Hall's marriage theorem

Hall's marriage theorem, from graph theory, is an excellent example of induction. It is called the "marriage theorem" because traditionally it is stated in terms of men and women, but we'll instead use robots and batteries.

Say we have a set R of n robots and a set B of n batteries. Each robot submits a list of batteries it likes. The question we're trying to answer is when we can pair off robots and batteries so that each robot gets assigned a battery it likes. Let's think about some examples.

Example 1. Say we have robots r_1, r_2 and r_3, and batteries b_1, b_2 and b_3. Robot r_1 likes batteries b_1 and b_2, robot r_2 likes battery r_2 and robot r_3 likes batteries r_1 and r_3. Let's depict this as follows.

$$
\begin{tikzcd}
r_1 \arrow[r] \arrow[dr] & b_1 \\
r_2 \arrow[r] & b_2 \\
r_3 \arrow[r] \arrow[uur] & b_3
\end{tikzcd}
$$

Here, we can find a matching. We have to give battery r_2 to b_2, then we have to give battery b_1 to r_1, since b_2 has already been taken, which means we have to give battery b_3 to r_3.

Example 2. Say we have the following situation.

$$
\begin{tikzcd}
r_1 \arrow[r] & b_1 \\
r_2 \arrow[r] \arrow[ur] & b_2 \\
r_3 \arrow[ur] & b_3
\end{tikzcd}
$$

Here, we can't have a matching: there are 3 robots, but they collectively like only 2 batteries, so there's no way there can be a matching.

Example 3. Here's a slightly less transparent situation where a matching doesn't exist.

$$
\begin{tikzcd}
r_1 \arrow[r] & b_1 \\
r_2 \arrow[r] \arrow[ur] & b_2 \\
r_3 \arrow[ur] & b_3 \\
r_4 \arrow[r] \arrow[ur] & b_4
\end{tikzcd}
$$

Here, every battery is liked by some robot. But if we look at the set of batteries liked by robots r_1, r_2 and r_3, it consists of only b_1 and b_2, so there's no way we can assign to each robot a battery that it likes because there's no way to share 2 batteries amongst 3 robots.

It turns out that the problem described in examples 2 and 3 is the only thing that can go wrong. More precisely,

Theorem 4 (Hall's marriage theorem)**.** *A matching between robots and batteries exists if and only if, for every subset $X \subseteq B$ of robots, the set of batteries $Y \subseteq B$ liked by robots in X is at least as big as X.*

The condition of the theorem is often called the "matching condition." We'll abbreviate it to MC.

Proof. We induct on the number n of robots and batteries. More precisely, let $P(n)$ be the statement that if we have n robots and n batteries and MC is satisfied, then a matching exists.

The base case, $P(1)$, is easy. If we have only one robot, then MC says that that robot must like at least one battery, so we match up our lone robot with our lone battery and we're done.

We now proceed by strong induction. We're assuming $P(1), \ldots, P(k)$ and trying to prove $P(k+1)$, so suppose we have $k+1$ robots and $k+1$ batteries and that MC is satisfied. Let's split up into two cases.

Case 1 is when every proper subset of robots like a strictly larger number of batteries. See example 5 below for an example of this situation. Pick any robot r and pick a battery b that r likes. Pair them off. Now we want to show that $R - \{r\}$ and $B - \{b\}$ also satisfy MC, and then we can apply $P(k)$ to conclude that there is a matching between these two sets too. To see that these satisfy MC, let $X \subseteq R - \{r\}$ and let Y be the set of batteries liked by some robot in X. Our assumption in this case is that $|X| < |Y|$. Notice that $|Y - \{b\}|$ is either equal to $|Y| - 1$ or equal to $|Y|$, depending on whether or not $b \in Y$. In any case, this means that $|X| \leq |Y - \{b\}|$. This means that MC is satisfied for $R - \{r\}$ and $B - \{b\}$, so we can find a matching between these robots and batteries using $P(k-1)$. We've already paired off r and b, so we've constructed a matching between R and B.

Case 2, then, is when there is a proper subset S of robots that likes exactly as many batteries (we know that MC is satisfied, so every subset likes at least as many batteries, and if it's not strictly greater, then it has to be equal). (Example 1 is of this form, since the set of robots $\{r_2\}$ has only one element, and likes only one battery.) Let $|S| = m$ and let T be the set of m batteries liked by robots in S. Clearly MC is satisfied for S and T, so by $P(m)$ we know that there is a matching between these two sets.

So now we need to find a matching between $R - S$ and $B - T$. Both of these sets have $(k+1) - m$ elements, so we want to use $P(k+1-m)$, but to do that we need to check that MC is also satisfied by these two sets. So let $X \subseteq R - S$ be a set of robots in $R - S$. Let Y be the set of robots liked by X excluding those that are in T. Now notice that $Y \cup T$ is precisely the set of batteries liked by robots in $X \cup S$. Indeed, every battery in Y is liked by some robot in X, and every battery in T is liked by some robot in S, and conversely, all robots in X like batteries that are either in Y or in T, and all robots in S like batteries in S. So the MC tells us that

$$|X \cup S| \leq |Y \cup T|.$$

But notice that X and S are disjoint sets, so $|X \cup S| = |X| + |S|$, and similarly Y and T are disjoint too so $|Y \cup T| = |Y| + |T|$. Since $|S| = |T| = m$, we can subtract m from both sides of our equation and get that

$$|X| \leq |Y|.$$

This shows that MC is satisfied for $R - S$ and $B - T$. So by $P(k+1-m)$, a matching exists between these two sets also. We now put together the matching of S and T with the matching of $R - S$ and $B - T$ to get a matching of all of R with all of B. $\square$

Example 5. Consider the following situation.

$$\begin{tikzcd} r_1 \arrow[r] \arrow[dr] & b_1 \\ r_2 \arrow[ur] \arrow[dr] & b_2 \\ r_3 \arrow[r] \arrow[ur] & b_3 \end{tikzcd}$$

This is an example of case 1. Each single robot likes 2 batteries, and each pair of robots like 3 batteries, which means that all proper subset of robots like a strictly larger number of batteries.