The Theorem on the Six Pedals / Darij Grinberg

In the following, I will use the abbreviation "*pedal*" of a point P on a line l for the foot of the perpendicular from P to l.

In studying the two Brocard triangles, I have currently (September 2003) found the following result which is helpful in finding new triangle centers and relationships between known centers (cf. Hyacinthos messages #8021, #8030, #8034 etc.).

Theorem on the Six Pedals.

Let ABC be a triangle and P and Q two points. We construct the perpendiculars from Q to the lines BC, CA, AB. Let X, Y, Z be the pedals of the point P on these perpendiculars. On the other hand, let X', Y', Z' be the pedals of the point Q on the lines AP, BP, CP. Then, the lines XX', YY', ZZ' concur.

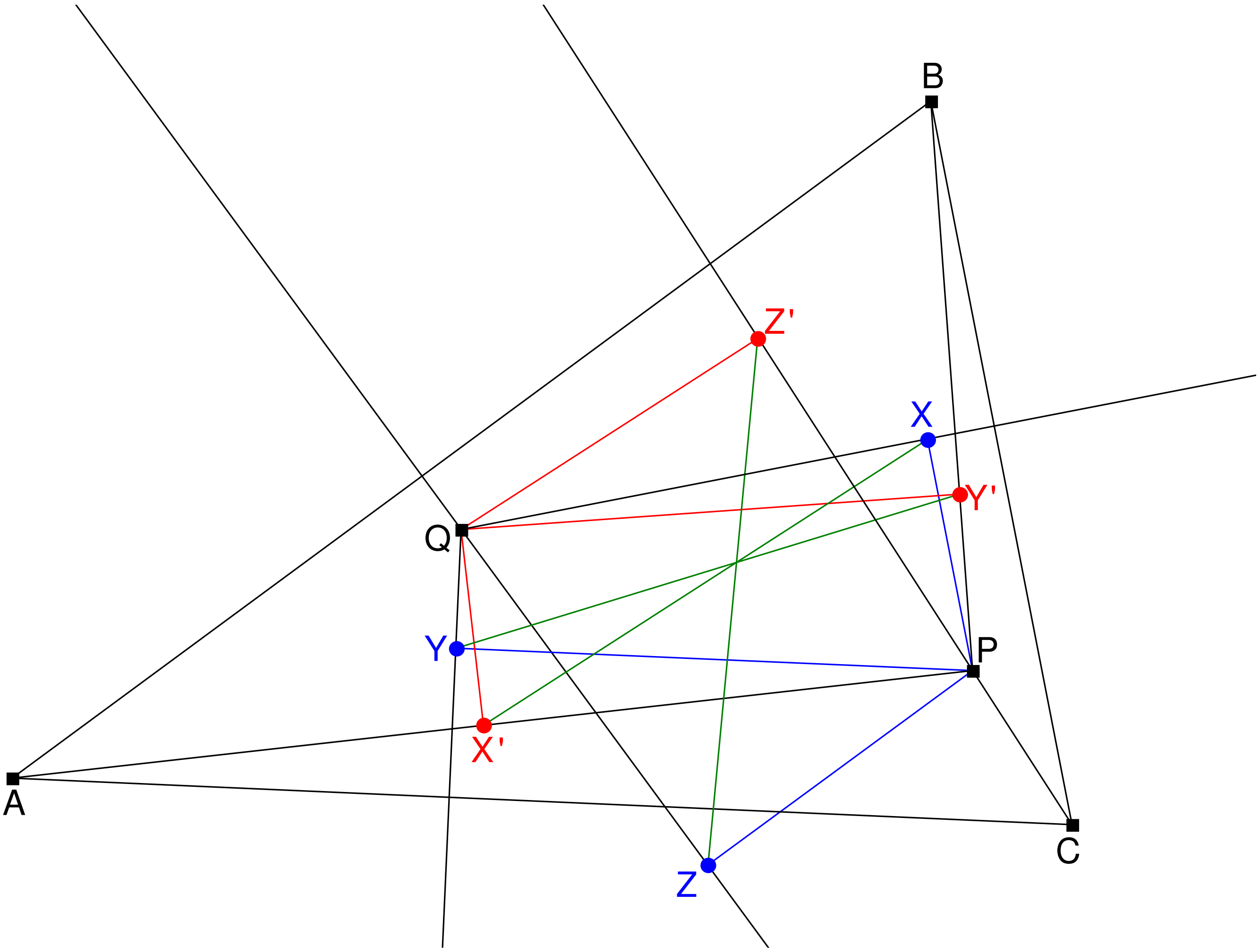

Fig. 1

Proof. The Sine Law in triangles BPC, CPA and APB yields

$$\frac{\sin\measuredangle BCP}{\sin\measuredangle PBC} = \frac{BP}{CP}; \qquad \frac{\sin\measuredangle CAP}{\sin\measuredangle PCA} = \frac{CP}{AP}; \qquad \frac{\sin\measuredangle ABP}{\sin\measuredangle PAB} = \frac{AP}{BP}.$$

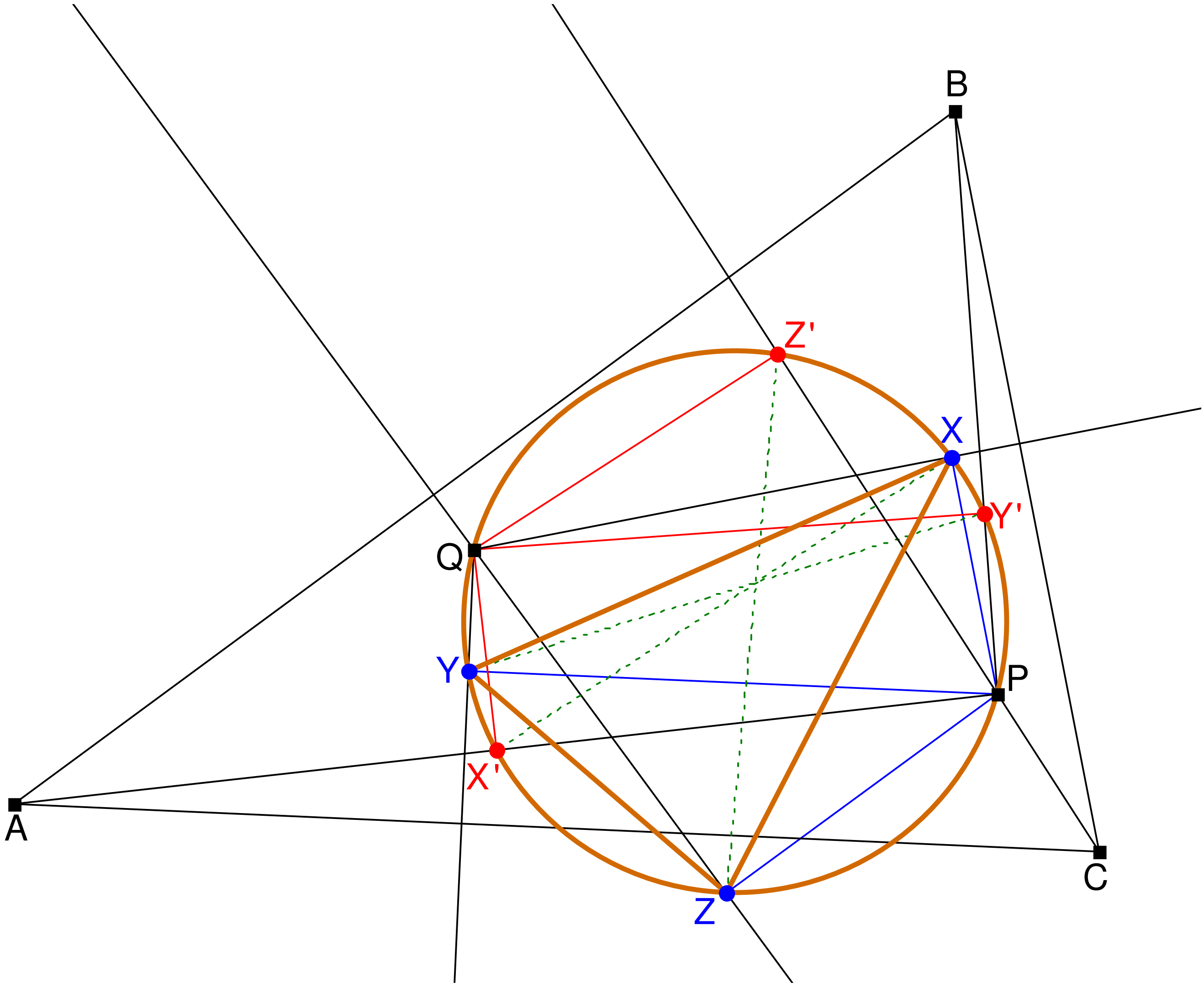

Fig. 2

(See Fig. 2.) Because of $\measuredangle PXQ = 90°$, $\measuredangle PYQ = 90°$, $\measuredangle PZQ = 90°$, $\measuredangle PX'Q = 90°$, $\measuredangle PY'Q = 90°$ and $\measuredangle PZ'Q = 90°$, the points X, Y, Z, X', Y' and Z' lie on the circle with diameter PQ. In other words, the eight points P, Q, X, Y, Z, X', Y' and Z' are concyclic. Thus, $\measuredangle XZZ' = \measuredangle XPZ'$. But $PX \perp QX$ and $QX \perp BC$ yield $PX \parallel BC$, and hence $\measuredangle XPZ' = \measuredangle BCP$. Consequently, $\measuredangle XZZ' = \measuredangle BCP$ and $\sin\measuredangle XZZ' = \sin\measuredangle BCP$. (Note that some other arrangement cases may also occur where, e. g., we have not $\measuredangle XZZ' = \measuredangle BCP$ any more, but $\measuredangle XZZ' = 180° - \measuredangle BCP$; but then, the equation $\sin\measuredangle XZZ' = \sin\measuredangle BCP$ still holds.) By the same reasoning, $\sin\measuredangle Z'ZY = \sin\measuredangle PCA$. Herewith,

$$\frac{\sin\measuredangle XZZ'}{\sin\measuredangle Z'ZY} = \frac{\sin\measuredangle BCP}{\sin\measuredangle PCA}.$$

Analogously,

$$\frac{\sin\measuredangle ZYY'}{\sin\measuredangle Y'YX} = \frac{\sin\measuredangle ABP}{\sin\measuredangle PBC} \qquad \text{and} \qquad \frac{\sin\measuredangle YXX'}{\sin\measuredangle X'XZ} = \frac{\sin\measuredangle CAP}{\sin\measuredangle PAB},$$

and thus

$$\begin{aligned}\frac{\sin\measuredangle XZZ'}{\sin\measuredangle Z'ZY} \bullet \frac{\sin\measuredangle ZYY'}{\sin\measuredangle Y'YX} \bullet \frac{\sin\measuredangle YXX'}{\sin\measuredangle X'XZ} &= \frac{\sin\measuredangle BCP}{\sin\measuredangle PCA} \bullet \frac{\sin\measuredangle ABP}{\sin\measuredangle PBC} \bullet \frac{\sin\measuredangle CAP}{\sin\measuredangle PAB} \\ &= \frac{\sin\measuredangle BCP}{\sin\measuredangle PBC} \bullet \frac{\sin\measuredangle CAP}{\sin\measuredangle PCA} \bullet \frac{\sin\measuredangle ABP}{\sin\measuredangle PAB} \\ &= \frac{BP}{CP} \bullet \frac{CP}{AP} \bullet \frac{AP}{BP} = 1.\end{aligned}$$

After the Ceva theorem in the trigonometric form (applied to triangle XYZ), the lines XX', YY', ZZ'

concur. This proves the Theorem on the Six Pedals.

A new proof of the Ceva Theorem / Darij Grinberg

A well-known theorem that can be shown in several different ways is the Ceva Theorem (we treat it here without the converse):

Ceva Theorem. Let ABC be an arbitrary triangle. Further, let A', B', C' be points on its sides BC, CA, AB, for which the lines AA', BB', CC' concur. Then (with directed segments)

$$\frac{AC'}{C'B} \bullet \frac{BA'}{A'C} \bullet \frac{CB'}{B'A} = 1.$$

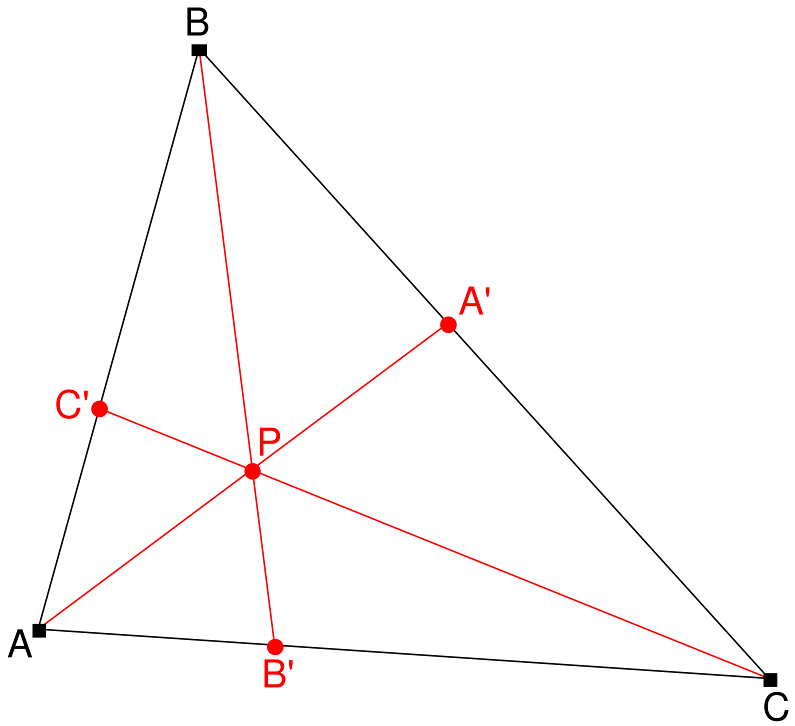

Fig. 1

Here I present a probably new proof of this result. Denote by P the intersection of the lines AA', BB', CC'. The parallel to BC through P meets CA at B_a and AB at C_a. The parallel to CA through P meets AB at C_b and BC at A_b. The parallel to AB through P meets BC at A_c and CA at B_c.

As segments on parallels,

$$\frac{AC'}{C'B} = \frac{B_cP}{PA_c}.$$

On the other hand,

$$\frac{B_cP}{AB} = \frac{PB'}{BB'} \qquad \text{and} \qquad \frac{PA_c}{AB} = \frac{PA'}{AA'},$$

hence

$$\frac{B_cP}{AB} : \frac{PA_c}{AB} = \frac{PB'}{BB'} : \frac{PA'}{AA'}, \qquad \text{i. e.} \qquad \frac{B_cP}{PA_c} = \frac{PB'}{BB'} : \frac{PA'}{AA'}.$$

Consequently,

$$\frac{AC'}{C'B} = \frac{PB'}{BB'} : \frac{PA'}{AA'}.$$

Similarly,

$$\frac{BA'}{A'C} = \frac{PC'}{CC'} : \frac{PB'}{BB'} \qquad \text{and} \qquad \frac{CB'}{B'A} = \frac{PA'}{AA'} : \frac{PC'}{CC'}.$$

Now

$$\frac{AC'}{C'B} \bullet \frac{BA'}{A'C} \bullet \frac{CB'}{B'A} = \left(\frac{PB'}{BB'} : \frac{PA'}{AA'}\right) \bullet \left(\frac{PC'}{CC'} : \frac{PB'}{BB'}\right) \bullet \left(\frac{PA'}{AA'} : \frac{PC'}{CC'}\right) = 1,$$

what proves the Ceva Theorem.

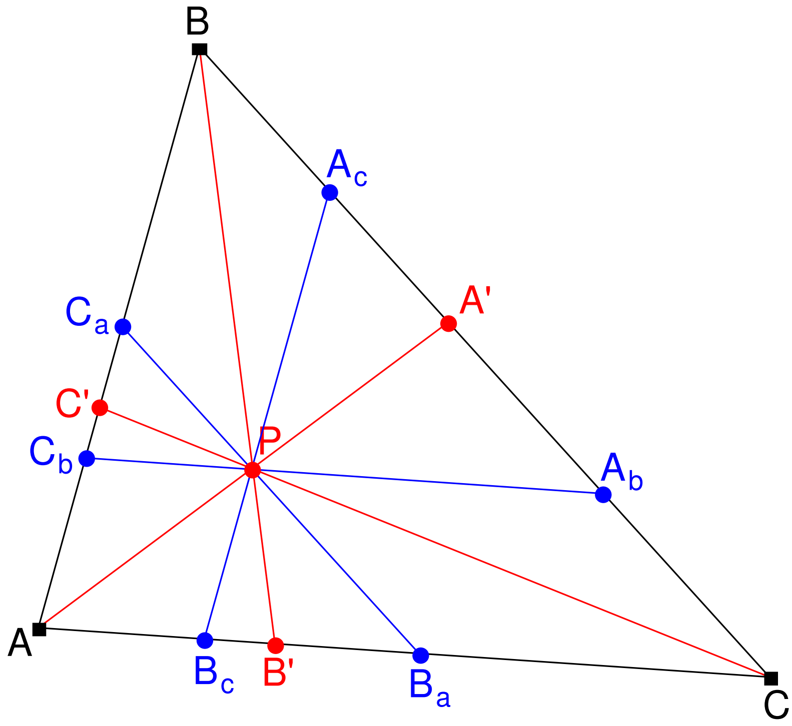

Fig. 2

The Mitten point as radical center / Darij Grinberg

Abstract

The Mitten point of a triangle, defined as the perspector of the medial and excentral triangles, is shown to be the radical center of a variable circle triad.

1. Introduction

Let ΔABC be a triangle, M_a, M_b and M_c the midpoints of its sides BC, CA, AB, respectively, and I_a, I_b, I_c the excenters opposite to the vertices A, B, C.

The lines I_aM_a, I_bM_b and I_cM_c concur at one point M, which is called **Mittenpunkt** or **middlespoint** of triangle ABC. For reasons of homogenity (compared with the Gergonne point, Nagel point, median point etc.), we shall call it **Mitten point** throughout this note. In Clark Kimberling's list of triangle centers [2], the Mitten point is the center X_9.

The usual proof of the concurrence of the lines I_aM_a, I_bM_b and I_cM_c is by identifying these lines as the symmedians of triangle $I_aI_bI_c$. In this note, we shall give another proof and obtain the Mitten point as the radical center of a family of circle triads.

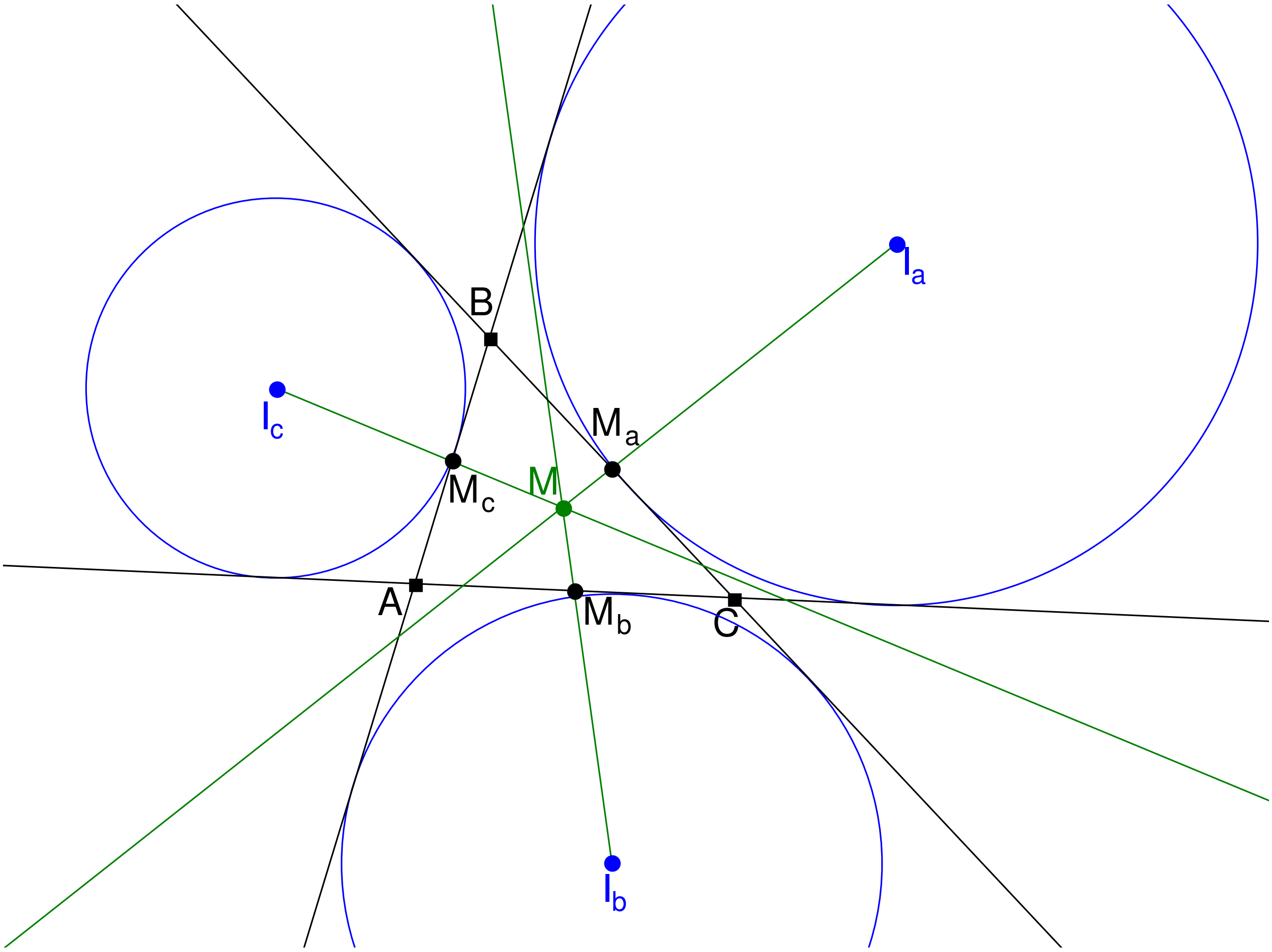

Fig. 1

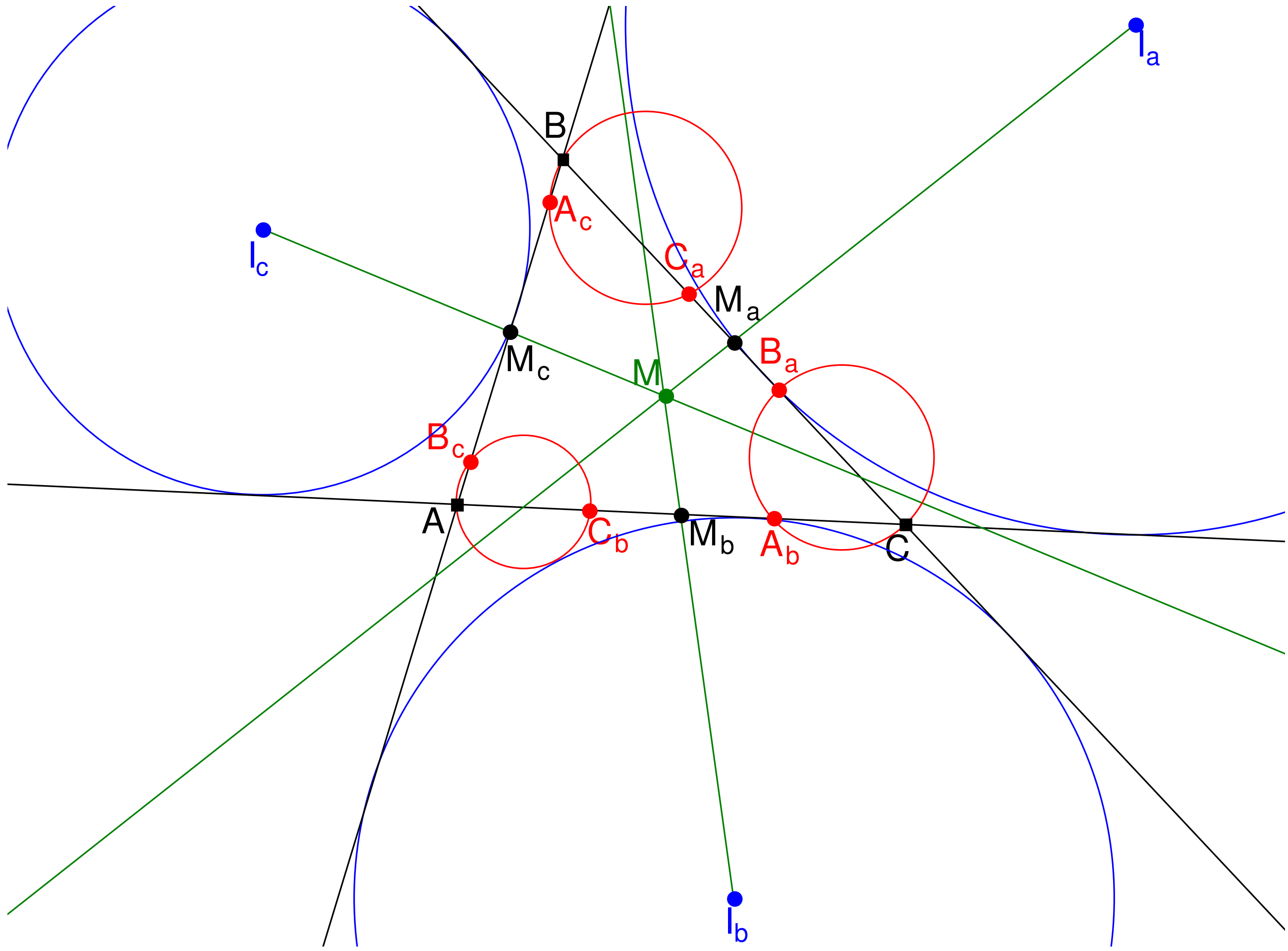

Fig. 2

2. The circle triads

We direct the sides BC, CA and AB of triangle ABC in such way that the segments BC, CA and AB are positive (and the segments CB, AC and BA are negative). Further let A_b, A_c, B_c, B_a, C_a, C_b be points on the lines CA, AB, AB, BC, BC, CA, respectively, fulfilling

$$BB_a = C_aC = CC_b = A_bA = AA_c = B_cB = d$$

for some real d.

Then, we are going to prove:

Theorem 1. The pairwise radical axes of the circles AB_cC_b, BC_aA_c and CA_bB_a are the lines I_aM_a, I_bM_b and I_cM_c.

Since the pairwise radical axes of three circles concur at the radical center, this yields:

Theorem 2. The lines I_aM_a, I_bM_b and I_cM_c concur at one point. This point is the Mitten point of triangle ABC and is the radical center of the circles AB_cC_b, BC_aA_c and CA_bB_a.

With this result, we have obtained a new proof of the existence of the Mitten point.

3. Proof of Theorem 1

Let's concentrate on the proof of Theorem 1 (Fig. 3).

Since the excenters I_b and I_c lie on the external angle bisector of the angle CAB, the line I_bI_c passes through A. Let X be the intersection of this line with the circle ABC, different from A. [By the way, X is the midpoint of I_bI_c; however, we won't need this property in the further proof.] We will show:

Lemma 3. This point X lies on the circle AB_cC_b.

Proof (Fig. 3). Since the points I_b and I_c lie on the external bisector of the angle CAB, we have $\measuredangle I_cAB = \measuredangle I_bAC$, i. e. $\pi - \measuredangle XAB = \measuredangle XAC$. This indicates that the chordal angles of the chords XB and XC in the circle ABC are equal (in fact, $\pi - \measuredangle XAB = \measuredangle XCB$ is the chordal angle of XB, and $\measuredangle XAC$ is the chordal angle of XC). Thus, the two chords are equal themselves: $XB = XC$. On the other hand, $B_cB = CC_b = d$. Furtheremore, $\measuredangle B_cBX = \measuredangle C_bCX$ (since $\measuredangle ABX = \measuredangle ACX$ as chordal angles). Consequently, triangles B_cBX and C_bCX are congruent, and we get $\measuredangle XB_cB = \measuredangle XC_bC$. Because of $\measuredangle AB_cX = \pi - \measuredangle XB_cB$ and $\measuredangle AC_bX = \pi - \measuredangle XC_bC$ this gives $\measuredangle AB_cX = \measuredangle AC_bX$; thus, the point X lies on the circle AB_cC_b. Herewith, Lemma 3 is proven.

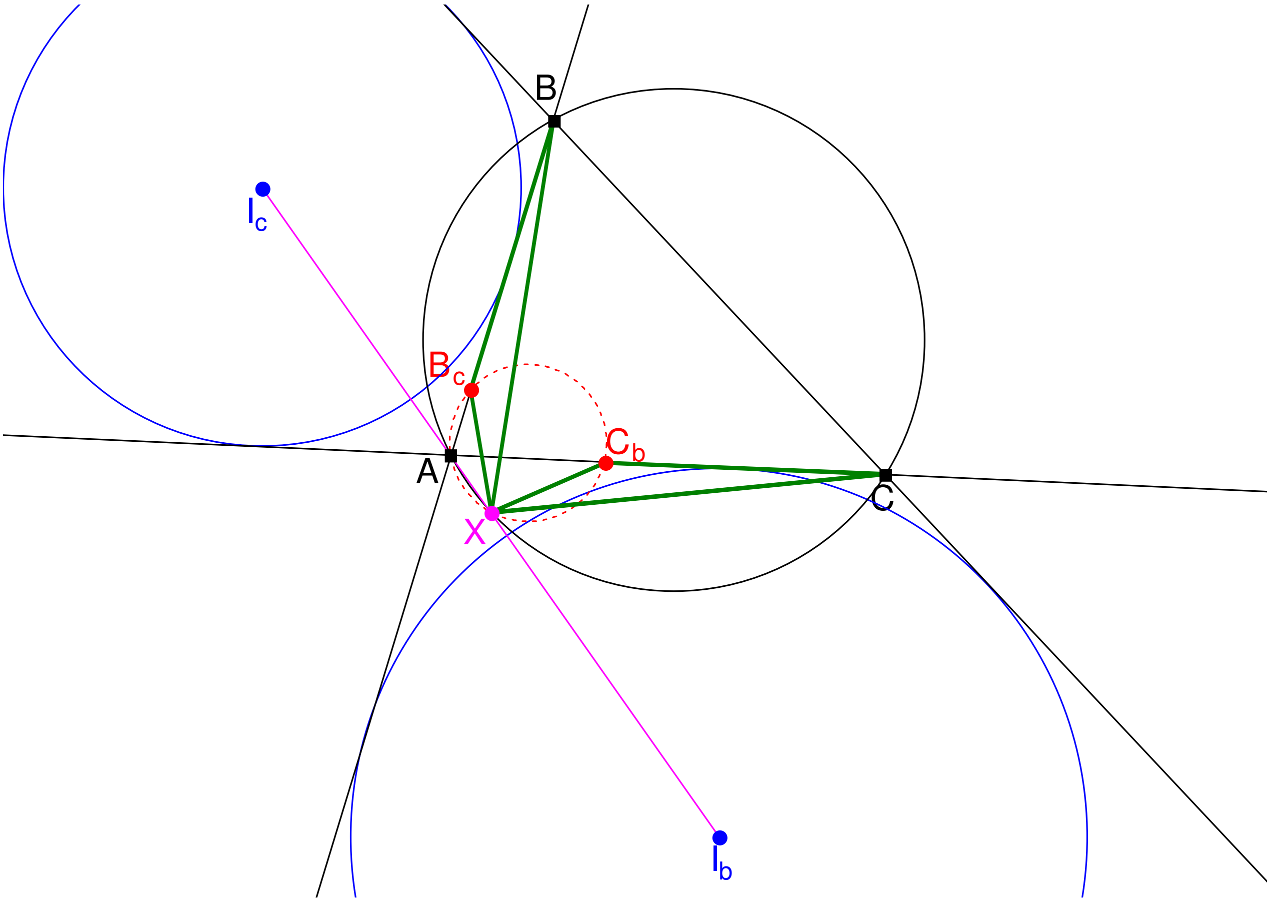

Fig. 3

Hence, the common points of circles ABC and AB_cC_b are A and X. The radical axis of the circles ABC and AB_cC_b turns out to be the line AX, i. e. the line I_bI_c. [If the points A and X coincide, the circles ABC and AB_cC_b touch each other.

In fact, in that case, the line I_bI_c, i. e. the external bisector of the angle CAB, must be a tangent to the circle ABC. Hence, this line makes angles B and C with the sides AC and AB, respectively; but since it is the external bisector, these angles must be equal. Therefore, the angles B and C are equal, and triangle ABC is isosceles. The tangency of the circles ABC and AB_cC_b now follows by symmetry.]

Analogously, the radical axis of the circles ABC and BC_aA_c is the line I_cI_a.

The two radical axes intersect at the point I_c, which therefore must be the radical center of the three circles ABC, AB_cC_b and BC_aA_c. Hence, I_c also lies on the radical axis of the circles AB_cC_b and BC_aA_c.

Finally consider the midpoint M_c of AB. The power of M_c with respect to the circle AB_cC_b is $M_cA \bullet M_cB_c$; the power of M_c with respect to the circle BC_aA_c is $M_cB \bullet M_cA_c$. But since M_c is the midpoint of AB, we have $M_cB = -M_cA$ (directed edges!), and from $AA_c = B_cB = d$ it follows that

$M_cB_c = M_cB - B_cB = -M_cA - AA_c = -M_cA_c$. Therefore,

$$M_cA \bullet M_cB_c = (-M_cB) \bullet (-M_cA_c) = M_cB \bullet M_cA_c.$$

Thus, the powers of M_c with respect to the circles AB_cC_b and BC_aA_c are equal. The point M_c must lie on the radical axis of the two circles. But we also know that I_c lies on this radical axis. Hence, the radical axis of the circles AB_cC_b and BC_aA_c is the line I_cM_c.

Analogously, the radical axis of the circles BC_aA_c and CA_bB_a is the line I_aM_a, and the radical axis of the circles CA_bB_a and AB_cC_b is the line I_bM_b.

Theorem 1 is proven.

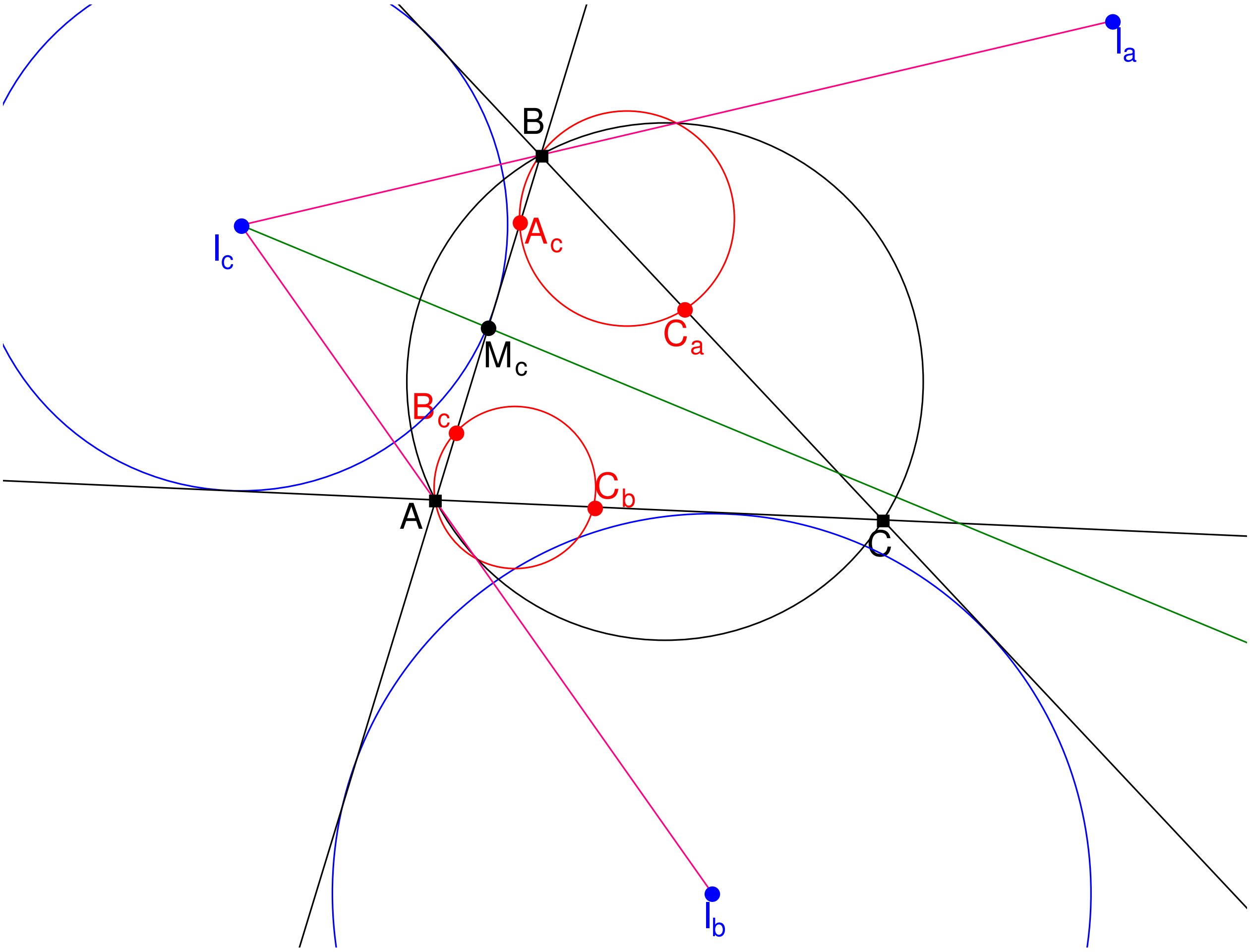

Fig. 4

Theorem 1 was given by Paul Yiu in [3] with a redundant condition; an analytic proof by means of barycentric coordinates was done by Michel Garitte [1].

References

[1] M. Garitte, *Hyacinthos message* #6588.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`

[3] P. Yiu, *Hyacinthos message* #2346.

On the Lemoine circumcevian triangle / Darij Grinberg

Let L be the symmedian point of an arbitrary triangle ΔABC. The circumcircle of ΔABC intersects AL at X, BL at Y and CL at Z. Then, the triangle XYZ is the circumcevian triangle of the point L; we call it the **Lemoine circumcevian triangle**. Obviously, we have:

Theorem 1. The triangle ABC and the Lemoine circumcevian triangle XYZ have the same circumcenter.

We intend to prove another theorem ([1]):

Theorem 2. The triangle ABC and the Lemoine circumcevian triangle XYZ have the same symmedian point, i. e. the point L is also the symmedian point of ΔXYZ.

First, we note:

Theorem 3. The triangles ΔALC and ΔZLX are similar.

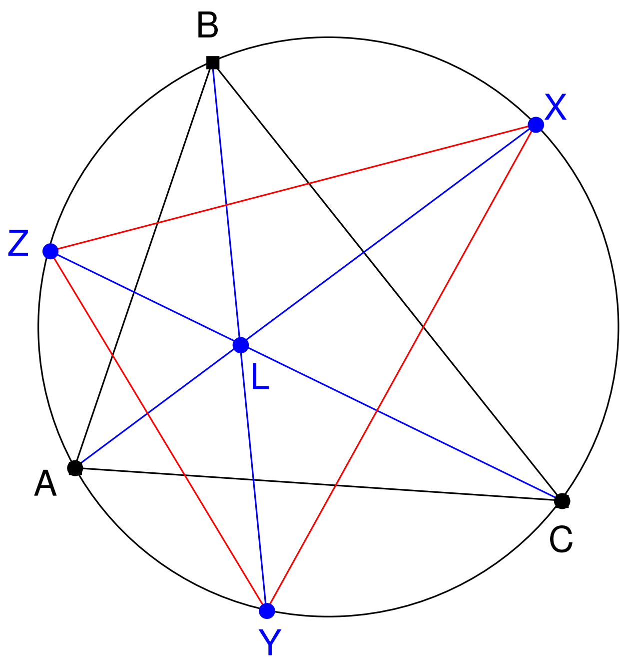

Fig. 1

In fact,

$$\measuredangle ALC = \measuredangle ZLX \qquad \text{and}$$
$$\measuredangle LCA = \measuredangle ZCA = \measuredangle ZXA \qquad \text{(cyclic)}$$
$$= \measuredangle LXZ,$$

what gives $\Delta ALC \sim \Delta ZLX$.

From this similarity, we obtain that the altitudes of triangles ΔALC and ΔZLX are proportional to the corresponding sides. Hence, if we denote by $d(P;\ g)$ the distance of an arbitrary point P from a line g, then we have

$$\frac{d(L;\ ZX)}{d(L;\ CA)} = \frac{ZX}{CA},$$

i. e.

$$d(L;\ ZX) = ZX \bullet \frac{d(L;\ CA)}{b}.$$

But we know that the symmedian point L has homogeneous trilinear coordinates $L(a : b : c)$ with respect to the original triangle, i. e. there exists a real k for which

$$d(L;\ BC) = ka; \qquad d(L;\ CA) = kb; \qquad d(L;\ AB) = kc.$$

Thus,

$$d(L;\ ZX) = ZX \bullet \frac{d(L;\ CA)}{b} = ZX \bullet k.$$

Analogously, $d(L;\ XY) = XY \bullet k$ and $d(L;\ YZ) = YZ \bullet k$. Thus, the point L has homogeneous trilinear coordinates $L(YZ : ZX : XY)$ with respect to ΔXYZ. Consequently, L is the symmedian point of ΔXYZ, what completes the proof.

Referring to this property, the triangle XYZ is called **cosymmedian triangle** of ΔABC.

As a corollary, we get:

Theorem 4. The triangle ABC and the Lemoine circumcevian triangle XYZ have a common Brocard axis.

Indeed, the two triangles have a common circumcenter and a common symmedian point, and therefore they have a common Brocard axis (since the Brocard axis joins the circumcenter with the symmedian point).

Now we are going to show another property:

Theorem 5. Let

$$1 = YZ \cap CA; \qquad 2 = YZ \cap AB;$$
$$3 = ZX \cap AB; \qquad 4 = ZX \cap BC;$$
$$5 = XY \cap BC; \qquad 6 = XY \cap CA.$$

Then, the lines 14, 25 and 36 pass through the point L (Fig. 2).

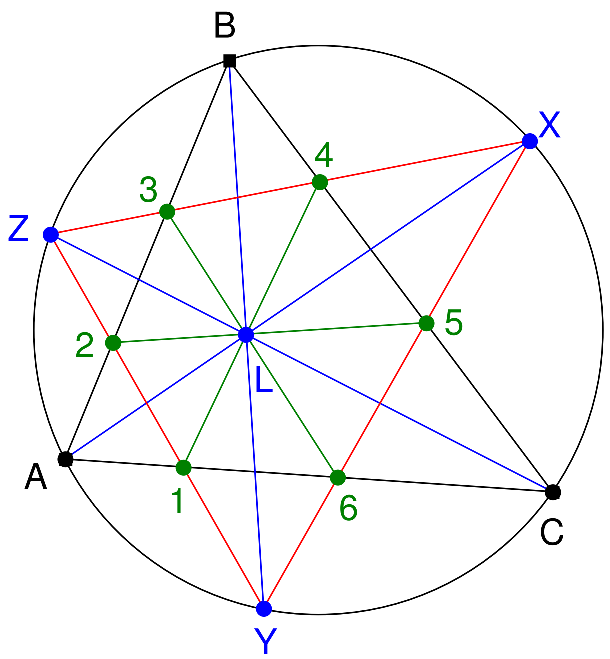

Fig. 2

After a bit of thinking, this result turns out to be quite simple and independent from the presumption that L is the symmedian point. In fact, L can be an arbitrary point. The proof (Fig. 3) uses the Pascal Theorem, applied to the inscribed hexagon $ABCZYX$, yielding that the intersections of opposite sides, i. e. the points

$$AB \cap ZY = 2; \qquad BC \cap YX = 5; \qquad CZ \cap XA = L$$

are collinear. Hence, L lies on 25; analogously, L lies on 14 and on 36.

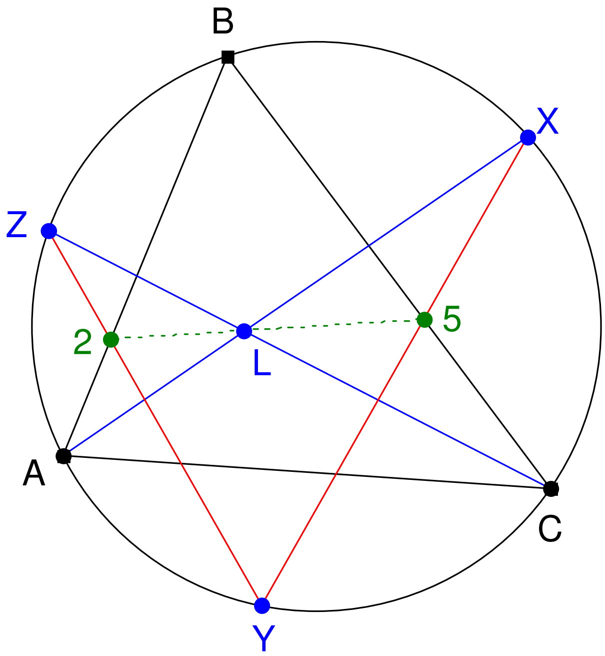

Fig. 3

References

[1] Ross Honsberger: *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, USA 1995.