FEURBACH's THEOREM

A NEW PURELY SYNTHETIC PROOF

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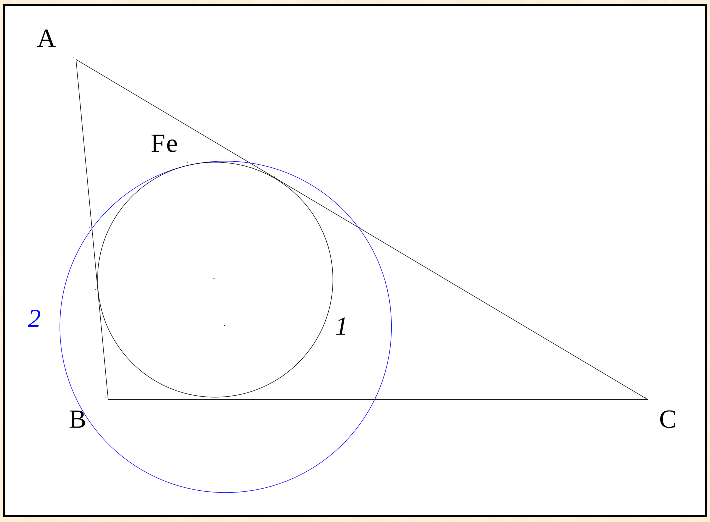

Abstract. We present a new purely synthetic proof of the Feuerbach's theorem, and a brief biographical note on Karl Feuerbach.
The figures are all in general position and all cited theorems can all be demonstrated synthetically.

Summary

A. A short biography of Karl Feuerbach	2
B. Four lemmas	3
1. Lemma **1**	
Example **1**	
Example **2**	
2. Lemma **2**	
3. Lemma **3**	
4. Lemma **4**	
C. Proof of Feuerbach's theorem	7

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1 St-Denis, Île de la Réunion (France), le 25/11/2010.
2 El teorema de Feuerbach : una demonstration puramente sintética, *Revistaoim* **26** (2006) ; http://www.oei.es/oim/revistaoim/index.html
3 http://geometry.ru/articles.htm

A. A SHORT BIOGRAPHY

OF

KARL FEUERBACH


[4]

Karl Feuerbach, the brother of the famous philosopher Ludwig Feuerbach, was born on the 30th of May 1800 in Jena (Germany). Son of the jurist Paul Feuerbach and of Eva Troster, he was a brilliant student at the Universities of Erlangen and Freiburg, and received his doctorate at the age of 22 years. He started teaching mathematics at the Gymnasium of Erlangen in 1822 but he continued to frequent a circle of his ancient fellow' students known for their debauchery and debts. In the same year, he published in Nürnberg, a small book of 62 pages with a long and confusing title[5] in which he presented his discovery about the most

beautiful theorem of elementary geometry ever seen since the time of Euclid [6]

on page 38,

the incircle is tangent to the nine-points circle

which he proved analytically.

In 1824, he was arrested with 19 students and imprisoned for one year in München (Bavaria, Germany) because of his political opinions.

After that he became very depressed as he felt responsible for the entire situation. In order to save his fellows, he tried an attempt on his life once by opening his veins and once more by jumping from a window. After his liberation, he returned to his family for a while and finally began teaching again in Hof, thanks to the king's intervention for him, but a new depression made him leave this place as well. In 1828, after health amelioration, he took up his teaching job for the third time in Erlangen until the day he drew a sword in his classroom threatening to cut of the heads of those pupils who could not solve the equation written on the blackboard. From that day, he had to give up teaching permanently and he decided to live a cloistered life. During six years, he let himself go, growing his hair, beard and nails. In his loneliness, he contemplated the painting of his nephew Anselm Feuerbach. This impulsive and perturbed teacher died in Erlangen on the 12th of March 1834.

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4 The MacTutor History of Mathematics archive ; http://www-groups.dcs.st-and.ac.uk/~history/BiogIndex.html

5 Feuerbach K., *Eigenschaften einiger merkwurdigen Punkte des geradlinigen Dreiecks, und mehrerer durch sie bestimmten Linien und Figuren.*
Johnson R. A., *Advanced Euclidean Geometry*, Dover, New York (1960) 200-205.

6 Coolidge, The Heroic Age of Geometry, *Bulletin of the American Mathematical Society* **35**, 1229.

B. FOUR LEMMAS

Auguste Miquel was already known in 1836 when he still was a pupil at the Barbet Institute in Paris thanks to his publishing in the short-lived mathematical journal *Le Géomètre*. In this journal, founded by Guillard, he presented some proofs of Steiner's theorems which had not been proved up to this day. In 1838, in the *Journal de Liouville*, Auguste Miquel [7] published the *Six circles* theorem.

1. Lemma 1.

The circles that have for chords the sides of a cyclic quadrilateral
cut again in
four concyclics points.

We point out that Isaac Moisevitch Yaglom [8] considered that

this quite elegant theorem doesn't look particularly fertile ;

however, he added that

the consequences of this simple theorem,
as we find often in geometry,
could be qualified as remarkable without exaggeration.

From the foregoing lemma, we derive this particular case :

the circles which have for diameters the sides of a cyclic quadrilateral,
cut again in
four concyclics points

in other words,

the four projections of the vertices of a cyclic quadrilateral upon the two diagonals,
are concyclics.

We may now study two illustrations of this particular case, in the geometry of the triangle.

EXAMPLE 1.

The German historian Max Simon [9] attributes the following result to the Artillery lieutenant Urbain Victor Calabre and to the professor Raphaël Malloizel [10] of Sainte.-Barbe School in Paris :

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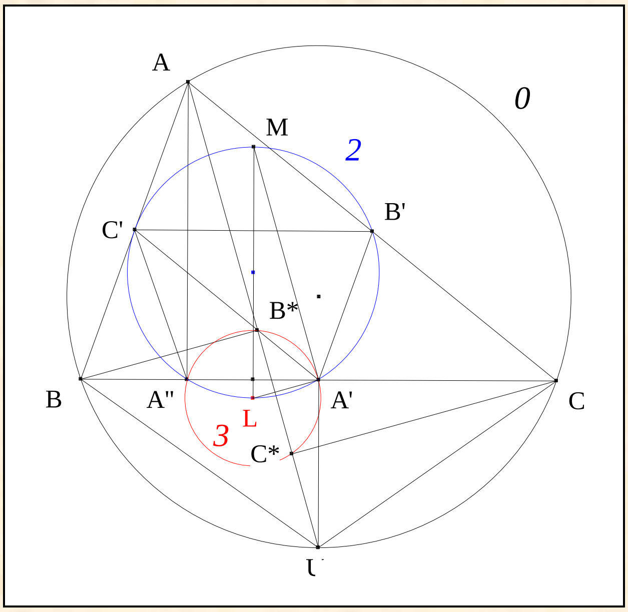
[7] Miquel A., Théorème sur les intersections des cercles et des sphères, *Journal de Liouville*, Sér. I, 3 (1838) 517-522 ;
F. G.-M., Théorème 142, *Exercices de Géométrie* 6-ième édition, Rééditions Jacques Gabay, Paris (1991) 298 :
Ayme J.-L., Du théorème de Reim au théorème des six cercles, G.G.G. vol. **2** ; http://perso.orange.fr/jl.ayme

[8] Yaglom I. M., *Géométrie des nombres complexes*, Éditions Mir, Moscou (1973) 35.

[9] Simon M., *Über die Entwicklung der Elementar Geometrie im XIX-Jahrhundert*, Leipzig, Teubner (1906) 128.

[10] Malloizel R., *Journal de Mathématiques Elémentaire* de Bourget (1878) 97 ;
Lalesco T., *La Géométrie du Triangle*, Rééditions Jacques Gabay, Paris (1987) 7 ;
Problem #11006, *Amer. Math. Montly* **110** (2003) 340 ;
Ayme J.-L., Des six cercles de Miquel aux cercles de Morel, Calabre Malloizel, G.G.G. vol. **2** ; http://perso.orange.fr/jl.ayme



the feet B* *and* C* *of the vertices* B *and* C *on the* A-*bisector of a triangle* ABC,
the midpoint A' *of* BC *and the foot* A" *of the* A-*altitude,*
are four concyclics points ;
the center of this circle
is
the midpoint L *of the arc* A'A" *of the nine-point circle which doesn't contain the midpoint of* AB.

The first part is a direct application of lemma **1**.

Proof

Let	*0*	be the circumcircle of ABC,
	U	be the point of intersection of the A-bisector with *0*,
	B', C'	be the midpoints of the sides AC, AB,
	2	be the nine-point circle passing through A', B', C', A"
	3	be the circle passing through the points A', B*, A",
and	L, M	be the points of intersection of the perpendicular bisector of the chord A'A" with *2*.

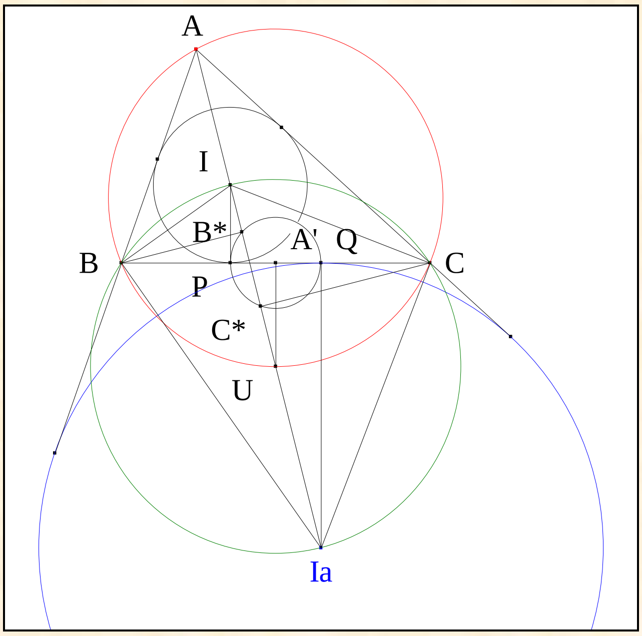
The line B'C' joining the midpoints B' and C' of the sides AC and AB of ABC is parallel to BC.
According to the definition of the nine-point circle, the quadrilateral A"A'B'C' is a cyclic trapezoid and is isosceles.
It follows that LM is the perpendicular bisector of B'C' and hence a diameter of *2*.
Therefore, M is the midpoint of the arc B'C', and A'M is the bisector of <C'A'B'.
Since the quadrilateral A'B'AC' is a parallelogram, the bisectors A'M and AU of the two opposite angles <A' and <A, are parallel.
Since the triangle A'ML is inscribed in a half-circle, A'M ⊥ A'L.
Therefore, A'L ⊥ AP , A'L // BB*, and A'L // CC*.
Furthermore, A' being the midpoint of BC, A'L is the perpendicular bisector of B*C*.

Conclusion

LM being the perpendicular bisector of A'A", and B*, C*, A' and A" being four concyclics points,
L is the center of *3*.

EXAMPLE 2.

The American geometer Nathan Altshiller-Court[11] attributes the following results to Eugène Catalan :



(a) *If* I *is the incenter and* Ia *the* A*-excenter of a triangle* ABC,
the circle which has for diameter IIa *passes through the vertices* B *and* C*, and has center* U.

(b) *The points of contact* P *and* Q *of the side* BC *of* ABC *with the incircle and the* A*-excircle,*
are two isotomic points.[12]

By applying the particular case of the *Six circles* theorem to the cyclic quadrilateral IBIaC,
the points P, B*, Q and C* lie on a circle which has for center A'.

In 1813, the French geometer Louis Gaultier, during his studies at Ecole Polytechnique in Paris, wrote a memoir entitled *Les contacts des cercles* [13] in which he pointed out a remarkable property of the radical axis, i.e. the common chord of two intersecting circles:

2. Lemma 2.

(a) *a circle orthogonal to two given circles has its center on the radical axis of the two given circles.*

(b) *If a circle has its center on the radical axis of two circles and is orthogonal to one of them,*
it is orthogonal to the other circle also.

At the beginning of the 20th century, the Christian Brother Gabriel-Marie (civil name, Edmond Brunhes) presented the Reim's result [14] in his *Exercices de Géométrie*, result from which I present a converse theorem :

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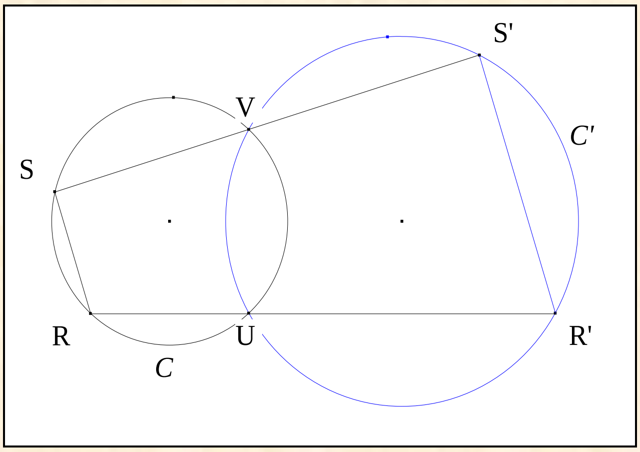
11 Altshiller-Court N., Theorem **453**, *College Geometry*, Richmond (1923) 75-77.
Catalan E., Théorème **21**, *Théorèmes et problèmes de Géométrie Élémentaire*, Dunod, Paris (1879) 46.

12 Gaultier L. (de Tours), Les contacts des cercles, *Journal de l'École Polytechnique Cahier* **16** (1812) 124- 214.
Altshiller-Court N., Theorem **453**, *College Geometry*, Richmond (1923) 205.

13 See reference 10.

14 F. G.-M., Théorème **124**, *Exercices de Géométrie*, sixième édition (1920), Rééditions Jacques Gabay, paris (1991) 283.

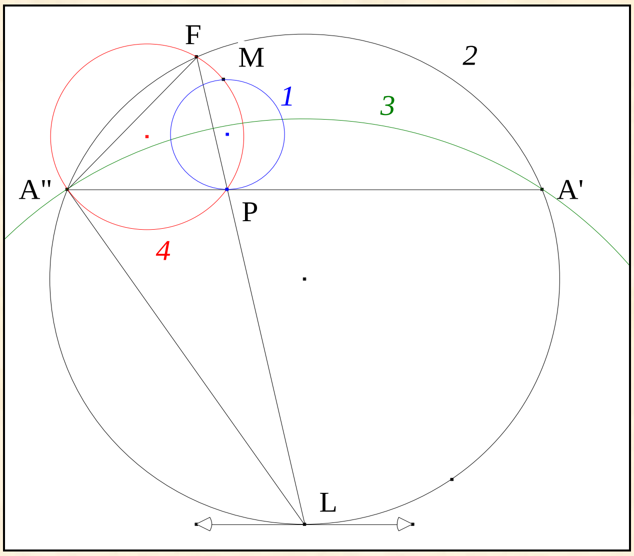


3. Lemma 3.

let C, C' be two intersecting circles, and U, V *their two points of intersection.*
Let any line passing through U, *cut C and C' at* R *and* R' *respectively. Draw parallel lines through* R *and* R' *to cut their respective circles at* S *and* S'. *Then* S, V, *and* S' *are collinear.*

If the points S and V coincide, then the line SVS' is the tangent to *C* at V.

At the same time, the *Leybourn's Mathematical repository* [15] proposed a theorem from which I present another converse theorem:



4. Lemma 4.

let 2 be a circle, A"A' *one of its horizontal chords, and* L *its south pole. Let 3 be the circle with center at* L *and passing through* A"*and* A'. *Let* P *be a point on* A"A', *and let* F *be the second point of intersection of the line* LP *with 2. Let 1 be a north-circle tangent to* A"A' *at* P *and orthogonal to 3. Then 1 is tangent to 2 at* F.

Proof.

Let *4* be the circle passing through the points A", P, and F, and M the second point of intersection of *4* and *1*. According to lemma 3 applied to the circles *2* and *4*, since the line PA" is parallel to the tangent to *2* at L, line A"L is tangent to *4* at A".

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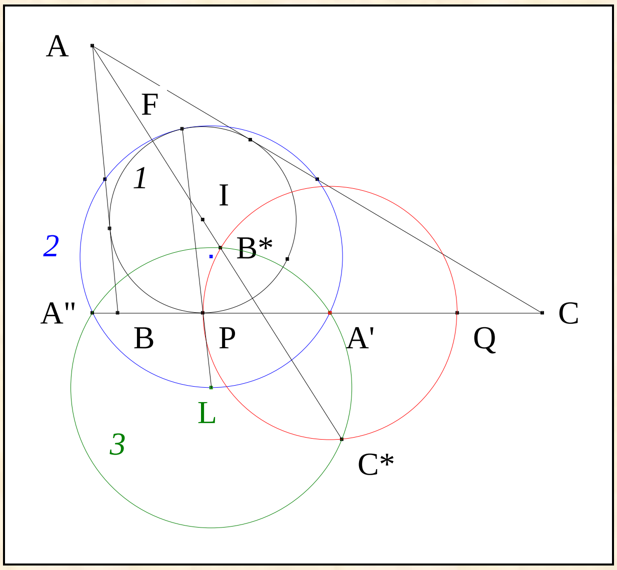
[15] *Leybourn's Mathematical repository* (New series) **6** tome I, p. 209.
Shirali Shailesh, On the generalized Ptolemy theorem, *Crux Mathematicorum*, **2**, vol. 22 (1996) 48-53.

Consequently, the tangent to *3* at A" is a diameter of *4*, and *4* is orthogonal to *3*.
According to the lemma **2-a**, *3* being orthogonal to *1* and *4*, the center L of *3* lies on their radical axis PM.
Thus, M coincides with F, i.e. *1* and *2* pass through F.

Conclusion

the tangents to *1* at P and to *2* at L being parallel, *1* is tangent to *2* at F.

C. PROOF OF FEUERBACH's THEOREM



Let ABC *be a triangle, 1 its incircle, and 2 its nine-point circle. Then 1 is tangent to 2.*

Proof.

The notations are always the same.
According to lemma **1**-example **2**, the points P, B*, Q, and C* lie on a circle which, by definition, is orthogonal to *1*.
According to lemma **1**-example **1** and lemma **2-b**, the points B*, C*, A', and A", lie on a circle which is orthogonal to *1*.

Conclusion

P being a point on the chord A'A" of *2*, according to the lemma **4**, *1* is tangent to *2*.
The point of tangency F named Feuerbach's point is the second point of intersection of the line LP with the circle *2*.