Complex Geometry

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BGW-COMPLEX, OTIS*

§1 Reading

This is a bit heavy on the reading this time.

§1.1 Handouts to read

- Either of the following two:
  - Preferred: Chapter 6 of EGMO, on complex numbers.
  - Alternatively: *Bashing Geometry with Complex Numbers*, §1, §2, §3, §5. (Available from my website.)
- My blog post https://wp.me/p3jiSp-gq, on arc midpoints. This fixes a major error in my book and introduces new content.

§1.2 Degrees of freedom

Roughly, the number of **degrees of freedom** is the number of real numbers which are needed to specify the entire figure. This is hard to make precise[1], but easy to get the hang of.

Convention: In OTIS materials, we will consider figures only distinct up to translation and rotation, but *not* up to scaling. Thus for example,

- A triangle has three degrees of freedom[2]: it is uniquely determined by its side lengths. Or, it is uniquely determined by a side and two angles.
- A cyclic quadrilateral has four degrees of freedom; for example, it is uniquely determined by its side lengths.
- But a generic quadrilateral has *five* degrees of freedom. For example, it is uniquely determined by its four side lengths and the length of one of its diagonals.

[1] "Dimension of moduli space" is the way you make it precise.

[2] Fans of Cartesian and trig often prefer two degrees of freedom, because they use a different convention.

It's intuitively true (but hard to prove even if defined) that the number of degrees of freedom is well-defined: if you try to count the degrees of freedom in two different ways, you will get the same answer. This makes it a valuable invariant for setting up complex numbers.

An important consideration when setting up variables in a problem is the number of degrees of freedom. Ideally, you want the number of variables to match the number of degrees of freedom. The rules are:

- Every variable you set on the unit circle encodes one degree of freedom; so a typical abc triangle setup has three degrees of freedom in variables.
- Every variable you set which is *not* on the unit circle will encode *two* degrees of freedom: if x is a point, then your calculation will involve x and $\overline{x}$, so it effectively will have two variables in it. This makes sense: it takes two real numbers to specify a random point in $\mathbb{R}^2$.

It is possible to have your variables with more degrees of freedom than the problem allows. However, this means that your variables will not be "independent" of each other, and will be linked by some relation.

More precisely, if your variables encode N degrees of freedom, but the problem has only M degrees of freedom, there will be $N-M$ relations between them. It's nice to have $N=M$ (equivalently $N-M=0$) but sometimes nothing can be done about it.

§2 Lecture notes

§2.1 Formulas you must know

- Foot from any point Z to a chord $\overline{AB}$ of the unit circle is $\frac{1}{2}(a+b+z-ab\overline{z})$.
- For AB a chord of the unit circle, the equation for $P \in \overline{AB}$ is $p + ab\overline{p} = a+b$.
- Intersection of two chords AB and CD of the unit circle is $\frac{ab(c+d)-cd(a+b)}{ab-cd}$. (This can be applied even if $A=B$ or $C=D$.)

  In particular, if a and b are on the unit circle the two tangents meet at $\frac{2ab}{a+b}$. (I call this an *ice-cream cone* for obvious reasons; this formula is very useful.)

The first implies the second (foot equals itself!), the second implies the third (by Cramer's rule, say).

§2.2 On the importance of setting up

The most important part of any complex solution is the part that comes BEFORE the calculation starts. I cannot emphasize this enough. This is not really a "bashing" unit.

Complex numbers has a reputation as a brute-force method where you grind through long calculations, and succeed if you have enough fortitude. To me, this is exactly backwards. Rather, your success is already mostly determined before you begin the main calculation — the fate is set by how you chose to set up the calculation. In that sense, even though we *finish* the problem by algebraic work, the bulk of the solving is still done with your synthetic skills.

For each problem, you should decide first on the following:

- The choice of unit circle (if any)

- The choice of free variables
- By which method you will compute the rest of the points.

I call this the *setup* of the problem.

The most important thing I have to say is that the *amount of time taken is a function of the setup*, rather than a function of the problem. I attribute my personal success with complex numbers to my ability to simply find better setups than most contestants. (Here is a story: in MOP 2019 I found a setup for 2018 G7 that took me half a page to execute. I then had the misfortune of having to grade the contestant papers; the average complex numbers solution was eight pages long. This means that the students spent 16 times as effort because they picked their setups poorly.)

This means a couple things about using complex numbers.

(i) You need to learn to be able to estimate how long a calculation will take as a function of your setup. This is actually not a hard skill to learn; it comes with experience.

(ii) Don't start actually doing the calculation until after you have a setup and know it will finish in reasonable time.

(iii) Always be on the look-out for a better setup than the one you already have. Even merely rephrasing the problem might lead you to save a factor of 2 in the calculation; and a few synthetic observations might save you factor of 5 as well.

Therefore it is vital that you **do not turn off your brain**. As I said complex numbers has a reputation as mindless, and I think this stereotype is completely wrong. The dominating term in the success of this method is actually the ability to pick a setup well, and this is a skill that requires much more geometric intuition than it gets credit for.

§2.3 Advice for actually doing calculations (execution)

With all that said, some specific advice about actually doing the calculations:

0. You should basically never try to solve a quadratic.
1. Always keep expressions factored unless expanding is necessary to proceed.
2. When checking an expression E is equal to its conjugate or negative conjugate, always compute and factor the entire expression E first before trying to take the conjugate.
3. When comparing two polynomial expressions, factor rather than expand. The irreducible decompositions should always match, factor by factor. (Analogy in $\mathbb{Z}$: $144 \cdot 245 = 420 \cdot 84$. Multivariable polynomials also a UFD.)
4. Always do the calculations on official solution paper, neatly the first time, and **turn everything in**. Do not do the calculation once badly on scratch paper, then submit an executive summary to the graders — the graders should have access to your entire calculation. (I know examples of students who did the former and lost nearly all points.)
5. *Homogenity*: in many situations, the formulas you use will preserve the degrees of points as one, provided you treat $\overline{z}$ as having degree -1. For example, consider the complex foot formula:
$$\frac{a+b+z-ab\overline{z}}{2}.$$

All terms have degree $+1$ (the last term is $(+1)+(+1)+(-1)=1$). In general, you will often find that any expression for a "point" will have degree exactly 1. Similarly, ratios will often need to have degree exactly zero.

This gives a good way to check your work: if you find that your expressions are not the right degree (especially if e.g. you are adding two expressions of different degrees), then make sure that there is a reason for this.

This advice doesn't apply if, e.g. you decide that to set e.g. a point to have coordinate $+1$.

6. Consider **turning your paper sideways** (this was suggested to me by Michael Ma). This is useful for increasing the amount of terms you can have on a line.

§2.4 Example problems

Example 2.1 (Newton)

In quadrilateral $ABCD$ with incircle ω, prove that the midpoint of $\overline{AC}$ and the midpoint of $\overline{BD}$ are collinear with the center of ω.

Walkthrough. We give the setup to you in this walkthrough. We set ω as a unit circle and denote by $WXYZ$ the tangency points. Let w, x, y, z be corresponding "free" variables on ω.

(a) Assume $A = \overline{WW} \cap \overline{ZZ}$. Find the coordinates of a.

(b) Write down the coordinates of b, c, d.

(c) Prove that $\frac{\frac{1}{2}(a+c)}{\frac{1}{2}(b+d)} = \frac{(w+x)(y+z)}{(z+w)(x+y)}$. (Remember to keep things factored in calculations!)

(d) Show that the quantity in (c) is a real number. As long as you keep everything factored, this should take almost no calculation at all; do not expand anything.

Example 2.2 (Simson)

Let ABC be a triangle with orthocenter H, and P a point on its circumcircle. Show that the three feet of the perpendiculars from P to the sides are collinear with the midpoint of $\overline{PH}$.

Walkthrough.

(a) Decide on the setup: a choice of unit circle and variables, and a plan for how to compute each point and establish the conclusion.

(b) Compute x. (This should involve almost no calculation; remember your formulas.)

(c) Write down y and z by symmetry.

(d) Show that x, y, z are collinear. (Either use a determinant from complex shoelace, or $\frac{z-x}{z-y}$ works too.)

(e) Show that x and y are collinear with the midpoint of $\overline{PH}$.

Example 2.3 (JMO 2011/5)

Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$. Prove that $\overline{BE}$ bisects $\overline{AC}$.

Walkthrough.

(a) Show that this problem has three degrees of freedom.

(b) Show that $d/c = a/e$ from the parallel condition.

(c) Of course, we use ω as the unit circle. Come up with a setup, using exactly three of the variables on ω as free variables. (There are multiple possible choices.)

(d) Carry out the setup.

Example 2.4 (Romania 2003)

Prove that the midpoints of the altitudes of a nondegenerate triangle are collinear if and only if the triangle is right.

Walkthrough. This problem is not difficult, but some meta-considerations in this problem give a few nice lessons. We'll use (ABC) as the unit circle.

(a) Find the coordinates of X, the midpoint of the A-altitude.

(b) Write down the coordinates of Y, Z by symmetry.

(c) Using the complex shoelace formula, write down the 3×3 determinant which is equivalent to X, Y, Z being collinear.

Now, in principle we want to show the determinant of (c) is zero iff $\triangle ABC$ is right. Here is how one can do so.

(d) What is the degree of that determinant as a polynomial in terms of a, b, c?

(e) If $\angle A = 90^\circ$, what should be true about b and c?

(f) Conclude that $b + c$ should divide the determinant.

(g) By considering a degenerate case where $B = C$ along ω, show that $b - c$ should also divide the determinant.

(h) Come up with a guess for how the determinant in (c) factors, up to constant factors.

(i) Verify that this factorization is correct (you should work out the constant factors here).

(j) Conclude.

§3 Problems

Instructions: Solve [20♣]. If you have time, solve [30♣]. Problems with red weights are mandatory.

> Dumby? What is that, Dumbo and Barbie?
>
> Zuming Feng at MOP

Page limits: For this unit, there is a *page limit* associated to some problems here, to ensure you stay "on the rails" in terms of the efficiency of your setup..

This page limit is very generous: it is always at least 3 times the length of my typeset solution (not including diagrams). Therefore, you should try to remain comfortably within the page limit.

[2♣] **Problem 1** (AIME 2012)**.** Complex numbers a, b and c are the zeros of a polynomial $P(z) = z^3 + qz + r$, and $|a|^2 + |b|^2 + |c|^2 = 250$. The points corresponding to a, b, and c in the complex plane re the vertices of a right triangle with hypotenuse h. Find h^2.

[2♣] **Problem 2.** In triangle ABC with circumcenter O, point X is the reflection of O across $\overline{BC}$. Define Y, Z similarly. Prove that AX, BY, CZ are concurrent.

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[2♣] **Problem 3** (USA TST 2014/5)**.** Let $ABCD$ be a cyclic quadrilateral, and let E, F, G, and H be the midpoints of AB, BC, CD, and DA respectively. Let W, X, Y and Z be the orthocenters of triangles AHE, BEF, CFG and DGH, respectively. Prove that the quadrilaterals $ABCD$ and $WXYZ$ have the same area.

[3♣] **Problem 4** (PUMaC Finals 2016 A3)**.** On a cyclic quadrilateral $ABCD$, let M and N denote the midpoints of $\overline{AB}$ and $\overline{CD}$. Let E be the projection of C onto $\overline{AB}$ and let F be the reflection of N over the midpoint of $\overline{DE}$. Assume F lies in the interior of quadrilateral $ABCD$. Prove that $\angle BMF = \angle CBD$.

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[3♣] **Problem 5** (PUMaC 2013)**.** Let γ and I be the incircle and incenter of triangle ABC. Let D, E, F be the tangency points of γ to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and let D' be the reflection of D about I. Assume EF intersects the tangents to γ at D and D' at points P and Q. Show that $\angle DAD' + \angle PIQ = 180^\circ$.

[3♣] **Problem 6** (China TST 2011)**.** Let Γ be the circumcircle of a triangle ABC. Assume $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ are diameters of Γ. Let P be a point inside ABC other than the circumcenter and let D, E, F be the feet from P to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Let X be the reflection of A' across D; define Y and Z similarly. Prove that $\triangle XYZ \sim \triangle ABC$.

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[3♣] **Problem 7** (China TST 2006)**.** Let H be the orthocenter of triangle ABC. Let D, E, F lie on the circumcircle of ABC such that $\overline{AD} \parallel \overline{BE} \parallel \overline{CF}$. Let S, T, U respectively denote the reflections of D, E, F across $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Prove that points S, T, U, H are concyclic.

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[3♣] **Problem 8** (JMO 2011/5)**.** Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$. Prove that $\overline{BE}$ bisects $\overline{AC}$.

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[3♣] **Problem 9** (Taiwan TST 2014)**.** In $\triangle ABC$ with incenter I, the incircle is tangent to $\overline{CA}$, $\overline{AB}$ at E, F. The reflections of E, F across I are G, H. Let Q be the intersection of $\overline{GH}$ and $\overline{BC}$, and let M be the midpoint of $\overline{BC}$. Prove that $\overline{IQ}$ and $\overline{IM}$ are perpendicular.

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[3♣] **Problem 10** (Shortlist 1998 G7)**.** Let ABC be a triangle such that $\angle ACB = 2\angle ABC$. Let D be the point on the side BC such that $CD = 2BD$. The segment AD is extended to E so that $AD = DE$. Prove that

$$\angle ECB + 180^\circ = 2\angle EBC.$$

[3♣] **Problem 11** (USA TST 2000)**.** Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to $\overline{AB}$ and $\overline{CD}$, respectively. Prove that $\overline{EF}$ is perpendicular to the line through the midpoints of $\overline{AD}$ and $\overline{BC}$.

[3♣] **Problem 12** (China 1996)**.** Triangle ABC has orthocenter H. The tangents from A to the circle with diameter $\overline{BC}$ are AP and AQ. Prove that H, P, Q are collinear.

[3♣] **Problem 13** (Math Prize for Girls 2017)**.** Give an example of an equilateral 13-gon, convex and nondegenerate, whose internal angle measures are all multiples of 20°.

[3♣] **Problem 14** (RMM 2019/2)**.** Let $ABCD$ be an isosceles trapezoid with $\overline{AB} \parallel \overline{DC}$. Let E be the midpoint of $\overline{AC}$. Denote by Γ and Ω the circumcircles of triangles ABE and CDE, respectively. The tangent to Γ at A and the tangent to Ω at D intersect at point P. Prove that $\overline{PE}$ is tangent to Ω.

Page limit: 1 page, with moderate synthetic observations.

[5♣] **Problem 15** (Schiffler)**.** Let ABC be a non-equilateral triangle with incenter I. Prove that the Euler lines of $\triangle AIB$, $\triangle BIC$, $\triangle CIA$, $\triangle ABC$ are concurrent.

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[5♣] **Problem 16** (Shortlist 2018 G4)**.** A point T is chosen inside a triangle ABC. Let A_1, B_1, C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of $\triangle A_1B_1C_1$. The lines A_1T, B_1T, C_1T meet Ω again at A_2, B_2, C_2, respectively. Prove that lines AA_2, BB_2, CC_2 meet on Ω.

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[1♣] **Mini Survey.** Answer the following optional questions.

(a) About how many hours did the problem set take?

(b) Name any problems that stood out (e.g. especially nice, instructive, boring, or unusually easy/hard for its placement).

Any other thoughts are welcome too. Examples: suggestions for new problems to add, things I could explain better in the notes, overall difficulty or usefulness of the unit.

§4 Solutions to the walkthroughs

§4.1 Solution 2.1, Newton

Set unit circle to be ω. Let W, X, Y, Z be the tangency points on ω. Let w, x, y, z be corresponding free variables on ω.

If $A = \overline{WW} \cap \overline{ZZ}$, then

$$\begin{aligned}
a &= \tfrac{2zw}{z+w} \\
b &= \tfrac{2wx}{w+x} \\
c &= \tfrac{2xy}{x+y} \\
d &= \tfrac{2yz}{y+z}. \\
\implies \frac{1}{2}(a+c) &= \frac{1}{\overline{z}+\overline{w}} + \frac{1}{\overline{x}+\overline{y}} \\
\implies \frac{1}{2}(b+d) &= \frac{1}{\overline{w}+\overline{x}} + \frac{1}{\overline{y}+\overline{z}} \\
\text{quotient} &= \frac{(w+x)(y+z)}{(z+w)(x+y)} \in \mathbb{R} \\
\overline{\text{quotient}} &= \frac{(\frac{1}{w}+\frac{1}{x})(\frac{1}{y}+\frac{1}{z})}{(\frac{1}{z}+\frac{1}{w})(\frac{1}{x}+\frac{1}{y})} \\
&= \frac{(w+x)(y+z)}{(z+w)(x+y)} \in \mathbb{R}.
\end{aligned}$$

§4.2 Solution 2.2, Simson

Here is a solution using complex numbers. Let (ABC) be the unit circle, of course. Then it would be sufficient to show that the three points

$$\begin{aligned}
x &= \frac{1}{2}(a+b+p-ab/p) \\
y &= \frac{1}{2}(a+c+p-ac/p) \\
z &= \frac{1}{2}(a+b+c+p)
\end{aligned}$$

are collinear. Note that

$$\frac{z-x}{z-y} = \frac{c+ab/p}{b+ac/p} = \frac{pc+ab}{pb+ac}$$

so

$$\left(\overline{\frac{z-x}{z-y}}\right) = \frac{\frac{1}{pc}+\frac{1}{ab}}{\frac{1}{pb}+\frac{1}{ac}} = \frac{pc+ab}{pb+ac} = \frac{z-x}{z-y}$$

hence $\frac{z-x}{z-y} \in \mathbb{R}$.

Also, the midpoint M of PH (here $h = a+b+c$) is given by

$$m = \frac{1}{2}(p+h) = \frac{1}{2}(p+a+b+c).$$

We want to check that X, Y, M are collinear, equivalently $\frac{m-x}{m-y} \in \mathbb{R}$. This is also routine:

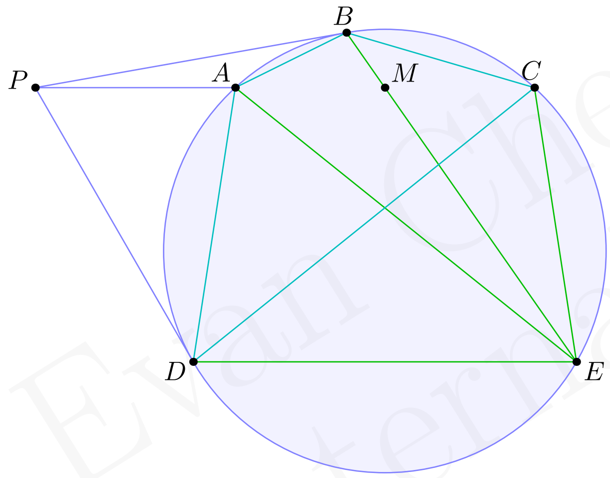
$$\frac{m-x}{m-y} = \frac{\frac{ab}{p} - c}{\frac{ac}{p} - b} = \frac{ab - cp}{ac - bp}.$$

$$\frac{\frac{1}{ab} - \frac{1}{cp}}{\frac{1}{ac} - \frac{1}{bp}} = \frac{m-x}{m-y}.$$

§4.3 Solution 2.3, JMO 2011/5

We present two solutions.

First solution using harmonic bundles Let $M = \overline{BE} \cap \overline{AC}$ and let ∞ be the point at infinity along $\overline{DE} \parallel \overline{AC}$.



Note that $ABCD$ is harmonic, so

$$-1 = (AC; BD) \overset{E}{=} (AC; M\infty)$$

implying M is the midpoint of $\overline{AC}$.

Second solution using complex numbers (Cynthia Du) Suppose we let b, d, e be free on unit circle, so $p = \frac{2bd}{b+d}$. Then $d/c = a/e$, and $a + c = p + ac\overline{p}$. Consequently,

$$ac = de$$

$$\frac{1}{2}(a+c) = \frac{bd}{b+d} + de \cdot \frac{1}{b+d} = \frac{d(b+e)}{b+d}.$$

$$\frac{\frac{1}{2}(a+c)}{2ac} = \frac{(b+e)}{e(b+d)}.$$

From here it's easy to see

$$\frac{a+c}{2} + \frac{a+c}{2ac} \cdot be = b + e$$

which is what we wanted to prove.

§4.4 Solution 2.4, Romania 2003

This is actually easy with casework by drawing in the medial triangle. But it is also an instructive exercise in complex numbers.

Let a, b, c be on the unit circle in the usual manner. The midpoint of the A-altitude, which denote by X, then has coordinates

$$\begin{aligned} x &= \frac{1}{2}\left[a + \frac{1}{2}\left(a+b+c-\frac{bc}{a}\right)\right] \\ 4x-(a+b+c) &= 2a - \frac{bc}{a} = \frac{2a^2-bc}{a} \\ \overline{4x-(a+b+c)} &= \frac{\frac{2}{a^2}-\frac{1}{bc}}{1/a} = \frac{2bc-a^2}{abc}. \end{aligned}$$

Define y and z similarly. Now x, y, z are collinear iff $4x-(a+b+c)$, et cetera, are collinear. So the desired collinearity is equivalent to

$$\begin{aligned} 0 = \det \begin{bmatrix} \frac{2a^2-bc}{a} & \frac{2bc-a^2}{abc} & 1 \\ \frac{2b^2-ca}{b} & \frac{2ca-b^2}{abc} & 1 \\ \frac{2c^2-ab}{c} & \frac{2ab-c^2}{abc} & 1 \end{bmatrix} &= \frac{1}{(abc)^2} \det \begin{bmatrix} 2a^2-bc & 2abc-a^3 & a \\ 2b^2-ca & 2abc-b^3 & b \\ 2c^2-ab & 2abc-c^3 & c. \end{bmatrix} \\ &= \frac{1}{(abc)^2}\sum_{\text{cyc}} \left(c\left((2a^2-bc)(2abc-b^3)-(2b^2-ca)(2abc-a^3)\right)\right) \\ &= \frac{1}{(abc)^2}\sum_{\text{cyc}} \left[4a^3bc^2-2ab^2c^3-2a^2b^3c+b^4c^2-4ab^3c^2+2a^2bc^3+2a^3b^2c-a^4c^2\right] \\ &= \frac{1}{(abc)^2}\sum_{\text{cyc}} \left[b^4c^2-a^4c^2\right] = \frac{-(a^2-b^2)(b^2-c^2)(c^2-a^2)}{(abc)^2} \\ &= \frac{-(a-b)(b-c)(c-a)(a+b)(b+c)(c+a)}{(abc)^2}. \end{aligned}$$

Thus we're done.

> **Remark.** Indeed, the factorization could have been known in advance, since $(a+b)(b+c)(c+a)$ must be a factor, as is $(a-b)(b-c)(c-a)$, since if two vertices coincide then two of the midpoints coincide. Degree counting then tells us the final result beforehand.