completes the induction.

2.3.26 For the base case, $1*2*3$ is divisible by 6. Now suppose that $n(n+1)(n+2)$ divisible by 6. We wish to show that $(n+1)(n+2)(n+3)$ is divisible by 6. To do so, we show that $(n+1)(n+2)(n+3)$ is divisible by 2 and 3. If n was divisible by 3, then $n+3$ is divisible by 3. otherwise, one of $n+1$ and $n+2$ is divisible by 3. If n was divisible by 2, then $n+2$ is divisible by 2. Otherwise, one of $n+1$ and $n+2$ is divisible by 2. Therefore, the product $(n+1)(n+2)(n+3)$ is divisible by 6.

2.3.27 Let's proceed by induction. For the base case, the null set has only one subset (itself). For the inductive step, suppose that a set with n elements has 2^n subsets. A set with $n+1$ elements has these same 2^n subsets, plus another 2^n subsets formed by adding the $n+1$st element to each of the original 2^n subsets. Therefore, it has 2^{n+1} subsets.

2.3.28

$$\begin{aligned}1+a+a^2+...+a^{n-1} &= S\\ a+a^2+...+a^n &= Sa\\ 1-a^n &= S(1-a)\\ \frac{1-a^n}{1-a} &= S\end{aligned}$$

2.3.29 We first prove the triangle inequality:

$$|x+y|\leq|x|+|y|$$

Let $x=a+bi$ and $y=c+di$. Then

$$\begin{aligned}|x+y| &= |(a+c)+(b+d)i|\\ &= \sqrt{(a+c)^2+(b+d)^2}\end{aligned}$$

We wish to prove that this is less than

$$\begin{aligned}|x|+|y| &= |a+bi|+|c+di|\\ &= \sqrt{a^2+b^2}+\sqrt{c^2+d^2}\end{aligned}$$

By Cauchy,

$$\begin{aligned}ac+bd &\leq \sqrt{(a^2+b^2)(c^2+d^2)}\\ a^2+2ac+c^2+b^2+2bd+d^2 &\leq a^2+b^2+c^2+d^2+2\sqrt{(a^2+b^2)(c^2+d^2)}\\ \sqrt{(a+c)^2+(b+d)^2} &\leq \sqrt{a^2+b^2}+\sqrt{c^2+d^2}\end{aligned}$$

Which completes the proof.

We now wish to extend the result to an arbitrary amount of numbers. As shown, the base case is true. For the inductive step, suppose that

$$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$$

Let $z = |a_1 + a_2 + ... + a_n|$. Then by the triangle inequality,

$$|z + a_{n+1}| \leq |z| + |a_{n+1}|$$

Substituting,

$$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n| + |a_{n+1}|$$

2.3.31 In modular arithmetic language, our goal is to show that

$$7^n - 1 = 0 \pmod 6$$

Since $7 \equiv 1 \pmod 6$, this translates to

$$1^n - 1 = 0 \pmod 6$$

Since $1^n \equiv 1 \pmod 6$, the result follows.

2.3.32 We proceed by strong induction. By definition, $x + \frac{1}{x}$ is an integer. Suppose that $x^k + \frac{1}{x^k}$ is an integer for all k less than n. Consider

$$\left(x + \frac{1}{x}\right)^n = x^n + \frac{1}{x^n} + \binom{n}{1}\left(x^{n-1} + \frac{1}{x^{n-1}}\right) + \binom{n}{2}\left(x^{n-2} + \frac{1}{x^{n-2}}\right)...$$

$\left(x + \frac{1}{x}\right)^n$ is an integer, and so is $\binom{n}{1}\left(x^{n-1} + \frac{1}{x^{n-1}}\right) + \binom{n}{2}\left(x^{n-2} + \frac{1}{x^{n-2}}\right)...$ by hypothesis. Therefore, $x + \frac{1}{x^n}$ is an integer. This completes the induction.

2.3.33 We proceed by induction on n. For the base case,

$$(1 + x)^2 = x^2 + 2x + 1 > 1 + 2x$$

Suppose that the theorem holds for $n = m$. Then

$$(1 + x)^m > 1 + mx$$

Multiplying both sides by $(1 + x)$,

$$(1 + x)^m > (1 + mx)(1 + x) = 1 + (m + 1)x + mx^2 > 1 + (m + 1)x$$

Which completes the induction.

2.3.34 Let $b(n)$ be the number of ones in the base 2 representation of n. We claim that $f(n) = b(n)$. Let us proceed by strong induction. The base case $f(1) = 1$ is given. Suppose that the result holds for $1, 2, ...n - 1$. We consider two cases:

Case 1: $n - 1$ even. Then n has one more 1 than $n - 1$, since adding 1 changes the last 0 to a 1. Also, we are given that $f(n) = f(n - 1) + 1$. By inductive hypothesis, $f(n - 1) = b(n - 1)$. Therefore, $f(n) = b(n)$

for even $n-1$.

Case 2: $n-1$ odd. Then n ends in a 0, as adding 1 to the last 1 in the base 2 representation of $n-1$ changes it from a 1 to a 0. Then n has the same number of ones as $\frac{n}{2}$, so $b(n) = b\left(\frac{n}{2}\right)$. It is given that $f(n) = f\left(\frac{n}{2}\right)$. By inductive hypothesis, $f\left(\frac{n}{2}\right) = b\left(\frac{n}{2}\right)$. Therefore, $f(n) = b(n)$ for odd $n-1$.

Since the inductive step holds for both even and odd $n-1$, the induction is complete.

2.3.35 (a) Proceed by induction. The base case is trivially true: $f(1) = f(1)$. Suppose that

$$f_1 + f_3 + ... + f_{2n-1} = f_{2n}$$

holds for n. By the definition of Fibonacci numbers,

$$\begin{aligned} f_{2n+2} &= f_{2n+1} + f_{2n} \\ &= f_{2n+1} + (f_{2n-1} + f_{2n-3} + ... + f(1)) \end{aligned}$$

which completes the induction.

(b) Again we proceed by induction. The base case is true: $f_2 = f(3) - 1$. Suppose that

$$f_2 + f_4 + ... + f_{2n} = f_{2n+1} - 1$$

holds.

$$\begin{aligned} f_{2n+3} &= f_{2n+2} + f_{2n+1} \\ &= f_{2n+2} + (f_{2n} + f_{2n-2} + ... + f_2 + 1) \\ f_{2n+3} - 1 &= f_2 + f_4 + ... + f_{2n+2} \end{aligned}$$

which completes the induction.

(c) Proceed by induction on n. For the base case, $f_1 < 2^1$ and $f_2 < 2^2$. For the inductive step,

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} \\ &< 2^{n-1} + 2^{n-2} \\ &< 2^{n-1} + 2^{n-1} \\ &< 2^n \end{aligned}$$

completing the induction.

(f) We proceed by induction. The base case is true:

$$f_3 f_1 - f_2^2 = 2 - 1 = (-1)^2$$

For the inductive step, assume that

$$f_{n+1} f_{n-1} - f_n^2 = (-1)^n$$

Then

$$\begin{aligned} f_{n+1} f_{n-1} - f_n^2 &= f_{n+1}(f_{n+1} - fn) - f_n^2 \\ &= f_{n+1}^2 - f_n f_{n+1} - f_n^2 \\ &= f_{n+1}^2 - f_n(f_{n+1} + f_n) \\ &= f_{n+1}^2 - f_n(f_{n+2}) \\ &= -\left(f_n f_{n+2} - f_{n+1}^2\right) \end{aligned}$$

Therefore, $f_n f_{n+2} - f_{n+1}^2 = -(-1)^n = -1^(n+1)$, as desired.

(g)

2.3.36 We will first prove Pick's Theorem for rectangles, then right triangles, then an arbitrary triangle, then any polygon.

Lemma 1: The area of any lattice rectangle with sides parallel to the coordinate axes is $I + \frac{B}{2} - 1$, where I is the number of interior points and B is the number of boundary points.

Proof: Consider an $x \times y$ lattice rectangle. It has a total of $(x+1)(y+1)$ lattice points: $(x-1)(y-1)$ interior point and $2x+2y$ boundary points. The area of the rectangle is xy, which is equal to $(x-1)(y-1)+\frac{2x+2y}{2}-1$. This completes the lemma.

Lemma 2: The area of any right triangle with legs parallel to the coordinate axes is $I + \frac{B}{2} - 1$, where I is the number of interior points and B is the number of boundary points.

Proof: Consider an $x \times y$ right triangle inscribed in an $x \times y$ rectangle. The triangle has $x + y + 1$ lattice points in common with the boundary of the rectangle. Let the number of interior points in the triangle be I_Δ and the number of boundary points on the triangle but interior to the rectangle be B_Δ. Then the total number of boundary points on the triangle is $B_\Delta + x + y + 1$. By Lemma 1,

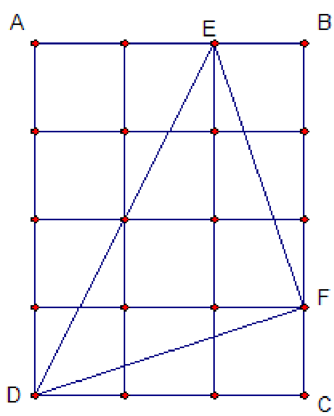
$$xy = 2I_\Delta + B_\Delta + \frac{2x + 2y}{2} - 1$$

Thus,

$$\frac{xy}{2} = I_\Delta + \frac{B_\Delta + x + y + 1}{2} - 1$$

Which is precisely want we wanted to show.

Lemma 3: The area of any lattice triangle is $I + \frac{B}{2} - 1$, where I is the number of interior points and B is the number of boundary points.



Proof: Inscribe the triangle in a rectangle with sides parallel to the coordinate axes. We have 3 vertices of the triangle to place on 4 sides of the rectangle, so by the Pigeonhole there will be at least one vertex of the triangle on a vertex of the rectangle. Therefore, the rectangle is divided into three right triangles and the arbitrary lattice triangle. Let I_Δ and B_Δ be the number of lattice points interior to all three right triangles, respectively, and I_* and B_* the number of lattice points interior to ΔDEF. We wish to show that $[DEF] = I_* + B_*/2 - 1$.

Let I be the total number of interior points of $ABCD$. Then

$$I = I_* + I_\Delta + B_* - 3$$

Let B be the total number of boundary points of $ABCD$. Then

$$B_\Delta = B + B_*$$

Note: D, E, and F are double-counted on both sides.

We substitute these equations into an expression for the area of ΔBEF using Pick's theorem:

$$\begin{aligned}
[BEF] &= [ABCD] - ([AED] + [EBF] + [FCD]) \\
&= \left(I + \frac{B}{2} - 1\right) - \left(I_\Delta + \frac{B_\Delta}{2} - 3\right) \\
&= (I_* + I_\Delta + B_* - 3) + \left(\frac{B_\Delta - B_*}{2}\right) - 1 - I_\Delta - \frac{B_\Delta}{2} + 3 \\
&= I_* + I_\Delta + B_* - 3 + \frac{B_\Delta}{2} + -\frac{B_*}{2} - 1 - I_\Delta - \frac{B_\Delta}{2} + 3 \\
&= I_* + \frac{B_*}{2} - 1
\end{aligned}$$

Which completes the lemma.

Finally, we are ready to move on to the general Pick's Theorem, which states that the area of any lattice polygon is $I + \frac{B}{2} - 1$, where I is the number of interior lattice points and B is the number of boundary lattice points of the polygon. Number the vertices of the polygon from 1 to n and triangulate the polygon by drawing $n-1$ line segments from vertex 1 to vertices 2 through n. The area of the polygon is the sum of the areas of the individual triangles. Each interior point of the polygon is included once, since the points that are both interior to the polygon and the triangles are all included once and the points that are interior to the polygon and boundary to the triangles are included twice at half value, as each internal diagonal is shared by two triangles.

Let the number of boundary points but not vertices of the polygon be B_*. Then

$$B = B_* + n$$

and each lattice point in B_* is included with weight $\frac{1}{2}$ in the area.

Vertex 1 is included $n-2$ times (in all $n-2$ triangles) with weight $\frac{1}{2}$, vertices 2 and n are included only once with weight $\frac{1}{2}$, and vertices 3 through $n-1$ are included twice with weight $\frac{1}{2}$. Therefore, the vertices are counted $(n-2)+1+1+2(n-3) = 3n-6$ times with weight $\frac{1}{2}$. Thus, the area is equal to

$$\begin{aligned}
A &= I_p + \frac{B_*}{2} + \frac{3n-6}{2} - (n-2) \\
&= I_p + \frac{B_*}{2} + \frac{3n-6}{2} - \frac{2n-4}{2} \\
&= I_P + \frac{B_*}{2} + \frac{n}{2} - \frac{2}{2} \\
&= I_p + \frac{B}{2} - 1
\end{aligned}$$

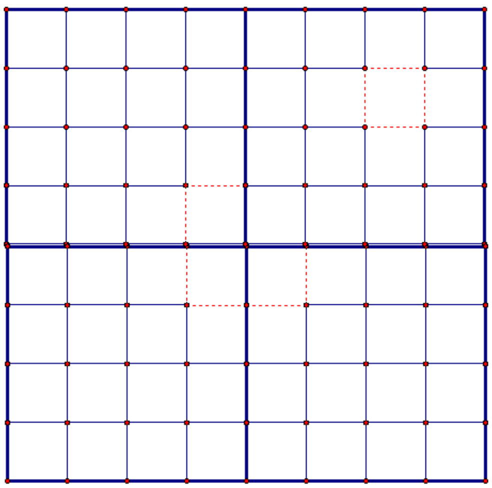
which completes the proof.

2.3.37 (a)

(b)

(c)

2.3.38 We proceed by induction. For the base case, we can trivially cover a 2×2 square with one square taken out with an ell (the resulting figure is an ell). Now suppose that we can always cover a $2^n \times 2^n$ square with one square taken out.



Now consider a $2^{n+1} \times 2^{n+1}$ and divide it into four $2^n \times 2^n$ grids. Without loss of generalitiy, assume that the removed square is from the upper right $2^n \times 2^n$ grid. By inductive hypothesis, this can be tiled by ells. Now we can color the center of the grid with an ell, removing one square from each of the remaining $2^n \times 2^n$ grids. By inductive hypothesis, all of these can also be tiled by ells. We conclude that all grids of the form $2^k \times 2^k$ can be tiled with ells.

2.3.39 We claim that silver matrices exist for all $2^k \times 2^k$ arrays. We proceed by induction. For the base case, the matrix

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

satisfies the conditions. For the inductive hypothesis, suppose that there exists a silver matrix M_k for a $2^k \times 2^k$ array. Let B_k be the matrix formed by adding 2^k to all the elements of M_k, and let A_k be the matrix formed by replacing all instances of $2^{k+1}+1$ in B_k with 2^{k+1}. Then we claim that the matrix

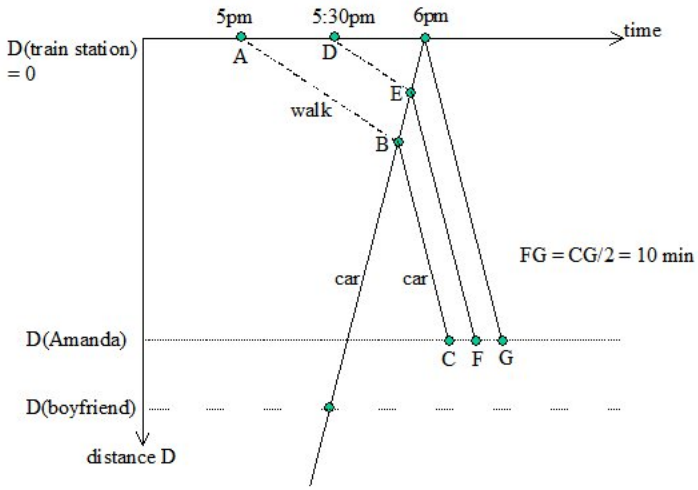
$$\begin{bmatrix} M_k & B_k \\ A_k & M_k \end{bmatrix}$$

is a silver matrix. Consider the set of numbers G included in i^{th} row and i^{th} column of this matrix. If $1 \leq i \leq 2^n$, then it includes the i^{th} row of B_k and the i^{th} column of A_k; otherwise, it includes the $i - 2^k$th row of A_k and the $i - 2^k$th column of B_k. By our definition of A_k and B_k, this means that all the numbers 2^{k+1} through $2^{k+2} - 1$ are in G. Also, the i^{th} row and the i^{th} column include the i or $i - 2^k$th row and the i or $i - 2^k$th column of M_k, so by the definition of M_k, the numbers 1 through $2^{k+1} - 1$ are also in G. Therefore, all the numbers 1 through $2^{k+2} - 1$ are in G, so we have constructed a silver matrix for a $2^{k+1} \times 2^{k+1}$ array. This completes the induction.

We conclude that all array of the form $2^n \times 2^n$ have silver matrixes, and there are clearly infinitely many of these.

2.4: Other Important Strategies

2.4.7



2.4.8 We first present an algebraic solution. Let x be the time between sunrise and noon, D be the distance between A and B, and s_1 and s_2 the speeds, respectively, of Pat and Dana. We have the following equations:

$$\begin{aligned}\frac{D}{s_1+s_2} &= x\\ \frac{D}{s_1} &= x+5\\ \frac{D}{s_2} &= x+11.25\end{aligned}$$

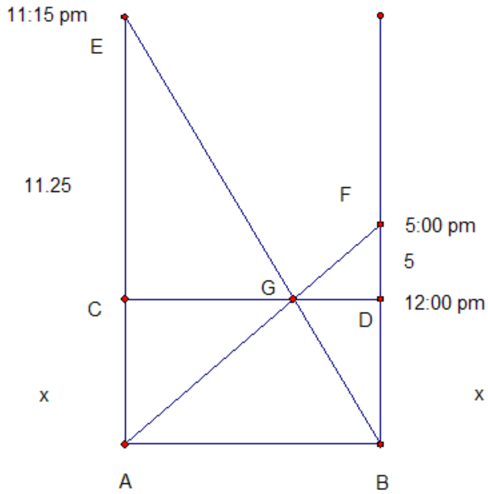
We can invert each equation to make it easier to solve:

$$\begin{aligned}\frac{s_1+s_2}{D} &= \frac{1}{x}\\ \frac{s_1}{D} &= \frac{1}{x+5}\\ \frac{s_2}{D} &= \frac{1}{x+11.25}\end{aligned}$$

Now, solving for x:

$$\begin{aligned}\frac{1}{x} &= \frac{1}{x+5}+\frac{1}{x+11.25}\\ (x+5)(x+11.25) &= x(x+5+x+11.25)\\ x^2+16.25x+56.25 &= 2x^2+16.25x\\ 56.25 &= x^2\\ \boxed{7.5} &= x\end{aligned}$$

Now, we will recast the problem as a pure geometry problem by using a distance-time graph:



Since $AE||BF, \Delta EGA \sim \Delta BGF$, so

$$\frac{EG}{BG} = \frac{AG}{GF}$$

We also have $\Delta EGC \sim \Delta BGD$, so

$$\frac{EG}{BG} = \frac{EC}{BD}$$

Additionally, $\Delta AGC \sim \Delta FGD$, so

$$\frac{AG}{GF} = \frac{AC}{FD}$$

Putting these together,

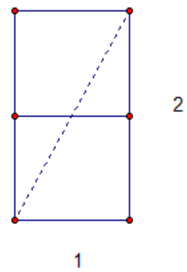
$$\begin{aligned}
\frac{EC}{BD} &= \frac{AC}{FD} \\
\frac{11.25}{x} &= \frac{x}{5} \\
x^2 &= 56.25 \\
x &= \boxed{7.5}
\end{aligned}$$

2.4.9

2.4.10 Relative to an observer, in one hour, the hour hand travels one twelfth of the distance that the minute hand travels, and the minute hand travels a full 360° degrees. Then, relative to the hour hand, the minute hand travels at 11/12 its speed, and thus requires $\frac{12}{11}$ of the time to traverse the full 360°. Thus, the time required is $\boxed{\frac{12}{11}}$ of an hour.

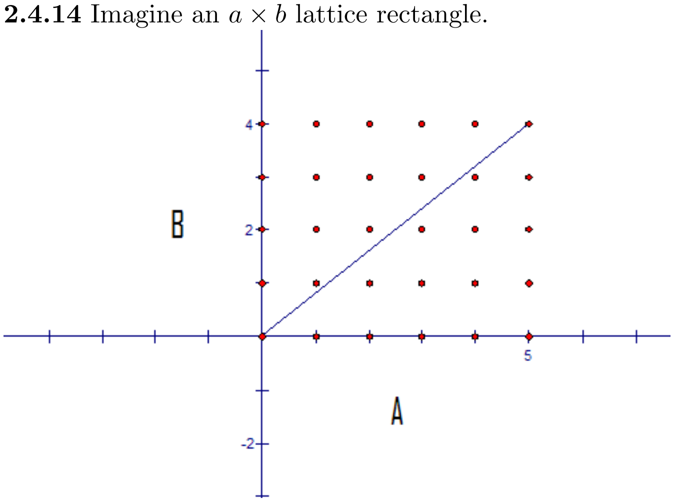
2.3.12 One method is to use partial derivatives, but consider the following simpler approach: we can interpret the sum as the square of the distance between $(u, \sqrt{2-u^2})$ and $(v, \frac{9}{v})$. These are just the parametric equations of a semicircle of radius $\sqrt{2}$ and a hyperbola with vertices $(3,3)$ and $(-3,-3)$. (NEEDS PROOF) The shortest distance occurs between the points $(1,1)$ and $(3,3)$ and have value $\boxed{8}$.

2.4.13 We can unfold the cube as if it were make out of paper.



It is clear that the shortest distance is a straight line between the two points, of length $\boxed{\sqrt{5}}$.

2.4.14 Imagine an $a \times b$ lattice rectangle.



We can interpret

$$\sum_{i=1}^{b-1} \left\lfloor \frac{ai}{b} \right\rfloor$$

as the number of lattice points to the left of the diagonal but not on the coordinate axes. We can interpret

$$\sum_{j=1}^{a-1} \left\lfloor \frac{bj}{a} \right\rfloor$$

as the number of lattice points under the diagonal but not on the coordinate axes. By symmetry, these are clearly equal.

2.4.15 Consider

$$\begin{aligned}
a_n &= \sqrt{1-a_{n-1}} \\
&= \sqrt{1-\sqrt{1-a_{n-2}}} \\
&= \sqrt{1-\sqrt{1-\sqrt{1-a_{n-3}}}} \\
&= \sqrt{1-\sqrt{1-...\sqrt{1-a_1}}}
\end{aligned}$$

Now as $n \to \infty$, we have

$$\begin{aligned} a_n^2 &= 1 - \sqrt{1 - \sqrt{1 - ...\sqrt{1 - a_1}}} \\ &= 1 - a_n \end{aligned}$$

Therefore,

$$\begin{aligned} a_n^2 &= 1 - a_n \\ a_n^2 + a_n - 1 &= 0 \\ a_n &= \frac{-1 \pm \sqrt{5}}{2} \end{aligned}$$

Since the sequence always takes the positive root, $\frac{-1-\sqrt{5}}{2}$ is not a viable solution, so the answer is $\boxed{\dfrac{-1+\sqrt{5}}{2}}$.

2.4.16 We can view

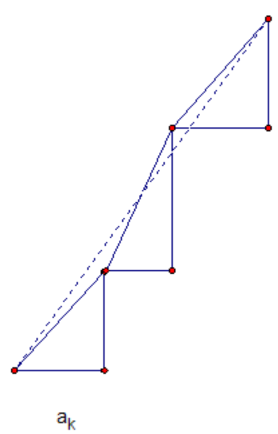
$$\sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2}$$

as the sum of the hypotenuses of n right triangles with parallel legs.

2.4.16 We can view

$$\sqrt{(2k-1)^2 + a_k^2}$$

as the hypotenuse of a right triangle with base $2k - 1$ and height a_k. WLOG, we position the triangles (rotating and reflecting if necessary) such that the a_k are all parallel to a horizontal base and the $2k - 1$ are all parallel along a vertical height.



The given sum forms a path of hypotenuses from the bottom triangle to the top triangle. Clearly, the sum is minimized when the path is a straight line, in which case it is the hypotenuse of a right triangle with legs $\sum_{k=1}^{n} a_k = 17$ and $\sum_{k=1}^{n} 2k - 1 = n^2$. For $n = 10$, the minimum sum is then $\sqrt{17^2 + 100^2} = \boxed{\sqrt{10289}}$.

2.4.17 Substitute $y = x_2$, and consider $(ax + by, cx + dy)$ as the endpoint of the vector sum $(a, c)x + (b, d)y$. Then number of solutions to $ax + by, cx + dy, 0 \leq x, y < 1$ integers is the number of lattice points inside the parallelogram with vertices $(0, 0), (a, c), (a + b, c + d), (b, d)$. Taking

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

the area of this parallelogram is $ad - bc = k$. Also, the area is

$$I + \frac{B}{2} - 1$$

by Pick's theorem. Since $\gcd(a, b) = \gcd(c, d) = 1$, there are no lattice points on the boundary of the parallelogram except for the 4 vertices. Setting these two values equal,

$$ad - bc = k = I + 1$$

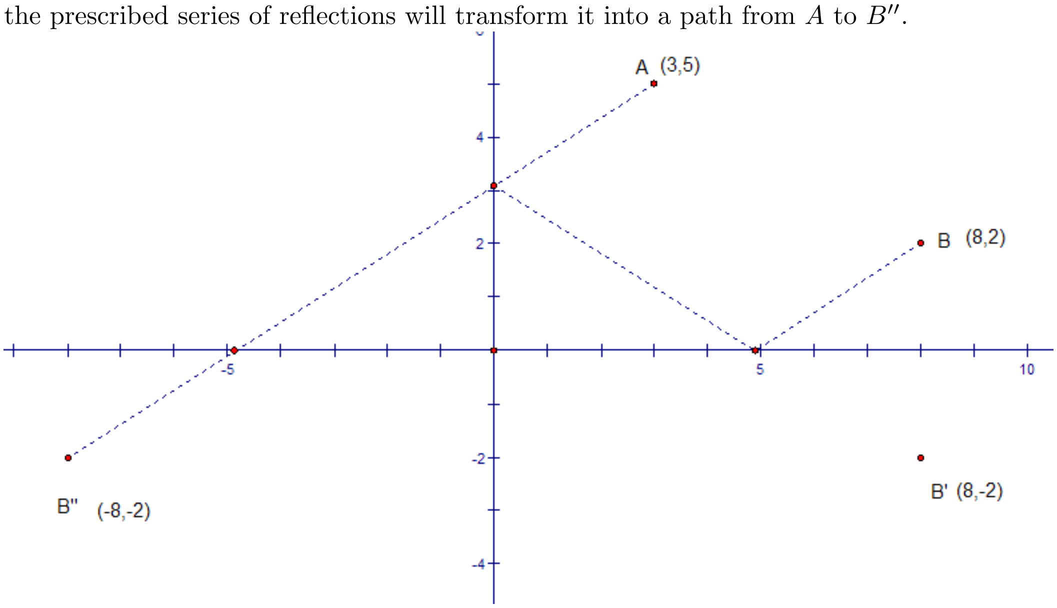
so $I = k - 1$. Since x_1 and x_2 are allowed to be 0, this provides one more solution, for a total of $\boxed{k}$ solutions as desired.

2.4.18 Let's change our point of view on an elastic collision. Suppose the marbles are indistinguishable. Then the marbles colliding inelastically and bouncing back with unchanged speed and reversed direction looks the same as if they had just passed through eachother. Therefore, the set of beginning points and orientations is the same as the ending points and orientations.

Chapter 3: Fundamental Tactics for Solving Problems

3.1: Symmetry

3.1.13 Reflect B across the x axis to a point B' and then across the y axis to a point B''. Draw the straight line from A to B''. Reflecting this across the y axis, and then the sub x-axis portion across the x-axis produces the optimal path. To see this, draw any other path that touches the x axis and y axis, and the prescribed series of reflections will transform it into a path from A to B''.



3.1.14 Consider the product of the arithmetic sequence

$$\begin{aligned}(a-k)(a)(a+k)(a+2k) &= (a^2-k^2)(a^2+2ak) \\ &= a^4 + 2a^3k - a^2k^2 - 2ak^3\end{aligned}$$

This symmetry in these terms inspires us to consider $(a^2 + ak + k^2)^2$. However, clearly one of these terms must be negative. Since we know that $(a+b+c)^2 = \sum_{cyc} a^2 + \sum_{cyc} 2ab$, the positive $2a^3k$ terms motivates us to try $-k^2$.

$$\begin{aligned}(a^2+ak-k^2)^2 &= a^4+a^2k^2+k^4+2(a^3k-ak^3-a^2k^2)\\ &= a^4+2a^3k-a^2k^2-2ak^3+k^4\end{aligned}$$

Which we see is just k^4 more than our original expression. Thus, $(a-k)(a)(a+k)(a+2k)$ is k^4 less than a perfect square, and in the special case where $k=1$, $k^4=1$ as desired.

3.1.15 If n is not a perfect square, then its $d(n)$ divisors can be paired into $\frac{d(n)}{2}$ pairs with product n. Therefore, the product of the divisors is $\boxed{n^{\frac{d(n)}{2}}}$. In fact, this formula still works if n is a perfect square, because although the square root has no complement, it contributes exactly $n^{\frac{1}{2}}$ to the total product.

3.1.16 (a)The result follows from simply multiplying $(x+y+z)^2$ out.

(b) We have already shown the identity

$$x^3+y^3+z^3=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)+3xyz$$

$(x+y+z)=s_1$, $-(xy+yz+zx)=-s_2$, and $3xyz=3s_3$. In addition, we have already shown that $x^2+y^2+z^2=s_1^2-2s_2$. Putting these together,

$$\begin{aligned}x^3+y^3+z^3 &= s_1(s_1^2-2s_2-s_2)+3s_3\\ &= s_1^3-3s_1s_2+3s_3\end{aligned}$$

(c) We note that $(x+y+z)^3 = x^3+y^3+z^3+3(x+y)(y+z)(z+x)$. Since we already know the symmetric functions for the other two expressions we can find the third:

$$\begin{aligned}(x+y)(y+z)(z+x) &= \frac{(x+y+z)^3-(x^3+y^3+z^3)}{3}\\ &= \frac{s_1^3-(s_1^3-3s_1s_2+3s_3)}{3}\\ &= \frac{3s_1s_2+s_3}{3}\\ &= s_1s_2+s_3\end{aligned}$$

(d) We factor: $x^4(y+z)+y^4(z+x)+z^4(x+y)$.

(e)

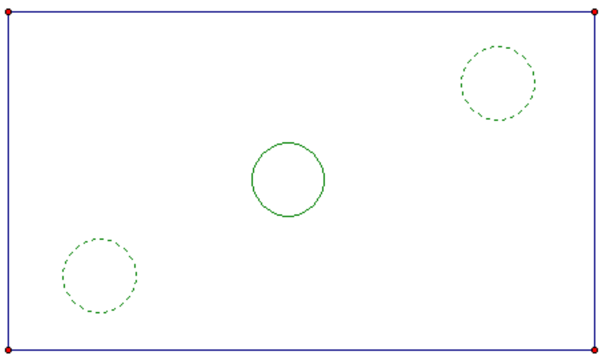
(f)

(g)

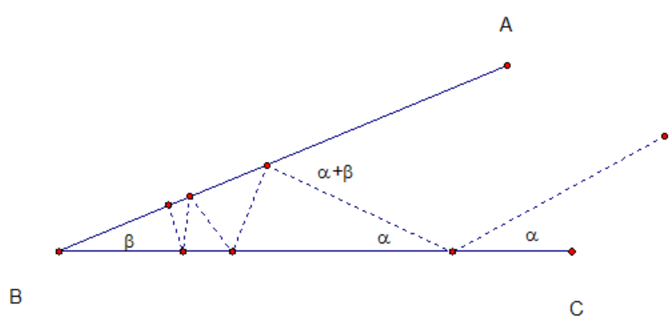
(h)

3.1.17 By symmetry, the bugs always maintain the configuration of a square. After a certain amount of time t, they form a square with half the side length of the original and have turned a certain number of degrees x. After another $\frac{t}{2}$ seconds, they will have turned another x degrees. Since the time required to turn x° always halves, they turn an infinite number of degrees.

3.1.18 The first player should place his first penny at the center of the table. Afterwards, he should 'mirror' the second player's move by reflecting it over this center penny. This guarantees that he always has a move if the second player does by the symmetry of the configuration.



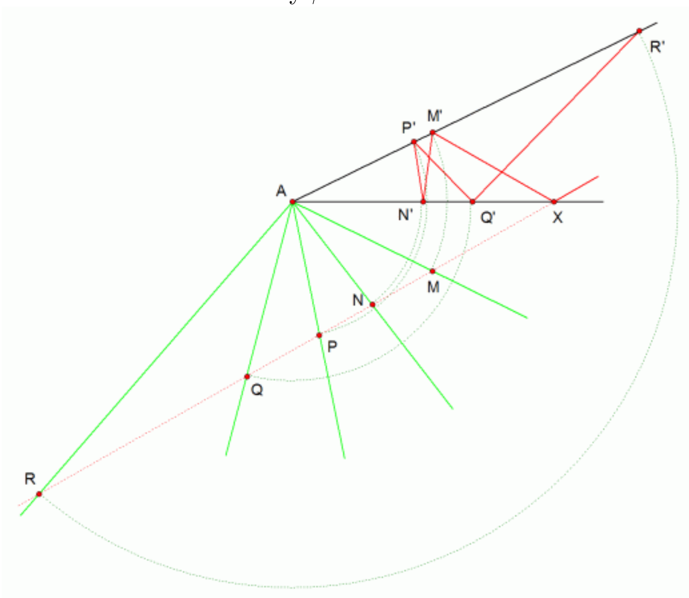
3.1.19 Every time the billiard ball bounces, the angle of incidence increases by β (by the exterior angle theorem). When the ball begins to bounce backwards, we can interpret this as an obtuse angle of incidence.



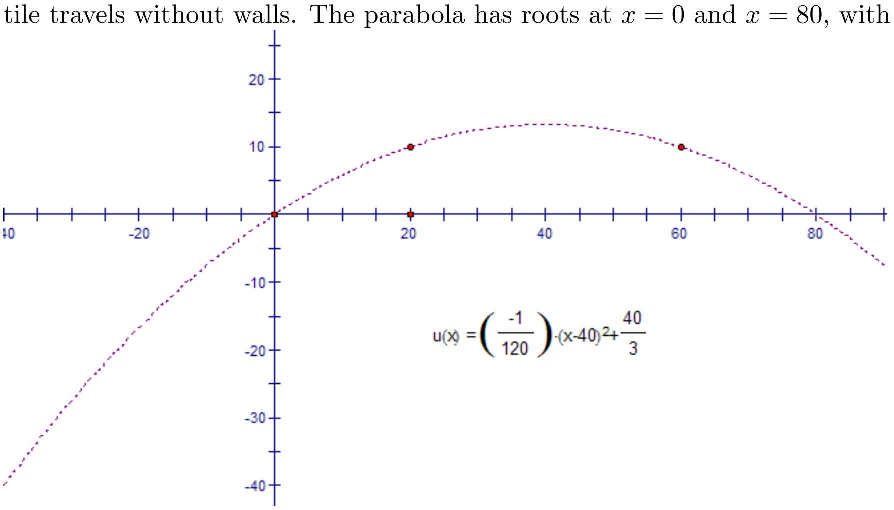
When the angle exceeds $180 - \beta$, the ball will not hit the walls again. Suppose it bounces k times, including the first bounce at C. Then we want the minimum k such that

$$\begin{aligned}
\alpha + (k-1)\beta &\geq 180 - \beta \\
k\beta &\geq 180 - \alpha \\
k &\geq \frac{180 - \alpha}{\beta} \\
k &= \boxed{\left\lceil \frac{180 - \alpha}{\beta} \right\rceil}
\end{aligned}$$

3.1.20 Rotate the angle β. Each time the billiard ball bounces corresponds to the original line hitting one of the rotations. Now the formula $\boxed{\left\lceil \dfrac{180-\alpha}{\beta} \right\rceil}$ becomes obvious, as the angle with which the line hits the rotations decreases by β each time.



3.1.20 If we reflect the path of the ball across the wall, the reflected path is the trajectory the projectile travels without walls. The parabola has roots at $x = 0$ and $x = 80$, with $y = 10$ at $x = 20$ and $x = 60$.



The equation of this parabola is $-\frac{1}{120}(x-40)^2 + \frac{40}{3}$, so the vertex is located at $\left(40, \frac{40}{3}\right)$ and the maximum height if $\boxed{\dfrac{40}{3}}$ feet.

3.1.21

3.1.22

3.1.23 We wish to impose symmetry somehow, but the function T is not symmetrical, so how do we proceed? Let's consider

$$\begin{aligned}
T(x,y,z)+T(y,z,x)+T(z,x,y) &= \sum_{cyc}(x+y)^2+\sum_{cyc}(y-z)^2 \\
&= \sum_{cyc}(x^2+2xy+y^2)+\sum_{cyc}(y^2-2yx+z^2) \\
&= 4\sum_{cyc}x^2
\end{aligned}$$

Since the planet is a sphere with radius 20, $\sum_{cyc} x^2 = 20^2 = 400$. Now let's examine our result. We have shown that any point (x,y,z), when taken together with (y,z,x) and (z,x,y), has an average of $\frac{1600}{3}^{\circ}$. Therefore, the average temperature is $\boxed{\dfrac{1600}{3}^{\circ}}$.

3.1.24 First, we simplify the given expression:

$$\frac{1}{1+(\tan x)^{\sqrt{2}}} = \frac{(\cos x)^{\sqrt{2}}}{(\cos x)^{\sqrt{2}}+(\sin x)^{\sqrt{2}}}$$

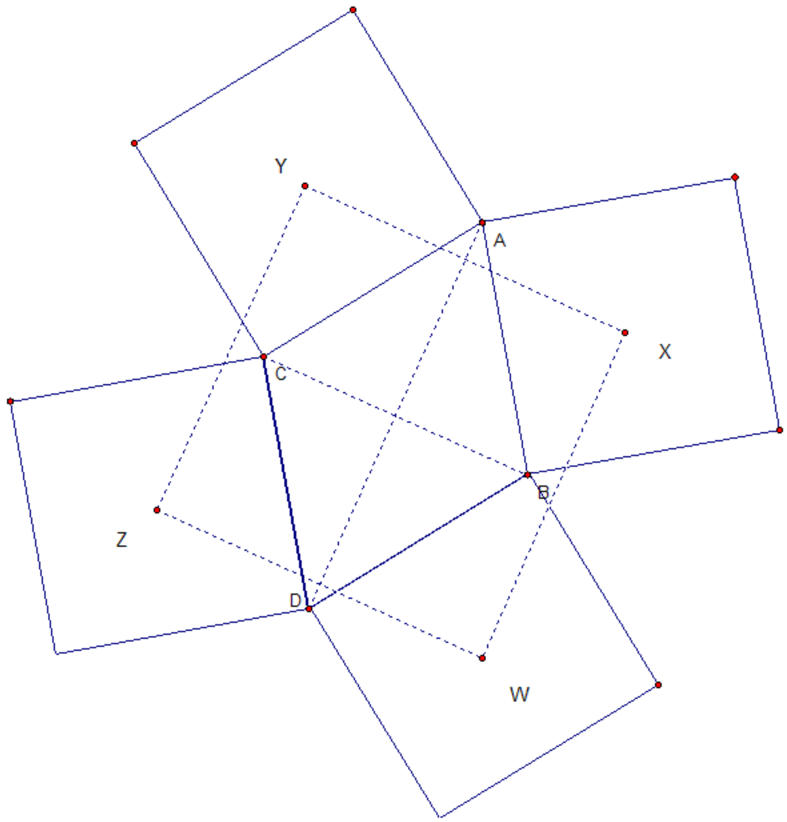
Now we remember that $\cos x = \sin(\pi/2 - x)$, which allows us to impose symmetry in the problem:

$$\begin{aligned}
\int_0^{\pi/2}\frac{(\cos x)^{\sqrt{2}}}{(\cos x)^{\sqrt{2}}+(\sin x)^{\sqrt{2}}}dx &= \int_{\pi/2-\pi/2}^{\pi/2-0}\frac{(\cos x)^{\sqrt{2}}}{(\cos x)^{\sqrt{2}}+(\sin x)^{\sqrt{2}}}dx \\
&= \int_{\pi/2}^{0}-\frac{(\cos(\pi/2-x))^{\sqrt{2}}}{(\cos(\pi/2-x))^{\sqrt{2}}+(\sin(\pi/2-x))^{\sqrt{2}}}dx \\
&= \int_{\pi/2}^{0}\frac{(\sin x)^{\sqrt{2}}}{(\sin x)^{\sqrt{2}}+(\cos x)^{\sqrt{2}}}dx \\
&= \int_{0}^{\pi/2}\frac{(\sin x)^{\sqrt{2}}}{(\sin x)^{\sqrt{2}}+(\cos x)^{\sqrt{2}}}dx
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^{\pi/2}\frac{(\cos x)^{\sqrt{2}}}{(\cos x)^{\sqrt{2}}+(\sin x)^{\sqrt{2}}}dx &= \frac{1}{2}\int_0^{\pi/2}\frac{(\cos x)^{\sqrt{2}}}{(\cos x)^{\sqrt{2}}+(\sin x)^{\sqrt{2}}}+\frac{(\sin x)^{\sqrt{2}}}{(\sin x)^{\sqrt{2}}+(\cos x)^{\sqrt{2}}}dx \\
&= \frac{1}{2}\int_0^{\pi/2} 1dx \\
&= \boxed{\frac{\pi}{4}}
\end{aligned}$$

3.1.25 A 90° rotation of the rhombus sends W to X, X to Y, Y to Z, and Z to W, implying that $WXYZ$ is a quadrilateral with equal and orthogonal diagonals. Therefore, $WXYZ$ is a square.



3.1.26

3.1.27

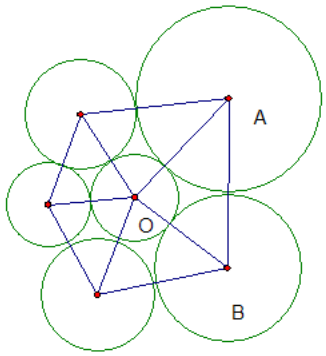
3.1.28

3.2: The Extreme Principle

3.2.7 Since all the squares contain positive integers, we can choose the square with the minimum number. If any of the squares around it have a larger number, then their average will be greater, which is impossible. Therefore, all the squares have the same number.

3.2.8 Since there are a finite number of points, we can again choose the smallest number. If any of the adjacent numbers is greater, then the average will be greater.

3.2.9 Pick the coin with the smallest diameter. Suppose it is tangent to more than 5 other coins.



Consider the triangle formed by the center O of this smallest circle with the centers A and B of two other tangent circles. By definition, the circle with center O has the smallest radius, so $OA, OB < AB$. This implies that $\angle AOB > 60$. Therefore, if the coin is tangent to at least six other coins, the total angle around O is greater than $360°$, clearly impossible.

3.2.10 First, we note that if anybody is hit more than once, then somebody must be left dry. Therefore, assume that nobody is hit more than once. Consider the two people the least distance apart. These two people shoot eachother, and by assumption nobody else shoots them. Therefore, we can eliminate them from our consideration. We can again choose the two closest people and eliminate them. Eventually, we are reduced to one person, who must remain dry.

If n is even, the argument fails at the last step. Indeed, it is not hard to see that n groups of 2 people placed far apart will result in everybody being shot.

3.2.11 We proceed by induction. For two numbers a and b, the hypothesis holds, since they become $\mathrm{GCD}(a, b)$ and $\mathrm{LCM}(a, b)$. Suppose that the hypothesis holds for n. We have already shown that for $n+1$ numbers, there is a least number l_k which divides all the other numbers. We can throw this number out. By hypothesis, the remaining n number can be arranged in such a chain, and the induction is complete.

3.2.12 Suppose we do have such a problem produced by writing the numbers from 1 to n in order, with $10^k \leq n < 10^{k+1}$. Consider the longest string of 0's (which will have length k) and choose the one with the least number in front of it (which will be a 1) in case there are ties. Since a number cannot start with 1, we know the string

$$1 \ \underbrace{0 \ 0 \ \ldots \ 0}_{k \text{ times}}$$

is part of a single number m. Additionally, since it is $k+1$ digits long, it must in fact be the entire number. Suppose first that this string occurs entirely on one side of the number. Then there must be a string

$$\underbrace{0 \ 0 \ \ldots \ 0}_{k \text{ times}} \ 1$$

On the other side of the number, but this is implies that the string

$$1 \ \underbrace{0 \ 0 \ \ldots \ 0}_{k \text{ times}}$$

occurs on the other side as well, clearly impossible. Now suppose that the string occurs in the middle of the number, on point of symmetry. However, this is impossible, since we know that it is preceded by 9 and followed by 0:

$$9 \ 1 \ \underbrace{0 \ 0 \ \ldots \ 0}_{k \text{ times}} \ 1 \ 0$$

Therefore, we conclude that no such palindrome can exist.

3.2.13 Consider the smallest and largest values, 1 and n^2. Their difference is $n^2 - 1$. Additionally, there exists a path of at most $n-1$ steps from 1 to n^2. Therefore, at least one of the steps has value at least $\frac{n^2-1}{n-1} = n+1$.

3.2.14 First, it is clear that any graph not in one of the two categories cannot be drawn in the specified manner, since if a vertex has odd degree, it is impossible to both get in and get out every time. Therefore, if a graph has vertices of odd degree, they must be the starting and ending vertices.

Now we will prove that any graph in with all edges of even degree or only two edges of odd degree, in which case the starting and ending vertices must have odd degree, can be drawn in such a way. Call an edge *open* if it has already been traveled and *closed* otherwise. Consider the longest possible path; we claim that this path traverses each edge once and only once. Suppose otherwise; then at the end of this longest path, we are at a vertex with no open edges. This implies that we are at a vertex with odd degree or even degree and the begining vertex, since all the other vertices have even degree and if we get to them, we can always get away as well. By supposition, there exists at least one pair of vertices somewhere that are connected by open edges. If they are connected by more than one open edge, then we could easily have increased the length of the path by traveling back and forth the appropriate number of times, thus contradicting the maximality of the path. Therefore, they are connected by only one open edge, implying that they are both of odd degree. But our current vertex is either of odd degree or even degree and the beginning vertex, so this is impossible! We conclude that any such path can be drawn in the the desired manner.

3.2.15 (a)Consider the values of $b - at$ as t ranges over the positive integers. We claim that the minimum nonnegative value r satisfies $0 \leq r < a$. Otherwise, the minimum value is greater than or equal to a. But subtracting a from this produces a smaller nonnegative integer. Contradiction!

(b) $a|b$ implies $b = k_1 a$. Likewise, $a|c$ implies $c = k_2 a$. Putting these two together, $bx + cy = ak_1x + ak_2y = a(k_1x + k_2y)$.

(c) Consider $a = \prod p_i^{e_i}$ and $b = \prod p_i^{f_i}$. Then $\text{LCM}(a, b) = \prod p_i^{\max(e_i, f_i)}$. Additionally, let $m = \prod p_i^{d_i}$. Then $a|m \implies d_i > e_i$ and $b|m \implies d_i > f_i$, so $d_i > \max(e_i, f_i)$. Therefore, $\text{LCM}(a, b)|m$.

(d)Consider the minimum positive value d of $ax + by$. By the division algorithm,

$$\begin{aligned} a &= dp + r \\ b &= dq + s \end{aligned}$$

with $0 \leq r, s < d$. Then r and s are expressable as linear combinations of a and b, with $0 \leq r, s < d$. Since d is the minimum possible positive integer expressable as a linear combination of a and b, $r = s = 0$. Therefore, $d|a$ and $d|b$, so $d|\gcd(a, b)$. Also, as proved earlier, $\gcd(a, b)|ax + by$. Therefore, $\gcd(a, b)|d$ and $d|\gcd(a, b)$, implying that they are equal.

3.2.16 Consider P in the field $Z[q]$. Since q divides all the coefficients except for a_n, P reduces to $a_n x^n$. If this is reducible, it must be into at least two polynomials, both without constant terms in the field $Z[q]$, implying that $q^2|a_0$, which is impossible. Contradiction.

3.3: The Pigeonhole Principle

3.3.10 Suppose that all of the holes contain at most $\lceil p/h \rceil - 1$ pigeons. Then the total number of pigeons is at most

$$\begin{aligned} h(\lceil p/h \rceil - 1) &< h(p/h + 1 - 1) \\ &< p \end{aligned}$$

which is impossible.

3.3.11 Suppose that there are n people. Then the maximum possible number of people that anyone can know is $n-1$, and the minimum is 0. However, it is no possibility for one person to know everyone and one person to know nobody. Therefore, there are only $n-1$ possibilities for the number of people known and n people. By the pigeonhole principle, two people know the same number of people.

3.3.12 Suppose that the statement never holds; then the total number of pigeons is less than or equal to $(a_1 - 1) + (a_2 - 1) + ... + (a_n - 1) = a_1 + a_2 + ... + a_n - n$. However, there are $a_1 + a_2 + ... + a_n - n + 1$ pigeons, so this is impossible.

3.3.13 There is no need for the first 4 points to be maximally separated.

3.3.14

3.3.16 Divide the hexagon into six equilateral triangles. By the pigeonhole principle, two points are in the same triangle, so they are at most 1 unit apart.

3.3.16 Split the square into 50 rectangles, each of area $\frac{1}{50}$. By the pigeonhole principle, three points lie in the same rectangle. Therefore, they have area at most $\frac{1}{100} = .001$.

3.3.17 If any two of the integers are congruent mod $2n$, then their difference is divisible by $2n$. Therefore, suppose that no two are congruent mod $2n$. Split the residues into $n+1$ sets:

$$\{0\}, \{1, 2n-1\}, \{2, 2n-2\}, ...\{n-1, n+1\}, \{n\}$$

There are at least n numbers to put in the $n-1$ sets

$$\{1, 2n-1\}, \{2, 2n-2\}, ...\{n-1, n+1\}$$

By the pigeonhole principle, at least two are in the same set, so their sum is a multiple of $2n$.

3.3.18 Split the set into pairs:

$$\{1, 2\}, \{3, 4\}, ...\{2n-1, 2n\}$$

By the pigeonhole principle, there must be two numbers in the same pair, which implies that they are relatively prime.

3.3.19 Suppose that there are n people. Rotate the table $n-1$ times; each person has the right dish once. Therefore, there are n occasions where a person has the right dish and $n-1$ rotations; by the pigeonhole principle, there must be a rotation where at least two people have the correct dish.

3.3.20 Let the sequence be $a_1 a_2 ... a_N$, where a_i represent digits and let the n distinct integers be $x_1, x_2, ... x_n$. Consider the products

$$\begin{array}{r} a_1 \\ a_1 a_2 \\ a_1 a_2 a_3 \\ ... \\ a_1 a_2 ... a_N \end{array}$$

Consider the 'prime factorization' of each product $x_1^{e_1} x_2^{e_2} ... x_n^{e_n}$, and consider the exponents $e_1, e_2, ... e_n$ in base 2. If all of them are even, then we are done, for the product is a perfect square. Otherwise, there are $2^n - 1$ possible odd-even combinations and 2^n products. By the pigeonhole principle, there must be two with the same odd-even exponent combination. Their quotient is of the form $a_j a_{j+1} ... a_k$ and has all even quotients, so it is a perfect square.

This can be tightened by replacing n in $N > 2^n$ with the number of actual primes involved.

3.3.21 The greatest possible value attainable is $m(1+\sqrt{2})$. Split the number line from 0 to $m(1+\sqrt{2})$ into $(m+1)^2 - 1$ different intervals. There are $(m+1)^2$ possible values of $a + b\sqrt{2}$ with $0 \le a \le m$ and $0 \le b \le m$. By the Pigeonhold principle, there must be two, $x_1 + y_1\sqrt{2}$ and $x_2 + y_2\sqrt{2}$, which lie in the same interval. Their difference is between 0 and

$$\frac{m(1+\sqrt{2})}{m^2+2m} = \frac{1+\sqrt{2}}{m+2}$$

Additionally, they are of the form $a + b\sqrt{2}$, with $-m \le a, b \le m$.

3.3.22 Consider

$$\begin{array}{r} 7 \\ 77 \\ 777 \\ ... \\ \underbrace{7777...7}_{n+1 \text{times}} \end{array}$$

We have $n+1$ numbers and n possible remainders when divided by n; by the pigeonhole principle, there must be two numbers congruent mod 7. There difference is divisible by 7 and consists of only 0's and 7's.

3.3.23 $\{1, 2, 4, 8, 16\}$; $\{3, 6, 12, 24\}$; $\{5, 10, 20\}$; $\{7, 14\}$; $\{9, 18\}$; $\{11, 22\}$; $\{13\}$; $\{15\}$; $\{17\}$; $\{19\}$; $\{21\}$; $\{23\}$; $\{25\}$

3.3.24 Let a_i be the total number of games that the chess player plays up until the ith day. Consider the union of the two sets

$$\{a_1, a_2, ... a_{56}\} \cup \{a_1 + 23, a_2 + 23, ... a_{56} + 23\}$$

a_1 is at least 1, and $a_{56} + 23$ is at most $88 + 23 = 111$. Also, the numbers in each set are distinct, since the player has at least one chess game per day. Therefore, the union has at most 111 elements. Each of the 56 days of the tournament contributes 2 values, one to the first set and one to the second set, for a total of 112 values. By the Pigeonhole principle, there must be two that are equal, so there is some number $a_j + 23$ in the second set which is equal to some number a_k in the first set. Then from the jth day to the kth day, the chess player plays exactly 23 games.

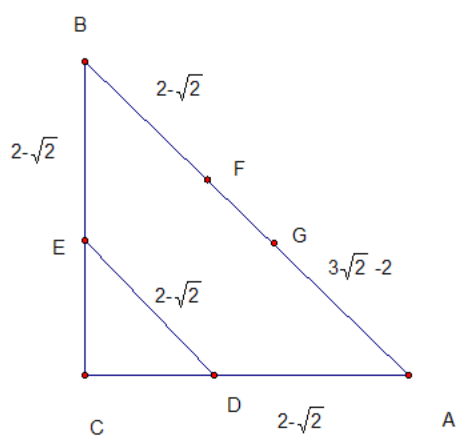
3.3.25 (note: the question in the book omits the fact that the triangle is right) Construct the points D and E such that D is $2-\sqrt{2}$ away from A on AC and E is $2-\sqrt{2}$ away from D on CB. Then $EC = CD = \sqrt{2}-1$, so $EB = 2-\sqrt{2}$. Lastly, construct F and G so that F is $2-\sqrt{2}$ away from B on AB and G is $2-\sqrt{2}$ away from A on AB. Also,

$$\begin{aligned}
DF^2 = EG^2 &= (2-\sqrt{2})^2 + (3\sqrt{2}-2)^2 - 2(2-\sqrt{2})(3\sqrt{2}-2)\cos 45^\circ \\
&= 6 - 4\sqrt{2} + 22 - 12\sqrt{2} - \sqrt{2}(6\sqrt{2} - 6 - 4 + 2\sqrt{2}) \\
&= 28 - 16\sqrt{2} - (16 - 10\sqrt{2}) \\
&= 12 - 6\sqrt{2} \\
&> 6 - 4\sqrt{2} \\
&= (2-\sqrt{2})^2
\end{aligned}$$

Therefore, $DF = EG > 2-\sqrt{2}$. Finally,

$$\begin{aligned}
CF^2 = CD^2 &= 1^2 + (2-\sqrt{2})^2 - 2(2-\sqrt{2})(1)\cos 45 \\
&= 1 + 6 - 4\sqrt{2} - \sqrt{2}(2-\sqrt{2}) \\
&= 7 - 4\sqrt{2} - 2\sqrt{2} + 2 \\
&= 9 - 6\sqrt{2} \\
&> 6 - 4\sqrt{2} \\
&= (2-\sqrt{2})^2
\end{aligned}$$

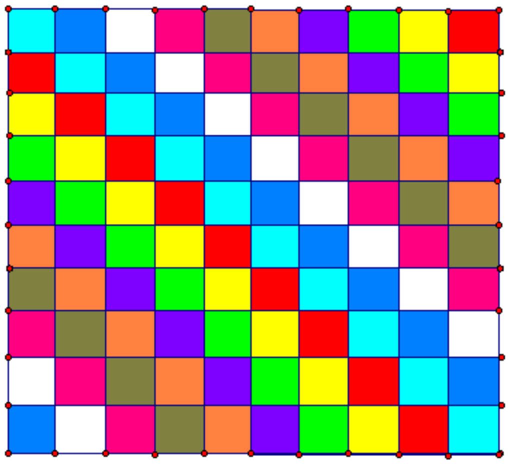
So $DF = EG > 2-\sqrt{2}$.



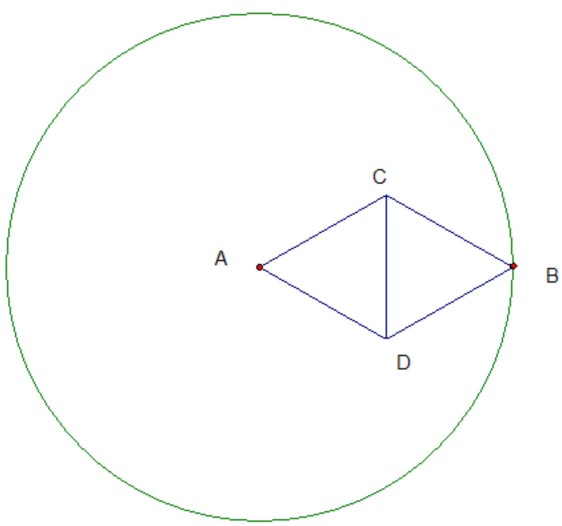
Since A and B are at least $2-\sqrt{2}$ away from any of the points C, D, E, F, G, they take up two colors. Therefore, if the triangle can be 4-colored, the C, D, E, F, G must be 2-colored. Suppose F is color A. Then D must be color B, since it is further than $2-\sqrt{2}$ away from F. Then E must be color A, since it is $2-\sqrt{2}$ away from D. Finally, G must be B, since it is further than $2-\sqrt{2}$ away from E. But C is further than $2-\sqrt{2}$ from both F and G, so it cannot be color A or B. Thus, the points cannot be 2-colored and the triangle cannot be 4-colored.

3.3.26 Two rooks attack eachother iff they are in the same row or the same column. Let x_k be the column of a given square R_k and y_k be the row. Then we color the diagonals according to the rule $x_{k+1} = x_k + 1$

and $y_{k+1} = y_k + 1$, where we subtract 10 if the sum exceeds 10. Then we have covered the chessboard with 10 diagonals, all with on different columns and rows. By the pigeonhole principle, there must be at least one row with at least 5 rooks.



3.3.27 (a) Draw a circle about a point A, WLOG colored red, with radius $\sqrt{3}$. If all the points on the circle are red, then we can select a cord of length 1 and we are done. Otherwise there exists a point B colored, without loss of generality, green. We can draw two equilateral triangles of side length 1 that share a base, with opposite vertices at A and B since we set the radius to be $\sqrt{3}$, twice the height of the triangle. Let the shared edge be CD. If either C or D is red or green, then we are done; otherwise, they are both the third color, WLOG blue, and we are also done.



(b) Let the colors be red and blue. Assume to the contrary, then there exist distances m and n, WLOG $m < n$ (as we have already shown the equality case), such that there are no two blue points m apart and no two red points n apart. There must be a red point somewhere, or the problem is trivial. Draw a circle of radius n around this point. If any point on the circle is red, then we have two red points n apart, a contradiction. Otherwise, the entire circle is blue. Now draw a circle with radius $m + n$ around the red point. If any point on this circle is blue, then we have two blue points a distance of m apart, so the circle must be red. But then we can easily find a chord of length n. Contradiction.

(c)

(d)

(e) Draw 7 parallel horizontal lines in the plane. Consider three vertical lines that intersect all 13 lines at the same x-coordinates. For each line, by the pigeonhole principle, two of the intersections have the same color. There are $\binom{3}{2} = 3$ ways to select a pair of points with the same color, and there are two possible colors. Again by the pigeonhole principle, at least two of the 7 lines must have the same color configuration. Connecting these points forms a rectangle as desired.

3.4: Invariants

3.4.17 Let the consecutive integers run from x_1 to x_n, with $x_i + 1 = x_{i+1}$. Then we have the sum

$$\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}$$

First, we claim that there is a unique number $x_k, 1 \leq k \leq n$ such that $p_2(x_k)$ is maximal. Otherwise, suppose that there are two numbers $2^j a$ and $2^j b$ which have the maximal p-adic valuation, with a and b both odd and $b > a$. But then the number $2^j(a+1)$ is also included, and $a+1$ is even, so $p_2(2^j(a+1)) > p_2(2^j a)$, contradicting our hypothesis.

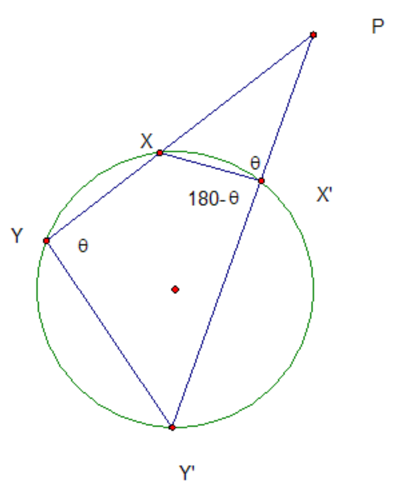
$$\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n} = \frac{\sum_{j=1}^n \prod_{i \neq j} x_i}{\prod_{j=1}^n x_j}$$

Let $m := p_2(\prod_{j=1}^n x_j)$. Then $p_2(\prod_{i \neq j} x_i) = m - p_2(x_j)$. Since there exists a unique x_j such that $p_2(x_j)$ is maximal, there is a term $\prod_{i \neq j} x_i$ in the numerator such that $p_2(\prod_{i \neq j} x_i)$ is minimal. If we divide out by this power of two, then we have the sum of several even numbers and one odd number in the numerator, which is clearly odd.

3.4.18 Since $\angle PX'X$ and $\angle PYY'$ are both supplementary to $\angle XX'Y'$, they are equal. By AA, $\Delta PX'X \sim \Delta PYY'$, so

$$\frac{PX'}{PX} = \frac{PY}{PY'}$$

Multiply out gives $PX'(PY') = PX(PY)$ as desired.



3.4.19 We note that $11 \equiv -1 \pmod{10}$. Suppose $n = a_1 * 10^n + a_2 * 10^{n-1} + ... + a_0$. Then $n \equiv a_1(-1)^n + a_2 * (-1)^{n-1} + ... + a_0 \pmod{10}$. Therefore, if we define $S(n)$ as the alternating sum of the digits of n, then $n - S(n)$ is divisible by 11.

3.4.20 No. We claim that, for any two points (x, y) in the set produced by the three operations on $(7, 29)$, $11|(x-y)$. First, $11|(7-29)$. Clearly, if $11|(x-y)$, then $11|((x+1)-(y+1)|$, so the first operation preserves this property. If $11|(x-y)$, and x and y are both even, then $11|(\frac{x-y}{2})$, so the second operation also preserves this property. Finally, if $11|(x-y)$ and $11|(y-z)$, then $11|(x-y+y-z = x-z)$. Since $1999 - 3 = 1996$, which is not divisible by 11, we can never get from $(7, 29)$ to $(3, 1999)$.

3.4.21 Factor: $uv+u+v = (u+1)(v+1)-1$. Now we see that each new term will be of this form. For example, if we choose to combine this with another element w, then we have $((u+1)(v+1)-1+1))(w+1)-1 = (u+1)(v+1)(w+1)-1$. Therefore, the last term will always be $(1+1)(2+1)...(100+1)-1 = \boxed{101!-1}$.

3.4.22 Since $a-b|b-c$ and $b-c|c-a$, $a-b|c-a$. Also, $c-a|a-b$. This implies that $c-a = a-b$. Similarly, $b-c = c-a = a-b$. Imagine the three numbers a, b, c on a number line. WLOG, $a > b > c$. Then a is some distance k to the right of b and c is some distance k to the left. But $c - a = 2k = k$, which is impossible.

3.4.23 Consider the function $f(\{x_1, x_2, x_3\}) = x_1^2 + x_2^2 + x_3^2$. Then $f(\{3, 4, 12\}) = 3^2 + 4^2 + 12^2$. Additionally,

$$(.6a - .8b)^2 + (.8a - .6b)^2 = .36a^2 + .64a^2 + .36b^2 + .64b^2 - 2(.6)(.8)ab + 2(.6)(.8)ab = a^2 + b^2$$

Therefore, the operation preserves the value of f on the set. Since $f(\{3, 4, 12\}) < f(\{4, 6, 12\})$, $\{3, 4, 12\}$ cannot be transformed into $\{4, 6, 12\}$.

3.4.24 Originally, the chocolate bar is in one piece. At the end, it will be in 48 pieces. Each cut increases the number of pieces by 1. Therefore, the second player always makes the last move. The solution is similar for an $m \times n$ bar.

3.4.25

3.4.26

3.4.27 Let a be the number of outside entrance doors and b be the number of even entrance doors. If we add up all the doors per room, then our sum is $a + 2b$, since every door in a is counted once and every door in b, because it connects two rooms, is counted twice. Additionally, we have $a + 2b$ is even, since every room has an even number of doors, so a is even.

3.4.28 We will prove that all the players' weights must be congruent mod 2^n for all n. This implies that they all have the same weight. We proceed by induction. For the base case, all the weights must have the same parity; otherwise, we can choose a referee so that there are an odd number of odd weights. Then clearly it is impossible to divide the players into two different teams. We conclude that all the players' weights can be written as $2k$ or $2k+1$, with k an integer. Therefore, they are all congruent mod 2. Then the property still holds if we divide by 2 (in the former case) or subtract 1 and divide by 2 (in the latter case). Therefore, they are all congruent mod 4. Continuing in this manner, the players' weights must be congruent mod 2^n for all n, as desired.

3.4.29 (Note: we assume that the problem omits the fact that the boy must move less than n spaces; otherwise, the problem is trivial) We claim that only even n work. First, we will prove that no odd n work. If the boy rides the same horse twice, then he cannot ride all n horses. The boy moves a total of $n-1$ times. He clearly cannot move a distance of 0, otherwise he rides the same horse twice. Therefore, he must move $1, 2, ...n-1$ in some order. However, this sums to $\frac{n(n-1)}{2}$, and since n is odd, this is divisible by n. Therefore, at the last move, he ends up on the same pony as he started on.

Now it suffices to show that there is a sequence of moves that always guarantees him to ride all horses for even n. We claim that the sequence $1, n-2, 3, n-4, 5, ...4, n-3, 2, n-1$. First, it is clear that all the terms in the sequence are distinct, because every other term starting with the first runs through the odd numbers $1, 3, 5, ...n-1$ and every other term starting with the second runs through the even numbers $n-2, n-4, ...2$. Let a_i be the sum of the first i terms mod n. Then $a_1 = 1, a_2 = n-1, a_3 = 2, a_4 = n-2, ...a_{n-2} = n/2+1, a_{n-1} = n/2$. The boy will ride all the horses.

3.4.30 There are 9 points and 8 possible odd-even states of the coordinates. By the Pigeonhole principle, two have the same odd-even combination for their coordinates. Then the midpoint of these two points is an integer.

3.4.31 Write the first few terms in mod 2:

$$0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1$$

Since the sequence can be explicitly constructed from 7 terms, the sequence $0, 1, 0, 1, 0, 1, 1$ repeats. Therefore, it is impossible for $1, 3, 5, 7, 9$ to occur.

Alternate Solution: The function $2x_1 + 4x_2 + 6x_3 + 8x_4 + 10x_5 + 12x_6$ is clearly invariant mod 10.

3.4.34 By the Pigeonhole principle, at least $\lfloor 1978/6 \rfloor = 330$ members must come from the same country. Let this country be A. Then it has at least 330 members $a_1, a_2, ...a_{330}$. Consider the 329 differences

$$a_2 - a_1, a_3 - a_1, ...a_{330} - a_1$$

If any of these differences are in A, then we are done. Otherwise, by the pigeonhole principle, there are at least $\lfloor 329/5 \rfloor = 66$ in some country B. Then B has at least 66 members $b_1, b_2, ...b_{66}$ of the form $a_k - a_1$. If any of the 65 differences

$$b_2 - b_1, b_3 - b_1, ...b_{65} - b_1$$

is in B, then we are done. Furthermore, if any of them are in A, then we are also done, since $b_k - b_1 = (a_k - a_1) - (a_2 - a_1) = a_k - a_2$. Therefore, they must be in the other four countries. By the pigeonhole principle, there must be $\lfloor 65/4 \rfloor = 17$ in some country C, with at least 17 members $c_1, c_2, ...c_{17}$ of the form $b_k - b_1$. We consider the 17 differences

$$c_2 - c_1, c_3 - c_1, ...c_{17} - c_1$$

If any of these are in C, then we are done. If any are in B, then we are also done since $c_k - c_1 = (b_k - b_1) - (b_2 - b_1) = b_k - b_2$. If any are in A, we are also done because $c_k - c_1 = b_k - b_2 = (a_k - a_1) - (a_3 - a_1) = a_k - a_3$. So there exists a country D with at least $\lfloor 17/3 \rfloor = 6$ members $d_1, d_2, ...d_6$ of the form $c_k - c_1$. We consider the 5 differences

$$d_2 - d_1, d_3 - d_2, ...d_6 - d_1$$

By a similar argument, none of them can be in D, C, B, or A. Then there exists a country E with at least $\lfloor 5/2 \rfloor = 3$ members e_1, e_2, e_3 of the form $d_k - d_1$. Consider the 2 differences

$$e_2 - e_1, e_3 - e_1$$

By a similar argument, neither of these can be in E, D, C, B, or A, so they must both be in the last country F. But then $f_2 - f_1$ cannot be in F, E, D, C, B, or A, so at least one country must have one member who is the sum of two other members.

3.4.35 Consider listing the elements of the matrix in two ways. First, we count along all the rows and then all the columns. Then every entry is listed twice.

Second, we count the first row and the first column, the second row and the second column, etc. Clearly, this is just a reordering of the first counting method. By the definition of the silver matrix, each time we count the ith row and the ith column, we encounter all the elements $1, 2, ...2n-1$, plus a repeat of the elements on the main diagonal. If n is odd, then we encounter the $2n-1$ numbers n times each, plus n repeats on the main diagonal. At least $n-1$ of the numbers are not repeated, and so appear an odd number of times, which is impossible.

3.4.36

3.4.37 When we remove an edge, the two divided faces are merged, so the number of faces is reduced by one as well, preserving the equation. When we remove a vertex, we remove all the emanating edges and combine all the surrounding faces into one. Since there are the same number of surrounding edges and faces, this also preserves the equality.

Chapter 4: Three Important Crossover Tactics

4.1: Graph Theory

4.1.2 The forest has k components, each with one more vertex than edge, so $e = v - k$.

4.1.3 Suppose otherwise. Then the graph is a forest so $e < v$. But since $e \geq v$, this is a contradiction; therefore, it contains a cycle.

4.1.4 We claim that each vertex must have the same number of edges leading into it as out, or there can be two vertices, one with exactly one more leading out and one with exactly one more leading in, and we start at the former and end at the latter. First, these conditions are necessary, otherwise there exists a vertex where we can get in but not out.

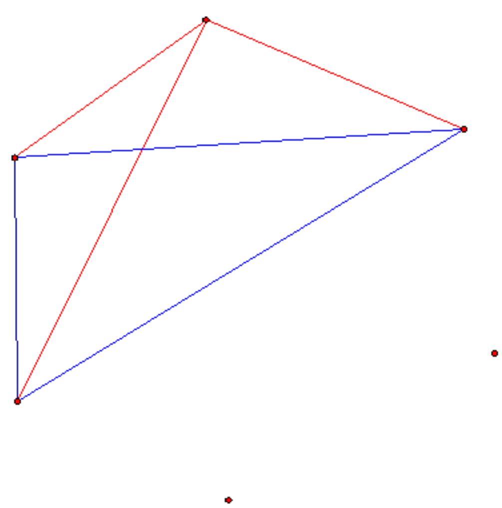
Suppose for the sake of contradiction that there exists a graph satisfying these conditions but not containing an Eulerian path. We pick the longest feasible path. Call an edge *open* if it has been traversed. Eventually, we must get stuck at a vertex while there exists some pair of vertices connected by an open edge. The vertex we are stuck at must have one more edge leading in than out or it must be the starting vertex; otherwise, we would not be stuck. Suppose the two vertices are connected by more than one open edge. Since the number of edges leading into a vertex is at most one greater than the number leading out, then some of these open edges point in both directions. However, this means that we could have increased the length of the path simply by traveling back and forth, contradicting the definition of our current path. Therefore, there is only one open edge. This implies that these two vertices are both of odd degree. But the vertex that we are stuck at either has odd degree, a contradiction, or it is the starting vertex, in which case none of the vertices have odd degree. Again, we reach a contradiction.

4.1.6 We first prove that G is connected. Suppose otherwise; then there exists two disconnected subgraphs. Since the degree of every vertex in each graph is at least $v/2$, then there must be at least $v/2$ other points in each. Since there are only v vertices, this is clearly an impossibility.

Now, consider the longest possible cycle $p_1p_2p_3...p_n$. We claim that this is a Hamiltonian cycle. Suppose otherwise; then there exists a point Q not in this cycle. Consider the n pairs $p_1p_2, p_2p_3, p_3p_4, ...p_np_1$, wheer n is at most $v-1$. If, in each of these pairs p_ip_{i+1}, Q is connected to both p_i and p_{i+1}, then Q can be added into the cycle by inserting it between p_i and p_{i+1}. Therefore, Q can only be connected to one point per pair, for a total of at most $\frac{v-1}{2}$ points. However, Q is connected to at least $\frac{v}{2}$ points, so this is impossible. Contradiction.

4.1.8 This problem is equivalent to 3.3.11.

4.1.9 We can interpret this as a graph with six vertices, with every pair of vertices connected by an edge. Color the edge red if they know eachother and blue otherwise. Then our goal is to show that there exists some triangle completely of one color. Consider one of the points. It is connected to five other points with red or blue, so by the pigeonhole principle it is connected to at least three points with (WLOG) red edges. If any of these three points are connected by red edges, then we have a red triangle. Otherwise, they are all connected by blue edges, in which case we hav ea blue triangle.



4.1.10 We can interpret this as a graph with seventeen vertices, with every pair of vertices connected by an edge. Color the edge red if the corresponding people haven't met, blue if they are good friends, or green if they hate eachother. If there exists a triangle all of one color, then we are done. Consider one point. By the pigeonhole principle, it must be connected to at least 6 other points with one color, WLOG red. Among these six points, if any are red, then we are done. Therefore, they must be colored with blue and green. Here the proof proceeds as in 4.1.9.

4.1.11 Suppose otherwise. Then there are at least two disconnected subgraphs, and every vertex in each is connected to at least $(v-1)/2$ other vertices. This implies that there are at least $\lfloor(v-1)/2\rfloor + 1 \leq \frac{v+1}{2}$ vertices in each, clearly impossible.

4.1.12 We claim that the minimum is $\binom{n-1}{2} + 1$. First, it is clear that no amoount smaller than this suffices, since we can simply form two graphs, one with $n-1$ vertices all connected to eachother and one point with degree 0. Now, we will prove that $\binom{n-1}{2}$ is always sufficient. By the pigeonhole principle, there must be at least $\frac{n-1}{2}$ vertices with degree at least $\frac{n-2}{2}$.

Suppose that there are two disconnected subgraphs. If there are more than two, simply add edges until we have two subgraphs. Then the total number of edges is at most

$$\begin{aligned}
\binom{m}{2} + \binom{n-m}{2} &= \frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2} \\
&= \frac{m^2-m}{2} + \frac{n^2-nm-n-nm+m^2+m}{2} \\
&= \frac{n^2-2mn+2m^2-n}{2}
\end{aligned}$$

We wish to show that this is less than $\binom{n-1}{2}$.

$$\begin{aligned}
\frac{n^2-2mn+2m^2-n}{2} &\leq \frac{n^2-n)}{2} \\
2m^2 &\leq 2mn
\end{aligned}$$

And this is clearly true.

4.1.13 Let the rooms be denoted by vertices and the doors by edges. It is sufficient to prove that, given a graph, it is always possible to find a path that travels from the outside vertex to a vertex of odd degree. Suppose otherwise; then consider the subgraph of points connected by some path to the outside vertex. This outside vertex has degree one, the entrance door. By hypothesis, all of the other vertices have even degree. Then the total number of edges is half the sum of the degrees of each vertex, which is a sum of several even numbers and 1. This is clearly an odd number, which is impossible.

4.1.14

4.1.15 For the only if part, suppose that the bipartite graph does have some odd cycle. Let the two partitions be U and V. Pick any arbitrary point on the cycle and trace the edges as if they were directed. Then subsequent edges are in opposite directions. Since there are an odd number of edges, the last edge is in the same direction as the first edge, WLOG $U \to V$. However, this is impossible since the first point is in U.

For the if statement,

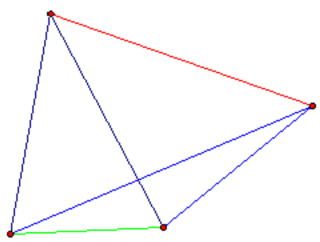
4.1.16 The tournament is equivalent to a K_n, where there are n people. Indicate a victory of p_k over p_j by drawing the directed edge p_kp_j. Then we wish to prove that there exists a path from p_1 to p_{100}. We claim that the longest path following the directed segments includes all the players. If not, there exists some vertex P that is not connected to any of the other vertices. Consider the *last* vertex along the path that is directed towards P. Let this vertex be p_k. Then p_k beat P and P beat p_{k+1}, so we can insert P between p_k and p_{k+1}, contradicting the maximality of the path. If this p_k does not exist, then P must have beaten everybody, so we can insert it in front of p_1, again contradicting the maximality of the path.

(b) A directed complete path possesses a Hamiltonian path.

(c) No. Just consider if the directed graph forms a cycle $p_1p_2...p_np_1....$ We can start at any point on this cycle.

4.1.17 In graph theory language, the problem is equivalent to finding the minimum complete graph such that all edges are colored red, green, or blue and no triangle exists which is three different colors or one color. First, we claim that no vertex can be connected with three other vertices by the same color, WLOG red. Suppose otherwise; then none of these three vertices can be connected by red. Also, they cannot all be connected by blue or green. Therefore, they must be connected by both blue or green- but then one of the vertices is connected to blue, green, and red, which is also not allowed.

We claim that the maximum number of communities is 4. The following graph shows that 4 is possible.



It suffices to show that 5 communities is impossible. Consider all the edges emanating from one vertex. From the problem statement, they cannot consist of three differenct colors, so by the pigeonhole principle some three of them are the same. We have already proved that this implies the existence of a monochromatic or trichromatic triangle, so this is impossible.

4.1.18

4.1.19

Chapter 5: Algebra

5.1: Sets, Numbers, and Functions

5.1.1 (a) Yes. Pair $x \in A$ with $x + 1 \in B$, and vice versa.

(b) Yes. Pair $x \in Z$ with $2x \in A$, and vice versa.

5.1.2(a) If $x \in A$ but not B, then $1_A(x)1_B(x) = 0$ and $1_{A\cup B} = 0$. If $x \in A, B$, then $1_A(x)1_B(x) = 1$ and $1_{A\cup B} = 1$. If x is not in A and B, then $1_A(x)1_B(x) = 0$ and $1_{A\cup B}(x) = 0$.

(b) If $x \in A$, then $1_A(x) = 1$ and $1_{\bar{A}}(x) = 0$. Otherwise, $1_A(x) = 0$ and $1_{\bar{A}}(x) = 1$.

5.1.3 False. The null set contains no elements but the set of the null set contains the null set.

5.1.4 (a) The statement is equivalent to proving that there is an irrational number between any rational number and 0.

(b) The statement is equivalent to proving that there is an irration number between 0 and a rational or irrational number.

5.1.5 For each element in A, we need to pick a unique element in B to pair it with. There are then n^m functions from A to B.

5.1.6 $f(84) = f(f(84+5)) = f(86)$. Continuing in this way, $f(84) = f(1000) = \boxed{997}$.

5.1.7 $0 = f(0) = f(7+(-7)) = f(7-(-7)) = f(14)$. Also, $0 = f(0) = f(2+(-2)) = f(2-(-2)) = f(4)$. Finally, $0 = f(7+(-3)) = f(7-(-3)) = f(10)$. This shows us that the 0's are periodic mod 10 and are located at 0 and 4. Then the total number of roots is

$$\frac{2000}{10} * 2 + 1 = \boxed{401}$$

5.1.8 Let $f(x) = \lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 6x \rfloor + \lfloor 8x \rfloor$. Then

$$f\left(x + \frac{1}{2}\right) = \lfloor 2x+1 \rfloor + \lfloor 4x+2 \rfloor + \lfloor 6x+3 \rfloor + \lfloor 8x+4 \rfloor = f(x) + 10$$

So, all we need to do is to find possible integral values of $f(x)$ between 0 and 1. If $x < 1/8$, then $f(x) = 0$. Since the floor function changes at integers, we need only to test multiples of $\frac{1}{8}$ and $\frac{1}{6}$.

$$\begin{aligned} f\left(\frac{3}{24}\right) &= 1 \\ f\left(\frac{4}{24}\right) &= 2 \\ f\left(\frac{6}{24}\right) &= 4 \\ f\left(\frac{8}{24}\right) &= 5 \\ f\left(\frac{9}{24}\right) &= 6 \end{aligned}$$

Therefore, we can obtain all numbers congruent to $1, 2, 4, 5, 6$ mod 10. There are a total of $\boxed{500}$ such numbers.

5.1.9 True. Let $x = k^2 + r$, where k is a positive integer and r is a positive real less than $2k+1$. Then $\lfloor \sqrt{x} \rfloor = k$. Also, $k \leq \sqrt{\lfloor x \rfloor} < k+1$.

5.1.10 It will be sufficient to prove that

$$\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$$

Since $4n+2$ can never be a perfect square.

$$\begin{aligned}\sqrt{4n+1} &< \sqrt{n}+\sqrt{n+1} \\ 4n+1 &< 2n+1+2\sqrt{n^2+n} \\ 2n &< 2\sqrt{n^2+n} \\ 4n^2 &< 4n^2+4n\end{aligned}$$

$$\begin{aligned}\sqrt{n}+\sqrt{n+1} &< \sqrt{4n+2} \\ 2n+1+2\sqrt{n^2+n} &< 4n+2 \\ 2\sqrt{n^2+n} &< 2n+1 \\ 4n^2+4n &< 4n^2+4n+1\end{aligned}$$

5.1.12 The terms $\frac{n(n-1)}{2}+1$ through $\frac{n(n+1)}{2}$ are all equal to n- in other words, the $\Delta_{n-1}+1$ through Δn numbers (where Δn represents the nth triangular number $n(n+1)/2$). We would like to create a 'reverse-triangulate' function that gives us the original number from a triangle number. Note that $\frac{n^2+n}{2} = \frac{1}{8}(4n^2+4n)$. Then $\sqrt{8\Delta n+1} = 2n+1$, so

$$n = \frac{\sqrt{8\Delta n+1}-1}{2}$$

Therefore, the nth element is

$$\left\lfloor \frac{\sqrt{8n+1}-1}{2} \right\rfloor$$

5.1.13 Let's proceed by induction. For the base case

$$\left\lfloor \frac{1+1}{2} \right\rfloor = 1$$

For the inductive hypothesis, suppose that

$$\left\lfloor \frac{n+2^0}{2} \right\rfloor + \left\lfloor \frac{n+2^1}{2^2} \right\rfloor + \left\lfloor \frac{n+2^2}{2^3} \right\rfloor + \dots + \left\lfloor \frac{n+2^{n-1}}{2^n} \right\rfloor = n$$

holds for n. We note that as n changes to $n+1$, the k-th term in the sequence increases by 1 iff n satisifies

$$n \equiv 2^{k-1} \pmod{2^k}$$

Solution 2: We rewrite the expression as:

$$\left\lfloor \frac{n}{2}+\frac{1}{2} \right\rfloor + \left\lfloor \frac{n}{2^2}+\frac{1}{2} \right\rfloor + \left\lfloor \frac{n}{2^3}+\frac{1}{2} \right\rfloor + \left\lfloor \frac{n}{2^k}+\frac{1}{2} \right\rfloor + \cdots$$

Now we will use Hermite's identity, which states that

$$\lfloor x \rfloor + \left\lfloor x+\frac{1}{n} \right\rfloor + \left\lfloor x+\frac{2}{n} \right\rfloor + \dots + \left\lfloor x+\frac{n-1}{n} \right\rfloor = n$$

Particularly, $\lfloor x \rfloor + \lfloor x+\frac{1}{2} \rfloor = \lfloor 2x \rfloor$. Then $\lfloor x+\frac{1}{2} \rfloor = \lfloor 2x \rfloor - \lfloor x \rfloor$. Substituting,

$$\begin{aligned}\left\lfloor \frac{n}{2}+\frac{1}{2} \right\rfloor + \left\lfloor \frac{n}{2^2}+\frac{1}{2} \right\rfloor + \left\lfloor \frac{n}{n}+\frac{1}{2} \right\rfloor &= \left(\lfloor n \rfloor - \left\lfloor \frac{n}{2} \right\rfloor\right) + \left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2^2} \right\rfloor\right) + \dots + \left(\left\lfloor \frac{n}{2^{n-2}} \right\rfloor - \left\lfloor \frac{n}{2^{n-1}} \right\rfloor\right) \\ &= n\end{aligned}$$

5.2: Algebraic Manipulation Revisited

5.2.24 We know that $ab + a + b$ can be factored. The given equation is not exactly in this form, but it is close. We move the $30x^2$ to the other side to get

$$3x^2y^2 - 30x^2 + y^2 = 517$$

And this factors nicely:

$$(3x^2 + 1)(y^2 - 10) = 507$$

Keeping in mind that x and y are integers, we examine the factors of 507: $\{1, 3, 13, 39, 169, 507\}$. The 39 catches our attention because $39 + 10 = 49 = 7^2$. Solving $3x^2 + 1 = 13$, we have $x^2 = 4$, also a perfect square. Therefore, $y = 7$ and $x = 2$, so $3x^2y^2 = \boxed{588}$.

5.2.25 We factor:

$$(x + y)(x - y) = 20$$

Since $x+y > x-y$, there exists one solution for every pair of factors of 20, which are $\pm(1, 20), \pm(2, 10), \pm(4, 5)$. Since $x + y + x - y = 2x$, we add and divide by 2 to find x; similarly, we subtract and divide by 2 to find y. Since the sum of two negative numbers is negative and the sum of a negative and an odd is odd, the only pair that works is $(2, 10)$. Therefore, $x = 6$ and $y = 4$.

5.2.26 This is another chance for us to use our factoring trick:

$$(x + 3)(y + 5) = 215$$

The factors of 215 are $\pm\{1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 200\}$. Clearly, the negative factors will not work. The pairs $(1, 200)$ and $(2, 100)$ do not work because one of x, y is negative in each case. For $(4, 50)$, we must have $x + 3 = 4$ and $y + 5 = 50$. For the other 3 pairs, we have two solutions each.

5.2.27 Let this smallest integer be n. Then n must clearly end in 2. We can write $n = 10a + 2$. Then

$$n^3 = 1000a^3 + 600a^2 + 120a + 8$$

$1000a^3$ does not affect the last 3 digits, so we can ignore that term. We want the tens digit of this to be 8. The only term that affects the tens digit is $120a$. Since we want this to be 8, we need a to end in 4 or 9. We then analyze these cases separately:

Case 1: a ends in 4, so we can write $a = 10b + 4$. We want the hundreds digit to be 8, so $600(10b + 4)^2 + 120(10b + 4)$ must have 8 as a hundreds digit. Multiplying this out,

$$600(100b^2 + 80b + 16) + 1200b + 480 = 600(100)b^2 + 48000b + 9600 + 1200b + 480$$

must have 8 as a hundreds digit. We can ignore the first two terms. We then have $1200b$ with a hundreds digit of 8, so the least b is 4. Then $a = 44$ and $n = 442$.

Case 2: a ends in 9, so we can write $a = 10b + 9$. We want the hundreds digit of

$$600(10b + 9)^2 + 120(10b + 9) = 600(100)b^2 + 2(600)(90)b + 48600 + 1200b + 1080$$

to be 8. We can throw out the first two terms. Then we want $1200b$ to have a hundreds digit of 2, so $b = 1$. Then $a = 19$ and $n = 192$.

Since $192 < 442$, the smallest such positive integer is $\boxed{192}$.

5.2.28 $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$. Therefore,

$$xy + yz + zx = \frac{(x + y + z)^2 - 1}{2}$$

Since $(x + y + z)^2 \geq 0$, the minimum is $-\frac{1}{2}$. To see that this is possible, set $x = -\frac{\sqrt{2}}{2}, y = \frac{\sqrt{2}}{2}, z = 0$.

5.2.29 Make the substitution $a = x+y$ and $b = xy$. Then $xy+x+y = a+b$ and $x^2y+xy^2 = xy(x+y) = ab$. Also note that $x^2 + y^2 = a^2 - 2b$. Now, we can solve for a and b:

$$\begin{aligned} a + b &= 71 \\ a^2 + b^2 &= 5041 \\ a^2 + b^2 - 4ab &= 1521 \\ a - b &= \pm 39 \end{aligned}$$

We must proceed with a little caution. $a - b = +39$ yields $a = 55, b = 16$, which is impossible if x and y are to be integers. Thus, $a - b = -39$, yielding $a = 16, b = 55$. Then $a^2 - 2b = 146$.

5.2.30 Note that $n^4 + 2n^3 + 2n^2 + 2n + 1 = (n^2 + n + 1)^2 - n^2 = (n^2 + 2n + 1)(n^2 + 1) = (n + 1)^2(n^2 + 1)$. For this to be a perfect square, $n^2 + 1$ must be a perfect square, implying that n is 0. Then $m = \pm 1$.

5.2.31 Noting the coefficients are staggered perfect squares, our plan is to use successive differences to find the next row.

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1 \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12 \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123 \end{aligned}$$

We subtract the first row from the second to obtain

$$3x_1 + 5x_2 + ... + 15x_7 = 11$$

Next we subtract the second row from the third

$$5x_1 + 7x_2 + ... + 17x_7 = 111$$

And subtracting these from eachother yields

$$2x_1 + 2x_2 + ... + 2x_7 = 100$$

Now we add this back to $5x_1 + 7x_2 + ... + 17x_7$ to obtain

$$7x_1 + 9x_2 + ... + 19x_7 = 211$$

Adding this to the final row in the original system gives

$$16x_1 + 25x_2 + ... + 100x_7 = 123 + 211 = \boxed{334}$$

Alternate Solution: We wish to find numbers a, b, and c such that

$$an^2 + b(n+1)^2 + c(n+2)^2 = (n+3)^2$$

We view this as a polynomial in n, then expand in equate coefficients:

$$\begin{aligned} a+b+c &= 1 \\ 2b+4c &= 6 \\ b+4c &= 9 \end{aligned}$$

Solving, $a = 1, b = -3$ and $c = 3$. Therefore, we should add the first equation with three times the third equation and then subtract the second equation to obtain the answer. Doing so gives $\boxed{334}$.

5.2.32 We will show that the number $\underbrace{444...4}_{n+1\text{times}}\underbrace{888...8}_{n\text{times}}1$ is always a perfect square. We can express this as a sum of powers of ten, as encoded by the decimal system:

$$9 + 80(10^{n-1} + 10^{n-2} + ... + 1) + 4 * 10^{n+1}(10^n + 10^{n-1} + ... + 1)$$

Using the formula for geometric series, this is equal to

$$\begin{aligned} 9 + 80\left(\frac{10^n - 1}{9}\right) + 4 * 10^{n+1}\left(\frac{10^{n+1} - 1}{9}\right) &= \frac{4 * 10^{2n+2} - 4 * 10^{n+1} + 8 * 10^{n+1} - 80 + 81}{9} \\ &= \frac{(2 * 10^{n+1})^2 + 2 * (2 * 10^{n+1}) + 1}{3^2} \\ &= \left(\frac{2 * 10^{n+1} + 1}{3}\right)^2 \end{aligned}$$

Therefore, the square root of $\underbrace{444...4}_{n+1\text{times}}\underbrace{888...8}_{n\text{times}}1$ is $\frac{2*10^{n+1}+1}{3}$. Since $10 \equiv 1 \pmod 3$, $2^1 0^{n+1} \equiv 2 \pmod 3$ so $2 * 10^{n+1} + 1$ is divisible by 3.

5.2.33 The squares look hopeful. We extract $-4n^2$ from $-20n^2$ to complete the square:

$$n^4 - 4n^2 + 4 - 16n^2 = (n^2 - 2)^2 - (4n)^2$$

Then we can factor the result:

$$(n^2 - 4n - 2)(n^2 + 4n - 2)$$

We we need to find when these are greater than 1:

$$\begin{array}{rl} n^2 - 4n - 2 > 1 & n^2 + 4n - 2 > 1 \\ n^2 - 4n - 3 > 0 & n^2 + 4n - 3 > 0 \\ (n-1)(n-3) > 0 & (n - .6)(n + 4.6) > 0 \end{array}$$

The first equation is greater than 1 for values less than 1 and greater than 3. The second is greater than 1 for values less than -4 and greater than 0. Therefore we only need to try n from -4 to 3. Doing so shows that they are composite.

5.2.34 We will use the identity $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$. We know all of these values except $xy + yz + zx$, which we can find this by squaring $x + y + z$:

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$$

Since $(x+y+z)^2 = 49$ and $x^2+y^2+z^2 = 49$, we know that $xy+yz+zx = 0$. Then

$$\begin{aligned} xyz &= \frac{x^3+y^3+z^3-(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}{3} \\ &= \frac{7-(7)(49)}{3} \\ &= \boxed{-112} \end{aligned}$$

5.2.35 Substitute $a = 16x^2-9$ and $b = 9x^2-16$. Then $a^3+b^3 = (a+b)(a^2-ab+b^2)$. Since $a^2+b^2-2ab \geq 0, a^2+b^2-ab \geq 0$ with equality iff $a = b = 0$. Substituting $16x^2-9$ for a and $9x^2-16$ for b, we find that this is impossible. Therefore, the only roots occur when $a+b = 0$. Solving,

$$25x^2 - 25 = 0$$

Therefore, $x = \boxed{\pm 1}$.

5.2.7 Factor to get

$$(x-y)(x^2+xy+y^2) = 721$$

There are two pairs of factors of $721 : (1, 721)$ and $(7, 103)$. $(x^2+xy+y^2)-(x-y)^2 = 3xy$. Since x, y must be positive, we must have $x-y$ as the smallest element in each pair. Now we solve each equation:

$$\begin{array}{rl} x^2+xy+y^2 = 721 & x^2+xy+y^2 = 103 \\ x-y = 1 & x-y = 7 \\ xy = 240 & xy = 54 \\ x^2+2xy+y^2 = 961 & x^2+2xy+y^2 = 157 \\ x+y = 31 & x+y = \sqrt{157} \\ x^2-2xy+y^2 = 1 & \\ x-y = 1 & \\ x = 16, y = 15 & \end{array}$$

We can eliminate the second option because $x+y$ is not rational. Therefore, $(x, y) = \boxed{(16, 15)}$.

5.2.37 We note that $(x+y+z)^3 = x^3+y^3+z^3+3(x+y)(y+z)(z+x)$. Since

$$x^3+y^3+z^3 = x^3+y^3+z^3+3(x+y)(y+z)(z+x)$$

If any of two of x, y, z are negative opposites, then the equation is satisfied.

5.2.38 Noting that $324 = 18^2$, we are immediately alerted to manipulation of perfect squares. Consider

$$\begin{aligned} x^4+18^2 &= x^4+2(x^2)(18)+18^2-2(x^2)(18) \\ &= (x^2+18)^2-(6x)^2 \\ &= (x^2-6x+18)(x^2+6x+18) \\ &= ((x-3)^2+9)((x+3)^2+9) \end{aligned}$$

We hope to use this identity to telescope some of the terms. Indeed, every term in the denominator is 6 less than its corresponding term in the numerator. The product is then

$$\begin{aligned}\frac{(7^2+9)(13^2+9)(19^2+9)...(61^2+9)}{(1^2+9)(7^2+9)(13^2+9)...(58^2+9)} &= \frac{61^2+9}{1^2+9}\\ &= \frac{3730}{10}\\ &= \boxed{373}\end{aligned}$$

5.3: Sums and Products

5.3.6 Let the geometric series be

$$a, ar, ar^2, ...ar^{n-1}$$

The product of the terms is $a^n r^{1+2+..n-1} = \boxed{a^n r^{n(n+1)/2}}$.

5.3.7 We will use a similar tactic as the only used to derive the original geometric series formula. Let

$$S = r + 2r^2 + 3r^3 + ...nr^n$$

Then

$$rS = r^2 + 2r^3 + ... + (n-1)r^n + nr^{n+1}$$

And

$$S - rS = r + r^2 + r^3 + ... + r^n - nr^{n+1} = \frac{r(1-r^n)}{1-r} - nr^{n+1}$$

Therefore,

$$S = \frac{\frac{r(1-r^n)}{1-r} - nr^{n+1}}{1-r} = \boxed{\frac{r-(n+1)r^{n+1}+nr^{n+2}}{(1-r)^2}}$$

5.3.8 Ideally, we would like to express this as a sum of partial fractions to telescope the terms. Let

$$\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$$

Clearing denominators,

$$1 = A(n+1)(n+2) + B(n)(n+2) + C(n)(n+1)$$

We can now use a clever tool to find A, B, C. Viewing this as a polynomial in n, we set $n = 0$. Then we have $A(1)(2) = 1$, so $A = \frac{1}{2}$. Setting $n = -1$, we have $1 = B(-1)(1)$, so $B = -1$. Setting $n = -2$, we have $1 = C(-2)(-1)$, so $C = \frac{1}{2}$. Now our sum becomes

$$\frac{1}{2*1} - \frac{1}{2} + \frac{1}{2*3} + \frac{1}{2*2} - \frac{1}{3} + \frac{1}{2*4}... + \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2n+4}$$

Now let's consider the coefficient of the term $\frac{1}{k}$. Except for $k = 1, 2, n-1, n$, each such fraction is added three times- once each with A, B, C. since $B = -1$ and $A = C = 1/2$, it contributes 0 to the total sum.

Now we simply need to analyze $k = 1, 2, n-1, n$. $\frac{1}{1}$ is added with weight $1/2$. $\frac{1}{2}$ is added with weight $1/2 - 1 = -1/2$. $\frac{1}{n-1}$ is added with weight $-1 + \frac{1}{2} = -\frac{1}{2}$. $\frac{1}{n}$ is added with weight $1/2$. Therefore, the sum is

$$\frac{1}{1} - \frac{1}{2*2} - \frac{1}{2*n-1} + \frac{1}{2n} = \boxed{\frac{3}{4} - \frac{1}{2n(n-1)}}$$

5.3.9 Note that $\frac{n(n+1)(n+2)}{3*2*1} = \binom{n+2}{3}$. Let S be the given sum. Then

$$\begin{aligned} \frac{S}{6} &= \sum_{k=1}^{n} \binom{k+2}{3} \\ &= \binom{n+3}{4} \\ S &= \boxed{\frac{1}{6}\binom{n+3}{4}} \end{aligned}$$

5.3.10 Partition the subsets of $\{1, 2, ...n\}$ into two sets A and B: the set of subsets without the element n and the set of subsets with the element n. For any set in A, we can simply add the element n to establish a bijection with an element in B. Let S denote the alternating sum of a set. Consider $S(q \subset A) + S(q \cup n)$. This sum is equal to n, since all the other elements cancel out. Similarly, the sum of each of the elements of A and its corresponding element in B is n, so the desired sum is simply n times the number of elements in A, which is half the number of subsets in $\{1, 2, ...n\}$: $\boxed{n(2^{n-1})}$.

5.3.11 For each $x \in A$, $1_A(x) = 1$. Therefore, each element of A contributes a sum of 1. For each element not in A, $1_A(x) = 0$, so it contributes 0. Therefore $\sum\limits_{x \in U} 1_A(x) = |A|$.

5.3.12 We can write $n * n! = (n + 1 - 1)n! = (n + 1)! - n!$. Then

$$\sum_{k=1}^{n} kk! = \sum_{k=1}^{n}(k+1)! - \sum_{k=1}^{n} k! = \boxed{(n+1)! - 1}$$

5.3.13 We write

$$\frac{k}{(k+1)!} = \frac{(k+1)-1}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}$$

Then we can telescope the sum

$$\begin{aligned} \sum_{k=1}^{n} \frac{k}{(k+1)!} &= \sum_{k=1}^{n} \left(\frac{1}{k!} - \frac{1}{(k+1)!}\right) \\ &= \sum_{k=1}^{n} \frac{1}{k!} - \sum_{k=1}^{n} \frac{1}{(k+1)!} \\ &= 1 - \frac{1}{(n+1)!} \end{aligned}$$

5.3.14 We note that $\cos\theta(2\sin\theta) = 2\cos\theta\sin\theta = \sin 2\theta$. We hope to use this as a catalyst for the product. Multiplying the product by $2^{n+1}\sin\theta$ yields

$$\begin{aligned}
S(2^{n+1}\sin\theta) &= 2^{n+1}\sin\theta\cos\theta\cos 2\theta\cos 4\theta ... \cos 2^n\theta \\
&= 2^n \sin 2\theta \cos 2\theta \cos 4\theta ... \cos 2^n\theta \\
&= 2^{n-1}\sin 4\theta\cos 4\theta ... \cos 2^n\theta \\
&... \\
&= 2\sin 2^n\theta\cos 2^n\theta \\
&= \sin 2^{n+1}\theta
\end{aligned}$$

Therefore, $S = \boxed{\dfrac{\sin(2^{n+1}\theta)}{2^{n+1}\sin\theta}}$.

5.3.15 We note that $\frac{1}{\log_k u} = \log_u k$. Then

$$\begin{aligned}
\sum_{k=2}^{n}\frac{1}{\log_k u} &= \sum_{k=2}^{n}\log_u k \\
&= \log_u \prod_{k=2}^{n} k \\
&= \boxed{\log_u n!}
\end{aligned}$$

5.3.16 By symmetry, the average value of $\sum_{n=1}^{5}|a_{2n-1} - a_{2n}|$ is five times the average of $|a_1 - a_2|$. For $a_1 = k$, this sum takes on the values $k-1, k-2, ...1, 1, ...10-k$. This sum is equal to $\frac{k(k-1)}{2} + \frac{(10-k)(11-k)}{2} = k^2 - 11k + 55$. Summing this from $k = 1$ to 10,

$$\begin{aligned}
\sum_{k=1}^{10}(k^2 - 11k + 55) &= \sum_{k=1}^{10}k^2 - 11\sum_{k=1}^{10}k + 55\sum_{k=1}^{10} \\
&= \frac{10(11)(21)}{6} - 11\left(\frac{10(11)}{2}\right) + 55(10) \\
&= 385 - 11(55) + 550 \\
&= 330
\end{aligned}$$

Since there are 9 values for each of the 10 possibilities for a_1, the average value of $|a_1 - a_2|$ is $\frac{330}{90} = \frac{11}{3}$. Therefore, the average value of $|a_1 - a_2| + |a_3 - a_4| + |a_5 - a_6| + |a_7 - a_8| + |a_9 - a_{10}|$ is $5\left(\frac{11}{3}\right) = \boxed{\dfrac{55}{3}}$.

5.2.17 If $\Delta(\Delta A) = (1, 1, ...)$, then $\Delta A = (k, k+1, k+2 ...)$ and $A = (x, x+k, x+2k+1, ...)$. Particularly, the nth term of A is $x + (n-1)k + \frac{(n-1)(n-2)}{2}$. Then we have

$$\begin{aligned}
x + 18k + \frac{17(18)}{2} &= 0 \qquad & x + 93k + \frac{93(92)}{2} &= 0 \\
31x + 31(18)k + \frac{31(17)(18)}{2} &= 0 \qquad & 6x + 6(93)k + \frac{6(93)(92)}{2} &= 0
\end{aligned}$$

Subtracting,

$$\begin{aligned}
25x &= \frac{6(93)(92) - 31(18)(17)}{2} \\
25x &= 9(31)(92) - 9(31)(17) \\
x &= \frac{9(31)(75)}{25} \\
&= \boxed{837}
\end{aligned}$$

5.2.18

5.2.19 We will prove the more interesting result that the bug always reaches the wall if the rubber band is of any arbitrary finite length and stretches by a finite length. Let the rubber band stretch k inches per minute. After one minute, the bug travels $\frac{1}{k}$ of the distance. As the rubber band is stretched, this ratio is preserved. At two minutes, the bug travels $\frac{1}{2k}$ of the distance. Particularly, at n minutes, the bug travels $\frac{1}{nk}$ of the distance. Therefore, after n minutes, the bug travels a total of $\frac{1}{k}\left(\frac{1}{1} + \frac{1}{2} + ... + \frac{1}{n}\right)$ of the way. Since the harmonic series diverges, this will exceed any k at some point.

5.2.20 There are 9^n different numbers with n nonzero digits. Each is greater than or equal to 10^{n-1}, so the reciprocal is less than or equal to $\frac{1}{10^{n-1}}$. Therefore, the sum of the reciprocals of the n-digit numbers is less than $\frac{9^n}{10^{n-1}}$. Summing this from 1 to ∞,

$$\begin{aligned}
S &< \sum_{n=1}^{\infty} \frac{9^n}{10^{n-1}} \\
&= \frac{9}{1 - 9/10} \\
&= \frac{9}{1/10} \\
&= 90
\end{aligned}$$

So the series converges.

5.3.21 First, we show that $\zeta(2) < 2$.

$$\begin{aligned}
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}... &< \frac{1}{1^2} + \frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4}... \\
&= \frac{1}{1} + \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3}... \\
&= 2
\end{aligned}$$

We will use the same principles, with a little more precision, to show that $\zeta(2) < \frac{7}{4}$.

$$\begin{aligned}
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}... &< \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{2*3} + \frac{1}{3*4}... \\
&= \frac{1}{1} + \frac{1}{4} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4}... \\
&= \frac{1}{1} + \frac{1}{4} + \frac{1}{2} \\
&= \frac{7}{4}
\end{aligned}$$

5.3.22

$$\begin{aligned}
\prod_{k=i}^{n} \frac{k^3-1}{k^3+1} &= \prod_{k=i}^{n} \frac{k-1}{k+1} \prod_{k=i}^{n} \frac{k^2+k+1}{k^2-k+1} \\
&= \prod_{k=i}^{n} \frac{k-1}{k+1} \prod_{k=i}^{n} \frac{k(k+1)+1}{(k-1)k+1} \\
&= \left(\frac{(i-1)i}{n(n+1)}\right)\left(\frac{n(n+1)+1}{(i-1)i+1}\right) \\
&= \frac{(i-1)i}{(i-1)i+1}\left(\frac{n(n+1)+1}{n(n+1)}\right)
\end{aligned}$$

as $n \to \infty$ the factor in parentheses becomes 1, so our answer is

$$\frac{(i-1)i}{(i-1)i+1} = \frac{2}{3}$$

5.3.24 We want to establish a range of numbers whose fourth roots will round to n.

$$\left(n+\frac{1}{2}\right)^4 = n^4 + 2n^3 + \frac{3}{2}n^2 + \frac{1}{2}n + \frac{1}{16}$$

Therefore, the fourth root of numbers up to $n^4 + 2n^3 + \frac{3}{2}n^2 + \frac{1}{2}n$, which is always an integer, will round to n.

$$\left(n-\frac{1}{2}\right)^4 = n^4 - 2n^3 + \frac{3}{2}n^2 - \frac{1}{2}n + \frac{1}{16}$$

Therefore, the fourth root of numbers greater than $n^4 - 2n^3 + \frac{3}{2}n^2 - \frac{1}{2}n$, which is always an integer, will round to n. Thus there are a total of $4n^3+n$ numbers whos fourth roots round to n. Since $6^4 - 2(6^3) + \frac{3}{2}(6^2) - \frac{1}{2}(6) = 1785$,

$$\begin{aligned}
\sum_{k=1}^{1785} \frac{1}{f(k)} &= \sum_{k=1}^{6} \frac{1}{k}(4k^3+k) \\
&= \sum_{k=1}^{6} 4k^2 + 1 \\
&= 4\sum_{k=1}^{6} k^2 + \sum_{k=1}^{6} 1 \\
&= 370
\end{aligned}$$

For the numbers from 1785 to 1995, $f(k) = 7$. Since there are 210 of these, they contribute $\frac{1}{7}(210) = 30$ to the sum. Therefore,

$$\sum_{k=1}^{1995} \frac{1}{f(k)} = 370 + 30 = \boxed{400}$$

5.4: Polynomials

5.4.6 Let our polynomial be $P(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_0$. By the Fundamental Theorem of Algebra, this has a root r_1. We can factor this root out to get $P(x) = (x - r_1)(b_nx^n + b_{n-1}x^{n-1} + ... + b_0)$. We can repeat the process, reducing the degree by 1 each time. Eventually, we have $P(x) = (x - r_1)(x - r_2)(x - r_3)...(x - r_n)c$, where c is some constant. Since c cannot have a root unless it is the zero polynomial, P has at most n distinct roots.

5.4.7 Suppose that f and g have degree n and are equal at $n + 1$ points. Then consider the polynomial $H(x) := f(x) - g(x)$. The degree of H is at most n, and it has $n + 1$ roots, implying that it is the zero polynomial. Therefore $f(x) - g(x) = 0$ for all x, and they are the same polynomial.

5.4.8 We first verify that conjugation distributes over multiplication and addition.

$$\overline{a + bi + c + di} = a + c - (b + d)i = a - bi + c - di = \overline{a + bi} + \overline{c + di}$$

And vice versa.

$$\overline{(a + bi)(c + di)} = ac - bd + (ad + bc)i = (a - bi)(c - di) = (\overline{a + bi})(\overline{c + di})$$

And vice versa. Our goal is to prove that if $P(z) = 0$, then $P(\overline{z}) = 0$. Let $P = a_nx^n + a_{n-1}x^{n-1} + ... + a_0$. Then

$$\begin{aligned} P(\overline{z}) &= a_n\overline{z}^n + a_{n-1}\overline{z}^{n-1} + ... + a_0 \\ &= a_n\overline{z^n} + a_{n-1}\overline{z^{n-1}} + ... + a_0 \\ &= \overline{a_nz^n} + \overline{a_{n-1}z^{n-1}} + ... + \overline{a_0} \\ &= \overline{a_nz^n + a_{n-1}z^{n-1} + ... + a_0} \\ &= \overline{P(z)} \end{aligned}$$

And since $P(z) = 0, \overline{P(z)} = 0$, so $P(\overline{z}) = 0$ and $\overline{z}$ is a root of P, as desired.

5.4.9 By the remainder theorem, the remainder occurs when we let $x^3 - x = 0$, or $x = x^3$. Substituting, the expression simplifies to $5x$.

Alternate solution: We first divide the polynomial by x. Then

$$\begin{aligned} P(x) &:= x^{80} + x^{48} + x^{24} + x^6 + 1 \\ P(x) &= (x^2 - 1) \cdot q(x) + r(x) \\ \deg r &< \deg(x^2 - 1) = 2 \implies r := ax + b \\ P(x) &= (x^2 - 1) \cdot q(x) + ax + b \end{aligned}$$

Now we substitute values to find a and b:

$$\begin{aligned} P(1) &= 5 = a + b \\ P(-1) &= 5 = a - b \end{aligned}$$

So $a = 5, b = 0$ and $r = 5x$.

5.4.10 If we have a root of the form $a + \sqrt{5}$, where a is rational, then $a - \sqrt{5}$ must also be a root for the polynomial to have rational coefficients. Even though $\sqrt{2}$ is not rational, we employ this idea, so $\sqrt{2}-\sqrt{5}$ is a root. Then we have

$$(x-(\sqrt{2}+\sqrt{5}))(x-(\sqrt{2}-\sqrt{5})) = x^2 - 2\sqrt{2} - 3$$

If we substitute $y = x^2$, then we have the factor $y-(3+2\sqrt{2})$. Now we know that adding the factor $3-2\sqrt{2}$ will produce a polynomial with rational coefficients. We can then multiply by a constant if necessary.

$$(x^2-(3+2\sqrt{2}))(x^2-(3-2\sqrt{2})) = x^4-(3+2\sqrt{2}+3-2\sqrt{2})x^2+(3-2\sqrt{2})(3+2\sqrt{2}) = x^4-6x^2+11$$

Since this polynomial has integer coefficients, we are done.

5.4.11 Let a root of the monic polynomial $x^n + a_{n-1}x^{n-1} + ... + a_0$ be $\frac{p}{q}$, with p and q relatively prime. Then

$$\begin{aligned}\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + ... + a_0 &= 0\\ p^n + a_{n-1}p^{n-1}q + ... + a_0q^n &= 0\end{aligned}$$

All the terms are divisible by q except p^n, and 0 is divisible by q. Therefore, q must be 1. Then $\frac{p}{q}$ is an integer.

5.4.12 Consider the polynomial mod 2. If x is odd, then $P(x) \equiv P(1) \equiv 1 \pmod 2$. If x is even, then $P(x) \equiv P(0) \equiv 1 \pmod 2$. Therefore, P is odd at all integer values.

5.4.13 Suppose that p can be factored into a product of two such polynomials Q and R. Then

$$Q(x)R(x) = P(x)$$

Without loss of generality, let $\deg Q < \deg R$. Then $\deg Q \leq 999$. Since both Q and R have integer coefficients, they must output integers if x is an integer. Particularly, they must output integers at the 1999 values of x that produce ± 1 for P. Therefore, Q outputs ± 1 1999 times, and by the pigeonhole principle, it must output either 1 or -1 1000 times, which is impossible because it is of degree at most 999.

5.4.14 Since r and s are roots of the first equation, we have $r+s = a+d$ and $rs = ad-bc$. If we can show that $r^3+s^3 = a^3+d^3+3abc+3bcd$ and $r^3s^3 = (ad-bc)^3$, then we are done. Since $rs = ad-bc$, $r^3s^3 = (rs)^3 = (ad-bc)^3$ as desired. To find r^3+s^3, we use the identity

$$r^3+s^3 = (r+s)(r^2-rs+s^2)$$

We have $r+s = a+d$. Squaring, $r^2+s^2+2rs = a^2+d^2+2ad$. Subtracting $3rs$, $r^3+s^3-rs = a^2+d^2-ad+3bc$.

$$\begin{aligned}r^3+s^3 &= (r+s)(r^2-rs+s^2)\\ &= (a+d)(a^2+d^2-ad+3bc)\\ &= a^3+d^3+(a+d)(3bc)\\ &= a^3+d^3+3abc+3bcd\end{aligned}$$

which completes the proof.

5.4.15 Suppose that the polynomial can be factored into two polynomials f and g with integer coefficients. One of these must be linear and the other must be quadratic. Without loss of generality, let f be linear. We have

$$f(x)g(x) = (x-a)(x-b)(x-c)-1$$

Since f and g both have integer coefficients, they must output integers for integral values of x. Particularly, they must output integers at a, b, and c. $f(x)g(x) = -1$ for $x = a, b, c$. Then f is either 1 or -1 at a, b, and c. By the pigeonhole principle, it must be either 1 or -1 for at least two values, which is impossible. Therefore, $(x-a)(x-b)(x-c) - 1$ cannot be factored in such a way.

5.4.16 We claim that we need $n+1$ points. Then the nth level of successive differences is an integer, implying that all the consecutive integers will produce integral values of $p(x)$. To see that n points is not sufficient, we can simply pick a non-integer for the $n+1$st point and use Lagrange Interpolation.

5.4.17

5.4.18 Let $p = a+b, q = c+d, r = ab, s = cd$. We will express Vieta's formulas using these equations, with the goal of showing that $r^6 + r^4 + r^3 - r^2 - 1 = 0$.

$$\begin{aligned}
-1 = a+b+c+d &= p+q \\
0 = ab+ac+ad+bc+bd+cd &= r+s+ac+ad+bc+bd \\
&= r+s+(a+b)(c+d) \\
&= r+s+pq \\
0 = abc+abd+acd+bcd &= rc+rd+as+bs \\
&= r(c+d)+s(a+b) \\
&= rq+sp \\
-1 = abcd &= rs
\end{aligned}$$

Now $q = -(p+1)$ and $s = -1/r$. Substituting into $r+s+pq = 0$,

$$\begin{aligned}
r - \frac{1}{r} - p(p+1) &= 0 \\
\frac{r^2-1}{r} &= p(p+1)
\end{aligned}$$

Substituting into $rq+sp = 0$ yields

$$\begin{aligned}
r(-p-1) - \frac{1}{r}(p) &= 0 \\
r^2(-p-1) - p &= 0 \\
-pr^2 - r^2 - p &= 0 \\
p(-r^2-1) &= r^2 \\
p &= \frac{-r^2}{1+r^2} \\
p(p+1) &= \frac{-r^2}{1+r^2}\left(\frac{1}{1+r^2}\right) \\
&= \frac{-r^2}{(1+r^2)^2}
\end{aligned}$$

Now equating these,

$$\begin{aligned}
\frac{r^2-1}{r} &= \frac{-r^2}{(1+r^2)^2} \\
-r^3 &= (r^4-1)(r^2+1) \\
-r^3 &= r^6+r^4-r^2-1 \\
0 &= r^6+r^4+r^3-r^2-1
\end{aligned}$$

as desired.

5.4.19 Consider $F(x) := P(x) - 5$. Then $F(a) = F(b) = F(c) = F(d) = 0$, so $F(x) = (x-a)(x-b)(x-c)(x-d)G(x)$, where $G(x)$ is a polynomial with integer coefficients. If $\exists k$ s.t. $P(k) = 8$, then $F(k) = 3$. Since $G(k)$ is an integer, at least 3 of $x-a, x-b, x-c, x-d$ must be ± 1. By the pigeonhole principle, two must be equal, but this is impossible since a, b, c, d are distinct.

5.5: Inequalities

5.5.11 Since $a_i^2 \geq 0$ with equality iff $a_i = 0$, then $\sum a_i^2 \geq 0$ with equality iff $a_1 = a_2 = ... = a_n = 0$.

5.5.12 Suppose that not all the a_i are equal. If all of the a_i are less than S/n, then $a_1 + a_2 + ... + a_n < n(S/n) = S$, which is impossible. Similarly, not all the a_i can be greater than S/n.

5.5.13 $1999!^{(2000)} = (1999!)!^{(1999)} > (2000)!^{(1999)}$.

5.5.14

$$\begin{aligned}
\frac{10^{1999}+1}{10^{2000}+1} &< \frac{10^{1998}+1}{10^{1999}+1} \\
\iff (10^{1999}+1)(10^{1999}+1) &< (10^{1998}+1)(10^{2000}+1) \\
\iff 2 * 10^{1999} &< 10^{1998} + 10^{2000} \\
\iff 2 * 10 &< 1 + 100
\end{aligned}$$

We conclude that $\frac{10^{1999}+1}{10^{2000}+1} < \frac{10^{1998}+1}{10^{1999}+1}$.

5.5.15 By difference of squares,

$$\begin{aligned}
2(1998) &\leq 1000^2 \\
3(1997) &\leq 1000^2 \\
&... \\
999(1001) &\leq 1000^2 \\
1000(1999)(2000) &\leq 1000^4
\end{aligned}$$

Therefore, $2000! \leq 1000^{2000}$.

5.5.16

5.5.24 Let's massage the sum a bit. $\frac{1}{n!} < \frac{1}{n(n-1)}$. Apply this beginning with $\frac{1}{2!}$:

$$\begin{aligned}
1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}... &< 1 + 1 + \frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4}... \\
&= 2 + \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3}... \\
&= 3
\end{aligned}$$

5.5.25 Since x and $x+2$ have the same sum as $x+1$ and $x+1$ but are farther apart, their product is smaller.

5.5.26 (a) We have already shown the base case $n=2$. For inductive step, suppose that AM-GM holds for 2 and 2^k. We wish to show that

$$\frac{x_1+x_2+x_3...+x_{2^k}+y_1+y_2+...+y_{2^k}}{2^{k+1}} \geq \sqrt[2^{k+1}]{x_1x_2...x_{2^k}y_1y_2...y_{2^k}}$$

By inductive hypothesis,

$$\frac{x_1+x_2+...+x_{2^k}}{2^k} \geq \sqrt[2^k]{x_1x_2...x_{2^k}}$$

and

$$\frac{y_1+y_2+...+y_{2^k}}{2^k} \geq \sqrt[2^k]{y_1y_2...y_{2^k}}$$

And by the two variable AM-GM,

$$\begin{aligned}
\frac{\frac{x_1+x_2+...+x_{2^k}}{2^k}+\frac{y_1+y_2+...+y_{2^k}}{2^k}}{2} &\geq \frac{\sqrt[2^k]{x_1x_2...x_{2^k}}+\sqrt[2^k]{y_1x_2...x_{2^k}}}{2} \\
&\geq \sqrt{\sqrt[2^k]{x_1x_2...x_{2^k}}\sqrt[2^k]{x_1x_2...x_{2^k}}} \\
&= \sqrt{\sqrt[2^k]{x_1x_2...x_{2^k}y_1y_2...y_{2^k}}} \\
&= \sqrt[2^{k+1}]{x_1x_2...x_{2^k}y_1y_2...y_{2^k}}
\end{aligned}$$

(b) We would like to transform $\frac{a+b+c+d}{4}$ into $\frac{a+b+c}{3}$. We can accomplish this by a clever choice of d. If we let d be the average of a,b,c, then the average of a,b,c and the average is the average:

$$\frac{a+b+c+\frac{a+b+c}{3}}{4}=\frac{a+b+c}{3}$$

And by the four variable AM-GM,

$$\begin{aligned}
\frac{a+b+c+\frac{a+b+c}{3}}{4} &\geq \sqrt[4]{abc\left(\frac{a+b+c}{3}\right)} \\
\frac{a+b+c}{3} &\geq \sqrt[4]{abc\left(\frac{a+b+c}{3}\right)} \\
\left(\frac{a+b+c}{3}\right)^4 &\geq abc\left(\frac{a+b+c}{3}\right) \\
\left(\frac{a+b+c}{3}\right)^3 &\geq abc \\
\frac{a+b+c}{3} &\geq \sqrt[3]{abc}
\end{aligned}$$

(c) We will use this strategy to prove that if AM-GM holds for n, then it holds for $n-1$. By inductive hypothesis,

$$\frac{x_1+x_2+...+x_n}{n} \geq \sqrt[n]{x_1x_2...x_n}$$

Choose $x_n = \frac{x_1+x_2+...+x_{n-1}}{n-1}$. Then

$$\begin{aligned}
\frac{x_1+x_2+...+x_{n-1}+\frac{x_1+x_2+...+x_{n-1}}{n-1}}{n} &\geq \sqrt[n]{x_1x_2...x_{n-1}\left(\frac{x_1+x_2+...+x_{n-1}}{n-1}\right)} \\
\frac{x_1+x_2+...+x_{n-1}}{n-1} &\geq \sqrt[n]{x_1x_2...x_{n-1}\left(\frac{x_1+x_2+...+x_{n-1}}{n-1}\right)} \\
\left(\frac{x_1+x_2+...+x_{n-1}}{n-1}\right)^n &\geq x_1x_2...x_{n-1}\left(\frac{x_1+x_2+...+x_{n-1}}{n-1}\right) \\
\left(\frac{x_1+x_2+...+x_{n-1}}{n-1}\right)^{n-1} &\geq x_1x_2...x_{n-1} \\
\frac{x_1+x_2+...+x_{n-1}}{n-1} &\geq \sqrt[n-1]{x_1x_2...x_{n-1}}
\end{aligned}$$

5.5.27 Note that

$$\frac{1}{\sqrt{1,000,000}+\sqrt{999,999}} = \sqrt{1,000,000}-\sqrt{999,999}$$

Therefore, we need to find the closest n to $\sqrt{1,000,000}+\sqrt{999,999}$. Since $1,000,000$ is a perfect square and $999 < \sqrt{999,999} < 1,000,000$, we just need to know whether 999.5^2 is less than $999,999$ or not. We find that it is, so the answer is $\boxed{2000}$.

5.5.28 Taking the square root,

$$\begin{aligned}
\sqrt{n!} &< \left(\frac{n+1}{2}\right)^{\frac{n}{2}} \\
\iff \sqrt{1*2*...*n} &< \left(\frac{n+1}{2}\right)^{\frac{n}{2}}
\end{aligned}$$

Since $\sqrt{k(n+1-k)} \leq \frac{n+1}{2}$ by AM-GM, this inequality is true.

5.5.29 If the product contains $k > 4$, then we can replace k by $3(k-3) = 3k-9$, which is greater than k for all k greater than 4. If the product contains 4, we can replace it by $2*2$ leaving the product unchanged. Since $2+2+2 = 3+3$ and $2^3 < 3^2$, if we have more than 3 twos we can replace them with 2 threes, thereby increasing the product. Therefore, the product is maximized when we have the maximal number of threes. Since $1974/3 = 658$, the maximum product is $3^{658}*2$.

5.2.30 We can write $a^2+b^2+c^2-ab-bc-ac = \frac{1}{2}((a-b)^2+(b-c)^2+(c-a)^2)$. Since a,b,c are positive, it follows that $\frac{1}{2}(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2)$ is nonnegative (with equality iff $a=b=c$), so $a^2+b^2+c^2 > 3abc$.

5.2.31 Square the quantity. We can decrease the product by replacing $(2k+1)^2$ in the denominator by

$(2k)(2k+2)$, except for the first term:

$$\begin{aligned}\left(\frac{1*1}{2*2}\right)\left(\frac{3*3}{4*4}\right)...\left(\frac{(2n-1)*(2n-1)}{2n*2n}\right) &\geq \left(\frac{1}{2*2}\right)\left(\frac{2*4}{4*4}\right)...\left(\frac{(2n-2)*(2n)}{2n*2n}\right)\\ &= \frac{1}{4n}\end{aligned}$$

Therefore,

$$\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)...\left(\frac{2n-1}{2n}\right) \geq \frac{1}{\sqrt{4n}}$$

For the upper bound, we will use the same strategy to decrease the denominators, thereby decreasing the product:

$$\begin{aligned}\left(\frac{1*1}{2*2}\right)\left(\frac{3*3}{4*4}\right)...\left(\frac{(2n-1)*(2n-1)}{2n*2n}\right) &< \left(\frac{1}{1*3}\right)\left(\frac{3*3}{3*5}\right)...\left(\frac{(2n-1)*(2n-1)}{(2n-1)*(2n+1)}\right)\\ &= \frac{1}{2n+1}\\ &< \frac{1}{2n}\end{aligned}$$

Therefore,

$$\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)...\left(\frac{2n-1}{2n}\right) < \frac{1}{\sqrt{2n}}$$

5.5.32 We can write

$$x+\frac{a_1+a_2+...+a_n}{n} = \frac{(x+a_1)+(x+a_2)+...+(x+a_n)}{n}$$

Then by AM-GM,

$$\left(\frac{(x+a_1)+(x+a_2)+...+(x+a_n)}{n}\right)^n \geq (x+a_1)(x+a_2)...(x+a_n)$$

5.5.33

5.5.34 We can interpret $\sqrt{a_i^2+b_i^2}$ as the hypotenuse of a right triangle with legs a_i and b_i. The RHS represents the hypotenuse of a right triangle with legs $a_1+a_2+...+a_n$ and $b_1+b_2+...+b_n$. Rotating and reflecting as necessary, the LHS represents the path of hypotenuses from one endpoint of this hypotenuse to the other. By the triangle inequality, we have

$$\sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}+...+\sqrt{a_n^2+b_n^2} \geq \sqrt{(a_1+a_2+...+a_n)^2+(b_1+b_2+...+b_n)^2}$$

as desired.

5.5.35 Since the dot product is $|a||b|\cos\theta$, it is always less than or equal to $|a||b|$. Let $\overrightarrow{a} = (a_1\ a_2\ ...\ a_n)$ and $\overrightarrow{b} = (b_1\ b_2\ ...\ b_n)$. Then $|a||b| = (a_1^2+a_2^2+...a_n^2)(b_1^2+b_2^2+...+b_n^2)$ and the dot product is $(a_1b_1+a_2b_2+...+a_nb_n)$.

5.5.36 Since the sum of squares is always positive, there are no real roots. Therefore, the discriminant must be negative. The discriminant is equal to

$$\left(2\sum_{i=1}^{n} a_i b_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Setting this as less than 0 then yields Cauchy.

5.5.37 By Cauchy,

$$(a_1 + a_2 + ... + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right) \geq \left(\underbrace{1 + 1 + .. + 1}_{n \text{ times}}\right)^2 = n^2$$

5.5.38 By Cauchy,

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\underbrace{1 + 1 + ... + 1}_{n \text{ times}}\right) \geq \sum_{i=1}^{n} a_i = 1$$

Therefore,

$$\left(\sum_{i=1}^{n} a_i^2\right) \geq \frac{1}{n}$$

5.5.39 By AM-GM, $a^2b + b^2c + c^2a \geq 3abc$. Similarly, $ab^2 + bc^2 + ca^2 \geq 3abc$. Therefore,

$$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$$

5.5.40 By Cauchy,

$$(3)(a + b + c) \geq (\sqrt{a} + \sqrt{b} + \sqrt{c})^2$$

The result then follows from taking the square root.

5.5.41 (Note: the result follows immediately from Titu's Lemma form of Cauchy). Write $x_1 = \frac{1^2}{a}, x_2 = \frac{1^2}{b}, x_3 = \frac{2^2}{c}, x_4 = \frac{4^2}{d}$. By Cauchy,

$$(x_1 + x_2 + x_3 + x_4)(a + b + c + d) \geq (1 + 1 + 2 + 4)^2$$

Dividing by $a + b + c + d$ yields the result.

5.5.42 Let r_1, r_2, r_3, r_4, r_5 be the roots of the equation. By Vieta's sums, $-a = \sum r_i$ and $b = \sum r_i r_j$. Then

$$a^2 - 2b = \left(\sum r_i\right)^2 - 2\left(\sum r_i r_j\right) = \sum r_i^2$$

Therefore,

$$2a^2 - 4b = 2\sum r_i^2$$

By AM-GM, $\frac{r_i^2}{2} + \frac{r_j^2}{2} \geq r_i r_j$. Splitting $2\sum r_i^2 = 4\sum \frac{r_i^2}{2}$, we see that we can pair each r_i with each r_j exactly once. Then by AM-GM,

$$2a^2 - 4b = 2\sum r_i^2 \geq \sum r_i r_j = b$$

So $2a^2 > 5b$ if all the r_i^2 are positive. Since we are given that $2a^2 < 5b$, this is impossible.

5.5.43 By Cauchy,

$$\begin{aligned} x^2+y^2+z^2 &\geq (x+y+z)\left(\frac{x+y+z}{3}\right) \\ x^2+y^2+z^2 &\geq (x+y+z)(\sqrt[3]{xyz}) \\ &\geq x+y+z \end{aligned}$$

5.5.44 We claim that the minimum is $\boxed{\frac{3}{2}}$.

$$\begin{aligned} \frac{x}{y+z} + \frac{y}{z+x}\frac{z}{x+y} &\geq \frac{3}{2} \\ \frac{x+y+z}{y+z} + \frac{x+y+z}{z+x} + \frac{x+y+z}{x+y} &\geq \frac{9}{2} \\ \frac{(x+y)+(y+z)+(z+x)}{y+z} + \frac{(x+y)+(y+z)+(z+x)}{z+x} + \frac{(x+y)+(y+z)+(z+x)}{x+y} &\geq 9 \\ \big((x+y)+(y+z)+(z+x)\big)\left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x}\right) &\geq 9 \end{aligned}$$

Which follows from Cauchy.

5.5.45 For the LHS, $yz + zx + xy \geq 3\sqrt[3]{x^2y^2z^2}$. Since $x, y, z > 0$ and $x + y + z = 1$, $x, y, z < 1$. Therefore, $\sqrt[3]{x^2y^2z^2} \geq \sqrt{x^2y^2z^2} = xyz$, so $yz + zx + xy \geq 3xyz \geq 2xyz$ with equality iff two of the x, y, z are 0 and the other is 1.

For the RHS, consider $(1-2x)(1-2y)(1-2z) = 1 - 2(x+y+z) + 4(xy+yz+zx) - 8xyz = -1 + 4(xy+yz+zx) - 8xyz$. Then

$$xy + yz + zx - 2xyz = \frac{1}{4}((1-2x)(1-2y)(1-2z)+1)$$

But by AM-GM,

$$\sqrt[3]{(1-2x)(1-2y)(1-2z)} \leq \frac{1-2x+1-2y+1-2z}{3} = \frac{1}{3}$$

So $(1-2x)(1-2y)(1-2z) \leq \frac{1}{27}$. Then

$$xy + yz + zx - 2xyz \leq \frac{1}{4}\left(\frac{1}{27}+1\right) = \frac{7}{27}$$

5.5.46 Let n be the degree of the polynomial.

Case 1: $n = 1$. The only polynomials of degree one with coefficients equal to ± 1 are $\pm(x+1)$ and $\pm(x-1)$.

Case 2: $n = 2$. The only polynomials of degree two with coefficients equal to ± 1 are $\pm(x^2+x+1), \pm(x^2-x+1), \pm(x^2+x-1), \pm(x^2-x-1)$. Of these, only the latter two have positive discriminants, and thus all real roots.

Case 3: $n \geq 3$. Let the polynomial by $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_0$. Let the roots be $r_1, r_2, ...r_n$. Therefore $a_{n-1} = \sum_{i=1}^{n} r_i$ and $a_{n-2} = \sum_{1 \leq i < j \leq n} r_i r_j$. Then $a_{n-1}^2 - 2a_{n-2} = \sum_{i=1}^{n} r_i^2$. By AM-GM,

$$n \sum_{i=1}^{n} r_i^2 \geq n \sqrt[n]{\prod_{i=1}^{n} r_i^2} = n \sqrt[n]{a_0^2} = n \sqrt[n]{1} = n$$

But $a_{n-1}^2 - 2a_{n-2} \leq 1 - (-2) = 3$, so there are no solutions for $n > 3$. For $n = 3$, we must have a_1 the opposite sign of a_3. Then it is easy to check that

$$\pm(x^3 + x^2 - x - 1), \pm(x^3 - x^2 - x + 1)$$

Are the only solutions.

5.5.47 By AM-GM,

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + ... + \frac{a_n}{b_n} \geq n \sqrt[n]{\frac{a_1 a_2 ... a_n}{b_1 b_2 ... b_n}}$$

Since $b_1, b_2, ...b_n$ is a permutation of $a_1, a_2, ...a_n$, $n\sqrt[n]{\frac{a_1 a_2 ... a_n}{b_1 b_2 ... b_n}} \geq n$ with equality iff $\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n}$.

5.5.48 We prove the more general Rearrangement Inequality, which states that if $a_1, a_2, ...a_n$ and $b_1, b_2, ...b_n$ are two nondecreasing sequences, then the sum

$$\sum a_i b_j$$

is maximizied when $i = j$ and minimized when $i = n + 1 - j$, with equality iff at least one of the sequences is constant. We proceed by induction. For 2,

$$(a_1 - a_2)(b_1 - b_2) \geq 0$$

Therefore,

$$a_1 b_1 + a_2 b_2 \geq a_1 b_2 + a_2 b_1$$

Now assume that the result holds for n. We show that it holds for $n + 1$. Let

$$a_1 x_1 + a_2 x_2 + ... + a_1 x_n$$

be the maximum value of the sum. Suppose there exists an $i < j$ such that $x_i > x_j$. Then

$$a_i x_i + a_j x_j \leq a_i x_j + a_j x_i$$

By the two variable rearrangement, with equality iff $a_i = a_j$. Then we can simply interchange consecutive values a_i and a_{i+1} until $x_1 \leq x_2 \leq ... \leq x_n$.

Chapter 6: Combinatorics

6.1: Introduction to Counting

6.1.20 (a) The LHS can be interpreted as the number of ways to choose two people from a group of n women and n men. Another way to count this: we can choose two men in $\binom{n}{2}$ ways, two women in $\binom{n}{2}$ ways, and a man and a women in $n * n$ ways, which is precisely the RHS.

(b) We interpret the LHS as the number of ways to pick $n+1$ people from a group of $2n+2$ people. Now we count this in another way: fix two people, Adam and Bob. Then the group either contains both Adam and Bob, neither of them, or exactly one of them. There are $\binom{2n}{n+1}$ ways to pick a group of $n+1$ people with neither Adam nor Bob. There are $\binom{2n}{n}$ ways to pick a group with Adam but not Bob, and vice versa. There are $\binom{2n}{n-1}$ ways to pick a group with both Adam and Bob. This sum is precisely the RHS.

6.1.21 (a) Each divisor is of the form

$$d = p_1^{f_1} p_2^{f_2} ... p_t^{f_5}$$

We must have $f_i \leq e_i$. Then for each f_i, there are $e_i + 1$ possibilities (0 to e_i).

(b) A product of integers is odd if and only if all the factors are odd. Since the number of factors of n is

$$(e_1 + 1)(e_2 + 1)...(e_t + 1)$$

n has an odd number of factors iff each $e_i + 1$ is odd, in which case each e_i is even, implying that n is a perfect square.

6.1.22 (a) We note that these are the coefficients of $(1+1)^n = 2^n$.

(b) These are the coefficients of $(1-1)^n = 0$.

6.1.23 For each subset, a certain element can either by in or out. Therefore, if the set has n elements, there are 2^n possible subsets.

6.1.24 (a) We see that the LHS is just the number of subsets of a set with n elements, which we have already shown to be 2^n.

(b)

6.1.25 There are 10^3 perfect squares and 10^2 perfect cubes. We can pair a square and a cube in $10^3 * 10^2 = 10^5$ ways. Even though we may be overcounting, this is far less than half of 10^6.

6.1.26 There are $\binom{49}{2} = 1176$ different ways to choose the two yellow squares. There are 24 colorings in which the squares are diametrically opposite, and thus are counted only twice. Of the remaining $1176 - 24 = 1152$

colorings, each coloring is counted four times, once for each rotation. Therefore, the answer is

$$\frac{1152}{4} + \frac{24}{2} = \boxed{300}$$

6.1.27 Suppose we want to count the number of appearances of $x_1^{a_1} x_2^{a_2} ... x_n^{a_n}$. This is equivalent to counting the number of ways of ordering $a_1 x_1' s, a_2 x_2' s, ... a_n x_n' s$. Therefore, the answer is

$$\frac{r!}{a_1! a_2! ... a_n!}$$

6.2: Partitions and Bijections

6.2.8 By the Pigeonhole Principle, at least two cards will be of the same suit. Since there are only 13 denominations, one of these will be cyclically 6 above the other. Return this card. Using the remaining three cards, we can designate one of 6 numbers using their 6 possible orderings, so we can indicate the exact card returned to the deck.

6.2.9 There are four possibilities for each person: nothing, ice cream, cookie, both. Therefore, there are 4^8 possible combinations. However, we must subtract the number of ways in which nobody gets both an ice-cream and a cookie. There are 3 possibilities for each person: nothing, ice cream, cookie. Therefore, the answer is $\boxed{4^8 - 3^8}$.

6.2.10 The sum of all the elements if $\frac{30(31)}{2} = 465$. Therefore, if the sum of the elements of a subset is greater than 232, the sum of the elements of its complement is less than or equal to 232. Then the number of subsets with sum greater than 232 is half the total number of subsets: $\boxed{2^{29}}$.

6.2.11 There is a bijection between such sequences and subsets of $\{1, 2, ...1000\}$ containing 1 and 1000. Therefore, the answer is 2^{998}.

6.2.12 We have

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$$

Applying the identity to the last term, we get

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n-1}{r} + \binom{n-1}{r+1}$$

We see that this telescopes to the Hockey Stick Identity.

6.2.13 We showed earlier that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} ... + (-1)^n \binom{n}{n} = 0$$

The negative terms are the number of subsets with an odd number elements and the positive terms are the number of subsets with an even number of elements.

6.2.14 Consider the coordinate system (a, b), which designates that a piece is in row a, column b. In order to ensure that two pieces are in different rows and different columns, both their coordinates must be different. There are $\binom{8}{2} = 28$ ways to choose 2 different row coordinates and $\binom{8}{2} = 28$ ways to choose 2 different column coordinates. Given the row coordinates and column coordinates, there are 2 different ways to pair them into two ordered pairs. Therefore, the answer is $2 * 28 * 28 = \boxed{1568}$.

6.2.15 There are $\binom{8}{2}$ ways to choose the two horizontal sides. There are $\binom{8}{2}$ ways to choose the two vertical sides. There are exactly 4 intersections, so they uniquely determine the points. Therefore, the answer if $28 * 28 = \boxed{784}$.

6.2.16 We place one red ball and one white ball in each ball first to satisfy the requirement. Then the problem is one of counting the number of ways to place $r - n$ red balls and $w - n$ white balls in n boxes. The ball-and-urn formula yields the answer $\boxed{\binom{r-1}{r-n}\binom{w-1}{w-n}}$.

6.2.17 We count the number of ways that the vehicle cannot park. First place the 8 cars. There are 9 spots bordering these to place the 4 empty spots. Therefore, there are a total of $\binom{9}{4} = 126$ ways such that the car cannot park. There are $\binom{12}{8} = 495$ ways to position the other 8 cars in total. Thus, the probability of being able to park is $\frac{495-126}{495} = \boxed{\dfrac{41}{55}}$.

6.2.18 We claim that the sum is equal to $\binom{2n}{n}$. Consider the coefficient of x^n in

$$(1+x)^n(1+x)^n = (1+x)^{2n}$$

By the binomial theorem, it is $\binom{2n}{n}$. However, it is also equal to the coefficient of 1 times the coefficient of x^n, the coefficient of x^1 times the coefficient of x^{n-1}, etc. Since $\binom{n}{k}^2 = \binom{n}{k}\binom{n}{n-k}$,

$$\sum_{k=1}^{n}\binom{n}{k}^2 = \sum_{k=1}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$$

6.2.19 We consider n balls and $n - 1$ divider slots between them:

$$* - * - * - \ldots * - *$$

Then the number of ways of partitioning n into an ordered sum of k integers is the number of ways to fill the slots with dividers. Each slot can be filled or unfilled, so there are a total of $\boxed{2^{n-1}}$ ways.

6.2.20 (a) We have 17 slots:

$$- - - - \ldots -$$

There are $\binom{17}{8}$ ways to place 8 biscuits in the slots, or $\binom{17}{9}$ ways to put up 9 separaters to divide the biscuits into 10 categories (dogs).

(b) For each biscuit, there are 10 different dogs that can eat it. Therefore, there are 10 combinations of biscuit-dog consumption.

6.2.21 The answer is the number of ways to partition 8 into a sum of 10 nonnegative numbers, where order does not matter. In other words, it is just the number of partitions of 8, which is $\boxed{22}$.

6.2.22 For each term $x^a y^b z^c$, we must have $a + b + c = 1999$. Therefore, the number of different terms is the number of nonnegative solutions to $a + b + c = 1999$, which is $\boxed{\binom{2001}{2}}$.

6.2.23 Suppose we have n balls and $n - 1$ dividers:

$$* - * - * - \ldots * - *$$

Then the number of ways to partition this into a sum of three positive numbers if the number of ways to put 2 dividers in the $n - 1$ slots, which is $\boxed{\binom{n-1}{2}}$.

6.2.24 Consider each element. Either it is in subset A, or subset B, or both. Therefore, there are 3^n possible pairs *if we take order into account*. In this counting, each pair is counted twice except S, S whose reverse is itself. Therefore, the total number of pairs is $\boxed{\frac{3^n+1}{2}}$.

6.2.25 We know that $1, 2, 3, 4, \ldots 7$ have already been placed on the pile. For each k element subset of $\{1, 2, 3, \ldots 7\}$, there are $k + 2$ possible places for 9: in the $k + 1$ slots between the elements of the subset, or not at all. Therefore the answer

$$\begin{aligned}
\sum_{k=0}^{7}(k+2)\binom{7}{k} &= \sum_{k=0}^{7} k\binom{7}{k} + 2\sum_{k=0}^{7}\binom{7}{k} \\
&= 7(2^6) + 2(2^7) \\
&= \boxed{704}
\end{aligned}$$

6.2.26 The probability is preserved if we remove the distinction between maple and oak trees. We can see this because it does not affect the condition of birch trees being adjacent, so it simply multiplies both the total number of cases and the individual cases by a scalar. Then the problem reduces to that of 6.2.16: how many ways can we place 5 trees in 12 slots so that no two are adjacent? First place the 7 oaks/maples. There are 8 slots bordering these where we need to place 5 boys. Therefore, the total number of possibilities is $\binom{8}{5} = 56$. There are $\binom{12}{5} = 792$ total possibilities, so the probability is $\boxed{\frac{7}{99}}$.

6.2.27 We consider two cases: all three people are adjacent, or two are adjacent and the other is not.

Case 1: All three adjacent. There are 25 possible seatings for the rightmost person, who uniquely determines the position of the other two.

Case 2: Two adjacent. There are 25 possible seatings for the rightmost person of the adjacent pair and 21 possible locations for the third person. The number of ways is $25 * 21$.

Since there are $\binom{25}{3} = 2300$ total ways, the probability is

$$\frac{25 * 22}{2300} = \boxed{\frac{11}{46}}$$

6.2.28 For every choice of 4 points, there is one intersection. Therefore, there are a total of $\binom{n}{4}$ intersections.

6.2.29 We first consider the case where the girls are indistinguishable and the boys are indistinguishable. First place the g girls. There are $g+1$ empty seats between girls, and we need to place the b boys in these $g+1$ seats. There are $\binom{g+1}{b}$ ways of doing this.

Now consider the case where the girls are distinguishable. First place the g girls; this can be done in $g!$ ways. There are $g+1$ empty seats between the girls for us to place the b boys. There are $b!\binom{g+1}{b}$ ways of doing this. Therefore, the total number of ways is $g!b!\binom{g+1}{b}$.

6.2.30 Let's consider the meaning of a single index r. $\binom{k}{r}$ is the number of ways we can choose r colors to get. Consider the h hats separates by $h-1$ slots:

$$* - * - * - * - ...*$$

Now that we have decided upon our colors, we need to choose the number of hats for each color. The number of ways to do this is $\binom{h-1}{r-1}$. Therefore, the number of ways to order h hats from a store with k different kinds is also

$$\sum_{r=1}^{k} \binom{k}{r}\binom{h-1}{r-1}$$

6.2.31 We interpret the RHS as the number of ways to select a committee of k members from $n+m$ candidates. The LHS is the number of ways to select j members from a body of n candidates and $k-j$ members from a body of m candidates, as j ranges from 0 to k. Therefore, it is also the number of ways to select a committee of k members from $n+m$ total candidates.

6.2.32 We double overcount every time three boys get a combination of toys. Therefore, we need to subtract $2\binom{4}{3}3! = 48$.

6.2.33

6.2.34 Let S denote $\{1, 2, ...n\}$ and S_i denote $\{1, 2, ...m\} - \{i\}$. We note that we can find S_{ij} by the identity $f(S_i \cup S_j) = f(S_{ij}) = \min\{f(S_i), f(S_j)\}$. Therefore, we can the value of all subsets with two elements missing. Similarly, we can find all subsets with three elements missing. Therefore, we can uniquely determine f as long as we know $f(S_1), f(S_2), ...f(S_n)$ as well as $f(S)$. The given identity implies that if $A \subset B$, then $f(A) \leq f(B)$. So for each of the m possible values of $f(S)$, $f(S_i)$ can take on the values 1 to m. Therefore, for $f(S) = j$, there are j^n possible functions $f : P_n \to \{1, 2, ...m\}$. Thus

$$c(n, m) = \sum_{j=1}^{m} j^n$$

6.3: The Principle of Inclusion-Exclusion

6.3.7 The total number of orders is the number of ways to put 10 balls into 31 urns- that is, $\binom{40}{10}$. The number of orders with distinct flavors is $\binom{31}{10}$. Therefore, the number of orders with duplication is

$$\binom{40}{10} - \binom{31}{10}$$

6.3.8 The solution does not account for the possibility that the number will be higher depending on whether or not a certain spouse has already been used.

6.3.9 We will proceed by a combination of counting the complement and using PIE. Let A denote the set of integers from $1 - 1000$ divisible by 2, B denote the set of integers from $1 - 1000$ divisible by 3, and C denote the set of integers from $1 - 1000$ divisible by 5. We wish to find $|A \cup B \cup C|$. By PIE,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

$$\begin{aligned}
|A| &= \left\lfloor \tfrac{1000}{2} \right\rfloor = && 500 \\
|B| &= \left\lfloor \tfrac{1000}{3} \right\rfloor = && 333 \\
|C| &= \left\lfloor \tfrac{1000}{5} \right\rfloor = && 200 \\
|A \cap B| &= \left\lfloor \tfrac{1000}{6} \right\rfloor = && 166 \\
|B \cap C| &= \left\lfloor \tfrac{1000}{15} \right\rfloor = && 66 \\
|C \cap A| &= \left\lfloor \tfrac{1000}{10} \right\rfloor = && 100 \\
|A \cap B \cap C| &= \left\lfloor \tfrac{1000}{30} \right\rfloor = && 33
\end{aligned}$$

Now substituting these values into our formula, we find that

$$|A \cup B \cup C| = 500 + 333 + 200 - 166 - 66 - 100 + 33 = 734$$

Then there are $1000 - 734 = \boxed{266}$ integers from 1 to 1000 not divisible by $2, 3$, or 5.

6.3.10 We will use complementary counting again. In order for the product to be divisible by 10, it must be divisible by 2 and 5. Let A be the set of integers generated not divisible by 2 and B be the set of integers generated not divisible by 5. Then we wish to find $|A \cup B|$. By PIE,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For a sequence of integers to be in A, they must be odd. Therefore,

$$|A| = 5^n$$

For a sequence of integers to be in B, they must not be 5. Therefore,

$$|B| = 8^n$$

For a sequence of integers to be in $A \cap B$, they must be odd but not 5. Therefore,

$$|A \cap B| = 4^n$$

Then

$$|A \cup B| = 5^n + 8^n - 4^n$$

Since there are 9^n total possibilities, the desired probability is

$$\boxed{1 - \frac{5^n + 8^n - 4^n}{9^n}}$$

6.3.11 We note that the number of solutions in nonnegative integers to $a + b + c + d = 17$ is the number of ways to put 17 ball into 4 urns, which is $\binom{20}{3} = 1140$. Similarly, the number of solutions in nonnegative integers to $a + b + c = k$ is $\binom{k+2}{2}$. Now we let d range from 0 to 12, so k ranges from 17 to 5. Then the number of solutions is

$$\begin{aligned} \sum_{k=5}^{17} \binom{k+2}{2} &= \sum_{k=7}^{19} \binom{k}{2} \\ &= \sum_{k=2}^{19} \binom{k}{2} - \sum_{k=2}^{6} \binom{k}{2} \\ &= \binom{20}{3} - \binom{7}{3} \\ &= \boxed{1105} \end{aligned}$$

6.3.12 We will count the complement. Let A_i be the set of permutations that fixes the element a_i. We wish to find $|A_1 \cup A_2 \cup A_3 ... \cup A_n|$. By PIE,

$$|A_1 \cup A_2 \cup A_3 ... \cup A_n| = \sum_{i=1}^{n} (-1)^{n-i} S_i$$

Where

$$S_i = \sum_{1 \le x_1 < x_2 < ... < x_i \le n} |A_{x_1} \cup A_{x_2} \cup A_{x_3} ... \cup A_{x_n}|$$

The set $A_{x_1} \cup A_{x_2} \cup A_{x_3} ... \cup A_{x_n}$ is the set of the permutations that fix $a_{x_1}, a_{x_2}, ... a_{x_n}$. There are $n - i$ ways to order the $n - i$ unfixed elements and $\binom{n}{i}$ ways to choose the elements to be fixed. Therefore,

$$S_i = \binom{n}{i} (n - i)!$$

Then

$$\begin{aligned}
|A_1 \cup A_2 \cup A_3 ... \cup A_n| &= \sum_{i=1}^{n} (-1)^{i-1} S_i \\
&= \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} (n-i)! \\
&= \sum_{i=1}^{n} (-1)^{i-1} \frac{n!}{i!(n-i)!} (n-i)! \\
&= \sum_{i=1}^{n} (-1)^{i-1} \frac{n!}{i!} \\
&= n!(1 - \frac{1}{1!} + \frac{1}{2!} - ... + (-1)^n \frac{1}{n!})
\end{aligned}$$

6.3.13 The left hand side is the number of permutations of an ordered sequence of n objects. We interpret the term for an individual index i of the right hand side as the number of derangements with exactly i fixed points. There are $\binom{n}{i}$ ways of choosing the fixed points and D_{n-i} ways of arranging the other terms so that none are fixed points. As i ranges from 0 to n, we see that this covers all permutations.

6.3.14 Again, we count the complement. Let A_i be the set of permutations so that the ith person has the same neighbor to the right. We wish to find $|A_1 \cup A_2 \cup A_3 ... \cup A_n|$. By PIE,

$$|A_1 \cup A_2 ... \cup A_n| = \sum_{i=1}^{n} (-1)^{i-1} S_i$$

where

$$S_i = \sum_{1 \le x_1 < x_2 < ... < x_i \le n} |A_{x_1} \cup A_{x_2} \cup A_{x_3} ... \cup A_{x_i}|$$

$A_{x_1} \cup A_{x_2} \cup ... \cup A_{x_i}$ is the set of permutations that leaves $x_1, x_2, ... x_i$ with the same neighbor to the right. We can combine x_k and $x_k + 1$ into a single block, encoding the condition that x_k retains the neighbor to the right. Note that if we have i people with fixed neighbors to the right, then we always have $n - j$ blocks of length one or more to be arranged in $(n-i)!$ ways. Since there are $\binom{n}{i}$ ways to choose which people will have fixed neighbors, $S_i = \binom{n}{i}(n-i)!$. Then

$$\begin{aligned}
|A_1 \cup A_2 ... \cup A_n| &= \sum_{i=1}^{n} (-1)^{i-1} S_i \\
&= \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} (n-i)! \\
&= \sum_{i=1}^{n} (-1)^{i-1} \frac{n!}{i!}
\end{aligned}$$

Since this is the complement, the number of ways with different neighbors is

$$n! - \sum_{i=1}^{n} \frac{n!}{i!}$$

Which we note is equal to D_n.

6.3.15 We count the complement. Let A_i be the set of orders that do not use flavor i. We want to find $|A_1 \cup A_2 \cup ... \cup A_k|$. By PIE, this is equal to

$$|A_1 \cup A_2 \cup ... \cup A_k| = \sum_{i=1}^{n} (-1)^{i-1} S_i$$

Where

$$S_i = \sum_{1 \leq x_1 < x_2 < ... < x_i \leq n} |A_{x_1} \cup A_{x_2} \cup A_{x_3} ... \cup A_{x_i}|$$

Since i flavors cannot be used, all there are $k-i$ options for all the other choices and thus $(k-i)^n$ different orders. There are $\binom{k}{i}$ ways of choosing the i flavors. Therefore,

$$S_i = \binom{k}{i}(k-i)^n$$

So

$$\begin{aligned} |A_1 \cup A_2 \cup ... \cup A_k| &= \sum_{i=1}^{n} (-1)^{i-1} S_i \\ &= \sum_{i=1}^{n} (-1)^{i-1} \binom{k}{i} \\ &= \binom{k}{1}(k-1)^n - \frac{k}{2}(k-2)^n ... (-1)^{k-1}\binom{k}{k} 0^n \end{aligned}$$

And since this is the complement, the desired number is

$$k^n - \binom{k}{1}(k-1)^n + \frac{k}{2}(k-2)^n ... (-1)^k \binom{k}{k} 0^n$$

as desired.

6.3.16 We count the complement. A_i be the set of permutations such that i and $i+n$ are adjacent. By PIE,

$$|A_1 \cup A_2 \cup ... \cup A_k| = \sum_{i=1}^{n} (-1)^{i-1} S_i$$

Where

$$S_i = \sum_{1 \leq x_1 < x_2 < ... < x_i \leq n} |A_{x_1} \cup A_{x_2} \cup A_{x_3} ... \cup A_{x_i}|$$

We can combine x_k and $x_k + n$ into a brick. We have $i + 2n - 2i = 2n - i$ bricks, which can be arranged in $(2n-i)!$ ways. Additionally, there are 2^i ways of arranging the order within the two-element bricks. There are $\binom{n}{i}$ ways to choose $x_1, x_2, ... x_i$. Therefore,

$$S_i = \binom{n}{i}(2n-i)! 2^i$$

We note that

$$
\begin{aligned}
\frac{s_{i+1}}{s_i} &= \frac{\binom{n}{i+1}(2n-i-1)!2^{i+1}}{\binom{n}{i}(2n-i)!2^i} \\
&= \frac{\frac{n!}{(i+1)!(n-i-1)!}(2n-i-1)!2^{i+1}}{\frac{n!}{i!(n-i)!}(2n-i)!2^i} \\
&= \frac{2(n-i)}{(2n-i)(i+1)} \\
&< \frac{1}{1} * \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}
$$

Therefore, $S_i \geq \frac{1}{2}S_{i+1}$. Therefore,

$$
\begin{aligned}
|A_1 \cup A_2 \cup ... \cup A_k| &= \sum_{i=1}^{n}(-1)^{i-1}S_i \\
&= S_1 - S_2 + S_3... \\
&> S_1 - S_2 \\
&> \frac{S_1}{2} \\
&= \frac{n * (2n-1)! * 2}{2} \\
&= \frac{2n!}{2}
\end{aligned}
$$

6.4: Recurrence

6.4.4 Let $\sigma(n)$ denote the number of different messages that can be typed with n keystrokes. All messages either begin with a one letter word or not. In the former case, the next keystroke is a space and thereafter we have $\sigma(n-2)$ possibilities. In the latter case, all messages are the letter prefixed onto one of the combinations of $\sigma(n-1)$. Therefore, $\sigma(n) = \sigma(n-1) + \sigma(n-2)$. Since $\sigma(1) = 1$ and $\sigma(2) = 1$, $\sigma(n) = f_n$.

6.4.5 Let R_n be the set of subsets of $S_n := \{1, 2, ...n\}$ such that $S \cup S + 1 = \{1, 2, ...n\}$. All elements of R_n fall into two categories: those containing n and those not containing n. In the former case, $n \in S \cup (S + 1)$ and then we just need to make sure that that $S - \{n\} \cup S - \{n\} + 1 = \{1, 2, ...n-1\}$, which is just S_{n-1}. In the latter case, $n - 1 \in S$ otherwise we don't have the element n in $S \cup S + 1$. Then we just need $S - \{n-1, n\} \cup S - \{n-1, n\} + 1 = \{1, 2, ...n-2\}$, which is just S_{n-2}. therefore, $S_n = S_{n-1} + S_{n-2}$.

6.4.6 Let $P(n)$ denote the set of subsets of $\{1, 2, ...n\}$ that contain no two consecutive elements of $\{1, 2, ...n\}$. These can be partitioned into two categories: those that contain the element n, and those that not. In the former case, the subset cannot contain $n-1$, and thereafter all subsets are of the form $\{S|S \subset P(n-2)\} \cup |n|$. In the latter case, the subsets are just the elements of $P(n-1)$. Therefore, $P(n) = P(n-2) + P(n-1)$.

6.4.7

6.4.8 Clearly all x_k are of the form $a^{d_k}b^{e_k}c^{f_k}$. Let's examine the exponents of a, b, c. Since $x_{n+1} = cx_nx_{n-1}$, $d_{n+1} = d_n + d_{n-1}$. Similarly, $e_{n+1} = e_n + e_{n-1}$ and $f_{n+1} = f_n + f_{n-1} + 1$.

6.4.9 Let C_n denote the number of legal arrangements for $2n$ parentheses. The first partheses must be a (. Consider the complement) to this parentheses.

$$(A)B$$

A must be a legal arrangement of parentheses of length $0, 2, 4, ...2(n-1)$ and B must be a legal arrangement of parentheses of length $2(n-1), 2(n-2), ...0$. If A has $2k$ parentheses, then B has $2n-2k-2$ and there are $C_k * C_{n-k-1}$ ways of choosing these. Therefore,

$$C_n = C_0C_{n-1} + C_1C_{n-2} + ..C_{n-1}C_0 = \sum_{u+v=n-1} C_uC_v$$

And this is the Catalan recurrence.

6.4.10

6.4.11 We claim that a set A is has no selfish proper subsets iff all its elements are greater than or equal to $|A|$. If this is the case, then all the proper subsets have cardinality less than $|A|$ and least element greater or equal to $|A|$, so they cannot be selfish. For the converse, suppose that A has no selfish subsets with at least one element k less than $|A|$. But then we can take k with $k-1$ other elements, since we have at least $k+1$ elements, and this subset is selfish.

Let S_n be the set of minimal selfish subsets of $\{1, 2, ...n\}$. The elements of S_n can be classified under two categories: those containing n and those not containing n. In the former case, all elements are minimal selfish subsets of $\{1, 2, ...n-1\}$ and therefore are precisely the elements of S_{n-1}. In the latter case, we perform the following operation on an set $x \in S_n$: remove n and reduce all remaining terms by 1. We will prove that this is a bijective mapping from to S_{n-2}. Since we have reduced $|x|$ by 1, if x is minimal selfish then reducing the minimal term by 1 will maintain this property. For the converse, take an element y of S_{n-2}, increase all terms by 1, and add n. Since we have increased the minimal term by 1, if y is minimal selfish then adding n maintains this property. Finally, the largest possible term is n. Therefore, $S_n = S_{n-1} + S_{n-2}$. Since $S_2 = 1$ and $S_3 = 2$, $S_n = f_n$.

6.4.12 It will be sufficient to show that the sum of the rows satisfy the fibonacci relation $f_n = f_{n-1} + f_{n-2}$. Fill in all the empty terms of each row with 0's. Then each term is the sum of the two terms above it. The key insight is that the above term to the left is in dotted line $n-2$ and the above term to the right is in dotted line $n-1$. Therefore, each term of line n is the sum of a term in line $n-1$ and a term in line $n-2$, so the sum satisfies the fibonacci relation.

6.4.13

6.4.14 We will show that this problem is equivalent to 6.4.9, and therefore w_n is the nth Catalan number. We view each customer as either a (or) parentheses. A 5 dollar bill corresponds to (and a 10 dollar bill corresponds to). Since each 5 dollar bill adds one 5 dollar bill to the cashier and each 10 dollar bill takes one away, we need to make sure that there are always at least as many people who have paid with 5 dollar bills as 10 dollar bills at any times. This is equivalent to showing that there are at least as many (as) at any time. At the end, we know that there are no 5 dollar bills, so the total number of (is equal to the total number of). Therefore, there is a bijection between the payment of the $2n$ customers and legal sequences of $2n$ parentheses.

6.4.15 The n object must be in one of the other $n-1$ places, and there are $n-1$ ways of choosing which one. Suppose that n is in place k. There are two possibilities: k is in place n or not. In the former case, there are D_{n-2} ways of arranging the remaining terms. In the latter case, k takes on the role of n in that it cannot be in the nth place. All other aspects of the sequence are the same. Therefore, there are D_{n-1} ways of arranging the remaining $n-1$ objects. Therefore,

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

We will use this to show the formula

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^n \frac{1}{n!}\right)$$

by induction. The base case $n = 1$ yields $D_1 = 0$, which is true. Also, $n = 2$ yields $D_2 = 1$, which is also true. Now suppose the formula holds for all numbers less than n. Then

$$\begin{aligned}
D_n &= (n-1)(D_{n-1} + D_{n-2}) \\
&= (n-1)\left[(n-1)!\left(1 - \frac{1}{1!} + \dots + (-1)^{n-1}\frac{1}{(n-1)!}\right) + (n-2)!\left(1 - \frac{1}{1!} + \dots + (-1)^{n-2}\frac{1}{(n-2)!}\right)\right] \\
&= (n-1)\left[(n-1)!\left(1 - \frac{1}{1!} + \dots + (-1)^{n-1}\frac{1}{(n-1)!}\right)\right] + \left[(n-1)!\left(1 - \frac{1}{1!} + \dots + (-1)^{n-2}\frac{1}{(n-2)!}\right)\right] \\
&= (n-1)!\left[(n-1)\left(1 - \frac{1}{1!} + \dots + (-1)^{n-1}\frac{1}{(n-1)!}\right) + \left(1 - \frac{1}{1!} + \dots + (-1)^{n-2}\frac{1}{(n-2)!}\right)\right] \\
&= (n-1)!\left[n\left(1 - \frac{1}{1!} + \dots + (-1)^{n-1}\frac{1}{(n-1)!}\right) - (-1)^{n-1}\frac{1}{(n-1)!}\right] \\
&= n!\left(1 - \frac{1}{1!} + \dots + (-1)^{n-1}\frac{1}{(n-1)!}\right) + (-1)^n \\
&= n!\left(1 - \frac{1}{1!} + \dots + (-1)^{n-1}\frac{1}{(n-1)!} + (-1)^n\frac{1}{n!}\right)
\end{aligned}$$

6.4.16 (a) We establish a bijection between the partitions of $2n+1$ and the partitions of $2n$. For any partition of $2n$, we simply add one and we have a partition of $2n+1$. In the opposite direction, every partition of $2n+1$ must have a 1, otherwise it is the sum of several even numbers and cannot be odd. Then we can simply remove the 1 to arrive at a partition of $2n$.

(b) Consider the partitions of $2n$. They can be placed in two categories: those containing at leaset one 1, and those not. The first category has a bijection with the partitions of $B(2n-1)$, since we can add or remove a 1 to go in either direction. The second category has a bijection with the partitions of $B(n)$, since we can divide or multiply by 2 to go in either direction.

(c) We proceed by induction. For the base case, $B(2) = 2, B(3) = 2$. It will be sufficient to prove that $B(n)$ is even for all even $2n \geq 2$, since $2n+1 = 2n$. Suppose that the property holds for all $k, 2 \leq k \leq 2n-1$. From (b), $B(2n) = B(2n-1)+B(n)$. By hypothesis, $B(2n-1)$ and $B(n)$ are even as long as $2n-1 \geq 2, 2n \geq 4$ which is true for all $n \geq 2$.

6.4.17 (a) There is only one partition of a set into one set, which is the set itself.

(b) For each element, we can choose to place it in subset A or subset B, resulting in 2^n ways. Each arrangement is double counted, since this takes the order of A and B into consideration. We then subtract 1 for the empty set. Therefore, there are $2^{n-1} - 1$ ways.

(c) Each subset must contain at least one element. Since there are $n-1$ subsets, exactly one must contain two elements, and there are $\binom{n}{2}$ ways to choose the elements that will be put together.

6.4.18 Either $n+1$ is in a partition by itself or it is not. In the former case, there are $\left\{ \begin{matrix} n \\ k-1 \end{matrix} \right\}$ ways to partition the remaining n elements. In the latter case, there are k subsets to place $n+1$ in, and they are precisely the k-partitions of n.

6.4.19 Arrange the n kids in a line. There are $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ ways to partition the n kids into k subsets according to the flavors that they will be receiving and $k!$ ways of assigning the flavors.

6.4.20 Since we have counted the same thing in two ways,

$$k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{r=0}^{k} (-1)^r \binom{k}{r} (k-r)^n$$

Dividing by $k!$,

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{r=0}^{k} (-1)^r \binom{k}{r} (k-r)^n$$

6.4.22 (a) Each of the n kids has k options, so there are k^n ways.

(b) There are $\left\{ \begin{matrix} n \\ r \end{matrix} \right\}$ ways of partitioning the n kids into r subsets according to the flavors they will be receiving and $P(k,r)$ ways of assigning these flavors.

6.4.23 As shown in the previous problem,

$$x^n = \sum_{r=1}^{x} |A_r| = \sum_{r=1}^{x} \left\{ \begin{matrix} n \\ r \end{matrix} \right\} P(x,r).$$

Also, if $x \geq r$, then $x^{\underline{r}} = P(x,r)$. Define $P(x) = x^n$ and $Q(x) = \sum_{r=1}^{n} \left\{ \begin{matrix} n \\ r \end{matrix} \right\} x^{\underline{r}}$. These are both polynomials in x. Since they are equal at all $x \geq r$ and have finite degree n, they are equal for all values.

Chapter 7: Number Theory

7.1: Primes and Divisibility

7.1.1 We proceed by strong induction to prove that all $n \geq 2$ can be factored completely into primes. For the base case, $2 = 2$. For the inductive hypothesis, suppose that all $k, 2 \leq k \leq n-1$ can be factored completely into primes. Now consider n. It can either be factored into the product of two numbers besides 1 and itself or not. If not, then it is prime by definition. Otherwise, it can be factored into two numbers $\leq n$, which by hypothesis can both be factored completely, completing the induction.

7.1.4 (b) Consider $b = 14, a = 4$. In Z, $14 = 3(4) + 2$. But we don't have 3 in E, so there do not exist such integers q, r.

7.1.9 Select the least n that can be prime factorized in two different ways:

$$p_1^{e_1} p_2^{e_2} ... p_n^{e_n} = q_1^{f_1} q_2^{f_2} ... q_m^{f_m}$$

Note that this assumption ensures that no prime appears on both sides of the equation. Since $p_1 | n = q_1^{f_1} q_2^{f_2} ... q_m^{f_m}$, $p_1 | q_1^{f_1}$ or $p_1 | q_2^{f_2} q_3^{f_3} ... q_m^{f_m}$. Since $p_1 | q_1^{f_1}$ implies that $p_1 | q_1$, which is impossible, the latter must be true. If we continue in this way, we either have $p_1 | q_i$, which is impossible, or $p_1 | 1$, also impossible.

7.1.10 Let $a := n^3 + 2n$ and $b := n^4 + 3n^2 + 1$. If we can find a linear combination $ax + by = 1$, then we are done. Our first goal is to eliminate the n^4 term:

$$b - na = n^4 + 3n^2 + 1 - n^4 - 2n^2 = n^2 + 1$$

Now we multiply by n and subtract from a to eliminate the n^3 term:

$$\begin{aligned} a - n(b - na) &= n^3 + 2n - n^3 - n \\ (n^2 + 1)a - nb &= n \end{aligned}$$

And now we can multiply as necessarily to create the polynomial $n^4 + 3n^2$:

$$\begin{aligned} b - n^3((n^2+1)a - nb) + 3n((n^2+1)a - nb) &= n^4 + 3n^2 + 1 - n^4 - 3n^2 \\ b - (n^5 + n^3)a + n^4 b + (3n^3 + 3n)a - 3n^2 b &= 1 \\ (-n^5 + 2n^3 + 3n)a + (n^4 - 3n^2 + 1)b &= 1 \end{aligned}$$

7.1.12 It will suffice to show that the algorithm produces a linear combination of a and b which divides both a and b. We have a chain of steps:

$$\begin{aligned} b &= q_1 a + r_1 \\ a &= q_2 r_1 + r_2 \\ r_1 &= q_3 r_2 + r_3 \\ &... \\ r_{n-3} &= q_{n-1} r_{n-2} + r_{n-1} \\ r_{n-2} &= q_n r_{n-1} \end{aligned}$$

Now r_{n-2} is a multiple of r_{n-1}, so $r_{n-1}|r_{n-2}$. Then $r_{n-1}|r_{n-3}$, and in general $r_{n-1}|r_i, i \leq n-1$. Therefore, $r_{n-1}|a$ and $r_{n-1}|b$. Also, reversing these equations,

$$\begin{aligned} r_{n-1} &= r_{n-3} - q_{n-1}r_{n-2} \\ &= r_{n-3} - q_{n-1}(r_{n-4} - q_{n-2}r_{n-3}) \\ &= r_{n-3}(1 + q_{n-1}q_{-2}) - r_{n-4}(q_{n-1}) \end{aligned}$$

continuing in this way, we eventually express r_{n-1} as a linear combination of a and b.

7.1.13 (a) Let's examine the coefficient of t. It is added $17 * 11$ times in $17(2 + 11t)$ and subtracted $11 * 17$ times in $11(-3 - 17t)$.

(b) Suppose there exists a solution not of the form $(2 + 11t, -3 - 17t)$. Let this solution be $(2 + k, -3 - j)$. Then $17(2 + k) + 11(-3 - j) = 1$, so $17k - 11j = 0$. Then $17k = 11j$, with $k, j \in Z$. This implies that $17|j$ and $11|k$, so $j = 17j'$ and $k = 11k'$. Substituting, we have $j' = k'$, but this means that $j = 17t$ and $k = 11t$, contradicting our assumption.

(c)

$$\begin{aligned} 1 &= 7 - 2 * 3 \\ &= 7 - 2(17 - 2 * 7) \\ &= 5 * 7 - 2 * 17 \\ &= 5(24 - 1 * 17) - 2 * 17 \\ &= 5 * 24 - 7 * 17 \\ &= 5 * 24 - 7(89 - 3 * 24) \\ &= 26 * 24 - 7 * 89 \end{aligned}$$

(d) Suppose there exists a solution not of the form $(2u + 11t, -3u - 17t)$. We can write this solution as $(2u + k, -3u - j)$. Then $17(2u + k) - 11(3u + j) = 0$, so $17k - 11j = 0$, where $u, j \in Z$. Now we proceed similarly.

7.1.14 (a) Let $\alpha = a_1 + b_1\sqrt{-6}$ and $\beta = a_2 + b_2\sqrt{-6}$. Then

$$N(\alpha)N(\beta) = (a_1^2 + 6b_1^2)(a_2^2 + 6b_2^2) = (a_1a_2)^2 + 6\left((a_1b_2)^2 + (a_2b_1)^2\right) + 36(b_1b_2)^2$$

and

$$\begin{aligned} N(\alpha\beta) &= N(a_1 + b_1\sqrt{-6})(a_2 + b_2\sqrt{-6}) \\ &= N(a_1a_2 + (a_1b_2 + b_1a_2)\sqrt{-6} - 6b_1b_2) \\ &= (a_1a_2)^2 - 12a_1a_2b_1b_2 + 36(b_1b_2)^2 + 6((a_1b_2)^2 + (b_1a_2)^2 + 2a_1a_2b_1b_2) \\ &= (a_1a_2)^2 + 6((a_1b_2)^2 + (b_1a_2)^2) + 36(b_1b_2)^2 \end{aligned}$$

(c) Suppose that $\exists \alpha, \beta \in F$ such that $\alpha\beta = 2$. Then $N(\alpha\beta) = N(2) = 4 = N(\alpha)N(\beta)$. Since the norm always maps to an integer, if α and β are both integers then 4 must be the product of two perfect squares other than 1 and itself. Therefore, at least one of α, β is not an integer. WLOG let this be α. But then $N(\alpha) \geq 6$, a contradiction.

(d) Note that $7 = (1+\sqrt{-6})(1-\sqrt{-6})$ and $31 = (5+\sqrt{-6})(5-\sqrt{-6})$.

(e) $N(2-\sqrt{-6}) = 4+6 = 10$. Suppose that $\alpha\beta = 2-\sqrt{-6}$. Then $N(\alpha\beta) = 10$. Neither α nor β can be an integer, since that would imply that the normal is a perfect square. However, this implies that their normal is greater than or equal to 6, so $N(\alpha\beta) > 10$. Contradiction.

(f) $10 = 2 * 5 = (2-\sqrt{-6})(2+\sqrt{-6})$.

7.1.15 By the Euclidean Algorithm, $f_n - f_{n-1} = f_{n-2}$. Proceeding in this way, $\gcd(f_n, f_{n-1}) = \gcd(f_2, f_1) = 1$.

7.1.16 Consider $\frac{ab}{\gcd(a,b)}$. Dividing this by a yields $\frac{b}{\gcd(a,b)}$, an integer. Dividing by b yields $\frac{a}{\gcd(a,b)}$, also an integer. Therefore, $\frac{ab}{\gcd(a,b)}$ is a multiple of a, b and hence is greater than or equal to $\operatorname{lcm}(a,b)$.

Now consider $\frac{ab}{\operatorname{lcm}(a,b)}$. Dividing this into a yields $\frac{\operatorname{lcm}(a,b)}{b}$, which is an integer. Dividing this into b yields $\frac{\operatorname{lcm}(a,b)}{a}$, which is also an integer. Therefore, $\frac{ab}{\operatorname{lcm}(a,b)}$ divides both a and b and therefore is less than or equal to $\gcd(a,b)$. From the first result,

$$ab \geq \gcd(a,b)\operatorname{lcm}(a,b)$$

and from the second,

$$ab \leq \operatorname{lcm}(a,b)\gcd(a,b)$$

so they must be equal.

7.1.17 Let $a = \prod p_i^{d_i}, b = \prod p_i^{e_i}, c = \prod p_i^{f_i}$. It will be sufficient to show that the exponent of every prime p_k is equal on both sides. WLOG, let $d_k \geq e_k \geq f_k$. We will use the p-adic notation $p_q(x)$ to denote the exponent of q in the prime factorization of x. Then

$$\begin{aligned} p_{p_k}([a,b,c]^2) &= 2d_k \\ p_{p_k}([a,b]) &= d_k \\ p_{p_k}([b,c]) &= e_k \\ p_{p_k}([c,a]) &= d_k \end{aligned}$$

Therefore,

$$p_{p_k}\left(\frac{[a,b,c]^2}{[a,b][b,c][c,a]}\right) = -e^k$$

For the right side,

$$\begin{aligned} p_{p_k}((a,b,c)^2) &= 2f_k \\ p_{p_k}((a,b)) &= e_k \\ p_{p_k}((b,c)) &= f_k \\ p_{p_k}((c,a)) &= f_k \end{aligned}$$

Therefore,

$$p_{p_k}\left(\frac{(a,b,c)^2}{(a,b)(b,c)(c,a)}\right) = -e^k$$

7.1.18 Suppose that there exist two such consecutive primes whose average is a prime. Now we have two consecutive primes with a prime between them, which contradicts our assumption.

7.1.19 Consider the $n-1$ integers $n!+2, n!+3, ...n!+n$. These are divisible by $2, 3, ...n$ respectively and are therefore composite. Thus, we can create a string of composite numbers of any arbitrary length.

7.1.20 Of the integers $1, 2, ...n$, there must exist a unique element divisible by the highest power of two. Otherwise, there are at least two numbers $2^k x$ and $2^k y$, with x and y odd that are divisible by the highest power of two. However, $2^k(x+1)$ is divisible by at least 2^{k+1}, a contradiction. Now consider putting the fractions over a common denominator. We will show that the resulting fraction always has even denominator and odd numerator when put in simplest form. The terms in the numerator are of the form $n!/a$, as a ranges from 1 to n. Since we have shown that there is an a divisible by a maximal power of 2, there is a term in the numerator divisible by a minimal power of 2. We can divide by the appropriate power of 2 to get the sum of several even numbers and an odd number over an even number, which is always and odd number over an even number and hence not an integer.

7.1.21

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$

This is clearly an integer. The numerator is divisible by p and the denominator is not, or p would be the product of factors besides 1 and itself. Therefore, $\binom{p}{k}$ is always divisible by p.

7.1.22 A number has a 0 for each factor of 2 and 5. We know that $n!$ has many more factors of 2 than 5, so it is just a matter of counting the factors of 5. There is one 0 for each multiple of 5, another for each multiply of 25, etc. Therefore, $n!$ ends in $\boxed{\sum_{k=1}^{\infty}\left\lfloor\frac{n}{5^k}\right\rfloor}$ zeroes.

7.1.23

$$\binom{p^r}{k} = \frac{p^r!}{k!(p^r-k)!}$$

Let's count the factors of p in the numerator and the denominator. From the previous problem, there are

$$\left\lfloor\frac{p^r}{p}\right\rfloor + \left\lfloor\frac{p^r}{p^2}\right\rfloor + ... + \left\lfloor\frac{p^r}{p^r}\right\rfloor$$

factors of p in the numerator. This is equal to $1+p+p^2+...+p^{r-1}$.

$$\left\lfloor\frac{k}{p}\right\rfloor + \left\lfloor\frac{p^r-k}{p}\right\rfloor + \left\lfloor\frac{k}{p^2}\right\rfloor + \left\lfloor\frac{p^r-k}{p^2}\right\rfloor + ... + \left\lfloor\frac{k}{p^{r-1}}\right\rfloor + \left\lfloor\frac{p^r-k}{p^{r-1}}\right\rfloor$$

factors of p. We can ignore the terms after p^{r-1} since we know that both k and p^r-k are less than p^r. This is less than or equal to

$$\frac{k}{p} + \frac{p^r-k}{p} + \frac{k}{p^2} + \frac{p^r-k}{p^2} + ... + \frac{k}{p^{r-1}} + \frac{p^r-k}{p^{r-1}}$$

and this is equal to $p + p^2..p^{r-1}$. Therefore, there is at least one more factor of p in the numerator. Since $\binom{p^r}{k}$ is always an integer, it is always divisible by p.

7.1.24 Since $a+b|a^n+b^n$, if $n|f_k$ then $n|f_{ak}$ for all positive integers a from Binet's formula.

7.1.25 Suppose that there are only finitely many: $p_1, p_2, ...p_n$. Then take $4p_1p_2...p_n - 1$. This is of the form $4k+3$ and it is not divisible by $p_1, p_2, ...p_n$. Since no primes are of the form $4k$ or $4k+2$, it must be divisible by only primes of the form $4k+1$. However, a product of primes of the form $4k+1$ is still of the form $4k+1$, so it must have at least one prime factor of the form $4k+3$, a contradiction.

7.1.26 Suppose there are only finitely many: $p_1, p_2, ...p_n$. Take $6p_1p_2...p_n + 5$. This is not divisible by $p_1, p_2, ...p_n$. Since no primes are of the form $6k, 6k+2, 6k+3$, or $6k+4$, it must be a product of primes of the form $6k+1$. However, a product of primes of the form $6k+1$ is still of the form $6k+1$, so it must have at least one prime of the form $6k-1$, a contradiction.

7.1.27 (a) Suppose that there exists a product of two primitive polynomials that is not primitive. Let these polynomials be $P(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_0$ and $Q(x) = b_mx^m + b_{m-1}x^{m-1} + ... + b_0$. Suppose that some prime p divides $P(x)Q(x)$. Let k be the greatest integer such that a_k is not divisible by p and j be the greatest integer such that b_j is not divisible by p. The coefficient of x^{j+k} is

$$a_kb_j + a_{k+1}b_{j-1} + a_{k+2}b_{j-2} + ... + a_{k-1}b_{j+1} + a_{k-2}b_{j+2}...$$

Then all the terms except the first are divisible by p, so the sum cannot be divisible by p. Contradiction.

(b) Without loss of generality, assume that $f(x)$ is primitive. Let $f(x) = q(x)r(x)$, where $q(x)$ and $r(x)$ are polynomials with rational coefficients. Then there exist integers a and b such that $a * q(x)$ and $b * r(x)$ are primitive. Also, $ab * q(x)r(x) = abf(x)$ is primitive, so $a, b = \pm 1$. Therefore, $q(x)$ and $r(x)$ are primitive.

7.1.28 (a) If the product of two relatively prime integers is a perfect square, they must both be perfect squares. However, n and $n+1$ are not both perfect squares for any positive integer n.

(b) Let the three integers be $n-1, n, n+1$. The product of the first and third terms is $n^2 - 1$. This is relatively prime to n since $n \geq 2$. Therefore, $n^2 - 1$ and n must both be perfect squares, which is impossible.

7.1.29 We first show that $i|a_i \forall i$. We are given that

$$\gcd(a_i, a_{2i}) = \gcd(i, 2i) = i$$

so $i|a_i$. Now suppose that $i \neq a_i$ for some i. Then $a_i = ki$. We have

$$\gcd(a_i, a_{ki}) = \gcd(i, ki) = i$$

But $ki|a_{ki}$ and $ki|a_i$, which implies that $\gcd(i, ki) = ki$. We are forced to conclude that $k = 1$ and $a_i = i \forall i$.

7.1.30 Proceed by contradiction. Let $\sqrt[3]{p} = a$, $\sqrt[3]{q} = a + md$, $\sqrt[3]{r} = a + nd$. Then

$$\frac{\sqrt[3]{r} - \sqrt[3]{p}}{\sqrt[3]{q} - \sqrt[3]{p}} = \frac{n}{m}$$

Multiplying out,

$$m\sqrt[3]{r} - m\sqrt[3]{p} = n\sqrt[3]{q} - n\sqrt[3]{p}$$
$$(n-m)\sqrt[3]{p} = n\sqrt[3]{q} - m\sqrt[3]{r}$$

Now cubing,

$$\begin{aligned}(n-m)^3 p &= n^3 q - m^3 r - 3(n^2 m\sqrt[3]{q^2 r} - m^2 n\sqrt[3]{r^2 q}) \\ &= n^3 q - m^3 r - 3mn(\sqrt[3]{qr})(n\sqrt[3]{q} - m\sqrt[3]{r}) \\ &= n^3 q - m^3 r - 3mn\sqrt[3]{qr}\sqrt[3]{p} \\ &= n^3 q - m^3 r - 3mn\sqrt[3]{pqr}\end{aligned}$$

Now we have reached a contradiction since the LHS is rational but the RHS is not.

7.2: Congruence

7.2.5 We can write $ar = br + km$. If $r \perp m$, then $r|k$. Then we can divide by r, preserving the relation $a \equiv b \pmod{m}$.

7.2.6 If $a, b \equiv 1 \pmod 3$ then $a^2 + b^2 = c^2 \equiv 2 \pmod 3$. By Fermat's, $x^2 \equiv 0, 1 \pmod 3$, so this is impossible. Therefore, one of a, b must be a multiple of three.

7.2.7 The only cubic residues mod 7 are $0, 1, 6$. There is no way for two to add to the other if none are 0, so at least one must be a multiple of 7.

7.2.8 If $x \equiv 1, 2 \pmod 3$, then $x^2 \equiv 1 \pmod 3$ so $x^2 + 2$ is divisible by 3. Thus $x^2 + 2$ is divisible by 3. Therefore, $x^2 + 2$ is composite for all x not divisible by 3 except 1.

7.2.9 Let the number be

$$\underline{a}\ \ \underline{b}\ \ \underline{c}\ \ \underline{d}\ \ \underline{e}\ \ \underline{f}\ \ \underline{g}\ \ \underline{h}\ \ \underline{i}$$

e must be 5 since all numbers divisible by 5 end in 0 or 5. We know that b, d, f, and h are the only even numbers. Furthermore, $3|a+b+c$ and $3|a+b+c+d+e+f$, so $3|d+e+f$. Since $e = 5$ and d and f are both even, we know that $d+e+f$ is an odd multiply of 3. Since 9 is too small and 21 is too big, $d+e+f = 15$, so $d + f = 10$. Since cd is divisible by 4 and c is odd, we must have $d = 2$ or $d = 6$. Since fgh is divisible by 8 and $f = 4$ or 8, $fgh = 432, 472, 816, 896$. By analyzing each case, we conclude that 183654729 is the only such number.

7.3.10 (a) We prove that the sequence is monotonically decreasing until it reaches a one-digit number, at which point it will be constant. In base 10 representation, $n = a_k 10^k + a_{k-1}10^{k-1} + ... + a_0$ and $f(n) = a_k + a_{k-1} + ... + a_0$. Therefore $f(n) < n$ as long as n has more than two digits.

(b) Consider the residues mod 9: $0, 1, 2, 3, 4, 5, 6, 7, 8$. No prime greater than 3 can be $0, 3, 6 \pmod 9$ and twin primes have one residue between them. Therefore, the only possible pairs are $(2, 4), (5, 7), (8, 1)$, all which multiply to $8 \pmod 9$. Since a number is congruent to the sum of its digits mod 9, the digital sum of the product of any two twin primes is 8.

(c) Let $g(n)$ denote the number of digits of n. Then $f(n) \leq 9 * g(n)$. $g(4444^{4444}) \leq g\left((10^4)^{4444}\right) = 10^{17776}$. Therefore, $f(4444^{4444}) \leq 9 * 17776 = 159984$. Therefore $g\left(f(4444^{4444})\right) \leq 6$ and $f\left(f(4444^{4444})\right) \leq 54$. The largest digital sum of any number less than 54 is 49, which has digital sum 13. Therefore, we know that $f(f(f(n))) \leq 13$. Now note that $n \equiv f(n) \pmod 9$. Since $4444^{4444} \equiv 7^{4444} \equiv (7^3)^{1481} * 7 \equiv 7 \pmod 9$, $f(f(f(n))) \equiv 7 \pmod 9$. Therefore, $f(f(f(n))) = 7$.

7.2.11 We know that $a^{p-1} \equiv 1 \pmod p$ by Fermat's Little Theorem. Suppose that k is the order of a mod p, so k is the smallest positive integer such that $a^k \equiv 1 \pmod p$. Now suppose that k does not divide p. Then $p - 1 = qk + r$, where $0 < r < k$. Since we have $a^{qk+r} \equiv 1 \pmod p$ and $a^{qk} \equiv 1 \pmod p$, then $a^r \equiv 1 \pmod p$. But $r < k$, contradicting the minimality of k. Therefore, $k|p - q$.

7.2.12

7.2.13 a_0 must be prime or the sequence fails at the first number. Note that the nth term of the sequence is $2^n a_0 + 2^n - 1$. If $a_0 \neq 2$, then $2^{a_0-1} \equiv 1 \pmod{a_0}$, so $2^n - 1$ is divisible by a_0 and so is $2^n a_0$, so the sequence contains composite numbers. If $a_0 = 2$, then it is easy to verify that the sequence contains a composite:

$$2, 5, 11, 23, 47, 95, \ldots$$

7.2.14 For the base case, $1^p \equiv 1 \pmod p$. Now suppose that $a^p \equiv a \pmod p$. Then $(a+1)^p = a^p + \binom{p}{1}(a+1)^{p-1} + \ldots + 1^p$. All the terms except the first and last have a coefficient of the form $\binom{p}{k}, 0 < k < p$ and are therefore divisible by p. Then $(a+1)^p \equiv a^p + 1 \equiv a + 1 \pmod p$.

7.2.16 (a) Suppose we want to find solutions to

$$\begin{aligned} w &\equiv k \pmod x \\ w &\equiv j \pmod y \end{aligned}$$

and we are given that $ax + by = 1$ (note that this implies that $\gcd(x, y) = 1$). Then $by \equiv 1 \pmod x$ and $ax \equiv 1 \pmod y$. Consider $ajx + bky \pmod{xy}$. Taken mod x, any auxiliary multiple of xy will disappear. Therefore,

$$ajx + bky \equiv bky \equiv (by)k \equiv k \pmod x$$

and similarly,

$$ajk + bky \equiv ajx \equiv (ax)j \equiv j \pmod y$$

(b) A simple counterexample: suppose we want to find x such that

$$x \equiv 1 \pmod 2$$

$$x \equiv 2 \pmod 4$$

This is clearly impossible because the first equation requires that x be odd while the second requires that x be even.

(c) Suppose we have two different y, z such that

$$\begin{array}{ll} x_1 \equiv y \pmod{mn} & x_2 \equiv z \pmod{mn} \\ x_1 \equiv a \pmod{m} & x_2 \equiv a \pmod{m} \\ x_1 \equiv b \pmod{n} & x_2 \equiv b \pmod{n} \end{array}$$

Then $x_1 - x_2 \equiv 0 \pmod{m}$, so $m|(x_1 - x_2)$. Also, $x_1 - x_2 \equiv 0 \pmod{n}$, so $n|(x_1 - x_2)$. But since $\gcd(m, n) = 1$, $mn|(x_1 - x_2)$ so $x_1 - x_2 \equiv 0 \pmod{mn} \implies x_1 \equiv x_2 \pmod{mn}$, contradicting our assumption that they were different.

(d) Suppose that we want to solve the system

$$\begin{array}{rcll} x & \equiv & a_1 & \pmod{m_1} \\ & \equiv & a_2 & \pmod{m_2} \\ & \equiv & a_3 & \pmod{m_3} \\ & ... & & \\ & \equiv & a)n & \pmod{m_n} \end{array}$$

Let $M = \prod_{i=1}^{n} m_i$ and let $k_i = \frac{M}{m_i}$. By the Euclidean Algorithm, there exists integers w_i, y_i, such that

$$w_i m_i + y_i k_i = 1$$

Then $y_i k_i \equiv 1 \pmod{m_i}$. We see that if we let $x \equiv a_i y_i k_i$, then $x \equiv a_i \pmod{m_i}$. Now we let

$$x \equiv \sum_{i=1}^{n} a_i y_i k_i \pmod{M}$$

Mod m_i, all terms disappear except $a_i y_i k_i$, so we know that this satisfies all the congruences.

(e) Let p_1, p_2, p_3 be three distinct primes. Then by the Chinese Remainder theorem, $\exists x$ such that

$$\begin{array}{rcll} x & \equiv & 0 & \pmod{p_1^{1999}} \\ x & \equiv & -1 & \pmod{p_2^{1999}} \\ x & \equiv & -2 & \pmod{p_3^{1999}} \end{array}$$

And this guarantees that $x, x+1, x+2$ are three consecutive numbers, each divisible by the 1999th power of an integer.

(f) Let $p_1, p_2, ...p_{1999}$ be 1999 distinct primes. By the Chinese Remainder theorem, $\exists x$ such that

$$\begin{array}{rcll} x & \equiv & 0 & \pmod{p_1^3} \\ x & \equiv & -1 & \pmod{p_2^3} \\ x & \equiv & -3 & \pmod{p_3^3} \\ & ... & & \\ x & \equiv & -(1998) & \pmod{p_{1998}^3} \end{array}$$

Then $x, x+1, ...x+1998$ is a string of 1999 numbers, each of which is divisible by the cube of an integer.

7.2.17 Let b_n be the nth number written in base $p-1$ and read in base p. Then b_n is the sequence of integers which do not have the digit $p-1$ when expressed in base p. We first show that no arithmetic sequence of length p exists. Suppose to the contrary that such a sequence does exist; let the last digit of the first term in base p be h, and let the constant difference be k. Then the last digits of the other terms of the sequence will be $h+k, h+2k, h+3k, ...h+(p-1)k \pmod p$. Together with h, these form a complete class of residues mod p, so one of them must be $p-1$, which is impossible. We conclude that no arithmetic sequences exist.

Now we show that b_n is the least integer greater than a_{n-1} not forming an arithmetic sequence of length p with any preceeding terms. Proceed by induction. The inductive hypothesis holds trivially for $n=0$. Suppose that it holds for n. If n ends in k such that $0 \leq k < p-2$ in base $p-1$, then $b_n = b_{n-1}+1$ and is clearly the least integer greater than b_n with the desired properties. Otherwise, n ends in $p-2$ in base $p-1$. Then $b_n = b_{n-1}+2$, so we just need to show that $b_{n-1}+1$ forms an arithmetic sequence with $p-1$ preceding terms. But since n ends in $p-2$, the $p-1$ numbers $b_{n-1}, b_n-2, ...b_n-(p-2)$ are in the sequence, so $b_{n-1}+1$ forms an arithmetic sequence of length p with these terms.

7.2.18 Let $d = d_1 d_2 ... d_9$ (where the d_i are digits) and define e and f similarly. We are given that $d - 10^{9-k} d_k + 10^{9-k} e_k$ is divisible by 7. Summing over values of k from 1 to 9, we have $7 | 9d - d + e$ so $7 | 8d + e$.

Similarly, $7 | e - 10^{9-k} e_k + 10^{9-k} f_k$. Adding this to $7 | d - 10^{9-k} d_k + 10^{9-k} e_k$ gives $7 | d + e + 10^{9-k} f_k - 10^{9-k} d_k$. Since we have $7 | d + e$ this reduces to $7 | 10^{9-k}(f_k - d_k)$. Because $\gcd(7, 10) = 1$, this implies that $7 | f_k - d_k$.

7.2.19

7.3: Number Theoretic Functions

7.3.1 Plugging in $a = b = 1$, we have

$$f(1) = f(1)^2$$

Therefore, $f(1) = 1$.

7.3.4 Take $a = 6, b = 12$. Then $d(6 * 12) = 12$ but $d(6)d(12) = 24$.

7.3.5 (a)The only divisors of p^r are $1, p, p^2 ... p^r$. This sum is equal to

$$\frac{p^{r+1}-1}{p-1}$$

(b) The only divisors of pq are $1, p, q, pq$.

$$1 + p + q + pq = (p+1)(q+1)$$

7.3.6 Consider each prime p_i dividing d. Since $p_i | ab$ and $a \perp b$, $p_i | a$ or $p_i | b$. We can perform this for each prime to get u and v. In the reverse direction, $uv | ab$ by definition. Since $a \perp b$, $u'v' = uv \implies u' = u$ and $v' = v$.

7.3.7 Consider each divisor d of n. Fix v; then u ranges through the divisors of a and the sum is $v(\sigma(u))$. Now as we let v range through the divisors of b, the sum is $\sigma(v)\sigma(u)$.

7.3.10 (a) Since all numbers less than p are relatively prime to it, $\phi(p) = p - 1$.

(b) All numbers except the $p^{r-1} - 1$ multiples of p are relatively prime to p^r, so $\phi(p^r) = p^r - 1 - (p^{r-1} - 1) = p^r - p^{r-1}$.

(c) All numbers except the $p - 1$ multiples of q and $q - 1$ multiples of p are relatively prime to pq, so $\phi(pq) = pq - 1 - (p - 1 + q - 1) = pq + p + q + 1 = (p - 1)(q - 1)$.

(d) There are $p^{r-1}q^s - 1$ numbers divisible by p, $q^{s-1}p^r - 1$ divisible by q, and $p^{r-1}q^{s-1} - 1$ divisible by pq. By PIE, $\phi(p^r q^s) = p^r q^s - 1 - \left[p^{r-1}q^s + q^{s-1}p^r - p^{r-1}q^{r-1} - 1\right] = p^r q^s - p^{r-1}q^s - p^{s-1}q^r + p^{r-1}q^{s-1} = p^{r-1}q^{s-1}(p-1)(q-1)$.

7.3.11 We want to show that $\mu(mn) = \mu(m)\mu(n)$. If n or m equals 1, then this is trivial. If $p^2|n$ or m, then $\mu(m)\mu(n) = 0$ and $\mu(mn) = 0$. Otherwise, $m = p_1 p_2 ... p_r$ and $n = q_1 q_2 ... q_s$, where q_i and p_i are distinct primes. Then $\mu(m)\mu(n) = -1^{r+s}$. Also, $\mu(mn) = \mu(p_1 p_2 ... p_r q_1 q_2 ... q_s) = -1^{r+s}$.

7.3.12 If $n = 1$, then $_{d|n}\mu(d) = \mu(1) = 1$. Otherwise, let $n = \prod_{i=1}^{r} p_i^{e_i}$. We need only to add the nonzero factors of n, which are the factors of $\prod p_i$. This sum is

$$\binom{r}{0} - \binom{r}{1} + \binom{r}{2} + ... + (-1)^r \binom{r}{r} = 0$$

7.3.14 Suppose we have $\phi(n) = n\left(\frac{p_1 - 1}{p_1}\right)\left(\frac{p_2 - 1}{p_2}\right) ... = 14$. Clearling the denominators, we have $n'(p_1 - 1)(p_2 - 1) ... = 14$. We note that the possible values of $p_i - 1$ are $1, 2, 4, 6, 10, 12$. The only ones that divide 14 are $1, 2$. Therefore, the factor of 7 must come from n'. But this implies that one of the p_i is 7, which is impossible.

7.3.15 The opposite applies: that is, $\sigma(ab) > \sigma(a)\sigma(b)$.

7.3.16 Clearly $\phi(n) < 6$ if $n < 7$, and $\phi(7) = 6$, so the smallest n is $\boxed{7}$.

7.3.17 We have $(e_1 + 1)(e_2 + 1) ... (e_k + 1) = 10$. We want to minimize $p_1^{e_1} ... p_k^{e_k}$. Therefore, we want minimal values for e, giving $e_1 = 5, e_2 = 2$. We then pick the smallest possible primes in that order, so $n = 2^4 3 = \boxed{48}$.

7.3.18 $\mu(33) + \mu(34) + \mu(35) = 3$.

7.3.19 Multiply out to get $\sigma_r(n) = n^r\sigma_{-r}(n)$. Now consider a divisor d of n. $n^r d^{-4} = \frac{n^r}{d^r} = \left(\frac{n}{d}\right)^r$, which is in $\sigma_r(n)$. Thus, they are equal.

7.3.20 (a)

$$\begin{array}{ccccc} \omega(1)=1 & \omega(2)=1 & \omega(3)=1 & \omega(4)=1 & \omega(5)=1 \\ \omega(6)=2 & \omega(7)=1 & \omega(8)=1 & \omega(9)=1 & \omega(10)=2 \\ \omega(11)=1 & \omega(12)=2 & \omega(13)=1 & \omega(14)=2 & \omega(15)=2 \\ \omega(16)=1 & \omega(17)=1 & \omega(18)=2 & \omega(19)=1 & \omega(20)=2 \\ \omega(21)=2 & \omega(22)=2 & \omega(23)=1 & \omega(24)=2 & \omega(25)=1 \end{array}$$

(b) $\omega(17!)$ consists of all the primes less than 17, which are $2, 3, 5, 7, 11, 13, 17$, for a total of $\boxed{7}$.

(c) It can be shown by counterexamples in a that ω is not multiplicative. Instead, we have $\omega(a)+\omega(b) = \omega(ab)$ for $a \perp b$.

7.3.21(a)

$$\begin{array}{ccccc} \Omega(1)=1 & \Omega(2)=1 & \Omega(3)=1 & \Omega(4)=2 & \Omega(5)=1 \\ \Omega(6)=2 & \Omega(7)=1 & \Omega(8)=3 & \Omega(9)=2 & \Omega(10)=2 \\ \Omega(11)=1 & \Omega(12)=3) & \Omega(13)=1 & \Omega(14)=2 & \Omega(15)=2 \\ \Omega(16)=4 & \Omega(17)=5 & \Omega(18)=3 & \Omega(19)=1 & \Omega(20)=3 \\ \Omega(21)=2 & \Omega(22)=2 & \Omega(23)=1 & \Omega(24)=4 & \Omega(25)=2 \end{array}$$

(b)For example, $\Omega(2)\Omega(4) \neq \Omega(8)$.

(c) Suppose $\Omega(a) = \prod p_i^{e_i}$ and $\Omega(b) = \prod q_i^{f_i}$. Then $\Omega(a) + \Omega(b) = \sum e_i + \sum f_i$ and $\Omega(ab) = \Omega(\prod e_i \prod f_i) = \sum e_i + \sum f_i$.

7.3.22

7.3.24 In the if direction, $\phi(p) = p - 1$ and $\sigma(p) = p + 1$, so the sum is $2n$. For a composite n, it will be sufficient to show that $\sigma(n) > 2n$. We induct on the number of primes m.

$$\sigma(n) \geq (p_1+1)(p_2+1)...(p_m+1)$$

Since n is composite, there are at least two primes. Let $n = \prod p_i^{e_i}$. $\sigma(n)$ is greater than $(p_1^{e_1}+1)(p_2^{e_2}+1)...(p_m^{e_m}+1)$ and $\phi(n)$ is $\left(\prod p_i^{e_i-1}\right)(p_1-1)(p_2-1)...(p_m-1)$. Upon expansion, this is clearly greater than $2\prod p_i^{e_i}$.

7.3.25 Let $\{x | \gcd(x,n) = 1\}$. This is the set

$$\{x_1, x_2, ...x_{\phi(n)}\}$$

Consider a relatively prime to n. Then

$$\{ax_1, ax_2, ...ax_{\phi(n)}\} = \{x_1, x_2, ...x_{\phi(n)}\} \pmod{p}$$

since all the ax_i are still relatively prime to n and if $ax_1 \equiv ax_2 \pmod{n}$, then $x_1 \equiv x_2 \pmod{n}$. Multiplying all the elements of both sets, we have

$$a^{\phi(n)} \prod x_i \equiv \prod x_i \pmod{p}$$

Therefore,

$$a^{\phi(n)} \equiv 1 \pmod p$$

7.3.26

7.3.27 Write $9 = 10 - 1$. We want to find $(10-1)^{99} \pmod{100}$. Then

$$(10-1)^{99} = 10^{99} - \binom{99}{1}10^{98} + ... + \binom{99}{98}10 * (-1)^{98} + \binom{99}{99}(-1)^{99}$$

All the terms except the last two disappear mod 100. This is $99 * 10 - 1 = 989$. Therefore, the last two digits are 89.

7.3.28

7.3.29 $f(k)$ appears $g(d)$ times for each $k|(n/d)$. $k|(n/d) \longleftrightarrow d|(n/k)$. Therefore, $f(k)$ appears $g(d)$ times for each $d|(n/k)$. Therefore, it appears

$$\sum_{d|(n/k)} g(d)$$

times in total. Therefore,

$$\sum_{d|n} g(d)F\left(\frac{n}{d}\right) = \sum_{k|n} f(k) \sum_{u|(n/k)} g(u)$$

7.3.31 Since μ encodes pie,

$$p(n) = \sum_{d|n} \mu(d)26^{n/d}$$

subtracts all words of a length dividing n, since these can be replicated to form a nonprime word of length n.

Diophantine Equations

7.4.5 (a) Let $u = \prod p_i^{e_i}$ and $v = \prod q_j^{f_j}$. The only way for a number to be a perfect square is for all the exponents of its prime factorization to be even. Since $u \perp v$, they share no primes. Therefore, the prime factorization is $\prod p_i^{e_i} \prod q_j^{f_j}$. If this is a perfect square, than all the e_i and f_j are even, implying that u and v are perfect squares.

(b) We can write $u = pa$ and $v = pb$, with $a \perp b$. We are given that $uv = p^2ab = x^2$, so a and b must both be perfect squares. Thus $u = pr^2$ and $v = ps^2$.

7.4.6 $-5^4 + 5^5 = 5^4(4) = (2 * 5^2)^2$. Suppose $-5^4 + 5^5 + 5^n = x^2$. Using difference of squares, $5^n = (x+50)(x-50)$. Then $x+50$ and $x-50$ are powers of 5 that differ by 100, so $n = 3$.

$2^4 + 2^7 = 2^4(9) = (3 * 2^2)^2$. Suppose $2^4 + 2^7 + 2^n = x^2$. Then $2^n = (x+12)(x-12)$. Then $x + 12$ and $x - 12$ are powers of 2 that differ by 24, so $n = 8$.

7.4.7 WLOG, let $a \geq b \geq c$. Then $\left(1 + \frac{1}{c}\right)^3 \geq 2$, so $c < 4$. Clearly $c > 1$, so we have two possibilities: $c = 2$ and $c = 3$.

Case 1: $c = 2$. Then $\frac{3}{2}\left(1 + \frac{1}{b}\right)^2 \geq 2$, so $b < 7$. Also, $\frac{3}{2}\left(1 + \frac{1}{b}\right) < 2$, so $b > 3$. Then $a = 15, 9$, and 7 respectively.

Case 2: $c = 3$. By similar reasoning, $b = 3, 4$ and $a = 8, 5$ respectively.

7.4.8 $2^8 + 2^{11} = 2^8(9) = (3 * 2^4)^2$. Suppose $2^8 + 2^{11} + 2^n = x^2$. Then $2^n = (x + 48)(x - 48)$. Then $x + 48$ and $x - 48$ are two powers of 2 that differ by 96, so $n = 12$.

7.4.9 Clear denominators to get $xy = nx + ny$. This can be factored:

$$(x - n)(y - n) = n^2$$

Therefore, there exists a solution for each ordered pair of factors of n^2.

7.4.10 Note that $1599 \equiv 15 \pmod{14}$. Since all fourth powers are 0 or 1 (mod 17), no sum of fourteen quartics can equal 1599.

7.4.11 WLOG, let $a \geq b \geq c$. If $b, c \geq 2$ then we have $abc - 2 \geq 4a - 2 \geq 3a$, with equality iff $a = b = c = 2$. Otherwise $c = 1$, so we have $ab - 2 = a + b + 1$. Simplifying, $(a-1)(b-1) = 4$. Then we obtain the solutions $a = 3, b = 3$ and $a = 5, b = 2$.

7.4.12 We note that $(x+3)^3 = x^3 + 9x^2 + 27x + 27 > x^3 + 8x^2 - 6x + 8$. Now, $x^3 + 8x^2 - 6x + 8 - (x+2)^3 = x^3 + 8x^2 - 6x + 8 - x^3 - 6x^2 - 12x - 8 = 2x^2 - 18x$. For $x > 9$, this will be positive, so for $x > 9$, the expression is strictly between two perfect cubes and cannot be a perfect cube. Therefore, we only need to check $x = 0$ through 9, yielding the solutions $(0, 2)$, and $(9, 11)$.

7.4.13

7.4.14 WLOG, let us concentrate on primitive solutions. Clearly not both x and y can be even, otherwise z will be even. Also, x and z cannot both be even, since y is odd and we have $0 + 2 \equiv 0 \pmod{4}$. Now subtract x^2 and factor as a difference of squares:

$$2y^2 = (z + x)(z - x)$$

Since $z + x$ and $z - x$ are the same parity, and the right hand side is even, both are even. Let $z + x = 2u$ and $z - x = 2v$. This forces y to be even, so let $y = 2w$. Then

$$2w^2 = uv$$

Now we will show that u and v are relatively prime. Suppose not; then for some integer $d > 1$ $d|u$ and $d|v$, so $d|(u - v) = x$ and $d|(u + v) = z$. But then $d|y$, contradicting our assumption that the triple was relatively prime. Therefore, u and v are relatively prime. Then $u = 2r^2$ and $v = s^2$, or vice versa. Then $x = 2r^2 - s^2$ or $s^2 - 2r^2$ (whichever is positive) and $z = 2r^2 + s^2$, and $y = 2y' = 2rs$. Finally, we note that x and z are odd, so s is odd. Thus, all triples are of the form $\boxed{k(2r^2 - s^2, 2rs, s^2 - 2r^2)}$.

7.4.16

7.4.19 (c) $(3u + 4v)^2 - 2(2u + 3v)^2 = 9u^2 + 16v^2 + 24uv - 2(4u^2 + 9v^2 + 12uv) = u^2 - 2v^2 = 1$.

(d) We note that another solution is $17, 12$. Since we want a recursion of the form $(a_1u + b_1v, a_2u + b_2v)$ and we must have $(a_1u)^2 - 2(a_2u)^2 = 1$, it makes sense to include $17, 12$ as a_1, b_1 respectively. Then pick $b_2 = 17, b_1 = 24$; this will ensure that the uv terms cancel. Luckily, $24 - 2(17)^2 = 2$, so the recursion works. Therefore, if (u, v) is a solution, then so is $\boxed{(17u + 24v, 12u + 17v)}$.

7.4.20 Suppose we have $x^2 - Dy^2 = 1$. This can be factored as $(x + \sqrt{D}y)(x - \sqrt{D}y) = 1$. Certainly, if we square both factors, the equality will be preserved.

$$(x + \sqrt{D}y)^2 = x^2 + 2xy\sqrt{D} + y^2 = (x^2 + y^2) + \sqrt{D}2xy$$

$$(x - \sqrt{D}y)^2 = x^2 - 2xy\sqrt{D} + y^2 = (x^2 + y^2) - \sqrt{D}2xy$$

And multiplying these together, we get

$$(x^2 + y^2)^2 - d(2xy)^2 = 1$$

So we have found a new solution.

7.4.22 We claim that all solutions of the Pell's equation $x^2 - 2y^2 = 1$ generated by the recursion in 7.4.19 are square-free. It is clear that x^2 and y^2 are square-free, and x^2 and $2y^2$ are consecutive. It will suffice to prove that y^2 is always even. We proceed by induction. This holds for the base case $1, 0$. For the inductive step, if v is even and u is odd, then $2u + 3v$ is also even. Since this recursion generates infinitely many solutions, there are infinitely many consecutive square-full integers.

7.5: Miscellaneous Instructive Exmaples

7.5.3 Consider a positional argument. If $s(U) = U$, then $a_i = a_{i+1}$ for $i = 1$ through 7, where we take $a_8 = a_1$. But this means that $a_1 = a_2 = ... = a_7 = a_1$.

7.5.4 Similarly, we have the relation $a_i = a_{i+k}$. Then $a_i = a_{i+2k} = a_{i+3k}...$. We know that all values will appear because $k, 2k, 3k, ...7k$ form a complete set of residues mod 7.

7.5.5 We have shown that if $s^r(U) = U$ for any $0 < r < 7$, then U is boring. Otherwise, all shifts of the word are different. Therefore, all sororities have either 1 or 7 members.

7.5.6 There are 26 boring words and 26^7 total words, so $26^7 - 26$ 7-member sororities. Therefore, $26^7 - 26$ is divisible by 7.

7.5.7 The only change we need for the generalization is that $k, 2k, 3k, ...pk$ form a complete set of residues mod p.

7.5.9 (a) We induct on k. Obviously, $k = 1$ is true. Suppose $(u+p)^n - u^n$ is divisible by p. Multiplying by $u+p$, we have $(u+p)^{n+1} - u^{n+1} - up$ is divisible by p. But since up is divisible by p, $(u+p)^{n+1} - u^{n+1}$ is divisible by p.

(b) We can factor

$$(u+p)^k - u^k = (p)((u+p)^{k-1} + (u+p)^{k-2}u + ... + u^{k-1})$$

and this is clearly divisible by p.

7.5.10 By the binomial theorem,

$$(a+b)^p = a^p + \binom{p}{1}a^{p-1}b + ... + b^p$$

All the terms disappear mod p except $a^p + b^p$, so this is congruent to $a^p + b^p \pmod{p}$.

7.5.11 By the multinomial theorem,

$$a^p = \sum \frac{p!}{x_1!x_2!...x_a!}$$

If all of the a_i are less than p, then $\frac{p!}{x_1!x_2!...x_a!}$ is divisible by p, so it vanishes mod p. Otherwise, there are a ways the denominator can be p, and those are $x_1 = p, x_2 = p, ...x_a = p$. Therefore, $a^p \equiv a \pmod{p}$.

7.5.12 We prime factorize: $10^{40} = 2^{40}5^{40}$ and $20^{30} = 2^{60}5^{30}$. There are $41^2 = 1681$ factors of 10^{40} and $61(31) = 1891$ factors of 20^{30}. Factors of both 10^{40} and 20^{30} are factors of $2^{40}5^{30}$, and there are $41(31) = 1271$ of these. By PIE, there are a total of $1681 + 1891 - 1271 = \boxed{2301}$ factors of either.

7.5.13 Since $\text{lcm}(m,n)\gcd(m,n) = mn$, we have two pairs of numbers with the same sum and same product. Therefore, they are roots of the same quadratic equation, so they are equal. WLOG, $n \leq m$. Then $\gcd(m,n) = n$, implying that $n|m$.

7.5.14 There are 25 primes from 1 to 100, so this looks like a good candidate for pigeonhole. It will be sufficient to prove that no more than two primes can occur in any geometric progression. WLOG, let the first prime of the geometric progression be p and the second be q. Let the common ratio be $\frac{r}{s}$, where $\gcd(r,s) = 1$. Then

$$p\left(\frac{r}{s}\right)^k = q$$

And

$$pr^k = qs^k$$

Since $p \neq q$, $q|r$ and $p|s$. But then p will be in the denominator of all future terms of the geometric sequence and not the numerator, so no other terms will be integers.

7.5.15 First Solution: The power of a prime p dividing the numerator is $\sum \lfloor \frac{2m}{p} \rfloor + \lfloor \frac{2n}{p} \rfloor$ and the denominator is $\sum \lfloor \frac{m}{p} \rfloor + \lfloor \frac{n}{p} \rfloor + \lfloor \frac{m+n}{p} \rfloor$.