

# Geometry Problem Set

National Camp 2016

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## 1. Basic Stuffs

**Problem 1.1.** Prove that the diagonals of rhombus are perpendicular.

**Problem 1.2.**  $L, M$  be the midpoint of  $BC$  and  $CA$  of  $\triangle ABC$  resp. Prove that  $AL = BM \iff AC = BC$ .

**Problem 1.3.** Let  $P, Q, R, S$  be 4 points on a plane. Prove  $Pr \perp QS \iff PQ^2 - QR^2 = PS^2 - RS^2$ .

**Problem 1.4.** Let  $P, Q, R$  be points on sides  $BC, CA, AB$  of  $\triangle ABC$ . Prove that the perpendiculars to the sides at these points are concurrent if and only if  $BP^2 + CQ^2 + AR^2 = PC^2 + QA^2 + RB^2$ .

**Problem 1.5.** Let  $P$  and  $Q$  be arbitrary points on sides  $BC$  and  $CA$  resp. Let the internal bisectors of  $\angle CAP$  and  $\angle CBQ$  meet at  $R$ . Prove that  $\angle AQB + \angle APB = 2\angle ARB$ .

**Problem 1.6.** In  $\triangle ABC$ ,  $P$  lies on  $BC$ . Prove that

$$\frac{BP}{CP} = \frac{AB \times \sin \angle BAP}{AC \times \sin \angle PAC}$$

**Problem 1.7.** Let  $D, E, F$  are the midpoints of  $BC, CA, AB$  resp. Prove that  $\angle CAD = \angle ABE \iff \angle AFC = \angle ADB$ .

**Problem 1.8.** Let circles  $S_1$  and  $S_2$  meet at points  $A$  and  $B$ . An arbitrary line passing through  $A$  intersects  $S_1$  and  $S_2$  at  $P$  and  $Q$  resp. Prove  $\frac{BP}{BQ}$  is constant.

**Problem 1.9.** Let  $BO \cap \odot ABC = Q$ . Prove that  $AQCH$  is a parallelogram where  $O$  and  $H$  are the circumcenter and orthocenter of  $ABC$  resp.

**Problem 1.10.** Let the angle bisector of  $\angle BAC$  meets  $\odot ABC$  at  $A$  and  $X$  resp. Prove that  $XI = XB = XC = XI_a$  where  $I$  is the incenter and  $I_a$  is the excenter opposite to  $A$  of  $\triangle ABC$ .

**Problem 1.11.** Let  $L, M, N$  are the midpoints of  $BC, CA, AB$  and  $AD, BE, CF$  are altitudes of  $\triangle ABC$ . Prove that

- $O$  is the orthocenter of  $\triangle LMN$ .
- $H$  the incenter of  $\triangle DEF$ .
- $D, E, F, L, M, N$  all lie on a circle.

- The center of this circle is the midpoint of  $OII$ .

**Problem 1.12.** In  $\triangle ABC$ ,  $\angle BAC = 90^\circ$ ,  $AD$  is an altitude. The circle with center  $A$  and radius  $AD$  meets  $\triangle ABC$  at  $U$  and  $V$  resp. Prove that  $UV$  passes through the midpoint of  $AD$ .

**Problem 1.13.** (a) Let the incircle and excircle (opposite to  $A$ ) of  $\triangle ABC$  meet  $BC$  at  $D$  and  $E$  resp. Suppose  $F$  is the antipode of  $D$  wrt the incircle. Prove that  $A, F, E$  are collinear. (b)  $M$  be the midpoint of  $DE$ . Prove that  $MI$  meets  $AD$  at its midpoint.

**Problem 1.14.** Let the incircle of  $\triangle ABC$  meet  $AB$  and  $AC$  at  $X$  and  $Y$  resp.  $BI$   $CI$  meet  $XY$  at  $P$  and  $Q$  resp. Prove that  $BYXC$  is cyclic.

**Problem 1.15.** If four points  $A, B, C, D$  on a line satisfy the property that  $\frac{AC}{BC} = \frac{AD}{BD}$ , then  $A, B, C, D$  are in harmonic order. Prove that if  $A, B, C, D$  are in harmonic order and  $M$  is the midpoint of  $AB$ , then

(a)  $MA^2 = MC \cdot MD$  and  $DA > DB = DC \cdot DM$ .

(b)  $P$  is a point s.t  $\angle APB = 90^\circ$ , then  $PA$  and  $PB$  are two bisectors of  $\angle CPD$ .

**Problem 1.16.**  $AD$  is an altitude of  $\triangle ABC$ .  $E, F$  are on  $AC, AB$  so that  $AD, BE, CF$  are concurrent. Prove  $\angle EDA = \angle FDA$ .

**Problem 1.17.** Let  $O$  be the circumcenter of  $\triangle ABC$  and  $A', B', C'$  are reflections of  $A$  on  $BC, CA, AB$  resp. Prove that  $AA', BB', CC'$  are concurrent.

**Problem 1.18.** Let  $D, E$  are on sides  $AC, AB$  of  $\triangle ABC$  resp. such that  $BE = CD$ . Let  $\odot ABC \cap \triangle ADE = P$ . Prove that  $PB = PC$ .

**Problem 1.19.** Let a line  $PQ$  touches circle  $S_1$  and  $S_2$  at  $P$  and  $Q$  resp. Prove that the radical axis of  $S_1$  and  $S_2$  passes through the midpoint of  $PQ$ .

**Problem 1.20.** Two equal-radius circles  $\omega_1$  and  $\omega_2$  are centered at points  $O_1$  and  $O_2$ . A point  $X$  is reflected through  $O_1$  and  $O_2$  to get points  $A_1$  and  $A_2$ . The tangents from  $A_1$  to  $\omega_1$  touch  $\omega_1$  at points  $P_1$  and  $Q_1$ , and the tangents from  $A_2$  to  $\omega_2$  touch  $\omega_2$  at points  $P_2$  and  $Q_2$ . If  $P_1Q_1$  and  $P_2Q_2$  intersect at  $Y$ , prove that  $Y$  is equidistant from  $A_1$  and  $A_2$ .

**Problem 1.21.** Let  $BD, CE$  are altitudes of  $\triangle ABC$  and  $M$  be the midpoint of  $BC$ .  $MH \cap \odot ABC = L$ . Prove that  $AK, BC, DE$  are concurrent.

**Problem 1.22.** Two circles  $\omega$  and  $\Gamma$  touch one another internally at  $P$  with  $\omega$  inside of  $\Gamma$ . Let  $AB$  be a chord of  $\Gamma$  which touches  $\omega$  at  $D$ . Let  $PD \cap \Gamma = Q$ . Prove that  $QA = QB$ .

**Problem 1.23.** Let  $AD$  be a symmedian of  $\triangle ABC$  with  $D$  on  $\odot ABC$ . Let  $M$  be the midpoint of  $AD$ . Prove that  $\angle BMD = \angle CMD$  and  $A, M, O, D$  are cyclic where  $O$  is the circumcenter of  $\triangle ABC$ .

## 2. Olympiad Problems

**Problem 2.1.** Let  $PB$  and  $PC$  are tangent to  $\odot ABC$ . Let  $D, E, F$  are projection of  $A$  on  $BC, PB, PC$  resp. Prove that  $AF^2 = AE \times AF$ .

**Problem 2.2.** Let  $D$  and  $E$  are on  $AB$  and  $AC$  s.t  $DE \parallel BC$ .  $P$  is an arbitrary point inside  $\triangle ADE$ .  $PB, PC \cap DE = F, G$ . Let  $\odot PDG \cap \odot PGE = Q$ . Prove that  $A, P, Q$  are collinear.

**Problem 2.3.** Let  $AB$  and  $CD$  be chords in a circle of center  $O$  with  $A, B, C, D$  distinct, and with the lines  $AB$  and  $CD$  meeting at a right angle at point  $E$ . Let also  $M$  and  $N$  be the midpoints of  $AC$  and  $BD$  respectively. If  $MN \perp OE$ , prove that  $AD \parallel BC$ .

**Problem 2.4.** Circles  $C_1$  and  $C_2$  intersect at  $A$  and  $B$ . Let  $M \in AB$ . A line through  $M$  (different from  $AB$ ) cuts circles  $C_1$  and  $C_2$  at  $Z, D, E, C$  respectively such that  $D, E \in ZC$ . Perpendiculars at  $B$  to the lines  $EB, ZB$  and  $AD$  respectively cut circle  $C_2$  in  $F, K$  and  $N$ . Prove that  $KF = NC$ .

**Problem 2.5.** Let  $D$  be a point on side  $AC$  of triangle  $ABC$ . Let  $E$  and  $F$  be points on the segments  $BD$  and  $BC$  respectively, such that  $\angle BAE = \angle ACF$ . Let  $P$  and  $Q$  be points on  $BC$  and  $BD$  respectively, such that  $EP$  and  $FQ$  are both parallel to  $CD$ . Prove that  $\angle BAP = \angle CAQ$ .

**Problem 2.6.** In the non-isosceles triangle  $ABC$  an altitude from  $A$  meets side  $BC$  in  $D$ . Let  $M$  be the midpoint of  $BC$  and let  $N$  be the reflection of  $M$  in  $D$ . The circumcircle of triangle  $AMN$  intersects the side  $AB$  in  $P \neq A$  and the side  $AC$  in  $Q \neq A$ . Prove that  $AN, BQ$  and  $CP$  are concurrent.

**Problem 2.7.** In triangle  $ABC$ , the interior and exterior angle bisectors of  $\angle BAC$  intersect the line  $BC$  in  $D$  and  $E$ , respectively. Let  $F$  be the second point of intersection of the line  $AD$  with the circumcircle of the triangle  $ABC$ . Let  $O$  be the circumcentre of the triangle  $ABC$  and let  $D'$  be the reflection of  $D$  in  $O$ . Prove that  $\angle D'FE = 90^\circ$ .

**Problem 2.8.** Let  $ABCD$  be a convex quadrilateral such that the line  $BD$  bisects the angle  $ABC$ . The circumcircle of triangle  $ABC$  intersects the sides  $AD$  and  $CD$  in the points  $P$  and  $Q$ , respectively. The line through  $D$  and parallel to  $AC$  intersects the lines  $BC$  and  $BA$  at the points  $R$  and  $S$ , respectively. Prove that the points  $P, Q, R$  and  $S$  lie on a common circle.

**Problem 2.9.** The incircle of triangle  $ABC$  touches  $BC, CA, AB$  at points  $A_1, B_1, C_1$ , respectively. The perpendicular from the incenter  $I$  to the median from vertex  $C$  meets the line  $A_1B_1$  in point  $K$ . Prove that  $CK$  is parallel to  $AB$ .

**Problem 2.10.** Let  $X$  be an arbitrary point inside the circumcircle of a triangle  $ABC$ . The lines  $BX$  and  $CX$  meet the circumcircle in points  $K$  and  $L$  respectively. The line  $LK$  intersects  $BA$  and  $AC$  at points  $E$  and  $F$  respectively. Find the locus of points  $X$  such that the circumcircles of triangles  $AFK$  and  $AEL$  touch.

**Problem 2.11.** Let  $BD$  be a bisector of triangle  $ABC$ . Points  $I_a, I_c$  are the incenters of triangles  $ABD, CBD$  respectively. The line  $I_aI_c$  meets  $AC$  in point  $Q$ . Prove that  $\angle DBQ = 90^\circ$ .

**Problem 2.12.** Given right-angled triangle  $ABC$  with hypotenuse  $AB$ . Let  $M$  be the midpoint of  $AB$  and  $O$  be the center of circumcircle  $\omega$  of triangle  $CMB$ . Line  $AC$  meets  $\omega$  for the second time in point  $K$ . Segment  $KO$  meets the circumcircle of triangle  $ABC$  in point  $L$ . Prove that segments  $AL$  and  $KM$  meet on the circumcircle of triangle  $ACM$ .

**Problem 2.13.** Let  $BN$  be median of triangle  $ABC$ .  $M$  is a point on  $BC$ .  $S$  lies on  $BN$  such that  $MS \parallel AB$ .  $P$  is a point such that  $SP \perp AC$  and  $BP \parallel AC$ .  $MP$  cuts  $AB$  at  $Q$ . Prove that  $QB = QP$ .

**Problem 2.14.** Let  $ABCD$  be a convex quadrilateral with  $AB$  parallel to  $CD$ . Let  $P$  and  $Q$  be the midpoints of  $AC$  and  $BD$ , respectively. Prove that if  $\angle ABP = \angle CBD$ , then  $\angle BCQ = \angle ACD$ .

**Problem 2.15.** Point  $P$  lies inside a triangle  $ABC$ . Let  $D, E$  and  $F$  be reflections of the point  $P$  in the lines  $BC, CA$  and  $AB$ , respectively. Prove that if the triangle  $DEF$  is equilateral, then the lines  $AD, BE$  and  $CF$  intersect in a common point.

**Problem 2.16.** Let  $\triangle ABC$  be an acute angled triangle. The circle with diameter  $AB$  intersects the sides  $AC$  and  $BC$  at points  $E$  and  $F$  respectively. The tangents drawn to the circle through  $E$  and  $F$  intersect at  $P$ . Show that  $P$  lies on the altitude through the vertex  $C$ .

**Problem 2.17.** Let  $\gamma$  be circle and let  $P$  be a point outside  $\gamma$ . Let  $PA$  and  $PB$  be the tangents from  $P$  to  $\gamma$  (where  $A, B \in \gamma$ ). A line passing through  $P$  intersects  $\gamma$  at points  $Q$  and  $R$ . Let  $S$  be a point on  $\gamma$  such that  $BS \parallel QR$ . Prove that  $SA$  bisects  $QR$ .

**Problem 2.18.** Given is a convex quadrilateral  $ABCD$  with  $AB = CD$ . Draw the triangles  $ABE$  and  $CDF$  outside  $ABCD$  so that  $\angle ABE = \angle DCF$  and  $\angle BAE = \angle FDC$ . Prove that the midpoints of  $\overline{AD}$ ,  $\overline{BC}$  and  $\overline{EF}$  are collinear

**Problem 2.19.** Let  $P$  be a point out of circle  $C$ . Let  $PA$  and  $PB$  be the tangents to the circle drawn from  $C$ . Choose a point  $K$  on  $AB$ . Suppose that the circumcircle of triangle  $PBK$  intersects  $C$  again at  $T$ . Let  $P'$  be the reflection of  $P$  with respect to  $A$ . Prove that

$$\angle PBT = \angle P'KA$$

**Problem 2.20.** Consider a circle  $C_1$  and a point  $O$  on it. Circle  $C_2$  with center  $O$ , intersects  $C_1$  in two points  $P$  and  $Q$ .  $C_3$  is a circle which is externally tangent to  $C_2$  at  $R$  and internally tangent to  $C_1$  at  $S$  and suppose that  $RS$  passes through  $Q$ . Suppose  $X$  and  $Y$  are second intersection points of  $PR$  and  $OR$  with  $C_1$ . Prove that  $QX$  is parallel with  $SY$ .

**Problem 2.21.** In triangle  $ABC$  we have  $\angle A = \frac{\pi}{3}$ . Construct  $E$  and  $F$  on continue of  $AB$  and  $AC$  respectively such that  $BE = CF = BC$ . Suppose that  $EF$  meets circumcircle of  $\triangle ACE$  in  $K$ . ( $K \neq E$ ). Prove that  $K$  is on the bisector of  $\angle A$

**Problem 2.22.** In triangle  $ABC$ ,  $\angle A = 90^\circ$  and  $M$  is the midpoint of  $BC$ . Point  $D$  is chosen on segment  $AC$  such that  $AM = AD$  and  $P$  is the second meet point of the circumcircles of triangles  $\triangle AMC, \triangle BDC$ . Prove that the line  $CP$  bisects  $\angle ACB$

**Problem 2.23.** Let  $C_1, C_2$  be two circles such that the center of  $C_1$  is on the circumference of  $C_2$ . Let  $C_1, C_2$  intersect each other at points  $M, N$ . Let  $A, B$  be two points on the circumference of  $C_1$  such that  $AB$  is the diameter of it. Let lines  $AM, BN$  meet  $C_2$  for the second time at  $A', B'$ , respectively. Prove that  $A'B' = r_1$  where  $r_1$  is the radius of  $C_1$ .

**Problem 2.24.** Given a triangle  $ABC$ , let  $P$  lie on the circumcircle of the triangle and be the midpoint of the arc  $BC$  which does not contain  $A$ . Draw a straight line  $l$  through  $P$  so that  $l$  is parallel to  $AB$ . Denote by  $k$  the circle which passes through  $B$ , and is tangent to  $l$  at the point  $P$ . Let  $Q$  be the second point of intersection of  $k$  and the line  $AB$  (if there is no second point of intersection, choose  $Q = B$ ). Prove that  $AQ = AC$ .

**Problem 2.25.** Let  $ABCD$  be a cyclic quadrilateral in which internal angle bisectors  $\angle ABC$  and  $\angle ADC$  intersect on diagonal  $AC$ . Let  $M$  be the midpoint of  $AC$ . Line parallel to  $BC$  which passes through  $D$  cuts  $BM$  at  $E$  and circle  $ABCD$  in  $F$  ( $F \neq D$ ). Prove that  $BCEF$  is parallelogram

**Problem 2.26.** The side  $BC$  of the triangle  $ABC$  is extended beyond  $C$  to  $D$  so that  $CD = BC$ . The side  $CA$  is extended beyond  $A$  to  $E$  so that  $AE = 2CA$ . Prove that, if  $AD = BE$ , then the triangle  $ABC$  is right-angled

**Problem 2.27.**  $ABCD$  is a cyclic quadrilateral inscribed in the circle  $\Gamma$  with  $AB$  as diameter. Let  $E$  be the intersection of the diagonals  $AC$  and  $BD$ . The tangents to  $\Gamma$  at the points  $C, D$  meet at  $P$ . Prove that  $PC = PE$

**Problem 2.28.** The quadrilateral  $ABCD$  is inscribed in a circle. The point  $P$  lies in the interior of  $ABCD$ , and  $\angle PAB = \angle PBC = \angle PCD = \angle PDA$ . The lines  $AD$  and  $BC$  meet at  $Q$ , and the lines  $AB$  and  $CD$  meet at  $R$ . Prove that the lines  $PQ$  and  $PR$  form the same angle as the diagonals of  $ABCD$

**Problem 2.29.** Let  $ABCD$  be a cyclic quadrilateral with opposite sides not parallel. Let  $X$  and  $Y$  be the intersections of  $AB, CD$  and  $AD, BC$  respectively. Let the angle bisector of  $\angle AXD$  intersect  $AD, BC$  at  $E, F$  respectively, and let the angle bisectors of  $\angle AYB$  intersect  $AB, CD$  at  $G, H$  respectively. Prove that  $EFGH$  is a parallelogram.

**Problem 2.30.** Triangle  $ABC$  is given with its centroid  $G$  and circumcentre  $O$  is such that  $GO$  is perpendicular to  $AG$ . Let  $A'$  be the second intersection of  $AG$  with circumcircle of triangle  $ABC$ . Let  $D$  be the intersection of lines  $CA'$  and  $AB$  and  $E$  the intersection of lines  $BA'$  and  $AC$ . Prove that the circumcentre of triangle  $ADE$  is on the circumcircle of triangle  $ABC$ .

**Problem 2.31.** Let  $M$  be the midpoint of the side  $AC$  of  $\triangle ABC$ . Let  $P \in AM$  and  $Q \in CM$  be such that  $PQ = \frac{AC}{2}$ . Let  $(ABQ)$  intersect with  $BC$  at  $X \neq B$  and  $(BCP)$  intersect with  $BA$  at  $Y \neq B$ . Prove that the quadrilateral  $BXMY$  is cyclic.

**Problem 2.32.** Let be given a triangle  $ABC$  and its internal angle bisector  $BD$  ( $D \in BC$ ). The line  $BD$  intersects the circumcircle  $\Omega$  of triangle  $ABC$  at  $B$  and  $E$ . Circle  $\omega$  with diameter  $DE$  cuts  $\Omega$  again at  $F$ . Prove that  $BF$  is the symmedian line of triangle  $ABC$ .

**Problem 2.33.**  $\triangle ABC$  is a triangle such that  $AB \neq AC$ . The incircle of  $\triangle ABC$  touches  $BC, CA, AB$  at  $D, E, F$  respectively.  $H$  is a point on the segment  $EF$  such that  $DH \perp EF$ . Suppose  $AH \perp BC$ , prove that  $H$  is the orthocentre of  $\triangle ABC$ .