Some Own Problems

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August 26, 2016

1 Problem 1

[1] **Part 1:** Let ABC be a triangle with all angles$> 45^\circ$ and circumcenter O. D, E, F are the feet of the altitudes from A, B, C. G is the centroid. Intersection of DG and (ABC) is A'. Define B' and C' analogously.$A'B'$ intersects with AC at L_{ac} and $A'B'$ with BC at L_{bc}. Define the L's with other subscripts similarly. Let (O_a) be the circle passing through A and L_{ac} and tangent to $A'B'$. Let (O'_a) be the circle through A and L_{ab} and tangent to $C'A'$. Here we denote by (X) a circle with center as X. Define the circles with the other subscripts in the same manner. Let the circumcentre of $AO_aO'_a$ be O''_a. Define other centres similarly, using a cyclic shift of variable names. Extend $A'O''_a$, $B'O''_b$ and $C'O''_c$ to form a triangle $A_1B_1C_1$. Let the mixtilinear incircle touch points with its circumcircle be X, Y, and Z. Prove that the cevians A_1X etc concur at the point P such that the circumcentre of the cevian triangle of the isotomic conjugate of isogonal conjugate of P with respect to $A_1B_1C_1$ is the circumcentre of ABC.

Part 2: Prove that the point O is the radical center of the circles $(O_aO_bO'_aO'_b)$ etc, $(O_aO'_aA')$ etc, $(O_aO'_aB')$ etc, and $(O_aO'_aC')$ etc. after showing that they exist.

2 Problem 2

Let the perimeter of a triangle ABC be 2 and let BC be the smallest side. Let P and Q be on AC and AB such that $AP + PB = AQ + QC = 1$. A line parallel to the internal angle bisector of B through P meets the perpendicular bisector of BC at T. BP intersects QC at W. Prove that A, W, T are collinear iff $AB = AC$.

[1] A bit hard, so you may consider solving this later.

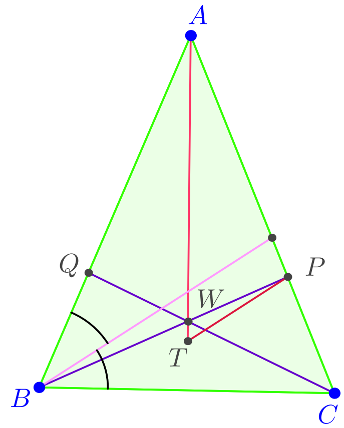

Figure 1: Problem 2

3 Problem 3

Let H be the orthocenter of a $\triangle ABC$, T being its isotomic conjugate. Let the cevian triangle of T be DEF with D, E, F on $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. Let the reflections of A, B, C over the centroid G be D_1, E_1, F_1 respectively. Let the ray $\overline{DD_1}$ intersect the circumcircle of BHC at D_2. Let the circumcenter of ABC be O. Let the midpoint of OD_2 be D_3. Define E_3, F_3 in a similar manner. Prove that $\angle BD_3C - \angle E_3D_3F_3 = \angle BAC$.

4 Problem 4

Let H be the orthocenter of a triangle ABC. Let M be the midpoint of BC, and let E, F be the feet of the B and the C altitudes onto the opposite sides. Let X be the intersection of ray MA with the circumcircle of BHC. Prove that HX, EF and BC concur at a point and also show that the line joining that point and A is perpendicular to the line HM.

5 Problem 5

Let I be the incenter of a triangle ABC and let the A, B and C mixtilinear incircles touch the circumcircle of ABC at T_A, T_B and T_C respectively. Let IT_A, IT_B, IT_C cut BC, CA, AB at X, Y and Z respectively. Prove that AX, BY and CZ concur at a point joining the centroid of ABC to the Gergonne point of ABC.

6 Problem 6

Let ABC be a triangle with the midpoints of the sides $\overline{BC}, \overline{CA}, \overline{AB}$ being A', B', C' respectively. Let the circle tangent to the circumcircle of $AB'C'$ internally, to the circumcircles of

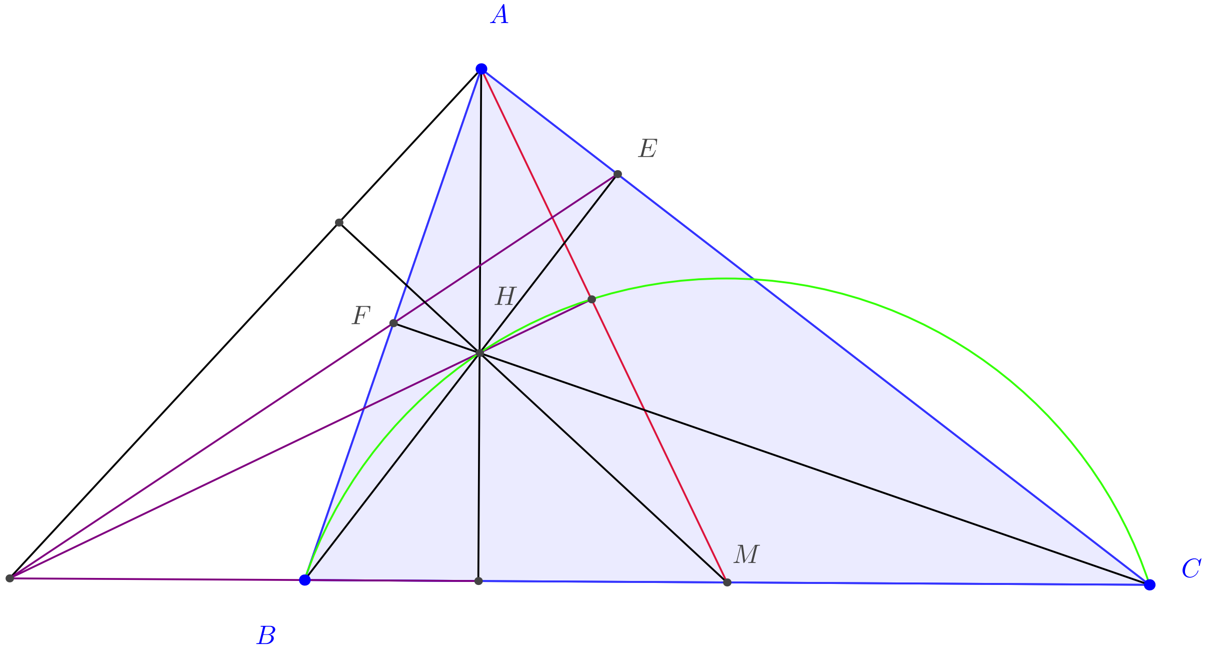

Figure 2: Problem 4

$BC'A'$ and $CA'B'$ externally be tangent to the circumcircle of $AB'C'$ at X. Define Y, Z similarly. Prove that $A'X$, $B'Y$ and $C'Z$ are concurrent.

7 Problem 7

Let $\triangle ABC$ be a triangle with circumcircle Γ. Let M_A, M_B, M_C be the midpoints of the arcs BC, CA, AB of Γ containing exactly 2 points of the triangle. Let the reflections of A, B, C over M_A, M_B, M_C respectively be D, E, F. Let BC meet DE, DF in K, L. Let $KF \cap LE = X$. Define Y, Z similarly. Prove that DX, EY, FZ concur at a point.

8 Problem 8

[2] Find all functions $f : [\mathbb{R}^+ \cup \{0\}]^2 \longrightarrow [\mathbb{R}^+ \cup \{0\}]^2$ in 2 variables satisfying the following properties:

1. $f(\sqrt{ab}, f(\frac{a-b}{2}, x)) = f(x, \frac{a+b}{2})\ \ \forall a, b \in \mathbb{R}^+ \cup \{0\}$ with $a > b$.
2. $f(\alpha, 0) = \alpha\ \forall \alpha \in \mathbb{R}^+ \cup \{0\}$
3. $f(x, y)$ is continuous as a function of x and as a function of $y\ \forall x, y \in \mathbb{R}^+ \cup \{0\}$.

[2] This functional equation can be used to give a proof of the Pythagoras' Theorem too!

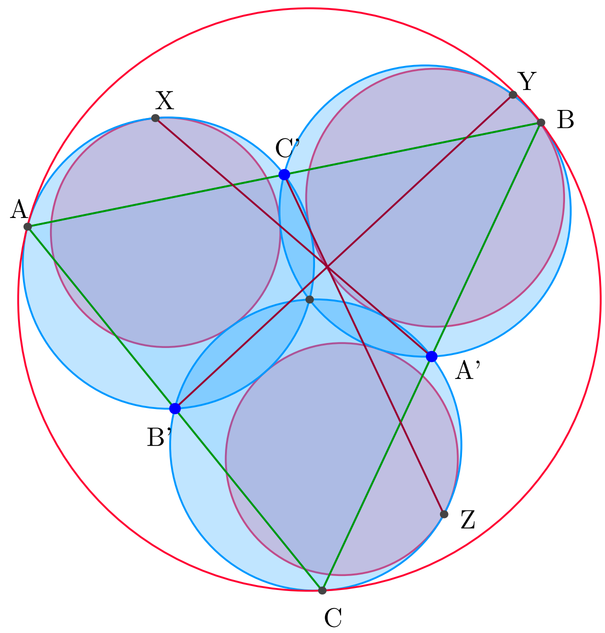

Figure 3: Problem 6

4. $f(x, y) = f(y, x)\ \forall x, y \in \mathbb{R}^+ \cup \{0\}$.
5. $f(x, f(y, z)) = f(f(x, y), z)\ \forall x, y, z \in \mathbb{R}^+ \cup \{0\}$.

9 Problem 9

Part 1:(Along with Sutanay Bhattacharya) Suppose n is a composite number. Does there exist a function $f : \{1, 2, \cdots n\} \longrightarrow \{1, 2, \cdots n\}$ satisfying all the following conditions?

1. $\sum_{i=1}^{n} f(i)^3 = \left(\sum_{i=1}^{n} f(i)\right)^2$
2. f is non decreasing
3. f has a unique minimum and a unique maximum
4. f is neither injective nor surjective [3]

Part 2: The third condition in the above is changed to "f has a unique minimum and a non-unique maximum".

[3]This dashes all the solver's hopes...

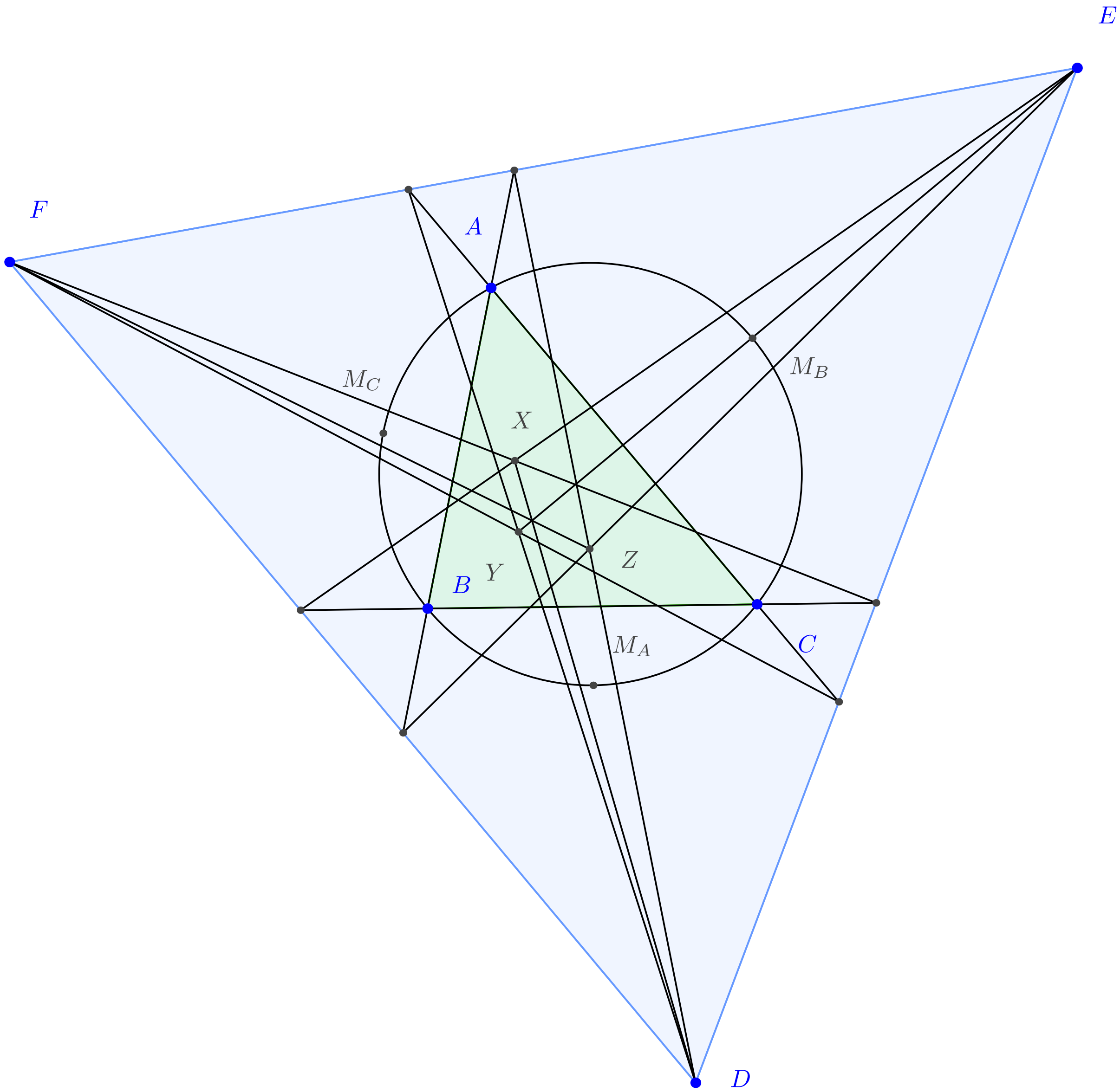

Figure 4: Problem 7