Spain

Cono Sur 2010

Reply

uglysolutions

#1 Jun 17, 2010, 1:01 am • 1

Por estos días es la XXI Olimpíada Matemática del Cono Sur, en Brasil... hoy fue la primera prueba, si hay alguien en este foro que esté participando podría postear los problemas por favor? Gracias!! =)

cyshine

#2 Jun 18, 2010, 6:42 pm

http://www.opm.mat.br/conesul2010/provas.php

uglysolutions

#3 Jun 19, 2010, 12:54 am

Obrigado Shine!

Ahora a esperar los resultados =P

mavropnevma

#4 Jun 19, 2010, 6:33 pm

CONO SUR, Day 2, Problem 3. Does it exist a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma + \textrm{id}_{\mathbb{N}}$ be injective and $\textrm{Im}(\sigma + \textrm{id}_{\mathbb{N}}) = \mathbb{P}$, the set of all positive primes?

Hint

Solution

BlAcK_CaT

#5 Jun 19, 2010, 9:17 pm

Hola.

Si no es mucha la molestia, alguien puede poner el medallero de la OMCS porfa? En especial quiero saber como le fue a Chile

De antemano gracias =)

Luis González

#6 Jun 19, 2010, 10:25 pm

Día 2, problema 5. El incírculo (I) de $\triangle ABC$ toca los lados BC, CA y AB en D, E y F, respectivamente. Sean ω_a, ω_b y ω_c los circuncírculos de los triángulos EAF, DBF, DCE, respectivamente. Las rectas DE y DF cortan ω_a en $E_a \not\equiv E$ y $F_a \not\equiv F$. Sea r_A la recta E_aF_a. Se definen r_B y r_C de modo análogo. Demostrar que las rectas r_A, r_B y r_C determinan un triángulo cuyos vértices pertenecen a los lados del triángulo ABC.

Solución

uglysolutions

#7 Jun 19, 2010, 11:28 pm

> *BlAcK_CaT wrote:*
> Hola.
>
> Si no es mucha la molestia, alguien puede poner el medallero de la OMCS porfa? En especial quiero saber como le fue a Chile
>
> De antemano gracias =)

No tengo el medallero completo, pero te puedo decir que una medalla de plata fue para Chile :) , me lo contaron ayer los del equipo argentino. No sé si sacaron premios los demás del equipo chileno.

Hubo tres medallas de oro:
Ariel Zylber - Argentina - 48 puntos
Julián Mejía - Perú - 47 puntos
João Lucas Camelo Sá - Brasil - 42 puntos

Hubo (al menos, no sé si hubieron más) 6 medallas de plata:
Gabriel Lopes - Brasil - 40 puntos
Raúl Chávez Sarmiento - Perú - 40 puntos
José García Sulca - Perú - 34 puntos
? - Chile - ? puntos
Melanie Sclar - Argentina - 31 puntos
Maria Clara Mendes - Brasil - 31 puntos

De los bronces ya no sé mucho, sólo sé que el corte de bronce fue menor a 20 puntos.

En cuanto al ranking de países, los tres primeros puestos los ocuparon Perú (151 puntos), Brasil (no sé, porque me falta saber el puntaje del chico que sacó bronce), Argentina (129 puntos).

BlAcK_CaT
#8 Jun 19, 2010, 11:40 pm

> *uglysolutions wrote:*

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Muchas gracias por la información

Puedo cooperar con el hecho que el corte de plata fue 31 puntos, ya que Jesus Ccopa (de Perú) sacó bronce con 30 ptos

uglysolutions

#9 Jun 20, 2010, 12:53 am

Sí, me olvidé de decir eso, Mariano Bonifacio de Argentina también sacó bronce con 30...

uglysolutions

#10 Jun 21, 2010, 7:24 am

Acá subieron el medallero a la página oficial:

http://www.opm.mat.br/conesul2010/premiados.php

Quick Reply

High School Olympiads

symmetry. try to do this

dap

#1 Jun 18, 2010, 9:17 pm

given a triangle ABC, 3 altitude AA0,BB0,CC0 , orthocenter H an a point M in the triangle.
A1 is the symmetry of M through BC
B1 is the symmetry of M through AC
C1 is the symmetry of M through AB
Prove that A0A1, B0B1, C0C1 are concurrent

Please help me find some things on problems, which involves the symmetry

jayme

#2 Jun 18, 2010, 10:02 pm

Dear Mathlinkers,
for beginnig consider the M-anticevian triangle of ABC ; then some harmonic quadruple...
Sincerely
Jean-Louis

Luis González

#3 Jun 18, 2010, 10:50 pm

Let $(u:v:w)$ be the barycentric coordinates of M with respect to $\triangle ABC$. Reflection A_1 of M across BC has coordinates $A_1(-a^2u:a^2v+2S_Cu:a^2w+2S_Bu)$. Since $A_0(0:S_C:S_B)$, the equation of the line τ_a passing through A_1, A_0 is:

$$\tau_a \equiv (vS_B-wS_C)x+uS_By-uS_Cz=0$$

By cyclic permutation of a,b,c we get the equations of τ_b, τ_c as

$$\tau_b \equiv -vS_Ax+(wS_C-uS_A)y+vS_Cz=0$$

$$\tau_c \equiv wS_Ax-wS_By+(uS_A-vS_B)z=0$$

Lines τ_a, τ_b, τ_c concur at a point P with coordinates

$$P \equiv (u(vS_B+wS_C-uS_A):v(uS_A+wS_C-vS_B):w(uS_A+vS_B-wS_C))$$

skytin

#4 Jun 19, 2010, 1:15 am

lines A_0M B_0M C_0M intersects at point M easy to see that line A_0A_1 is simmetric to MA_0 wrt AA_0 = biss of angle B_0A_0C_0 ... so A_1M B_1M C_1M intersect in isogonal point M' wrt A_0B_0C_0

dap

#5 Jun 19, 2010, 9:59 am

> skytin wrote:
> lines A_0M B_0M C_0M intersects at point M easy to see that line A_0A_1 is simmetric to MA_0 wrt AA_0 = biss of angle B_0A_0C_0 ... so A_1M B_1M C_1M intersect in isogonal point M' wrt A_0B_0C_0

I don't understand . please give me a good geometry solution ,not coordinate

Luis González

#6 Jun 19, 2010, 11:34 am

> *skytin wrote:*
>
> lines A_0M, B_0M, C_0M intersects at point M easy to see that line A_0A_1 is simmetric to MA_0 wrt AA_0 = biss of angle $B_0A_0C_0$. So A_1M, B_1M, C_1M intersect in isogonal point M' wrt $A_0B_0C_0$

> *dap wrote:*
>
> I don't understand. please give me a good geometry solution, not coordinate

dap, the solution offered by skytin is the simplest if you manage the concept of Isogonal conjugates. Lines A_0M and A_0A_1 are obviously symmetrical across AA_0, but since the altitudes of $\triangle ABC$ become internal/external bisectors of $\triangle A_0B_0C_0$, then $\angle(A_0M, A_0C_0) = \angle(A_0A_1, A_0B_0) \Longrightarrow A_0M, A_0A_1$ are isogonal lines with respect to $\angle B_0A_0C_0$.

Quick Reply

High School Olympiads

Segments relation 1 (Sangaku)

Reply

77ant

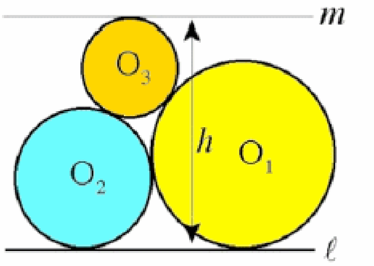
#1 Jun 19, 2010, 1:57 am

Hi.
I got an interesting problem related to Sangaku.
Here it is.

Prove that $8r_1r_2r_3h=\{(r_1+r_2)h-2r_1r_2\}^2$

Thanks.

Attachments:



Luis González

#2 Jun 19, 2010, 3:48 am

Let M,N be the contact points of $(O_1),(O_2)$ with ℓ and L the contact point of (O_3) with $m.$ Denote X,Y the orthogonal projections of O_1,O_2 onto $m.$ By Pythagorean theorem for the right trapezoids XLO_3O_1 and YLO_3O_2 we obtain:

$$XL=\sqrt{(r_3+r_1)^2-(h-r_1-r_3)^2}=\sqrt{2h(r_1+r_3)-h^2} \ \ (1)$$

$$YL=\sqrt{(r_3+r_2)^2-(h-r_2-r_3)^2}=\sqrt{2h(r_2+r_3)-h^2} \ \ (2)$$

On the other hand, since $(O_1),(O_2)$ are externally tangent, it follows that $MN=2\sqrt{r_1r_2}.$ But $MN=XL+YL,$ together with $(1),(2)$ we get:

$$2\sqrt{r_1r_2}=\sqrt{2h(r_1+r_3)-h^2}+\sqrt{2h(r_2+r_3)-h^2}$$

After squaring both sides of the equation and simplifying we get:

$$8hr_1r_2r_3=4(r_1r_2)^2+h^2(r_1+r_2)^2-4hr_1r_2(r_1+r_2)$$

Right hand side is a perfect square binomial, therefore:

$$8hr_1r_2r_3=\left[(r_1+r_2)h-2r_1r_2\right]^2$$

Furthermore, we can solve the latter expression as a quadratic equation in h for positive solution

$$h=\frac{2r_1r_2\left(r_1+r_2+2r_3+2\sqrt{r_3(r_1+r_2+r_3)}\right)}{(r_1+r_2)^2}$$

77ant

#3 Jun 19, 2010, 4:12 am

Hi, luisgeometra.
Thank you so much for interests and reply.

Virgil Nicula

#4 Jun 20, 2010, 11:38 pm

Interesting, nice and easy **problem** ! I"ll present a proof which is similarly with the **luis**'method.

Denote $T_1\in (O_1)\cap l$, $T_2\in (O_2)\cap l$ and the points U , V for which $O_3\in UV$, $UV\parallel l$, $UO_1\perp l$, $VO_2\perp l$.

Observe that $2\sqrt{r_1r_2}=MN=O_3U+O_3V$. Appling the **Pitagora**'s theorem in the triangles O_1O_3U and O_2O_3V

obtain $\sqrt{(r_1+r_3)^2-[h-(r_1+r_3)]^2}+\sqrt{(r_2+r_3)^2-[h-(r_2+r_3)]^2}=2\sqrt{r_1r_2}\iff$

$\sqrt{2h(r_1+r_3)-h^2}+\sqrt{2h(r_2+r_3)-h^2}=2\sqrt{r_1r_2}\iff$
$\sqrt{2h(r_1+r_3)-h^2}-\sqrt{2h(r_2+r_3)-h^2}=\frac{h(r_1-r_2)}{\sqrt{r_1r_2}}$.

Thus, $\sqrt{2h(r_1+r_2)-h^2}=\sqrt{r_1r_2}+\frac{h(r_1-h_2)}{2\sqrt{r_1r_2}}\iff 2h(r_1+r_2)-h^2=r_1r_2+h(r_1-r_2)+\frac{h^2(r_1-r_2)^2}{4r_1r_2}$
$\iff$

$(r_1+r_2)^2\cdot h^2-4r_1r_2(r_1+r_2+2r_3)\cdot h+4r_1^2r_2^2=0\iff \boxed{[(r_1+r_2)\cdot h-2r_1r_2]^2=8r_1r_2r_3\cdot h}$.

77ant

#5 Jun 21, 2010, 1:40 am

Dear **Virgil Nicula**.
Thanks a lot. Metric calculation of Sangaku problems seems not easy **only to me**.

Quick Reply

High School Olympiads

help !! NEED THE SOLUTION OF THIS NOW

Reply

dap

#1 Jun 17, 2010, 10:24 am

Triangle ABC, H is its orthocenter, (O,R) is the circumcircle of ABC .
A1 is the symmetry of A through BC
B1 is the symmetry of B through CA
C1 is the symmetry of C through AB
Prove that A1,B1,C1 are collinear if only OH=2R

Luis González

#2 Jun 17, 2010, 11:22 am

Please, use more meaningful subjects next time.

Let G, N be the centroid and 9-point center of $\triangle ABC$. Reflection triangle $\triangle A_1B_1C_1$ of $\triangle ABC$ is homothetic to the pedal triangle $\triangle A_0B_0C_0$ of N WRT $\triangle ABC$ under the homothety with center G and coefficient $4 \Longrightarrow$ Points A_1, B_1, C_1 are collinear iff points A_0, B_0, C_0 are collinear. Then $A_0B_0C_0$ becomes a Simson line with pole $N \Longleftrightarrow N \in (O)$, i.e. $ON = R = \frac{1}{2}OH$.

Luis González

#3 Jun 18, 2010, 9:16 pm

> Quote:
> Let G, N be the centroid and 9-point center of $\triangle ABC$. Reflection triangle $\triangle A_1B_1C_1$ of $\triangle ABC$ is homothetic to the pedal triangle $\triangle A_0B_0C_0$ of N WRT $\triangle ABC$ under the homothety with center G and coefficient 4.

Let D be the foot of the A-altitude, M be the midpoint of BC and $T \equiv AD \cap (O)$, different from A. It is well-known that $\overline{OM} = \frac{1}{2}AH$ and $\overline{HD} = \frac{1}{2}\overline{HT}$, since NA_0 becomes the median of the right trapezoid $HOMD$. Then we have

$$\overline{NA_0} = \frac{\overline{OM} + \overline{HD}}{2} = \frac{\overline{AH} + \overline{HT}}{4} = \frac{\overline{HA_1}}{4} \Longrightarrow \frac{\overline{HA_1}}{\overline{NA_0}} = 4$$

Because of $\frac{\overline{GH}}{\overline{GN}} = 4 \Longrightarrow G, A_0, A_1$ are collinear $\Longrightarrow \frac{\overline{GA_1}}{\overline{GA_0}} = 4$

Similarly, $\frac{\overline{GB_1}}{\overline{GB_0}} = \frac{\overline{GC_1}}{\overline{GC_0}} = 4 \Longrightarrow \triangle A_0B_0C_0 \sim \triangle A_1B_1C_1$ are homothetic under the homothety with center G and factor 4.

Quick Reply

High School Olympiads

MA.BC=MB.AC=MC.AB. too hard!

Reply

dap

#1 Jun 17, 2010, 7:12 pm

M is in a triangle ABC (is not a issoceles triangle): MA.BC=MB.AC=MC.AB. (I),(I1),(I2),(I3) is the incircle of ABC,MBC,MCA,MAB. Prove that AI1,BI2,CI3 are concurrent at a point on IM.

Luis González

#2 Jun 17, 2010, 10:20 pm

Rename A', B', C' the incenters of $\triangle MBC, \triangle MCA$ and $\triangle MAB$. M is a common point of the three Apollonius circles of $\triangle ABC \Longleftrightarrow M$ satisfies $AM \cdot BC = BM \cdot CA = CM \cdot AB$ $(*)$, i.e. M is an Isodynamic point $\Longleftrightarrow$ angle bisectors of $\angle BMC, \angle CMA, \angle AMB$ pass through the feet of the angle bisectors of $\angle BAC, \angle ABC, \angle BCA$. Thus, let V be the common foot of the internal angle bisectors of $\angle BAC, \angle BMC$ and let $X \equiv MI \cap AA'$. By Menelaus' theorem for $\triangle MIV$ cut by the transversal AXA', keeping in mind the relation $(*)$ we have:

$$\frac{XI}{MX} = \frac{AI}{AV} \cdot \frac{VA'}{A'M} = \frac{AB+AC}{AB+BC+CA} \cdot \frac{BC}{MB+MC}$$

$$\frac{XI}{MX} = \frac{AC}{MC} \cdot \frac{BC}{AB+BC+CA} \quad (1)$$

Similarly if $X' \equiv BB' \cap MI$, by Menelaus' theorem for the triangle formed by M, I and the foot of B-angle bisector and the transversal $BX'B'$, we obtain the relation

$$\frac{X'I}{MX'} = \frac{BC+AB}{AB+BC+CA} \cdot \frac{AC}{MA+MC} = \frac{AC}{MC} \cdot \frac{BC}{AB+BC+CA} \quad (2)$$

From (1) and (2), it follows that $X \equiv X' \Longrightarrow$ lines AA', BB', MI concur at X. Analogously $X \in CC'$.

Luis González

#3 Jun 17, 2010, 11:02 pm

Points M, A, B, C can be non-coplanar and the result is still true, even more general.

Proposition: $ABCD$ is a tetrahedron whose edges satisfy $AB \cdot CD = AC \cdot BD = AD \cdot BC$. Then the lines connecting the vertices A, B, C, D with the incenters I_a, I_b, I_c, I_d of its opposite faces are concurrent.

The condition $AB \cdot CD = AC \cdot BD = AD \cdot BC$ $(*)$, implies that pairs of lines AI_a, DI_d and AI_a, BI_b are coplanar. Hence, let $X \equiv AI_a \cap BI_b$ and $Y \equiv AI_a \cap DI_d$. Thus, we shall prove $X \equiv Y$. By Menelaus' theorem for $\triangle API_a$ and $\triangle AQI_a$ cut by the transversals DYI_d, BXI_b, keeping in mind $(*)$, we have

$$\frac{AY}{YI_a} = \frac{DP}{DI_a} \cdot \frac{I_dA}{PI_d} = \frac{(BD+DC+CB)}{DB+DC} \cdot \frac{(AB+AC)}{BC}$$

$$\frac{AY}{YI_a} = \frac{AC}{BC} \cdot \frac{(BD+DC+CB)}{CD} \, , (1)$$

$$\frac{AX}{XI_a} = \frac{BQ}{BI_a} \cdot \frac{I_bA}{QI_b} = \frac{(BD+DC+CB)}{DB+BC} \cdot \frac{(AC+AD)}{DC}$$

$$\frac{AX}{XI_a} = \frac{AD}{BD} \cdot \frac{(BD+DC+CB)}{CD} \, , (2)$$

Since $AC \cdot BD = BC \cdot AD$, from the expressions (1) and (2) it follows that $Y \equiv X \equiv AI_a \cap BI_b \cap DI_d$. Similarly, we'll obtain $X \in CI_c$.$\square$

Quick Reply

High School Olympiads

concurent 2.

Reply

skytin
#1 Jun 17, 2010, 1:48 am

given trangle ABC median AM intersect (O) at point A' if H is othocenter then line MH meet arc BC (which contein point A) at A" points B' B" C' C" have same construction prove that lines A'A" B'B" and C'C" are concurent.

Luis González
#2 Jun 17, 2010, 7:00 am • 1

Denote D the foot of the A-altitude. Orthocenter H of $\triangle ABC$ is the center of the negative inversion taking the 9-point circle into its circumcircle $(O) \Longrightarrow$ Pairs D, A and M, A'' are inverse points, thus A, A'', D, M are concyclic $\Longrightarrow \angle AA''M$ is right $\Longrightarrow MA''$ passes through the antipode A_0 of A WRT (O). Likewise, line connecting the midpoint of AC with H passes through the antipode B_0 of B WRT (O). By Pascal theorem for the hexagon $AA'B_0BB'A_0$, the intersections $G \equiv AA' \cap BB', U \equiv A'B_0 \cap B'A_0$ and $O \equiv BB_0 \cap AA_0$ are collinear $\Longrightarrow U$ lies on Euler line $e \equiv OGH$. Again, by Pascal theorem for the hexagon $A''A'B_0B''B'A_0$, the intersections $P \equiv A'A'' \cap B'B'', U \equiv A'B_0 \cap B'A_0$ and $H \equiv B_0B'' \cap A_0A''$ are collinear $\Longrightarrow P \in e$. Analogous reasoning yields $P \equiv e \cap A'A'' \cap C'C'' \Longrightarrow$ Lines $A'A'', B'B''$ and $C'C''$ concur at a point P lying on the Euler line of $\triangle ABC$.

mathVNpro
#3 Aug 2, 2010, 3:28 pm

> Quote:
>
> given trangle ABC median AM intersect (O) at point A' if H is othocenter then line MH meet arc BC (which contein point A) at A" points B' B" C' C" have same construction prove that lines A'A" B'B" and C'C" are concurent.

The above problem is exactly them same as the *lemma* 2 that I used it to prove a nice problem last year. Luis's proof is fantastic! I have another proof which can be seen at this link: http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1505003#p1505003

Quick Reply

High School Olympiads

how to find

Reply

jemima
#1 Jun 12, 2010, 8:41 pm

is there any formula to find distence between speiker centre and orthocentre, if so please give me.

frenchy
#2 Jun 16, 2010, 5:01 am

see here http://mathworld.wolfram.com/SpiekerCenter.html
$SH = \frac{1}{4r}\sqrt{(a^4 - ba^3 - ca^3 + bca^2 - b^3a - c^3a + bc^2a + b^2ca + b^4 + c^4 - bc^3 - b^3c)}$
its like this if i am not mistaken

Luis González
#3 Jun 16, 2010, 9:30 am

The expression looks more simplified in terms of the circumradius R, inradius r and semiperimeter s of $\triangle ABC$. Denote O, H, G, I, S the circumcenter, orthocenter, centroid, incenter and Spieker point of $\triangle ABC$. By using *Leibniz theorem* for the circumcenter O we obtain the relation:

$$OG^2 = \tfrac{1}{3}(OA^2 + OB^2 + OC^2) - \tfrac{1}{9}(a^2 + b^2 + c^2) = R^2 - \tfrac{1}{9}(a^2 + b^2 + c^2)$$

Since $OG = \frac{1}{3}OH$, it follows that:

$$OH^2 = 9R^2 - (a^2 + b^2 + c^2) \ , \ HG^2 = 4R^2 - \tfrac{4}{9}(a^2 + b^2 + c^2)$$

Using the identity $a^2 + b^2 + c^2 = 2s^2 - 2r^2 - 8Rr \ (*)$ we get:

$$HG^2 = 4R^2 + \tfrac{8}{9}(r^2 - s^2 + 4Rr) \ \ (1)$$

Incircle (I) and 9-point circle (N) of $\triangle ABC$ are internally tangent $\Longrightarrow IN = \frac{1}{2}R - r$. Note that IN becomes the I-median of $\triangle IOH$, thus by Stewart theorem we get:

$$IN^2 = \tfrac{1}{2}(IO^2 + IH^2) - \tfrac{1}{4}OH^2 \Longrightarrow IH^2 = 2IN^2 + \tfrac{1}{2}OH^2 - IO^2$$

$$IH^2 = 2\left(\frac{R}{2} - r\right)^2 + \tfrac{9}{2}R^2 - \tfrac{1}{2}(a^2 + b^2 + c^2) - (R^2 - 2Rr)$$

$$IH^2 = 4R^2 + 2r^2 - \tfrac{1}{2}(a^2 + b^2 + c^2) = 4R^2 + 3r^2 + 4Rr - s^2 \ \ (2)$$

Again by *Leibniz theorem*, this time for the incenter I, we obtain the relation:

$$IG^2 = \tfrac{1}{3}(IA^2 + IB^2 + IC^2) - \tfrac{1}{9}(a^2 + b^2 + c^2)$$

Using the well-known formulas:

$$IA = \sqrt{\frac{bc(s-a)}{s}} \ , \ IB = \sqrt{\frac{ac(s-b)}{s}} \ , \ IC = \sqrt{\frac{ab(s-c)}{s}}$$

After expanding, simplifying terms and using the identity $(*)$ we get:

$IG^2 = \frac{1}{9}(s^2 + 5r^2 - 16Rr) \ \ (3)$

The homothety with center G and factor $-\frac{1}{2}$ takes the incenter of $\triangle ABC$ into the incenter of its medial triangle, i.e. the Spieker point $S \Longrightarrow \overline{GI} : \overline{GS} = -2 : 1.$ Therefore, by Stewart theorem for the cevian HS in $\triangle HIG$ we get:

$HS^2 = \frac{3}{2}HG^2 + \frac{3}{4}IG^2 - \frac{1}{2}IH^2$

Plugging the distances HG, IH, IG from the expressions $(1), (2), (3)$ into the latter one and simplifying terms, we obtain

$$HS = \sqrt{4R^2 + \frac{1}{4}r^2 - \frac{3}{4}s^2 + 2Rr}$$

jemima

#4 Jun 16, 2010, 8:50 pm

excellent, but i solved in different way without using any nine point centre property or homothey, i used exclusively stewarts theorem i derived a formula like leibnitz formula (centroid to any point in the plane) skpeiker centre to any point in the plane in that i substitued instead of M as orthocentre, i will give that generilasition in my next post, in any body have generalisation please give i will verify whether mine is correct or not.

Quick Reply

High School Olympiads

can you prove it

Reply

dap

#1 Jun 15, 2010, 10:44 am

Given four points A,B,C,D sastify (ACBD)=-1. Prove that exists a circle that it contacts with 4 circles with diameter AB,BC,CD,DA.

This post has been edited 2 times. Last edited by dap, Jun 16, 2010, 8:02 pm

Luis González

#2 Jun 15, 2010, 10:48 pm

Let $\omega_1, \omega_2, \omega_3$ and ω_4 denote the circles with diameters AB, BC, CD, AD and let M be the midpoint of AC. By Newton's theorem $\overline{MA}^2 = \overline{MC}^2 = \overline{MB} \cdot \overline{MD} \Longrightarrow$ The inversion $\mathcal{I}$ in the circle with diameter AC takes B into D while A, C are double points, thus M becomes the exsimilicenter of ω_2, ω_3 and the insimilicenter of ω_1, ω_4. Circle ω internally tangent to ω_4 and externally tangent to ω_2, ω_3 at U, V transforms into itself under $\mathcal{I}$, since of U, V, M are collinear due to Monge & d'Alembert theorem and $\overline{MC}^2 = \overline{MB} \cdot \overline{MD} = \overline{MU} \cdot \overline{MV}$. Since $\mathcal{I} : \omega_2 \mapsto \omega_3, \omega_1 \mapsto \omega_4$ and ω is double, by conformity it follows that ω is tangent to $\omega_1, \omega_2, \omega_3$ and ω_4.

pldx1

#3 Jun 16, 2010, 3:17 pm

Hello,

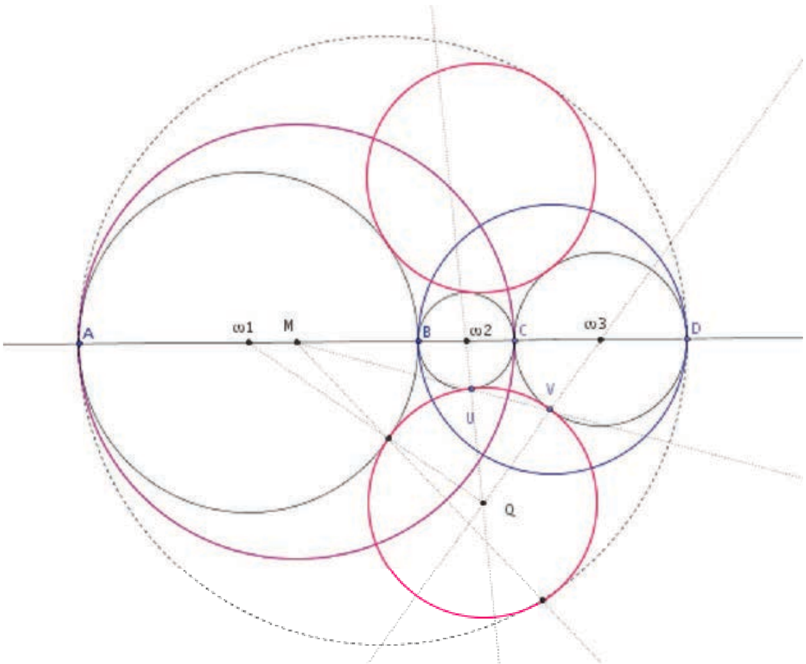
Forget, for the moment, circle ω_4 and $(A, C, B, D) = -1$. Since circles $\omega_1,\ \omega_2,\ \omega_3$ are not through the same point, it exists eight circles that are tangent to them. These circles, real or imaginary, appear by pairs that are inverse in the common orthogonal cycle of $\omega_1,\ \omega_2,\ \omega_3$ (here, the line AB).

These eight circles are real here: ω_2 (twice), circle with diameter $[AC]$ (twice), circle with diameter $[BD]$ (twice) and the red circles $\omega,\ \omega'$. Introduce now ω_4. Then circles with diameter $[AC]$ or $[BD]$ are ever tangent to the four ω_j.

Moreover, we have the equivalence: ω is tangent to ω_4 if and only if $(A, C, B, D) = -1$ (by the luisgeometria's proof).

Best regards, Pierre.

Attachments:



Quick Reply

High School Olympiads

tangent and orthocenter

Reply

bui_haiha2000

#1 Jun 24, 2006, 11:56 am

Let triangle ABC, and incircle I is tangle BC,CA,AB at X,Y,Z respectively.
Let AA', BB' , CC' is altitude from vertex A,B,C to BC,CA,AB respectively/
Let M,N,P is incircle of triangle AB'C' , BC'A' , CA'B'
1)Prove that M,N,P is orthocenter of triangle AYZ,triangle BZX,triangle CXY
2)Prove that the common tangent outside difference BC of circle (N) and circle (P) pass the orthocenter of triangle XYZ

Luis González

#2 Jun 15, 2010, 11:24 am • 1

1) From the similar triangles $\triangle AB'C' \sim \triangle ABC,$ it follows that: $\frac{AM}{AI} = \frac{B'C'}{BC} = \cos A.$ Since AI is the circumdiameter of $\triangle AYZ$ issuing from $A,$ we deduce that M is the orthocenter of $\triangle AYZ.$ Analogously, N, P are the orthocenters of $\triangle BZX$ and $\triangle CXY$

2) M, N, P are the reflections of I across $YZ, ZX, XY.$ If U denotes the orthocenter of $\triangle XYZ,$ then M, N, P are the circumcenters of $\triangle YUZ, \triangle ZUX$ and $\triangle XUY,$ i.e. sidelines of $\triangle MNP$ are the perpendicular bisectors of UX, UY, UZ $\Longrightarrow$ reflection of U across NP lies on $BC,$ i.e. U lies on the common external tangent of $(N), (P),$ different from $BC.$ Hence, common external tangents of $(M), (N), (P),$ different from $BC, CA, AB,$ concur at $U.$

jayme

#3 Mar 15, 2011, 9:36 pm

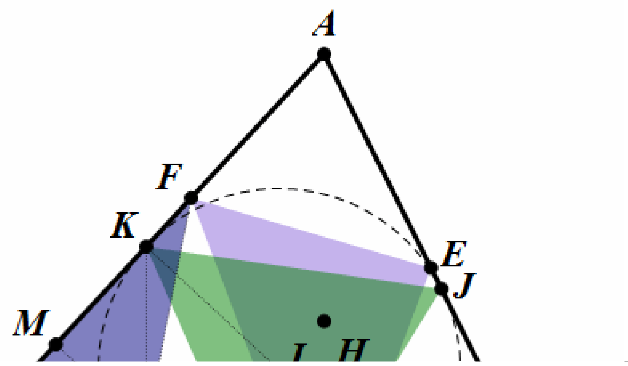
Dear Luis and mathlinkers,
thank Dear Luis for given this link for a future article.
You can have a look on
http://perso.orange.fr/jl.ayme vol. 5 the Feuerbach-Ayme theorem
Very sincerely for your help
Jean-Louis

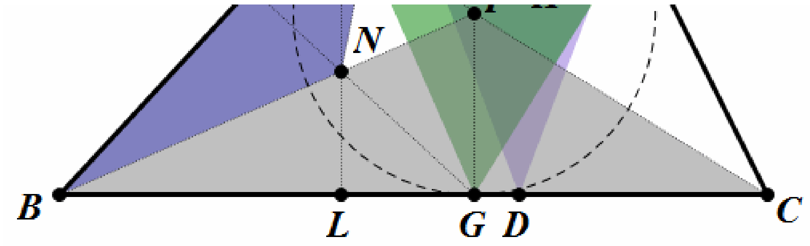
Headhunter

#4 Mar 16, 2011, 1:42 am

(1)
Let N be the orthocenter of $\triangle BGK$ and then $\triangle NBM \sim \triangle IBG$
$\triangle GBM \sim \triangle CBF \implies BN/BI = BM/BG = BF/BC$ and $\triangle NBF \sim \triangle IBC$
$\angle BFN = \angle ICB$, $\angle BFD = \angle BCA$ and then FN bisect $\angle BFD$
Thus, N is the incenter of $\triangle BFD$ and likewise, the same is true for the others.

Attachments:





mudok

#5 Jan 24, 2014, 12:04 am

> *Luis González wrote:*
> $\frac{AM}{AI} = \cos A$. Since AI is the circumdiameter of $\triangle AYZ$ issuing from A, we deduce that M is the orthocenter of $\triangle AYZ$.

Why?

Arab

#6 Jan 24, 2014, 9:40 pm • 1

@mudok

Because in an arbitrary $\triangle ABC$ with orthocenter H and circumradii R, $AH = 2R\cos A$ or $AH = 2R\cos(180^\circ - A)$.

I think that in his solution, Luis assumed that $A < 90^\circ$, which is without loss of generality, so M is the orthocenter of $\triangle AYZ$.

Quick Reply

High School Olympiads

touching circles and concurrency

immodestius

#1 Jun 11, 2010, 3:21 pm

Let ABC be an acute triangle. A circle k_A touches AB, AC (internally) and the the circle with diameter BC, the point of tangency being A'. Definie B' and C' similar. Prove that AA', BB' and CC' are concurrent.

Luis González

#2 Jun 11, 2010, 9:57 pm • 1

$(K) \equiv k_A$ and X, Y are the orthogonal projections of the midpoint M of BC onto AB, AC. P, Q are the tangency points of (K) with AB, AC and U, V are the orthogonal projections of A' onto AB, AC, respectively. In the trapezoid $XMKP$, we have

$$A'U = \frac{XM \cdot KA' + KP \cdot MA'}{MK} = \frac{KA' \cdot (XM + MA')}{MK} = \frac{KA' \cdot (a + h_c)}{2MK}$$

Analogously, in the trapezoid $YMKQ$, we have $A'V = \dfrac{KA' \cdot (a + h_b)}{2MK}$

$$\Longrightarrow \frac{A'U}{A'V} = \frac{a + h_c}{a + h_b} = \frac{1 + \sin B}{1 + \sin C} = \frac{\sin \widehat{BAA'}}{\sin \widehat{CAA'}}$$

By cyclic exchange we get the expressions

$$\frac{\sin \widehat{CBB'}}{\sin \widehat{ABB'}} = \frac{1 + \sin C}{1 + \sin A}, \quad \frac{\sin \widehat{ACC'}}{\sin \widehat{BCC'}} = \frac{1 + \sin A}{1 + \sin B}$$

$$\frac{\sin \widehat{BAA'}}{\sin \widehat{CAA'}} \cdot \frac{\sin \widehat{CBB'}}{\sin \widehat{ABB'}} \cdot \frac{\sin \widehat{ACC'}}{\sin \widehat{BCC'}}$$

$$= \frac{(1 + \sin B)}{(1 + \sin C)} \cdot \frac{(1 + \sin C)}{(1 + \sin A)} \cdot \frac{(1 + \sin A)}{(1 + \sin B)} = 1$$

Hence, by trigonometric Ceva's theorem we conclude that AA', BB', CC' concur. $\square$

Further, the concurrency point is a well-known Kimberling center. Let D be the intersection of the ray AA' with BC. Then

$$\frac{BD}{CD} = \frac{c \cdot \sin \widehat{BAA'}}{b \cdot \sin \widehat{CAA'}} = \frac{c(1 + \sin B)}{b(1 + \sin C)}$$

Using barycentric coordinates with respect to $\triangle ABC$ at this point, we obtain:

$D\left(0 : \dfrac{b}{1 + \sin B} : \dfrac{c}{1 + \sin C}\right)$, or equivalently $D\left(0 : \dfrac{1}{ac + S} : \dfrac{1}{ab + S}\right)$

AA', BB', CC' concur at $X_{1123}\left(\dfrac{1}{bc + S} : \dfrac{1}{ac + S} : \dfrac{1}{ab + S}\right)$, Paasche point.

• This gives us a particular construction of k_A, k_B and k_C denoted by ω_A, ω_B and ω_C in the attachment below and an alternate construction of X_{1123}. This is: Two congruent circles, touching externally at P_a, are tangent to BC and the opposite rays of CA, BA. Define similarly P_b, P_c, then $AP_a \equiv AA'$, $BP_b \equiv BB'$ and $CP_c \equiv CC' \Longrightarrow X_{1123} \equiv AP_a \cap BP_b \cap CP_c$

CA, BA. Define similarly Γ'_b, Γ'_c, then $A\Gamma'_a = AA''$, $B\Gamma'_b = BB''$ and $C\Gamma'_c = CC'' \Longrightarrow X_{1123} = A\Gamma'_a \cap B\Gamma'_b \cap C\Gamma'_c$.

Attachments:
X1123.pdf (33kb)

This post has been edited 2 times. Last edited by Luis González, Mar 11, 2015, 10:23 am

sunken rock
#3 Jun 13, 2010, 4:45 pm

See here too:
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=202363

Best regards,
sunken rock

Luis González
#4 Jun 13, 2010, 9:36 pm

> *sunken rock wrote:*
> See here too: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=202363

Dear Stan, you have misread the problem. Circles k_A, k_B, k_C are not the mixtilinear incircles, but k_A is tangent to AB, AC and the circle with diameter BC (externally) at A'. Similarly for k_B, k_C.

Luis González
#5 Jun 14, 2010, 10:02 am

There exists at most four circles tangent to the rays AB, AC and the circle Γ_a with diameter AB. Two circles $(K), (K')$ touch Γ_a externally at A', A'', but the inversion through pole A with power $p(A, \Gamma_a)$ takes Γ_a into itself and $(K) \mapsto (K') \Longrightarrow \eta_a \equiv AA' \equiv AA''$. As it was proved before, for this case we have $X_{1123} \equiv \eta_a \cap \eta_b \cap \eta_c$. Likewise, two circles $(T), (T')$ touch Γ_a internally at A_0, A_1, but $\tau_a \equiv AA_0 \equiv AA_1$. For this case, analogous calculations show that $X_{1336} \equiv \tau_a \cap \tau_b \cap \tau_c$.

mathreyes
#6 Feb 20, 2013, 3:27 am

Al first glance, this interesting property might be tackled by Monge's theorem considerations.

Actually I haven't a solid idea, but the drawing by Luis makes me think that X_{1123} is an insimilicenter.

Quick Reply

High School Olympiads

Two nice parallels (own)

Reply

jayme
#1 Jun 13, 2010, 12:58 pm

Dear Mathlinkers,
let ABC be a triangle, M the midpoint of BC, I the incenter, V the meetpoint of MI and the A-altitude,
Ib the B-excenter, A', B' the points of contact of the B-excercle with BC, CA,
U the meetpoint of A'Ib and MI, W the meetpoint of UC and B'Ib.
Prove: VW is parallel to BC.
A synthetic proof would be nice.
Sincerely
Jean-Louis

Luis González
#2 Jun 14, 2010, 6:57 am

We continue with the trend of the topic Orthocenter. Circumcenter S of $\triangle ACI_b$ is the midpoint of $II_b,$ since A,C lie on the circle with diameter $II_b.$ Distance from W to C is twice the distance from S to $AI_b.$ If N denotes the midpoint of $AI_b,$ we have that $NS=\frac{1}{2}AI=\frac{1}{2}CW,$ but since $AIDV$ is a parallelogram, $DV=CW=\frac{1}{2}SN.$ Together with $DV \parallel CW,$ we deduce that $VDCW$ is a parallelogram $\Longrightarrow VW \parallel BC.$

Quick Reply

High School Olympiads

Orthocenter (Own)

Reply

jayme
#1 Jun 13, 2010, 1:53 pm

Dear Mathlinkers,
let ABC be a triangle, M the midpoint of BC, I the incenter,
Ib the B-excenter, A', B' the points of contact of the B-excercle with BC, CA,
U the meetpoint of A'Ib and MI, W the meetpoint of UC and B'Ib.
Prove: W is the orthocenter of ACIb.
A synthetic proof would be nice.
Sincerely
Jean-Louis

Luis González
#2 Jun 14, 2010, 6:39 am

Let D,F be the tangency points of the incircle (I) and A-excircle (I_a) with BC. Antipode D' of D WRT (I) lies on AF and M is also midpoint of $DF \Longrightarrow IM$ is the D-midline of $\triangle DFD'$. Hence, $AVID'$ is a parallellogram, i.e. $AV = ID' = ID \Longrightarrow AIDV$ is a parallelogram $\Longrightarrow VD \parallel AI$, i.e. $VD \perp AI_b$. From the topic Two interesting parallels, we know that $VD \parallel UC$, thus $UC \perp AI_b$. As a result, $W \equiv I_bB' \cap UC$ is the orthocenter of $\triangle ACI_b$.

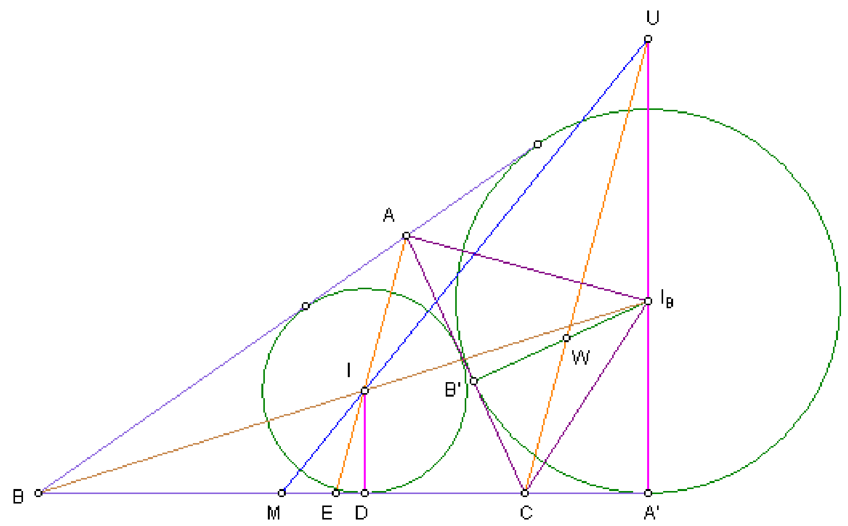
77ant
#3 Jun 14, 2010, 9:31 am

Dear jayme. Here my poor proof is.

From $\triangle MID \sim \triangle MUA'$, $UA' = \left(\frac{b+c}{c-b}\right) r$. From $\frac{ED}{ID} = \frac{CA'}{UA'}$ (a little calculation) , $\triangle IDE \sim \triangle UA'C$.

$\therefore AE \parallel UC$ and $UC \perp AI_b$.

Attachments:



Quick Reply

High School Olympiads

a fixed point and the locus

Reply

Source: Indonesia TST 2009 Second Stage Test 2 P3

Raja Oktovin

#1 Apr 5, 2009, 5:31 pm

Let C_1 be a circle and P be a fixed point outside the circle C_1. Quadrilateral $ABCD$ lies on the circle C_1 such that rays AB and CD intersect at P. Let E be the intersection of AC and BD.
(a) Prove that the circumcircle of triangle ADE and the circumcircle of triangle BEC pass through a fixed point.
(b) Find the the locus of point E.

livetolove212

#2 Apr 5, 2009, 8:40 pm

> Raja Oktovin wrote:
> Let C_1 be a circle and P be a fixed point outside the circle C_1. Quadrilateral $ABCD$ lies on the circle C_1 such that rays AB and CD intersect at P. Let E be the intersection of AC and BD.
> (a) Prove that the circumcircle of triangle ADE and the circumcircle of triangle BEC pass through a fixed point.
> (b) Find the the locus of point E.

a, Let PK, PQ are two tangents from P to $(C_1), PO \cap KQ = \{H\}, CH \cap (C_1) = \{J\}, BH \cap (C_1) = \{F\}$
We have $HA.HO = HK.HQ = HC.HJ$
so PJOC is cyclic we get $\angle OJH = \angle OPC (1)$
On the other hand, $OH.OP = R^2 = OD^2$ then $\frac{OH}{OD} = \frac{OD}{OA}$
therefore $\Delta HOD \sim \Delta DOP$ then $\angle OPC = \angle ODH (2)$
From (1) and (2) we have $\angle HJO = \angle HDO$ so $HO \perp JD$
therefore $JD//KQ$
Similarly, $AF//KQ//JD$ then $arcAD = arcJF$
We get $\angle BHC = \angle BEC$ then BHEC is cyclic, similarly AHED is cyclic
So the circumcircle of triangle ADE and the circumcircle of triangle BEC pass through a fixed point H
b, We have $180^o - \angle BHE = \angle BCE = 1/2(arcBK + arcKA) = 1/2(arcBK + arcQF)$
$= \angle BHK$
Then K,H,E is collinear
So $E \in [KQ]$

Luis González

#3 Jun 13, 2010, 5:09 am

Let O be the center of C_1 and τ be the polar of P WRT C_1 (Locus of E), $F \equiv OP \cap \tau$. Denote ω_1, ω_2 the circumcircles of $\triangle EBC, \triangle PBC$, respectively. Line $AD \equiv a$ is both image of ω_1 and ω_2 through the inversions with poles E, P and powers $p(E, C_1), p(P, C_1)$, respectively. Hence, by conformity $\angle(a, C_1) = \angle(\omega_1, C_1) = \angle(\omega_2, C_1)$. Since B, C are double points in the inversion $\mathcal{I}$ about $C_1(O)$, then ω_1 and ω_2 are homologous under $\mathcal{I}$. Keeping in mind that $\mathcal{I} : F \mapsto P$, it follows that $F \in \odot(BEC)$. By analogous reasoning we get that $F \in \odot(AED)$.

Quick Reply

Spain

geometria

Reply

Tony2006
#1 May 29, 2007, 9:24 am

Demuestra que si se construyen cuadrados exteriormente sobre los lados de un trapecio isosceles, entonces el cuadrilatero formado por los cuatro centros de los cuadrados tiene diagonales de igual longitud y perpendiculares.

cuenca
#2 May 30, 2007, 8:52 am

es mas general: para todo cuadrilatero(no necesariamente convexo)

Jutaro
#3 May 30, 2007, 9:01 am

> *cuenca wrote:*
> es mas general: para todo cuadrilatero(no necesariamente convexo)

para ser exacto ese es el teorema de van aubel: http://mathworld.wolfram.com/vanAubelsTheorem.html

fabycv
#4 Jun 3, 2007, 7:04 am

considere un semicirculo de centro O y de diametro AB . Una linea interseca a AB en M y al semicirculo en C y D tal que $MB < MA$ y $MD < MC$. Los circuncirculos de los triangulos de AOC y DOB se intersecan en el segundo punto k . Demuestre que MK y KO son perpendiculares.

resuelvanlo
faby

conejita
#5 Jun 5, 2007, 9:39 pm

Sugerencia al problema propuesto por fabycv:
Invertir con centro O y radio OA. Un problema muy bonito!!!

aev5peru
#6 Jun 12, 2007, 9:46 am

> *fabycv wrote:*
> considere un semicirculo de centro O y de diametro AB . Una linea interseca a AB en M y al semicirculo en C y D tal que $MB < MA$ y $MD < MC$. Los circuncirculos de los triangulos de AOC y DOB se intersecan en el segundo punto k . Demuestre que MK y KO son perpendiculares.
>
> resuelvanlo
> faby

Considerando al semi circulo como circunferencia de inversion (Circuferencia de magnus), las rectas BD y AC se invierten en las circunferencias que pasan por BDO y ACO respectivamente , donde el segundo punto de interseccion es K, entonces las rectas AC Y BD deben de cortarse pues este punto sera el inverso de K, llamemosle K', luego sea T el punto de interseccion de AD con BC, R1 y R2 los puntos de tangencia de las rectas trasadas desde K' a la semiscircunferencia, entonces la polar de M es TK', , La polar de R1 es K'R1, de R2 es K'R2, como concurren las polares entonces M, K1, K2, son colineales. Por otra parte como K es el inverso de K', entonces R1, R2, K soncolineales, entonces M, K, R1, R2, COLINEALES, entonces <NKK' =90.

Luis González

#7 Apr 18, 2009, 8:40 am

De hecho, existe una generalización para n-gonos cuya demostración es algo dificil, la solución que pude observar usaba números complejos. Adicional al teorema de Van Aubel, si se hace esta construcción interiormente entonces también resulta otro cuadrado. El área de ABCD es entonces igual a la diferencia del área de estos dos cuadrados y a este resultado se le llama (por analogía al triángulo) teorema extendido de Napoleón.

Luis González

#8 Apr 27, 2010, 4:18 am

> *fabycv wrote:*
> Considere un semicírculo de centro O y de diámetro $AB.$ Una linea interseca a AB en M y al semicírculo en C y D tal que $MB < MA$ y $MD < MC.$ Los circuncírculos de los triángulos de AOC y DOB se intersecan en el segundo punto $K.$ Demuestre que MK y KO son perpendiculares.

La inversión en el círculo con centro O y radio OA transforma a las circunferencias $\odot(OCA)$ y $\odot(ODB)$ en las rectas AC, DB y a la recta DC en la circunferencia $\mathcal{N} \equiv \odot(ODC).$ $P \equiv AC \cap BD$ y $E \equiv \mathcal{N} \cap AB$ diferente de O son pues los inversos de $K, M,$ pero por ser D, C las proyecciones ortogonales de A, B en PB, PA y O el punto medio de $AB,$ se sigue que $\mathcal{N}$ es el círculo de los 9 puntos de $\triangle PAB \Longrightarrow PE \perp BA.$ Luego como E, M y K, P son pares inversos, entonces los puntos P, K, E, M son concíclicos $\Longrightarrow MK \perp OK.$

Luis González

#9 Jun 12, 2010, 11:32 pm

El problema anterior propuesto, puede ser facilmente generalizado:

A, B, C, D son cuatro puntos en una circunferencia $(O).$ Los circuncírculos de $\triangle ODA$ y $\triangle OCB$ se cortan en O y $K.$ Si $M \equiv DC \cap AB,$ entonces probar que $MK \perp OK.$

La inversión en (O, R) transforma a los círculos $\odot(ODA), \odot(OCB)$ en las rectas DA, CB y por tanto $P \equiv AD \cap BC$ es el inverso de $K \Longrightarrow P$ yace en $OK.$ Si OK corta a (O) en $L, N,$ resulta que $R^2 = OL^2 = ON^2 = OK \cdot OP \Longrightarrow (L, N, O, P) = -1,$ por tanto MK es la polar de P con respecto a $(O) \Longrightarrow MK \perp OK.$

Quick Reply

High School Olympiads

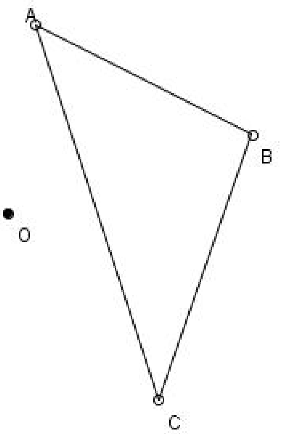
the foci of the conic

Reply

jrrbc

#1 Nov 7, 2009, 5:21 am

Attachments:


Find the foci of the conic centred at the given point O and circumscribed to the given triangle ABC

castigioni

#2 Jun 9, 2010, 8:37 pm

Is the B angle right?

Luis González

#3 Jun 12, 2010, 7:07 am

Reflections P, Q, R of A, B, C about O obviously lie on the wanted conic $\mathcal{H}$. Isogonal conjugation WRT $\triangle ABC$ takes circumcircle $\mathcal{K}$ of $\triangle ABC$ into the line at infinity and $\mathcal{H}$ into a line η passing through the isogonal conjugates P^{-1}, Q^{-1} of P, Q WRT $\triangle ABC$. So, the fourth intersection U of $\mathcal{K}, \mathcal{H}$ is the isogonal conjugate of the infinite point of η, this is: Draw the parallel τ from A to η, the reflection of τ across the angle bisector of $\angle bc$ cuts $\mathcal{K}$ again at U. If $D \equiv BC \cap AU$, then angle bisectors u, v of $\angle(AD, BC)$ are parallel to the axes of the conic $\Longrightarrow$ parallels from O to u, v are the corresponding focal axis f and minor axis e.

Construct a tangent line to $\mathcal{H}$ by using Pascal theorem. For instance, by Pascal theorem for the non-convex hexagon $PPABCR$, the intersections $X \equiv PP \cap BC, O \equiv PA \cap CR$ and $Y \equiv AB \cap RP$ are collinear $\Longrightarrow \lambda \equiv XP$ is tangent to $\mathcal{H}$ at P, where $X \equiv OY \cap BC$.

Denote by F, F_0 the foci of $\mathcal{H}$. The tangent line λ and its normal $\lambda' \perp \lambda$ through P are angle bisectors of $\angle(PF, PF_0)$. Hence if λ, λ' cut f at M, N, then $(F, F_0, M, N) = -1 \Longrightarrow \overline{OF}^2 = \overline{OF_0}^2 = \overline{OM} \cdot \overline{ON} \Longrightarrow$ circle ω centered at O whose radius equals the geometric mean of the known segments OM, ON cuts f at F, F_0.

Quick Reply

High School Olympiads

Two pairs of lines and circumcircle of ABC (own).

Reply

Virgil Nicula

#1 Jun 11, 2010, 9:40 am

Let ABC be a triangle with the circumcircle w . Denote $\left\|\begin{array}{ccc} D\in (BC) & , & \widehat {DAB}\equiv\widehat {DAC}\\\\ S\in AD\cap w & , & S\ne A\end{array}\right\|$ and the midpoints M , N , P of (AB) , $AD)$, (AC) respectively. Prove that $(SM\cap CN)\cup (SP\cap BN)\subset w$.

This post has been edited 1 time. Last edited by Virgil Nicula, Jun 11, 2010, 10:26 am

Luis González

#2 Jun 11, 2010, 10:08 am

Tangent line τ of ω at S is parallel to $BC.$ Thus, let A_{∞} be the infinity point of $BC \parallel \tau.$ By Pascal theorem for hexagon $SSACBU,$ where $U \equiv SP \cap \omega,$ the intersections $A_{\infty} \equiv BC \cap \tau,$ $N' \equiv SA \cap BU$ and $P \equiv AC \cap US$ are collinear $\Longrightarrow PN' \parallel BC \Longrightarrow N \equiv N' \Longrightarrow U \equiv BN \cap SP \in \omega.$ Similarly, if $V \equiv SM \cap \omega,$ different from $S,$ by Pascal theorem for $SSABCV$ we'll have $V \equiv CN \cap SM \in \omega.$

This post has been edited 1 time. Last edited by Luis González, Jun 11, 2010, 10:26 am

Virgil Nicula

#3 Jun 11, 2010, 10:25 am

My proof. Denote $X\in SM\cap CN$ and $Y\in SP\cap BN$.

$$\left\|\begin{array}{ccccc} \triangle ABS\sim\triangle ADC & \Longrightarrow & \widehat {ASM}\equiv\widehat {ACN} & \Longrightarrow & X\in w\\\\ \triangle ACS\sim\triangle ADB & \Longrightarrow & \widehat {ASP}\equiv\widehat {ABN} & \Longrightarrow & Y\in w\end{array}\right\|\ \Longrightarrow\ \{X,Y\}\subset w\ .$$

sunken rock

#4 Jan 5, 2011, 12:14 am

Generalization: The problem is valid for any M, N, P collinear and $MP \parallel BC$!

1) $AB\cdot AC=AD\cdot AS$ (1) ($\triangle ABD\sim\triangle ASC$).
2) $\frac{AM}{AB}=\frac{AN}{AD}=\frac{AP}{AC}=k$, i.e. $AM=k\cdot AB$, etc, divide (1) by k, get:
$AM\cdot AC=AN\cdot AS$, i.e. $\triangle AMS\sim\triangle ANC$, hence $\angle ASM=\angle ACN$, similarly for SP and BN.

Best regards,
sunken rock

Quick Reply

High School Olympiads

collinearity of strange defined points.

Reply

immodestius

#1 Jun 11, 2010, 4:18 am

Let ABC be a triangle witch circumcircle (O). The medians AA', BB' and CC' cut (O) in A and A'', B and B'', C and C'', resectively. The line of tangency to (O) through A'' meets the perpendicular to AO through A' in X. Define Y and Z similar. Prove that X, Y and Z are collinear.

Luis González

#2 Jun 11, 2010, 5:45 am

Centroid $G \equiv AA' \cap BB' \cap CC'$ of $\triangle ABC$ is the insimilicenter between its circumcircle (O) and 9-point circle (N). A, A' are homologous under such homothety $\Longrightarrow$ tangent η of (N) at A' is parallel to the tangent τ of (O) at $A \Longrightarrow \eta$ is antiparallel to $BC \Longrightarrow A'X \equiv \eta$. The lines τ, XA'' and AA'' bound an isosceles triangle, therefore $\triangle XA'A''$ is isosceles with apex X $\Longrightarrow X$ has equal power with respect to $(O), (N) \Longrightarrow X$ lies on the radical axis of $(O), (N)$, i.e. the orthic axis of $\triangle ABC$. Analogously, Y, Z lie on the orthic axis of $\triangle ABC$.

immodestius

#3 Jun 11, 2010, 3:17 pm

Nice Solution!

Quick Reply

High School Olympiads

Concurrent Lines

Reply

trigg

#1 Jun 11, 2010, 1:44 am

Let **ABCD** be a convex quadrilateral.Suppose that there exists a circle **w** tangent to ray **BA** beyond **A** and to the ray **BC** beyond **C** , which is also tangent to the lines **AD** and **CD**.Let **E** be the intersection point of **BA** and **CD** and **F** be the intersection point of **BC** and **AD**.Let **K** be the intersection point of angle bisector of **EAD** and **ED** and **L** be the intersection point of angle bisector of **DCF** and **DF**.Prove that **AC** , **KL** and the angle bisector of **CDF** are concurrent.

Luis González

#2 Jun 11, 2010, 4:55 am

This merely follows from the well-known Monge & d'Alembert theorem.

Denote O the center of ω and let $(U),(V)$ be the incircles of $\triangle DAE$ and $\triangle DCF$. Then UV is the bisector of $\angle CDF$. $K \equiv UO \cap DE$ and $L \equiv VO \cap DF$ are the insimilicenters of $(U),(O)$ and $(V),(O)$, while A,C are their corresponding exsimilicenters. By Monge & d'Alembert theorem, AC,KL,UV concur at the exsimilicenter of $(U),(V)$.

Quick Reply

High School Olympiads

I hope some one can solve this problem

Reply

dap

#1 Jun 10, 2010, 3:56 pm

Given a triangle ABC. (I) is the incircle of ABC.(I) meets BC at A1. AA2 is the altitude of ABC and A3 is the midpoint of AA2. Similarly,we have B1B3,C1C3. Prove that A1A3,B1B3,C1C3 are concurrent

Luis González

#2 Jun 10, 2010, 6:59 pm

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=331144
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=265276
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=42412
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=48453

jayme

#3 Jun 10, 2010, 9:19 pm

Dear Mathlinkers,
after proving like Luis that A1A3 goes through the A-excenter of ABC,
I think that we have the result by considering the triangle ABC, the excentral triangle of ABC and the incentral triangle of ABC and applying the cevian nest theorem...
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Equality of angles [Iran TST 2010]

Reply

Omid Hatami

#1 Jun 9, 2010, 7:06 am • 6

Circles W_1, W_2 intersect at P, K. XY is common tangent of two circles which is nearer to P and X is on W_1 and Y is on W_2. XP intersects W_2 for the second time in C and YP intersects W_1 in B. Let A be intersection point of BX and CY. Prove that if Q is the second intersection point of circumcircles of ABC and AXY

$$\angle QXA = \angle QKP$$

This post has been edited 1 time. Last edited by Omid Hatami, Jun 10, 2010, 3:10 am

frenchy

#2 Jun 9, 2010, 4:22 pm

do you mean the intersection of the circumscribed cercles of ABC and AXY??

goodar2006

#3 Jun 9, 2010, 11:01 pm • 2

up to this point I have reached that $AXKC$ and $AYKB$ are concyclic . is it useful ?

Luis González

#4 Jun 10, 2010, 2:45 am • 9

Let D be the second intersection of $\odot(ABC)$ with ω_1 and E the second intersection of DC with ω_1. For convenience, denote $\angle AXY = \angle AYX = \theta$.

By Reim's theorem for $\odot(ABC)$ and ω_1 with common chord BD, it follows that $XE \parallel AC \Longrightarrow \angle BXE = \pi - 2\theta$. Since $\angle BDX = \theta$, then $\angle XDE = \angle XYA = \theta \Longrightarrow X, Y, C, D$ are concyclic. Therefore, AQ, XY, DC concur at the radical center O of $\odot(AXY), \odot(ABC)$ and $\odot(XYC)$, which becomes the exsimilicenter of ω_1, ω_2, since O is also center of the positive inversion taking ω_1, ω_2 into each other. Thus, the tangent of $\odot(XYK) \equiv \omega_3$ at K passes through O. Further, AB, AC are tangent lines of ω_3 at points X, Y. Then KA is the polar of O WRT ω_3 cutting ω_3 again at R such that OR is tangent to ω_3 at R. If $P \equiv AK \cap XY$, then the cross ratio (X, Y, P, O) is harmonic.

Consequently, if $M \equiv PK \cap XY$ (midpoint of XY), it follows that $OR^2 = OK^2 = OQ \cdot OA = OP \cdot OM$. Since K, P, R, A are collinear, being K,R double points, K, M, R, Q, O are concyclic, yielding $\angle QKP = \angle QOY$. But because of $AY^2 = AR \cdot AK = AQ \cdot AO$, we deduce that $\angle QOY = \angle QYA = \angle QXA \Longrightarrow \angle QXA = \angle QKP$.

RaleD

#5 Feb 3, 2011, 4:24 am • 2

> goodar2006 wrote:
> up to this point I have reached that $AXKC$ and $AYKB$ are concyclic . is it useful ?

This is useful in my solution. After noticing this and that $AX = AY$ we make an inversion with center in A. After some angle chasing we actually got following problem:
Let tangents in K, X, Y on circumcircle of KXY intersect sides XY, KY, KX in Q, C, B respectively. Show that Q, C, B are collinear
(in problem we have that Q, B, C are collinear and need to get that QK is tangent but this is same

I don't have nice explanation for this, however it follows from Ceva's theorem.

goodar2006

#6 Feb 3, 2011, 9:06 pm • 1

> *RaleD wrote:*
>
> > *goodar2006 wrote:*
> > up to this point I have reached that $AXKC$ and $AYKB$ are concyclic . is it useful ?
>
> This is useful in my solution. After noticing this and that $AX = AY$ we make an inversion with center in A. After some angle chasing we actually got following problem:
> *Let tangents in K, X, Y on circumcircle of KXY intersect sides XY, KY, KX in Q, C, B respectively. Show that Q, C, B are collinear*
> (in problem we have that Q, B, C are collinear and need to get that QK is tangent but this is same
> I don't have nice explanation for this, however it follows from Ceva's theorem.

beautiful solution! thanks.

paul1703

#7 Feb 13, 2011, 3:14 am

> *frenchy wrote:*
> do you mean the intersection of the circumscribed cercles of ABC and AXY??

actualy the point you have reached with the cyclic quadrilaterals folows from the fact that k is the miquel point of the quadrilateral AXPY

vladimir92

#8 Feb 13, 2011, 11:56 pm • 6

As always, Iran have nice problems to propose for contestants.
My solution

wsjradha

#9 Apr 21, 2011, 1:25 am • 15

Here is a solution with spiral similarity.

Note that K is the center of spiral similarity that takes BX to YC. In particular, $BXK \sim YCK$. But, we know that
$\angle KYC = \angle KBX = \angle KXY$
since XY is tangent to ω_1. Similarly, $\angle KXB = \angle KCY = \angle KYX$. Thus, we have that
$BXK \sim XYK \sim YCK$,
and so there is a spiral similarity at K that takes BX to XY to YC. Let $\angle BKX = \angle XKY = \angle YKZ = \theta$. Then, we have
$\angle BAC = 180 - \angle AXY - \angle AYX = 180 - \theta - \theta = 180 - 2\theta$.
Let M, N be the midpoints of BX, CY, respectively. Then, the spiral similarity at K taking BX to YC takes M to N, or
$\angle MKN = \angle BKY = 2\theta$.
Since $\angle MAN = \angle BAC = 180 - 2\theta$, we have that $AMKN$ is a cyclic quadrilateral.

We know that Q is the center of spiral similarity that takes BX to CY, so $\angle MQN = \angle XQY = \angle XAY = 180 - 2\theta$.
Thus, the five points A, M, K, N, Q are concyclic.

Let $PK \cap XY = Z$. Then, by radical-axis, we know that Z is the midpoint of segment XY. By the spiral similarity at K, we know that $\angle ZKN = \theta$, or
$\angle QKP = \theta - \angle QKN$.
But, we have
$\angle QKN = \angle QAN = \angle QAY = \angle QXY = \angle QAX - \angle QXA = \theta - \angle QXA$.
It follows that $\angle QXA = \angle QKP$, as desired.

vntbqpqh234
#10 Jun 2, 2011, 9:58 pm • 6

That is my proof.
Let $(O), (O_4)$ be the circumcircle of $\triangle KXY, \triangle AXY$
And M is the intersection of QO with XY
KP meets XY at N.
Easy to see:
$\angle AXY = \angle AYX = \angle XKY$
then AX, AY touch with $(O$ then
O lies on O_4 and O, A, N conllinear.
And KA is symmedian of $\triangle XKY$
On the other hand: $\triangle KBX, KXY, KYC$ are similar
then:
$\frac{XM}{MY} = \frac{QX}{QY} = \frac{BX}{BY} = \frac{KX^2}{KY^2}$
Hence K, M, A is conllinear.
Easy to see M, N, Q, A lie on circle.(*)
Have $OK^2 = OX^2 = ON.OA$ (**)
when (*) and (**) have
$\angle OQN = \angle KAO = \angle OKN$ then K, O, N, Q lie on a circle
Hence $\angle QKA = \angle QOA = \angle QXA$
QED

dragon96
#11 Jul 16, 2011, 6:05 am

> *wsjradha wrote:*
> We know that Q is the center of spiral similarity that takes BX to CY, so $\angle MQN = \angle XQY = \angle XAY = 180 - 2\theta$. Thus, the five points A, M, K, N, Q are concyclic.

Sorry to revive, but I'm not following why Q is the center of spiral similarity. Also, how does the fact that $\angle MQN = \angle XQY$ follow from that? Thanks.

EDIT: Nevermind.

antimonyarsenide
#12 Apr 3, 2012, 9:41 pm • 1

It's amazing how many geometry problems this paper helps you solve:

http://web.mit.edu/yufeiz/www/olympiad/cyclic_quad.pdf

Unfortunately I'd forgotten almost everything in there, so it was hard getting this problem to work D:

Pedram-Safaei
#13 Apr 4, 2012, 1:23 am

use of the two centers of rotational homothety and two midpoints of two side...

sayantanchakraborty
#14 Oct 17, 2014, 12:54 am

There is also a nice solution using inversion.Just invert with center A and arbitrary radius.Let X' be the image of X under the inversion.Then $X'P'KB'$ and $KPY'C'$ are cyclic.So
$\angle X'P'Y' = \angle X'P'K' + \angle Y'PK = \angle K'B'A + \angle K'C'A = \angle BKA + \angle CKA = 180 - \angle BAC$

so $AX'PY'$ is also cyclic.Now some trivial steps yeild you $\angle AQ'X' = \angle AQ'K' + \angle AP'K'$ which means $\angle QXA = \angle QKP$ as desired.

stephcurry

#15 Jun 5, 2015, 10:42 pm

dang this question took me forever, but it was nice

First, WLOG let Q be closer to Y than X. Let $\angle BKX = \angle 1, \angle XBY = \angle 2,$ and $\angle XCY = \angle 3$. Since K is the second intersection of the circumcircles of $\triangle PXB$ and $\triangle PYC$, there is a spiral similarity centered at K that takes $\triangle KBX$ to $\triangle KYC$. Notice that this spiral similarity also takes $\triangle KBY$ to $\triangle KXC$, so those triangles are similar as well. Thus, $\angle KYC = \angle 1$ as well. By angle chasing, we find that $\angle CXY = \angle 2$, $\angle BYX = \angle 3$, and $\angle AXY = \angle AYX = \angle 1$. By exterior angles of $\triangle BXY$, we see that $\angle 2 + \angle 3 = \angle 1$. Thus, $\angle XKY = \angle XKP + \angle PKY = \angle 2 + \angle 3 = \angle 1$. Also, we have $\angle KXY = \angle KXC + \angle CXY = \angle KBY + \angle 2 = \angle KBX$, so $\triangle KBX \sim \triangle KXY$, and the spiral similarity that sends $\triangle KBY$ to $\triangle KXC$ also sends $\triangle KBX$ to $\triangle KXY$. It also sends $\triangle KXY$ to $\triangle KYC$ by similar reasoning.

Since Q is the second intersection of the circumcircles of $\triangle AXY$ and $\triangle ABC$, Q is the center of spiral similarity that sends $\triangle QXY$ to $\triangle QBC$, which implies that $\triangle QXY \sim \triangle QBC$ are similar. Since quadrilateral $AQYX$ is cyclic, $\angle XQY = \angle XAY = 180 - 2\angle 1$.

Let M and N be midpoints of BX and CY, respectively.

Lemma 1: Quadrilateral $QMKN$ is cyclic
Proof: By MGT, spirally similar triangles QXY and QBC are similar to $\triangle QMN$, so $\angle MQN = \angle$XQY$= 180 - 2\angle 1$. By MGT again, spirally similar triangles KBY and KXC are similar to $\triangle KMN$ are similar, so $\angle MKN = \angle BKY = \angle BKX + \angle XKY = 2\angle 1$. This implies that quad $QMKN$ is cyclic because $\angle MQN + \angle MKN = 180$.

Thus by Lemma 1, we have $\angle QKN = \angle QMN = \angle QXY$. By angle chasing, $\angle QKP = \angle PKN - \angle QKN = \angle PKN - \angle QXY$. Now we extend KP to intersect XY at a point Z.

Lemma 2: Z is the midpoint of XY
Proof: As KP is the radical axis of circles W_1 and Q_2, Z lies on the radical axis. This implies that it has equal power with respect to both circles, so $ZX^2 = ZY^2$, which implies that $ZX = ZY$, so Z is the midpoint of XY. Finally we have $\triangle KXZ \sim \triangle KYN$, where Z is midpoint of XY.

By Lemma 2, the spiral similarity that takes $\triangle KXY$ to $\triangle KYC$ also takes Z to N, as both are corresponding midpoints, so $\triangle KZN \sim \triangle KYC$ by spiral similarity. Thus, $\angle ZKN = \angle YKC = \angle 1$, so $\angle QKP = \angle 1 - \angle QXY = \angle QXA$, as desired. QED

tranquanghuy7198

#16 Jun 6, 2015, 8:48 pm • 2

My solution:
Notice that $\triangle QXB \sim \triangle QYC$ (same direction)
We construct $\triangle QKL \sim \triangle QXB \sim \triangle QYC$ (same direction)
$\Rightarrow \exists S_{(Q,\alpha,k)}$ which is a spiral similarity that maps:
$X \mapsto B, Y \mapsto C, K \mapsto L$
$\Rightarrow \triangle BCL \sim \triangle XYK$
$\Rightarrow \angle BLC = \angle XKY = 180 - \angle XPY = 180 - \angle BPC$
$\Rightarrow BPCL$ is cyclic
$\Rightarrow \angle CPL = \angle CBL = \angle YXK = \angle CPK$ (because XY is tangent to ω_1)
$\Rightarrow \overline{P, K, L}$
$\Rightarrow \angle QKP = 180 - \angle QKL = 180 - \angle QXB = \angle QXA$
Q.E.D

gavrilos

#17 Aug 23, 2015, 3:41 pm

My solution:

Let T be the second intersection point of W_2 and the circumcircle of $\triangle ABC$.Also,suppose that $R \equiv TY \cap AB$.

We will first show that $\triangle AXY$ is isosceles.

Indeed,$\angle AXY = 180^\circ - \angle BXY = 180^\circ - \angle PXY - \angle BXP = 180^\circ - \angle PBX - \angle BXP = \angle BPX = \angle CP$

$= 180^\circ - \angle PCY - \angle PYC = 180^\circ - \angle PYX - \angle PYC = 180^\circ - \angle XYC = \angle AYX$ q.e.d.

We will go on with showing that $XYTB$ is cyclic.Indeed,$\angle CTY = 180^\circ - \angle YPC = 180^\circ - \angle AXY$

and $\angle CTB = \angle CAB = \angle YAX = 180^\circ - 2\angle AXY$.

Thus,$\angle YTB = \angle CTY - \angle CTB = 180^\circ - \angle AXY - (180^\circ - 2\angle AXY) = \angle AXY$ which is what we wanted.

Hence,$RY \cdot RT = RX \cdot RB$ implying that R has equal powers wrt W_1, W_2.Thus,it lies on their radical axis which is PK.

Thus,P, K, R are collinear.We have $\angle QTR = \angle QTY = \angle QTC - \angle CTY =$

$= 180^\circ - \angle QAY - (180^\circ - \angle YPC) = \angle YPC - \angle QAY = \angle AXY - \angle QAY = \angle QXA = \angle QXR$

whence we get that $QRXT$ is cyclic (1).

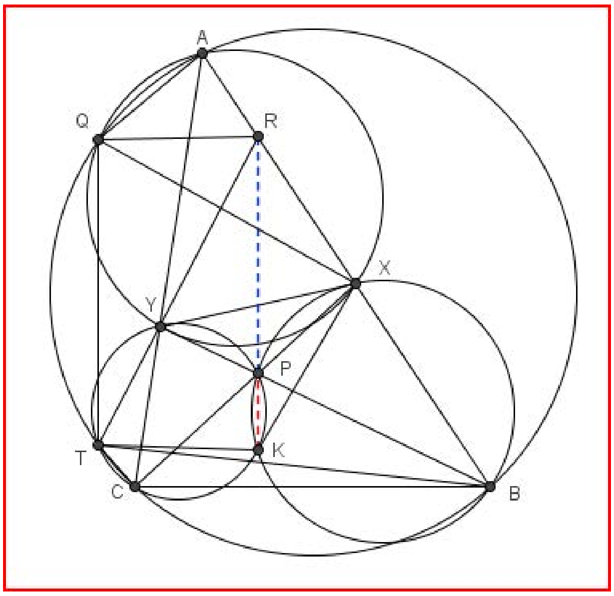
Also,$\angle QTK = \angle QTY + \angle YTK = \angle QXA + \angle KPB =$

$= \angle QXA + \angle KXB = 180^\circ - \angle QXK$ which gives that $QXKT$ is also cyclic (2).

From $(1), (2)$ we conclude that Q, R, X, K, T all lie on the same circle.

Thus,$\angle QXA + \angle QXR = \angle QKR = \angle QKP$ which ends the proof.

Attachments:



SmartClown

#18 Feb 12, 2016, 9:13 pm

By easy angle chasing we get $BKYA$ and $CKXA$ are cyclic. Now let $AQ \cap XY = L$ and $PK \cap XY = R$ and R is the midpoint of XY. Notice that by inversion at A and radius $\sqrt{XY}$ point Q goes to L so we have $\angle QXA = \angle QLR$ so we need to prove that $QLKR$ is cyclic so it is enough to prove that $\angle KRY = \angle AQK$. Let $\angle XYP = \angle YKR = x$ and $\angle XYP = \angle XKR = y$. Because AX and AY are tangents to the circumcircle of KXY we have KA is the symmedian in $\triangle KXY$. So we have $\angle AKY = y$. Now let the tangent at K of the circumcircle KXY cut AY at S. We get $\angle SKA = \angle YRK$ so we need to prove that $\angle SKA = \angle AQK$ which is equivalent to SK being the tangent to the circle AKQ so we need to prove that circumcircles KXY and KQA are tangent to each other. Now we invert at A. Now we let the picture of some point X be X_1. Now K_1 is the point inside the isosceles triangle AXY such that $\angle AY_1K_1 = \angle K_1X_1Y_1$ and the equivalent when we switch X_1 and Y_1. Now because $BKYA$ and $XKCA$ are cyclic we have that $B_1 = AX_1 \cap K_1Y_1$ and $C_1 = AY_1 \cap X_1K_1$ and because $AQBC$ and $AQXY$ are cyclic we have $Q_1 = X_1Y_1 \cap B_1C_1$. Now we only need to prove that Q_1K_1 is tangent to circumcircle XKY. Now because B is the intersection of one side of the triangle and the tangent and the same holds for C_1 and we need to prove that the same holds for Q_1 it is enough to prove that intersections of tangents at K, X, Y and XY, KY, KX respcetively in any triangle KXY are collinear and it follows from easy Menelaus so we are finished.
These angles are like this in my configuration but the idea is the same in any.

Abubakir

#19 Mar 3, 2016, 9:13 pm

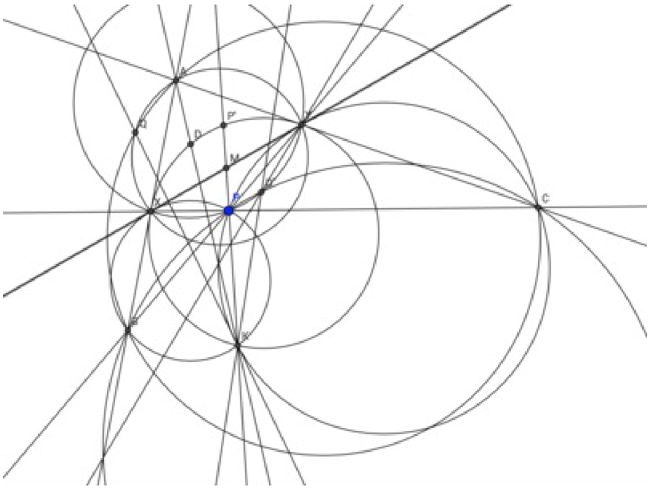
> *RaleD wrote:*

Let tangents in K, X, Y on circumcircle of KXY intersect sides XY, KY, KX in Q, C, B respectively. Show that Q, C, B are collinear
(in problem we have that Q, B, C are collinear and need to get that QK is tangent but this is same
I don't have nice explanation for this, however it follows from Ceva's theorem.

Actually, this called Pascal's theorem for triangle. Beautiful solution!

ABCDE

#20 Apr 8, 2016, 8:16 am



First, note that $\angle XKY = \angle XKP + \angle PKY = \angle XBP + \angle YCP = \angle XBP + \angle XYP = \angle AXY$ and similarly $\angle XKY = \angle AYX$, so the circumcircle of XKY is tangent to AX and AY. Let I be the incenter of ABC. Note that $\angle BPC = 180^\circ - \angle XPB = 180^\circ - \angle AXY = 90^\circ + \angle XAY = \angle BIC$, so $BPIC$ is cyclic. This means that the circumcircle of BPC is symmetric with respect to the angle bisector of $\angle BAC$. Let M be the midpoint of XY, D be the harmonic conjugate of K with respect to X and Y (well-known that D is reflection of P across XY), P' be the reflection of P over M, and D' be the reflection of D over M. Note that $DPD'P'$ is a rectangle that is symmetric with respect to BC and the perpendicular bisector of BC.

Now, perform a $\sqrt{bc}$ inversion with respect to KXY. Note that the circumcircle of KPX maps to AY and the circumcircle of KQY maps to AX. This means that A maps to P and B maps to C. Now, Q maps to the intersection of the circumcircles of PBC and PXY. Note that both circles are symmetric with respect to the angle bisector of $\angle BAC$, so Q maps to D'. Now, we have $\angle QKP = \angle AKD'$ and $\angle QXA = \angle QXK - \angle AXK = \angle YD'K - \angle YPK = \angle PYD' + \angle PKD'$. Hence, it suffices to show that $\angle AKP = \angle PYD'$. But $\angle AKP = \angle DKP'$, and the circumcircles of XYK and XYP are reflections of each other across XY. But since $P'D$ is the reflection of PD' across XY, the corresponding arcs and thus angles are equal as desired.

Quick Reply

High School Olympiads

Application of the Pascal's theorem.

Reply

Virgil Nicula

#1 Jun 9, 2010, 10:08 pm

Let ABC be a triangle with circumcircle $w = C(O)$ and incircle (I) . The circle (I) touches BC in D and AI , AO meet again w in M , S respectively. Denote $X \in BS \cap CM$, $Z \in BM \cap CS$, $\{M, P\} = DM \cap w$, $Y \in AB \cap CP$ and $T \in BP \cap AC$. Prove that $I \in XY \cap ZT$.

This post has been edited 1 time. Last edited by Virgil Nicula, Jun 11, 2010, 8:06 pm

Luis González

#2 Jun 9, 2010, 11:25 pm

Let N be the midpoint of the arc BAC. Inversion in the circle (M) with radius $MB = MI$ swaps (O) and the sideline BC. The incenter I is double and $D \mapsto P$, thus $\odot(IDP)$ is double $\Longrightarrow MI$ is tangent to $\odot(IDP)$ at I. Hence, $\angle IPD = \angle DIM = \angle AMN = \angle APN$, but $\angle MPN$ is right since M, N are antipodal. Then $\angle API = \angle MPN = 90^\circ$, which implies that chords PS and AM meet at I. By Pascal theorem for the non-convex hexagon $BMACSP$, the intersections $Z \equiv BM \cap CS, I \equiv MA \cap SP$ and $T \equiv AC \cap BP$ are collinear $\Longrightarrow I \in ZT$. Again, by Pascal theorem for the non-convex hexagon $BAMCPS$, the intersections $Y \equiv BA \cap CP, I \equiv AM \cap PS$ and $X \equiv MC \cap SB$ are collinear $\Longrightarrow I \in XY$. As a result, $I \equiv XY \cap ZT$.

Quick Reply

High School Olympiads

Euler line through Incenter

Reply

vaibhav2903
#1 Jun 8, 2010, 11:43 pm

prove that if the Euler line passes through incenter. then,the triangle is an isosceles triangle

Luis González
#2 Jun 9, 2010, 5:07 am

If I, G, O denote the incenter, centroid and circumcenter of $\triangle ABC$ with inradius r, then:

$$\boxed{[IGO] = \frac{|(a-b)(b-c)(c-a)|}{24r}}$$

This formula has been discussed before, so you could try a search with some keywords. A straightforward proof is using areal/barycentric coordinates WRT $\triangle ABC$. As a result, $I \in OG \Longleftrightarrow [IGO] = 0 \Longleftrightarrow a = b, b = c$ or $c = a$.

yetti
#3 Jun 9, 2010, 6:46 am

$H = O^*$ is isogonal conjugate of $O \Longrightarrow$ isogonal conjugate of the Euler line $e \equiv OH$ is a hyperbola $\mathcal{E}$ through A, B, C, O, H. Suppose $I \in OH \Longrightarrow I = I^* \in \mathcal{H} \Longrightarrow \mathcal{H}$ degenerates to the lines $AOIH, BC$ or cyclic exchange $\Longrightarrow$ $AB = AC$ or cyclic exchange.

vaibhav2903
#4 Jun 9, 2010, 10:02 am

very nice proof luis

vaibhav2903
#5 Jun 9, 2010, 10:19 am

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=323445&p=1735161&hilit=area+of+IGO#p1735161

77ant
#6 Dec 5, 2010, 9:02 am

The following is not mine. I have seen it somewhere.
Assume that it's not isosceles.
AI, BI meet (O) at D, E. OI/IH=OD/AH=OE/BH. Therefore AH=BH.
Thus BC=CA. It contradict to assumption.

Quick Reply

High School Olympiads

Two interesting parallels (own)

jayme
#1 Jun 6, 2010, 3:36 pm

Dear Mathlinkers,
let ABC be a triangle, I the incenter, D the point of contact of the incircle with BC, V the meetpoint of MI and the A-altitude, Ib the B-excenter, A' the point of contact of the B-excercle with BC,
and U the meetpoint of IbA' and MI.
Prove: UC is parallel to DV.
A synthetic proof would be nice.
Sincerely
Jean-Louis

gold46
#2 Jun 6, 2010, 4:01 pm

What is M?
centroid?

jayme
#3 Jun 6, 2010, 4:20 pm

Sorry,
M is the midpoint of BC
Jean-Louis

Luis González
#4 Jun 7, 2010, 2:09 am

Let F be the tangency point of the A-excircle (I_a, r_a) with BC. If E is the foot of the A-altitude of ABC, then the midpoint L of AE lies on I_aD, since A, D and the antipode of F WRT (I_a) are collinear. Therefore

$$\frac{ED}{DF} = \frac{h_a}{2r_a} = \frac{s-a}{a} \Longrightarrow \frac{ED}{DM} = \frac{2(s-a)}{a} = \frac{CA'}{CM}$$

Since $\triangle VME \sim \triangle UMA'$ are centrally similar with center M, it follows that their cevians VD and UC are homologous $\Longrightarrow UC \parallel VD$. Note that UV can be an arbitrary line through M.

High School Olympiads

triangle inequality

Reply

vaibhav2903

#1 Jun 6, 2010, 5:29 pm

ABC is a triangle in which I is its incentre.the angle bisectors of A, B, C meet the circumcircle at X, Y, Z respectively.prove the following triangle inequalities:

1. $AI + BI + CI \leq IX + IY + IZ$

2. $\frac{1}{IX} + \frac{1}{IY} + \frac{1}{IZ} \geq \frac{3}{R}$

im sorry,i wrote the problem wrong.this is the actual question.

This post has been edited 1 time. Last edited by vaibhav2903, Jun 6, 2010, 9:25 pm

frenchy

#2 Jun 6, 2010, 7:34 pm

$1)$
i hope i am not wrong
we got $AX = AI + IX$
from easy angle chasing we get $IX = XC = XB$
so X is the center of the cercle passing through points I, C, B
now we need to show
$3R \geq BI + CI$
where $R = IX$
$\angle IBC = \frac{\angle B}{2}, \angle ICB = \frac{\angle C}{2}, \angle BIC = \angle A + \frac{\angle B + \angle C}{2}$
so we get $BI = 2R \cdot sin\angle IBC, IC = 2R \cdot sin\angle ICB$
so we need to prove
$\frac{3}{2} \geq sinIBC + sinICB$
but $90 \geq \angle IBC + \angle ICB$
so it is easy to show using derivates that the maximum of $sinIBC + sinICB$
is obtained for one of the angle 60 and the other 30
then we get $1.5 \geq sinIBC + sinICB = 1.36$
so we are done
P.S.It is strange .Is the problem correct for sure??

This post has been edited 1 time. Last edited by frenchy, Jun 7, 2010, 1:35 am

Luis González

#3 Jun 6, 2010, 11:20 pm

Let (O) be the circumcircle of $\triangle ABC$ and I_a, I_b, I_c the excenters against A, B, C. Incenter I and circumcircle (O) become orthocenter and 9-point circle of the acute $\triangle I_aI_bI_c$. Therefore, X, Y, Z are midpoints of II_a, II_b, II_c. By Erdös-Mordell inequality for $\triangle I_aI_bI_c \cup I$, we get

$$II_a + II_b + II_c = 2(IX + IY + IZ) \geq 2(IA + IB + IC)$$

$$\Longrightarrow IX + IY + IZ \geq IA + IB + IC.$$

By Erdös-Mordell inequality for $\triangle ABC \cup I$, we get $IA + IB + IC \geq 6r$. Power of I with respect to the circumcircle (O) is $IA \cdot IX = IB \cdot IY = IC \cdot IZ = 2Rr$. Then the previous expression can be re-written as

$$\frac{2Rr}{IX} + \frac{2Rr}{IY} + \frac{2Rr}{IZ} \geq 6r \Longrightarrow \frac{1}{IX} + \frac{1}{IY} + \frac{1}{IZ} \geq \frac{3}{R}.$$

Quick Reply

High School Olympiads

the perpendicular bisector of median AK halves O1O2

Reply

ridgers

#1 Jun 6, 2010, 1:22 am

In the side BC of the triangle ABC we choose the point D such that D is not the midpoint ofBC.We denote by O_1 and O_2 the centers of the circumscribed circles of the triangles ABD and ADC. Prove that the perpendicular bisector of the median AK of the triangle ABC divides the segment O_1O_2 in two equal parts.

yetti

#2 Jun 6, 2010, 2:40 am

Lemma

Let O be circumcenter of $\triangle ABC$. Let L be midpoint of AK and let perpendicular bisector of AK cut O_1O_2 at M.Perpendicular to BC at D cuts the perpendicular bisector O_1O_2 of AD at P. Then $PD = PA$ and O_1PO_2 bisects $\angle APD \Longrightarrow O_2O_2$ is tangent at P to parabola with focus A and directrix BC. Likewise, perpendicular bisectors O_1O, O_2O, LM of AB, AC, AK are tangent to the same parabola. Let N be the direction of O_1O_2, a point at infinity. 4 fixed tangents OO_1, OO_2, LM and the line at infinity cut the fifth variable tangent O_1O_2 at O_1, O_2, M, N and the cross ratio $\frac{\overline{NO_1}}{\overline{NO_2}} \cdot \frac{\overline{MO_2}}{\overline{MO_1}} = \frac{\overline{MO_2}}{\overline{MO_1}} = \text{const}$ for any fifth tangent. When D coincides with the A-altitude foot of $\triangle ABC$, then O_1, O_2 become midpoints of AB, AC and L, M coincide with the common midpoint of $AK, O_1O_2 \Longrightarrow \frac{\overline{MO_2}}{\overline{MO_1}} = -1$ for any $D \in BC$.

Luis González

#3 Jun 6, 2010, 3:03 am

> ridgers wrote:
>
> In the side BC of the triangle ABC we choose the point D such that D is not the midpoint of BC. We denote by O_1 and O_2 the centers of the circumscribed circles of the triangles ABD and ADC. Prove that the perpendicular bisector of the median AK of the triangle ABC divides the segment O_1O_2 in two equal parts.

The result can be generalized for two points equally characterized in the similar $\triangle ABC$ and $\triangle AO_1O_2$. Take a look at the topic Find locus of D, precisely post # 4 and subsequent replies.

oneplusone

#4 Jun 6, 2010, 9:30 pm • 1

$\angle O_1AB = \angle ADB - 90 = 90 - \angle ADC = \angle O_2AC$. Let M, N, P be midpoints of AB, AC, AK, then $\triangle O_1AM \sim \triangle O_2AN$ so by spiral symmetry $\triangle AO_1O_2 \sim \triangle AMN$. Let X be the midpoint of O_1O_2. Note that P is the midpoint of MN, thus by more spiral symmetry, $\triangle XPA \sim \triangle O_1MA$ thus $\angle XPA = 90$ and we conclude that XP is the perpendicular bisector of AK.

phuongtheong

#5 May 15, 2012, 7:22 pm

We can solve this problem as the way below:
Denote O is the midpoint of O_1O_2.
We have: $\overline{MD}.\overline{MB} + \overline{MD}.\overline{MC} = 0$
$\Leftrightarrow MO_1^2 - O_1A^2 + MO_2^2 - O_2A^2 = 0$
$\Leftrightarrow 2(M0_1^2 + MO_2^2) - O_1O_2^2 = 2(AO_1^2 + AO_2^2) - O_1O_2^2$
$\Leftrightarrow OA = OM$

Quick Reply

High School Olympiads

externally tangent circles

Reply

grobber
#1 Sep 9, 2003, 6:01 am

2 circles C1 and C2 are externally tangent at C and have centers O1 and O2 respectively. Another circle is tangent to C1 and C2 at M and N respectively. AB is the diameter of this third circle which is perpendicular to the common tangent of C1 and C2 which contains C. Call this tangent t and assume that A and O1 are on the same side of t. Prove that AO2, BO1, and t are concurrent.

Arne
#2 Sep 9, 2003, 9:30 pm

I like this one ! Bulgaria 96. Solution in

http://www.unl.edu/amc/a-activities/a4-for-students/mc96-97-01feb.pdf

or in one of Valentin's pdf documents.

grobber
#3 Sep 9, 2003, 10:16 pm

Well, I couldn't say I liked it so much.. I only have a computational proof and I don't like computations

Luis González
#4 Jun 5, 2010, 3:10 am

Let O_3 be the center of the circle $\mathcal{C}_3$ tangent to $\mathcal{C}_1, \mathcal{C}_2$ at $M, N.$ From the parallel radii $O_1C \parallel O_3A$ and $O_2C \parallel O_3B,$ it follows that AC and BC go through the exsimilicenters M, N of $\mathcal{C}_1 \sim \mathcal{C}_3$ and $\mathcal{C}_2 \sim \mathcal{C}_3,$ respectively. Lines AN, BM and t concur at the orthocenter H of $\triangle ABC$ and let $T \equiv MN \cap CH.$ Then AT is the polar of B with respect to $\odot(MNC) \Longrightarrow$ Tangent lines of $\odot(MNC)$ at C, N meet on $AT \Longrightarrow O_2 \in AT.$ By the same argument we have $O_1 \in BT \Longrightarrow$ $T \equiv BO_1 \cap AO_2 \cap t.$

Quick Reply

High School Olympiads

Albania IMO TST 2009 Question 1

ridgers

#1 Jun 4, 2010, 9:02 pm

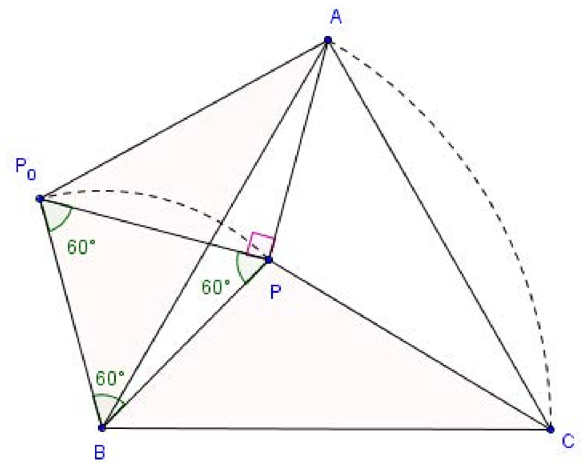
An equilateral triangle has inside it a point with distances 5,12,13 from the vertices . Find its side.

Luis González

#2 Jun 4, 2010, 9:20 pm

This problem has been posted many times with different numerical examples. For instance, see easy problem (equilateral triangle). Again, the distances from the point to the vertices of the equilateral triangle form a pythagorean triple. See the attachment below.

Attachments:



frenchy

#3 Jun 5, 2010, 1:15 am • 1

note the point as P and manage $12^2+5^2=13^2$
let $PA=12, PB=5, PC=13$
rotate the $\triangle BPC$ on 60 degrees around C
so P goes to X
B goes to A
so we have $PX=XC=PC=12$
and $AX=BP=5$
so we have $\angle AXP=90$
so we have $AX=12, XC=5, \angle APC=150$
so $AC^2=AX^2+XC^2+2AX\cdot XC\cdot cos60$
so $AC=\sqrt{229}$
so we are done
P.S.sorry for my first solution it was completely incorect
hope now it is allright

This post has been edited 5 times. Last edited by frenchy, Jun 5, 2010, 4:24 am

ridgers
#4 Jun 5, 2010, 1:22 am

> *frenchy wrote:*
> note the point as P and manage $12^2+5^2=13^2$
> let $PA=12, PB=5, PC=13$
> now on the extension of PC take a point X such that $XC=13$
> now we have $XA=BP=5$
> and we also have $BX=PA=12$
> so $BXAP$ is a rectangle ($\angle XAP=\angle XBP=90$ and $BP\parallel XA$)
> so we get $AB=13$
> analogues we get $BC=AC=13$
> so we are done

I think your solution is wrong!

yetti
#5 Jun 7, 2010, 2:24 pm

3 concentric circles $\mathcal{X},\mathcal{Y},\mathcal{Z}$ have given radii $a=5, b=12, c=13.$ A ray from the common center P cuts $\mathcal{X},\mathcal{Z}$ at X, Z_0. Let $\triangle OPX, \triangle XY_0Z_0$ be equilateral, their vertices on the opposite sides of the ray $(OX.$ Rotate/scale $\triangle XY_0Z_0$ around X into $\triangle XYZ,$ while keeping $Z\in\mathcal{Z}.$ $\triangle OXY\cong\triangle PXZ$ are congruent by SAS $\Longrightarrow OY=PZ=c, Y$ is on a circle $\mathcal{O}\cong\mathcal{Z}$ with center O and radius $c.$ Let $Y_1,Y_2\in\mathcal{O}\cap\mathcal{Y}$ and let $Z_1,Z_2\in\mathcal{Z}$ be the corresponding vertices of equilateral $\triangle XY_1Z_2, \triangle XY_2Z_2.$ Then $PX:PY_1:PZ_1=PX:PY_2:PZ_2=a:b:c.$ Triangles $\triangle OPY_1\cong\triangle OPY_2$ have sides a,b,c and if these are Pythagorean triples, the calculation is particularly easy. Suppose they are not.

$$XY^2=PX^2+PY^2-2\ PX\cdot PY\cos(\widehat{OPY}\pm 60^\circ)=\frac{1}{2}(a^2+b^2+c^2)\pm 2\sqrt{3}[abc]$$

Using Heron formula $16[abc]^2=2(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4),$ we get $x^2=XY^2$ as roots of quadratic equation

$$x^4-(a^2+b^2+c^2)x^2+(a^4+b^4+c^4)-(b^2c^2+c^2a^2+a^2b^2)=0$$

For Pythagorean triple $a=5, b=12, c=13$, $x=\sqrt{c^2\pm ab\sqrt{3}}=\sqrt{169\pm 60\sqrt{3}}.$ The larger root corresponds to P inside of the equilateral $\triangle XYZ.$

Cassius
#6 Jun 2, 2012, 6:05 pm

A more general fact, which is easily established using analytical geometry: let a, b, c be the distances of the point from the vertices and L the sidelenght of the triangle. Then we always have

$$3(a^4+b^4+c^4+L^4)=(a^2+b^2+c^2+L^2)^2.$$

In this particular case, $L=\sqrt{169+60\sqrt{3}}.$

Quick Reply

High School Olympiads

Two circles are tangent

Reply

bvarici
#1 Jun 4, 2010, 6:18 pm

Two circles are internally tangent at N. The chords BA and BC of the larger circle are tangent to the smaller circle at K and M respectively. Q and P are midpoint of arcs AB and BC respectively. Circumcircles of triangles BQK and BPM are intersect at L. Show that $BPLQ$ is a parallelogram.

oneplusone
#2 Jun 4, 2010, 8:18 pm • 2

First note that NKQ and NMP are straight lines. $\angle BLK+\angle BLM=360-\angle BQN-\angle BPN=180$ so KLM is a straight line. $\angle BPL=\angle BML=\angle KNM=180-\angle QBP$ thus $QB\parallel LP$. Similarly $QL\parallel BP$ and we are done.

Luis González
#3 Jun 4, 2010, 8:30 pm

> bvarici wrote:
> Two circles are internally tangent at N. The chords BA and BC of the larger circle are tangent to the smaller circle at K and M respectively. Q and P are midpoint of arcs AB and BC respectively. Circumcircles of triangles BQK and BPM are intersect at L. Show that $BPLQ$ is a parallelogram.

A more general result was proposed by Petrisor in the topic XUYV is a parallelogram.

sincostan
#4 Jun 14, 2010, 9:26 pm • 1

Ayme's favourite theorem $\longrightarrow$ NKQ collinear, NMP collinear.
pivot theorem $\longrightarrow$ MKL collinear.
Consider segment NB intersecting the smaller touching circle at Y again.
reim's theorem $\longrightarrow$ YM // BP, mutatis mutandis, KY // Q'B. anton reim's theorem $\longrightarrow$ points Y,Q', and second intersection of the circles apart from K (we call D) are collinear. mutatis mutandis, P,Y and second intersection (Z say) also collinear.
By reim QL // YM, mutatis mutandis, LP// KY...... translates to a parallelogram
$Quod\ Erat\ Demonstrandum$

Alternatively an angle chase, as oneplusone, gave is a method which is more practical and easier to come by.
IS there any other synthetics available in this problem? please inform me. Perhaps the generalisation(as indicated by luisgeometria) any pure synthesis there? please inform me.

exmath89
#5 Jul 20, 2013, 3:17 am

Solution

Quick Reply

High School Olympiads

Find locus of point

Reply

shepv186

#1 Jun 2, 2010, 10:29 pm

Let 3 permanent points A, B, C. Find locus of point M such that $MB^2 + MC^2 = MA^2$

Luis González

#2 Jun 3, 2010, 11:44 pm

D is the midpoint of BC and E is the reflection of A about D. MD is common M-median of triangles $\triangle MAE$ and $\triangle MBC$. By Stewart theorem, we have:

$$MD^2 = \frac{1}{2}(MA^2 + ME^2) - AD^2 \ , \ MD^2 = \frac{1}{2}(MB^2 + MC^2) - \frac{1}{4}BC^2$$

Since $MA^2 = MB^2 + MC^2$, it follows that

$$MA^2 - 2MD^2 = \frac{1}{2}BC^2 \Longrightarrow ME^2 = 2AD^2 - \frac{1}{2}BC^2 = CA^2 + AB^2 - BC^2$$

Locus of M is then a circumference with center E and radius $\sqrt{CA^2 + AB^2 - BC^2}$ as long as $CA^2 + AB^2 - BC^2 \geq 0$, i.e. $\angle BAC \leq 90^\circ$. When $\angle BAC = 90^\circ$ the circumference degenerates into the point E.

Quick Reply

High School Olympiads

Old but Nice (related to 2a=b+c)

Reply

77ant

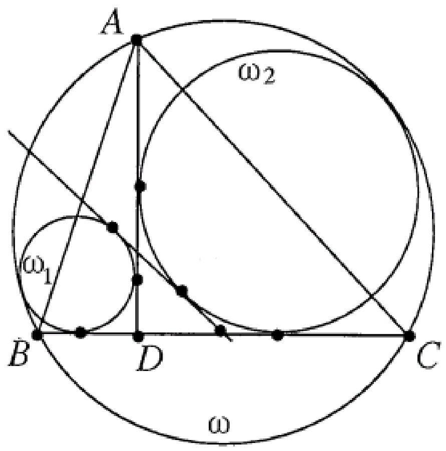
#1 Jun 2, 2010, 10:19 pm

Hi, everyone.
I ask you for help because a proof ,which I have but is not mine, seems not good.
It may be summarized as follows.

Let ABC be an acute triangle with $AB \neq CA$. Let D be the foot of the altitude from A to BC and let ω be the circumcircle of the triangle ABC. Let ω_1 be the circle that is tangent to AD, BD, and ω. Let ω_2 be the circle that is tangent to AD, CD, and ω. Finally let l be the common internal tangent to ω_1 and ω_2 that is not AD. Prove that l passes through the midpoint of BC if and only if $2BC$=AB+CA.

Thanks.

Attachments:



Luis González

#2 Jun 3, 2010, 1:14 am

Let the common internal tangent of ω_1, ω_2, different from BC, cut BC at M. Incircle (I) of $\triangle ABC$ touches BC at P. By Thebault theorem we easily deduce that I is the midpoint of the segment connecting the centers of ω_1, ω_2, thus P is midpoint of $\overline{DM}$. Consequently, if M is the midpoint of $\overline{BC}$, then I is on the perpendicular bisector of $\overline{DM}$. Keeping in mind that D, M and the midpoints N, L of AC, AB are vertices of an isosceles trapezoid with $DM \parallel NL$, then $IN = IL \Longrightarrow$ $|BC - AC| = |BC - AB| \Longrightarrow$ either $AB = AC$ or $AB + AC = 2BC$.

Quick Reply

High School Olympiads

a sphere touches the plane

Reply

Source: Vietnam NMO 1996, problem 2

mr.danh

#1 Sep 5, 2008, 4:24 pm

Given a trihedral angle Sxyz. A plane (P) not through S cuts Sx,Sy,Sz respectively at A,B,C. On the plane (P), outside triangle ABC, construct triangles DAB,EBC,FCA which are confruent to the triangles SAB,SBC,SCA respectively. Let (T) be the sphere lying inside Sxyz, but not inside the tetrahedron SABC, toucheing the planes containing the faces of SABC. Prove that (T) touches the plane (P) at the circumcenter of triangle DEF.

hendrata01

#2 May 31, 2010, 10:38 am

I don't have a complete solution, but I believe that this is almost a complete proof.

Let O be the center of T. Let α be the plane that contains SAB and let Q be the point where T touches α.

Suppose R is the intersection of AB and SR. Let U be the point where T touches P.

We know that $DR = SR$ because ABD and SAB are congruent. We also know that $RU = RQ$ because they're both tangent lines to the sphere.

I *believe* (but cannot prove) that U, R, D are collinear, so that $UD = UR + RD = RQ + SR = SQ$.

Similarly, if we label $Q' and Q''$ as the points where T touches the planes containing SBC and SCA, then $UE = SQ'$ and $UF = SQ''$. But $SQ = SQ' = SQ''$ because they're all tangent lines, so $UD = UE = UF$ which means U is the circumcenter of DEF.

Now, for the proof that U, R, D are collinear, I don't have the proof myself, but from many2 times visualizing, I almost believe that the following is correct:
1. $SR \perp AB$
2. If β is the plane containing O, U, Q then $\alpha \perp \beta$ and $\beta \perp P$.
3. $S \in \beta$

If we can prove any of the three points above, then it's not hard to establish that U, R, D are collinear.

Luis González

#3 Jun 1, 2010, 12:41 am

Denote by $\delta, \alpha, \beta, \gamma$ the planes containing the the faces of the tetrahedron $SABC$ opposite to S, A, B, C. Since $\triangle SBC \cong \triangle EBC$, it follows by obvious specular symmetry that the angle bisector (plane) of the dihedral angle α, δ, which pass through the S-excenter T of $SABC$, is also the perpendicular bisector of $\overline{ES}$. Likewise, the angle bisector of the dihedral angle γ, δ is the perpendicular bisector of $\overline{DS} \Longrightarrow TE = TS = TD \Longrightarrow T$ lies on the perpendicular bisector m of $\overline{DE}$. Therefore, orthogonal projection P of T on δ lies on the intersection $m \cap \delta$, i.e. the perpendicular bisector (line) of DE in $\delta \Longrightarrow PE = PD$. Similarly, $PE = PF \Longrightarrow P$ is the circumcenter of $\triangle DEF$.

Quick Reply

High School Olympiads

A metric relation regarding the Anticenter of ABCD

Luis González

#1 May 29, 2010, 11:19 pm

Convex quadrilateral $ABCD$ has circumcircle (O,R). Define $P\equiv AC\cap BD, E\equiv AB\cap DC$ and $F\equiv AD\cap BC$. If δ denotes the distance from the Anticenter T of $ABCD$ to the line EF, then prove the relation:

$$4PO\cdot\delta=AC^2+BD^2-4R^2$$

yetti

#2 May 31, 2010, 11:25 am • 1

Let $OP\perp EF$ cuts EF at Q and $OP\cdot OQ=R^2$. Then the equation is $\frac{AB^2}{R^2}+\frac{BD^2}{R^2}-4=4\frac{\delta}{OQ}$.
WLOG, A,B are arbitrary complex numbers on the unit circle (O) and P arbitrary imaginary number inside (O), so that $|P|<1$ and $\Re(P)=0\iff P=-\overline{P}$. Lines AP,BP cut (O) again at C,D, respsctively. Using $A\overline{A}=1$, etc.

$$\text{LHS}=|A-C|^2+|B-D|^2-4=-(A\overline{C}+\overline{A}C+B\overline{D}+\overline{B}D)=4-|A+C|^2-|B+D|^2$$

Line BP with equation $z(\overline{B}-\overline{P})-\overline{z}(B-P)=\overline{B}P-B\overline{P}$ cuts the unit circle $z\overline{z}=1$ at B,D, the roots of

$$z^2-\frac{\overline{B}P-B\overline{P}}{\overline{B}-\overline{P}}z-\frac{B-P}{\overline{B}-\overline{P}}=0\Longrightarrow$$

$$B+D=\frac{\overline{B}P-B\overline{P}}{\overline{B}-\overline{P}},\ |B+D|^2=\frac{|\overline{B}P-B\overline{P}|^2}{|B-P|^2}=\frac{|P|^2|B+\overline{B}|^2}{|B-P|^2}$$

In exactly the same way,

$$A+C=\frac{\overline{A}P-A\overline{P}}{\overline{A}-\overline{P}},\ |A+C|^2=\frac{|\overline{A}P-A\overline{P}|^2}{|A-P|^2}=\frac{|P|^2|A+\overline{A}|^2}{|A-P|^2}$$

Quadrilateral centroid is $G=\frac{1}{4}(A+B+C+D)$ and its anticenter $T=2G=\frac{1}{2}(\frac{\overline{A}P-A\overline{P}}{\overline{A}-\overline{P}}+\frac{\overline{B}P-B\overline{P}}{\overline{B}-\overline{P}})$. Since $Q=\frac{1}{\overline{P}}$,

$$\text{RHS}=4(1-\frac{\Im(T)}{\Im(Q)})=4(1-\Im(P)\cdot\Im(T))=$$

$$=4-2\Im(P)\cdot\Im(\frac{\overline{A}P-A\overline{P}}{\overline{A}-\overline{P}}+\frac{\overline{B}P-B\overline{P}}{\overline{B}-\overline{P}})=$$

$$=4-2\Im(P)\cdot\frac{(A+\overline{A})\cdot\Im(P(A-P))}{|A-P|^2})-2\Im(P)\cdot\frac{(B+\overline{B})\cdot\Im(P(B-P))}{|B-P|^2})=$$

$$=4-2\Im(P)\cdot\frac{(A+\overline{A})\cdot\Im(P)\cdot\Re(A)}{|A-P|^2})-2\Im(P)\cdot\frac{(B+\overline{B})\cdot\Im(P)\cdot\Re(B)}{|B-P|^2})=$$

$$=4-\frac{|P|^2|A+\overline{A}|^2}{|A-P|^2})-\frac{|P|^2|B+\overline{B}|^2}{|B-P|^2})=$$

$= 4 - |A+C|^2 - |B+D|^2 \implies \text{LHS} = \text{RHS}.$

Luis González

#3 May 31, 2010, 1:13 pm • 1

Dear Vladimir, thanks for your proof, but I have no experience with complex numbers in geometry, so I wish I could understand your nice proof fully. Here is mine

Let τ_a, τ_b, τ_c and τ_d be the tangent lines of (O) at A, B, C, D, respectively. Line EF is the polar of P with respect to (O, R), thus $U \equiv \tau_b \cap \tau_d$ and $V \equiv \tau_a \cap \tau_c$ lie on EF. Let M, N, L, K be the orthogonal projections of P on $\tau_c, \tau_b, \tau_a, \tau_d$. For simplicity denote $\delta(X)$ the distance from point X to EF. By Salmon's theorem for the pairs of polars $(UV, \tau_a), (UV, \tau_b), (UV, \tau_c), (UV, \tau_d)$ and their corresponding pairs of poles WRT (O), we have

$$\frac{PO}{PL} = \frac{R}{\delta(A)}, \ \frac{PO}{PK} = \frac{R}{\delta(D)}, \ \frac{PO}{PM} = \frac{R}{\delta(C)}, \ \frac{PO}{PN} = \frac{R}{\delta(B)}$$

$$\implies \frac{PM+PN+PL+PK}{PO} = \frac{\delta(A)+\delta(B)+\delta(C)+\delta(D)}{R} \ (*)$$

Anticenter T of $ABCD$ is the reflection of O about the centroid G of $ABCD$, therefore

$$\delta(G) = \frac{\delta(O)+\delta}{2} = \frac{\delta(A)+\delta(B)+\delta(C)+\delta(D)}{4} \ (1)$$

Let A', B' be the projections of A, B on VC, UD. Since $\triangle VAC$ and $\triangle UBD$ are isosceles with apices V, U we get

$$PM + PL = AA' = \frac{AC^2}{2R}, \ PN + PK = BB' = \frac{BD^2}{2R}$$

$$\implies PM + PN + PL + PK = \frac{AC^2 + BD^2}{2R} \ (2)$$

Combining the expressions (1) and (2) with $(*)$ yields

$$AC^2 + BD^2 = 4R \cdot \frac{PO \cdot \delta + PO \cdot \delta(O)}{R} = 4PO \cdot \delta + 4R^2.$$

This post has been edited 1 time. Last edited by Luis González, May 31, 2010, 6:22 pm

yetti

#4 May 31, 2010, 1:37 pm

Dear Luis, I was really curious about your solution.

The only thing not exactly obvious in the calculation with complex numbers is that equation of a line through arbitrary points P, Q is

$$\begin{vmatrix} z & \overline{z} & 1 \\ P & \overline{P} & 1 \\ Q & \overline{Q} & 1 \end{vmatrix} = 0 \iff$$

$$z(\overline{P} - \overline{Q}) - \overline{z}(P - Q) = \overline{P}Q - P\overline{Q}$$

and perhaps that inversion of P in unit circle is $Q = \dfrac{1}{\overline{P}}$.
$\overline{z}$ is complex conjugate of z, etc., and $\Re(z), \Im(z)$ are real and imaginary parts of z.

yetti

#5 Jun 1, 2010, 2:07 am

Rewriting the calculation without complex numbers:

Let U, V be midpoints of AC, BD. EF is polar of P WRT $(O) \Longrightarrow OP \perp EF$. Let $Q \equiv OP \cap EF \Longrightarrow OP \cdot OQ = R^2$. Let p be a line through O, such that $p \perp OP \Longrightarrow p \parallel EF$. Let X, Y be feet of perpendiculars to p from U, V. From $\triangle POU \sim \triangle OUX, OU^2 = OP \cdot UX$ and likewise, $OV^2 = OP \cdot VY$. By Pythagorean theorem for $\triangle OUA, \triangle OVB$,

$$AC^2+BD^2-4R^2=4R^2-4(OU^2+OV^2)=4R^2-4OP\cdot(UX+VY)$$

Let $S \in p$ be foot of perpendicular to p from the quadrilateral anticenter T. Since $OUTV$ is a parallelogram, its diagonals UV, OT intersecting at the quadrilateral centroid $G \Longrightarrow TS=UX+VY$ and

$$AC^2+BD^2-4R^2=4R^2-4OP\cdot TS=4R^2-4OP\cdot(OQ-\delta)=4OP\cdot\delta$$

Quick Reply

High School Olympiads

A metrical relation in a circle (own ?!).

Reply

Virgil Nicula

#1 May 31, 2010, 12:25 am

Proposed problem.

Let $ABCD$ be a convex quadrilateral which is inscribed in the circle w . Denote $M\in DD\cap AB$,

$N\in CC\cap AB$ si $P\in CD\cap AB$. Prove that $\frac {AM\cdot AN}{BM\cdot BN}=\left(\frac {PA}{PB}\right)^2$ and $\frac {MA\cdot MB}{NA\cdot NB}=\left(\frac {PM}{PN}\right)^2$.

I denoted XX - the tangent line in the point $X\in w$ to w.

This post has been edited 1 time. Last edited by Virgil Nicula, May 31, 2010, 2:33 am

Goutham

#2 May 31, 2010, 2:15 am

That is a cool problem. Thank you.

I invert with (D, k^2) for sufficient $k\neq 1$

Let $\overleftrightarrow{AB}=p, \overleftrightarrow{NC}=q$

Let the images of $A, B, C, P, M, N, w, p, q$ be $A', B', C', P', M', N', w', p', q'$ respectively.

The image consists of

$1)$ p', q' being two intersecting circles.

$2)$ DN' being a common chord of p', q'

$3)$ w' being a line intersecting p' at A', B' and q' at C' tangentially.

$4)$ P' being the intersection point of DC' with p'

Now, we have the ratios

$$AM=\frac{A'M'\times DA\times DM}{k^2}$$

$$AN=\frac{A'N'\times DA\times DN}{k^2}$$

$$BM=\frac{B'M'\times DB\times DM}{k^2}$$

$$BN=\frac{B'N'\times DB\times DN}{k^2}$$

$$PA=\frac{P'A'\times DP\times DA}{k^2}$$

$$PB = \frac{P'B' \times DP \times DB}{k^2}$$

Substituting all of these in the required expression and simplifying, we are required to prove that

$$\frac{A'M' \times A'N'}{B'M' \times B'N'} = \left(\frac{P'A'}{P'B'}\right)^2$$

Now, A', B', P', D, M' are concyclic as they lie on p'

So, $\angle A'M'B' = \angle A'P'B' = \angle A'N'B'$

So, the angle bisectors of $\angle A'M'B', \angle A'P'B', \angle A'N'B'$ meet $A'B'$ at a single point, say O

Now, by angle bisector theorem, $\dfrac{A'O}{OB'} = \dfrac{A'M'}{M'B'} = \dfrac{A'P'}{P'B'} = \dfrac{A'N'}{N'B'}$

So, $\dfrac{A'M' \times A'N'}{B'M' \times B'N'} = \left(\dfrac{A'O}{OB'}\right)^2 = \left(\dfrac{P'A'}{P'B'}\right)^2$ as required.

Hence, proved.

Virgil Nicula

#3 May 31, 2010, 2:36 am

Thanks for yours interest. I edited a second metrical relation in initial enunciation of this problem.

Luis González

#4 May 31, 2010, 3:12 am

Let U, V and X, Y be the orthogonal projections of A, B onto CN, DM. Then

$$\frac{AM \cdot AN}{BM \cdot BN} = \frac{AU \cdot AV}{BX \cdot BY}$$

Let R be the radius of ω. Since any chord in a circle is the geometric mean between the diameter and the distance from one of its endpoints to the tangent line through the another one, we rewrite the latter expression as

$$\frac{AM \cdot AN}{BM \cdot BN} = \frac{AU \cdot AV}{BX \cdot BY} = \frac{AC^2 \cdot AD^2 \cdot 2R}{BC^2 \cdot BD^2 \cdot 2R} = \left(\frac{AC \cdot AD}{BC \cdot BD}\right)^2$$

From $\triangle PAD \sim \triangle PCB$ and $\triangle PDB \sim \triangle PAC$ we have the proportions

$$\frac{AD}{BC} = \frac{PA}{PC}, \ \frac{AC}{BD} = \frac{PC}{PB}$$

$$\Longrightarrow \frac{AM \cdot AN}{BM \cdot BN} = \left(\frac{PA}{PC} \cdot \frac{PC}{PB}\right)^2 = \left(\frac{PA}{PB}\right)^2$$

Virgil Nicula

#5 May 31, 2010, 6:06 am

> Quote:
>
> **Proposed problem.** In a cyclical convex quadrilateral $ABCD$ denote $M \in DD \cap AB$, $N \in CC \cap AB$
>
> si $P \in CD \cap AB$. Prove that $\dfrac{AM \cdot AN}{BM \cdot BN} = (\dfrac{PA}{PB})^2$ si $\dfrac{MA \cdot MB}{NA \cdot NB} = (\dfrac{PM}{PN})^2$.
>
> I used the notation xx for the tangent line in a point x of the circumcircle.

I used the notation XX for the tangent line in a point X of the circumcircle.

Proof. Denote $AB=a$, $BC=b$, $CD=c$, $DA=d$, $AC=e$, $BD=f$.

Apply a remarkable **Haruki**'s lemma for the points C , D :

$$\left\{\begin{array}{c} X:=C\implies\left\{\begin{array}{c}\frac{PA\cdot NB}{PN}=\frac{bd}{c}\\ \frac{PB\cdot NA}{PN}=\frac{ef}{c}\end{array}\right.\\ X:=D\implies\left\{\begin{array}{c}\frac{MA\cdot PB}{MP}=\frac{bd}{c}\\ \frac{MB\cdot PA}{MP}=\frac{ef}{c}\end{array}\right.\end{array}\right\|\implies\left\{\begin{array}{c}\frac{PA\cdot NB}{PN}=\frac{MA\cdot PB}{MP}\\ \frac{PB\cdot NA}{PN}=\frac{MB\cdot PA}{MP}\end{array}\right\|\implies\left\{\begin{array}{c}\frac{AM\cdot AN}{BM\cdot BN}=(\frac{PA}{PB})^2\\ \frac{MA\cdot MB}{NA\cdot NB}=(\frac{PM}{PN})^2\end{array}\right. .$$

TelvCohl

#6 Feb 16, 2015, 9:55 pm

My solution:

Since $(N,P;A,B)=C(C,D;A,B)=D(C,D;A,B)=(P,M;A,B)$,

so we get $\frac{NA\cdot PB}{NB\cdot PA}=\frac{PA\cdot MB}{PB\cdot MA}\Longrightarrow\boxed{\frac{MA\cdot NA}{MB\cdot NB}=\frac{PA^2}{PB^2}}$

Since $(N,P;A,B)=(M,P;B,A)$,
so P is the double point of the involution determined by $\{A,B\},\{M,N\}$,

hence we get $(M,N;A,P)=(N,M;B,P)\Longrightarrow\boxed{\frac{MA\cdot MB}{NA\cdot NB}=\frac{PM^2}{PN^2}}$

Q.E.D

MariusBocanu

#7 Feb 17, 2015, 12:06 am

> *Goutham wrote:*
> Now, A',B',P',D,M' are concyclic as they lie on p'
>
> So, $\angle A'M'B'=\angle A'P'B'=\angle A'N'B'$
>
> So, the angle bisectors of $\angle A'M'B'$, $\angle A'P'B'$, $\angle A'N'B'$ meet $A'B'$ at a single point, say O

I am not sure you are right. Actually, I am sure you are wrong. The three bisectors intersect in the midpoint of the arc $A'B'$ of p'.

Quick Reply

High School Olympiads

A problem, its easy extension, its generalization a.s.o.

Reply

Virgil Nicula

#1 May 28, 2010, 7:21 pm

Proposed problem. Let w be a circle and let P be an exterior point w.r.t. the circle w . Denote the points $\{A,B\}\subset w$ for which

$P\in AA\cap BB$, the midpoint M of $[AP]$, $R\in BM\cap w$, $S\in PR\cap w$ where $S\not\equiv R$. Prove that $BS\parallel AP$.

An easy extension. Let ABC be a triangle with the circumcircle w . Denote $P\in BC\cap AA$,

the midpoint M of $[AP]$, $R\in BM\cap w$, $S\in PR\cap w$ where $S\not\equiv R$. Prove that $CS\parallel AP$.

Generalization. Let $ABCD$ be a convex quadrilateral inscribed in the circle w . Denote $P\in AB\cap CD$, the harmonical conjugate

Q of P w.r.t. $\{A,B\}$, the midpoint M of $[PQ]$, $R\in MD\cap w$ and $S\in PR\cap w$ where $S\not\equiv R$. Prove that $CS\parallel AP$.

Luis González

#2 May 30, 2010, 12:56 pm

> *Virgil Nicula wrote:*
> **Generalization.** Let $ABCD$ be a convex quadrilateral inscribed in the circle $w.$ Denote $P\in AB\cap CD,$ the harmonical conjugate Q of P w.r.t. $\{A,B\},$ the midpoint M of $[PQ],$ $R\in MD\cap w$ and $S\in PR\cap w$ where $S\not\equiv R.$ Prove that $CS\parallel AP.$

By Newton's theorem we have $MQ^2=MP^2=MB\cdot MA=p(M,\omega)=MR\cdot MD\Longrightarrow$ The circumference $\odot(MDP)$ is tangent to line ABP at $P\Longrightarrow\angle APS=\angle RDC=\angle RSC\Longrightarrow CS\parallel ABP.$

Quick Reply

High School Olympiads

excircle and circumcircle

Reply

vaibhav2903

#1 May 30, 2010, 1:08 am

denote by (O,R) and (I,R_a), the circumcircle and the excircle opposite to vertex A of $\triangle ABC$ respectively.prove that

$IA.IB.IC=4R.R_a^2$

Luis González

#2 May 30, 2010, 4:53 am

Let X,Y,Z be the tangency points of (I,R_a) with BC,CA,AB. J denotes the incenter of $\triangle ABC$ and M denotes the midpoint of $\overline{IJ}$, which lies on (O), because of $J,(O)$ become orthocenter and 9-point circle of the excentral triangle of $\triangle ABC$. The inversion in (I,R_a) takes the 9-point circle of $\triangle XYZ$ into (O). Therefore, we have the relation:

$$\frac{R_a}{2R}=\frac{{R_a}^2}{p(I,(O))} \Longrightarrow p(I,(O))=IM\cdot IA=\tfrac{1}{2}IJ\cdot IA=2R\cdot R_a \ (*)$$

On the other hand, IX and IJ become the altitude and circumdiameter issuing from the vertex I of $\triangle IBC$. Hence, $IB\cdot IC=IX\cdot IJ=R_a\cdot IJ$. Combining with $(*)$ we obtain $IA\cdot IB\cdot IC=4R\cdot {R_a}^2$.

Quick Reply

High School Olympiads

Points on orthic triangle

jgnr

#1 May 29, 2010, 8:41 am

Let AA_1, BB_1, CC_1 be the altitudes of an acute triangle ABC. Points M and N are on segments B_1C_1 and C_1A_1 respectively such that $\angle MAN = \angle B_1AA_1$. Prove that line NA bisects $\angle MNC_1$.

I'm looking for a solution without trigonometry.

Luis González

#2 May 29, 2010, 9:38 am

It is well-known that sidelines of the acute $\triangle ABC$ are external angle bisectors of $\triangle A_1B_1C_1 \Longrightarrow C_1A$ is the external bisector of $\angle MC_1N$. N-excenter of $\triangle NMC_1$ is the unique point on the ray C_1A such that $\angle AC_1M + \angle MAN = 90^\circ$. Indeed,

$\angle MAN + \angle AC_1B_1 = 90^\circ - \angle C + \angle C = 90^\circ$

$\Longrightarrow A$ is the N-excenter of $\triangle NMC_1 \Longrightarrow NA$ bisects $\angle MNC_1$.

jgnr

#3 May 29, 2010, 3:15 pm

Let $\angle ANC_1 = x$ and $\angle AMN = y$. By sine law on triangle AC_1N, AMN, AMB_1 respectively we get $\frac{AC_1}{\sin x} = \frac{AN}{\sin C}$, $\frac{AN}{\sin(90^\circ + C - y)} = \frac{AM}{\sin y}$, $\frac{AM}{\sin B} = \frac{AB_1}{\sin(90^\circ + C - x)}$. Multiplying these equations and simplifying we get $\frac{\cos(C-x)}{\sin x} = \frac{\cos(C-y)}{\sin y}$, $\cos C \cot x - \sin C = \cos C \cot y - \sin C$, $x = y$, as desired.

High School Olympiads

Perpendicular

Reply

Source: Taiwan MO 2000, Indian TST 2011

Akashnil
#1 May 22, 2010, 11:02 pm • 2

Let ABC be a triangle in which $BC < AC$. Let M be the mid-point of AB, AP be the altitude from A on BC, and BQ be the altitude from B on to AC. Suppose that QP produced meets AB (extended) at T. If H is the orthocenter of ABC, prove that TH is perpendicular to CM.

frenchy
#2 May 23, 2010, 1:08 am • 1

it is a very easy question if you know polars
and a hard one if you do not know
let $CC' \perp AB$ so the cercle around the point B, A, Q, P have the center at M
so C' is the harmonical conjugate of T versus the circle
so C is on the polar of T
using the duality principle we get that T is on the polar of C
so $TH \perp CM$ as M is the center

This post has been edited 1 time. Last edited by frenchy, May 25, 2010, 12:28 am

frenchy
#3 May 23, 2010, 2:47 am

another nice solution
using the notes from first solution we now take K on CM such that $HK \perp CM$
so now $C'MHK, QPHKC, QPC'M$ are cyclic the last is cyclic cose all 4 point are on the Eulres circle
but from radical axis or Monge theorem we get that $QP, C'M, HK$ are concurent but $QP, C'M$ are concurent at T so we get that $HT \perp CM$
so we are done

Joao Pedro Santos
#4 May 24, 2010, 5:24 am

Even though the statement seems nice, this problem can be trivially solved using Brockard's Theorem...
Since $\angle APB = \angle AQB = 90^\circ$, A, B, P, Q are concyclic, so by Brockard's Theorem M is the orthocentre of CHT, therefore $CM \perp TH$.

mahanmath
#5 May 24, 2010, 6:01 am

Just harmonic division . It was also Brazil TST (2001 or 2002)

dgreenb801
#6 May 24, 2010, 7:04 am • 4

Let K be the perpendicular from H to CM, let CH meet AB at N. Then as $\angle HNM = \angle HKM = 90$, $KHNM$ is cyclic. Also, as $\angle HKC = \angle HQC = \angle HPC = 90$, $KHPCQ$ is cyclic. Finally, as P,Q,M, and N lie on the Nine-point circle, $PQMN$ is cyclic. By the radical axis theorem, the radical axes of these three circles, PQ, MN, and KH, must concur, so KH passes through T. So $TH \perp CM$.

Many other olympiad problems where we wish to prove perpendicularity from an outside point, can be solved similarly, by dropping a perpendicular from the middle of the three points to the desired line, finding three cyclic quadrilaterals (often using the nine-point circle), and then using the radical axis theorem. I've seen it in three or four different problems.

EDIT: I now see that this is the same solution as frenchy's but in more detail. But I see he should have written C' is the perpendicular from C to AB rather than AC, that was why I was confused.

Virgil Nicula

#7 May 24, 2010, 7:12 pm • 2

Old problem ... Very nice and well-known the **frenchy**'s or **dgreenb801**'s proof
without polars or proiective geometry. I''ll come back soon with another proof.

Raja Oktovin

#8 May 26, 2010, 9:30 am

Suppose a, b, p, q is on the unit circle of a complex plane. We can see that
$m = 0$, $h = \dfrac{ap(b+q) - bq(a+p)}{ap - bq}$, $t = \dfrac{ab(p+q) - pq(a+b)}{ab - pq}$ and $c = \dfrac{aq(b+p) - bp(a+q)}{aq - bp}$.
also $\overline{t} = \dfrac{b+q-a-p}{bq-ap}$, $\overline{t} = \dfrac{p+q-a-b}{pq-ab}$, and $\overline{c} = \dfrac{b+p-a-q}{bp-aq}$.
we need to prove that $\dfrac{t-h}{\overline{t}-\overline{h}} = -\dfrac{c-m}{\overline{c}-\overline{m}}$.
the computation is messy but at least you know where it will solve the problem. $m = 0$ is very helpful though.

Virgil Nicula

#9 May 26, 2010, 9:49 pm

See here http://www.artofproblemsolving.com/Forum/viewtopic.php?t=46146 ...

Virgil Nicula

#10 May 27, 2010, 4:14 am • 4

Generalization. Let ABC be a triangle. For a point P consider the points $D\in BC$, $E\in CA$, $F\in AB$ so that

$BFPD$, $CDPE$ are cyclically. Denote $m(\angle PDC)=m(\angle PEA)=m(\angle PFB)=\phi$ and $L\in EF\cap BC$.

The sideline BC cut again the circumcircle of $\triangle DEF$ in the point K . Prove that $m\left(\widehat{LP,AK}\right)\in\{\phi,\pi-\phi\}$.

Particular cases.

PC1. Let ABC be a triangle. For a point P denote its projections $D\in BC$, $E\in CA$, $F\in AB$ on the sidelines of $\triangle ABC$.

Denote $L\in EF\cap BC$. The sideline BC cut again the circumcircle of $\triangle DEF$ in the point K . **Prove that** $LP\perp AK$.

PC2. Let $\triangle ABC$. Denote the midpoint M of the side $[BC]$, the projections E , F of the orthocenter H

to the sidelines AC , AB respectively and the intersection $L\in BC\cap EF$. **Prove that** $LH\perp AM$.

PC3. Let ABC be a triangle with incircle $C(I)$ which touches the sides of $\triangle ABC$ in

$D\in BC$, $E\in CA$, $F\in AB$. Denote $T\in BC\cap EF$. **Prove that** $TI\perp AD$.

PC4. Let ABC be a triangle with exincircle $C(I_a)$ which touches the sides of $\triangle ABC$ in

$D'\in BC$, $E'\in CA$, $F'\in AB$. Denote $T'\in BC\cap E'F'$. **Prove that** $T'I_a\perp AD'$.

Luis González

#11 May 29, 2010, 3:30 am • 1

Nice generalization, dear Virgil

Circles $\odot(BDF), \odot(CDE), \odot(AFE)$ concur at their Miquel point P WRT $\triangle ABC$. Let $U \equiv AK \cap LP$ and V, U' the second intersections of $\odot(AFE)$ with LA, LP, respectively. Then P becomes Miquel point of $\odot(AVE), \odot(LDV)$, $\odot(CED)$ WRT $\triangle ALC$, in other words, P, V, L, D are concyclic. In the inversion through pole L with power $LP \cdot LU'$, we have that $V \mapsto A, U' \mapsto P$ and $K \mapsto D \Longrightarrow$ points A, U', K are collinear $\Longrightarrow U \equiv U'$. Then $\angle(LP, KA) = \angle(FP, FA) = \phi \pmod{\pi}$.

Rijul saini

#12 Sep 17, 2010, 11:27 pm

> Akashnil wrote:
>
> Let ABC be a triangle in which $BC < AC$. Let M be the mid-point of AB; AP be the altitude from A on BC; and BQ be the altitude from B on to AC. Suppose QP produced meets AB (extended) in T. If H is the ortho-center of ABC, prove that TH is perpendicular to CM.

A different solution (OWN) can be found here

RSM

#13 Mar 27, 2011, 2:06 pm

Let ABC be a triangle in which $AC < AC$. Let M be the mid-point of BC, CF be the altitude from C on AB, and BE be the altitude from B on to AC. Suppose that EF produced meets BC (extended) at T. If H is the orthocenter of ABC, prove that TH is perpendicular to CM.

I did not read the other solutions. So I don't know if it has been posted before.

Clearly, $(TBDC) = -1$
From this easy to see that $TD.TM = TB.TC$
Consider the circles $\odot ADM$ and $\odot BEFC$. Both of their centres lie on AM.
H lie on their radical axis since $BH.HE = AH.HD$
T lies on the radical axis too because $TD.TM = TB.TC$
So $TH \perp AM$

---

sax

#14 Aug 30, 2011, 3:19 pm

Another nice solution using the following lemma:
Lemma: Let $ABCD$ is a quadrilateral inscribed in a circle centred at O. If $AB \cap DC = R, AD \cap BC = Q, AC \cap BD = P$, then $OP \perp QR$.(Also, $OP \cap QR$ is the Miquel point of $ABCD$).
Applying this lemma to the problem, $MH \perp TC$, therefore H is the orthocentre of $\triangle TCM$ and the result follows

---

waver123

#15 Aug 30, 2011, 6:06 pm

> sax wrote:
>
> Another nice solution using the following lemma:
> Lemma: Let $ABCD$ is a quadrilateral inscribed in a circle centred at O. If $AB \cap DC = R, AD \cap BC = Q, AC \cap BD = P$, then $OP \perp QR$.(Also, $OP \cap QR$ is the Miquel point of $ABCD$).

can you post a proof for your lemma?

---

sax

#16 Sep 1, 2011, 2:04 am • 1

See this http://web.mit.edu/yufeiz/www/olympiad/cyclic_quad.pdf. It is the fact 15..

---

waver123

#17 Sep 2, 2011, 1:48 pm

> sax wrote:
> See this http://web.mit.edu/yufeiz/www/olympiad/cyclic_quad.pdf. It is the fact 15..

thanks, a very interesting and useful article.do you have more such articles?

oty

#18 Mar 18, 2012, 7:35 am

My solution : let N intersection of (TH) and (CM) , E intersection of (TH) and (BP) , I the orthogonal projection of M on (BP) by thales we have I is the Midelpoint of $[BP]$. we need to prove that the quadrilateral $HNQC$ is cocyclique , equivalent to $\widehat{NCQ}=x$ and $\widehat{NHQ}=y$ are equal . we have $\widehat{BHP}=\widehat{AHQ}=\widehat{C}$ and $\widehat{EHP}=\widehat{C}-y$, it's easy to calculate $tan(C-y)=\frac{tan(C)tan(B)}{tan(C)+2tan(2B)}$ (using the fact of I is the Midelpoint of [BP]) (*) , and to calculate $tan(C-x)$ we have the points T,P,Q and T , E ,H are alligned , then by Menelaus on the triangles ABC and BAP , we can calculate $BE=\frac{BP.tan(B)}{tan(C)+2tan(B)}$, or Click to reveal hidden text$tan(C-x)=\frac{BE}{HP}=\frac{BP}{HP}.(\frac{tan(B)}{tan(C)+2tan(B)})$, hence $tan(C-x)=\frac{tan(C)tan(B)}{tan(C)+2tan(B)}$ and so by (*) we have $x=y$, done!

sayantanchakraborty

#19 Mar 13, 2014, 11:34 pm

Another nice solution:

Let X be the foot of altitude from C to AB.Then
Radical axis(nine point circle,$\odot CXM$)=AB.
Radical axis(nine point circle,$\odot BPQA$)=PQ.

As $PQ\cap AB=T$ we get that T is a point on the radical axis of $\odot CXM,\odot BPQA$.
As $HC*HX=HP*AH$,H is also a point on the radical axis of $\odot CXM,\odot BPQA$.
TH is the radical axis of the two circles.
As the center of $\odot CXM$ lies on CM and the center of $\odot ABPQ$ is M,and radical axis of two circles is perpendicular to the line joining centres,we obtain $TH\perp CM$.

Bye...

Sayantan....

PRO2000

#20 Oct 6, 2015, 10:33 pm

$ABPQ$ is $Concyclic$ with center at M , radius $=\frac{AB}{2}$.
Denote this circle by ω.
$AB\cap PQ=T$.
$AQ\cap PB=C$.
$AP\cap QB=H$.
So , TCH is self polar triangle with center of ω as its orthocentre.
As M is center of ω ,
$TH\perp CM$.

Quick Reply

High School Olympiads

Concyclic

Reply

Aquarious

#1 May 28, 2010, 1:45 pm

Let H be the orthocenter of the triangle ABC and P an arbitrary point on circumcircle of triangle. BH meets AC at E. $PAQB$ and $PARC$ are two parallelograms and AQ meets HR at X. Show that $EX \parallel AP$.

Luis González

#2 May 28, 2010, 6:40 pm

Didn't you post the same problem few days ago?. Please do not double post.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=336749
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=65343 (note the similarity with this problem)

Quick Reply

High School Olympiads

sum of 2 altitudes = 3rd altitude

Reply

Aquarious

#1 May 28, 2010, 1:45 pm

Let ABC be an acute triangle with centroid G and angular bisectors AM,BN and CK. Prove that one of the altitudes of triangle ABC equals the sum of the remaining two if and only if G lies on a side of
triangle MNK

Luis González

#2 May 28, 2010, 6:32 pm

The straight line MK is the geometric locus of the points whose sum of the oriented distances to AC and AB equals the oriented distance to BC. Note that its equation in trilinear coordinates with respect to $\triangle ABC$ is given by $\beta+\gamma-\alpha=0.$ Thus if X,Y,Z are the orthogonal projections of G onto $BC,CA,AB,$ we get

$$\overline{GX}=\overline{GY}+\overline{GZ} \Longleftrightarrow G \in \overline{MK} \Longleftrightarrow h_a=h_b+h_c.$$

Quick Reply

High School Olympiads

Triple incircle

Reply

oneplusone

#1 May 27, 2010, 7:50 pm

In a circumscribed quadrilateral $ABCD$ (there exists a circle tangent to all sides), let E be a point on AB. Prove that there exists a common external tangent to the 3 incircles of $\triangle AED, \triangle BEC, \triangle CED$.

Luis González

#2 May 27, 2010, 9:16 pm • 1

$(I), (J), (K), (L)$ are the incircles of $ABCD, \triangle EBC, \triangle EAD, \triangle EDC$ and τ is the common external tangent of $(J), (K)$ different from AB. Directed common tangents AB, BC, CD, DA of $(K), (J)$ and degenerate circles $(C), (D)$ are tangent to the directed circle (I). Therefore, the directed common tangents τ, EC, CD, DE of $(K), (J), (C), (D)$ are tangent to another directed circle, i.e. circle (L). For a proof of the general case, see the highlighted theorem and its proof here.

Quick Reply

Spain

Geometria hard

Reply

Solving
#1 May 5, 2010, 4:19 pm

Sea $\triangle ABC$ un triángulo acutángulo . M, N y P son las proyecciones ortogonales del baricentro de $\triangle ABC$ en sus lados. Demostrar que $\frac{2}{9}<\frac{[MNP]}{[ABC]}\leq\frac{1}{4}$.

Without Latex

Sea ABC un triángulo y acutangula M, N y P, el ortogonal de la baricentro
ABC en sus costados. Demostrar que 2 / 9 <[MNP] / <[ABC] = <1 / 4.

Help!!

This post has been edited 1 time. Last edited by Luis González, May 6, 2010, 4:03 am

Luis González
#2 May 27, 2010, 11:49 am

Sean H, O, G el ortocentro, baricentro y circuncentro del triángulo acutángulo $\triangle ABC$ con circunradio $R.$ Usando el teorema de Euler para el triángulo pedal $\triangle MNP$ de G respecto a $\triangle ABC,$ la desigualdad pedida es equivalente a:

$$\frac{2}{9}<\frac{[MNP]}{[ABC]}=\frac{p(G,(O))}{4R^2}=\frac{OG^2-R^2}{4R^2}\leq\frac{1}{4}$$

La desigualdad $[MNP]\leq\frac{1}{4}[ABC]$ es inmediata considerando $OG\geq 0$ y por otro lado $OG^2-R^2>\frac{8}{9}R^2\Longrightarrow 9OG^2<R^2$ equivale a probar que $OH<R$ en vista que $OH=3OG.$ Esta última es obvia para triángulos acutángulos $ABC,$ ya que H es interior a su circuncírculo.

BIAcK_CaT
#3 Aug 8, 2010, 5:48 am

> *Solving wrote:*
> Sea $\triangle ABC$ un triángulo acutángulo . M, N y P son las proyecciones ortogonales del baricentro de $\triangle ABC$ en sus lados. Demostrar que $\frac{2}{9}<\frac{[MNP]}{[ABC]}\leq\frac{1}{4}$.

Sea $D\in BC: AD\perp BC$. Notemos que $AB=2R\cdot sen(\angle C)$ (donde R corresponde al circumradio del $\triangle ABC$),y obteniendo identidades análogas, se cumple que:

$$AD=\frac{2[ABC]}{BC}=\frac{AB\cdot AC}{2R}=2R\cdot sen(\angle B)\cdot sen(\angle C)$$

Como $\triangle ADA'\sim\triangle GMA'$ y $AG=3\cdot GA'$ (donde A' es el punto medio de BC), obtenemos que $GM=\frac{2}{3}R\cdot sen(\angle B)\cdot sen(\angle C)$. Análogamente $GY=\frac{2}{3}R\cdot sen(\angle C)\cdot sen(\angle A)$ y $GZ=\frac{2}{3}R\cdot sen(\angle A)\cdot sen(\angle B)$.

Por otra parte es sencillo ver que los cuadriláteros $ANGP, BMGP, CMGN$ son cíclicos y a partir de esto obtenemos que $\angle MGN=180-\angle C$, $\angle NGP=180-\angle A$ y $\angle PGM=180-\angle B$. Luego usamos que

$\angle MGN = 180 - \angle C$, $\angle NGP = 180 - \angle A$ y $\angle PGM = 180 - \angle B$. Luego, veamos que $[MNP] = [MGN] + [NGP] + [PGM]$.

Como $[MGN] = \frac{1}{2} GM \cdot GN \cdot sen(\angle MGN) = \frac{2}{9} R^2 sen^2(\angle A) sen(\angle B) sen(\angle C)$, obtenemos que

$\frac{[MNP]}{[ABC]} = \frac{1}{9}(sen^2(\angle A) + sen^2(\angle B) + sen^2(\angle C))$. Por la desigualdad de Jensen aplicada a la función $f(x) = sen^2(x)$, obtenemos que $\frac{[MNP]}{[ABC]} = \frac{f(\angle A) + f(\angle B) + f(\angle C)}{9} \geq \frac{1}{3} f(\frac{\pi}{3}) = \frac{1}{4}$.

Para la otra cota de la desigualdad, notemos que como el $\triangle ABC$ es acutángulo, entonces $cos(\angle A), \cos(\angle B), cos(\angle C)$ son mayores que cero, y entonces

$$\sum sen^2(\angle A) = 3 - \sum \cos^2(\angle A)$$

$$> 3 - 2\cos(\angle A)\cos(\angle B)\cos(\angle C) - \sum \cos^2(\angle A) = 2$$

Y se deduce la desigualdad.

High School Olympiads

involving absolute value

Reply

∫FaiLurE□

#1 May 23, 2010, 6:03 pm

$ABCD$ is a cyclic quadrilateral. Let M,N be midpoints of diagonals AC,BD respectively. Lines BA,CD intersect at E and lines AD,BC intersect at F. Prove that:
$|\frac{AC}{BD}-\frac{BD}{AC}|=\frac{2MN}{EF}$

Luis González

#2 May 23, 2010, 9:57 pm

Let L be the midpoint of EF lying on the Newton line MN of the complete quadrangle $ABCD$. Since E,F are conjugate points with respect to the circumcircle (O) of $ABCD$, then the circle (L) with diameter $\overline{EF}$ belows to the orthogonal pencil defined by (O) and the axis $EF \Longrightarrow (L)$ is orthogonal to (O) and the circle (O') with diameter OK, where $K \equiv AC \cap BD$ is the pole of EF WRT $(O) \Longrightarrow$ power of L to (O') equals $LE^2=LF^2$. Therefore

$$LE^2=LN \cdot LM=(LM+MN)LM \Longrightarrow \frac{MN}{LE}=\frac{LE}{LM}-\frac{LM}{LE} \ \ (\star)$$

Let P,Q,R be the midpoints of BC,CE,EB. It is clear that $ERPQ$ is a parallelogram and $L \in RQ$. By Menelaus theorem for $\triangle PQR$ cut by the Newton line of $ABCD$, we have:

$$\frac{LM}{LN} \cdot \frac{RN}{RP} \cdot \frac{PQ}{MQ}=1 \Longrightarrow \frac{LM}{LN}=\frac{EC}{ED} \cdot \frac{EA}{EB}=\left(\frac{EA}{ED}\right)^2=\left(\frac{AC}{BD}\right)^2.$$

Combining with the power of L to (O'), we get $\frac{LM}{LE}=\frac{AC}{BD}$

Substituting the previous ratio into the expression $(\star)$ yields

$$\frac{MN}{LE}=\frac{BD}{AC}-\frac{AC}{BD} \Longrightarrow \frac{MN}{EF}=\frac{1}{2}\left|\frac{AC}{BD}-\frac{BD}{AC}\right|$$

Quick Reply

High School Olympiads

ABC is atriangle,prove the inequality

Reply

Amir Hossein
#1 May 22, 2010, 11:09 pm

ABC is a triangle, with inradius r and circumradius R. Show that:

$$\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)+\sin\left(\frac{B}{2}\right)\cdot\sin\left(\frac{C}{2}\right)+\sin\left(\frac{C}{2}\right)\cdot\sin\left(\frac{A}{2}\right)\leq\frac{5}{8}+\frac{r}{4\cdot R}.$$

Luis González
#2 May 23, 2010, 12:26 am

Substitute the following well-known identities into the required inequality

$$\frac{r}{R}=4\cdot\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}$$

$$\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}=\frac{1}{2}-\frac{1}{2}\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)$$

$$\sum\sin\frac{A}{2}\cdot\sin\frac{B}{2}=\frac{1}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2-\frac{1}{2}\sum\sin^2\frac{A}{2}$$

The inequality becomes then $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\leq\frac{3}{2}$,

which is true by Jensen's inequality for the concave function $\sin x$, $\forall\ x\in(0,\pi)$

Amir Hossein
#3 May 23, 2010, 11:18 am

Thank you!
Very nice solution

Quick Reply

High School Olympiads

Radical axis, Apollonius and concurrency

Reply

Source: Mine

juancarlos

#1 Aug 29, 2006, 5:04 am • 1

Construct the circles $(O_1), (O_2), (O_3)$ with diameters the sides BC, AC, AB of the acute triangle ABC, if we denote as U, V, W the cut points between the radical axis of the inner Apollonius circle (O_0) of $(O_1), (O_2), (O_3)$ and the circumcircle (O) of ABC with the lines BC, AC, AB respectively. From U draw the tangent UX to the arc BC of the circumcircle (O) that not contain A, similarly drawn the tangents VY and WZ to the circumcircle (O). Prove that AX, BY, CZ concur.
I can't prove that: If X', Y', Z' are the touch points of (O_0) with $(O_1), (O_2), (O_3)$ respectively, then AX', BY', CZ' concur.

Luis González

#2 May 19, 2010, 6:58 pm • 1

This is a very interesting configuration, but the concurrency of the straight lines AX, BY, CZ is true for any line cutting the extentions of the side-segments BC, CA, AB.

Lemma. A', B', C' are three points on the arcs BC, CA, AB of the circumcircle (O) of $\triangle ABC$. Tangents of (O) at A', B', C' cut BC, CA, AB at A_0, B_0, C_0. Then A_0, B_0, C_0 are collinear iff AA', BB', CC' concur.

Since $\angle BA'A_0 = \angle BAA'$ and $\angle AA'A_0 = \angle CAA' \pmod \pi$, it follows that

$$\frac{A_0B}{A_0C} = \frac{A'B}{A'C} \cdot \frac{\sin \widehat{BAA'}}{\sin \widehat{CAA'}} = \left(\frac{\sin \widehat{BAA'}}{\sin \widehat{CAA'}}\right)^2$$. Similarly, we have the expressions

$$\frac{B_0C}{B_0A} = \left(\frac{\sin \widehat{CBB'}}{\sin \widehat{ABB'}}\right)^2 , \quad \frac{C_0A}{C_0B} = \left(\frac{\sin \widehat{ACC'}}{\sin \widehat{BCC'}}\right)^2$$

$$\Longrightarrow \frac{A_0B}{A_0C} \cdot \frac{B_0C}{B_0A} \cdot \frac{C_0A}{C_0B} = \left(\frac{\sin \widehat{BAA'}}{\sin \widehat{CAA'}} \cdot \frac{\sin \widehat{CBB'}}{\sin \widehat{ABB'}} \cdot \frac{\sin \widehat{ACC'}}{\sin \widehat{BCC'}}\right)^2$$

By Menelaus' theorem and trig-Ceva we conclude that A_0, B_0, C_0 are collinear if and only if AA', BB', CC' concur.

> *juancarlos wrote:*
> I can't prove that: If X', Y', Z' are the touch points of (O_0) with $(O_1), (O_2), (O_3)$ respectively, then AX', BY', CZ' concur.

Pairwise circles $(O_1), (O_2), (O_3)$ cut at the feet A', B', C' of the A-,B- and C- altitude of $\triangle ABC$. Inversion through pole A with power $\overline{AC} \cdot \overline{AB'} = \overline{AB} \cdot \overline{AC'}$ transforms (O_1) into itself and $(O_2), (O_3)$ into the lines CC', BB'. Thus (O_0) is taken into the circle (O_a) tangent to CC', BB' and internally tangent to (O_1) at the inverse X'' of $X' \Longrightarrow X'' \in AX'$.

Let H be the orthocenter of $\triangle ABC$. In the figure $\triangle HBC \cup (O_1) \cup (O_a)$, it's well-known that the angle bisector of $\angle BX''C$ goes through the incenter of $\triangle HBC$ and the incenter of triangle bounded by BC, CB', BC', i.e. incenter of $\triangle ABC$, for a proof see R,S and T are collinear, very nice!. Thus, IX'' goes through the midpoint V_a of the semicircle $BC'B$, i.e center of the inner Vecten square with side length BC. Similarly, IY'' and IZ'' go through the centers V_b, V_c of the inner Vecten squares CA, AB. In this configuration, AX'', BY'', CZ'' concur at Kimberling center $X_{1336} \Longrightarrow$ Lines AX', BY', CZ' concur at X_{1336}.

This post has been edited 2 times. Last edited by Luis González, May 19, 2010, 7:12 pm

jayme

#3 May 19, 2010, 7:09 pm

Dear Mathlinkers,
A thought for the beloved Juan Carlos Salazar who died the 30 march 2088 and was a great contributor to Mathlinks.
See: http://perso.orange.fr/jl.ayme vol. 2 In memorian Juan Carlos Salazar.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Determine triangle ABC.

Reply

jaydoubleuel

#1 May 16, 2010, 8:01 pm

Determine $\triangle ABC$ (such as equilateral, isoceles, etc) when it satisfies the following
--Let D, E denote the intersections of angle bisectors of $\angle A, \angle B$ and its opposite sides.
--Also, let F denote the incenter of $\triangle ABC$. and M the midpoint of $\overline{DE}$
--Then, $\overline{MF}$ meets perpendicular with $\overline{AB}$

yetti

#2 May 17, 2010, 2:23 am

Incircle (F) of $\triangle ABC$ touches BC, CA, AB at X, Y, Z. Let $\triangle X'Y'Z'$ be anticomplementary to $\triangle XYZ$. DE is polar of Z' WRT (F), $FZ' \perp DE$. XY cuts AB at W. $F(A, B, Z, W) \equiv F(D, E, M, W)$ is harmonic bundle $\Longrightarrow FW \parallel DE$. AB, XY, FW are concurrent at $W \Longrightarrow$ their poles Z, C, and direction of FZ' WRT (F) are collinear $\Longrightarrow CZ \parallel FZ'$. F is 9-point circle center of $\triangle X'Y'Z' \Longrightarrow$ 9-point circle center G of $\triangle XYZ$ is on its Z-symmedian $ZC \Longrightarrow$ either C is reflection of its circumcenter F in XY or $\triangle XYZ$ is isosceles with $ZX = ZY \Longrightarrow$ either $\angle BCA \equiv \angle XCY$ is right or $\triangle ABC$ is isosceles with $CA = CB$.

Luis González

#3 May 17, 2010, 9:14 am

Let P be the tangency point of the incircle (F, r) with AB. FP passes through $M \Longleftrightarrow [\triangle PFD] = [\triangle PFE]$

$$\frac{[\triangle PFD]}{[\triangle PFA]} = \frac{FD}{FA} \Longrightarrow [\triangle PFD] = \frac{ra(s-a)}{2(b+c)}$$

$$\frac{[\triangle PFE]}{[\triangle PFB]} = \frac{FE}{FB} \Longrightarrow [\triangle PFE] = \frac{rb(s-b)}{2(a+c)}$$

$$\Longrightarrow \frac{a(s-a)}{b+c} = \frac{b(s-b)}{a+c} \Longrightarrow a(ab+bc+c^2-a^2) = b(ab+ac+c^2-b^2)$$

$\Longrightarrow (a-b)(c^2-a^2-b^2) = 0$. Therefore, either $a = b$ or $\angle ACB = 90^\circ$.

yetti

#4 May 17, 2010, 9:48 am

Let I, J, K be its excenters of $\triangle ABC$ against A, B, C and O its circumcenter. Let (P) be circumcircle of $\triangle IJF$. (O) is its 9-point circle, K its orthocenter, POK its Euler line. $IBFC, JAFC$ are cyclic $\Longrightarrow \overline{DC} \cdot \overline{DB} = \overline{DI} \cdot \overline{DF}$ and $\overline{EC} \cdot \overline{EA} = \overline{EJ} \cdot \overline{EF} \Longrightarrow DE \perp POK$ is radical axis of $(P), (O)$.
Bisector CFK of $\angle C$ cuts AB at Q. Let the incircle (F) and excircle (K) touch AB at Z, Z'. N is common midpoint of AB, ZZ'. C, Q are external/internal similarity centers of $(F), (K) \Longrightarrow \frac{CF}{CK} = -\frac{QF}{QK} = -\frac{QZ}{QZ'} \Longrightarrow \frac{QZ}{QN} = \frac{QC}{QK} \Longrightarrow$ $CZ \parallel NK$.
Let (F) touch CB, CA at X, Y and let XY cut AB at W. Cross ratio $(A, B, Z, W) = -1$ is harmonic. Assume $MF \perp AB$. Then $F(A, B, Z, W) \equiv F(D, E, M, W)$ is harmonic bundle $\Longrightarrow FW \parallel DE$. AB, XY, FW concur at $W \Longrightarrow$ their poles WRT (F), i.e., Z, C and direction $\perp FW$ are collinear $\Longrightarrow CZ \perp (FW \parallel DE) \Longrightarrow CZ \parallel OK \Longrightarrow NK \parallel OK \Longrightarrow$ either $O \equiv N$ or $OK \perp AB \Longrightarrow$ either $\angle BCA$ is right or $CB = CA$.

Quick Reply

Spain

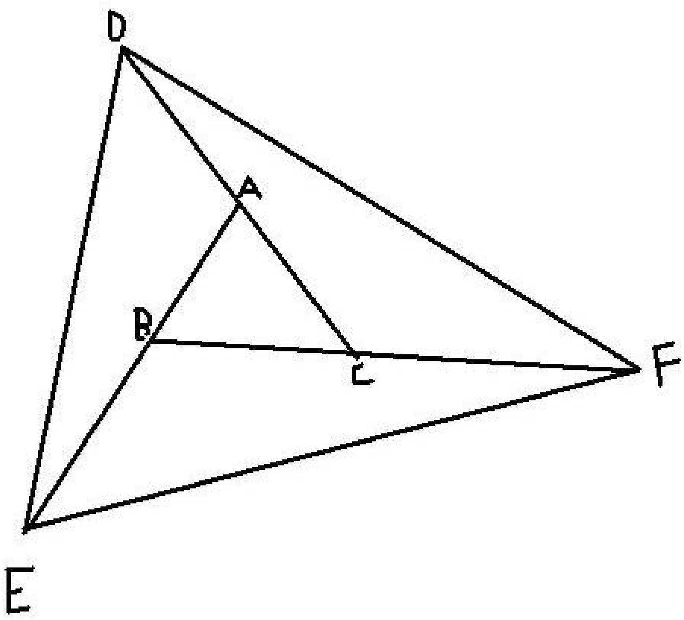
geometria - Otro aquí

Reply

Solving
#1 May 5, 2010, 4:48 pm

En la figura el área del triángulo ABC es equilátero S. ¿Cuál es el área del triángulo DEF y AD = BE = FC = AB?

Attachments:



Amir Hossein
#2 May 5, 2010, 6:03 pm

Hi!
Could you please write your problem English ?
Thanks!

Solving
#3 May 5, 2010, 6:40 pm

> *amparvardi wrote:*
> Hi!
> Could you please write your problem English ?
> Thanks!

In the figure the area of triangle ABC is equilateral S. What is the area of triangle DEF if AD = BE = FC = AB?

Luis González

#4 May 17, 2010, 12:45 am

Generalización: Sobre las semirectas opuestas a AC, CB, BA de un $\triangle ABC$ se definen los puntos C', B', A', tales que AC', CB', BA' equivalen a una fracción $\frac{1}{n}$ de $AC, CB, BA.$ Hallar la razón de áreas entre $\triangle A'B'C'$ y $\triangle ABC.$

Usamos que la razón de areas entre dos triángulos con un par de ángulos iguales/suplementarios es igual a la razón que existe entre el producto de los lados adyacentes a tales ángulos, así

$$\frac{[AC'A']}{[ABC]} = \frac{AC' \cdot AA'}{AC \cdot AB} = \frac{\frac{1}{n}AC(AB + \frac{1}{n}AB)}{AC \cdot AB} = \frac{n+1}{n^2}$$

Similarmente se tiene las expresiones:

$$\frac{[BA'B']}{[ABC]} = \frac{n+1}{n^2}, \ \frac{[CB'C']}{[ABC]} = \frac{n+1}{n^2}$$

$$[A'B'C'] = [ABC] + [AC'A'] + [BA'B'] + [CB'C'] = [ABC]\left(1 + \frac{3(n+1)}{n^2}\right)$$

$$\Longrightarrow \frac{[A'B'C']}{[ABC]} = \frac{n^2+3n+3}{n^2}$$

● Notar particularmente que para $n = 1$ (el problema propuesto) la razón de áreas es 7.

Quick Reply

High School Olympiads

Midpoint for pleasure

jayme
#1 May 15, 2010, 4:03 pm

Dear Mathlinkers,
ABC a triangle, (1) the incircle of ABC, DEF the contact triangle of ABC, U the second point pf intersection of AD with (1), U' the antipode of U wrt (1), Tu' the tangent to (1) at U',
Y, Z the point of intersection of Tu' with UC, UB.
Prove AD bisect YZ.
Sincerely
Jean-Louis

skytin
#2 May 15, 2010, 10:42 pm

look my post Chinese IMO Team Selection Test 2008
if W lie on AD and CW=CD and if line CW intersect UB at point Q then CW=WQ easy to show that CW is paralel to tangent from U to (1) so YZ is paralel to CQ and trangles CUQ and YUZ ar homotety so YD = DZ

Luis González
#3 May 15, 2010, 11:04 pm

Let M,N be the second intersections of UC,UB with (I). The quadrilateral $UMDN$ is harmonic $\Longrightarrow$ Tangents to (I) through M,N and $UD \equiv AD$ concur at a point V. Using the configuration discussed in the article *A propos du théorème de Boutin. Version á partir du triangle de contact*, for $\triangle UMN$ and the antipode U' of U on its circumcircle, it follows that $UV \equiv AD$ bisects YZ.

High School Olympiads

Constant angle

Reply

Pain rinnegan

#1 May 14, 2010, 1:38 am

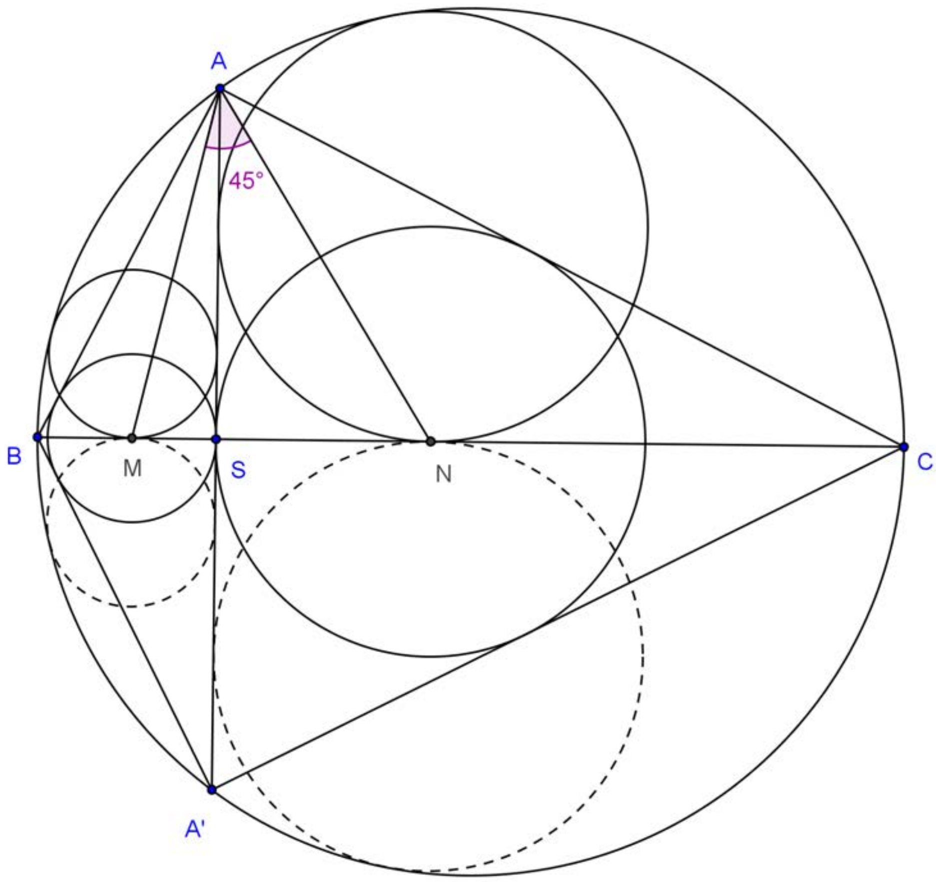
Let S be a point on the diameter BC of a semicircle. The perpendicular from S on BC meets the semicircle in A. The circles inscribed in the curved triangle ABS and ACS are tangent to BC at M and N, respectively. Show that the measure of $\sphericalangle MAN$ is constant with respect to S.

Luis González

#2 May 14, 2010, 1:58 am

Let A' be the reflection of A across BC. Thebault's theorem for the isosceles ABA' and ACA' reveals that M,N are their incenters, i.e. AM,AN bisect <BAS and <CAS.

Attachments:



Quick Reply

High School Olympiads

Nice problem

gold46
#1 May 10, 2010, 11:02 am

I solved this problem.Do you know that? Is it old problem?
A, B, C, D points lie on a circle. Prove that incenters of ABC, BCD, CDA, DAB triangles are concylic.

jayme
#2 May 10, 2010, 11:19 am

Dear Mathlinkers,
you can see for a synthetic proof, references and converse,
http://perso.orange.fr/jl.ayme vol. 4 Le rectangle de Ryokan Maruyama
Sincerely
Jean-Louis

lajanugen
#3 May 10, 2010, 11:42 am

The incenters are the vertices of a rectangle. Can be easily proved using the fact that in any $\triangle PQR$ the internal angular bisector of $\angle P$ bisects the arc QR of the circumcircle which doesn't contain P

oneplusone
#4 May 10, 2010, 12:56 pm

There is also a similar question:

Let the diagonals of convex quadrilateral $ABCD$ intersect at P. Prove that the incentres of the triangles APB, BPC, CPD, DPA are cyclic if and only if there exists a circle inscribed in $ABCD$.

Luis González
#5 May 12, 2010, 9:11 am

> *oneplusone wrote:*
> Let the diagonals of convex quadrilateral $ABCD$ intersect at P. Prove that the incentres of the triangles APB, BPC, CPD, DPA are cyclic if and only if there exists a circle inscribed in $ABCD$.

See the topic http://www.artofproblemsolving.com/Forum/viewtopic.php?t=21758.

livetolove212
#6 May 12, 2010, 11:46 am

First we introduce a lemma:
Given triangle ABC with incircle (I). Let D be an arbitrary point on BC. Let I_a, I_b be the incenter of triangles ADB, ADC, E be the projections of I on BC. Then I_1, I_2, D, E are concyclic.
Proof:
Let G, H be the projections of I_1, I_2 on BC. We have $DH = \dfrac{AD+DC-AC}{2}$,
$EG = BE - BG = \dfrac{BC+AB-AC-AB-BD+AD}{2} = \dfrac{AD+DC-AC}{2}$. Therefore $DH = EG$. Let M

be the midpoint of $I_1I_2, MX \perp BC$ then X is the midpoint of GH. But $DH = EG$ so X is the midpoint of ED, which follows that $MD = ME$. Hence $MI_1 = MI_2 = MD = ME$. We are done.

Back to our problem:

If $ABCD$ is a circumscribed quadrilateral. Let I_1, I_2, I_3, I_4 be the incenters of triangles APB, BPC, CPD, DPA, respectively, O be the incenter of $ABCD$, X, Y, Z, T be the incenters of triangles ABD, ABC, BCD, CDA.

Since AX, CY, BZ concur at O then applying Desargues theorem we obtain $(AY \cap BX), (BZ \cap CY), (XZ \cap AC)$ are collinear or I_1I_2, AC, XZ are concurrent. (1)

Similarly I_3I_4, AC, XZ are concurrent. Therefore the intersection T of I_1I_2 and I_3I_4 lies on AC.

Let L be the projection of Y on AC then $CL = \dfrac{AC + BC - AB}{2}$, L' be the projection of Z on AC then $CL' = \dfrac{AC + CD - AD}{2}$.

But $AB + CD = AD + BC$ thus $CL = CL'$ or $L \equiv L'$.

According to the lemma above we claim I_1LPI_2 and I_3LPI_4 are cyclic quadrilateral. (2)

From (1) and (2) we deduce that $TI_1.TI_2 = TL.TP = T_3.TI_4$. The result follows then.

I am not able to prove the inserve.

Attachments:

picture164.pdf (7kb)

livetolove212

#7 May 12, 2010, 11:51 am

Another similar problem:

Given a cyclic quadrilateral $ABCD$. $AC \cap BD = \{L\}$. Prove that the symmedian points of triangles ALB, BLC, CLD and DLA are concyclic if and only if $AC \perp BD$ or $ABCD$ is a trapezoid.

The theorem of Ryokan Maruyama can be reversed: Let I_1, I_2, I_3, I_4 be the incenters of triangles ABC, CBD, CDA, DAB. If $I_1I_2I_3I_4$ is a rectangle then $ABCD$ is a cyclic quadrilateral.

oneplusone

#8 May 12, 2010, 12:56 pm

> *livetolove212 wrote:*
>
> **Another similar problem:**
>
> Given a cyclic quadrilateral $ABCD$. $AC \cap BD = \{L\}$. Prove that the symmedian points of triangles ALB, BLC, CLD and DLA are concyclic if and only if $AC \perp BD$ or $ABCD$ is a trapezoid.

Let E, F, G, H be the midpoints of AB, BC, CD, DA and E_1, F_1, G_1, H_1 be the symmedians. Assume $AC \perp BD$. Note that $\angle E_1LF_1 = 180 - \angle ELF = 180 - \angle EBF$ thus E_1LF_1B is cyclic and similarly $F_1LG_1C, G_1LH_1D, H_1LE_1A$ are cyclic. So now, $\angle H_1G_1F_1 = \angle H_1G_1L + \angle LG_1F_1 = \angle LCF_1 + \angle LDH_1 = \angle LCB + \angle LDA$. Similarly, $\angle H_1E_1F_1 = \angle LBC + \angle LAD$ and they both add up to 180, thus $E_1F_1G_1H_1$ is cyclic.

The inverse is always harder, I'll think about it later. By the way, this is turning into a marathon?

Quick Reply

High School Olympiads

The relationship between the area of 3 triangles

Reply

creatorvn

#1 May 10, 2010, 5:11 pm

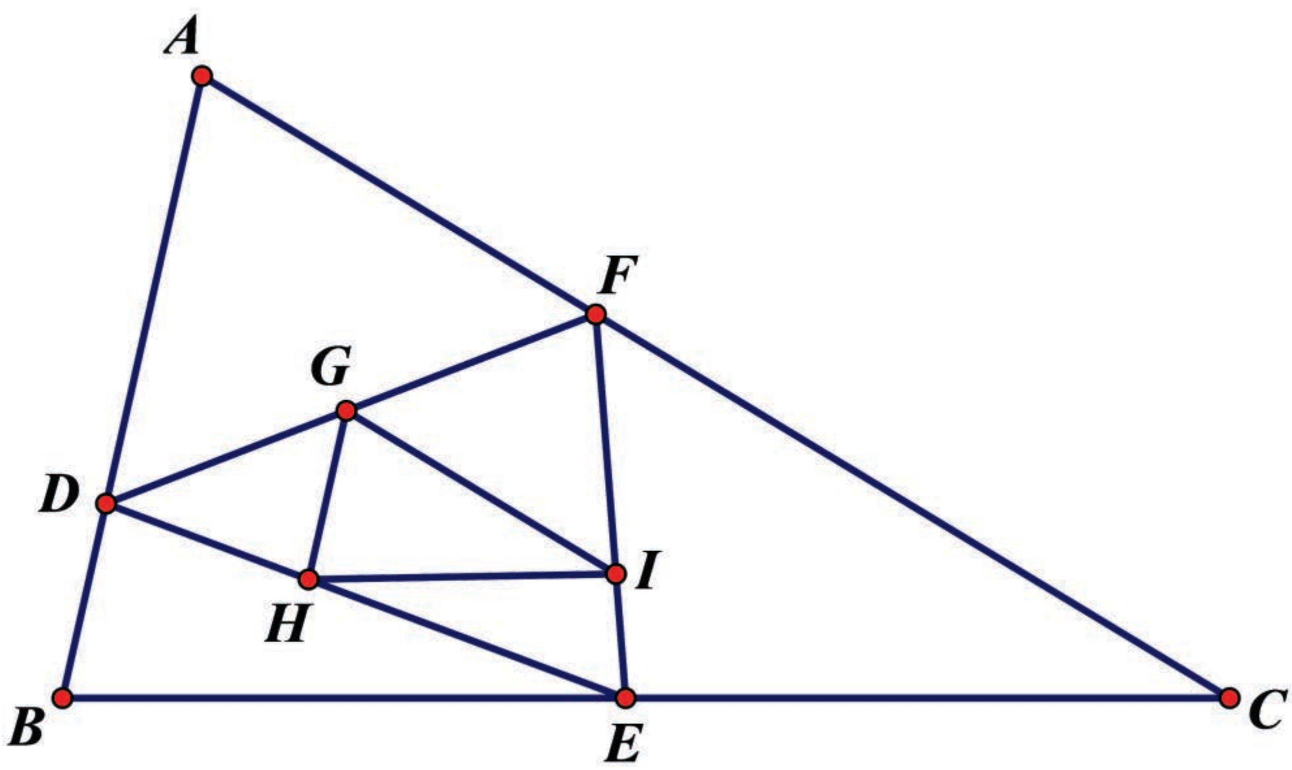
In the figure,
D lies on the segment AB
E lies on the segment BC
F lies on the segment CA

G lies on the segment DF
H lies on the segment DE
I lies on the segment EF

GH // AB
GI // AC
HI // BC

Prove that:
$S(DEF)^2 = S(ABC).S(GHI)$

Attachments:



This post has been edited 1 time. Last edited by yetti, May 28, 2010, 2:40 pm
Reason: copying figure fromphotobucket to mathlinks

Luis González

#2 May 10, 2010, 11:50 pm

Let $O \equiv AG \cap BH \cap CI$ be the homothetic center of $\triangle ABC$ and $\triangle GHI$. Let X, Y, Z be the orthogonal projections of O onto BC, CA, AB. Then

$[DEF] = [OIEH] + [OHDG] + [OGFI]$, (1)

$[ABC] = [OBC] + [OCA] + [OAB] = \dfrac{OX \cdot BC + OY \cdot CA + OZ \cdot AB}{2}$, (2)

On the other hand, we have the expressions

$[OIEH] = \dfrac{OX \cdot HI}{2}$, $[OHDG] = \dfrac{OZ \cdot GH}{2}$, $[OGFI] = \dfrac{OY \cdot IG}{2}$

Combining with (1), we obtain

$[DEF] = \dfrac{OX \cdot HI + OY \cdot IG + OZ \cdot GH}{2}$

If k stands for the similarity coefficient of $\triangle ABC$ and $\triangle GHI$, it follows that

$[DEF] = \dfrac{OX \cdot BC + OY \cdot CA + OZ \cdot AB}{2k}$,

Together with (2), we have that $[DEF] = \dfrac{[ABC]}{k}$

Since $[ABC] = k^2 \cdot [GHI] \Longrightarrow [DEF] = \sqrt{\dfrac{[GHI]}{[ABC]}} \cdot [ABC]$

$\Longrightarrow [DEF]^2 = [ABC] \cdot [GHI]$.

creatorvn

#3 May 11, 2010, 10:16 am

Please tell me why AG, BH, CI are concurrent lines, please ?
Thank you

High School Olympiads

Relation between the areas

Reply

Pain rinnegan

#1 May 9, 2010, 4:14 pm

Consider a triangle ABC and let $A' \in (BC),\ B' \in (CA),\ C' \in (AB)$ such that lines $AA',\ BB',\ CC'$ are concurrent. Let $\sigma,\ \sigma_A,\ \sigma_B,\ \sigma_C,\ \sigma_O$ be the areas of triangles $ABC,\ AB'C',\ BC'A',\ CA'B',\ A'B'C'$, respectively. Prove that:

$$\sigma_O^2 \cdot \sigma = 4\sigma_A \cdot \sigma_B \cdot \sigma_C$$

Luis González

#2 May 10, 2010, 5:46 am

AA', BB', CC' concur at P with barycentric coordinates $(u : v : w)$ WRT $\triangle ABC$

$$\frac{[\triangle A'B'C']}{[\triangle ABC]} = \frac{2uvw}{(v+w)(u+w)(u+v)}, \frac{[\triangle AB'C']}{[\triangle ABC]} = \frac{vw}{(u+v)(u+w)}$$

$$\frac{[\triangle BA'C']}{[\triangle ABC]} = \frac{uw}{(v+w)(u+v)}, \frac{[\triangle CA'B']}{[\triangle ABC]} = \frac{uv}{(u+w)(v+w)}$$

$$[\triangle AB'C'] \cdot [\triangle BA'C'] \cdot [\triangle CA'B'] = \left[\frac{uvw}{(v+w)(u+w)(u+v)}\right]^2 \cdot [\triangle ABC]^3$$

$$[\triangle AB'C'] \cdot [\triangle BA'C'] \cdot [\triangle CA'B'] = \frac{[\triangle A'B'C']^2 \cdot [\triangle ABC]}{4}$$

Quick Reply

High School Olympiads

lexmarkx125

#1 May 7, 2010, 1:20 pm

Let P be a point on circumcircle of triangle ABC. D and E are the feet of perpendicular from P to BC and CA respectively. L is mid-point of AD and M is mid-point of BE. Prove that DE prependicular to LM.

jayme

#2 May 7, 2010, 4:24 pm

Dear Mathlinkers,
1. LM is the Gauss line of the complete quadrilateral ABDE
2. the Steiner's line of ABDE is parallel to the Simson line DE
3. We know that this two lines are perpendicular. (see for example http://perso.orange.fr/jl.ayme vol. 4 La droite de Gauss et de Steiner)
Sincerely
Jean-Louis

lexmarkx125

#3 May 7, 2010, 9:43 pm

> *jayme wrote:*
> Dear Mathlinkers,
> 1. LM is the Gauss line of the complete quadrilateral ABDE
> 2. the Steiner's line of ABDE is parallel to the Simson line DE
> 3. We know that this two lines are perpendicular. (see for example http://perso.orange.fr/jl.ayme vol. 4 La droite de Gauss et de Steiner)
> Sincerely
> Jean-Louis

ER....I still don't understand wt you mean....can you explain by using complex number/vector/coordinate geometry??

Luis González

#4 May 7, 2010, 10:08 pm

> *lexmarkx125 wrote:*
> Let P be a point on circumcircle of triangle ABC. D and E are the feet of perpendicular from P to BC and CA respectively. L is mid-point of AD and M is mid-point of BE. Prove that DE prependicular to LM.

The same problem was posted earlier by 77ant here. My solution is exactly the same as Jayme's, but you may check the another solution offered.

jayme

#5 May 8, 2010, 1:10 am

Dear Luis,
thank you for your reference.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Converse of Butterfly??

Reply

Pikachu!!!
#1 May 5, 2010, 12:12 am

Does the converse of the Butterfly Theorem is true? or we have other interesting details?

Luis González
#2 May 5, 2010, 1:15 am

We prove a stronger version of the Bufferfly Theorem, then its converse will follows easily.

Proposition. Let AB be any chord in a circle (O). Let P be a point on AB. Two chords MN, RS are drawn through P. Let $X \equiv AB \cap MS$ and $Y \equiv AB \cap RN$. Then we have

$$\frac{1}{AP} - \frac{1}{XP} = \frac{1}{PB} - \frac{1}{PY}$$

Perform the inversion through pole P and power k^2 that transforms the circumcircle (O) of $MRNS$ into itself. It follows that $MS \mapsto \odot(PRN)$ and $RN \mapsto \odot(PMS)$. Then the line AB cuts $\odot(PRN)$ and $\odot(PMS)$ again at the inverse images U, V of X, Y, respectively. From $\angle MUP = \angle NRP = \angle SMP = \angle SVP$, we deduce that if $T \equiv UN \cap VS$, then $\triangle TUV$ is isosceles with apex T. If VS cuts (O) again at L, we have that $\angle TLN = \angle NRS = \angle UVL \Longrightarrow NL \parallel UV$, which means that the center of (O) lies on the perpendicular bisector of segment UV. By obvious symmetry it folllows that $UB = VA \Longrightarrow PU - PB = PV - PA$. It remains to use the power k^2 to plug the corresponding primitive distances, this is:

$$\frac{k^2}{PX} - \frac{k^2}{PA} = \frac{k^2}{PY} - \frac{k^2}{PB} \Longrightarrow \frac{1}{PA} - \frac{1}{PX} = \frac{1}{PB} - \frac{1}{PY}.$$

Quick Reply

High School Olympiads

Proof of aa'=bb'+cc'

Reply

Kunihiko_Chikaya
#1 May 4, 2010, 8:15 am

Given two triangles $ABC,\ A'B'C'$ such that $\angle A$ and $\angle A'$ are supplementary angles and $\angle B=\angle B'$.
Prove that we have $aa'=bb'+cc'$, where $a,\ b,\ c$ denote the side lengths of $BC,\ CA,\ AB$ respectively
and $a',\ b',\ c'$ denote the side lengths of $B'C',\ C'A',\ A'B'$ respectively.

1956 Tokyo Institute of Technology entrance exam

SOURBH
#2 May 4, 2010, 6:15 pm

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=347669

Zhero
#3 May 4, 2010, 6:21 pm

Click to reveal hidden text

Luis González
#4 May 4, 2010, 7:01 pm • 1

Consider a convex quadrilateral $ABCD$ such that $\angle ADB+\angle ACB=180^\circ$ and let $M\equiv AC\cap BD$. The two triangles $\triangle ADM$ and $\triangle BCM$ satisfy the wanted condition. Thus, we shall prove that $MA\cdot MB=AD\cdot BC+CM\cdot DM$.

Take a point P on the ray $\overrightarrow{MD}$ such that $AP\parallel BC$. Then $\triangle MPA\sim\triangle MBC, \triangle ADP\sim\triangle MCB$. Let k be the similarity coefficient of $\triangle MCB$ to $\triangle ADP$

$$\frac{AD}{MC}=k\Longrightarrow\frac{AD}{k}\cdot\frac{MA}{MC}=MA\Longrightarrow\frac{AD}{k}\cdot\frac{PM}{MB}=MA$$

$$\Longrightarrow MA\cdot MB=\frac{AD\cdot PD}{k}+\frac{AD\cdot DM}{k}=AD\cdot BC+CM\cdot DM.$$

Kunihiko_Chikaya
#5 May 4, 2010, 10:52 pm

The problem has been solved completely.

My solution is almost same as luisgeometria's solution. I remarked the similarity as well.

> SOURBH wrote:
> http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=347669

Oh! Problem that creatorvn posted is exactly same as the problem posed 54 years ago, which is miraculous.

Just curiousity, is this problem very famous?

yetti

#6 May 6, 2010, 8:32 am

Take arbitrary $\triangle ABA'$ with circumcircle (O). Internal bisector of $\angle ABA'$ cuts (O) again at C and AA' at K. $\triangle ABC$ with $a = BC, b = CA, c = AB$ and $\triangle A'BC$ with $a' = BC, b' = CA', c' = A'B$ satisfy the conditions.
$\triangle ABC \sim \triangle KBA'$ are similar, having equal angles $\Longrightarrow CA \cdot CA' = CA \cdot CA = BC \cdot KC$. $\triangle ABC \sim \triangle KAC$ are similar, having equal angles $\Longrightarrow AB \cdot A'B = BC \cdot BK$. As a result,
$bb' + cc' = CA \cdot CA' + AB \cdot A'B = BC \cdot KC + BC \cdot BK = BC \cdot BC = aa'$.

sunken rock

#7 Jan 6, 2011, 3:08 am

Construct the triangle $\triangle ABC$, and its altitude CD (D on BA); take B'' the image of B in D. Take then B'' on $(AB''$ s.t. $AB' = c'$ and draw $B'C' \parallel B''C$. $\triangle ABC$ and $\triangle AB'C'$ are our triangles. For our purpose, let's count $AB'' = c''$.
See that, from Thales, $\frac{AC'}{AC} = \frac{AB'}{AB''} = \frac{B'C'}{B''C}$, i.e. $\frac{b'}{b} = \frac{c'}{c''} = \frac{a'}{a} = k$ (because, obviously, $B''C = BC = a$), hence $AB'' = c'' = \frac{c'}{k}$, apply Menelaos in $\triangle BB''C$ with CA:
$BC^2 \cdot AB'' + B''C^2 \cdot AB = AC^2 \cdot BB'' + BB'' \cdot AB \cdot AB''$, or
$a^2 \cdot \left(c + \frac{c'}{k}\right) = b^2 \cdot \left(c + \frac{c'}{k}\right) + \frac{c \cdot c' \left(c + \frac{c'}{k}\right)}{k}$, or $a^2k + b^2k = c \cdot c'$, but $a \cdot k = a'$ and $b \cdot k = b'$, done.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Collinear points - BxMO 2010

Reply

Martin N.

#1 May 2, 2010, 10:49 pm

On a line l there are three different points A, B and P in that order. Let a be the line through A perpendicular to l, and let b be the line through B perpendicular to l. A line through P, not coinciding with l, intersects a in Q and b in R. The line through A perpendicular to BQ intersects BQ in L and BR in T. The line through B perpendicular to AR intersects AR in K and AQ in S.
(a) Prove that P, T, S are collinear.
(b) Prove that P, K, L are collinear.

(2nd Benelux Mathematical Olympiad 2010, Problem 3)

Luis González

#2 May 4, 2010, 12:21 am

By Pascal theorem for the degenerate cyclic hexagon $AALKBB$, the intersections $S \equiv a \cap BK, T \equiv b \cap AL$ and $P' \equiv LK \cap AB$ are collinear. Let $C \equiv AL \cap BK$ and D be the projection of C onto AB, then obviously CD passes through the orthocenter H of $\triangle ABC$ and $(A,B,D,P') = -1$. But from $QA \parallel HD \parallel RB$, it follows that $(A,B,D,P) = -1 \Longrightarrow P \equiv P' \Longrightarrow P,T,S$ and P,K,L are collinear.

Pikachu!!!

#3 May 4, 2010, 11:41 pm

Luis's solution is similar to mine.
Here is a bit difference.

By Pascal Theorem,(too)
1. for cyclic $AALKBB$, we have ST, KL, AB are concurent,
2. for cyclic $AAKLBB$, we have QR, KL, AB are concurent.

Hence, we get that ST, KL must pass through P

eze100

#4 May 5, 2010, 9:07 am

What does it mean $(A,B,D,P') = -1$???

Dimitris X

#5 May 6, 2010, 12:58 am

> eze100 wrote:
> What does it mean $(A,B,D,P') = -1$???

It means that (A,B,D,P') is a harmonic 4-tumble

Dimitris

Lyub4o

#6 Aug 18, 2011, 8:57 pm

I don't really understand how did **Pikachu!!!** use ABLK to prove that QR;KL;AB at the same are concurent.

Quick Reply

High School Olympiads

IMO ShortList 2002, geometry problem 7

Source: IMO ShortList 2002, geometry problem 7

orl

#1 Sep 28, 2004, 7:00 pm • 1

The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let AD be an altitude of triangle ABC, and let M be the midpoint of the segment AD. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N.

Attachments:

This post has been edited 1 time. Last edited by orl, Oct 25, 2004, 6:16 am

orl

#2 Sep 28, 2004, 7:00 pm

Please post your solutions. This is just a solution template to write up your solutions in a nice way and formatted in LaTeX. But maybe your solution is so well written that this is not required finally. For more information and instructions regarding the ISL/ILL problems please look here: introduction for the IMO ShortList/LongList project and regardingsolutions :)

grobber

#3 Sep 29, 2004, 7:20 am • 1

I don't like this solution, but I couldn't find a better one this late at night (or this early in the morning; it's 4:15 AM here :)).

Let $S=KA\cap\Omega$, and let T be the antipode of K on Ω. Let X,Y be the touch points between Ω and CA,AB respectively.

The line AD is parallel to KT and is cut into two equal parts by KS,KN,KD, so $(KT,KN;KS,KD)=-1$. This means that the quadrilateral $KTSN$ is harmonic, so the tangents to Ω through K,S meet on NT. On the other hand, the tangents to Ω through the points X,Y meet on KS, so $KXSY$ is also harmonic, meaning that the tangents to Ω through K,S meet on XY.

From these it follows that BC,XY,TN are concurrent. If $P=XY\cap BC$, it's well-known that $(B,C;K,P)=-1$, and since $\angle KNP=\angle KNT=\frac{\pi}{2}$, it means that N lies on an Apollonius circle, so NK is the bisector of $\angle BNC$.

From here the conclusion follows, because if $B'=NB\cap\Omega,\ C'=NC\cap\Omega$, we get $B'C'||BC$, so there's a homothety of center N which maps Ω to the circumcircle of BNC.

darij grinberg

#4 Sep 30, 2004, 1:58 pm

Grobber, I like your solution! Just to clarify a few points which took me some time to understand:

> *grobber wrote:*
> and since $\angle KNP=\angle KNT=\frac{\pi}{}$

and since $\angle KMT = \angle KNT = \frac{\pi}{2}$,

This is because the segment KT is a diameter of Ω.

> grobber wrote:
> From here the conclusion follows, because if $B' = NB \cap \Omega,\ C' = NC \cap \Omega$, we get $B'C' \| BC$,

Is this trivial? The only explanation I have is to use the intersecting secant and tangent theorem, which yields $BB' \cdot BN = BK^2$ and $CC' \cdot CN = CK^2$, from what we conclude $\frac{BB' \cdot BN}{CC' \cdot CN} = \frac{BK^2}{CK^2}$, but since the line NK bisects the angle BNC, we have $\frac{BK}{CK} = \frac{BN}{CN}$, so that we get $\frac{BB' \cdot BN}{CC' \cdot CN} = \frac{BN^2}{CN^2}$, and thus $\frac{BB'}{CC'} = \frac{BN}{CN}$, what immediately implies B'C' || BC.

As for another solution of the problem, see http://www.mathlinks.ro/Forum/viewtopic.php?t=14741 .

Darij

This post has been edited 1 time. Last edited by darij grinberg, Sep 30, 2004, 5:19 pm

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sprmnt21

#5 Sep 30, 2004, 3:27 pm

I like very mutch Grobber's solution too.

Another way to see that B'C'//BC, once we know that <BNK = <CNK is the following: <B'C'K = <B'NK = <C'NK = <C'KC.

---

darij grinberg

#6 Sep 30, 2004, 5:19 pm

Indeed, I was stupid...

Darij

Agr_94_Math

#7 May 1, 2010, 4:46 pm

An alternate solution using some computation and inversion :
Let us prove that for a circle through $B < C$ tangent to the incircle of triangle ABC at point N', K, M, N' are collinear.
Now, it is enough if we prove that angles BKM and BKN' are equal.

After a few computations, we get $\tan BKM = \frac{cos(\frac{B}{2})cos(\frac{C}{2})}{sin(\frac{B-C}{2})}$

Now apply an inversion with center K and radius BK.

Thereagain, after a few computations, we get $tan(BKN') = tan(BKM) = \frac{cos(\frac{B}{2})cos(\frac{C}{2})}{sin(\frac{B-C}{2})}$.

So we are done.

---

jayme

#8 May 1, 2010, 7:05 pm

Dear Mathlinkers,
this problem was already posted, but where?
In order to have a complete synthetic proof, we can observe that NK goes through the A-excenter...
Sincerely
Jean-Louis

Agr_94_Math

#9 May 1, 2010, 8:26 pm

Dear jayme,
I too on trying for a synthetic solution, tried by the same collinearity.

That is the midpoint of the altitude , tangency point of the incircle with the corresponding side and the corresponding excenter are collinear.
This is true by considering the diametrically opposite point of the tangency point of the excircle with BC and drawing a parallel through it to BC and using homothety and semiprojection result.

But I was not able to finish the problem using this synthetic idea.
So, could you please tell your complete solution?

Virgil Nicula

#10 May 2, 2010, 8:00 am • 1

I observed now this **old and nice problem**. I"ll search its synthetical proof and return soon. Yes **Jayme**,

your remark is very interesting. Indeed, if denote the point L where the A-exincircle touches the side $[BC]$,

then $M \in I_aK \cap IL$ because $\left\{\begin{array}{c}\frac{KD}{KL} = \frac{s-a}{a} = \frac{h_a}{2r_a} = \frac{MD}{LI_a} \\ \frac{LD}{LK} = \frac{s}{a} = \frac{h_a}{2r} = \frac{MD}{IK}\end{array}\right.$.

Luis González

#11 May 3, 2010, 3:11 am • 1

Let N' be the tangency point of the circle ω passing through B, C with Ω. Let U denote the antipode of K WRT Ω and V the tangency point of the A-excircle (I_a) with BC. According to this topic, $N'U, N'K$ bisects $\angle AN'V$ internally and externally. Let N'' be the image of N' under the homothety with center A that takes Ω and (I_a) into each other. Then $UN' \parallel VN'' \Longrightarrow N'K \perp N''V \Longrightarrow \triangle N'VN''$ is isosceles with apex N', which implies that $I_a \in N'K$, due to $I_aV = I_aN'' = r_a$. But since $M \in KI_a$, we deduce that $N \equiv N'$.

ThinkFlow

#12 Dec 30, 2011, 7:49 am

Here is a somewhat longer solution...

Solution

SnowEverywhere

#13 Jan 19, 2012, 5:43 am

Let I be the incenter and I_A be the A-excenter of triangle $\triangle ABC$. Let ω be the incircle, Γ be the A-excircle and let Ω be the circle with diameter AD. The homothety with center A sending ω to Γ takes K to K' where the tangent at K' to Γ is parallel to BC, perpencular to AD and hence parallel to the tangent to Ω at A. Hence the negative homothety taking Ω to Γ takes A to K' and therefore has center on AK'. The center of this negative homothety also lies on the common tangent BC to Ω and Γ and therefore the center of the negative homothety is K. This implies that M, I_A and K are collinear. Now let the circle γ with diameter II_A intersect ω at X and Y. Note that this implies that XI_A and YI_A are tangent to ω and hence that XY is the polar of I_A with respect to ω. Now let XY intersect BC at T. Let N' denote the point on ω such that TN' is tangent to ω and $N' \neq K$. Now note that since T lies on the polar of I_A with respect to ω, I_A lies on the polar of T with respect to ω, which is $N'K$. Hence I_A, N', K and M are collinear and thus $N = N'$. Since BI_A and CI_A bisect the exterior angles of the triangle at B and C, respectively, γ passes through B and C. Hence $BCYX$ is cyclic and, by power of a point with respect to γ and ω, $TN^2 = TX \cdot TY = TB \cdot TC$. This implies that the circumcircle of $\triangle BCN$ is tangent to ω at N, as desired.

Zhero

#14 Mar 21, 2012, 6:44 am

WLOG, let $AB < AC$ (the case $AB = AC$ is trivial.) Let $a = BC$, $b = CA$, $c = AB$. Let I be the the center of Ω, let r be its radius, let Ω be tangent to AB and AC at X and Y, respectively, and let Z be the point where XY meets BC. Let ω_A be the A-excircle of ABC, let I_A be its center, r_A be its radius, and let J be its tangency point with BC, We first claim that M, K, and J are collinear.

Let J' be its antipode in ω_A. A homothety centered at A taking Ω to ω_A sends K to J'. Thus, a homothety centered at K mapping Ω to ω_A must send A to J' as well. Line AD is sent to a line parallel to it passing through J', i.e., line JJ'. Since J, D both line on BC, the homothety must map D to J. Thus, the homothety must map the midpoint of AD to the midpoint of JJ', so M, K, and I_A must be collinear.

Now let P be the midpoint of ZK. Because (Z, B, K, C) is harmonic, we must have $PK^2 = PB \cdot PC$ (this can also be verified by computing the lengths directly, given ZB, which can be found through Menelaus.) Consequently, if the tangent to Ω from P distinct from PB is tangent to Ω at N', we have $PN'^2 = PK^2 = PB \cdot PC$, whence PN' is tangent to the circumcircle of $\triangle N'BC$, i.e., N' is the tangency point of the circle through B, C tangent to Ω. We wish to show that N', K, and I_A are collinear.

We claim that $\triangle PIK \sim \triangle I_aKJ$. The result would then follow, for we would have $\angle I_aKJ = \angle ZIK = 90^\circ - \angle N'KI = \angle N'KD$. Since both triangles are right, it suffices to prove $PK/KI = I_aJ/KJ \iff PK \cdot KJ = rr_A$. Now, $rr_A = \dfrac{K^2}{s(s-a)} = (s-b)(s-c)$, where K is the area and s is the semiperimeter of $\triangle ABC$. Also, $KJ = a - BK - CJ = a - 2(s-b) = b - c$, and

$$\frac{ZB}{ZB+a} = \frac{KB}{KC} = \frac{s-b}{s-c} \implies ZK = ZB + s - b = \frac{(s-b)(a+b-c)}{b-c}.$$

Thus,

$$PK \cdot KJ = \frac{ZK}{2} \cdot KJ = \frac{(s-b)(s-c)}{b-c} \cdot (b-c) = (s-b)(s-c) = rr_A,$$

so we are done.

v_Enhance

#15 Aug 21, 2012, 12:45 am • 8

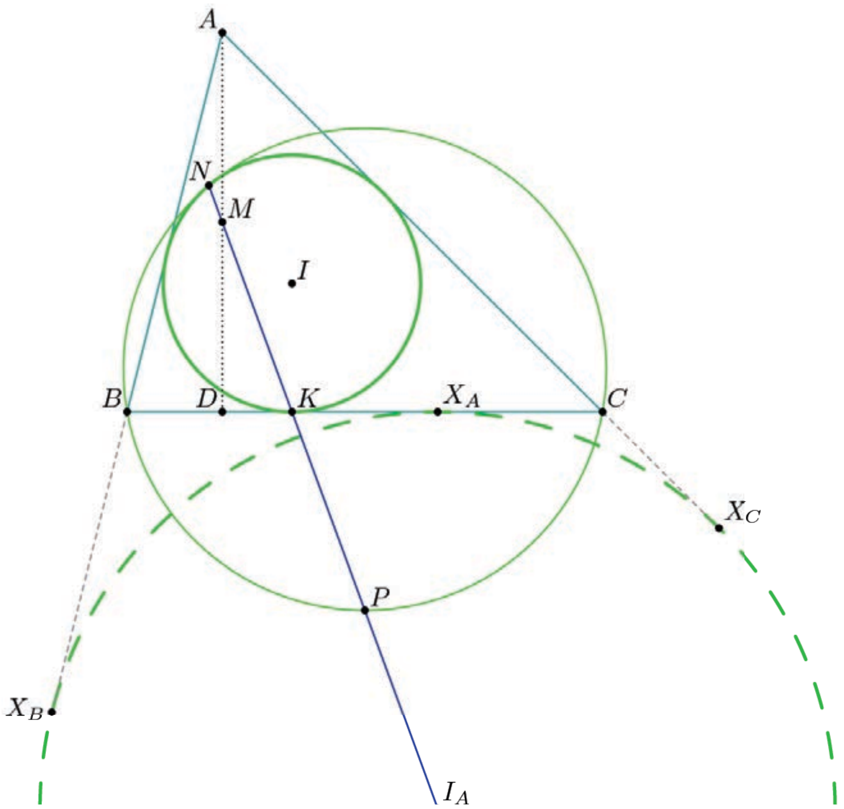
Let I_A be the A-excenter with tangency points X_A, X_B, and X_C to BC, CA and AB, respectively. Define P to be the midpoint of KI_A. Let r be the radius of the incircle and R the radius of the A-excircle.

It is well-known that M, K and I_A are collinear. We claim that $NBPC$ is cyclic; it suffices to prove that $2BK \cdot KC = 2KP \cdot KN = KN \cdot KI_A$. On the other hand, by Power of a Point we have that

$$I_AK\left(I_AK + KN\right) = II_A^2 - r^2 \implies KN \cdot KI_A = II_A^2 - r^2 - I_AK^2$$

Now we need only simplify the right-hand side using the Pythagorean Theorem; it is $\left((r+R)^2 + KX_A^2\right) - r^2 - \left(R^2 + KX_A^2\right) = 2Rr$. So it suffices to prove $Rr = (s-b)(s-c)$, which is not hard.

Now, since P is the midpoint of minor arc $\widehat{BC}$ of (NBC) (via $BK = CX_A$), while the incircle is tangent to segment BC at K, the conclusion follows readily.



sjaelee

#16 Aug 21, 2012, 10:45 pm • 1

Solution

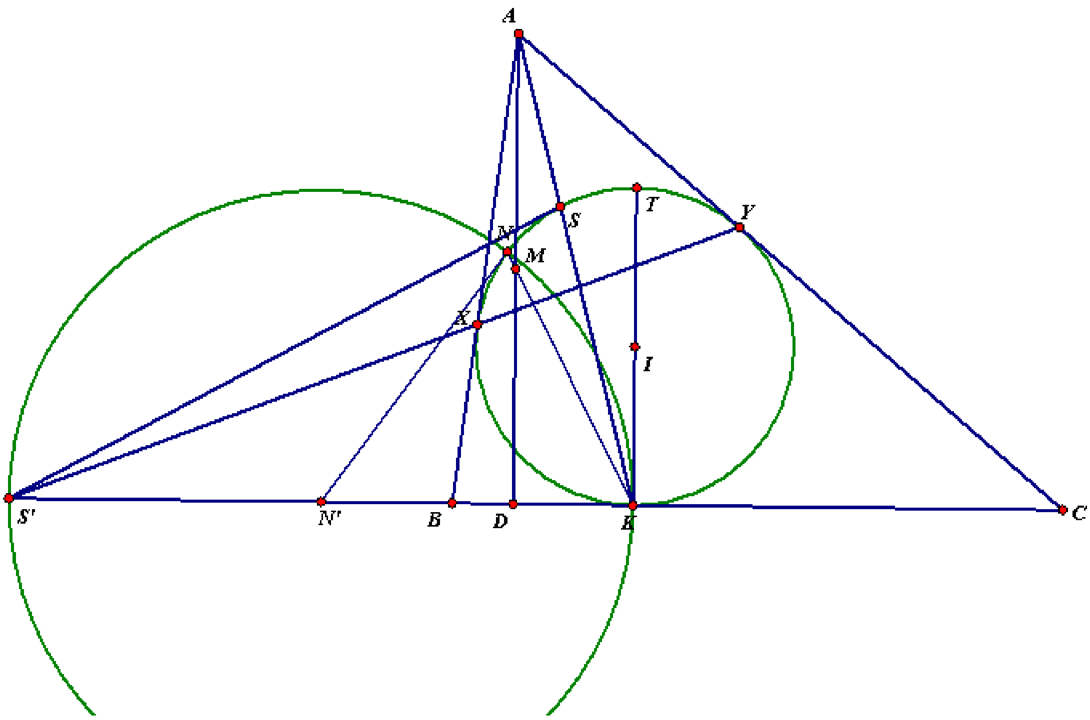
polya78

#17 Sep 8, 2012, 10:12 pm • 2

Interestingly, this problem can be solved using almost exclusively harmonic ratios. Again, let X, Y be the tangent points of Ω with AB and AC respectively and S, T be the intersections of Ω with AK and KI (extended).

Let V be the point at infinity on AD. Then $(V, M; A, D) = -1$, so by projection through K onto Ω, we have, as previously noted, that $KNST$ is a harmonic quadrilateral. Let N', S', T' be the intersections of the tangents at N, S, T with BC. Then $(T', N'; S', K) = -1$, or N' is the midpoint of $S'K$. A lies on the pole of S', so S' lies on the pole of A, which is XY. Thus $(S', K; B, C) = -1$ as well. This means that B and C are inverses with respect to the circle with diameter $S'K$, or that $N'B * N'C = N'K^2 = N'N^2$, which makes $N'N$ a tangent of circle BNC.

Attachments:



math154

#18 Mar 8, 2013, 1:47 am

> *ThinkFlow wrote:*
>
> Let the tangent to Ω at N intersect line BC at P. It suffices to show that aopsnowrap][math=inline]$PN^2 = PB\cdot BC$[/math].[/aopsnowrap

Here's a nice way to finish, inspired by the official solution to RMM 2013 Problem 3.

Let $Q = PI \cap NMK$ be the inverse of P w.r.t. (I), so $PQ \cdot PI = PK^2 = PN^2$, whence it's enough to show $BQIC$ is cyclic. On the other hand, it's well-known that II_a is a diameter of (BIC) and M, K, I_a are collinear, so $\angle IQI_a = 90^\circ$ implies that Q lies on the circle with diameter II_a, as desired.

Rijul saini

#19 Jun 3, 2013, 4:26 pm

> *ThinkFlow wrote:*

Thinkflow wrote:

> Let the tangent to Ω at N intersect line BC at P. It suffices to show that $PN^2 = PB \cdot BC$.

The way I did this was to let R be the inverse of M w.r.t. (I), $S = PR \cap EF, Q = EF \cap BC$. Let L be the point diametrically opposite to K, and let the tangent at L intersect EF at T. Let U be $AL \cap BC$.

By the properties of harmonics, it suffices to prove P is the midpoint of QK.

Now, KT is the polar of U, hence $KT \perp IU$.

On the other hand, P is the pole of KN, so the polar of M passes through P. As it also passes through R, so PR is the polar of M. Hence, the pole of AM is $EF \cap PR$, which is S. Hence, $IS \perp AM \implies IS||BC$. The other thing to notice is that PR is the polar of M implies $PR \perp IM \implies PS \perp IU$ because I, M, U are collinear.

Together with $KT \perp IU$ we get $PS \perp KT$. Now, we need to prove P is the midpoint of QK, so it suffices to prove S is the midpoint QT. But this is clear, as the perpendicular distance from S to BC is r (as $IS||BC$), and the perp distance from T is $2r$ (as $LT||BC$).

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thecmd999

#20 Dec 5, 2013, 11:48 am

Solution

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v_Enhance

#21 Dec 5, 2013, 11:53 am • 1

> thecmd999 wrote:
>
> By this (thanks v_Enhance), ...

You're welcome!

Wolstenholme

#22 Aug 4, 2014, 9:55 am

Let $J = AK \cap \Omega$ and let K' be the antipode of K with respect to Ω. Let R and S be the tangency points of Ω with lines AB and AC respectively and let $X = RS \cap BC$. Let A_∞ be the point at infinity on line AD.

It is clear that $(A, D; M, A_\infty) = -1$ so taking perspective at K and projecting this harmonic division onto Ω we have that quadrilateral $KNJK'$ is harmonic. Moreover since the tangents from R and S to Ω intersect at A and since A, J, K are collinear this implies that quadrilateral $KRJS$ is harmonic as well. Since BC is the tangent from K to Ω this means that X is on the tangent from J to Ω. Therefore X, N, K' are collinear. Therefore $\angle XNK = 180 - \angle K'NK = 90$.

It is well-known that $(X, K; B, C) = -1$ so the fact that $\angle XNK = 90$ implies that NK bisects $\angle BNC$. Now by Archimedes' Lemma this implies that the circumcircle of $\triangle BNC$ is tangent to Ω at N as desired, so we are done.

Now I want to discuss the motivation for this solution. We want to prove that two circles are tangent and we have that one of them is tangent to a chord of the other - this immediately means we can rephrase the problem into proving an angle bisection using Archimedes' Lemma. Now, we have a midpoint of an altitude, as well as an angle we want to prove is bisected and these two things imply that we should look at harmonics and projective geometry. When we have an incircle and tangency points and want a harmonic division, constructing X is always a good idea. The problem then boils down to proving $\angle XNK = 90$ - in other words, that K', N, X are colllinear (we phrase it like this because we want to put things in terms of points on Ω). Now we have lots of tangents and we are thinking about harmonics so finding the right harmonic quadrilaterals is not too arduous a task. All in all, with the right motivation this problem can take 10 minutes!

---

sayantanchakraborty

#23 Sep 4, 2014, 2:00 pm

Let I_a denote the excenter opposite to A,R_a denote the radius of (I_a) and X be the foot of perpendicular from I_a on BC.

Claim 1:M, K, I_a are collinear.

*Proof:*Note that $MD = \dfrac{AD}{2} = \dfrac{\triangle}{a}$ and $I_aX = \dfrac{\triangle}{s-a}$.So $\dfrac{DM}{I_aX} = \dfrac{s-a}{a} = \dfrac{b+c-a}{2a}$.

$DX = KB - BD = \frac{a+c-b}{2} - c\cos B = \frac{a+c-b}{2} - \frac{a^2+c^2-b^2}{2a} = \frac{(b-c)(b+c-a)}{2}$ and
$KX = BX - BK = (s-c)-(s-b) = b-c$.So $\frac{DK}{KX} = \frac{b+c-a}{2a}$.

Combing these we get $\frac{DK}{KX} = \frac{DM}{I_aX}$ so we are done.

Claim 2:If J is the midpoint of KI_a then $BNCJ$ is a cyclic quadrilateral.

Proof:It suffices to show that $NK \cdot KJ = BK \times KC$ or $NK \cdot KI_a = 2(s-b)(s-c)$.Now equating the powers of I_a wrt Ω we get

$I_aK(I_aK+KN) = II_a{}^2 - r^2 \implies I_aK \cdot KN = II_a{}^2 - r^2 - I_aK^2 = IC^2 + CI_a{}^2 - r^2 - R^2 - KX^2 = KC^2 + XC^2 - KX^2 = (s-c)^2 + (s-b)^2 - (b-c)^2 = 2(s^2 - sc - sb + bc) = 2(s-b)(s-c)$
so the claim is proven.

Now it is quite straightforward:By dropping perpendicular from J to BC and using the fact that $BK = CX$ we see that $JB = JC$.So NK bisects $\angle BNC$ and now everything is fine.

jayme
#24 Sep 4, 2014, 4:52 pm

Dear Mathlinkers,
see also
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=455436

Sincerely
Jean-Louis

TelvCohl
#25 Nov 23, 2014, 9:42 pm • 3

I found an interesting and nice proof :

My solution:

Let I_a be $A-$ excenter of $\triangle ABC$.
Let Ω touch CA, AB at P, Q, respectively .
Let W, X, Y, Z be the midpoint of BQ, BK, CK, CP, respectively .
Let T be the tangent point of (I_a) with BC and T' be the reflection of T in I_a .

From homothety with center A send Ω to (I_a) we get A, K, T' are collinear .

Since M, I_a is the midpoint of AD, TT', respectively ,
so we get M, K, I_a are collinear .
Since WX, YZ is the radical axis of $\{B, \Omega\}, \{C, \Omega\}$, respectively .
so $R = WX \cap YZ$ is the radical center of $\{B, C, \Omega\}$.
From homothety with center K send $\triangle XRY$ to $\triangle BI_aC$ we get $R \in KI_a$,
so we get N, M, K, R, I_a are collinear .

Invert with respect to $(R, RB) \equiv (R, RC)$.

Since $RK \cdot RN = RB^2 = RC^2$,
so the image of (BNC) is BC which is tangent to Ω ,
hence we get (BNC) is tangent to Ω at N .

Q.E.D

EulerMacaroni
#26 Jul 16, 2015, 10:25 am

Here's a (bad) solution I found, which appears to be similar to **TelvCohl**'s (I was trying to practice inversion, so my calculation is pretty sparsely written out). Without further ado pretty sparsely written out). Without further ado...

Let T be the midpoint of minor arc BC in the circumcircle of $\triangle BCN$.

Lemma
T lies on the radical axis of Ω and the circle with radius zero centered at B.

Proof of Said Lemma
It suffices to show that the perpendicular bisector of $\overline{BC}$, line KM, and the radical axis of Ω and the circle at B concur. The proof is by barycentric coordinates. The three lines have the following three equations:

$$KM \implies x(b-c)(-a+b+c)-a(a-b+c)y+a(a+b-c)z=0$$

$$\text{Radical axis} \implies x(-a+b+3c)-(a-b+c)y+(3a+b-c)z=0$$

$$\perp \text{ bisector of } B \implies x(c^2-b^2)-a^2y+a^2z=0$$

Line KM is found easily by the coordinates of K and M; the radical axis is found by subtracting the equations of the two circles (the incircle has equation $\sum_{\text{cyc}} a^2yz=(\sum_{\text{cyc}}(s-a)^2x)(x+y+z)$ and the degenerate circle has equation $\sum_{\text{cyc}} a^2yz=(c^2x+a^2z)(x+y+z)$ which is found by setting the equation of the circumcircle equal to the tangent to the circumcircle at B), and the perpendicular bisector equation is well-known. It now suffices to check that

$$\det\begin{pmatrix}(b-c)(-a+b+c) & -a(a-b+c) & a(a+b-c)\\ c^2-b^2 & -a^2 & a^2\\ -a+b+3c & -(a-b+c) & 3a+b-c\end{pmatrix}=0$$

It is straightforward to check that this is true (for example, by subtracting the second row from the first row, then subtracting the new first row from the third row, and then subtracting a times the first row from the second row (this causes two zeros to appear, but I'm too lazy to write it out here)).

Now, invert about T with radius $TB=TC$; it is clear that this inversion swaps segment $\overline{BC}$ and the circumcircle of $\triangle NBC$ and also swaps K and N, by Power of a Point. However, since $\overline{BC}$ is already tangent to $\triangle NBC$ at K, the conclusion follows.

This post has been edited 1 time. Last edited by EulerMacaroni, Jul 17, 2015, 4:22 am
Reason: unhide sol

AMN300
#27 Apr 11, 2016, 11:54 am

Let's do synthetic. Define $NN\cap BC=X$. Let $\odot(ABC)=\Gamma$, the midpoint of minor $\widehat{BC}$ be Y, and the A-excircle be ω. Let ω be centered at I_A, touch BC at T, and finally define the antipode of T wrt ω as T'. Finally, let $NK\cap XI=P$

First, we claim M,K,I_A are collinear. $AD\parallel TT'$ evidently. Combined with A,K,T' and D,K,T collinear, there exists a homothety centered at K takes $\overline{AD}$ to $\overline{TT'}$. M and I_A are midpoints of those respective lines, so the claim is proven.

Now, I claim that $BPIC$ is cyclic. It is clear that $\angle KPI=\angle I_API=90$ from looking at quadrilateral $XNIK$ and the previous claim. Since I and I_A are antipodes wrt $\odot(BIC)$, we have $BPIC$ cyclic as requested.

$BPIC$ cyclic implies $XP\cdot XI=XB\cdot XC$. But from right triangle XNI, similar triangles yield $XN^2=XP\cdot XI$. So $XN^2=XP\cdot XI=XB\cdot XC$, thus PoP yields the desired result and we are done.

This post has been edited 1 time. Last edited by AMN300, Apr 14, 2016, 10:36 am
Reason: forgot to define P

Virgil Nicula
#28 Apr 14, 2016, 8:29 am

See and **here**.

ABCDE

#29 May 18, 2016, 11:05 am

Let I_a be the A-excenter. It is well-known and can be shown by homothety that M, K, I_a are collinear. Let the incircle be tangent to AB at J and AC at L. Let B_1, B_2, C_1, C_2 be the respective midpoints of BJ, BK, CK, CL. Note that B_1B_2 and C_1C_2 are respectively the radical axes of B and the incircle and C and the incircle. Hence, they meet at R, the radical center of B, C, and the incircle. Note that a homothety centered at K with factor 2 sends B_1B_2 and C_1C_2 to the external angle bisectors of $\angle B$ and $\angle C$, so it sends R to I_a. Hence, R is the midpoint of KI_a so it lies on KM.

Now, invert about R with radius RB. Note that this fixes B, C, and the incircle, so it takes K to the second intersection of RK with the incircle, which is N. But since this inversion fixes B and C, line BKC goes to the circumcircle of BNC, and since BKC is tangent to the incircle so is the circumcircle of BNC as desired.

Quick Reply

High School Olympiads

ratio of radii

AndrewTom

#1 Apr 30, 2010, 3:24 pm

Find the ratio of the radius of an escribed circle of a triangle to the radius of the circumscribing circle, in terms of the angles of the triangle.

Prove that the ratio of the radii of the two circles which touch the inscribed circle and the sides AB, AC of a triangle ABC is $\tan^4 \frac{1}{4}(B+C)$>

Luis González

#2 May 2, 2010, 1:16 am

1) Denote R and ϱ_a the circumradius and A-exradius of $\triangle ABC$, respectively. From the well-known identities

$$[\triangle ABC] = 2R^2 \cdot \sin A \cdot \sin B \cdot \sin C \ , \ [\triangle ABC] = \varrho_a (s-a)$$

$$\Longrightarrow \frac{R}{\varrho_a} = \frac{b+c-a}{4R \cdot \sin A \cdot \sin B \cdot \sin C} = \frac{\sin B + \sin C - \sin A}{2 \cdot \sin A \cdot \sin B \cdot \sin C}.$$

2) The radii R_i of a chain of circles (O_i, R_i) tangent to the sides of an angle $\angle(p,q)$ such that (O_i, R_i) is externally tangent to (O_{i-1}, R_{i-1}) and (O_{i+1}, R_{i+1}) form a decreasing geometric progression with ratio $\frac{1-\sin\theta}{1+\sin\theta}$, where θ stands for $\frac{1}{2}\angle(p,q)$.

Therefore, by denoting R_1, R_2 $(R_2 > R_1)$ the radii of the two circles externally tangent to the incircle (I, r) and tangent to the rays AB, AC, we obtain

$$\frac{R_1}{r} = \frac{1-\sin\frac{A}{2}}{1+\sin\frac{A}{2}} \ , \ \frac{r}{R_2} = \frac{1-\sin\frac{A}{2}}{1+\sin\frac{A}{2}}$$

$$\Longrightarrow \frac{R_1}{R_2} = \left(\frac{1-\sin\frac{A}{2}}{1+\sin\frac{A}{2}}\right)^2 = \tan^4\left(\frac{\pi}{4} - \frac{A}{4}\right) = \tan^4\left(\frac{B+C}{4}\right).$$

High School Olympiads

On the hyperbola tangent (me).

Reply

Luis González

#1 Apr 30, 2010, 9:06 pm

Prove the following result synthetically:

$\mathcal{H}$ is a hyperbola with foci A, B and center O. P is an arbitrary point on $\mathcal{H}$ and let the tangent of $\mathcal{H}$ through P cut its asymptotes at M, N. Then $PA + PB = OM + ON$.

yetti

#2 May 1, 2010, 1:51 pm

WLOG, P is on hyperbola branch around the focus B. Let $(Q), (R)$ be circumcircles of $\triangle ABP, \triangle MNO$, respectively. Hyperbola tangent MN bisects the angle $\angle BPA$ internally and P is midpoint of $MN \Longrightarrow$ perpendicular bisector p of MN bisects the angle $\angle APB$ externally. AB bisects the angle $\angle NOM$ internally and perpendicular bisector $q \equiv OQ$ of AB bisects this angle externally. Let $S \equiv p \cap OQ$ and $T \equiv p \cap AB \Longrightarrow S \in (Q) \cap (R)$ and $T \in (R)$. Let C be reflection of B in $p \equiv PS \Longrightarrow PB = PC. \angle BCA = \frac{1}{2}\angle BPA = \frac{1}{2}\angle BSA \Longrightarrow S$ is circumcenter of $\triangle ABC. \triangle APS \sim \triangle TPB$ are similar, having equal angles $\Longrightarrow PA \cdot PC = PA \cdot PB = PS \cdot PT = PM \cdot PN \Longrightarrow AMCN$ is cyclic. Isoceles trapezoid $MBCN$ is also cyclic $\Longrightarrow AMBCN$ is cyclic with circumcircle (S) of $\triangle ABC$. Let ON cut (S) again at K, the reflection of M in $q \equiv OQ \Longrightarrow OK = OM$ and $AKMB$ is an isosceles trapezoid $\Longrightarrow AKCN$ is also isosceles trapezoid and $PA + PB = PA + PC = AC = KN = OK + ON = OM + ON$.

Quick Reply

High School Olympiads

AB+AC > PA + PB + PC , old problem

Reply

yzymat
#1 Apr 30, 2010, 12:55 pm

a triangle $ABC, AB > AC > BC$, a point P Inside the triangle , prove that
$AB + AC > PA + PB + PC$

Thanks!

FelixD
#2 Apr 30, 2010, 2:54 pm

> *yzymat wrote:*
> Let P be a point inside triangle $\triangle ABC$, such that $\overline{BC} < \overline{CA} < \overline{AB}$. Prove that
> $$\overline{AB} + \overline{AC} > \overline{PA} + \overline{PB} + \overline{PC}.$$

Draw parallels g and h through P, such that $AB \parallel g$ and $BC \parallel h$. Moreover, let the intersection of g and AC be C_1, the intersection of g and BC be C_2, the intersection of h and AC be A_1 and the intersection of h and AB be A_2. Since $\overline{AB} > \overline{AC}$, we have $\overline{AP} < \overline{AA_2}$. Moreover, since $\overline{AC} > \overline{BC}$, we have $\overline{CP} < \overline{CC_1}$ and $\overline{C_1A} < \overline{C_2B} = \overline{PA_2}$.
Summing up,

$$\overline{AP} + \overline{BP} + \overline{CP} < \overline{AA_2} + \overline{CC_1} + \overline{BP}.$$

Thus, it's left to prove that

$$\overline{PA_2} + \overline{A_2B} < \overline{BP}$$

holds true, which is the triangle inequality in $\triangle PA_2B$.

Luis González
#3 Apr 30, 2010, 8:08 pm

> *yzymat wrote:*
> Let P be a point inside triangle $\triangle ABC$, such that $\overline{BC} < \overline{CA} < \overline{AB}$. Prove that
> $$\overline{AB} + \overline{AC} > \overline{PA} + \overline{PB} + \overline{PC}.$$

See also the previous lemma in the topic Inequality in the acute triangle.

Quick Reply

Site Support

New Skin

Reply

Hamster1800
#1 Apr 19, 2010, 4:51 am • 2

I noticed that with the maintenance today the skin from AoPS 2.0 has replaced the old skin on both the main AoPS site and mathlinks.ro. However, I much prefer the older skin and have found that there is no option to change the way the site and/or forum is displayed. In the past there has always been a choice between the AoPS and mathlinks skin. Is there any plan to put this functionality back in? If so, when would it be expected to be incorporated? If not, please consider it for those of us who prefer the older skins.

Additionally, I have noticed that with the update $\LaTeX$ in the forums is failing to display. I am sure that this is not intentional, so is it safe to assume that we can expect another site update soon?

EDIT: The LaTeX issue is only when using mathlinks.ro rather than artofproblemsolving.com.

jonathanchou711
#2 Apr 19, 2010, 5:07 am

I agree with the option to switch to different skins. But I really don't care much about my layout.

archimedes1
#3 Apr 19, 2010, 6:10 am • 3

I agree, I prefer the older mathlinks skin.

rrusczyk
#4 Apr 19, 2010, 6:16 am • 1

We're unlikely to add the old skins. It's hard enough maintaining one site, which is part of the reason we essentially merged the views of the old ML and AoPS.

AIME15
#5 Apr 19, 2010, 6:32 am

Tags (bold/italics/etc.) are not available in the Mafia forum. Which is somewhat annoying.

Kent Merryfield
#6 Apr 19, 2010, 7:46 am

I haven't yet figured out how to add or delete members of a restricted forum that I moderate. The general access seems to be through my own profile page, but I don't see a way to add members.

Kunihiko_Chikaya
#7 Apr 19, 2010, 7:50 am

Please see this: http://www.mathlinks.ro/Forum/resources.php?c=87&cid=79&year=1990

When we download the PDF file, we can see typos.

rrusczyk

#8 Apr 19, 2010, 8:06 am

> *Kent Merryfield wrote:*
> I haven't yet figured out how to add or delete members of a restricted forum that I moderate. The general access seems to be through my own profile page, but I don't see a way to add members.

Profile->Usergroups->Manage groups->Manage users

(Took me a little while to find it, too.)

---

aleph0

#9 Apr 19, 2010, 8:40 am

Like AIME15 said, none of the formatting things work in the Mafia forum. Also, I deleted my post and it showed up very strangely. Also, is it possible to get rid of all the Re:s?

Luis González

#10 Apr 19, 2010, 8:41 am • 1

I personally don't like this Aops skin and I think that the majority of us were used to the *Mathlinks skin*. By the way, where is the favorite topic list? Is it the *Manage bookmarks* option?, but it is completly empty. Does this mean that I lost hundreds of important topics added during an entire year?.

BOGTRO

#11 Apr 19, 2010, 9:05 am

Minor problems: bold/italic don't seem to work.

Moderators don't show up in green (on the bottom of the page)

and I like the AoPS skin like 1349357932472398 times better

---

rrusczyk

#12 Apr 19, 2010, 9:09 am

Interesting -- **bold** and *italic* appear to work for me. Were there other ways to do these on the old site that don't work now?

BOGTRO

#13 Apr 19, 2010, 9:10 am

And also, when I try to post in the Anonymous manner in PreCalculus, it says "Error: You are not enrolled in this class". Of course, it might have something to do with the fact that the class is over, but there is still a challenge set and the transcripts.

BOGTRO

#14 Apr 19, 2010, 9:11 am

Well, **bold test** and *italic test* doesn't seem to work on the Mafia forum, which isn't good (although I'm not actually playing in any mafia games), because votes are typcially bolded to distinguish suspicions. Apparently they work here though...

EDIT: Someone should test if there is still a 10-minute wait between consecutive posts. I was going to post that it wasn't in effect anymore, but then I realized that I'm a mod (something I forget a lot -.-)

---

levans

#15 Apr 19, 2010, 9:27 am

Bold, etc. should work on Mafia now. Please let us know of any other forums this may be an issue.

We'll look into the classroom forums, and a host of other issues, beginning on Monday.

Looks like the conversion script that phpBB provides to convert from 2.0 to 3.0 isn't all that great. Feel free to yell at them :)

High School Olympiads

goemetry

lovemathvn

#1 Jan 23, 2010, 3:51 pm

Let ABCD be a quadrilateral where $\widehat{ABC}=\widehat{ADC}=90^0$ and $\widehat{BCD}<90^0$.Choose a point E on the opposite ray of AC such that DA is the angle-besector of BDE.Let M be the chosen arbitrarily between D and E.choose another point N on the opposite ray of BE such that $\widehat{NCB}=\widehat{MCD}$.Prove that MC is the angle-besector of DMN

my solution for this problem uses cyclic quadrilateral.Now could you find some another solutions for me?thanks:D

lovemathvn

#2 Feb 3, 2010, 7:20 pm

> lovemathvn wrote:
>
> Let ABCD be a quadrilateral where $\widehat{ABC}=\widehat{ADC}=90^0$ and $\widehat{BCD}<90^0$.Choose a point E on the opposite ray of AC such that DA is the angle-besector of BDE.Let M be the chosen arbitrarily between D and E.choose another point N on the opposite ray of BE such that $\widehat{NCB}=\widehat{MCD}$.Prove that MC is the angle-besector of DMN
>
> my solution for this problem uses cyclic quadrilateral.Now could you find some another solutions for me?thanks:D

this topic hasn't had any responds yet:(

Luis González

#3 Apr 29, 2010, 5:24 am

Let $P \equiv AD \cap BC$ and $Q \equiv BA \cap DC.$ Then A becomes the orthocenter of the acute $\triangle CPQ$ and $\triangle BED$ is its orthic triangle. Thus, EC bisects $\angle BED$ and C becomes the E-excenter of $\triangle EBD.$ If $\angle MCD=\angle NCB,$ it follows that $\angle MCN=\angle DCB \Longrightarrow \angle DCB=\angle MCN=\angle 90^{\circ}-\frac{1}{2}\angle BED,$ which implies that C is common E-excenter of $\triangle BED$ and $\triangle NEM \Longrightarrow MC$ bisects $\angle DMN.$

High School Olympiads

The circumcenter of a variable triangle is a fix point

Petry

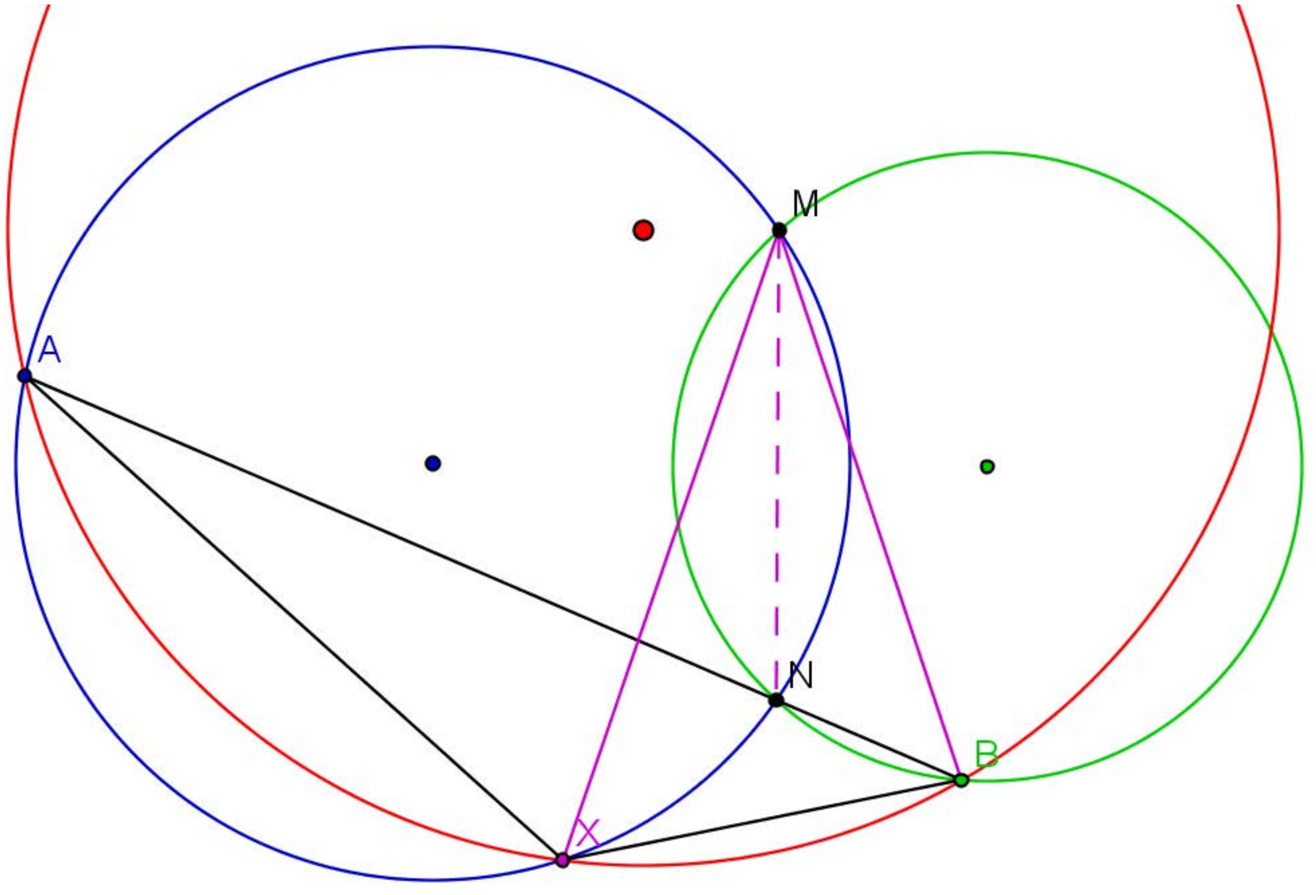
#1 Apr 27, 2010, 5:17 am

Hello!

Let two circles Γ_1 and Γ_2 intersect in points M and N. A is a variable point on the circle Γ_1 and AN intersects the circle Γ_2 again at B. If a point X lies on the circle Γ_1 such that $(MN$ is the bisector of the angle BMX then prove that the circumcenter of the triangle AXB is a fix point.

Best regards,
Petrisor Neagoe

Attachments:



Luis González

#2 Apr 27, 2010, 5:59 am

If the perpendicular line to MN through M cuts Γ_1 and Γ_2 again at $U, V,$ then the circumcenter of $\triangle AXB$ is the midpoint of segment $UV.$ See the topic Concyclic.

Site Support

Favorites

Reply

rrusczyk
#1 Apr 21, 2010, 12:07 am

We have recovered the Favorites -- they are in the profile under "Manage Bookmarks". We will place a link in the side navigation for them in the next week or so.

jjfun1
#2 Apr 21, 2010, 5:24 am

Will past friends/foes be recovered, too?

rrusczyk
#3 Apr 21, 2010, 9:22 pm

We'll look into that soon.

Luis González
#4 Apr 27, 2010, 3:30 am

Thanks for recovering our favorite topics! but , Is there any way to sort bookmarks in the order they were added (As it used to be). Since these are being stored by date (descending) which I think is neither practical nor easy to manage.

rrusczyk
#5 Apr 27, 2010, 6:52 am

Currently there is not, but we'll add it to the list of items to consider adding. (The new bookmarks table is not indexed the same way as the old favorites table was.)

Quick Reply

High School Olympiads

a problem about h_max

Reply

qwerty777
#1 Apr 24, 2010, 3:19 am

Let ABC be an acute triangle, H be orthocenter and h_max be the longest height.
Prove that AH + BH + CH <= 2h_max

I proved that AH + BH + CH <= (a^2+b^2+c^2)^1/2. But I don't know what to do after that.
plz help

Luis González
#2 Apr 24, 2010, 3:49 am

Since AH+BH+CH=2(R+r) for an acute ABC, the problem is equivalent to the one proposed in the topic Geometric inequality with h,r,R. I'm sure this problem has been posted several times in this forum, so perhaps you could try a search with some keywords to check additional solutions.

Quick Reply

High School Olympiads

X , Y , Z Are Collinear.

Reply

seifi-seifi
#1 Apr 23, 2010, 6:28 pm

In Right Triangle ABC With $\angle A=90$, B',C' Are Midpoini Of AC,AB And Incircle Of ABC Touch AB,AC,CB At Q,P,R. PQ Intersect Circumcircle of ABC At E,F Such that E Is Near To Q.If $BE\cap RQ=X$ And $CF\cap RP=Y$ And $PQ\cap B'C'=Z$ Prove That : X,Y,Z Are Collinear.

Luis González
#2 Apr 23, 2010, 7:38 pm • 1

It is well-known that in any triangle the polar of a vertex with respect to the incircle, the inner angle bisector issuing from a second vertex and the midline relative to the third one are concurrent, for instance you may read the article *Another unlikely concurrence* available at http://pagesperso-orange.fr/jl.ayme/. As a result, if A',I denote the midpoint of BC and the incenter of $\triangle ABC$, then the lines $PQ,BI,A'B'$ concur as well as $PQ,CI,A'C'$ do. But since $A'B'$ and $A'C'$ are the perpendicular bisectors of segments AC,AB, it follows that $E\equiv PQ\cap BI\cap A'B'$ and $F\equiv PQ\cap CI\cap A'C'$. By the same reasoning we have that $X\equiv BE\cap RQ\cap B'C'$ and $Y\equiv CF\cap RP\cap B'C'\Longrightarrow X,Y,Z$ are collinear.

shoki
#3 Apr 23, 2010, 8:10 pm

because of the well-known theorem X is on the radical axis of $(O),(I)$ where $(O)=(ABC)$. it's easy to see that Y,Z have the same property. thus X,Y,Z are collinear and the line passing thru them is perpendicular to OI.

Luis González
#4 Apr 23, 2010, 8:57 pm

> *shoki wrote:*
> because of the well-known theorem X is on the radical axis of $(O),(I)$ where $(O)=(ABC)$. it's easy to see that Y,Z have the same property. thus X,Y,Z are collinear and the line passing thru them is perpendicular to OI.

Which figure are you considering?. Line passing through X,Y,Z is not the radical axis of (I),(O), but the A-midline of ABC.

Edit: Sorry, I just realized that I considered P,Q on AB,AC instead of AC,AB. Points X,Y,Z are collinear as well, but it's another configuration. I guess that I've solved a different problem

Quick Reply

High School Olympiads

Points in triangle

Reply

indybar

#1 Feb 26, 2006, 6:16 pm

In $\triangle ABC$, let I be the incenter, O be the circumcenter, H be the orthocenter, R be the circumradius, E be the midpoint of OH, r be the inradius, and s be the semiperimeter.

(a) Find the distance IH in form of R, r, s
(b) Find the distance IE in form of R, r

indybar

#2 Mar 1, 2006, 12:39 pm

Anybody can solve it?

Luis González

#3 Apr 22, 2010, 11:37 pm

Using Leibniz theorem for the circumcenter O, we obtain the relation

$$OG^2 = \frac{1}{3}(OA^2 + OB^2 + OC^2) - \frac{1}{9}(a^2 + b^2 + c^2) = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$$

Since $OG = \frac{1}{3}OH$, it follows that $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$.

Incircle (I) and 9-point circle (E) of $\triangle ABC$ are internally tangent $\Longrightarrow IE = \frac{1}{2}R - r$. Notice that IE becomes the I-median of $\triangle IOH$, therefore

$$IE^2 = \frac{1}{2}(IO^2 + IH^2) - \frac{1}{4}OH^2 \Longrightarrow \ IH^2 = 2IE^2 + \frac{1}{2}OH^2 - IO^2$$

$$IH^2 = 2\left(\frac{R}{2} - r\right)^2 + \frac{9}{2}R^2 - \frac{1}{2}(a^2 + b^2 + c^2) - (R^2 - 2Rr)$$

$$IH^2 = 4R^2 + 2r^2 - \frac{1}{2}(a^2 + b^2 + c^2)$$

Because of $a^2 + b^2 + c^2 = 2s^2 - 2r^2 - 8Rr \Longrightarrow IH = \sqrt{4R^2 + 3r^2 + 4Rr - s^2}$.

Quick Reply

High School Olympiads

Circles

Reply

Source: nice

mr.danh

#1 May 19, 2008, 3:46 pm • 1

ABC is a non-isosceles triangle. The incircle tangents to BC, CA, AB at D, E, F respectively. M, N are on lines EF such that $BMNC$ is a cyclic quadrilateral. Denote K be the midpoint of side BC. Prove that M, N, D, K lie on a circle.

Zagros

#2 Jul 4, 2008, 3:03 pm • 1

Assume that $AB < AC$. Let EF intersect BC in point G. Let $BC = a$, $AE = AF = x$, $BF = BD = y$ and $CD = CE = z$.

Based on Menelaus theorem, we have $\left(\frac{CE}{EA}\right)\left(\frac{AF}{FB}\right)\left(\frac{BG}{GC}\right) = 1$ or

$\left(\frac{z}{x}\right)\left(\frac{x}{y}\right)\left(\frac{BG}{BG+a}\right) = 1 \Rightarrow \frac{BG+a}{BG} = \frac{z}{y} \Rightarrow 1 + \frac{a}{BG} = \frac{z}{y} \Rightarrow BG = \frac{ay}{z-y}\ (1)$.

$BMNC$ is a cyclic quadrilateral $\Rightarrow GM.GN = GB.GC\ (2)$.

To show that $MDKN$ is a cyclic quadrilateral, we have to prove that $GD.GK = GM.GN$. By using (1) and (2), we have

$MDKN$ is a cyclic quad $\iff GD.GK = GB.GC \iff (GB+y)(GB+\frac{a}{2}) = GB(GB+a) \iff$

$GB(\frac{a}{2}-y) = \frac{ay}{2} \iff \left(\frac{ay}{z-y}\right)\left(\frac{a}{2}-y\right) = \frac{ay}{2} \iff a = y+z.$

The proof is complete.

Luis González

#3 Apr 22, 2010, 4:07 am

> *mr.danh wrote:*
>
> ABC is a non-isosceles triangle. The incircle tangents to BC, CA, AB at D, E, F respectively. M, N are on lines EF such that $BMNC$ is a cyclic quadrilateral. Denote K be the midpoint of side BC. Prove that M, N, D, K lie on a circle.

Let line EF cut BC at P. Then the cross ratio (B, C, D, P) is harmonic $\Longrightarrow \overline{PD} \cdot \overline{PK} = \overline{PB} \cdot \overline{PC} = \overline{PN} \cdot \overline{PM} \Longrightarrow$ Points M, N, D, K are concyclic.

Quick Reply

High School Olympiads

Similar triangles

Reply

oneplusone
#1 Apr 20, 2010, 7:40 pm

Let A', B', C' be points on BC, AC, AB of triangle ABC such that $\triangle A'B'C' \sim \triangle ABC$. Prove that the orthocentres of both triangles are equidistant from the circumcentre of $A'B'C'$.

NO COMPLEX

Luis González
#2 Apr 21, 2010, 7:50 am • 2

Circles $\odot(AB'C'), \odot(BA'C'), \odot(CB'A')$ concur at the Miquel point O of $\triangle ABC \cup A'B'C'$. Since $\angle B'OA' = \angle ACB = \angle A'C'B'$ and $\angle C'O'A' = \angle ABC = \angle A'B'C' \pmod \pi$, then O is the orthocenter of $\triangle A'B'C'$. Moreover $\angle OCA = \angle OA'B' = \angle OC'B' = \angle OAC \Longrightarrow OA = OC$. Likewise we have $OA = OB \Longrightarrow O$ is circumcenter of $\triangle ABC$. If D, E, F denote the midpoints of BC, CA, AB, then O becomes center of the spiral similarity that swaps $\triangle A'B'C'$ and $\triangle DEF$. If O', N are the circumcenters of $\triangle A'B'C'$ and $\triangle DEF$ (9-point center of ABC), then O', N are homologous under the referred spiral similarity as well as $C', F \Longrightarrow \triangle ONO' \sim \triangle OFC' \Longrightarrow \angle ONO'$ is right $\Longrightarrow O'$ lies on the perpendicular bisector of the segment connecting the circumcenter O and orthocenter H of $\triangle ABC \Longrightarrow OO' = HO'$.

Quick Reply

High School Olympiads

how we can draw a circle such that .... .

Reply

seifi-seifi
#1 Apr 20, 2010, 10:59 pm

in triangle ABC draw a circle such that its internal tangent to circumcircle of ABC and tangent to AB and AC.

im sorry if it posted before.

epsilonist
#2 Apr 20, 2010, 11:31 pm

> *seifi-seifi wrote:*
> in triangle ABC draw a circle such that its internal tangent to circumcircle of ABC and tangent to AB and AC.
>
> im sorry if it posted before.

I think you did mean **construct** by using only a compass and a straightedge, right?

If so, I'll show you here.

There is a well-known result that if the A-mixtilinear circle touches the sides AB and AC at points P and Q, respectively, then the midpoint of PQ is the incenter of the triangle ABC.

First, we use those tools to construct three internal angle bisectors cutting at point I, the incenter.
From I, construct the line perpendicular to AI at I cutting AB and AC, indeed, at P and Q, respectively.
Construct the lines passing through P, Q perpendicular to AB, AC, respectively, cutting at point K.

This K is, indeed, the center of the A-mixtilinear circle.
Hence the circle can be constructed centered at K with radius KP.

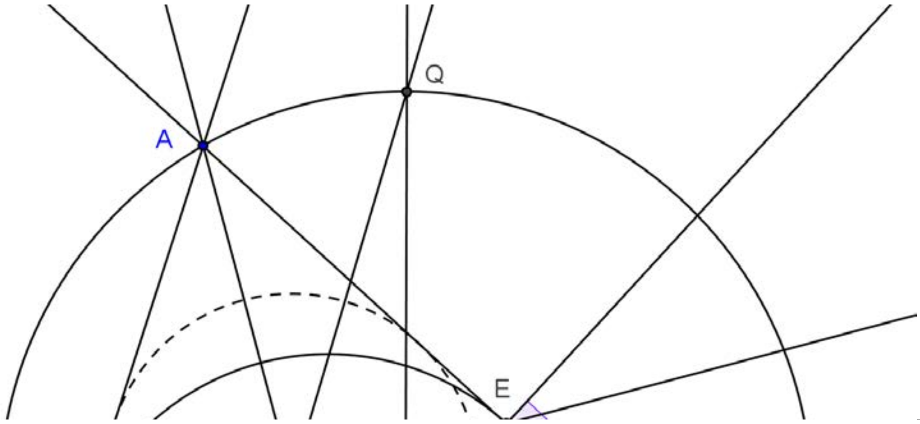
We're finished.

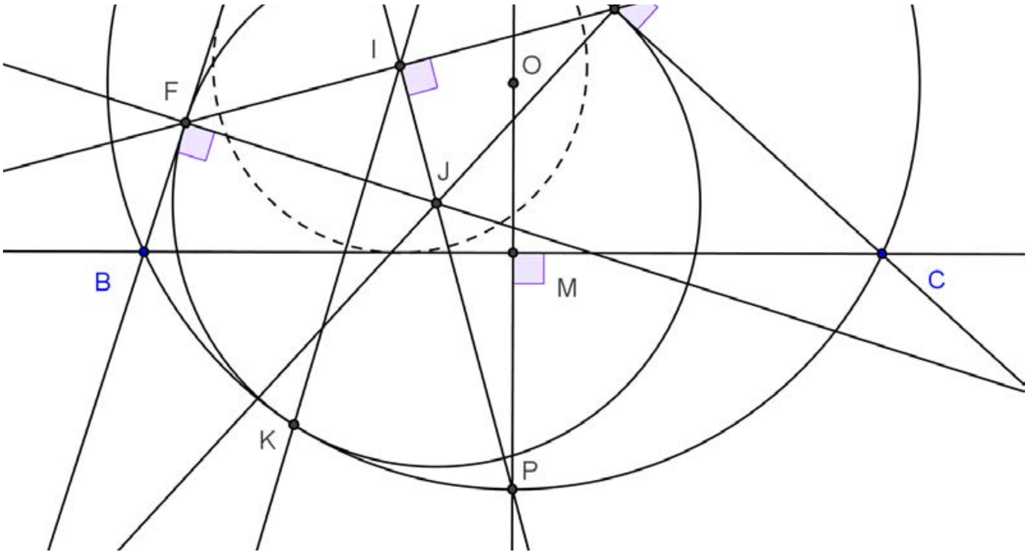
This post has been edited 1 time. Last edited by epsilonist, Apr 20, 2010, 11:35 pm

Luis González
#3 Apr 20, 2010, 11:32 pm

See the attachment below

Attachments:





Quick Reply

High School Olympiads

Tagent circles

Reply

Source: own

livetolove212
#1 Apr 17, 2010, 12:24 pm • 1

Given triangle ABC and its circumcircle (O). Construct a circle (O') which is tangent to (O) at A and tangent to BC. The second tangents of (O') through B, C intersect at D. DA cuts (O) again at E. Two tangents of (O) through E intersect BC at G, H. Prove that (EGH) is tangent to (O).

Luis González
#2 Apr 19, 2010, 12:02 pm • 1

Let ω be the circle tangent to DB, DC and internally tangent to (O) at E'. D is exsimilicenter of $(O') \sim \omega$, A is exsimilicenter of $(O) \sim (O')$ and E' is exsimilicenter of $(O) \sim \omega$. By Monge & d'Alembert theorem, D, A, E' are collinear $\Longrightarrow E \equiv E'$. In this configuration, it's well-known that EO' bisects $\angle BEC$, for instance see the article *Un remarquable résultat de Vladimir Protassov* at http://pagesperso-orange.fr/jl.ayme/. Therefore, if EG, EH cut (O) again at G', H', then the arcs BG' and CH' are congruent $\Longrightarrow GH \parallel G'H'$, which implies that $\odot(EGH)$ is tangent to (O) at E.

litongyang
#3 Apr 19, 2010, 4:21 pm

> livetolove212 wrote:
> Given triangle ABC and its circumcircle (O). Construct a circle (O') which is tangent to (O) at A and tangent to BC. The second tangents of (O') through B, C intersect at D. DA cuts (O) again at E. Two tangents of (O) through E intersect BC at G, H. Prove that (EGH) is tangent to (O).

E is a point on (O), how can we make two angents of (O) through E?

Concyclicboy
#5 Apr 21, 2012, 12:54 pm

> luisgeometra wrote:
> it's well-known that EO' bisects $\angle BEC$, for instance see the article *Un remarquable résultat de Vladimir Protassov* at http://pagesperso-orange.fr/jl.ayme/

In the article the circles are externally tangents, in the problem are internally tangents. In the other case when the circles are internally tangents I understand that... for example the bisector of angle $\angle BEC$ pass through the excenter I_E of BEC or the bisector of $\angle BAC$ pass through the excenter I_D

But i can't understand why EO' bisects $\angle BEC$, or why E, O' and I_E are collinear.

Quick Reply

High School Olympiads

An interesting geometric problem

Source: not hard

mr.danh
#1 Sep 28, 2008, 3:03 pm

Let a triangle ABC and I is its incenter. AI cuts the incircle (I) at D. Prove that the tangent of (I) at D and the external bisector of angle BIC meet on BC.

Luis González
#2 Apr 18, 2010, 9:54 am

Denote X, Y, Z the tangency points of (I) with BC, CA, AB and let the internal angle bisector of $\angle BIC$ and the tangent of (I) at D cut BC at $V, P,$ respectively. We shall prove that IP is the external bisector of $\angle BIC.$ Midpoint M of the arc BC of the circumcircle $\odot(ABC)$ is circumcenter of $\triangle BIC.$ Thus, IX and $IM \equiv IA$ are isogonals with respect to $\angle BIC \Longrightarrow IV$ bisects $\angle MIX.$ Analogously, if R is the projection of X on $YZ,$ the rays XI, XR are isogonals with respect to $\angle YXZ.$ Lines IM, XR are parallel since they are both perpendicular to $YZ \Longrightarrow$ angle bisectors of $\angle MIX$ and $\angle RXI$ are parallel $\Longrightarrow XD \parallel IV,$ but $IP \perp XD \Longrightarrow IP \perp IV \Longrightarrow IP$ is the external bisector of $\angle BIC.$

jayme
#3 Nov 10, 2014, 1:58 pm

Dear Mathlinkers,
with my notation

Dear Mathlinkers,

1. ABC a triangle
2. (I) the incircle
3. DEF the contact triangle
4. X the point of intersection of the segment AI with (I)
5. M the foot of the I-inner bissector of the triangle BIC
6. A* the point of intersection of the tangent to (I) at X and BC.

Proof

1. According to http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=612981
DX // IM
2. A*I is perpendicular to DX
3. A*I is then perpendicular to IM
and we are done…

Sincerely
Jean-Louis

Dukejukem
#4 Nov 11, 2014, 1:26 am

Solution

Quick Reply

High School Olympiads

Fixed point

Reply

Arne
#1 Jul 1, 2004, 4:19 pm

Given is a halfplane with points A and C on its edge. For every point B of this halfplane, consider the squares $ABKL$ and $BCMN$ lying outside the triangle ABC. Prove that the lines LM (as B varies) pass through a fixed point.

castigioni
#2 Apr 15, 2010, 4:44 am

Is this problem correct?I drawing the conditions and the line doesn't have a fixed point

Bictor717
#3 Apr 16, 2010, 8:56 am

I constructed the figure using Geometer's Sketchpad, and there is a fixed point. Make sure that you're fixing A and C, and that the line you're considering connects the vertices of the squares that are furthest apart.

Unfortunately I haven't made much progress on the solution.

Luis González
#4 Apr 16, 2010, 9:35 am

According to problem 2 in the topic 3 geometry problems, the lines AN,CK,LM concur at a point P such that <APC is right and LM is its external bisector. P moves on the semicircumference with diameter AC constructed in the given half-plane, thus LM always passes through its midpoint Q, obviously fixed.

Quick Reply

High School Olympiads

problem with Triangle

Reply

Source: Prove that AN is prep to NM

Abdek

#1 Apr 16, 2010, 2:59 am

Problem:
Let ABC be a triangle and D the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BE, and AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of line segments BC and EF, respectively. Prove that AN is perpendicular to NM.

Luis González

#2 Apr 16, 2010, 6:49 am

Since $AEBD$ and $ADFC$ are inscribed in the circles with diameters $AB, AC,$ we get $\angle(EA,ED) = \angle(BA,BD)$ and $\angle(FA,FD) = \angle(CA,CD) \Longrightarrow \triangle AEF$ and $\triangle ABC$ are directly similar and their common vertex A is center of the spiral similarity. Since $\frac{\overline{NE}}{\overline{NF}} = \frac{\overline{MB}}{\overline{MC}} = -1,$ it follows that M, N are homologous under the referred spiral similarity. Then $\angle NAM$ equals the rotational angle $\Longrightarrow \angle(AN,AM) = \angle(AE,AB) = \angle(DE,DB) \Longrightarrow A, M, D, M$ are concyclic. Therefore, $\angle ANM = \angle ADM = 90^\circ.$

Quick Reply

High School Olympiads

an acute triangle

Reply

Source: Prove that MN'P' is equilateral

Abdek

#1 Apr 16, 2010, 3:05 am

Let ABC be an acute triangle, and let T be a point in its interior such that $\angle ATB = \angle BTC = \angle CTA$. Let M, N, and P be the projections of T onto BC, CA, and AB, respectively. The circumcircle of the triangle MNP intersects the lines BC, CA, and AB for the second time at M', N', and P', respectively. Prove that the triangle $M'N'P'$ is equilateral.

Luis González

#2 Apr 16, 2010, 6:21 am

T is the 1st Fermat point of $\triangle ABC$ which sees BC, CA, AB under 120°. The perpendicular lines dropped from A, B, C to NP, PM, MN concur at its isogonal conjugate T'. Let X, Y, Z be the orthogonal projections of T' onto BC, CA, AB. From the cyclic quadrilaterals $T'XBZ, T'YCX$ and the pairs of isogonal rays BT, BT' and CT, CT', we have

$$\angle ZXT' = \angle T'BZ = \angle TBC \ , \ \angle YXT' = \angle T'CY = \angle TCB$$

$$\Longrightarrow \angle ZXY = 180^\circ - \angle TBC - \angle TCB = 60^\circ$$

Similarly, we'll get $\angle YZX = \angle ZYX = 60^\circ \Longrightarrow \triangle XYZ$ is equilateral. But, since two isogonal conjugates have the same pedal circle, we deduce that $X \equiv M', Y \equiv N'$ and $Z \equiv P'$, i.e. $\triangle M'N'P'$ is equilateral.

MJ GEO

#3 Apr 17, 2010, 11:35 pm

Its interesting that this triangle has the less perimeter form the equilateral triangles that thier vertexs are on sides of ABC.

ridgers

#4 Feb 24, 2011, 1:54 am

Is there any other more simple way without using isogonal conjugate?

vladimir92

#5 Feb 25, 2011, 3:56 am

Dear Louis,
Your solution is really nice, but we can also do it this way:

> *ridgers wrote:*
> Is there any other more simple way without using isogonal conjugate?

Of course there is: By proving the propreties used by Louis related to isogonal conjugates and use them as lemmas.
fact 1 . The perpendicular lines dropped from A , B and C to sides (in the usual order) of the pedal triangle of a point P with respect to $\triangle ABC$ concur at the isogonal conjugates of P with respect to $\triangle ABC$.
fact 2. Two isogonal conjugates with respect to $\triangle ABC$ have the same podar circle.

Important fact: The triangle $\triangle M'N'P'$ is the equilateral which have vertices in sides of $\triangle ABC$ for which the area $[M'N'P']$ is minimum.

Quick Reply

High School Olympiads

equality in the triangle ABC

Reply

aktyw19

#1 Mar 31, 2010, 5:08 pm

Chords $AA',\ BB'\ CC'$ circle with O center intersect at X. Show that if $OX\perp MX$, where M - the center of gravity of the triangle ABC, then there is equality:

$$\frac{AX}{A'X}+\frac{BX}{B'X}+\frac{CX}{C'X}=3$$

Luis González

#2 Apr 16, 2010, 1:38 am

By Leibniz theorem for X,O in $\triangle ABC$ with its centroid $M,$ we get

$$XA^2+XB^2+XC^2=3XM^2+\tfrac{1}{3}(a^2+b^2+c^2)$$

$$3R^2=3OM^2+\tfrac{1}{3}(a^2+b^2+c^2)$$

$$\Longrightarrow XA^2+XB^2+XC^2=3(XM^2+R^2-OM^2)$$

Since $OX\perp MX,$ we have $OM^2=XM^2+XO^2$

$$XA^2+XB^2+XC^2=3(R^2-OX^2)$$

Indeed the RHS equals three times the power of X to $(O),$ hence

$$\frac{XA^2}{XA\cdot XA'}+\frac{XB^2}{XB\cdot XB'}+\frac{XC^2}{XC\cdot XC'}=3\Longrightarrow\frac{XA}{XA'}+\frac{XB}{XB'}+\frac{XC}{XC'}=3.$$

Quick Reply

High School Olympiads

incircle

Reply

Aquarious
#1 Feb 21, 2010, 1:28 pm

The incircle of ABC touches BC at D ,and the excircle opposite to B touches BC at E suppose

that AD= AE prove that 2C-B=180'

Luis González
#2 Apr 15, 2010, 8:04 am

O,H,I are the circumcenter, orthocenter and incenter of $\triangle ABC.$ I_a,I_b,I_c are the three excenters against $A,B,C.$ If $AD=AE,$ then AH becomes the midline of the trapezoid $EDII_b \Longrightarrow AH$ passes through the midpoint B' of $II_b,$ but since $I,(O)$ become orthocenter and 9-point circle of the $\triangle I_aI_bI_c,$ it follows that $B' \in (O).$ We deduce then $\gamma > \frac{\pi}{2}$ and $\angle(AC,AH)=\angle(BC,BI) \Longrightarrow \gamma-\frac{\pi}{2}=\frac{\beta}{2} \Longrightarrow 2\gamma-\beta=\pi.$

Aquarious
#3 Apr 15, 2010, 10:33 am

"but since I,(O) become orthocenter and 9-point circle of the excentral triangle \triangle I_aI_bI_c , it follows that B' \in (O) "

I don't understand

Quick Reply

High School Olympiads

Circle

Reply

Aquarious

#1 Apr 12, 2010, 11:00 am • 1

Two circles intersect at D and E .They are tangent to the sides AB and AC of ABC at B and C ,respectively . If D is the midpoint of BC , prove that (DA)(DE)=DC^2

zzz123

#2 Apr 13, 2010, 8:44 am

Let $AB = x, AC = y, BD = DC = \frac{z}{2}$ since B, D are points of tangency we have $\angle ABD = \angle DEB$ and $\angle ACD = \angle CED$ so,$ABEC$ is cyclic we have $\triangle EDB \sim \triangle ECA \to EC = ED(\frac{2y}{z})$ and $\triangle BAE \sim \triangle DCE \to EB = ED(\frac{2x}{z})$ Now,since D is the midpoint of BC Using Stewart's theorem in $\triangle ABC$ and $\triangle BEC$ for which AD and ED are median we have $AD = \frac{1}{2}\sqrt{2x^2 + 2y^2 - z^2}$ and $ED = (\frac{z^2}{4})(\frac{2}{\sqrt{2x^2 + 2y^2 - z^2}})$which immediately gives $(AD)(DE) = DC^2$

Batominovski

#3 Apr 13, 2010, 10:01 am

Consider the triangle A_1BD and A_2CD constructed as follows:
1) A_1 lies on the ray BA and A_2 lies on the ray CA;
2) The triangles A_1BD and BED are directly similar;
3) The triangles A_2CD and CED are directly similar.
Thus, $\frac{A_1D}{BD} = \frac{BD}{DE}$ and $\frac{A_2D}{CD} = \frac{CD}{DE}$. This means

$$A_1D = \frac{BD^2}{DE} = \frac{CD^2}{DE} = A_2D\,.$$

Since $\angle BDA_1 + \angle CDA_2 = \angle EDB + \angle EDC = \pi$, we see that D, A_1, and A_2 lie on the same straight line ℓ, with A_1 and A_2 on the same side of ℓ with respect to D. Therefore, $A_1 = A_2 = A$. This establishes

$$AD \cdot DE = DC^2\,.$$

Luis González

#4 Apr 14, 2010, 2:14 am

Since $\angle(AB, AC) = \angle(BE, BC) + \angle(CB, CE) = \angle(EC, EB)$ (mod 180), we get that $ACEB$ is cyclic and harmonic, due to $\angle(ED, EB) = \angle(BC, BA) = \angle(EC, EA) \Longrightarrow EA$ is both E- and A- symmedian of $\triangle EBC$ and $\triangle ABC$.

$$\Longrightarrow\ DE \cdot AE = EB \cdot EC\ ,\ DA \cdot AE = AB \cdot AC$$

$$\Longrightarrow\ DA \cdot DE = \frac{EB \cdot EC \cdot AB \cdot AC}{AE^2}$$

The fact that $ACEB$ is harmonic yields $AC \cdot EB = AB \cdot EC$, hence we obtain

$$DA \cdot DE = \left(\frac{AC \cdot BE}{AE}\right)^2 = DC^2 = DB^2.$$

Quick Reply

High School Olympiads

M,N are equidistant from O

Reply

Source: Mathematical Reflections

saeedghodsi

#1 Apr 12, 2010, 11:34 pm

M,N are two points inside the circle $C(O)$ such that O is the midpoint of MN . A is an arbitary point on this circle and AM,AN intersect this circle for the second time at points B,C respectively . the two tangents to the circle at points B,C meet each other at D . prove that the perpendicular bisector of MN passes through the midpoint of AD

Luis González

#2 Apr 13, 2010, 7:33 am • 1

Let A' be the antipode of A WRT (O) and τ the tangent line of (O) at A'. If B' denotes the antipode of B WRT (O), it follows that $ABA'B'$ is a rectangle. Thus, reflection N of M about O lies on the line $A'B' \Longrightarrow N \equiv AC \cap A'B' \cap OM$. By Pascal theorem for the degenerate cyclic hexagon $A'A'ACBB'$, the intersections $P \equiv BC \cap \tau, O \cap AA' \cap BB'$ and $N \equiv A'B' \cap AC$ are collinear, in other words, MN, BC and τ concur at P. Therefore, polar of P with respect to (O) passes through A' and the intersection of the tangents of (O) at $B,C \Longrightarrow DA'$ is the polar of P with respect to $(O) \Longrightarrow DA' \perp MN$. If U denotes the midpoint of AD, then we have $UO \parallel DA'$, i.e. $UO \perp MN$.

Quick Reply

High School Olympiads

simple collinear (Own)

Reply

¬[ƒ(Gabriel)³²¹º]¼

#1 Jan 26, 2010, 8:33 am

Let a triangle ABC and a point P. Call A' the second intersection of the circle with center B from P and the circle with center C from P. Define B' and C' cyclically. Call O the circumcenter of A'B'C'. Prove that OP, AA', BB' cuncur iff AA', BB', CC' concur.

Luis González

#2 Apr 12, 2010, 7:16 am

A', B', C' are obviously the reflections of P about $BC, CA, AB,$ thus A, B, C become circumcenters of $\triangle PB'C'$, $\triangle PC'A'$ and $\triangle PA'B' \Longrightarrow$ perpendicular lines dropped from A, B, C to $B'C', C'A'$ and $A'B'$ concur at $O.$ Assume that AA', BB', CC' concur at a point $Q.$ Then $\triangle ABC$ and $\triangle A'B'C'$ are perspective through Q and orthologic through the orthology centers $P, O.$ By Sondat's theorem, P, Q, O are collinear $\Longrightarrow AA', BB', CC', OP$ concur. Conversely, assume that AA', BB', OP concur at a point $Q.$ Let $C'' \equiv CQ \cap PC'.$ Then $\triangle ABC$ and $\triangle A'B'C''$ are perspective through Q and orthologic, where P is one of the orthology centers. Thus by Sondat's theorem, it follows that the perpendicular line dropped from C to $A'B'$ have to cut PQ at their second othology center, in other words, O is their second orthology center. Consequently, $BO \perp A'C''$ and $AO \perp B'C''$, which implies that C' and C'' are necessarily identical.

Quick Reply

High School Olympiads

Challenging Geometry Proof

Reply

ShocK n AwE
#1 Apr 11, 2010, 10:20 pm

[geogebra]7308ee9201641a5b2ae6af864a4ec94c944da001[/geogebra]

Let AB be any chord in a circle O. Let P be a point on AB.

Draw two chords through P, MN and RS. Let the intersection of AB and MS be X and the intersection of AB and RN be Y.

Prove that $\frac{1}{AP}-\frac{1}{XP}=\frac{1}{PB}-\frac{1}{PY}$

Luis González
#2 Apr 12, 2010, 12:26 am

Inversion with pole P and power k^2 transforms the circumcircle (O) of $MRNS$ into itself and MS, RN go into $\odot(PRN)$ and $\odot(PMS)$. Then AB cuts $\odot(PRN)$ and $\odot(PMS)$ again at the inverses U, V of $X, Y,$ respectively. From $\angle MUP=\angle NRP=\angle SMP=\angle SVP,$ we deduce that if $T \equiv UN \cap VS,$ then $\triangle TUV$ is isosceles with apex $T.$ If VS cuts (O) again at $L,$ we have $\angle TLN=\angle NRS=\angle UVL \Longrightarrow NL \parallel UV,$ which means that the center of (O) lies on the perpendicular bisector of the segment $UV.$ By obvious symmetry, it folllows that $UB=VA \Longrightarrow$ $PU-PB=PV-PA.$ Now, it remains to use the inversion power k^2 to plug the corresponding primitive distances

$$\frac{k^2}{PX}-\frac{k^2}{PA}=\frac{k^2}{PY}-\frac{k^2}{PB} \Longrightarrow \frac{1}{PA}-\frac{1}{PX}=\frac{1}{PB}-\frac{1}{PY}.$$

dgreenb801
#3 Apr 12, 2010, 1:02 am

This is equivalent to Haruki's Lemma! :) See proof #11 here Butterfly Theorem

ShocK n AwE
#4 Apr 12, 2010, 5:14 am

Awesome link! My solution was very similar to the area one (Proof 8). Since you posted the link, no need for another solution on top of 18 more.

Quick Reply

High School Olympiads

Three lines concur on the Euler line

Reply

Source: 0

Luis González
#1 Mar 13, 2010, 9:59 pm

Let O,H be the circumcenter and orthocenter of triangle $\triangle ABC$. $\triangle H_aH_bH_c$ is its orthic triangle and $\triangle A'B'C'$ is the antipedal triangle of O with respect to $\triangle H_aH_bH_c$. A',B',C' against $H_a,H_b,H_c,$ respectively. Show that lines $A'H_a,B'H_b$ and $C'H_c$ concur on the Euler line of $\triangle ABC$.

jayme
#2 Mar 14, 2010, 3:03 pm

Dear Luis and Mathlinkers,
very nice result. I think you have a proof. A synthetic one ?
I see only that O and H are pole of orthology wrt A'B'C' and HaHbHc.
We have to prove that one of these points are a PC point... But I don't see how to prove it. Then we are done...
Sincerely
Jean-Louis

Luis González
#3 Mar 14, 2010, 10:14 pm

Dear Jean-Louis, yes you are right. The idea is to prove that HaHbHc and A'B'C' are perspective and orthologic through the orthology centers O (which is obvious) and H (which may be identified by seeing the sidelines of A'B'C' as tangent lines of the MacBeath inconic). More general ideas can be seen in the topic Concurrent 2 (PC-point again).

jayme
#4 Mar 15, 2010, 8:07 pm

Dear Luis and Mathlinkers,
thank for your answer.
I have a question : can we arrive to the solution whithout calculus?
Sincerely
Jean-Louis

jayme
#5 Mar 16, 2010, 9:56 pm

Dear Luis and Mathlinkers,
Continuing my idea, H is the incenter and O the Bevan point of HaHbHc.
It is know that the Bevan point is a PC-point with the Nagel point being the associate point.
What do you think?
Sincerely
Jean-Louis

Luis González
#6 Mar 16, 2010, 10:18 pm

Dear Jean-Louis, thanks for your interest. That's right, we can formulate the problem from your point of view. To prove that triangles A'B'C' and HaHbHc are perspective, we could use the following lemma: Antipedal triangle of P with respect to ABC is perspective with ABC if and only if its pedal triangle with respect to ABC is perspective with ABC. This can be proved easily with

Ceva's theorem. Indeed O is the Bevan point of HaHbHc which is a PC-point in HaHbHc associated with its Nagel point as you pointed out, thus A'Ha, B'Hc, C'Hc concur at a point Q.

To show that Q lies on OH, we may use Sondat's theorem, but we need to identify H as the second orthology center of A'B'C' and HaHbHc. I don't have a proof of this result without using conic tangents, any simpler ideas?

jayme

#7 Mar 16, 2010, 10:56 pm

Dear Luis and Mathlinkers,
yes, I was to quickly. Having prove synthetically the Sondat theorem, I keep always in mind that any theorem can be proved in the shape of the elementary Geometry.
I have no idea for H... a new challenge.
Sincerely
Jean-Louis

jayme

#8 Mar 16, 2010, 11:14 pm

Dear Luis and Mathlinkers,
the begining of an idea:
1. H is the center of HaHbHc
2. consider the H-pedal triangle wrt HaHbHc quoted A"B"C"
3. We can think to the cevian nest theorem applied to HaHbHc, A"B"C" and A'B'C'
4. the line A'A" and cyclically are concurrent
5. We have to prove that A'A" is orthogonal to HbHc...
But it seem to be difficult
Sincerely
Jean-Louis

armpist

#9 Mar 17, 2010, 3:41 am

Triangle of reflections of Lemoine pt. of ABC in ABC sides is
perspective with orthic of ABC and with our antipedal triangle
of O. This common persp. center is Isog. cong of Lemoine point in orthic
has to be on Euler line.

M.T.

jayme

#10 Mar 17, 2010, 2:12 pm

Dear Luis and Mathlinkers,
are we done with the proof of Kostas Vittas : http://www.mathlinks.ro/Forum/viewtopic.php?t=338742
Sincerely
Jean-Louis

Luis González

#11 Apr 11, 2010, 6:40 am

I noticed that the proposed problem can be easily generalized, but firt of all let us introduce two previous lemmata.

Lemma 1: P is a point on the plane of $\triangle ABC$ and let $\triangle XYZ$ and $\triangle A'B'C'$ be the pedal and antipedal triangle of P with respect to $\triangle ABC. X, Y, Z$ and A', B', C' against $A, B, C,$ respectively. Then $\triangle ABC$ and $\triangle XYZ$ are perspective if and only if $\triangle ABC$ and $\triangle A'B'C'$ are perspective.

It's clear that $\triangle PAC' \sim \triangle PZB$ and $\triangle PAB' \sim \triangle PYC$ since $\angle PC'A = \angle PBA$ and $\angle PB'A = \angle PCA.$ Then

$$\frac{PA}{PZ} = \frac{AC'}{BZ} \ , \ \frac{PA}{PY} = \frac{AB'}{CY} \Longrightarrow \frac{AB'}{AC'} \cdot \frac{PZ}{PY} \cdot \frac{CY}{BZ} \ (1)$$

By cyclic exchange we get the relations

$\frac{BC'}{BA'} = \frac{AZ}{CX} \cdot \frac{PX}{PZ}$ (2) , $\frac{CA'}{CB'} = \frac{BX}{AY} \cdot \frac{PY}{PX}$ (3)

Multiplying $(1), (2)$ and (3) together yields

$$\frac{AB'}{AC'} \cdot \frac{BC'}{BA'} \cdot \frac{CA'}{CB'} = \frac{PZ}{PY} \cdot \frac{PX}{PZ} \cdot \frac{PY}{PX} \cdot \frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = \frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ}$$

By Ceva's theorem we conclude that AA', BB', CC' concur if and only if AX, BY, CZ concur, i.e. $\triangle ABC$ and $\triangle XYZ$ are perspective if and only if $\triangle ABC$ and $\triangle A'B'C'$ are perspective.

Lemma 2: If P, Q are two isogonal conjugates WRT $\triangle ABC$, then there exists an inconic $\mathcal{K}$ in $\triangle ABC$ with foci P, Q.

Without loss of generality assume that point P lies inside $\triangle ABC$, then so is Q. Let P_1, P_2, P_3 and Q_1, Q_2, Q_3 be the perpendicular feet from P, Q on BC, CA, AB. It's well-known that the six points $P_1, P_2, P_3, Q_1, Q_2, Q_3$ lie on a circle centered at the midpoint K of PQ, thus let ϱ be its radius. Let Q_0 be the reflection of Q across BC and $A_0 \equiv PQ_0 \cap BC$. Then BC is external bisector of $\angle PA_0Q$. Since K, Q_1 are the midpoints of PQ, QQ_0, we obtain $A_0P + A_0Q = 2\varrho$. Similarly, there exists B_0, C_0 on CA, AB such that CA, AB are external bisectors of $\angle PB_0Q, \angle PC_0Q$, respectively and whose sum of distances to P, Q equals $2\varrho \Longrightarrow$ There exists an inellipse $\mathcal{K}$ with foci P, Q and pedal circle (K, ϱ). The proof when P is outside $\triangle ABC$ is analogous, but in such a case $\mathcal{K}$ becomes either hyperbola or parabola when $P \in \odot(ABC)$. Conversely, the foci of a conic $\mathcal{K}$ are isogonal conjugate with respect to any triangle bounded by three tangent lines of $\mathcal{K}$.

Generalization: P is a PC-point with respect to $\triangle ABC$ and Q is its isogonal conjugate. $\triangle P_1P_2P_3$ is the pedal triangle of P with respect to $\triangle ABC$. P_1, P_2, P_3 against A, B, C. Let $\triangle A'B'C'$ be the anti-pedal triangle of Q with respect to $\triangle P_1P_2P_3$. A', B', C' against P_1, P_2, P_3. Then the four lines $A'P_1, B'P_2, C'P_3$ and PQ concur.

Let M, N, L be the orthogonal projections of A, B, C on P_2P_3, P_3P_1 and P_1P_2. Then $Q \equiv AM \cap BN \cap CL$. Since $\triangle P_1P_2P_3$ is cevian triangle of some point, by Cevian Nest Theorem, it follows that MP_1, NP_2, LP_3 concur at a point $V \Longrightarrow$ Q is a PC-point with respect to $\triangle P_1P_2P_3$ with V as its associated point. Using the Lemma 1, we deduce that $\triangle A'B'C'$ and $\triangle P_1P_2P_3$ are perspective through $U \Longrightarrow U \equiv A'P_1, B'P_2, C'P_3$. Hence, it suffices to prove that $U \in PQ$.

From Lemma 2, let $\mathcal{K}$ be the inconic with foci P, Q. This has the circumcircle of $\triangle P_1P_2P_3$ as its pedal circle, hence the sidelines of $\triangle A'B'C'$ are tangents of $\mathcal{K} \Longrightarrow P, Q$ are isogonal conjugates with respect to $\triangle A'B'C'$, which means that the perpendicular lines dropped from A', B', C' to P_2P_3, P_3P_1 and P_1P_2 concur at $P \Longrightarrow \triangle A'B'C'$ and $\triangle P_1P_2P_3$ are perspective through U and orthologic through orthology centers P, Q. By Sondat's theorem, we conclude that P, Q, U are collinear and our proof is completed.

Quick Reply

High School Olympiads

diagonals of a hexagon inscibed in a cirlce!

Reply

ridgers

#1 Apr 11, 2010, 2:09 am

Prove that diagonals AD, BE and CF of the $ABCDEF$ hexagon inscribed in a circle intersects in a point if and only if $AB.CD.EF = BC.DE.FA$.

Luis González

#2 Apr 11, 2010, 5:16 am • 1

Discussed before. For instance, take a look at the topic http://www.mathlinks.ro/viewtopic.php?t=295796. By the way, use the command \cdot instead of the ordinary dot to represent product of segments, it looks better.

Quick Reply

High School Olympiads

Determine the locus of intersections of lines

Reply

Source: Sharygin contest 2008. The correspondence round. Problem 16

Doctor A

#1 Sep 3, 2008, 8:01 pm

(A.Zaslavsky, 9--11) Given two circles. Their common external tangent is tangent to them at points A and B. Points X, Y on these circles are such that some circle is tangent to the given two circles at these points, and in similar way (external or internal). Determine the locus of intersections of lines AX and BY.

pohoatza

#2 Sep 4, 2008, 2:23 am

Assume without loss of generality that the third circle Γ is tangent internally to the first two given ones (the other case can be treated analogously). Denote by X', and by Y' the intersections of the line XY with the first, and second circle respectively, and let O_1, O_2 be the centers of these first two given circles γ_1, and γ_2; finally let Q be the center of Γ. Since $\angle O_1XX' = \angle O_1X'X = \angle QXY = \angle QYX = \angle O_2YY' = \angle O_2Y'Y$, we have that $O_1X \| O_2Y'$, and $O_1Y \| O_1X'$. Thus, $\angle AXX' = \angle BY'Y$, and since AB is tangent to γ_2, we conclude that $\angle AXX' = \angle BY'Y = \angle ABY$. This shows that the points A, B, X, and Y lie on a same circle.

Luis González

#3 Apr 10, 2010, 9:31 pm

Let ω_1, ω_2 denote the two given circles and ω denotes the third circle internally/externally tangent to ω_1, ω_2 at X, Y. Let O be the exsimilicenter of $\omega_1 \sim \omega_2, X, Y$ are the exsimilicenters/insimilicenters of $\omega \sim \omega_1$ and $\omega \sim \omega_2$. Therefore, by Monge & d'Alembert theorem XY passes through $O \Longrightarrow X, Y$ and A, B are inverse points under the direct inversion through pole O that takes ω_1 and ω_2 into each other $\Longrightarrow A, B, X, Y$ are concyclic $\Longrightarrow P \equiv AX \cap BY$ moves on the radical axis of ω_1, ω_2.

sunken rock

#4 Jul 2, 2010, 8:02 pm

First, let's see that $\{T\} \equiv AX \cap BY$ belongs to the third circle.

Proof: if ω_1 and ω_2 are the first two circles and Ω - the third one, then X is the similicenter (in- or ex-) of ω_1 and Ω and, obviously, if $\{X, T'\} \equiv AX \cap \Omega$, then the tangent to Ω at T' is parallel to AB, the same for B and T", T" being the intersection of BY with Ω, hence $T" \equiv T' \equiv T$.

Next, if $(TZ$ is the direction $(BA$ of this tangent, we have $\widehat{BAT} = \widehat{XTZ}$ (1), but from the circle Ω: $\widehat{XTZ} = \widehat{XYT}$, (2), from (1) and (2) getting that $ABYX$ is cyclic, i.e. T belongs to the radical axis of ω_1 and ω_2.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Special quadrilateral

Reply

Source: Own, not difficult

sunken rock

#1 Apr 14, 2009, 9:01 pm

ABCD is a convex quadrilateral for which <ACB + <ADB = 180 degs. If M is the common point of AC and BD, then the following equality holds:
AM*MB = AD*BC + CM*DM.

Best regards,
sunken rock

Ahiles

#2 Apr 14, 2009, 9:25 pm

Denote $\angle AMB = \beta$, $\angle BCM = \alpha$.

Out relation is equivalent to

$$\frac{AM}{AD} \cdot \frac{MB}{BC} = 1 + \frac{CM}{BC} \cdot \frac{DM}{AD}$$

$$\frac{\sin\alpha}{\sin\beta} \cdot \frac{\sin\alpha}{\sin\beta} = 1 + \frac{\sin(\alpha+\beta)}{\sin\beta} \cdot \frac{\sin(\alpha-\beta)}{\sin\beta}$$

$$\sin^2\alpha = \sin^2\beta + (\sin\alpha\cos\beta + \sin\beta\cos\alpha)(\sin\alpha\cos\beta - \sin\beta\cos\alpha)$$

$$\sin^2\alpha = \sin^2\beta + \sin^2\alpha\cos^2\beta - \sin^2\beta\cos^2\alpha$$

$$\sin^2\alpha(1-\cos^2\beta) = \sin^2\beta(1-\cos^2\alpha)$$

$$\sin^2\alpha\sin^2\beta = \sin^2\alpha\sin^2\beta$$

sunken rock

#3 Apr 21, 2009, 9:03 pm

First synthetic solution
If <ADB = <ACB = 90 degs, the problem is quite trivial from triangles AMD and BMC similar.
Suppose, w.l.o.g. that <ACB > 90 degs and draw BE and AF perpendicular on AC and BD respectively, with E and F on AC and BD respectively, see that triangles BCE and ADF are similar, same for BME and AMF; from their similitude ratio we get:
DF/CE = AF/BE = AD/BC = AM/BM = MF/ME = k (1)
The relation to prove becomes:
AM*MB = AD*BC + (MF + DF)*(ME - CE); substituting AM = k*BM, AD = k*BC and other similar relations from (1) we shall finally get
BM^2 = BE^2 + ME^2, obviously, true.

Best regards,
sunken rock

Luis González

#4 Apr 9, 2010, 10:37 pm

Take a point P on the ray $\overrightarrow{MD}$ such that $AP \parallel BC$. Then $\triangle MPA \sim \triangle MBC$, $\triangle ADP \sim \triangle MCB$ and let k denote the

Take a point P on the ray MD such that $AP \parallel BC$. Then $\triangle MPA \sim \triangle MBC, \triangle ADP \sim \triangle MCB$ and let k denote the similarity coefficient of $\triangle MCB$ to $\triangle ADP$

$$\frac{AD}{MC} = k \Longrightarrow \frac{AD}{k} \cdot \frac{MA}{MC} = MA \Longrightarrow \frac{AD}{k} \cdot \frac{PM}{MB} = MA$$

$$\Longrightarrow MA \cdot MB = \frac{AD \cdot PD}{k} + \frac{AD \cdot DM}{k}$$

$$PD = k \cdot BC \ , \ AD = k \cdot CM \Longrightarrow MA \cdot MB = AD \cdot BC + CM \cdot DM.$$

Other metric relations of this quadrilateral $ABCD$ come from the fact that triangles $\triangle ADM$ and $\triangle BCM$ are pseudo-similar $\Longrightarrow MA \cdot BC = MB \cdot AD$. If $P \equiv AD \cap BC$, then P, C, M, D lie on a circle ω. Hence (A, B) are conjugate points with respect to ω, which implies that

$$p(A, \omega) + p(B, \omega) = AB^2 \Longrightarrow AM \cdot AC + BM \cdot BD = AB^2.$$

sunken rock

#5 Jan 6, 2011, 1:05 am

2nd synthetic solution - idea.
Continuing the work from here: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=202546&p=1114247&hilit=finding+a+theorem#p1114247, we shall find another nice solution.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Midpoint of the segment joining two centers

Reply

Moonmathpi496
#1 Apr 9, 2010, 9:34 am

ABC is a triangle and M is the midpoint of BC. D is any point on BC. Let O_1 and O_2 denote the circumcenters of triangles ADB and ACD respectively. Let X denote the intersection of O_1O_2 and the perpendicular bisector of AM.
Show that X is the midpoint of O_1O_2.

Luis González
#2 Apr 9, 2010, 10:23 am

Equivalent formulation: Two circles $(O_1), (O_2)$ intersect at A, D and a line through D cuts $(O_1), (O_2)$ at B, C, respectively. Then the midpoint M of BC lies on the circumference centered at the midpoint X of O_1O_2 with radius $XA = XD$. For a proof you may see the problem 5 (post #6) in the thread Some problems.

Moonmathpi496
#3 Apr 9, 2010, 12:39 pm

In this problem I don't understand how you concluded that: M moves on a circle coaxal with ω_1 and ω_2 centered at the midpoint of the segment connecting their centers.

Quick Reply

High School Olympiads

3 circumcenters are collinear

Reply

eivos

#1 Apr 8, 2010, 6:58 pm

Let ABC be an acute triangle.
H is the orthocenter of $\triangle ABC$.
K is an arbitary point in the plane of $\triangle ABC$.
Let $KA_0 \perp HA (A_0 \in HA)$ and $HA_1 \perp KA$.
We define B_0, C_0, B_1, C_1 similarly.
Prove that the circumcenter of $\triangle AA_0A_1, \triangle BB_0B_1, \triangle CC_0C_1$ are collinear.

Luis González

#2 Apr 9, 2010, 12:04 am

Clearly $A_0, B_0, C_0, A_1, B_1, C_1$ lie on the circle (U) with diameter HK and from $KC_0 \parallel AB, KB_0 \parallel AC$, it follows that $\angle B_0A_0C_0 = \angle B_0KC_0 = \angle BAC$. Likewise $\angle A_0B_0C_0 = \angle ABC$, thus $\triangle ABC$ and $\triangle A_0B_0C_0$ are similar. Moreover if A' denotes the foot of the A-altitude of $\triangle ABC$ onto the sideline BC, we have that

$$\angle KBC = \angle A'HB_1 = \angle A_0B_0B_1 \ , \ \angle KBA = \angle C_0HB_1 = \angle C_0B_0B_1.$$

Therefore, if K' denotes the homologous of K in $\triangle A_0B_0C_0$, then B_0B_1 is the isogonal ray of B_0K' with respect to $\angle A_0B_0C_0$. Similarly, A_0A_1 and C_0C_1 are the isogonal rays of A_0K' and C_0K' with respect to $\angle B_0A_0C_0$ and $\angle A_0C_0B_0$ $\Longrightarrow$ Lines A_0A_1, B_0B_1, C_0C_1 concur at the isogonal conjugate V of K' WRT $\triangle A_0B_0C_0$. Consequently, midpoints A'', B'', C'' of A_0A_1, B_0B_1, C_0C_1 lie on circle with diameter UV. Since (U) is orthogonal to $\odot(AA_0A_1), \odot(BB_0C_1)$ and $\odot(CC_0C_1)$, it follows that the inversion in the circle (U) takes A'', B'', C'' into their respective centers, i.e. the circumcenters O_a, O_b, O_c of $\triangle AA_0A_1, \triangle BB_0C_1$ and $\triangle CC_0C_1$. Since A'', B'', C'', U are concyclic, then O_a, O_b, O_c are collinear.

Quick Reply

High School Olympiads

The Euler line parallel to the bisector

Source: Sharygin contest 2008. The correspondence round. Problem 14

Doctor A
#1 Sep 3, 2008, 7:56 pm

(V.Protasov, 9--10) The Euler line of a non-isosceles triangle is parallel to the bisector of one of its angles. Determine this angle (There was an error in published condition of this problem).

castigioni
#2 Dec 2, 2009, 9:33 pm

The angle is 120

Luis González
#3 Apr 8, 2010, 7:10 am

Let O, H, N be the circumcenter, orthocenter and 9-point center of $\triangle ABC$. Assume that OH is parallel to the internal bisector of $\angle BAC$. Which implies that HO bisects $\angle BHC$, since HB, HC and the external bisector of $\angle BAC$ bound an isosceles triangle with apex H. Hence if U, V denote the midpoints of HC, HB, then the quadrilateral $HUNV$ is cyclic, where N is the midpoint of the arc UV of its circumcircle. By simple angle chase we obtain then

$$\angle UNV + \angle BHC = 180^\circ \Longrightarrow 2\angle BHC + \angle BHC = 180^\circ$$

$$\Longrightarrow \angle BHC = 60^\circ \Longrightarrow \angle BAC = 120^\circ.$$

MJ GEO
#4 Apr 9, 2010, 6:41 pm

easy problem.just see that HO bisects BHC and $BO = CO$ so $BHCO$ is cyclic.so $360 - 2A + 180 - A = 180$ so $A = 120$

High School Olympiads

cyclic

Reply

Source: ver nice

MJ GEO

#1 Apr 6, 2010, 9:18 pm

in cyclic quadrilateral $ABCD$,AB,CD and AD,BC intersect at P,Q.let L be the intersection of bisecters of APC and AQC.let M,N be the midpoint of diagnols AC,BD.prove that L,M,N are colinear.
(i have good solution,but i realy intersted for another one.)

Luis González

#2 Apr 6, 2010, 9:32 pm

There's more to say about this figure, thus I'll restate the enunciation

Proposition: $ABCD$ is a cyclic quadrilateral. E is the intersection of AB,CD and F is the intersection of $AD,BC.$ Then the angle bisectors of $\angle AED$ and $\angle DFC$ cut orthogonally at a point lying on the Newton line of $ABCD.$

Let M,N,L be the midpoints of AC,BD,EF lying on the Newton line n of $ABCD. P,Q$ are two points on n such that $LE=LF=LP=LQ$ (EP is internal ray of <BEC). $EPFQ$ is obviously a rectangle. We shall prove that EP bisects $\angle AED$ and similarly we'll have that FP bisects $\angle AFB.$

The circle (L) with diameter EF belows to the orthogonal pencil defined by the axis EF and the circumcircle (O) of $ABDC.$ Hence, (L) is orthogonal to (O) and the circle ω with diameter OK where $K \equiv AC \cap BD$ is the pole of EF with respect to $(O).$ If (L) and ω meet at X',Y', then the pencil of rays through the vertex L of the convex angle $\angle X'LY$ cut $(L),\omega$ at four points harmonically separeted. But from $ON \perp KN$ and $OM \perp KM,$ it follows that M,N lie on $\omega \Longrightarrow$ $(M,N,P,Q)=-1.$ Since $QE \perp EP,$ we deduce that EP is the angle bisector of $\angle NEM$ $(\star).$ On the other hand, from $\triangle EBD \sim \triangle ECA,$ due to $\angle EBD=\angle ECA,$ it follows that $\triangle EBN \sim \triangle ECM \Longrightarrow \angle BEN=\angle CEM.$ Together with $(\star),$ we have $\angle BEP=\angle CEP \Longrightarrow EP$ is the angle bisector of $\angle AED.$ Likewise, FP is the angle bisector of $\angle AFB$ and the proof is completed.

MJ GEO

#3 Apr 6, 2010, 10:34 pm

thank you very much for your nice proof,its realy interesting.but i cant understand this improtant part that (M,N,P,Q)=-1 :(
.please help.here is my idea.
let M,N be the midpoints of AC,BD.we have QL is per pendicular to PL easily.let X,Z be the intersections of PI with AD,BC and Y,T be the intersections of QI with AB,CD.so $XYZT$ is rhombus.so for vector ML we have $2ML=AY+CZ=k(AB+CD)$.and similiary we have $2NL=k(DA+BC)$.but $(CD+DA+AB+BC)=0$ so M,N,L are colinear. :)

Quick Reply

High School Olympiads

circle and parallelogram

Reply

barasawala

#1 Dec 29, 2006, 1:43 pm

In acute triangle ABC, w is the circumcircle and O the circumcenter. w_1 is the circumcircle of triangle AOC, and OQ is the diameter of w_1. Let M, N be on AQ, AC respectively such that $AMBN$ is a parallelogram. Prove that MN, BQ intersect on w_1.

barasawala

#2 Dec 31, 2006, 4:34 pm

Anybody got this?

Luis González

#3 Apr 6, 2010, 11:10 am

Let L be the midpoint of AB and P be the second intersection of ω_1 with BQ. Then $\angle APQ = \angle BNA = \angle ABC$. Thus if $R \equiv BQ \cap AC$ and $D \equiv BN \cap AP$, then $PDNR$ is cyclic. But notice that PQ bisects $\angle APC$ since Q is the midpoint of the arc AC of ω_1. As a resut, $\angle BPC = 180^\circ - \angle ABC = \angle BNC \Longrightarrow BPNC$ is cyclic $\Longrightarrow \angle NPR = \angle BCA$, but since $PDNR$ is cyclic, we obtain $\angle NPR = \angle NDR = \angle BCA = \angle ABN \Longrightarrow DR \parallel BA$. Therefore, the cevian NP of $\triangle BNA$ goes trough the midpoint L of $AB \Longrightarrow P \equiv BQ \cap MN \in \omega_1$.

Quick Reply

High School Olympiads

7 concurrent lines

Reply

Source: own

livetolove212

#1 Apr 4, 2010, 7:45 am

Given triangle ABC, its circumcircle (O) and its incircle (I). Let D, E, F be the tangencies of (I) and BC, CA, AB, respectively. DI, EI, FI intersect (I) again at D', E', F'. Construct a circle which is internal tangent to (I) at D' and tangent to (O) at A_1. Let A_2 be the tangency of A-Mixtilinear incircle and (O), A_3 be the intersection of the circle with diameter AI and (O). Similarly we define $B_1, B_2, B_3, C_1, C_2, C_3$. Prove that 7 lines $AA_1, BB_1, CC_1, A_2A_3, B_2B_3, C_2C_3$ and OI are concurrent.

Luis González

#2 Apr 5, 2010, 4:55 am • 1

Dear Linh, this is such a nice problem, but first off all, let me introduce a previous lemma in order to make the proof clearer.

Lemma: Incircle (I) of $\triangle ABC$ is tangent to BC at D. M is the midpoint of the arc BC of its circumcircle (O) not containing A. Ray MD cuts (O) at T. Then $\angle ATI$ is right.

Let P be the midpoint of the arc ABC of the circumcircle. Inversion in the circle (M) with radius $MB = MC = MI$ swaps (O) and the sideline BC, I is double and $D \mapsto T$. Thus, $\odot(IDT)$ is double $\Longrightarrow MI$ is tangent to $\odot(IDT)$. Hence, it follows that $\angle ITD = \angle DIM = \angle AMP = \angle ATP$ but $\angle MTP$ is right since M, P are antipodal, then we have $\angle ATI = MTP = 90^\circ$ and the proof is completed.

Back to our problem, let M, P denote the midpoints of the arcs BC and BAC of the circumcircle (O). From the upper lemma, we get that A_3, D, M are collinear. Let Γ_a and ω_a denote the A-mixtilinear incircle and the circle internally tangent to (I) at D' and tangent to (O) at A_1. D' is the exsimilicenter of $(I) \sim \omega_a$ and A_1 is the exsimilicenter of $\omega_a \sim (O)$, by Monge and d'Alembert theorem, $D'A_1$ passes through the exsimilicenter U of $(I) \sim (O)$. On the other hand, A is exsimilicenter of $(I) \sim \Gamma_a$ and A_2 is the exsimilicenter of $\Gamma_a \sim (O)$, then AA_2 passes through $U \Longrightarrow IO, AA_2, D'A_1$ concur at U, but from the parallel radii OP, ID' and OM, ID, it follows that P, D', A_1 and M, U, A_3 are collinear. It's well-known that the tangency point A_2 of $(O), \Gamma_a$ is the second intersection of PI with (O). Therefore, by Pascal theorem for the cyclic hexagon $MA_3A_2PA_1A$, the intersections $U \equiv MA_3 \cap PA_1, A_3A_2 \cap AA_1$ and $I \equiv AM \cap PA_2$ are collinear, i.e. AA_1, A_2A_3, IO concur at a point V.

On the other hand, by Pascal theorem for the cyclic hexagon $A_2A_3BB_2B_3A$, the intersections $V' \equiv A_2A_3 \cap B_2B_3$, $R \equiv BA_3 \cap AB_3$ and $U \equiv BB_2 \cap AA_2$ are collinear. If N is the midpoint of the arc AC of (O) not containing B, again by Pascal theorem for the hexagon MA_3BNB_3A, the intersections $U \equiv MA_3 \cap NB_3, R \equiv AB_3 \cap BA_3$ and $I \equiv BN \cap AM$ are collinear $\Longrightarrow O, I, U, V'$ are collinear such that $V' \equiv IO \cap A_2A_3 \Longrightarrow V \equiv V'$. Analogously, C_2C_3 passes through V as well as BB_1 and CC_1 do $\Longrightarrow AA_1, BB_1, CC_1, A_2A_3, B_2B_3, C_2C_3, IO$ concur at V.

skytin

#3 Apr 5, 2010, 9:59 pm

What mean A-Mixtilinear incircle ?

livetolovemath030894

#5 Apr 6, 2010, 10:09 pm

First we'll express 2 lemma (I'll use notation $\overline{A; B; C}$ to tell that $A; B; C$ are collinear)
+Lemma 1:Let A_4 be antipode of A_2 wrt (O) then $\overline{A_4; A_1; I}$
Proof:
Let $L; M$ be the midpoint of arc BAC and BC.It's easy to see that $\overline{L; D'; A_1}$ and $\overline{L; I; A_2}$.2 tangent of (I) through A_2 cut

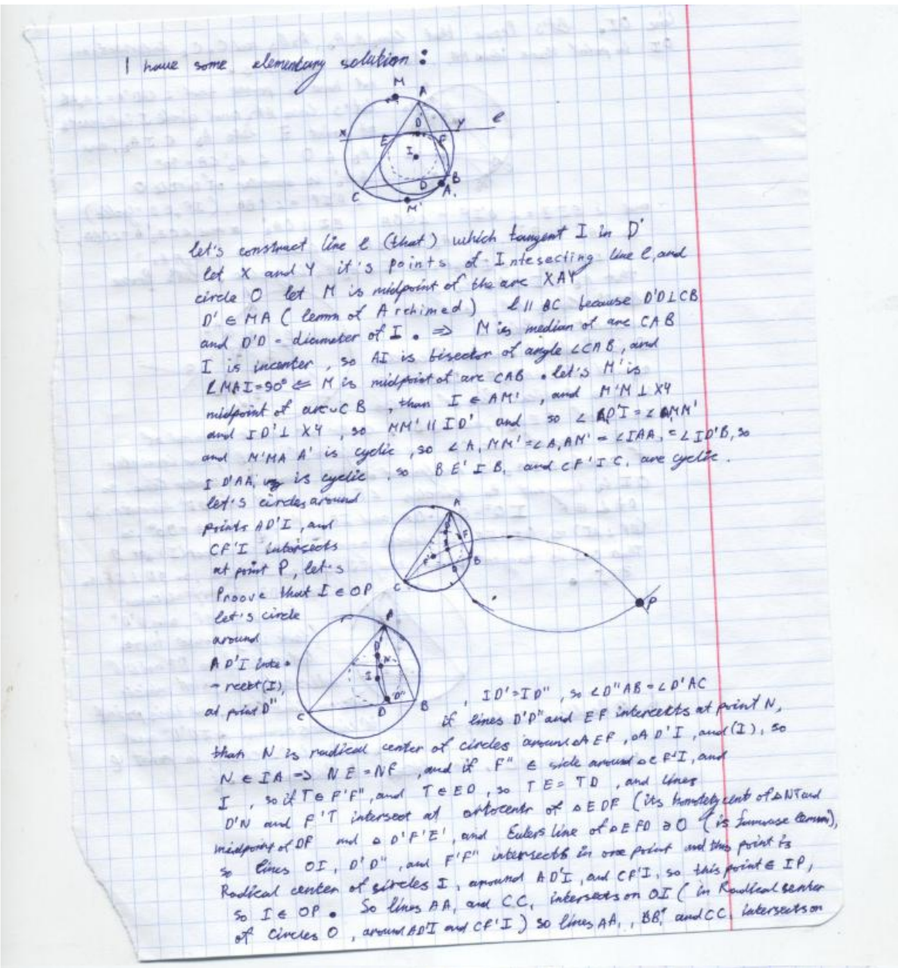
(O) at $N;T$.Applying Poncelet's theorem we'll get (I) is incircle of $\triangle A_2NT$.It's follow that $LI=LT=LN$.Morever,$LN^2=LD'.LA_1 \Rightarrow \overline{LI}$ be the tangent of $(D'IA_1) \Rightarrow \angle LA_1I=\angle LID'=\angle MLA_2 \Rightarrow \overline{A_4;A_1;I}$
+Lemma 2:$AA_2;CC_2;OI$ concurrent
Proof:
We can prove that $\overline{A_3;D;M}$.Let $E;F$ be tangency point of (I) wrt $AC;AB.J;K$ be the midpoint of arc $AB;AC$(not contain $C;B$).It's easy to see that $\triangle MJK$ and $\triangle DEF$ is homethetic.It's implies that A_3M meet OI at the external center of similitude of $(O);(I)(i)$.Applying the Pascal's theorem for hexagon $LAA_3A_2MA_5$ We'll get $\overline{O;I;P'}(ii)(P'\equiv A_3M\cap AA_2)$.From $(i);(ii)$We'll get the result
I'll post solution of the problem later.I'm sorry Nor I'm busy

Attachments:
L1.pdf (15kb)
L2.pdf (15kb)

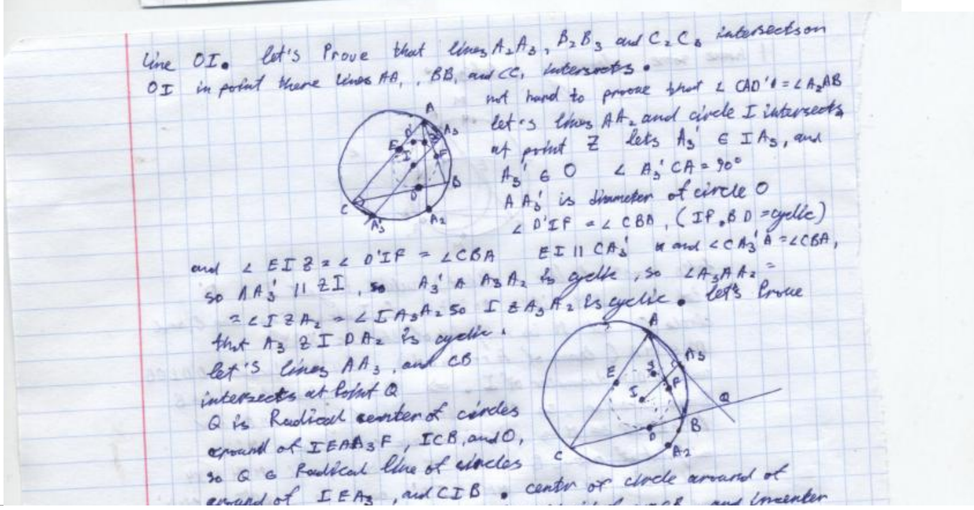
skytin
#6 Apr 7, 2010, 12:41 am • 1

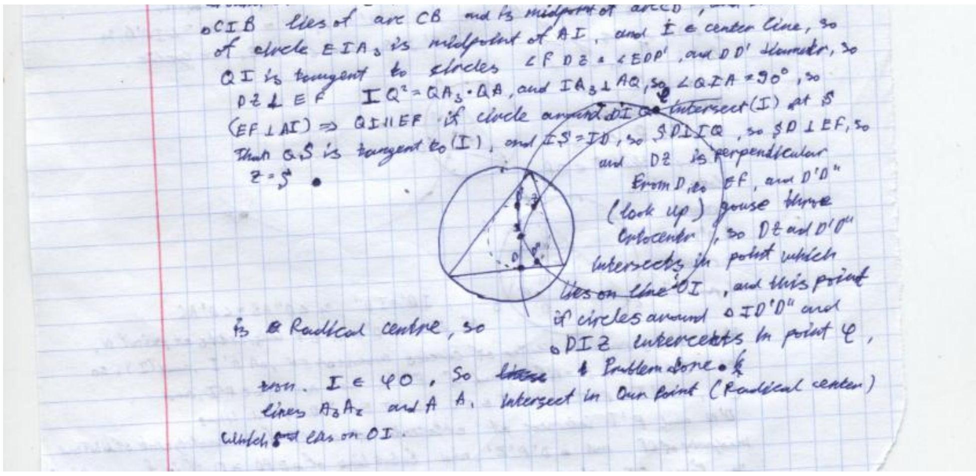
look 2 images

Attachments:


I have some elementary solution:

let's construct line ℓ (that) which tangent I in D'
let X and Y it's points of Intesecting line ℓ, and circle O let M is midpoint of the arc XAY
$\ell \parallel BC$ because $D'D \perp CB$
$D' \in MA$ (lemm of Archimed)
and $D'D$ - diameter of I. $\Rightarrow$ M is median of arc CAB
I is incenter, so AI is bisector of angle $\angle CAB$, and
$\angle MAI = 90° \Leftarrow$ M is midpoint of arc CAB. let's M' is midpoint of arc CB, than $I \in AM'$, and $M'M \perp XY$
and $ID' \perp XY$, so $MM' \parallel ID'$ and so $\angle A_1D'I = \angle A_1MM'$
and M'MA A' is cyclic, so $\angle A_1MM' = \angle A_1AM' = \angle IAA_1 = \angle ID'B$, so
$ID'AA_1$ is cyclic, so $BE'IB_1$ and $CF'IC_1$ are cyclic.
let's circles around points AD'I, and CF'I intersects at point P, let's Proove that $I \in OP$
let's circle around AD'I inte-rsect (I), at point D''
, $ID' = ID''$, so $\angle D''AB = \angle D'AC$
if lines D'D'' and EF intersects at point N,
than N is radical center of circles around $\triangle AEF$, $\triangle AD'I$, and (I), so $N \in IA \Rightarrow NE = NF$, and if $F'' \in$ circle around $\triangle CF'I$, and
I, so if $T \in F'F''$, and $T \in ED$, so $TE = TD$, and lines
D'N and F'T intersect at ortocenter of $\triangle EDF$ (its homotety cent of $\triangle NT$ and midpoint of DF and $\triangle D'F'E'$, and Eulers line of $\triangle EFD \ni O$ (is famouse lemm),
so lines OI, D'D'', and F'F'' intersects in one point and this point is
Radical center of circles I, around AD'I, and CF'I, so this point $\in IP$,
so $I \in OP$. So lines AA_1 and CC_1 intersects on OI (in Radical center of circles O, around AD'I and CF'I) so lines AA_1, BB_1 and CC_1 intersects on


line OI. let's Prove that lines A_2A_3, B_2B_3 and C_2C_3 intersects on
OI in point there lines AA_1, BB_1 and CC_1 intersects.
not hard to prove that $\angle CAD' = \angle A_2AB$
let's lines AA_2 and circle I intersects at point Z lets $A_3' \in IA_3$, and
$A_3' \in O$ $\angle A_3'CA = 90°$
AA_3' is diameter of circle O
$\angle D'IF = \angle CBA$, (IF, BD = cyclic)
$EI \parallel CA_3'$ and $\angle CA_3'A = \angle CBA$,
and $\angle EIZ = \angle D'IF = \angle CBA$
so $AA_3' \parallel ZI$, so $A_3'AA_3A_2$ is cyclic, so $\angle A_3AA_2 =$
$= \angle IZA_2 = \angle IA_3A_2$ so IZA_3A_2 is cyclic. let's Prove
that A_3ZIDA_2 is cyclic.
let's lines AA_3, and CB intersects at point Q
Q is Radical center of circles around of $IEAA_3F$, ICB, and O,
so $Q \in$ Radical line of circles around of IEA_3, and CIB. center of circle around of
... and incenter



ΔCIB lies of arc CB and is midpoint of arcCD,
of circle EIA3 is midpoint of AI, and I ∈ center line, so
QI is tangent to circles ∠FDZ = ∠EDD', and DD' diameter, so
DZ ⊥ EF IQ² = QA3·QA, and IA3 ⊥ AQ, so ∠QIA = 90°, so
(EF ⊥ AI) ⇒ QI ∥ EF if circle around DI intersect (I) at S
Then QS is tangent to (I), and IS = ID, so SD ⊥ IQ, so SD ⊥ EF, so
Z = S.
and DZ is perpendicular
From D to EF, and D'D"
(look up) pouse three
Ortocenter, so DZ and D'D"
intersects in point which
lies on line OI, and this point
is a Radical centre, so
if circles around ΔID'D" and
ΔDIZ intercepts in point φ,
& Problem done.
tion. I ∈ φO, So lines
lines A3A2 and A A1 intersect in Our Point (Radical center)
which lies on OI.

$\zeta(p)=\sum_{n=1}^{\infty}\frac{1}{n^p}$

livetolovemath030894

#7 Apr 7, 2010, 11:22 am • 1

The solution of Linh's Problem

+Let C_4 be antipode of C wrt (O).Applying Pascal's theorem for $C_2A_3ACC_3A_2 \Rightarrow \overline{K;P;G}(1)$.Morever,According to Pascal's theorem for $A_3C_4ACA_5C_3 \Rightarrow \overline{O;I;G}(2)$.From $(1);(2)$ and lemma 2.We'll get $A_3A_2;C_2C_3;OI$ are concurrent.Anagolously,$A_2A_3;B_2B_3;C_2C_3;OI$ are concurrent
+According to lemma 1.Applying the Pascal theorem $A_3AA_4A_5A_2A_1 \Rightarrow \overline{O;I;K'}$.It's easy to see that $K\equiv K' \Rightarrow AA_1;A_2A_3;OI$ are concurrent.Similarly,We'll get $AA_1;BB_1;CC_1;A_2A_3;B_2B_3;C_2C_3;OI$ are concurrent.Our proof is compleleted

Attachments:

P1.pdf (15kb)
P2.pdf (14kb)

shoki

#8 Apr 7, 2010, 4:47 pm

Let M_a be the midpoint of arc BC not containing A . Let M'_a be the antipode of M_a.Define similarly M_b, M_c, M'_b, M'_c . Let A' be the antipode of A. Define similarly B', C'.Let X be the homothetic center of $(I),(O)$. we have some lemmas for which i will give a hint instead of a complete proof.

Lemma 1:
A_3, M_a, D are collinear.
hint
Lemma 2:
$A'I \cap M_aD$ lies on (O) .
hint
Lemma 3:
A', I, A_3 are collinear .
hint
Lemma 4:
$X \in M_aA_3$.
hint
Lemma 5:
$X \in M'_aA_1$.
hint
Lemma 6:
$X \in AA_2$.
hint
Lemma 7:
$I \in A_2M'_a$.
hint

Now the rest is done by just using the lemmas and three times pascal's theorem.
at first due to the hexagon $AA_1M'_aA_2A_3M_a$ and pascal's theorem, the collinearity of $Y=(AA_1,A_2A_3),I,X$ is obtained.
since X is the homothetic center of $(I),(O)$ we have the collinearity of X,I,O,Y.
thus the ten lines $OI, A_1M'_a, B_1M'_b, C_1M'_c, A_3M_a, B_3M_b, C_3M_c, AA_2, BB_2, CC_2$ are all concurrent at X.
now it remains to prove that $Y=(A_2A_3,B_2B_3)$. then it's pretty clear that by repeating the whole story for C_2C_3 we'll get the concurrency of the seven lines $AA_1, BB_1, CC_1, A_2A_3, B_2B_3, C_2C_3$ and OI at Y.
assume $Y'=(A_2A_3,B_2B_3)$. also let $Q=(B_2A,M_bA_3)$.
now consider the two hexagons $M_bA_3A_2AB_2B_3$ and $M'_bM_bA_3A'AB_2$.
the first hexagon implies the collinearity of X,Q,Y' and the second one the collinearity of O,Q,I .

since O, I, X are collinear so we get the collinearity of O, I, X, Q, Y' so in fact we got $(OI, A_2A_3) = Y'$ but since $(OI, A_2A_3) = Y$ we get $Y' = Y$. we are done !

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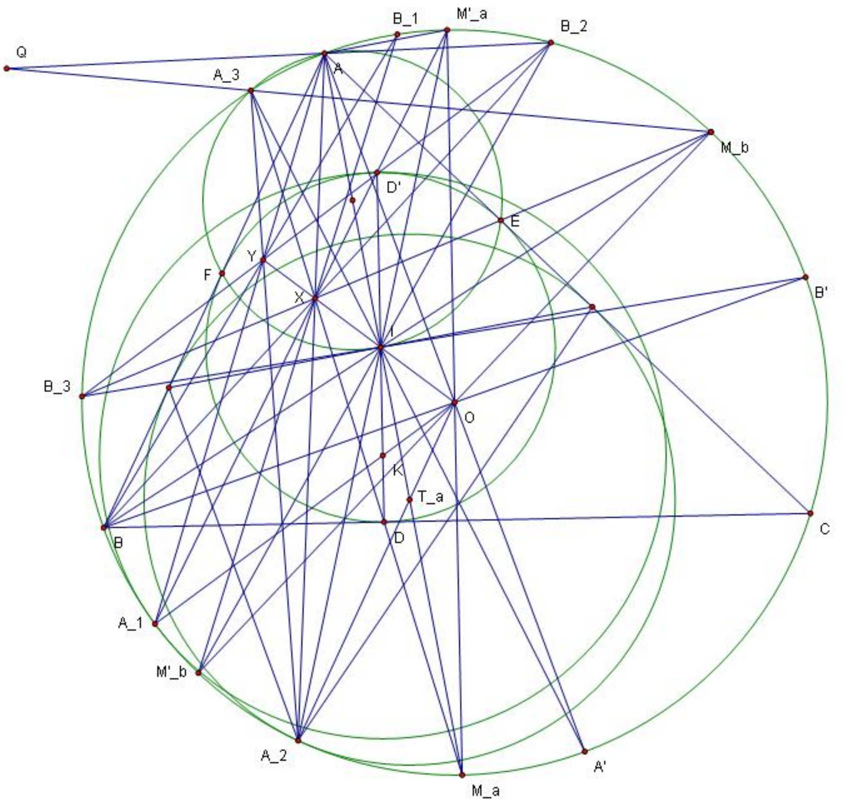


High School Olympiads

very nice

Reply

MJ GEO

#1 Apr 5, 2010, 3:48 am

let B_1, C_1 be the midpoints of AB, AC of triangle ABC.$(ABC_1, ACB_1) = P$ and $(AP, AB_1C_1) = Q$. prove that $2AP = 3AQ$.

Luis González

#2 Apr 5, 2010, 4:10 am

See http://www.artofproblemsolving.com/viewtopic.php?t=276858.

P.S. Next time, use meaningful subjects. Descriptions such as: very nice, very hard, cool problem, etc do not tell anything about the content of the thread and makes searching more difficult.

MJ GEO

#3 Apr 5, 2010, 4:19 am

thank you for link.

Quick Reply

High School Olympiads

Congruent Opposite Sides of Skew Quadrilateral

Reply

Source: 1977 USAMO Problem 4

Brut3Forc3

#1 Apr 4, 2010, 8:54 am

Prove that if the opposite sides of a skew (non-planar) quadrilateral are congruent, then the line joining the midpoints of the two diagonals is perpendicular to these diagonals, and conversely, if the line joining the midpoints of the two diagonals of a skew quadrilateral is perpendicular to these diagonals, then the opposite sides of the quadrilateral are congruent.

Luis González

#2 Apr 4, 2010, 11:08 am

Label $ABCD$ the given quadrilateral. M, N denote the midpoints of the diagonals $AC, BD,$ respectively.

• Assume that $AD = CB$ and $AB = CD.$ Then $\triangle ADC \cong \triangle CBA$ by SSS criterion $\Longrightarrow$ their medians DM and BM are congruent. Hence $\triangle DMB$ is isosceles with apex $M.$ The median MN is identical to the altitude on $DB \Longrightarrow$ $MN \perp DB.$ Likewise, $\triangle ADB \cong \triangle CBD,$ then $\triangle ANC$ is isosceles with apex $N \Longrightarrow NM \perp AC.$

• Conversely, if $MN \perp DB$ and $NM \perp AC,$ the triangles $\triangle MDB$ and $\triangle NAC$ are isosceles with legs $MB = MD$ and $NA = NC,$ respectively, which implies that

$$CD^2 + CB^2 = AB^2 + AD^2 \ , \ AD^2 + CD^2 = AB^2 + CB^2.$$

Substracting and adding both expressions yields $AD = CB$ and $AB = CD.$

Quick Reply

High School Olympiads

problem

Reply

Aquarious

#1 Feb 21, 2010, 1:23 pm

1.Let D be a point on the side AC of ABC . Let E and F be point on the segment BD and BC

,respectively such that angle BAE=CAF .Let P and Q be point on BC and BD respectively , such that

EP parallel DC and FQ parallel CD Prove that angle BAP=CAQ

Luis González

#2 Apr 4, 2010, 5:17 am

Let $U \equiv AP \cap BD, V \equiv AF \cap BD$. Then $(B,D,U,V) = (B,C,P,F)$

$$\Longrightarrow \frac{BU}{UD} \cdot \frac{VD}{BV} = \frac{BP}{PC} \cdot \frac{FC}{BF}$$

But since $DC \parallel EP \parallel QF$, we get $\frac{BQ}{QD} = \frac{BF}{FC}$, $\frac{BP}{PC} = \frac{BE}{ED}$ (Thales theorem)

$$\Longrightarrow \frac{BU}{UD} \cdot \frac{BQ}{QD} = \frac{BE}{ED} \cdot \frac{BV}{VD}$$

By Steiner theorem we conclude that, if rays AE, AV are isogonals WRT $\angle BAD$, then the rays AU, AQ are isogonals WRT $\angle BAD$ as well $\Longrightarrow \angle BAP = \angle CAQ$.

Quick Reply

High School Olympiads

A PROBLEM WITH ruler-and-compass construction

Reply

simon89889

#1 Apr 3, 2010, 10:57 pm

Give you a triangle ABC
Find point E on line AB,point F on line AC such that BE=EF=FC

Please help me XD

Luis González

#2 Apr 3, 2010, 11:04 pm

Lemma: $\triangle ABC$ is given and let A' be the reflexion of A about the midpoint M of $BC.$ I is the incenter of $\triangle ABC$ and the line through A' and parallel to the inner angle bisector of $\angle A,$ intersects the circumcircle of $\triangle BIC$ at $P.$ Then $BD = DE = EC,$ where $D \equiv AB \cap CP$ and $E \equiv AC \cap BP.$

Assume there exists the points D and E on BA and AC such that $BD = DE = EC$ and $P' \equiv BE \cap CD.$ We'll prove that P and P' are necessarily identical. It's easy to see that P' lies on $\odot(IBC),$ due to

$$\angle DP'B = \angle EDC + \angle DEB = \frac{1}{2}(\angle AED + \angle ADE) \Longrightarrow$$

$\angle DP'B = \frac{1}{2}(\pi - \alpha) = \pi - \angle BIC \Longrightarrow BIP'C$ is cyclic.

On the other hand, it is well-known that the locus of the intersection of the diagonals of the convex quadrilateral $BD'E'C$ satisfying $BD' = E'C$ is the A'-angle bisector of the antimedial triangle $\triangle A'B'C'$, which is nothing but the parallel line to AI from A'. This was discussed in the topic plane geometry 2, hence $P' \equiv P$ and the proof is completed.

Quick Reply

High School Olympiads

Extension of a problem from APMO, 1999 (posted by Shobber).

Reply

Source: Own (I think it is interesting !).

Virgil Nicula

#1 Apr 29, 2009, 9:58 am

> Virgil Nicula wrote:
> **An easy extension.** Let w_1 and w_2 be two secant circles with $w_1 \cap w_2 = \{P, Q\}$. The tangent in P to w_1 touches again w_2 in C. The tangent t to w_1 in a point $A \in w_1$ cut w_2 in B_1, B_2 so that the line $t = \overline{AB_1B_2}$ doesn't separate P, Q and it is closer to P. Denote $R_1 \in AP \cap CB_1$, $R_2 \in AP \cap CB_2$. Prove that :
>
> $1 \blacktriangleright$ PB_2 , R_1B_1 are tangent to the circumcircle of the triangle PQR_1.
>
> $2 \blacktriangleright$ PB_1 , R_2B_2 are tangent to the circumcircle of the triangle PQR_2.
>
> $3 \blacktriangleright$ $\widehat{R_1QR_2} \equiv \widehat{B_1PB_2}$ and $\triangle\ B_1QR_1 \sim \triangle\ B_2QR_2 \sim \triangle\ AQP$.
>
> $4 \blacktriangleright$ If AP cut again w_2 in S, then B_1S is tangent to the circumcicle of B_1AQ and B_2S is tangent to the circumcircle of B_2AQ.

See here a particular case - a problem from **APMO, 1999** posted by **Shobber**.

Virgil Nicula

#2 May 11, 2009, 1:49 am

I don't understand why this problem hasn't at least one reply.

Luis González

#3 Apr 3, 2010, 7:41 am

We use directed angles modulo π in order to embrace all the configurations of ω_1, ω_2 together with the tangent B_1B_2.

1) Let B be an arbitrary point of ω_1 on the arc PQ not containing A. Thus $\angle(BQ, BP) = \angle(PQ, \tau) = \angle(B_1Q, B_1C)$ and $\angle(BQ, BP) = \angle(AQ, AP) \Longrightarrow \angle(AQ, AP) = \angle(B_1Q, B_1C_1) \Longrightarrow A, Q, B_1, R_1$ are concyclic. Then we deduce that $\angle(R_1Q, R_1P) = \angle(B_1Q, B_1A) = \angle(PQ, PB_2) \Longrightarrow PB_2$ is tangent to $\odot(PQR_1)$. Consider $M \equiv AQ \cap PB_2$ and $N \equiv AP \cap QB_1$. Hence, we get $\angle(QM, QN) = \angle(AB_1, AP) - \angle(B_2B_1, B_2P) = \angle(PN, PM) \Longrightarrow P, Q, M, N$ are concyclic. Then it follows that $\angle(QA, QB_1) = (PB_2, PA) = \angle(QP, QR_1)$ and $\angle(QA, QB_1) = \angle(R_1A, R_1B_1)$ $\Longrightarrow \angle(R_1P, R_1B_1) = \angle(QP, QR_1) \Longrightarrow R_1B_1$ is tangent to $\odot(PQR_1)$.

2) Since $\angle(AQ, AP) = \angle(B_1Q, B_1C) = \angle(B_2Q, B_2C)$, we deduce that the points Q, A, R_2, B_2 are concyclic $\Longrightarrow$ $\angle(R_2P, R_2Q) = \angle(B_2B_1, B_2Q) = \angle(PB_1, PQ)$, which implies that PB_1 is tangent to $\odot(PQR_2)$ and for the same reason that Q, A, R_2, B_2 are concyclic, we get $\angle(R_2Q, R_2B_2) = \angle(AQ, AB_2) = \angle(PQ, PR_2)$, i.e. R_2B_2 is tangent to $\odot(PQR_2)$.

3) $\angle(QR_2, QR_1) = \angle(QA, QR_1) + \angle(QR_2, QA) = \angle(PB_2, PB_1)$ and since $\angle(QP, QR_1) = \angle(QA, QB_1)$, we have consequently $\angle(QA, QP) = \angle(QB_2, QR_2)$, which clearly implies that $\triangle B_1QR_1 \sim \triangle B_2QR_2 \sim \triangle AQP$.

4) $\angle(B_1B_2, B_1S) = \angle(PB_2, PA) = \angle(QA, \angle QB_1) \Longrightarrow B_1S$ is tangent to $\odot(B_1AQ)$. On the other hand, we have $\angle(B_2S, B_2C) = \angle(PS, PC) = \angle(QB_2, QR_2) \Longrightarrow B_2S$ is tangent to $\odot(B_2AQ)$.

Quick Reply

High School Olympiads

Ineq-Ligouras-275

Source: Triangle - by P. Ligouras - 2010

Ligouras
#1 Apr 1, 2010, 2:15 pm

Let ABC be an equilateral triangle, and W its incircle. If D and E are points of the sides AB and AC, respectively, such that DE is tangent to W, show that

$$\frac{DB}{DA}+\frac{EC}{EA}\geq 4$$

.

Luis González
#2 Apr 1, 2010, 10:01 pm

Let M,N be the tancency points of the incircle $\mathcal{W}$ with AB,AC and WLOG assume that points D,E lie on the segments AM,AN. Since $DECB$ is tangential, by Newton's theorem BE,CD and MN concur at a point Q. If $P \equiv AQ \cap BC$, then Q is the midpoint of the cevian AP, since MN is the A-midline of $\triangle ABC$.

By Menelaus' theorem for $\triangle ABP$, $\triangle ACP$ cut by $\overline{CQD}$ and $\overline{BQE}$ we obtain

$$\frac{DB}{DA}=\frac{BC}{PC}, \ \frac{EC}{EA}=\frac{BC}{PB} \Longrightarrow \frac{DB}{DA}+\frac{EC}{EA}=BC\left(\frac{1}{PB}+\frac{1}{PC}\right)=\frac{BC^2}{PB\cdot PC}$$

By AM-GM we get $PB+PC=BC\geq 2\sqrt{PB\cdot PC} \Longrightarrow \frac{BC^2}{PB\cdot PC}\geq 4$

Therefore $\frac{DB}{DA}+\frac{EC}{EA}\geq 4.$

Ligouras
#3 Apr 2, 2010, 4:01 pm

these are the resolutions of whom loves the classical geometry!!!!!. Thanks dear friend :)

High School Olympiads

Ho Quang Vinh

Reply

saeedghodsi
#1 Mar 31, 2010, 2:35 pm

the excircle of triangle ABC in side angle A touches side BC at $A^{'}$. points $B^{'}$ and $C^{'}$ are defined similarly. the lines $AA^{'}, BB^{'}, CC'$ are concurrent at point N . let D, E, F be the orthogonal projections of N onto the sides BC, CA, AB, respectively . suppose that R, r are circumradius and inradius of triangle ABC respectively . prove that :

$$\frac{S_{DEF}}{S_{ABC}} = \frac{r}{R}(1 - \frac{r}{R})$$

which S_{XYZ} denotes the area of triangle XYZ

Luis González
#2 Apr 1, 2010, 2:50 am

$\triangle A_0B_0C_0$ is the anticomplementary triangle of $\triangle ABC$. Then its circumcircle (O) becomes 9-point circle of $\triangle A_0B_0C_0$ and Nagel point $N_a \equiv AA' \cap BB' \cap CC'$ of $\triangle ABC$ becomes incenter of $\triangle A_0B_0C_0 \Longrightarrow$ Incircle (N_a) and and 9-point circle (O) of $\triangle A_0B_0C_0$ are internally tangent at F'. If F'' denotes the antipode of F', we have

$$p(N_a, (O)) = \overline{N_aF'} \cdot \overline{N_aF''} = 2r(2R - 2r) = 4r(R - r)$$

By Euler's theorem for the pedal triangle of N_a WRT $\triangle ABC$, we get then

$$\frac{[\triangle DEF]}{[\triangle ABC]} = \frac{p(N_a, (O))}{4R^2} = \frac{4r(R - r)}{4R^2} = \frac{r}{R}\left(1 - \frac{r}{R}\right).$$

saeedghodsi
#3 Apr 1, 2010, 2:23 pm

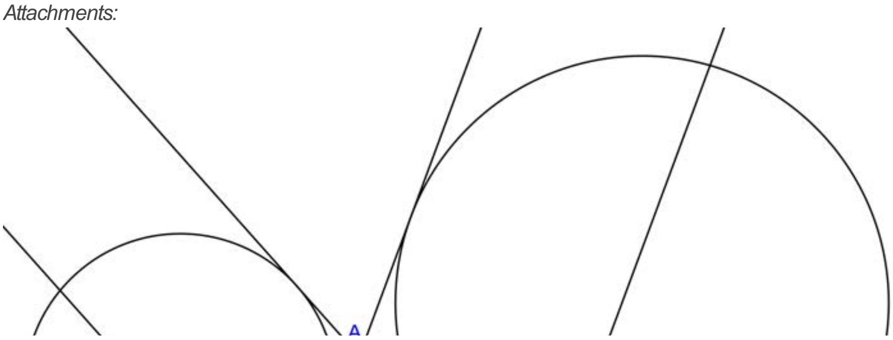
> *Quote:*
> $\triangle A_0B_0C_0$ is the anticomplementary triangle of $\triangle ABC$

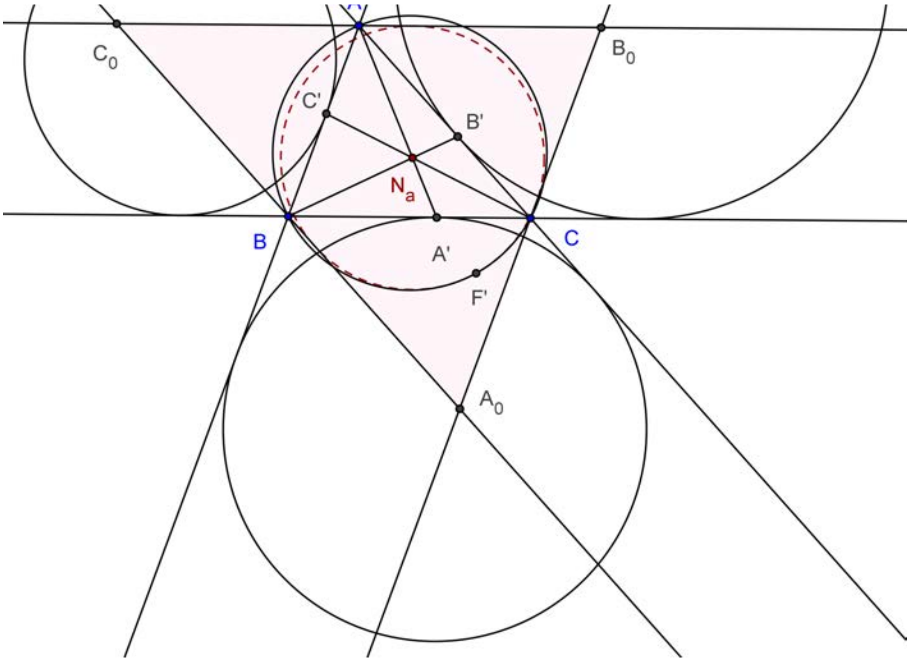
what does it mean?

Luis González
#4 Apr 1, 2010, 9:11 pm

It's the triangle bounded by the parallels to the sides BC,CA,AB passing through the vertices A,B,C, respectively. It is usually named as antimedial triangle of ABC as well.

Attachments:





Quick Reply

High School Olympiads

Perpendicular of circle

Source: very nice!

tmath

#1 Apr 1, 2010, 9:09 am

D, E are points on the AC, AB sides of $\triangle ABC$ such that $DE||BC$. Let P is arbitrary point which exists inside of the $\triangle AED$. Let $PB \cap DE = F$, $PC \cap DE = G$. If O_1 and O_2 are circumcenters of PGD and PFE, then prove that $AP \perp O_1O_2$

Luis González

#2 Apr 1, 2010, 10:08 am

There is a little typo in the enunciation. O_1, O_2 should be circumcenters of $\triangle PGE$ and $\triangle PFD$, otherwise the problem is not true. Let the cevian AP cut ED and BC at U, V respectively. From $\triangle AED \sim \triangle ABC$ and $\triangle PFG \sim \triangle PBC$ we obtain

$$\frac{UG}{UF} = \frac{VC}{VB}, \frac{VC}{VB} = \frac{UD}{UE} \Longrightarrow UG \cdot UE = UF \cdot UD$$

$\Longrightarrow AP$ is the radical axis of $\odot(PGE)$ and $\odot(PFD) \Longrightarrow AP \perp O_1O_2$.

jgnr

#3 Apr 1, 2010, 10:16 am

I think there is some typos, I assume O_1 and O_2 are circumcenters of triangles PFD and PGE respecitvely.

Extend AP to intersect DE at R and BC at S. So $\frac{RF}{RG} = \frac{SB}{SC} = \frac{RE}{RD}$ and $RF \cdot RD = RE \cdot RG$. Thus the power of R wrt the two circles are equal, so R lies on the radical axis of the circle. This implies that $PR \perp O_1O_2$. Since line PR coincides with AP, we get $AP \perp O_1O_2$.

High School Olympiads

incircles

Reply

MMNN

#1 Apr 1, 2010, 9:31 am

on AB, AC, BC we choose three point E, F, G such that $r(AEF) = r(CFG) = r(BEG)$. prove that : $r(AEF) + r(EFG) = r(ABC)$

Luis González

#2 Apr 1, 2010, 9:42 am

For convenience, I'll restate the proposition as follows

Proposition: $\triangle ABC$ is acute and there exists points D, E, F on BC, CA, AB, such that the inradii of $\triangle AEF, \triangle BDF, \triangle CDE$ are all equal to r_0. If the inradii of $\triangle DEF$ and $\triangle ABC$ are given by ϱ and r, then $\varrho + r_0 = r$.

Let O_1, O_2, O_3 denote the centers of the incircles of $\triangle AFE, \triangle BDF, \triangle CED$. $(O_1), (O_2), (O_3)$ touch EF, FD, DE, at M, N, L. $(O_2), (O_3)$ touch BC, at P, Q. $(O_3), (O_1)$ touch CA at R, S and $(O_1), (O_2)$ touch AB at T, U. Since O_1O_2UT, O_2O_3QP and O_3O_1SR are rectangles and O_1, O_2, O_3 lie on the internal bisectors of A, B, C, it follows that $\triangle ABC$ and $\triangle O_1O_2O_3$ are centrally similar through their common incenter I. The similarity coefficient equals the ratio between their perimeters/inradii

$$\frac{r - r_0}{r} = \frac{O_1O_2 + O_2O_3 + O_3O_1}{a + b + c} = \frac{PQ + RS + TU}{a + b + c}$$

$$\frac{r - r_0}{r} = \frac{DN + DL + EL + EM + FM + FN}{a + b + c} = \frac{DE + EF + FD}{a + b + c} \quad (1)$$

On the other hand, we have

$$r(a + b + c) = r_0(a + b + c + DE + EF + FD) + \varrho(DE + EF + FD)$$

$$\Longrightarrow r - r_0 = \frac{(DE + EF + FD)(r_0 + \varrho)}{a + b + c} \quad (2)$$

Substituting $(r - r_0)$ from (1) into (2) yields $r = r_0 + \varrho$.

Quick Reply

High School Olympiads

ABC is an equilateral triangle

Source: Nguyen Minh Ha

saeedghodsi
#1 Mar 31, 2010, 2:41 pm

let M be a point in the interior of equilateral triangle ABC and X, Y, Z be the reflections of M across the sides BC, CA, AB respectively . prove that triangles ABC and XYZ have the same centroid

Luis González
#2 Apr 1, 2010, 4:02 am

Lemma: $\triangle BDC, \triangle CEA$ and $\triangle AFB,$ are 3 similar isoceles triangles constructed outwardly on the sides of $\triangle ABC.$ Then $\triangle ABC$ and $\triangle DEF$ share the same centroid.

Let $\angle BCD = \angle CAE = \angle ABF = \theta.$ Using Conway's notation, the barycentric coordinates of D, E, F with respect to $\triangle ABC$ are given by

$D \ (-a^2 : S_C + S_\theta : S_B + S_\theta)$

$E \ (S_C + S_\theta : -b^2 : S_A + S_\theta)$

$F \ (S_B + S_\theta : S_A + S_\theta : -c^2)$

Since the sum of the homogeneous barycentric coordinates of D, E, F equals $2S_\theta$, due to the identities $S_B + S_C = a^2$, $S_A + S_C = b^2$ and $S_A + S_B = c^2$, it follows that $\triangle ABC$ and $\triangle DEF$ have the same centroid.

● Back to the problem, since $\overline{AM} = \overline{AY} = \overline{AZ}$ and $\angle ZAY = 2\angle BAC = 120^\circ \Longrightarrow \triangle AZY, \triangle BXZ, \triangle CXY$ are 3 similar isosceles triangles erected outwardly on the sides of $\triangle XYZ,$ now using the above lemma, the conclusion follows.

Quick Reply

High School Olympiads

triangle inequality

Reply

vaibhav2903

#1 Mar 31, 2010, 8:16 pm

consider a point P and let the distance from P to the sides of the ABC be x, y, z ito a, b, c resp.let the distances from P to the vertices of the triangle be u, v, w to A, B, C resp.prove that

$$au + bv + cw \geq 4\triangle ABC$$

$$ux + vy + cz \geq 2(xy + yz + zx)$$

Luis González

#2 Mar 31, 2010, 11:17 pm

Let M, N, L be the midpoints of BC, CA, AB. In the quadrangles $PLAN, PMBL, PNCM$, we have the inequalities:

$$[PLAN] \leq \frac{PA \cdot NL}{2} = \frac{PA \cdot BC}{4}$$

$$[PMBL] \leq \frac{PB \cdot LM}{2} = \frac{PB \cdot CA}{4}$$

$$[PNCM] \leq \frac{PC \cdot MN}{2} = \frac{PC \cdot AB}{4}$$

Adding the previous inequalities yields $PA \cdot BC + PB \cdot CA + PC \cdot AB \geq 4[ABC]$.

vaibhav2903

#3 Apr 7, 2010, 2:45 pm

what about the second one?. anyone could do it.

Quick Reply

High School Olympiads

Equilateral triangle

Reply

jarod

#1 Jan 10, 2006, 5:25 pm

Let ABC be an isosceles triangle $(BA = BC)$. (O, R) is the circumcircle of $\triangle ABC$. It's known that : There exists a point D inside (O) such that $\triangle BCD$ is an equilateral triangle . AD intersects (O) at E . Prove that : $DE = R$

Luis González

#2 Mar 31, 2010, 5:40 am • 1

Since $BA = BD = BC,$ it follows that $\triangle BAD$ is isosceles with apex $B.$ Thus, $\angle BDA = \angle BAE = \pi - \angle BCE \Longrightarrow$ $BCED$ is a kite $\Longrightarrow BE$ is the perpendicular bisector of $DC.$ Consequently, $\angle COE = 2\angle CBE = 60^\circ \Longrightarrow \triangle OCE$ is equilateral with side lenght $R \Longrightarrow DE = EC = EO = R.$

Quick Reply

High School Olympiads

Power of point

Reply

Aquarious

#1 Mar 29, 2010, 8:12 pm

The circle of ABC is tangent to BC CA and AB at D E F respectively. Suppose the incircle intersect AD again at a point X

such that AX=XD . XB and XC intersect the circle again at point Y and Z respectively .

Show that EY=FZ

Luis González

#2 Mar 29, 2010, 9:26 pm

Rays XB and XC become the X-symmedians of triangles $\triangle XFD$ and $\triangle XED.$ Then if M and N denote the midpoints of FD and $ED,$ it follows that $\angle FXB=\angle MXD$ and $\angle EXC=\angle NXD.$ Because of $MX \parallel BA$ and $NX \parallel CA,$ we have $\angle BAD=\angle FXB$ and $\angle CAD=\angle EXC.$ On the other hand, since BF and CE are both tangent to $(I),$ it follows that $\angle BAD=\angle FXB=\angle BFY, \angle CAD=\angle EXC=\angle CEZ \Longrightarrow AD \parallel FY \parallel EZ.$ Consequently, quadrilateral $FEZY$ is an isosceles trapezoid $\Longrightarrow$ its diagonals EY and FZ are congruent.

This post has been edited 1 time. Last edited by Luis González, Mar 30, 2010, 6:36 pm

dgreenb801

#3 Mar 30, 2010, 10:03 am

Does this mean that AD must be an angle bisector in this case?

Luis González

#4 Mar 30, 2010, 10:14 am

dgreenb801, no, it doesn't. The fact that X is the midpoint of AD does not imply that Gergonne ray AD is identical to the inner angle bisector of <BAC. For instance, revert the construction: Consider the circle (I), two arbitrary points D,X on (I) and let A be the reflection of D about X. Tangent of (I) at D and tangents from A to (I) bound a triangle with the desired conditions which is not necessarily isosceles. Is that what you mean?.

dgreenb801

#5 Mar 30, 2010, 10:55 am

Yes, that is what I meant. Here is why I thought that:
If $FEZY$ is an isoceles trapezoid then $FY=EZ$, so $\angle FXY=\angle EXZ$ since they intercept the same arc. But you proved $\angle FXB=\angle BAD$ and $\angle EXC=\angle CAD$, which would mean AD is an angle bisector. Where did I go wrong?

Aquarious

#6 Mar 30, 2010, 11:02 am

why\angle FXB = \angle MXD and \angle EXC = \angle NXD ?

dgreenb801

#7 Mar 30, 2010, 11:05 am

That is a well-known propetry of a symmedian (reflection of the median over the angle bisector). See http://www.cut-the-knot.org/Curriculum/Geometry/Symmedian.shtml#explanation for a good proof.

dgreenb801

#8 Mar 30, 2010, 11:21 am

Thank you, it took me a while to realize it.

Quick Reply

High School Olympiads

Turkey 2010 TST Second Day Question 2

Source: Turkey 2010 TST Second Day Question 2

zeyd

#1 Mar 29, 2010, 6:04 pm

Let D be a point inside a triangle ABC; BD intersects AC at point E and CD intersects AB at point F. If points A, E, D, F are concyclic, we call the circle which passes through the points A, E, D, F to be $S(D)$. Show that all circles $S(D)$ have a common point different from A.

Luis González

#2 Mar 30, 2010, 8:35 am

$B \equiv AF \cap ED$ and $C \equiv AE \cap FD$ are conjugate points WRT the circumcircle $\mathcal{S}$ of the convex $AEDF$. Then it follows that the circle (M) with diameter BC is orthogonal to $\mathcal{S}$. If H is the orthocenter of $\triangle ABC$ and A', B', C' the feet of the altitudes on BC, CA, AB, the inversion through pole A with power $\overline{AA'} \cdot \overline{AH}$ transforms (M) into itself and takes $\mathcal{S}$ into a straight line s orthogonal to (M) due to the conformity $\Longrightarrow M \in s$. Hence, the circle $\mathcal{S}$ passes through the image of M under the referred inversion, i.e. the orthogonal projection of H on the median AM.

Ahwingsecretagent

#3 Mar 30, 2010, 5:50 pm

What is the definition for conjugates with respect to a cyclic quadrilateral? Thanks.

gauravpatil

#4 Apr 16, 2010, 10:15 pm

On inverting through A We see that the Line $E'F'$ is a polar of a point that varies on $B'C'$ WRT the circumcircle of $AB'C'$.
$E'F'$ can be considered well using the fact $\pi = \angle AEB + \angle AFC = \angle AB'E' + \angle AC'F'$
Therefore,all lines $E'F'$ are concurrent at pole of $B'C'$ which is constant point that is mapped to a point on the median in triangle ABC.

High School Olympiads

metric relation

Reply

Source: baltic way, 2006

pohoatza

#1 May 1, 2007, 8:25 pm

Let the medians of the triangle ABC intersect at point M. A line d through M intersects the circumcircle ABC at X and Y so that A and C lie on the same side of d. Prove that $BX \cdot BY = AX \cdot AY + CX \cdot CY$.

Luis González

#2 Mar 28, 2010, 11:04 pm

Let N be the midpoint of AC and A', B', C', N' the orthogonal projections of A, B, C, N on the line d. Segment NN' becomes the median of the trapezoid $ACC'A' \Longrightarrow NN' = \frac{1}{2}(AA' + CC')$. But from $\triangle MBB' \sim \triangle MNN'$, we get the proportion $\frac{BB'}{NN'} = \frac{BM}{NM} = 2$. Hence, it follows that $BB' = AA' + CC'$ $(\star)$.

On the other hand, if R denotes the circumradius of $\triangle ABC$, we have the relations

$$BX \cdot BY = 2R \cdot BB' \ , \ AX \cdot AY = 2R \cdot AA' \ , \ CX \cdot CY = 2R \cdot CC'.$$

Combining these expressions with $(\star)$ yields

$$\frac{BX \cdot BY}{2R} = \frac{AX \cdot AY}{2R} + \frac{CX \cdot CY}{2R} \Longrightarrow BX \cdot BY = AX \cdot AY + CX \cdot CY.$$

Quick Reply

High School Olympiads

Short and hard

Reply

oneplusone
#1 Mar 27, 2010, 2:20 pm

Let X be on the extension of BC of triangle ABC. Let the incircles of ABX and ACX meet at 2 points P and Q. Prove that PQ passes through a fixed point independent of X.

Hint: Let the excircle opposite A meet BC at D, then PQ passes through the midpoint of AD, I think.

[geogebra]50a85773e6a9c2efa6f2bca419c90b3b01483d1e[/geogebra]

Luis González
#2 Mar 28, 2010, 5:36 am

Let U,V and M,N denote the tangency points of the incircles of $\triangle ACX$ and $\triangle ABX$ with $BC, AX,$ respectively. It is well-known that in any triangle the midline referred to a vertex, the inner angle bisector issuing from another vertex and the polar of the remaining vertex with respect to its incircle concur. Consequently, $R \equiv UV \cap ED$ lies on the external bisector of $\angle ACB$ and $S \equiv MN \cap ED$ lies on the internal bisector of $\angle ABC \Longrightarrow S, R$ are both fixed on the line $ED.$ Therefore, the radical axis PQ of the incircles of $\triangle ACX, \triangle ABX,$ which becomes the midline of the trapezoid $MUVN,$ passes through the midpoint of $SR.$

livetolove212
#3 May 9, 2010, 11:35 pm

Actually, this problem was proposed in IMO Shortlisted. For an example, you can see it in my article:
http://nguyenvanlinh.wordpress.com/2009/11/21/the-simple-geometry-problem-and-its-application-own/

Quick Reply

High School Olympiads

cyclic pentagram

Reply

Source: own

livetolove212

#1 Mar 27, 2010, 10:14 am • 1

Given a circle (O). Let 5 points A, B, C, D, E lie on (O) and $(O_1), (O_2), (O_3), (O_4), (O_5)$ pass through $A, B; B, C; C, D; D, E; E, A$, respectively. Let A', B', C', D', E' be the second intersections of (O_1) and (O_2), (O_2) and (O_3), (O_3) and (O_4), (O_4) and (O_5), (O_5) and (O_1). If AA', BB', CC', DD', EE' concur at S, then prove that A', B', C', D', E' are concyclic and the center of $(A'B'C'D'E')$ lies on OS.

Note that: This problem is the generalization of Pentagram open problem 1

Attachments:
picture.PDF (21kb)

Luis González

#2 Mar 27, 2010, 11:54 am • 1

S is common radical center of the circles $(O_1), (O_2), (O_3), (O_4), (O_5)$. Thus, inversion $\mathcal{I}$ through center S with power equal to the power of S with respect to $(O_1), (O_2), (O_3), (O_4), (O_5)$ transforms them into themselves and maps $A \mapsto A'$, $B \mapsto B'$, $C \mapsto C'$, $D \mapsto D'$ and $E \mapsto E'$. If A, B, C, D, E are concyclic, then A', B', C', D', E' are concyclic on another circumference (O'), namely the inverse image of (O) under $\mathcal{I}$. Since the inverse of a circumference is also its homothetic image through the same center of inversion, it follows that S, O, O' are collinear.

borislav_mirchev

#3 Mar 27, 2010, 6:37 pm

Very interesting problem and solution. I think you both are great geometers. Greetings guys!

Quick Reply

High School Olympiads

triangle problem

vaibhav2903

#1 Mar 27, 2010, 12:31 am

$\triangle ABC$ and $\triangle DAC$ are two isosceles triangle such that $\angle BAC$=20 and $\angle ACD$=100.P.T.$AB = BC + CD$

Luis González

#2 Mar 27, 2010, 7:25 am

I think that the bases should be $BC, AC,$ respectively and $\angle ADC = 100^\circ$ instead of $\angle ACD = 100^\circ$.

By simple angle chase we have $\angle ABC = 80^\circ$ and $\angle DAC = 40^\circ \Longrightarrow \angle DAB = 60^\circ$. Hence, if we take a point P on $\overline{AB}$ such that $AP = AD$, then $\triangle ADP$ is equilateral, which implies that $DA = DP = DC \Longrightarrow \triangle PDC$ is isosceles with apex D. Then $\angle DPC = \frac{1}{2}(180^\circ - 40^\circ) = 70^\circ$. Therefore, $\angle BPC = 180^\circ - 60^\circ - 70^\circ = 50^\circ \Longrightarrow \triangle BPC$ is isosceles with apex $B \Longrightarrow BP = BC$. Then it follows that $AB = BP + AP = BC + AD = BC + CD$.

sunken rock

#3 Feb 22, 2012, 11:53 pm

Remark: Seeing that $ABCD$ is cyclic may show faster that $BCDP$ (P as below) is a kite!

Best regards,
sunken rock

High School Olympiads

Altitudes of triangle

Reply

Source: Own

tmath

#1 Mar 26, 2010, 9:05 am

AD, BE, CF are altitudes of triangle ABC. If A_1, B_1, C_1 are orthocenters of $\triangle AEF, \triangle BDF, \triangle CDE$ respectively, then prove that DA_1, EB_1, FC_1 are congruent.

This post has been edited 1 time. Last edited by tmath, Mar 26, 2010, 9:48 am

jgnr

#2 Mar 26, 2010, 9:34 am

After doing some angle chasing, we get $\angle DEA_1 = 90 - (B - A)$, so

$\sin EDA_1 = \frac{EG}{DA_1} \sin DEA_1 = \frac{2r \cos AEF}{DA_1} \cos(B - A) = \frac{2r \cos B}{DA_1} \cos(B - A)$. Similarly we get

$\sin FDA_1 = \frac{2r \cos C}{DA_1} \cos(C - A)$. Hence $ \frac{\sin\EDA_1}{\sin FDA_1=\frac{\cos B\cos(B-A)}{\cos C\cos(C-A)}$. By the same way, we can calculate the other angle proportions and by Ceva we have done.

Luis González

#3 Mar 26, 2010, 10:24 pm

Let H be the orthocenter of $\triangle ABC.$ Since $EA_1 \parallel FH$ and $FA_1 \parallel EH,$ it follows that HFA_1E is a parallelogram. Similarly, HDB_1F and HEC_1D are parallelograms $\Longrightarrow A_1, B_1, C_1$ are the reflections of H about the midpoints of segments $EF, FD, DE.$ Hence if U denotes the centroid of $\triangle DEF,$ then $\triangle A_1B_1C_1$ is the image of $\triangle DEF$ under the composition of homotheties $\mathcal{H}(U, -\tfrac{1}{2}) \circ \mathcal{H}(H, 2) \Longrightarrow \triangle DEF$ and $\triangle A_1B_1C_1$ are centrally symmetric. Therefore, $DA_1,$ EB_1, FC_1 concur at the symmetry center of $\triangle DEF$ and $\triangle A_1B_1C_1$ lying on the line $UH.$

Quick Reply

High School Olympiads

A , A' , M are collinear

Reply

seifi-seifi

#1 Mar 26, 2010, 11:33 am

in triangle XYZ the incircle touch YZ, ZX, XY at L, S, R. the bisector of $\angle SLR$ intersect incircle at A. from I we draw a line parallel toAL and it intersect RS at A'. if M be midpoint of YZ prove that A, A', M are collinear.

(I be incenter)

im sorry for my bad inglish.

Luis González

#2 Mar 26, 2010, 6:40 pm • 1

For convenience, rename $\triangle LSR \longrightarrow \triangle ABC, \triangle XYZ \longrightarrow \triangle A_0B_0C_0$.

$\triangle A_0B_0C_0$ is the tangential triangle of $\triangle ABC$ and N is the midpoint of BC. Tangent line of (O) at A' cuts A_0C_0 and A_0B_0 at $X, Y,$ respectively. By Newton's theorem for the tangential quadrilateral XYB_0C_0, we have $D \equiv XB_0 \cap YC_0 \cap BC$. Let R be the midpoint of DA_0. Since A' is the midpoint of XY, then due to obvious symmetry about OA_0, it follows that R, A' and the midpint M_1 of B_0C_0 are collinear on the Newton's line of the complete quadrangle $XDYA_0$. On the other hand, let the parallel to RA'' from A_0 cut BC, XY and DA' at M, P, Q. $A'A''$ and PA'' become midlines of $\triangle MQD$. Thus

$\frac{PA_0}{PQ} = \frac{PA_0}{PM} = \frac{A'A_0}{A'N}$. But since BC is the polar of A_0 WRT (O), we have

$\frac{A'A_0}{A'N} = \frac{OA_0}{OA'} \Longrightarrow \frac{OA_0}{OA'} = \frac{PA_0}{PQ} \Longrightarrow O$ lies on the line PA''.

Thus, $A'A''$ passes through M_1 and similarly $B'B''$ and $C'C''$ pass through the midpoints M_2, M_3 of C_0A_0 and $A_0B_0 \Longrightarrow$ $A'A'', B'B'', C'C''$ concur at the 2nd mid-arc point of $\triangle A_0B_0C_0$ and the proof is completed.

Quick Reply

High School Olympiads

Midpoint lies on the radical axis of two incircles (2)

Source: own

livetolove212

#1 Mar 24, 2010, 4:27 pm

Given triangle ABC. Let P be a point inside ΔABC such that $AB+BP=AC+CP$. $BP\cap AC=\{Y\}$, $CP\cap AB=\{Z\}$. Prove that the midpoint of BC lies on the radical axis of two incircles of two triangles YCP and ZBP.

vittasko

#2 Mar 25, 2010, 6:38 pm

Let $(I_1),\ (I_2)$ be, the incircles of the triangles $\triangle ZBP,\ \triangle YCP$ respectively and let $L,\ N$ be, their tangency points to the side-segments $AB,\ AC,$ of the given triangle $\triangle ABC.$

Because of $AB+BP=AC+CP$, (1) we have also that $AZ+ZP=AY+YP$, (2) as well and not difficult to prove.

From $(2)\Longrightarrow AZ-YP=AY-ZP$ (3)

From (3) we conclude that there is a circle so be it $(K),$ outwardly to $AZPY,$ which tangents to all its sidelines.

From $(1),\ (2)$ we can easy to prove that $BL=CN$ (4)

Based on the below **Lemma 1,** we have that $BD=CE$, $(5)\Longrightarrow MD=ME$, (6) where $D,\ E,$ are the orthogonal projections of the centers $I_1,\ I_2$ of $(I_1),\ (I_2)$ respectively, on $BC.$

● In order to be the midpoint M of BC on the radical axis of $(I_1),\ (I_2),$ it is enough to prove that $(MI_1)^2-R_1^2=(MI_2)^2-R_2^2$, (7) where $R_1,\ R_2$ are the radii of $(I_1),\ (I_2),$ respectively.

$(MI_1)^2-R_1^2=(MD)^2+(I_1D)^2-(BI_1)^2+(BL)^2=(MD)^2-(BD)^2+(BL)^2$, (8)

Similarly, $(MI_2)^2-R_2^2=(ME)^2-(CE)^2+(CN)^2$, (9)

From $(4),\ (5),\ (6),\ (8),\ (9)\Longrightarrow(7)$ and the proof is completed.

LEMMA 1. - A triangle $\triangle ABC$ is given and let P be, an arbitrary point on its internal angle bisector of $\angle A.$ The lines through $B,\ C$ and perpendicular to $BP,\ CP$ respectively, intersect the external angle bisector of $\angle A,$ at points $D,\ E,$ respectively. Prove that the midperpendicular of $BC,$ bisects the segment $DE.$

Kostas Vittas.

PS. I will post here later the proof of the **Lemma 1** I have in mind and some more details for the proof of the result (4) I mentioned.

Attachments:

t=340482.pdf (7kb)

This post has been edited 1 time. Last edited by vittasko, Mar 26, 2010, 12:35 am

vittasko

#3 Mar 25, 2010, 10:03 pm

> *vittasko wrote:*
>
> **LEMMA 1. - A triangle $\triangle ABC$ is given and let P be, an arbitrary point on its internal angle bisector of $\angle A.$ The**

lines through $B,\ C$ and perpendicular to $BP,\ CP$ respectively, intersect the external angle bisector of $\angle A$, at points $D,\ E$, respectively. Prove that the midperpendicular of BC, bisects the segment DE.

PROOF OF THE LEMMA 1. Through vertices $B,\ C$ of the given triangle $\triangle ABC$, we draw the lines perpendicular to $AB,\ AC$ respectively and we denote as $F,\ Z$, their points of intersection, from the external angle bisector of $\angle A$.

Because of the similarity of the right triangles $\triangle BAF,\ \triangle CAZ$, we have that the midperpendicular of BC, bisects the segment FZ (it is a well known result, which has been proved some times in this forum) and it is easy to show that the midpoint N of FZ, is coincided with the midpoint of the arc BC of the circumcircle (O) of $\triangle ABC$, containing the vertex A.

Because of $FB \perp BA$ and $DB \perp BP$ we have that $\angle FBD = \angle ABP$ and so, the triangles $\triangle FBD,\ \triangle ABP$ are similar, because they have also $\angle BFD = \angle BAP = \frac{\angle A}{2}$.

So, we have that $\frac{BF}{BA} = \frac{FD}{AP} \Longrightarrow FD = AP \cdot \frac{BF}{BA}, (1)$

Similarly, the triangles $\triangle ZCE,\ \triangle ACP$ are also similar and then, we have that $\frac{CZ}{CA} = \frac{ZE}{AP} \Longrightarrow ZE = AP \cdot \frac{CZ}{CA}, (2)$

But, from the similarity of the right triangles $\triangle BAF,\ \triangle CAZ$, we have that $\frac{BF}{CZ} = \frac{BA}{CA} \Longrightarrow \frac{BF}{BA} = \frac{CZ}{CA}, (3)$

From $(1),\ (2),\ (3) \Longrightarrow FD = ZE\,, (4)$

From (4) and from $NF = NZ$, we conclude that $ND = NE$ and the proof of the **Lemma 1** is completed.

• It is obvious that we have also that $MD' = ME'$, where $D',\ E'$ are the ortogonal projections of $D,\ E$ respectively, on BC.

Kostas Vittas.

Attachments:
t=340482(a).pdf (5kb)

This post has been edited 1 time. Last edited by vittasko, Mar 26, 2010, 12:38 am


livetolove212
#4 Mar 25, 2010, 11:11 pm

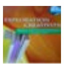
Such a nice solution dear **Kostas**! . Let S be the intersection of YZ and AP. We can also prove that SM is the radical axis of (I_1) and (I_2) and $SK \perp BC$.


Luis González
#5 Mar 26, 2010, 2:35 am

If $AB + BP = AC + CP$, then convex quadrilateral $AZPY$ is A-circumscriptible in a circle ω with center O. Let ω_1 and ω_2 be the incircles of $\triangle ZPB$ and $\triangle YPC$. U, V and R, S are the tangency points of ω_1, ω_2 with PZ, PY and PB, PC, respectively. UR, VS and PO cut BC at M, N, D, respectively. Obviously, $UR \parallel VS \parallel OP$ since OP bisects $\angle BPC$. Then $\triangle BRM \sim \triangle BPD$ and $\triangle CSN \sim \triangle CPD$

$$\frac{BR}{BP} = \frac{BM}{BD}\ ,\ \frac{CS}{CP} = \frac{CN}{CD} \Longrightarrow \frac{BR}{CS} \cdot \frac{CP}{BP} = \frac{BM}{CN} \cdot \frac{CD}{BD}\ (\star)$$

But by angle bisector theorem in $\triangle BPC$ we get $\frac{CP}{BP} = \frac{CD}{BD}$. If B', C' are the tangency points of ω with PB, PC, we have $PB' = PC' = BR = CS$. Thus, from $(\star)$, it follows that $BM = CN$, which implies that midpoint of BC is also midpoint of $MN \Longrightarrow$ Midline of trapezoid $UVSR$ (radical axis of ω_1, ω_2) passes through the midpoint of BC.


livetolove212
#6 Mar 26, 2010, 5:26 pm

Thanks **Luis** for your nice proof! Let me introduce my proof:
Without lost of generality, we can asume that $PB > PC$.
Let M be the midpoint of BC. We know that there is a circle (O) which tangents to BP at B', CP at C'. Let $J, K; S, R$ be the tangencies of (O_1) and (O_2) with PZ, PB, respectively. Construct a line through M and parallel to PO, intersects PB at T. Then TM is the cleaver of triangle PBC.
We get $TB = TP + PC$.

We will show that $TR = TK$ iff $TP + PS = TK$
$\Leftrightarrow TB - BK = TP + PC - CS \Leftrightarrow BK = CS$
$\Leftrightarrow PB' = PC'$ (right)
So TM is the midline of trapezoid JKSR and we are done.

Quick Reply

Spain

Problemas:: GEOMETRIA

Reply

M4RI0

#1 Dec 8, 2006, 9:03 pm • 1

Aca abro el topic para geo

$Problema\ 1$

Sea $ABCD$ un cuadrilátero cíclico, y sea O la intersección de sus diagonales. Suponer que:

$(AB)(OC) = (AC)(OB) = (BC)(OA)$

Demostrar que $BD \geq 4BO$ (MOP, 1998)

aev5peru

#2 Dec 11, 2006, 4:47 am

SOLUCION.
Como $(AB)(OC) = (AO)(BC) = (BO)(AC)...(1)$,
ENTONCES $< ABO =< OBC$, por el inscriptible $< CAD =< ACD \Rightarrow AD = CD = n$
Sea $AB = a \rightarrow AO = ak$, $BC = b \rightarrow OC = bk$, $BO = c$, despejando c en (1), en funcion de a y b,
$c = ab/(a+b)...(2)$
por la semejansa entre los triangulos AOD y BOC: $(c)(d) = (ak)(bk)$, reemplazando (2)
tenemos que $d = (a+b)k^2...(3)$
Aplicando el Teorema de Ptolomeo.
$(c+d)(a+b)k = n(a+b) \rightarrow (c+d)k = n...(4)$
Por la semejansa en los triangulos AOB y BAD, tenemos que
$n^2 = d(c+d)$, reemplazando (4).
$k^2(c+d) = d$,reemplazando (3)
$c+d = a+b \rightarrow d = a+b-c$ reemplazando (3) y (4).
$(a+b)k^2 = (a+b) - ab/(a+b) \Rightarrow k^2 = (a^2+b^2+ab)/(a+b)^2$
reemplazando en lo que nos pide.
$BD \geq 4BO$
$<=> d \geq 3c$
$<=> k^2(a+b) \geq 3ab/(a+b)$
$<=> (a^2+b^2+ab)/(a+b)^2 \geq 3ab/(a+b)^2$
$<=> (a-b)^2 \geq 0$, lo cual es verdadero.
$Q.E.D.$

mhuarancca

#3 Dec 15, 2006, 8:22 am

p_1)(nivel 2 , rioplatense 2006)

sea ABC un triangulo rectangulo en A . considere todos los triangulos XYZ rectangulos isosceles en X, donde esta sobre el segmento BC , Y sobre el segmento AB, y Z sobre el segmentos AC.
determinar el lugar geometrico de los puntos de las hipotenusas YZ de tales triangulos XYZ

M4RI0

#4 Dec 16, 2006, 10:52 am

Aqui mi solucion para el problema 1:

Denotemos: $AB = m$, $BC = n$, $OA = a$, $OB = b$, $OC = c$

Denotemos: $AB = m, BC = n, OA = a, OB = b, OC = c$

Luego:$(m)(c) = (a+c)(b) = (n)(a) \Rightarrow \frac{m}{a} = \frac{n}{c} \Rightarrow \measuredangle ABO = \measuredangle OBC \Rightarrow \boxed{b^2 = mn - ac}$

Además:$\frac{c}{a+c} = \frac{b}{m}$ y $\frac{a}{a+c} = \frac{b}{n} \Rightarrow \frac{b}{m} + \frac{b}{n} = 1 \Rightarrow \boxed{b(m+n) = mn}$

ABCD es cíclico, de lo cual tenemos: $\boxed{ac = bd}$

Por MA y MH:$\frac{m+n}{2} \geq \frac{2}{\frac{1}{m}+\frac{1}{n}} = 2b$

Usaremos las tres relaciones encontradas para llegar a la desigualdad pedida:

$\Rightarrow m+n \geq 4b \Rightarrow b(m+n) \geq 4b^2 \Rightarrow mn \geq 4b^2$
$\Rightarrow b^2 + ac \geq 4b^2 \Rightarrow ac \geq 3b^2 \Rightarrow bd \geq 3b^2 \Rightarrow d \geq 3b \Rightarrow b+d \geq 4b$
$\Rightarrow BD \geq 4BO$

carlosbr

#5 Dec 17, 2006, 4:45 am

Problema 03:

Sea ABC un triangulo acutangulo y sean D, E, F los pies de las alturas trazadas desde A, B, C.
Sean P, Q, R los pies de las perpendicuares desde A, B, C hacia EF, FD, DE, respectivamente.

Probar que las lineas AP, BQ, CR son concurrentes.

Carlos Bravo

mhuarancca

#6 Dec 17, 2006, 6:07 am

demostremos que $< ABE =< QBD$, vemos que si esto se cumple , esto se hara analogo para los puntos A y C , y como AD, BE, FC concurren se cumple el ceva trigonometrico , con ello tambein se cumplira para BQ, AP, CR, vemos que $< ABE + < BAE = 90$, y por inscriptible $< BAE =< BDQ$ ademas, $< BDQ + < QBD = 90$ porlo tanto $< ABE =< QBD$,
QED

aev5peru

#7 Dec 17, 2006, 7:23 am

> carlosbr wrote:
> Problema 03:
>
> Sea ABC un triangulo acutangulo y sean D, E, F los pies de las alturas trazadas desde A, B, C.
> Sean P, Q, R los pies de las perpendicuares desde A, B, C hacia EF, FD, DE, respectivamente.
>
> Probar que las lineas AP, BQ, CR son concurrentes.
>
> Carlos Bravo

Bueno, como dice marco, como son isogonales, concurren en el circunetro de ABC.

mhuarancca

#8 Dec 19, 2006, 7:45 am

bueno tuve un pequeño error al tipear el problema :

P_1) sea ABC un triangulo rectangulo en A. considere todos los triangulos XYZ rectangulos isosceles en X, donde X esta sobre el segmento BC, Y sobre el segmento AB, y Z sobre el segmento AC.
determinar el lugar geometrico de los puntos medios de las hipotenusas YZ de tales triangulos XYZ.

M4RI0

#9 Dec 19, 2006, 10:16 am

Ahora si tiene sentido. Para el anterior el lugar creo q era todo el triangulo

M4RI0

#10 Dec 27, 2006, 8:21 am

$Problema\ 4$

Una circunferencia es tangente al circuncírculo de un triangulo ABC y ademas tangentes a los lados AB, AC en P, Q respectivamente. Demostrar que el punto medio de PQ es el incentro del triangulo ABC.

Pueden encontrar este problema en el Matescopio con el numero 200.4

aev5peru

#11 Dec 28, 2006, 10:42 pm

> M4RI0 wrote:
>
> $Problema\ 4$
>
> Una circunferencia es tangente al circuncírculo de un triangulo ABC y ademas tangentes a los lados AB, AC en P, Q respectivamente. Demostrar que el punto medio de PQ es el incentro del triangulo ABC.
>
> Pueden encontrar este problema en el Matescopio con el numero 200.4

Este problema sale rapido, usando el teorema de Pascal.

M4RI0

#12 Dec 30, 2006, 6:00 am

Postea tu solucion por favor. Yo tengo una ubicando primero un punto X sobre AB tal que XM es perpendicular a AB, luego con trigonometria hallamos AX que resulta ser $p - a$(p : semiperimetro) y como ademas M se encuentra sobre la bisectriz del angulo A, concluimos que M es el incentro de ABC.

cuenca

#13 Jan 31, 2007, 9:07 am

Mas vale tarde que nunca.....Sea X el punto de tangencia entre el circuncirculo y la otra circunferencia. XP interseca a la circunferencia circunscrita C en P_1, XQ interseca a C en Q_1, es sencillo demostrar que P_1 es el punto que divide al arco AB en partes iguales(por homotecia entre las circunferencias), luego CP_1, BQ_1 son bisectrices y por T. PASCAL en P_1, A, Q_1, C, X, B, las bisectrices se cortan en un punto sobre PQ, luego como $AQ = AP$, al trazar la bisectriz de A, cortara a PQ en su punto medio.

mhuarancca

#14 May 16, 2007, 8:41 am

algunos problemas q pueden salir por potencia de un punto :

1) Sean S_1 y S_2 dos circulos y sea O el punto de corte de las dos tangentes externas comunes a los dos circulos . demostrar que la potencia de O con respecto a cualquier circulo tangente externamente a S_1 y S_2 es constante.

2) Sea S un circulo ; sean A y B dos puntos . una recta variable a traves de A corta a S en M y N . demostar que los circuncirculos de los triangulos MNB pasan por otro punto fijo , distinto de B

3) En un triangulo ABC , sea I el incentro y sean X , Y y Z los puntos de tangencia del incirculo con los lados BC , AC y AB respectivamente . la recta AX corta al incirculo nuevamente en P , y la recta AI corta a YZ en Q . demostrar que X , I , Q y P estan sobre una misma circunferencia

aev5peru
#15 May 17, 2007, 4:23 am

> *mhuarancca wrote:*
> 3) En un triangulo ABC , sea I el incentro y sean X , Y y Z los puntos de tangencia del incirculo con los lados BC , AC y AB respectivamente . la recta AX corta al incirculo nuevamente en P , y la recta AI corta a YZ en Q . demostrar que X , I , Q y P estan sobre una misma circunferencia

A esto le falta algo, ya que si $AB = AC$, entonces, P, Q, I, X seran colineales, considerando AB diferente de AC.
Tomando a la circunferencia inscrita al triangulo ABC como circunferencia de Magnus (inversion), invirtiendo la recta AX. A, P, X, se invierten en Q, P, X respectivamente, como I es el centro de esta circunferencia. P, Q, I, X, son puntos conciclicos.

mhuarancca
#16 Jun 1, 2007, 5:57 am

en un triangulo ABC , un circulo corta a BC en L y L" , a AC en M y M" y a AB en N y N" . demostrar que si AL , BM y CN son concurrentes, entonces AL" , BM" y CN" tambien son concurrentes

skuge
#17 Sep 2, 2007, 12:29 pm • 1

Voy a mostrar un lema más general del cual sigue el problema inmediatamente:

<u>**Lema:**</u> *Si* (AL, BM, CN) *y* (AL', BM', CN') *son dos trios de cevianas concurrentes con* L, L', M, M', N, N' *sobre los lados de un triangulo. Los puntos* L, L', M, M', N, N' *están sobre una cónica.*

<u>*Demostración:*</u> Sean $X \in MN \cap BC, Y \in LN' \cap AC$ y $Z \in L'M' \cap AB$
Por Ceva:
$$(\frac{BN}{NA})(\frac{AM}{MC})(\frac{CL}{LB}) = 1$$
$$(\frac{BN'}{N'A})(\frac{CL'}{L'B})(\frac{AM'}{M'C}) = 1$$

Por Menelao:
$$\frac{BX}{XC} = (\frac{BN}{NA})(\frac{AM}{MC})$$
$$\frac{CY}{YA} = (\frac{BN'}{N'A})(\frac{CL}{LB})$$
$$\frac{AZ}{ZB} = (\frac{CL'}{L'B})(\frac{AM'}{M'C})$$

Entonces
$$(\frac{BX}{XC})(\frac{CY}{YA})(\frac{AZ}{ZB}) = (\frac{BN}{NA})(\frac{AM}{MC})(\frac{BN'}{N'A})(\frac{CL}{LB})(\frac{CL'}{L'B})(\frac{AM'}{M'C}) = 1$$
Que por Menelao X, Y, Z son colineales.
Por el teorema de Pascal, como los lados opuestos del hexagono $LL'MM'NN'$ se cortan en puntos colineales los puntos L, L', M, M', N, N' están sobre una cónica.

En el problema original como cinco puntos definen una cónica los otro tres puntos de corte del ciruclo deben determinar cevianas concurrentes.

aev5peru
#18 Sep 3, 2007, 5:37 am

> *skuge wrote:*
> Voy a mostrar un lema más general del cual sigue el problema inmediatamente:
>
> <u>**Lema:**</u> *Si* (AL, BM, CN) *y* (AL', BM', CN') *son dos trios de cevianas concurrentes con* L, L', M, M', N, N' *sobre los lados de un triangulo. Los puntos* L, L', M, M', N, N' *están sobre una cónica.*
>
> <u>*Demostración:*</u> Sean $X \in MN \cap BC, Y \in LN' \cap AC$ y $Z \in L'M' \cap AB$.

creo q podemos reducir la demostracion.
Vemos que $A, N', B, Zytambien, X, B, L, C$, son cuaterna armonica. Ahora desde Y tenemos: -YC, YL, YB, YX y -YA, YN', YB, YZ son rayos armonicos...entonces X, Z, Y COLINEALES!!.

skuge
#19 Sep 3, 2007, 5:52 am

En qué número de problema vamos?? Bueno acá va el siguiente, uno fácil de Estonia:

Sean M, N y K los puntos de tangencia del incirculo de $\triangle ABC$. Si Q es el centro del circulo que pasa por los puntos medios de MN, NK y KM,
Muestre que Q está sobre la linea que une el incentro y el circuncentro de $\triangle ABC$.

aev5peru
#20 Sep 5, 2007, 6:50 am

> skuge wrote:
> En qué número de problema vamos?? Bueno acá va el siguiente, uno fácil de Estonia:
>
> Sean M, N y K los puntos de tangencia del incirculo de $\triangle ABC$. Si Q es el centro del circulo que pasa por los puntos medios de MN, NK y KM,
> Muestre que Q está sobre la linea que une el incentro y el circuncentro de $\triangle ABC$.

SOL.
Consideremos a el incirculo de ABC como la circunferencia de inversion.
vemos que los puntos medios de MN, NK, KM, son los puntos inversos de A, B, C, entonces la circunferencia q pasa por ABC se invierte en la circunferencia q pasa por los puntos medios de MN, NK, KM, ademas sabemos que cuando se invierten una circunferencia T ,sea $T*$ la circunferencia invertida respecto de la de inversion, entonces los centros de la circunferencia de inversion, de T y $T*$ son colineales. Entonces Q, incirculo y circuncirculo son colineales.

skuge
#21 Sep 5, 2007, 10:13 pm

Bien, esa esta linda... cuando tenga tiempo la proxima semana pongo mi solución...
Otro problemita de calentamiento.

En un triangulo ABC con $\angle A = 60$ muestre que si la linea de euler corta los lados BA y CA en P y Q respec. $HP = OQ$

(con H y O la notacion de siempre)

cuenca
#22 Sep 6, 2007, 9:26 am

buen prob. ahi esta mi sol.
Click to reveal hidden text

aev5peru
#23 Sep 6, 2007, 11:14 am

> skuge wrote:
> Bien, esa esta linda... cuando tenga tiempo la proxima semana pongo mi solución...
> Otro problemita de calentamiento.
>
> En un triangulo ABC con $\angle A = 60$ muestre que si la linea de euler corta los lados BA y CA en P y Q respec. $HP = OQ$
>
> (con H y O la notacion de siempre)

Consideraremos sin perder generalidad $AB < AC$ (ya que si $AB = AC$, entonces ABC equilatero y su recta de euler no esta definida).
Ahora $< BHC =< BOC = 120$, entonces $BHOC$ inscriptible. Sabemos que la recta de euler pasa por H y O (

P, O, HyQ colineales). Ahora $< HBA = 30$, Como $BHOC$ inscriptible, $< PHB =< OCB = HCA$ (lo ultimo puesto que OyH son isogonales), pero $< HCA = 30$, luego $< PHB = 30$, entonces, $< APH = 60$, ENTONCES AQP equilatero. Luego $< PAH =< OAC$, entonces APH y AQO son congruentes $(A.L.A.)$, Con lo cual $PH = OQ$ que es lo que queriamos demostrar.

skuge

#24 Sep 6, 2007, 2:17 pm

Mi solución es parecida a la de cuenca:
Es conocido que la reflexión de la altura sobre la bicectriz pasa por el circuncírculo y es fácil mostrar que $AH = AO$ entonces $\triangle AHO$ es isóseles y $AI \perp HO$, mirnado angulitos es obvio que PQA es equilatero y esto termina el problema porque la bicectriz pasa por el punto medio.

skuge

#25 Sep 6, 2007, 2:28 pm

Muestre que la reflexión sobre los lados opuestos de las cevianas que pasan por el punto de fermat concurren.

aev5peru

#26 Sep 7, 2007, 9:06 am

> skuge wrote:
> Muestre que la reflexión sobre los lados opuestos de las cevianas que pasan por el punto de fermat concurren.

No entiendo....reflexion respecto de que?....Bueno si fuera de la bisectriz del angulo respectivo de donde sale la ceviana.Entonces creo que con Ceva Trigonometrico (la relacion de lados que nos indica el teorema de ceva lo expresamos en relacion de senos de el angulo formado por las cevianas..). tendriamos lo pedido.

skuge

#27 Sep 7, 2007, 2:27 pm • 1

> aev5peru wrote:
> No entiendo....reflexion respecto de que?....Bueno si fuera de la bisectriz del angulo respectivo de donde sale la ceviana.Entonces creo que con Ceva Trigonometrico (la relacion de lados que nos indica el teorema de ceva lo expresamos en relacion de senos de el angulo formado por las cevianas..). tendriamos lo pedido.

Ok. El problema es el siguiente:
Si F es le punto de fermat del $\triangle ABC$ sean x, y y z las reflexiones de las rectas AF, BF y CF sobre los lados BC, CA y AB respectivamente.
~Mostrar que x, y, z concurren.

cuenca

#28 Sep 12, 2007, 9:57 am

Lo primero que se me ocurrio fue coord. baricentricas, pero me parecio muy operativo; demostrare por coordenadas trilineales que x, y, z concurren en el conjugado isogonal del punto de fermat.
A_1 es el reflejo de A en BC, por un calculo simple $A_1 = (-1 : 2cosC : 2cosB)$, ademas es conocido que el punto de fernat es $(\frac{1}{sen(A+60)} : \frac{1}{sen(B+60)} : \frac{1}{sen(C+60)})$, luego si A_2 es la interseccion de la rescta AF(F:punto de fermat) con BC, entonces $A_2 = (0 : \frac{1}{sen_(B+60)} : \frac{1}{sen(C+60)})$, luego para hallar la ecuacion de la recta A1A2, se necesita que el determinante de:
$\begin{pmatrix} x & y & z \\ 0 & y_1 & z_1 \\ -1 & 2cosC & 2cosB \end{pmatrix}$ es 0(y1=csc(b+60),z1=csc(c+60)); de esto:
$xsen(C-B) + ysen(B+60) - zsen(C+60) = 0$, por la simetria en las demas ecuaciones, lo que se puede hacer es hallar el determinante del sistema y demostrar que es 0(demora mucho), yo intui que era el conj. isogonal de F, y lo grafique en computadora y en efecto lo era ; para finalizar basta probar que el conj. isogonal de
F=$F_1(sen(A+60) : sen(B+60) : sen(C+60))$ verifica la ecuacion, y en efecto: como:

sen(A+60)=sen(B+C-60), la ecuacion queda: $-cos(B+C).sen(B-C)+senBcosB-senCcosC=0$ lo cual es evidente por identidades de arcos compuestos. Luego cada recta reflejada pasa por un punto fijo, luego todas concurren.

Nota: Esto es muy operativo, ¿has encontrado algo sintetico, skuge?
P.D. Si hay algo que no se entiende o alguna correcion, por favor decirme para corregirlo o explicarlo mejor

cuenca

#29 Sep 19, 2007, 8:39 am

Mi turno de proponer un problema:
En un triangulo ABC, sea M un punto arbitrario, demostrar la siguiente desigualdad:
$AMsenBMC+BMsenAMC+CMsenAMB\leq s$, donde s es el semiperimetro del triangulo ABC y averiguar para que punto M se da la igualdad.

conejita

#30 Sep 22, 2007, 12:40 am

Mi solucion:

Primero tracemos las perpendicular a MA, MB , MC por los pntos A, B,C respectivamente. Dichas perpendiculares forman con sus intersecciones un triangulo TUV (T es el vertice opuesto a A, U opuesto a B, y V opuesto a C).
Ahora, el problema se transforma en demostrar que:
$AMsenT+BMsenU+CMsenVleqs$ lo cual es equivalente a
AM.VU/2 + BM.VT/2 CM.UT/2 $/leq$ R(AB+BC+CA) (R=circunradio de TUV) y esto es
(ABC) $/leq$ R(AB+BC+CA).
Ahora, como sabemos que el triangulo pedal de menor perimetro es el ortico, entonces si demostramos que
(ABC) $/leq$ R(DE+EF+FD) (donde D, E, F son los pies de las alturas del triangulo TUV; D es opuesto a T, E es opuesto a U, y F es opuesto a V) ya habremos acabado.
Pero esta ultima desigualdad, de hecho es una igualdad. Por lo que el problema ha sido probado.

Un problema muy interesante.

conejita

#31 Sep 22, 2007, 7:25 pm

Ah, se me olvidaba. La igualdad se da cuanto M es ortocentro de TUV, es decir, cuando es incentro de ABC.

Luis González

#32 Mar 24, 2010, 11:56 pm

> *mhuarancca wrote:*
> Sea ABC un triángulo rectángulo en $A.$ Considere todos los triángulos XYZ rectángulos isosceles en $X,$ donde X esta sobre el segmento BC,Y sobre el segmento AB y Z sobre el segmento $AC.$ Determinar el lugar geométrico de los puntos medios de las hipotenusas YZ de tales triángulos $XYZ.$

Como $XY=XZ$ y $\angle YXZ=\angle YAZ=90^{\circ},$ se sigue que el cuadrilátero $AYXZ$ es cíclico y que los arcos XY y XZ de su circuncírculo son conguentes $\Longrightarrow AX$ es bisectriz de $\angle BAC.$ Así X es fijo en BC siendo el pie de la A-bisectriz interior.

Si P es la proyección ortogonal de X en el cateto AB y M el punto medio del segmento $YZ,$ el cuadrilátero $MYPX$ es cíclico en vista que los ángulos $\angle YPX$ y $\angle YMX$ son ambos rectos. Luego se tendrá $\angle XPM=\angle XYZ=45^{\circ}.$

El lugar geométrico de M es pues el segmento de recta que conecta los pies de las perpendiculares de X a $AB,AC.$

Luis González

#33 Mar 25, 2010, 2:33 am

> *mhuarancca wrote:*
> Sean $\mathcal{S}_1$ y $\mathcal{S}_2$ dos círculos y sea O el punto de corte de las dos tangentes externas comunes a los dos círculos. Demostrar que la potencia de O con respecto a cualquier circulo tangente externamente a $\mathcal{S}_1$ y $\mathcal{S}_2$ es constante.

Sea $\mathcal{S}$ una circunferencia arbitraria tangente externamente a $\mathcal{S}_1$ y $\mathcal{S}_2$ en $A, B,$ respectivamente. Los puntos A y B son los centros de homotecia negativa que transforman $\mathcal{S}_1 \mapsto \mathcal{S}$ y $\mathcal{S}_2 \mapsto \mathcal{S}$, respectivamente. O es por supuesto el centro de la homotecia positiva que transforma $\mathcal{S}_1 \mapsto \mathcal{S}_2$, así por el teorema de Monge-d'Alembert los puntos A, B, O están alineados. A, B son homólogos en la inversión que trasforma $\mathcal{S}_1$ en $\mathcal{S}_2$ con mismo centro O, por consiguiente la potencia de O con respecto a la circunferencia $\mathcal{S}$ es constante e igual a la potencia de inversión que transforma $\mathcal{S}_1$ en $\mathcal{S}_2$.

Luis González

#34 Mar 25, 2010, 3:15 am

> *mhuarancca wrote:*
> En un triángulo ABC, sea I el incentro y sean X, Y y Z los puntos de tangencia del incírculo con los lados BC, AC y AB, respectivamente. La recta AX corta al incírculo nuevamente en P y la recta AI corta a YZ en Q. Mostrar que X, I, Q y P están sobre una misma circunferencia.

Por teorema del cateto en el triángulo $\triangle AYI$ se tiene $AY^2 = AQ \cdot AI$, pero la potencia de A con respecto al incírculo (I) es $AY^2 = AP \cdot AX \Longrightarrow AQ \cdot AI = AP \cdot AX \Longrightarrow I, Q, P, X$ son concíclicos.

Luis González

#35 Mar 25, 2010, 7:24 am

> *skuge wrote:*
> Muestre que la reflexión sobre los lados opuestos de las cevianas que pasan por el punto de fermat concurren.

Ya que tal concurrencia es también cierta para el segundo punto de Fermat, se puede enunciar el problema así:

Proposición: F es un punto de Fermant de $\triangle ABC$ y F' el correspondiente punto Isodinámico. F_a, F_b, F_c son los simétricos de F en BC, CA, AB y $\triangle A''B''C''$ es el triángulo ceviano de F. Entonces $\triangle A''B''C''$ y $\triangle F_aF_bF_c$ son perspectivos con centro de perspectividad F'.

Como el triángulo pedal de F' respecto a $\triangle ABC$ es equilátero, entonces los puntos simetricos A', B', C' de F' respecto a BC, CA, AB forman otro triángulo equilátero homotético al triángulo pedal de F'. Las perpendiculares desde A, B, C a los lados del triángulo pedal de F' concurren en su conjugado isogonal F. Entonces $AX \perp B'C'$, $BY \perp A'C'$ y $CZ \perp A'B'$ (estando X,Y,Z en B'C',C'A', A'B') concurren en F. Note que BC, CA, AB pasan a ser mediatrices de $F'A, F'B, F'C \Longrightarrow AX, BY, CZ$ son mediatrices de $B'C', C'A', A'B'$. Consiguientemente resulta $A' \in AF, B' \in BF$ y $C' \in CF$. Las simétricas F_aF', F_bF', F_cF' de las rectas AF, BF, CF con respecto a BC, CA, AB pasan entonces por A'', B'', C'' $\Longrightarrow \triangle A''B''C''$ y $\triangle F_aF_bF_c$ son perspectivos a través de F'.

Quick Reply

High School Math

interesting geometry

Reply

ElChapin

#1 Apr 18, 2008, 12:37 am

In a triangle ABC the midpoint of BC is M. Let L and N be the intersections of AM with the incircle (with center I) of ABC.

Prove that if $CA = AB + AM$ then $\angle NIL = \frac{2\pi}{3}$

ElChapin

#2 Apr 23, 2008, 9:58 am

hint

Luis González

#3 Mar 24, 2010, 10:56 pm

Assume that $AC > AB$ and let P be the projection of I onto $\overline{AM}$. If r denotes the inradius of $\triangle ABC$, we have

$$[\triangle ABM] = \tfrac{1}{2}[\triangle ABC] = \tfrac{1}{2}IP \cdot AM + \tfrac{1}{2}r \cdot BM + \tfrac{1}{2}r \cdot AB$$

$$[\triangle ABM] = \tfrac{1}{2}(IP \cdot AM + \tfrac{1}{2}r \cdot BC + r \cdot AB)$$

$$\tfrac{1}{2}r(AB + AC + BC) = IP \cdot AM + \tfrac{1}{2}r \cdot BC + r \cdot AB$$

$$\Longrightarrow \tfrac{1}{2}r(AC - AB) = IP \cdot AM \Longrightarrow \ IP = \tfrac{1}{2}r = \tfrac{1}{2}IL$$

Therefore, in the right $\triangle IPL$, we have $\angle ILP = 30^\circ \Longrightarrow \angle NIL = 120^\circ$.

Quick Reply

High School Olympiads

IG\perp BC

Reply

Source: Own (level **)

Virgil Nicula

#1 Dec 24, 2007, 11:39 pm

Let ABC be a triangle for which $AB\ne AC$.
Denote the its centroid G and the its incircle $C(I,r)$.
Prove that $IB\cdot IC=r\cdot IA\Longleftrightarrow IG\perp BC$.

Luis González

#2 Mar 24, 2010, 1:31 am

Let M be the midpoint of BC,D the tangency point of the incircle (I) with BC and P the second intersection of AI with the circumcircle (O) of $\triangle ABC.$ It's well-known that P is the circumcenter of $\triangle IBC.$ Thus, IP,ID are isogonals with respect to $\angle BIC \Longrightarrow IB \cdot IC=2ID \cdot IP=2r \cdot IP \ (\star).$ Obviously, $IG \perp BC \Longleftrightarrow IG \parallel MP.$ Hence by Thales theorem we claim that $IG \perp BC \Longleftrightarrow \frac{AI}{IP}=\frac{AG}{GM}=2.$ Then, from the expression $(\star),$ it follows that $IG \perp BC \Longleftrightarrow$ $IB \cdot IC=r \cdot IA.$

Quick Reply

High School Olympiads

relation in triangle

Reply

livetolove212

#1 Mar 23, 2010, 3:15 pm

Given triangle ABC and its circumcircle (O). Let BE, CF be the bisectors of angle B and C. Ray EF cuts (O) at X, ray FE cuts (O) at Y. Prove that:

$$\frac{1}{BX}+\frac{1}{CY}=\frac{1}{BY}+\frac{1}{CX}+\frac{1}{AX}+\frac{1}{AY}$$

Luis González

#2 Mar 23, 2010, 9:29 pm

From http://www.artofproblemsolving.com/viewtopic.php?t=326249, we have

$$\frac{1}{BX}=\frac{1}{AX}+\frac{1}{CX}, \frac{1}{CY}=\frac{1}{AY}+\frac{1}{BY}$$

$$\Longrightarrow \frac{1}{BX}+\frac{1}{CY}=\frac{1}{BY}+\frac{1}{CX}+\frac{1}{AX}+\frac{1}{AY}.$$

Quick Reply

High School Olympiads

equality

aktyw19

#1 Mar 21, 2010, 11:39 pm

In triangle
$A+B+C=180^o$
prove that
$sin^2\frac{A}{2}+sin^2\frac{B}{2}+sin^2\frac{C}{2}-1=-2sin\frac{A}{2}*sin\frac{B}{2}*sin\frac{C}{2}$

This post has been edited 1 time. Last edited by aktyw19, Mar 22, 2010, 12:54 am

Luis González

#2 Mar 21, 2010, 11:43 pm

Let R,r,s be the circumradius, inradius and semiperimeter of $\triangle ABC.$ Then

$$4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}=4\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}=\frac{4(s-a)(s-b)(s-c)}{abc}=\frac{r}{R}$$

Using the identity $\cos A+\cos B+\cos C=1+\frac{r}{R}$ yields

$$1-2\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}=\frac{3}{2}-\frac{1}{2}(\cos A+\cos B+\cos C), \ \ (\ast)$$

Now the result of summing up the identities

$$\sin^2\frac{A}{2}=\frac{1-\cos A}{2} \ , \ \sin^2\frac{B}{2}=\frac{1-\cos B}{2} \ , \ \sin^2\frac{C}{2}=\frac{1-\cos C}{2}$$

equals the RHS of the expression $(\ast).$ This is:

$$1-2\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}=\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}.$$

High School Olympiads

Nice problem from a friend: prove G is the centroid of ABP

Reply

Source: Central American Olympiad 2007, Problem 6

Jutaro

#1 Jun 12, 2007, 9:48 am

Consider a circle S, and a point P outside it. The tangent lines from P meet S at A and B, respectively. Let M be the midpoint of AB. The perpendicular bisector of AM meets S in a point C lying inside the triangle ABP. AC intersects PM at G, and PM meets S in a point D lying outside the triangle ABP. If BD is parallel to AC, show that G is the centroid of the triangle ABP.

Arnoldo Aguilar (El Salvador)

Aldo Pacchiano

#2 Jun 13, 2007, 6:13 am • 1

Be Q the midpoint of AM, T the intersection between the perpendicular bisector of AM and AP, and R the foot of the perpendicular drawn from B to AG. $QT \parallel PM$ and Q is midpoint of $AM \rightarrow T$ is midpoint of AP. and C is midpoint of $AG \rightarrow AC = CG \rightarrow MC \parallel BG \rightarrow BG = AD$ (because $AM = MB$) $\rightarrow ADBG$ is a parallelogram and $AG = GB = BD = DA \rightarrow \angle ADP = \angle PDB = \angle BGD = \angle DGA = \alpha$ and $\angle BAG = \angle BAD = \angle ABG = \angle ABD = 90 - \alpha$ as AP and PB are tangents to $S \rightarrow \angle BAP = \angle ABP = 2\alpha$. $\angle ABR = 90 - \angle BAR = 90 - (90 - \alpha) = \alpha$ Be X the intersection point of PM and S $\angle ABX = \angle ADX = \alpha$ then R, X, and B are colineal. $AMXR$ is cyclic, so $\angle AMR = \angle AXR = \angle ADB = 2\alpha$ be $T1$ intersection between MR and $AP \rightarrow AMT1$ is isoceles $\rightarrow T1$ lies on the perpendicular bisector of $AM \rightarrow T1 = T$. Also, $\angle MRA = \angle MXA = 90 - \angle MAX = 90 - \alpha \rightarrow MR = AM$. $QCRB$ is cyclic $\rightarrow \angle BQR = \angle BCR = 2\alpha \rightarrow QR \parallel AT$ $(\angle BQR = \angle BAT = 2\alpha) \rightarrow$ as $QM = QA, MR = RT \rightarrow AT = AB$. Be T' the point where BG touches AP by a little angle chasing we get: $\angle ABT' = \angle AT'B = 90 - \alpha \rightarrow T' = T$. Then G is the intersection of PM and BT (M and T being midpoints of AB and AP respectively) so it's the gravicenter.

massnet

#3 Dec 19, 2007, 11:43 pm

> *Aldo Pacchiano wrote:*
>
> Be T' the point where BG touches AP by a little angle chasing we get: $\angle ABT' = \angle AT'B = 90 - \alpha \rightarrow T' = T$.

I can't see how this implies $T' = T$ (we still don't know $\angle ATB$, do we?)

Luis González

#4 Mar 21, 2010, 11:02 am

Let I be the second intersection of PD with S. Since $\angle PBI = \angle IDB = \angle IBM$, it follows that I is the incenter of $\triangle PAB \Longrightarrow BD$ is the external bisector of $\angle PBA \Longrightarrow$ Parallel AC to BD cuts PB at N such that $\triangle BNA$ is isosceles with apex B. Let $\angle ABD = \theta$. Since $BNCM$ and $ABIC$ are both cyclic quadrilaterals, we have $\angle ACB = \angle AIB = 2\theta$ and

$$\angle BCM = \angle BNM = \angle ACB - \angle ACM = 2\theta - (\pi - 2\theta) = 4\theta - \pi.$$

On the other hand, if O is the center of S, we obtain

$$\angle PAB = \angle AOI = \pi - 2\theta \Longrightarrow \angle APB = \pi - 2(\pi - 2\theta) = 4\theta - \pi = \angle BNM$$

Thus, $NM \parallel PA \Longrightarrow N$ is the midpoint of segment PB, which implies that $G \equiv PM \cap AC$ is the centroid of $\triangle PAB$.

Quick Reply

High School Olympiads

Easy but nice

Reply

wya
#1 Mar 20, 2010, 10:35 pm

Given an acute and scalene triangle ABC. Segments AD and BE are altitudes. Determine angle BCA if the midpoint of segment DE lies on the Euler's line of triangle ABC.

Luis González
#2 Mar 20, 2010, 11:28 pm

Let H,O be the orthocenter and circumcenter of $\triangle ABC$ and M,O' denote the midpoint of segment DE and orthocenter of $\triangle CDE$, respectively. Since $EO' \parallel HD$ and $DO' \parallel HE$, it follows that quadrilateral $EHDO'$ is a parallelogram. Keeping in mind that $CO \perp DE$, since DE is antiparallel to AB, we get that $O' \equiv HM \cap CO \equiv O \Longrightarrow \triangle CDA$ is isosceles and right at $D \Longrightarrow \angle BCA = 45^\circ$.

wya
#3 Mar 21, 2010, 1:55 am

Nice, luis.
Here is my solution:

Let H,O,N be the orthocenter, circumcenter and nine-point center of $\triangle ABC$ and denote M as the midpoint of segment DE.

Case I: $N \neq M$.
Because D,E lies on the nine-point circle, we have
Euler's line $=$ line $MN =$ the perpendicular bisector of DE
$\Rightarrow HD = HE \Rightarrow HA = HB \Rightarrow AC = BC$,
contradiction with the condition ABC is the scalene triangle.

Case II: $N = M$.
Because H,N,O lies on a line and $HN = NO$,we have
$DO \parallel HE \Rightarrow DO \perp AC \Rightarrow DO$ is the perpendicular bisector of AC
$\Rightarrow DA = DC \Rightarrow$ angle $BCA = ACD = 45^\circ$.

MJ GEO
#4 Apr 15, 2010, 12:25 am

nice :) .let M,H be the midpoint of DE and orthcenter.ABH is similiar to DHM.from this we coclude that HM is symedian of triangle ABH.if O lies on HM then $a=b$,but triangle is scalene.so O is the intersection of tangents to circle (ABH) in A,B.so $180-2C=2C$ and $C=45$. :)

Quick Reply

High School Olympiads

Construction of a trapezoid

Source: Iberoamerican Olympiad 1992, Problem 5

Jutaro

#1 May 14, 2007, 10:39 pm

Given a circle Γ and the positive numbers h and m, construct with straight edge and compass a trapezoid inscribed in Γ, such that it has altitude h and the sum of its parallel sides is m.

pontios

#2 May 15, 2007, 7:30 pm • 1

Suppose that we have constructed the isosceles trapezoid $ABCD$ with $AB \parallel CD$

Let $\theta = \angle CAB$ and $k = \dfrac{m}{2}$

Let C' be the projection of C to the line AB. Then $AC' = \dfrac{AB}{2} + \dfrac{CD}{2} = k$

From the right triangle $AC'C$ we have $\tan\theta = \dfrac{CC'}{AC'} = \dfrac{h}{k}$

Construction

Let $\Gamma = (O, R)$ be the given circle
We construct a right triangle with two perpendicular sides h and k and hypotenuse w.
Then the angle between k and w is θ
We construct an arc BC on the circle such that $\dfrac{BC}{2R} = \dfrac{h}{w}$. In other words it is $BC = 2R\cdot\sin\theta$

We construct a triangle $BC'C$, right at C', with $CC' = h$ $^{\star(1)}$

We extend BC' to meet the circle at A
Through C we bring a parallel to AB which meets the circle at D $^{\star(2)}$

Proof

If the above constructed quadrilateral $ABCD$ is convex, then it is a trapezoid with altitude h, it is inscribed in Γ and $\angle BAC$ satisfies the property $\tan(\angle BAC) = \dfrac{h}{k}$, so we have $AB + CD = 2k$

Discussion

(1)st condition $h < 2R\cdot\sin\theta \iff h < 2R\cdot\dfrac{h}{w} \iff \boxed{w < 2R}$

(2)nd condition
The quadrilateral $ABCD$ must be convex. So, the points A, D must be on the same side of BC
So, if d is the perpendicular bisector of AB then C have to be in the same half-plane with B

So $AC > BC \iff$

$\angle ABC > \angle BAC \iff$

$\sin(\angle ABC) > \sin(\angle BAC) \iff$

$\dfrac{CC'}{BC} > \sin\theta \iff$

BC

$\frac{h}{BC} > \sin\theta \iff$

$h > 2R \cdot \sin^2\theta \iff$

$h > 2R\frac{h^2}{w^2} \iff$

$\boxed{w^2 > 2Rh}$

Note
Another way to construct the point A, after finding the arc BC is to use the fact that $BA = w$

Luis González

#3 Mar 20, 2010, 6:14 am

Let K, L, M, N be the midpoints of $AB, BC, CD, DA.$ It is clear that $ABCD$ is an isosceles trapezoid $\Longrightarrow KLMN$ is a rhombus with known diagonals $KM = h$ and $LN = \frac{1}{2}(AB + CD) = \frac{1}{2}m.$ The construction of a rhombus congruent to $KLMN$ is immediate and then we obtain the measure of the diagonal AC which is twice the length of its side. Triangle $\triangle ACD$ with side length $AC = 2KL,$ altitude h onto DC and circumcircle Γ is constructible, this is: Fix the chord AC in Γ and draw the circumference with diameter $AC.$ Circumference centered at A with radius h cuts the circumference with diameter AC at the orthogonal projection H of A onto $DC,$ then ray CH cuts Γ at $D.$ The parallel line to CD passing through A cuts Γ again at $B,$ which completes the trapezoid.

Quick Reply

High School Olympiads

triangle and point

Reply

matha

#1 Mar 18, 2010, 1:34 am

let ABC be a triangle with sides a,b,c and P a point in its interior. Denote the dinstances of the point P to the vertices $A,B,C$by x,y,z and the distances from the sides BC,CA,AB by p,q,r. Prove that

$$apx^2+bqy^2+crz^2=8R^2A',$$

where R is the circumradius of ABC and A' the area of the pedal triangle of the point P.

matha

#2 Mar 18, 2010, 1:11 pm

I read somewhere that a proof can be found in an article of M. Klamkin with the title " Triangle inequalities via transforms" but i cannot find it.

Luis González

#3 Mar 18, 2010, 11:26 pm

Let X,Y,Z be the othogonal projections of P onto BC,CA,AB and let $(u:v:w)$ be the barycentric coordinates of P WRT $\triangle ABC.$ By Lagrange theorem, we have

$$u\cdot PA^2+v\cdot PB^2+w\cdot PC^2=\frac{vwa^2+uwb^2+uvc^2}{u+v+w}$$

Then, we substitute the following well known identities:

$$u=\frac{BC\cdot PX\cdot (u+v+w)}{2[\triangle ABC]}$$

$$v=\frac{CA\cdot PY\cdot (u+v+w)}{2[\triangle ABC]}$$

$$w=\frac{AB\cdot PZ\cdot (u+v+w)}{2[\triangle ABC]}$$

$$\frac{BC\cdot PX\cdot PA^2+CA\cdot PY\cdot PB^2+AB\cdot PZ\cdot PC^2}{2[\triangle ABC]}=\frac{vwa^2+uwb^2+uvc^2}{(u+v+w)^2}$$

By Euler's theorem for the pedal triangle $\triangle XYZ$ of $P,$ we obtain

$$\frac{[\triangle XYZ]}{[\triangle ABC]}=\frac{vwa^2+uwb^2+uvc^2}{4R^2\cdot (u+v+w)^2}$$

$$\Longrightarrow BC\cdot PX\cdot PA^2+CA\cdot PY\cdot PB^2+AB\cdot PZ\cdot PC^2=8R^2\cdot [\triangle XYZ].$$

Quick Reply

High School Olympiads

Product of ratios

Reply

Source: Central American Olympiad 2006, Problem 6

Jutaro

#1 Apr 30, 2007, 10:48 am

Let $ABCD$ be a convex quadrilateral. $I=AC\cap BD$, and E, H, F and G are points on AB, BC, CD and DA respectively, such that $EF\cap GH=I$. If $M=EG\cap AC$, $N=HF\cap AC$, show that

$$\frac{AM}{IM}\cdot\frac{IN}{CN}=\frac{IA}{IC}.$$

Virgil Nicula

#2 Apr 8, 2009, 8:37 am

> *Jutaro wrote:*
> Let $ABCD$ be a convex quadrilateral and $I=AC\cap BD$, $E\in (AB)$, $H\in (BC)$, $F\in (CD)$, $G\in (DA)$ such that $I\in EF\cap GH$. Denote $M\in EG\cap AC, N\in HF\cap AC$. Prove that $\frac{MA}{MI}\cdot\frac{NI}{NC}=\frac{IA}{IC}$.

Lemma 1. Let ABC be a triangle and $D\in (BC)$, $E\in (CA)$, $F\in (AB)$ and $P\in AD\cap EF$. Then $\frac{PD}{PA}\cdot BC=\frac{EC}{EA}\cdot DB+\frac{FB}{FA}\cdot DC$.

Proof.

Lemma 2. Let $ABCD$ be a convex quadrilateral and $I\in AC\cap CD$, $E\in (AB)$, $F\in (CD)$ such that $I\in EF$. Then $\frac{EB}{EA}\cdot\frac{FC}{FD}=\frac{IB}{ID}\cdot\frac{IC}{IA}$.

Proof.

Proof of the proposed problem. $\left\|\begin{array}{c} I\in HG\ \stackrel{(lemma\ 2)}{\Longrightarrow}\ \frac{HB}{HC}\cdot\frac{GA}{GD}=\frac{IB}{ID}\cdot\frac{IA}{IC}\ \Longrightarrow\ \frac{GD}{GA}\cdot IB=\frac{HB}{HC}\cdot\frac{IC}{IA}\cdot ID\\ I\in EF\ \stackrel{(lemma\ 2)}{\Longrightarrow}\ \frac{EB}{EA}\cdot\frac{FC}{FD}=\frac{IB}{ID}\cdot\frac{IC}{IA}\ \Longrightarrow\ \frac{EB}{EA}\cdot ID=\frac{FD}{FC}\cdot\frac{IC}{IA}\cdot IB\end{array}\right\|$

$\bigoplus\ \Longrightarrow$

$\frac{GD}{GA}\cdot IB+\frac{EB}{EA}\cdot ID=\frac{IC}{IA}\cdot\left(\frac{HB}{HC}\cdot ID+\frac{FD}{FC}\cdot IB\right)\stackrel{(lemma\ 1)}{\Longrightarrow}\frac{MI}{MA}\cdot BD=\frac{IC}{IA}\cdot\left(\frac{NI}{NC}\cdot BD\right)\Longrightarrow$ $\boxed{\frac{MA}{MI}\cdot\frac{NI}{NC}=\frac{IA}{IC}}$.

Virgil Nicula

#3 Apr 10, 2009, 8:54 am

Dedicate to **Armpist** ...

> *Jutaro wrote:*
> Let $ABCD$ be a convex quadrilateral and $I=AC\cap BD$, $E\in (AB)$, $H\in (BC)$, $F\in (CD)$, $G\in (DA)$

Let $ABCD$ be a convex quadrilateral and $I = AC \cap BD$, $E \in (AB)$, $H \in (BC)$, $F \in (CD)$, $G \in (DA)$

such that $I \in EF \cap GH$. Denote $M \in EG \cap AC$, $N \in HF \cap AC$. Prove that $\frac{MA}{MI} \cdot \frac{NI}{NC} = \frac{IA}{IC}$.

Lemma 1. Let ABC be a triangle and $D \in (BC)$, $E \in (CA)$, $F \in (AB)$ and $P \in AD \cap EF$. Then $\frac{PD}{PA} \cdot BC = \frac{EC}{EA} \cdot DB + \frac{FB}{FA} \cdot DC$.

Proof.

Lemma 2. Let $ABCD$ be a convex quadrilateral and $I \in AC \cap BD$, $E \in (AB)$, $F \in (CD)$ such that $I \in EF$. Then $\frac{EB}{EA} \cdot \frac{FC}{FD} = \frac{IB}{ID} \cdot \frac{IC}{IA}$.

Proof.

Proof of the proposed problem.
$$\left\|\begin{array}{ccccc} I \in HG & \overset{(lemma\ 2)}{\Longrightarrow} & \frac{HB}{HC} \cdot \frac{GA}{GD} = \frac{IB}{ID} \cdot \frac{IA}{IC} & \Longrightarrow & \frac{GD}{GA} \cdot IB = \frac{HB}{HC} \cdot \frac{IC}{IA} \cdot ID \\ I \in EF & \overset{(lemma\ 2)}{\Longrightarrow} & \frac{EB}{EA} \cdot \frac{FC}{FD} = \frac{IB}{ID} \cdot \frac{IC}{IA} & \Longrightarrow & \frac{EB}{EA} \cdot ID = \frac{FD}{FC} \cdot \frac{IC}{IA} \cdot IB \end{array}\right\|$$
$\bigoplus \implies$

$$\frac{GD}{GA} \cdot IB + \frac{EB}{EA} \cdot ID = \frac{IC}{IA} \cdot \left(\frac{HB}{HC} \cdot ID + \frac{FD}{FC} \cdot IB\right) \overset{(lemma\ 1)}{\Longrightarrow} \frac{MI}{MA} \cdot BD = \frac{IC}{IA} \cdot \left(\frac{NI}{NC} \cdot BD\right) \Longrightarrow \boxed{\frac{MA}{MI} \cdot \frac{NI}{NC} = \frac{IA}{IC}}.$$

Luis González

#4 Mar 18, 2010, 10:24 am

Notice that triangles $\triangle AEG$ and $\triangle CFH$ are perspective through I, thus by Desargues theorem, the intersections $X \equiv AD \cap BC, Y \equiv AB \cap DC$ and $P \equiv EG \cap HF$ are collinear. Let $Q \equiv AC \cap XY$. Then by Menelaus' theorem for $\triangle MQP$ and $\triangle NQP$ cut by the straight lines AB and CD, we have

$$\frac{YQ}{YP} \cdot \frac{PE}{EM} \cdot \frac{MA}{QA} = 1 \ \ , \ \ \frac{YQ}{YP} \cdot \frac{PF}{NF} \cdot \frac{NC}{QC} = 1$$

$$\Longrightarrow \frac{MA}{NC} \cdot \frac{PE}{EM} \cdot \frac{NF}{PF} = \frac{QA}{QC} \ (\star)$$

But by Menelaus' theorem for $\triangle MNP$ cut by $\overline{FIE}$, we have $\frac{PE}{EM} \cdot \frac{NF}{PF} = \frac{IN}{IM}$

Combining the latter expression with $(\star)$ yields $\frac{MA}{NC} \cdot \frac{IN}{IM} = \frac{QA}{QC}$

Since $(A, C, I, Q) = -1$, it follows that $\frac{AM}{IM} \cdot \frac{IN}{CN} = \frac{QA}{QC} = \frac{IA}{IC}$.

Quick Reply

High School Olympiads

ABC in a isosceles with AB=AC, and C a point in line BC

Reply

Source: prove that sum of two radii is a constant value

ridgers
#1 Mar 17, 2010, 4:12 pm

We analyze the triangle ABC, where $AB = AC$. In the line BC we take the changeable point M(i.e. M is not fixed) such that B is between M and C. Prove that the sum of the radius of the inscribed circle of triangle AMB and the radius of the escribed circle of triangle AMC (AC , the extension of MA and the extension of MC are tangent with this circle), is a constant value.

Luis González
#2 Mar 17, 2010, 10:04 pm • 1

Let the incircle (I_1, r_1) of $\triangle ABM$ and the M-excircle (I_2, r_2) of $\triangle ACM$ touch BC at X, Y, respectively. Let φ denote the base angle of the isosceles $\triangle ABC$.

$$r_1 = BX \cdot \cot \frac{\varphi}{2} \ , \ r_2 = CY \cdot \cot \frac{\varphi}{2} \Longrightarrow \ r_1 + r_2 = \cot \frac{\varphi}{2}(BX + CY)$$

Since $BX = \frac{1}{2}(AB + BM - AM)$, $CY = \frac{1}{2}(AM + AC - CM)$

$$\Longrightarrow r_1 + r_2 = \frac{1}{2} \cot \frac{\varphi}{2}(2AB - BC) = \text{dist}(A, BC).$$

inio
#3 Mar 18, 2010, 2:05 pm

easily we have the triangle BPN is similar with CQA

So $\frac{R_{(Q)}}{R_{(P)}} = \frac{AC}{BN}$

and $\frac{R_{(P)}}{R_{(Q)} + R_{(P)}} = \frac{BN}{BN + AB}\ ^{(1)}$

We have that $R_{(P)} = \frac{2AB.BN.cos^2\frac{\widehat{ABM}}{2}}{AB + BN}\ ^{(2)}$

from $_{(1)}$ and $_{(2)}$ we get

$$R_{(Q)} + R_{(P)} = 2AB.cos^2\frac{\widehat{ABM}}{2}$$

Quick Reply

High School Olympiads

4 concylic points

Reply

Source: Swiss Math Olympiad 2010 - final round, problem 2

Martin N.

#1 Mar 17, 2010, 2:06 am

Let $\triangle ABC$ be a triangle with $AB \neq AC$. The incircle with centre I touches BC, CA, AB at D, E, F, respectively. Furthermore let M the midpoint of EF and AD intersect the incircle at $P \neq D$.
Show that $PMID$ ist cyclic.

Luis González

#2 Mar 17, 2010, 6:41 am

In the right triangle $\triangle AEI, \overline{AM}$ is the orthogonal projection of the leg $\overline{AE}$ onto its hypotenuse $\overline{AI} \Longrightarrow AE^2 = AM \cdot AI,$ but from the power of A with respect to the incircle (I), we have $AE^2 = AP \cdot AD \Longrightarrow AM \cdot AI = AP \cdot AD \Longrightarrow$ Four points P, M, I, D are concyclic.

livetolovemath030894

#3 Mar 17, 2010, 7:25 pm

you can see here http://www.mathlinks.ro/Forum/viewtopic.php?t=149394

Mathias_DK

#4 Mar 21, 2010, 5:38 am

> *Martin N. wrote:*
> Let $\triangle ABC$ be a triangle with $AB \neq AC$. The incircle with centre I touches BC, CA, AB at D, E, F, respectively. Furthermore let M the midpoint of EF and AD intersect the incircle at $P \neq D$.
> Show that $PMID$ ist cyclic.

An inversion in the incircle will transform M into A and fix both P and D. Since A, P, D are colinear the conclusion follows.

Quick Reply

High School Olympiads

Three circles have a common point

Reply

Source: Swiss Math Olympiad 2010 - final round, problem 9

Martin N.

#1 Mar 17, 2010, 2:56 am

Let k and k' two concentric circles centered at O, with k' being larger than k. A line through O intersects k at A and k' at B such that O seperates A and B. Another line through O intersects k at E and k' at F such that E separates O and F.
Show that the circumcircle of $\triangle OAE$ and the circles with diametres AB and EF have a common point.

Luis González

#2 Mar 17, 2010, 6:12 am

Moreover the circumcircle of $\triangle OBF$ also passes through such a common point. Consider the inversion through pole O and power $\overline{OE}\cdot\overline{OF}$. Circle with diameter EF is double and the opposite rays of OB, OA cut k' and $k,$ respectively at the inverse images A', B' of A, B. Then $\odot(OBF) \mapsto EB', \odot(OEA) \mapsto FA'$ and circle with diameter AB is transformed into the circle with diameter $A'B'$. Let $P \equiv FA' \cap EB'$. Since $EB' \perp FA' \Longrightarrow P$ is common point of FA', EB' and the circles with diameters $A'B'$ and EF. Hence, $\odot(OEA), \odot(OBF)$ and circles with diameters AB, EF concur at the inverse P' of P.

Martin N.

#3 Mar 18, 2010, 3:19 am • 1

Ma solution is the following:
Denote k_1, k_2 the circles with diametres AB, DE, respectively. Moreover, $k_1 \cap k_2 = \{M, N\}$, where M is in the inner of angle $\angle AOE$, and $N_1 = AE \cap k_1$ and $N_2 = AE \cap k_2$.
As $\angle AN_1B = \angle FN_2E = 90^\circ$ due to Thales' theorem, and $\triangle OBF$ is isosceles with $\overline{OB} = \overline{OF}$ or $\angle FBO = \angle OFB$, we must have $N = N_1 = N_2$.
Thus, we have

$$\angle FEM = \angle FNM = 90^\circ - \angle MNA = 90^\circ - \angle MBA = \angle BAM = \angle OAM,$$

implying that $AOEM$ is cyclic. :)

sankha012

#4 Feb 5, 2011, 4:45 pm

I have made it too much complicated :P
Let m denote the circle with radius AB and n denote the circle with radius EF.Draw tangents to m at A and B.
The tangent at E meets the tangent at A at M and The tangent at F meets the tangent at B at N.The points M and N have equal powers w.r.t m and n.So MN is the radical axis of m and n.Our required point must lie on MN.Observe that M lies on the circumcircle of $\triangle OAE$.But this is not the required point,since its power w.r.t n is $MF^2 > 0$.Let P be the projection of O on MN.This P is the second intersection of the circle OAE with MN.We claim this one is the required point.
$\angle OPE = \angle EAO$ since A, O, E, P, M are concyclic.and since $OA = OE$ we have $\angle OEA = \angle OAE = \angle OPE$.
Also,$\angle FPN = \angle FON = \angle NOB$ since $PONF$ is cyclic and $FN = NB$.Thus $\angle EOB = 2\angle OPE = 2\angle FPN$.So $\angle FPE = \angle EPN + \angle FPN = \angle EPN + \angle OPE = \frac{\pi}{2}$.So $P \in n$ and since MN is the radical axis,P is also on m.
QED

Quick Reply

Spain

Reto de la Semana #6

Reply

Pascual2005
#1 Oct 8, 2006, 10:21 pm

Poco tarde pero finalmente:

1. Hallar todas las funciones de los naturales en los naturales tales que $f(m^2+n^2)=f^2(m)+f^2(n)$ para todo (m,n) naturales.

(los naturales son $0,1,2...$)

2. Sea S un conjunto de $n\geq 3$ puntos en el plano. Pruebe que existe un conjunto T de $2n-5$ puntos tales que cualquier triangulo determinado por vértices en S contiene un punto de T.

3. Sea ABC un triángulo acutángulo. Sea M el punto medio del lado AB y D pie de la altura desde C. Una línea por M intersecata a las líneas CA, CB en K, L respectivamente con $CK=CL$. Sea O el cincuncentro del triángulo CKL. Demostrar que $OD=OM$.

4. Sean $p>5$ un primo y n un entero par. Demostrar que

Si $L_n\equiv 2\pmod p$, entonces $L_n\equiv 2\pmod{p^2}$;

Si $L_n\equiv -2\pmod p$, entonces $L_n\equiv -2\pmod{p^2}$.

tipe
#2 Oct 9, 2006, 3:01 am

Solo una aclaración:
$L_1=1, L_2=3, L_{n+1}=L_n+L_{n-1} (n\geq 2)$
es la sucesión de Lucas.

$Tipe$

Pascual2005
#3 Oct 16, 2006, 4:18 am

quedan pocos dias para enviar sus soluciones!!! donde estan todos???

campos
#4 Oct 16, 2006, 8:06 am

jeje, hoy no es el ultimo dia?

Pascual2005
#5 Oct 20, 2006, 9:49 am

se amplia el plazo a petición de varios participantes...

R.G.A.M.

#6 Oct 21, 2006, 8:58 am

Gracias, en serio. Sucede que, en mi colegio, estamos en plena etapa de exámenes y recién el 20 de noviembre finaliza. Ya saben que cuando los estudios priman no hay mucho tiempo para resolver problemas matemáticos…

R.G.A.M.

#7 Nov 8, 2006, 7:15 pm

¿Significa $f^2(x)$ lo mismo que $(f(x))^2$ o que $f(f(x))$?

Por el momento, estuve utilizando la primera acepción, pero existen materiales que utilizan la otra.

aev5peru

#8 Nov 9, 2006, 8:51 am

> R.G.A.M wrote:
> ¿Significa $f^2(x)$ lo mismo que $(f(x))^2$ o que $f(f(x))$?
>
> Por el momento, estuve utilizando la primera acepción, pero existen materiales que utilizan la otra.

bueno en algunos libros, consideran como aplicacion de la funcion n veces, pero entonces indican, en el problema supongo que consideran la pimera acepcion.
chau,
PD: cuando vence el plazo??

Luis González

#9 Mar 16, 2010, 9:16 pm

> Pascual2005 wrote:
> 3. Sea ABC un triángulo acutángulo. Sea M el punto medio del lado AB y D pie de la altura desde C. Una línea por M intersecta a las líneas CA, CB en K, L respectivamente con $CK = CL$. Sea O el circuncentro del triángulo CKL. Demostrar que $OD = OM$.

Como $MA = MB$ y $CK = CL$, al aplicar el teorema de Menelao en el $\triangle ABC$ cortado por la transversal KML se tendrá $AK = BL$. Así $E \equiv \odot(CKL) \cap \odot(ABC) \not\equiv C$ es el centro de la rotación que lleva AK en BL y como $EA = EB$, entonces E es el punto medio del arco AB de $\odot(ABC)$. Desde luego, E yace en la bisectriz interior de $\angle ACB \Longrightarrow O \in CE$.

Siendo CE el circundiámetro de $\triangle CKL$ perpendicular a KL, se sigue que O es el punto medio de CE y si A', B' son los puntos medios de los lados CB, CA, entonces O es el corte de CE con la mediatriz de $A'B'$ puesto que la homotecia positiva con centro C y coeficiente $\frac{1}{2}$ transforma $\overline{AB}$ en $\overline{A'B'}$, E en O y la mediatriz EM de AB en la mediatriz de $A'B'$.

Por último resta observar que el cuadrilátero $A'B'DM$ es un trapecio isósceles en el cual O yace en su eje de simetría, ya que en efecto $\triangle MA'B'$ y $\triangle DB'A'$ son simétricos con respecto a la mediatriz de $A'B'$, por tanto $OD = OM$.

Quick Reply

High School Olympiads