Variations of the Steinbart Theorem / Darij Grinberg

We begin with citing the so-called Steinbart Theorem (Oliver Funck [1]; [2] gives a partial converse):

Let ΔABC be a triangle and $\Delta A'B'C'$ its tangential triangle. This means: At the points A, B and C we draw tangents to the circumcircle of ΔABC; the intersection of the tangents at B and at C is called A', and the points B' and C' are defined similarly.

Further let A'', B'' and C'' be points on the circumcircle of ΔABC satisfying the condition that the lines AA'', BB'' and CC'' concur. Then the Steinbart Theorem says that the lines $A'A''$, $B'B''$ and $C'C''$ also concur. (Fig. 1)

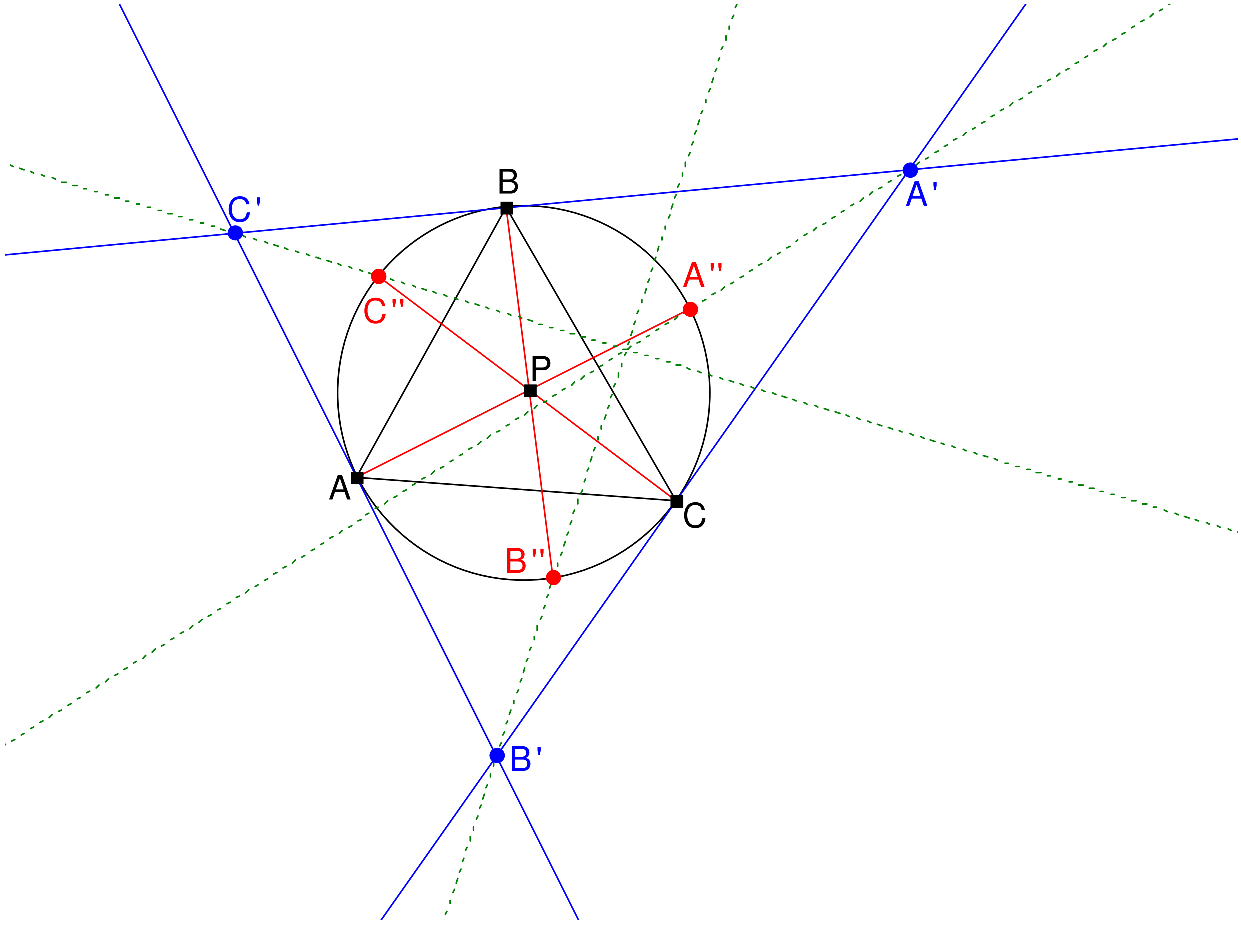

Fig. 1

Now, one will ask himself if the condition that the lines AA'', BB'' and CC'' concur is also necessary to make the lines $A'A''$, $B'B''$ and $C'C''$ concur.

The answer is surprising – No! There are other positions of the points A'', B'' and C'' for which the lines $A'A''$, $B'B''$ and $C'C''$ concur. Namely, this is also valid if the points $AA'' \cap BC$, $BB'' \cap CA$ and $CC'' \cap AB$ are collinear. (Fig. 2). (The sign $\cap$ means "intersection".)

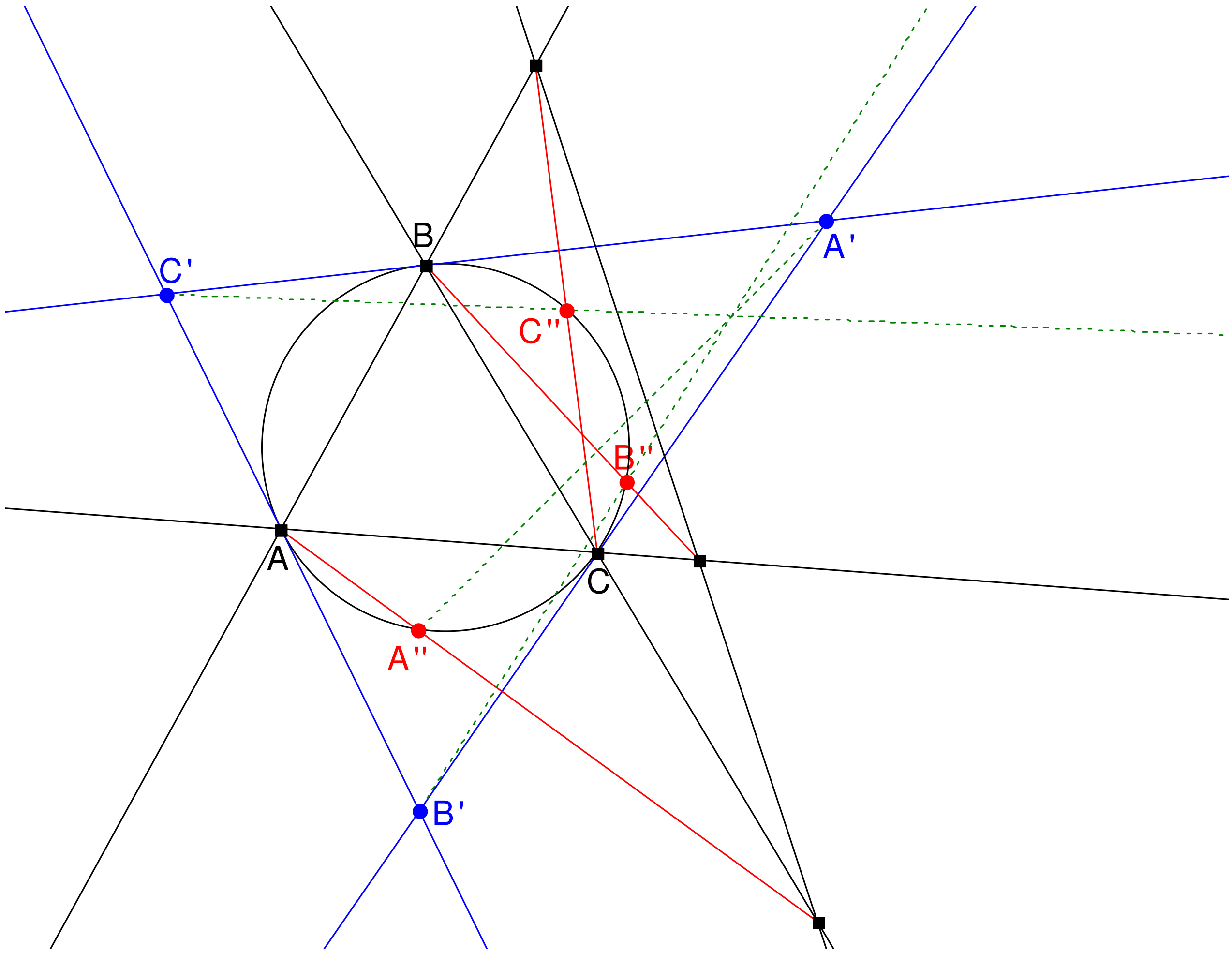

Fig. 2

So we are going to prove the following theorem:

Theorem 1, the Extended Steinbart Theorem. Let A'', B'' and C'' be any three points on the circumcircle of ΔABC. The lines $A'A''$, $B'B''$ and $C'C''$ concur if and only if either the lines AA'', BB'' and CC'' concur or the points $AA'' \cap BC$, $BB'' \cap CA$ and $CC'' \cap AB$ are collinear.

The *proof* will be essentially an extension of the one given for a partial converse of the Steinbart Theorem in [2]. We denote $K = AA'' \cap BC$, $L = BB'' \cap CA$ and $M = CC'' \cap AB$. (Fig. 3)

Now we can rewrite Theorem 1: The lines $A'A''$, $B'B''$ and $C'C''$ concur if and only if either the lines AK, BL and CM concur or the points K, L and M are collinear.

We want to prove this.

From the Ceva theorem, the lines AK, BL and CM concur if and only if

$$\frac{BK}{KC} \bullet \frac{CL}{LA} \bullet \frac{AM}{MB} = 1.$$

From the Menelaos theorem, the points K, L and M are collinear if and only if

$$\frac{BK}{KC} \bullet \frac{CL}{LA} \bullet \frac{AM}{MB} = -1.$$

It follows that *either* the lines AK, BL and CM concur *or* the points K, L and M are collinear if and only if

$$\left(\frac{BK}{KC} \bullet \frac{CL}{LA} \bullet \frac{AM}{MB}\right)^2 = 1.$$

From the Ceva theorem in the trigonometric form, the lines $A'A''$, $B'B''$ and $C'C''$ concur if and only if

$$\frac{\sin\measuredangle C'A'A''}{\sin\measuredangle A''A'B'} \bullet \frac{\sin\measuredangle A'B'B''}{\sin\measuredangle B''B'C'} \bullet \frac{\sin\measuredangle B'C'C''}{\sin\measuredangle C''C'A'} = 1.$$

Thus, it remains to establish that

$$\frac{\sin\measuredangle C'A'A''}{\sin\measuredangle A''A'B'} \bullet \frac{\sin\measuredangle A'B'B''}{\sin\measuredangle B''B'C'} \bullet \frac{\sin\measuredangle B'C'C''}{\sin\measuredangle C''C'A'} = 1$$

is valid if and only if

$$\left(\frac{BK}{KC} \bullet \frac{CL}{LA} \bullet \frac{AM}{MB}\right)^2 = 1$$

is valid.

In fact, we will show the equation

$$\frac{\sin\measuredangle C'A'A''}{\sin\measuredangle A''A'B'} \bullet \frac{\sin\measuredangle A'B'B''}{\sin\measuredangle B''B'C'} \bullet \frac{\sin\measuredangle B'C'C''}{\sin\measuredangle C''C'A'} = \left(\frac{BK}{KC} \bullet \frac{CL}{LA} \bullet \frac{AM}{MB}\right)^2. \tag{1}$$

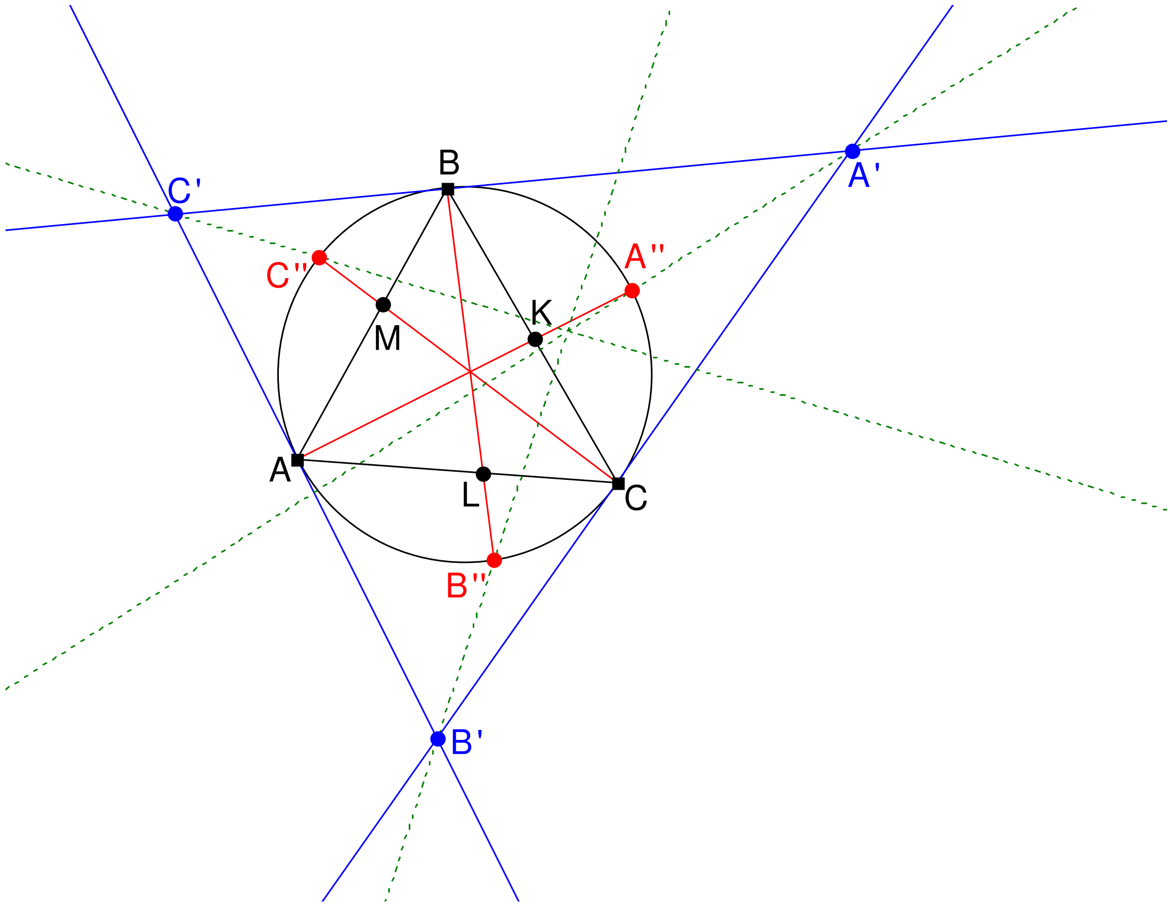

Fig. 3

At first, the Sine Law gives

$$\frac{BK}{KC} = \frac{\sin\measuredangle BAK \bullet AB : \sin\measuredangle AKB}{\sin\measuredangle KAC \bullet CA : \sin\measuredangle AKC}.$$

But $\measuredangle AKB + \measuredangle AKC = 180°$ and $\sin\measuredangle AKB = \sin\measuredangle AKC$, what leads to

$$\frac{BK}{KC} = \frac{\sin\measuredangle BAK \bullet AB}{\sin\measuredangle KAC \bullet CA} = \frac{\sin\measuredangle BAK}{\sin\measuredangle KAC} \bullet \frac{c}{b} = \frac{\sin\measuredangle BAA''}{\sin\measuredangle A''AC} \bullet \frac{c}{b}.$$

Analogously,

$$\frac{CL}{LA} = \frac{\sin\measuredangle CBB''}{\sin\measuredangle B''BA} \bullet \frac{a}{c} \qquad \text{and} \qquad \frac{AM}{MB} = \frac{\sin\measuredangle ACC''}{\sin\measuredangle C''CB} \bullet \frac{b}{a};$$

hence,

$$\begin{aligned}\frac{BK}{KC} \bullet \frac{CL}{LA} \bullet \frac{AM}{MB} &= \left(\frac{\sin\measuredangle BAA''}{\sin\measuredangle A''AC} \bullet \frac{c}{b}\right) \bullet \left(\frac{\sin\measuredangle CBB''}{\sin\measuredangle B''BA} \bullet \frac{a}{c}\right) \bullet \left(\frac{\sin\measuredangle ACC''}{\sin\measuredangle C''CB} \bullet \frac{b}{a}\right) \\ &= \left(\frac{\sin\measuredangle BAA''}{\sin\measuredangle A''AC} \bullet \frac{\sin\measuredangle CBB''}{\sin\measuredangle B''BA} \bullet \frac{\sin\measuredangle ACC''}{\sin\measuredangle C''CB}\right) \bullet \left(\frac{c}{b} \bullet \frac{a}{c} \bullet \frac{b}{a}\right),\end{aligned}$$

and

$$\frac{BK}{KC} \bullet \frac{CL}{LA} \bullet \frac{AM}{MB} = \frac{\sin\measuredangle BAA''}{\sin\measuredangle A''AC} \bullet \frac{\sin\measuredangle CBB''}{\sin\measuredangle B''BA} \bullet \frac{\sin\measuredangle ACC''}{\sin\measuredangle C''CB}. \tag{2}$$

Furthermore

$$\begin{aligned}\frac{\sin\measuredangle C'A'A''}{\sin\measuredangle A''A'B'} &= \frac{\sin\measuredangle BA'A''}{\sin\measuredangle A''A'C} \\ &= \frac{BA'' \bullet \sin\measuredangle A''BA' \ : \ A'A''}{A''C \bullet \sin\measuredangle A''CA' \ : \ A'A''} \qquad \text{(Sine Law)} \\ &= \frac{BA'' \bullet \sin\measuredangle A''BA'}{A''C \bullet \sin\measuredangle A''CA'} = \frac{BA''}{A''C} \bullet \frac{\sin\measuredangle A''BA'}{\sin\measuredangle A''CA'}.\end{aligned}$$

Now, we have the chordal-tangent angles $\measuredangle A''BA' = \measuredangle BAA''$ and $\measuredangle A''CA' = \measuredangle A''AC$. We conclude

$$\frac{\sin\measuredangle C'A'A''}{\sin\measuredangle A''A'B'} = \frac{BA''}{A''C} \bullet \frac{\sin\measuredangle BAA''}{\sin\measuredangle A''AC}.$$

Since a chord in a circle is equal to the double radius of the circle times the sine of the chordal angle of the chord, we have $BA'' = 2r\sin\measuredangle BAA''$ and $A''C = 2r\sin\measuredangle A''AC$, where r is the circumradius of ΔABC. Thus,

$$\frac{\sin\measuredangle C'A'A''}{\sin\measuredangle A''A'B'} = \frac{2r\sin\measuredangle BAA''}{2r\sin\measuredangle A''AC} \bullet \frac{\sin\measuredangle BAA''}{\sin\measuredangle A''AC} = \left(\frac{\sin\measuredangle BAA''}{\sin\measuredangle A''AC}\right)^2.$$

Similarly,

$$\frac{\sin\measuredangle A'B'B''}{\sin\measuredangle B''B'C'} = \left(\frac{\sin\measuredangle CBB''}{\sin\measuredangle B''BA}\right)^2 \qquad \text{and}$$

$$\frac{\sin\measuredangle B'C'C''}{\sin\measuredangle C''C'A'} = \left(\frac{\sin\measuredangle ACC''}{\sin\measuredangle C''CB}\right)^2;$$

multiplication gives

$$\begin{aligned}&\frac{\sin\measuredangle C'A'A''}{\sin\measuredangle A''A'B'} \bullet \frac{\sin\measuredangle A'B'B''}{\sin\measuredangle B''B'C'} \bullet \frac{\sin\measuredangle B'C'C''}{\sin\measuredangle C''C'A'} \\ &= \left(\frac{\sin\measuredangle BAA''}{\sin\measuredangle A''AC} \bullet \frac{\sin\measuredangle CBB''}{\sin\measuredangle B''BA} \bullet \frac{\sin\measuredangle ACC''}{\sin\measuredangle C''CB}\right)^2,\end{aligned}$$

and after (2) this becomes

$$\frac{\sin\measuredangle C'A'A''}{\sin\measuredangle A''A'B'} \bullet \frac{\sin\measuredangle A'B'B''}{\sin\measuredangle B''B'C'} \bullet \frac{\sin\measuredangle B'C'C''}{\sin\measuredangle C''C'A'} = \left(\frac{BK}{KC} \bullet \frac{CL}{LA} \bullet \frac{AM}{MB}\right)^2,$$

what proves the formula (1). This establishes Theorem 1.

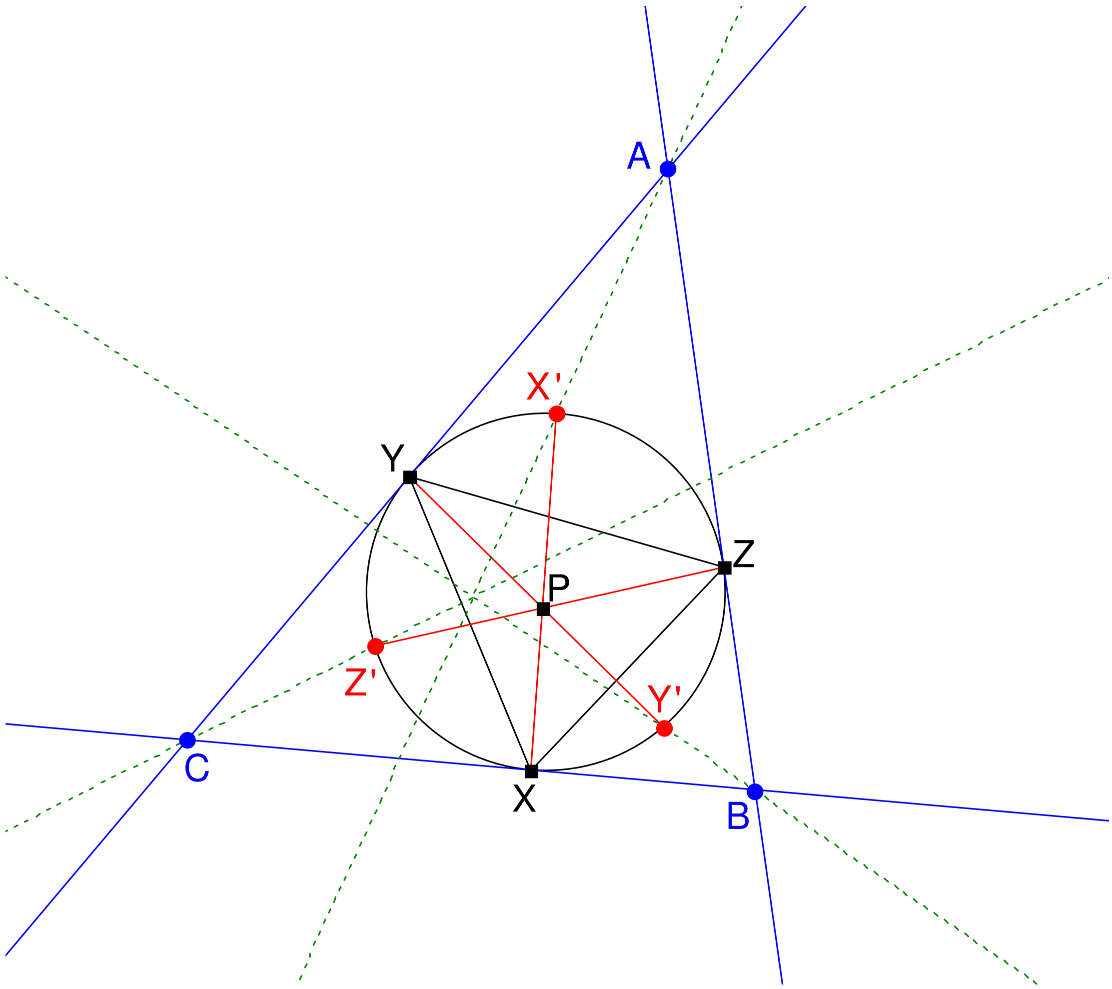

Fig. 4

Let ΔABC an arbitrary triangle; the incircle of this triangle touches the sides BC, CA and AB at the points X, Y and Z, respectively. Then, the triangle XYZ is called **Gergonne triangle** of triangle ABC. (It is also called **contact triangle** or **intouch triangle**.) Obviously, the incircle of ΔABC is the circumcircle of ΔXYZ, and the triangle ΔABC itself is the tangential triangle of ΔXYZ. This indicates that we can apply Theorem 1 to triangle XYZ, and get:

Theorem 2. Let X', Y' and Z' be any three points on the incircle of ΔABC. The lines AX', BY' and CZ' concur if and only if either the lines XX', YY' and ZZ' concur or the points $XX' \cap YZ$, $YY' \cap ZX$ and $ZZ' \cap XY$ are collinear.

(See Fig. 4 for the case when the lines XX', YY' and ZZ' concur, and Fig. 5 for the case when the points $XX' \cap YZ$, $YY' \cap ZX$ and $ZZ' \cap XY$ are collinear.)

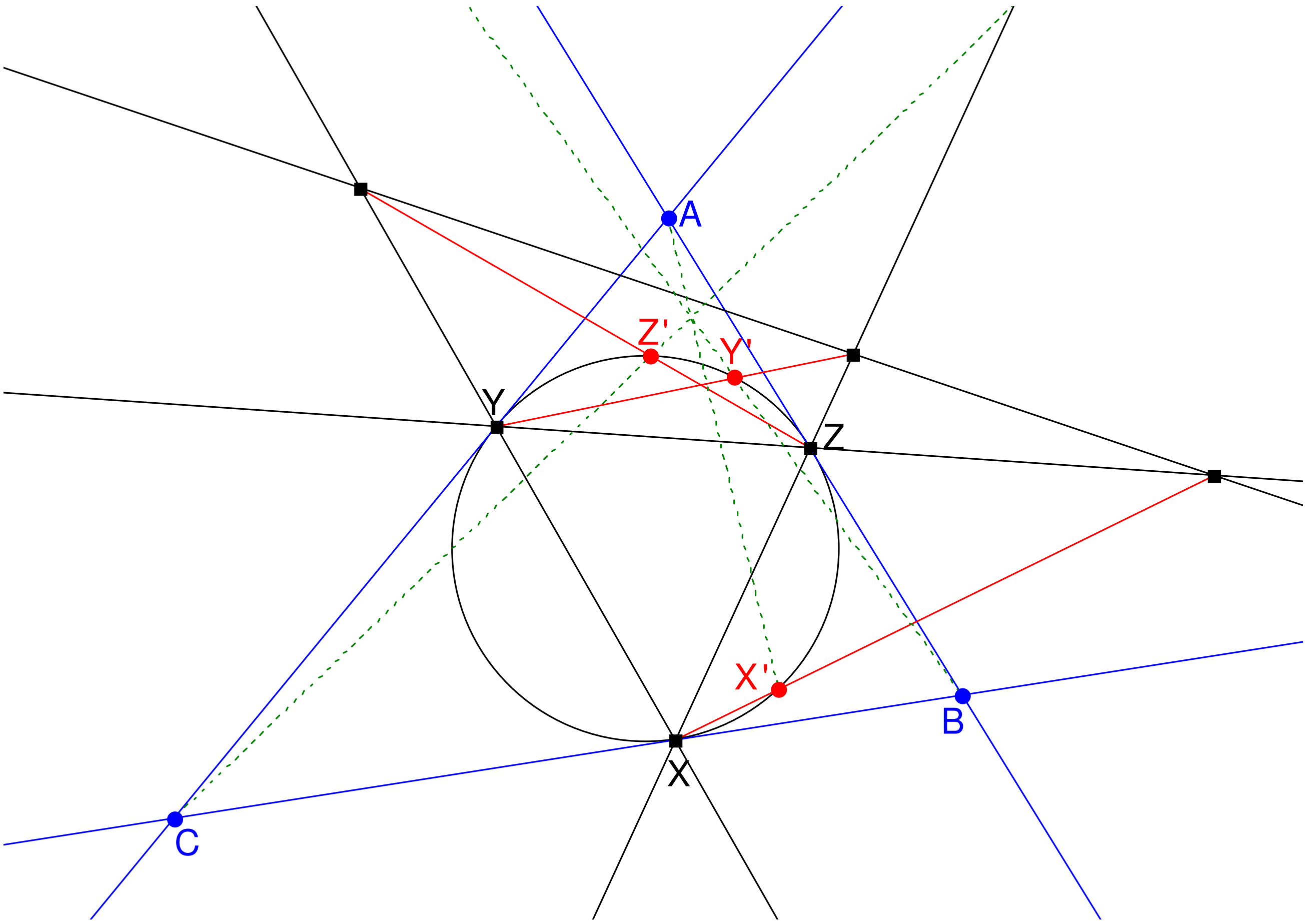

Fig. 5

This theorem has an interesting corollary.

Theorem 3. Let ΔABC be a triangle and ΔXYZ its Gergonne triangle. Further, let D, E and F be any three points on the sides BC, CA and AB of triangle ABC. From D, E and F, draw the tangents to the incircle of ΔABC (different from the triangle's sides BC, CA, AB); these tangents touch the incircle at the points X', Y' and Z', respectively. Then the lines AX', BY' and CZ' concur if and only if either the points D, E and F are collinear or the lines AD, BE and CF concur.

(See Fig. 6 for the case when the lines AD, BE and CF concur, and Fig. 7 for the case when the points D, E and F are collinear.)

Remark that a part of this theorem (namely that if the lines AD, BE and CF concur, the lines AX', BY' and CZ' concur), was found by Jean-Pierre Ehrmann. See Hyacinthos message #6966. It can be generalized for any inscribed conic instead of the incircle.

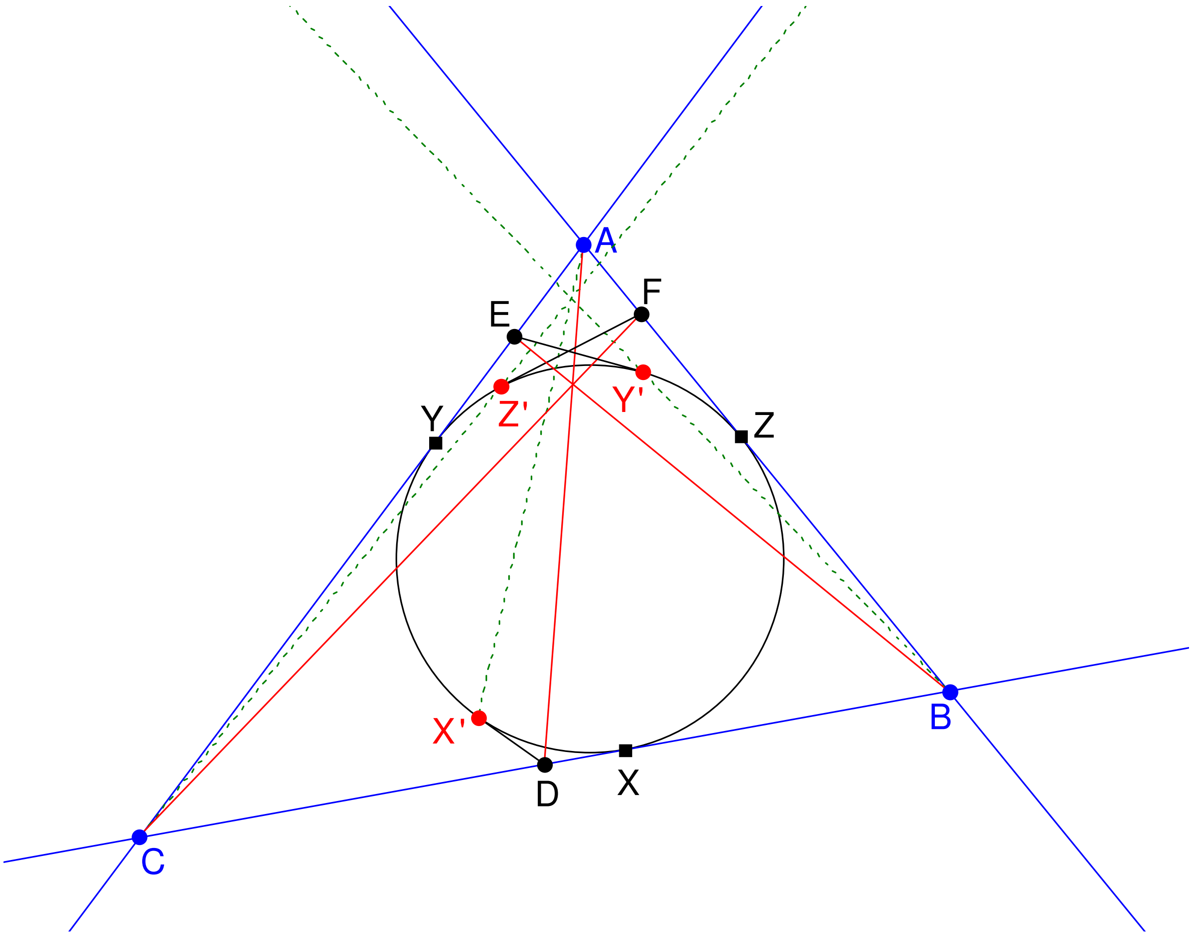

Fig. 6

We proceed to show Theorem 3 for the incircle using the notions of poles and polars with respect to circles.

Remember that if a point lies P outside of a circle k, the polar of this point with respect to k is the line joining the two points where the tangents from P to k touch k. From this, we have:

The polars of the points A, B and C with respect to the incircle of ΔABC are the lines YZ, ZX and XY; the polars of the points D, E and F with respect to the incircle of ΔABC are the lines XX', YY' and ZZ'. Hence, the poles of the lines AD, BE and CF are the points $XX' \cap YZ$, $YY' \cap ZX$ and $ZZ' \cap XY$.

The points X', Y' and Z' lie on the incircle of ΔABC. After Theorem 2, the lines AX', BY' and CZ' concur if and only if either the lines XX', YY' and ZZ' concur or the points $XX' \cap YZ$, $YY' \cap ZX$ and $ZZ' \cap XY$ are collinear. The lines XX', YY' and ZZ' are the polars of the points D, E and F and concur if and only if the points D, E and F are collinear. The points $XX' \cap YZ$, $YY' \cap ZX$ and $ZZ' \cap XY$ are the poles of the lines AD, BE and CF and are collinear if and only if the lines AD, BE and CF concur.

This proves Theorem 3.

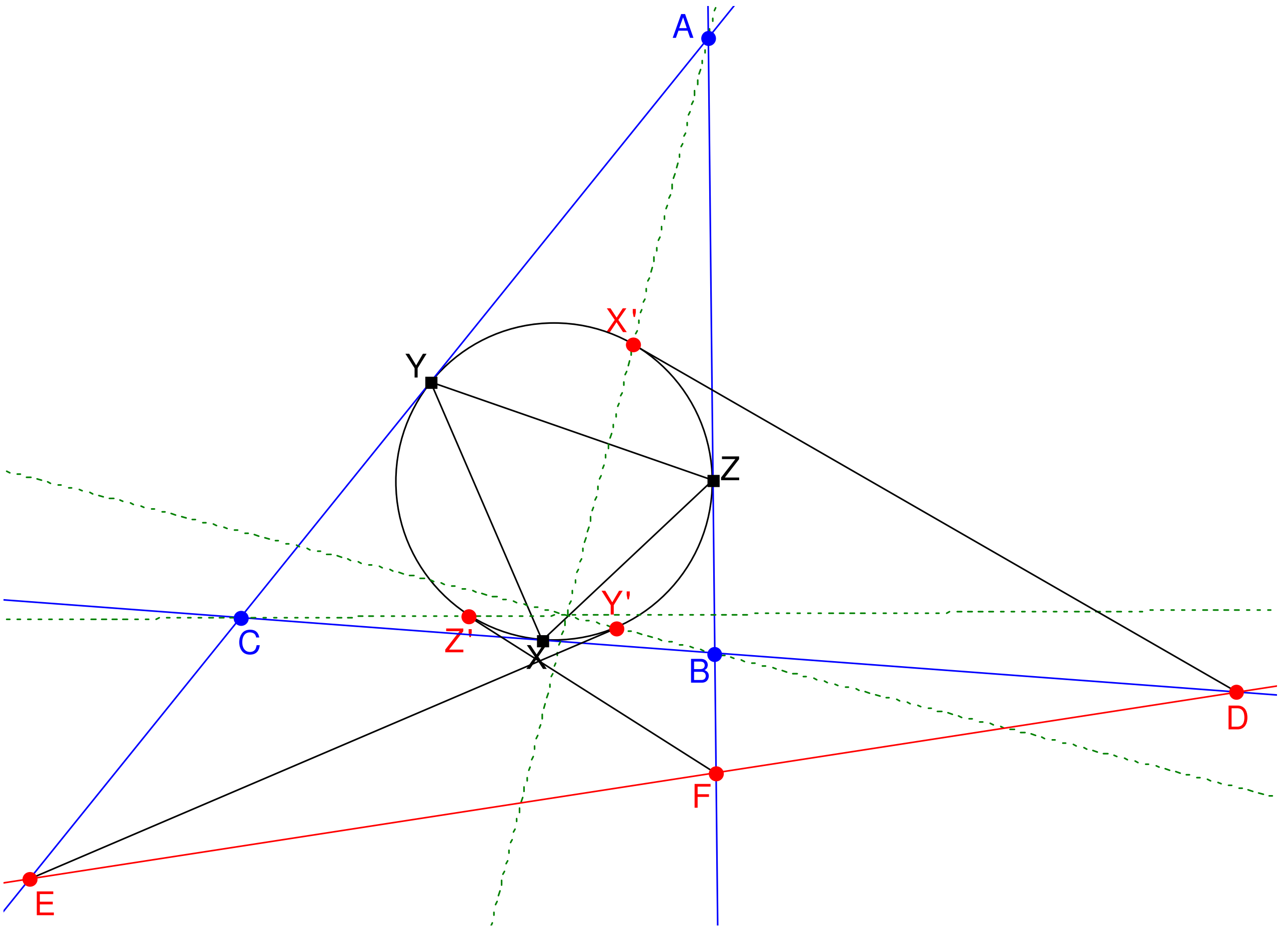

Fig. 7

References

[1] Oliver Funck: *Geometrische Untersuchungen mit Computerunterstützung*, `http://www.uni-duisburg.de/SCHULEN/STG/Wettbewerbe/jufo2.html`

[2] Stanley Rabinowitz, Francisco Bellot, María Ascensión López, René De Vogelaere: *Solution of Problem* 1364 (*Points of Rabinowitz*), Mathematics Magazine 1/65 (1992) pages 59-61.