High School Olympiads

Incenter - Radical Axes

Reply

Source: Hyacinthos #22579

rodinos

#1 Sep 25, 2014, 4:24 am • 1

Let A'B'C' be the cevian triangle of the incenter I.
Denote:
Ab, Ac = the orthogonal projections of B',C' on AA', resp.
(Oab), (Oac) = the circumcircles of BB'Ab, CC'Ac, resp.
Similarly (Obc), (Oba), (Oca), (Ocb)
Ra = the radical axis of (Oab), (Oac). Similarly Rb, Rc
A* = Ra /\ BC, B* = Rb /\ CA, C* = Rc /\ AB.
1. Ra, Rb, Rc are concurrent.
2. ABC, A*B*C* are perspective.
(ie AA*,BB*,CC* are concurrent)

APH

Luis González

#2 Oct 1, 2014, 7:25 am • 1

Let (O) denote the circumcircle of $\triangle ABC$ and let I_a, I_b, I_c be the excenters againts $A, B, C,$ respectively. Since $\angle I_aBB' = \angle I_aA_bB' = 90^\circ$, then $I_a \in \odot(BB'A_b) \Longrightarrow \odot(BB'A_b)$ passes through the projection U of I_a on $B'C'$. Similarly $\odot(CC'A_c)$ passes through I_a and $U \Longrightarrow r_a \equiv I_aU$ is radical axis of $\odot(BB'A_b)$ and $\odot(CC'A_c)$. But it is well-known that $I_aO \perp B'C'$, note that $B'C'$ is the radical axis of $\odot(II_bI_c)$ and (O), hence $O \in r_a$ and analogously O is on radical axes r_b and r_c similarly defined , i.e. $O \equiv r_a \cap r_b \cap r_c$.

The concurrency of AA^*, BB^*, CC^* simply follows from the Cevian Nest Theorem on the excentral $\triangle I_aI_bI_c$, but this meetpoint is specially the Schiffler point of $\triangle ABC$. See post #10 at Concurrence of 4 Euler lines [Schiffler point].

Quick Reply

High School Olympiads

Show that 2MN=BM+CN

Reply

Source: P2: Bosnia and Herzegovina TST 2002

Sayan

#1 Sep 30, 2014, 8:12 am

Triangle ABC is given in a plane. Draw the bisectors of all three of its angles. Then draw the line that connects the points where the bisectors of angles ABC and ACB meet the opposite sides of the triangle. Through the point of intersection of this line and the bisector of angle BAC, draw another line parallel to BC. Let this line intersect AB in M and AC in N. Prove that $2MN = BM + CN$.

Luis González

#2 Sep 30, 2014, 9:16 am

Internal bisectors of $\angle ABC$ and $\angle ACB$ cut CA, AB at E, F and EF cuts internal bisector of $\angle BAC$ at P. It's well-known that any point on $\overline{EF}$ verifies that the sum of its distances to CA, AB equals its distance to BC. Thus if X, Y, Z denote the distances from P to BC, CA, AB, we have $PX = PY + PZ = 2 \cdot PY$.

$[BMNC] = [PBM] + [PCN] + [PBC] = \frac{1}{2}PY \cdot (BM + NC) + \frac{1}{2}PX \cdot BC.$

But the area of trapezoid $BMNC$ is $[BMNC] = \frac{1}{2}PX \cdot (MN + BC) \Longrightarrow$

$PX \cdot (MN + BC) = PY \cdot (BM + NC) + PX \cdot BC \Longrightarrow$

$PX \cdot MN = PY \cdot (BM + CN) \Longrightarrow 2 \cdot MN = BM + CN.$

jayme

#3 May 16, 2015, 11:10 am

Dear Mathlinkers,
another approach based on

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=387538
http://www.artofproblemsolving.com/community/c6h1089575_a_point_as_incenter

Sincerely
Jean-Louis

tranquanghuy7198

#4 May 16, 2015, 2:37 pm

E, F, P are defined as in Mr.Luis's post. I is the incenter of $\triangle ABC$, $AI \cap BC = D$
$MN \cap IB, IC = K, L$
$\frac{KL}{BC} = \frac{IP}{ID} = \frac{AP}{AD} = \frac{MN}{BC}$
$\Rightarrow KL = MN$
$\Rightarrow 2MN = MN + KL = MK + NL = BM + CN$. Q.E.D

Quick Reply

High School Olympiads

Concurrent lines

Reply

Source: Imo training

Sardor

#1 Sep 29, 2014, 2:30 pm

Let ABC be a triangle with incenter I. Fix a line l tangent to the incircle of ABC (not BC, CA nor AB). Let A_0, B_0, C_0 be points on l such that $\angle AIA_0 = \angle BIB_0 = \angle CIC_0 = 90^o$. Show that AA_0, BB_0, CC_0 are concurrent.

Luis González

#2 Sep 29, 2014, 6:38 pm

Posted before (and probably elsewhere) at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=281632

For a generalization see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=455814

Sardor

#3 Sep 29, 2014, 8:10 pm

Thank you very much great Luis González.

Quick Reply

High School Olympiads

Locus when cyclic

Reply

Source: Own

Bandera

#1 Sep 29, 2014, 2:04 am

Two intersecting lines, k and l, form an angle with vertex A. Two points, B and C, are given inside this angle. We choose every pair of points, K and L, lying on k and l respectively, such that the quadrilateral $AKPL$, where P is the point of intersection of the lines BL and CK, is cyclic. If M is the midpoint of KL, prove that the locus of M is a straight line.

Luis González

#2 Sep 29, 2014, 5:55 am • 2

Let BC cut k at S and let the circumcircle of $AKPL$ cut the circumcircle ω of $\triangle PBC$ again at N. As $\angle BPC = \pi - \widehat{A} = \text{const}$, then ω is fixed. Since $\angle NBC = \angle NPC = \angle SAN \Longrightarrow N \in \odot(ASB) \Longrightarrow N$ is fixed on ω. Now $\angle NLK = \angle NAS$ and $\angle KNL = \angle KPL$ are constant $\Longrightarrow$ all $\triangle NKL$ are similar with corresponing median $NM \Longrightarrow \angle(NK, NM)$ is constant and $\frac{NK}{NM}$ is constant $\Longrightarrow$ locus of M is the line image of k under the spiral similarity with center N, rotational angle $\angle(NK, NM)$ and coefficient $\frac{NK}{NM}$.

Bandera

#3 Sep 29, 2014, 9:49 pm • 1

My solution, which provides a symmetric way for constructing the locus.

Let D and E be the points on l and k respectively such that $\angle BDA = \angle CAE$ and $\angle CEA = \angle BAD$. Then $\triangle BDA \sim \triangle CAE$. If F is the point where AC and BD meet, then $\angle BFA = \angle FDA + \angle DAF = \angle FAE + \angle DAF = \angle LAK$. Thus, $\angle BFC = \pi - \angle CPB$ and $PBFC$ is cyclic $\Longrightarrow \angle DBL = \angle ACK$. So, $\triangle LBD \sim \triangle KCA$ and $\dfrac{EK}{KA} = \dfrac{AL}{LD}$. Thus, if N is a point on ED such that $KN \parallel AD$, then $AKNL$ is a parallelogram. So, M is halfway between A and (constant) line DE.

Quick Reply

High School Olympiads

Parallel lines

Reply

Source: Iranian 3rd round Geometry exam P3 - 2014

wiseman

#1 Sep 28, 2014, 6:33 pm

Distinct points B, B', C, C' lie on an arbitrary line ℓ. A is a point not lying on ℓ. A line passing through B and parallel to AB' intersects with AC in E and a line passing through C and parallel to AC' intersects with AB in F. Let X be the intersection point of the circumcircles of $\triangle ABC$ and $\triangle AB'C'(A \neq X)$. Prove that $EF \parallel AX$.

Luis González

#2 Sep 29, 2014, 12:05 am • 1

Let $X_{\infty}, Y_{\infty}, Z_{\infty}, A_{\infty}, B_{\infty}, C_{\infty}$ denote the infinite points of BE, EF, FC, BC, CA, AB. Parallel from A to EF cuts BC at P. By Desargues involution theorem, the opposite sidelines of the complete quadrangle $BEFC$ form an involution on the line at infinity $\Longrightarrow A(X_{\infty}, Y_{\infty}, C_{\infty}) \barwedge A(Z_{\infty}, A_{\infty}, B_{\infty}) \Longrightarrow (B', P, B) \barwedge (C', A_{\infty}, C) \Longrightarrow P$ is center of this involution $\Longrightarrow PB \cdot PC = PB' \cdot PC' \Longrightarrow P$ is on radical axis AX of $\odot(ABC)$ and $\odot(AB'C') \Longrightarrow EF \parallel AXP$, as desired.

Mathematicalx

#3 Oct 30, 2014, 1:00 am • 1

Let D is intersection point of $|AX|$ and $|FC|$. Then we have $\angle DB'A = \angle FBE$ and we have $DB'/FB = AB'/EB$. So we have $AX||EF$.

Quick Reply

High School Olympiads

Nice seven circles problem

Reply

Source: Own

buratinogigle

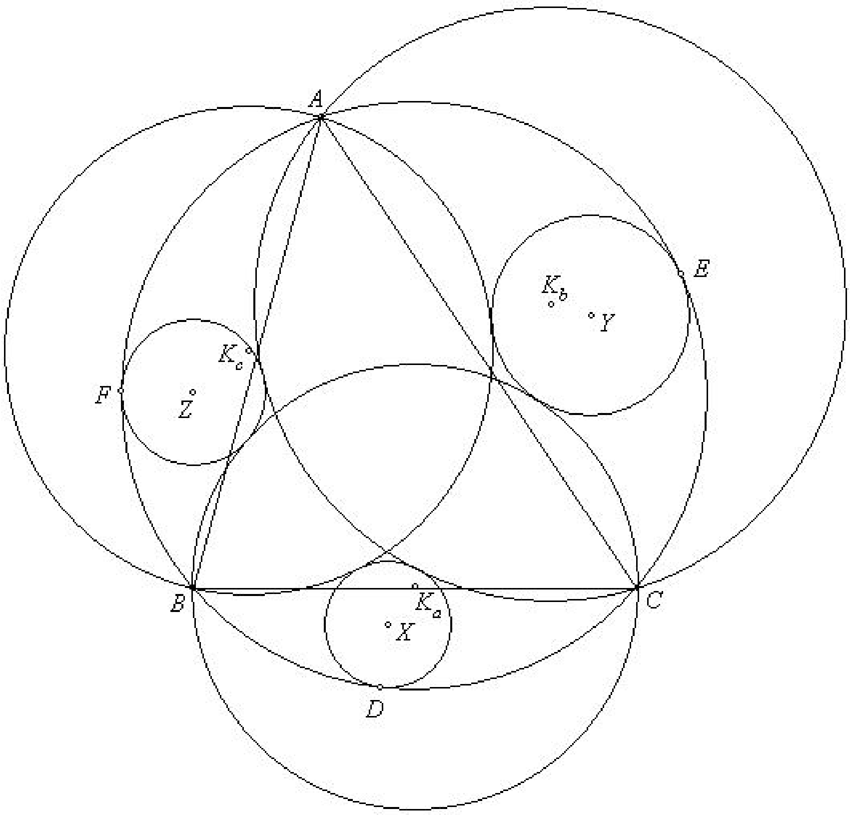
#1 Sep 26, 2014, 6:44 pm • 1

$(K_a), (K_b), (K_c)$ are three arbitrary circles passing through $B, C; C, A; A, B$.

a) Prove that AD, BE, CF are concurrent.

b) Prove that AX, BY, CZ are concurrent.

Attachments:



ThirdTimeLucky

#2 Sep 28, 2014, 3:25 am • 3

Soultion for part (a):

Lemma 1: For a cyclic convex hexagon $ABCDEF$, its diagonals AB, CD, EF are concurrent $\iff$ $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

This is well known, so no proof.

Lemma 2 : Let ABC be a triangle. Triangles BDC, CEA, AFB are constructed such that $\angle FAB = \angle EAC$, $\angle CBD = \angle ABF$ and $\angle BCD = \angle ACE$. The incircles of $\triangle BDC, \triangle CEA, \triangle AFB$ are $(I_A), (I_B), (I_C)$ respectively. (I_A) touches BC at A_1, (I_B) touches CA at B_1 and (I_C) touches AB at C_1. Then AA_1, BB_1, CC_1 are respectively. (I_A) touches BC at A_1, (I_B) touches CA at B_1 and (I_C) touches AB at C_1. Then AA_1, BB_1, CC_1 are concurrent.

Proof : Since $\angle CBD = \angle ABF$, we have $\angle I_ABC = \angle I_CBA$, so $\triangle I_CBC_1 \sim \triangle I_ABA_1 \implies \frac{BA_1}{BC_1} = \frac{I_AA_1}{I_CC_1} = \frac{r_A}{r_C}$.
Similarly, $\frac{CB_1}{CA_1} = \frac{r_B}{r_A}, \frac{AC_1}{AB_1} = \frac{r_C}{r_B}$. Multiplying,

$$\frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} \cdot \frac{AC_1}{C_1B} = 1$$

So the Converse of Ceva's theorem guarantees that AA_1, BB_1, CC_1 are concurrent.

Proof of problem : Let $(K_b) \cap (K_c) = \{A, P\}, (K_c) \cap (K_a) = \{B, Q\}, (K_a) \cap (K_b) = \{C, R\}$. Consider the transformation which is a composition of an inversion with center A and power $AB \cdot AC$ and flip over the angle bisector of $\angle A$. In this transformation, look at circles $(K_b), (K_c), (X)$. $B \mapsto C, C \mapsto B$ and suppose $P \mapsto P'$. Then $(K_b) \mapsto \overline{CP'}$, $(K_c) \mapsto \overline{BP'}$, $(O) \mapsto \overline{BC}$. Therefore, $(X) \mapsto$ the incircle of $\triangle BP'C$ which touches BC at D', the image of D. Since inversion preserves angles, we have $\angle BCP' = \angle((O), (K_C)) = \angle C^*$ and $\angle CBP' = \angle((O), (K_b)) = \angle B^*$. Now,

$$\frac{BD}{D'C} = \frac{AB}{AD'}, \frac{CD}{D'B} = \frac{AC}{AD'}$$

$$\implies \frac{BD}{DC} = \frac{AB}{AC} \cdot \frac{CD'}{D'B}$$

Repeating the same procedure, but now starting with B, we will have (Y) mapping to the incircle of $\triangle CQ'A$, with $\angle ACQ' = \angle C^*$ and $\angle CAQ' = \angle A^*$, and $\frac{CE}{EA} = \frac{BC}{BA} \cdot \frac{AE'}{E'C}$

Similarly, $\frac{AF}{FB} = \frac{CA}{CB} \cdot \frac{BF'}{F'A}$, where $\triangle AR'B$ will be defined analogously.

Multiplying, we get

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{CD'}{D'B} \cdot \frac{BF'}{F'A} \cdot \frac{AE'}{E'C} = 1$$

where the last inequality follows from Lemma 2. This implies $BD \cdot CE \cdot AF = DC \cdot EA \cdot FB$, and by Lemma 1 for the hexagon $AFBDCE$, AD, BE, CF are concurrent.

Luis González

#3 Sep 28, 2014, 11:46 am • 2

Denote by (O) the circumcircle of $\triangle ABC$ and let U, V be the tangency points of $(Y), (Z)$ with (K_a). E and F are the exsimilicenters of $(O) \sim (Y)$ and $(O) \sim (Z) \Longrightarrow EF$ goes through the exsimilicenter A_0 of $(Y) \sim (Z)$. Likewise, U and V are the insimilicenters of $(K_a) \sim (Y)$ and $(K_a) \sim (Z) \Longrightarrow UV$ passes through A_0. $A_0 \equiv EF \cap UV \cap YZ$ is also center of the inversion that swaps (Y) and (Z), hence E, F, U, V are concyclic $\Longrightarrow EF, UV, BC$ are pairwise radical axes of $(O), (K_a), \odot(EFVU)$ concurring at their radical center A_0.

By similar reasoning, exsimilicenters B_0 and C_0 of $(Z) \sim (X)$ and $(X) \sim (Y)$ lie on CA, AB respectively $\Longrightarrow \triangle ABC$ is perspective with both $\triangle DEF$ and $\triangle XYZ$ through the homothety axis $\overline{A_0B_0C_0}$ of $(X), (Y), (Z)$. Hence, by Desargues theorem AD, BE, CF concur and AX, BY, CZ concur.

Quick Reply

High School Olympiads

A point on the angle bisector

Reply

ricarlos

#1 Sep 27, 2014, 7:54 pm

Let ABC be a triangle and M, N, Q the midpoints of AB, BC and AC, respectively. Let P be a point on the interior angle bisector of $\angle BCA$.
$D = AP \cap MN$
$E = BP \cap MQ$,
prove that $MD = ME$

Luis González

#2 Sep 28, 2014, 12:00 am • 2

Let I be the incenter of $\triangle ABC$ and AI, BI cut MN, MQ at $X, Y,$ respectively. Since $\angle MXA = \angle CAX = \angle MAX,$ then $\triangle MAX$ is isosceles with $MA = MX$ and similarly $MY = MB \Longrightarrow MX = MY,$ i.e. $\triangle MXY$ is M-isosceles. Now $A(C, D, X, M) = A(C, P, I, B) = B(C, P, I, A) = B(C, E, Y, M) \Longrightarrow (\infty, D, X, M) = (\infty, E, Y, M)$ $\Longrightarrow DE \parallel XY \Longrightarrow \triangle MDE$ is M-isosceles.

ricarlos

#3 Sep 28, 2014, 6:35 am

> *Luis González wrote:*
> Now $A(C, D, X, M) = A(C, P, I, B) = B(C, P, I, A) = B(C, E, Y, M) \Longrightarrow$ $(\infty, D, X, M) = (\infty, E, Y, M) \Longrightarrow DE \parallel XY \Longrightarrow \triangle MDE$ is M-isosceles.

Quick Reply

High School Olympiads

Two isotomic points

Reply

jayme
#1 Sep 26, 2014, 6:44 pm

Dear Mathlinkers,

1. ABC a triangle
2. G the median point of ABC
3. P, Q two points so that G is the midpoint of the segment PQ
4. P*, Q* the resp. isotomics of P, Q wrt ABC.

Prove synthetically : P*Q* is parallel to PQ.

Sincerely
Jean-Louis

mihai miculita
#2 Sep 26, 2014, 11:21 pm

Using barycentric coordinate, if $P(\alpha;\beta;\gamma);\alpha+\beta+\gamma=1,$
view my attachment in pdf!!!

Attachments:
Jaime_TWO_ISOTOMIC_POINTS.doc (30kb)
Jaime_TWO_ISOTOMIC_POINTS.pdf (72kb)

Luis González
#3 Sep 27, 2014, 2:10 am • 1

Fix the line $\ell \equiv PQ$ through the centroid $G.$ As P,Q vary, their isotomic conjugates P^* and Q^* run on the isotomic circum-conic $\mathcal{C}$ of ℓ WRT $\triangle ABC$ tangent to ℓ at the fixed point G of the mapping $P \mapsto P^*.$ Since $P \barwedge P^*$ and $P \mapsto Q$ involutive, then it follows that $P^* \mapsto Q^*$ is an involution fixing $\mathcal{C} \Longrightarrow$ all P^*Q^* go through the pole of the involution.

Let ℓ cut BC,CA,AB at A_0,B_0,C_0 and let A_1 be the reflection of A_0 on $G.$ AA_1 cuts BC at M and the parallel to ℓ through A cuts BC at $N.$ If D is the midpoint of $BC,$ we have $\frac{DN}{DA_0}=\frac{DA}{DG}=3$ and by Menelaus' theorem for $\triangle GDA_0$ cut by $AA_1M,$ we get

$$\frac{MD}{MA_0}=\frac{AD}{AG}\cdot\frac{A_1G}{A_1A_0}=\frac{3}{2}\cdot\frac{1}{2}=\frac{3}{4}\Longrightarrow\frac{DM}{DA_0}=-3\Longrightarrow DM=-DN,$$

which means that AN is the isotomic of $AA_1.$ Hence when $P\equiv A_0,Q\equiv A_1,$ then P^* goes to A and Q^* goes to a point on $AN \Longrightarrow PQ \parallel P^*Q^*$ and the same happens when P coincides with B_0 or $C_0.$ Hence $PQ \parallel P^*Q^*$ holds or all positions of $P.\blacksquare$

Remark: Using the previous result and keeping in mind that $Q^* \mapsto P$ is a stereographic projection of $\mathcal{C}$ onto ℓ with double point $G,$ we show that all lines PQ^*,QP^* go through a fixed point; the 4th intersection of $\mathcal{C}$ and the Steiner circum-ellipse of ABC.

jayme
#4 Sep 27, 2014, 11:01 am

Dear all,
thank you for your interest in this old problem...
I research always a synthetic proof whithout any calculation or fonction.... I come to ask me this question/ perhaps are we on the border of the elementary geometry?

What do you think?
Thank again dear Luis and Mihai for your contribution.
Jean-Louis

mihai miculita
#5 Sep 27, 2014, 2:17 pm

Full my solution! (I'm sorry, is in Romanian). View in attachment!
EDIT: I'm sorry, in The solution was computing an error has occurred. We have corrected this error!

Attachments:
jayme_TWO_ISOTOMIC_POINTS(Sol_Miculita).doc (62kb)

jayme
#6 Sep 27, 2014, 3:29 pm

Dear Mihai and Mathlinkers,
thank but I research a synthetic one with no calculation... since a long time..
Very sincerely for your help with barycentric coordinates and your nice work
Jean-Louis

Ashutoshmaths
#7 Sep 29, 2014, 2:17 pm • 1

What are isotomic points?

jayme
#8 Sep 29, 2014, 2:35 pm

Dear Mathlinkers,
you can see

http://jl.ayme.pagesperso-orange.fr/ vol. 5 Gohierre de Longchamps dans les journaux scientifiques p. 14-17

Sincerely
Jean-Louis

utkarshgupta
#9 Oct 6, 2014, 11:23 pm

Something in English

Quick Reply

High School Olympiads

Hard and beautiful geometry

Reply

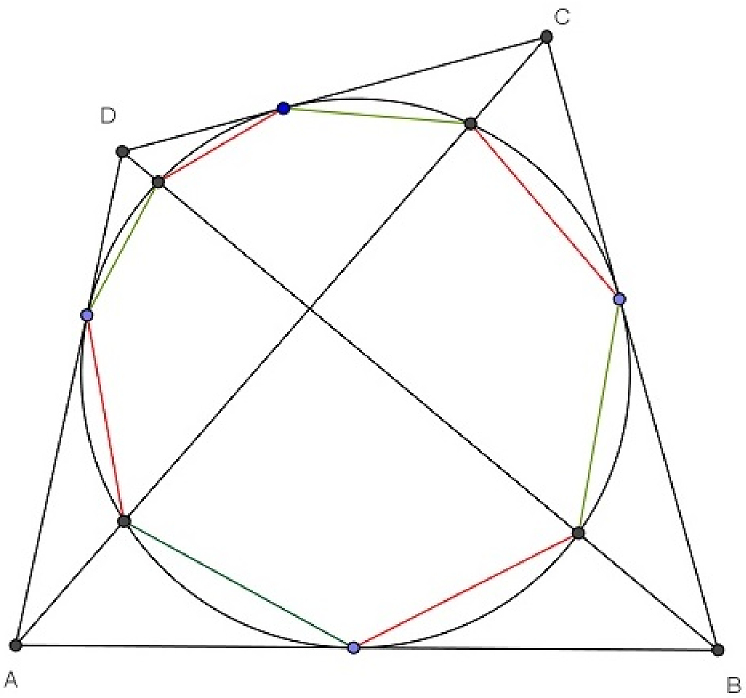
Source: (mpdb)

borislav_mirchev

#1 Sep 25, 2014, 1:17 am • 4

Let $ABCD$ is a circumsrcibed quadrilateral. l_1, l_3, l_5, l_7 are the lengths of the 4 green segments. l_2, l_4, l_6, l_8 are the lengths of the 4 red segments. Prove that $l_1l_3l_5l_7 = l_2l_4l_6l_8$.

Attachments:



Luis González

#2 Sep 26, 2014, 5:55 am • 6

Let $K \equiv AC \cap BD.$ Incircle (I,r) touches DC, CB, BA, AD at $P, Q, R, S.$ AC cuts (I) at U, V (U is between C,V) and BD cuts (I) at X, Y (X is between D,Y). Let $\mathrm{d}(J,\ell)$ denote the distance from a point J to a line $\ell.$ We use that the length of a chord in a circle is the geometric mean between the diameter and the distance from one of its endpoints to the tangent of the circle at the other:

$$PU^2 = 2r \cdot \mathrm{d}(U,DC)\ ,\ UQ^2 = 2r \cdot \mathrm{d}(U,CB) \Longrightarrow$$

$$\frac{PU}{UQ} = \sqrt{\frac{\mathrm{d}(U,DC)}{\mathrm{d}(U,CB)}} = \sqrt{\frac{\mathrm{d}(K,DC)}{\mathrm{d}(K,CB)}}.$$

Now, multiplying the cyclic expressions together gives

$$\frac{PU}{UQ} \cdot \frac{QY}{YR} \cdot \frac{RV}{VS} \cdot \frac{SX}{XP} = \sqrt{\frac{\mathrm{d}(K,DC)}{\mathrm{d}(K,CB)} \cdot \frac{\mathrm{d}(K,CB)}{\mathrm{d}(K,BA)} \cdot \frac{\mathrm{d}(K,BA)}{\mathrm{d}(K,AD)} \cdot \frac{\mathrm{d}(K,AD)}{\mathrm{d}(K,DC)}} = 1.$$

georgi111

#3 Sep 26, 2014, 8:19 pm

It is very nice solution ...

ThirdTimeLucky

#4 Sep 27, 2014, 8:04 pm • 3

Another solution:

Let AB, BC, CD, DA touch the incircle at P, Q, R, S respectively. Let AC intersect the incircle at A_1, C_1 (A_1 closer to A) and similarly define B_1, D_1. $PQ \cap RS = E$, $PS \cap QR = F$, $PR \cap QS = G$.

With respect to the incircle, the polar of D is RS and the polar of B is PQ, so the polar of $E = RS \cap PQ$ is BD. By Brokard's Theorem, the polar of E is also FG, so F, G, B, D are collinear. Similarly, E, G, A, C are also collinear.

Use similar triangles to get

$$\frac{RC_1}{SA_1} = \frac{ER}{EA_1} = \frac{EC_1}{ES}$$

$$\frac{PA_1}{QC_1} = \frac{EA_1}{EQ} = \frac{EP}{EC_1}$$

$$\implies \frac{PA_1 \cdot RC_1}{SA_1 \cdot QC_1} = \frac{ER}{EQ} = \frac{EP}{ES} = \sqrt{\frac{EP \cdot ER}{EQ \cdot ES}}$$

Similarly, we get

$$\frac{QB_1 \cdot SD_1}{PB_1 \cdot RD_1} = \sqrt{\frac{FQ \cdot FS}{FR \cdot FP}}$$

Therefore,

$$\frac{PA_1 \cdot QB_1 \cdot RC_1 \cdot SD_1}{SA_1 \cdot PB_1 \cdot QC_1 \cdot RD_1} = \sqrt{\frac{PE}{EQ} \cdot \frac{QF}{FR} \cdot \frac{RE}{ES} \cdot \frac{SF}{FP}}$$

The right hand side of the above equals one, by Menelau's Theorem for the quadrilateral $PQRS$ and line EF and so $l_1 l_3 l_5 l_7 = l_2 l_4 l_6 l_8$ as required.

buratinogigle

#5 Sep 28, 2014, 8:10 am • 1

Actually, this problem is true for circumscribed 2n-gon and the nice solutions of Luis and ThirdTimeLucky is available in this case, also.

borislav_mirchev

#6 Sep 28, 2014, 1:28 pm • 2

This problem have at least three more solutions, I think some of them involve lots of calculations and computer may be needed to finish them. Thanks to @buratinogigle I got the following idea for generalizations - similar statements can be true for tangential 2n-gon with intersecting "main" diagonals and circle - concentric to its incircle. I created the problem on my own. Is it a well known statement?

buratinogigle

#7 Sep 28, 2014, 2:35 pm • 1

Thank you very much dear Mr Borislav for the nice problem. One of the most beautiful problem that I know.

TelvCohl

#8 Oct 9, 2014, 3:36 pm • 3

My solution:

Let (O) be the incircle of ABCD.
Let R be the intersection of AC and BD.
Let W', Y' be the intersection of BD and TS, PQ, respectively.
Let X', Z' be the intersection of AC and SP, QT, respectively.
Let P, Q, T, S be the intersection of (O) and AB, BC, CD, DA, respectively.
Let W, X, Y, Z be the intersection of (O) and RD, RA, RB, RC, respectively.

By Newton theorem
we get P R T are collinear and Q R S are collinear.
It's easy to see that RTW'S~RQY'P and RSX'P~RTZ'Q,
so we get (TW'/W'S)*(PY'/Y'Q)=1 and (SX'/X'P)*(QZ'/Z'T)=1

Since WSYT is a harmonic quadrilateral,
so WY is W-symmedian line of triangle WTS,
hence we get TW/WS=sqrt(TW'/W'S).
Similarly we can get SX/XP=sqrt(SX'/X'P), PY/YQ=sqrt(PY'/Y'Q), QZ/ZT=sqrt(QZ'/Z'T).

so (TW/WS)(SX/XP)(PY/YQ)(QZ/ZT)
= sqrt((TW'/W'S)(SX'/X'P)(PY'/Y'Q)(QZ'/Z'T))=1
ie. TW*SX*PY*QZ=WS*XP*YQ*ZT

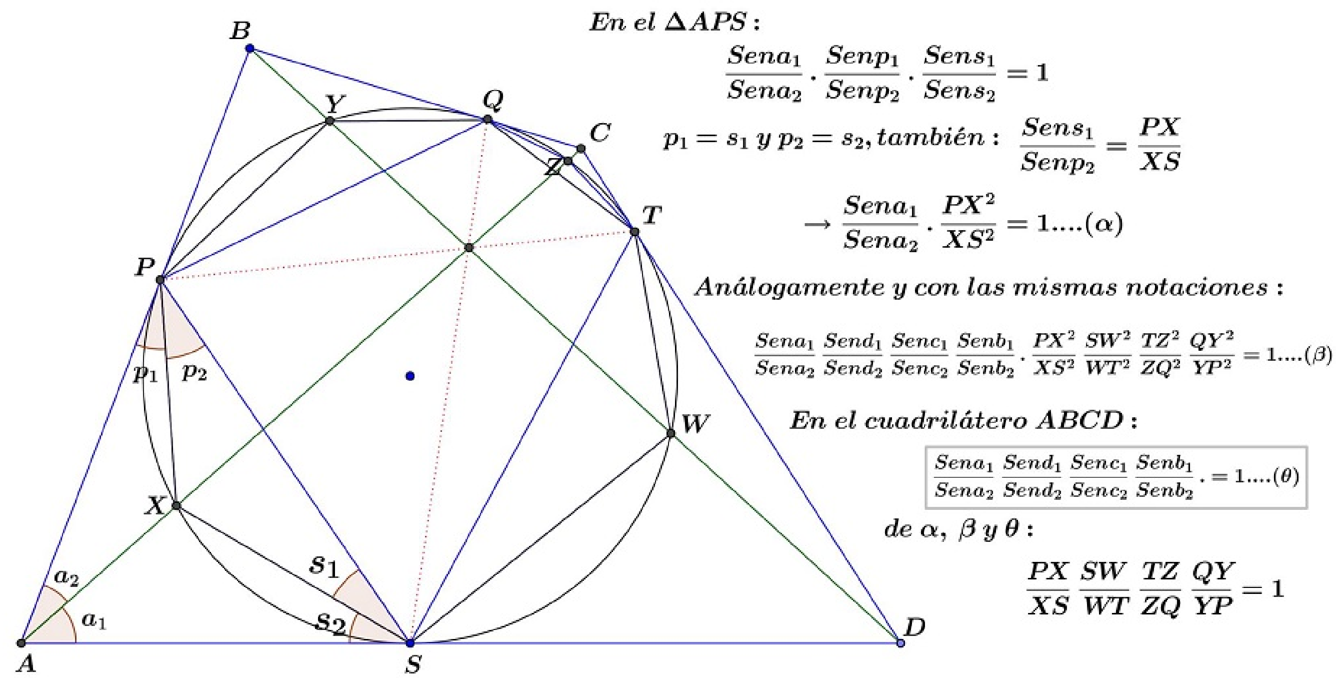
Q.E.D

borislav_mirchev

#9 Oct 11, 2014, 1:23 am • 1

A cool solution - Julio Orihuela from Peru Geometrico group. This problem have even more solutions.

Attachments:



borislav_mirchev

#10 Oct 15, 2014, 2:34 am • 1

http://www.artofproblemsolving.com/blog/101479 there is one more interesting solution in Mr. Virgil Nicula's blog - PP15.

Sardor

#11 Nov 29, 2014, 1:31 am • 1

Here my solution(I think , it's easy and nice):

Let AB, BC, CD, DA touch the incircle at X, Y, Z, T respectively. Let AC intersect the incircle at A_1, C_1 (A_1 closer to A) and similarly define B_1, D_1.Let $XY \cap BB_1 = B_2$.We know that B_1B_2 is symmedian of the triangle XB_1Y, then by Steiner's theorem we have $(\frac{XB_1}{YB_1})^2 = \frac{XB_2}{YB_2} = \frac{XB}{YB} \cdot \frac{sin\angle ABD}{sin\angle CBD} = \frac{sin\angle ABD}{sin\angle CBD}$ and ...

Hence we must prove that $\frac{sin\angle ABD}{sin\angle CBD} \cdot \frac{sin\angle BCD}{sin\angle ACD} \cdot \frac{sin\angle CDB}{sin\angle BDA} \cdot \frac{sin\angle DAC}{sin\angle CAB} = 1$, but it's very easy :

Let $AC \cap BD = P$, then we have $\frac{sin\angle ABD}{sin\angle CBD} = \frac{AP}{CP} \cdot \frac{BC}{AB}$ and etc.

So we are done !

(My solution works for tangential 2n-gon)

borislav_mirchev

#12 Dec 6, 2014, 11:31 pm • 1

In case if it is a new result and have no its own name - you can call it Adi's theorem. (Adi is the short name of the girl I love.)

Quick Reply

High School Olympiads

Source: Iranian 3rd round Geometry exam P1

AHZOLFAGHARI

#1 Sep 25, 2014, 10:16 pm

In the circumcircle of triange $\triangle ABC$, AA' is a diameter.
We draw lines l' and l from A' parallel with Internal and external bisector of the vertex A.
l' Cut out AB, BC at B_1 and B_2.
l Cut out AC, BC at C_1 and C_2.
Prove that the circumcircles of $\triangle ABC$ $\triangle CC_1C_2$ and $\triangle BB_1B_2$ have a common point.

(20 points)

Luis González

#2 Sep 26, 2014, 2:51 am

Let M and N be the midpoints of the arcs BC and BAC of $\odot(ABC)$. MB_2 cuts $\odot(ABC)$ again at P.
$\angle BB_1B_2 = \angle BAM = \angle BPM \Longrightarrow P \in \odot(BB_1B_2)$. But since
$A'(B, C, N, M) = A'(B, C, B_2, C_2) = A(B, C, N, M) = -1$, it follows that PB_2, PC_2 bisect $\angle BPC$ internally and externally, i.e. N, P, C_2 are collinear $\Longrightarrow \angle CPC_2 = \angle CAN = \angle CC_1M \Longrightarrow P \in \odot(CC_1C_2)$. Hence $\odot(BB_1B_2)$, $\odot(CC_1C_2)$ and $\odot(ABC)$ concur at P, as desired.

AHZOLFAGHARI

#3 Sep 28, 2014, 1:41 am

Here is my solution !
M is the midpoint of the arc BC.
MB_2 cuts the circle at P.
$\angle BB_1B_2 = \angle BAM = \angle BPM$
so P is on the circle of triangle BB_1B_2.
so we must prove that P is on the circle of triangle CC_1C_2.
$\angle AC_1M = \frac{180 - CM}{2} = 90 - \angle\frac{A}{2}$
so $C_1C_2C = \angle C - 90 + \angle\frac{A}{2} = \frac{C - B}{2}$
$\angle B_2PA' = \frac{C - B}{2}$
so P, C_2, A', B_2 are on a circle . so $\angle B_2PC_2 = 90$
$\angle CPC_2 = 90 + \angle\frac{A}{2}$
so $\angle CC_1C_2 + \angle CPC_2 = 180$
so P is on the circle triangle CC_1C_2
so three circle have a common point .)P

jayme

#4 Oct 22, 2014, 9:23 pm

Dear Mathlinkers,

1. U, V the midpoints of the arcs BC which contain A, not A
2. P, Q second points of intersection of (ABC) wrt (CC1C2), (BB1B2)
3. according to the Reim's theorem, U, P and C2 collinear, V, Q, B2 collinear
4. B2 orthocenter of UVC2

and we are done.

Sincerely
Jean-Louis

High School Olympiads

Prove that 4 points lie on a circumference

Reply

Source: Iberoamerican Olympiad 2014, Problem 5

Leicich

#1 Sep 25, 2014, 5:56 am

Let ABC be an acute triangle and H its orthocenter. Let D be the intersection of the altitude from A to BC. Let M and N be the midpoints of BH and CH, respectively. Let the lines DM and DN intersect AB and AC at points X and Y respectively. If P is the intersection of XY with BH and Q the intersection of XY with CH, show that H, P, D, Q lie on a circumference.

Luis González

#2 Sep 25, 2014, 8:55 am • 1

If L is the reflection of H on D, then DM is H-midline of $\triangle BHL \Longrightarrow DX \parallel LB$ and similarly $DY \parallel LC \Longrightarrow \triangle DXY$ and $\triangle LBC$ are homothetic with center $A \Longrightarrow XY \parallel BC$. Since $\triangle MBD$ is obviously M-isosceles, then it follows that $DBXP$ is an isosceles trapezoid $\Longrightarrow \angle PDB = \angle ABC$ and similarly $\angle QDC = \angle ACB$. Thus, $\angle PDQ = 180^\circ - \angle ABC - \angle ACB = \angle BAC = 180^\circ - \angle PHQ \Longrightarrow PHQD$ is cyclic.

IDMasterz

#3 Sep 25, 2014, 9:59 am

To prove $XY \parallel BC$, let $XY \cap AH = L$. Consider $X(D,H;L,A) \cap BH = Y(D,H;L,A) \cap CH \implies XY \parallel BC$. Now, $MB = MD$ so $PXBD$ is an isosceles trapezoid, hence $DP \cap AB = M'$, the midpoint of AB. So, $\angle MDY = 90 = \angle NDX \implies 180 - \angle PDQ = \angle MDN = \angle PHQ$.

juckter

#4 Sep 26, 2014, 3:06 am

Applying Menelaus' theorem twice:

$$\frac{AX}{XB} \cdot \frac{BM}{MH} \cdot \frac{HD}{DA} = 1 = \frac{AY}{YC} \cdot \frac{CN}{NH} \cdot \frac{HD}{DA}$$

From M and N being the midpoints of BH and CH, it follows that $\frac{AX}{XB} = \frac{AY}{YC}$, hence $XY \parallel BC$. The solution then easily follows.

This post has been edited 1 time. Last edited by juckter, Sep 27, 2014, 12:20 am

Vo Duc Dien

#5 Sep 26, 2014, 6:55 am

$XY \parallel BC$ because $AXDY$ is cyclic and $PQDH$ is cyclic because both $BDPX$ and $CDQY$ are isosceles trapezoids.

sayantanchakraborty

#6 Oct 4, 2014, 11:00 pm

It is a very easy problem.

Note that P being the midpoint of hypotenuse BH is the circumcenter of $\odot BDH$,so $\angle BMD = 2\angle C$.Also note that
$\angle ABH = 90 - \angle A$.So $\angle BXD = 2\angle C + \angle A - 90 = 90 + \angle C - \angle B$
Similar angle chasing will give $\angle DYA = 90 + \angle C - \angle B$.Thus $DXAY$ is cyclic,so
$\angle QYD = \angle XYD = \angle XAD = \angle BAD = 90 - B = \angle HCD = \angle QCD \implies QYCD$ is cyclic.
So $\angle PQD = \angle YCD = \angle C$.Also note that $\angle PHD = \angle BHD = \angle C$.Thus $PHQD$ is a cyclic quadrilateral as desired.

MillenniumFalcon

#7 Oct 6, 2014, 12:47 pm

@sayantanchakraborty: I think you meant M is the midpoint of BH.
Also, DXAY cyclic can be shown easily by:
Angle XDY= 180 - XDB - YDC = 180 - (90-C) - (90-B) = 180 - A = 180 - XAY.

utkarshgupta

#8 Oct 6, 2014, 2:23 pm

Lemma : $\angle ADY = \angle B$

$\angle DHB = \angle B$
Now since N is the midpoint of hypotenuse of right triangle HDC,
$\implies NH = ND$
$\implies$ Our lemma

Similarly $\angle XDA = \angle C$

$\implies AXDY$ is concyclic

$\implies \angle XYA = \angle C$

$\implies XY || BC$
$PQ||BC$

Now it is easy.

utkarshgupta

#9 Oct 6, 2014, 2:53 pm

> *MillenniumFalcon wrote:*
> @sayantanchakraborty: I think you meant M is the midpoint of BH.

Quite obviously as it is given in the question

bcp123

#10 Oct 6, 2014, 4:50 pm

$\triangle XMB$ and $\triangle YNC$ are perspective by line AD so XY,MN,BC concur $\Rightarrow XY \parallel BC$. $BM = MD$ and
$DN = NC$ so $XPDB$ and $YQDC$ are isosceles trapezoids. $\angle DPQ = \angle B$ and
$\angle PQD = \angle C \Rightarrow \angle PDQ = \angle A = \pi - \angle BHC$ hence the result.

Quick Reply

High School Olympiads

Beautiful property regarding the Euler line

Reply

hqdhftw

#1 Aug 20, 2014, 10:39 pm • 1

Given an acute triangle ABC, d is its Euler line. M, N is the reflection of B, C via d. P is an arbitrary point on d.
1. PM intersects AC at E, PN intersects AB at F. S is the reflection of H-the orthocenter of triangle ABC via EF. Prove that S lies on the circumcircle of triangle ABC.
2. Prove that PS passes through a fixed point.

Luis González

#2 Sep 9, 2014, 7:12 am • 3

Let (O) be the circumcircle of $\triangle ABC$ and L the reflection of A on OH. $d \equiv OH$ cuts BC, CA, AB at A', B', C'. When P varies on d, then $MP \mapsto NP$ is a perspectivity inducing a homography $E \mapsto F$ between AC and $AB \Longrightarrow EF$ envelopes a fixed conic that touches AC, AB. When P coincides with A', B', C', then EF coincides with the reflections MN, NL, LM of BC, CA, AB on OH, respectively, all tangent to the inconic $\mathcal{C}$ with foci O, H by obvious symmetry $\Longrightarrow$ all EF touch $\mathcal{C} \Longrightarrow$ projection T of H on EF is on pedal circle of $\mathcal{C}$: the 9-point circle (N_9) of $\triangle ABC \Longrightarrow$ reflection S of H about T is then on the image (O) of (N_9) under homothethy with center H and coefficient 2. $\blacksquare$

Let EF cut BC at D and let AH cut (O) again at U. Since BC and EF are perpendicular bisectors of $\overline{HU}$ and $\overline{HS}$, then D is the circumcenter of $\triangle HUS \Longrightarrow OD \perp US$ is perpendicular bisector of $\overline{US}$, hence the application $OD \mapsto US$ is homographic, but as $D \mapsto E$ is a homography between BC, CA and $E \mapsto P$ a perspectivity between CA, d, we get $S \barwedge P$ with double points at $\{d \cap (O)\} \Longrightarrow S \mapsto P$ is a stereographic projection of (O) onto $d \Longrightarrow$ all PS go through a fixed point $X \in (O)$. Observing the cases where P coincides with A', B' or C', bearing in mind that H is also orthocenter of $\triangle MNL$, we deduce that this fixed X is the Euler's reflection point of $\triangle MNL$. $\blacksquare$

Quick Reply

High School Olympiads

The ratio of the areas is independent of the points E and F

Reply

nkalosidhs
#1 Aug 24, 2014, 7:30 pm

Let $ABCD$ be a cyclic quadrilateral. Point E and F on sides AB and CD respectively, such that $\frac{AE}{EB}$=$\frac{CF}{FD}$. Le P be a point on EF such that $\frac{PE}{PF}$=$\frac{AB}{CD}$. Show that the ratio of the areas of the triangles $\triangle APD$ and $\triangle BPC$ is independent of the choice of the points E and F.

Luis González
#2 Sep 8, 2014, 2:54 am

If S denotes the center of the spiral similarity that swaps $\overline{AB}$ and $\overline{CD},$ then all $\triangle SEF$ are similar with corresponding cevian $SP,$ because of $PE:PF$ is constant. Therefore $\angle(SE,SP)$ and $\tfrac{SE}{SP}$ are constant $\Longrightarrow P$ moves on the image of AB under spiral similarity with center $S,$ rotational angle $\angle(SE,SP)$ and coefficient $\tfrac{SE}{SP}.$

When $E \equiv A,$ then $F \equiv C$ and P goes to the point U on $\overline{AC},$ satisfying $UA:UC=AB:CD.$ But if $M \equiv AD \cap BC,$ from $\triangle MAB \sim \triangle MCD,$ we get $AB:CD=MA:MC \Longrightarrow UA:UC=MA:MC \Longrightarrow U$ is on internal bisector of $\angle AMB.$ Similarly when $E \equiv B,$ then $F \equiv D$ and P goes to the intersection of BD with the internal bisector of $\angle AMB$ $\Longrightarrow P$ is on the internal bisector of $\angle AMB \Longrightarrow P$ is equidistant from AD and $BC \Longrightarrow$ $[APD]:[BPC]=AD:BC=\text{const},$ as desired.

Luis González
#3 Sep 8, 2014, 4:10 am

Even simpler: Again let $M \equiv AD \cap BC.$ Since $EA:EB=FC:FD,$ then $\triangle MAB$ and $\triangle MCD$ are similar with corresponding cevians ME and $MF.$ Hence, we get $\angle AME=\angle CMF$ (ME,MF are isogonals WRT $\angle AMB$) and $\tfrac{ME}{MF}=\tfrac{AB}{CD}=\tfrac{PE}{PF} \Longrightarrow MP$ is internal bisector $\angle EMF \Longrightarrow MP$ is also internal bisector of $\angle AMB \Longrightarrow P$ is equidistant from AD and $BC \Longrightarrow [APD]:[BPC]=AD:BC=\text{const},$ as desired.

nkalosidhs
#4 Sep 13, 2014, 8:01 pm

Very nice solution...

Sardor
#5 Sep 13, 2014, 9:03 pm • 1

It's very old problem.See here please:

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=124390&sid=911df099b7e91b8fcd3fc1aae9bc3cc5#p124390

Quick Reply

High School Olympiads

Orthogonal projections on external bisectors

Reply

Source: Bulgarian MO 2002 4th round day 1 problem 2

mitrov

#1 Sep 6, 2014, 6:54 pm

Consider the orthogonal projections of the vertices A, B and C of triangle ABC on external bisectors of $\angle ACB$, $\angle BAC$ and $\angle ABC$, respectively. Prove that if d is the diameter of the circumcircle of the triangle, which is formed by the feet of projections, while r and p are the inradius and the semiperimeter of triangle ABC, prove that $r^2+p^2=d^2$

Proposed by Alexander Ivanov

Luis González

#2 Sep 6, 2014, 10:15 pm • 1

Let A_1,A_2 be the projections of A on the external bisectors of $\angle ACB, \angle ABC$, resp. The pairs of points $\{B_1,C_1\}$ and $\{C_1,C_2\}$ are defined similarly. Let D,E,F be the midpoints of BC,CA,AB and let J be the incenter of $\triangle DEF$.

$\triangle ACA_1$ is right with circumcenter $E \Longrightarrow \angle AEA_1=2\angle ACA_1=180^\circ-\angle ACB \Longrightarrow EA_1 \parallel BC \Longrightarrow A_1 \in EF$ and similarly $A_2 \in EF$. Now, if P denotes the projection of J on EF (tangency point of incircle (J) of DEF with EF), we get $PA_1=PE+EA_1=PE+EA=\frac{1}{2}(p-b)+\frac{1}{2}b=\frac{1}{2}p$ and similarly $PA_2=\frac{1}{2}p$. Hence by Pythagorean theorem for $\triangle JPA_1$, we obtain

$$JA_1=JA_2=\sqrt{PJ^2+PA_1{}^2}=\frac{1}{2}\sqrt{r^2+p^2},$$

which means that all points A_1,A_2,B_1,B_2,C_1,C_2 lie on a same circle with center J and diameter $d=\sqrt{r^2+p^2}$.

Quick Reply

High School Olympiads

Tangent to the incircle

Reply

jayme
#1 Sep 5, 2014, 4:13 pm

Dear Mathlinkers,

1. ABC a triangle
2. (I) the incircle of ABC
3. DEF the contact triangle of ABC
4. M, N the midpoints of AC, AB
5. A* the point of intersection of MN and EF
6. B', C' the point of intersection of BA* and AC, CA* and AB.

Prove : B'C' is tangent to (I)

Sincerely
Jean-Louis

Luis González
#2 Sep 6, 2014, 1:39 am • 1

This is valid for any points M, N on AC, AB, not necessarily their midpoints.

Let $P \equiv EF \cap BC, A' \equiv B'C' \cap BC$ and $L \equiv AA' \cap CC'$. From the complete quadrangle $BB'CC'$, we get $(C, C', L, A^*) = -1$, but from the complete quadrangle $BCEF$, we get $(C, B, D, P) = -1 \Longrightarrow$ $F(C, C', D, A^*) = -1$, which forces $FL \equiv FD$, i.e. $L \equiv AA' \cap CC' \cap DF$. Therefore, by the converse of Brianchon theorem, there is a conic tangent to BC, CA, AB at D, E, F, the incircle (I), and tangent to $A'C' \Longrightarrow B'C'$ touches (I), as desired.

jayme
#3 Sep 6, 2014, 11:11 am

Dear Luis and Mathlinkers,
for the problem, I have considered the Feuerbach point Fe of ABC. According to the Hamilton's line, D, A* and Fe are collinear. I finish withe the Newton's theorem...
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Radical center lies on Euler line.

Locked

Sardor

#1 Sep 6, 2014, 12:31 am

Let ABC be a triangle and let D, E, F be the tangency points of its incircle (I) with BC, CA, AB, respectively. Let X_1 and X_2 be the intersections of line EF with the circumcircle (O) of triangle ABC. Similarly define Y_1, Y_2 and Z_1, Z_2. Prove that the radical center of the circles DX_1X_2, EY_1Y_2 and FZ_1Z_2 lies on line OI.

Luis González

#2 Sep 6, 2014, 12:41 am

Discussed before at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=253661
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=297000
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=385158

For the general mapping see post #6 at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=297002

High School Olympiads

coaxal circles

Locked

Sardor

#1 Sep 6, 2014, 12:20 am

Let K be an arbitrary circle in the plane of a given triangle ABC. Let K'_A and K''_A be the circles through B and C which are tangent to K at X' and X'', respectively. Similarly define K'_B, K''_B, K'_C, K''_C and their tangency points with K, Y', Y'', Z' and Z'', respectively. Prove that the the circumcircles of triangles $AX'X'', BY'Y'$ and $CZ'Z'$ are coaxal .

Luis González

#2 Sep 6, 2014, 12:27 am

Firstly, do not double post, there's no need to resubmit the problem when you can edit it. Secondly, take your time to search before posting. Topic locked.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=482840
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=506238
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=595152

High School Olympiads

An Intersting locus problem:)

Reply

mathVNpro

#1 Mar 15, 2009, 11:13 pm

Let B, C be two fixed points on the circle (O). Let M be the middle point of the smaller arc of BC. Let D be an changing point on the smaller arc of BC. DM intersects BC at H. Let H' be the reflection of H by DC. Find the locus of H'.

mathVNpro

#2 Mar 16, 2009, 10:38 pm

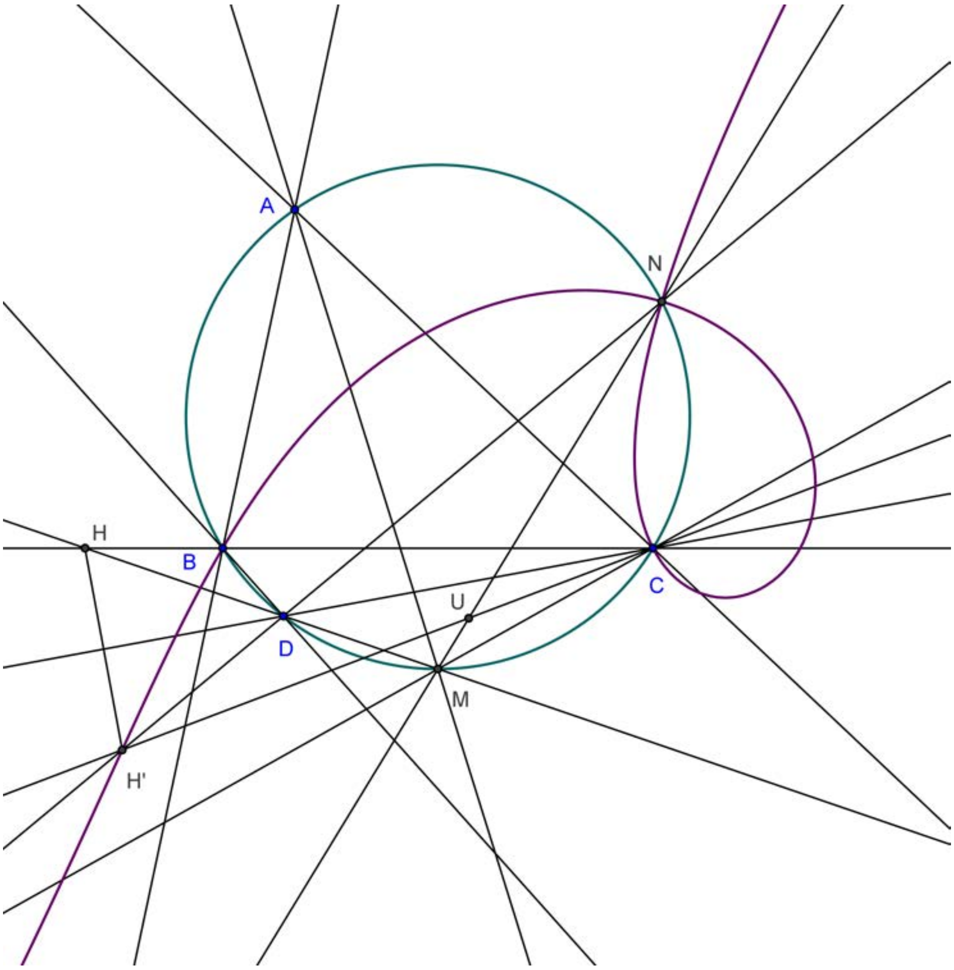
nobody can??? What a pity???

Luis González

#3 Sep 5, 2014, 7:13 am

If DH' cuts (O) again at N, then DC bisects $\angle HDH' \equiv \angle MDN \Longrightarrow$ arcs CM and CN of (O) are congruent $\Longrightarrow N$ is fixed. DM bisects $\angle BDC \Longrightarrow \angle CDM = \angle BDH$ and since $\angle DBH = \angle DMC$, then it follows that $\angle DHC = \angle DCM \Longrightarrow \angle DNM = \angle DCM = \angle DHC = \angle DH'C$. Hence, if $U \equiv CH' \cap MN, \triangle UNH'$ is U-isosceles, or $UN = UH' \Longrightarrow H'$ runs on an oblique strophoid with pole C, cusp N and construction line NM.

Attachments:



Iliopoulos

#4 Sep 5, 2014, 8:35 pm

What happens when D varies all over the circle?

Quick Reply

High School Olympiads

Going to the third power

Reply

Source: Indonesian Mathematical Olympiad 2014 Day 2 Problem 6

chaotic_iak

#1 Sep 4, 2014, 9:07 pm

Let ABC be a triangle. Suppose D is on BC such that AD bisects $\angle BAC$. Suppose M is on AB such that $\angle MDA = \angle ABC$, and N is on AC such that $\angle NDA = \angle ACB$. If AD and MN intersect on P, prove that $AD^3 = AB \cdot AC \cdot AP$.

Luis González

#2 Sep 4, 2014, 11:22 pm

Since $\angle ADN + \angle ADM = \angle ACB + \angle ABC = 180^\circ - \angle MAN$, then $AMDN$ is cyclic, being D the midpoint of its circumcircle arc MN. Hence $\angle ADN = \angle AMN$ and $\angle DAN = \angle DAM$ gives $\triangle ADN \sim \triangle AMP \Longrightarrow AM \cdot AN = AP \cdot AD$ (1). But $\angle MDA = \angle ABC \Longrightarrow AD^2 = AM \cdot AB$ and similarly $AD^2 = AN \cdot AC$, hence $AD^4 = AB \cdot AC \cdot AM \cdot AN$ (2). Combining (1) and (2) yields $AD^3 = AB \cdot AC \cdot AP$.

Quick Reply

High School Olympiads

Nice geometry

Reply

Source: maybe old

Sardor
#1 Aug 16, 2014, 9:10 pm

Consider triangle ABC and let $(I_a),(I_b),(I_c)$ be the excircles corresponding to the vertices A,B,C respectively. Let P a point in the interior of the triangle ABC and consider its cevian AA',BB',CC'. Denote by X,Y,Z the tangents from A',B',C' to the excircles $(I_a),(I_b),(I_c)$, respectlively, such that $X\notin BC, Y\notin CA, Z\notin AB$. Prove that the lines AX,BY,CZ are concurrent.

Sardor
#2 Aug 17, 2014, 9:06 pm

I have one solution with ugly calculations,there are another nice solution, please !

Sardor
#3 Aug 19, 2014, 5:28 pm

Any nice solution ? Please help me !!!

mathuz
#4 Aug 22, 2014, 4:17 pm

We have the triangles ABC and $A'B'C'$ are perspective triangles. It's not hard to see: the triangles $A'B'C'$ and $I_aI_bI_c$ are also perspective triangles. It's very good idea!!

Luis González
#5 Sep 4, 2014, 10:30 am • 2

Dear mathuz, do your remarks lead to a solution?. If yes, please enlighten me.

Let $(I_a),(I_b),(I_c)$ touch BC,CA,AB at D,E,F and AX,BY,CZ cut BC,CA,AB at U,V,W, respectively. By dual of Desargues theorem for the fourline $(BC,BC,A'X,A'X)$ circumscribed to (I_a), it follows that AA' is fixed under involution $\{AB,AX\mapsto AC,AD\} \Longrightarrow (B,C,U,A')\barwedge(C,B,D,A')\Longrightarrow$

$$\frac{\overline{UB}}{\overline{UC}}\cdot\frac{\overline{A'C}}{\overline{A'B}}=\frac{\overline{DC}}{\overline{DB}}\cdot\frac{\overline{A'B}}{\overline{A'C}}\Longrightarrow\frac{\overline{UB}}{\overline{UC}}=\frac{\overline{DC}}{\overline{DB}}\cdot\left(\frac{\overline{A'B}}{\overline{A'C}}\right)^2.$$

Now, multiplying the cyclic expressions together, bearing in mind that AA',BB',CC' and AD,BE,CF concur resp, we obtain $\frac{\overline{UB}}{\overline{UC}}\cdot\frac{\overline{VC}}{\overline{VA}}\cdot\frac{\overline{WA}}{\overline{WB}}=1$, which means that $AX\equiv AU, BY\equiv BV$ and $CZ\equiv CW$ concur.

Quick Reply

High School Olympiads

Prove right angle

Locked

Source: Croatia TST 2004 Problem 3

niraekjs

#1 Sep 4, 2014, 5:26 am

A line intersects a semicircle with diameter AB and center O at C and D, and the line AB at M, where $MB < MA$ and $MD < MC$. If the circumcircles of the triangles AOC and DOB meet again at K, prove that $\angle MKO$ is right.

Luis González

#2 Sep 4, 2014, 7:17 am

It's originally a problem from the ARMO 1995, posted many times before.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=4919
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=220651
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=221141
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=559097

High School Olympiads

Impossible incircle problem

stani95

#1 Aug 29, 2014, 7:41 pm

Given triangle ABC with incentre I and midpoint of the arc ACB is M. $MI \cap k(ABC) = H$. An arbitrary point J is chosen on the arc ACB. The tangent from J to the incircle intersects the segment AB in point L. The angle bisector of $\angle LHJ$ intersects LJ at S. Prove that $SI \parallel JM$.

Luis González

#2 Sep 3, 2014, 9:45 pm • 1

Let the 2nd tangent from J to the incircle (I) cut AB at N (WLOG assume that A is between L and N). If JL, JN cut circumcircle $\odot(ABC)$ again at $P, Q,$ then by Poncelet porism, PQ touches (I). Circles $\odot(JNL)$ and $\odot(JPQ)$ meet then at J and the Miquel point H' of the tangential $PQNL,$ hence from the problem discussed at Circumscribed quadrilateral with inversion (check 1st paragraph in post #2), we get that $H'I$ bisects $\angle PHN.$ But $\angle HAN = \angle HPB$ and $\angle HNL = \angle HJL \equiv \angle HJP = \angle HBP$ implies $\angle AHN = \angle BHP \Longrightarrow H'I$ also bisects $\angle AH'B,$ i.e. $H'I$ passes through the midpoint M of the arc $ACB \Longrightarrow H \equiv H'$.

From the aforementioned problem, we also get that NI touches $\odot(PIH)$. Hence if D is the midpoint of the arc JL of $\odot(JNL),$ then HS passes through D and by incenter property $DJ^2 = DL^2 = DI^2 = DS \cdot DH \Longrightarrow S \in \odot(PIH)$ $\Longrightarrow \angle JSI = \angle PHI \equiv \angle PHM = \angle SJM \Longrightarrow IS \parallel JM.$

High School Olympiads

very beautiful problem

Reply

fandogh

#1 Aug 24, 2014, 10:50 am

ABC is a triangle. The externally incircle which is tangent to segment AB at E_1 is tangent to BC and AC at D_1 and F_1. And the externally incircle which is tangent to segment AC at F_2 is tangent to BC and AB at D_2 and E_2. If X be the intersection of D_1E_1 and D_2F_2 and Y be the intersection of D_1E_2 and D_2F_1, prove that XY is Parallel to the internally angle bisector af A.

Luis González

#2 Sep 2, 2014, 10:42 pm

Let the internal bisector of $\angle BAC$ cut BC at $D.$ Parallels from D_1 and D_2 to AD cut AB and $AC,$ respectively at M and $N.$ From $\triangle ABD \sim \triangle MBD_1,$ we get

$$\frac{BM}{c}=\frac{s-a}{BD} \Longrightarrow BM=c\cdot(s-a)\cdot\frac{b+c}{a\cdot c}=\frac{(s-a)(b+c)}{a}.$$

Similarly, we get the same expression for the length of $CN \Longrightarrow BM=CN.$ Now since $BE_1=CF_2$ and $E_1E_2=F_1F_2,$ it follows that $(B,M,E_1,E_2) \cong (C,N,F_2,F_1) \Longrightarrow D_1(B,M,E_1,E_2)=D_2(C,N,F_2,F_1) \Longrightarrow$ intersections $X \equiv D_1E_1 \cap D_2F_2, Y \equiv D_1E_2 \cap D_2F_1$ and $D_1M \cap D_2N$ (at infinity) are collinear $\Longrightarrow XY \parallel AD.$

TelvCohl

#3 Oct 27, 2014, 9:26 pm

My solution:

Since $\frac{D_1C}{BD_1}\cdot\frac{D_2C}{BD_2}\cdot\frac{F_1A}{CF_1}\cdot\frac{F_2A}{CF_2}\cdot\frac{E_1B}{AE_1}\cdot\frac{E_2B}{AE_2}=1,$
so from Carnot theorem we get $D_1, E_1, F_1, D_2, E_2, F_2$ lie on a conic ,
hence from Pascal theorem (for $D_1E_1F_1D_2F_2E_2$) we get XY is parallel to the internally bisector of $\angle BAC$.

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Jan 14, 2016, 1:33 pm

Gibby

#4 Nov 2, 2014, 12:14 am

Can somebody clarify the problem for me? I'm not sure if he's mistranslating the problem.

Quick Reply

High School Olympiads

Reflection wrt Euler line

Reply

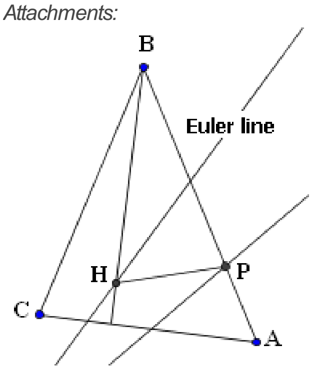
Source: own

ricarlos

#1 Aug 25, 2014, 12:24 am

Let ABC be a triangle with orthocenter H and circumcenter O.
Let P be the intersection of the side AB with the reflection of side BC over HO.
Prove that $\angle PHO = \angle ABC$.
Click to reveal hidden text

Attachments:



Luis González

#2 Sep 1, 2014, 8:44 pm

Let D, E, F be the feet of the altitudes on BC, CA, AB and $Q \equiv OH \cap BC$. Let $\mathcal{C}$ be the inconic with foci O, H whose pedal circle is the 9-point circle $\odot(DEF)$ of $\triangle ABC$. By symmetry, reflection QP of BC across HO also touches $\mathcal{C} \Longrightarrow$ projection X of H on PQ is on $\odot(DEF)$. Hence, from cyclic quadrilaterals $HDQX$ and $HFPX$, we get $\widehat{FXH} = \widehat{FPH}$ and $\widehat{DXH} = \widehat{DQH} \Longrightarrow$

$$\widehat{FPH} + \widehat{DQH} = \widehat{FXH} + \widehat{DXH} = \widehat{FXD} = \pi - \widehat{DEF} = 2 \cdot \widehat{ABC} \Longrightarrow$$

$$\widehat{PHO} \equiv \widehat{PHQ} = \widehat{FPH} + \widehat{DQH} - \widehat{PBQ} = 2 \cdot \widehat{ABC} - \widehat{ABC} = \widehat{ABC}.$$

THVSH

#3 Feb 27, 2015, 9:29 pm • 1

My solution: (not use conic)

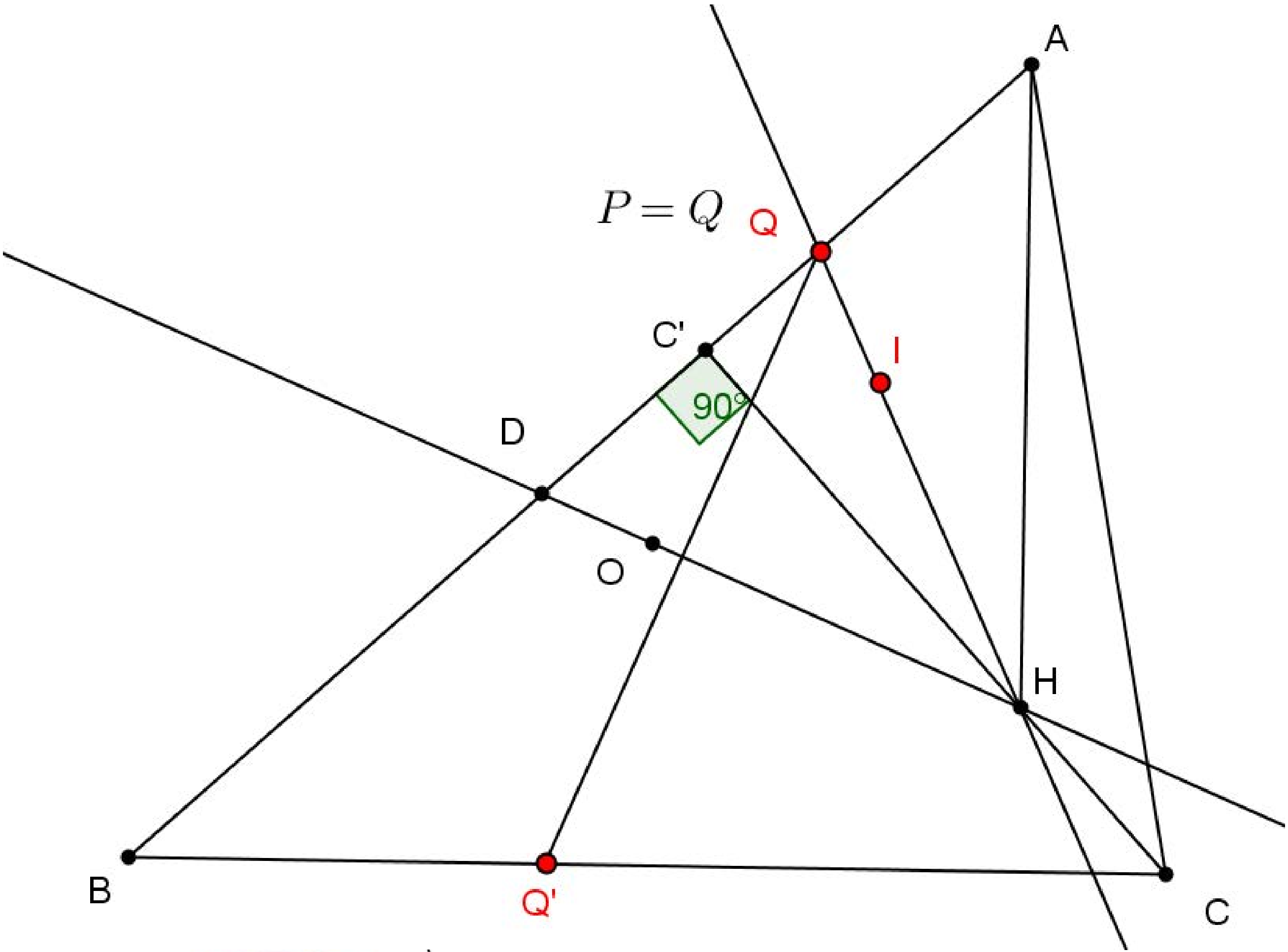
Lemma: Let ABC be a triangle with orthocenter H and circumcenter O. O' is the reflection of O wrt BC. Let $M = AO' \cap BC$. O_1 is the circumcenter of (ABM). Let $N = AO_1 \cap BC$. Prove that the reflection of N wrt AO' lies on CH.

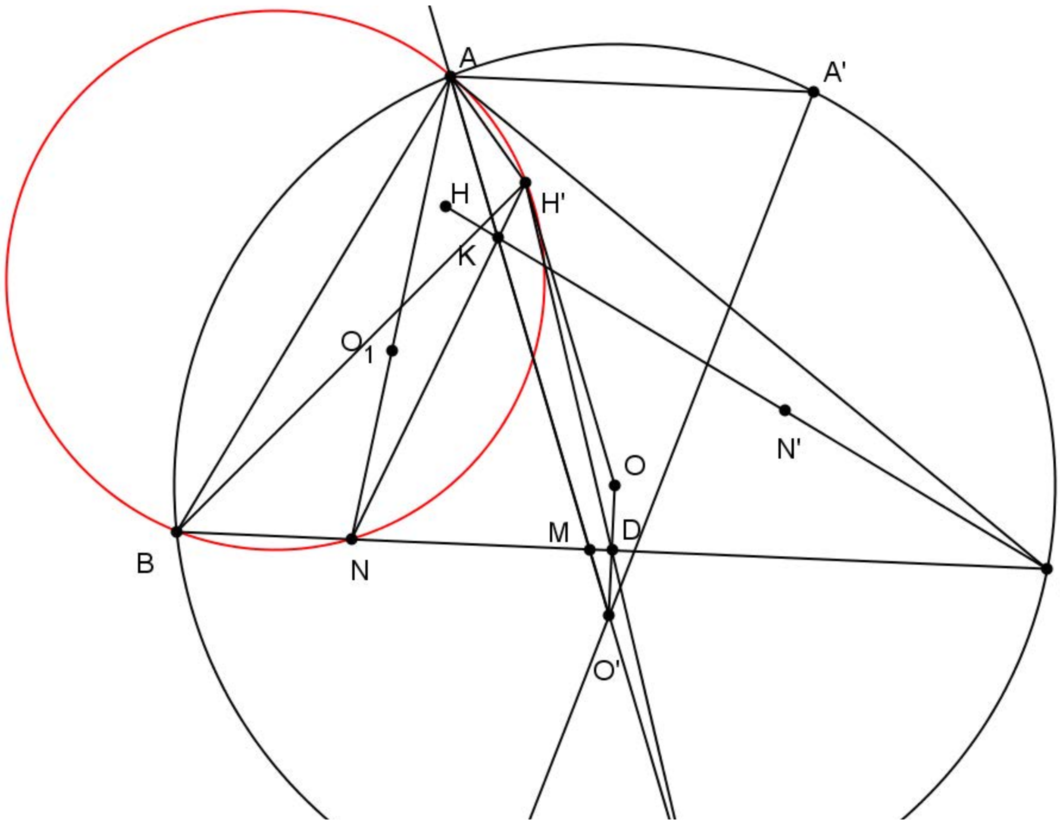
Proof:
.$A' \in (O)$ such that $AA' \parallel BC$Let D be the midpoint of BC.
H' is the reflection of H wrt AO'. H_1, E is the reflection of H' wrt BC, D. Let $K = NH' \cap AO'$
$*$ We have: O' is the center of (BHC), so $H' \in (BHC) \longrightarrow H_1 \in (O)$.
$*$ We see that $OH' \parallel O'E$
In the other side, It is easy to see that $OH' \parallel AO' \longrightarrow E \in AO' \longrightarrow H_1, O', A'$ are collinear.
So H_1A, H_1O' are isogonal conjugates in $\angle BH_1C$.
$\longrightarrow \angle BH'O = \angle BH_1O' = \angle CH_1A = \angle ABC$
$\longrightarrow \angle AH'B = \angle AH'O - \angle BH'O = 180 - \angle H'AM - \angle ABC$
$= 180 - \angle HAM - \angle ABC = 180 - \angle BAN - \angle ABC = \angle ANB$

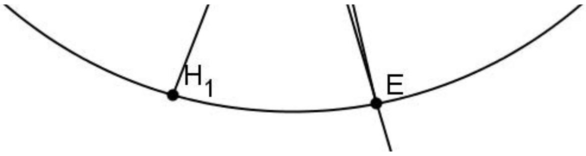
Thus, A, B, N, H' are concyclic. So $\angle AHK = \angle AH'K = 180 - \angle ABC = \angle AHC$
$\longrightarrow K \in HC$. It also means that the reflection of N wrt AO' lies on $CH.$ Q.E.D

Back to our problem:
Let $C' = CH \cap AB$
Let $OH \cap AB = D, I$ is center of (AHD), $HI \cap AB = Q$
From lemma (use for $\triangle HAB$), we get: the reflection of Q wrt OH lies on BC.
So the reflection of BC wrt OH pass through $Q \longrightarrow P = Q$
Thus, $\angle PHO = \angle QHO = \angle C'HA = \angle ABC$ Q.E.D

Attachments:







TelvCohl

#4 Feb 28, 2015, 12:21 am

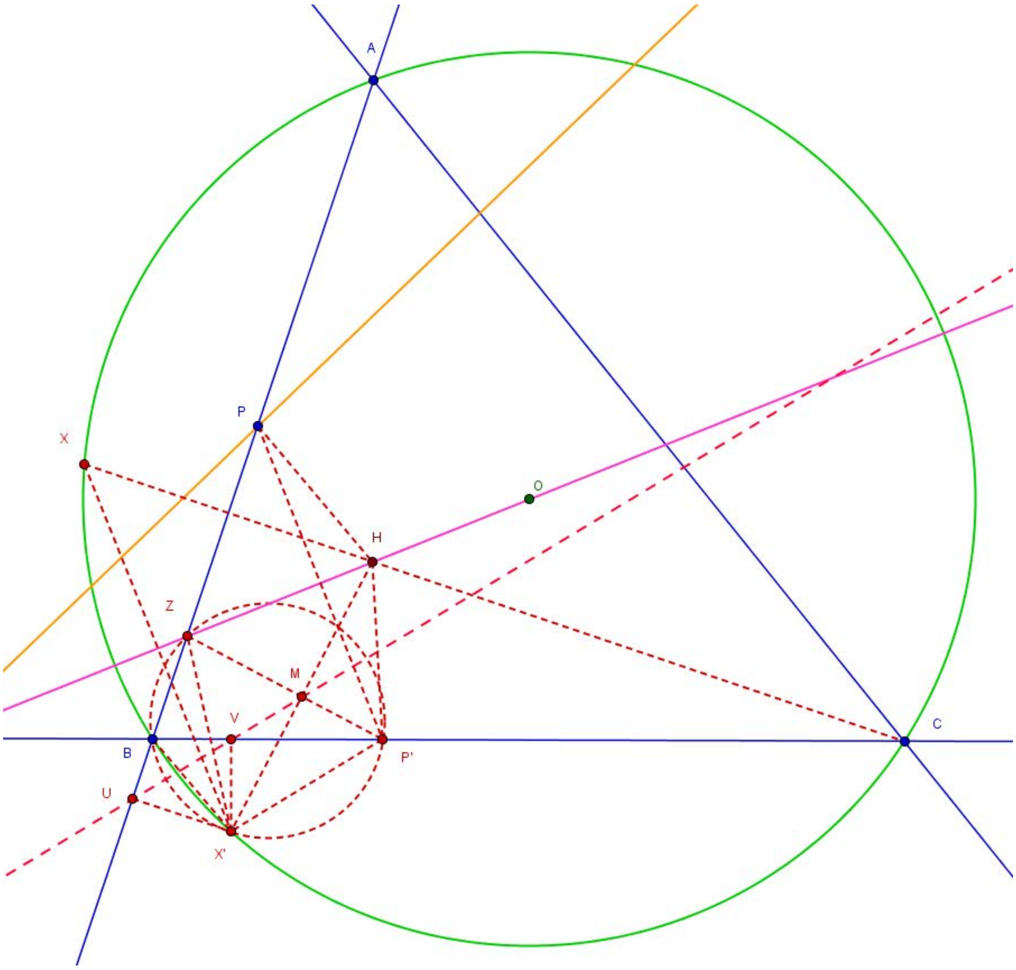
My solution (without conic) :

Let $P^* \in BC$ be the reflection of P in OH .
Let $X = CH \cap \odot(ABC)$ and $X^* \in \odot(ABC)$ be the reflection of X in OH .
Let $Z = OH \cap AB$ and U, V be the projection of X^* on AB, BC, respectively .

Since H, X are symmetry WRT AB ,
so H, X^* are symmetry WRT $ZP^* \Longrightarrow M \equiv ZP^* \cap HX^*$ is the midpoint of HX^* .
From Steiner theorem we get M, U, V are collinear at the Simson line of X^* WRT $\triangle ABC$,
so from the converse of Simson theorem we get $X^* \in \odot(BP^*Z) \Longrightarrow \measuredangle PHO = \measuredangle OHP^* = \measuredangle P^*X^*Z = \measuredangle CBA$.

Q.E.D

Attachments:



tranquanghuy7198

#5 Mar 31, 2015, 7:06 am

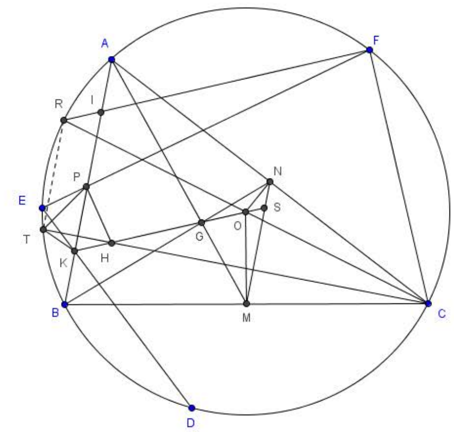
Here's my solution:

First, we state without proving the following lemma, which is the extension of the Butterfly theorem: Let $ABCD$ be the quadrilateral inscribed in the circle (O) and let the arbitrary line intersect AD, BC, AC, BD at P, Q, R, S, respectively. Then, $OP = OQ$ iff $OR = OS$. Moreover, when both those conditions hold, let the line which passes through A and is parallel to the arbitrary line mentioned above intersect (O) again at W then we receive the inscribed quadrilateral $BWQS$.
The proof, which bases on angle chasing, is very easy and no more point needs constructing. Now we turn back to our main problem:

Let D, E, F be the points reflection of A, B, C WRT OH

Let D, E, F be the points reflecting A, B, C WRT OH.
EF intersects AB at P. We will prove that $(HP, HO) = (BC, BA)$. Indeed:
CH intersects (O) at T (different from C). R is diametrical opposite to C. RF intersects AB at I. M, N are the midpoints of CB, CA, then AM, BN, OH concur at the centroid G of triangle ABC. MN intersects OH at S. DE, AB, OH concur at K.
First, $RF \parallel OH$ (perpendicular to CF). Consider the homothety $H(C, 2)$:
$N \longrightarrow A, M \longrightarrow B, \longrightarrow RF, S \longrightarrow I$, so we have: $\frac{AI}{IB} = \frac{NS}{SM}$.
We also have: $\frac{NS}{SM} = \frac{BK}{KA}$ because AM, BN, KS concur, therefore: $\frac{AI}{IB} = \frac{BK}{KA}$, and $OI = OK$.
Now we consider the quadrilateral $DERF$: the line AB intersects DE, RF at K, I, respectively and $OI = OK$. Apply the lemma with the notice that $RT \parallel AB$, we receive the inscribed quadrilateral $KTEP$. It means that:
$(HP, HO) = (HP, HK) = (TK, TP) = (EK, EP) = (ED, EF) = (BC, BA)$.
The proof finishes here.

Attachments:



Quick Reply

High School Olympiads

Nice

Reply

Source: unknown

Iliopoulos

#1 Aug 31, 2014, 3:24 pm

Let O, I are the circumcenter and the incenter of a triangle ABC. If AB is not equal to AC, D, E are the midpoints of AB, AC espectively and $BC = \dfrac{AB+AC}{2}$, then prove that the angle bisector of $\angle CAB$ is vertical to OI.

Luis González

#2 Sep 1, 2014, 5:41 am

Let Y, Z be the tangency points of the incircle (I) with AC, AB (projections of I on AC,AB). Then $AB + AC = 2 \cdot BC \Longrightarrow AC - BC = BC - AB \Longrightarrow DZ = EY \Longrightarrow$ the right triangles $\triangle IZD$ and $\triangle IYE$ are congruent by SAS $\Longrightarrow ID = IE$. Together with $\angle DAI = \angle EAI$ (keeping in mind that $AE \neq AD$), it follows that I is the midpoint of the arc DOE of circle $\odot(ADE) \Longrightarrow \angle AIO = \angle ADO = 90^\circ$, i.e. internal bisector of $\angle BAC$ is perpendicular to OI.

gavrilos

#3 Sep 1, 2014, 6:20 am

The bisector of $B\hat{A}C$ intersects BC at F and let I be the incenter of $\triangle ABC$

Let now $H \equiv AF \cap (O)$ where (O) the circumcircle of $\triangle ABC$

Ptolemy's theorem gives $AH \cdot BC = AB \cdot CH + AC \cdot BH$.

It is known that $BH = CH$ thus the latter relation becomes

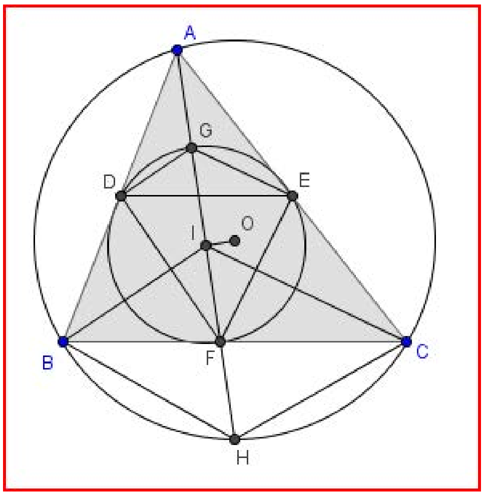
$AH \cdot BC = (AB + BC) \cdot CH = 2BC \cdot CH \Leftrightarrow AH = 2CH$

Taking account of $IH = BH = CH$ the latter is equivalent to $AI + IH = 2IH \Leftrightarrow AI = IH$.

Thus I is the midpoint of AH and the proof is over.

I hope my proof is correct.

Attachments:



Iliopoulos

#4 Sep 2, 2014, 9:34 pm

Solution(standar notation):

We know that:

$$AI^2 = \frac{bc(s-a)}{s}$$

$$Rr = \frac{abc}{4s}$$

By Euler's formula:

$$OI^2 = R^2 - 2Rr$$
$$\implies OI^2 = R^2 - \frac{abc}{2s}$$

$$a = \frac{b+c}{2}$$
$$\implies 2s = 3a$$

And:

$$AI^2 = \frac{bc(s-a)}{s} = \frac{abc}{2s}$$

So:

$$OI^2 + AI^2 = R^2 = OA^2$$

So, by the converse of Pythagor's theorem, AI is vertical to OI.

bvdsf

#5 Sep 11, 2014, 3:18 pm

very nice solution

jayme

#6 Sep 26, 2014, 11:04 am

Dear Mathlinkers,
after nice proofs, is there another synthetic approach for this problem?
Sincerely
Jean-Louis

IDMasterz

#7 Sep 28, 2014, 5:24 pm

^^Of course, let external angle bisector of A meet (O) at H'. Since $OI \parallel AH'$ and H', O, H are collinear (O bisects HH') then I bisects AH. So $\frac{AI}{AI_A} = 3 = \frac{s}{s-a} \implies 2a = b + c$.

Quick Reply

High School Olympiads

Incenter of a triangle = Nagel point of reference triangle

Source: ETC - X(5692)

mohohoho

#1 Aug 19, 2014, 10:11 pm • 1

Let ABC be a triangle. Reflect the incenter of ABC around the three perpendicular bisectors of ABC, given another new triangle.

Prove that the incenter of the new triangle so formed is the Nagel Point of the reference triangle ABC.

Luis González

#2 Aug 31, 2014, 4:32 am • 2

Let (I) and (O) be the incircle and circumcircle of $\triangle ABC.$ D,E,F are the midpoints of BC,CA,AB and J is the incenter of $\triangle DEF;$ Spieker point of $\triangle ABC.$ Since the Nagel point of $\triangle ABC$ is the reflection of I on J (well-known), then it is enough to show that J is the incenter of the triangle formed by projections X,Y,Z of I on perpendicular bisectors $OD,OE,OF.$

Clearly X,Y,Z lie on circle ω with diameter $\overline{OI}.$ Since $IY \parallel AC$ and $IZ \parallel AB,$ then $\angle AIY=\angle AIZ=\tfrac{1}{2}\angle BAC,$ i.e. IA bisects $\angle YIZ,$ hence 2nd intersection L of ω with AI (projection of O on AI) is the midpoint of the arc YZ of $\omega \Longrightarrow XL$ is internal bisector of $\angle YXZ.$

Let (I) and the A-excircle (I_a) touch BC at U and $V.$ U' is the antipode of U WRT $(I).$ A is the exsimilicenter of $(I) \sim (I_a)$ $\Longrightarrow A,V,U'$ are collinear and since D is also midpoint of $\overline{UV},$ then $\triangle IDX$ becomes the medial triangle of $\triangle UVU',$ i.e. $ID \parallel AX.$ Hence if M is the midpoint of the arc BC of $(O),$ we obtain $\tfrac{XD}{XM}=\tfrac{AI}{AM}=\tfrac{\frac{1}{2}AI}{\frac{1}{2}AM}=\tfrac{DJ}{ML} \Longrightarrow X,J,L$ are collinear, i.e. XJ bisects $\angle YZX$ and similar reasoning gives that YJ and ZJ bisect $\angle ZYX$ and $\angle XZY,$ resp $\Longrightarrow J$ is incenter of $\triangle XYZ,$ as desired.

TelvCohl

#3 Jan 18, 2015, 1:27 pm • 1

Generalization :

Let O be the circumcenter of $\triangle ABC$.
Let T be the pole of the Simson line with direction parallel to OP WRT $\triangle ABC$.
Let P_a, P_b, P_c be the reflection of P in the perpendicular bisector of BC, CA, AB, respectively .
Let P' be a point such that $\triangle ABC \cup P \sim \triangle P_aP_bP_c \cup P'$ (easy to see $\triangle ABC \sim \triangle P_aP_bP_c$) .

Then P' is the image of T under the inversion WRT $\odot(O, OP)$.

Proof :

Easy to see O is the center of $\odot(PP_aP_bP_c)$.

Since $\angle POP_a = 180^\circ - 2\angle(BC, OP) = 180^\circ - 2\angle TAO = \angle AOT$,
(Notice that the isogonal conjugate of AT WRT $\angle BAC$ is perpendicular to OP)
so we get $\triangle ABC \cup T \sim \triangle P_aP_bP_c \cup P$.

Since $\triangle OP'P \sim \triangle OPT$,
so we get $P' \in OT$ and $OP' \cdot OT = OP^2$.
i.e. P' is the image of T under the inversion WRT $\odot(O, OP)$

Done

For the original problem (P is the incenter of $\triangle ABC$) :

Since T is the Anti-complement of Feuerbach point of $\triangle ABC$,
so we get the Nagel point N_a lie on OT and $ON_a = \mathrm{R} - 2\mathrm{r}$ (R, r is the radius of $\odot(O), \odot(P)$, respectively .) ,
hence combine with $OP = \sqrt{\mathrm{R}(\mathrm{R} - 2\mathrm{r})}$ and $OT = \mathrm{R}$ we get $ON_a \cdot OT = OP^2 \Longrightarrow N_a$ is the incenter of $\triangle P_aP_bP_c$.

buratinogigle

#4 Jan 18, 2015, 1:54 pm

A remark. If P is a point on OI line of triangle ABC. D, E, F are reflections of P through perpendicular bisector of BC, CA, AB. K is incenter of DEF. Na, O are Nagel point and circumcenter of ABC. Then O, K, Na are collinear and $INa \parallel PK$.

Quick Reply

High School Olympiads

Triangle similarity

Reply

Source: Rioplatense Olympiad 2011, Level 3, Problem 2

Leicich

#1 Aug 30, 2014, 2:56 am

Let ABC an acute triangle and H its orthocenter. Let E and F be the intersection of lines BH and CH with AC and AB respectively, and let D be the intersection of lines EF and BC. Let Γ_1 be the circumcircle of AEF, and Γ_2 the circumcircle of BHC. The line AD intersects Γ_1 at point $I \neq A$. Let J be the feet of the internal bisector of $\angle BHC$ and M the midpoint of the arc $\overset{\frown}{BC}$ from Γ_2 that contains the point H. The line MJ intersects Γ_2 at point $N \neq M$. Show that the triangles EIF and CNB are similar.

Luis González

#2 Aug 30, 2014, 11:24 am

EF is radical axis of $\Gamma_1, \odot(BCEF)$ and BC is radical axis of $\odot(BCEF), \odot(ABC) \Longrightarrow AD$ is radical axis of $\odot(ABC), \Gamma_1 \Longrightarrow I \in \odot(ABC) \Longrightarrow I$ is center of the spiral similarity that swaps $\overline{CE}$ and $\overline{BF} \Longrightarrow$ $IF : IE = BF : CE = HB : HC.$ But since HJ and NJM bisect $\angle BHC$ and $\angle BNC,$ then by angle bisector theorem $NB : NC = JB : JC = HB : HC \Longrightarrow NB : NC = IF : IE.$ Together with $\angle CNB = \angle EHC = \angle EIF,$ it follows that $\triangle EIF \sim \triangle CNB$ by SAS.

Quick Reply

High School Olympiads

radical center

Reply

HQN

#1 Aug 18, 2014, 3:29 pm

Given triangle ABC. (O),(I) respectively are its cirmcumscribed circle and incircle. D,E,F is tagency points of (I) to BC,CA,AB. H is orthorcenter of DEF. EF,DF,DE respectively intersect (O) at A_1, A_2; B_1, B_2; C_1, C_2. Prove that H is the radical center of $(DA_1A_2), (EB_1B_2), (FC_1C_2)$

Luis González

#2 Aug 28, 2014, 11:39 am

Posted before at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=253661
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=297000

For a generalization, see post #6 at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=297002

Quick Reply

High School Olympiads

Cono Sur Olympiad 2005, Problem 4

Reply

Davi Medeiros

#1 Aug 20, 2014, 2:48 am

Let ABC be a isosceles triangle, with $AB=AC$. A line r that pass through the incenter I of ABC touches the sides AB and AC at the points D and E, respectively. Let F and G be points on BC such that $BF=CE$ and $CG=BD$. Show that the angle $\angle FIG$ is constant when we vary the line r.

Luis González

#2 Aug 28, 2014, 9:39 am

F is image of E under reflection across CI followed by symmetry about the midpoint M of BC and similarly G is image of D under reflection across BI followed by symmetry about $M.$ Since $D \mapsto E$ is a perspectivity between $AB, AC,$ then it follows that $\mathbb{H}: F \mapsto G$ is a homography clearly with no fixed points $\Longrightarrow$ it is induced by two superposed congruent pencils, or the sides of a rigid angle whirling around its vertex $P.$

Parallel from I to AB cut AC at $L.$ U is reflection of L on CI and V is reflection of U about $M,$ then U,V are the limiting points of $\mathbb{H} \Longrightarrow P$ is on the perpendicular bisector AI of $UV.$ When $DE \parallel BC,$ then $IECG$ and $IDBF$ become rhombi, thus by symmetry, we have $\angle FIG=\angle ILE=\angle IUV,$ which implies that $I \equiv P \Longrightarrow \angle FIG=\angle BAC=\text{const}$ for all $r.$

Davi Medeiros

#3 Sep 6, 2014, 10:14 am

Nice Solution, Luis González! One can also solve this problem with similar triangles.

Let P and Q be the orthogonal projections of I on BC and AB, respectively. Since I is the incenter, we have $QB=BP=PC=a$, and because $BD=CG=b$, we get $DQ=GP=a-b$. So, the triangles IDQ and IGP are congruent, since $IP=IQ$ and $\angle GPI=\angle DQI=90$. Thus, we have $\angle IDQ=\angle IGP$, or $\angle ADE=\angle IGF$. Analogously, $\angle AED=\angle IFG$ Hence, trinangles ADE and IGF are similar, which means that $\angle FIG=\angle EAD=\angle BAC$, which is a fixed angle, solving the problem.

Quick Reply

High School Olympiads

Perpendicular in cyclic quadrilateral.

Reply

Source: Own

CTK9CQT
#1 Aug 12, 2014, 5:51 am

Hopefully this would be a new problem.
Let $ABCD$ be a cyclic quadrilateral inscribed in (O). $E = AD \cap BC$, $F = AC \cap BD$. S is on FO. The line passing through S and parallel to BD cuts DA at Q and the line passing through S and parallel to AC cuts BC at P. X is the midpoint of PQ. Prove that $OX \perp CD$.

Attachments:
ch3.pdf (8kb)

Luis González
#2 Aug 12, 2014, 8:48 am • 1

As S varies on OF, the application $S \mapsto X$ is an affine homography (see the general configuration at Beautiful locus 3), so the locus of X is another line. When $S \equiv F$, then X coincides with the midpoint of $\overline{CD}$, thus it suffices to show that when $S \equiv O$, the midpoint of $\overline{PQ}$ lies on the perpendicular bisector ℓ of $\overline{CD}$.

Let U, V denote the projections of O on BC, DA and K the projection of F on CD. Let ℓ cut PQ at Y. From $OP \parallel FC$, $OQ \parallel FD$ and $OY \parallel FK$, we get

$$\frac{OP}{OQ} = \frac{OU}{OV} = \frac{\cos \widehat{BDC}}{\cos \widehat{ACD}} = \frac{\sin \widehat{DFK}}{\sin \widehat{CFK}} = \frac{\sin \widehat{QOY}}{\sin \widehat{POY}},$$

which means that OY is the O-median of $\triangle OPQ$, i.e. $Y \equiv X$ is the midpoint of $\overline{PQ}$, as desired.

CTK9CQT
#3 Aug 12, 2014, 9:29 am

To show that when S varies on OF then X runs on OY, we can use the E.R.I.Q lemma.

Sardor
#4 Aug 12, 2014, 10:52 am

Really nice problem,thank youCTK9CQT.What is E.R.I.Q lemma?

CTK9CQT
#5 Aug 12, 2014, 11:35 am • 1

It's Equal Ratios in Quadrilateral, you can see at: http://livetolove94.files.wordpress.com/2010/04/eriq.pdf

CTK9CQT
#6 Nov 15, 2014, 2:43 pm

@Luis: can you show me how the configuration at Beautiful locus 3 can be applied to my problem, I mean, in the configuration at Beautiful locus 3, P varies in a conic and in my problem, S varies in the line FO. Do you think in case of line, the proof remains the same? And where can I find more about this affine homography? Thank you in advance.

Quick Reply

High School Olympiads

Spiral similarity

Locked

Source: Zhao

Legend-crush

#1 Aug 9, 2014, 3:56 am

Let ABCD be a quadrilateral. LEt diagonals AC and BD meet at P. Let O_1 and O_2 be the circumcenters of APD and BPC. Let M,N and O be the midpoints of AC,BD and O_1O_2 .
Show that O is the circumcenter of MNP

Luis González

#2 Aug 9, 2014, 5:58 pm • 1

Posted before. It actually holds for points M,N,O dividing AC,DB and O_1O_2 in the same ratio (see the 1st link below).

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=441718
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=514278
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=580130

Seekrit

#1 Aug 8, 2014, 4:49 pm

In cyclic quadrilateral $ABCD$, $BC=CD$. Let ω be the circle centred at C and tangent to BD. Let I be the incenter of triangle ABD.
Prove the line through I parallel to AB is tangent to ω.

wiseman

#2 Aug 8, 2014, 9:00 pm

Lemma:
I is the incenter of $\triangle ABC$. If AI intersects with the circumcircle of $\triangle ABC$ in M,

then $BM=CM=MI$.

Proof of the lemma:$\angle MBI=\angle IBC+\angle CBM=\angle\frac{B}{2}+\angle\frac{A}{2}=\angle IBA+\angle IAB=\angle BIM$.

$\Longrightarrow BM=MI$. Like above we can prove that $MI=CM\Rightarrow MI=MB=MC$.

Proof of the problem: Let E and F be the feet of perpendiculars from C to

BD and line$d$(line passing through I). we only have to prove that CE and CF.

$\longrightarrow CE=\sin(\angle CDB).CD$.

$\longrightarrow CF=\sin(\angle CIF).CI$.

But we know that $\angle CDB=\angle CAB=\angle CIF$ and $CD=CI$(by lemma).

$\Longrightarrow CE=CF\Longrightarrow$ Circle centered at C and tangent to BD is tangent

to line d and we are done. $\blacksquare$.

professordad

#3 Aug 8, 2014, 9:32 pm • 1

Click to reveal hidden text

Luis González

#4 Aug 8, 2014, 11:56 pm • 1

I remarked this result as a lemma in the paper On a triad of circles tangent to the circumcircle and the sides at their midpoints, Forum Geometricorum, 11 (2011) 145--154 (see lemma 3). But I found a much better solution:

(I,ϱ) and (J,ϱ_a) denote the incircle and A-excircle of $\triangle ABD$. Let X,Y be the projections of C on AB,BD and let Z be the projection of I on CX. Using that the midpoint C of the arc BD is also midpoint of $\overline{IJ}$ (well-known), we get $CX=\tfrac{1}{2}(\varrho+\varrho_a)$ and $CY=\tfrac{1}{2}(\varrho_a-\varrho)\Longrightarrow CX-\varrho=CY\Longrightarrow CX-ZX=CZ=CY\Longrightarrow IZ$ is a tangent of $\odot(C,CY)$ parallel to AB.

Quick Reply

High School Olympiads

equal angles

Locked

daDangminh

#1 Aug 8, 2014, 3:04 pm

Given a circle with center O. On a line outside the circle, draw perpendicular OP to the line. From an arbitrary point Q on the line, draw tangents QA, QB. AB intersects OP at K. M, N is the feet of perpendiculars from P to QA, QB respectively. prove that MN goes through the midpoint of KP

Luis González

#2 Aug 8, 2014, 7:20 pm • 1

Does the subject have anything to do with the problem?, plus the point P is not well defined. Anyways, it is clearly IMO Shortlist 1994, G5.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=5198
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=294580

High School Olympiads

Three points are collinear iff

Reply

Source: IMO 2013 P4

buratinogigle

#1 Jul 20, 2014, 3:38 pm

Let ABC be a triangle with P,Q are two isogonal conjugate points inside triangle. D,E,F are projection of P on BC,CA,AB, resp. K is projection of Q on BC. R is a point on side BC. M,N lie on circle $(RBF),(RCE)$, resp, such that $RM\parallel PC$, $RN\parallel PB$. Prove that M,N,P are collinear iff $R=D$ or $R=K$.

Luis González

#2 Aug 8, 2014, 1:12 am • 1

Since $\widehat{CEM}=\widehat{CRM}=\widehat{CBP}$ and $\widehat{BFN}=\widehat{BRN}=\widehat{BCP}$ are constant, then as R varies on BC, M and N move on two fixed lines m,n passing through E,F, respectively. Since RN and RM have fixed directions, then $M\mapsto N$ is clearly an affine homography between m and $n\Longrightarrow MN$ would pass through P twice at most, i.e. for two positions of R and nowhere else. Hence, it suffices to show that it holds for $R\equiv D$ and $R\equiv K$ as claimed in the proposition.

Assume that $R\equiv D$. Thus from cyclic $PDBNF$, we get $\widehat{BPN}=\widehat{BDN}=\widehat{BCP}\Longrightarrow N$ is on the tangent of $\odot(PBC)$ at P and similarly M is on the tangent of $\odot(PBC)$ at $P\Longrightarrow P,M,N$ are collinear.

Assume that $R\equiv K$. J is the Miquel point of KEF WRT $\triangle ABC$; common point of the circles $\odot(AEF),\odot(BFK)$, $\odot(CKE)$ and KJ cuts $\odot(AEF)$ again at X. $\widehat{FAX}=\widehat{FJA}=\widehat{FBK}$, i.e. $AX\parallel BC$. If U,V denote the projections of Q on AC,AB, then D,E,F,K,U,V are concyclic (well-known) $\Longrightarrow$ $\widehat{KJN}=\widehat{KFN}=\widehat{BKN}+\widehat{BFK}=\widehat{KUV}+\widehat{PCB}$. But since $PA\perp UV$ and $PC\perp UK$, we get $\widehat{KUV}=\pi-\widehat{APC}=\widehat{PAC}+\widehat{PCA}\Longrightarrow\widehat{KJN}=\widehat{PCB}+\widehat{PAC}+\widehat{PCA}=\widehat{ACB}+\widehat{PAC}$. Now, on the other hand, using the cyclic $PJAX$ and $AX\parallel BC$ gives $\widehat{KJP}=\pi-\widehat{PAX}=\widehat{PAC}+\widehat{ACB}$, consequently $\widehat{KJP}=\widehat{KJN}\Longrightarrow P,J,N$ are collinear and similar reasoning gives that P,J,M are collinear $\Longrightarrow P,M,N$ are collinear.

Quick Reply

High School Olympiads

Conjecture

Reply

Source: Own

mohohoho

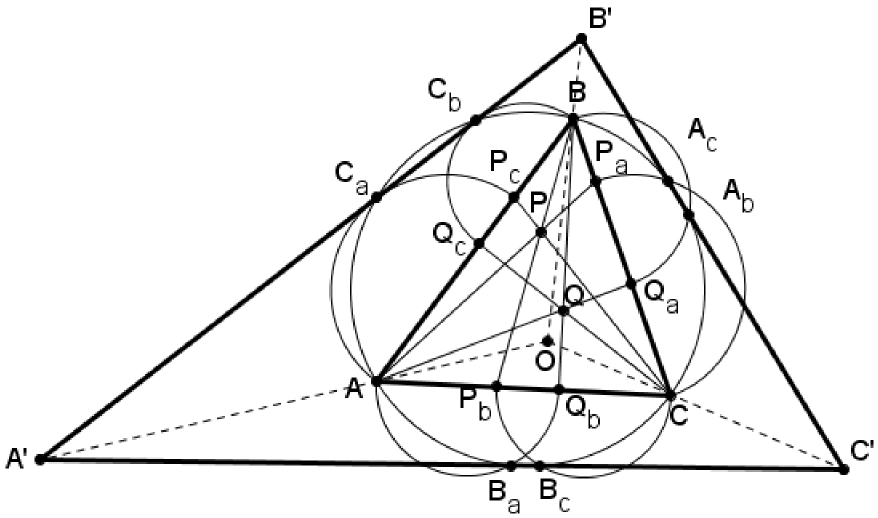
#1 Aug 6, 2014, 4:28 am

Let ABC be a triangle. Let P, Q be two points inside of triangle ABC. Let P_aP_bP_c be the cevian triangle wrt P. Let Q_aQ_bQ_c be the cevian triangle wrt Q. Draw semicircles on segments BQ_a, AQ_b, BQ_c, CP_a, CP_b, AP_c. Denote the circumcircle (ABC). Let:

A_b = (ABC) ∩ (BQ_a)
A_c = (ABC) ∩ (CP_a)
B_c = (ABC) ∩ (CP_b)
B_a = (ABC) ∩ (AQ_b)
C_a = (ABC) ∩ (AP_c)
C_b = (ABC) ∩ (BQ_c)
A' = C_aC_b ∩ B_aB_c
B' = C_aC_b ∩ A_cA_b
C' = A_cA_b ∩ B_aB_c

Prove that AA', BB', CC' are concurrent.

Attachments:



Luis González

#2 Aug 6, 2014, 10:30 am • 2

Let $A_bA_c, B_aB_c, C_bC_a,$ cut BC, CA, AB at $A_0, B_0, C_0,$ respectively. Inversion with center A_c and power equal to the power of A_c WRT $\odot(ABC)$ takes B, A_b into $C, A_c,$ thus the circle with diameter $\overline{BQ_a}$ goes to the circle with diameter $\overline{CP_a}$ $\Longrightarrow A_c$ coincides with the center of the direct inversion that swaps these circles, i.e. A_c is center of the involution $\{B, P_a\} \mapsto \{C, Q_a\} \Longrightarrow (A_0, B, C, P_a) = (\infty, C, B, Q_a) \Longrightarrow (B, C, A_0, P_a \cdot Q_a) = -1,$ in other words A_c is on the trilinear polar τ of the ceva product $P \cdot Q.$ Analogously, B_0 and C_0 lie on $\tau \Longrightarrow \triangle ABC$ and $\triangle A'B'C'$ are perspective through $\tau.$ By Desargues theorem AA', BB' and CC' concur.

mohohoho

#3 Aug 6, 2014, 10:51 am

Thank you for your proof, Luis.

Quick Reply

High School Olympiads

Incenters and cyclic quadrilateral.

Locked

CTK9CQT

#1 Aug 5, 2014, 9:47 pm

Let acute triangle ABC inscribed in (O) and M, N be the midpoints of the small arcs AB, AC respectively. D be the midpoint of MN. G be an arbitrary point on small arc BC and I, J, K be the incenters of $\triangle ABC, \triangle ABG, \triangle ACG$ respectively. $E = DI \cap (O)$. Prove that $GIJK$ is a concyclic quadrilateral.

Attachments:

ms.pdf (6kb)

Luis González

#2 Aug 5, 2014, 11:13 pm • 2

It's Iran PPCE 1997 P2 over again. So topic locked.

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=349634
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=313870
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=393618

High School Olympiads

Fixed circle

Reply

Source: Own

buratinogigle

#1 Jul 20, 2014, 4:29 pm

Let ABC be a triangle with incircle (I). (K) is a circle contain (I). P is a point on (I). Tangent at P of (I) cuts (K) at M,N. The second tangents at M,N of (I) intersect at Q. QM,QN cut BC at E,F, resp. Circle (QMN) and (QEF) intersect again at R. Prove that R always lies on a fixed circle when P move.

Luis González

#2 Aug 5, 2014, 10:55 am • 3

Let (I) touch BC,QM,QN at $X,Y,Z,$ respectively. Inversion WRT (I) takes M,N,Q,E,F into the midpoints of $YP,PZ,ZY,ZX,XY,$ respectively, hence circles $\odot(QMN)$ and $\odot(QEF)$ go to 9-point circles of $\triangle PYZ$ and $\triangle XYZ$ $\Longrightarrow$ the inverse of R is then the 2nd intersection of the 9-point circles of $\triangle PYZ$ and $\triangle XYZ,$ i.e. the anticenter T of the cyclic $PYXZ,$ which is then the midpoint between X and the orthocenter H of $\triangle PYZ.$ So it is enough to find the locus of $H.$

Let the perpendicular from I to PH cut MN at $S.$ It's easy to see that SI is the tangent of $\odot(IMN)$ at $I \Longrightarrow$ $SI^2=SM \cdot SN \Longrightarrow S$ has equal power WRT (K) and $I \Longrightarrow S$ runs on the radical axis τ of $(K),I,$ perpendicular to $KI.$ But since PH is the polar of S WRT $(I),$ then it follows that PH passes through the fixed pole J of τ WRT (I) $(\star).$

If YZ cuts MN at $V,$ from the harmonic cross ratio $(M,N,P,V)=-1,$ we deduce that the circle ω with diameter PV is orthogonal to both (I) and $(K) \Longrightarrow$ all ω form an elliptic pencil passing through the limiting points $\{L_1,L_2\}$ of $(I),(K).$ Hence, if $L \in \omega$ is the foot of the perpendicular from P to $YZ,$ we have, keeping in mind $(\star),$ $JP \cdot JL=JL_1 \cdot JL_2=\text{const} \Longrightarrow$ locus of L is the inverse circle (K_1) of (K) under inversion with center J and power $JL_1 \cdot JL_2.$ Consequently J is also exsimilicenter of $(K) \sim (K_1),$ since it is inside $(K).$ Now if PL cuts (I) again at U (reflection of H on YZ), we get:

$$JL=\tfrac{1}{2}(JH+JU) \Longrightarrow \frac{JH}{JU}=2 \cdot \frac{JL}{JU}-1.$$

$\tfrac{JL}{JU}$ is constant and equal to the coefficient of the homothety with center J that takes (K) to $(K_1) \Longrightarrow \tfrac{JH}{JU}$ is constant $\Longrightarrow H$ describes a circle (K_2) homothetic to (K) under a homothety with center $J.$ Therefore, the locus of T is a circle $(K_3),$ image of (K_2) under homothety with center X and coefficient $\tfrac{1}{2} \Longrightarrow$ locus of R is a circle (K_4) image of (K_3) under inversion WRT $(I).$

Quick Reply

High School Olympiads

Midpoint of tangent intersections

Locked

jlammy

#1 Aug 4, 2014, 2:57 am

In $\triangle ABC$, let B', C' be the reflections of B, C in CA, AB respectively. Let the line tangent to circle $AB'C'$ at A intersect $BC, B'C'$ at X, Y. Prove that A is the midpoint of XY.

Luis González

#2 Aug 4, 2014, 3:49 am

This has been discussed at least 3 times before.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=522363
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=568915
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=570916

High School Olympiads

Prove that line PQ bisects diagonal BD

Reply

Source: Saudi Arabia BMO TST Day II Problem 3

TheMaskedMagician

#1 Aug 3, 2014, 10:27 pm

Let $ABCD$ be a parallelogram. A line ℓ intersects lines $AB,\ BC,\ CD,\ DA$ at four different points $E,\ F,\ G,\ H,$ respectively. The circumcircles of triangles AEF and AGH intersect again at P. The circumcircles of triangles CEF and CGH intersect again at Q. Prove that the line PQ bisects the diagonal BD.

Luis González

#2 Aug 4, 2014, 3:33 am

Let $\odot(AGH)$ cut CD again at U and let AU cut CB at V. Using $AD \parallel CB$ and the cyclic $AUGH$, we get $\angle AVC = \angle DAU = \angle HGU \Longrightarrow UVGF$ is cyclic. Therefore, AP, UG and FV are pairwise radical axes of $\odot(AEF)$, $\odot(AGH)$ and $\odot(UVGF)$ concurring at their radical center $C \equiv UG \cap FV$, i.e. $P \in AC$ and analogously we'll have that $Q \in AC \Longrightarrow PQ \equiv AC$ bisects the diagonal BD.

Quick Reply

High School Olympiads

Prove that lines PQ,BC, and MT are concurrent

Reply

Source: Saudi Arabia BMO TST Day III Problem 5

TheMaskedMagician

#1 Aug 3, 2014, 10:43 pm

Let ABC be a triangle. Circle Ω passes through points B and C. Circle ω is tangent internally to Ω and also to sides AB and AC at $T,\ \ P$, and Q, respectively. Let M be midpoint of arc $\widehat{BC}$ (containing T) of Ω. Prove that lines $PQ,\ \ BC$, and MT are concurrent.

Luis González

#2 Aug 3, 2014, 10:59 pm

Proof using Casey's chord theorem:
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=399496

Proof using Sawayama and Pascal theorem:
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=397123

Sardor

#3 Aug 4, 2014, 9:39 am

see also
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=867524&sid=2c302b55efbd9a049d3640d339136c70#p867524

Quick Reply

High School Olympiads

Ratio of angles.

Reply

Source: own

ricarlos

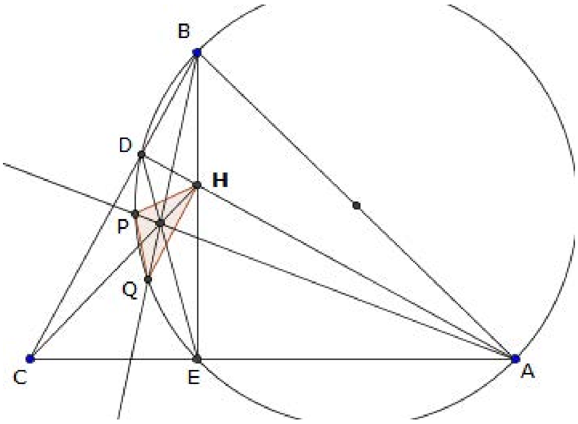
#1 Aug 2, 2014, 11:33 pm • 1

Let ABC an acute triangle with orthocenter H.
Let D, E and Γ, the feet of altitudes from A and B and a circle with diameter AB (center in midpoint of AB), respectively.

$L = CH \cap DE$,
$P = AL \cap \Gamma$,
$Q = BL \cap \Gamma$.

Suppose that the circumcenter of ΔPQH lie on the altitude BE.
Calculate $\frac{\angle ABC}{\angle CAB}$

Attachments:



Luis González

#2 Aug 3, 2014, 4:52 am

From cyclic $CDHE$, we have $LC \cdot LH = LD \cdot LE$, thus the inversion with center L and power equal to the power of L WRT Γ takes P, Q, H, D into A, B, C, E, respectively $\Longrightarrow \odot(PQH)$ goes to the circumcircle $(O) \equiv \odot(ABC)$ and line BE goes to $(J) \equiv \odot(CQLD)$. By conformity, if BE and $\odot(PQH)$ are orthogonal (circumcenter of PQH on BE), then (O) and (J) are orthogonal $\Longrightarrow J$ is on the tangent of (O) at C. In other words, CJ and the perpendicular CO from C to DE are isogonals WRT $\widehat{DCL} \Longrightarrow \widehat{JCB} = \widehat{OCH} \Longrightarrow \widehat{A} = |\widehat{B} - \widehat{A}| \Longrightarrow \widehat{B} = 2 \cdot \widehat{A}$.

ricarlos

#3 Aug 11, 2014, 2:02 am

Let
$\angle CAB = \delta, \angle QBE = \beta, \angle DAB = \alpha$.
$F = AD \cap BQ$,
$G = BE \cap AP$,
O =circumcenter of ΔPQH.

(DH, FA) is a harmonic division and ΔFQA is a right triangle $\rightarrow QB$ is bisector of $\angle DQH$.
$\angle DQB = \angle HQB = \angle DAB = \alpha$ (BDQA is cyclic)

(EH, GB) is a harmonic division and ΔGPB is a right triangle $\rightarrow PA$ is bisector of $\angle EPH$.
$\angle EPA = \angle HPA = \angle EBA = 90 - \delta$ (BPEA is cyclic)

$\angle QPE = \angle QBE = \beta$ (BPQE is cyclic)
$\angle QPH = 180 - 2\delta + \beta$
$\angle QOH = 360 - 2\angle QPH = 360 - 360 + 4\delta - 2\beta$
$\angle QHO = 90 - (\angle QOH)/2 = 90 - 2\delta + \beta$ (1)

$\angle QHO = \alpha + \beta$ (2) ($\angle QHO$ is exterior angle of ΔQHB)

$(1) = (2) = 90 - 2\delta + \beta = \alpha + \beta \to 90 - \alpha = 2\delta$

$\angle ABC = 90 - \alpha = 2\delta$

$\frac{\angle ABC}{\angle CAB} = \frac{2\delta}{\delta} = 2.$

(comentario: no se quien cambio la notacion pero creo que en mi post original $O = CH \cap DE$)

Quick Reply

High School Olympiads

Prove the F,E, X with on the straight line.

Reply

nvthe_cht

#1 Aug 2, 2014, 9:11 am

Let ABC be a traingle, non-isosceles at vertex A. Let O, H denote its circumcenter and orthocenter respectively. The line through A and perpendicular to OH intersects BC at D. Let E be the foot of the perpendicular from B to AC; F be the foot of the perpendicular from C to AB. Let X be midpoint of AD. Prove that: F, E, X with on the straight line.

Luis González

#2 Aug 2, 2014, 10:04 am

If Q,R denote the midpoints of AC,AB, then the problem is equivalent to show that EF,QR and the perpendicular from A to OH are concurrent. This is precisely a problem from a TST Cono Sur (2006) with a generalization for a pair of isogonal conjugates.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=82266
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=383037

Quick Reply

High School Olympiads

Midpoint on Radical Axis

Reply

Source: ELMO 2014 Shortlist G6, by Yang Liu

v_Enhance

#1 Jul 24, 2014, 7:35 pm

Let $ABCD$ be a cyclic quadrilateral with center O.
Suppose the circumcircles of triangles AOB and COD meet again at G, while the circumcircles of triangles AOD and BOC meet again at H.
Let ω_1 denote the circle passing through G as well as the feet of the perpendiculars from G to AB and CD.
Define ω_2 analogously as the circle passing through H and the feet of the perpendiculars from H to BC and DA.
Show that the midpoint of GH lies on the radical axis of ω_1 and ω_2.

Proposed by Yang Liu

XmL

#2 Jul 25, 2014, 6:49 am

Prove that G, H are the poles of the intersections of the opposite sides of the cyclic quad. wrt (O), and then we get a well known orthocenter configuration.

Wolstenholme

#3 Jul 27, 2014, 6:16 am

First, let $E = AB \cap CD$ and $F = BC \cap DA$ and denote the circumcircle of $ABCD$ by ω. Let O_1, O_2 be the centers of ω_1, ω_2 respectively. Let M be the midpoint of GH.

Consider the inversion about ω. It is clear that line AB goes to ω_1 and that line CD goes to ω_2 so E goes to G. Similarly F goes to H. Moreover, ω_1 and ω_2 are the circles with diameters EG and FH respectively. Now since M is the midpoint of GH and since O_1 is the midpoint of GE we have that $O_1M \parallel HE$ and similarly $O_2M \parallel GF$.

But since GF is the polar of E with respect to ω we have that $GF \perp OE$ so $O_2M \perp OE$ which implies $O_2M \perp OO_1$. Similarly $O_1M \perp OO_2$ and so M is the orthocenter of triangle OO_1O_2. This means that $OM \perp O_1O_2$ so to show that M is on the radical axis of ω_1 and ω_2 it suffices to show that O is on this radical axis.

However, this is clear since both ω_1 and ω_2 are orthogonal to ω, so we are done.

Interestingly after doing the inversion step, a complex number solution is somewhat doable as well.

buratinogigle

#4 Jul 27, 2014, 7:58 am • 1

This problem is true for all point P lies on line OQ when Q is intersection of AC and BD.

Luis González

#5 Aug 2, 2014, 7:28 am • 1

Here is a proof of the generalization mentioned by buratinogigle:

Let $X \equiv AD \cap CB$ and $Y \equiv AB \cap CD. \omega_1$ and ω_2 are the circles with diameters $\overline{GY}$ and $\overline{HX}$. AB, CD, PG are pairwise radical axes of $(O), \odot(PAB)$ and $\odot(PCD)$, concurring at their radical center $Y \Longrightarrow YA \cdot YB = YP \cdot YQ \Longrightarrow$ G is on the inverse of OQ under the inversion with center Y and power equal to the power of Y WRT (O); the circle with diameter $\overline{XY} \Longrightarrow \angle XGY = 90^\circ$ and similarly we have $\angle XHY = 90^\circ$. Hence if $R \equiv XG \cap YH$, the radical axis of ω_1, ω_2 is just the line passing through P and the orthocenter Z of of $\triangle RGH$, which bisects $\overline{GH}$ since $PHZG$ is clearly parallelogram.

Quick Reply

High School Olympiads

3 circumcenters collinear

Reply

Source: vankhea

vankhea
#1 Jul 19, 2014, 12:20 pm

Let D,E,F be points on the sides BC,CA,AB such that $DE\perp CA; EF\perp AB; FD\perp BC$. Let P,Q,R be midpoints of DC,EA,FB. Prove that circumcenter of $\Delta ABC,\Delta DEF,\Delta PQR$ collinear.

kaszubki
#2 Jul 20, 2014, 2:12 am

Let's denote centers of these 3 circles by O,X,Y respectively.

I have no idea how to solve this problem without notice that Click to reveal hidden text

Sketch of proof:
Click to reveal hidden text

vankhea
#3 Jul 20, 2014, 8:55 pm

Click to reveal hidden text
Thanks you kaszubki
But if you let Y is midpoint of OX so the problem is O,X,Y collinear without proof.
the spacial of this problem is need to prove that Y is midpoint of OX

mathuz
#4 Jul 21, 2014, 12:06 am

very nice @vankhea!
It's nice problem, i have solution by homothety.
Let the circles $(AFE),(BFD),(CDE)$ intersect at one point M. Then M is Brocard point of the triangles ABC, DEF, PQR.

vankhea
#5 Aug 1, 2014, 5:56 pm

> mathuz wrote:
> very nice @vankhea!
> It's nice problem, i have solution by homothety.
> Let the circles $(AFE),(BFD),(CDE)$ intersect at one point M. Then M is Brocard point of the triangles ABC, DEF, PQR.

can you give me a full solution ?

Luis González
#6 Aug 1, 2014, 11:47 pm

Denote by O,Y and Z the circumcenters of $\triangle ABC,\triangle DEF$ and $\triangle PQR$, resp.

Let J be the Miquel point of DEF WRT $\triangle ABC.$ Since $\angle DEF = \angle CAB, \angle EFD = \angle ABC$ and $\angle FDE = \angle BCA,$ it follows that $\odot(AEF), \odot(BFD), \odot(CDE)$ are tangent to $DE, EF, FD,$ resp $\Longrightarrow J$ is 2nd Brocard point of $\triangle DEF.$ Therefore, $\triangle ABC \cup O$ and $\triangle EFD \cup Y$ are spirally similar with center their common 2nd Brocard point J and rotational angle $\angle AJE = \angle CJD = \angle BJF = 90^\circ \Longrightarrow \angle OJY = 90^\circ.$ Since $\triangle JAQ, \triangle JCP$ and $\triangle JBR$ are similar isosceles triangles with base angle the Brocard angle of $\triangle ABC,$ it follows that $\triangle ABC$ and $\triangle QRP$ are similar with common 2nd Brocard point J and $\triangle JZO \sim \triangle JQA \Longrightarrow ZJ = ZO.$

On the other hand, as D, E, F are the reflections of J on $RP, PQ, QR,$ then Y becomes isogonal conjugate of J WRT $\triangle PQR,$ i.e. 1st Brocard point of $\triangle PQR \Longrightarrow ZJ = ZY.$ Hence, we conclude that Z is circumcenter of the right $\triangle OJY$ $\Longrightarrow Z$ is midpoint of $OY.$

High School Olympiads

Locus of centers.

MexicOMM

#1 Jul 20, 2014, 1:19 am

Let ABC be an acute triangle.
Find the locus of the centers of the rectangles which have their vertices on the sides of ABC.

Luis González

#2 Aug 1, 2014, 7:53 am • 1

Label $PQRS$ the vertices of these rectangles and $K \equiv PR \cap QS$ their centers. There are exactly two possibilities to inscribe $PQRS$ in $\triangle ABC,$ namely two consecutive vertices P,S on BC or two opposite vertices P,R on BC and cyclically for CA and $AB.$ The former case trivially gives the locus of K as the union of the sidelines $BC,CA,AB.$

For the remaining case, animate Q,R on $AB,AC.$ The application sending Q to the midpoint K of PR is an affine homography, even for any fixed directions, not necessarily $QR \parallel BC$ and $QP \perp BC$ (check the general configuration at Beautiful locus 3), therefore K runs on a line. Considering limiting cases $Q \equiv R \equiv A$ and $(Q \equiv B, R \equiv C),$ it follows that this line passes through the midpoint of BC and the midpoint of the A-altitude.

Hence summing up, the locus of K is the union of the 3 sidelines of $\triangle ABC$ and the 3 lines joining the midpoint of each side with the midpoint of its correpnding altitude.

MexicOMM

#3 Aug 1, 2014, 8:10 am

Additional question. Is it possible that one point is the center of 3 rectangles?

Luis González

#4 Aug 1, 2014, 9:22 am • 2

MexicOMM, yes it is the symmedian point L of ABC. This follows from the fact the lines joining the midpoints of the sides with the midpoints of their correpnding altitudes are concurrent.

Let $\triangle A'B'C'$ be the tangential triangle of $\triangle ABC.$ $L \equiv AA' \cap BB' \cap CC'$ is the symmedian point of $\triangle ABC.$ X is the foot of the A-altitude and D,M are the midpoints of $BC,AX,$ resp. DM is the D-median of $\triangle DAX$ and $DA' \parallel AX$ (both perpendicular to BC) $\Longrightarrow D(A,X,M,A')=-1.$ Hence, if AA' cuts DM,BC at $L',U,$ we get $(A,U,L',A')=-1,$ but from the complete quadrilateral $BCB'C',$ we have $(A,U,L,A')=-1 \Longrightarrow L \equiv L',$ i.e. DM passes through $L.$ Similarly the lines joining the midpoints of CA and AB with midpoints of the B- and C- altitude pass through $L.$

utkarshgupta

#5 Jan 20, 2015, 8:39 pm

Canadian MO 2006

Quick Reply

High School Olympiads

Centroid of projection points on the sides of a triangle

Reply

Source: Kürschák 1999, problem 2

randomusername
#1 Jul 15, 2014, 9:32 pm

Given a triangle on the plane, construct inside the triangle the point P for which the centroid of the triangle formed by the three projections of P onto the sides of the triangle happens to be P.

Luis González
#2 Aug 1, 2014, 4:48 am

Overkill: In general, if A', B', C' are the projections of P on $BC, CA, AB,$ then the image of P under the affine homography $\{A'B'C'\} \mapsto \{ABC\}$ is nothing but the isogonal conjugate of P WRT $\triangle ABC.$ Thus, P is centroid of $\triangle A'B'C' \Longleftrightarrow P$ is the isogonal conjugate of the centroid of $\triangle ABC,$ i.e. its symmedian point.

P.S. This is a rather old problem, for other proofs see
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=209057
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=258869

Quick Reply

High School Olympiads

Reflect the vertices of a triangle on the opposite sides

Reply

Source: Kürschák 1985, problem 3

randomusername

#1 Jul 27, 2014, 5:30 pm

We reflected each vertex of a triangle on the opposite side. Prove that the area of the triangle formed by these three reflection points is smaller than the area of the initial triangle multiplied by five.

Luis González

#2 Aug 1, 2014, 2:03 am

Label $\triangle ABC$ the given triangle with circumcircle (O,R) and A',B',C' denote the reflections of A,B,C on BC,CA,AB. $\triangle A'B'C'$ is homothetic to the pedal triangle $\triangle XYZ$ of its 9-point center N through the homothety with center G, the centroid of $\triangle ABC$, and coefficient 4 (this has been posted before) $\Longrightarrow [A'B'C'] = 16[XYZ]$. Hence, it suffices to prove that $16[XYZ] < 5[ABC]$. By Euler's theorem for $\triangle XYZ$, we get

$$\frac{[XYZ]}{[ABC]} = \frac{|p(N,(O))|}{4R^2} \Longrightarrow 4 \cdot |p(N,(O))| < 5R^2.$$

Assume that N lies outside (O), otherwise the inequality is trivial. Then $4(ON^2 - R^2) < 5R^2 \Longrightarrow ON^2 < \frac{9}{4}R^2$. Since $ON = \frac{3}{2}OG$, it follows that $\frac{9}{4}OG^2 < \frac{9}{4}R^2 \Longrightarrow OG < R$, which is certainly true as G is always inside (O).

Quick Reply

High School Olympiads

Cyclic quadrilateral

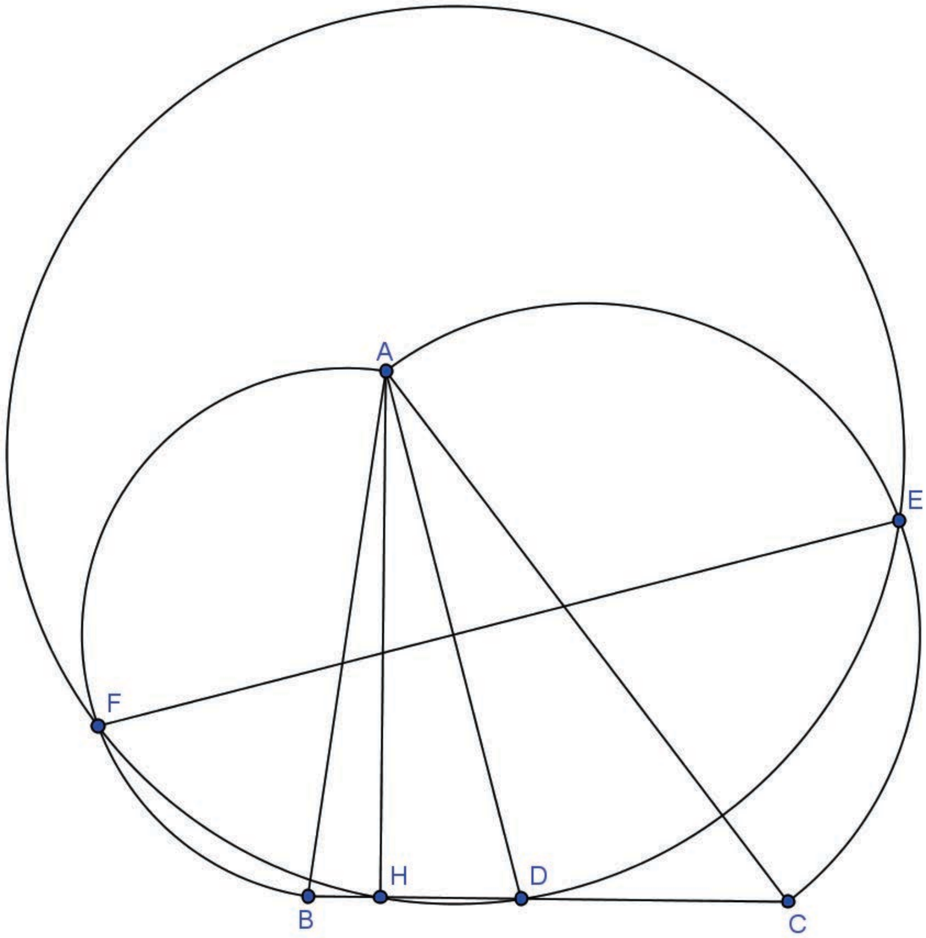
Reply

CTK9CQT

#1 Jul 31, 2014, 7:49 am

Let $\triangle ABC$, AH is an altitude, AD is a bisector. Perpendicular bisector of AD cut the semicircles with diameter AB, AC at F, E outside the triangle, respectively. Prove that $EFHD$ is concyclic.

Attachments:



Luis González

#2 Jul 31, 2014, 12:21 pm

Let Q, R be the midpoints of AC, AB. $M \equiv AD \cap QR$ is the midpoint of AD. By angle bisector theorem, we have $MQ : MR = QA : RA \Longrightarrow M$ is the insimilicenter of the circles $(Q), (R)$ with diameters AC, AB. Hence, if L is the second intersection of AD with (R), we have $AE \parallel FL \Longrightarrow \angle EAM = \angle FLA = \angle FBA$. Since $\angle AFB = \angle AME = 90^\circ$, then $\angle AEF = \angle FAB \Longrightarrow \angle DEF = \angle AEF = \angle FAB = \angle FHB \Longrightarrow EFHD$ is cyclic, as desired.

jayme

#3 Jul 31, 2014, 1:02 pm

Dear Mathlinkers,
see also
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=545791
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Mixtilinear Collinearity

Source: ELMO 2014 Shortlist G7, by Robin Park

v_Enhance

#1 Jul 24, 2014, 7:49 pm

Let ABC be a triangle inscribed in circle ω with center O; let ω_A be its A-mixtilinear incircle, ω_B be its B-mixtilinear incircle, ω_C be its C-mixtilinear incircle, and X be the radical center of $\omega_A, \omega_B, \omega_C$. Let A', B', C' be the points at which ω_A, ω_B, ω_C are tangent to ω. Prove that AA', BB', CC' and OX are concurrent.

Proposed by Robin Park

Luis González

#2 Jul 25, 2014, 4:41 am

Let (I) be the incircle of $\triangle ABC.$ A is the exsimilicenter of $(I) \sim \omega_A$ and A' is the exsimilicenter of $(O) \sim \omega_A,$ thus by Monge and d'Alembert theorem, AA' hits OI at the exsimilicenter of $(O) \sim (I)$ and similarly BB', CC', i.e. AA', BB', CC' concur on $OI.$ As for the radical center X of $\omega_A, \omega_B, \omega_C,$ it is the midpoint X_{999} between I and the homothety center X_{57} of the intouch and excentral triangle, lying on OI (for a synthetic solution see the problem mixtilinear incircles).

High School Olympiads

Collinear Points [Conjecture]

Reply

Source: Hyacinthos #22478

rodinos

#1 Jun 30, 2014, 10:52 pm

Let ABC be a triangle and I its incenter.

Let La, Lb, Lc be the Euler lines and Na, Nb,Nc the Nine Point Circle Centers of IBC,ICA,IAB, resp.

The perpendicular to La at Na intersects BC at Aa. Similarly, the perpendicular to Lb at Nb intersects CA at Bb and the perpendicular to Lc at Nc intersects AB at Cc.

Conjecture: The points Aa, Bb, Cc are collinear.

aph

Luis González

#2 Jul 15, 2014, 3:11 am • 2

Let $\triangle I_aI_bI_c$ be the excentral triangle of $\triangle ABC.$ Common circumcenter O_a of $\triangle IBC$ and $\triangle I_aBC$ is the midpoint of the arc BC of $(O) \equiv \odot(ABC)$ and the orthocenter H_a of $\triangle IBC$ is the reflection of I_a on the midpoint M of $BC,$ due to the parallelogram $BH_aCI_a.$ Thus if D is the reflection of O_a on $M,$ by symmetry, it follows that the perpendicular bisector of I_aD cuts BC at the reflection P_a of A_a on $M \Longrightarrow \frac{A_aB}{A_aC}=\frac{P_aC}{P_aB}.$ P_b and P_c are defined cyclically $\Longrightarrow$

$$\frac{A_aB}{A_aC} \cdot \frac{B_bC}{B_bA} \cdot \frac{C_bA}{C_cB}=\frac{P_aC}{P_aB} \cdot \frac{P_bA}{P_bC} \cdot \frac{P_cB}{P_cA}. \ (1)$$

Perpendiculars to OI_a,OI_b,OI_c at O cut $I_bI_c,I_cI_a,I_aI_b,$ resp, at 3 collinear points X,Y,Z on orthotransversal of O WRT $\triangle I_aI_bI_c.$ Since the midpoint of I_aD is the 9-point center of $\triangle I_aBC,$ then we deduce that $\triangle I_aBC \cup P_a \sim \triangle I_aI_bI_c \cup X$ $\Longrightarrow \frac{P_aC}{P_aB}=\frac{XI_c}{XI_b}.$ Multiplying the cyclic expressions together gives

$$\frac{P_aC}{P_aB} \cdot \frac{P_bA}{P_bC} \cdot \frac{P_cB}{P_cA}=\frac{XI_c}{XI_b} \cdot \frac{YI_a}{YI_c} \cdot \frac{ZI_b}{ZI_a}=1. \ (2)$$

Thus, from (1) and $(2),$ we get $\frac{A_aB}{A_aC} \cdot \frac{B_bC}{B_bA} \cdot \frac{C_bA}{C_cB}=1,$ which means that $A_a,$ B_b and C_c are collinear.

buratinogigle

#3 Jul 15, 2014, 9:09 am • 1

Here is another conjecture

Let ABC be a triangle with circumcircle (O). P is a point such that Euler lines of triangle PBC,PCA,PAB are concurrent. Let PA,PB,PC cuts (O) again at D,E,F. Let X,Y,Z be isogonal conjugates of D,E,F with respect to triangle PBC,PCA,PAB, respectively. Prove that perpendicular bisectors of DX,EY,FZ intersect BC,CA,AB, respectively, on three collinear points.

Quick Reply

High School Olympiads

Collinearity Problem

Reply

Source: ?

RocketSingh

#1 Jun 17, 2014, 4:44 am

Let ABC be a triangle. Let the incircle of the triangle be I and the incircle contact sides AB,BC and CA at F,D and E respectively. Let X be the intersection of the Euler line of DEF with EB. Let Y be the foot of the altitude dropped from F to DE. Let Z be the intersection of FC and ID. Show that X,Y and Z are collinear.

Luis González

#2 Jul 13, 2014, 9:23 pm

Let T and G_e denote the orthocenter and symmedian point of $\triangle DEF$ (Gergonne point of ABC). Isogonal conjugate of G_e WRT $\triangle DEF$ is its centroid lying on Euler line $IT \Longrightarrow D, E, F, I, T, G_e$ lie on a same conic, isogonal conjugate of IT WRT $\triangle DEF$. Now, by Pascal theorem for hexagon G_eEDITF, the intersections $X \equiv EG_e \equiv EB \cap IT, Y \equiv ED \cap TF$ and $Z \equiv ID \cap FG_e \equiv FC$ are collinear.

mathuz

#3 Jul 13, 2014, 11:10 pm

I have another way. This problem is equivalent to :

Given a triangle ABC with H, O are orthocenter and circumcenter. The tangent line to (O) at the point C intersects with the tangent lines to (O) at the points A, B at Q, P. Let $BQ \cap OH = Y$, $AP \cap OC = Z$ and $AH \cap BC = X$. Then the points X, Y, Z are collinear.

The solution will be short and synthetic. Use by

$$\frac{HY}{OY} = \frac{BH}{OQ}$$

and Menelaus's theorem. :)

IDMasterz

#4 Jul 13, 2014, 11:28 pm

^^Isogonal conjugate of a line is a conic, thats all Luis has done, then used pascal theorem :) It is still synthetic in my opinion.

XmL

#5 Jul 14, 2014, 12:59 am

Let $L \in DE$ such that $FL \parallel BC$, Since $DI \perp (BC \parallel FL)$, hence $FY \cap DI = K$ is the orthocenter of $DLF \Rightarrow LK \parallel BI \parallel EH$ where H is the othocenter of DEF. Clearly FLZ, CBI are homothetic, hence $L = BZ \cap DE$, now apply Desarge's theorem on EHY, BZI to get ZY, IH, EB are concurrent and we are done.

Quick Reply

High School Olympiads

Nice perpendicular

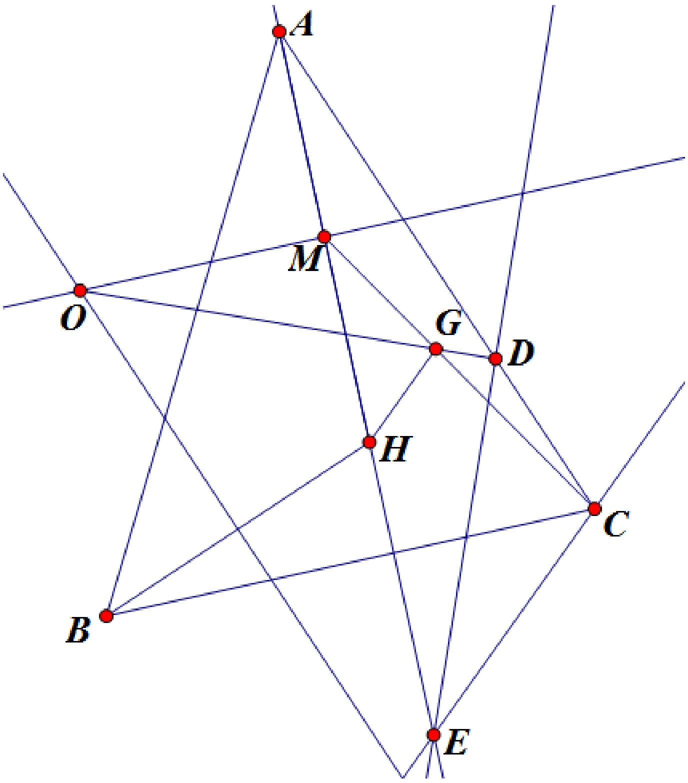
Reply

Source: China

thrive_vn
#1 Jul 12, 2014, 8:31 pm • 1

Dear mathlinkers,
1.Let ABC be a triangle with H is its orthocenter and $BH \cap AC \equiv D$.
2.O is the circumcenter of $\triangle ABH$,M is midpoint of AH.
3.$OD \cap CM \equiv G$,$E \in AH, CE \parallel HG$.
Prove that $OD \perp DE$.
Proposed by PHCP
Click to reveal hidden text

Attachments:



Luis González
#2 Jul 13, 2014, 8:53 am • 1

Since $\angle OHM = 90^\circ - \angle ABH = 90^\circ - \angle ACH = \angle CHD$ and $\angle OMH = \angle CDH = 90^\circ$, then by Jacobi's theorem in the $\triangle HMD$, it follows that DO, MC and its H-altitude concur, i.e $HG \perp MD \Longrightarrow EC$ is perpendicular to MD at P. Thus, if $F \equiv AH \cap BC$, then $FMPC$ is cyclic $\Longrightarrow EF \cdot EM = EC \cdot EP$, i.e. E has equal power WRT the circle (S) with diameter $\overline{CD}$ and the 9-point circle $(N) \equiv \odot(MDF)$ of $\triangle ABC \Longrightarrow DE$ is radical axis of (N) and (S) $\Longrightarrow DE \perp NS$. But since N is midpoint of $\overline{CO}$, then NS is C-midline of $\triangle COD \Longrightarrow NS \parallel OD \Longrightarrow OD \perp DE$.

XmL
#3 Jul 13, 2014, 9:47 am • 1

Lemma: ABC, ADE are two inversely similar right triangles with right angles at B, D. If F is on AD such that $CF \perp BD$, then F, E, B, D are concyclic.

Proof: Let X, Y be the feet of perpendiculars drawn from E, C on to BD. It's well known that $DX = BY$, hence the reflection of E,denoted by E' over the perpendicular bisector of BD lies on FY. Since $\angle E'BY = \angle EDX = \angle DFY$, hence B, D, F, E' are concyclic so F, E, B, D, E' are concyclic.

Original problem:

Notice the two inversely similar right triangles HDC, HMO with right angles at D, M. It's well known that $HG \perp MD$(Jacobi's). Hence $CE \perp MD$, since $E \in MH$ using our lemma gives E, D, M, O are concyclic$\Rightarrow$ $\angle ODE = \angle OME = 90$ and we done.

Btw I used to post problems composed by PHCP too. He's a really nice guy.

Quick Reply

High School Olympiads

A line, a median and a bisector have a common point

Source: Tuymaada 2014, Day 2, Problem 3, Senior League

Aiscrim
#1 Jul 12, 2014, 3:18 pm

A parallelogram $ABCD$ is given. The excircle of triangle $\triangle ABC$ touches the sides AB at L and the extension of BC at K. The line DK meets the diagonal AC at point X; the line BX meets the median CC_1 of trianlge $\triangle ABC$ at Y. Prove that the line YL, median BB_1 of triangle $\triangle ABC$ and its bisector CC' have a common point.

(A. Golovanov)

IDMasterz
#2 Jul 12, 2014, 11:35 pm

Lemma ABC be a triangle, C excircle be C and tangency points be A', B', C' with centre I_C, radius r. The C median cuts $A'B'$ at P. Incentral triangle of ABC be $A_1B_1C_1$ and orthic triangle be $A_2B_2C_2$. Then, $PC_1 \parallel CC'$, so that $\frac{r}{CC_2} = \frac{I_CP}{I_CC'}$

Proof Let the intouch triangle be $A_3B_3C_3$. Note $C_3I \cap \odot A_3B_3C_3 \in CC'$, so it follows $CC' \parallel IM$, where M is the midpoint of AB and I is the incentre, as the gergonne and nagel points are isotomic conjugates. Now, also, P is the pole of the parallel through C to AB, so it lies on I_CC'. Hence, it suffice to show $\frac{MC'}{C'C_2} = \frac{C_1C_3}{C_1C_2}$. Now note $(I_C, I; C_1, C) = -1$ so dropping perpendiculars is follows $(C_2, C_1; C_3, C') = -1$. So, $(MC')^2 = MC_1 \cdot MC_2 \implies \frac{MC_2}{MC'} = \frac{MC'}{MC_1} \implies \frac{C'C_2}{MC_3} = \frac{C'C_1}{MC_1} = \frac{C_1C_2}{C_1C_3}$, so take inverse we get what we want.

Main Proof We must show BYC and $I_CC'D$ are perspective. This means that if $I_CC' \cap CY = T,\ YB \cap C'D = U,\ CB \cap I_CD = V$, then T, U, V are collinear by Desagues theorem. Let M_B be the midpoint of AC. The cross ratio (AM_B, XC), by taking persepctive from D, is $\frac{BC}{BA'} = \frac{a}{s-a}$, so if $Z = BC \cap LD$ then:

$$\frac{UC'}{UD} \cdot \frac{YD}{YC'} = \frac{UC'}{UD} \cdot \frac{c}{s-a} = \frac{a}{s-a} \implies \frac{UC'}{UD} = \frac{a}{c}$$

Now, height from D to BC is equal to height from A to BC so if r is the C-exradius, then $\frac{DV}{DI_C} = \frac{bc}{2Rr}$ where R is the radius of $\odot ABC$. By the lemma, we have $\frac{I_CP}{I_CC'} = \frac{2Rr}{ab}$ so:

$$\frac{UC'}{UD} \cdot \frac{I_CP}{I_CC'} \cdot \frac{I_CP}{I_CC'} = \frac{a}{c} \cdot \frac{2Rr}{ab} \cdot \frac{bc}{2Rr} = 1$$

Therefore TUV is a line by Menelaus theorem.

Luis González
#3 Jul 13, 2014, 12:10 am • 1

Denote $BC = a, CA = b, AB = c, s = \frac{1}{2}(a+b+c)$ (b>a). Let CC', CC_1, CL cut AD at E, F, G, resp. Clearly

$AE = b$ and $AF = a \Longrightarrow FE = b - a$ and $\triangle LBC \sim \triangle LAG$ gives

$\dfrac{GA}{a} = \dfrac{s-b}{s-a} \Longrightarrow FG = a - a \cdot \dfrac{s-b}{s-a} = \dfrac{a(b-a)}{s-a} \Longrightarrow$

$\dfrac{FE}{FG} = (b-a) \cdot \dfrac{s-a}{a(b-a)} = \dfrac{s-a}{a} = \dfrac{BK}{BC}$

$\Longrightarrow D(C,B,K,A) = C(G,F,E,B) \Longrightarrow D(C,B_1,X,A) = B(C,B_1,X,A) = C(B,C',C_1,L) \Longrightarrow$
$CC' \cap BB_1, Y \equiv BX \cap CC_1$ and L are collinear, i.e. YL, BB_1 and CC' concur.

bonciocatciprian

#4 Jul 13, 2014, 2:11 am

We solve the problem using barycentric coordinates:

We have $A = (1,0,0)$; $B = (0,1,0)$; $C = (0,0,1)$. Since $ABCD$ is a parallelogram, we get

$$D = A + C - B = (1,-1,1)$$

Because $BK = BL = p - a$, we have

$$K = (0 : p : a - p) L = (p - a : p - b : 0)$$

In order to find out the coordinates of X, we have to deduce the equation of the line DK:

$$D \in DK \Rightarrow u - v + w = 0 \ (1) K \in DK \Rightarrow vp + w(a-p) = 0 \ (2)$$

From (1) and (2) the equation follows:

$$ax + (a-p)y - pz = 0$$

Now, since the equation of the AC is obviously $y = 0$, the coordinates of X are:

$$X = (p : 0 : a)$$

Looking for the coordinates of Y, we write the equations of the lines XB and CC_1:

$$XB : \frac{x}{z} = \frac{p}{a} \ (3) CC_1 : x = y \ (4)$$

From (3) and (4), we obtain the coordinates of Y:

$$Y = (p : p : a)$$

Let $\{T\} = \text{bis } (\angle ACB) \cap BB_1$. We must show that $T \in LY$. Since the equation of bis $(\angle ACB)$ is $\dfrac{x}{y} = \dfrac{a}{b}$, and the equation of BB_1 is $x = z$, we get that

$$T = (a : b : a)$$

In order to find out the equation of LY, we must solve the system

$$Y \in LY \Rightarrow up + vp + wa \ (5) L \in LY \Rightarrow u(p-a) + v(p-b) = 0 \ (6)$$

After some computations, one can find:

$$LY : a(p-b)x + a(a-p)y + p(b-a)z = 0$$

Plugging in the coordinates of T, we finally get, after some other calculations, that $T \in LY$, namely the three lines are concurrent. $\square$

Quick Reply

High School Olympiads

equal angles in convex quadrilateral

Locked

Source: old but not easy

Sardor

#1 Jul 11, 2014, 2:28 am

Let $ABCD$ be a convex quadrilateral with AB not parallel to CD, let X be a point inside $ABCD$ such that $\angle ADX = \angle BCX < 90$ and $\angle DAX = \angle CBX$.If Y is the point of intersection of the perpendicular bisectors of AB and CD , prove that $\angle AYB = 2\angle ADX$.

Luis González

#2 Jul 12, 2014, 9:13 am

Do not repost shortlisted problems; go to the contest section if you already know the source. This is IMOSL 2000 G6.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=59215&
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=61826
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=557966 (post #2)

High School Olympiads

Concyclic circumcentres and orthocentres

Reply

Source: Indian TST Day 3 Problem 1

hajimbrak
#1 Jul 11, 2014, 3:23 pm

In a triangle ABC, with $AB \neq AC$ and $A \neq 60^0, 120^0$, D is a point on line AC different from C. Suppose that the circumcentres and orthocentres of triangles ABC and ABD lie on a circle. Prove that $\angle ABD = \angle ACB$.

Luis González
#2 Jul 12, 2014, 1:49 am

Let O, H denote the circumcenter and orthocenter of $\triangle ABC$ and O_1, H_1 denote the circumcenter and orthocenter of $\triangle ABD$. Since $\frac{AH}{AO} = \frac{AH_1}{AO_1} = 2\cos \widehat{A}$ and AO, AO_1 are the isogonals of AH, AH_1 WRT $\widehat{BAC}$, we deduce that $\triangle AHH_1$ and $\triangle AOO_1$ are inversely similar by SAS. Hence if $M \equiv BH \cap AO$ and $N \equiv OO_1 \cap AH$, then $\triangle AHM$ and $\triangle AON$ are inversely similar with corresponing cevians AH_1 and AO_1 or $OHNM$ is cyclic and $\overline{O_1O} : \overline{O_1N} = \overline{H_1H} : \overline{H_1M}$. Thus if $P \equiv ON \cap HM$, then all $\odot(PO_1H_1)$ go through the center K of the spiral similarity that swaps ON and HM; 2nd intersection of $\odot(PMN)$ and $\odot(POH)$.

If OH_1O_1H is cyclic $\Longrightarrow O_1H_1, KP, OH$ and MN concur at the common radical center Q of $\odot(OH_1O_1H)$, $\odot(PO_1H_1), \odot(POH)$ and $\odot(OHNM)$, i.e. O_1H_1 must pass through Q. But as $O_1 \mapsto H_1$ is an affine homography between ON and HM, that only happens at $O_1H_1 \equiv NM, O_1H_1 \equiv OH$ and nowhere else. The former case occurs when $D \equiv C$ (contradiction) and the 1st case clearly happens when BD is antiparallel to BC WRT AB, AC and the conclusion follows.

mathuz
#3 Jul 16, 2014, 4:15 pm

let O, H are circumcenter and orthocenter of the triangle ABC, anologuosly $O', H' \in \triangle ABD$.

Let $AO \cap BH = Z$, $AH \cap OO' = Y$ and $BH \cap OO' = X$. Suppose that $AB < AC$ and $H \in BZ$. Obviously, $XH' > HH'$.

If $Z \in HH'$ then we have that $OYHZ$ is cyclic and obviously, $O' \in OY$. Then $2\angle ADB = \angle AO'B < \angle AYB = 2\angle B$, $180^\circ - \angle ABD = \angle AH'D > \angle A + \angle AZH' = \angle A + \angle B$. So $\angle C > \angle ABD$ and $\angle B > \angle ADB$. Contradiction.

If $H' \in HZ$ then $Y \in OO'$. Again, a contradiction.

polya78
#4 Sep 23, 2014, 12:46 am

Let O, H, O', H' be the relevant points, W its circumcircle and l the angle bisector of angle A. Let X be the reflection of W about l, Y the dilation of X about A with factor AO/AH and Z the reflection of Y in l. Then O, O' are the two points of intersection of W, Y. The two points of intersection of X, Z are reflections of O, O' in l, as well as their inverses with respect to a circle centered at A (in which X, Y are inverses, as are W, Z). Thus AO, AO' are reflections in l of each other, and the desired result follows easily.

Quick Reply

High School Olympiads

Line joining incentre and midpoint perpendicular to a line

Reply

Source: Indian TST Day 2 Problem 1

hajimbrak

#1 Jul 11, 2014, 3:18 pm

In a triangle ABC, let I be its incenter; Q the point at which the incircle touches the line AC; E the midpoint of AC and K the orthocenter of triangle BIC. Prove that the line KQ is perpendicular to the line IE.

Luis González

#2 Jul 11, 2014, 9:31 pm • 1

Let F be the midpoint of AB and let (I) touch BC at P. If BI cuts EF at L, we have $\widehat{FLB} = \widehat{LBC} = \widehat{FBL} \Longrightarrow \triangle FBL$ is F-isosceles $\Longrightarrow FA = FB = FL \Longrightarrow \widehat{ALB} = 90^\circ \Longrightarrow AQLI$ is cyclic $\Longrightarrow \widehat{CQL} = \widehat{AIL} = 90^\circ - \frac{1}{2}\widehat{ACB} = \widehat{CQP} \Longrightarrow L \in PQ$, i.e. L is on the polar of C WRT $(I) \Longrightarrow CK \perp IL$ is the polar of L WRT $(I) \Longrightarrow ELF \perp IK$ is the polar of K WRT $(I) \Longrightarrow KQ$ is the polar of E WRT $(I) \Longrightarrow KQ \perp IE$, as desired.

mathuz

#3 Jul 13, 2014, 10:11 pm • 1

let P is tangency point of B-excircle to AC. Then $\angle KIQ = \angle C = \angle BCP$ and

$$\frac{BC}{CP} = \frac{KI}{IQ}.$$

So $\triangle CBP \sim \triangle QKI$ and $\angle IKQ = \angle CBP \Rightarrow KQ \perp BP$.

We have that $BP \parallel IE$, now $KQ \perp IE$.

sayantanchakraborty

#4 Jul 13, 2014, 11:19 pm

This problem can also be solved using barycentric coordinates:using the fact that $I = (a, b, c)$; $I_a = (-a, b, c)$ we see that the circumcenter of $\triangle BIC$ has coordinates $\frac{I + I_a}{2}$.Then $K = B + C + I - 2O$ where $B = (0, 1, 0)$, $C = (0, 0, 1)$ and O is the circumcenter of $\triangle BIC$.Also note that $Q = (s - c, 0, s - b)$; $E = (1, 0, 1)$.Thus the coordinates of points K, Q, I, E are known to us,and all that is required is testing the criteria for $KQ \perp IE$.
(I have wrtten all the coordinates in normalized form)

This post has been edited 1 time. Last edited by sayantanchakraborty, Nov 9, 2014, 10:16 pm

IDMasterz

#5 Jul 14, 2014, 11:04 am

Since I solved this as I woke up, Ill make it intuitive.

So, we want the polar of K to contain E, equivalent is that its inverse K' lies on the midline. If ID intersects the parallel to BC at A to make P, then we must show K' is the midpoint of PD. P_∞ be the point at infinity on $ID \implies (P, D; E, P_\infty) = -1$ or $(P', D; E, I) = -1$. Obviously, $P' = ID \cap EF$, which means if the perpendiculars from B to IC and C to IB are X, Y, then $XY \in EF$. But every good geometer should have in his repertoire that, if $IC \cap EF = X'$, then $CX'F \sim CIB \implies CX'B \sim CFI \implies X' = X$. Similarly for Y, so we are done.

Ashutoshmaths

#6 Jul 25, 2014, 9:06 pm • 1

my solution

sayantanchakraborty

#7 Nov 9, 2014, 10:14 pm

I have another geometrical approach to this.

The circumdiameter of $\triangle BKC$ is $\frac{a}{cos\frac{A}{2}}$ and consequently $KI = atan\frac{A}{2}$.Thus $\frac{KI}{IQ} = \frac{atan\frac{A}{2}}{r} = \frac{tan\frac{A}{2}}{tan\frac{B}{2}} + \frac{tan\frac{A}{2}}{tan\frac{C}{2}} = \frac{s-b}{s-a} + \frac{s-c}{s-a} = \frac{a}{s-a}$.

Now let $AI \cap BC = Y$ and $IE \cap BC = X$.Applying Menelaus' theorem in $\triangle AYC$ with transversal XIE we get $\frac{CX}{YX} \cdot \frac{YI}{AI} \cdot \frac{AE}{CE} = 1$

$$\implies \frac{YX}{CX} = \frac{YI}{AI} = \frac{BY}{AB} = \frac{a}{b+c}$$

$$\implies \frac{CY}{CX} = \frac{b+c-a}{b+c}$$

$$\implies CX = \frac{ab}{b+c-a}$$

So $\frac{CX}{CE} = \frac{2a}{b+c-a} = \frac{a}{s-a} = \frac{KI}{IQ}$.Also note that $\angle KIQ = \angle BCE = \angle C$.Thus $\triangle XCE \sim \triangle KIQ$ and the result is immediate.

utkarshgupta

#8 Nov 10, 2014, 4:51 pm

What do you think will be the rating of this problem on ISL scale

Like G1, G2 or what ?

IDMasterz

#9 Nov 10, 2014, 8:50 pm

From what I have seen recently, probably G4 or G3 :)

jayme

#10 Jun 21, 2015, 2:07 pm

Dear Mathlinkers,
a proof involving only two harmonic pencils with three couples of perpendicular rays leeds to the result.

Sincerely
Jean-Louis

MillenniumFalcon

#11 Jun 22, 2015, 9:44 am

> mathuz wrote:
>
> $$\frac{BC}{CP} = \frac{KI}{IQ}.$$

Can you explain more on this step? I do not get it.

anantmudgal09

#12 Sep 17, 2015, 8:51 am

Finally bashed it...
Take in-circle to be the unit circle and then note that
$K \longrightarrow k = \dfrac{2d^2(q+f)}{(q+d)(f+d)}$
Now its simple since we know
$E \longrightarrow e = \dfrac{qf}{q+f} + \dfrac{qd}{q+d}$ and so
$z = \dfrac{e-k}{q-i}$ and we get
$z = \dfrac{(q+f)(q-d)}{q(d+f)}$(after some smart manipulations)
which is clearly imaginary after conjugating. :)
(Note:- $I, F \longrightarrow i, f$ where F is the in-touch point on AB respectively.)

This post has been edited 1 time. Last edited by anantmudgal09, Sep 17, 2015, 8:53 am

Dukejukem

#13 Sep 17, 2015, 9:00 am

Just a note: Using the well-known Lemma 8 here, one can show that the polar of K w.r.t. the incircle is the A-midline l of $\triangle ABC.$ Consequently, $E \equiv l \cap AC$ is the intersection of the polars of K and Q, which is precisely the pole of $KQ.$ The desired result follows.

kapilpavase

#14 Sep 20, 2015, 12:41 pm

Let QI intersect the incircle at X.Let BX further intersect the circle at Y.It is well known that $BY \parallel IE$.Let incircle touch BC at P .As $QY \perp XY$ it is sufficient to prove that $PI \cap YQ = K'$ is indeed K.But this is very easy,as we get $K'YPB$ to be cyclic,so $\angle BKY = \angle YPC = \angle PQY$ which further implies $K'B \parallel PQ$.Now as $CI \perp PQ, \Rightarrow CI \perp K'B$ and indeed K' is the orthocentre of BIC,which is K.QED :)

This post has been edited 2 times. Last edited by kapilpavase, Sep 20, 2015, 12:43 pm

utkarshgupta

#15 Mar 5, 2016, 10:45 pm

Coordinate :P

Let P, Q, R be the A, B, C intouch points.
Let Q' be the reflection of Q in I.

It is well known that $IE||BQ'$
So showing
Let the incentre I be $(0,0)$ and P be $(0,-1)$
Let $Q = (\cos(x), \sin(x))$ and $R = (\cos(y), \sin(y))$

Then it is obvious that $C = (\dfrac{1+\sin x}{\cos x}, -1)$ and $B = (\dfrac{1+\sin y}{\cos y}, -1)$

Since $IP \perp BC, K \in IP$
That is the $x-$coordinate of K is zero.
Let K be $(0, m)$
Since K is the orthocentre $KB \perp IC$,

$$(\frac{m+1}{-\frac{1+\sin y}{\cos y}}) \cdot (\frac{-1}{\frac{1+\sin x}{\cos x}}) = -1$$

This yields,

$$m = -(\frac{(1+\sin x)(1+\sin y)}{\cos x \cos y} + 1)$$

To show now that $KQ \perp BQ'$ is just easy computation.

This post has been edited 2 times. Last edited by utkarshgupta, Mar 5, 2016, 10:47 pm

wwwrqnojcm

#17 Apr 3, 2016, 11:56 pm

> *Luis González wrote:*
>
> Let F be the midpoint of AB and let (I) touch BC at P. If BI cuts EF at L, we have $\widehat{FLB} = \widehat{LBC} = \widehat{FBL} \Longrightarrow \triangle FBL$ is F-isosceles $\Longrightarrow FA = FB = FL \Longrightarrow \widehat{ALB} = 90^\circ \Longrightarrow AQLI$ is cyclic $\Longrightarrow \widehat{CQL} = \widehat{AIL} = 90^\circ - \frac{1}{2}\widehat{ACB} = \widehat{CQP} \Longrightarrow L \in PQ$, i.e. L is on the polar of C WRT $(I) \Longrightarrow CK \perp IL$ is the polar of L WRT $(I) \Longrightarrow ELF \perp IK$ is the polar of K WRT $(I) \Longrightarrow KQ$ is the polar of E WRT $(I) \Longrightarrow KQ \perp IE$, as desired.

Could someone specify how the last step is reached when we already have QLP collinear and two perpendicular relationship?

wwwrqnojcm

#20 Apr 6, 2016, 9:00 am

\bmp.....

Quick Reply

High School Olympiads

Nine points circle is midcircle

Reply

Source: Own

buratinogigle

#1 Jun 26, 2014, 10:48 am • 1

Let $ABCD$ be cyclic quadrilateral with circumcircle (O). AD cuts BC at E. AB cuts CD at F. AC cuts BD at G. Prove that nine points circle of triangle GEF passes through centroid of $ABCD$ and it is midcircle of (O) and cirmcumcirle of triangle GEF.

Note. Let (K) and (L) be two circles then midcircle of (K) and (L) is circle center at midpoint of KL and passing through intersections of (K) and (L).

Luis González

#2 Jul 11, 2014, 1:19 am • 2

Let P,Q,R,S be the midpoints of AB,BC,CD,DA, resp. $K\equiv PR\cap QS$ is the centroid of $\{A,B,C,D\}$. P,Q,R,S,E,F,G lie on a same conic $\mathcal{C}$: the 9-point conic of $ABCD$, whose center is K (due to the parallelogram PQRS). This is the locus of the centers of all conics through A,B,C,D, for a proof see About Nine-point Conic (post #7 for a synthetic solution), thus particularly $O\in\mathcal{C}$. By Brokart's theorem, O is orthocenter of $\triangle GEF$, hence $\mathcal{C}$ is a rectangular circum-hyperbola of $\triangle GEF \Longrightarrow$ its center K is on 9-point circle of $\triangle GEF$.

As $\triangle GEF$ is self-polar WRT (O), then $\{E,F\},\{F,G\}$ and $\{G,E\}$ are conjugate points WRT (O), i.e. circles with diameters $\overline{EF},\overline{FG}$ and $\overline{GE}$ are orthogonal to (O). Hence, if M,N,L are the midpoints of EF,FG,GE, we have $\overline{ME}\cdot\overline{MF}=-p(M,(O))$ and similarly for N,L. In other words, powers of M,N,L WRT (O) and $\odot(GEF)$ are in the same ratio -1 $\Longrightarrow$ 9-point circle $\odot(MNL)$ of $\triangle GEF$ is the midcircle of $(O),\odot(GEF)$.

Quick Reply

High School Olympiads

Locus and Envelope (Concentric Circles)

Reply

Source: Hyacinthos #22512

rodinos

#1 Jul 10, 2014, 3:15 pm

Let (K) be a given circle and a,b,c three given lines passing
through the center K of the circle.
Let L be a line tangent to (K) and La,Lb,Lc its reflections
in a,b,c, resp. and A'B'C' the triangle bounded byLa,Lb,Lc.

Which is the locus of the circumcenters of A'B'C' and which is
the envelope of the circumcircle of A'B'C' as L moves being tangent
to (K)?

Applications to triangle:

1. (K) = the incircle (I) and a,b,c = AI,BI,CI, resp.

2. (K) = the NPC (N) snd a,b,c = AN.BN,CN, resp.

APH

Luis González

#2 Jul 10, 2014, 8:45 pm • 1

Clearly the reflections of L on a,b,c are tangent to $(K) \Longrightarrow$ all $\triangle A'B'C'$ have incircle (K,r). If L cuts a,b at $X,Y,$ then K becomes C'-excenter of $\triangle XYC' \Longrightarrow \angle XKY=90^\circ-\frac{1}{2}\angle A'C'B'$. Since $\angle XKY$ is constant, then $\angle A'C'B'$ is constant and similarly $\angle B'A'C'$ and $\angle C'B'A'$ are constant $\Longrightarrow$ all $\triangle A'B'C'$ remain congruent $\Longrightarrow$ their circumcircles (O) have constant radius R. Thus, by Euler's theorem $KO=\sqrt{R^2-2Rr}=\text{const} \Longrightarrow$ locus of the circumcenter O of $\triangle A'B'C'$ is the circumference with center K and radius $\sqrt{R^2-2Rr}$ and the envelope of (O) is the union of the circumferences with center K and radius $R \pm \sqrt{R^2-2Rr}$.

Quick Reply

High School Olympiads

Geometric Construction Problem

Reply

rodinos

#1 Jul 9, 2014, 3:47 pm • 1

To find the intersection of two non-parallel line segments without extending them.

https://www.facebook.com/photo.php?fbid=670777126331720&set=a.237986886277415.57448.100001983178784&type=1&theater

aph

ThirdTimeLucky

#2 Jul 10, 2014, 1:58 am • 1

I don't know whether this is the intended solution, but here is a try.

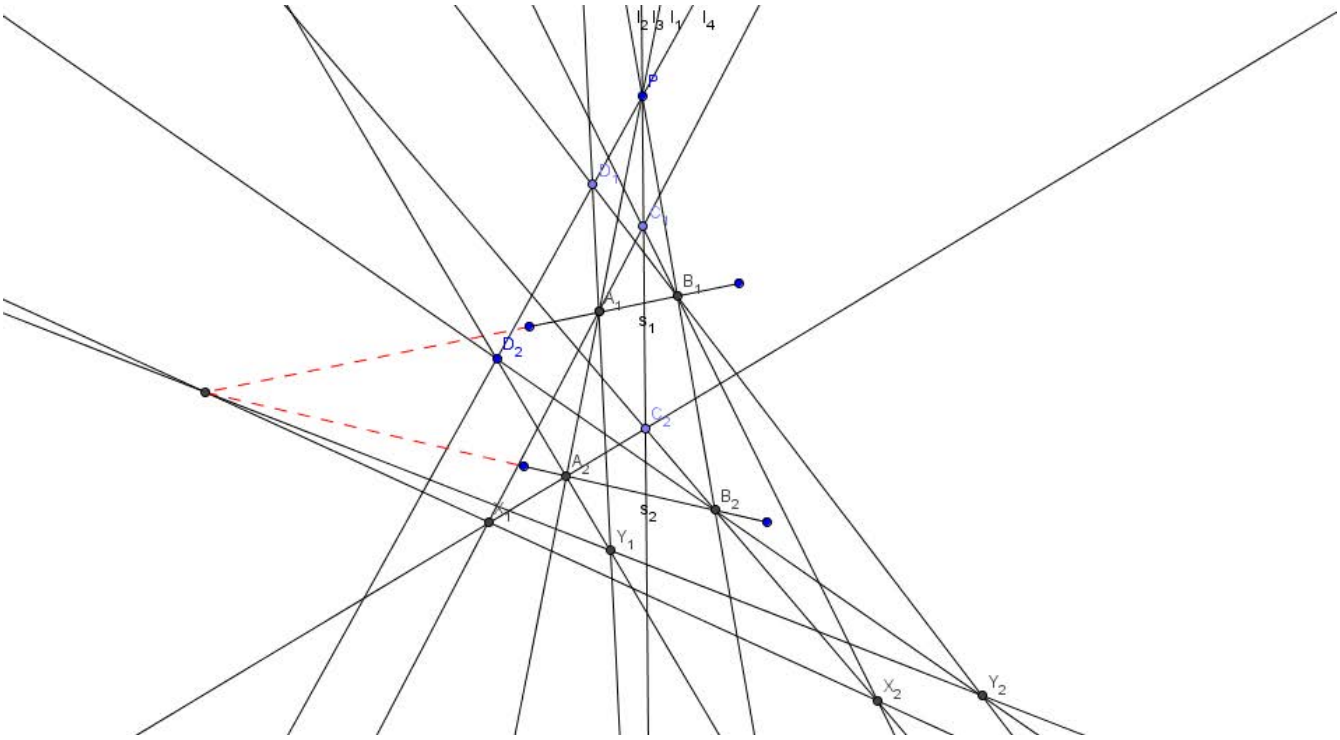
Let the segments be s_1 and s_2 and take a point P not on s_1 or s_2. Draw two arbitrary lines through P, ℓ_1, ℓ_2 which intersect both s_1, s_2. Define $\ell_1 \cap s_1 := A_1, \ell_1 \cap s_2 := A_2, \ell_2 \cap s_1 := B_1, \ell_2 \cap s_2 := B_2$.

$(\star)$ Now draw a third line, ℓ_3 through P and take arbitrary points C_1, C_2 on ℓ_3. Finally, let $A_1C_1 \cap A_2C_2 := X_1, B_1C_1 \cap B_2C_2 := X_2$.

Repeat $(\star)$ with a fourth line, ℓ_4 to get analogously the points Y_1, Y_2. Then $\overline{X_1X_2} \cap \overline{Y_1Y_2}$ is the desired intersection.

Proof : Desargues Theorem.

Attachments:



rodinos

#3 Jul 10, 2014, 2:07 am • 1

@ThirdTimeLucky: Good !

How about a solution with circles (and lines)? It is quite simple

Luis González

#4 Jul 10, 2014, 2:29 am • 2

In general, given a conic arc ω and a line ℓ in the plane (disjoint if you mind), the intersections of ℓ with the conic $\mathcal{C}$ containing the arc ω can be constructed with ruler and compass: Picking 5 arbitrary points A, B, C, D, E on ω the intersections $\{P_1, P_2\} \equiv \mathcal{C} \cap \ell$ (real or imaginary) are just the double points of the homography that the pencils $A(C, D, E)$ and $B(C, D, E)$ induce on ℓ leaving $\mathcal{C}$ invariant.

rodinos

#5 Jul 10, 2014, 4:16 am

and my solution

https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/22510

Quick Reply

High School Olympiads

Nice circles in cyclic quadrilateral.

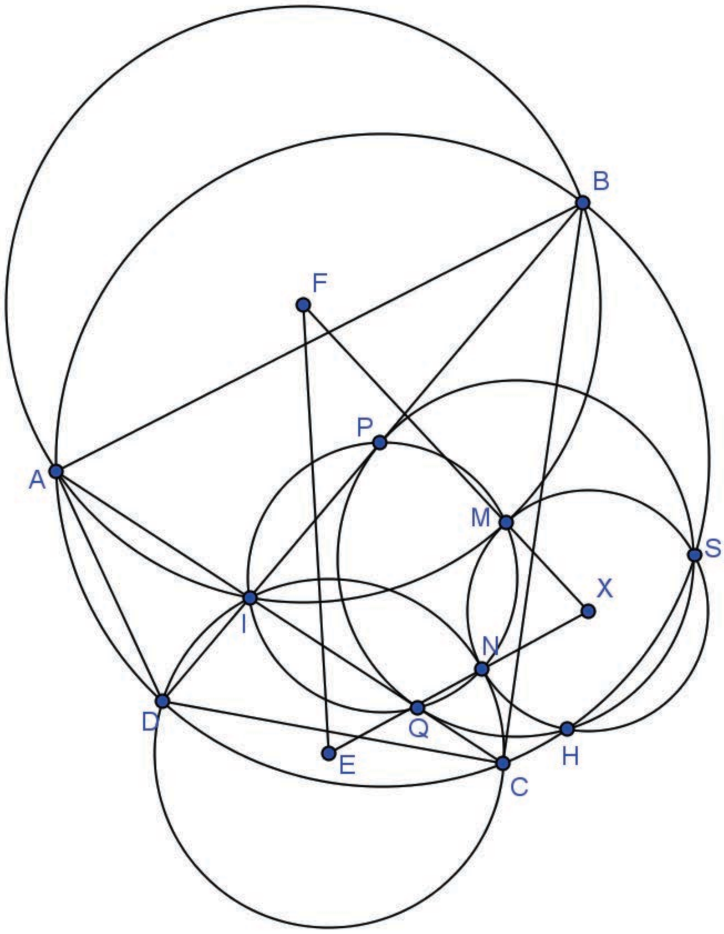
Reply

CTK9CQT

#1 Jul 9, 2014, 2:52 pm

Let $ABCD$ is a cyclic quadrilateral in circle (O) and I be the intersection of two diagonals AC and BD. A circle (X) touching two circles (IAB) at M and (ICD) at N cuts (O) at H, S. Circle (IMN) cuts BD, AC at P, Q respectively. Prove that:
a) $PHSQ$ is a cyclic quadrilateral.
b) Circle $(PHSQ)$ touches BD and AC.

Attachments:



Luis González

#2 Jul 9, 2014, 11:55 pm • 1

Inversion with center I and power $\overline{IC}\cdot\overline{IA}=\overline{IB}\cdot\overline{ID}$ leaves (O) fixed and carries $\odot(IAB)$ and $\odot(ICD)$ into lines CD and AB, resp. Thus circle (X) goes to a circle (Y) touching CD, AB at the inverses M', N' of M, N and cutting (O) at the inverses H', S' of H, S. $\odot(IMN)$ goes to line $M'N'$, cutting BD and AC at the inverses P', Q' of P, Q.

As AC and BD are antiparallel WRT AB, CD, they are equally inclined to the external bisector τ of $\angle(BA, CD)$, thus it follows that $M'N' \parallel \tau$ forms equal angles with AC, BD, i.e $\triangle IP'Q'$ is I-isosceles or $\angle AQ'N' = \angle DP'M'$. Since $\angle BAC = \angle BDC$, then $\triangle AQ'N' \sim \triangle DP'M' \Longrightarrow \frac{AQ'}{AN'} = \frac{DP'}{DM'}$. Thus if (Z) is the circle tangent to IQ', IP' at P', Q', the latter expression means that the ratio of powers of A and D WRT (Y) and (Z) are equal $\Longrightarrow (O), (Y), (Z)$ form a pencil. In other words, P', Q', S', H' lie on same circle touching $AC, BD \Longrightarrow P, Q, S, H$ lie on a same circle touching AC, BD.

Quick Reply

High School Olympiads

A Radical Axis

Reply

rodinos

#1 Jul 8, 2014, 5:18 am

Let ABC be a triangle and A'B'C' the cevian triangle of I
(ie AA',BB',CC' are the internal angle bisectors of ABC)
Let Ab, Ac be the reflections of A' in CC', BB', resp.

Which is the radical axis of the Nine Point Circles of the triangles AbBC and AcBC?

aph

Luis González

#2 Jul 8, 2014, 12:32 pm • 2

Let M, Y, Z be the midpoints of BC, BA_c, CA_b, respectively and let E, F be the projections of B, C on AC, AB. $\odot(MZE)$ and $\odot(MYF)$ are the 9-point circles of $\triangle A_bBC$ and $\triangle A_cBC$.

Let U, V be the midpoints of $\overline{A'A_c}, \overline{A'A_b}$ (projections of A' on IB,IC). Then $VZ \parallel CA'$ and $UY \parallel BA'$. But from cyclic quadrilateral $IUA'V$ (right angles at U,V), we get $\widehat{IVU} = \widehat{IA'U} = 90^\circ - \widehat{BIA'} = \frac{1}{2}\widehat{ACB} = \widehat{ICB} \Longrightarrow UV \parallel BC \Longrightarrow YZ \parallel BC \Longrightarrow \widehat{AFE} = \widehat{ACB} = \widehat{AZY} \Longrightarrow EFYZ$ is cyclic $\Longrightarrow AY \cdot AF = AZ \cdot AE$, in other words, A has equal power WRT $\odot(MZE)$ and $\odot(MYF) \Longrightarrow$ A-median AM is the radical axis of $\odot(MZE), \odot(MYF)$.

Quick Reply

High School Olympiads

Radical axis

Reply

Source: Own

buratinogigle

#1 Jul 7, 2014, 8:05 pm

Let ABC be a triangle circumcircle (O). D,E,F are midpoints of BC,CA,AB, respectively. (K_a) is A-excircle of triangle EAF. ℓ_a is radical axis of (K_a) and (O). Similarly we have ℓ_b,ℓ_c.

a) Prove that ℓ_a,ℓ_b,ℓ_c cut BC,CA,AB, respectively, at three collinear points.

b) Prove that ℓ_a,ℓ_b,ℓ_c cut EF,FD,DE, respectively, at three collinear points.

Luis González

#2 Jul 8, 2014, 11:06 am • 1

a) Let (K_a) touch AB,AC at A_b,A_c. The pairs B_c,B_a and C_a,C_b are defined similarly. $A_0 \equiv BC \cap A_bA_c$, $P \equiv BA_c \cap CA_b$ and $A_1 \equiv AP \cap BC$. The pairs B_0,B_1 and C_0,C_1 are defined similarly. From the complete quadrangle BA_bA_cC, the pencil $A(P,B,C,A_0)$ is harmonic $\Longrightarrow AP$ is the polar of A_0 WRT (K_a), thus $\{A_0,A_1\}$ are conjugate points WRT (K_a) and $(O) \Longrightarrow$ the circle with diameter $\overline{A_0A_1}$ is orthogonal to both (K_a) and $(O) \Longrightarrow$ radical axis ℓ_a of $(O),(K_a)$ passes through the midpoint A_2 of $\overline{A_0A_1}$. Similarly ℓ_b and ℓ_c go through the midpoints B_2 and C_2 of $\overline{B_0B_1}$ and $\overline{C_0C_1}$.

By Menelaus' theorem for $\triangle ABC$, cut by transversal $\overline{A_0A_bA_c}$, keeping in mind that $AA_b=AA_c$, we get $\frac{A_0B}{A_0C}=\frac{BA_b}{CA_c}$. Multiplying the cyclic expressions together, we get

$$\frac{A_0B}{A_0C}\cdot\frac{B_0C}{B_0A}\cdot\frac{C_0A}{C_0B}=\frac{BA_b}{CA_c}\cdot\frac{CB_c}{AB_a}\cdot\frac{AC_a}{BC_b}\ (\star).$$

But since $DC_b=DB_c=\frac{1}{2}(s-a) \Longrightarrow D$ is also midpoint of $\overline{C_bB_c} \Longrightarrow BC_b=CB_c$. Likewise, E and F are midpoints of $\overline{A_cC_a}$ and $\overline{B_aA_b}$, i.e. $CA_c=AC_a$ and $AB_a=BA_b$. Hence the RHS of the expresion $(\star)$ equals 1 $\Longrightarrow A_0,B_0,C_0$ are collinear by the converse of Menelaus' theorem $\Longrightarrow A_2,B_2,C_2$ are then collinear on the Newton line of the quadrangle bounded by the sidelines of $\triangle A_1B_1C_1$ and $\overline{A_0B_0C_0}$. $\blacksquare$

b) $R_a \equiv \ell_b \cap \ell_c$ is the radical center of $(O),(K_b),(K_c)$ and since D is also midpoint of the common tangent $\overline{C_bB_c}$ of $(K_b),(K_c)$, it follows that DR_a is the radical axis of $(K_b),(K_c)$. Hence defining R_b and R_c cyclically, $\triangle DEF$ and $\triangle R_aR_bR_c \equiv \triangle(\ell_a,\ell_b,\ell_c)$ are perspective through the radical center of $(K_a),(K_b),(K_c)$. By Desargues theorem, ℓ_a,ℓ_b,ℓ_c cut EF,FD,DE, resp, at three collinear points. $\blacksquare$

Quick Reply

High School Olympiads

paralell

Reply

huuthieu

#1 Jul 8, 2014, 6:41 am

Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not paralell to CD) intersect at point P.Points O_1 and O_2 are circumcenters and points H_1 and H_2 are the orthocenters of triangles ABP and CDP, respectively. The perpendicular from O_1 on CD and the perpendicular from O_2 on AB intersect at F. Prove that PF is paralell to H_1H_2.

Luis González

#2 Jul 8, 2014, 8:48 am • 1

Since PO_1, PH_1 and PO_2, PH_2 are isogonals WRT $\widehat{APB}$, then it follows that $\widehat{O_1PO_2} = \widehat{H_1PH_2}$. But in the $\triangle PAB$ and $\triangle PCD$, we have the relations $PH_1 = 2 \cdot PO_1 \cdot \cos \widehat{APB}$ and $PH_2 = 2 \cdot PO_2 \cdot \cos \widehat{APB} \Longrightarrow \frac{PH_1}{PH_2} = \frac{PO_1}{PO_2} \Longrightarrow$ $\triangle PO_1O_2$ and $\triangle PH_1H_2$ are inversely similar by SAS. Since $(FO_1 \parallel PH_2) \perp CD$ and $(FO_2 \parallel PH_1) \perp AB$, then $\widehat{O_1FO_2} = \widehat{H_1PH_2} = \widehat{O_1PO_2} \Longrightarrow PFO_1O_2$ is cyclic $\Longrightarrow \widehat{PFO_2} = \widehat{PO_1O_2} = \widehat{PH_1H_2}$, which implies that $PF \parallel H_1H_2$, as desired.

Quick Reply

High School Olympiads

Properties with Square.

Reply

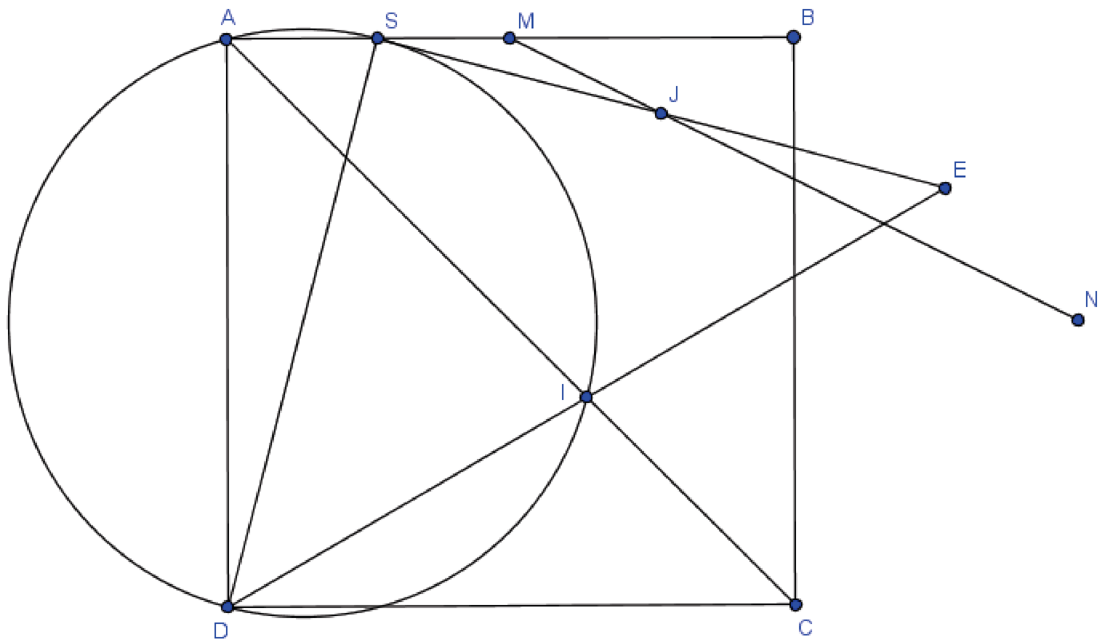
Source: Maybe own.

CTK9CQT

#1 Jul 7, 2014, 1:27 pm

Let $ABCD$ be a square and S be a point on segment $[AB]$. Circle with diameter DS cut $[AC]$ at I. Let E be the symmetric point of D through I, J be the midpoint of SE and M be the midpoint of AB.
a) Prove that if $S \equiv M$ then $J \in [BC]$
b) Find the locus of J when S moves on $[AB]$.

Attachments:



Tsikaloudakis

#2 Jul 7, 2014, 9:22 pm

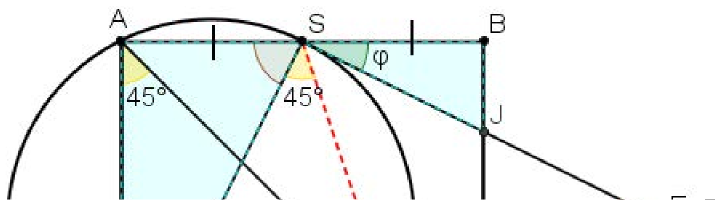
$$a)\quad If\ \ AS = SB\ \ then\ :$$

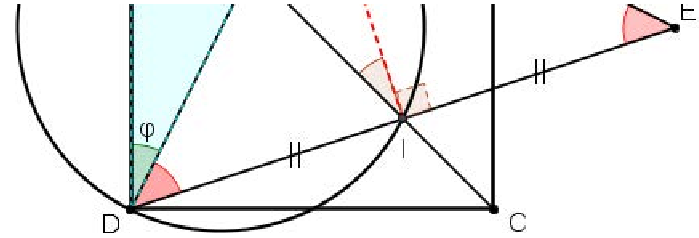
$$DI = IE = SI \ \Rightarrow\ D\hat{S}I = S\hat{D}I = S\hat{E}I = 45^O \ \Rightarrow$$

$$A\hat{D}S = B\hat{S}J = \varphi \ \Rightarrow\ tr.SBJ \approx tr.SAD \ \Rightarrow$$

$$\frac{SJ}{SE} = \frac{SJ}{SD} = \frac{SB}{AD} = \frac{1}{2} \ \Rightarrow\ SE = 2SJ$$

Attachments:





This post has been edited 1 time. Last edited by Tsikaloudakis, Jul 8, 2014, 2:36 am

Luis González

#3 Jul 7, 2014, 10:17 pm

Clearly E runs on the parallel to AC through $B \Longrightarrow \angle ABE = 135^\circ$. From cyclic $ASID$, we get $\angle SDE = \angle SAI = 45^\circ$ $\Longrightarrow BEDS$ is cyclic $\Longrightarrow \angle SED = \angle SBD = 45^\circ \Longrightarrow \triangle SED$ is isosceles right with apex $S \Longrightarrow \triangle DSJ$ are all similar $\Longrightarrow$ locus of J is the image ℓ of AB under the spiral similarity with center D, rotational angle $\angle(DS, DJ)$ and coefficient $\frac{DS}{DJ}$.

When $S \equiv A$, J goes to M and when $S \equiv B$, J goes to the midpoint of BP, where $P \equiv BE \cap CD \Longrightarrow \ell$ is the B-midline of $\triangle BPA$. Thus, when $S \equiv M$, then J is on the B-median BC of $\triangle BPA$.

Tsikaloudakis

#4 Jul 7, 2014, 10:29 pm

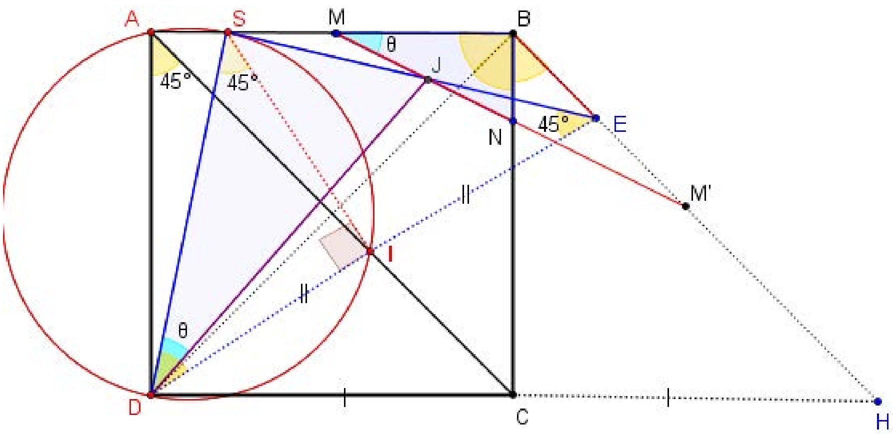
$$b)\quad If\ \ AM = MB\,,\ BN = \frac{1}{4}NC\,,\ \ DH = 2DC\ \ and\ \ M' = BH \cap MN\,,\ then\ :$$

$$tr.SDJ \approx tr.MBN$$

$$So\ :\ J \in MM'$$

The locus of J is the segment MM'

Attachments:



Quick Reply

High School Olympiads

geometry about Inversion

Locked

zuanpi
#1 Jul 6, 2014, 5:38 pm

Let PQ be the diameter of semicircle H. Circle O is internally tangent to H and tangent to PQ at C. Let A be a point on H and B a point on PQ such that AB PQ and is tangent to O. Prove that AC bisects PAB.

Luis González
#2 Jul 7, 2014, 9:40 pm

It's a problem from the Israeli Math Olympiad (1995).

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=17446
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=146656
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=416223

High School Olympiads

Tangents circles

Reply

Source: It looks so simple ...

BlackSelena

#1 Jun 26, 2014, 1:37 am

Let circle (O) with constant chord AB. 2 circles (O_1), (O_2) are externally tangent and internally tangent with (O). O_1 and O_2 on the same side w.r.t AB and tangent with AB.
Let C, D are the tangency point of (O_1) and (O_2) with (O) respectively (assume that $ACDB$ formed a quadrilateral in that order)
Prove that: $\frac{CD}{AC.BD} = const$

Luis González

#2 Jul 7, 2014, 10:33 am

Let X, Y be the tangency points of $(O_1), (O_2)$ with AB. The inversion with center the midpoint M of the arc AB of (O) and radius $MA = MB = \varrho$ swaps (O) and AB, leaving $(O_1), (O_2)$ fixed $\Longrightarrow X, Y$ go to C, D and the common point I of $(O_1), (O_2)$ is double, i.e. $I \in (M, \varrho)$. Hence

$$\frac{AC}{AX} = \frac{\varrho^2}{\varrho \cdot MX} = \frac{\varrho}{MX}, \ \frac{BD}{BY} = \frac{\varrho^2}{\varrho \cdot MY} = \frac{\varrho}{MY}, \ \frac{CD}{XY} = \frac{\varrho^2}{MX \cdot MY} \Longrightarrow$$

$$\frac{CD}{AC \cdot BD} = \frac{\varrho^2 \cdot XY}{MX \cdot MY} \cdot \frac{MX \cdot MY}{\varrho^2 \cdot AX \cdot BY} = \frac{XY}{AX \cdot BY} \quad (1).$$

Let IX, IY cut (M, ϱ) again at U, V. Since $\angle XIY$ is clearly right and the radical axis MI of $(O_1), (O_2)$ bisects $\overline{XY}$, it follows that UV is a diameter of (M, ϱ) parallel to AB, i.e. $ABVU$ is a fixed isosceles trapezoid with legs $AU = BV \Longrightarrow \angle AIU = \angle BIV$ and since $\angle IAB = \angle IVB$, then $\triangle IAX \sim \triangle IVB$ and likewise $\triangle IBY \sim \triangle IUA$, yielding

$$\frac{AX}{BV} = \frac{IX}{IB}, \ \frac{BY}{AU} = \frac{IY}{IA} \Longrightarrow \frac{AX \cdot BY}{AU^2} = \frac{IX \cdot IY}{IA \cdot IB}.$$

But if P is the projection of I on AB, we have in the triangles $\triangle IXY$ and $\triangle IAB$, the relations $IP \cdot XY = IX \cdot IY$ and $IA \cdot IB = 2 \cdot IP \cdot \varrho$. Thus, in the latter relation, we have

$$\frac{AX \cdot BY}{AU^2} = \frac{XY \cdot IP}{2 \cdot IP \cdot \varrho} = \frac{XY}{2\varrho} \quad (2).$$

Combining (1) and (2) gives $\frac{CD}{AC \cdot BD} = \frac{2\varrho}{AU^2} = \text{const}.$

TelvCohl

#3 Oct 22, 2014, 8:16 am • 1

My solution

Let Y, Z be the tangent point of $(O_1), (O_2)$ and AB .
Let X be the midpoint of arc AB and $V = (O_1) \cap (O_2)$.
Let $Y' = VY \cap (X, r), Z' = VZ \cap (X, r)$ ($r = XA = XB = XV$).

Easy to see C, Y, X are collinaer and D, Z, X are collinear .

Invert with respect to (X, r) ,
then $A, Y, Z, B \longleftrightarrow A, C, D, B$

so $\dfrac{CD}{AC \cdot BD} = \dfrac{YZ}{AY \cdot BZ}$ ... (1)

Since $\angle YVZ = 90$ and the radical axis VX of $((O_1), (O_2))$ pass through the midpoint of YZ and $Y'Z'$,
so $Y'Z'$ is a diameter of (X, r) which is parallel to AB , ie. Y', Z' are two fixed points on (X, r)
hence we get $\dfrac{AB \cdot YZ}{AY \cdot BZ} = (A, Z; B, Y) = (A, Z'; B, Y') = \text{const.}$ ... (2)

From (1) and (2) we get $\dfrac{CD}{AC \cdot BD}$ is const..

Q.E.D

Quick Reply

High School Olympiads

Reflection of center over a chord

Reply

mefeke

#1 Jul 6, 2014, 6:10 pm

Let O be the center and AB be a chord of given circle.
Let O' be the reflection of O over AB.
Let P be a point on line AB.
Let $O'P \cap OA = Q$.
Let $QR \parallel OB$ and $R \in AB$.
Let the circle through R and B be tangent to OP at S.
Show that $AP = PS$.

Luis González

#2 Jul 6, 2014, 11:12 pm • 1

Let the parallel to PQ through O cut AB at C. Isosceles triangles $\triangle OAB$ and $\triangle QAR$ are homothetic with corresponding cevians OC and $QP \Longrightarrow \frac{PA}{PR} = \frac{CA}{CB}$. But since O' is the reflection of O on PC, it follows that $OCO'P$ is a rhombus, i.e. P and C are symmetric about the midpoint of $\overline{AB} \Longrightarrow \frac{PB}{PA} = \frac{CA}{CB}$. Therefore, $\frac{PA}{PR} = \frac{PB}{PA} \Longrightarrow PA^2 = PR \cdot PB$, but $PS^2 = PR \cdot PB \Longrightarrow PA = PS$.

Quick Reply

High School Olympiads

An easy locus

Reply

rodinos
#1 Jul 6, 2014, 5:37 am

Let ABC be a triangle, P a point and A'B'C' the pedal triangle of P.
Let Q be a point on PA' and A* the intersection of the perpendicular through B to QC' and the perpendicular through C to QB'

Which is the locus of A* as Q moves on PA'?

aph

Luis González
#2 Jul 6, 2014, 6:55 am • 1

Actually, if B', C' are arbitrary points on AC, AB and Q runs on the perpendicular from P to BC, the locus of A^* still have the same nature; it's a line ℓ perpendicular to $B'C'$. When B', C' coincide with the projections of P on AC, AB, as the problem states, ℓ becomes the isogonal of AP WRT $\angle BAC$.

As Q varies, the pencils $B'Q$ and $C'Q$ are perspective, hence the pencils BA^* and CA^*, formed by perpendiculars from B and C to $C'Q$ and $B'Q$, are projective with double ray BC when Q is at infinity, thus they are perspective $\Longrightarrow A^*$ moves on a fixed line ℓ perpendicular to $B'C'$, because when $Q \in B'C'$, A^* goes to the infinity point of $\perp B'C'$.

rodinos
#3 Jul 6, 2014, 6:30 pm

Thanks, Luis :-

These kinds of loci are "useful" in the geometry of triangle. We get three lines respective to A,B,C and ask whether the triangle they bound is perspective (or orthologic or parallelogic) with ABC (or other triangles: medial, orthic, etc).
In the case of the pedal triangle, we get concurrent lines at the isogonal conjugate of P.
Other case: A'B'C' = cevian triangle of P. For which P's the three lines are concurrent?
(or more generally they bound a triangle perspective with ABC).

On the other hand, we can take P, Q fixed and Ba,Ca as variable points. That is:

Let P, Q be two fixed points (special case: PQ is perpendicular to BC).
A line L passes through P and intersects AC,AB at Ba,Ca, resp.
Let A* be the intersection of the perpendicular through B to QCa and the perpendicular through C to QBa
Which is the locus of A* as the line L moves around P?

Note: If we apply it to geometry of triangle, we have to take three same points Q1,Q2,Q3 (for example: Q1, Q2, Q3 = the reflections of P in BC,C,AB, resp). Or we can take P,Q having a relation (for example: Q is the isogonal or isotomic conjugate of P)

Quick Reply

High School Olympiads

Supplementary Angles ???

Reply

Source: BrMO 2005 R2

utkarshgupta
#1 Jul 5, 2014, 8:34 pm

Let ABC be a triangle with $AC > AB$. The point X lies on the side BA extended through A, and the point Y lies on the side CA in such a way that $BX = CA$ and $CY = BA$. The line XY meets the perpendicular bisector of side BC at P. Show that
$\angle BPC + \angle BAC = 180$.

Luis González
#2 Jul 5, 2014, 11:33 pm

Let M be the midpoint of the arc BC of the circumcircle $\odot(ABC)$, not containing A. AM and XY cut BC at D and Z, respectively. By Menelaus' theorem for $\triangle ABC$ cut by $\overline{XYZ}$, we get

$$\frac{BZ}{ZC}=\frac{YA}{CY}\cdot\frac{XB}{XA}=\frac{AC-AB}{CY}\cdot\frac{XB}{AC-AB}=\frac{AC}{AB}=\frac{CD}{DB},$$

which means that D and Z are symmetric WRT the midpoint of $\overline{BC}$, i.e. $\triangle MDZ$ is M-isosceles. Since $AX=AY=AC-AB$, then $XYZ \parallel AM \Longrightarrow PZMD$ is a rhombus, i.e. P is reflection of M on $BC \Longrightarrow$ $\widehat{BPC}=\widehat{BMC}=180^\circ-\widehat{BAC}$.

jlammy
#3 Jul 6, 2014, 3:03 am

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=575260
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=327416
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=482947
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=470946
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=254072&p=1389325
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=125&t=72694&p=421058

Please make some effort to search.

Quick Reply

High School Olympiads

prove it please

Reply

Source: AMMEL OMGE 4102 P1

nawaites

#1 Jul 5, 2014, 7:17 pm

Let D and E be points in the interiors of sides AB and AC , respectively, of a triangle ABC, such that $DB = BC = CE.)$. Let the lines CD and BE meet at F If the incentre I of triangle ABC, the orthocentre H of triangle DEF and Denote by M_1, M_2 the midpoints of BD, CE respectively. Prove $IH \perp M_1M_2$

Luis González

#2 Jul 5, 2014, 9:21 pm

Further, IH is the perpendicular bisector of $\overline{M_1M_2}$.

Since $\triangle BCD$ and $\triangle CBE$ are isosceles with apices B, C, the internal bisectors BI, CI of $\angle DBC, \angle ECB$ are perpendicular to $CD, BE \Longrightarrow I$ is orthocenter of $\triangle FBC \Longrightarrow IH$ is Steiner line of the quadrangle $BEDC$, perpendicular to its Newton line M_1M_2 (this is well-known). If M is the midpoint of BC, obviously $\triangle BMM_1$ is B-isosceles with symmetry axis $BI \Longrightarrow IM = IM_1$ and similarly we have $IM = IM_2 \Longrightarrow IM_1 = IM_2 \Longrightarrow IH$ is then perpendicular bisector of $\overline{M_1M_2}$.

nawaites

#3 Jul 6, 2014, 1:15 pm

How to prove this one?
"IH is Steiner line of the quadrangle $BEDC$, perpendicular to its Newton line M_1M_2 (this is well-known)"

This is not too wrllknown for some people yhat is beginner like me i am sorry can another proof it?

IDMasterz

#4 Jul 14, 2014, 12:29 pm

If H' is the orthocentre of BFC, the considering circles with diametres BD, CE, the radical axis is perpendicular to M_1M_2 and contains both H', H, so then it suffice to show $I \in HH'$. HH' perpendiculaly bisects M_1M_2, and note is M is the external angle bisector meets circumcircle then MM_1M_2A are concyclic since $MADE$ are concyclic. If midpoint of BC is M_A, then the reflection of it over angle bisectors makes M_1, M_2. Since I is preserved over such transformations, $IM_1 = IM_2$ and the result follows.

nawaites

#5 Jul 14, 2014, 1:32 pm

I am sorry but i think I is orthocenter of triangle BFC????????

Quick Reply

High School Olympiads

Equal parallel line segments

jlammy

#1 Jul 5, 2014, 2:34 am • 1

In all these questions, the definition of P is the same.

(a) Let P be an interior point of $\triangle ABC$ such that if $\ell_a \parallel BC$ is a line segment through P with end points on CA and AB (ℓ_b, ℓ_c are defined similarly), then $\ell_a = \ell_b = \ell_c$. Prove that

$$\ell_a = \frac{2abc}{ab+bc+ca}.$$

(b) Let the incircle touch BC, CA, AB at X, Y, Z. Let $\triangle A'B'C'$ be such that $A'B' \parallel AB$ contains C, with analogous definitions for $B'C', C'A'$. Prove that $\triangle XYZ$ and $\triangle A'B'C'$ are in perspective, and the perspector is P.

(c) Let J be the isotomic conjugate of the incentre I. Show that J, G, P are collinear, where G is the centroid, and find the ratio $JP : PG$.

Luis González

#2 Jul 5, 2014, 9:42 am • 1

Proposition **(b)** is false in general, $\triangle XYZ$ should be the incentral triangle of $\triangle ABC$ and not its intouch triangle.

Let h_a, h_b, h_c denote the length of the altitudes issuing from A, B, C and let $\mathrm{d}(X, \tau)$ denote the distance from a point X to a line τ. If AP cuts BC at D, we get

$$\frac{\ell_a}{a} = \frac{AP}{AD} = \frac{AD-PD}{AD} = 1 - \frac{PD}{AD} = 1 - \frac{[PBC]}{[ABC]} = 1 - \frac{\mathrm{d}(P,BC)}{h_a} =$$

$$= \frac{h_a - \mathrm{d}(P,BC)}{h_a} = \frac{\mathrm{d}(P,B'C')}{h_a} \ (1).$$

Similarly, we get the expressions

$$\frac{\ell_b}{b} = 1 - \frac{[PCA]}{[ABC]} = \frac{\mathrm{d}(P,C'A')}{h_b} \ (2)$$

$$\frac{\ell_c}{c} = 1 - \frac{[PAB]}{[ABC]} = \frac{\mathrm{d}(P,A'B')}{h_c} \ (3).$$

Therefore from (2) and (3), we have $\ell_b = \ell_c \Longleftrightarrow \dfrac{\mathrm{d}(P,A'C')}{\mathrm{d}(P,A'B')} = \dfrac{c}{b} \cdot \dfrac{h_b}{h_c} = \text{const},$

which means that the locus of the points P fulfilling $\ell_b = \ell_c$ is a line passing through A' and the foot of the A-internal bisector of $\triangle ABC$, clearly satisfying the locus condition. Similarly, locus of the points P satisfying $\ell_a = \ell_c$ is the line passing through B' and the foot of the B-internal bisector of $\triangle ABC \Longrightarrow P$ is then the cevian quotient I/G, anticomplement of J WRT $\triangle ABC$ $\Longrightarrow P \in JG$ and $\overline{PJ} : \overline{PG} = 3 : 2$.

On the other hand, adding the expressions $(1), (2), (3)$ together gives

$$\ell_a\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 3 - \frac{[PBC]+[PCA]+[PAB]}{[ABC]} = 2 \Longrightarrow \ell_a = \frac{2abc}{bc+ca+ab}.$$

jlammy

#3 Jul 5, 2014, 3:40 pm • 1

Oops, I meant to say the incentral triangle, not intouch.

@Luis González

Could you show how you established that JGP was a straight line?

Also, could you attempt **(b)**, knowing that $\triangle XYZ$ is the incentral triangle?

Quick Reply

High School Olympiads

An area equality

Source: European Mathematical Cup 2013, Junior Division, P2

joybangla

#1 Jul 3, 2014, 3:09 pm • 1

Let P be a point inside a triangle ABC. A line through P parallel to AB meets BC and CA at points L and F, respectively. A line through P parallel to BC meets CA and BA at points M and D respectively, and a line through P parallel to CA meets AB and BC at points N and E respectively. Prove

$$[PDBL]\cdot[PECM]\cdot[PFAN]=8\cdot[PFM]\cdot[PEL]\cdot[PDN]$$

Proposed by Steve Dinh

Luis González

#2 Jul 3, 2014, 9:00 pm

$$\frac{[PEL]}{[PFN]}=\frac{PE\cdot PL}{PF\cdot PN},\ \frac{[PFM]}{[PDL]}=\frac{PF\cdot PM}{PD\cdot PL},\ \frac{[PDN]}{[PEM]}=\frac{PD\cdot PN}{PE\cdot PM}\Longrightarrow$$

$$\frac{[PEL]}{[PFN]}\cdot\frac{[PFM]}{[PDL]}\cdot\frac{[PDN]}{[PEM]}=\frac{PE\cdot PL}{PF\cdot PN}\cdot\frac{PF\cdot PM}{PD\cdot PL}\cdot\frac{PD\cdot PN}{PE\cdot PM}=1.$$

But from paralellograms $PDBL, PECM, PFAN,$ we have $[PFN]=\tfrac{1}{2}[PFAN], [PDL]=\tfrac{1}{2}[PDBL]$ and $[PEM]=\tfrac{1}{2}[PECM]$. Thus, the latter expression becomes

$[PDBL]\cdot[PECM]\cdot[PFAN]=8\cdot[PFM]\cdot[PEL]\cdot[PDN]$.

High School Olympiads

Special locus of points

Reply

Source: British FST-2

hajimbrak

#1 Jun 30, 2014, 10:36 pm

Let ABC be a triangle. For a point P of the plane, let A' be the foot of the perpendicular dropped from P to BC. Points B' and C' are defined analogously. Find the locus of points P in the plane such that

$$PA' \cdot PA = PB' \cdot PB = PC' \cdot PC$$

Bandera

#2 Jul 2, 2014, 1:43 am

Answer

Luis González

#3 Jul 3, 2014, 10:52 am

Posted before. It's also a problem from Vietnam NMO 1994.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=588656
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=234384

Quick Reply

High School Olympiads

Line IO parallel to a side

Reply

ThirdTimeLucky

#1 Jun 15, 2014, 11:02 pm

In acute $\triangle ABC$, let H, I, O be the orthocenter, incenter and circumcenter respectively. Suppose that $IO \parallel BC$ and let BH intersect AC at E. Suppose P is a point on segment BC such that $CP = CE$. Prove that $HP \parallel AI$.

Luis González

#2 Jun 17, 2014, 4:04 am

Let R and r denote the radii of the circumcircle (O) and incircle (I). F is the foot of the C-altitude and AI cuts BC at V. $OI \parallel BC \Longrightarrow \text{dist}(O, BC) = \text{dist}(I, BC) \Longrightarrow r = R \cdot \cos A \Longrightarrow \cos A = \frac{r}{R} = \cos A + \cos B + \cos C - 1 \Longrightarrow \cos B + \cos C = 1$. Substituting $\cos B = \frac{BF}{BC}$ and $\cos C = \frac{CE}{BC}$ gives $BC = CE + BF = CP + BF \Longrightarrow BP = BF$. Therefore $\frac{BP}{CP} = \frac{BF}{CE} = \frac{HB}{HC} \Longrightarrow HP$ bisects $\widehat{BHC}$. Now simple angle chase yields

$$\widehat{HPB} = \tfrac{1}{2}\widehat{BHC} + \widehat{HCB} = 90^\circ - \tfrac{1}{2}\widehat{A} + 90^\circ - \widehat{B} = \widehat{C} + \tfrac{1}{2}\widehat{A} = \widehat{AVB},$$

which implies that $HP \parallel AI$, as desired.

Quick Reply

High School Olympiads

A regular hexagon

Source: Romanian District Olympiad 2014, Grade 8, P3

joybangla

#1 Jun 15, 2014, 7:11 pm • 1

Let $ABCDEF$ be a regular hexagon with side length a. At point A, the perpendicular AS, with length $2a\sqrt{3}$, is erected on the hexagon's plane. The points $M, N, P, Q,$ and R are the projections of point A on the lines $SB, SC, SD, SE,$ and SF, respectively.

a. Prove that the points M, N, P, Q, R lie on the same plane.
b. Find the measure of the angle between the planes (MNP) and (ABC).

Luis González

#2 Jun 16, 2014, 2:48 am

Since $SA^2 = SM \cdot SB = SN \cdot SC = SP \cdot SD = SQ \cdot SE = SR \cdot SF$, then M, N, P, Q, R are images of B, C, D, E, F under the inversion with center S and power SA^2. Thus they lie on the inverse ω' of the circumcircle hexagon ω: the intersection of the inverse sphere of the plane ABC and the inverse plane of the sphere through S, ω, i.e. M, N, P, Q, R are coplanar.

Clearly the tangent of ω at A is also the tangent of ω' at A. Hence, the angle θ between their planes ABC, MNP is the angle formed by their diameters AX and AY. So, we have a $\triangle SAX$ right at A where Y is projection of A on $SX \Longrightarrow$ $\theta = \angle YAX = \angle ASX$. Since $SA = 2a\sqrt{3}$ and $AX = 2a$, then $\tan\theta = \frac{2a}{2a\sqrt{3}} = \frac{1}{\sqrt{3}} \Longrightarrow \theta = 30^\circ$.

High School Olympiads

Find measure of angle

Source: Romanian District Olympiad 2014, Grade 7, P3

joybangla
#1 Jun 15, 2014, 6:38 pm • 1

Let ABC be a triangle in which $\measuredangle A = 135^{\circ}$. The perpendicular to the line AB erected at A intersects the side BC at D, and the angle bisector of $\angle B$ intersects the side AC at E.
Find the measure of $\measuredangle BED$.

professordad
#2 Jun 15, 2014, 10:48 pm

Using trig

Wait this is better

Luis González
#3 Jun 15, 2014, 11:47 pm

The orthogonal projection F of D on BE is on circle $\odot(BAD)$ with diameter $\overline{BD}$. Further, it is the midpoint of its arc AD, because BE bisects $\angle ABD$. Since AC is clearly external bisector of $\angle BAD$, then E becomes B-excenter of $\triangle ABD \Longrightarrow$ F is equidistant from A, D, E (valid for any triangle), i.e. triangle $\triangle DEF$ is isosceles right with legs $FD = FE \Longrightarrow$ $\angle FED \equiv \angle BED = 45^{\circ}$.

sunken rock
#4 Jun 17, 2014, 11:28 am • 1

Clearly AC is the external bisector of the angle BAD, hence E is the $B-$excenter of $\triangle ABD$. If I is the incenter of $\triangle ABD$, it follows that $AIDE$ is cyclic, for any triangle, hence $m(\angle BED) = \dfrac{m(\widehat{BAD})}{2} = 45^{\circ}$.

Best regards,
sunken rock

High School Olympiads

Proving a perpendicularity

Reply

Source: Romanian District Olympiad 2014, Grade 7, P4

joybangla
#1 Jun 15, 2014, 6:45 pm • 1

Let $ABCD$ be a square and consider the points $K \in AB, L \in BC,$ and $M \in CD$ such that ΔKLM is a right isosceles triangle, with the right angle at L. Prove that the lines AL and DK are perpendicular to each other.

Luis González
#2 Jun 15, 2014, 9:50 pm

Let O be the midpoint of $\overline{KM}$ (projection of L on KM). Then quadrilaterals $BKOL$ and $CMOL$ are cyclic $\Longrightarrow$ $\angle OBL = \angle OKL = 45^\circ$ and $\angle OCL = \angle OML = 45^\circ \Longrightarrow O$ concides with the center of $ABCD \Longrightarrow$ $O \equiv AC \cap BD.$ If $\odot(BOL)$ cuts AL again at $P,$ we have $\angle KPL = \angle KOL = \underline{90}^\circ$ and $\angle OPL = \angle OBL = 45^\circ = \angle ODA \Longrightarrow P$ is on circle $\odot(OAD)$ with diameter $\overline{DA} \Longrightarrow \angle DPA = 90^\circ \Longrightarrow D, P, K$ are collinear $\Longrightarrow DK \perp AL.$

sunken rock
#3 Jun 17, 2014, 1:41 pm

$\angle BKL + \angle KLB = \angle CLM + \angle CML = \angle BLK + \angle CLM = 90^\circ$, hence $\angle BKL = \angle CLM$ and $\angle BLK = \angle CML$; with $LK = LM$ we get $\triangle BLK \cong \triangle CML$, hence $BK = CL, BL = CM$, which implies $AK = BL$, i.e. $\triangle KAD \cong \triangle LBA$, or $AL \perp KD$.

Best regards,
sunken rock

bonciocatciprian
#4 Jun 18, 2014, 2:37 pm

It is easy to notice that $\Delta KBL \equiv \Delta LCM(ASA)$. Therefore, $AK = BL$, so $\Delta ABL \equiv \Delta DAK(SAS) \Rightarrow m(\widehat{AEK}) = \frac{\pi}{2}$.

Quick Reply

High School Olympiads

Congruent angles

Reply

Source: Romanian District Olympiad 2014, Grade 6, P3

joybangla

#1 Jun 15, 2014, 6:20 pm • 1

The points M, N, and P are chosen on the sides BC, CA and AB of the ΔABC such that $BM = BP$ and $CM = CN$. The perpendicular dropped from B to MP and the perpendicular dropped from C to MN intersect at I. Prove that the angles $\measuredangle IPA$ and $\measuredangle INC$ are congruent.

Luis González

#2 Jun 15, 2014, 9:11 pm

Perpendicular bisectors BI, CI of $\overline{MP}, \overline{MN}$ are obviously internal bisectors of $\angle PBM$ and $\angle NCM$, meeting at the incenter I of $\triangle ABC$. Now since $IP = IM = IN$ and AI bisects $\angle PAN$, then either I is the midpoint of the arc PN of $\odot(APN)$ or $AP = AN$, in which case P, N become the projections of I on BA, AC. Thus, quadrilateral $IPAN$ is always cyclic $\Longrightarrow \angle IPA = \angle INC$.

Quick Reply

High School Olympiads

Always twice

Reply

Nemo-Mnemo
#1 Jun 14, 2014, 3:57 pm

An isosceles triangle has vertex A and base BC. Through a point D on AB, we draw a perpendicular on AB to meet BC extended at E, such that $AD = CE$. If DE meets AC at F, show (without using trigonometry or Menelaus' Theorem) that the area of triangle ADF is twice that of triangle CFE.

professordad
#2 Jun 15, 2014, 12:47 am

Click to reveal hidden text

Luis González
#3 Jun 15, 2014, 3:01 am

If H denotes the projection of F on BC, it suffices to prove that $FD = 2 \cdot FH$.

Miquel point $N \equiv \odot(ABC) \cap \odot(ADF) \cap \odot(CEF)$ of $BCFD$ is center of the rotation that swaps the congruent segments $\overline{AD}$ and $\overline{CE} \Longrightarrow NA = NC$, i.e. N is midpoint the arc AC of $\odot(ABC)$. If $\odot(ADF)$ cuts the A-altitude again at I, we have $\widehat{AIF} = \widehat{ADF} = 90^\circ \Longrightarrow IF \parallel BC \Longrightarrow \widehat{ANI} = \widehat{AFI} = \widehat{ACB} \Longrightarrow B, I, N$ are collinear, i.e. I is incenter of $\triangle ABC$. Now let P, R be the projection of I on CB, BA and let FI cut AB at G. Since I is obviously midpoint of $\overline{FG}$, we have $FH = IP = IR = \frac{1}{2}FD$, as desired.

Quick Reply

High School Olympiads

Geometry Problem

Locked

Jul

#1 Jun 14, 2014, 7:55 pm

Let (O) is a circle has center O and l is a line. P is a point on l. $OQ \perp l \quad (Q \in l)$. PE, PF are two tangent of (O) and $E, F \in (O)$. $QA \perp PE, QB \perp PF \quad (A \in PE, B \in PF)$. Suppose $OQ \cap EF = I$ and $AB \cap OQ = H$. Prove that : $QH = IH$

Luis González

#2 Jun 14, 2014, 9:42 pm

Discussed before. It's IMO Shortlist 1994, G5

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=5198
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=294580

High School Olympiads

hard problem

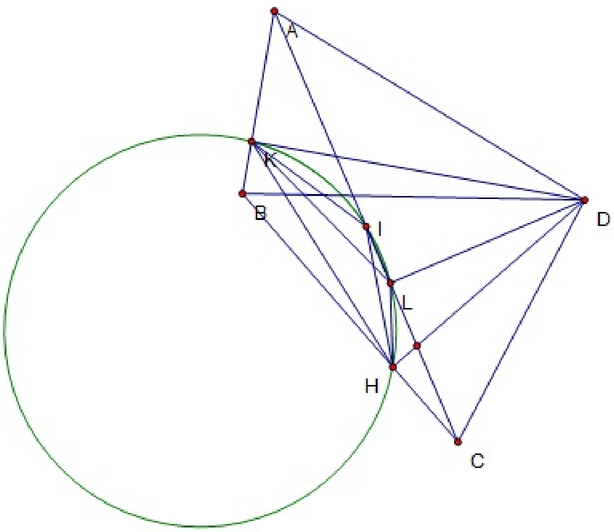
Locked

mathandyou

#1 Jun 14, 2014, 8:36 am

Given quadrilateral $ABCD$, $\angle BAD = \angle BCD$. $DH \perp BC$, $DL \perp CA$, $DK \perp AB$. (HLK) cuts AC at I. Prove that I is midpoint of AC

Attachments:



Luis González

#2 Jun 14, 2014, 10:15 am • 1

Firstly, give your posts meaningful subjects. Secondly, the problem has been posted before. Topic locked.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=111617
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=359120
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=558486

High School Olympiads

Triangle Construction Problem

Reply

Source: Math. Chronicles (Athens, Aug. - Sept. 1969, p. 141)

rodinos

#1 Jun 13, 2014, 2:12 pm

To construct triangle ABC if are given the second intersections of the internal angle bisectors and the circumcircle (Circumcevian triangle of I).

aph

Luis González

#2 Jun 13, 2014, 8:08 pm • 1

Denote by I and I_a, I_b, I_c the incenter and 3 excenters of $\triangle ABC$. AI, BI, CI cut the circumcircle (O) again at P, Q, R (known). Since I and (O) are the orthocenter and 9-point circle of $\triangle I_aI_bI_c$, then P, Q, R are the midpoints of II_a, II_b, II_c $\Longrightarrow \triangle PQR$ is homothetic to $\triangle I_aI_bI_c$ under a homothety with center $I \Longrightarrow I$ is also orthocenter of $\triangle PQR$. Consequently, A is found as the second intersection of $\odot(PQR)$ with the perpendicular from P to RQ and similarly B and C.

rodinos

#3 Jun 13, 2014, 10:19 pm

Hello Luis and Thanks :)

The solution appeared in the magazine the problem was published in, is based also on homothety:
the tangents to (known) circumcircle of PQR at P,Q,R [I am using your notation] bound a triangle homothetic to ABC.

This construction leads to a locus problem, I have already posted (*):

Let ABC be a triangle, P a point, A'B'C' the circumcevian triangle
of P and A"B"C" the antipedal triangle of O wrt triangle A'B'C'
(tangential triangle of A'B'C').
Which is the locus of P such that ABC, A"B"C" are perspective?
(I is on the locus)

(*)
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/22423

Luis González

#4 Jun 14, 2014, 12:14 am

You're welcome dear Antreas. Regarding the locus problem, it is simply the whole plane, i.e. the lines AA",BB",CC" concur for any point P. This is usally referred as "Steinbart theorem".

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=249
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=488425
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=569985

Quick Reply

High School Olympiads

Determination in a convex quadrilateral.

Reply

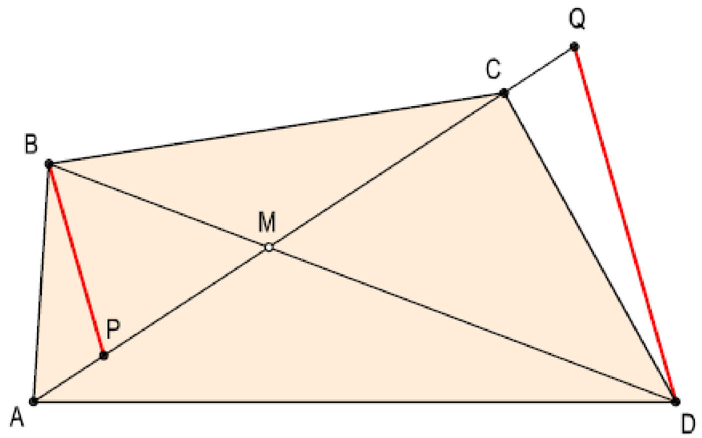
vittasko

#1 Jun 13, 2014, 2:45 pm

The diagonal BD of a given convex quadrilateral passes through the midpoint M of the other diagonal AC. Determine two points $P,\ Q$ on the line AC, such that $AP = CQ$ and $BP \parallel DQ$.

Kostas Vittas.

Attachments:



Luis González

#2 Jun 13, 2014, 8:46 pm

It's very easy Kostas, we can try a more general problem: Given 4 points A, B, C, D in general position and a fixed line ℓ on its plane, find a point X on ℓ such that XB, XD cut AC at points P, Q satisfying that the segments $\overline{AP}$ and $\overline{CQ}$ are congruent and equally oriented.

As X runs on ℓ, the pencils BX, DX are perspective inducing a homography $\mathbb{H}_1$ on AC and as P, Q vary on AC, such that $\overline{AP}$ and $\overline{CQ}$ are congruent and equally oriented, then $\mathbb{H}_2 : P \mapsto Q$ is another homography on AC. Thus, the desired points Q are the fixed points of the homography $\mathbb{H}_3 \equiv \mathbb{H}_1 \circ \mathbb{H}_2$. This gives rise to at most 2 solutions when $\mathbb{H}_3$ is hyperbolic or any solution if $\mathbb{H}_3$ is elliptic.

Quick Reply

High School Olympiads

Interesting concurrency 1

Reply

wiseman

#1 Jun 11, 2014, 1:42 pm

Ellipse W is tangent to segments BC , CA and AB of triangle ABC in D,E and F

respectively. Line d_a is tangent to the circumcircle of triangle ABC and passes

through A. Lines d_b and d_c are defined respectively. Let d_b and d_c intersect

each other in A'. Points B' and C' are defined respectively.

1) Prove that A'D, B'E and C'F re concurrent.

2) Prove that if the concurrency point is K, then K is the Lemoine point of

triangle A'B'C' if and only if the foci of the ellipse W are O and H.

(O is the circumcenter and H is the orthocenter of triangle ABC).

Luis González

#2 Jun 13, 2014, 5:31 am • 1

AD, BE, CF concur at the perspector P of the inconic WRT $\triangle ABC$ (this follows by Brianchon theorem in its degenerate form), thus by Cevian Nest Theorem $A'D, B'E, C'F$ concur at the cevian quotient P/K where $K \equiv AA' \cap BB' \cap CC'$ is the symmedian point of $\triangle ABC$. Since any inconic is unambiguosly defined by its perspector, then it suffices to show that the symmedian point of $\triangle A'B'C'$ is the cevian quotient between the perspector of the MacBeath inconic of $\triangle ABC$ (the inconic with foci O,H) and K.

By reflective property, BC bisects $\widehat{HDO} \Longrightarrow$ reflection M of O on BC is on HD. If AO cuts BC at D', then from parallelogram $AHMO$, we get $\widehat{DOM} = \widehat{HMO} = \widehat{HAO} = \widehat{D'OM} \Longrightarrow \triangle ODD'$ is O isosceles, i.e. D and D' are symmetric WRT the midpoint of BC. Thus, by symmetry $A'D$ and $A'D'$ are isogonals WRT $\widehat{BA'C}$.

Perpendicular ℓ_A from A' to OD' is the polar of D' WRT $(O) \Longrightarrow$ pencil formed by lines $\ell_A, A'B, A'C$ and $A'D'$ is harmonic, but since $\ell_A \parallel B'C'$, then it follows that $A'D'$ is the A'-median of $\triangle A'B'C' \Longrightarrow A'D$ is the A'-symmedian of $\triangle A'B'C'$. Similarly, $B'E$ and $C'F$ are the B'- and C'- symmedians of $\triangle A'B'C'$ and the conclusion follows.

Quick Reply

High School Olympiads

Three equilateral triangles

Source: Centroamerican Olympiad 2014, problem 5

Jutaro

#1 Jun 12, 2014, 2:01 am

Points A, B, C and D are chosen on a line in that order, with AB and CD greater than BC. Equilateral triangles APB, BCQ and CDR are constructed so that P, Q and R are on the same side with respect to AD. If $\angle PQR = 120^\circ$, show that

$$\frac{1}{AB} + \frac{1}{CD} = \frac{1}{BC}.$$

Luis González

#2 Jun 12, 2014, 3:37 am

Let $E \equiv PB \cap CR$. $\triangle EBC$ is obviously equilateral. Since $\widehat{PQR} = 120^\circ = 90^\circ + \frac{1}{2}\widehat{PER}$ and EQ bisects $\widehat{PER}$ internally, we deduce that Q is the incenter of $\triangle EPR$. Hence, if $M \equiv PQ \cap ER$ and $N \equiv RQ \cap EP$, then by angle bisector theorem, using $QB \parallel ER$, we get $\frac{BC}{AB} = \frac{BQ}{PB} = \frac{EM}{PE} = \frac{RM}{PR}$. Similarly $\frac{BC}{CD} = \frac{PN}{PR}$, hence it is enough to show that $PN + RM = PR$.

Clearly Q is midpoint of the arc MN of $\odot(EMN)$, i.e. $\triangle QMN$ is Q-isosceles with $\widehat{MQN} = 120^\circ$. Thus, reflection X of M on RQ forms equilateral triangle $\triangle XMN$ with center $Q \Longrightarrow X$ is also reflection of N on PQ. Hence, $RM = RX$ and $PN = PX \Longrightarrow PR = PX + RX = PN + RM$, as desired.

Jutaro

#3 Jun 12, 2014, 3:49 am

Nice solution, Luis. Here is a trigonometric proof.

Set $x = AB, y = BC, z = CA$. By the law of cosines we have

$$\begin{aligned} PQ^2 &= x^2 + y^2 - 2xy\cos 60^\circ = x^2 + y^2 - xy \\ QR^2 &= y^2 + z^2 - 2yz\cos 60^\circ = y^2 + z^2 - yz. \end{aligned}$$

Moreover, a quick application of the Pythagorean theorem allows us to write PR in terms of x, y and z:

$$\begin{aligned} PR^2 &= \left(y + \frac{x+z}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}(z-x)\right)^2 \\ &= x^2 + y^2 + z^2 + xy + yz - xz. \end{aligned}$$

On the other hand, the condition $\angle PQR = 120^\circ$ together with the law of cosines yields the following constraint on PQ, QR and PR:

$$PR^2 = PQ^2 + QR^2 - 2PQ\cdot QR\cos 120^\circ = PQ^2 + QR^2 + PQ\cdot QR.$$

Substituting the first three equations we get

$$x^2+y^2+xy+yz-zx = x^2+y^2-xy+y^2+z^2-yz+\sqrt{(x^2+y^2-xy)(y^2+z^2-yz)}$$

which is equivalent to

$$2xy+2yz-xz = \sqrt{(x^2+y^2-xy)(y^2+z^2-yz)}.$$

Squaring both sides of the last equation and rearranging we arrive at

$$3x^2y^2+3y^2z^2+9y^2zx-3x^2yz-3xy^3-3xyz^2-3y^3z=0.$$

The left-hand side factorizes as $3y(x-y+z)(xy+yz-zx)$, which means that either $y=0$, $x-y+z=0$ or $xy+yz-zx=0$. The first case is discarded since $y>0$, and the second one is impossible given the condition $x,z>y$. Hence $xy+yz-zx=0$, i.e. $1/x+1/z=1/y$, which is just what we want.

Concyclicboy

#4 Jun 12, 2014, 11:38 am • 1

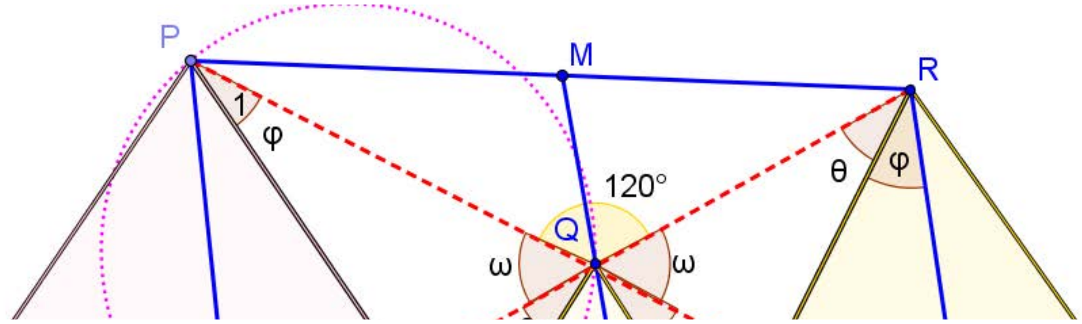
1. The circuncircle of QCR meets AB at E
2. Note that QRE is equilateral, then by Ptolemy's theorem $CR=QC+CE$ or $CD=BC+CE$
3. Since $\widehat{PQR}=120^\circ$ we have $\triangle QMN\sim\triangle CRE \Longrightarrow CE=BC*CD/AB$
4. Then $CD=BC+BC*CD/AB \Longrightarrow \dfrac{1}{AB}+\dfrac{1}{CD}=\dfrac{1}{BC}$

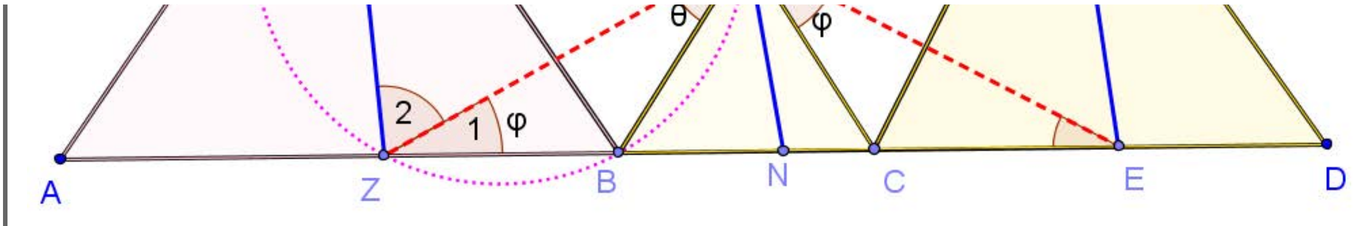
Tsikaloudakis

#5 Jun 14, 2014, 1:06 am

$$\begin{aligned}
&Let\ E=PQ\cap AD\ and\ Z=RQ\cap AD.\\
&we\ have\ :\\
&Z\hat{Q}E=120^o\ \Rightarrow\ \theta+\varphi=60^o\\
&\hat{Z}_1+\theta=60^o\ \Rightarrow\ \hat{Z}_1=\varphi\\
&\hat{Z}_1=\hat{\mathrm{P}}_1\ \Rightarrow\ P,Z,B,Q\ is\ concyclic\ \Rightarrow\ \hat{Z}_2=60^o\ \Rightarrow\ ZP||ER\ \Rightarrow\\
&EZPR\ is\ trapezium.\\
&Let\ MN||ZP\ then\ tr.BZR\approx tr.CQN\approx tr.EDR\ and\ QM=QN\ \Rightarrow\\
&\left.\begin{matrix}\dfrac{BC}{CD}=\dfrac{ZQ}{ZR}\\ \dfrac{BC}{AB}=\dfrac{QR}{ZR}\end{matrix}\right\}\ \Rightarrow\ \dfrac{BC}{CD}+\dfrac{BC}{AB}=1\ \Rightarrow\ \dfrac{1}{CD}+\dfrac{1}{AB}=\dfrac{1}{BC}
\end{aligned}$$

Attachments:





panamath

#6 Feb 1, 2015, 2:55 am

Let $m\angle BQP = \alpha$ and $m\angle CQR = \beta$. It's easy to see that $\alpha + \beta = 180^\circ$.

By sine law on $\triangle BPQ$ we have

$$\frac{BQ}{BP} = \frac{BC}{AB} = \frac{\sin\angle BPQ}{\sin\angle BQP} = \frac{\sin(120-\alpha)}{\sin\alpha}$$

And on $\triangle CQR$ we have

$$\frac{CQ}{CP} = \frac{BC}{CD} = \frac{\sin\angle CRQ}{\sin\angle CQR} = \frac{\sin(120-\beta)}{\sin\beta}.$$

Thus

$$\frac{BC}{AB}+\frac{BC}{CD} = \frac{\sin(120-\alpha)}{\sin\alpha}+\frac{\sin(120-\beta)}{\sin\beta} = \frac{\sin(\alpha+60)}{\sin\alpha}+\frac{\sin(\alpha-60)}{\sin(180-\alpha)}$$

$$= \frac{2\sin\alpha\cos 60^\circ}{\sin\alpha} = 1$$

.

Hence the result.

Quick Reply

High School Olympiads

Very hard

malgnaig
#1 Jun 11, 2014, 5:21 pm

Given triangle ABC with circumcircle (O) and incircle (I). Call M, N, P, Q, R, S, respectively the symmetric point of B, C, C, A, A, B through the line IC, IB, IA, IC, IB, IA. Call X, Y, Z is the center of the circumscribed circle of triangle AMN, BPQ, CRS.
1. Prove I is the cirumcenter of triangle XYZ
2. Assuming fixed BC, A change in the (O). Prove Euler line of triangle XYZ passing a fixed point.

Luis González
#2 Jun 12, 2014, 12:39 am • 1

1) WLOG assume that $AC > CB > BA.$ Let the incircle (I,r) touch BC, CA, AB at D, E, F and P, Q denote the midpoints of the arcs BC and BAC of the circumcircle $(O,R).$ Clearly $BN = CM = BC$ (M between A,C and A between B,N) and since $QB = QC, \angle QCM = \angle QBN,$ it follows that $\triangle QCM \cong \triangle QBN$ by SAS $\Longrightarrow \angle QMC = \angle QNA$ $\Longrightarrow AMQN$ is cyclic with circumcircle $(X).$

Since $\angle PAQ$ is right, (X) cuts AP again at the antipode U of $Q.$ From $\triangle UPQ \sim \triangle MCQ$ (due to $\angle QUP = \angle QMC,$ $\angle UPQ = \angle MCQ$) and the similar isosceles $\triangle PBC \sim \triangle OQC$ (due to $\angle BPC = \angle QOC = 180^\circ - \angle BAC$), we get

$$\frac{PI}{BC} = \frac{PC}{BC} = \frac{R}{CQ} \Longrightarrow 2 \cdot \frac{PI}{CM} = \frac{PQ}{CQ} = \frac{PU}{CM} \Longrightarrow PU = 2 \cdot PI,$$

thus I is midpoint of $\overline{PU} \Longrightarrow IX$ is U-midline of $\triangle UPQ \Longrightarrow IX \parallel PQ \perp BC \Longrightarrow X \in ID$ and $IX = \frac{1}{2}PQ = R.$ Analogously, Y, Z are on the extensions of $EI, FI,$ such that $IY = IZ = R \Longrightarrow I$ is circumcenter of $\triangle XYZ.$

2) $\triangle XYZ$ is then image of $\triangle DEF$ under a homothety with center $I,$ which leaves all lines through I fixed. IO is Euler line of $\triangle DEF$ (well-known), thus IO is also Euler line of $\triangle XYZ.$ Therefore, the fixed point is none other than $O.$

TelvCohl
#3 Nov 1, 2014, 12:02 pm

My solution:

Let I_a be the A-excenter of $\triangle ABC$.
Let A', B', C' be the midpoint of arc CAB, arc ABC, arc BCA .
Let M', N', X' be the reflection of M, N, X in AI , respectively .
Let A'', B'', C'' be the point on (I, IO) satisfy $OA''//BC, OB''//CA, OC''//AB$.

Since $A'B = A'C, BN = CM, \angle A'BN = \angle A'CM$,
so $\triangle A'BN \cong \triangle A'CM$,
hence we get $\angle A'NA = \angle A'MA$. ie. $A' \in (AMN)$
Similarly, we can prove $B' \in (BPQ), C' \in (CRS)$.

Since $AM' = OC'', AN' = OB''$,
so $\triangle AM'N' \cap X' \cong \triangle OC''B'' \cap I$,
hence we get $IX'AO$ is a parallelogram and $IX = IX' = OA = R$.
Similarly, we can prove $IY = R$ and $IZ = R$,
so I is the circumcenter of $\triangle XYZ$.

Since $I_a = BB' \cap CC', AS = AB, AP = AC$,
so we get $I_aB * I_aB' = I_aC * I_aC'$ and $AB * AP = AC * AS$.
ie. AI is the radical axis of $\{(Y), (Z)\}$ and $AI \perp YZ$

Since OX is the perpendicular bisector of AA' ,
so we get $XO//AI$ and $XO \perp YZ$.
Similarly, we can prove $YO \perp ZX$ and $ZO \perp XY$,
so O is the orthocenter of $\triangle XYZ$,
hence the **Euler line** of $\triangle XYZ$ pass through a fixed point O when A varies on (O) .

Q.E.D

Quick Reply

High School Olympiads

geogemetry

malgnaig

#1 Jun 11, 2014, 5:14 pm

Given triangle ABC inscribed (O). X lies in any triangle. AX, BX, CX cut (O) in A ', B', C '. Tangent of (O) in A ', B', C 'cut BC, CA, AB at M, N, P. prove M, N, P are collinear.

Luis González

#2 Jun 11, 2014, 9:21 pm

It's a particular case of the following configuration:

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=274286
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=420593

High School Olympiads

Easy geo

Locked

Source: Romania JBMO TST 2014 Problem 3

junioragd
#1 Jun 11, 2014, 5:03 am

Let ABC be an acute triangle and let O be its circumcentre. Now, let the diameter PQ of circle ABC intersects sides AB and AC in their interior at D and E, respectively. Now, let F and G be the midpoints of CD and BE. Prove that <FOG=<BAC

Also, it is pretty strange that only 3 olympiad team members solved it.

Luis González
#2 Jun 11, 2014, 5:19 am

I find it amusing how this problem appears in competitions over and over again. Even a generalization has been discussed before in this forum.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=16567
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=218486
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=291269
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=439734
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=487476

High School Olympiads

Perspector on the Euler Line

Reply

Source: Hyacinthos #22404

rodinos

#1 Jun 10, 2014, 5:11 pm

Let ABC be a triangle and A'B'C' the pedal triangle of H (orthic triangle).

Let A*, B*, C* be the other than O intersections of the lines A'O, B'O, C'O and the circles BOC, COA, AOB, respecvtively.

The triangles ABC, A*B*C* are perspective.

Generalizations (Loci)
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/22404

Luis González

#2 Jun 10, 2014, 8:51 pm • 1

Inversion in (O) leaves A, B, C fixed and takes lines BC, CA, AB into the circles $\odot(OBC), \odot(OCA), \odot(OAB) \Longrightarrow$ A^*, B^*, C^* are the inverse images of A', B', C' under the referred inversion. Lines AA^*, BB^*, CC^* go to the circles $\odot(AOA'), \odot(BOB'), \odot(COC')$, thus it is enough to prove that these meet at a 2nd point (other than O). Indeed, if H denotes the orthocenter of $\triangle ABC$, then $HA \cdot HA' = HB \cdot HB' = HC \cdot HC'$, i.e. H has equal power WRT $\odot(AOA'), \odot(BOB'), \odot(COC') \Longrightarrow$ they are coaxal with radical axis $OH \Longrightarrow$ they meet again at a point $X \in OH$. Therefore, AA^*, BB^*, CC^* meet at the inverse X^* of X in the circumcircle, also lying on OH.

Quick Reply

High School Olympiads

Ellipse

Reply

wiseman

#1 May 22, 2014, 12:53 am

In triangle ABC,I is the incenter.The incircle is tangent to BC,AC,AB in D,E,F respectively.

We draw the ellipse **W_a** with foci A and I which passes through E and F. ellipses **W_b**

and **W_c** are defined like **W_a**. Now take **A'** as the intersection point of **W_b**

and **W_c**. Points B' and C' are defined respectively. prove that AA' , BB' and CC' are

concurrent.

Luis González

#2 Jun 8, 2014, 10:34 pm • 1

Arc $FA'D$ of ellipse $\mathcal{W}_b$ is the locus of the centers of the circles externally tangent to the circle $\odot(B,BD)$ and internally tangent to (I). Likewise arc $EA'D$ of ellipse $\mathcal{W}_c$ is the locus of the centers of the circles externally tangent to the circle $\odot(C,CD)$ and internally tangent to $(I) \Longrightarrow A'$ is the center of the circle (A') externally tangent to $\odot(B,BD), \odot(C,CD)$ and internally tangent to (I). B', C' are redefined similarly.

Let EF, FD, DE cut BC, CA, AB at X, Y, Z. Line $\overline{XYZ}$ is the trilinear polar of the Gergonne point $AD \cap BE \cap CF$. X is clearly the exsimilicenter of $\odot(B,BD)$ and $\odot(C,CD)$, which is also center of direct inversion that interchange them. This inversion takes $\odot(B,BD)$ into $\odot(C,CD)$ and leaves (I) and $\odot(A,AE)$ fixed, because $XD^2 = XE \cdot XF$. Thus by conformity, the inverse of (B') is none other than $(C') \Longrightarrow X \in B'C'$. Analogously, $Y \in C'A'$ and $Z \in A'B' \Longrightarrow \triangle ABC$ and $\triangle A'B'C'$ are perspective through $\overline{XYZ}$. By Desargues theorem AA', BB', CC' concur.

Quick Reply

High School Olympiads

Reverse of a known theorem of geometry

Reply

rodinos
#1 Jun 7, 2014, 5:33 am

Let ABC be a triangle and I its incenter. A circle passing through I intersects the lines
AI, BI, CI at A',B',C' resp. other than I. Prove that the reflections of BC in B'C', CA in C'A', AB in A'B'
coincide in one line.

aph

Luis González
#2 Jun 7, 2014, 8:38 am • 1

WLOG assume that $B' \in \overline{IB}, C' \in \overline{IC}$ and A' is on the extension of $\overrightarrow{AI}$. The remaining configurations are treated similarly. Let $B'C', C'A', A'B'$ cut BC, CA, AB at $X, Y, Z.$ From cyclic $IB'A'C'$, we get $\widehat{ZA'Y} = 180^\circ - \widehat{BIC} = 90^\circ - \tfrac{1}{2}\widehat{YAZ}$, and since AA' bisects $\widehat{YAZ}$ internally, then it follows that A' is the A-excenter of $\triangle AYZ \Longrightarrow YZ$ is the reflection of CA, AB on $C'A', B'A'$, resp. Similarly, since $\widehat{XC'Y} = \widehat{BIA} = 90^\circ + \tfrac{1}{2}\widehat{XCY}$ and CC' bisects $\widehat{XCY}$ internally, it follows that C' is incenter of $\triangle CXY \Longrightarrow XY$ is reflection of BC, CA on $B'C', C'A'$, resp. Therefore, X, Y, Z are collinear and this line coincides with the reflections of BC, CA, AB on $B'C', C'A', A'B'$, as desired.

jayme
#3 Jun 7, 2014, 9:58 am

Dear Mathlinkers,
which is the "the known theorem of Geometry" or its name?
Sincerely
Jean-Louis

rodinos
#4 Jun 7, 2014, 12:51 pm

Reverse of the Generalized Collings Theorem
Hyacinthos #22387
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/22387

jayme
#5 Jun 7, 2014, 2:07 pm

Dear A. and Mathlinkers,
why the Collings theorem is not attribuate to Steiner? the lines concur on the anti Steiner point...
Sincerely
Jean-Louis

rodinos
#6 Jun 7, 2014, 2:24 pm

It is a result by Collings.
cf:
"We begin with a RESULT by S. N. Collings [1]................" (capitals mine)

Darij Grinberg, Anti-Steiner points with respect to triangle.
Zipped PDF file here:
http://web.mit.edu/~darij/www/geometry2.html

jayme
#7 Jun 7, 2014, 3:47 pm

Dear A. and Mathlinkers,
thank for your various references.
Can you precise the reference where you have proposed for the time your nice result (incenter on the circumcircle).
Sincerely
Jean-Louis

rodinos
#8 Jun 7, 2014, 11:06 pm

Cosmin Pohoata, in reply to me, wrote:
"Indeed, I used the fact you mentioned as a main lemma in the proof of
my problem."
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/16732

Cosmin refers to his paper on Parry point:
http://forumgeom.fau.edu/FG2008volume8/FG200806index.html

Luis González
#9 Jun 8, 2014, 3:14 am

It's worth remarking that this configuration and other related results were discussed before. See the following topics

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=379391
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=421236
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=541836

jayme
#10 Jun 8, 2014, 12:16 pm

Dear A., Luis and Mathlinkers,
thank very much for your reference which permit me to have an overlook of this problem...
Sincerely
Jean-Louis

rodinos
#11 Jun 9, 2014, 11:47 pm

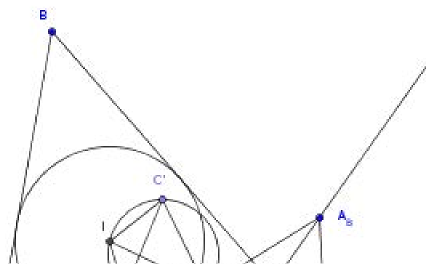
Application to geometry of triangle
(trilinear pole of the line)

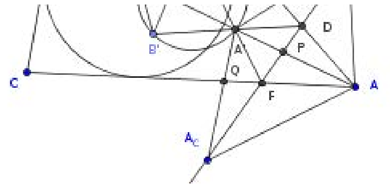
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/22394

Particle
#12 Jun 18, 2014, 11:30 am

Proof

Attachments:





Quick Reply

High School Olympiads

Beautiful locus 2

Reply

wiseman

#1 May 28, 2014, 12:38 pm

(I) is the incircle of triangle ABC. The arbitrary point D is lying on segment BC.

(O) and (O') are the incircles of triangles ABD and ACD respectively.

Let K be the radical center of Circles (I),(O) and (O'). Find the locus of K.

Luis González

#2 Jun 6, 2014, 8:08 am • 1

(I) touches CA, AB at Y, Z, (O) touches BA at U and (O') touches CA at V. Radical axis τ_B of $(I), (O)$ is the line through the midpoint M of $\overline{ZU}$ perpendicular to BI and radical axis τ_C of $(I), (O')$ is the line through the midpoint N of $\overline{YV}$ perpendicular to CI. $K \equiv \tau_B \cap \tau_C$ is the radical center of $(I), (O), (O')$.

In the problem Beautiful locus 4, we proved that $O \mapsto O'$ is a homography between IB, IC. Since the directions of OU and $O'V$ remain fixed, then $U \mapsto V$ is a homography between $AB, AC \Longrightarrow M \mapsto N$ is consequently a homography between AB, AC as well. Since the directions of KM, KN remain fixed (perpendicular to IB,IC), it follows that K moves on a fixed hypebola $\mathcal{K}$ with asymptotes perpendicular to IB, IC. As D is restricted between B and C, the locus of K is a part of $\mathcal{K}$.

Quick Reply

High School Olympiads

Beautiful locus 3

Reply

wiseman

#1 May 28, 2014, 7:59 pm

(O) is the circumcircle of triangle ABC. D is an arbitrary point lying on the circle (O).

E and F are the perpendicular feet from D to AB and AC respectively. Let G be the

midpoint of segment EF. find the locus of G.

wiseman

#2 May 29, 2014, 9:16 pm • 1

Hint : the locus is an ellipse.

Luis González

#3 Jun 6, 2014, 4:00 am • 2

The problem is a particular case of the following configuration:

e, f are two fixed lines meeting at A and $\mathcal{C}$ is a fixed conic on its plane. P runs on $\mathcal{C}$ and $E \in e, F \in f$, such that the directions of PE and PF are fixed. Then the locus of the midpoint G of $\overline{EF}$ is another conic of the same kind.

Proof: Let the parallel from F to PE cut e at U and Q is the midpoint of $\overline{FU}$. Since all $\overline{FU}$ remaing parallel, then Q runs on a fixed line q through A. If M is the midpoint of $\overline{PE}$ and GM cuts AQ, FU at N, V, then all $\triangle QVG \cup N$ remain similar, because $QG \parallel e, VGM \parallel PF, FQ \parallel PE$ and $\angle AQU = \text{const} \Longrightarrow \overline{GN} : \overline{GV} = \overline{GN} : \overline{GM}$ is constant. Therefore $P \mapsto M$ is an affine homology fixing e and $M \mapsto G$ is an affine homology fixing $q \Longrightarrow$ the application $P \mapsto G$ is therefore an affine homography. This fixes the line at infinite, so if P describe a conic $\mathcal{C}$, then G describe another conic of the same kind.

Quick Reply

High School Olympiads

Beautiful locus 4

Reply

wiseman

#1 May 31, 2014, 8:27 pm

In triangle ABC, D is an arbitrary point on segment BC. E and F are the incenters

of triangles ABD and ACD respectively. Let G be a point lying on segment EF such

that (EG)/(GF) is a constant amount. Find the locus of G.

(Sorry if it's repeated!)

Luis González

#2 Jun 5, 2014, 10:09 pm

Let $I \equiv BE \cap CF$ be the incenter of $\triangle ABC$. Incircles $(E), (F)$ of $\triangle ABD, \triangle ACD$ touch BC at $U, V,$ resp and their internal common tangent, different from $AD,$ cuts BC at $X.$ Then we have
$BX = BU + UX = BU + VD = \frac{1}{2}(AB + BD - AD) + \frac{1}{2}(AD + DC - AC) \Longrightarrow$
$BX = \frac{1}{2}(AB + BC - AC) \Longrightarrow X$ coincides with the tangency point of the incircle (I) with $BC.$ Thus,
$\angle EXF = \angle EDF = 90^\circ.$

As E, F vary, the pencils XE, XF are then congruent $\Longrightarrow E \mapsto F$ is a homography betwen the lines IB and $IC.$ If the parallels from G to IB, IC cut IC, IB at $M, N,$ then $\frac{IM}{MF} = \frac{EG}{GF} = \frac{EN}{NI} \Longrightarrow M \mapsto N$ is a homography between IB, IC as well, thus G is on a fixed conic through the infinite points of $IB, IC,$ i.e. a hyperbola $\mathcal{H}$ with asymptotes parallel to $IB, IC.$ As D is restricted between B and $C,$ the locus of G is just part of $\mathcal{H}.$

Quick Reply

High School Olympiads

Beautiful locus 6

wiseman

#1 Jun 5, 2014, 1:28 am

Points A and B and circle W are given. Let C be an arbitrary

point on circle W. Let K be the Lemoine point of triangle ABC.

(Lemoine point of a triangle is the concurrency point of the

triangle's symmedians)

Find the locus of K.

Luis González

#2 Jun 5, 2014, 8:12 am • 1

Let τ_A, τ_B, τ_C be the tangents of $\mathcal{W}$ at $A, B, C,$ respectively. $A' \equiv \tau_B \cap \tau_C$ and $B' \equiv \tau_A \cap \tau_C \Longrightarrow K \equiv AA' \cap BB'$ is the symmedian point of $\triangle ABC.$ As C varies, τ_C envelopes $\mathcal{W} \Longrightarrow A' \mapsto B'$ is a homography between the fixed lines τ_A, τ_B inducing then a homography between AA' and $BB' \Longrightarrow K$ moves on a conic through $A, B,$ which is clearly an ellipse $\mathcal{E},$ because K is always ordinary (never at infinity). In addition, $\mathcal{E}$ is bitangent to $\mathcal{W}$ through $A, B,$ since τ_A and τ_B are the images of the rays BA and $AB,$ respectively, in the aforementioned homography.

High School Olympiads

Points on Vertices to Incenter

Locked

Source: MOP 2005 Homework - Black Group #16

Konigsberg

#1 May 24, 2014, 10:08 am

Let I be the incenter of triangle ABC, and let A_1, B_1, and C_1 be arbitrary points lying on segments AI,BI, and CI, respectively. The perpendicular bisectors of segments AA_1, BB_1, and CC_1 form triangles $A_2B_2C_2$. Prove that the circumcenter of triangle $A_2B_2C_2$ coincides with the circumcenter of triangle ABC if and only if I is the orthocenter of triangle $A_1B_1C_1$.

Luis González

#2 May 24, 2014, 10:31 pm

Konigsberg, please use the search before posting contest problems.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=380390
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=270902
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=231881
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=193159
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=5761

High School Olympiads

Draw tangents to incircle

Reply

Source: own

rodinos

#1 May 17, 2014, 11:10 am

We are given: Triangle ABC, its circumcircle, and a point P outside the triangle.
To construct with ruler only the tangents to incircle through P.
Is the circumcircle (or any other given circle) necessary for the construction?

Luis González

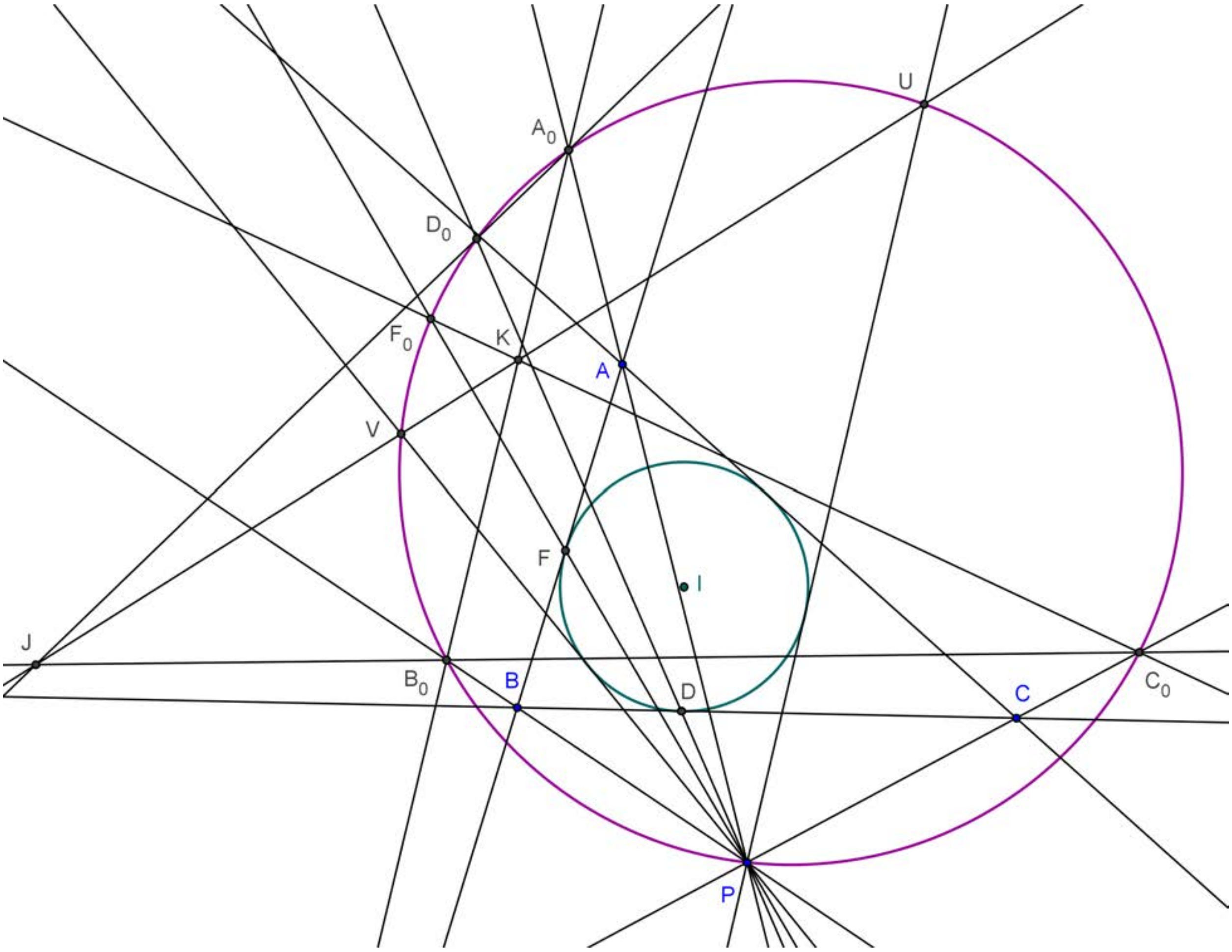
#2 May 20, 2014, 4:06 am • 1

In general, I think the circumcircle $\odot(ABC)$ is useless. Tangents from P to the incircle (I) or any inconic are double rays of 2 superposed homographic pencils with center $P,$ hence any conic not passing through P becomes useless to construct the double elements of the homography. We can use an arbitrary circle ω (or conic) through $P.$

Animate a tangent of $(I) \Longrightarrow$ its intersections with $AB,BC,$ for example, form homographic series, thus tangents from P to (I) are just the double rays of the referred homography $\mathbb{H}.$ If (I) touches BC,BA at $D,F,$ then we have $\mathbb{H}: C \mapsto A, B \mapsto F, D \mapsto B,$ just enough to define the homography. If we know a circle ω through P and the tangency points $D,F,$ then the construction of the double rays of $\mathbb{H}$ uses no more than a straight edge.

PA,PB,PC,PD,PF cut ω again at $A_0,B_0,C_0,D_0,F_0.$ Line through $J \equiv A_0D_0 \cap B_0C_0$ and $K \equiv A_0B_0 \cap C_0F_0$ is the axis of the homography $C_0 \mapsto A_0, B_0 \mapsto F_0, D_0 \mapsto B_0,$ cutting ω at its double points U,V as long as $\mathbb{H}$ is not elliptic, i.e. P outside $(I).$ Hence PU,PV are the desired tangents.

Attachments:



Quick Reply

High School Olympiads

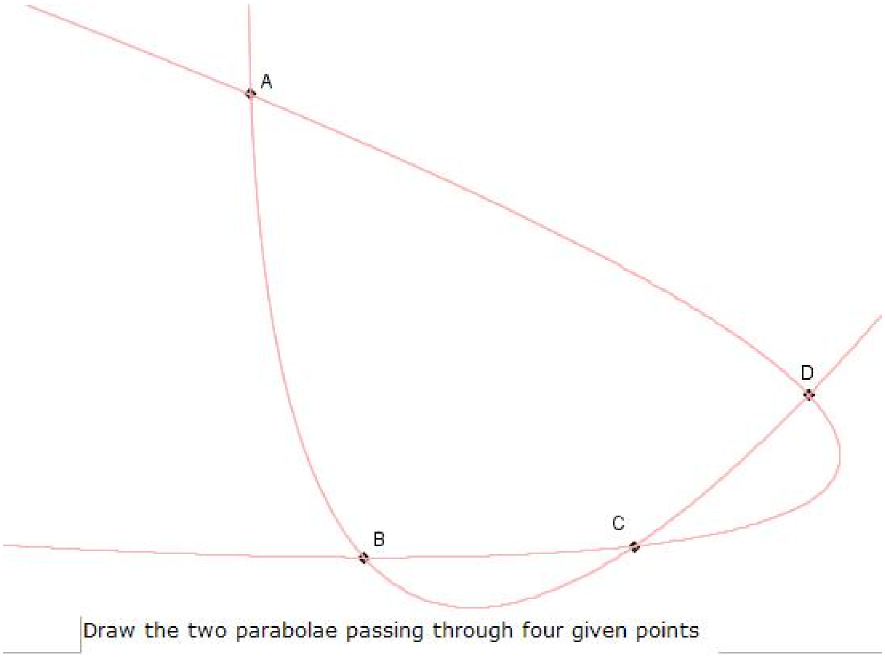
i want 2 points: 1 on a parabola and 1 on another parabola

Reply

jrrbc

#1 Jun 11, 2008, 3:38 am

Attachments:



Luis González

#2 May 19, 2014, 10:30 am

Denote $A_{\infty}, B_{\infty}, C_{\infty}, D_{\infty}$, the intersections of AB, BC, CD, DA with the line τ_{∞} at infinity. By Desargues theorem, the pencil of conics through A, B, C, D induce an involution on τ_{∞}, thus the parabolas $\mathcal{P}_1, \mathcal{P}_2$ through A, B, C, D touch τ_{∞} at the fixed points X_{∞}, Y_{∞} of the involution $\mathbf{I} : \{A_{\infty} \mapsto C_{\infty}, B_{\infty} \mapsto D_{\infty}\}$. One way two construct these directions is as follows: Let $P \equiv AB \cap CD$ and let u, v be the parallels through P to AD, BC, resp. If $U \equiv u \cap BC$, then $\mathbf{I} : \{B \mapsto C, U \mapsto B_{\infty}\} \Longrightarrow U$ is center of the involution on BC, therefore the circle centered at U and orthogonal to $\odot(PBC)$ (it can be any circle through B,C) cuts BC at the fixed points M, N on $BC \Longrightarrow PM, PN$ have the desired directions X_{∞}, Y_{∞}.

Now, technically the 2 parabolas $\mathcal{P}_1, \mathcal{P}_2$ are defined by 5 points $\{A, B, C, D, X_{\infty}\}$ and $\{A, B, C, D, Y_{\infty}\}$, but we can construct their foci and focal axes easily. For instance, using Pascal theorem for $AABCDX_{\infty}$, we have a construction of the tangent τ_A of $\mathcal{P}_1$ at A and similarly its tangents τ_B, τ_C, τ_D. Miquel point F_1 of the quadragle $(\tau_A, \tau_B, \tau_C, \tau_D)$ is the focus of $\mathcal{P}_1$ and the parallel from F_1 to X_{∞} determines its focal axis. Similarly, we construct the focus and focal axis of $\mathcal{P}_2$.

Quick Reply

High School Olympiads

Perpendiculars Bisectors concurrent at N

Reply

Source: Hyacinthos, message 22295

rodinos

#1 May 18, 2014, 1:56 pm

Let ABC be a triangle and A'B'C' the cevian triangle of O.

Denote:

Lb, Lc = the reflections of the AA' line in AB,AC, resp.

Ab, Ac = the orthogonal projections of A' on Lb,Lc, resp.

Similarly Bc,Ba and Ca,Cb.

The perpendicular bisectors of AbAc, BcBa, CaCb
are concurrent at the NPC center N.

Synthetic prof?

https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/22295

Luis González

#2 May 19, 2014, 12:27 am

Since $\angle AA_bA' = \angle AA_cA' = 90^\circ$, then the circumcircle (U) of the cyclic $AA_bA'A_c$ cuts BC again at the foot D of the A-atitude. Since $\angle BAA_b = \angle BAO$ and $\angle BAD = \angle CAO$, it follows that $\angle DAA_b = \angle BAC$ and likewise $\angle DAA_c = \angle BAC \Longrightarrow \angle DAA_b = \angle DAA_c \Longrightarrow D$ is midpoint of the arc $A_bA'A_c$ of $(U) \Longrightarrow UD$ is perpendicular bisector of $\overline{A_bA_c}$. Thus, it is enough to prove that the 9-point center N of $\triangle ABC$ lies on UD.

Let H be the orthocenter of $\triangle ABC$. Inversion with center H and power $\overline{HA} \cdot \overline{HD}$ swaps the circumcircle (O) and 9-point circle (N), leaving (U) fixed. Since (U) is internally tangent to (O) at A, then it is also tangent to (N) at D, by conformity $\Longrightarrow$ $N \in UD$, as desired.

Quick Reply

High School Olympiads

show that PQ is diameter

Reply

Source: Middle European Mathematical Olympiad 2013 T-6

syk0526
#1 May 17, 2014, 2:57 pm

Let K be a point inside an acute triangle ABC, such that BC is a common tangent of the circumcircles of AKB and AKC. Let D be the intersection of the lines CK and AB, and let E be the intersection of the lines BK and AC . Let F be the intersection of the line BC and the perpendicular bisector of the segment DE. The circumcircle of ABC and the circle k with centre F and radius FD intersect at points P and Q.
Prove that the segment PQ is a diameter of k.

Luis González
#2 May 18, 2014, 8:43 pm • 1

AK is common radical axis of $\odot(AKB), \odot(AKC)$ cutting its common tangent $\overline{BC}$ at its midpoint $M \Longrightarrow AK$ is the A-median of $\triangle ABC \Longrightarrow DE \parallel BC$ for any K on AM, thus $\widehat{DEK} = \widehat{CBK} = \widehat{DAK} \Longrightarrow ADKE$ is cyclic. Hence, by Miquel's theorem $\odot(BDK)$ and $\odot(CEK)$ meet again at $X \in BC$. Hence $\widehat{KDX} = \widehat{KBC} = \widehat{KAD} \Longrightarrow$ $\widehat{EDX} = \widehat{KAD} + \widehat{KDE} = \widehat{BAC}$ and similarly $\widehat{DEX} = \widehat{BAC}$, thus $\triangle XED$ is X-isosceles $\Longrightarrow X \equiv F$.

Since $\widehat{DFB} = \widehat{EFC}$ and $\widehat{BDF} = \widehat{BKF} = \widehat{ECF}$, then $\triangle DBF \sim \triangle CEF \Longrightarrow \frac{FD}{FC} = \frac{FB}{FE} \Longrightarrow FD^2 = FB \cdot FC$ $\Longrightarrow$ power of F WRT $\odot(ABC)$ equals $-FD^2 \Longrightarrow \odot(ABC)$ bisects $\odot(F, FD)$, i.e. PQ is a diameter of $\odot(F, FD)$.

Mikasa
#3 Jun 10, 2014, 12:28 pm

> *Luis González wrote:*
> AK is common radical axis of $\odot(AKB), \odot(AKC)$ cutting its common tangent $\overline{BC}$ at its midpoint $M \Longrightarrow AK$ is the A-median of $\triangle ABC \Longrightarrow DE \parallel BC$ for any K on AM, thus $\widehat{DEK} = \widehat{CBK} = \widehat{DAK} \Longrightarrow ADKE$ is cyclic.

I did the same thing at the start of my proof. Now wlog assume that $AB > AC$.

Now, let AF' be the A-symmedian of $\triangle ABC$. Then it is a well known fact that $\dfrac{BF'}{F'C} = \dfrac{AB^2}{AC^2}$. Now, applying sine law in $\triangle ABK$ we get that $KB^2 = AB^2 \cdot \dfrac{\sin\angle BAK}{\sin\angle AKB}$.
Similarly, $KC^2 = AC^2 \cdot \dfrac{\sin\angle CAK}{\sin\angle AKC}$. So,

$$\frac{KB^2}{KC^2} = \frac{AB^2}{AC^2} \cdot r, \text{ where } r = \frac{\sin\angle BAK \cdot \sin\angle AKC}{\sin\angle AKB \cdot \sin\angle CAK}.$$

Now we know that,
$\angle BAK = \angle KBC, \angle CAK = \angle KCB$ and $\angle AKC = 180^\circ - \angle CKM, \angle AKB = 180^\circ - \angle BKM$. Also, $\dfrac{\sin\angle KBC}{\sin\angle KCB} = \dfrac{KC}{KB}$, so we get that,

$$r = \frac{KC \cdot \sin\angle CKM}{KB \cdot \sin\angle BKM}$$

We introduce a lemma here:
Lemma: In $\triangle XYZ$, the point W lies on YZ (Possibly on the extension). Then

Lemma: In $\triangle XYZ$, the point W lies on YZ (Possibly on the extension). Then,

$$\frac{YW}{WZ}=\frac{XY\cdot\sin\angle YXW}{XZ\cdot\sin\angle ZXW}.$$

Proof: Applying sine law in $\triangle YXW, \triangle ZXW$ finishes it.

Now by this lemma, we get that $r=\frac{CM}{MB}=1$. So, $\frac{KB^2}{KC^2}=\frac{AB^2}{AC^2}=\frac{BF'}{F'C}$.
Thus KF' is the K-symmedian of $\triangle KBC$. Now $\angle CKF=\angle BKM=\angle EKA=\angle EDA=\angle DBF$ i.e. $F'\in\odot BKD$. Similarly $F'\in\odot CKE$. Now, $\angle DEK=\angle KBC, \angle F'EK=\angle KCB$. So, $\angle F'ED=\angle KBC+\angle KCB=\angle BKD=\angle BAC$. Similarly, $\angle F'DE=\angle BAC$. Thus we have that $F\equiv F'$. Also, FD, FE are tangents to $\odot ADKE$.

Some simple angle chasing now provides that $\angle BDF=\angle ACB, \angle CDF=\angle KBC$. So by applying sine law in $\triangle DBF, \triangle DCF$ we get that $\frac{DF}{FB}=\frac{\sin\angle ABC}{\sin\angle ACB}=\frac{AC}{AB}$ and that $\frac{FC}{DF}=\frac{\sin\angle KBC}{\sin\angle KCB}=\frac{KC}{KB}$. But we proved before that $\frac{AC^2}{AB^2}=\frac{KC^2}{KB^2}$. So we get that $\frac{DF}{FB}=\frac{FC}{DF}\Rightarrow FD^2=FP^2=BF\cdot FC$. Since F is the center of $\odot DEP$ we have that F is on the radical axis, PQ, of $\odot ABC, \odot DEP$. Thus PQ is a diameter of k.

Mikesar

#4 Jun 10, 2014, 7:06 pm

> *Luis González wrote:*
>
> Hence, by Miquel's theorem $\odot(BDK)$ and $\odot(CEK)$ meet again at $X\in BC$. Hence $\widehat{KDX}=\widehat{KBC}=\widehat{KAD}\Longrightarrow\widehat{EDX}=\widehat{KAD}+\widehat{KDE}=\widehat{BAC}$ and similarly $\widehat{DEX}=\widehat{BAC}$, thus $\triangle XED$ is X-isosceles $\Longrightarrow X\equiv F$.

We can also avoid angle chasing using the well known fact that if O is the circumcenter of $ADKE$ then $OX\perp BC$. But $OF\perp BC$ since O, F belong to the perpendicular bisector of DE and $DE\parallel BC$. The rest of my proof is the same.

saturzo

#5 Jun 12, 2014, 5:23 pm

Let AK, the radical axis of $\odot AKB$ and $\odot AKC$, intersect BC and DE at M and M', respectively; so $MB^2=MC^2\implies M$ is the midpoint of $BC\implies\pi-\angle DKE=\angle MBK+\angle MCK=\angle BAK+\angle KAC=\angle DAE$.
$\therefore K\in\odot ADE\implies\angle EDC=\angle CAK=\angle BCD\implies DE\parallel BC$. So, M' is the midpoint of DE. Now note that, in $\odot ADE, BC$ is the polar of the pole M' and $M'F\perp BC[\because M'$ is the midpoint of $DE]$. So, F is the inverted point of M' w.r.t. $\odot ADE$. And if O is the center of $\odot ADE$, then $OM'\perp DE$ as M' is the midpoint of chord the DE. So, FD and FE touch $\odot ADE$, simple angle chasing implies $\angle BDE=\angle ECF, \angle DBF=\angle CEF\implies\triangle DBF\sim\triangle CEF$. So, $FB.FC=FD.FE=FP^2$.
So if FP intersects $\odot ABC$ again at Q', then, $FQ'=FP\implies Q'\in k\implies Q'\equiv Q\implies PQ$ is the diameter of k - done!

Particle

#6 Jun 20, 2014, 1:13 pm

Proof

Quick Reply

High School Olympiads

euler line and simson line

Reply

tanerkalyoncu52
#1 May 18, 2014, 3:03 am

I have some problems. If you send geometric proofs of these problems, I have gratitude to you.

P1. ABC is a triangle and Ia, Ib, Ic are excenters. Prove that Euler lines of the triangles IaBC, IbCA, IcAB are concurrent at a point P. What are the properties of the point P.

P2. Let DEF be orthic triangle of ABC. Let's draw 9 points circle of ABC. Prove that Simson line of DEF respect to point P is parallel to Euler line of ABC.

Thanks for your efforts. Best regards … Taner Kalyoncu

Luis González
#2 May 18, 2014, 9:12 am • 1

This has been discussed before. For P1, Euler lines of IaBC, IbCA, IcAB concur at a point P on the circumcircle (ABC), which is the Feuerbach point X(100) of the antimedial triangle of ABC. See the links below for a proof and some more properties of P. As for P2, it is just the effect of X(100) being the isogonal conjugate of the perpendicular direction of the line connecting the circumcenter and incenter of ABC.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=50&t=378803
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=370155

Quick Reply

High School Olympiads

Isosceles triangle

Reply

Source: Middle European Mathematical Olympiad 2013 I-3

syk0526

#1 May 17, 2014, 2:53 pm

Let ABC be an isosceles triangle with $AC = BC$. Let N be a point inside the triangle such that $2\angle ANB = 180^\circ + \angle ACB$. Let D be the intersection of the line BN and the line parallel to AN that passes through C. Let P be the intersection of the angle bisectors of the angles CAN and ABN. Show that the lines DP and AN are perpendicular.

Luis González

#2 May 18, 2014, 7:06 am

Clearly $\widehat{ANB} = 90^\circ + \frac{1}{2}\widehat{ACB}$ means that N is on circle ω through A, B and tangent to CA, CB. Hence $\widehat{PAN} = \frac{1}{2}\widehat{CAN} = \frac{1}{2}\widehat{NBA} = \widehat{NBP} \Longrightarrow P$ is midpoint of the arc NA of $\omega \Longrightarrow P$ is on perpendicular bisector of $\overline{AN}$. $CD \parallel AN$ gives $\widehat{CDB} = \widehat{AND} = 90^\circ - \frac{1}{2}\widehat{ACB} = \widehat{CAB} \Longrightarrow D \in \odot(ABC) \Longrightarrow$ $\widehat{ADN} = \widehat{ADB} = \widehat{ACB} = 90^\circ - \frac{1}{2}\widehat{DNA} \Longrightarrow \triangle DAN$ is D-isosceles, therefore DP is perpendicular bisector of $\overline{AN}$ $\Longrightarrow DP \perp AN$, as desired.

Quick Reply

High School Olympiads

Two circles

Source: Tournament of Towns, Fall 2002, Senior A Level, P5

joybangla

#1 May 17, 2014, 10:09 pm

Two circles Γ_1, Γ_2 intersect at A, B. Through B a straight line ℓ is drawn and $\ell \cap \Gamma_1 = K, \ell \cap \Gamma_2 = M$ $(K, M \neq B)$. We are given $\ell_1 \parallel AM$ is tangent to Γ_1 at Q. $QA \cap \Gamma_2 = R$ $(\neq A)$ and further ℓ_2 is tangent to Γ_2 at R.
Prove that:

- $\ell_2 \parallel AK$
- ℓ, ℓ_1, ℓ_2 have a common point.

Luis González

#2 May 18, 2014, 5:03 am

If ℓ cuts ℓ_1 at $S,$ then it suffices to show that RS is tangent to Γ_2 and parallel to $AK.$

Angle chasing using the tangency between ℓ_1, Γ_1 and the cyclic quadrilateral $ARMB$ gives $\angle QBA = \angle SQR \equiv \angle SQA = \angle QAM = \angle SBR \Longrightarrow \angle ABR = \angle QBS$ and $QBRS$ is cyclic. Thus, from cyclic quadrilaterals $QBRS$ and $QBAK,$ we get $\angle QAK = \angle QBK \equiv \angle QBS = \angle QRS \Longrightarrow RS \parallel AK$ and $\angle ABR = \angle QBS = \angle QRS \Longrightarrow RS$ is tangent to $\Gamma_2,$ as desired.

jayme

#3 May 20, 2014, 7:36 pm

Dear Mathlinkers,
another approach...

1. X the point of intersection of KM and the tangent l1
2. According to Reim's theorem KQ // MR
3. According to a converse of the little Pappus theoreme applied to the hexagon XQKAMRX insert in the lines QAR and KBM, RX // AQ
4. by an simple angle chasing, we prove that RX is tangent to Gamma 2 at R.

Sincerely
Jean-Louis

High School Olympiads

Concurrent lines

Reply

Source: North Korea Team Selection Test 2013 #1

syk0526
#1 May 17, 2014, 3:41 pm

The incircle of a non-isosceles triangle ABC with the center I touches the sides BC, CA, AB at A_1, B_1, C_1 respectively. The line AI meets the circumcircle of ABC at A_2. The line B_1C_1 meets the line BC at A_3 and the line A_2A_3 meets the circumcircle of ABC at $A_4 (\neq A_2)$. Define B_4, C_4 similarly. Prove that the lines AA_4, BB_4, CC_4 are concurrent.

nima1376
#2 May 17, 2014, 8:43 pm

$(A_4, A, B, C) = -1 \Rightarrow AA_4, BB_4, CC_4$ are concurrent in lemone poin of ABC

Luis González
#3 May 17, 2014, 10:21 pm

nima1376, you are wrong, the quadrilateral ABA_4C is not harmonic in general. The concurrency point is not the Lemoine point of $\triangle ABC$ but its center X_{57}, i.e. the perspector of $\triangle ABC$ and the orthic triangle of $\triangle A_1B_1C_1$.

Let A_0, B_0, C_0 be the projections of A_1, B_1, C_1 on B_1C_1, C_1A_1, A_1B_1. D and M denote the midpoints of $\overline{BC}$ and the arc BAC of the circumcircle (O).

From $(B, C, A_1, A_3) = -1$, we get $A_3D \cdot A_3A_1 = A_3B \cdot A_3C = A_3A_4 \cdot A_3A_2 \Longrightarrow DA_1A_4A_2$ is cyclic $\Longrightarrow$ $\angle A_2A_4A_1 = \angle A_2DA_1 = 90^\circ \Longrightarrow M \in A_1A_4$. Furthermore, $A_1A_0 \perp A_3A_0$ implies that B_1C_1 bisects $\angle BA_0C$ externally or $\angle BA_0C_1 = \angle CA_0B_1$. Since $\angle BC_1A_0 = \angle CB_1A_0$, then $\triangle BA_0C_1 \sim \triangle CA_0B_1 \Longrightarrow$ $C_1A_0 : B_1A_0 = BC_1 : CB_1 = BA_1 : CA_1$, therefore isosceles $\triangle AB_1C_1 \sim \triangle MCB$ are similar with corresponding cevians AA_0 and $MA_1 \Longrightarrow \angle MA_1C = \angle AA_0B_1$. Hence since $A_0A_1A_4A_3$ is cyclic on account of the right angles at A_0, A_4, we deduce that A, A_0, A_4 are collinear and analogously $B_0 \in BB_4$ and $C_0 \in CC_4$. By Cevian Nest Theorem the conclusion follows.

nima1376
#4 May 17, 2014, 10:49 pm

> *Luis González wrote:*
> nima1376, you are wrong, the quadrilateral ABA_4C is not harmonic in general. The concurrency point is not the Lemoine point of $\triangle ABC$ but its center X_{57}, i.e. the perspector of $\triangle ABC$ and the orthic triangle of $\triangle A_1B_1C_1$.

sorry for my bad mistake
another solution
let O is circumcircle of triangle ABC
$A_2A_1 \cap O = K$.
$(B, C, A_1, A_4) = -1 \Rightarrow (K, A_4, B, C) = -1 \Rightarrow \frac{BA_4}{A_4C} = \frac{KB}{KC}$
but K is center of spiral similar goes BC_1 to $CB_1 \Rightarrow \frac{BK}{CK} = \frac{BC_1}{CB_1}$
so we are done with ceva...

laFiesta
#5 May 20, 2014, 6:47 am

North Korea is NOT Korea! "Korea" means South Korea. Why is this in Korea category?

Mikasa

#6 May 20, 2014, 12:13 pm

First we claim that $\frac{A_4B}{A_4C}=\frac{A_3B}{A_3C}$. For the sake of argument, let us assume that A_3 is on the ray BC.

Now $\angle BA_4A_3=180^\circ-\frac{\angle A}{2}$.
So, $\angle A_4A_3C=\angle A_4A_3B=180^\circ-\angle BA_4A_3-\angle A_4BA_3$
$=\frac{\angle A}{2}-\angle A_4BA_3=\frac{\angle A}{2}-\angle A_4BC=\angle A_2AC-\angle A_4AC$
$=\angle A_2AA_4=\angle A_2BA_4=\angle A_2BA_4$.

This means A_2B is tangent to the circle BA_3A_4 and A_2C is tangent to the circle CA_3A_4. From these information, we have that $A_4B=\frac{A_2B\times A_3B}{A_2A_3}$ and $A_4C=\frac{A_2C\times A_3C}{A_2A_3}$. Since $A_2B=A_2C$ we have $\frac{A_4B}{A_4C}=\frac{A_3B}{A_3C}$ as we claimed.

By similar arguments, we can show that, $\frac{B_4C}{B_4A}=\frac{B_3C}{B_3A}$ and $\frac{C_4A}{C_4B}=\frac{C_3A}{C_3B}$.

Now apply Menelaus's theorem for the lines A_3B_1, B_3C_1, C_3A_1. Then we have three equations:
1) $\frac{BA_3}{A_3C}\cdot\frac{CB_1}{B_1A}\cdot\frac{AC_1}{C_1B}=1$

2) $\frac{CB_3}{B_3A}\cdot\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}=1$

3) $\frac{AC_3}{C_3B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A}=1$

Multiply (1),(2),(3) and use the fact $AC_1=AB_1, BC_1=BA_1, CA_1=CB_1$ to get that,

$$\frac{A_3B}{A_3C}\cdot\frac{B_3C}{B_3A}\cdot\frac{C_3A}{C_3B}=1$$

Thus by our previous argument,

4) $\frac{A_4B}{A_4C}\cdot\frac{B_4C}{B_4A}\cdot\frac{C_4A}{C_4B}=1$

Now,by sine law, we have that $\frac{A_4B}{A_4C}=\frac{\sin\angle BAA_4}{\sin\angle CAA_4}$. Deriving similar expressions for $\frac{B_4C}{B_4A},\frac{C_4A}{C_4B}$, and using them in (4), we get that ,

$$\frac{\sin\angle BAA_4}{\sin\angle CAA_4}\cdot\frac{\sin\angle CBB_4}{\sin\angle ABB_4}\cdot\frac{\sin\angle ACC_4}{\sin\angle BCC_4}=1.$$

This the trigonometric form of Ceva's theorem. So AA_4, BB_4, CC_4 are concurrent.

jayme

#7 May 23, 2014, 4:34 pm

Dear Mathlinkers,
the problem have something to do with the A-mixtilinear incircle of ABC...
see
http://perso.orange.fr/jl.ayme , A new mixtilinear incircle adventure I, G.G.G. vol. 4, p. 20-21.
Sincerely
Jean-Louis

Konigsberg

#8 Dec 13, 2014, 7:22 pm

is this too hard for a problem 1 on an olympiad?

IDMasterz

#9 Dec 13, 2014, 9:05 pm

To Konigsberg; depends :) To jayme, yes indeed, my solution shows this directly. Let the perpendicular to I through AI meet BC at A_5, then note $A(A_5, A_4; B, C) = -1$. We know that $A_5B_5C_5$ are collinear and perpendicular to HI, so AA_4, BB_4, CC_4 are concurrent.

TelvCohl

#10 Dec 13, 2014, 9:18 pm • 1

My solution (for the original problem) :

Since AA_1, BB_1, CC_1 are concurrent at X_7 of $\triangle ABC$,
so from Desargue theorem we get A_3, B_3, C_3 are collinear . ... $(\star)$
Since A_4A_2, B_4B_2, C_4C_2 is the external bisector of $\angle CA_4B, \angle AB_4C, \angle BC_4A$, respectively ,

so combine with $(\star)$ we get $\frac{CA_4}{A_4B} \cdot \frac{AB_4}{B_4C} \cdot \frac{BC_4}{C_4A} = \frac{CA_3}{A_3B} \cdot \frac{AB_3}{B_3C} \cdot \frac{BC_3}{C_3A} = 1$.

ie. AA_4, BB_4, CC_4 are concurrent

Q.E.D

My solution (for $AA_4 \cap BB_4 \cap CC_4 \equiv X_{57}$) :

Let D be the projection of A_1 on B_1C_1 and X be the midpoint of arc BAC .

Since $(B, C; A_1, A_3) = -1$,

so $\frac{CA_4}{A_4B} = \frac{CA_3}{A_3B} = \frac{CA_1}{A_1B}$,

hence we get A_4A_1 is the bisector of $\angle CA_4B$ and $X \in A_4A_1$.

Since $D(B, C; A_1, A_3) = -1$,
so DA_1 is the bisector of $\angle BDC$,
hence we get $\triangle DC_1B \sim \triangle CB_1D$.

Since $\frac{B_1D}{DC_1} = \frac{CD}{DB} = \frac{CA_1}{A_1B}$,

so we get $\triangle AC_1B_1 \cap D \sim \triangle XBC \cap A_1$,
hence from $\angle BAD = \angle BXA_1 = \angle BAA_4$ we get $D \in AA_4$.
ie. X_{57} of $\triangle ABC$ lie on AA_4

Similarly, we can prove $X_{57} \in BB_4$ and $X_{57} \in CC_4$,
so we get AA_4, BB_4, CC_4 are concurrent at the X_{57} of $\triangle ABC$.

Q.E.D

IDMasterz

#11 Dec 15, 2014, 4:06 pm

To prove AA_4, BB_4, CC_4 are concurrent at X_{57} from my post, we can do this:

Let ℓ be the radical axis of $\odot ABC, \odot A_1B_1C_1$. Let the medial triangle of $A_1B_1C_1$ be $A_6B_6C_6$ and note that ℓ is the radical axis of $\odot A_6B_6C_6$ by harmonic conjugates (fact 1). So, note $AA_5 \cap B_1C_1 \in \ell$ for obvious reasons, and because of the fact 1 we conclude AA_5 meets B_1C_1 at the same point where the orthic axis of $A_1B_1C_1$ meets B_1C_1, so AA_4 contains the foot of the A altitude of $A_1B_1C_1$.

Aiscrim

#13 Dec 18, 2015, 2:34 pm

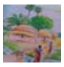
This problem is really weak in the sense that we don't need A_1, B_1, C_1 to be the tangency points of the incircle with the sides of $\triangle ABC$; it is enough to have AA_1, BB_1, CC_1 concurrent.

Let $\{X_A\} = A_1A_2 \cap (ABC)$. As $(A_3, A_1, B, C) = -1$, by perspectivity from A_2 we get that X_ABA_4C is harmonic. This yields $\dfrac{A_4B}{A_4C} = \dfrac{X_AB}{X_AC}$, but X_AA_2 is the bisector of $\widehat{BX_AC}$, whence $\dfrac{A_4B}{A_4C} = \dfrac{A_1B}{A_1C}$.

Writing the analogous relations and multiplying them we get that

$$\frac{A_4B}{A_4C} \cdot \frac{B_4C}{B_4A} \cdot \frac{C_4A}{C_4B} = \frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1$$

which is equivalent to the fact that AA_4, BB_4, CC_4 are concurrent.



Ankoganit

#14 May 23, 2016, 11:10 am

Let (J_a) be a circle tangent to BC at A_1 and tangent to (ABC) at some point, say T_a. Define $J_b, J_c; T_b, T_c$ similarly. We claim that $A_4 \equiv T_a$ etc.

Let X_a denote the midpoint of arc BAC of (ABC). If $\mathcal{H}$ denotes the homothety centered at T_a that takes (J_a) to (ABC), then it takes BC to the line ℓ parallel to BC and tangent to (ABC), clearly at X_a. Then $\mathcal{H}$ takes A_1 to X_a; so T_a, A_1, X_a are collinear. Since X_aA_2 is a diameter of (ABC), we have $X_aT_a \perp T_aA_2 \implies A_1T_a \perp T_aA_2$ $(\star)$.

Next, observe that $(B, C, A_1, A_3) = -1 \implies T_a(B, C, A_1, A_3)$ is a harmonic pencil. But T_aX_a and hence T_aA_1 is the bisector of $\angle BT_aC$, so we have $A_1T_a \perp T_aA_3$. Combining this with $(\star)$ gives A_3, T_a, A_2 are collinear, so $T_a \equiv A_4$.

But from Concurrency on OI we have AT_a etc. are concurrent at the Isogonal Mittenpunkt, as desired. $\blacksquare$

This post has been edited 1 time. Last edited by Ankoganit, May 23, 2016, 11:14 am

Quick Reply

High School Olympiads

Orthocentre's unique property

Reply

Source: Tournament of Towns, Fall 2002, Junior A Level, P4

joybangla

#1 May 17, 2014, 3:18 pm

Point P is chosen in the plane of triangle ABC such that $\angle ABP$ is congruent to $\angle ACP$ and $\angle CBP$ is congruent to $\angle CAP$. Show P is the orthocentre.

Luis González

#2 May 17, 2014, 9:05 pm

Solution 1. Let PA, PB, PC cut BC, CA, AB at $X, Y, Z,$ respectively. Then $\angle YCZ = \angle ZBY \Longrightarrow BCYZ$ is cyclic and $\angle XBY = \angle YAX \Longrightarrow ABXY$ is cyclic. Since P is on radical axis BY of $\odot(BCYZ)$ and $\odot(ABXY)$, then $CAZX$ is also cyclic. Hence $\angle BXZ = \angle BAC = \angle CXY$ and $\angle AXY = \angle ABY = \angle ACZ = \angle AXZ \Longrightarrow XA, BC$ bisect $\angle YXZ \Longrightarrow AX \perp BC$ and similarly, $BY \perp CA, CZ \perp AB$ and the conclusion follows.

Solution 2. If Q is the isogonal conjugate of P WRT $\triangle ABC$, we have $\angle CBQ = \angle ABP = \angle ACP = \angle BCQ \Longrightarrow QB = QC$ and $\angle ABQ = \angle CBP = \angle CAP = \angle BAQ \Longrightarrow QA = QB \Longrightarrow QA = QB = QC \Longrightarrow Q$ is circumcenter of $\triangle ABC \Longrightarrow P$ is orthocenter of $\triangle ABC$.

Quick Reply

High School Olympiads

Four concyclics points

Reply

Source: Mathley problem 2

jayme

#1 May 16, 2014, 7:53 pm

Dear Mathlinkers,

if I have correct translated this problem in vietnamiese…

1. ABC a triangle
2. (O), (I) the resp. circumcircle, incircle of ABC
3. D the second point of intersection of AI with (O)
4. M the point of intersection of the perpendicular to AD through I with BC
5. H the foot of the perpendicular to OI through M.

Prove : A, D, M and H are concyclic.

Sincerely
Jean-Louis

Please can any one precise the reference of this problem?

Ashutoshmaths

#2 May 16, 2014, 8:02 pm • 1

Geogebra says that it's false if et means and.

jayme

#3 May 16, 2014, 8:07 pm

Dear Mathlinkers,
I haven't the proof... but my programm says that they are concyclic..
If some other one can verify, welcome..
Sincerely
Jean-Louis

IMI-Mathboy

#4 May 16, 2014, 10:58 pm

Dear mathlinkers (from my solution) i think this one is correct and beautiful!

mathuz

#5 May 17, 2014, 12:26 am • 1

we have diffirent solutions!

Here one of them:
Let T is A-mixtilinear point of ABC, and we have M,T,D are collinear. Let O' is circumcenter of the triangle AMD. We have the triangles $OO'D$ and AMT are relatively similar and

$$OO'=\frac{DO\cdot MT}{AT}=\frac{MI}{2}.$$

So since $OO' \parallel MI$ we get that if $MO' \cap IO = X$ then $MO' = XO'$ and X lies on (O'). Hence H lies on the circle with diameter MX which the circumcircle of AMD.

Luis González

#6 May 17, 2014, 12:39 am • 1

(I) touches BC, CA, AB at $P, Q, R. T$ is the orthocenter of $\triangle PQR$ and L is the foot of the P-altitude of $\triangle PQR. PL$ is clearly the polar of M WRT (I), thus M is on the polar of T WRT $(I) \Longrightarrow MH \perp OIT$ is then the polar of T WRT $(I) \Longrightarrow M, H$ are the inverses of $K \equiv IM \cap PL$ and T WRT (I). Since the inverse of A WRT (I) is the midpoint U of QR, then it suffices to show that $\odot(UTK)$ cuts IA again at the inverse X of D WRT (I). Indeed, from isosceles trapezoid $UTKX$, we get $IX = TL$, i.e. $ILTX$ is parallelogram $\Longrightarrow X$ is the antipode of L WRT 9-point circle of $\triangle PQR$, the inverse of (O) WRT $(I) \Longrightarrow X$ is the inverse of D WRT (I).

IMI-Mathboy

#7 May 17, 2014, 2:54 am

Let K,T be intersection of (O) with MD and MA. Then we know $IT \perp MD$, $IK \perp MA$ and $IH \perp DA$.$(*)$.Let (O_1) be circumcircle of MAD with diametr MP then O is midpoint of PI (it is by the facts $(*)$), hence it is not difficult to see that H is on (O_1). SO we are done.

mathuz

#8 May 17, 2014, 3:02 am

wow, wonderful solution! :

Thanks, Luis.

jayme

#9 May 17, 2014, 2:56 pm

Dear Mathlinkers,
I have revisited the proof of Mathboy where there is some typos...
For me it is a nice synthetic proof because it involves K as the point of contact of the A-mixtilinear incircle of ABC, the line KI (Lauvernay's line), some converse of the Reim's theorem and finally the Brahmagupta's theorem...
Thank you for your outline of your proof.
Sincerely
Jean-Louis

Particle

#10 Jun 23, 2014, 1:23 pm • 1

Solution due to me and saturzo
Click to reveal hidden text

Quick Reply

High School Olympiads

incircle and excircles

Reply

Source: 2013 All-Russian Olympiad Final Round Grade 10 Day 2 P7

micliva

#1 May 16, 2014, 8:47 pm • 1

The incircle of triangle ABC has centre I and touches the sides BC, CA, AB at points A_1, B_1, C_1, respectively. Let I_a, I_b, I_c be excentres of triangle ABC, touching the sides BC, CA, AB respectively. The segments I_aB_1 and I_bA_1 intersect at C_2. Similarly, segments I_bC_1 and I_cB_1 intersect at A_2, and the segments I_cA_1 and I_aC_1 at B_2. Prove that I is the center of the circumcircle of the triangle $A_2B_2C_2$.

L. Emelyanov, A. Polyansky

Luis González

#2 May 16, 2014, 10:20 pm

Denote by r, R the radii of the incircle (I) and circumcircle (O) of $\triangle ABC$. If P is the circumcenter of $\triangle II_bI_c$, we have $I_bI_c \parallel B_1C_1, PI_c \parallel IB_1 \perp AC, PI_b \parallel IC_1 \perp AB \Longrightarrow \triangle IB_1C_1$ and $\triangle PI_cI_b$ are homothetic with homothety center $A_2 \equiv I_cB_1 \cap I_bC_1 \cap IP$. Hence using that $\odot(II_bI_c) \cong \odot(I_aI_bI_c)$ (ther radii equal $2R$), we get then

$$\frac{PA_2}{IA_2} = \frac{I_bI_c}{B_1C_1} = \frac{2R}{r} \Longrightarrow \frac{IP}{IA_2} = \frac{2R}{IA_2} = \frac{2R+r}{r} \Longrightarrow IA_2 = \frac{2R \cdot r}{2R+r},$$

which is obviously a symmetric expression, thus $IA_2 = IB_2 = IC_2 = \dfrac{2R \cdot r}{2R+r}$ and the conclusion follows.

thecmd999

#3 Sep 25, 2014, 11:08 am

Solution

TelvCohl

#4 Nov 29, 2014, 10:30 am

My solution **:**

Since $\triangle A_1B_1C_1$ and $\triangle I_aI_bI_c$ are homothetic ,
so we get $\dfrac{B_2I_a}{B_2C_1} = \dfrac{I_aI_c}{A_1C_1} = \dfrac{I_aI_b}{A_1B_1} = \dfrac{C_2I_a}{C_2B_1}$.
i.e. B_2 and C_2 are symmetry WRT AI

Similarly, we can prove C_2 and A_2 are symmetry WRT BI ,
so $IA_2 = IB_2 = IC_2 \Longrightarrow I$ is the circumcenter of $\triangle A_2B_2C_2$.

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Nov 28, 2015, 7:22 am

anantmudgal09

#5 Nov 28, 2015, 7:07 am

Solution:-

It is obvious that $\triangle A_1B_1C_1$ and $\triangle I_aI_bI_c$ are homothetic.

Now, it is also clear that C_2 is the center of the negative homothety $\mathbb{H}$ mapping A_1B_1 to I_bI_a.

Let C_3 be the image of I in $\mathbb{H}$. Then since the ratios of all the three such homotheties are the same(becuase the triangles are homothetic), it is equivalent to proving that $IC_3 = IA_3 = IB_3$.

Now, it is clear that in triangle $I_aI_bI_c$,I is the orthocenter and C_3 the reflection of the circumcenter of $I_aI_bI_c$ in I_aI_b. Now, this indirectly means that IC_3 is of the same length as the circumradius and hence we conclude.$\blacksquare$

This post has been edited 5 times. Last edited by anantmudgal09, Nov 28, 2015, 7:12 am

Quick Reply

High School Olympiads

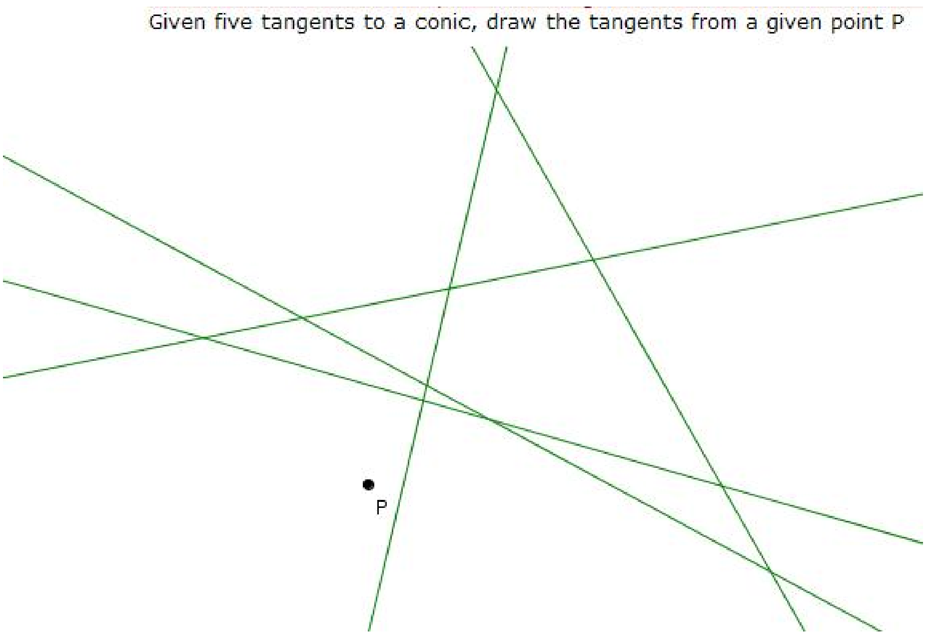
tangents from a point

Reply

jrrbc

#1 May 8, 2008, 3:57 am

Attachments:



yetti

#2 May 11, 2008, 11:09 am • 1

Relabel the given point to construct another tangent from as J.

Label the lines a, b, c, d, e and their intersections $A \equiv a \cap b, B \equiv b \cap c, C \equiv c \cap d, D \equiv d \cap e, E \equiv e \cap a.$ Using Brianchon theorem for the pentagon $ABCDE$ tangential to the conic, find the tangency points A_t, B_t, C_t, D_t, E_t with $AB, BC, CD, DE, EA.$ That is, diagonals AC, EB from A, B meet at A' and DA' cuts AB at its tangency point $A_t.$ The tangency points B_t, C_t, D_t, E_t are constructed in the same way. Given five points A_t, B_t, C_t, D_t, E_t of a conic (using the same notation), the conic tangents a, b, c, d, e at these points can be constructed similarly, using Pascal theorem.

If the pentagon $ABCDE$ is convex, the inscribed conic is an ellipse. However, if all tangency points A_t, B_t, C_t, D_t lie on a semi-ellipse or on one hyperbola branch, the conic type may not be obvious. To decide, if the conic is an ellipse or a hyperbola, construct the circumcircle (O_t) of the $\triangle A_tB_tC_t$ and find isogonal conjugate points D_t^*, E_t^* of D_t, E_t with respect to this triangle. If the line $D_t^*E_t^*$ cuts (O_t) at two points, the conic has two points at infinity and it is a hyperbola. If $D_t^*E_t^*$ does not cut $(O_t),$ all points of the conic are finite and it is an ellipse. To construct just another tangent from J is a slim result, we can construct the conic main axes and foci. Construction of another conic tangent from the point J is then trivial.

Ellipse. Let A_m, B_m be midpoints of the ellipse chords $E_tA_t, A_tB_t.$ The lines $a_m \equiv AA_m, b_m \equiv BB_m$ meet at the ellipse center $I,$ because the ellipse $\mathcal{K}$ inscribed in $ABCDE$ can always be projected to an incircle of $A'B'C'D'E'$ by a parallel projection, which preserves segment ratios on the corresponding lines. Since a_m cuts the ellipse chord E_tA_t at its midpoint, it also cuts any other ellipse chord parallel to E_tA_t at its midpoint. Let a_i be parallel to E_tA_t through $I.$ Then a_m, a_i are directions of two conjugate diameters of the ellipse.

Given the ellipse center $I,$ the directions of two conjugate diameters a_m, a_i and an ellipse tangent a with its tangency points $A_t,$ we can construct endpoints U_1, U_2 and V_1, V_2 of the conjugate diameters with these directions. The parallel to the conjugate diameter a_i through A_t (this is E_tA_t) cuts AI at A_m and $IU_1^2 = IA \cdot IA_m.$ Again, parallel projection of the ellipse to a circle makes $E_tA_t \perp IA'$, IU_1 becomes radius of the projected circle, and A_m is inversion of A' is this circle. The other end of the diameter with the direction a_m is the reflection U_2 of U_1 in I. In exactly the same way, we can construct endpoints V_1, V_2 of the conjugate diameter with the direction a_i.

Given two conjugate diameters U_1U_2, V_1V_2 of the ellipse $\mathcal{K}$, we can construct both lengths and directions of its main axes $p = P_1P_2, q = Q_1Q_2$: Let a perpendicular to the conjugate diameter V_1V_2 at I cut the circle centered at I with radius $IV_1 = IV_2$ at W_1, W_2, so that $IW_1 = IW_2 = IV_1 = IV_2$. Let M be the midpoint of U_1W_1, let $\mathcal{M}$ be a circle with center M and radius MI, and let XY be its diameter through U_1, M, W_1. Then IX, IY are the directions of the ellipse major and minor axes P_1P_2, Q_1Q_2 and $\frac{p}{2} = XW_1 = YU_1, \frac{q}{2} = XU_1 = YW_1$.

Lengths of the main axes.

Directions of the main axes.

I may do a hyperbola later, the construction up to a pair of conjugate diameters is similar and the rest is even simpler than for an ellipse.

yetti

#3 May 12, 2008, 8:56 am • 1

Hyperbola. Let A_m, B_m be midpoints of the hyprtbola chords E_tA_t, A_tB_t. The lines $a_m \equiv AA_m, b_m \equiv BB_m$ meet at the hyperbola center I, because the $\triangle IE_tA_t$ or the $\triangle IA_tB_t$ can alway be projected to an isosceles $\triangle IE'_tA'_t$ or $\triangle IA''_tB''_t$ by parallel projections, which preserve segment ratios on the corresponding lines. By symmetry, $A'A'_m$ then passes through I, hence AA_m also passes through I. Since a_m cuts the hyperbola chord E_tA_t at its midpoint, it also cuts any other hyperbola chord parallel to E_tA_t at its midpoint. Let a_i be parallel to E_tA_t through I. Then a_m, a_i are directions of two conjugate diameters of the hyperbola. Only one of the lines a_m, a_i can intersect the given hyperbola $\mathcal{K}$ (with the tangents a, b, c, d, e) at the enpoints of one conjugate diameter, the other line intersects its conjugate hyperbola $\mathcal{K}^*$, i.e., the hyperbola with the major and minor axes exchanged.

Given the hyperbola center I, the directions of two conjugate diameters a_m, a_i and a hyperbola tangent a with its tangency points A_t, we can construct endpoints U_1, U_2 and V_1, V_2 of the conjugate diameters with these directions. Assume that a_m cuts the given hyperbola $\mathcal{K}$ and a_i its conjugate hyperbola $\mathcal{K}^*$. The parallel to the conjugate diameter a_i through A_t (this is E_tA_t) cuts AI at A_m and $IU_1^2 = IA \cdot IA_m$. Again, the $\triangle IE_tA_t$ can be projected to an isosceles $\triangle IE'_tA'_t$ by a parallel projection, which projects its conjugate diameter U_1U_2 in the direction a_i into the major axis $U'_1U'_2$ of the projected hyperbola. If equation of the projected hyperbola is $\dfrac{x^2}{p^2} - \dfrac{y^2}{q^2} = 1$, the equations of its tangents at $E'_t = (x_{et}, y_{et}), A'_t = (x_{at}, y_{at})$ are $\dfrac{x_{et}x}{p^2} - \dfrac{y_{et}y}{q^2} = 1, \dfrac{x_{at}x}{p^2} - \dfrac{y_{at}y}{q^2} = 1$. By symmetry, $x_{et} = x_{at}, y_{et} = -y_{at}$ and $A'_m = (x_{at}, 0)$. Solving for the x-coordinate x_a of their intersection A', $\dfrac{x_{at}x_a}{p^2} = 1$ or $IA' \cdot IA'_m = IU_1'^2$. The other end of the diameter with the direction a_m is the reflection U_2 of U_1 in I. In exactly the same way, we can construct endpoints V_1, V_2 of the conjugate diameter with the direction a_i.

Given two conjugate diameters U_1U_2, V_1V_2 of the hyperbola $\mathcal{K}$, we can construct both lengths and directions of its main axes $p = P_1P_2, q = Q_1Q_2$: Let u_1, u_2 be parallels to the conjugate diameter V_1V_2 through U_1, U_2 and let v_1, v_2 be parallels to the conjugate diameter U_1U_2 through V_1, V_2. The lines u_1, v_1, u_2, v_2 form a parallelogram $KLMN$ with $K \equiv v_2 \cap u_1$, $L \equiv u_1 \cap v_1, M \equiv v_1 \cap u_2, N \equiv u_2 \cap v_2$ with the conjugate diameters U_1U_2, V_1V_2 as its midlines, i.e., U_1, V_2, U_2, V_2 are midpoints of KL, LM, MN, NK. Since u_1, u_2 are tangents of the given hyperbola $\mathcal{K}$ at U_1, U_2, while v_1, v_2 are tangents to the conjugate hyperbola $\mathcal{K}^*$ at V_1, V_2, the diagonals KM, LN of the parallelogram $KLMN$ are their common asymptotes. The hyperbola main axes bisect the angles between its asymptotes. Any tangent to the given hyperbola $\mathcal{K}$ has its tangency point in the plane quadrant bisected by the major axis. The hyperbola tangent a with its tangency point A_t is the internal bisector of the angle $\angle F_2A_tF_1$, where F_1, F_2 are the hyperbola foci. Thus a intersects the perpendicular bisector of F_1F_2, the hyperbola minor axis, at X_a on the circumcircle $\mathcal{O}_a$ of the $\triangle A_tF_1F_2$. The perpendicular bisector of A_tX_a cuts the minor axis at the circumcenter O_a and a circle centered at O_a with radius $O_aA_t = O_aX_a$ cuts the hyperbola major axis at F_1, F_2. Length of the hyperbola major axis is given by $p = P_1P_2 = |P_1F_2 - P_1F_1| = |A_tF_2 - A_tF_2|$.

Luis González

#4 May 16, 2014, 9:57 am • 1

Here is another construction that works for any conic.

Denote by a, b, c, d, e the given tangents. a, b, c cut d at A, B, C, respectively and a, b, c cut e at A', B', C', respectively. $\{A, B, C\} \mapsto \{A', B', C'\}$ is then a homography between d and e inducing a homography between the superposed pencils $P(A, B, C)$ and $P(A', B', C')$. Hence tangents from P to the desired conic are the double rays in the referred homography. That is, draw arbitrary circle ω through P intersecting the pencils again at two circular series, their homography axis cuts ω at $U, V \Longrightarrow$ lines PU, PV are the desired tangents.

Quick Reply

High School Olympiads

Maybe hard?

Reply

Source: Unknown

toto1234567890

#1 May 15, 2014, 4:00 pm

Given a triangle ABC. Show that the Newton line of the quadrilateral made by the tangents from the inner and outer 4 Feurbach point to the Euler nine point circle coincides with the Euler line of triangle ABC.

nsato

#2 May 15, 2014, 9:02 pm • 1

As a reference, this problem was published in the American Mathematical Monthly as problem 4549, Vol. 61 (1954).

Luis González

#3 May 15, 2014, 10:36 pm • 3

I also proposed this problem for Mathley contest Round 13 (2012). Here is my solution

Lemma: Arbitrary line through centroid G of $\triangle ABC$ cut BC, CA, AB at P, Q, R and P', Q', R' are the reflections of P, Q, R on the midpoints D, E, F of BC, CA, AB. Then the line $\overline{P'Q'R'}$ touches the Steiner inellipse of $\triangle ABC$.

An affine homography taking $\triangle ABC$ into an equilateral $\triangle A'B'C'$ transforms its Steiner inellipse into the incircle (G') of $\triangle A'B'C'$, thus it's suffices to prove the lemma for an equilateral $\triangle ABC$ with incircle (G).

WLOG assume that E is between A, Q and F is between A, R. Remaining configurations are treated similarly. From the G-isosceles $\triangle GQQ'$ and $\triangle GRR'$, we get $\widehat{RGR'} = 90^\circ - \frac{1}{2}\widehat{ARQ}$ and $\widehat{QGQ'} = 90^\circ - \frac{1}{2}\widehat{AQR} \Longrightarrow$ $\widehat{R'GQ'} = \frac{1}{2}(\widehat{ARQ} + \widehat{AQR}) = 60^\circ = 90^\circ - \frac{1}{2}\widehat{Q'AR'}$. Since AG bisects $\widehat{Q'AR'}$, then we deduce that G coincides with the A-excenter of $\triangle AQ'R' \Longrightarrow (G)$ touches $R'Q'$ and similarly (G) touches $P'Q' \Longrightarrow \overline{P'Q'R'}$ touches (G).

Back to the problem, let D, E, F be the midpoints of BC, CA, AB and $G \equiv AD \cap BE \cap CF$ the centroid. $\mathcal{G}$ is the Steiner inellipse with center G and $(N) \equiv \odot(DEF)$ is the 9-point circle of $\triangle ABC$ touching its incircle (I) at F_e and the excircles at F_a, F_b, F_c. $\tau_e, \tau_a, \tau_b, \tau_c$ denote the tangents of (N) at F_e, F_a, F_b, F_c.

If IG cuts BC, CA, AB at P, Q, R, then according to the problem Common midpoint (see post #7 for a synthetic approach), τ_e goes through the reflections P', Q', R' of P, Q, R on D, E, F, therefore by the previous lemma, τ_e touches $\mathcal{G}$. By extraversion of the incircle case, τ_a, τ_b, τ_c also touch $\mathcal{G} \Longrightarrow (N)$ and $\mathcal{G}$ are both inscribed in the same quadrangle $(\tau_e, \tau_a, \tau_b, \tau_c)$, thus by Newton's theorem their centers N, G lie on its Newton line, i.e. Euler line NG is Newton line of quadrangle $(\tau_e, \tau_a, \tau_b, \tau_c)$.

toto1234567890

#4 May 16, 2014, 3:55 pm

Thanks for your solution!!

Quick Reply

High School Olympiads

A geometry problem

Reply

wiseman

#1 May 15, 2014, 12:04 am

In triangle ABC, we draw a circle which is tangent to segments AB and AC and internally

tangent to the circumcircle of triangle ABC in X. points Y and Z are defined like X. (Y lies on

arcAC and Z lies on arc AB).prove that if we take the midpoint of arc AC as M,then XZ, AC

and YM are concurrent. (Excuse me if it's repeated!)

Luis González

#2 May 15, 2014, 11:07 am • 2

I have come across this concurrency when solving other problems (see the links below for two different approaches). Sorry I'm lazy to write a solution again.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=510926 (1st two paragraphs of the solution)
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=573657 (1st two paragraphs of the solution)

Quick Reply

High School Olympiads

Forming an equilateral hexagon

Reply

Source: Tournament of Towns,Spring 2002, Senior A Level, P5

joybangla

#1 May 14, 2014, 9:23 pm

Let AA_1, BB_1, CC_1 be the altitudes of acute ΔABC. Let O_a, O_b, O_c be the incentres of $\Delta AB_1C_1, \Delta BC_1A_1, \Delta CA_1B_1$ respectively. Also let T_a, T_b, T_c be the points of tangency of the incircle of ΔABC with BC, CA, AB respectively. Prove that $T_aO_cT_bO_aT_cO_b$ is an equilateral hexagon.

Luis González

#2 May 14, 2014, 9:54 pm

Denote (I,r) the incircle of $\triangle ABC$. From $\triangle ABC \sim \triangle AB_1C_1$, we get

$$\frac{AO_a}{AI} = \frac{AC_1}{AC} = \cos \widehat{A} = \cos \widehat{T_bAT_c}.$$

Since AI is circumdiameter of the A-isosceles $\triangle AT_bT_c$, then the latter expression means that O_a coincides with the orthocenter of $\triangle AT_bT_c \Longrightarrow T_bO_a \parallel IT_c$ (both perpendicular to AB) and $T_cO_a \parallel IT_b$ (both perpendicular to AC) $\Longrightarrow$ $IT_bO_aT_c$ is a rhombus $\Longrightarrow O_aT_b = O_aT_c = r$ and analogously we'll get $O_bT_c = O_bT_a = O_cT_a = O_cT_b = r \Longrightarrow$ hexagon $T_aO_cT_bO_aT_cO_b$ has equal sides.

Quick Reply

High School Olympiads

Lb Lc r

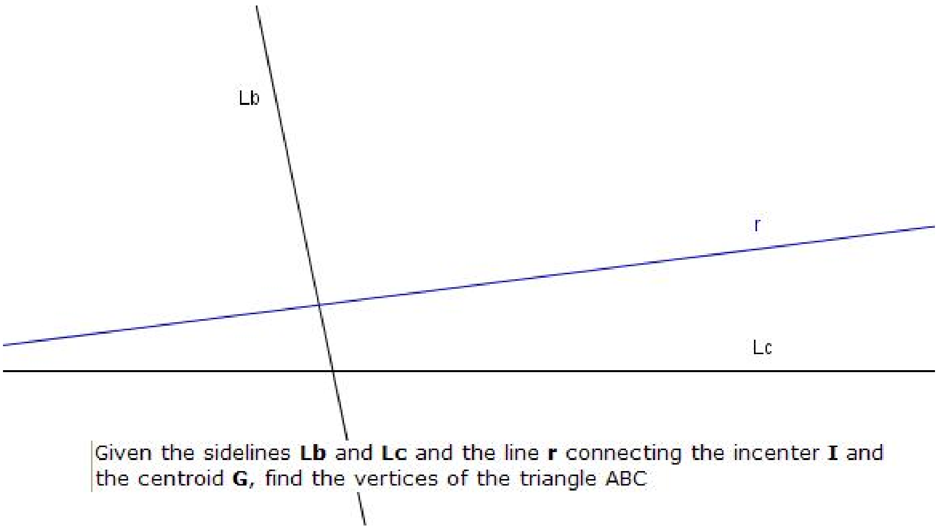
Reply

jrrbc
#1 Sep 4, 2008, 3:56 am

Lb side
Lc side
r line conecting I with G

i want the triangle

Attachments:



Luis González
#2 May 14, 2014, 9:16 am

This is very similar to the problem Lb Lc IH.

Construct incircle (I) tangent to ℓ_b,ℓ_c and denote by M,N,L the unknown midpoints of $BC,CA,AB.$ As $BC \equiv \ell_a$ envelopes $(I), B \mapsto C$ is a homography between ℓ_b and $\ell_c,$ thus $L \mapsto N$ is clearly a homography. Since the directions $LM \parallel \ell_b$ and $NM \parallel \ell_c$ remain fixed, then M describes a hyperbola $\mathcal{M}$ with asymptotes parallel to $\ell_b,\ell_c \Longrightarrow G$ describes the hyperbola $\mathcal{G}$ homothetic to $\mathcal{M}$ under homothety with center A and coefficient $\frac{2}{3}.$ Hence, $G \equiv \mathcal{G} \cap r \Longrightarrow G$ are the double points of a homography that $\mathcal{G}$ induces on $r.$ Once G is determined the construction of $\triangle ABC$ is straightforward.

Quick Reply

High School Olympiads

Hard Geo!

Locked

Sardor

#1 May 13, 2014, 8:56 pm

Let ABC be a given triangle and let p_A,p_B,p_C be the lines through the vertices A,B,C and parallel to the Euler line OH, where O and H are the circumcenter and orthocenter of ABC. Let X be the intersection of p_A with the sideline BC. The points Y,Z are defined analogously. If I_a,I_b,I_c are corresponding excenters of triangle ABC, then the lines XI_a,YI_b,ZI_c are concurrent on the circumcircle of triangle $I_aI_bI_c$.

Luis González

#2 May 13, 2014, 9:51 pm

Please give your posts meaningful subjects. The problem has been posted before; the concyclicity of the perspector and the 3 excenters holds for any direction, not necessarily the Euler infinite point (see the 2nd link below).

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=222199
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=575157

High School Olympiads

cyclic and circumscribed

Source: Tournament of Towns,Spring 2002, Junior O Level, P4

joybangla
#1 May 13, 2014, 4:48 pm

Quadrilateral $ABCD$ is circumscribed about a circle Γ and K,L,M,N are points of tangency of sides AB,BC,CD,DA with Γ respectively. Let $S\equiv KM\cap LN$. If quadrilateral $SKBL$ is cyclic then show that $SNDM$ is also cyclic.

Luis González
#2 May 13, 2014, 9:04 pm

By Newton's theorem, or degenerate Brianchon theorem, the diagonals AC and BD pass through S. Since $BK=BL,$ then it follows that B is the midpoint of the arc KL of $\odot(SKL) \Longrightarrow BD$ bisects $\angle KSL \equiv \angle MSN.$ Together with $DM=DN,$ we deduce that D is either the midpoint of the arc MN of $\odot(SMN)$ or $SM=SN,$ which gives $SL=SK$ (contradiction), hence $SNDM$ is cyclic.

High School Olympiads

Bulgaria 1997

Locked

Domination1998
#1 May 13, 2014, 6:08 pm

Let ABC be a triangle and M,N the feet of the angle bisectors of B,C, respectively. Let D be the intersection of the ray MN with the circumcircle of ABC. Prove that 1/BD=1/AD+1/CD.

Luis González
#2 May 13, 2014, 7:37 pm

If you already know the source of the problem, then use the search or go to Olympiad Resources. Topic locked.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=3807
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=220067
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=372791
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=374198

High School Olympiads

Equation about Incircles radius

Locked

BlackSelena

#1 May 13, 2014, 10:12 am

Found it in a document (without proof)
Let acute triangle ABC. D lie on segment BC. Let r_1, r_2, r be the incircle radius of triangle ABD,ACD, ABC respectively. h_a is the length of attidude from A.
Prove that $r_1 + r_2 - r = \frac{2r_1r_2}{h_a}$

Luis González

#2 May 13, 2014, 10:40 am

The so-called "infamous inradii problem". Posted many times before.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=373086
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=123243
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=19468

High School Olympiads

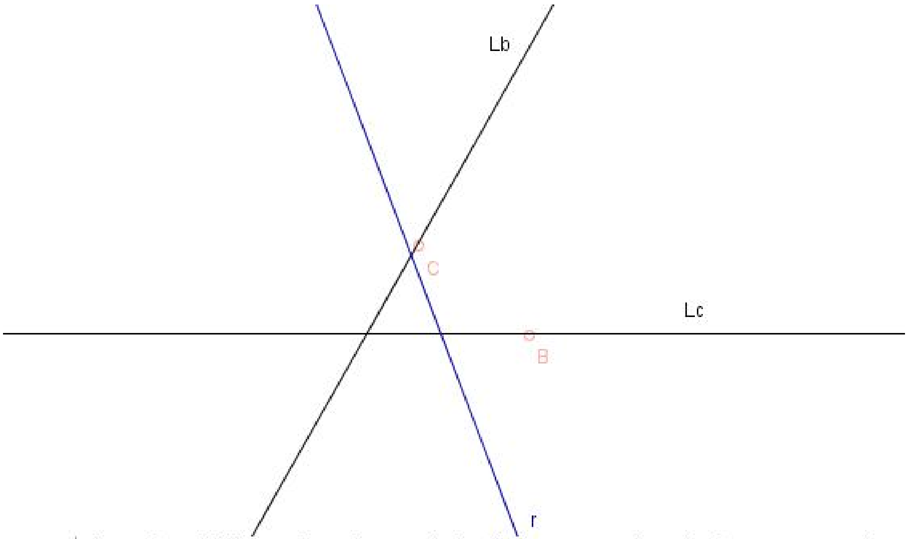
LB Lc IH

Reply

jrrbc

#1 Sep 6, 2008, 3:52 am

Attachments:



Given the sidelines **Lb** and **Lc** and the line **r** connecting the incenter **I** and the orthocenter **H**, find the vertices of the triangle ABC

Luis González

#2 May 13, 2014, 9:03 am

Internal bisector of $\angle(\ell_b, \ell_c)$ cuts r at the incenter I. Construct incircle (I) centered at I and tangent to ℓ_b, ℓ_c.

Animate the third side ℓ_a tangent to $(I) \Longrightarrow B \mapsto C$ is then a homography between ℓ_b and ℓ_c. Since the directions $CH \perp \ell_c$ and $BH \perp \ell_b$ remain fixed, the locus of H is a hyberbola $\mathcal{H}$ with asymptotes perpendicular to ℓ_b, ℓ_c, therefore $H \equiv \mathcal{H} \cap r$, giving rise to at most two solution triangles $\triangle ABC$. These intersections are then the double points of the homography that $\mathcal{H}$ induces on r, that is, construct five points H_1, H_2, H_3, H_4, H_5 of $\mathcal{H}$ (5 positions of ℓ_a), then H are the double points of the homography $H_1(H_2, H_3, H_4) \mapsto H_5(H_2, H_3, H_4)$ induced on r. Once H is determined, the construction of $\triangle ABC$ is straightforward; ℓ_a is the incirle tangent perpendicular to AH, leaving A, H in the same side.

rodinos

#3 May 13, 2014, 11:55 pm

A different construction by Angel Montesdeoca:

https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/22273

Quick Reply

High School Olympiads

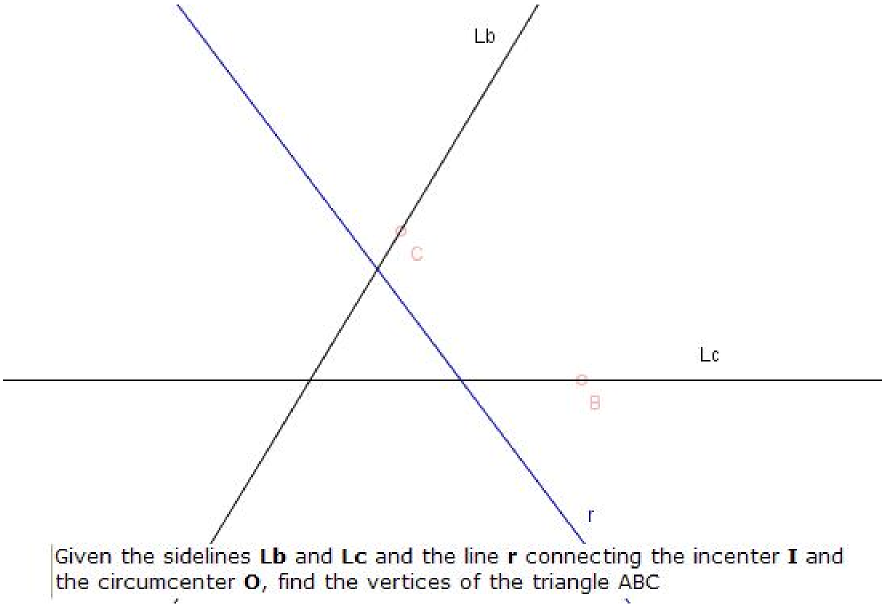
Lb Lc IO

Reply

jrrbc

#1 Sep 6, 2008, 4:00 am

Attachments:



Luis González

#2 May 13, 2014, 6:48 am

Internal bisector of $\angle(\ell_b, \ell_c)$ cuts r at the incenter I. Construct incircle (I) centered at I and tangent to ℓ_b, ℓ_c at E, F and let D be the unknown tangency point of (I) with $\ell_a \equiv BC$.

It's known that r is the Euler line of $\triangle DEF$ containing its unknown orthocenter T. Since the reflection U of T on EF is on (I), then the reflection of r towards EF will intersect (I) at U and the anti-Steiner point of r WRT $\triangle DEF$, i.e. the Feuerbach point of $\triangle ABC$, giving rise to 2 solution triangles $\triangle ABC$. Perpendicular from U to EF cuts (I) again at D and the tangent ℓ_a of (I) at D completes $\triangle ABC$.

rodinos

#3 May 13, 2014, 5:30 pm

Variation:
Lb, Lc and HI line (instead of OI ine).

rodinos

#4 May 13, 2014, 10:32 pm

My "natural" (after the original problem) variation was already solved by Luis with projective geometry.
The locus of the orthocenter of the triangle ABC, as BC moves touching the known incircle, is a hyperbola.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=224690

I am wondering if there is a construction with no use of conics.

Note: I guess that the locus of the circumcenter is a hyperbola as well.

And now another construction which can be done without conics:

Let Lb, Lc be the lines of AC,AB, resp., Lo, Lh two other lines containing the circumcenter, orthocenter, resp.
To construct the triangle. (As I said, it is possible to construct it without conics).
Note: We can replace Lo with any curve containing O..... :)

Quick Reply

High School Olympiads

Point P in interior of ABCD

Reply

Source: 2014 CMO #4

TheMaskedMagician

#1 May 12, 2014, 5:01 am

The quadrilateral $ABCD$ is inscribed in a circle. The point P lies in the interior of $ABCD$, and $\angle PAB = \angle PBC = \angle PCD = \angle PDA$. The lines AD and BC meet at Q, and the lines AB and CD meet at R. Prove that the lines PQ and PR form the same angle as the diagonals of $ABCD$.

Luis González

#2 May 12, 2014, 7:34 am

Let $(O_1), (O_2), (O_3), (O_4)$ denote the circumcircles of $\triangle PAB, \triangle PBC, \triangle PCD, \triangle PDA,$ resp. Since $QA \cdot QD = QB \cdot QC$, then Q has equal power WRT $(O_2), (O_4) \Longrightarrow QP$ is radical axis of $(O_2), (O_4) \Longrightarrow QP \perp O_2O_4$ and likewise $RP \perp O_1O_3 \Longrightarrow \angle(PQ, PR) = \angle(O_2O_4, O_1O_3)$, hence it suffices to show that $\angle(O_2O_4, O_1O_3) = \angle(AC, BD)$.

Since (O_1) is tangent to BC, we have $\angle APB = \angle ABQ = \angle ADC$, but since $O_1O_2 \perp PB$ and $O_1O_4 \perp PA$, then $\angle O_4O_1O_2 = \pi - \angle APB = \pi - \angle ADC = \angle ABC$. By similar reasoning, using that $(O_2), (O_3), (O_4)$ are tangent to CD, DA, AB, resp, we get $\angle O_1O_2O_3 = \angle BCD, \angle O_2O_3O_4 = \angle CDA$ and $\angle O_3O_4O_1 = \angle DAB \Longrightarrow O_4O_1O_2O_3$ is cyclic and similar to $ABCD \Longrightarrow \angle(O_2O_4, O_1O_3) = \angle(AC, BD)$.

sayantanchakraborty

#3 Aug 29, 2014, 10:55 am

Let $\angle PAB = \angle PBC = \angle PCD = \angle PDA = \theta$.Without loss of generality let $\angle A > \angle B$.Then $\angle PDA = \angle PBC = \theta \implies P, B, Q, D$ concyclic.Thus $\angle DPQ = \angle DBQ = \angle DBC$.

Similarly $\angle RCA = \angle PAB = \theta \implies$ points P, A, R, C concyclic.Thus $\angle RPA = \angle RCA = \angle DCA$.By easy angle chasing one also has $\angle DPA = \angle C$.Thus $\angle DPR = \angle C - \angle RPA = \angle C - \angle DCA = \angle ACB$.

Thus $\angle QPR = \angle QPD + \angle RPD = \angle DBC + \angle ACB$ and the result follows.

Quick Reply

High School Olympiads

Ex-Feuerbach Points and A Concurrence

Reply

Source: Hyacinthos, message #22235

rodinos

#1 May 7, 2014, 8:29 am

In triangle ABC, H is the orthocenter, Ia,Ib,Ic the excenters and Fa,Fb,Fc the ex-Feuerbach points.

It is known that the Nine Point Circles of the triangles AHIa, BHIb, CHIc pass throuch Fa,Fb,Fc, resp.
(Just the extraversion of the main theorem: The Nine Point Circle of the triangle AHI - where I = Incenter - passes through the Feuerbach point of the triangle. Reference? I know it is old but I do not have an old reference)

Now, let A' be the other than Fa intersection of the Nine Point Circle of AHIa and the excircle (Ia) and similarly B', C'.

The lines AA', BB', CC' are concurrent. The point they concur is a new point (ie not listed in the Kimberling's ETC).

Synthetic proof?

Luis González

#2 May 11, 2014, 9:30 am • 2

9-point circles of $\triangle AIH, \triangle BIH$ and $\triangle CIH$ passing through the Feuerbach point F_e of $\triangle ABC$ is just the effect of the rectangular hyperbola through A, B, C, H, I having center the Feuerbach point F_e. The same holds for the excircles by extraversion.

Let the incircle (I) touch BC at D and D' is the antipode of D WRT $(I). U, V$ are the midpoints of $\overline{IH}, \overline{IA}$ and DU cuts (I) again at $L. VD'$ cuts (I) again at F_e (for a proof see the general configuration at Intersect on circle post #3), hence since $UV \parallel AH \parallel ID'$, we get $\widehat{ULF_e} \equiv \widehat{DLF_e} = \widehat{DD'F_e} = \widehat{UVF_e} \Longrightarrow L$ is on 9-point circle $\odot(UVF_e)$ of $\triangle AHI$. Now if the A-excircle (I_a) touch BC, CA, AB at X, Y, Z, then by extraversion of the previous result, A' is the 2nd intersecion of (I_a) with the line through X and the midpoint K of $\overline{HI_a}$.

H and K become then X_{155} and X_{1147} of the extouch $\triangle XYZ$, thus the barycentrics of K WRT $\triangle XYZ$ are then $(\sin 4\widehat{X} : \sin 4\widehat{Y} : \sin 4\widehat{Z}) = (\sin 2\widehat{A} : \sin 2\widehat{B} : \sin 2\widehat{C})$. Therefore, we have

$$\frac{A'Y}{A'Z} = \frac{[KXY]}{[KXZ]} \cdot \frac{XZ}{XY} = \frac{\sin 2C}{\sin 2B} \cdot \frac{\sin \frac{B}{2}}{\sin \frac{C}{2}} \Longrightarrow$$

$$\left(\frac{A'Y}{A'Z}\right)^2 = \frac{\text{dist}(A'AC)}{\text{dist}(A', AB)} = \left(\frac{\sin 2C}{\sin 2B} \cdot \frac{\sin \frac{B}{2}}{\sin \frac{C}{2}}\right)^2.$$

Considering the cyclic expressions, we conclude that AA', BB' and CC' concur at a point P whose trilinear coordinates WRT $\triangle ABC$ are then

$$P \equiv \left(\sin^2 \frac{A}{2} \cdot \csc^2 2A : \sin^2 \frac{B}{2} \cdot \csc^2 2B : \sin^2 \frac{C}{2} \cdot \csc^2 2C\right),$$

which indeed is not listed in the current edition of ETC.

rodinos

#3 May 11, 2014, 9:47 am

Also, If I is the incenter, and A",B",C" the other than Feuerbach point Fe intersections of the Nine Point Circles of AHI, BHI,CHI with incircle (I),resp.
then AA",BB",CC" are concurrent.

In general we have the locus problem:

In general we have:

Let ABC be a triangle and P a point.
The Nine Point Circles of AHP,BHP,CHP,ABC and the pedal circle of P are concurrent at a point P*
(center of the r. c/hyperbola through P)

Now, let A*,B*,C* be the other than P* intersections of the Nine Point Circles of AHP,BHP,CHP
with the pedal circle of P, resp.

Which is the locus of P such that ABC, A*B*C* are perspective?

(Hyacinthos #22250
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/22250)

Luis González

#4 May 11, 2014, 11:25 am • 2

Let $\triangle DEF$ be the pedal triangle of P WRT $\triangle ABC.$ 9-point circles of $\triangle AHP, \triangle BHP$ and $\triangle CHP$ pass through the center P^* of the rectangular hyperbola through $A,B,C,P,H,$ i.e. the Poncelet point P^* of $A,B,C,P,$ which also lies on the pedal circle $\odot(DEF)$ of $P.$

Again using the result found in Intersect on circle, we decude that A^*, D and the midpoint U of $\overline{PH}$ are collinear and the same holds for B^* and $C^* \Longrightarrow \triangle A^*B^*C^*$ is the circumcevian triangle of U WRT $\triangle DEF.$ Now from here, it follows that $\triangle ABC$ and $\triangle A^*B^*C^*$ are perspective iff H is on the line connecting P with the isogonal conjugate of P WRT $\triangle ABC \Longrightarrow$ locus of P is a isogonal circum-cubic with pivot $H;$ the orthocubic $\text{pK}(X_6,X_4)$ of $\triangle ABC.$

Quick Reply

High School Olympiads

Isogonal Conjugate Construction

Reply

Source: Hyacinthos message #22168

rodinos
#1 May 4, 2014, 5:01 am

Let ABC be a triangle, P a point and A'B'C' the cevian triangle of P. Denote:
Ab, Ac = the reflections of A' in BB', CC', resp., A2, A3 = the reflections of Ab, Ac in AC, AB, resp.
Similarly B2,B3 and C2,C3. The perpendicular bisectors of A2A3, B2B3, C2C3 concur at the isogonal conjugate of P.
Synthetic proof?

jayme
#2 May 4, 2014, 11:11 am

Dear Mathlinkers,
who want to propose a figure?
Sincerely
Jean-Louis

Luis González
#3 May 10, 2014, 10:11 am • 1

I believe there's a typo in the proposition. A2, A3 should be the reflections of Ab, Ac in AB, AC, respectively and not AC, AB.

Clearly B and C become circumcenters of $\triangle A_bA_2A'$ and $\triangle A_cA_3A'$, respectively. Hence, $\angle A_2BA' = 2\angle A_2A_bA' = 2\angle ABP = 2\angle CBQ$, where Q is the isogonal conjugate of P, thus BQ is symmetry axis of the B-isosceles $\triangle BA_2A'$, i.e. BQ is perpendicular bisector of $\overline{A_2A'}$. Similarly, CQ is perpendicular bisector of $\overline{A_3A'}$, thus Q is circumcenter of $\triangle A_2A_3A'$, lying on the perpendicular bisector of $\overline{A_2A_3}$ and similarly Q lies on perpendicular bisectors of $\overline{B_2B_3}$ and $\overline{C_2C_3}$.

rodinos
#4 May 10, 2014, 3:50 pm

Luis: Sorry for the typo and thanks for the solution!!

But ΟΥΔΕΝ ΚΑΚΟΝ ΑΜΙΓΕΣ ΚΑΛΟΥ (ancient Greek proverb) : Every bad situation has some good aspect to it

How about the locus of P such that the perpendicular bisectors of the segments in the "wrong" definition are concurrent?
ie
Let ABC be a triangle, P a point and A'B'C' the cevian triangle of P. Denote:
Ab, Ac = the reflections of A' in BB', CC', resp., A2, A3 = the reflections of Ab, Ac in AC, AB, resp.
Similarly B3,B1 and C1,C2. Which is the locus of P such that the perpendicular bisectors of A2A3, B3B1, C1C2 are concurrent?

rodinos
#5 May 11, 2014, 12:48 am

The locus in question is a "nonic through the points A, B, C, X(1), X(4), X(13), X(14), vertices of triangle orthic"
Hyacinthos, message #22254

Quick Reply

High School Olympiads

incentres and other concyclic points

Reply

Source: Kyiv mathematical festival 2014

rogue

#1 May 10, 2014, 3:06 am

Let AD, BE be the altitudes and CF be the angle bissector of acute non-isosceles triangle ABC and $AE+BD=AB$. Denote by I_A, I_B, I_C the incentres of triangles AEF, BDF, CDE respectively. Prove that points D, E, F, I_A, I_B and I_C lie on the same circle.

Luis González

#2 May 10, 2014, 3:48 am

Perpendicularities $AD \perp BC$ and $BE \perp AC$ are unnecessary; we only need points D, E on BC, AC, such that $AEDB$ is cyclic and $AE+BD=BC$.

Using the converse of 9th ibmo - brazil 1994/q2 (i), it follows that the circle with center F tangent to CB, CA also touches DE $\Longrightarrow F$ becomes C-excenter of $\triangle CDE \Longrightarrow D, E$ are on circle with diameter $\overline{FI_C}$. Angle chase gives $\widehat{DI_BF} = 90^\circ + \frac{1}{2}\widehat{DBA} = 90^\circ + \frac{1}{2}(180^\circ - \widehat{DEA}) = 180^\circ - \widehat{DEF} \Longrightarrow I_B \in \odot(DEF)$. Likewise $I_A \in \odot(DEF)$, hence D, E, F, I_A, I_B, I_C are concyclic.

Quick Reply

High School Olympiads

Prove is the lines of centers

Reply

StanleyST
#1 May 9, 2014, 3:32 pm

Let ABC be an acute triangle. Let D,E,F be the tangency points of the incircle of this triangle with sides BC,AC,AB respectively. Let X be a point on the segment BC and let Y on the segment AX such that $AE=AY$. Prove that the perpendicular bisector of segment DY is the line of the incenters of triangles ABX and ACX .

Luis González
#2 May 9, 2014, 9:53 pm

Let (I_1) and (I_2) denote the incircles of $\triangle ABX$ and $\triangle ACX,$ resp. According to problem Locus of intersection, Y is the intersection of the common external tangent of $(I_1),(I_2)$ (other than BC) with $AX.$

$(I_1),(I_2)$ touch BC at U,V and their common internal tangent (other than AX) cuts BC at $Z.$ Then Y,Z lie on the circle with diameter $\overline{I_1I_2}$ and $BZ=BU+UZ=BU+XV=\tfrac{1}{2}(AB+BX-AX)+\tfrac{1}{2}(AX+CX-AC) \Longrightarrow$ $BZ=\tfrac{1}{2}(AB+BC-AC)=BD \Longrightarrow Z \equiv D.$ Now, since I_1 and I_2 are the midpoints of the arcs YD and YXD of $\odot(I_1DXI_2Y),$ then I_1I_2 is the perpendicular bisector of $\overline{DY}.$

Mikasa
#3 Jun 12, 2014, 11:19 am

Wlog assume that X is on the segment CD. Let the incenter of $\triangle ABC$ be I. Let I_1,I_2 be the incenters of $\triangle ABX, \triangle ACX$ respectively. Now $AF=AY=AE$. So both the triangle $\triangle AEY,\triangle AFY$ are isosceles. Since AI_1 bisects $\angle FAY$, AI_1 is the perpendicular bisector of FY. Similarly, AI_2 is the perpendicular bisector of EY. Thus, $I_1Y=I_1F$ and $I_2Y=I_2E$.
Now $I_1 \in BI$ and BI is the perpendicular bisector of FD. So $I_1D=I_1F=I_1Y$. So I_1 is on the perpendicular bisector of DY. Similarly we can show that $I_2Y=I_2D$ i.e. I_2 is on the perpendicular bisector of DY. Thus the perpendicular bisector of the segment DY is I_1I_2 as desired.

saturzo
#4 Jun 12, 2014, 7:54 pm

Easy! Let I_1 and I_2 be the incenters of $\triangle ABX$ and $\triangle ACX$. Now in $\triangle ABX$, $AF=AE=AY$ and AI_1 is the internal bisector of $\angle BAX$; so, by symmetry, $I_1F=I_1Y$ and in the same way, $I_1F=I_1D \implies I_1D=I_1Y$, simmilarly in $\triangle AXC$, $I_2D=I_2Y$; so I_1I_2 must be perpendicular bisector of DY.

Quick Reply

High School Olympiads

locus of points P

mathzz

#1 May 9, 2014, 12:12 am

Let ABC be a triangle. For a point P of the plane, let A' be the foot of the perpendicular dropped from P to BC.

Points B' and C' are defined analogously. Find the locus of points P in the plane such that

$PA.PA' = PB.PB' = PC.PC'$

Luis González

#2 May 9, 2014, 9:24 am

Let Q denote the isogonal conjugate of P WRT $\triangle ABC$. From $PB \cdot PB' = PC \cdot PC'$, we obtain

$$\frac{PB}{PC} = \frac{PC'}{PB'} = \frac{PB \cdot \sin \widehat{PBA}}{PC \cdot \sin \widehat{PCA}} = \frac{PB \cdot \sin \widehat{QBC}}{PC \cdot \sin \widehat{QCB}} \Longrightarrow \sin \widehat{QBC} = \sin \widehat{QCB}.$$

Hence, either $\widehat{QBC} = \widehat{QCB} \Longrightarrow Q$ is on perpendicular bisector of $\overline{BC}$ or $\widehat{QBC} = \pi - \widehat{QCB} \Longrightarrow Q$ is at infinity. Combined with the the other condition $PA \cdot PA' = PC \cdot PC'$, it follows that Q is either at infinity or it is the circumcenter O of $\triangle ABC \Longrightarrow P$ either lies on the circumcircle (O) or it is the orthocenter H of $\triangle ABC$.

High School Olympiads

Similar to P:11.2

Reply

Source: All Russian 2014 Grade 9 Day 2 P2

mathuz

#1 May 3, 2014, 8:53 pm

Let $ABCD$ be a trapezoid with $AB \parallel CD$ and Ω is a circle passing through A, B, C, D. Let ω be the circle passing through C, D and intersecting with CA, CB at A_1, B_1 respectively. A_2 and B_2 are the points symmetric to A_1 and B_1 respectively, with respect to the midpoints of CA and CB. Prove that the points A, B, A_2, B_2 are concyclic.

I. Bogdanov

Luis González

#2 May 9, 2014, 4:32 am • 1

> *mathuz wrote:*
> Let $ABCD$ is trapezoid with $AB \parallel CD, \Omega$ is circle passes through $A, B, C, D. \omega$ is circle passes through C, D and intersects with CA, CB at A_1, B_1 respectively. A_2 and B_2 are the points symmetric to A_1 and B_1 respectively, with respect to the midpoints of CA and CB. Prove that the points A, B, A_2, B_2 are concyclic.

Typo corrected in red color. This is proved in the solution of the problem All Russian-2014, Grade 11, day 2, P2.

mathuz

#3 May 17, 2014, 2:08 am

you are right!
Thank you Luis.

nima1376

#4 May 18, 2014, 12:29 pm

D is a center of spiral similar which goes BB_1 to $AA_1 \Rightarrow \dfrac{AA_1}{BB_1} = \dfrac{AD}{BD} = \dfrac{BC}{CA}$
$AA_1.AC = BB_1.BC \Rightarrow CA_2.AC = CB_2.BC$
so A_2B_2BA is cycle.
done

saturzo

#5 May 19, 2014, 5:02 pm • 1

$ABCD$ is cyclic in Ω. So, $\measuredangle BAC = \measuredangle DCA \Rightarrow BC = AD$
Similarly $BD = AC$.
Now let $\{D, D'\} = AD \cap \omega$. And by symmetry, $AD' = BB_1$
Now A_1CDD' is cyclic(in ω) and $A_1C \cap DD' = A$. So(using power of point), $AA_1.AC = AD'.AD$.
$\therefore CA_2/CB_2 = AA_1/BB_1 = AA_1/AD' = AD/AC = CB/CA \Rightarrow CA_2.CA = CB_2.CB$
$\therefore A_2, B_2, A, B$ are concyclic.

$[QED]$

thecmd999

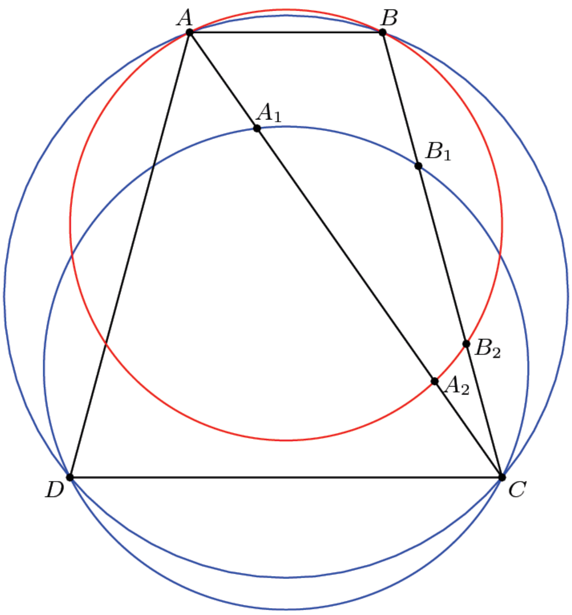
#6 Sep 23, 2014, 12:50 am

Solution

v_Enhance

#7 Dec 1, 2014, 7:36 am

What a nice illustration of spiral similarity. Though I would have just said "isosceles trapezoid" in the problem statement.



We have $\triangle DAA_1 \sim \triangle DBB_1$, but $DA = CB$ and $DB = AC$. So $AA_1 \cdot AC = BB_1 \cdot BC$, implying that $CA_2 \cdot CA = CB_2 \cdot CB$.

utkarshgupta

#8 Jan 1, 2015, 1:48 pm

Let ω intersect BD in A'
Then it is easy to see $AA_1 = BA'$
But since $A'B_1CD$ are concyclic,
$BB_1 \cdot BC = BA' \cdot BD$
$\implies BB_1 \cdot CB = AA_1 \cdot CA$
$\implies CB_2 \cdot CB = CA_2 \cdot CA$
QED

aditya21

#9 Mar 22, 2015, 1:55 pm

easy!! but still posting!

let ω intersect AD in K
than quite easily $\angle AKB_1 = \angle ACD = 180 - \angle ABB_1$
and hence ABB_1K is isosceles trapezium.
now by POP
we have $AD.AK = AA_1.AC = CA_2.AC$
on other note $AD.AK = BC.AK = BC.BB_1 = BC.BB_2$
and hence $BC.BB_2 = CA_2.CA$
and hence by POP we have ABB_2A_2 is cyclic quad.
thus we are done

anantmudgal09

#10 Oct 23, 2015, 6:46 pm

Another solution:

Let the circle AA_2B intersect AB again at B'.

Now, AB is the radical axis of $(ABCD);(AA_2B)$ and CD is the radical axis of $(ABCD);(DCA_1B_1)$.

Now, $AB \parallel CD$ and so $AB \parallel CD \parallel l$ where l is the radical axis of $(AA_2B);(DCA_1B_1)$

Let M and N be the mid points of CA, CB respectively. It is evident that $MN \parallel l$ and also,

$MA_1.MC = MA_2.MA$

so M lies on l. Therefore, N lies on l too and so by power of a point $B_2 \equiv B'$ thus, the result holds.

Quick Reply

High School Olympiads

with the Euler's line

Reply

Source: unknown

jayme

#1 May 3, 2014, 8:16 pm

Dear Mathlinkers,

1. ABC a triangle
2. (E) the Euler's line of ABC
3. P a point on (E)
4. Ma, Mb, Mc the perpendicular bissectors of BC, CA, AB
5. X, Y, Z the symetrics of P wrt Ma, Mb, Mc

Prove : AX, BY and CZ are concurrent.

Sincerely
Jean-Louis

jayme

#2 May 4, 2014, 4:05 pm

Dear Mathlinkers,
any ideas?
Sincerely
Jean-Louis

mathuz

#3 May 4, 2014, 4:26 pm

what is mean Ma, ..?
Are points? Or lines?

jayme

#4 May 4, 2014, 4:32 pm

Dear mathuz and Mathlinkers,
Ma is the perpendicular bissector (or mediatrix) of the segment BC; it is a line...
Sincerely
Jean-Louis

Luis González

#5 May 9, 2014, 12:24 am • 3

Posted before by Darij at Point on Euler line reflected in perpendicular bisectors. In addition the isogonal conjugate of the concurrency point $Q \equiv AX \cap BY \cap CZ$ is on the euler line e of $\triangle ABC.$

Another proof: Animate P on the Euler line $e. X, Y, Z$ move on the reflections of e on perpendicular bisectors of $BC, CA, AB,$ hence the application $Y \mapsto Z$ is clearly homographic $\Longrightarrow Q_A \equiv BY \cap CZ$ is on a fixed conic $\mathcal{J}$ though $B, C.$ When P coincides with the circumcenter $O,$ then obviously $Q_A \equiv O$ and when P coincides with the orthocenter $H,$ then Q_A becomes the Prasolov point Pr of $\triangle ABC.$ Furthermore, it's easy to see that there exists a P verifying that its reflections on the perpendicular bisectors of AC, AB lie on $AB, AC,$ resp, thus $A \in \mathcal{J} \Longrightarrow \mathcal{J}$ is then the Jerabek hyperbola of $\triangle ABC$ passing through $A, B, C, O, Pr.$ Analogously, $Q_B \equiv CZ \cap AX$ is on $\mathcal{J} \Longrightarrow Q_B \equiv Q_A \equiv Q \Longrightarrow$ $Q \equiv AX \cap BY \cap CZ$ and its isogonal conjugate is then on the isogonal e of $\mathcal{J}.$

Quick Reply

High School Olympiads

on a sphere

Reply

Source: All Russian 2014 Grade 11 Day 2 P2

mathuz

#1 Apr 30, 2014, 10:19 am

The sphere ω passes through the vertex S of the pyramid $SABC$ and intersects with the edges SA, SB, SC at A_1, B_1, C_1 other than S. The sphere Ω is the circumsphere of the pyramid $SABC$ and intersects with ω circumferential, lies on a plane which parallel to the plane (ABC).
Points A_2, B_2, C_2 are symmetry points of the points A_1, B_1, C_1 respect to midpoints of the edges SA, SB, SC respectively. Prove that the points A, B, C, A_2, B_2, and C_2 lie on a sphere.

Luis González

#2 May 8, 2014, 8:58 am

Plane SAB cuts Ω along a circle (O) and cuts the circumference $\Omega \cap \omega$ again at P. If M, N denote the midpoints of $\overline{SA}, \overline{SB}$, then $SMON$ is clearly cyclic with circumcircle (K), the midcircle of $(O_1) \equiv \odot(SA_1B_1P)$ and $(O_2) \equiv \odot(SA_2B_2)$, whose center is then the midpoint of $\overline{O_1O_2} \Longrightarrow SO_1OO_2$ is a parallelogram with diagonal intersection $K \Longrightarrow SO_2 \parallel OO_1 \perp SP \parallel AB$, i.e. O_2 is on S-altitude of $\triangle SAB$, implying that A_2B_2 is antiparallel to AB WRT $SA, SB \Longrightarrow SA \cdot SA_2 = SB \cdot SB_2$ and similarly $SB \cdot SB_2 = SC \cdot SC_2 \Longrightarrow SA \cdot SA_2 = SB \cdot SB_2 = SC \cdot SC_2$ $\Longrightarrow A, B, C, A_2, B_2, C_2,$ lie on a same sphere.

Quick Reply

High School Olympiads

St.Peterburg, P6 Grade 9, 2013

Reply

mathuz

#1 Apr 22, 2014, 1:24 am

Given $ABCD$ quadrilateral with $AB=BC=CD$. Let $AC \cap BD=O$, X,Y are symmetry points of O respect to midpoints of BC, AD and Z is intersection point of lines, which perpendicular bisects of AC, BD. Prove that X,Y,Z are collinear.

Luis González

#2 Apr 22, 2014, 2:24 am • 1

We only need a quadrilateral $ABCD$ satisfying $AB=CD.$

Let P,Q,M,N denote the midpoints of $AC,BD,BC,AD,$ respectively. Clearly, it suffices to prove that the midpoint L of OZ lies on $MN.$ Indeed $PN=QM=\tfrac{1}{2}CD=\tfrac{1}{2}AB=QN=MP \Longrightarrow PMQN$ is a rhombus $\Longrightarrow MN$ is perpendicular bisector of $PQ,$ passing through the circumcenter L of $\triangle OPQ,$ i.e. $L \in MN,$ as desired.

Quick Reply

High School Olympiads

External tangent lines

zarengold
#1 Mar 12, 2014, 9:31 pm

Given is a triangle ABC and points D, E on side BC. Denote the point of intersection of external mutual tangent lines of the incircles of triangles $\triangle ABD$ and $\triangle ACE$ as X. Denote the point of intersection of external mutual tangent lines of the incircles of triangles $\triangle ABE$ and $\triangle ACD$ as Y. Prove that X coincides with Y.

Luis González
#2 Apr 21, 2014, 10:14 am • 1

Let $(I_1), (I_2), (J_1), (J_2)$ denote the incircles of $\triangle ABD, \triangle ACE, \triangle ABE, \triangle ACD$, respectively. It suffices to show that I_1I_2, J_1J_2 and BC concur.

$\widehat{BAI_1} = \frac{1}{2}\widehat{BAD} = \frac{1}{2}(\widehat{BAC} - \widehat{CAD}) = \widehat{CAI} - \widehat{CAJ_2} = \widehat{IAJ_2}$. Similarly, we have $\widehat{CAI_2} = \widehat{IAJ_1}$ and since $\widehat{IAB} = \widehat{IAC}$, we get $A(I, J_1, I_1, B) = A(C, I_2, J_2, I) \Longrightarrow (I, J_1, I_1, B) = (C, I_2, J_2, I) = (I, J_2, I_2, C) \Longrightarrow$ lines I_1I_2, J_1J_2 and BC concur, as desired.

High School Olympiads

Easy geometry

Reply

Source: Lithuanian TST 2014 #1

Zaltad

#1 Apr 21, 2014, 1:04 am

Circle touches parallelogram's $ABCD$ borders AB, BC and CD respectively at points K, L and M. Perpendicular is drawn from vertex C to AB . Prove, that the line KL divides this perpendicular into two equal parts (with the same length).

Luis González

#2 Apr 21, 2014, 1:33 am

It's nothing but a limiting case of the following configuration: $\triangle ABC$ is scalene with incirce (I,r) and B-excircle (I_b) touching its sidelines BC, CA at $M,N.$ Then MN cuts the A-altitude at a point $X,$ such that $AX=r.$

This is an old problem posted many times before, e.g Inradius and altitude and elsewhere.

Zaltad

#3 Apr 21, 2014, 1:19 pm

Can somebody post a solution using complex numbers just for learning purposes? :)

Mikasa

#4 Apr 22, 2014, 10:13 am • 1

Let O be the midpoint of KM. Let the perpendicular bisector of KM intersect KL at N. Let CN intersect AB at Q.
Note that $ON \parallel MC \parallel KQ$. Thus, $\angle CMN = \angle MNO = \angle KNO = \angle QKN = \angle BKL = \angle BLK$
$= \angle CLN$ i.e. $CNLM$ is a cyclic quad. But $\angle MCN = 180^\circ - \angle MLN = 90^\circ$. Thus $OQ \perp AB$.
But since $MKQC$ is a trapezoid, and ON is the perpendicular bisector of KM, and $ON \parallel KQ$, N must be the midpoint of CQ.

Quick Reply

High School Olympiads

3 mixtilinear incircles have a common tangent

Reply

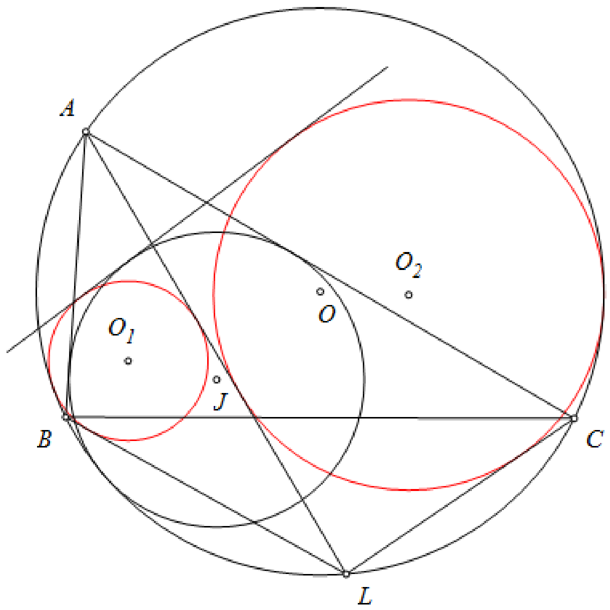
Source: own

livetolove212

#1 Apr 15, 2014, 4:00 pm • 2

Given triangle ABC with its circumcircle (O). Let L be an arbitrary point on arc BC which does not contain A. Prove that A-mixtilinear incircle of triangle ABC, L-mixtilinear incircles of triangles LAB and LAC have a common tangent.

Attachments:



mathuz

#2 Apr 15, 2014, 7:35 pm

Casey's theorem killed!

jayme

#3 Apr 15, 2014, 7:54 pm • 1

Dear Mathlinkers,
I think that this problem can be solved without any calculation...
Thank to mathus for the first idea...
Sincerely
Jean-Louis

livetolove212

#4 Apr 16, 2014, 9:34 am

Dear mathuz, how can you use Casey's theorem?

aopsermath

#5 Apr 17, 2014, 4:41 pm

Livetolove212, do you have any proof?

Luis González

#6 Apr 20, 2014, 1:11 pm • 5

Lemma: AB and CD are two intersecting chords in a circle (O). P is a variable point on the arc AD that does not contain B, C and R, S are the incenters of $\triangle PAB, \triangle PCD,$ resp. Then the circle $\odot(PRS)$ cuts (O) at a fixed point T belowing to the incircle of the curvilinear triangle bounded by AB, CD and the arc $AD.$

Proof: PR, PS cut (O) again at the midpoints X, Y of the arcs $AB, CD,$ obviously fixed and by incenter property, we have $XA = XB = XR$ and $YC = YD = YS.$ Second intersection T of (O) with $\odot(PRS)$ is center of the spiral similarity swaping $\overline{XR}$ and $\overline{YS} \Longrightarrow \frac{TR}{TS} = \frac{TX}{TY} = \frac{XR}{YS} = \text{const}$ and since $\angle RTS = \angle RPS \equiv \angle XPY$ is constant, then it follows that $\triangle TRS$ are all similar $\Longrightarrow \angle TRS = \angle TPS \equiv \angle TPY$ is constant $\Longrightarrow T$ is fixed.

When $P \equiv T,$ then $\odot(PRS)$ is internally tangent to (O) at T and $RS \parallel XY.$ If $X' \equiv TX \cap AB$ and $Y' \equiv YT \cap CD,$ we have then $XA^2 = XX' \cdot XT$ and $YD^2 = YY' \cdot YT \Longrightarrow$

$$\frac{XX' \cdot XT}{YY' \cdot YT} = \frac{XA^2}{YD^2} = \frac{XT^2}{YT^2} \Longrightarrow \frac{XT}{YT} = \frac{XX'}{YY'} \Longrightarrow X'Y' \parallel XY \parallel RS.$$

Tangents of $\odot(TRS)$ at S, R are clearly parallel to $AB, CD,$ hence the homothety with center $T,$ carrying R, S to $X', Y',$ takes $\odot(TRS)$ into the circle $\odot(TX'Y')$ tangent to (O) at T and tangent to $AB, CD,$ i.e. the incircle of the curvilinear triangle bounded by AB, CD and the arc $AD.$ $\blacksquare$

Back to the problem, let D, E be the tangency points of $(O_1), (O_2)$ with their common tangent ℓ that leaves L, O_1, O_2 in the same side. ℓ cuts the arcs AC, AB of (O) at $P, Q.$ If $(O_1), (O_2)$ touch AL at $U, V,$ then by Thebault-Sawayama lemma, $K \equiv O_1O_2 \cap DU \cap EV$ is the incenter of $\triangle LPQ,$ thus $\angle EKD = 90^\circ \Longrightarrow O_1D, O_2E$ are tangents of $\odot(KED).$ Again by Sawayama lemma, incenters I_1, I_2 of $\triangle LAB, \triangle LAC$ are the projections of U and V on LO_1 and $LO_2,$ resp. Hence if $\odot(KED)$ cuts O_1O_2 again at $F,$ we have $O_1D^2 = O_1U^2 = O_1L \cdot O_1I_1 = O_1F \cdot O_1K \Longrightarrow I_1 \in \odot(LFK)$ and likewise $I_2 \in \odot(LFK) \Longrightarrow L, I_1, I_2, K$ lie on a same circle $\omega.$

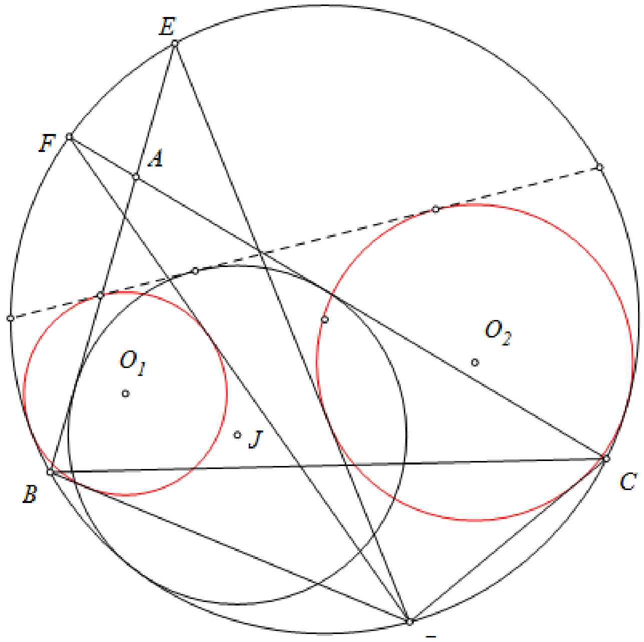
Using the lemma for the chords AB, PQ and the point L on the arc $BCP,$ it follows that 2nd intersection W of $\omega \equiv \odot(LKI_1)$ with (O) belows to the incircle Ω_1 of the curvilinear triangle bounded by AB, PQ and the arc $BCP.$ Again using the lemma for the chords AC, PQ and the point L on the arc $CBQ,$ it follows that 2nd intersection W of $\omega \equiv \odot(LKI_2)$ with (O) belows to the incircle Ω_2 of the curvilinear triangle bounded by AC, PQ and the arc $CBQ.$ But there's only one circle internally tangent to (O) at W and tangent to $PQ \Longrightarrow \Omega_1 \equiv \Omega_2$ coincide with the A-mixtilinear incircle (J) of $\triangle ABC$ and the conclusion follows.

livetolove212

#7 Apr 20, 2014, 2:48 pm • 3

Thanks Luis for your nice proof which is base on the generalization of Iran 1998. This generalization is belong to Tran Quang Hung.

Attachments:



P

Quick Reply

High School Olympiads

fixed circle

Reply

phuong

#1 Apr 18, 2014, 4:54 pm

Let ABC be a triangle and D, E, F are the midpoints of BC, CA, AB, respectively. K is a fixed point on segment EF. The circle with diameter AD cuts the arbitrary line through K at M, N. The lines MF, NF cut the circle with diameter AD at P, Q. Prove the midpoint of PQ belongs to the fixed circle.

Luis González

#2 Apr 18, 2014, 9:45 pm • 1

There's a lot of things here that we actually don't need. Only a circle (O), any fixed line ℓ and fixed points $K, F \in \ell$ are sufficient.

As MN spins around K, then clearly $P \mapsto Q$ is a homography on (O). When P coincides with one of the intersections of (O) with ℓ, then Q will go to the other intersection and vice versa, thus $P \mapsto Q$ is involutive $\Longrightarrow PQ$ goes through a fixed point $L \in \ell$. Hence if X denotes the midpoint of $\overline{PQ}$, then $OX \perp LX \Longrightarrow X$ is on the circle with diameter $\overline{OL}$.

In general, when $(O) \equiv \mathcal{O}$ is a conic with center O, the locus of X is another conic passing through O and L. Since OX and PQ have conjugate directions WRT $\mathcal{O}$, then they induce a homography between the pencils LX, OX and the result follows.

phuong

#3 Apr 19, 2014, 4:54 am

> Luis González wrote:
> As MN spins around K, then clearly $P \mapsto Q$ is a homography on (O).

Luis González! What is homography transformation in geometry? Could you explain some thing?

Luis González

#4 Apr 19, 2014, 8:08 am

phuong, two planar figures are said to be homographic, iff they correspond to each other point to point and line to line, such that the incidence condition between points and lines is preserved. It can be proved that homography preserves cross ratios and if it has 4 fixed points then it's the identity. Homographies also form a projective group, because: the inverse of a homography is a homography, the composition of two homographies is also homography and the identity is a homography itself.

In the problem, we have a homography leaving the circle (O) invariant, thus it has an axis τ cutting (O) at the double points (real or imaginary). Since, it is involutive (double correspondence), then all PQ go through a fixed point I, which is in fact the pole of τ WRT (O). This is easy to prove resorting to the construction of homologous points using the axis τ.

phuong

#5 Apr 20, 2014, 1:25 pm

I know point K, the intersection point of EF and PQ, is the fixed point. But i can't prove it by elementary proof. Anyone can prove it?

Quick Reply

High School Olympiads

A problem from a forgotten contest!

Reply

tempusername
#1 Apr 16, 2014, 3:59 pm

Let ω be a circle tangent to the side BC and the circumcircle of the triangle ΔABC, with center O, such that O lies on the internal angle bisector of A. Let ω touches side BC and the circumcircle in P and Q, respectively. Prove that $\widehat{BAP} = \widehat{CAQ}$.

I read this problem somewhere, but I can't remember. Would you mind telling where were it from? Thanks

jayme
#2 Apr 16, 2014, 4:30 pm

Dear Mathlinkers,
is ABC A-isocèles?
Sincerely
Jean-Louis

sunken rock
#3 Apr 18, 2014, 1:29 am

> *jayme wrote:*
> Dear Mathlinkers,
> is ABC A-isocèles?
> Sincerely
> Jean-Louis

No, Q and A lie on different sides of BC.

Best regards,
sunken rock

Luis González
#4 Apr 18, 2014, 3:10 am

Angle bisector AO of $\angle BAC$ cuts circumcircle $(K) \equiv \odot(ABC)$ again at the midpoint N of its arc $BQC \Longrightarrow KN$ is the perpendicular bisector of BC and clearly K, O, Q are collinear $\Longrightarrow \angle NAQ = \frac{1}{2}\angle QKN = \frac{1}{2}\angle POK$. But since $\triangle POQ$ is O-isosceles, we have $\angle QPO = \angle PQO = \frac{1}{2}\angle POK \Longrightarrow \angle NAQ = \angle QPO \Longrightarrow A, P, O, Q$ are concyclic, being O the midpoint of the arc $PQ \Longrightarrow AO$ also bisects $\angle PAQ$ and the conclusion follows.

Quick Reply

High School Olympiads

Six distances

Reply

Source: Own

Bandera

#1 Mar 27, 2014, 5:34 am

Let $A_1A_2A_3$ be a triangle with incenter I. $M_1M_2M_3$ and $I_1I_2I_3$ are the medial and the contact (intouch) triangles of $A_1A_2A_3$, respectively. c_1, c_2, c_3 are distances from I to the sides of $M_1M_2M_3$ and d_1, d_2, d_3 are distances from I to the sides of $I_1I_2I_3$. Prove that

$$c_1c_2c_3 = d_1d_2d_3.$$

Sardor

#2 Mar 30, 2014, 9:26 pm

I have a trigonometric solution,any another solutions?

Luis González

#3 Apr 18, 2014, 2:12 am • 1

Notation is a bit nasty to me, so rename $\triangle ABC \equiv \triangle A_1A_2A_3$ and let $\text{d}(P,\ell)$ denote the distance from point P to the line ℓ. Incircle (I) touches BC, CA, AB at X, Y, Z and D, E, F are the midpoints of BC, CA, AB.

If J is the projection of B on AI, then we have $\widehat{BFJ} = 2 \cdot \widehat{BAJ} = \widehat{BAC} \Longrightarrow FJ \parallel AC \Longrightarrow J \in DF$. From cyclic $IJXB$ (due to right angles at J,X), we have $\widehat{JXC} = \widehat{BIJ} = \widehat{YXC} \Longrightarrow J \in XY$. Consequently, we have

$$\frac{\text{d}(I,FD)}{\text{d}(I,XY)} = \frac{\sin \widehat{IJF}}{\sin \widehat{IJY}} = \frac{\sin \widehat{BAJ}}{\sin \widehat{IBX}} = \frac{\sin \frac{A}{2}}{\sin \frac{B}{2}}.$$

Multiplying cyclic expressions together gives

$$\frac{\text{d}(I,FD)}{\text{d}(I,XY)} \cdot \frac{\text{d}(I,DE)}{\text{d}(I,YZ)} \cdot \frac{\text{d}(I,EF)}{\text{d}(I,ZX)} = \frac{\sin \frac{A}{2}}{\sin \frac{B}{2}} \cdot \frac{\sin \frac{B}{2}}{\sin \frac{C}{2}} \cdot \frac{\sin \frac{C}{2}}{\sin \frac{A}{2}} = 1 \Longrightarrow$$

$$\text{d}(I,FD) \cdot \text{d}(I,DE) \cdot \text{d}(I,EF) = \text{d}(I,XY) \cdot \text{d}(I,YZ) \cdot \text{d}(I,ZX).$$

Bandera

#4 Apr 21, 2014, 11:34 am

Actually, $c_1c_2 = d_3^2$ (and, similarly, $c_2c_3 = d_1^2$ and $c_3c_1 = d_2^2$). It follows from the fact that, if we take projections of two vertices of the initial triangle on a bisector of the third angle, the resulting points (like J in the notation of Luis González) are inverses of each other with respect to incircle of the triangle.

Quick Reply

High School Olympiads

Locus of the point

Reply

joybangla

#1 Apr 16, 2014, 3:46 pm

In a triangle ABC an interior point P is such that $BD+DC=BE+EC$ where $D=BP\cap AC, E=CP\cap AB$. Find the locus of P.

Luis González

#2 Apr 17, 2014, 12:10 am

$BD+DC=BE+EC \Longrightarrow D,E$ lie on ellipse $\mathcal{E}$ with foci $B,C.$ By property of conic tangents, external bisectors e,d of $\widehat{BEC}$ and $\widehat{BDC}$ are tangents of $\mathcal{E}$ at $E,D.$ If $J \equiv e \cap d,$ again by conic tangents property, BJ bisects $\widehat{EBD} \Longrightarrow J$ is B-excenter of $\triangle BPE \Longrightarrow AEPD$ is tangential with incenter $J \Longrightarrow AE+PD=AD+PE \Longrightarrow$ $AB-BE+BD-PB=AC-DC+EC-PC.$ Combined with $BD+DC=BE+EC$ gives $PC-PB=AC-AB \Longrightarrow$ locus of P is a branch of the hyperbola with foci B,C that goes through $A.$

Quick Reply

Parallelogram and Locus (Own)

Arab
#1 Mar 15, 2014, 2:53 pm

O, I are the circumcenter and incenter of $\triangle ABC$ with fixed circumcircle ω and fixed side BC. $R \in BI, S \in CI$ such that $RE \parallel SF \parallel OI$, where E, F are the pedals of I on CA, AB respectively. Prove that

(1) the quadrilateral $REFS$ is a parallelogram,
(2) $IT \perp BC$, where $T = RF \cap SE$.

Find the locus of T when A varies on $\widehat{BC}$ of ω (All these As are on the same side of BC).

XmL
#2 Mar 16, 2014, 3:18 am

Let D be the pedal of I on BC. Construct H the orthocenter $\triangle DEF$. It's well known that H, I, O are collinear. Since $FH, SC \perp DE$, therefore $FSHI$ is a parallelogram, so is $HIRE$ similarly and there goes (1).

For (2), we construct I' on the extension of SI such that I is the midpoint of SI'. Hence $FH = SI = II'$. let M be the midpoint of DE, since it's well known that $FH = 2IM$, therefore M is the midpoint of $II' \Rightarrow ID \parallel EI' \parallel IT \Rightarrow I, D, T$ are collinear and there goes (2).

For the locus, we can use this result from above:$TD = \frac{3}{2}ID$.

Arab
#3 Mar 16, 2014, 8:24 am

Alternatively,$P = BE \cap OI, Q = CF \cap OI \implies \frac{EP}{PB} = 2\sin^2\frac{B}{2}, \frac{FQ}{QC} = 2\sin^2\frac{C}{2} \implies \frac{SI}{RI} = \frac{BI}{CI} \implies$ B, C, R, S are concyclic $\implies RS \parallel EF \implies$ the quadrilateral $REFS$ is a parallelogram.

The locus is part of an ellipse.

XmL
#4 Mar 16, 2014, 10:04 am

What I've written above wasn't my first "instinct", I introduced H so that it flows with (2).
For (1) we could also consider the intersection of $(OE, RE), (OF, SF)$ and then ratio chase.

Arab
#5 Mar 16, 2014, 10:41 am

What do you mean by $(OE, RE), (OF, SF)$?

Luis González
#6 Apr 14, 2014, 9:35 pm

Let L and X be the centroid and orthocenter of $\triangle DEF$, lying on its Euler line OI. Since $ER \parallel XL \equiv OI$ and $EX \parallel IR$ (both perpendicular to DF), then $REXI$ is a parallelogram $\Longrightarrow ER = XI$. Similarly $SFXI$ is a parallelogram, i.e. $FS = XI$, therefore $REFS$ is a parallelogram. If M is the midpoint of $\overline{EF}$ and $T' \in ID$, such that $\overline{IT'} : \overline{ID} = -1 : 2$, then we get $MT' = \frac{3}{2}LI = \frac{3}{2} \cdot \frac{1}{3}XI = \frac{1}{2}XI = \frac{1}{2}FS$ and $MT' \parallel LI \parallel FS,$ which implies that E, T', S are collinear and T' is midpoint of $\overline{ES} \Longrightarrow T \equiv T' \Longrightarrow TID \perp BC.$

As A varies on ω, I describes an arc of a circle ω_A passing through $B, C,$ because $\widehat{BIC} = 90^\circ + \frac{1}{2}\widehat{A} = \text{const}.$ Since $\overline{DI} : \overline{DT} = 2 : 3,$ then the affine homology accross BC with characteristic $\frac{2}{3}$ and affine direction $\perp BC$ will transform ω_A into an ellipse Ω_A passing through $B, C \Longrightarrow$ locus of T is an arc BC of this ellipse. If P is the midpoint of $\overline{BC}$ and J the midpoint of the arc BC of ω (center of ω_A), then the center K of Ω_A lies on PJ and verifies $\overline{JK} : \overline{JP} : -1 : 2.$

Quick Reply

High School Olympiads

Find the locus

Locked

StanleyST

#1 Apr 14, 2014, 3:51 pm

Let D be a variable point on the side BC of triangle ABC. We consider the incircles of triangles ABD and ACD and let d be the common tangent of these circles different from BC. d and AD meet at point M. Find the locus of points M

Luis González

#2 Apr 14, 2014, 7:53 pm

Posted before. If the incircle touches AC,AB at Y,Z, the locus of M is the circular arc with center A and radius AY=AZ.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=28329
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=385764
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=448435

High School Olympiads

two circles

Reply

Source: unknown

Blitzkrieg97

#1 Apr 14, 2014, 2:14 am

Given $ABCD$ and $A_1B_1C_1D_1$ quadrilaterals, which have same inscribed and circumscribed circles. Prove that AC, BD, A_1C_1 and B_1D_1 intersects in one point

Luis González

#2 Apr 14, 2014, 5:00 am • 1

If (I,r) and (O,R) denote the common incircle and circumcircle of $ABCD$ and $A_1B_1C_1D_1$, then $P \equiv AC \cap BD$ is on IO and verifies $OP = \frac{2R^2 \cdot IO}{R^2 - IO^2}$ (for a proof see the thread Bicentric quadrilateral 4). This is obviously independent of the quadrilateral $ABCD \Longrightarrow P \equiv AC \cap BD \cap A_1C_1 \cap B_1D_1$. $\square$

Now a generalization: If $ABCD$ and $A_1B_1C_1D_1$ are both inscribed and circumscribed to two same conics, then AC, BD, A_1C_1 and B_1D_1 concur.

Label $\mathcal{I}$ and $\mathcal{O}$ the inscribed and circumscribed conic, respectively. By Desargues theorem, the tangents A_1B_1, A_1D_1 from A_1 to $\mathcal{I}$ below to the involution $A_1A \mapsto A_1C$, $A_1B \mapsto A_1D$ that transforms $\mathcal{O}$ into itself $\Longrightarrow P \equiv AC \cap BD$ is the center of the homographic involution $\Longrightarrow B_1D_1$ goes through P and similarly A_1C_1 goes through P.

Blitzkrieg97

#3 Apr 14, 2014, 4:38 pm

very beautiful solution, thanks

Quick Reply

High School Olympiads

Locus of a point

Reply

pandypanda1234

#1 Mar 12, 2014, 3:16 pm

For any triangle ABC on a plane, find the locus of all points P in its interior such that AP bisects angle BPC.

Luis González

#2 Apr 13, 2014, 9:56 pm • 2

If AP cuts BC at D, then by angle bisector theorem, AP bisects $\angle BPC$ either internally or externally $\Longleftrightarrow \frac{PB}{PC}=\left|\frac{DB}{DC}\right|$. Thus, if $(x:y:z)$ are the barycentric coordinates of P WRT $\triangle ABC$, we have

$$PB^2=\frac{a^2z^2+2S_Bzx+c^2x^2}{(x+y+z)^2} \ , \ PC^2=\frac{b^2x^2+2S_Cxy+a^2y^2}{(x+y+z)^2} \Longrightarrow$$

$$\frac{a^2z^2+2S_Bzx+c^2x^2}{b^2x^2+2S_Cxy+a^2y^2}=\frac{z^2}{y^2} \Longrightarrow \mathcal{K} \equiv x(c^2y^2-b^2z^2)+2S_By^2z-2S_Cyz^2=0.$$

Locus $\mathcal{K}$ is a cubic through $A \equiv (1:0:0), B \equiv (0:1:0), C \equiv (0:0:1)$ (circum-cubic), the foot of the A-altitude $H_A \equiv (0:S_C:S_B)$ and the Fermat points $X_{14}, X_{15} \equiv \left(\frac{1}{\sqrt{S_A} \pm S}\right)$. Traces of $\mathcal{K}$ on lines AC, AB are only A, C and A, B, respectively and its traces on the line at infinity $x+y+z=0$, setting $(x:y:z) \longrightarrow (-\mu-1:\mu:1)$, are $(-i \cdot S \pm S_B : i \cdot S \pm S_A : \pm c^2)$, always imaginary and $(-2:1:1)$, which is the infinite point of the A-median.

TelvCohl

#3 Oct 11, 2014, 3:03 pm • 1

The locus of P is the isogonal conjugate of A-Apollonius circle WRT $\triangle ABC$:

Let P' be the isogonal conjugate of P WRT $\triangle ABC$. Since $\frac{P'B}{P'C}=\frac{\sin \angle P'CB}{\sin \angle CBP'}=\frac{\sin \angle ACP}{\sin \angle PBA}=\frac{AB}{AC}$, so P' lies on the A-Apollonius circle of $\triangle ABC$.

This post has been edited 1 time. Last edited by TelvCohl, Dec 20, 2015, 7:01 pm

Quick Reply

High School Olympiads

Circumscribed Quadrilateral

Reply

muscleman

#1 Apr 2, 2014, 8:47 pm

Let I be the incenter of a circumscribed quadrilateral ABCD. The tangents to circle AIC at points A, C meet at point X. The tangents to circle BID at points B, D meet at point Y . Prove that X, I, Y are collinear.

fmasroor

#2 Apr 5, 2014, 7:28 am

This is one of the ones I was convinced I could crack with symmedians. But I didn't.

Luis González

#3 Apr 13, 2014, 4:28 am

Let the incircle (I) of $ABCD$ touch AB, BC, CD, DA at $P, Q, R, S,$ respectively. $A' \equiv IA \cap SP$ and $C' \equiv IC \cap QR$ are midpoints of $\overline{SP}, \overline{QR}.$ Since $IP^2 = IA \cdot IA' = IC \cdot IC'$, then $ACC'A'$ is cyclic $\Longrightarrow A'C'$ is antiparallel to AC WRT $IA, IC \Longrightarrow$ I-symmedian IX of $\triangle IAC$ becomes I-median of $\triangle IA'C' \Longrightarrow IX$ cuts $\overline{A'C'}$ at its midpoint $V,$ which is none other than the center of the Varignon's parallelogram of $PQRS.$ Similarly Y falls on the line $IV \Longrightarrow I, X, Y$ are collinear.

Quick Reply

High School Olympiads

show that PA+PB=PC

Source: Singapore TST round 1

daDangminh
#1 Apr 12, 2014, 6:52 am

Given triangle ABC, with E,F on AC,AB respectively such that BE,CF bisect angle B and C respectively. P,Q are on the minor arc AC of the circumcircle of triangle ABC such that AC//PQ and BQ//EF. Show that PA+PB=PC

XmL
#2 Apr 12, 2014, 9:37 am

A fun proof:

Let I be the incenter of ABC and $CF \cap (ABC) = G$.

Lemma: E lies on the radical axis of $(AGF), (FBI)$

Proof: Let $(AGF) \cap AC = K$. Since $\angle KFI = \angle CAG = \angle CFB$ and CG bisects $\angle KCB$. Therefore by ASA congruence B, K are reflexive over $CF \Rightarrow \angle CKI = \angle IBC = \angle ABI \Rightarrow A, K, I, B$ are concyclic and hence I lies on the radical axis of $(AGF), (FBI)$.

Main problem: Reflect B over the perpendicular bisector of AC which lands on (ABC), denote it B'. Hence $PB + PA = PC \iff QB' + QC = QA$. Now let L be a point of segment AQ such that $AL = QB'$ Through L construct a line parallel to BQ which intersects AC at B''. Since $\angle ALB'' = \angle AQB = \angle QCB'$ and $\angle QAC = \angle QB'C$, therefore by ASA $\triangle ALB'', \triangle B'QC$ are congruent$\Rightarrow AB'' = CB' = AB, CQ = LB''$, hence we now just need to prove $QL = AQ - AL = QA - BQ' = QC \iff QB''$ bisects $\angle AQB \iff Q, B'', F$ are collinear.

Let $QG \cap FE = J$, since $\angle GQB = \angle GJF = \angle GAF$, therefore A, G, F, J are concyclic. By our lemma we know that B, I, J, F are concyclic. Hence $\angle FIB = \angle FIB = 90 - \angle A/2 = \angle AB''B \Rightarrow J, E, B'', B$ are concyclic. Since $\angle EJB'' = \angle EBB'' = 90 - \angle A/2 - \angle B/2 = \angle C/2 = \angle GJF$, therefore G, J, B'', Q are collinear and we are done.

Luis González
#3 Apr 12, 2014, 11:44 am

There is more to say about this configuration; P is actually one of the Feuerbach points of the antimedial triangle of ABC. Thus, it follows that either PC=PA+PB or PB=PA+PC, according to whether P is on arc AC or arc AB. This is an extraversion of the incircle case point [Feuerbach point of a triangle; FY + FZ = FX].

Let $\triangle XYZ$ be the antimedial triangle of $\triangle ABC$ (X,Y,Z againts A,B,C) and its X-excircle (J) touches its 9-point circle $(O) \equiv \odot(ABC)$ at its Feuerbach point F_X. Parallel from F_X to AC cuts (O) again at $D \Longrightarrow BD, BF_X$ are isogonals WRT $\triangle ABC \Longrightarrow BD$ has the direction of the isogonal conjugate of F_X WRT $\triangle ABC$.

It's known that EF is perpendicular to the line connecting O and the A-excenter of $\triangle ABC$, hence EF is also parallel to the line connecting the circumcenter K of $\triangle XYZ$ with J. It's also known that JK is the Steiner line of F_X WRT $\triangle ABC$ (this is again an extraversion of the incircle case), thus $\perp JK$ is direction of the isogonal conjugate of F_X WRT $\triangle ABC \Longrightarrow BD \perp JK \Longrightarrow BD \parallel EF \Longrightarrow D \equiv Q, F_X \equiv P$, as desired.

High School Olympiads

russian problem

Reply

max123
#1 Apr 11, 2014, 1:00 pm

ABC is a traingle, let the incircle of it touch AB and AC in X,Y. Let K be the midpoint of the arc AB of the circumcenter of the traingle. If XY bisects AK, find the possible measures of the angle <BAC.

junioragd
#2 Apr 12, 2014, 7:21 am

Do you mean arc BC

Luis González
#3 Apr 12, 2014, 7:43 am

The problem clearly specifies the arc AB. I don't think there's a typo.

Let U and V be the reflections of A on X and Y. If XY passes through the midpoint of $\overline{AK}$, then U, V, K must be collinear. Since $UV \parallel XY \perp AI$ and $KA = KI$ (valid for any ABC), then it follows that KVU is perpendicular bisector of $\overline{AI}$, which makes the I-isosceles $\triangle IAV$ and $\triangle IAU$ both equilateral $\Longrightarrow \angle UAV = \angle BAC = 120^\circ$.

Quick Reply

High School Olympiads