Adventitious Quadrangles: A Geometrical Approach
Author(s): J. F. Rigby
Source: *The Mathematical Gazette*, Vol. 62, No. 421 (Oct., 1978), pp. 183-191
Published by: Mathematical Association
Stable URL: http://www.jstor.org/stable/3616687
Accessed: 15-11-2015 10:36 UTC

Monsky deals with Tripp's other generalisation to "N-adventitious angles" —solutions of the isosceles triangle problem in which the angles are integral multiples of $180/N$ degrees for values of N other than factors of 180. He shows that, apart from kites and fans (which clearly give infinitely many solutions), there are only 8 other solutions. These, together with the $53-(2\times 13+1) = 26$ of Tripp's adventitious sets which are not kites or fans, give in all a total of 34 N-adventitious sets—17 pairs of cyclic complements— which we list here; the columns containing Monsky's solutions are indicated with an asterisk.

			*								*	*					*
N	18	18	24	30	30	30	30	30	30	36	42	42	60	60	60	60	84
a	2	2	6	2	2	2	12	12	20	4	2	12	4	4	24	24	24
b	6	7	6	7	11	12	7	8	4	13	13	8	19	23	13	17	23
c	3	5	5	3	7	7	4	4	2	5	11	5	11	7	7	13	19
θ	1	1	4	2	2	1	2	1	1	1	8	4	5	1	4	3	13
a'	2	2	6	2	2	2	12	12	20	4	2	12	4	4	24	24	24
b'	6	7	6	7	11	12	7	8	4	13	13	8	19	23	13	17	23
c'	5	6	2	5	9	11	5	7	3	12	5	4	14	22	9	14	10
θ'	3	2	1	4	4	5	3	4	2	8	2	3	8	16	6	4	4

Rigby, in his discussion of the quadrangle problem, brings together a number of unexpected geometrical relationships. It seems appropriate to give his account in full in a separate article, which follows this summary. In his §9 he makes further reference to Monsky's results.

D.A.Q.

Adventitious quadrangles: a geometrical approach

J. F. RIGBY

1. Introduction

A quadrangle has four vertices, of which no three are collinear, and six sides joining the vertices in pairs. If the angle between each pair of the six sides is an integral multiple of π/n radians, n being an integer, the quadrangle is said to be *n-adventitious* [1]. A quadrangle is *adventitious* if it is n-adventitious for some n. For example, the quadrangle $BCDE$ in Fig. 1 (the original adventitious quadrangle from which all the discussion started in [1]) is 18-adventitious. Various problems are posed in [1]; in a suitably generalised form these problems can be summarised as: find all adventitious quadrangles *and prove their existence by elementary geometry.*

To obtain a complete list of adventitious quadrangles we must use algebraic methods. These methods incidentally provide simple proofs of the various results on adventitious angles discussed in the *Gazette* [1, 2]. The author has found geometrical proofs of the existence of *nearly all* adventitious quadrangles. We can do no more here than summarise the results given in a longer paper [3], copies of which can be obtained from the author. That paper was the result of collaboration between the author, Dr P. A. B. Pleasants and Dr N. M. Stephens, of University College Cardiff. Further references to related work can be found there.

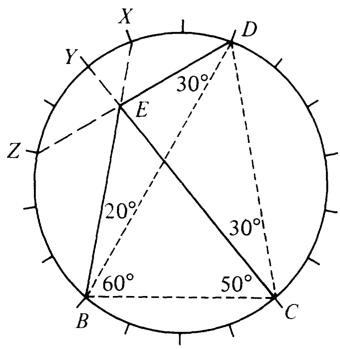

FIGURE 1.

2. Triplets of concurrent diagonals

In Fig. 1, in which we have drawn the circumcircle of BCD, it is easily seen that B, C, D, X, Y, Z are six vertices of a regular 18-gon inscribed in the circle; this is because all the angles are multiples of $10^\circ = \pi/18$ radians. Thus we have three intersecting diagonals BX, CY, DZ of a regular 18-gon. In general, the problem of finding all n-adventitious quadrangles is equivalent to the problem of finding all triple intersections of diagonals of a regular n-gon. All such triple intersections, or *triplets*, were found forty years ago by the Dutch mathematician Gerrit Bol [4], but the problem remains of proving their existence by elementary geometry. We shall think in terms of n points equally spaced round a circle, rather than of a regular n-gon.

We start by showing two geometrical methods of obtaining one triplet from another; we then apply these methods, starting with triplets of a simple type, to obtain a large number of triplets (but not quite all of them).

3. Conjugation

In Fig. 2a let $AX; BY; CZ$ be any triplet; don't worry at the moment about how the particular triplet in the figure was obtained. Let AX' be the line such that $\angle BAX' = \angle XAC$ (it is important that these two angles should have the same sign) and define BY' and CZ' similarly, as in the figure. Then X', Y', Z' are vertices of the polygon, since equal angles on a circle are

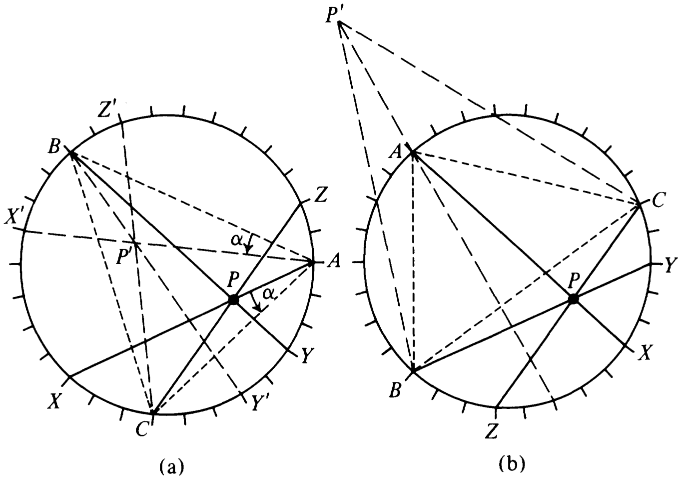

FIGURE 2.

subtended by equal arcs; moreover AX', BY', CZ' are concurrent in a point P' called the *isogonal conjugate* of P with respect to triangle ABC. The existence of this isogonal conjugate is a standard result in the geometry of triangles, and can be proved in various ways.

This process of obtaining one triplet from another is called *isogonal conjugation*, and the second triplet AX'; BY'; CZ' is an *isogonal conjugate* of the first. (The process is akin to Tripp's "cyclic complementation" in [1].) Since each diagonal has two ends, we can choose the triangle ABC in $2^3 = 8$ ways when a triplet is given. (See Fig. 2b for instance, where we start with the same triplet as in Fig. 2a.) Hence a triplet will in general have eight isogonal conjugates. Each of these in turn has eight conjugates, but surprisingly we can generate a total of only fifteen triplets at most by applying successive isogonal conjugations to a given triplet†. These triplets form a *conjugacy class*; any triplet of the class can be used as the initial triplet from which the others are generated.

4. Substitution

If a conjugacy class contains a triplet with two equal diagonals, as in Fig. 3, these diagonals will have a line of symmetry that is also a line of symmetry of the polygon. If we reflect the third diagonal in this line of symmetry, we obtain four intersecting diagonals, or five if the line of symmetry is itself a (diametral) diagonal. From these four or five diagonals

† Triplets that can be obtained from each other by rotation or reflection are regarded as being the same.

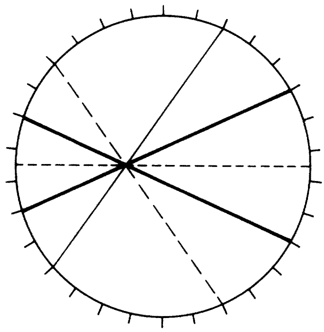

FIGURE 3.

we can select, in one or more ways, three diagonals forming a triplet that will in general belong to a different conjugacy class. This process is called *substitution*.

We may also apply the process of substitution if one diagonal of the initial triplet is a diameter, in which we can reflect.

5. *Orthic classes*

If n is even, let PQR be a triangle of vertices of the n-gon. The altitudes of PQR are seen to be diagonals of the n-gon (Fig. 4). They form an *orthic triplet* of diagonals, which generates an *orthic* (*conjugacy*) *class*. An orthic class in general consists of four orthic triplets, each counted twice, six symmetric trivial triplets (consisting of a diameter and two diagonals symmetric with respect to the diameter) and a central triplet (three diameters meeting in the centre). Note that, as in Fig. 4, if J is the orthocentre of PQR, it is also the incentre or an excentre of ABC.

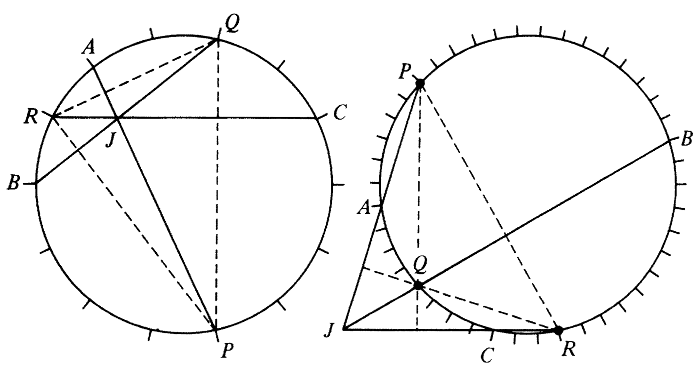

FIGURE 4.

6. *Generation of triplets*

In §5, if n is a multiple of 6 and if the arc QR is one-third of the circumference of the circle, we see that the altitudes through Q and R give equal diagonals of the n-gon. Thus we can perform a substitution to obtain a new triplet. This triplet will have isogonal conjugates, and we may be able to perform a substitution on one of these conjugates; the process can be continued. Triplets that can be obtained in this way are said to be *accessible*; their existence can be proved by geometry. Before finding out how many triplets this process yields, we must do some algebra.

7. *Algebraic representation of triplets*

If we take our circle to be the unit circle in the Argand diagram, and one vertex on the circle as the point 1, we can represent the vertices of the n-gon by the n complex nth roots of unity. Let the ends of three diagonals XY, ZT, UV be represented by $x, y; z, t; u, v$. It can be shown that the three diagonals are concurrent if and only if

$$\begin{aligned} xyz + xyt + ztu + ztv + uvx + uvy \\ - uvt - uvz - xyv - xyu - zty - ztx = 0. \end{aligned} \tag{1}$$

We have here a sum of twelve roots of unity equalling 0. Since any one of the vertices of the n-gon can be chosen as the point 1, it is only the ratios of these twelve roots of unity with which we are concerned. It is interesting and important that the same twelve roots of unity, or more precisely the same ratios, are obtained from all the fifteen triplets in a conjugacy class. To obtain all classes of triplets we have to find all solutions of (1). This was the approach of Stephens and Pleasants; Bol considered an equivalent trigonometrical equation.

8. *Isomorphism and multiplication*

We can also represent the vertices of the n-gon by integers modulo n, labelling one vertex 0 and the other vertices 1, 2, 3,... in order, ending up with $n - 1$. Suppose that n_1, n_2; n_3, n_4; n_5, n_6 are integers (mod n) representing the ends of the diagonals of a triplet, and let k be an integer coprime to n. It can be shown, using equation (1) and the theory of automorphisms of fields generated by roots of unity, that the integers kn_1, kn_2;...; kn_5, kn_6 also represent the ends of the diagonals of a triplet (Fig. 5). This triplet is said to be *isomorphic* to the original triplet, and is obtained from it by *multiplying by k*. This idea is easily extended to multiplication of a conjugacy class by k to obtain another conjugacy class; two such conjugacy classes are *isomorphic*.

When making a list of all triplets, we need list only one triplet from each conjugacy class†. Moreover, we need list only one conjugacy class from each set of isomorphic conjugacy classes. This shortens the list considerably.

† Such a list, based on Bol's work, is given in [3].

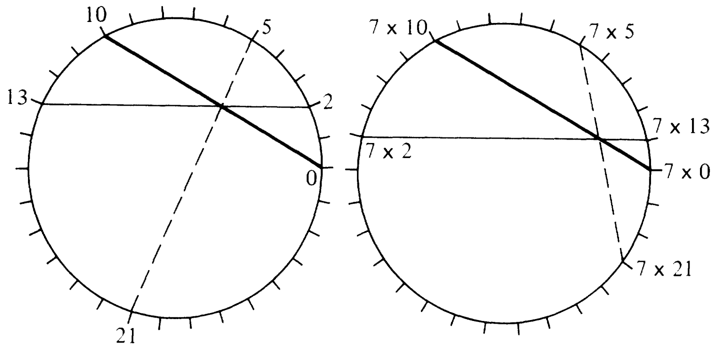

FIGURE 5.

9. Bol's list of triplets

Bol [4] showed that there are no triplets when n is odd, and only orthic classes of triplets if n is not divisible by 6. If $n = 6m$, there are three more types of class, called *general classes*, for each m, and further *sporadic classes* of triplet for certain special values of n. If we regard isomorphic classes of triplets as being the same, the number of sporadic classes for each special value of n is as follows:

value of n	30	42	60	84	90	120	210
number of sporadic classes	4	1	6	2	1	1	1

Dr Paul Monsky, of Brandeis University, Mass., has communicated to the editor the results of his recent work on this topic. His algebraic calculations repeat the work of Bol; Pleasants performed similar calculations before discovering Bol's paper (see §7). Monsky deduces that, in addition to the various types of quadrangle derived from orthic classes of triplets, there are 120 one-parameter families of general quadrangles, and 1830 sporadic quadrangles; these numbers can be deduced from various results in [3] and [4].

10. Inaccessible classes

It can be shown that all the general and sporadic classes, *with two exceptions*, can be obtained by the process described in §6; in other words they are accessible. The two exceptions are the sporadic classes when $n = 90$ and 210. One triplet from each of these classes is listed below, the ends of the diagonals being given as integers modulo n:

$n = 90$ *0, 37; 12, 38; 29, 43*

$n = 210$ *0, 60; 31, 101; 45, 162*

It is easily seen from the definition of accessibility in §6 that, if a conjugacy class is accessible, one triplet in the class must contain either two equal diagonals or a diameter. We can check that no triplet in these two sporadic classes contains two equal diagonals or a diameter. This is why the classes are not accessible; they cannot be obtained by the geometrical processes described in §6, though their existence is easily checked using equation (1) of §7. To see how this checking is carried out, let us consider $n = 210$; the reader can try $n = 90$ for himself.

If we write $\omega = \exp(2\pi i/3)$, $\xi = \exp(2\pi i/5)$, $\eta = \exp(2\pi i/7)$, so that $\omega^3 = \xi^5 = \eta^7 = 1$, the ends of the diagonals of the above sporadic 210-triplet are $1, \eta^2; -\omega\xi^3\eta^5, -\omega^2\xi^3\eta^5; -\eta^5, \xi\eta^4$.
When we substitute in the left-hand side of (1) we obtain

$$-\omega\xi^3 - \omega^2\xi^3 - \xi\eta + \xi^2 - \xi\eta^2 - \xi\eta^4$$
$$-\omega^2\xi^4 - \omega\xi^4 - \xi\eta^6 + 1 - \xi\eta^5 - \xi\eta^3.$$

To show that this expression is equal to 0, we simply use the well known identities

$$1 + \omega + \omega^2 = 1 + \xi + \xi^2 + \xi^3 + \xi^4 = 1 + \eta + \eta^2 + \eta^3 + \eta^4 + \eta^5 + \eta^6 = 0.$$

We now describe three further unsuccessful ideas for obtaining the inaccessible triplets geometrically.

11. Quadruplets, and other multiple intersections

Four concurrent diagonals are called a *quadruplet*. All quadruplets with a point of intersection inside the circle were found by Bol [4]; they are also listed in [3]. The quadruplet of Fig. 6 is obtained by combining the orthic triplet indicated by unbroken lines and the general triplet *0, 15; 3, 36; 4, 44*. We can then extract two sporadic triplets from the quadruplet. This gives a new method of obtaining sporadic triplets from orthic and general triplets, a generalisation of the method of substitution. However, it does not help us to obtain inaccessible triplets from accessible triplets, since no inaccessible triplet belongs to a quadruplet.

In addition to quadruplets, there are symmetric intersections of six

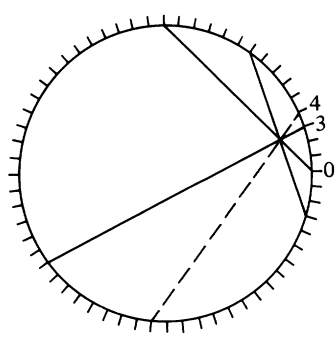

FIGURE 6.

diagonals when $n = 30$, and of seven diagonals when $n = 30$ and 60; there is also an asymmetric intersection of five diagonals when $n = 42$. All these were found by Bol and are listed in [3]. There are no internal intersections of more than seven diagonals.

12. Pascal's theorem

In Fig. 7, BC; EF; XY is an orthic triplet and AB; DE; XY is a general triplet. The hexagon $ABCDEF$ is inscribed in a circle, so by Pascal's theorem (which is proved in most books on projective geometry) the three points of intersection of opposite sides of this hexagon, namely P, Q, R, lie on a line. Hence R lies on XY and we have a triplet AF; CD; XY; this triplet is sporadic. However, it can be shown that it is not possible to put together two accessible triplets in this way in order to obtain an inaccessible triplet using Pascal's theorem.

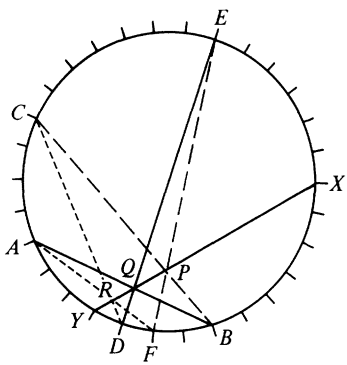

FIGURE 7.

13. Adventitious 5-points

A *5-point* consists of five points, no three being collinear, joined by ten lines. If the angle between each pair of lines is an integral multiple of π/n, the 5-point is *adventitious.*

Using the connection between triplets and adventitious quadrangles described in §2, we can define *orthic*, *general*, *sporadic*, *accessible* and *inaccessible* quadrangles. In Fig. 8, the angles are marked as multiples of $\pi/30$ radians; T is the incentre of triangle PQS, and R is an excentre of PTS. We easily show that Q is an excentre of RST, so all the angles in the figure can now be found. It turns out that P is an excentre of QRS, but the quadrangle $PQRT$ is sporadic.

What we have done here is fit together three orthic quadrangles to form an adventitious 5-point, from which we then extract the sporadic quadrangle $PQRT$. Can we fit together three accessible quadrangles to form an adven-

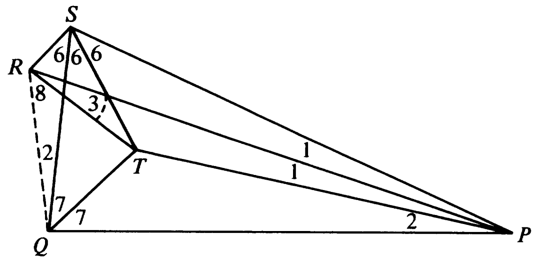

FIGURE 8.

titious 5-point and then extract an inaccessible quadrangle? The answer turns out to be "no".

A few interesting 5-points are given in [3], but no attempt has been made to produce a complete list.

14. Other geometrical methods

Various interesting geometrical methods for obtaining one adventitious quadrangle from another are described in the preceding article in this issue [5], and are illustrated in Fig. 3 of that article. These methods can be generalised in the sense that the original triangle in Fig. 3 need not be isosceles. Fig. 3(2) illustrates cyclic complementation, which is equivalent to isogonal conjugation as we remarked in §3; the other methods can be applied only to certain types of accessible quadrangle, and the resulting new quadrangle is still accessible.

15. Conclusion

Three problems (at least) remain unsolved.

(a) Prove the existence of the two inaccessible classes by elementary geometry.

(b) Make a list of all those quadruplets, with intersection point outside the circle, that are not isomorphic (in the sense of §8) to a quadruplet with internal intersection point.

(c) Find all adventitious 5-points.

References

1. C. E. Tripp, Adventitious angles, *Mathl Gaz.* **59**, 98–106 (No. 408, June 1975).
2. D. A. Quadling (editor), The adventitious angles problem: a progress report, *Mathl Gaz.* **61**, 55–58 (No. 415, March 1977).
3. J. F. Rigby, Multiple intersections of diagonals of regular polygons, and related topics, *Geom. Dedicata* (to appear).
4. G. Bol, Beantwoording van prijsvraag no. 17, *Nieuw Archf Wisk.* (2) **18**, 14–66 (1936).
5. D. A. Quadling (editor), Last words on adventitious angles, *Mathl Gaz.* **62**, 174–183 (No. 421, October 1978).

J. F. RIGBY

Department of Pure Mathematics, University College, P.O. Box 78, Cardiff CF1 1XL