About geometric problem in Sharygin contest 2015

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1 Problem and its expands

On the Sharygin geometric contest 2015 [1] there is geometric problem suggested by authors **A.Rudenko** and **D.Khilko** as following

Problem 1. Let the triangle ABC and its altitudes AD, BE, CF. M is the midpoint of EF. AM cuts DE at K. Prove that K is laying on the midline of the triangle ABC respectively B.

We represent two proof suggested in the answer

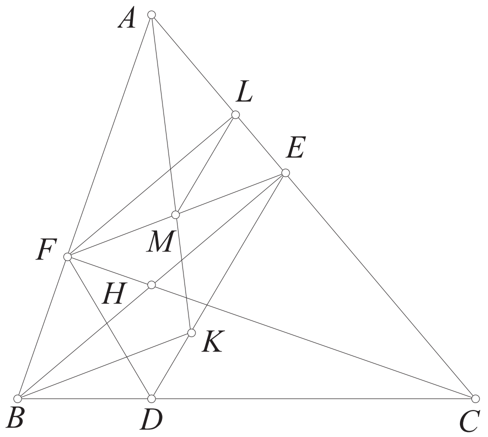

Figure 1.

Proof. Call by L the projection of F on CA. The right triangle FLE has the median LM so $ML = ME$. Then $\angle MLE = \angle MEL = \angle DEC$, note that the last equality expression deduce from $\angle FEB = \angle DEB$. Deduce $ML \parallel DE$. Then $\frac{AM}{AK} = \frac{AL}{AE} = \frac{AF}{AB}$, deduce $FM \parallel BK$. So two triangles FML and BKE have two respectively parallel edges, and the triangle FML is isosceles at M so $KB = KE$ then K belong to the perpendicular bisector BE in other way K is laying on the midline of the triangle ABC respectively B. $\square$

Remark. This is interesting problem based on one simple figuration of the three concurrent altitudes; we have interesting and meaning result. We rightly seen AM is symmedian of the triangle ABC, so the perpendicular bisector of BE cuts DE at one point laying on the symmedian of the triangle ABC. Then if the perpendicular bisector of BE, CF cut DE, DF at two points then the line connected this two point will bisect EF. This is interesting consequently problem. More general problem firstly suggested by the author and the pupil **Nguyen Duc Bao** of 10 grade math, special school Phan Boi Chau, Nghe An generalizes as following in [2,3]

Problem 2. Let the quadrileteral $ABCD$ and AC cuts BD at E. The circumcircle of the triangle EAD and EBC cut each other on F differently from E. The perpendicular bisectors of AC, DB cut FA, FB at M, N respectively. Prove that MN bisect AB.

The following proof based on the idea of the pupil **Huynh Bach Khoa** of 10 grade math of special school Tran Hung Dao, Binh Thuan trong [3]

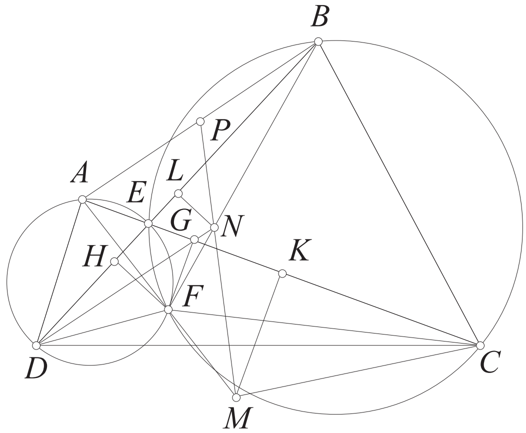

Figure 2.

Proof. Call by G, H the projection of F on CA, BD and K, L the projection of M, N on CA, BD, then K, L are the midpoints of AC, DB. The triangles FDB and FAC are similar and FH, FG are the altitudes respectively and FL, FK are the medians respectively. Then apply Thales theorem,we have $\frac{NF}{NB} = \frac{LH}{LB} = \frac{GK}{KC} = \frac{KG}{KA} = \frac{MF}{MA}$. And MN cuts AB at P. Apply Menelaus theorem for the triangle FAB, deduce $PA = PB$ or MN bisects AB. □

The above problem is expand once more by the author as following

Problem 3. Let the quadrileteral $ABCD$ and P is the point on AB. AC cuts BD at E. The circle $(EAD), (EBC)$ cut each other at F differently from E. On CA, BD to get the points Q, R such that $PQ \parallel BC$ and $PR \parallel AD$. On FB, FA to get S, T such that $RS \perp BD$ and $QT \perp AC$. Prove that P, S, T are collinear.

This proof suggested by the author using the idea of the above proof

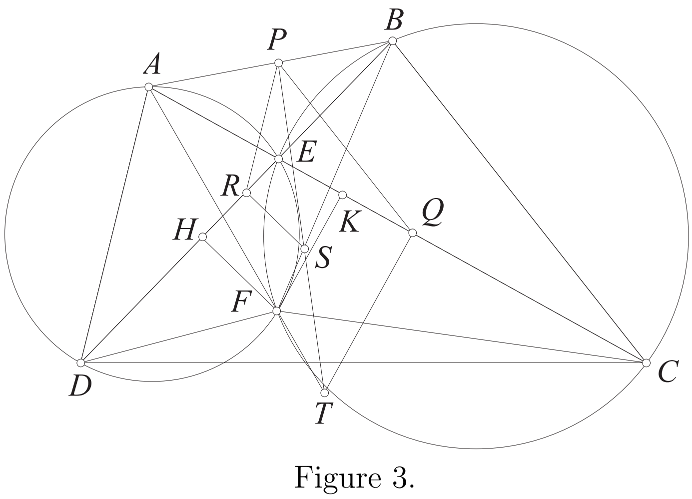

Figure 3.

Proof. Call by H, K the projection of F on DB, AC. Easily seen $\frac{RB}{RD} = \frac{PB}{PA} = \frac{QC}{QA}$. Then two triangles FDB and FAC are similar and the altitudes FH, FK and the points R, Q device the edges DB and AC in the same ratio. Apply Thales theorem, we have $\frac{SF}{SB} = \frac{RH}{RB} = \frac{QK}{QC} = \frac{QK}{QA}.\frac{QA}{QC} = \frac{TF}{TA}.\frac{PA}{PB}$. Then deduce $\frac{SB}{SF}.\frac{TF}{TA}.\frac{PA}{PB} = 1$, Apply Menelaus theorem for the triangle FAB we have P, S, T collinear. □

Return to the remark in the first problem: the intersection of the perpendicular bisector BE and DF is laying on the symmedian of A, we have an expanding as following

Problem 4. Let the triangle ABC. One circle (K) through B, C cuts CA, AB at E, F differently from C, B. BE cuts CF at H. D is projection of K on AH. The perpendicular bisector BE cuts DE at S. Prove that S is laying on the symmedian of the triangle ABC.

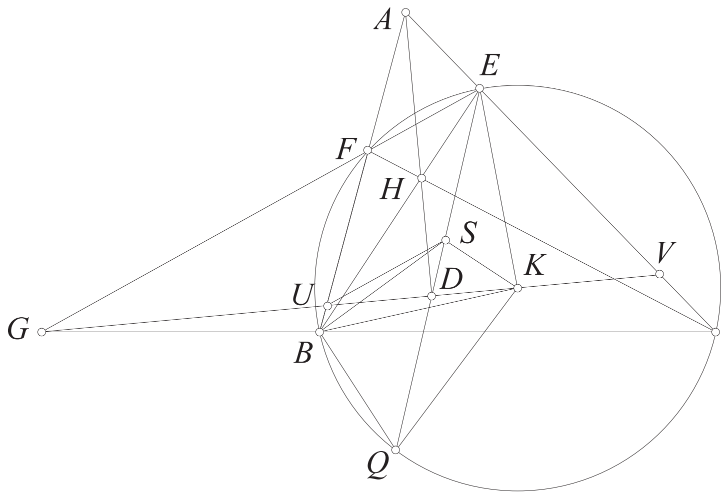

Figure 4.

Proof. Easily seen KD, BC, EF are concurrent at G. According to the familiar problem on Miquel point, we have AH perpendicular to KG at D and the quadrileteral $EFDK$ is inscribed. We need prove AS bisects EF. Indeed, ES cuts (K) at Q differently from E. KD cuts AB, AC at U, V. We have $\angle BSQ = 2\angle BEQ = \angle BKQ$, then the quadrileteral $BSKQ$ inscribed. Also we have $\angle DKQ = \angle EDK - \angle DQK = \angle EFK - \angle DEK = \angle FEK - \angle DEK = \angle FES = 180° - \angle FBQ$. Then the quadrileteral $BUKQ$ is inscribed. From two inscribed quadrileteral then the quadrileteral $BUSQ$ is inscribed, deduce $\angle FUS = \angle BQS = \angle AFE$ or $US \parallel EF$. We also have $(UV, DG) = A(UV, DG) = D(FE, AG) = -1$. Then $S(FE, UA) = E(FS, UV) = (GD, UV) = -1$. Combine with $US \parallel EF$ deduce AS bisects EF or AS is symmedian of the triangle ABC. □

Remark. When (K) is a circle with the diameter BC, we have the problem 1. From this idea of first problem, the author suggest the following problem in [4]

Problem 5. Let the triangle ABC. One circle (K) trough B, C cuts CA, AB at E, F differently from C, B. BE cuts CF at H. D is projection of K on AH. On DE, DF to get the points M, N such thet $BM \perp BE, CN \perp CF$. Prove that the the symmedian of the triangle ABC bisects MN.

Although this problem were seen as the consequence of the previous problem, but the present in the symmetric way so we have some other interesting proof suggested, the first proof was used in the above problem

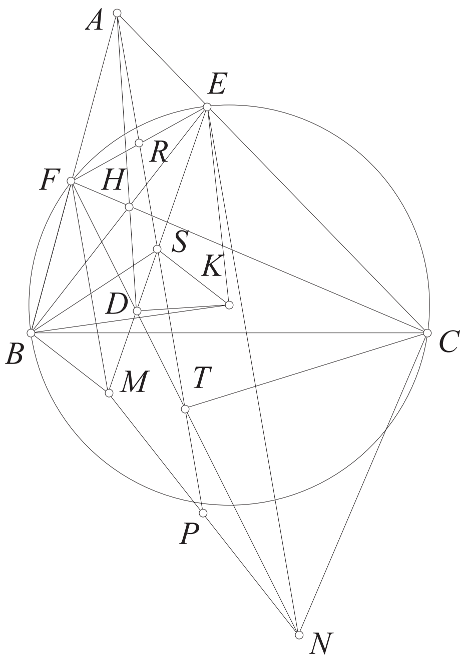

Figure 5.

First proof. Call by S, T the midpoints of ME, FN. Firstly seen S, T is laying on the perpendicular bisector of BE, CF so S, T is laying on the symmedian through A of the triangle ABC. So A, S, T, R are collinear and R is the midpoints of EF. From the median property of the triangle, deduce $FM \parallel EN \parallel AR$, so according to the median property of the trapezoid then AR bisects MN. □

The following second proof is also interesting when apply lemma E.R.I.Q , it was suggested by **ao Vu Quang** the pupil of 12 grade math of the special school Hanoi-Ams

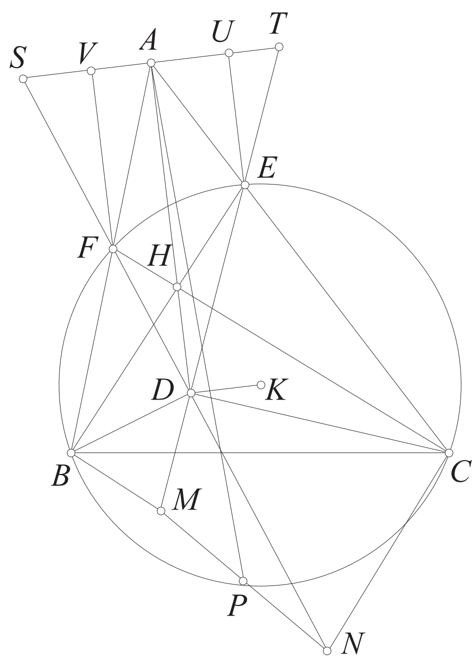

Figure 6.

Second proof. The straight line A parallel to DK cuts DE, DF at T, S respectively. According to the family problem on Miquel point, so the quadrilaterals $BDHF$ and $CDHE$ are inscribed, so $\angle FDH = \angle FBH = \angle ECH = \angle EDH$, then deduce the triangle DST is isosceles. Call by U, V the projection of E, F on ST. Easily seen the quadrilaterals $AFDC$ is inscribed, so $\angle AEU = \angle EAD = \angle CFD$, so the right triangles EAU and FNC are similar. Analogously, the triangles FAV and EMB are similar. Then, $\frac{ME}{NF} = \frac{ME}{BE} \cdot \frac{BE}{CF} \cdot \frac{CF}{NF} = \frac{FA}{FV} \cdot \frac{AE}{AF} \cdot \frac{EU}{EA} = \frac{EU}{AD} \cdot \frac{AD}{FV} = \frac{DS}{SF} \cdot \frac{ET}{TD} = \frac{SF}{TE}$, Note that, the last equations deduced from the triangle DST is isosceles. Then we have $\frac{ET}{EM} = \frac{FS}{FN}$. According to the E.R.I.Q lemma so A and the midpoints of EF, MN are collinear. $\square$

The third proof is pure geometric, suggested by **Nguyen Tien Dung** the student of K50 of the institute of foreign trade.

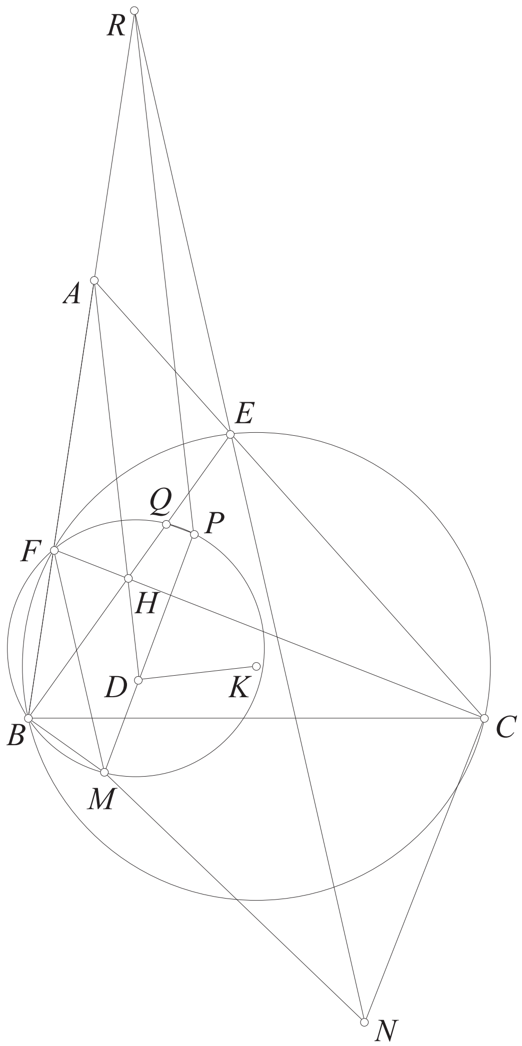

Figure 7.

The third proof. According to the familiar proof, the quadrilaterals $BDHF, CDHE, ABDE, ACDF$ are inscribed. We have the couple of the similar triangles DBE and DFC, DBM and DNC. Then easily seen $DM.DN = DB.DC = DE.DF$ so $\frac{DM}{DE} = \frac{DF}{DN}$ then $EN \parallel FM$. Call by P, Q the second intersection of EM, EB and the circle (BMF) respectively. EN, AB cut each other at R. Because $\angle BPM = \angle BFM = \angle BRE$ so the quadrileteral $BPER$ is inscribed. Then

$\angle BAD = \angle BEP = \angle BRP$ so $AD \parallel PR$. Because of $BM \perp BE$ so easily seen $PQ \perp ME$. Then $\angle FPQ = \angle FBQ = \angle RPE = 90° - \angle RPQ$ so $PF \perp PR$. Note that $AD \parallel PR$, DA is the bisector of $\angle PDF$ so DA is the perpendicular bisector of PF, then A is the midpoint of FR. Then the median from the edge A of the triangle AEF bisects MN. □

Remark. Three proof have their separate specify. The third proof is nice and pure geometric, the second proof with using E.R.I.Q lemma is interesting idea to help proving the relevance problem, the first proof is also the way to create this problem.

To specify this problem when (K) is one circle with the diameter BC, we have the following problem

Problem 6. Let the triangle ABC with the altitudes AD, BE, CF. On the DE, DF to get the points M, N such that $BM \parallel CA$ and $CN \parallel AB$. Call by S, T the midpoints of EF, MN. Prove that A, S, T are collinear.

Besides using problem 1 directly for analogously proving as above, mr. **Nguyen Tien Dung** the pupil of K50 of institute of foreign trade suggested other pure geometric proof as following

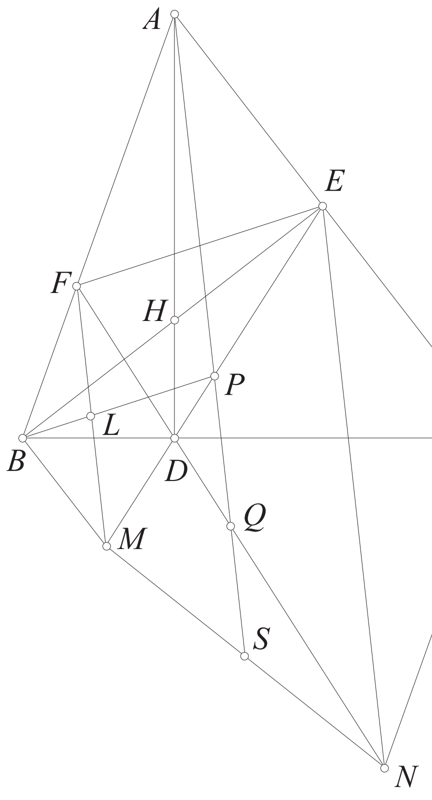

Figure 8.

Proof. We have $BM \parallel AC \perp BE$, $CN \parallel AB \perp CF$. Then $\frac{DM}{DE} = \frac{DB}{DC} = \frac{DF}{DN}$ nn $EN \parallel FM$. P, Q are the midpoints of EM, FN respectively. Call by H the orthocenter of the triangle ABC. BP, FM cut each other at L. We have the couples of similar triangle BAH and BEF, AHF and EMB. Then $\frac{PL}{PB} = \frac{PL}{PM} = \frac{EF}{EM} = \frac{EF}{AH} \cdot \frac{AH}{EM} = \frac{EB}{AB} \cdot \frac{AF}{EB} = \frac{AF}{AB}$ so $AP \parallel FM$. Analogously $AQ \parallel EN$ so A, P, Q are collinear on the straight line, which bisects EF, MN is also the symmedian of the triangle ABC. □

This problem is generalized as following in [4]

Problem 7. Given ABC and any P. PA, PB, PC cut BC, CA, AB at D, E, F. On DE, DF to get the points M, N such that $BM \parallel CA$ and $CN \parallel AB$. Call by S, T the midpoints of EF, MN. Prove that A, S, T are collinear.

The proof using E.R.I.Q Lemma, see [4]

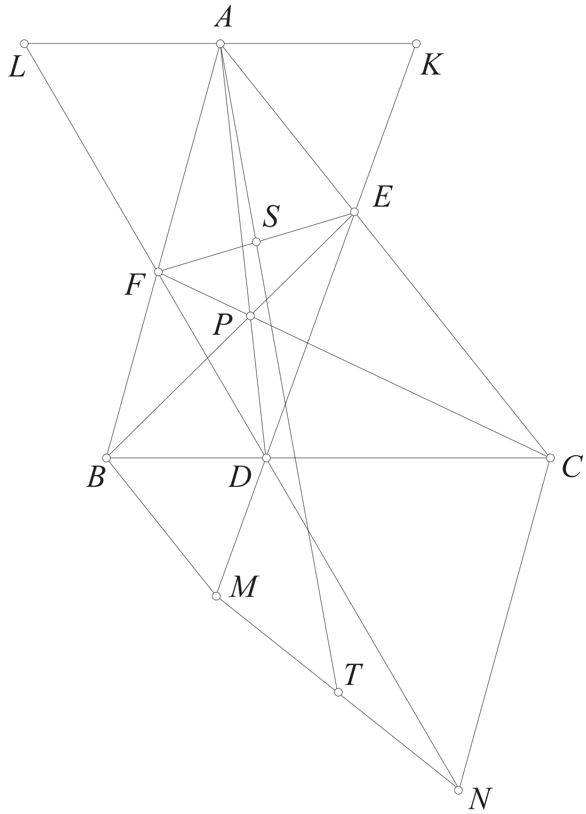

Figure 9.

Proof. The straight line parallel to BC through A cuts DE, DF at K, L respectively. We have $\frac{EM}{EK} = \frac{EM}{ED} \cdot \frac{ED}{EK} = \frac{CB}{CD} \cdot \frac{EC}{EA}$. Analogously, $\frac{FN}{FL} = \frac{BC}{BD} \cdot \frac{FB}{FA}$. According to Ceva Theorem we have $\frac{DB}{DC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = 1$. Then, $\frac{EM}{EK} = \frac{FN}{FL}$. In other hand, easily seen A is the midpoints of KL. According to E.R.I.Q Lemma then A, S, T are collinear. $\square$

Remark. The above problem presented reflect and apply proof used E.R.I.Q lemma is very interesting. If use the result of this problem and note that it is easily proving $FM \parallel EN \parallel ST$ so ST bisects ME, NF then the intersection of ST with ME, NF are laying on the median respectively with B and C. We suggest the general problem 1 as following

Problem 8. Given ABC and any P. PA, PB, PC cut BC, CA, AB at D, E, F. Call by M the midpoint of EF. AM cuts DE at K. Prove that K is laying on the median respectively with B of the triangle ABC.

The above problem may generalize more as following, in [4]

Problem 9. Let the triangle ABC and D, E, F are laying on the edges BC, CA, AB. On DE, DF to get the points M, N such that $BM \parallel CA$ and $CN \parallel AB$. Call by S, T the midpoints of EF, MN. Prove that the straight lines through A, B, C parallel to ST, DF, DE respectively will be concurrently.

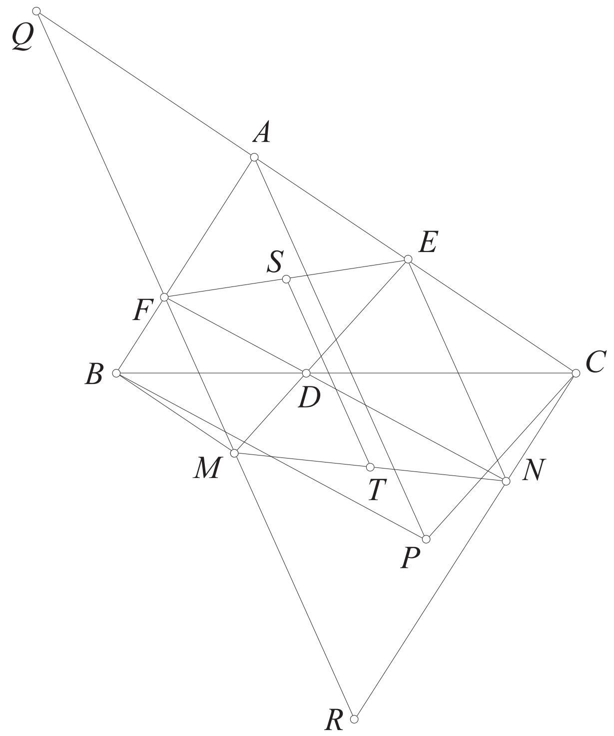

Figure 10.

Proof. We have $\frac{DM}{DE} = \frac{DB}{DC} = \frac{DF}{DN}$ so $FM \parallel EN \parallel ST$. The straight lines through A parallel to ST cuts the straight lines through C parallel to DE at P. We will prove $PB \parallel DF$, Indeed, easily seen $\triangle APC \sim \triangle QME$ so $\frac{PA}{QM} = \frac{AC}{EQ} = \frac{AC}{CQ} \cdot \frac{CQ}{QE} = \frac{FR}{RQ} \cdot \frac{CR}{NR} = \frac{FR}{NR} \cdot \frac{RC}{RQ} = \frac{FR}{NR} \cdot \frac{FA}{FQ} = \frac{FR}{NR} \cdot \frac{AB}{QM}$. Then we have $\frac{PA}{AB} = \frac{FR}{NR}$, and note that $PA \parallel FR$ and $AB \parallel NR$ so $PB \parallel FN$. $\square$

Remark. Use the above proof we can rewrite this problem in other way and it is al so one more expand of 1 problem.

Problem 10. Let the triangle ABC and D, E, F are laying on the edges BC, CA, AB. On DE, DF to get the points M, N such that M, N are laying on the median respectively B, C of the triangle ABC. Prove that the straight lines through A, B, C parallel to MN, DF, DE respectively, will be concurrently.

2 Some application

There are a lot of application from the above problem, use these results to do the following exercises

Problem 11. Let the quadrileteral $ABCD$ and AC cuts BD at E. The circumcircles of the triangle EAD and EBC cut each other at F differently from E. The perpendicular bisector AC cuts FA, FC at M, N. The straight lines through M parallel to AB cuts FB at P. Prove that PN bisects BC.

Problem 12. Let the triangle ABC construct the bisectors BE, CF and the parallelogram $ABDC$. The perpendicular bisector BE, CF cuts BC at K, L. On the segments DB, DC to get the points M, N such that $BM = BK$ and $CN = CL$. Call by P the midpoint of MN. Prove that AP bisects EF.

Problem 13. Let the triangle ABC and D is on BC. The circumcircles of the triangle DAB, DAC cut CA, AB at E, F differently from A. On DE, DF to get the points M, N such that M, N are laying on the median respectively B, C of the triangle ABC. Prove that MN parallel to the symmedian of the triangle ABC.

We can recognize well-known lemma, it seem the consequence of the problem 8

Problem 14. Let the triangle ABC has the incircle (I) touched BC, CA, AB at D, E, F. IB, IC cut EF at M, N respectively. Prove that M, N are laying on the median respectively C, B of the triangle ABC.

Problem 15. Let the triangle ABC with the altitudes AD, BE, CF. M, N are the midpoints of DF, DE. BM, CN cut DE, DF at K, L respectively. Prove that $KL \parallel BC$.

Problem 16. Let the triangle ABC and any P. PA, PB, PC cut BC, CA, AB at D, E, F respectively. M, N are the midpoints of DF, DE. BM, CN cut DE, DF at K, L respectively. Prove that $KL \parallel BC$.

Problem 17. Let the triangle ABC with the incircles center I. The perpendicular bisector IA cuts CA, AB at E, F. M, N are the midpoints of CA, AB. MF cuts NE at P. Prove that IP bisects BC.

In the end, I would like to give the thanks to **Nguyen Tien Dung** the student K50 of the University of the Foreign Trade, who contribute a lot of proof and help me to complete this article.

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[4] buratinogigle, Topic Symmedian bisects segment
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