High School Olympiads

Tangent line in a circumscriptible quadrilateral

Reply

Source: me

pohoatza

#1 Apr 20, 2009, 6:48 pm

Let $ABCD$ be a circumscriptible quadrilateral and let γ be an arbitrary line tangent to its incircle. Let A', B', C', D' be the orthogonal projections of A, B, C, D on γ, respectively. Prove that

$$\frac{AA' \cdot CC'}{BB' \cdot DD'} = \frac{AO \cdot CO}{BO \cdot DO}.$$

Luis González

#2 Apr 21, 2009, 1:10 am • 1

Lemma: $ABCD$ is a cyclic quadrilateral and P is a point lying on its circumcircle (O). Let X, Y, Z, V the orthogonal projections of P onto sidelines AD, AB, BC, CD. Then $PX \cdot PZ = PY \cdot PV$.

$$\frac{PX}{PY} = \frac{\sin \widehat{PAX}}{\sin \widehat{PAB}} \ , \ \frac{PZ}{PV} = \frac{\sin \widehat{PCB}}{\sin \widehat{PCD}}$$

Since $\angle PAB = \angle PCD$ and $\angle PAX = \angle PCD \pmod \pi$, it follows that $PX \cdot PZ = PY \cdot PV$ and the proof of the lemma is completed. $\square$

Let M, N, L, Q be the tangency points of the incircle (O) with DA, AB, CB, CD. Let X, Y, Z, V be the projections of P onto MN, NL, LQ, QM, where $P \equiv \gamma \cap (O)$. γ is the polar of P WRT (O) and MN, NL, LQ, QM are the polars of vertices A, B, C, D, respectively. By Salmon's theorem, we deduce that

$$\frac{AA'}{AO} = \frac{PX}{R} \ , \ \frac{CC'}{CO} = \frac{PZ}{R}$$

$$\frac{BB'}{BO} = \frac{PY}{R} \ , \ \frac{DD'}{DO} = \frac{PV}{R}$$

But according to the previous lemma, we have $PX \cdot PZ = PY \cdot PV$

$$\Longrightarrow \frac{AA' \cdot CC'}{BB' \cdot DD'} = \frac{AO \cdot CO}{BO \cdot DO}$$

pohoatza

#3 Apr 21, 2009, 7:55 pm

Very nice Luis, thanks. My proof was way uglier (and by this I mean really computational).

Luis González

#4 Apr 21, 2009, 8:35 pm

With the same conditions, but now $ABCD$ is bicentric. Show that

$$\frac{AA' \cdot CC'}{BB' \cdot DD'} = \frac{[ABCD]}{BO \cdot DO} - 1 \ , \ \frac{BB' \cdot DD'}{AA' \cdot CC'} = \frac{[ABCD]}{AO \cdot OC} - 1$$

Quick Reply

High School Olympiads

Inscribing Triangles

Reply

Source: Concurrent?

hamjam

#1 Apr 21, 2009, 8:58 am

Take a triangle and inscribe an ellipse in it. Join each vertex with the point of tangency (of the ellipse with the triangle) of the opposite side. Do the three lines meet at one point? Always?

Luis González

#2 Apr 21, 2009, 10:17 am

Yes, this is a degenerate case of the Brianchon theorem: A hexagon $ABCDEF$ is circumscribed around a conic section, if and only if the lines AD, BE and CF concur. Alternatively, for this particular case, project the conic into a circle by central projection. The subject lines obviously concur at the Gergonne point of the projected triangle, so the three primitive lines also concur.

Quick Reply

High School Olympiads

easy,old and beautiful

Reply

Source: imo 81

v235711

#1 Apr 21, 2009, 9:36 am

3 congruent circles have a common point P and are in the interior of triangle ABC.Each circle is tangent to 2 sides of the triangle.Prove that the incenter , the circuncenter of ABC and P are collinear.

Luis González

#2 Apr 21, 2009, 9:57 am

If O_1, O_2, O_3 denote the centers of the said circles, then it's clear that $\triangle O_1O_2O_3$ is homothetic to $\triangle ABC$ through a homothety with center I. But P is the circumcenter of $\triangle O_1O_2O_3$ since P is equidistant from O_1, O_2, O_3. Hence, circumcenters of $\triangle ABC$ and $\triangle O_1O_2O_3$ are collinear with the homothetic center I. Further, P is X_{55} of $\triangle ABC$.

Quick Reply

Spain

Trisección

Reply

M4RI0
#1 Jun 1, 2007, 8:33 pm

Indicar los pasos para trisecar usando regla y compás un ángulo de 27°

fanicha
#2 Jun 2, 2007, 1:51 am

Para hacer eso no hay que inscribirle un ángulo recto primero??

R.G.A.M.
#3 Jun 3, 2007, 5:35 am

Trácese un ángulo de 90° en cualquier parte.
Luego transcríbase tres veces el ángulo de 27° sobre el anterior, de forma que estos tres ángulos sean adyacentes, y uno tenga un lado común con el ángulo de 90°.
Sobrará una sección del ángulo, que medirá $90^\circ - 3 \times 27^\circ = 9^\circ$.
Es fácil, por último, transcribir este último ángulo obtenido en el dado por el enunciado.

M4RI0
#4 Jun 5, 2007, 9:08 am

Muy bien, si saben de otros ángulos que se puedan trisecar porfavor posteen.

R.G.A.M.
#5 Jun 6, 2007, 3:04 am

Aplicando el método análogo (casi idéntico, en realidad), bien puede trisecarse ángulos de $9^\circ; 18^\circ; 36^\circ; 45^\circ; 54^\circ; 63^\circ; 72^\circ; 81^\circ$. Ha de haber más ángulos, pero la lista es bastante extensa (en efecto, no la terminé, pues corresponden a ángulos múltiplos de 9).

Jutaro
#6 Jun 6, 2007, 8:13 am

ahora q me acuerdo hay un problema relacionado con todo esto, de la usamo 1981 si no me equivoco, que dice asi:

Si n no es múltiplo de 3, probar que el ángulo $\frac{\pi}{n}$ puede ser trisecado con regla y compás.

espero sea de su interes :)

S. E. Puelma Moya
#7 Jun 15, 2008, 6:44 am

> Jutaro wrote:
> ahora q me acuerdo hay un problema relacionado con todo esto, de la usamo 1981 si no me equivoco, que dice asi:
>
> Si n no es múltiplo de 3, probar que el ángulo $\frac{\pi}{n}$ puede ser trisecado con regla y compás.

n

espero sea de su interes :)

No todos los ángulos con medida $\frac{\pi}{n}$ pueden ser construídos con regla y compás (por ejemplo, $\frac{\pi}{7}$), entonces es necesario suponer que el ángulo con medida $\frac{\pi}{n}$ ya está construído.

Es dado un ángulo $\angle AOB$, cuya medida es $\frac{\pi}{n}$. Construya un ángulo $\angle A'O'B'$, cuya medida es $\frac{\pi}{3}$. Existen números enteros positivos: a,b, tales que an-3b=1, determinados de la siguiente manera:

- Si n=3k+1, entonces a=1, b=k
- Si n=3k+2, entonces a=2, b=2k+1

Son construídos ángulos con medidas $\frac{a\pi}{3}$ y $\frac{b\pi}{n}$. Finalmente, es construído un ángulo cuya medida sea igual a la diferencia: $\frac{a\pi}{3} - \frac{b\pi}{n} = \frac{an - 3b}{3n} = \frac{1}{3n}$. Copiando dos veces esta medida en el ángulo original, éste será trisecado.

Luis González

#8 Apr 18, 2009, 8:58 am

En general trisecar un ángulo no es posible con regla y compas, salvo ciertos ángulos particulares. La operación regla-compas equivale analiticamente a la resolución de ecuaciones lineales, cuadraticas y hasta algunas bicuadraticas. La trisección del ángulo en general involucra la resolución de una ecuacion cubica irreducible, por lo tanto no es en general posible. Una tecnica no-convencional para la trisección gráfica del ángulo es la llamada "operación" deslizar la regla y el uso de ciertas curvas algebraicas como la trisectriz de Maclaurin, la trisectriz de Newton, la concoide de Nicomedes, entre otras.

Spain

3er. problema de Sangaku

Reply

Hetidemek
#1 Jan 2, 2007, 3:50 am

Hola amigos, mi 1er. post.

Les dejo el 3er. problema de Sangaku que en verdad no es difícil.

Image not found

Se tienen 3 circunferencias de radios a, b y c como se muestra en la figura, todas tangentes entre sí y tangentes a una línea horizontal.

Pruebe que $\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}=\frac{1}{\sqrt{c}}$

Saludos.

Kunihiko_Chikaya
#2 Jan 2, 2007, 5:46 am

Denote $L(R,\ r)$ by the length of the common external tangent line for two circles with radii $R,\ r$, by Pythagorian theorem we have that $(R+r)^2-(R-r)^2=\{L(R,\ r)\}^2\Longleftrightarrow L(R,\ r)=2\sqrt{Rr}.$
Since $L(c,\ a)+L(b,\ c)=L(a,\ b)\Longleftrightarrow 2\sqrt{ca}+2\sqrt{bc}=2\sqrt{ab}$
$\Longleftrightarrow \sqrt{c}(\sqrt{a}+\sqrt{b})=\sqrt{ab}$, yielding $\frac{1}{\sqrt{c}}=\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}$. Q.E.D.

Hetidemek
#3 Jan 2, 2007, 6:27 am

Excelente.

Saludos.

Hetidemek
#4 Jan 2, 2007, 8:08 am

Expondré una demostración en español para que así sea todo más expedito.

Sean A, B y C los puntos de contacto (de izquierda a derecha) de las circunferencias con la respectiva línea tangente.

De acuerdo con la propiedad de la tangente común a dos circunferencias tangentes exteriormente, se tiene que:

\[\begin{equation*}\begin{aligned}\overline{AB}&= 2\sqrt{a \cdot c}\hfill \\ \overline{BC}&= 2\sqrt{b \cdot c}\hfill \\ \overline{AC}&= 2\sqrt{a \cdot b}\hfill \\ \end{aligned}\end{equation*}\]

Pero $\overline{AC}=\overline{AB}+\overline{BC}$ (suma de segmentos), y por tanto:

\[\begin{equation*}\begin{aligned}2\sqrt{a \cdot b}&= 2\sqrt{a \cdot c}+2\sqrt{b \cdot c}\hfill \\ \sqrt{ab}&= \sqrt{ac}+\sqrt{bc}\hfill \\ \frac{{\sqrt{ab}}}{{\sqrt{abc}}}&= \frac{{\sqrt{ac}}}{{\sqrt{abc}}}+\frac{{\sqrt{bc}}}{{\sqrt{abc}}}\hfill \\ \frac{1}{{\sqrt c }}&= \frac{1} {{\sqrt b }}+\frac{1}{{\sqrt a }}\hfill \\ \end{aligned}\end{equation*}\]

{{\sqrt b }}+\frac{1}{{\sqrt a }}\hfill \\ \end{aligned}\end{equation*}\]

Luis González

#5 Apr 18, 2009, 8:46 am

Este es el conocido Primer teorema de Mikami y Kobayashi. La formula extendida teniendo en cuenta la tangencia externa del círculo con radio c es la siguiente:

$$\left|\frac{1}{\sqrt{r_b}} \pm \frac{1}{\sqrt{r_a}}\right| = \frac{1}{\sqrt{r_c}}$$

Quick Reply

High School Olympiads

Bicentric quadrilateral 5

Reply

juancarlos

#1 Aug 30, 2006, 7:56 am

Let $ABCD$ be a bicentric quadrilateral with incircle (I). If K,M,L,N are the touch points of (I) with AB,BC,CD,DA respectively and from I draw the altitude IQ to BD. Prove that: $[ABD].[QML]=[BCD].[KQN]$

Luis González

#2 Apr 18, 2009, 5:45 am

Let (O,R) be the circumcircle of $ABCD$. $\triangle KQN$ is pedal triangle of I WRT $\triangle ABD$ and $\triangle QML$ is pedal triangle of I WRT $\triangle BDC$. By Euler's theorem we have:

$$\frac{|\triangle KQN|}{|\triangle ABD|}=\frac{p(I,(O))}{4R^2} \ \ , \ \ \frac{|\triangle QML|}{|\triangle BCD|}=\frac{p(I,(O))}{4R^2}$$

$$\Longrightarrow \ |\triangle ABD|\cdot|\triangle QML|=|\triangle BCD|\cdot|\triangle KQN|$$

Quick Reply

High School Olympiads

An IOH smart and interesting problem

Reply

manlio
#1 Jan 8, 2004, 4:40 am

Let I,O,H denote respectively the incenter,the circumcenter and the orthocenter of a triangle with sides a,b,c and the inradius r. Prove that the area A of the triangle IOH is

A=|(a-b)(b-c)(c-a)|/8r

amfulger
#2 Jan 8, 2004, 10:52 am

I'll use vectors to solve this problem.
Let R be the radius of the circumcircle.
In the following, MN denotes the vector with its origin in M and endpoint in N, not the segment joining M and N.
It is known that OH=OA+OB+OC and OI=(aOA+bOB+cOC)/(a+b+c)=(aOA+bOB+cOC)/(2p).
I'll denote by MN*PQ the vector product (not the scalar product) of vectors MN and PQ.
It is known that is we have a triangle ABC then 2*area[ABC]=|AB*AC|, where |MN| is the length of the segment supporting MN.

So 2*area[IOH]=|OH*OI|=(1/2p)*|(OA+OB+OC)*(aOA+bOB+cOC)|=(1/(2p))|OA*OB*(b-a)+OB*OC*(c-b)+OC*OA*(a-c)| (1).

OA, OB, OC are coplanar so OA*OB, OB*OC, OC*OA are all vectors perpendicular to the plane of OABC, so they are colinear. Because I make all the vector products in the same sense (clockwise or counterclockwise - I prefer to draw the triangle counterclockwise), all the vector products have the same sense (they are on the same side of the plane ABC). So there is a versor (unit vector- |u|=1) u such that OA*OB=|OA*OB|*u, OB*OC=|OB*OC|*u, OC*OA=|OC*OA|*u, so (1) comes to:
2*area=(1/(2p))*|R ²sin(2C)*u+...|=(R ²/(2p))|(b-a)sin(2C)+(c-b)sin(2A)+(a-c)sin(2B)|=(R ²/(2p)) *|(2(b-a)sinCcosC+...|=(R/(2p))*|(b-a)ccosC+(c-b)acosA+(a-c)bcosB|. (I used the Sine Law for the last)

Let T=(b-a)cosC+(c-b)cosA+(a-c)cosB
We have T=(b-a)c(a ² +b ² -c ²)/(2ab)+...=(1/(2abc))*[(b-a)cc(aa+bb-cc)+(c-b)aa(bb+cc-aa)+(a-c)bb(aa+cc-bb)].
So 2*area=(R/(4abcp))|(b-a)cc(aa+bb-cc)+(c-b)aa(bb+cc-aa)+(a-c)bb(aa+cc-bb)|**(2)**.

Let S=(b-a)cc(aa+bb-cc)+(c-b)aa(bb+cc-aa)+(a-c)bb(aa+cc-bb).
We have a-c=-(b-a)-(c-b), so
S=(b-a)(aacc-cccc-aabb+bbbb)+(c-b)(aacc-bbcc-aaaa+bbbb)=(b-a)(bb-cc)(bb+cc-aa)+(c-b)(bb-aa)(bb+aa-cc)=(b-a)(b-c)[(b+c)(bb+cc-aa)-(b+a)(bb+aa-cc)]=(b-a)(b-c)[(aa+bb+cc)(c-a)-2aa(b+c)+2cc(a+b)|=(b-a)(b-c)[(aa+bb+cc)(c-a)-2ac(a-c)-2b(a-c)(a+c)]=(b-a)(b-c)(c-a)(aa+bb+cc+2ab+2ac+2bc)=(b-a)(b-c)(c-a)(a+b+c) ²=4p ²(b-a)(b-c)(c-a).

So back in (2) we have 2*area=(Rp/abc)|(a-b)(b-c)(c-a)|. we have to prove 2*area=2*|(a-b)(b-c)(c-a)|/(8r).
I suffices to show (I say suffices and not equivalent because we have problems if the triangle is isosceles) that Rp/(abc)=1/(4r) <=> pr=abc/(4R). The LSH and thr RHS are equal because they both equal area[ABC].

So area[IOH]=|(a-b)(b-c)(c-a)|/(8r).

manlio
#3 Jan 8, 2004, 11:53 pm

Your proof is correct and very nice.

I would like to know if it is possible to prove this result using geometric transformations.

amfulger

#4 Jan 13, 2004, 1:25 am

Let w be the midpoint of OH. w is the center of the nine points circle. By Feurbach, we get Iw=R/2-r.
Formulas for OH and OI are well known.
From these a formula for IH can de found and so we know the lenghts of all the sides of OIH and we can found the area from Peron's formula.

The proof for Feurbach requires inversion. The fact that w is the center of the 9 points center can be proved by similarity (I don't know if homothety is a word). The formula for OI can be found by inversion, so there are lots of geometric transformations in the computations of the sidelengths for OIH, but I think that the computations for the area are uglier than mine.

OH ² =9R ² -aa-bb-cc.
OI ² =R ² -2rR.
Iw=R/2-r and is the median in OIH. From this we will get
HI ² =4RR+2rr-(1/2)(aa+bb+cc).

Now lets look at Perron's formula for the area of a triangle whose sidelengths are a,b,c.
It is 1/4 \sqrt ((a+b+c)(-a+b+c)(a-b+c)(a+b-c))=1/4 \sqrt (((b+c) ²-a ²)(a ² -(b-c) ²) =...=1/4 \sqrt (2aabb+2aacc+2bbcc-aaaa-bbbb-cccc), so to compute the area it sufices to know the squares of the sidelengths.

I don't want to finish the computations because my previous solution had plenty of it.

Luis González

#5 Apr 18, 2009, 12:17 am

Let us use areal coordinates (normalized barycentric coordinates) with respect to $\triangle ABC$. Coordinates of the centroid, orthocenter and incenter are given by:

$$G\left(\frac{1}{3}:\frac{1}{3}:\frac{1}{3}\right)\ ,\ H(\cot B\cot C:\cot A\cot C:\cot A\cot B)\ ,\ I\left(\frac{a}{2p}:\frac{b}{2p}:\frac{c}{2p}\right)$$

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \cot B\cot C & \cot A\cot C & \cot A\cot B \\ \frac{a}{2p} & \frac{b}{2p} & \frac{c}{2p} \end{pmatrix} = \frac{[\triangle IGH]}{[\triangle ABC]} \Longrightarrow$$

$$6p\cdot\frac{[\triangle IGH]}{[\triangle ABC]} = \sum_{\text{cyclic}} a(\cot A\cot C-\cot A\cot B)$$

Make the substitution:

$$\cot A=\frac{b^2+c^2-a^2}{4[ABC]}\ ,\ \cot B=\frac{a^2+c^2-b^2}{4[ABC]}\ ,\ \cot C=\frac{a^2+b^2-c^2}{4[ABC]}$$

$$\frac{48[\triangle IHG]\cdot[\triangle ABC]^2}{r} = \sum_{\text{cyclic}} a^4(c-b)+a^3(c^2-b^2)$$

$$48[\triangle IHG]\cdot[\triangle ABC]^2 = 2rp[a^3(c-b)+b^3(a-c)+c^3(b-a)]$$

$$\Longrightarrow\ [\triangle IHG]=\frac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{24[\triangle ABC]}$$

Since $[\triangle IGO]=\frac{1}{2}[\triangle IHG] \Longrightarrow\ [\triangle IGO]=\frac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{48[\triangle ABC]}$

Using difference of cubes and substituting $[\triangle ABC]=rp$ gives

$$[\triangle IGO]=\frac{|(a-b)(b-c)(c-a)|}{24r} \Longrightarrow\ [\triangle IOH]=\frac{|(a-b)(b-c)(c-a)|}{8r}$$

Quick Reply

High School Olympiads

i find it difficult [Euler line of intouch triangle]

Source: Iran 1995; Hungary-Israel Competition 2000; 97th KÃ¼rschÃ¡k Competition 1997

galois
#1 Jun 16, 2004, 10:12 am

Given a triangle ABC. Let M, N, P be the points where the incircle of triangle ABC touches its sides BC, CA, AB, respectively. Prove that the orthocenter of triangle MNP (this is the point where the altitudes of triangle MNP concur), the incenter of triangle ABC and the circumcenter of triangle ABC are collinear.

vinoth_90_2004
#2 Jun 16, 2004, 12:04 pm

this problem is very nice
(replace M,N,P by A',B',C' just bcoz im too lazy too remember which corresponds to which). Drop perpendiculars from H to BC, CA and AB, meeting at A'', B'' and C'' respectively. Now i think i read somewhere in the proposed problem section the following:
For any point P in ABC, drop perps from P to the sides meeting at A', B', C'. define f(P)=BA'+CB'+AC'. Then for some 2 points P and Q, f(P)=f(Q) iff PQ is parellel to the line through the incentre and circumcenter. Now using this, It would suffice to prove (with this notation) f(H)=f(I)=s. After some work, this reduces to proving (with directed segments) A'A''+B'B''+C'C''=0. I cant find a nice proof of this :(:(but we can do it with some trigonometry. It reduces to proving that
sum (cyclic) of sin a * (tan b / tan c) * sin (c - b) = 0, where a = 1/2 <CAB, b=1/2 <CBA, etc. this isnt too hard (i dont feel like posting it for some reason :D :D).

sam-n
#3 Jun 16, 2004, 12:47 pm • 1

oh vinoth_90_2004 , you use the lemma which its proof is really harder than the orginal problem. this problem is easily solvable with homothety:

assume that the angel bisector of A and B and C intersect the circumcircle at M' and N' and P'. it is trivial that P'M'||PM and you can conclude that $H_T^k(\Delta PMN) = \Delta P'M'N'$ s.t T is a center of homothety and k is its ratio. we know that $H_T^k(H_{\Delta PMN}) = H_{\Delta P'M'N'}$ and $O_{\Delta P'M'N'} = O, O_{\Delta PMN} = I$ so O,I,T are collinear also $H_{\Delta P'M'N'} = I, H_{\Delta PMN} = H'$ so I ,H' ,T are collinear so O,H',I are collinear.

this solves the problem. also i have seen another proof with inversion.

vinoth_90_2004
#4 Jun 16, 2004, 1:27 pm

hmm the lemma isn't hard to prove in this case - i.e. any point K on IO satisfies f(K)=s. Obviously f(I)=f(O)=s. Since K is on O, after some ratio calculations it follows that f(K)=c1*f(I) + c2*f(O) for some constants c1,c2, c1+c2=1 and the conclusion follows. (in the general case i suppose it may be much harder tho - but we only need this case :)).

This post has been edited 1 time. Last edited by vinoth_90_2004, Jun 16, 2004, 1:33 pm

grobber
#5 Jun 16, 2004, 3:09 pm • 1

Here's another approach:

Fix the circumcircle and the incircle and move A around on the circumcircle. The midpoint of NP is the inverse of A wrt the incircle, meaning that the locus of the midpoint of NP is a circle (the inverse of the circumcircle wrt the incircle). This means that as we move A along the circumcircle, the nine-point circle of MNP is invariant, so its center is also invariant, but its center is the midpoint of the segment formed by I and the orthocenter of MNP, so the orthocenter of MNP is also invariant. Now is the midpoint of the segment formed by I and the orthocenter of MNP, so the orthocenter of MNP is also invariant.. Now take A in the intersection of IO and the fixed circumcircle of ABC. We will have $AB = AC,\ MN = MP$, so the orthocenter of MNP will be on OI, but this means that it's always there, Q.E.D.

You might have noticed that I've used a particular case of Poncelet's theorem: Given two circles (O) and (I), if (I) is the incircle of a triangle ABC inscribed in (O), then no matter where we choose A on (O), if the tangents from A to (I) cut (O) in $B,\ C$ then BC is also tangent to (I). This is a really famous result and I won't prove it here. I'm sure it can be used in contests without proof (it was even needed in one of the Mathlinks contests, but I proved it there just to make sure I get the points:)).

darij grinberg

#6 Jun 16, 2004, 4:41 pm • 3

I remember having solved the above problem for the olympiad 2002 project (Bulgaria 35). I use slightly different notations.

Problem. Let I be the incenter of a non-equilateral triangle ABC and X, Y, Z be the tangency points of the incircle with the sides BC, CA, AB, respectively. Prove that the orthocenter of the triangle XYZ lies on the line OI, where O is the circumcenter of the triangle ABC.

Solution. Let H be the orthocenter of triangle XYZ. We want to show that H lies on OI.

Let D, E, F be the feet of the altitudes of triangle XYZ. The points E and F lie on the circle with diameter YZ; hence, < ZYF = 180° - < ZEF, or, in other words, < ZYX = < XEF. But as a secant-chord angle, < BXZ = < ZYX. Thus, < BXZ = < XEF, and EF || BC. Similarly, FD || CA and DE || AB; hence, the triangles DEF and ABC are homothetic. Let the center of homothety be S. There is a homothety with center S mapping triangle DEF to triangle ABC. This homothety must map the incenter of triangle DEF to the incenter of triangle ABC. But the incenter of triangle DEF is the orthocenter H of triangle XYZ (since triangle DEF is the orthic triangle of triangle XYZ, but the orthocenter of any acute triangle is the incenter of its orthic triangle), and the incenter of triangle ABC is I. Hence, our homothety maps H to I; therefore, S, H and I are collinear. Now, H is the orthocenter and I is the circumcenter of triangle XYZ; hence, the line HI is the Euler line of triangle XYZ. For this reason, we can say that S lies on the Euler line of triangle XYZ.

On the other hand, our homothety maps the circumcenter of triangle DEF to the circumcenter of triangle ABC. The circumcenter G of triangle DEF is the center of the nine-point circle of triangle XYZ (since triangle DEF is the orthic triangle of triangle XYZ, but the nine-point circle of any triangle is the circumcircle of its orthic triangle), and the circumcenter of triangle ABC is O. Hence, our homothety maps G to O, and consequently, S, G and O are collinear. But S lies on the Euler line of triangle XYZ (as we know), and G also lies on the Euler line of triangle XYZ (since G is the nine-point center of triangle XYZ). Hence, O must also lie on the Euler line of triangle XYZ. In other words, O lies on the line HI, and H lies on OI, qed..

This solution used homothety as the basic idea; other solutions use inversion instead.

Darij

This post has been edited 2 times. Last edited by darij grinberg, Apr 30, 2006, 8:46 pm

wellknown

#7 Jun 16, 2004, 7:39 pm

does anyone happen to know an elementary proof of Poncelet's porism for triangles ? i mean without using inversion.

treegoner

#8 Jul 3, 2004, 5:22 am

I just want to show another idea, which is a little bit different from Darij :
Let X', Y', Z' are midpoints of YZ, ZX, XY. Thus I, X' A are collinear.
Let W is the circumcircle of X'Y'Z'.Now using the inversion (I, r^2), it will transform X' to A, Y' to B, Z' to C. Hence I, W, O are collinear.
On the other hand, IW is the Euler line of XYZ. Therefore, the orthocenter of XYZ must lie on OI.

darij grinberg

#9 Jul 3, 2004, 2:17 pm

Indeed. Note that in your solution, you use the

Lemma. If a circle with center O is inverted in a circle with center M, then the image is a circle whose center lies on the line MO.

I'd bet 50% of all IMO participants don't know this one ;-)

Darij

Darij

Sailor
#10 Sep 15, 2004, 5:40 pm

I doubt. Almost everyone from Moldova solved this problem using your lemma.

darij grinberg
#11 Sep 15, 2004, 7:44 pm

No wonder that Moldova was one of the better-placed countries on the IMO 2004. But here in Germany, the lemma is pretty unknown. Most people know that an inversion maps circles to circles, and that the center usually doesn't get mapped to the corresponding center of the image circle, but they don't know about the collinearity.

Darij

paul_mathematics
#12 Jan 25, 2005, 8:40 pm

> *treegoner wrote:*
> I just want to show another idea, which is a little bit different from Darij :
> Let X', Y', Z' are midpoints of YZ, ZX, XY. Thus I, X' A are collinear.
> Let W is the circumcircle of X'Y'Z'.Now using the inversion (I, r^2), it will transform X' to A, Y' to B, Z' to C. Hence I, W, O are collinear.
> On the other hand, IW is the Euler line of XYZ. Therefore, the orthocenter of XYZ must lie on OI.

Why is IW the Euler line of XYZ?

darij grinberg
#13 Jan 25, 2005, 11:12 pm

> *paul_mathematics wrote:*
>> *treegoner wrote:*
>> I just want to show another idea, which is a little bit different from Darij :
>> Let X', Y', Z' are midpoints of YZ, ZX, XY. Thus I, X' A are collinear.
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>> On the other hand, IW is the Euler line of XYZ. Therefore, the orthocenter of XYZ must lie on OI.
>
> Why is IW the Euler line of XYZ?

Well, the points X', Y', Z' are the midpoints of the sides YZ, ZX, XY of triangle XYZ, and thus lie on the nine-point circle of triangle XYZ. Hence, the circumcircle of triangle X'Y'Z' is the nine-point circle of triangle XYZ, and its center W must be the nine-point center of triangle XYZ. Also, it is clear that the point I is the circumcenter of triangle XYZ (since the incircle of triangle ABC is the circumcircle of triangle XYZ). Now, the line joining the circumcenter with the nine-point center of a triangle is the Euler line of this triangle. Thus, the line IW is the Euler line of triangle XYZ.

Darij

armpist
#14 Feb 18, 2006, 4:23 am

Make points O,I, H the intersection points of sides of an inscribed hexagon

on a Pascal line.

T.Y.

M.T.

pohoatza

#15 Apr 22, 2007, 3:37 pm

We will use absolute barycentric coordinates. Firstly, consider G', O' the centroid, respectively the circumcenter of the triangle MNP and O, I the circumcenter, respectively the incenter of ABC. Since

$$G' = (\frac{1}{3}(\frac{p-c}{b}+\frac{p-b}{c}) \ : \frac{1}{3}(\frac{p-a}{c}+\frac{p-c}{a}) \ : \frac{1}{3}(\frac{p-b}{a}+\frac{p-a}{b})),$$

$$O' = I = (\frac{a}{2p} \ : \frac{b}{2p} \ : \frac{c}{2p}),$$

$$O = (\frac{a^2(b^2+c^2-a^2)}{2\sum a^2b^2 - \sum a^4} \ : \frac{b^2(c^2+a^2-b^2)}{2\sum a^2b^2 - \sum a^4} \ : \frac{c^2(a^2+b^2-c^2)}{2\sum a^2b^2 - \sum a^4}),$$

the collinearity of the centroid of MNP with O and I, is equivalent to proving that

$$\begin{vmatrix} a & b & c \\ a\cos A & b\cos B & c\cos C \\ a[b(p-b)+c(p-c)] & b[c(p-c)+a(p-a)] & c[a(p-a)+b(p-b)] \end{vmatrix} = 0$$

$$\iff \begin{vmatrix} 1 & 1 & 1 \\ \cos A & \cos B & \cos C \\ b(p-b)+c(p-c) & c(p-c)+a(p-a) & a(p-a)+b(p-b) \end{vmatrix} = 0$$

$$\iff \begin{vmatrix} 1 & 1 & 1 \\ a(b^2+c^2-a^2) & b(c^2+a^2-b^2) & c^2(a^2+b^2-c^2) \\ b^2+c^2-bc & c^2+a^2-ac & a^2+b^2-ab \end{vmatrix} = 0,$$

which is true. Hence, G' lies on OI and since IG' is the Euler line of MNP, which also contains its orthocenter, we deduce that the orthocenter lies also on the line OI.

This post has been edited 1 time. Last edited by pohoatza, Nov 15, 2007, 10:43 pm

andyciup

#16 Apr 23, 2007, 12:50 am

Consider an inversion that keeps the incircle of triangle ABC as the circle of inversion.
The points A, B, C become the midpoints of the triangle MNP, and thus the circumcircle of triangle ABC becomes the nine-point circle of triangle MNP. This means that the circumcenter of triangle ABC becomes the nine-point-center for triangle MNP. But because our inversion is of center I, this means that the circumcenter of ABC, the circumcenter of ABC after the inversion, and the incircle of triangle ABC are coliniar, which is equivalent to saying that the circumcenter of ABC belongs to the line conecting the circumcenter of MNP with the nine-point center of MNP, that is, the Euler line of MNP.
QED.

edriv
#17 Apr 23, 2007, 1:54 am

> Quote:
> This means that the circumcenter of triangle ABC becomes the nine-point-center for triangle MNP

This is not true, inversion does not send centers to centers. But it's true that the image of a point is collinear with the center of inversion, and by symmetry, the two centres are collinear with I.

Virgil Nicula
#18 Apr 25, 2007, 5:47 am

> Equivalent version of the problem wrote:
> Let a nonequilateral $\triangle ABC$. The incircle $C(I)$ is tangent to BC, CA, AB at D, E, F respectively and G' is the centroid of $\triangle DEF$.
> Prove that $G' \in OI$, where O- is the circumcenter of ΔABC .

Proof I (metrical). Suppose w.l.o.g. $b > c$. Denote the incircle $C(I, r)$ and the circumcircle $C(O, R)$ of the given triangle. Define the middlepoint M of the side $[BC]$ and the intersections $L \in AI \cap EF$, $K \in AI \cap BC$, $S \in G'I \cap BC$. Prove easily

the following relations :

$$\left\{\begin{array}{c} abc = 4RS\ ,\ S = pr \\ a = 2R\sin A\ ,\ r = (p-a)\tan\frac{A}{2} \\ \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}\ ,\ \cos A = 1 - 2\sin^2\frac{A}{2} \end{array}\right. \quad ; \quad \left\{\begin{array}{c} DM = \frac{b-c}{2}\ ,\ DK = \frac{(b-c)(p-a)}{b+c} \\ AK^2 = \frac{4bcp(p-a)}{(b+c)^2}\ ,\ IL\cdot IA = r^2 \\ OM = R\cdot|cosA|\ ,\ \frac{IA}{b+c} = \frac{IK}{a} = \frac{AK}{2p} \end{array}\right. .$$

$1 \blacktriangleright \dfrac{IK}{IL} = \dfrac{IK\cdot IA}{IL\cdot IA} = \dfrac{a(b+c)}{4p^2r^2}\cdot AK^2 = \dfrac{a(b+c)}{4p^2r^2}\cdot\dfrac{4bcp(p-a)}{(b+c)^2} = \dfrac{abc(p-a)}{pr^2(b+c)} \Longrightarrow \boxed{\dfrac{IK}{IL} = \dfrac{4R(p-a)}{r(b+c)}}$ (1).

$2 \blacktriangleright$ Apply the Menelaus' theorem to the transversal $\overline{SG'I}$ and $\triangle LDK$: $\dfrac{SD}{SK}\cdot\dfrac{IK}{IL}\cdot\dfrac{G'L}{G'D} = 1 \Longrightarrow$
$\boxed{\dfrac{SD}{SK} = \dfrac{r(b+c)}{2R(p-a)}}$ (2).

$3 \blacktriangleright \dfrac{SD}{r(b+c)} = \dfrac{SK}{2R(p-a)} = \dfrac{DK}{2R(p-a)-r(b+c)} = \dfrac{(b-c)(p-a)}{(b+c)[2R(p-a)-r(b+c)]} \Longrightarrow$
$SD = \dfrac{r(b-c)(p-a)}{2R(p-a)-r(b+c)}$ and $SM = SD + DM = \dfrac{r(b-c)(p-a)}{2R(p-a)-r(b+c)} + \dfrac{b-c}{2} \Longrightarrow$
$\boxed{SM = (b-c)\cdot\dfrac{2R(p-a)-ar}{2[2R(p-a)-r(b+c)]}}$ (3).

$4 \blacktriangleright$ Therefore, $\dfrac{SM}{SD} = \dfrac{2R(p-a)-ar}{2r(p-a)} = \dfrac{2R(p-a)-a(p-a)\tan\frac{A}{2}}{2r(p-a)} = \dfrac{2R-a\tan\frac{A}{2}}{2r} =$
$\dfrac{2R-2R\sin A\tan\frac{A}{2}}{2r} =$
$\dfrac{R}{r}\cdot(1-2\sin\dfrac{A}{2}\cos\dfrac{A}{2}\tan\dfrac{A}{2}) = \dfrac{R}{r}\cdot(1-2\sin^2\dfrac{A}{2}) = \dfrac{R\cos A}{r} = \dfrac{OM}{ID} \Longrightarrow S \in OI \Longrightarrow O \in SI \equiv G'I \Longrightarrow$
$G' \in OI$.

gemath
#19 Apr 25, 2007, 4:34 pm

I think this problem was well know. A proof using vectors:
The orthocenter H' of triangle MNP lies on Euler line of triangle MNP, but the Euler line of MNP is line OI. Indeed let intersection of lines IA, IB, IC with (O) be A', B', C' easily seen I is orthocenter of

$A'B'C' \Rightarrow \vec{OI} = \vec{OA'} + \vec{OB'} + \vec{OC'}$ but easily seen $\vec{OA'} = \frac{R}{r}\vec{IM} \Rightarrow \vec{OI} = \frac{R}{r}(\vec{IM} + \vec{IN} + \vec{IP}) = \frac{R}{r}\vec{IH'}$

Thus we not only show $H' \in OI$ but also we show $IH' = \frac{r}{R}OI$ nice result

Virgil Nicula

#20 Apr 26, 2007, 8:07 am

Lemma. Let ABC be a triangle with the orthocenter H and the circumcircle $C(O)$. Then there is the **Sylvester's identity** $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2 \cdot \overrightarrow{HO}$ which is equivalently with $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$.

Remark. If we"ll choose O as the origin of the vectorial system, i.e. $X \equiv \overrightarrow{OX}$ and $\overrightarrow{XY} = Y - X$, then Sylvester's identity becomes $\boxed{A + B + C = H}$.

Proof III (Gemath). Denote the second intersections A', B', C' of the lines IA, IB, IC with the circumcircle of $\triangle ABC$. Prove easily that the incenter I of $\triangle ABC$ is the orthocenter of $\triangle A'B'C'$, i.e. $A' + B' + C' = I$. From the evident relations $\frac{OA'}{ID} = \frac{R}{r}$ and $OA' \parallel ID$ obtain $r \cdot \overrightarrow{OA'} = R \cdot \overrightarrow{ID}$, i.e. $r \cdot A' = R \cdot (D - I)$ a.s.o. Therefore, $r \cdot I = r \cdot (A' + B' + C') = R \cdot (D + E + F - 3 \cdot I) = 3R \cdot (G' - I)$, i.e. $r \cdot \overrightarrow{OI} = 3R \cdot \overrightarrow{IG'}$., what means $G' \in OI$. If H' is the orthocenter of $\triangle DEF$, then $3 \cdot \overrightarrow{IG'} = \overrightarrow{IH'}$ (the line $\overline{H'G'I}$ is the Euler's line for $\triangle DEF$) and the previous relation becomes $\boxed{r \cdot \overrightarrow{OI} = R \cdot \overrightarrow{IH'}}$.

Virgil Nicula

#21 Apr 26, 2007, 11:46 am

Preliminary study. Let $\triangle ABC$ with the orthocenter H, the circumcircle $C(O, R)$ and the incircle $C(I, r)$. For a point X which belongs to the circle w

denote XX as the tangent in the point X to the circle w and $pr_d[XY]$ as the length of projection of the segment $[XY]$ to the line d. Then :

$1 \blacktriangleright OI^2 = R(R - 2r)$ and $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$ (well-known).

$2 \blacktriangleright pr_{AA}[OH] = |b \cdot \cos B - c \cdot \cos C|$.

Indeed, $pr_{AA}[OH] = pr_{AA}[AH] = AH \cdot |\sin(B - C)| = 2R|\cos A \sin(B - C)| =$
$R|2\cos(B + C)\sin(B - C)| =$

$R|\sin 2B - \sin 2C| = |2R\sin B \cdot \cos B - 2R\sin C \cdot \cos C| = |b \cdot \cos B - c \cdot \cos C|$.

Proof IV. Denote the orthocenter H' of $\triangle DEF$. Apply the above relations to $\triangle DEF$ and we"ll obtain :

$$1' \blacktriangleright H'I^2 = 9r^2 - \sum EF^2 = 9r^2 - \sum \left[2(p - a)\sin\frac{A}{2}\right]^2 = 9r^2 - \sum 4(p - a)^2 \cdot \frac{(p - b)(p - c)}{bc} =$$

$$9r^2 - \frac{4(p - a)(p - b)(p - c)}{abc} \cdot \sum a(p - a) = 9r^2 - \frac{4pr^2}{4Rpr} \cdot 2r(4R + r) = 9r^2 - \frac{2r^2(4R + r)}{R} \Longrightarrow$$

$$\boxed{H'I^2 = \frac{r^2(R - 2r)}{R}}.$$

$$2' \blacktriangleright pr_{BC}[H'I] = \left|DE \cdot \cos\widehat{DFE} - DF \cdot \cos\widehat{DEF}\right| = \left|2(p - c)\sin^2\frac{C}{2} - 2(p - b)\sin^2\frac{B}{2}\right| =$$

$$\frac{2(p - a)(p - b)(p - c)}{abc} \cdot |b - c| =$$

$$\frac{2pr^2}{4Rpr} \cdot |b - c| \Longrightarrow \boxed{pr_{BC}[H'I] = \frac{r}{2R} \cdot |b - c|}$$. Observe that $pr_{BC}[OI] = \frac{1}{2} \cdot |b - c|$ and $\boxed{\frac{H'I}{OI} = \frac{r}{R}}$. Therefore,

$$\frac{pr_{BC}[H'I]}{pr_{BC}[OI]} = \frac{\frac{r}{2R} \cdot |b - c|}{\frac{|b - c|}{2}} = \frac{r}{R} = \frac{H'I}{OI} \Longrightarrow \boxed{\frac{pr_{BC}[H'I]}{pr_{BC}[OI]} = \frac{H'I}{OI}} \quad (1).$$

$$\frac{DH'}{DO}\cdot\frac{\sin\widehat{IDH'}}{\sin\widehat{IDO}}=\frac{2r\cos\widehat{EDF}}{DO}\cdot\frac{\sin\frac{|B-C|}{2}}{\cos\widehat{ODM}}=\frac{2r\cos\frac{B+C}{2}\sin\frac{|B-C|}{2}}{DM}=\frac{r|\sin B-\sin C|}{\frac{|b-c|}{2}}=$$

$\frac{r|b-c|}{2R\cdot\frac{|b-c|}{2}}=\frac{r}{R}=\frac{IH'}{IO}\Longrightarrow\boxed{\frac{DH'}{DO}\cdot\frac{\sin\widehat{IDH'}}{\sin\widehat{IDO}}=\frac{IH'}{IO}}$ (2). From the relations (1) and (2) results that $H'\in OI$.

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mathVNpro

#22 Apr 10, 2009, 9:45 pm

Lemma 1: Let ABC be a triangle. P is a point inside triangle ABC. AP,BP,CP intersects BC, CA, AB respectively at A-1,B-1,C-1. Let H be the orthocenter of triangle A-1B-1C-1. A-1B-1 intersects AB at C-2. Define the same with B-2, C-2. We call A-3 is the midpoint of A-1A-2, define the same with B-3, C-3.
Then A-3, B-3, C-3 are collinear and the line A-3B-3C-3 is perpendicular with OH (O is the center of circumcircle of triangle ABC).
Proof: Radical axis and Harmonic divsion can kill this lemma easily.
Back to our problem, apply the above lemma, we have OH is perpendicular to XYZ (H is the orthocenter of triangle MNP, O is the center of circumcircle of triangle ABC, X, Y, Z respectively are the midpoints of MK,NQ,PR- K is the intersection of NP and BC, define the same for Q, R).
Now we have to prove that IO is perpendicular to XYZ. This becomes a very well- known problem, we can solve this by using inversion or Radical axis and Harmonic divsion, they are all OK.
Then we complete our proof.

Luis González

#23 Apr 16, 2009, 10:41 pm

The barycentric coordinates of the orthocenter X of $\triangle MNP$ are easily computed since the M-altitude passes through $M(s-b:0:s-a)$ and the infinite point of the A- angle bisector $cy-bz=0.$ Thus

$$X\equiv X_{65}\left(\frac{a(b+c)}{b+c-a}:\frac{b(c+a)}{c+a-b}:\frac{c(a+b)}{a+b-c}\right)$$

Now it is easy to check that $I(a:b:c)$, $O(a^2S_A:b^2S_B:c^2S_C)$ and X_{65} are collinear:

$$\begin{vmatrix} a & b & c \\ a^2S_A & b^2S_B & c^2S_C \\ \frac{a(b+c)}{b+c-a} & \frac{b(c+a)}{c+a-b} & \frac{c(a+b)}{a+b-c} \end{vmatrix}=0$$

Quick Reply

High School Olympiads

Never been solved

gopherhole112
#1 May 24, 2005, 6:32 am

I'm sure there's an easy way to do this, but apparently its never been solved. Draw a circle and a diameter (Circle A). Along the diameter draw another circle (Circle B) such that it is internally tangent to the larger circle, and that A and B share part of a diameter. From the end of the smaller cicle draw an isosceles triangle with a base on the diameter of A. In the space at the top, draw another circle (Circle C) tangent to the side of the triangle, internally tangent to A, and externally tangent to B. Draw a line to the point where the triangle meets B on the diameter of A. Prove that this is perpendicular to the diameter.

probability1.01
#2 May 24, 2005, 6:35 am

The wording is hard to understand. Perhaps a diagram?

gauss202
#3 May 24, 2005, 9:54 am

This problems seems suitable for an attack by inversion.

karthik_k
#4 May 24, 2005, 2:12 pm

> *Quote:*
> Draw a circle and a diameter

What do you mean ??

JBL
#5 May 24, 2005, 9:20 pm

That means, draw a circle with one of its diameters.

I followed this much:

> *Quote:*
> Draw a circle and a diameter (Circle A). Along the diameter draw another circle (Circle B) such that it is internally tangent to the larger circle, and that A and B share part of a diameter.

Now, I don't have any idea what you mean when you say the next part.

An unrelated question: how do you know either that there is an easy way to do this or that it has never been done before?

gopherhole112
#6 May 25, 2005, 4:06 am

I'd be happy to show a diagram if someone can show me how to put one on the site. I didn't see it on the internet, so I would have

I'd be happy to show a diagram if someone can show me how to put one on the site. I didn't see it on the internet, so I would have to send it from my computer. In the meantime, lets try this again

1. Draw a circle (circle A) and one of its diameters
2. Draw another circle (circle B) internally tangent to circle A at the point where the drawn diameter meets circle A.
3. It should be that the diameter of circle A goes through the center of circle B and that a diameter of circle B can be on top of the diameter of circle A
4. Draw an isosceles triangle such that it its base is on the diameter, completely within circle A and completely outside of circle B. It should take up the rest of the diameter of circle A that circle B did not. The vertex of the triangle should be extended upward on the circumfrence of circle A.
5. A third circle should be drawn (circle C) such that it is tangent internally to circle A, tangent externally to circle B, and tangent to the closest leg of the right triangle.
6. From the center of circle C we draw a line to the point where the the triangle meets circle B.

Prove that this line is perpendicular to the drawn diameter of circle A

Someone mentioned inversion. What is that? and how would you use it?
Everything I have tried has bit analytic geometry type stuff, but it's gotten me nowhere.

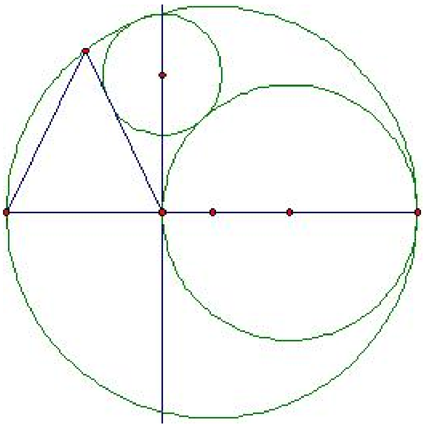
probability1.01
#7 May 25, 2005, 4:13 am

Where did the right triangle come from in step 5?

JBL
#8 May 25, 2005, 4:37 am

He meant isosceles there. And the one neglected piece of information is that the vertex of the triangle is on the circle. A diagram is attached.

Attachments:



gauss202
#9 May 25, 2005, 5:00 am

> *gopherhole112 wrote:*
> Someone mentioned inversion. What is that? and how would you use it?
> Everything I have tried has bit analytic geometry type stuff, but it's gotten me nowhere.

Inversion is a process of transforming geometric figures into "inverse" geometric figures. Its not really easy to describe how this is done, but you can probably find it in an advanced geometry book like Geometry Revisited by Coxeter.

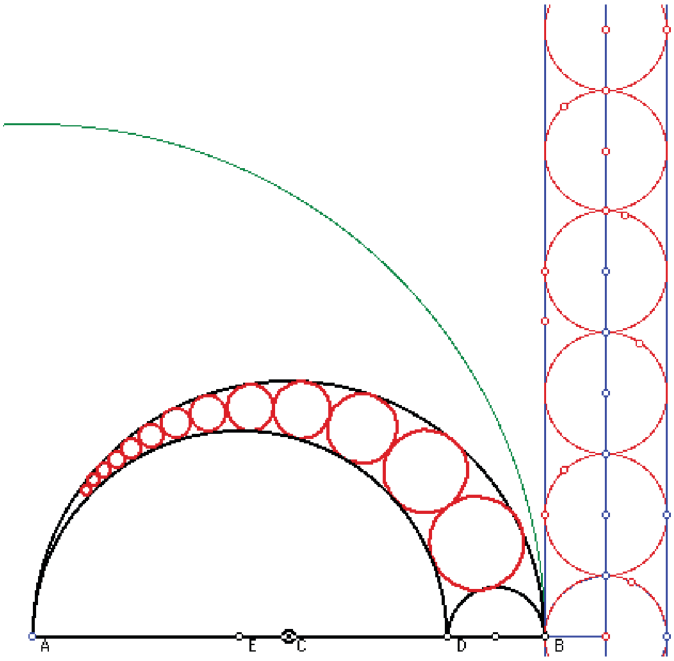
Inversion has the nice property that it preserves circles (and also lines, with lines being regarded as circles of infinite radius). It also preserves the angles between lines and circles. So often, if you choose your inversion carefully, you can transform a figure with lots of circles and lines where a relationship between them (like, say, perpendicularity) is hard to see, into one where that property is very easy to see.

For a classic example of how to use inversion on a problem, look up something called a Pappus Chain. I posted a problem about it a lonnng time ago. I think I titled it A Difficult Geometry Problem.

Edit - I attached a figure showing the inversion of a pappus chain. It transforms the set of (dark) red circles on the left (through the green circle) into the set of (light) red circles on the right. Notice that the semi-circle E is transformed into the right-most verticle blue line, and semi-circle C is transformed into the left-most verticle blue line.

I bet if you put this problem in the Olympiad Geometry section someone there will solve it very quickly. Inversion is a geometric technique often learned by Olympiad students.

Attachments:



gopherhole112

#10 May 25, 2005, 9:08 am

yeah,
JBL's diagram is right, althought I usually put the triangle on the right side, it doesn't change the problem

grodij

#11 May 25, 2005, 9:22 am

Can anyone draw a picture for it?
I didn't undestand the problem.

yetti

#12 May 25, 2005, 4:28 pm • 1

Let $(A), (B)$ be 2 internally tangent circles with radii $a > b$. Let the circles $(A), (B)$ intersect their center line AB at points C, D different from their tangency point O. Let the normal to the center line AB at the midpoint M of the segment CD intersect the larger circle (A) at a point E (forming an isosceles triangle $\triangle CDE$ together with the points C, D). Let (P) be a circle tangent to the line DE and to the circles $(A), (B)$ internally and externally. Prove that the line PD is perpendicular to the center line $AB \equiv CO$.

Denote $\phi = \dfrac{\angle CED}{2} = \angle CEM = \angle DEM$. Obviously,

$$EM = \sqrt{CM \cdot OM} = \sqrt{(a-b)(a+b)} = \sqrt{a^2 - b^2}$$

$$CE = \sqrt{CM^2 + EM^2} = \sqrt{(a-b)^2 + a^2 - b^2} = \sqrt{2a(a-b)}$$

$$\tan\phi = \frac{CM}{EM} = \frac{a-b}{\sqrt{a^2-b^2}} = \sqrt{\frac{a-b}{a+b}}$$

$$\cos\phi = \frac{EM}{CE} = \sqrt{\frac{a^2-b^2}{2a(a-b)}} = \sqrt{\frac{a+b}{2a}}$$

To construct the circle (P), invert the figure in a circle (O) centered at the tangency point O and with radius $r = 2\sqrt{ab}$. Since the circles $(A), (B)$ both pass through the inversion center O, they are carried into straight lines l_A, l_B perpendicular to the center line AB. Let these 2 lines intersect the center line at points A', B'. By the basic properties of inversion,

$$OA' = \frac{r^2}{OC} = \frac{4ab}{2a} = 2b = OD, \quad OB' = \frac{r^2}{OD} = \frac{4ab}{2b} = 2a = OC$$

Thus the points $A' \equiv D,\ B' \equiv C$ are identical and the lines l_A, l_B are normals to the center line AB at the points D, C, respectively. The line DE is carried into a circle passing through the inversion center O and through the intersections U, V of this line with the inversion circle (O), i.e., into the circumcircle (Q) of the isosceles triangle $\triangle UVO$. Since the point D is carried into the point $B' \equiv C$, the circle (Q) passes through the point C and its center is on a normal to the center line AB at the midpoint A of the diameter OC of circle (A). Let the line DE intersect the circle (B) at a point F different from D. Since OD is a diameter of this circle, the angle $\angle OFD = 90^o$ is right. Let the line OF intersect the circle (A) at a point F'. The tangency point O of the circles $(A), (B)$ is their homothety center and also the homothety center of the right angle triangles $\triangle OF'C \sim \triangle OFD$. Hence, the lines $CF' \parallel DF$ are parallel. Consequently, the angles $\angle OCF' = \angle ODF = \angle CDE = \angle DCE$ are all equal and the point F' is a reflection of the point E in the center line AB. The angles $\phi = \dfrac{\angle CED}{2} = \angle CEF' = \angle COF' \equiv AOF$ spanning the same arc CF' of the circle (A) are equal. As a result,

$$AQ = OA \tan \widehat{AOF} = a \tan \phi = a\sqrt{\frac{a-b}{a+b}}$$

and the radius of the circles (Q) is

$$r_Q = \sqrt{OA^2 + AQ^2} = \sqrt{a^2 + a^2 \tan^2 \phi} = \frac{a}{\cos \phi} = a\sqrt{\frac{2a}{a+b}}$$

The circle (P) touching the circles $(A), (B)$ and the line DE is carried in our inversion into a circle (P') touching the parallel lines $l_A \parallel l_B$ and the circle (Q). Hence, its radius is $r'_P = \dfrac{CD}{2} = \dfrac{2a - 2b}{2} = a - b$. Consider the lines l_A, l_B and the circles $(P'), (Q)$ directed. A directed line is considered tangent to a directed circle iff they have the same directions at their tangency point and similar rule applies to tangent directed circles. For example, let the circle (P') be directed counter-clockwise, i.e., $r'_P = a - b > 0$. Then the line l_A has to be directed up, the line l_B down and the circle (Q) clockwise, i.e., $r_Q = -\dfrac{a}{\cos \phi} < 0$ (radii of circles directed clockwise are considered negative). Expand the figure (the circles $(P'), (Q)$ and the lines l_A, l_B) by the distance $d = |r_Q| > 0$. Directed lines are carried into directed lines parallel to the original lines at the distance d right to the original lines, when looking down their directions. Directed circles are carried into directed circles with the same centers, but with radii increased by the distance d. Hence, the directed lines $l_A, l_B \perp AB$ are carried into directed lines $l'_A, l'_B \perp AB$ at points D', C', such that $DD' = d, CC' = d$,

$$C'D' = CD + 2d = 2(a-b) + \frac{2a}{\cos \phi} = 2(a-b) + 2a\sqrt{\frac{2a}{a+b}}$$

The circle (Q) is carried into the circle centered at the point Q and with radius $r'_Q = r_Q + d = r_Q + |r_Q| = 0$. i.e., into the point Q (recall that $r_Q < 0$ is considered negative). The circle (P') is carried into the circle $(P', r''_P = P'Q)$ centered at the point P', tangent to 2 parallel lines l'_A, l'_B and passing through the point Q. Hence, the point P' is the intersection of the axis EM of the parallel lines l_A, l_B or l'_A, l'_B with a circle centered at the point Q and with radius

$$r''_P = P'Q = \frac{C'D'}{2} = a - b + \frac{a}{\cos \phi} = a - b + a\sqrt{\frac{2a}{a+b}}$$

Let the axis EM intersect a line through the point Q parallel to the center line AB at a point N. From the rectangle $AMMQ$,

$$QN = AM = b, \quad MN = AQ = a\sqrt{\frac{a-b}{a+b}}$$

and using the Pythagorean theorem for the right angle triangle $\triangle QNP'$,

$$P'N^2 = P'Q^2 - QN^2 = a^2 - 2ab + 2a(a-b)\sqrt{\frac{2a}{a+b}} + \frac{2a^3}{a+b}$$

Let the circle (P') intersect the circle (A) at a point X and consider the triangle $\triangle AP'X$.

$$AX = a, \quad P'X = r'_P = a - b, \quad AX^2 + P'X^2 = a^2 + (a-b)^2$$

From the right angle triangle $\triangle AMP'$,

$$AP'^2 = AM^2 + P'M^2 = AM^2 + (P'N - MN)^2 =$$

$$= QN^2 + P'N^2 - 2P'N \cdot MN + MN^2 = P'Q^2 + AQ^2 - 2P'N \cdot AQ$$

$$P'Q^2 + AQ^2 = a^2 - 2ab + b^2 + 2a(a-b)\sqrt{\frac{2a}{a+b}} + \frac{2a^3}{a+b} + \frac{a^2(a-b)}{a+b} =$$

$$= \frac{a^3 + a^2b - 2a^2b - 2ab^2 + 2a^3 + a^3 - a^2b}{a+b} + 2a(a-b)\sqrt{\frac{2a}{a+b}} =$$

$$= \frac{2a^3 + 2a^2(a-b) - b^2(a-b)}{a+b} + 2a(a-b)\sqrt{\frac{2a}{a+b}}$$

$$2P'N \cdot AQ = 2a\sqrt{\frac{a-b}{a+b}} \cdot \sqrt{a^2 - 2ab + 2a(a-b)\sqrt{\frac{2a}{a+b}} + \frac{2a^3}{a+b}} =$$

$$= 2a\sqrt{\frac{a^2(a^2-b^2) - 2ab(a^2-b^2) + 2a^3(a-b)}{(a+b)^2} + \frac{2a(a-b)^2}{a+b}\sqrt{\frac{2a}{a+b}}} =$$

$$= 2a\sqrt{\frac{a^4 - 2a^3b + a^2b^2}{(a+b)^2} + \frac{2a^4 - 2a^3b - 2a^2b^2 + 2ab^3}{(a+b)^2} + \frac{2a(a-b)^2}{a+b}\sqrt{\frac{2a}{a+b}}} =$$

$$= 2a\sqrt{\frac{a^2(a^2 - 2ab + b^2)}{(a+b)^2} + \frac{2a(a^3 - a^2b - ab^2 + b^3)}{(a+b)^2} + \frac{2a(a-b)^2}{a+b}\sqrt{\frac{2a}{a+b}}} =$$

$$= 2a\sqrt{\frac{a^2(a-b)^2}{(a+b)^2} + \frac{2a(a^2-b^2)(a-b)}{(a+b)^2} + \frac{2a(a-b)^2}{a+b}\sqrt{\frac{2a}{a+b}}} =$$

$$= 2a\sqrt{\frac{a^2(a-b)^2}{(a+b)^2} + \frac{2a(a-b)^2}{(a+b)} + \frac{2a(a-b)^2}{a+b}\sqrt{\frac{2a}{a+b}}} =$$

$$= 2a\left(\frac{a(a-b)}{a+b} + (a-b)\sqrt{\frac{2a}{a+b}}\right)$$

$$AP'^2 = P'Q^2 + AQ^2 - 2P'N \cdot AQ =$$

$$= \frac{2a^3 + 2a^2(a-b) - b^2(a-b)}{a+b} + 2a(a-b)\sqrt{\frac{2a}{a+b}} -$$

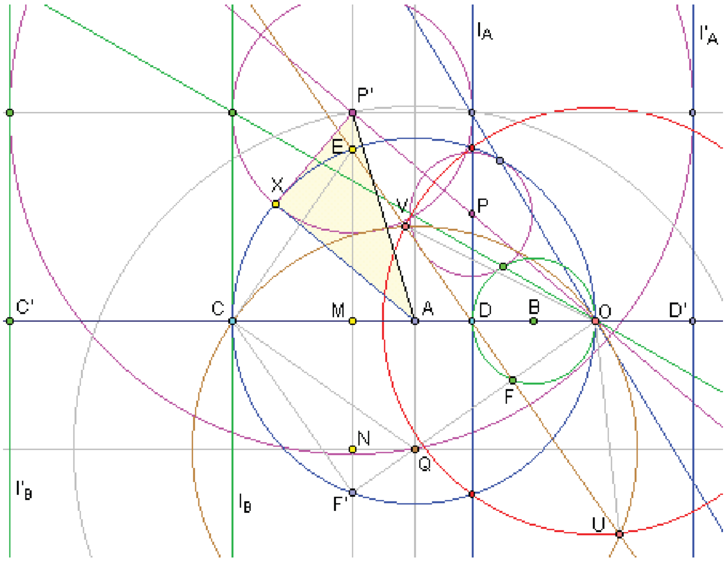
$$-\frac{2a^2(a-b)}{a+b} - 2a(a-b)\sqrt{\frac{2a}{a+b}} =$$

$$= \frac{2a^3 - b^2(a-b)}{a+b} = \frac{a^3 + b^3 + a(a^2-b^2)}{a+b} =$$

$$= a^2 - ab + b^2 + a(a-b) = (a-b)^2 + a^2$$

Consequently, $AP'^2 = a^2 + (a-b)^2 = AX^2 + P'X^2$ and by Pythagorean theorem, the angle $\angle AXP' = 90^o$ is right. Since $AX, P'X$ are radii of the circles $(A), (P')$ at their intersection point X, the circles $(A) \perp (P')$ are perpendicular to each other. Inversion is an involution (a transformation that is its own inverse) and therefore, the line l_A is carried in our inversion in the circle (O) into the circle (A). Since the circles $(A) \perp (P')$ are perpendicular to each other and since inversion preserves angles between curves, the circle (P) is perpendicular to the line l_A. This is possible only if the center P of this circle lies on the line l_A. As a result, the line $PD \equiv l_A$ is perpendicular to the center line $AB \equiv OD$.
:spider:
Never is a long time !!!

Attachments:



This post has been edited 4 times. Last edited by yetti, Oct 16, 2005, 5:46 pm

Magnara

#13 May 25, 2005, 10:09 pm

Holy mother of....

gauss202

#14 May 25, 2005, 10:25 pm

I bow before your perseverence and LaTeX'ing skills Yetti.

And also your knowledge of geometry!

3X.lich

#15 May 25, 2005, 10:36 pm

> *Magnara wrote:*
>
> Holy mother of....

That's got to be one of the longest post ever posted in here.
Great one, yetti

blahblahblah

#16 May 25, 2005, 10:58 pm

That is not a long post for yetti. Actually, it's not a particularly long post for darij, either.

Out of curiosity, do you write things down first or do all your calculations on the screen, yetti?

Amir.S

#17 May 25, 2005, 11:04 pm

COOL!

U R realy good at calculating YETTI

yetti

#18 May 26, 2005, 2:18 am

Thanks to all of you for your kind words.

> *gauss202 wrote:*

> gauss202 wrote:
>
> I bow before your perseverence and LaTeX'ing skills Yetti. ...

> *blahblahblah wrote:*
>
> ...Out of curiosity, do you write things down first or do all your calculations on the screen, yetti?

It was worse than that. I did the calculations on the screen, not knowing the solution, with a pile of paper next to me to try things out. When I finished the post and hit the <Submit> button, I got this silly message: "Only members with special permission are allowed to post in this forum." and the whole typed page expired. This was because somebody moved the thread in the middle of my work. Since this was not the first time something like this happened to me, I occassionally copy the partial text of the intended post into the Notepad and save it. Hence, I did not loose everything, just everything from the first row of .

Regards, Yetti.

This post has been edited 1 time. Last edited by yetti, Jan 16, 2009, 9:20 pm

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gopherhole112

#19 May 26, 2005, 7:20 am

You took all the fun out of that problem. Sitting here in awe is a lot less fun than trying to solve it. While your at it, why don't you prove that the real part of all *s* is 1/2 for...
1/(1-1/(2^s))*1/(1-1/(3^s))*1/(1-1/(5^s))*1/(1-1/(2^s))*1/(1-1/(7^s))*1/(1-1/(11^s))*...=0

---

bomb

#20 May 26, 2005, 10:26 am

No!! The BOMB begs to differ. Sitting here watching you mock the GREAT YETTI is soemthing which THE BOMB takes lightly not of. As regards your problem, you should post it in a new thread and Bomb realises your problem bears stark resemblance to te unsolved Riemann zeta-function hypothesis. So either Bomb or you is crazy to try and solved a conjecture which has defied the might of math giants like Erdos, Gauss, Hilbert, Polya etc for over a century. Why do you not try to solve this gem, prove there exist arbitrarily long APs of primes. (Hint- It was first proved by a new beast of this mellenia, one whose records in IMO stand unparalled to this very day)

Cheers

-Bomb

---

darij grinberg

#21 Jun 5, 2005, 5:10 am

It's a bit late, but I just wanted to remark that the problem is rather old and was solved in http://www.mathlinks.ro/Forum/viewtopic.php?t=18266 .

Darij

---

armpist

#22 Apr 17, 2006, 11:20 pm

As usual, there is more to say about this problem, and this is the purpose of this

post. One interesting development stands out :

in http://www.artofproblemsolving.com/Forum/topic-69620.html

sprmnt21 claims that

....

Some equivalent thesys of that claimed in this problem can be used to give an one line solution of the Sangaku problem treated at

http://www.cut-the-knot.org/Curriculum/Geometry/CirclesAndRegularTriangle.shtml

Starting from this result one can also solve the problem of the 17-th Iranian Mathematical Olympiad 1999/2000 posed at

http://www.artofproblemsolving.com/Forum/topic-65854.html

....

M.T.

This post has been edited 1 time. Last edited by armpist, Dec 16, 2013, 4:44 pm

Virgil Nicula

#23 Apr 18, 2006, 6:51 am

Here is a short analytical proof.

llthigore

#24 Apr 20, 2006, 1:57 am

What is the difference between an inversion and a conformal map? I don't know much about either, but they seem pretty similar (preserves angles and circles). Is inversion a specific variety of conformal mapping?

Virgil Nicula

#25 Apr 22, 2006, 4:35 am

What is your opinion about the my above proof ? Has nobody some objections ? I am waiting the your point of view.

armpist

#26 Apr 16, 2009, 5:24 am

Dear Virgil

Thank you very much for your thoughtful solution.

I especially liked the formula for the angle <AOC at the end:

simple, yet with plenty of geometrical taste.

Sorry my reply is so late, but formulas usually take a long time to cause an

emotion in moi.

M.T.

Luis González

#27 Apr 16, 2009, 7:59 am

For convenience, I'll restate the problem as follows

Proposition. Circle $C_1(r_1)$ with diameter AB and circle $C_2(r_2)$ with diameter AC are internally tangent through $A.$ A circle $C_3(r_3)$ is internally tangent to the semicircle C_1 with diameter AB and its center lies in the perpendicular to AB through $C.$ Tangent CE from C to C_3 cuts C_1 at D and G lies on C_4 such that $\triangle GCB$ is isosceles with base $CB.$ Then D and G are identical.

From the tangencies between C_1, C_2 and $C_1, C_3,$ letting $O_3C=d,$ we have

$d^2+r_2^2=(r_2+r_3)^2 \quad , \quad d^2+(2r_2-r_1)^2=(r_1-r_3)^2$

$ \Longrightarrow r_3 = \frac {2r_2(r_1 - r_2)}{r_1 + r_2} \ \ , \ \ d^2 = \frac {8r_1{r_2}^2(r_1 - r_2)^3}{(r_1 + r_2)^2}$

Therefore, $\frac{O_3E^2}{O_3C^2} = \frac{{r_3}^2}{d^2} = \frac{r_1 - r_2}{2r_1} \ (1)$

But note that $\frac{CF^2}{CG^2} = \frac{(r_1 - r_2)^2}{(r_1 - r_2)^2 + {r_1}^2 - {r_2}^2} = \frac{r_1 - r_2}{2r_1} \ (2)$

From the expressions (1) and (2), it follows that

$$\frac{CF}{CG} = \frac{O_3E}{O_3C} \Longrightarrow \triangle CO_3G \sim \triangle GCF \Longrightarrow \angle DCH + \angle O_3CE = 90^\circ \Longrightarrow D \equiv G.$$

yetti

#28 Apr 16, 2009, 8:35 am

After all these years, you still miss the point. Hint: No formulas are necessary, just find an **incenter**. (Posted in another thread on Mathlinks.)

Luis González

#29 Apr 16, 2009, 9:33 am

> *yetti wrote:*
> After all these years, you still miss the point. Hint: No formulas are necessary, just find an **incenter**. (Posted in another thread on Mathlinks.)

Thanks for the observation dear yetti, but I don't know how to prove this problem without calculations.

yetti

#30 Apr 16, 2009, 9:43 am

I did not mean you, Luis, but the corypheus of all science. I stopped thinking about the problem the moment I hit the <Submit> button. Until it was posted again. I have nothing against formulas or your solution, but pointing out a limiting case in the post above does not solve the problem.

jayme

#31 Apr 16, 2009, 7:47 pm

Dear Yetti and Mathlinkers,
finally, you can see a completely synthetical proof of this nice san Gaku on my site :
http://perso.orange.fr/jl.ayme vol. 4 La fameuse san Gaku de Ima prefecture de Gumma (1803)
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Area

Reply

toiyeutoan

#1 Mar 9, 2006, 6:14 pm

Let E and F be the midpoints of the sides AB and AD of a parallelogram ABCD. The lines CE and BF meet at K, and M is a point on the line EC such that the lines BM and KD are parallel. Prove that the area of triangle KFD equals to the area of quadrilateral KBMD.

Luis González

#2 Apr 15, 2009, 10:51 pm

We use barycentric coordinates WRT $A\ (1:0:0)$, $B\ (0:1:0)$, $D\ (0:0:1)$. Then

$C\ (-1:1:1)$, $E\ (1:1:0)$, $F\ (1:0:1)$, $K\ (1:3:1)$, $M\ (-1:7:4)$

$$\frac{[\triangle KFD]}{[\triangle ABD]}=\frac{3}{(1+3+1)(1+1)}=\frac{3}{10}$$

$$\frac{[KMBD]}{[\triangle ABD]}=\frac{7+3}{(7+4-1)(1+3+1)}+\frac{1+4}{(7+4-1)(1+3+1)}=\frac{3}{10}$$

$$\Longrightarrow [\triangle KDF]=[KBMD]=\frac{3}{20}[ABCD]$$

Quick Reply

High School Olympiads

The line passes through the incenter

Reply

Source: Croatia TST 2009

djuro

#1 Apr 15, 2009, 8:07 pm • 1

A triangle ABC is given with $|AB| > |AC|$. Line l tangents in a point A the circumcirle of ABC. A circle centered in A with radius $|AC$ cuts AB in the point D and the line l in points E, F (such that C and E are in the same halfplane with respect to AB). Prove that the line DE passes through the incenter of ABC.

Luis González

#2 Apr 15, 2009, 8:45 pm • 1

I is the incenter and DI cuts ℓ at E_0. Since $\angle BIC = \angle BDC = 90^\circ + \frac{1}{2}\angle BAC$, it follows that B, I, C, D are concyclic $\Longrightarrow \angle E_0IC = \angle ABC$. Then $\angle ABC = \angle E_0AC \Longrightarrow A, I, C, E_0$ are concyclic. Since $\angle AID = \angle AIC$, then chords $AC = AE_0$ are equal $\Longrightarrow E \equiv E_0$, as desired.

Remark: Similarly, we show that DF passes through the A-excenter of $\triangle ABC$.

windrock

#3 Apr 20, 2009, 9:20 am • 1

Very easy. Call I is the incenter of ΔABC. We have:
$\angle ADE = \pi - frac\angle A + \angle B2 = frac\angle C2 =$
$angleACI = angleADI =$
Hence we get the result.

linboll

#4 Apr 20, 2009, 7:03 pm • 3

In fact Where I first met this problem was in China's National High-school Math Competition(I don't konw the official English name of this contest, I just translated the words) in year 2005.
It was in the second test.The second test contains 3 olympiad-type problems, each problem worths 50 points.120 minutes are allowed in the second test.
I did this problem easily even though I was still a Junior student at that time.

Virgil Nicula

#5 Apr 20, 2009, 9:20 pm • 1

> djuro wrote:
>
> Let $\triangle ABC$ with the incenter I and $c > b$. Denote $D \in (AB)$ for which $AD = b$ and $E \in AA$ (the tangent in a point A
>
> to the circumcircle of $\triangle ABC$) so that $AE = b$ and the sideline AB doesn't separate C , E . Prove that the line $I \in DE$.

Proof. $X \in AI \cap DE$ and $Y \in BC \cap AI \quad \Longrightarrow \quad m(\angle BAE) = A + B$, $YD = YC = \frac{ab}{b+c}$, $m(\angle ADY) = C$.

Thus, $DY \parallel AE \quad \Longrightarrow \quad \frac{XA}{[illegible]} = \frac{AE}{[illegible]} = \frac{b}{[illegible]} \quad \Longrightarrow \quad \frac{XA}{[illegible]} = \frac{b+c}{[illegible]} \overset{\text{Van Aubel}}{\Longrightarrow} X := I$. In conclusion, $I \in DE$.

XY DY $\frac{ab}{b+c}$ XY a

MariusBocanu

#6 May 6, 2011, 12:32 pm

Denote $\widehat{BAC} = 2a, \widehat{ACB} = 2c, \widehat{ABC} = 2b$(after angle chasing) we have $\widehat{EAC} = 2b$(using $\triangle ACE, \triangle ACD$ are isosceles we have $\widehat{ADE} = 90 - a - b$,so$\widehat{EDC} = b, \widehat{ACI} = c, \widehat{ICD} = 90 + a - c$ and all we have to do is to aplt Ceva's reciprocal theorem in tre trigonometric form. The concluision follows only using thing like
$sinasinb = -\frac{1}{2}(cos(a+b) - cos(a-b)$

Microtarx

#7 May 11, 2011, 5:00 am

Let $\angle CAB = 2\alpha$, $\angle ABC = 2\beta$, $\angle ACB = 2\gamma$. Suppose that $G \in BC$ such that AG is bisector of $\angle A$. Let I be the intersection point of DE with AG. By hypothesis $\angle EAC = 2\beta$, also the triangles ADC, ADE and ACE are isosceles, therefore making some calculations, $\angle AED = \gamma$, $\angle EDC = \alpha$ and $\angle IAC = \alpha$, hence AICE is cycle and $\angle ACI = \angle AEI = \angle AED = \gamma$, then CI is bisector of $\angle C$ thus I is the incenter.

[geogebra]5e812a8a2647e28f7ba0108b9e11c654a4a76154[/geogebra]

littletush

#8 Nov 13, 2011, 11:09 am

> *linboll wrote:*
> In fact Where I first met this problem was in China's National High-school Math Competition(I don't konw the official English name of this contest, I just translated the words) in year 2005.
> It was in the second test.The second test contains 3 olympiad-type problems, each problem worths 50 points.120 minutes are allowed in the second test.
> I did this problem easily even though I was still a Junior student at that time.

yes,it is.and it can be second-killed by just some angle substitutions.
by the way,the official name is China Second Round.

genxium

#9 Nov 25, 2011, 12:57 am

It can be also easily done with $S_{\triangle ADE} = S_{\triangle AID} + S_{\triangle AIE}$

Quick Reply

High School Olympiads

Proportion of Triangles

Reply

Source: Point in interior of triangle ABC

triplebig

#1 Apr 14, 2009, 6:34 am

Let P be an interior point of a triangle with sides a, b, c in which lines parallel to the sides of the triangle are traced. THe segments of the parallels delimited by the sides of the triangle have the same length. Find its value.

Answer

Farenhajt

#2 Apr 14, 2009, 6:54 am

Let the desired segment be d, and let the perpendiculars from P to the sides be n_a, n_b, n_c respectively. Then

$$\frac{d}{a} = \frac{h_a - n_a}{h_a} = 1 - \frac{n_a}{h_a}$$

Summing those up, we get $d\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 3 - \left(\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}\right)$

The expression in the parentheses on the RHS is equal to 1, which is easily proven if h_x is substituted by $\frac{2A}{x}$, where $x \in \{a, b, c\}$ and A is the area of the triangle. After that, the result follows.

Luis González

#3 Apr 14, 2009, 7:17 am

Let $(u : v : w)$ be the normalized barycentric coordinates of P WRT $\triangle ABC$. Then lengths of the paralell sections from P to BC, CA, AB are given by

$$d_a = a(1-u) \ \ , \ \ d_b = b(1-v) \ \ , \ \ d_c = c(1-w)$$

Solving $a(1-u) = b(1-v) = c(1-w)$, together with $u + v + w = 1$ gives

$$P \equiv \left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a} : \frac{1}{a}+\frac{1}{c}-\frac{1}{b} : \frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right) \Longrightarrow$$

$$X_{192} \equiv 3 \cdot X_2 - 2 \cdot X_1{}^{-1} \equiv \left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a} : \frac{1}{a}+\frac{1}{c}-\frac{1}{b} : \frac{1}{b}+\frac{1}{a}-\frac{1}{c}\right)$$

This post has been edited 1 time. Last edited by Luis González, Apr 14, 2009, 10:25 am

triplebig

#4 Apr 14, 2009, 7:58 am

Thanks a lot guys, that is very helpful.

luis, I am not accustomed to this notation you use but I would like to understand how you solved this problem.

Is this really a simple solution that I would understand if I understood the notation or does it involve more than basic concepts?

Quick Reply

High School Olympiads

a nice problem

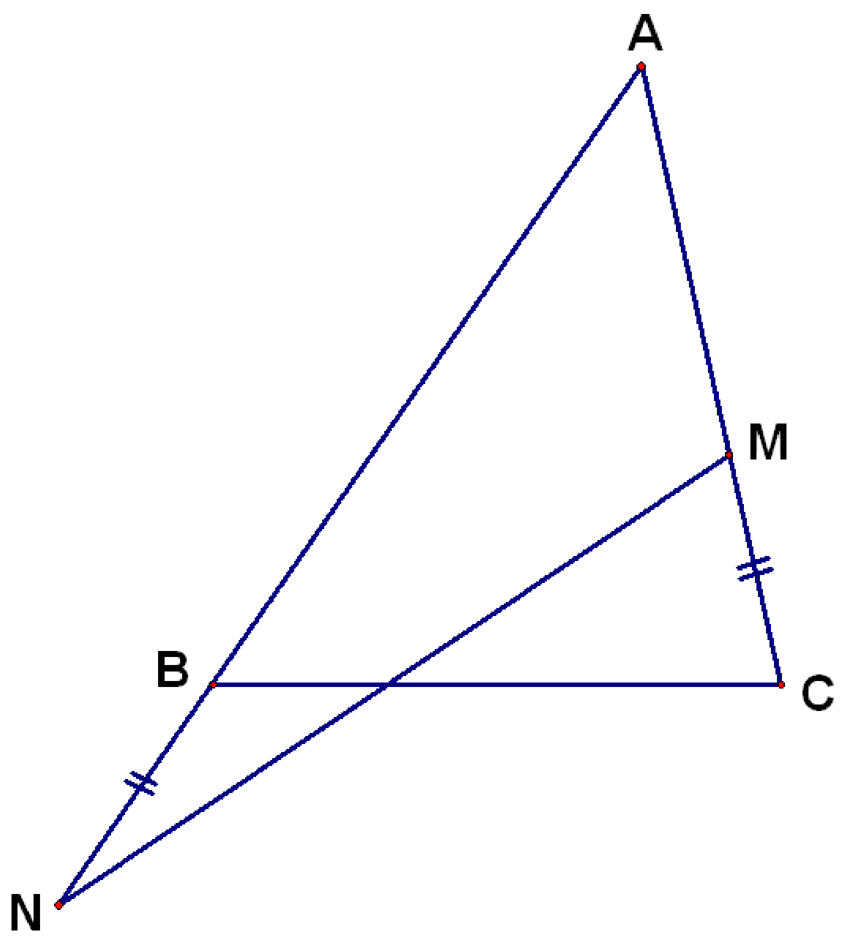
Reply

thanhnam2902
#1 Apr 13, 2009, 8:33 pm

Let ABC is a triangle and let $AB > AC$. Let M is a change point and M lie on side AC. Let N is a change point and N lie on the line pass two point A, B and N lie out region of $\triangle ABC$.

Know that $BN = CM$. Prove that circle (AMN) always pass a fixed point other point A.

Attachments:



Luis González
#2 Apr 14, 2009, 6:09 am

Let P be the 2nd intersection of (O) and $\odot(AMN)$. Note that $\angle MPN = \angle BPC = 180^\circ - \angle A \Longrightarrow \angle BPN = \angle CPM$ and since $ABPC$ is cyclic $\Longrightarrow \angle PMC = \angle NBP$, which implies that $\triangle PBN$ and $\triangle PCM$ are congruent, because $CM = BN$. As a result, $PB = PC \Longrightarrow P$ is the midpoint of the arc BC of circumcircle (O), obviously fixed.

sunken rock
#3 Apr 14, 2009, 12:37 pm

It is a well known property:

if M, N lie on (AC), (AB) so that CM = BN, then the triangle MNP is isosceles, P being the midpoint of the arc BC containing A of (ABC), with the angle <MPN = <BAC, while when M and N lie as in subject problem, then the triangle MQN is isosceles, Q being the midpoint of the arc BC not containing A of (ABC).

Proof: If P is the midpoint of arc BAC, then BP = PC and <ABP = <ACP = |(<B - <C)|/2 and, with CM = BN we get PM = PN and <BPN = <CPM, which gives <MPN = <A.

For 2nd part (our problem), just to change the solution, see that a (180 degs - <A) rotation around Q will map C to B and M to N, so ANQM is cyclic.

Best regards,
sunken rock

High School Olympiads

Nice power of point problem

Reply

Rose_joker

#1 Apr 13, 2009, 7:01 pm

Given triangle ABC with the shortest side BC and let P be a point of AB such that PCB=BAC and Q be a point on AC such that QBC=BAC. Prove that the line through the centers of the circumcircles of triangle ABC and APQ is perpendicular to BC

Luis González

#2 Apr 13, 2009, 11:05 pm

$\angle BAC = \angle PCB = \angle QBC$ implies that circles $\odot(AQB)$ and $\odot(APC)$ are tangent to BC through B, C. Hence, $BC^2 = BP \cdot BA$ and $BC^2 = BQ \cdot CA \Longrightarrow B, C$ have equal power with respect to $\odot(APQ) \Longrightarrow B, C$ are equidistant from the center of $\odot(APQ)$. Centers of $\odot(ABC)$ and $\odot(APQ)$ lie on the perpendicular bisector of BC.

Quick Reply

High School Olympiads

four perpendiculars concurrent

Reply

hollandman

#1 Apr 11, 2009, 8:09 am

Let $ABCD$ be a cyclic quadrilateral, and M, N, O, P the midpoints of sides AB, BC, CD, DA respectively. Prove that the perpendiculars from M to CD, from N to DA, from O to AB, and from P to BC are concurrent.

erudito

#2 Apr 11, 2009, 8:38 am

This is the fifth problem from OBM (Brazilian Mathematics Olympiad in 2001).

mathVNpro

#3 Apr 11, 2009, 5:13 pm

This problem is very classic. Here is my outline solution about this problem:

LEMMA: Let ABCD be the cyclic quadrilateral, AC intersects BD by the point E. If M is the midpoint of AB then ME is perpendicular to CD.

proof: Leave for the readers (This is very easy, just simple angle chasing then you will get the result 😄)
Apply this lemma 4 times in the problem, we shall get the result.

hollandman

#4 Apr 11, 2009, 7:12 pm

> *mathVNpro wrote:*
> This problem is very classic. Here is my outline solution about this problem:
>
> **LEMMA**: Let ABCD be the cyclic quadrilateral, AC intersects BD by the point E. If M is the midpoint of AB then ME is perpendicular to CD.
>
> *proof*: Leave for the readers (This is very easy, just simple angle chasing then you will get the result 😄)
> Apply this lemma 4 times in the problem, we shall get the result.

I don't think your lemma is correct...

mihai miculita

#5 Apr 11, 2009, 7:42 pm

Let Ω the circumcenter of cyclic quadrilateral ABCD and $\{G\} = MO \cap NP$, then the perpendiculars from to M, to CD, from N to DA, and from O to AB, and to P to BC are concurrent in the symetric of point Ω respect to G.

mathVNpro

#6 Apr 11, 2009, 7:42 pm

Oops 😳 . I am so sorry, I have thought that these diagnols of this ABCD are perpendicular to each other 😁 . Sorry for my big mistake

mistake.

livetolove212

#7 Apr 11, 2009, 8:20 pm

> hollandman wrote:
>
> Let $ABCD$ be a cyclic quadrilateral, and M, N, O, P the midpoints of sides AB, BC, CD, DA respectively. Prove that the perpendiculars from M to CD, from N to DA, from O to AB, and from P to BC are concurrent.

Let I be the intersection of MO and PN. E be the circumcenter of ABCD, H be the symetric of E respect to I
Since $MNOP$ is a parallelogram, we get $MI = OI$, moreover $HI = EI$ then $MHOE$ is a parallelogram
Hence $MH \perp DC$. Similarly, $NH \perp AD, OH \perp AB, PH \perp BC$

Luis González

#8 Apr 13, 2009, 7:24 pm

Let the perpendiculars from M to CD and O to AB meet at U. If K is the circumcenter of $ABCD$, then $KM \perp AB$ and $KO \perp CD$ implies that $KMUO$ is a paralelogram $\Longrightarrow U$ is the reflection of K about the midpoint G of OM. But G is also the midpoint of NP. Hence, perpendiculars from N to DA and P to BC also pass through U.

Remark: It's easy to see that U coincides with the Euler point of ABCD.

bloops

#9 Apr 16, 2009, 8:16 pm

> hollandman wrote:
>
> Let $ABCD$ be a cyclic quadrilateral, and M, N, O, P the midpoints of sides AB, BC, CD, DA respectively. Prove that the perpendiculars from M to CD, from N to DA, from O to AB, and from P to BC are concurrent.

1) The point of concurrency, the mid points of opposite sides and the circumcentre form a parallelogram.
Let the circumcentre be G and the point of concurrency be F. Since we can drop perpendiculars from G to AB, CD at their midpoints M and O respectively, $MGOF$ is a parallelogram. Similarly, $NGPF$ is also a parallelogram.

2) Moving on to the Argand plane, with circumcentre as the origin, denote z_1, z_2, z_3, z_4 as the vertices of $ABCD$. The point of concurrency is nothing but the vector sum of the midpoints of opposite sides.

So, $F = \frac{z_1 + z_3}{2} + \frac{z_2 + z_4}{2}$ which is the same point as $\frac{z_1 + z_4}{2} + \frac{z_2 + z_3}{2}$. Thus, we arrive at the same point for each pair of opposite sides.

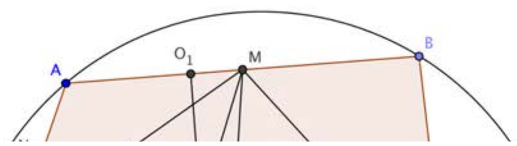
I think this proof can be generalized to a cyclic n-gon, n being even, if I'm not mistaken.

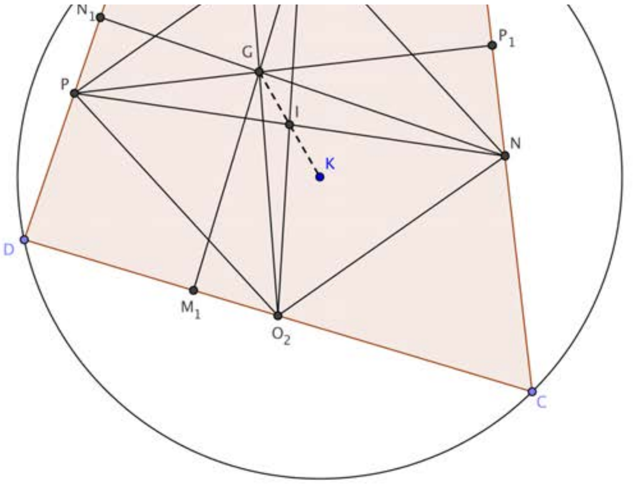
mathVNpro

#10 Apr 21, 2009, 7:19 am

Here is my solution:
Denote M_1, N_1, O_1, P_1 the projections of M, N, O, P onto CD, DA, AB, BC, respectively. Now, It is easy to see that $MNOP$ is parallelogram. Let I be the intersection of its diagonals, K be the circumcenter of $ABCD$.
The symmetry through point I maps K into G, maps M into O, hence, the line KM is maped in to the line OG. But KM is perpendicular to AB, therefore, OG is also perpendicular to AB.
With the same argument, we also have PG, MG, NG, respectively, is perpendicular to BC, CD, AD.
Then we get the result of this problem.
Our proof is completed.

Attachments:





jayme

#11 Apr 21, 2009, 3:40 pm

Dear Mathlinkers,
can some one explain why the meetpoint is called "anticenter".
Who gives this name?
Any reference after Honsberger?
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

A Circle Inside an Equilateral Triangle

Reply

Source: An old problem

sunken rock

#1 Apr 13, 2009, 10:49 am

The circle O lies entirely inside the equilateral triangle ABC. If AM, BN and CP are tangents to O, (M, N and P lying on O), then one can construct a triangle having as sides the lengths AM, BN and CP.

Best regards,
sunken rock

Luis González

#2 Apr 13, 2009, 11:24 am

The idea is to use Purser's theorem, which is none other than a degenerate case of Casey's theorem, coming from Ptolemy's theorem. For instance, if circle (O) touches the circumcircle $\mathcal{K}$ of $\triangle ABC$ at the small arc BC, we have $AM = BN + CP$. But if (O) does not touch $\mathcal{K}$, then Ptolemy's inequality holds and Casey/Purser relation become inequality in exactly the same way. Therefore, $BN + CP \geq AM \Longrightarrow AM, BN, CP$ are sides of a nondegenerate triangle.

Quick Reply

High School Olympiads

simson line

Reply

boogie master

#1 Apr 13, 2009, 6:16 am

If P is a point from the circumcircle ot ABC and H is the orthocentre, prove that the Simson line from P bisects HP.

Luis González

#2 Apr 13, 2009, 7:21 am

This follows from a more general configuration: Given 4 coplanar points A,B,C,D with no 3 of them collinear, then the pedal circle of each point with respect to the triangle formed by the remaining points concur at the Poncelet point of A,B,C,D. Now, if A,B,C,D are concyclic, then the pedal circle of D with respect to ABC becomes the Simson line d of D passing through the Poncelet point of the cyclic ABCD, i.e. the center of symmetry of ABCD and the quadrilateral formed by the orthocenters of BCD,CDA,DAB,ABC. As a result, d bisects the segment connecting D with the orthocenter of ABC.

Quick Reply

High School Olympiads

Two segments have the same lenght

Reply

Source: This problem can be done by using projective geometry (I have a solution)

erudito

#1 Apr 11, 2009, 1:19 am

Geometry problem of a brazilian selection test to the Cone-Sul Olympiad.

EDIT

Consider a circle α tangent to two paralell lines l_1 and l_2 at A and B, respectively. Call C any point of l_1, since it's not A. Let D and E be two points in the circunference, such that they are not coincident and are at the same side of AB as C. Let $CD \cap \alpha = \{F, D\}$ and $CE \cap \alpha = \{G, E\}$. Let $AD \cap l_2 = H$ and $AE \cap l_2 = I$.Let $AF \cap l_2 = J$ and $AG \cap l_2 = K$. Show that $JK = HI$.

This post has been edited 2 times. Last edited by erudito, Apr 11, 2009, 7:28 am

Luis González

#2 Apr 11, 2009, 6:04 am

Something in the enunciation must be incorrect. For instance, consider the situation where the chords DE and AB are parallel and very close to each other. Then the projection of segment $\overline{DE}$ from A onto ℓ_2 tends to ∞, while projection of $\overline{DE}$ from C onto ℓ_2 tends to 0. So $FG = HI$ makes no sense whatsoever.

erudito

#3 Apr 11, 2009, 7:21 am

I'm terribly sorry by a mistake in the exercise. The problem was edited.

Luis González

#4 Apr 13, 2009, 4:00 am

Assume that $JK = HI$. Since (J, K) and (H, I) are the inverse of (F, G) and (D, E) under the inversion with center A and power AB^2, it follows that

$$\frac{JK}{FG} = \frac{AB^2}{AF \cdot AG}, \ \frac{HI}{ED} = \frac{AB^2}{AE \cdot ED} \Longrightarrow \frac{FG}{ED} = \frac{AF \cdot AG}{AE \cdot AD}$$

On the other hand, projecting the quadrilateral $FGED$ on ℓ_2 gives

$$\frac{AF \cdot AG}{AE \cdot AD} = \frac{FG}{ED} = \sqrt{\frac{CF \cdot CG}{CD \cdot DE}}$$

Since (D, E) and (F, G) are inverse under the inversion with center C and power CA^2

$$\Longrightarrow \frac{FG}{ED} = \frac{CA^2}{CD \cdot CE}$$

$$\Longrightarrow \frac{CA^2}{CD \cdot CE} = \sqrt{\frac{CF \cdot CG}{CD \cdot DE}}$$

Since $CA^2 = CD \cdot CF = CE \cdot CG \Longrightarrow CD \cdot CE = CD \cdot CE$, which is true.

erudito

#5 Apr 14, 2009, 3:54 am

I didn't know a solution like yours. Very good strategy. I would like to see more ideas and solutions to this problem, and if somebody want I will post my solution, that uses great tools of projective geometry.

Quick Reply

High School Olympiads

Areal coordinates of the orthocenter

Laplace

#1 Apr 11, 2009, 12:39 am

I need to show that areal coordinates of H are
H = (cotB cotC, cotA cotC, cotA cotB), where H is the orthocenter of a triangle ABC.

I know that the areal coordinates of point H are:

Image not found

However I'm not sure how to show that the two are equal.

Luis González

#2 Apr 12, 2009, 12:09 am

Areal coordinates of a point are its normalized barycentric coordinates:

$$H\left(\frac{\tan A}{\tan A+\tan B+\tan C}:\frac{\tan B}{\tan A+\tan B+\tan C}:\frac{\tan C}{\tan A+\tan B+\tan C}\right)$$

Now use the identity $\tan A\cdot\tan B\cdot\tan C=\tan A+\tan B+\tan C$

$$H\ (\cot B\cot C:\cot A\cot C:\cot B\cot C)$$

High School Olympiads

Easy Geometry

Reply

Source: Taiwan 1st TST, 1st independent study, question 1

k2c901_1

#1 Aug 12, 2005, 9:25 am

Consider a circle O_1 with radius R and a point A outside the circle. It is known that $\angle BAC = 60^\circ$, where AB and AC are tangent to O_1. We construct infinitely many circles O_i $(i = 1, 2, \dots)$ such that for $i > 1$, O_i is tangent to O_{i-1} and O_{i+1}, that they share the same tangent lines AB and AC with respect to A, and that none of the O_i are larger than O_1. Find the total area of these circles.

I know this problem was easy, but it still appeared in the TST, and so I posted it. It was kind of a disappointment for me.

Luis González

#2 Apr 11, 2009, 7:26 am

Generalization: Γ_i is a circle of radius r_i inscribed in an angle of measure 2α such that each Γ_i is externally tangent to Γ_{i+1} and $r_{i+1} < r_i$. Then the sum of the areas of the circles Γ_i equals the area of a circle of radius $r = \frac{1}{2} r_0(\sqrt{\sin\alpha} + \sqrt{\csc\alpha})$.

Let a, b the sides of the subject angle with vertex P and denote X_i the tangency point of $\Gamma_i(O_i, r_i)$ with line a. From the similar $\triangle POX_0 \sim \triangle PO_1X_1$ we get

$$\frac{O_1X_1}{O_0X_0} = \frac{r_1}{r_0} = \frac{PO_1}{PO_0} = \frac{PO_0 - O_0O_1}{PO_0} = \frac{PO_0 - r_0 - r_1}{PO_0}$$

Plugging $PO_0 = r_0 \cdot \csc\alpha$ yields $\dfrac{r_1}{r_0} = \dfrac{\csc\alpha - 1}{\csc\alpha + 1} = \dfrac{1 - \sin\alpha}{1 + \sin\alpha}$

Radii $r_0, r_1, r_2, r_3 \ldots$ form a decreasing geometric progression with ratio $k = \dfrac{1 - \sin\alpha}{1 + \sin\alpha} \Longrightarrow$ Areas S_i of Γ_i form another decreasing geometric progession with ratio k^2.

$$\sum_{i=0}^{\infty} S_i = \frac{S_0}{1 - k^2} = \frac{\pi {r_0}^2}{1 - \left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^2} = \left(\frac{1 + \sin\alpha}{2\sqrt{\sin\alpha}}\right)^2 \cdot \pi {r_0}^2$$

Let ϱ be the radius of the circle equivalent to the chain of circles Γ_i. Then

$$\pi\varrho^2 = \left(\frac{1 + \sin\alpha}{2\sqrt{\sin\alpha}}\right)^2 \cdot \pi {r_0}^2 \implies \varrho = \frac{1}{2} r_0(\sqrt{\sin\alpha} + \sqrt{\csc\alpha}).$$

Quick Reply

High School Olympiads

Find the area of a triangle

Reply

moldovan

#1 Apr 10, 2009, 4:18 pm

Consider the triangle ABC and M a point inside it. Denote by G_1, G_2, G_3 the centroids of MAB, MBC, MCA and $G_1G_2 = 13, G_2G_3 = 20, G_3G_1 = 21$. Find the area of ABC.

mihai miculita

#2 Apr 10, 2009, 4:51 pm

1).If denote: $a = |G_1G_2| = 13;\ b = |G_2G_3| = 20;\ c = |G_1G_3| = 21 \Rightarrow$
$\Rightarrow p = \dfrac{a+b+c}{2} = \dfrac{13+20+21}{2} = 27 \Rightarrow$
$\Rightarrow S_{G_1G_2G_3} = \sqrt{p.(p-a).(p-b).(p-c)} =$
$= \sqrt{27.(27-13).(27-20).(27-21)} = 126.$
2).Let M_1, M_2, M_3 the midpoint of sides $[BC], [CA]$ and $[AB]$, then:
$S_{ABC} = 4.S_{M_1M_2M_3} = 4.\dfrac{9}{4}.S_{G_1G_2G_3} = 9.S_{G_1G_2G_3} = 9.126 = 1134.$

Luis González

#3 Apr 11, 2009, 4:15 am

$\triangle G_1G_2G_3$ is the image of $\triangle ABC$ under the composition $(P, \frac{2}{3}) \circ (G, -\frac{1}{2})$

$$\Longrightarrow \frac{[\triangle ABC]}{[\triangle G_1G_2G_3]} = \left(\frac{2}{3} \cdot \frac{1}{2}\right)^2 = 9 \Longrightarrow [\triangle ABC] = 9 \cdot 126 = 1134$$

Quick Reply

High School Olympiads

Inequality on real numbers

Reply

Luis González
#1 Apr 9, 2009, 9:43 am

If x,y are real numbers, prove the inequality

$$(x^7+y^7)^3\le(x^3+y^3)^7$$

fjwxcsl
#2 Apr 9, 2009, 10:01 am

$$(x^3+y^3)^7-(x^7+y^7)^3$$

$$=(x+y)[7y^3x^3(x-y)^{14}+91y^4x^4(x-y)^{12}+462y^5x^5(x-y)^{10}$$

$$+1176y^6x^6(x-y)^8+1614x^7y^7(x-y)^6+1182y^8x^8(x-y)^4$$

$$+423x^9y^9(x-y)^2+60y^{10}x^{10}]\ge 0,$$

it is holds only for $x,y\ge 0$!

Attachments:
xtkx-09-29(1)-026.rar (307kb)

Potla
#3 Apr 9, 2009, 12:42 pm

> *fjwxcsl wrote:*
>
> $$(x^3+y^3)^7-(x^7+y^7)^3$$
>
> $$=(x+y)[7y^3x^3(x-y)^{14}+91y^4x^4(x-y)^{12}+462y^5x^5(x-y)^{10}$$
>
> $$+1176y^6x^6(x-y)^8+1614x^7y^7(x-y)^6+1182y^8x^8(x-y)^4$$

$$+423x^5y^5(x-y)^2+60y^{10}x^{10}]\geq 0,$$

it is holds only for $x, y \geq 0$!

er, translation? Please help!

Mathias_DK

#4 Apr 9, 2009, 3:09 pm

The generalisation also holds(and are pretty easy to prove):

$$\left(\sum_{i=0}^{n} x_i\right)^a \geq \sum_{i=0}^{n} x_i^a$$

for $a > 1, x_i > 0$.

With equality iff at most one of x_i is positive.

In your example we have $n = 2, x_1 = x^3, x_2 = y^3, a = \frac{7}{3}$..

Luis González

#5 Apr 10, 2009, 10:57 am

Using Muirhead's theorem, the required inequality is equivalent to

$$7[14,0]+7[12,2]+14[11,3]+17[9,5]+18[8,6] \geq 7[13,1]+17[10,4]+9[7,7]$$

Which follows from the majorizations

$$7[14,0] \succ 7[13,1]$$

$$7[12,2] \succ 7[10,4]$$

$$10[11,3] \succ 10[10,4]$$

$$4[11,3] \succ 4[7,7]$$

$$5[9,5] \succ 5[7,7]$$

Mathias_DK

#6 Apr 10, 2009, 10:13 pm

Just a note on notation:

We don't have $7[14,0] \succ 7[13,1]$. What we do have is $(14,0) \succ (13,1)$ which implies $7[14,0] \geq 7[13,1]$

Quick Reply

High School Olympiads

strange problem [ellipse inscribed in triangle ABC]

sxcxc

#1 Feb 21, 2005, 8:54 am

Let AD, BE, CF be the interior angle bisectors of $\triangle ABC$. P is a point inside $\triangle ABC$. Denote h_a, h_b, h_c the distance of P to the sides of $\triangle ABC$. If $\sqrt{h_a}, \sqrt{h_b}, \sqrt{h_c}$ is the sides of a triangle, then prove
(1) the locus of point P is the interior of an ellipse Γ, and BC, CA, AB tangents Γ at point D, E, F respectively;
(2) the area of Γ satisfies $\frac{4\sqrt{3}\pi}{9}S_{\triangle DEF} \le S_\Gamma \le \frac{\sqrt{3}\pi}{9}S_{\triangle ABC}$.

Luis González

#2 Apr 10, 2009, 5:08 am

1) $x = h_a$, $y = h_b$, $h_c = z$. Thus, trilinear coordinates of P WRT $\triangle ABC$ are $(x : y : z)$. Firstly, let us consider the points P such that triangle with side length $\sqrt{x}, \sqrt{y}, \sqrt{z}$ is degenerate. Squaring $\sqrt{x} + \sqrt{y} = \sqrt{z}$ twice yields

$$\mathcal{E} \equiv x^2 + y^2 + z^2 - 2(xy + yz + zx) = 0$$

Which is the trilinear equation of the inellipse $\mathcal{E}$ with intouch triangle $\triangle DEF$. Cyclic permutation of the squares roots of x, y, z gives the same inconic. Now, every point inside the inellipse $\mathcal{E}$ satisfies the inequality $x^2 + y^2 + z^2 - 2(xy + yz + zx) \ge 0$, which is clearly equivant to $\sqrt{x} + \sqrt{y} \ge \sqrt{z} \Longrightarrow (x, y, z)$ are sides of a non-degenerate triangle. Points outside $\mathcal{E}$ satisfy $\sqrt{x} + \sqrt{y} \le \sqrt{z}$ and the triangle does not exist.

High School Olympiads

triangle ABC_area

Source: Terence Tao

communist

#1 Apr 9, 2009, 4:52 pm

Let ABC be a triangle. Let D be the point on AB such that AD=AB/3, let E be the point on BC such that BE=BC/3, and let F be the point on CA such that CF=CA/3. The line segments CD,AE, and BF divide ABC into 3 outer triangles, 3 quad. and one inner triangle. Show that the inner triangle has area equal to one seventh of area of ABC.

Mathias_DK

#2 Apr 9, 2009, 6:43 pm

> *communist wrote:*
>
> Let ABC be a triangle. Let D be the point on AB such that AD=AB/3, let E be the point on BC such that BE=BC/3, and let F be the point on CA such that CF=CA/3. The line segments CD,AE, and BF divide ABC into 3 outer triangles, 3 quad. and one inner triangle. Show that the inner triangle has area equal to one seventh of area of ABC.

Make an affine transformation that transforms $\triangle ABC$ into the equilateral $\triangle PQR$ with sidelength 3. (It's wellknown that one exists). In an affine transformation the ratio between areas are the same, so we just have to prove that the inner triangle in $\triangle PQR$ is one seventh of the area of $\triangle PQR$. From here it is very easy, we can just look at some similar triangles :)

Luis González

#3 Apr 9, 2009, 8:44 pm

Based on given ratios, the barycentric coordinates of D, E, F WRT $\triangle ABC$ are

$D \ (2:1:0) \ , \ E \ (0:2:1) \ , \ F \ (1:0:2)$

AE, CF, CD bound a $\triangle XYZ$ whose vertices are $(2:1:4), (4:2:1), (1:4:2)$

$$\frac{|\triangle XYZ|}{|\triangle ABC|} = \frac{1}{7}\begin{pmatrix} 2 & 1 & 4 \\ 4 & 2 & 1 \\ 1 & 4 & 2 \end{pmatrix} = \frac{1}{7} \Longrightarrow |\triangle ABC| = 7|\triangle XYZ|$$

High School Olympiads

Extrema problems

Reply

Luis González

#1 Apr 9, 2009, 7:21 am

1) Let x, y, z be three real numbers such that $x, y, z \geq -\frac{1}{4}$ and $x + y + z = 1$. Find the maximum value of $\sqrt{4x+1} + \sqrt{4y+1} + \sqrt{4z+1}$.

2) Let a, b, c, d, e be real numbers such that $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$. Find the maximum value of e.

3) a, b are real numbers. Find the maximum and minimum value of $a\cos\phi + b\sin\phi$.

Potla

#2 Apr 9, 2009, 12:49 pm

Solution to 3)

Let $r > 0$ and θ be two reals such that $a = r\cos\theta, b = r\sin\theta$.
We have,
$a^2 + b^2 = r^2(\sin^2\theta + cos^2\theta) = r^2 \implies r = \sqrt{a^2+b^2}$.
Now, $A = a\cos\phi + b\sin\phi = r\cos\theta\cos\phi + r\sin\theta\sin\phi = r\cos(\theta - \phi)$
We have, $-1 \leq \cos(\theta - \phi) \leq 1 \implies -r \leq A \leq r$.
$\implies -\sqrt{a^2+b^2} \leq A \leq \sqrt{a^2+b^2}$.
[edited to fix a mistake]

Potla

#3 Apr 9, 2009, 1:09 pm • 1

Solution to 1)

From cauchy's inequality,
$A^2 = (\sum \sqrt{4x+1})^2 \leq 3(4+3) = 21$ $ (\as a + b + c = 1)$
Hence $A \leq \sqrt{21} \implies A_{max} = \sqrt{21}$
Click to reveal hidden text

Dr Sonnhard Graubner

#4 Apr 9, 2009, 7:21 pm

hello, we have $\sqrt{4x+1} + \sqrt{4y+1} + \sqrt{4z+1} \leq \sqrt{21}$ and the equal sign holds if $x = y = z = \frac{1}{3}$.
Sonnhard.

peine

#5 Apr 9, 2009, 8:38 pm

for the first problem, we have the function $f(a) = \sqrt{a}$ is concave the by Jensen,
$\sum \sqrt{4a+1} \leq \sqrt{21}$ with equality if $a = b = c = \frac{1}{3}$
for the second problem, it's really very nice,
by Cauchy-Schwarz,
$4(a^2 + b^2 + c^2 + d^2) \geq (a + b + c + d)^2$
$\Leftrightarrow 4(16 - e^2) \geq (8 - e)^2$
$\Leftrightarrow 5e^2 - 16e \leq 0$
$\Leftrightarrow e \leq \frac{16}{5}$

Quick Reply

High School Olympiads

a + b + c ≤ 2sqrt(S(tan A + tan B + tan C))

Reply

Source: Nice

SUPERMAN2

#1 Apr 8, 2009, 7:22 pm

Let ABC is an acute triangle.a,b,c are its sides.S is its area.Prove the ineq:
$a+b+c\le 2\sqrt{S(tanA+tanB+tanC)}$

Luis González

#2 Apr 9, 2009, 6:57 am

If p,r are the semiperimeter and inradius of $\triangle ABC$, then the inequality becomes

$$\tan A+\tan B+\tan C\ge\frac{p}{r}\Longleftrightarrow\tan A\cdot\tan B\cdot\tan C\ge\frac{p}{r}$$

Let d_a,d_b,d_c be the distances from the circumcenter O to the sidelines of $\triangle ABC$.

$$\tan A=\frac{BC}{2d_a}\ ,\ \tan B=\frac{CA}{2d_b}\ ,\ \tan B=\frac{AB}{2d_c}\Longrightarrow\frac{BC\cdot CA\cdot AB}{8\cdot d_a\cdot d_b\cdot d_c}\ge\frac{p}{r}$$

Using $4Rrp=BC\cdot CA\cdot AB$ and $R\ge 2r$ gives $R^3\ge 8\cdot d_a\cdot d_b\cdot d_c$

Which is Erdos-Mordell inequality for circumcenter O of the acute $\triangle ABC$.

SUPERMAN2

#3 Apr 11, 2009, 7:31 am

I have a different solution.We know the ineq:(Japan MO)
$(a+b-c)^2(b+c-a)^2(c+a-b)^2\ge(a^2+b^2-c^2)(b^2+c^2-a^2)(a^2+c^2-b^2)$
Substitute $a^2+b^2-c^2=2abcosC$,$b^2+c^2-a^2=2bccosA$,$c^2-a^2-b^2=2cacosB$ and
$(a+b-c)(b+c-a)(c+a-b)=\frac{16S^2}{a+b+c}$,we get
$(\frac{16S^2}{a+b+c})^2\ge 8a^2b^2c^2cosAcosBcosC$(*)
=>$32S^4\ge a^2b^2c^2(a+b+c)^2cosAcosBcosC$
Noting that
$S^3=\frac{1}{8}a^2b^2c^2sinAsinBsinC$
=>$32S^4=4Sa^2b^2c^2sinAsinBsinC$(**)
From (*) and (**),we have:
$4SsinAsinBsinC\ge(a+b+c)^2cosAcosBcosC$
=>$4StanAtanBtanC\ge(a+b+c)^2$
=>$2\sqrt{S(tanA+tanB+tanC}\ge a+b+c$.

Marian Dinca

#4 Apr 18, 2009, 4:39 pm

Another proof the inequality

$ \mbox{triangle ABC acut } a^2 = b^2 + c^2 - 2bc.\cos{A} = b^2 + c^2 - 4S.\ctg{A} \Rightarrow \\ \Rightarrow ctg{A} = \frac {b^2 + c^2 - a^2}{4S} > 0 \Rightarrow tg{A} = \frac {4S}{a^2 + b^2 - c^2}; \\ \mbox{analog: } tg{B} = \frac {4S}{a^2 + c^2 - b^2}; tg{C} = \frac {4S}{a^2 + b^2 - c^2}. \\ \mbox{Let: } b^2 + c^2 - a^2 = x > 0; a^2 + c^2 - b^2 = y > 0; a^2 + b^2 - c^2 = z > 0\Rightarrow \\

\Rightarrow \left\{\begin{array}{c} a^2 = \frac {y + z}{2} \\ b^2 = \frac {x + z}{2} \\ c^2 = \frac {x + y}{2} \end{array} \right \Rightarrow \left\{\begin{array}{c} a = \sqrt {\frac {y + z}{2}} \\ b = \sqrt {\frac {x + z}{2}} \\ c = \sqrt {\frac {x + y}{2}} \end{array} \right , \mbox{ and } \\ 16.S^2 = 2.\sum{a^2b^2} - \sum{a^4} = \sum{(a^2 + b^2 - c^2).c^2} = \sum{z.\left(\frac {x + y}{2}\right)} = \sum{xy}. \\ \sqrt {\left(\sum{tg{A}}\right).S} = \sqrt {\sum{\frac {4S}{b^2 + c^2 - a^2}}} = \sqrt {4S^2.\sum{\frac {1}{x}}} = \sqrt {\frac {\sum{xy}}{4}.\sum{\frac {1}{x}}} = \\ = \frac{1}{2}.\sqrt {\frac {\sum{x.(y + z)}}{2}.\sum{\frac {1}{x}}}\ge\frac{1}{2}.\sum{\sqrt {\frac {y + z}{2}}} = \sum{a}. \\ \mbox{Aconding Caucy - Buniakowsky inequality for: } \\ a_1 = \frac {x.(y + z)}{2}; a_2 = \frac {y.(x + z)}{2}; a_3 = \frac {z.(x + y)}{2}; b_1 = \frac {1}{x};b_2 = \frac {1}{y}; b_2 = \frac {1}{z} \\ \Rightarrow {\sum{a_i}}.{\sum{b_i}}\ge\left(\sum{\sqrt {a_i.b_i}\right)^2.$

Quick Reply

High School Olympiads

Inequality on three positive numbers

Reply

Luis González
#1 Apr 9, 2009, 3:58 am

If three positive numbers a, b, c satisfy $a+b+c=1,$ then show that

$$\sqrt{9a+1}+\sqrt{9b+1}+\sqrt{9c+1}\leq 6$$

peine
#2 Apr 9, 2009, 4:41 am

the function $f(x)=\sqrt{x}$is concave then by Jensen
$\sum\sqrt{9a+1}\leq 3\sqrt{\frac{9(a+b+c)+3}{3}}=6$

ElChapin
#3 Apr 9, 2009, 8:57 am

By AMGM we have that
$\sum\sqrt{9a+1}\leq\frac{1}{2}\sum\frac{4+(9a+1)}{2}=\frac{9}{4}(a+b+c)+\frac{15}{4}=6$

:)

secrets
#4 Apr 9, 2009, 10:42 am

dear friend,don't post old problem on box.

Potla
#5 Apr 9, 2009, 11:44 am

We have,
$(\sum\sqrt{9a+1})^2\leq 3(9(a+b+c)+3)=36$ (from Cauchy.) :)
Hence $\sum\sqrt{9a+1}\leq 6$

Potla
#6 Apr 9, 2009, 12:01 pm

Another solution to your problem: :D
From the RMS-AM inequality or the QM-AM inequality, we have:.
$\frac{\sum\sqrt{9a+1}}{3}\leq\sqrt{\frac{9+3}{3}}$ (as a+b+c=1)
$\implies\sum\sqrt{9a+1}\leq 3\sqrt{4}=6.$
QED :) :D

Quick Reply

High School Olympiads

Areas in a triangle.

Reply

Source: Own.

Virgil Nicula

#1 Feb 1, 2009, 8:32 pm

Consider in a triangle ABC the points $M\in (BC)$, $N\in (CA)$, $P\in (AB)$ and $Q\in BN\cap CP$,

$R\in BN\cap MP$, $S\in CP\cap MN$. Prove that $\boxed{[MRQS]=[NQP]\iff\frac{MB\cdot MC}{AN\cdot AP}=\frac{a^2}{bc}}$.

Luis González

#2 Apr 9, 2009, 2:18 am

By Menelaus' theorem for $\triangle ABN, \triangle APC$ cut by transversals $\overline{PCQ}, \overline{BQN}$ we get

$$\frac{CQ}{PC}\cdot\frac{AP}{AB}\cdot\frac{BN}{QN}=1 \ , \ \frac{BQ}{BN}\cdot\frac{AN}{AC}\cdot\frac{PC}{PQ}=1$$

$$\Longrightarrow \frac{CQ\cdot BQ\cdot AP\cdot AN}{AB\cdot AC\cdot PQ\cdot QN}=1$$

$$\frac{AP\cdot AN}{AB\cdot AC}=\frac{MB\cdot MC}{BC^2}\Longrightarrow\frac{CQ\cdot BQ}{PQ\cdot QN}=\frac{BC^2}{MB\cdot MC}$$

$$\frac{[\triangle BQC]}{[\triangle PQN]}=\frac{CQ\cdot BQ}{PQ\cdot QN}\Longrightarrow\frac{[\triangle BQC]}{[\triangle PQN]}=\frac{BC^2}{MB\cdot MC}$$

We must have: $\frac{[MRQS]}{[\triangle BQC]}=\frac{MB\cdot MC}{BC^2}$, which is not true for arbitrary M on BC.

Virgil Nicula

#4 Apr 9, 2009, 8:43 pm

Sorry again. Here is the correct enunciation. Now I am surely that it is right. Success !

> Virgil Nicula wrote:
> Let $\triangle ABC$, the points $M\in (BC)$, $N\in (CA)$, $P\in (AB)$ so that $\underline{Q\in AM\cap BN\cap CP}$ and
>
> $R\in BN\cap MP$, $S\in CP\cap MN$. Prove that $\boxed{[MRQS]=[NQP]\iff\frac{MB\cdot MC}{AN\cdot AP}=\frac{a^2}{bc}}$.

Virgil Nicula

#5 Apr 10, 2009, 5:04 am

Dedicated to **Luis González**.

> Virgil Nicula wrote:
> Let $\triangle ABC$, the points $M\in (BC)$, $N\in (CA)$, $P\in (AB)$ so that $\underline{Q\in AM\cap BN\cap CP}$ and

$R\in BN\cap MP$, $S\in CP\cap MN$. Prove that $\boxed{[MRQS]=[NQP]\iff\frac{MB\cdot MC}{AN\cdot AP}=\frac{a^2}{bc}}$.

Proof. Denote $\boxed{\begin{array}{c}\frac{NA}{n}=\frac{NC}{1}=\frac{AC}{n+1}\\ \frac{PA}{p}=\frac{PB}{1}=\frac{AB}{p+1}\end{array}}$. Apply **Ceva**'s theorem to $Q/ABC\ :\ \frac{MB}{MC}\cdot\frac{NC}{NA}\cdot\frac{PA}{PB}=1$, i.e. $\boxed{\frac{MB}{n}=\frac{MC}{p}=\frac{BC}{n+p}}$.

Apply **Van Aubel**'s relation : $\frac{QA}{QM}=\frac{PA}{PB}+\frac{NA}{NC}$, i.e. $\frac{QA}{QM}=n+p\implies\boxed{\frac{QA}{n+p}=\frac{QM}{1}=\frac{AM}{n+p+1}}$. Apply **Menelaus'** theorem

to mentioned transversals/triangles : $\left\|\begin{array}{c}\overline{BQN}/APC\ :\ \frac{BP}{BA}\cdot\frac{NA}{NC}\cdot\frac{QC}{QP}=1\implies\frac{n}{p+1}\cdot\frac{QC}{QP}=1\\ \overline{CQP}/ANB\ :\ \frac{CN}{CA}\cdot\frac{PA}{PB}\cdot\frac{QB}{QN}=1\implies\frac{p}{n+1}\cdot\frac{QB}{QN}=1\end{array}\right\|\implies$ $\boxed{\begin{array}{c}\frac{QC}{p+1}=\frac{QP}{n}=\frac{CP}{n+p+1}\\ \frac{QB}{n+1}=\frac{QN}{p}=\frac{BN}{n+p+1}\end{array}}$.

Apply **Routh**'s relation : $\left\|\begin{array}{c}\overline{MRP}\ :\ \frac{RP}{RM}=\frac{BP}{BM}\cdot\frac{NA}{NC}\cdot\frac{BC}{BA}\implies\frac{RP}{RM}=\frac{n+p}{p+1}\\ \overline{MSN}\ :\ \frac{SN}{SM}=\frac{CN}{CM}\cdot\frac{PA}{PB}\cdot\frac{CB}{CA}\implies\frac{SN}{SM}=\frac{n+p}{n+1}\end{array}\right\|\implies\boxed{\begin{array}{c}\frac{RP}{n+p}=\frac{RM}{p+1}=\frac{PM}{n+2p+1}\\ \frac{SN}{n+p}=\frac{SM}{n+1}=\frac{NM}{n+p+1}\end{array}}$.

Denote $S=[ABC]$.

$\frac{[PQN]}{S}=\frac{[PQN]}{BQC]}\cdot\frac{[BQC]}{ABC]}=\frac{QP}{QC}\cdot\frac{QN}{QB}\cdot\frac{MQ}{MA}=$
$\frac{n}{p+1}\cdot\frac{p}{n+1}\cdot\frac{1}{n+p+1}\implies\boxed{[PQN]=\frac{np}{(n+1)(p+1)(n+p+1)}\cdot S}$.

${ \left\|\begin{array}{c} \frac {[RQM]}{[ACB]} = \frac {[RQM]}{[PQM]}\cdot\frac {[PQM]}{[PCM]}\cdot\frac {[PCM]}{[PCB]}\cdot\frac {[PCB]}{[ACB]} = \frac {MR}{MP}\cdot\frac {PQ}{PC}\cdot\frac {CM}{CB}\cdot\frac {BP}{BA} = \frac {np}{(n + p)(n + p + 1)(n + 2p + 1)} \\ \\{ \frac {[SQM]}{[ABC]} = \frac {[SQM]}{[NQM]}\cdot\frac {[NQM]}{[NBM]}\cdot\frac {[NBM]}{[NBC]}\cdot\frac {[NBC]}{[ABC]} = \frac {MS}{MN}\cdot\frac {NQ}{NB}\cdot\frac {BM}{BC}\cdot\frac {CN}{CA} = \frac {np}{(n + p)(n + p + 1)(2n + p + 1)}}\end{array}\right\|$ $\implies\boxed{\begin{array}{c}\frac{[RQM]}{S}=\frac{np}{(n+p)(n+p+1)(n+2p+1)}\\ \frac{[SQM]}{S}=\frac{np}{(n+p)(n+p+1)(2n+p+1)}\end{array}}$.

Thus, $[MRQS]=[RQM]+[SQM]=\frac{np}{(n+p)(n+p+1)}\cdot\left(\frac{1}{n+2p+1}+\frac{1}{2n+p+1}\right)\cdot S\implies$

$\boxed{[MRQS]=\frac{np[3(n+p)+2]}{(n+p)(n+p+1)(n+2p+1)(2n+p+1)}\cdot S}$. Denote $\sigma=n+p\ ,\ \pi=np$. Thus, $\boxed{[MRQS]=[PQN]}\iff$

$\frac{3(n+p)+2}{(n+p)(n+2p+1)(2n+p+1)}=\frac{1}{(n+1)(p+1)}\iff(3\sigma+2)(\sigma+\pi+1)=\sigma(2\sigma^2+3\sigma+\pi+1)\iff$

$\sigma^3-\sigma\pi-2\sigma-\pi-1=0\iff(\sigma+1)(\sigma^2-\sigma-\pi-1)=0\iff\sigma^2=$
$\sigma+\pi+1\iff\boxed{(n+p)^2=(n+1)(p+1)}\iff$

$\frac{n}{n+p}\cdot\frac{p}{n+p}=\frac{n}{n+1}\cdot\frac{p}{p+1}\iff\frac{MB}{BC}\cdot\frac{MC}{BC}=\frac{AN}{AC}\cdot\frac{AP}{AB}\iff\boxed{\frac{MB\cdot MC}{AN\cdot AP}=\frac{a^2}{bc}}$.

Remark. Show easily that $[MRQS]=[PQN]\iff\left(\frac{QA}{QM}\right)^2=\frac{CA}{CN}\cdot\frac{BA}{BP}$. Here are some particular cases :

▶ **IF** $Q := I$ is incenter of $\triangle ABC$ **THEN**

$\boxed{[MRQS] = [PQN] \iff (b+c)^2 = (a+b)(a+c) \iff \frac{RP \cdot SN}{RM \cdot SM} = 1}$ (**nice !**).

▶ **IF** $Q := N$ is **Nagel**'s point of $\triangle ABC$ **THEN** $\boxed{[MRQS] = [PQN] \iff bc = a^2}$ (**nice !**).

▶ **IF** $Q := \Gamma$ is **Gergonne**'s point of $\triangle ABC$ **THEN**

$\boxed{[MRQS] = [PQN] \iff bc = a(p-a) \iff (a-b)(a-c) + bc = 0}$ (**nice !**).

This post has been edited 6 times. Last edited by Luis González, Mar 21, 2015, 11:50 am

Quick Reply

High School Olympiads

concurrent

Reply

Source: Unknown

Parkdoosung

#1 Jul 22, 2005, 12:51 am

Circles C1 and C2 are tangent to each other at K and are tangent to circle C at M and N. External tangent of C1 and C2 intersect C at a and B.
AK and BK intersct with circle C at E and F if AB is diameter of C.
Prove EF and MN and OK are concurrent. (O is center of circle C)

Luis González

#2 Apr 8, 2009, 12:28 am

Inversion with center K and power equal to the power of K with respect to C_0 transforms C_0 into itself and C_1, C_2 into two parallel lines ℓ_1, ℓ_2 tangent to C_0 through $M', N' \Longrightarrow M', N'$ are antipodal points on C_0. Line MN is taken into the circle $\odot(KM'N')$ and line EF is taken into the circle $\odot(KAB)$. Note that O has equal power WRT $\odot(KAB)$ and $\odot(KM'N')$, due to $\overline{OM'} \cdot \overline{ON'} = \overline{OB} \cdot \overline{OA}$. Hence, $\odot(KAB)$ and $\odot(KM'N')$ meet again on the line KO. If $\odot(KM'N')$, $\odot(KAB)$ and KO concur $\Longrightarrow MN, EF$ and OK concur as well.

Quick Reply

High School Olympiads

A polynomial root

Reply

Source: 0

Luis González

#1 Apr 6, 2009, 8:05 am

Prove that $\sin\frac{\pi}{14}$ is a root of the polynomial equation

$$8x^3-4x^2-4x+1=0$$

quocbao153

#2 Apr 6, 2009, 1:47 pm

Oh,by the graph method, this problem may be very easy that the proof is the intersection point of the function graph $y=8x^3-4x^2-4x+1$ and $y=0$. May this solution be optimal?

ith_power

#3 Apr 6, 2009, 3:19 pm

not so evident though...
Let

$\theta=\pi/14$
$\rightarrow \sin 14\theta=0$
$\rightarrow 14Cos\theta^{13}Sin\theta-364Cos\theta^{11}Sin\theta^3+2002Cos\theta^9Sin\theta^5-3432Cos\theta^7Sin\theta^7+2002Cos\theta^5Sin\theta^9-364Cos\theta^3Sin\theta^{11}+14Cos\theta Sin\theta^{13}=0$

which on factorising gives:
$2x(\sqrt{1-x^2})\boxed{(1-4x-4x^2+8x^3)}(-1-4x+4x^2+8x^3)(-7+56x^2-112x^4+64x^6)=0$ where $x=sin\theta$

Xantos C. Guin

#4 Apr 7, 2009, 4:29 am • 1

Click to reveal hidden text

Luis González

#5 Apr 7, 2009, 7:49 am

The proposed problem is equilavent to show that: If a is the side-lenght of a regular 14-gon, then its circumradius R is a real positive solution of $R^3+a^3-a^2R-2aR^2=0$.

Let O be the center of the 14-gon and B,C two consecutive vertices. Thus $\angle BOC=\frac{180^\circ}{7}$. There exists two points P,Q on OC,OB such that $BP=PQ=QO=a$. Draw parallels $QT=x$ and $PS=y$ to BC. Then $\triangle CBP$ and $\triangle QOT$ are congruent $\Longrightarrow PC=QT=x$, but $\triangle BCP$ and $\triangle OBC$ are similar

$$\Longrightarrow \frac{PC}{BC}=\frac{BC}{R}\Longrightarrow x=\frac{a^2}{R}\ (1)$$

$QTPS$ is a trapezoid with $PS=QS=y$ and since $\triangle OSP\sim\triangle OBC$, we get:

$SP \quad OS \quad y \quad y+a \quad a^2$

$$\frac{}{BC} = \frac{}{OB} \implies \frac{}{a} = \frac{}{R} \implies y = \frac{}{R-a} \; (2)$$

$$QS = TP = y \implies TP + PC = OC - OT \implies \; y + x = R - a \; (3)$$

Combining (1), (2) and (3) yields:

$$\frac{a^2}{R-a} + \frac{a^2}{R} = R - a \implies R^3 + a^3 = a^2R + 2aR^2.$$

Quick Reply

High School Olympiads

Cyclic trapezoid

Reply

Source: 0

Luis González

#1 Apr 7, 2009, 7:21 am

A trapezoid $ABCD$ with parallel bases $\overline{AB}$ and $\overline{DC}$ is inscribed in a circle $(O).$ O is outside $ABCD.$ Diagonals of $ABCD$ intersect at $E.$ Consider the circle $\mathcal{C}(r)$ tangent to the diagonals and internally tangent to the arc BC of $(O).$ M,N are midpoints of $\overline{AB}$ and the arc CD of $(O),$ respectively. Show that

1) $\frac{1}{r}=\frac{1}{EM}+\frac{1}{EN}$

2) If $\overline{AB}$ is a diameter, find the ratio $\frac{EM}{EN}$ such that $\mathcal{C}(r)$ and the circle tangent to the diagonals of the trapezoid and internally tangent to the small arc CD are congruent.

yetti

#2 Apr 12, 2009, 12:47 pm

For (1), see http://www.mathlinks.ro/viewtopic.php?t=106321. For (2), let $C(r')$ be (the smaller) circle through N tangent to $AC, BD.$ $\frac{EN}{1+\frac{EA}{MA}}=r'=r=\frac{EN}{1+\frac{EN}{EM}}\Longleftrightarrow\frac{EA}{MA}=\frac{EN}{EM}.$ When $MA=R=EM+EN,$ then $x^2=\frac{EN^2}{EM^2}=\frac{EA^2}{R^2}=\frac{EM^2}{R^2}+1=\frac{1}{(1+\frac{EN}{EM})^2}+1=\frac{1}{(1+x)^2}+1\Longleftrightarrow x^4+2x^3-2x-2=0.$ Resolvent cubic $y^3+4y+4=0$ of this quartic does not have constructible roots and neither has the quartic.

Quick Reply

High School Olympiads

PEQ = 90 (very, very, nice)

Reply

Ashegh
#1 Oct 3, 2005, 11:43 pm • 1

let E be a point and (C) be a circle . and EC and ED are tangent to (C). let F and G be the midpoints of EC and ED.

let Q be a variable point on line FG. QA and QB are tangents to (C). and AB intersects line GF in point P.

prove that :

angle PEQ=90º

lomos_lupin
#2 Oct 4, 2005, 1:30 am

Ashegh ,you should post this in the Proposed problems ,cause our teacher has solved it for us like this:
Solution

Ashegh
#3 Oct 5, 2005, 3:43 pm

i want a powerfull solution

pooyan_seva
#4 Oct 7, 2005, 1:09 am

that was the most beautifull problem that i have ever seen .
and i'm going to solve it as soon as possible...

pooyan_seva
#5 Oct 7, 2005, 1:12 am

ee... bacheha che bahal to in safhe hame az iran post kardan.

jensen
#6 Oct 7, 2005, 2:45 am

yes i am agree with pooyan in everything and i will think about this problem either and i'll post my solution as soon as i can(my powerful solution)
and another more thing,Ashegh wher do you get these beautiful problems?

Ashegh
#7 Oct 8, 2005, 10:22 pm

you know it better jensen.from u teacher Mr.vakily

livetolove212

#8 Apr 6, 2009, 3:25 pm

> lomos_lupin wrote:
> Ashegh ,you should post this in the Proposed problems ,cause our teacher has solved it for us like this:
> Solution

I don't know what is radical axis? Can you explain for me?
Here is my solution:
Let H be the intersection of OE and CD (O is the center of (C)), (Q,QE) intersects (O) at A' and B'
We have $OA'^2 = OB'^2 = OD^2 = OH.OE$ and H lies on (Q) so OB and OA are two tangents of (Q), it's equivalent to $\angle QB'O = \angle QA'O = 90^o$ or $A \equiv A', B \equiv B'$
Let J,K be the intersection of AB and CD, QO and AB
We wish to have $\angle QEP = \angle QHP = 90^o$
Because $PF//CD$ then $\angle QPK = \angle KJH = \angle KOH (1)$
On the other side, $QK.QO = QB^2 = QH^2$ or $\frac{QK}{QH} = \frac{QH}{QO}$
Hence ΔOQH~ΔHQK then $\angle QOH = \angle QHK (2)$
From (1) and (2) we get $\angle QHK = \angle QPK$ therefore QKHP is cyclic
So $\angle QHP = \angle QKP = 90^o$

Luis González

#9 Apr 7, 2009, 6:01 am

FG is radical axis of the said circle (O) and degenerate circle $E \Longrightarrow PE^2 = PB \cdot PA$ and $QE = QB = QA.$ But power of P to the circumference with center Q passing through A, B is $PB \cdot PA = PQ^2 - QA^2 = PQ^2 - QE^2 \Longrightarrow EQ^2 = PQ^2 - PE^2$ and the conclusion follows.

Remark: If $M \equiv AB \cap CD,$ then we have $OM \parallel PE.$

Quick Reply

High School Olympiads

ratio of areas

Reply

Source: Plane Euclidean Geometry - Ex2c - Q12

maliksaim

#1 Dec 26, 2008, 2:03 pm

Given triangle ABC, points P, Q, R are taken on BC, CA, AB so that

$$BP:PC = CQ:QA = AR:RB = m:n$$

Prove that,

$$area(PQR):area(ABC) = m^3+n^3:(m+n)^3$$

mihai miculita

#2 Dec 26, 2008, 4:56 pm

1). If barycentric coordinates of $P(0;m;n);Q(p;0;q);R(s;t;0);m+n=p+q=s+t=1$, then:
$\frac{area(PQR)}{area(ABC)}=\begin{vmatrix}0 & m & n\\ p & 0 & q\\ s & t & 0\end{vmatrix}.$
2). If $\frac{|PB|}{|PC|}=\frac{|QC|}{|QA|}=\frac{|RA|}{|RB|}=\frac{m}{n}\Rightarrow P(0;\frac{n}{m+n};\frac{m}{m+n});\dots$

mihai miculita

#3 Dec 26, 2008, 5:15 pm

1). If barycentric coordinates of $P(0;m;n);Q(p;0;q);R(s;t;0);m+n=p+q=s+t=1$, then:
$\frac{area(PQR)}{area(ABC)}=\begin{vmatrix}0 & m & n\\ p & 0 & q\\ s & t & 0\end{vmatrix}.$
2). If $\frac{|PB|}{|PC|}=\frac{|QC|}{|QA|}=\frac{|RA|}{|RB|}=\frac{m}{n}\Rightarrow P(0;\frac{n}{m+n};\frac{m}{m+n});\dots$

maliksaim

#4 Dec 26, 2008, 5:30 pm

i'm not familiar with barycentric coordinates... could u please prove using elementary similarity theorems...

maliksaim

#5 Dec 27, 2008, 4:50 pm

hello... anybody?????

the problem is actually posed in the second chapter of the book which is about similar figures so it is actually supposed to be proved using only elementary similarity theorems.....

sunken rock

#6 Apr 6, 2009, 10:48 pm

See that S(ARQ)/S(ABC) = S(BPR)/S(ABC) = S(CPQ)/S(ABC) = mn/[(m + n)^2], then

S(ARQ) + S(BPR) + S(CPQ) + S(PQR) = S(ABC), hence
3mn/[(m + n)^2] + S(PQR)/S(ABC) = 1, i.e. S(PQR)/S(ABC) = (m^2 + n^2 - mn)/[(m + n)^2] = (m^3 + n^3)/[(m + n)^3].

I used S(ARQ)/S(ABC) = AR.AQ.sin(<BAC)/AB.AC.sin(<BAC) and x^3 + y^3 = (x + y)(x^2 + y^2 - xy)

Here S(XYZ) means the area of the triangle XYZ.

To prove that the triangles ARQ, BPR and CPQ have the same area:
Draw RR' || BC, R' on AC, QQ' || AB, Q on BC, see that BP = CQ', CQ = AR' and RQ' || AC, thence the triangles BPR and CQ'R' have the same area, but the triangles CQ'R' and AQQ' have the same area as well, so for AQQ' and ARQ.
Similarly for S(ARQ) = S(CPQ).

Best regards,
sunken rock

Virgil Nicula
#7 Apr 6, 2009, 11:33 pm

Lemma. Let ABC be a triangle. Consider the points $\left\| \begin{array}{ccc} M\in (BC) & , & \frac{MB}{MC}=m \\ N\in (CA) & , & \frac{NC}{NA}=n \\ P\in (AB) & , & \frac{NA}{NB}=n \end{array}\right\|$.

Then the area $[MNP]=\frac{1+mnp}{(1+m)(1+n)(1+p)}\cdot S$, where $S=[ABC]$.

Proof. $\left\| \begin{array}{c} \frac{MB}{m}=\frac{MC}{1}=\frac{BC}{1+m} \\ \frac{NC}{n}=\frac{NA}{1}=\frac{CA}{1+n} \\ \frac{PA}{p}=\frac{PB}{1}=\frac{AB}{1+p} \end{array}\right.$. Observe that $\frac{[MNP]}{[ABC]}=\frac{[ABC]-[APN]-[BMP]-[CNM]}{[ABC]}=$

$$1-\frac{AP}{AB}\cdot\frac{AN}{AC}-\frac{BM}{BC}\cdot\frac{BP}{BA}-\frac{CN}{CA}\cdot\frac{CM}{CB}=1-\frac{p}{1+p}\cdot\frac{1}{1+n}-\frac{m}{1+m}\cdot\frac{1}{1+p}-\frac{n}{1+n}\cdot\frac{1}{1+m}=$$

$$\frac{(1+m)(1+n)(1+p)-p(1+m)-m(1+n)-n(1+p)}{(1+m)(1+n)(1+p)}=\frac{1+mnp}{(1+m)(1+n)(1+p)}.$$

Remark. I used often that if $X\in AB$ (sideline !) and $Y\in AC$ (sideline !), then $\frac{[AXY]}{[ABC]}=\frac{AX}{AB}\cdot\frac{AY}{AC}$.

Particular case. $m=n=p=\frac{r}{s}\implies\frac{[MNP]}{[ABC]}=\frac{1+m^3}{(1+m)^3}=\frac{r^3+s^3}{(r+s)^3}$ (the proposed problem).

Otherwise, $\frac{MB}{MC}=\frac{NC}{NA}=\frac{PA}{PB}=\frac{r}{s}\implies [APN]=[BMP]=[CNM]=\frac{rs}{(r+s)^2}\cdot [ABC]\implies$

$$\frac{[MNP]}{[ABC]}=1-\frac{3rs}{(r+s)^2}=\frac{r^2+s^2-rs}{(r+s)^2}\implies [MNP]=\frac{r^3+s^3}{(r+s)^3}\cdot [ABC].$$

This post has been edited 1 time. Last edited by Virgil Nicula, Apr 7, 2009, 12:23 am

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pohoatza
#8 Apr 7, 2009, 12:18 am

Here's a nice (actually immediate) application of this formula: Let D, E, F be points lying on the sides BC, CA, and AB, respectively (of a given triangle ABC). Let M, N, P be the midpoints of the segments AD, BE, CF, respectively. Then, $[DEF]=4[MNP]$.

Try to give a synthetic proof for the above fact :). It's quite nice.

Virgil Nicula

#9 Apr 7, 2009, 2:23 am

> *Virgil Nicula wrote:*
>
> $$\triangle\ ABC\ \wedge\ \boxed{\begin{array}{ccc} D\in (BC)\ ;\ \frac{DB}{DC}=m & & X\in (BC)\ ;\ \frac{XB}{XC}=x & & M\in AD\cap YZ \\ E\in (CA)\ ;\ \frac{EC}{EA}=n & \wedge & Y\in (CA)\ ;\ \frac{YC}{YA}=y & \wedge & N\in BE\cap ZX \\ F\in (AB)\ ;\ \frac{FA}{FB}=p & & Z\in (AB)\ ;\ \frac{ZA}{ZB}=z & & P\in CF\cap XY \end{array}}\implies$$
>
> $$\boxed{[MNP]=\frac{(1+xyz)(1+mnpxyz)}{[mz(1+y)+(1+z)]\cdot [nx(1+z)+(1+x)]\cdot [py(1+x)+(1+y)]}\cdot [ABC]}\ .$$
>
> **Particular case.** $x=y=z=1\implies 4\cdot [MNP]=[DEF]$ (the **Pohoatza**'s problem).

Proof. Apply the **Routh**'s relation :

$$\left\|\begin{array}{ccccc} M\in YZ\cap AD & \implies & \frac{MY}{MZ}=\frac{DC}{DB}\cdot\frac{AY}{AZ}\cdot\frac{AB}{AC} & \implies & u=\frac{MY}{MZ}=\frac{1+z}{mz(1+y)} \\ N\in ZX\cap BE & \implies & \frac{NZ}{NX}=\frac{EA}{EC}\cdot\frac{BZ}{BX}\cdot\frac{BC}{BA} & \implies & v=\frac{NZ}{NX}=\frac{1+x}{nx(1+z)} \\ P\in XY\cap CF & \implies & \frac{PX}{PY}=\frac{FB}{FA}\cdot\frac{CX}{CY}\cdot\frac{CA}{CB} & \implies & w=\frac{PX}{PZ}=\frac{1+y}{py(1+x)} \end{array}\right\|\implies uvw=\frac{1}{mnpxyz}\text{ and}$$

$$[MNP]=\frac{1+uvw}{(1+u)(1+v)(1+w)}\cdot [XYZ]=\frac{1+uvw}{(1+u)(1+v)(1+w)}\cdot\frac{1+xyz}{(1+x)(1+y)(1+z)}\cdot [ABC]\implies$$

$$[MNP]=\frac{1+\frac{1}{mnpxyz}}{\frac{[mz(1+y)+(1+z)]\cdot [nx(1+z)+(1+x)]\cdot [py(1+x)+(1+y)]}{mnpxyz(1+x)(1+y)(1+z)}}\cdot\frac{1+xyz}{(1+x)(1+y)(1+z)}\implies$$

$$[MNP]=\frac{(1+xyz)(1+mnpxyz)}{[mz(1+y)+(1+z)]\cdot [nx(1+z)+(1+x)]\cdot [py(1+x)+(1+y)]}\cdot [ABC]\ .$$

This post has been edited 4 times. Last edited by Virgil Nicula, Apr 7, 2009, 10:07 am

Luis González

#10 Apr 7, 2009, 2:50 am

Using Routh theorem for three equal ratios $\frac{BP}{PC}=\frac{CQ}{QA}=\frac{AR}{RB}=\frac{m}{n}$ yields

$$\frac{|\triangle PQR|}{|\triangle ABC|}=\frac{1+\frac{m}{n}\cdot\frac{m}{n}\cdot\frac{m}{n}}{(1+\frac{m}{n})(1+\frac{m}{n})(1+\frac{m}{n})}=\frac{m^3+n^3}{(m+n)^3}$$

Quick Reply

High School Olympiads

Point inside triangle

Reply

Source: Cut equal segments

armpist
#1 Jan 31, 2008, 3:40 am

Construct a point inside a triangle such that cevians of it cut on the sides equal disjoint segments.

T.Y.
M.T.

armpist
#2 Apr 6, 2009, 2:05 am

Anyone?

M.T.

Luis González
#3 Apr 6, 2009, 2:20 am

Do you mean the points of equal cevians?. i.e the foci of the Steiner circumellipse

http://www.xtec.cat/~qcastell/ttw/ttweng/definicions/d_Bickart_p.html
http://www.xtec.cat/~qcastell/ttw/ttweng/resultats/r163.html

armpist
#4 Apr 6, 2009, 3:20 am

No. The disjoined parts of the sides have equal lengths.The cevians are AX,BY,CZ.

XC = YA = ZB (or the other).

T.Y.
M.T.

Luis González
#5 Apr 6, 2009, 3:34 am

Let $\overline{XC} = \overline{YA} = \overline{ZB} = m$. Then by Ceva's theorem we have

$$\frac{\overline{BX} \cdot \overline{CY} \cdot \overline{AZ}}{m^3} = 1 \Longrightarrow m^3 = (a-m)(b-m)(c-m) \Longrightarrow$$

$2m^3 - m^2(a+b+c) + m(ab+bc+ca) - abc = 0 \Longrightarrow m$ is not constructible.

Quick Reply

High School Olympiads

Very nice and interesting

Reply

Source: By Zhonghao Ye, a distinguished Chinese professor

Fang-jh

#1 Mar 25, 2009, 4:11 pm • 1

Prove that the nine-point center of a given triangle is equidistant from the circumcenter and the orthocenter of the triangle formed by arbitrary three Simson lines of the given triangle .

yetti

#2 Apr 5, 2009, 1:34 am • 1

H, O, N are orthocenter, circumcenter and 9-point circle center of a $\triangle ABC.$ Let $P, Q, R \in (O)$ be poles of 3 arbitrary Simson lines x, y, z intersecting at $X \equiv y \cap z, Y \equiv z \cap x, Z \equiv x \cap y.$ Let H_S, O_S, N_S be orthocenter, circumcenter and 9-point circle center of the $\triangle XYZ.$ Let K, L, M be midpoints of $HP, HQ, HR.$ As $(N), (O)$ are similar with similarity center H and coefficient $\frac{1}{2}$ and $\angle RPQ = \frac{1}{2}\angle ROQ = \angle(z, y) = \angle ZXY,$ etc., (N) is circumcircle of $\triangle KLM$ and $\triangle KLM \sim \triangle PQR \sim \triangle XYZ$ are directly similar. But $K \in x \equiv YZ, L \in y \equiv ZX, M \in z \equiv XY,$ which means that $\triangle KLM$ is obtained from the pedal triangle $\triangle X'Y'Z' \sim \triangle XYZ$ of $\triangle XYZ$ WRT $O_S,$ i.e., from its medial triangle, by a spiral similarity with center $O_S. O_S$ is therefore the common orthocenter of the $\triangle X'Y'Z', \triangle KLM,$ while N_S, N are their respective circumcenters. Consequently, $\triangle O_SN_SN \sim \triangle O_SX'K$ are spirally similar $\Longrightarrow$ the angle $\angle O_SN_SN = \angle O_SX'K$ is right, so that N is on the perpendicular bisector of O_SH_S through $N_S.$

PDatK40SP

#3 Apr 5, 2009, 5:51 pm

Very nice problem and solution :) There's a nice (and easy by angle chase) result:
Let M, N, P are arbitrary poiints in the circumcircle of $\triangle ABC.$ Let d_M is the line pass through M and parallel to the Simson line of M wrt $\triangle ABC.$ d_M intersects (ABC) at $M'.$ Define d_N, d_P, N', P' similarly. Let $X = d_N \cap d_P, Y = d_P \cap d_M, Z = d_M \cap d_N,$ and T, U are circumcenter and orthocenter of $\triangle XYZ.$ Then: $T = (XNP) \cap (YPM) \cap (ZMN)$ and $U = (XN'P') \cap (YP'M') \cap (ZM'N')$
This result can lead to the initial problem, but it isn't a nice solution.

armpist

#4 Apr 6, 2009, 12:50 am

Dear friends,

M.T.

Historical note:

I suspect I. Newton new about this property, for he knew mechanics very well.

L. Euler knew this set-up for sure, he knew too much about his circle and even more

about his line.

This post has been edited 1 time. Last edited by armpist, Dec 16, 2013, 4:52 pm

Luis González

#5 Apr 6, 2009, 1:17 am

We'll use two well-known wesults

1) Directed angle between two Simson lines WRT $\triangle ABC$ with poles P,Q equals half of the arc PQ of the circumcircle.

2) Circumcenters of the set of similar triangles $\triangle A'B'C' \sim \triangle ABC$ with A',B',C' on BC,CA,AB lie on the perpendicular bisector of the segment connecting the orthocenter and circumcenter of $\triangle ABC.$

Let P,Q,R be the poles of the Simson lines p,q,r WRT $\triangle ABC.$ From (1) we deduce that $\triangle XYZ$ bounded by p,q,r is similar to $\triangle PQR.$ Sidelines of $\triangle XYZ$ bisect the segments OP,OR,OQ at $M,N,L,$ since a Simson line is equidistant from its pole and the orthocenter of $\triangle ABC.$ Therefore, $\triangle MNL$ is homothetic to $\triangle PQR$ through the homothety $(H,\frac{1}{2}) \Longrightarrow$ 9-point center of $\triangle ABC$ is circumcenter of $\triangle MNL.$ Together with $\triangle XYZ \sim \triangle MNL,$ the conclusion follows from assertion (2).

Quick Reply

High School Olympiads

Reflections of a point M.

Reply

Source: Old / new ?

PDatK40SP

#1 Apr 5, 2009, 5:03 pm • 1

I've found this problem accidentally, but I don't know it's old or new. Could somebody tell me if it was proposed before ?
Problem: Given a triangle $\triangle ABC$. Let M_1, M_2, M_3 be the reflections of M about BC, CA, AB. Assume that AM_1, BM_2, CM_3 concur at a point N. Then M, N and the isogonal conjugate of M with respect to $\triangle ABC$ lie on a line d which is parallel to the Euler line of $\triangle ABC$

Luis González

#2 Apr 5, 2009, 9:36 pm

AM_1, BM_2 and CM_3 concur $\Longleftrightarrow M$ lies on the Neuberg cubic $\mathcal{N}$ of $\triangle ABC.$ By Sondat's theorem for the orthologic triangles $\triangle ABC$ and $\triangle M_1M_2M_3,$ we conclude that line $d \equiv MN$ passes through the isogonal conjugate of M WRT $\triangle ABC.$ To prove $d \parallel OH,$ let us use barycentric coordinates WRT $\triangle ABC.$

Equation of d passing through $M \equiv (u:v:w)$ and $M' \equiv \left(\frac{a^2}{u}:\frac{b^2}{v}:\frac{c^2}{w}\right)$ is:

$$d \equiv u(c^2v^2-b^2w^2)x+v(a^2w^2-c^2u^2)y+w(b^2u^2-a^2v^2)z=0$$

d is parallel to the Euler line of $\triangle ABC \Longleftrightarrow d$ goes through its infinity point X_{30}

$$(S_AS_B+S_AS_C-2S_BS_C : S_BS_C+S_BS_A-2S_CS_A : S_CS_A+S_CS_B-2S_AS_B)$$

Hence, setting $(u,v,w) \longrightarrow (x,y,z),$ locus $f(x,y,z)=0$ of M is

$\sum_{\text{cyclic}} (S_AS_B+S_AS_C-2S_AS_B)(c^2y^2-b^2z^2)x=0,$ i.e. the Neuberg cubic $\mathcal{N}.$

PDatK40SP

#3 Apr 5, 2009, 10:01 pm

Thank you :) Actually I don't know much about triangle cubics, but I've searched on Wolfram... So this and some problems I thought "mine" have been proposed for at least 50 years :D
And how about the problem MN parallel to the Euler line of $\triangle ABC$? I also found that the Euler line of $\triangle M_1M_2M_3$ passes through the circumcenter of $\triangle ABC$

Quick Reply

High School Olympiads

Perpendicular

Reply

Source: Chinese TST 2009 P1

Fang-jh

#1 Apr 4, 2009, 5:39 pm • 2

Let ABC be a triangle. Point D lies on its sideline BC such that $\angle CAD = \angle CBA.$ Circle (O) passing through B,D intersects AB,AD at E,F, respectively. BF meets DE at G.Denote by$M$ the midpoint of $AG.$ Show that $CM\perp AO.$

Luis González

#2 Apr 5, 2009, 12:59 pm • 2

Let $N \equiv AO \cap BC, V \equiv AO \cap EF$ and $Q \equiv EF \cap BC.$ Let T be projection of C on AN and $P \equiv AQ \cap CT.$ $\angle CAD = \angle CBA$ implies that $EF \parallel AC.$ Therefore, $\triangle ATC$ and $\triangle VZQ$ are similar (Z is the projection of Q on AN). By Menelaus' theorem for $\triangle ANQ$ cut by the transversal $\overline{TPC}$, we obtain:

$\frac{AP}{PQ} = \frac{CN}{CQ} \cdot \frac{TA}{TN}$. But $\frac{CN}{NQ} = \frac{AN}{AV}$, due to $VQ \parallel AC.$

On the other hand, $\frac{AN}{AV} = \frac{ZN}{ZV}$ (A,V,Z,N are harmonically separeted).

$\frac{AP}{PQ} = \frac{ZN}{ZV} \cdot \frac{TA}{TN}$. But $\frac{TA}{ZV} = \frac{TC}{ZQ} = \frac{TN}{ZN} \Longrightarrow AP = PQ$

$\Longrightarrow P$ is the midpoint of $AQ.$ Consequently, T is the midpoint of $AZ \Longrightarrow AT$ passes through G and the conclusion follows.

pohoatza

#3 Apr 7, 2009, 12:54 pm • 4

A more synthetic approach: Notice that from the hypothesis we know that CA is tangent to the circumcircle Γ of ABD. Consider the point A as a degenerated circle, and thus, since the radical axis of A, Γ and $(BDEF)$ are concurrent at the circles' radical center, we deduce that C is the radical center of the three circles. Thus, we are left to show that M lies on the radical axis of A and $(BDEF)$. But this is immediate, since G lies on the polar of A wrt. $(BDEF)$ and so, if we denote by X, Y the intersections of AG with $(BDEF)$, we have that $XA/XG = YA/YG$, which is equivalent with $MG^{2}= MA^{2}= MX\cdot MY$.
In conclusion, M lies on AO since MA^{2} is the power of M wrt. A and $MX\cdot MY$ the power of M wrt. $BDEF$.

hollandman

#4 Apr 8, 2009, 8:08 am • 1

It might be interesting to formulate the problem this way:

Given a cyclic quadrilateral $BDFE$. Rays BD and EF meet at R, and rays BE and DF meet at A. Let a line through A parallel to EF meet BD at C. Let FB and ED intersect at G, and GR and AC intersect at L. Show that $AC = CL$.

Would this inspire a different approach?

vittasko

#5 Apr 9, 2009, 2:44 am • 1

hollandman wrote:

> hollandman wrote:
>
> It might be interesting to formulate the problem this way:
>
> Given a cyclic quadrilateral $BDFE$. Rays BD and EF meet at R, and rays BE and DF meet at A. Let a line through A parallel to EF meet BD at C. Let FB and ED intersect at G, and GR and AC intersect at L. Show that $AC = CL$.
>
> Would this inspire a different approach?

The pencil $R.BAFG$, is in harmonic conjugation as well.

So, because of $AL \parallel RF$, we conclude that $AC = CL$, (1)

Similarly, we have that $AZ = AN$, (2) where Z is the point of intersection of the line segment BF, from the line through A and parallel to DE and $N \equiv GR \cap AZ.$

From $(1), (2) \Longrightarrow CZ \parallel GR$, (3)

But, it is well known that $OA \perp GR$, (4)

(The line connecting the circumcenter of a cyclic quadrilateral, with the point of intersection of its diagonals, is perpendicular to the line connecting the points of intersection of its opposite sidelines.)

Hence, from $(3), (4)$, we conclude that $OA \perp CM$ and the proof of the proposed problem is completed.

REFERENCE. - http://www.mathlinks.ro/Forum/viewtopic.php?t=110887

Kostas Vittas.

Attachments:
t=268931.pdf (7kb)

windrock
#6 Apr 19, 2009, 6:05 am • 1

Call the intersection of BC and EF is P, PG and AD, AC are X and T. We know that G is the orthorcenter of ΔABC, and $(AXDF) = -1$, $AT//PE$ hence $CA = CT$. But $PG \perp AO$, so $AO \perp CM$.
An old idea for the problem!

jayme
#7 Apr 19, 2009, 7:40 pm

Dear Fang-jh, Cosmin and Mathlinkers,
can some one draw the figure... I don't arrive to the conclusion which is asked.
Thank you
Sincerely
Jean-Louis

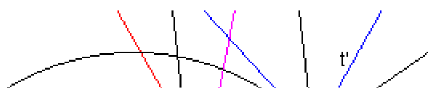
jayme
#8 Apr 20, 2009, 9:24 pm

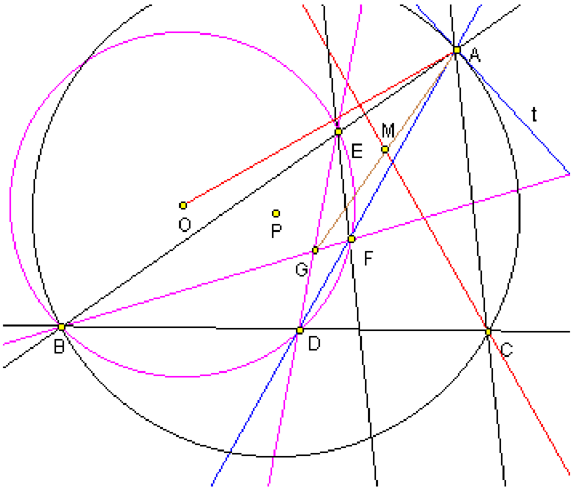
Dear Mathlinkers,
pleasr can some one draw the figure of the initial, problem...
Is there a typo in the text of the problem?
Sincerely
Jean-Louis

yetti
#9 Apr 20, 2009, 11:13 pm

If t is tangent to the circumcircle (P) of triangle ABC at A and t' reflection of t in AC, then t' cuts BC at D, such that <CAD = <ABC = -<CBA (not <CAD = <CBA).

Attachments:





jayme

#10 Apr 20, 2009, 11:59 pm

Dear Yetti,
now it is clear for me... because I was taking the synthetic result of Cosmin...
Thank you very much
Sincerely
Jean-Louis

mathVNpro

#11 Apr 25, 2009, 11:08 am

Here is my solution (using Inversion:D):
Let P be the intersection of EF and BC. Then, it is very well- known that G is the orthocenter of triangle OPA (This result can be proved by using **Pole-Polar Theory**). Let A_1, P_1, O_1 be the intersections of (AG, OP), (PG, OA), (OG, AP), respectively. Consider the inversion through pole O, power R^2, where R is the radius of (O), we get, $I(O,R) : B \mapsto B$, $D \mapsto D$, $E \mapsto E$, $F \mapsto F$. Therefore, $I(O,R) : BD \mapsto (OBC)$. Hence, $P \mapsto A_1$ and $A_1 \in (OBC)$. Now, it is easy to notice that $\overline{PD}.\overline{PB} = \overline{PA_1}.\overline{PO} = \overline{PG}.\overline{PP_1}$, which implies BP_1GD is concyclic quadrilateral, let (I) be the circumcircle of BP_1GD. $I(O,R) : G \mapsto O_1$, $P_1 \mapsto A$, $B \mapsto B$, $D \mapsto D$. Therefore, $I(O,R) : (I) \mapsto (ABD)$. In the other hand, $I(O,R) : C \mapsto C'$, then $C' \in (OBC)$, hence, $I(O,R) : CP_1 \mapsto (OAC')$. Now, it is easy to notice that CA is the tangent of (ABD), which implies that $CA^2 = \overline{CD}.\overline{CB} = \overline{CC'}.\overline{CO}$, which also implies that CA is also tangent of (AOC'). Therefore, (ADB) internally tangents (AOC'). Therefore, CP_1 is also a tangent of (I). Therefore, $CP_1^2 = \overline{CD}.\overline{CB} = CA^2$, hence $CP_1 = CA$, which implies that C lies on the bisector of AP_1, M also lies on the bisector of AP_1 (Because M is considered the circumcircle of (AP_1G)). Therefore, CM is the bisector of AP_1, hence, $CM \perp AO$.
Our proof is completed 😁

vladimir92

#12 Aug 24, 2010, 11:15 am

Sorry to revive this old thread but I have an alternative proof to this wonderfull problem. In my proof, Q is the midpoint of AG.
[geogebra]623d3e5ef774564f959c2dd8f96536adbf98793b[/geogebra]
My solution

TheReds

#13 Dec 28, 2010, 5:02 am

> *Fang-jh wrote:*
> Let ABC be a triangle. Point D lies on its sideline BC such that $\angle CAD = \angle CBA.$ Circle (O) passing through B, D intersects AB, AD at E, F, respectively. BF meets DE at G.Denote by$M$ the midpoint of $AG.$ Show that $CM \perp AO.$

When D **lies on its sideline** BC then D lies on between B and C or not ?

TheReds

#14 Dec 29, 2010, 12:41 pm

Please reply me ! I isn't good at English.

AnonymousBunny

#15 Oct 7, 2014, 1:30 pm

Solution

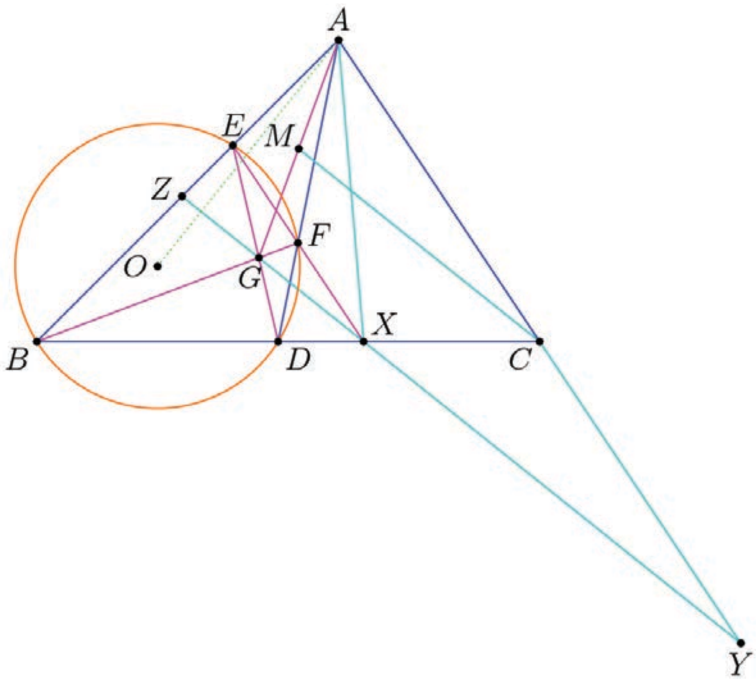
anantmudgal09

#16 Sep 13, 2015, 9:30 am

Nice problem.
By **Brokard's Theorem**, and angle chasing it suffices to showing that if AG meets EF, BC at P, T respectively then observe that $(A, G; P, T) = -1$ and so $TP.TM = TG.TA$ and the angle condition means EF parallel to CA and we want CM parallel to HG. Now this is trivial by converting these angles into ratios.

Ankoganit

#17 May 20, 2016, 11:18 am



Suppose $EF \cap BC = X, XG \cap AC = Y, XG \cap AB = Z$. Note that $\angle FAC = \angle DAC = \angle ABC = \angle EBD = \angle EFA$, so that $FE||AC$; let P_∞ be the point at infinity in this direction.
In the triangle XBE, XZ, ED, BF are concurrent cevians and $DF \cap BE = A$, so we have $(E, B; A, Z) - 1 \implies X(E, B; A, Z) = -1$. Intersecting this pencil with AC we have $(A, Y; C, P_\infty) = -1 \implies C$ is the midpoint of AY. But M is the midpoint of AG, so $CM||YG \implies CM||XG...(\star)$. But applying BROKARD'S THEOREM to the cyclic quadrilateral $DFEB$, we XG is the polar of A w.r.t. $(O) \implies XG \perp AO$. Combining this with $(\star)$, we arrive at $CM \perp AO$, as desired. $\blacksquare$

This post has been edited 2 times. Last edited by Ankoganit, May 20, 2016, 11:19 am

Quick Reply

High School Olympiads

cyclic incenter

Reply

hoangclub

#1 Apr 4, 2009, 5:48 am

Let $ABCD$ be a quadrilateral and (I) be the incircle of it.Let E be the intersection of two diagonals AC and BD.Let M, N, P, Q be the incenter of triangleEAB, EBC, ECD, ED respectively.Prove that M, N, P, Q are cyclic.

Luis González

#2 Apr 4, 2009, 11:52 pm

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=21758
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=64627
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=199861
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=228291

Quick Reply

High School Olympiads

Concurrence in a triangle.

Reply

Source: Own.

Virgil Nicula

#1 Feb 27, 2009, 2:36 pm

Let $\triangle\ ABC$ and a point P for which $\left|\begin{array}{c} D\in AP\ \cap BC\\ E\in BP\ \cap\ CA\\ F\in CP\ \cap\ AB\end{array}\right|$. Construct the parallelograms

$PEXF$, $PFYD$, $PDZE$. Prove that the lines AX, BY si CZ are concurrently.

vittasko

#2 Feb 28, 2009, 3:27 am

Click to reveal hidden text

Kostas Vittas.

No Reason

#3 Feb 28, 2009, 10:05 pm

My solution (similar to vittasko's hint)

Denote the points like my figure.

We have:

$\frac{AM}{ME}=\frac{AF}{FB}$

$\frac{AN}{NF}=\frac{AE}{EC}$

Apply Ceva's Theorem to triangles AFE and ABC,we have:

$\frac{MA}{ME}\cdot\frac{NF}{NA}\cdot\frac{GE}{GF}=1$

$\frac{FA}{FB}\cdot\frac{EC}{EA}\cdot\frac{DB}{DC}=1$

So: $\frac{GE}{GF}=\frac{DB}{DC}$

Similar:

$\frac{HF}{HD}=\frac{EC}{EA}$

$\frac{ID}{IE}=\frac{FA}{FB}$

So:

$\frac{GE}{GF}\cdot\frac{HF}{HD}\cdot\frac{ID}{IE}=1$

According to Ceva's Theorem,we see that DG,EH,FI are concurrent.

Because AD,BE,CF are concurrent,then apply the Ceva Nest Theorem,we see that: AG,BH,CI are concurrent.

Q.E.D

Image not found[/img]

Luis González

#4 Apr 1, 2009, 10:40 pm

Let us use barycentric coordinates WRT $\triangle ABC$. Then $P\equiv (u:v:w)$

$D \equiv (0 : v : w)$, $E \equiv (u : 0 : w)$, $F \equiv (u : v : 0)$

$AD \equiv vz - wy = 0$, $BE \equiv wx - uz = 0$, $CF \equiv uy - vx = 0$

Line ℓ_1 passing through E and parallel to CF has equation

$\ell_1 \equiv uvz + (uw + u(u+v))y - vwx = 0$

Line ℓ_2 passing through F and parallel to BE has equation

$\ell_2 \equiv uwy + (uv + u(u+w))z - vwx = 0$

ℓ_1 and ℓ_2 meet at the point $X(*** : uvw(u+w) : uvw(u+v))$

Hence the foot X_0 of the A-cevian of X is $X_0 \equiv (0 : u+v : u+v)$

Likewise, $Y_0 \equiv (v+w : 0 : v+u)$, $Z_0 \equiv (w+v : w+u : 0)$

AX, BY, CZ concur at the complement $Q \equiv (w+v : u+w : u+v)$ of P.

Virgil Nicula

#5 Apr 2, 2009, 12:24 am

Thanks to all ! Here is a similar nice problem (more easily !) **with very many applications** :

> Virgil Nicula wrote:
>
> Let $\triangle\ ABC$ and a point P for which denote the points $\left|\begin{array}{ccc} D\in BC & ; & PD\perp BC\\ E\in\ CA & ; & PE\perp CA\\ F\in AB & ; & PF\perp AB\end{array}\right|$. Construct
>
> the parallelograms $PEXF$, $PFYD$, $PDZE$. Prove that the lines AX , BY si CZ are concurrently.

Luis González

#6 Apr 4, 2009, 8:36 am

> Quote:
>
> Let $\triangle\ ABC$ and a point P for which denote the points $\left|\begin{array}{ccc} D\in BC & ; & PD\perp BC\\ E\in\ CA & ; & PE\perp CA\\ F\in AB & ; & PF\perp AB\end{array}\right|$. Construct
>
> the parallelograms $PEXF$, $PFYD$, $PDZE$. Prove that the lines AX , BY si CZ are concurrently.

AX,BY,CZ concur at the isogonal conjugate of P WRT ABC.

Quick Reply

High School Olympiads

Geometric Inequality

Reply

apollo

#1 Apr 3, 2009, 6:33 pm

Let $ABCD$ be a convex quadrilateral, Prove that $AB^2+BC^2+CD^2+DA^2\geq AC^2+BD^2$

sunken rock

#2 Apr 3, 2009, 7:30 pm

There is a formula, due to Newton, if I am not mistaken, stating:

Sum of squares of sides of a quadrilateral equals sum of squares of main diagonals plus 4 times the square of the segment joining the midpoints of these diagonals

Hence the inequality is trivial.

Best regards
sunken rock

Luis González

#3 Apr 3, 2009, 7:53 pm

Let X,Y,Z,W be the midpoints of AB,BC,CD,DA and M,N the midpoints of $AC,BD.$ It is easy to se that $XYZW,$ $XMZN,WMYN$ are parallelograms.

$2(XN^2+NZ^2)=XZ^2+MN^2\ ,\ (WM^2+MY^2)=WY^2+MN^2$

$2(XN^2+NZ^2+WM^2+MY^2)=XZ^2+2MN^2+WY^2$

$XZ^2+WY^2=2(WX^2+WY^2)$

$\Longrightarrow \ XN^2+NZ^2+WM^2+MY^2=WX^2+WY^2+MN^2.$

Substituting $XN=\frac{1}{2}BC\ ,\ NZ=\frac{1}{2}DA\ ,\ WM=\frac{1}{2}AB\ ,\ MY=\frac{1}{2}CD$

$\Longrightarrow \ AB^2+BC^2+CD^2+DA^2=AC^2+BD^2+4MN^2$

quykhtn-qa1

#4 Apr 3, 2009, 10:23 pm

Prove that: $AB+BC+CD+DA\geq AC+BD+2MN$.Try it

Quick Reply

High School Olympiads

Vecten's figure2

Reply

Source: own?

jayme

#1 Apr 2, 2009, 10:41 am

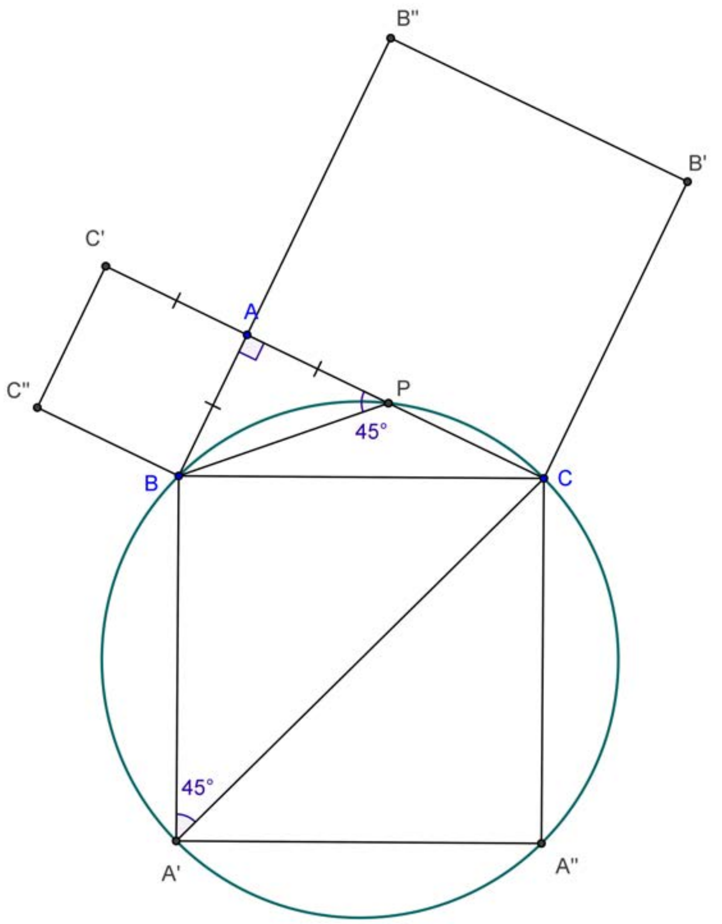
Dear Mathlinkers,
Let ABC be a rectangular triangle at A,
CB'B"A, AC'C"B, BA'A"C three squares erected externally on the sides of ABC
Prove that the symetric of C' and B" wrt A is on the circumcircle of BA'A"C.
Any reference?
Sincerely
Jean-Louis

Luis González

#2 Apr 2, 2009, 11:19 am

See the diagram below for a PWW (proof without words).

Attachments:



Virgil Nicula

#3 Apr 3, 2009, 7:21 am

> jayme wrote:
>
> Let ABC be a rectangular triangle at A, $CB'B"A$, $AC'C"B$, $BA'A"C$ three squares erected externally on the sides of ABC . Prove that the symetric F, E of C' and $B"$ wrt A is on the circumcircle of $BA'A"C$.

The quadrilateral $EBFC$ is an isosceles trapezoid ($BF \parallel EC$) with $m(\angle BEC) = 45^\circ$ and $m(\angle BFC) = 135^\circ$.

Quick Reply

High School Olympiads

A concurrence with very many applications..

Reply

Source: Own.

Virgil Nicula
#1 Apr 1, 2009, 7:04 am

Given are $\triangle ABC$ and $D\in (BC)$, $E\in (CA)$, $F\in (AB)$ so that $AD\cap BE\cap CF\ne\emptyset$. Consider

$X\in (EF)$, $Y\in (FD)$, $Z\in (DE)$. Prove that $AX\cap BY\cap CZ\ne\emptyset\iff DX\cap EY\cap FZ\ne\emptyset$.

Luis González
#2 Apr 1, 2009, 9:01 am

Project the perspectrix of $\triangle ABC, \triangle DEF$ to infinity and label projected points with primes. Thus, D, E, F go to the midpoints D', E', F' of $B'C'$, $C'A'$, $A'B'$ and X, Y, Z go to $X', Y'Z'$ on $E'F', F'D', D'E'$. Now, get rid of the primes. Since the lines AX, BY and CZ obviously divide the pairs of sides $(FE, BC), (DF, CA)$ and (ED, AB) into the same ratios, then it follows by the converse of Ceva's theorem that AX, BY, CZ concur $\Longleftrightarrow DX, EY, FZ$ concur.

jayme
#3 Apr 1, 2009, 11:03 am

Dear Virgil, Luis and Mathlinkers,
this is a nice formulation of the cevian nests theorem.
A synthetic proof of this result can be seen on my site :
http://perso.orange.fr/jl.ayme vol. 3
Sincerely
Jean-Louis

jayme
#4 Apr 1, 2009, 11:15 am

Dear Virgil and Mathlinkers,
some interesting applications can also be seen on my site :
http://perso.orange.fr/jl.ayme
vol. 3 The cross-cevian point
Produit et quotient cévien de deux points
Sincerely
Jean-Louis

sunken rock
#5 Apr 1, 2009, 12:29 pm

One can use the following **Lemma**:

Let D be a point on BC; a straight line l intersects AB, AC and AD at M, N and P respectively. The following relation occurs: (AB/AC).(AN/AM) = (BD/CD).(NP/MP)

The lemma can easily be proven: draw parallels to BC through M and N; they will intersect AD at M' and N', etc.
As it can be seen, it's not restrictive [i.e. D on (BC), etc.].

It appeared in Romanian **Gazeta Matematica** well before 1980. Is it called now "Transversal Theorem"?

Best regards,

sunken rock

Virgil Nicula

#6 Apr 1, 2009, 7:28 pm

Thank you for your commentaries of this nice problem. I agree the elementary proof of **Sunkenrock**. Here is his proof.

Lemma. Let ABC be a triangle. Then $D\in (BC)$, $E\in (CA)$, $F\in (AB)$ and $X\in AD\cap EF\implies$ $\frac{XF}{XE}=\frac{DB}{DC}\cdot\frac{AF}{AE}\cdot\frac{AC}{AB}$.

Proof.

> *Virgil Nicula wrote:*
> Given are $\triangle ABC$ and $D\in (BC)$, $E\in (CA)$, $F\in (AB)$ so that $AD\cap BE\cap CF\ne\emptyset$. Consider
> $X\in (EF)$, $Y\in (FD)$, $Z\in (DE)$. Prove that $AX\cap BY\cap CZ\ne\emptyset\iff DX\cap EY\cap FZ\ne\emptyset$.

Proof. Apply the **Ceva**'s theorem to the point $S\in AD\cap BE\cap CF$ and $\triangle ABC\implies\frac{DB}{DC}\cdot\frac{EC}{EA}\cdot\frac{FA}{FB}=1$.

Denote $\left\|\begin{array}{c} M\in AX\cap BC\\ N\in BY\cap CA\\ P\in CZ\cap AB\end{array}\right\|$. Using the upper lemma obtain $\left\|\begin{array}{c} \frac{XF}{XE}=\frac{MB}{MC}\cdot\frac{AF}{AE}\cdot\frac{AC}{AB}\\ \frac{YD}{YF}=\frac{NC}{NA}\cdot\frac{BD}{BF}\cdot\frac{BA}{BC}\\ \frac{ZE}{ZD}=\frac{PA}{PB}\cdot\frac{CE}{CD}\cdot\frac{CB}{CA}\end{array}\right\|\ \bigodot\implies$

$\frac{XF}{XE}\cdot\frac{YD}{YF}\cdot\frac{ZE}{ZD}=\frac{MB}{MC}\cdot\frac{NC}{NA}\cdot\frac{PA}{PB}$. In conclusion, $AX\cap BY\cap CZ\ne\emptyset\iff$ $\frac{MB}{MC}\cdot\frac{NC}{NA}\cdot\frac{PA}{PB}=1\iff\frac{XF}{XE}\cdot\frac{YD}{YF}\cdot\frac{ZE}{ZD}=1\iff DX\cap EY\cap FZ\ne\emptyset$.

Rijul saini

#7 Jun 9, 2010, 2:01 am

> *Virgil Nicula wrote:*
> Given are $\triangle ABC$ and $D\in (BC)$, $E\in (CA)$, $F\in (AB)$ so that $AD\cap BE\cap CF\ne\emptyset$. Consider
> $X\in (EF)$, $Y\in (FD)$, $Z\in (DE)$. Prove that $AX\cap BY\cap CZ\ne\emptyset\iff DX\cap EY\cap FZ\ne\emptyset$.

See here http://www.artofproblemsolving.com/Forum/blog.php?u=50172&b=33721

Quick Reply

High School Olympiads

fast one

Source: imo 95 problem 1

v235711

#1 Apr 1, 2009, 5:49 am

Let A,B,C,D be distinct points on a line , in that order .The circles with diameters AC and BD intersect at X and Y . O is an arbitrary point on the line XY but not on AD. CO intersects the circle with diameter AC again at M , and BO intersects the other circle at N.Prove that the lines AM,DN ,XY are concurrent. :)

Luis González

#2 Apr 1, 2009, 7:01 am

Let H, H' be the orthocenters of $\triangle AOC$ and $\triangle DOB$, respectively. $H \equiv AM \cap XY$ and $H' \equiv DN \cap XY$. If $T \equiv XY \cap AD$, it follows that $OT \cdot HT = TC \cdot TA$ and $OT \cdot H'T = TB \cdot TD$. Therefore, we conclude that $H \equiv H'$, due to $TA \cdot TC = TB \cdot TD \Longrightarrow$ Lines XY, AM, DN concur.

High School Olympiads

Three tangent circles

Reply

juancarlos

#1 Dec 27, 2006, 3:56 am • 1

Let S be exsimilicenter of the excircles $(I_b), (I_c)$ of $ABC(AC > AB)$. Drawn the small circle (O) tangent to the excircles $(I_b), (I_c)$ and BC at T. If H and M are the foot of the altitude from A and the midpoint of BC respectively, prove that:
$\frac{1}{ST} = \frac{1}{TH} - \frac{1}{TM}$

yetti

#2 Dec 27, 2006, 10:49 pm • 1

Let $T_b,\ T_c$ be tangency points of $(I_a),\ (I_b)$ with BC, $r_b,\ r_c$ their radii, and s the triangle semiperimeter. Since $T_bB = CT_c = s - a,$ M is also the midpoint of T_bT_c. A is the internal similarity center of $(I_b),\ (I_c),$ the intersection of their common internal tangents.

$$T_bT_c = T_bC + CT_c = s - a + s = b + c$$

$$T_bM = \frac{T_bT_c}{2} = \frac{b+c}{2}$$

$$T_bH = T_bT_c \cdot \frac{r_b}{r_b + r_c} = (b+c) \cdot \frac{r_b}{r_b + r_c}$$

$$\frac{r_b}{r_c} = \frac{T_bS}{T_cS} = \frac{T_bT_c + T_cS}{T_cS} = \frac{b+c}{T_cS} + 1$$

$$\frac{r_c}{r_b} = \frac{T_cS}{T_bS} = \frac{T_bS - T_bT_c}{T_bS} = 1 - \frac{b+c}{T_bS}$$

$$T_cS = (b+c) \cdot \frac{r_c}{r_b - r_c},\quad T_bS = (b+c) \cdot \frac{r_b}{r_b - r_c}$$

Invert the 3 tangent circles $(I_b),\ (I_c),\ (O)$ in a circle (T) with arbitrary radius ρ and centered at the tangency point T of (O) with BC. The line BC passing through the inversion center T is carried into itself, the circle (O) tangent to BC at the inversion center T goes into a line $o \parallel BC$ and the excircles $(I_b),\ (I_c)$ into 2 congruent circles $(I'_b),\ (I'_c),$ both tangent to the parallel lines $o \parallel BC, r'_b = r'_c$ being thair radii and $T'_b,\ T'_c$ their tangency points with BC. The inversion center T is a similarity center of $(I'_b) \sim (I_b)$ and $(I'_c) \sim (I_c),$ with similarity coefficients equal to the power of inversion ρ^2 divided by the power of the inversion center T to the original circle, respectively.

$$\frac{r'_b}{r_b} = \frac{TT'_b}{TT_b} = \frac{\rho^2}{TT_b^2},\quad \frac{r'_c}{r_c} = \frac{TT'_c}{TT_c} = \frac{\rho^2}{TT_c^2},\quad r'_b = r'_c \implies \frac{TT_b}{TT_c} = \sqrt{\frac{r_b}{r_c}}$$

In addition, $TT_b + TT_c = T_bT_c = b + c.$ Calculating $TT_b,\ TT_c$ from their ratio and sum:

$$TT_b = (b+c) \cdot \frac{\sqrt{r_b}}{\sqrt{r_b} + \sqrt{r_c}},\quad TT_c = (b+c) \cdot \frac{\sqrt{r_c}}{\sqrt{r_b} + \sqrt{r_c}}$$

Calculating the segment lengths TS, TM, TH:

$$TS = T_bS - TT_b = (b+c)\left(\frac{r_b}{r_b - r_c} - \frac{\sqrt{r_b}}{\sqrt{r_b} + \sqrt{r_c}}\right) = (b+c) \cdot \frac{\sqrt{r_br_c}}{r_b - r_c}$$

$$TM = TT_b - T_bM = (b+c)\left(\frac{\sqrt{r_b}}{\sqrt{r_b} + \sqrt{r_c}} - \frac{1}{2}\right) = \frac{b+c}{2} \cdot \frac{\sqrt{r_b} - \sqrt{r_c}}{\sqrt{r_b} + \sqrt{r_c}}$$

$$TH = T_bH - TT_b = (b+c)\left(\frac{r_b}{r_b+r_c} - \frac{\sqrt{r_b}}{\sqrt{r_b}+\sqrt{r_c}}\right) =$$

$$= (b+c) \cdot \frac{\sqrt{r_b r_c}}{r_b+r_c} \cdot \frac{\sqrt{r_b}-\sqrt{r_c}}{\sqrt{r_b}+\sqrt{r_c}}$$

Subtracting the reciprocal segment lengths:

$$\frac{1}{TH} - \frac{1}{TM} = \frac{1}{b+c} \cdot \frac{\sqrt{r_b}+\sqrt{r_c}}{\sqrt{r_b}-\sqrt{r_c}} \cdot \left(\frac{r_b+r_c}{\sqrt{r_b r_c}} - 2\right) =$$

$$= \frac{1}{b+c} \cdot \frac{(\sqrt{r_b}+\sqrt{r_c})(\sqrt{r_b}-\sqrt{r_c})}{\sqrt{r_b r_c}} = \frac{1}{b+c} \cdot \frac{r_b - r_c}{\sqrt{r_b r_c}} = \frac{1}{TS}$$

juancarlos

#3 Jan 1, 2007, 7:58 pm

If X, Y are the touch point of the small circle with $(Ic), (Ib)$ respectively, now: $\angle HXT = \angle TYM$

Luis González

#4 Mar 31, 2009, 10:03 pm • 1

Exsimilicenter S of $(I_c) \sim (I_b)$ is also center of the positive inversion $\mathcal{I}$ taking $(I_c), (I_b)$ into each other. If P, Q denote the tangency points of (O) with (I_c) and (I_b), then PQ passes through S, since the set of circles externally/internally tangent to $(I_b), (I_c)$ transforms into itself under $\mathcal{I}$. The 9-point circle of $\triangle ABC$ belows to this set indeed

$$SQ \cdot SP = ST^2 = SM \cdot SH \Longrightarrow ST^2 = (ST+TM)(ST-TH)$$

$$ST^2 = ST^2 - ST \cdot TH + ST \cdot TM - TH \cdot TM$$

$$ST = \frac{TM \cdot TH}{TM - TH} \Longrightarrow \frac{1}{ST} = \frac{1}{TH} - \frac{1}{TM}$$

Quick Reply

High School Olympiads

Quadrilateral (relatively simple i suppose)

Reply

sebs0r

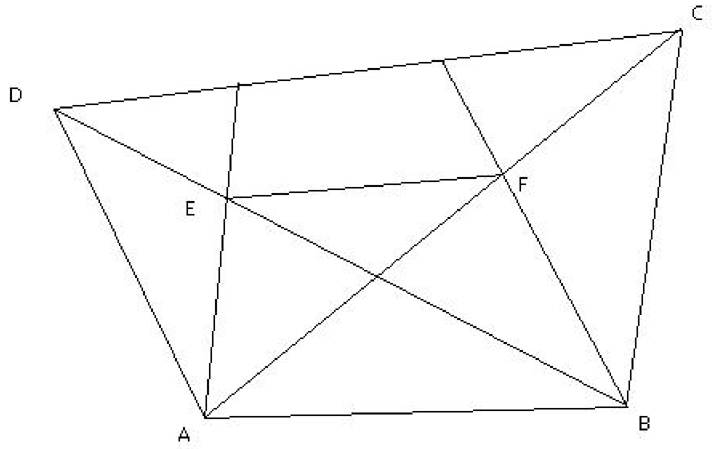
#1 Mar 31, 2009, 10:23 am

Hi folks, im having a little difficulty with this (seemingly) simple quadrilateral problem

ABCD is a quadrilateral with the angles DAB and ABC obtuse. AC and BD are diagonals meeting at O. Point E lies on BD such that line AE is parallel to BC. Point F lies on AC such that line BF is parallel to AD. Prove that EF is parallel to DC

Would really appreciate some help on this one.
Cheers,

Attachments:



Luis González

#2 Mar 31, 2009, 10:41 am

Very simple problem. Note that $\triangle BOF \sim \triangle EOA$ and $\triangle AEO \sim \triangle COB$ gives

$$\frac{OF}{OA} = \frac{OB}{OD}, \frac{OA}{OC} = \frac{OE}{OB} \Longrightarrow \frac{OD}{OE} = \frac{OC}{OF} \Longrightarrow EF \parallel DC.$$

sebs0r

#3 Mar 31, 2009, 2:25 pm

cheers buddy - I knew it was simple but didn't think it was THAT easy. Feel like a bit of a nob now. Anywho, thanks for the quick reply. Have a good rest of the day

jayme

#4 Mar 31, 2009, 3:48 pm

Dear Lym and Mathlinkers,
here is also an application of the Pappus 's theorem:
consider the hexagone AEFBCDA built on the lines AC and BD.
Qincerely
Jean-Louis

Quick Reply

High School Olympiads

What is the trajectory Of point P

Source: Own

lym

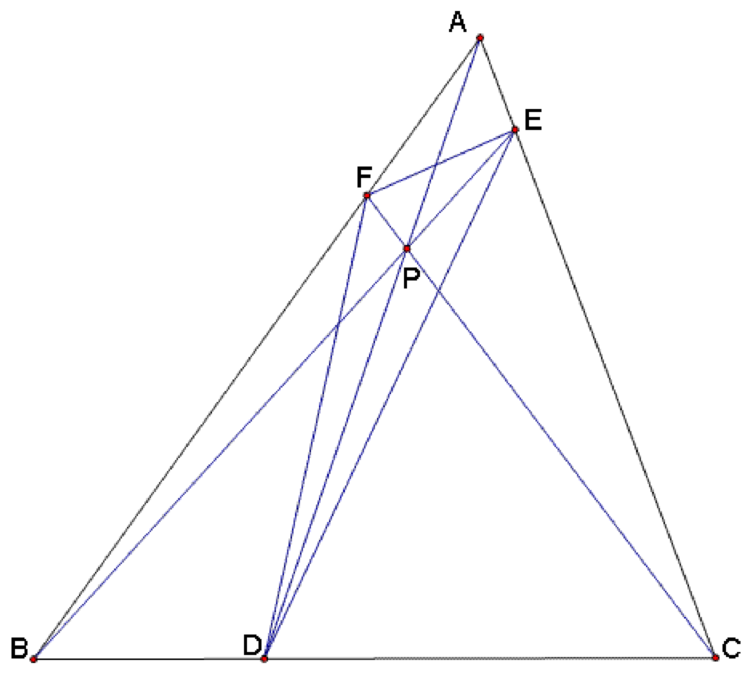
#1 Mar 29, 2009, 10:32 pm

Let ABC be an arbitrary triangle. The point P is inside of $\triangle ABC$.

$$\frac{[ADF]}{[CDF]}=M\ ,\ \frac{[ADE]}{[BDE]}=N\ ,\ \sqrt{M}+\sqrt{N}=1.$$

what is the trajectory of point P ?

Attachments:



resurrection

#2 Mar 30, 2009, 6:28 am

Sorry, but I don't understand what you mean by the "trajectory" of point P. Could you explain to me?

lym

#3 Mar 30, 2009, 7:08 am

it means “locus” and this locus is a conic, an elliptical arc.

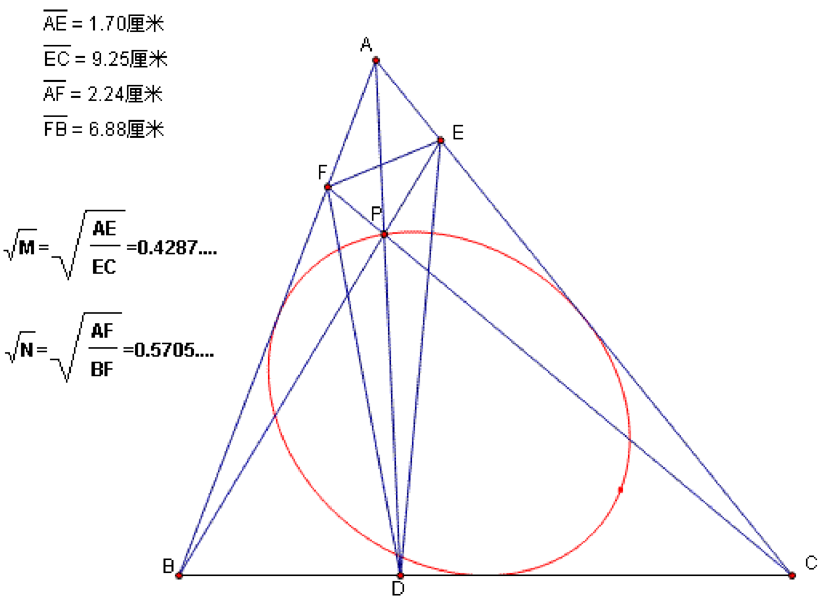
lym

#4 Mar 30, 2009, 8:30 am

This is the picture here

Attachments:

Attachments:



yetti

#5 Mar 30, 2009, 10:02 pm

Parallel project arbitrary P inside $\triangle ABC$ to triangle orthocenter. Label all projected points with primes. Area ratios are invariant.

$$\frac{S(\triangle A'D'F')}{S(\triangle A'P'B')} = \frac{A'D' \cdot A'F'}{A'P' \cdot A'B'} = \frac{C'D' \cdot C'F'}{C'P' \cdot C'B'} = \frac{S(\triangle C'D'F')}{S(\triangle C'P'B')} \Longrightarrow$$

$\dfrac{S(\triangle ADF)}{S(\triangle CDF)} = \dfrac{S(\triangle APB)}{S(\triangle CPB)}$ and similarly, $\dfrac{S(\triangle ADE)}{S(\triangle BDE)} = \dfrac{S(\triangle APC)}{S(\triangle BPC)}$.

Let K, L, M be midpoints of BC, CA, AB. Parallel project $\triangle ABC$ to equilateral triangle. Label all projected points with asterisks. Area ratios are again invariant. Let X, Y, Z be orthogonal projections of P^* to B^*C^*, C^*A^*, A^*B^*. Then

$\dfrac{S(\triangle A^*D^*F^*)}{S(\triangle C^*D^*F^*)} = \dfrac{P^*Z}{P^*X}$ and $\dfrac{S(\triangle A^*D^*E^*)}{S(\triangle B^*D^*E^*)} = \dfrac{P^*Y}{P^*X}$.

$\sqrt{P^*Z} + \sqrt{P^*Y} = \sqrt{P^*X} \iff P^*$ is on incircle arc L^*M^* of equilateral $\triangle A^*B^*C^*$ opposite to K^*. See http://www.mathlinks.ro/viewtopic.php?t=106446. P is therefore on Steiner inellipse arc LM of $\triangle ABC$ opposite to K.

Luis González

#6 Mar 30, 2009, 10:38 pm

Let $(x:y:z)$ be the barycentric coordinates of P with respect to $\triangle ABC$. Thus we consider oriented areas throughout the calculations.

$$D\ (0:y:z)\ \ ,\ \ E\ (x:0:z)\ \ ,\ \ F\ (x:y:0)$$

$$\frac{|\triangle ADF|}{|\triangle ABC|} = \frac{yz}{(x+y)(y+z)}\ \ ,\ \ \frac{|\triangle CDF|}{|\triangle ABC|} = \frac{xy}{(x+y)(y+z)}$$

$$M = \frac{|\triangle ADF|}{|\triangle CDF|} = \frac{z}{x}\ \ ,\ \ N = \frac{|\triangle ADE|}{|\triangle BDE|} = \frac{y}{x} \Longrightarrow$$

$$\sqrt{\frac{y}{x}} + \sqrt{\frac{z}{x}} = 1 \Longrightarrow\ x^2 + y^2 + z^2 - 2(xy + yz + zx) = 0$$

Locus of P is the Steiner inellipse of $\triangle ABC$ centered at its centroid $G\ (1:1:1)$.

lym

#7 Apr 7, 2009, 1:06 am

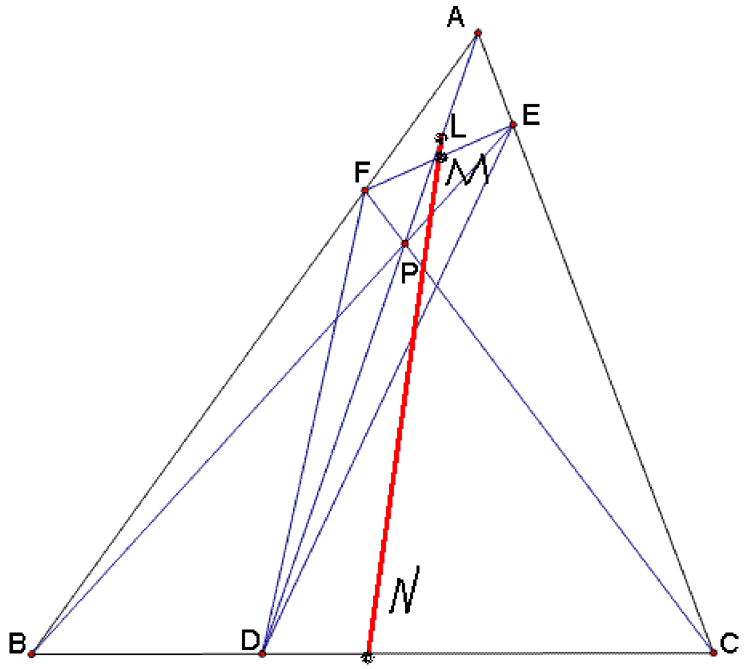
Let me tell you how I fabricate this problem，See that picture，I give a important conclusion：
http://www.aoshoo.com/bbs1/dispbbs.asp?boardid=43&id=13143&page=0&star=1（second page，13th floor）

The circle ⊙AP,⊙EF,⊙BC have the same axis，so the $Newton-theorem$ become easy .
The $Orthocenter-Group, four-orthopole, four-symmetrical$ point of $Mique$ totally 12 points are the axis of the Complete quadrilateral $AEPF$ and ：

$S\Delta BEF/S\Delta ABP = MN/LN = S\Delta CEF/S\Delta ACP = r$；
$S\Delta AEF/S\Delta ABC = LM/LN = S\Delta PEF/S\Delta PBC = s$；
$S\Delta APE/S\Delta BCE = ML/MN = S\Delta APF/S\Delta BCF = t$，$r+s=1, s/r=t$.

this is only basic conclusion，there are some Special circumstances，such as Orthogonal ，use this can solve **Bulgaria 46th Olympiad** ，**Bulgaria 46th Olympiad** is the Special circumstances of three circle Orthogonal.

Attachments:



Quick Reply

High School Olympiads

Vecten's figure

Reply

Source: own

jayme

#1 Mar 29, 2009, 1:27 pm

Dear Luis and Mathlinkers,
it seem that the Vecten's figure is inexhaustible..
Let ABC be a triangle,
CB'B"A, AC'C"B, BA'A"C three squares erected externally on the sides of ABC,
and X, Y respectively the points of intersection of AA' and CC', AC and BB".
Prove that XY is parallel to BB'.
Sincerely
Jean-Louis

Luis González

#2 Mar 30, 2009, 1:21 pm

We use barycentric coordinates WRT $\triangle ABC$, keeping in mind Conway's notation

$$B'' \ (S_C + S : -b^2 : S_A) \ , \ C' \ (S_B + S : S_A : -c^2)$$

$$A' \ (-a^2 : S_C + S : S_B) \ , \ B' \ (S_C : -b^2 : S_A + S)$$

$$AA' \equiv S_B y - (S_C + S)z = 0 \ , \ CC' \equiv (S_B + S)y - S_A x = 0$$

$$BB'' \equiv S_A x - (S_C + S)z = 0 \ , \ BB' \equiv (S_A + S)z - S_C z = 0$$

Infinite point of BB'' is then $T_\infty(S_C : S + S_A + S_C : S_A + S)$

$$X \equiv CC' \cap AA' \equiv ((S_C + S)(S_B + S) : S_A(S_C + S) : S_B S_A)$$

$$Y \equiv AC \cap BB'' \equiv (S_C + S : 0 : S_A)$$

$$\Longrightarrow \ XY \equiv -S_A x + Sy + (S_C + S)z = 0$$

Infinite point of XY is $P_\infty(S_C : S + S_A + S_C : S_A + S) \Longrightarrow \ T_\infty \equiv P_\infty$.

jayme

#3 Mar 30, 2009, 3:13 pm

Dear Luis and Mathlinkers,
I found a synthetical proof based on Desargues and Pappus that I will write next.
Sincerely
Jean-Louis

lym

#4 Mar 30, 2009, 9:40 pm

Dear jayme□this problem is very easy .
Actually as long as we draw a complete map□use the methods we already have□maybe we can Infer many conclusions.
We can solved it with several methods and deal with different ways □you know.
Next□I use area and sine function what is I often use to solved□

$AR/A'R = AC/A'C * sin\square ACC'/sin\square DCA'$.......(1).

$AQ'/A''Q' = AB/A''B * sin\square ABB''/sin\square DBA''$.......(2).

$(1)/(2) =>$

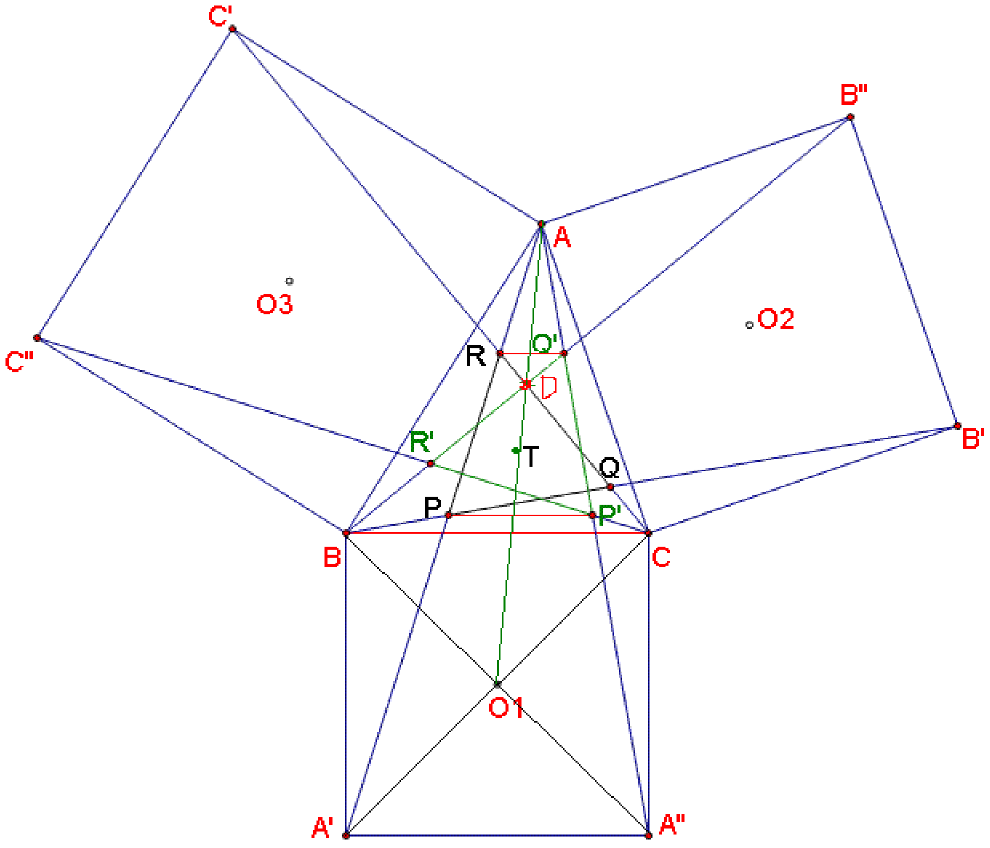
$AR/A'R * A''Q'/AQ' = AC/AB * sin\square ACC'/sin\square ABB''$

$= AC/AB * sin\square ACC'/sin\square AC'C = AC/AB * AB/AC = 1$.

So□$AR/A'R = AQ'/A''Q'$.

We also can think aboat Macro approach□such as “$Desarguestheorem$”.

Attachments:



This post has been edited 1 time. Last edited by lym, Mar 30, 2009, 9:46 pm

jayme

#5 Mar 30, 2009, 9:49 pm

Dear Lym and Mathlinkers,
Luis is right, but now if you applied the Pappus's theorem, you will find my result. Nice you have found the beginning.
Sincerely
Jean-Louis

lym

#6 Mar 30, 2009, 11:02 pm

Sorry□Dear jayme□I don't know the Pappus's theorem. Can you introduce to me□
I will use different methods to solve it.

lym

#7 Apr 1, 2009, 12:38 am

I didn't found a good way to prove it□
I just use Menelaus theorem to solve it(easy)□and I have a Desargues's method□but is not very simple.
If anybody need□I can put it.

So□I think useing Pappus's theorem is very ingenious□I look forward to jayme's method□

msecco

#8 Apr 1, 2009, 11:32 pm

I will tell you Pappus'Theorem.
If you have 2 lines r and s, and choose 3 points in each: A, B and C at r and D, E and F at s in that order, then the intersections of AE and BD, AF and CD and BF and CE are collinears.

Regards!

vittasko

#9 Apr 2, 2009, 1:40 am

The **Pappos theorem** is also true, for any order of the triads of arbitrary points, considered on two given lines.

So, if $A,\ B,\ C,$ are three arbitray points on the line (e) in any order and $A',\ B',\ C',$ are three arbitrary also points on the line (e') in any order , the **Pappos theorem** states that the points $X\equiv AB'\cap A'B$ and $Y\equiv AC'\cap A'C$ and $Z\equiv BC'\cap B'C,$ are collinear.

Kostas Vittas.

jayme

#10 Apr 29, 2010, 4:13 pm

Dear Mathlinkers,
an article concerning the Vecten's figure and its developpement can be seeing on my site
http://perso.orange.fr/jl.ayme , la figure de Vecten vol. 5 p. 42
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

concyclic with altitudes

Reply

hollandman

#1 Mar 20, 2009, 10:13 pm

In the triangle ABC, denote by A_1, B_1, and C_1 the feet of altitudes. Let P be the perpendicular foot C_1 on line A_1B_1, and let Q be that point of line A_1B_1 for which $AQ = BQ$. Show that $\angle PAQ = \angle PBQ$, and find the size of these two angles in terms of angles A, B, C.

hollandman

#2 Mar 29, 2009, 11:40 pm

Anybody? :)

Luis González

#3 Mar 30, 2009, 11:20 am

Let X be the intersection of A_1C_1 with AB. If M is the midpoint of AB, then QM is the perpendicular bisector of AB. Note that $PQMC_1$ is cyclic $\Longrightarrow XP \cdot XQ = XM \cdot XC_1$, but X lies on the orthic axis of $\triangle ABC$, i.e. the radical axis of the circumcircle and the 9-point circle $\odot(A_1B_1C_1) \Longrightarrow XA \cdot XB = XM \cdot XC_1 \Longrightarrow XP \cdot XQ = XA \cdot XB \Longrightarrow PQAB$ is cyclic. Since $(X, A, C_1, B) = -1$ and $PC_1 \perp PX$, it follows that PC_1 bisects $\angle APB$. As a result, PC and the perpendicular bisector CM meet at Y lying on $\odot(PQAB)$. Hence, $\angle PAQ = |\angle A - \angle B|$.

Quick Reply

High School Olympiads

Nice problem about concurrent:D

mathVNpro
#1 Mar 27, 2009, 11:51 am

Let (I) be the incircle of triangle ABC. Let M, N be the reflections of I through AB, AC respectively. IM, IN intersect (I,IM) by the points M', N' respectively. Prove that BN' intersects CM' at a point wich lies on the circle (I,IM)

mathVNpro
#2 Mar 27, 2009, 5:12 pm

No idea??? What a pity???

Mathias_DK
#3 Mar 27, 2009, 8:07 pm • 1

> mathVNpro wrote:
> Let (I) be the incircle of triangle ABC. Let M, N be the reflections of I through AB, AC respectively. IM, IN intersect (I,IM) by the points M', N' respectively. Prove that BN' intersects CM' at a point wich lies on the circle (I,IM)

I have a solution by complex numbers.
The main idea in the proof:
Let I be the center of the complex plane, and let m'', n'', p'' be the points where (I) touches AB, AC, BC respectively, and assume wlog $|n''| = |m''| = |p''| = 1$. Then notice that $m' = -2m''$ and $n' = -2n''$ and that the circle (I, IM) has radius 2. We can also get a, b, c by $a = \frac{2n''m''}{n''+m''}$, and similar with b, c. Then let z be the intersection point of BN' and CM'.
Then we have $(b-n')\times(z-n') = 0$ and $(c-m')\times(z-m') = 0$. And after some computations we get
$z = 2\frac{n''m''+m''p''+p''n''}{n''+m''+p''}$. So $|z| = 2 \iff |n''m''+m''p''+p''n''| = |n''+m''+p''|$, but that is obvious since
$|n''m''+m''p''+p''n''| = |\overline{n''m''+m''p''+p''n''}| = \left|\frac{n''+m''+p''}{n''m''p''}\right|$
$= |n''+m''+p''| \iff |z| = 2 \iff z \in (I, IM)$. QED

mathVNpro
#4 Mar 27, 2009, 10:10 pm

Huh?? I have never thought that this problem can be solved in complex number.... I used inversion to solve this. Is there anyone who can solve this beautiful geometry problem just using pured plane geometry (transformation....., etc.). The problem still wait for you.

Mathias_DK
#5 Mar 27, 2009, 10:59 pm

> mathVNpro wrote:
> Huh?? I have never thought that this problem can be solved in complex number.... I used inversion to solve this. Is there anyone who can solve this beautiful geometry problem just using pured plane geometry (transformation....., etc.). The problem still wait for you.

I would like to see the solution by inversion.

mathVNpro

#6 Mar 28, 2009, 9:59 am

Still no solution??? Try hard

mathVNpro

#7 Mar 28, 2009, 4:35 pm

Still no solution for this problem, my dear mathlinkers????

nsato

#8 Mar 28, 2009, 10:09 pm

BN' and CM' intersect at the reflection of the incentre through the Feuerbach point. It follows from this result:

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=221027

mathVNpro

#9 Mar 28, 2009, 10:22 pm

> *nsato wrote:*
> BN' and CM' intersect at the reflection of the incentre through the Feuerbach point. It follows from this result:
>
> http://www.artofproblemsolving.com/Forum/viewtopic.php?t=221027

Can you make yourself clearer, in detail.

mathVNpro

#10 Mar 29, 2009, 7:34 am

Hello, still no answer...)

Mashimaru

#11 Mar 29, 2009, 9:00 pm

OK, there is a good-looking solution now. And I was extremely surprised since this one was came up with by a friend of mine who is studying specialized in Literature class of my school...

Let (I) tangents to BC, CA, AB at A_0, B_0, C_0 respectively. Consider the homothety with centre I which turns $B \mapsto B_2, C \mapsto C_2, M' \mapsto B_1, N' \mapsto B_2$, thus B_0B_1, C_0C_1 are two diameters of (I). We are to prove that B_1B_2 and C_1C_2 intersect each other at a point $F \in (I)$.

We have $A_0B_1 \perp A_0B_0, IC \perp A_0B_0$ then $A_0B_1 // IC$ and so $A_0(B_1C_2IC) = -1$. But $A_0I \perp A_0C$ so they are inner and outer angle bisector of $\widehat{B_1A_0C_2}$. Similarly, A_0I and A_0C are inner and outer angle bisector of $\widehat{B_2A_0C_1}$. This implies that $(A_0B_1, A_0C_1) \equiv (A_0B_2, A_0C_2) \pmod{\pi}$.

We also have $(C_1B_1, C_1A_0) \equiv (B_0B_1, B_0A_0) \equiv (CI, CA_0) \equiv (C_2B_2, C_2A_0) \pmod{\pi}$ so $\triangle A_0B_1C_1 \triangle A_0B_2C_2$ and we have $\dfrac{A_0B_1}{A_0B_2} = \dfrac{A_0C_1}{A_0C_2}$, which implies $\triangle A_0B_1B_2A_0C_1C_2$ and since $(B_1B_2, B_1A_0) \equiv (C_1C_2, C_1A_0) \pmod{\pi}$, which means B_1B_2 and C_1C_2 intersect (I) at the same point. That's what we have to prove.

This post has been edited 3 times. Last edited by Mashimaru, Mar 29, 2009, 9:55 pm

mathVNpro

#12 Mar 29, 2009, 9:37 pm

Look like you have mistyped something, Mashimaru, can you check it again???

Mashimaru

#13 Mar 29, 2009, 9:39 pm

I have edited the solution posted above.

Luis González

#14 Mar 30, 2009, 1:31 am

Lemma: $\triangle ABC$ has circumcircle (O) and M, N denote the midpoints of AC, AB. Let M' and N' be points on rays MO and NO, such that $OM' = 2OM$ and $ON' = 2ON$. Then circles $\odot(OBM')$ and $\odot(OCN')$ meet at a point lying on (O).

Proof: Note that the distance from the orthocenter H to B is twice the distance from O to AC and similarly for A, C. Hence, $BM'OH$ and $CN'OH$ are parallelograms $\Longrightarrow \angle BPO = 180^\circ - \angle BHO$ and $\angle CPO = 180^\circ - \angle CHO \Longrightarrow$ $\angle BPC = 360^\circ - \angle BHO + \angle CHO = 180^\circ - \angle A \Longrightarrow P \in (O)$.

Problem: Incircle (I) of $\triangle ABC$ is tangent to AB, AC, BC through X, Y, Z. X' and Y' are the antipodes of X, Y WRT (I) and M and N denote the midpoints of BI and CI. Show that MY' and NX' meet at a point P lying on (I).

Let V and W be the midpoints of YZ and XZ. Negative inversion WRT (I) takes X' into X, Y' into Y and M, N into the points D, E on rays WI and VI, such that $ID = 2IW$ and $IE = 2IV$, since this inversion is the composition of a direct inversion and a central symmetry about I. Hence, lines MY' and NX' are taken into the circles $\odot(YID)$ and $\odot(XIE)$. According to the previous lemma, the latter circles intersect a point on (I), hence their primitive lines intersect on (I) as well.

P.S. Unfortunately, I'm not able to describe the intersection with this reasoning.

---

mathVNpro

#15 Mar 30, 2009, 1:02 pm

Your solution is so like mine. But I ignore the lemma and consider it as a "destination" I have to come up with while I am solving this problem. This problem is pretty nice, isn't it, huh??

---

aev5peru

#16 Apr 1, 2009, 6:55 am

> *mathVNpro wrote:*
>
> Let (I) be the incircle of triangle ABC. Let M, N be the reflections of I through AB, AC respectively. IM, IN intersect (I,IM) by the points M', N' respectively. Prove that BN' intersects CM' at a point wich lies on the circle (I,IM)

Nice problem!!
My teacher solved the problem , but the other form.
He proofed.
Lema .- Let ABC , and I be the incenter of ABC, and the incircle touch to AB,BC and CA at C',A' and B' , respectively.
Let A_0 in IA'such that $IA_0 = k.A'A_0$.
B_0,C_0 , similarly.
Then AA_0,BB_0,CC_0 are concurrent at X

the problem is a case of lemma.

If E the point of tangency of incircle with the nine circle of ABC . Then I,E and X are collinear and $IE = EX$.

PD: my teacher sayed "the lemma is the theorem karilla ".

---

hxy09

#17 Apr 4, 2009, 6:33 pm

> *nsato wrote:*
>
> BN' and CM' intersect at the reflection of the incentre through the Feuerbach point. It follows from this result:
>
> http://www.artofproblemsolving.com/Forum/viewtopic.php?t=221027

Could you please show us how to apply the previous result?I think your solution is very interesting
I have made two figures for you

Attachments:
Doc1.doc (24kb)

nsato
#18 Apr 4, 2009, 8:58 pm

Take this result in http://www.artofproblemsolving.com/Forum/viewtopic.php?t=221027, and then apply a homothety with centre I and dilation factor 2.

Also, note the result in post #5.

hxy09
#19 Apr 5, 2009, 11:47 am

Oh,I see,Thank you for your nice proof

hxy09
#20 Apr 12, 2009, 1:58 pm

> *nsato wrote:*
> Take this result in http://www.artofproblemsolving.com/Forum/viewtopic.php?t=221027, and then apply a homothety with centre I and dilation factor 2.
>
> Also, note the result in post #5.

To nsato,do you know any synthetic proof for it?(without using the property of X_{80})
To luigeometria,can you come up with a solution for point P in your post is Feuerbach's point
And I have a proof using a bit long calculation,though I think it looks ugly,but I like it very much.

Attachments:

Problem 1: Let I be the incircle of triangle ABC and ⊙I touches AB BC CA at Z, X, Y, assuming the radius of ⊙I is r. Let ⊙I' be the circle of radius 2r and the center of ⊙I' is I, then $YI \cap ⊙I' = M$ $ZI \cap ⊙I' = N$

Proof: CN intersects BM at a point X_{80}, which lies on ⊙I' and X_{80} is the reflection of I with respect of Feuerbach's point

Problem 2: (Antreas Hatzipolakis) Given △ABC and its incircle ⊙I and AA_c is the altitude of A wrt IC, AA_b is the altitude of A wrt IB, similarly B_aB_c C_aC_b then, the Euler line of $\triangle AA_cA_b$ $\triangle BB_aB_c$ $\triangle CC_aC_b$ concur at Fuerbach's point of △ABC

We can easily prove that two problems are almost equivalent, the key to both of them is to show X_{80} ($X_{80} = CN \cap BM$) is the reflection of I wrt Feuerbach's, now we prove our theorem:

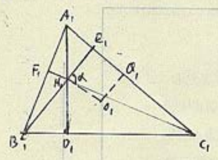
hxy09

#21 Apr 12, 2009, 2:11 pm

my proof 🙂

Attachments:

As for $\angle(IH_1, H_1T)$



I is the circumcenter of $\triangle A_1B_1C_1$

$\therefore \angle(IH_1, H_1T) = \angle O_1H_1E_1$

$H_1E_1 = 2R\cos A_1\cos C_1 \qquad O_1Q_1 = R\cos B_1$

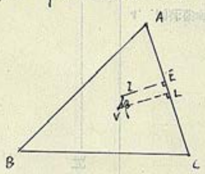
$E_1Q_1 = \frac{1}{2}A_1C_1 - A_1E_1 = R\cdot\sin B_1 - 2R\sin C_1\cos A_1$

$\therefore \tan\alpha = \dfrac{\sin B_1 - 2\sin C_1\cos A_1}{2\cos A_1\cos C_1 - \cos B_1}$

Clearly $\angle A_1 = \frac{1}{2}(\angle B + \angle C)$ ...

$\therefore \tan\alpha = \dfrac{\cos\frac{B}{2} - 2\sin\frac{A}{2}\cos\frac{C}{2}}{2\sin\frac{A}{2}\sin\frac{C}{2} - \sin\frac{B}{2}} = \dfrac{\sin\frac{C-A}{2}}{\cos\frac{C-A}{2} - 2\sin\frac{B}{2}}$

As for $\angle(YI, VI)$



V is the center of nine-point circle

so V is the midpoint of O and H

$\therefore IE = r \qquad VL = \frac{1}{2}(R\cos B + 2R\cos A\cos C)$

$EL = AL - AE = R(\sin C\cos A - \frac{1}{2}\sin B + \sin A - \sin C)$

$\therefore \tan\beta = \dfrac{\sin C\cos A - \frac{1}{2}\sin B + \sin A - \sin C}{\frac{1}{2}(\cos B + 2\cos A\cos C) - \frac{r}{R}}$

$= \dfrac{\sin(C-A) + 2(\sin A - \sin C)}{\cos(A-C) - \frac{2r}{R}}$

$\Leftarrow \beta = 2\alpha \Leftarrow \tan\beta = \tan 2\alpha$

$= \dfrac{2\sin\frac{C-A}{2}(\cos\frac{C-A}{2} - 2\sin\frac{B}{2})}{(\cos\frac{C-A}{2} - 2\sin\frac{B}{2})^2 - \sin^2\frac{C-A}{2}}$

$= \dfrac{\sin(C-A) + 2(\sin A - \sin C)}{\cos(C-A) + 2 - 2(\cos A + \cos B + \cos C)} \qquad (*)$

$(*) \Leftarrow \sum\cos A = \frac{r}{R} + 1$ which is trivial

which prove our Theorem

QED.

hxy

Proof of Problem 1: Consider the homothety with center I and dilation fator $\frac{1}{2}$

let N_1, M_1 be the midpoint of IN and IM

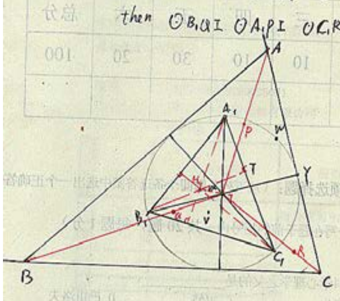
C_1', B_1' be the midpoint of IC and IB

We only need to prove $C_1N_1 \cap B_1M_1 =$ Feuerbach's

Lemma (luigeometria): consider the inversion $I(I, r^2)$

which turns B' to Q, C' to R, A' to P

then $\odot B_1QI$, $\odot A_1PI$, $\odot C_1RI$, $\odot I$ concur at W



proof: $QI = 2d(I, A_1C_1)$

let H_1 be the orthocenter of $\triangle A_1B_1C_1$

then $B_1H_1 = 2d(I, A_1C_1)$ (I is the circumcenter of $\triangle A_1B_1C_1$)

$\because A_1C_1 \parallel AC$ (easy to prove)

Hence $B_1H_1 \mathrel{\underline{\parallel}} QI \quad A_1H_1 \mathrel{\underline{\parallel}} PI \quad C_1H_1 \mathrel{\underline{\parallel}} RI$

let W be the intersection of $\odot B_1QI$ and $\odot I$ different from B_1

then $\angle B_1WI = \pi - \angle B_1QI = \pi - \angle B_1H_1I = \alpha$

$\because \angle A_1WI = \angle A_1WB_1 + \alpha = \angle C_1 + \alpha = \angle A_1H_1T + \alpha = \angle AH_1I = \angle A_1PI$

$\therefore A_1PWI$ are on the same circle

similarly $W \in \odot C_1RI$

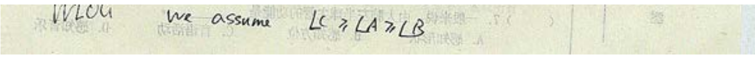
$\therefore \odot B_1QI \cap \odot A_1PI \cap \odot C_1RI \cap \odot I = W$ prove our lemma.

We are sufficient to show W is the Fuerbach's

let V be the center of nine point circle $\Leftrightarrow VIW$ collinear

and $\angle B_1WI = \alpha \Leftrightarrow \angle(YI, VI) = 2\alpha \Leftrightarrow \angle(YI, VI) = 2\angle(IH_1, H_1T)$

All we left is calculation



hxy09

#22 Apr 14, 2009, 7:29 am

After one night thought,I came up with a solution using complex number just like in post 3 (without much calculation)
I pesonally think complex number is a good choice :)

hqthao

#23 Jan 16, 2010, 2:11 am

dear Mathias_DK, I don't think your computation is right, I have the coordinate of z is more complicated. :)

Mathias_DK

#24 Feb 3, 2010, 6:40 pm • 1 👍

> hqthao wrote:
> dear Mathias_DK, I don't think your computation is right, I have the coordinate of z is more complicated. :)

I will show you all of my computations:

We can prove that if Z is the intersection of AB and CD then:
$z = \frac{\frac{\overline{a}b-a\overline{b}}{a-b} - \frac{\overline{c}d-c\overline{d}}{c-d}}{\frac{\overline{a-b}}{a-b} - \frac{\overline{c-d}}{c-d}}$.

(In the case where $|a| = |b| = |c| = |d| = 1$ this gives the wellknown $z = \dfrac{ab(c+d)-cd(a+b)}{ab-cd}$)

In this case z is the intersection of BN' and CM'.
$b = \dfrac{2m''p''}{m''+p''}, c = \dfrac{2n''p''}{n''+p''}, n' = -2n'', m' = -2m''$
$z = \frac{\frac{\overline{b}n'-b\overline{n'}}{b-n'} - \frac{\overline{c}m'-c\overline{m'}}{c-m'}}{\frac{\overline{b-n'}}{b-n'} - \frac{\overline{c-m'}}{c-m'}}$.

$$b - n' = 2\frac{m''p''+n''m''+n''p''}{m''+p''}$$
$$c - m' = 2\frac{m''p''+n''m''+n''p''}{n''+p''}$$

So:
$$\frac{\overline{b-n'}}{b-n'} = \frac{m''+p''+n''}{n''(m''p''+p''n''+n''m'')}$$

$$\frac{\overline{c-m'}}{c-m'} = \frac{m''+p''+n''}{m''(m''p''+p''n''+n''m'')}$$

$$\overline{b}n' - b\overline{n'} = \frac{4}{(m''+p'')n''}(m''p''-n''^2)$$

$$\overline{c}m' - c\overline{m'} = \frac{4}{(n''+p'')m''}(n''p''-m''^2)$$

$$\frac{\overline{b}n' - b\overline{n'}}{b-n'} = \frac{2(m''p''-n''^2)}{n''(m''p''+p''n''+n''m'')}$$

$$\frac{\overline{c}m' - c\overline{m'}}{c-m'} = \frac{2(n''p''-m''^2)}{m''(m''p''+p''n''+n''m'')}$$

$$\frac{\overline{b}n' - b\overline{n'}}{b - n'} - \frac{\overline{c}m' - c\overline{m'}}{c - m'} = \frac{2(m'' - n'')}{n''m''}$$

$$\frac{\overline{b - n'}}{b - n'} - \frac{\overline{c - m'}}{c - m'} = \frac{m'' - n''}{m''n''} \frac{m'' + p'' + n''}{(m''p'' + p''n'' + n''m'')}$$

So: $z = 2\frac{m''p'' + p''n'' + n''m''}{m'' + n'' + p''}$

Do you agree now ?

livetolove212

#25 Feb 7, 2010, 5:56 pm

See the generalization at here:
http://www.mathlinks.ro/viewtopic.php?t=325489

hqthao

#26 Feb 7, 2010, 8:06 pm

oh, sorry Mathias_DK, I had count again and you right. I think that time I had used some wrong formula (because I had checked so carefully).
@ Mathias_DK: can you tell me the relative of K and $\overline{K}$ when K is a point belong to a line thought M and perpendicular/paralel to a line contain A and B (I had read in TiTu book, and doesn't know so much and the way he talk about the line in complex plane not like in IMO compedium)

Mathias_DK

#27 Feb 7, 2010, 9:07 pm • 1

> *hqthao wrote:*
> oh, sorry Mathias_DK, I had count again and you right. I think that time I had used some wrong formula (because I had checked so carefully).
> @ Mathias_DK: can you tell me the relative of K and $\overline{K}$ when K is a point belong to a line thought M and perpendicular/paralel to a line contain A and B (I had read in TiTu book, and doesn't know so much and the way he talk about the line in complex plane not like in IMO compedium)

If $K(k)$ lies on a line which includes the point $M(m)$ and has direction α, we can say the following about K:
$\frac{k-m}{\overline{\overline{k-m}}} = \frac{\alpha}{\overline{\alpha}}$.
This is because $k-m$ can be written on the form $k \cdot e^{iv}$ for some $k \in \mathbb{R}, v \in [0; 2\pi]$. And so v denotes the direction of the line from M to K. Then $\frac{k-m}{\overline{\overline{k-m}}} = e^{i2v}$, and if $\alpha = k_2 \cdot e^{iu}$ we have $\frac{\alpha}{\overline{\alpha}} = e^{i2u}$. So $\frac{k-m}{\overline{\overline{k-m}}} = \frac{\alpha}{\overline{\alpha}}$ iff $v \equiv u \bmod \pi$. That is $k-m$ and α has the same direction.
In Titu's book he writes that a is parallel to b iff $a \times b = 0 \iff \frac{1}{2}(\overline{a}b - a\overline{b}) = 0$. This is of course equivalent to $\frac{a}{\overline{a}} = \frac{b}{\overline{b}}$ if we assume $a, b \neq 0$. (When i say that $A(a)$ and $B(b)$ are parallel I mean that $\vec{OA}$ and $\vec{OB}$ are parallel)
If you want to know if to a is perpendicular to b you just check if $a \times (ib) = 0 \iff \frac{1}{2}i(\overline{a}b + a\overline{b}) = 0$ which is equivalent to $a \cdot b = 0 \iff \frac{1}{2}(\overline{a}b + a\overline{b}) = 0$ or $\frac{a}{\overline{a}} = -\frac{b}{\overline{b}}$.

I'm not quite sure what you asked exactly but I hope this answers it.

hqthao

#28 Feb 8, 2010, 2:43 pm

oh, thanks Mathias_DK. but May I have a question for you, direction α ? how can we dertermine it ? (for example, we have it when the point go through two point or something else....) thanks so much

Mathias_DK

#29 Feb 8, 2010, 10:03 pm • 1

> *hqthao wrote:*
> oh, thanks Mathias_DK. but May I have a question for you, direction α ? how can we dertermine it ? (for example, we have it when the point go through two point or something else....) thanks so much

In the case were it should be parallel to AB we would put $\alpha = a - b$ and in the case where it should be perpendicular $\alpha = e^{i\frac{\pi}{2}}(a-b) = i(a-b)$

hqthao

#30 Feb 9, 2010, 12:17 am

oh, I know. thanks very much. I must learn a lot about complex number in geometry

Quick Reply

High School Olympiads

prove that it passes through a fix point...

Reply

v235711
#1 Mar 29, 2009, 8:28 am

we are given a circle segment .A cincle inscribed in tis circle segment is tangent at A and B.
prove that AB passes through a fixed point , independently of the inscribed circle.

Luis González
#2 Mar 29, 2009, 8:51 am

Γ is the fixed circle and C, D are two fixed points on it. Denote ω the variable circle internally tangent to Γ at A and tangent to $\overline{CD}$ at B. Lines AC, AD cut ω again at C', D'. Since A is the exsimilicenter of $\omega \sim \Gamma$, it follows that $CD \parallel C'D' \Longrightarrow$ Arcs BC' and BD' of ω are equal $\Longrightarrow AB$ bisects $\angle CAD$ internally. Hence, AB always passes through the midpoint of the arc CD of Γ not containing A.

v235711
#3 Mar 29, 2009, 8:56 am

why shgould BA be the bisector?

v235711
#5 Mar 29, 2009, 9:12 am

i proved that AB passes through the midpoint of arc CD using a dilatation with center A,
carrying the inscribed circle at the "big" circle. I found it very cool, despite that it is easy.

Kenny O
#6 Mar 29, 2009, 9:42 am

AB passes through the midpoint of arc CD..easy by angle chase

Quick Reply

High School Olympiads

Concurrent lines in the regular 18-gon

Reply

Source: 0

Luis González

#1 Mar 29, 2009, 7:46 am

$P_1P_2P_3P_4...P_{18}$ are consecutive vertices of a regular 18-gon. Prove synthetically that the following lines concur:

a) $P_{11}P_2$, P_1P_8 , P_3P_{14} , P_4P_{16} , $P_{18}P_6$

b) P_3P_{12} , P_1P_6 , P_2P_8

c) P_8P_{18} , P_4P_{15} , P_1P_{10}

d) P_3P_{12} , P_6P_{17} , P_2P_{10}

nsato

#2 Mar 30, 2009, 1:31 pm

Click to reveal hidden text

Quick Reply

High School Olympiads

Maximum area in the right-angled triangle

Reply

Source: 0

Luis González

#1 Mar 27, 2009, 1:05 am

Let Ω be a semicircumference with diameter $\overline{BC}$ and A is a variable point on Ω. The medians BM_b and CM_c of $\triangle ABC$ issuing from B and C and the internal angle bisector of $\angle BAC$ bound a triangle $\triangle XYZ$. Prove that the area of $\triangle XYZ$ is maximum when the altitude from A to BC equals the fifth part of $\overline{BC}$.

Virgil Nicula

#2 Mar 28, 2009, 3:38 am

> Quote:
>
> **Lemma.** In $\triangle ABC$ consider the points $\left\|\begin{array}{ccc} M\in (BC) & ; & \frac{MB}{MC}=m\\ N\in (CA) & ; & \frac{NC}{NA}=n\\ P\in (AB) & ; & \frac{PA}{PB}=p\end{array}\right\|$ for which denote $\left\|\begin{array}{c} X\in BN\cap CP\\ Y\in CP\cap AM\\ Z\in AM\cap BN\end{array}\right\|$.
>
> Then the area of $\triangle XYZ$ is given by the relation $[XYZ]=\frac{(1-mnp)^2}{(1+m+mn)(1+n+np)(1+p+pm)}\cdot [ABC]$.

Proof. Apply the **Menelaus' theorem** to the following transversals in the mentioned triangles :

$\left\|\begin{array}{ccc} \overline{BZN}/ACM & : & \frac{BM}{BC}\cdot\frac{NC}{NA}\cdot\frac{ZA}{ZM}=1\iff\frac{ZA}{m+1}=\frac{ZM}{mn}=\frac{AM}{1+m+mn}\\ \overline{CXP}/BAN & : & \frac{CN}{CA}\cdot\frac{PA}{PB}\cdot\frac{XB}{XN}=1\iff\frac{XB}{n+1}=\frac{XN}{np}=\frac{BN}{1+n+np}\\ \overline{AYM}/CBP & : & \frac{AP}{AB}\cdot\frac{MB}{MC}\cdot\frac{YC}{YP}=1\iff\frac{YC}{p+1}=\frac{YP}{pm}=\frac{CP}{1+p+pm}\end{array}\right\|$. Prove easily that

$$[XYZ]=[ABC]-[ABM]-[BCN]-[CAP]+[CXN]+[AYP]+[BZM]\Longrightarrow$$

$$\frac{[XYZ]}{[ABC]}=1-\frac{BM}{BC}-\frac{CN}{CA}-\frac{AP}{AB}+\frac{MB}{BC}\cdot\frac{ZM}{AM}+\frac{NC}{CA}\cdot\frac{XN}{BN}+\frac{PA}{AB}\cdot\frac{YP}{CP}\Longrightarrow$$

$$\frac{[XYZ]}{[ABC]}=1-\sum\frac{BM}{BC}\cdot\left(1-\frac{ZM}{AM}\right)=1-\sum\frac{BM}{BC}\cdot\frac{ZA}{AM}=1-\sum\frac{m}{m+1}\cdot\frac{m+1}{1+m+mn}=$$

$$1-\sum\frac{m}{1+m+mn}\Longrightarrow\frac{[XYZ]}{[ABC]}=1-\frac{m}{1+m+mn}-\frac{n}{1+n+np}-\frac{p}{1+p+pm}\Longrightarrow$$

$$\ldots\ldots\Longrightarrow\boxed{[XYZ]=\frac{(1-mnp)^2}{(1+m+mn)(1+n+np)(1+p+pm)}\cdot [ABC]}\ .$$

> Luis González wrote:
>
> Let ABC be a A - right triangle with two fixed vertices B , C and a mobile vertex A on the its circumcircle . Denote
>
> the midpoints N , P of the sides $[CA]$, $[AB]$ respectively and the point $M\in (BC)$ for which $\widehat{MAB}\equiv\widehat{MAC}$.

The lines AM , BN , CP delimit a triangle XYZ . Prove that the area of XYZ is maximum if and only if $\boxed{h_a = \frac{a}{5}}$.

Proof. Suppose w.l.o.g. $a = 1$, i.e. $b^2 + c^2 = 1$ and $x = \frac{c}{b} \ge 1$. Observe that $[ABC] = \frac{bc}{2(b^2+c^2)} = \frac{x}{2(x^2+1)}$.

Apply the upper lemma for $n = p = 1$, $m = x$ and obtain $[XYZ] = \frac{x(1-x)^2}{6(m+2)(2m+1)(m^2+1)}$, i.e.
$[XYZ] = \frac{1}{6} \cdot f(x)$,

where $\boxed{fx) = \frac{x(1-x)^2}{(2x^2+5x+2)(x^2+1)}\ ,\ x \ge 1}$. Thus, $[XYZ]$ is maximum iff $f(x)$ is maximum on $[1, \infty)$. Prove easily that

$f'(x)\ .s.s.\ -2(x-1)\left(x^5-3x^4-7x^3-7x^2-3x+1\right) = -2(x-1)(x+1)\left(x^4-4x^3-3x^2-4x+1\right)\ .s.s.$
$-\left(x^2+x+1\right)\left(x^2-5x+1\right)\ .s.s.\ -\left(x^2-5x+1\right)$. Therefore, $f(x)\ ,\ x \ge 1$ is maximum iff
$\boxed{x := x_{\max} = \frac{5+\sqrt{21}}{2}}$.

On other hand, $5h_a = a \iff 5ah_a = a^2 \iff 5bc = b^2 + c^2 \iff x^2 - 5x + 1 = 0$, i.e. $x = \frac{5+\sqrt{21}}{2}$. In conclusion,

the area of the triangle $[XYZ]$ is maximum $\iff 5h_a = a \iff \frac{c}{b} = \frac{5+\sqrt{21}}{2} \iff$
$\boxed{\frac{a}{2\sqrt{5}} = \frac{b}{\sqrt{7}-\sqrt{3}} = \frac{c}{\sqrt{7}+\sqrt{3}}}$.

> Virgil Nicula wrote:
> **An easy extension.** Let ABC be a triangle with two fixed vertices B , C and a mobile vertex A on the its circumcircle, where $A \le 90^\circ$.
>
> Denote the midpoints N , P of the sides $[CA]$, $[AB]$ respectively and the point $M \in (BC)$ for which $\widehat{MAB} \equiv \widehat{MAC}$. The lines AM ,
>
> BN , CP delimit a triangle XYZ . Prove that the area of XYZ is maximum if and only if
> $\boxed{\left(3\sqrt{2} + 4\sin\frac{A}{2}\right) \cdot h_a = 2\cos\frac{A}{2} \cdot a}$.

Luis González
#3 Mar 28, 2009, 3:49 am

Now, I would like to present my solution. By Routh theorem for the triangle $\triangle GMN$ bounded by BM_b, CM_c and the A-internal bisector, we have:

$$\frac{|\triangle GMN|}{|\triangle ABC|} = \frac{(\frac{c}{b}-1)^2}{(1+\frac{c}{b}+\frac{c}{b})(1+1+\frac{c}{b})(1+1+1)} = \frac{b^2+c^2-2bc}{6(b^2+c^2)+15bc}$$

Substituting $a^2 = b^2 + c^2$ (Pythagorean theorem) and $2|\triangle ABC| = a \cdot h_a = bc$ yields

$$|\triangle GMN| = f(h_a) = \frac{a^2h_a - 2a{h_a}^2}{12a+30h_a} \Longrightarrow f'(h_a) = \frac{60a{h_a}^2 + 48h_aa^2 - 12a^3}{(12a+30h_a)^2}.$$

Solving the quadratic $f'(h_a) = 0$ reveals the maximum of $f(h_a)$ at $h_a = \frac{1}{5}a.$

Virgil Nicula
#4 Mar 28, 2009, 4:55 am

Thank you. I didn't know it.
See above my proof of your nice problem.
The proof of the general case is similarly.
In the case $A > 90^\circ$ is a bit of studing, but isn't hardly.
I am in Florida. Here is the hour 6:50 PM.
And in your country (**Venezuela**, I think !)) is same time ?!

Quick Reply

High School Olympiads

Concurrence

Reply

Source: very hard

Fang-jh

#1 Feb 8, 2009, 6:42 pm

Given a triangle ABC. Let A_1, B_1, C_1 be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by $Y = CA \cap C_1A_1, Z = AB \cap A_1B_1$. Point P lies in the plane of triangle ABC. Let P_1, P_2, P_3 be the reflections of P in sidelines BC, CA, AB, respectively. Let l_1, l_2 be lines through Y, Z, respectively, parallel to P_3P_1, P_1P_2. Prove that if AP_1, BP_2, and CP_3 are concurrent, then so are l_1, l_2, and AP_1.

Luis González

#2 Mar 28, 2009, 2:16 am

> Quote:
> Let P_1, P_2, P_3 be the reflections of P in sidelines BC, CA, AB, respectively

> Quote:
> AP_1, BP_2, and CP_3 are concurrent $\Longleftrightarrow \ell_1, \ell_2$, and AP_1 are concurrent

Hint

Petry

#3 Mar 29, 2009, 12:48 am

Very nice problem and ... very hard!

A synthetic proof for the Sondat theorem is here:
http://pagesperso-orange.fr/jl.ayme/Docs/Le%20theoreme%20de%20Sondat.pdf

The triangles ABC and $A_1B_1C_1$ are orthologic and the points O and H are the orthologic centers of these triangles. The lines AA_1, BB_1, CC_1 are concurrent at the point H. The Sondat theorem implies that $OH \perp YZ$. (1)

Let P^* be the isogonal conjugate of the point P wrt ABC, $\{Y'\} = P_1P_3 \cap CA$, $\{Z'\} = P_1P_2 \cap AB$.
The triangles ABC and $P_1P_2P_3$ are orthologic and the points P and P^* are the orthologic centers of these triangles. The lines AP_1, BP_2, CP_3 are concurrent at a point Q.
The Sondat theorem implies that the points P, P^*, Q are collinear and $PP^* \perp Y'Z'$. (2)

Now, P_1, P_2, P_3 are the reflection points of P in BC, CA, AB, respectively and the lines AP_1, BP_2, CP_3 are concurrent. From here http://pagesperso-orange.fr/bernard.gibert/Exemples/k001.html
we have that the point P lies on the Neuberg cubic, so $PP^* || OH$. (3)
I don't have a synthetic solution to prove that $PP^* || OH$ or $PQ || OH$. 🙂

(1),(2),(3) $\Rightarrow YZ || Y'Z'$.

$\{M\} = P_1P_3 \cap YZ$, $\{N\} = P_1P_2 \cap YZ$.
$MN || Y'Z' \Rightarrow \frac{P_1M}{P_1Y'} = \frac{P_1N}{P_1Z'}$ (4)
Let V' be the intersection point of AP_1 with the parallel through M to AC,
and $V"$ be the intersection point of AP_1 with the parallel through N to AB.
$MV' || Y'A \Rightarrow \frac{P_1M}{P_1Y'} = \frac{P_1V'}{P_1A}$ (5) and $NV" || Z'A \Rightarrow \frac{P_1N}{P_1Z'} = \frac{P_1V"}{P_1A}$ (6)

(4),(5),(6) $\Rightarrow \dfrac{P_1V'}{P_1A} = \dfrac{P_1V"}{P_1A} \Rightarrow V' = V" = V$.

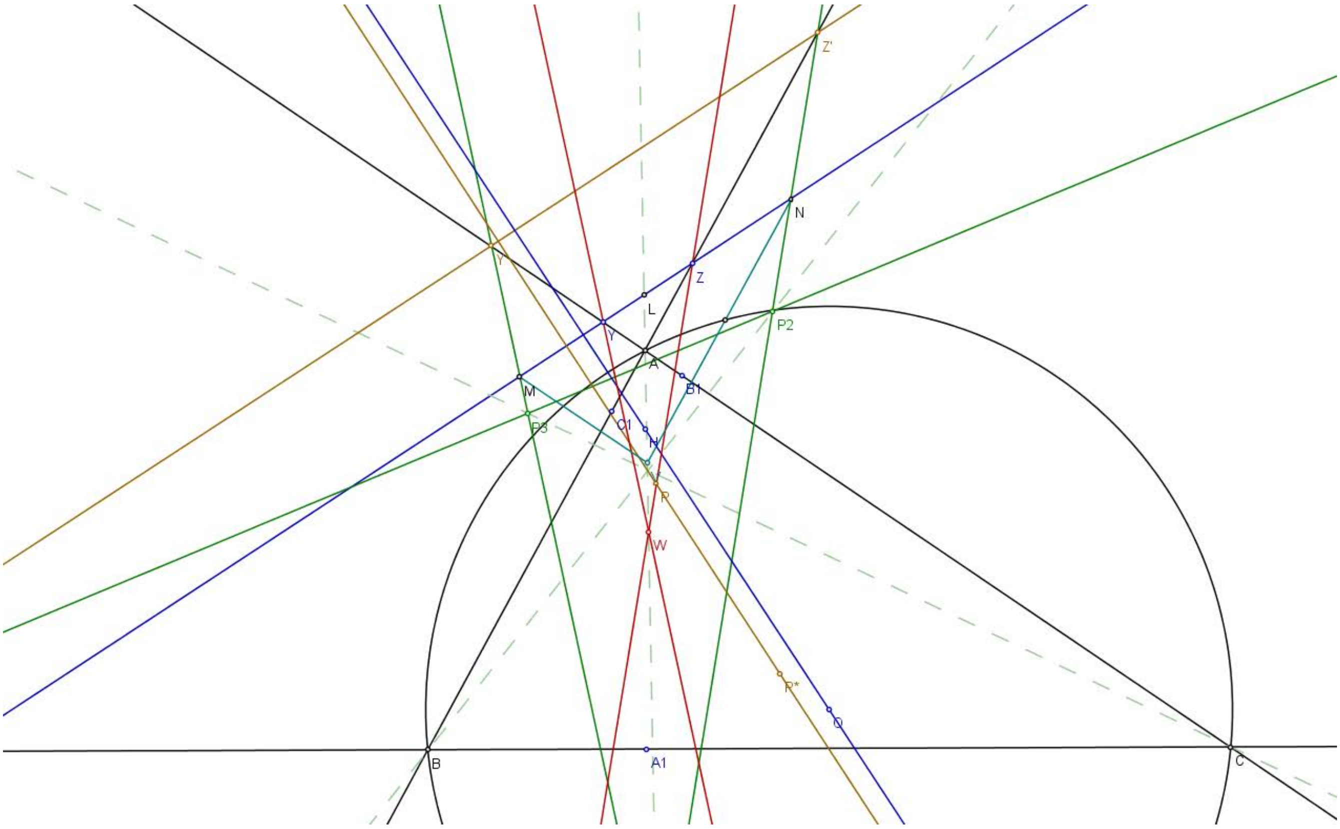
$\{L\} = AP_1 \cap YZ$
$YA||MV \Rightarrow \dfrac{LY}{LM} = \dfrac{LA}{LV}$ (7) and $ZA||NV \Rightarrow \dfrac{LZ}{LN} = \dfrac{LA}{LV}$ (8)
(7),(8) $\Rightarrow \dfrac{LY}{LM} = \dfrac{LZ}{LN}$ (9)

$\{W'\} = l_1 \cap AP_1$ and $\{W"\} = l_2 \cap AP_1$.
$YW'||MP_1 \Rightarrow \dfrac{LY}{LM} = \dfrac{LW'}{LP_1}$ (10) and $ZW"||NP_1 \Rightarrow \dfrac{LZ}{LN} = \dfrac{LW"}{LP_1}$ (11)
(9),(10),(11) $\Rightarrow \dfrac{LW'}{LP_1} = \dfrac{LW"}{LP_1} \Rightarrow W' = W" = W$.

So, the lines l_1, l_2, AP_1 are concurrent at W.

Best regards, Petrisor Neagoe :)

Attachments:



Petry

#4 Mar 29, 2009, 9:32 am

Dear luis,

The points P_1, P_2, P_3 are the REFLECTION of P in sidelines BC, CA, AB.
The triangle $P_1P_2P_3$ IS NOT THE PEDAL TRIANGLE of the point P.

Best regards

Quick Reply

High School Olympiads

Unique point X

Reply

Source: IMO Shortlist 1995, G

thaithuan_GC
#1 Aug 3, 2008, 5:10 pm • 1

Let A, B and C be non-collinear points. Prove that there is a unique point X in the plane of ABC such that

$$XA^2+XB^2+AB^2 = XB^2+XC^2+BC^2 = XC^2+XA^2+CA^2.$$

Mathias_DK
#2 Mar 27, 2009, 10:14 pm • 1

> thaithuan_GC wrote:
> Let A, B and C be non-collinear points. Prove that there is a unique point X in the plane of ABC such that
>
> $$XA^2+XB^2+AB^2 = XB^2+XC^2+BC^2 = XC^2+XA^2+CA^2.$$

It's easily killed with complex numbers. Consider the complex plane with center at the circumcenter of $\triangle ABC$, and let a, b, c, x be the corresponding complex numbers to A, B, C, X respectively. (Then $|a| = |b| = |c|$)
Besides let X' be a 180° rotation of X, so $x' = -x$.
Let $\cdot$ be the real product, then we have:
$XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 \iff$
$XA^2 + AB^2 - XC^2 - BC^2 = 0 \iff$
$(x-a)\cdot(x-a) + (a-b)\cdot(a-b) - (x-c)\cdot(x-c) - (b-c)\cdot(b-c) = 0 \iff$ (Using that $|a| = |b| = |c|$)
$(a-c)\cdot(b-(-x)) = (a-c)\cdot(b-x') = 0 \iff$
$AC \perp BX'$
In the same way we get, $AB \perp CX'$, $BC \perp AX'$, so there is equality iff X' is the orthocenter.
So the equation is fulfilled if and only if X is the orthocenter rotated 180° around the circumcenter :)

Luis González
#3 Mar 27, 2009, 11:33 pm

The first condition $XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2$ is equivalent to $XA^2 - XC^2 = BC^2 - AB^2$. Which clearly means that X belows to the reflection of the B-altitude with respect to the perpendicular bisector of AC. By similar reasoning, we conclude that X is the reflection of the orthocenter H with respect to the circuncenter O. In other words, X is the De Longchamps point X_{20} of $\triangle ABC$.

kyyuanmathcount
#4 Jul 12, 2009, 7:17 am

Draw the following circles:
Center C, radius segment AB
Center B, radius segment AC
Center A, radius segment BC.

From the equations, it is clear that the unique point is the radical center of these circles.

jayme
#5 Jul 12, 2009, 9:34 am

Dear kyyuanmathcount,
this is the way that de Longchamps quoted by Luis has taken...
Sincerely
Jean-Louis

viperstrike
#6 Mar 26, 2016, 9:05 am

I think maybe a problem which can be so easily done with coordinates should not be on ISL.

WLOG $B(-1,0), C(1,0), A(a,b), X(x,y)$. Since A, B, C not collinear we have $b \neq 0$. We have

$BC^2 = 4$
$AB^2 = (a+1)^2 + b^2$
$AC^2 = (a-1)^2 + b^2$

$XB^2 = (x+1)^2 + y^2$
$XC^2 = (x-1)^2 + y^2$
$XA^2 = (x-a)^2 + (y-b)^2$

The given condition is there exists a real number K with:

I. $K = (x-a)^2 + (y-b)^2 + (x+1)^2 + y^2 + (a+1)^2 + b^2$
II. $K = (x+1)^2 + y^2 + (x-1)^2 + y^2 + 4$
III. $K = (x-1)^2 + y^2 + (x-a)^2 + (y-b)^2 + (a-1)^2 + b^2$

Define $N = K - 2(x^2 + y^2)$. From (II), we have $N = 6$. Then (I) and (III) are of the form:

I. $\alpha_1 x + \beta_1 y + \gamma_1 = 6$
III. $\alpha_3 x + \beta_3 y + \gamma_3 = 6$

Note that $\beta_3 = -2b \neq 0$. Multiply (I) by β_3 and (III) by β_1, and subtract:

$x(\alpha_1\beta_3 - \alpha_3\beta_1) + (\gamma_1\beta_3 - \gamma_3\beta_1) = 6(\beta_3 - \beta_1)$

Therefore there exists a unique solution to this system iff $\alpha_1\beta_3 \neq \alpha_3\beta_1$. But $\alpha_1 = 2 - 2a$,$\beta_1 = -2b$,$\alpha_3 = -2a - 2$, $\beta_3 = -2b$, so this is obviously the case.

Quick Reply

High School Olympiads

Five points collinear

Reply

libra_gold
#1 Nov 26, 2006, 8:18 am

A quadrangle $PQRS$ has an in-center O (which implies $PQ+RS=QR+SP$). $T=PR\cap QS$. H_A, H_B, H_C, H_D are respectively the orthocenters of POQ, QOR, ROS, SOP. Prove that T, H_A, H_B, H_C, H_D are collinear.

Hawk Tiger
#2 Nov 26, 2006, 10:48 am

I have solved it by Meallous(: Idon't know how to spell) Theorem,But very long.

libra_gold
#3 Nov 26, 2006, 2:08 pm

Can you give the main idea by using "Menelaus" theorem?

Hawk Tiger
#4 Nov 26, 2006, 2:49 pm

I have found another way to solve it .Please wait me and I will try to complete my solution,and post it.I am busy now....

libra_gold
#5 Nov 27, 2006, 6:44 am

Thank you. And I'm looking forward to it.

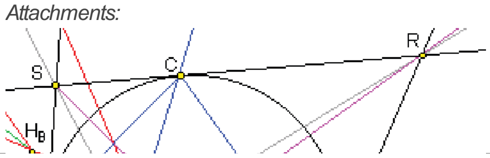
yetti
#6 Nov 27, 2006, 12:19 pm

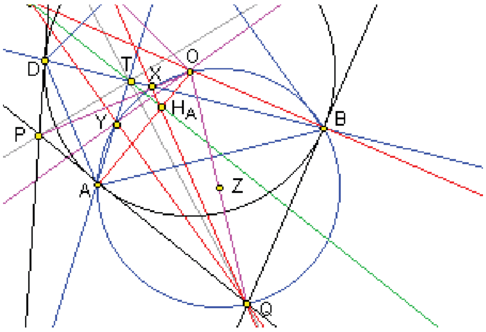
The proof is quite short. Let A, B, C, D be the tangency points of PQ, QR, RS, SP with the quadrilateral incircle (O). The tangential quadrilateral PQRS and its contact cyclic quadrilateral ABCD have the same diagonal intersection $T \equiv PR \cap QS \equiv AC \cap BD$. The tangency points A, B of PQ, QR with the incircle (O) are feet of the O-altitudes of the triangles $\triangle POQ,\ \triangle QOR$, respectively. The quadrilateral AOBQ is a cyclic kite on account of OA = OB, QA = QB, and the right angles $\angle OAQ = \angle OBQ = 90^\circ$, let (Z) be the its circumcircle with diameter OQ. Let X, Y be feet of the Q-altitudes of the triangles $\triangle POQ,\ \triangle QOR$, respectively. The angles $\angle OXQ = \angle OYQ = 90^\circ$ are both right, hence $X,\ Y \in (Z)$. Since AOBQ is a kite with OA = OB, OQ bisects $\angle AOB$ and

$$\angle ADB = \frac{\angle AOB}{2} = \angle QOB = \angle QXB$$

Since in addition $QX \parallel AD$ (both perpendicular to OP), the points B, X, D are collinear and the lines $BX \equiv BD$ are identical. Similarly, we can show that the points A, Y, C are collinear and the lines $AY \equiv AC$ are identical. Thus $T \equiv AY \cap BX$. By Pascal's theorem for the cyclic hexagon AQBOXY, the intersections $H_A \equiv OA \cap QX, H_B \equiv OB \cap QY, T \equiv AY \cap BX$ are collinear. In exactly the same way, we can show that $H_B,\ H_C,\ T$ are collinear, $H_C,\ H_D,\ T$ are collinear, and $H_D,\ H_A,\ T$ are collinear. As a result, all 5 points $H_A,\ H_B,\ H_C,\ H_D,\ T$ are collinear.

Attachments:





libra_gold
#7 Nov 28, 2006, 7:26 am

Great!!! Thank you very much.

darij grinberg
#8 Sep 13, 2008, 5:08 pm

This is now Theorem 18 **a)** in "Circumscribed quadrilaterals revisited". The proof is more elementary than Yetti's above (Pascal is not used), but more configuration-dependent, so I prefer Yetti's. What was your proof, Hawk Tiger?

darij

This post has been edited 1 time. Last edited by darij grinberg, Nov 18, 2008, 6:52 pm

April
#9 Sep 16, 2008, 9:25 am

It's known that $\frac{PT}{TR}=\frac{PA}{RC}$, $\frac{QT}{TS}=\frac{QB}{SD}$. We have $\angle AOQ=\angle BOQ$, so $\angle APH_A=\angle BRH_B$. It implies that $\triangle PAH_A\sim\triangle RBH_B$, i.e. $\frac{PA}{RC}=\frac{PH_A}{RH_B}$. And notice that $PH_A\parallel RH_B$ and $\frac{PA}{RB}=\frac{PT}{RT}$, so by the Thales' theorem, we have H_A, T and H_B are collinear.

Similarly, H_B, T and H_C are collinear; H_C, T and H_D are collinear. Hence H_1, H_2, H_3, H_4 and E are lie on a line. And our solution is completed.

No Reason
#10 Mar 25, 2009, 6:16 pm

This line has a very interesting property:
Let d the line pass through O,midpoint of PR and midpoint of PS and d' be the line pass through $T, H_1H_2H_3H_4$.Then $d\perp d'$

Luis González
#11 Mar 27, 2009, 11:35 am

Lemma. If M and N are the midpoints of the sides AB, AC of $\triangle ABC$, then MN is the polar of the orthocenters of the triangles $\triangle IBC, \triangle I_aBC$ with respect to the incircle (I) and the A-excircle (I_a), respectively.

Let ω denote the incircle of $PQRS, X\equiv PQ\cap SR$ and $Y\equiv QR\cap PS.$ Line XY is the polar of T WRT ω and according to the lemma, polars of H_A, H_B, H_C, H_D are the lines p, q, r, s joining the midpoints of $(YQ, YP), (XQ, XR),$ $(YR, YS), (XP, XS).$ Since p, q, r, s, XY concur at the midpoint U of $XY,$ then H_A, H_B, H_C, H_D, T are collinear on the polar of U WRT $\omega,$ as desired.

Quick Reply

High School Olympiads

bisectors in quadrilateral

Reply

hollandman

#1 Mar 26, 2009, 11:22 pm

Let $ABCD$ be a convex quadrilateral such that $AD||BC$ and AC is perpendicular to BD. Let the angle bisectors of $\angle A, \angle B$ intersect at Y, those at $\angle C, \angle D$ intersect at X. Prove that $XY||AD$.

Luis González

#2 Mar 27, 2009, 6:27 am

Let M and N be the points where the inner angle bisectors of $\angle BAD$ and $\angle ADC$ meet the sideline BC. It's easy to see that $\angle AYX$ and $\angle DXC$ are right due to

$\angle DAB + \angle ABC = 180^\circ$ and $\angle ADC + \angle DCB = 180^\circ$.

Hence, triangles $\triangle ABN$ and $\triangle CDN$ are isosceles with bases $\overline{AN}$ and $\overline{DM} \Longrightarrow X, Y$ are midpoints of $DM, AN \Longrightarrow X, Y$ lie on the midparallel of the trapezoid.

Quick Reply

High School Olympiads

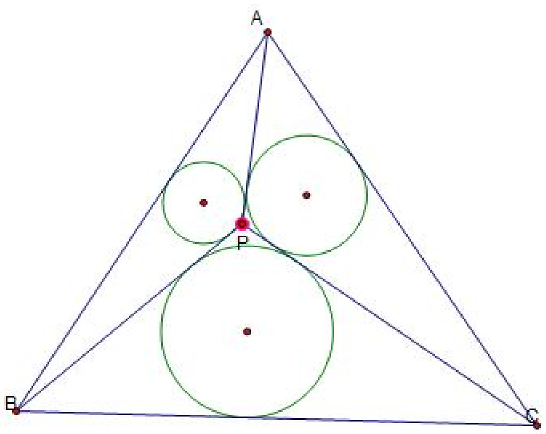
radius problem.

Reply

Jianxing113725
#1 Mar 17, 2009, 8:34 pm

Find the point P in triangle ABC,to let the triangle PBC PAC and PAB have the same radius.
My English is so poor!!!

Attachments:



Jianxing113725
#2 Mar 25, 2009, 9:20 pm

No one to help me...

Luis González
#3 Mar 26, 2009, 9:37 pm

I think that P is not constructible with ruler-compass since it must satisfy

$$\frac{[\triangle APC]}{PC+PA+AC}=\frac{[\triangle BPC]}{PB+PC+BC},\ \frac{[\triangle APB]}{PA+PB+AB}=\frac{[\triangle BPC]}{PB+PC+BC}$$

Both loci are not constructible by R & C.

Jianxing113725
#4 Mar 27, 2009, 8:32 pm

I got this problem from a math magazine....I guess the point maybe the internal point...
Thank you all the same...

Quick Reply

High School Olympiads

A nice and usual metrical relation in a triangle.

Reply

Source: Own.

Virgil Nicula

#1 Mar 25, 2009, 9:27 pm

Let ABC be a triangle with the circumcircle w and two points $\{P, Q\} \subset w$

so that $PQ \parallel BC$ and the line BC separates A , P . Prove that

$$\left\| \begin{array}{c} PA \cdot QA = AB \cdot AC + PB \cdot PC \\ \\ PA^2 = AB \cdot AC + PB \cdot PC \iff AB = AC \quad \vee \quad \widehat{PAB} \equiv \widehat{PAC} \end{array} \right\| .$$

See **here** .

Luis González

#2 Mar 26, 2009, 8:55 pm

Let h_a, x, y be the distances from A to BC, A to PQ and P to BC. If $PQ \parallel BC$, it follows that $x = h_a + y$. But since $\triangle ABC, \triangle APQ$ and $\triangle BPC$ have the same circumcircle (O, R), we deduce that

$$AB \cdot AC = 2R \cdot h_a \ , \ \ AP \cdot AQ = 2R \cdot x \ , \ \ PB \cdot PC = 2R \cdot y$$

$$\Longrightarrow \frac{AP \cdot AQ}{2R} = \frac{AB \cdot AC}{2R} + \frac{PB \cdot PC}{2R} \Longrightarrow AP \cdot AQ = AB \cdot AC + PB \cdot PC$$

Virgil Nicula

#3 Mar 26, 2009, 9:36 pm

Nice proof, **Luisgeometra !**

> *Virgil Nicula wrote:*
> Let ABC be a triangle with the circumcircle w and two points $\{P, Q\} \subset w$
>
> so that $PQ \parallel BC$ and the line BC separates A , P . Prove that
>
> $$\boxed{\begin{array}{c} PA \cdot QA = AB \cdot AC + PB \cdot PC \\ \\ PA^2 = AB \cdot AC + PB \cdot PC \iff AB = AC \quad \vee \quad \widehat{PAB} \equiv \widehat{PAC} \end{array}} .$$

Proof. Suppose w.l.o.g. that $PB \le PC$. Denote $m(\angle PAB) = m(\angle QAC) = x$.

Observe that $\left\| \begin{array}{ccc} m(\angle APQ) & = & B + x \\ m(\angle AQP) & = & C + x \end{array} \right\|$ and $AP \cdot AQ = AB \cdot AC + PB \cdot PC \iff$

$\sin(C + x) \cdot \sin(B + x) = \sin C \cdot \sin B + \sin x \cdot \sin(A - x) \iff$

$\cos(C - B) - \cos(B + C + 2x) = \cos(C - B) - \cos(C + B) + \cos(2x - A) - \cos A \iff$

$\cos(C - B) + \cos(2x - A) = \cos(C - B) + \cos A + \cos(2x - A) - \cos A$, what is evidently.

The mentionned equivalence from the conclusion is also evidently.

sunken rock

#4 Nov 4, 2009, 2:30 am

See that BCQP is an isosceles trapezoid, apply twice Ptolemy, then cosine theorem, ready. For the 2nd relation see that, if AB = AC then obviously AP = AQ and, from 1st relation, done.

If $\angle PAB = \angle PAC$, then P≡Q and, from 1st relation, done again.

Best regards
sunken rock

Quick Reply

High School Olympiads

Sum of distances from NPC to the sidelines

Reply

Source: 0

Luis González

#1 Mar 21, 2009, 12:18 pm

$\triangle ABC$ is acute with 9-point center N and $\triangle XYZ$ is the pedal triangle of N with respect to $\triangle ABC$. R, r, r_0 stand for the radii of the circumcircle, incircle and incircle of the orthic triangle of $\triangle ABC$. Show that:

$$NX + NY + NY = \frac{1}{2}\left(R + 3r + r_0 + \frac{r^2}{R}\right)$$

Luis González

#2 Mar 26, 2009, 8:51 am

O, H are the circumcenter and orthocenter of $\triangle ABC$. M_a, M_b, M_c denote the midpoints of BC, CA, AB and H_a, H_b, H_c denote the feet of the A-,B- and C- altitudes. Since PX becomes the median of the right trapezoid OHH_aM_a, we get $NX = \frac{1}{2}(OM_a + HH_a)$. Thus, adding the symmetric expressions together gives

$$NX + NY + NZ = \tfrac{1}{2}(OM_a + OM_b + OM_c + HH_a + HH_b + HH_c) =$$

$$= \tfrac{1}{2}[R + r + h_a + h_b + h_c - (HA + HB + HC)] = \tfrac{1}{2}(h_a + h_b + h_c - R - r)$$

According to this topic, we have $h_a + h_b + h_c = 2R + 4r + r_0 + \frac{r^2}{R}$. Thus

$$NX + NY + NZ = \frac{1}{2}\left(R + 3r + r_0 + \frac{r^2}{R}\right).$$

High School Olympiads

Two circles and two tangents

Source: Polish MO 2009 Round 2

hollandman

#1 Mar 26, 2009, 7:31 am

Disjoint circles o_1, o_2, with centers I_1, I_2 respectively, are tangent to the line k at A_1, A_2 respectively and they lie on the same side of this line. Point C lies on segment I_1I_2 and $\angle A_1CA_2 = 90^\circ$. Let B_1 be the second intersection of A_1C with o_1, and let B_2 be the second intersection of A_2C with o_2. Prove that B_1B_2 is tangent to the circles o_1, o_2.

Luis González

#2 Mar 26, 2009, 8:44 am

We use the degenerate Pappus theorem. Since I_1, C, I_2 are collinear, then the parallel through I_1 to CA_2, (perpendicular to CA_1) and the parallel through I_2 to CA_1 (perpendicular to CA_2) concur at a point Q lying on $\overline{A_1A_2}$. Let τ denote the tangent to (I_1) through B_1. Note that τ passes through Q, because CQ is the perpendicular bisector of $\overline{A_1B_1} \Longrightarrow QI_2$ is the Q-external bisector of the isosceles triangle $\triangle QB_1A_1 \Longrightarrow (I_2)$ is tangent to τ.

vittasko

#3 Mar 26, 2009, 11:46 pm

For an alternative approach, see the problem Common polar with tangents

Kostas Vittas.

High School Olympiads

prove that it is bisector...

v235711

#1 Mar 26, 2009, 5:02 am

in triangle ABC let A1 , B1 ,C1 be the foot of the bisectors of A ,B ,C . let D be the intersection of AA1 and C1B1 and E the intersection of CC1 and A1B1 . prove that BB1 is the bisector of angle <DBE

Luis González

#2 Mar 26, 2009, 6:34 am

Let W,K,L be the feet of the angle bisectors issuing from A,B,C and D,E,F the points where CL,AW,BK meet $WK,KL,LW,$ respectively. By cevian nest theorem, we deduce that $\triangle EFD$ and $\triangle ABC$ are perspective and the perpective axis coincides with the perspective axis of $\triangle ABC$ and $\triangle WKL.$ Hence, the lines LK,DF and BC concur at the foot P of the A-external bisector. If $H \equiv AW \cap DF,X \equiv AF \cap BC,Y \equiv AD \cap BC,$ then $(F,H,D,P)=-1$ implies that $A(X,W,Y,P)=-1.$ Since $AP \perp AW,$ then AW bisects $\angle XAY.$

Virgil Nicula

#3 Mar 26, 2009, 10:01 am

> v235711 wrote:
> In $\triangle ABC$ let D , E , F be the feet of the bisectors from A , B , C . Denote
>
> $X\in BE\cap DF$, $Y\in CF\cap DE$. Prove that AD is the bisector of $\widehat{XAY}$.

Proof. Denote $\left\|\begin{array}{ccc} m(\angle XAB) & = & x\\ m(\angle YAC) & = & y\end{array}\right\|$. Prove easily that $\left\|\begin{array}{ccc} \frac{BD}{BF} & = & \frac{b+a}{b+c}\\ \\ \frac{CD}{CE} & = & \frac{c+a}{c+b}\\ \\ \frac{AE}{AF} & = & \frac{a+b}{a+c}\end{array}\right\|$. Therefore,

$$\left\|\begin{array}{c} \frac{b+a}{b+c}=\frac{BD}{BF}=\frac{XD}{XF}=\frac{AD}{AF}\cdot\frac{\sin\left(\frac{A}{2}-x\right)}{\sin x}\\ \\ \frac{c+a}{c+b}=\frac{CD}{CE}=\frac{YD}{YE}=\frac{AD}{AE}\cdot\frac{\sin\left(\frac{A}{2}-y\right)}{\sin y}\end{array}\right\|\ (:)\ \implies \frac{a+b}{a+c}=\frac{AE}{AF}\cdot\frac{\sin\left(\frac{A}{2}-x\right)}{\sin x}\cdot\frac{\sin y}{\sin\left(\frac{A}{2}-y\right)}\implies$$

$$\sin\left(\frac{A}{2}-x\right)\cdot\sin y=\sin\left(\frac{A}{2}-y\right)\cdot\sin x\implies\cos\left(\frac{A}{2}+y-x\right)=\cos\left(\frac{A}{2}+x-y\right)\implies x=y\ .$$

v235711

#4 Mar 26, 2009, 8:25 pm

my solution uses MD and NE , parallel to BB1, with M and N on sides AB and CB , respectively.
the goal is to prove that triangles MDB and NEB are similar.
then it´s just simple computations 😁

Virgil Nicula

#5 Mar 26, 2009, 9:20 pm • 1 👍

#5 Mar 26, 2009, 9:29 pm • 1

An easy extension.

Let ABC be a triangle and denote the foot $D\in (BC)$ of the A - bisector.

For a point $P\in (AD)$ denote $E\in BP\cap AC$, $F\in CP\cap AB$ and

$X\in BE\cap DF$, $Y\in CF\cap DE$. Prove that $\widehat{DAX}\equiv\widehat{DAY}$.

A strong extension.

Giving are $\triangle ABC$ and an interior point P . Denote $\left\|\begin{array}{c} D\in AP\cap BC\\\\ E\in BP\cap AC\\\\ F\in CP\cap AB\end{array}\right\|$, $\left\|\begin{array}{c} X\in BE\cap DF\\\\ Y\in CF\cap DE\end{array}\right\|$

and $\left\|\begin{array}{ccc} m(\angle BAX)=x & ; & m(\angle CAY)=y\\\\ m(\angle PAX)=u & ; & m(\angle PAY)=v\end{array}\right\|$. Prove that $\frac{\sin (x+u)}{\sin (y+v)}=\frac{\sin x\sin v}{\sin y\sin u}$.

For $x+u=y+v=\frac{A}{2}$ obtain the easy extension.

Mateescu Constantin

#6 Aug 22, 2009, 5:52 pm

> Virgil Nicula wrote:
>
> **A strong extension.**
>
> Giving are $\triangle ABC$ and an interior point P . Denote $\left\|\begin{array}{c} D\in AP\cap BC\\\\ E\in BP\cap AC\\\\ F\in CP\cap AB\end{array}\right\|$, $\left\|\begin{array}{c} X\in BE\cap DF\\\\ Y\in CF\cap DE\end{array}\right\|$
>
> and $\left\|\begin{array}{ccc} m(\angle BAX)=x & ; & m(\angle CAY)=y\\\\ m(\angle PAX)=u & ; & m(\angle PAY)=v\end{array}\right\|$. Prove that $\frac{\sin (x+u)}{\sin (y+v)}=\frac{\sin x\sin v}{\sin y\sin u}$.

Apply **Ceva**'s theorem (trigonometrical form) to :

$ \left\| \begin{array}{cccccc} X/\triangle ABD & : & \frac {\sin x}{\sin u}\cdot\frac {\sin(\angle ADF)}{\sin(\angle BDF)}\cdot\frac {\sin(\angle DBP)}{\sin(\angle PBA)} = 1 & \Longleftrightarrow & \frac {\sin x}{\sin u} = \frac {\sin(\angle BDF)}{\sin(\angle ADF)}\cdot\frac {\sin(\angle ABE)}{\sin(\angle CBE)} & (1) & \\ Y/\triangle ADC & : & \frac {\sin v}{\sin y}\cdot\frac {\sin(\angle ACP)}{\sin(\angle DCP)}\cdot\frac {\sin(\angle CDE)}{\sin(\angle EDA)} = 1 & \Longleftrightarrow & \frac {\sin v}{\sin y} = \frac {\sin(\angle BCF)}{\sin(\angle ACF)}\cdot\frac {\sin(\angle ADE)}{\sin(\angle CDE)} & (2) & \\ P/\triangle ABC & : & \frac {\sin(x + u)}{\sin(y + v)}\cdot \frac {\sin(\angle ABE)}{\sin(\angle CBE)}\cdot\frac {\sin(\angle BCF)}{\sin(\angle FCA)} = 1 & \Longleftrightarrow & \frac {\sin(x + u)}{\sin(y + v)} = \frac {\sin(\angle ABE)}{\sin(\angle CBE)}\cdot\frac {\sin(\angle BCF)}{\sin(\angle FCA)} & (3) & \end{array}\right\|$

$$\overset{(1)\ \odot\ (2)}{\Longrightarrow}\ \frac{\sin x}{\sin u}\cdot\frac{\sin v}{\sin y}=\frac{\sin(\angle ABE)}{\sin(\angle CBE)}\cdot\frac{\sin(\angle BCF)}{\sin(\angle FCA)}\cdot\frac{\sin(\angle BDF)}{\sin(\angle ADF)}\cdot\frac{\sin(\angle ADE)}{\sin(\angle CDE)}$$

$$\overset{(3)}{\Longleftrightarrow}\ \frac{\sin x}{\sin u}\cdot\frac{\sin v}{\sin y}=\frac{\sin(x+u)}{\sin(y+v)}\cdot\frac{\sin(\angle BDF)}{\sin(\angle ADF)}\cdot\frac{\sin(\angle ADE)}{\sin(\angle CDE)}\quad (4)$$

Also we have :

$\left\|\ \frac{BF}{FA}=\frac{BD}{AD}\cdot\frac{\sin(\angle BDF)}{\sin(\angle ADF)}\ \right\|$ $\frown$ $\frac{BF}{}\ \frac{AE}{}\ \ \frac{BD}{}\ \frac{\sin(\angle BDF)}{}\ \frac{\sin(\angle ADE)}{}$

$$\left\| \frac{AE}{EC} = \frac{AD}{DC} \cdot \frac{\sin(\angle ADE)}{\sin(\angle CDE)} \right\| \odot \Longrightarrow \frac{}{FA} \cdot \frac{}{EC} = \frac{}{DC} \cdot \frac{}{\sin(\angle ADF)} \cdot \frac{}{\sin(\angle CDE)}$$

$$\overset{Ceva}{\Longleftrightarrow} \frac{\sin(\angle BDF)}{\sin(\angle ADF)} \cdot \frac{\sin(\angle ADE)}{\sin(\angle CDE)} = 1 \quad (5)$$

$$\overset{(4)\ \wedge\ (5)}{\Longrightarrow} \boxed{\frac{\sin x}{\sin u} \cdot \frac{\sin v}{\sin y} = \frac{\sin(x+u)}{\sin(y+v)}}\ .$$

Quick Reply

High School Olympiads

same area

moldovan

#1 Mar 26, 2009, 12:50 am

In the parallelogram $ABCD$ consider M on CD such that $M\neq C$ and $M\neq D$. Denote by N the intersection between AM and BC. Prove that the triangles DMN and BCM have the same area.

Luis González

#2 Mar 26, 2009, 5:27 am

$$\frac{[\triangle DMN]}{[\triangle ADM]}=\frac{MN}{AM} \ \ , \ \ \angle ADM+\angle BCM=180^\circ$$

$$\frac{[\triangle ADM]}{[\triangle BCM]}=\frac{AD\cdot DM}{BC\cdot MC}=\frac{DM}{MC}$$

$$\frac{MC}{DM}=\frac{MN}{AM}\Longrightarrow\frac{[\triangle ADM]}{[\triangle BCM]}=\frac{[\triangle ADM]}{[\triangle DMN]}\Longrightarrow [\triangle BCM]=[\triangle DMN]$$

Virgil Nicula

#3 Mar 26, 2009, 8:43 am

> moldovan wrote:
> Let $ABCD$ be a parallelogram. Consider the point $M\in CD-\{C,D\}$
>
> and the intersection $N\in AM\cap BC$. Prove that $[DMN]=[BCM]$.

Proof. $[DNC]\stackrel{(AD\|NC)}{=}[ANC]\ \Longrightarrow [DNC]\pm [MNC]=[ANC]\pm [MNC]\ \Longrightarrow$

$[DMN]=[ACM]\stackrel{(AB\|CM)}{=}[BCM]\ \Longrightarrow [DMN]=[BCM]$.

High School Olympiads

Cyclic quadrilaterals

Reply

Source: Easy

rachid

#1 Mar 22, 2009, 11:27 am

Let $\triangle ABC$,and $D=(K)\cap(L)$; $E=(L)\cap(M)$ and $F=(M)\cap(K)$ where (K), (L) and (M) are the external bisectors of $\angle BAC$, $\angle ABC$

and $\angle CAB$ respectively.A', B' and C' are the circumcenters of $\triangle EBC$, $\triangle AFC$ and $\triangle ADB$ respectively.$C'B'$ meets AB at E' and

AC at F', $A'C'$ meets AB at $J$and BC at I, $A'B'$ meets AC at G and BC at H.

Prove that $B'HIC'$, $E'B'A'J$ and $F'GA'C'$ are cyclic.

Click to reveal hidden text

This post has been edited 2 times. Last edited by rachid, Mar 22, 2009, 8:42 pm

grn_trtle

#2 Mar 22, 2009, 1:43 pm

It's not possible for $C'B'$ to meet AB at E, E being the intersection of L and M. Same for $C'B'$ meeting AC at F.

Is there a mistake in the problem?

My work on the diagram thus far:
[geogebra]2cdc82a783281498609fd43786dc7709d4d867dc[/geogebra]

I think you have two Es and two Fs, which are making the statement that we need to prove ambiguous.

rachid

#3 Mar 22, 2009, 8:02 pm

Yes,sorry for the typo,I meant $C'B'\cap AB=E'$ and $C'B'\cap AC=F'$.

rachid

#4 Mar 25, 2009, 8:06 am

Any solutions ??!

Luis González

#5 Mar 25, 2009, 9:10 am

Circles $\odot AFC$, $\odot CBE$ and $\odot BDA$ concur at the incenter I of $\triangle ABC$ and $\triangle A'B'C'$ has parallel sides to the intouch triangle of $\triangle ABC \Longrightarrow \angle E'B'A'=90^\circ-\frac{1}{2}A$ and $\angle C'JE'=\angle BJA'=90^\circ-\frac{1}{2}A$. Therefore, $\angle E'B'A'=\angle C'JE' \Longrightarrow E'B'A'J$ is cyclic. The same reasoning is used for the remaning quadrilaterals.

Quick Reply

High School Olympiads

If and only if

Reply

Source: Chinese TST 2009 1st quiz P1

Fang-jh
#1 Mar 21, 2009, 2:49 pm

Given that circle ω is tangent internally to circle Γ at S. ω touches the chord AB of Γ at T. Let O be the center of ω. Point P lies on the line AO. Show that $PB \perp AB$ if and only if $PS \perp TS$.

This post has been edited 1 time. Last edited by Fang-jh, Mar 21, 2009, 4:19 pm

Luis González
#2 Mar 24, 2009, 11:17 am • 1

We restate the problem as follow

Proposition. We are given a triangle $\triangle ABC$ $(AC > AB)$. Circle ω with center X and radius ρ is tangent to the circumcircle (O) and $\overline{BC}$ through A, W, respectively. Perpendicular to BC through C cuts the external bisector of $\angle BAC$ at P. Then B, X, P are colllinear.

Obviously, AW bisects $\angle BAC$ and $X \in AO$, hence $\angle XAW = \frac{1}{2}(\angle B - \angle C)$

$$\Longrightarrow \ \rho = \frac{AW}{2\cos(\frac{\angle B - \angle C}{2})} \ (1)$$

Since $APCW$ is cyclic $\Longrightarrow \angle CPW = \angle CAW = \frac{1}{2}\angle A \Longrightarrow PC = CW \cdot \cot \dfrac{A}{2}$

On the other hand, $CW = AW \cdot \dfrac{\sin\frac{A}{2}}{\sin C}$ (2). Then, combining (1) and (2) yields

$$\frac{PC}{\rho} = \frac{2\cos\frac{(\angle B - \angle C)}{2} \cdot \cos\frac{A}{2}}{\sin C}$$

Combining this latter expression with the identities

$$\frac{\cos\frac{\angle B - \angle C}{2}}{\sin\frac{A}{2}} = \frac{b+c}{a} \text{ and } \sin A = 2\sin\frac{A}{2} \cdot \cos\frac{A}{2} \Longrightarrow \frac{PC}{\rho} = \frac{b+c}{c}.$$

But $\dfrac{b+c}{c} = \dfrac{CB}{WB} \Longrightarrow \dfrac{PC}{\rho} = \dfrac{CB}{WB} \Longrightarrow P, X, B$ are collinear.

vittasko
#3 Mar 25, 2009, 10:23 pm • 2

Let K be, the center of the circle (K), instead of ω and let O be, the center of the circle (O), instead of Γ and we denote as X, the midpoint of the arc AB, not containing the point S.

From $KT \parallel OX$ and $\dfrac{SK}{SO} = \dfrac{KT}{OX}$, we conclude that the points $S,\ T,\ X$, are collinear.

So, the line segment ST, is the angle bisector of the angle $\angle S$, of the triangle $\triangle SAB$.

We denote the point P, as the intersection point of the line segments through vertices $S,\ B$ and perpendicular to $ST,\ AB$

respectively (SP is the external bisector of $\angle S$) and it is enough to prove that the line segment BP, passes through the point K.

EQUIVALENT PROBLEM. – A triangle $\triangle ABC$ is given with circumcircle (O) and let $T,\ X$ be, the points of intersections of $BC,\ (O)$ respectively, from the angle bisector of the angle $\angle A$. Through the vertex C, we draw the line perpendicular to BC, which intersects the external bisector of $\angle A$, at point so be it P and let be the point $K\equiv OA\cap BP$. Prove that $KT\perp BC$.

PROOF. – [**t=265876**] - Through the vertex B, we draw the line perpendicular to BC, which intersects the line segment AP, as the external bisector of $\angle A$, at point so be it Q and let be the point $A'\equiv (O)\cap AO$.

It is easy to show that the line segments $BQ,\ BA'$, are isogonal conjugates with respect to the angle $\angle B$, from $QB\perp BC$ and $AB\perp BA'$.

Similarly, the line segments $CP,\ CA'$, are isogonal conjugates with respect to the angle $\angle C$.

Because of now, the external angle bisector of $\angle A$, is a particular case of two lines through the vertex A, isogonal conjugates with respect to the angle $\angle A$, based on the **Jacobi theorem** (I like better to say the **Isogonic theorem**), we conclude that the line segments $AA'\equiv AO,\ BP,\ CQ$, are concurrent at one point so be it K and we will prove that $KT\perp BC$.

• From $BQ\parallel CP\Longrightarrow \dfrac{KB}{KP}=\dfrac{BQ}{CP}$, (1)

Because of the triangles $\triangle AQB,\ \triangle APC$ have $\angle BAQ=\angle CAP$ and $\angle AQB+\angle APC=180^o$, we conclude that $\dfrac{BQ}{CP}=\dfrac{AB}{AC}$, (2)

Because of AT is the angle bisector of $\angle A$, we have that $\dfrac{AB}{AC}=\dfrac{TB}{TC}$, (3)

From $(1),(2),(3)\Longrightarrow \dfrac{KB}{KP}=\dfrac{TB}{TC}$, (4)

From (4) we conclude that $KT\parallel PC\Longrightarrow KT\perp BC$ and the proof of the equivalent problem is completed.

REMARK. - It is easy to show that the circle (K) centered at K with radius $KA=KT$ (easy to prove), tangents internaly to (O) at point A and also tangents to BC, at point T and then, we have the configuration as the proposed problem states.

REFERENCE. - http://www.mathlinks.ro/Forum/viewtopic.php?t=154396

Kostas Vittas.

Attachments:
t=265876.pdf (6kb)

Akashnil
#4 Apr 9, 2009, 9:28 am • 2

Let the tangents at S,T of ω meet at Q

For any point X denote by h_X the distance of the point X from line QO

We have $QT^2=QA\cdot QB\implies \dfrac{QB}{QT}=\dfrac{QT}{QA}\implies \dfrac{QB}{QT}=\dfrac{QT}{QA}=\dfrac{QB-QT}{QT-QA}=\dfrac{TB}{TA}$

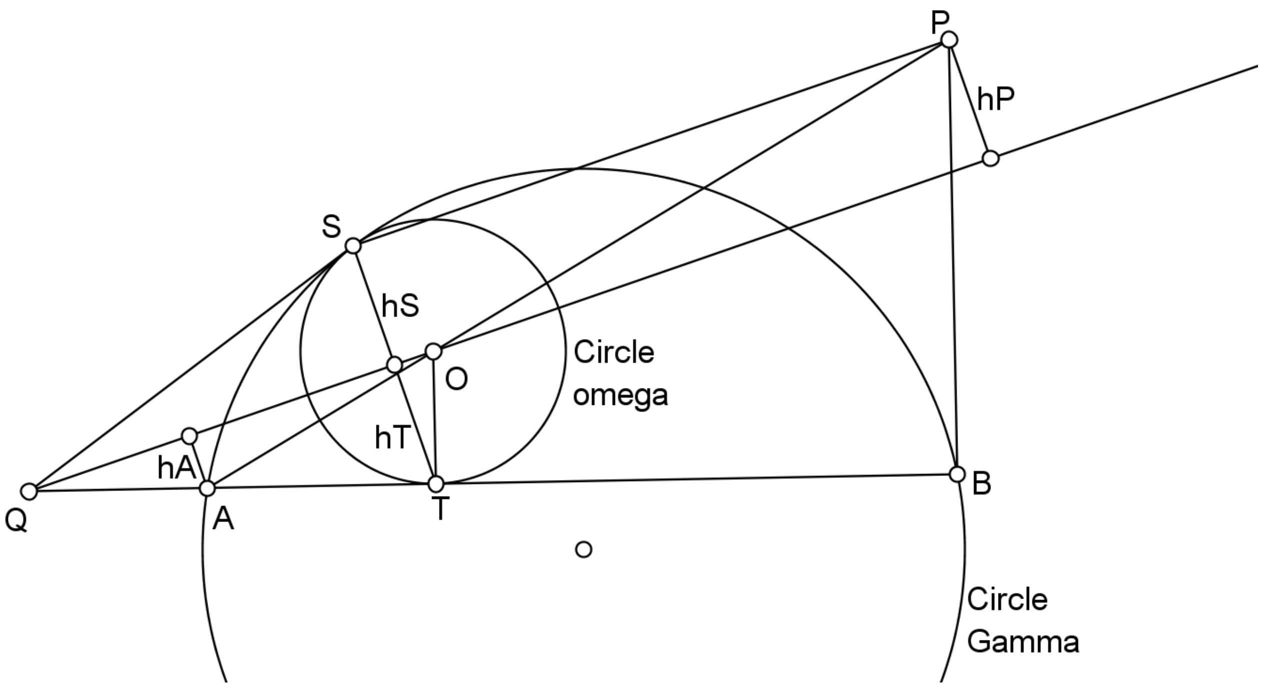
And since $\dfrac{h_T}{h_A}=\dfrac{QT}{QA}$ we have $\dfrac{h_T}{h_A}=\dfrac{TB}{TA}$

Since $QO\perp TS$

$$PS\perp TS\iff PS||QO\iff h_P=h_S\iff h_A\cdot\frac{PO}{AO}=h_T\iff$$
$$\frac{PO}{AO}=\frac{h_T}{h_A}\iff\frac{PO}{AO}=\frac{TB}{TA}\iff\Delta PAB\sim\Delta OAT\iff PB\perp AB$$

QED

Attachments:



jayme

#5 Apr 9, 2009, 4:37 pm

Dear Mathlinkers,
1. (0) the circle going through S, A and B
2. (1) the circle tangent to AB and internally to 0
2. X the midpoint of the arc AB, not containing the point S
3. Tx the tangent to 0 at X
4. According to Reim's theorem applied to (0) and (1), Tx // ATB
5. 2 the circle with diameter PT
6. According to a converse of Reim's theorem aplied to (0) and (2), X, T and the second intersection of 0 and 2à (other than B) are collinear.
7. We are done...
Sincerely
Jean-Louis

Akashnil

#6 Apr 10, 2009, 9:41 am

Dear Jean-Louis,

Can you please state the Reim's theorem?
(I searched in google and almost all the hits refer to ML-AOPS)

I know and can prove (without Reim's theorem) that X, T, S are collinear. But could not use it to solve this problem when I tried. So I want to understand your solution.

Thank you very much
Akashnil

jayme

#7 Apr 10, 2009, 2:07 pm

Dear Mathlinkers,
for the Reim's theorem (general case and particular case), for the two converses, see
http://perso.orange.fr/jl.ayme then: à propos
Sincerely
Jean-Louis

Akashnil

#8 Apr 10, 2009, 6:42 pm

I do not understand the language ...

Virgil Nicula

#9 Apr 11, 2009, 12:33 am

> vittasko wrote:
>
> Let ABC be a triangle with the circumcircle $w = C(O, R)$ and let T , X be the intersections of BC , w respectively with the bisector of the angle $\angle A$. Through the vertex C we draw the perpendicular line to BC which intersects the external bisector of $\angle A$ in the point P . Denote $K \in OA \cap BP$. Prove that $KT \perp BC$.

Proof. Suppose w.l.o.g. $b > c$. Observe that $\left\|\begin{array}{c} m(\angle KAB) = 90^\circ - C \\ m(\angle KAP) = 90^\circ - \frac{B-C}{2} \end{array}\right\|$. Denote $\left\|\begin{array}{c} D \in BC \\ AD \perp BC \end{array}\right\|$. Therefore, $m(\angle APC) = 90^\circ - \frac{B-C}{2}$

$\frac{KB}{KP} = \frac{AB}{AP} \cdot \frac{\sin \widehat{KAB}}{\sin \widehat{KAP}} = \left|\frac{c \cdot \sin(90^\circ - C)}{AP \cdot \cos \frac{B-C}{2}}\right| = \frac{c \cdot |\cos C|}{DC} = \frac{c}{b}$. In conclusion, $\frac{KB}{KP} = \frac{TB}{TC} \Longrightarrow KT \parallel PC \Longrightarrow KT \perp BC$.

Virgil Nicula

#10 Apr 12, 2009, 1:55 am • 1

> vittasko wrote:
>
> Let ABC be a triangle with the circumcircle $w = C(O, R)$ and let T , X be the intersections of BC , w respectively with the bisector of the angle $\angle A$. Through the vertex C we draw the perpendicular line to BC which intersects the external bisector of $\angle A$ in the point P . Denote $K \in OA \cap BP$. Prove that $KT \perp BC$.

Proof (proiectively). Denote $L \in BC \cap AP$, the diameter $[XY]$, the line d for which $K \in d$, $d \parallel PC$, i.e. $d \perp BC$ and the points $U \in AP \cap d$, $V \in PT \cap LK$, $W \in LK \cap PC$, $T_1 \in d \cap AX$, $T_2 \in d \cap BC$. Observe that $KU = KT_1$ because $K \in AO$ - the A-median in $\triangle AXY$ and $\overline{UKT_1} \parallel \overline{XOY}$. Thus, the division $\{B, C; L, T\}$ is harmonically $\Longleftrightarrow$ the pencil $P\{B, C; L, T\}$ is harmonically $\Longleftrightarrow$ the divison $\{K, W; L, V\}$ is harmonically. Therefore, $KT_2 \parallel WC \Longrightarrow KT_2 = KU \Longrightarrow KT_1 = KT_2 \Longrightarrow T_1 \equiv T_2 \equiv T \Longrightarrow KT \perp BC$.

---

dgreenb801

#11 Apr 12, 2009, 2:12 am

Akashnil, Reim's theorem is stated here:http://alt1.mathlinks.ro/viewtopic.php?t=35337

Akashnil

#12 Apr 12, 2009, 1:52 pm

Thank you dgreenb801.
But I still can't understand the way it is used here. Can anyone help me?

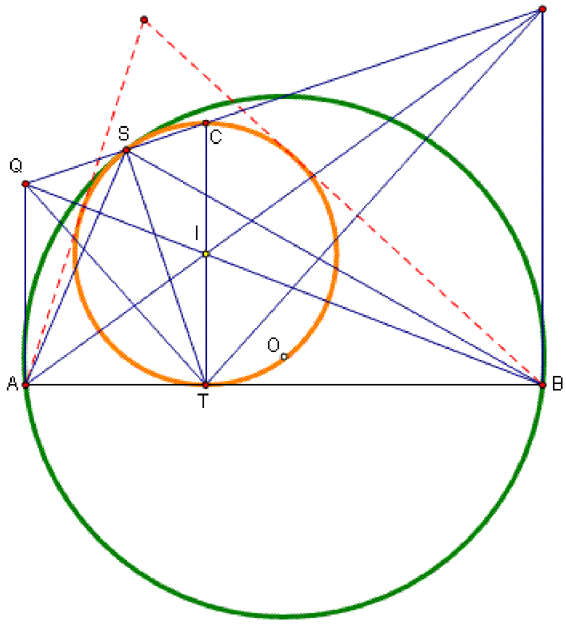
---

lym

#13 Apr 12, 2009, 3:13 pm • 1

This problem have many good geometry method http://www.aoshoo.com/bbs1/dispbbs.asp?boardid=15&Id=15330
see one of then

Attachments:



not_trig

#14 Apr 14, 2009, 4:14 am

It is sufficient to show that the perpendicular to ST at S, the perpendicular to AB at B, and AO concur. We show this by trig Ceva on triangle SAB:

Let the perpendicular to ST at S meet AO at P, and the perpendicular to AB at B meet AO at P'. Then it suffices to show that

$$\frac{\sin\angle BSP}{\sin\angle ASP}\frac{\sin\angle SAP}{\sin\angle BAP}\frac{\sin\angle ABP}{\sin\angle SBP}=1$$

Let $\angle SAB=\alpha, \angle SBA=\beta$. We prove a lemma:

Lemma. $\angle TSA=\angle TSB$.

This is fairly well-known; extend ST to hit the outer circle at R, and let AS intersect ω at A'. Then, $\angle ATA'=\angle TSA'$ because AT is tangent to ω. However, by the homothety centered at S taking ω to the outer circle, $A'T||AR$. Hence, $\angle ATA'=\angle TAR$, which equals $\angle RSB=\angle TSB$ because they intercept the same arc. ■

So, we may let $\angle TSB=\angle TSA=\gamma$.

Now, $\angle BSP=90+\gamma, \angle aSP=90-\gamma$ because $SP\perp TS$. Hence, $\dfrac{\sin\angle BSP}{\sin\angle ASP}=1$. Also, $\angle SAP=90-\alpha$, $\angle BAP=90$.

It suffices to show that $ \displaystyle \frac{\sin{\angle ABP}}{\sin{\angle SBP}} = \frac{1}{\sin{90-\alpha}$. But this is evident from considering triangles BTO, BSO: $\sin\angle ABP=\dfrac{TO}{BO}=\dfrac{SO}{BO}=\dfrac{\sin\angle SBP}{\sin\angle OSB}$, but $\angle OSP$ = $90-\angle OSM=90-\angle SAB=90-\alpha$, where M is an arbitrary point on the common tangent to the circles at S on the same side of OS as B. So, we are done.

Zhero

#15 Aug 4, 2010, 7:36 am • 1

First, I will show that if $PB\perp PA$ implies that $SP\perp ST$, then $SP\perp ST$ implies $PB\perp PA$. If $SP\perp ST$, define Q on AO such that $P'B\perp BA$. By assumption, we must have $SQ\perp ST$. Hence, both P and Q lie on the intersection of OS and the perpendicular to ST through S, yielding $P=Q$. The result follows.

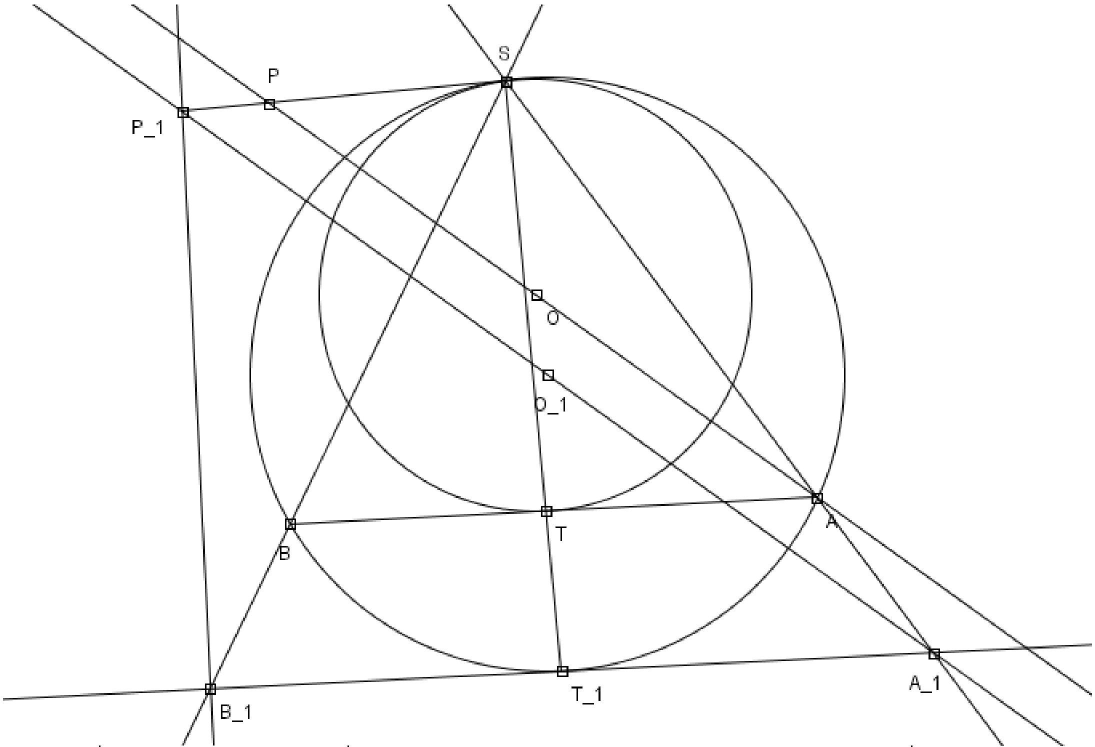
Consider the homothety centered at S that maps ω to Γ. Let the images of A, B, O_1, P_1, and T under this homothety be A_1, B_1, O_1, P_1, and T_1, respectively. The tangent to Γ at T_1 is parallel to AB, so T is the midpoint of arc AB, whence $\angle B_1ST_1=\angle T_1SA_2$.

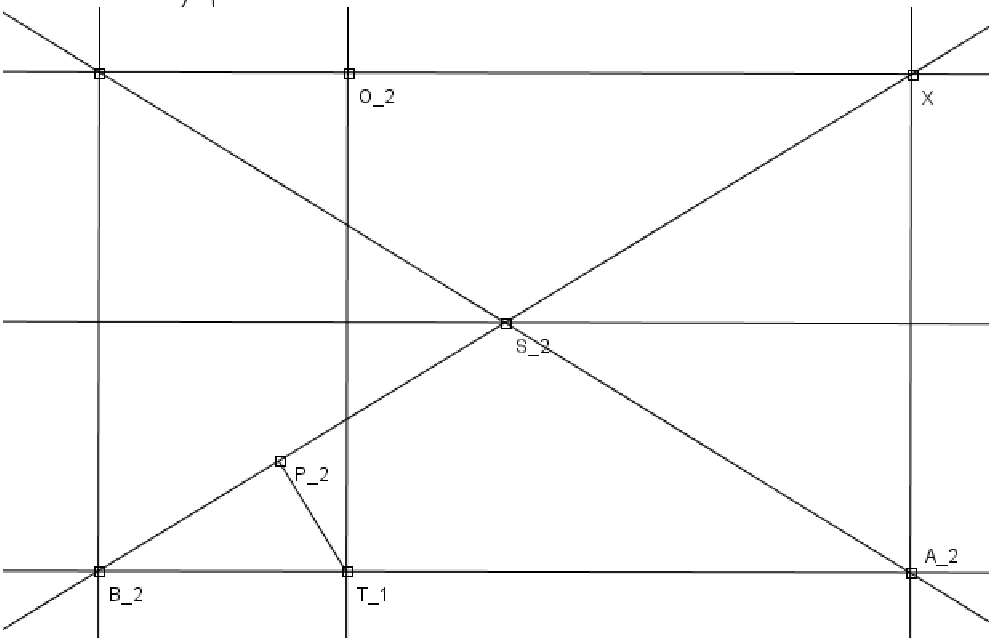
Let P_1' be the intersection of the perpendicular to ST_1 through S and the perpendicular to A_1B_1 through B_1. It is sufficient to show that P_1', O_1, and A_1 are collinear, for then both P_1 and P_1' must lie on AO and the perpendicular to B_1A_1 through B, which implies they are equal.

Perform an inversion centered at T_1 with arbitrary radius; let the images of P_1', S, A_1, O_1, and B_1 be P_2, S_2, A_2, O_2 and B_2, respectively. $\angle T_1S_1B_1 = \angle T_1S_1A_1$, yielding $\angle T_1B_2S_2 = \angle T_1A_2S_2$, whence $S_2A_2 = S_2B_2$. O_1 is sent to the reflection of T_1 across the line through S_2 parallel to A_2B_2. P_2 lies on B_2S_2 and $P_2T_1 \perp B_2S_2$ (since $T_1B_1P_1S$ is cyclic, and $T_1B_1 \perp P_1B_1$.) We wish to show that $T_1P_2O_2A_2$ is cyclic.

Let X be the intersection of B_2S_2 and the line through A_2 perpendicular to A_2B_2. Note that $XO_2||A_2T_1$. It follows that $XP_2T_1A_2$ and $T_1P_2O_2X$ are cyclic, so $O_2P_2T_1A_2$ is cyclic, as desired.

Attachments:





goldeneagle

#16 May 11, 2011, 6:46 pm

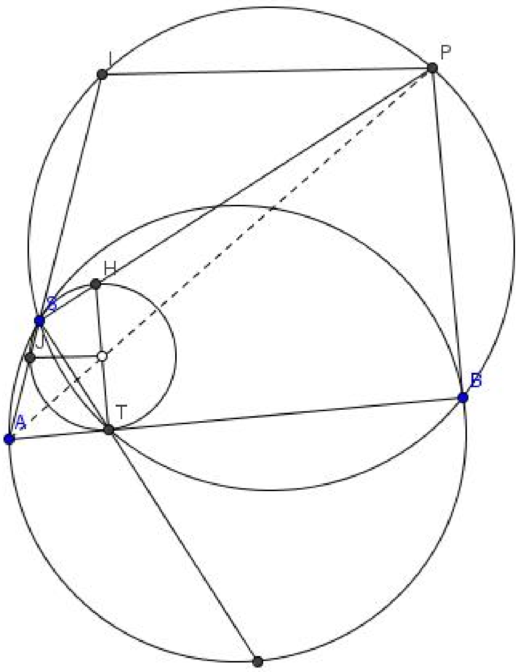
my solution: consider M and M' are midpoint of arc AB and ASB, respectively. as we know S, T, M are colliner(a useful lemma!). i want to prove that if P is a point on AO such that $PS\perp ST$ then $PB\perp BT$.
$PS\perp ST$ so P, S, M' are colliner. so it's sufficient to prove that$STBP$ is cyclic. for this i will prove that $\angle BSP = \angle BTP$ but this is equivalence to $TP||AM'$. so i define L as a line through T parall to AM'. so i will prove that L, AO, SM' are concurrent. prove this by using ceva(sinus form) in triangle STA.

skytin

#17 May 30, 2011, 7:52 pm

I have proved this problem on this site yet , but i don't remember where
Let TH is diameter of omega
Let point P such that angle PBT = TSP = 90
let's prove that P is on line AO
Let line AS intersect w at points J and S
Let line AS intersect (BTS) at points S and I
Well known that TS is inner angle bissector of BSA , so angle BSP = PSI , so BP = IP
After Reim's Theorem easy to see that TJ || BI
TO || BP and TO = JO , so A is Homotety center of triangles TJO and BIP . done

Attachments:



math154

#18 Jan 23, 2013, 3:11 am

Let O_1, r_1, r be the center of Γ, radius of Γ, and radius of ω, respectively. Let $M = ST \cap \Gamma$; it's well-known that M is the midpoint of arc AB not containing S.

Let Q be the intersection of tangent BB and (STB); then $QB \perp AB$ and $QS \perp TS$, and we want to show that A, O, Q are collinear. Now $\angle BQT = \angle BST = \alpha$, so

$$\frac{AT}{OT}\frac{AB}{QB} = \frac{AT}{r}\frac{TB\cot\alpha}{2r_1\sin 2\alpha} = \frac{TS\cdot TM}{4rr_1\sin^2\alpha}.$$

But $\triangle MTA \sim \triangle MAS$ by simple angle chasing, so $MT\cdot MS = MA^2 = 4r_1^2\sin^2\alpha$. We thus obtain

$$\frac{AT}{OT}\frac{AB}{QB} = \frac{TS}{MS}\frac{MT\cdot MS}{4rr_1\sin^2\alpha} = \frac{OS}{O_1S}\frac{r_1}{r} = 1,$$

whence A, O, Q are collinear, as desired.

TelvCohl

#19 Dec 25, 2014, 7:17 am • 1

My solution:

Let another tangent of ω through A cut Γ at C and tangent to ω at R.

From incenter of triangle we get the midpoint H of TR lie on (TBS),
so $AB \perp BP \iff S, H, T, B, P$ are concyclic $\iff TS \perp SP$.

Q.E.D

jayme

#20 Dec 26, 2014, 5:57 pm

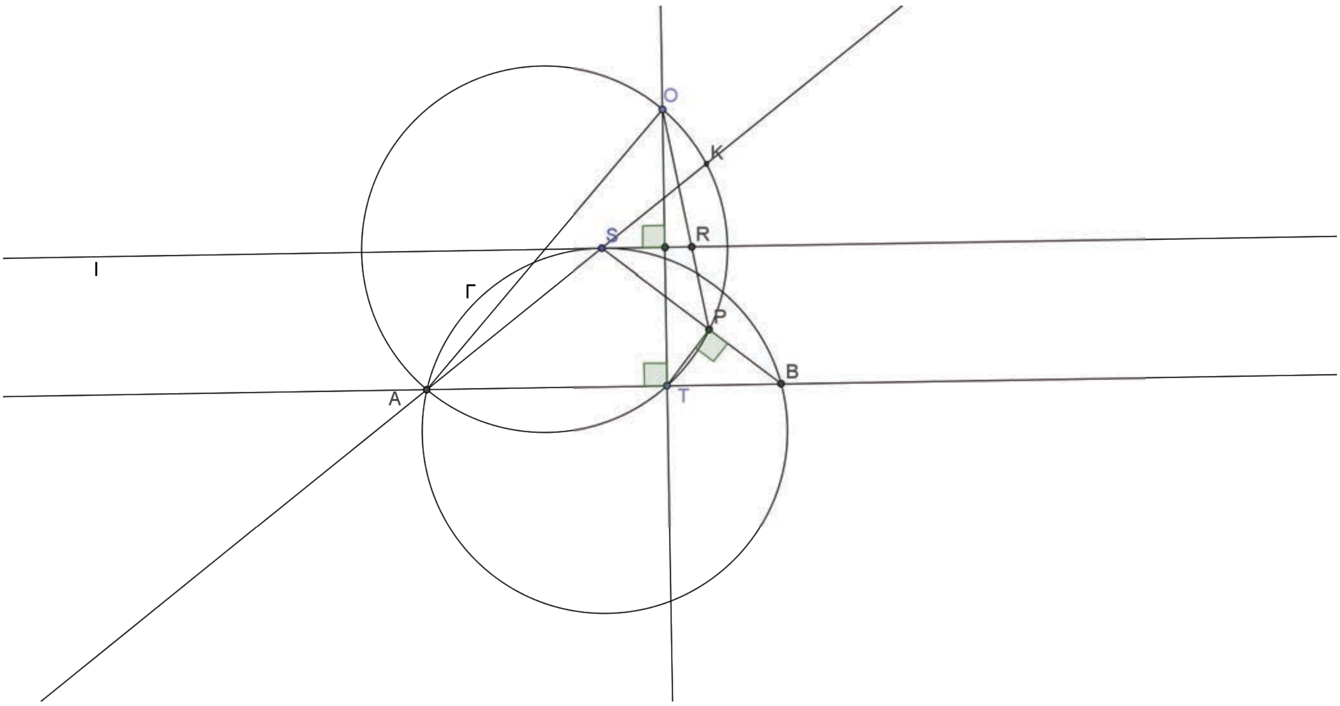
Dear Mathlinkers,
with harmonic division we have also a proof...
Sincerely
Jean-Louis

izaya-kun

#21 Mar 10, 2015, 12:17 am

Take an inversion with center T and any circle. Then ω circle transforms into line ℓ and the Γ circle tranforms into a circle tangent to line ℓ.And the line ℓ bisects OT (which is under inversion) .The line AO tranforms into the circle passing through points A, O, T(which are under inversion). The circle that passes through points S, T, B tranforms into the line that passes through S and B(which are under inversion) . Point P under inversion transforms into the intersection between SB and the circle that passes through points A, O, T.
Our condition is equvalent to showing $TP \perp SB$. Let line through A, S intersect the circle that passes through points A, O, T the second time at point K.Let OP intersect ℓ at point R.Then since $\ell \parallel AB$ and $\triangle ABS$ is isosceles(which is obvious) we find that $\angle OAK = \angle PAT = \angle TOP$.
$\angle SPO = \angle SRO - \angle RSB = 90^\circ - \angle TOP - \angle SAT = 90^\circ - \angle OAT = \angle AOT = \angle APT$ .And because $\angle ATO = \angle APO = 90^\circ$, we find $\angle TPS = 90^\circ$. So $TP \perp SB$ and we are done.

Attachments:



drmzjoseph

#22 Jun 17, 2015, 11:03 am

Z is the antipode of T at ω and $SZ \cap AB \equiv X \Rightarrow (A, B, X, T) = -1$
$PS \perp TS \iff P \in SZ \iff P(Z, T, A, B) = -1 \iff PB \parallel ZT \iff PB \perp AB$

Dukejukem

#23 Aug 15, 2015, 8:29 am

#23 Aug 15, 2015, 6:29 am

Let ST cut Γ for a second time at M, and let the perpendicular to AB through B cut AO at Q. Let the tangent to ω at S meet AB at Z, and denote $O_1 \equiv OZ \cap QT$. We will prove that $QS \perp TS$.

First, note that the homothety with center S that takes $\omega \mapsto \Gamma$ also takes $T \mapsto M$. It follows that this homothety takes AB to the line tangent to Γ at M. Therefore, the tangent to Γ at M is parallel to AB, which implies that M is the midpoint of arc $\widehat{AB}$. Hence, ST bisects $\angle ASB$. Now, from the Menelaus Theorem for the transversal ZOO_1 w.r.t. $\triangle AQT$, we obtain

$$1 = \frac{ZT}{ZA} \cdot \frac{OA}{OQ} \cdot \frac{O_1Q}{O_1T}.$$

Observe that $\triangle AOT \sim \triangle AQB$ and $\triangle ZAS \sim \triangle ZSB$. It follows from these two relations and the Angle Bisector Theorem that $OA : OQ = TA : TB = SA : SB = ZA : ZS$. Because $ZS = ZT$ by equal tangents, we obtain $O_1Q = O_1T$. Hence, O_1 is the midpoint of $\overline{QT}$ in right triangle $\triangle BQT$. It follows that $O_1B = O_1Q = O_1T$. Moreover, since O_1 lies on OZ, the perpendicular bisector of $\overline{ST}$, we have $O_1S = O_1T$. Thus, B, Q, T, S are concyclic, and QT is a diameter of their circumcircle. Hence, $QS \perp TS$ as desired. $\square$

Wolowizard

#24 Oct 2, 2015, 9:53 pm • 1

My solution :
Considering the equivalent problem :set intersection of BP and circumcircle of ABC X , and set $\angle XAC = x$. Now we can calculate that angles $\angle KAO = (B-C)/2\angle OAP = 90-(b-c)/2\angle AKn = (B-C)/2$
$\angle APB = x-(B-C)/2\angle BPK = 90-x-a/2\angle nKP = a/2$(n is perpendicular to BC at K) which now implies that Cevas theorem hold for triangle AKP. So we are done.

This post has been edited 1 time. Last edited by Wolowizard, Oct 2, 2015, 9:54 pm
Reason: Typo

Abubakir

#25 Jan 26, 2016, 11:58 pm

My solution, take P such that $\angle TSP = \angle PBA = 90$. Then, we will prove that A, O and P are collinear,. Set $SP \cap AB = X$ and $SP \cap \omega = Y$. Considering homotethy centred at S which brings ω to Γ gives us that ST bissects $\angle ASB$. Notice that $\angle XST$ is adjacent to $\angle TSP$, so $\angle XST = 90$ meaning that (X, T, A, B) are harmonic quadruple. Also, $\angle YST \equiv \angle TSP = 90$. So O is the midpoint of YT. we will prove that PA bissects YT, which completes solution. Indeed, consider harmonic pencil (PX, PT, PA, PB) intersecting line YT. PB is perpendicular to AB from our choice of P, YT is perpendicular to AB from is diametr of ω and AB is tangent to ω at T, so PB and YT are parallel, $PX \cap YT = Y$, $PT \cap YT = T$, so PA intersects YT at the midpoint, which is O, QED.

Quick Reply

High School Olympiads

concyclic?

Reply

sfsreporter
#1 Mar 24, 2009, 12:04 am

i have a situation in which i have three line segments having perpendicular bisectors concurrent.

can call the 6 points concyclic?

sfsreporter
#2 Mar 24, 2009, 8:06 am

what other condition should they satisfy so that they can be called concyclic?

Luis González
#3 Mar 24, 2009, 9:16 am

We can't tell exactly as it merely depends on the problem conditions. But for instance, if we have six points A,B,C,D,E,F such that perpendicular bisectors of AB,CD,EF concur at some point O, then A,B,C,D,E,F are concyclic if and only if the perpendicular bisectors of another pair of distinct segments pass through O.

For more results see also Cyclic Hexagon, Fuhrmann's Theorem and elsewhere.

Quick Reply

High School Olympiads

An interesting Sangaku problem

Reply

Source: 0

Luis González

#1 Mar 14, 2009, 12:25 am

$ABCD$ is a square with side length $a.$ P and Q are two points on segment $\overline{AB},$ such that P is between A,Q and Q is between $B,P.$ Let $\mathcal{C}_1(r_1)$ and $\mathcal{C}_2(r_2)$ be the incircles of triangles $\triangle APD, \triangle BQC$ and let $\mathcal{C}_3(r_3)$ be the circle tangent to segments $\overline{PD}, \overline{QC}$ and $\overline{DC}.$ Show that:

$$\frac{1}{r_3}=\frac{1}{a-2r_1}+\frac{1}{a-2r_2}$$

yetti

#2 Mar 17, 2009, 9:58 am

4 different circles are tangent to PD, QC, DC. If $E \equiv DP \cap CQ$ and A, P, Q, B follow on AB in the described order, the equation holds for incircle $\mathcal{C}_3$ of $\triangle CDE.$ M, N are midpoints of AD, BC. Circles $\mathcal{C}_1, \mathcal{C}_2$ are tangent to AD, BC at S, T. $\angle C, \angle D, \angle E$ are angles of $\triangle CDE.$

$$\frac{1}{a-2r_1}+\frac{1}{a-2r_2}=\frac{1}{2MS}+\frac{1}{2NT}=\frac{1}{PD-AP}+\frac{1}{QC-BQ}=$$

$$=\frac{1}{\frac{a}{\sin D}-\frac{a}{\tan D}}+\frac{1}{\frac{a}{\sin C}-\frac{a}{\tan C}}=\frac{1}{a}\left(\frac{1}{\tan\frac{D}{2}}+\frac{1}{\tan\frac{C}{2}}\right)=\frac{1}{r_3}$$

Prove that circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ have common tangent.

ith_power

#3 Mar 17, 2009, 10:22 am

> yetti wrote:
>
> 4 different circles are tangent to PD, QC, DC. If $E \equiv DP \cap CQ$ and A, P, Q, B follow on AB in the described order, the equation holds for incircle $\mathcal{C}_3$ of $\triangle CDE.$ M, N are midpoints of AD, BC. Circles $\mathcal{C}_1, \mathcal{C}_2$ are tangent to AD, BC at S, T. $\angle C, \angle D, \angle E$ are angles of $\triangle CDE.$
>
> $$\frac{1}{a-2r_1}+\frac{1}{a-2r_2}=\frac{2}{MS}+\frac{2}{NT}=\frac{1}{PD-AP}+\frac{1}{QC-BQ}=$$
>
> $$=\frac{1}{\frac{a}{\sin D}-\frac{a}{\tan D}}+\frac{1}{\frac{a}{\sin C}-\frac{a}{\tan C}}=\frac{1}{a}\left(\frac{1}{\tan\frac{D}{2}}+\frac{1}{\tan\frac{C}{2}}\right)=\frac{1}{r_3}$$
>
> ---
>
> Prove that circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ have common tangent.

the first line should be $\frac{1/2}{MS}$

PD-AP=2*MS.

yetti
#4 Mar 17, 2009, 10:37 am

Right. How about the common tangent ?

yetti
#6 Mar 20, 2009, 10:43 pm

This is consequence of the square ABCD having an incircle (I) and of the following theorem: If 4 directed common tangents of 4 directed circles touch a single directed circle, then their other 4 directed common tangents also touch a single directed circle. In this case, 2 of the 4 directed circles are $\mathcal{C}_1, \mathcal{C}_2$ and the other 2 are of zero radius (points C, D). Directed common tangents CD, DC of the points C, D coincide in position, but they still have opposite directions. One is directed tangent of directed (I), the other of directed $\mathcal{C}_3$. See http://www.mathlinks.ro/viewtopic.php?t=247411.

sunken rock
#7 Mar 21, 2009, 5:33 pm

Let OA ∩ CD = X, reflection of C and D in OA be C' and D' respectively. Due to symmetry, CD, OA and C'D' meet at X, so B is on the polar of X to the circle O, and we can conclude that AX = OA, and it's quite easy to see that, since OE and DX are perpendicular, E belongs to the circle (A, AO), which is the reflection of circle O in B, hence BE=BF.

Best regards,
sunken rock

armpist
#8 Mar 22, 2009, 11:10 pm

It would be terribly nice to know who deleted my post with two

solutions for this nice Sangaku and for what personal reasons?

If there is an interest I can always resubmit. I remember the

solutions, yet I believe that my post can be restored from the backup.

M.T.

yetti
#9 Mar 23, 2009, 1:01 am

You can resubmit. However, if any of your posts contain least air of personal attacks, they will be deleted again in their entirety.

Virgil Nicula
#10 Mar 23, 2009, 10:46 am

Nice and easy problem ! Thank you.

> luisgeometria wrote:
> $ABCD$ is a square $(AB=a)$. Let $\{P\ ,\ Q\}\subset (AB)$ be two points so that $P\in (AQ)$.
>
> Let $C_1(r_1)\ ,\ C_2(r_2)$ be the incircles of APD , BQC respectively. Let $C_3(I,r_3)$ be the circle
>
> which is tangent to (PD) , (QC) and (DC) . Show that $\frac{1}{r_3}=\frac{1}{a-2r_1}+\frac{1}{a-2r_2}$.

Proof. Denote $\{U\ ,\ V\}\subset (CD)$ for which $PU\parallel QV\perp CD$. Observe that $\left\|\begin{array}{c} a-2r_1=DP-PA\\ a-2r_2=CQ-QB\end{array}\right\|$.

Thus, $\frac{1}{a-2r_1}+\frac{1}{a-2r_2}=\frac{1}{DP-PA}+\frac{1}{CQ-QB}=\frac{DP+PA+CQ+QB}{a^2}$. Therefore,

$\frac{1}{r_3}=\frac{1}{a-2r_1}+\frac{1}{a-2r_2}\Longleftrightarrow r_3\cdot(DP+PA+CQ+QB)=a^2\Longleftrightarrow$

$r_3\cdot(DP+DU+CQ+CV)=a^2\Longleftrightarrow 2\left([DIP]+[DIU]+[CIQ]+[CIV]\right)=a^2\Longleftrightarrow$

$2\left([DPQC]-[PIQ]-[UIV]\right)=a^2\Longleftrightarrow a\cdot(a+PQ)-a\cdot PQ=a^2$, what is truly.

This post has been edited 3 times. Last edited by Virgil Nicula, Mar 24, 2009, 4:23 am

Luis González

#11 Mar 23, 2009, 11:25 am

Thanks everyone for your nice solutions. Mine is too long so, I wont post it but I'd like to attach another solution for this problem yet not elementary. See pages *33* and *63*

Attachments:

Sangaku problems.pdf (657kb)

Virgil Nicula

#12 Mar 24, 2009, 4:24 am

> *Virgil Nicula wrote:*
>
> **An easy extension.**
>
> $ABCD$ is a rectangle $(AB=a\ ,\ AD=b)$. Let $\{P\ ,\ Q\}\subset(AB)$ be two points so that $P\in(AQ)$.
>
> Let $C_1(r_1)\ ,\ C_2(r_2)$ be the incircles of APD , BQC respectively. Let $C_3(I,r_3)$ be the circle which
>
> is tangent to (PD) , (QC) and (DC) . Show that $\frac{a}{r_3}=\frac{b}{b-2r_1}+\frac{b}{b-2r_2}$ and $\frac{2r_3}{a}\ +\ \frac{r_1+r_2}{b}\ \le\ 1$.

Three particular cases.

P1. $ABCD$ is a square $(AB=a)$. Let $\{P\ ,\ Q\}\subset(AB)$ be two points so that $P\in(AQ)$ .Let $C_1(r_1)\ ,\ C_2(r_2)$ be the incircles of APD , BQC respectively. Let $C_3(I,r_3)$ be the circle which is tangent to (PD) , (QC) and (DC) .

Show that $\frac{1}{r_3}=\frac{1}{a-2r_1}+\frac{1}{a-2r_2}$ and $r_1+r_2+2r_3\ \le\ a$ (the proposed problem).

P2. $ABCD$ is a rectangle $(AB=a\ ,\ AD=2a)$. Let $\{P\ ,\ Q\}\subset(AB)$ be two points so that $P\in(AQ)$.

Let $C_1(r_1)\ ,\ C_2(r_2)$ be the incircles of APD , BQC respectively. Let $C_3(I,r_3)$ be the circle which

is tangent to (PD) , (QC) and (DC) . Show that $\frac{1}{r_3}=\frac{1}{a-r_1}+\frac{1}{a-r_2}$ and $2r_3\ +\ \frac{r_1+r_2}{2}\ \le\ a$.

P3. $ABCD$ is a rectangle $(AB=2b\ ,\ AD=b)$. Let $\{P\ ,\ Q\}\subset(AB)$ be two points so that $P\in(AQ)$.

Let $C_1(r_1)\ ,\ C_2(r_2)$ be the incircles of APD , BQC respectively. Let $C_3(I,r_3)$ be the circle which

is tangent to (PD) , (QC) and (DC) . Show that $\frac{2}{r_3}=\frac{1}{b-2r_1}+\frac{1}{b-2r_2}$ and $r_1+r_2+r_3\le b$.

armpist

#13 Jun 19, 2009, 7:21 am

This trivial Sangaku problem needs a PWW solution.

Let me put just the dedication words.
This note is for my North American friends.

Friendly,
M.T.

Attachments:
ML 5.doc (24kb)

jayme

#14 Jun 19, 2009, 4:35 pm

Dear Mathlinkers,
I want only to give a reference for this nice figure, san Gaku (1877)

Fukagawa H., Pedoe D., 3.2.4. Japanese Temple Geometry Problems (1989) 40.

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

locus of isogonal conjugate point

Leonhard Euler

#1 Feb 13, 2009, 9:44 pm

Let ABC be a triangle and X, Y be the variable points on AC, AB such that direction of XY is constatnt. Let P be the intersection of BX, CY. Prove that isogonal conjugate point of X wrt triangle ABC lie on constant line.

Luis González

#2 Mar 23, 2009, 6:00 am

Generalization: Point U lies in the plane of $\triangle ABC$. Variable line through U cuts AB, AC at M, N and $P \equiv BN \cap BM$. Then the isogonal conjugate P' of P with respect to $\triangle ABC$ moves on a fixed line.

Let $\mathcal{H}$ be the circumconic of $\triangle ABC$ tangent to lines BU, CU and define $Q \equiv AP \cap BC$. Since $N(B, C, Q, M) \equiv N(P, A, Q, M) = -1$, it follows that MU is the polar of Q WRT $\mathcal{H} \Longrightarrow P \in \mathcal{H}$. Hence, P' moves on the isogonal line ℓ of $\mathcal{H}$ with respect to $\triangle ABC$.

High School Olympiads

CF is bisector

Source: Russia 2003

stergiu
#1 Aug 7, 2008, 1:24 pm

In a triangle ABC we have $AB = AC$. The incircle of $\triangle ABC$ touches BC at M.Let N the midpoint of AB and E the common point of the incircle and MN, but not point M. If the tangent to the incircle at E meets side AB at F , prove that CF bisects angle C.

Babis Stergiou

Akashnil
#2 Aug 7, 2008, 2:26 pm

My solution is total angle chase.

Denote I incenter. Draw the line AIM. Let $IX \perp AB$. Join IE. Let Bisector of $\angle C$(going through I) meet AB at F'

Let $\angle BAM = \angle CAM = \theta \implies \angle IME = \angle IEM = \theta \implies \angle AIE = 2\theta$
But $\angle AIX = 90^0 - \theta \implies \angle XIE = \angle AIE - \angle AIX = 2\theta - (90^0 - \theta) =$
$3\theta - 90^0 \implies \angle XIF = \dfrac{3\theta - 90^0}{2}$
$\angle XFI = 90^0 - \dfrac{3\theta - 90^0}{2} = 135^0 - \dfrac{3\theta}{2}$
But $\angle XF'I = \angle AF'C = 180^0 - 2\angle A - \dfrac{\angle C}{2} = 180^0 - 2\theta - \dfrac{\frac{180^0 - \angle A}{2}}{2} = 135^0 - \dfrac{3\theta}{2}$
So $\angle XFI = \angle XF'I \implies F \equiv F'$

jayme
#3 Aug 7, 2008, 4:16 pm

Dear Mathlinkers,
let 1 the incircle, I the center of 1, D the point of contact of 1 with AB.
1. E is the symmetric of D wrt FI.
2. ABC being A-isoceles, M is the Feuerbach' s point of ABC.
3. It is known that if F is the foot of the C-bissector, then E is on NM
4. By reasoning in absurdum….
Sincerely
Jean-Louis

vittasko
#4 Aug 7, 2008, 7:14 pm

Let D be, the tangency point of the incircle (I) of $\triangle ABC$, with AB and we denote the point $K \equiv AM \cap DE$.

Because of the line segment FI, where I is the center of (I), is the midperpendicular of the segment DE, it is enough to prove that this line passes through the point C.

So, let be the point $C' \equiv BC \cap FI$ and we will prove that $C' \equiv C$.

It is easy to show that $\triangle DKM$, is an isosceles triangle with $DK = DM$, because of

$\angle DKM = \angle DAM + \angle ADK = \angle AMN + \angle NMD = \angle DMK \Longrightarrow \angle DKM = \angle DMK = \angle IBM$, (1)

From $C'I \perp DE$ and $C'M \perp AM$ we conclude that $\angle DKM = \angle IC'M$, (2)

From $(1), (2) \Longrightarrow \angle IC'M = \angle IBM = \angle ICM$, (3)

Hence, from $(3) \Longrightarrow C' \equiv C$ and the proof is completed.

Kostas Vittas.

Attachments:
t=219246.pdf (5kb)

Virgil Nicula
#5 Aug 10, 2008, 8:13 am

An easy extension. Let ABC be a triangle with the incircle $w = C(I)$. Define the point $M \in (BC)$ so that $MB = \frac{b(p-b)}{c}$. Denote the point $N \in (AB)$ for which $MN \parallel AC$ and the intersection E of w with MN nearer by N . The tangent to w at E meets side $[AB]$ at F . Prove that $I \in CF$.

Remark. For $b = c$ the point M becomes the midpoint of $[BC]$ and obtain the **Stergiu**'a proposed problem.

sunken rock
#6 Mar 22, 2009, 3:20 am

1) Call G=FE∩AC. MN being parallel to AC is easy to see MCGE isosceles trapezoid, hence ÐG=ÐC=ÐB(1) and GE = MC. If P is tangency point of AC with circle I, then we have BM=MC=CP=GE=GP, so CG=BC (2).
BCGF is circumscriptible, so from BF+CG=BC+GF , with (2) we get BF = GF and the triangles BCF and GCF are congruent (s.a.s), hence the result.

Here Ð sign stands for angle sign.

2) Concerning Virgil's extension, it seems CM=b(p-b)/c.

Best regards,
sunken rock

Luis González
#7 Mar 22, 2009, 7:05 am

Pole of AB WRT incircle (I) is the tangency point X of (I) with AB, pole of the angle bisector CI WRT (I) is the infinite point T_{∞} of direction $\perp CI$. E is the pole of line FE WRT (I), but note that FE is the reflection of BC about IY, i.e. $Y \equiv (I) \cap AC \Longrightarrow EX \perp CI \Longrightarrow E, X, T_{\infty}$ are collinear $\Longrightarrow$ Lines CI, EF and AB concur.

dgreenb801
#8 Mar 22, 2009, 7:15 pm

Let FE meet BC at J. Like sunken rock, if we extend EF and CA to meet at G, then since $ME \parallel AC$, $\triangle CJG \sim \triangle ABC$ but since they have the same incircle, they are congruent. Is it this easy?

Quick Reply

High School Olympiads

midpoint and two right triangles

Reply

hollandman

#1 Mar 22, 2009, 2:18 am

Let ABC be a triangle. Right triangles ADB and AEC are constructed outwardly such that $\angle ADB = \angle AEC$ are right and $\angle DAB = \angle EAC$. Let M be the midpoint of BC. Prove that $\angle MDE = \angle DAB$.

Luis González

#2 Mar 22, 2009, 5:42 am

Let N, L be the midpoints of AB, AC. Denote $\angle DAB = \angle EAC = \varphi$. It's easy to see that $\triangle NMD$ and $\triangle LEM$ are congruent, because of $\angle MLE = \angle MND = \angle A + 2\varphi$ and $NM = LE = \frac{1}{2}b, LM = ND = \frac{1}{2}c$. Hence, $MD = ME \Longrightarrow \triangle MED$ is isosceles with apex M. Therefore

$$\angle DME = \angle A + \angle LME + \angle NMD = \angle A + 180^\circ - \angle MLE = 180^\circ + 2\varphi$$

$$\angle DEM = \angle EDM = 90^\circ - \frac{1}{2}(180^\circ - 2\varphi) = \varphi$$

sunken rock

#3 Mar 22, 2009, 8:32 pm

Take P on the perpendicular bisector of BC such that angle MBP = angle DBA, with P and A separated by BC. There is a spiral similarity around B mapping D to M and A to P and another one around C mapping E to M and A to P, both similarities having the same arguments; we got DM = EM, the angle between them 2* angle ABD, the conclusion follows.

Best regards,
sunken rock

Quick Reply

High School Olympiads

perpendicular trapezoid

Reply

hollandman

#1 Mar 22, 2009, 12:14 am

Let $ABCD$ be a trapezoid such that AB is perpendicular to both BC and AD. The circle with AB as a diameter cut CD at R and S. Let l_1 be the line passing through D and parallel to BR. Let l_2 be the line passing through C and parallel to AR. Prove that l_1 and l_2 intersect on AB.

Luis González

#2 Mar 22, 2009, 1:02 am

The solution is quite straightforward using Pappus theorem.

Lines AD and BC intersect at the infinite point of $\perp AB$, lines ℓ_1, RB intersect at the infinite point of RB and lines ℓ_2, RA meet at the infinite point of RA. If R, C, D are collinear, then A, B and the intersection $\ell_1 \cap \ell_2$ are collinear too.

hollandman

#3 Mar 22, 2009, 1:43 am

Yes, that's a nice solution!

What about without Pappus?

terryjohn

#4 Mar 22, 2009, 2:20 am

Let l_1 intersect AB at T and AR at K. Extend AR to meet BC at Q.
Since $TD||BR$ we have that $AK \perp TD$. Moreover $\triangle DAT \sim \triangle ABQ$ thus

$$\frac{DK}{KT} = \frac{AR}{RQ} = [\text{since } AD||BC] = \frac{DR}{RC},$$

which proves that $CT||AR$, as desired.

Quick Reply

High School Olympiads

On the Steiner inellipse

Source: 0

Luis González

#1 Mar 18, 2009, 11:06 am

Let M be the midpoint of the side BC of $\triangle ABC.$ τ is a line on its plane which intersects sides AB, AC at X and $Y,$ respectively. Let S_1 and S_2 be the areas of $\triangle MBX$ and $\triangle MCY.$ Show that

$$\frac{6}{[\triangle ABC]}=\frac{1}{S_1}+\frac{1}{S_2}\Longleftrightarrow \tau \text{ is tangent to the Steiner inellipse of } \triangle ABC$$

Luis González

#2 Mar 21, 2009, 11:50 pm

Let us use barycentric coordinates WRT $\triangle ABC.$ $px+qy+rz=0$ is the barycentric equation of $\tau.$ Then τ cuts AB and AC at $X:(q:-p:0)$ and $Y:(r:0:-p)$

$$\frac{[\triangle XBM]}{[\triangle ABC]}=\frac{q}{2(q-p)} \ \ , \ \ \frac{[\triangle YCM]}{[\triangle ABC]}=\frac{r}{2(r-p)}$$

$$\Longrightarrow \frac{[\triangle XBM]+[\triangle YMC]}{[\triangle XBM]\cdot[\triangle YCM]}=\frac{4qr+2pr-2pq}{qr}[\triangle ABC]$$

$$\Longrightarrow \frac{4qr-2pr-2pq}{qr}=6 \Longrightarrow pq+pr+qr=0$$

τ satisfies the wanted condition $\Longleftrightarrow pq+pr+qr=0,$ which indeed means that τ is tangent to Steiner inellipse $\mathcal{E}\equiv x^2+y^2+z^2-2xy-2xz-2yz=0,$ since $\tau\equiv[p,q,r]$ is tangent to $\mathcal{E}\Longleftrightarrow$ Point $(p:q:r)$ lies on the dual conic $\mathcal{D}\equiv xy+xz+yz=0$ of $\mathcal{E}.$

High School Olympiads

Hungary-Israel Binational 1991_2

Reply

April

#1 Oct 28, 2008, 12:00 pm

The vertices of a square sheet of paper are A, B, C, D. The sheet is folded in a way that the point D is mapped to the point D' on the side BC. Let A' be the image of A after the folding, and let E be the intersection point of AB and $A'D'$. Let r be the inradius of the triangle EBD'. Prove that $r = A'E$.

sunken rock

#2 Mar 21, 2009, 5:41 pm

Take D' a variable point on (BC); the perpendicular bisector of DD' intersects AD at P. D'P is the image of AD when folding to bring D on (BC). Take A' on D'P so that A'D' = AD. The triangle PDD' is isosceles and, since its altitude from D' equals the side of the square, so will the one from D, hence we conclude that PD' is the tangent from D' to the circle (D,DC).
If M is its tangency point to this circle we see that D'M = D'C and EM =EA, so the perimeter of the triangle BD'E is 2a (a – the side of the square) and, being right angled at B, its inradius will be r = p - D'E; but p= AD = A'D', hence r = A'D' - D'E = A'E.

Best regards,
sunken rock

Luis González

#3 Mar 21, 2009, 9:24 pm

This Sangaku problem is known as "The problem of the coin and the sheet". See pages 29 and 98 in the below attachment.

Attachments:

Sangaku problems.pdf (657kb)

Quick Reply

High School Olympiads

tangent to the circumcircle

Reply

Source: Bulgarian Olympiad, 1995

mr.danh

#1 Aug 21, 2008, 8:03 pm

Let triangle ABC has semiperimeter p. E,F are located on AB such that $CE = CF = p$. Prove that the C-excircle of triangle ABC touches the circumcircle (EFC).

The QuattoMaster 6000

#2 Aug 21, 2008, 10:41 pm

> mr.danh wrote:
> Let triangle ABC has semiperimeter p. E,F are located on AB such that $CE = CF = p$. Prove that the C-excircle of triangle ABC touches the circumcircle (EFC).

Solution

sunken rock

#3 Mar 21, 2009, 5:21 pm

Apply Casey to C, E, F and C-excircle.

Best regards,
sunken rock

Luis González

#4 Mar 21, 2009, 9:19 pm

If excircle (I_c) is tangent to CB, CA through X, Y, then it is known that $CX = CY = p$. Inversion with center C and power p^2 swaps circle $\odot(CEF)$ and the sideline BC and takes C-excircle (I_c) into itself. Since BC and (I_c) are tangent, then by conformity $\odot(CEF)$ and (I_c) are tangent as well.

Quick Reply

High School Olympiads

An old problem

Reply

Source: circumcenters are collinear

mr.danh

#1 Aug 21, 2008, 8:03 pm

The incircle (I) of triangle ABC touches BC,CA,AB at A',B'C'. Prove that the circumcenters of triangles AIA',BIB',CIC' are collinear.

tchebytchev

#2 Aug 21, 2008, 8:40 pm

nice problem
let C'' the intersection of AB and the circumcircle of CIC' , his center is the midle of $[IC'']$,and we define with the same way the points B'' and A'' .
to conclude it suffice to prove that A'', B'' and C'' are collinear .
from the fact that CC'' is an exterior bissector of $< BCA$ we have $\frac{C''A}{C''B}=\frac{CA}{CB}$.
and with the same way for $\frac{B''A}{B''C}$ and $\frac{A''B}{A''C}$ and by menalus we get the desired result.

Erken

#3 Aug 21, 2008, 8:50 pm

If I am not mistaken,this problem was posted at least twice before.
I remember that **pohoatza** proposed outstanding solution with using inversion(exactly my solution,but I was the one who posted the problem) and several other solutions were posted as well,one of them contains using of **Menelaus** theorem.
Hmm...It looks like I was wrong ,but nevertheless,the problem mentioned above was generalization of this problem,if I am not mistaken,the goal was to prove that these three circles intersect at most in two points,what is obviously stronger result than the one posted above by **mr.danh**.
Proof:
Invert with center in I.The rest follows from the fact that three medians in a triangle are concurrent.
Edited(after 15 minutes)
That's a miracle: I've just casually found this problem,here is the link:
http://www.mathlinks.ro/viewtopic.php?search_id=1663197197&t=181827

mr.danh

#4 Aug 22, 2008, 7:38 am

This problem is a problem of Singaporean TST 1998 and the author's solution is the proof with **using inversion**.

jayme

#5 Aug 22, 2008, 3:16 pm

Dear Mathlinkers,
the circumcircles of triangles AIA',BIB',CIC' are concurrent at the Schröder's point of ABC (See a nice paper of Darij Grinberg on his website) ; it result that the circumcenters are collinear.
Note that this point of concurrence is on the OI line of ABC.
Sincerely
Jean-Louis

sunken rock

#6 Mar 21, 2009, 5:17 pm

The external angle bisectors of triangle ABC cut the opposite sides at 3 collinear points. A homothety center I and ratio ½ brings A, B, and C to M, N and P, the midpoints of AI, BI and CI respectively.
The center of excircle of triangle AIA' is the intersection of the perpendicular bisector of AI with MN and the centers of the other 2 triangles are obtained similarly. Due to above mentioned homothety, these three points are collinear.

Best regards,
sunken rock

Luis González

#7 Mar 21, 2009, 8:58 pm

Inversion WRT incircle (I) takes A' into itself and A into the midpoint M of $\overline{B'C'}$. Hence, circles $\odot(AIA')$, $\odot(BIB')$ and $\odot(CIC')$ are taken into the medians of the intouch triangle $\triangle A'B'C'$. Thereby, the said circles pass through the incenter I and the inverse of the centroid G' of $\triangle A'B'C'$ under inversion WRT (I), i.e. they are coaxal.

Virgil Nicula

#8 Mar 28, 2009, 8:11 am

> Quote:
>
> **An easy extension.** Let P be a point from the plane of the triangle ABC . Denote
>
> the projections X , Y , Z of the point P to the sidelines BC , CA , AB respectively.
>
> Prove that the circumcenters of the triangles APX , BPY , CPZ are collinearly.

An indication.

Quick Reply

High School Olympiads

Though construction

Reply

Kenny O
#1 Mar 21, 2009, 2:11 am

Construct a triangle ABC given $b-c$,the median m_a and the internal bisector w_a

hurdler
#2 Mar 21, 2009, 2:50 am

horrendously ugly bash

alternatively, just an idea:

edit: Click to reveal hidden text

Luis González
#3 Mar 21, 2009, 7:23 am

Length w_a can be expressed as ${w_a}^2 = AB \cdot AC - BV \cdot CV$, where V is the foot of the A-angle bisector. Since $BV \cdot CV$ is the power of V WRT the semicircle (M) with diameter $\overline{BC}$, we rewrite $AV^2 = AB \cdot AC - \frac{1}{4}BC^2 + VM^2$.

$\Longrightarrow AV^2 = \frac{1}{2}(AC^2 + AB^2) - \frac{1}{2}(AC - AB)^2 - \frac{1}{4}BC^2 + VM^2 \ \ (1)$

By Apollonius theorem $AM^2 = \frac{1}{2}(AB^2 + AC^2) - \frac{1}{4}BC^2 \ \ (2)$

Combining (1) and (2) gives $(AC - AB)^2 = 2(AM^2 - AV^2 + VM^2)$,

which permits to construct $\triangle AMV$ with three known sides, then it remains to determinate the vertices B and C. Note that circle with center M and radius $\frac{1}{2}|AC - AB|$ cuts AV at the orthogonal projections of B, C onto $\overline{AV}$. Hence, this gives the construction of B, C completing the triangle $\triangle ABC$.

hurdler
#4 Mar 21, 2009, 9:03 am

nice solution!

Is there a nice geometry proof of how to get $bc - BW \times WC = w_a$?

Luis González
#5 Mar 21, 2009, 9:46 am

Ray AV cuts circumcircle (O) at P. From $\triangle ABV \sim \triangle APC$ we get

$AP \cdot AV = AB \cdot AC \Longrightarrow (AV + PV)AV = AB \cdot AC$

But $AV \cdot VP = BV \cdot CV \Longrightarrow AV^2 = AB \cdot AC - BV \cdot CV$

Quick Reply

High School Olympiads

Circumcircle of triangle PQR passes through midpoint of BC

Reply

Source: IMO Shortlist 1997, Q18

orl

#1 Aug 10, 2008, 9:24 am

The altitudes through the vertices A, B, C of an acute-angled triangle ABC meet the opposite sides at D, E, F, respectively. The line through D parallel to EF meets the lines AC and AB at Q and R, respectively. The line EF meets BC at P. Prove that the circumcircle of the triangle PQR passes through the midpoint of BC.

The QuattoMaster 6000

#2 Aug 10, 2008, 10:23 am

> orl wrote:
> The altitudes through the vertices A, B, C of an acute-angled triangle ABC meet the opposite sides at D, E, F, respectively. The line through D parallel to EF meets the lines AC and AB at Q and R, respectively. The line EF meets BC at P. Prove that the circumcircle of the triangle PQR passes through the midpoint of BC.

Solution

jayme

#3 Aug 10, 2008, 5:28 pm

Dear Mathlinkers,
this problem has also being proposed at : O.M. IRAN 1998 and Hubei Math Contest 1994.
Thanks for your nice proof.
I saw another solution based on power in the book of Mohamed Assila.
Sincerely
Jean-Louis

orl

#4 Aug 10, 2008, 6:42 pm

Please post those further solutions you know for this problem in this thread. You are also invited to send your solutions for the other posted IMO Shortlist problems. Thanks.

jayme

#5 Aug 10, 2008, 8:29 pm

Dear "Orl",
sorry, I made a confusion in the name. Here is the reference of the book that I have given.
Soulami Tarik Belhaj, Les Olympiades de mathématiques, Ellipses.
I will send my proof that I have in mind.
Sincerely
Jean-Louis

dgreenb801

#6 Aug 10, 2008, 8:40 pm

Quatto, what is point T? Do you mean F instead? Also, can you explain why $MH \perp AP$?

The QuattoMaster 6000

#7 Aug 11, 2008, 7:31 am

Yes, sorry **dgreenb801**, F is the same as T; sorry for the error. [Moderator edit: Error fixed.]

Apply the following lemma to cyclic quadrilateral $ABCD$ to get that $MH \perp AP$:

Consider cyclic quadrilateral $ABCD$ with center O and intersection of diagonals being P. Let AB and CD intersect at X and AD and BC meet at Y. Then, $OP \perp XY$.

This can be proven as follows:

Let the intersection between the circumcircles of $\triangle ABY$ and DCY be Z. Let the feet of the perpendiculars from P to AD and BC be Q and R respectively and the midpoints of AD and BC be M and N respectively. Now, $\angle ADZ = \angle ZCB$ and $\angle ZBC = 180 - \angle ZBY = 180 - \angle YAZ = \angle ZAD$, so $\triangle ZAD \sim \triangle ZBC$. Since M and N are the respective midpoints of AD and BC, we see that $\triangle ZAM \sim \triangle ZBN$, so $\angle YMZ = \angle YNZ$, so $YZNM$ is cyclic. Since O is the circumcenter of $ABCD$, we see that $\angle YMO + \angle YNO = 90 + 90 = 180$, so $YNOM$ is cyclic, so $YZNOM$ is cyclic. Thus, $\angle YZO = \angle YNO = 90$, so $OZ \perp YZ$. By Miguel's Theorem, we see that Y, Z, and X are collinear, so $OZ \perp YX$. Now, notice that since $ABCD$ is cyclic, $\triangle PAD \sim \triangle PBC$, so $\triangle PAQ \sim \triangle PBR$, so $\frac{AQ}{BR} = \frac{AP}{PB} = \frac{AD}{BC}$, which means that $\triangle ZAQ \sim \triangle ZBR$, which implies that $\angle ZQY = \angle ZRY$, so $ZRQY$ is cyclic. Since $\angle PRY + \angle PQY = 90 + 90 = 180$, we see that $YZRPQ$ is cyclic, so $\angle PZY = \angle PRY = 90$, which implies that P and O lie on the perpendicular through Z to YX. Thus, $OP \perp YX$.

dgreenb801

#8 Aug 11, 2008, 8:06 am

Thank you !

April

#9 Sep 11, 2008, 7:18 am • 1

I think this problem is posted many times on MathLinks, but I can't get the links, so let me present the simplest solution I have in mind (surely that it's not new)

We have quadrilateral $BCEF$ is cyclic and $EF \parallel QR$, so the quadrilateral $BQCR$ is also cyclic. Therefore $DQ \cdot DR = DB \cdot DC$ (1)

On the other hand, $(PBDC)$ is a harmonic division, so $DB \cdot DC = DM \cdot DP$ (2), where M is the midpoint of BC.

From (1) and (2), we conclude that M lies on the circumcircle of triangle PQR.

limes123

#10 Nov 12, 2008, 1:36 am

> *The QuattoMaster 6000 wrote:*
> Consider cyclic quadrilateral $ABCD$ with center O and intersection of diagonals being P. Let AB and CD intersect at X and AD and BC meet at Y. Then, $OP \perp XY$.

We can prove, that O is orthocenter of XYP. We see that $(A,D;F,Y) = (B,C;G,Y) = (D,A;F,Y)$ => (A,D;F,Y)=(B,C;G,Y)=1. Hence PX is polar of Y => $PX \perp OY$. Analogously $XO \perp YP$ QED (I think it's known as Brokard's theorem)

sunken rock

#11 Mar 21, 2009, 1:42 am

I have a simpler way to prove this lemma: it is well known (Pascal to the 'hexagons' ABCCDA and ABBCDD) that the tangents to circle (ABCD) at A and C concur on XY, let their common point be K, same the ones at B and D, which concur at L. Then since AP.PC = BP.PD it follows that P belongs to radical axis of the circles (K,KA) and (L,LB). But O belongs to that line too [the tangents from O to both circles are radii of (ABCD)], therefore OP and KL are parallel, but KL and XY are perpendicular.

Best regards,
sunken rock

This post has been edited 1 time. Last edited by sunken rock, Mar 21, 2009, 11:08 am

Luis González

#12 Mar 21, 2009, 6:35 am

Lines PR and PQ cut circle $\mathcal{T} \equiv \odot(RCQB)$ again at X and Y. Notice that P is radical center of $\mathcal{T}$, the 9-point circle (N) and the circle with diameter $\overline{BC}$. Inversion with center P and power $\overline{PB} \cdot \overline{PC}$ takes $D \mapsto M, R \mapsto X$ and $Q \mapsto Y$. In other words, circle $\odot(PQR)$ is taken to the line XY. But $(B, C, D, P) = -1$ and $P \equiv QY \cap XR$ is on the polar of the intersection of the diagonals of the quadrilateral $RXQY$ WRT $\mathcal{T} \Longrightarrow XY$ passes through D. If XY passes through D, then its inverse $\odot(PQR)$ passes through the inverse M of D and the proof is completed.

sayantanchakraborty

#13 Jan 23, 2014, 11:34 am

Let O be the midpoint of BC.

We note that $\angle PFB = \angle AFE = \angle C$

$\angle FPB = \angle B - \angle C$

Applying the law of sines in triangle PFB.

$$PB = \frac{BFsinC}{sin(B-C)} = \frac{acosBsinC}{sin(B-C)}$$

$$PD = PB + BD = \frac{acosBsinC}{sin(B-C)} + ccosB$$

$$OD = OB - BD = \frac{a}{2} - ccosB$$

$\angle QFD = \angle BFD = \angle C$

$\angle FQD = \angle AFE = \angle C$

So $QD = FD = \dfrac{csin2B}{2sinC}$

Similarly applying the law of sines in triangle DRC.

$$DR = \frac{DCsinC}{sinB} = \frac{bsin2C}{2sinB}$$

Now points P,Q,O,R lie on a circle

$\Leftrightarrow PD * OD = QD * DR$

$$\Leftrightarrow (\frac{acosBsinC}{sin(B-C)} + ccosB)(\frac{a}{2} - ccosB) = \frac{bcsin2Bsin2C}{4sinBsinC}$$

$\Leftrightarrow 4sinBcosBsinCcosC = 4sinBcosBsinCcosC$

which is true.Hence the proof.

Maths is the doctor of science.

AnonymousBunny

#14 Jun 5, 2014, 6:51 pm

Darn I failed to come up with a synthetic solution, and tried trig bash instead, which I believe is the more natural approach. My solution is probably similar to that posted by **sayantanchakraborty**, although I didn't read it fully.

Diagram

Diagram
Solution

Quick Reply

High School Olympiads

cyclic quadrilateral

Reply

Cassi

#1 Aug 13, 2008, 11:27 pm

Let $ABCD$ be a cyclic quadrilateral, inscribed in a circle ω. Let A', B', C', D' be the points where the tangents at A and B, at B and C, at C and D and at D and A, respectively, intersect. Prove that the lines AC, BD, $A'C'$ and $B'D'$ are concurrent, that is, they intersect at one point.

not_trig

#2 Aug 14, 2008, 2:33 am

This is just Brianchon's theorem applied to the degenerate hexagons $A'AD'C'CB'$ and $A'BB'C'DD'$.

sunken rock

#3 Mar 20, 2009, 10:35 pm

However, this is called …Newton's Theorem "In a circumscribed quadrilateral the diagonals and the lines joining the points of contact of opposite sides with the inscribed circle are concurrent". (A'B'C'D' is the quadrilateral, A'C' and B'D' its diagonals, AC and BD the lines joining the contact points of opposite sides with the inscribed circle) and has the following proof as well:

For faster writing: let M, N, P and Q be the contact points of AB, BC, CD and DA with the incircle and, let {R} = AC∩MP.
Apply sine theorem to DARM and DCRN and, seeing that ÐAMR=180º-ÐCPR (the 2 angles count half of the measure of the arcs MQP and PNM) and ÐARM=ÐPRC, we get AR/CR=AM/CP (1).
Let {R'} = AC∩NQ. After similar considerations we get AR'/CR'=AQ/CN (2), but AQ=AM and CN=CP, and from (1) and (2) we get R≡R'.
Similarly we shall get that MP and NQ intersect BD at the same point, thence AC, BD, MP and NQ are concurrent.

Best regards,
sunken rock

Luis González

#4 Mar 21, 2009, 5:30 am

> Cassi wrote:
> Let $ABCD$ be a cyclic quadrilateral, inscribed in a circle ω. Let A', B', C', D' be the points where the tangents at A and B, at B and C, at C and D and at D and A, respectively, intersect. Prove that the lines $AC, BD, A'C'$ and $B'D'$ are concurrent, that is, they intersect at one point.

See http://www.artofproblemsolving.com/Forum/viewtopic.php?t=265111

Quick Reply

High School Olympiads

square

Reply

Source: Circumcircle

Ulanbek_Kyzylorda KTL

#1 Mar 15, 2009, 10:36 am

P is the point on the circumcircle of the square ABCD between points C and D (on arc CD). Prove that PA^2-PB^2=PB*PD-PA*PC.

Mathias_DK

#2 Mar 19, 2009, 7:16 pm

> *Ulanbek_Kyzylorda KTL wrote:*
> P is the point on the circumcircle of the square ABCD between points C and D (on arc CD). Prove that PA^2-PB^2=PB*PD-PA*PC.

Let a, b, c, d, p be the corresponding complex numbers to A, B, C, D, P. Wlog assume $a = 1, b = i, c = -1, d = -i$, then $p = e^{i\alpha} = \cos\alpha + i\sin\alpha$, where $\alpha \in (\pi; \frac{3\pi}{2})$.
Then obviously $|PA| = \sqrt{2-2\cos\alpha}$, $|PB| = \sqrt{2-2\sin\alpha}$, $|PC| = \sqrt{2+2\cos\alpha}$, $|PD| = \sqrt{2+2\sin\alpha}$. Inserting gives:
$|PA|^2 - |PB|^2 = |PB|\cdot|PC| - |PA|\cdot|PC| \iff$
$\sin\alpha - \cos\alpha = \sqrt{1-\sin^2\alpha} - \sqrt{1-\cos^2\alpha}$.
But since $\alpha \in (\pi; \frac{3\pi}{2})$: $\sqrt{1-\sin^2\alpha} = -\cos\alpha$ and $\sqrt{1-\cos^2\alpha} = -\sin\alpha$. Inserting yields the desired :) QED

Luis González

#3 Mar 20, 2009, 10:50 am

Lemma: $\triangle ABC$ is isosceles with $AB = AC = L$ and P is a point on the arc BC of its circumcircle that does not contain A. Then $PA^2 - L^2 = PB \cdot PC$.

Use the previous lemma for the isosceles $\triangle ABC$ and $\triangle BAD$

$PB^2 - a^2 = PA \cdot PC$, $PA^2 - a^2 = PB \cdot PD$

$\Longrightarrow PB^2 - PA^2 = PA \cdot PC - PB \cdot PD.$

Mathias_DK

#4 Mar 20, 2009, 3:50 pm

Is that lemma easier to prove than proving the problem directly ? :)

Mathias_DK

#5 Mar 20, 2009, 8:02 pm

Okay I have a solution by complex numbers, but I would like to see your solution - which I guess is more classic? :)

Virgil Nicula

#6 Mar 25, 2009, 10:01 pm

#6 Mar 25, 2009, 10:01 pm

> Luis González wrote:
> **Lemma:** $\triangle ABC$ is isosceles with $AB = AC = L$ and P is a point on the arc BC of its circumcircle that does not contain $A.$ Then $PA^2 - L^2 = PB \cdot PC.$

See **here** an usual and nice extension of this lemma.

Quick Reply

High School Olympiads

Segment in interior....

Source: JBMO Shortlist 2000 problem 14

Ahiles

#1 Mar 19, 2009, 1:46 am

Let ABC be a triangle. Find all segments XY, which are situated in the interior of the triangle, such that XY and five from the segments XA, XB, XC, YA, YB, YC divide area of ABC in five equivalent parts. Prove that all found segments are concurrent.

Luis González

#2 Mar 20, 2009, 6:07 am

$[\triangle AXB]=[\triangle CXB]=[\triangle AYX]=[\triangle CYX]=[\triangle CYA] \Longrightarrow X$ lies on the B-median BM_b and Y coincides with the centroid of $\triangle AXC$. Further, B, X, Y are collinear such that $BX=\frac{2}{5}BM_b$ and $YM_b=\frac{1}{5}BM_b$. By similar reasoning, we conclude that such three segments concur at the centroid G of $\triangle ABC$.

High School Olympiads

How many solutions can you offer for this beautiful problem

mathVNpro
#1 Mar 18, 2009, 11:53 pm

Let (I) be the incircle of triangle ABC. D, E, F are the tagency points of (I) with BC, CA, AB, respectively. M, N, P respectively are the projections of A, B, C onto BC, CA, AB. We call X, Y, Z are the midpoints of AM, BN, CP. Prove that: DX, EY, FZ intersect at a point.

@ My solution uses pole and polar with radical axes.

nsato
#2 Mar 19, 2009, 1:40 am

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=42412
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=48453

Luis González
#3 Mar 19, 2009, 9:20 pm

Let H_a, H_b, H_c be the feet of the altitudes issuing from A, B, C and X_a, X_b, X_c the tangency points of the incircle (I) with $BC, CA, AB.$ Using barycentric coordinates WRT $\triangle ABC$ we have

$$I \equiv (a : b : c) \ , \ O \equiv (a^2S_A : b^2S_B : c^2S_C)$$

$$X_a \equiv (0 : s-c : s-b) \ , \ H_a \equiv (0 : S_C : S_B)$$

Since $S_B + S_C = a^2,$ then the coordinates of the midpoint N of segment $\overline{AH_a}$ are $N \equiv (a^2 : S_C : S_B).$ Therefore, equation of line NX_a is given by:

$$\ell_a \equiv (S_C(s-b) - S_B(s-c))x - a^2(s-b)y + a^2(s-c)z = 0$$

$OI \equiv bc(bS_B - cS_C)x + ac(cS_C - aS_A)y + ab(aS_A - bS_B)z = 0$ cuts ℓ_a at

$$X_{57}\left(\frac{a}{b+c-a} : \frac{b}{c+a-b} : \frac{c}{a+b-c}\right)$$

High School Olympiads

concurrent in triangle

Reply

hollandman

#1 Mar 19, 2009, 1:50 am

In triangle ABC, let X, Y, Z be points on sides BC, CA, AB respectively. Assume that the perpendicular to BC at X, the perpendicular to CA at Y, and the perpendicular to AB at Z are concurrent. Prove that the perpendicular from A to YZ, the perpendicular from B to ZX, and the perpendicular from C to XY are concurrent.

Luis González

#2 Mar 19, 2009, 5:54 am

Let P be the concurrency point of the perpendiculars from X, Y, Z to BC, CA, AB. Let M be the orthogonal projection of A onto YZ. Note that $AYPZ$ is cyclic on account of $\angle PYA = \angle PZA = 90^\circ$. Therefore, $\angle ZAM = \angle PZY = \angle PAY$ $\Longrightarrow$ Rays AP, AM are isogonal WRT $\angle BAC$. By similar reasoning, we conclude that perpendiculars from A, B, C to YZ, ZX, XY concur at the isogonal conjugate of P.

sunken rock

#3 Mar 19, 2009, 8:10 pm

Wasn't that the Carnot Theorem?

Best regards,
sunken rock

Luis González

#4 Mar 19, 2009, 8:14 pm

I think Carnot theorem states something different: Given three points on the plane of $\triangle ABC$, then the perpendiculars from X, Y, Z to BC, CA, AB concur $\Longleftrightarrow$

$$(XB^2 - XC^2) + (YC^2 - YA^2) + (ZA^2 - ZB^2) = 0$$

Quick Reply

High School Olympiads

prove that it is parallelogram

Reply

silouan
#1 Sep 4, 2006, 6:45 pm

On the sides AB, BC, CD, DA of a quadrilateral $ABCD$ we construct alternately to outside and inside similar triangles with vertices Y, W, X, Z respectively .Show that $YWXZ$ is a parallelogram .

yetti
#2 Sep 5, 2006, 8:01 am

Denote $\phi = \angle AYB,\ k = \frac{BY}{AY}$. Spiral similarity $\mathcal{S}_Y$ with center Y, rotation angle ϕ and similarity coefficient k carries the vertex A into B. Likewise spiral similarity $\mathcal{S}_W$ with center W, rotation angle $-\phi$ and similarity coefficient 1/k carries B into C, spiral similarity $\mathcal{S}_X$ with center X, rotation angle ϕ and similarity coefficient k carries C into D, and spiral similarity $\mathcal{S}_Z$ with center Z, rotation angle $-\phi$ and similarity coefficient 1/k carries D into A. The similarity transformation $S_Z \circ S_X \circ S_W \circ S_Y$ is a translation, because the sum of all rotation angles is zero and the product of all similarity coefficients one. Since this translation has the fixed point A, it is an identity transformation. On the other hand, $S_W \circ S_Y = \mathcal{T}_1,\ S_Z \circ S_X = \mathcal{T}_2$ are also translations, because in both cases, the sum of rotation angles is zero and product of similarity coefficients one. Since $\mathcal{T}_2 \circ \mathcal{T}_1 = I$ is identity transformation, translations $\mathcal{T}_1,\ \mathcal{T}_2$ are by segments of the same length with opposite directions. Spiral similarity $\mathcal{S}_Y$ leaves Y in place, hence translation $\mathcal{T}_1$ carries Y into the same point Y' as spiral similarity S_W, so that the triangles $\triangle YWY' \sim \triangle BWC$ are spirally similar. Likewise, spiral similarity $\mathcal{S}_X$ leaves X in place, hence translation $\mathcal{T}_2$ carries X into the same point X' as spiral similarity S_Z, so that the triangles $\triangle XZX' \sim \triangle DZA$ are also spirally similar. Since the triangles $\triangle BWC \sim \triangle DZA$ are directly similar, the triangles $\triangle YWY' \sim \triangle XZX'$ are directly similar as well. From $\mathcal{T}_2 \circ \mathcal{T}_1 = I$ and $C = \mathcal{T}_1(A),\ A = \mathcal{T}_2(C)$, we have $YY' = AC = -CA = -XX'$ and $YY' \parallel AC \parallel XX'$, so that the last 2 triangles are centrally similar (congruent) with similarity coefficient -1 and similarity center $O \equiv YX \cap WZ \cap Y'X'$, the common midpoint of YX, WZ, Y'X'. Since the diagonals YX, WZ of YWXZ cut each other in half at O, it is a parallelogram.

This post has been edited 1 time. Last edited by yetti, Sep 10, 2006, 12:42 am

silouan
#3 Sep 6, 2006, 7:35 pm

Thank you very much very nice solution .
My solution was by complex numbers .

Luis González
#4 Mar 19, 2009, 11:36 am

B is the center of similitude of triangles $\triangle BYA$ and $\triangle BWC$. Such a similitude is the product of the rotation $(B, \angle ABC)$ and the direct homothecy $\nabla(B, \frac{YW}{AC})$. This takes A into C and Y into $W \Longrightarrow \frac{YW}{AC} = \frac{BY}{BA} = k$

Similarly, we'll have: $\frac{ZX}{AC} = \frac{XD}{CD} = k \Longrightarrow YW = XZ$

Mutatis mutandis, $YZ = WX \Longrightarrow$ Quadrilateral $XWYZ$ is a parallelogram.

yetti
#5 Mar 19, 2009, 12:58 pm

Is spiral similarity of triangles BYA and BWC with center B, which takes $A \to C, Y \to W$ (as you say), really product of the rotation $(B, \angle ABC)$ and a direct homothety $\nabla(B, \frac{BY}{BA})$?

Luis González

#6 Mar 19, 2009, 8:04 pm

I apologize for such nonsense, I'm pretty drowsy right now.

What I'm trying to explain is that $\triangle BYW$ and $\triangle BAC$ are similar, then $\frac{YW}{AC}=\frac{BY}{BA}=k$ due to SAS criterion. This similitude with coefficient $\frac{BA}{BC}$ takes A into C and Y into W.

vittasko

#7 Mar 20, 2009, 7:42 pm

Based on the problem Prove APRQ is a parallelogram, we have that the angle formed by the line segments $YW,\ AC,$ is equal to $\angle\omega$ and that $YW=u\cdot AC,$ where $u=\frac{YB}{AB}=\frac{WB}{BC}.$

Similarly, we have also that the angle formed by the line segments $XZ,\ AC,$ is also equal to $\angle\omega$ and that $XZ=u\cdot AC,$ where $\frac{XD}{CD}=\frac{DZ}{AD}=u=\frac{YB}{AB}=\frac{WB}{BC}.$

Hence, we conclude that $YW\parallel=XZ$ and the proof is completed.

Kostas Vittas.

Attachments:

t=109435.pdf (15kb)

Quick Reply

High School Olympiads

Minimum Value problem

Reply

Source: From Crux

shaaam

#1 Mar 19, 2009, 1:34 pm

Triangle ABC has sides 8, 15 and 17. A point P is inside the triangle. Find the minimum value of $PA^2+PB^2+PC^2$.

Please post your solutions in detail :)

Thanks.

Luis González

#2 Mar 19, 2009, 7:37 pm

Minimun value occurs when P is the centroid of $\triangle ABC$, due to Leibniz theorem.

$PA^2+PB^2+PC^2=\frac{1}{3}(a^2+b^2+c^2)+3\cdot PG^2$

$PA^2+PB^2+PC^2$ is minimum $\Longleftrightarrow \ PG=0 \Longleftrightarrow \ P\equiv G$

$\Longrightarrow \ PA^2+PB^2+PC^2=\frac{578}{3}$

Quick Reply

High School Olympiads

Perpendiculars from A to line QR

Reply

Source: USA TST 2008, Day 1, Problem 2

orl

#1 Sep 5, 2008, 4:06 pm • 1

Let P, Q, and R be the points on sides BC, CA, and AB of an acute triangle ABC such that triangle PQR is equilateral and has minimal area among all such equilateral triangles. Prove that the perpendiculars from A to line QR, from B to line RP, and from C to line PQ are concurrent.

mr.danh

#2 Sep 5, 2008, 4:58 pm • 1

Obviously, triangle DEF is the pedal triangle of isodynamic point. So, these line are concurrent at Fermat's point (which point satisfying $\angle APB = \angle BPC = \angle CPA = 120^o$)

hollandman

#3 Mar 19, 2009, 6:32 am

> mr.danh wrote:
> Obviously, triangle DEF is the pedal triangle of isodynamic point.

What is this point, and why?

Luis González

#4 Mar 19, 2009, 7:15 am • 1

Let $\triangle A'B'C'$ be an arbitrary circumscribed equilateral triangle in $\triangle ABC$. Circumcircles $(X), (Y), (Z)$ of $\triangle A'CB$, $\triangle B'AC, \triangle C'BA$ concur at its Miquel point WRT $\triangle A'B'C'$, i.e. the 1st Fermat point F of $\triangle ABC$. Lines FA, FB, FC are pairwise radical axes of the circles $(X), (Y), (Z)$. Then $\triangle XYZ$ is equilateral an its sidelines are the perpendicular bisectors of FA, FB, FC. Let M, N be the projections of Z, Y on $B'C'$ and Z' the projection of Y on ZM. If $Z \not\equiv Z'$, then $YZ \geq YZ' = MN = \frac{1}{2} B'C'$ and we get similar expressions cyclically. Thus, $\triangle A'B'C'$ attains its maximum area (perimeter) if and only if $\triangle A'B'C'$ is centrally similar to $\triangle XYZ$, that is, if $\triangle A'B'C'$ is centrally similar to the pedal triangle $\triangle F_1F_2F_3$ of the 1st Isodynamic point F' of $\triangle ABC$.

For each inscribed equilateral triangle $\triangle DEF$ in $\triangle ABC$ there exists another equilateral triangle $\triangle D'E'F'$ circumscribed in $\triangle ABC$ and centrally similar to $\triangle DEF$. By Gergonne-Ann theorem we have $[ABC]^2 = [DEF] \cdot [D'E'F'] \Longrightarrow [DEF]$ attains its minimum iff $[D'E'F']$ attains its maximum, in other words, if $\triangle D'E'F'$ is centrally similar to the pedal triangle of F'. Then $\triangle DEF$ is identical with $\triangle F_1F_2F_3$.

silversheep

#5 Apr 8, 2009, 7:52 pm • 4

By Miquel's Theorem, the circumcircles of AQR, BPR, CPQ intersect at a point M.
Claim: $\angle MPC = \angle MQA = \angle MRB = 90^\circ$
Sketch of proof: Suppose otherwise. Let P', Q'. R' be feet of perpendiculars from M to BC, CA, AB. From $\Delta MPP' \cong \Delta MQQ' \cong \Delta MRR'$ we get $\Delta P'Q'R' \cong \Delta PQR$, with $\Delta P'Q'R'$ smaller, a contradiction.
So the perpendiculars through P, Q, R to BC, CA, and AB intersect. The triangles ABC, PQR are orthologic, hence the perpendiculars through A, B, C to QR, RP, PQ intersect (This can be proved with Carnot's Theorem and Pythagorean Theorem)

P.S. It seems that the proof works even if "PQR is equilateral" is replaced by "PQR is similar to a given triangle"

math154

#6 May 27, 2011, 12:57 am • 3

Place a, b, c on the unit circle. WLOG $\triangle PQR$ (which has the same orientation as $\triangle ABC$) is oriented such that $p + q\omega + r\omega^2 = 0$, where $\omega = e^{2\pi i/3}$ satisfies $\omega^2 + \omega + 1 = 0$. Then $|p - q| = |q - r| = |r - p|$ is the side length of $\triangle PQR$.

Taking the conjugate of this equation and using $P \in BC$, etc.

$$\frac{b+c-p}{bc} + \frac{c+a-q}{ca}\omega^2 + \frac{a+b-r}{ab}\omega = 0,$$

so plugging in $p = -q\omega - r\omega^2$, we have

$$q - r = \frac{bc + ca\omega + ab\omega^2 + r(a + b\omega^2 + c\omega)}{\omega(a - b\omega)}.$$

Let $r = ta + (1-t)b$ for some $t \in \mathbb{R}$. Then

$$\begin{aligned} |q - r| &= \left| \frac{bc + ca\omega + ab\omega^2 + t(a-b)(a + b\omega^2 + c\omega) + b(a + b\omega^2 + c\omega)}{\omega(a - b\omega)} \right| \\ &= \left| \frac{(c-b)(a - b\omega)\omega + t(a-b)(a + b\omega^2 + c\omega)}{\omega(a - b\omega)} \right|. \end{aligned}$$

Thus to minimize $|q - r$ (which clearly also minimizes $|r - p| = |p - q| = |q - r|$!), we must have

$$\begin{aligned} t = \Re\left(\frac{\omega(b-c)(a-b\omega)}{(a-b)(a+b\omega^2+c\omega)}\right) \implies r &= b + (a-b)\Re\left(\frac{\omega(b-c)(a-b\omega)}{(a-b)(a+b\omega^2+c\omega)}\right) \\ &= b + \frac{\omega(b-c)(a-b\omega)(bc-a^2)}{2(a+b\omega^2+c\omega)(bc+ca\omega+ab\omega^2)}. \end{aligned}$$

Thus

$$\begin{aligned} q - r &= \frac{bc + ca\omega + ab\omega^2 + r(a + b\omega^2 + c\omega)}{\omega(a - b\omega)} \\ &= \frac{bc + ca\omega + ab\omega^2}{\omega(a - b\omega)} + \frac{b(a + b\omega^2 + c\omega)}{\omega(a - b\omega)} + \frac{(b-c)(bc-a^2)}{2(bc + ca\omega + ab\omega^2)} \\ &= (c - b) + \frac{(b-c)(bc-a^2)}{2(bc + ca\omega + ab\omega^2)} \end{aligned}$$

and if $AX \perp QR$ and $BX \perp RP$, then

$$\begin{aligned} -\frac{x-a}{\overline{x} - \frac{1}{a}} = \frac{q-r}{\overline{q-r}} &= \frac{-(b-c) + \frac{(b-c)(bc-a^2)}{2(bc+ca\omega+ab\omega^2)}}{\frac{b-c}{bc} + \frac{(b-c)(bc-a^2)}{2abc(a+b\omega^2+c\omega)}} \\ &= \frac{abc(a + b\omega^2 + c\omega)}{bc + ca\omega + ab\omega^2} \frac{bc - a^2 - 2(bc + ca\omega + ab\omega^2)}{bc - a^2 + 2a(a + b\omega^2 + c\omega)} \\ &= -\frac{abc(a + b\omega^2 + c\omega)}{bc + ca\omega + ab\omega^2} \end{aligned}$$

and similarly,

$$\frac{x-b}{\overline{x} - \frac{1}{b}} = \frac{bca(b + c\omega^2 + a\omega)}{ca + ab\omega + bc\omega^2} = \omega^2 \frac{abc(a + b\omega^2 + c\omega)}{bc + ca\omega + ab\omega^2}.$$

Solving for x (or $\overline{x}$), we get an equation invariant under the cyclic rotation $a \mapsto b \mapsto c \mapsto a$, as desired.

Edit: $\LaTeX$ error

This post has been edited 1 time. Last edited by math154, May 29, 2013, 10:29 am

mathskid

#7 Jul 5, 2011, 2:19 pm

> mr.danh wrote:
>
> Obviously, triangle DEF is the pedal triangle of isodynamic point. So, these line are concurrent at Fermat's point (which point satisfying $\angle APB = \angle BPC = \angle CPA = 120^o$)

Page 6 of the following document proves this. (Theorem 2.5)
http://awesomemath.org/wp-content/uploads/reflections/2010_6/Isodynamic_moon_c.pdf

jayme

#8 Jul 5, 2011, 5:40 pm

Dear Mathlinkers,
accepting that the point in question is the first isodynamic point of ABC,
the first Napoleon triangle is homothetic to PQR
see : http://perso.orange.fr/jl.ayme vol. 2 the Lester's circle p. 14
This Napoleon triangle being orthologic to ABC, we are done
Sincerely
Jean-Louis

Dukejukem

#9 Oct 18, 2015, 9:05 pm

Generalization: Fix a triangle $\triangle A_1B_1C_1$. Let $\triangle ABC$ be a triangle and let A_2, B_2, C_2 lie on BC, CA, AB so that $\triangle A_2B_2C_2 \sim \triangle A_1B_1C_1$, and $[A_2B_2C_2]$ is minimal among all such triangles. Then $\triangle A_2B_2C_2$ and $\triangle ABC$ are orthologic.

Proof: Let X, Y, Z vary on BC, CA, AB, respectively, so that $\triangle XYZ \sim \triangle A_1B_1C_1$. We claim that as X, Y, Z vary, the Miquel Point M of $\{X, Y, Z\}$ WRT $\triangle ABC$ is fixed.

Indeed, from cyclic quadrilaterals, we deduce that

$$\measuredangle ZXY = \measuredangle ZXM + \measuredangle MXY = \measuredangle ZBM + \measuredangle MCY = -\measuredangle BMC - \measuredangle CAB,$$

where the last step follows from examining quadrilateral $MBAC$. Thus, since $\measuredangle ZXY$ and $\measuredangle CAB$ are fixed, it follows that $\measuredangle BMC$ is fixed as well. Similar arguments show that $\measuredangle CMA$ and $\measuredangle AMB$ are fixed, and it follows that M is fixed.

Now, if R_A is the circumradius of $\triangle AYZ$, the Law of Sines yields $YZ = 2R_A \sin \angle YAZ$. But since $\angle YAZ$ is fixed, it follows that YZ is minimal when R_A is minimal. Because $\overline{AM}$ is a fixed chord on $\odot(AYZ)$, we see that R_A is minimized when $\overline{AM}$ is a diameter of $\odot(AYZ)$, in which case $\angle AYM = \angle AZM = 90^\circ$. Thus, Y, Z are just the projections of M onto AC, AB, respectively. Meanwhile, $\measuredangle MXB = \measuredangle MYB = 90^\circ$, implying that X is the projection of M onto BC.

Hence, $\triangle A_2B_2C_2$ is the pedal triangle of M WRT $\triangle ABC$, and it follows that $\triangle A_2B_2C_2$ and $\triangle ABC$ are orthologic, as desired. $\square$

Remark: It follows from a well-known property of isogonal conjugates that the perpendiculars from A, B, C to B_2C_2, C_2A_2, A_2B_2, respectively, concur at the isogonal conjugate of M WRT $\triangle ABC$.

Quick Reply

High School Olympiads

Poincare theorem

Reply

BG Yoda

#1 Mar 17, 2009, 2:23 pm

Can somebody give me a hint what implies the Poincare theorem?

I know it is something about an inscribed and described $n-$polygon... but the main statement is incomprehensible to me

puuhikki

#2 Mar 18, 2009, 1:55 am

Do you mean Poncelet's porism?

BG Yoda

#3 Mar 19, 2009, 2:34 am

Exactly Thank you **puuhikki**

Is there a geometric proof by the way ?

Luis González

#4 Mar 19, 2009, 5:39 am

The general theorem (due to Poncelet) is as follows:

Suppose circles ω_i belong to the same pencil and A_0 is a point on ω_0. Tangent to ω_1 from A_0 cuts ω_0 again at A_1, tangent to ω_2 from A_1 cuts ω_0 again at A_2, etc., the tangent to ω_{i+1} from A_i cuts ω_0 again at A_{i+1}. Suppose that for some n, the points A_n and A_0 coincide. Then for any B_0 on ω_0, the similarly constructed point B_n coincides with B_0.

Quick Reply

High School Olympiads

quadrilateral and incircle

Reply

jgnr

#1 Mar 18, 2009, 6:05 am

Let $ABCD$ be a quadrilateral which has an incircle. The incircle touches sides AB, BC, CD, DA at E, F, G, H respectively. Prove that AC, BD, EG, FH concur.

Luis González

#2 Mar 18, 2009, 6:09 am

The well-known Newton's theorem. Posted many times before

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=50&t=19549
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=147923
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=208680

jgnr

#3 Mar 18, 2009, 6:30 am

can you show some proof?

Luis González

#4 Mar 18, 2009, 6:56 am

Let $S \equiv EG \cap HF. X \equiv BC \cap AD$ is the pole of line HF WRT incircle (I) and $Y \equiv AB \cap DC$ is the pole of EG WRT $(I) \Longrightarrow XY$ is the polar of S WRT (I). Thus, points $W \equiv EF \cap HG$ and $V \equiv EH \cap FG$ lie on $XY. W$ and V are the poles of DB and AC WRT (I). If X, Y, V, W are collinear, then EG, HF, AC, DB concur.

sunken rock

#5 Mar 26, 2009, 2:28 am

Actually the proof is quite simple: let's call P the intersection of AC and EG. See that AE = AH (1), CG = CF (2), angle AEP + angle CGP = 180 degs (3). Applying the sine theorem to the triangles AEP and PCG, then to the triangles AHP and CFP we get that both GE and FH intersect AC at the same point. Similarly we can prove they are concurrent on BD.

Best regards,
sunken rock

Mathias_DK

#6 Mar 26, 2009, 3:35 am

> *Johan Gunardi wrote:*
> Let $ABCD$ be a quadrilateral which has an incircle. The incircle touches sides AB, BC, CD, DA at E, F, G, H respectively. Prove that AC, BD, EG, FH concur.

Let a, b, c, d, e, f, g, h be the corresponding complex numbers to A, B, C, D, E, F, G, H, and consider the complex plane with the same center as the incircle. (Assume wlog $|e| = |f| = |g| = |h| = 1$)
Then we easily get:
$a = \frac{2he}{h+e}, b = \frac{2ef}{e+f}, c = \frac{2fg}{f+g}, d = \frac{2gh}{g+h}$.
AC, EG, FH are concurrent iff:
$((a-c)-(e-g))\overline{((e-g)-(f-h))} = \overline{((a-c)-(e-g))}((e-g)-(f-h))$, which should be true

(I don't feel like doing it by hand right now, but notice that $\overline{a} = \frac{2}{h+e}, \overline{c} = \frac{2}{f+g}$... It should be pretty straightforward..)

Quick Reply

High School Olympiads

Construction

Reply

Source: IMO Longlist

jgnr

#1 Mar 17, 2009, 8:52 pm

Construct the circle that is tangent to three given circles.

Luis González

#2 Mar 17, 2009, 11:35 pm

This is the celebrated Apollonius' problem (CCC) that can be reduced to the case (LCC), transforming one of the circles into a line by an inversion, or the simpler case (PCC) by a Vieta's transformation: adding or substracting (according to the desired tangency) the smaller radius to the bigger radii.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=46945
http://www2.washjeff.edu/users/mwoltermann/Dorrie/32.pdf
http://www.cut-the-knot.org/pythagoras/Apollonius.shtml

Quick Reply

High School Olympiads

concyclic in parallelogram

Reply

hollandman

#1 Mar 17, 2009, 3:31 am

Given a parallelogram $ABCD$ ($BD > AC$). A circle with diameter AC cut BD at P,Q ($BQ > BP$) A tangent to that circle at C meet AB, AD at X,Y respectively. Prove that P,Q,X,Y are concyclic.

Luis González

#2 Mar 17, 2009, 6:19 am

Let O be the center of the parallelogram and $M \equiv DB \cap XY$. C' is the reflection of C about Y. Then $AC' \parallel DB$ and let P_∞ be their infinite point. Since $(D,B,O,P_\infty) = -1$, it follows that $(X,C,Y,C') = -1 \Longrightarrow MC^2 = MY \cdot MX$, but MC^2 is the power of M to the circle with diameter $\overline{AC} \Longrightarrow MY \cdot MX = MP \cdot MQ$.

P.S. We also can go for a couple of similar triangles to get the same relation.

hollandman

#3 Mar 17, 2009, 6:45 am

> Quote:
> We also can go for a couple of similar triangles to get the same relation.

I don't think it's that easy. Can you show it?

Luis González

#4 Mar 17, 2009, 6:56 am • 1

$$\triangle MBY \sim \triangle MDC \Longrightarrow \frac{MB}{MD} = \frac{MY}{MC} \quad (1)$$

$$\triangle MBC \sim \triangle MDX \Longrightarrow \frac{MD}{MB} = \frac{MX}{MC} \quad (2)$$

Combining (1) and (2) yields $MC^2 = MX \cdot MY$, and the conclusion follows.

Quick Reply

High School Olympiads

Perpendicularity in cyclic quadrilateral

Reply

Source: Moldova TST 2009 Day 1 problem 3

Ahiles
#1 Mar 14, 2009, 8:26 pm

Quadrilateral $ABCD$ is inscribed in the circle of diameter BD. Point A_1 is reflection of point A wrt BD and B_1 is reflection of B wrt AC. Denote $\{P\} = CA_1 \cap BD$ and $\{Q\} = DB_1 \cap AC$. Prove that $AC \perp PQ$.

Luis González
#2 Mar 14, 2009, 10:16 pm

Denote (O, ϱ) the circumcircle of $ABCD$ and $R \equiv AC \cap DB.$ Clearly $\angle BPC = \angle BAC - \angle DAA_1 \Longrightarrow$ $\angle BPC = \angle BAC - \angle BAO = \angle OAC \Longrightarrow AOCP$ is cyclic. Thus, $AR \cdot RC = PR \cdot OR = \varrho^2 - OR^2 \Longrightarrow$

$$(PO - OR) \cdot OR = \varrho^2 - OR^2 \Longrightarrow PO \cdot OR = \varrho^2 = OD^2 \ \ (1)$$

If H is the intersection of lines BB_1 and AC, then OH is a midline of $\triangle BDB_1$

$$\Longrightarrow OH \parallel DQ \Longrightarrow \triangle DQR \sim \triangle OHR \Longrightarrow \frac{DR}{OR} = \frac{DQ}{OH} \ \ (2)$$

From (1) and (2) we get $\frac{PD}{OD} = \frac{DQ}{OH} \Longrightarrow \frac{PD}{OB} = \frac{DQ}{OH},$

which implies that $\triangle QDP \sim \triangle BHO \Longrightarrow PQ \parallel BH \perp AC.$

vittasko
#3 Mar 15, 2009, 1:56 am

I think the formulation of the problem it's OK and easy to prove.

Kostas Vittas.

Ahiles
#4 Mar 15, 2009, 10:27 pm

Here is my solution from TST:
There are lot of cases (I mean drawings) and because of this I thnk I will lose some points.
Suppose that everything is as in the figure (other cases are simmilliar).

Image not found

We wanna prove that $PQ \parallel BB'$ or

$$\frac{DQ}{QB_1} = \frac{DP}{PB}$$

Let $\angle ABD = \alpha$, $\angle DBC = \beta$.

Then

$\angle BPC = 90^\circ - \angle PA_1A = 90^\circ - \angle CDA = 90^\circ - (180^\circ - \angle ABC) = \alpha + \beta - 90^\circ.$
$\angle PCB = \angle A_1AB = 90^\circ - \angle ABD = 90^\circ - \alpha.$

$BC = BD\sin\angle BDC = BD\cos\beta.$

So in $\triangle BPC$:

$$\frac{PB}{BC} = \frac{\sin\angle PCB}{\sin\angle BPC} = \frac{\cos\alpha}{\sin(\alpha+\beta-90^\circ)}$$

$$\frac{PB}{BD} = \frac{\cos\alpha\cos\beta}{-\cos(\alpha+\beta)} \iff \frac{BD}{PB} = \frac{-\cos(\alpha+\beta)}{\cos\alpha\cos\beta}$$

$$\frac{BD+PB}{PB} = \frac{-\cos(\alpha+\beta)+\cos\alpha\cos\beta}{\cos\alpha\cos\beta}$$

$$\boxed{\frac{DP}{PB} = \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

Let N be projection of D onto AC. Then $ND \parallel B_1M$, so

$$\frac{QB_1}{QD} = \frac{B_1M}{DN}$$

But

$$B_1M = BM = AB\sin\angle MAB = AB\sin\angle CDB = AB\cos\beta = BD\cos\alpha\cos\beta$$

$DN = CD\sin\angle ACD = CD\sin\angle ABD = BD\sin\alpha\sin\beta$, so

$$\boxed{\frac{QB_1}{CD} = \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

And we are done.

Luis González

#5 Mar 16, 2009, 12:02 am

Here is another solution:

Since B_1 is reflection of B on AC, then QC bisects $\angle BQC$ internally. Tangents of $\odot(ABCD)$ at A, A_1 meet on BD due to obvious symmetry, hence ABA_1D is harmonic $\Longrightarrow C(A_1, A, B, C) = -1 \Longrightarrow (P, E, B, D) = -1,$ where $E \equiv AC \cap BD.$ Therefore QP is the external bisector of $\angle BQD \Longrightarrow PQ \perp QC \equiv AC.$

vittasko

#6 Mar 16, 2009, 1:59 am

Let be the point $C' \equiv (O) \cap PA$ and then, because of $AE = EA_1,$ where $E \equiv BD \cap AA_1,$ we have $\angle C'PB = \angle BPC$ (because of symmetry) $\Longrightarrow \angle C'AB = \angle BAC$, (1)

So, because of $AD \perp AB,$ we conclude that the pencil $A.PDRB,$ where $R \equiv BD \cap AC,$ is in harmonic conjugation.

Because of now, the pencil $Q.PDRB,$ is also in harmonic conjugation and $\angle DQR = \angle RQB,$ from the isosceles triangle $\triangle QBB_1$ with $QB = QB_1,$ we conclude that $PQ \perp QR \equiv AC$ and the proof is completed.

Kostas Vittas.

Attachments:

t=264432.pdf (6kb)

This post has been edited 4 times. Last edited by vittasko, Mar 16, 2009, 2:44 pm

Ahiles

#7 Mar 16, 2009, 2:28 am

Cool...
That's the nicest way to solve this nice problem (it was also the official solution).

Virgil Nicula
#8 Mar 16, 2009, 5:16 pm

> Ahiles wrote:
> $ABCD$ is inscribed in the circle with diameter $[BD]$. Point A_1 is reflection of A w.r.t. BD and B_1 is reflection of B w.r.t. AC . Denote $P\in CA_1\cap BD$ and $Q\in DB_1\cap AC$. Prove that $AC\perp PQ$.

Proof. Denote $S\in AC\cap BD$. Observe that $\left\|\begin{array}{c} CB\perp CD\\\\ \widehat{BCP}\equiv\widehat{BCS}\end{array}\right\|\Longrightarrow\ \{P,S\}$ are conjugate w.r.t. $\{B,D\}$.

Since $\widehat{SQB}\equiv\widehat{SQD}$ and $\{P,S\}$ are conjugate w.r.t. $\{B,D\}$ obtain $QS\perp QP$, i.e. $AC\perp PQ$.

limes123
#9 Nov 2, 2009, 6:01 pm

We can also prove that B is incenter of $\triangle ACP$ and D is P-excenter in this triangle. Then the result is quite easy to prove.

CCMath1
#10 Nov 8, 2009, 6:06 pm

Stragely , it is the same problem for Mongolian TST 2008, day3 problem 2
. See http://www.mathlinks.ro/viewtopic.php?p=1135126#1135126

Quick Reply

High School Olympiads

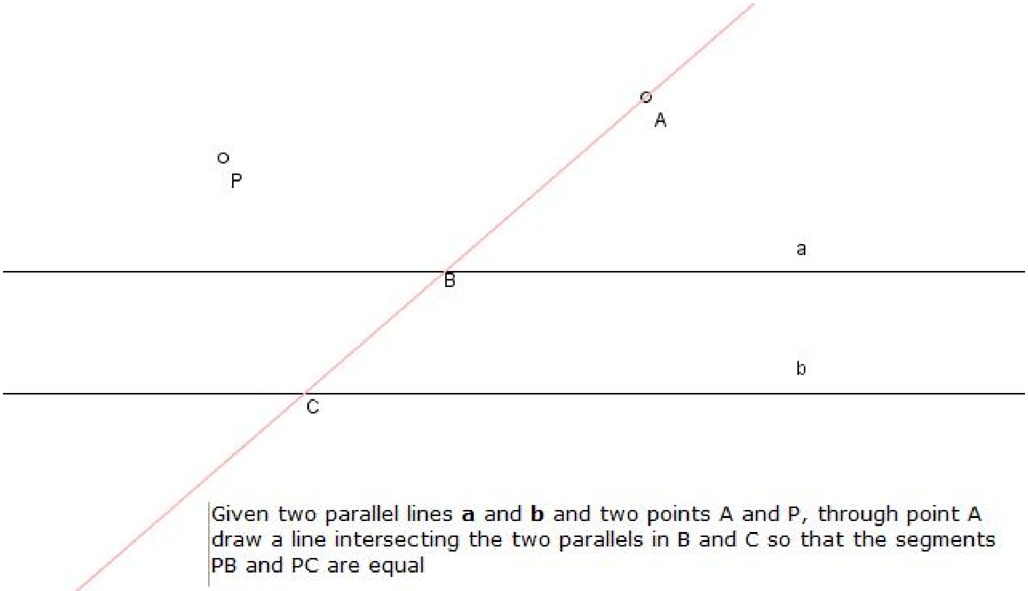
draw a line

Reply

jrrbc

#1 Mar 15, 2009, 6:37 am

Attachments:

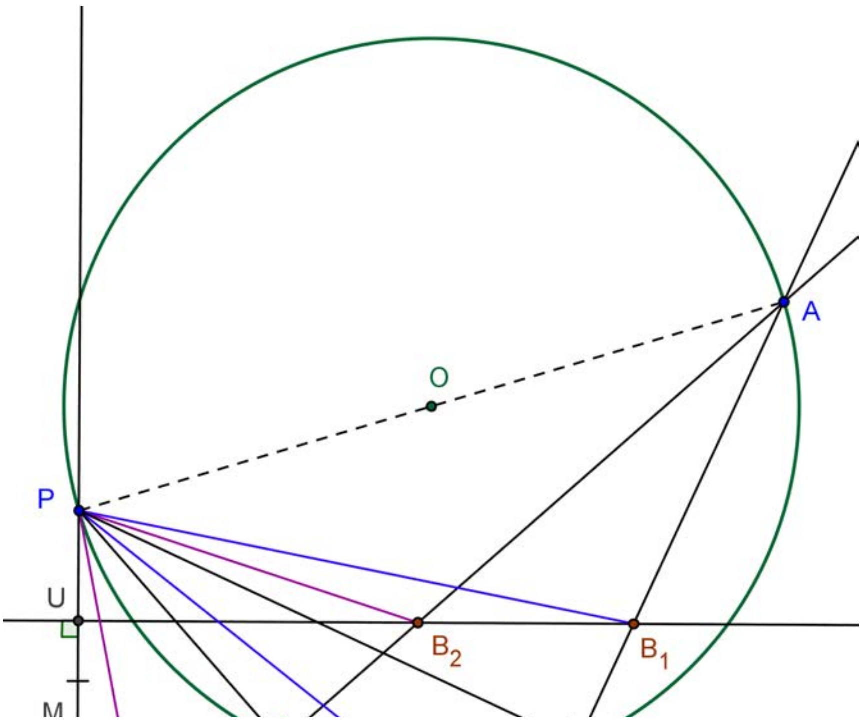


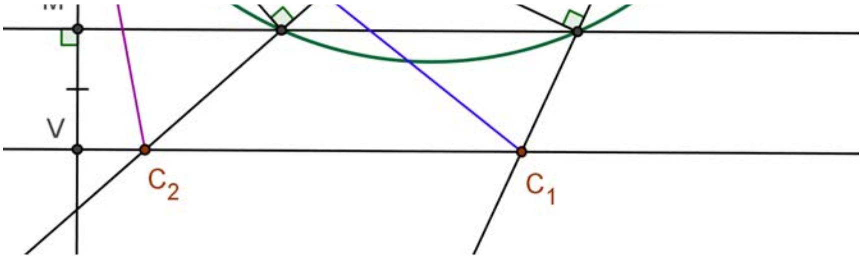
Luis González

#2 Mar 15, 2009, 8:28 am

Pretty simple. See the sketch below for a construction without words.

Attachments:

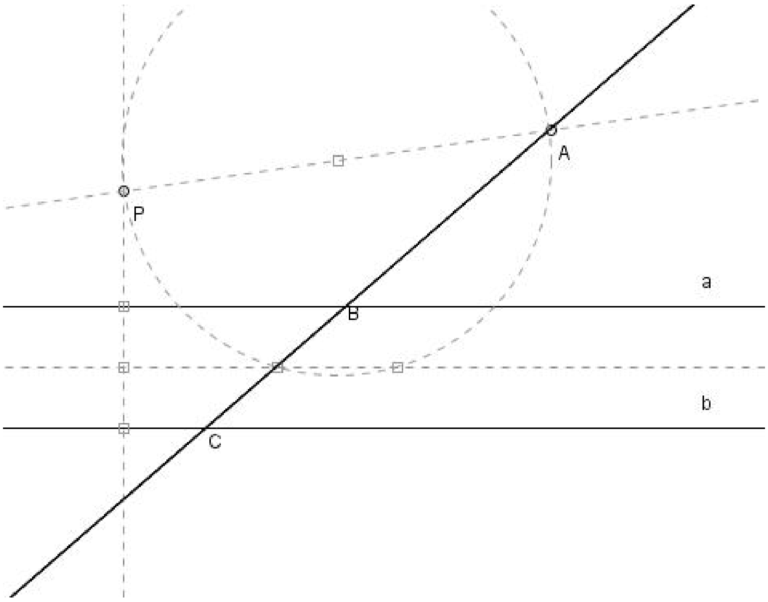




jrrbc

#3 Mar 15, 2009, 2:19 pm

Attachments:



Quick Reply

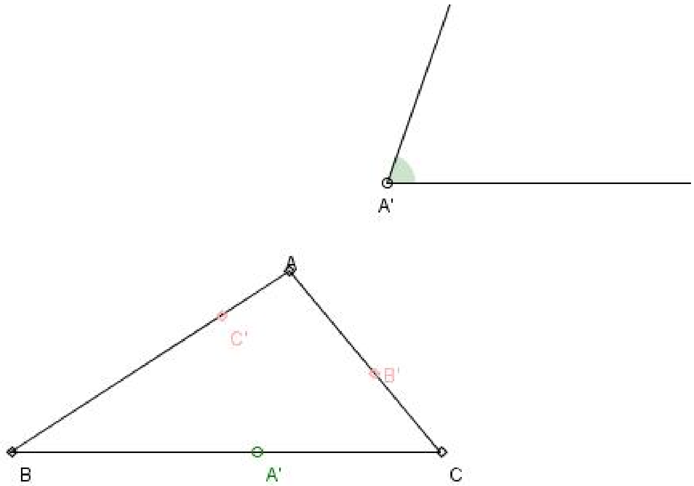
High School Olympiads

isosceles triangle

Reply

jrrbc
#1 Aug 31, 2008, 6:25 am

Attachments:



Inscribe into the triangle ABC an isosceles triangle A'B'C', with sides A'B' and A'C' equal, given the vertex A' and the angle A'

sunken rock
#2 Mar 15, 2009, 6:20 am

It seems that, constructing the triangle BTC with BT = TC, angle BTC=angle A' and A and T on each side of BC, AT will intersect BC at A', then parallel lines drawn through A' to BT and CT will intersect AB and AC respectively at C' and B'.
Similar constructions for CA and AB, provided the intersection point (A' for BC is on the segment BC), therefore A' cannot be any given point on the border of the triangle ABC, but maximum three points.

Best regards,
sunken rock

Luis González
#3 Mar 15, 2009, 8:15 am

Solution 1. Perform a rotation with center A' and angle equal to the given angle $\widehat{B'A'C'}$. Such a rotation takes the line $c \equiv AB$ into a line c', thus $B' \equiv AC \cap c'$. Now the construction of the isosceles $\triangle A'B'C'$ is straightforward.

Solution 2. Circles $\odot(AB'C'), \odot(BC'A'), \odot(CA'B')$ concur at Miquel point P. Angle chasing using the cyclic quadrilaterals $PA'BC'$ and $PA'CB'$ gives $\widehat{BPC} = \widehat{BAC} + \widehat{B'A'C'}$ and similarly $\widehat{APC} = 90^\circ - \frac{1}{2}\widehat{B'A'C'} + \widehat{ABC}$. Thus knowing the measure of $\widehat{B'A'C'}$, we can construct the point P intersecting the circular arcs that see BC, CA under the referred angles. The construction of B' and C' follows.

Quick Reply

High School Olympiads

nice inequality

Reply

Source: Ukrainian journal contest, problem 345, by Maria Rozhkova

rogue

#1 Mar 13, 2009, 9:15 pm

Let I be the incenter of a triangle ABC. Points P and R, T and K, F and Q are chosen on sides AB, BC, and AC respectively such that $TQ\|AB$, $RF\|BC$, $PK\|AC$ and the lines TQ, RF, and PK are concurrent at the point I. Prove that $TK+QF+PR\geq KF+PQ+RT$.

Luis González

#2 Mar 13, 2009, 11:45 pm

Let r denote the inradius of $\triangle ABC$. Substituting $TK=r\cdot(\cot B+\cot C)$, $PQ=r\cdot\sec\frac{A}{2}$ and the cyclic expressions, the object inequality becomes:

$$2\cdot(\cot A+\cot B+\cot C)\geq\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}.$$

Quick Reply

High School Olympiads

About circles and common tangents

Source: 0

Luis González

#1 Mar 9, 2009, 3:44 am

Let $ABCD$ be a convex quadrilateral and ω_1, ω_2 denote the incircles of $\triangle ABC$ and $\triangle ADC$, the latter one is tangent to $\overline{AC}$ at F. Assume that there exists a circle ω (outside ABCD) tangent to lines AB, BC, CD and CA. Show that

a) Common external tangents of ω_1, ω_2 intersect on ω.

b) Common external tangents of ω, ω_1 intersect at a point Y lying on BF.

c) If ω and ω_1 are tangent to line BA at M, N and the circle wih diameter $\overline{MN}$ cuts ω at X, such that F lies on XB, then show that Y lies on circle ω_1.

kaka__2004

#2 Mar 12, 2009, 6:29 am

who can ? It seems nice

Luis González

#3 Mar 13, 2009, 12:56 am

Questions a) & b)

If ω_1 is tangent to AC through E, then it's easy to figure out that $AF = CE$. Let X be the intersection of the center line of ω_1, ω_2 with line $AC \Longrightarrow X$ is the insimilicenter of ω_1, ω_2, B is the exsimilicenter of ω, ω_1 and D is the insimilicenter of ω, ω_2. By Monge's theorem, it follows that D, X, B are collinear. Let O, O_1, O_2 be the centers of ω, ω_1 and ω_2. Since $AF = CE$, then B-excircle of $\triangle ABC$ is tangent to $\overline{AC}$ through F and D-excircle of $\triangle ADC$ is tangent to $\overline{AC}$ through E. Rays BF and DE pass through the antipodes E', F' of E, F. B-excircle is homothetic to ω through B. Thus, ray BF cuts ω at the homologous H of E' under such homothety $\Longrightarrow OH \parallel EE'$. Likewise, D-excircle, ω and ω_2 are homothetic through $D \Longrightarrow DE$ cuts ω at the homologous H' of $F' \Longrightarrow OH' \parallel FF'$. Now, if $FF' \parallel EE' \Longrightarrow OH \parallel OH'$, i.e. $H \equiv H' \Longrightarrow$ Exsimilicenter H of ω_1, ω_2 lies on ω. Intersection Y of the common external tangents of ω, ω_2 is their exsimilicenter. Hence, by Monge's theorem, we conclude that exsimilicenters of $\omega_1, \omega_2, \omega$ are collinear $\Longrightarrow Y, B, H$ are collinear.

Question c)

Clearly, $X \equiv H \Longrightarrow \angle MHN = 90^\circ$. Hence, line NH cuts ω at the antipode M' of M, which is the homologous of N under the negative homothety ω, ω_1. Since this line passes through D, then it cuts ω at H, different from M', and cuts ω_1 at E', because of $EE' \parallel OH$. In other words, line HM' goes through $D, E, N \Longrightarrow \triangle CNE$ is isosceles with apex C. Circle with diameter $\overline{MN}$ also passes through Q', which is the inverse of H through the inversion with pole B, thus H, Q, N, M are conclyclic. If $P' \equiv HN \cap QN \Longrightarrow \angle HQP' = \angle HNM$. Since $\triangle CNE$ is C-isosceles, it follows that

$\angle CEN = \angle HMN = \angle HQP' \Longrightarrow \angle P'EF = \angle FQ'P' \Longrightarrow FQEP'$ is cyclic.

Which implies that $P \equiv P'$. Now, from parallel radii $O_2P \parallel OM \parallel O_1N$ and $O_2F \parallel OH$, it follows that $Q \equiv PM \cap FH$ is the exsimilicenter of ω_2, ω and our proof is completed.

High School Olympiads

Collinear

Reply

encyclopedia
#1 Mar 10, 2009, 11:28 pm

Let ABC be a triangle and a point P, $A'B'C'$ is Cevian triangle of P, $PA\cap B'C'=A''$, M is projection of A'' on BC circumcricle $(ANC)\cap AB=\{E\}, (ANB)\cap AC=\{F\}$, let circumcircle $(BME)\cap(CMF)=\{M,P'\}$ prove that M,P,P' are collinear.

Luis González
#2 Mar 10, 2009, 11:43 pm

There is a typo in your problem, N is not previously defined, but I believe N=M.

By Miquel theorem, $AEP'F$ is cyclic, thus AE, FP', CM concur at the radical center B of $\odot(AEP'F), \odot(CFP'M)$ and $\odot(AEMC)$, i.e. B, F, P' are collinear. Hence $\angle P'MC=\angle P'FA\equiv\angle BFA=\angle AMB\Longrightarrow MA', MA''$ bisect $\angle AMP'\Longrightarrow M(A,P',A'',A')=-1$. But $M(A,P,A'',A')=-1\Longrightarrow M,P,P'$ are collinear.

encyclopedia
#3 Mar 11, 2009, 8:14 am

> encyclopedia wrote:
> Let ABC be a triangle and a point P, $A'B'C'$ is Cevian triangle of P, $PA\cap B'C'=A''$, M is projection of A'' on BC circumcricle $(ANC)\cap AB=\{E\}, (ANB)\cap AC=\{F\}$, let circumcircle $(BME)\cap(CMF)=\{M,P'\}$ prove that M,P,P' are collinear.

Sorry it must be

Let ABC be a triangle and a point P, $A'B'C'$ is Cevian triangle of P, $PA\cap B'C'=A''$, M is projection of A'' on BC circumcricle $(AMC)\cap AB=\{E\}, (AMB)\cap AC=\{F\}$, let circumcircle $(BME)\cap(CMF)=\{M,P'\}$ prove that M,P,P' are collinear.

vittasko
#4 Mar 12, 2009, 6:45 pm

Let $(O_1),\ (O_2)$ be, the circumcircles of the triangles $\triangle AMB,\ \triangle AMC$ respectively and also let $(O_3),\ (O_4)$ be, the ones, of the triangles $\triangle BME,\ \triangle CMF$ respectively.

It is easy to show that the quadrilateral $AEP'F$, where $P'\equiv(O_3)\cap(O_4)$, is cyclic, from $\angle BEP'=\angle CMP'=\angle AFP'$, (1)

Also it is easy to show that the line segment EP', passes through the vertex C of $\triangle ABC$, from $\angle FP'P''=\angle A=\angle FMC$, (2) where P'' is an arbitrary point on the extension of the line segment EP', towards P'.

Similarly, the line segment FP', passes through the point B.

• We have now, the configuration of the complete quadrilateral $AEP'FBC$, where $AEP'F$ is cyclic with circumcircle so be it (O) and let be the point $N\equiv EF\cap AP'$.
The point M, as the common point of the circumcircles $(O_1),\ (O_2),\ (O_3),\ (O_4)$ of the triangles $\triangle ABF,\ \triangle ACE,\ \triangle BEP',\ \triangle CFP'$ respectively, is the **Miquel point**, of the complete quadrilateral $AEP'FBC$, which lies on BC as well, because of cyclic $AEP'F$.

Also, we have the result that the points $O,\ N,\ M,$ are collinear and $ONM\perp BC$ as well and easy to prove elementary or shortest by polar theory (the line segment $BC,$ is the polar of N with respect to circumcircle $(O),$ of $AEP'F$).

So, we conclude that the point $A''\equiv B'C'\cap AP,$ lies on the line segment $ONM\perp BC.$

• Let be the point $N'\equiv BC\cap AP'$ and from the complete quadrilateral $AEP'FBC$ we have that the points $A,\ N,\ P',\ N',$ are in harmonic conjugation and then, the pencil $M.ANP'N',$ is also in harmonic conjugation.

Similarly, from the complete quadrilateral $AB'PC'BC$ we have that the points $A,\ A'',\ P,\ A',$ are in harmonic conjugation and then, the pencil $M.AA''PA',$ is in harmonic conjugation too.

So, because of the harmonic pencils $M.ANP'N',\ M.AA''PA',$ have the line segment MA as their common ray and also they have $MN\equiv MA''$ and $MN'\equiv MA',$ we conclude that they have also $MP'\equiv MP\ ,(3)$

Hence, from (3) we conclude that the points $M,\ P,\ P',$ are collinear and the proof is completed.

Kostas Vittas.

Attachments:
t=263663.pdf (12kb)

encyclopedia
#5 Mar 13, 2009, 3:52 pm

Nice solution dear vittasko, here is my solution

Because $(APA''A')$ is harmonic and $MA''\perp MA'$ thus MA'' is bisector of $\angle PMA$ similar if $B'C'\cap BC=A'''$ then $(BCMA''')$ is harmonic then MA'' is bisector of $\angle B'MC'$, this is succifient to show $\angle AMB=\angle PMC$, let $BF\cap(MFC)=\{P'\}$ we have $\angle BMA=\angle BFA$ but $\angle BMA=\angle PMC$ so that $MP'FC$ is cyclicmso we can prove $BEP'M$ is cyclic, too. We are done. It is the way I created this problem!

mathVNpro
#6 Apr 8, 2009, 8:36 pm

First I want to sorry if my solution below is same as anyone's .

Here is my solution:
Let D be the intersection of B'C' with BC. It is easy to see that (D,A',B,C)=-1.
Therefore (D,A",C',B')=-1. Then M(D,A",C',B')=-1, but MA" is perpendicular with MD. So MA" is the bisector of (MC',MB'). With the same argument, we are also able to prove that A"M is the bisector of (MA,MP). As the result, we have (MC',MA)=(MP,MB').
Now it is easy to prove that E, M', C are collinear (Directed angle can kill it easily).
Therefore, (MM',MB')=(MM',MF)+(MF,MC)+(MC,MB')=(CM',CA)+(AC,AB)+(MC',MB)=(ME,MA)+(MB,ME)+(MC',MB)=(MC',MA).
Then, (MM',MB')=(MP,MB').
This prove that M, M', P are on the same line.
Our proof is completed.

Quick Reply

High School Olympiads

A convex polygon is inscribed in a circle

Reply

Source: Viet Nam TST 1985 problem 4

tdl

#1 Mar 10, 2009, 8:59 am

A convex polygon $A_1A_2...A_n$ is inscribed in a circle with center O and radius R so that O lies inside the polygon. Let the inradii of triangles $A_1A_2A_3, A_1A_3A_4, ..., A_1A_{n-1}A_n$ be denoted by $r_1, r_2, ..., r_{n-2}$. Show that:

$r_1 + r_2 + ... + r_{n-2} \leq R\left(n\cos\frac{\pi}{n} - n + 2\right)$.

Luis González

#2 Mar 10, 2009, 9:45 am

By Mikami and Kobayashi theorem for the cyclic polygon $A_1A_2A_3A_4 \ldots A_n$, we have

$$r_1 + r_2 + r_3 + r_4 + \ldots + r_{n-2} = \delta_1 + \delta_2 + \delta_3 + \delta_4 + \ldots + \delta_n - R \cdot (n-2)\ , (*)$$

Where $\delta_1, \delta_2, \delta_3, ..., \delta_n$ denote the distances from O to $A_1A_2, A_2A_3, A_3A_4 \ldots A_nA_1$. Let $\phi_1, \phi_2, \phi_3, \ldots, \phi_n$ denote the corresponding central angles of the chords A_1A_2, A_2A_3, etc $\Longrightarrow \delta_1 = R \cdot \cos\frac{\phi_1}{2}$ and similarly for $\delta_2, \delta_3, \ldots, \delta_n$

Substituting δ_n into $(*)$, the subsequent sum of the radii equals

$$R \cdot \left(\cos\frac{\phi_1}{2} + \cos\frac{\phi_2}{2} + \cos\frac{\phi_3}{2} + ... + \cos\frac{\phi_n}{2}\right) + R \cdot (2-n)$$

Since O is inside the polygon, then $\frac{\phi_n}{2}$ is not greater than $\frac{\pi}{2}$. Thus, by Jensen's inequality

$$\cos\frac{\phi_1}{2} + \cos\frac{\phi_2}{2} + \ldots + \cos\frac{\phi_n}{2} \leq n \cdot \cos\frac{\pi}{n}$$

$$\Longrightarrow\ r_1 + r_2 + r_3 + r_4 + \ldots + r_{n-2} \leq R\left(n \cdot \cos\frac{\pi}{n} + 2 - n\right)$$

Quick Reply

High School Olympiads

Metric relation regarding Jenkins circles.

Reply

Source: 0

Luis González

#1 Mar 5, 2009, 10:52 am

Let $\triangle ABC$ be an acute triangle with inradius r and circumradius R. r_0 is the inradius of its orthic triangle, $\lambda_a,\lambda_b,\lambda_c$ are the radii of its three Jenkins circles and ϕ is the radius of its Conway circle. Show the following relation:

$$\phi^2\left(\frac{1}{\lambda_a}+\frac{1}{\lambda_b}+\frac{1}{\lambda_c}\right)=8R+4\left(r+r_0+\frac{r}{R^2}\right)$$

Luis González

#2 Mar 10, 2009, 12:01 am

Let S and I be the Spieker point and incenter of $\triangle ABC$. ϱ denotes the radius of the Apollonius circle. We know that

$$\varrho=\frac{p^2+r^2}{4r}\ \ (1)\ \ ,\ \ \phi^2=p^2+r^2\ \ (2)$$

Inversion with center S and power k^2 equal to the power of S with respect to the excircles takes the sidelines BC,CA,AB into the Jenkins circles $\mathcal{J}_a,\mathcal{J}_b,\mathcal{J}_c$. Their centers lie then on the perpendiculars SX,SY,SZ to BC,CA,AB. Such inversion also takes the 9-point circle (N) into the Apollonius circle (U,ϱ).

$$\frac{2\varrho}{R}=\frac{2k^2}{Rr}\Longrightarrow\varrho=\frac{k^2}{r}\ \ (2.1)$$

On the other hand, we have $k^2=SX\cdot2\lambda_a=SY\cdot2\lambda_b=SZ\cdot2\lambda_c$

But distance SX is half the distance from the incenter I to the parallel through A to BC. Similarly SY and SY, hence

$$k^2=(h_a-r)\cdot\lambda_a=(h_b-r)\cdot\lambda_b=(h_c-r)\cdot\lambda_c$$

Combining this latter expression with (2.1) yields

$$\varrho\left(\frac{1}{\lambda_a}+\frac{1}{\lambda_b}+\frac{1}{\lambda_c}\right)=\frac{h_a+h_b+h_c}{r}-3$$

But according to [1] we have $h_a+h_b+h_c=2R+4r+r_0+\dfrac{r}{R^2}$

$$\Longrightarrow\ \left(\frac{1}{\lambda_a}+\frac{1}{\lambda_b}+\frac{1}{\lambda_c}\right)=\frac{2R+4r+r/R^2}{r}-3$$

Combining this latter relation with (2) gives:

$$\phi^2\left(\frac{1}{\lambda_a}+\frac{1}{\lambda_b}+\frac{1}{\lambda_c}\right)=8R+4\left(r+r_0+\frac{r}{R^2}\right)$$

[1] http://www.artofproblemsolving.com/Forum/viewtopic.php?t=259185

Quick Reply

High School Olympiads

A perpendicularity problem

Reply

Mashimaru

#1 Mar 5, 2009, 7:04 pm

Let M be an arbitrary point on (O) and P be a point outside (O). Circle (ω) with center I and diameter PM intersects (O) at the second point N. Tangent of (ω) at P intersects MN at Q. The line passes through Q and perpendicular to OP intersects PM at A. AN intersects (O) again at B. BM intersects OP at C. Prove that $AC \perp OQ$.

kaka__2004

#2 Mar 7, 2009, 10:17 am

It's so complicated
no one can?

yetti

#3 Mar 8, 2009, 9:26 am • 1

PM cuts (O) again at R. MN is radical axis of $(O), \omega$, tangent PQ of ω at P is radical axis of $P, \omega \Longrightarrow Q$ is radical center of $(O), P, \omega$ and $QA \perp OP$ is radical axis of $(O), P$. The angle $\angle MNP$ is right, let be circle with diameter PQ, passing through N. Tangent PA of at P is radical axis of $P,$ $\Longrightarrow A$ is radical center of $(O), P,$ $\Longrightarrow AN$ is radical axis of $(O),$ and B their intersection other than $N \Longrightarrow AP^2 = \overline{AB} \cdot \overline{AN} = \overline{AR} \cdot \overline{AM}$. From inversion in circle with center A and radius AP, we get $\angle PBR = \angle PBA + \angle ABR = \angle APN + \angle AMN = 90^\circ$. Circumcircle of the right $\triangle PBR$ is centered at the midpoint of PR and therefore tangent to PQ at P. Again, Q is radical center of $(O), P,$ and radical axis BR of $(O),$ goes through Q. $A \equiv MR \cap NB, Q \equiv MN \cap BR$ are intersections of the opposite sides of cyclic $MNBR$. Its diagonal intersection $C \equiv MB \cap NR$ is pole of $AQ \perp OP$ WRT (O), therefore O, C, P are collinear. Intersection $Q \equiv MN \cap BR$ is pole of AC WRT (O), therefore $AC \perp OQ$.

Luis González

#4 Mar 9, 2009, 10:45 am • 1

Let X the be second intersection of $\overline{MP}$ with (O). Inversion with center Q and power QP^2 takes (O) and ω into themselves and takes circle (T) with diameter $\overline{PQ}$ into line PM. Hence, X, B, Q are collinear and $B \in (T)$. $XMNB$ is cyclic and line AQ passing through the intersection of its opposite sidelines is the polar of the intersection of its diagonals WRT $(O) \Longrightarrow$ N, C, X are collinear $\Longrightarrow AC$ is the polar of Q WRT $(O) \Longrightarrow AC \perp OQ$.

RSM

#5 Jun 21, 2011, 8:35 pm

Suppose, $QA \cap OP = X$. Note that $QX.QA = QP^2 = QR^2$ where QR and QS is tangent to (O). Note that $QXOR$ is cyclic. So A lies on RS. Suppose, the second intersection point of QB and (O) is B'. Note that, polar of Q pass through the intersection point of $B'M$ and BN, but it also pass through A which lies on BN. So $B'M \cap BN = A$. So B', M, A are collinear. Suppose, $BM \cap B'N = C'$. Note that polar of C' is AQ. Also pole of AQ lies on OP since $OP \perp AQ$. So $C' \equiv C$. Pole of AC is Q. So $OQ \perp AC$.

skytin

#6 Jun 22, 2011, 1:52 pm

Let QA intersect OP at point X
QX*QA = QP*QP = QN*QM , so XNMA is cyclic
angle MXO = 90 - AXM = 90 - ANM = MBO , so BOMX is cyclic
Let Perpendicular from O on CA is OH
Easy to see that XHOA is cyclic , so HC*CA = OC*CX = MC*BC , so BHMA is cyclic
angle MHA = MBA = 90 - MNO , so HOMN is cyclic
Q is Radical center of (HOMN) (O) and (XHOA) , so H is on OQ C is Orthocenter of triangle QOA . done

Quick Reply

High School Olympiads

Twin inequalities

Source: 0

Luis González

#1 Mar 4, 2009, 4:30 am

Prove that in any $\triangle ABC$ at least one of the following two inequalities is true:

$$\frac{9}{4}R^2+4r^2+7Rr\geq p^2 \quad , \quad 9R^2-5r^2-2Rr\geq p^2$$

Luis González

#2 Mar 8, 2009, 8:42 am

Let I, O, G, N be the incenter, circumcenter, centroid and nine-point center of $\triangle ABC$. $\overline{IG}$ is an internal cevian of $\triangle ION$, since G is between N and O. Hence, either $IN \geq IG$ or $IO \geq IG$. Now, we use the well-known formulas

$$IN=\frac{1}{2}R-r$$

$$IO=\sqrt{R^2-2Rr}$$

$$IG=\sqrt{\frac{1}{9}(p^2+5r^2-16Rr)}$$

$$IN\geq IG \Longrightarrow \frac{1}{2}R-r\geq\sqrt{\frac{1}{9}(p^2+5r^2-16Rr)} \Longrightarrow \frac{9}{4}R^2+4r^2+7Rr\geq p^2$$

$$IO\geq IG \Longrightarrow R^2-2Rr\geq\frac{1}{9}(p^2+5r^2-16Rr) \Longrightarrow 9R^2-5r^2-2Rr\geq p^2$$

High School Olympiads

Polar of orthocenter WRT incircle

yetti
#1 Feb 26, 2009, 12:02 am

I is the incenter and H orthocenter of a $\triangle ABC.$ $H_a \in BC, H_b \in CA, H_c \in AB$ are feet of altitudes $AH, BH, CH.$ U, V, W are circumcenters of $\triangle AIH_a, \triangle BIH_b, \triangle CIH_c.$ Show that U, V, W are collinear and the line UVW is polar of H WRT the incircle $(I).$

nsato
#2 Feb 27, 2009, 8:00 am

Here's what I have: Let X' denote the inverse of X in the incircle. Since ABH_aH_b is a cyclic quadrilateral, $A'B'H'_aH'_b$ is a cyclic quadrilateral. Similarly, $A'C'H'_aH'_c$ and $B'C'H'_bH'_c$ are also cyclic quadrilaterals. The radical axes of their circumcircles, taken in pairs, are $A'H'_a$, $B'H'_b$, and $C'H'_c$, so they concur at some point X.

It follows that the circumcircles of triangles AIH_a, BIH_b, and CIH_c all pass through I and X'. Therefore, the circumcircles are coaxial, which means their circumcentres are collinear.

Furthermore, the points A, H, and H_a are collinear, so $A'H'H'_aI$ is a cyclic quadrilateral. Similarly, $B'H'H'_bI$ is a cyclic quadrilateral. The radical axes of the circumcircles of $A'B'H'_aH'_b$, $A'H'H'_aI$, and $B'H'H'_bI$, taken in pairs, are $A'H'_a$, $B'H'_b$, and IH'. Since X is the intersection of $A'H'_a$ and $B'H'_b$, X also lies on IH'.

To prove that UVW is the polar of H, I need to prove that X is the midpoint of IH, but I have a feeling this approach will not be enough.

yetti
#3 Feb 27, 2009, 9:00 am

I did not have complete proof when posting the problem. This is what I've got since:

To show that U, V, W are collinear is simple: Powers of I to $(U), (V), (W)$ are all equal to zero and powers of H to $(U), (V), (W)$ are all equal to half the power of H to the circumcircle $(O).$ Consequently, $(U), (V), (W)$ are coaxal and their centers collinear. Collinearity of U, V, W also follows from the fact that their polars WRT (I) concur at H (overkill and to be proved).

Incircle (I) touches BC at $D.$ AI cuts the circumcircle (O) again at $X.$ The A-altitude AH cuts (O) again at $P.$ XD cuts AH at M and the circumcircle (O) again at $N.$ (X) is a circle with center X and radius $XB = XC = XI.$ Inversion in (X) takes the line BC into the circumcircle $(O),$ hence D goes to N and $\overline{XD} \cdot \overline{XN} = XI^2.$ This means that $\angle IAM = \angle XAP = \angle XID = \angle INX = \angle INM,$ which makes $AIMN$ cyclic. Let NI cut AO again at $A'.$ Then $\angle A'AX = \angle OAX = \angle XAH = \angle IAM = \angle INM = \angle A'NX,$ which makes $AA'XN$ cyclic, with circumcircle $(O).$ Therefore, $A' \in (O),$ AA' is a diameter of $(O).$ The angles $\angle ANA' = \angle ANI = \angle AMI$ are then right and $MH_a = ID = r$ (the inradius). This was also discussed at [1].

A-excircle (I_a) touches BC at $D_a.$ In exactly the same way, XD_a cuts the circumcircle (O) again at N_a and N_aI_a cuts (O) again at $A',$ such that AA' is a diameter of $(O).$ As D_a is a reflection of D in the perpendicular bisector OX of $BC,$ so is N_a reflection of N in $OX.$ Let $J \in ID$ be reflection of I_a in $OX,$ then NJ goes through the reflection P of A' in $OX.$

$\triangle AIU$ is isosceles with $UA = UI$ and IN is its I-altitude. Let perpendicular to BC through N cut (O) again at $Q.$ Then $\angle IAQ = \angle XAQ = \angle XNQ = \angle XMP = \angle NMA = \angle NIA.$ It follows that AQ is the A-altitude of the isosceles $\triangle AIU$ and $AQ \perp IU.$ AQ is therefore parallel to the polar $u \perp IU$ of U WRT $(I).$ The circumcenter U of the $\triangle AIH_a$ is on the perpendicular bisectors k of $AH_a,$ the A-midline of $\triangle ABC,$ hence the polar u of U goes through the pole K of this midline, identical with the reflection of I_a in the midpoint of $BC,$ or with the reflection of J in the line BC [2]. Let u cut AH at $H'.$ Then $\angle JKH' = \angle KH'A = \angle AQN = \angle APN = \angle KJP.$ This means that $u \equiv KH'$ is a reflection of JP in BC and $H' \equiv H$ is a reflection of P in $BC.$ As a result, the orthocenter H is on the polar u of U WRT $(I).$ (Note the proof that u is the Steiner line of $\triangle ABC$ with the pole N.) In exactly the same way, H is on the polars v, w of V, W WRT (I), therefore the Steiner line of $\triangle ABC$ with the pole N.) In exactly the same way, H is on the polars v, w of V, W WRT (I), therefore, U, V, W are collinear and UVW is a polar of H WRT (I).

References:

[1] http://www.mathlinks.ro/viewtopic.php?t=256552
[2] http://www.mathlinks.ro/viewtopic.php?t=260772

This post has been edited 1 time. Last edited by yetti, Feb 27, 2009, 10:08 am

vittasko

#4 Mar 5, 2009, 4:52 am

Let $A',\ B',\ C'$ be, the midpoints of the side-segments $BC,\ AC,\ AB$ respectively and also let $A'',\ B'',\ C''$ be, the midpoints of the arcs $BC,\ AC,\ AB$ respectively (not containing the third vertex), of the circumference (O), of the given triangle $\triangle ABC.$

We denote the point $U\equiv B'C'\cap B''C''$, which is the circumecenter of the triangle $\triangle AIH_{a}$, because of the line segments $B'C',\ B''C''$, are the midperpendiculars of $AH_{a},\ AI$, respectively.

Similarly, the points $V\equiv A'C'\cap A''C'',\ W\equiv A'B'\cap A''B''$, are the circumcenters of the triangles $\triangle BIH_{b},\ \triangle CIH_{c}$, respectively.

It is easy to show the collinearity of $U,\ V,\ W$, because of the triangles $\triangle A'B'C'\ \ \triangle A''B''C''$ are perspective, because of $O\equiv A'A''\cap B'B''\cap C'C''$, where O is the circumcenter of $\triangle ABC.$

● The orthocenter H of $\triangle ABC$, is the radical center of the circles $(U),\ (V)$ and the circle (A'), with diameter $BC.$

So, we denote as R, the intersection point of the circles $(U),\ (V)$, the other that I and then we have that this point is constant, from $(HB)\cdot (HH_{b}) = (HC)\cdot (HH_{c}) = (HI)\cdot (HR)$, (1)

We denote the points $R'\equiv (V)\cap (W)$ and $R''\equiv (U)\cap (W)$, the other than I and by the same way as before, we conclude that $(HI)\cdot (HR) = (HI)\cdot (HR') = (HI)\cdot (HR'')$ and then the result of $R''\equiv R'\equiv R$, it follows.

That is the circles $(U),\ (V),\ (W)$, are coaxial and let be the point $P\equiv UVW\cap IR.$

● From $(IH)\cdot (HR) = (HB)\cdot (HH_{b}) = \frac {a^{2}}{4} - (HA')^{2}$

we have that $(IH)\cdot (2IP - IH) = \frac {a^{2}}{4} - (HA')^{2}\Longrightarrow 2(IH)\cdot (IP) - (IH)^{2} = \frac {a^{2}}{4} - (HA')^{2}$, (2)

Because of $IP\perp UVW$, In order to be the line segment UVW, as the polar of H with respect to the incircle (I) of $\triangle ABC$, by **Newton theorem**, we must prove that $(IH)\cdot (IP) = r^{2}$, (3) where r is the radius of $(I).$

From $(2),\ (3)$, it is enough to prove that $(HA')^{2} - (HI)^{2} = \frac {a^{2}}{4} - 2r^{2}$, (4)

But, the equality (4) is true, based on the problem A useful equality and the proof is completed.

Kostas Vittas.

Attachments:
t=262208.pdf (12kb)

Luis González

#5 Mar 6, 2009, 4:54 am

Incircle (I) touches AC, AB, BC at X, Y, Z and $Q\equiv AI\cap XY.$ Let M, N be the midpoints of $\overline{BC}$ and the arc BC of the circumcircle. Let H_1 denote the orthocenter of $\triangle IBC.$ It's well-known that:

$$AH = BC\cdot\cot A\ \ (1)\ \ ,\ \ MN = BC\cdot\frac{1-\cos A}{2\sin A}\ \ (2)$$

Circumcenter O_1 of $\triangle AIH_a$ is the intersection of the A-midline with the perpendicular bisector τ of $IA.$ The pole of the A-midline WRT (I) is the orthocenter H_1 of $\triangle IBC$ and the pole of τ WRT (I) is the reflection J of I about $Q\Longrightarrow JH_1$ is the polar of O_1 WRT $(I).$ Hence, it suffices to show that $H\in JH_1.$ If the perpendicular to BC through Z cuts HJ at H', then $\triangle AJH$ and $\triangle IJH'$ are similar, which implies that

$AH \quad AJ \quad 1 \quad AI \quad\quad AI \quad\quad 1$

$$\frac{}{IH'} = \frac{}{JI} = \frac{}{2} \cdot \frac{}{QI} - 1 \ , \quad \frac{}{QI} = \frac{}{2\sin^2 \frac{A}{2}}$$

$$\Longrightarrow \ \frac{AH}{IH'} = \frac{1}{2\sin^2 \frac{A}{2}} - 1 = \frac{\cos A}{1 - \cos A}$$

Combining this latter expression with (1) and (2) yields

$$IH' = \frac{1 - \cos A}{\sin A} \cdot BC \Longrightarrow IH' = 2 \cdot MN \quad (3)$$

Since M is the circumcenter of $\triangle IBC,$ from (3) we deduce that H' is the orthocenter of $\triangle IBC \Longrightarrow H' \equiv H_1,$ and the conclusion follows.

Quick Reply

High School Olympiads

Problem 215b from Hadamard's Elementary Geometry (misprint?)

Reply

Markelov

#1 Mar 5, 2009, 5:09 pm

Problem 215b from Hadamard's Elementary Geometry.

We are given two lines, two points A, B on these lines and a point O. Through point O, draw a line which intersects the given line in points M, N such that the ratio of AM to BN is equal to a given number.

The problem occurs at the end of the section on similarity and homothecy, but before the section including Ceva, Menelaus, Poles and Polars, or Inversion in a circle. So these tools should not be used. Of course, I need a synthetic solution, not an analytic one. So for instance the number referred to in the problem can be given as the ratio of two segments.

Any suggestions?

Thank you!
Sergey Markelov

Luis González

#2 Mar 6, 2009, 12:50 am • 1

Let a and b be the two given lines, A and B lie on a and $b.$ Take two points M and N on a and b such that $\overline{AM} : \overline{BN} = k$ is the given ratio. Construct the center H of the spiral similarity that takes the oriented segments $\overline{AM}$ and $\overline{BN}$ into each other. We'll obtain a solution by constructing the two homologous points X and Y collinear with the given point $O.$ From the similar triangles $\triangle HXY \sim \triangle HAB$ (SAS criterion), the angle $\angle HXO$ is constructible $\Longrightarrow \angle HXO = \angle HAB \Longrightarrow X, Y$ are then constructible $\Longrightarrow XY$ is the wanted line. Note that M and N can be chosen on differents sides of A, B on the lines $a, b,$ which yields more than one solution in general.

Markelov

#3 Mar 10, 2009, 5:11 pm

I got your idea, thank you very much, Luis !!

Quick Reply

High School Olympiads

Complete quadrilateral

Reply

stergiu

#1 Mar 4, 2009, 1:36 pm

A retired mathematics teacher asked me to find a proof for this problem(he has not found a solution till now) :

PROBLEM

The centers of the Euler circles of the four tringles in complete quadrilateral are concyclic !

Is this a theorem ? I have never seen it before !

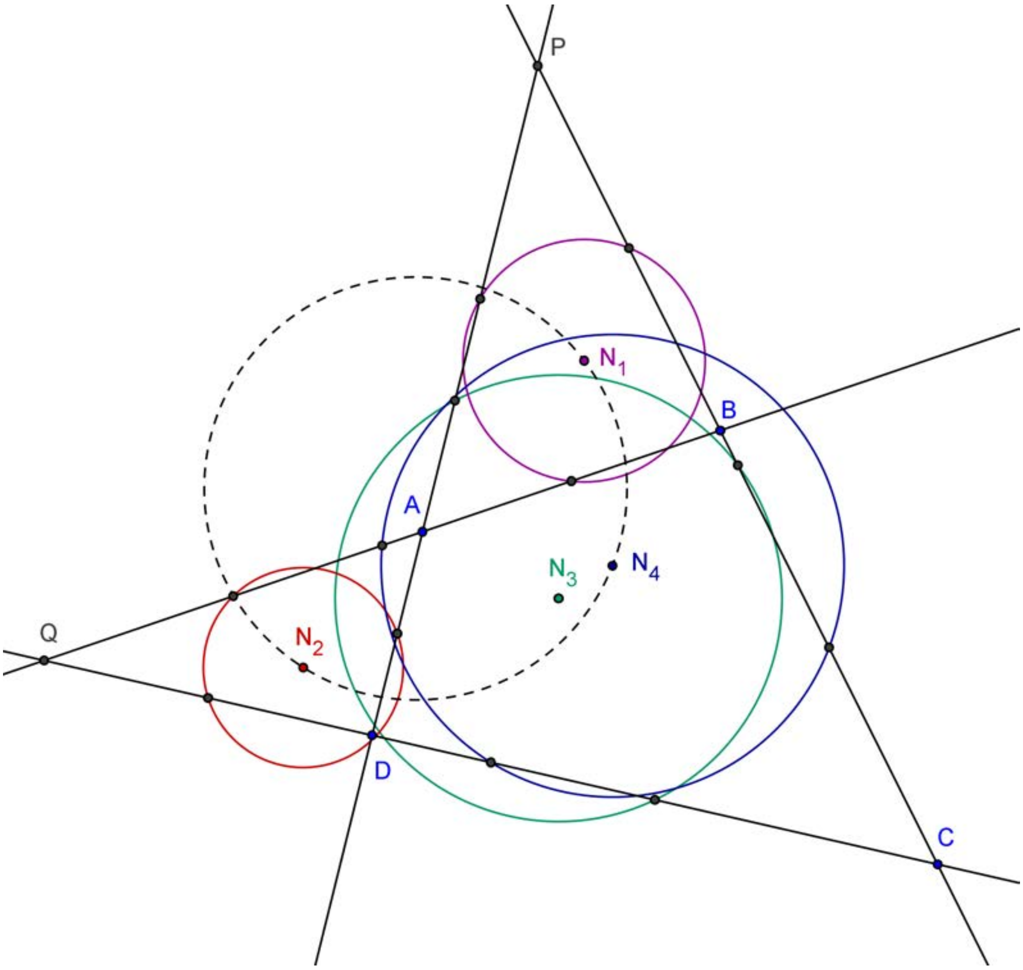
Babis

Luis González

#2 Mar 5, 2009, 12:26 am

According to my sketch the proposition is false.

Attachments:



yetti

#3 Mar 5, 2009, 12:49 am

I does not seem right. Suppose lines AB, CD intersect at X, and lines BC, DA intersect at Y. 9-point circle centers of triangles

I does not seem right. Suppose lines AB, CD intersect at X, and lines BC, DA intersect at Y. 9-point circle centers of triangles YAB, YCD, XBC, XDA are not concyclic. Which four triangles of this complete quadrilateral do you have in mind ?

stergiu

#4 Mar 5, 2009, 2:12 am

Well. first thanks for your answers .Second, I wrote to Bil to send me the source of the problem !
The only theorem I have found is that if the quadrilateral is cyclic , then the Euler centers of 4 triangles (I will write tomorrow which triangles are these, but surely they are not those of the complete one) are concyclic.

Babis

Quick Reply