Chapter 2

Complex Numbers: Geometry

As is well known, the complex field can be identified with $\mathbb{R}^2$ via the map

$$z = x + iy \mapsto (x, y).$$

An important new feature with respect to real analysis is the introduction of the point at infinity, which leads to the compactification of $\mathbb{C}$. These various aspects, and some others, such as Moebius maps, are considered in this chapter.

2.1 Geometric interpretation

Exercise 2.1.1. *Describe the geometric object whose vertices are defined by the roots of order n of unity.*

To have a good understanding of some forthcoming notions (for instance, limit at infinity, or the notion of pole of an analytic function), it is better to be able to leave the complex plane, and go one step further and add a point, called infinity, and denoted by the symbol ∞ (without sign, in opposition to real analysis, where you have $\pm\infty$), in such a way that the extended complex plane $\mathbb{C} \cup \{\infty\}$ is compact. The set

$$\mathbb{C} \cup \{\infty\}$$

is called the extended complex plane. See Section 13.1 for a reminder of the notion of compactness. For the topological details of the construction, see Section 13.4. In the next exercise we discuss the geometric interpretation of the point at infinity, by identifying the extended complex plane with the Riemann sphere

$$\mathbb{S}_2 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \, ; x_1^2 + x_2^2 + x_3^2 = 1\right\}.$$

Exercise 2.1.2. *For $(x_1, x_2, x_3) \in \mathbb{S}_2 \setminus \{(0, 0, 1)\}$, define $\varphi(x_1, x_2, x_3)$ to be the intersection of the line defined by the points $(0, 0, 1)$ and (x_1, x_2, x_3) with the complex*

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plane. Show that

$$\varphi(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}, \tag{2.1.1}$$

and that φ *is a bijection between* $\mathbb{S}_2 \setminus \{(0,0,1)\}$ *and* $\mathbb{C}$*, with inverse given by*

$$\varphi^{-1}(u + iv) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right). \tag{2.1.2}$$

Setting $z = u + iv$, (2.1.2) may be rewritten as

$$\varphi^{-1}(z) = \left(\frac{z + \overline{z}}{|z|^2 + 1}, \frac{z - \overline{z}}{i(|z|^2 + 1)}, \frac{|z|^2 - 1}{|z|^2 + 1} \right).$$

The map (2.1.1) is called the *stereographic projection.*

The geometrical interpretation of the point at infinity is as follows: The map φ is extended to the point $(0, 0, 1)$ by

$$\varphi(0, 0, 1) = \infty, \tag{2.1.3}$$

and going to ∞ on the complex plane means going to $(0, 0, 1)$ on the Riemann sphere. More precisely, recall that, by definition, a sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ tends to infinity if

$$\lim_{n\to\infty} |z_n| = +\infty, \tag{2.1.4}$$

that is, if and only if

$$\lim_{n\to\infty} \varphi^{-1}(z_n) = (0, 0, 1), \tag{2.1.5}$$

where this last limit can be understood in two equivalent ways: The first, and simplest, is just to say that the limit is coordinate-wise in $\mathbb{R}^3$. The second is to view $\mathbb{S}_2$ as a topological manifold, and see the limit in the corresponding topology. See also Exercise 13.1.5, where φ allows us to define a metric on the Riemann sphere, called the stereographic metric.

The intersection of $\mathbb{S}_2$ with a (non-tangent) plane is a circle. Note that the *projection* of a circle of the Riemann sphere on the plane will not be a circle in general. For instance the projection of the circle

$$\begin{aligned} x_1 &= x_3, \\ x_1^2 + x_2^2 + x_3^2 &= 1 \end{aligned}$$

onto the plane is the ellipse $2x_1^2 + x_2^2 = 1$. But we have:

Exercise 2.1.3. (see [18, Exercise 19, pp. 16–17]).

(a) *There is a one-to-one correspondence between circles on the Riemann sphere and lines or circles on the plane.*

(b) *Let S be a circle on the Riemann sphere. Then, $\varphi(S \setminus \{(0,0,1)\})$ is a circle on the plane if and only if $(0,0,1) \notin S$ and is a line otherwise.*

To summarize, via the map φ^{-1} the point at infinity in the extended complex plane should be seen as any other point of the complex plane. Furthermore, there is no difference between lines and circles in the extended complex plane. A line is a circle whose image under φ^{-1} goes via the point $(0,0,1)$.

The notion of *simply-connected set* is central in complex function theory. In [13, Theorem 4.1], eleven equivalent definitions for an open connected set to be simply-connected are given. See also [28, Theorem 4.65, p. 113] for a similar result. In this book, we focus most of the time on the much simpler notion of star-shaped set, but we will give a number of equivalent characterizations of simply-connected sets. Recall first:

Definition 2.1.4. A set $\Omega \subset \mathbb{C}$ is called *star-shaped* if there is a point $z_0 \in \Omega$ such that, for every $z \in \Omega$, the interval

$$[z_0, z] = \{tz_0 + (1-t)z,\ t \in [0,1]\}$$

lies in Ω.

The point z_0 need not be unique. For instance, a convex set is star-shaped with respect to each of its points.

Theorem 2.1.5. *An open star-shaped subset of $\mathbb{C}$ is simply-connected.*

The first definition of a simply-connected set, which is condition (d) in [13, Theorem 4.1], is as follows:

Definition 2.1.6. A connected open subset Ω of the complex plane is *simply-connected* if the set $\mathbb{S}_2 \setminus \varphi^{-1}(A)$ is connected (in the topology of $\mathbb{S}_2$).

It is enough to check that $\mathbb{S}_2 \setminus \varphi^{-1}(A)$ is arc-connected.

Exercise 2.1.7. (a) *Show that the punctured plane $\mathbb{C}\setminus\{0\}$ is not simply-connected.*

(b) *Show that the set*

$$\Omega = \mathbb{C} \setminus \{\{x \in \mathbb{R}; |x| \geq 1\} \cup \{iy; y \in \mathbb{R},\ |y| \geq 1\}\}$$

is simply-connected.

The Riemann sphere can also be identified with the projective line. This last object is introduced in the next exercise:

Exercise 2.1.8. *In $\mathbb{C}^2 \setminus \{(0,0)\}$ define the equivalence relation:*

$$(z_1, z_2) \sim (w_1, w_2) \quad \iff \quad (z_1, z_2) = c(w_1, w_2) \tag{2.1.6}$$

for some non-zero complex number c.

(1) *Show that* $\sim$ *indeed defines an equivalence relation. We denote by* $\overset{\circ}{z}$ *the equivalence class of* $(z_1, z_2) \in \mathbb{C}^2$ *and by* $\mathbb{P}$ *the set of the equivalence classes.*

(2) *Let* $\overset{\circ}{(z_1, z_2)} \in \mathbb{P}$*. Show that the elements in the equivalence class have all at the same time either non-zero second component or zero second component.*

We denote by $\mathbb{A}$ *(*$\mathbb{A}$ *stands for affine) the set of equivalence classes for which the second component in any of its representative is non-zero. Show that the map*

$$\psi(\overset{\circ}{z}) = \frac{z_1}{z_2}$$

is a one-to-one correspondence from $\mathbb{A}$ *onto* $\mathbb{C}$*, and that its inverse is given by*

$$\psi^{-1}(u) = \overset{\circ}{(u, 1)} \quad u \in \mathbb{C}. \tag{2.1.7}$$

The projective line $\mathbb{P}$ is the set of the equivalence classes of $\sim$.

Exercise 2.1.9. *Prove the claim made in the proof of Exercise 2.1.3 on the intersection of the plane and* $\mathbb{S}_2$*, that is, prove that equation (2.4.2) in Section 2.4 below is a necessary and sufficient condition for the plane and the Riemann sphere to intersect, and that the plane is tangent to the Riemann sphere if and only if equality holds in (2.4.2).*

2.2 Circles and lines and geometric sets

We recall now the formulas for equations of lines and circles in the complex plane. In the plane $\mathbb{R}^2$, a line is the set of points $M = (x, y)$ such that

$$ax + by + c = 0,$$

where $(a, b, c) \in \mathbb{R}^3$ and $(a, b) \neq (0, 0)$. Setting

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i},$$

we get

$$\overline{\alpha} z + \alpha \overline{z} + \beta = 0, \tag{2.2.1}$$

with

$$\alpha = a - bi \in \mathbb{C} \setminus \{0\} \quad \text{and} \quad \beta = 2c \in \mathbb{R}.$$

Conversely, any expression (2.2.1) with $(\alpha, \beta) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{R}$ is the equation of a real line.

Similarly, a circle in $\mathbb{R}^2$ is the set of points $M = (x, y)$ such that

$$x^2 + y^2 - 2ax - 2by + c = 0,$$

where a, b, c are real and such that $a^2+b^2-c>0$. The center of the circle is the point (a,b) and its radius is $\sqrt{a^2+b^2-c}$. In the complex plane we obtain

$$|z|^2 - a(z+\overline{z}) + ib(z-\overline{z}) + c = 0,$$

that is,

$$|z|^2 - z(a-ib) - \overline{z}(a+ib) + c = 0,$$

that is, using (1.6.2),

$$|z-(a+ib)|^2 = a^2+b^2-c.$$

We have just seen two analytic expressions, one for lines and one for circles. There is an alternative way to write the equations of lines and circles in a unified manner as

$$|z-z_0| = \lambda|z-z_1|,$$

where $z_0 \neq z_1$ and $\lambda > 0$. This expression describes a circle when $\lambda \neq 1$, this is an Apollonius circle, and a line when $\lambda = 1$.

Exercise 2.2.1. *Show that the set of points*

$$|z-z_0| = \lambda|z-z_1| \tag{2.2.2}$$

where $\lambda > 0$, $\lambda \neq 1$ and $z_0 \neq z_1$ is the circle with center and radius

$$\frac{z_0-\lambda^2 z_1}{1-\lambda^2} \quad \textit{and} \quad \frac{\lambda}{|1-\lambda^2|}|z_0-z_1|$$

respectively. Show that, conversely, a line or a circle is of the form (2.2.2) for some choice of λ and of z_0, z_1.

Exercise 2.2.2. *Characterize and draw the sets of points in the plane $\mathbb{R}^2$ such that:*

(a) $|z-1+i| = 1$.

(b) $z^2+\overline{z}^2 = 2$.

(c) $|z-i| = |z+i|$.

(d) $|z|^2+3z+3\overline{z}+10 = 0$.

(e) $|z|^2+3z+\overline{z}+5 = 0$.

(f) $z^2+3z+3\overline{z}+5 = 0$.

(g) $|z| > 1 - \mathrm{Re}\ z$.

(h) $\mathrm{Re}\ (z(1-i)) < \sqrt{2}$.

For a question similar to the last one, see [48, p. 13].

The following is the last exercise of the book [98].

Exercise 2.2.3. *Find the image of the unit circle under the map $z \mapsto w(z) = z - z^n/n$ where $n = 2, 3, \ldots$*

2.3 Moebius maps

Recall that a Moebius map is a transformation of the form

$$\varphi(z) = \frac{az+b}{cz+d}$$

with $ad - bc \neq 0$. Such transformations are also called *linear fractional transformations*, and *linear transformations* in the older literature. See for instance Ford's book [58]. We recall that the image under a Moebius map of a line or a circle in the complex plane is still a line or a circle. We have already met a special case of Moebius maps in Section 1.1, namely the Blaschke factors; see (1.1.42), (1.1.43) and (1.1.44). Finite or infinite products of Blaschke factors (of the same kind) are considered in Exercise 3.6.14.

The formula

$$\varphi(z) - \varphi(w) = \frac{(ad-bc)(z-w)}{(cz+d)(cw+d)}, \tag{2.3.1}$$

that is, for $z \neq w$,

$$\frac{\varphi(z) - \varphi(w)}{z-w} = \frac{(ad-bc)}{(cz+d)(cw+d)},$$

will prove useful in the sequel.

The first exercise expresses the fact that Moebius maps form a group isomorphic to the group $GL(\mathbb{C}, 2)/(\mathbb{C} \setminus \{0\})$ of 2×2 invertible matrices with complex entries factored out by the invertible numbers.

Exercise 2.3.1. *Let*

$$\varphi_\ell(z) = \frac{a_\ell z + b_\ell}{c_\ell z + d_\ell}, \quad \ell = 1, 2,$$

be two Moebius transforms. Show that

$$\varphi_1(\varphi_2(z)) = \frac{az+b}{cz+d}, \tag{2.3.2}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}. \tag{2.3.3}$$

Sometimes it is convenient to use the following notation: Setting

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we define for $w \in \mathbb{C}$

$$T_M(w) = \frac{aw+b}{cw+d}. \tag{2.3.4}$$

Equation (2.3.2) can be then rewritten as

$$T_{M_1}(T_{M_2}(w)) = T_{M_1M_2}(w). \tag{2.3.5}$$

This equation suggests that infinite products of matrices should be considered, when infinite compositions of Moebius transforms come into play. See Theorem 3.6.2 for the first issue and Section 11.5 for the second one. The matrices in these products are usually normalized. Indeed we have:

Exercise 2.3.2. *Let φ be a possibly degenerate Moebius map, and let*

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad \textit{and} \quad M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

be such that

$$\varphi(z) = T_{M_1}(z) = T_{M_2}(z),$$

for every z in their common domain of definition. Show that there is a complex number $\lambda \neq 0$ such that

$$M_2 = \lambda M_1.$$

Using (2.3.3) we can study in particular the compositions of Blaschke factors of the form (1.1.42).

Exercise 2.3.3. *Let u and v be in the open unit disk $\mathbb{D}$. Show that*

$$w = \frac{u+v}{1+u\overline{v}} \in \mathbb{D}, \tag{2.3.6}$$

and that

$$b_u(b_v(z)) = \frac{1+u\overline{v}}{1+v\overline{u}}\, b_w(z). \tag{2.3.7}$$

In Exercise 2.3.13 we compute the n-th iterate of a Blaschke factor of the form (1.1.42).

Exercise 2.3.4. (1) *Let $w \in \mathbb{D}$ (resp. $w \in \mathbb{C}_r$). Prove that b_w (resp. B_w) is a one-to-one map from $\mathbb{D}$ (resp. $\mathbb{C}_r$) onto $\mathbb{D}$.*

(2) *What happens if $|w| > 1$ in the first case and $\operatorname{Re} w < 0$ in the second case?*

Exercise 2.3.5. (see [120, p. 25], [48, Exercice 33.15, p. 301]). *Given two triples of complex numbers (z_1, z_2, z_3) and (w_1, w_2, w_3) such that*

$$z_\ell \neq z_j \quad \textit{and} \quad w_\ell \neq w_j, \quad \textit{for} \quad \ell, j = 1, 2, 3, \quad \ell \neq j,$$

show that the map $z \mapsto w$ defined by

$$\frac{w-w_1}{w-w_2} \cdot \frac{w_3-w_2}{w_3-w_1} = \frac{z-z_1}{z-z_2} \cdot \frac{z_3-z_2}{z_3-z_1} \tag{2.3.8}$$

is a Moebius map such that $w(z_\ell) = w_\ell$ for $\ell = 1, 2, 3$.

Suitably interpreted, the formula (2.3.8) still makes sense when one of the w_j or/and one of the z_j is equal to ∞. For instance, when $w_1 = z_2 = \infty$ we have

$$\frac{w_3 - w_2}{w - w_2} = \frac{z - z_1}{z_3 - z_1},$$

that is

$$w = w_2 + \frac{(w_3 - w_2)(z_3 - z_1)}{z - z_1}.$$

Exercise 2.3.6. *Show that four points are on the same complex circle (or on the same complex line) if and only if the number*

$$\frac{\left(\frac{(z_1 - z_2)}{(z_1 - z_3)}\right)}{\left(\frac{(z_2 - z_4)}{(z_3 - z_4)}\right)} \tag{2.3.9}$$

is real.

The number (2.3.9) is called the *cross-ratio*. See for instance [131, Theorem 2, p. 3].

Hint to the solution. Consider the case of a circle.

(1) Prove that the result is true for the unit circle.

(2) Prove that any circle can be mapped onto the unit circle by an appropriate Moebius map.

(3) Prove that (2.3.9) is invariant under Moebius transforms.

Exercise 2.3.7. *For which $k \in \mathbb{R}$ is the image of the circle $|z - 1| = k$ under the Moebius map $f(z) = \dfrac{z-3}{1-2z}$ a line? Find the equation of the line.*

The next result appears in [99]; see also [83, Theorem 7, p. 67]. A natural approach would be to use the Schur algorithm. See (6.4.6). This approach does not seem to lead anywhere, and the pedestrian approach leads to a quite short proof.

Exercise 2.3.8. *Show that the non-trivial Moebius map $\varphi(z) = \dfrac{az+b}{cz+d}$ maps the open unit disk into itself if and only if*

$$|\overline{a}c - \overline{b}d| + |ad - bc| \le |d|^2 - |c|^2. \tag{2.3.10}$$

We note the following: For the Blaschke factor (1.1.42), we have

$$a = 1, \quad b = -w, \quad c = -\overline{w}, \quad d = 1,$$

and inequality (2.3.10) reads as

$$|-1(\overline{w})+\overline{w}1|+|1-|w|^2|\leq 1-|w|^2,$$

and thus becomes an equality. We note that equality in (2.3.10) may hold even when the image of the open unit disk is included in, but different from, the open unit disk. For instance, take the function

$$\frac{z+1}{-z+3},$$

which appears in the solution of Exercise 6.4.6.

We send the reader to Exercise 14.3.10 for a related exercise. As suggested by our colleague Dr Izchak Lewkowicz, we propose:

Exercise 2.3.9. *Let* $\varphi(z)=\dfrac{az+b}{cz+d}$ *be a non-trivial Moebius map. Find necessary and sufficient conditions on* a,b,c,d *for* φ *to map the open left half-plane into itself.*

Exercise 2.3.10. *For which values of* z_0 *does the function*

$$\varphi(z)=\frac{z-1}{z-z_0}$$

map the open unit disk into itself?

Exercise 2.3.11. *Let* $\varphi(z)=\dfrac{az+b}{cz+d}$ *be a non-trivial Moebius map, and assume that the equation* $\varphi(z)=z$ *has two distinct solutions* z_1 *and* z_2*.*

(a) *Prove that there is a number* $k\in\mathbb{C}$ *such that*

$$\frac{\varphi(z)-z_1}{\varphi(z)-z_2}=k\frac{z-z_1}{z-z_2}. \tag{2.3.11}$$

(b) *Give a formula for the* n*-th iterate*

$$\underbrace{\varphi\circ\varphi\circ\cdots\varphi}_{n-\text{times}}.$$

(c) *Compute the* n*-th iterate of*

$$\varphi(z)=\frac{1-3z}{z-3}.$$

Remark 2.3.12. The number k is called the multiplier of φ (see [58, (12), p. 10]. For relations and applications to the theory of automorphic functions, see for instance [58].

We now look at the special case where $\varphi(z) = b_w(z)$, where b_w is the Blaschke factor (1.1.42).

Exercise 2.3.13. (a) *Compute the n-th iterate of the Blaschke factor* (1.1.42).

(b) *What is*

$$\lim_{n\to\infty} \underbrace{b_w \circ b_w \circ \cdots \circ b_w}_{n\text{-}times}.$$

In relation with the following exercise, see the Herglotz formula for functions holomorphic in the open upper half-plane, and with a positive real part there; see formula (5.5.21).

Exercise 2.3.14. *Let $w \in \mathbb{C} \setminus \mathbb{R}$. What is the image of the real line under the Moebius transform $\frac{zw+1}{z-w}$?*

2.4 Solutions

Solution of Exercise 2.1.1. By formula (1.1.14) with $\rho = 1$ and $\theta = 0$ we see that the roots of order n of unity are

$$z_k = \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n}), \quad k = 0, \ldots, n-1.$$

The points $M_k = (\cos(\frac{2k\pi}{n}), \sin(\frac{2k\pi}{n}))$, $k = 0, \ldots, n-1$, are on the unit circle, and are the vertices of a regular polygon of order n, the first vertex being the point $(1, 0)$. $\square$

Solution of Exercise 2.1.2. The equation of the line passing through the points $(0, 0, 1)$ and (x_1, x_2, x_3) is

$$(u, v, w) = (0, 0, 1) + t(x_1, x_2, x_3 - 1) \quad t \in \mathbb{R}.$$

We want $w = 0$, and thus

$$1 + t(x_3 - 1) = 0, \quad \text{that is,} \quad t = \frac{1}{1 - x_3}.$$

The result follows. The map is clearly one-to-one. To show that it is onto, let $u+iv \in \mathbb{C}$ be given. A point $(x_1, x_2, x_3) \in \mathbb{S}_2 \setminus \{(0, 0, 1)\}$ is such that $\varphi(x_1, x_2, x_3) = u + iv$ if and only if

$$x_1 = u(1 - x_3) \quad \text{and} \quad x_2 = v(1 - x_3).$$

Thus

$$u^2 + v^2 = \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{1 - x_3^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3}.$$

Thus

$$x_3 = \frac{u^2+v^2-1}{u^2+v^2+1}.$$

So,

$$1 - x_3 = \frac{2}{u^2+v^2+1},$$

and the formulas for x_1 and x_2 follow. □

Solution of Exercise 2.1.3. Let

$$ax_1 + bx_2 + cx_3 = d$$

be the equation of a plane P in $\mathbb{R}^3$. Note that $(0,0,1) \in P$ if and only if

$$c = d. \tag{2.4.1}$$

Using elementary analytic geometry one sees that the plane intersects the Riemann sphere if and only if

$$a^2 + b^2 + c^2 \geq d^2, \tag{2.4.2}$$

and that it is non-tangent if and only if the inequality is strict in (2.4.2):

$$a^2 + b^2 + c^2 > d^2. \tag{2.4.3}$$

These facts are proved in Exercise 2.1.9 at the end of this section. Consider now the image of $S \cap P$ under φ:

$$(u,v) \in \varphi(S \cap P) \iff \varphi^{-1}(u,v) \in S \cap P,$$

that is, if and only if it holds that

$$2au + 2bv + c(u^2+v^2-1) = d(u^2+v^2+1),$$

that is, if and only if

$$(c-d)(u^2+v^2) + 2au + 2bv = c + d. \tag{2.4.4}$$

If $c = d$ we obtain the equation of a line. If $c \neq d$, we rewrite (2.4.4) as

$$(u + \frac{a}{c-d})^2 + (v + \frac{b}{c-d})^2 = \frac{c+d}{c-d} + \frac{a^2}{(c-d)^2} + \frac{b^2}{(c-d)^2} = \frac{a^2+b^2+c^2-d^2}{(c-d)^2},$$

which is the equation of a circle since (2.4.3) is in force.

The converse direction is done as follows: Given a line

$$au + bv = e,$$

we consider the plane

$$ax_1 + bx_2 + ex_3 = e.$$

Given a circle

$$(u - a_0)^2 + (v - b_0)^2 = R^2,$$

we may assume that $c - d = 1$ in the equation of the plane, and take the plane

$$a_0x_1 + b_0x_2 + cx_3 = d,$$

where c and d satisfy

$$c - d = 1 \quad \text{and} \quad R^2 - a_0^2 - b_0^2 = c + d. \qquad \square$$

Solution of Exercise 2.1.7. (a) The set

$$\mathbb{S}_2 \setminus \varphi^{-1}\{\mathbb{C} \setminus \{0\}\}$$

consists of two points, namely $(0, 0, -1)$ and $(0, 0, 1)$, and therefore is not connected.

(b) Consider the points

$$\begin{aligned} A_1 &= \varphi^{-1}(1, 0) = (1, 0, 0),\\ A_2 &= \varphi^{-1}(-1, 0) = (-1, 0, 0),\\ A_3 &= \varphi^{-1}(0, 1) = (0, 1, 0),\\ A_4 &= \varphi^{-1}(0, -1) = (0, -1, 0) \end{aligned}$$

in $\mathbb{S}_2$. With Ω as in the statement of the exercise, $\varphi^{-1}(\Omega)$ consists of the point $(0, 0, 1)$ and of four non-intersecting lines which link the points each of the point A_j to $(0, 0, 1)$. $\square$

We note that the above set is in fact star-shaped with respect to the origin. In this book, we concentrate almost uniquely on star-shaped sets, and therefore have given a somewhat informal proof of the above exercise.

Solution of Exercise 2.1.8. Let $\overset{\circ}{z} \in \mathbb{P}$ and (z_1, z_2) and (w_1, w_2) be two elements in $\overset{\circ}{z}$. Since $z_1 = cw_1$ for $c \neq 0$, we see that z_1 and w_1 are simultaneously both zero or both non-zero. The map ψ is well defined; indeed, let (z_1, z_2) and (w_1, w_2) be two elements in $\overset{\circ}{z}$. If $z_1 = 0$, then $w_1 = 0$ and $\psi(\overset{\circ}{z}) = 0$. On the other hand, if $z_1 \neq 0$, then $w_1 \neq 0$ and it follows from (2.1.6) that

$$\frac{z_2}{z_1} = \frac{w_2}{w_1},$$

and so ψ is well defined. The map is one-to-one. Indeed, let $\overset{\circ}{z}$ and $\overset{\circ}{w}$ be two elements of $\mathbb{P}$, and assume that $\overset{\circ}{z} \neq \overset{\circ}{w}$. Let $(z_1, z_2) \in \overset{\circ}{z}$ and $(w_1, w_2) \in \overset{\circ}{w}$. Then, if

$z_1 = 0$ we have that $w_1 \neq 0$ and so $\psi(\overset{\circ}{z}) \neq \psi(\overset{\circ}{w})$. If both z_1 and w_1 are different from 0,

$$\frac{z_2}{z_1} \neq \frac{w_2}{w_1},$$

and in this case too, $\psi(\overset{\circ}{z}) \neq \psi(\overset{\circ}{w})$. Finally, the formula (2.1.7) follows from the fact that $u \in (\overset{\circ}{u}, 1)$. $\square$

Solution of Exercise 2.1.9. The point with coordinates

$$\begin{aligned}(x_0, y_0, z_0) &= \left(\frac{da}{a^2+b^2+c^2}, \frac{db}{a^2+b^2+c^2}, \frac{dc}{a^2+b^2+c^2}\right)\\ &= \frac{d}{a^2+b^2+c^2}(a, b, c)\end{aligned}$$

belongs to the plane. Let u and v be a pair of unit and orthogonal vectors in $\mathbb{R}^3$, which are moreover orthogonal to (a, b, c). A point (x, y, z) is in $P \cap \mathbb{S}_2$ if and only if

$$x^2 + y^2 + z^2 = 1$$

and

$$(x - x_0, y - y_0, z - z_0) = tu + sv \quad \text{for some} \quad t, s \in \mathbb{R}.$$

Equivalently,

$$(x, y, z) = \frac{d}{a^2+b^2+c^2}(a, b, c) + tu + sv,$$

and therefore, taking norms of both sides of this equality,

$$1 = \frac{d^2}{a^2+b^2+c^2} + t^2 + s^2.$$

Thus

$$\frac{a^2+b^2+c^2-d^2}{a^2+b^2+c^2} = t^2 + s^2. \tag{2.4.5}$$

Thus, the intersection will be non-empty if and only if (2.4.2) is in force, and reduced to a point if and only if

$$a^2 + b^2 + c^2 = d^2.$$

Assume now that (2.4.3) holds, and let

$$R = \sqrt{\frac{a^2+b^2+c^2-d^2}{a^2+b^2+c^2}}.$$

It follows from (2.4.5) that there exists $\theta \in [0, 2\pi)$ such that

$$s = R\cos\theta \quad \text{and} \quad t = R\sin\theta.$$

It follows that the intersection of the plane P and of the Riemann sphere is the circle with center $\frac{d}{a^2+b^2+c^2}(a,b,c)$ and radius $R = \sqrt{\frac{a^2+b^2+c^2-d^2}{a^2+b^2+c^2}}$. ☐

Solution of Exercise 2.2.1. Equality (2.2.2) is equivalent to

$$|z|^2 + |z_0|^2 - 2\text{Re}\ z\overline{z_0} = \lambda^2(|z|^2 + |z_1|^2 - 2\text{Re}\ z\overline{z_1}),$$

that is, since $\lambda > 0$ and $\lambda \neq 1$,

$$|z|^2 - 2\text{Re}\ z\overline{\frac{z_0 - \lambda^2 z_1}{1-\lambda^2}} = \frac{\lambda^2|z_1|^2 - |z_0|^2}{1-\lambda^2}.$$

Completing the square we obtain

$$\left|z - \frac{z_0 - \lambda^2 z_1}{1-\lambda^2}\right|^2 - \left(\frac{|z_0 - \lambda^2 z_1|}{1-\lambda^2}\right)^2 = \frac{\lambda^2|z_1|^2 - |z_0|^2}{1-\lambda^2}, \tag{2.4.6}$$

and hence we get the circle of center $\frac{z_0-\lambda^2 z_1}{1-\lambda^2}$ and radius R defined by

$$R^2 = \left(\frac{|z_0 - \lambda^2 z_1|}{1-\lambda^2}\right)^2 + \frac{\lambda^2|z_1|^2 - |z_0|^2}{1-\lambda^2} = \left(\frac{\lambda}{|1-\lambda^2|}|z_0 - z_1|\right)^2.$$

We now study the converse, and focus only on the case of a circle. Let

$$|z - \Omega| = R$$

be the circle of center Ω and radius $R > 0$. We are looking for $\lambda > 0$ and $z_0, z_1 \in \mathbb{C}$ (with $z_0 \neq z_1$) such that

$$\Omega = \frac{z_0 - \lambda^2 z_1}{1-\lambda^2},$$
$$R = \frac{\lambda}{|1-\lambda^2|}|z_0 - z_1|.$$

From the first equation we get

$$z_0 = (1-\lambda^2)\Omega + \lambda^2 z_1.$$

Plugging this expression in the second equation we obtain

$$R = \lambda|\Omega - z_1|.$$

We take

$$z_1 = \Omega + \frac{R}{\lambda}.$$

Then

$$z_0 = (1-\lambda^2)\Omega + \lambda^2\Omega + \lambda R = \Omega + \lambda R,$$

which ends the proof. ☐

Solution of Exercise 2.2.2. For (a) we have the circle with center $(1,-1)$ and radius 1. Equation (b) can be rewritten as $x^2-y^2=1$, and so we obtain an hyperbole. Case (c) is the line orthogonal to the interval $(0,1)$ and $(0,-1)$ and passing by the middle of this interval, i.e., it is just the real line. More misleading are (d), (e) and (f). The equations look like the equation of a circle but this is not the case. For (d) we have the empty set. Indeed, we have

$$\begin{aligned}|z|^2+3z+3\overline{z}+10=0 &\iff |z+3|^2+10-9=0\\ &\iff |z+3|^2+1=0.\end{aligned}$$

Equation (e) becomes in cartesian coordinates

$$x^2+y^2+4x+y+5+iy=0.$$

Equating real and imaginary parts to 0 we obtain

$$x^2+y^2+4x+y+5=0 \quad \text{and} \quad y=0.$$

The equation $x^2+4x+5=0$ has no real solution, and so (e) also corresponds to the empty set. We leave (g) and (h) to the student, and turn to (f). Condition (f) is equivalent to

$$\sqrt{x^2+y^2}>1-x. \tag{2.4.7}$$

If $x\geq 1$, every y meets this condition. Assume now that $x<1$. Equation (2.4.7) is then equivalent to

$$x^2+y^2>(1-x)^2,$$

that is, to

$$y^2>1-2x. \tag{2.4.8}$$

We already know that $x<1$. If $x\in(\frac{1}{2},1)$, every y meets this condition. If $x\leq\frac{1}{2}$, we get the points strictly outside the parabola defined by (2.4.8) and for which $x\leq 1/2$. All together, the set is the complement of the points inside or on the parabola $y^2=1-2x$. $\square$

Solution of Exercise 2.2.3. Write $z(t)=e^{it}$, with $t\in[0,2\pi]$, and $w(t)=x(t)+iy(t)$. We obtain

$$\begin{aligned}x(t)&=\cos t-\frac{1}{N}\cos(Nt),\\ y(t)&=\sin t-\frac{1}{N}\sin(Nt).\end{aligned}$$

These are the parametric equations of an epicycloid, described by a point on a circle of radius $\frac{1}{N}$ rolling over a circle of radius $1-\frac{1}{N}$. See also for instance [108, Exercise 7, p. 421]. $\square$

Solution of Exercise 2.3.1. Indeed,

$$\begin{aligned}
\varphi_1(\varphi_2(z)) &= \frac{a_1\varphi_2(z)+b_1}{c_1\varphi_2(z)+d_1}\\
&= \frac{a_1\dfrac{a_2z+b_2}{c_2z+d_2}+b_1}{c_1\dfrac{a_2z+b_2}{c_2z+d_2}+d_1}\\
&= \frac{a_1(a_2z+b_2)+b_1(c_2z+d_2)}{c_1(a_2z+b_2)+d_1(c_2z+d_2)}\\
&= \frac{(a_1a_2+b_1c_2)z+a_1b_2+b_1d_2}{(c_1a_2+d_1c_2)z+c_1b_2+d_1d_2}\\
&= \frac{az+b}{cz+d},
\end{aligned}$$

where

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}. \qquad \square$$

Solution of Exercise 2.3.2. We have for all $z \in \mathbb{C}$ where both functions are defined:

$$\varphi(z) = \frac{a_1z+b_1}{c_1z+d_1} = \frac{a_2z+b_2}{c_2z+d_2}. \tag{2.4.9}$$

Hence,

$$z^2(a_1c_2-a_2c_1)+z(a_1d_2-a_2d_1+c_2b_1-b_2c_1)+b_1d_2-b_2d_1 \equiv 0.$$

One can proceed by remarking that the coefficients of the above polynomial are all equal to 0, and then by distinguishing various cases. We will chose another avenue to solve the problem (we will make use of limit at infinity; technically speaking, we will see this notion really only in the next chapter). With

$$X(z) = \frac{c_1z+d_1}{c_2z+d_2},$$

and taking into account (2.4.9) we have

$$\begin{aligned}
\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}\begin{pmatrix} z\\ 1\end{pmatrix} &= (c_1z+d_1)\begin{pmatrix} \varphi(z)\\ 1\end{pmatrix}\\
&= X(z)(c_2z+d_2)\begin{pmatrix} \varphi(z)\\ 1\end{pmatrix}\\
&= X(z)\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}\begin{pmatrix} z\\ 1\end{pmatrix}.
\end{aligned}$$

Take now two points u and v, with $u \neq v$, at which all the expressions make sense. We have

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} u & v \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} uX(u) & vX(v) \\ X(u) & X(v) \end{pmatrix}.$$

Thus

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} uX(u) & vX(v) \\ X(u) & X(v) \end{pmatrix} \frac{1}{u-v} \begin{pmatrix} 1 & -v \\ -1 & u \end{pmatrix} \tag{2.4.10}$$

$$= \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} \frac{uX(u)-vX(v)}{u-v} & \frac{uv(X(v)-X(u))}{u-v} \\ \frac{X(u)-X(v)}{u-v} & \frac{uX(v)-vX(u)}{u-v} \end{pmatrix}. \tag{2.4.11}$$

It will appear from the proof that $X(z)$ is a constant, but we do not know this at this stage. Without loss of generality, we may assume that

$$\lim_{z\to\infty} X(z) \neq \infty.$$

Indeed, if

$$\lim_{z\to\infty} X(z) = \infty,$$

then

$$\lim_{z\to\infty} \frac{1}{X(z)} = 0,$$

and replacing X by $1/X$ amounts to interchange the indices $_1$ and $_2$. Letting $u \to \infty$, and then $v \to \infty$ in (2.4.11) we obtain

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = X(\infty) \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Hence, $X(\infty) \neq 0$ (otherwise, φ will not be defined) and we obtain the result by setting $\lambda = \frac{1}{X(\infty)}$. □

Solution of Exercise 2.3.3. It suffices to note that

$$\begin{pmatrix} 1 & -u \\ -\overline{u} & 1 \end{pmatrix} \begin{pmatrix} 1 & -v \\ -\overline{v} & 1 \end{pmatrix} = \begin{pmatrix} 1+u\overline{v} & -(u+v) \\ -(\overline{u}+\overline{v}) & 1+\overline{u}v \end{pmatrix}.$$

Then, the associated transformation is

$$\frac{(1+u\overline{v})z-(u+v)}{-z(\overline{u}+\overline{v})+1+\overline{u}v} = \frac{1+u\overline{v}}{1+v\overline{u}} b_w(z).$$ □

Solution of Exercise 2.3.4. It follows from (1.1.49) with $z = v$ that b_w sends the open unit disk into itself. It follows from (2.3.6) and (2.3.7) that

$$b_w(b_{-w}(z)) = z,$$
$$b_{-w}(b_w(z)) = z,$$

for $z \in \mathbb{D}$. The first equation shows that b_w is onto and the second equation shows that b_w is one-to-one.

When $|w| > 1$, the map b_w is one-to-one from $\mathbb{D} \setminus \{\frac{1}{\overline{w}}\}$ onto $\{z; |z| > 1\}$. The limiting case $w = \infty$ corresponds to $b(z) = \frac{1}{z}$, which is a one-to-one map from $\mathbb{D} \setminus \{0\}$ onto $\{z; |z| > 1\}$.

The case of B_w is treated in a similar way. $\square$

Solution of Exercise 2.3.5. Solving w in function of z in the above expression leads to

$$\begin{aligned}
w(z) &= \frac{w_1 - w_2 \dfrac{z - z_1}{z - z_2} \cdot \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{w_3 - w_1}{w_3 - w_2}}{1 - \dfrac{z - z_1}{z - z_2} \cdot \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{w_3 - w_1}{w_3 - w_2}} \\
&= \frac{(z - z_2)w_1 - w_2(z - z_1) \cdot \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{w_3 - w_1}{w_3 - w_2}}{(z - z_2) - (z - z_1) \cdot \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{w_3 - w_1}{w_3 - w_2}}.
\end{aligned}$$

Hence, $z \mapsto w(z)$ is indeed a Moebius map. One checks that $w(z_\ell) = z_\ell$ for $\ell = 1, 2, 3$ by direct computation:

$$\begin{aligned}
w(z_1) &= \frac{(z_1 - z_2)w_1 - w_2(z_1 - z_1) \cdot \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{w_3 - w_1}{w_3 - w_2}}{(z_1 - z_2) - (z_1 - z_1) \cdot \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{w_3 - w_1}{w_3 - w_2}} \\
&= \frac{(z_1 - z_2)w_1}{(z_1 - z_2)} \\
&= w_1. \\
w(z_2) &= \frac{(z_2 - z_2)w_1 - w_2(z_2 - z_1) \cdot \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{w_3 - w_1}{w_3 - w_2}}{(z_2 - z_2) - (z_2 - z_1) \cdot \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{w_3 - w_1}{w_3 - w_2}} \\
&= \frac{-w_2(z_2 - z_1) \cdot \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{w_3 - w_1}{w_3 - w_2}}{-(z_2 - z_1) \cdot \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{w_3 - w_1}{w_3 - w_2}} \\
&= w_2. \\
w(z_3) &= \frac{(z_3 - z_2)w_1 - w_2(z_3 - z_1) \cdot \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{w_3 - w_1}{w_3 - w_2}}{(z_3 - z_2) - (z_3 - z_1) \cdot \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{w_3 - w_1}{w_3 - w_2}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(z_3-z_2)w_1 - w_2(z_3-z_2)\cdot \dfrac{w_3-w_1}{w_3-w_2}}{(z_3-z_2)-(z_3-z_2)\cdot \dfrac{w_3-w_1}{w_3-w_2}} \\
&= \frac{w_1 - w_2 \cdot \dfrac{w_3-w_1}{w_3-w_2}}{1-\dfrac{w_3-w_1}{w_3-w_2}} \\
&= \frac{w_1(w_3-w_2)-w_2(w_3-w_1)}{w_3-w_2-w_3+w_1} \\
&= \frac{w_3(w_1-w_2)}{w_1-w_2} \\
&= w_3. \qquad \square
\end{aligned}
$$

Solution of Exercise 2.3.6. We follow the hints given after the exercice, and focus on the case of a circle. We first assume that the four points are on a common circle. For the unit circle, we can always assume that one of the points is $z_1 = 1$, and so we have to check that for any t_2, t_3 and t_4 in $(0, 2\pi)$,

$$
\frac{(1-e^{it_2})(e^{it_3}-e^{it_4})}{(1-e^{it_3})(e^{it_3}-e^{it_4})}
$$

is real, that is, to check that

$$
\frac{(1-e^{it_2})(e^{it_3}-e^{it_4})}{(1-e^{it_3})(e^{it_3}-e^{it_4})} = \frac{(1-e^{-it_2})(e^{-it_3}-e^{-it_4})}{(1-e^{-it_3})(e^{-it_3}-e^{-it_4})}.
$$

This is checked by multiplying both the numerator and denominator of the right side by $e^{it_2}e^{it_3}e^{it_4}$.

Let now $|z - z_0| = R$ be the equation of another circle. The Moebius map $\varphi(z) = \frac{z-z_0}{R}$ maps this circle onto the unit circle. We now check the invariance of the cross ratio using (2.3.1). Indeed using this equation we have

$$
\begin{aligned}
\varphi(z_1)-\varphi(z_2) &= \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)}, \\
\varphi(z_1)-\varphi(z_3) &= \frac{(ad-bc)(z_1-z_3)}{(cz_1+d)(cz_3+d)}, \\
\varphi(z_2)-\varphi(z_4) &= \frac{(ad-bc)(z_2-z_4)}{(cz_2+d)(cz_4+d)}, \\
\varphi(z_3)-\varphi(z_4) &= \frac{(ad-bc)(z_3-z_4)}{(cz_4+d)(cz_3+d)},
\end{aligned}
$$

and hence the result by a direct computation.

Still for the case of a circle, we consider the converse statement: Let there be therefore four pairwise points for which (2.3.9) is real. The first three points determine uniquely a circle, which we move, via a Moebius map, to be the unit circle. The corresponding quotient (2.3.9) does not change. So we are left with the following question: Given four pairwise different points for which the quotient is real, three of them being on the unit circle, show that the fourth is also on the unit circle. We set $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2}$, $z_3 = e^{i\theta_3}$, where $\theta_1, \theta_2, \theta_3 \in [0, 2\pi)$ are pairwise different. We have thus:

$$\frac{(e^{-i\theta_1} - e^{-i\theta_2})(e^{-i\theta_1} - e^{-i\theta_3})}{(e^{-i\theta_2} - e^{-i\theta_4})(e^{-i\theta_3} - \overline{z_4})} = \frac{(e^{i\theta_1} - e^{i\theta_2})(e^{i\theta_1} - e^{i\theta_3})}{(e^{i\theta_2} - e^{i\theta_4})(e^{i\theta_3} - z_4)}.$$

Applying the previous exercise with the Moebius map $\varphi(z) = 1/z$ we see that

$$\varphi(e^{i\theta_j}) = e^{-i\theta_j}, \quad j = 1, 2, 3,$$

and so $\overline{z_4} = \varphi(z_4)$, i.e., $|z_4| = 1$. This concludes the proof. $\square$

Solution of Exercise 2.3.7. Set $w = \frac{z-3}{1-2z}$. Then, $z = \frac{w+3}{1+2w}$. The condition $|z-1| = k$ becomes thus $\left|\frac{w+3}{1+2w} - 1\right| = k$, i.e.,

$$|w + 3 - (1 + 2w)| = k|1 + 2w|,$$

which can be rewritten as

$$|2 - w| = 2k|w + \frac{1}{2}|.$$

Hence, we obtain a line if and only if $k = \frac{1}{2}$. The equation of the line is $|2 - w| = |w + \frac{1}{2}|$, i.e., $x = \frac{3}{4}$.

If one wants only the value of k but not the equation of the line, a shorter way is as follows: We know that the image of the circle is either a line or a circle. It will be a line if and only if it is not a bounded set, i.e., if and only if $z = \frac{1}{2}$ belongs to the circle $|z - 1| = k$, i.e., $|\frac{1}{2} - 1| = k$. Hence $k = \frac{1}{2}$. $\square$

Solution of Exercise 2.3.8. By hypothesis, $ad - bc \neq 0$ and thus the map $z \mapsto w$ is invertible, and its inverse is given by

$$z = \frac{wd - b}{a - cw}.$$

We know that $|z| < 1$ and want to find a necessary and sufficient condition for the set of images w to be in the open unit disk. We have

$$\begin{aligned} |z| < 1 &\iff |z|^2 < 1 \\ &\iff |wd - b|^2 < |a - wc|^2 \\ &\iff |w|^2|d|^2 + |b|^2 - 2\mathrm{Re}\,(\overline{b}dw) < |w|^2|c|^2 + |a|^2 - 2\mathrm{Re}\,(\overline{a}cw) \\ &\iff |w|^2(|d|^2 - |c|^2) - 2\mathrm{Re}\,\left\{(\overline{b}d - \overline{a}c)w\right\} + |b|^2 - |a|^2 < 0. \end{aligned} \tag{2.4.12}$$

At this stage we pause and remark that, necessarily, $|c| < |d|$. Indeed, if $|d| = |c|$, the above can be rewritten as

$$-2\mathrm{Re}\,\left\{(\overline{b}d - \overline{a}c)w\right\} + |b|^2 - |a|^2 < 0, \tag{2.4.13}$$

which is an unbounded set (in fact, a half-plane). Note that necessarily

$$(\overline{b}d - \overline{a}c) \neq 0.$$

If it is equal to 0, then on the one hand (2.4.13) leads to $|b| < |a|$ and on the other hand we have

$$\begin{aligned} \overline{b}d - \overline{a}c = 0 &\Longrightarrow |b||d| = |a||c| \\ &\Longrightarrow |d| = |c|\frac{|a|}{|b|} \end{aligned}$$

which together with $|d| = |c|$ leads to $|a| = |b|$.

If $|c| > |d|$, the point $-\frac{d}{c}$ is in $\mathbb{D}$, and φ has a pole at this point. Thus the image of $\mathbb{D}$ by φ cannot be bounded. So $|c| < |d|$. We divide both sides of (2.4.12) by $|d|^2 - |c|^2$ and obtain

$$|w|^2 - 2\mathrm{Re}\,\left\{\frac{\overline{b}d - \overline{a}c}{|d|^2 - |c|^2}w\right\} + \frac{|b|^2 - |a|^2}{|d|^2 - |c|^2} < 0.$$

Completing the square we obtain

$$\left|w - \frac{\overline{b}d - a\overline{c}}{|d|^2 - |c|^2}\right|^2 < -\frac{|b|^2 - |a|^2}{|d|^2 - |c|^2} + \left|\frac{\overline{b}d - \overline{a}c}{|d|^2 - |c|^2}\right|^2.$$

We have

$$\begin{aligned} -\frac{|b|^2 - |a|^2}{|d|^2 - |c|^2} + \left|\frac{\overline{b}d - \overline{a}c}{|d|^2 - |c|^2}\right|^2 &= \frac{-(|b|^2 - |a|^2)(|d|^2 - |c|^2) + |\overline{b}d - \overline{a}c|^2}{(|d|^2 - |c|^2)^2} \\ &= \frac{|ad - bc|^2}{(|d|^2 - |c|^2)^2}. \end{aligned}$$

Thus the image of the open unit disk is the open disk of center

$$w_0 = \frac{\overline{b}d - a\overline{c}}{|d|^2 - |c|^2}$$

and radius

$$r_0 = \frac{|ad - bc|}{|d|^2 - |c|^2}.$$

This open disk will be included in the open unit disk if and only if

$$|w_0| + r_0 \le 1,$$

which can be rewritten as

$$\frac{|\overline{b}d - \overline{a}c|}{|d|^2 - |c|^2} + \frac{|ad - bc|}{|d|^2 - |c|^2} \le 1.$$

Multiplying both sides by the strictly positive number $|d|^2 - |c|^2$ we obtain (2.3.10). □

Solution of Exercise 2.3.9. The map

$$z \mapsto \varphi_0(z) = \frac{1-z}{1+z}$$

is one-to-one and onto from the open unit disk onto the open left half-plane, and is equal to its inverse. Therefore the map $\varphi(z) = \dfrac{az+b}{cz+d}$ maps the open left half-plane into itself if and only if $\varphi_0 \circ \varphi \circ \varphi_0$ maps the open unit disk into itself. In view of (2.3.3) the coefficients of $\varphi_0 \circ \varphi \circ \varphi_0$ can be chosen to be

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -a-b-c+d & -a-b+c+d \\ -a+b-c+d & a+b+c+d \end{pmatrix}.$$

Applying (2.3.10), we obtain the condition

$$|(\overline{b}-\overline{d})(a+c)+(\overline{a}-\overline{c})(b+d)|+2|ad-bc| \le (a+c)(\overline{b}+\overline{d})+(\overline{a}+\overline{c})(b+d). \quad (2.4.14)$$

We note that, in (2.4.14), equality will hold in particular for

$$\varphi(z) = z + w \quad \text{and} \quad \varphi(z) = \frac{1}{z+w},$$

where $w + \overline{w} \ge 0$. □

Solution of Exercise 2.3.10. Of course, from (2.3.10) with

$$a = 1, \quad b = -1, \quad c = 1, \quad \text{and} \quad d = -z_0,$$

we know that the condition is

$$2|1 - z_0| < |z_0|^2 - 1. \tag{2.4.15}$$

We reproduce this result directly as follows: Set

$$w = \frac{z-1}{z-z_0}.$$

Then

$$z = \frac{1-wz_0}{1-w},$$

and the condition $|z| < 1$ becomes

$$|1 - wz_0| < |1 - w|, \quad \text{that is,} \quad |1 - wz_0|^2 < |1 - w|^2.$$

This in turn can be rewritten as

$$|w|^2|z_0|^2 - 2\mathrm{Re}\,(wz_0) + 1 < |w|^2 - 2\mathrm{Re}\,(w) + 1,$$

i.e.,

$$|w|^2(|z_0|^2 - 1) - 2\mathrm{Re}\,w(z_0 - 1) < 0. \tag{2.4.16}$$

Now, if $|z_0| = 1$ the above inequality defines a half-plane, and cannot be inside the unit disk. Assume now that $|z_0| > 1$. Then (2.4.16) becomes

$$|w|^2 - 2\mathrm{Re}\,w\left(\frac{z_0 - 1}{|z_0|^2 - 1}\right) < 0,$$

i.e.,

$$\left|w - \frac{\overline{z_0} - 1}{|z_0|^2 - 1}\right|^2 < \left|\frac{\overline{z_0} - 1}{|z_0|^2 - 1}\right|^2.$$

Thus the image of the open unit disk is the open disk with center $C = \frac{\overline{z_0}-1}{|z_0|^2-1}$ and radius $R = |\frac{\overline{z_0}-1}{|z_0|^2-1}|$. This disk will be inside the open unit disk if and only if $|C| + R < 1$, that is

$$2\frac{|z_0 - 1|}{|z_0|^2 - 1} < 1,$$

which is (2.4.15) since $|z_0| > 1$.

Assume now $|z_0| < 1$. Then (2.4.16) becomes

$$|w|^2 - 2\mathrm{Re}\,\left(w\left(\frac{\overline{z_0} - 1}{|z_0|^2 - 1}\right)\right) > 0.$$

This can be rewritten as

$$\left|w - \frac{\overline{z_0} - 1}{|z_0|^2 - 1}\right|^2 > \left|\frac{\overline{z_0} - 1}{|z_0|^2 - 1}\right|^2,$$

and thus defines an unbounded set, and cannot be in the open unit disk. Thus z_0 cannot be of modulus strictly less than 1, and the necessary and sufficient condition is (2.4.15). $\square$

Solution of Exercise 2.3.11. (a) In view of formula (2.3.1) we have

$$\begin{aligned}\frac{\varphi(z)-z_1}{\varphi(z)-z_2} &= \frac{\varphi(z)-\varphi(z_1)}{\varphi(z)-\varphi(z_2)}\\ &= \frac{\dfrac{(z-z_1)(ad-bc)}{(cz+d)(cz_1+d)}}{\dfrac{(z-z_2)(ad-bc)}{(cz+d)(cz_2+d)}}\\ &= k\frac{z-z_1}{z-z_2},\end{aligned}$$

with

$$k=\frac{cz_2+d}{cz_1+d}. \tag{2.4.17}$$

(b) Denote by $\varphi^{\circ n}(z)$ the n-th iterate of φ. Iterating (2.3.11) we have

$$\frac{\varphi^{\circ n}(z)-z_1}{\varphi^{\circ n}(z)-z_2}=k^n\frac{z-z_1}{z-z_2}, \tag{2.4.18}$$

and hence

$$\varphi^{\circ n}(z)=\frac{z_1-z_2k^n\dfrac{z-z_1}{z-z_2}}{1-k^n\dfrac{z-z_1}{z-z_2}}=\frac{z_1(z-z_2)-z_2k^n(z-z_1)}{z-z_2-k^n(z-z_1)}.$$

Thus

$$\varphi^{\circ n}(z)=\frac{(z_1-z_2k^n)z-(1-k^n)z_1z_2}{(1-k^n)z-(z_2-z_1k^n)}. \tag{2.4.19}$$

(c) When $\varphi(z)=(1-3z)/(z-3)$ the equation

$$\varphi(z)=z$$

has two distinct roots, namely $z_1=1$ and $z_2=-1$. We have $c=1$, $d=-3$ and

$$k=\frac{cz_2+d}{cz_1+d}=2.$$

Therefore, from (2.4.19) we have

$$\varphi^{\circ n}(z)=\frac{(1+2^n)z+1-2^n}{(1-2^n)z+1+2^n}.$$ $\square$

We note the following: When $|k| > 1$, equation (2.4.18) implies that, for $z \neq z_1$,

$$\lim_{n\to\infty} \varphi^{\circ n}(z) = z_2,$$

while we have

$$\lim_{n\to\infty} \varphi^{\circ n}(z) = z_1$$

for $z \neq z_2$ when $|k| < 1$.

Solution of Exercise 2.3.13. We use the preceding exercise. For $w \neq 0$ (which is the only case of interest), the equation

$$b_w(z) = z,$$

that is,

$$z - w = z - z^2\overline{w}$$

has two distinct solutions, say z_1 and z_2. Set $w = \rho e^{i\theta}$. Then

$$z_1 = e^{i\theta} \quad \text{and} \quad z_2 = -e^{i\theta}.$$

The multiplier k is given by formula (2.4.17), and hence equal to

$$k = \frac{-\overline{w}z_2 + 1}{-\overline{w}z_1 + 1} = \frac{1+\rho}{1-\rho}.$$

In particular, we see that k is real and belongs to $(1, +\infty)$. Formula (2.4.19) gives

$$\underbrace{b_w \circ b_w \circ \cdots \circ b_w}_{n\text{-times}}(z) = \frac{z - w_n}{1 - z\overline{w_n}},$$

where

$$w_n = e^{i\theta}\frac{k^n - 1}{k^n + 1}.$$

(*b*) We have

$$\lim_{n\to\infty} w_n = e^{i\theta},$$

and thus

$$\lim_{n\to\infty} \underbrace{b_w \circ b_w \circ \cdots \circ b_w}_{n\text{-times}}(z) = e^{i\theta}.$$

In particular the limit is independent of $z \in \mathbb{D}$. □

Solution of Exercise 2.3.14. Let $\lambda = \frac{zw+1}{z-w}$. Then $z = \frac{\lambda w+1}{\lambda - w}$. The number z is real if and only if it holds that

$$\frac{\lambda w + 1}{\lambda - w} = \frac{\overline{\lambda}\overline{w} + 1}{\overline{\lambda} - \overline{w}},$$

which can be rewritten as

$$|\lambda|^2 - \lambda\overline{\alpha} - \overline{\lambda}\alpha + 1 = 0,$$

with

$$\alpha = \frac{|w|^2 + 1}{\overline{w} - w}.$$

Completing the square we get

$$|\lambda - \alpha|^2 = |\alpha|^2 - 1.$$

It is readily checked that

$$|\alpha|^2 - 1 = \left(\frac{|w^2 + 1|}{|w - \overline{w}|}\right)^2,$$

and we get a circle of center α and radius

$$R = \frac{|w^2 + 1|}{|w - \overline{w}|}.$$

The circle reduces to a point when $w = \pm i$. $\square$

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