No! Extra Math Olympiad!

6 April 2019

Problem 1. Find all functions $f : R \to R$ such that: $f(x+y) = f(x) + f(y) + f(xy)$ $\forall x, y \in R$

Problem 2. For a pair of integers a and b, with $0 < a < b < 1000$, set $S \subseteq \{1, 2, \ldots, 2003\}$ is called a skipping set for (a, b) if for any pair of elements $s_1, s_2 \in S$, $|s_1 - s_2| \notin \{a, b\}$. Let $f(a, b)$ be the maximum size of a skipping set for (a, b). Determine the maximum and minimum values of f.

Problem 3. $\odot O_1$ and $\odot O_2$ touches each other externally at a point T, quadrilateral $ABCD$ is inscribed in $\odot O_1$, and the lines DA, CB are tangent to $\odot O_2$ at points E and F respectively. Line BN bisects $\angle ABF$ and meets segment EF at N. Line FT meets the arc $\widehat{AT}$ (not passing through the point B) at another point M different from A. Prove that M is the circumcenter of $\triangle BCN$.

Problem 4. Given a real sequence $\{x_n\}_{n=1}^{\infty}$ with $x_1^2 = 1$. Prove that for each integer $n \geq 2$,

$$\sum_{i|n} \sum_{j|n} \frac{x_i x_j}{\operatorname{lcm}(i, j)} \geq \prod_{\substack{p \text{ is prime} \\ p|n}} \left(1 - \frac{1}{p}\right).$$