Inversion

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Introduction

An *inversion* with respect to a given circle (sphere) is the map sending each point A other than the center O of the circle to the point A' on ray OA such that $OA' = r^2/OA$. What makes this map useful is the fact that it preserves angles and maps lines and circles onto lines or circles. Thus appropriate inversions can reduce the number of unpleasant circles (mapping them to lines) and often even turn a difficult problem into a quite simple one, as we show on a number of solved problems. Problems range from Ptolemy's inequality, to Feuerbach's theorem, and some of the hardest problems appearing on math competitions.

General properties

Inversion Ψ is a map of a plane or space without a fixed point O onto itself, determined by a circle k with center O and radius r, which takes point $A \neq O$ to the point $A' = \Psi(A)$ on the ray OA such that $OA \cdot OA' = r^2$. From now on, unless noted otherwise, X' always denotes the image of object X under a considered inversion.

Clearly, map Ψ is continuous and inverse to itself, and maps the interior and exterior of k to each other, which is why it is called ``inversion". The next thing we observe is that $\triangle P'OQ' \sim \triangle QOP$ for all point $P, Q \neq O$ (for $\angle P'OQ' = \angle QOP$ and $OP'/OQ' = (r^2/OP)/(r^2/OQ) = OQ/OP$), with the ratio of similitude $\frac{r^2}{OP \cdot OQ}$. As a consequence, we have

$$\angle OQ'P' = \angle OPQ \quad \text{and} \quad P'Q' = \frac{r^2}{OP \cdot OQ} PQ.$$

What makes inversion attractive is the fact that it maps lines and circles into lines and circles. A line through O (O excluded) obviously maps to itself. What if a line p does not contain O? Let P be the projection of O on p and $Q \in p$ an arbitrary point of p. Angle $\angle OPQ = \angle OQ'P'$ is right, so Q' lies on circle k with diameter OP'. Therefore $\Psi(p) = k$ and consequently $\Psi(k) = p$. Finally, what is the image of a circle k not passing through O? We claim that it is also a circle; to show this, we shall prove that inversion takes any four concyclic points A, B, C, D to four concyclic points A', B', C', D'. The following angles are regarded as oriented. Let us show that $\angle A'C'B' = \angle A'D'B'$. We have $\angle A'C'B' = \angle OC'B' - \angle OC'A' = \angle OBC - \angle OAC$ and analogously $\angle A'D'B' = \angle OBD - \angle OAD$, which implies $\angle A'D'B' - \angle A'C'B' = \angle CBD - \angle CAD = 0$, as we claimed. To sum up:

- A line through O maps to itself.
- A circle through O maps to a line not containing O and vice-versa.
- A circle not passing through O maps to a circle not passing through O (not necessarily the same).

Remark. Based on what we have seen, it can be noted that inversion preserves angles between curves, in particular circles or lines. Maps having this property are called *conformal*.

When should inversion be used? As always, the answer comes with experience and cannot be put on a paper. Roughly speaking, inversion is useful in destroying ``inconvenient" circles and angles on a picture. Thus, some pictures ``cry" to be inverted:

- There are many circles and lines through the same point A. Invert through A.

Example (IMO 2003, Shortlist)

Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

Show solution.

- There are many angles $\angle AXB$ with fixed A, B. Invert through A or B.

Example (IMO 1996, problem 2)

Let P be a point inside $\triangle ABC$ such that $\angle APB - \angle C = \angle APC - \angle B$. Let D, E be the incenters of $\triangle APB, \triangle APC$ respectively. Show that AP, BD, and CE meet in a point.

Show solution.

Caution: Inversion may also bring new inconvenient circles and angles.

Problems

Problem 1

Circles k_1, k_2, k_3, k_4 are such that k_2 and k_4 each touch k_1 and k_3. Show that the tangency points are collinear or concyclic.

Hide solution.

Let k_1 and k_2, k_2 and k_3, k_3 and k_4, k_4 and k_1 touch at A, B, C, D, respectively. An inversion with center A maps k_1 and k_2 to parallel lines k_1' and k_2', and k_3 and k_4 to circles k_3' and k_4' tangent to each other at C' and tangent to k_2' at B' and to k_4' at D'.

It is easy to see that B', C', D' are collinear. Therefore B, C, D lie on a circle through A.

Problem 2

Prove that for any points A, B, C, D, $AB \cdot CD + BC \cdot DA \geq AC \cdot BD$, and that equality holds if and only if A, B, C, D are on a circle or a line in this order. (*Ptolemy's inequality*)

Hide solution.

Applying the inversion with center A and radius r gives $AB = \frac{r^2}{AB'}$, $CD = \frac{r^2}{AC' \cdot AD'} C'D'$, etc. The required inequality reduces to $C'D' + B'C' \geq B'D'$.

Problem 3

Let ω be the semicircle with diameter PQ. A circle k is tangent internally to ω and to segment PQ at C. Let AB be the tangent to k perpendicular to PQ, with A on ω and B on segment CQ. Show that AC bisects the angle $\angle PAB$.

Hide solution.

Invert through C. Semicircle ω maps to the semicircle ω' with diameter $P'Q'$, circle k to the tangent to ω' parallel to $P'Q'$, and line AB to a circle l centered on $P'Q'$ which touches k (so it is congruent to the circle determined by ω'). Circle l intersects ω' and $P'Q'$ in A' and B' respectively. Hence $P'A'B'$ is an isosceles triangle with $\angle PAC = \angle A'P'C = \angle A'B'C = \angle BAC$.

Problem 4

Points A, B, C are given on a line in this order. Semicircles $\omega, \omega_1, \omega_2$ are drawn on AC, AB, BC respectively as diameters on the same side of the line. A sequence of circles (k_n) is constructed as follows: k_0 is the circle determined by ω_2 and k_n is tangent to $\omega, \omega_1, k_{n-1}$ for $n \geq 1$. Prove that the distance from the center of k_n to AB is $2n$ times the radius of k_n.

Hide solution.

Under the inversion with center A and squared radius $AB \cdot AC$ points B and C exchange positions, ω and ω_1 are transformed to the lines perpendicular to BC at C and B, and the sequence (k_n) to the sequence of circles (k_n') inscribed in the region between the two lines. Obviously, the distance from the center of k_n' to AB is $2n$ times its radius. Since circle k_n is homothetic to k_n' with respect to A, the statement immediately follows.

Problem 5

A circle with center O passes through points A and C and intersects the sides AB and BC of the triangle ABC at points K and N, respectively. The circumscribed circles of the triangles ABC and KBN intersect at two distinct points B and M. Prove that $\measuredangle OMB = 90^\circ$. (*IMO 1985-5.*)

Hide solution.

Invert through B. Points A', C', M' are collinear and so are K', N', M', whereas A', C', N', K' are on a circle. What does the center O of circle $ACNK$ map to? *Inversion does not preserve centers.* Let B_1 and B_2 be the feet of the tangents from B to circle $ACNK$. Their images B_1' and B_2' are the feet of the tangents from B to circle $A'C'N'K'$, and since O lies on the circle BB_1B_2, its image O' lies on the line $B_1'B_2'$ - more precisely, it is at the midpoint of $B_1'B_2'$. We observe that M' is on the polar of point B with respect to circle $A'C'N'K'$, which is nothing but the line B_1B_2. It follows that $\angle OBM = \angle BO'M' = \angle BO'B_1' = 90^\circ$.

Problem 6

Let p be the semiperimeter of a triangle ABC. Points E and F are taken on line AB such that $CE = CF = p$. Prove that the circumcircle of $\triangle EFC$ is tangent to the excircle of $\triangle ABC$ corresponding to AB.

Hide solution.

The inversion with center C and radius p maps points E and F and the excircle to themselves, and the circumcircle of $\triangle CEF$ to line AB which is tangent to the excircle. The statement follows from the fact that inversion preserves tangency.

Problem 7

Prove that the nine-point circle of triangle ABC is tangent to the incircle and all three excircles. (*Feuerbach's theorem*)

Hide solution.

We shall show that the nine-point circle ϵ touches the incircle k and the excircle k_a across A. Let A_1, B_1, C_1 be the midpoints of BC, CA, AB, and P, Q the points of tangency of k and k_a with BC, respectively. Recall that $A_1P = A_1Q$; this implies that the inversion with center A_1 and radius A_1P takes k and k_a to themselves. This inversion also takes ϵ to a line. It is not difficult to prove that this line is symmetric to BC with respect to the angle bisector of $\angle BAC$, so it also touches k and k_a.

Problem 8

The incircle of a triangle ABC is tangent to BC, CA, AB at M, N and P, respectively. Show that the circumcenter and incenter of $\triangle ABC$ and the orthocenter of $\triangle MNP$ are collinear.

Hide solution.

The incenter of $\triangle ABC$ and the orthocenter of $\triangle MNP$ lie on the Euler line of the triangle ABC. The inversion with respect to the incircle of ABC maps points A, B, C to the midpoints of NP, PM, MN, so the circumcircle of ABC maps to the nine-point circle of $\triangle MNP$ which is also centered on the Euler line of MNP. It follows that the center of circle ABC lies on the same line.

Problem 9

Points A, B, C are given in this order on a line. Semicircles k and l are drawn on diameters AB and BC respectively, on the same side of the line. A circle t is tangent to k, to l at point $T \neq C$, and to the perpendicular n to AB through C. Prove that AT is tangent to l.

Hide solution.

An inversion with center T maps circles t and l to parallel lines t' and l', circle k and line n to circles k' and n' tangent to t' and l' (where $T \in n'$), and line AB to circle a' perpendicular to l' (because an inversion preserves angles) and passes through $B', C' \in l'$; thus a' is the circle with diameter $B'C'$. Circles k' and n' are congruent and tangent to l' at B' and C', and intersect a' at A' and T respectively. It follows that A' and T are symmetric with respect to the perpendicular bisector of $B'C'$ and hence $A'T \parallel l'$, so AT is tangent to l.

Problem 10

Let $A_1A_2A_3$ be a nonisosceles triangle with incenter I. Let C_i, $i = 1, 2, 3$, be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (the addition of indices being mod 3). Let B_i, $i = 1, 2, 3$, be the second point of intersection of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles $A_1B_1I, A_2B_2I, A_3B_3I$ are collinear. (*IMO 1997 Shortlist*)

Hide solution.

The centers of three circles passing through the same point I and not touching each other are collinear if and only if they have another common point. Hence it is enough to show that the circles A_iB_iI have a common point other than I. Now apply inversion at center I and with an arbitrary power. We shall denote by X' the image of X under this inversion. In our case, the image of the circle C_i is the line $B_{i+1}'B_{i+2}'$ while the image of the line $A_{i+1}A_{i+2}$ is the circle $IA_{i+1}'A_{i+2}'$ that is tangent to $B_i'B_{i+2}'$, and $B_i'B_{i+2}'$. These three circles have equal radii, so their centers P_1, P_2, P_3 form a triangle also homothetic to $\triangle B_1'B_2'B_3'$. Consequently, points A_1', A_2', A_3', that are the reflections of I across the sides of $P_1P_2P_3$, are vertices of a triangle also homothetic to $B_1'B_2'B_3'$. It follows that $A_1'B_1', A_2'B_2', A_3'B_3'$ are concurrent at some point J', i.e., that the circles A_iB_iI all pass through J.

Problem 11

If seven vertices of a hexahedron lie on a sphere, then so does the eighth vertex.

Hide solution.

Let $AYBZ, AZCX, AXDY, WCXD, WDYB, WBZC$ be the faces of the hexahedron, where A is the ``eighth'' vertex. Apply an inversion with center W. Points B', C', D', X', Y', Z' lie on some plane π, and moreover, C', X', D'; D', Y', B'; and

B', Z', C' are collinear in these orders. Since A is the intersection of the planes YBZ, ZCX, XDY, point A' is the second intersection point of the spheres $WY'B'Z', WZ'C'X', WX'D'Y'$. Since the circles $Y'B'Z'$, $Z'C'X'$, $X'D'Y'$ themselves meet at a point on plane π, this point must coincide with A'. Thus $A' \in \pi$ and the statement follows.

Problem 12

A sphere with center on the plane of the face ABC of a tetrahedron $SABC$ passes through A, B and C, and meets the edges SA, SB, SC again at A_1, B_1, C_1, respectively. The planes through A_1, B_1, C_1 tangent to the sphere meet at a point O. Prove that O is the circumcenter of the tetrahedron $SA_1B_1C_1$.

Hide solution.

Apply the inversion with center S and squared radius $SA \cdot SA_1 = SB \cdot SB_1 = SC \cdot SC_1$. Points A and A_1, B and B_1, and C and C_1 map to each other, the sphere through A, B, C, A_1, B_1, C_1 maps to itself, and the tangent planes at A_1, B_1, C_1 go to the spheres through S and A, S and B, S and C which touch the sphere $ABCA_1B_1C_1$. These three spheres are perpendicular to the plane ABC, so their centers lie on the plane ABC; hence they all pass through the point $\overline{S}$ symmetric to S with respect to plane ABC. Therefore $\overline{S}$ is the image of O. Now since $\angle SA_1O = \angle S\overline{S}A = \angle\overline{S}SA = \angle OSA_1$, we have $OS = OA_1$ and analogously $OS = OB_1 = OC_1$.

Problem 13

Let KL and KN be the tangents from a point K to a circle k. Point M is arbitrarily taken on the extension of KN past N, and P is the second intersection point of k with the circumcircle of triangle KLM. The point Q is the foot of the perpendicular from N to ML. Prove that $\angle MPQ = 2\angle KML$.

Hide solution.

Apply the inversion with center M. Line MN' is tangent to circle k' with center O', and a circle through M is tangent to k' at L' and meets MN' again at K'. The line $K'L'$ intersects k' at P', and $N'O'$ intersects ML' at Q'. The task is to show that $\angle MQ'P' = \angle L'Q'P' = 2\angle K'ML'$.

Let the common tangent at L' intersect MN' at Y'. Since the peripheral angles on the chords $K'L'$ and $L'P'$ are equal (to $\angle K'L'Y'$), we have $\angle L'O'P' = 2\angle L'N'P' = 2\angle K'ML'$. It only remains to show that L', P', O', Q' are on a circle. This follows from the equality $\angle O'Q'L' = 90^\circ - \angle L'MK' = 90^\circ - \angle L'N'P' = \angle O'P'L'$ (the angles are regarded as oriented).

Problem 14

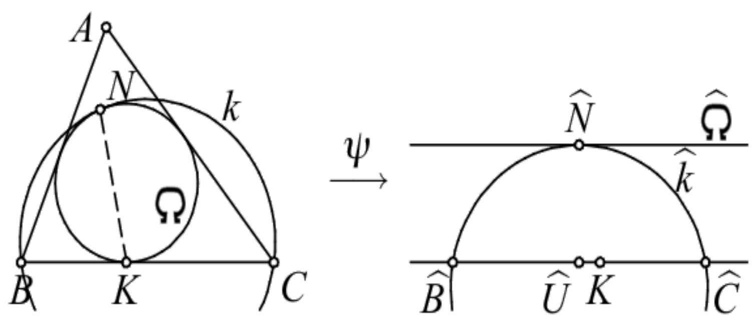
The incircle Ω of the acute-angled triangle ABC is tangent to BC at K. Let AD be an altitude of triangle ABC and let M be the midpoint of AD. If N is the other common point of Ω and KM, prove that Ω and the circumcircle of triangle BCN are tangent at N. (*IMO 2002 Shortlist*)

Hide solution.

Let k be the circle through B, C that is tangent to the circle Ω at point N'. We must prove that K, M, N' are collinear. Since the statement is trivial for $AB = AC$, we may assume that $AC > AB$. As usual, $R, r, \alpha, \beta, \gamma$ denote the circumradius and the inradius and the angles of $\triangle ABC$, respectively.

We have $\tan\angle BKM = DM/DK$. Straightforward calculation gives $DM = \frac{1}{2}AD = R\sin\beta\sin\gamma$ and $DK = \frac{DC-DB}{2} - \frac{KC-KB}{2} = R\sin(\beta-\gamma) - R(\sin\beta - \sin\gamma) = 4R\sin\frac{\beta-\gamma}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$, so we obtain

$$\tan\angle BKM = \frac{\sin\beta\sin\gamma}{4\sin\frac{\beta-\gamma}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}} = \frac{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\sin\frac{\beta-\gamma}{2}}.$$



To calculate the angle BKN', we apply the inversion ψ with center at K and power $BK \cdot CK$. For each object X, we denote by $\widehat{X}$ its image under ψ. The incircle Ω maps to a line $\widehat{\Omega}$ parallel to $\widehat{B}\widehat{C}$, at distance $\frac{BK\cdot CK}{2r}$ from $\widehat{B}\widehat{C}$. Thus the point $\widehat{N'}$ is the projection of the midpoint $\widehat{U}$ of $\widehat{B}\widehat{C}$ onto $\widehat{\Omega}$. Hence

$$\tan\angle BKN' = \tan\angle\widehat{B}K\widehat{N'} = \frac{\widehat{U}\widehat{N'}}{\widehat{U}K} = \frac{BK\cdot CK}{r(CK-BK)}$$

Again, one easily checks that $KB\cdot KC = bc\sin^2\frac{\alpha}{2}$ and $r = 4R\sin\frac{\alpha}{2}\cdot\sin\frac{\beta}{2}\cdot\sin\frac{\gamma}{2}$, which implies

$$\tan\angle BKN' = \frac{bc\sin^2\frac{\alpha}{2}}{r(b-c)} \& = \frac{4R^2\sin\beta\sin\gamma\sin^2\frac{\alpha}{2}}{4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\cdot 2R(\sin\beta-\sin\gamma)} = \frac{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\sin\frac{\beta-\gamma}{2}}.$$

Hence $\angle BKM = \angle BKN\prime$, which implies that $K, M, N\prime$ are indeed collinear; thus $N\prime \equiv N$.