Recurrence and Summation Techniques

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1 Recurrence and Summation Techniques

1. (BUET Freshers' Math Mania) Bristy has an infinite list of numbers in her hand. The first, second and third numbers on the list are $16, 41$ and 68 respectively. The list has the property that if you were to take any four consecutive numbers a, b, c, d from it in that order, you'd have $d - a = 3(c - b)$. How many perfect squares are on that list?

2. A permutation $(a_1, a_2, a_3, \cdots, a_n)$ of the numbers $(1, 2, 3, \cdots, n)$ is called *almost-sorted* if there exists exactly one $i \in \{1, 2, 3, \cdots, n-1\}$ such that $a_i > a_{i+1}$. How many *almost-sorted* permutations of the numbers $(1, 2, 3, \cdots, n)$ are there?

3. Use a summation factor to solve the recurrence

$$\begin{aligned} T_0 &= 5; \\ 2T_n &= nT_{n-1} + 3 \cdot n!, \quad \text{for } n > 0. \end{aligned}$$

4. Use the perturbation method to find a closed form for the sum $\sum_{k=0}^{n} k2^k$.

5. Try to evaluate $\sum_{k=0}^{n} kH_k$ by the perturbation method but deduce the value of $\sum_{k=0}^{n} H_k$ instead.

6. Use the perturbation method to find a closed form for the general arithmetico-geometric series $\sum_{k=0}^{n} A_k G_k$ where $\{A_k\}$ and $\{G_k\}$ are arithmetic and geometric progressions with common difference and ratio d and r respectively.

7. A permutation $(a_1, a_2, a_3, \cdots, a_n)$ of the numbers $(1, 2, 3, \cdots, n)$ is called *somewhat-sorted* if there exists exactly two $i \in \{1, 2, 3, \cdots, n-1\}$ such that $a_i > a_{i+1}$. How many *somewhat-sorted* permutations of the numbers $(1, 2, 3, \cdots, n)$ are there?

2 Miscellaneous Problems

1. (BUET CSE Fest Math Olympiad 2019) You have a pet frog named Croak and he loves candies. For a frog however, he displays quite an extensive range of emotions. For example, he is capable of feeling happiness, sadness and even anger. There are two types of candies he likes you to feed him: green and red. However, he reacts in a variety of ways depending on the order in which you decide to feed him.

   - Whenever he's happy, feeding him any number of green candies keeps him happy. However, a single red candy makes him go sad.
   - Whenever he's sad, a red candy, strangely enough, makes him go back to being happy again. A green candy, however, sends him to an angered state.
   - Whenever he's angry, feeding him any number of red candies keeps him angry. But a green candy makes him go sad.

   You decide to feed Croak a total of 10 candies. Given that he was happy both before and after the feeding process, how many ways could the feeding process have gone? What is the answer if 10 is replaced by any arbitrary number n? You may assume that you have an endless supply of both types of candies.

2. Compute $\sum_{i=0}^{2019} k^2 \binom{2019}{i}$.

3. Show that no function $f : A \to 2^A$ is surjective if 2^A is the power set of A.

3 Things to Do

1. Go through the WOOT handout for more linear recurrence problems from different Olympiads.

2. See `https://brilliant.org/wiki/recurrence-relations-method-of-summation-factors/` for more summation factor problems.

3. See chapter 2, *Concrete Mathematics* by Donald Knuth for even more summation techniques.