# Write Down Formulas!

**Based on a MOP class of the same name by Yang P. Liu**

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## §1 Lecture notes

Sometimes, you should just write things out.

This unit isn't quite like the Grinding unit. In Grinding, there is a lot more set-up: you often have to figure out how you want to set up cases to minimize the amount of calculation you want to do. In this Formulas unit, you will often not really have cases at all. You still have to be careful, but often not because of casework, but because you're manipulating a double or triple sum that's not that intuitive. Often the computation feels *linear*.

Here's a picture in my head to describe the difference. The casework from Grinding is *wide but shallow*: each individual case is easy and self-contained, but there's a lot of them. But here the computations from Formulas is *narrow but deep*: you write down one big formula and then go after it for a page or two. Complex numbers in geometry feels more like the latter, if you are used to it.

Expect plenty of $\sum$ signs to join the party. A lot of these problems would also fit in Sums unit, but often here the sums aren't set up for you.

### §1.1 Walkthroughs

There isn't any particular theory I need to cover, so rather I want to use these walkthroughs to make a rather political statement: Cayley's formula is best proved by formulas.

The theorem statement is:

**Theorem** (Cayley's formula)
The number of labeled trees on $n$ vertices is $n^{n-2}$.

Most proofs of this theorem that you will see in the literature are bijective[1] or quote some theorems from enumerative combinatorics (Kirchhoff's matrix tree theorem for



[1] For example, https://usamo.wordpress.com/2017/09/04/joyals-proof-of-cayleys-tree-formula/ is the cleanest proof I know.

example). For *you*, an olympiad high school student, I don't think either of these are the easiest to come up with.

Instead, here are two problems which I definitely think you can solve yourself, and which imply the result.

**Example 1.1**

Let $G$ be a simple graph with $k$ connected components, which have $a_1, \dots, a_k$ vertices, respectively. Count the number of ways to add $k-1$ edges to $G$ to form a connected graph.

**Walkthrough.** We let $g(a_1, \dots, a_k)$ denote the answer. (Cayley's formula is then the assertion that $g(1, \dots, 1) = k^{k-2}$.)

**(a)** Compute $g(a, b)$.

**(b)** Show that $g(a, b, c) = abc(a+b+c)$.

**(c)** Prove that

$$g(a, b, c, d) = \sum_{\text{sym}} \left[ \frac{1}{6} a^3 bcd + \frac{1}{2} a^2 b^2 cd \right].$$

Factor the resulting expression.

**(d)** Come up with a conjecture for $g(a_1, \dots, a_k)$ (keeping in mind the desired answer $k^{k-2}$ for $g(1, \dots, 1)$).

For the induction, it's actually easier to do the casework if you assume the edges have some order (hence multiplying by a factor of $(k-1)!$). So let $f(a_1, \dots, a_k)$ be the number of ways to add $k-1$ edges, *in order*.

**(e)** Prove considering where to place the first edge gives you the recursion

$$f(a_1, \dots, a_k) = \sum_{1 \le i < j \le k} a_i a_j f(a_i + a_j, \underbrace{a_1, \dots, a_k}_{\text{missing } a_i \text{ and } a_j})$$

**(f)** Now just power through the calculation to complete the induction.

**Example 1.2**

Let $d_1, \dots, d_n$ be positive integers with $\sum_i d_i = 2n - 2$. Find the number of labeled trees on $n$ vertices such that the degree of the $i$th vertex is $d_i$.

**Walkthrough.** I'll even tell you what the answer is this time:

$$\frac{(n-2)!}{(d_1 - 1)! \dots (d_n - 1)!}.$$

Prove it by induction, in the same way as the previous example.

The latter result also implies Cayley's formula by the multinomial theorem.

## §1.2 Commentary

So the reflexes I want you to pick up from this unit are *when* it's possible to solve a problem simply by doing enough calculations (somewhat different from casework). This is more often possible than people suspect. The previous walkthroughs are thus good examples in the sense that you look at them, and once you have the conjectured answer (and maybe even before that), you *feel* like the natural recursion you have *should be* good enough to just solve the problem.

The issue of course is that, well, you can't tell by looking, because the calculation is probably long enough that it won't fit your memory. No worries, *that's what paper is for*!

Some untested philosophy:

- I think roughly you can think of these as problems that would be easy if you had infinite working memory (which is different from infinite computational power). I don't think these calculations are long in the same sense that a complex numbers calculation is long. My opinion is that actually they are "short" in the number of equals signs. They are just hard to think about because most people are not really able to manipulate $n$-variable multi-sums in their head, which is why being able to write is so helpful here.

- You will actually not swap the order of summation as often you as you might expect from the Sums unit.

- Recursion and strengthened induction go well with this unit. It's almost like, you can actually describe what you want completely, it just takes a lot of symbols? For example in the previous problems, we had a real recursion that did what we wanted; it just happens to take a fairly dense notation to write down, but the essence is fairly routine.

  From a linguistic point of view, this is why mathematical notation is so powerful, because it lets you concisely express content that would not be possible in the English language. It is less natural this way, but better than not expressible at all.

## §1.3 A longer example (due to Michael Ren)

**Example 1.3** (USAMO 2017/2)

Let $m_1, m_2, \dots, m_n$ be a collection of $n$ positive integers, not necessarily distinct. For any sequence of integers $A = (a_1, \dots, a_n)$ and any permutation $w = w_1, \dots, w_n$ of $m_1, \dots, m_n$, define an $A$-inversion of $w$ to be a pair of entries $w_i, w_j$ with $i < j$ for which one of the following conditions holds:

- $a_i \ge w_i > w_j$,
- $w_j > a_i \ge w_i$, or
- $w_i > w_j > a_i$.

Show that, for any two sequences of integers $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$, and for any positive integer $k$, the number of permutations of $m_1, \dots, m_n$ having exactly $k$ $A$-inversions is equal to the number of permutations of $m_1, \dots, m_n$ having exactly $k$ $B$-inversions.

**Walkthrough.** It suffices to prove the result for $B = (0, \dots, 0)$ by transitivity; such pairs $(i, j)$ (satisfying $i < j$ and $w_i > w_j$) are *inversions*; see https://en.wikipedia.org/wiki/Inversion_(discrete_mathematics).

There isn't a good closed form for the number of permutations on $\{1, \dots, n\}$ with exactly $k$ inversions, but there is a good generating function for it (which is even better for our purposes, since we only want to show equality!), and we'll use that to do all our computation:

**(a)** Let $n$ be a positive integer. Prove that the coefficient of $q^s$ in the generating function

$$n!_q = 1 \cdot (1+q) \cdot (1+q+q^2) \cdot \dots \cdot (1+q+q^2+\dots+q^{n-1})$$

is equal to the number of permutations on $\{1, \dots, n\}$ with exactly $s$ inversions. (Most likely, you will use induction on $n$. This is called the *$q$-factorial*.)

Unfortunately, in the present problem the given numbers we are permuting need not be distinct and this will be a significant headache. We take our given multiset $M$ of $n$ positive integers, we suppose the distinct numbers are $\theta_1 < \theta_2 < \dots < \theta_m$. (In what follows we count permutations on $M$ with multiplicity: so $M = \{1, 1, 2\}$ still has $3! = 6$ permutations total.) We let $e_i$ be the number of times $\theta_i$ appears. Therefore the multiplicities $e_i$ should have sums

$$e_1 + \dots + e_m = n$$

and $m$ denotes the number of distinct elements.

Finally, we let

$$F(e_1, \dots, e_m) = \sum_{\text{permutations} \sigma} q^{\text{number inversions of } \sigma}$$

be the associated generating function for the number of inversions. For example, the first claim we proved says that $F(1, \dots, 1) = n!_q$.

**(b)** Optional warm-up: compute $F(2, \underbrace{1, \dots, 1}_{n-2})$. Solving this will make the next part more natural.

**(c)** Compute $F(e_1, \dots, e_m)$. This is an overcounting argument: try to show you have an overcount of $\frac{e_i!_q}{e_i!}$ for each $i$, by imagining what happens if you arbitrarily force some order on those $e_i$ equal elements (in one of $e_i!$ ways).

**(d)** Verify that

$$\frac{e_i \cdot F(e_1, \dots, e_i - 1, \dots, e_m)}{F(e_1, \dots, e_m)} = \frac{1-q^{e_i}}{1-q^n}$$

to check your work; we'll use this later in our calculation.

So, we now have the generating function that counts the answer when $B = (0, \dots, 0)$. The problem tells us to prove the same generating function works for any random sequence $A$. The proof is again an ugly induction on $n$.

For the inductive step, fix $A$, and assume the first element satisfies $\theta_k \le a_1 < \theta_{k+1}$ (so $0 \le k \le m$; we for convenience set $\theta_0 = -\infty$ and $\theta_m = +\infty$). We count the permutations based on what the first element $\theta_i$ of the permutation is.

**(e)** Show that if $1 \le i \le k$ we get a contribution of

$$q^{e_1+\dots+e_{i-1}+\dots+e_{k+1}+\dots+e_m} \cdot e_i \cdot F(e_1, \dots, e_i - 1, \dots, e_m)$$

**(f)** Show that if $k+1 \le i \le n$ we get a contribution of

$$q^{e_k+\dots+e_{i-1}} \cdot e_i \cdot F(e_1, \dots, e_i - 1, \dots, e_m).$$

So, we have two massive sums over $i$, and we want to show the sum equals $F(e_1, \dots, e_m)$.

**(g)** Divide out by $F(e_1, \dots, e_m)$ to get some identity, and verify it telescopes.

## §2 Practice problems

*Instructions*: Solve [35♣]. If you have time, solve [42♣]. Problems with red weights are mandatory.

> My brethren, hear me! For there is little time left. All that remains of our race, our civilization, are those that stand beside you now. Trust in each other. Strike as one will! Let our last stand burn a memory so bright, we will be remembered forever! En taro Tassadar!
>
> Artanis in *Wings of Liberty* campaign

[2♣] **Problem 1** (USAMO 1977/3)**.** If $a$ and $b$ are two distinct roots of $x^4 + x^3 - 1 = 0$, prove that $ab$ is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

[2♣] **Problem 2** (IMO 2013/1)**.** Let $k$ and $n$ be positive integers. Prove that there exist positive integers $m_1, \dots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right) \dots \left(1 + \frac{1}{m_k}\right).$$

[2♣] **Problem 3** (USAMO 2010/5)**.** Let $q = \frac{3p-5}{2}$ where $p$ is an odd prime, and let

$$S_q = \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 6 \cdot 7} + \dots + \frac{1}{q(q+1)(q+2)}.$$

Prove that if $\frac{1}{p} - 2S_q = \frac{m}{n}$ for integers $m$ and $n$, then $m - n$ is divisible by $p$.

[3♣] **Problem 4** (Putnam 2018 A6)**.** Four points are given in the plane, with no three collinear, such that the squares of the $\binom{4}{2} = 6$ pairwise distances are all rational. Show that the ratio of the areas between any two of the $\binom{4}{3} = 4$ triangles determined by these points is also rational.

[3♣] **Problem 5** (China TST 2016/2/4)**.** For every positive integer $m = 2^k t$, where $k \ge 0$ and $t$ is odd, we let $f(m) = t^{1-k}$. Prove that for any positive integers $a \le m$ with $a$ odd, the number $f(1)f(2)\dots f(m)$ is an integer divisible by $a$.

[3♣] **Problem 6** (Shortlist 2005 N3)**.** Let $a$, $b$, $c$, $d$, $e$, $f$ be positive integers. Assume that $S = a+b+c+d+e+f$ divides both $abc + def$ and $ab + bc + ca - de - ef - fd$. Prove that $S$ is composite.

[3♣] **Problem 7** (PRIMES 2017 M2)**.** Let $n \ge 1$ be a fixed positive integer. An $n \times n$ matrix $M$ is generated as follows: for each $1 \le i, j \le n$, we randomly write either $i$ or $j$ in the $(i, j)$th entry, each with probability $\frac{1}{2}$. Find the expected value of $\det M$.

[3♣] **Problem 8** (Math Prize for Girls 2018)**.** A *smooth number* is a positive integer of the form $2^m 3^n$ for $m, n \in \mathbb{Z}_{\ge 0}$. Consider the set $S$ of triples $(a, b, c)$ of smooth numbers such that $\gcd(a, b)$, $\gcd(b, c)$, $\gcd(c, a)$ are pairwise distinct. Evaluate

$$\sum_{(a,b,c) \in S} \frac{1}{abc}.$$

[3♣] **Problem 9** (EGMO 2015/5)**.** Let $m, n$ be positive integers with $m > 1$. Anastasia partitions the integers $1, 2, \dots, 2m$ into $m$ pairs. Boris then chooses one integer from each pair and finds the sum of these chosen integers. Prove that Anastasia can select the pairs so that Boris cannot make his sum equal to $n$.

[3♣] **Problem 10** (Shortlist 2016 A5)**.** Consider fractions $\frac{a}{b}$ where $a$ and $b$ are positive integers.

(a) Prove that for every positive integer $n$, there exists such a fraction $\frac{a}{b}$ such that $\sqrt{n} \le \frac{a}{b} \le \sqrt{n+1}$ and $b \le \sqrt{n}+1$.

(b) Show that there are infinitely many positive integers $n$ such that no such fraction $\frac{a}{b}$ satisfies $\sqrt{n} \le \frac{a}{b} \le \sqrt{n+1}$ and $b \le \sqrt{n}$.

[3♣] **Problem 11** (Putnam 2015 A4)**.** For a real number $x$, let $S_x$ denote the set of positive integers $n$ such that $\lfloor nx \rfloor$ is even. Find the value of

$$\inf_{x \in [0,1)} \left( \sum_{n \in S_x} \frac{1}{2^n} \right).$$

[5♣] **Problem 12** (TSTST 2018/6)**.** Let $S = \{1, \dots, 100\}$, and for every positive integer $n$ define

$$T_n = \{(a_1, \dots, a_n) \in S^n \mid a_1 + \dots + a_n \equiv 0 \pmod{100}\}.$$

Determine which $n$ have the following property: if we color any 75 elements of $S$ red, then at least half of the $n$-tuples in $T_n$ have an even number of coordinates with red elements.

[3♣] **Problem 13** (Shortlist 2018 N2)**.** Let $n > 1$ be a positive integer. Each cell of an $n \times n$ table contains an integer which is $1 \pmod n$. It is given that the sum of the numbers in any row, as well as the sum of numbers in any column, is congruent to $n \pmod{n^2}$. Denote by $R_i$ the product of the numbers in the $i^{\text{th}}$ row and by $C_j$ the product of the numbers in the $j^{\text{th}}$ column. Prove that

$$R_1 + \dots + R_n \equiv C_1 + \dots + C_n \pmod{n^4}.$$

[5♣] **Required Problem 14** (Shortlist 2014 N6)**.** Let $a_1 < a_2 < \dots < a_n$ be pairwise coprime positive integers with $a_1$ being prime and $a_1 \ge n+2$. On the segment $I = [0, a_1 a_2 \cdots a_n]$ of the real line, mark all integers that are divisible by at least one of the numbers $a_1, \dots, a_n$. These points split $I$ into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by $a_1$.

[9♣] **Problem 15** (USA TST 2019/2)**.** Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of integers considered modulo $n$ (hence $\mathbb{Z}/n\mathbb{Z}$ has $n$ elements). Find all positive integers $n$ for which there exists a bijective function $g \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, such that the 101 functions

$$g(x), \quad g(x)+x, \quad g(x)+2x, \quad \dots, \quad g(x)+100x$$

are all bijections on $\mathbb{Z}/n\mathbb{Z}$.

[9♣] **Problem 16** (USAMO 2018/3)**.** Let $n \ge 2$ be an integer, and let $\{a_1, \dots, a_m\}$ denote the $m = \varphi(n)$ integers less than $n$ and relatively prime to $n$. Assume that every prime divisor of $m$ also divides $n$. Prove that $m$ divides $a_1^k + \dots + a_m^k$ for every positive integer $k$.

[9♣] **Problem 17** (Putnam 2017 B6)**.** Find the number of ordered 64-tuples $(x_0, \dots, x_{63})$ of distinct elements of $\{1, \dots, 2017\}$ which satisfy

$$x_0 + x_1 + 2x_2 + 3x_3 + \dots + 63x_{63} \equiv 0 \pmod{2017}.$$

[1♣] **Mini Survey.** At the end of your submission, answer the following questions.

(a) About how many hours did the problem set take?

(b) Name any problems that stood out (e.g. especially nice, instructive, boring, or unusually easy/hard for its placement).

Any other thoughts are welcome too. Examples: suggestions for new problems to add, things I could explain better in the notes, overall difficulty or usefulness of the unit.

# §3 Solutions to the walkthroughs

## §3.1 Solution 1.1

The answer is

$$a_1 \dots a_k (a_1 + \dots + a_k)^{k-2}$$

which generalizes Cayley's formula!

We will show that

$$f(a_1, \dots, a_k) = k!(a_1 \dots a_k)(a_1 + \dots + a_k)^{k-2}$$

counts the number of ways to pick $k-1$ edges, *in order*. The proof is by induction on $k$, with $k = 1$ being clear. If we add an edge between the first and second connected components, there are $a_1 a_2$ ways to do so, and the number of ways to finish is $f(a_1 + a_2, a_3 \dots, a_k)$. So

$$\begin{aligned} f(a_1, \dots, a_k) &= \sum_{1 \le i < j \le k} a_i a_j f(a_i + a_j, \underbrace{a_1, \dots, a_k}_{\text{missing } a_i \text{ and } a_j}) \\ &= (k-1)! \sum_{1 \le i < j \le k} (a_i + a_j)(a_1 \dots a_k)(a_1 + \dots + a_k)^{k-2} \\ &= (k-1)!(a_1 \dots a_k)(a_1 + \dots + a_k)^{k-2} \sum_{1 \le i < j \le k} (a_i + a_j) \\ &= k!(a_1 \dots a_k)(a_1 + \dots + a_k)^{k-1}. \end{aligned}$$

## §3.2 Solution 1.2

The answer is

$$\frac{(n-2)!}{(d_1 - 1)! \dots (d_n - 1)!}.$$

This can be proven by induction, with the base case $n = 2$ being trivial.

Now assume $n \ge 3$. Since $d_i$ are positive integers with average $2 - 2/n < 2$, we may WLOG assume $d_n = 1$ (a leaf of the tree). So by deleting the leaf, the trees on $\{1, \dots, n\}$ with a leaf at $n$ are in bijection with trees on $\{1, \dots, n-1\}$ where one of the vertices $k$ has its degree decreased by one.

Hence, by induction hypothesis, the number of trees is

$$\begin{aligned} &\sum_{k=1}^{n-1} \frac{(n-3)!}{(d_1 - 1)! \dots (d_{k-1} - 1)!(d_k - 2)!(d_{k+1} - 1)! \dots (d_{n-1} - 1)!} \\ &= \sum_{k=1}^{n-1} \frac{(n-3)!}{(d_1 - 1)! \dots (d_{n-1} - 1)!} \cdot (d_k - 1) \\ &= \frac{(n-3)!}{(d_1 - 1)! \dots (d_{n-1} - 1)!} \sum_{k=1}^{n-1} (d_k - 1) \\ &= \frac{(n-3)!}{(d_1 - 1)! \dots (d_{n-1} - 1)!} [(2n - 2) - n] \\ &= \frac{(n-2)!}{(d_1 - 1)! \dots (d_{n-1} - 1)!} = \frac{(n-2)!}{(d_1 - 1)! \dots (d_n - 1)!}. \end{aligned}$$

Here we adopt the convention that $(-1)! = \infty$, corresponding to the cases where $d_k = 1$ above.

## §3.3 Solution 1.3, USAMO 2017/2

The following solution was posted by Michael Ren, and I think it is the most natural one (since it captures all the combinatorial ideas using a $q$-generating function that is easier to think about, and thus makes the problem essentially a long computation).

Denote by $M$ our multiset of $n$ positive integers. Define an *inversion* of a permutation to be pair $i < j$ with $w_i < w_j$ (which is a $(0, \dots, 0)$-inversion in the problem statement); this is the usual definition (see https://en.wikipedia.org/wiki/Inversion_(discrete_mathematics)). So we want to show the number of $A$-inversions is equal to the number of usual inversions. In what follows we count permutations on $M$ with multiplicity: so $M = \{1, 1, 2\}$ still has $3! = 6$ permutations.

We are going to do what is essentially recursion, but using generating functions in a variable $q$ to do our book-keeping. (Motivation: there's no good closed form for the number of inversions, but there's a great generating function known — which is even better for us, since we're only trying to show two numbers are equal!) First, we prove two claims.

**Claim** — For any positive integer $n$, the generating function for the number of permutations of $(1, 2, \dots, n)$ with exactly $k$ inversions is

$$n!_q \overset{\text{def}}{=} 1 \cdot (1+q) \cdot (1+q+q^2) \cdot \ldots (1+q+\dots+q^{n-1}).$$

Here we mean that the coefficient of $q^s$ above gives the number of permutations with exactly $s$ inversions.

*Proof.* This is an induction on $n$, with $n = 1$ being trivial. Suppose we choose the first element to be $i$, with $1 \le i \le n$. Then there will always be exactly $i - 1$ inversions using the first element, so this contributes $q^i \cdot (n-1)!_q$. Summing $1 \le i \le n$ gives the result. □

Unfortunately, the main difficulty of the problem is that there are repeated elements, which makes our notation much more horrific.

Let us define the following. We take our given multiset $M$ of $n$ positive integers, we suppose the distinct numbers are $\theta_1 < \theta_2 < \dots < \theta_m$. We let $e_i$ be the number of times $\theta_i$ appears. Therefore the multiplicities $e_i$ should have sums

$$e_1 + \dots + e_m = n$$

and $m$ denotes the number of distinct elements. Finally, we let

$$F(e_1, \dots, e_m) = \sum_{\text{permutations } \sigma} q^{\text{number inversions of } \sigma}$$

be the associated generating function for the number of inversions. For example, the first claim we proved says that $F(1, \dots, 1) = n!_q$.

**Claim** — We have the explicit formula

$$F(e_1, \dots, e_m) = n!_q \cdot \prod_{i=1}^{m} \frac{e_i!}{e_i!_q}.$$

*Proof.* First suppose we perturb all the elements slightly, so that they are no longer equal. Then the generating function would just be $n!_q$.

Then, we undo the perturbations for each group, one at a time, and claim that we get the above $e_i!_q$ factor each time. Indeed, put the permutations into classes of $e_1!$ each where permutations in the same classes differ only in the order of the perturbed $\theta_1$'s (with the other $n-e_1$ elements being fixed). Then there is a factor of $e_1!_q$ from each class, owing to the slightly perturbed inversions we added within each class. So we remove that factor and add $e_1! \cdot q^0$ instead. This accounts for the first term of the product.

Repeating this now with each term of the product implies the claim. $\square$

Thus we have the formula for the number of inversions in general. We wish to show this also equals the generating function the number of $A$-inversions, for any fixed choice of $A$. This will be an induction by $n$, with the base case being immediate.

For the inductive step, fix $A$, and assume the first element satisfies $\theta_k \le a_1 < \theta_{k+1}$ (so $0 \le k \le m$; we for convenience set $\theta_0 = -\infty$ and $\theta_m = +\infty$). We count the permutations based on what the first element $\theta_i$ of the permutation is. Then:

- Consider permutations starting with $\theta_i \in \{\theta_1, \dots, \theta_k\}$. Then the number of inversions which will use this first term is $(e_1 + \dots + e_{i-1}) + (e_{k+1} + \dots + e_m)$. Also, there are $e_i$ ways to pick which $\theta_i$ gets used as the first term. So we get a contribution of
$$q^{e_1+\dots+e_{i-1}+\dots+e_{k+1}+\dots+e_m} \cdot e_i \cdot F(e_1, \dots, e_i - 1, \dots, e_m)$$
in this case (with inductive hypothesis to get the last $F$-term).

- Now suppose $\theta_i \in \{\theta_{k+1}, \dots, \theta_m\}$. Then the number of inversions which will use this first term is $e_{k+1} + \dots + e_{i-1}$. Thus by a similar argument the contribution is
$$q^{e_k+\dots+e_{i-1}} \cdot e_i \cdot F(e_1, \dots, e_i - 1, \dots, e_m).$$

Therefore, to complete the problem it suffices to prove

$$\begin{aligned}
&\sum_{i=1}^{k} q^{e_1+\dots+e_{i-1}+e_{k+1}+\dots+e_m} \cdot e_i \cdot F(e_1, \dots, e_i - 1, \dots, e_m) \\
&+ \sum_{i=k+1}^{m} q^{e_{k+1}+\dots+e_{i-1}} \cdot e_i \cdot F(e_1, \dots, e_i - 1, \dots, e_m) \\
&= F(e_1, \dots, e_m).
\end{aligned}$$

Now, we see that

$$\frac{e_i \cdot F(e_1, \dots, e_i - 1, \dots, e_m)}{F(e_1, \dots, e_m)} = \frac{1 + \dots + q^{e_i - 1}}{1 + q + \dots + q^{n-1}} = \frac{1 - q^{e_i}}{1 - q^n}$$

so it's equivalent to show

$$1 - q^n = q^{e_{k+1}+\dots+e_m} \sum_{i=1}^{k} q^{e_1+\dots+e_{i-1}}(1 - q^{e_i}) + \sum_{i=k+1}^{m} q^{e_{k+1}+\dots+e_{i-1}}(1 - q^{e_i})$$

which is clear, since the left summand telescopes to $q^{e_{k+1}+\dots+e_m} - q^n$ and the right summand telescopes to $1 - q^{e_{k+1}+\dots+e_m}$.

**Remark.** Technically, we could have skipped straight to the induction, without proving the first two claims. However I think the solution reads more naturally this way.