30^{th}

Iranian Mathematical Olympiad

Selected Problems and Their Solutions

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Iranian Team Members in the 54^{th} IMO (Santa Marta-Colombia)

From left to right:

- Tina Torkaman
- Aref Sadeghi
- Alek Bedroya
- Seyyed Mohammadhossein Seyyedsalehi
- Mohammad Javad Shabani Zahraei
- Mahan Tajrobehkar

Contents

Preface

The 30^{th} Iranian National Mathematical Olympiad consisted of four rounds. The first round was held on 24 February 2012 all over the country. The exam consisted of 10 short-answer questions and 15 multiple-choice questions to be solved in 3 hours. In total more than 25000 students participated in the exam and 1446 of them were validated for the next round.

The second round was held on 30 April and 1 May 2012. In each day, participants were given 3 problems to be solved in 4.5 hours. In this round, from a total of 1446 participants, 43 of them were chosen to participate in the third round.

The examination of the third round consisted of five separate exams, and a final exam containing 8 questions, each having it's specified time to be solved in. In the end, 10 people were awarded Bronze Medal, 20 people were awarded Silver Medal, and 13 people were awarded Gold Medal, which the following list represents the names of the Gold Medalists:

Jalal Ahmadi
Mohammad Javad Baferooni
Alek Bedroya
Hossein Hazrati
Mohammad Amin Ketabchi
Hadi Khodabande
Aref Sadeghi
Seyyed Mohammadhossein Seyyedsalehi
Mohammad Javad Shabani Zahraei
Mahan Tajrobehkar
Seyyed Alireza Tavakoli
Tina Torkaman
Mobin Yahyazade Jelodar

The Team Selection Test was held on 6 days, each day having 3 problems to be solved in 4.5 hours. In the end, the top 6 participants were chosen to participate in the 54^{th} IMO as members of the Iranian Team.

In this booklet, we present 6 problems of the Second Round, 8 problems of the final exam of the Third Round, and 18 problems of the Team Selection Test, together with their solutions.

It's a pleasure for the authors to offer their grateful appreciation to all the people who have contributed to the conduction of the 30^{th} Iranian Mathematical Olympiad, including the National Committee of Mathematics Olympiad, problem proposals, problem selection teams, exam preparation teams, coordinators, editors, instructors and all who have shared their knowledge and effort to increase the Mathematics enthusiasm in our country, and assisted in various ways to the conduction of this scientific event.

Problems

Second Round

1. (Morteza Adl) Consider a circle C_1 and a point O on it. Circle C_2 with center O, intersects C_1 in two points P and Q. C_3 is a circle which is externally tangent to C_2 at R and internally tangent to C_1 at S and suppose that RS passes through Q. Suppose X and Y are second intersection points of PR and OR with C_1. Prove that QX is parallel to SY. ($\rightarrow$ p.21)

2. (Morteza Saghafian) Suppose n is a natural number. In how many ways can we place numbers $1, 2, ..., n$ around a circle such that each number is a divisor of the sum of it's two adjacent numbers? ($\rightarrow$ p.21)

3. (Mahyar Sefidgaran, Amin Nejatbakhsh) Prove that if t is a natural number then there exists a natural number $n > 1$ such that $(n, t) = 1$ and none of the numbers $n + t, n^2 + t, n^3 + t, ...$ are perfect powers. ($\rightarrow$ p.23)

4. (Morteza Saghafian) a) Do there exist 2-element subsets $A_1, A_2\ , A_3, ...$ of natural numbers such that each natural number appears in exactly one of these sets and also for each natural number n, sum of the elements of A_n equals $1391 + n$?
b) Do there exist 2-element subsets $A_1, A_2, A_3, ...$ of natural numbers such that each natural number appears in exactly one of these sets and also for each natural number n, sum of the elements of A_n equals $1391 + n^2$? ($\rightarrow$ p.23)

5. (Sahand Seifnashri) Consider the second degree polynomial $x^2 + ax + b$ with real coefficients. We know that the necessary and sufficient condition for this polynomial to have roots in real numbers is that its discriminant, $a^2 - 4b$, be greater than or equal to zero. Note that the discriminant is also a polynomial with variables a and b. Prove that the same story is not true for polynomials of degree 4: Prove that there does not exist a 4 variable polynomial $P(a, b, c, d)$ such that the fourth degree polynomial $x^4 + ax^3 + bx^2 + cx + d$ can be written as the product of four 1st degree polynomials if and only if $P(a, b, c, d) \geq 0$. (All the coefficients are real numbers.) ($\rightarrow$ p.24)

6. (Mehdi E'tesami Fard) The incircle of triangle ABC, is tangent to sides BC, CA and AB at D, E and F respectively. The reflection of F with respect to B and the reflection of E with respect to C are T and S respectively. Prove that the incenter of triangle AST is inside or on the incircle of triangle ABC. ($\rightarrow$ p.24)

Third Round

1. (Morteza Saghafian) Let G be a simple undirected graph with vertices $v_1, v_2, ..., v_n$. We denote the number of acyclic orientations of G with $f(G)$.
a) Prove that $f(G) \leq f(G - v_1) + f(G - v_2) + \cdots + f(G - v_n)$.
b) Let e be an edge of the graph G. Denote by G' the graph obtained by omitting e and making it's two endpoints as one vertex. Prove that $f(G) = f(G - e) + f(G')$.
c) Prove that for each $\alpha > 1$, there exists a graph G and an edge e of it such that $\frac{f(G)}{f(G-e)} < \alpha$. 90 minutes (→ p.26)

2. (Ali Khezeli) Suppose S is a convex figure in plane with area 10. Consider a chord of length 3 in S and let A and B be two points on this chord which divide it into three equal parts. For a variable point X in $S - \{A, B\}$, let A' and B' be the intersection points of rays AX and BX with the boundary of S. Let S' be those points of X for which $AA' > \frac{1}{3}BB'$. Prove that the area of S' is at least 6.
105 minutes (→ p.26)

3. (Amirhossein Gorzi) Prove that for each $n \in \mathbb{N}$ there exist natural numbers $a_1 < a_2 < \cdots < a_n$ such that $\phi(a_1) > \phi(a_2) > \cdots > \phi(a_n)$. 50 minutes (→ p.28)

4. (Hamidreza Ziarati) We have n bags each having 100 coins. All of the bags have 10 gram coins except one of them which has 9 gram coins. We have a balance which can show weights of things that have weight of at most 1 kilogram. At least how many times shall we use the balance in order to find the different bag? 75 minutes (→ p.28)

5. (Mostafa Einollahzade, Erfan Salavati) We call the three variable polynomial P cyclic if $P(x, y, z) = P(y, z, x)$. Prove that cyclic three variable polynomials P_1, P_2, P_3 and P_4 exist such that for each cyclic three variable polynomial P, there exists a four variable polynomial Q such that

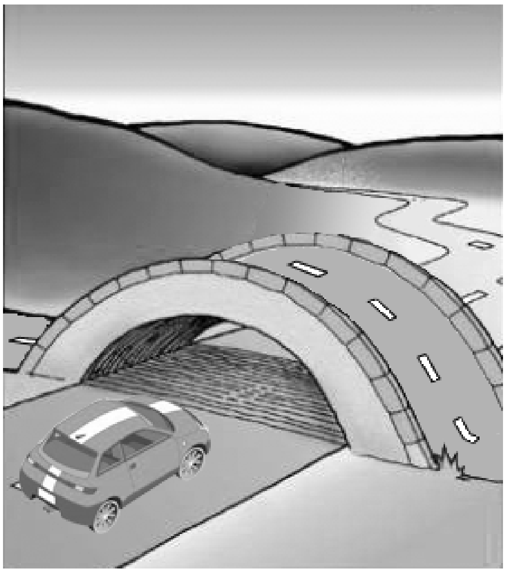
$$P(x, y, z) = Q(P_1(x, y, z), P_2(x, y, z), P_3(x, y, z), P_4(x, y, z)).$$

60 minutes (→ p.29)

6. (Mostafa Einollahzade) a) Prove that a real number $a > 0$ exists such that for each natural number n, there exists a convex n-gon P in plane with lattice points as vertices such that the area of P is less than an^3.
b) Prove that there exists $b > 0$ such that for each natural number n and each n-gon P in plane with lattice points as vertices, the area of P is not less than bn^2.
c) Prove that there exist $\alpha, c > 0$ such that for each natural number n and each n-gon P in plane with lattice points as vertices, the area of P is not less than $cn^{2+\alpha}$.
105 minutes (→ p.30)

7. (Erfan Salavati) The city of Bridge Village has some highways. Highways are closed curves that have intersections with each other or themselves in 4-way crossroads. Mr.Bridge Lover, mayor of the city, wants to build a bridge on each crossroad in order to decrease the number of accidents. He wants to build the bridges in such a way that in each highway, cars pass above and under a bridge alternatively. By knowing the number of highways determine whether this action is possible or not.

50 minutes (→ p.33)



8. (Morteza Saghafian) a) Does there exist an infinite subset S of the natural numbers, such that $S \neq \mathbb{N}$, and such that for each natural number $n \notin S$, exactly n members of S are coprime with n?

b) Does there exist an infinite subset S of the natural numbers, such that for each natural number $n \in S$, exactly n members of S are coprime with n? 75 minutes (→ p.34)

Team Selection Test

1. (Mehdi E'tesami Fard) In acute triangle ABC, let H be the foot of the perpendicular from A to BC and also let J and I be the B-excenter of triangle ABH and C-excenter of triangle ACH, respectively. If P is the point of tangency of the incenter of triangle ABC with side BC, prove that points I, J, P and H lie on the same circle. (→p.36)

2. (Ali Khezeli) At most how many subsets of the set $\{1, 2, ..., n\}$ can be selected in such a way that if A and B are two selected subsets and $A \subset B$, then $|B - A| \geq 3$? (By |X| we mean number of elements of the set X.) (→p.37)

3. (Morteza Saghafian) For nonnegative integers m and n, the sequence $a(m, n)$ of real numbers is defined as follows: $a(0, 0)$ is equal to 2, and for each natural number n, $a(0, n) = 1$ and $a(n, 0) = 2$. Also for $m, n \in \mathbb{N}$:

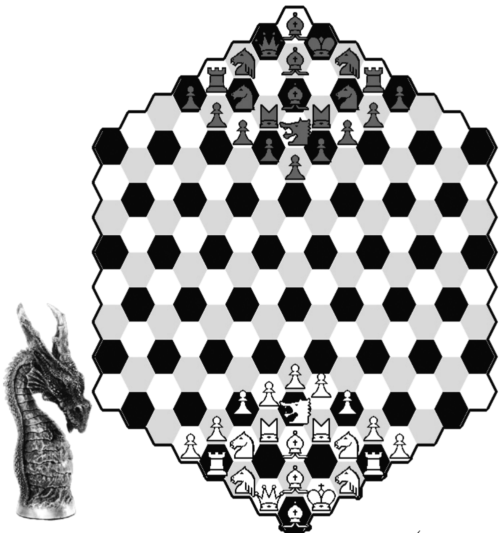
$$a(m, n) = a(m - 1, n) + a(m, n - 1)$$

Prove that for each natural number k, all roots of the polynomial $P_k(x) = \sum_{i=0}^{k} a(i, 2k + 1 - 2i)x^i$ are real numbers. (→p.37)

4. (Shayan Dashmiz) Suppose m and n are two nonnegative integers. In the Philosopher's Chess, the chessboard is an infinite array of same regular hexagon cells. The Phoenix piece, which is a special piece in this kind of chess, moves as follows:

At first, the Phoenix selects one of the six directions and moves m cells in that direction. Then it turns 60 degrees clockwise and moves n cells in that new direction to get to the final point.

At most how many cells of the Philosopher's Chess' chessboard exist in such a way that one cannot start from one of them and reach another one with a finite number of movements of the Phoenix piece?



(→p.38)

5. (Mahan Malihi) Do there exist natural numbers a, b and c such that $a^2 + b^2 + c^2$ is divisible by $2013(ab + bc + ca)$? (→p.39)

6. (Ali Khezeli) Let X, Y, X' and Y' be four points on a line l with the same order as written. Consider arcs ω_1 and ω_2 with end points $\{X, Y\}$ and arcs ω'_1 and ω'_2 with end

points $\{X', Y'\}$ on one side of l such that ω_1 is tangent to ω_1' and ω_2 is tangent to ω_2'. Prove that the external common tangent of ω_1 and ω_2' and the external common tangent of ω_1' and ω_2 meet on l. (→p.41)

7. (Mohammad Ali Karami) Nonnegative real numbers $p_1, p_2, ..., p_n$ and $q_1, q_2, ..., q_n$ are given such that

$$p_1 + p_2 + ... + p_n = q_1 + q_2 + ... + q_n.$$

Among all matrices with nonnegative real entries for which sum of entries of the ith row is p_i and sum of entries of the jth column is q_j, find the maximum value that the trace of the matrix can have. (→p.44)

8. (Yahya Motevassel) Find all infinite arithmetic progressions $a_1, a_2, ...$ for which there exists natural number $N > 1$ such that for each $k \in \mathbb{N}$:

$$a_1 a_2 \cdots a_k | a_{N+1} a_{N+2} \cdots a_{N+k}.$$

(→p.46)

9. (Mohammad Jafari) Find all functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ such that f is an increasing function and also for each $x, y \in \mathbb{R}^+$:

$$f(f(x) + 2g(x) + 3f(y)) = g(x) + 2f(x) + 3g(y)$$
$$g(f(x) + y + g(y)) = 2x - g(x) + f(y) + y.$$

(→p.46)

10. (Morteza Saghafian) To each edge of a graph, we have assigned a real number in a way that for each even walk of this graph, sum of the numbers on edges of the walk, with minus and plus signs assigned alternatively, is zero. Prove that we can assign a real number to each vertex so that for each edge, the number assigned to it is the sum of numbers assigned to it's endpoints. (A walk is a closed path of edges that can have repeated number of vertices or edges.) (→p.47)

11. (Amirhossein Gorzi) Suppose we have a triangle with side lengths a, b and c such that $a \geq b \geq c$. Prove that

$$\sqrt{a(a + b - \sqrt{ab})} + \sqrt{b(a + c - \sqrt{ac})} + \sqrt{c(b + c - \sqrt{bc})} \geq a + b + c.$$

(→p.48)

12. (Ali Zamani) Quadrilateral $ABCD$ is inscribed in circle ω. Suppose that I_1 and I_2 are incenters of triangles ACD and ABC and r_1 and r_2 are inradiuses of triangles ACD and ABC, respectively. We know that $r_1 = r_2$. Circle ω' is tangent to sides AB and AD and also is tangent to circle ω at T. Tangents at T and A to circle ω intersect

each other in K. Prove that I_1, I_2 and K are collinear. (→p.48)

13. (Ali Zamani) Point P is an arbitrary point inside acute triangle ABC and A_1, B_1 and C_1 are reflections of P with respect to sides of triangle ABC. Prove that the centroid of triangle $A_1B_1C_1$ is inside triangle ABC. (→p.50)

14. (Amirhossein Gorzi) We have drawn n rectangles in plane where n is a natural number. Prove that among $4n$ resulting right angles, at least $\lfloor\sqrt{4n}\rfloor$ distinct right angles exist. (→p.50)

15. (Amirhossein Gorzi) a) Does there exist a sequence of natural numbers $a_1 < a_2 < \cdots$ for which there exists a natural number N in a way that for each natural number $m \geq N$, a_m is divisible by exactly $d(m) - 1$ terms of the sequence? ($d(n)$ is the number of divisors of natural number n.)

b) Does there exist a sequence of natural numbers $a_1 < a_2 < \cdots$ for which there exists a natural number N in a way that for each natural number $m \geq N$, a_m is divisible by exactly $d(m) + 1$ terms of the sequence? (→p.50)

16. (Amirhossein Gorzi) The function $f : \mathbb{Z} \to \mathbb{Z}$ has the property that for all integers m and n

$$f(m) + f(n) + f(f(m^2 + n^2)) = 1.$$

We know that integers a and b exist such that $f(a) - f(b) = 3$. Prove that integers c and d can be found such that $f(c) - f(d) = 1$. (→p.51)

17. (Mahan Malihi) AD and AH are angle bisector and altitude of vertex A, respectively. The perpendicular bisector of AD, intersects semicircles with diameters AB and AC which are drawn outside triangle ABC in X and Y, respectively. Prove that the quadrilateral $XYDH$ is concyclic. (→p.51)

18. (Ali Khezeli) A special kind of parallelogram tile is made up by attaching the legs of two right isosceles triangles of side length 1. We want to put a number of these tiles on the floor of an $n \times n$ room such that the distance from each vertex of each tile to the sides of the room is an integer and also no two tiles overlap. Prove that at least an area n of the room will not be covered by the tiles. (→p.52)

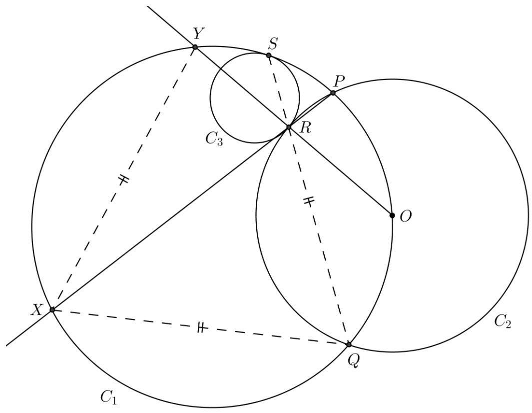
Solutions

Second Round

1 . Let l be the common tangent of C_1 and C_3. Then:

$$\left.\begin{array}{ccccccc} \overset{\frown}{SR} & = & 2\angle RSl & = & 2\angle QSl & = & \overset{\frown}{SQ} \\ \overset{\frown}{SR} & = & \overset{\frown}{RQ} & = & 2\angle RPQ & = & \overset{\frown}{XQ} \end{array}\right\}$$

$$\Rightarrow \overset{\frown}{SQ} = \overset{\frown}{XQ} \Rightarrow SQ = XQ. \quad (1)$$



So it is sufficient to prove that $QX = XY$. Notice that O is the center of C_2, so:

$$OP = OR \Rightarrow \angle OPR = \angle ORP \Rightarrow \overset{\frown}{OX} = \overset{\frown}{OP} + \overset{\frown}{XY}$$
$$\Rightarrow \overset{\frown}{OQ} + \overset{\frown}{QX} = \overset{\frown}{OP} + \overset{\frown}{XY} \Rightarrow \overset{\frown}{QX} = \overset{\frown}{XY}. \quad (2)$$

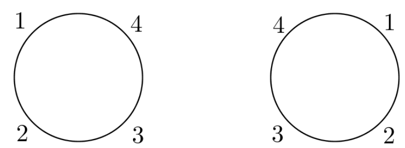
$$(1),(2) \Rightarrow SQ = XQ = XY \Rightarrow SY \| QX.$$

□

2 . We call an arrangement ***good*** if it satisfies the problem requirements. Consider one of these arrangements. n should be a divisor of sum of it's two adjacent numbers, but the sum of these two numbers is less than $n + n = 2n$. So sum of the two adjacent numbers of n must be n. Thus for a natural number k, $1 \leq k \leq n-1$, these two numbers are k and $n-k$. As a result, we see that by omitting n, the condition of the problem is still satisfied, because $n \equiv n-k \pmod{k}$ and $n \equiv k \pmod{n-k}$, which means k and $n-k$ still divide sum of their two adjacent numbers. So by omitting n, we reach to a *good* arrangement of numbers $1, 2, ..., n-1$. For $n = 1$ and $n = 2$, the answer to the question is 1; and for $n = 3$ the answer is 2. Now, by using induction, we prove that the answer

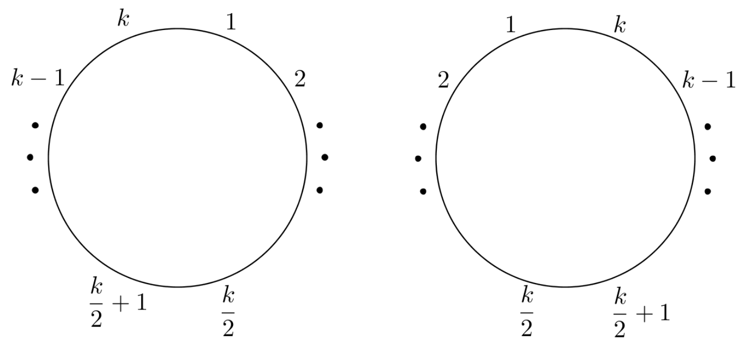
to the question is 4 for odd natural numbers n greater than 3, and 2 for even natural numbers n greater than 2.

Base case of the induction: $n = 4$



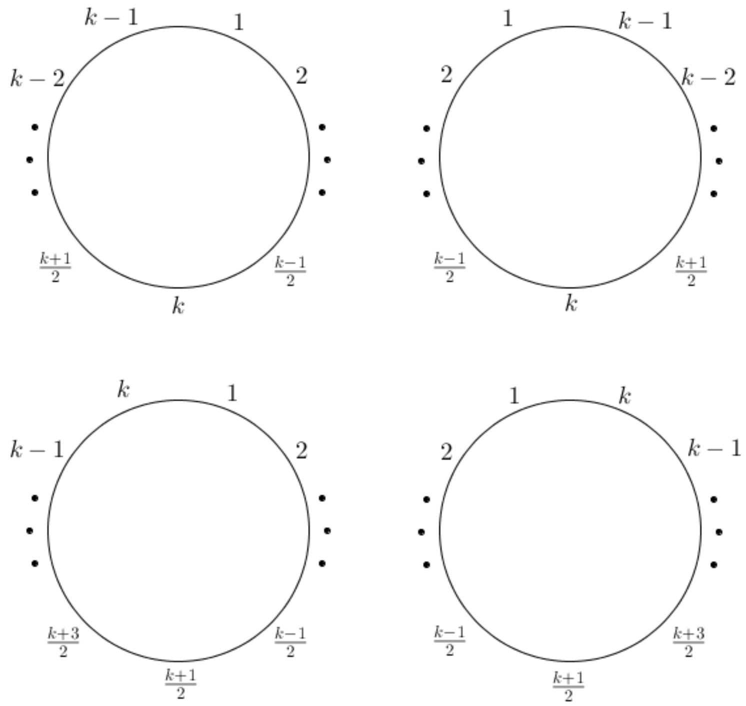
Suppose that our assumption holds for $n = k$. A *good* arrangement of numbers $1, 2, ..., k + 1$ will become a *good* arrangement of numbers $1, 2, ..., k$ by omitting $k + 1$.

If $k+1$ is odd, by the induction hypothesis *good* arrangements of numbers $1, 2, ..., k$ are



and $k + 1$ could only be placed between numbers 1 and k or between numbers $\frac{k}{2}$ and $\frac{k}{2} + 1$ which have a sum of $k + 1$. Thus we have 4 *good* arrangements in this case.

If $k + 1$ is even, by the induction hypothesis *good* arrangements of numbers $1, 2, ..., k$ are



and $k+1$ could only be placed between the two numbers 1 and k in the two bottom-most arrangements. So we have 2 *good* arrangements in this case.

In conclusion, The answer to the problem for $n = 1$ and $n = 2$ is 1, for $n = 3$ and for all other even numbers is 2, and for all other odd numbers is 4.

$\square$

3 . Let p be a prime factor of $t + 1$ and let $p^m || t + 1$. Define $K = m!\phi(p^{m+1})$ and $n = (t(t + 1) + 1)^K$. Now, we have $n^r + t \equiv 1 + t \pmod{p^{m+1}}$, so if $n^r + t = b^s$ then $s \le m$. On the other hand we have

$$n^r = (t(t+1)+1)^{Kr} = \left((t(t+1)+1)^{\frac{K}{s}r}\right)^s = {n_0}^s, \qquad \text{(Because } s|m!|K)$$

so

$$t = b^s - n^r = b^s - {n_0}^s \ge (n_0+1)^s - {n_0}^s \ge sn_0 \ge 2n_0 > t$$

which is a contradiction.

$\square$

4 . a) Let $A_i = \{x_i, y_i\}$. So $x_i + y_i = 1391 + i$, but:

$$\begin{aligned} 1391^2 + 1391 \times 696 &= \sum_{i=1}^{1391}(x_i + y_i) \\ &= x_1 + x_2 + \cdots + x_{1391} + y_1 + y_2 + \cdots + y_{1391} \\ &\ge 1 + 2 + \cdots + 2 \times 1391 = 1391 \times 2783. \end{aligned}$$

Contradiction!

b) We construct such A_i's inductively. Let $A_1 = \{1, 1391\}$. Now, assume that $A_1, A_2, \ldots$ and A_k are constructed. Let x_{k+1} be the least positive integer which is not in any of A_i's ($1 \leq i \leq k$). Obviously $x_{k+1} \leq 2k+1$, so the number

$$y_{k+1} = 1391 + (k+1)^2 - x_{k+1} \geq 1391 + k^2$$

is not in any of A_i's too (because sum of the elements of each A_i, ($1 \leq i \leq k$) is at most $1391 + k^2$ until now). So we can have $A_{k+1} = \{x_{k+1}, y_{k+1}\}$, and in these A_i's, each positive integer has appeared exactly once (because of the method of choosing x_{k+1} in each step).

□

5 . If we put $a = c = 0$, polynomial $x^4 + bx^2 + d$ can be written as product of four linear terms if and only if quadratic polynomial $y^2 + by + d$ has two nonnegative roots. Therefore $P(0, b, 0, d) \geq 0$ if and only if $b \leq 0$, $d \geq 0$ and $b^2 - 4d \geq 0$. For a fixed $b \leq 0$ let $Q_b(d) = P(0, b, 0, d)$. Now, $Q_b(d) \geq 0$ if and only if $0 \leq d \leq \frac{b^2}{4}$ and hence by continuity of Q_b, $Q_b(b^2/4)$ must be zero. This implies that for all $b \leq 0$, one variable polynomial $P(0, b, 0, \frac{b^2}{4}) = 0$, and hence this polynomial is always zero. This means that polynomial $x^4 + bx^2 + \frac{b^2}{4}$ has four real roots for all values of b . Contradiction!

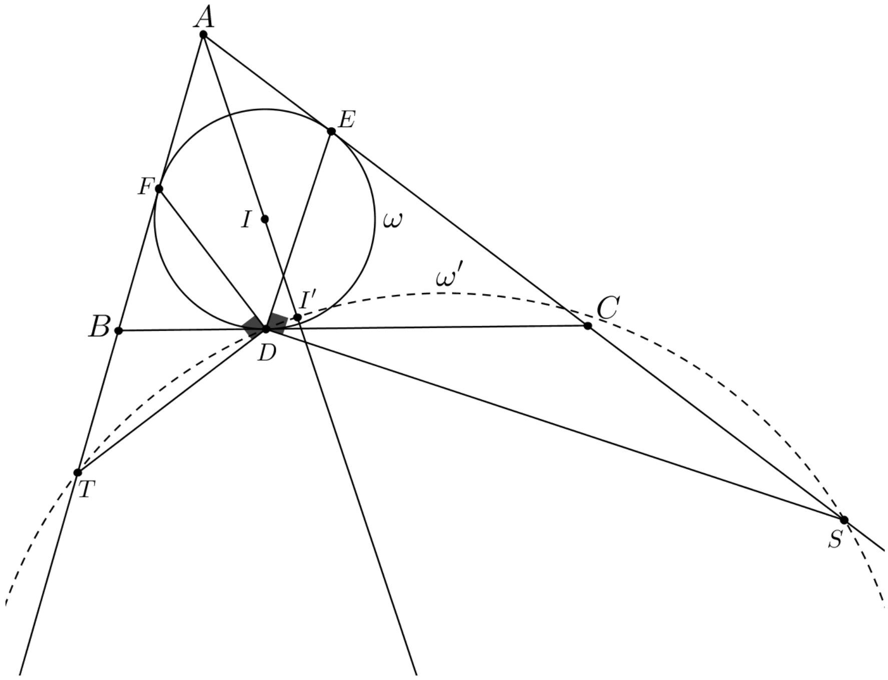
□

6 . Let I and I' be the incenters of triangles ABC and ATS respectively, and ω and ω' be the incircle of triangle ABC and circumcircle of $TI'S$ respectively.

We have

$$\begin{aligned}
\angle TDF &= \angle SDE = 90^\circ \\
\angle EDF &= \frac{1}{2}\angle EIF = 90^\circ - \frac{\angle A}{2} \\
\Rightarrow \angle TDS &= 360^\circ - 90^\circ - 90^\circ - (90^\circ - \frac{\angle A}{2}) \\
&= 90^\circ + \frac{\angle A}{2}.
\end{aligned}$$

On the other hand we have $TI'S = 90^\circ + \frac{\angle A}{2}$; therefore, $TDI'S$ is cyclic and D is on ω'. We know that, center of ω' is the midpoint of $\overset{\frown}{TS}$ of the circumcircle of triangle ATS, so it is on the line II'. Hence two circles ω, ω' intersect at D, and I' is the intersection of ω' and the line joining the two centers. Thus I' is inside ω.



A
E
F
I
ω
ω′
I′
C
B
D
T
S

□

Third Round

1 . a) In each acyclic orientation of G, there is at least one vertex (say v_i) such that all edges connected to that are directed toward v_i, because if not the orientation will have a cycle which contradicts the fact that the orientation is acyclic. By omiting this vertex and all edges connected to it, we obtain an acyclic orientation for the graph $G - v_i$.

The number of acyclic orientations of $G - v_i$ is $f(G - v_i)$, but for some orientations of G there exists more than one vertex with the property discussed. Therefore, the assertion is proved.

b) Every acyclic orientation of $G - e$ results in at least one acyclic orientation of G because if v_i and v_j are two endpoints of edge e, then if e cannot be oriented from v_i to v_j, it means that there is a path (in $G - e$) from v_j to v_i. Similarly, if e cannot be oriented from v_j to v_i, it means that there is a path (in $G - e$) from v_i to v_j.We conclude that because of two directed paths from v_i to v_j and from v_j to v_i, there exists a directed cycle containing v_i and v_j in $G - e$, which is a contradiction.

So each acyclic orientation of $G - e$ results in one or two acyclic orientaions of G. Now, if edge e can be oriented in two directions, then in $G - e$ there is no directed path either form v_j to v_i or from v_i to v_j. So if we omit e to get G', as stated in the problems statement, an acyclic orientation of G' can be found, and the assertion is followed.

c) It suffices to consider the complete graph with n vertices. In this case $f(G) = n!$, and according to part (b) we have

$$f(G - e) = n! - (n - 1)! = (n - 1)(n - 1)!.$$

Therefore $\frac{f(G)}{f(G-e)} = \frac{n!}{(n-1)(n-1)!} = \frac{n}{n-1}$. For each $\alpha > 1$, there is a natural number n such that

$$\frac{n}{n-1} = 1 + \frac{1}{n-1} < \alpha$$

and this completes the proof.

□

2 . The idea is to remove some neighborhoods of A and B, because near these points we cannot bound $\frac{AA'}{BB'}$. Let Z be the intersection of the ray AB with the boundary of S and let ZB' intersect AA' in A''. A'' is between A and A' since S is convex. So, if we let S'' be the set of those points X such that $AA'' > \frac{1}{3}BB'$, we have $S'' \subseteq S'$. Hence, it is enough to prove that the area of S'' is at least 6.

By *Menelaus's theorem*, we have

$$\begin{aligned} &\frac{AA''}{A''X} \cdot \frac{XB'}{B'B} \cdot \frac{BZ}{ZA} = 1 \\ &\Rightarrow \frac{AA''}{A''X} \cdot \frac{XB'}{B'B} = 2. \end{aligned}$$

To omit of $A''X$, we write $\frac{AA''}{A''X}$ in terms of AA'' and AX and we treat $B'X$ similarly:

$$\begin{aligned}
&\Rightarrow \frac{A''X}{AA''} = \frac{1}{2} \cdot \frac{XB'}{B'B} \\
&\Rightarrow \frac{AX}{AA''} = 1 - \frac{1}{2}(1 - \frac{BX}{BB'}) \\
&\Rightarrow \frac{AX}{AA''} = \frac{1}{2} \cdot \frac{BX}{BB'} + \frac{1}{2}.
\end{aligned}$$

Let D_1 be the set of those points X such that $\frac{1}{2} \cdot \frac{BX}{BB'} + \frac{1}{2} \geq \alpha \frac{BX}{BB'}$ for some constant α (for suitable α, D_1 is a neighborhood of B). If $X \notin D_1$, then

$$\begin{aligned}
\frac{AX}{AA''} &< \alpha \frac{BX}{BB'} \\
\Rightarrow \frac{AA''}{BB'} &> \frac{1}{\alpha} \cdot \frac{AX}{BX}.
\end{aligned}$$

Let D_2 be the set of those points X in the plane such that $\frac{AX}{BX} < \frac{\alpha}{3}$ (for suitable α, D_2 is a neighborhood of A). If X is not in either of D_1 or D_2, then

$$\frac{AA''}{BB'} > \frac{1}{\alpha} \cdot \frac{\alpha}{3} = \frac{1}{3}$$

as desired. So

$$S - D_1 - D_2 \subseteq S'' \subseteq S'.$$

Now, we calculate the areas of D_1 and D_2. If $\alpha < 3$, then D_2 is the interior part of the Apolonius circle of A and B. Also,

$$X \in D_1 \quad \Leftrightarrow \quad \frac{1}{2} \cdot \frac{BX}{BB'} + \frac{1}{2} \geq \alpha \frac{BX}{BB'} \Leftrightarrow \frac{BX}{BB'} < \frac{1}{2\alpha - 1}.$$

So, if $\alpha > 1$ then D_1 is a neighborhood of B similar to S. Now, we take $\alpha = \frac{3}{2}$. We conclude that the area of D_1 is $(\frac{1}{2})^2$ times the area of S which is $\frac{10}{4} = 2.5$. Also, if C and D are the intersection points of the boundary of D_2 with the line AB and C is between A and B, we have

$$\begin{aligned}
\frac{AD}{DB} = \frac{1}{2} \Rightarrow \frac{AD}{AD+1} = \frac{1}{2} \Rightarrow AD = 1 \\
\frac{AC}{CB} = \frac{1}{2} \Rightarrow \frac{AC}{1 - AC} = \frac{1}{2} \Rightarrow AC = \frac{1}{3}.
\end{aligned}$$

So the diameter of D_2 is $1 + \frac{1}{3} = \frac{4}{3}$ and hence, the area of D_1 is $(\frac{2}{3})^2\pi < 1.5$. So

$$\begin{aligned}
T_X &\geq T_{S-D_1-D_2} \\
&\geq 2T_S - T_{D_1} - T_{D_2} \\
&> 10 - 2.5 - 1.5 = 6.
\end{aligned}$$

where T_K represents the area of figure K in the plane. Hence, the assertion is proved.

□

3 . First we prove a lemma.

Lemma. *Let $a > b$ be two positive integers such that $\frac{a}{b} > 4$. Then there exist some positive integer l such that $b < \phi(2^l) = 2^{l-1} < 2^l < a$.*

Proof of lemma. There exists some positive integer m such that $2^{m-1} \le b < 2^m$. So $a > 4b \ge 2^{m+1}$ and hence for $l = m+1$ we have $b < 2^{l-1} < 2^l < a$, as claimed.

□

We construct the sequence in the problem inductively. The assertion for $n = 1$ is obvious. Now, suppose that $a_1 < a_2 < \cdots < a_n$ is a sequence such that $\phi(a_1) > \phi(a_2) > \cdots > \phi(a_n)$. Let p_N be the greatest prime factor of a_i's (p_j is the jth prime number), $1 \le i \le n$. Since for every x with prime factors greater than p_N, the greatest common divisor of x and a_i is 1 for all $1 \le i \le n$, we deduce that the sequence $a_1x < a_2x < \cdots < a_nx$ also satisfies the condition of the problem (because $\phi(a_ix) = \phi(a_i)\phi(x)$). Now, our goal is to find such x and some positive integer y such that $y < a_1x$ and $\phi(y) > \phi(a_1x)$. First we claim that there is a positive integer x with prime factors greater that p_N such that $\frac{a_1x}{\phi(a_1)\phi(x)} > 4$ or equivaletly, $\frac{x}{\phi(x)} > 4\frac{\phi(a_1)}{a_1}$.
To prove this, let $x_m = p_{N+1}p_{N+2}\cdots p_{N+m} (m \in \mathbb{N})$. We have

$$\frac{x_m}{\phi(x_m)} = \prod_{i=N+1}^{N+m} \frac{p_i}{p_i - 1} = \prod_{i=N+1}^{N+m} \left(1 + \frac{1}{p_i - 1}\right) \ge \sum_{i=N+1}^{N+m} \frac{1}{p_i - 1}.$$

The sum $\sum_{i=N+1}^{\infty} \frac{1}{p_i - 1} > \sum_{i=N+1}^{\infty} \frac{1}{p_i}$ diverges by a well-known fact, so we can find m such that $\frac{x_m}{\phi(x_m)} > 4\frac{\phi(a_1)}{a_1}$. We let $x = x_m$. Now, since $\frac{a_1x}{\phi(a_1)\phi(x)} > 4$, according to the lemma there is some positive integer l such that $\phi(a_1x) < \phi(2^l) < 2^l < a_1x$. So putting $y = 2^l$ leads to the new sequence

$$y < a_1x < a_2x < \cdots < a_nx$$

which has $n+1$ terms and satisfies the problem's condition.

□

4 . Obviously, if we are able to find the different bag among n bags with k times using the balance, for less than n bags also k times using the balance is enough. Denote by $f(k)$ the maximum number of bags such that the different bag among them can be found with at most k times using the balance. Our goal is to find $f(k)$ for all $k \in \mathbb{N}$.

(i) $f(1) = 14$. If $n = 14$, for $1 \le i \le 14$ we put $i - 1$ coins from the ith bag on the balance, and if the resulting weight is $910 - k$ ($91 = 1 + 2 + \cdots + 13$) grams, then the $(k+1)$st bag is different, so $f(1) \ge 14$. To prove the converse inequality, suppose that we have a number of bags and we want to use the balance only once. Let a_i be the number of bags for which we take exactly i coins from them. We have

$$a_1 + 2a_2 + \cdots + ma_m \leq 100. \quad (*)$$

Note that for $i \geq 0, a_i \leq 1$ because if we take the same number of coins from two different bags, we cannot distinguish between them. Now, we want to maximize $\sum_{i=0}^{m} a_i$. Since the coefficient of a_k in $(*)$ equals k, it's best for a_0 to be maximized, then a_1 should be maximized and so on. We conclude that the maximum value will be 14. So $f(1) \leq 14$ and consequently $f(1) = 14$.

(ii) $f(2) = 60$. To find $f(2)$, again $(*)$ should be satisfied. Also, for any integer $i \geq 0$, we must have $a_i \leq 14$. Here we want to maximize $\sum_{i=0}^{m} a_i$. The maximum value will be gained when $a_0 = a_1 = a_2 = a_3 = 14$ and $a_4 = 4$. Therefore $f(2) = 60$.

(iii) $f(3) = 140$. By a similar argument, here $(*)$ should be satisfied, and also for each positive integer $i \geq 0$, we must have $a_i \leq 60$, so the maximum value will be gained when $a_0 = a_1 = 60$ and $a_2 = 20$; therefore, $f(3) = 140$.

(iv) Finally, if $k > 3$, then $a_i \leq 100$ and we must have $a_0 \leq f(k-1)$, so we get the maximum value of $\sum_{i=0}^{m} a_i$ when $a_0 = f(k-1)$ and $a_1 = 100$. Hence $f(k) = f(k-1) + 100$.

Upshot.

$n \in$	$[1, 14]$	$[15, 60]$	$[61, 140]$	$[100k + 40, 100k + 140], k \in \mathbb{N}$
	1	2	3	$k + 3$

□

5 . Let

$$Q(x, y, z) = P(x, y, z) + P(y, x, z)$$
$$R(x, y, z) = P(x, y, z) - P(y, x, z).$$

By these definitions it is obvious that Q is a symmetric polynomial and R is an antisymmetric one. Note that $R(x, x, z) = 0$, so R is divisible by $x - y$. Similarly, $y - z$ and $z - x$ also divide R. Therefore there is a polynomial S such that

$$R(x, y, z) = (x - y)(y - z)(z - x)S(x, y, z).$$

And it's obvious that S is a symmetric polnomial. We conclude that

$$P = \frac{1}{2}Q + \frac{1}{2}(x - y)(y - z)(z - x)S.$$

Every symmetric polynomial in three variables can be written as a polynomial of elementary symmetric polynomials $P_1 = x + y + z$, $P_2 = xy + yz + zx$ and $P_3 = xyz$. If we put $P_4 = (x - y)(y - z)(z - x)$ (which obviously is cyclic) the proof is complete. □

6 . We call a convex n-gon $A_1A_2\cdots A_n$ with vertices from lattice points *good* if for each integer $1 \le i \le n-1$, vector $\overrightarrow{A_iA_{i+1}}$ lies in the first quadrant of the cartesian plane. Throughout the solution all polygons are considered to have vertices from lattice points. We denote the least possible value for the area of a convex n-gon and the area of a good convex n-gon by $f(n)$ and $g(n)$, respectively. Furthermore, in such polygons let $u_i = \overrightarrow{A_iA_{i+1}}$ $(1 \le i \le n-1)$. We claim that $g(\frac{n}{4}) \le f(n) \le g(n)$. The rightmost inequality is obvious, since good polygons are a subset of all such polygons. To prove the leftmost inequality, note that for each convex n-gon P the rightmost, leftmost, topmost and bottommost vertices divide the perimeter of the polygon into at most 4 parts, so at least one part has at least $\frac{n}{4}$ veritces and the area of the convexhull of these vertices is at least $g(\frac{n}{4})$ so the claim is proved.

Now, it suffices to prove the assertions for $g(n)$. $(g(n) = O(n^2), ...)$

Lemma 1. *The area of a good n-gon P with vectors $u_1, u_2, \dots$ and u_{n-1} equals $\frac{1}{2}\sum_{i<j} |u_i \times u_j|$.*

Proof of lemma. Suppose $P = A_1A_2\cdots A_n$ and A_1 is the origin. Then the area of triangle $A_1A_iA_{i+1}$ equals $\frac{1}{2}|u_i \times A_i| = \frac{1}{2}\sum_{j=1}^{i-1} |u_i \times u_j|$ (Note that all the cross products have the same sign). The area of P equals sum of the area of these triangles, so by summing these areas the lemma is proved. □

Now let us return to the main problem:

a) Let $u_i = (1, i)$, Since the slope of vectors $u_1, u_2, \dots$ and u_{n-1} are increasing, the n-gon that they determine is good. Applying lemma 1 implies

$$\begin{aligned} S &= \frac{1}{2}\sum_{i<j} |u_i \times u_j| = \frac{1}{2}\sum_{i<j} (j-i) = \frac{1}{2}\sum_{j} \frac{j(j-1)}{2} \\ &\le \frac{1}{4}\sum_{j} j^2 = \frac{n(n+1)(2n+1)}{24} \end{aligned}$$

and this implies the assertion.

b) According to the lemma, the area of each good n-gon is at least $\binom{n-1}{2}$, because $|u_i \times u_j|$ is a positve integer, so $g(n) \ge \binom{n-1}{2}$.

c) It suffices to prove the following claim.

Claim 1. *For sufficiently large n, the area of each good $(n+2)$-gon is at least $\frac{1}{24}n^{\frac{5}{2}}$.*

By lemma 1, the following claim implies claim 1.

Claim 2. *For sufficiently large n and integer vectors $u_1, u_2, \dots u_{n+1}$ with distinct slopes we have:*

$$\sum_{i<j} |u_i \times u_j| \ge \frac{1}{12}n^{\frac{5}{2}}.$$

If $\sum_j |u_i \times u_j| \geq \frac{1}{6} n^{\frac{3}{2}}$ for $1 \leq i \leq n+1$, summing these expressions implies the claim. So without loss of generality we can assume that

$$\sum_i |u_i \times u_{n+1}| \leq \frac{1}{6} n^{\frac{3}{2}}.$$

We do not have the assumption of monotonocity of slopes any further. For each i, let $C_i = \{u_l : |u_l \times u_{n+1}| = i\}$ and $k_i = |C_i|$. Obviously $\sum k_i = n$. Elements of C_i lie on two lines parallel to u_{n+1}, so for each $j \leq n$, at most 4 numbers among the values $|u_j \times u_l| \quad (u_l \in C_j)$ can be equal (except number 0 when $u_j \in C_i$ which is unique).

Since these values are all integer, we have

$$\sum_{u_l \in C_i} |u_j \times u_l| \geq 4 \binom{\frac{k_i}{4}}{2}.$$

Therefore,

$$\sum_{l \leq n} |u_j \times u_l| \geq 4 \sum_i \binom{\frac{k_i}{4}}{2} = \frac{1}{8} \sum_i k_i^2 - \frac{n}{2},$$

and

$$\sum_{j,l} |u_j \times u_l| \geq \frac{n}{8} \sum_i k_i^2 - \frac{n^2}{2},$$

but $\sum k_i = n$ and $\sum i k_i \leq \frac{1}{6} n^{\frac{3}{2}}$. Now, according to the following lemma we have $\sum k_i^2 \geq \frac{3}{2} n^{\frac{3}{2}}$, and therefore for sufficiently large n

$$\sum_{j,l} |u_j \times u_l| \geq \frac{3}{16} n^{\frac{5}{2}} - \frac{n^2}{2} \geq \frac{1}{6} n^{\frac{5}{2}}$$

and this completes the proof of the claim.

Lemma 2. *Let $k_1, k_2, \ldots, k_n$ be nonnegative real numbers such that $\sum k_i = a$ and $\sum i k_i \leq b$. Show that $\sum k_i^2 \geq \frac{a^3}{4b}$.*

Proof of lemma. Suppose that values a and b are constants and $k_1, k_2, \ldots, k_n$ are variables satisfying the conditions for which $\sum k_i^2$ is minimum. We conclude that k_i's, $1 \leq i \leq n$, are in decreasing order because if $k_i < k_j$ for some $i < j$ then we can substitute both of them with $\frac{1}{2}(k_i + k_j)$ and $\sum k_i^2$ will have a smaller value.

We claim that $k_1, k_2, \ldots, k_n$ must be a block of an arithmetic progression unless they are zero. Suppose that k_{j-1}, k_j and k_{j+1} are nonzero and $k_j \neq \frac{k_{j-1}+k_{j+1}}{2}$. If we change these three to $k_{j-1}+x, k_j - 2x, k_{j+1}+x$, it is easy to check that this new sequence also satisfies the conditions. Thus for the difference of the value $\sum k_i^2$ for these two sequences we have

$$\Delta = ((k_{j-1}+x)^2 + (k_j - 2x)^2 + (k_{j+1}+x)^2) - (k_{j-1}^2 + k_j^2 + k_{j+1}^2)$$

$$= 6x^2 + 2x(k_{j-1} + k_j + k_{j+1}).$$

Note that the coefficient of x is nonzero. Hence we can choose a small value of x having a different sign from it's coefficient for Δ to be less than 0. Therefore, each three consecutive nonzero numbers among k_i's must form an arithmetic progression. By modifying n if necessary, we can assume that the sequence $k_1, k_2, \dots$ and k_n is a decreasing arithmetic progression of nonzero numbers, so $k_i = r - si$ for constants $r, s \geq 0$. We have

$$\begin{array}{rcl} a &=& nr - s\sum i \\ c &=& r\sum i - s\sum i^2 \quad \leq \quad b. \end{array}$$

Hence

$$\sum k_i^2 = nr^2 - 2rs\sum i + s^2\sum i^2 = r(nr - s\sum i) - s(r\sum i - s\sum i^2),$$

so

$$\sum k_i^2 = ra - sc.$$

The values of r and s can be calculated from the linear system of equations above in terms of a, b and c:

$$r = \frac{a\sum i^2 - c\sum i}{n\sum i^2 - (\sum i)^2}\ , \quad s = \frac{a\sum i - nc}{n\sum i^2 - (\sum i)^2}.$$

By *Cauchy-Schwarz Inequality* we obtain that

$$\sum k_i^2 \geq \frac{(\sum k_i)^2}{n} = \frac{a^2}{n}.$$

Since $r - sc$ is negative we have

$$\begin{array}{rc} & a\sum i^2 - c\sum i - an\sum i + n^2c \geq 0 \\ \Rightarrow & a(\frac{n^2(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}) \leq c(n^2 - \frac{n(n+1)}{2}) \leq \frac{cn(n+1)}{2} \\ \Rightarrow & a(n - \frac{2n+1}{3}) \leq c \\ \Rightarrow & n \leq 3\frac{c}{a} + 1 \leq 4\frac{c}{a}. \end{array}$$

The last inequality follows from the fact that $c \geq a$ (because $c = \sum ik_i \geq \sum k_i = a$). Finally we have

$$\sum k_i^2 \geq \frac{a^2}{n} = \frac{a^3}{an} \geq \frac{a^3}{4c} \geq \frac{a^3}{4b}$$

as desired.

$\square$

Claim 3. *The area of such good $(n+2)$-gon is at least $\frac{1}{200}n^3$ and therefore $f(n) = \Theta(n^3)$.*

To prove this claim it suffices to prove that for each set of integer vectors $u_1, u_2, \ldots$ and u_{n+1} with distinct slopes, we have

$$\sum_{i<j} |u_i \times u_j| \geq \frac{1}{100} n^3.$$

Similar to the previous claim, it's enough to assume $\sum_i |u_i \times u_{n+1}| \leq \frac{1}{50} n^2$. C_i and k_i are defined as before. In this part we must prove a better bound for $\sum_{u_l \in C_i} |u_j \times u_l|$. If $u_l, u_{l'} \in C_i$ lie on a line parallel to u_{n+1}, then $u_l - u_{l'}$ is an integer multiple of u_{n+1} (we can suppose u_{n+1} is not an integer multiple of any other vector, if else we can change it with a shorter vector pointing at the same direction). So $u_j \times u_l - u_j \times u_{l'}$ is a multiple of $u_j \times u_{n+1}$. Note that elements of C_i lie on two lines parallel to u_{n+1}. Consequently

$$\sum_{u_l \in C_i} |u_j \times u_l| \geq 4J \binom{\frac{k_i}{4}}{2}$$

where we have assumed $u_j \in C_J$.

Summing these inequalities on index i implies

$$\sum_{l \leq n} |u_j \times u_l| \geq 4J \sum_i \binom{\frac{k_i}{4}}{2}.$$

Then by summing on index j we have

$$\sum_{j,l} |u_j \times u_l| \geq 4(\sum i k_i) \sum_i \binom{\frac{k_i}{4}}{2} = (\sum i k_i) \left(\frac{1}{8} \sum k_i^2 - \frac{n}{2} \right).$$

Since $\sum k_i = n$ and $\sum i k_i \leq \frac{1}{50} n^2$, by applying lemma 2 we obtain $\sum k_i^2 \geq \frac{25}{2} n$. Hence

$$\begin{aligned} \sum_{j,l} |u_j \times u_l| &\geq (\tfrac{1}{8} - \tfrac{1}{25})(\sum i k_i)(\sum k_i^2) \\ &\geq \tfrac{1}{4}(\tfrac{1}{8} - \tfrac{1}{25})(\sum k_i)^3 \quad \geq \tfrac{1}{50} n^3. \end{aligned}$$

□

7 . We show that bridges satisfying the problem's conditions can be built no matter what the configuration of the highways is.

Throughout the solution we denote the highways by $C_1, C_2, \ldots$ and C_n and by G the graph determined by those highways (crossroads are vertices of the graph and highways between two crossroads are edges of the graph).

Lemma. *Faces of G can be colored by two colors in a way that the colors of every two adjacent faces is different (Two faces are adjacent if they have an edge in common).*

Proof of lemma. We proceed by induction on number of edges of graph G. We call a coloring of faces of G satisfying the mentioned property, a *good* coloring. Suppose that C is the shortest path in G that lies in just one highway and assume $C \subset C_1$. Note that because of minimality, C cannot cross itself, so is a simple curve, therefore dividing the the plane into two regions. Now, $C - C_1$ is a closed or maybe an empty curve.

Suppose that H is the graph consisting of $C - C_1, C_2, \dots$ and C_n. Since the number of edges of H is less than G, H has a good coloring according to the induction hypothesis. Now, we add curve C to H and change the color of faces inside C. It is very easy to check that this is a good coloring of G. The base case is when G is empty. In this case we have only one region which obviously has a good coloring!

□

Finally consider a good coloring (by colors black and white) for the graph G and an arbitrary orientation for each highway. We have two types of edges. For some edges when we are moving through ine (in it's specified direction) our right face is white. We call such edges, edges of type 1 and others, edges of type 2. Build a bridge in the way of each edge of type 1.

Note that in each crossroad exactly two of the edges are directed into the crossroad and only one of those two is of type 1. Hence our construction do not cause any contradictions.

□

8 . a) We claim that a set cannot exist with this property. Assume to the contrary that there exists a set S satisfying the problems condition and let $n \in \mathbb{N}$, $n \notin S$. Thus exactly n elements of S are coprime to n. Therefore, there exist infinitely many prime numbers which do not belong to S. Let p and q be two of them. Since $p \notin S$ exactly p elements of S are not divisible by p (coprime to p), for each $\alpha > 1$, $p^\alpha \in S$. This is true because if $p^\alpha \notin S$, there must be p^α elements of S coprime to p^α (and therefore coprime to p), but we have shown that S has exactly p elements of this kind. Finally, the greatest common divisor of p^α and q is 1, so infinitely many elements of S are coprime to $q \notin S$, a contradiction!

b) We construct set the S in the following way:

Suppose $a_1, a_2, a_3, \dots$ are elements of S and at first all a_i's, $i \in \mathbb{N}$, are 1. In each step we multiply a_i by some primes.

For $n \in \mathbb{N}$ in step n:

(i) We multiply a_n by some new primes p_{2n-1} and p_{2n} so that a_n become greater than a_{n-1} (Note that p_n is the nth prime number).

(ii) Some elements among $\{a_1, a_2, \dots, a_{n-1}\}$ are coprime to a_n. let b_n be the number of such elements, so $b_n < n < a_n$. Let $t_n = a_n - b_n$.

(iii) Because of the type of construction, infinitely many number of a_i's, $i > n$, are coprime to a_n. We leave first t_n and assume that the t_nth number is a_m.

(iv) In the sequence $a_{m+1}, a_{m+2}, \dots$ multiply the first 2^{n-1} terms by p_{2n-1}, the next 2^{n-1} terms by p_{2n} and so on.

Therefore at the end of step n:

(i) Exactly a_n numbers are coprime to a_n.

(ii) For each sequence $(q_1, q_2, \dots, q_n)$ of prime numbers such that $q_i \in \{p_{2i-1}, p_{2i}\}$, there exist infinitely many numbers k such that a_k has exactly these prime factors. So for each a_i there exist infinitely many numbers k such that a_k has prime factors different from those of a_i and consequently is copirme to a_i.

$\square$

Team Selection Test

1 . Let J' and I' be foot of the perpendicular lines from J and I to line BC, respectively. Furhtermore, throughout the solution we denote by $p(XYZ)$ and $S(XYZ)$ the semiperimeter and area of triangle XYZ, respectively.

J' is the tangency point of B-excircle of triangle ABH and line BH, so we have $BJ' = p(ABH) = \frac{AB+BH+AH}{2}$. Also we have $BP = \frac{AB+BC+CA}{2}$ and hence $PJ' = BJ' - BP = \frac{AH+CA-CH}{2}$. II' is the radius of C-excircle of triangle ACH, therefore

$$II' = \frac{S(ACH)}{p(ABH) - AH} = \frac{AH.CH}{AC + CH - AH}.$$

We claim that $PJ' = II'$.

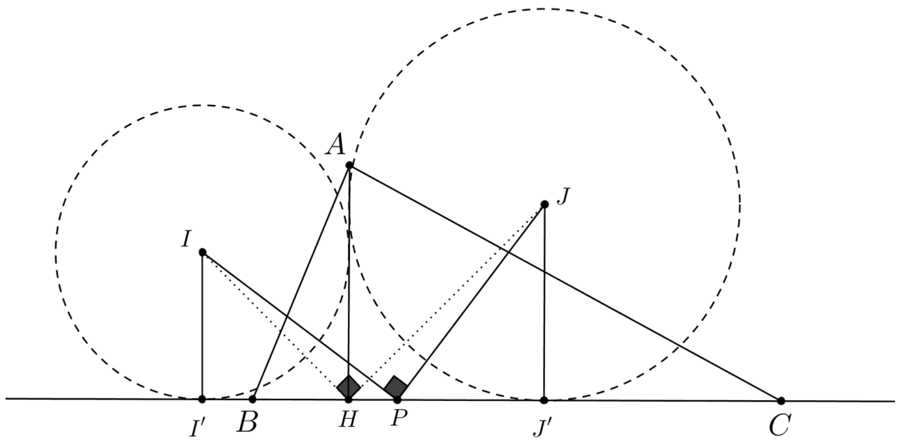
$$PJ' = II' \Leftrightarrow \frac{AH + CA - CH}{2} = \frac{AH.CH}{AC + CH - AH}$$

$$\Leftrightarrow 2AH.CH = CA^2 - (CH - AH)^2 = CA^2 - CH^2 - AH^2 + 2CH.AH$$

$$\Leftrightarrow CA^2 = CH^2 + AH^2.$$

Which is true because of the *Pythagorean Theorem.* By a similar argument we get $PI' = JJ'$. Therefore, the right-angled triangles PII' and PJJ' are congruent, so

$$\angle IPI' = \angle PJJ' = 90° - \angle JPJ' \Rightarrow \angle IPJ = 90°$$



HI and HJ are angle bisectors of $\angle AHB$ and $\angle AHC$, respectively, and therefore $\angle IHJ = \frac{1}{2}(\angle AHB + \angle AHC) = 90°$. Since $\angle IPJ = \angle IHJ = 90°$, we deduce that I, P, J and H lie on the same circle.

□

2 . We claim that the maximum number of such sets is $\lceil \frac{2^n}{3} \rceil$.
We proceed by induction on n. For the base case we must check the claim for $n = 1, 2$ which is obvious. Suppose that the claim is true for n. For $n + 2$ let $\mathcal{G}$ be the family of those selected subsets of $\{1, 2, \ldots, n + 2\}$. We divide $\mathcal{G}$ into four sets with respect to their intersection with $\{n + 1, n + 2\}$.

$$\mathcal{G}_0 = \{A \in \mathcal{G} | n + 1 \notin A, n + 2 \notin A\}$$
$$\mathcal{G}_1 = \{A \in \mathcal{G} | n + 1 \in A, n + 2 \notin A\}$$
$$\mathcal{G}_2 = \{A \in \mathcal{G} | n + 1 \in A, n + 2 \in A\}$$
$$\mathcal{G}_3 = \{A \in \mathcal{G} | n + 1 \notin A, n + 2 \in A\}$$

For $i = 0, 1, 2$ let $\mathcal{F}_i$ be the family of $A - \{n + 1, n + 2\}$ for each $A \in \mathcal{G}_i$. We claim that for each $B \subset \{1, 2, \ldots, n\}$, B is in at most one of the sets $\mathcal{F}_0, \mathcal{F}_1$ and $\mathcal{F}_2$, Because if $B \in \mathcal{F}_i, \mathcal{F}_j$, where $i < j$, it means that there exist B_i and B_j in $\mathcal{G}$ such that

$$B_i - \{n + 1, n + 2\} = B,\ B_j - \{n + 1, n + 2\} = B.$$

Now, $B_i \subset B_j$ but $|B_j - B_i| = |B_j| - |B_i| = j - i < 3$ which is a contradiction, so $|\mathcal{G}_0| + |\mathcal{G}_1| + |\mathcal{G}_2| \leq 2^n$. The number of elements of $\mathcal{G}_3$ is at most $\lceil \frac{2^n}{3} \rceil$ according to the induction hypothesis, so

$$|\mathcal{G}| \leq 2^n + \lceil \frac{2^n}{3} \rceil = \lceil \frac{2^{n+2}}{3} \rceil.$$

To construct an example for this upper bound, for $i = 0, 1, 2$ let $\mathcal{S}_i$ be the family of subsets $A \subset \{1, 2, \ldots, n\}$ such that $|A| \equiv i \pmod 3$. We have

$$|\mathcal{S}_0| + |\mathcal{S}_1| + |\mathcal{S}_2| = 2^n.$$

Since these sets are disjoint, one of them has at least $\lceil \frac{2^n}{3} \rceil$ elements which obviously satisfies the problem condition.

$\square$

3 . Let $Q_k(x) = \sum_{i=0}^{k} a(i, 2k - 2i)x^i$. According to the recurrence relation for $a(m, n)$, we get

$$P_k(x) = xP_{k-1}(x) + Q_k(x)$$
$$Q_k(x) = xQ_{k-1}(x) + P_{k-1}(x).$$

So $Q_k(x) = P_k(x) - xP_{k-1}(x)$ and therefore, $P_k(x) - xP_{k-1}(x) = x(P_{k-1}(x) - xP_{k-1}(x)) + P_{k-1}(x)$. Finally, we get

$$P_k(x) = (2x + 1)P_{k-1}(x) - x^2 P_{k-1}(x).$$

Hence we get a recurrence relation for $P_{k+1}(x)$ where $k \geq 2$, $P_1(x) = 3x + 1$ and $P_2(x) = 5x^2 + 5x + 1$.

Claim. *For each positive integer $k \geq 2$ all of the roots of $P_k(x)$ and $P_{k-1}(x)$ are real and distinct. Furthermore, if $a_1 < a_2 < \cdots < a_{k-1}$ and $b_1 < b_2 < \cdots < b_k$ are roots of P_{k-1} and P_k, respectively, we have*

$$b_1 < a_1 < b_2 < a_2 < \cdots < a_{k-1} < b_k.$$

Proof. We proceed by induction. For the base case $k = 2$, $\frac{-1}{3}$ is the only root of P_1 and $P_2(\frac{-1}{3}) < 0$ so $\frac{-1}{3}$ lies between the two roots of $P_2(x)$.

Suppose that $P_{k-1}(x)$ and $P_k(x)$ satisfy the induction hypothesis. We know $P_{k+1}(x) = (2x+1)P_k(x) - x^2P_{k-1}(x)$. Now, we consider the signs of P_k and P_{k-1} on different real numbers. First suppose that k is even.

x	$-\infty$	b_1	a_1	b_2	a_2	$\cdots$	b_{k-1}	a_{k-1}	b_k	$+\infty$
P_k	$+$	0	$-$	0	$+$	$\cdots$	0	$-$	0	$+$
x^2P_{k-1}	$-$	$-$	0	$+$	0	$\cdots$	$-$	0	$+$	$+$

Since $P_{k+1}(x) = (2x+1)P_k(x) - x^2P_{k-1}(x)$, for the sign of P_{k+1} in respective b_i's we have

x	$-\infty$	b_1	a_1	b_2	a_2	$\cdots$	b_{k-1}	a_{k-1}	b_k	$+\infty$
P_{k+1}	$-$	$+$		$-$		$\cdots$	$+$		$-$	$+$

According to the change of signs of $P_{k+1}(x)$ and by the *Mean Value Theorem*, for each $1 \leq i \leq k-1$, P_{k+1} has a root between b_i and b_{i+1}. Also, it has one root less than b_1 and one root greater than b_k, which completes the proof for even values of k. The case where k is an odd number is similar, the only difference being the signs of b_1 and $-\infty$.

Note that

(i) 0 is not a root of any of the P_k's. Because $P_k(0) = a(0, 2k+1) = 1$, $P_{k-1}(x)$ and $x^2P_{k-1}(x)$ have the same sign.

(ii) The coefficient of x^k in $P_k(x)$ equals $a(2k, 1) > 0$, so $P_k(+\infty) > 0$ and the sign of $P_k(-\infty)$ is related to the parity of k.

$\square$

4 . By considering centers of the hexagons, we get the following equivalent problem:

"Two vertices α and β of the triangular lattice are called equivalent if $\alpha - \beta$ is equal to sum of finitely many vectors from the set:

$$A = \{\pm(n\vec{i} - m\vec{j}), \pm(n\vec{k} + m\vec{i}), \pm(n\vec{j} + m\vec{k})\}$$

Where $\vec{i} = (1, 0), \vec{j} = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $\vec{k} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ according to the normal Cartesian coordinates.

What is the maximum number of nonequivalent vertices?"

Note that $\vec{k} = \vec{i} + \vec{j}$, thus we can determine vectors of set A using only $\vec{i}$ and $\vec{j}$:

$$A = \{\pm(n\vec{i} - m\vec{j}), \pm((n+m)\vec{i} + n\vec{j}), \pm(m\vec{i} + (m+n)\vec{j})\}.$$

Since the first vector $n\vec{i} - m\vec{j}$ is the difference of the two others, two vertices α and β are equivalent iff their difference $\alpha - \beta$ can be written as a linear combination of vectors $(n+m)\vec{i} + n\vec{j}$ and $m\vec{i} + (m+n)\vec{j}$ with integer coefficients.

Note that for integers a, b, x and y the space generated by the vectors $\{a\vec{i}+b\vec{j}, x\vec{i}+y\vec{j}\}$ (using integer coefficients) equals the space generated by vectors $\{(a-x)\vec{i}+(b-y)\vec{j}, x\vec{i}+y\vec{j}\}$. Therefore, we can change former vectors to get simpler vectors to work with. Note that in this change the value of $ay - bx$ (determinant of the matrix formed by vectors (a, b) and (x, y) as rows) is invariant, because $ay - bx = (a-x)y - (b-y)x$. Now, in a similar way to Euclidean Algorithm, if $a \geq x$ we change $\{a\vec{i} + b\vec{j}, x\vec{i} + y\vec{j}\}$ by $\{(a-x)\vec{i} + (b-y)\vec{j}, x\vec{i} + y\vec{j}\}$. Repeating this process several times leads to vectors of the form $s\vec{i} + r\vec{j}$ and $0\vec{i} + t\vec{j}$. It means that we can suppose the coefficient of $\vec{i}$ for one of them is zero. hence, if we want to go from one vertex to another, the difference of the coefficients of $\vec{i}$ must be a multiple of s, since $t\vec{j}$ does not affect the coefficient of $\vec{i}$. As a result, for two equivalet vertices, we can first use vector $s\vec{i} + r\vec{j}$ several times to match their first coordinate. Then, we must use $t\vec{j}$ to match the second coordinate. Hence we have st nonequivalent vertices. But

$$st = \det \begin{bmatrix} s & r \\ 0 & t \end{bmatrix}$$

which is invariant during the process.

$$\det \begin{bmatrix} s & r \\ 0 & t \end{bmatrix} = \det \begin{bmatrix} m+n & m \\ n & m+n \end{bmatrix} = m^2 + mn + n^2$$

So the maximum number of nonequivalent vertices is $m^2 + mn + n^2$.

□

5 .

Lemma 1. *Let $A \equiv 2 \pmod 3$ be a positive integer. Then there exists a prime numbr p such that $p \equiv 2 \pmod 3$ and $p^\alpha || A$ where α is an odd integer.*

Proof of lemma. Assume to the contrary that there is not such a prime number p. Therefore, if $A = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime factorization of A, we have two cases for p_i's, $1 \leq i \leq k$, to consider:

Case 1. $p_i \equiv 1 \pmod 3 \ \Rightarrow \ p_i^{\alpha_i} \equiv 1 \pmod 3$.

Case 2. $p_i \equiv 2 \pmod 3$ and $2|\alpha_i \Rightarrow p_i^{\alpha_i} \equiv (p_i^2)^{\frac{\alpha_i}{2}} \equiv 1 \pmod 3$.

As a result, we must have $A \equiv 1 \pmod 3$ which is a contradiction.

□

Lemma 2. *Let $p \equiv 2 \pmod 3$ be a prime number. Show that $\{0^3, 1^3, \ldots, (p-1)^3\}$ is a complete residue system modulo p.*

Proof of lemma. Obviously $i^3 \equiv 0^3 \pmod p$ iff $i \equiv 0 \pmod p$. Suppose that $p \nmid i, j$. Our goal is to show that $i^3 \equiv j^3 \pmod p$ iff $i \equiv j \pmod p$. One part of the proof is obvious. To prove the other part, suppose that $p = 3t + 2$. Then by *Fermat's Little Theorem*, we have $i^{3t+1} \equiv j^{3t+1} \equiv 1 \pmod p$. Hence, we have

$$i^{3t} i \equiv i^{3t+1} \equiv j^{3t+1} \equiv (j^3)^t j \equiv i^{3t} j \pmod p.$$

Since $(i, p) = 1$, we get $i \equiv j \pmod p$. □

Now we are ready to solve the main problem. We claim that there is no such triple. Assume to the contrary that

$$a^2 + b^2 + c^2 = 2013k(ab + bc + ca)$$

for some positive integer k.
First, without loss of generality we can suppose that a, b and c have no common factor, because if $(a, b, c) = d > 1$, we can divide them by d to get a new triple with no common factor. We have $(a + b + c)^2 = (2013k + 2)(ab + bc + ca)$. $2013k + 2 \equiv 2 \pmod 3$, so by lemma 1 there is some prime number $p \equiv 2 \pmod 3$ such that $p^{2n+1} || 2013k + 2$ $(n \geq 0)$.

$$p^{2n+1} || 2013k + 2 \Rightarrow p^{2n+1} | (a + b + c)^2 \Rightarrow p^{2n+2} | (a + b + c)^2$$

$$\Rightarrow p^{2n+2} | (2013k + 2)(ab + bc + ca) \Rightarrow p | ab + bc + ca.$$

As a result, $p | a + b + c$ and $p | ab + bc + ca$. Hence

$$0 \equiv ab + bc + ca \equiv ab + c(a + b) \equiv ab + c(-c) \pmod p \Rightarrow ab \equiv c^2 \pmod p$$

$$\Rightarrow c^3 \equiv abc \pmod p.$$

By a similar argument, $a^3 \equiv b^3 \equiv abc \pmod p$, so by lemma 2 we deduce $a \equiv b \equiv c \pmod p$ and since $p | a+b+c$ and $3 \nmid p$, we find that p divides a, b and c, which contradicts our assumption that $(a, b, c) = 1$.

□

6 . Without loss of generality, we can suppose that ω_1 is shorter than ω_2. Let l' be the common external tangent of ω_1' and ω_2 and let it intersects l at O. Let OA be tangent to ω_1 at A, $B := \omega_1 \cap \omega_1'$, $C := \omega_1' \cap l'$, $D := l' \cap \omega_2$, $E := \omega_2 \cap \omega_2'$ and let OF be tangent to ω_2' at F. We must prove that A, O and F are collinear.
Let the common internal tangent of ω_1 and ω_1' meet l at O', so

$$O'X.O'Y = O'B^2 = O'X'.O'Y'.$$

Thus the power of O' with respect to ω_2 and ω_2' is equal to this value. Hence the circle C_1 with center O' and radius $O'B$ passes through E and is orthogonal to all arcs ω_1, ω_2, ω_1' and ω_2'. We also have $OA^2 = OX \cdot OY = OD^2$ and $OF^2 = OX' \cdot OY' = OC^2$, So the circle C_2 with center O and radius OA is orthogonal to both ω_1 and ω_2 and also the circle C_3 with center O and radius OF is orthogonal to both ω_1' and ω_2'.

We use the following classic lemma without proving it.

Lemma 1. *Let circle ω be orthogonal to circles C_1 and C_2 and intersect them in $\{A_1, B_1\}$ and $\{A_2, B_2\}$, respectively. Let O, O_1 and O_2 be the centers of ω, C_1 and C_2, respectively. Suppose that the right angles $\angle OA_1O_1$ and $\angle OA_2O_2$ have different orientations; i.e. one of them is clockwise and the other one is not. Then the pairs $\{A, A'\}$ and $\{B, B'\}$ are direct anti-homologous points and pairs $\{A, B'\}$ and $\{A', B\}$ are inverse anti-homologous points for C_1 and C_2.* □

Using the lemma for circle C_2, we deduce that A and D are inverse anti-homologous points for ω_1 and ω_2 (we do not check the conditions here for brevity). Using the lemma again for circle C_1, we get that B and E are also inverse anti-homologous points for ω_1 and ω_2. So, the quadrilateral $ADEB$ is cyclic.

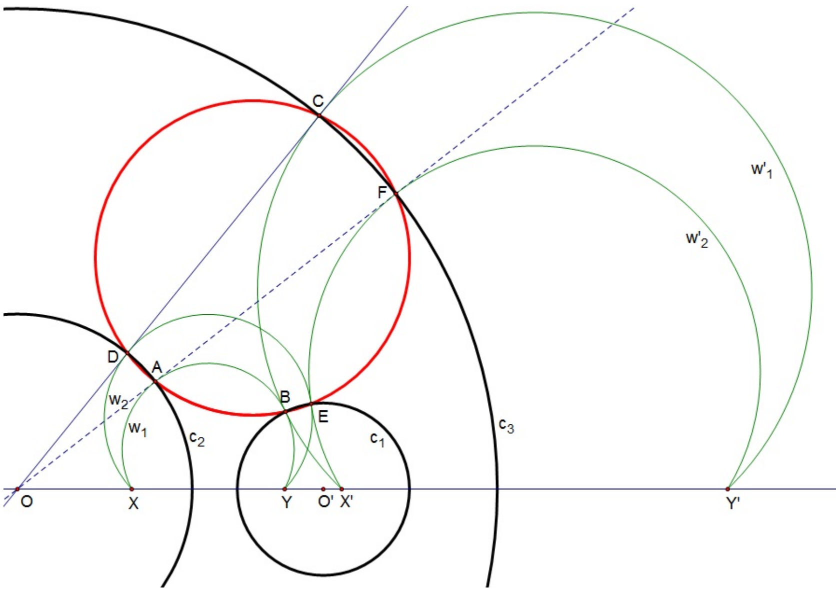
Similarly, by using the lemma twice for circles C_1 and C_3, we get that pairs $\{B, E\}$ and $\{C, F\}$ are inverse anti-homologous points for ω_1' and ω_2'. So the quadrilateral $BEFC$ is also cyclic.

According to the lemma for circle C_1, points B and E are direct anti-homologous points for ω_1' and ω_2, and so are C and D. Hence, the quadrilateral $BCDE$ is also cyclic.

By the last three paragraphs, we deduce that all points $A, B, \ldots, F$ lie on the circumcircle of triangle BCD, which we call it C_4 (checking distinctness of the points is not mentioned here). Now, circles C_2, C_3 and C_4 have a line of symmetry, so their intersections and centers also have a line of symmetry. It follows that O, A and F are collinear and the problem is solved.

Second Solution. To prove that $A, B, \ldots, F$ lie on a circle, we can use the following lemma.

Lemma 2. *If a cricle ω is tangent to circles C_1 and C_2, then the points of tangency are anti-homologous points for C_1 and C_2. If none of C_1 and C_2 contains the other and ω is tangent to them both internally or both externally, then the points of tangency are direct anti-homologous points.* □



Note that these lemmas remain true if one of the circles is a line (considered as a circle with infinite radius).

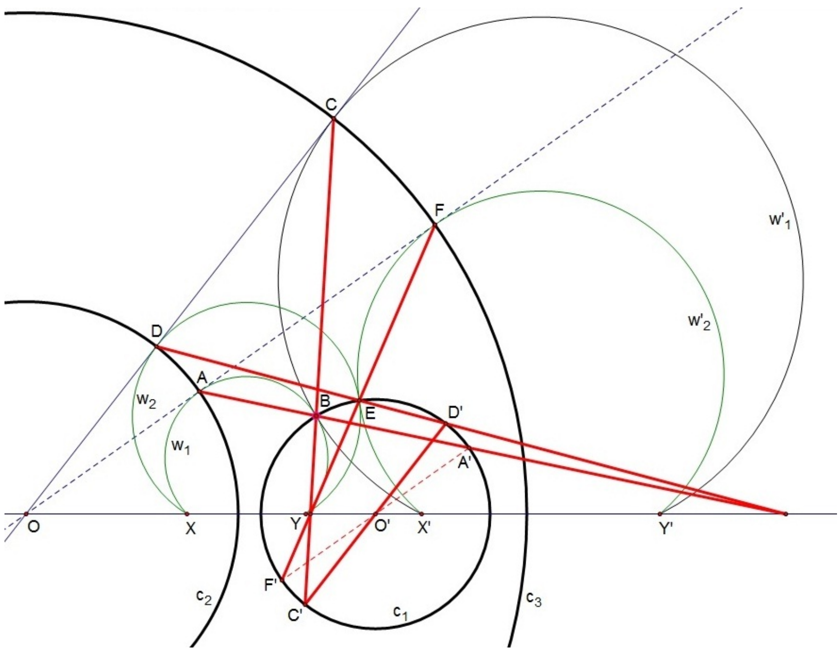
If we suppose that interior of l' (considered as a circle) is the half-plane not containing X, then according to lemma 2 for circle ω_1', points B and C are direct anti-homologous for circles ω_1 and l'. By applying lemma 1 for circle C_2, points A and D are also anti-homologous points for ω_1 and l'. Using the fact that the arcs are all in one side of l, it is easily seen that A and D are also direct anti-homologous points, so $ABCD$ is a cyclic quadrilateral.

Similarly, by considering anti-homologous points for pairs of circles $\{\omega_2, \omega_1'\}$, $\{l', \omega_2'\}$ we conclude that quadrilaterals $BCDE$ and $CDEF$ are also cyclic, so points A, B, C, D, E and F lie on a circle which we call it C_4.

The rest of the proof is the same as the first solution. $\square$

Third Solution. Define points $A, B, \ldots, F$ and circles C_1, C_2 and C_3 similar to the first solution. We can suppose that X is between X' and O, without loss of generality. Thus, C_1 and C_2 are disjoint and C_3 contains both of them. According to lemma 1 and the fact that ω_1 lies entirely on one side of l, points A and B are direct anti-homologous points for C_2 and C_1. Similarly, B and C are inverse anti-homologous points for C_1 and C_3 and points C and D are direct anti-homologous points for C_3 and C_2. Pairs of points $\{D, E\}$ and $\{E, F\}$ are also anti-homologous in a similar manner as are pairs $\{A, B\}$ and $\{B, C\}$, respectively.

Our motivation is that two anti-homologous points are related to each other by an inversion depending only on the two circles. So the problem says that the composition of six particular inversions is the identity. With this intuition in mind, we move all points to C_1. Let A', C', D' and F' be points on C_1 such that pairs $\{A, A'\}$ and $\{D, D'\}$ are



direct homologous points for circles C_1 and C_2, and pairs $\{C, C'\}$ and $\{F, F'\}$ are inverse homologous points for circles C_1 and C_3, respectively.

We have

$$O'D' \parallel OD \parallel OC \parallel O'C',$$

so, $A'D'$ passes through O'. The lines $A'B$ and $D'E$ pass through the direct homothetic center of C_1 and C_2, which lies on l. The lines BC' and EF' pass through the inverse homothetic center of C_1 and C_3, which also lies on l. So, by Pascal's theorem for the cyclic hexagon $A'BC'D'EF'$, we conclude that $A'F'$ also passes through O'. Hence, we similarly have

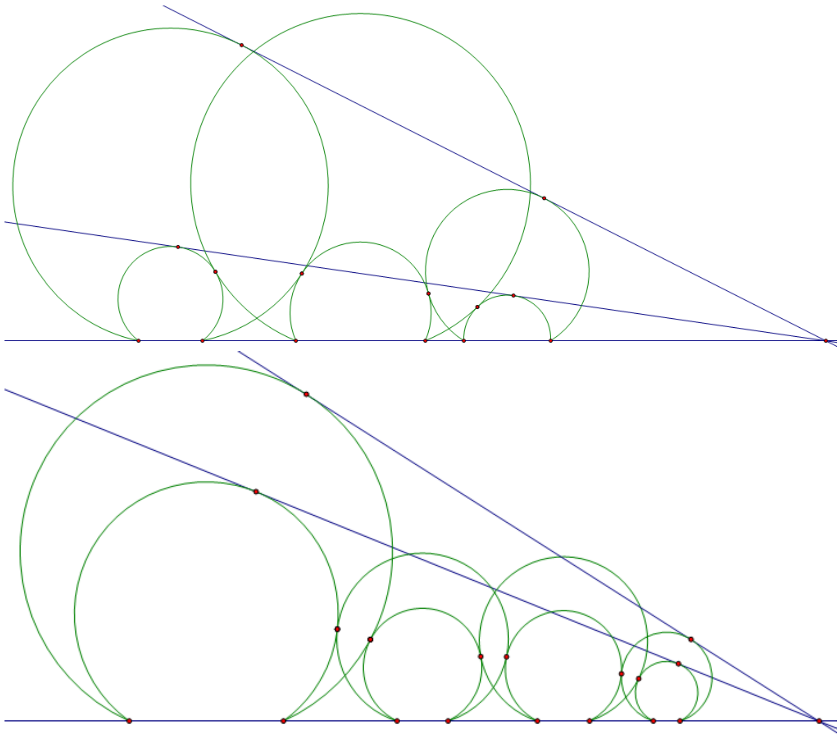
$$OA \parallel O'A' \parallel O'F' \parallel OF,$$

so O, A and F are collinear and we are done. □

Comment 1. If we extend the arcs to perfect circles, then the other external tangent of ω_1 and ω_2' and the other external tangent of ω_2 and ω_1' also meet on l.

Comment 2. If we let ω_1 and ω_2 be in different sides of l, then the assertion is no longer right. Instead, the corresponding internal common tangents meet on l. Also an assertion like that of the first comment holds.

Comment 3. We can solve the problem using hyperbolic geometry. If we let the half-plane be the Poincaré model, then the arcs and the common tangents are equidistant curves; i.e. the locus of points with a constant distance from a given line. This solution gives interesting generalizations of the problem. For example, see the following figures.



$\square$

7 . ***First Solution.*** We use the notation A for matrices and A_{ij} for the entry of row i and column j.

First, observe that $A_{ii} \le p_i$, since the sum of the ith row is p_i and the entries are nonnegative. Similarly, $A_{ii} \le q_i$. Thus, $A_{ii} \le \min(p_i, q_i)$ and it follows that trace$(A) \le \min(p_1, q_1) + \min(p_2, q_2) + ... + \min(p_n, q_n)$. It remains to prove there is a matrix with the property that $A_{ii} = \min(p_i, q_i)$ for every $1 \le i \le n$.

We proceed by induction on n. The case $n = 1$ is obvious, since $p_1 = q_1$. Assume the assertion holds for $n = 1, 2, \dots, k$ and consider the case $n = k + 1$. We can suppose $q_1 \le p_1$ without loss of generality. Let $A_{11} = q_1$ and $A_{i1} = 0$ for $2 \le i \le n$. We claim that there exist nonnegative real numbers $\beta_i := A_{1i}$ for $2 \le i \le n$ such that these conditions hold:

$$\sum_{i=2}^{n} \beta_i = p_1 - q_1,$$
$$\beta_i \le q_i - \min(p_i, q_i) \text{ for every } 2 \le i \le n. \tag{1}$$

To verify our claim, first we prove a lemma.

Lemma. *Let $a, b_1, b_2, \dots, b_k$ be nonnegative real numbers such that $a \le b_1 + b_2 + \cdots + b_k$. Then there exist nonnegative real numbers $a_1, a_2, \dots, a_k$ such that $a_1 + a_2 + \cdots + a_k = a$ and $a_i \le b_i$ for every $1 \le i \le k$.*

Proof of lemma. We use induction on k. For $k = 1$ the lemma is obvious. If $b_k > a$, we can set $a_1 = a_2 = \cdots = a_{k-1} = 0$ and $a_k = a$, so suppose that $b_k \le a$. Let $a_k := b_k$. Now, we must have $a_1 + a_2 + \cdots + a_{k-1} = a - b_k$. but $a - b_k \le b_1 + b_2 + \cdots + b_k - b_k = b_1 + b_2 + \cdots + b_{k-1}$ and the assertion follows by the induction hypothesis. $\square$

We have

$$\begin{aligned}\sum_{i=2}^{n}(q_i - \min(p_i, q_i)) &= \sum_{i=1}^{n}(q_i - \min(p_i, q_i)) = \sum_{i=1}^{n}(p_i - \min(p_i, q_i)) \\ &\ge p_1 - \min(p_1, q_1) = p_1 - q_1.\end{aligned}$$

Hence, by the lemma we can find nonnegative real numbers $\beta_2, \beta_3, \ldots, \beta_n$ such that equation 1 holds. Let $A_{1i} = \beta_i$ for $2 \le i \le n$. The remaining entries of A should satisfy

- $A_{i2} + A_{i3} + \cdots + A_{in} = p_i$ for $2 \le i \le n$,
- $A_{2i} + A_{3i} + \cdots + A_{ni} = q_i - \beta_i$ for $1 \le i \le n$.

We have $q_i - \beta_i \ge \min(p_i, q_i)$ by equation 1, so $\min(p_i, q_i - \beta_i) = \min(p_i, q_i)$. Hence, the rest of the matrix can be filled using the induction hypothesis. $\square$

Second Solution. We can change the order of $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ arbitrarily by changing corresponding rows and columns of the matrix. So we can suppose $p_1 \le q_1, p_2 \le q_2, \ldots, p_k \le q_k$ and $p_{k+1} \ge q_{k+1}, \ldots, p_n \ge q_n$. Let $A_{ii} = \min(p_i, q_i)$ for $1 \le i \le n$. Also let $A_{ij} = 0$ if $i \ne j$ and either $i \le k$ or $j > k$. The remaining entries $\{A_{ij} : i > k, j \le k\}$ should satisty:

- $A_{i1} + A_{i2} + \cdots + A_{ik} = p_i - q_i$ for $i > k$,
- $A_{(k+1)i} + \cdots + A_{ni} = q_i - p_i$ for $i \le k$.

i.e. we should find a matrix with nonnegative entries for the given row sums and column sums. We can construct such a matrix using induction on the number of rows as follows.

By the lemma above, we can find $A_{n1}, A_{n2}, \ldots, A_{nk}$ such that their sum is $p_n - q_n$ and $A_{ni} \le q_i - p_i$ for each $i \le k$. Then, we delete the n'th row and replace $q_i - p_i$ by $q_i - p_i - A_{ni}$ for each $i \le k$. We can proceed by using induction and the desired matrix will be constructed. $\square$

Comment. This problem originally arises from probability theory. Suppose $\sum_i p_i = 1$ and we want to have two random variables X, Y such that $\mathbb{P}(X = i) = p_i$ and $\mathbb{P}(Y = i) = q_i$ for $1 \le i \le n$. The problem wants the maximum possible value of $\mathbb{P}(X = Y)$ over all possible joint distributions of X and Y.

$\square$

8 . We denote the problem's condition by $*$. Let d be the common difference and $a+d$ be the initial term of this arithmetic proression, where a and d are integers such that $a+nd>0$ for each $n \in \mathbb{N}$ (Note that a may be negative but $a+d, a+2d, \ldots$ are all positvie). Hence, d is positive and $d>-a$.

Putting $k=N+m$ for nonnegative integer m in $*$ gives us

$$a_1 a_2 \cdots a_{N+m} | a_{N+1} a_{N+2} \cdots a_{2N+m},$$

so by omitting equal terms:

$$a_1 a_2 \cdots a_N | a_{k+1} a_{k+2} \cdots a_{k+N}$$

for each $k \geq N$ $(m \geq 0)$. We denote this relation by $**$. Let $n_0>N$ be a natural number congruent to -1 modulo $P=a_1 a_2 \cdots a_N$. Then, by $**$ we have:

$$P|(a+(n_0+1)d)(a+(n_0+2)d) \cdots (a+(n_0+N)d).$$

Since $P|n_0+1$, we get

$$P|a(a+d) \cdots (a+(N-1)d).$$

Since $P=(a+d)(a+2d) \cdots (a+Nd)$, we get $a+Nd|a$. Now, if $a>0$, $a+Nd \leq a$, so $Nd \leq 0$. Because $N>1$ this implies $d=0$, so $a_n=a$ for all natural numbers n. Any constant sequence is an answer.
If $a=0$, $a_n=a+nd=nd$, and this is another sequence satisfying the problem's condition.
If $a<0$, $a+Nd \leq |a|=-a$, so $Nd \leq -2a$. Since $N \geq 2$ and $d>-a$ we have $-2a<Nd \leq -2a$ which is a contradiction.
Finally, $a_n=nd$ for $d>0$ and constant sequences, are the only sequences satisfying $*$.

$\square$

9 . f is an increasing function so $f(x)+x$ is strictly increasing and therefore injective. Let $A=f(x)+2g(x)+3f(y)$ and $B=f(y)+2g(y)+3f(x)$. We have

$$f(A)+A=3(f(x)+f(y)+g(x)+g(y)),$$

$$f(B)+B=3(f(y)+f(x)+g(y)+g(x)).$$

Since $f(A)+A=f(B)+B$ and $f(x)+x$ is injective, we deduce that $A=B$:

$$f(x)+2g(x)+3f(y)=f(y)+2g(y)+3f(x).$$

This implies that $f(x)-g(x)=f(y)-g(y)$ for all $x, y>0$. Thus, $f(x)-g(x)$ equals some contant value c. Putting $x=y$ in the second equation gives $g(*)=3x+c>c$, so g takes any value greater than c and consequently, f takes any value greater than $2c$. By substituting $g(x)=f(x)-c$ in the first equation we get

$$f(3(f(x)+f(y))-2c)=3(f(x)+f(y)).$$

Therefore, $f(x) = x - 2c$ for large values of x. Thus, we can find values of x and y such that $f(x) = x - 2c$ and $f(y) = y - 2c$. Therefore, from the second equation we have

$$g(f(x) + y + g(y)) = g(x - 2c + y + y - 3c) = x + 2y - 8c$$

$$2x - g(x) + f(y) + y = 2x - x + 3c + y - 2c + y = x + 2y + c.$$

Hence, $c = 0$ and $f(x) = g(x)$ for all $x \in \mathbb{R}^+$. Furthermore, we have $f(x) = g(x) = x$ for large values of x. Now, let y be sufficiently large such that $f(y) = y$ and let x be an arbitrary positive real number in the second equation.

$$g(f(x) + y + g(y)) = g(f(x) + 2y) = f(x) + 2y$$

$$2x - g(x) + f(y) + y = 2x - x + y + y = x + 2y.$$

Therefore, $f(x) = x$ for all $x \in \mathbb{R}^+$, so we get $f(x) = g(x) = x$ for all $x \in \mathbb{R}^+$ and it's easy to check that this is indeed a solution.

□

10 . Denote by G the graph in the problem. We proceed by induction on the number of edges of the complement graph $\overline{G}$.
The base case is when G is a complete graph.
Let v_1, v_2, v_i and v_j be some four vertices of the complete graph G. By the problem's condition for the 4-cycle $v_1 v_i v_j v_2 v_1$ we get

$$N(v_i v_1) - N(v_i v_2) = N(v_j v_1) - N(v_j v_2)$$

Where by $N(vv')$ we mean the number assigned to the edge vv'. Thus, for each i, the number $N(v_i v_1) - N(v_i v_2)$ is constant. We denote by K this constant value. Now, let n_i be the number we want to assign to the vertex v_i. We must have

$$\begin{aligned} n_1 - n_2 = &\quad N(v_1 v_i) - N(v_2 v_i) = K \\ n_1 + n_2 = &\quad N(v_1 v_2). \end{aligned}$$

By solving this system of equations we get the numbers due to the vertices v_1 and v_2. Now, for each i let $n_i = N(v_i v_1) - n_1$. We claim that these numbers satisfy the assertion for complete graphs. For this, we must show that for every i and j, $n_i + n_j = N(v_i v_j)$. For $j = 2$ this is equivalent to the second relation in the above system.

For other values of j, consider 4-cycle $v_1 v_i v_j v_2 v_1$. we have

$$N(v_1 v_2) + N(v_i v_j) = N(v_1 v_i) + N(v_2 v_j).$$

Thus,

$$n_1 + n_2 + N(v_i v_j) = n_1 + n_i + n_j + n_2 \Rightarrow N(v_i v_j) = n_i + n_j.$$

This finishes the proof of the base case.

Suppose that G does not contain edge v_iv_j.
We claim that we can add edge v_iv_j and assign a number to it such that the assigned number satisfies the problem's condition for $G \cup \{v_iv_j\}$ and therefore, according to the induction hypothesis the statement holds for $G \cup \{v_iv_j\}$.
If the edge v_iv_j is not included in any even walk, we can choose an arbitrary number for $N(v_iv_j)$, and if the edge v_iv_j is included in some even walk, the assigned number is determined uniquely.
Furthermore, if the edge v_iv_j is included in two even walks C_1 and C_2, $C_1 \cup C_2 - \{v_iv_j\}$ is also an even walk which satisfies the problem's condition. Therefore, the assigned numbers to the edge v_iv_j by C_1 and C_2 are equal.

$\square$

11 . Let $a = x^2$, $b = y^2$ and $c = z^2$ for $a \geq y \geq z \geq 0$. We have

$$x^2 \leq y^2 + z^2 \leq (y+z)^2 \;\Rightarrow\; x \leq y + z,$$

so x, y and z are sides of a triangle.
Set $A_x = \sqrt{y^2 + z^2 - yz}$, $A_y = \sqrt{z^2 + x^2 - zx}$ and $A_z = \sqrt{x^2 + y^2 - xy}$. Then,

$$A_x \leq A_y \;\Leftrightarrow\; y^2 + z^2 - yz \leq x^2 + z^2 - xz \;\Leftrightarrow\; (x-y)(x+y-z) \geq 0$$

$$A_y \leq A_z \;\Leftrightarrow\; x^2 + z^2 - xz \leq x^2 + y^2 - xy \;\Leftrightarrow\; (y-z)(y+z-x) \geq 0.$$

Hence, $A_x \leq A_y \leq A_z$.
Therefore,

$$xA_z + yA_y \geq yA_z + xA_y \;\Leftrightarrow\; (A_z - A_y)(x-y) \geq 0$$

$$yA_y + zA_x \geq zA_y + yA_x \;\Leftrightarrow\; (A_y - A_x)(y-z) \geq 0.$$

For the main problem we have

$$2(xA_z + yA_y + zA_x) = (xA_z + yA_y) + (yA_y + zA_x) + xA_z + zA_x \geq$$

$$(yA_z + xA_y) + (zA_y + yA_x) + xA_z + zA_x = (y+z)A_x + (x+z)A_y + (x+y)A_z.$$

By *Cauchy-Schwarz Inequality* we have

$$(x+y)A_z = (x+y)\sqrt{x^2 + y^2 - xy} = \sqrt{(x+y)(x^3+y^3)} \geq x^2 + y^2,$$

and similar inequalities hold for $(x+z)A_y$ and $(y+z)A_x$. Finally, summing these three inequlities implies the assersion.

$\square$

12 .

Lemma. *Let ABD be a triangle with circumcircle ω. Suppose that ω' is a circle tangent to sides AB and AD at E and F, respectively, and tangent to ω at T. M and N are midpoints of shorter arcs $\overset{\frown}{AB}$ and $\overset{\frown}{AD}$, respectively. Then tangents to ω at A and T, and line MN are concurrent.*

Proof of lemma.

$$\angle NAF = \frac{1}{2}\overset{\frown}{ND} = \frac{1}{2}\overset{\frown}{NA} = \angle ATN,$$

so triangles TAN and AFN are similar, so $\frac{NF}{NA} = \frac{NA}{NT}$, and consequently $NA^2 = NF.NT$. In a similar way we can show that $MA^2 = ME.MT$. Thus,

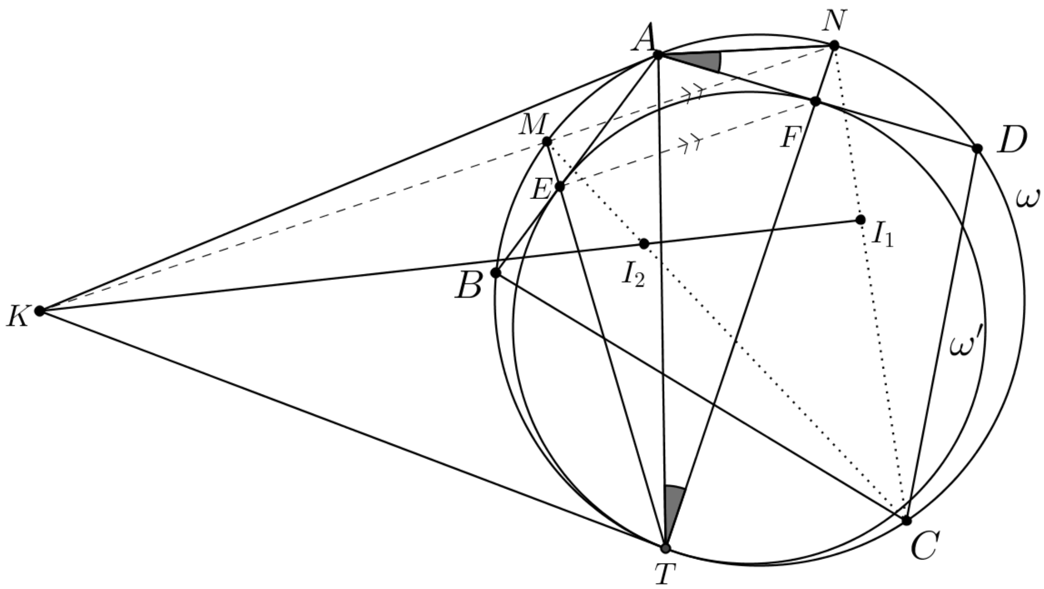
$$\left(\frac{MA}{AN}\right)^2 = \frac{ME.MT}{NF.NT}. \quad (*)$$

T is the external homethetic center of ω and ω'. This homothety maps E and F to M and N, respectively, so $EF||MN$ and by *Thales' Theorem* $\frac{ME}{MF} = \frac{MT}{NT}$. This and $(*)$ imply

$$\left(\frac{MA}{NA}\right)^2 = \frac{ME}{NF}.\frac{MT}{NT} = \left(\frac{MT}{NT}\right)^2 \Rightarrow \frac{MA}{NA} = \frac{MT}{NT}.$$

Therefore, M is the harmonic conjugate of N with respect to A and T, and hence tangents to ω at A and T meet on the line MN.

$\square$



We keep using the notations introduced in the lemma. We showed that K lies on the line MN and now we want to show that K lies on the line I_1I_2. We have

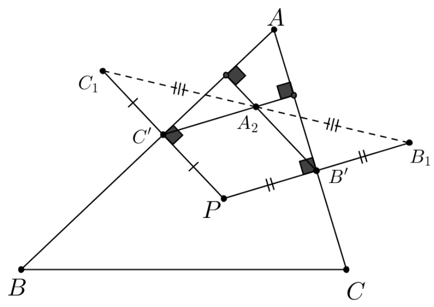
$$\frac{KM}{KN} = \frac{AM.\sin(\angle MAK)}{AN.\sin(\angle NAK)} = \frac{AM}{AN}.\frac{\sin(\angle MCA)}{\sin(\angle NCA)} =$$

$$\frac{MI_2}{NI_1}\cdot\frac{\sin(\angle MCA)}{r_2}\cdot\frac{r_1}{\sin(\angle NCA)}=\frac{MI_2}{NI_1}\cdot\frac{CI_1}{CI_2}.$$

As a result, $\frac{MK}{KN}\cdot\frac{NI_1}{I_1C}\cdot\frac{CI_2}{I_2M}=1$ and by *Menelaus' Theorem*, K, I_1 and I_2 are collinear.

□

13 . Let A_1, B_1 and C_1 be reflections of P with respect to BC, AC and AB, respectively. Moreover, we denote by A_2 and A' the midpoints of B_1C_1 and PA_1, respectively. Points B_2, C_2, B' and C' are defined similarly. Note that $A_2C' \parallel PB_1$ and $A_2B' \parallel PC_1$, so $A_2C' \perp AC$ and $A_2B' \perp AB$. Therefore, A_2 is the orthocenter of triangle $AB'C'$.



$$\angle AB'C' \leq \angle AB'P = 90°.$$

Similarly, $\angle AC'B' \leq 90°$. Furthermore, $\angle B'AC' = \angle BAC < 90°$ because triangle ABC is acute. Therefore, triangle $AB'C'$ is acute, so its orthocenter A_2 lies inside it and therefore, inside triangle ABC. By similar arguments points B_2 and C_2 lie inside trianlgle ABC. Hence, the centroid of triangle $A_2B_2C_2$ lies inside triangle ABC, but the centroid of triangle $A_2B_2C_2$ coincides with that of triangle $A_1B_1C_1$, and this completes the proof.

□

14 . First suppose that all rectangles have sides parallel to coordinate axis. There are four types of right angles: ⌜, ⌝, ⌞ and ⌟.

Suppose that there are a number of angles ⌜ and b number of angles ⌟. Since each rectangle is determined by these two angles, we get $ab \geq n$, and by $AM-GM$ inequality we get $\left(\frac{a+b}{2}\right)^2 \geq ab \geq n$, so $a+b \geq 2\sqrt{n}$. Similarly, if there are c number of angles ⌝ and d number of angles ⌞, we can deduce that $c+d \geq 2\sqrt{n}$. Thus we have at least $4\sqrt{n}$ right angles.

If there are different directions for rectangles, we denote by $n_1, n_2, \ldots$ and n_k the number of rectangles in each direction. Since $n=\sum_i n_i$, we get

$$4\sqrt{n} \leq 4\sqrt{n_1}+4\sqrt{n_2}+\cdots+4\sqrt{n_k} \leq \text{Number of angles}.$$

□

15 . a) It suffices to find a sequence a_n such that $a_1 \nmid a_n$ for all natural numbers $n>1$ and $i|j$ if and only if $a_i|a_j$ for all natural numbers $i,j \geq 2$. Since $(2^m-1, 2^n-1) = 2^{(m,n)}-1$, the sequence defined by $a_1=2$ and $a_n=2^n-1$ for $n>1$ has this property.

b) We prove a stronger assertion.
For each function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq n$ for $n > M$, where $M, n \in \mathbb{N}$, there exists a sequence $a_1 < a_2 < a_3 < \cdots$ such that a_n is divisible by exactly $f(n)$ terms of the sequence for sufficiently large positive integers n.
Let $a_1, a_2, \ldots, a_M$ be a strictly increasing sequence of prime numbers. Then, for $k > M$, set $a_k = [a_1, a_2, \ldots, a_{f(k)-1}]q_k$ where q_k is a new prime number specifically for a_k. Obviously, we can choose q_k large enough so that a_k became greater than a_{k-1}.
By this construction a_k is divisible by $a_1, a_2, \ldots, a_{f(k)-1}$ and itself which are $f(k)$ numbers. Furthermore, a_k is not divisible by a_j for some other j, because each a_j has its specific prime number q_j which has not appeared in the definition of a_k.

□

16 . Let $A = \{f(x) - f(y) | x, y \in \mathbb{Z}\}$. By the problem's condition we know $3 \in A$, and our goal is to show that $1 \in A$.
Obviously, A is closed under multiplication by -1 $(-(f(x) - f(y)) = f(y) - f(x))$. We claim that A is closed under summation.
Suppose that $\alpha, \beta \in A$, so there are some integers m_1, m_2, n_1 and n_2 such that $\alpha = f(m_1) - f(m_2)$ and $\beta = f(n_1) - f(n_2)$. We have

$$f(m_1) + f(n_1) + f(f(m_1^2 + n_1^2)) = 1$$

$$f(m_2) + f(n_2) + f(f(m_2^2 + n_2^2)) = 1.$$

Subtracting these two equalities gives

$$\alpha + \beta = f(f(m_2^2 + n_2^2)) - f(f(m_1^2 + n_1^2)) \in A$$

It is a well-known fact that any subset of $\mathbb{Z}$ closed under summation and multiplication by -1 is $k\mathbb{Z}$ for some positive integer k. Since $3 \in A$, $k|3$, so $k = 1$ or $k = 3$. If $k = 3$, then for each $x, y \in \mathbb{Z}$, $f(x) - f(y)$ is divisible by 3 and thus all numbers in the form $f(x)$ have the same remainder modulo 3. We denote this common remainder by r. Hence,

$$1 \equiv f(m) + f(n) + f(f(m^2 + n^2)) \equiv r + r + r \equiv 3r \equiv 0 \pmod 3,$$

which is a contradiction. Therefore, $k = 1$, so $A = \mathbb{Z}$. Thus, $1 \in A$, so there exist integers c and d such that $f(c) - f(d) = 1$.

□

17 . Let M, N and P be the midpoints of segments AB, AC and AD, respectively. Since M and N are the centers of circles with diameters AB and AC, respectively, and AD is the bisector of $\angle BAC$, we have

$$\frac{MP}{NP} = \frac{BD}{CD} = \frac{AB}{AC} = \frac{AM}{AN} = \frac{MX}{NY}.$$

Hence, $\angle PXM + \angle PYN = 180°$, or $\angle PXM = \angle PYN$. However, $\angle PXM < \angle BXA = 90°$, and $\angle PYN < \angle AYC = 90°$; therefore, the first case cannot hold, so $\angle PXM =$

$\angle PYN$ (1). Furthermore, $\angle MPX = \angle NPY$ (2). By (1) and (2) we deduce that triangles PXM and PYN are similar, So

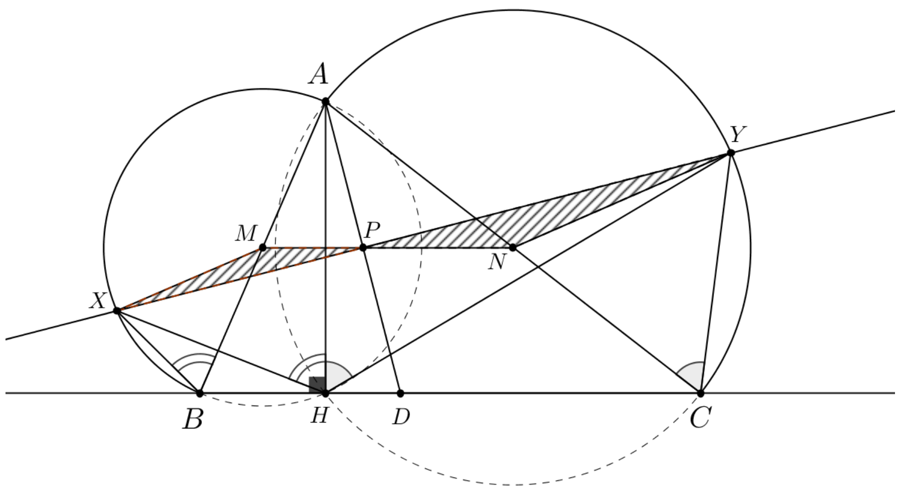
$$\angle PMX = \angle PNY \Rightarrow 180^\circ - \angle B + 180^\circ - 2\angle ABX = \angle C + 2\angle ACY$$

$$\Rightarrow \angle ABX + \angle ACY = 180^\circ - \angle B - \angle C + 90^\circ - \angle ABX + 90^\circ - \angle ACY$$

$$\Rightarrow \angle ABX + \angle ACY = \angle BAC + \angle XAB + \angle CAY \quad (3).$$

Since $\angle AHB = \angle AHC = 90^\circ$, H lies on the circumcircles of triangles AXB and AYC, so $\angle ABX = \angle AHX$ (4) and $\angle ACY = \angle AHY$ (5). Thus,

$$\angle XHY = \angle XHA + \angle AHY = \angle ABX + \angle ACY = \angle YAX \quad \text{By } 3, 4, 5.$$



Since XY is the perpendicular bisector of segment AD, we get $\angle XAY = \angle XDY$. Therefore, $\angle YHX = \angle YDX$, so the quadrilateral $XHDY$ is cyclic.

□

18 . Suppose that the room in the problem is the square $[0, n] \times [0, n]$ in the Cartesian plane. Furthermore, throughout the solution we use these definitions:

- A lattice point: A point in the plane with integer coordinates.
- A lattice square: A square with lattice points as vertices and side lengths one.
- A lattice edge: A side of a lattice square.
- A lattice triangle: A right isosceles triangle of side lenght 1 and lattice points as vertices.

- A small triangle: A right isosceles triangle of side lenght $\frac{\sqrt{2}}{2}$ and a lattice edge as its hypotenuse.

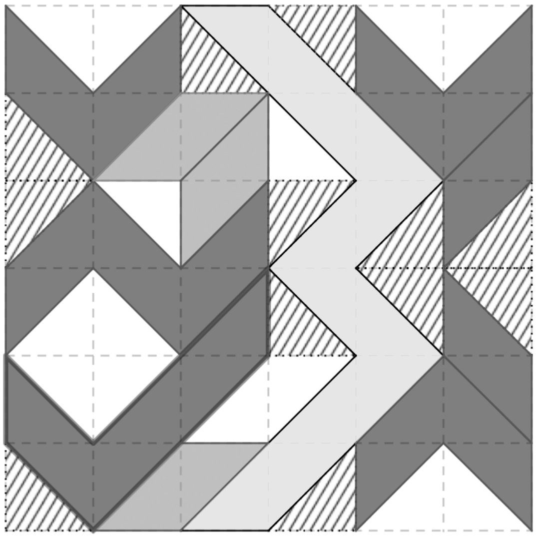
The boundary of each tile contains two lattice edges and two diameters of two lattice squares, so we have two types of tiles. By these definitions the following lemma is obvious.

Lemma. *Suppose that e is a lattice edge of the lattice square S. Furthermore, assume that e is not included in the boundary of any tile that lies in the same side of e as S. By these conditions, there exist a lattice triangle and a small triangle containing e which is contained in S that are not covered by any tiles.* □

Consider a graph with lattice edges as it's vertices. Two vertices are connected if and only if their corresponding edges are on the boundary of the same tile. In this graph the degree of each vertex is at most two. Furthermore, two connected vertices correspond to parallel edges, and if the degree of a vertex corresponding to a horizontal (vertical) edge is 2, one of the edges connected to this edge in the graph is above (right of) it and the other lies below (left of) it. Hence, each connected component of the graph is a path containing parallel lattice edges, and the order of vertices in this path coincides with the order of one of the coordinates of the corresponding edges (x-coordinate for vertical edges and y-coordinate for horizontal edges).
Thus, lattice edges on the line $y = n$ lie on different paths. We denote these paths by $U_1, U_2, \ldots, U_n$. Note that some of U_i's might contain only one vertex. The bottommost lattice edges of these paths are different. Let SU_i be the lattice square below the bottommost edge of U_i. If SU_i lies above the line $y = 0$, according to the lemma one of the lattice triangles in SU_i is not covered. We denote this triangle by TU_i. Furthermore, the upper small triangle in SU_i is not covered. We denote this small triangle by QU_i. In the same manner, we denote by $D_1, D_2, \ldots$ and D_n the paths containing lattice edges on the line $y = 0$, by $L_1, L_2, \ldots$ and L_n the paths containing the edges on the line $x = 0$, and finally by $R_1, R_2, \ldots$ and R_n the paths containing lattice edges on the line $x = n$. Triangles TD_i, TL_i, TR_i, QD_i, QL_i and QR_i are defined similarly. We have three cases:

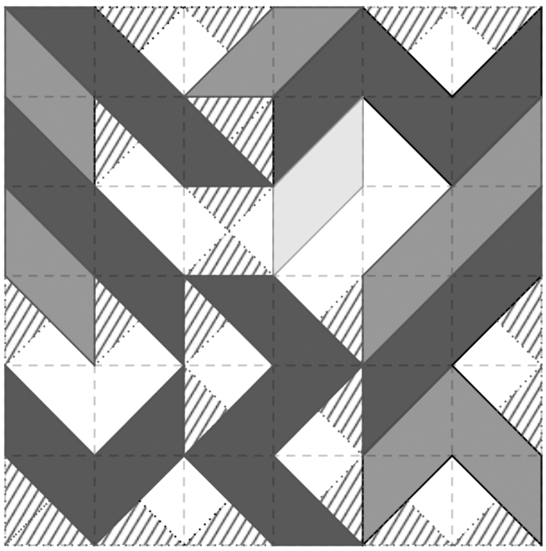
Case1. Suppose that there exists a path like P containing one lattice edge of the lines $x = 0$ and $x = n$. In this case all the tiles of paths $U_1, U_2, \ldots$ and U_n are above the tiles of the path P and tiles of the paths $D_1, D_2, \ldots$ and D_n lie below the tiles of the path P. Now, $2n$ paths $D_1, D_2, \ldots$ and D_n, and $U_1, U_2, \ldots$ and U_n are all distinct, so the triangles $TU_1, TU_2, \ldots$ and TU_n, and $TD_1, TD_2, \ldots$ and TD_n can be defined and are disjoint. Since these triangles are not covered, and the area of each of them is $\frac{1}{2}$, the sum of the area of these triangles is n. Hence, the assertion is proved in this case.

Case2. There exists a path P containing one lattice edge of the lines $y = 0$ and $y = n$. This case is similar to case 1.



An example for case 2.

Case3. There are not any paths of cases 1 or 2, so small triangles QD_i, QU_i, QL_i and QR_i, for $1 \leq i \leq n$, are all defined and are disjoint. Since these $4n$ triangles are not covered by any tiles and the area of each is $\frac{1}{4}$, the sum of the areas of them is n, so the proof is complete.



An example for case 3.

$\square$

The Holy Month of Ramadan and Muslim Mathematicians

This year the IMO and the month of Ramadan are coincident.The holy month of Ramadan, which is called as the month of mercy and forgiveness by Allah, is the ninth month of the lunar months. According to the Islamic regulations, Muslims should fast every day of this month from sunrise to sunset and refrain from eating and drinking, except for those who are ill or traveling.

> *O, believers! Prescribed to you is the fast, as it was prescribed to those before you, that you may guard against evil. (2-183)*
> *For a certain number of days (you have to fast) but if any one of you be sick, or on a journey, then (he shall fast a like) number of other days, and those who are not able to do it, (for being too old, or permanently sick) can be redeemed by feeding a poor. He that does good of his own accord, it is better for him. And it is better for you to fast, if you could knew it. (2-184)*

The base of the lunar months is the rotation of the moon around the Earth. Each lunar month begins when the new moon is sighted. Every twelve lunar months forms a lunar year, which is almost equal to 354 or 355 days. Islamic regulations like the law of fasting and also the important Islamic ceremonies like the Hadj and the religious fetes are based on the lunar months, which explains why determining the lunar month duration has been so important among the Muslims. The need for sighting the moon crescent along with the need for orientation specially for determining the direction of Kiblah (which is the Ka'ba in the city of Mecca) has caused the progress of Astronomy in the Islamic countries.

> "The prediction of the visibility of the moon crescent has been one of the most serious problems Muslims have faced, because the moon in its rotation around the Earth is located near the sun from the view point of an observer on Earth, at the beginning and the ending of the lunar month. Therefore the crescent sighting is only possible in a short period of time near sunset. Important factors in sighting the crescent are the location of the moon and the sun relative to the horizon of the observer, the pause of the moon after the sunset, the location of the moon in its orbit relative to the Earth, the brightness of the crescent and the weather. Muslim astronomers knew that determining the probability of sighting the crescent requires solving complicated spherical astronomy problems, so they could obtain the relative positions of the sun and the moon on the horizon of the observer. The method used was in a way with which the observer, using his calculations, knew the location whe-

re he was meant to search for the thin and dim crescent; therefore observers tried to sight the moon crescent in the sunset of the 29^{th} day of the lunarmonth in a serene and an unhindered horizon, and they would repeat their sightings the night after in case their sightings were unsuccessful. One of the simplest criteria in sighting the crescent has been brought up by Kharazmi (Mohammad ibn Mūsā al-Khwārizmī), the Iranian mathematician and astronomer. The basis of this criterion was the calculation of the time difference between the moonset and the sunset at the local horizon. If this difference reaches 12 degrees (equal to 48 minutes of time) the moon crescent will be visible. As time passed, Muslim astronomers obtained more complicated and precise criteria for the prediction of the visibility of the crescent."

Translation Resource: *Astronomy in Iran and the World of Islam*(In Persian), Ali Akbar Velayati, pp 237-241.