The Perpendicularity Lemma

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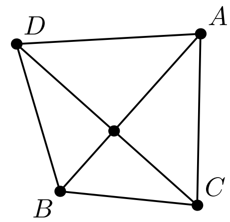
Abstract

In this note, we introduce the Perpendicularity Lemma, a simple result that, when applied, can reduce quite a few complex olympiad geometry problems to mere computations. Credits to pohoatza for the lemma and many of the problems in this handout.

1 The Lemma

Lemma 1.1. *Given points A, B, C, D in the plane, $AB \perp CD$ if and only if $AC^2 - AD^2 = BC^2 - BD^2$.*



Proof. This is a direct application of the Pythagorean Theorem for the "if" direction and the Law of Cosines for the "only if" direction after rearranging the equation to $AC^2 + BD^2 = BC^2 + AD^2$. □

Of course, with such a simple proof and such a simple statement, one could easily be tricked into underestimating its power. However, the lemma provides a very simple and relatively computationally light method of translating perpendicularity into a metric computation. Don't take it from us; take it from the problems!

2 Example Problems

Example 2.1: (*Carnot's Theorem*) Let M, N, P be points on sides BC, CA, AB, respectively, of triangle ABC. The the perpendicular lines through M, N, P concur if and only if $(BM^2 - CM^2) + (CN^2 - AN^2) + (AP^2 - BP^2) = 0$.

Proof. To begin, we prove the "only if" direction. Let the concurrency point be X and let the lines be perpendicular; we would like to show that the metric condition is true. However, the lemma yields

$$BM^2 - CM^2 = XB^2 - XC^2,$$
$$CN^2 - AN^2 = XC^2 - XA^2,$$
$$AP^2 - BP^2 = XA^2 - XB^2,$$

and summing the three easily gives the result.

For the "if" direction, assume that the metric condition holds. Consider the intersection of the perpendiculars from M and N, and let it be point X'. We will show that $X'P \perp AB$.

From the Lemma, $X'B^2 - X'C^2 = BM^2 - CM^2$, as $X'M \perp BC$. Similarly, we find that $X'C^2 - X'A^2 = CN^2 - AN^2$, as $X'N \perp AC$. Summing these two yields $X'B^2 - X'A^2 = BM^2 - CM^2 + CN^2 - AN^2$, which reduces to $BP^2 - AP^2$ from the given condition, and the result follows using the converse of our Lemma. □

Example 2.2: In non-isosceles triangle ABC, let I, G denote the incenter and centroid, respectively. Prove that $IG \perp BC$ if and only if $AB + AC = 3BC$.

Proof. Assume that $b+c=3a$. From the Lemma, $IG \perp BC$ if and only if $IB^2 - IC^2 = GB^2 - GC^2$. From the formula $IB^2 = \frac{ac(s-b)}{s}$, where s denotes the semi-perimeter of the triangle, we find that

$$\begin{aligned} IB^2 - IC^2 &= \frac{a^2c + ac^2 - abc}{2s} - \frac{a^2b + ab^2 - abc}{2s} \\ &= \frac{a^2c + ac^2 - a^2b - ab^2}{a+b+c}. \end{aligned}$$

Since $b+c=3a$ and $a+b+c=4a$, we have

$$\frac{a^2c + ac^2 - a^2b - ab^2}{a+b+c} = \frac{(2c^2 - bc) - (2c^2 - bc)}{6} = \frac{c^2 - b^2}{3}.$$

Now, using the median formula, we compute

$$GB^2 = \frac{2}{3} \cdot \frac{2a^2 + 2c^2 - b^2}{4} = \frac{2a^2 + 2c^2 - b^2}{6},$$

implying that

$$\begin{aligned} GB^2 - GC^2 &= \frac{2a^2 + 2c^2 - b^2}{6} - \frac{2a^2 + 2b^2 - c^2}{6} \\ &= \frac{c^2 - b^2}{3}. \end{aligned}$$

Thus, $IB^2 - IC^2 = GB^2 - GC^2$, or $IG \perp BC$ if $b+c=3a$.
If $IG \perp BC$, then it follows that

$$\frac{a^2c + ac^2 - a^2b - ab^2}{a+b+c} = \frac{c^2 - b^2}{3}.$$

Note that this equation is linear in a, implying that there is at most one solution for a. Once we note that $a = \frac{b+c}{3}$ satisfies this, the result follows. □

Example 2.3: In triangle ABC, let BB_1 and CC_1 be altitudes. If $BC \cap B_1C_1 = T$ and H is the orthocenter, prove TH is perpendicular to the C-median.

Proof. Let M denote the midpoint of side BC, and let A_1 be the foot of the altitude from C to AB. Since we want to show that $TH \perp AM$, it suffices to prove that H is the orthocenter of triangle TAM, or that $HM \perp AT$.

From the Lemma, it suffices to show that $HT^2 - HA^2 = MT^2 - MA^2$. It is a well known result that $HA = 2R\cos a$ (why?). Using the median formula, $AM^2 = \frac{2b^2+2c^2-a^2}{4}$. To compute HT^2, we may use the Pythagorean Theorem to find $HT^2 = HA_1^2 + TA_1^2$, using $HA_1^2 = 4R^2\cos^2 b \cos^2 c$ and $BT = \frac{ac\cos b}{b\cos c - c\cos b}$. We can now apply the lemma and conclude; the computations are left to the reader.[1] □

Example 2.4: *(IMO 1994)* Let ABC be an isosceles triangle with $AB = AC$. M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB. Q is an arbitrary point on BC different from B and C. E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear. Prove that OQ is perpendicular to EF if and only if $QE = QF$.

Proof. If $OQ \perp EF$, then $BEQO$ and $QOFC$ are cyclic. Thus $\angle EOQ = \angle EBQ = \angle ABC$ and $\angle QOF = \angle ACB$. Since ABC is isosceles, it follows that $\angle EOQ = \angle QOF$, so triangle EOF is isosceles and $EQ = QF$.

If $QE = QF$, by the Lemma, it suffices to show that $OE^2 - OF^2 = EQ^2 - QF^2 = 0$, or that $OE = OF$. Using the Law of Sines in triangles BEQ and QFC, we find that

$$QE = \frac{BE}{\sin BQE} \cdot \sin ABC,$$

and

$$QF = \frac{CF}{\sin CQF} \cdot \sin ACB.$$

Dividing the two, we find that $1 = \frac{QE}{QF} = \frac{CF}{BE}$, or, equivalently, $CF = BE$. Consider right triangles BOE and COF. By symmetry, $BO = CO$, and as we showed, $BE = CF$. Thus, triangles BOE and COF are congruent, and it follows that OE and OF have equal length. We may conclude. [2] □

[1] This problem is a good example of how choosing what you want to prove carefully can drastically reduce computations; trying to show that $HT \perp AM$ directly is much more computationally heavy. Also, this problem is not difficult with poles and polars, but was included to show the versatility of this lemma.

[2] Of course, though powerful on its own, the lemma is much more powerful with synthetic observations!

3 Problems

Here we present some problems that can be solved using this lemma and results that follow from it. The problems are arranged in (roughly) increasing order of difficulty. Happy solving!

1. Prove that medians the AA', BB' in triangle ABC are perpendicular if and only if $a^2 + b^2 = 5c^2$.

2. Find, with proof, the point P in the interior of an acute triangle ABC for which $BL^2 + CM^2 + AN^2$ is a minimum, where L, M, N are the feet of the perpendiculars from P to BC, CA, AB respectively.

3. Let ABC be a triangle and D, E, F the feet of the altitudes. Let X, Y, Z be the midpoints of the segments EF, FD, DE, respectively. Prove that the lines perpendicular from X, Y, Z to BC, CA, AB, respectively, are concurrent.

4. (*106 Geometry Problems*) In triangle ABC, let BC be the longest side. Point X is chosen on side AB such that $BX = BC$. Similarly, point Y is chosen on AC such that $CY = BC$. Prove that OI is perpendicular to XY, where O and I are the circumcenter and incenter, respectively, of triangle ABC.

5. Let ABC be a triangle, O its circumcenter, and I_a its A-excenter. Let E, F be the feet of the angle bisectors from verticies B, C, respectively. Show that $EF \perp OI_a$. [3]

4 Hints

Here we present some hints to the problems in the previous section.

1. Use the median formula and the fact that $AG/GA' = 2$, where G is the centroid.

2. Carnot's Theorem!

3. Look for medians, and use Carnot.

4. Don't be afraid to apply the Lemma directly!

5. Repeat Hint 4.

[3]"The Legendary Geometry Problem!" ∼ DrMath