High School Olympiads

Lithuania TS 2009

cashaz

#1 Dec 29, 2009, 3:39 pm

In triangle ABC, angle A=30, points S and I are ABC circumcentre and incentre respectively. Points D and E are on sides BA and CA respectively such that BD=CE=BC. Prove that SI is perpendicular and equal to DE.

cashaz

#2 Jan 2, 2010, 12:27 am

so if there is anyone who can help me and post the solution?

Virgil Nicula

#3 Jan 2, 2010, 1:53 am

> *cashaz wrote:*
> Let ABC be a triangle with the circumcentre O and the incentre I . Suppose that exist $D\in (AB)$
>
> and $E\in (AC)$ so that $BD=CE=BC$. Prove that $\|\begin{array}{cc} 1. & OI\perp DE\\ 2. & A=30^{\circ}\stackrel{?!}{\implies} DE=OI\end{array}$.

sunken rock

#4 Jan 2, 2010, 2:34 am

1) $DE\perp OI$ is easy to prove, and already old; I think the easiest way is to prove
$DI^2-EI^2=DO^2-EO^2=a\cdot (c-b)$, by drawing the perpendiculars from I and O onto AB and AC.
2) It's easy as well to see that $\angle DIE=45^\circ$ and $\angle DOE=135^\circ$, hence O is the orthocenter of $\triangle DIE$, done.

Best regards,
sunken rock

Luis González

#5 Jan 2, 2010, 4:24 am

There's a more general result coming from this configuration.

Proposition. Let $\triangle ABC$ be a scalene triangle and M,N are two points on $AC,AB,$ such that $BN=CM=BC.$ Then MN is perpendicular to $OI,$ where O,I are the circumcenter and incenter of $\triangle ABC$ and $\frac{MN}{BC}=\frac{OI}{R}.$

Proof. In the isosceles triangles $\triangle OAC$ and $\triangle OAB,$ we have

$$R^2-OM^2=AM\cdot MC, \ R^2-ON^2=AN\cdot BN$$

$$\Longrightarrow ON^2-OM^2=AM\cdot MC-AN\cdot BN$$

$$\Longrightarrow ON^2-OM^2=(CA-BC)BC-(AB-BC)BC=BC(CA-AB)$$

Since $\triangle MIC$ and $\triangle MIB$ are isosceles, we have $IM=IB,IN=IC.$ Then

$$IN^2-IM^2=IC^2-IB^2=BC(CA-AB)$$

$\Longrightarrow ON^2 - OM^2 = IN^2 - IM^2 \Longrightarrow IO \perp MN$

On the other hand, $\angle NBM = \angle OCI = 90^\circ - \frac{1}{2}\angle C - \angle A$. If $L \equiv CI \cap BM$, then it follows that $\angle NMB$ and $\angle OIC$ are supplementary, which implies that $\triangle IOC$ and $\triangle MNB$ are pseudo-similar, thus

$\frac{MN}{OI} = \frac{BN}{OC} \Longrightarrow \frac{MN}{BC} = \frac{OI}{R}$.

Note that $\angle A = 30^\circ \iff BC = R \iff DE = OI$ (the proposed problem).

cashaz

#6 Jan 4, 2010, 3:56 pm

> Quote:
>
> sunken rock Re: Lithuania TS 2009
> Nice solution
>
> ---
>
> 1) is easy to prove, and already old; I think the easiest way is to prove , by drawing the perpendiculars from and onto and .
> 2) It's easy as well to see that and , hence is the orthocenter of , done.
>
> Best regards,
> sunken rock

can you explain me how you get DIE=45 and all yout second part of the solution.

Thnx luis, very interesting solution.

MJ GEO

#7 Jan 4, 2010, 4:22 pm

its interesting and easy to prove that $OI = R_A DE$

Mathias_DK

#8 Jan 4, 2010, 6:23 pm

In $\triangle ABC, \angle A = 30^\circ$, the points O and I are the circumcentre and incentre of $\triangle ABC$ respectively. Points D and E are on the sides BA and CA respectively such that $BD = CE = BC$. Prove that OI is perpendicular and equal to DE.
(I'm used to O instead of S 😋)

I have a solution using trigonometry:

Let $\angle DBC = \beta, \angle ECB = \gamma$, then $\beta + \gamma = 150^\circ$. We easily find $\angle OBI = 60^\circ - \frac{\beta}{2} = \frac{\gamma}{2} - 15^\circ = \angle DCE$. Let X be the point on DC, such that $BI = CX$. Then $\triangle CXE$ is congruent to $\triangle BIO$. Let $\angle ECD = u$. Then
$IO^2 = EX^2 = CX^2 + EC^2 - 2CX \cdot EC \cdot \cos u$, $DE^2 = DC^2 + EC^2 - 2DC \cdot EC \cdot \cos u$.
So $IO^2 = DE^2 \iff$
$BI^2 + EC^2 - 2BI \cdot EC \cdot \cos u = DC^2 + EC^2 - 2DC \cdot EC \cdot \cos u \iff$
$(BI - DC)(BI + DC - 2EC \cdot \cos u) = 0$
I will prove $BI + DC = 2EC \cdot \cos u$, which implies $IO = DE$.
Let $BD = BC = CE = BO = OC = t$.

From the law of sines relation we get: $BI = \frac{\sin \frac{\gamma}{2}}{\sin 75^\circ} t$, since $\angle BCI = \frac{\gamma}{2}$, and $\angle BIC = 105^\circ$.

Using the law of sines once again: $CD = \frac{\sin \beta}{\sin 90^\circ - \frac{\beta}{2}} t = 2\cos\left(\frac{\beta}{2}\right) t$.

Letting $x = \frac{\beta}{2}$, and using $\beta + \gamma = 150^\circ$, we see that $BI + DC = 2EC \cdot \cos u \iff$
$\frac{\sin 75^\circ - x}{\sin 75^\circ} + 2\cos x = 2\cos 60^\circ - x$.
But using the relations $\sin a + b = \sin a \cos b + \sin b \cos a$, $\cos a + b = \cos a \cos b - \sin a \sin b$, and $\cot 75^\circ = 2 - \sqrt{3}$ it is easy to see that both sides equal $\sin x + \sqrt{3}\cos x$.
Hence $DE = IO$. But then we also have $DE = EX$. Let

Hence $DE = IO$. But then we also have $DE = EX$. Let
$v = \angle EDC = angle EDX = \angle DXE = 180^\circ - \angle EXC = 180^\circ - \angle BID$.
EC can be obtained by rotating and moving BC with an angle of $ 180^\circ - \ECB = 180^\circ - \gamma$ in positive direction.
DE can be obtained by rotating, moving, and scaling CE with an angle of $180^\circ - \angle DEC = \angle EDC + \angle DCE$, so DE can be obtained by rotationg, moving, and scaling BC with an angle of $90^\circ + \left(v + \frac{\beta}{2}\right)$
Similarily OI can be obtained by rotating, moving and scaling BC with an angle of $v + \frac{\beta}{2}$. It follows that ED can be obtained by rotating, moving, and scaling OI with and angle of 90°, and hence $ED \perp OI$. And we are done :)

sunken rock

#9 Jan 4, 2010, 11:03 pm

To expand little bit my compressed solution:
Due to symmetry the following angle equalities hold:
$\angle BID = \angle BIC = \angle CIE = 90^\circ + \frac{1}{2} \cdot m(\angle A) = 105^\circ \implies \angle DIE = 360^\circ - 3 \cdot 105^\circ = 45^\circ$.
$\angle BOC = 60^\circ$, as $\triangle BOC$ is equilateral; $\angle OBD + \angle OCE = 180^\circ - \angle A - \angle OBC - \angle OCB = 30^\circ$. The triangles OBD and OCE are isosceles, hence $\angle BOD = 90^\circ - \frac{1}{2} \cdot \angle OBD$, $\angle COE = 90^\circ - \frac{1}{2} \cdot \angle OCD$, so
$\angle BOD + \angle COE = 180^\circ - \frac{\angle OBD + \angle OCE}{2} = 165^\circ$, consequently
$\angle DOE = 360^\circ - (\angle BOD + \angle COE + \angle BOC) = 135^\circ$. Under the circumstances O is the orthocenter of $\triangle DIE$ (O is placed on the altitude from I and $\angle DIE + \angle DOE = 180^\circ$, hence it's true, as claimed).

I hope to be all clear now!

Best regards,
sunken rock

cashaz

#10 Jan 5, 2010, 3:49 pm

> *Quote:*
> I hope to be all clear now!

yes its all clear now, thanks a lot

behdad.math.math

#11 Jan 7, 2010, 2:26 am

$\triangle ABC$, D is on AC and E is on BC such that $AD = AB = BE.O,I$ are circumcenter and incenter of $\triangle ABC$ respectively.prove that
1)$OI \perp DE$
2)the radius of circumcircle of $\triangle CDE$ is equal to OI.

solution 1)
M is the midpoint of arcBC , K is the midpoint of BC , AM intersect BD at N , R is the radius of circumcircle of $\triangle ABC$.
MI=$2R \sin \frac{A}{2}$.
BD=$2BN$=$2AB \sin \frac{A}{2}$.
Now it is easy to compute that $\frac{BD}{MI}$ = $\frac{BE}{OM}$ (1)
$\triangle EDB$ and $\triangle OIM$: $OM \perp BE$, $IM \perp DB$ (2)
(1),(2) $\Longrightarrow \triangle IMO \sim \triangle DBE$
Now $\triangle IMO$ and $\triangle DBE$ are similar $\Longrightarrow IM \perp BE$, $BE \perp OM \Longrightarrow OI \perp DE$
solution 2)
$\triangle IMO \sim \triangle DBE \longrightarrow \frac{DE}{OI}$ = $\frac{BE}{OM}$ = $\frac{AB}{R}$
$\frac{AB}{R}$ = $2 \sinh \Longrightarrow \frac{DE}{OI}$=$2 \sinh$. radius of circumcircle CDE=$ R^$.
$ \frac{DE}{R^}$=$2 \sinh$, hence $\frac{DE}{OI}$ = $ \frac{DE}{R^}$ $\Longrightarrow$ $ OI=R^$

jayme

#12 Apr 30, 2011, 5:39 pm

Dear Mathlinkers,
an article concerning some results about “two equal segments on two sides of a triangle” has been put on my website
http://perso.orange.fr/jl.ayme vol. 6 Deux segments égaux sur deux côtés d'un triangle p. 23
You can use Google translator
Sincerely
Jean-Louis

Quick Reply

Spain

carlosbr

#1 Mar 26, 2006, 8:38 pm

$12nd$ Iberoamerican Olympiad
Guadalajara, MEXICO. [1997]

Edited by djimenez

Carlos Bravo

Attachments:
1997.pdf (35kb)

Luis González

#2 Dec 31, 2009, 10:26 am

> Quote:
> **Problema 1.** En un triángulo $\triangle ABC$, una circunferencia centrada en su incentro I corta a BC en D, P (estando D mas cerca de B) corta a CA en E, Q (estando E mas cerca C) y corta a AB en F, R (estando F mas cerca de A). S, T y U son las intersecciones de las diagonales de los cuadrilátero convexos $EQFR, FRDP$ y $DPEQ$. Probar que las circunferencias circunscritas a $\triangle FRT, \triangle DPU$ y $\triangle EQS$ concurren.

Como I equidista de BC, CA, AB, es claro que las cuerdas interseptadas DP, EQ y FR son congruentes. Así $EQFR$, $FRDP$ y $DPEQ$ son trapecios isosceles $\Longrightarrow QR = EF, FD = RP, DE = PQ$. Ello implica que $\triangle DEF$ y $\triangle PQR$ son congruentes teniendo circuncentro común I y ademas que $SFRT, TDPU$ y $UEQS$ son cíclicos, debido a que $\angle EFD = \angle QRP, \angle FDE = \angle RPQ$ y $\angle DEF = \angle PQR$.

I es evidentemente centro de la rotación que lleva $\triangle DEF$ en $\triangle PQR$ y como los puntos medios M y N de los lados QR, EF son homólogos en tal rotación, el ángulo $\angle MIN$ vale precisamente en ángulo de rotación. Así $\angle MIN = \angle RIF$, pero como $IMSN$ es cíclico, se tiene $\angle FSR = \angle FTR = \angle FIR \Longrightarrow I \in \odot(FRT)$. Entonces por analogía se concluye que $I \equiv \odot(FRT) \cap \odot(DPU) \cap \odot(EQS)$.

Luis González

#3 Jan 1, 2010, 10:43 pm

Problema 4. (Generalización). Sean P y Q dos puntos isogonales en $\triangle ABC$. Los rayos AP, AQ cortan su circunferencia circunscrita en P', Q'. Se definen $M \equiv BC \cap QP', N \equiv PQ' \cap BC$ y $L \equiv PM \cap QN$. Mostrar que $APLQ$ es un paralelogramo.

Sea $\triangle A'B'C'$ el triángulo pedal de P con respecto a $\triangle ABC$. Como P y Q son isogonales se tiene $QA \perp B'C'$ y $QB \perp C'A'$. Así

$$\angle AQB = 180^\circ - \angle A'C'B' = 180^\circ - (\angle PBA' + \angle PAB') = 180^\circ - \angle PBP'$$

$$\Longrightarrow \frac{\overline{P'M}}{\overline{MQ}} = \frac{\overline{BP'}}{\overline{BQ}} \cdot \frac{\sin \widehat{P'BC}}{\sin \widehat{CBQ}}, \quad \frac{\overline{P'P}}{\overline{PA}} = \frac{\overline{BP'}}{\overline{BA}} \cdot \frac{\sin \widehat{PBP'}}{\sin \widehat{ABP}}$$

Por ley del seno en $\triangle ABQ$ resulta $\dfrac{\overline{BA}}{\overline{BQ}} = \dfrac{\sin \widehat{AQB}}{\sin \widehat{BAQ}}$

Combinando con las dos anteriores teniendo en cuenta que $\angle AQB = 180^\circ - \angle PBP', \angle BAQ = \angle P'BC$ y $\angle CBQ = \angle ABP$, se obtiene $\frac{\overline{P'M}}{\overline{MQ}} = \frac{\overline{P'P}}{\overline{PA}} \Longrightarrow PM \parallel AQ$ y analogamente se prueba que $QN \parallel AP$.

Quick Reply

High School Olympiads

12nd ibmo - mexico 1997/q2.

Source: Spanish Communities

carlosbr

#1 Apr 23, 2006, 2:18 am

In a triangle ABC, it is drawn a circumference with center in the incenter I and that meet twice each of the sides of the triangle: the segment BC on D and P (where D is nearer two B); the segment CA on E and Q (where E is nearer to C); and the segment AB on F and R (where F is nearer to A).

Let S be the point of intersection of the diagonals of the quadrilateral $EQFR$. Let T be the point of intersection of the diagonals of the quadrilateral $FRDP$. Let U be the point of intersection of the diagonals of the quadrilateral $DPEQ$.

Show that the circumcircle to the triangle $\triangle FRT$, $\triangle DPU$ and $\triangle EQS$ have a unique point in common.

yetti

#2 Apr 23, 2006, 4:34 am • 2

Let r be the inradius and r_0 radius of the given circle centered at the incenter I. Let A', B', C' be the tangency points of the incircle (I, r) with the sides BC, CA, AB. The triangles $\triangle DEF \cong \triangle PQR$ are congruent and both are obtained from the contact triangle $\triangle A'B'C'$ by a spiral similarity with the center I, the same similarity coefficient $\frac{r_0}{r}$ and rotational angles $\angle DIA' = \angle EIB' = \angle FIC' = \phi$ resp. $\angle PIA' = \angle QIB' = \angle RIC' = -\phi$ of the same magnitude and opposite signs. Consequently,

$$2\phi = \angle DIP = \angle EIQ = \angle FIR =$$

$$\angle(DE, PQ) = \angle(EF, QR) = \angle(FD, RP) =$$

$$= \angle DUP = \angle QSE = \angle RTF =$$

$$= \angle QUE = \angle RSF = \angle DTP$$

Thus the circumcircles of the triangles $\triangle FRT, \triangle DPU, \triangle EQS$ all meet at the incenter I and moreover, these circumcircles are congruent to each other.

Luis González

#3 Jan 1, 2010, 12:28 am • 2

Since I is equidistant from BC, CA, AB, then it's clear that the intersected chords DP, EQ and FR are congruent, thus $EQFR, FRDP$ and $DPEQ$ are isosceles trapezoids $\Longrightarrow QR = EF, FD = RP, DE = PQ$. This implies that $\triangle DEF$ and $\triangle PQR$ are congruent having I as their common circumcenter. Moreover, $SFRT, TDPU$ and $UEQS$ are cyclic due to $\angle EFD = \angle QRP, \angle FDE = \angle RPQ$ and $\angle DEF = \angle PQR$. I is obviously center of the rotation taking $\triangle DEF$ into $\triangle PQR$ and since the midpoints M, N of the sides QR, EF are homologous under such rotation, the measure of the angle $\angle MIN$ is precisely the rotational angle $\Longrightarrow \angle MIN = \angle RIF$. But since $IMSN$ is cyclic, we have $\angle MIN = \angle FSR = \angle FTR = \angle FIR \Longrightarrow I \in \odot(FRT)$. By similar reasoning, we conclude that $I \equiv \odot(FRT) \cap \odot(DPU) \cap \odot(EQS)$.

Quick Reply

High School Olympiads

Isosceles triangle APT where T is Feurbach point (internal)

Agr_94_Math

#1 Dec 31, 2009, 11:17 am

Given a triangle ABC with $\angle A = 60$ degrees, prove that $AP = AT$ where $P = IDAA'$ where I is the incenter, D midpoint of BC and AA' is the altitude from A to BC, $A' \in BC$ and T is the point of contact of the Nine point circle and incircle.

Luis González

#2 Dec 31, 2009, 12:26 pm

Let X, Y be the tangency points of the incircle (I, r) and the A-excircle (I_a) with BC. Let X' denote the antipode of X WRT (I). A, Y, X' are clearly collinear $\Longrightarrow DI$ is the X-midline of $\triangle AXY \Longrightarrow APIX'$ is a parallelogram, hence $AP = IX' = r$. On the other hand, let N be the 9-point center of $\triangle ABC$ and M, L the midpoints of AC, AB. Simple angle chase gives

$$\angle NLB = 180^\circ - \angle B - 30^\circ = 60^\circ + \angle B + \angle C - \angle B - 30^\circ = \angle NMA$$

Since $NM = NL = \frac{R}{2}$, we deduce that N is equidistant from $AC, AB \Longrightarrow N \in IA$. So the Feuerbach poin F_e of $\triangle ABC$ lies on AI as well. Further, F_e is midpoint of AI, since $AI = 2r \Longleftrightarrow \angle A = 60^\circ$, hence $AP = AF_e = r$.

Agr_94_Math

#3 Dec 31, 2009, 9:49 pm

Well Luis, my friend, actually I wanted to prove $AT = r$ using another approach.

Anyway, I have two beautiful ways for proving $AT = r$ and $AP = r$. (Yours was extremely good as well)

Lemma 1: $AT = r$

Proof:
Consider the nine point circle. We know by definition that the nine point circle is the circumcircle of triangle ABC. So assume D, E, F are midpoints of BC, CA, AB respectively. Therefore, circumcircle of triangle DEF is the nine point circle.

Consider a point K such that $Ak = r$, on AI. First, since $AI = r\dfrac{A}{2} = r\dfrac{\pi}{6} = 2r$ where r is the inradius.
So, k will be the midpoint of AI.

By a homothety of center G and ratio $\dfrac{-1}{2}$, triangle ABC maps to triangle DEF. (G is the centroid)
Therefore, $\angle EDF = \dfrac{\pi}{3}$.

By a homothety of center A and ratio $\dfrac{1}{2}$, triangle BIC maps to triangle FKE. This implies $\angle FKE = \angle BIC = \dfrac{2\pi}{3}$.
This is because $\angle BIC = \dfrac{\pi}{2} + \dfrac{\pi}{6} = \dfrac{2\pi}{3}$.

Therefore, $\angle FKE + \angle FDE = \dfrac{2\pi + \pi}{3} = \pi$. This implies $FKED$ is a cyclic quadrilateral.

This implies K lies on the nine point circle .

But $IK = r$ (Since K is midpoint of AI) . Therefore, K lies on the incircle as well.

Thus, we have that K lies on both the incircle and nine point circle .

By Feurbach Theorem, we know that the incircle is internally tangent to the nine point circle.

Since K lies on both incircle and nine point circle, K must be the internal point of tangency of the incircle and nine point circle.

This implies $K = T$. Since $AK = r$, it implies that $AT = r$.

Lemma 2: $AP = r$.

In a triangle ABC, let D be the midpoint of BC, I as incenter, P is the intersection of DI with the altitude AX of triangle ABC.

Consider the circumcircle of triangle ABC with circumcenter O. Now, if AI meets the circle at some point X, X is the midpoint of arc BC. This implies that OX is the perpendicular bisector of BC.

Let the point of tangency of the incircle with side AC be Q. So, IQ is perpendicular to AC.

Now, $\angle IAQ = \angle BAX = \angle BCX = \frac{1}{2}\angle A$.

This implies that the right triangles AIQ and CXD are similar. Also, since $\angle IAP = \angle IXD$ and $\angle AIP = \angle XID$, triangles AI and XID are also similar.

Hence, $\frac{AP}{XD} = \frac{AI}{IX}$ and $\frac{AI}{XC} = \frac{IQ}{XD}$.

Since $IX = XC$, we have $AP = AQ$.

Well, actually wanted to prove $AT = r$ using another approach and so I thought of using $AP = r$ and trying to prove triangle ATR is isoceles.

(Should have put it in proposed problems section).

Can someone please post a solution for $AP = AT$ by angle chase using the fact that $AP = r$ and not using $AT = r$?Like an angle chase etc?

mathVNpro

#4 Jan 1, 2010, 12:26 pm

> *Quote:*
>
> Given a triangle $\triangle ABC$ with $\angle A = 60^\circ$. Let (I) be the incircle of $\triangle ABC$ and D be the midpoint of BC. Denote A' by the projection of A onto BC and P by the intersection of ID and AA'. Let T be the *Feurebach point* wrt $\triangle ABC$. Prove that $AP = AT$.

Let D, E, F respectively be the midpoints of BC, CA, AB and A'', B'', C'' respectively be the tangency points of (I) with BC, CA, AB. Let T' be the midpoint of AI, then it is followed that T' is the circumcenter of $\triangle AB''C''$. But since $\angle A = 60^\circ$, hence $\angle B''T'C'' = 120^\circ$. But $\angle B''A''C'' = 120^\circ$, thus $\angle(T'B'', T'C'') = \angle(A''B'', A''C'')$, which implies $T' \in (A''B''C'') \equiv (I)$. Further, since $T'F \| IB$ and $T'E \| IC$, which yields that $\angle(T'F, T'E) = \angle(IB, IC)$. In the other hand, due to the fact that $IB, IC, A''C''$ and $A''B''$ form a concyclic quadrilateral, then $\angle(IB, IC) = \angle(A''C'', A''B'')$. Also from the beginning, we have known that $\angle(A''C'', A''B'') = \angle(AB, AC)$ and with a notice that $AFDE$ is parallelogram. As the result, $\angle(T'F, T'E) = \angle(DF, DE)$. Therefore, T' also lies on (DEF), which is also the $9-$ point circle wrt $\triangle ABC$. As the result, T' is the *Feurebach point* wrt $\triangle ABC \Longrightarrow T' \equiv T$. Now, by the *Newton theorem* for the "quadrilateral" $ABA''C$ with (I) its inscribed circle, we obtain that ID also passes through the midpoint M of AA''. Then it is followed that $TM \| IA'' \| AP$, but T has already been the midpoint of IA, thus M is also the midpoint of IP. The diagonals of quadrilateral $APA''I$ intersect each other at their midpoints $\Longrightarrow APA''I$ is parallelogram. Therefore, $AP = IA'' = IT = AT$. Our proof is completed then. $\square$

Attachments:

mathlinks26.pdf (18kb)

Quick Reply

High School Olympiads

the striangle

shortlist
#1 Dec 29, 2009, 5:28 pm

Give a cute straingle $\triangle ABC$. I is Tolicelli of it . Prove Euler line of $\triangle IBC$, $\triangle IAB$, $\triangle IAC$ current (cut at a point)

shortlist
#2 Dec 29, 2009, 6:28 pm

Who can help me?
I thoungt it pay for much times

pco
#3 Dec 29, 2009, 7:01 pm

> shortlist wrote:
> Who can help me?
> I thoungt it pay for much times

1) You posted a geometry problem in an algebra forum so less interested readers.

2) you posted in a "proposed and own" forum, so you claimed that you have the solution and ask for a second one. So nobody is on hurry to answer you.

Maybe a better choice would be to post in "geometry unsolved problems"

Luis González
#4 Dec 31, 2009, 12:33 am

The notation and wording is inconvenient. Try to use I for the incenter.

> Quote:
> T is Toricelli point of $\triangle ABC.$ Prove Euler lines of $\triangle TBC, \triangle TAB, \triangle TAC$ concur.

Let $\triangle A'BC, \triangle B'CA$ and $\triangle C'AB$ be the three equilateral triangles erected outwardly on $BC, CA, AB.$ Let X, Y, Z be their circumcenters, thus $T \equiv (X) \cap (Y) \cap (Z)$ and $T \equiv AA' \cap BB' \cap CC'.$ Let G_1, G_2, G_3 denote the centroids of $\triangle TBC, \triangle TCA, \triangle TAB \Longrightarrow XG_1, YG_2$ and ZG_3 are the Euler lines of $\triangle TBC, \triangle TCA, \triangle TAB.$ If M is the midpoint of BC and G is the centroid of $\triangle ABC,$ we get

$$\frac{MX}{MA'} = \frac{MG_1}{MT} = \frac{MG}{MA} = \frac{1}{3}$$

It follows that X, G_1, G are collinear on a parallel line to $AA'.$ Hence, Euler lines of $\triangle TBC, \triangle TCA, \triangle TAB$ concur at the centroid G of $\triangle ABC.$

High School Olympiads

An equilateral triangle

Ulanbek_Kyzylorda KTL

#1 May 1, 2009, 11:31 pm

An equilateral triangle ABC is inscribed ina circle k. On the sides AC and AB,the points M and N are chosen respectively so that AM=2MC and AN=NB. The ray MN meets the circle k at P. Prove that PA/PB - PB/PA=1/2

Luis González

#2 Dec 30, 2009, 7:26 am

Trilinear coordinates $(\alpha:\beta:\gamma)$ of M and N WRT $\triangle ABC$ can be expressed as $M\ (1:0:2)$ and $N\ (0:1:1)$. Therefore $MN \equiv \gamma+2\beta-2\alpha=0$. Let X,Y,Z denote the orthogonal projections of P on BC,CA,AB, respectively. From the latter equation (in terms of oriented distances) we obtain

$$2(\overline{PY}-\overline{PX})=\overline{PZ} \Longrightarrow \frac{\overline{PY}}{\overline{PZ}}-\frac{\overline{PX}}{\overline{PZ}}=\frac{1}{2} \ \ (\star)$$

Since $\angle PCX=\angle PAZ$, $\angle PBZ=\angle PCY$ we have

$$\triangle PCX \sim \triangle PAZ \text{ and } \triangle PBZ \sim \triangle PCY \Longrightarrow \frac{\overline{PC}}{\overline{PA}}=\frac{\overline{PX}}{\overline{PZ}} \ , \ \frac{\overline{PC}}{\overline{PB}}=\frac{\overline{PY}}{\overline{PZ}}$$

Combining with $(\star)$ yields $\frac{\overline{PC}}{\overline{PB}}-\frac{\overline{PC}}{\overline{PA}}=\frac{1}{2}$

Since $\overline{AB}=\overline{BC}=\overline{CA}$, by Ptolemy's theorem for cyclic quadrilateral $PACB$, we get

$$\overline{PC}=\overline{PA}+\overline{PB} \Longrightarrow \frac{\overline{PA}+\overline{PB}}{\overline{PB}}-\frac{\overline{PA}+\overline{PB}}{\overline{PA}}=\frac{1}{2} \Longrightarrow \frac{\overline{PA}}{\overline{PB}}-\frac{\overline{PB}}{\overline{PA}}=\frac{1}{2}$$

High School Olympiads

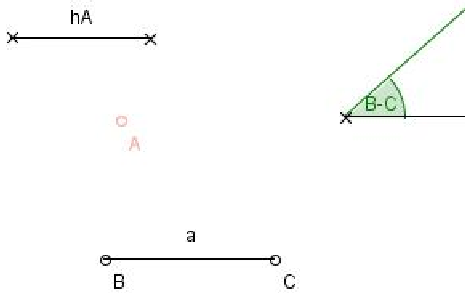
ha BC <(B-C)

Reply

jrrbc
#1 Dec 26, 2009, 4:06 am

Attachments:

Draw the triangle ABC given the length **hA** of the altitude relative to the side BC, the length **a** of side BC and the difference **B-C** of the angles B and C



MJ GEO
#2 Dec 26, 2009, 9:48 pm

this problem is equivalent to this.we have a and h_a and $l-a$ and i think its open.

jrrbc
#3 Dec 26, 2009, 11:55 pm

"this problem is equivalent to this." ?

Luis González
#4 Dec 29, 2009, 8:17 am

Let V, D, M denote the unknown feet of the altitude, angle bisector and median on BC, respectively. The angle between the altitude h_a and the angle bisector AV is presicely half of the given difference $|\beta-\gamma|$. Thus, choose D on a fixed line a (the sideline BC), construct the A-vertex on the perpendicular to a at D with the given distance h_a and draw the angle bisector of $\angle BAC$, which is a A-ray v such that $\angle(v, h_a) = \frac{1}{2}|\beta-\gamma| \Longrightarrow V \equiv v \cap a$. Perpendicular to AV through A is the external bisector of $\angle BAC$ meeting a at V'. The circle with diameter $\overline{VV'}$ is the A-Apollonius circle, which is orthogonal to the circle (M) with diameter $\overline{BC}$, since $(B, C, V, V') = -1$. Hence, if N is the midpoint of VV' and P is one of the intersections of (N) and $(M) \Longrightarrow NP \perp MP$. The right $\triangle MPN$ is constructible since its catheti NP and MP are known as half of the distances VV' and $BC \Longrightarrow$ length of the hypotenuse gives the position of M on the sideline a. Circle centered at M with radius $\frac{1}{2}BC$ cuts a at B, C and $\triangle ABC$ is completed.

jrrbc
#5 Dec 29, 2009, 4:16 pm

"construct the A-vertex on the perpendicular to a at D with the given distance h_a and draw the angle bisector of $\angle BAC$ which is a A-ray v such that $\angle(v, h_a) = \frac{1}{2}(\beta-\gamma) \Longrightarrow V \equiv v \cap a$ "

There's two feet of altitude onto BC: D and V ?

Luis González

#6 Dec 29, 2009, 10:45 pm

> *Luis González wrote:*
>
> choose D on a fixed line a (the sideline BC), construct the A-vertex on the perpendicular to a at D with the given distance h_a and draw the angle bisector of $\angle BAC$, which is a A-ray v such that $\angle(v,h_a)=\frac{1}{2}|\beta-\gamma|\Longrightarrow V\equiv v\cap a.$

Read carefully, we have to fix a sideline a and then choose a point D as the foot of the A-altitude. But if you mean that vertices B and C are previously fixed, then there's no problem either, because we can construct $\triangle A'B'C'\cong\triangle ABC$. Then $AB=A'B'$ and $AC=A'C'$.

jrrbc

#7 Dec 30, 2009, 5:01 am

thanks

Quick Reply

High School Olympiads

Cyclic quadrangle

Reply

behdad.math.math

#1 Dec 28, 2009, 12:39 am

In cyclic quadrangle $ABCD$, M and N are midpoints of AC and BD respectively.AC is bisector of $\angle BMD$.prove that BD is the bisector of $\angle ANC$ too.

behdad.math.math

#2 Dec 29, 2009, 12:40 am

Nobody want to help?

Luis González

#3 Dec 29, 2009, 9:56 am

Let $P \equiv DA \cap CB, \ Q \equiv AB \cap DC$ and $E \equiv AC \cap BD.$ It's known that E is the pole of PQ WRT the circumcircle (O) of $ABCD \Longrightarrow OE \perp PQ.$ We have $OM \perp AC$ and let $X \equiv OM \cap PQ.$ Since $(B, D, E, X) = -1$ and CA bisects $\angle BMD,$ then it follows that $X \equiv DB \cap OM \cap PQ.$ Hence if $Y \equiv CA \cap PQ,$ then E is the othocenter of $\triangle OXY \Longrightarrow DB \perp OY$ and due to $(C, A, E, Y) = -1,$ we deduce that DB bisects $\angle ANC.$

sunken rock

#4 Dec 29, 2009, 1:16 pm

See here:
http://www.mathlinks.ro/viewtopic.php?search_id=2088370956&t=6557

Best regards,
sunken rock

Quick Reply

High School Olympiads

About symmedian point

Source: 0

Luis González

#1 Dec 26, 2009, 12:55 am

Incircle (I) of $\triangle ABC$ is tangent to BC, CA, AB at D, E, F. The A-excircle is tangent to BC at P. Let Q be the antipode of D WRT (I) and define the points $M \equiv QE \cap BC, N \equiv QF \cap BC$ and $R \equiv IA \cap EF$. Prove that the lines QR and PI meet at the symmedian point of $\triangle QMN$.

hophinhan

#2 Dec 26, 2009, 5:30 pm

Solution:

$\boxed{\widehat{QMN}} = 90^o - \widehat{QDE} = \boxed{\widehat{QFE}} = \boxed{\widehat{QEA}} = \widehat{QDE} = \boxed{\widehat{ICD}}$

$\Longrightarrow EFMN$ is concyclic ;$QM \parallel IP$; $QN \parallel IB$; $\triangle NBF$ is isosceles ; $\triangle MCE$ is isosceles

Because $I_aB \perp IB$; $IB \parallel QN$ and $\triangle NBF$ is isosceles $\Longrightarrow I_aB$ is the midpenperdicular line of NF . Similarly, I_aC is the midpenperdicular line of ME. It's mean I_a is the center of circle $(MNFE)$.

$\Longrightarrow P$ is midpoint of MN

$\triangle QEF \sim \triangle QNM$ and R is midpoint of EF . So the symmedian point of $\triangle QMN$ lies on QR (the reflection of QP throught the angle bisector of angle Q of $\triangle QMN$)

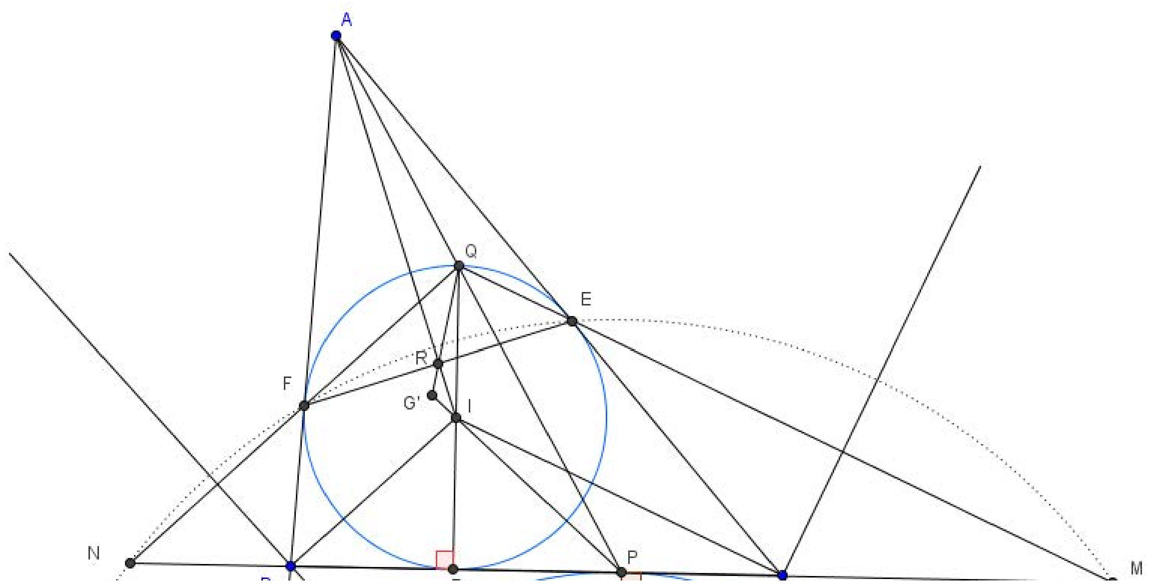
We also knew that the the symmedian point of $\triangle QMN$ is collinear with I (midpoint of Q-altitude of $\triangle QMN$) and P (the midpoint of side MN)

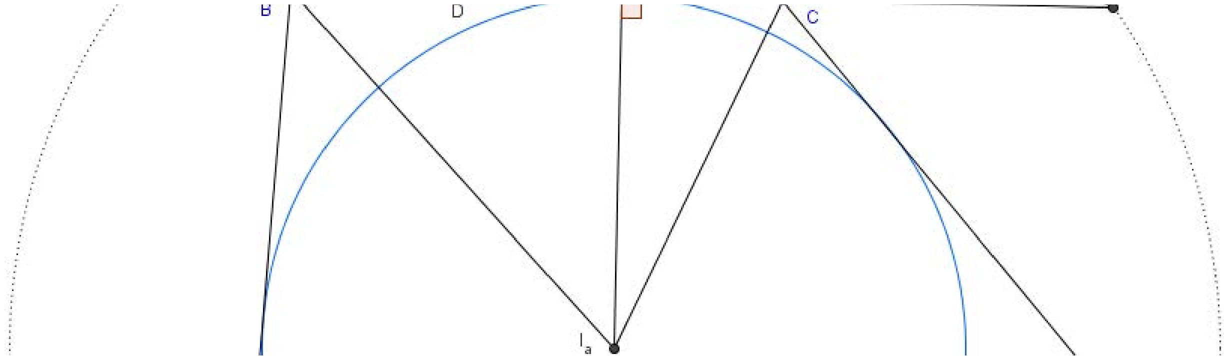
As the consequence, $G' \equiv IP \cap QR$ is the symmedian point of $\triangle QMN$

We end the proof .

Click to reveal hidden text

Attachments:





This post has been edited 1 time. Last edited by hophinhan, Dec 27, 2009, 11:27 am

Luis González

#3 Dec 27, 2009, 8:52 am

Thanks for your solution hophinhan, mine is similar but I used inversion to prove that P is the midpoint of $MN.$

Inversion with respect to the circumference centered at Q with radius QD takes the incircle (I) into the sideline $BC.$ Thus M,N are inverse images of E,F and the circle (A,AF) orthogonal to (I) is transformed into the circle with diameter MN $\Longrightarrow$ its center must lie on $QA \Longrightarrow P \equiv QA \cap BC$ is midpoint of $MN.$ It's easy to see that QR is the Q-symmedian of $\triangle QMN,$ since R is the midpoint of the antiparallel section EF to $MN.$ Then the line connecting the midpoint I of its Q-altitude QD and the midpoint P of MN is a Schwatt line of $\triangle QMN$ containing its symmedian point, i.e. $J \equiv QR \cap PI$ is the symmedian point of $\triangle QMN$ and the proof is completed.

lym

#4 Dec 27, 2009, 11:17 am

“Schwatt line”□I first heard that□it's good□thanks□you guys .

Quick Reply

High School Olympiads

easy and hard problems

Reply

MJ GEO

#1 Dec 26, 2009, 9:25 pm

1.(easy) in triangle ABC AD, BE, CF are altitudes and concur at H.M is the midpoint of AH.EM and FD intersect AB and BE at Q and P.prove that PQ is perpendicular to BC
2.(hard) if AD, CF, BE arent altitude problem is corect again.

Luis González

#2 Dec 27, 2009, 2:12 am

Problem. Let P be a point on the plane of $\triangle ABC$. X, Y, Z are the intersections of PA, PB, PC with BC, CA, AB. M is the midpoint of AP. Line YM meets AB at N and XZ meets BP at L. Then prove that $NL \parallel AP$.

By Menelaus' theorem for $\triangle ABP$ cut by the transversal $\overline{NMY}$, we get

$$\frac{\overline{PY}}{\overline{BY}} \cdot \frac{\overline{BN}}{\overline{NA}} \cdot \frac{\overline{AM}}{\overline{MP}} = 1 \Longrightarrow \frac{\overline{BN}}{\overline{NA}} = \frac{\overline{BY}}{\overline{PY}}$$

On the other hand, since $(B, P, L, Y) = -1$, we have $\frac{\overline{BY}}{\overline{PY}} = \frac{\overline{BL}}{\overline{LP}}$

$$\Longrightarrow \frac{\overline{BL}}{\overline{LP}} = \frac{\overline{BN}}{\overline{NA}} \Longrightarrow NL \parallel AP.$$

When P is identical to the orthocenter H of $\triangle ABC$, we get $NL \perp BC$.

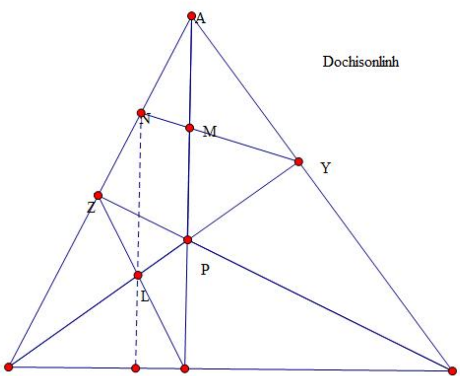
Victory.US

#3 Jan 2, 2010, 8:43 am

> MJ GEO wrote:
> 1.(easy) in triangle ABC AD, BE, CF are altitudes and concur at H.M is the midpoint of AH.EM and FD intersect AB and BE at Q and P.prove that PQ is perpendicular to BC
> 2.(hard) if AD, CF, BE arent altitude problem is corect again.

it's quite easy problem.

Attachments:



Virgil Nicula

#4 Jan 2, 2010, 2:55 pm

> *luisgeometra wrote:*
> Let P be a point in the plane of ΔABC and X, Y, Z be the meetpoints of PA, PB, PC with BC, CA, AB respectively. Denote the midpoint M of AP , $N \in YM \cap AB$ and $L \in XZ \cap BP$. Prove that $NL \parallel AP$.

Proof (similarly with **luisgeometra**'s). Denote $N' \in AB$ for which $LN' \parallel AP$ and $R \in LN' \cap AC$. Since the pencil $A(B, L, P, Y)$ is harmonically obtain for the transversal $\overline{RN'L}$ that $N'L = N'R$, i.e. $M \in YN'$. In conclusion, $N' \equiv N$.

Agr_94_Math

#5 Apr 9, 2010, 10:06 pm

> *MJ GEO wrote:*
> 1.(easy) in triangle ABC AD, BE, CF are altitudes and concur at H.M is the midpoint of AH.EM and FD intersect AB and BE at Q and P.prove that PQ is perpendicular to BC

The 1st part has an easy angle chase solution.

EM is the median of right triangle AEH. Therefore, $EM = MH = AM$. THis implies triangle EMH is isoscles. Thus, angles $EMH = EHM$
By vertically opposite angles, $BHD = MHE$.
Sicne $BFHD$ is a cyclic quadrilateral, angles $BFD = BHD = MHE = MEH$.
Also, by the same cyclic quadrilateral, angles $FBH = FDH$. Since angles $BFD = PEQ$, it implies that $FPEQ$ is a cylic quadrilateral.
Therefore, angles $PQF = PEF$. This implies that $FQP = FDH$.
THis implies $PQ || AD$. Therefore, PQ perpendicular to BC.

jayme

#6 Apr 10, 2010, 1:50 pm

Dear Mathlinkers,
the problem 1 has been posted but where?
I gave a oroof base on an application of Pascal's theorem.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

nice and strange

Reply

MJ GEO
#1 Dec 26, 2009, 9:32 pm

AE, CF, BD are concur at Q in triangle $ABC.AQ$ is perpendicular to DF at P. $BPDC$ is cyclic.prove that $BD=CD$

Luis González
#2 Dec 27, 2009, 1:40 am

$R \equiv FD \cap BC$ and let M and M' be the midpoint of BC an the orthogonal projection of D onto BC. We shall prove that $M \equiv M'$, then $\triangle BDC$ is isosceles with base BC. It's enough to see that $(B, C, E, R)=-1$ and that the quadrilateral $PDM'E$ is cyclic on account of the right angles at P, M'. Then we have

$\overline{RB} \cdot \overline{RC} = \overline{RE} \cdot \overline{RM}$ and $\overline{RP} \cdot \overline{RD} = \overline{RE} \cdot \overline{RM'} = \overline{RB} \cdot \overline{RC}$

$\Longrightarrow \overline{RM} = \overline{RM'} \Longrightarrow M \equiv M'.$

Victory.US
#3 Jan 1, 2010, 11:15 pm

Nice! my solution is like your! quite simple:)

shoki
#4 Jan 25, 2010, 1:34 am

we use the following well-known lemma:
let AD be the altitude of ΔABC and let the lines CF, BE intersect each other on AD then AD is the angle bisector of EDF.
by the above lemma it follows that AP is the angle bisector of BPC .now let this line intersect $(BPDC)$ at S it implies that $\angle BDS = \angle SDC$.but we know that $\angle SPD = 90^\circ$ or in other words DS is also the diameter.
so DS is the diameter and also the angle bisector of $\angle BDC$ so we get $BD=CD$.

Quick Reply

High School Olympiads

P,Q,A lie on a line

Reply

MJ GEO

#1 Dec 26, 2009, 9:16 pm

M, N lie on AB, AC such that MN is parallel to BC. K, L lie MN such that BL, CK intersect at P. Circles MPK and NPL intersect at P, Q. Prove that points P, Q, A lie on a line.

Luis González

#2 Dec 27, 2009, 12:30 am

If $M' \equiv MQ \cap BC$ and $N' \equiv NQ \cap BC$, then $QPCM', QPBN'$ are both cyclic since $\angle PCM' = \angle LKP = \angle PQM$ and $\angle PBN' = \angle KLP = \angle PQN$. Thus $\odot(MQN)$ cuts AC, AB at D, E lying respectively on $\odot(QPC)$ and $\odot(QPB)$, due to $\angle NDQ = \angle NMQ = \angle QPC$ and $\angle MEQ = \angle MNQ = \angle BPQ$. From power of A WRT $\odot(MQN)$ and $\triangle ABC \sim \triangle AMN$, we have

$$AD \cdot AN = AE \cdot AM \Longrightarrow AD \cdot \frac{AC \cdot MN}{BC} = AE \cdot \frac{AB \cdot MN}{BC}$$

$\Longrightarrow AD \cdot AC = AE \cdot AB \Longrightarrow A$ is on the radical axis PQ of $\odot(QPC), \odot(QPB)$.

Victory.US

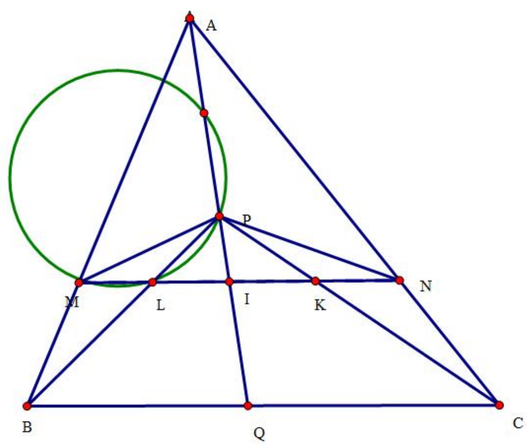
#3 Jan 1, 2010, 10:29 pm

here is simple solution

$AP \cap MN, BC = I, Q'$. then $\dfrac{IL}{IK} = \dfrac{BQ'}{BC} = \dfrac{IM}{IN} \rightarrow \overline{IL}.\overline{IN} = \overline{IM}.\overline{IK} \Rightarrow P_{I/(NPL)} = P_{I/(PKM)}$

then AI is radical axis , follow P, Q, A lie on a line.

Attachments:



Quick Reply

High School Olympiads

very very hard,nice,hard,nice,...(without a theory)

Reply

MJ GEO
#1 Dec 26, 2009, 9:40 pm

3 circles are tangent to each other and the sides of ABC.if the tangency of circles with each other named A_1,B_1,C_1 prove that AA_1,BB_1,CC_1 concur at one point.

Luis González
#2 Dec 26, 2009, 10:37 pm

Let (O_a) be the circle tangent to AB,AC and denote similarly $(O_b),(O_c).$ Then, $A_1 \equiv (O_b) \cap (O_c), B_1 \equiv (O_a) \cap (O_c)$ and $C_1 \equiv (O_a) \cap (O_b).$ B_1 and C_1 are the insimilicenters of $(O_a) \sim (O_c)$ and $(O_a) \sim (O_b).$ Then by Monge-d'Alembert theorem, it follows that $A'' \equiv BC \cap B_1C_1$ is the exsimilicenter of $(O_b) \sim (O_c).$ Analogously, intersections $B'' \equiv A_1C_1 \cap AC$ and $C'' \equiv A_1B_1 \cap AB$ are the exsimilicenters of $(O_a) \sim (O_c)$ and $(O_a) \sim (O_b) \Longrightarrow A'',B'',C''$ are collinear on the homothety axis of $(O_a),(O_b),(O_c).$ Now, based on Desargues theorem, the triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ are perspective through $A''B''C'' \Longrightarrow AA_1,BB_1,CC_1$ concur.

MJ GEO
#3 Dec 26, 2009, 10:48 pm

thanks.my solution is exactly like you and the theory that i writed in title is Deasarge.but i want another solution without this theory.is there anybody that can help me?

mathVNpro
#4 Dec 26, 2009, 11:57 pm

> MJ GEO wrote:
> 3 circles are tangent to each other and the sides of ABC.if the tangency of circles with each other named A_1,B_1,C_1 prove that AA_1,BB_1,CC_1 concur at one point.

Another interesting result:

Given $\triangle ABC$ and three circles (O_1), (O_2), (O_3) externally tangents to each other pairwise and (O_1) tangents to AB and AC, similarly for (O_2), (O_3). Let A_1 is the tangency point of (O_2) and (O_3), I_a is the $A-$ excircle center. Similarly for I_b, I_c and B_1, C_1. Then I_aA_1, I_bB_1, I_cC_1 are concurrent.

MJ GEO
#5 Dec 27, 2009, 7:02 pm

anybody? i am sure you can find another solution

MJ GEO
#6 Jan 1, 2010, 4:59 pm

anybody?please just write a idea

Quick Reply

High School Olympiads

Show angleMKN=90

Reply

Source: Nice Geometry

dgreenb801
#1 Dec 24, 2009, 2:10 am

Two circles intersect at A and B. A line through A meets the circles at C and D. M and N are the midpoints of arcs BC and BD, both not containing A. K is the midpoint of CD. Show that $\angle MKN = 90$.

Luis González
#2 Dec 24, 2009, 4:31 am

Let B' be the inverse of B under the inversion with center A and power $\overline{AC} \cdot \overline{AD}$. Circles $(O), (O')$ go to lines DB' and $CB' \Longrightarrow M' \equiv AM \cap DB'$ and $N' \equiv AN \cap CB'$ are the inverses of M, N, respectively. Let $U \equiv M'N' \cap CD$. Since the pencil $A(C, B, M, N)$ is harmonic, it follows that pencil $B'(M', N', A, U)$ is also harmonic $\Longrightarrow (C, D, A, U)$ is harmonic $\Longrightarrow \overline{AC} \cdot \overline{AD} = \overline{AK} \cdot \overline{AU}$, i.e. U is the inverse of A under the referred inversion. Since U, M', N' are collinear, then K, M, N, A are concyclic and the conclusion follows.

sunken rock
#3 Mar 4, 2012, 1:47 pm

Take a random point X on $(AB$ produced (to see that $\angle MBX = \angle ACM\ (*)$ and $\angle NBX = \angle ADN\ (**)$).
Rotate triangle $\triangle MBN$ about M of angle $\alpha = \angle BAC$; B will go to C and N to a point N' so that $\angle MCN' = \angle MBN$, implying, from $(*), (**)$ that $CN'' \parallel DN$; with $CN' = ND$ we get that N' is the reflection of N in K, done (do not forget, the rotation preserves the distances, hence $MN' = MN$).

Best regards,
sunken rock

Quick Reply

High School Olympiads

Collinear points

Reply

Source: Îœ,Îœ11

stergiu

#1 Dec 21, 2009, 11:48 pm

Let a quadrilateral $ABCD$. The perpendicular from A to BD meets the perpendicular from B to CD at point

P .The perpendicular from C to BD meets the perpendicular form D to AB at Q. Prove that the line PQ

passes through the point of intersection of the diagonals AC and BD.

Babis

Luis González

#2 Dec 24, 2009, 4:07 am

Let E,F be the feet of the perpendiculars from D,B on AB,CD and M,N the feet of the perpendiculars from A,C on BD. Since $AEDM$ and $MFPD$ are cyclic, we have $\angle BAM=\angle QDN$ and $\angle BPM=\angle BDF$. Thus $\triangle BAM\sim\triangle QDN$ and $\triangle BPM\sim BDF$. Then

$$\frac{\overline{CN}}{\overline{DN}}=\frac{\overline{BM}}{\overline{MP}}\ ,\ \frac{\overline{QN}}{\overline{DN}}=\frac{\overline{BM}}{\overline{AM}}\Longrightarrow\frac{\overline{QN}}{\overline{CN}}=\frac{\overline{MP}}{\overline{AM}}$$

$$\frac{\overline{QN}+\overline{CN}}{\overline{CN}}=\frac{\overline{MP}+\overline{AM}}{\overline{AM}}\Longrightarrow\frac{\overline{AM}}{\overline{CN}}=\frac{\overline{AP}}{\overline{QC}}$$

Define the intersections $T\equiv AC\cap BD$ and $T'\equiv PQ\cap AC$.

From $\triangle T'QC\sim\triangle T'PA$ we obtain $\frac{\overline{AP}}{\overline{QC}}=\frac{\overline{AT'}}{\overline{CT'}}$

But $\frac{\overline{AM}}{\overline{CN}}=\frac{\overline{AT}}{\overline{CT}}\Longrightarrow\frac{\overline{AT'}}{\overline{CT'}}=\frac{\overline{AT}}{\overline{CT}}\Longrightarrow T\equiv T'$ and the conclusion follows.

Quick Reply

High School Olympiads

Vietnam NMO 2000_2

Reply

April

#1 Oct 26, 2008, 6:15 am

Two circles (O_1) and (O_2) with respective centers O_1, O_2 are given on a plane. Let M_1, M_2 be points on (O_1), (O_2) respectively, and let the lines O_1M_1 and O_2M_2 meet at Q. Starting simultaneously from these positions, the points M_1 and M_2 move clockwise on their own circles with the same angular velocity.
(a) Determine the locus of the midpoint of M_1M_2.
(b) Prove that the circumcircle of $\triangle M_1QM_2$ passes through a fixed point.

je4ko

#2 Dec 20, 2009, 1:41 pm

For a) I managed to prove that the locus is the circle centered at the midpoint of O_1O_2 , call it O and pasing through the midpoint of M_1M_2, call it M, i.e. with radius OM.

Luis González

#3 Dec 23, 2009, 7:41 am

Assume that M_1, M_2 are starting points on $(O_1), (O_2)$, both fixed, and X, Y their positions when they have described an angle θ clockwise. We prove that if $Q \equiv XO_1 \cap YO_2$, then the circumcircle of $\triangle XYQ$ goes through a fixed point. Since $\triangle XM_1O_1, \triangle YM_2O_2$ are similar isosceles triangles, the spiral similarity taking O_1M_1 into O_2M_2 also carries $\triangle XM_1O_1 \cup (O_1)$ into $\triangle YM_2O_2 \cup (O_2)$. If $P' \equiv OM_1 \cap OM_2$, then the center P of this spiral similarity is the second intersection of the circles $\odot(P'O_1O_2)$ and $(P'M_1M_2)$. Therefore $\angle XPY = \angle O_1PO_2 = \angle XQY \Longrightarrow QPXY$ is cyclic $\Longrightarrow \odot(XYQ)$ passes through the fixed P.

Let M, N be the midpoints of XY and O_1O_2. Since $\triangle PXY \sim \triangle PO_1O_2$, the angle between their medians PM, PN equals the angle between their homologous sides $PX, PO_1 \Longrightarrow \triangle PMN$ and $\triangle PXO_1$ are similar. Thus

$$\frac{NM}{O_1X} = \frac{PN}{PO_1} \Longrightarrow NM = \frac{O_1X \cdot PN}{PO_1} = \text{const.}$$

Hence, locus of M is the circle centered at N whose radius is the segment joining N and the midpoint of M_1M_2

Quick Reply

Spain

17th - IBERO - EL SALVADOR 2002.

Reply

carlosbr

#1 Mar 26, 2006, 8:59 pm

$17th$ Iberoamerican Olympiad
San Salvador, EL SALVADOR. [2002]

Edited by djimenez

Carlos Bravo

Attachments:
2002.pdf (28kb)

Luis González

#2 Dec 22, 2009, 11:39 am

> *Quote:*
> **Problema 3.** Sea P un punto interior al triángulo equilátero $\triangle ABC$ tal que $\angle APC = 120^\circ$. Sea M la intersección de CP con AB y N la intersección de AP y BC. Hallar el lugar geométrico del circuncentro de $\triangle BMN$ al variar P.

Si O es el centro de $\triangle ABC$, es claro que P se mueve en el arco de circunferencia AOC que mira a AC bajo 120°. Entonces si $\odot(BMN)$ corta a BO en O', observando el cuadrilátero cíclico $BNPO'$, se tiene que $\angle O'PA = \angle O'BN = 30^\circ$. Lo cual implica $O \equiv O'$, puesto que $\angle OPA = \angle OCA = 30^\circ \Longrightarrow$ La circunferencia $\odot(BMN)$ pertenece al haz que pasa por los puntos fijos B, O. El lugar de su centro es pues la mediatriz de BO.

Luis González

#3 Dec 22, 2009, 12:28 pm

> *Quote:*
> **Problema 4.** En el triángulo $\triangle ABC$ con lados diferentes, sea D el pie de la bisectriz interior del $\angle ABC$ y E, F las proyecciones ortogonales de A, C sobre la recta BD. Si M es la proyección ortogonal de D en BC, mostrar que $\angle EMD = \angle DMF$.

Basta ver que si D' es el pie de la bisectriz exterior del $\angle BAC$, se tiene $BD' \parallel AE \parallel CF$ y esto significa que la cuaterna (B, D, E, F) es la proyección de la cuaterna (D', D, A, C) sobre la bisectriz del $\angle BAC$ desde el punto del infinito de su gradiente ortogonal. Como (D', D, A, C) es armónica, entonces (B, D, E, F) también lo es. Por propiedad básica de las cuaternas armónicas si $DM \perp BC$, entonces se deduce que BC y DM son las bisectrices exterior e interior del $\angle EMF$.

Quick Reply

High School Olympiads

Quadrilateral and circumcircle

Reply

imedeen

#1 Dec 21, 2009, 10:25 am • 1

Let Quadrilateral $ABCD$, AC and BD meet at O.The circumcircle of triangle AOD and BOD meet at M. OM meet the circumcircle of triangle AOB and COD at S,T. Prove that $MS=MT$

Luis González

#2 Dec 22, 2009, 7:48 am • 1

There's a little typo in the enuntiation, I believe it should be:

> Quote:
> Let quadrilateral $ABCD, AC$ and BD meet at $O.$ Circles AOD and BOC meet at $M. OM$ meet the circles AOB and COD at $S,T.$ Prove that $MS=MT$

Inversion centered at O with arbitrary power k^2 takes the circles $\odot(OAB), \odot(ODC), \odot(OBC), \odot(OAD)$ into the lines $\ell_1, \ell_2, \ell_3, \ell_4,$ such that $A' \equiv \ell_1 \cap \ell_4, B' \equiv \ell_1 \cap \ell_3, C' \equiv \ell_2 \cap \ell_3, D' \equiv \ell_2 \cap \ell_4, M' \equiv \ell_3 \cap \ell_4, S' \equiv \ell_1 \cap OM$ and $T' \equiv \ell_2 \cap OM$ are the inverse images of A, B, C, D, M, S and T respectively. Since cross ratio (M', O, S', T') is harmonic, it follows that $\frac{M'S'}{M'T'} = -\frac{OS'}{OT'}$. Hence

$$\frac{\frac{k^2 \cdot \overline{MS}}{\overline{OM} \cdot \overline{OS}}}{} \cdot \frac{\overline{OM} \cdot \overline{OT}}{k^2 \cdot \overline{MT}} = -\frac{k^2 \cdot \overline{OT}}{k^2 \cdot \overline{OS}} \Longrightarrow \overline{MS} = -\overline{MT}.$$

Quick Reply

High School Olympiads

Concyclic points

Reply

Source: M,M13

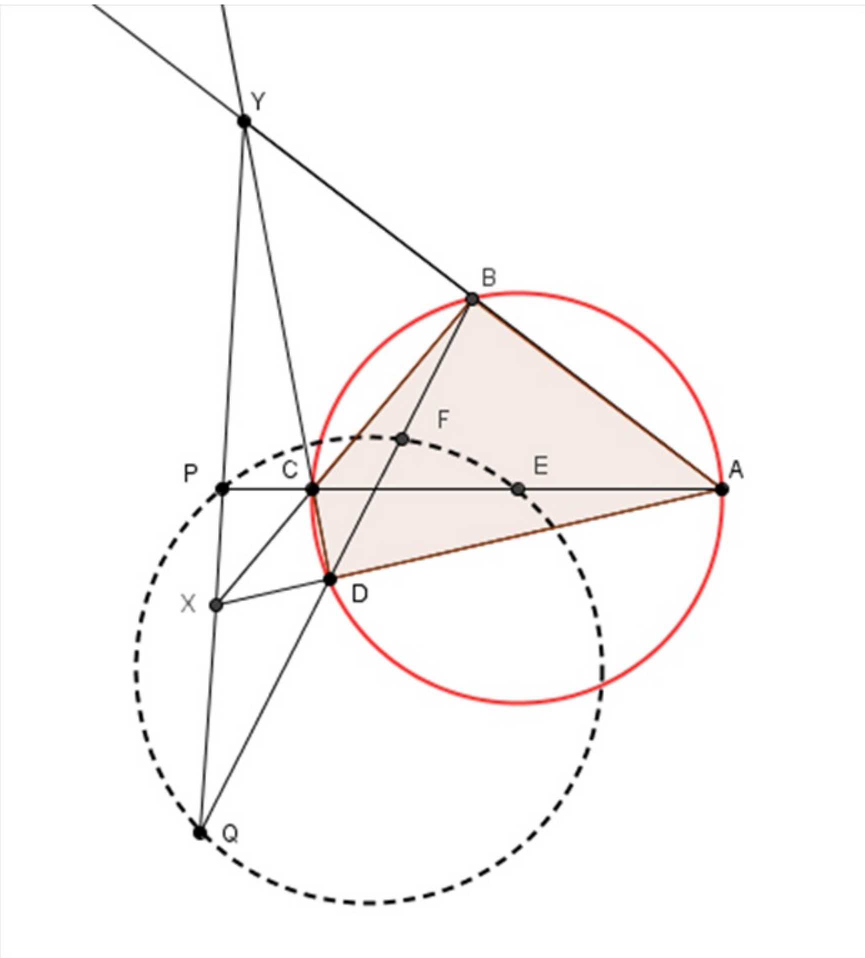
stergiu

#1 Dec 20, 2009, 10:23 pm

Let a cyclic quadrilateral $ABCD$. Lines AD, BC meet at X and lines AB, CD meet at Y. The lines AC, BD meet line XY at P, Q respectively. The midpoints of AC, BD are E, F respectively.Prove that points E, F, P, Q are concyclic.

Babis

Attachments:



Luis González

#2 Dec 21, 2009, 12:02 am

Midpoints F, E, M of AC, BD, XY are collinear on the Newton's line of the complete quadrangle $ABCD$. Circle (M) with diameter XY belows to the othogonal pencil defined by the axis XY and the circumcircle (O) of $ABCD$. Thus (M) is also orthogonal to the circle ω with diameter OZ, where $Z \equiv AC \cap BD$ is the pole of XY WRT (O). ω obviously passes through the orthogonal projections E, F of O on BD, AC

$\Longrightarrow MX^2 = MY^2 = p(M, \omega) = ME \cdot MF.$

On the other hand, notice that since cross ratio (X, Y, P, Q) is harmonic, due to Newton's theorem, we have $MX^2 = MY^2 = MP \cdot MQ$. Hence $ME \cdot MF = MP \cdot MQ$, i.e. E, F, P, Q are concyclic.

Petry

#3 Dec 21, 2009, 2:32 am

Hello!

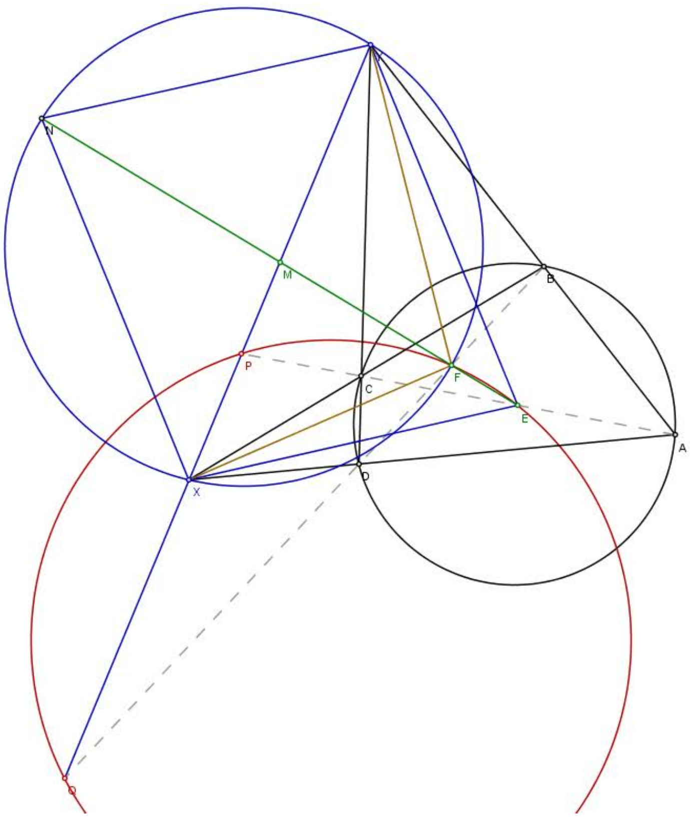
$ABCD$ is cyclic $\Rightarrow \angle DBX = \angle CAX \Rightarrow \Delta DBX \sim \Delta CAX \Rightarrow \dfrac{XD}{XC} = \dfrac{DB}{CA}$ and $\angle XDB = \angle XCA \Rightarrow$
$\Rightarrow \dfrac{XD}{XC} = \dfrac{DF}{CE}$ and $\angle XDF = \angle XCE \Rightarrow \Delta XDF \sim \Delta XCE \Rightarrow \angle XFD = \angle XEC$ (1)
Similarly $\angle YFB = \angle YEC$ (2)
(1),(2) $\Rightarrow \angle XFY = 180^\circ - (\angle XFD + \angle YFB) = 180^\circ - (\angle XEC + \angle YEC) =$
$= 180^\circ - \angle XEY \Rightarrow \angle XFY = 180^\circ - \angle XEY$ (3)

Let M be the midpoint of $[XY]$. It's known that the points E, F, M are collinear.
Construct the point N such that $XEYN$ is parallelogram $\Rightarrow$ M is the midpoint of $[EN]$
(3)$\Rightarrow \angle XFY = 180^\circ - \angle XEY = 180^\circ - \angle XNY \Rightarrow XFYN$ is cyclic $\Rightarrow$
$\Rightarrow \angle MXF = \angle YXF = \angle YNF = \angle MEX \Rightarrow \angle MFX = \angle MXE$ (4)

(1),(4)$\Rightarrow \angle EFQ = \angle EPQ \Rightarrow$ the points E, F, P, Q are concyclic.

Best regards, Petrisor Neagoe

Attachments:



Quick Reply

High School Olympiads

concyclic and perpendicular segments

Reply

stergiu
#1 Dec 17, 2009, 2:11 am

In thw figure below ΑΒΓΔ is iscribed , Λ is the midpoint of ΑΓ and Μ the midpoint of ΒΔ. Let Κ the orthocenter of triangle ΜΡΣ.Prove that

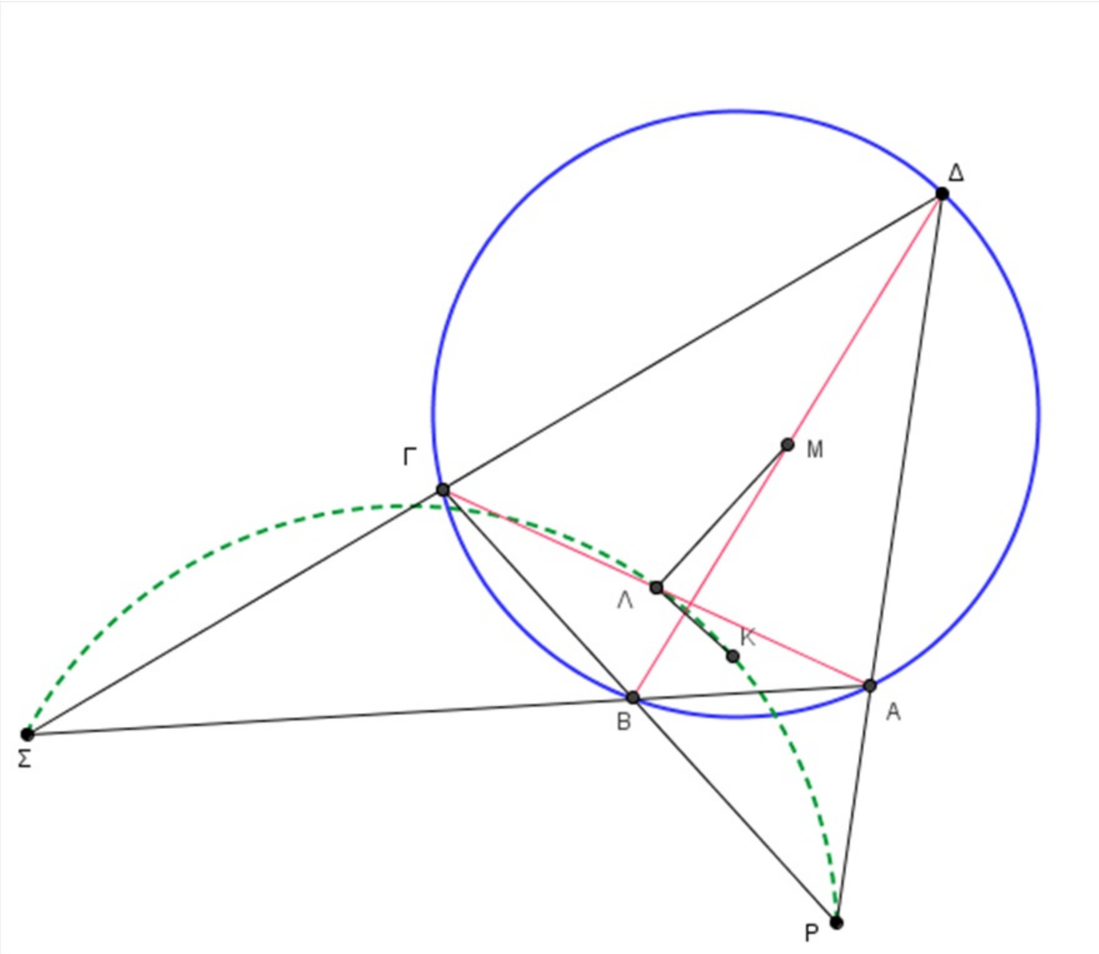
a) points Ρ, Σ, Κ,Λ are concyclic

b) Angles ΡΜΛ , ΛΚΣ are equal,

c) $\angle KLM = 90$, where L= Λ

Babis

Attachments:



Petry
#2 Dec 20, 2009, 2:21 am

Hello!

Let $ABCD$ be a cyclic quadrilateral, $\{P\} = AD \cap BC$, $\{Q\} = AB \cap CD$ and L, M are the midpoints of $[AC]$, $[BD]$. If K is the orthocenter of the triangle ΔMPQ then prove:
a) the points P, Q, K, L are concyclic

a) the points P, Q, K, L are concyclic
b) $\angle PML = \angle LKQ$
c) $\angle KLM = 90^\circ$

Solution:

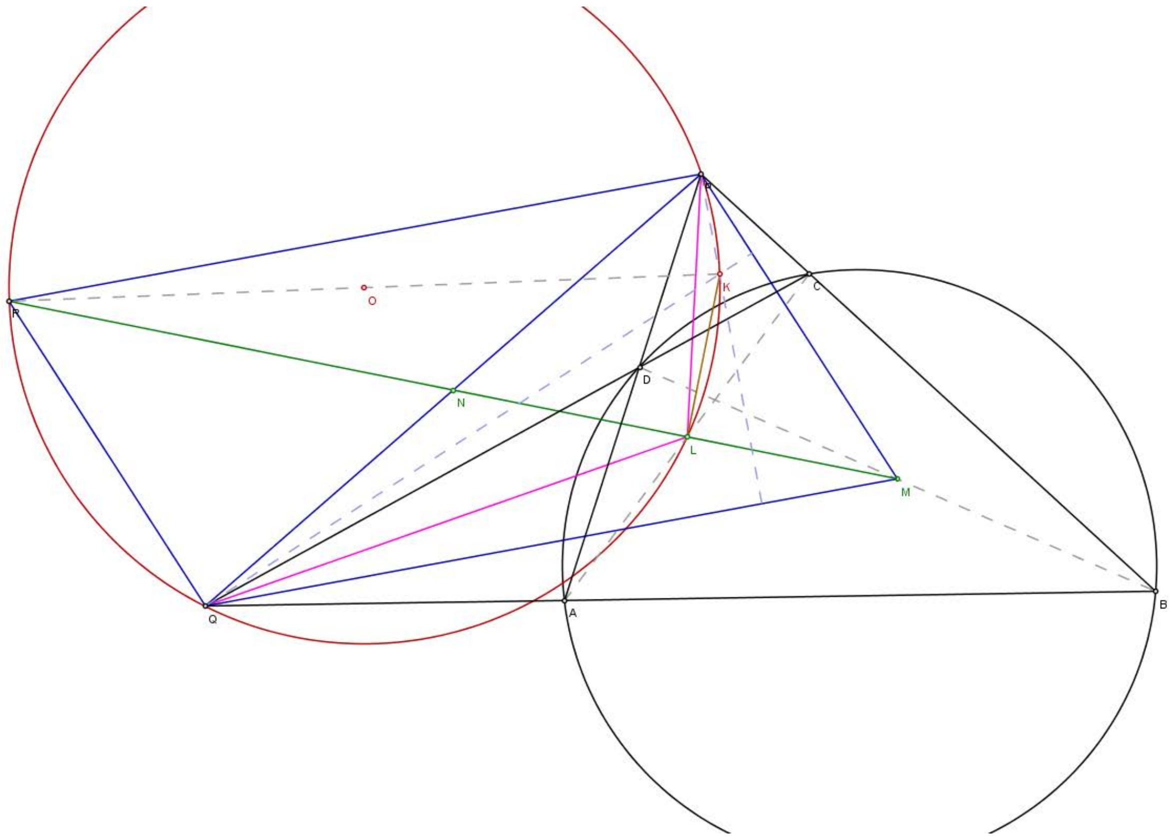
a)
$ABCD$ is cyclic $\Rightarrow \angle CAP = \angle DBP \Rightarrow \Delta CAP \sim \Delta DBP \Rightarrow \dfrac{PC}{PD} = \dfrac{CA}{DB}$ and $\angle PCA = \angle PDB \Rightarrow$
$\Rightarrow \dfrac{PC}{PD} = \dfrac{CL}{DM}$ and $\angle PCL = \angle PDM \Rightarrow \Delta PCL \sim \Delta PDM \Rightarrow \angle PLC = \angle PMD$ (1)
$ABCD$ is cyclic $\Rightarrow \angle ACQ = \angle DBQ \Rightarrow \Delta ACQ \sim \Delta DBQ \Rightarrow \dfrac{QA}{QD} = \dfrac{AC}{DB}$ and $\angle QAC = \angle QDB \Rightarrow$
$\Rightarrow \dfrac{QA}{QD} = \dfrac{AL}{DM}$ and $\angle QAL = \angle QDM \Rightarrow \Delta QAL \sim \Delta QDM \Rightarrow \angle QLA = \angle QMD$ (2)
(1),(2) $\Rightarrow \angle PLQ = 180^\circ - (\angle PLC + \angle QLA) = 180^\circ - (\angle PMD + \angle QMD) =$
$= 180^\circ - \angle PMQ \Rightarrow \angle PLQ = 180^\circ - \angle PMQ$ (3)
K is the orthocenter of $\Delta MPQ \Rightarrow \angle PKQ = 180^\circ - \angle PMQ$ (4)
(3),(4) $\Rightarrow \angle PLQ = \angle PKQ \Rightarrow$ the points P, Q, L, K are concyclic.

b)
Let N be the midpoint of $[PQ]$. It's known that the points L, M, N are collinear.
The points P, Q, L, K lie on the circle $\Gamma\,(O)$ and $\{R, K\} = KO \cap \Gamma$
$KP \perp PR$ and $KP \perp MQ \Rightarrow PR || MQ$ (5)
$KQ \perp QR$ and $KQ \perp MP \Rightarrow QR || MP$ (6)
(5),(6) $\Rightarrow MPRQ$ is parallelogram and N is the midpoint of $[MR]$
$\angle PML = \angle LRQ = \angle LPQ = \angle LKQ \Rightarrow \angle PML = \angle LKQ$.

c)
$\angle KLR = 90^\circ \Rightarrow \angle KLM = 90^\circ$.

Best regards, Petrisor Neagoe

Attachments:



Luis González

#3 Dec 20, 2009, 5:54 am

Lemma: In any triangle $\triangle ABC$, the orthocenter H, its projection onto the median AM and the vertices B, C are concyclic.

Let D, E be the feet of the altitudes from B and C. Denote $Q \equiv DE \cap BC$. Line AQ is polar of H WRT the circle (M) with
diameter BC, thus QH is the polar of A WRT (M) $\Longrightarrow AM \perp QH$. Let $P \equiv AM \cap QH$. Note that H is common

diameter BC, thus QH is the polar of A WRT $(M) \Longrightarrow AM \perp QH$. Let $R \equiv AM \cap QH$. Note that H is common orthocenter of $\triangle AQM$ and $\triangle ABC$, thus $\overline{RH} \cdot \overline{HQ} = \overline{DH} \cdot \overline{HB} = \overline{CH} \cdot \overline{HE}$. Hence, inversion with center H and power $\overline{HR} \cdot \overline{HQ}$ takes R, B, C into the collinear points $Q, D, E \Longrightarrow B, C, H, R$ are concyclic.

Midpoints M, L, N of BD, AC, PQ are collinear on the Newton's line of $ABCD$ and the circle (N) with diameter PQ belows to the orthogonal pencil defined by the axis PQ and the circumcircle (O) of $ABCD$. Thus (N) is orthogonal to the circle ω whose diameter is the segment joining O and the pole of PQ WRT (O). Obviously, ω passes through the orthogonal projections M, L of O on BD, AC. Therefore, we have $NP^2 = NQ^2 = NL \cdot NM \Longrightarrow \odot(MLP)$ and $\odot(MLQ)$ are tangent to PQ through P, Q, this is $\angle LPQ = \angle NMP$ and $\angle LQP = \angle NMQ \Longrightarrow$ $\angle PLQ = 180^\circ - (\angle NMP + \angle NMQ) = \angle PKQ \Longrightarrow L \in \odot(PQK) \Longrightarrow \angle PML = \angle LKQ$. Using the previous lemma in $\triangle MPQ$, L is identical to the projection of K onto MN.

Quick Reply

High School Olympiads

Pedal, Cevian and radical center

Reply

Source: Own

livetolove212

#1 Aug 25, 2009, 4:41 pm

Given triangle ABC with circumcircle $(O).P$ lies inside triangle ABC, AP, BP, CP intersect BC, CA, AB at A', B', C' respectively such that $A'B'C'$ is pedal triangle. $B'C' \cap (O) = \{A_1, A_2\}$, similar for $B_1, B_2, C_1, C_2.P'$ is isogonal conjugate of P wrt ABC. Prove that the radical center of three circles $(A'A_1A_2), (B'B_1B_2), (C'C_1C_2)$ lies on OP'

TRAN THAI HUNG

#2 Aug 25, 2009, 5:13 pm

What does "isogonal conjugate " mean?

livetolove212

#3 Aug 26, 2009, 3:05 pm

You can find it with google.

mihai miculita

#4 Aug 26, 2009, 3:55 pm

If R radical center, then: $R \notin OP'$.Something 's wrong, view figure...

Attachments:

Radical center....pdf (29kb)

vittasko

#5 Aug 26, 2009, 6:09 pm

I think that the problem is true since it states that $\triangle A'B'C$, as the **Cevian triangle** of P, is also the **Pedal triangle** of another point so be it Q, which (I see by the drawing) lies also on the line segment OP'.

An interested problem here is how to construct in general, points with the property as the problem states (such that their **Cevian triangle**, to be also the **Pedal triangle** of another point).

The only known to me points with this property, in the configuration of a given triangle, are the **Orthocenter** and the **Centroid** and the **Gegrgone point**.

Also, for every point which has the property as the proposed problem states, its **Isotomic point** with respect to the given triangle, has also the same property (easy to prove by **Ceva theorem** and **Carnot theorem**).

Kostas Vittas.

Attachments:

t=297530.pdf (21kb)

This post has been edited 1 time. Last edited by vittasko, Aug 27, 2009, 12:47 am

mihai miculita

#6 Aug 26, 2009, 8:57 pm

OK!, or if P=G(centroid), then $\triangle A'B'C'$ -is the pedal triangle of O.

Luis González
#7 Aug 26, 2009, 9:21 pm

> *vittasko wrote:*
> An interested problem here is how to construct in general, points with the property as the problem states (such that their **Cevian triangle**, to be also the **Pedal triangle** of another point.

Dear Kostas, the locus of the points that satisfy that their pedal triangle is also a cevian triangle is known as the Darboux cubic of the triangle ABC.

livetolove212
#8 Aug 27, 2009, 4:36 pm

> *mihai miculita wrote:*
> If R radical center, then: $R \notin OP'$.Something 's wrong, view figure...

Dear mihai miculita,
I think in your figure $A'B'C'$ is not pedal triangle!
Two special cases of this problem are at here:
P is **Gergone** point: http://www.mathlinks.ro/viewtopic.php?t=253661
P is **centroid**: http://www.mathlinks.ro/viewtopic.php?t=297152

vittasko
#9 Aug 27, 2009, 8:29 pm

Thank you dear **Luis** for the reference about the **Darboux cubic**.

So, there are some well known points outwardly of a given triangle and their **Isotomic points**, with the property as we want. But, is there an elementary way to construct points of this cubic?

Sometimes, although a locus is an unusual curve, however we can define points belong to it, by elementary way.

Best regards, Kostas Vittas.

Luis González
#10 Sep 3, 2009, 3:19 am • 1

Let $T \equiv B'C' \cap BC$ and let M_a be the midpoint of BC. From $(B, C, A', T) = -1$, it follows that $TA_1 \cdot TA_2 = TB \cdot TC = TM_a \cdot TA' \Longrightarrow M_a$ lies on $\odot(A'A_1A_a)$. Similarly M_b, M_c lie on the circles $\odot(B'B_1B_2)$ and $\odot(C'C_1C_2)$. Center O_a of $\odot(A'A_1A_2)$ is the intersection of the perpendicular bisector of $A'M_a$ with the perpendicular to A_1A_2 from O. Thus, O_a is the midpoint of OA'', where A'' is the intersection of the perpendicular to A_1A_2 from O with the perpendicular to BC at A'. Cyclically, O_b, O_c are midpoints of $OB'', OC'' \Longrightarrow \triangle O_aO_bO_c$ and $\triangle A''B''C''$ are centrally similar with factor 2.

Radical axis $\mathcal{L}_c$ of $(O_a), (O_b)$ goes through C' and is orthogonal to $O_aO_b \parallel A''B''$. Similarly, $\mathcal{L}_a, \mathcal{L}_b$ go through A', B' orthogonally to $B''C''$ and $A''C''$. If Q is the common point of the perpendiculars to BC, CA, AB at A', B', C', then $\triangle A'B'C'$ and $\triangle A''B''C''$ are perspective with perspector Q and orthologic, one orthology center is O and the other is the radical center of $(O_a), (O_b), (O_c) \Longrightarrow R \equiv \mathcal{L}_a \cap \mathcal{L}_b \cap \mathcal{L}_c$. By Sondat's theorem O, Q, R are collinear. ■

P.S. It remains to prove the following lemma:

Lemma: P is a point on the plane of $\triangle ABC$. $\triangle A'B'C'$ is the cevian triangle of P such that $\triangle A'B'C'$ is pedal triangle of Q. Then the isogonal conjugate P' of P lies on OQ, where O is the circumcenter of $\triangle ABC$.

Luis González
#11 Dec 20, 2009, 12:49 am

> *Quote:*
> Lemma: P is a point on the plane of △ABC. △A′B′C′ is the cevian triangle of P such that △A′B′C′ is pedal triangl

Lemma: P is a point on the plane of $\triangle ABC$. $\triangle A'B'C'$ is the cevian triangle of P such that $\triangle A'B'C'$ is pedal triangle of Q. Then the isogonal conjugate P' of P lies on OQ, where O is the circumcenter of $\triangle ABC$.

See the proof of this supporting lemma as a remark in the topic Concurrent 2 (PC point again). Hence the proof of the proposed problem above is finally completed.

Quick Reply

High School Olympiads

IMO Shortlist 2008, Geometry problem 4

Reply

Source: IMO Shortlist 2008, Geometry problem 4, German TST 5, P3, 2009

April

#1 Jul 10, 2009, 4:21 am • 3

In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the point A anf F and tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that lines PE and QF intersect on the circumcircle of triangle AEF.

Proposed by Davood Vakili, Iran

mr.danh

#2 Jul 10, 2009, 9:16 am • 5

> April wrote:
> In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the point A anf F and tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that lines PE and QF intersect on the circumcircle of triangle AEF.

Solution

Click to reveal hidden text

arash71

#3 Jul 19, 2009, 11:02 pm • 1

I have same soloutions as Mr.danh's.
I have a question everybody : Isnt G4 a little easier than G3?

kasym

#4 Dec 8, 2009, 3:34 pm

Thanks to Davood Vakili for a good problem.

mathVNpro

#5 Dec 17, 2009, 11:25 pm • 1

> Quote:
> In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the point A anf F and tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that lines PE and QF intersect on the circumcircle of triangle AEF.
>
> *Proposed by Davood Vakili, Iran*

Let H be the orthocenter of $\triangle ABC$ and D is the projection of A onto BC. Then consider the inversion through pole H, power $k = \overline{HA}.\overline{HD} = \overline{HE}.\overline{HB} = \overline{HC}.\overline{HF}$. We have $\mathcal{I}(H,k)$ maps P into P' and Q is taken into Q', combine with the result that $\mathcal{I}(H,k)$ maps BC into AEF. Thus, P' and Q' both lie on (AEF). Therefore, obviously, H will be also the orthocenter of $\triangle AQP$, and Q', P' respectively are the projections of P, Q onto AQ and AP.

Since AF is the radical axes of the two circles that pass through A, F and tangent to BC at P, Q respectively. Therefore, AF will goes to midpoint B of PQ. In the other hand, we also have $BP^2 = BQ^2 = \overline{BF}.\overline{BA} = \overline{BD}.\overline{BC}$. Then if C' is the intersection of $P'Q'$ with PQ, we have $(QPDC')$ is harmonic. And by *MacLaurin identity*, $BQ^2 = BP^2 = \overline{BD}.\overline{BC'}$, which implies C must coincide with C'. Or we can say P', Q' and C are collinear. From $(CDPQ)$ is harmonic, then $\overline{CA}.\overline{CE}$ $= \overline{CP}.\overline{CQ} = \overline{CD}.\overline{CB} \Longrightarrow E \in (AQP)$. The same argument holds for $F \in (HPQ)$.

Finally, we have $\angle(FQ, EP) = \angle(FQ, QP) + \angle(QP, EP) = \angle(HF, HP') + \angle(AQ, AE)$ $= \angle(AF, AP') + \angle(AP', AE) = \angle(AF, AE)$. Thus, if W is the intersection of EP with FQ, then $W \in (AEF)$. Our proof is completed then. $\square$

Luis González

#6 Dec 18, 2009, 6:17 am • 3

Let H be the orthocenter of $\triangle ABC$ and let D the foot of the A-altitude. B lying on the radical axis AF of both circles has equal power with respect to them. Thus $BQ^2 = BP^2 = BF \cdot BA = BH \cdot BE = BD \cdot BC \Longrightarrow$ circle $\odot(PHE)$ is tangent to BC at P, yielding $\angle EPQ = \angle BHP$ and also due to Newton's theorem, the cross ratio (D, C, P, Q) is harmonic $\Longrightarrow$ $CP \cdot CQ = CD \cdot CB = CH \cdot CF$, i.e. $FHPQ$ is cyclic $\Longrightarrow \angle FQP = \angle CHP$. Therefore if $R \equiv QF \cap PE$, we have $\angle QRP = \angle BAC \Longrightarrow R \in \odot(AFE)$.

RaleD

#7 Sep 7, 2010, 3:30 am • 1

Here is my solution:
First inversion with center A and radius AF. Now it is enough to prove that circumcircles of AP_1E_1 and AQ_1F intersects at point R which belongs to the line E_1F. This is equivalent to
$\angle AP_1E_1 = \angle AQ_1F$. We see $\angle AP_1E_1 = \angle P_1AF + \angle AFP_1 + \angle P_1E_1F + \angle P_1FE_1$
$= \angle B_1P_1F + \angle B_1FP_1 + \angle LP_1F + \angle P_1FL$

(here we used that FP is tangent to circumcircle of AQB and then it is also tangent to circumcircle of ELP_1 ; L belongs to E_1F_1 and AL is perpendicular to E_1F_1)
Because $\angle AQ_1P_1 = \angle B_1P_1F + \angle B_1FP_1$. So we see that we need to show
that P_1Q_1FL is cyclic. That is true and we'll prove this if we take S as midpoint of AH (H is orthocenter of AEF). It holds that $\angle SP_1F = \angle SQ_1F = 90 = \angle SLF$. Done!

AlexanderMusatov

#8 Jul 7, 2011, 3:17 pm • 1

We have to prove that $\widehat{BFQ} = \widehat{PEC}$.We have $BQ^2 = BA \cdot BF$,triangles BFQ, BQA are similar therefore $\widehat{BFQ} = \widehat{BQA}$.It remains to prove that $\widehat{PEC} = \widehat{BQA}$. Since $BP = BQ$ and $BP^2 = BD \cdot BA$,D is inverse of C wrt circle (B, BP) so we have $(C, D; P, Q) = -1$.By a known theorem we have $CP \cdot CQ = CD \cdot CB$,but $EABD$ is cyclic so $CE \cdot CA = CD \cdot CB = CP \cdot CQ$.Thus, $PEAQ$ is cyclic,and $\widehat{PEC} = \widehat{BQA}$.

jayme

#9 Jul 10, 2011, 6:00 pm

Dear Mathlinkers,
in order to have a more upper view for each problem, we can image that each proposed problem are built on a squeletton i.e. based on a well known situation.
In this case, I have though to the Boutin's theorem which originated a chain of well known results and conduct to the result in question.
This is only a personnal... to be developped...
Sincerely
Jean-Louis

Bertus

#10 Aug 15, 2011, 12:04 am

My Solution :
Consider X such that $X \equiv (FQ) \cap (PE)$. If X lies in (AEF) then $\angle XEA = \angle XFA \Leftrightarrow \angle CEP = \angle BFQ$.
Since (AFQ) is tangent to BC and (AFP) si tangent to BC then we have : $BQ^2 = BF.BA = BP^2$. Otherwise, we have : $\angle BFQ = \angle ABC - \angle FQC = \angle ABC - \angle BAQ$ and $\angle CEP = \pi - \angle ACB - \angle CPE$. Hence we are left to prove that : $\angle CAQ = \angle CPE$ which means it suffices to prove that $\triangle ACQ \sim \triangle CPE$.
From the fact that $\triangle CFB \sim \triangle CDH$ and $\triangle BHD \sim \triangle BEC$, we have : $\frac{BH}{\quad} = \frac{BD}{\quad}$ and $\frac{CH}{\quad} = \frac{CD}{\quad}$ which give us :

$BH.BE + CH.CF = BC(CD + BD) = BC^2$. But we have : $BP^2 = BF.BA = BH.BE$ and $CH.CF = CE.CA$, then we get : $CE.CA = BC^2 - BP^2 = (CB - BP)(CB + BQ) = CP.CQ$ and hence : $\frac{CA}{CQ} = \frac{CP}{CE}$ which means that $\triangle CAQ \sim \triangle CPE$ which give us the desired result.

Swistak

#11 Jun 7, 2012, 1:25 am • 1

It's obvious that if we prove that AEPQ is cyclic then we will be done.
Now take inversion in A which maps circle BCEF to itself and our thesis is equivalent to that Q', P', C are collinear, but this is well-known result about polars.

This post has been edited 1 time. Last edited by Swistak, Jun 8, 2012, 2:41 am

r1234

#12 Jun 7, 2012, 9:58 pm

Let us invert the figure wrt A with power $AF.AC = AE.AB$.So BC goes to $\odot AEF$.Now $\odot APF, \odot AQF$ goes to the tangents drawn from C to $\odot AEF$.So the inverses of P, Q (suppose P', Q').So we just need to show that the circles $\odot AP'B, \odot AQ'C$ concur on BC.For this note that $P'Q'$ is the polar of C wrt $\odot AEF$.So B, P', Q' are collinear. Now the rest is just angle chasing.

subham1729

#13 Jun 7, 2012, 10:29 pm

Let M be the intersection point of QF and PE. We need to prove that □QMP = □BAC. Since □MQP = □QAB (QB is a tangent to the circle around □QFA),
it is enough to prove that □QAB+□BAC = □QMP+□MQP, or equivalently,
□QAE =□EPC. Thereforewe need to prove that AQPE is a cyclic quadrilateral.
From $BQ^2 = BF.BA = BP^2$ we get BP = BQ. Adding $BF.BA = BP^2$ to AF · AB = AE ·AC (which holds since BCEF is cyclic) we get $AB^2 = AE.AC + BP^2$.
From the Pythagorean theorem we have $AB^2 = AE^2 + BE^2 = AE^2 + BC^2 - CE^2$,
from which we get $BC^2 - CE^2 = AE.EC + BP^2$ This implies that $BC^2 - BP^2 = CE2 + AE.EC$, or equivalently CE . (CE +AE) = (BC+BP)(BC−BP) =CQ·CP.
Thus CE ·CA =CP·CQ and QPEA is cyclic.

Bigwood

#14 Jun 11, 2012, 5:35 pm

I used trigonometry to prove that $QD : DH = AD : DP \Leftrightarrow QH \perp AP$, so H is the orthocenter of $\triangle APQ$, thus $PH \perp AQ$. $BP^2 = BF \cdot BA = BH \cdot BE$ implies $\angle BPH = \angle BEP$, and saecula saeculorum says $\angle QFB = \angle AQB$. Let the intersection be X, and get $\angle XFH = \frac{\pi}{2} - \angle AQB = \angle HPB = \angle BEP$. Thus F, H, E, X cyclic.

AnonymousBunny

#15 May 30, 2014, 10:48 am

Click to reveal hidden text

nima1376

#16 Jun 5, 2014, 4:27 pm • 1

let D is foot of perpendicular from A on BC.
$BQ^2 = BF.BA = BD.BA = BC^2 - CD.BC \Rightarrow CE.CA = CD.BC = BC^2 - BQ^2 = BC^2 - BP^2 = CP.CQ$
so $AEPQ$ is cycle.
$\widehat{QAB} = \widehat{FQB}, \widehat{EPC} = \widehat{A} + \widehat{QAB} \Rightarrow$ lines PE and QF intersect on the circumcircle of triangle AEF.
done

JuanOrtiz

#17 Jun 12, 2014, 9:00 pm • 1

Define H as orthocenter and D such that AD is altitude. Let H' be reflection of H across BC. Firstly notice that by power of point we have

$BF \times BA = BD \times BC = DB \times (DC + DB) \Rightarrow BP^2 - BD^2 = (BF \times BA) - BD^2 = DB \times DC = DH' \times DA$

but $BQ = BP$ because $BP^2 = BF \times BA = BQ^2$. So $BP^2 - BD^2 = PD \times DQ$ and so

$PD \times DA = DA \times DH'$. From this we see that H is the orthocenter of APQ.

Notice that $BH \times BE = BF \times BA = BP^2$ and so the circumcircle of PEH is tangent to BC. Therefore

$\angle EPB = 180 - \angle EHP$.

The rest is just angle chasing. Also, we see $\angle BQF = \angle FAQ$. Since $PH \perp AQ$ and $BE \perp AC$ we see that

$\angle EPB = 180 - \angle CAQ = 180 - \angle CAB - \angle FQB$

and so the lines PE and QF for an angle of $\angle CAB$, so we are done.

shinichiman

#18 Jun 14, 2014, 11:55 am

Suppose that QF intersects (AEF) at L. We will prove E, L, P are collinear. We have
$BQ^2 = BP^2 = BF \cdot BA = BD \cdot BC$ so $BQ = BP$ and $BP^2 = BD \cdot BC$.
Let K be the orthorcenter of triangle ABC and AD is the altitude. $AL \cap BC = R$ then $AR \perp KL$. Therefore $KDRL$ is cyclic. Hence $AL \cdot AR = AK \cdot AD = AF \cdot AB = AE \cdot AC$. It follows that $BFLR$ and $LECR$ are cyclic.
Since $BFLR$ is cyclic then by power of point we get $QO^2 - OK^2 = QF \cdot QL = QB \cdot QR$. We have

$$\begin{aligned} QO^2 - OK^2 &= QO^2 - (OD - DK)^2 \\ &= QD^2 + DK \cdot DA \\ &= (QB + BD)^2 + DB \cdot DC \\ &= QB^2 + 2QB \cdot BD + BD \cdot BC \\ &= 2QB \cdot BD \end{aligned}$$

Thus, $2BD = QR$ or D is the midpoint of QR.
Now we need to prove $\overline{PR} \cdot \overline{PC} = PO^2 - OK^2$. Similarly, we obtain $PO^2 - OK^2 = PD^2 - \overline{DB} \cdot \overline{DC}$.
Hence,

$$\begin{aligned} \overline{PR} \cdot \overline{PC} = PO^2 - OK^2 &\iff \overline{PR} \cdot \overline{PC} = PD^2 - \overline{DB} \cdot \overline{DC} \\ &\iff (\overline{PR} + \overline{DP}) \cdot \overline{PC} = \overline{DP} \cdot (\overline{DP} + \overline{PC}) - \overline{DB} \cdot \overline{DC} \\ &\iff \overline{DR} \cdot \overline{PC} = \overline{BP} \cdot \overline{DC} \\ &\iff \overline{QD} \cdot \overline{PC} = \overline{QB} \cdot \overline{DC} \\ &\iff (\overline{QB} + \overline{BD}) \cdot \overline{PC} = \overline{QB} \cdot \overline{DC} \\ &\iff \overline{BD} \cdot \overline{PC} = \overline{QB} \cdot \overline{DP} \qquad (1) \end{aligned}$$

Since B, D are the mid point of QP and QR then $\overline{PR} = \overline{QR} - \overline{QP} = 2(\overline{DR} - \overline{BP}) = 2(\overline{PR} - \overline{BD})$. It follows that $\overline{PR} = 2\overline{BD}$.
Thus, $(1) \iff \overline{PR} \cdot \overline{PC} = \overline{QP} \cdot \overline{DP}$.
We have

$$\begin{aligned} BP^2 = \overline{BD} \cdot \overline{BC} &\implies 2BP^2 = \overline{PR} \cdot (\overline{BP} + \overline{PC}) \\ &\iff 2\overline{BP}(\overline{BP} - \overline{BD}) = \overline{PR} \cdot \overline{PC} \\ &\iff \overline{PR} \cdot \overline{PC} = \overline{QP} \cdot \overline{DP} \end{aligned}$$

So (1) is true or $\overline{PR} \cdot \overline{PC} = PO^2 - OK^2$.
$PE \cap (AEF) = L'$ then $\overline{PL'} \cdot \overline{PE} = \overline{PR} \cdot \overline{PC} = PO^2 - OK^2$. It follows that $EL'RC$ is cyclic. Thus, $L' \equiv L$ or E, L, P are collinear.

bonciocatciprian

#19 Jun 17, 2014, 11:34 pm • 1

Let $\{M\} = QF \cap EP$. We observe that $\odot AFE = \odot FEH$, since $AFHE$ is cyclic. So, we need to prove that $FEMH$ is cyclic, namely that $\angle HEM \equiv \angle HFM$. Because $\odot FAP$ and $\odot FAQ$ are tangent to BC, we get $BQ^2 = BP^2 = BF \cdot BA \Rightarrow BP^2 = BH \cdot BE \Rightarrow \Delta BHP \sim \Delta BPE \Rightarrow \angle BEP \equiv \angle HPB$. So, we must have $\angle HPQ \equiv \angle MFH$, namely that $QFHP$ is cyclic. This reduces to
$CH \cdot CF = QC \cdot PC \Leftrightarrow BC \cdot AC \cos(\widehat{ACB}) = QC \cdot PC$, which is equivalent to
$BC \cdot AC \cos(\widehat{ACB}) = (BC + BP)(BC - BP) = BC^2 - BF \cdot BA \Leftrightarrow$
$BC \cdot AC \cos(\widehat{ACB}) = BC^2 - BC \cdot BA \cos(\widehat{ABC})$, which is obvious (after reducing the factor BC).

junioragd

#20 Aug 2, 2014, 2:14 am

First,observe that BQ and BP are tangents,so we have that <BQA=<BFQ.Now,we need to prove that <BFQ+<PEA=180,whic is equivalent showing that <BQA+PEA=<PBA+PEA=180,so we prove that QPEA is a cyclic.Now,let D be the foot of the perpendicular of A on to BC.CP*CQ=CB*CB-BP*BP=CB*CB-BF*BA=CB*CB-BD*CB=CB*CD=CE*CA,so we are finished.

sayantanchakraborty

#21 Oct 24, 2014, 11:26 am

Note that
$CP \cdot CQ = (CB - BP)(CB + BQ) = CB^2 - BP^2 = CB^2 - BF \cdot BA = CB^2 - BD \cdot BC = CD \cdot BD = CE \cdot CA$
so $PQAE$ is cyclic.So $\angle PQA = \angle PEC = \angle AEJ$.Also we have $\angle PQA = \angle BQA = \angle AFJ$.Thus $AFEJ$ is also cyclic proving the statement of the problem.

Gibby

#22 Dec 3, 2014, 8:04 am

Can somebody please explain to me why (DCPQ) is harmonic?

Legend-crush

#23 Jan 11, 2015, 3:51 am

My solution:
it suffices to show that $\angle PTQ = \angle A \Leftrightarrow \angle EPQ = \pi - \angle EAQ \Leftrightarrow \ AEPQ$ cyclic
Consider inversion of center A and of radius $\sqrt{AE.AC} = \sqrt{AF.AB}$. the image of M is M'
(BC) becomes $(AB'C'P'Q') = \omega$. circles (APF) and AFQ become lines tangeant to ω at P' and Q' . Let D' be the reflexion of A wtrt the center of ω .
We have to prove that E', P', Q' are collinear.
since $E' = AC' \cap B'D'$ and $F' = C'D' \cap AB'$. E' lies in the polar of F'. but obviously F' is the intersection of tangeants from P' and Q'. hence this polar is just $P'Q'$
Thus E', P', Q' are collinear, equivalently A, E, P, Q is cyclic $\Leftrightarrow$ $AETF$ cyclic

aditya21

#24 Apr 3, 2015, 4:56 pm

my solution =

we have AF as radical axis of $\odot AFQ$ and $\odot AFP$ so B has equal powers with both
and hence $BF.BA = BP^2 = BQ^2$

now let H be orthocentre of triangle ABC.
than $BF.BA = BH.BE$ as $AFHE$ is cylic quad.

now $CP.CQ = BC^2 - BP^2 = CE^2 + BE^2 - BH.BE = 4R^2 sinAsinBcosC$ after using some trigonometry with
$BE = 2RsinAsinC, BH = 2RcosB, CE = 2RsinAcosC$

similarly we get $CE.CA = 4R^2 sinAsinBcosC$ again using some trigonometry.

so $CE.CA = CP.CQ$ so by POP we have $AEPQ$ is cyclic quad.

now also let $QF \cap PE = M$
so $\angle QAC = \angle MPC = \angle QMP + \angle MQP$

also $\angle QAC = \angle QAB + \angle BAC = \angle QMP + \angle MQP$
since BQ is tangent to $\odot AFQ$
so $\angle QAB = \angle FQB$

thus we get $\angle BAC = \angle QMP = \angle EMF$
and thus A, F, E, M are concyclic.

and we are done

This post has been edited 1 time. Last edited by aditya21, Apr 3, 2015, 4:59 pm
Reason: ed

watosla
#25 Apr 4, 2015, 1:19 am

thank you mr vakili

jayme
#26 Apr 4, 2015, 3:19 pm

Dear Mathlinkers,

1. (1) the circle with center at B and orthogonal to circle (AEF)
2. U, V the points of intersection of (1) and (AEF)
3. according to a degenerated case of the Pascal theorem, UV goes through C
4. according to the three chords theorem, A, E, P, Q are concyclic
5. according to the pivot theorem we are done.

Sincerely
Jean-Louis

utkarshgupta
#27 Apr 4, 2015, 5:25 pm

Let $S = QF \cap PE$

Lemma : $PEAQ$ is cyclic
$CE \cdot CA = a\cos C \cdot b = ab \cdot \cos C$
$CP \cdot CQ = (CB - CP) \cdot (CB + BQ) = (a + \sqrt{BF \cdot BA})(a - \sqrt{BF \cdot BA} = a^2 - BF \cdot BA = a^2 - ac\cos B$
$\implies CE \cdot CA = CP \cdot CQ$
$\implies PEAQ$ is cyclic.

Now let $\angle QFB = x$ and $\angle BQF = y$
$\implies BAQ = y$ (alternate segment theorem)
$\implies \angle CPE = A + y$

Now using angle sum property in triangle BQF and CPE
$\implies \angle BFQ = \angle CEP = x$
$\implies \angle SFA = \angle SEA$
$\implies AFES$ is concyclic

Thus S lies on $\odot AFE$

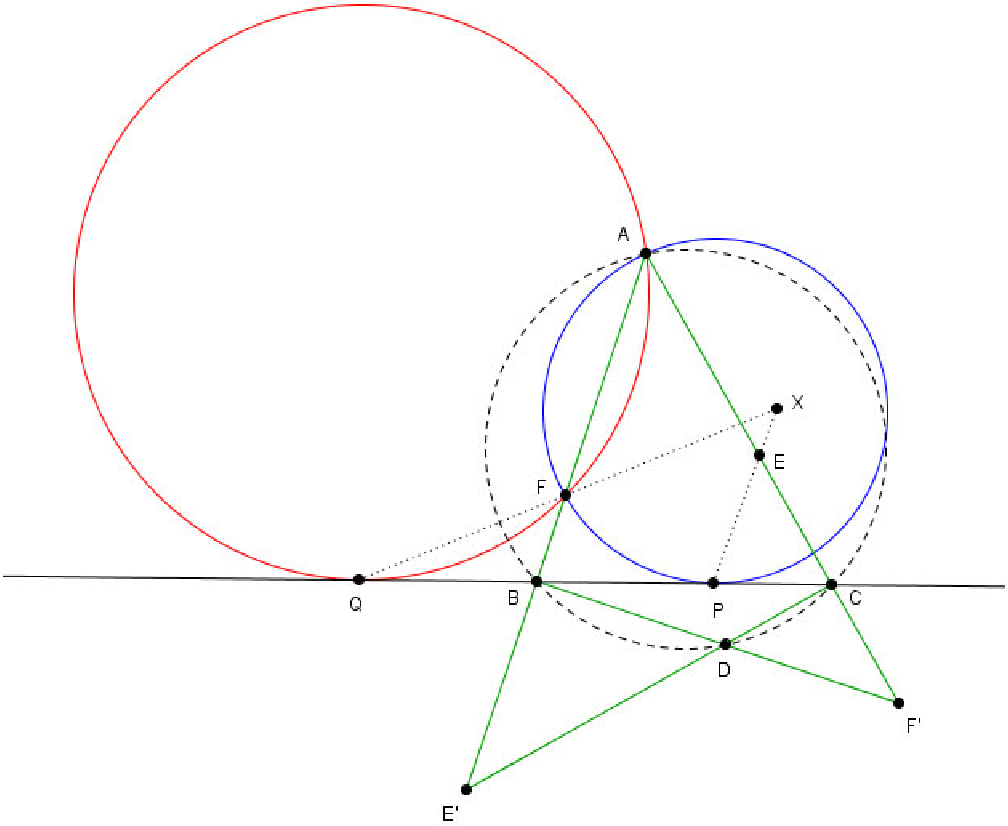
Dukejukem
#28 May 8, 2015, 10:13 am

Denote $X \equiv PE \cap QF$ and let the lines through B, C perpendicular to AB, AC meet AC, AB at F', E', respectively. Let BF' and CE' meet at D, and note that D is the antipode of A w.r.t. $\odot(ABC)$ because $AB \perp DB$ and $AC \perp DC$.

Note that since PQ is tangent to $\odot(AFQ)$, we have $\measuredangle AFQ = \measuredangle AQP$, where the angles are directed. Now, we claim that A, E, P, Q are concyclic. For the moment, let us assume this is the case, so that we may write

$$\measuredangle AFX = \measuredangle AFQ = \measuredangle AQP = \measuredangle AEP = \measuredangle AEX,$$

which implies that the points A, E, F, X are concyclic, as desired. Therefore, we need only show that A, E, P, Q are concyclic.



To see this, consider a transformation $\mathcal{T} : Z \mapsto Z'$, composed of an inversion with radius $\sqrt{bc}$, combined with a reflection in the A-angle bisector. It is easy to see that $B' = C$ and $C' = B$. Therefore, $\mathcal{T}$ takes the line BC to $\omega \equiv \odot(ABC)$. Furthermore, it is clear that E' lies on AB, and satisfies $\angle ACE' = \angle AB'E' = \angle AEB = 90^\circ$, so we have already marked the image of E under $\mathcal{T}$ on our diagram. Similarly, we have already marked the image of F. Now, because $\odot(AFQ)$ is tangent to BC, it follows under inversion that $F'Q'$ is tangent to ω. Similarly, $F'P'$ is tangent to ω. Therefore, the line $P'Q'$, which is just the image of $\odot(APQ)$ under $\mathcal{T}$, is nothing other than the polar of F' w.r.t. ω. Thus, in order to prove that $E \in \odot(APQ)$, we need only prove that $E' \in P'Q'$, or equivalently, that E' lies on the polar of F' w.r.t. ω. But this follows immediately from Brokard's Theorem (page 3) applied to cyclic quadrilateral $ABDC$. $\square$

This post has been edited 2 times. Last edited by Dukejukem, May 8, 2015, 10:20 am
Reason: Minor edits

Dukejukem

#29 May 8, 2015, 8:02 pm

Here is an even more efficient solution: Denote $X \equiv PE \cap QF$, and let D be the foot of the altitude from A to BC. Then note that $AD \perp CD$ and $AF \perp CF$, so A, C, D, F are inscribed in a circle of diameter $\overline{AC}$. Because B lies on the radical axis of $\odot(AFQ)$ and $\odot(AFP)$, it follows from Power of a Point that

$$BP^2 = BQ^2 = BF \cdot BA = BD \cdot BC.$$

Then since B is the midpoint of $\overline{PQ}$, it is well-known (Lemma 1.5) that this relation implies that $(Q, P; D, C)$ is a harmonic division. Because A, B, D, E are inscribed in a circle of diameter $\overline{AB}$, we may use Lemma 1.5 again to find that

$$CP \cdot CQ = CD \cdot CB = CE \cdot CA,$$

where the last equality follows from Power of a Point. Therefore, A, E, P, Q are concyclic, and we may proceed as in the above solution:

> *Dukejukem wrote:*
> Note that since PQ is tangent to $\odot(AFQ)$, we have $\measuredangle AFQ = \measuredangle AQP$, where the angles are directed. Now, we claim that A, E, P, Q are concyclic. For the moment, let us assume this is the case, so that we may write
>
> $$\measuredangle AFX = \measuredangle AFQ = \measuredangle AQP = \measuredangle AEP = \measuredangle AEX,$$
>
> which implies that the points A, E, F, X are concyclic, as desired. Therefore, we need only show that A, E, P, Q are concyclic.

DrMath

#30 Apr 10, 2016, 2:44 am • 1

Note that $BP^2 = BF \cdot BA = BD \cdot BC$ where D is the foot of the altitude from A to BC. Note that by radical axis, $BP = BQ$. On the other hand,
$CP \cdot CQ = (CB - BP)(CB + BP) = BC^2 - BP^2 = BC(BC - BD) = CD \cdot CB = CE \cdot CA$. Thus $AEPQ$ is cyclic.

What we want to show is $\angle FQB + \angle BPE = 180 - A$. Indeed, if the intersection of PE and QF is X, this means $\angle EFX = 180 - A$. But note $\angle FQB + \angle BPE = \angle QAF + 180 - \angle QAC = 180 - \angle BAC$ as desired.

Quick Reply

High School Olympiads

the most nice problem in the world

Reply

MJ GEO

#1 Dec 16, 2009, 11:17 pm

prove that in the all of equailateral triangles that the vertex of this triangles lies on AB, AC, BC the $F_1F_2F_3$ has the less perimeter that F_1 is the projection of conjugate of F that F is the Fermat point in the triangle ABC (and F_2, F_3 like that) (Fermat point is a piont that sees the sides of triangle with equal angles)

Luis González

#2 Dec 16, 2009, 11:56 pm

MJ GEO, please give your posts meaningful subject. The subject has to say what the problem is about in best way possible. This is in fact a requirement for being able to search old problems. As for this problem check issue #13 of IMO Shortlist 93 for a proof and further results (http://www.artofproblemsolving.com/Forum/viewtopic.php?t=80880).

MJ GEO

#3 Dec 17, 2009, 8:11 pm

thank for your refrence,but i cant find anything between too problems.now see my solution.first its easy to prove that $F_1F_2F_3$ is equailateral and then prove that F_1F_2 is per perpendicular to AF and the others like that (F_1 is on AB and F_2 is on AC) now if F be the Fermat point then $S(ABC) = S(AF_1FF_2) + S(BF_1FF_3) + S(CF_3FF_2)$ so $S(ABC) = (AF + BF + CF)(x)$ that x is F_1F_2, F_2F_3, F_1F_3 and its trivial that $AF + BF + CF < AK + BK + CK$ that K is an arbitray point in triangle and the problem is solve.oh im sorry too much because of writing wrong problem,we must prove $F - 1F_2F_3$ has the less surface

Luis González

#4 Dec 18, 2009, 3:11 am

Let $\triangle A'B'C'$ be an arbitrary equilateral triangle circumscribed in $\triangle ABC$. Circumcircles $(X), (Y), (Z)$ of $\triangle A'CB$, $\triangle B'AC, \triangle C'BA$ concur at the Miquel point of $\triangle A'B'C' \cup ABC$, i.e. the 1st Fermat point F of $\triangle ABC$. Lines FA, FB, FC are pairwise radical axes of the circles $(X), (Y), (Z)$. Then $\triangle XYZ$ is equilateral whose sidelines are the perpendicular bisectors of FA, FB, FC. Let M, N be the projections of Z, Y on $B'C'$ and Z' the projection of Y on ZM. If $Z \not\equiv Z'$, then $YZ \geq YZ' = MN = \frac{1}{2}B'C'$ and we get similar expressions cyclically. Thus, $\triangle A'B'C'$ attains its maximum area (perimeter) if and only if $\triangle A'B'C'$ is centrally similar to $\triangle XYZ$. That is, if $\triangle A'B'C'$ is centrally similar to the pedal triangle $\triangle F_1F_2F_3$ of the 1st Isodynamic point F' of $\triangle ABC$.

For each equilateral triangle $\triangle DEF$ inscribed in $\triangle ABC$ there exists another equilateral triangle $\triangle D'E'F'$ circumscribed in $\triangle ABC$ and centrally similar to $\triangle DEF$. By Gergonne-Ann theorem we have $[ABC]^2 = [DEF] \cdot [D'E'F'] \Longrightarrow [DEF]$ attains its minimum iff $[D'E'F']$ attains its maximum, i.e. if $\triangle D'E'F'$ is centrally similar to the pedal triangle of F' WRT $\triangle ABC$. Then $\triangle DEF$ is identical with $\triangle F_1F_2F_3$.

Quick Reply

High School Olympiads

Geometry marathon!

Reply

Source: Everyone likes geometry! Right?

Poincare

#1 Dec 15, 2009, 5:39 am

This is just what is says in the title; a geometry marathon.

NOTE: This is a geometry marathon, not a trig marathon, pure trig problems (as in trig sequences, proving trig equations) aren't allowed, but you are very welcome to use trig in your solutions. Also, please number the problems since every 15 problems, I'm going to make a PDF out of em.

If a problem is clogging up the marathon for more than 2 days, and the poster hasn't posted a solution despite a PM (which will be sent out after the first day), I'll post a new problem.

1. In a triangle ABC, median BD is such that angles A and DBC are equal, and angle ADB = 45°, prove that angle A = 30°.

Teensy hint

mathwizarddude

#2 Dec 15, 2009, 6:12 am

Man I suck at Euclidean geometry... I stared at the diagram trying to do some construction work but failed...
trig bash
A synthetic solution is appreciated
Problem 2: Try the last two unsolved problems in this thread http://www.artofproblemsolving.com/Forum/viewtopic.php?t=310548

Poincare

#3 Dec 15, 2009, 6:32 am

> mathwizarddude wrote:
>
> Man I suck at Euclidean geometry... I stared at the diagram trying to do some construction work but failed...
> trig bash
> A synthetic solution is appreciated
> Problem 2: Try the last two unsolved problems in this thread http://www.artofproblemsolving.com/Forum/viewtopic.php?t=310548

One problem please. (those look a bit hard, are you sure they fit into intermediate? One may think they're the ones clogging your marathon!)

Luis González

#4 Dec 15, 2009, 7:28 am

Alternate solution to problem 1.

Because of $\triangle BDC \sim \triangle ABC$, we have $\frac{BC}{DC}=\frac{AC}{BC} \Longrightarrow BC^2=DC\cdot AC=2\cdot DC^2 \Longrightarrow BC=\sqrt{2}DC$. If X,Y denote the orthogonal projections of D and B onto BC and AC, we have $\frac{BC}{DC}=\frac{BY}{DX}=\sqrt{2}$. Since $\triangle BYD$ is isosceles with apex Y, then $BY=\frac{\sqrt{2}}{2}BD \Longrightarrow DX=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{2}BD=\frac{1}{2}BD \Longrightarrow \angle CBD=\angle BAC=30^\circ$.

Poincare

#5 Dec 15, 2009, 8:10 am

New problem:

http://www.gogeometry.com/problem/problem002.htm

Luis González

#6 Dec 16, 2009, 6:23 am

> gogeometry wrote:
>
> Prove the following: if, in a triangle $\triangle ABC, \angle BAO = \angle OAC = 20^\circ, \angle ACO = 10^\circ$, and $\angle OCB = 30^\circ$, then $\angle OBC = 80^\circ$. Try to use elementary geometry (Euclid's Elements.)

Let the reflection of CA across CO cut BA at P and let $Q \equiv AO \cap BC$. Then we have $\angle OCP = \angle OCA = 10^{\circ}$ and $\angle PCB = \angle QAB = 20^\circ$. Since $\triangle BAC$ is isosceles with apex B, it's clear that $PQ \parallel AC \Longrightarrow \angle QPC = \angle PCA = 20^\circ$. On the other hand, since CO and AO are bisectors of $\angle PCA$ and $\angle PAC$, it follows that O is the incenter of $\triangle APC \Longrightarrow PO$ bisects $\angle APC \Longrightarrow \angle APO = \frac{1}{2}(180^\circ - 40^\circ - 20^\circ) = 60^\circ$. But $\angle BQA = 40^\circ + 20^\circ = 60^\circ$, thus the quadrilateral $PBQO$ is cyclic $\Longrightarrow \angle OBC = \angle OPQ = 60^\circ + 20^\circ = 80^\circ. \blacksquare$

Problem 3: $ABCD$ is a rhombus such that $\triangle ABD$ is equilateral. An arbitrary line ℓ through C cuts the extension of its sides AB and AD at M and N, respectively. Show that the lines BN and DM meet on the circumcircle of $\triangle ABD$.

Hint for problem 3

Poincare

#7 Dec 17, 2009, 9:30 am

Okay. I promise we'll come back to your problem, I don't have time to solve it, for now. I'll solve it as soon as I get home from school tomorrow.

New no. 3:
in right triangle ABC, BF is a median from B (that is where the right angle is) to the hypotenuse, BE bisects angle ABC and BD is an altitude from B to the hypotenuse (AC). Prove angle DBE equals angle EBF.

Hint

Yavor Tsvetanov 52

#8 Dec 17, 2009, 8:19 pm

solution

Problem 4.:
The incircle of a triangle ABC is tangent to BC,CA,AB at M,N and P, respectively. Show that
the circumcenter and incenter of tr.ABC and the orthocenter of tr.MNP are collinear.

Luis González

#9 Dec 17, 2009, 10:25 pm

> Yavor Tsvetanov 52 wrote:
>
> Problem 4.: The incircle of a triangle ABC is tangent to BC,CA,AB at M,N and P, respectively. Show that the circumcenter and incenter of tr.ABC and the orthocenter of tr. MNP are collinear.

The inversion with respect to the incircle (I) of $\triangle ABC$ takes the vertices A, B, C into the midpoints A', B', C' of MP, PM, MN. Hence the circumcircle (O) is taken into the circle $\odot(A'B'C')$, which is the 9-point circle of $\triangle MNP$. Since the center of the inversion is collinear with the centers of the circle and its inverse image, it follows that the 9-point center of $\triangle MNP$ lies on $IO \Longrightarrow IO$ is Euler line of $\triangle MNP$, thus the orthocenter of $\triangle MNP$ lies on IO.

Next problem is the unsolved problem 3. A hint can be seen in the post number #6

Quick Reply

High School Olympiads

Perpendicular segments

Reply

Source: Baltic way - 2009

stergiu

#1 Dec 16, 2009, 9:54 pm

In a quadrilateral $ABCD$ we have $AB||CD$ and $AB = 2CD$. A line ℓ is perpendicular to CD and contains the point C. The circle with centre D and radius DA intersects the line ℓ at points P and Q. Prove that $AP \perp BQ$.

Luis González

#2 Dec 17, 2009, 8:59 am

Let E be the reflection of D across ℓ. The quadrilateral $ABED$ is clearly a parallelogram. Thus, (D) and (E) centered at D, E with radii DA, EB are symmetrical about their radical axis $\ell \equiv PQ$. If M is the second intersection of (D) with AB, then PQ bisects $\angle MQB$. Hence $\angle PQB = \angle PQM = \angle PAM$. If $R \equiv PQ \cap AB$ and $S \equiv AP \cap BQ$, then $ARSQ$ is cyclic $\Longrightarrow \angle ARQ = \angle ASQ = 90^\circ$.

dgreenb801

#3 Dec 21, 2009, 3:48 am

Extend AD and BC to meet at E. So $AD = DE, CE = CB$, and AE is the diameter of the circle with center D. Thus, $AP \perp PE$, so we have to show $PE \parallel BQ$. But $CP = PQ$ (any perpendicular from the center of a circle to a chord bisects the chord), $CE = CB$, and $\angle PCE = \angle BCQ$, so $\triangle PCE \cong \triangle QCB$, so $PE \parallel BQ$.

[geogebra]87ff73cbcae5d9a74c74ee0c1d5e7225df25739c[/geogebra]

Quick Reply

High School Olympiads

hard and very nice

Reply

MJ GEO

#1 Dec 16, 2009, 10:43 pm

the circle w is tangent to BA, CA and circumcirle of triangle ABC at P, Q, T.if X_a be the tangency of excircle of vertex A prove that the angles BAT and $CAX-a$ are equal

Luis González

#2 Dec 17, 2009, 12:39 am • 3

Since A is the exsimilicenter of the incircle (I) of $\triangle ABC$ and the A-mixtilinear incircle $\omega,$ it follows that A is also the center of the positive inversion $\mathcal{I}$ taking (I) into $\omega,$ hence $\mathcal{I}$ takes the circumcircle (O) of $\triangle ABC$ into a tangent line ℓ of (I) at the inverse T' of $T.$ ℓ is then orthogonal to $AO \Longrightarrow \ell$ is antiparallel to $BC.$ Thus, $\triangle ABC$ and $\triangle(AC, AB, \ell)$ are similar with corresponding Nagel cevians AX_a and $AT' \Longrightarrow \angle BAX_a = \angle CAT' \Longrightarrow AT$ and AX_a are isogonals WRT $\angle BAC,$ as desired.

MJ GEO

#3 Dec 17, 2009, 7:46 pm

first thank you for your solution.but i want to write another solution.its easy to prove that $(a+b+c)AP = bc$ then is trivial that TP, TQ are bisectors of BTA, TAC angles and we can say $(AP)(TB) = (BP)(TA)$ and in the other triangle like that.then we know that $2(CX_a) = a+c-b, 2(BX_a) = a+b-c)$ and when we analise it with last its trivial that $sin(TAB) = sin(CAX_a)$ and the problem solve

MBGO

#4 Dec 5, 2012, 6:32 pm

typo:$(a+b+c)AP = 2bc$

MathPanda1

#5 Nov 14, 2014, 2:26 am

> MJ GEO wrote:
>
> first thank you for your solution.but i want to write another solution.its easy to prove that $(a+b+c)AP = bc$ then is trivial that TP, TQ are bisectors of BTA, TAC angles and we can say $(AP)(TB) = (BP)(TA)$ and in the other triangle like that.then we know that $2(CX_a) = a+c-b, 2(BX_a) = a+b-c)$ and when we analise it with last its trivial that $sin(TAB) = sin(CAX_a)$ and the problem solve

Could you please write with a little more detail? For example, I don't know why TP, TQ are bisectors of BTA, TAC angles from $(a+b+c)AP = bc$ (It could just be that I am being stupid). Thanks a lot!

MathPanda1

#6 Nov 22, 2014, 10:15 am • 1

Here is another solution: Consider the following transformation: a composition of a reflection across the bisector of $\angle BAC$ and then inversion with ratio $\sqrt{AB*AC}$. It takes B to C and vice versa. thus, it takes w to BC and vice versa. Since the mixitilinear circle is tangent to w, it's image is tangent to BC. Also, it is inscribed in $\angle BAC$, so it is the excircle. Hence, this transformation takes T to X_a as they are corresponding tangents. Therefore, angles BAT and CAX_a are equal.

TelvCohl

#7 Nov 22, 2014, 10:22 am • 1

See circle tangent to BA,CA and circumcircle&isogonal conjugate

This post has been edited 1 time. Last edited by TelvCohl, Aug 21, 2015, 1:06 am

Quick Reply

High School Olympiads

Concurrency again

Reply

Sassha

#1 Dec 16, 2009, 1:56 am

Let I be the incenter of $\triangle ABC$. The incircle touches the triangle in points A', B', C'. ($A' \in BC, B' \in AC, C' \in AB$). Let l be an arbitrary line through I and let A'', B'', C'' be the reflections of A', B', C' in l. Prove that AA'', BB'', CC'' are concurrent.

Luis González

#2 Dec 16, 2009, 5:50 am • 1

None other than a Steinbart point sent to the line at infinite.

Reflections A'', B'', C'' of A', B', C' across the straight line ℓ obviously lie on the incircle (I). Thus, by Steinbart's theorem, AA'', BB'', CC'' concur if and only if $A'A'', B'B'', C'C''$ concur. Indeed the lines $A'A'', B'B'', C'C''$ meet at the infinite point of $\perp \ell$.

Agr_94_Math

#3 Dec 16, 2009, 12:48 pm

Consider triangle ABC with the points of tangency of the incircle with the sides as A', B', C' respectively. We know that AA', BB', CC' are concurrent at the Nagel point of triangle ABC.. Now all of $A"A', B"B', C"C'$ are parallel due to reflections upon the line which contains the diameter of the incircle as its segment. So let $A"A', B"B', C"C'$ meet at infinity. That will be the spiral similarity center for $ABC, A"B"C"$. (they are congruent due to reflections). So by property of transformations, even $AA", BB", CC"$ concur.

Quick Reply

High School Math

circle and line go through fixed points

Reply

Unidranix

#1 Dec 14, 2009, 2:17 pm

Given a fixed circle (O,R) and a fixed point M outside (O), a fixed point N inside (O). $A,B\in(O)$ such that AB goes through N. MA and MB meet (O) at the second point C,D respectively.
part 1
part 2

Luis González

#2 Dec 14, 2009, 10:35 pm

Let ray MN cut $\odot(MAB)$ at M'. From power of N to $\odot(MAB)$ we get $\overline{NM}\cdot\overline{NM'}=\overline{NA}\cdot\overline{NB}=p(N,(O))$. Since M,N are both fixed, then M' is fixed $\Longrightarrow\odot(MAB)$ goes trough M and its inverse image M' through the inversion with center N and power $\overline{NA}\cdot\overline{NB}$. Inversion $\mathcal{I}$ with center M and power equal to the power of M with respect to (O) takes (O) into itself and the line CD into the circle $\odot(MAB)$, passing through the fixed point M'. Thus CD goes through the fixed inverse image of M' under $\mathcal{I}$.

Quick Reply

High School Olympiads

Two Circle

Reply

Source: Moroccan Mathematical olympiad 2009, First test , Problem 4

Abdek

#1 Dec 14, 2009, 6:32 am

The circle C_1 meets a circle C_2 at A and B .The tangent of C_2 at A meets C_1 at C and the tangent of C_1 at A meets C_2 at D. Let (d) be a line pass through A and C the circum-circle of ΔACD. (d) meets C_1, C_2 and C at M, N and P respectively. Prove that $AM = NP$

Luis González

#2 Dec 14, 2009, 6:55 am

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=311308
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=313905
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=246834
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=187699

Quick Reply

High School Olympiads

harmonic

Reply

Source: independent

77ant

#1 Jul 26, 2009, 11:39 pm

Hi, everyone

Prove the following:
if R be any point on a circle, A and B fixed points on a diameter and equidistant from the center,
the envelope of a line which cuts harmonically the two circles with A, B as centers and AR, BR as radii
is independent of the position of R on the circle.

Thanks
Please your answers. I am studying materials related to this post nowadays.

yetti

#2 Jul 27, 2009, 7:06 am

Let (O) be the circle with R on it. Let a line l cut the circle (A) at X, Y and the circle (B) at Z, T. Assume that the cross ratio $\frac{\overline{XZ}}{\overline{XT}} \cdot \frac{\overline{YZ}}{\overline{YT}} = -1$ is harmonic. Let M, N be midpoints of XY, ZT and $(M), (N)$ circles with diameters XY, ZT, intersecting at P, P'. Since $PX \perp PY$, these lines bisect $\angle ZPT$ internally and externally, (M) is P-Apollonius circle of $\triangle PZT$ with circumcircle $(N) \Longrightarrow (M) \perp (N)$. Inversion in (M) takes (N) to itself and Z to $T \Longrightarrow$ $\overline{MZ} \cdot \overline{MT} = -\overline{MX} \cdot \overline{MY} \Longrightarrow \frac{p(M,(B))}{p(M,(A))} = -1$. In exactly the same way, $\frac{p(N,(B))}{p(N,(A))} = -1$. Since powers of M, N to $(A), (B)$ are in the same ratio $r = -1$, M, N are on a circle coaxal with $(A), (B)$, centered on AB and passing through R. As $\frac{\overline{AO}}{\overline{BO}} = r = -1, \odot(MNR) \equiv (O)$. The feet M, N of perpendiculars $AM \perp l, BN \perp l$ from the fixed points A, B to the arbitrary line l are therefore on the fixed circle $(O) \Longrightarrow l$ is tangent to a conic with foci A, B and pedal circle (O).

Luis González

#3 Dec 14, 2009, 5:57 am

Suppose that A, B lie inside the circle (O) centered at the midpoint of AB with radius $OR = \varrho$. Let the variable line ℓ cut (A) at P, Q and (B) at X, Y. Let M denote the orthogonal projection of A on ℓ, which is the midpoint of the chord PQ. Since $(P, Q, X, Y) = -1$, by Newton's theorem we have $MP^2 = MX \cdot MY \Longrightarrow p(M, (A)) = p(M, (B))$. Hence M lies on the circle centered at the midpoint of AB and coaxal with (A) and (B), i.e. $M \in (O)$. Let A' be the reflection of A across ℓ and let $N \equiv \ell \cap BA'$. A' moves on the homothetic circle $(B, 2\varrho)$ of (O) through the homothety centered at A with coefficient 2. Therefore $NA + NB = NA' + NB = 2\varrho = \text{const} \Longrightarrow$ Locus of N is the ellipse $\mathcal{E}$ with foci A, B and major axis 2ϱ. Since ℓ is the external bisector of $\angle ANB$, it follows that ℓ is tangent to $\mathcal{E}$. When A, B lie outside (O), analogous reasoning yields that ℓ is tangent to the hyperbola with foci A, B and major axis 2ϱ.

Quick Reply

High School Olympiads

PAa +PBb +PCc > 4(ABC) help

Reply

Source: [Mexican IMO training 2008 homework]

chingonkan999
#1 Apr 9, 2008, 3:23 am

Prove that for any point P in the interior of a triangle ABC, the sum $PA \cdot a + PB \cdot b + PC \cdot c$ is bigger or equal than $4(ABC)$.

Remark. Here, (ABC) denotes the area of triangle ABC.

BaBaK Ghalebi
#2 Apr 9, 2008, 9:41 pm

you have posted the same problem here:

http://www.mathlinks.ro/viewtopic.php?t=198639

posting the same problem again doesn't help you get the solution,so stop it.

chingonkan999
#3 Apr 9, 2008, 11:06 pm

I know i posted the problem mate, but because it turned out to be in spanish i think people stop givin the solution, so please give the solution anyone that can.

Thank you

BaBaK Ghalebi
#4 Apr 9, 2008, 11:11 pm

posting spanish in the other topic was a wrong thing to do,anyway please don't double post again it's spamming.

chingonkan999
#5 Apr 10, 2008, 12:38 am

i wont do it again, sorry,

Does anyone have the solution?

BaBaK Ghalebi
#6 Apr 10, 2008, 1:00 am

let $PA = x, PB = y, PC = z$ now let m, n, p denote the distances from point P to the sides BC, CA, AB respectively,now we have to prove that:

$ax + by + cz \geq 4S$

where S denotes the area of $\triangle ABC$.

now let H be a point on side BC such that $AH \perp BC$ and let $h_a = AH$,now we obviously have:

$x + m \geq h_a$

$\Rightarrow a(x+m) \geq ah_a = 2S$

$\Rightarrow ax \geq 2S - am$

in the same way we conclude that:

$by \geq 2S - bn$

and

$cz \geq 2S - cp$

thus:

$ax + by + cz \geq 6S - (am + bn + cp)$ (*)

now note that in triangle $\triangle PBC$ we have:

$\frac{1}{2}.ma = S_{\triangle PBC} \Rightarrow am = 2S_{\triangle PBC}$

similary:

$bn = 2S_{\triangle PCA}$

and

$cp = 2S_{\triangle PAB}$

so according to (*) we get that:

$ax + by + cz \geq 6S - 2\left(S_{\triangle PBC} + S_{\triangle PCA} + S_{\triangle PAB}\right)$

$\Rightarrow ax + by + ca \geq 6S - 2S = 4S$

QED

paqui_2590

#7 Apr 12, 2008, 5:40 am

Please do not answer posts like this. This problem is part of a homework that Chingonkan999 must do alone. The homework is part of Mexico's training program.

Maybe mexican trainers haven't noticed, but is a little illegal to do what chingonkan999 did.

chingonkan999

#8 Apr 13, 2008, 11:08 pm

I am sorry paqui but this is not illegal at all,

I already told the main mexican trainer and he even said i could ask for help from the trainer that gave us the homework.

It is not the first time i upload a problem, it is not even the first time someone uploads a problem of the homework. Even you uploaded a problem of the homework, the one of the 21agon so please do not say things like that.

Thankyou

Altheman

#9 Apr 14, 2008, 12:23 am

It's not like this is a new problem or anything...I'm sure it could be found a book or something.

Luis González

#10 Dec 12, 2009, 1:05 am

Let M,N,L be the midpoints of $BC,CA,AB.$ In the quadrangles $PLAN, PMBL, PNCM,$ we have the inequalities:

$$[PLAN] \le \frac{PA \cdot NL}{2} = \frac{PA \cdot BC}{4}$$

$$[PMBL] \le \frac{PB \cdot LM}{2} = \frac{PB \cdot CA}{4}$$

$$[PNCM] \le \frac{PC \cdot MN}{2} = \frac{PC \cdot AB}{4}$$

Adding the 3 previous inequalities gives $PA \cdot BC + PB \cdot CA + PC \cdot AB \ge 4[ABC]$

Quick Reply

High School Olympiads

area 2

Reply

Source: triangle

yildobekir

#1 Dec 10, 2009, 12:33 am

ABC triangle has sides a,b,c and P is a point in ABC triangle and $m(\widehat{APB}=m(\widehat{APC}=m(\widehat{BPC}=120^\circ$. If A(ABC)=S

$|AP|+|BP|+|CP|=\sqrt{\frac{a^2+b^2+c^2}{2}}+2S\sqrt{3}$

Luis González

#2 Dec 10, 2009, 8:02 am • 1

Construct outwardly on the sides BC, CA, AB of $\triangle ABC$ the equilateral triangles BCA', CAB', ABC' whose centers are $X, Y, Z.$ Then it's well-known that $P \equiv (X) \cap (Y) \cap (Z)$ and $P \equiv AA' \cap BB' \cap CC'.$ The lines PA, PB, PC are pairwise radical axes of $(X), (Y), (Z).$ Thus, the sidelines of $\triangle XYZ$ are perpendicular to $PA, PB, PC,$ respectively $\Longrightarrow$ $\triangle XYZ$ is equilateral and A, B, C are the reflections of P across $YZ, ZX, XY.$ If $X', Y'Z'$ denote the projections of P onto $YZ, ZX, XY,$ then $PA+PB+PC=2(PX'+PY'+PZ').$

Let L be the side-lenght of $\triangle XYZ.$ By Viviani's theorem, the sum $(PX'+PY'+PZ')$ equals the altitude of $\triangle XYZ \Longrightarrow$ $PA+PB+PC=\sqrt{3}L.$ Thus, it remains to find the side-lenght L of $\triangle XYZ$ in terms of $AB, AC, BC.$

By cosine law in $\triangle AYZ,$ kepping in mind that AY, AZ are circumradii of the equilateral $\triangle CAB', \triangle ABC',$ we have

$$L^2=AY^2+AZ^2-AY \cdot AZ \cdot 2\cos(A+60^\circ)$$

$$L^2=\frac{AB^2+AC^2}{3}+\frac{AB \cdot AC \cdot (\sqrt{3}\sin A-\cos A)}{3}$$

Using the identities

$$AB \cdot AC \cdot \sin A=2S \ , \ BC^2=AB^2+AC^2-AB \cdot AC \cdot 2\cos A$$

$$\Longrightarrow \ L^2=\frac{AB^2+AC^2+BC^2}{6}+\frac{2\sqrt{3}}{3}S$$

$$L=\sqrt{\frac{AB^2+AC^2+BC^2}{6}+\frac{2\sqrt{3}}{3}S}$$

$$PA+PB+PC=\sqrt{3}L=\sqrt{\frac{AB^2+AC^2+BC^2}{2}+2\sqrt{3}S}$$

Quick Reply

High School Olympiads

Inscribed quadrilateral

Reply

Source: from D.Monk 's book , UKMT

stergiu

#1 Dec 9, 2009, 10:09 pm

Let $ABCD$ an inscribed quadrilateral in a circle with center O . Lines BC, AD meet each other at P and lines

BA, CD inersect at Q. The line through Q and perpendiculat to AC meets OP at X.Prove that $\angle ABX = 90$

Babis

Luis González

#2 Dec 9, 2009, 11:46 pm

Let $E \equiv AC \cap BD$ and R the projection of Q on AC. Since EQ is the polar of P WRT (O), it follows that line EQ is perpendicular to OP at a point T. Similarly, line PE is the polar of Q WRT (O), thus PE is perpendicular to OQ at a point S. From the quadrilateral $TESO$ inscribed in the circle with diameter TS, we have $QE \cdot QT = QS \cdot QO$. The line passing through the contact points of the tangents from Q to (O) is identical to its polar PE. Then $QB \cdot QA = QS \cdot QO \Longrightarrow QB \cdot QA = QE \cdot QT = QR \cdot QX \Longrightarrow ABRX$ is cyclic, i.e. $\angle ABX = 90^\circ$.

Quick Reply

High School Olympiads

Equi-distance

Reply

Source: own

TRAN THAI HUNG

#1 Dec 8, 2009, 9:21 pm

LetABC is a triangle. (I_B) , (I_C) are the escribed circles. (I_B) meets BC,AC atM,N. (I_C) meet BC,AB at P,Q. I_BI_C meetPQ,MNat H,K.
D is the mid point ofBC.
Prove that$DH=DK$:)

mathVNpro

#2 Dec 8, 2009, 11:02 pm

> *TRAN THAI HUNG wrote:*
> LetABC is a triangle. (I_B) , (I_C) are the escribed circles. (I_B) meets BC,AC atM,N. (I_C) meet BC,AB at P,Q. I_BI_C meetPQ,MNat H,K.
> D is the mid point ofBC.
> Prove that$DH=DK$:)

Very nice and quite easy problem.

It is well- known that $CK\perp AI_b$ and $BH\perp AI_c\Longrightarrow K,H$ respectively are the projections of C, and B onto I_bI_c. Let (I_a) is the center of the excircle of $\triangle ABC$ wrt A. Then it is followed that $\triangle ABC$ is the orthic triangle wrt $\triangle I_aI_bI_c$. Then if we call M is the midpoint of $BC\Longrightarrow MD$ is the perpendicular bisector of BC. Thus $\angle MCD=\angle MBD$. But note that $CDMK$ and $MDBH$ are two concyclic quadrilaterals, then $\angle MCD=\angle MKD, \angle MBD=\angle MHD$, which implies $\angle MHD=\angle MKD$. As the result, $\triangle DKH$ is $D-$ isoleces triangle $\Longrightarrow DK=DH.\Box$

Luis González

#3 Dec 8, 2009, 11:11 pm

Let E be the midpoint of AC and $H'\equiv DE\cap PQ.$ Since $\triangle PBQ$ is isosceles and $DE \parallel AB,$ then $\triangle PDH'$ is also isosceles $\Longrightarrow DH'=DP=\frac{1}{2}(AB+AC) \Longrightarrow EH'=\frac{1}{2}AC.$ On the other hand, denote $H''\equiv I_bI_c\cap DE.$ From the isosceles $\triangle EAH''$ we get $EH''=EA=EH'=\frac{1}{2}AC \Longrightarrow H\equiv H'\equiv H''.$ Therefore DH and DK are the midlines of AB and $AC \Longrightarrow$ Lines DK,DH and I_bI_c bound an isosceles triangle.

Quick Reply

High School Olympiads

Lines Concurrent

Source: Own result?

mathVNpro

#1 Dec 6, 2009, 11:21 am

Let ABC be a triangle inscribed in circumcircle (O). Denote A_1, B_1, C_1 respectively be the projections of A,B,C onto BC, CA, AB. Let A_2, B_2, C_2 respectively be the intersections of AO, BO, CO with BC, CA, and AB. A circle Ω_a passes through A_1, A_2 and tangents to the arc of BC that doesn't contain A of (O) at T_a. The same definition holds for T_b, T_c. Prove that AT_a, BT_b and CT_c are concurrent.

Best regard,
mathVNpro

Luis González

#2 Dec 6, 2009, 11:44 am

Since the circle $\odot(AA_1A_2)$ is tangent to (O) at A, then the tangent ℓ_1 to (O) at A is the radical axis of $\odot(AA_1A_2)$ and (O). The sideline BC is the radical axis of $\odot(AA_1A_2)$, Ω_a and the tangent ℓ_2 to (O) at T_a is the radical axis of Ω_a and (O). Hence, lines BC, ℓ_1, ℓ_2 concur at the radical center O_a of circles $(O), \Omega_a$ and $\odot(AA_1A_2)$. It's known that $O_a \equiv \ell_1 \cap BC$ is the center of the A-Apollonius circle of $\triangle ABC$, which is orthogonal to $(O) \Longrightarrow AT_a$ is the A-symmedian. By similar reasoning we conclude that lines AT_a, BT_b, CT_c concur at the symmedian point of $\triangle ABC$.

k.l.l4ever

#3 Dec 7, 2009, 10:09 pm

The generalization of this problem also holds:(instead of H is orthocenter and O is circumcenter we have H, O are arbitary isogonal conjugates points).
Proof:Let (AA_1A_2) cut AB at A_3 and AC at A_4.Then $\angle A_3A_4A_2 = \angle A_4A_3A_1 \rightarrow A_1A_2A_3A_4$ is isoceles trapezium.From that $A_3A_4 || BC \rightarrow (AA_3A_4)$ is tangent to (ABC).The back of this proof is same of luisgeometria's proof.

High School Olympiads

Concurrent lines in a triangle.

Source: Own (?!).

Virgil Nicula
#1 Dec 5, 2009, 3:25 am

> Quote:
> Construct the triangles XBC , AYC , ABZ outside of the triangle ABC so that
>
> $XBC\sim AYC\sim ABZ$. Prove that the lines AX , BY , CZ are concurrently.

Remark. We can obtain new and interesting problems by studying some particular cases of this problem.

Luis González
#2 Dec 5, 2009, 8:59 am

In fact, this is Jacobi's theorem. A proof with barycentric coordinates goes this way:

$$\angle BXC=\angle YAC=\angle ZAC=\omega$$

$$\angle XBC=\angle AYC=\angle ABZ=\theta$$

$$\angle BCX=\angle ACY=\angle AZB=\varphi$$

Using Conway's formula the barycentric coordinates of X,Y,Z are given by

$$X\ (-a^2:S_C+S_\varphi:S_B+S_\theta)$$

$$Y\ (S_C+S_\varphi:-b^2:S_A+S_\omega)$$

$$Z\ (S_B+S_\theta:S_A+S_\omega:-c^2)$$

Hence, AX,BY,CZ concur at a point J (Jacobi's perspector) with barycentric coordinates

$$J\left(\frac{1}{S_A+S_\omega}:\frac{1}{S_B+S_\theta}:\frac{1}{S_C+S_\varphi}\right)$$

livetolove212
#4 Dec 6, 2009, 11:51 am

Dear Mathlinkers,
A simple proof of this problem is to use Ceva-sine theorem and some area ratios. :)

Virgil Nicula
#5 Dec 7, 2009, 2:43 pm

Indeed, my proposed problem is **Jakobi's theorem**. Sorry, I didn't know it. Thank you, **Luis** !

> Quote:
> Construct the triangles XBC , AYC , ABZ outside of the triangle ABC so that
>
> $XBC\sim AYC\sim ABZ$. Prove that the lines AX , BY , CZ are concurrently.

Method I (**V.N.** & **livetolove212**). Denote $\left\|\begin{array}{c} M\in AX\cap BC\ ,\ N\in BY\cap CA\ ,\ P\in CZ\cap AB\\ m(\angle BAZ)=m(\angle CAY)=x\\ m(\angle CBX)=m(\angle ABZ)=y\\ m(\angle ACY)=m(\angle BCX)=z\\ x+y+z=180^{\circ}\end{array}\right\|$. Observe that

$\frac{BX}{CX}=\frac{\sin\widehat{BCX}}{\sin\widehat{CBX}}=\frac{\sin z}{\sin y}$. Therefore, $\frac{MB}{MC}=\frac{[ABX]}{[ACX]}=\frac{BA\cdot BX\cdot\sin\widehat{ABX}}{CA\cdot CX\cdot\sin\widehat{ACX}}\implies$ $\boxed{\frac{MB}{MC}=\frac{c}{b}\cdot\frac{\sin z}{\sin y}\cdot\frac{\sin(B+y)}{\sin(C+z)}}$.

Prove analogously that $\frac{NC}{NA}=\frac{a}{c}\cdot\frac{\sin x}{\sin z}\cdot\frac{\sin(C+z)}{\sin(A+x)}$ and $\frac{PA}{PB}=\frac{b}{a}\cdot\frac{\sin y}{\sin x}\cdot\frac{\sin(A+x)}{\sin(B+y)}$. In conclusion,

$\frac{MB}{MC}\cdot\frac{NC}{NA}\cdot\frac{PA}{PB}=1$ and using the **Ceva**'s theorem obtain that the lines AX, BY, CZ are concurrently.

Method II (V.N.). $J\in BY\cap CZ\implies AYC\sim ABZ\implies\left\|\begin{array}{c}\frac{AY}{AC}=\frac{AB}{AZ}\\ \widehat{CAZ}\equiv\widehat{YAB}\end{array}\right\|\implies AYB\sim ACZ\implies$ $\left\|\begin{array}{c}\widehat{ABY}\equiv\widehat{AZC}\\ \widehat{AYB}\equiv\widehat{ACZ}\end{array}\right\|\implies$

$\left\|\begin{array}{c}\widehat{ABJ}\equiv\widehat{AZJ}\\ \widehat{AYJ}\equiv\widehat{ACJ}\end{array}\right\|\implies AJBZ$, $AJCY$ are cyclically $\implies\left\|\begin{array}{c} m(\angle ZJA)=m(\angle ZBA)=y\\ m(\angle YJA)=m(\angle YCA)=z\\ (x+y+z=180^{\circ})\end{array}\right\|\implies$ $m(\angle CJY)=x\implies BXCJ$

is cyclically $\implies m(\angle CJX)=m(\angle CBX)=y\implies m(\angle AJX)=x+y+z=180^{\circ}\implies J\in AX\implies$ $AX\cap BY\cap CZ\neq\emptyset$.

This post has been edited 1 time. Last edited by Luis González, Feb 21, 2016, 10:27 pm
Reason: Fixing LaTeX

Quick Reply

High School Olympiads

orthocenters and concurrent lines

Reply

Source: Ukrainian journal contest, problem 360, by Maria Rozhkova

rogue
#1 Dec 4, 2009, 9:17 pm

Let AA_1, BB_1, CC_1 be the altitudes of an acute triangle ABC. Denote by A_2, B_2 and C_2 the orthocenters in triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the straight lines A_1A_2, B_1B_2 and C_1C_2 are concurrent.

Luis González
#2 Dec 4, 2009, 11:04 pm

It is enough to see that $A_2C_1HB_1$, $B_2C_1HA_1$ and $C_2B_1HA_1$ are parallelograms $\Longrightarrow \triangle A_2B_2C_2$ is centrally similar to the medial triangle of $\triangle A_1B_1C_1$ with similarity center H and coefficient 2. Hence $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are centrally symmetric $\Longrightarrow$ Straight lines A_1A_2, B_1B_2, C_1C_2 concur at their symmetry center.

Martin N.
#3 Dec 5, 2009, 12:08 am

The lines AA_1, BB_1, CC_1 and AA_2, BB_2, CC_2 respectively are isogonal conjugates. As the first three lines intersect at the orthocenter, the second three are concurrent as well; their point of intersection is the circomcentre of $\triangle ABC$ (the isogonal conjugate of the orthocentre).

Luis González
#4 Dec 5, 2009, 12:12 am

> *Martin N. wrote:*
> The lines AA_1, BB_1, CC_1 and AA_2, BB_2, CC_2 respectively are isogonal conjugates. As the first three lines intersect at the orthocenter, the second three are concurrent as well; their point of intersection is the circomcentre of $\triangle ABC$ (the isogonal conjugate of the orthocentre).

Dear Martin N, the problem is not about the concurrency of AA_2, BB_2, CC_2 (clearly obvious) but the concurrency of the straight lines A_1A_2, B_1B_2, C_1C_2.

Martin N.
#5 Dec 5, 2009, 12:35 am

ouups sorry you are right, luisgeometria

Quick Reply

High School Olympiads

Two circles

Reply

AndrewTom
#1 Dec 4, 2009, 7:18 pm

Two circles, of different radius, with centres at B and C, touch externally at A. A common tangent, not through A, touches the first circle at D and the second at E. The line through A which is perpendicular to DE and the perpendicular bisector of BC meet at F. Prove that $BC = 2AF$.

Luis González
#2 Dec 4, 2009, 11:54 pm

Let $P \equiv DE \cap BC$ and Q the orthogonal projection of A on DE. Since P and A are the exsimilicenter and insimilicenter of $(B) \sim (C)$, we have that $(B, C, A, P) = -1 \Longrightarrow DE, QA$ bisects $\angle BQC$ externally and internally $\Longrightarrow QA$ and the perpendicular bisector of BC meet on the circumcircle of $\triangle BQC \Longrightarrow F \in \odot(BQC)$.

In the right trapezoid $DECB$ with bases DB, EC, we have

$$\overline{AQ} = \frac{\overline{BD} \cdot \overline{AC} + \overline{CE} \cdot \overline{AB}}{\overline{AB} + \overline{AC}} = \frac{2 \cdot \overline{AB} \cdot \overline{AC}}{\overline{AB} + \overline{AC}} \quad (1)$$

From power of A to $\odot(BQC)$, we have $\overline{AF} = \dfrac{\overline{AB} \cdot \overline{AC}}{\overline{AQ}} \quad (2)$

From (1) and (2) we obtain $\overline{AF} = \frac{1}{2}(\overline{AB} + \overline{AC}) = \frac{1}{2}\overline{BC}$, as desired.

lajanugen
#3 Dec 5, 2009, 2:23 am

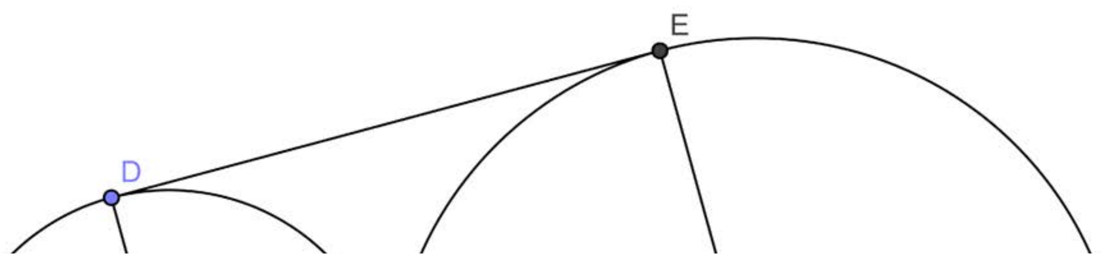
Let D, N, E be the projections of B, M, C respectivley on the common tangent.
Since $BM = MC, DN = NE$. We know that the common tangent at A to the circles bisects DE. Since N bisects DE, we conclude that AN is the tangent at A and hence, $AN//FM$.
$MN//FA$ since both are perpendicular to tangent DE
Hence, $ANMF$ is a parallelogram and $AF = NM = \dfrac{1}{2}(BD + CE) \to BC = 2.AF$

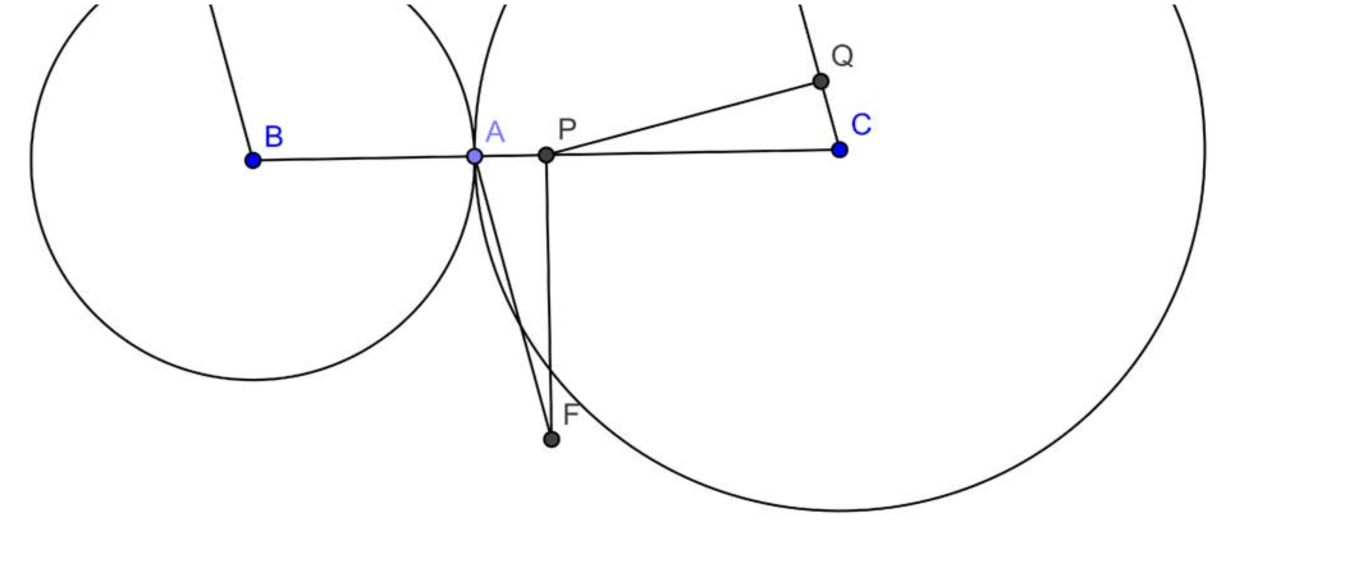
Image not found

jgnr
#4 Dec 5, 2009, 11:54 am

Let P be the midpoint of BC and Q is its foot of perpendicular to line CE. Clearly $\triangle PAF$ and $\triangle CQP$ are similar, and since $AP = \dfrac{r_2 - r_1}{2} = CQ$, they are congruent. So $AF = PC = \dfrac{1}{2}BC$. Q.E.D.

Attachments:





Quick Reply

High School Olympiads

Prove that MN passes through the midpoint of PQ

Reply

Source: Ukrainian journal contest, problem 362, by Igor Nagel

rogue

#1 Dec 4, 2009, 9:14 pm

Let ω_1 be the incircle of a triangle ABC. The circle ω_1 has center I and touches the sides AB and AC at points M and N. A circle ω_2 passes through points A and I and intersects the sides AB and AC at points Q and P respectively. Prove that the line segment MN passes through the midpoint of line segment PQ.

Luis González

#2 Dec 4, 2009, 10:31 pm

Since $IM = IN$ and $IP = IQ$, it follows that the right $\triangle IMP$ and $\triangle INQ$ are congruent $\Longrightarrow \overline{MP} = \overline{NQ}$. Let $T \equiv PQ \cap MN$. By Menelaus' theorem for $\triangle APQ$ cut by the transversal MTN, we have:

$$\frac{\overline{NQ}}{\overline{NA}} \cdot \frac{\overline{AM}}{\overline{MP}} \cdot \frac{\overline{PT}}{\overline{TQ}} = 1$$

Since $\overline{AM} = \overline{NA}$ and $\overline{MP} = \overline{NQ} \Longrightarrow \overline{PT} = \overline{TQ} \Longrightarrow MN$ bisects PQ

lajanugen

#3 Dec 4, 2009, 11:02 pm

Assume that the second circle isnt the one that passes through A, M, I, N

We have $IQ = IP$. Let the circle with center at I intersect AC at R(different from P)
Let MN and PQ intersect at S. Since QR and MN are both perpendicular to IA, they are parallel.
In triangle PQR, since $PN = NR$ and $RQ//NS$, we're done.

Image not found

mathVNpro

#4 Dec 4, 2009, 11:03 pm

> *rogue wrote:*
>
> Let ω_1 be the incircle of a triangle ABC. The circle ω_1 has center I and touches the sides AB and AC at points M and N. A circle ω_2 passes through points A and I and intersects the sides AB and AC at points Q and P respectively. Prove that the line segment MN passes through the midpoint of line segment PQ.

Let S' be the projection of I onto PQ. Then by the *Simpson line* theorem, we obtain that M, N, S' are collinear, which implies that $S' \equiv S \Longrightarrow IS \perp PQ$. Also note that AI is the internal bisector of $\angle PAQ$, thus $IP = IQ$. As the consquence, S is the midpoint of PQ. Our proof is completed then. $\square$

sunken rock

#5 Jun 14, 2010, 11:26 pm

Since AI is an angle bisector, we get $PI = QI$, hence $\Delta QMI \equiv \Delta PNI$, hence $MQ = NP$. With $AN = AM$, applying the theorem of Menelaos to the ΔPAQ with the transversal MN, we get that MN passes through the midpoint of PQ.

Best regards,
sunken rock

Quick Reply

High School Olympiads

geometric mean

Reply

aadil

#1 Jul 11, 2009, 5:29 pm

in a triangle ABC prove that there is a point D on side AB such that CD is the geometric mean of AD and DB if and only if $sinA*sinB<=sin^2(C/2)$

chrischris

#2 Jul 11, 2009, 8:52 pm

not sure..

Luis González

#3 Dec 4, 2009, 9:17 am

Let D' be the second intersection of the ray CD with the circumcircle (O) of $\triangle ABC$. From the power of D WRT (O), we have $CD \cdot DD' = AD \cdot BD = CD^2 \Longrightarrow CD = DD'$. Hence, D' lies on the homologous line ℓ of AB under the homothety with center C and coefficient 2. Thus, there exist such a D on BC if and only if ℓ cuts (O).

Let M, N be the midpoints of AB and the arc AB and H, H' the orthogonal projections of C, D' on AB. Since $\triangle CHD \cong \triangle D'H'D \Longrightarrow CH = D'H'$. Thereby, there exists at most two points D if and only if $CH < MN$, there exists one point D if and only if $D' \equiv N$, i.e. ℓ is tangent to (O) and there is no such a point D if $CH > MN$. Therefore, the necessary condition for the existence of at least one solution is that $CH \le MN$.

Since $\angle NAM = \angle NCB = \frac{1}{2}\angle C$, we have $MN = \frac{1}{2}AB \cdot \tan\frac{C}{2} \Longrightarrow$

$$CH \le \frac{1}{2}AB \cdot \tan\frac{C}{2} \Longrightarrow \frac{2CH}{AB} \cdot \cos\frac{C}{2} \cdot \sin\frac{C}{2} \le \sin^2\frac{C}{2} \Longrightarrow$$

$$\frac{CH}{AB} \cdot \sin C \le \sin^2\frac{C}{2} \Longrightarrow \sin A \cdot \sin B \le \sin^2\frac{C}{2}.$$

Quick Reply

High School Olympiads

Concurrent 1

Reply

Source: own

livetolove212

#1 Dec 3, 2009, 11:36 am

Given triangle ABC and its incircle $(I).(I)$ touches BC, CA, AB at A_1, B_1, C_1, respectively. Let A_2, B_2, C_2 be the points on IA_1, IB_1, IC_1 such that $\frac{\overline{IA_2}}{\overline{IA_1}} = \frac{\overline{IB_2}}{\overline{IB_1}} = \frac{\overline{IC_2}}{\overline{IC_1}}$. Prove that 3 circles $(AIA_2), (BIB_2), (CIC_2)$ concur at the point $Q \neq I$.

Luis González

#2 Dec 4, 2009, 3:32 am • 1

Denote $\frac{\overline{IA_2}}{\overline{IA_1}} = \frac{\overline{IB_2}}{\overline{IB_1}} = \frac{\overline{IC_2}}{\overline{IC_1}} = k \implies \overline{IA_2} = \overline{IB_2} = \overline{IC_2} = kr.$

Inversion with respect to the incircle (I) takes the vertices A, B, C into the midpoints M, N, L of $\overline{B_1C_1}, \overline{C_1A_1}, \overline{A_1B_1}$. If A_3, B_3, C_3 are the inverse points of A_2, B_2, C_2, it follows that circles $\odot(AIA_1), \odot(AIB_1)$ and $\odot(AIC_1)$ are transformed into the lines MA_3, NB_3, LC_3. Moreover, we get $\overline{IA_3} \cdot \overline{IA_2} = \overline{IA_3} \cdot kr = r^2 \Longrightarrow \overline{IA_3} = \frac{r}{k}$. By similar reasoning we have that $\triangle A_3B_3C_3$ and $\triangle A_1B_1C_1$ are centrally similar with similarity coefficient $\frac{1}{k}$, which also implies that the medial triangle $\triangle MNL$ of $\triangle A_1B_1C_1$ and $\triangle A_3B_3C_3$ are centrally similar with similarity coefficient $-\frac{2}{k} \Longrightarrow MA_3, NB_3, LC_3$ concur at their similarity center. Hence the primitive figures, namely, $\odot(AIA_1), \odot(AIB_1)$ and $\odot(AIC_1)$ concur at another point Q lying on the diacentral line IO of $\triangle ABC$.

Quick Reply

High School Olympiads

Concurrent

Reply

Source: with inversion

mathson

#1 Nov 30, 2009, 1:50 pm

Two circles Γ and Γ' intersect at A and D. A line is tangent to Γ and Γ' at E and F , respectively. The line BC contains the point D parallel to the line EF such that C in Γ and B in Γ'. Show that the circumcircles of $\triangle BDE$ and $\triangle CDF$ intersect again on the line AD.

Luis González

#2 Nov 30, 2009, 6:05 pm

Consider the positive inversion through center D and arbitrary radius. Let B', C', E', F' be the images of B, C, E, F. The circumferences $\odot(BDE)$ and $\odot(CDF)$ are taken into the lines $B'E'$ and $C'F'$ intersecting at the inverse image A' of A and the circles Γ and Γ' are taken into the lines $A'C'$ and $A'B'$. Since $BC \parallel EF$, the common tangent EF of Γ, Γ' is transformed into the incircle ω of $\triangle A'B'C'$ tangent to $B'C'$, $C'A'$, $A'B'$ at D, E', F'. The lines $B'E'$, $C'F'$ and $A'D$ concur at the Gergonne point of $\triangle A'B'C'$, thus the primitive figures, namely the circles $\odot(BDE), \odot(CDF)$ and the double line AD concur at another point besides D.

shoki

#3 Nov 30, 2009, 6:55 pm

another solution is to prove that $O_1S_1 = O_2S_2$ and then obviously we r done!
(O_1, S_1, O_2, S_2 are the centers of $\Gamma, (CDF), \Gamma', (BDE)$)

mathson

#4 Dec 2, 2009, 2:14 pm

> *luisgeometria wrote:*
> The circumferences $\odot(BDE)$ and $\odot(CDF)$ are taken into the lines $B'E'$ and $C'F'$ intersecting at the inverse image A' of A

I think the intersection of $B'E'$ and $C'F'$ is not always A'. Check the image.
If I'm wrong, I want to know why?

Attachments:
F1.pdf (31kb)

Luis González

#5 Dec 3, 2009, 12:12 am

> *mathson wrote:*
> I think the intersection of $B'E'$ and $C'F'$ is not always A'. Check the image.

Dear mathson, I did not claim that the lines B'E' and C'F' meet at A', but they meet at the Gergonne point of A'B'C'.

mathson

#6 Dec 3, 2009, 2:46 am

I'm really sorry, I didn't understand you at the first time.

> Quote:
> the common tangent EF to Γ and Γ' is transformed into the incircle ω of $\triangle A'B'C'$

Is there any generalization for this? Or theorem?

Quick Reply

High School Olympiads

Geometry with concurency

Reply

soruz
#1 Dec 2, 2009, 12:29 am

In the convex quadrilateral $ABCD$,the points G_1, G_2, G_3, G_4 are center of gravity for next triangles BCD, ACD, ABD and ABC.Prove that lines AG_1, BG_2, CG_3, DG_4 have a common point.

Luis González
#2 Dec 2, 2009, 1:45 am

Let M, N, P be the midpoints of $AC, BD, DC.$ Since $\frac{NG_3}{NA} = \frac{NG_1}{NC} = \frac{1}{3}$, it follows that $G_1G_3 \parallel AC$ and $G_1G_3 = \frac{1}{3}AC.$ Similarly, $G_2G_4 \parallel BD$ and $G_2G_4 = \frac{1}{3}BD.$

Since $\frac{PG_2}{PA} = \frac{PG_1}{PB} = \frac{1}{3} \Longrightarrow G_1G_2 \parallel AB, G_1G_2 = \frac{1}{3}AB.$ By similar reasoning, we conclude that quadrilaterals $G_1G_2G_3G_4$ and $ABCD$ are centrally similar with similarity coefficient $-\frac{1}{3} \Longrightarrow AG_1, BG_2, CG_3, DG_4$ concur at their similarity center.

Quick Reply

High School Olympiads

soruz

#1 Dec 1, 2009, 8:30 pm

Along the sides of triangle ABC built outside ,equilateral triangles ABD, BCE and ACF.Show that triangles ABC and DEF have the same center of gravity.

Luis González

#2 Dec 1, 2009, 9:41 pm

According to the Kiepert's theorem, the result can be generalized. Construct outwardly on BC, CA, AB three similar isosceles triangles BDC, CEA and AFB. Then $\triangle ABC$ and $\triangle DEF$ share the same centroid.

Let $\angle BCD = \angle CAE = \angle ABF = \theta$. Using Conway's notation, the barycentric coordinates of D, E, F with respect to $\triangle ABC$ are given by

$$D\ (-a^2 : S_C + S_\theta : S_B + S_\theta)$$

$$E\ (S_C + S_\theta : -b^2 : S_A + S_\theta)$$

$$F\ (S_B + S_\theta : S_A + S_\theta : -c^2)$$

Since the sum of the homogeneous barycentric coordinates of D, E, F equals $2S_\theta$, due to the identities $S_B + S_C = a^2$, $S_A + S_C = b^2$ and $S_A + S_B = c^2$, it follows that $\triangle ABC$ and $\triangle DEF$ have the same centroid.

soruz

#3 Dec 2, 2009, 1:50 am

I found a elementary solution with vectors but I have a doubt!
Let be G the center of gravity of the triangle ABC. We have
$\vec{GD} + \vec{GE} + \vec{GF} = \vec{GA} + \vec{AD} + \vec{GB} + \vec{BE} + \vec{GC} + \vec{CF}$.Let be M symmetric of point E in relation to BC.
How can I show that $\vec{FM} = \vec{AD}$.
It looks easy that $\vec{MC} = \vec{BE}$.Results that$\vec{AD} + \vec{BE} + \vec{CF} = \vec{FM} + \vec{MC} + \vec{CF} = \vec{0}$, so $\vec{GD} + \vec{GE} + \vec{GF} = \vec{0}$.
So how can I get that $\vec{FM} = \vec{AD}$.

vittasko

#4 Dec 2, 2009, 4:57 pm

As said me a friend of mine **Nikos Kyriazis**, a related more general problem, is coming from the past.

GENERAL PROBLEM. - A triangle $\triangle ABC$ is given and let $\triangle A'BC,\ \triangle B'AC,\ \triangle C'AB$ be, three similar triangles erected on its side-segments $BC,\ AC,\ AB$ respectively, outwardly or inwardly to it, with $\angle A' = \angle B' = \angle C'$ and $\angle A'BC = \angle B'CA = \angle C'AB$. Prove that the triangles $\triangle ABC,\ \triangle A'B'C'$, have the same centroid.

This general problem, is appeared as **Theorem 352 - III** in F.G.M book of Geometry.

The proof there, is very elementary and I post here now only the schema, because I think it is enough.

Click to reveal hidden text
Kostas Vittas.

Attachments:
t=315336.pdf (6kb)

Quick Reply

High School Olympiads

Another way, of a well known construction problem.

Reply

Source: To prove that BD = DE = EC.

vittasko

#1 Nov 30, 2009, 2:04 am

A triangle $\triangle ABC$ is given and let A' be, the reflexion of A, with respect to the midpoint M of its side-segment BC. The line through A' and parallel to the internal angle bisector of $\angle A$, intersects the circumcircle (K) of the triangle $\triangle BIC$, where I is the incenter of $\triangle ABC$, at point so be it P. Prove that $BD = DE = EC$, where $D \equiv AB \cap CP$ and $E \equiv AC \cap BP$.

Kostas Vittas.

Attachments:

t=314930.pdf (5kb)

Luis González

#2 Nov 30, 2009, 2:39 am

Assume there exists the points D and E on BA and AC such that $BD = DE = EC$ and $P' \equiv BE \cap CD$. We'll prove that P and P' are necessarily identical. It's easy to see that P' lies on $\odot(IBC)$, due to

$$\angle DP'B = \angle EDC + \angle DEB = \frac{1}{2}(\angle AED + \angle ADE) \implies$$

$$\angle DP'B = \frac{1}{2}(\pi - \alpha) = \pi - \angle BIC \implies BIP'C \text{ is cyclic.}$$

On the other hand, it is well-known that the locus of the intersection of the diagonals of the convex quadrilateral $BD'E'C$ satisfying $BD' = E'C$ is the A'-angle bisector of the antimedial triangle $\triangle A'B'C'$, which is the parallel line to AI from A'. This was discussed in the topic plane geometry 2, hence $P' \equiv P$ and the proof is completed.

sunken rock

#3 Nov 30, 2009, 4:23 am

Remark:

If $\widehat{BAC} = 60^\circ$, then $P \equiv O$, where O is the circumcenter of $\triangle ABC$.

Best regards,
sunken rock

Quick Reply

High School Olympiads

4 concurrent circles

Reply

Source: own result?

mathVNpro
#1 Nov 28, 2009, 11:01 pm • 1

Let (O) be the circumcircle of triangle $\triangle ABC$. Denote M_a, M_b and M_c respectively by the antipoles of A, B, C with respect to (O). Let O_a, O_b and O_b respectively be the reflections of O across to BC, CA, AB. Prove that the circumcircles of triangles $\triangle OO_aM_a$, $\triangle OO_bM_b$ and $\triangle OO_cM_c$ go through a common point F, such that F is different from O and $F \in (O)$.

Best regard
mathVNpro

Luis González
#2 Nov 29, 2009, 3:23 am • 1

Consider the positive inversion with respect to (O). Vertices A, B, C and their antipodes transform into themselves, the midpoints M, N, L of BC, CA, AB are taken into the vertices of the tangential triangle $A'B'C'$ and reflections of O about BC, CA, AB are taken into the midpoints U, V, W of OA', OB', OC', respectively. Therefore the circles $\odot(OO_aM_a)$, $\odot(OO_bM_b), \odot(OO_cM_c)$ are transformed into the lines UM_a, VM_b, WM_c. Circle with diameter UA is the A'-Garitte circle of $\triangle A'B'C'$ passing through its Feuerbach point. Since M_a is the antipode of A WRT (O), we deduce that UM_a cuts (O) again at the Feuerbach point F of $\triangle A'B'C' \Longrightarrow UM_a, VM_b, WM_c$ concur at F. Hence, $\odot(OO_aM_a), \odot(OO_bM_b)$, $\odot(OO_cM_c), (O)$ concur at the Feuerbach point F of the tangential triangle $\triangle A'B'C'$.

livetolove212
#3 Nov 29, 2009, 6:54 am

Denote Q the Anti-Steiner point of Euler line of triangle ABC.
We have $\angle OO_aQ = \angle HOO_a = \angle AHO = \angle AB'L = \angle AM_aQ$. Therefore OQO_aM_a is cyclic. Similarly we get $(OO_aM_a), (OO_bM_b), (OO_cM_c)$ concur at Q which lies on (O).

Attachments:
picture79.pdf (11kb)

Mashimaru
#4 Nov 29, 2009, 4:12 pm

Let H be the orthocenter of $\triangle ABC$ and $H_a \equiv AH \cap (O)$, $F \equiv (OO_aM_a) \cap (O)$. We have H_a is symmetry to M_a wrt OO_a so $(FO, FO_a) \equiv (M_aO, M_aO_a) \equiv (H_aO_a, H_aH_aO) \equiv (FO, FO_a) \pmod \pi$ so $F \equiv O_aH_a \cap (O)$. On the other hand, it is well known that O_aH_a, O_bH_b, O_cH_c, the symmetry of the Euler line of $\triangle ABC$ wrt BC, CA, AB concurs at a point on (O), this point must be F and so (OO_bM_b) and (OO_cM_c) also passes through F, qed.

mathVNpro
#5 Nov 29, 2009, 7:16 pm

Thank you for all yours' solution, my friend. But personally, I think that the problem can be solved in the simplest way.
Here is my solution

Let F be the intersecitons of (OO_bM_b) and (OO_bM_c). We will prove these two following claims.

• *Claim 1. F is a point lies on (O).*

Indeed, we have $\angle M_bFM_c = \angle M_bFO + \angle OFM_c$ $(*)$. But since $F \in (OO_bM_b)$ and also $\in (OO_cM_c)$, thus, we have $\angle M_bFO = 180^\circ - \angle M_bO_bO$ and $\angle OFM_c = 180^\circ - \angle M_cO_cO$. Let H be the orthocenter of $\triangle ABC$. It is well- known that M_cOHO_c and M_bOHO_b are parallelogram and also with the notice that $HB \| OO_b$, $HC \| OO_c$, we obtain $\angle M_bO_bO = \angle OHB$ and $\angle M_cO_cO = \angle OHC$. Thus, from $(*)$, we have $\angle M_bFM_c = 360^\circ - \angle OHB - \angle OHC$ $= \angle BHC = 180^\circ - \angle BAC = 180^\circ - \frac{1}{2}\angle BOC = 180^\circ - \angle M_cBM_b$, which implies that $F \in (O)$.

● *Claim 2. F also lies on the circumcircle of $\triangle OO_aM_a$.*

Indeed, $\angle M_aFO = \angle M_aFM_b - \angle OFM_b$. But $\angle M_aFM_b = 180^\circ - \angle M_aCM_b = 90^\circ - \angle ACM_b = 90^\circ - \angle ABO$ $= \angle ACB$. As in the proof of the first claim, we have known that $\angle OFM_b = 180^\circ - \angle OHB$. Thus $\angle M_aFO = \angle ACB -$ $180^\circ + \angle OHB$. Besides, with also a notice that OHO_aM_a is a parallelogram and $OO_a \| AH$, thus $\angle OO_aM_a = \angle OHA$. We need to show that $\angle ACB - 180^\circ + \angle OHB = \angle OHA$. But this is equivalent to an obvious fact $\angle ACB = 180^\circ - \angle AHB$. As the consequences, $F \in (OO_aM_a)$, also.

From the two above claims, we are able to conclude that (OO_aM_a), (OO_bM_b) and (OO_cM_c) go through a common point F $\in (O)$. Our proof is completed then. $\square$

Remark. Thanks to this problem and the solutions of **Mashimaru**, **Livetolove212** and mine, we have another inversive proof for the *Feurebach* concurrency of UM_a, VM_b, WM_c in **Luisgeometria**'s post. First, let me restaste this concurrency property again:

Problem.Let ABC be a triangle with (I) its incircle. Denote A_0, B_0, C_0 respectively be the tangency points of (I) with BC, CA, AB. Let A_1, B_1, C_1 be the antipodes of A_0, B_0, C_0 with respect to (I). A_2, B_2, C_2, respectively, are the midpoints of IA, IB, IC. Prove that A_1A_2, B_1B_2 and C_1C_2 concurrent at F_e, the *Feurebach point* wrt $\triangle ABC$.

Proof

All angles are directed modulo π.

Consider the inversion through pole I, power $k = r^2$, where r is the radii of (I). We have $\mathcal{I}(I, k)$ maps A_1, B_1, C_1 into themselves and A_2, B_2, C_2 is mapped into A_2', B_2', C_2' which respectively are the reflections of I across to B_0C_0, C_0A_0 and A_0B_0. Therefore, $\mathcal{I}(I, k) : A_1A_2 \mapsto (IA_1A_2')$, $B_1B_2 \mapsto (IB_1B_2')$ and $C_1C_2 \mapsto (IC_1C_2')$. But according to the problem from this topic, we have (IB_1B_2'), (IA_1A_2') and (IC_1C_2') go through a point $F_e \in (I)$, note that F_e is mapped into itself through $\mathcal{I}(I, k) \Longrightarrow A_1A_2, B_1B_1, C_1C_2$ go through $F_e \in (I)$ $(*)$. We claim that F_e $\in$ the three $9-$ point Euler circles wrt $\triangle IBC$, $\triangle ICA$ and $\triangle IAB$. $(**)$

Indeed, first, we have $A_0B_1 \perp A_0B_0; CT \perp A_0B_0$, thus $A_0B_1 \| CI$, but C_2 is the midpoint of IC. Hence, $(A_0B_1, A_0C_2, A_0I, A_0C) = -1$. Further, since $A_0I \perp A_0C \Longrightarrow A_0I$ and A_0C respectively are the internal and external bisector with respect to $\angle B_1A_0C_2$. Similarly, we also have A_0I and A_0C respectively are the internal and external bisector of $\angle B_2A_0C_1$. And for that reasons, we obtain $\angle B_1A_0C_1 = \angle B_2A_0C_2$. But since $F_e \in (I) \Longrightarrow \angle B_1F_eC_1 = \angle B_1A_0C_1$. Therefore, $\angle B_2F_eC_2 = \angle B_2A_0C_2$, which implies $F_e \in (A_0B_2C_2)$, which is also the $9-$ point Euler circle of $\triangle IBC$. Argue the same, we obtain that the three $9-$ point Euler circle of $\triangle IBC$, $\triangle ICA$ and $\triangle IAB$ go through $F_e \in (I)$. The result $(**)$ is lead as followed. We claim that F_e also $\in$ the $9-$ point Euler circle wrt $\triangle ABC$. $(***)$

Indeed, let M_a, M_b, M_c respectively be the midpoints of BC, CA, AB. We have $\angle M_cF_eM_a = \angle M_cF_eB_1 + \angle B_1F_eC_2$ $+ \angle C_2F_eM_a = \angle M_cA_2B_2 + \angle B_2M_aC_2 + \angle C_2B_2M_a = \angle IBA + \angle CIB \angle BCI = \angle CBA$. But $\angle CBA$ $= \angle M_cM_bM_a$ (With a notice that $M_bM_c \| BC$) $\Longrightarrow \angle M_cF_eM_a = \angle M_cM_bM_a$, which implies $F_e \in (M_aM_bM_c)$. We obtain the result $(***)$.

F_e lies on (I), F_e also lies on the $9-$ point circle wrt $\triangle ABC$. Hence, F_e is the *Feurebach point* wrt $\triangle ABC$. Then combine $(*)$ and $(***)$, we obtain the result of the porblem. Our proof is completed then. $\square$

This post has been edited 5 times. Last edited by mathVNpro, Nov 30, 2009, 11:09 am

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mathVNpro

#6 Nov 29, 2009, 11:54 pm

> *mathVNpro wrote:*
>
> Let (O) be the circumcircle of triangle $\triangle ABC$. Denote M_a, M_b and M_c respectively by the antipoles of A, B, C with respect to (O). Let O_a, O_b and O_b respectively be the reflections of O across to BC, CA, AB. Prove that the circumcircles of triangles $\triangle OO_aM_a$, $\triangle OO_bM_b$ and $\triangle OO_cM_c$ go through a common point F, such that F is different from O and $F \in (O)$.

From this problem, there is some nice things realted to it. Hope you enjoy it and solve it with a nice solution!

With the same notations of objects in the above result.

(1) Let d be the line contains the centers of (OO_aM_a), (OO_bM_b) and (OO_cM_c). Then d always tangents to a fixed circle, particularly the circle $(O, \frac{R}{2})$, where R is the radii of (ABC).

(2) Let d_a, d_b, d_c be perpendicular bisectors of OM_a, OM_b, OM_c. Let $C' \equiv d_a \cap d_b$, $B' \equiv d_a \cap d_c$ and $C' \equiv d_a \cap d_b$. Then AA', BB', CC' are concurrent at a point F' which is also a point lies on (O), futher F' is the antipode of F wrt (O).

Luis González

#7 Nov 30, 2009, 1:46 am • 2

> mathVNpro wrote:
> (1) Let d be the line contains the centers of (OO_aM_a), (OO_bM_b) and (OO_cM_c). Then d always tangents to a fixed circle, particularly the circle $(O, \frac{R}{2})$, where R is the radii of (ABC).

This follows easily by noticing that the circumferences $\odot(OO_aM_a), \odot(OO_bM_b)$ and $\odot(OO_cM_c)$ are coaxal with common radical axis the straight line $OF \Longrightarrow$ their centers lie on the perpendicular bisector of OF.

> mathVNpro wrote:
> (2) Let d_a, d_b, d_c be perpendicular bisectors of OM_a, OM_b, OM_c. Let $C' \equiv d_a \cap d_b, B' \equiv d_a \cap d_c$ and $C' \equiv d_a \cap d_b$. Then AA', BB', CC' are concurrent at a point F' which is also a point lies on (O), futher F' is the antipode of F wrt (O).

Using the same notation in my previous post, the central symmetry with center O maps F into its antipode F' on (O), A into M_a and the perpendicular bisectors of OM_b, OM_c into the perpendicular bisectors of OB, OC obviously meeting at the midpoint U of OA'. Since $F \in UM_a$, then $AA' \in F'$. By similar reasoning the result follows.

jayme

#8 Dec 7, 2014, 1:16 pm

Dear Mathlinkers,
concerning the message #3, the Reim's theorem gives the result....
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Pentagram open problem 1

Reply

Source: Is it a famous theorem

borislav_mirchev

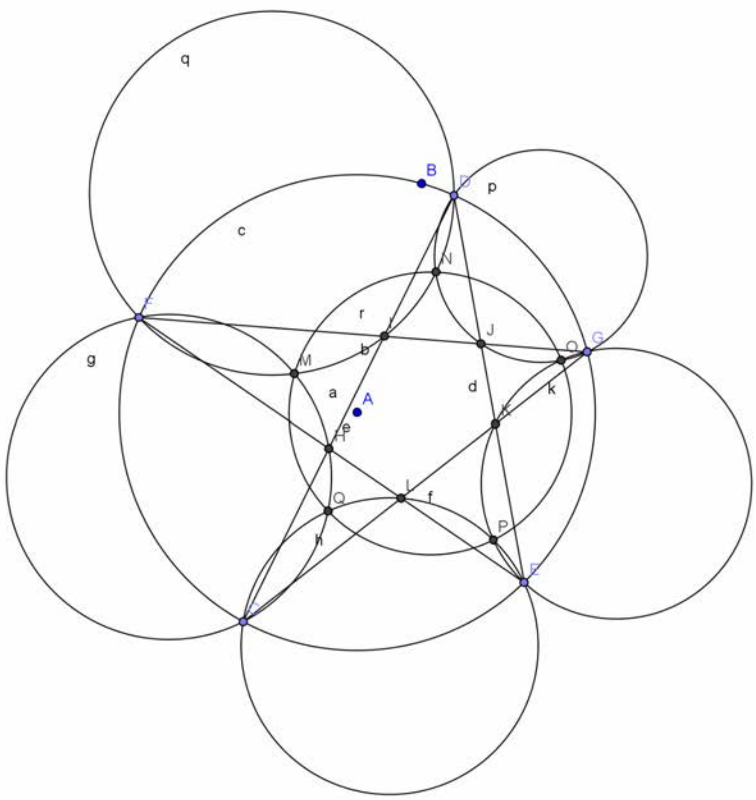
#1 Nov 9, 2009, 4:21 am

I found very interesting Miquel theorems. Most interesting for me was this one:
http://www.gogeometry.com/miquel_pentagram1.htm
Inspired of that theorem using geogebra I found the configuration on the picture.

To be easier for you - take a look at the file attached.
It is given a pentagram (five rays star). The vertices of the pentagram lies on a circle. There are constructed five smaller circles. Prove that their second intersecting points lies on a circle.

My questions are:
1. Is it a famous theorem or a problem from math competition?
2. How to prove that statement?
3. What is its level of difficulty?
4. Is it a beatiful problem?

Attachments:



borislav_mirchev

#2 Nov 15, 2009, 5:23 am

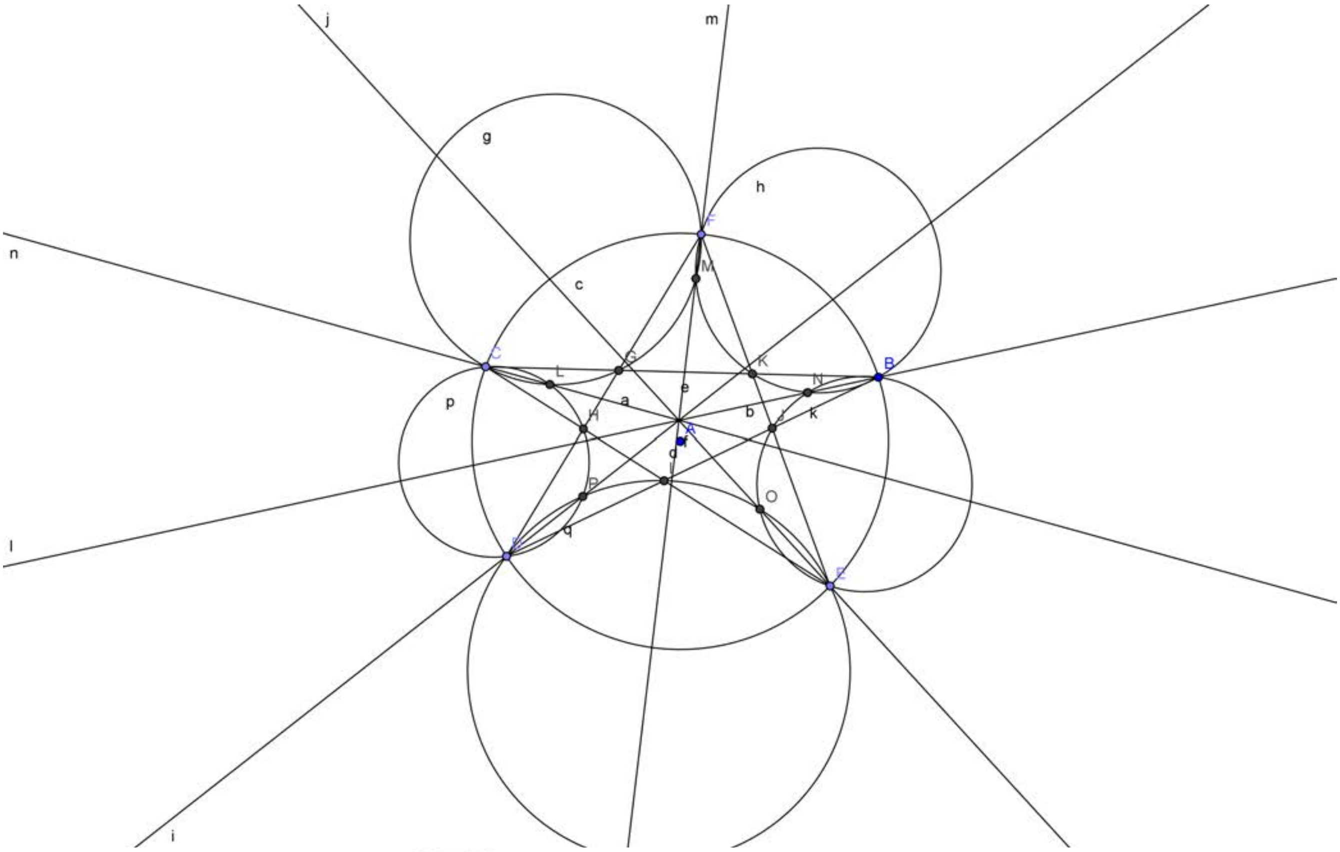
I also don't know how to solve the problem. I think it can be solved only by using inscribed angle/quadrilateral properties and additional constructions.

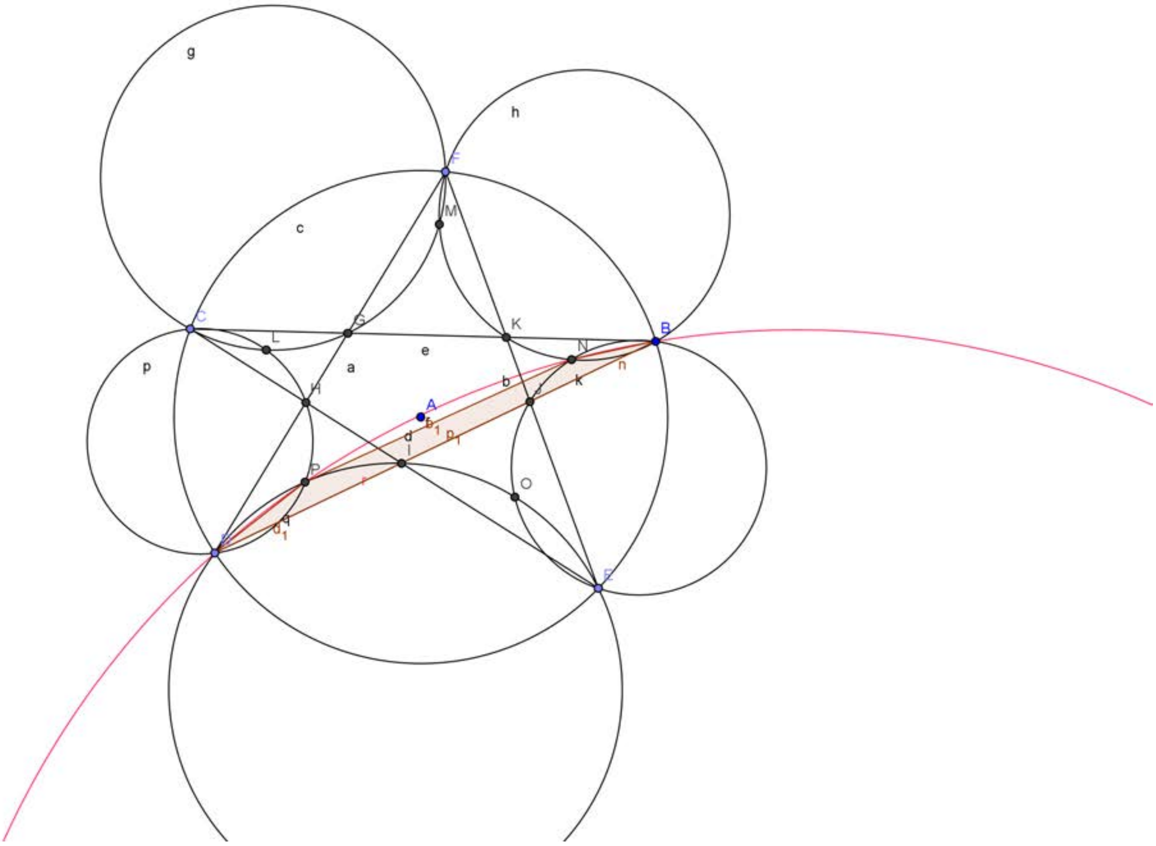
If you can solve the problems in the attached pictures our initial problem is solved.

1. In the IntersectingLines.png - you should prove the all five lines intersects at a common point.
2. In the InscibedQuad.png - you should prove the quadrilateral marked is inscribed.

If you have some questions - please ask me.

Attachments:





Luis González

#3 Nov 29, 2009, 10:41 am

> *borlislav_mirchev wrote:*

It is given a pentagram (five rays star). The vertices of the pentagram lies on a circle. There are constructed five smaller circles. Prove that their second intersecting points lies on a circle.

See *Les pentagrammes de Miquel et de Morley*, Volume (4) here.

borislav_mirchev

#4 Nov 30, 2009, 1:24 pm

Thank you very much these materials they are very interesting ... I made mistakes in all my solutions ... it seems that the problem is very hard or incorrect. But I believe it is simply a hard problem.

borislav_mirchev

#5 Dec 29, 2009, 7:49 pm

I'm sorry all my attempts to solve the problems failed ... my friends also failed ... I'll give 50 euros for first correct solution with matherial used in high school.
The tallent have no price ... but it is just a bonus for invested time.
Proof that the statements are not correct is also a solution.

livetolove212

#6 Feb 20, 2010, 1:54 pm

Dear Mathlinkers,
Here is the solution for 50 Euros

Attachments:
solution.PDF (85kb)

skytin

#7 Mar 12, 2011, 11:02 pm • 1

See my solution here :
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=393621

skytin

#8 Apr 10, 2011, 8:55 pm • 1

This problen is from :
International Mathematical Olympiad 2009
Hong Kong Team Selection Test 1
See here:
http://gifted.hkedcity.net/Gifted/Download/0809IMOTest/IMO-Test%201-August-2008.pdf

borislav_mirchev

#9 Apr 12, 2011, 2:15 am

Dear skytin
Thank you very much for very useful information and the excellent solution. The strange thing is I discovered the statement at my own.
Some of the statements I rediscovered appeared also at Sharygin MO 2010 and Romania TST 1993 other were some well known theorems but not for me

Quick Reply

High School Olympiads

orthocenter problem

Reply

tang zy

#1 Nov 28, 2009, 8:15 pm

Let H be the orthocenter of triangle $\triangle ABC$, O be the midpoint of segment BC.
AD is a line penpendicular to OH and D be the intersection of line OH and AD.
Prove that A, B, C, D are in a circle.

Luis González

#2 Nov 28, 2009, 8:48 pm

P, Q, R are feet of the altitudes onto $BC, CA, AB.$ Negative inversion with center H an power $HA \cdot HP$ maps the circumcircle (O) of $\triangle ABC$ into its nine-point circle $(N).$ From the cyclic quadrilateral $ADPM$ on account of the right angles $\angle ADP$ and $\angle AMP,$ we have $HA \cdot HP = HM \cdot HD \Longrightarrow M$ is the inverse image of D through $(H, -k^2),$ since $M \in (N).$ Then D lies on $(O),$ as desired.

lajanugen

#3 Nov 28, 2009, 9:05 pm

Let O be the circumcenter of triangle ABC. It is well known that $AH = 2.OM$ (The homothety with center G(the centroid) and ratio $-1/2$ carries OM into HA)
Let AO and HM meet at point P . Since AH and OM are parallel and $AH = 2.OM$, $AO = OP$ whiCh implies that P is on the circumcenter and AP is a diameter of the circumcircle. Since $\angle ADP$ is right, we're done

dgreenb801

#4 Nov 29, 2009, 3:54 am

Let P,Q,R be the feet of the perpendiculars to BC,CA, AB. Let S be the foot of the perpendicular from H to AO. Then since $\angle HSO = \angle HPO = 90$, HPOS is cyclic. Also, since $\angle ADH = \angle ARH = \angle ASH = \angle AQH = 90$, $ADRHSQ$ is cyclic. Next, P,O,Q, and R are on the nine-point circle, so $POQR$ is cyclic. So by the radical axis theorem, PO, HS and QR concur at a point X. Also, since $\angle ADO = \angle APO = 90$, $ADPO$ is cyclic. Thus, the radical axes of $ADPO$, $HSPO$, and $ADRHSQ$ concur, so AD passes through X. But since AD, HS, and PO concur, by the converse of the radical axis theorem, A,B,C, and D are concyclic (end of problem).

Note that since $\angle BQC = \angle BRC = 90$, $BQRC$ is cyclic and since QR,HS, and BC concur, by the converse of the radical axis theorem, B,H,S and C are concyclic. So we have 7 cyclic quadrilaterals!!!!! (see picture)
[geogebra]8b3d249524d402aa7591fd1eda1b5d9849bc1b55[/geogebra]

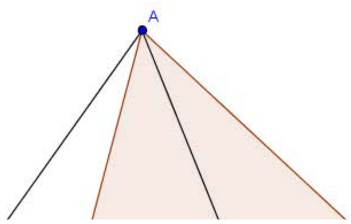
But we also have $ARPC$,$CPRA$, $BRHP$, $DHRX$ and $PHQC$ are cyclic, so I guess you could make that 12.

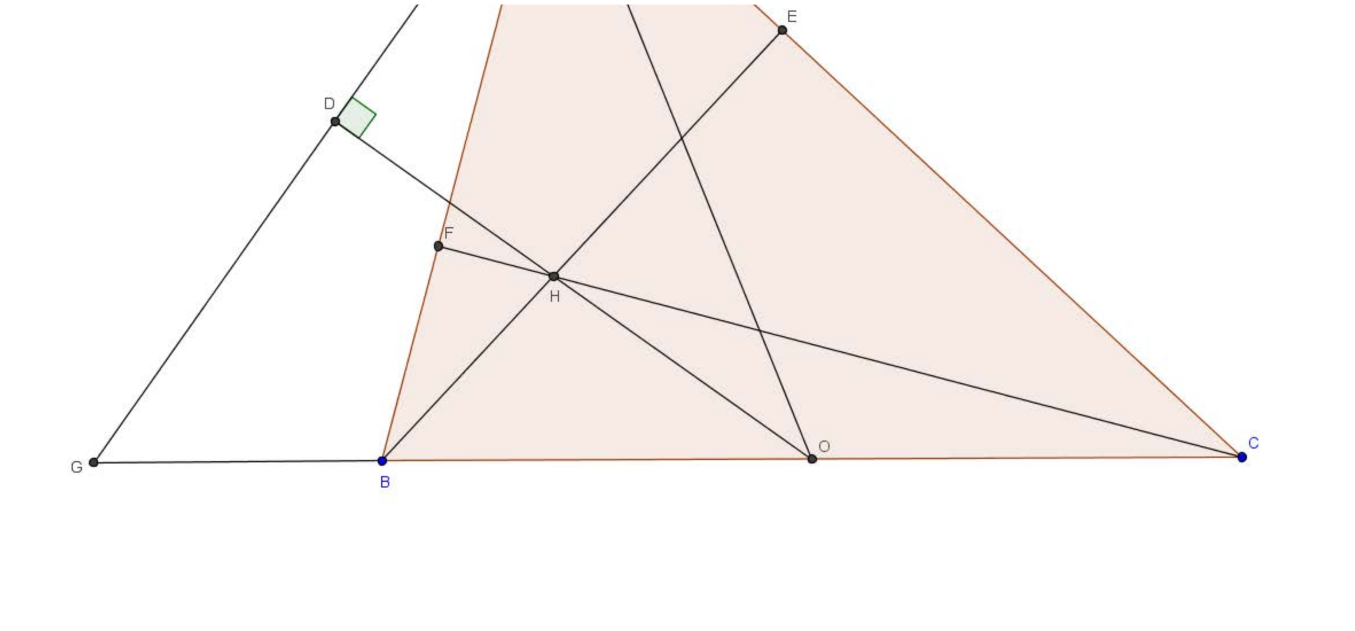
jgnr

#5 Nov 29, 2009, 8:13 am

Click to reveal hidden text

Attachments:





tang zy

#6 Nov 29, 2009, 12:40 pm

Nice solution of all of you!
Let P, Q, R be the feet of the perpendiculars to BC, CA, AB .Let G be the intersection of line BC and QR,We can also prove that A, D, G are in a line.

jgnr

#7 Nov 29, 2009, 12:48 pm

> *tang zy wrote:*
> Nice solution of all of you!
> Let P, Q, R be the feet of the perpendiculars to BC, CA, AB .Let G be the intersection of line BC and QR,We can also prove that A, D, G are in a line.

I used this fact to prove the original problem.

Mashimaru

#8 Nov 29, 2009, 3:50 pm

A very simple solution.

Let AK be the diameter of the circumcircle of $\triangle ABC$ then $KB \perp AB$ and $CH \perp AB$ so $KB \parallel CH$. Analogously, $KC \parallel BH$ so $KBHC$ is a parallelogram. This means that KH passes through O. Let $D' \equiv KH \cap (ABC)$ then $AD' \perp OH$ and so $D' \equiv D$. Therefore A, B, C, D are concylic, QED.

mathson

#9 Nov 30, 2009, 12:55 pm

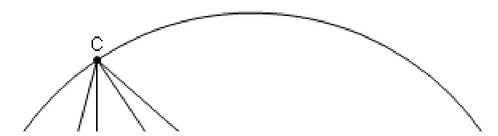
Why is the map (O) under $I(H, k)$ is (N)?

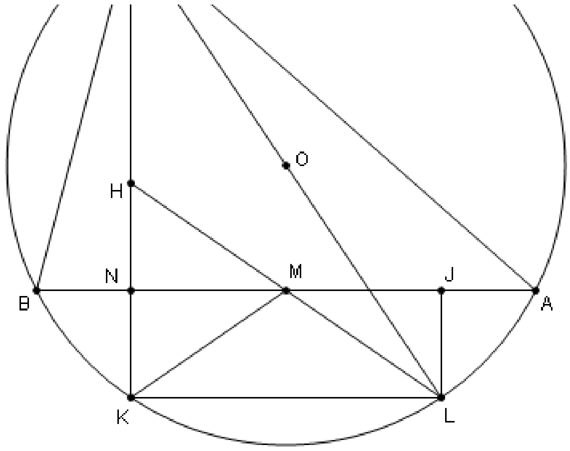
jgnr

#10 Nov 30, 2009, 2:33 pm

Another solution inspired by my friend:

Attachments:





Quick Reply

High School Olympiads

Help!Help

Yafmuzz

#1 Nov 23, 2009, 7:32 pm

PROBLEM 1: $\triangle ABC$. AD, BE, CF are three altitudes and H is the orthocenter. $X \in XD, Y \in BE, Z \in CF$ such that $\frac{XA}{AD} + \frac{YB}{BE} + \frac{ZC}{CF} = 1$. Prove that: HXYZ is cyclic
PROBLEM 2: Let a, b, c be three sides of a triangle. R and r are the radius of circumcircle and incircle. prove that $a + b + c < 4(r + R)$

shoki

#2 Nov 24, 2009, 10:27 pm

i think that the sum of fractions in the first problem must be 2,(sorry if i'm wrong)

at first note that the equation can be transformed to:
$\frac{XA}{\frac{2S_{\Delta ABC}}{BC}} + \frac{YB}{\frac{2S_{\Delta ABC}}{AC}} + \frac{ZC}{\frac{2S_{\Delta ABC}}{AB}} = 2$

or in other words :

$XA.BC + YB.AC + ZC.AB = 4S_{\Delta ABC}$

replace XA by $AD - XD$ and similarly for YB, ZC to get

$2S_{\Delta ABC} = XD.BC + YE.AC + ZF.AB = 2S_{\Delta BXC} + 2S_{\Delta ACY} + 2S_{\Delta ABZ}$

$S_{\Delta ABC} = S_{\Delta BXC} + S_{\Delta ACY} + S_{\Delta ABZ}$
now see here
the second problem must be well-known...

Yafmuzz

#3 Nov 25, 2009, 4:47 pm

I have just edited the problem.
What about the problem 2. Can anyone give me the solution?

Luis González

#4 Nov 27, 2009, 6:15 am

> Quote:
> $\triangle ABC. AD, BE, CF$ are three altitudes and H is the orthocenter. $X \in XD, Y \in BE, Z \in CF$ such that $\frac{XA}{AD} + \frac{YB}{BE} + \frac{ZC}{CF} = 2$. Prove that $HXYZ$ is cyclic

Assume WLOG that Y lies on the ray HE. By Gergonne-Euler theorem for H, we have

$$\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2 \Longrightarrow \frac{AX + XH}{AD} + \frac{YB - YH}{BE} + \frac{CZ + ZH}{CF} = 2$$

$$\frac{XH}{AD} + \frac{ZH}{CF} - \frac{YH}{BE} = 0 \Longrightarrow BC \cdot XH + AB \cdot ZH = AC \cdot YH \ (\star)$$

Let the circle passing through H, X, Z cut the ray HE at Y'. We'll prove that $Y \equiv Y'$.

Since $\angle Y'XZ = \angle EHC = \angle BAC$ and $\angle Y'ZX = \angle EHA = \angle ACB$, we have that $\triangle ABC \sim \triangle XY'Z$ and let k be their similarity coefficient. Thus

$$XY' \cdot HZ = k \cdot AB \cdot HZ$$

$$Y'Z \cdot HX = k \cdot BC \cdot HX$$

$$Y'H \cdot XZ = k \cdot AC \cdot Y'H$$

By Ptolemy's theorem for $HXY'Z \Longrightarrow k(AB \cdot HZ + BC \cdot HX) = k \cdot AC \cdot YH' \Longrightarrow$ $AB \cdot HZ + BC \cdot HX = AC \cdot YH'$.

Due to $(\star)$, we deduce that $Y \equiv Y'$ and the proof is completed.

> *Yafmuzz wrote:*
>
> Let a, b, c be three sides of a triangle. R and r are the radii of circumcircle and incircle, prove that $a + b + c < 4(r + R)$

Assume that $\triangle ABC$ is acute and let H be the orthocenter of $\triangle ABC$. We use that the sum of the distances from H to the vertices of the acute triangle $\triangle ABC$ equals the sum of the diameters of its incircle and circumcircle

By triangle inequality for $\triangle HAB, \triangle HBC$ and $\triangle HCA$, we obtain

$$HA + HB > AB \ , \ HB + HC > BC \ , \ HC + HA > AC$$

$$\Longrightarrow 2(HA + HB + HC) = 4(R + r) > a + b + c.$$

If $\triangle ABC$ is right the inequality is trivial, but if $\triangle ABC$ is obtuse, then the argument fails.

Yafmuzz
#5 Nov 27, 2009, 2:24 pm

> *Luis González wrote:*
>
> We'll use that the sum of the distances from H to the vertices of the acute triangle $\triangle ABC$ equals the sum of the diameters of its incircle and circumcircle

Can you prove this?
The sum of the fractions in problem 1 is 1

Yafmuzz
#6 Nov 27, 2009, 7:08 pm

The sum is 1.
My teacher has just given me the solution of problem 1.
Anyway, thanks for your solution.

---

shoki
#7 Nov 27, 2009, 7:29 pm

sorry,it seems that i wasn't thinking right but...
but there is sth wrong again(correct me if i'm wrong!),
all the equations in my proof here are $\Longleftrightarrow$!!plz take a look at it!
which implies that if $HXYZ$ is cyclic then the sum of the fractions must be 2!

Nam Luu
#8 Nov 28, 2009, 10:06 am

Here is my solution if the sum of the fractions is 1:
Let N be the point in the triangle such that $NZ \perp CF$, $NY \perp BE$, $AN \cap BC \equiv L$, $CN \cap AB \equiv M$, $BN \cap CA \equiv K$.

We have: $\frac{CZ}{CF}=\frac{CN}{CM},\frac{BN}{BK}=\frac{BY}{BE}$ and notice that $\frac{AN}{AL}+\frac{BN}{BK}+\frac{CN}{CM}=1=\frac{XA}{AD}+\frac{YB}{BE}+\frac{ZC}{CF}$.

Hence $\frac{AN}{AL}=\frac{AX}{AD}\implies\frac{AN}{AL}=\frac{AX}{AD}\implies NX\parallel BC\implies NX\perp AD$.

Thus $\angle NYH=\angle NZH=\angle NXH=90^o$ and $HXYZ$ is cyclic.
If N is not in the triangle, may be my solution is wrong.
Can anyone show me how to post image in the message. I'm a new here.

This post has been edited 1 time. Last edited by NamLuu, Nov 28, 2009, 8:35 pm

Luis González

#9 Nov 28, 2009, 11:28 am

> *Nam Luu wrote:*
> $\frac{AN}{AL}+\frac{BN}{BK}+\frac{CN}{CM}=1=\frac{XA}{AD}+\frac{YB}{BE}+\frac{ZC}{CF}$

This relation is not true, it actually should be

$$\frac{AN}{AL}+\frac{BN}{BK}+\frac{CN}{CM}=\frac{XA}{AD}+\frac{YB}{BE}+\frac{ZC}{CF}=2.$$

Quick Reply

High School Olympiads

Concurrent

Reply

Source: three circles and two circles

77ant

#1 Nov 27, 2009, 10:28 am

Dear everyone.

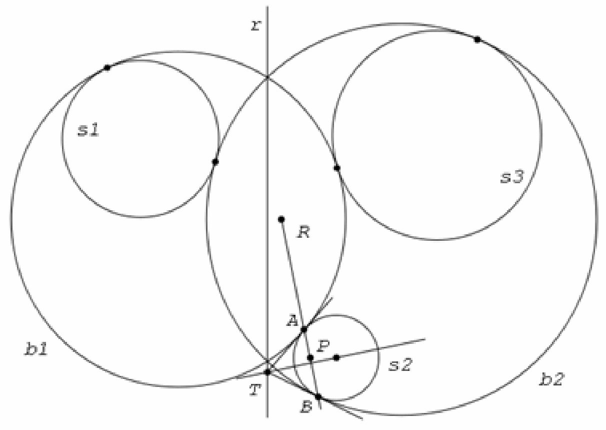
Please read the following.

for non intersecting circles s1, s2, s3 there are two circles b1, b2 tangent to all of them.
b1, b2 are tangent to s2 at A, B respectively. the radical axis of b1, b2 is r.
b1, b2 are tangent to s1 at C, D respectively.
b1, b2 are tangent to s3 at E, F respectively.

(1) prove that two tangents to s2 at A, B and r are concurrent.
(2) prove that AB, CD, EF are concurrent at the radical center of s1, s2, s3

May I ask you a little help? Thank you.

Attachments:



livetolove212

#2 Nov 27, 2009, 8:17 pm

a, Two tangents to s_2 at A, B and r concur at the radical center of three circles b_1, b_2, s_2.
b, **Lemma:** Given two circle (O_1) and (O_2) are tangent at point A. Construct two lines l_1, l_2 through A. Denote X_1, X_2, Y_1, Y_2 the intersections of l_1, l_2 and $(O_1), (O_2)$.Then $X_1X_2//Y_1Y_2$.
For the proof: Construct the tangent of two circles ...
Back to our problem:
See the figure below.
J the intersection of AB and EF, X, Y the intersections of BE and s_3, s_2; Q the intersection of FX and b_1 then $XO_3//QI_1$.
On the other side, we have $\frac{EO_3}{EI_2} = \frac{XO_3}{BI_2} \Rightarrow XO_3//BI_2$.
$\Rightarrow QI_1//BO_2$. But $\frac{QI_1}{BO_2} = \frac{I_1A}{O_2A}$ therefore A, B, Q are collinear.
Similarly, denote P the intersection of EF and b_1 then A, Y, P are collinear.
Applying the lemma above we get $PQ//EB$.But $PQFA$ is cyclic then applying Reim's theorem we obtain $AFEB$ is cyclic.
$\Rightarrow J$ lies on the radical axis of s_2 and s_3. Similarly we are done.

Attachments:
picture78.pdf (12kb)

Luis González

#3 Nov 28, 2009, 5:59 am

R is the insimilicenter of $(B_1) \sim (B_2), C$ is the exsimilicenter of $(S_1) \sim (B_1)$ and D is the insimilicenter of $(S_1) \sim (B_2)$. By Mongue & d'Alembert theorem, the straight line CD goes through R and similarly the straight lines AB, EF go through R. Since (B_1) and (B_2) have common points, then R is center of positive inversion with power k^2 carrying (B_1) and (B_2) into each other. Let C' be the inverse of C. Then it suffices to prove that C and C' are necessarily identical. $(B_2), (B_1)$ are taken into each other and due to the conformity, the circle (S_1) is taken into a circle tangent to $(B_1), (B_2)$ through $C', D \Longrightarrow C \equiv C'$. Hence, (R, k^2) carries $(S_1), (S_2), (S_3)$ into themselves $\Longrightarrow R$ is the radical center of $(S_1), (S_2), (S_3)$.

Mashimaru

#4 Nov 30, 2009, 5:12 pm

As in the figure, let T denote the intersection of the two tangents at A, B of (S_2). It is clearly that $\mathcal{P}_{T/(b_1)} = TA^2 = TB^2 \mathcal{P}_{T/(b_2)}$ so T lies on r, which is the radical axis of (b_1) and (b_2). The same argument holds for (S_1) and (S_3).

Now let TA', TB' be the tangents from T of $(b_1), (b_2)$, which is different of TA, TB. Denote $R \equiv A'B' \cap AB$. We have A, B, A', B' are concyclic (they all lie on the circle with center T) so $\overline{RA} \cdot \overline{RB} = \overline{RA'} \cdot \overline{RB'}$, hence R is the center of the inversion takes (b_1) to (b_2) and thus R is also the center of the homothety takes (b_1) to (b_2). Thus, AB passes through the inner center R of homothety of (b_1) and (b_2). The same thing holds for CD and EF, so they AB, CD, EF concurs at R. By the arguments above we also have $\overline{RA} \cdot \overline{RB} = \overline{RC} \cdot \overline{RD} = \overline{RE} \cdot \overline{RF}$, which means that the power of R with respect to $(S_1), (S_2), (S_3)$ is equal to each other. Therefore, R is the radical center of $(S_1), (S_2), (S_3)$, QED.

Quick Reply

High School Olympiads

Geometry problem

Reply

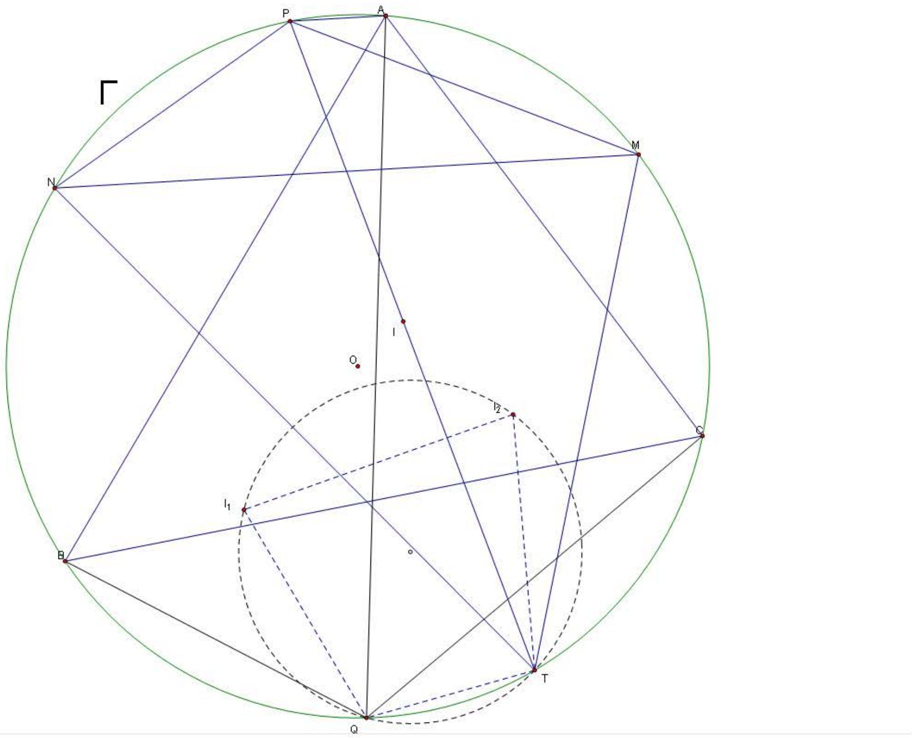
CCMath1

#1 Nov 23, 2009, 7:50 pm

Let O, I be the circumcenter and the incenter of $\triangle ABC$. Let Γ be the circumcircle of $\triangle ABC$, M, N is the midpoint of arc AC, AB (Doesn't contain point B and C) $AP//MN$ .P is on the circumcircle .let $PI \cap \odot O = T$. Then

(1) Prove that : $PM * MT = PN * NT$;
(2) Take point $Q \in arcBTC$ let I_1, I_2 be the incenters of $\triangle ABQ$ and $\triangle ACQ$, prove that :points I_1, I_2, Q, T are concyclic.

Attachments:



livetolove212

#2 Nov 23, 2009, 11:55 pm

a, It's easy to prove that P is the midpoint of arc BC which contains A.
$\Rightarrow T$ is the tangency of A-Mixtilinear incircle of triangle ABC with (O).
Denote $K = TN \cap AB, L = TM \cap AC \Rightarrow I$ is the midpoint of KL.
But $KL \perp AI, MN \perp AI \Rightarrow KL//MN \Rightarrow TI$ is the median of triangle MNT.
By the law of sine we get $\frac{\sin \angle NMP}{\sin \angle MNP} = \frac{\sin \angle NTP}{\sin \angle MTP} = \frac{\sin \angle MNT}{\sin \angle NMT}$
$\Rightarrow \frac{NP}{MP} = \frac{MT}{NT} \Rightarrow NT.NP = MT.MP$
b, We have N, I_1, Q are collinear, M, I_2, Q are collinear.
$\frac{NT}{MT} = \frac{PM}{PN} = \frac{AN}{AM} = \frac{NI_1}{MI_2}$
$\Rightarrow \Delta TNI_1 \sim \Delta TMI_2 \Rightarrow \angle I_1TI_2 = \angle NTM = \angle I_1QI_2 \Rightarrow QED$

Attachments:

Attachments:
picture71.pdf (11kb)

Luis González

#3 Nov 24, 2009, 6:17 am • 1

Let L be the midpoint of the arc $BAC.$ It is well-known that I is the orthocenter of $\triangle MNL,$ hence antipode P of L WRT the circumcircle forms the parallelogram $PMIN \Longrightarrow TP$ is the T-median of $\triangle TMN.$ From the isosceles trapezoid $PAMN,$ we get $\angle MTA=\angle NTP,$ which implies that TA is the T-symmedian of $\triangle TMN.$ Then its T-Apollonius circle cuts its circumcircle (O) at $A \Longrightarrow \frac{TN}{TM}=\frac{AN}{AM}=\frac{PM}{PN}.$

Circles centered at the midpoints N,M of the arcs AB,AC with radii NB and MC contain the incenters I_1,I_2 of $\triangle ABQ$ and $\triangle AQC \Longrightarrow NI_1=AN$ and $MI_2=AM.$ Thus, in the latter proportion we have $\frac{TN}{TM}=\frac{NI_1}{MI_2}.$ Since $\angle I_2MT=\angle I_1NT,$ we deduce that $\triangle I_1NT \sim \triangle I_2MT \Longrightarrow \angle QI_1T=\angle QI_2T \Longrightarrow TQI_1I_2$ is cyclic.

This post has been edited 1 time. Last edited by Luis González, Nov 24, 2009, 5:30 pm

mathVNpro

#4 Nov 24, 2009, 6:53 am

See here: http://forum.mathscope.org/showthread.php?t=8757
Or here for more solutions: http://forum.mathscope.org/showthread.php?t=9366.

I think that this is a problem which belongs to Iran MO 1997,so check it out here also:
http://www.mathlinks.ro/viewtopic.php?p=349634#349634

CCMath1

#5 Nov 24, 2009, 3:54 pm • 1

Here is my solution for this problem:[Sorry for my bad English]
1.Easy to get that B,I,M and C,I,N are collinear , With I the incentre of $\triangle ABC$we easily get $IM=MA$ and $IN=NA$ With equal arcs we get that $APNM$ is an isosceles trapzoid , then $PM=AN=IN,PN=AM=MI$ $\rightarrow MINP$ is a parallelogram , let $NM\cap IP=K$ we get $NK=KM$ with $\frac{PM}{NT}=\frac{PK}{NK}=\frac{PK}{KM}=\frac{PN}{MT}\Rightarrow$ $\boxed{PM*MT=PN*NT}$.
2.Only need to prove that :$\angle I1I2T=\angle TMN$ We only need to prove that $\triangle I1I2T$ □ $\triangle NMT\Leftrightarrow\triangle NI1T$ □ $\triangle MI2T$ we have $\angle QNT=\angle QMT$, We only need to prove :$\frac{NI1}{NT}=\frac{MI2}{MT}$ ....@. With $NPMB$ is an isosceles trapzoid (Easy to get) and $NI1=NB$(the property of incentre) we finally get that $NI1=PM$, similary , We get $MI2=PN$ so with the conclusion of (1) we get @ , So we can get $\angle I1QT+\angle TI2I1=\pi\rightarrow I[1],I[2],Q,T$ are concyclic. QED.
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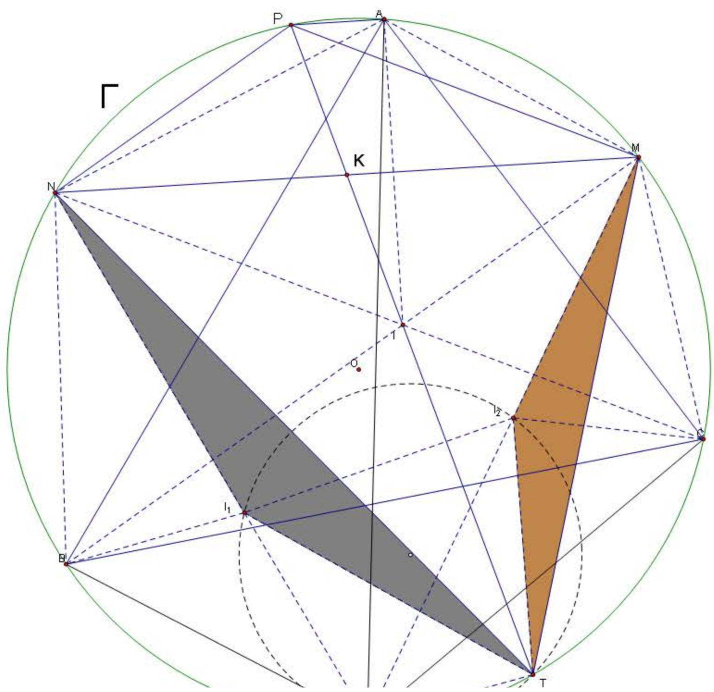
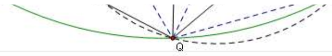
Attachments:





Petry

#6 Nov 25, 2009, 11:42 pm • 1

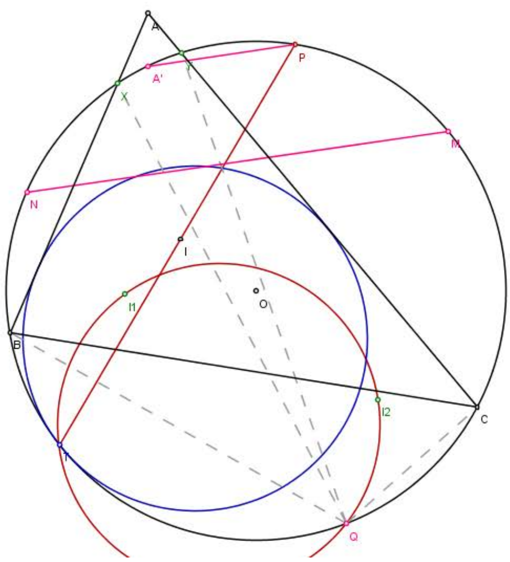
Hello!

The point (2) of the proposed problem can be generalized:
Let I be the incenter of ΔABC, Ω is a circle through the points B, C and $\{X, B\} = AB \cap \Omega$, $\{Y, C\} = AC \cap \Omega$.
A', M, N are the midpoints of the arcs $\widehat{XY}$, $\widehat{CY}$, $\widehat{BX}$ ($A' \notin \widehat{XBY}$, $M \notin \widehat{CBY}$, $N \notin \widehat{BCX}$), $P \in \Omega$ such that $A'P||MN$ and $\{T, P\} = PI \cap \Omega$. If $Q \in \widehat{BTC}$ and I_1, I_2 are the incenters of ΔQXB, ΔQYC then prove that the points I_1, I_2, Q, T are concyclic.

Solution:
It's easy to prove that P is the midpoint of the arc $\widehat{BA'C}$.
Let Γ be the circle that tangent to the lines AB, AC and to the circle Ω (the circles Γ and Ω are internally tangent) and $\{T'\} = \Gamma \cap \Omega$ (the points T' and A are on different sides of BC).
From here http://www.artofproblemsolving.com/Forum/viewtopic.php?t=284806 $\Rightarrow T' = T$.
The message #4 from here http://www.artofproblemsolving.com/Forum/viewtopic.php?t=56605 $\Rightarrow$ the points I_1, I_2, Q, T are concyclic.

Best regards, Petrisor Neagoe :)

Attachments:



Quick Reply

High School Olympiads

Inequality in triangle

Reply

Mateescu Constantin

#1 Nov 23, 2009, 4:01 pm

Show that in any triangle ABC the following inequality holds: $\frac{b^2+c^2}{m_a}+\frac{c^2+a^2}{m_b}+\frac{a^2+b^2}{m_c}\le 12R$.

Luis González

#2 Nov 23, 2009, 6:06 pm • 1

Let M denote the midpoint of BC and let the median AM cut the circumcircle (O) of $\triangle ABC$ at P. From the power of M to (O) we get $AM\cdot MP=BM\cdot MC=\frac{1}{4}a^2$

$$\Longrightarrow AP=AM+MP=\frac{AM^2+\frac{1}{4}a^2}{AM}=\frac{\frac{1}{2}(b^2+c^2)-\frac{1}{4}a^2+\frac{1}{4}a^2}{m_a}=\frac{b^2+c^2}{2m_a}$$

Since a diameter is greater or equal than the chord AP, we have $2R\ge\frac{b^2+c^2}{2m_a}$

By cyclic exchange we have: $2R\ge\frac{a^2+c^2}{2m_b}$, $2R\ge\frac{a^2+b^2}{2m_c}$

$$\Longrightarrow 12R\ge\frac{b^2+c^2}{m_a}+\frac{a^2+c^2}{m_b}+\frac{a^2+b^2}{m_c}$$

Quick Reply

Spain

Mas lugares sobre puntos notables

Reply

Luis González

#1 Apr 30, 2009, 10:12 am

Un triángulo $\triangle ABC$ tiene sus lados AB y AC sobre las semirectas fijas Ax y Ay. Hallar con demostración los lugares geométricos del circuncentro, ortocentro y baricentro del mismo si la suma $AB+AC$ es constante.

Luis González

#2 Nov 22, 2009, 10:47 pm

B', C' son dos puntos en Ax, Ay tales que $AB'=AC'=\frac{1}{2}(AB+AC)$. Se tendrá entonces

$$BB'=CC'=\frac{1}{2}|AB-AC|$$

Las circunferencias $(O)\equiv\odot(ABC)$ y $\odot(AB'C')$ se cortan pues en el centro P de la rotación que lleva $\overrightarrow{BB'}$ sobre $\overrightarrow{CC'}$, por consiguiente $PB=PC\Longrightarrow P$ es un punto fijo siendo este el antipodal de A en la circunferencia $\odot(AB'C')$. La circunferencia circunscrita variable de $\triangle ABC$ forma pues un haz con eje radical (cuerda) común $AP\Longrightarrow$ El lugar geométrico del circuncentro O es la mediatriz ℓ de AP.

Por otro lado si G es baricentro del $\triangle ABC, M, N$ las proyecciones de G en AC, AB y h_b, h_c las alturas desde B y C, como la distancia del baricentro a un lado vale un tercio de la distancia de su vertice opuesto a el, tenemos:

$$GM+GN=\frac{1}{3}(h_b+h_c)=\frac{1}{3}\sin A(AB+AC)=\text{const}$$

Entonces el lugar del baricentro G es una paralela ℓ_1 a la mediatriz ℓ de P, ya que es sabido que el lugar de los puntos cuya suma de distancias a dos rectas fijas es una cantidad constante, son los lados de un rectángulo cuyas diagonales son tales rectas fijas. Como $GO:HG=1:2$, el lugar geométrico del ortocentro H del $\triangle ABC$ es otra recta ℓ_2 paralela a las descritas por O y G de modo que las distancia de ℓ_2 a ℓ_1 es el doble de la distancia de ℓ a ℓ_1.

Quick Reply

High School Olympiads

Quadrilateral and midpoints

Reply

small

#1 Nov 20, 2009, 10:33 pm

Let ABCD be a convex quadrilateral and let E and F be the points of intersections of the lines AB, CD and AD, BC, respectively. Prove that the midpoints of the segments AC, BD, and EF are collinear.

Dimitris X

#2 Nov 21, 2009, 12:20 am

Thats the gauss line of the complete quadrialateral.....

proof
First of all we have to use a theorem that tells us that the circles with diameters the diagonals of the qomplete quadrilateral form a bundle....

Let C_1, C_2, C_3 be the circles with diameter BD, FE, AC.

These 3 circles form a bundle.

Let l be the radical axis of these tree circles,and K, L, N the midpoints of AC, BD, EF.

From well known thereom we now that $KL \perp l$ and $KN \perp l$ (note that K, L, N are the centers of C_1, C_2, C_3).

But from a point there is only one perpendicular line to a given line.
So K, L, N are collinear....

Dimitris

sunken rock

#3 Nov 21, 2009, 12:25 am

Ask Jean-Louis, he will provide you a link where to find 30 various proofs of this theorem.

Best regards,
sunken rock

Luis González

#4 Nov 21, 2009, 11:17 am

Let M, N, L be the midponts of AC, BD, EF and P, Q, R the midpoints of BC, CE, EB. It's is clear that L, M, N lie on the sidelines RQ, QP, PR of the medial triangle $\triangle PQR$ of $\triangle BCE$, hence:

$$\frac{\overline{LD}}{\overline{LR}} = \frac{\overline{FC}}{\overline{FB}} \ , \ \frac{\overline{RN}}{\overline{NP}} = \frac{\overline{ED}}{\overline{DC}} \ , \ \frac{\overline{PM}}{\overline{MD}} = \frac{\overline{BA}}{\overline{AE}} \Longrightarrow$$

$$\frac{\overline{LD}}{\overline{LR}} \cdot \frac{\overline{RN}}{\overline{NP}} \cdot \frac{\overline{PM}}{\overline{MD}} = \frac{\overline{FC}}{\overline{FB}} \cdot \frac{\overline{BA}}{\overline{AE}} \cdot \frac{\overline{ED}}{\overline{DC}} = 1$$

By the converse of Menealus' theorem, the points M, N, L are collinear.

sunken rock

#5 Nov 23, 2009, 4:19 pm

For other (probably) new solution, please, see http://www.mathlinks.ro/viewtopic.php?p=1114247#1114247.

Best regards,
sunken rock

Quick Reply

High School Olympiads

collinear

Source: need help

KDS

#1 Nov 20, 2009, 6:33 pm

1) Let M be a point inside the $\triangle ABC$. The line l_a is perpendicular to MA at M. l_a cuts line BC at A_1. We define B_1, C_1 similarly. Prove that A_1, B_1, C_1 are collinear.

2) Let M be a point inside the $\triangle ABC$. Denote by H the orthocenter of $\triangle ABC$. The line d_a passes through H and is perpendicular to MA. d_a cuts BC at A_0. We define B_0, C_0 similarly. Prove that A_0, B_0, C_0 are collinear.

Luis González

#2 Nov 21, 2009, 5:17 am

> KDS wrote:
> 1) Let M be a point inside the $\triangle ABC$. The line l_a is perpendicular to MA at $M. l_a$ cuts line BC at A_1. We define B_1, C_1 similarly. Prove that A_1, B_1, C_1 are collinear.

Let H be the orthocenter of $\triangle ABC$ and H_a, H_b, H_c the feet of the altitudes onto BC, CA, AB. Circles $(A') \equiv \odot(MAA_1), (B') \equiv \odot(MBB_1)$ and $(C') \equiv \odot(MCC_1)$ with diameters AA_1, BB_1, CC_1 pass through H_a, H_b, H_c, respectively and since $AH \cdot HH_a = BH \cdot HH_b = CH \cdot HH_c$, it follows that $(A'), (B'), (C')$ are coaxal with common radical axis the straight line $HM \Longrightarrow A', B', C'$ lie on a perpendicular to HM. Since A', B', C' are the midpoints of AA_1, BB_1, CC_1, then A', B', C' lie on the sidelines EF, FD, DE of the medial triangle $\triangle DEF$ of $\triangle ABC$, respectively. Therefore

$$\frac{\overline{A'F}}{\overline{A'E}} = \frac{\overline{A_1B}}{\overline{A_1C}}\ ,\ \frac{\overline{B'D}}{\overline{B'F}} = \frac{\overline{B_1C}}{\overline{B_1A}}\ ,\ \frac{\overline{C'E}}{\overline{C'D}} = \frac{\overline{C_1A}}{\overline{C_1B}}$$

$$\frac{\overline{A_1B}}{\overline{A_1C}} \cdot \frac{\overline{B_1C}}{\overline{B_1A}} \cdot \frac{\overline{C_1A}}{\overline{C_1B}} = \frac{\overline{A'F}}{\overline{A'E}} \cdot \frac{\overline{C'E}}{\overline{C'D}} \cdot \frac{\overline{B'D}}{\overline{B'F}} = 1$$

By the converse of Menelaus' theorem, the points A_1, B_1, C_1 are collinear.

> KDS wrote:
> 2) Let M be a point inside the $\triangle ABC$. Denote by H the orthocenter of $\triangle ABC$. The line d_a passes through H and is perpendicular to MA. d_a cuts BC at A_0. We define B_0, C_0 similarly. Prove that A_0, B_0, C_0 are collinear.

Let $X \equiv d_a \cap MA$ and define cyclically Y, Z. Using the same notations above, since AXH_aA_0 is a cyclic quadrilateral on account of the right angles at X and H_a, we have $\overline{HX} \cdot \overline{HA_0} = \overline{AH} \cdot \overline{HH_a}$ and by cyclic exchange:

$$\overline{HY} \cdot \overline{HB_0} = \overline{BH} \cdot \overline{HH_b}\ \ ,\ \ \overline{HZ} \cdot \overline{HC_0} = \overline{CH} \cdot \overline{HH_c}$$

$$\Longrightarrow \overline{HX} \cdot \overline{HA_0} = \overline{HY} \cdot \overline{HB_0} = \overline{HZ} \cdot \overline{HC_0} = k^2$$

Since X, Y, Z lie on the circle with diameter $\overline{HM}$, it follows that A_0, B_0, C_0 lie on the inverse line of the circle with diameter $\overline{HM}$ under the negative inversion with center H and radius k.

mathVNpro

#3 Nov 21, 2009, 1:28 pm

> Quote:
> 1)Let M be a point inside the $\triangle ABC$. The line l_a is perpendicular to MA at M. l_a cuts line BC at A_1. We define B_1, C_1 similarly. Prove that A_1, B_1, C_1 are collinear.

Solution

Consider the inversion through pole M, with any power k. Let A', B', C' and A_1', B_1', C_1' respectively be the image of A, B, C and A_1, B_1, C_1 through $\mathcal{I}(I, k)$. Therefore, we have $\mathcal{I}(I, k) : \overline{ABC_1} \mapsto (A'B'C_1'M)$, $\overline{ACB_1} \mapsto (A'C'B_1'M)$ and $\overline{BCA_1} \mapsto (B'C'A_1'M)$. Then $MC' \perp MC_1'$, $MB' \perp MB_1'$ and $MA' \perp MA_1'$. We claim that M, B_1', C_1', A_1' are concyclic.

Indeed, let O_1, O_2, O_3 respectively be the circumcenters of $(A'C'B_1'M)$, $(B'C'A_1'M)$ and $(A'B'C_1'M)$. Since $O_2O_3 \perp MB'$, $MB_1' \perp MB'$, thus $O_2O_3 || MB_1'$. Therefore, the perpendicular bisector of MB_1' will be the line d_1 passes O_1 and $d_1 \perp O_2O_3$. With the same argument for O_1O_2 and O_1O_3. As the consequence, d_1, d_2, d_3 repsectively will be the perpendiculars from O_1, O_2, O_3 wrt $\triangle O_1O_2O_3$, obviously, they are concurrent at the orthocenter of $\triangle O_1O_2O_3$, which implies that M, B_1', A_1', C_1' are concyclic. Therefore, A_1, B_1, C_1 are collinear. $\square$

shoki

#4 Nov 25, 2009, 8:22 pm

problem 1)

in fact we have $\dfrac{BA_1}{A_1C} = \dfrac{MB}{MC}.\dfrac{\sin \angle BMA_1}{\sin \angle CMA_1}$

if we write the same thing for the other ratios and then multiply them (menelaus ' theorem)
we'll see that each of the sides MA, MB, MC will appear one time as the denominator and another time as the numerator so we have to prove the equality which is consisted of some sinuses of the angles,this part is the same as the previous one, i mean by using the equalities :
$\sin \angle BMA_1 = \sin \angle AMB_1$ and similarly for the other four angles for which their sin become equal with one of the other and u can see that in this way we'll obtain what we wanted.
so we are done.
problem 2)
let $d_a \cap AM = M_a$ and similarly for the other lines.
for this part we do the same thing for computing the ratio $\dfrac{CA_0}{BA_0}$ in fact we have

$$\frac{CA_0}{BA_0} = \frac{CH}{BH}.\frac{\sin \angle CHA_0}{\sin \angle BHA_0}$$

by the same reasoning for the problem one and by using the equality (use the fact that $AFHM_aE$ is cyclic wher F, E are the feet of altitudes)
$\sin \angle BHA_0 = \sin \angle M_aAC = \sin \angle MAC \quad ; \quad \sin \angle CHA_0 = \sin \angle M_aAB = \sin \angle MAB$

and similarly for the other angles we must prove that :
$$\frac{\sin \angle MAB}{\sin \angle MAC}.\frac{\sin \angle CBM}{\sin \angle MBA}.\frac{\sin \angle MCA}{\sin \angle MCB} = 1$$

which is obvious since the lines AM, BM, CM are concurrent at M (ceva's theorem,sinus form)
so we are done!
in fact in this way we have also proven this (the converse statement):
suppose that A_1, B_1, C_1 are some points on BC, CA, AB respectively.
let the perpendicular from H to AA_1 meet BC at A_c and similarly for the other sides.
then prove that A_0, B_0, C_0 are collinear if and only if AA_1, BB_1, CC_1 are concurrent.
here is an added problem similar to problem 1:
added problem (it may be posted before):
let l be a line which intersects the sides of ABC at A_1, B_1, C_1 respectively.
consider a point P on the plane of ABC and let the perpendicular from P to PA_1 intersects l at A_2.
define similarly B_2, C_2.
prove the concurrency of AA_2, BB_2, CC_2.
hint

livetolove212

#5 Nov 26, 2009, 1:19 pm

Problem 2: Denote A'', B'', C'' the projections of A, B, C on BC, CA, AB, respectively. $C'H \cap CMatI$.

We have $C'M^2 - HM^2 = C'I^2 - HI^2 = 2C'H.HI + C'H^2$
$\Rightarrow C'M^2 - C'H^2 = HM^2 + 2C'H.HI = HM^2 + 2HC.HC''$ (because $C'C''IC$ is cyclic)
Similarly $A'M^2 - A'H^2 = HM^2 + 2HA.HA''$, $B'M^2 - B'H^2 = HM^2 + 2HB.HB''$.
Note that $HA.HA'' = HB.HB'' = HC.HC''$
$\Rightarrow A'M^2 - A'H^2 = B'M^2 - B'H^2 = C'M^2 - C'H^2$.
So A', B', C' lie on a line which is perpendicular to HM.

Attachments:
picture76.pdf (5kb)

Quick Reply

High School Olympiads

cyclic quadrilateral

Reply

stvs_f
#1 Oct 8, 2009, 9:11 am

let ABCD is a Cyclic quadrilateral that: (AB,CD)=E and (AD,BC)=F.
M and N are midpoints of AC and BD .if AC>BD prove that:

$$\frac{MN}{EF}=\frac{1}{2}(\frac{AC}{BD}-\frac{BD}{AC})$$

stvs_f
#2 Oct 10, 2009, 9:00 am

any solution?????

77ant
#3 Oct 10, 2009, 9:15 am

please see http://www.mathlinks.ro/viewtopic.php?search_id=1795840187&t=64864

Luis González
#4 Nov 20, 2009, 6:09 am

Let L be the midpoint of EF, which lies on the Newton line MN of the complete quadrangle $ABCD$. Since E,F are conjugate points WRT the circumcircle (O) of $ABCD$, then the circle (L) with diameter $\overline{EF}$ belows to the orthogonal pencil defined by (O) and the axis EF. Thus (L) is orthogonal to $(O) \Longrightarrow (L)$ is also orthogonal to the circle (O') with diameter OK, where $K \equiv AC \cap BD$ is the pole of EF WRT $(O) \Longrightarrow$ power of L to (O') equals $LE^2=LF^2$. Therefore

$$LE^2=LN\cdot LM=(LM+MN)LM \Longrightarrow \frac{MN}{LE}=\frac{LE}{LM}-\frac{LM}{LE} \ (\star)$$

Let P,Q,R be the midpoints of BC,CE,EB. It is clear that $ERPQ$ is a parallelogram and $L \in RQ$. By Menelaus theorem for $\triangle PQR$ cut by the Newton line of $ABCD$, we have

$$\frac{LM}{LN}\cdot\frac{RN}{RP}\cdot\frac{PQ}{MQ}=1 \Longrightarrow \frac{LM}{LN}=\frac{EC}{ED}\cdot\frac{EA}{EB}=\left(\frac{EA}{ED}\right)^2=\left(\frac{AC}{BD}\right)^2$$

Combining with the power of L to (O'), we get $\frac{LM}{LE}=\frac{AC}{BD}$

Substituting the previous ratio into the expression $(\star)$ yields

$$\frac{MN}{LE}=\frac{BD}{AC}-\frac{AC}{BD} \Longrightarrow \frac{MN}{EF}=\frac{1}{2}\left|\frac{AC}{BD}-\frac{BD}{AC}\right|.$$

Quick Reply

High School Olympiads

incircle of a quadrilateral

Reply

Source: Indonesia IMO 2007 TST, Stage 2, Test 5, Problem 1

Raja Oktovin

#1 Nov 15, 2009, 12:52 pm

Let $ABCD$ be a cyclic quadrilateral and O be the intersection of diagonal AC and BD. The circumcircles of triangle ABO and the triangle CDO intersect at K. Let L be a point such that the triangle BLC is similar to AKD (in that order). Prove that if $BLCK$ is a convex quadrilateral, then it has an incircle.

Luis González

#2 Nov 16, 2009, 9:15 am

Let $P \equiv OK \cap BC$. From the cyclic quadrilaterals $ABKO, DCKO$ it's easy to see that KP bisects $\angle BKC$ since $\angle BKP = \angle BAO = \angle ODC = \angle CKP$. Let the angle bisectors of $\angle KBL$ and $\angle KCL$ cut LC, LB at B', C'.

$$\angle KCL = \angle KCB + \angle ADK = \angle ACB - \angle KCO + \angle ADB + \angle ODK$$

$$\Longrightarrow \angle KCL = 2\angle BCA = \angle BCC' = \angle KCA$$

Similarly, $\angle CBB' = \angle KBD$, which implies that CA, CC' and BD, BB' are isogonals WRT $\angle KCB$ and $\angle KBC$, respectively. Hence if $I \equiv BB' \cap CC'$, then O, I are isogonal conjugates WRT $\triangle BKC$, but angle bisector KO of $\angle BKC$ is self-isogonal $\Longrightarrow I \in KP \Longrightarrow$ internal angle bisectors of the convex quadrilateral $BLCK$ concur at I.

Quick Reply

High School Olympiads

Find the circumcenter

Reply

Source: Own, old

sunken rock

#1 Nov 15, 2009, 4:11 pm

Let P be a random point on the circle circumscribed to the trapezoid $ABCD$ with $AB \parallel CD$ whilst K, L, M and N respectively being the projections of P onto AC, BD, BC and AD respectively.
Prove that K, L, M and N lie on a circle; find the center of this circle.

Best regards,
sunken rock

This post has been edited 2 times. Last edited by sunken rock, Nov 16, 2009, 12:18 pm

Luis González

#2 Nov 16, 2009, 2:58 am

Let Q, R be the projections of P onto $DC, AB.$ $(N, L, R), (M, K, R), (N, Q, K)$ are collinear on the Simson lines with pole P with respect to the triangles $\triangle ABD, \triangle ABC$ and $\triangle ACD.$ From the cyclic pentagons $DLQPN$ and $CKQPM,$ it follows that $\overline{PQR}$ is the radical axis of their circumcircles $\Longrightarrow RK \cdot RM = RL \cdot RN \Longrightarrow$ Quadrilateral $KLNM$ is cyclic. From the collinearity of N, Q, K and the cyclic $LQPN, KQPM$ we have

$\angle KMP = \angle RQK = \angle NQP = \angle NLP \Longrightarrow LRMP$ is cyclic.

Note that PR bisects $\angle LPK,$ because of $\angle BDC = \angle LPR = \angle ACD = \angle KPR.$ Hence, if O is the circumcenter of $KLMN,$ we have $\angle LOK = 2\angle LMK = \angle LPK \Longrightarrow LKOP$ is cyclic. Thus, circumcenter of $KLMN$ is the midpoint of the arc $LPK.$

P.S. However, I cannot see O as an incenter of some triangle, could you clarify what you mean with "find the incenter" ?.

nsato

#3 Nov 16, 2009, 12:15 pm

Let O_1 be the projection of P onto the perpendicular bisector of AB (which is the same as the perpendicular bisector of CD).
Let $S = AD \cap BC$, and let $T = AC \cap BD$.

It is not hard to see that S, M, N, O_1, and P are concyclic. Furthermore, $\angle NSO_1 = \angle MSO_1$, so O_1 lies on the perpendicular bisector of MN. Similarly, T, K, L, O_1, and P are concyclic, and O_1 lies on the perpendicular bisector of KL.
Hence, O_1 is the circumcenter of quadrilateral $KLMN$.

This post has been edited 1 time. Last edited by nsato, Nov 16, 2009, 11:31 pm

sunken rock

#4 Nov 16, 2009, 2:44 pm

To Luis:
Sorry for the wrong wording, I edited (the title).
For the first step, my proof was exactly as yours, for the 2nd far more complicated than **nsato**'s, using a homothecy centered P and ratio 1/2, i.e. O' being on the circle diameter OP, that is, the midpoint of the chord PP' || AB.

Remark: *The projections of P', the other end of the chord parallel to the trapezoid bases onto the same sides and diagonals lie on the same circle..*

Best regards,
sunken rock

Quick Reply

High School Olympiads

triangles- |EASY|

Reply

Rijul saini
#1 Nov 15, 2009, 9:45 pm

Prove that the triangle formed by the points of contact of the sides of a given triangle with the ex circles corresponding to these sides is equivalent to the triangle formed by the points of contact of the sides of the triangle with the inscribed circle.

Luis González
#2 Nov 15, 2009, 11:50 pm

There is a more general result, which states that the cevian triangles of two isotomic points with respect to a given triangle have the same area. Note that the Nagel point and the Gergonne point are obviously isotomic conjugates. Thus, the intouch and extouch triangle have equal area. For a proof of the referred property see the topic collinear-shariguin.

P.S. Alternatively, note that the extouch triangle is the pedal triangle of the Bevan point, which is the reflection of the incenter about the circumcenter. Then use Euler's theorem.

Quick Reply

High School Olympiads

collinear

stvs_f

#1 Aug 16, 2009, 2:56 pm

let <AOB is an angle that $A_1,...,A_4$ are 4point on AO and $B_1,...,B_4$ are 4point on BO.
$(A_1B_1,A_2B_2)=N$ and $(A_4B_4,A_3B_3)=M$.prove that:O,N,M are collinear if and only if:

$$\frac{OB_1}{OB_3}\cdot\frac{OB_2}{OB_4}\cdot\frac{B_3B_4}{B_1B_2}=\frac{OA_1}{OA_3}\cdot\frac{OA_2}{OA_4}\cdot\frac{A_3A_4}{A_1A_2}$$

Luis González

#2 Nov 15, 2009, 7:56 am

Let P and Q be the intersections of the ray passing through O and $A_1B_2\cap A_2B_2$ with A_2B_2 and A_4B_4 respectively. From the harmonic pencil $O(A_2,B_2,P,N)$, we get

$$O,M,N \text{ are collinear} \Longleftrightarrow (A_4,B_4,Q,M)=-1 \Longleftrightarrow \frac{QA_4}{QB_4}=\frac{MA_4}{MB_4}\ (\star)$$

By Menelaus' theorem for $\triangle OA_4B_4$ cut by the tranversal $\overline{A_3B_3M}$ we get

$$\frac{MA_4}{MB_4}=\frac{A_4A_3}{OA_3}\cdot\frac{OB_3}{B_3B_4}\quad (1)$$

On the other hand, by Menelaus' theorem for $\triangle OA_2B_2$ cut by the tranversal $\overline{A_1B_1N}$, we deduce that

$$\frac{QA_4}{QB_4}=\frac{OA_4}{OB_4}\cdot\frac{\sin\widehat{AOQ}}{\sin\widehat{BOQ}}=\frac{OA_4}{OB_4}\cdot\frac{PA_2}{PB_2}\cdot\frac{OB_2}{OA_2}$$

$$\Longrightarrow \frac{QA_4}{QB_4}=\frac{OA_4}{OB_4}\cdot\frac{OB_2}{OA_2}\cdot\frac{A_1A_2}{OA_1}\cdot\frac{OB_1}{B_1B_2}\quad (2)$$

Substituting (1) and (2) into $(\star)$ gives

$$\frac{OA_4}{OB_4}\cdot\frac{OB_2}{OA_2}\cdot\frac{A_1A_2}{OA_1}\cdot\frac{OB_1}{B_1B_2}=\frac{A_4A_3}{OA_3}\cdot\frac{OB_3}{B_3B_4}$$

$$\Longrightarrow \frac{OB_1}{OB_3}\cdot\frac{OB_2}{OB_4}\cdot\frac{B_3B_4}{B_1B_2}=\frac{OA_1}{OA_3}\cdot\frac{OA_2}{OA_4}\cdot\frac{A_3A_4}{A_1A_2}$$

High School Olympiads

inradii 3 times!

Source: Indonesia IMO 2010 TST, Stage 1, Test 3, Problem 4

Raja Oktovin

#1 Nov 12, 2009, 10:06 am

Let ABC be an acute-angled triangle such that there exist points D,E,F on side BC,CA,AB, respectively such that the inradii of triangle AEF,BDF,CDE are all equal to r_0. If the inradii of triangle DEF and ABC are r and R, respectively, prove that

$$r+r_0=R.$$

Soewono, Bandung

math10

#2 Nov 12, 2009, 10:13 am

> *Raja Oktovin wrote:*
> Let ABC be an acute-angled triangle such that there exist points D,E,F on side BC,CA,AB, respectively such that the inradii of triangle AEF,BDF,CDE are all equal to r_0. If the inradii of triangle DEF and ABC are r and R, respectively, prove that
>
> $$r+r_0=R.$$
>
> *Soewono, Bandung*

Indonesia IMO 2010 TST : :
it is too early to choose team IMO 2010 .

Raja Oktovin

#3 Nov 12, 2009, 11:15 am

cool enough uh?

fyi,
regional selection test was in april 2009,
national olympiad was in august 2009,
about 30 contestants go to 1st stage
the 1st stage of tst is in october-november 2009,
about 12 contestants go to 2nd stage
the 2nd stage of tst will be in about march 2010 (together with APMO 2010),
the 2nd stage is to choose 6 IMO 2010 contestants
the 3rd stage is the last preparation to IMO.

Luis González

#4 Nov 13, 2009, 11:47 pm

Let O_1,O_2,O_3 denote the centers of the incircles of $\triangle AFE, \triangle BDF, \triangle CED$. $(O_1),(O_2),(O_3)$ touch EF,FD,DE, at M,N,L. $(O_2),(O_3)$ touch BC, at P,Q. $(O_3),(O_1)$ touch CA at R,S and $(O_1),(O_2)$ touch AB at T,U. Since

O_1O_2UT, O_2O_3QP and O_3O_1SR are rectangles and O_1, O_2, O_3 lie on the internal bisectors of $A, B, C,$ it follows that $\triangle ABC$ and $\triangle O_1O_2O_3$ are centrally similar through their common incenter I. The similarity coefficient equals the ratio between their perimeters/inradii

$$\frac{r-r_0}{r}=\frac{O_1O_2+O_2O_3+O_3O_1}{a+b+c}=\frac{PQ+RS+TU}{a+b+c}$$

$$\frac{r-r_0}{r}=\frac{DN+DL+EL+EM+FM+FN}{a+b+c}=\frac{DE+EF+FD}{a+b+c} \ (1)$$

On the other hand, we have:

$$r(a+b+c)=r_0(a+b+c+DE+EF+FD)+\varrho(DE+EF+FD)$$

$$\implies r-r_0=\frac{(DE+EF+FD)(r_0+\varrho)}{a+b+c} \ (2)$$

Substituting $(r-r_0)$ from (1) into (2) yields $r=r_0+\varrho$.

Quick Reply

High School Olympiads

O1, O2, O3 are collinear

Reply

Source: Indonesia IMO 2010 TST, Stage 1, Test 4, Problem 2

Raja Oktovin
#1 Nov 12, 2009, 10:09 am

Circles Γ_1 and Γ_2 are internally tangent to circle Γ at P and Q, respectively. Let P_1 and Q_1 are on Γ_1 and Γ_2 respectively such that P_1Q_1 is the common tangent of P_1 and Q_1. Assume that Γ_1 and Γ_2 intersect at R and R_1. Define O_1, O_2, O_3 as the intersection of PQ and P_1Q_1, the intersection of PR and P_1R_1, and the intersection QR and Q_1R_1. Prove that the points O_1, O_2, O_3 are collinear.
Rudi Adha Prihandoko, Bandung

Agr_94_Math
#2 Nov 12, 2009, 8:30 pm

By Desargue's Theorem, we just have to prove that PP_1, QQ_1, RR_1 are concurrent .
So we have to prove that meet of PP_1 and QQ_1 lies on RR_1. RR_1 is the radical axis of Γ_1 and Γ_2. So we have to prove that X lies on the the radical axis of Γ_1 and Γ_2, where X is the intersection of PP_1 and QQ_1. That is, X has equal power with respect to the two circles Γ_1 and Γ_2. We prove this by the definition of power of points, or in other words, quadrilateral PQQ_1P_1 is cyclic. Using powers of O_1, O_2, O_3, we can easily get the cyclic condition. Hence done.

Luis González
#3 Nov 13, 2009, 9:43 am

Since P and Q are exsimilicenters of $\Gamma_1 \sim \Gamma$ and $\Gamma_2 \sim \Gamma$, it follows that $O_1 \equiv PQ \cap P_1Q_1$ is the exsimilicenter of Γ_1, Γ_2. O_1 is also center of the inversion with power O_1R^2 taking Γ_1 and Γ_2 into each other, therefore the pairs P_1, Q_1 and P, Q are inverse points $\Longrightarrow PQQ_1P_1$ is cyclic. Let Γ_3 denote its circumcircle. RR_1, PP_1, QQ_1 concur at the radical center S of Γ_1, Γ_2, Γ_3. By Desargues theorem, the triangles $\triangle PRQ$ and $\triangle P_1R_1Q_1$ are perspective through $S \Longrightarrow$ Intersections $O_1 \equiv PQ \cap P_1Q_1, O_2 \equiv PR \cap P_1R_1, O_3 \equiv RQ \cap R_1Q_1$ are collinear, as desired.

nawaites
#4 Jul 17, 2014, 6:42 pm

How to prove PQQ1P1 by powerpoibt since your explanation sir Agr_94_math is not very clear we cant saya tge intersection is at the radical axis,isnt it???

Quick Reply

High School Olympiads

Croatian Mathematical Olympiad 2006

Reply

Source: problem 3

yuyang

#1 Nov 11, 2009, 7:05 pm

The circles Γ_1 and Γ_2 intersect at the points A and B. The tangent line to Γ_2 through the point A meets Γ_1 again at C and the tangent line to Γ_1 through A meets Γ_2 again at D. A half-line through A, interior to the angle $\angle CAD$, meets Γ_1 at M, meets Γ_2 at N, and meets the circumcircle of $\triangle ACD$ at P. Prove that $|AM| = |NP|$.

Luis González

#2 Nov 12, 2009, 5:02 am

Inversion with center A and radius k takes the circles Γ_1 and Γ_2 into the lines γ_1 and γ_2, respectively parallel to the double lines AD and AC. Hence, $A' \equiv \gamma_1 \cap \gamma_2, D' \equiv AD \cap \gamma_2, C' \equiv AC \cap \gamma_1$ are the inverses of B, D, C. Circle $\odot(ACD)$ is taken into the line $C'D' \Longrightarrow P' \equiv AP \cap C'D', M' \equiv AM \cap \gamma_1, N' \equiv AN \cap \gamma_2$ are the inverses of P, M, N. From this topic, we know that the circle centered at P' with radius $P'A$ is orthogonal to the circle $\odot(A'M'N')$

$$\Longrightarrow P'A^2 = P'M' \cdot P'N' = (AM' - AP')(AN' - AP')$$

Using the power of inversion k^2, we obtain

$$\frac{k^4}{AP^2} = \left(\frac{k^2}{AM} - \frac{k^2}{AP}\right) \cdot \left(\frac{k^2}{AN} - \frac{k^2}{AP}\right)$$

$$\Longrightarrow \frac{1}{AP^2} = \left(\frac{1}{AM} - \frac{1}{AP}\right) \cdot \left(\frac{1}{AN} - \frac{1}{AP}\right) \Longrightarrow AM \cdot AN = PM \cdot PN$$

Which implies that P and A equidistant from the midpoint of $\overline{MN} \Longrightarrow AM = NP.$

Quick Reply

High School Olympiads

Croatian Mathematical Olympiad 2006

Reply

Source: problem 1

yuyang

#1 Nov 11, 2009, 6:59 pm

Prove that three tangents to a parabola always form the sides of a triangle whose altitudes intersect on the directrix of the parabola.

Luis González

#2 Nov 11, 2009, 8:38 pm

Let $\mathcal{P}$ be a parabola with focus F and directrix $p.$ $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are three arbitrary tangents of $\mathcal{P}.$ Define the intersections $A \equiv \mathcal{L}_1 \cap \mathcal{L}_2, B \equiv \mathcal{L}_1 \cap \mathcal{L}_3, C \equiv \mathcal{L}_2 \cap \mathcal{L}_3.$ A', B', C' are the orthogonal projections of F on $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3.$ Reflections of focus F across A', B', C' lie on $d,$ thus A', B', C' are collinear $\Longrightarrow F \in \odot(ABC).$ Therefore, the directrix $p,$ which is the image of $\overline{A'B'C'}$ under the dilatation with center F and coefficient $2,$ becomes Steiner line with pole F WRT $\triangle ABC \Longrightarrow p$ passes through the orthocenter of $\triangle ABC.$

Quick Reply

High School Olympiads

Prove KG = AB

Source: Nice problem

thanhnam2902
#1 Nov 6, 2009, 11:24 pm

Let circle (O) with AB is a diameter, let (d) is tangent of circle at point B. Let C is a point lie on circle and let H is perpendicular foot of point C on AB. Let D is a point lie on CH ray pass point H such that $HD = AC$. Two lines pass point D and contact with circle (O) meet line (d) at K and G respectively, with K lie on segment BG. Prove that $KG = AB$.

Attachments:
kG=2R.pdf (9kb)

Petry
#2 Nov 7, 2009, 9:26 pm

Hello!
Let R be the radius of the circle (O).
$DE = DF$, $GE = GB$ and $KF = KB$.

$DE^2 = DO^2 - OE^2 = HD^2 + OH^2 - R^2 = AC^2 - (R^2 - OH^2) =$
$= AC^2 - CH^2 = AH^2 \Rightarrow DE = AH$.
So $DE = DF = AH$.

$T \in GK$ such that $DT \perp GK \Rightarrow DT = HB$ and $BT = HD = AC$.

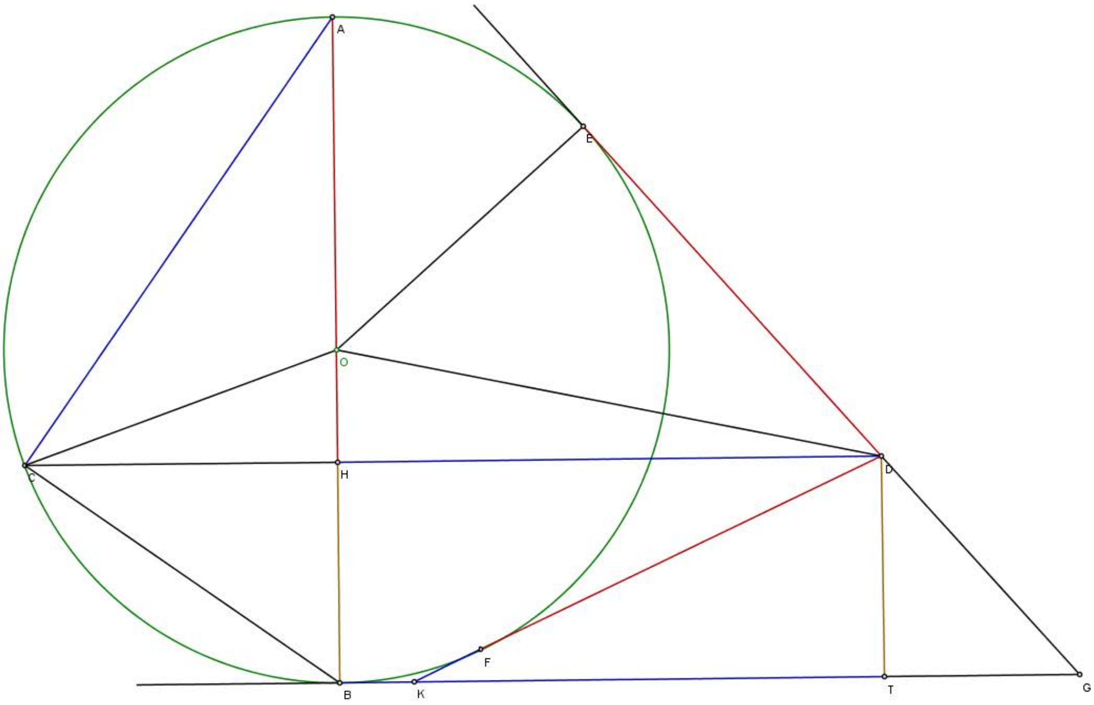
$GD^2 = GT^2 + DT^2 \Rightarrow (GE - DE)^2 = (GB - BT)^2 + HB^2 \Rightarrow$
$\Rightarrow (GB - AH)^2 = (GB - AC)^2 + HB^2 \Rightarrow$
$\Rightarrow GB^2 - 2 \cdot GB \cdot AH + AH^2 = GB^2 - 2 \cdot GB \cdot AC + AC^2 + HB^2 \Rightarrow$
$\Rightarrow 2 \cdot GB \cdot AC - 2 \cdot GB \cdot AH = HB^2 + AC^2 - AH^2 \Rightarrow$
$\Rightarrow 2 \cdot GB \cdot (AC - AH) = HB^2 + HC^2 \Rightarrow GB = \dfrac{BC^2}{2 \cdot (AC - AH)} \Rightarrow$
$\Rightarrow GB = \dfrac{2 \cdot R \cdot HB}{2 \cdot (AC - AH)} \Rightarrow GB = \dfrac{R \cdot HB}{AC - AH}$ (*)

$DK^2 = DT^2 + KT^2 \Rightarrow (DF + KF)^2 = HB^2 + (BT - KB)^2 \Rightarrow$
$\Rightarrow (AH + KB)^2 = HB^2 + (AC - KB)^2 \Rightarrow$
$\Rightarrow AH^2 + 2 \cdot AH \cdot KB + KB^2 = HB^2 + AC^2 - 2 \cdot AC \cdot KB + KB^2 \Rightarrow$
$\Rightarrow 2 \cdot AC \cdot KB + 2 \cdot AH \cdot KB = HB^2 + AC^2 - AH^2 \Rightarrow$
$\Rightarrow 2 \cdot KB \cdot (AC + AH) = HB^2 + HC^2 \Rightarrow KB = \dfrac{BC^2}{2 \cdot (AC + AH)} \Rightarrow$
$\Rightarrow KB = \dfrac{2 \cdot R \cdot HB}{2 \cdot (AC + AH)} \Rightarrow KB = \dfrac{R \cdot HB}{AC + AH}$ (**)

(*), (**) $\Rightarrow GK = GB - KB = \dfrac{R \cdot HB}{AC - AH} - \dfrac{R \cdot HB}{AC + AH} =$
$= R \cdot HB \cdot \left(\dfrac{1}{AC - AH} - \dfrac{1}{AC + AH}\right) = R \cdot HB \cdot \dfrac{2 \cdot AH}{AC^2 - AH^2} =$
$= 2 \cdot R \cdot \dfrac{AH \cdot HB}{CH^2} = 2 \cdot R = AB \Rightarrow GK = AB.$

Best regards, Petrisor Neagoe

Attachments:



thanhnam2902

#3 Nov 10, 2009, 5:32 pm

Thank you very much. I have a solution for this problem but it's very long same your solution. Who can solve this problem by other way? Thank you so much.

Luis González

#4 Nov 10, 2009, 9:38 pm

Problem: In $\triangle ABC$, let D the tangency point of the B-excircle (I_b, r_b) with BC and Q the antipode of D WRT (I_b). Perpendicular from A to DQ cuts DQ and (I_b) at P, E, respectively such that E, A lie on different sides of DQ. If $EQ = AP$, then $a = 2r_b$.

In the right $\triangle EQD$ we have $EQ^2 = PQ \cdot DQ = 2r_b(2r_b - h_a) \quad (1)$

By Pythagorean theorem in $\triangle API_b$ we obtain

$$AP^2 = AI_b{}^2 - PI_b{}^2 = r_b{}^2 + (s-c)^2 - (r_b - h_a)^2$$

$$AP^2 = (s-c)^2 + 2h_a r_b - h_a{}^2 \quad (2)$$

Since $EQ = AP$, from the equations (1) and (2) we obtain

$$2r_b(2r_b - h_a) = (s-c)^2 + h_a(2r_b - h_a) \Longrightarrow 2r_b - h_a = s - c$$

Using the identity $2r_b = \dfrac{a \cdot h_a}{s-b}$, the above expression simplifies as

$$h_a\left(\frac{a+b-s}{s-b}\right) = s - c \Longrightarrow h_a = s - b = \frac{2r_b(s-b)}{a} \Longrightarrow a = 2r_b$$

Quick Reply

High School Olympiads

Collinearity + Perpendicularity.

Reply

Luis González

#1 Nov 8, 2009, 9:23 pm

In the scalene $\triangle ABC,$ let I, N_a, J be its incenter, Nagel point and symmedian point of its excentral triangle $\triangle I_aI_bI_c.$ D, E, F are the midpoints of BC, CA, AB and X, Y, Z are the tangency points of (I) with $BC, CA, AB.$ Perpendiculars from X, Y, Z to AN_a, BN_a, CN_a cut EF, FD, DE at $A', B', C'.$ Show that A', B', C' lie on a perpendicular to $IJ.$

nsato

#2 Nov 9, 2009, 9:51 am

Are you sure? On the triangles I've tried, A', B', and C' are not collinear.

yetti

#3 Nov 9, 2009, 11:04 am

I, Na are incenter and Nagel point of the reference $\triangle ABC,$ while J is symmedian point of the excentral $\triangle I_aI_bI_c,$ I trust.

I_bI_cBC is cyclic, BC with midpoint D is antiparallel of I_bI_c WRT $\angle I_cI_aI_b \Longrightarrow I_aD$ is A-symmedian of $\triangle I_aI_bI_c.$ Similarly, I_bE, I_cF are its B-, C_symmedians $\Longrightarrow I_aD, I_bE, I_cF$ concur at $J.$ Let H_a be reflection of I_a in the midpoint $D,$ the orthocenter of $\triangle IBC. ZX, EF, CI$ concur at P and XY, EF, BI concur at Q (well known) $\Longrightarrow$ the altitudes BH_a, CH_a are polars of P, Q WRT incircle (I) and $EF \equiv PQ$ is polar of H_a WRT $(I).$ Similarly, reflections H_b, H_c of I_b, I_c in the midpoints E, F are polars of FD, DE WRT $(I).$ From central similarity $(I_a) \sim (I)$ with similarity center $A, ID \parallel ANa,$ the perpendicular $p_a \perp ANa$ from X is polar of the midpoint D WRT $(I),$ and the A-symmedian $DH_a \equiv I_aJ$ is polar of the intersection $A' \equiv p_a \cap EF$ WRT $(I).$ Similarly, perpendiculars $p_b \perp BNa, p_c \perp CNa$ from Y, Z are polars of the midpoints E, F WRT (I) and the B-, C-symmedians $EH_b \equiv I_bJ, FH_c \equiv I_cJ$ are polars of the intersections $B' \equiv p_b \cap FD, C' \equiv p_c \cap DE$ WRT $(I) \Longrightarrow A', B', C'$ are collinear, $A'B'C'$ is polar of J WRT (I) and $A'B'C' \perp IJ.$

nsato

#4 Nov 9, 2009, 9:46 pm

> yetti wrote:
>
> I, Na are incenter and Nagel point of the reference $\triangle ABC,$ while J is symmedian point of the excentral $\triangle I_aI_bI_c,$ I trust.

Darn, that's why I couldn't get it to work.

Luis González

#5 Nov 10, 2009, 9:04 am • 1

Thanks for your solution Vladimir, my approach is very similar to yours.
To nsato, sorry for the inconvenience, perhaps the enunciation was a little ambiguous.

Since I is the Nagel point of the medial triangle $\triangle DEF,$ we have that $DI \parallel AN_a, EI \parallel BN_a$ and $FI \parallel CN_a \Longrightarrow XA' \perp ID, YB' \perp IE$ and $ZC' \perp IF.$ Hence, XA', YB', ZC' are the polars of D, E, F WRT $(I).$ On the other hand, let M, N be the orthogonal projections of C, B on $BI, CI. A'' \equiv CM \cap BN$ is the othocenter of $\triangle BIC.$ Since $A''C \parallel BI_a$ and $A''B \parallel CI_a \Longrightarrow A''CI_aB$ is a parallelogram $\Longrightarrow A'' \in DI_a$ and analogously $B'' \in EI_b, C'' \in FI_c.$

Lines $A''I_a, B''I_b, C''I_c$ issuing from the vertices of $\triangle I_aI_bI_c$ and passing through the midpoints of the antiparallel sections BC, CA, AB to its opposite sides are presicely the symmedians of $\triangle I_aI_bI_c$ meeting at $J,$ and it is well-known that A'', B'', C'' are the poles of the sidelines EF, FD, DE of the medial triangle WRT $(I).$ Thus A', B', C' lie on the polar of J WRT $(I) \Longrightarrow A'B'C'$ is perpendicular to $IJ.$

Quick Reply

High School Olympiads

Collinear iff

77ant

#1 Nov 8, 2009, 1:31 am

Dear everyone.

Please think of the following.

A circle with center O is internally tangent to two circles inside it, with centers O1 and O2, at points S and T respectively.
Suppose the two circles inside intersect at points M, N with N closer to ST.
Show that S, N, T are collinear if and only if SO1/OO1 = OO2/TO2.

Thank you for reading.

Luis González

#2 Nov 8, 2009, 9:19 am

O,O_1,S and O,O_2,T are collinear and triangles OST,O_1SN,O_2NT are isosceles whose legs are the radii of $(O),(O_1),(O_2)$, respectively. If S,N,T are collinear, from $\triangle O_1SN \sim \triangle O_2NT$, due to $\angle OST=\angle OTS$ and $\angle O_1NS=\angle O_2NT$, we obtain $\frac{NO_1}{TO_2}=\frac{SO_1}{NO_2}$. But it's easy to see that OO_1NO_2 is a parallelogram $\Longrightarrow NO_1=OO_2$ and $NO_2=OO_1 \Longrightarrow \frac{OO_2}{TO_2}=\frac{SO_1}{OO_1}$.

Assume that $\frac{OO_2}{TO_2}=\frac{SO_1}{OO_1}$, which implies $\frac{OO_2+TO_2}{TO_2}=\frac{SO_1+OO_1}{OO_1}$

$\Longrightarrow TO_2=OO_1=NO_2$ and similarly we'll have $OO_2=SO_1=NO_1 \Longrightarrow OO_1NO_2$ is a parallelogram.

High School Olympiads

Open problem

Reply

Source: Is it a famous theorem

borislav_mirchev

#1 Nov 8, 2009, 5:03 am

I would like to ask you the following question for the problem below:
Is it a famous theorem or a problem from a competition?

It is given an acute angled triangle ABC and a random point P inside it.
Ga, Gb, Gc are respectively the centers of the gravity of the triangles:
BCP, CAP, ABP. Prove that the lines: AGa, BGb, CGc
intersects at a common point.

The statement is true even if ABC is any triangle and P is outside it.

I'm sorry - the problem is not for this section - it is too easy. I solved it in a minutes. Please don't solve it.

Luis González

#2 Nov 8, 2009, 5:51 am • 1

G is the centroid of $\triangle ABC$ and let M be the midpoint of BC. AM cuts GP at P'. By Menelaus' theorem for $\triangle GPM$ cut by $\overline{AP'G_a}$, we have

$$\frac{\overline{AG}}{\overline{AM}} \cdot \frac{\overline{MG_a}}{\overline{G_aP}} \cdot \frac{\overline{PP'}}{\overline{P'G}} = 1 \Longrightarrow \frac{\overline{P'G}}{\overline{P'P}} = -\frac{2}{3} \cdot \frac{1}{2} = -\frac{1}{3}.$$

Analogously, we show that BG_b, CG_c pass through the same point P'.

Quick Reply

High School Olympiads

Find M in triangle ABC

Reply

Source: Help please!!

muroanxfile019

#1 Nov 7, 2009, 8:16 pm

Give $\triangle ABC$, find M in triangle so that $MA^2 = MB^2 + MC^2$ and $\angle AMC = 90^\circ$

shoki

#2 Nov 8, 2009, 1:24 am

one of the problems of the book : An Intro. To The Modern Geometry Of The Triangle And The Circle,Nathan Altshiller Court, says that we can find the locus of the points M for which we have $AM^2 = BM^2 + CM^2$.
if we solve this question then the rest is trivial...
then we find the intersection(s) of this locus with the circle with diameter CA and we r done!
the part in which this question appears it explains the medians property and prove this well-know property of the triangle in which for all points M in the plane of ΔABC :
$MA^2 + MC^2 + MB^2 = AG^2 + BG^2 + CG^2 + 3MG^2$ where G is the centroid of ΔABC.
that will be kind if someone proves this problem that i mentioned without (x, y) coordinates....
thx in advance

my solution:
we use (x, y) coordinates and define $A : (0, a); B : (b, 0); C : (c, 0)$ where $a, c \geq 0 \geq b$.
assuming $M : (x, y)$ the above formula give us :

$$2MA^2 - 3MG^2 = const \implies 2(x)^2 + 2(y-a)^2 - 3(x - \frac{b+c}{3})^2 - 3(y - \frac{a}{3})^2 = const$$

$$\implies 2x^2 + 2y^2 + 2a^2 - 4ay - 3x^2 - 3(\frac{b+c}{3})^2 + 2x(b+c) - 3y^2 - 3(\frac{a}{3})^2 + 2ay = const$$

$$\implies -x^2 - y^2 + \frac{5a^2}{3} - 2ay + 2x(b+c) - \frac{(b+c)^2}{3} = const$$

$$\implies -(x^2 + (b+c)^2 - 2x(b+c)) - (y^2 + a^2 + 2ay) = const - \frac{2(b+c)^2}{3} - \frac{8a^2}{3}$$

$$\implies -(x - (b+c))^2 - (y+a)^2 = const - \frac{2(b+c)^2}{3} - \frac{8a^2}{3}$$

$$\implies (x - (b+c))^2 + (y+a)^2 = -const + \frac{2(b+c)^2}{3} + \frac{8a^2}{3}$$

note that the left side is representing a circle with center coordinates $((b+c), -a)$ and radius $\sqrt{-const + \frac{2(b+c)^2}{3} + \frac{8a^2}{3}}$ also it's obvious that the radius of this circle is constant since a, b, c are constant.
so we have proven that the locus of the point M is a circle centered at $((b+c), -a)$ with a specific radius which is

$$const = AG^2 + BG^2 + CG^2 = \frac{1}{3}(AB^2 + BC^2 + CA^2)$$

$$= \frac{1}{3}((a^2 + b^2) + (b-c)^2 + (a^2 + c^2))$$

$$\implies -const + \frac{2(b+c)^2}{3} + \frac{8a^2}{3} = \frac{b^2 + c^2}{3} + 2a^2 + 2bc$$

so the radius of this circle is $\sqrt{\frac{b^2 + c^2}{3} + 2a^2 + 2bc}$

and the center of this circle can be found as follows:
we can assume that $CH > BH$ where H is the projection of A on BC.then u take the point T on BC such that T belongs to the segment BC and $CT = BH$.then we draw a line parralel to BC passing through A and a line perpendicular to BC at T.if the intersections of this two line is L then the reflection of L wrt BC called L' will be the center of the circle on which M varies.(i mean $L'B = LB, L'C = LC, L' \neq L$)

so finally the center of this circle is L' ,as defined,and its radius is $\sqrt{\frac{b^2+c^2}{3} + 2a^2 + 2bc}$.

note that in order to find all M asked in the topic we must consider both cases $BH > CH$ and $BH < CH$.
now we can easily find the intersection(s) of this circle with the circle with diameter CA.the rest is a matter of computations.

Luis González

#3 Nov 8, 2009, 4:29 am

N is midpoint of BC and P is the reflection of A about N. MN is common M-median of $\triangle AMP$ and $\triangle MBC$.

$$MB^2 + MC^2 = 2MN^2 - \tfrac{1}{2}a^2 \ , \ MN^2 = \tfrac{1}{2}(MA^2 + MP^2) - AN^2.$$

Combining both expresions yields

$$MB^2 + MC^2 - MA^2 = MP^2 - 2AN^2 + \tfrac{1}{2}a^2$$

$$MB^2 + MC^2 - MA^2 = MP^2 - 2\left[\tfrac{1}{2}(b^2 + c^2) - \tfrac{1}{4}a^2\right] + \tfrac{1}{2}a^2$$

$$MB^2 + MC^2 - MA^2 = MP^2 - b^2 - c^2 + a^2.$$

Hence, locus of M that satisfies $MA^2 = MB^2 + MC^2$ is a circumference (P) centered at the reflection of A about the midpoint of BC with radius $\sqrt{b^2 + c^2 - a^2}$, as long as $b^2 + c^2 \geq a^2$. Now, draw this circumference and cut it with the circumference with diameter AC to get the wanted points M.

muroanxfile019

#4 Nov 8, 2009, 9:46 am

Thanks for your solutions,thank **shoki** and **luisgeometria** very much!I'm very glad!

High School Olympiads

Equality of segments

Reply

Source: segments of tetrahedron

77ant

#1 Jun 4, 2008, 1:22 pm

For a tetrahedron SABC,the incircle of □ABC has the incenter I and touches BC,CA,AB at points E,F,D respectively.let A',B',C' be points on SA,SB,SC s.t. AA'=AD,BB'=BE,CC'=CF and let S' be the point diametrically opposite to S on the circumsphere of SABC.assume that the line SI is an altitude of SABC.show that S'A'=S'B'=S'C'.

Please

Luis González

#2 Nov 7, 2009, 8:54 am

Let $\mathcal{T}_a, \mathcal{T}_b, \mathcal{T}_c$ be the spheres with centers A, B, C passing through $(D, F, A'), (D, E, B'), (E, FC')$. I is the radical center of the circles centered at A, B, C passing through $(F, D), (D, E), (E, F)$. Therefore, the perpendicular line to the face ABC through I is the radical axis of $\mathcal{T}_a, \mathcal{T}_b, \mathcal{T}_c \Longrightarrow$ powers of S to $\mathcal{T}_a, \mathcal{T}_b, \mathcal{T}_c$ equal k^2.

$$SB^2 - (BB')^2 = SC^2 - (CC')^2 = SA^2 - (AA')^2 = k^2$$

Power of B with respect to the circumsphere (O, R) is $R^2 - (B'O)^2 = BB' \cdot B'S$

Note that in the triangle $\triangle B'SS'$, the segment $B'O$ is the median issuing from B'. Thus

$$(B'O)^2 = \frac{1}{2}(SB')^2 + \frac{1}{2}(S'B')^2 - \frac{1}{4}(2R)^2$$

Substituting $(BO')^2$ from the previous expression yields

$$(S'B')^2 = 4R^2 - (SB')^2 - 2BB' \cdot B'S$$

$$(S'B')^2 = 4R^2 - SB^2 - (BB')^2 + 2BS \cdot BB' - 2BB' \cdot B'S$$

$$(S'B')^2 = 4R^2 - SB^2 - (BB')^2 + 2BB'(BS - B'S)$$

$$(S'B')^2 = 4R^2 - SB^2 + (BB')^2$$

$$(S'B')^2 = 4R^2 - k^2$$

Since this latter expression is independent of the chosen vertex, we conclude that

$$S'A' = S'B' = S'C' = \sqrt{4R^2 - k^2}$$

Quick Reply

High School Olympiads

New open problem

Source: Is it a famous theorem

borislav_mirchev

#1 Nov 5, 2009, 2:12 am

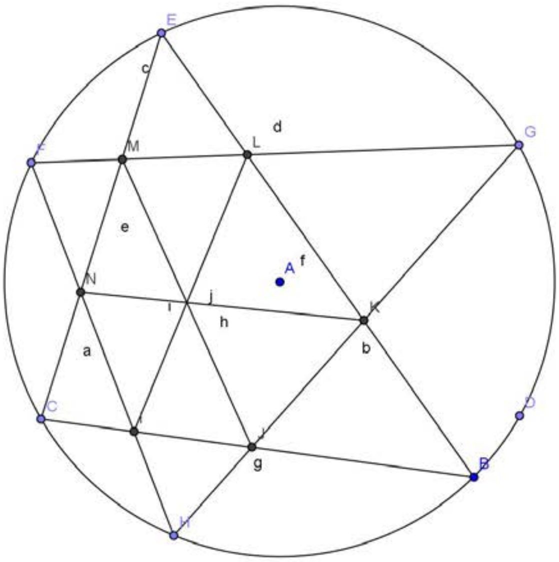
Hello!
To be easy for you - just see the attachment: There are given six points on a circle BHCFEG (in this order). KLMNIJ are the vertices of the hexagon formed by the intersecting lines. Prove that the opposite diagonals of KLMNIJ - KN, JM, IL intersects at a common point.

I discovered this statement using Geogebra. The software says it it is true. My questions are:
1. Is it a famous theorem/problem?
2. How to prove it.

I would like to ask you - if it is possible please send me the proofs in a private message and if it possible, please use as simple techniques as possible. If you would like to share your proofs to all the members of the forum - it is not a problem.

Thank you very much!

Attachments:



vittasko

#2 Nov 5, 2009, 5:01 am • 1

I don't know if it is a famous theorem dear **Borislav**, but surely it is coming from the past.

So, there is another more geberal configuration of an arbitrary convex hexagon, as follows :

GENERAL THEOREM. - Let $ABCDEF$ be, an arbitrary convex hexagon and we denote the points $K \equiv AB \cap CD$ and $L \equiv BC \cap DE$ and $M \equiv CD \cap EF$ and $N \equiv DE \cap AF$ and $P \equiv EF \cap AB$ and $P \equiv AF \cap BC$. Prove that :

(a) - If the diagonals $AD,\ BE,\ CF$ of the given hexagon, are concurrent at one point then, the line segments $KN,\ LP,\ MQ$, taken per two of them, intersect each other at three points $X,\ Y,\ Z$, lie on the above diagonals. The inverse is also true (schema t=310049).

(b) - If the line segments $KN,\ LP,\ MQ$ are concurrent at one point then, the diagonals $AD,\ BE,\ CF$ of the given hexagon taken per two of them, intersect each other at thre points $R,\ S,\ T$, lie on the above line segments. The inverse is also true (schema t=310049(a)).

● Return now in to your problem (schema **t=310049(b)**), based on the **Pascal theorem**, it is easy to show that the points $A,\ R,\ D$, are collinear (consider the non-convex cyclic hexagon $PKMQNL$).

That is, the point $R \equiv LP \cap MQ$, lies on the diagonal AD of the convex hexagon $ABCDEF$ and similarly, the points $S \equiv KN \cap LP$ and $T \equiv MQ \cap KN$, lie on the diagonals $CF,\ BE$, respectively.

Hence, based on the part **(a)** of the above general theorem, we conclude that the diagonals $AD,\ BE,\ CF$ of $ABCDEF$, are concurrent at one point and the proof of the proposed problem is completed.

Best regards, Kostas Vittas.

PS. I will search my archive and I will post here next time the (not difidult) proofs I have in mind, of the parts **(a), (b),** of the General theorem I mentioned.

Attachments:
t=310049(b).pdf (6kb)
t=310049(a).pdf (4kb)
t=310049.pdf (5kb)

This post has been edited 1 time. Last edited by vittasko, Nov 5, 2009, 5:43 am

borislav_mirchev
#3 Nov 5, 2009, 5:43 am

Very nice proof dear Kostas!

Thank you very much!
Do you like the problem?
What is the level of difficulty of that problem?
Everyone says it is probably a result from the past but no one said who is the inventor. Probably it is not very famous result.
What if we don't know these properties of the hexagon and Pascal theorem and we start from school level /we know similarities, inscribed angles, chord proportions, Sine law and at most Ceva and Menelaus/?

vittasko
#4 Nov 5, 2009, 5:53 am

> *borislav_mirchev wrote:*
> Very nice proof dear Kostas!
>
> Thank you very much!
> Do you like the problem?
> What is the level of difficulty of that problem?
> Everyone says it is probably a result from the past but no one said who is the inventor. Probably it is not very famous result.
> What if we don't know these properties of the hexagon and Pascal theorem and we start from school level /we know similarities, inscribed angles, chord proportions, Sine law and at most Ceva and Menelaus/?

Dear **Borislav**, I have also (re)discovered the general theorem some years ago and I don't know any reference about it.

But, I think that it is not a new result.

Best regards, Kostas Vittas.

This post has been edited 1 time. Last edited by vittasko, Nov 5, 2009, 11:42 am

borislav_mirchev
#5 Nov 5, 2009, 6:01 am

Dear Kostas,

It is not my first interesting result i rediscovered ... for example I rediscovered a generalization of the Butterfly theorem and Romania TST 1993 problem /the equilateral triangle/ as well as the formulae sin(a+b). Anyway I'll be happy the people spent good time in solving the problem.

The problem is probably true if the circle is an ellipse as the Pascal theorems. It probably can be solved in many ways - with analytical geometry / complex numbers / trigonometry, similarities. Any other solutions are appreciated and aswers of the questions above.

borislav_mirchev
#6 Nov 6, 2009, 1:17 am

I'm sorry for my double post but can we see the proofs of the lemas you use? I like these statements.

vittasko
#7 Nov 6, 2009, 2:03 am

> *borislav_mirchev wrote:*
> I'm sorry for my double post but can we see the proofs of the lemas you use? I like these statements.

Dear **Borislav**, I searched a little today morning but, I didn't find the proofs (I don't remember them) and I am too lazy to try again to solve the **Lemmas** I mentioned.

Give me please a little time and I hope that I wiil find them.

Kostas Vittas.

shoki
#8 Nov 6, 2009, 2:37 am • 1

see here:
http://www.mathlinks.ro/viewtopic.php?t=295936
(note that the proof that i posted works also for this).

borislav_mirchev
#9 Nov 6, 2009, 3:10 am

Dear Kostas,

Thank you for the nice materials ... there is not need to hurry ... I'll try to solve the original problem. I hardly start anything but when I start I hardly stop ... who know maybe I can prove the lemmas Smile ... but if you find the materials, please post for all mathlinkers or send me them in a PM.

In the weekend I'll try to find some time for thinking. Why you write my name in bold?

Shoki, your solution is simpler. It is a nice application of the Pascal theorem. Thank you! Indeed may I ask you to post a picture with exact letters on it, to replace the word "obvious" with your exact arguments and every time when you apply Pascal theorem to say the vertices of the hexagram you apply it for.

I like both solutions.

Luis González
#10 Nov 6, 2009, 5:15 am • 1

> *borislav_mirchev wrote:*
> There are given six points on a circle BHCFEG (in this order). KLMNIJ are the vertices of the hexagon formed by the intersecting lines. Prove that the opposite diagonals of KLMNIJ - KN, JM, IL intersects at a common point

Due to Brianchon's theorem the problem is equivalent to show that that the convex hexagon LMNIJK is circumscribed in a conic. Take a look at the topic Two triangles on a circle.

shoki

#11 Nov 6, 2009, 4:59 pm

> *borislav_mirchev wrote:*
>
> ....
> to replace the word "obvious" with your exact arguments and every time when you apply Pascal theorem to say the vertices of the hexagram you apply it for.
> ...

i used papus theorem twice in the second line.

borislav_mirchev

#12 Nov 7, 2009, 1:37 am

shoki's and yetti's solutions use similar arguments. yetti's solution is very clear and elegant with excellent descriptions it makes the problem to looks as a simple game. i think the result i posted is great and beautiful and i had the luck to find it on my own. in the weekend i'll try to solve Kostas' lemmas and two more people promised me to show their solutions.

vittasko

#13 Nov 7, 2009, 4:29 am • 1

> *vittasko wrote:*
>
> **GENERAL THEOREM. - Let $ABCDEF$ be, an arbitrary convex hexagon and we denote the points $K\equiv AB\cap CD$ and $L\equiv BC\cap DE$ and $M\equiv CD\cap EF$ and $N\equiv DE\cap AF$ and $P\equiv EF\cap AB$ and $P\equiv AF\cap BC$. Prove that :**
>
> **(a) - If the diagonals $AD,\ BE,\ CF$ of the given hexagon, are concurrent at one point then, the line segments $KN,\ LP,\ MQ,$ taken per two of them, intersect each other at three points $X,\ Y,\ Z,$ lie on the above diagonals. The inverse is also true (schema t=310049).**

● **PROOF**. - Let be the points $O\equiv AD\cap BE\cap CF$ and $R\equiv CP\cap EQ$ (**schema t=310049(c)**).

Applying the **Pappos theorem**, for the triads of points $P,\ F,\ E$ and $Q,\ B,\ C,$ on the line segments $PM,\ QL$ respectively, we have that the points $A\equiv PB\cap QF$ and $R\equiv PC\cap QE$ and $O\equiv FC\cap BE,$ are collinear.

Because of now the collinearity of the points $A,\ O,\ D,$ we conclude that the $R,$ lies on the diagonal AD of the given hexagon $ABCDEF.$

● Applying again the **Pappos theorem**, for the triads of the points $P,\ E,\ M$ and $Q,\ C,\ L,$ we have that the points $R,\ X\equiv PL\cap QM$ and $D\equiv EL\cap CM,$ are collinear and so, we conlude that the point $X,$ lies also on $AD.$

Similarly, the points $Y\equiv PL\cap KN$ and $Z\equiv QM\cap KN,$ lie on the diagonals $CF,\ BE$ and the proof of the part **(a)** of the general theorem, is completed.

Kostas Vittas.

Attachments:

t=310040(c).pdf (5kb)

This post has been edited 6 times. Last edited by vittasko, Nov 8, 2009, 3:17 am

borislav_mirchev

#14 Nov 7, 2009, 4:42 am

Dear Kostas,

Thank you very much! You always show us the beauty of the clear geometry approaches.
These theorems are one of the many proofs of my words. A person can learn many things from you and the other mathlinkers.
You were right - it was an old problem but maybe a little forgotten.

You were right - it was an old problem but maybe a little forgotten!

vittasko

#15 Nov 7, 2009, 1:19 pm • 1

> vittasko wrote:
>
> **GENERAL THEOREM. - Let $ABCDEF$ be, an arbitrary convex hexagon and we denote the points $K\equiv AB\cap CD$ and $L\equiv BC\cap DE$ and $M\equiv CD\cap EF$ and $N\equiv DE\cap AF$ and $P\equiv EF\cap AB$ and $P\equiv AF\cap BC$. Prove that :**
>
> **(b) - If the line segments $KN,\ LP,\ MQ$ are concurrent at one point then, the diagonals $AD,\ BE,\ CF$ of the given hexagon taken per two of them, intersect each other at thre points $R,\ S,\ T,$ lie on the above line segments. The inverse is also true (schema t=310049(a)).**

● - (**schema t=310049(d)**) - Because of the concurrency of $O\equiv PL\cap KN\cap MQ,$ based on the **Desarques theorem**, we conclude that the triangles $\triangle PKM,\ \triangle LNQ$ are perspective and so, we have that the points $S\equiv PK\cap LN$ and $U\equiv PM\cap LQ$ and $V\equiv KM\cap NQ$ are collinear.

We consider now these points as the points of intersection of the opposite sidelines of the given hexagon $ABCDEF$ and based on the **Pascal theorem** and because of their collinearity, we conclude that this hexagon is inscribed in a conic.

Applying again the **Pascal theorem**, in the non-convex hexagon $ADEFCB$ inscribed in that conic, we conclude that the points $P\equiv EF\cap BA$ and $X\equiv AD\cap FC$ and $L\equiv DE\cap CB$ are collinear.

That is, the point $X,$ as the point of intersection of the diagonals $AD,\ CF$ of the given convex hexagon $ABCDEF,$ lies on the line segment $PL.$

Similarly, the points $Y\equiv BE\cap CF$ and $Z\equiv AD\cap BE,$ lie on the line segments $KN,\ MQ$ respectively and the proof of the part **(b)** of the general theorem is completed.

Kostas Vittas.

Attachments:

t=310049(d).pdf (6kb)

Quick Reply

High School Olympiads

Triangle AB>AC

Reply

Source: pr1

Ulanbek_Kyzylorda KTL

#1 Nov 4, 2009, 9:57 pm

In triangle ABC(AB>AC) AD is median and AF is bisector. FN is parallel to AC (N is on AB) and it intersects AD at L. Prove that CL is perpendicular to AF.

Luis González

#2 Nov 5, 2009, 8:47 am

Let $K \equiv AF \cap CL$ and $E \equiv AC \cap DK.$ Since $LF \parallel AC,$ it follows that the cevians AF and CL of $\triangle ADC$ cut on its D-median $\Longrightarrow E$ is the midpoint of $AC \Longrightarrow DE \parallel BA.$ Consequently, $\angle AKF = \angle DKF = \angle BAF = \angle KAE \Longrightarrow EA = EK = EC \Longrightarrow \angle AKC = 90^\circ$ as desired.

Quick Reply

High School Olympiads

Inequality in the acute triangle

Reply

Source: posted several times but still no solution

Nebraska boy

#1 Nov 2, 2009, 7:52 pm

Show that in the acute triangle there's always a pair of sides whose sum is greater than the sum of the diameteres of its incircle and circumcircle

Sincerely
T.H

Luis González

#2 Nov 4, 2009, 6:17 pm

Lemma: Point M lies inside $\triangle ABC$ and $a = \min(a, b, c)$. Then we have $b + c > MA + MB + MC$.

Draw the parallel to BC passing through M intersecting AC and AB at X, Y, respectively. Draw the altitude AH and w.l.o.g assume that M lies inside $\triangle AHB$. We have that $YA > MA \ (1)$ and since $AC > CB \Longrightarrow AX > XY$, due to $\triangle ABC \sim \triangle AXY$. Thus $AX + XC = b > XY + XC \ (2)$

By triangle inequality $MX + XC > MC \ , \ MY + YB > MB$

$\Longrightarrow CX + XY + YB > MB + MC \ (3)$

Adding inequalities $(1), (2), (3)$ together yields

$b + YA + YB + XC + XY > MA + MB + MC + XY + XC$

$\Longrightarrow b + c > MA + MB + MC$

The proof is analogous when M either lies inside $\triangle AHC$ or on the altitude AH.

Assume that $CA \geq AB \geq BC$. Using the above result when M is identical to the orthocenter H of the acute ABC, we get $b + c > HA + HB + HC = 2(R + r)$.

Quick Reply

High School Olympiads

Nice

Reply

cnyd
#1 Oct 19, 2009, 10:05 pm

The incircle of ABC touches $[BC], [CA], [AB]$ at D, E, F, respectively.$[AK] \perp [BC], [AK] \cap [EF] = H$. If $[DH] \perp [EF]$, show that H is the orthocenter of ABC.

This post has been edited 1 time. Last edited by cnyd, Oct 20, 2009, 12:26 pm

cnyd
#2 Oct 27, 2009, 8:30 pm

I solved

$\angle AEF = \angle AFE = a$

$\angle KAC = b, \angle BAK = 180 - 2a - b$

$[BH] \cap [AC] = P, [CH] \cap [AB] = Q$

$\angle BHF = c \implies \angle FBC = a - c$

$\angle EHC = d \implies \angle HCE = a - d$

$|DC| = |EC| = x, |BD| = |BF| = y$

$\frac{x}{sin(90-d)} = \frac{|HC|}{sin(a+b)}, \frac{|HC|}{sina} = \frac{x}{sind} \implies$

$sindsin(a+b) = sin(90-n).sina$,similarly;

$\frac{y}{sinc} = \frac{|BH|}{sina}, \frac{y}{sin(90-c)} = \frac{|BH|}{sin(a+b)}$

$\implies sinasin(90-c) = sincsin(a+b) \implies tanc = tand \implies c = d$

in $\triangle ABC$ $sinb.sin(90+c-b-a).sin(a-c) = sin(2a+b).sin(a+b+c-90).sin(a-c)$

$\implies sinb.sin(90+c-b-a) = sin(2a+b).sin(a+b+c-90)$

$a+c = 90+u \implies sinb.sin(2a+b-u) = sin(2a+b).sin(b+u)$

$\implies cos(2a+2b-u) - cos(2a-u) = cos(2a+2b+u) - cos(2a-u) \implies u = 0$

$a+c = 90 \implies [BP] \perp [AC], [CQ] \perp [AB]$ QED $\square$

mr.danh
#3 Nov 1, 2009, 9:21 pm • 2

Nice soln?

mathVNpro
#4 Nov 1, 2009, 11:51 pm • 1

> Quote:
> The incircle of ABC touch $[BC],[AC],[AB]$ at D,E,F respectively.$[AK]\perp[BC],[AK]\cap[EF]=H$ if $[DH]\perp[EF]\implies H$ is orthocenter of ABC,show that

Denote r by the radii of incircle (I) of $\triangle ABC$. Consider the inversion through pole I, power $k=r^2$, we have $\mathcal{I}(I,k)$: $A\mapsto M_a$, $B\mapsto M_b$, $C\mapsto M_c$, where M_a,M_b,M_c respectively are midpoints of EF,FD,DE. Therefore, through $\mathcal{I}(I,k)$, $(ABC)\mapsto(M_aM_bM_c)$, which is also the $9-$ point circle wrt $\triangle DEF$. Since H is the projection of D onto EF. Thus, $H\in(M_aM_bM_c)$. Hence, if H^* is the image of H through $\mathcal{I}(I,k)$, $H\in(ABC)$. In the other hand, $H\in EF$ and $\mathcal{I}(I,k):EF\mapsto(AEF)\Longrightarrow H^*\in(AEF)$, which implies $\angle IH^*A=90^\circ=\angle HH^*A$, which implies $H^*\in([AH])$.

Let $K\equiv AH^*\cap BC$, we have $\overline{KH^*}.\overline{KA}=\overline{KB}.\overline{KC}$. Now let $M\equiv HI\cap BC$ and $A_1\equiv AH\cap BC$, we have $\angle AA_1M=\angle AH^*M$, which implies A,H^*,A_1,M are concyclic. Thus $\overline{KH^*}.\overline{KA}=\overline{KA_1}.\overline{KM}\Longrightarrow\overline{KA_1}.\overline{KM}=\overline{KB}.\overline{KC}$ (1). Let K' be the pole of AH wrt (I), since AH passes A and H, thus, $K'\in EF$- the polar of A and AH^*- the polar of H, wrt (I). Then $(K'HFE)=-1\Longrightarrow(AK',AH,AB,AC)=-1$ or $(AK,AA_1,AB,AC)=-1$, which implies $(KA_1BC)=-1$. Combine with (1), we conclude M is the midpoint of BC.

Let O_1,O_2,O respectively are the circumcenters of $([AH]),([AI])$ and (ABC). As we have proved in the beginning, we have AH^* is the radical axes of these 3 circles. Hence, $\overline{OO_1O_2}\perp AH^*$. But $\overline{HIM}\perp AH^*$. Therefore, $O_1O\|HM$, we have $HO_1\|OM$ (since both $\perp BC$). As the result, O_1HMO is parallelogram $\Longrightarrow HA=HO_1=OM$ or we can say $OM=\dfrac{1}{2}AH$, which leads to the result that H is the orthocenter of $\triangle ABC$. Our proof is completed then. $\square$

This post has been edited 3 times. Last edited by mathVNpro, Nov 2, 2009, 10:04 am

Luis González

#5 Nov 2, 2009, 2:28 am • 1

Let B',C' be the feet of the cevians BH,CH and $P\equiv EF\cap BC$. From the harmonic cross ratio (B,C,D,P) and $DH\perp EF$, it follows that EF bisects $\angle BHC$ externally $\Longrightarrow\angle B'HE=\angle C'HE$. Since $\angle AEF=\angle AFE$, we have that $\angle HB'E=\angle HC'F\Longrightarrow$ quadrilateral $B'C'BC$ is cyclic. If L,M are the feet of the altitudes from B,C, then $B'C'$, LM and BC concur at the harmonic conjugate of K with respect to B,C. But since $B'C'$ and LM are both antiparallel to BC, it follows that LM and $B'C'$ are necessarily identical $\Longrightarrow H$ is orthocenter of $\triangle ABC$.

This post has been edited 1 time. Last edited by Luis González, Nov 2, 2009, 5:38 pm

thaithuan_GC

#6 Nov 2, 2009, 3:23 pm

What are "inversion" and "harmonic cross ratio"? They aren't elementary.

mathVNpro

#7 Nov 2, 2009, 7:38 pm

> thaithuan_GC wrote:
> What are "inversion" and "harmonic cross ratio"? They aren't elementary.

I totally don't agree with you! And beside, please don't post any post without a constructive mood in it! It won't help for the topic!

This problem reminds me a problem that I have solved about 3 months ago.

PROBLEM- Let ABC be a triangle with (I) its incircle. Let D,E,F respectively be the tangency points of (I) with BC,CA,AB. Prove that the orthocenter H of $\triangle ABC$ lies on EF if and only if HI goes through the midpoint of BC.

Luis González

#8 Nov 4, 2009, 7:27 am

> mathVNpro wrote:
> **PROBLEM- Let ABC be a triangle with (I) its incircle. Let D,E,F respectively be the tangency points of (I) with BC,CA,AB. Prove that the orthocenter H of $\triangle ABC$ lies on EF if and only if HI goes through the**

midpoint of BC.

Let M be the midpoint of BC, D' the tangency point of the A-excircle with BC and $H' \equiv MI \cap AK$, where K is the foot of the A-altitude. The ray AD' cuts (I) at the antipode D'' of D and since M is also the midpoint of DD', it follows that $MI \parallel AD'' \Longrightarrow AH'ID''$ is a parallelogram $\Longrightarrow AH' = ID = r$ $(\star)$. On the other hand, from the original topic we have proved that $H \in EF \Longleftrightarrow DH \perp EF$. Therefore, $AIDH$ is also a parallelogram $\Longrightarrow AH = ID = r$. From $(\star)$ we deduce that H and H' coincide and the proof is completed.

mathVNpro

#9 Nov 4, 2009, 10:05 am

> *Quote:*
>
> **PROBLEM- Let ABC be a triangle with (I) its incircle. Let D, E, F respectively be the tangency points of (I) with BC, CA, AB. Prove that the orthocenter H of $\triangle ABC$ lies on EF if and only if HI goes through the midpoint of BC.**

I will prove the $(\Longrightarrow)$ side of this statement, which is: **If H is the orthocenter of $\triangle ABC$ then HI goes through the midpoint of BC.**. The $(\Longleftarrow)$ side is the problem of this topic.

Proof

Let d be the line through H such that d is parallel with BC. According to my proof of the problem of this topic, we have already had $(K'HFE) = -1$, which implies that $(AK', AH, AF, AE) = -1$. Since $AH \perp BC \Longrightarrow AH \perp d$. $AK' \perp HI$, $AF \perp HC$ and $AE \perp HB$. As the result, $(d, HI, HC, HB) = -1$, but $d || BC$, we conclude that HI passes the midpoint of BC. $\square$

Quick Reply

High School Olympiads

A sphere is inscribed in tetrahedron ABCD

Reply

Source: Polish MO Round 3 2009 problem 5

math10

#1 Nov 3, 2009, 11:17 am

A sphere is inscribed in tetrahedron $ABCD$ and is tangent to faces BCD, CAD, ABD, ABC at points P, Q, R, S respectively. Segment PT is the sphere's diameter, and lines TA, TQ, TR, TS meet the plane BCD at points A', Q', R', S'. respectively. Show that A is the center of a circumcircle on the triangle $S'Q'R'$.

Luis González

#2 Nov 3, 2009, 7:12 pm

Consider the stereographic projection on the insphere ω with pole T (the antipode of P) and stereographic plane $\pi \equiv BDC.$ This projection maps S, Q, R into S', Q', R', P is a double point and circle $\odot(SQR)$ on the surface of ω is taken into the circle $\odot(S'Q'R')$ in the plane $\pi.$ Due to the conformity of the stereographic projection, the sphere k with center A and orthogonal to $\omega,$ which intersects ω along the circumference $\odot(SQR),$ is taken into the sphere k' containing its image $\odot(S'Q'R')$ and orthogonal to the plane $BCD \Longrightarrow$ the center of k' coincides with the circumcenter of $\triangle S'Q'R'.$ Since the pole T of the stereographic projection (inversion) is collinear with the center of the sphere and its image, it follows that $A' \equiv \pi \cap AT$ is the circumcenter of $\triangle S'Q'R'.$

Quick Reply

High School Olympiads

L,M,N are collinear

Reply

Source: maybe posted

hxy09
#1 Oct 21, 2009, 6:37 pm

$D.E.F$ are the midpoints of BC, CA, AB
DL, EN, FM are tangent to the inscribecircle of ABC
$L \in EF$
$M \in DE$
$N \in DF$
Prove that L, M, N are collinear

livetolove212
#2 Oct 21, 2009, 11:45 pm • 1

Lemma: Given triangle ABC and circle $(O).(O) \cap BC, CA, AB = \{A_1, A_2, B_1, B_2, C_1, C_2\}$. Then AA_1, BB_1, CC_1 are concurrent iff AA_2, BB_2, CC_2 are concurrent.
Proof: We can prove this lemma easily by using power of a point and Ceva's theorem.
Return to our problem:
See the names of points on the figure.
A theorem about circumscribed quadrilateral tells us ZB_1, YC_1, EF concur at R.
Denote $A' = C_1Z \cap B_1Y.(EFR)$ is the polar of A' wrt (I) then $A'I \perp EF$ or A', I, A_1 are collinear.
Similarly we get $A'A_1, B'B_1, C'C_1$ are concurrent.
Applying the lemma above we get $A'X, B'Y, C'Z$ concur at K.
L lies on the polar of A' then A' lies on the polar of L. Therefore $A'X$ is the polar of L.
$\Rightarrow L$ lies on the polar of K.
Similarly we obtain L, M, N lie on the polar of K. Our proof is completed then.
Remark: AX, BY, CZ concur at **Nagel** point of triangle ABC.

Attachments:
picture50.pdf (6kb)

Luis González
#3 Oct 22, 2009, 10:05 am • 1

Let X, Y, Z be the tangency points of (I) with BC, CA, AB and D', E', F' the contact points of the tangents from D, E, F to (I), different from X, Y, Z. The poles A', B', C' of EF, FD, DE WRT (I) lie on IX, IY, IZ, respectively due to $EF \parallel BC, FD \parallel AC$ and $DE \parallel AB$. Thus, polars of D, E, F WRT (I), which are the lines $B'C', C'A', A'B'$, are identical with $XD', YE', ZF' \Longrightarrow L, N, M$ are the poles of $A'D', B'E', C'F'$. Since $A'X, B'Y$ and $C'Z$ concur at I, then $A'D', B'E'$ and $C'F'$ concur at the cyclocevian I' of I in $\triangle A'B'C' \cup (I)$. Therefore, L, N, M are collinear on the polar of I' WRT (I).

● Another result coming from this configuration:

$\triangle ABC$ is scalene and let X, Y, Z be the tangency points of the excircles $(I_a), (I_b), (I_c)$ with BC, CA, AB. Perpendicular bisectors of AX, BY, CZ meet BC, CA, AB at A', B', C', respectively. Then A', B', C' are collinear.

vittasko
#4 Oct 22, 2009, 7:27 pm

This nice problem, has already been posted before, as Interesting and hard, by **cvix**.

> *livetolove212 wrote:*

Lemma: Given triangle ABC and circle (O).$(O) \cap BC, CA, AB = \{A_1, A_2, B_1, B_2, C_1, C_2\}$. Then AA_1, BB_1, CC_1 are concurrent iff AA_2, BB_2, CC_2 are concurrent.

There are some (not well known I think) interesting concurrencies and one collinearity, in the configuration of an arbitrary circle (K), cutting the side-segments $BC,\ AC,\ AB$ of a given triangle $\triangle ABC$, at pairs of points $D,\ D'$ and $E,\ E'$ and $F,\ F'$, respectively (the points $D,\ E,\ F$ are adjoining to the vertices $B,\ C,\ A,$).

• **The line segments $AA',\ BB',\ CC'$, are concurrent at one point so be it P, where $A' \equiv BE \cap CF'$ and $B' \equiv AD' \cap CF$ and $C' \equiv AD \cap BE'$.**

• **The line segments $AA'',\ BB'',\ CC''$, are concurrent at one point so be it Q, where $A'' \equiv BE' \cap CF$ and $B'' \equiv AD \cap CF'$ and $C'' \equiv AD' \cap BE$.**

• **The line segments $A'A'',\ B'B'',\ C'C''$, are concurrent at one point so be it R, lying on the line segment PQ.**

Best regards, Kostas Vittas.

Attachments:
t=307278.pdf (9kb)

hxy09
#5 Oct 24, 2009, 3:11 pm

Thank you for precious comment
I paste four different solutions below(very sorry if repeated)
The first is purely algebraic(I like calculating all the time)
The third used **Brianchon theorem**
The second and the last based on the theorem of radical axis
Wish you like them!

Attachments:

$LHS = \frac{a}{\sin 2\alpha}(\sin\alpha + c)\cos\alpha$

$= \frac{a \sin(\alpha + c)}{2\sin\alpha}$

$\Leftrightarrow a(\cos c + \cot\alpha \sin c) = \frac{1}{2}(b + c - a)$

$\Leftrightarrow (\frac{c-a}{2r}\sin c + \cos c)a = \frac{1}{2}(b + c - a)$

$LHS = (\frac{\sin\frac{C-A}{2}\cos\frac{1}{2}C}{\sin\frac{1}{2}A} + \cos c)\sin A$

$= 2\cos\frac{1}{2}A\cos\frac{1}{2}c\sin\frac{C-A}{2} + \cos c \sin A$

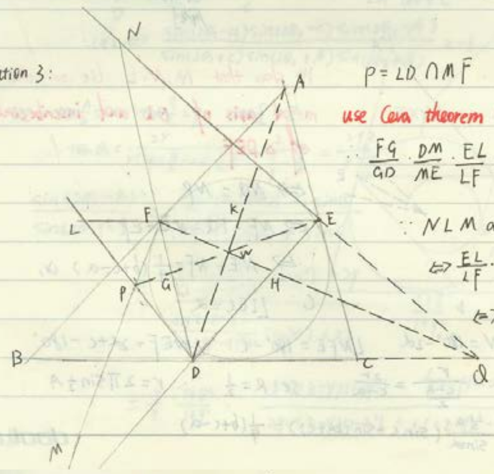
$= (\sin\frac{B}{2} + \cos\frac{1}{2}(A-c))\sin\frac{C-A}{2} + \cos c \sin A$

$= \sin\frac{B}{2}\sin\frac{C-A}{2} + \frac{1}{2}\sin(c+A)$

$= \frac{1}{2}(\sin B + \sin c - \sin A) = RHS$

QED.
本证法属于常规解

Solution 3:



$P = LD \cap MF$

use Ceva theorem (P and $\triangle EFD$)

$\because \frac{FG}{GD}\cdot\frac{DM}{ME}\cdot\frac{EL}{LF} = 1$

$\because$ N L M are collinear

$\Leftrightarrow \frac{EL}{LF}\cdot\frac{FN}{ND}\cdot\frac{DM}{ME} = 1$

$\Leftrightarrow \frac{FG}{GD} = \frac{FN}{ND}$ (*)

$= 3(z-y)(1-y^2) - ((z-y)^2 - 4y^2z^2)$

$= (z-y)(2z - z + y) - 2y^2 z(z-y) + 4y^2z^2$

$= (z-y)(z+y) + 2y^2z^2 + 2y^3z$

$= (z+y)(z-y+2y^2z)$

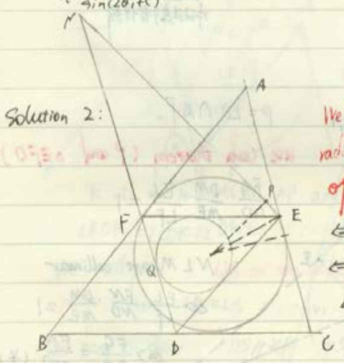
$2y^2z + z - y$

$$= 2y^2z + z^2(x+y) + xyz - xyz - y^2z - y^2x$$
$$= y^2z - y^2x + z^2x + z^2y$$
$$= yz(z+y) + x(z-y)(z+y)$$
$$= (y+z)(yz+xz-xy)$$

$$\therefore \frac{\sin(2\theta_1-B)}{\sin(2\theta_1+C)} = \frac{y}{z}\cdot\frac{1+z^2}{1+y^2}\cdot\frac{(y+z)^2(yz+xz-xy)}{(y+z)^2(xz-xy-yz)}$$

$$\therefore \prod \frac{\sin(2\theta_1-B)}{\sin(2\theta_1+C)} = -1$$

Q.E.D.



Solution 2: We show that M, N, L lie on the radical axis of $\odot I$ and inscribed circle of $\triangle DEF$

$\Leftrightarrow NQ = NP$

$\Leftrightarrow NF - FQ = NE - EP$

$\Leftrightarrow NE - NF = \frac{1}{4}(b+c-a)$ $(*)$

$\angle IEC = \alpha$

$\therefore \angle N = 180^\circ - 2\alpha \quad \angle NFE = 180^\circ - C \quad \angle NEF = 2\alpha + C - 180^\circ$

$\tan\alpha = \frac{r}{\frac{c-a}{2}} = \frac{2r}{c-a} \quad$ set $R = \frac{1}{2} \quad \therefore r = 2\prod \sin\frac{1}{2}A$

$(*) \Leftrightarrow \frac{\frac{1}{2}a}{\sin 2\alpha}\left(\sin C + \sin(2\alpha + C)\right) = \frac{1}{4}(b+c-a)$

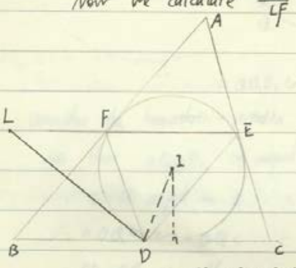
48. D, E, F are the midpoints of BC, CA, AB, DL, EN, FM is tangent to the incircle of $\triangle ABC$ $\quad L\in EF \quad M\in DE \quad N\in DF$

Prove that L, M, N are collinear

Solution 1: According to Menelaus theorem L, M, N are collinear

$\Leftrightarrow \frac{EL}{LF}\cdot\frac{FN}{ND}\cdot\frac{DM}{ME} = 1 \quad (*)$

Now we calculate $\frac{EL}{LF}$



$$\frac{EL}{LF} = \frac{[EDL]}{[FDL]} = \frac{DE\cdot\sin\angle EDL}{DF\cdot\sin\angle FDL}$$

$\angle IDC = \theta_1$, θ_2, θ_3 respectively

$\therefore \angle EDL = 2\theta_1 - B$

$\angle FDL = 2\theta_1 - B - A$

$= 2\theta_1 - 180^\circ + C$

$$\therefore (*) \Leftrightarrow \frac{\sin(2\theta_1-B)\sin(2\theta_2-C)\sin(2\theta_3-A)}{\sin(2\theta_1+C)\sin(2\theta_2+A)\sin(2\theta_3+B)} = -1$$

set $\tan\frac{A}{2} = x \quad \tan\frac{B}{2} = y \quad \tan\frac{C}{2} = z \qquad \because \sum xy = 1$

$$\therefore \tan\theta_1 = \frac{2r}{r(\cot\frac{B}{2} - \cot\frac{C}{2})} = \frac{2}{\frac{1}{y}-\frac{1}{z}} = \frac{2yz}{z-y}$$

$$\frac{\sin(2\theta_1-B)}{\sin(2\theta_1+C)} = \frac{\tan2\theta_1\cdot\cos B - \sin B}{\tan2\theta_1\cdot\cos C + \sin C}$$

$$= \frac{\frac{4yz(z-y)}{(z-y)^2-4y^2z^2}(1-y^2) - 2y}{\frac{4yz(z-y)}{(z-y)^2-4y^2z^2}(1-z^2) + 2z}\cdot\frac{1+z^2}{1+y^2}$$

$$= \frac{y}{z}\cdot\frac{1+z^2}{1+y^2}\cdot\frac{2z(z-y)(1-y^2) - \left((z-y)^2 - 4y^2z^2\right)}{2y(z-y)(1-z^2) + \left((z-y)^2 - 4y^2z^2\right)}$$

hxy09

#6 Oct 24, 2009, 3:24 pm

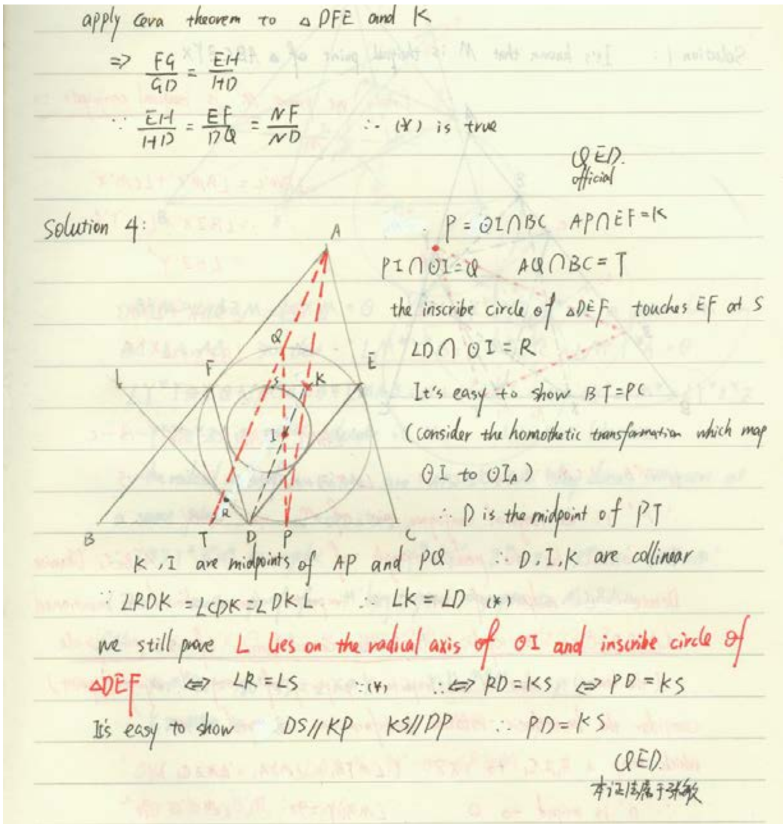
The last photo

Attachments:

$Q = NE \cap BC$

apply Brianchon theorem to $FPDQEA$

$\therefore FQ, AD, PE$ intersect at W



apply Ceva theorem to $\triangle DFE$ and K

$\Rightarrow \frac{FG}{GD} = \frac{EH}{HD}$

$\because \frac{EH}{HD} = \frac{EF}{DQ} = \frac{NF}{ND}$ $\therefore$ (*) is true

QED. official

Solution 4: $P = OI \cap BC$ $AP \cap EF = K$

$PI \cap OI = Q$ $AQ \cap BC = T$

the inscribe circle of $\triangle DEF$ touches EF at S

$LD \cap OI = R$

It's easy to show $BT = PC$

(consider the homothetic transformation which map OI to OI_A)

$\therefore D$ is the midpoint of PT

K, I are midpoints of AP and PQ $\therefore D, I, K$ are collinear

$\because \angle RDK = \angle CDK = \angle DKL$ $\therefore LK = LD$ (*)

we still prove L lies on the radical axis of OI and inscribe circle of $\triangle DEF$ $\Leftrightarrow LR = LS$ $\because$ (*) $\therefore \Leftrightarrow RD = KS \Leftrightarrow PD = KS$

It's easy to show $DS // KP$ $KS // DP$ $\therefore PD = KS$

QED

本证法属于张敏

Mashimaru

#7 Oct 28, 2009, 11:41 pm • 1

Here is my solution.

Let A_0 be the point of tangency of (I) and BC, A_0A_3 is the diameter of (I), $A_0' \equiv AA_3 \cap BC$, $A_5 \equiv AA_3 \cap B_1C_1$ and A_4 is the point of tangency of (I) and DL. It is known that A, A_3, A_5, A_0' are collinear. We have $DA_0 = DA_4 = DA_0'$, which yields $A_0A_4 \perp A_4A_0'$. On the other hand, $DI \perp A_0A_4$ and D, I are the midpoints of A_0A_0' and A_0A_3 so $DI \parallel A_3A_0'$. Hence $A_3A_0' \perp A_0A_4$. Therefore $A_4 \in \overline{AA_3A_5A_0'}$. Moreover, $\angle LA_5A_4 = \angle DA_0'A_4 = \angle DA_4A_0' = \angle LA_4A_4$ so $LA_4^2 = LA_5^2$, meanwhile L lies on the radical axis of (I) and (I'). Use the same argument of M, N, we have L, M, N are collinear since they are all belong to the radical axis of (I) and (I'), QED.

Generalized problem (Stated by PDatK40SP)
Let ABC be a triangle with incircle (I) and A_1, B_1, C_1 are three arbitrary points on BC, CA, AB. Denoted by A_2 the intersection of B_1C_1 with the tangent of (I) from A_1 but different of BC. Prove that the A_2, B_2, C_2 are collinear iff AA_1, BB_1, CC_1 concur.

livetolove212

#8 Oct 29, 2009, 4:41 pm

> vittasko wrote:
>
> There are some (not well known I think) interesting concurrencies and one collinearity, in the configuration of an arbitrary circle (K), cutting the side-segments $BC,\ AC,\ AB$ of a given triangle $\triangle ABC$, at pairs of points $D,\ D'$ and $E,\ E'$ and $F,\ F'$, respectively (the points $D,\ E,\ F$ are adjoining to the vertices $B,\ C,\ A,$).
>
> • **The line segments $AA',\ BB',\ CC'$, are concurrent at one point so be it P, where $A' \equiv BE \cap CF'$ and $B' \equiv AD' \cap CF$ and $C' \equiv AD \cap BE'$.**
>
> • **The line segments $AA'',\ BB'',\ CC''$, are concurrent at one point so be it Q, where $A'' \equiv BE' \cap CF$ and $B'' \equiv AD \cap CF'$ and $C'' \equiv AD' \cap BE$.**
>
> • **The line segments $A'A'',\ B'B'',\ C'C''$, are concurrent at one point so be it R, lying on the line segment PQ.**
>
> Best regards, Kostas Vittas.

Dear **Kostas**,
I have other similar concurrencies.
Given triangle $ABC.(O) \cap BC, CA, AB = \{A_1, A_2, B_1, B_2, C_1, C_2\}.X_1 = A_1B_1 \cap A_2C_2$, similar for $Y_1, Z_1.X_2 = C_2B_2 \cap C_1B_1$, similar for $Y_2, Z_2.A_3 = A_1C_1 \cap A_2B_2$, similar for B_3, C_3. Prove that:
a, X_1X_2, Y_1Y_2, Z_1Z_2 concur at D.
b, A_3X_1, B_3Y_1, C_3Z_1 concur at E.
c, A_3X_2, B_3Y_2, C_3Z_2 concur at F.
d, D, E, F are collinear.

Attachments:
picture58.pdf (11kb)

vittasko
#9 Oct 30, 2009, 4:04 am

> *livetolove212 wrote:*
> Given triangle $ABC.(O) \cap BC, CA, AB = \{A_1, A_2, B_1, B_2, C_1, C_2\}.X_1 = A_1B_1 \cap A_2C_2$, similar for $Y_1, Z_1.X_2 = C_2B_2 \cap C_1B_1$, similar for $Y_2, Z_2.A_3 = A_1C_1 \cap A_2B_2$, similar for B_3, C_3. Prove that:
> a, X_1X_2, Y_1Y_2, Z_1Z_2 concur at D.
> b, A_3X_1, B_3Y_1, C_3Z_1 concur at E.
> c, A_3X_2, B_3Y_2, C_3Z_2 concur at F.
> d, D, E, F are collinear.

Thank you dear **Linh** for another group of familiar concurrencies and collinearity, new to me.

I think that the key of the solution is the collinearity of $A_3,\ A,\ X_1$ (similarly of the triads $B_3,\ B,\ Y_1$ and $C_3,\ C,\ Z_1$) and the concurrency of $F \equiv A_3X_2 \cap B_3Y_2 \cap C_3Z_2$.

We can after to complete the proof, applying the **Desmic theorem**.

I will check my notes and if I am not mistaken, I will post here later more deatails (it is a crazy dancing of the **Pascal** and **Desarques**, theorems :)).

Best regards, Kostas Vittas.

This post has been edited 1 time. Last edited by vittasko, Oct 30, 2009, 6:00 pm

livetolove212
#10 Oct 30, 2009, 11:39 am

Dear **Kostas**,
I found an interesting concurrency:
Denote $A_4 = BB_2 \cap CC_1, A_5 = BB_1 \cap CC_2$, similar for B_4, B_5, C_4, C_5. Prove that 6 lines $AX_1, BY_1, CZ_1, A_4A_5, B_4B_5, C_4C_5$ concur at E.

Attachments:
picture59.pdf (11kb)

vittasko
#11 Oct 30, 2009, 6:45 pm

> *livetolove212 wrote:*
> Given triangle $ABC.(O) \cap BC, CA, AB = \{A_1, A_2, B_1, B_2, C_1, C_2\}.X_1 = A_1B_1 \cap A_2C_2$, similar for $Y_1, Z_1.X_2 = C_2B_2 \cap C_1B_1$, similar for $Y_2, Z_2.A_3 = A_1C_1 \cap A_2B_2$, similar for B_3, C_3. Prove that:
> a, X_1X_2, Y_1Y_2, Z_1Z_2 concur at D.
> b, A_3X_1, B_3Y_1, C_3Z_1 concur at E.
> c, A_3X_2, B_3Y_2, C_3Z_2 concur at F.
> d, D, E, F are collinear.

● Applying the **Pascal theorem** in cyclic hexagon $A_1A_2B_1B_2C_1C_2$, we have that the points $P \equiv A_1A_2 \cap B_2C_1$ and $Q \equiv B_1B_2 \cap A_1C_2$ and $R \equiv A_2B_1 \cap C_1C_2$, are collinear.

Applying again the **Pascal theorem** in cyclic non-convex hexagon $A_1C_1C_2B_2A_2B_1$, we have that the points $B_3 \equiv C_2B_2 \cap B_1A_1$ and $A_3 \equiv A_1C_1 \cap B_2A_2$ and $R \equiv C_1C_2 \cap A_2B_1$, are collinear and similarly, we have the collinearities of the points $C_3,\ B_3,\ P$ and $C_3,\ A_3,\ Q$

• Let be the points $A' \equiv A_2B_1 \cap A_1C_2,\ B' \equiv A_2B_1 \cap B_2C_1,\ C' \equiv A_1C_2 \cap B_2C_1$ and because of the collinearity of the points $P,\ Q,\ R,$ based on the **Desarques theorem**, we conclude that the triangles $\triangle A_3B_3C_3,\ \triangle A'B'C'$ are perspective and then, we have that $F \equiv A_3A' \cap B_3B' \cap C_3C'$, (1)

We consider now, the cyclic non-convex hexagon $A_1C_1B_1A_2B_2C_2,$ and based on the **Pascal theorem**, we have that the collinearity of the points $A_3 \equiv A_1C_1 \cap A_2B_2$ and $X_2 \equiv C_1B_1 \cap B_2C_2$ and $A' \equiv B_1A_2 \cap C_2A_1,$ are collinear.

That is, the point $X_2,$ lies on the line segment A_3A' and similarly, the points $Y_2,\ Z_2,$ lie on the line segments $B_3B',\ C_3C',$ respectively.

• So, we have now the configuration of the concurrent segments $A_3X_2,\ B_3Y_2,\ C_3Z_2$ and based on the **Desmic theorem** (it is what I have called as **Second theorem of concurrent segments**), we conclude the concurrencies of $E \equiv A_3X_1 \cap B_3Y_1 \cap C_3Z_1$ and $D \equiv X_1X_2 \cap Y_1Y_2 \cap Z_1Z_2$ and the collinearity of the points $F,\ D,\ E.$

Applying last time **Pascal theorem** in the cyclic non-convex hexagon $A_1C_1C_2A_2B_2B_1,$ we have the collinearity of the points $A_3,\ A,\ X_1.$

That is, the point A lies on the line segment A_3X_1 and similarly, the points $B,\ C,$ lie on the line segments $B_3Y_1,\ C_3Z_1$ and the proof is completed.

Kostas Vittas.

Attachments:
t=307278(a).pdf (18kb)

vittasko
#12 Oct 30, 2009, 10:45 pm

> *livetolove212 wrote:*
> Denote $A_4 = BB_2 \cap CC_1, A_5 = BB_1 \cap CC_2,$ similar for B_4, B_5, C_4, C_5. Prove that 6 lines $AX_1, BY_1, CZ_1, A_4A_5, B_4B_5, C_4C_5$ concur at E.

It has been posted before in Hyacinthos Forum, by **Neagoe Petrisor**.

Please see at:
http://tech.groups.yahoo.com/group/Hyacinthos/message/14694?threaded=1&l=1

http://f1.grp.yahoofs.com/v1/8l7qSiVX4nPCMuMD9la0jLzuvgh_5iLlK3e6TvCc2JkLHDX2p4jDWN4tTtKOM7qanEJLECNHGDt-B8zKNmV33rXnP_GV/concurent%20line.pdf

Best regards, Kostas Vittas.

Luis González
#13 Oct 31, 2009, 8:12 am • 1

> *Mashimaru wrote:*
> **Generalized problem** (Stated by PDatK40SP) Let ABC be a triangle with incircle (I) and A_1, B_1, C_1 are three arbitrary points on $BC, CA, AB.$ Denoted by A_2 the intersection of B_1C_1 with the tangent of (I) from A_1 but different of $BC.$ Prove that the A_2, B_2, C_2 are collinear iff AA_1, BB_1, CC_1 concur.

Let D, E, F be the tangency points of (I) with BC, CA, AB and D', E', F' the contact points of the tangents from A_1, B_1, C_1 to $(I),$ different from $D, E, F.$ Poles A_0, B_0, C_0 of the lines B_1C_1, C_1A_1, A_1B_1 WRT (I) form a triangle circumscribed in $\triangle DEF$ and $\triangle D'E'F'$, since DD', EE', FF' are identical to the polars of A_1, B_1, C_1 WRT $(I) \Longrightarrow$ A_0D', B_0E' and C_0F' are the polars of $A_2, B_2, C_2.$

On the other hand, DA_0, EB_0, FC_0 are polars of $A' \equiv BC \cap B_1C_1,\ B' \equiv CA \cap C_1A_1$ and $C' \equiv AB \cap A_1B_1.$ By Desargues theorem, the points A', B', C' are collinear if and only if AA_1, BB_1, CC_1 concur at a point $P \Longrightarrow A_0D, B_0E$ and C_0F concur at the pole Q of the trilinear polar of P WRT $(I).$ Thus, A_0D', B_0E' and C_0F' concur at the cyclocevian Q' of Q WRT $\triangle A_0B_0C_0 \Longrightarrow A_2, B_2, C_2$ lie on the polar of Q' WRT $(I).$

Quick Reply

High School Olympiads

Parallel

Reply

Source: Own?

livetolove212
#1 Oct 29, 2009, 3:53 pm

Given triangle ABC and two isogonal conjugate points E and F. Denote $A_1B_1C_1, A_2B_2C_2$ the pedal triangles of triangle ABC wrt E and $F. B_1C_1\cap B_2C_2=\{A_3\}$, similar for B_3, C_3. Prove that $AA_3//BB_3//CC_3$

Luis González
#2 Oct 29, 2009, 7:02 pm

It's well known that A_1,B_1,C_1,A_2,B_2,C_2 lie on a same circle ω whose center is the midpoint of $\overline{EF}$. Let T be the 2nd intersections of $\odot(AB_1C_1)$ and $\odot(AB_2C_2)$, i.e. the projection of A on EF. Then AT, B_1C_1, B_2C_2 are pairwise radical axes of $\omega, \odot(AB_1C_1), \odot(AB_2C_2)$ concurring at ther radical center A_3, i.e $AA_3 \perp EF$. Similarly, BB_3 and CC_3 are perpendicular to EF. The conclusion follows.

livetolove212
#3 Oct 30, 2009, 11:56 am

Thanks you dear **Luis** for your nice solution.

Mashimaru
#4 Nov 1, 2009, 3:44 pm

Lemma.
Let X, Y, Z be three points in a plane and l_1, l_2 be two lines in plane (XYZ). Denote X_i, Y_i, Z_i the orthogonal projections of X, Y, Z on l_i where $i=1,2$. Then X, Y, Z are collinear iff $\frac{\overline{X_1Y_1}}{\overline{X_1Z_1}}=\frac{\overline{X_2Y_2}}{\overline{X_2Z_2}}$.

The lemma is quite trivially so I will not present its proof.

Now to the solution.

It is known that $A_1, A_2, B_1, B_2, C_1, C_2$ lies on a circle whose center is the midpoint of EF. Define that circle (O).

Let $C_0 \equiv A_1B_2\cap A_2B_1$ and C_a, C_b be the orthogonal projections of C_0 on CA, CB. We have A_1, A_1, B_1, B_2 are concyclic so $\triangle C_0A_1A_2$ and $\triangle C_0B_1B_2$ are similar. This yields $\frac{\overline{C_aA_1}}{\overline{C_aA_2}}=\frac{\overline{C_bB_1}}{\overline{C_bB_2}}$ so by the lemma, O, C_0, E, F are collinear. Now in the complete quadrilateral $(A_1A_2B_1B_2, CC_3)$, the pencil $(C_3A_1, C_3B_1, C_3C_0, C_3C)$ is harmonic, so C_0 is the pole of CC_3 with respect to (O). From this we deduce that $OC_0\perp CC_3$, but O, C, C_0, E, F are collinear so $EF\perp CC_3$. Analogously $EF\perp AA_3$ and $EF\perp BB_3$. Therefore $AA_3 \parallel BB_3 \parallel CC_3$. QED.

Quick Reply

High School Olympiads

Old problem about Jerabek point

Reply

k.l.l4ever
#1 Oct 24, 2009, 7:13 pm

Prove that the Euler circle of $\Delta ABC, AHO, BHO, CHO$ concurrent at the Jerabek point of ΔABC,where H and O is the orthocenter and the circumcenter of ΔABC,respectively.
(the Jerabek point of triangle ABC is the antipole of X wrt the Euler circle (E) of triangle ABC,where X is the Anti-Steiner point of the Euler line wrt (E))

PS:I've edited it.Thanks nsato ^^

This post has been edited 1 time. Last edited by k.l.l4ever, Oct 24, 2009, 11:47 pm

nsato
#2 Oct 24, 2009, 10:01 pm

It's Jerabek, not Jebarek.

livetolove212
#3 Oct 25, 2009, 9:56 am

Denote A', B', C' the midpoints of BC, CA, AB, X the Anti-Steiner of triangle ABC wrt OH, E the center of Nine-point circle of triangle ABC, J, Q the midpoints of AO, AH.
Since $C'QB'O$ is a parallelogram we get $S = QO \cap B'C'$ is the midpoint of $B'C'$.
$\Rightarrow S, E, J$ are collinear.
But S is the midpoint of EJ then (EJQ) lies on $B'C'$.
Denote P the projection of A onto OH, H_a the projection of A onto BC, X' the reflection of P across the line $B'C'$, A'' the reflection of O across the line $B'C'$.
We have H_a, E, A'' are collinear.
$\Rightarrow 90^o = \angle APO = \angle H_aX'A''$
$\Rightarrow X'$ lies on (E) or $X' \equiv X$.
Therefore X is the reflection of P across the line $B'C'$. But $P \in (EJQ)$ so $X \in (EJQ)$.
Similarly we are done.

Attachments:
picture53.pdf (12kb)

Luis González
#4 Oct 26, 2009, 12:36 am

9-point circle (N_1) of $\triangle AHO$ goes through the 9-point center N of $\triangle ABC$, i.e. the midpoint F of AH and the orthogonal projection A' of A on OH. If M, L are the midpoints of AC, AB and D, E the projections of B, C on AC, AB, then $J \equiv ML \cap ED$ is the A-vertex of the side triangle determined by the pedal triangles of O, H. Since they are isogonal conjugates WRT $\triangle ABC$, then A, J, A' are collinear. Let P be the second intersection of (N_1) and (N) and $Q_1 \equiv LD \cap EM$. Note that AJ is the polar of Q_1 WRT (N), hence $Q_1A' \cdot Q_1N = Q_1D \cdot Q_1L \Longrightarrow Q_1$ lies on the radical axis of $(N_1), (N) \Longrightarrow Q_1 \in FP$. From the harmonic pencil $A'(F, A, P, Q_1)$ and $AA' \perp OH$, we have that $A'A$ bisects $\angle FA'P \Longrightarrow \angle PA'A = \angle FA'A = \angle FAA' \Longrightarrow PA' \parallel AH$. From cyclic quadrilateral $AA'OS$, where S is the second intersection of AH with (N_1), we get that $\angle ASO$ is right $\Longrightarrow OS \parallel ML$. On the other hand, since F is the orthocenter of $\triangle ALM$ and O is the antipode of A in the circumcircle $\odot(ALM)$, it follows that ML bisects $FO \Longrightarrow ML$ is the symmetry axis of the isosceles trapezoid $FPA'S$. Hence, $(N_1), (N)$ and the reflection of OH onto ML concur. By similar reasoning the result follows.

Quick Reply

High School Olympiads

Nice

Reply

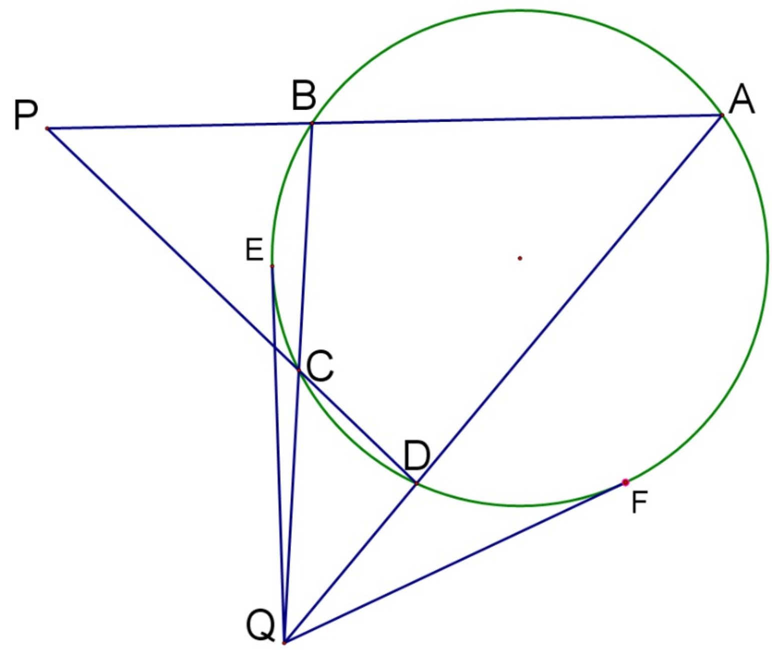
Source: ChinaMO-12

CCMath1

#1 Oct 25, 2009, 2:58 pm

Let $ABCD$be inscribed in $\odot O$,let $AB\cap DC=P$ and $BC\cap AD=Q$ QE,QF are tangent to $\odot O$ prove that pointsP,E,F are collinear.

Attachments:



Luis González

#2 Oct 25, 2009, 9:02 pm

$R\equiv BD\cap AC$ and X,Y are the intersections of line PR with the sidelines $BC,AD.$ From the hamonic cross ratios $(B,C,X,Q)=(A,D,Y,Q)=-1,$ it follows that XY is the polar of Q with respect to $(O).$ But the line joining the contact points of the tangents from Q to (O) is indeed the polar of $Q\Longrightarrow P,E,F$ are collinear.

Alternate approach

thanhnam2902

#3 Oct 25, 2009, 9:15 pm

Let AC meet BD at point $I\implies QI$ is polar line of point P with circle $(O)\implies P$ lie on polar line of point Q with circle (O).
But EF is polar line of point Q. Hence point P lie on line EF.
Q.E.D.

Quick Reply

Spain

V Olimpiada del Cono Sur - Uruguay 1994

Reply

carlosbr

#1 Feb 5, 2006, 12:39 pm

Estoy colgando el examen de las olimpiadas de un nivel aceptable para esta seccion que es de nive de inciacion en olimpiadas .
espero les agrade...

Carlos Bravo

V Olimpiada del Cono Sur
Uruguay - 1994

▶ Problema 1.
El entero positivo n tienen 1994 d\'igitos. De esos, 14 son iguales a 0 y las cantidades de veces que los digitos $1, 2, 3, 4, 5, 6, 7, 8, 9$ aparecen son respectivamente proporcionales a $1, 2, 3, 4, 5, 6, 7, 8, 9$. Mostrar que n no puede ser un cuadrado perfecto.

▶ Problema 2.
Se considera una circunferencia C, de di\'ametro $AB = 1$. Se toma un punto cualquiera P_0 en C, distinto de A, y a partir de ella se costruy\'o una secuencia de puntos P_0, P_1, P_2,... en C del siguiente modo: Q_n es el sim\'etrico de A en relaci\'on a P_n y la recta que contiene a B y Q_n intersecta a C en B y $P_(n+1)$ (no necesariamente distintos). Muestre que es posible escojer P_0 de manera que las dos condiciones a seguir sean satisfechas:
(a). $P_0\hat{A}B < 1^\circ$.
(b). Existen enteros $0 \leqslant r < s$ tales que el tri\'angulo AP_rP_s sea equil\'atero.

▶ Problema 3.
Sea a un real positivo dado. Determine el valor m\'inimo posible de $x^3 + y^3$, donde x y y son reales positivos tales que $xy(x + y) = a$.

▶ Problema 4.
Pedro y Cecilia participan de un juego con las siguientes reglas: Pedro escoge un entero positivo a y Cecilia gana si puede encontrar un entero positivo b, primo relativo con a y tal que la descompocición en factores primos de $a^3 + b^3$ contenga al menos tres primos distintos. Muestre que Cecilia siempre puede ganar.

▶ Problema 5.
Halle todas las ternas (x, y, z) de enteros, con x e y primos entre s\'i - PESI y tales que $x^2 + y^2 = 2z^2$.

▶ Problema 6.
Sea ABC un triángulo restángulo en C. Sobre el lado AB tomamos un punto D de modo que $CD = k$ y los radios de las circunferencias inscritas en los triángulos ACD y BCD sean iguales. Mostrar que $S_{ABC} = k^2$.

$\LaTeX ed$ by Carlos Bravo 2006.

carlosbr

#2 Feb 5, 2006, 12:50 pm

Haber ..
aki cargo la version en pdf
y el source en latex ..

para q les sea mas facil descargarlos...

Saludos

Carlos Bravo
Lima - Peru

carlosbr

#3 Feb 5, 2006, 12:55 pm

Haber ..
aki cargo la version en pdf
y el source en latex ..

para q les sea mas facil descargarlos...

Saludos

Carlos Bravo
Lima - Peru

Attachments:
Conosur1994.zip (38kb)

Luis González

#4 Oct 24, 2009, 8:01 am • 1

> *carlosbr wrote:*
> Sea ABC un triángulo rectángulo en C. Sobre el lado AB tomamos un punto D de modo que $CD=k$ y los radios de las circunferencias inscritas en los triángulos ACD y BCD sean iguales. Mostrar que $[ABC]=k^2$.

Por conveniencia denotemos $AC=b, CB=a, AB=c, AD=m, BD=n, r$ el inradio, h la altura sobre la hipotenusa BC, p el semiperímetro y ϱ el inradio igual de $\triangle ADC$ y $\triangle BDC$. Usando el Teorema de Sakabe Kohan para $n=2$ se tiene:

$$1-\frac{c}{p}=\left(1-\frac{2\varrho}{h}\right)^2 \ .(1)$$

Como el área un triangulo es igual al producto de su inradio por su semiperímetro, el área $[ABC]$ puede ser expresada

$$[ABC]=\tfrac{1}{2}\varrho(b+k+m)+\tfrac{1}{2}\varrho(a+k+n)=\varrho(p+k) \implies \frac{2\varrho}{h}=\frac{c}{p+k} \ . (2)$$

Teniendo en cuenta que $\triangle ABC$ es rectángulo en $C \Longleftrightarrow r=p-c$. Combinando las expresiones (1) y (2) tenemos

$$\frac{r}{p}=\left(1-\frac{c}{p+k}\right)^2 \implies k=\frac{p\sqrt{r}-r\sqrt{p}}{\sqrt{r}-\sqrt{p}} \implies$$

$$k^2=\frac{rp^2+pr^2-2pr\sqrt{rp}}{r+p-2\sqrt{rp}}=\frac{rp(r+p-2\sqrt{rp})}{r+p-2\sqrt{rp}}=rp=[ABC]$$

Quick Reply

High School Olympiads

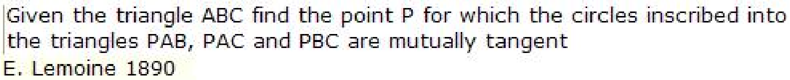
Lemoine point

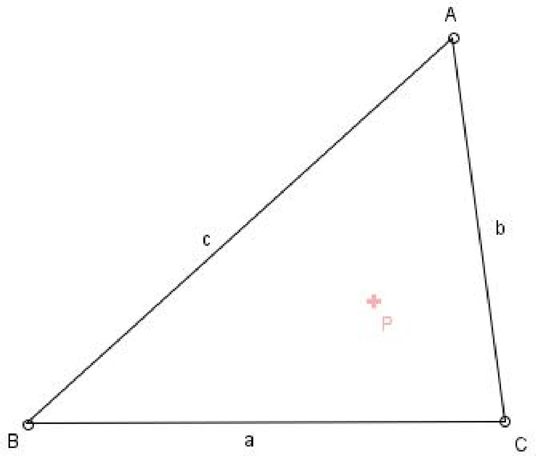
Reply

jrrbc

#1 Oct 21, 2009, 1:40 am

Attachments:


Given the triangle ABC find the point P for which the circles inscribed into the triangles PAB, PAC and PBC are mutually tangent
E. Lemoine 1890



Luis González

#2 Oct 21, 2009, 5:49 am

Let M, M' denote the tangency points of the incircles of $\triangle PAB$ and $\triangle PAC$ with $PA.$ Both circles are tangent $\Longleftrightarrow M$ and M' are identical $\Longleftrightarrow AM = AM'.$

$\Longleftrightarrow AP - PB + c = AP - PC + b \Longleftrightarrow PB - PC = c - b$

$\Longleftrightarrow P$ lies on the hyberbola $\mathcal{H}_a$ with foci B, C passing through $A,$ precisely on the branch passing through $A.$ Therefore, the wanted point P is the first Soddy's point S_0 of $\triangle ABC.$ The construction of S_0 was discussed here.

Quick Reply

High School Olympiads

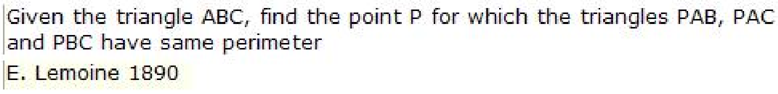
Lemoine point

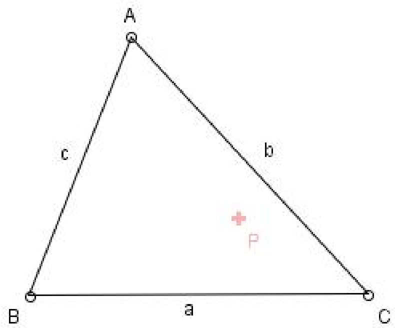
Reply

jrrbc

#1 Oct 21, 2009, 1:44 am

Attachments:





Luis González

#2 Oct 21, 2009, 5:31 am

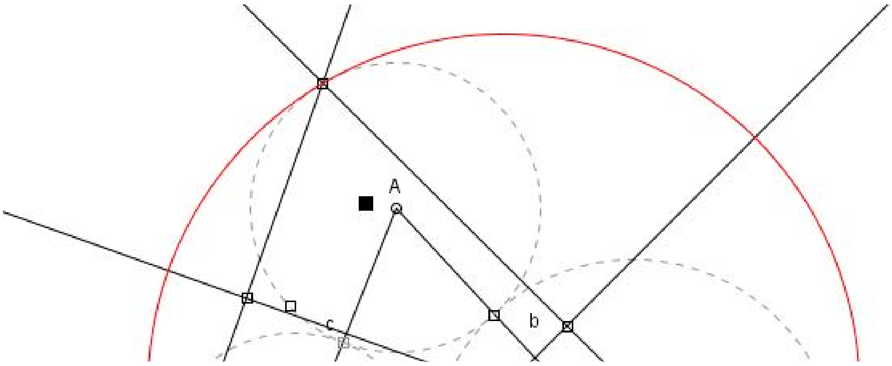
The condition can be expressed as $PB-PC=b-c$, $PA-PC=a-c$, which implies that P is one of the intersections of the three hyperbolas $\mathcal{H}_a, \mathcal{H}_b, \mathcal{H}_c$ with foci in two vertices and passing through the third one. Precisely, it is the intersection of the three branches that do not pass through A, B, C. In order to construct such point, let X, Y, Z be the tangency points of the incircle (I) with BC, CA, AB and let $(A), (B), (C)$ be the circles centered at A, B, C with radii AZ, BX, CY, respectively. Indeed, centers S' of the circles internally tangent to $(B), (C)$ satisfy $S'B-S'C=CX-BX=b-c \Longrightarrow S' \in \mathcal{H}_a$. Hence the wanted point P is identical to the center of the second Soddy's circle $(S_0{}')$, which is internally tangent to $(A), (B), (C)$.

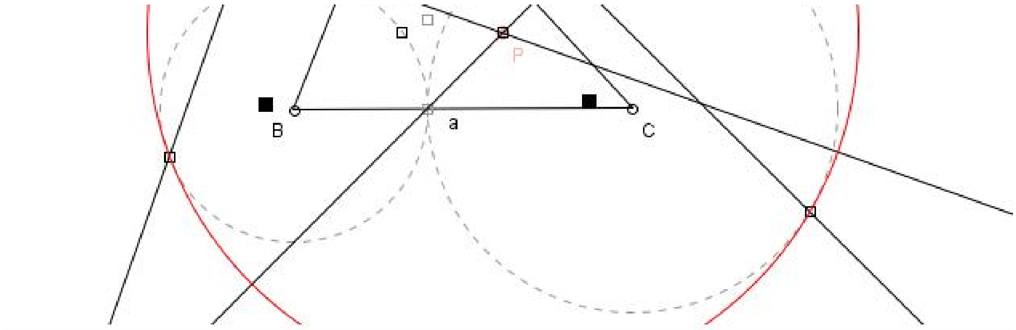
jrrbc

#3 Oct 21, 2009, 8:42 am

could you tell me please what's wrong whith this Soddy construction?

Attachments:





Quick Reply

High School Olympiads

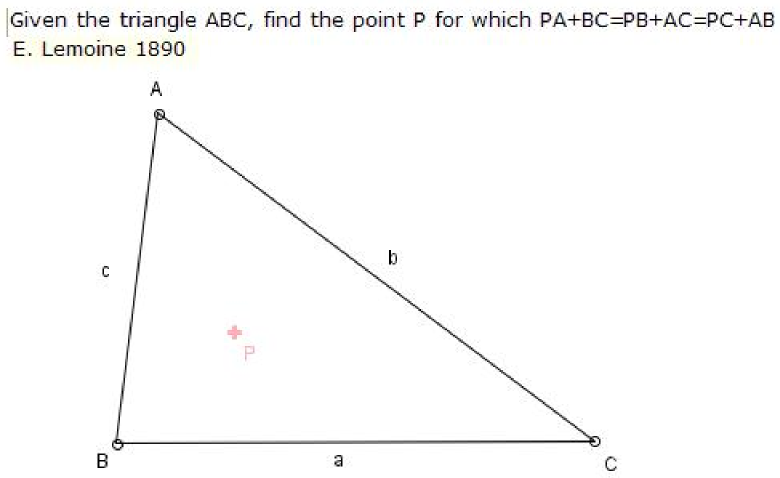
Lemoine point

Reply

jrrbc

#1 Oct 21, 2009, 1:35 am

Attachments:



Luis González

#2 Oct 21, 2009, 5:07 am

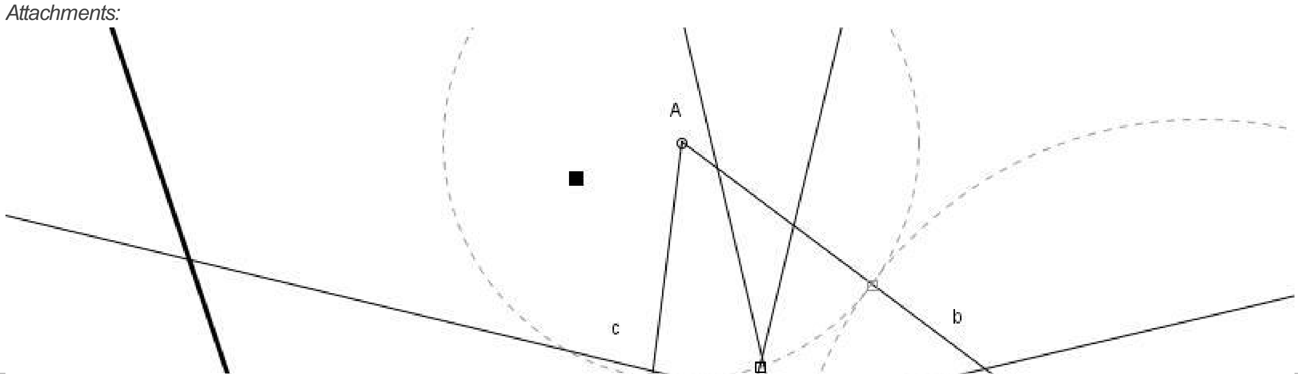
The condition can be expressed as $PB - PC = c - b\ ,\ PA - PB = b - a,$ which implies that P is one of the intersections of the three hyperbolas $\mathcal{H}_a, \mathcal{H}_b, \mathcal{H}_c$ with foci in two vertices and passing through the third one. Particularly, the intersection of the three branches passing through $A, B, C,$ respectively. In order to construct such point, let X, Y, Z be the tangency points of the incircle (I) with BC, CA, AB and let $(A), (B), (C)$ be the circles centered at A, B, C with radii $AZ, BX, CY,$ respectively. Indeed, centers S of the circles externally tangent to $(B), (C)$ satisfy $SB - SC = BX - CX = c - b \Longrightarrow S \in \mathcal{H}_a.$ Therefore, the wanted point P is identical to the center of the first Soddy's circle $(S_0),$ which is externally tangent to $(A), (B), (C).$

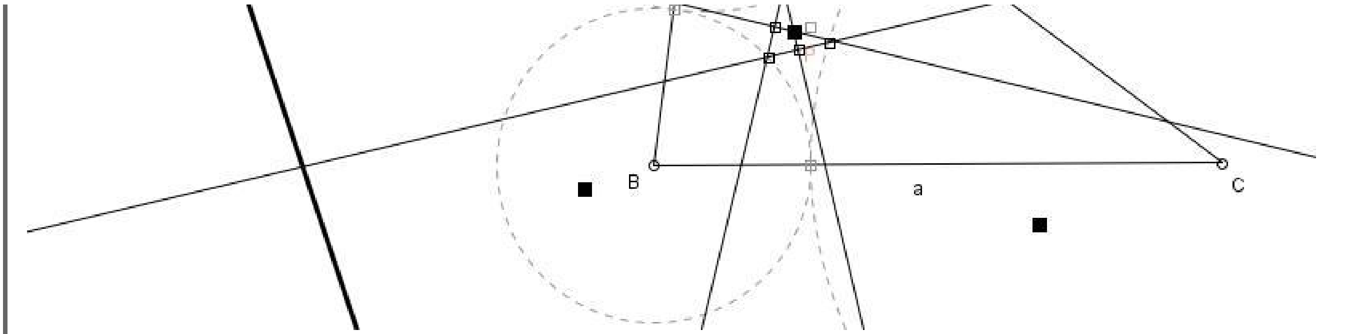
jrrbc

#3 Oct 21, 2009, 8:18 am

could you help me tell me what's wrong in this Soddy construction?

Attachments:





Quick Reply

High School Olympiads

area problem

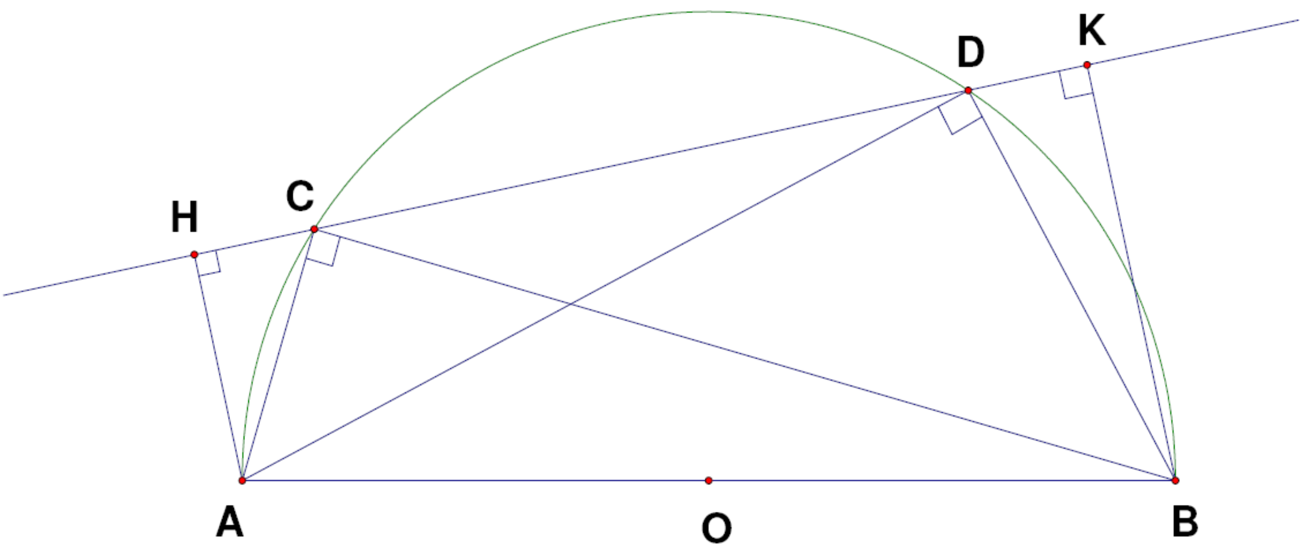
Reply

inversionA007

#1 Oct 18, 2009, 4:50 pm

Let a haft circle with AB is diameter as picture. Let CD is a chore of haft circle. Let H, K are foots of perpendicular line from A, B to CD respectively. Let S_1 is area of $AHKB$ quadrilateral, let S_2 is area of ACB triangle and S_3 is area of ADB triangle. Prove that $S_1 = S_2 + S_3$.

Attachments:



Luis González

#2 Oct 19, 2009, 1:47 am

$$\angle HAC = \angle KCB = \angle DAB \Longrightarrow \triangle AHC \sim \triangle CKB \sim \triangle ADB \Longrightarrow$$

$$\frac{[\triangle AHC]}{[\triangle ADB]} = \frac{AC^2}{AB^2} \ , \ \frac{[\triangle CKB]}{[\triangle ADB]} = \frac{BC^2}{AB^2}$$

$$\frac{[\triangle AHC] + [\triangle CKB]}{[\triangle ADB]} = \frac{AC^2 + BC^2}{AB^2} \Longrightarrow [\triangle AHC] + [\triangle CKB] = [\triangle ADB]$$

$$\Longrightarrow [AHKB] = [\triangle AHC] + [\triangle CKB] + [\triangle ACB] = [\triangle ADB] + [\triangle ACB].$$

Quick Reply

Spain

Cuadrilatero circunscrito

Reply

jonfv

#1 Oct 25, 2006, 1:04 pm

Considere un cuadrillatero $ABCD$ circunscrito a una circunferencia Γ de centro O. Sean P y Q los puntos de contacto de Γ con AB y BC ,respectivamente. Si X es el punto de interseccion de PQ y AC, pruebe que:
a) XO es ortogonal a BD.
b) Los puntos donde las tangentes a Γ que pasan por X interceptan a Γ estan sobre BD.

Obs: Si R y S son los puntos de contacto de Γ con CD y DA , "creo que" P, B, Q, R, D y S estan sobre una elipse.

conejita

#2 Oct 26, 2006, 7:57 pm

Click to reveal hidden text

conejita

#3 Oct 28, 2006, 7:58 pm

Bueno, aqui mi solucion:
Pirmero llamemos T a la interseccion de PR, QS, AC y BD(es conocido que concurren).
Utilizando polos y polares (o algun menelao por alla), observamos que PQ, AC y RS concurren en X. Analogamente observamos que PS, QR y BD concurren en un punto Y.
a) Bueno, ahora tenemos un cuadrilatero PQRS ciclico. Ahora usaremos el siguiente teoremita que dice:
"Sea PQRS un cuadilatero ciclico de centro O, donde PQ y RS se intersectan en X, PS y QR se intersectan en Y, PR y QS se intersectan en T. Entonces O es ortocentro del triangulo XTY" Vemos que aplicando este teorema al problema ya demostramos que XO es perpendicular a BD.

b)Ahora, usaremos el sig. teorema:
"Sea PQRS un cuadrilatero ciclico, donde PQ y RS se intersectan en X, PS y QR se intersectan en Y, PR y QS se intersectan en T. Trazamos la recta YT y a los puntos donde corte a la circunferencia circuscrita de PQRS les llamamos J y K. Entonces XJ y XK son las tangentes desde X a la circunferencia"
Vemos que aplicando esto al problema obtenemos un resultado rapido.

conejita

#4 Oct 28, 2006, 8:07 pm

Aqui pongo un link donde se puede encontrar la demostracion del teorema usado para el inciso a).
http://www.mathlinks.ro/Forum/viewtopic.php?t=108432
Jeje

Luis González

#5 Oct 18, 2009, 8:27 am

Definamos $M\equiv AD\cap BC, N\equiv AB\cap DC, X\equiv PQ\cap SR, Y\equiv PS\cap QR.$ Por teorema de Newton se tiene que SQ, PR, AC, DB concurren en un punto, llámese E y como SQ y PR son las polares de M y N respecto a (O), se sigue que MN es la polar de E respecto a $(O).$ En el cuadrilátero cíclico completo $PQRS$ la diagonal XY es polar de su circuncírculo (O) respecto a $E\equiv PR\cap SQ$, así pues M, N, X, Y estan alineados. Las rectas PQ y SR se cortan en el polo X de la recta DB, por consiguiente $OX\perp DB$ y los puntos de contacto de las tangentes desde X a (O) están en $DB.$

> jonfv wrote:
> Obs: Si R y S son los puntos de contacto de Γ con CD y DA, "creo que" P, B, Q, R, D y S estan sobre una elipse.

Esta conjetura es cierta y facil de demostrar. Los puntos P, B, Q, R, D, S estan en una misma cónica ya que los lados opuestos del hexágono $PBQRDS$ se cortan en los tres puntos alinedos $M, N, Y.$ Por el teorema de Pascal $PBQRDS$ está inscrito en una cónica en la cual MN es una recta de Pascal.

Quick Reply

High School Olympiads

Nice geometry

Reply

dgreenb801

#1 Oct 17, 2009, 8:24 pm

In a parallelogram ABCD with $\angle A < 90$, the circle with diameter AC meets the lines CB and CD again at E and F, respectively, and the tangent to this circle at A meets BD at P. Show that P, F, and E are collinear.

Luis González

#2 Oct 18, 2009, 4:19 am • 1

Let O be the center of the parallelogram and Q the orthogonal projection of A on the diagonal DB. Since $AF \perp DC$ and $AE \perp CB$, the quadrilaterals $ABEQ$ and $ADFQ$ are both cyclic $\Longrightarrow \angle DQF = \angle DAF = \angle BAE = \angle BQE$. Therefore, AQ and DB are the internal and external bisector of $\angle EQF$, thus the perpendicular bisector of EF meets DB at the midpoint of the arc EFQ of $\odot(EFQ) \Longrightarrow OEFQ$ is cyclic. EF is the radical axis of $\odot(OEFQ)$ and $\odot(O)$, DB is the radical axis of $\odot(OEFQ)$ and $\odot(AQO)$, the tangent of $\odot(O)$ at A is the radical axis of (O) and $\odot(AQO) \Longrightarrow$ AP, DB, EF concur at the radical center P of $(O), \odot(AQO), \odot(OEFQ) \Longrightarrow P, F, E$ are collinear.

Quick Reply

High School Olympiads

A geometry problem

Reply

can_hang2007
#1 Oct 13, 2009, 10:07 pm

Let A and B be two fixed points in the circle (O,R). Determine the position of point M $(M \in (O))$ such that the sum $MA+MB$ attains the maximum value.

Luis González
#2 Oct 14, 2009, 12:13 am

It is easy to see that M must be the midpoint of the bigger arc AB of the circle (O). Take B' on the ray AM such that $MB=MB'$. The triangle $\triangle MBB'$ is isosceles with base $BB' \Longrightarrow \angle MB'B=\frac{1}{2}\angle AMB$, therefore B' moves on the circumference (N) centered at the midpoint N of the arc AB and radius $NA=NB$.

The length of the chord AB', $AB'=MA+MB$ obviously attains its maximum when it is identical to the diameter of (N) passing through A. Hence, it follows that $M \equiv N$.

tuanpham
#3 Oct 14, 2009, 10:20 am

Sorry, Why do you know B' moves on (N;NA)

Mashimaru
#4 Oct 14, 2009, 11:07 pm

Dear **Mr.luis**, the problem first stated that A, B are two points **in** circle (O), not **on** circle (O)

Quick Reply

High School Olympiads

Again with the Feuerbach's point

Source: own ?

jayme

#1 Aug 11, 2008, 6:43 pm

Dear Mathlinkers,
let ABC a triangle, A' the foot of the A-altitude of ABC, I the incenter of ABC, X the point of intersection of the perpendicular issue from I with BC, X' the symmetric of X wrt AI and Fe the Feuerbach's point of ABC.
Prove synthetically that the circle through Fe, X' and A' pass through the midpoint of AI.
Any reference?
Sincerely
Jean-Louis

Jan

#2 Aug 11, 2008, 10:31 pm • 1

Dear Jean-Louis Ayme,

it seems you're investigating the Feuerbach point quite actively...
Here is a solution to your problem, synthetic as always, I hope you like it.

In a recent topic on a problem of yours, I proved that D, X', F_e are collinear, where D is the midpoint of B and C. You can find both proof and reference to a problem in Mathemtical Reflections there.

Proof: We call D, E, F the midpoints of BC, AC, AB and S the intersection of AI and BC. We denote M for the midpoints of A and I.

Because $IX'SX$ is a cyclic quadrilateral, it follows that $\measuredangle X'SD = \measuredangle X'IX = \measuredangle OAA'$. Now, if we call N the midpoint of H and A, it is well known that N lies on the nine-point circle of ΔABC. A homothecy centered at H with factor 2 maps N to A, it maps D to the antipode of A in the circumcircle of ΔABC, and it maps A' to the reflection of H in BC, which evidently lies on the circumcircle of ΔABC. We can now see that ND is parallel to AO, which proves that $\measuredangle OAA' = \measuredangle DNA' = \measuredangle DF_eA'$. We conclude that $F_eX'SA'$ is cyclic.

Now, call A_1 and A_2 the projections of A onto BI and CI. Clearly, A_1A_2 is parallel to BC (angle chase). Now, call H' the orthocenter of ΔAA_1A_2, and X'' the antipode of X in the incircle of ΔABC. It is crystal clear that M is the circumcenter of ΔAA_1A_2. We have that $AH' = 2.AM.\cos A_1AA_2 = AI.\cos\dfrac{B+C}{2} = AI.\sin\dfrac{A}{2} = r$. Of course r is the radius of the incircle as always. This shows that X'' is the reflection of H' in M, and hence it lies on the Euler line of ΔA_1AA_2. Now, it follows from Hatzipolakis' theorem that MX'' passes through F_e.

Because F_e is the external center of similitude of the incircle and NPC of ΔABC, it follows that F_eX passes through the midpoint of arc $A'D$ in the NPC of ΔABC. It follows that F_eX'' is the external angle bisector of $\measuredangle A'F_eD$, which is still equal to $\angle A'AO$. It follows that $\measuredangle MF_eA' = \measuredangle ASA'$. This shows that $F_eSA'M$ is cyclic. We already proved that $F_eX'SA'$ is cyclic, so we are done.

Comments:

As to avoid any misinterpretations, the proof of X'' being the De Longchamps point of ΔAA_1A_2 is due to treegoner, and can be found in Hyacinthos message 10913. In fact, using some earlier results of Darij Grinberg, he proved Hatzipolakis' theorem.

For those who don't know Hatzipolakis' theorem: Don't worry. This theorem was found in 2004 by Antreas Hatzipolakis, and first proven by Darij Grinberg. You can find it on Hyacinthos. The theorem states that the Euler line of ΔAA_1A_2 passes through the Feuerbach point of ΔABC.

Two recent topics, all related to this configuration:

http://www.mathlinks.ro/viewtopic.php?t=219830
http://www.mathlinks.ro/viewtopic.php?t=219598

Figure:

Image not found

Luis González

#3 Oct 12, 2009, 10:42 am

Let V denote the foot of the A-angle bisector, M the midpoint of BC, D the midpoint of AI and D' the projection of D onto BC. From the right trapezoid $AIXA'$, it follows that DD' bisects $\angle DAX \Longrightarrow DX'$ and DA' are the reflections of DX across AI and DD', respectively. Hence

$\angle A'DX' = 2\angle VDD' = 2\angle VAA' \Longrightarrow$

$\angle A'VX' + \angle A'DX' = 2(\angle VAA' + \angle AVA') = 180^\circ$

$\Longrightarrow D, X', V, A'$ are concyclic $(\star)$

Since $(X, Y, V, A) = -1$, where Y is the tangency point of the A-excircle (I_a) with BC and M is also midpoint of XY, we have $MX^2 = MV \cdot MA'$. But the inversion with center M and power MX^2 takes the 9-point circle (N) into the common external tangent of $(I), (I_a)$, differerent from BC, which is the reflection of BC about $AV \Longrightarrow F_e \in MX'$. Power of M to (I) is $MX^2 = MV \cdot MA' = MX' \cdot MF_e$. Because of $(\star)$, it follows that the five points D, X', V, A', F_e are concyclic and the proof is completed.

jayme

#4 Oct 12, 2009, 5:13 pm

Dear Mathlinkers,
1. let (1) be the incircle of ABC, A" the midpoint of AI
(2) the circle with diameter A"X , (Garitte's circle it goes through Fe; see http://perso.orange.fr/jl.ayme vol. 4 Symmetric of OI p. 14)
(3) the circle passing through A', A" ,X'
M the meetpoint of AI with BC and N the second point of intersection of AI with (2).
2. The tangent to (1) at X' goes through M.
3. By a converse of the pivot theorem applied to triangle X'MN with (1), (3), (2),
(3) goes through Fe.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

nice

Reply

unt

#1 Oct 8, 2009, 12:55 pm

Given a triangle $ABC. M$ is a point on the $AC. \omega_1, \omega_2$ is a circumcircle of triangles ABM and CBM. Bisector of $\angle AMB$ intercept ω_1 in C_1; bisector of $\angle CMB$ intercept ω_2 in A_1. L -- midpoint of AC.

Prove that $\angle C_1 L A_1 = 90^\circ$.

plane geometry

#2 Oct 8, 2009, 3:13 pm

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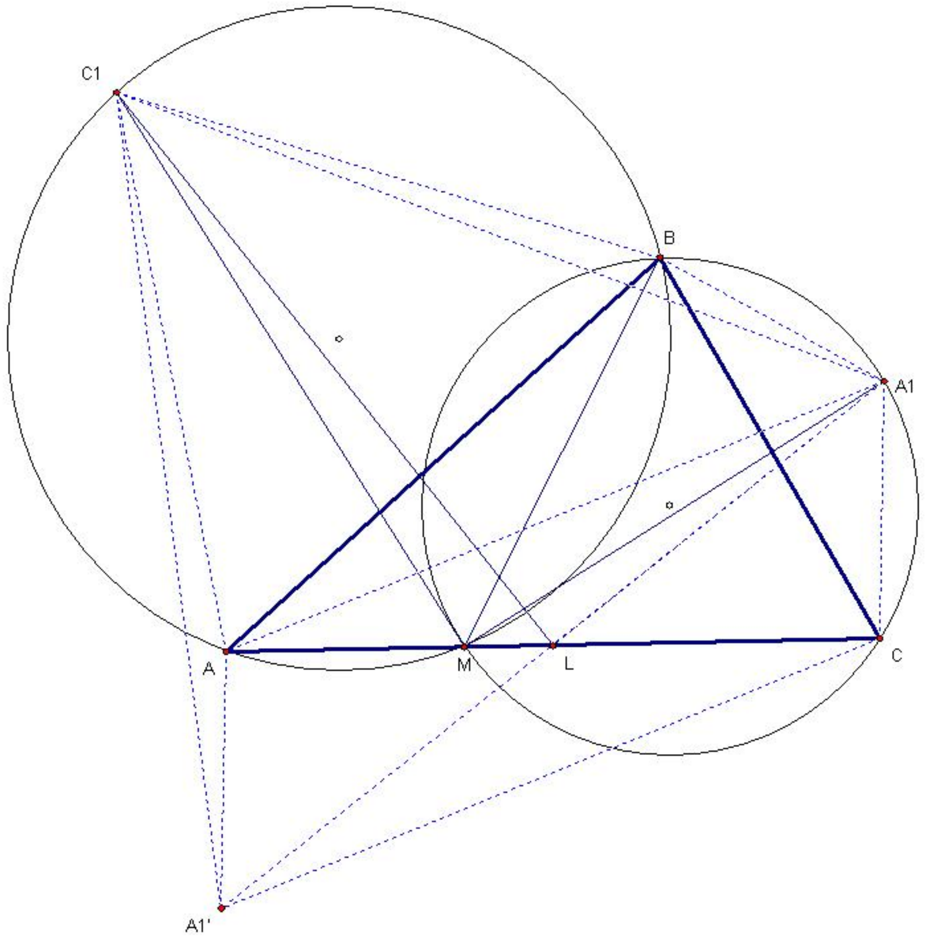
Attachments:



unt

#3 Oct 8, 2009, 3:55 pm

Nice solution!
thanks !!!

Elementaryyy

#4 Oct 9, 2009, 5:00 pm

Use inversion center at M with radius $MA.MC$

Luis González

#5 Oct 11, 2009, 11:09 pm

> *Elementaryyy wrote:*
> Use inversion center at M with radius $MA \cdot MC$

Consider the negative inversion with center M and power $\overline{MA} \cdot \overline{MC}$. Let B' be the inverse of B. Circles ω_1, ω_2 are taken into the lines $B'C, B'A$ respectively and angle bisectors of $\angle(BM, AC)$ are double $\Longrightarrow A_2 \equiv MA_1 \cap B'A$ and $C_2 \equiv MC_1 \cap B'C$ are the inverse of A_1, C_1. Let $N \equiv A_2C_2 \cap BB'$ and $L' \equiv A_2C_2 \cap AC$. From the harmonic cross ratio (A_2, C_2, N, L'), it follows that (A, C, M, L) is harmonic as well $\Longrightarrow \overline{ML} \cdot \overline{ML'} = \overline{MA} \cdot \overline{MC} \Longrightarrow L$ and L' are inverse under the referred inversion. Since A_2, C_2, L' are collinear, then $\odot(A_1C_1L)$ goes through M.

Concyclicboy

#6 Oct 13, 2009, 11:07 am

> *plane geometry wrote:*
> Click to reveal hidden text

I did not understand your solution, but I think you this:

Let A_1' point such that CA_1AA_1' is a parallelogram.
You can prove that $\angle C_1AA_1' = \angle C_1BA_1$
Then, because $AA_1' = A_1C$ and $BC_1 = AC_1$, the triangles $A_1'AC_1$ and C_1BA_1 are congruents
Then, $A_1'MA_1$ is isosceles, and K is the midpoint and finish.

Quick Reply

High School Olympiads

Parallelogram perpendicularity

Reply

Source: Crux

dgreenb801
#1 Oct 11, 2009, 6:50 am

In parallelogram $ABCD$, with center O and $\angle AOB > 90$, B_1 is on the extension of DB through B and A_1 is on the extension of CA through A such that $A_1B_1 \parallel AB$ and $2\angle A_1B_1C = \angle ABC$. Prove that $A_1D \perp B_1C$.

Luis González
#2 Oct 11, 2009, 11:02 am • 1

Let $E \equiv A_1D \cap B_1C$ and F the intersection of the inner bisector of $\angle ABC$ with DC. We shall prove that $\triangle EA_1B_1$ is necessarily right. Since $\angle A_1B_1C = \frac{1}{2}\angle ABC$ and $A_1B_1 \parallel AB$, it follows that $BF \parallel CB_1$. Let $M \equiv EO \cap A_1B_1$. From $DC \parallel A_1B_1$, it follows that cevians A_1C and B_1D of $\triangle EA_1B_1$, meet on its E-median. Thus, EO passes through the midpoints of A_1B_1 and $AB \Longrightarrow EO$ is the midline of the parallels AD, BC. Therefore, $\triangle MB_1E$ is homothetic to the isosceles $\triangle CBF$, due to $BF \parallel CB_1, DC \parallel A_1B_1, EM \parallel CB \Longrightarrow \triangle EMB_1$ is isosceles with legs $ME = MB_1$ and M is the midpoint of $A_1B_1 \Longrightarrow \triangle EA_1B_1$ is right.

Concyclicboy
#3 Oct 12, 2009, 12:06 am

Similar like Luis you can prove, using Ceva, that M is the midpoint... blah blah blah
Is easy note that $\angle A_1M'E = \angle ABC = 2\angle A_1B_1C = \angle A_1B_1C + \angle M'EB_1$, and then M'FE is isosceles and finish.
But I have a Question: ¿Do you think there is a solution without using Ceva?
I say that because a lot of problem don't have solution (not that I know) without this idea.

Immanuel Bonfils
#4 Oct 12, 2009, 6:00 am

Please, where is M' ?

Concyclicboy
#5 Oct 12, 2009, 10:39 am

> *Immanuel Bonfils wrote:*
> Please, where is M' ?

Sorry I should have said M, my M' is the M of Luis

Quick Reply

High School Olympiads

Cyclic Quadrilateral

Reply

Source: Problem 2, Centroamerican Olympiad 2009

Concyclicboy
#1 Oct 8, 2009, 4:19 am

\item Two circles Γ_1 and Γ_2 intersect at points A and B. Consider a circle Γ contained in Γ_1 and Γ_2, which is tangent to both of them at D and E respectively. Let C be one of the intersection points of line AB with Γ, F be the intersection of line EC with Γ_2 and G be the intersection of line DC with Γ_1. Let H and I be the intersection points of line ED with Γ_1 and Γ_2 respectively. Prove that F, G, H and I are on the same circle.

livetolove212
#2 Oct 8, 2009, 5:12 pm

Construct a tangency l of (O) and (O_1) then $\angle GHD = \angle(l, DG) = \angle CED$ thus $GH//CE$
Similarly, $FI//DC$
But $CG.CD = CF.CE$ then $DGFE$ is cyclic
Therefore $\angle GHD = \angle CED = \angle CGF = \angle CFI$, which follows that G, I, H, F are concyclic.

Attachments:
picture37.pdf (10kb)

Luis González
#3 Oct 10, 2009, 5:32 am

Inversion with center C and power $CB \cdot CA$ takes Γ_1 and Γ_2 into themselves and Γ into a common external tangent γ to $\Gamma_1, \Gamma_2 \Longrightarrow F, G$ are the contact points of γ with Γ_1, Γ_2. Since Γ is internally/externally tangent to Γ_1 and Γ_2 (both cases hold), it follows that D, E are inverse points through a positive inversion (O, k^2) taking Γ_1 into Γ_2. Hence, H, I are inverse points and so are F, G obviously, thus $OI \cdot OH = OG \cdot OF = k^2 \Longrightarrow F, G, H, I$ are concyclic.

jayme
#4 Oct 10, 2009, 11:06 am

Dear Mathlinkers,
I can not resist to applied my favourite theorem..
1. Let (0) be the initial circle, (1) the one tangent at D, (2) the one tangent at E.
2. Let (3) the circle passing through D, G, E.
3. According to Monge or d'Alembert's theorem, it goes through F.
4. By Reim's theorem applied to (0) and (1), IF//DC
5. By a converse of the pivot theorem applied to (0), (1) and (3), GF is tangent to (1) at G
6. By a converse of Reim's theorem applied to (1) with IF//DG, F, G, H, I are cocyclics.
Sincerely
Jean-Louis

Moonmathpi496
#5 Nov 23, 2009, 10:57 pm

> *Concyclicboy wrote:*
> \item Two circles Γ_1 and Γ_2 intersect at points A and B. Consider a circle Γ contained in Γ_1 and Γ_2, which is tangent to both of them at D and E respectively. Let C be one of the intersection points of line AB with Γ, F be the intersection of line EC with Γ_2 and G be the intersection of line DC with Γ_1. Let H and I be the intersection points of line ED with Γ_1 and Γ_2 respectively. Prove that F, G, H and I are on the same circle.

It is enough to prove that $\angle GHI = \angle GFI$.

$DGFE$ is is a cyclic quadrilateral, as $CG \cdot CD = CA \cdot CB = CF \cdot CE$. So, $\angle CFG = \angle D$
We know that a homothety with center D maps $\Gamma_1 \to \Gamma$. So, $GH \parallel EC$, and for the same reason $FI \parallel FD$.
We have, $\angle GHI = \angle E$, and $\angle GFI = \pi - \angle CFG - \angle IFE = \pi - \angle D - \angle C = \angle E$.
Q.E.D.

diegu
#6 Dec 5, 2009, 1:53 am

> *livetolove212 wrote:*
> Construct a tangency l of (O) and (O_1) then $\angle GHD = \angle(l, DG) = \angle CED$ thus $GH//CE$
> Similarly, $FI//DC$
> But $CG.CD = CF.CE$ then $DGFE$ is cyclic
> Therefore $\angle GHD = \angle CED = \angle CGF = \angle CFI$, which follows that G, I, H, F are concyclic.

why if $CG.CD = CF.CE$ then $DGFE$ is cyclic?
and on the picture where is the tangency l?
and why $\angle GHD = \angle(l, DG) = \angle CED$?

thanks

diegu
#7 Dec 5, 2009, 5:09 pm

> *diegu wrote:*
>> *livetolove212 wrote:*
>> Construct a tangency l of (O) and (O_1) then $\angle GHD = \angle(l, DG) = \angle CED$ thus $GH//CE$
>> Similarly, $FI//DC$
>> But $CG.CD = CF.CE$ then $DGFE$ is cyclic
>> Therefore $\angle GHD = \angle CED = \angle CGF = \angle CFI$, which follows that G, I, H, F are concyclic.
>
> why if $CG.CD = CF.CE$ then $DGFE$ is cyclic?
> and on the picture where is the tangency l?
> and why $\angle GHD = \angle(l, DG) = \angle CED$?
>
> thanks

please...I don't understand

Joao Pedro Santos
#8 Mar 10, 2010, 2:08 am

> *diegu wrote:*
>> *diegu wrote:*
>>> *livetolove212 wrote:*
>>> Construct a tangency l of (O) and (O_1) then $\angle GHD = \angle(l, DG) = \angle CED$ thus $GH//CE$
>>> Similarly, $FI//DC$
>>> But $CG.CD = CF.CE$ then $DGFE$ is cyclic
>>> Therefore $\angle GHD = \angle CED = \angle CGF = \angle CFI$, which follows that G, I, H, F are concyclic.
>>
>> why if $CG.CD = CF.CE$ then $DGFE$ is cyclic?
>> and on the picture where is the tangency l?
>> and why $\angle GHD = \angle(l, DG) = \angle CED$?

thanks

please...I don't understand

The cyclicity of $DGFE$ is a direct consequence of the Power of a Point Theorem's converse...
$\angle GHD = \angle(l, DG) = \angle CED$ is a direct consequence of the Half-Inscribed Angle Theorem...
But I think Γ is inside Γ_1 and Γ_2, and not the opposite... Anyway, the difficulty is the same.
Here's my solution:

Let O, O_1 and O_2 be the centers of Γ, Γ_1 and Γ_2.
From the tangencies between the circles, we get the colinearities $D-O-O_1$ and $E-O-O_2$.
Since C lies on the radical axis of Γ_1 and Γ_2, the points D, E, F, G are concyclic.
With some angle chasing, it's easy to see it suffices to prove $\angle DGH = \angle EFI$.
If we have the colinearity $O_1 - E - O - D - O_2$, it's trivial, since $\angle DGH = \angle EFI = 90^\circ$.
Otherwise it suffices to prove $\angle DO_1H = \angle EO_2I$.
Let D' be the reflection of D w.r.t. O.
Let D'' be the reflection of D w.r.t. O_1.
It's easy to see $[DD'E]$ and $[DD''H]$ are similar, so we have $\frac{DE}{DH} = \frac{DD'}{DD''} = \frac{DO}{DO_1}$, therefore $[DOE]$ and $[DO_1H]$ are similar, so $\angle DOE = \angle DO_1H$.
Analogously we prove $\angle DOE = \angle EO_2I$, so $\angle DO_1H = \angle EO_2I$, QED.

Quick Reply

High School Olympiads

Parallel Lines

Reply

Source: Problem 5, Centroamerican Olympiad 2009

Concyclicboy

#1 Oct 8, 2009, 4:25 am

Given an acute and scalene triangle ABC, let H be its orthocenter, O its circumcenter, E and F the feet of the altitudes drawn from B and C, respectively. Line AO intersects the circumcircle of the triangle again at point G and segments FE and BC at points X and Y respectively. Let Z be the point of intersection of line AH and the tangent line to the circumcircle at G. Prove that HX is parallel to YZ.

This post has been edited 1 time. Last edited by Concyclicboy, Oct 9, 2012, 9:01 pm

Luis González

#2 Oct 8, 2009, 5:27 am

Rays AO, AH are isogonal WRT $\angle BAC \Longrightarrow AX \perp EF$. Since EF and BC are antiparallel WRT AB,AC, then HX and GD are antiparallel WRT $AO, AH \Longrightarrow \angle AHX = \angle AGD$. But the quadrilateral $DYGZ$ is cyclic on account of the right angles $\angle YDZ = \angle YGZ = 90^\circ$. Therefore, $\angle AHX = \angle AGD = \angle AZY \Longrightarrow HX \parallel YZ$.

ma 29

#3 Oct 8, 2009, 2:30 pm

Denote $S = AH \cap EF; D = AH \cap BC; A, V = AH \cap (O)$

We have :$(AHSD)$ is harmonic ,so $\dfrac{SA}{SH} = \dfrac{DA}{DH} = \dfrac{DA}{DV} = \dfrac{YA}{YG}$

$=> SY//HG$ (1)

On other hand,we see that $SX//ZG$ (2)

Since (1) ,(2) and the Pappus theorem then $HX//YZ$

:)

jayme

#4 Oct 8, 2009, 8:28 pm

Dear Mathlinkers,

1. like Luis observes the quadrilateral DYGZ is cyclic
2. after Luis's obsevation the quadrilateral HXGD is also cyclic
3. use now the Reim's theorem and we ae done

Sincerely

Jean-Louis

Dimitris X

#5 Oct 9, 2009, 12:57 am

> *jayme wrote:*
> 3. use now the Reim's theorem and we ae done

Can you tell me what this theorem says dear **jayme**.

jayme
#6 Oct 9, 2009, 4:51 pm

Dear Dimitris and Mathlinkers,
you will see what you want on
http://perso.orange.fr/jl.ayme/
Sincerely
Jean-Louis

Elementaryyy
#7 Oct 10, 2009, 3:52 pm

We have
$AX = sinC.cosA.AC$
$AY = \frac{sinC}{cosKAX}.AC$
$AH = cosA.2R$
$AZ = \frac{2R}{cosKAX}$
then $\frac{AX}{AY} = \frac{AH}{AZ}$
done.

thanhnam2902
#8 Oct 11, 2009, 9:58 pm • 1

This is my solution:

+) Easy we get $AG \perp EF$ because $\angle AEX + \angle XAE = \angle AHF + \angle CBG = \angle ABC + \angle CBG = 90^0$.

+) Because $XFBG$ and $FBKH$ are cyclic $\Longrightarrow AX.AG = AF.AB = AH.AK \Longrightarrow HKGX$ is cyclic $\Longrightarrow \angle HXY = \angle ZKG$

+) But $ZKXG$ is cyclic $\Longrightarrow \angle ZKG = \angle ZYG \Longrightarrow XH//YZ$, **Q.E.D.**

Attachments:
Parallel Problem - Quy-CQT.pdf (9kb)

ElChapin
#9 Oct 12, 2009, 12:53 am

Another solution.

Click to reveal hidden text

epsilonist
#10 Jan 28, 2010, 9:23 pm

Wow, I don't understand why but I think this problem is really nice.
Thanks to the author.

eps0519
#11 Dec 6, 2011, 5:30 am

Spelling mistake.

> *thanhnam2902 wrote:*
> **This is my solution:**
>
> +) Easy we get $AG \perp EF$ because $\angle AEX + \angle XAE = \angle AHF + \angle CBG = \angle ABC + \angle CBG = 90^0$.
>
> +) Because $XFBG$ and $FBKH$ are cyclic $\Longrightarrow AX.AG = AF.AB = AH.AK \Longrightarrow HKGX$ is cyclic
> $\Longrightarrow \angle HXY = \angle ZKG$

$\Longrightarrow \angle HXY = \angle ZKG$

+) But $ZKYG$ is cyclic $\Longrightarrow \angle ZKG = \angle ZYG \Longrightarrow XH//YZ$, **Q.E.D.**

Quick Reply

Spain

Paralelas

Reply

conejita

#1 Oct 26, 2006, 7:13 pm

Sea ABCD un cuadrilatero convexo. Sean P y Q sobre AD y BC respectivamente tales que cumplan:
$BP/PC = AQ/QD = AB/CD$.
Si BA y CD se intersectan en R, demostrar que la bisectriz l del angulo BRC y la recta PQ son paralelas.

jonfv

#2 Oct 28, 2006, 2:16 am

Deberia ser P sobre BC y Q sobre AD.

Luis González

#3 Oct 7, 2009, 7:05 am

Sean X, Y dos puntos en las diagonales BD, CA tales que $\frac{BP}{PC} = \frac{BX}{XD} = \frac{AY}{YC} = \frac{AQ}{QD}$. De este modo tenemos $PX \parallel QY \parallel RD, PY \parallel QX \parallel RA$ y por otro lado:

$$\frac{PX}{CD} = \frac{PB}{BC}, \frac{PY}{AB} = \frac{PC}{BC} \Longrightarrow \frac{PX}{PY} = \frac{CD}{AB} \cdot \frac{PB}{PC} = \frac{AB}{CD} \cdot \frac{CD}{AB}$$

$\Longrightarrow \ PX = PY \Longrightarrow \ PXQY$ es un rombo.

Por consiguiente la diagonal PQ de $PXQY$ es bisectriz de $\angle XPY$ y como las rectas PX, PY son paralelas a RC, RB, entonces sus bisectrices PQ y l también lo son.

Quick Reply

High School Olympiads

Geometry Marathon

Reply

apratimdefermat
#1 Sep 30, 2009, 7:00 pm

Hi Everyone,

This is an attempt to make a nice **Geometry** marathon at the Pre-Olympiad level.

If you post a problem, please don't forget to indicate its **number** and if you write a solution please indicate to which problem it is and also quote the original problem.Please post a new problem after the pending problem is solved.

Please show a **complete solution** and try to include figures wherever needed.

Lastly, I was inspired by the ongoing inequalities marathon.I hope it will also be a success like its counterpart.

Btw , its advisable to install Java as it is needed to view and edit geogebra applets.Here's the link
http://java.sun.com/javase/downloads/index.jsp

Now to the first problem:

Problem 1:
In a triangle ΔABC, the sides AB and AC are tangent to a circle with diameter along BC at the points Q and P respectively.Let E and F be the extremities of the diameter along BC. EP and FQ intersect at K.Prove that K lies on the altitude from A to BC of ΔABC.

Problem 2:
$ABCD$ is a cyclic quadrilateral and there exists a circle centered on the side AB that is tangent to BC, CD, DA. Prove that $AB = AD + BC$.

apratimdefermat
#2 Oct 1, 2009, 2:47 pm

Solution to Problem 2

Problem 3:
A cyclic quadrilateral $ABCD$ is given. The lines AD and BC intersect at E, with C between B and E; the diagonals AC and BD intersect at F. Let M be the midpoint of the side CD, and let $N \neq M$ be a point on the circumcircle of the triangle ΔABM such that $AN/BN = AM/BM$. Prove that the points E, F and N are collinear.

Problem 4:
Let (O) be a circumference, P an internal point and AA' a chord passing through P. Show that the circles that go through P tangent to (O) at A, A', respectively and the circle with diameter OP are coaxal (they concur at another point P).

apratimdefermat
#3 Oct 5, 2009, 5:26 pm

Solution to Problem 4

Let circles (O_1) and (O_2) be tangent to the circle (O) at A and A' respectively.
Let $(O_1)$and (O_2) meet at Q other than P.
We then have, PO_1OO_2 is a parallelogram.
So, OP is bisected by O_1O_2 at O'(say).
Again O_1O_2 bisects the radical axis PQ of $(O_1)$and (O_2) at right angles at R(say).So, $O'R \,||\, OQ$.

And thus, $\angle OQP = \frac{\pi}{2}$.
Therefore, the circle through OP as diameter passes through Q.

Problem 5:
Let I be the incenter of a given ΔABC and let D, E, F where the incircle touches the sides BC, CA, AB of ΔABC . Now let X, Y, Z be three points on the lines ID, IE, IF such that directed segments IX, IY, IZ are congruent. Prove that the lines AX, BY, CZ are concurrent.

PS. I waited for about 3 days, but no one seems interested in the Geometry Marathon

Luis González

#4 Oct 6, 2009, 8:34 am

1. http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=276882
2. http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=86774
3. http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=39093
4. http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=258361
5. http://www.artofproblemsolving.com/Forum/viewtopic.php?p=186725

Spain

Tony2006
#1 Oct 1, 2006, 11:54 pm

Sea M el punto medio de la mediana AD del triangulo ABC(D pertenece al lado BC) la recta BM corta al lado AC en el punto N. Demuestre que AB es tangente a la circunferencia circunscrita al triangulo NBC si, y solamente si, se verifica $BM/MN = BC^2/BN^2$

Virgil Nicula
#2 Oct 2, 2006, 2:33 am

Enunciation. Given is a triangle ABC. Denote the middlepoints D, M of the segments $[BC]$, $[AM]$ and the point $N \in BM \cap AC$.

Prove that the line AB is tangent to the circumcircle of the triangle $BCN \iff \frac{MB}{MN} = \left(\frac{BC}{BN}\right)^2 \iff AC = AB\sqrt{3}$.

Proof. Aplly the Menelaus' theorem to the transversal $\overline{BMN}$ and the triangle ADC : $\frac{BD}{BC} \cdot \frac{NC}{NA} \cdot \frac{MA}{MD} = 1 \iff$ $\frac{NC}{2} = \frac{NA}{1} = \frac{b}{3}$.

Apply the Menelaus' theorem to the transversal $\overline{AMD}$ and the triangle BNC : $\frac{AN}{AC} \cdot \frac{DC}{DB} \cdot \frac{MB}{MN} = 1 \iff$ $\frac{MB}{3} = \frac{MN}{1} = \frac{BN}{4}$.

Apply the teorem of the B- median in the triangle ABD : $4 \cdot BM^2 = 2\left(c^2 + \frac{a^2}{4}\right) - m_a^2$

and $4m_a^2 = 2(b^2 + c^2) - a^2 \Longrightarrow 16 \cdot BM^2 = 3a^2 - 2b^2 + 6c^2 \Longrightarrow 9 \cdot BN^2 = 3a^2 - 2b^2 + 6c^2$.

1. ▶ $\frac{MB}{MN} = \left(\frac{BC}{BN}\right)^2 \iff \sqrt{3} = \frac{a}{BN} \iff BN = \frac{a\sqrt{3}}{3} \iff 3a^2 - 2b^2 + 6c^2 = 3a^2 \iff b = c\sqrt{3}$.

2. ▶ The line AB is tangent to the circumcircle of the triangle $NBC \iff \triangle ABN \sim \triangle ACB \iff \frac{c}{b} = \frac{BN}{a} = \frac{NA}{c}$ $\iff$

$BN = \frac{ac}{b}$ and $NA = \frac{c^2}{b} \iff BN = \frac{ac}{b}$ and $b = c\sqrt{3}$, i.e. $BN = \frac{a\sqrt{3}}{3} \iff b = c\sqrt{3}$.

Luis González
#3 Oct 5, 2009, 4:10 am

Es bien sabido que la recta BM divide a AC en dos partes siendo una el doble de la otra $\Longrightarrow CN = 2AN$. Por consiguiente al aplicar teorema de Menelao en el $\triangle ADC$ cortado por la transversal $\overline{BMN}$ se obtiene $\frac{BM}{MN} = 3$.

Haciendo uso del Teorema de Stewart en la ceviana BN del $\triangle ABC$ resulta

$$AC \cdot BN^2 = AN \cdot BC^2 + CN \cdot BA^2 - AN \cdot CN \cdot AC$$

$$\Longrightarrow \ BN^2 = \tfrac{1}{3}BC^2 + \tfrac{2}{3}BA^2 - \tfrac{2}{9}AC^2 \Longrightarrow \frac{BC^2}{BN^2} = \frac{BC^2}{\frac{1}{3}BC^2 + \frac{2}{3}BA^2 - \frac{2}{9}AC^2}$$

$$\frac{BM}{MN} = \frac{BC^2}{BN^2} \iff \frac{BC^2}{BN^2} = \frac{BC^2}{\frac{1}{3}BC^2 + \frac{2}{3}BA^2 - \frac{2}{9}AC^2} = 3$$

$\iff \ BA^2 = \frac{1}{3}AC^2 = AN \cdot AC \iff AB$ es tangente a la circunferencia $\odot(NBC)$.

Quick Reply

High School Olympiads

Classical problem

Reply

Source: Very nice

nguyenvuthanhha

#1 Oct 3, 2009, 8:37 am

Let AC ; BD be two diagonals of the convex quadrilateral $ABCD$; $\angle BAD = \angle BCD = 90^o$; $AC \cap BD = \{E\}$

M; N are circumcenters of two triangles ABE; CDE respectively

Prove that if K is the midpoint of MN then K is in the line BD

Luis González

#2 Oct 3, 2009, 11:14 am

Let O be the circumcenter of $ABCD$. Note that $\angle AME + \angle AOE = 2(\angle ABD + \angle ADB) = \pi \Longrightarrow AMEO$ is cyclic, thus $\angle MOE = \angle MAE = \angle ADB = \angle ACB = \angle NED \Longrightarrow OM \parallel AD \parallel EM$. Mutatis mutandis, the quadrilateral $CNOE$ is cyclic $\Longrightarrow ON \parallel BC \parallel EM$. Therefore, $MONE$ is a parallelogram $\Longrightarrow$ diagonals $EO \equiv BD$ and MN bisect each other.

nguyenvuthanhha

#3 Oct 3, 2009, 12:30 pm

Can you explain more how you get $\angle MAE = \angle ADB$

dgreenb801

#4 Oct 6, 2009, 5:29 am

> *nguyenvuthanhha wrote:*
> *Can you explain more how you get $\angle MAE = \angle ADB$*

Since M is the circumcenter of $\triangle ABE$, $\angle AME = 2\angle ABE$, but $\triangle AME$ is isoceles (AM and ME are equal radii).

$\angle MAE = \frac{1}{2}(180 - \angle AME) = \frac{1}{2}(180 - 2\angle ABE) = 90 - \angle ABE$

$= 90 - (90 - \angle ADB)$ (since $\angle BAD = 90$) $= \angle ADB$.

nguyenvuthanhha

#5 Oct 6, 2009, 9:04 am

*Thanks **dgreenb801 ; luisgeometria** ; nice and neat solution*

vittasko

#6 Oct 7, 2009, 1:23 am

• Let $K,\ L$ be, the circumcenters of the triangles $\triangle AED,\ \triangle BEC$ respectively and we denote as O', the midpoint of the diagonal AC, where $OO' \perp AC$ as well.

The lines through $A,\ D$ and perpendicular to $AC,\ BD$ respectively, intersect each other at point so be it F.

Similarly, the lines through $B,\ C$ and perpendicular to $BD,\ AC$ respectively, intersect each other at point so be it Z.

● We have now the configuration of the points $F,\ O,\ Z,$ of which their orthogonal projections on the line segments $AC,\ BD$ (the point O is coincided with its orthogonal projection on BD), form segments with equal ratios ($\frac{OD}{OB} = \frac{O'A}{O'C} = 1$) and so, we conclude that these points are collinear with the same ratio as well and then, we have $OF = OZ.$

Because of now, the midpoints $K,\ L$ of the segments $EF,\ EZ$ respectively, we have that The point $P\equiv KL\cap EO$ is the midpoint of KL (from $KL\parallel FZ$).

● Hence, from the parallelogram $KNLM,$ we conclude that the point P is also the midpoint of $MN,$ lying on BD and the proof is completed.

Kostas Vittas.

Attachments:
t=303929.pdf (5kb)

Quick Reply

Spain

GEOMETRIA: P1: Colinealidad

Reply

carlosbr
#1 Nov 15, 2005, 10:49 am

Rusia 1997

Sea Γ un circulo con centro O e inscrito al triangulo ABC,
al cual toca en sus lados AC, AB, BC en los puntos K, M, N respectivamente. La mediana BB_1 del triangulo ABC corta al segmento MN en D. Mostrar que los puntos O, D, K son colineales.

(Sugerencia)
Hint

Carlos Bravo
Lima -PERU

marko avila
#2 Nov 20, 2005, 11:07 pm

he encontrado una solución bastante fea para el problema. utilizando ley de senos, teorema de la bisectriz generalizada , identidades trigonométricas probé que la mediana y la linea OK dividen al segmento MN en razón AB : BC Y por lo tanto MN,OK Y BB_1 concurren . pero me gustaría ver una solución mas bonita (la de semejanza)
puedes colgarla carlos?? gracias

carlosbr
#3 Nov 23, 2005, 10:47 am

Bueno el Hint realmente contiene el key ..
aqui envio la solucion referida en la sugerencia ...

Solucion:

Sea $L = KO \cap MN$, y sea A_1, C_1 los puntos de interseccion de AB y BC respectivamente con la paralela trazada a AC que pase por L, portanto tenemos $A_1L = LC_1$, lo cual implica que L es el punto medio de BB_1 y entonces $L = D$.

Notar que: $\angle A_1MO$ y $\angle A_1LO$ son rectos, entonces el cuadrilatero MA_1OL es ciclico, entonces $\angle MLA_1 = \angle MOA_1$ como tambien $\angle C_1LN = \angle C_1ON$

(de manera analoga, notemos que $\angle MLA_1$ y $\angle C_1LN$ son rectos, entonces $\angle MOA_1 = \angle C_1ON$). De aqui tenemos que los triangulos OMA_1 y ONC_1 son congruentes, entonces $OA_1 = OC_1$ con lo cual concluimos que $A_1L = LC_1$.

Carlos Bravo
Lima -PERU

Luis González
#4 Jun 21, 2009, 11:17 pm

Supongamos que KO corta a MN en $D.$ Tenemos en el triangulo isósceles $\triangle BMN$

$$\frac{\sin\angle DBC}{\sin\angle DBA} = \frac{DN}{DM} = \frac{KN}{KM} \cdot \frac{\sin\angle DKN}{\sin\angle DKM} = \frac{\cos(\frac{C}{2})\sin(\frac{C}{2})}{\cos(\frac{A}{2})\sin(\frac{A}{2})} = \frac{\sin C}{\sin A} = \frac{c}{a}$$

$\Longrightarrow D$ yace en la mediana BB_1.

Luis González

#5 Oct 3, 2009, 8:05 am

Denotemos T_∞ el punto del infinito de la recta AC, el cual es polo de KD respecto a (O). B es el polo de MN respecto a (O) y el polo de BB_1 es el punto P en el rayo MN tal que $(M, N, D, P) = -1$, pero como $(A, C, B_1, T_\infty) = -1 \Longrightarrow PB \parallel AC$. Como los polos B, P, T_∞ estan alineados, entonces las polares MN, BB_1, DK concurren $\Longrightarrow$ los puntos O, D, K estan alineados.

Quick Reply

High School Olympiads

Concurrent in hexagon

Reply

buratinogigle

#1 Sep 30, 2009, 11:16 pm • 1

Let $ABCDEF$ be a cyclic hexagon and point P. Let A',B',C',D',E',F' be circumcenter of triangles PAB,PBC,PCD,PDE,PEF,PFA. Prove that $A'D',B'E',C'F'$ are concurrent.

Luis González

#2 Oct 1, 2009, 2:38 am • 4

Lemma: Let P be a point on the plane of a circle (O,r). Q is a variable point on (O) and $\mathcal{L}$ is the perpendicular line to PQ through Q. Then the envelope of $\mathcal{L}$ is a conic $\mathcal{H}$.

Proof: Assume that P lies inside (O) and let P' be the reflection of P about Q. P' lies on the homothetic circle (O') of (O) under the homothety with center P and coefficient 2. $\mathcal{L}$ becomes the perpendicular bisector of PP' and let $K \equiv \mathcal{L} \cap O'P'$. We have $PK+KO'=P'K+KO'=2r=\text{const}$. Thus, K moves on the ellipse $\mathcal{H}$ with foci P,O' and center O. Since $\mathcal{L}$ is the external bisector of $\angle PKO'$, it follows that $\mathcal{L}$ is tangent to $\mathcal{H}$ at K.

Hexagon $A'B'C'D'E'F'$ is homothetic to the hexagon formed by the perpendicular lines $\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4,\mathcal{L}_5,\mathcal{L}_6$ to PA,PB,PC,PD,PE,PF through its vertices under the homothety with center P and coefficient $\frac{1}{2}$. From the the above lemma, the hexagon bounded by $\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4,\mathcal{L}_5,\mathcal{L}_6$ is circumscribed in a conic $\mathcal{H}$, thus $A'B'C'D'E'F'$ is also circumscribed in a conic $\mathcal{H}'$. By Brianchon's Theorem, the diagonals $A'D',B'E'$ and $C'F'$ of the tangential hexagon $A'B'C'D'E'F'$ concur.

livetolove212

#3 Nov 9, 2011, 2:09 pm • 5

Consider the inversion I with center P, power ω.
I: $A\mapsto A'',B\mapsto B'',C\mapsto C'',D\mapsto D'',E\mapsto E'',F\mapsto F''$.
$(PAB)\mapsto A''B'',(PBC)\mapsto B''C'',(PCD)\mapsto C''D'',(PDE)\mapsto D''E'',(PEF)\mapsto E''F'',(PFA)\mapsto F''A''$.
Let A_1,B_1,C_1,D_1,E_1,F_1 be the reflections of P wrt $A''B'',B''C'',C''D'',D''E'',E''F'',F''A''$, respectively.
Then $A'\mapsto A_1,B'\mapsto B_1,C'\mapsto C_1,D'\mapsto D_1,E'\mapsto E_1,F'\mapsto F_1$.
So $A'D',B'E',C'F'$ are concurrent iff 3 circles $(PA_1D_1),(PB_1E_1),(PC_1F_1)$ are coaxal. $(*)$
Since A,B,C,D,E,F are concyclic we get A'',B'',C'',D'',E'',F'' are concyclic. By Pascal's theorem, the intersections of $A''B''$ and $D''E''$; $B''C''$ and $E''F''$; $A''F''$ and $C''D''$ are collinear, which follows that the centers of 3 circles $(PA_1D_1),(PB_1E_1),(PC_1F_1)$ are collinear. Therefore $(*)$ is true. We are done.

daothanhoai

#4 Mar 3, 2015, 2:55 pm

A generalization in file attachment

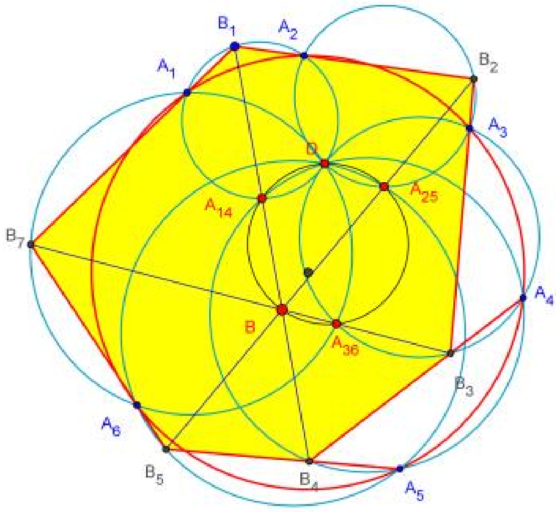
Attachments:

Problem 2 (A generalization Brianchon theorem). *Let 6 points $A_1,A_2,A_3,A_4,...,A_n$ lie on a circle, let a point* D *on the plan. Let B_1 be any point on the (A_1DA_2), define the line B_iA_{i+1} meets the circle $A_{i+1}DA_{i+2}$ at B_{i+1}* with $i=1,2,3,3,4,5,6$ and A_7 at A_1. Show that:

1-Three line B_1B_4,B_2B_5,B_3B_6 are concurrent, denote at new Brianchon's point.

- When D at circumcenter of $(A_1A_2A_3A_4A_5A_6)$ and B_1A_2 is tangent line of $(A_1A_2A_3A_4A_5A_6)$ this problem is Brianchon's theorem.

2-The circle (A_1DA_2) meets the circle (A_4DA_5) again at A_{14}, The circle (A_2DA_3) meets the circle (A_5DA_6) again at A_{25}, The circle (A_3DA_4) meets the circle (A_6DA_1) again at A_{36}. Show that new Brianchon's point, A_{14}, A_{25}, A_{36} lie on a circle.



TelvCohl

#5 Mar 3, 2015, 4:39 pm

> *daothanhoai wrote:*
> A generalization in file attachment

See the topic Six-points circle and one point problem (Part **B**) for **1** and my post at the same topic for **2**

Quick Reply

High School Olympiads

4 circles tangent to 2 others iff the rdical axis concur

Reply

Source: Own - shortlist of the first round of the second edition of the oliforum contest

¬[ƒ(Gabriel)³²¹⁰]¼

#1 Sep 28, 2009, 9:31 pm

Let a circle G and four circle G1, G2, G3, G4 internally tangent to G. Call (1,2) the radical axis of G1 and G2 and so on.

► $1)$ (1,2), (2,3), (3,4), (4,1) concur iff G1, G2, G3, G4 are tangent exernally to a circle G'(different from G).
► $2)$ call P the point of cuncurrence of (1,2), (2,3), (3,4), (4,1), call C the centre of G and C' the centre of G'. So prove that P, C and C' are collinear.

vittasko

#2 Sep 28, 2009, 11:14 pm

LEMMA. - A circle (O) is given with diameter AC and let O' be, an arbitrary point, between $A,\ O,$ where O is the center of $(O).$ We draw an arbitrary circle $(O'),$ withe the center $O,$ which intersects $AC,$ at points so be it $B,\ D$ ($B,$ between $A,\ O'$) and let (K) be, an arbitrary circle tangents externally to (O') at point E and also tangents internally to $(O),$ at point $F.$ Prove that $P\equiv AC\cap EF,$ is a fixed point.

PROOF. Let be the point $Z\equiv (O')\cap PF$ the other than E and it is easy to show that $O'Z\parallel KF$, (1) where K is the center of the circle $(K),$ because of the point E is the internal homothety center of the circles (O') and $(K),$ as well.

From $(1),$ and because of $KF\equiv OF,$ we conclude that the point $P\equiv OO'\cap ZF$ is fixed, as the internal homothety center of the circles $(O),\ (O')$ and the proof of the **Lemma** is completed.

REMARK. - It is useful for the solution of the proposed problem, to say that P is the point of intersection of $AC,$ from the radical axis of the circles $(O),\ (O').$

Also, it is easy to show that the points $C,\ D,\ E,\ F,$ are concyclic (from $O'Z\parallel OF\Longrightarrow\angle BO'Z=\angle COF\Longrightarrow\angle ZBD=\angle FCD\Longrightarrow\angle ZED=\angle FCD$). Similarly, the points $A,\ B,\ E,\ F,$ are also concyclic.

Hence, we conclude that $(PE)\cdot(PF)=(PA)\cdot(PB)=(PC)\cdot(PD)=u^2$, (2) (the circles $(O),\ (O'),$ are inverse each other with pole P and power u^2).

● Return now in to the proposed problem, based on the above **Lemma**, we have that the line segments $E_1F_1,\ E_2F_2,\ E_3F_3,\ E_4F_4,$ pass through the point $P,$ where $E_1,\ F_1,$ are the tangency points of the circle (K_1) (instead of (G_1)), with (O') externally and also with (O) internally, respectively and so on.

Because of now the remark above, we have also $(PE_1)\cdot(PF_1)=(PE_2)\cdot(PF_2)=(PE_3)\cdot(PF_3)=(PE_4)\cdot(PF_4)=u^2$ and so, we conclude that P is the point of intersection of the radical axes of the circles $(K_1),\ (K_2),\ (K_3),\ (K_4),$ taken per two of them and the proof is completed.

NOTE : - It is clear that when two circles $(O),\ (O')$ as the problem states are given, we can easy to construct the circle $(K),$ tangented to (O') exteranally and also tangented to (O) internally, by the definition of the point $P,$ as the point of intersection of the line segment OO', from the radical axis of the given circles.

Kostas Vittas.

Attachments:

t=303216.pdf (7kb)

This post has been edited 6 times. Last edited by vittasko, Sep 29, 2009, 8:17 pm

Luis González

#3 Sep 28, 2009, 11:16 pm

Assume that $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ are externally tangent to a single circle $\mathcal{G}'$. Let P be the radical center of $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$. Inversion with center P and power equal to the power of P with respect to these three circles, transforms $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ into themselves and due to the conformity, the inverse of $\mathcal{G}$ is $\mathcal{G}'$. Thus, circle $\mathcal{G}_4$ is double $\Longrightarrow$ P is the radical center of $\mathcal{G}_3, \mathcal{G}_4$ and $\mathcal{G}_1$ as well $\Longrightarrow$ radical axes $\mathcal{L}_{12}, \mathcal{L}_{23}, \mathcal{L}_{34}$ concur.

Similarly, if we suppose that four radical axes $\mathcal{L}_{12}, \mathcal{L}_{23}, \mathcal{L}_{34}$ and $\mathcal{L}_{41}$ concur, then inversion with center P and power equal to the power of P with respect to them, takes $\mathcal{G}$ into a circle $\mathcal{G}'$ externally tangent to $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$, due to the conformity. We know that the centers of $\mathcal{G}$ and its inverse $\mathcal{G}'$ must be collinear with the center of inversion P.

vittasko

#4 Sep 30, 2009, 4:56 pm

Sorry dear all my friends but, as said a friend of mine **Kostas Elatiotis**, there are some mistakes (that colored red) in what I wrote.

> *vittasko wrote:*
> **REMARK.** - It is useful for the solution of the proposed problem, to say that P is the point of intersection of AC, from the radical axis of the circles $(O),\ (O')$.

The correct is that P, is also the point of intersection of AC, from the radical axis of the circles with diameters AB and CD, because $A,\ B$ and $C,\ D$, are the pairs of points inverse each other, in the mentioned inversion.

> *vittasko wrote:*
> **NOTE :** - It is clear that when two circles $(O),\ (O')$ as the problem states are given, we can easy to construct the circle (K), tangented to (O') exteranally and also tangented to (O) internally, by the definition of the point P, as the point of intersection of the line segment OO', from the radical axis of the given circles.

The correct is, by the definition of the point P, as the internal homothety center of the given circles (the definition of P as the point of intersection of AC, from the radical axis of the circles with diameters AB and CD is also correct, but it is more difficult to construct).

Sorry again, Kostas Vittas.

Quick Reply

High School Olympiads

collinearity in bicentric quardilateral

Reply

scarface

#1 Sep 26, 2009, 11:48 pm

Let $ABCD$ bicentric quadrilateral, intersection of diagonals AC and BD is E, center of incircle is I, center of circumcircle is O. Prove that points E, I, O are collinear.

Luis González

#2 Sep 27, 2009, 12:06 am

Let $X \equiv AD \cap BC, Y \equiv AB \cap DC$ and M, N, L, P are the tangency points of (I) with AB, BC, CD, DA. By Newton's theorem for the tangential quadrilateral $ABCD$, we have $E \equiv ML \cap NP$, which implies that XY is the polar of E WRT $(I) \Longrightarrow EI \perp XY$. But since the quadrilateral $ABCD$ is also cyclic, then XY is the polar of its diagonal intersection E WRT its circumcircle $(O) \Longrightarrow EO \perp XY$. Therefore, we deduce that E, I, O are collinear on a perpendicular to XY.

scarface

#3 Sep 27, 2009, 12:45 am

Sorry, I don't understood polarity. why $EI \perp XY$ and $EO \perp XY$. Can you explain? thank you very much your interest.

vittasko

#5 Sep 27, 2009, 4:28 am

It is a well known classical problem. For some elementary approaches, you can see :
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=31693 - Proof (the simplest I think) of the collinearity as the proposed problem states.
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=110887 - Proof of the perpenticularity mentioned by **Luis**
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=139564 - Proof of a result of the polar theory.

Kostas Vittas.

Quick Reply

High School Olympiads

Concurrent 1

Reply

buratinogigle

#1 Sep 26, 2009, 1:06 pm

Let ABC be a triangle and take three similar rectangles inscribed ABC (a rectangle inscribed ABC has two vertices on side a and two remaining vertices on two other sides of ABC). Let A', B', C' be center of these rectangles (cyclic with A, B, C), respectively.

a) Prove that AA', BB', CC' are concurrent.

b) Let A'', B'', C'' are reflections of A', B', C' through BC, CA, AB, resp. Prove that AA'', BB'', CC'' are concurrent.

Note that, part a) is generalization of the problem on the post IMO ShortList 2001, geometry problem 1.

Luis González

#2 Sep 26, 2009, 11:36 pm

● Let $PQRS$ be the rectangle relative to A such that $P \in AB$ and $Q \in AC$. Construct outwardly on BC the rectangle BCC_aB_a similar to $PQRS$. Homothety with center A and coefficient $\frac{AQ}{AC}$ takes $PQRS$ into $BCC_aB_a \Longrightarrow$ their centers A' and A_1 are collinear with the homothetic center $A \Longrightarrow A_1 \in AA'$. Likewise, B_1, C_1 lie on BB', CC'.

$\triangle BCA_1, \triangle CAB_1, \triangle ABC_1$ are 3 similar isosceles triangles erected outwardly on the sides of $\triangle ABC$. By Kiepert theorem, AA', BB', CC' concur at the outer Kiepert perspector relative to $\angle CBA_1 = \angle ACB_1 = \angle BAC_1 = \phi$.

● Homothety with center A, taking $PQRS$ into BCC_aB_a, takes the reflection A'' of A' about BC into the reflection A_2 of A_1 about B_aC_a. Let M_a denote the midpoint of BC.

$$\tan \widehat{M_aCA_2} = \frac{M_aA_2}{M_aC} = \frac{3A_1M_a}{M_aC} = 3\tan\phi$$

Therefore, $\triangle A_2BC, \triangle B_2CA$ and $\triangle C_2AB$ are three similar isosceles triangles erected outwardly on the sides of $\triangle ABC$ $\Longrightarrow AA'', BB'', CC''$ concur at the outer Kiepert perspector relative to $\tan^{-1}(3\tan\phi)$.

Quick Reply

High School Olympiads

1/Ra+1/Rb+1/Rc=2/r

Reply

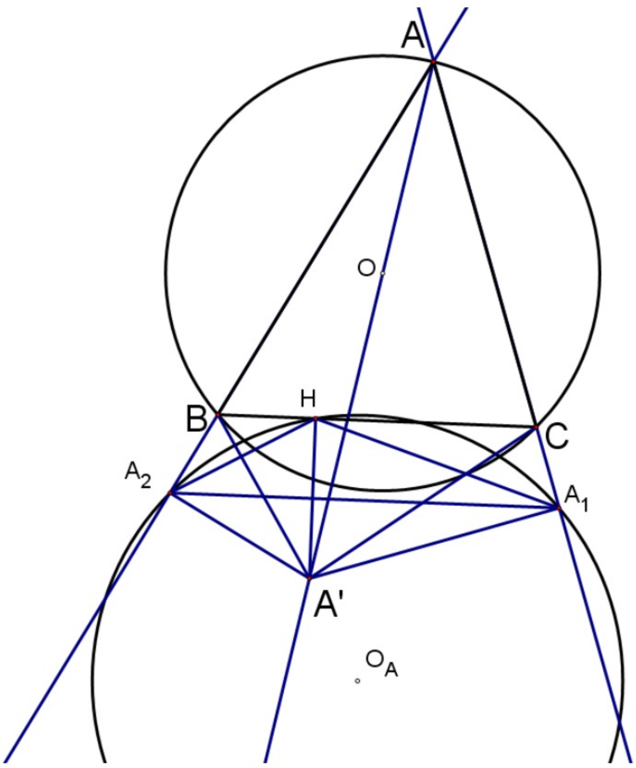
Source: CMO 2008

CCMath1

#1 Sep 25, 2009, 7:22 pm

In an acute triangle let circle O be its circumcircle and A' is on ray AO with $\angle BA'A = \angle CA'A$ and $A'A[1]$ □ AC ; $A'A[2]$□ AB;$A'H$ □ BC , the foot are points $A1, A2, H$. $R[A]$ is radius of the circumcircle of triangle $HA1A2$, $R[B], R[C]$ are defined similarly . Prove that : $\frac{1}{R[A]} + \frac{1}{R[B]} + \frac{1}{R[C]} = \frac{2}{R}$;(R is radius of the circumcircle of triangle ABC).

Attachments:



Luis González

#2 Sep 26, 2009, 2:17 am • 1

Let $N_a \equiv AA' \cap BC$ and define the points N_b, N_c cyclically. Since $A'A$ is the bisector of $\angle BA'C$ and O lies on the perpendicular bisector of BC, it follows that O is the midpoint of the small arc BC of the circumcircle $\odot(BA'C)$. Therefore, (O) cuts AA' at the A'-excenter A of $\triangle BA'C$ and its incenter $I' \Longrightarrow BA$ and CA become external bisectors of $\angle A'BC$ and $\angle A'CB \Longrightarrow \angle A'CA_1 = \angle ACB, \angle A'BA_2 = \angle ABC.$

Quadrilaterals $A'HCA_1$ and $A'HBA_2$ are cyclic on account of the right angles at H and A_1, A_2, respectively. Thus

$$\angle A_1HA' = \angle A'CA_1 = \angle ACB \ , \ \angle A_2HA' = \angle A'BA_2 = \angle ABC$$

$$\Longrightarrow \ \angle A_1HA_2 = \angle ABC + \angle ABC = \pi - \angle BAC = \angle A_2A'A_1$$

This implies that circumcircles of $\triangle HA_1A_2$ and $\triangle A'A_1A_2$ are congruent. Hence, the diameter AA' of $\odot(A'A_1A_2)$ is twice the length of the radius R_a. Now, from the harmonic division (A, I', N_a, A'), we obtain

$$OA^2 = R^2 = ON_a \cdot OA' \implies AA' = OA' + R = \frac{R^2}{ON_a} + R$$

$$\implies \frac{R}{AA'} = \frac{R}{2R_a} = \frac{ON_a}{AN_a}$$

The cyclic sum yields:

$$\frac{R}{R_a} + \frac{R}{R_b} + \frac{R}{R_c} = 2\left(\frac{ON_a}{AN_a} + \frac{ON_b}{BN_b} + \frac{ON_c}{CN_c}\right) = \frac{2}{R}$$

Quick Reply

High School Olympiads

Geometry (1)

Reply

shoki

#1 Sep 24, 2009, 11:28 pm

Let ABC be triangle and denote by (I) its incircle and M the midpoint of BC. Let $\{N, P\} = AM \cap (I)$. Draw the parallel lines with BC through N and P and let them intersect (I) at S, T respectively. Let $AS \cap BC = X$ and $AT \cap BC = Y$. Prove that $MX = MY$.

Dimitris X

#2 Sep 25, 2009, 3:30 am

Draw the parallel line l from the vertex A to BC. The quadrilateral F, N, F', P is harmonic. The tangent at N, P are intersecting on the line l. From Pascals theorem we can prove that the intersection points of the lines PF', FN and $F'N, PF$ are on the l. We consider the complete quadrilateral $NF'PFJV$ where J, V are the he intersection points of the lines PF', FN and $F'N, PF$. So from brockard's theorem we know that the IS' is perpendicular at l and of course on the parallel line from P. So now if we consider the isosceles trapezium $NSPT$ the S' will be the intersection point of its diagonals TS and PN, because S' is on the midlebisector of TP (and of course NS).
So we prove that S, S', T are collinear.

Luis González

#3 Sep 25, 2009, 9:31 am • 2

Let D, E, F be the tangency points of (I) with BC, CA, AB. It is well-known that AM, DI and EF concur, i.e. ST, AM and EF concur at a point K.

$$\frac{MX}{NS} = \frac{AM}{AN}, \ \frac{MY}{PT} = \frac{AM}{AP} \Longrightarrow \frac{MX}{MY} = \frac{NS}{PT} \cdot \frac{AP}{AN} = \frac{KN}{KP} \cdot \frac{AP}{AN}$$

Since EF is polar of A WRT (I), it follows that $(P, N, K, A) = -1 \Longrightarrow$

$$\frac{MX}{MY} = \frac{KN}{KP} \cdot \frac{AP}{AN} = 1 \Longrightarrow MX = MY.$$

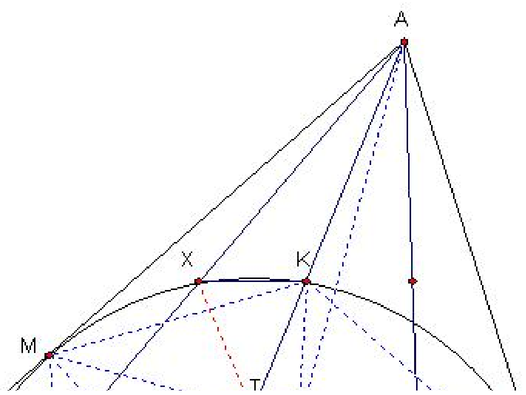
This post has been edited 1 time. Last edited by Luis González, Sep 25, 2009, 11:13 am

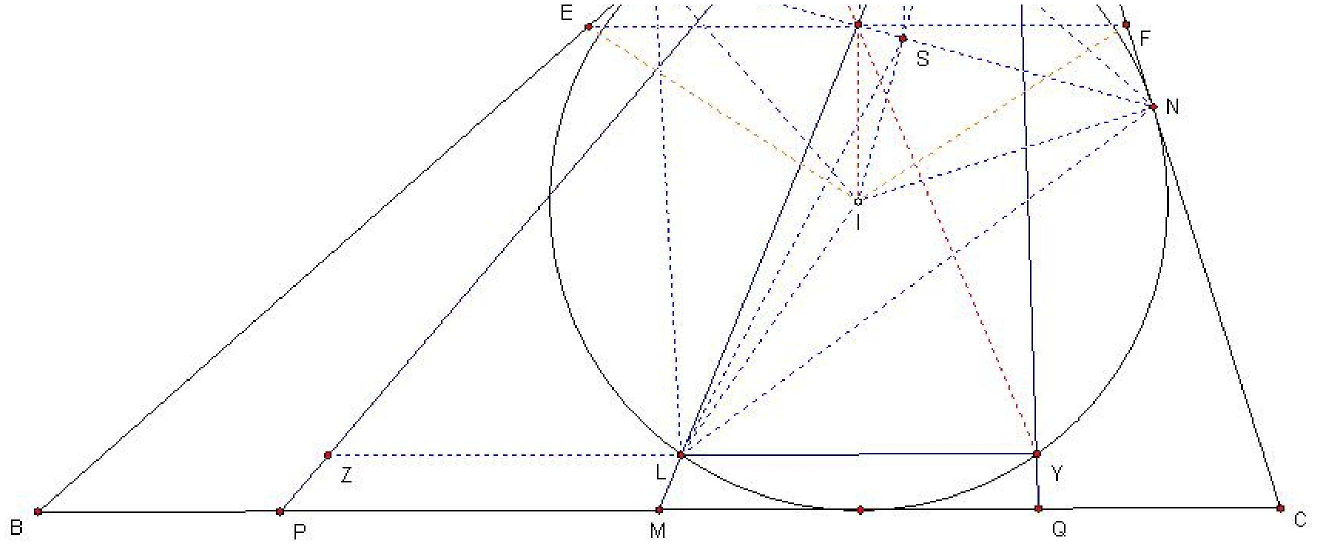
plane geometry

#4 Sep 25, 2009, 10:36 am

notice the fact that AKTL are harmonious division we can conclude that X,T,Y are collinear
then the rest part are trivial

Attachments:





shoki

#5 Sep 25, 2009, 11:28 am

it is one of the geometry exercises given to the students participating in iran3rd round 2009.i will try to post most of them.their topics will be named like this topic(geometry(2),etc).

Petry

#6 Sep 26, 2009, 7:46 pm

The proposed problem can be generalized:

Let ABC be a triangle, $\Gamma\left(J\right)$ is a circle that tangent to the lines AB, AC (the point J lies on the bisector of the angle $\angle BAC$),
M is the midpoint of BC and $\{N, P\} = AM \cap \Gamma$. The parallel lines with BC through the points N, P intersect Γ at the points
S, T, respectively. If $\{X\} = AS \cap BC$ and $\{Y\} = AT \cap BC$ then prove that $MX = MY$.

The proof is similarly to the Luis' proof.

Best regards, Petrisor Neagoe

Quick Reply

Greece

Γεωμετρία - Μία ζόρικη καθετότητα.

Reply

vittasko

#1 Sep 11, 2009, 12:40 am

Δίνεται τρίγωνο $\triangle ABC$ και έστω $D,\ E,$ τα σημεία επαφής του εγγεγραμμένου κύκλου του $(I),$ στις πλευρές $BC,\ AC,$ αντιστοίχως. Δια του σημείου K επί της $BC,$ όπου $BD = KC,$ φέρνουμε την κάθετη ευθεία επί την $BC,$ η οποία τέμνει την DE το σημείο έστω $L.$ Αποδείξτε ότι$BN\perp AK,$ όπου Nείναι το μέσον του$KL.$

Κώστας Βήττας.

Attachments:

t=300345.pdf (4kb)

vittasko

#2 Sep 11, 2009, 12:55 pm

Δείτε μία όμορφη λύση του **dimitris pap** Εδώ, που δείχνει ότι αυτή η καθετότητα, δεν είναι και τόσο ζόρικη...

Όπως λέει και ο **Nick** κάπου αλλού, τις ζόρικες (γυναίκες, ή ασκήσεις), άμα τις παιδέψεις θα βγούνε. Όσον αφορά βέβαια στους γητευτές των ασκήσεων, μιλάμε για στριπτίζ. :)

Να είστε γεροί και δυνατοί, Κώστας Βήττας.

Luis González

#3 Sep 22, 2009, 11:10 am

Proposition. In a triangle $\triangle ABC$, let D,E the tangency points of the incircle (I) with the sides $BC,AC.$ Let K be a point on BC such that $BD=KC.$ The perpendicular to BC at K meets DE at L and the perpendicular from B to AK cuts LK at $N.$ Then N is the midpoint of $LK.$

Let M be the midpoint of $BC,$ which is also the midpoint of $DK.$ Since MI is the M-Nagel ray of the medial triangle of $\triangle ABC,$ we have $MI \parallel AK.$ Therefore $MI \perp BN,$ which implies that $\angle MID=\angle NBK.$ From the similar triangles $\triangle BKN \sim \triangle IDM$ and $\triangle LKD \sim \triangle CDI$ (AA criterion) we have

$$\frac{NK}{BK}=\frac{DM}{ID} \ , \ \frac{LK}{CD}=\frac{DK}{ID}$$

Since $BK=CD$ and $DM=\frac{1}{2}DK,$ we get $NK=\frac{1}{2}LK \Longrightarrow N$ is midpoint of $LK.$

Quick Reply

High School Olympiads

prove!

Reply

undead

#1 Sep 15, 2009, 2:24 pm

let triangle ABC, AB=AC. $P \in$ triangle ABC
prove that:
$PA^2 + PB.PC \leq AB^2$

Duelist

#2 Sep 18, 2009, 10:43 am

If P is on BC

If P is on AB or AC

Arrange your tan

#3 Sep 18, 2009, 1:00 pm

Unfinished business

Does this problem not also include showing the inequality is true
if point P belongs to the interior of triangle ABC as well,
(still with AB = AC)?

undead

#4 Sep 18, 2009, 8:23 pm

but if P inside triangle ABC ???

dgreenb801

#5 Sep 19, 2009, 11:07 pm

Hint

aadil

#6 Sep 20, 2009, 4:04 pm

dgreen can you tell me what you mean by power of B and C?

Bugi

#7 Sep 20, 2009, 4:10 pm

Power of a point
http://en.wikipedia.org/wiki/Power_of_a_point

Luis González

#8 Sep 21, 2009, 6:28 am

Let M be the midpoint of BC and WLOG assume that P lies inside $\triangle ABM.$ Let $Q \in \overrightarrow{CP}$ such that $PB = PQ.$ The circumcenter O' of $\triangle QBC$ lies on AM such that $\angle BO'C = 2\angle BQC = \angle BPC.$ Thus the power of P WRT (O') is $PC \cdot PQ = PC \cdot PB = O'C^2 - O'P^2.$

$S \equiv PC \cap AM$ and N is the midpoint of $CP.$ For each P lying on $AB,$ the midpoint of the cevian CP lies on the C-midline. Therefore, for each P inside $\triangle ABM,$ the midpoint N of CP lies inside $\triangle AMC \Longrightarrow S$ is between $P, N.$

On the other hand, since O' is the midpoint of the arc $BPC,$ the ray $\overrightarrow{CP}$ is internal to $\angle O'CB \Longrightarrow O'$ is between S and $A.$ Then it follows that the orthogonal projection X of O' onto CP lies between C and the orthogonal projection Y of A onto $CP.$ Thus

$$AC^2 - AP^2 = YC^2 - YP^2 = 2PC \cdot NY$$

$$O'C^2 - O'P^2 = XC^2 - XP^2 = 2PC \cdot NX$$

$$NY \geq NX \Longrightarrow AC^2 - AP^2 \geq O'C^2 - O'P^2 \Longrightarrow AB^2 - PA^2 \geq PB \cdot PC.$$

Quick Reply

High School Olympiads

Iran(3rd round)2009

Reply

Source: Problem 5 Geometry

shoki

#1 Sep 13, 2009, 6:31 pm

5-Two circles S_1 and S_2 with equal radius and intersecting at two points are given in the plane.A line l intersects S_1 at B,D and S_2 at A,C(the order of the points on the line are as follows:A,B,C,D).Two circles W_1 and W_2 are drawn such that both of them are tangent externally at S_1 and internally at S_2 and also tangent to l at both sides.Suppose W_1 and W_2 are tangent.Then PROVE $AB=CD$.

shoki

#2 Sep 17, 2009, 10:01 pm

nobody? at least, it isn't ugly...

Luis González

#3 Sep 18, 2009, 9:54 pm

$P \equiv \omega_1 \cap \omega_2$ and M,N are the tangency points of ω_1 and ω_2 with a common external tangent. Inversion with center P and power $PB \cdot PD$ takes (S_1) and the line l into themselves, the circles ω_1 and ω_2 go to two parallel lines k_1 and k_2 tangent to (S_1) and the circle (S_2) goes to a circle (S_2') tangent to k_1,k_2. Hence, (S_2) is congruent to its inverse (S_2'). Further, $(S_2),(S_2')$ are symmetrical about $P \Longrightarrow PC \cdot PA = PB \cdot PD.$

By Casey's theorem for $(\omega_1,\omega_2,D,B,S_1)$ and $(\omega_1,\omega_2,A,C,S_2)$, we get

$$DB=\frac{2PB \cdot PD}{MN} \ , \ AC=\frac{2PA \cdot PC}{MN}$$

Since $PC \cdot PA = PB \cdot PD \Longrightarrow AC=BD \Longrightarrow AB=CD.$

Quick Reply

High School Olympiads

the circle is inscribed into the angle

Source: Ukrainian journal contest, problem 354, by Igor Nagel

rogue

#1 Sep 15, 2009, 7:23 pm

Point M is chosen at the diagonal BD of parallelogram $ABCD.$ The straight line AM intersects the side CD and the straight line BC at points K and N respectively. Let ω_1 be the circle with centre M and radius MA and ω_2 be the circumcircle of triangle $KNC.$ Denote by P and Q the intersection points of circles ω_1 and $\omega_2.$ Prove that the circle ω_2 is inscribed into the angle $QMP.$

Luis González

#2 Sep 16, 2009, 12:06 am

Let O be the center of the parallelogram and A' the reflection of A about $M.$ Since O is midpoint of $AC \Longrightarrow DB \parallel CA'.$ Let T_{∞} be the infinity point of this direction. Since $(D,B,O,T_{\infty})=-1,$ it follows that the pencil $C(D,B,O,A')$ is harmonic $\Longrightarrow (K,N,A,A')=-1.$ By Newton's theorem, we have then $MA^2=MN \cdot MK \Longrightarrow$ Circle ω_1 centered at M with radius MA is orthogonal to $\omega_2.$