mousavi
222 posts

Feb 20, 2011, 6:12 pm • 4 PM #1

Let's start a marathon for polynomial problems.
If a person solved a problem,he/she should put a new problem.

problem 1

Find all polynomials $P(x)$ such that there is a unique polynomial $Q(x)$:

$Q(0)=0$, $\forall\, x,y$: $x+Q(y+P(x))=y+Q(x+P(y))$

MathBaron
20 posts

Feb 23, 2011, 10:40 pm • 3 PM #2

First assume $P(x)=c \quad \forall x\in\mathbb{R}$, the original equation became $x+Q(y+c)=y+Q(x+c)$
Let then $x\mapsto x-c$ and $y\mapsto 0$, then we have $Q(x)=x-c+Q(c)$, and since $Q(0)=0$ and $Q(0)=0+c-Q(c)$ we have $Q(c)=c$ and so unique solution $Q(x)=x$. Hence $\forall c\in\mathbb{R}$, $P(x)=c$ gives unique Q.

Now look for P such that $deg(P)\geq 1$.
Plugging $y=0$ in the original equation we have $x+Q(P(x))=Q(x+P(0))$, so
$deg(x+Q(P(x)))=deg(Q(x+P(0))) \quad (*)$
but also we have $deg(x+Q(P(x)))=max\{1;deg(Q)\cdot deg(P)\}$ and $deg(Q(x+P(0)))=deg(Q)$.
From the latest three equation we have $deg(Q)\geq 1$ and since we are assuming $deg(P)\geq 1$ we have
$deg(x+Q(P(x)))=deg(Q)\cdot deg(P)$ and so $(*)$ becames

$$deg(Q)\cdot deg(P)=deg(Q)$$

and since $deg(Q)\geq 1$ we have $deg(P)=1$, therefore $P(x)=ax+b \text{ for } a\neq 0; b\in\mathbb{R}$.
So the original equation become

$$x+Q(y+ax+b)=y+Q(x+ay+b) \quad (1)$$

Plugging $y=0$ in the (1) we have

$$x+Q(ax+b)=Q(x+b)$$

If $deg(Q)=n>1$ and q_0 is the leading coefficient of Q, the leading coefficient of $x+Q(ax+b)$ is a^nq_0 and the leading coefficient of $Q(x+b)$ is q_0, so we need that $a^nq_0=q_0$ and so $a=1$, therefore $P(x)=x+b$ and (1) becomes

$$x+Q(x+y+b)=y+Q(x+y+b)$$

and so no solution.
So we have $deg(Q)=1$ and since $Q(0)=0$ we have $Q(x)=kx$.
Then (1) become

$$x+ky+akx+kb=y+kx+aky+kb$$

which is equivalent to

$$(ak+1)x+ky=kx+(ak+1)y \quad \forall x,y\in\mathbb{R}$$

So we need that $ak+1=k$ and so $k=\dfrac{1}{a-1}$ which gives a unique polynomial Q which indeed is solution.

So the solutions are:
$P(x)=c$ whit $c\in\mathbb{R}$
$P(x)=ax+b$ whit $a\neq 1\in\mathbb{R}$ and $b\in\mathbb{R}$

I have no problem to submit. Anybody can submit one instead of me.

mousavi
222 posts

Feb 25, 2011, 1:21 am • 1 PM #3

problem 2

Find all polynomials $P(x)$ with real coefficient such that:

$$P(0)=0, \quad \text{and} \quad \lfloor P\lfloor P(n)\rfloor\rfloor+n=4\lfloor P(n)\rfloor \quad \forall n\in\mathbb{N}.$$

Batominovski
1602 posts

Feb 25, 2011, 5:15 am • 2 PM #4

> mousavi wrote:
> **problem 2**
>
> Find all polynomials $P(x)$ with real coefficient such that:
>
> $P(0)=0, [P[P(n)]]+n=4[P(n)]$, $(n\in N)$
>
> **note** : [ ] is the floor function

Let $p := \deg(P)$. So, $\max\{p^2, 1\} = p$ or $p = 1$. Because $P(0) = 0$, $P(x) = \alpha x$ for some real α. Consider

$$4\alpha = \lim_{n\to\infty} \frac{4\lfloor P(n)\rfloor}{n} = \lim_{n\to\infty} \frac{\left\lfloor P\left(\lfloor P(n)\rfloor\right)\right\rfloor}{n} + 1 = \alpha^2 + 1\,.$$

Therefore, $\alpha = 2 \pm \sqrt{3}$. It is easy to see that $\alpha \neq 2 - \sqrt{3}$ by checking the case $n = 1$. So, $\alpha = 2 + \sqrt{3}$, and write $\bar{\alpha} = 2 - \sqrt{3}$. Now, for every positive integer n,

$$\begin{aligned} 4\lfloor P(n)\rfloor - \left\lfloor P\left(\lfloor P(n)\rfloor\right)\right\rfloor &= 4\lfloor \alpha n\rfloor - \lfloor \alpha\lfloor \alpha n\rfloor\rfloor \\ &> 4\lfloor \alpha n\rfloor - \alpha\lfloor \alpha n\rfloor = \bar{\alpha}\lfloor \alpha n\rfloor \\ &> \bar{\alpha}(\alpha n - 1) = n - \bar{\alpha} > n - 1\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} 4\lfloor P(n)\rfloor - \left\lfloor P\left(\lfloor P(n)\rfloor\right)\right\rfloor &= 4\lfloor \alpha n\rfloor - \lfloor \alpha\lfloor \alpha n\rfloor\rfloor \\ &< 4\lfloor \alpha n\rfloor - \alpha\lfloor \alpha n\rfloor + 1 = \bar{\alpha}\lfloor \alpha n\rfloor + 1 \\ &< \bar{\alpha}(\alpha n) + 1 = n + 1\,. \end{aligned}$$

Thus, the equality $\left\lfloor P\left(\lfloor P(n)\rfloor\right)\right\rfloor + n = 4\lfloor P(n)\rfloor$ is satisfied for all $n \in \mathbb{N}$. Hence, $P(x) = \left(2 + \sqrt{3}\right)x$ satisfies the condition.

No problem to submit. Maybe I will think of one later.

---

nguyenhung
559 posts

Feb 25, 2011, 8:43 pm — PM #5

Find all polynomial satisfy these condition:

$$P\left(2\right) = 2$$
$$P\left(x^2\right) = x^2\left(x^2+1\right)P\left(x\right)$$

---

pco
15396 posts

Feb 25, 2011, 8:57 pm — PM #6

> nguyenhung wrote:
>
> Find all polynomial satisfy these condition:
>
> $$P\left(2\right) = 2$$
> $$P\left(x^2\right) = x^2\left(x^2+1\right)P\left(x\right)$$

$P(x)$ is non constant (else it would be 0 and the first condition is no longer true)
Degree n is such that $2n = n + 4$ and so $n = 4$
$P(x)$ is even
$P(0) = 0$ (set $x = 0$) and so $P(x)$ is divisible by x^2
$P(-1) = 0$ (set $x = i$) and so $P(1) = 0$ and so $P(x)$ is divisible by $x^2 - 1$

So $P(x) = ax^2(x^2 - 1)$ and the first condition gives $a = \frac{1}{6}$

And so $\boxed{P(x) = \frac{x^4 - x^2}{6}}$ which indeed is a solution.

---

makar
1547 posts

Feb 25, 2011, 9:09 pm — PM #7

> nguyenhung wrote:
>
> Find all polynomial satisfy these condition:
>
> $$P\left(2\right) = 2$$
> $$P\left(x^2\right) = x^2\left(x^2+1\right)P\left(x\right)$$

Let degree of $P(x)$ be $n \implies 2n = n + 4 \implies n = 4$

Also $P(0) = P(1) = P(-1) = 0$and $P(x) = P(-x)$

$$\implies P(x) = ax^2(x^2 - 1) \implies a = \frac{1}{6}$$

$$\implies P(x) = \frac{1}{6}(x^4 - x^2)$$

Edit

> beaten

Problem 4:

> If all the roots of the polynomial equation $x^3 + ax^2 + bx + c = 0$ are real, show that $12ab + 27c \leq 6a^3 + 10(a^2 - 2b)^{3/2}$. When does equality hold?

Mar 3, 2011, 1:50 pm • 1 — PM #8

mousavi

mousavi
222 posts

> makar wrote:
>
> > nguyenhung wrote:
> >
> > Find all polynomial satisfy these condition:
> >
> > $$P\left(2\right)=2$$
> > $$P\left(x^2\right)=x^2\left(x^2+1\right)P\left(x\right)$$
>
> Let degree of $P(x)$ be $n\implies 2n=n+4\implies n=4$
>
> Also $P(0)=P(1)=P(-1)=0$and $P(x)=P(-x)$
>
> $$\implies P(x)=ax^2(x^2-1)\implies a=\frac{1}{6}$$
>
> $$\implies P(x)=\frac{1}{6}(x^4-x^2)$$
>
> Edit
>
> Problem 4:

$a=-(x+y+z),b=xy+yz+zx,c=-xyz$

$(12ab+27c-6a^3)^2\le 100(a^2-2b)^3\iff (x-y)^2(y-z)^2(z-x)^2\ge 0$

mousavi
222 posts
Mar 3, 2011, 1:56 pm • 1 PM #9

problem 5

values of polynomial $P(x)$ for three consecutive integer numbers,are divisible by 3.prove that $P(m)$is divisible by 3 for each integer number m.

pco
15396 posts
Mar 3, 2011, 3:55 pm PM #10

> mousavi wrote:
>
> **problem 5**
>
> values of polynomial $P(x)$ for three consecutive integer numbers,are divisible by 3.prove that $P(m)$is divisible by 3 for each integer number m.

You did not say $P(x)\in\mathbb Z[X]$ and so this is obviously wrong : choose $P(x)=\frac{x^3-x}3$ such that $P(-1)$, $P(0)$ and $P(1)$ all are divisible by 3 while $P(2)$ is not

If $P(x)\in\mathbb Z[X]$ and $f(a)\equiv f(a+1)\equiv f(a+2)\equiv 0\pmod 3$ with $a\in\mathbb Z$, then :
If $f(u)\not\equiv 0\pmod 3$ for some $u\in\mathbb Z$:
$u\ne a$ and $u-a|f(u)-f(a)\not\equiv 0\pmod 3$ and so $u-a\not\equiv 0\pmod 3$
$u\ne a+1$ and $u-a-1|f(u)-f(a+1)\not\equiv 0\pmod 3$ and so $u-a\not\equiv 1\pmod 3$
$u\ne a+2$ and $u-a-2|f(u)-f(a+2)\not\equiv 0\pmod 3$ and so $u-a\not\equiv 2\pmod 3$

And so impossibility
And so $f(u)\equiv 0\pmod 3$ $\forall u\in\mathbb Z$

makar
1547 posts
Mar 3, 2011, 11:43 pm PM #11

Problem 6:

Find all pairs of integers $m>2,n>2$ such that there are infinitely many positive integers k for which (k^n+k^2-1) divides (k^m+k-1).

mousavi
222 posts
Mar 7, 2011, 7:58 pm • 1 PM #12

problem 7

$Q(x)$ is a nonzero polynomial.prove that for each natural n,polynomial $P(x)=(x-1)^nQ(x)$,in minimum,has $n+1$ nonzero coefficient.

Rijul saini
799 posts
Mar 18, 2011, 1:52 pm PM #13

> mousavi wrote:
>
> **problem 7**
>
> $Q(x)$ is a nonzero polynomial.prove that for each natural n,polynomial $P(x)=(x-1)^nQ(x)$,in minimum,has $n+1$ nonzero coefficient.

Please don't post a new problem until the previous one has been solved.

mousavi
222 posts
Mar 20, 2011, 1:54 pm PM #14

> Rijul saini wrote:
>
> > mousavi wrote:
> >
> > **problem 7**
> >
> > $Q(x)$ is a nonzero polynomial.prove that for each natural n,polynomial $P(x)=(x-1)^nQ(x)$,in minimum,has $n+1$ nonzero coefficient.

Please don't post a new problem until the previous one has been solved.

it's not good a problem stops the marathon.

lightest
51 posts

Mar 22, 2011, 9:19 am

PM #15

> Rijul saini wrote:
>
> > mousavi wrote:
> >
> > **problem 7**
> >
> > $Q(x)$ is a nonzero polynomial.prove that for each natural n,polynomial $P(x)=(x-1)^nQ(x)$,in minimum,has $n+1$ nonzero coefficient.
>
> Please don't post a new problem until the previous one has been solved.

But Problem 6 is not a polynomial problem. It is just a problem which incidentally contains two polynomials.

abhinavzand...
419 posts

Apr 15, 2011, 8:47 pm

PM #16

$EMERGENCY$!!!

$MARATHON\ DYING$!!!

$GOTTA\ GIVE\ A\ PROBLEM$!!!

Considering all that lets just skip the previous problem.

Problem 8
Just give a polynomial whose square has less number of terms than itself.

I know its a famous polynomial but this is the only way the marathon won't die.

pco
15396 posts

Apr 15, 2011, 10:24 pm

PM #17

> abhinavzandubalm wrote:
>
> **Problem 8**
> **Just give a polynomial whose square has less number of terms than itself.**

My humble opinion is that it is **quite quite unfair** to claim that this is an olympiad level problem.

This is know as "Sparse Polynomial Square" problem.

Least degree is 12

See some examples here : http://mathworld.wolfram.com/SparsePolynomialSquare.html

And nobody can solve this problem in an olympiad contest !

nguyenvutha...
474 posts

Apr 15, 2011, 10:47 pm

PM #18

Find all polynomials $P(x)\in\mathbb{R}[x]$ such that :

$$P(x).P(x+1)=P(x^2+2)\quad\forall x\in\mathbb{R}$$

In my opinion ;This is an hard problem

It appeared in Mathematical and Youth megazine about 4 years ago and there are just several correct solutions submitted

pco
15396 posts

Apr 16, 2011, 2:49 pm • 3

PM #19

> nguyenvuthanhha wrote:
>
> Find all polynomials $P(x)\in\mathbb{R}[x]$ such that :
>
> $$P(x).P(x+1)=P(x^2+2)\quad\forall x\in\mathbb{R}$$

Looking for solutions in the form ax^2+bx+c, it's immediate to find :
$P(x)=0\ \forall x$
$P(x)=1\ \forall x$
$P(x)=x^2-x+2\ \forall x$

P,Q solutions implies PQ solution and so we get also $(x^2-x+2)^n$

r real root of $P(x)$ implies $r^2+2>|r|$ real root too and so no real root for non constant solutions and so non constant solutions must have even degrees.

If highest summand of non constant solution is a_nx^n, we get $a_n^2=a_n$ and so non constant solutions must be monic polynomials with even degrees.

Suppose now that two different non constant monic solutions exist for a given even degree $p>0$ and let $P(x)$ one of them and $P(x)+Q(x)$ the other with degree of $P=p$ and degree of $Q=q<p$ (since both P and $P+Q$ are monic) :

$(P(x)+Q(x))(P(x+1)+Q(x+1))=P(x^2+2)+Q(x^2+2)$ and so :

$P(x)Q(x+1)+P(x+1)Q(x)+Q(x)Q(x+1)=Q(x^2+2)$

But degree of $LHS=p+q$ while degree of $RHS=2q$ and so this equality is impossible.

So there is no non constant solution for odd degrees and at most one non constant solution for even degrees.

And since we already found $(x^2-x+2)^n$, this is the unique one.

Hence the answer :
$P(x)=0\,\forall x$
$P(x)=1\,\forall x$
$P(x)=(x^2-x+2)^n\,\forall x$ and for any positive integer n

---

abhinavzand...
419 posts

Apr 16, 2011, 6:10 pm — PM #20

> pco wrote:
>
> > abhinavzandubalm wrote:
> >
> > **Problem 8**
> > **Just give a polynomial whose square has less number of terms than itself.**
>
> My humble opinion is that it is **quite quite unfair** to claim that this is an olympiad level problem.
>
> This is know as "Sparse Polynomial Square" problem.
>
> Least degree is 12
>
> See some examples here : http://mathworld.wolfram.com/SparsePolynomialSquare.html
>
> And nobody can solve this problem in an olympiad contest !

NO NO NO
It was not meant to be solved
i just didnt want this marathon to go down
youll notice that there is quite a gap between the last problem and mine so just wanted to get a new post on this topic

---

Rijul saini
799 posts

Apr 17, 2011, 3:19 pm • 1 — PM #21

Find all polynomials $P(x)\in\mathbb{C}[x]$ such that

$$P(x^2)=P(x)\cdot P(x+1)$$

---

pco
15396 posts

Apr 17, 2011, 4:12 pm • 2 — PM #22

> Rijul saini wrote:
>
> Find all polynomials $P(x)\in\mathbb{C}[x]$ such that
>
> $$P(x^2)=P(x)\cdot P(x+1)$$

Constant solutions are $P(x)=0\,\forall x$ and $P(x)=1\,\forall x$
If r is a real root of non constant $P(x)$, so is r^2 and so the ony real roots of $P(x)$ may be $0,1$.

Let then $P(x)=x^m(x-1)^nQ(x)$ with m,n non negative integers and $Q(0)Q(1)\neq 0$

The equation is $x^{2m}(x^2-1)^nQ(x^2)=x^m(x-1)^nQ(x)(x+1)^mx^nQ(x+1)$

Looking at power of x, we get $2m=m+n$ else either $Q(0)=0$, either $Q(1)=0$

So we get $Q(x^2)=Q(x)Q(x+1)$ and $Q(x)$ is either constant either of even degree (since without any real root).
We found the solutions $x^n(x-1)^n$

It's easy to see that any non constant solution is monic.

Suppose then that it exist two distinct non constant monic solutions with even degree p. Let $P(x)$ one of them and $P(x)+R(x)$ the other with degree of $P=p$ and degree of $R=r<p$

Then $(P(x)+R(x))(P(x+1)+R(x+1))=P(x^2)+R(x^2)$

and so $P(x)R(x+1)+P(x+1)R(x)+R(x)R(x+1)=R(x^2)$
But degree of LHS is $p+r$ while degree of RHS is $2r$ and so this is impossible

So there is at most one solution for even degrees and since we already found $x^n(x-1)^n$, this is the only one.

Hence the solutions :
$P(x)=0\,\forall x$
$P(x)=1\,\forall x$
$P(x)=x^n(x-1)^n\,\forall x$ and for any $n\in\mathbb{N}$

Rijul saini
799 posts

Apr 17, 2011, 6:03 pm • 1 — PM #23

> pco wrote:
>
> > Rijul saini wrote:

> Find all polynomials $P(x) \in \mathbb{C}[x]$ such that
> $$P(x^2) = P(x) \cdot P(x+1)$$
>
> $Q(x)$ is either constant either of even degree (since without any real root).

How could you conclude that if the coefficents are complex?

pco
15396 posts

Apr 17, 2011, 10:40 pm • 1 PM #24

> Rijul saini wrote:
>
> > pco wrote:
> >
> > > Rijul saini wrote:
> > > Find all polynomials $P(x) \in \mathbb{C}[x]$ such that
> > > $$P(x^2) = P(x) \cdot P(x+1)$$
> >
> > $Q(x)$ is either constant either of even degree (since without any real root).
>
> How could you conclude that if the coefficents are complex?

You're right

So maybe I missed some solutions.
I'll look further.

pco
15396 posts

Apr 18, 2011, 12:32 pm • 2 PM #25

> Rijul saini wrote:
> Find all polynomials $P(x) \in \mathbb{C}[x]$ such that
> $$P(x^2) = P(x) \cdot P(x+1)$$

Next trial :
Only constant solutions are $P(x) = 0$ and $P(x) = 1$
Considering then non constant solutions :
Let r real root of $P(x)$: we get that r^2 is real root too and so the only possible real roots are $-1,0,1^$
We can exclude -1 since then $P(4) = P(-2)P(-1) = 0$ and 4 would be real root too.

So the only possible real roots are $0, 1$

Let z a non real complex root of $P(x)$: we get that z^2 is root too and so $|z| = 1$
We also get that $(z-1)^2$ is root and so $|z-1| = 1$

But $|z| = |z-1| = 1$ implies $z = \pm e^{i\frac{\pi}{3}}$ and none may be a root since then z^2 is not $\pm e^{i\frac{\pi}{3}}$

So the only roots of $P(x)$ may be $0, 1$ and $P(x) = cx^n(x-1)^m$

Plugging this in original equation, we get $c = 1$ and $m = n$

Hence the only solutions :
$P(x) = 0\ \forall x$
$P(x) = 1\ \forall x$
$P(x) = x^n(x-1)^n\ \forall x$ and for any $n \in \mathbb{N}$

Batominovski
1602 posts

Apr 18, 2011, 12:49 pm • 1 PM #26

Problem 11: Let q be a positive real number such that $q \neq 1$. Define $[x]_q := \dfrac{1-q^x}{1-q}$ for all $x \in \mathbb{R}$. Also, we define the q-factorials $[0]_q! := 1$ and $[n]_q! := [n]_q \cdot [n-1]_q!$ for every $n \in \mathbb{N}$. Finally, the q-binomial coefficient $\binom{n}{k}_q$ is

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!},$$

for every $n \in \mathbb{N}_0$ and $k \in \mathbb{Z}$, $0 \leq k \leq n$. (The set $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.)

Prove that the q-binomial coefficients are symmetric polynomials in q.

PS: I know a combinatorial approach. So, it would be great if someone comes up with a purely algebraic solution.

This post has been edited 1 time. Last edited by Batominovski, Apr 18, 2011, 10:56 pm

pco
15396 posts

Apr 18, 2011, 2:04 pm • 5 PM #27

> Batominovski wrote:
> **Problem:** Let q be a positive real number such that $q \neq 1$. Define $[x]_q := \dfrac{1-q^x}{1-q}$ for all $x \in \mathbb{R}$. Also, we define the q-factorials $[0]_q! := 1$ and $[n]_q! := [n]_q \cdot [n-1]_q!$ for every $n \in \mathbb{N}$. Finally, the q-binomial coefficient $\binom{n}{k}_q$ is
>
> $$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!},$$

for every $n \in \mathbb{N}_0$ and $k \in \mathbb{Z}$, $0 \le k \le n$. (The set $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.)

Prove that the q-binomial coefficients are symmetric polynomials in q.

I consider the extension of $\binom{n}{k}_q$ where $\binom{n}{k}_q = 0$ if $k<0$ or $k>n$

It's easy to check (and it's the key) that $\boxed{\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k\binom{n-1}{k}_q}$

And so that $\binom{n}{k}_q$ is indeed a polynomial in q with integer coefficients and with degree $k(n-k)$ (easy induction)

Proving then that this polynomial is symetric is to prove that $q^{k(n-k)}\binom{n}{k}_{\frac{1}{q}} = \binom{n}{k}_q$

If this is true for $n-1, k-1$, then :

$q^{k(n-k)}\binom{n}{k}_{\frac{1}{q}} = q^{k(n-k)}\binom{n-1}{k-1}_{\frac{1}{q}}\frac{1-q^{-n}}{1-q^{-k}}$

$= q^{(k-1)((n-1)-(k-1))}\binom{n-1}{k-1}_{\frac{1}{q}} q^{n-k}\frac{1-q^{-n}}{1-q^{-k}}$

$= \binom{n-1}{k-1}_q \frac{q^n-1}{q^k-1}$

$= \binom{n}{k}_q$

So this is true for n, k. And since this is true for $n, 0$ it's true for $n+k, k$ and so for any n, k where $n \ge k$

Q.E.D.

Apr 18, 2011, 9:39 pm • 1 PM #28

Batominovski
1602 posts

Problem 12: Fix $k \in \mathbb{N}$, $k \ge 2$. Prove that every polynomial in $\mathbb{Z}[x]$ can be represented as a sum of k irreducible polynomials over $\mathbb{Z}$.

PS: I think I saw this problem sometime ago in mathlinks, but can't exactly remember by whom. If you own this problem, please let me know and I shall add the credit to you.

Problem 13: Let p be a prime number. Also, $k \in \mathbb{N}$ and b_1, b_2, $\ldots$, b_k are integers ($b_k \neq 0$). Consider a sequence $\{a_n\}_{n\in\mathbb{Z}}$ of integers satisfying the linear recurrence

$$a_n - b_1a_{n-1} + b_2a_{n-2} - \cdots + (-1)^k b_k a_{n-k} = 0$$

for every $n \in \mathbb{Z}$. Prove that $\{a_n\}$ is periodic modulo p with a period less than p^k.

PS: This is a well known result, but have you ever tried to prove it?

This post has been edited 2 times. Last edited by Batominovski, Apr 18, 2011, 11:48 pm

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Apr 18, 2011, 10:39 pm • 1 PM #29

mousavi
222 posts

please write the number of problems

Apr 18, 2011, 10:52 pm • 1 PM #30

Batominovski
1602 posts

> *mousavi wrote:*
>
> please write the number of problems

I lost track. Sorry.

BTW, I saw nothing wrong with Problem 7. Why did somebody say it was a bad problem? I think it's a good one. Let me requote it.

> *mousavi wrote:*
>
> **problem 7**
>
> $Q(x)$ is a nonzero polynomial.prove that for each natural n,polynomial $P(x) = (x-1)^nQ(x)$,in minimum,has $n+1$ nonzero coefficient.

Jun 5, 2011, 10:10 pm • 1 PM #31

PhilAndrew
207 posts

> *Batominovski wrote:*

> **Problem 12:** Fix $k \in \mathbb{N}, k \geq 2$. Prove that every polynomial in $\mathbb{Z}[x]$ can be represented as a sum of k irreducible polynomials over $\mathbb{Z}$.
>
> PS: I think I saw this problem sometime ago in mathlinks, but can't exactly remember by whom. If you own this problem, please let me know and I shall add the credit to you.

It doesn't matter too much since it follows immediately from Perron's criterion - let $f \in \mathbb{Z}[X]$, $m = \deg f$ and take $g(X) + f(X)$ and $-g(X)$ for $k = 2$, where $g(X) = X^{m+2} + \alpha X^{m+1} + 1$ for α large enough to ensure irreducibility, adding 1 to $g(X)$ if $X|g(X) + f(X)$. For $k = 3$, let $h(X) = X^2 + X + 1$ and take $g + f - h$, $-g$ and h, adjusting α, m and the free term of g as required. For larger k, keep adding pairs of opposite irreducibles.

FAMILY
44 posts

Jun 14, 2011, 6:54 pm • 1 PM #32

Please help me:

$\boxed{1}$ Given f, g are polymials with one variable and h is polymial with two variables such that $f(x) - f(y) = h(x, y)(g(x) - g(y)) \forall x, y \in R$. Prove that exist q is polymial such that $f(x) = q(g(x))$.

$\boxed{2}$ Given f, g be two polymials with integer cofficients. We said $f \equiv g(mod m)$ if $(f - g)$ have all of coefficients are divisible by m. Let f, g, h, r, s be four polymials with integer cofficients such that $rf + sg \equiv 1(mod p)$ and $fg \equiv h(mod p)$. Prove that: exist F, G are two polymials with integer cofficients such that $F \equiv f(mod p)$, $G \equiv g(mod p)$ and $FG \equiv h(mod p^n)$ for $n \in \mathbb{N}$.

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abhinavzand...
419 posts

Jun 14, 2011, 9:54 pm • 1 PM #33

I think that these problems have been double posted.
So I will give a guessing problem.
It's interesting but I don't have a method.
Problem (I don't know but will edit if somebody tells me).
Find a polynomial in two variables such that it never takes the value 0 but can approach infinitely close to 0.

PhilAndrew
207 posts

Jun 14, 2011, 11:47 pm • 1 PM #34

> abhinavzandubalm wrote:
>
> **Problem.**
> Find a polynomial in two variables such that it never takes the value 0 but can approach infinitely close to 0.

You're right; this comes from an IMC Longlist, yet it must be folklore. Consider $f(X, Y) = (XY - 1)^2 + X^2$. Then $f(x, 1/x) = x^2$ which can be made arbitrarily small for $x > 0$, yet it is clear that $f(x, y) \geq 0$ and there is no equality case.

Zeus93
64 posts

Jul 4, 2011, 9:23 pm • 1 PM #35

Problem 14:
a, b are roots of $P(x) = x^4 + x^3 - 1$
Prove that $a.b$ is root of $Q(x) = x^6 + x^4 + x^3 - x^2 - 1$

---

pco
15396 posts

Jul 5, 2011, 6:20 pm • 1 PM #36

> Jumong4958 wrote:
>
> Problem 14:
> a, b are roots of $P(x) = x^4 + x^3 - 1$
> Prove that $a.b$ is root of $Q(x) = x^6 + x^4 + x^3 - x^2 - 1$

Posted many many times
Let $a + b = s$ and $ab = p \neq 0$

Write $x^4 + x^3 - 1 = (x^2 - sx + p)(x^2 + ax + b) = (\frac{1}{p}x^2 - \frac{s}{p}x + 1)(px^2 + apx + bp)$
$= (\frac{1}{p}x^2 - ux + 1)(px^2 + vx + w)$

Identification of constant terms implies $w = -1$ and the equation is $x^4 + x^3 - 1 = (\frac{1}{p}x^2 - ux + 1)(px^2 + vx - 1)$

Identification of x terms implies $v = -u$ and the equation is $x^4 + x^3 - 1 = (\frac{1}{p}x^2 - ux + 1)(px^2 - ux - 1)$

Identification of x^3 terms implies $-\frac{u}{p} - up = 1$ and so $u = -\frac{p}{1 + p^2}$

Identification of x^2 terms implies $p - \frac{1}{p} + u^2 = 0$

and so $\frac{1}{p} - p = u^2 = \frac{p^2}{p^4 + 2p^2 + 1}$ which implies $(p^2 - 1)(p^4 + 2p^2 + 1) = -p^3$

And so $\boxed{p^6 + p^4 + p^3 - p^2 - 1 = 0}$

Q.E.D.

BigSams
1581 posts

Jul 7, 2011, 11:37 pm • 1 PM #37

Problem 15.
For each quadratic polynomial $p(x)$, prove that there exist quadratic polynomials $q(x)$ and $r(x)$ such that $p(x) \cdot p(x + 1) = q(r(x))$.

Goutham
3128 posts

Jul 7, 2011, 11:44 pm • 1 PM #38

> BigSams wrote:
>
> **Problem 15.**
> For each quadratic polynomial $p(x)$, prove that there exist quadratic polynomials $q(x)$ and $r(x)$ such that $p(x) \cdot p(x + 1) = q(r(x))$.

Suppose α, β are the roots of $p(x)$, then we can see that

$p(x) \cdot p(x+1) = ((x-\alpha)(x-\beta)+x-\alpha)((x-\alpha)(x-\beta)+x-\beta)$ and we can choose $q(x) = p(x), r(x) = (x-\alpha)(x-\beta)+x$.

I have no problems to submit.

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abhinavzand...
419 posts

Jul 31, 2011, 2:13 pm — PM #39

This is not good , you know.
This marathon must not die.
Heres the next problem.
Problem 16
Find all polynomials P such that
$P(x,y) = P(x+1,y+1)$

This post has been edited 1 time. Last edited by abhinavzandubalm, Aug 1, 2011, 10:52 am

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Kingofmath101
2192 posts

Aug 1, 2011, 1:52 am — PM #40

Attempt...

Problem #16 Solution

> We assume that the condition works for all values of x and y. We let $x, y = -1$:
>
> $P(-1,-1) = P(0,0)$
>
> So, we must find a polynomial such that $P(x-1) = P(x)$.
>
> With n, p, z, and k being constants, we have
>
> $0(nx+py+z) = 0(nx+py+z+k)$ for all values of x and y.
>
> Essentially, the polynomial 0 works for any problem like this one.

Problem #17

> (Idea: Art of Problem Solving's Intermediate Algebra; I'm pretty sure the problem was from Britain's national olympiad or the Duke Math Meet) What is the remainder when x^4-1 is divided into $x^{200}-x^{196}+x^{192}-.........+x^4-1$?

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abhinavzand...
419 posts

Aug 1, 2011, 10:22 am — PM #41

@kingofmath101
nice try, but please try again.
So this is still pending.

> abhinavzandubalm wrote:
> **Problem 16**
> Find all polynomials P such that
> $P(x,y) = P(x+1,y+1)$

Problem 17 Solution

> I don't quite understand what you mean here.
> What I do know is that
> $x^{200}-x^{196}+x^{192}-...+x^4-1 = (x^4-1)(x^{196}+x^{188}+x^{180}+...+x^8+1)$
> What do you need me to do further?

---

abhinavzand...
419 posts

Aug 1, 2011, 10:57 am — PM #42

I understand that **Problem 16** is hard so until it is solved I would like users to post this on their posts.
Still pending.

> abhinavzandubalm wrote:
> **Problem 16**
> Find all polynomials P such that
> $P(x,y) = P(x+1,y+1)$

Problem 18
We have a 2010 degree polynomial P such that
$P(n) = 2^n \ \forall \ n \ \in \ \{1,2,3,...,2011\}$
Find $P(2012)$.

Batominovski
1602 posts

Aug 1, 2011, 11:46 am • 1 — PM #43

Problem 16:
Define for each pair $(x,y) \in \mathbb{C}^2$ the polynomial $Q_{x,y}(t) \in \mathbb{C}[t]$ via $Q_{x,y}(t) := P(x+t, y+t)$. We know that $Q_{x,y}$ is constant for all integer values of t, which implies that $Q_{x,y}$ is indeed a constant polynomial. Therefore, $P(x,y) = P(x+t,y+t)$ for every $t \in \mathbb{C}$. Replacing t by $-y$, we conclude that $P(x,y) = P(x-y,0)$ is a monovariate polynomial in $x-y$.

Problem 18:

Obviously, $P(x) = 2\left(\sum_{k=0}^{2010}\binom{x-1}{k}\right)$. Thus, $P(2012) = 2\left(\sum_{k=0}^{2010}\binom{2011}{k}\right) = 2^{2012}-2$.

PS: No problems to submit.

This post has been edited 2 times. Last edited by Batominovski, Aug 3, 2011, 4:34 am

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abhinavzand...
419 posts

Aug 1, 2011, 12:27 pm — PM #44

Problem 19.
Find all polynomials P such that

$P(P(x))P(x^2-1) = P(3x)^3 - P(x)$

Farenhajt
5167 posts

Aug 1, 2011, 12:48 pm
PM #45

Problem 19.

If $n = \deg P$, then $n^2 + 2n = 3n \iff n = 0 \vee n = 1$

(i) If $P(x) \equiv c$, then $c^3 - c^2 - c = 0 \iff c \in \left\{0, \frac{1 \pm \sqrt{5}}{2}\right\}$

(ii) If $P(x) = ax + b$, then the leading term on the LHS is $a^2x \cdot ax^2 = a^3x^3$, while the leading term on the RHS is $(3ax)^3 = 27a^3x^3$, hence $a = 0$ and we're back at (i).

No problem to submit, anyone's free to go.

abhinavzand...
419 posts

Aug 1, 2011, 1:25 pm
PM #46

You people really have no idea that it's quite hard for ME to make problems.
Problem 20.
Let $\overline{a_n a_{n-1} ... a_1 a_0}$ be an $n+1$ digit prime number.
Prove that the polynomial

$$P(x) = \sum_{i=0}^{i=n} a_i x^i$$

is irreducible over integers.

This post has been edited 1 time. Last edited by abhinavzandubalm, Aug 19, 2011, 7:30 pm

ArefS
300 posts

Aug 1, 2011, 9:22 pm • 1
PM #47

> abhinavzandubalm wrote:
> You people really have no idea that it's quite hard for ME to make problems.
> **Problem 20.**
> Let $a_n a_{n-1} ... a_1 a_0$ be an $n+1$ digit prime number.
> Prove that the polynomial
> $$P(x) = \sum_{i=0}^{i=n} a_i x^i$$
> is irreducible over integers.

assume that $P(x) = Q(x)R(x)$ where $Q(x), R(x)$ are polynomials of a degree greater or equal to 1 with integer coefficients.
we have that:$P(10) = \overline{a_n a_{n-1} \dots a_0}$ is a prime.
since $a_i \in \{0, 1, \dots, 9\}$ we get that for each root of $P(x)$, say z, we have that $Re(z) < 0$ (because of Descartes' rule of signs) so $P(x)$ has got no roots in the disk: $D = \{z \in \mathbb{C} \mid \|z - 10\| \leq 1\}$
we have that $P(10) = Q(10)R(10)$ WLOG assume that: $Q(10) = 1$.
we get that Q has got no roots in D. so $Q(x + 10) = f(x)$ has got no roots in the unit circle. that is, if $R = \{c_1, c_2, \dots, c_k\}$ is the set of all (complex) roots of the polynomial $f(x)$ then $\forall i \in \mathbb{N}_k; \|c_i\| > 1$.
it is a contradiction since $\|f(0)\| = 1 = \|\Pi_i c_i\| = \Pi_i \|c_i\| > 1$
PS: $\Pi_i c_i = c_1 c_2 \dots c_n$

Kingofmath101
2192 posts

Aug 2, 2011, 5:14 am
PM #48

Problem #21

> Find all possible values of x, and all values of x for which the product is undefined:
> $$\prod_{n=1}^{x} \frac{n^5 - n^3 - n^2 + 1}{n^5 - 4n^3 - n^2 + 4} = 0$$

carlosmath
169 posts

Aug 2, 2011, 5:54 am • 1
PM #49

> Kingofmath101 wrote:
> Problem #21

Solution

> We know:
> $$\frac{n^5 - n^3 - n^2 + 1}{n^5 - 4n^3 - n^2 + 4} = \frac{(n+1)(n-1)(n-1)(n^2+n+1)}{(n+2)(n-2)(n-1)(n^2+n+1)} = \frac{(n+1)(n-1)}{(n+2)(n-2)}$$
> Where $n \neq 1, 2$ also $x \neq 1, 2$.
> Now, simplifying fractions we get
> $$\prod_{n=1}^{x} \frac{n^5 - n^3 - n^2 + 1}{n^5 - 4n^3 - n^2 + 4} = \frac{4(x-1)}{(x+2)} = 0$$
> Finally, there aren't x which satisfy the equation.

Problem #22

> In the expansion of
> $$Q_{(x;y;z)} = \left(x^3 + xy + z^2\right)^n$$
> where $n \in \mathbb{Z}^+$, we find a term which has the form: $px^{11}y^2z^{n+4}$

Find the value of $p-n$

abhinavzand...
419 posts

Aug 2, 2011, 9:45 am PM #50

> carlosmath wrote:
> Problem #22

@carlosmath
Here you should have stated that n is given as even.

Solution 22

The trinomial theorem states that
$$(a+b+c)^n=\sum_{k_1+k_2+k_3=n}\frac{n!}{k_1!k_2!k_3!}a^{k_1}b^{k_2}c^{k_3}$$

Therefore , here
$a=x^3, b=xy, c=z^2$
The required term is of the form
$px^{11}y^2z^{n+4}$
Therefore
$$k_1=3, k_2=2, k_3=n-2-3=\frac{n+4}{2}$$
Therefore
$n=14$

The term here is
$\frac{14!}{2!3!9!}x^{11}y^2z^{18}$
Therefore
$p=20020, n=14$
Therefore
$p-n=20020-14=20006.$

I don't have any problem to submit right now.I'll think of something in the next $5-10$ minutes.

abhinavzand...
419 posts

Aug 2, 2011, 10:03 am • 1 PM #51

I really see no reason to hide the *problems*.

This is a very old problem,from $Putnam\ 1938$, i.e. , the very first Putnam competition , so it is quite easy.
Problem 23.
Let $P(y)=Ay^2+By+C$ be a quadratic polynomial in y.If the roots of the quadratic equation $P(y)-y=0$ are a and $b(a\neq b)$, show that a and b are the roots of the biquadratic equation $P(P(y))-y=0$.Hence write down a quadratic equation which will give the other two roots, c and d , of the biquadratic.Apply this result to solving the following biquadratic equation
$(y^2-3y+2)^2-3(y^2-3y+2)+2-y=0.$

This is an easy question as it is **B.1** of the very first exam.

ArefS
300 posts

Aug 2, 2011, 11:21 am PM #52

> abhinavzandubalm wrote:

> I really see no reason to hide the *problems*.
>
> This is a very old problem,from $Putnam\ 1938$, i.e. , the very first Putnam competition , so it is quite easy.
> **Problem 23.**
> Let $P(y)=Ay^2+By+C$ be a quadratic polynomial in y.If the roots of the quadratic equation $P(y)-y=0$ are a and $b(a\neq b)$, show that a and b are the roots of the biquadratic equation $P(P(y))-y=0$.Hence write down a quadratic equation which will give the other two roots, c and d , of the biquadratic.Apply this result to solving the following biquadratic equation
> $(y^2-3y+2)^2-3(y^2-3y+2)+2-y=0.$
>
> This is an easy question as it is **B.1** of the very first exam.

$$A(Ay^2+By+C)^2+B(Ay^2+By+C)+C-y=(Ay^2+By+C-y)(A(Ay^2+By+C+y)+B+1)$$

for this one:($(y^2-3y+2)^2-3(y^2-3y+2)+2-y=0$): $A=1, B=-3, C=2$

abhinavzand...
419 posts

Aug 2, 2011, 11:44 am PM #53

Problem 24.
The polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zero's but possibly different multiplicities.The same is true for $P(z)+1$ and $Q(z)+1$.
Prove that $P(z)\equiv Q(z)$.

abhinavzand...
419 posts

Aug 2, 2011, 11:46 am PM #54

The last problem that I have submitted is from the 16'th Putnam Competition.
That was the hardest problem of that competition.
B.7
Take it as a punishment for never submitting problems.

abhinavzand...
419 posts

Aug 2, 2011, 12:10 pm • 1 PM #55

> abhinavzandubalm wrote:
>
> **Problem 24.**
> The polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zero's but possibly different multiplicities.The same is true for $P(z)+1$ and $Q(z)+1$.
> Prove that $P(z)\equiv Q(z)$.

The solution has a little calculus involved so I am posting a solution here

Problem 25.
Let τ be the set of all polynomials in x with integer coefficients.For $f(x)$ and $g(x)$ in τ and m a positive integer, let $f\equiv g \bmod m$, mean that every coefficient of $[f-g](x)$ is divisible by m.
Let n and p be two positive integers with p a prime.Given that f,g,h,r,s are in τ with $rf+sg\equiv 1 \bmod p$ and $fg\equiv h \bmod p$, prove that there exist $F(x),G(x)$ in τ with $F\equiv f \bmod p$ and $G\equiv g \bmod p$, and $FG\equiv h \bmod p^n$.

abhinavzand...
419 posts

Aug 10, 2011, 8:31 pm • 1 PM #56

I don't want the marathon to die.
But you people are not giving solutions.
Solution 25.

> Lets prove it by induction.
> We will prove the existence of $F_k(x),G_k(x)$ such that
> $F_k\equiv f \bmod p, G_k\equiv g \bmod p$, and $F_kG_k\equiv h \bmod p$
> We also have $F_1=f,G_1=g$.
> For the inductive step we assume that we have $F_k(x),G_k(x)$ and try to prove the existence of F_{k+1},G_{k+1}.
> $h-F_kG_k=p^k\sigma$ where $\sigma\in\tau$
> We will try $F_{k+1}=F_k+p^k\Delta_1, G_{k+1}=G_k+p^k\Delta_2$ for some $\Delta_1,\Delta_2\in\tau$ which are not yet chosen.
> Then
> $\boxed{F_{k+1}\equiv F_k\equiv f \bmod p}$ and $\boxed{G_{k+1}\equiv G_k\equiv g \bmod p}$.
> $F_{k+1}G_{k+1}=F_kG_k+p^k(\Delta_2F_k+\Delta_1G_k)+p^{2k}\Delta_1\Delta_2$
> $\equiv F_kG_k+p^k(\Delta 2F_k+\Delta_1G_k) \bmod p^{k+1}$
> If we chose $\Delta_1=\sigma r$ and $\Delta_2=\sigma s$, then
> $\Delta_2F_k+\Delta_1G_k\equiv\sigma rf+\sigma sg=\sigma(rf+sg)\equiv\sigma \bmod p$
> So $p^k(\Delta_2F_k+\Delta_1G_k)\equiv p^k\sigma \bmod p^{k+1}$, and
> $F_{k+1}G_{k+1}\equiv F_kG_k+p^k\sigma=h \bmod p^{k+1}$.
> $\boxed{F_{k+1}G_{k+1}\equiv h \bmod p^{k+1}}$
> QED.

Problem 26
Let $P(x)=\sum_0^n a_ix^i$.
Let $\lambda:\mathbb{R}[x]\to\mathbb{R}$.
We define $\lambda(P(x))=\sum_0^n {a_i}^2$.
Let $f(x)=3x^2+7x+2$.Find , with proof, a polynomial $g(x)\ (\neq f(x))$ such that
(i) $g(0)=1$
(ii)$\lambda(f(x)^n)=\lambda(g(x)^n)\quad\forall\ n\ \in\ \mathbb{N}$.

LaChouetteRi...
33 posts

Aug 13, 2011, 4:05 am • 1 PM #57

> abhinavzandubalm wrote:
>
> **Problem 26**
> Let $P(x)=\sum_0^n a_ix^i$.
> Let $\lambda:\mathbb{R}[x]\to\mathbb{R}$.
> We define $\lambda(P(x))=\sum_0^n {a_i}^2$.
> Let $f(x)=3x^2+7x+2$.Find , with proof, a polynomial $g(x)\ (\neq f(x))$ such that
> (i) $g(0)=1$
> (ii)$\lambda(f(x)^n)=\lambda(g(x)^n)\quad\forall\ n\ \in\ \mathbb{N}$.

I think this problem was also given as a Putnam some time ago...
Anyways, I managed to find a solution again:
Solution 26.

> First, a little calculus shows that
> $\lambda(f)=\int_0^1 f(\exp(2i\pi\theta))f(\exp(-2i\pi\theta))d\theta=\int_0^1|f(\exp(2i\pi\theta))|^2d\theta$
> In particular, you would like to find a function g such that $|g(x)|=|f(x)|$ when $|x|=1$. If such a condition is certainly not necessary a priori, it is nevertheless sufficient (together with $g(0)=1$) for our purposes.
> Then, the trick is to factor $f(x)=(x+2)(3x+1)$. Indeed, if we consider $g(x)=(2x+1)(3x+1)$, note that $g(0)=1$ and that, for $|x|=1$, $|g(x)|=|x(2+1/x)(3x+1)|=|x||2+\overline{x}||3x+1|=|2+x||3x+1|=|f(x)|$.
> Therefore, $g(x)=(2x+1)(3x+1)=6x^2+5x+1$ satisfies the constraint of Problem 26.
>
> PS : asserting that g must be different from f is useless, since we already have $g(0)=1\neq 2=f(0)$.

I have just thought of some problem, which may not be appropriate here... But here it is:
Problem 27

Consider an integer n and a prime number p such that p divides $3n^2+3n+1$. Prove that 3 divides $p-1$.

jjax
108 posts

Aug 13, 2011, 8:05 pm — PM #58

> LaChouetteRieuse wrote:
> **Problem 27**
> Consider an integer n and a prime number p such that p divides $3n^2+3n+1$. Prove that 3 divides $p-1$.

This belongs in the NT forum. Nonetheless,
Click to reveal hidden text

Note that $p \neq 2, p \neq 3$ since $6|3n^2+3n$ implies that 6 doesn't divide $3n^2+3n+1$.
Now consider $p \geq 5$. $p|3n^2+3n+1 \implies p|36n^2+36n+12$
$\iff -3 \equiv (6k+3)^2 (\mod p)$
Thus, $\left(\frac{-3}{p}\right) = 1$. (-3 is a quadratic residue mod p)
Legendre symbol is multiplicative, so $\left(\frac{3}{p}\right)\left(\frac{-1}{p}\right) = 1$.

Euler's criterion tells us that $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$.
Quadratic reciprocity theorem tells us that $\left(\frac{3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{\left(\frac{3-1}{2}\right)\left(\frac{p-1}{2}\right)} = (-1)^{\frac{p-1}{2}}$.
Multiplying these gives $\left(\frac{-1}{p}\right)\left(\frac{3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{\frac{p-1}{2}+\frac{p-1}{2}} = 1$.
Since $\left(\frac{3}{p}\right)\left(\frac{-1}{p}\right) = 1$, we get $\left(\frac{p}{3}\right) = 1$, or p is a quadratic residue mod 3.

Since 3 does not divide p, and 1 is the only quadratic residue mod 3, we must have $p \equiv 1 (\mod 3)$.

Problem 28
Find all polynomials with rational coefficients, satisfying the following condition:
For any x satisfying $|x| \leq 1$, we have $P(x) = P\left(\frac{-x+\sqrt{3-3x^2}}{2}\right)$.

abhinavzand...
419 posts

Aug 14, 2011, 12:32 am — PM #59

Solution 28

Suppose that $deg\{P(x)\} > 0$.

$$x = 0 \implies P(0) = P\left(\frac{\sqrt{3}}{2}\right).$$

x divides $P(x) - P(0)$.

$x - \frac{\sqrt{3}}{2}$ divides $P(x) - P\left(\frac{\sqrt{3}}{2}\right)$.

Therefore

$x\left(x - \frac{\sqrt{3}}{2}\right)$ divides $P(x) - P(0)$.

Therefore $0, \frac{\sqrt{3}}{2}, \frac{-\sqrt{3}}{2}$ are roots.

Therefore $3x - 4x^3$ is a factor of $P(x) - P(0)$.
Here is an interesting fact that I found while playing with the numbers, I was stuck here for the longest time, which helped me a lot later.

$$3\left(\frac{-x+\sqrt{3-3x^2}}{2}\right) - 4\left(\frac{-x+\sqrt{3-3x^2}}{2}\right)^3 = 3x - 4x^3.$$

Let
$P(x) = P(0) + f_1(x)(3x - 4x^3)$.

Changing $x \to \frac{-x+\sqrt{3-3x^2}}{2}$ we get

$$f(x) = f\left(\frac{-x+\sqrt{3-3x^2}}{2}\right)$$

Thus we reduced $P(x)$ by a polynomial with a smaller degree , namely $f_1(x)$ where $deg\{f_1(x)\} = deg\{P(x)\} - 3$.
Thus by induction we get

$P(x) = P(0) + (3x - 4x^3)f_1(0) + (3x - 4x^3)^2 f_2(0) + (3x - 4x^3)^3 f_3(0) + ...$
$... + (3x - 4x^3)^n f_n(x)$.

Here we have that $deg\{f_n(x)\} \leq 2$ ....(i)

and $f_n(x)$ satisfies the original equation.
As $3x-4x^3$ divides $f_n(x) \implies deg\{f_n(x)\} \geq 3$ or 0..........(ii)
Hence the common point of (i) and (ii) is $f_n(x)$ is constant.
Hence the form of $P(x)$ is

$$\boxed{P(x)=\sum_{i=0}^{i=n} a_i(3x-4x^3)^i}.$$

Problem 29.
Find a polynomial $P(x)$ such that
$P(\alpha_i)=\beta_i \ \ \forall \ i \in \ \{1,2,3,...,n-1,n\}$ with all of the β_i being unique.
Prove that this polynomial is not unique.
Find condition for uniquity.

Problem is for basic algebra solvers as I thought that recently the problems were getting too tough for High-Schoolers, many apologies from me.
BTW the last problem that I posted (not this one) had a non-calculus solution but very much in the same essence of thinking.
So please do forgive me if you thought that these problems were not meant to be here.
This one is quite simple btw.

---

abhinavzand...
419 posts
Aug 18, 2011, 10:56 pm PM #60

I don't want to believe that nobody is willing to post solutions here.
I will wait for another $2-3$ days before I give the solution to this and the next problem or someone else does.

---

chronondecay
64 posts
Aug 21, 2011, 9:00 pm PM #61

Isn't this just...Click to reveal hidden text

Lagrange Interpolation?

Clearly P is not unique, since adding multiples of $(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ does not change the values that the polynomial takes at points $\alpha_1,\alpha_2,\ldots,\alpha_n$. However, adding restrictions such as $\deg P\leq n-1$ makes P unique.

I would also like to point out that Problem 24 was in the IMO Shortlist '81, which I had the misfortune of trying it before reading the (calculus-based) solution. I'm quite interested in a non-calculus proof, though, if anyone bothers to come up with one.

I don't have a problem right now. Could someone post one?

---

abhinavzand...
419 posts
Aug 22, 2011, 2:26 pm • 1 PM #62

@chronondecay
Yes it was what you said.
Problem 30.
Prove that the sum
$$\sum_{i=1}^{1000}\sqrt{(1000+i)^2+1}$$
is irrational.

People please try to submit problems.

Kingofmath101
2192 posts
Aug 22, 2011, 10:03 pm • 1 PM #63

At last, a nice induction problem!
Solution to Problem #30

We proceed by induction.

Base case: This is just the expression

$$\sqrt{(1000+1)^2+1}$$

To evaluate it:

$$\sqrt{1001^2+1}$$

It is known that the only two pairs of numbers that have consecutive integer outputs when squared are -1 and 0 and 0 and 1. (For $f(x)=x^2$, $f(-1)=1$, $f(0)=0$, and $f(1)=1$, as you all probably know.)

So it is true for the base case $n=1$. (where we have $\sum_{i=1}^{n}$)

Inductive assumption: If

$$\sum_{i=1}^{k}\sqrt{(1000+i)^2+1}\notin Q$$

then

$$\sum_{i=1}^{k+1}\sqrt{(1000+i)^2+1}\notin Q$$

Induction: We must prove that

$\sum_{i=1} \sqrt{(1000+i)^2+1}, \sqrt{(1000+1001)^2+1} \notin Q$

Is this is true, then our induction would be complete.

We again use the fact that the only two pairs of numbers that have consecutive integer perfect square outputs (into the function $f(x) = x^2$) are the pairs -1 and 0 and 0 and 1. Therefore, every term in the sum

$$\sum_{i=1}^{k+1} \sqrt{(1000+i)^2+1}$$

is irrational. This is because each term, when it is in the form

$$\sqrt{t_i}$$

where i is a positive integer, has the relationship to ρ, the domain of perfect squares, that

$$t_1, t_2, t_3, t_4, t_5, ..., t_k, t_{k+1} \notin \rho$$

The sum of two irrational numbers is an irrational number, so the sum of $k+1$ irrational numbers (k is still a positive integer.) is also irrational (though I don't know how to prove this 🙁).

Therefore, since 1000 is a positive integer, the sum is equivalent to an irrational number, and by induction, it is irrational for all positive integers n, so our induction is complete. $\square$

Remark If the problem had asked, we could try to proceed from here to evaluate the sum, though that would be more difficult to do then just to prove that it is irrational.

Irrelevant-to-polynomials question- Problem #31

Prove that the sum of two or more irrational numbers is irrational.

---

tenniskidper...
2376 posts

Aug 22, 2011, 10:11 pm PM #64

Actually that is not true, and hence your solution is invalid. For example, $\sqrt{2}$ and $2-\sqrt{2}$ are each irrational, but their sum is the rational number 2. Therefore, your conclusion does not follow, even though the problem is probably still true.

Question 30 is still open.

---

abhinavzand...
419 posts

Aug 22, 2011, 10:17 pm PM #65

> *Kingofmath101 wrote:*
> At last, a nice induction problem!
> Solution to Problem #30
>
> Irrelevant-to-polynomials question- Problem #31

🙁 🙁 🙁

Sad but Counter Example

Let $x = 0.1010010001000010000010\cdots$, $y = 0.0101101110111101111101\cdots$.
Here both x, y are irrational.But the sum is $x+y = 0.111111111111\cdots$ which is a rational.
I also understand that you probably meant irrational number of types $\sqrt[m]{n}$, but still your claim must be false.
As we can get $\sqrt[m]{n} + \sqrt[m]{-n} = 0$ if m is odd.
Or of the form $\sqrt[n]{x}, m - \sqrt[n]{x}$.
So please try again.

EDIT:- Got beaten!!!

This post has been edited 1 time. Last edited by abhinavzandubalm, Aug 28, 2011, 5:11 pm

---

Batominovski
1602 posts

Aug 22, 2011, 11:23 pm PM #66

Problem 30 has been posted a long time ago: here!

A more generalized version can be found here: http://www.thehcmr.org/issue2_1/mfp.pdf.

This post has been edited 1 time. Last edited by Batominovski, Aug 27, 2011, 4:17 am

---

abhinavzand...
419 posts

Aug 23, 2011, 12:04 am PM #67

Sorry for that. 😳 😳
Many apologies.
Next....
Problem 31.
Find a polynomial $P(x) \in \mathbb{Z}[x]$ such that $P(\sqrt{2}+\sqrt{5}) = 0$.

---

costantin07
200 posts

Aug 23, 2011, 12:37 am PM #68

solution 31

let $x = \sqrt{2}+\sqrt{5}$ we get: $x^2 = 7+2\sqrt{10}$
and: $x^4 = 89+28\sqrt{10} = -9+14x^2$
let's denote $P_0(X) = X^4 - 14X^2 + 9$
we have $P_0 \in \mathbb{Z}[X]$ and $P_0(\sqrt{2}+\sqrt{5}) = 0$

problem 32
find polynomials P wich are periodic.

Batominovski
1602 posts

Aug 23, 2011, 12:46 am PM #69

> costantin07 wrote:
> problem 32
> find polynomials P wich are periodic.

No nonconstant polynomials may be periodic, otherwise there exists $c\in\mathbb{R}$ such that $P(x)=c$ has infinitely many solutions.

costantin07
200 posts

Aug 23, 2011, 12:51 am PM #70

> Batominovski wrote:
>> costantin07 wrote:
>> problem 32
>> find polynomials P wich are periodic.
>
> No nonconstant polynomials may be periodic, otherwise there exists $c\in\mathbb{R}$ such that $P(x)=c$ has infinitely many solutions.

yes you are right! but why?u could give us the c and the eventual infinite set of roots, to end up with this (i know it's simple,it's just that i didn't have a problem to submit)

Kingofmath101
2192 posts

Aug 23, 2011, 3:42 am PM #71

I question if it is invalid, as f the following statement is true, then I would just need to make some minor revisions to my proof. The statement is:

For any natural numbers n_1 and n_2, where $\sqrt{n_1},\sqrt{n_2}\notin\mathbb{Q}$, then $\sqrt{n_1}+\sqrt{n_2}\notin\mathbb{Q}$.

If this is true, then I could use that instead of the false statement that the sum of any two irrational numbers is an irrational number.

tenniskidper...
2376 posts

Aug 23, 2011, 4:08 am PM #72

I think that statement is true, but still, to make the induction work you'd have to jump from both numbers being radicals to one being a radical and another being a sum of radicals, in which case your statement doesn't apply. You'd have to prove that if $\sqrt{a},\sqrt{b},\sqrt{c},\ldots\sqrt{d}\notin\mathbb{Q}$, then $\sqrt{a}+\sqrt{b}+\sqrt{c}+\ldots+\sqrt{d}\notin\mathbb{Q}$ which is basically the whole question.

abhinavzand...
419 posts

Aug 23, 2011, 12:25 pm PM #73

> costantin07 wrote:
> solution 31
> problem 32
> find polynomials P wich are periodic.

I think that this question has been solved so I hereby give the next problem.

Problem 33.
$P(x)\in\mathbb{Z}[x]\quad P(0)=P(1)=1999$.Prove that $P(x)$ has no integer roots.

EDIT:Made a silly typo error.@LaChouetteRieuse Thanks for pointing it out..

This post has been edited 1 time. Last edited by abhinavzandubalm, Aug 23, 2011, 8:07 pm

LaChouetteRi...
33 posts

Aug 23, 2011, 7:35 pm PM #74

[Edit: The above message contained an error that was corrected, so my former remark is now useless.]

This post has been edited 2 times. Last edited by LaChouetteRieuse, Aug 24, 2011, 12:20 am

abhinavzand...
419 posts

Aug 23, 2011, 8:07 pm PM #75

Sorry for the mistake.
I edited the question now so it should be fine I think.

LaChouetteRi...
33 posts

Aug 23, 2011, 8:58 pm PM #76

solution 33

costantin07
200 posts

Aug 23, 2011, 9:10 pm PM #77

solution 33

> let n be an integer root of P
> so : $n-0/P(n)-P(0)=-1999 \implies n/1999$
> and : $n-1/P(n)-P(1)=-1999 \implies n-1/1999$
> 1999 is a prime number so:$(n-1,n)\in \updownarrow D=\{-1999,-1,1,1999\}^2$ which is impossible,since there is no successive integers in $\updownarrow D$
> so P doesn't have any integer root!

problem 34
find all polynomials $P\in\mathbb{R}[X]$ such that:
$P(0)=0$ and $P(x^2+1)=P(x)^2+1,\forall\ x\in\mathbb{R}$
edit: beaten...

abhinavzand...
419 posts

Aug 25, 2011, 1:26 pm PM #78

Solution 34

> We know that if two polynomials are coincident on infinitely many points then they are identical.
> Here $P(x)$ agrees with $Q(x)=x$ on infinitely many points which can be constructed as $0,1,2,5,26,677,\cdots,\infty$.

Hence $P(x)=x$.

Problem 35.
Let r,s be the roots of
$x^2-(a+d)x+ad-bc=0$
Prove that r^3,s^3 are the mroots of
$y^2-(a^3+d^3+3abc+3bcd)y+(ad-bc)^3=0$.

costantin07
200 posts

Aug 25, 2011, 5:29 pm
PM #79

Hi!
solution 35

we have: $r+s=a+d$ and $rs=ad-bc$ so: $r^3s^3=(ad-bc)^3$ and:
$r^3+s^3=(r+s)^3-3rs(r+s)=(a+d)^3-3(ad-bc)(a+d)=a^3+d^3+3ad(a+d)-3(ad-bc)(a+d)=a^3+d^3+3abc+3bcd$
,so we are done.

problem 36
Determine all polynomials P for which $P(x)^2-1=P(x^2-4x+1)$, $\forall\ x\in\mathbb{R}$.

LaChouetteRi...
33 posts

Aug 26, 2011, 3:05 pm
PM #80

Solution 36

Consider $Q(X)=P(X+2)$.
We have $Q(X)^2=P(X+2)^2=1+P(X^2+4X+4-4X-8+1)=1+P(X^2-3)=1+Q(X^2-5)$.

First, suppose that Q is constant. Then, we have $P(X)=Q(X)=Q(0)=Q(0)^2-1$, and therefore
$$P(X)=\frac{1+\sqrt{5}}{2}\text{ or }P(X)=\frac{1-\sqrt{5}}{2},$$

which indeed verify $P(x)^2-1=P(x^2-4x+1),\forall x\in\mathbb{R}$.

Now, suppose Q is not constant: there exists a maximum $k\geq 0$ such that we can write $Q(X)=R(X^k)$, where R is some polynomial. Then, $R(X^k)^2=Q(X)^2=1+Q(X^2-5)=1+R(X^{2k}-5)$, so that $R(X)^2=1+R(X^2-5)$. Then, $R(X)^2$ is even, so that $R(X)$ is either even or odd. Since we cannot write $R(X)$ as $S(X^2)$ for any polynomial S, it follows that $R(X)$ is odd and that we can write $R(X)=XS(X^2)$ for some polynomial S .

Then, we have $X^2S(X^2)^2=1+(X^2-5)S(X^4-5)$, so that we get $XS(X)^2=1+(X-5)S(X^2-5)$. Note that
$$0=0\times S(0)^2=1-5S(-5),\text{ so that }S(-5)=\frac{1}{5}.$$
$$\text{Then, }-\frac{1}{5}=-5S(-5)^2=1-10S(20),\text{ so that }S(20)=\frac{3}{25}.$$

Now, consider some $x\geq 14$ such that $|2S(x)|\leq 1$.
$$|S(x^2-5)|=\frac{|xS(x)^2-1|}{x-5}\leq\frac{x/4+1}{x-5}=\frac{1}{4}+\frac{9/4}{x-5}\leq\frac{1}{4}+\frac{9/4}{9}=\frac{1}{2}.$$
Therefore, $|2S(x^2-5)|\leq 1$ and $x^2-5\geq 14^2-5\geq 14$.

Consider the sequence (u_n) such that $u_0=20$ and $u_{n+1}=u_n^2-5$. It is obvious that $u_n\to\infty$
Moreover, $u_0\geq 14$ and $|2S(u_0)|\leq 1$, so that $|2S(u_n)|\leq 1$ for every integer n.
In particular, it follows that S must be constant, which is not possible since $S(20)\neq S(-5)$.

Overall, it follows that Q was constant, and that we had
$$P(X)=\frac{1+\sqrt{5}}{2}\text{ or }P(X)=\frac{1-\sqrt{5}}{2}.$$

Problem 37
Prove that the polynomial $X^6+X^4+X^2+3X+4$ is irreducible over $\mathbb{Z}[X]$.

abhinavzand...
419 posts

Aug 29, 2011, 1:05 pm
PM #81

Solution 37

We easily see that there are no real roots as $P(x)=x^6+x^4+x^2+3x+4>0\ \forall x\in\mathbb{R}$.

Hence it has 3 quadratic factors.
Let
$P(x)=x^6+x^4+x^2+3x+4=(x^2+ax+b)(x^2+cx+d)(x^2+ex+f)$
$bdf=4$
Now by trail on their values we easily see that there is no integer solutions.
If you want the full solution of the trial then just say so.
@LaChouetteRieuse
If anybody has a better solution please write it here or at least give the general idea.

Problem 38.
Let $P(x)\in\mathbb{Z}[x]$ .Show that there exists $Q(x)\in\mathbb{Z}[x]$ such that all the coeffeceints of $P(x)Q(x)$ are divisible by 10^9 .

anonymousl...
1142 posts

Aug 29, 2011, 1:35 pm
PM #82

for example all the coefficients of $Q(x)$ be divisible by 10^9....

abhinavzand...
419 posts

Aug 29, 2011, 2:09 pm
PM #83

Problem 39.
$P(x)$ be a polynomial of degree $3n$ such that
$P(0)=P(3)=\cdots=P(3n)=2$

$P(0) = P(3) = \cdots = P(3n) = 2$
$P(1) = P(4) = \cdots = P(3n-2) = 1$
$P(2) = P(5) = \cdots = P(3n-1) = 0$
and $P(3n+1) = 730$
Find n.

anonymousl...
1142 posts

Aug 29, 2011, 6:37 pm — PM #84

$P(a) - P(b)$ is divisible by $a - b$.
suppose $n > 1$ then $P(0) - P(4) = 1$ bu this isn't divisible by 4 so $n = 1$.

anonymousl...
1142 posts

Aug 29, 2011, 6:40 pm — PM #85

Problem 40
The polynomial f has degree n and $f(k) = 2^k$ for $0 <= k <= n$.
Find $f(n+1)$.

abhinavzand...
419 posts

Aug 29, 2011, 7:39 pm — PM #86

Already posted
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=2381919#p2381919

So here
Problem 41.
Let

$$P(x) = x^n + \sum_{i=n-1}^{0} a_i x^i$$

be a polynomial with integer coeffeceints.Suppose that there exist four different integers a, b, c, d such that $P(a) = P(b) = P(c) = P(d) = 5$ .Prove that there is no integer such that $P(k) = 8$.

costantin07
200 posts

Aug 29, 2011, 7:49 pm — PM #87

> anonymouslonely wrote:
> $P(a) - P(b)$ is divisible by $a - b$.

if $P \in \mathbb{Z}[X]$

abhinavzand...
419 posts

Aug 29, 2011, 8:45 pm — PM #88

Yeah, man I am sorry I didn't notice it earlier but your solution is wrong......
Sorry about that but please try again,
So Problem **39 , 41** are still left.

Learner94
635 posts

Aug 29, 2011, 9:59 pm • 1 — PM #89

problem 41

> P is a $n-$th degree monic polynomial over Z. $P(x) - 5$ has 4 distinct integral roots namely a, b, c, d. So $P(x) - 5 = (x-a)(x-b)(x-c)(x-d)g(x)$. $g(x)$ is also a polynomial over Z
>
> let $\exists k \in Z$ such that $P(k) = 8$, that means $(k-a)(k-b)(k-c)(k-d)g(k)$ must be equal to 3. But its easy to see that this not possible

abhinavzand...
419 posts

Aug 29, 2011, 11:43 pm — PM #90

> abhinavzandubalm wrote:
> **Problem 39.**
> $P(x)$ be a polynomial of degree $3n$ such that
> $P(0) = P(3) = \cdots = P(3n) = 2$
> $P(1) = P(4) = \cdots = P(3n-2) = 1$
> $P(2) = P(5) = \cdots = P(3n-1) = 0$
> and $P(3n+1) = 730$
> Find n.

The only problem left now....

Learner94
635 posts

Aug 30, 2011, 2:15 am • 1 — PM #91

Problem 39
The answer is 4
proof

> By Lagrange Interpolation Formula $f(x) = 2\sum_{p=0}^{n}\left(\prod_{0\le r\ne 3p\le 3n}\frac{x-r}{3p-r}\right) + \sum_{p=1}^{n}\left(\prod_{0\le r\ne 3p-2\le 3n}\frac{x-r}{3p-2-r}\right)$
>
> and hence $f(3n+1) = 2\sum_{p=0}^{n}\left(\prod_{0\le r\ne 3p\le 3n}\frac{3n+1-r}{3p-r}\right) + \sum_{p=1}^{n}\left(\prod_{0\le r\ne 3p-2\le 3n}\frac{3n+1-r}{3p-2-r}\right)$
>
> after some calculations we get $f(3n+1) = \left(\binom{3n+1}{0} - \binom{3n+1}{3} + \binom{3n+1}{6} - \ldots\right)(2.(-1)^{3n} - 1) + 1$
>
> Given $f(3n+1) = 730$ so we have to find n such that
> $\left(\binom{3n+1}{0} - \binom{3n+1}{3} + \binom{3n+1}{6} - \ldots\right)(2.(-1)^{3n} - 1) = 729$
>
> **Lemma:** If p is even $\binom{p}{0} - \binom{p}{3} + \binom{p}{6} - \cdots = \frac{2^{p+1} sin^p\left(\frac{\pi}{3}\right)(i)^p\left(cos\left(\frac{p\pi}{3}\right)\right)}{3}$
>
> and if p is odd $\binom{p}{0} - \binom{p}{3} + \binom{p}{6} - \cdots = \frac{-2^{p+1} sin^p\left(\frac{\pi}{3}\right)(i)^{p+1}\left(sin\left(\frac{p\pi}{3}\right)\right)}{3}$

i is $\sqrt{-1}$
Using above lemmas we do not get any solution when n is odd, but when n is even $3n+1=13$ satisfies the required condition, hence $n=4$

Learner94
635 posts

Aug 30, 2011, 2:32 am • 1 PM #92

Please post more problems

---

mr10123
33 posts

Aug 30, 2011, 5:26 am • 1 PM #93

Let S contain all numbers such that x+7/(2x) is a positive integer. Only one element of S is in the form b+(√(2011))/2. Compute b.

LaChouetteRi...
33 posts

Aug 30, 2011, 1:34 pm PM #94

The solution of Problem 37 does not work:

> abhinavzandubalm wrote:
> We easily see that there are no real roots as $P(x)=x^6+x^4+x^2+3x+4>0\ \forall x\in\mathbb{R}$.
>
> Hence it has 3 quadratic factors.

Yes, it has 3 quadratic factors in $\mathbb{R}[x]$, but not necessarily in $\mathbb{Z}[x]$. It could be of the form $P(x)=(ax^2+bx+c)(d^4+cx^3+ex^2+fx+g)$ with ax^2+bx+c and $d^4+cx^3+ex^2+fx+g$ irreducible polynomials over $\mathbb{Z}[x]$.

Learner94
635 posts

Aug 30, 2011, 1:41 pm • 1 PM #95

> mr10123 wrote:
> Let S contain all numbers such that x+7/(2x) is a positive integer. Only one element of S is in the form b+(√(2011))/2. Compute b.

This is a polynomial marathon.

Problem 42

Let f be a polynomial over R of degree n with leading coefficient 1 and let $x_0<x_1<x_2<\cdots<x_n$ be some integers. Prove that there exists $k\in\{0,1,\cdots,n\}$ such that $|f(x_k)|\geq\frac{n!}{2^n}$

This post has been edited 1 time. Last edited by darij grinberg, Sep 2, 2011, 6:43 pm
Reason: $k\in \left \{1,2,\cdots,n \right \}$ replaced by $k\in \left \{0,1,\cdots,n \right \}$ to match the fact that the $x$es start with x_0

LaChouetteRi...
33 posts

Sep 1, 2011, 3:23 am • 3 PM #96

Solution 42

> Suppose, for the sake of contradiction, that the statement of Problem 42 is false, and consider a monic polynomial f of minimal degree n, as well as integers $x_0<x_1<\cdots<x_n$ that falsify it, and such that x_n-x_0 is minimal. Note that we must have $n\neq 0$, because otherwise we would have $f(X)=1$ and thus
> $|f(x_0)|=1=\frac{0!}{2^0}$.
>
> Consider the roots $r_1,\ldots,r_n$ of f, counted with multiplicity. Since f is monic, we can write
> $|f(x_k)|=\prod_{i=1}^n|x_k-r_i|$.
> Now, suppose that we have $x_{k+1}>x_k+1$ for some k such that $0\leq k<n$. Then, consider the complex numbers $r'_i=r_i$ if $\Re(r_i)<x_k+1$ and $r'_i=r_i-1$ if $\Re(r_i)\geq x_k+1$, as well as $x'_i=x_i$ if $i\leq k$ and $x'_i=x_i-1$ if $i\geq k+1$. One checks easily that $|x'_i-r'_j|\leq|x_i-r_j|$ for every i and j. Therefore, if we define
> $F(X)=\prod_{i=1}^n X-r'_i$,
> we still have F monic of degree n and
> $|F(x'_k)|\leq\frac{n!}{2^n}$.
> Moreover, we have $x'_n-x'_0=x_n-x_0-1$, which cannot be, by minimality of x_n-x_0. Therefore, we have $x_k=x_0+k$ for $k\in\{0,1,\ldots,n\}$.
>
> Now, consider the polynomial
> $g(X)=\frac{1}{n}(f(X+1)-f(X))$,
> which is monic of degree $n-1$. By minimality of n, we know that
> $|g(x_k)|\geq\frac{(n-1)!}{2^{n-1}}$
> for at least one integer x_k such that $0\leq k\leq n-1$. Then,
> $|g(x_k)|=\frac{1}{n}|f(x_{k+1})-f(x_k)|\leq\frac{1}{n}(|f(x_{k+1})|+|f(x_k)|)<2\frac{n!}{n2^n}=\frac{(n-1)!}{2^{n-1}}$,
> which is impossible.
>
> Therefore, our very first assumption was false, and the statement of Problem 42 was true.
>
> Nice problem, by the way!

I do not propose any problem, since Problem 37 is still open.

---

mr10123
33 posts

Sep 7, 2011, 1:42 am PM #97

I see my problem was deemed "not polynomial enough". Oh well.

Let P1(x) be the polynomial x-1/2, P2(x) be the polynomial (x-1/2)(x-1/3), and in general, Pn(x) is the polynomial with degree n, that has a leading coefficient of 1, and has the reciprocals of the first n prime numbers as roots. As n tends to infinity, find with proof the value of Pn(1).

LaChouetteRi...
33 posts

Sep 7, 2011, 3:55 am — PM #98

Anyways, there is certainly a typo in your problem. I would rather expect something like "Only one element of S is in the form $\frac{b+\sqrt{2011}}{2}$" instead of $b+\frac{\sqrt{2011}}{2}$ as you wrote. And, even if it is implicit, you should state that b must be integer (otherwise I do not know what "in the form [...]" should mean). However, I can still solve my version of the problem, then I will just have to pick the even numbers I found and divide them by 2.

> mr10123 wrote:
>
> Let S contain all numbers such that x+7/(2x) is a positive integer. Only one element of S is in the form b+(√(2011))/2. Compute b.

Solution of this problem

> Consider $b \in \mathbb{Z}$ such that $x = \frac{b+\sqrt{2011}}{2} \in S$. Note that 2011 is no perfect square, so that $x + \frac{7}{2x} = \frac{b+\sqrt{2011}}{2} + \frac{7}{b+\sqrt{2011}} = \frac{b+\sqrt{2011}}{2} + \frac{7(b-\sqrt{2011})}{b^2-2011}$ is an integer, and thus rational, number. In particular, since $\sqrt{2011}$ is not rational, we have $b^2 - 2011 = 2 \times 7$, and therefore $b = \pm\sqrt{2025} = \pm 45$.
>
> Conversely, if $b = \pm 45$, then $x + \frac{7}{2x} = b$ is integer.

In particular, it appears that your version of the problem had no solution, which is why I thought that there was a typo in it and replaced it by my slightly modified version.

This post has been edited 1 time. Last edited by LaChouetteRieuse, Sep 7, 2011, 12:40 pm

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LaChouetteRi...
33 posts

Sep 7, 2011, 4:18 am — PM #99

A for a solution of Problem 43

> > mr10123 wrote:
> >
> > I see my problem was deemed "not polynomial enough". Oh well.
> >
> > Let P1(x) be the polynomial x-1/2, P2(x) be the polynomial (x-1/2)(x-1/3), and in general, Pn(x) is the polynomial with degree n, that has a leading coefficient of 1, and has the reciprocals of the first n prime numbers as roots. As n tends to infinity, find with proof the value of Pn(1).

here it is:

Solution 43

> Consider some integer n, as well as the first n prime numbers $p_1 < p_2 < \cdots < p_n$ as well as, for every integer $k \in \{1,\dots,n\}$, the biggest integer α_k such that $p_k^{\alpha_k} \le p_n$. Remark that $\prod_{k=1}^{n}(\sum_{i=0}^{\alpha_k} p_k^{-i}) \ge \sum_{i=1}^{p_n} \frac{1}{i} \ge \sum_{i=1}^{n} \frac{1}{i} \ge \ln(n)$.
>
> Moreover, for every integer k, we have
> $$1 + \frac{1}{p_k - 1} = \frac{1}{1-\frac{1}{p_k}} \ge \sum_{i=0}^{\alpha_k} p_k^{-i}.$$
> It follows that
> $$\prod_{k=1}^{n}(1 + \frac{1}{p_k-1}) \ge \ln(n)$$
> and that
> $$\sum_{k=1}^{n} \ln(1 + \frac{1}{p_k-1}) \ge \ln(\ln(n)).$$
> Since $\frac{1}{p_k-1} \to 0$ when $k \to \infty$, it follows that $\ln(1+\frac{1}{p_k-1})$ is equivalent to $-\ln(1-\frac{1}{p_k})$ when $k \to \infty$.
>
> Therefore, $\sum_{k=1}^{n} ln(1-\frac{1}{p_k}) \to -\infty$ when $n \to \infty$, and $P_n(1) = \exp(\sum_{k=1}^{n} \ln(1-\frac{1}{p_k})) \to \exp(-\infty) = 0$.

Problem 37

> Prove that the polynomial $X^6 + X^4 + X^2 + 3X + 4$ is irreducible over $\mathbb{Z}[X]$.

is still open.

---

anonymousl...
1142 posts

Sep 7, 2011, 9:08 am — PM #100

> LaChouetteRieuse wrote:
>
> A for a solution of Problem 43 here it is:
>
> Solution 43
>
> Problem 37 is still open.

can somebody tell me how can we prove that a polynomial is irreducible?

---

mr10123

Sep 8, 2011, 5:01 am — PM #101

I apologize for the ambiguity on my part

33 posts

I apologize for the ambiguity on my part.

There are such things as "irreducibility criteria" that are often used to prove that a polynomial is irreducible

---

Batominovski
1602 posts

Sep 8, 2011, 9:59 am • 1

PM #102

> LaChouetteRieuse wrote:
> Prove that the polynomial $X^6+X^4+X^2+3X+4$ is irreducible over $\mathbb{Z}[X]$.

Let $P(X)$ be the polynomial in question. Define

$$Q(x):=P(x+1)=x^6+6x^5+16x^4+24x^3+22x^2+15x+10\,.$$

We see that $Q(x)=x^6+x=x(x+1)\left(x^4+x^3+x^2+x+1\right)$ is the factorization into irreducible factors of $Q(x)$ over $\mathbb{F}_2$. We conclude that, if $P(x)$ were to be reducible over $\mathbb{Z}$, it then would have a linear or a quadratic factor. It is easy to verify that $P(x)>0$ as $x^4+4x+3\geq 0$ and $x^2-x+1>0$ for every real x. Thus, a linear factor is impossible. We now look for a quadratic factor x^2+bx+c where b is odd and c is even. Note that $c>0$ must be satisfied; for P has no real roots.

We can observe Q modulo 5 and find that the irreducible factorization is $Q(x)=x^2\left(x^4+x^3+x^2+4x+2\right)$. From this, we deduce that $c=10$ and $5\mid b$. Thus, for some integers p, q, and r,

$$\begin{aligned}x^6+6x^5+16x^4+24x^3+22x^2+15x+10&=Q(x)\\&=\left(x^2+bx+10\right)\left(x^4+px^3+qx^2+rx+1\right)\,.\end{aligned}$$

The second-degree coefficient is then $22=1+br+10q\equiv 1\pmod 5$, a contradiction.

PSs:
1) Why should $x^4+x^3+x^2+x+1$ be irreducible over $\mathbb{F}_2$, and why should $x^4+x^3+x^2+4x+2$ be so over $\mathbb{F}_5$?
2) I don't think there is a general way to show that a polynomial is irreducible over a ring. There are even polynomials which are reducible over all finite fields, but are indeed irreducible over $\mathbb{Q}$. A useful criterion would be the well known theorem of Eisenstein.

Problem 44: I realized this when trying to solve Problem 37. Define, as before,

$$P(X):=X^6+X^4+X^2+3X+4\,.$$

Basically, the problem is $P(X)$ may be factorized into two nontrivial factors over $\mathbb{F}_{11}$, one being a linear factor $X+3$ and the other being a quintic (namely, it has degree 5). Prove that this quintic polynomial is irreducible over $\mathbb{F}_{11}$.

Problem 45: Give an example of polynomials with integer coefficients that are irreducible over $\mathbb{Z}$ but reducible over every $\mathbb{F}_p$, where p is prime. Are there infinitely many such polynomials (which are not related by affine transformations such as $f(x)\mapsto f(ax+b)$)?

This post has been edited 1 time. Last edited by Batominovski, Sep 8, 2011, 6:34 pm

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LaChouetteRi...
33 posts

Sep 8, 2011, 3:15 pm • 1

PM #103

Here is a solution for problem 44, which I do not like very much because it uses too boring calculations:
Solution 44

> $P(X)=(X+3)Q(X)$, where $Q(X)=X^5+8X^4+10X^3+3X^2+3X+5$. If $Q(X)$ is not irreducible, then it has some prime factor of degree at most 2, which must divide $X^{11^2}-X$. Therefore, we just need to compute the GCD of $Q(X)$ and $X^{121}-X$.
> One checks, by some tedious calculation I let my favorite math software do, that
> $X^{121}-X\equiv 7X^4+2X^3+9X^2+5X+10$ mod. $Q(X)$
> $Q(X)\equiv X+7$ mod. $7X^4+2X^3+9X^2+5X+10$
> $7X^4+2X^3+9X^2+5X+10\equiv 4$ mod. $X+7$
> Therefore, $Q(X)$ and $X^{121}-X$ are relatively prime, which proves that $Q(X)$ is irreducible over $\mathbb{F}_{11}$.

Besides, here is also a solution for problem 45, which I like much more:
Solution 45

> Consider the polynomial $P(X)=X^4+1$ and some integer $k\in\mathbb{N}$.
> Note that $P(X^{2^k})$ is the 2^{k+2}-th cyclotomic polynomial, and therefore is irreducible over $\mathbb{Z}$.
>
> Now, consider some prime number p. We have three cases:
> if -1 is a square in $\mathbb{F}_p$, let ω be one of its square roots. Then, $P(X)=(X^2-\omega)(X^2+\omega)$.
> if 2 is a square in $\mathbb{F}_p$, let τ be one of its square roots. Then, $P(X)=(X^2-\tau X+1)(X^2+\tau X+1)$.
> if -2 is a square in $\mathbb{F}_p$, let θ be one of its square roots. Then, $P(X)=(X^2-\theta X-1)(X^2+\theta X-1)$.
>
> Note that, if $p\neq 2$ and neither -1 nor 2 is a square in $\mathbb{F}_p$, then -2 is a square in $\mathbb{F}_p$;
> if $p=2$, then -1 is a square in $\mathbb{F}_p$.
> Therefore, we must be in one of the above three cases, and we can write $P(X)=A(X)B(X)$ in $\mathbb{F}_p$, where A and B have degree 2.
> In particular, $P(X^{2^k})=A(X^{2^k})B(X^{2^k})$ is not irreducible in $\mathbb{F}_p$.
>
> And, the infinitely many polynomials $P(X^{2^k})$ having different degrees, they cannot be obtained from each other by affine transformations.

Batominovski
1602 posts

Sep 8, 2011, 6:39 pm

PM #104

> LaChouetteRieuse wrote:
> Here is a solution for problem 44, which I do not like very much because it uses too boring calculations:

Well, I didn't have a better solution either. I posted here because I was wondering if there would be a solution which didn't involve finding the GCD of $P(X)$ and the huge polynomial $X^{121}-X$. The other solution I've got is working by trial-and-error and checking that $P(X)$ has no quadratic factor. Excellent solution to Problem 45 anyway.

Learner94
635 posts

Sep 11, 2011, 2:57 am • 2 — PM #105

Alternative solution to problem 42: Let f be a polynomial over $\mathbb{R}$ of degree n with leading coefficient 1 and let $x_0 < x_1 < x_2 < \cdots < x_n$ be some integers. Prove that there exists $k \in \{0, 1, \cdots, n\}$ such that $|f(x_k)| \geq \frac{n!}{2^n}$

Solution:

Since the polynomial is monic, we have $\sum_{k=0}^{n} \left(\frac{f(x_k)}{\prod_{0\leq k\neq j\leq n}(x_k - x_j)} \right) = 1$ (Consequence of Lagrange's Interpolation)

By triangle inequality $\sum_{k=0}^{n} \left(\frac{|f(x_k)|}{\prod_{0\leq k\neq j\leq n}|(x_k - x_j)|} \right) \geq 1$

Note that x_k are integers and $x_0 < x_1 < x_2 ... < x_n$, so we have $\prod_{0\leq k\neq j\leq n} |x_k - x_j| \geq k!(n-k)!$

and obviously $\sum_{k=0}^{n} \frac{|f(x_k)|}{k!(n-k)!} \geq \sum_{k=0}^{n} \left(\frac{|f(x_k)|}{\prod_{0\leq k\neq j\leq n}|(x_k - x_j)|} \right) \geq 1$ (*)

From (*) we have $n! \leq \sum_{k=0}^{n} |f(x_k)| \binom{n}{k}$ (*)(*)

Now we assume on the contrary that $\forall k \in \{0, 1 \cdots n\}$, $|f(x_k)| < \frac{n!}{2^n}$

Then $\sum_{k=0}^{n} |f(x_k)| \binom{n}{k} < \frac{n!}{2^n} \sum_{k=0}^{n} \binom{n}{k} = n!$

Which contradicts (*)(*)

Hence our assumption was false , and there must exist some $k \in \{0, 1 \cdots n\}$ such that $|f(x_k)| \geq \frac{n!}{2^n}$

This post has been edited 1 time. Last edited by Learner94, Oct 3, 2012, 3:36 pm

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Learner94
635 posts

Sep 11, 2011, 3:11 am — PM #106

@LaChouetteRieuse

Click to reveal hidden text

> I request you to post a new problem. :)

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abhinavzand...
419 posts

Sep 11, 2011, 11:56 am — PM #107

Problem 46.
Let $f(x) = x^n + 5x^{n-1} + 3$. Prove that $f(x)$ is irreducible in $\mathbb{Z}[x]$.

Learner94
635 posts

Sep 11, 2011, 1:52 pm • 2 — PM #108

Note that $f(x)$ has no integral root. The prime 3 divides coefficient of $a_k x^k$, i.e a_k where $0 \leq k \leq n-2$. But 3 does not divide coefficient of x^{n-1} i.e 5, and 3^2 does not divide 3. So by Eisenstein's criterion $f(x)$ can have an irreducible factor of degree at least $n-1$. Let the irreducible factor of degree $n-1$ be called $P(x)$ and $f(x) = P(x)Q(x)$ But then $Q(x)$ is a linear polynomial over Z , so must have an integral root. That implies $f(x)$ irreducible over Z

Learner94
635 posts

Sep 13, 2011, 2:51 pm — PM #109

Problem 47

Prove that $x^5 - x + a$ is irreducible over Z if $5 \nmid a$.

---

hvaz
148 posts

Sep 28, 2011, 6:35 am — PM #110

> Learner94 wrote:
> Problem 47
>
> Prove that $x^5 - x + a$ is irreducible over Z if $5 \nmid a$.

Suppose by sake of contradiction that $P(x) = x^5 - x + a = f(x)g(x)$, where $deg(f) \geq 1$ and $deg(g) \geq 1$.

We have two cases:

$i)\, deg(f) = 1$ and $deg(g) = 4$.

Firstly, notice that in f we have $[x] = 1$ and in g we have $[x^4] = 1$, because P is monic and $f, g \in \mathbb{Z}[x]$.

In this case, let $f(x) = x - t$ and $g(x) = x^4 + px^3 + qx^2 + rx + s$. We have that $P(x) = f(x)g(x) = x^5 + x^4(p-t) + x^3(q-pt) + x^2(r-qt) + x(s-rt) - ts = x^5 - x + a \longrightarrow p = t \longrightarrow q = p^2 \longrightarrow r = p^3 \longrightarrow s = p^4 - 1 \longrightarrow -(p^5 - p) = a$. Yet, notice that $p^5 \equiv p \pmod 5$, what is a contradiction, since $5 \nmid a$.

$ii)\, deg(f) = 2$ and $deg(g) = 3$.

Again, notice that in f we have $[x^2] = 1$ and in g we have $[x^3] = 1$, because P is monic and $f, g \in \mathbb{Z}[x]$.

In this case, let $f(x) = x^2 - px + q$ and $g(x) = x^3 + sx^2 + tx + w$. We have that $P(x) = f(x)g(x)$ $= x^5 + x^4(s-p) + x^3(t-ps+q) + x^2(w-pt+qs) + x(qt-pw) + qw = x^5 - x + a \longrightarrow s = p \longrightarrow$

$t = p^2 - q \longrightarrow w = p^3 - 2pq \longrightarrow 3p^2q - q^2 = p^4 - 1 \longrightarrow q(3p^2 - q) = p^4 - 1\ (I)$ and $qw = a \longrightarrow pq(p^2 - 2q) = a\ (II)$.

If $5 \mid p$ or $5 \mid q$, using (II) we get that $5 \mid a$, contradiction. Therefore, by (I) we have that $5 \mid 3p^2 - q$, because $5 \mid p^4 - 1$, and then $5 \mid p^2 - 2q$, implying that $5 \mid a$, another contradiction.

So we can conclude that P is irreducible over $\mathbb{Z}[x]$.

EDIT: We can also have in $i)$ that $[x] = -1$ in f and that $[x^4] = -1$ in g or in $ii)$ that $[x^2] = -1$ in f and that $[x^3] = -1$ in g, but it is exactly the same.

This post has been edited 2 times. Last edited by hvaz, Sep 28, 2011, 7:34 am

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hvaz
148 posts

Sep 28, 2011, 6:41 am — PM #111

Problem 48

Is it true that there are infinitely many polynomials $P(x)$ such that $P(x^2+1) = (P(x))^2$ for all real x ?

Batominovski
1602 posts

Sep 28, 2011, 10:39 am • 1 — PM #112

> Learner94 wrote:
> Problem 47
>
> Prove that $x^5 - x + a$ is irreducible over Z if $5 \nmid a$.

A shorter solution (but the idea is pretty much the same as the former solution):

We shall instead show that $x^5 - x + a$ is not reducible over $\mathbb{F}_5$ (on which all subsequent work shall be done). If $x^5 - x + a$ were to be reducible over $\mathbb{F}_5$, then it must be divisible by some quadratic $x^2 + px + q \in \mathbb{F}_5[x]$. (A linear factor is trivially impossible. Why? Consult Fermat's Little Theorem.) However, the remainder of $x^5 - x + a$ divided by $x^2 + px + q$ is $\left(p^4 - 3p^2q + q^2 - 1\right)x + a + pq\left(p^2 - 2q\right)$. We must therefore have

$$pq\left(p^2 - 2q\right) = -a \neq 0 \quad (1)$$

and

$$p^4 - 3p^2q + q^2 = 1\,. \quad (2)$$

From Equation (1), we see that $p \neq 0$, $q \neq 0$, so

$$p^2 \neq 2q\,. \quad (3)$$

Since $p \neq 0$, $p^4 = 1$ by Fermat's Little Theorem. Apply the last result into Equation (2), and get $-3p^2q + q^2 = 0$, or $q = 3p^2$ (recall that $q \neq 0$, so division by q is righteous). Consequently, $2q = 6p^2 = p^2$, which contradicts Equation (3).

---

tenniskidper...
2376 posts

Oct 4, 2011, 9:06 am • 1 — PM #113

Problem 48

Is it true that there are infinitely many polynomials $P(x)$ such that $P(x^2+1) = (P(x))^2$ for all real x?

Solution: No it is not.

For constant $P(x)$, we have $P(x) \equiv 0$ and $P(x) \equiv 1$ are solutions. Assume now that $P(x)$ has degree greater than or equal to 1.

Lemma 1: This polynomial is either even or odd.

Proof: Plug $-x$ in for x. Then $P(-x)^2 = P(x)^2$. Hence for all values of x, either $P(x) = P(-x)$ or $P(x) = -P(-x)$. Consider the polynomials $P(x) - P(-x)$ and $P(x) + P(-x)$. One of these has an infinite number of roots, and hence is identically 0. Hence either $P(x) = P(-x)$ or $P(x) = -P(-x)$, and $P(x)$ is either even or odd.

Lemma 2: If the polynomial is odd, then it has infinitely many roots.

Proof: Plug in 0 for x into $P(x) = -P(-x)$. Thus $P(0) = 0$. We can show that $P(1) = P(0^2+1) = 0$ and that $P(2) = P(1^2+1) = 1 - 1 = 0$, and more generally, for all terms in the sequence $x_n = x_{n-1}^2 + 1$, $p(x_n) = p(x_{n-1})^2 = 0$. Hence it has an infinite number of roots.

Lemma 3: If the polynomial is even, then there is some polynomial $Q(x)$ so that $P(x) = Q(x)^2$.

Proof: Because the polynomial is even, it can be written as a polynomial in x^2. So if we plug in $\sqrt{x-1}$ in for x in $P(x)$, we get a polynomial in $x - 1$, and hence in x. Call this polynomial $Q(x)$, so that $P(\sqrt{x-1}) = Q(x)$. If we plug it in to the given identity, we have that $P(x) = P(\sqrt{x-1}^2 + 1) = P(\sqrt{x-1})^2 = Q(x)^2$.

Lemma 4: The $Q(x)$ defined above also satisfies $Q(x^2+1) = Q(x)^2$.

Proof: We know that $Q(x)^2 = P(x)$, and also that $Q(x) = P(\sqrt{x-1})$. Plug in x^2+1 for x in the last equation and hence it is true.

Using these 4 lemmas, we can show that since $P(x)$ has degree greater than zero, we can continue finding $Q(x)$'s from these $P(x)$'s and replacing $P(x)$ with this new $Q(x)$ until we find a P with an odd degree, which then has to be identically zero. So there are no solutions with $\deg P(x) \geq 1$, and hence there are (much) fewer solutions than infinitely many.

<u>**Problem 49**</u>: Let $a(x,y)$ be the polynomial $x^2y + xy^2$, and $b(x,y)$ the polynomial $x^2 + xy + y^2$. Prove that we can find a polynomial $p_n(a,b)$ which is identically equal to $(x+y)^n + (-1)^n(x^n + y^n)$. For example, $p_4(a,b) = 2b^2$.

KittyOK
349 posts

Oct 30, 2011, 10:21 pm • 3 — PM #114

Solution to Problem 49

Define $p_n(a,b)$ inductively by $p_1(a,b) = 0$, $p_2(a,b) = 2b$, $p_3(a,b) = 3a$ and $p_n(a,b) = bp_{n-2}(a,b) + ap_{n-3}(a,b)$ for $n \geq 4$. It can be easily shown by induction on n that this works.

Problem 50
Let $n \geq 3$ be an integer and $a_n, a_{n-1}, ..., a_2$ be real numbers such that $a_n > 0$.
Show that there exist real numbers r, s such that when we define the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + rx + s$$

then there exist real numbers x_1, y_1 such that the sequence $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}$ defined by $x_{k+1} = P(x_k), y_{k+1} = P(y_k)$ for $k \geq 1$ is strictly decreasing and strictly increasing respectively and $P\left(\frac{x_1+y_1}{2}\right) = \frac{x_1+y_1}{2}$.

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jatin
547 posts
Jan 17, 2012, 2:36 pm PM #115
KittyOK, please post the solution to Problem 50 and let us continue the marathon.

Learner94
635 posts
Jan 17, 2012, 2:52 pm PM #116
Perhaps he doesn't have solution to that problem, see here http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=440741&p=2573864#p2573864 ... So I request someone to post a new problem and continue the marathon

---

abhinavzand...
419 posts
Jan 17, 2012, 11:38 pm PM #117
Problem 51
Let $P(x) \in \mathbb{F}_p$, where p is a prime and the symbol denotes the finite number ring of remainders modulo p.
Find a very simplified recursion for the number of irreducible polynomials $P(x)$ of degree n.

I thought that I would give a problem which is not very hard but also is not trivial.
I think that this problem might be a little hard for this form but the people might still get it.
Hint

Think of strong induction.

Batominovski
1602 posts
Jan 19, 2012, 4:00 pm PM #118

> *abhinavzandubalm wrote:*
> **Problem 51**
> Let $P(x) \in \mathbb{F}_p$, where p is a prime and the symbol denotes the finite number ring of remainders modulo p.
> Find a very simplified recursion for the number of irreducible polynomials $P(x)$ of degree n.

Let μ be the arithmetic Möbius function. Note that each irreducible polynomial of degree n in variable x is a factor of $x^{p^n} - x$.
Let $t_p(n)$ be the number of such polynomials. It is obvious to see that $\sum_{d|n} d \cdot t_p(d)$ is exactly p^n. Therefore, using the Möbius inversion formula, we get $n \cdot t_p(n) = \sum_{d|n} \mu(d)p^{n/d}$, or

$$t_p(n) = \frac{1}{n}\sum_{d|n}\mu(d)p^{n/d}.$$

---

dinoboy
2903 posts
Jan 20, 2012, 4:53 am PM #119
Batominovski's solution cites a few nontrivial results and does not fully justify a few things (i.e. why no other polynomials divide $x^{p^n} - x$) but that problem is such a well-known theorem that it doesn't really matter.

Problem 52: Show that if $P(x) \in \mathbb{Z}[x]$, then P is square-free iff $P'(x)$ and $P(x)$ have no non-trivial common divisors in $\mathbb{Z}[x]$.

---

abhinavzand...
419 posts
Jan 20, 2012, 5:06 pm PM #120
I think that this calculus question does not belong here.

So please do not post calculus questions here.

Solution 52

The first part is very straight forward so I will try to prove the reverse.

Suppose that $P(x)$ and $P'(x)$ have a factor $(x-a)$ in common.

Then $P(x) = (x-a)Q(x)$ and $P'(x) = (x-a)R(x)$.

$$\implies \frac{dP(x)}{dx} = Q(x) + (x-a)Q'(x)$$

$$\implies P'(x) = (x-a)R(x) = Q(x) + (x-a)Q'(x)$$

Hence $(x-a)$ divides $Q(x)$.

Hence $P(x)$ has a factor of the form $(x-a)^2$.

Problem 53.
Prove that if $P(x) \in \mathbb{R}[x]$ of degree n and let $R = \{r_1, r_2, \cdots, r_n\}$ be the roots of the polynomial.
If $|r_{i+1} - r_i| \geq |r_i - r_{i-1}| \quad \forall\ i$ then prove that $|P(x)|$ attains its maximum in $[r_1, r_n]$ in the interval $[r_{n-1}, r_n]$.

---

mlm95
245 posts
Jan 20, 2012, 7:03 pm PM #121
sorry i don't know the marathon's rules can we give link to the problems?

abhinavzand...
419 posts

Jan 20, 2012, 9:41 pm — PM #122

If I or someone else has reposted an older problem then you can give a link.

tenniskidper...
2376 posts

Jan 21, 2012, 1:18 am — PM #123

We must assume that all roots are real, or else the interval $[r_1, r_n]$ doesn't make sense... And also we must assume the roots are increasing order or else we can have $r_{n-1} > r_n > r_1$ and the question is wrong. But with those in mind, the question transforms into this question.

NP #54: Show that the graph of the polynomial $p(x)$ is symmetric about the point (a, b) iff there is a polynomial $q(x)$ such that $p(x) = b + (x-1)q((x-a)^2)$.

dinoboy
2903 posts

Jan 21, 2012, 3:55 am — PM #124

The solution for 52 is wrong, common factors do not have to be linear polynomials :(But this is easily fixed through the use of minimum polynomials.
And plus the forward direction is considerably harder than the backward considering the fact that the forward direction is not even true for all number fields.

EDIT : oops, I remembered wrong and it's the backwards direction that's not always true. Yeah the forward direction is fairly trivial.
For those interested though, an easy counterexample to the backwards direction is take the polynomial ring $(\mathbb{Z}_2(y))[x]$. Take the polynomial $P(x) = yx^2 + 1$ and you have your counterexample :) P is irreducible if I remember correctly and get $\gcd(P(x), P'(x)) = yx^2 + 1 \neq 1$!

Lastly, it is not a calculus question if done correctly, but whatever.

And problem 54 is wrong because this would imply $p(1) = b$, which is clearly false if you simply take $p(x) = x$ and $(a, b) = (0, 0)$.
I'm not even sure how the $x - 1$ and $(x-a)^2$ factors arose, so I can't come up with how to fix this problem :(

Problem 55: Show that $x^n + p^2x^{n-1} + p^2x^{n-2} + \dots + p^2$ is irreducible over $\mathbb{Z}[x]$ for odd primes p and positive integer n.

This post has been edited 1 time. Last edited by dinoboy, Jan 21, 2012, 3:13 pm

tenniskidper...
2376 posts

Jan 21, 2012, 3:58 am — PM #125

I apologize, Kalva gave this question. I redact it and rewrite it as $p(x) = b + (x-a)q((x-a)^2)$ instead.

dinoboy
2903 posts

Jan 21, 2012, 4:10 am — PM #126

OK, that is valid. To move on here's the solution:

Solution 54

> Define $R(x) = p(x+a) - b$. Note that R is symmetric about the origin.
> Clearly $R(0) = 0$, and thus $x|R(x)$. When we divide out x clearly then $R(x)/x$ is symmetric about the y-axis and thus is even and the sum of x^{2n} for various n
> Then $R(x)/x = q(x^2)$ for some polynomial q and thus $R(x) = x \cdot q(x^2)$.
> Then plugging in $x = x - a$ we have $(x-a)q((x-a)^2) = p(x) - b \implies p(x) = b + (x-a)q((x-a)^2)$, giving one direction.
>
> To prove the converse, notice $p(x+a) - b = x \cdot q(x^2)$ and it immediately follows $p(x+a) - b$ is symmetric about the origin and the result follows.

tenniskidper...
2376 posts

Jan 21, 2012, 5:12 am — PM #127

Click to reveal hidden text

Question 56: The real numbers a, b, A, B satisfy $(B-b)^2 < (A-a)(Ba-Ab)$. Show that the quadratics $x^2 + ax + b = 0$ and $x^2 + Ax + B = 0$ have real roots and between the roots of each there is a root of the other.

mlm95
245 posts

Jan 22, 2012, 2:38 am — PM #128

> tenniskidperson3 wrote:
> **Question 56**: The real numbers a, b, A, B satisfy $(B-b)^2 < (A-a)(Ba-Ab)$. Show that the quadratics $x^2 + ax + b = 0$ and $x^2 + Ax + B = 0$ have real roots and between the roots of each there is a root of the other.

sorry I'm a little lazy to do the problem completely i just show that both of the quadratics have real roots.

Click to reveal hidden text

> assume for the sake of contradiction that $x^2 + ax + b = 0$ doesn't have real roots then:
>
> $a^2 < 4b$ thus b is positive $\Rightarrow$ $(B-b)^2 b < (Ab-ab)(Ba-Ab)$
>
> $\Rightarrow$ $2b(B-b)^2 + (Ab-ab)^2 + (Ba-Ab)^2 < (Ba-ba)^2$
>
> by cauchy-schwarz inequality we have $(Ab-ab)^2 + (Ba-Ab)^2 \geq \frac{(Ba-ab)^2}{2}$ and then the contradiction will be obvious

Stephen
402 posts

Feb 2, 2012, 7:33 pm — PM #129

Okay, I'll just post the next problem.

Problem 57

P, Q, R are integer-coffecient polynomials. P is irreducible in $Z[x]$.

If $R(x)|P(Q(x))$ is satisfied, prove that $degR$ is not less than $degP$.

(Of course, R is not a constant polynomial.)

mlm95
245 posts

Feb 9, 2012, 2:16 am — PM #130

problem 57

see this http://www.efnet-math.org/math_tech/BonusProbDec0806.pdf

can anyone produce an elementary proof for this problem and give the next one?

KittyOK
349 posts

Mar 10, 2012, 12:59 pm — PM #131

Problem 58
Find all polynomials $P(x)$ and $Q(x)$ with **real** coefficients such that for infinitely many positive integers n the equality $P(1)P(2)...P(n)=Q(n!)$ is satisfied.

utsab001
162 posts

May 23, 2012, 11:29 pm — PM #132

> *Batominovski wrote:*
>
> > *mousavi wrote:*
> > **problem 2**
> >
> > Find all polynomials $P(x)$ with real coefficient such that:
> >
> > $P(0)=0,[P[P(n)]]+n=4[P(n)],(n\in N)$
> >
> > **note** : [ ] is the floor function
>
> Let $p:=\deg(P)$. So, $\max\left\{p^2,1\right\}=p$ or $p=1$. Because $P(0)=0$, $P(x)=\alpha x$ for some real α. Consider
>
> $$4\alpha=\lim_{n\to\infty}\frac{4\lfloor P(n)\rfloor}{n}=\lim_{n\to\infty}\frac{\left\lfloor P\left(\lfloor P(n)\rfloor\right)\right\rfloor}{n}+1=\alpha^2+1\,.$$
>
> Therefore, $\alpha=2\pm\sqrt{3}$. It is easy to see that $\alpha\neq 2-\sqrt{3}$ by checking the case $n=1$. So, $\alpha=2+\sqrt{3}$, and write $\bar{\alpha}=2-\sqrt{3}$. Now, for every positive integer n,
>
> $$\begin{aligned}4\lfloor P(n)\rfloor-\left\lfloor P\left(\lfloor P(n)\rfloor\right)\right\rfloor&=4\lfloor\alpha n\rfloor-\lfloor\alpha\lfloor\alpha n\rfloor\rfloor\\&>4\lfloor\alpha n\rfloor-\alpha\lfloor\alpha n\rfloor=\bar{\alpha}\lfloor\alpha n\rfloor\\&>\bar{\alpha}(\alpha n-1)=n-\bar{\alpha}>n-1\,.\end{aligned}$$
>
> On the other hand,
>
> $$\begin{aligned}4\lfloor P(n)\rfloor-\left\lfloor P\left(\lfloor P(n)\rfloor\right)\right\rfloor&=4\lfloor\alpha n\rfloor-\lfloor\alpha\lfloor\alpha n\rfloor\rfloor\\&<4\lfloor\alpha n\rfloor-\alpha\lfloor\alpha n\rfloor+1=\bar{\alpha}\lfloor\alpha n\rfloor+1\\&<\bar{\alpha}(\alpha n)+1=n+1\,.\end{aligned}$$
>
> Thus, the equality $\left\lfloor P\left(\lfloor P(n)\rfloor\right)\right\rfloor+n=4\lfloor P(n)\rfloor$ is satisfied for all $n\in\mathbb{N}$. Hence, $P(x)=\left(2+\sqrt{3}\right)x$ satisfies the condition.
>
> No problem to submit. Maybe I will think of one later.

sorry to ask but why did you take $p(x)=\alpha x$?

This post has been edited 1 time. Last edited by utsab001, Jun 16, 2012, 3:25 am

swkratis
34 posts

May 30, 2012, 4:02 pm — PM #133

> *dinoboy wrote:*
> **Problem 55:** Show that $x^n+p^2x^{n-1}+p^2x^{n-2}+...+p^2$ is irreducible over $\mathbb{Z}[x]$ for odd primes p and positive integer n.

This is just an aplication of Eisenstein's creterion for the prime p.

utsab001
162 posts

May 30, 2012, 4:40 pm — PM #134

> *swkratis wrote:*
>
> > *dinoboy wrote:*
> > **Problem 55:** Show that $x^n+p^2x^{n-1}+p^2x^{n-2}+...+p^2$ is irreducible over $\mathbb{Z}[x]$ for odd primes p and positive integer n.
>
> This is just an aplication of Eisenstein's creterion for the prime p.

How so? $p^2|p^2$ = the constant term.. but the criterion says it should not..

swkratis
34 posts

May 31, 2012, 12:07 am — PM #135

ohh yes.You are right.My mistake.

Particle
179 posts

Jun 26, 2012, 11:44 am — PM #136

Seems like problem 58 has stopped the marathon.Shouldn't we restart it?Someone please post an easy problem.

subham1729

Jun 26, 2012, 12:33 pm — PM #137

1479 posts

> Particle wrote:
> Seems like problem 58 has stopped the marathon.Shouldn't we restart it?Someone please post an easy problem.

Ok, now an easy problem

Determine all polynomials $P(x)$ with rational coefficients such that for all $|x|\le 1$
$P(x)=P(\frac{\sqrt{(3-3x^2)}-x}{2})$

Jun 26, 2012, 2:37 pm PM #138

pco
15396 posts

> subham1729 wrote:
> Determine all polynomials $P(x)$ with rational coefficients such that for all $|x|\le 1$
> $P(x)=P(\frac{\sqrt{3-3x^2}-x}{2})$

Let us forget the condition about rational coefficients.

It's easy to check that any constant polynomial fit, no degree $1,2$ polynomials fit and that $x^3-\frac{3}{4}x$ is a solution.

Let then $P(x)$ a solution with degree ≥ 3 and $P(x)=(x^3-\frac{3}{4}x)Q(x)+ax^2+bx^2+c$ it's euclidian division by $x^3-\frac{3}{4}x$

From original property, we have $P(0)=P(\frac{\sqrt 3}{2})=P(-\frac{\sqrt 3}{2})$ and so $a=b=0$

So $P(x)=(x^3-\frac{3}{4}x)Q(x)+c$ and so $Q(x)$ is also a solution.

It's then easy to conclude $\boxed{P(x)=H(x^3-\frac{3}{4}x)}$ where $H(x)$ is any polynomial $\in\mathbb{Q}[X]$ (in order to add back the rational coefficients constraints, whose interest I dont understand).

Jul 8, 2012, 11:27 pm • 1 PM #139

dien9c
162 posts

Problem 60. Let $P(x)=a_0x^n+...+an-1x+a_0$ satisfy

$$P(x)\le 1 \qquad \text{with} x\in[-1;1]$$

Prove that

$$|a_0+a_1x+...+a_nx^n|\le 2^{n-1} \qquad \text{with } x\in[-1;1]$$

Jul 14, 2012, 8:50 am PM #140

dien9c
162 posts

> dien9c wrote:
> Problem 60. Let $P(x)=a_0x^n+...+an-1x+a_0$ satisfy
>
> $$P(x)\le 1 \qquad \text{with} x\in[-1;1]$$
>
> Prove that
>
> $$|a_0+a_1x+...+a_nx^n|\le 2^{n-1} \qquad \text{with } x\in[-1;1]$$

Source: Walter Janous, Crux mathematicorum

Jul 19, 2012, 8:53 pm • 2 PM #141

Learner94
635 posts

Please see http://www.artofproblemsolving.com/Forum/viewtopic.php?f=38&t=489722&p=2745557#p2745557 .

Jul 21, 2012, 10:08 am PM #142

dien9c
162 posts

Problem 61.[Mathlink Contest]. Let a be nonzero integer, and n be another integer, $n\ge 3$. Prove that $P(x)$ is irreducible with

$$P(x)=x^n+ax^{n-1}+...+ax-1$$

Aug 28, 2012, 12:39 pm • 1 PM #143

mlm95
245 posts

solution for problem 61

> I'll give you the sketch of the proof.(if you want more explanation I'll do)
> **lemma.** assume that $p(x)=a_nx^n+...+a_0$ such that $0<a_n\le a_{n-1}\le ...\le a_0$. so if α is a root of P then $|\alpha|>1$ or $\alpha=-1$
> now assume that a is positive otherwise consider $-x^nP(\frac{1}{x})$.then obviously P has a real root in the interval $(0,1)$ let it be β.so $P(x)=(x-\beta)Q(x)$.then compute coefficients of Q in terms of β and a. it's easy to see that if $Q(x)=x^{n-1}+b_{n-2}x^{n-2}+...+b_0$ then $1\le b_{n-2}\le ...\le b_0$ so Q satisfies the condition of the above lemma.also -1 can't be a root of Q hence all roots of Q are outside the unit circle with 0 as it's center.now if $P(x)=f(x)g(x)$ all roots of f or g must occur outside the unit circle.but this is impossible because $P(0)=f(0)g(0)=-1$ so P is irreducible.

Sep 2, 2012, 1:07 pm PM #144

lehungvietbao
1045 posts

oh! Please post new problems!
I like this topic .:)

Sep 4, 2012, 5:37 pm PM #145

gcshudder

Since no one posts questions..here is **question62**.

5 posts

prove that there exists a real number c>0 such that whenever each of the numbers $a_0, a_1, ..., a_n$ are 1 or -1 and the polynomial $(x-1)^k$ divides the polynomial $a_0 + a_1x + ... + a_nx^n$ then $k < cln^2(n+1)$

---

mlm95
245 posts

Oct 3, 2012, 12:23 am PM #146

here is a link for the solution of above problem but in hungarian. hope someone can explain it.

http://www.komal.hu/verseny/feladat.cgi?a=feladat&f=A532&l=en

---

alibez
357 posts

Oct 5, 2012, 8:31 pm PM #147

Please post new problems....

---

alibez
357 posts

Oct 11, 2012, 6:48 pm PM #148

63

> prove that $P(x)$ is irreducible
>
> $$p(x) = \sum_{i=1}^{p} ix^{p-i}$$

Johann Peter...
289 posts

Oct 12, 2012, 10:23 pm PM #149

> alibez wrote:
> 63

p is prime?

---

alibez
357 posts

Oct 13, 2012, 5:33 am PM #150

yes

Johann Peter...
289 posts

Oct 13, 2012, 11:10 pm PM #151

> alibez wrote:
> yes

Just thinking, no solution indeed

MBGO
315 posts

Oct 15, 2012, 12:51 am PM #152

> alibez wrote:
> 63

Click to reveal hidden text

> we want to show that $P(x)$ is irreducible by contradiction. so let us assume $P(x) = f(x).q(x)$.
> first of all let us find a way to show contradiction,we are going to calculate $P(x)$ and change it to a useful form; so :
> ======================
> $$\sum_{i=1}^{p} ix^{p-i} = (\sum_{i=0}^{p-1} x^i) + (\sum_{i=0}^{p-2} x^i) + ... + (x+1) + (1) = \frac{x^p - 1}{x-1} + \frac{x^{p-1} - 1}{x-1} + ... + \frac{x^2-1}{x-1} + \frac{x-1}{x-1} =$$
> ======================
> $$\frac{(\sum_{i=0}^{p} x^i) - (p+1)}{x-1} = \frac{x^p + \frac{x^p - 1}{x-1} - (p+1)}{x-1} = \frac{x^p(x-1) + x^p - 1 - (x-1)(p+1)}{(x-1)^2}$$
> ======================
> as we want to eliminate some terms and change $"thisPolynomial"$ into a $"simple-version"$ we put $x+1$ instead of x; hence we have :
> ======================
> $$\Rightarrow \frac{(x+1)^{p+1} - x(p+1) - 1}{x^2} = f(x).g(x) \rightarrow (x+1)^{p+1} - x(p+1) - 1 = \sum_{i=0}^{p+1} \binom{p+1}{i} x^{p+1-i} - 1 - x(p+1) = \sum_{i=0}^{p-1} x^{p+1-i} - 1 = f(x).g(x).x^2$$
> ======================
> it's obviouse that the right side is divisible by x^2 and the left side modulo x^2 is -1;thus we get contradiction here and $P(x)$ is irreducible. so $TIWWWTP$.

.

MBGO
315 posts

Oct 15, 2012, 1:00 am PM #153

problem 2^6

> according to the Polynomial $P(x) = a_nx^n + ... + a_1x + a_0$ we know $a_0 \neq 0$ and there exist an integer m such that $\binom{m}{n} \leq |\frac{a_m}{a_0}|$.
>
> prove that $P(x)$ has got a Zero in the field $|z| < 1$.

---

alibez
357 posts

Oct 15, 2012, 1:24 am PM #154

[hide="hint of problem 2^6"]

$p(x) = x^n p(\frac{1}{x})$

[/hide]

alibez
357 posts

Oct 16, 2012, 1:01 pm — PM #155

65

find the number of real roots of the following polynomial

$p(x) = nx^n - x^{n-1} - ... - 1$

Johann Peter...
289 posts

Oct 17, 2012, 8:10 pm — PM #156

> alibez wrote:
>
> 65

I think it's right...

Johann Peter...
289 posts

Oct 17, 2012, 8:53 pm — PM #157

Click to reveal hidden text

alibez
357 posts

Oct 17, 2012, 9:20 pm — PM #158

> alibez wrote:
>
> 65

my solution

$p(x)(x-1) = f(x)$ so we have $f(x) = nx^{n+1} - x^n(n+1) - 1$

Suppose $\Rightarrow Q(x) = x^n f(\frac{1}{x})$ so we have $Q(x) = -x^{n+1} - x(n+1) + n \Rightarrow Q'(x) = x^n(-n-1) - (n+1)$ if α is root of Q Then $\frac{1}{\alpha}$ is root of f and ...... (2 if n is even and 1 if n is odd)

Learner94
635 posts

Oct 17, 2012, 11:17 pm • 1 — PM #159

Solution to problem 66 (?)

First we do the linear transformation $x \to x - 1/2$, now it suffices to show that $f(x) = (x^2 - 1/4)^{2^n} + 1$ is irreducible over the field $\mathbb{Q}$. Now $f(x) = \prod_{r = 1}^{2^n} \left (x^2 - \frac{1}{4} - e^{\frac{i (2r-1) \pi} {2^n} \right)}$.

Let $t \in \mathbb{C}$ be such that $t^2 - \frac{1}{4} - e^{\frac{i(2r-1)\pi}{2^n}} = 0$. We easily see that $\mathbb{Q}(e^{\frac{i(2r-1)\pi}{2^n}}) \subset \mathbb{Q}(t)$.

Now since $x^{2^n} + 1$ is irreducible over $\mathbb{Q}$, $[\mathbb{Q}(e^{\frac{i(2r-1)\pi}{2^n}}) : \mathbb{Q}] = 2^n$. Since $\mathbb{Q}(t) \neq \mathbb{Q}(e^{\frac{i(2r-1)\pi}{2^n}})$ (*), we have $[\mathbb{Q}(t) : \mathbb{Q}(e^{\frac{i(2r-1)\pi}{2^n}})] \geq 2$.

Now by multiplicative property of degrees of field extensions we have $[\mathbb{Q}(t) : \mathbb{Q}] = [\mathbb{Q}(t) : \mathbb{Q}(e^{\frac{i(2r-1)\pi}{2^n}})][\mathbb{Q}(e^{\frac{i(2r-1)\pi}{2^n}}) : \mathbb{Q}] \geq 2^{n+1}$. Since $\deg(f(x)) = 2^{n+1}$, it follows that $f(x)$ is the minimal polynomial of t. So it is irreducible.

Johann Peter...
289 posts

Oct 19, 2012, 6:03 pm — PM #160

> Learner94 wrote:
>
> Solution to problem 66 (?)

No more elementary solution? Or the properrties used are elementary and I am only an ignorant on the subject?

mlm95
245 posts

Oct 19, 2012, 9:23 pm — PM #161

check this link to see another solution:

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=38&t=49799

and a generalization is here:

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=2753780&sid=fcdebbb051a63c27bf7bd4dfc3e55f73#p2753780

Johann Peter...
289 posts

Oct 20, 2012, 5:46 am — PM #162

Problem 65
Click to reveal hidden text

Let it $P(x) = 1 + x^{h_1} + x^{h_2} + x^{h_3} + \cdots + x^{h_n}$ where h_i is increasing. Show that all real roots of P are greater than $\frac{1-\sqrt{5}}{2}$

Farenhajt
5167 posts

Oct 20, 2012, 5:54 am — PM #163

> Johann Peter Dirichlet wrote:
>
> Problem 65
> Click to reveal hidden text

Doesn't seem right

$P(x) = 1 + x$ or $P(x) = 1 + x + x^2 + x^3$ have a real root $x = -1$, and $-1 < \frac{1-\sqrt{5}}{2}$

Johann Peter...
289 posts

Oct 20, 2012, 8:11 pm — PM #164

> Farenhajt wrote:
>
> > Johann Peter Dirichlet wrote:
> >
> > Problem 65
> > Click to reveal hidden text
>
> Doesn't seem right

Yes, you are right

dinoboy
2903 posts

Oct 23, 2012, 11:01 am • 2 — PM #165

The correct statement should be all roots have modulo greater than $1/\phi$, where ϕ is the golden ratio. This is a copy and paste for this problem so the notation is slightly wrong.

Solution 65

Let our polynomial $p(x) = 1 + x^{a_1} + x^{a_2} + ... + x^{a_n}$ have a root r satisfying $|r| < \frac{1}{\phi}$ where $0 < a_1 < a_2 < ... < a_n$
.

Now note that $|p(r)| \geq 1 - |r|^{a_1} - |r|^{a_2} - ... - |r|^{a_n}$
Suppose $a_1 \neq 1$. Then we immediately get $|p(r)| > 0$ because $1 - \phi^{-2} - \phi^{-3} - ... = 0$. But this is a contradiction, hence $a_1 = 1$.
Hence $p(x) = 1 + x + x^{a_2} + x^{a_3} + ... + x^{a_n}$
Observe that $p(x)(1-x)$ has a root at $x = r$ still.
However, $p(x)(1-x) = 1 - x^2 + (1-x)x^{a_2} + (1-x)x^{a_3} + ... + (1-x)x^{a_n}$
$= 1 + b_1x^{c_1} + b_2x^{c_2} + ... + b_mx^{c_m}$
Note that $c_1 \geq 2$ and $b_i \in \{-1, 0, 1\}$.
But then we have $|p(r)(1-r)| > 0$ again, a contradiction, hence $|r| > 1/\phi$ and thus we are done.

Star-Breaker
31 posts

Oct 23, 2012, 7:40 pm — PM #166

Problem 66

Let $P(x)$ be a polynomial with integer coefficients such that $P(0) = 0, 0 \leq P(1) \leq 10^7$ and there exist integers a, b such that $P(a) = 1999$, $P(b) = 2001$ then find $P(1)$.

subham1729
1479 posts

Oct 23, 2012, 8:32 pm • 1 — PM #167

Solution 66

First of all we can take $P(x) = xQ(x)$.
As $Q(x)$ integer so $Q(x) - Q(a) = (x-a)R(x)$. Also $r(x)$ integer so $R(x) - R(b) = (x-b)S(x)$.
Combining we get $P(x)= xQ(a) + x(x -a)r\R(b) + x(x -a)(x -b)S(x)$
Now note $|a - b| = 1, 2$ and $a|1999, b|2001$.
Now setting $x = 1$ it's easy to see we obtain
$P(1) \equiv -1 mod((2000.2002.S(1)), +1 mod((1998.2000.S(1)), -3331 mod(8S(1))$
Now it's easy to see any $S(1)$ satisfies this.

Johann Peter...
289 posts

Oct 30, 2012, 7:47 pm — PM #168

Problem 67:

Find all real polynomails such that $P(x)P(2x^2-1) = P(x^2)P(2x-1)$.

subham1729
1479 posts

Oct 30, 2012, 8:37 pm • 1 — PM #169

Solution 67

Letting $g(x) = \frac{P(2x-1)}{P(x)}$ we've $g(x^{2^{-n}})$ is constant for all $n \in \mathbb{N}$ and $x > 0$
Now as the sequence $\{x^{2^{-n}}\}$ goes to $x^0 = 1$ when $n \to \infty$. Thus $g(x^{2^{-n}}) \to g(1)$
So $P(x) = P(2x-1)$ for all $x > 0$ thus P has infinitely many roots which implies this is constant polynomial.

Edited:Ohh ya Ali you are correct, I did wrong writing $\frac{P(1)}{P(1} = 1$, if we consider the case $P(1) = 1$
Then $P(x) = (x-1)g(x)$ again get last equation but with g. So we can take $P(x) = (x-1)^n f(x)$ such that $(x-1)^n||P(x)$
Putting again we must have $f(x) = c$

Edited now

This post has been edited 2 times. Last edited by subham1729, Oct 30, 2012, 9:13 pm

alibez
357 posts

Oct 30, 2012, 8:57 pm — PM #170

> subham1729 wrote:
>
> Solution 67

it is wrong !!!!

it is wrong !!!!

hint 67

$p(x)=(x-1)^n$

we have
$p(x)=2^n p(x)+R(x)$
and prove that $R(x)\equiv 0$

axa
15 posts
Oct 31, 2012, 8:14 am
PM #171

Problem 69

Find all non constant polynomials $P(x),Q(x)$ with real coefficients such that $P(x)Q(x+1)=P(x+2012)Q(x)$

subham1729
1479 posts
Oct 31, 2012, 11:23 am • 1
PM #172

Solution 69

Taking $R(x)=\prod_{i=0}^{2011} P(x+i)$ we've $\frac{Q(x)}{R(x)}=\frac{Q(x+1)}{R(x+1)}$
Thus clearly we've $\frac{Q(x)}{R(x)}=\frac{Q(x+n)}{R(x+n)}=a_n$ for all $n\in\mathbb{N}$
Now $deg(Q(x)>deg(P(x))\implies a_n\to\infty, deg(Q(x))\le deg(P(x))\implies a_n\to c$ with c constant with $n\to\infty$
We must have a_n goes to a constant term cause x is fixed. So we must have $\frac{Q(x)}{R(x)}=c\implies Q(x)=c\prod_{i=0}^{2011}P(x+i)$

MBGO
315 posts
Nov 26, 2012, 10:21 pm
PM #173

problem 70

find a polynomial P which $P(x)$ is Integer Only for x a perfect squares.

zabihpourma...
54 posts
Nov 27, 2012, 2:26 pm • 1
PM #174

Solution 70

We prove doesn't exist .We have $p(x)=a_nx^n+a_{n-1}x^{n-1}...+a_0$.At first it is clear that $p(x)$ can't be constant.So $n>0$
We can assume $a_n>0$ so $\lim_{x\to+\infty} p(x)=+\infty$
Assume that $p(x)$ in $[r,\infty]$ is strictly increasing. (1) (We know that r exists)
and Assume $\lceil p(r)\rceil=m$(integer) so $m=p(n_0^2), m+1=p(n_1^2),...$

so:$p(n_0^2)+k=p(n_k^2)$

we know that:
$n_0+k\le n_k\Rightarrow (n_0+k)^2\le(n_k)^2\overset{(1)}{\to}p((n_0+k)^2)\le p((n_k)^2)=p(n_0^2)+k\Rightarrow p(n_0^2)\ge p((n_0+k)^2)-k=a_n((n_0+k)^2)^n+a_{n-1}((n_0+k)^2)^{n-1}...a_1((n_0+k)^2)+a_0-k=g(k)$

so:
$\begin{cases}\lim_{k\to+\infty} g(k)=+\infty\\ g(k)\le p(n_0^2)\end{cases}\Rightarrow Contradiction!$
Proved End

MBGO
315 posts
Nov 27, 2012, 5:14 pm
PM #175

> zabihpourmahdi wrote:
> Assume $\lceil p(r)\rceil=m$(integer) so $m=p(n_0^2), m+1=p(n_1^2),...$

why this assumption correct that for any $i>=m$ there exist a k such that $p(k)=i$?

zabihpourma...
54 posts
Nov 27, 2012, 7:34 pm
PM #176

Because we assume that $p(x)$ is strictly increasing in $[r,\infty]$ so $p(x)$ is surjective in $[r,\infty]$ so for all $i>=m$ there exist x such that $p(x)=i$ but as the question said when p(x) is integer we have x is a prefect square so for all $i>=m$(i is integer) there exist k which $p(k^2)=i$

MBGO
315 posts
Nov 27, 2012, 7:48 pm
PM #177

Does the matter"strictly increasing" cause "surjectivity"? does the polynomial P continuous?

This post has been edited 1 time. Last edited by MBGO, Nov 28, 2012, 10:44 pm

zabihpourma...
54 posts
Nov 27, 2012, 9:23 pm
PM #178

I mean from surjective in $[r,\infty]$ is that $p(x)$ covers[$\lceil p(r)\rceil$, ∞] in $[r,\infty]$ because it is strictly increasing.
@Mahan

MBGO
315 posts
Nov 28, 2012, 10:41 pm
PM #179

Your Solution is Completely Correct 🙂
@Mahdi

alibez
357 posts
Jan 14, 2013, 10:35 pm
PM #180

problem 71:

find all polynimial such that :

$p(x^2)+x(3p(x)+p(-x))=p(x)^2+2x^2$

p.s: please post a new problem

This post has been edited 1 time. Last edited by alibez, Jan 16, 2013, 7:25 pm

alibez
357 posts

Jan 15, 2013, 10:07 pm • 1 — PM #181

No one????

Batominovski
1602 posts

Jan 15, 2013, 11:18 pm — PM #182

Problem 71(?): Find all polynomials $p(x)$ such that

$$p\left(x^2\right)+x\big(3p(x)+p(-x)\big)=\big(p(x)\big)^2+2x^2\,.$$

I have found x, $x+x^{2k+1}$, and $2x+x^{2k}$ via stupid coefficient balancing (here, k is a nonnegative integer). It is not difficult to verify that, if p is of degree $n\geq 3$, then it is monic and the coefficients of the intermediate terms are all 0, except for the linear and the constant terms. Do you have a wiser solution?

Next Problem (72?): Find all nonnegative integers n such that the polynomial $x^n+x^2y+xy^2+y^n$ is reducible in $\mathbb{C}[x,y]$.

Batominovski
1602 posts

Jan 26, 2013, 7:40 am • 1 — PM #183

Problem 73(?): Find all pairs $(a,b)\in\mathbb{C}\times\mathbb{C}$ such that $(a,b)\neq(0,0)$ and the polynomial

$$P\left(z_1,z_2,z_3\right)=a\left(z_1^3+z_2^3+z_3^3\right)+b\left(z_1z_2z_3\right)$$

is reducible in the ring $\mathbb{C}\left[z_1,z_2,z_3\right]$. If you know anything about projective spaces, you can describe (a,b) via its identification to the point $[a:b]$ in the 1-dimensional complex projective space $\mathbb{PC}^1$.

mlm95
245 posts

Jan 27, 2013, 2:49 am • 2 — PM #184

solution 73

> if P is reducible over $\mathbb{C}\left[z_1,z_2,z_3\right]$ it has a singular point.so the equations below must have a non-zero solution.
> $\begin{cases}3az_1^2+bz_3z_2=0\\3az_2^2+bz_3z_1=0\\3az_3^2+bz_1z_2=0\end{cases}$
> and after some calculation we get that these equations have a common non-zero solution iff $27a^3+b^3$ or $a=0$

baysa
86 posts

Jan 29, 2013, 9:41 pm — PM #185

Problem 74. n is odd and $P_1(x),\ldots,P_n(x)$ are non constant polynomials. If $P_1(P_2(x))=P_2(P_3(x))=\ldots=P_{n-1}(P_n(x))=P_n(P_1(x))$ then prove that $P_1(x)=\ldots=P_n(x)$.

alibez
357 posts

Feb 27, 2013, 8:49 pm • 2 — PM #186

hint

> We assume : $S_i(x)=P_{i+1}(x)-P_i(x)$ so we have $degS_i=k\leq n-1$ .....

like this http://www.artofproblemsolving.com/Forum/resources.php?c=137&cid=39&year=2009&sid=59fa4b1b8475594c4ec5508734c46484 $(p3)$

lehungvietbao
1045 posts

Apr 15, 2013, 5:14 pm — PM #187

Problem 75
Suppose $P(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$ satisfying $|P(x)|\leq 1,\quad\forall x\in[-1,1]$.
Prove that $Q(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ has property $|Q(x)|\leq 2^{n-1},\quad\forall\in[-1,1]$.

Saint123
183 posts

May 2, 2013, 11:13 am — PM #188

#76
Let $P(x)$ be a polynomial with integer coefficients. Given that $P(x)>x$ for all positive integers x.
Define $x_1=1$ and for any $n\in\mathbb{N}$, and $n>1$, define $x_n=P(x_{n-1})$. Given that for any $N\in\mathbb{N}$, there exists k such that $N|x_k$. Show that $P(x)=x+1$

subham1729
1479 posts

May 2, 2013, 12:35 pm — PM #189

Solution 76: $a-b|P(a)-P(b)\implies x_n-x_{n-1}|x_{n+1}-x_n$ for all n.Now also $\{x_n\}$ is increasing.Certainly if $def(P)>1$ then for infinitely many very large n we've $k>n$ such that $x_n-x_{n-1}|x_k\implies x_n-x_{n-1}|x_{n-1}$ and so we get there are infinitely many n such that $P(n)-n\leq n\implies lim_{n\to\infty}\frac{P(n)}{n}<2$ which is impossible since $Deg(P)>1$. So $P(x)=ax+b$ but then $x_{n+2}=a^{n+1}+b(a^n+.....a+1)$. Suppose $a>1$ now so a large power of a prime p which divide can't divide x_n for all n since $v_p(a^{n+1})$ is not fixed. So we must have $a=1$ but then $x_{n+2}=b(n+1)+1$ now if $b>1$ then b can't divide x_k for all k so $b=1$. Actually your first problem was also correct.

Particle
179 posts

May 2, 2013, 9:08 pm — PM #190

> *Saint123 wrote:*
> #76
> Let $P(x)$ be a polynomial with integer coefficients. Given that $P(x)>x$ for all positive integers x.
> Define $x_1=1$ and for any $n\in\mathbb{N}$, and $n>1$, define $x_n=P(x_{n-1})$. Given that for any $N\in\mathbb{N}$, there exists k such that $N|x_k$. Show that $P(x)=x+1$

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=531187&p=3033804#p3033804

rafiki
3 posts

May 4, 2013, 9:10 pm — PM #191

Hi,
Problem 77 :

Solve $x^3+y^3+z^3-3xyz=0$

msaeids
109 posts

May 4, 2013, 9:51 pm — PM #192

very easy
use this
the one in the left is equal to
(x+y+z)(x^2+y^2+z^2-xy-yz-zx)

(x+y+z)(x^2+y^2+z^2-xy-yz-zx)
excuse me for not using LATEX

alibez
357 posts

May 9, 2013, 10:45 am • 1 — PM #193

$\sum a^3 - 3abc = (\sum a)(\sum a^2 - \sum ab)$

we know $\sum a^2 \geq \sum ab \Rightarrow \sum a^2 - \sum ab \geq 0$ so :

if $\sum a^2 - \sum ab = 0 \Rightarrow a = b = c$

.....

War-Hammer
661 posts

Jun 1, 2013, 8:24 pm — PM #194

Problem 78 :

Find all polynomials $p(x)$ such that :

$$p(x+2)(x-2) + p(x-2)(x+2) = 2xp(x)$$

pco
15396 posts

Jun 2, 2013, 6:26 pm — PM #195

> War-Hammer wrote:
>
> **Problem 78 :**
>
> Find all polynomials $p(x)$ such that :
>
> $$p(x+2)(x-2) + p(x-2)(x+2) = 2xp(x)$$

Let $q(x) = p(x+2) - p(x)$ and the equation is $(x-2)q(x) = (x+2)q(x-2)$ which is a very classical problem whose solution is $q(x) = cx(x+2)$

So $p(x+2) - p(x) = cx(x+2)$

And so $\boxed{p(x) = ax^3 - 4ax + b}$ whatever are $a, b \in \mathbb{R}$

War-Hammer
661 posts

Jun 2, 2013, 7:57 pm — PM #196

Problem 79 :

Find all polynomials $p(x), q(x)$ such that :

$$p^2(x) + q^2(x) = (3x - x^3).p(x).q(x)$$

$$\forall x \in (0, \sqrt{3})$$

pco
15396 posts

Jun 2, 2013, 10:36 pm — PM #197

> War-Hammer wrote:
>
> **Problem 79 :**
>
> Find all polynomials $p(x), q(x)$ such that :
>
> $$p^2(x) + q^2(x) = (3x - x^3).p(x).q(x)$$
>
> $$\forall x \in (0, \sqrt{3})$$

I suppose we are speaking about polynomials $\in \mathbb{R}[X]$
If equality is true $\forall x \in (0, \sqrt{3})$ it is true $\forall x \in \mathbb{R}$ since polynomials.

Setting $x = 0, \sqrt{3}, -\sqrt{3}$, we get $p(x) = q(x) = 0$ for these three values and so $p(x) = (3x - x^3)p_1(x)$ and $q(x) = (3x - x^3)q_1(x)$

Equality becomes then $p_1(x)^2 + q_1^2(x) = (3x - x^3)p_1(x)q_1(x)$ and so the unique solution $\boxed{p(x) = q(x) = 0}$ $\forall x$

This post has been edited 1 time. Last edited by pco, Jun 2, 2013, 10:53 pm

War-Hammer
661 posts

Jun 2, 2013, 10:53 pm — PM #198

Yes we speak $\in \mathbb{R}[X]$

Problem 80 :

The polynomial $p(x)$ is increasing and the polynomial $q(x)$ is deacring such that :

$$2p(q(x)) = p(p(x)) + q(x)$$

$$\forall x \in \mathbb{R}$$

Prove that there exist a $x_0 \in \mathbb{R}$ such that :

$$p(x_0)=q(x_0)=x_0$$

pco
15396 posts

Jun 3, 2013, 12:01 am PM #199

> War-Hammer wrote:
> Yes we speak $\in \mathbb{R}[X]$
>
> **Problem 80 :**
>
> The polynomial $p(x)$ is increasing and the polynomial $q(x)$ is deacring such that :
>
> $$2p(q(x))=p(p(x))+q(x)$$
>
> $$\forall x\in\mathbb{R}$$
>
> Prove that there exist a $x_0\in\mathbb{R}$ such that :
>
> $$p(x_0)=q(x_0)=x_0$$

Since $p(x)$ is an increasing polynomial and $q(x)$ is a decreasing polynomial, there exists a unique x_0 such that $p(x_0)=q(x_0)$

So $p(p(x_0))+q(x_0)=2p(q(x_0))=2p(p(x_0))$ and we get $p(p(x_0))=q(x_0)=p(x_0)$

Since increasing, $p(x)$ is injective an so $p(p(x_0))=p(x_0)\implies p(x_0)=x_0$

Q.E.D.

War-Hammer
661 posts

Jun 3, 2013, 12:42 am PM #200

Problem 81 :

Find all polynomials $p(x)$ such that for all nonzero real numbers x,y,z with $\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$ we have :

$$\frac{1}{p(x)}+\frac{1}{p(y)}=\frac{1}{p(z)}$$

pco
15396 posts

Jun 3, 2013, 10:44 am PM #201

> War-Hammer wrote:
> **Problem 81 :**
>
> Find all polynomials $p(x)$ such that for all nonzero real numbers x,y,z with $\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$ we have :
>
> $$\frac{1}{p(x)}+\frac{1}{p(y)}=\frac{1}{p(z)}$$

From equation, we get that $P(x)\ne 0\,\forall x\ne 0$. eLt then $f(x)$ defined as :
$f(0)=0$
$f(x)=\frac{1}{P(\frac{1}{x})}\,\forall x\ne 0$

Functional equation becomes $f(x+y)=f(x)+f(y)\,\forall x,y\ne 0$ and so $f(x)=xf(1)\,\forall x\in\mathbb{Q}^+$

So $P(x)=\frac{x}{f(1)}\,\forall x\in\mathbb{Q}^+$ and so $\boxed{P(x)=ax}$ $\forall x$ and whatever is $a\ne 0$, which indeed is a solution.

War-Hammer
661 posts

Jun 3, 2013, 5:48 pm PM #202

Problem 82 :

Find all polynomials $p(x)\in\mathbb{R}[x]$ such that :

$$p(x^2+2x+1)=(p(x))^2+1$$

pco
15396 posts

Jun 3, 2013, 8:04 pm PM #203

> War-Hammer wrote:
> **Problem 82 :**
>
> Find all polynomials $p(x)\in\mathbb{R}[x]$ such that :
>
> $$p(x^2+2x+1)=(p(x))^2+1$$

Let $Q(x)=P(x-1)$ so that the functional equation becomes $Q(x^2+1)=Q(x)^2+1$

From there, look at http://www.artofproblemsolving.com/Forum/viewtopic.php?f=38&t=382979

And the solutions are : $\boxed{P(x)=h^{[n]}(x+1)}$ $\forall n\in\mathbb N\cup\{0\}$ where $h(x)=x^2+1$

War-Hammer
661 posts

Jun 3, 2013, 11:32 pm PM #204

Problem 83 :

Find all polynomials $P(x)\in\mathbb R[x]$ that have only real roots , such that :

$$P(x^2-1)=P(x)P(-x)$$

pco
15396 posts

Jun 4, 2013, 1:14 pm PM #205

> War-Hammer wrote:
>
> **Problem 83 :**
>
> Find all polynomials $P(x)\in\mathbb R[x]$ that have only real roots , such that :
>
> $$P(x^2-1)=P(x)P(-x)$$

Constant solutions are $P(x)=0\,\forall x$ and $P(x)=1\,\forall x$
About non constant solutions :

Let $f(x)=x^2-1$
If a_1 is a root of $P(x)$, so is $a_2=f(a_1)$, and so is $a_3=f(a_2)$ ...
So the sequence $a_{n+1}=f(a_n)$ must end in a cycle $a_n=a_p$ for some $n>p\ge 1$

Setting then $P(x)=\left(\prod_{i=p}^{n-1}(a_i-x)\right)Q(x)$, functional equation becomes :

$$\left(\prod_{i=p}^{n-1}(a_i+1-x^2)\right)Q(x^2-1)=\left(\prod_{i=p}^{n-1}(a_i-x)(a_i+xi)\right)Q(x)Q(-x)$$

$$\left(\prod_{i=p}^{n-1}(a_i+1-x^2)\right)Q(x^2-1)=\left(\prod_{i=p}^{n-1}(a_{i+1}+1-x^2)\right)Q(x)Q(-x)$$

And so $Q(x^2-1)=Q(x)Q(-x)$

So non constant solutions are any products of $\prod_{k=0}^{n-1}(f^{[k]}(a)-x)$ where a is a root of $f^{[n]}(x)=x$

So the problem is only to find (a,n) such that $f^{[n]}(a)=a$ and it's easy to show that only three elementary cycles exist :

$(0,2)$ or $(-1,2)$ which gives $\{0,-1,0,-1,0,-1,...\}$

$(\frac{1-\sqrt5}{2},1)$ which gives $\{\frac{1-\sqrt5}{2},\frac{1-\sqrt5}{2},\frac{1-\sqrt5}{2}...\}$

$(\frac{1+\sqrt5}{2},1)$ which gives $\{\frac{1+\sqrt5}{2},\frac{1+\sqrt5}{2},\frac{1+\sqrt5}{2}...\}$

Hence the solutions :

$\boxed{P(x)=a\left(\frac{1+\sqrt5}{2}-x\right)^m\left(\frac{1-\sqrt5}{2}-x\right)^n x^p(x-1)^p}$ $\forall x$ and whatever are $a\in\{0,1\}$ and $m,n,p\in\mathbb N\cup\{0\}$

War-Hammer
661 posts

Jun 4, 2013, 2:01 pm PM #206

But $P(x)=x(x+1)$ is also solution.

pco
15396 posts

Jun 4, 2013, 2:05 pm PM #207

> War-Hammer wrote:
>
> But $P(x)=x(x+1)$ is also solution.

You are right. This is a typo : the cycle $0,-1$ gives $(0-x)(-1-x)=x(x+1)$ and not $x(x-1)$ as I previously wrote.

And so :

$\boxed{P(x)=a\left(\frac{1+\sqrt5}{2}-x\right)^m\left(\frac{1-\sqrt5}{2}-x\right)^n x^p(x+1)^p}$ $\forall x$ and whatever are $a\in\{0,1\}$ and $m,n,p\in\mathbb N\cup\{0\}$

Thansk for your remark.

War-Hammer
661 posts

Jun 4, 2013, 2:17 pm PM #208

Problem 84 :

For polynomials $P(x),Q(x)\in\mathbb R[x]$ such that :

$$[P(x^2+1)]\quad[Q(x^2+1)]$$

$$[P(x^2+1)] = [Q(x^2+1)]$$

Prove that :

$$P(x) = Q(x)$$

$$\forall x \in \mathbb{R}$$

hofamo
58 posts

Jun 4, 2013, 4:38 pm PM #209

We have $H(x) = P(x^2+1) - Q(x^2+1) \in (-1, 1)$.
But we know $H(x)$ after one k ($k > 1$) is bigger or smaller than $1, -1$ if $\deg(H) > 0$ so we have $\deg(H) = 0$.
Now its easy to prove $P = Q$ because polynomials are continuous.

This post has been edited 1 time. Last edited by lehungvietbao, Jun 22, 2013, 12:33 pm
Reason: Fixed Latex

War-Hammer
661 posts

Jun 4, 2013, 5:57 pm PM #210

Solution 84

> Hofamo's Solution :
>
> we have $H(x) = P(x^2+1) - Q(x^2+1)$ is in $(-1, 1)$ but we know $H(x)$ after one $k, (k > 1)$ is bigger or smaller than $1, -1$ if deg$(H) > 0$ now we have $deg(H) = 0$ .now its easy to prove $P(x) = Q(x)$ because polynomials are continuous.
>
> My Solution :
>
> We know for infinite $k \in \mathbb{Z}$ the line $y = k$ intersect polynomial for variables $x \in (1, \infty)$ (Like $z_1, z_2, ...$)
>
> So from continuous of two polynomials in point $z_1, z_2, ...$ we can conclude $P(z_i^2+1) = Q(z_i^2+1)(i = 1, 2, ...)$. So $P(x), Q(x)$ are equal on infinite points which means $P(x) = Q(x)$

Problem 85 :

Two polynomials $P(x), Q(x)$ get some variable on $[n-1.n]$ for some $x \in [0, 1]$ ($n \in \mathbb{N}$ is assumed variable). If $P(x)$ is decreasing polynomial such that :

$$P(Q(nx)) = nQ(P(x))$$

Prove that there exist a $x_0 \in [0, 1]$ such that :

$$Q(P(x_0)) = x_0$$

War-Hammer
661 posts

Jun 22, 2013, 2:32 am PM #211

Any solution ???

vanu1996
607 posts

Jun 28, 2013, 9:49 pm PM #212

War-Hammer,please post the solution of problem 85 or continue without solution of 85.

War-Hammer
661 posts

Jun 28, 2013, 10:06 pm PM #213

I rather problem 85 still without solution , we continue with new problem :

Problem 86 :

a, b, c, d, e are real numbers with $a, e > 0$ and $ad^2 + b^2e - 4ace < 0$. Prove that :

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e = 0$$

has no real root.

Batominovski
1602 posts

Jun 28, 2013, 11:46 pm PM #214

Prob 86: Writing

$$P(x) = a\left(x^2 + \frac{b}{2a}x\right)^2 + e\left(\frac{d}{2e}x + 1\right)^2 - \frac{x^2}{4ae}\left(ad^2 + b^2e - 4ace\right)$$

solves the problem.

War-Hammer
661 posts

Jun 29, 2013, 12:00 am PM #215

Nice.

Problem 87 :

Find all polynomials $P(x) \in \mathbb{R}[x]$ such that :

$$(P(x))^2 + P(-x) = P(x^2) + P(x)$$

Jun 29, 2013, 9:35 pm • 1 PM #216

alibez
357 posts

Jun 29, 2013, 9:33 pm • 1 #216

> War-Hammer wrote:
> **Problem 87 :**
>
> Find all polynomials $P(x) \in \mathbb{R}[x]$ such that :
>
> $$(P(x))^2+P(-x)=P(x^2)+P(x)$$

hint

$P(0)=0 \Rightarrow P(x)=xQ(x) \Rightarrow Q(x)=-Q(-x) \Rightarrow P(-x)=P(x) \Rightarrow P(x^2)=P(x)^2 \Rightarrow P(x)=x^n \Rightarrow P(x)=x^{2k} \ \ \forall k \in \mathbb{N}$

---

xxp2000
520 posts

Jun 30, 2013, 4:29 am PM #217

> alibez wrote:
>> War-Hammer wrote:
>> **Problem 87 :**
>>
>> Find all polynomials $P(x) \in \mathbb{R}[x]$ such that :
>>
>> $$(P(x))^2+P(-x)=P(x^2)+P(x)$$
>
> $P(0)=0 \Rightarrow P(x)=xQ(x) \Rightarrow Q(x)=-Q(-x) \Rightarrow P(-x)=P(x) \Rightarrow P(x^2)=P(x)^2 \Rightarrow P(x)=x^n \Rightarrow P(x)=x^{2k} \ \ \forall k \in \mathbb{N}$

It is incomplete. Another solution is $x^{2k-1}+1$.

---

alibez
357 posts

Jul 1, 2013, 9:15 pm • 1 PM #218

> xxp2000 wrote:
>> alibez wrote:
>>> War-Hammer wrote:
>>> **Problem 87 :**
>>>
>>> Find all polynomials $P(x) \in \mathbb{R}[x]$ such that :
>>>
>>> $$(P(x))^2+P(-x)=P(x^2)+P(x)$$
>>
>> $P(0)=0 \Rightarrow P(x)=xQ(x) \Rightarrow Q(x)=-Q(-x) \Rightarrow P(-x)=P(x) \Rightarrow P(x^2)=P(x)^2 \Rightarrow P(x)=x^n \Rightarrow P(x)=x^{2k} \ \ \forall k \in \mathbb{N}$
>
> It is incomplete. Another solution is $x^{2k-1}+1$.

i say if $P(0)=0 \Rightarrow .....$

$P(0)^2=P(0) \Rightarrow P(0)=0 \ \ or \ \ 1$

hint

$P(0)=1 \Rightarrow P(x)=xQ(x)+1 \Rightarrow xQ(x)^2+Q(x)-xQ(-x)=xQ(x^2) \Rightarrow Q(x)^2=Q(-x)^2 \Rightarrow .... \Rightarrow Q(x)=Q(-x) \Rightarrow Q(x)=x^{2k}$

---

vanu1996
607 posts

Jul 3, 2013, 10:23 am PM #219

88.$(1+x+x^2+...+x^n)^2-x^n$ is the product of two polynomials.

---

amatysten
73 posts

Jul 4, 2013, 11:47 am PM #220

@vanu1996
Can you clarify the problem.
Prove, that $88-(1+x+...+x^n)^2-x^n$ is a product of two non-zero degree polynomials, is that correct?
Or find those polynomials?

---

ssilwa
5454 posts

Jul 4, 2013, 11:58 am • 1 PM #221

I think 88 is the problem number so it should read $(1+x+\cdots+x^n)^2-x^n\cdots$

---

amatysten
73 posts

Jul 4, 2013, 12:51 pm PM #222

Ok, then.

> Quote:
> $(1+x+\cdots+x^n)^2-x^n$

0) It's not true for $n=1$.

1) $(1+x+\cdots+x^n)^2=x^{2n}+2x^{2n-1}+3x^{2n-2}+\cdots+(n+1)x^n+\cdots+3x^2+2x+1$, can be shown by induction.

2) $(1+x+\cdots+x^n)^2-x^n=x^{2n}+\cdots+nx^n+\cdots+1=$
$=(x^{n-1}+x^n+\cdots+1)(x^{n+1}+x^n+\cdots+1), n>1$.

Problem 89.
$K,T,S\in R[x]$ such, that $K(T(x))+K(S(x))=const$.
Show, that $K(x)=const$ or $T(x)+S(x)=const$.

---

alibez
357 posts

Jul 6, 2013, 7:39 pm • 1 PM #223

> amatysten wrote:
> **Problem 89.**
> $K,T,S\in R[x]$ such, that $K(T(x))+K(S(x))=const$.
> Show, that $K(x)=const$ or $T(x)+S(x)=const$.

hint

> if $K\neq const$
> we have : $degT\times degK=degK\times degS$
>
> so $K(x)=a_nx^n+...a_1x+a_0$, $T(x)=b_mx^m+...+b_0$, $S(x)=c_mx^m+...+c_0$
>
> $\Rightarrow a_nb_m^nx^{mn}+a_nc_m^nx^{mn}=0\Rightarrow b_m^n+c_m^n=0$
>
> now if $a_j\neq 0\Rightarrow ....$

---

amatysten
73 posts

Jul 7, 2013, 12:05 pm PM #224

@alibez
Please, try to provide more elaborate solution in the future. Your way is right, but details are crucial in such problems.
Also, try to post some new problem. It's not a great fun to search for problems, it's fun to solve them. You've received some fun from solving, try to give the same to us.

Problem 90.
Prove that there exists a polynomial $P(x)$ that is divided by $(x-1)^n$, whose coefficients only equal to $-1,0$ or 1 and degree is not more than 2^n.

EDIT. $P's$ degree is not zero.

This post has been edited 1 time. Last edited by amatysten, Jul 7, 2013, 7:00 pm

---

vanu1996
607 posts

Jul 7, 2013, 1:16 pm PM #225

> amatysten wrote:
> @alibez
> Please, try to provide more elaborate solution in the future. Your way is right, but details are crucial in such problems.
> Also, try to post some new problem. It's not a great fun to search for problems, it's fun to solve them. You've received some fun from solving, try to give the same to us.
>
> **Problem 90.**
> Prove that there exists a polynomial $P(x)$ that is divided by $(x-1)^n$, whose coefficients only equal to $-1,0$ or 1 and degree is not more than 2^n.

Divided or divisible.

---

vanu1996
607 posts

Jul 7, 2013, 1:55 pm PM #226

solution 90- I think $P(x)=0$ is the only solution.If we multiply all roots of $P(x)$ than it should be equal to 1 or -1.than think the sum of roots,which is also $1,-1or0$.impossible.

---

vanu1996
607 posts

Jul 7, 2013, 2:11 pm PM #227

problem 91- Find all polynomials f over C satisfying $f(x)f(-x)=f(x^2)$.

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amatysten
73 posts

Jul 7, 2013, 6:59 pm PM #228

$P(x)=0$ wasn't allowed, btw. Forgot to tell that. I'll have to edit.
You proof that $deg(P)=0$ wasn't correct, though. It can comprise multiples, that have no roots, like x^2+x+1.
Thus we have two unsolved problems now.

---

alibez
357 posts

Jul 7, 2013, 8:07 pm • 1 PM #229

> vanu1996 wrote:
> problem 91- Find all polynomials f over C satisfying $f(x)f(-x)=f(x^2)$.

hint

> if r is a non -zero root of f then r^2 is also a root of f . then r^{2^k} is also a root of f

> as f has finitely many roots .....
>
> so $|r| = 1$
>
> we have found thus a root w for wich $w^{2^{n-1}} = 1$ ....
>
> so $Q(x)$ divides f . ($Q(x) = (x-w)(x-w^2)...(x-w^{2^{n-1}}$)
>
> now $Q(x)Q(-x) = (x-w)(-x-w)...(-x-w^{2^{n-1}}) = (-1)^n(x^2-w^2)...(x^2-w^{2^n}) = (-1)^nQ(x^2)$

solutions : $f(x) = (-x^n)Q_1(x)....Q_k(x)$ where $Q_i(x) = (w-x)....(w^{2^{m-1}}-x)$

---

alibez
357 posts

Jul 7, 2013, 8:24 pm • 2 — PM #230

> *amatysten wrote:*
>
> @alibez
> Please, try to provide more elaborate solution in the future. Your way is right, but details are crucial in such problems.
> Also, try to post some new problem. It's not a great fun to search for problems, it's fun to solve them. You've received some fun from solving, try to give the same to us.

problem 92

let : $(P_n(x,y,z))_{n=0}^{\infty}$ such that :

$P_0(x,y,z) = 1$

$\forall n \geq 1 \;\; P_n(x,y,z) = (x+z)(y+z)P_{n-1}(x,y,z+1) - z^2P_{n-1}(x,y,z)$

prove that : $P_n(x,y,z) = P_n(x,z,y) = P_n(z,x,y) = P_n(z,y,x) = P_n(y,x,z) = P_n(y,z,x)$

---

amatysten
73 posts

Jul 8, 2013, 5:15 pm — PM #231

Solution for **Problem 92**.

1) Let's show by induction that $P_n(x,y,z) = P_n(y,x,z), \forall n \in \mathbb{N}$.
It's true for $n=0$. Assuming that it's true for $n < k+1$ we get
$P_{k+1}(y,x,z) = (y+z)(x+z)P_k(y,x,z+1) - z^2P_k(y,x,z) =$
$= (x+z)(y+z)P_k(x,y,z+1) - z^2P_k(x,y,z) = P_{k+1}(x,y,z)$.

2) Let's show by induction that $P_n(x,y,z) = P_n(z,y,x), \forall n \in \mathbb{N}$.
It's true for $n=0$.
$P_1(x,y,z) = (x+z)(y+z) - z^2 = (z+x)(y+x) - x^2 = P_1(z,y,x)$.

Assuming that it's true for $n < k+2$ we get
$P_{k+2}(z,y,x) = (z+x)(y+x)P_{k+1}(z,y,x+1) - x^2P_{k+1}(z,y,x) = (z+x)(y+x)P_{k+1}(x+1,y,z) - x^2P_{k+1}(x,y,z) = (z+x)(y+x)[(x+1+z)(y+z)P_k(x+1,y,z+1) - z^2P_k(x+1,y,z)] - x^2[(x+z)(y+z)P_k(x,y,z+1) - z^2P_k(x,y,z)] = (x+z)(y+z)[(z+1+x)(y+x)P_k(z+1,y,x+1) - x^2P_k(z+1,y,x)] - z^2[(z+x)(y+x)P_k(z,y,x+1) - x^2P_k(z,y,x)] = (x+z)(y+z)P_{k+1}(z+1,y,x) - z^2P_{k+1}(z,y,x) = (x+z)(y+z)P_{k+1}(x,y,z+1) - z^2P_{k+1}(x,y,z) = P_{k+2}(x,y,z)$

3) From the two proven we can get the rest.

Problem 90 is still not solved. Check it out before posting another one.

---

hilii
10 posts

Jul 10, 2013, 2:54 pm — PM #232

> *amatysten wrote:*
>
> Solution for **Problem 92**.
>
> **Problem 90 is still not solved.** Check it out before posting another one.

vanu1996 solved it

---

msaeids
109 posts

Jul 10, 2013, 3:42 pm — PM #233

his answer wasn't allowed!!!

---

amatysten
73 posts

Jul 10, 2013, 5:40 pm — PM #234

I added $P(x) \neq 0$ after he provided his answer. His prove that $P's$ degree $=0$ was wrong anyway. There are nonzero solutions, i promise 🙂

---

tenniskidper...
2376 posts

Jul 10, 2013, 8:52 pm • 1 — PM #235

Just take $(x^{2^{n-1}}-1)(x^{2^{n-2}}-1)\ldots(x^4-1)(x^2-1)(x-1)$. Clearly this is divisible by $(x-1)^n$, has degree 2^n-1, and has only coefficients 1 and -1.

Problem 93: Suppose $a,b,c \in \mathbb{C}$ so that the three roots of $z^3+az^2+bz+c=0$ have modulus 1. Show that $z^3+|a|z^2+|b|z+|c|$ has only roots of modulus 1 also.

War-Hammer
661 posts

Jul 17, 2013, 11:49 pm — PM #236

Since there is no solution for problem 93 we will continue marathon with new problem.

War-Hammer
661 posts

Jul 17, 2013, 11:54 pm PM #237

Problem 94 :

Find all polynomials $p(x) \in \mathbb{R}[x]$ such that :

$$P(x)P(2x^2-1) = P(x^2)P(2x-1)$$

For all $x \in \mathbb{R}$

---

xxp2000
520 posts

Jul 20, 2013, 7:49 pm PM #238

> War-Hammer wrote:
> **Problem 94 :**
>
> Find all polynomials $p(x) \in \mathbb{R}[x]$ such that :
>
> $$P(x)P(2x^2-1) = P(x^2)P(2x-1)$$
>
> For all $x \in \mathbb{R}$

We can find $N > 1$ such that $P(x) \neq 0$ when $x > N$.

Now we fix $x > N$, $f(x) = \dfrac{P(2x-1)}{P(x)} = \dfrac{P(2x^2-1)}{P(x^2)} = f(x^2) = \cdots f(x^{2^n}) = \lim_{x\to\infty} \dfrac{P(2x-1)}{P(x)} = c$, where c is constant.

Since P is polynomial, we have $P(2x-1) = cP(x)$ for all x.
If P has root $r \neq 1$, $1 + 2^n(r-1)$ has to be its root for all integer $n > 0$. Absurd!
So a non-constant $P(x)$ has only root 1. Now we can see the only solution is $P(x) = a(x-1)^n$, where $n \geq 0$ and a is constant.

---

xxp2000
520 posts

Jul 20, 2013, 8:03 pm PM #239

> tenniskidperson3 wrote:
> Problem 93: Suppose $a, b, c \in \mathbb{C}$ so that the three roots of $z^3 + az^2 + bz + c = 0$ have modulus 1. Show that $z^3 + |a|z^2 + |b|z + |c|$ has only roots of modulus 1 also.

Obviously $|c| = 1$. We now show -1 is one of its root. We need to prove
$|z_1 + z_2 + z_3| = |z_1z_2 + z_1z_3 + z_2z_3| = |\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}|$,
which is trivial by noticing $\dfrac{1}{z_i} = \bar{z}_i$.

Now the new equation becomes (z+1)(z^2+(|a|-1)z+1)=0. Since $-1 \leq |a| - 1 \leq 2$, the other two roots also have modulus 1.

War-Hammer
661 posts

Jul 20, 2013, 10:20 pm PM #240

Problem 95 :

Find all real numbers a, b such that for all $u, v \in \mathbb{R}$ we have :

$$\max|x^2 - ux - v| \geq \max|x^2 - ax - b|$$

For all $x \in [0, 1]$.

This post has been edited 1 time. Last edited by War-Hammer, Jul 21, 2013, 4:02 pm

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xxp2000
520 posts

Jul 21, 2013, 4:51 am PM #241

> War-Hammer wrote:
> **Problem 94 :**
>
> Find all real numbers a, b such that for all $u, v \in \mathbb{R}$ we have :
>
> $$\max|x^2 - ux - v| \geq \max|x^2 - ax - b|$$
>
> For all $x \in [0, 1]$.

It should be problem 95.

We define $f(u, v) = \max\limits_{0 \leq x \leq 1} |x^2 - ux - v|$. Suppose $f(a, b) = \min\limits_{u,v} f(u, v)$.
Let x_1, x_2 be the roots of $x^2 - ax - b$. If $x_1, x_2 = \alpha \pm \beta i$ are complex, $(x-\alpha)^2 < (x - x_1)(x - x_2)$ implies $f(2\alpha, -\alpha^2) < f(a, b)$. Absurd! So x_1, x_2 are reals.
We can reduce $f(a, b)$ by forcing $x_1 = 0$ when $x_1 < 0$ or forcing $x_1 = 1$ when $x_1 > 1$. So $0 \leq x_1, x_2 \leq 1$ and $0 \leq a = x_1 + x_2 \leq 2$.

Let $M = \max\limits_{x\in[0,1]} x^2 - ax$ and $m = \min\limits_{x\in[0,1]} x^2 - ax$. Obviously $f(a, b) = \dfrac{M-m}{2}$ with $b = \dfrac{M+m}{2}$.

With $a \in [0,2]$, $M = \max(0, 1-a)$ and $m = -\dfrac{a^2}{4}$.

When $a > 1$, $\dfrac{M-m}{2} = \dfrac{a^2}{8} > \dfrac{1}{8}$; when $0 < a < 1$, $\dfrac{M-m}{2} = \dfrac{(2-a)^2}{8} > \dfrac{1}{8}$.

When $a = 1$, $\dfrac{M-m}{2} = \dfrac{1}{8}$.

The answer is $a = 1, b = -\dfrac{1}{8}$.

War-Hammer
661 posts

Jul 21, 2013, 4:15 pm PM #242

Problem 96 :

For a non constant polynomial $P(x) \in \mathbb{R}[x]$ we know that :

$$P(x)P(2x^2) = P(3x^3 + x)$$

For all $x \in \mathbb{R}$. Prove that all of the root of $P(x)$ are complex.

This post has been edited 1 time. Last edited by War-Hammer, Jul 21, 2013, 9:16 pm

pco
15396 posts

Jul 21, 2013, 4:21 pm PM #243

> War-Hammer wrote:
> **Problem 96 :**
>
> For a polynomial $P(x) \in \mathbb{R}[x]$ we know that :
>
> $$P(x)P(2x^2) = P(3x^3 + x)$$
>
> For all $x \in \mathbb{R}$. Prove that all of the root of $P(x)$ are complex.

What about $P(x) = 0 \, \forall x$?

War-Hammer
661 posts

Jul 21, 2013, 5:27 pm PM #244

Sorry , I forgot to write $P(x)$ is not constant polynomial.

pco
15396 posts

Jul 21, 2013, 5:49 pm PM #245

> War-Hammer wrote:
> **Problem 96 :**
>
> For a **non constant** polynomial $P(x) \in \mathbb{R}[x]$ we know that :
>
> $$P(x)P(2x^2) = P(3x^3 + x)$$
>
> For all $x \in \mathbb{R}$. Prove that all of the root of $P(x)$ are complex.

Suppose $\exists a \in \mathbb{R}$ such that $P(a) = 0$

1) If $a = 0$
Let $P(x) = x^n Q(x)$ with $n > 0$ and $Q(0) \neq 0$. Equation becomes $x^{2n} 2^n Q(x) Q(2x^2) = (2x^2+1)^n Q(3x^3+x)$
Setting $x = 0$ in this equation, we get $Q(0) = 0$, contradiction.
And so $a \neq 0$

2) If $a > 0$
Setting $x = a$ in equation, we get $P(3a^3 + a) = 0$ and since $3a^3 + a > a > 0$, we get that $P(x)$ has infinitely many distinct positive roots, which is impossible since $P(x)$ is not the zero polynomial.
So $a \leq 0$

3) If $a < 0$
Setting $x = a$ in equation, we get $P(3a^3 + a) = 0$ and since $3a^3 + a < a < 0$, we get that $P(x)$ has infinitely many distinct negative roots, which is impossible since $P(x)$ is not the zero polynomial.
So $a \geq 0$

So no such $a \in \mathbb{R}$

Q.E.D.

War-Hammer
661 posts

Jul 21, 2013, 9:20 pm PM #246

Problem 97 :

Suppose that $P(x) = ax^2 + bx + c$ and we know that :

$$|P(x)| \leq 1$$

For $x = -1, 0, 1$. Prove that :

$$P(x) \leq 2x^2 - 1$$

For all $x \in \mathbb{Z}$ except 0.

vanu1996
607 posts

Jul 24, 2013, 6:20 pm — PM #247

hint- $|c| \leq 1$ and $|a| + |b| \leq 2$,now from this we can be easily find.

msaeids
109 posts

Jul 24, 2013, 6:31 pm — PM #248

It can be done using interpolation polynomial in the Lagrange form.
you can see it here:
http://en.wikipedia.org/wiki/Lagrange_polynomial

War-Hammer
661 posts

Jul 24, 2013, 10:26 pm — PM #249

Full solution please.

vanu1996
607 posts

Jul 24, 2013, 10:36 pm — PM #250

It's easy,it is clear that a, b are lies $[0, 2]$ for $p(x)$ is max,now we have $p(x) \leq (a+b)x^2 - 1 \leq 2x^2 - 1$.

Ponewor
9 posts

Jul 28, 2013, 8:45 pm — PM #251

full solution & new problem?

War-Hammer
661 posts

Jul 28, 2013, 8:55 pm — PM #252

Problem 97 still unsolved , "vanu1996"'s solution is not correct.

xxp2000
520 posts

Jul 28, 2013, 9:30 pm — PM #253

Problem 97:
We have $|c|, |a+b+c|, |a-b+c| \leq 1$. So $|a \pm b| \leq 2$ and $|2a| \leq 4$.
Case 1: $a = 2$
$|a \pm b| \leq 2$ implies $b = 0$. Then $|c|, |a+c| \leq 1$ implies $c = -1$. The result holds.

Case 2: $a < 2$.
Let $f(x) = 2x^2 - 1 - P(x) = (2-a)x^2 - bx - 1 - c$. We will show $f(x) \geq 0, |x| \geq 1$.
$f(x)$ is symmetric to $x_0 = \dfrac{b}{4-2a}$.
$|a \pm b| \leq 2$ implies $|a + |b|| \leq 2$, $|b| \leq 2 - a < 4 - 2a$, $|x_0| \leq 1$.

Since $f(x)$ is quadratic and $f(\pm 1) = 1 - P(\pm 1) \geq 0$,
$f(x) \geq f(-1) \geq 0, x \leq -1$ and $f(x) \geq f(1) \geq 0, x \geq 1$.
Q.E.D.

CaesarIIIC
21 posts

Jul 28, 2013, 9:37 pm — PM #254

Using Lagrange Interpolation Formula we have :
$P(x) = P(1).\dfrac{x(x+1)}{2} + P(0).\dfrac{(x-1)(x+1)}{-1} + P(-1).\dfrac{x(x-1)}{2}$
$\rightarrow |P(x)| \leq |\dfrac{x(x+1)}{2}| + |1 - x^2| + |\dfrac{x(x-1)}{2}|$
$\rightarrow |P(x)| \leq \dfrac{x(x+1)}{2} + (x^2 - 1) + \dfrac{x(x-1)}{2}$ For all $x \in \mathbb{Z}$ except $-1; 0; 1$
$\rightarrow |P(x)| \leq 2x^2 - 1$ For all $x \in \mathbb{Z}$ except $-1; 0; 1$
$\rightarrow P(x) \leq 2x^2 - 1$ For all $x \in \mathbb{Z}$ except 0

War-Hammer
661 posts

Jul 28, 2013, 9:49 pm — PM #255

Problem 98 :

Find all polynomials $P(x) \in \mathbb{R}[x]$ such that :

$$(x+1)^3 P(x-1) - (x-1)^3 P(x+1) = 4(x^2 - 1)P(x)$$

For all $x \in \mathbb{R}[x]$.

xxp2000
520 posts

Jul 28, 2013, 10:35 pm — PM #256

> War-Hammer wrote:
> **Problem 98 :**
>
> Find all polynomials $P(x) \in \mathbb{R}[x]$ such that :
>
> $$(x+1)^3 P(x-1) - (x-1)^3 P(x+1) = 4(x^2 - 1)P(x)$$
>
> For all $x \in \mathbb{R}[x]$.

$x = 1$ gives $P(0) = 0$. Let $P(x) = xQ(x)$, we have
$(x+1)^2 Q(x-1) - (x-1)^2 Q(x+1) = 4xQ(x)$, or
$(x+1)^2 (Q(x-1) - Q(x)) = (x-1)^2 (Q(x+1) - Q(x))$.
Let $h(x) = Q(x) - Q(x-1)$, we get $-(x+1)^2 h(x) = (x-1)^2 h(x+1)$.
If $h(x)$ is not constant zero, leading coefficients of both sides cannot match.

So $h(x) = 0$ implies Q is constant and $P(x) = cx$.

War-Hammer
661 posts

Jul 29, 2013, 12:24 am — PM #257

Problem 99 :

Suppose that p_1, p_2, p_3, p_4 are prime distinct numbers. Prove that there is no polynomial degree three with integer coefficient like $Q(x)$ such that :

$$|Q(p_1)| = |Q(p_2)| = |Q(p_3)| = |Q(p_4)| = 3$$

xxp2000
520 posts

Jul 29, 2013, 1:16 am — PM #258

> War-Hammer wrote:
> **Problem 99 :**
>
> Suppose that p_1, p_2, p_3, p_4 are prime distinct numbers. Prove that there is no polynomial degree three with integer coefficient like $Q(x)$ such that :
>
> $$|Q(p_1)| = |Q(p_2)| = |Q(p_3)| = |Q(p_4)| = 3$$

Case 1): If all $Q(p_i)$ have same sign, Q has degree at least 4. Absurd!

Case 2): $Q(p_1) = Q(p_2) = Q(p_3) = 3$ and $Q(p_4) = -3$.
Then $Q(x) = a(x - p_1)(x - p_2)(x - p_3) + 3$ and $(p_4 - p_1)(p_4 - p_2)(p_4 - p_3)|6$.
If $p_4 = 2$, then all $p_4 - p_i$ are odd and $p_4 - p_i = -1, 1, 3$ or $p_4 - p_i = -1, 1, -3$. Absurd!
If $p_4 \neq 2$, at least two $p_4 - p_i$ are even and $4|(p_4 - p_1)(p_4 - p_2)(p_4 - p_3)|6$. Absurd!

Case 3): $Q(p_1) = Q(p_2) = 3$ and $Q(p_3) = Q(p_4) = -3$.
Say p_1, p_2, p_3 are odd.
$Q(x) = (ax + b)(x - p_1)(x - p_2) + 3$ and $4|(p_3 - p_1)(p_3 - p_2)|6$. Absurd!

tenniskidper...
2376 posts

Jul 29, 2013, 1:45 am — PM #259

Problem 100: Let $P_0(x) = 1, P_1(x) = x + 1$, and $P_{n+1}(x) = P_n(x) + xP_{n-1}(x)$. Show that all $P_i(x)$ with $i \geq 1$ have only real roots.

xxp2000
520 posts

Aug 3, 2013, 8:28 am — PM #260

> tenniskidperson3 wrote:
> **Problem 100:** Let $P_0(x) = 1, P_1(x) = x + 1$, and $P_{n+1}(x) = P_n(x) + xP_{n-1}(x)$. Show that all $P_i(x)$ with $i \geq 1$ have only real roots.

It is not hard to show
$P_n(x) = \frac{(1 + \sqrt{1+4x})^{n+2} - (1 - \sqrt{1+4x})^{n+2}}{2^{n+2}\sqrt{1+4x}}$.

Also P_n has degree $d_n = [\frac{n+1}{2}]$.

Now we can easily find the d_n real solutions
$x = -\frac{1}{4}(1 + \tan^2(\frac{i}{n+2}\pi)), i = 1, \cdots, d_n$.

tenniskidper...
2376 posts

Aug 3, 2013, 9:38 am — PM #261

Problem 101: Let a be a real number. Define a sequence of polynomials $P_i(x)$ as $P_0(x) = 1$ and $P_{n+1}(x) = x \cdot P_n(x) + P_n(a \cdot x)$. Prove that P_i are reciprocal polynomials.

xxp2000
520 posts

Aug 3, 2013, 3:34 pm — PM #262

> tenniskidperson3 wrote:
> **Problem 101:** Let a be a real number. Define a sequence of polynomials $P_i(x)$ as $P_0(x) = 1$ and $P_{n+1}(x) = x \cdot P_n(x) + P_n(a \cdot x)$. Prove that P_i are reciprocal polynomials.

It holds for $a = 1$.

When $a \neq 1$, we can easily show $P_n(x) = \sum_{i=1}^{n} \prod_{j=1}^{i} \frac{a^{n+1-j} - 1}{a^j - 1} x^i + 1, n > 1$ by induction. Obviously the RHS is reciprocal polynomial.

War-Hammer
661 posts

Aug 3, 2013, 5:14 pm — PM #263

Problem 102 :

Find all polynomials $P(x) \in \mathbb{R}[x]$ such that :

$$2yP(x+y) + (x-y)(P(x) + P(y)) \geq 0$$

For all $x, y \in \mathbb{R}$.

xxp2000
520 posts

Aug 3, 2013, 8:30 pm — PM #264

> War-Hammer wrote:
> **Problem 102 :**
>
> Find all polynomials $P(x) \in \mathbb{R}[x]$ such that :
>
> $$2yP(x+y) + (x-y)(P(x) + P(y)) \geq 0$$

> For all $x, y \in \mathbb{R}$.

Obviously $P(x) = 0$ is the only constant solution. Now consider non-constant solutions.
Let $Q(x, y)$ be the principle.
$Q(x, x) : 2xP(2x) \geq 0$. So $P(x)$ shares sign with x. $P(0) = 0$.
Compare $Q(x, -x)$ with $Q(-x, x)$, P is odd function.
$Q(x, -x-1) : P(x+1) - P(x) \leq \dfrac{2x+2}{2x+1}P(1) < 2P(1)$ for $x > 0$.
If P has degree higher than 1, $P(x+1) - P(x)$ has no upper bound for large x.
So $P(x) = ax$ with $a > 0$ is the only non-constant solution.

War-Hammer
661 posts

Aug 3, 2013, 10:56 pm — PM #265

Problem 103 :

Let $P(x) = x^2 + ax + b$ be a polynomial with real coefficient such that $P(P(x)) = 0$ has four real roots such that the sum of the two of them is -1. Prove that : $b \leq -\dfrac{1}{4}$.

This post has been edited 2 times. Last edited by War-Hammer, Aug 10, 2013, 1:00 pm

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sirknightingfail
260 posts

Aug 3, 2013, 11:29 pm — PM #266

The above is false if $b = c = 0$ and $a = d = 1$, which results in P(-1)=0

War-Hammer
661 posts

Aug 3, 2013, 11:47 pm — PM #267

but 0 is not even number , we are assume that 0 is not even or odd , in the other word , none of them is equal to 0.

---

professordad
4625 posts

Aug 3, 2013, 11:54 pm — PM #268

Click to reveal hidden text

> I believe $x^3 + 2x^2 + 2x + 1$ has -1 as a root..
> I don't think that all the roots can be irrational. 2 can be irrational, but they must be conjugates, and the third doesn't have a conjugate.
> EDIT: actually the above (that not all the roots can be irrational) may not be true. oops

This post has been edited 2 times. Last edited by professordad, Aug 6, 2013, 1:50 am

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War-Hammer
661 posts

Aug 4, 2013, 12:16 am — PM #269

OK , sorry , maybe there is a typo in original problem.
I've posted new problem as problem 103.

War-Hammer
661 posts

Aug 7, 2013, 3:08 pm — PM #270

Any solution ???

---

xxp2000
520 posts

Aug 10, 2013, 7:33 am — PM #271

> *War-Hammer wrote:*
> **Problem 103 :**
>
> Let $P(x) = ax^2 + bx + c$ be a polynomial with real coefficient such that $P(P(x)) = 0$ has four real roots such that the sum of the two of them is -1. Prove that : $b \leq -\dfrac{1}{4}$.

A counterexample is $P(x) = 5x^2 + 5x$, $P(P(x))$ has four real roots, two of which are $0, -1$.

War-Hammer
661 posts

Aug 10, 2013, 12:59 pm — PM #272

My mistake , sorry.

Edited again.

---

xxp2000
520 posts

Aug 10, 2013, 7:16 pm — PM #273

> *War-Hammer wrote:*
> **Problem 103 :**
>
> Let $P(x) = x^2 + ax + b$ be a polynomial with real coefficient such that $P(P(x)) = 0$ has four real roots such that the sum of the two of them is -1. Prove that : $b \leq -\dfrac{1}{4}$.

$P(x) = 0$ has two roots $\alpha, \beta = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$.
$P(P(x)) = 0$ has four roots $\dfrac{-a \pm \sqrt{a^2 - 4b + 4\alpha}}{2}$ and $\dfrac{-a \pm \sqrt{a^2 - 4b + 4\beta}}{2}$.

First consider $a = 1$.
$a^2 - 4b + 4\alpha = -1 - 4b - 2\sqrt{a^2 - 4b} \geq 0$.
So $4b + 1 \leq 0$. We are done.

Now consider $a \neq 1$.
$-1 = \dfrac{-a \pm \sqrt{a^2 - 4b + 4\alpha}}{2} + \dfrac{-a \pm \sqrt{a^2 - 4b + 4\beta}}{2}$, or
$\pm(2a - 2) = \sqrt{a^2 - 4b + 4\alpha} \pm \sqrt{a^2 - 4b + 4\beta}$, or
$(2a-2)^2 = 2a^2 - 8b - 4a \pm 2\sqrt{(a^2 - 4b + 4\alpha)(a^2 - 4b + 4\beta)}$, or
$(a-1)^2 + 4b + 1 = \pm\sqrt{(a^2 - 4b + 4\alpha)(a^2 - 4b + 4\beta)}$

If the sign of RHS is negative, we already get $4b + 1 \leq 0$.

If the sign of RHS is negative, we already get $4b + 1 \leq 0$.
If the sign is positive, we notice $RHS \leq |a^2 - 4b + 2\alpha + 2\beta| = |(a-1)^2 - 4b - 1|$.
Now $(a-1)^2 + 4b + 1 \leq |(a-1)^2 - 4b - 1|$ implies $4b + 1 \leq 0$.
Q.E.D.

War-Hammer
661 posts

Aug 10, 2013, 9:07 pm PM #274

Problem 104 :

p, q are complex number such that $q \neq 0$. Suppose that the polynomial $P(x) = x^2 + px + q^2 = 0$ has two roots x_1, x_2 such that $|x_1| = |x_2|$

Prove that $\dfrac{p}{q}$ is real.

Pain rinnegan
1581 posts

Aug 10, 2013, 9:48 pm PM #275

> War-Hammer wrote:
> **Problem 104 :**
>
> p, q are complex number such that $q \neq 0$. Suppose that the polynomial $P(x) = x^2 + px + q^2 = 0$ has two roots x_1, x_2 such that $|x_1| = |x_2|$
>
> Prove that $\dfrac{p}{q}$ is real.

Viete gives $p = -x_1 - x_2$ and $q^2 = x_1 x_2$. To prove that $\dfrac{p}{q} \in \mathbb{R}$, we need to show $\dfrac{p}{q} = \dfrac{\overline{p}}{\overline{q}} \Leftrightarrow p\overline{q} = \overline{p}q \Leftrightarrow p|q|^2 = \overline{p}q^2$. We have:

$p|q|^2 = (-x_1 - x_2)|x_1 x_2| = -(x_1 + x_2)|x_1|^2 = -x_1|x_1|^2 - x_2|x_1|^2 =$
$-x_1|x_2|^2 - x_2|x_1|^2 = -x_1 x_2 \overline{x_2} - x_2 x_1 \overline{x_1} = -x_1 x_2(\overline{x_1} + \overline{x_2}) = \overline{p}q^2$

Problem 105: Let $p > 2$ be a prime. Prove that the polynomial $P(X) = (X-1)(X-2)\cdot\ldots\cdot(X-p) + X + p$ is irreducible in $\mathbb{Z}[X]$.

xxp2000
520 posts

Aug 11, 2013, 3:36 am PM #276

> Pain rinnegan wrote:
> Problem 105: Let $p > 2$ be a prime. Prove that the polynomial $P(X) = (X-1)(X-2)\cdot\ldots\cdot(X-p) + X + p$ is irreducible in $\mathbb{Z}[X]$.

We will show the Eisenstein's criterion holds. Denote the coefficient of x^k as a_k.
$a_0 = p(1 - (p-1)!)$. By Wilson's theorem, $p|a_0$ and $p^2 \not| a_0$.
Also by Wilson's theorem, $a_1 = (p-1)! + 1 = 0 \mod p$.

Now it is left to show the $p|a_k$ when $k < p-1$.

We first show $p|S_k = \sum_{i=1}^{p} i^k$ for $k < p-1$ by induction.

Obviously $p|S_1$.
$I_k = \sum_{i=1}^{p} i(i+1)\cdots(i+k-1) = \dfrac{1}{k+1}\sum_{i=1}^{p} i(i+1)\cdots(i+k) - (i-1)\cdots(i+k-1) = \dfrac{p(p+1)\cdots(p+k)}{k+1}$.
We see $p|I_k, k < p-1$.
Also we can write $I_k = S_k + \sum_{i=1}^{k-1} b_i S_i$, where b_i is some integer coefficient depending on k.
So we can establish $p|S_k, k < p-1$ by induction.

Obviously $p|e_1 = 1 + 2 + \cdots + p$.
Now we can use induction and Newton's identities to show $p|a_k, k < p-1$.

msaeids
109 posts

Aug 11, 2013, 6:30 pm PM #277

Problem 106:
$P(x)$ is a polynomial from degree n such that $P(k) = \dfrac{k}{k+1}$ for $k = 0, 1, 2, \ldots n$.Calculate $P(n+1)$.

pco
15396 posts

Aug 11, 2013, 8:32 pm • 1 PM #278

> msaeids wrote:
> **Problem 106**:
> $P(x)$ is a polynomial from degree n such that $P(k) = \dfrac{k}{k+1}$ for $k = 0, 1, 2, \ldots n$.Calculate $P(n+1)$.

So $Q(x) = (x+1)P(x) - x$ is a polynomial with degree $n+1$ such that $Q(k) = 0\ \forall k \in \{0, 1, 2, \ldots, n\}$

So $Q(x) = ax(x-1)(x-2)\ldots(x-n)$
$Q(-1) = 1$ and so $a = \dfrac{(-1)^{n+1}}{(n+1)!}$

So $P(x) = \dfrac{\frac{(-1)^{n+1}}{(n+1)!}x(x-1)(x-2)\ldots(x-n) + x}{x+1}$ (with continuity extension when $x = -1$)

So $P(n+1) = \dfrac{\frac{(-1)^{n+1}}{(n+1)!}(n+1)n(n-1)\ldots1 + n + 1}{n+2} = \boxed{\dfrac{(-1)^{n+1} + n + 1}{n+2}}$

vanu1996
607 posts

Aug 11, 2013, 8:56 pm

PM #279

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1810905&sid=0a14e4bcc5c347bd9153e928566e7d2c#p1810905

vanu1996
607 posts

Aug 11, 2013, 9:03 pm

PM #280

107-It is known of a polynomial over Z that $p(n)>n$ for every positive integer n.Consider $x_1=1,x_2=p(x_1)\ldots$. We know that,for any positive integer N,there exist a term of the sequence divisible by N,find all such polynomials.

xxp2000
520 posts

Aug 11, 2013, 11:44 pm

PM #281

> vanu1996 wrote:
> 107-It is known of a polynomial over Z that $p(n)>n$ for every positive integer n.Consider $x_1=1,x_2=p(x_1)\ldots$. We know that,for any positive integer N,there exist a term of the sequence divisible by N,find all such polynomials.

We know x_n is strictly increasing. Suppose there exists n such that $x_n-1>x_{n-1}$.
Let $N=x_n-1$.
$x_{n+1}=p(x_n)=p(1)=x_2 \mod N$.
We can show $x_{n+k}=x_{k+1} \mod N, \forall k$ by induction.
So the sequence can only have remainders among $x_1, x_2, \cdots, x_{n-1}$. None of them is zero. Absurd!

Now we proved $x_n=x_{n-1}+1, \forall n$. Obviously we have $p(x)=x+1$.

randomusern...
1035 posts

Aug 16, 2013, 10:01 pm • 1

PM #282

Problem 108. Does there exist a nonconstant polynomial P with real coefficients that satisfies $(P(x))^2-1=P(x^2+1)$ for all real x?

This post has been edited 1 time. Last edited by randomusername, Aug 24, 2013, 10:17 pm

xxp2000
520 posts

Aug 17, 2013, 5:19 am

PM #283

> randomusername wrote:
> Does there exist a nonconstant polynomial P with real coefficients that satisfies $(P(x))^2-1=P(x^2+1)$ for all real x?

The answer is no!
Suppose there exists such P. Let P have the least positive degree.
$P(x)^2=P(-x)^2$ implies P is either even or odd.
If P is odd, $P(0)=0$. We define sequence $\{x_0=0, x_{n+1}=x_n^2+1\}$. All x_{2n} are the roots of $P(x)=0$ and all x_{2n-1} are the roots of $P(x)=-1$. Absurd!
Now consider P is even. We can write $P(x)=f(x^2)$.
$f(x^2)^2-1=f((x^2+1)^2)$, or
$f(x)^2-1=f((x+1)^2)$, or
$f(x-1)^2-1=f(x^2)$.
We see $g(x)=f(x-1)$ also satisfies the original f.e. But g has smaller degree than P. Absurd!

randomusern...
1035 posts

Aug 20, 2013, 10:08 pm • 1

PM #284

Yes. It is very nice, is it not?

Another extremely interesting polynomial equation is $f(x)f(x+1)=f(x^2+x+1)$. (Problem 109)

This post has been edited 1 time. Last edited by randomusername, Aug 24, 2013, 10:17 pm

xxp2000
520 posts

Aug 24, 2013, 6:01 am

PM #285

> randomusername wrote:
> Yes. It is very nice, is it not?
>
> Another extremely interesting polynomial equation is $f(x)f(x+1)=f(x^2+x+1)$.

This should be problem 109. Please remember to add the serial number so that we can track the mileage of the marathon.

The constant polynomial are $f(x)=0$ and $f(x)=1$. Now consider non-constant case.

Suppose x is the root, then x^2+x+1 and $(x-1)^2+(x-1)+1=x^2-x+1$ are also the roots.
Obviously x cannot be real, otherwise $x^2+|x|+1>|x|$ implies f has infinitely many roots.
Let $x=a+bi$, then $a^2\pm a+1-b^2+(2a\pm1)bi$ are also the roots.
Define $a_{n+1}=a_n^2+|a_n|+1-b_n^2$ and $b_{n+1}=(2a_n+sign(a_n))b_n$, where $sign(x)=1$ for $x\ge0$ and -1 otherwise.
Now all $S=\{a_n+b_ni,\forall n\}$ are the roots.

Case 1) $|b_n|>1$
If $a_n\ne0$, $|b_{n+1}|>|b_n|$;
If $a_n=0$, $|b_{n+2}|>|b_n|$.
So $|S|$ is infinite, absurd!

Case 2) $|b_n|=1$, $a_n\ne0$.
$|b_{n+1}|>1$ reduces to case 1), absurd!

Case 3) $|b_n|<1$
$a_{n+2}>a_{n+1}>|a_n|$, $|b_{n+2}|\ge(2a_{n+1}+1)|b_{n+1}|$.
We can show by induction that $a_{n+k}>|a_{n+k-1}|$ and $|b_{n+k}|\ge(2a_{n+1}+1)|b_{n+k-1}$ as long as $|b_{n+k-1}|<1$.
Since $|S|$ is finite, we have $|b_{n+K}|\ge1, |b_{n+K-1}|<1$ for some K. Since $a_{n+K}>0$, we reduce to either case 1 or case 2, absurd!

Hence, the only possible roots are $\pm i$. If i is the root, so is $i^2-i+1=-i$. Vice versa.
Now we have $f(x)=(x+i)^n(x-i)^m$.

It is easy to check the only non-constant solution is $f(x)=(x^2+1)^n$.

War-Hammer

Aug 24, 2013, 3:17 pm • 1

PM #286

Problem 110 :

661 posts

Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ be a polynomial where $a_i\in\mathbb{Z}$. Let m be a natural number such that for all $a\in\mathbb{Z}$, $P(a)$ is divisible by m. Prove that $n!a_n$ is divisible by m.

---

xxp2000
520 posts

Aug 24, 2013, 4:28 pm — PM #287

> War-Hammer wrote:
> **Problem 110 :**
>
> Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ be a polynomial where $a_i\in\mathbb{Z}$. Let m be a natural number such that for all $a\in\mathbb{Z}$, $P(a)$ is divisible by m. Prove that $n!a_n$ is divisible by m.

$n!a_n=\sum_{i=0}^{n}(-1)^i\binom{n}{i}P(n-i)$. Done!

War-Hammer
661 posts

Aug 24, 2013, 4:42 pm • 1 — PM #288

Yes , I know it's obvious with this lemma , but do you have any other proof ???

---

randomusern...
1035 posts

Aug 27, 2013, 2:26 am • 1 — PM #289

Why search? The solution with the lemma is probably as simple as it gets. I don't see any possible ways to grip the divisibility relation, and methods like induction slip off easily. The lemma, on the other hand, is very well known and follows easily from PIE or Lagrange Interpolation.

So could someone post another problem please? (I have already posted my favorite two.)

---

alibez
357 posts

Sep 9, 2013, 5:40 pm • 1 — PM #290

problem 111

Is there a polynomial $p(x,y,z)\in\mathbb{Z}[x,y,z]$ such that :

$P(x^n,x^{n+1},x+x^{n+2})=x\quad(\forall x\in\mathbb{R})$?

---

xxp2000
520 posts

Sep 14, 2013, 5:11 am • 1 — PM #291

> alibez wrote:
> **problem** 111
>
> Is there a polynomial $p(x,y,z)\in\mathbb{Z}[x,y,z]$ such that :
>
> $P(x^n,x^{n+1},x+x^{n+2})=x\quad(\forall x\in\mathbb{R})$?

We will show we can find $P(x,y,z)$ such that $P(a^n,a^{n+1},a+a^{n+2})=a^m$ for any $m\in\mathbb{N}$ by induction.

Obviously when $m\ge n^2$, we can write $m=kn+r$ with $0\le r<n$. Then pick $P(x,y,z)=x^{k-r}y^r$.

Assume we can find such P for $m\ge k$. Now $(a+a^{n+2})^{k-1}=a^{k-1}+Q(a)$ where Q is polynomial of a and $a^{k+n}|Q(a)$. By the induction assumption, we can find $F(x,y,z)$ such that $F(a^n,a^{n+1},a+a^{n+2})=Q(a)$. We can let $P(x,y,z)=z^{k-1}-F(x,y,z)$. So the claim also holds for $m=k-1$.

Q.E.D.

---

alibez
357 posts

Sep 15, 2013, 1:13 pm • 1 — PM #292

please post a problem !

---

cause_im_ba...
43 posts

Oct 26, 2013, 8:31 pm • 1 — PM #293

problem 112
f is a polynomial such that f(x)f(1/x)=f(x)+f(1/x) for all x. If f(3)=28 , find f(5)

---

shadow10
205 posts

Nov 14, 2013, 10:16 pm — PM #294

$f(x)$ is a polynomial function. Let, $f(x)=a_nx^n+a_{n-1}x^{n-1}+.........+a_0(a_n\neq0)$
we are getting,
$(a_nx^n+a_{n-1}x^{n-1}+.........+a_0).\left(\frac{a_n}{x^n}+\frac{a_{n-1}}{x^{n-1}}+......+a_0\right)=(a_nx^n+a_{n-1}x^{n-1}+.........+a_0)+\left(\frac{a_n}{x^n}+\frac{a_{n-1}}{x^{n-1}}+......+a_0\right)$

comparing the co-efficients of x^n , $a_na_0=a_n\implies a_0=1$

comparing the co-efficients of x^{n-1} , $a_na_1+a_{n-1}a_0=a_{n-1}\implies a_1=0$
similarly we get, $a_{n-1}=a_{n-2}=.......=a_1=0$ and $a_n=\pm1(a_n^2=a_0)$
Hence. $f(x)=\pm x^n+1$
$\pm3^n+1=28$ which gives $n=3$. $f(x)=x^3+1$
$f(5)=5^3+1=126$
Click to reveal hidden text

please someone posts a new problem

---

gobathegreat
411 posts

Nov 15, 2013, 8:35 pm • 1 — PM #295

Problem 113
Let $P(x)$, $Q(x)$ and $R(x)$ be polynoms with real coefficients. It is known that two of them are third degree and one second degree. Also it is known $P(x)^2+Q(x)^2=R(x)^2$. Prove that one of two polynoms of third degree has all real roots.

---

joybangla
836 posts

Nov 21, 2013, 1:17 pm — PM #296

> gobathegreat wrote:
> Problem 113

Let $P(x)$, $Q(x)$ and $R(x)$ be polynoms with real coefficients. It is known that two of them are third degree and one second degree. Also it is known $P(x)^2+Q(x)^2=R(x)^2$. Prove that one of two polynoms of third degree has all real roots.

Now observe that WLOG we can assume $(P(X)$ is monic and that implies $R(X)$has leading coefficient=± 1 .Now the given condition translates to $Q(X)^2=(R(X)-P(X))(R(X)+P(X))$ these two factors can be one linear and one cubic.
Proof: Otherwise they are both quadratics and their sum will be another quadratic.But their sum is $2R(X)$ which is a cubic.Contradiction.
Now if leading terms of both of them are 1 then $(R(X)-P(X))=aX+b$ which is also a factor of $Q(X)$ thus $Q(X)$ also has real roots.So let $Q(X)=(aX+b)(cX+d)$ so we see $R(X)-P(X)=aX+b$ and $R(X)+P(X)=(aX+b)(cX+d)^2$ now subtracting we get $2R(X)=(aX+b)((cX+d)^2-1) \implies R(X)=(\frac{a}{2}X+\frac{b}{2})(cX+d-1)(cX+d+1)$ hence it has three real roots.If one of them,say $R(X)$ has leading coeffecient -1 then consider $-R(X)$ and the conditions remain same and it has leading coefficient 1.Now follow the previous method and voila!
Problem 114

Given the polynomial $P(X)=X^3-153X^2+111X+38$
[1]Prove the interval $[1,3^{2000}]$ contains at least 9 solutions to the equation $P(X)\equiv 0 \pmod{3^{2000}}$.
[2]Determine exactly how many solutions to the congruence are there in the interval $[1,3^{2000}]$

shivangjindal
676 posts

Nov 23, 2013, 6:37 pm • 1 ⌧PM #297

Problem 114 , is from Vietnam National olympiad , see the link : http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1293828&sid=2e3912f42c292687baa407db493243bc#p1293828

lehungvietbao
1045 posts

Dec 29, 2013, 4:56 pm • 1 ⌧PM #298

Problem 115
Find all polynomials $P(x)$ with real coefficients such that

$$(a+1)\,(b+1)\in\mathbb{Q}\Rightarrow P\,(a)+P\,(b)+P\,(ab)\in\mathbb{Q}\quad\forall a,b\in\mathbb{R}$$

amatysten
73 posts

Jan 1, 2014, 1:42 pm ⌧PM #299

> lehungvietbao wrote:
> **Problem 115**
> Find all polynomials $P(x)$ with real coefficients such that
> $$(a+1)\,(b+1)\in\mathbb{Q}\Rightarrow P\,(a)+P\,(b)+P\,(ab)\in\mathbb{Q}\quad\forall a,b\in\mathbb{R}$$

1. We put $a=0$ and get $P(b)+2c_0\in\mathbb{Q},\forall b\in\mathbb{Q}$, which mean P has rational coefficients.
2. Now put $a=-1,b=\pi$ and get $T(\pi)=P(\pi)+P(-\pi)\in\mathbb{Q}$, which is possible only if $degT=0$, since π is not algebraic over $\mathbb{Q}$. Thus $P(x)$ has only odd powers of x.
3. Putting $a=b=k\sqrt{3}-1$ we get $2P(k\sqrt{3}-1)+P(3k^2+1-2k\sqrt{3})=R(k)+\sqrt{3}S(k)$, where $R(k),S(k)\in\mathbb{Q}[k]$. $\forall kS(k)$ must be 0, but writing in decreasing degrees of k only those parts that contain $\sqrt{3}$ we get $2P(k\sqrt{3}-1)=Ak^n\sqrt{3}+\ldots;P(3k^2+1-2k\sqrt{3})=B(3k^2+1)^{n-1}(2k\sqrt{3})+\cdots=Ck^{2n-1}\sqrt{3}$.
4. $S(k)=0,\forall k\Rightarrow n\geq 2n-1\Rightarrow n\leq 1$.
5. Any $P(x)\in\mathbb{Q}[x]$ with $degP\leq 1$ matches.

nima-amini
185 posts

Jan 9, 2014, 3:57 am ⌧PM #300

problem 116
if p,q two monic polynomial

$$P(P(X))=Q(Q(X))$$

prove

$$P\equiv Q$$

problem117
find all polynomials

$$P(X),Q(X),R(X)$$

with real coefficients such that

$$\forall X\in\mathbb{R};\sqrt{P(X)}-\sqrt{Q(X)}=R(X)$$

nima-amini
185 posts

Jan 9, 2014, 2:35 pm ⌧PM #301

problem 118
find all polynomials

$$P(X)$$

with real coefficients such that

$$P(X)^2+P(-X)=P(X^2)+P(X)$$

nima-amini
185 posts

Jan 9, 2014, 8:15 pm ⌧PM #302

problem 119
find all polynomials

$$p(x)$$

with real coefficients such that

$$\forall n\in\mathbb{N};\lfloor p(\lfloor p(n)\rfloor)\rfloor=4\lfloor p(n)\rfloor-n$$

nima-amini
185 posts

Jan 9, 2014, 9:28 pm PM #303

problem 120

$$n \in \mathbb{N}$$

.find number of

$$(P,Q)$$

with real coefficients such that

$$P(X)^2+Q(X)^2=X^{2n}+1$$

pco
15396 posts

Jan 9, 2014, 9:29 pm • 1 PM #304

> nima-amini wrote:
>
> problem 118
> find all polynomials
>
> $$P(X)$$
>
> with real coefficients such that
>
> $$P(X)^2+P(-X)=P(X^2)+P(X)$$

If $P(x)=c$ is constant, we get $c=0$ and $c=1$ and so :
$\boxed{P(X)=0}$ $\forall x$
$\boxed{P(X)=1}$ $\forall x$

If $P(x)=ax^n$ is not constant ($a\ne 0$, $n>0$) and has just one summand, we get $a^2x^{2n}+a(-1)^nx^n=ax^{2n}+ax^n$ and so
$\boxed{P(x)=x^{2p}}$ $\forall x$ and whatever is $p\in\mathbb{N}$

If $P(x)=ax^n+bx^p+\dots$ is not constant and has at least two summmands, identification of two highest degree summands in both LHS and RHS gives :
$p=0$ and $a=1$ and degree odd :
$\boxed{P(x)=x^{2p-1}+1}$ $\forall x$ and whatever is $p\in\mathbb{N}$

nima-amini
185 posts

Jan 9, 2014, 9:39 pm PM #305

> pco wrote:
>
> If $P(x)=ax^n+bx^p+\dots$ is not constant and has at least two summmands, identification of two highest degree summands in both LHS and RHS gives :
> $p=0$ and $a=1$ and degree odd :
> $\boxed{P(x)=x^{2p-1}+1}$ $\forall x$ and whatever is $p\in\mathbb{N}$

please explain more

pco
15396 posts

Jan 9, 2014, 10:12 pm • 1 PM #306

> nima-amini wrote:
>
> > pco wrote:
> >
> > If $P(x)=ax^n+bx^p+\dots$ is not constant and has at least two summmands, identification of two highest degree summands in both LHS and RHS gives :
> > $p=0$ and $a=1$ and degree odd :
> > $\boxed{P(x)=x^{2p-1}+1}$ $\forall x$ and whatever is $p\in\mathbb{N}$
>
> please explain more

If $P(x)=ax^n+bx^p+\dots$ is not constant and has at least two summmands :
If $n>p>0$:
Two highest degree summands of LHS are $a^2x^{2n}+2abx^{n+p}$ while two highest degree summands of RHS are $ax^{2n}+..$ one summand of degree $2p$ or n or less, both $<n+p$, so impossible.

So $p=0$ and $P(X)=ax^n+b$ and equation is $a^2x^{2n}+2abx^n+b^2+a(-1)^nx^n+b=ax^{2n}+b+ax^n+b$
So $a^2=a$ and $2ab+(-1)^n=a$ and $b^2+b=2b$
SO $a=b=1$ and n odd

alibez
357 posts

Jan 9, 2014, 11:41 pm PM #307

> nima-amini wrote:
>
> problem 118
> find all polynomials
>
> $$P(X)$$
>
> with real coefficients such that
>
> $$P(X)^2+P(-X)=P(X^2)+P(X)$$

we have :
$x\to -x\Rightarrow P(x)^2=P(-x)^2\Rightarrow P(x)=P(-x)\Rightarrow P(x)=x^{2k}$ or $P(x)=-P(-x)$ know if α is root of $P\Rightarrow\alpha^2$ is root of P so

root of 1 so .....

pco
15396 posts

Jan 10, 2014, 12:43 am • 2

PM #308

alibez wrote:

nima-amini wrote:

problem 118
find all polynomials

$$P(X)$$

with real coefficients such that

$$P(X)^2+P(-X)=P(X^2)+P(X)$$

we have :
$x\to -x\Rightarrow P(x)^2=P(-x)^2$ .....

Wrong.

alibez
357 posts

Jan 10, 2014, 12:00 pm

PM #309

pco wrote:

alibez wrote:

nima-amini wrote:

problem 118
find all polynomials

$$P(X)$$

with real coefficients such that

$$P(X)^2+P(-X)=P(X^2)+P(X)$$

we have :
$x\to -x\Rightarrow P(x)^2=P(-x)^2$ .....

Wrong.

sorry for my bad mistake

thanks PCO .

pco
15396 posts

Jan 10, 2014, 3:17 pm

PM #310

nima-amini wrote:

problem 119
find all polynomials

$$p(x)$$

with real coefficients such that

$$\forall n\in\mathbb{N};\lfloor p(\lfloor p(n)\rfloor)\rfloor=4\lfloor p(n)\rfloor-n$$

Looking at equivalences when $n\to\infty$, we get that degree is 1 and $P(x)=ax+b$ with $a^2=4a-1$ and so $a=2\pm\sqrt3$

Let $f(n)=\lfloor P(n)\rfloor=\lfloor an+b\rfloor$ so that we have $f(f(n))=4f(n)-n$
Note that this implies that $f(n)$ is injective.

$a\notin\mathbb{Q}$ and so $\{an+b\}$ is dense in $[0,1]$
If $a=2-\sqrt3\in(0,1)$, this implies $\exists n$ such that $\{an+b\}\in[0,1-a)$ and so $f(n+1)=f(n)$, impossible since injective.

So $a=2+\sqrt3$

So problem is to find b such that $m+1>an+b\ge m$ implies $4m-n+1>am+b\ge 4m-n$

$\iff m+1>an+b\ge m$ implies $(4-a)m+1>n+b\ge(4-a)m$

$\iff$ (multiplying by a and remembering $a(4-a)=1$) : $m+1>an+b\ge m$ implies $m+a>an+ab\ge m$

$\iff$: $m+1>an+b\ge m$ implies $m+a+b-ab>an+b\ge m+b-ab$

And since $\{an+b\}$ is dense in $[0,1]$, this is equivalent to $m+a+b-ab\ge m+1$ and $m\ge m+b-ab$

$\iff b\in[0,1]$

Hence the answer : $\boxed{P(x)=(2+\sqrt3)x+b\text{, whatever is }b\in[0,1]}$ $\forall x$

Jan 10, 2014, 3:46 pm

PM #311

Jan 10, 2014, 3:40 pm PM #311

pco
15396 posts

> nima-amini wrote:
> problem117
> find all polynomials
> $$P(X), Q(X), R(X)$$
> with real coefficients such that
> $$\forall X \in \mathbb{R}; \sqrt{P(X)} - \sqrt{Q(X)} = R(X)$$

Note that $P(x), Q(x) \geq 0\,\forall x$

We also have $P(x) - Q(x) = R(x)(\sqrt{P(x)} + \sqrt{Q(x)})$ and so all complex zeroes of $R(x)$ are zeroes of $P - Q$ and so $P(x) - Q(x) = R(x)S(x)$ for some $S(x) \in \mathbb{R}[X]$

So (using continity around roots) $\sqrt{P(x)} + \sqrt{Q(x)} = S(x)$

So $\sqrt{P(x)} = \frac{R(x)+S(x)}{2} \in \mathbb{R}[X]$ and $\sqrt{Q(x)} = \frac{S(x)-R(x)}{2} \in \mathbb{R}[X]$

Hence the answer :
$\boxed{(P(x), Q(x), R(x)) = (U^2(x), V^2(x), U(x) - V(x)),\text{ where } U(x), V(x) \in \mathbb{R}[X] \text{ and } U(x), V(x) \geq 0\ \forall x}$

Jan 10, 2014, 4:30 pm PM #312

nima-amini
185 posts

problem 121
find all polynomials
$$P, Q$$
with real coefficients such that
$$P(Q(X)) = P(X)Q(X)$$

Jan 10, 2014, 5:27 pm • 1 PM #313

pco
15396 posts

> nima-amini wrote:
> problem 121
> find all polynomials
> $$P, Q$$
> with real coefficients such that
> $$P(Q(X)) = P(X)Q(X)$$

If $P(x) = c$ constant, equation is $c = cQ(x)$ and so :
Either $c = 0$,
Either $Q(x) = 1$

If $Q(x) = c$ constant, equation is $P(c) = cP(x)$ and so :
Either $c = 0$ and $P(0) = 0$,
Either $P(x) = 0\,\forall x$
Either $c = 1$ and $P(x) = 1\,\forall x$

If neither $P(x)$, neither $Q(x)$ are constant :
Let $z \in \mathbb{C}$ any root of $Q(x)$: setting $x = z$, we get $P(0) = 0$ and so $P(x) = xR(x)$ and equation is $R(Q(x)) = xR(x)\,\forall x$ (since polynomials)

Let r, q be degrees of $R(x), Q(x)$ and we get $qr = r + 1 \iff r = 1$ and $q = 2$
So $R(x) = ax + b$ with $a \neq 0$ and so $aQ(x) + b = ax^2 + bx$ and $Q(x) = \frac{ax^2 + bx - b}{a}$

Hence the solutions :
S1 : $P(x) = 0\,\forall x$ and $Q(x)$ is any polynomial in $\mathbb{R}[X]$

S2 : $P(x) = c$ and $Q(x) = 1\,\forall x$, where c is any real

S3 : $P(x) = xR(x)$ and $Q(x) = 0\,\forall x$ where $R(x)$ is any polynomial in $\mathbb{R}[X]$

S4 : $P(x) = ax^2 + abx$ and $Q(x) = x^2 + bx - b\,\forall x$ and whatever are $a \neq 0, b$

Jan 10, 2014, 5:40 pm PM #314

nima-amini
185 posts

problem 122
find all polynomials
$$P(X, Y, Z)$$
with real coefficients such that
$$abc = 1 \to p(a + \frac{1}{a}, b + \frac{1}{b}, c + \frac{1}{c}) = 0$$

Jan 10, 2014, 5:48 pm PM #315

nima-amini
185 posts

problem 123
find all polynomials f,g,h such that

$$|f(x)|-|g(x)+h(x)|=\{\begin{array}{cc}-1 & x<-1\\ 3x+2 & -1\le x<0\\ -2x+2 & x>0\end{array}$$

pco
15396 posts

Jan 10, 2014, 6:50 pm • 1

PM #316

> nima-amini wrote:
> problem 123
> find all polynomials f,g,h such that
>
> $$|f(x)|-|g(x)+h(x)|=\{\begin{array}{cc}-1 & x<-1\\ 3x+2 & -1\le x<0\\ -2x+2 & x>0\end{array}$$

Since $u=f,v=g+h$ solution implies $\pm u,\pm v$ solution, WLOG consider $\lim_{x\to-\infty}u(x)\ge 0$ and $\lim_{x\to-\infty}v(x)\ge 0$

So $u-v=-1$ and expression is $E(x)=|u(x)|-|u(x)+1|$

If $u(x)\ge 0\,\forall x$, then $E(x)=-1\,\forall x$, incorrect.
Let then a the littleest real such that $\exists b>a$ such that $u(x)\ge 0\,\forall x\le a$ and $-1<u(x)<0\,\forall x\in(a,b]$

If $a<-1$ we get $E(x)=-u(x)-u(x)-1$ over $(a,\min(b,-1))$ and so $u(x)=0\,\forall x$ (since polynomial) which is not a solution
If $a>-1$, we get $E(x)=-1$ over $[-1,a]$, which is wrong

So $a=-1$ and $-2u(x)-1=3x+2$ and so $u(x)=-\frac{3}{2}(x+1)$ which is not a solution.

So **no solution**.

nima-amini
185 posts

Jan 10, 2014, 6:59 pm

PM #317

problem 124
find all polynomials f,g,h such that

$$|f(x)|-|g(x)|+h(x)=\begin{cases}-1 & x<-1\\ 3x+2 & -1\le x<0\\ -2x+2 & x>0\end{cases}$$

nima-amini
185 posts

Jan 11, 2014, 5:43 pm

PM #318

no idea about problem 124?

pco
15396 posts

Jan 11, 2014, 9:06 pm

PM #319

> nima-amini wrote:
> problem 124
> find all polynomials f,g,h such that
>
> $$|f(x)|-|g(x)|+h(x)=\begin{cases}-1 & x<-1\\ 3x+2 & -1\le x<0\\ -2x+2 & x>0\end{cases}$$

Since f,g,h solution implies $\pm f,\pm g,h$ solution, WLOG consider $\lim_{x\to-\infty}f(x)\ge 0$ and $\lim_{x\to-\infty}g(x)\ge 0$
Note that $f(x)=g(x)\forall x$ is never a solution.

Setting $x\to-\infty$, we get $f(x)-g(x)+h(x)=-1$ and expression is $|f(x)|-f(x)+g(x)-|g(x)|-1$ which can at most take four different forms :
-1
$-2f(x)-1$
$2g(x)-1$
$2g(x)-2f(x)-1$

And so only 6 possibilities and very few cases to test :

$-2f(x)-1=3x+2$ and $2g(x)-1=-2x+2$ and so $(f,g,h)=(-\frac{3}{2}x-\frac{3}{2},-x+\frac{3}{2},\frac{1}{2}x+2)$ which is not a solution

$-2f(x)-1=3x+2$ and $2g(x)-2f(x)-1=-2x+2$ and so $(f,g,h)=(-\frac{3}{2}x-\frac{3}{2},-\frac{5}{2}x,-x+\frac{1}{2})$ which indeed is a solution

$2g(x)-1=3x+2$ and $-2f(x)-1=-2x+2$ and so $(f,g,h)=(x-\frac{3}{2},\frac{3}{2}x+\frac{3}{2},\frac{1}{2}x+2)$ which is not a solution

$2g(x)-1=3x+2$ and $2g(x)-2f(x)-1=-2x+2$ and so $(f,g,h)=(\frac{5}{2}x+2,\frac{3}{2}x+\frac{3}{2},-x-\frac{3}{2})$ which is not a solution

$2g(x)-2f(x)-1=3x+2$ and $-2f(x)-1=-2x+2$ and so $(f,g,h)=(x-\frac{3}{2},\frac{5}{2}x,\frac{3}{2}x+\frac{1}{2})$ which is not a solution

$2g(x)-2f(x)-1=3x+2$ and $2g(x)-1=-2x+2$ and so $(f,g,h)=(-\frac{5}{2}x,-x+\frac{3}{2},\frac{3}{2}x+\frac{1}{2})$ which is not a solution

Hence the four solutions : $\boxed{(f,g,h)=\left(\pm\frac{3}{2}(x+1),\pm\frac{3}{2}x,-x+\frac{1}{2}\right)}$

---

nima-amini
185 posts

Jan 11, 2014, 9:40 pm — PM #320

problem125
find all polynomial

$$p(x,y)$$

such that

$$\forall x,y\in\mathbb{R}:p(y+x,y-x)=p(x,y)$$

---

nima-amini
185 posts

Jan 11, 2014, 9:44 pm • 1 — PM #321

problem 126
find all polynomial

$$p(x)$$

with real coefficients such that

$$x\neq 0\rightarrow (p(x))^2+(p(\frac{1}{x}))^2=p(x^2)p(\frac{1}{x^2})$$

---

nima-amini
185 posts

Jan 11, 2014, 9:47 pm — PM #322

problem 127
find all polynomial

$$p(x)$$

with complex coefficients such that

$$p(2x^2-1)=\frac{(p(x))^2}{2}-1$$

---

nima-amini
185 posts

Jan 11, 2014, 9:51 pm — PM #323

problem 128
let

$$f(x)=x^2-2ax-a^2-\frac{3}{4}$$

find all (a) such that

$$\forall x\in[0,1]\rightarrow|f(x)|\leq 1$$

---

nima-amini
185 posts

Jan 11, 2014, 9:56 pm — PM #324

problem 129
find all

$$n\in\mathbb{N},r\in\mathbb{R}$$

such that

$$2x^2+2x+1\mid(x+1)^n-r$$

---

pco
15396 posts

Jan 11, 2014, 10:29 pm — PM #325

> nima-amini wrote:
>
> problem125
> find all polynomial
>
> $$p(x,y)$$
>
> such that
>
> $$\forall x,y\in\mathbb{R}:p(y+x,y-x)=p(x,y)$$

$P(2y,2x)=P((y+x)+(y-x),(y+x)-(y-x))=P(y+x,y-x)=P(x,y)$

So $P(4x,4y)=P(x,y)$ and so $P(x,y)=P(\frac{x}{4^n},\frac{y}{4^n})$

Setting $n\to+\infty$ and using continuity of polynomials, we get $\boxed{P(x,y)=c}$ constant, which indeed is a solution, whatever is $c\in\mathbb{R}$

---

nima-amini
185 posts

Jan 12, 2014, 8:35 pm — PM #326

problem 130
find all polynomial P with real coefficients that satisfies $(P(x))^2-1=P(x^2+1)$

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nima-amini
185 posts

Jan 15, 2014, 12:41 am — PM #327

nice problems

This post has been edited 2 times. Last edited by nima-amini, Jul 31, 2014, 8:41 pm

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BISHAL_DEB
270 posts

Jan 15, 2014, 2:53 am — PM #328

Solution to problem 130:

Suppose there exists such a polynomial satisfying the given equation and $P(\alpha)=0$ for some $\alpha\in\mathbb{C}$. Then $P(\alpha^2+1)=-1$ and $P((\alpha^2+1)^2+1)=0$. Hence we get an infinite sequence of roots $\alpha,(\alpha^2+1)^2+1,(((\alpha^2+1)^2+1)^2+1)^2+1$

and $P((a^2+1)^2+1)=0$. Hence we get an infinite sequence of roots $a, (a^2+1)^2+1, (((a^2+1)^2+1)^2+1)^2+1, \ldots$. From a certain point this sequence has to be periodic as because a polynomial cannot have infinitely many roots. Hence we get an infinite family of polynomials.

BISHAL_DEB
270 posts

Jan 15, 2014, 3:36 am PM #329

> *nima-amini wrote:*
> problem 129
> find all
> $$n \in \mathbb{N}, r \in \mathbb{R}$$
> such that
> $$2x^2+2x+1 \mid (x+1)^n - r$$

Solution:

We can write

$$(x+1)^n = (2x^2+2x+1)q(x) + r$$

Let a, b be $\dfrac{-1 \pm i}{2}$ the roots of the polynomial $2x^2+2x+1$. Then $a+b=-1$ and $ab = 1/2$.

Case 1: For n odd. Clearly $n>1$. Let $n = 2k+1$ for some $k \in \mathbb{N}$. Then

$$(a+1)^{2k+1} = r$$

$$(a^2+2a+1)^k \cdot (a+1) = r$$

$$r = (-a^2)^k \cdot (-b) = -(\frac{i}{2})^k \cdot b$$

But as $r \in \mathbb{R}$ this case is discarded.

Case 2: For n even. Let $n = 2k$ for some $k \in \mathbb{N}$. Then

$$(a+1)^{2k} = r$$

$$(a^2+2a+1)^k = r$$

$$r = (-a^2)^k = -(\frac{i}{2})^k$$

Hence the possible family of solutions for (n, r) is $(4k, \dfrac{(-1)^{k+1}}{2^{2k}})$ for all $k \in \mathbb{N}$

---

lehungvietbao
1045 posts

Jan 17, 2014, 8:47 am • 1 PM #330

Problem 131
Find all polynomials $P(x)$ with real coefficients such that

$$\begin{cases} P(a+b+c) = 7P(a)+4P(b)-5P(c) \\ (a+b)(b+c)(c+a) = 2a^3+b^3-2c^3 \end{cases}$$

---

amatysten
73 posts

Feb 8, 2014, 3:11 pm • 1 PM #331

> *lehungvietbao wrote:*
> **Problem 131**
> Find all polynomials $P(x)$ with real coefficients such that
> $$\begin{cases} P(a+b+c) = 7P(a)+4P(b)-5P(c) \\ (a+b)(b+c)(c+a) = 2a^3+b^3-2c^3 \end{cases}$$

1. If I understand it correctly, we need to find all such $P(x)$ satisfying the first condition for any a, b, c satisfying the second condition.

2. Let $a = \sqrt[3]{2}c, b = -\sqrt[3]{2}c$, then the second condition is satisfied $\forall c \in \mathbb{R} \Rightarrow 6P(c) = 7P(\sqrt[3]{2}c) + 4P(-\sqrt[3]{2}c)$.

3. All the degrees on the left and on the right must match. Let $P(x) = \Sigma a_n x^n$. If $n = 2k \Rightarrow 6a_nc^n = 7a_n(\sqrt[3]{2})^n c^n + 4a_n(\sqrt[3]{2})^n c^n$. If $a_n \neq 0 \Rightarrow \dfrac{6}{11} = (\sqrt[3]{2})^n$ yielding a contradiction. If $n = 2k+1 \Rightarrow 6a_nc^n = 7a_n(\sqrt[3]{2})^n c^n - 4a_n(\sqrt[3]{2})^n c^n$. If $a_n \neq 0 \Rightarrow 2 = (\sqrt[3]{2})^n \Rightarrow n = 3$.

4. Thus, $\boxed{P(x) = Cx^3}$ is the only possibility and it fits $\forall C \in \mathbb{R}$, since
$(a+b+c)^3 = 7a^3+4b^3-5c^3 \Leftrightarrow (a+b)(b+c)(c+a) = 2a^3+b^3-2c^3$

$(a+b+c)^3 = 7a^3+4b^3-5c^3 \Leftrightarrow (a+b)(b+c)(c+a) = 2a^3+b^3-2c^3$.

amatysten
73 posts

Feb 8, 2014, 4:08 pm #332

> nima-amini wrote:
> problem 128
> let
> $$f(x)=x^2-2ax-a^2-\frac{3}{4}$$
> find all (a) such that
> $$\forall x\in[0,1]\to|f(x)|\le 1$$

1. $x_0=a$ is the bottom of parabola. If $x_0=a\le 0$ then $|f(x)|\le 1 \Leftrightarrow \begin{cases} f(0)\ge -1 \\ f(1)\le 1 \end{cases}$.

It gives $a\in\left[-\frac{1}{2};0\right]$.

2. If $a\ge 1$ then $|f(x)|\le 1 \Leftrightarrow \begin{cases} f(0)\le 1 \\ f(1)\ge -1 \end{cases}$.

It gives $a\in\emptyset$.

3. If $0<a<1$ then $|f(x)|\le 1 \Leftrightarrow \begin{cases} f(0)\le 1 \\ f(1)\le 1 \\ f(a)\ge -1 \end{cases}$.

It gives $a\in\left(0;\frac{1}{2\sqrt{2}}\right]$.

4. The answer is $\boxed{\forall a\in\left[-\frac{1}{2};\frac{1}{2\sqrt{2}}\right]}$.

randomusern...
1035 posts

Mar 2, 2014, 8:21 pm • 1 #333

> nima-amini wrote:
> problem 130
> find all polynomial P with real coefficients that satisfies $(P(x))^2-1=P(x^2+1)$

> BISHAL_DEB wrote:
> Click to reveal hidden text
>
> Solution to problem 130:
>
> Suppose there exists such a polynomial satisfying the given equation and $P(\alpha)=0$ for some $\alpha\in\mathbb{C}$. Then $P(\alpha^2+1)=-1$ and $P((\alpha^2+1)^2+1)=0$. Hence we get an infinite sequence of roots $\alpha,(\alpha^2+1)^2+1,(((\alpha^2+1)^2+1)^2+1)^2+1,\ldots$. From a certain point this sequence has to be periodic as because a polynomial cannot have infinitely many roots. Hence we get an infinite family of polynomials.

No. See page 15 of the marathon.

> xxp2000 wrote:
> Click to reveal hidden text
>
>> randomusername wrote:
>> Does there exist a nonconstant polynomial P with real coefficients that satisfies $(P(x))^2-1=P(x^2+1)$ for all real x?
>
> The answer is no!
> Suppose there exists such P. Let P have the least positive degree.
> $P(x)^2=P(-x)^2$ implies P is either even or odd.
> If P is odd, $P(0)=0$. We define sequence $\{x_0=0, x_{n+1}=x_n^2+1\}$. All x_{2n} are the roots of $P(x)=0$ and all x_{2n-1} are the roots of $P(x)=-1$. Absurd!
> Now consider P is even. We can write $P(x)=f(x^2)$.
> $f(x^2)^2-1=f((x^2+1)^2)$, or
> $f(x)^2-1=f((x+1)^2)$, or
> $f(x-1)^2-1=f(x^2)$.
> We see $g(x)=f(x-1)$ also satisfies the original f.e. But g has smaller degree than P. Absurd!

joybangla
836 posts

May 26, 2014, 2:31 pm #334

Hey a polynomial marathon should not die. I am posting wayyyyyyyyyyyyyy late but here's a problem. Pretty easy.
Problem 132:
Find all such $P\in\mathbb{R}[X]$ such that the following holds :
$$P(a)\in\mathbb{Z}\implies a\in\mathbb{Z}$$

theMFailure

Jun 22, 2014, 12:39 am #335

66 posts

> joybangla wrote:
>
> Hey a polynomial marathon should not die. I am posting wayyyyyyyyyyyyyyy late but here's a problem. Pretty easy.
> Problem 132:
> Find all such $P \in \mathbb{R}[X]$ such that the following holds :
>
> $$P(a) \in \mathbb{Z} \implies a \in \mathbb{Z}$$

Only the polynomials with coefficients $\in \mathbb{Z}$ satisfy I guess..... am I wrong??

here's $problem 133$
Consider a polynomial $P(a, b, c)$ where $a, b.c$ are sides of a triangle.
Can you give a sufficient and necessary condition for which $P(a, b, c) \geq 0 \forall a, b, c$

---

joybangla
836 posts

Jun 22, 2014, 11:39 am • 1 — PM #336

@theMFailure Yes you are. $P(x) = \sqrt{2}$. If you have done something then post the solution, don't just make a wild guess. For your problem, K.B. Stolarsky, cubic triangle inequalities gives that condition but for homogeneous polynomials of degree 3. Results better than that are probably not discovered yet. So Problem 132 is still unsolved. And it is easy, but not trivial I think.

AkosS
89 posts

Jul 8, 2014, 6:37 am — PM #337

> nima-amini wrote:
>
> problem 127
> find all polynomial
>
> $$p(x)$$
>
> with complex coefficients such that
>
> $$p(2x^2 - 1) = \frac{(p(x))^2}{2} - 1$$

A straightforward bashing with derivatives:

$P(x) = 1 \pm \sqrt{3}$ are solutions, we will show that they are in fact the only ones.

As $P(x) \in C^{\infty}[\mathbb{C}]$, we can take the kth derivative of the equation for each $k \in \mathbb{N}$. Note that $2x^2 - 1 = x$, if $x = 1$ or $x = -\frac{1}{2}$. Let $h(x) = 2P(2x^2 - 1) - P(x)^2 + 2$, so $h^{(k)} \equiv 0 \, \forall k \in \mathbb{N}$ from the initial conditions.

$H_l(x) := \{h(x) = 0, \dots, h^{(l)}(x) = 0\}$. We will show, that the only two solution for $H_l(1)$ is $P(1) = 1 \pm \sqrt{3}$ and $P^{(l)}(1) = 0$, if $\deg P > l > 0$.

Observation 0.: $h^{(l)}(x)$ contains terms having derivatives as factors up to the l th.

Observation 1.: $h^{(l)}(x)$ is the sum of terms of the form $\alpha x^{\beta} P^{(i)}(\chi(x)) P^{(j)}(\chi(x))$, where $\beta, i, j \in \mathbb{N}$, $\alpha \in \mathbb{Z}$, and $\chi(x) \in \{x, 2x^2 - 1\}$. To prove this, take the derivative:

$$\alpha\beta x^{\beta-1} P^{(i)}(\chi(x)) P^{(j)}(\chi(x)) + \alpha x^{\beta} \chi'(x) \left(P^{(i+1)}(\chi(x)) P^{(j)}(\chi(x)) + P^{(i)}(\chi(x)) P^{(j+1)}(\chi(x))\right)$$

Which is trivially the sum of terms of the same form, with $\beta, i, j \in \mathbb{N}$, $\alpha \in \mathbb{Z}$, and $\chi(x) \in \{x, 2x^2 - 1\}$, and $h(x)$ has the above described form.

Observation 2.: If $k > 1$, in $h^{(k)}(x)$ $P^{(k)}(\chi(x))$ appears only in the following terms:
$2^k x^k P^{(k)}(2x^2 - 1)$ and $-2P(x)P^{(k)}(x)$. This is true for $k = 1$ as $h'(x) = 2xP'(2x^2 - 1) - P(x)P'(x)$. As $h^{(k+1)}(x) = (h^k(x))'$, the $k + 1$th derivative can only appear in a term, if in $h^{(k)}(x)$ there was a term of the form descibed in Obs. 1. with $i = k$ and $j = 0$. But we can assume that the observation is true for k, and thus for $k + 1$ as well.

Observation 3.:
Plug $-x$ into the equation:

$$P(x)^2 - 2 = 2P(2x^2 - 1) = P(-x)^2 - 2 \Rightarrow P(x)^2 - P(-x)^2 = 0$$

Which implies either $P(x) = P(-x)$ or $P(x) = -P(-x)$, as if $P(x) \neq 0$, and $P(x)$ is a polyinomial, thus continous, so $\exists f(x) > \varepsilon > 0, \delta > 0$:
$f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$, and , meaning on this interval, the two polinomials must be equal because of the continuity of the functions. But if two polynomials equal at infinitely many points, they are equal. This means that P is an even or an odd function, so if x is a root, than $-x$ is as well, and that's true for P' as well.

Now to prove the statement:

$$2P(2x^2 - 1) = P(x)^2 - 2 \Rightarrow 2xP'(2x^2 - 1) = P(x)P'(x)$$

Put $x = 1$:

$$2P(1) = P(1)^2 - 2 \Rightarrow P(1) = 1 \pm \sqrt{3}$$

$$4P'(1) = P(1)P'(1) \Rightarrow P'(1) = 0 \text{ as } P(1) \neq 2$$

So for $k = 1$, we proved our statement. Now assume we proved it for $n < k$, and add the kth equation to our stack. From the first $k = 1$, we have $P(1) = 1 \pm \sqrt{3}$ and $P^{(l)}(1) = 0$. From observation 1. and 2. we get that the last equation becomes of first $k-1$, we have $P(1) = 1 \pm \sqrt{3}$, and $P^{(i)}(1) = 0$. From observation 1. and 2. we get that the last equation becomes of the form $(c_1, c_2 \in \mathbb{Z}$, and $P(1) \notin \mathbb{Q}$, so from

$$P^{(k)}(1)(c_1 - c_2P(1)) = 0$$

We get $P^{(k)}(1) = 0$. If we put $x = \frac{-1}{2}$, we get that $P(\frac{-1}{2}) = 1 \pm \sqrt{3}$ and $P^{(l)}(\frac{-1}{2}) = 0$, if $\deg P > l > 0$. Now look at $P'(x)$. From our statement, we get that 1 and $\frac{-1}{2}$ is a root of multiplicity $n-2$, and from Observation 3, so is -1, and $\frac{1}{2}$, so we have at least $4(n-2)$ roots, but we can only have $n-1$, so $n \leq 2$. But if $n = 2$, then 1 and $\frac{-1}{2}$ are both roots of P', but that's impossible. If $P(x) = ax + b$, then we get the equation system $4a - a^2 = 0, -2ab = 0, 2 - 2a + 2b - b^2 = 0$, which has solutions only with $a = 0$. Thus we finished our proof.

AkosS
89 posts

Jul 8, 2014, 7:01 am PM #338

> *joybangla wrote:*
> Hey a polynomial marathon should not die. I am posting wayyyyyyyyyyyyyyy late but here's a problem. Pretty easy.
> Problem 132:
> Find all such $P \in \mathbb{R}[X]$ such that the following holds :
> $$P(a) \in \mathbb{Z} \implies a \in \mathbb{Z}$$

Suppose P is good.
If $degP = 0$, then either P is integer for every real, or for none, so it is not possible.
If $degP > 1$, then $\exists n_0 := \max(\sup\{|\lambda| : P'(\lambda) = 0\}, \sup\{x : P(x+1) - P(x)| < 2\}) < \infty$. Let $n \in \mathbb{Z}$ and $n > n_0$.
Because P is continous, and strictly monotonic on $[n_0, \infty]$, and $|f(n) - f(n+1)| \geq 2$, so there is an integer in $[(f(n), f(n+1))$, but there is none in $(n, n+1)$, contradiction.
This leaves us with $P(x) = ax + b$. But from $P(0) = b$ and $P(1) = ax + b$, we get that $a, b \in \mathbb{Z}$, and $a \neq 0$, we can consider $P(x) - b$, as if the statement is true for $P(x)$, than it must be true for $P(x) - b$. Now $P(x) = ax$, so a strictly monotonous function. $P(0) = 0$, and due to $a \in \mathbb{Z}$, so is $f(1) = a$. There must not be an integer in $(0, a)$, or there is a rational in $(0, 1)$, for which the function is integer. so $a = \pm 1$. So $P(x) = \pm x + b$

CTK9CQT
234 posts

Jul 8, 2014, 11:21 am PM #339

I have a good problem about the discussing topic, although it can be well-known:
Let $P(x) \in R[x]$, $degP = n$. Prove that if $P(a) \in \mathbb{Z}\,\forall a \in \{0, 1, 2, ..., n\}$ then $P(x) \in \mathbb{Z}\,\forall x \in \mathbb{Z}$.

AkosS
89 posts

Jul 10, 2014, 3:16 am • 1 PM #340

> *nima-amini wrote:*
> problem 130
> find all polynomial P with real coefficients that satisfies $(P(x))^2 - 1 = P(x^2 + 1)$

Same way as 127 went:

Observation 0.
As $x, x^2 + 1, P(x) \in C^\infty[\mathbb{C}]$, we can take the kth derivative of the equation for each $k \in \mathbb{N}$.

Observation 1.

$$(P(x)^2)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} P^{(i)}(x) P^{(k-i)}(x)$$

Observation 2.

$$P(x^2+1)^{(k)} = \sum_{i=\lceil k/2 \rceil}^{k} c_{ik} P^{(i)}(x^2+1) x^{2i-k}$$

Both are trivial by induction. Also note that $c_{ik} = 2c_{(i-1)(k-1)} + (2i-k)c_{(i-1)k}$, so all c_{ik} are integers, and $c_{kk} = 2^k x^k$

Observation 3.

If this holds for $x \in \mathbb{R}$, then it is true for $x \in \mathbb{C}$ as well.
Since $x^2 + 1 = x$ is true for $\omega = 2^{-1}(1 \pm i\sqrt{3})$, and plugging it into the equation we get:

$$P(\omega)^2 - 1 = P(\omega) \Rightarrow P(\omega) \in \left\{ \frac{1 \pm \sqrt{5}}{2} \right\} =: \phi$$

Note that $\{1, \omega, \phi\}$ are linearly independent over $\mathbb{Q}$ (we choose 1 element from ω and ϕ each, note that $\omega^3 = 1$).

Now to solve the problem:

We will take the derivatives at $x = \omega$, and prove that $P^{(k)}(\omega) = 0$, if $k > 0$.

$$2P(x)P'(x) - 2xP'(1 + x^2) \equiv 0$$

Now put $x = \omega$:

$$0 \equiv 2P(\omega)P'(\omega) - 2\omega P(\omega) = 2(\varphi - \omega)P'(\omega)$$

Because of the independence, the first term is not zero, so $P'(\omega) = 0$. Now by induction, suppose we proved $P^{(i)}(\omega) = 0$ for all $1 \leq i < k$. Now look at the kth derivative of the original equation, using Observation 1. and 2. :

$$0 \equiv \left(P(x)^2 - P(x^2+1) - 1\right)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} P^{(i)}(x)P^{(k-i)}(x) - \sum_{i=\lceil k/2 \rceil}^{k} c_{ik}P^{(i)}(x^2+1)x^{2i-k} =: T(k,x)$$

Now we put $x = \omega$ and use the result about $P^{(i)}(\omega)$:

$$0 = T(k,\omega) = 2P^{(k)}(\omega)P(\omega) - 2^k\omega^k P^{(k)}(\omega) = P^{(k)}(\omega)(2\phi - 2^k\omega^k)$$

The second term is not 0, so $P^{(k)}(\omega) = 0$.

Now suppose $\deg P = n$. Using our result for $k = 1, 2, \ldots, n-1$, we get that ω is a root of multiplicity at least $n-1$ in $P'(x)$. But we have two ω candidates, so we have $2n - 2 \leq n - 1$ roots, so $n \leq 0$, so $P(x) = 2^{-1}(1 \pm \sqrt{5})$

mihirb
1842 posts

Apr 5, 2015, 4:20 am PM #341

Can someone revive this?

sansae
82 posts

Oct 23, 2015, 12:47 pm PM #343

anyone can revive this please...?
It must not be died

utkarshgupta
2272 posts

Nov 7, 2015, 3:51 pm PM #344

This problem is from a contest of course but may help the marathon not die.

Problem 131
Let f and g be two polynomials with integer coefficients such that the leading coefficients of both the polynomials are positive. Suppose $\deg(f)$ is odd and the sets $\{f(a) \mid a \in \mathbb{Z}\}$ and $\{g(a) \mid a \in \mathbb{Z}\}$ are the same. Prove that there exists an integer k such that $g(x) = f(x+k)$.

lebathanh
404 posts

Jul 15, 2016, 9:06 pm PM #346

and it is die

ThE-dArK-lOrD
2714 posts

Oct 24, 2016, 11:26 pm PM #347

Let's continue the marathon

> *utkarshgupta wrote:*
> **Problem 131**
> Let f and g be two polynomials with integer coefficients such that the leading coefficients of both the polynomials are positive. Suppose $\deg(f)$ is odd and the sets $\{f(a) \mid a \in \mathbb{Z}\}$ and $\{g(a) \mid a \in \mathbb{Z}\}$ are the same. Prove that there exists an integer k such that $g(x) = f(x+k)$.

Since degree of f is odd, we get that $\{f(a) \mid a \in \mathbb{Z}\}$ is unbounded below, so degree of g is odd
Since leading coefficient of f and g are positive and degree of f and g are odd , there exist constant C that $f(n+1) > f(n) > f(x), 0$ and $g(n+1) > g(n) > g(x), 0$ for all $x < n$ and $n > C$
So suppose that $f(C)$ is i^{th} smallest positive number in $\{f(a) \mid a \in \mathbb{Z}\}$ and $g(C)$ is j^{th} smallest positive number in $\{g(a) \mid a \in \mathbb{Z}\}$
We get that $(i+n)^{th}$ smallest positive number in $\{f(a) \mid a \in \mathbb{Z}\}$ is $f(C+n)$ and $(j+n)^{th}$ smallest positive number in $\{g(a) \mid a \in \mathbb{Z}\}$ is $g(C+n)$ for all $n \in \mathbb{Z}^+$
So $f(C+k-i) = g(C+k-j)$ for all $k > \max\{i,j\}$, this give us the desired result.

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