Iranian National Mathematical Olympiad

1. Suppose the circle C_2 passes from the center of circle C_1 and intersects C_1 at M, N. Let AB be a diameter of C_1 and A', B' are intersections of AM, BN with C_2. Prove that $A'B'$ is equal to the radius of C_1

   Solution : Let O be the center of C_1. Then we have

   $$\angle AMO = \angle MAO = \angle BNM = \angle B'OM$$

   By using the fact that two angles seeing a common arc in a circle are equal.

   So we have $MA' || OB'$ and because $MA'B'O$ is cyclic we have $OM = A'B'$.

2. Find all polynomials $P(x,y) \in \mathbb{R}[x,y]$ for which we have the identity :

$$P(x+y, x-y) = 2P(x,y)$$

Solution : We have

$$P(x,y) = 2P(\frac{x+y}{2}, \frac{x-y}{2}) = 4P(\frac{x}{2}, \frac{y}{2})$$

But since a polynomial identity is also an identity in its coefficents we have that all terms of P are of total degree 2.

So $P(x,y) = ax^2 + bxy + cy^2$.

So $P(x+y, x-y) = (a+b+c)x^2 + 2(a-c)xy + (a+c-b)y^2$.

So we must have $a+b+c = 2a$ and $2(a-c) = 2b$ and $a+c-b = 2c$ but all of these equations are equal. So the answers are

$$P(x,y) = (a+b)x^2 + axy + by^2$$

3. Suppose we have n intervals in $\mathbb{R}$ with the property that , among any k of them there are 2 which intersect. Prove that there exist $k-1$ points in $\mathbb{R}$ for which, any of these intervals has one of these points.

   Solution : We use induction on n. If one interval contains the other just remove it and from the remaining intervals find the k points and then since the removed interval contains another interval, it also contains one of the points.

   So suppose that no interval contains another. Name the intervals as $I_1, \ldots, I_n$ such that they are sorted by their beginning points in $\mathbb{R}$.

   Because no interval contains another using this order, they are also sorted by their end points.

   Now choose the end point of I_1. Now $I_2, I_3, \ldots, I_{k_1}$ contain this point and I_{k_1+1} does not.

   Now choose the endpoint of I_{k_1+1}. Now the intervals $I_{k_1+2}, \ldots, I_{k_2}$ contain this point and I_{k_2+1} does not.

   Repeat the above algorithm to find some intervals like $J_1, J_2, \ldots, J_t$ where we chose their endpoints. By above algorithm no two of J_is intersect. So we must have $t < k$. So we are done.

Iranian National Mathematical Olympiad

1. For $m, n > 2$ prove there exist a sequence of natural numbers bigger than one like $a_0, \ldots, a_k$ for which $a_0 = m$ and $a_k = n$ and

$$a_i + a_{i+1} | a_i a_{i+1} + 1 \quad (i = 0, 1, \ldots, k-1)$$

Solution : Just not that if a is odd then

$$a + (a+2) = 2(a+1) | (a+1)^2 = a(a+2) + 1$$

Now suppose that $m \leq n$ or else just reverse the sequence

Now note that

$$m + (m^2 - m - 1) = m^2 - 1 | m^3 - m^2 - m + 1 = m(m^2 - m - 1) + 1$$

And observe that $m^2 - m - 1$ is increasing for $m > 1$ so $m^2 - m - 1 \leq n^2 - n - 1$, and also $m^2 - m - 1$ is alsways odd.

Now take the sequence to be

$$m, (m^2 - m - 1), (m^2 - m - 1) + 2, (m^2 - m - 1) + 4, \ldots, (n^2 - n - 1), n$$

2. Suppose that A, B, C, D are four cyclic points on the circle C with this order.

   Prove that there exist 4 points like M on W for which $\frac{MA}{MB} = \frac{MD}{MC}$ and these points make a quadrilateral with its diagonals perpendicular to each other.

   Solution : Rewrite our equation as

$$MA.MC = MB.MD$$

   Now if R is the radius of W, we have

$$MA.MC.AC = S_{MAC} * R$$

   and

$$MB.MD.BC = S_{MBD} * R$$

   So we shall have

$$\frac{S_{MAC}}{S_{MBD}} = \frac{AC}{BD}$$

   But writing the formula of Surface, from above we shall have the distances of M from AC and BD are equal.

   If X is the intersection of AC and BD, then M should be the one of intersections of the 4 bisectors of AXB, BXC, CXD, DXA with W.

   If these points are M_1, M_2, M_3, M_4 then M_1M3, M_2M_4 pass through X and $M_1M_3 \perp M_2M_4$ because they are internal and external bisectors of angle X.

3. We have n books piled on each other. We simply reverse the top $kmodn+1$ books at level k.

   Prove that at some level we arrive at our first position (when a book is reversed it differs from itself when not reversed)

   Solution : Just take S to be the set of all 2 tuples like (a, b) where a is an state of the books and b is the number of the level mod n.

   Then for each $x \in S$ we define $f(x)$ to be the 2 tuple (a, b) where a is the resulting state after reversing the top k books and b is $(k+1)modn$ (k is just the number of level-mod-n in x)

   Then this function $(f(x))$ is injective because if $f(y) = x$ then y can be derived from x by reversing the top $(k-1)modn$ books in the state of books in x (again k is the level-mod-n in x)

   So f is a permutation because S is finite.

   For a permutation we can compose it into cycles. So starting at any point in S we arrive at itself sometime. In that time not also the states of books shall be equal to first state, but also the level-mod-n shall be equal.

Iranian Prepration Exam

1. Find all natural numbers like n where there exists a unique $0 \leq a < n!$ where

$$n! | a^n - 1$$

Solution : Let $n! = p_1^{a_1} \dots p_k^{a_k}$ be the factorization of $n!$.

Then by chinese remainder theorem the number of solutions to $n!|a^n - 1$ is the product of the number of solutions to $p_i^{a_i}|a^n - 1$.

So the requested property becomes $a^n = 1$ has a unique solution modulo each $p_i^{a_i}$.

If $p_i \neq 2$ then we have a primitive root modulo $p_i^{a_i}$ so the number of solutions to $a^n = 1$ is exactly $(n, \varphi(p_i^{a_i}))$. So we must have $(n, \varphi(p_i^{a_i})) = 1$.

But if n is composite then take p to be one of primes that divides n. Then power of p in $n!$ is greater than 1 so $p|\varphi(p^k)$ where k is the power of p in $n!$. But then $(\varphi(p^k), n) \neq 1$ which says this n does not work. (If n is a power of 2 then observe that $2|\varphi(p^k)$ for odd p and hence we must have $n \leq 2$ which says $n = 2$)

So let n be an odd prime number. Then $(\varphi(p_i^{a_i}, n) = 1$ because n is greater than $\varphi(p_i^{a_i})$. Hence it remains to check we have a unique solution modulo 2^t.

Just take $f(x) = x^n - 1$. Then $f(x) = 0$ has a unique solution modulo 2. And for this solution (1) we have $f'(x) = 1 \neq 0$ modulo 2.

So by the famous theorem we have unique solution modulo 2^t for every t.

2. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which have the property

$$f(x)f(y) = 2f(x + yf(x)) \tag{1}$$

Solution : Put in our equation $x = y$ which says

$$f(x)^2 = 2f(x + xf(x)) \tag{2}$$

Now suppose that $\inf\{f(x)|x \in \mathbb{R}^+\} = k$. So there is a sequence of x_is in $\mathbb{R}^+$ such that $\lim_{i\to\infty} f(x_i) = k$. Now putting $x = x_i$ in (2) we get

$$f(x_i)^2 = 2f(x + xf(x)) \geq 2k$$

Taking limits we get

$$k^2 = \lim_{i\to\infty} f(x_i)^2 \geq 2k$$

The above inequality says $k \geq 2$. So we have $f(x) \geq 2$ for each $x \in \mathbb{R}^+$.

Now we want to say f is an increasing function, that is $f(x) \leq f(x + \epsilon)$ for $\epsilon \in \mathbb{R}^+$.

Just put y in (1) to be equal to $\frac{\epsilon}{f(x)}$ to get

$$2f(x) \leq f(x)f(y) = 2f(x + \epsilon)$$

Note that if f is not strictly increasing then for some x we have $f(x) = 2$.

Now suppose that $f(x) = 2$ for some x. Then the equation (1) will become

$$2f(y) = 2f(2y + x) \to f(y) = 2f(2y + x)$$

, but since f is increasing we have $y < t < 2y + x$ yields $f(t) = f(y)$ and so $f(2y) = f(y)$.

So using induction we get $f(2^k) = f(2^t)$ for $k, t \in \mathbb{Z}$.But then each z is just in an interval $[2^k, 2^{k+1}]$ so $f(z) = f(2^k) = f(1)$

So f is a constant function and hence $f(x) = 2$ for all x.

So now assume that f is strictly increasing and hence injective. Swapping x, y in (1) we get

$$2f(x + yf(x)) = f(x)f(y) = 2f(y + xf(y)) \to x + yf(x) = y + xf(y)$$

But letting $x \to 0$ in above we get

$$\lim_{x\to 0} yf(x) = \lim_{x\to 0} x + yf(x) = \lim_{x\to 0} y + xf(y) = y$$

But this would lead to $lim_{x\to 0} f(x) = 1$ which is contradiction because $f(x) > 2$ for all x.

So the only solution is $f(x) = 2$.

3. Suppose that $a_1, \dots, a_n \in \mathbb{Z}$ such that $n|a_1+a_2+\dots+a_n$. Prove that there exist two permutations b_is and c_is of $1, 2, \dots, n$ such that for $1 \leq i \leq n$ we have :

$$n|a_i - b_i - c_i$$

Solution :

Iranian Team Selection Test

1. Suppose that p is a prime number. Find all natural numbers n such that $p|\varphi(n)$ and for all a such that $(a,n)=1$ we have
$$n|a^{\frac{\varphi(n)}{p}}-1$$

Solution : For a number like n we say k is a good number for n if and only if we have
$$(a,n)=1\to n|a^k-1$$
Then if we let $ord_n a$ be the smallest number such that $n|a^{ord_n a}-1$ then clearly the necessary and sufficient condition for k to be good for n is
$$lcm\{ord_n a|(a,n)=1\}|k$$
For a number like n let the smallest good number be $f(n)$.

Then using chinese remainder theorem we have $f(mn)=lcm(f(m),f(n))$ if $(m,n)=1$.

For a power of prime like p^k such that p is odd we have $f(p^k)=\varphi(p^k)$ because we have primitive root modulo p^k.

For powers of 2 we have $f(2)=1$ and $f(4)=2$ and we want to say $f(2^k)|2^{k-2}$ for $k\geq 3$.

Bacuse we don't have primitive root modulo 2^k for $k\geq 3$ then clearly $f(2^k)|2^{k-2}$.

Now back to the main problem, if $n=2^t p_1^{a_1}\dots p_k^{a_k}$ then we have
$$f(n)=lcm(f(2^t),\varphi(p_1^{a_1}),\dots,\varphi(p_k^{a_k}))$$
But $\varphi(n)=\varphi(2^t)\varphi(p_1^{a_1})\dots\varphi(p_k^{a_k})$.

For $p\neq 2$ both $\varphi(2^t)$ and $f(2^t)$ are not divisible by p so the necessary and sufficient condition for this case is that two of $\varphi(p_i^{a_i})$ are divisible by p.

For $p=2$ if we have two odd primes then their f values are divisible by 2 and n has the property.

If we have only one odd prime power so $n=2^t p^k$ then if $t>1$ both $f(2^t)$ and $f(p^k)$ are divisible by 2 and we are done (Also note that $f(2^t)|\varphi(2^t)$). For $t=1$ we have a primitive root and n does not have the property.

Also for $n=2^t$ we have the property if and only if $t\geq 3$ like we discussed $f(2^t)$.

2. Suppose n coins are available and we don't know their mass.

   We have a pair of balances and every time we can choose an even number of coins and put half of them on one side and the other on the other side of the balance, therefore a comparison will be done.

   Our aim is to determine, the mass of all coins are equal or not.

   Show that at least $n-1$ comparisons are needed.

   Solution : Suppose that with k comparisons we know the answer. Just think that all comparisons returned the result of equality.

   Then we want to say that all coins can be equal and can be not equal such that results of comparisons are correct.

   If we have the coin weights are $x_1, \ldots, x_n$ then each result of comparison is just a linear equation in these variables.

   If $k < n-1$ then the space of the answers for these equations has dimension greater than 1 so we have an answer with non-equal weights (or else the dimension would be 1)

   Take this answer for $x_1, \ldots, x_n$. Then add very large amount like c to every one of the x_is. Then clearly the equations are satisfied yet, but we can choose c so that x_is become positive.

   So we have an answer with non equal weights andf we are done.

3. Suppose ABC is a triangle with M the midpoint of BC.

   Suppose that AM intersects the incircle at K, L.

   We draw parallel line from K and L to BC and name their second intersection point with incircle X and Y.

   Suppose that AX and AY intersect BC at P and Q. Prove that BP=CQ

   Solution : Let H_1, H_2 be the homotheties with respect to A which map K, L to M, M.

   Let H_1, H_2 map the incircle to w_1, w_2. Then clearly w_1, w_2 pass through P, Q.

   Now suppose that we proved that power of B with respect to w_1 is equal to power of C with respect to w_2. Then we have

$$BM.BP = CM.CQ \to BP = CQ$$

   So we just need to prove these two powers are equal.

   If w_1 touces AB, AC at D_1, E_1 and w_2 touches AB, AC at D_2, E_2, we need to prove that $BD_1 = CE_2$.

   If N_1, N_2 are the midpoints of D_1D_2 and E_1E_2 then N_1N_2 passes through M. In triangles MN_1B and MN_2C we have $MB = MC$ and $\angle MN_1B + \angle MN_2C = \pi$ so we have $BN_1 = CN_2$.

   But $D_1N_1 = E_2N_2$, subtracting it from $BN_1 = CN_2$ we get $BD_1 = CE_2$, just what we want.

Iranian Team Selection Test

1. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that

$$\sum_{i,j=1}^{n} |x_i + x_j| \geq n \sum_{i=1}^{n} |x_i|$$

Solution : Take $f(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} |x_i + x_j|$ and $g(x_1, \ldots, x_n) = n \sum_{i=1}^{n} |x_i|$.

We want to say that if we put the arithmetic mean of positive x_is instead of them then $f - g$ will not increase.

It is clear that by doing this, value of g does not change. Suppose that $x_1, \ldots, x_k$ are the positive ones. Then the value of $\sum_{i,j=1}^{k} |x_i + x_j|$ also does not change. And also the value of $\sum_{i,j=k+1}^{n} |x_i + x_j|$ does not change.

Now take $f_i(x_1, \ldots, x_k)$ to be $\sum_{j=1}^{k} |x_i + x_j|$ for $i > k$. Then because the function $|x_i + x|$ is convex, by jensen inequality putting arithmetic mean of $x_1, \ldots, x_k$ instead of them will not increase the value of $\sum_{j=1}^{k} |x_i + x_j|$. We know that

$$f = \sum_{i,j=1}^{k} |x_i + x_j| + \sum_{i,j=k+1}^{n} |x_i + x_j| + 2 \sum_{i=k+1}^{n} f_i(x_1, \ldots, x_k)$$

So value of f also will not increase.

So we may put the arithmetic mean of negative x_is instead of them. We may also do this for non-negative x_is (by similar arguments).

So we may assume that $x_1 = x_2 = \ldots = x_k = -a$ and $x_{k+1} = x_{k+2} = \ldots = x_n = b$ for some non-negative numbers a, b.

Then we shall prove that

$$2k^2 a + 2(n-k)^2 b + 2k(n-k)|a-b| \geq kna + (n-k)nb$$

We may assume that $a \geq b$ or else change a, b. Then the inequality will become

$$(2k^2 + 2k(n-k) - kn)a + (2(n-k)^2 - 2k(n-k) - (n-k)n)b \geq 0$$

Simplifying above we arrive at

$$kna + (n^2 + 4k^2 - 5nk) \geq 0$$

But since $a \geq b$ we have

$$kna + (n^2 + 4k^2 - 5nk)b \geq (kn + n^2 + 4k^2 - 5nk)b = (n - 2k)^2 b \geq 0$$

2. Let ABC be a triangle such that it's circumcircle radius is equal to the radius of outer inscribed circle with respect to A. Suppose that the outer inscribed circle with respect to A touches BC, AC, AB at M, N, L.

   Prove that O (Center of circumcircle) is the orthocenter of MNL.

   Solution : Let R be the radius of circumcircle.

   If I_a be center of the outer inscribed circle with respect to A.

   Then AI_a intersects the circumcircle at middle of the arc BC (the one without A). Let this point be M'.

   Then we have $OM' \perp BC$ and $I_aM \perp BC$ also $OM' = R = I_aM$ so $I_aM' || OM$, but I_aM' is AI_a which is perpendicular to NL. So $OM \perp NL$.

   Now BI_a intersects the circumcircle at middle of the arc AC (the one that includes C). Let this point be N.

   Then we have $ON' \perp AC$ and $I_aN \perp AC$ also $ON' = R = I_aN$ so $I_aN' || ON$, but I_aN' is BI_a which is perpendicular to ML. So $ON \perp ML$.

   The other one ($OL \perp MN$) is just similar as above.

3. Let G be a tournoment such that it's edges are colored either red or blue.

   Prove that there exists a vertex of G like v with the property that, for every other vertex u there is a mono-color directed path from v to u.

   Solution : We use induction on the number of vertices of G.

   We say v sees u red if and only if there is a red-color path from v to u.

   For $|G| = 2$ its obvious. So suppose that $|G| > 2$. Take a verex of G like v. Then removing v from G we will have a vertex in the remaining tournoment with requested property. Let this be u.

   Then the edge between u and v is directed from v or else u has the requested property in G too. Let vu be for example red.

   If $v_1, \ldots, v_k$ are the vertices that u sees them blue. Then in the tournoment composed of $v, v_1, \ldots, v_k$ there is a verex with the requested property like w.

   If $w = v$ then v sees $v_1, \ldots, v_k$ mono-colored and it also sees other verices red. (Beacuse vu is red)

   So $w \neq v$. If w sees v red, then w sees $v_1, \ldots, v_k, v$ mono-colored and sees the other vertices red.

   If w sees v blue, then u sees w blue, and w sees v blue, so u sees v blue, so u has the requested property in G.

   So in all cases we are done.

Iranian Team Selection Test

1. We have n points in the plane, no three on a line. We call k of them good if they form a convex polygon and there is no other point in the convex polygon. Suppose that for a fixed k the number of k good points is c_k. Show that the following sum is independent of the structure of points and only depends on n :

$$\sum_{i=3}^{n}(-1)^i c_i$$

Solution : Let S be the set of our n points. We calculate the following sum in two ways

$$\sum_{k\geq 3, v1,\ldots,v_k\in S}(-1)^k$$

First it does not depend on structure of n points, so we may say it is $f(n)$.

Now in the other way, calculate the sum where sum runs on all subsets whose convex hull is a fixed polygon like $v_1,\ldots,v_t$.

If there are k points in the polygon $v_1,\ldots,v_t$ this sum will be

$$(-1)^t\sum_{i=0}^{k}(-1)^i\binom{k}{i}$$

which equals zero if $k\geq 1$ and $(-1)^t$ otherwise.

So we may partition the terms in the first sum into parts, each being summed over the vertices with a fixed convex hull.

Here the sum of each part will include 0 if the convex hull is not good and otherwise $(-1)^v$ where v is the number of vertices of convex hull.

So this sum is equal to

$$\sum_{i=3}^{n}(-1)^i c_i$$

So

$$\sum_{i=3}^{n}(-1)^i c_i = f(n)$$

2. Let n be a fixed natural number.

a) Find all solutions to the following equation :

$$\sum_{k=1}^{n}[\frac{x}{2^k}] = x - 1$$

b) Find the number of solutions to the following equation (m is a fixed natural) :

$$\sum_{k=1}^{n}[\frac{x}{2^k}] = x - m$$

Solution : First note that in both equations the left hand side is a non-negative integer, so right hand side must be so, that is x is a natural number.

Now let a_t be the $t+1$th digit (from right) in the representation of x in base 2.

Then we have

$$[\frac{x}{2^k}] = a_k + 2a_{k+1} + 4a_{k+2} + \dots$$

So

$$\sum_{k=1}^{n}[\frac{x}{2^k}] = \sum_{k=1}^{n}\sum_{i=k}^{\infty} 2^{i-k} a_k =$$

$$\sum_{i=0}^{\infty}\left(a_k \sum_{k=max(0,k-n)}^{k-1} 2^k \right) =$$

$$(2^0-1)a_0+(2^1-1)a_1+\dots+(2^{n-1}-1)a_{n-1}+(2^n-2^0)a_n+(2^{n+1}-2^1)a_{n+1}+\dots$$

So we have

$$x - \sum_{k=1}^{n}[\frac{x}{2^k}] = a_0 + a_1 + \dots + a_{n-1} + a_n + 2a_{n+1} + 4a_{n+2} + \dots$$

which is equal to

$$\sum_{i=0}^{n-1} a_i + [\frac{x}{2^n}]$$

Note that the numbers $a_0, \dots, a_{n-1}, [\frac{x}{2^n}]$ uniquely determine the number x.

Now back to the main problem

a) We shall have $a_0 + \dots + a_{n-1} + [\frac{x}{2^n}] = 1$.

That is exactly one of $a_0, a_1, \dots, a_{n-1}, [\frac{x}{2^n}]$ is equal to 1 and others are 0. That is $x = 2^0$ or $x = 2^1$ or $\dots$ or $x = 2^{n-1}$ or $x = 2^n$.

b) We must have

$$\sum_{i=0}^{n-1} a_i + [\frac{x}{2^n}] = m$$

$[\frac{x}{2^n}]$ can be any nonnegative number and for each value of this number like t we must have $\sum_{i=0}^{n-1} a_i = m - t$ which has exactly $\binom{n}{m-t}$ answers.

So the total answer is

$$\sum_{t=0}^{\infty} \binom{n}{m-t} = \sum_{t=0}^{m} \binom{n}{t}$$

3. Let l, m be two parallel lines in the plane.

   Let P be a fixed point between them.

   Let E, F be variable points on l, m such that the angle EPF is fixed to a number like α where $0 < \alpha < \frac{\pi}{2}$. (By angle EPF we mean the directed angle)

   Show that there is another point (not P) such that it sees the segment EF with a fixed angle too.

   Solution : Let R, Q be two points on l, m where the bisector of RPQ is parallel to l and $RPQ = 2\alpha$.

   Take V to be the reflection of P with respect to RQ.

   We want to say Q has the requested property. Take two projective maps from l to m like f, g where f is the one that $f(E) = F$ and g is the one that maps a point like x to a point like y where $\angle xVy = \pi - \alpha$.

   These maps are projective because if $x_1, x_2, x_3, x_4 \in l$ then $(x_1x_2x_3x_4) = P(x_1x_2x_3x_4) = P(y_1y_2y_3y_4) = (y_1y_2y_3y_4)$ where $y_i = f(x_i)$. Similar arguments for g hold.

   We want to say these two maps are equal, proving that V sees EF with the angle $\pi - \alpha$.

   For these maps to be equal we shall say three points are mapped equal under these maps.

   Take one to be where $E = \infty$. Then $F = Q$ and $\angle EVF = \pi - \alpha$.

   Take the other to be where $F = \infty$. Then $E = R$ and $\angle EVF = \pi - \alpha$.

   Take the third to be where the bisector of VPQ is PF. Now PF is the bisector of VPQ and QF is the external bisector of PQV, so VF is the external bisector of PVQ so VF is the external bisector of VPQ. Using similar arguments we get VE is the external bisector of VPR.

   So we have $\angle EVF = 2\pi - 2\frac{\pi-\alpha}{2} - 2\alpha = \pi - \alpha$ and we are done.

Iranian Team Selection Test

1. Let n be a fixed natural number greater than one. Find all n tuples of natural pairwise distinct and coprime numbers like $a_1, a_2, \ldots, a_n$ such that for $1 \leq i \leq n$ we have

$$a_1 + a_2 + \ldots + a_n | a_1^i + a_2^i + \ldots + a_n^i$$

Solution : Let $\sigma_k = a_1^k + a_2^k + \ldots + a_n^k$.

We have $\sigma_1 | \sigma_1, \sigma_2, \ldots, \sigma_n$. We want to say that $\sigma_1 | \sigma_i$ for $i > n$ too.

Take $P(x) = (x - a_1)(x - a_2) \ldots (x - a_n) = x^n + c_{n-1}x^{n-1} + \ldots + c_0$.

So we have $P(a_1) = P(a_2) = \ldots = P(a_n) = 0$.

Now take the following sum

$$0 = a_1 P(a_1) + a_2 P(a_2) + \ldots + a_n P(a_n) = \sigma_{n+1} + c_{n-1}\sigma_n + \ldots + c_0\sigma_1$$

From above we have $\sigma_1 | \sigma_{n+1}$.

Using similar arguments and induction we have $\sigma_1 | \sigma_{n+i}$.

Now take p^k to be one of the prime powers in factorization of σ_1.

Then letting n to be a number that $m > k$ and $\varphi(p^k) | m$, we have $p^k | \sigma_m$.

But if one of a_is is divisible by p then $p^k | a_i^m$. So σ_m is equal to n or $n - 1$ modulo p^k (depending on wether any a_i is divisible by p or not)

So we have $p^k | n$ or $p^k | n - 1$.

So we have $\sigma_1 | n(n - 1)$, and hence $\sigma_1 \leq n(n - 1)$.

Since a_is are pairwise coprime and non-equal we have σ_1 is greater than or equal to sum of first $n - 1$ primes plus 1. (Because if one of a_is is not 1 putting one of its primes instead of it makes σ_1 to decrease)

If $n > 5$ then we have $\sigma_1 \geq 1+2+3+5+7+11+13+15+17+\ldots+(2n-1)$ (9 is not included in the sum)

So $\sigma_1 \geq n^2 - 9 + 2 = n^2 - 7$ and whenever $n > 7$ we have $n^2 - 7 > n^2 - n = n(n - 1)$, so for $n > 7$ we have no answer.

For $n = 4, 6$ we have sum of the first n primes plus 1 is exactly one less than $n(n-1)$, so we have no answer (If a_is are $\{1, 2, 3, 5, \ldots\}$ they can not divide $n(n-1)$ and for any other set of a_is their sum is at least two units greater, because putting one of primes of one a_i instead of it will decrease it by at least 2)

For $n = 5$ the sum is 2 units less than $n(n-1)$. So by similar arguments as above the only possible answer is $\{1, 3, 4, 5, 7\}$, which is not an answer because $\sigma_4 = 1 + 1 + 1 + 0 + 1 = 4(mod 5)$ and $5|\sigma_1$.

For $n = 7$ the sum is exactly $n(n-1)$, so the only possible answer is $\{1, 2, 3, 5, 7, 11, 13\}$, which is not an answer beause $\sigma_6 = 1 + 1 + 1 + 1 + 0 + 1 + 1 = 6(mod 7)$ but $7|\sigma_1$.

For $n = 3$ again the sum is exactly $n(n-1)$, so the only possible solution is $\{1, 2, 3\}$, which is not an answer because $\sigma_2 = 1 + 1 + 0 = 2(mod 3)$ but $3|\sigma_1$.

For $n = 2$ the sum is greater than $n(n-1)$. So we have no solution at all.

2. Let ABC be an acute angle triangle.

   Suppose that D, E, F are the feet of perpendicluar lines from A, B, C to BC, CA, AB.

   Let P, Q, R be the feet of perpendicular lines from A, B, C to EF, FD, DE. Prove that

$$2(PQ + QR + RP) \geq DE + EF + FD$$

   Solution :

3. Suppose we have a simple polygon (that is it does not intersect itself, but not necessarily convex).

   Show that this polygon has a diameter which is completely inside the polygon and the two arcs it creates on the polygon perimeter (the two arcs have 2 vertices in common) both have at least one third of the vertices of the polygon.

   Solution : Let us decompose the polygon into triangles (it can be done by a famous theorem).

   If n is the number of vertices of polygon then there are exactly $n-2$ triangles (for example by double counting the sum of angles of polygons)

   Now let us create a graph where each vertex is a triangle in our decomposition and two vertices are adjacent if and only if the two triangles have a common side.

   Observe that this graph has no cycles (or else our polygon would have two sides or would have one of its vertices inside itself!)

   So the graph is a tree (its connected too). Let $t = n-2$ be the number of its vertices.

   Now we want to say that there is an edge, which when we cut it, the two connected components have at least $\frac{v-4}{3}$ of the verices. Because then the two components show us the two arcs in the polygon,each has at least $\frac{n-2-4}{3}+2$ of vertices, that is $\frac{n}{3}$ of vertices.

   Observe that in our graph each vertex has at most three neighbors.

   Take one vertex like v. If we remove one outgoing edge of v then the component without v has either less than $\frac{v-4}{3}$ or more than $\frac{2v+4}{3}$ vertices (or else we are done).

   Now note that there is exactly one neighbor of v with its component having more than $\frac{2v+4}{3}$ vertices. Because if two has this property then the graph would have more than $2\frac{2v+4}{3} > v$ vertices! If none has this property, then the graph would have less than $3\frac{v-4}{3}+1 < v$ vertices!

   Name this neighbor of v as $f(v)$.$(v, f(v)$ are adjacent in the graph)

   Now take this sequence : $v, f(v), f(f(v)), \ldots$. This sequence is a walk in our graph, and because our graph is a tree, some time we should walk back the last edge we walked.

   So there is a vertex like v such that $f(f(v)) = v$. Now cutting the edge between v and $f(v)$ yields two components each having at least $\frac{2v+4}{3}$ vertices, so our graph has at least $2\frac{2v+4}{3} > v$ vertices, contradiction!