A new proof of the Ceva Theorem / Darij Grinberg

A well-known theorem that can be shown in several different ways is the Ceva Theorem (we treat it here without the converse):

Ceva Theorem. Let ABC be an arbitrary triangle. Further, let A', B', C' be points on its sides BC, CA, AB, for which the lines AA', BB', CC' concur. Then (with directed segments)

$$\frac{AC'}{C'B} \bullet \frac{BA'}{A'C} \bullet \frac{CB'}{B'A} = 1.$$

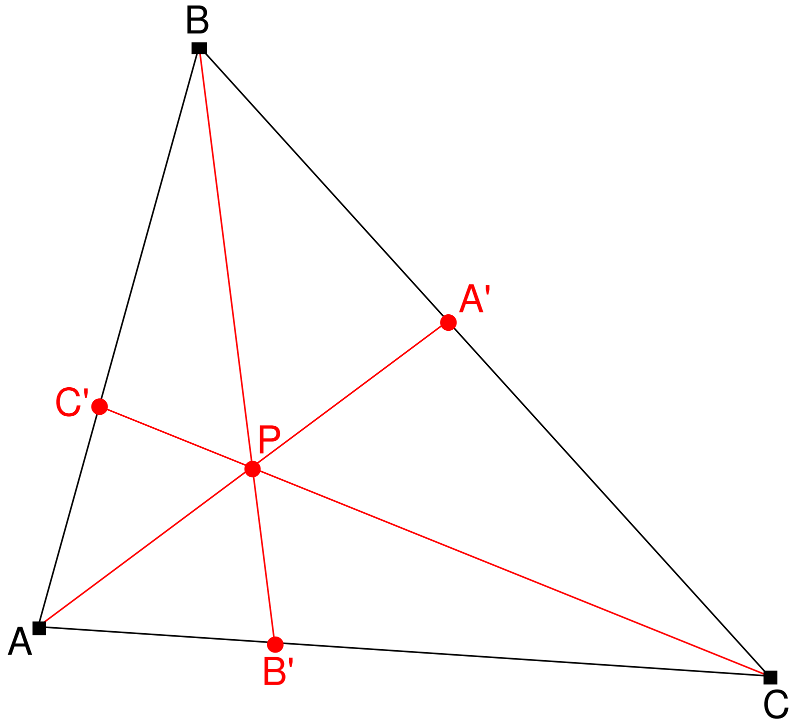

Fig. 1

Here I present a probably new proof of this result. Denote by P the intersection of the lines AA', BB', CC'. The parallel to BC through P meets CA at B_a and AB at C_a. The parallel to CA through P meets AB at C_b and BC at A_b. The parallel to AB through P meets BC at A_c and CA at B_c.

As segments on parallels,

$$\frac{AC'}{C'B} = \frac{B_cP}{PA_c}.$$

On the other hand,

$$\frac{B_cP}{AB} = \frac{PB'}{BB'} \qquad \text{and} \qquad \frac{PA_c}{AB} = \frac{PA'}{AA'},$$

hence

$$\frac{B_cP}{AB} : \frac{PA_c}{AB} = \frac{PB'}{BB'} : \frac{PA'}{AA'}, \qquad \text{i. e.} \qquad \frac{B_cP}{PA_c} = \frac{PB'}{BB'} : \frac{PA'}{AA'}.$$

Consequently,

$$\frac{AC'}{C'B} = \frac{PB'}{BB'} : \frac{PA'}{AA'}.$$

Similarly,

$$\frac{BA'}{A'C} = \frac{PC'}{CC'} : \frac{PB'}{BB'} \qquad \text{and} \qquad \frac{CB'}{B'A} = \frac{PA'}{AA'} : \frac{PC'}{CC'}.$$

Now

$$\frac{AC'}{C'B} \bullet \frac{BA'}{A'C} \bullet \frac{CB'}{B'A} = \left(\frac{PB'}{BB'} : \frac{PA'}{AA'}\right) \bullet \left(\frac{PC'}{CC'} : \frac{PB'}{BB'}\right) \bullet \left(\frac{PA'}{AA'} : \frac{PC'}{CC'}\right) = 1,$$

what proves the Ceva Theorem.

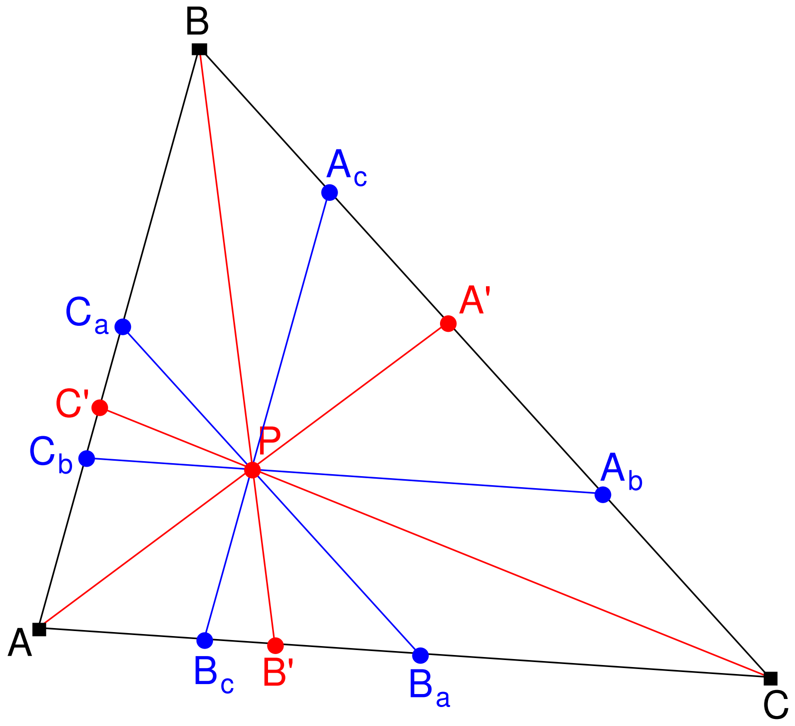

Fig. 2