Inequalities as Sums of Functions

Evan Chen

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§1 Reading

Read §2.5 of *The OTIS Excerpts*. Alternatively, you can use *A Brief Introduction to Olympiad Inequalities*, §2.

§2 Lecture notes

Techniques covered for this particular lecture:

- Jensen / Karamata
- Tangent line trick, $n-1$ EV
- Isolated fudging
- Smoothing

Example 2.1 (Shortlist 2009 A2)
Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=a+b+c$. Prove that:

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(a+2b+c)^2}+\frac{1}{(a+b+2c)^2}\le\frac{3}{16}.$$

Walkthrough. This is sort of a canonical Jensen problem.
The first step is almost forced upon us.

(a) Homogenize the inequality to eliminate the constraint.

It's not 100% true that we *always* want to homogenize right away, although it is quite often a good start. Sometimes there is some reason not to homogenize. But this is not the case here. The condition $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=a+b+c$ is not even tangentially related to the inequality we want to prove, and is in any case an abomination. So for this problem I think it would be hard to come up for a reason *not* to eliminate the constraint.

However, we will immediately turn around and recognize that if we set $a+b+c=3$, we can turn it into a sum of functions. And so we just follow through:

(b) De-homogenize the inequality in such a way that one can rewrite the inequality in the form $f(a)+f(b)+f(c) \le 0$ where $a+b+c=3$.

(c) Assuming you defined f correctly, show that (up to constant factors)

$$f''(x) = \frac{96}{(x+3)^4} - \frac{2}{x^3}.$$

(d) Prove that f is concave over the interval $[0,3]$.

(e) Finish by Jensen.

Example 2.2 (USAMO 2003/5)

Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

Walkthrough. This is the canonical tangent line trick problem.

(a) De-homogenize the inequality in such a way that the inequality can be written as a sum of functions, say $f(a)+f(b)+f(c) \le 8$, where $a+b+c$ is fixed.

(b) Optionally, check that the resulting function is *not* concave, so one cannot apply Jensen.

(c) Use the tangent line trick to approximate f at the equality point.

(d) Check that the approximation you found in (c) is valid over all positive real numbers, thus completing the problem.

Example 2.3 (MOP 2012)

Let a, b, c, d be positive real numbers with $a+b+c+d=4$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \ge a^2+b^2+c^2+d^2.$$

Walkthrough. This is annoying, and surprisingly pernicious. We write this as

$$\sum_{\text{cyc}} f(a) \ge 0 \qquad f(x) = \frac{1}{x^2} - x^2$$

where $f\colon \mathbb{R}_{>0} \to \mathbb{R}$.

(a) Optionally, check that f is not convex or concave, nor does the tangent line at $x=1$ work.

(b) Consider the behavior of f on $\mathbb{R}_{>0}$. Prove that there exists a unique constant M such that

- f is convex over $I_1 = (0, M)$ and
- f is concave over $I_2 = (M, \infty)$.

Moreover, determine the value of M.

Unfortunately, $n-1$ EV cannot be quoted directly here, because the variables have a constraint that they need to lie on $(0, \infty)$. So we need to repeat the argument.

(c) Show that if there are two or more points in I_2, we can smooth them so that all but one are equal to M.

(d) Show that we can smooth all the points in $[0, M]$ together.

(e) Deduce that we can smooth the points such that all but one are equal.

Thus, we are reduced to considering $3f(x) + f(4-3x) \ge 0$ for $0 < x < 4/3$.

(f) Factor the expression $3f(x) + f(4-3x)$. The numerator should have a double root at $x = 1$; why?

(g) Do some (long) calculation to conclude that the factored expression is nonnegative for all x. I think this boils down to proving that

$$4 + 2x + 24x^3 > 19x^2 + 9x^4 \qquad \forall\, 0 < x < 4/3.$$

§3 Practice Problems

Instructions: Solve [25♣]. If you have time, solve [32♣]. Problems with red weights are mandatory.

> I like maxims that don't encourage behavior modification.
>
> Calvin in *Calvin and Hobbes*

[2♣] **Problem 1** (Taiwan Quiz 2014)**.** Positive real numbers $a_1, a_2, \dots, a_n$ have sum 1. Prove that for any positive integer k,

$$\prod_{i=1}^{n} \left(a_i^k + \frac{1}{a_i^k} \right) \ge \left(n^k + \frac{1}{n^k} \right)^n.$$

[3♣] **Problem 2** (ELMO SL 2013 A6)**.** Let a, b, c be positive real numbers with $a+b+c=3$. Prove that

$$\frac{18}{(3-a)(4-a)} + \frac{18}{(3-b)(4-b)} + \frac{18}{(3-c)(4-c)} + 2(ab+bc+ca) \ge 15.$$

[3♣] **Problem 3** (Japan 1997)**.** Let a, b, c be positive reals. Prove that

$$\frac{(b+c-a)^2}{a^2+(b+c)^2} + \frac{(c+a-b)^2}{b^2+(c+a)^2} + \frac{(a+b-c)^2}{c^2+(a+b)^2} \ge \frac{3}{5}.$$

[3♣] **Problem 4** (Poland 1996)**.** Let a, b, c be real numbers such that $a+b+c=1$ and $a, b, c \ge -\frac{3}{4}$. Show that

$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \le \frac{9}{10}.$$

[5♣] **Required Problem 5** (MOP 2002)**.** Determine the possible values of

$$S = \left(\frac{2a}{b+c} \right)^r + \left(\frac{2b}{c+a} \right)^r + \left(\frac{2c}{a+b} \right)^r$$

over real numbers $a, b, c > 0$ for (i) $r = \frac{1}{2}$, and (ii) $r = \frac{2}{3}$.

[5♣] **Problem 6** (USAMO 2017/6)**.** Find the minimum possible value of

$$\frac{a}{b^3+4} + \frac{b}{c^3+4} + \frac{c}{d^3+4} + \frac{d}{a^3+4}$$

given that a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$.

[3♣] **Problem 7** (Korea 2011/4)**.** Find the maximal value of the expression

$$\frac{1}{a^2-4a+9} + \frac{1}{b^2-4b+9} + \frac{1}{c^2-4c+9}$$

if a, b, c are nonnegative real numbers with sum 1.

[3♣] **Problem 8.** Prove that if $a, b, c, d > 0$ and $abcd = 1$ then

$$a^3+b^3+c^3+d^3+8 \ge 3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right).$$

[5♣] **Required Problem 9** (Shortlist 2016 A1)**.** Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \ge 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \le \left(\frac{a+b+c}{3}\right)^2 + 1.$$

[5♣] **Problem 10** (PUMaC Finals 2013 A1)**.** Prove that

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \le \frac{1}{6ab+c^2} + \frac{1}{6bc+a^2} + \frac{1}{6ca+b^2}$$

for any positive real numbers a, b, c which obey $a^2+b^2+c^2=1$.

[5♣] **Problem 11** (CGMO 2007/3)**.** Let $n > 3$ be a fixed integer. Find the minimum possible value of

$$\frac{a_1}{a_2^2+1} + \frac{a_2}{a_3^2+1} + \dots + \frac{a_n}{a_1^2+1}$$

over all nonnegative reals $a_1, \dots, a_n$ which satisfy $a_1 + a_2 + \dots + a_n = 2$.

[1♣] **Mini Survey.** At the end of your submission, answer the following questions.

(a) About how many hours did the problem set take?

(b) Name any problems that stood out (e.g. especially nice, instructive, boring, or unusually easy/hard for its placement).

Any other thoughts are welcome too. Examples: suggestions for new problems to add, things I could explain better in the notes, overall difficulty or usefulness of the unit.

§4 Solutions to the walkthroughs

§4.1 Solution 2.1, Shortlist 2009 A2

Homogenize to get rid of constraint:

$$\sum_{\text{cyc}} \left(\frac{16}{(2a+b+c)^2} - \frac{3}{a(a+b+c)} \right) \le 0$$

To make this a sum of functions, we then *de-homogenize* with the condition $a+b+c=3$; thus we wish to show

$$\sum_{\text{cyc}} \left(\frac{16}{(a+3)^2} - \frac{1}{a} \right) \le 0 \qquad a+b+c=3.$$

Let $f(x) = 16/(x+3)^2 - 1/x$, so $f(1)=0$. Then

$$f''(x) = \frac{96}{(x+3)^4} - \frac{2}{x^3} \le 0$$

This is concave for $x \in [0,3]$ since for x in this interval we have $(x+3)^4 - 48x^3 = (x-3)(x^3-33x^2-45x-27) \ge 0$. (In fact $f''(3)=0$.) Consequently we are done as

$$f(a)+f(b)+f(c) \le 3f\left(\frac{a+b+c}{3}\right) = 3f(1) = 0$$

by Jensen.

§4.2 Solution 2.2, USAMO 2003/5

This is a canonical example of tangent line trick. Homogenize so that $a+b+c=3$. The desired inequality reads

$$\sum_{\text{cyc}} \frac{(a+3)^2}{2a^2+(3-a)^2} \le 8.$$

This follows from

$$f(x) = \frac{(x+3)^2}{2x^2+(3-x)^2} \le \frac{1}{3}(4x+4)$$

which can be checked as $\frac{1}{3}(4x+4)(2x^2+(3-x)^2) - (x+3)^2 = (x-1)^2(4x+3) \ge 0$.

§4.3 Solution 2.3, MOP 2012

This is annoying. Write this as

$$\sum_{\text{cyc}} f(a) \ge 0 \qquad f(x) = \frac{1}{x^2} - x^2$$

where $f\colon \mathbb{R}_{>0} \to \mathbb{R}$.

Note that $f''(x) = 6x^{-4} - 2$, so f is convex over $I_1 = (0, \sqrt[4]{3})$ and concave over $I_2 = (\sqrt[4]{3}, \infty)$. We can now repeat the argument from $n-1$ EV: first smooth any points in I_2 away from each other, then smooth the points in I_1 all together. In this way we can reduce to when $a=b=c$, say.

Now,

$$3f(x) + f(4-3x) = -\frac{12(x-1)^2\left(9x^4-24x^3+19x^2-2x-4\right)}{x^2(3x-4)^2}.$$

So it suffices to show that for $0<x<4/3$ we have

$$4+2x+24x^3 > 19x^2+9x^4.$$