.
And $\angle AP'B = \angle BPC = \angle BFC \implies \angle AP'R = \angle AFR$. So, A, P', F, R concyclic.
So, $\angle FRB = \angle BAP' = \angle CBP = \angle FEB [\because EF \perp BC] \implies EFBR$ concyclic.
So, $GF.GR = GE.GB = GA.GQ [\because ABQE$ concyclic $] \implies F, R, A, Q$ concyclic, i.e., A, F, P', Q, R concyclic.
Again note that $\triangle P'AB \sim \triangle PBC$ and F and D are, respectively, the midpoints of AB and BC. So $\triangle P'AF \sim \triangle PBD$
.
$\therefore \angle BPD = \angle AP'F = \angle AQF [\because A, F, P', Q$ concyclic]. $[QED]$

Quick Reply

High School Olympiads

Equal angles

Reply

socrates

#1 May 11, 2015, 4:18 am

Let ABC be an acute triangle. Let D be the foot of the altitude from A to BC, and F the foot from D to AB. E is the intersection between AC and the perpendicular to BC from B. Prove that $\angle BED = \angle DFC$.

Luis González

#2 May 11, 2015, 4:42 am

Let $P \equiv AB \cap DE$ and let K be the orthocenter of $\triangle PBD$. Since $AD \parallel EB \parallel PK$, all perpendicular to BC, then we have $P(B,D,K,C) = -1 \Longrightarrow CF$ goes through $N \equiv BK \cap PD$; the projection of B on ED. Therefore $\angle BED = \angle DBN = \angle DFN \equiv \angle DFC$.

TelvCohl

#3 May 11, 2015, 4:49 am

My solution :

Let X be the projection of D on AC .

Easy to see B, D, E, X are concyclic .

Since FX is anti-parallel to BC WRT $\angle BAC$,
so B, C, F, X are concyclic $\Longrightarrow \angle DFC = \angle BFC - 90^\circ = \angle BXC - 90^\circ = \angle BXD = \angle BED$.

Q.E.D

Quick Reply

High School Olympiads

Concurrent and collinear

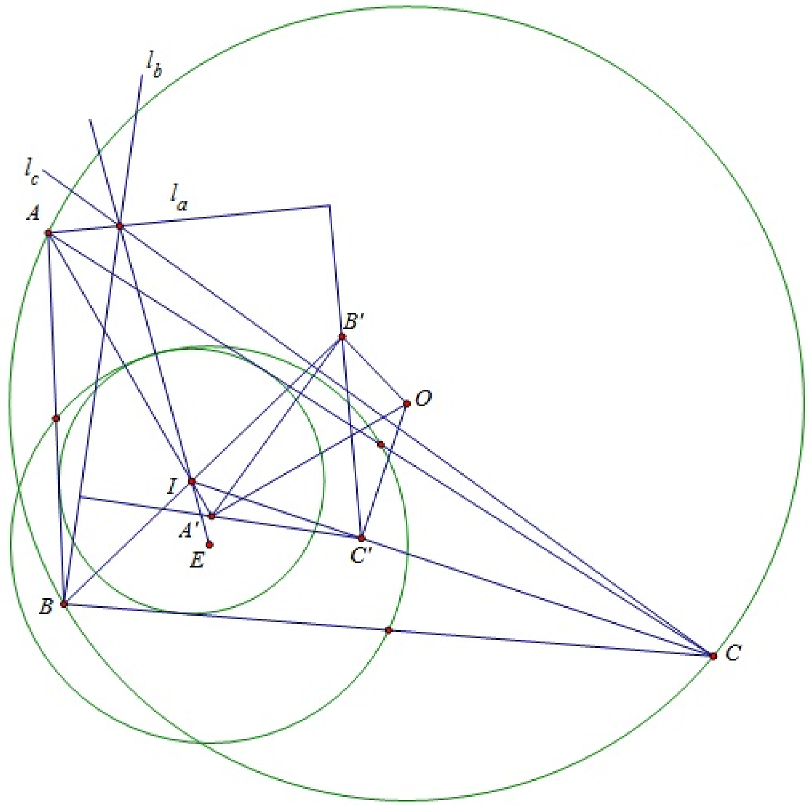
Source: Own

LeVietAn

#1 May 10, 2015, 6:49 pm

Dear Mathlinkers,
Let ABC be a non-isosceles traingle with O, I, E denote its circumcenter, incenter and center of the Euler circle respectively. Let A', B', C' be the orthogonal projections of point O on AI, BI, CI respectively. Let ℓ_a be a line through A and perpendicular to $B'C'$. The lines ℓ_b , ℓ_c are determined similarly to the way we construct the line ℓ_a. Prove that the lines ℓ_a, ℓ_b, ℓ_c intersect at a point on the line EI.

Attachments:



TelvCohl

#2 May 10, 2015, 7:12 pm

My solution :

Let A^*, B^*, C^* be the reflection of O in AI, BI, CI, respectively .

Since AA^* is A-altitude (isogonal conjugate of AO WRT $\angle A$) of $\triangle ABC$,
so $\text{dist}(A', AA^*) = \text{dist}(A', \tau_A)$ where τ_A is the perpendicular bisector of $BC \Longrightarrow EA' \perp BC$.
Similarly, we can prove $EB' \perp CA$ and $EC' \perp AB \Longrightarrow E$ is the orthology center of $\{\triangle A'B'C', \triangle ABC\}$.

Since $\ell_a \cap \ell_b \cap \ell_c$ is the orthology center of $\{\triangle ABC, \triangle A'B'C'\}$,
so from Sondat theorem we get ℓ_a, ℓ_b, ℓ_c are concurrent on EI .

Q.E.D

Luis González

#3 May 11, 2015, 3:01 am • 1

Reflection U of O on AI is on A-altitude $AH \Longrightarrow EA'$ is O-midline of $\triangle OUH \Longrightarrow EA' \parallel AH \perp BC$ and similarly $EB' \perp CA, EC' \perp AB \Longrightarrow \triangle ABC$ and $\triangle A'B'C'$ are orthologic with an orthology center $E \Longrightarrow X \equiv \ell_a \cap \ell_b \cap \ell_c$ is the other orthology center.

Since $\angle(XB, XC) = \angle(A'C', A'B') = \angle(IC, IB) \Longrightarrow X$ lies on reflection of $\odot(IBC)$ on BC and similarly X lies on reflections of $\odot(ICA)$ and $\odot(IAB)$ on $CA, AB \Longrightarrow X \equiv X_{80}$ is the antigonal conjugate of I, thus it lies on EI, being the reflection of I on the Feuerbach point.

Quick Reply

High School Olympiads

Easy Geometry

Source: Iran TST 2015, exam 1, day 1 problem 2

TheOverlord
#1 May 10, 2015, 7:18 pm • 1

I_b is the B-excenter of the triangle ABC and ω is the circumcircle of this triangle. M is the middle of arc BC of ω which doesn't contain A. MI_b meets ω at $T \neq M$. Prove that

$$TB \cdot TC = TI_b^2.$$

This post has been edited 2 times. Last edited by djmathman, Sep 10, 2015, 9:59 am
Reason: latex/formatting

Luis González
#2 May 10, 2015, 11:10 pm • 1

Let I and I_a, I_c be the incenter and the excenters of $\triangle ABC$ againts A, C. Since ω is 9-point circle of $\triangle I_aI_bI_c$, then ω cuts II_a, I_aI_b at their midpoints $M, N \Longrightarrow MN \parallel BI_b \Longrightarrow \angle BI_bT = \angle TMN = \angle TCI_b$. Since $\angle BTI_b = \angle CTI_b$ (TM bisects $\angle BTC$), then $\triangle TBI_b \sim \triangle TI_bC \Longrightarrow \frac{TB}{TI_b} = \frac{TI_b}{TC} \Longrightarrow TB \cdot TC = TI_b{}^2$.

aditya21
#3 May 18, 2015, 8:04 pm

nice and easy!
my solution = as $ABMT, ATCM$ are concylic hence
$\angle MTB = \angle MTC = \frac{\angle A}{2}$ which is due to AM bisecting $\angle BAC$
and thus $\angle CTI_b = \angle I_bTC = 180 - \frac{\angle A}{2}$
also now let $\angle TCI_b = x$ than $\angle TCB = 90 + \frac{\angle C}{2} - x$
and thus $\angle CBT = 180 - \angle TCB = 90 - \frac{\angle C}{2} + x$
and so $\angle TBI_b = \frac{\angle B}{2} - \angle CBT = \frac{\angle A}{2} - x = \angle CI_bT$
and thus $\triangle TBI_b \sim \triangle TI_bC \Longrightarrow TB.TC = TI_b{}^2$

so we are done

This post has been edited 2 times. Last edited by aditya21, May 19, 2015, 12:56 pm
Reason: e

tranquanghuy7198
#4 May 18, 2015, 9:41 pm

My solution:
Let D be the midpoint of arc BC containing A
$\Rightarrow D$ is the circumcenter of $\triangle I_bBC$ and MB, MC are tangent to (I_bBC)
$\Rightarrow I_bM$ is the symmedian of $\triangle I_bBC$
Moreover, T lies on the symmedian such that TI_b bisects $\angle BTC \Rightarrow \triangle TBI_b \sim \triangle TI_bC$ and the conclusion follows

colinhy
#5 May 23, 2015, 9:10 am

Here's a solution:

Observe that $\angle BTC = \angle A$ and since MI_B is an angle bisector of $\angle BTC$, $\angle BTI_B = \angle CTI_B = 180 - \frac{1}{2}\angle A$. Now, let D be the second intersection of the circumcircle τ of CTI_B and BC, so we have $\angle CDI_B = \frac{1}{2}\angle A$, which means that $\triangle BCI_B \sim \triangle BI_BD$ and BI_B is tangent to τ, so $\angle TCI_B = \angle TI_BB$, so by AA symmetry, $\triangle BTI_B \sim \triangle I_BTC$, and the result follows.

Here's a slightly more difficult problem based on the above:
Let I be the incenter of $\triangle ABC$. Show that it is possible to construct a right triangle with side lengths IM, MT, TI_B.

infiniteturtle

#6 May 23, 2015, 9:41 am

Extravert (mostly for convenience) to get

> *Quote:*
> ABC is a triangle with circumcircle w. Let I be the incenter, M be the midpoint of arc BC not including A, and let N be the midpoint of arc BAC. If $NI \cap w = T$, show that $TI^2 = TB \cdot TC$.

The problem now falls easily after drawing (BIC): Let $IT \cap (BIC) = X, CT \cap (BIC) = Y$. It's trivial by angle chasing that $BT = TY$ and that $IT = TX$, now PoP kills it.

Dukejukem

#7 Sep 10, 2015, 9:56 am

Let N be the midpoint of arc $\widehat{BAC}$ of ω and let I_c be the C-excenter. It is well-known that B, C, I_b, I_c are inscribed in the circle Γ of diameter $\overline{I_bI_c}$ with center N. Moreover, note that M is the midpoint of arc $\widehat{BC}$, implying that TM and TN are the internal and external bisectors of $\angle BTC$, respectively. Therefore, the inversion with power $TB \cdot TC$ combined with a reflection in TM swaps B and C fixes lines TM and TN. It follows that the center of Γ' (the image of Γ) is a point on line TN. However, the center of Γ' must also lie on the perpendicular bisector of $\overline{BC}$, implying that the center is N itself. Therefore, Γ is fixed under this inversion, and hence I_b is fixed as well. Thus, $TI_b^2 = TB \cdot TC$ as desired. $\square$

This post has been edited 1 time. Last edited by Dukejukem Sep 11, 2015, 3:54 am

hayoola

#8 Sep 10, 2015, 11:40 am

the interestion between IbC and w is thw midpoint of arc ABC

Quick Reply

High School Olympiads

b+c=2a

Reply

Source: continue

mr.danh
#1 Jul 4, 2008, 6:58 am

Let ABC be a triangle satisfying AB+AC=2BC, AH is the A-altitude. (1),(2) are two circles which intouches the circumcircle (ABC) and touches AH,BC. AB,AC,AH meet the external common tangent of (1) and (2) (other than BC) at M,N,K respectively. Prove that KM=KN.

pohoatza
#2 Jul 5, 2008, 6:31 pm

See here: http://www.mathlinks.ro/viewtopic.php?t=213165.

Moonmathpi496
#3 Jul 5, 2008, 7:12 pm

> *pohoatza wrote:*
> See here: http://www.mathlinks.ro/viewtopic.php?t=213165.

I think that the link you have given is nothing but the link of this post (213165)

yetti
#4 Jul 5, 2008, 7:27 pm

The correct links are:
http://www.mathlinks.ro/viewtopic.php?t=213046
http://www.mathlinks.ro/viewtopic.php?t=210518

The 2nd one also points to:
http://www.mathlinks.ro/viewtopic.php?t=118385
http://www.mathlinks.ro/viewtopic.php?t=88823

pohoatza
#5 Jul 6, 2008, 12:22 am

> *Moonmathpi496 wrote:*
>> *pohoatza wrote:*
>> See here: http://www.mathlinks.ro/viewtopic.php?t=213165.
>
> I think that the link you have given is nothing but the link of this post (213165)

I apologize. The link I was intending to give is: http://www.mathlinks.ro/viewtopic.php?p=518139#518139.

livetolove212
#6 May 8, 2015, 9:54 am

Let M_a be the midpoint of BC, T is the orthocenter of ABC. Ray M_aT cuts (O) at S. TM_a cuts (O) at G, AH cuts (O) at L. We have TM_a is the common internal tangent of (1) and (2) then $LG \parallel BC, AS \parallel MN$. Therefore $G(BCLS) = -1 = A(BCLS) = A(MNKS)$, we get K is the midpoint of MN.

Simple generalization: Given triangle ABC inscribed in (O). M is an arbitrary point on side BC. Let $(O_1), (O_2)$ be the Thebault circles of triangle ABC wrt AM, d be the common internal tangent of (O_1) and (O_2). d intersects (O) at D such that A and D lies on a half-plane wrt BC. DB, DC, d intersect the second common external tangent of (O_1) and (O_2) at U, V, T. Prove that $\frac{TU}{TV} = \frac{MC}{MB}$.

This post has been edited 1 time. Last edited by livetolove212, May 8, 2015, 9:54 am

Luis González

#7 May 10, 2015, 9:52 pm

> *livetolove212 wrote:*
> **Simple generalization:** Given triangle ABC inscribed in (O). M is an arbitrary point on side BC. Let $(O_1), (O_2)$ be the Thebault circles of triangle ABC wrt AM, d be the common internal tangent of (O_1) and (O_2). d intersects (O) at D such that A and D lies on a half-plane wrt BC. DB, DC, d intersect the second common external tangent of (O_1) and (O_2) at U, V, T. Prove that $\frac{TU}{TV} = \frac{MC}{MB}$.

Let $S \equiv UV \cap AM, N \equiv d \cap BC$ and $L \equiv AD \cap BC$. By Parallel tangent theorem we have $AD \parallel UV$. Since $MNST$ is obviously cyclic (isosceles trapezoid), then $MNAD$ is also cyclic $\Longrightarrow LM \cdot LN = LA \cdot LD = LB \cdot LC \Longrightarrow L$ is center of involution interchanging B, C and $M, N \Longrightarrow (L, B, C, N) = (\infty, C, B, M) = \tfrac{MC}{MB}$. But $(L, B, C, N) = D(A, U, V, T) = \tfrac{TU}{TV} \Longrightarrow \tfrac{MC}{MB} = \tfrac{TU}{TV}$.

TelvCohl

#8 May 10, 2015, 10:24 pm

> *livetolove212 wrote:*
> **Simple generalization:** Given triangle ABC inscribed in (O). M is an arbitrary point on side BC. Let $(O_1), (O_2)$ be the Thebault circles of triangle ABC wrt AM, d be the common internal tangent of (O_1) and (O_2). d intersects (O) at D such that A and D lies on a half-plane wrt BC. DB, DC, d intersect the second common external tangent of (O_1) and (O_2) at U, V, T. Prove that $\frac{TU}{TV} = \frac{MC}{MB}$.

My solution :

Let X_∞ be the infinity point on BC .
Let $N = d \cap \odot(ABC)$ and $R = AM \cap \odot(ABC)$.

From Parallel tangent theorem $\Longrightarrow NR \parallel BC, UV \parallel AD$,
so AN is the isogonal conjugate of $AR \equiv AM$ WRT $\angle BAC$,

hence $\frac{TU}{TV} = (T, \infty; U, V) = D(N, A; B, C) = A(N, A; B, C) = A(R, X_\infty; C, B) = (M, \infty; C, B) = \frac{MC}{MB}$.

Q.E.D

Quick Reply

High School Olympiads

AB+AC=2BC

Reply

Source: own

LeVietAn

#1 May 10, 2015, 1:30 pm • 1

Let ABC be a traingle with $AB+AC=2BC$. M,N are midpoints of AB,AC, respectively. Let I be incenter and G be centroid of traingle AMN. Prove that IG is tangent to nine points circle of traingle of ABC.

Luis González

#2 May 10, 2015, 2:02 pm

Let H be the orthocenter of $\triangle ABC$ and D,X the midpoints of BC and the arc BC of $\odot(ABC)$. Reflection P of X on D is midpoint of the arc BHC. Considering homothety with center A and coefficient 2, we only need to show that the line joining the incenter and centroid of $\triangle ABC$ touches $\odot(BHC)$.

For convenience relabel I,G the incenter and centroid of $\triangle ABC$. If $V \equiv AI \cap BC$, we have

$$\frac{IV}{AV}=\frac{a}{a+b+c}=\frac{a}{3a}=\frac{1}{3}=\frac{GD}{AD} \Longrightarrow IG \parallel BC.$$

$$\frac{IV}{VX}=\frac{a}{b+c-a}=1=\frac{PD}{DX} \Longrightarrow P \in IG.$$

Hence, we conclude that IG is the tangent of $\odot(BHC)$ at P, as desired.

Quick Reply

High School Olympiads

Lines intersect on circle and pass through incenters

Reply

Source: HSGS open olympiad 2015, Own

buratinogigle

#1 May 10, 2015, 11:39 am

Let $ABCD$ be a quadrilateral inscribed circle (O). Let I, J be incenters of triangles BAD, CAD. Let DI, AJ cut (O) again at S, T. IJ cuts AB, CD at M, N, respectively.

a) Prove that SM and TN intersect on (O).

b) Circumcircle of triangle ABN cuts CD again at P. Circumcircle of triangle CDM cuts AB again at Q. Prove that PQ passes through incenters of triangles ABC and DBC.

Luis González

#2 May 10, 2015, 12:15 pm • 1

a) From Sawayama's lemma we deduce that the circle ω_1 tangent to AB, CD and internally tangent to (O) through its arc AD at X touches AB, CD at M, N. Therefore $X \equiv SM \cap TN$.

b) Let ω_2 be the circle internally tangent to (O) through its arc BC and tangent to CD, AB at P', Q'. Inversion with center $O \equiv AB \cap CD$ and power $OA \cdot OB = OC \cdot OD$ fixes (O) and swaps ω_1 and $\omega_2 \Longrightarrow OA \cdot OB = ON \cdot OP' \Longrightarrow P \equiv P'$ and similarly $Q \equiv Q'$. Thus again by Sawayama's lemma, we deduce that PQ goes through the incenters of $\triangle ABC$ and $\triangle DBC$.

buratinogigle

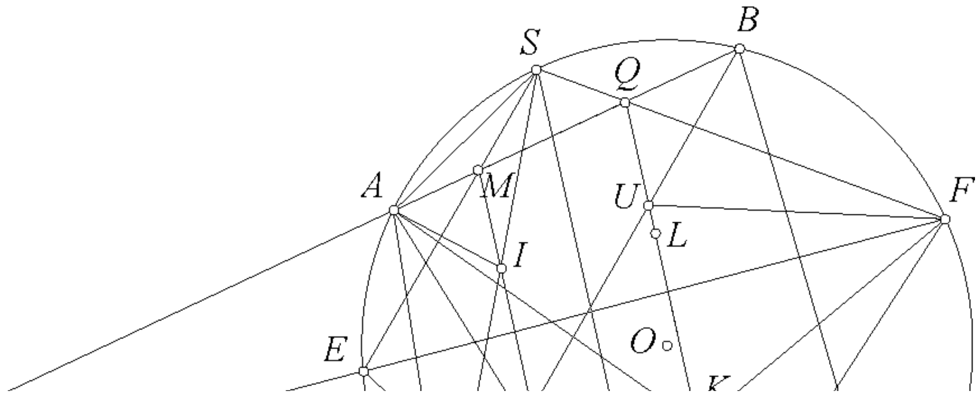
#4 May 10, 2015, 2:58 pm

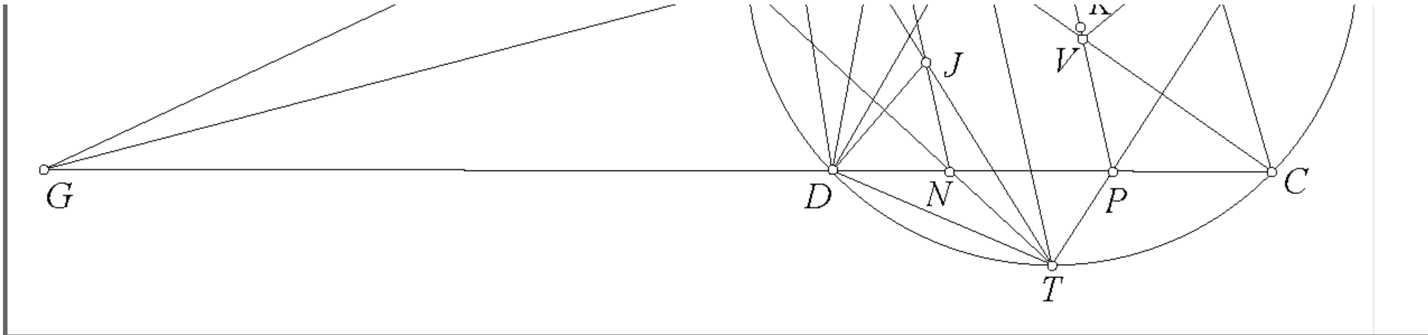
Thank dear Luis, here is my solution

a) We see SAI and DTJ are isocesles and $\angle ASI = \angle DTJ$ so they are similar. We see $AIJD$ is cyclic so $\angle MAI = \angle IAD = \angle DJN$ and $\angle NDJ = \angle JDA = \angle AIM$. From this two triangles MAI and NJD are similar. We deduce SMA and TNJ are similar. Therefore $\angle ASM = \angle NTJ$, so SM and TN intersect at E on (O).

b) Let AB cuts CD at G. GE cuts (O) again at F. We see $GC.GD = GE.GF = GM.GQ$. From this $MQFE$ is cyclic so $\angle QFE = \angle AME = \angle MAS + \angle MSA = \angle MBS + \angle AFE = \angle SFA + \angle ASE = \angle EFS$. We deduce S, Q, F are collinear. Similarly, T, P, F are collinear. From above we have SMA and TNJ so GMN is isoceles, thus $GM = GN$. We have $GM.GQ = GN.GP$ so $GP = GQ$ deduce $PQ \parallel MN \parallel ST$. From this circumcircle of triangle FPQ is tangent with (O). Follow Poncelet theorem PQ cuts DB, AC at U, V then circumcircle of triangle FUV is tangent to (O) and tangent to DB, AC. From this follow Thebault theorem then PQ passes through incenter of triangles ABC and DBC.

Attachments:





Quick Reply

High School Olympiads

Similar to Pascal: What's name of this?

Reply

Isogonics

#1 May 10, 2015, 11:27 am

Γ is a circle, and P, Q is fixed point on the plane.
A is moving points on Γ.
Define B, C, D such that
$B = AP \cap \Gamma, C = AQ \cap \Gamma, D = BQ \cap \Gamma$
where $B \neq A, C \neq A, D \neq B$.
Then CD passes a fixed point R, which lies on PQ.

Yeah, this is easy applying Pascal's theorem twice, but what I want to know is name of this property.
What's name of this property (or theorem)?

Luis González

#2 May 10, 2015, 11:52 am

It also can be seen as involution. If we let U, V be the intersections of PQ with Γ (either real or imaginary), then by Desargues involution theorem for $ABCD$, it follows that CD passes through the image of P under the involution that swaps U, V with double point Q, i.e. a fixed point on PQ.

For a more general point of view see Concurrent 7 (post #5).

Quick Reply

High School Olympiads

Concurrent 7

Reply

buratinogigle

#1 Feb 3, 2011, 8:49 pm • 1

Let ABC be a triangle, $A_1B_1C_1$ is pedal triangle of point P. (O) is circumcircle of triangle $A_1B_1C_1$. Q is a point on OP. QA_1, QB_1, QC_1 intersect (O) at the second points A_2, B_2, C_2, resp. PA_2, PB_2, PC_2 intersect (O) at the second points A_3, B_3, C_3, resp. R is a point on OP. RA_3, RB_3, RC_3 intersect (O) at the second points A_4, B_4, C_4, resp. Prove that AA_4, BB_4, CC_4 are concurrent.

Note that, this is generalization of the problems on the post Pedal circle and concurrent.

Happy lunar new year :)!

TelvCohl

#2 Mar 21, 2015, 12:48 am • 1

My solution:

Let M, N be the intersection of OP with $\odot(O)$.
Let Ψ be the Involution defined on line OP which send $M \longleftrightarrow N, Q \longleftrightarrow R$.

From Desargue involution theorem (for $A_1A_2A_3A_4$) we get $\Psi(P) \in A_4A_1$.
Similarly we can prove B_4B_1, C_4C_1 pass through $\Psi(P)$ which is lie on OP ,
so from the problem Concurrent 14 (b) we get AA_4, BB_4, CC_4 are concurrent .

Q.E.D

buratinogigle

#3 Mar 21, 2015, 2:06 pm

Thank you so much, I tried this for a long time. That's great work!

buratinogigle

#4 Mar 21, 2015, 2:10 pm

Actually, I have known the following theorem for a long time by my own, but I think it has name in geometry, does anybody know its name or source ?

Let ABC be a triangle with circumcircle (O) and P, Q, R are three collinear point. Let PA, PB, PC cut (O) again at A_1, B_1, C_1, reps. Let QA_1, QB_1, QC_1 cut (O) again at A_2, B_2, C_2, resp.. Let RA_2, RB_2, RC_2 cut (O) again at A_3, B_3, C_3, resp. Then AA_3, BB_3, CC_3 are concurrent.

Luis González

#5 Mar 21, 2015, 10:24 pm

The problem can be generalized as follows:

$P_1, P_2, P_3, \cdots, P_n$ are n points (n: odd) on a line ℓ. AP_1 cuts the circumcircle (O) again at A_1. A_1P_2 cuts (O) again at A_2, A_2P_3 cuts (O) again at A_3 and so on, until we get A_n. The points B_n and C_n are obtained similarly. Then AA_n, BB_n, CC_n and ℓ concur.

Proof: Just notice that A_n, B_n, C_n are the image of A, B, C under the composition of the n involutions with poles $P_1, P_2, P_3, \cdots, P_n$ fixing (O) and it interchanges the intersections U, V of ℓ with $(O) \Longrightarrow$ the homography $\{A, B, C, U\} \mapsto \{A_n, B_n, C_n, V\}$ is an involution $\Longrightarrow AA_n, BB_n, CC_n$ and $UV \equiv \ell$ concur at its pole.

Quick Reply

High School Olympiads

line which bisects the perimeter

Reply

Source: Kyiv mathematical festival 2015

rogue

#1 May 10, 2015, 1:11 am

Let O be the intersection point of altitudes AD and BE of equilateral triangle $ABC.$ Points K and L are chosen inside segments AO and BO respectively such that line KL bisects the perimeter of triangle $ABC.$ Let F be the intersection point of lines EK and $DL.$ Prove that O is the circumcenter of triangle $DEF.$

Luis González

#2 May 10, 2015, 2:30 am

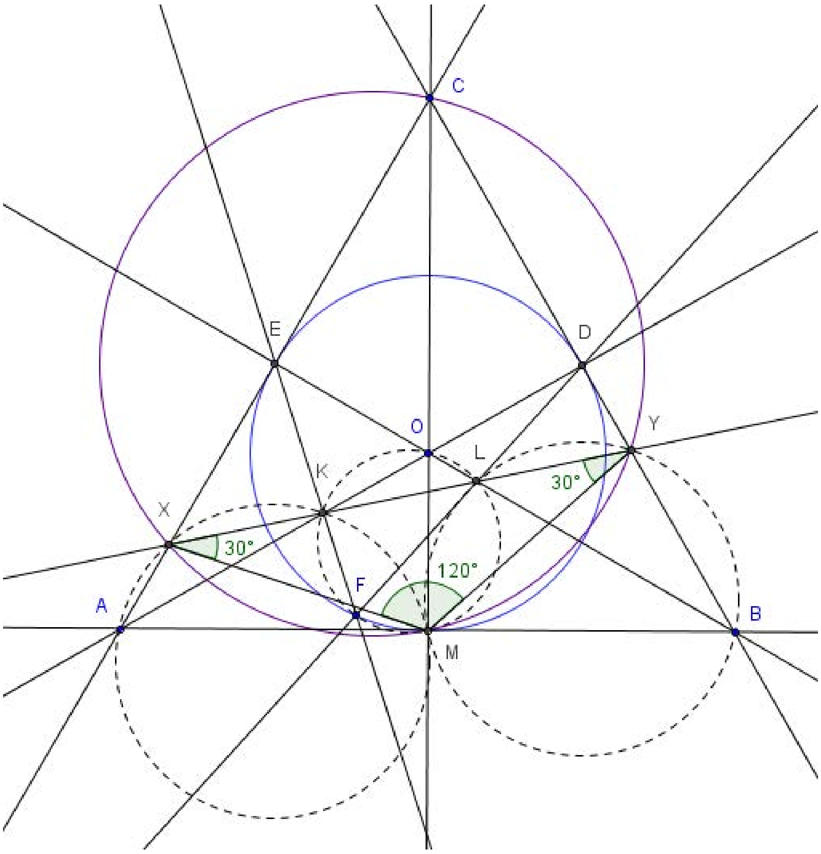
If KL cut CA,CB at $X,Y,$ then clearly X defines Y unambiguosly and O is the circumcenter of $\triangle DEF \Longleftrightarrow F$ is on the incircle $(O).$ Thus we can restate the problem as follows: F lies on (O) (inside of $\triangle OAB$). DF,EF cut OB,OC at L,K and LK cuts CA,CB at $X,Y.$ Then XY bisects the perimeter of $\triangle ABC.$

Let M be the midpoint of AB (tangency point of (O) with AB). Since $\angle KOL=120^{\circ}, \angle KFL=60^{\circ}$ and $\angle MFL=60^{\circ},$ it follows that $M \in \odot(FLOK)$ is the midpoint of its arc $KFL.$ Since $MOEA$ is cyclic, then M is then the Miquel point of KL WRT $\triangle OAE \Longrightarrow MKXA$ is cyclic $\Longrightarrow \angle MXK=\angle MAO=30^{\circ}$ and similarly $\angle MYK=30^{\circ}.$ As a result M is the midpoint of the arc XY of $\odot(CXY).$ Thus by Ptolemy's theorem for $CXMY,$ we get:

$$CX+CY=\frac{XY}{MX} \cdot CM=\sqrt{3} \cdot CM=\tfrac{3}{2}AB \Longrightarrow 2 \cdot (CX+CY)=3 \cdot AB \Longrightarrow$$

$$CX+CY=AB+(AB-CX)+(AB-CY)=AB+AX+BY.$$

Attachments:



Luis González

#3 May 10, 2015, 5:53 am

Okay, here is a more direct approach, working with the same previous notations.

From the condition that XY bisects the perimeter of $\triangle ABC,$ we deduce that $EX+DY=AX+BY$ and since $AE=DB \Longrightarrow AX=DY$ and $EX=BY \Longrightarrow X$ and Y are then homologous points under the rotation $(M,120^\circ)$ that swaps AE and $DB.$ So this makes $\triangle MXY$ isosceles with $\angle MXY=\angle MYX=30^\circ \Longrightarrow AXKM$ is cyclic $\Longrightarrow M$ is the Miquel point of $\triangle OAE$ WRT $KL \Longrightarrow MFKL$ is cyclic. But $\angle MKL=\angle MAX=60^\circ$ and similarly $\angle MLK=60^\circ \Longrightarrow \angle KFL=\angle KML=60^\circ \Longrightarrow F \in (O).$

Quick Reply

High School Olympiads

Regular Triangle

Reply

Relic_93

#1 May 9, 2015, 10:20 am

$\triangle ABC$ is a regular triangle, O is center of $\triangle ABC$.
and P is a arbitrary point inside circumcircle of $\triangle ABC$.

Let $PA=a, PB=b, PC=c, OP=x$, Find Area of $\triangle XYZ$ such that $YZ=a, ZX=b, XY=c$.

MillenniumFalcon

#2 May 9, 2015, 1:33 pm

So far I have the area is
(1/4*Sqrt[3]) (2L^2-a^2-b^2-c^2), where L is the side length of trangle ABC. How would one write L in terms of the other variables?

Luis González

#3 May 10, 2015, 4:38 am

Let L, R denote the side length and circumradius of $\triangle ABC$ and let $\triangle A'B'C'$ be the pedal triangle of P WRT $\triangle ABC$. We have $B'C'=PA\cdot\sin 60^\circ=\frac{\sqrt{3}}{2}a$ and similarly for the other sides.Thus $\triangle A'B'C'\sim\triangle XYZ$ with similarity coefficient $\frac{\sqrt{3}}{2}\Longrightarrow [XYZ]=(\frac{2}{\sqrt{3}})^2[A'B'C']=\frac{4}{3}[A'B'C']$. Now using Euler theorem, we obtain

$$[XYZ]=\frac{4}{3}[ABC]\cdot\frac{R^2-x^2}{4R^2}=\frac{4}{3}\cdot\frac{\sqrt{3}}{4}L^2\cdot\frac{\frac{1}{3}L^2-x^2}{4\cdot\frac{1}{3}L^2}=\frac{\sqrt{3}}{12}(L^2-3x^2).$$

But by Leibniz theorem, we have $a^2+b^2+c^2=L^2+3x^2\Longrightarrow$

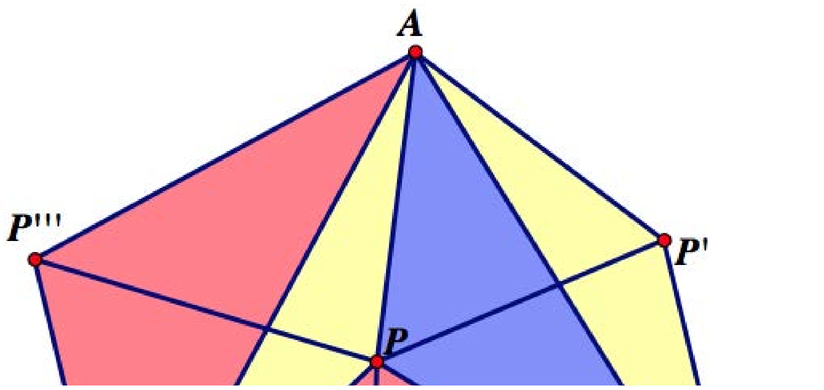
$$[XYZ]=\frac{\sqrt{3}}{12}(a^2+b^2+c^2-6x^2).$$

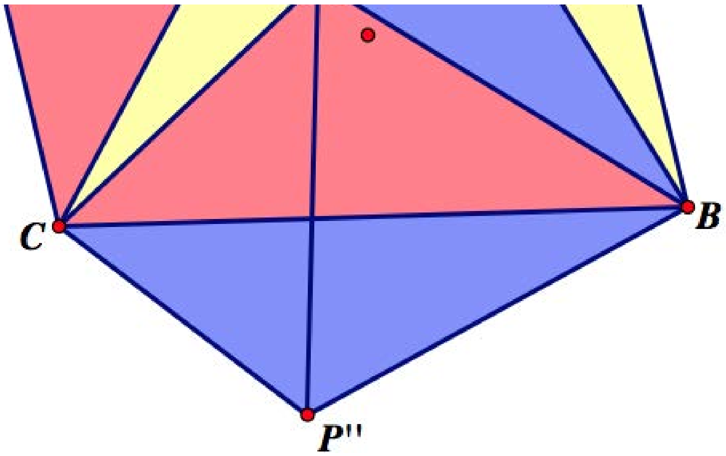
MillenniumFalcon

#4 May 11, 2015, 9:13 am

No need for those, consider rotating triangle APC so that the side AC now overlaps with side AB, and P is rotated to P'. and tada! APP' is equilateral!
Repeat for the other 2 triangles, we have:
Total area of 2*[ABC] and 3 equilateral triangles of side length a,b,c, and 3 of the areas that we want!
Then we quote Luis's leibniz theorem.

Attachments:





Quick Reply

High School Olympiads

May Olympiad 2012, Level 2, Problem 3

Locked

Source: http://www.oma.org.ar/enunciados/mayo2012.pdf

equacoesdiofantinas

#1 May 9, 2015, 9:36 am

In triangle ABC, we checked that $\hat{B} = 2\hat{C}$ and $\hat{A} > 90^\circ$. Let M the midpoint of the BC. The perpendicular by C alongside AC cuts the line AB in D. Show that $A\hat{M}B = D\hat{M}C$.

Luis González

#2 May 9, 2015, 9:47 am

Posted before at http://www.artofproblemsolving.com/community/c6h482414.

High School Olympiads

Perpendiculars and Angle Bisectors

Source: Olimpiada de Mayo

RobRoobiks
#1 Jun 5, 2012, 12:19 am

Given Triangle ABC, $\angle B = 2\angle C$, and $\angle A > 90^\circ$. Let M be midpoint of BC. Perpendicular of AC at C intersects AB at D. Show $\angle AMB = \angle DMC$

Click to reveal hidden text

yetti
#2 Jun 5, 2012, 1:05 am • 1

Perpendicular bisector of BC cuts AB at N. $\angle BCN = \angle CBN = 2\angle BCA \Longrightarrow CA$ bisects $\angle BCN$.
Together with $CD \perp CA \Longrightarrow$ cross ratio $(D, N, A, B) = -1$ is harmonic.
Since $MN \perp MB \Longrightarrow MN$ bisects $\angle AMD \Longrightarrow \angle AMB = \frac{\pi}{2} - \angle NMA = \frac{\pi}{2} - \angle NMD = \angle DMC$.

High School Olympiads

May Olympiad 2014, Level 2, Problem 2

Reply

Source: http://www.obm.org.br/export/sites/default/como_se_preparar/provas/Maio/docs/maio2014.pdf

equacoesdiofantinas

#1 May 9, 2015, 9:22 am

In a convex quadrilateral $ABCD$, be M, N, P and Q the midpoints of the sides AB, BC, CD and DA, respectively. If the segments MP and NQ divide $ABCD$ in four quads with the same area, show that $ABCD$ is a parallelogram.

Luis González

#2 May 9, 2015, 9:31 am

Let $O \equiv MP \cap NQ$. It's well-known that $MNPQ$ is a parallelogram for any $ABCD \Longrightarrow$
$[MON] = [NOP] = [POQ] = [QOM] \Longrightarrow [AMQ] = [BNM] = [CPN] = [DQP] \Longrightarrow$
$[ABD] = 4 \cdot [AMQ] = 4 \cdot [BNM] = [ABC] \Longrightarrow CD \parallel AB$ and similarly $AD \parallel BC \Longrightarrow ABCD$ is parallelogram.

Quick Reply

High School Olympiads

The line is tangent to mixtilinear incircle

Reply

Source: Own

buratinogigle
#1 May 6, 2015, 3:34 pm • 2

Let ABC be a triangle inscribed circle (O) and A-mixtilinear excircle (J). Two common external tangent of (O) and (J) touches (O) at M, N. Prove that MN is tangent to A-mixtilinear incircle of ABC.

TelvCohl
#2 May 6, 2015, 4:58 pm • 2

My solution :

Let V be the exsimilicenter of $\odot(O) \sim \odot(J)$.
Let $\odot(K)$ be A-mixtilinear incircle of $\triangle ABC$ and $\odot(K)$ touch $\odot(O)$ at P .
Let $\odot(J)$ touch $\odot(O)$ at T and $S = AT \cap \odot(K), U = VT \cap \odot(O)$ (see attachment) .

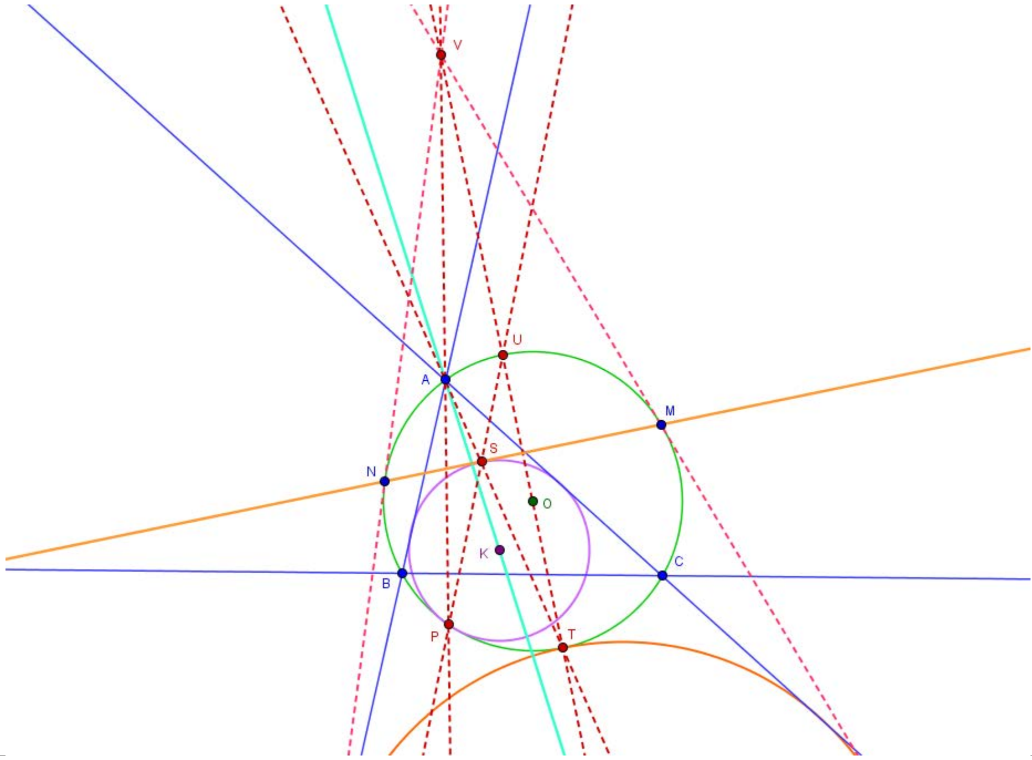
Let $\mathcal{T}(\odot, Q)$ ($Q \in \odot$) be the tangent of circle $\odot$ through point Q .

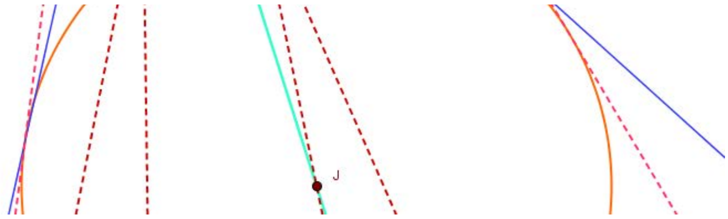
From D'Alembert theorem we get V, A, P are collinear .
From homothety with center V (map $\odot(J) \mapsto \odot(O)$) we get $\mathcal{T}(\odot(O), U) \parallel \mathcal{T}(\odot(J), T)$ . ... (1)
From homothety with center A (map $\odot(J) \mapsto \odot(K)$) we get $\mathcal{T}(\odot(K), S) \parallel \mathcal{T}(\odot(J), T)$ . ... (2)
From homothety with center P (map $\odot(K) \mapsto \odot(O)$) and (1), (2) we get P, S, U are collinear .

Since MN is the polar of V WRT $\odot(O)$,
so from cyclic quadrilateral $APUT$ we get $S \in MN$,
hence combine $\mathcal{T}(\odot(K), S) \parallel \mathcal{T}(\odot(J), T) \parallel MN \implies MN$ is tangent to $\odot(K)$ at S .

Q.E.D

Attachments:





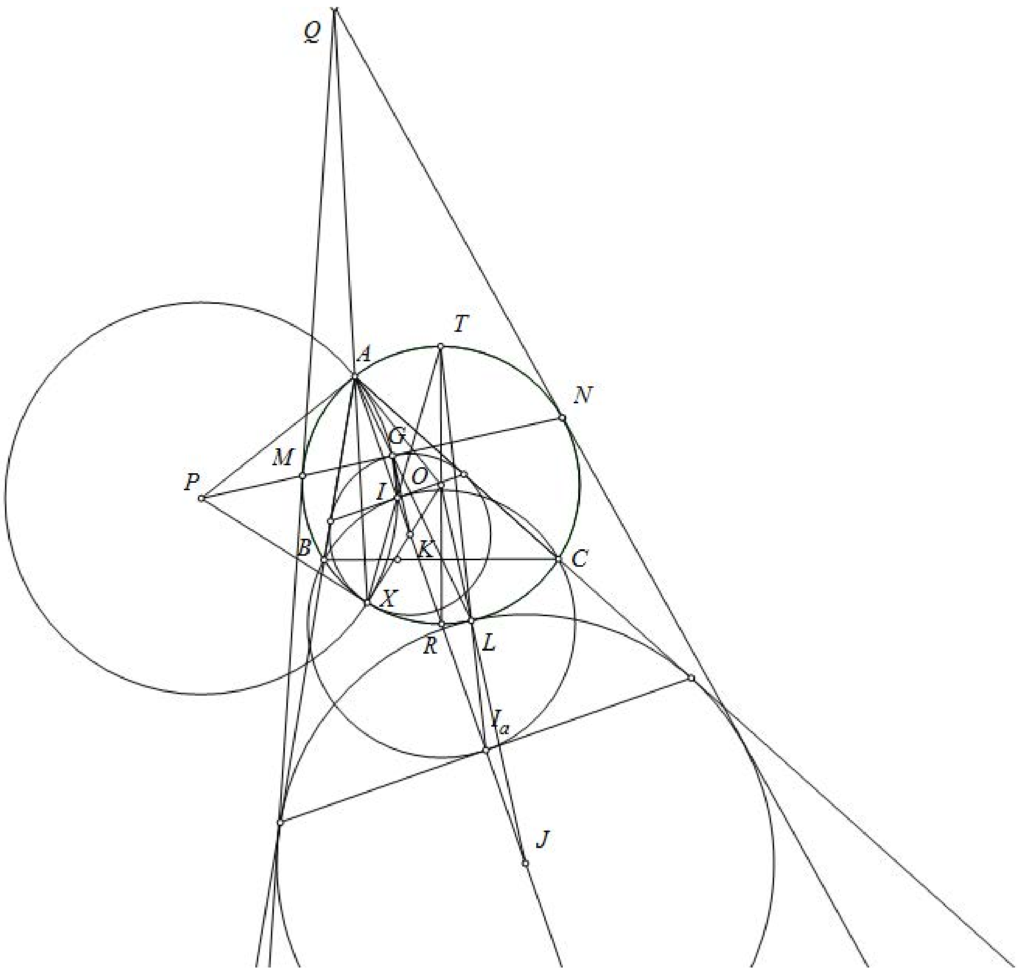
livetolove212

#3 May 8, 2015, 12:42 am • 1

My solution.

Let L be the tangency of (O) and (J), X be the tangency of (O) and (K)- the A-mixtilinear incircle of ABC, I, I_a be the incenter and A-excenter, R be the intersection of AI and (O). The homothety $\mathcal{H}_A^{\frac{AI}{AI_a}} : (J) \mapsto (K)$, so $L \mapsto G$ which lies on (K). We have $IG \parallel I_aL$. Since XI intersects LI_a at the midpoint T of arc BAC, then $IG \parallel LT$ or $AXIG$ is concyclic. We have $KI.KA = R^2_{(K)} = KX^2$ then (AIX) and (K) are orthogonal. On the other side, $\angle AXI = \angle ARO = \angle IAO$ then OA is tangent to (AIX) or (O) and (AIX) are orthogonal. Let P be the intersection of the tangents through A and X of (O), then P is the center of (AIX). We have PG is tangent to (K). Let M, N be the intersections of PG and (O). We get $AMXN$ is the harmonic quadrilateral. This means the intersection Q of the tangents through M and N of (O) lies on AX. According to Monge D'Alembert theorem for 3 circles $(K), (J), (O)$, the exsimilicenter of $(J), (O)$ lies on AX. But $MN \perp KG \parallel JL$ then the perpendicular bisector of MN cuts AX at the exsimilicenter of $(J), (O)$. This means Q is the exsimilicenter of $(J), (O)$, therefore M, N lie on the common external tangents of (O) and (J). We are done.

Attachments:



This post has been edited 1 time. Last edited by livetolove212, May 8, 2015, 12:45 am

Luis González

#4 May 9, 2015, 8:59 am • 1

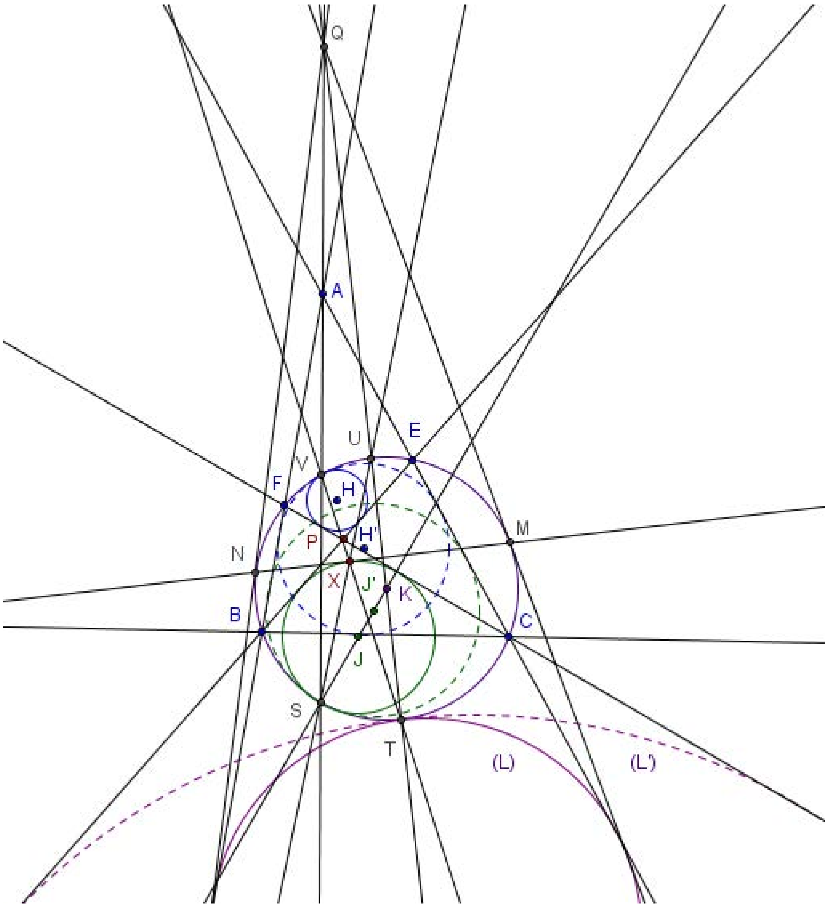
Generalization: Circle (K) though B, C cuts AC, AB at E, F and $P \equiv BE \cap CF$. Circle (J) is tangent to $\overline{PB}, \overline{PC}$ and internally tangent to (K) and circle (L) is tangent to AB, AC and externally tangent to (K) (see the diagram below for clearness). External common tangents of $(K), (L)$ touch (K) at M, N. Then MN is tangent to (J).

External common tangents of $(L), (K)$ intersect at Q. S, T are the tangency points of $(J), (L)$ with (K) and TKQ cuts (K) again at U. Circle (H) is tangent to $\overline{PE}, \overline{PF}$ and internally tangent to (K) at V.

According to 3 circles with common tangency point, circle (H') internally tangent to the arc EF of (K) and tangent to AB, AC touches (K) at V and the circle (J') internally tangent to the arc BSC of (K) and tangent to AB, AC touches (K) at S. By extraversion, circle (L') externally tangent to the arc BSC of (K) and tangent to PB, PC touches (K) at T. Thus by Monge & d'Alembert theorem we conclude that A, Q, V, S are collinear and V, P, T are collinear.

Let SU cut (J) again at X. Since S is the exsimilicenter of $(J) \sim (K)$, then $JX \parallel KU \equiv L'T \Longrightarrow JX$ and $L'T$ are parallel radii of (J) and $(L') \Longrightarrow XT$ passes through their exsimilicenter $P \Longrightarrow X \equiv VT \cap US$. Now, from the complete cyclic $STUV$, it follows that the polar MN of Q WRT (K) goes through X. Since U is obviously the midpoint of the arc MEN of (K), then it follows that (J) is tangent to MN at X.

Attachments:



High School Olympiads

Bicentric quadrilateral

Reply

Source: Vietnam IMO training 2015- Own

livetolove212

#1 May 8, 2015, 9:40 am • 1

Given triangle ABC with $AB+AC=3BC$. Let I_a be A-excenter. P lies on ray CB, Q lies on ray BC such that $CP=CA, BQ=BA$. (PBI_a) cuts AB again at M, (QCI_a) cuts AC again at N. Prove that $BCNM$ is a bicentric quadrilateral.

TelvCohl

#3 May 9, 2015, 12:01 am • 1

My solution :

Let I be the incenter of $\triangle ABC$.
Let $\triangle DEF$ be the intouch triangle of $\triangle ABC$.
Let $P' \equiv AP \cap CI, Q' \equiv AQ \cap BI$ be the midpoint of AP, AQ, respectively .
Let T be the midpoint of BC and $S = AI \cap \odot(ABC), Y = BI \cap AP, Z = CI \cap AQ$.

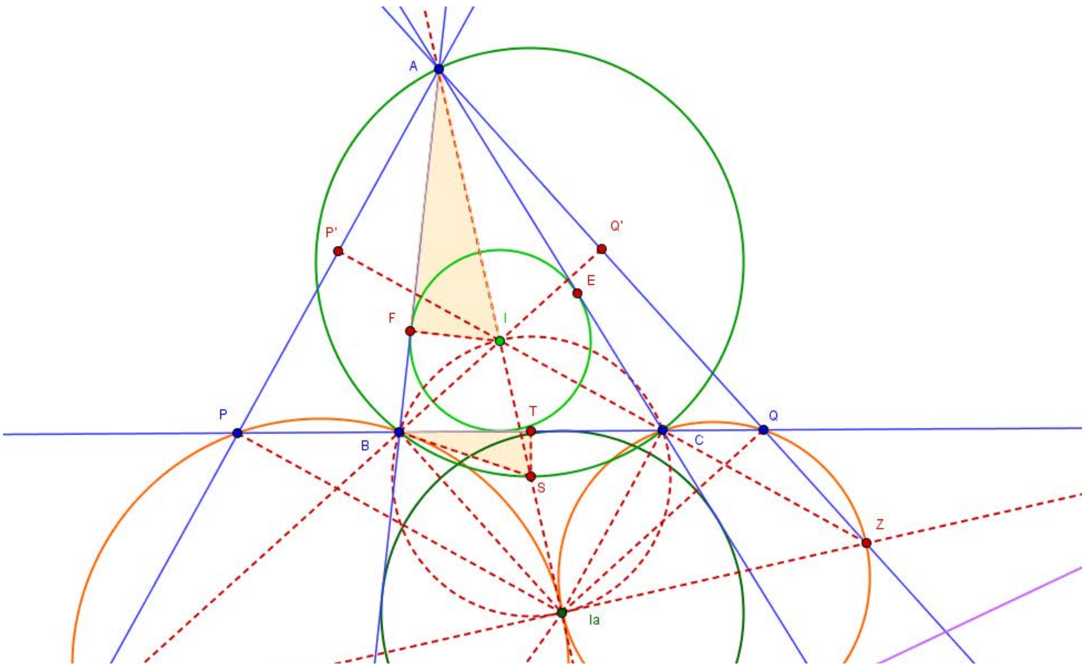
From $AB+AC=3BC \Longrightarrow AE=AF=BC$,
so from $\triangle AEI \sim \triangle BTS \Longrightarrow SI=SB=\frac{1}{2}AI \Longrightarrow AI=II_a$.

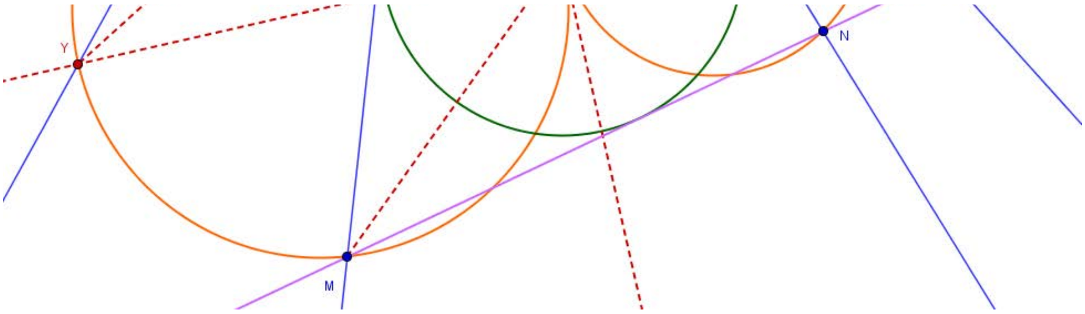
From $BQ' \parallel I_aQ \Longrightarrow BQ'QI_a$ is rectangle ,
so $\angle CI_aI=\angle CBI=\angle CQI_a \Longrightarrow AI_a$ is tangent to $\odot(CQI_a)$.
Similarly, AI_a is tangent to $\odot(BPI_a) \Longrightarrow AI_a$ is the common tangent of $\{\odot(BPI_a), \odot(CQI_a)\}$.

From $\angle YPI_a=\angle YBI_a=90^\circ \Longrightarrow I_aY$ is the diameter of $\odot(BPI_a)$.
Similarly, we can prove Z is the antipode of I_a in $\odot(CQI_a) \Longrightarrow Y, I_a, Z$ are collinear .
From $\angle YI_aM=\angle YBM=\frac{1}{2}\angle CBA=\angle PBY=\angle PI_aY \Longrightarrow M$ is the reflection of P in YZ .
Similarly, N is the reflection of Q in $YZ \Longrightarrow MN$ is tangent to $\odot(I_a) \Longrightarrow BCMN$ is a bicentric quadrilateral .

Q.E.D

Attachments:





Luis González

#4 May 9, 2015, 12:26 am • 1

Incircle (I,r) and A-excircle (I_a,r_a) touch BC at $D,E.$ $\frac{r}{r_a}=\frac{s-a}{s}=\frac{1}{2}\Longrightarrow I$ is midpoint of $AI_a.$ Since $EP=2\cdot CD$ and $r_a=2\cdot r,$ the right $\triangle IDC, \triangle I_aEP$ are similar by SAS $\Longrightarrow \widehat{EI_aP}=\widehat{CID}=90^\circ-\frac{1}{2}\widehat{C}\Longrightarrow \widehat{PI_aB}=\frac{1}{2}\widehat{A}\Longrightarrow$ $\widehat{PMB}=\widehat{BAI_a}\Longrightarrow AI_a\parallel PM$ is tangent of $\odot(PMB)$ through the midpoint I_a of its arc PM and similarly AI_a is tangent of $\odot(QNC).$ Thus if the perpendicular from I_a to $PM\parallel QN$ cuts BC at $X,$ then by symmetry XM,XN are the reflections of BC on $XI_a\Longrightarrow MN$ touches (I_a) and because of $AB\cdot AM=AI_a{}^2=AC\cdot AN\Longrightarrow BCNM$ is also cyclic.

TelvCohl

#5 May 9, 2015, 12:39 am

Sorry ... I misread the problem

I did not prove B,C,M,N are concyclic at my previous post .
Here is the proof of [B,C,M,N are concyclic] (I'll use the same picture) :

Since AI_a is perpendicular to YZ ,
so MN is anti-parallel to BC WRT $\angle A\Longrightarrow B,C,M,N$ are concyclic .

Done

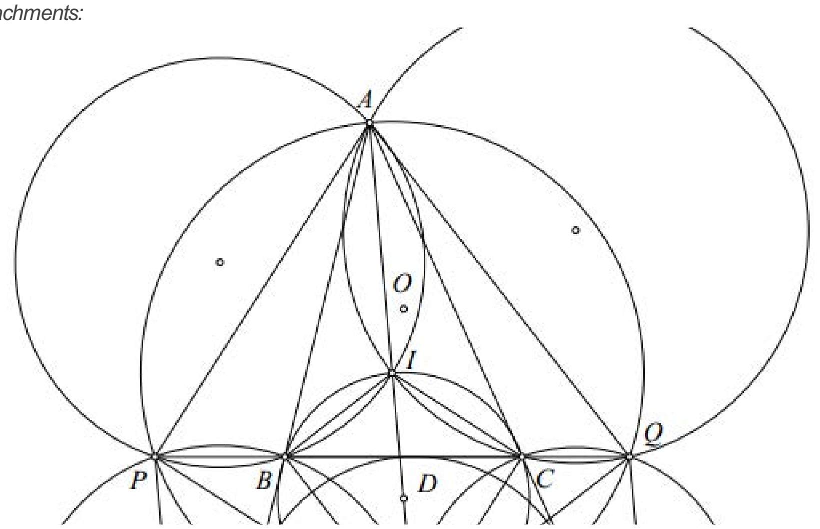
livetolove212

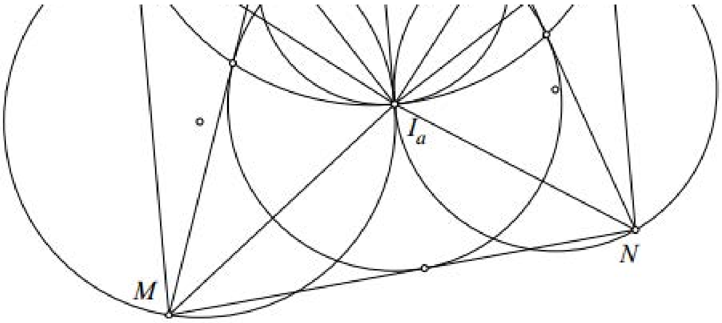
#6 May 9, 2015, 12:46 am

Thank you dear TelvCohl and Luis. This is my proof.

Let D be the midpoint of arc BC. It's easy to see that $IA=IQ=IP.$ Since $AB+AC=3BC$ then $AI=2ID$ or $IA=II_a.$ Therefore API_aQ is inscribed in the circle whose center is I. Then $AICQ, AIBP$ are concyclic.
We have $\angle PMB=\angle PI_aB=\angle PI_aA-\angle BI_aA=\angle AQB-\angle ICB=\angle I_aIC-\angle ICA=\angle IAC$ then
$\angle BI_aM=180^\circ-\angle BPM=\angle PBM+\angle PMB=\angle B+\dfrac{1}{2}\angle A.$
Similarly $\angle CI_aN=\angle C+\dfrac{1}{2}\angle A.$ We obtain $\angle BI_aM+\angle CI_aN=180^\circ.$
This means MN is tangent to (I_a) or $BCNM$ is circumscribed quadrilateral.
On the other side, $\angle CNM=2\angle CNI_a=2\angle PQI_a=2\angle PAI=2\angle IBC=\angle ABC$ or $BCNM$ is concyclic

Attachments:





This post has been edited 1 time. Last edited by livetolove212, May 9, 2015, 12:46 am

Quick Reply

High School Olympiads

A useful equality.

Reply

vittasko

#1 Mar 3, 2009, 8:09 pm

A triangle $\triangle ABC$ is given with incenter $I,$ orthocenter H and let M be, the midpoint of its side-segment $BC.$ Prove that $(HM)^2 - (HI)^2 = \frac{a^2}{4} - 2r^2,$ where $a = BC$ and $r,$ is the inradius of $\triangle ABC.$

Kostas Vittas.

Luis González

#2 Mar 4, 2009, 10:57 am

We use the well-known formulae:

$$IH^2 = 4R^2 - \tfrac{1}{2}(a^2+b^2+c^2) + 2r^2 \ \ (1)$$

$$OH^2 = 9R^2 - (a^2+b^2+c^2) \ \ (2)$$

Power of H WRT the circle with diameter $\overline{BC}$ is $\tfrac{1}{4}a^2 - HM^2 = k^2$ and k^2 is the power of the inversion taking the 9-point circle (N) of $\triangle ABC$ into its circumcirle $(O).$ From inversion properties, we get:

$$\frac{k^2}{OH^2 - R^2} = \frac{R}{2R} \Longrightarrow k^2 = \tfrac{1}{2}(R^2 - OH^2)$$

$$\Longrightarrow k^2 = \tfrac{1}{2}(a^2+b^2+c^2) - 4R^2 = \tfrac{1}{4}a^2 - HM^2 \ \ (3)$$

Combining the expressions (1) and (3) gives $HM^2 - IH^2 = \tfrac{1}{4}a^2 - 2r^2$

vittasko

#3 Mar 5, 2009, 2:12 am

Thank you dear **Luis**, for your interest and nice solution.

I like better your approach with the argument of the inversion power that takes the circucircle of $\triangle ABC,$ to its nine-point circle.

Since I am not good in metrical solutions, I had only an ugly approach, used some ready formulas from the bibliography I have in mind.

(1) - $(HI)^2 = 2r^2 - 4R^2 \cos A \cos B \cos C$

(2) - $(HM)^2 = \dfrac{2[(HB)^2 + (HC)^2] - a^2}{4}$

(3) - $HB = 2R\cos B, \ \ HC = 2R\cos C$

(4) - $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

From the wanted equality, $(HM)^2 - (HI)^2 = \dfrac{a^2}{4} - 2r^2,$ combining the above formulas, we obtain by calculations the equality $a = 2R\sin A,$ which is true.

● As an interesting remark, we can say that the radical axis of the circles $(M),$ with diameter BC and $(I'),$ centered at incenter $I,$ with radius $r' = \sqrt{2}r,$ passes through the orthocenter $H,$ of the given triangle $\triangle ABC.$

Thank you again, Kostas Vittas.

This post has been edited 1 time. Last edited by vittasko, Mar 5, 2009, 3:16 pm

Quick Reply

High School Olympiads

Nagel point is the radical center

Locked

Source: Own

livetolove212

#1 May 8, 2015, 9:20 am

Given triangle ABC inscribed in (O). Let X, Y, Z be the midpoints of arcs BAC, CBA, ACB. Prove that Nagel point of triangle ABC is the radical center of 3 circles $(X, XA), (Y, YB), (Z, ZC)$.

Luis González

#2 May 8, 2015, 10:02 am • 2

Posted before at http://www.artofproblemsolving.com/community/c6h521011.

High School Olympiads

Nagel point is radical center

Reply

Source: Own

buratinogigle

#1 Feb 16, 2013, 5:01 pm

Let ABC be a triangle with altitudes AD, BE, CF. X, Y, Z are midpoints of BC, CA, AB. Consider circles $(X, XD), (Y, YE), (Z, ZF)$. Prove that radical center of $(X), (Y), (Z)$ is Nagel point of triangle DEF.

Luis González

#2 Feb 16, 2013, 11:12 pm • 1

Let $\triangle D_0E_0F_0$ be the antimedial triangle of $\triangle DEF$, thus Nagel point of $\triangle DEF$ is the incenter I_0 of $\triangle D_0E_0F_0$.

FD_0 and ED_0 cut 9-point circle $(N) \equiv \odot(DEF)$ again at F', E'. Since $DEF'F$ is an isosceles trapezoid with bases $\overline{DE} \parallel \overline{FF'}$, then midpoint Z of the arc DFE is also midpoint of the arc $FF' \Longrightarrow ZF = ZF' \Longrightarrow F'$ is on circle (Z, ZF), i.e. FF' is radical axis of (N) and (Z, ZF). Similarly EE' is radical axis of (N) and $(Y, YE) \Longrightarrow D_0 \equiv EE' \cap FF'$ is radical center of (N), (Y, YE) and $(Z, ZF) \Longrightarrow D_0$ has equal power WRT (Y, YE) and (Z, ZF). Hence, the internal bisector D_0I_0 of $\angle E_0D_0F_0$, which is clearly perpendicular to YZ, is the radical axis of (Y, YE) and $(Z, ZF) \Longrightarrow I_0$ has equal power WRT (Y, YE) and (Z, ZF). Similarly I_0 has equal power WRT (Y, YE) and $(X, XD) \Longrightarrow I_0$ (Nagel point of DEF) is radical center of $(X, XD), (Y, YE)$ and (Z, ZF).

livetolove212

#3 May 8, 2015, 11:07 am

Well-known lemma: Given triangle ABC with incenter I. Let H, K, L be the projections of A on BC, of B, C on AI, M be the midpoint of BC then H, K, L, M lie on the circle whose center is the midpoint of arc MH of Euler circle.

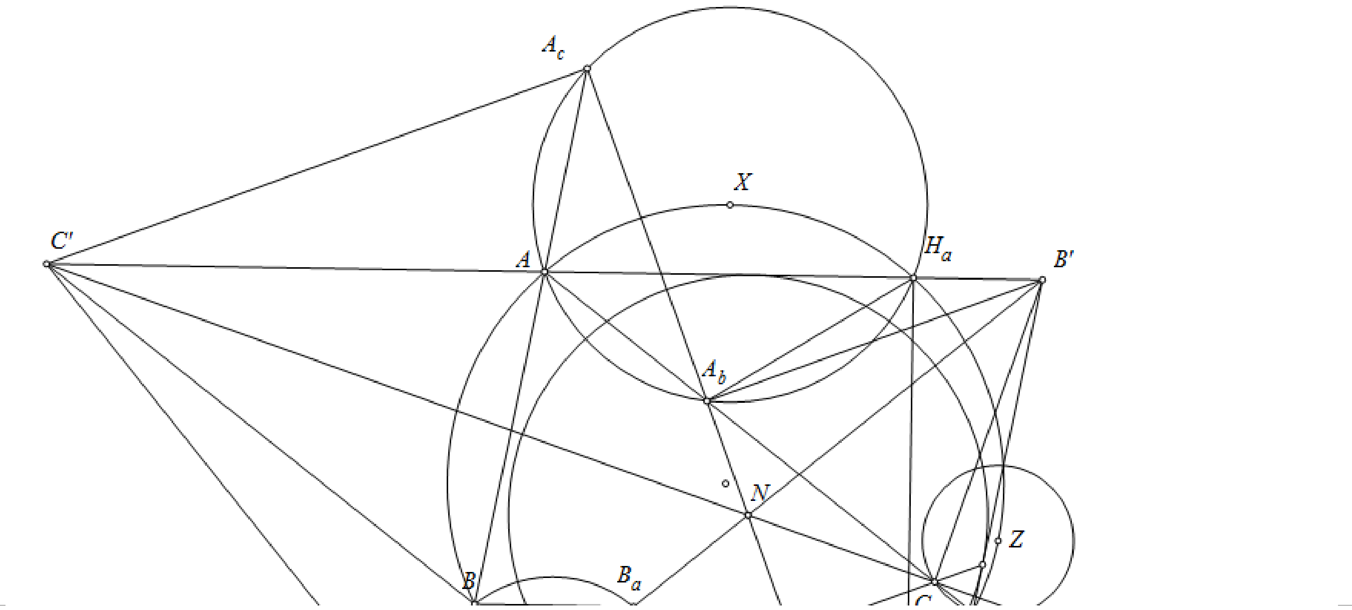
Back to our problem.

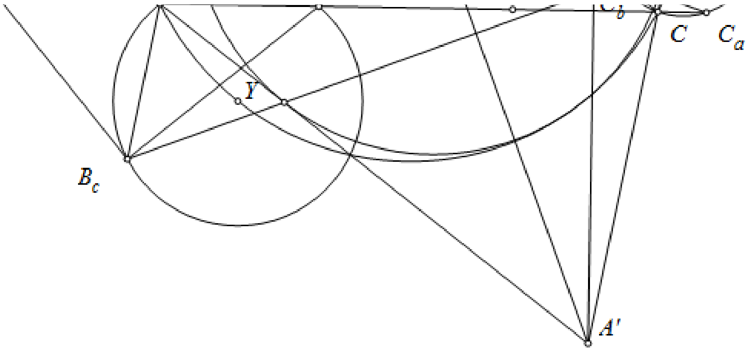
We rewrite the problem as

http://www.artofproblemsolving.com/community/c6t316f6h1086451_nagel_point_is_the_radical_center

Let $A'B'C'$ be the anti-medial triangle of ABC. Then N is the incenter of $A'B'C'$. Let A_c, A_b be the projections C', B' on $A'N$, similar with B_a, B_c, C_a, C_b, H_a be the projection of A' on BC. Then A_c, A_b, A, H_a lie on (X, XA). It's easy to prove that $A_b, A_c, B_a, B_c, C_a, C_b$ lie on 3 sides of triangle ABC. We have $\angle C'C_bB_c = \angle C'B'N = \angle B'B_aC$ then $B_cB_aC_bC_a$ is concyclic. Thus N lies on the radical axis of (Y, YB) and (Z, ZC). Analogously we are done.

Attachments:





This post has been edited 2 times. Last edited by livetolove212, May 30, 2015, 8:23 am

buratinogigle

#4 May 8, 2015, 4:52 pm

From the way livetolove212 demonstrates the problem, we can extend it as following

Let ABC be a triangle inscribed circle (O). D, E, F are midpoint of arc BC, CA, AB that contain A, B, C, resp. Let X, Y, Z devide OD, OE, OF in the same ratio. Prove that Nagel point of ABC is radical center of circles $(X, XA), (Y, YB), (Z, ZC)$.

Quick Reply

High School Olympiads

Parallel line

Reply

Source: Vietnam IMO team trainning summer 2015, Own

buratinogigle

#1 May 8, 2015, 12:17 am • 1

Let ABC be a triangle inscribed circle (O). Circle (C,CB) cuts BA again at D and cuts (O) again at E. DE cuts (O) again at F. CO cuts DE,AB at G,L, resp. Q is reflection of E through midpoint of GF. P lies on EF such that $PL\perp BC$. Circle (LPQ) cuts AB again at R. K is reflection of R through CF. Prove that $FK\parallel BC$.

Luis González

#2 May 8, 2015, 6:05 am • 1

By symmetry we have $BE \perp CO$ at S. C-altitude CZ cuts (O) again at F' and EF' cuts AB at D'. Then $\widehat{BEF'}=\widehat{BCZ}=\widehat{BSZ} \Longrightarrow SZ \parallel EF'$ is the B-midline of $\triangle BED' \Longrightarrow D'$ is reflection of B on $Z \Longrightarrow D \equiv D'$ and $F \equiv F'$. Redefining $K \in AB$, such that $FK \parallel BC$ and letting R be the reflection of K on Z, then we need to prove that $LPQR$ is cyclic.

Let PL cut CF,CB at U,J. Since K becomes orthocenter of $\triangle FUL$, its reflection R on UF is on its circumcircle. Moreover $\widehat{GFU}=\widehat{EBC}=\widehat{SLU} \Longrightarrow FGLUR$ is cyclic $\Longrightarrow DL \cdot DR=DG \cdot DF$. Hence $LPQR$ is cyclic $\Longleftrightarrow$ $DG \cdot DF=DP \cdot DQ \Longleftrightarrow DG \cdot DF=DP \cdot (DF+EG) \Longleftrightarrow DF \cdot (DG-DP)=DF \cdot PG=DP \cdot EG.$ So we only need to show $\tfrac{DF}{EG}=\tfrac{DP}{PG}.$

From $\triangle EGS \sim \triangle LBJ$ and $\triangle DFZ \sim \triangle CLJ,$ we get $\tfrac{EG}{LB}=\tfrac{BS}{LJ}$ and $\tfrac{DF}{LC}=\tfrac{BZ}{JC} \Longrightarrow$

$$\frac{DF}{EG}=\frac{LC}{LB} \cdot \frac{BZ}{JC} \cdot \frac{LJ}{BS}=\frac{LC}{LB} \cdot \frac{BC \cos \widehat{LBC}}{LC \cdot \cos \widehat{LCB}} \cdot \frac{LC}{BC}=\frac{LC}{LB} \cdot \frac{\cos \widehat{LBC}}{\cos \widehat{LCB}}.$$

But $\dfrac{DP}{PG}=\dfrac{LZ}{LS} \cdot \dfrac{\sin \widehat{BLJ}}{\sin \widehat{CLJ}}=\dfrac{LC}{LB} \cdot \dfrac{\cos \widehat{LBC}}{\cos \widehat{LCB}} \Longrightarrow \dfrac{DF}{EG}=\dfrac{DP}{PG}$, as desired.

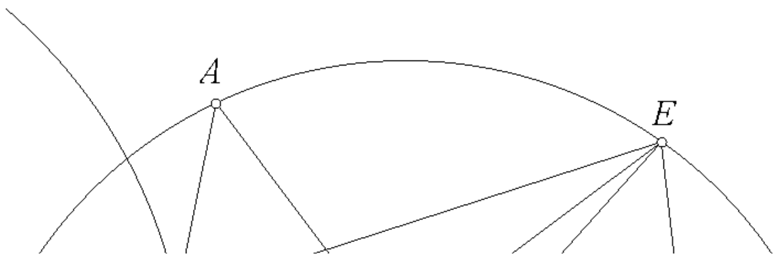
buratinogigle

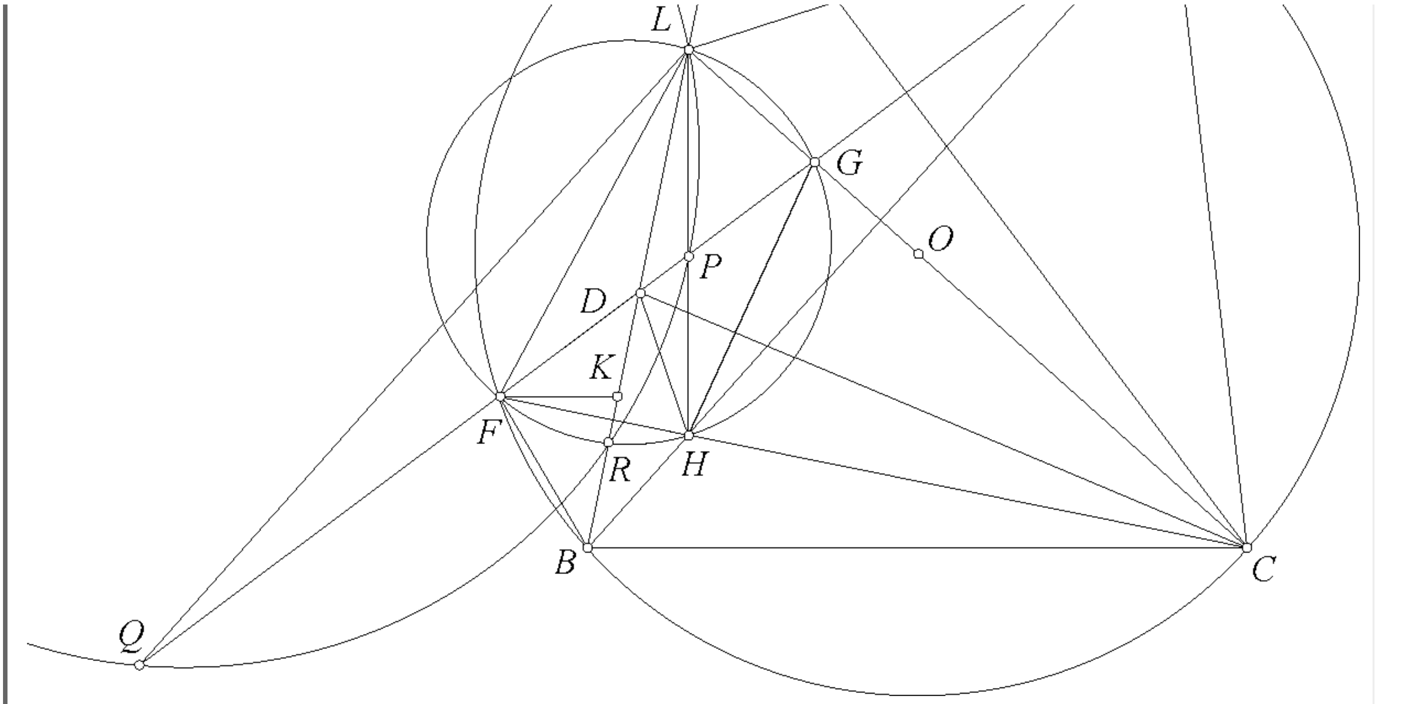
#3 May 8, 2015, 10:23 am • 1

Thank dear Luis, here is my solution

Let CF cut BE at H. We have $\angle FCB=\angle FEB=\frac{1}{2}\angle BDC$. So CF is bisector of $\angle BCD$ thus $\angle HCD=\angle HCB=\angle HED$, so $CHDE$ is cyclic. We see B,E are reflection about OC. From this $\angle BDC=\angle CBD=\angle LEC$ deduce $CELD$ is cyclic, thus C,E,L,D,H lie on circle (α). Therefore, $\angle HLG=\angle HEC=\angle CBE=\angle CFE$ deduce $HGLF$ inscribed circle(β). Easily seen H is orthocenter of BCL and LH is radical axis of (α) and (β). So if LH cuts EF at P then $PG.PF=PD.PE$. Thus by Desargues involution $(PD,GF)=(GF,E)$ or $(DP,FG)=(FG,Q)$ deduce $DR.DL=DP.DQ=DG.DF$. From this R lies on (β). But LR is altitude thus K is orthocenter of LFH. From this FK is perpendicular to LH and parrallel BC. We are done.

Attachments:

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High School Olympiads

Australia

#1 May 7, 2015, 10:16 am

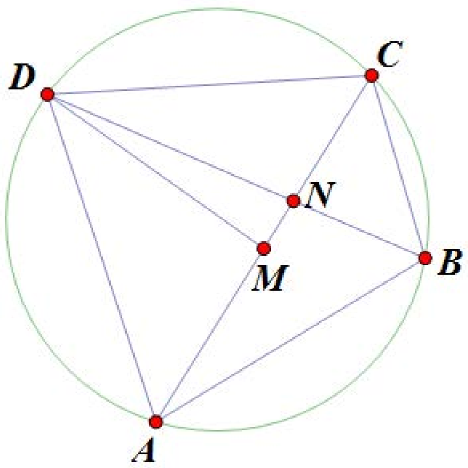
For quadrilaterals $ABCD$, $AC \bigcap BD = N$,and the midpoint M on AC,if such

$$\frac{BC^2}{CD^2} = \frac{BN}{DN}$$

show that

$$\frac{MN^2}{MC^2} + \frac{DN^2}{DM^2} = 1$$

Attachments:



Luis González

#2 May 7, 2015, 11:15 am

Since $\frac{BC^2}{CD^2} = \frac{BN}{ND} \Longrightarrow CN$ is the C-symmedian of $\triangle CDB \Longrightarrow ABCD$ is harmonic $\Longrightarrow DB$ is the D-symmedian of $\triangle ADC \Longrightarrow \triangle DAB \sim \triangle DMC$. Hence

$$\frac{AB^2}{AD^2} = \frac{CB^2}{CD^2} = \frac{NB}{ND} \Longrightarrow \frac{DN^2}{DM^2} = \frac{DN \cdot NB}{MC^2} = \frac{NA \cdot NC}{MC^2} \Longrightarrow$$

$$\frac{(MC - MN) \cdot (MC + MN)}{MC^2} = \frac{DN^2}{DM^2} \Longrightarrow \frac{MC^2 - MN^2}{MC^2} = \frac{DN^2}{DM^2}$$

$$\Longrightarrow \frac{MN^2}{MC^2} + \frac{DN^2}{DM^2} = 1.$$

Australia

#3 May 7, 2015, 11:44 am

It's very very nice!Thank you

High School Olympiads

K is the midpoint of XY

Reply

Source: Own

THVSH

#1 May 6, 2015, 11:14 pm • 1

Let ABC be a triangle. $\odot(K)$ is an arbitrary circle passing through B, C. $\odot(K)$ intersects CA, AB at E, F, respectively. $BE \cap CF = H$.
Let d_1 be the line passing through K and perpendicular to AH
Let d_2 be the line passing through A and perpendicular to BE
Let d_3 be the line passing through A and perpendicular to CF
$d_1 \cap d_2 = X; d_1 \cap d_3 = Y$
Prove that K is the midpoint of XY.

Luis González

#2 May 6, 2015, 11:30 pm • 1

Let $P \equiv BC \cap EF$. Then PH is the polar of A WRT $(K) \Longrightarrow PH \perp KA$. Since $H(E, F, P, A) = -1$, then the pencil formed by perpendiculars from A to HE, HF, HP, HA is also harmonic, i.e. $A(X, Y, K, \infty) = -1 \Longrightarrow K$ is midpoint of XY.

buratinogigle

#3 May 7, 2015, 12:00 am

I have an idea from this configuration

Let ABC be a triangle. $\odot(K)$ is an arbitrary circle passing through B, C. $\odot(K)$ intersects CA, AB at E, F, respectively. $BE \cap CF = H$.

Let d_1 be the line passing through K and perpendicular to AH. P is a point on d_1.
Let d_2 be the line passing through P and perpendicular to BE, d_2 cuts AB at X
Let d_3 be the line passing through P and perpendicular to CF, d_3 cuts AC at Y.

Prove that circle (AXY) has a fixed point other than A when P move.

When K is midpoint of BC, I known this is a well known problem but I can't find the link!

Luis González

#4 May 7, 2015, 12:29 am • 1

Dear buratinogigle, that actually comes from a more general configuration.

If we have, for example, a $\triangle ABC$ and a variable point $P \in BC$ and $X \in CA, Y \in AB$, such that the directions of PX, PY are fixed, then all circles $\odot(AXY)$ go through a fixed point, other than A. This is because the series X, Y are similar, thus XY envelopes a parabola $\mathcal{P}$ tangent to $AC, AB \Longrightarrow$ all circles $\odot(AXY)$ go through its focus.

THVSH

#5 May 8, 2015, 6:46 pm • 1

Dear **buratinogigle** and **Luis González**,
Thank you very much for your interest! Here is my solution for my problem (using butterfly theorem):

Let d be the line passing through A and perpendicular to AK. $d \cap BE = U; d \cap CF = V$.
From butterfly theorem, we get: A is the midpoint of UV.
We have: $\angle UAH = 90 + \angle HAK = \angle AKX; \angle UHA = \angle AXK$
$\Longrightarrow \triangle UAH \sim \triangle AKX \Longrightarrow \dfrac{KX}{AH} = \dfrac{AK}{UA} \Longrightarrow KX.UA = AH.AK$
Similarly, we have $KY.VA = AH.AK$
$\Longrightarrow KX = KY$ Q.E.D

Quick Reply

High School Olympiads

A property of the seven circles theorem configuration

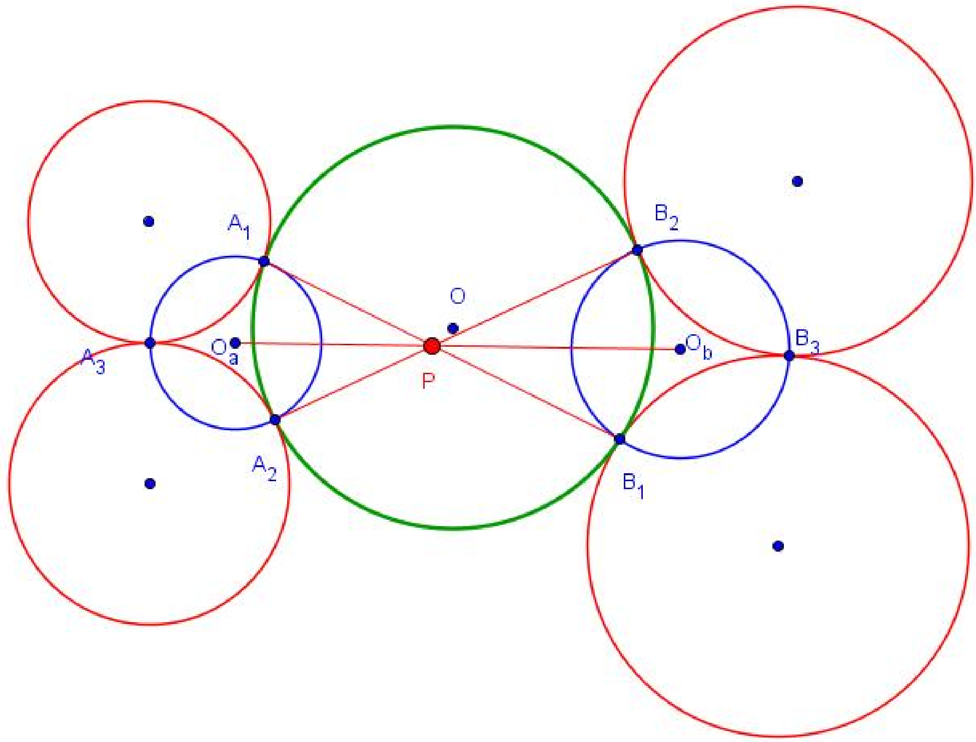
Reply

daothanhoai

#1 May 6, 2015, 1:30 pm

Let circle (1) touchs circle (2) at A_3 and circle (3) touchs circle (4) at B_3. Let circle (1),(2),(3),(4) touch a new circle at A_1, A_2, B_1, B_2 respectively (show in the figure). Let O_a = center of $(A_1A_2A_3)$; O_b=center of $(B_1B_2B_3)$ show that: A_1B_1, A_2B_2 and O_aO_b are concurrent

Attachments:



This post has been edited 2 times. Last edited by daothanhoai, May 6, 2015, 1:34 pm

TelvCohl

#2 May 6, 2015, 4:21 pm

My solution :

Let $X = A_1A_2 \cap B_1B_2, Y = A_1B_2 \cap A_2B_1$.
Let ℓ_A, ℓ_B be the common tangent of $\{\odot(1), \odot(2)\}, \{\odot(3), \odot(4)\}$, respectively .
Let $A_3A_1, A_3A_2, B_3B_1, B_3B_2$ cut $\odot(O)$ again at $A_1^*, A_2^*, B_1^*, B_2^*$, respectively .

From homothety with center A_1 (map $\odot(1) \mapsto \odot(O)$) $\Longrightarrow$ the tangent of $\odot(O)$ through A_1^* is parallel to ℓ_A .
Similarly, we can get the tangent of $\odot(O)$ through A_2^* is parallel to $\ell_A \Longrightarrow A_1^*A_2^*$ is the diameter of $\odot(O)$.

From $\angle OA_1A_2 = \angle OA_2A_1 = 90^\circ - \angle A_2A_3A_1 \Longrightarrow O_aA_1, O_aA_2$ are the tangents of $\odot(O)$.
Similarly, O_bB_1, O_bB_2 are the tangents of $\odot(O) \Longrightarrow O_aO_b$ is the polar of $A_1A_2 \cap B_1B_2 \equiv X$ WRT $\odot(O)$,
so we conclude that A_1B_1, A_2B_2, O_aO_b are concurrent at the pole of XY WRT $\odot(O)$.

Q.E.D

Luis González

#3 May 6, 2015, 10:36 pm

If $(O_1),(O_2)$ denote the circles (1),(2), then clearly (O_a) is the incircle of $\triangle OO_1O_2 \Longrightarrow (O_a)$ is orthogonal to (O) and similarly (O_b) is orthogonal to (O). Inversion with center P and power $\overline{PA_1}\cdot\overline{PB_1}=\overline{PA_2}\cdot\overline{PB_2}$ fixes (O) and swaps $(O_a),(O_b)$ due to conformity $\Longrightarrow O_a$ and O_b are collinear with the center of the inversion P.

daothanhoai

#4 May 7, 2015, 7:15 pm

The proof above is a new proof of the seven circles theorem?? Could you show detail?

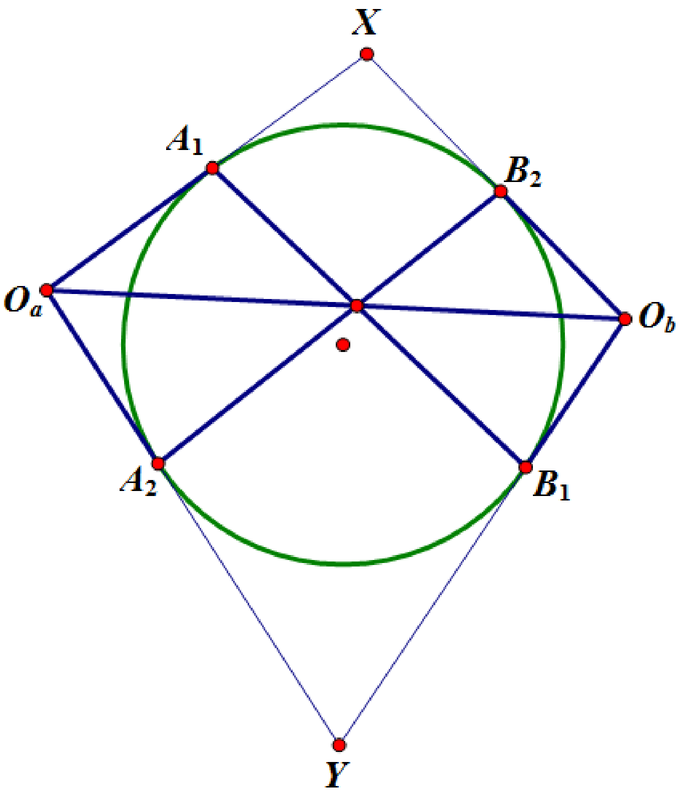
https://en.wikipedia.org/wiki/Seven_circles_theorem

yunxiu

#5 May 27, 2015, 5:40 pm

Denote $O_aA_1\cap O_bB_2=X$,$O_aA_2\cap O_bB_1=Y$. From Newton's Theorem A_1B_1, A_2B_2 and O_aO_b are concurrent.

Attachments:



Quick Reply

High School Olympiads

Angles between centers

Reply

manlio

#1 Jan 17, 2004, 10:05 pm

In a triangle ABC let H be the orthocenter, I the incenter, O the circumcenter, N the nine point center, r the inradius and R the circumradius. Prove that

$$\measuredangle HIO \geq \frac{\pi}{2} + \arcsin\sqrt{\frac{2r}{R}}$$

with equality iff $\measuredangle HIN = \frac{\pi}{2}$.

Proposed by Dan Sachelaire, ICCE , Bucharest,Romania in AMM 1997
as problem 10602.

Luis González

#2 May 6, 2015, 3:09 am • 1

Since IN is the I-median of $\triangle IOH$, we get $HN^2 = \frac{1}{2}(OI^2 + IH^2) - IN^2$. By Pythagorean theorem $\angle HIN = 90^\circ \Longleftrightarrow HN^2 = IN^2 + IH^2$. Combining these two expressions, keeping in mind that $IN = \frac{1}{2}R - r$, we get

$$OI^2 - IH^2 = 4 \cdot IN^2 \Longleftrightarrow \frac{IH^2}{OI^2} = 1 - 4 \cdot \frac{IN^2}{OI^2} = 1 - \frac{4(\frac{R}{2} - r)^2}{R^2 - 2Rr} = \frac{2r}{R}.$$

Therefore $\angle HIN = 90^\circ \Longleftrightarrow \sqrt{\frac{2r}{R}} \cdot OI = IH$. But from the problem angle is obtuse (see post #4), we have $\angle HIO \geq 90^\circ + \arcsin\sqrt{\frac{2r}{R}}$ with equality $\Longleftrightarrow \sqrt{\frac{2r}{R}} \cdot OI = IH$. The conclusion follows.

Quick Reply

High School Olympiads

angle is obtuse

Reply

Source: SAMC 2011

tchebytchev

#1 Apr 17, 2015, 10:57 pm

ABC is a non-isosceles triangle. O, I, H are respectively the center of its circumscribed circle, the inscribed circle and its orthocenter.
prove that $\widehat{OIH}$ is obtuse.

TelvCohl

#2 Apr 18, 2015, 12:08 am • 1

My solution:

Let R, r be the radius of $\odot(O), \odot(I)$, respectively .
Let N be the 9-point center of $\triangle ABC$ and T be the reflection of O in I .

Since $\overline{HT} = 2 \cdot \overline{NI} = \text{R} - 2\text{r}$,
so $\overline{HT} < \sqrt{\text{R}^2 - 2\text{Rr}} = \overline{OI} = \overline{TI} \Longrightarrow \angle HIT < 90^\circ \Longrightarrow \angle OIH > 90^\circ$.

Q.E.D

andria

#3 Apr 18, 2015, 12:43 am

I think the following is stronger: http://artofproblemsolving.com/community/c6h39447p246214

Luis González

#4 Apr 18, 2015, 1:07 am

More general, we prove that $\widehat{OIH} \geq 90^\circ + \arcsin\sqrt{\tfrac{2r}{R}}$. Where r and R denote the radii of the incircle and circumcircle of $\triangle ABC.$

Denoting by $\varrho = \cos A \cos B \cos C$, we get $OI^2 = R^2 - 2Rr, IH^2 = 2(r^2 - 2R^2\varrho)$ and $OH^2 = R^2(1 - 8\varrho)$. By cosine law for $\triangle OIH$, we get then

$$2 \cdot OI \cdot IH \cdot \cos\widehat{OIH} = OI^2 + IH^2 - OH^2 =$$

$$= -2(Rr - r^2 - 2R^2\varrho) = -\left(\frac{2r}{R} \cdot OI^2 + IH^2\right) \Longrightarrow$$

$$2 \cdot OI \cdot IH \cdot \sin(\widehat{OIH} - 90^\circ) = \frac{2r}{R} \cdot OI^2 + IH^2.$$

But $\dfrac{2r}{R}OI^2 + IH^2 \geq 2\sqrt{\dfrac{2r}{R}} \cdot OI \cdot IH$ with equality iff $\sqrt{\dfrac{2r}{R}} \cdot OI = IH$. Hence

$$\sin(\widehat{OIH} - 90^\circ) \geq \sqrt{\frac{2r}{R}} \Longrightarrow \widehat{OIH} \geq 90^\circ + \arcsin\sqrt{\tfrac{2r}{R}}.$$

TelvCohl

#5 Apr 18, 2015, 1:30 am

Moreover, if $\triangle ABC$ is an acute triangle we can prove $\angle OIH > 135^\circ$:

Let $\mu = \cos\angle A \cos\angle B \cos\angle C > 0$

We have to prove

$$\cos\angle OIH = \frac{OI^2 + HI^2 - OH^2}{2 \cdot OI \cdot HI} < \frac{-1}{\sqrt{2}}$$

$$\Longleftrightarrow (OI^2 + HI^2 - OH^2)^2 > 2 \cdot OI^2 \cdot HI^2$$

$$\Longleftrightarrow (\mathrm{Rr} - \mathrm{r}^2 - 2\mathrm{R}^2\mu)^2 > (\mathrm{R}^2 - 2\mathrm{Rr})(\mathrm{r}^2 - 2\mathrm{R}^2\mu)$$

$$\Longleftrightarrow (\mathrm{r}^2 - 2\mathrm{R}^2\mu)^2 + 2\mu(\mathrm{R}^2 - 2\mathrm{Rr})^2 > 0$$

Quick Reply

High School Olympiads

Saint Petersburg 2013

Reply

Source: Saint Petersburg 2013,grade 11

MRF2017

#1 May 5, 2015, 12:32 pm

In quadrilateral $ABCD$, M and N are midpoints of AB,CD,respectively.We know $AN=MD$, $CM=BN$.
Prove that $AC=BD$

TelvCohl

#3 May 5, 2015, 1:22 pm • 1

My solution:

From $AN=DM \Longrightarrow AN^2=DM^2 \Longrightarrow \frac{1}{2}AC^2+\frac{1}{4}AB^2=\frac{1}{2}BD^2+\frac{1}{4}CD^2$ . ... (1)

From $BN=CM \Longrightarrow BN^2=CM^2 \Longrightarrow \frac{1}{2}AC^2+\frac{1}{4}CD^2=\frac{1}{2}BD^2+\frac{1}{4}AB^2$ . ... (2)

From (1) and (2) $\Longrightarrow AC^2=BD^2 \Longrightarrow AC=BD$.

Q.E.D

Luis González

#4 May 5, 2015, 9:03 pm

Posted before at http://www.artofproblemsolving.com/community/c6h587168 and see also
http://www.artofproblemsolving.com/community/c6h585828 (same problem for a tetrahedron).

Quick Reply

High School Olympiads

St.Peterburg, P2 Grade 10, 2013

mathuz

#1 Apr 27, 2014, 10:27 am

in a convex quadrilateral $ABCD$, M, N are midpoints of BC, AD respectively. If $AM = BN$ and $DM = CN$ then prove that $AC = BD$.

S. Berlov

MariusBocanu

#2 Apr 27, 2014, 2:00 pm

$AM = BN$ means $2(AB^2 + AC^2) - BC^2 = 2(AB^2 + BD^2) - AD^2$ and $CN = DM$ means $2(CD^2 + CA^2) - AD^2 = 2(DB^2 + DC^2) - BC^2$ and summing these relations we obtain $AC^2 = BD^2$, done.

High School Olympiads

St.Peterburg, P3 Grade 11, 2013

Reply

mathuz

#1 Apr 17, 2014, 11:05 am

Let M and N are midpoint of edges AB and CD of the tetrahedron $ABCD$, $AN = DM$ and $CM = BN$. Prove that $AC = BD$.

S. Berlov

Luis González

#2 Apr 17, 2014, 12:27 pm • 1

Construct parallelograms $CMDY$ and $ANBZ \Longrightarrow DY = CM = BN = AZ, MY = NZ = 2 \cdot MN$ and since $AN = DM,$ then it follows that $\triangle DYM \cong \triangle AZN$ by SSS $\Longrightarrow DN = AM = BM.$ Now construct parallelograms $CBDU$ and $ACBV \Longrightarrow DU = BC, UN = BN = CM$ and since $DN = BM,$ it follows that $\triangle DUN \cong \triangle BCM$ by SSS $\Longrightarrow BD = VB = AC.$

Quick Reply

High School Olympiads

Iran 1998 colinearity problem

Locked

dragonx111

#1 May 5, 2015, 1:06 am

Let ABC be a triangle and D be the point on the extension of side BC past C such that CD = AC. The circumcircle of □ACD intersects the circle with diameter BC again at P. Let BP meet AC at E and CP meet AB at F. Prove that the points D, E, F are collinear.

Luis González

#2 May 5, 2015, 3:50 am

Posted before at http://www.artofproblemsolving.com/community/c6h302228.

High School Olympiads

collinear

Reply

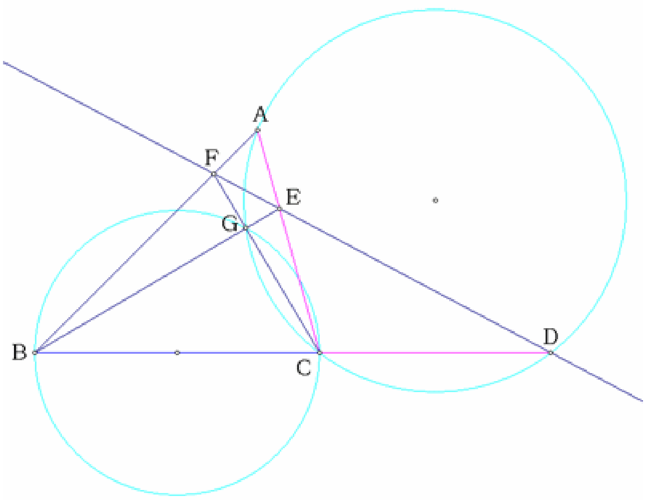
77ant

#1 Sep 22, 2009, 9:52 am

Dear everyone.

For an acute triangle ABC, draw a point D on the extension of C on BC such that AC=CD.
Let the circle with its diameter BC, the circumcircle of triangle ACD be O(1), O(2) respectively.
O(1) and O(2) meets each other at G, which is not C.
AB and CG meet at F. BG and CA meet at E.
Prove that F, E, D are collinear.

Attachments:



livetolove212

#2 Sep 22, 2009, 11:41 am

Let $AG \cap BC = \{J\}$
D, E, F are collinear iff $(BCJD) = -1 (1)$. But $\angle BGC = 90^o$ so $(1) \Leftrightarrow GC$ is the bisector of $\angle JGD$.
Since $AGCD$ is cyclic we get $\angle JGC = \angle CDA = \angle CAD = \angle CGD$. Therefore GC is the bisector of $\angle JGD$ and we are done!

Quick Reply

High School Olympiads

Orthotransversal and orthopole

Locked

Source: Own

VUThanhTung

#1 May 4, 2015, 9:18 pm

Consider a reference triangle ABC with orthocenter H. P is a point on the plane, d_P is the orthotransversal of P WRT $\triangle ABC$ and P' is the orthopole of d_P WRT $\triangle ABC$. Prove that:

1. H, P, P' are collinear.
2. if P lies on the circumcircle of $\triangle ABC$ then P' lies on the nine point circle of $\triangle ABC$.
3. if P lies on the nine point circle of $\triangle ABC$ then $P = P'$.

Luis González

#2 May 4, 2015, 9:29 pm • 1

1. See http://www.artofproblemsolving.com/community/c6h186883.
3. See http://www.artofproblemsolving.com/community/c6h497230.
2. Note that when P is on the circumcircle (O), its orthotransversal clearly passes through O, thus its orthopole is on the 9-point circle.

High School Olympiads

Orthopole and orthotransversal

Reply

buratinogigle
#1 Sep 5, 2012, 11:30 pm • 2

Prove that orthopole of orthotransversal of a point that lies on nine points circle of a triangle, also is that point.

Luis González
#2 Sep 6, 2012, 5:58 am • 4

See the lemma (second reply) in this topic.

P is a point on the 9-point circle (N) of $\triangle ABC.$ Perpendiculars to PA,PB,PC at P cut BC,CA,AB at $X,Y,Z,$ respectively. $\tau \equiv XYZ$ is orthotransversal of P WRT $\triangle ABC.$ Circles with diameters AX,BY,CZ are coaxal with radical axis passing through the orthocenter H of $\triangle ABC.$ They meet at P and the inverse Q of P under the inversion with center H that takes (N) into the circumcircle (O) of $\triangle ABC \Longrightarrow Q \in (O).$ Hence, QX,QY,QZ cut (O) again at the antipodes A_0,B_0,C_0 of $A,B,C.$ By Pascal theorem for $QB_0BACC_0,$ the intersections $Y \equiv CA \cap QB_0,$ $O \equiv BB_0 \cap CC_0$ and $Z \equiv AB \cap QC_0$ are collinear, i.e. $O \in \tau \Longrightarrow P$ is then the orthopole of τ WRT $\triangle ABC.$

Quick Reply

High School Olympiads

Concurrent lines

Reply

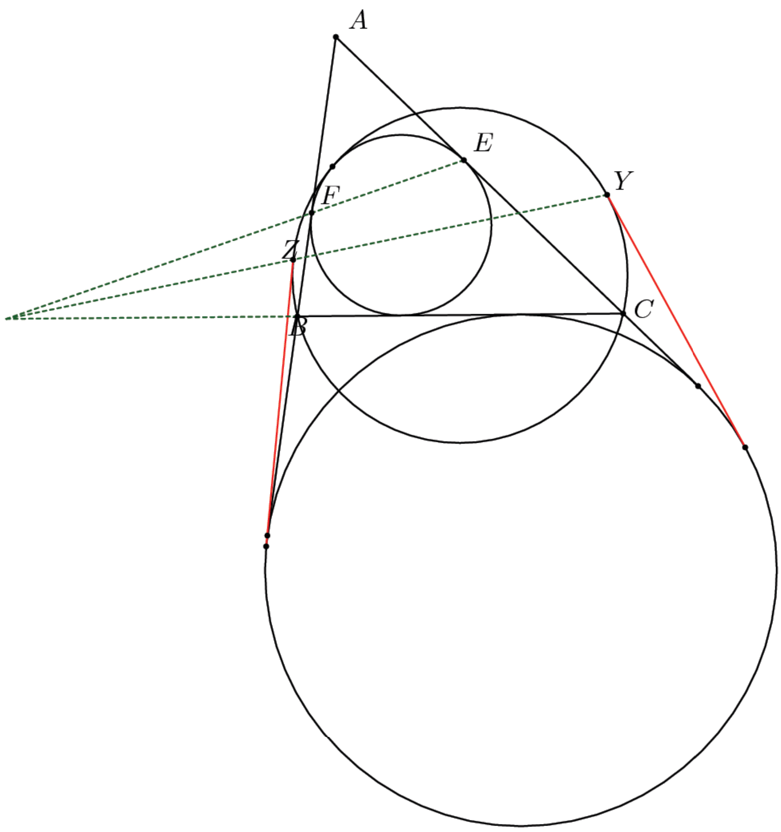
Source: Own invention

drmzjoseph

#1 May 3, 2015, 8:46 am • 4

I want to dedicate next problem, to my dear friends Luis González and Telv Cohl
(My post #100 *w*)

Let ABC be a triangle with incircle $\odot(I)$, that touch AB and AC at F and E respectively.
Let ω be the circle tangent to $\odot(I)$ and passes through B and C. The tangent lines to ω and the A-excircle of ABC, touch ω at Z and Y.
Prove that BC, YZ and EF are concurrent (or parallels).



This post has been edited 1 time. Last edited by drmzjoseph, May 3, 2015, 8:50 am

Isogonics

#2 May 3, 2015, 1:45 pm • 2

Very nice problem...
I'm not sure whether I'm allowed to post my solution, but An outline:

Let T be point of contact between $\odot(I)$ and ω. We will think concentrate on triangle TBC.
(It means: This will be the reference triangle.)
Let D be the point at which $\odot(I)$ and BC touches.
It's not difficult to get TD bisects $\angle BTC$, and $D, T, EF \cap BC$ forms apollonian circle.

Let S be center of external similitude for ω and A-excircle.
Here the Idea follows: ZY is polar of S wrt ω.
Also, let D' be the point at which A-excircle and BC touches,
and let M be midpoint of arc BTC, and let O, I_A be center of ω, A-excircle,
then S is $OI_A \cap MD'$.

Then using similar things to https://www.artofproblemsolving.com/community/c6h546178p3160582 for TBC
(Well, it's not directly applying, but it's not difficult with some variation)
We get O also lies on TD.

Then it's able to calculate where S and inversion of S wrt ω is (it's not so difficult..)
and to calculate where YZ and BC meets. Then the proof is done.
(Perhaps synthetic proof for this part can be found by others..)

TelvCohl

#4 May 3, 2015, 2:58 pm • 3

Thank you my dear friend **drmzjoseph**

My solution :

Let I_a be A-excenter of $\triangle ABC$.
Let K be the exsimilicenter of $\omega \sim \odot(I_a)$.
Let O be the center of ω and R be the midpoint of YZ .
Let D, N be the tangent point of BC with $\odot(I), \odot(I_a)$, respectively .
Let T be the tangent point of $\odot(I)$ with ω and $S = AT \cap \omega, J = EF \cap BC$.

It suffices to prove J, Y, Z are collinear .

First, let me state two well-known properties **:**
(1) AN pass through the antipode D' of D in $\odot(I)$.
(2) T lie on DI_a (for the proof you can see 2002 ISL G7 post #25) .

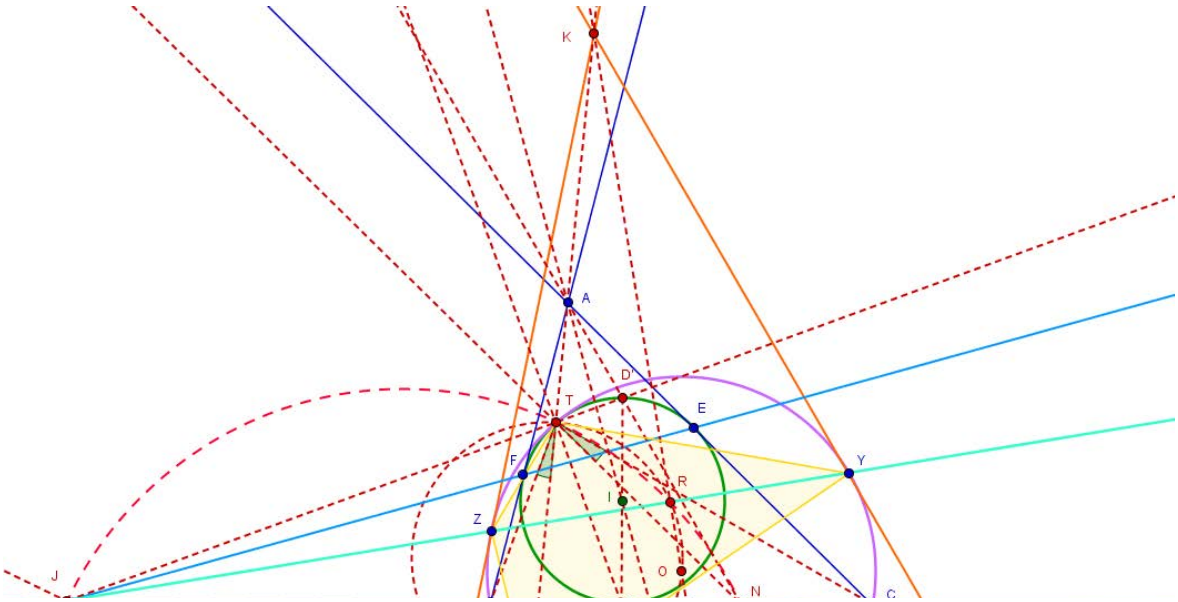
From D'Alembert theorem we get A, K, T are collinear ,
so $TYSZ$ is a harmonic quadrilateral $\Longrightarrow \angle TRZ = \angle TYS$.
From $KR \cdot KO = KY^2 = KZ^2 = KT \cdot KS \Longrightarrow O, R, T, S$ are concyclic .

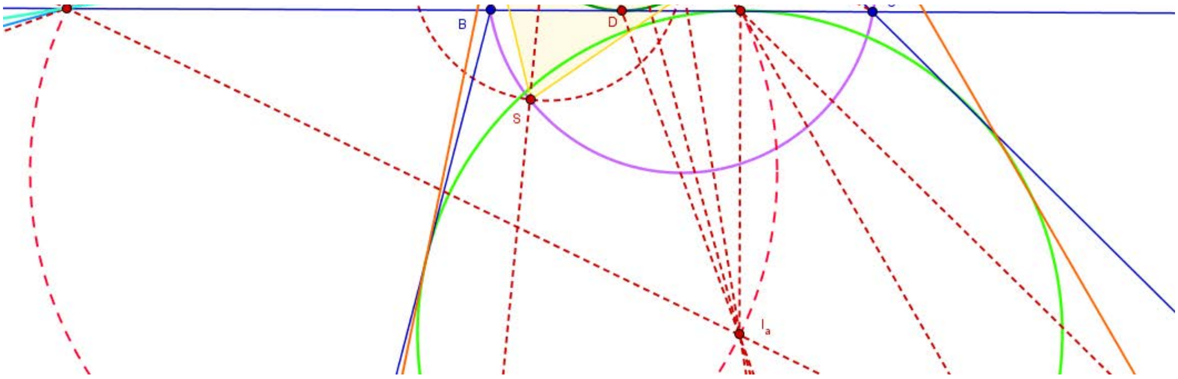
Since TD is the bisector of $\angle BTC$,
so from $(J, D; B, C) = -1$ we get TJ is the external bisector of $\angle BTC \Longrightarrow D' \in TJ$.
Since $T(S, N; D, J) = (A, N; D', TD \cap AN) = I_a(A, N; D', TD \cap AN) = I_a(I, N; D', D) = -1$,
so combine $JT \perp TD \Longrightarrow TD$ is the bisector of $\angle STN \Longrightarrow \angle BTS = \angle NTC \Longrightarrow \angle TNB = 180^\circ - \angle SBT$.

Easy to see J, T, N, I_a lie on a circle with diameter JI_a ,
so from $\angle TNJ = 180^\circ - \angle SBT = 180^\circ - \angle SZT = \angle TYS = \angle TRZ \Longrightarrow \angle TRI_a = \angle TNI_a$,
hence we get R lie on $\odot(JI_a) \Longrightarrow JR \perp I_aR \Longrightarrow J, Y, Z, R$ are collinear (notice that $\overline{YRZ} \perp I_aR$) .

Q.E.D

Attachments:





Luis González

#5 May 4, 2015, 1:22 am • 5

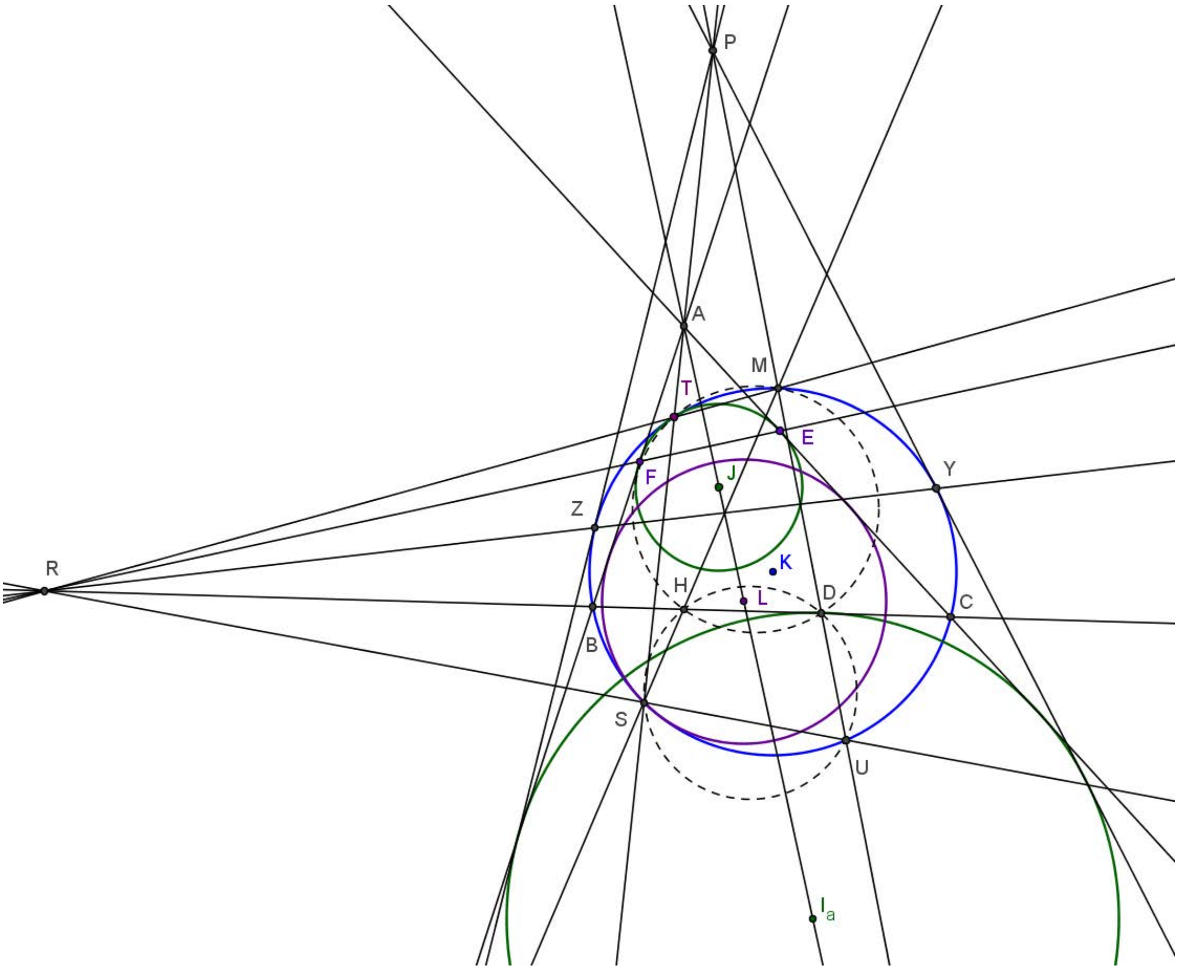
Thanks drmzjoseph for your dedication. Here is a generalization to this nice problem:

Generalization: Let (K) be a circle passing through B, C and (J) is a circle tangent to AC, AB at E, F and internally tangent to (K) (see the diagram below). Common external tangents of (K) and the A-excircle (I_a) touch (K) at $Y, Z.$ Then BC, YZ, EF concur.

T is the tangency point of $(J), (K)$ and M is the midpoint of the arc BTC, D is the tangency point of (I_a) with $BC. P$ is the intersection of the external common tangents of $(I_a), (K)$ (their exsimilicenter). Since A is the exsimilicenter of $(J) \sim (I_a)$ and T is the exsimilicenter of $(J), (K),$ then by Monge & d'Alembert theorem, P, A, T are collinear. Moreover, if (L) is the other circle tangent to AB, AC and internally tangent to (K) at $S,$ again by Monge & d'Alembert theorem, A, T, S are collinear. In this configuration it's known that TS, TD are isogonals WRT $\angle BTC,$ thus if $H \equiv MS \cap BC,$ we have $\angle TDC = \angle TBS = \angle TMH$ (mod 180º) $\Longrightarrow TMDH$ is cyclic, but if MD cuts (K) again at $U,$ then $HDUS$ is cyclic due to $MC^2 = MD \cdot MU = MH \cdot MS \Longrightarrow MT, BC, US$ are pairwise radical axes of $(K), \odot(TMDH), \odot(HDUS)$ concurring at their radical center $R.$

Since I_aD, KM are parallel radii of $(I_a), (K)$ (both perpendicular to BC), then MD goes through their exsimilicenter $P.$ Thus from the complete cyclic $TSUM,$ it follows that the polar YZ of P WRT (K) passes through $R,$ but according to Internally tangent circles and lines and concurrency, BC, EF, MT concur. Therefore, BC, YZ, EF concur at $R,$ as desired.

Attachments:



TelvCohl

#7 May 4, 2015, 6:23 am • 1

> *Luis González wrote:*
> Here is a generalization to this nice problem:
>
> **Generalization:** Let (K) be a circle passing through B, C and (J) is a circle tangent to AC, AB at E, F and internally tangent to (K) (see the diagram below). Common external tangents of (K) and the A-excircle (I_a) touch (K) at Y, Z. Then BC, YZ, EF concur.

My solution **:**

Let D be the tangent point of $\odot(I_a)$ with BC .
Let T be the tangent point of $\odot(J)$ with $\odot(K)$.
Let V be the exsimilicenter of $\odot(K) \sim \odot(I_a)$ and $S = AT \cap \odot(K)$.
Let R be the midpoint of YZ and M be the midpoint of arc BC in $\odot(K)$.
Let $G = AD \cap \odot(J)$ (see attachment) and $H = TI_a \cap \odot(J)$.

First, let me state two well-know fact **:**

(1) TI_a is the bisector of $\angle BTC$.
(2) TM, EF, BC are concurrent at X (see 2007 Chinese TST 3rd Quiz P1) .

It suffices to prove X, Y, Z are collinear .

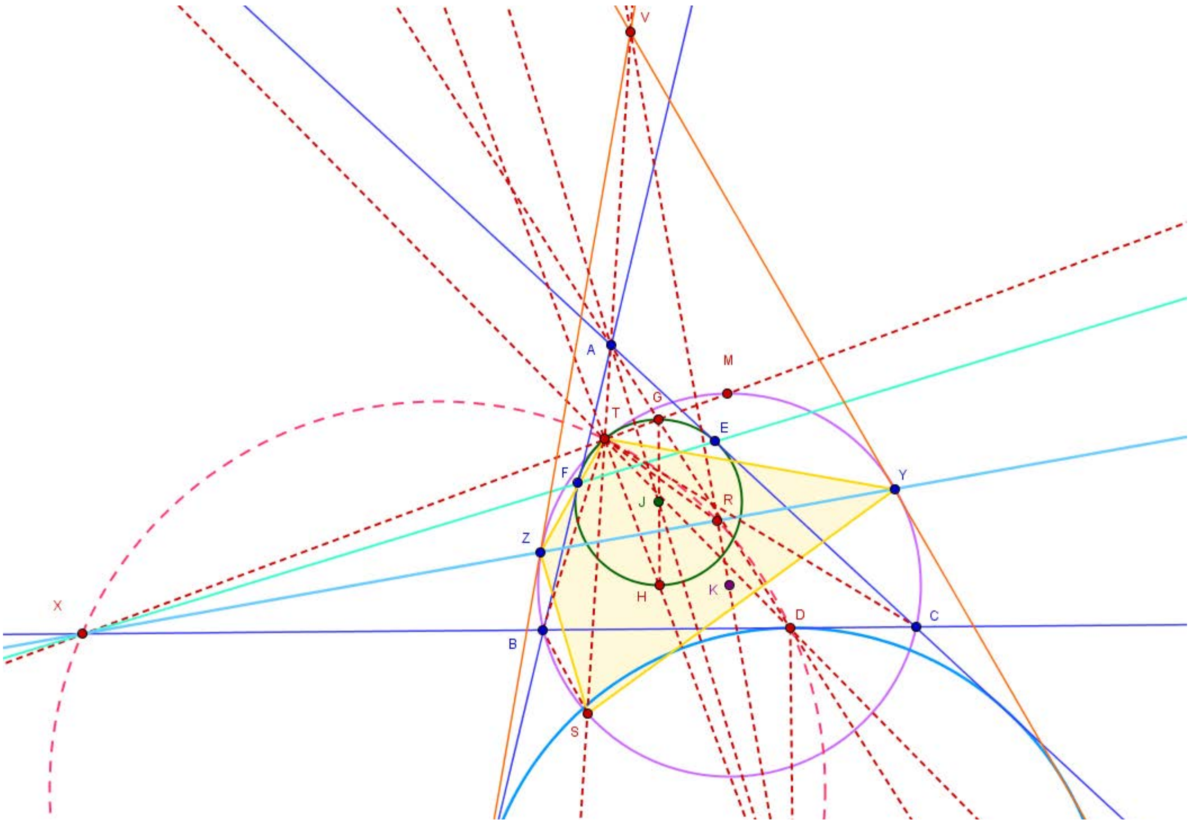
From homothety with center A map $\odot(I_a) \mapsto \odot(J) \Longrightarrow GJ \perp BC$.
From homothety with center T map $\odot(J) \mapsto \odot(K) \Longrightarrow G \in TM, HJ \perp BC$ (i.e. $H \in GJ$) .

Since $T(S, D; X; I_a) = (A, D; G, I_aT \cap AD) = I_a(J, D; G, H) = -1$,
so combine with $TX \perp TI_a$ we get TI_a is the bisector of $\angle STD \Longrightarrow \angle BTS = \angle DTC$,
hence $\angle TDB = 180^\circ - \angle SBT = 180^\circ - \angle SZT = \angle TVS$.

From D'Alembert theorem we get V, A, T are collinear ,
so $TYSZ$ is a harmonic quadrilateral $\Longrightarrow \angle TRZ = \angle TYS = \angle TDB$,
hence $\angle TRI_a = \angle TDI_a \Longrightarrow R \in \odot(DTXI_a) \equiv \odot(XI_a) \Longrightarrow RX \perp VI_a \Longrightarrow X \in YZ$.

Q.E.D

Attachments:



drmzjoseph

#8 Jun 23, 2015, 8:52 am

> *Luis González wrote:*
> Here is a generalization to this nice problem:
>
> **Generalization:** Let (K) be a circle passing through B, C and (J) is a circle tangent to AC, AB at E, F and internally tangent to (K) (see the diagram below). Common external tangents of (K) and the A-excircle (I_a) touch (K) at Y, Z. Then BC, YZ, EF concur.

Similar to solution of **Luis González**.
T is the tangency point of $(J), (K)$, and M is the midpoint of the arc BTC,, AT cut again to (K) at S, using the well-know result TS, TD are isogonals WRT $\angle BTC$,, and according to Internally tangent circles and lines and concurrency BC, EF, MTconcurrent at R. TD cut again to (K) at X so $BC \parallel SX \parallel \ell$ where ℓ is the tangent from M to (K) i.e. the exsimilicenter P between (K) and (I_a) belongs to MD, moreover by Pascal's theorem at five points T, U, M, M, S, X follows $TM \cap US \equiv R'$ then $R'D, \ell$ and XS are concurrent i.e. $R \equiv R'$. also by Monge & d'Alembert theorem P belongs TS i.e. $P \equiv TS \cap MU$ then P and R are conjugate points WRT (K) is sufficient.

Comment

This post has been edited 1 time. Last edited by drmzjoseph, Jun 23, 2015, 8:54 am

Quick Reply

High School Olympiads

hard problem

Reply

Source: Kazakhstan 2011

izat

#1 May 3, 2015, 7:53 pm

In an acute-angled triangle ABC the bisector AD, M and N - respectively the midpoints of AB and AC. Prove that the angle MDN is not less than the angle BAC

This post has been edited 1 time. Last edited by izat, May 3, 2015, 7:53 pm

Luis González

#2 May 3, 2015, 11:27 pm

Let L be the midpoint of BC and let P be the foot of the A-altitude. It's well-known that AD is always between AP and AL (for this note that AD and the perpendicular bisector of BC meet at the midpoint of the circumcircle arc BC). Thus D is always inside the 9-point circle $\odot(MNLP) \Longrightarrow \angle MDN > \angle MLN = \angle BAC$.

suli

#3 May 4, 2015, 2:46 am

Yes! This is actually a problem I can solve! :)

Without loss of generality let $AB \ge AC$. Let E be the foot of the altitude from A to BC, and F be the midpoint of BC. By the Angle Bisector Theorem and the fact that $\angle BAE \ge \angle CAE$ from an easy trig argument, we have that D is sandwiched between (or is possibly equal to) F and E. Therefore, D lies inside the nine-point circle which passes through M, N, E, F. As a result, $\angle MDN \ge \angle MEN = \angle BAC$.

Quick Reply

High School Olympiads

fixed point

Reply

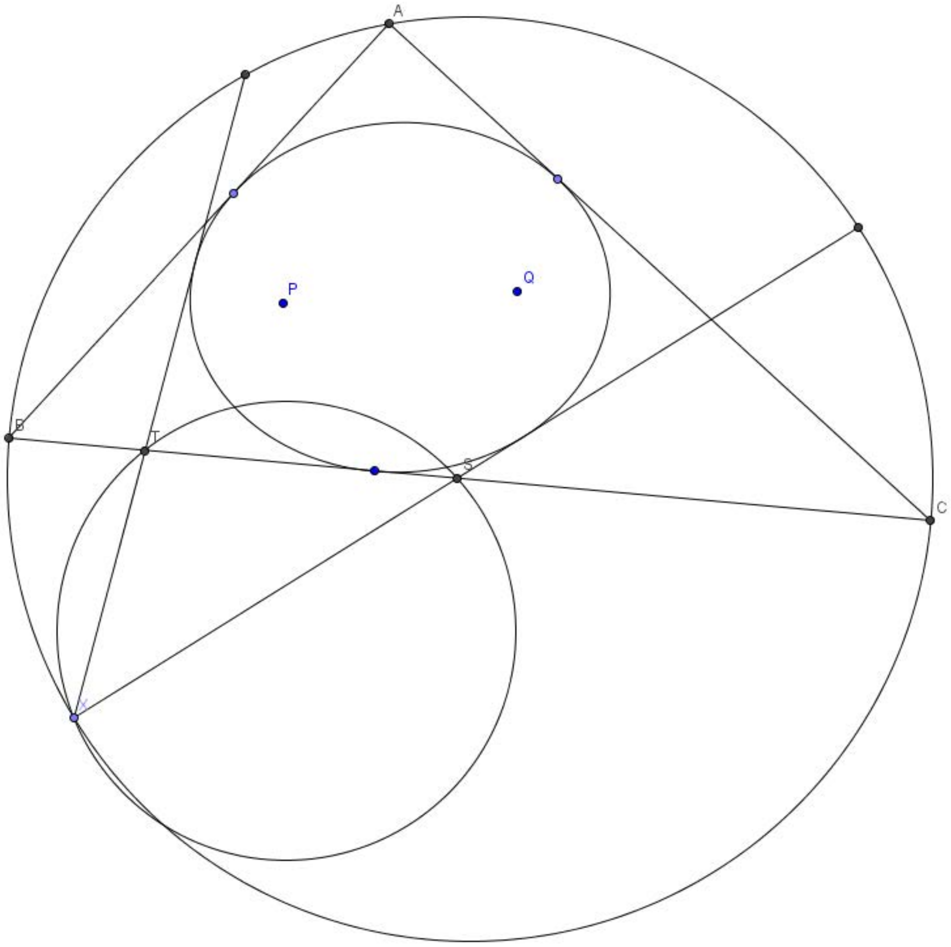
Source: own

andria

#1 May 3, 2015, 1:57 am

Points P is an arbitrary point inside the triangle $\triangle ABC$ Q is isognal conjugate of P WRT $\triangle ABC$ consider an ellipse W with two foci P, Q that touches BC, CA, AB point X is a variable point on the circumcircle of $\triangle ABC$ let the tangents from X to W intersect BC at S, T prove that circumcircle of $\triangle XST$ passes from a fixed point when X varies on the circumcircle of $\triangle ABC$.(may be it is old)

Attachments:



Eugenis

#2 May 3, 2015, 2:15 am

What was the inspiration for this problem?

Luis González

#3 May 3, 2015, 2:22 am

More general: take arbitrary point X and AX cuts $\odot(ABC)$ again X^*. Tangents from X^* to a fixed inconic $\mathcal{C}$ cut BC at S, T. Then all circles $\odot(X^*ST)$ go through a fixed point $K \in \odot(ABC)$.

For a proof see Cevian and mixtilinear incircle (post #12); the proof is exactly the same. If the parallel from A to BC cuts $\odot(ABC)$ again at M and $\mathcal{C}$ touches BC at U, then K is the 2nd intersection of MU with $\odot(ABC)$.

Quick Reply

High School Olympiads

concurrence

Reply

huyhoang
#1 Jul 7, 2011, 8:21 pm

Let ABC be a triangle. A_1, B_1, C_1 are the points in BC, CA, AB such that A_1, B_1, C_1 are collinear. Let M be an arbitrary point on the line $\overline{A_1B_1C_1}$ that does not coincide with A_1, B_1, C_1. A_2 is the reflection of A_1 with respect to M. We define B_2, C_2 analogously. Prove that AA_2, BB_2, CC_2 are concurrent.

Luis González
#2 Apr 30, 2015, 10:18 am

Let $P \equiv BB_2 \cap CC_2$ and $A_3 \equiv AP \cap B_1C_1$. By Desargues involution theorem for $ABCP$ cut by $\overline{A_1B_1C_1}$, it follows that $B_1 \mapsto B_2, C_1 \mapsto C_2, A_1 \mapsto A_3$ is an involution, but the symmetry on M is indeed involutive, forcing $A_2 \equiv A_3 \Longrightarrow AA_2, BB_2, CC_2$ concur at P.

TelvCohl
#3 Apr 30, 2015, 10:23 am

See Blaikie theorem (for all friends of involutions) :)

jayme
#4 Apr 30, 2015, 11:48 am

Dear Mathlinkers,
for a synthetic proof and more, you can see

http://jl.ayme.pagesperso-orange.fr/Docs/Papillon.pdf p. 139-141.

Sincerely
Jean-Louis

Luis González
#5 May 3, 2015, 12:45 am

Remark: P is on the circum-hyperbola whose asymptotes are $\ell \equiv \overline{A_1B_1C_1}$ and its isotomic line τ.

As M varies on ℓ, clearly $B_2 \mapsto C_2$ is a homography on $\ell \Longrightarrow$ pencils BB_2, CC_2 are homographic $\Longrightarrow P$ moves on a conic $\mathcal{H}$ with center O through B, C and A (when M is the midpoint of B_1C_1). When M is at infinity, then B_2, C_2 go to infinity as well $\Longrightarrow P$ is at infinity $\Longrightarrow \ell$ is an asymptote of $\mathcal{H}$.

Let D be the midpoint of BC. As the asymptotes are the fixed O-rays of the involution formed by conjugate diameters, then it follows that the other asymptote τ goes through O cutting BC at the reflection of A_1 on D. Similarly τ passes through the isotomics of B_1 and C_1 WRT CA, AB, as desired.

Quick Reply

High School Olympiads

Concyclic points iff angle is 60°

Reply

Source: My teacher

NEWDORMANTUSER
#1 May 2, 2015, 11:17 pm

Let PQR be a triangle. Let O, I, H be respectively the circumcenter, incenter, and orthocenter of the triangle. If $\angle RQP \geq \angle RPQ \geq \angle PRQ$, prove that O, I, H, R are concyclic if and only if $\angle RPQ = 60^\circ$.

Luis González
#2 May 2, 2015, 11:47 pm

This should be O, I, H, R are concyclic $\Longleftrightarrow$ either $\angle RPQ = 60^\circ$ or $\angle PQR = 60^\circ$.

If $\angle RPQ = 60^\circ$, then $\angle QHR = \angle QIR = \angle QOR = 120^\circ \Longrightarrow O, I, H, R, Q$ are concyclic and similarly when $\angle PQR = 60^\circ$. Conversely, if $ROIH$ is cyclic then $IO = IH$, because of $\angle IRO = \angle IRH$. Thus $IO = IH \Longrightarrow POIH$ is either cyclic or $PO = PH$ and the same for $QOIH$, but it's impossible that all $ROIH, POIH, QOIH$ are cyclic. If $QOIH$ is cyclic, then $QHOR$ is cyclic $\Longrightarrow \angle RPQ = 60^\circ$ and similarly if $POIH$ is cyclic, then $PHOR$ is cyclic $\Longrightarrow \angle PQR = 60^\circ$.

NEWDORMANTUSER
#3 May 18, 2015, 6:19 pm

> *Luis González wrote:*
> Thus $IO = IH \Longrightarrow POIH$ is either cyclic or $PO = PH$ and the same for $QOIH$,.
> If $QOIH$ is cyclic, then $QHOR$ is cyclic.

Can anyone explain why
1. $IO = IH \Longrightarrow POIH$ is either cyclic or $PO = PH$.

and why
2. If $QOIH$ is cyclic, then $QHOR$ is cyclic.

are true?

Thank you

NEWDORMANTUSER

NEWDORMANTUSER
#4 May 18, 2015, 11:43 pm

Anyone, please?

Thanks.

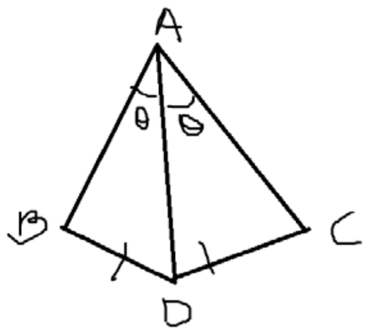
KudouShinichi
#5 May 19, 2015, 1:17 am • 1

PArt 1

see in the attachment that $\dfrac{BD}{sinBAD} = \dfrac{DC}{sinCAD} = \dfrac{AD}{sinABD} = \dfrac{AD}{sinACD}$

$\implies$ either $\angle DBA = \angle DCA$ or $\angle DBA + \angle DCA = 180$
$(sin\theta = sin180 - \theta)$

Attachments:



This post has been edited 1 time. Last edited by KudouShinichi, May 19, 2015, 1:18 am

KudouShinichi

#6 May 19, 2015, 1:26 am • 1

part 2

see that H and O are isogonal conjugates so if $QOIH$ are cyclic implies $IH = IO$ and since again (isogonal)
$\angle HRI = \angle ORI$ implies
(see part 1) $HIOR$ cyclic so we get that Q, H, I, O, R all are on the same circle(circumcircle of $\triangle HIO$) so $QHOR$ cyclic

This post has been edited 1 time. Last edited by KudouShinichi, May 19, 2015, 1:27 am

Quick Reply

High School Olympiads

Nice Geometry problem

Source: vankhea

vankhea

#1 May 2, 2015, 8:58 pm

Let K be point inside of triangle ΔABC. The rays AK, BK, CK cuts circumcircle of ΔABC at D, E, F respectively.
Prove that:
$$\frac{BD \times CE}{BC \times DE} + \frac{CE \times AF}{CA \times EF} + \frac{AF \times BD}{AB \times FD} = 1$$

Luis González

#2 May 2, 2015, 10:44 pm

Note that $\frac{BD \cdot CE}{BC \cdot DE} = \frac{\sin \widehat{BAD}}{\sin \widehat{BAC}} \cdot \frac{\sin \widehat{EAC}}{\sin \widehat{EAD}} = A(B, D, C, E)$ and similarly for the others. Thus the object expression is equivalent to:

$$A(B, D, C, E) + B(C, E, A, F) + C(A, F, B, D) = 1.$$

Since K is inside of (O), there exist a homology taking (O) into another circle with center the image of K. Cross ratios remain invariant, so it suffices to prove the relation for $K \equiv O$. Letting H be the orthocenter of $\triangle ABC$, we get then:

$$\frac{BD \cdot CE}{BC \cdot DE} = \frac{AE \cdot CE}{BC \cdot AB} = \frac{[CEA]}{[ABC]} = \frac{[HCA]}{[ABC]}.$$

Adding the cyclic expressions together gives:

$$\frac{BD \cdot CE}{BC \cdot DE} + \frac{CE \cdot AF}{CA \cdot EF} + \frac{AF \cdot BD}{AB \cdot FD} = \frac{[HBC] + [HCA] + [HAB]}{[ABC]} = 1.$$

Quick Reply

Spain

OMCC 2012, problema 1

Reply

Jutaro
#1 Jul 3, 2012, 7:34 am

Hallar todos los enteros positivos que sean iguales a 700 veces la suma de sus dígitos.

José Nieto (Venezuela)

crimeeee
#2 Jul 21, 2012, 3:22 am

Sea n el entero positivo y $s(n)$ la suma de sus dígitos. Para $n > 99999$, se tiene $s(n) < 55$, por lo que $700 * 54 = 37800$, es decir que n tiene como máximo 5 dígitos. Además n tiene que tener como mínimo 4 dígitos, ya que para 3 dígitos la única posibilidad es $700 * 1 = 700$, lo que es imposible, ya que en este caso $s(n) = 7$.

Para n de cuatro dígitos:

$700(a+b+c+d) = 1000a + 100b + 10c + d$, es decir $2b + 2.3c + 2.33d = a$. Como a es natural (al ser un dígito), entonces necesariamente b y c deben valer 0. Así $2b = a$, por lo que las soluciones son: (2100, 4200, 6300, 8400).

Para n de cinco dígitos:

$700(a+b+c+d+e) = 10000a + 1000b + 100c + 10d + e$, es decir $2c + 2,3d + 2,33e = 31a + b$. Como en el caso anterior, d y e deben valer 0, por lo que queda: $2c = 31a + b$. Como a vale como mínimo 1, entonces es imposible.

Así, las soluciones son: 2100, 4200, 3600 y 8400.

Luis González
#3 May 2, 2015, 2:03 am

Evidentemente no hay ninguno de éstos de una sola cifra. Si $n = 10a + b$ y $n = 7(a+b)$, entonces $3a = 6b$ y $a = 2b$. Resultan así para n los valores $21, 42, 63, 84,$ los cuales generan las soluciones $2100, 4200, 6300$ y 8400.

Ahora probaremos que no hay más soluciones. Si $n = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0$ con $k \geq 2$ y $n = 7(a_k + a_{k-1} + \cdots + a_0) \Longrightarrow (10^k - 7)a_k + (10^{k-1} - 7)a_{k-1} + \cdots + 3a_1 = 6a_0$, pero $6a_0 \geq 6 \cdot 9 = 54$, mientras que el miembro izquierdo es al menos $10^k - 7 \geq 93$. Entonces hay sólo las 4 soluciones mecionadas.

Quick Reply

High School Olympiads

Easy problem

Reply

Source: Own invention

drmzjoseph

#1 May 1, 2015, 11:49 pm

The A-excircle of $\triangle ABC$ touch BC, CA and AB at D, E and F respectively. Let M be the foot of perpendicular from D at EF, let N be the midpoint of EF.
Prove that $M \in \odot(ABC) \iff N \in \odot(ABC)$, and show that it is possible for some $\triangle ABC$

:)

Luis González

#2 May 2, 2015, 1:32 am

Inversion WRT the A-excircle (I_a) takes midpoints of EF, FD, DE into $A, B, C,$ i.e. it takes 9-point circle ω of $\triangle DEF$ through M, N into $\odot(ABC)$. Since the inversion circle is always coaxal with the inverse circles, then ω goes trough $\odot(ABC) \cap (I_a)$. Thus $N \in \odot(ABC) \Longleftrightarrow \odot(ABC) \equiv \omega \Longleftrightarrow M \in \odot(ABC)$.

PROF65

#3 May 2, 2015, 10:08 pm

Let T be the intersection of EF and BC then $(T, D; B, C) = -1$ ie $M(T, D; B, C)$ is harmonic thus MD is angle bisector of $\widehat{BMC}$ so we deduce that $\widehat{BMF} = \widehat{CME} \implies \widehat{MBF} = \widehat{MBE} \implies \widehat{ABM} = \widehat{ACM}$ therefore
$M \in (ABC) \iff \widehat{ABM} = \dfrac{\pi}{2} \implies N \in (ABC)$
if $N \in (ABC)$ then N is on the bisector of BC thus $NIDM$ is cyclic (I is the midpoint of BC) but
$TB.TC = TD.TI = TM.TN$ ie M is on the circle (ABC)

This post has been edited 1 time. Last edited by PROF65, May 2, 2015, 10:58 pm
Reason: more accurate

Quick Reply

High School Olympiads

tangential in rectangle

Reply

Peres123

#1 May 1, 2015, 2:59 pm

Given a circle in rectangle ABCD such that whole the circle inside the rectangle. Let AW, BX, CY, DZ are tangents from A, B, C, D to the circle with $AW = a, BX = b, CY = c$ Then in a, b, c terms $DZ = ...$

Luis González

#2 May 1, 2015, 9:13 pm • 1

Label (O, ϱ) the given circle. By Pythagorean theorem we have $AO^2 = \varrho^2 + a^2, BO^2 = \varrho^2 + b^2, CO^2 = \varrho^2 + c^2,$ $DO^2 = \varrho^2 + DZ^2$. But by British flag theorem, we have $AO^2 + CO^2 = BO^2 + DO^2$. Combined $DZ = \sqrt{a^2 + c^2 - b^2}$.

Quick Reply

High School Olympiads

very hard problem

Source: from my homework

quangminhltv99

#1 Apr 30, 2015, 8:55 pm

Let A', B', C' be the feet of the altitudes of nonright triangle ABC issued from A, B, C respectively. Let D, E, F be the incenters of triangles $AB'C'$, $BC'A'$ and $CA'B'$ respectively. Calculate the circumradius of triangle DEF in terms of $BC = a, CA = b, AB = c$

quangminhltv99

#2 May 1, 2015, 7:04 pm

Any solution?

Luis González

#3 May 1, 2015, 7:55 pm

Let the incircle (I, r) of $\triangle ABC$ touch BC, CA, AB at $X, Y, Z.$ Since $\triangle ABC \cup I \sim \triangle AB'C' \cup D,$ we get $\frac{AD}{AI} = \frac{B'C'}{BC} = \cos A.$ As I is the antipode of A in the circumcircle of the A-isosceles $\triangle AYZ,$ we deduce that D is orthocenter of $\triangle AYZ \Longrightarrow D$ is reflection of I across YZ and similarly E, F are the reflections of I across $ZX, XY \Longrightarrow \triangle DEF \cong \triangle XYZ \Longrightarrow$ circumradius of $\triangle DEF$ equals $r.$

quangminhltv99

#4 May 1, 2015, 8:08 pm

Excuse me but I don't know how to calculate r in terms of a, b, c

Luis González

#5 May 1, 2015, 8:58 pm

As stated, r is the inradius of $\triangle ABC.$ Its formula in terms of a, b, c is well-known

$$r = \frac{1}{2}\sqrt{\frac{(b+c-a)(c+a-b)(a+b-c)}{a+b+c}}$$

High School Math

Geometry problem

Reply

Higgsboson79

#1 Apr 30, 2015, 9:06 am

Let ABC be a triangle such that $AB=AC$, and let D be the midpoint of BC. Let $E \in AB$, such that $AB \perp DE$. Label with F the midpoint of DE. Prove that $AF \perp CE$.

Luis González

#2 Apr 30, 2015, 9:57 am • 1

If M is the midpoint of BD, we have $\angle MED=\angle MDE=\angle EAD \Longrightarrow ME$ is tangent of $\odot(AED) \Longrightarrow AM$ is the A-symmedian of $\triangle AED \Longrightarrow \angle DAF=\angle EAM.$ But $\angle BEM=\angle EBM=\angle ACM \Longrightarrow ACME$ is cyclic $\Longrightarrow$ $\angle ECD=\angle EAM=\angle DAF \Longrightarrow AF \perp CE.$

sunken rock

#3 Apr 30, 2015, 12:55 pm • 1

A generalization can be found at post #6 here

Best regards,
sunken rock

Quick Reply

High School Olympiads

Intersection of three lines

Reply

Source: Pythagoras Olympiade February 1981, Holland

rae306

#1 Apr 30, 2015, 1:08 am

Given is a not-right angled triangle ABC. D, E and F are the feet of the altitudes from A, B and C respectively. P, Q and R are the midpoints of line segments EF, FD and DE respectively. p is the line through P perpendicular to BC, q is the line through Q perpendicular to CA and r is the line through R perpendicular to AB.
Prove that p, q and r intersect.

This post has been edited 1 time. Last edited by rae306, Apr 30, 2015, 1:22 am

Luis González

#2 Apr 30, 2015, 1:21 am

Assuming that you mean that D,E,F are the feet of the **altitudes** from $A,B,C,$ then this simply follows by Carnot's theorem (orthologic triangles). Perpendiculars from A,B,C to $(EF \parallel QR),(FD \parallel RP),(DE \parallel PQ)$ concur at the circumcenter of $\triangle ABC,$ so p,q,r concur at the other orthology center.

Luis González

#3 Apr 30, 2015, 1:44 am

Or even simpler. It's well-known that DA,EB,FC bisect $\angle FDE,\angle DEF,\angle EFD,$ thus since $PR \parallel DF$ and $PQ \parallel DE,$ then $p \parallel DA$ bisects $\angle RPQ$ and similarly for $p,q,r.$ Therefore p,q,r concur at the incenter or an excenter of $\triangle PQR.$

jayme

#4 May 1, 2015, 11:55 am

Dear Mathlinkers,
the point of intersection is also the center of the Taylor's circle of ABC...
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

perpendicular

Reply

Source: Own

LeVietAn

#1 Apr 29, 2015, 8:58 pm

Dear Mathlinkers,
Let $ABCD$ be a rectangle inscribed circle (O). Let P be a point on the smaller arc of AB from the circle (O). AP cut BC at E and BP cut AD at F. Point H is the foot of the perpendicular from P to AB. Let M be midpoint of PH. Prove that OM is perpendicular to EF.

TelvCohl

#3 Apr 29, 2015, 9:30 pm • 1

My solution:

Let X be the reflection of H in O .
Let $T \equiv PH \cap CD$ be the projection of P on CD .
Let $Y \in BC, Z \in AD$ be the point such that $TY \perp PC, TZ \perp PD$.

Easy to see $PX \parallel MO$.

From $\angle APC = \angle BPD = 90^\circ \Longrightarrow TY \parallel PE, TZ \parallel PF$,
so $\triangle PEF$ and $\triangle TYZ$ are congruent and homothetic $\Longrightarrow EF \parallel YZ$.

Since X is the isotomic conjugate of T on CD ,
so from An extension of a problem of perpendicularity we get $PX \perp YZ \Longrightarrow OM \perp EF$.

Q.E.D

Luis González

#4 Apr 29, 2015, 11:19 pm • 1

Let X be the antipode of P on (O). Since $\frac{BF}{AE} = \frac{PF}{PA} = \frac{XB}{XA}$, the right $\triangle XBF$ and $\triangle XAE$ are similar $\Longrightarrow$ $\angle AEX = \angle BFX$. Hence if Y is the projection of X on EF (2nd intersection of $\odot(XBF)$ and $\odot(XAE)$), we have $\angle AYX = \angle AEX = \angle BFX = \angle BYX \Longrightarrow YX, EF$ bisect $\angle AYB \Longrightarrow YX$ goes through the harmonic conjugate H of $EF \cap AB$ WRT $A, B \Longrightarrow (OM \parallel XHY) \perp EF$.

jayme

#5 May 2, 2015, 4:35 pm

Dear Mathlinkers,
another approach... without any calculation

1. the Gauss line OM of the quadrilateral DPBH is perpendicular to the Steiner line which passes through P
2. by Desargues theorem, this Steiner line is parallel to EF...

Sincerely
Jean-Louis

jayme

#6 May 19, 2015, 5:46 pm

Dear Mathlinkers,
I come back with the idea which consists to consider the orthopole of the triangle PEF wrt EF... and we are done without any calculation...
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Problem in geometry

Reply

chicky

#1 Apr 23, 2010, 11:49 am

Let $\triangle ABC$. Line d (I is the incenter, $I \in d$) intersects with sides AB, AC, BC at points M, N, P. Prove that:

$$\frac{a}{\bar{B}P.\bar{P}C}+\frac{b}{\bar{C}N.\bar{N}A}+\frac{c}{\bar{A}M.\bar{M}B}=\frac{(a+b+c)^2}{abc}$$

BaronShadeNight

#2 Jul 2, 2010, 6:49 am

Mathematics and Youth Magazine....

Quick Reply

High School Olympiads

Incenter

Reply

nvhmath

#1 Jun 25, 2012, 1:19 pm

Let ABC be triangle. A line d passing through its incenter meets AB, BC, CA respectively at M, P, N. Prove that :
$\frac{a}{BP \cdot PC} + \frac{b}{CN \cdot NA} + \frac{c}{AM \cdot MB} = \frac{(a+b+c)^2}{abc}$

Nisan

#2 Jun 26, 2012, 11:38 am

Without loss of generality we can assume that
1) The incenter is 0 (in the complex plane).
2) d is the real line.
3) The Incircle of the triangle ABC is the unit circle.

Let ω be the unit circle.
Let $x = BC \cap \omega$, $y = AC \cap \omega$, and $z = AB \cap \omega$.

Lemma: Let x and y be two points on ω. Let l_x, l_y be the tangents to ω on x and y respectively. Then $l_x \cap l_y = \frac{2xy}{x+y}$.

Proof:
$l_x \cap l_y$ is the inverse of $\frac{x+y}{2}$ with respect to the circle ω. Therefore,
$l_x \cap l_y = (\overline{\frac{x+y}{2}})^{-1} = \frac{2}{\bar{x}+\bar{y}} = \frac{2}{x^{-1}+y^{-1}} = \frac{2xy}{x+y}$.

By the lemma,
$A = \frac{2yz}{y+z}$

$B = \frac{2xz}{x+z}$

$C = \frac{2xy}{x+y}$

How can we determine P?
P lies on $BC = l_x$
Can P lie also on $l_\bar{x}$?
Yes!
Because of symmetry $l_\bar{x}\capl_x$ lies on d, and therefore it must be P.
$P = l_{\bar{x}} \cap l_x = \frac{2x\bar{x}}{x+\bar{x}} = \frac{2xx^{-1}}{x+x^{-1}} = \frac{2x}{x^2+1}$

Similarly,
$N = \frac{2y}{y^2+1}$
and
$M = \frac{2z}{z^2+1}$.

We substitute the above quantities in $\frac{BC}{BP \cdot PC}$ and obtain $\frac{BC}{BP \cdot PC} = |\frac{\frac{2xz}{x+z} - \frac{2xy}{x+y}}{(\frac{2xz}{x+z} - \frac{2x}{x^2+1})(\frac{2x}{x^2+1} - \frac{2xy}{x+y})}|$.

The angle of $\dfrac{2xz}{x+z}-\dfrac{2xy}{x+y},\dfrac{2xz}{x+z}-\dfrac{2x}{x^2+1},\dfrac{2x}{x^2+1}-\dfrac{2xy}{x+y}$ is the angle of ix.

Therefore, $\dfrac{\frac{2xz}{x+z}-\frac{2xy}{x+y}}{(\frac{2xz}{x+z}-\frac{2x}{x^2+1})(\frac{2x}{x^2+1}-\frac{2xy}{x+y})}x$ is on the imaginary line.

Because $|x|=1$ we get that $\pm i\dfrac{BC}{BP\cdot PC}=\dfrac{\frac{2xz}{x+z}-\frac{2xy}{x+y}}{(\frac{2xz}{x+z}-\frac{2x}{x^2+1})(\frac{2x}{x^2+1}-\frac{2xy}{x+y})}x.$

I am not quite sure whether the sign in the above equation is positive or negative. However, this doesn't matter because the sign is fixed.

$$\pm i\left(\frac{a}{BP\cdot PC}+\frac{b}{CN\cdot NA}+\frac{c}{AM\cdot MB}\right)=\sum_{cyc}\frac{\frac{2xz}{x+z}-\frac{2xy}{x+y}}{\left(\frac{2xz}{(x+z)(z+y)}-\frac{2x}{x^2+1}\right)\left(\frac{2x}{x^2+1}-\frac{2xy}{z+y}\right)}x=-\frac{1}{2}\sum_{cyc}\frac{\frac{z}{x+z}-\frac{y}{x+y}}{\left(\frac{1}{x^2+1}-\frac{z}{x+z}\right)\left(\frac{1}{x^2+1}-\frac{y}{x+y}\right)}=-\frac{1}{2}\sum_{cyc}\frac{\frac{xz-xy}{(x+z)(x+y)}}{\frac{xz^2-z}{(x^2+1)(x+z)}\frac{xy^2-y}{(x^2+1)(x+y)}}=-\frac{1}{2}\sum_{cyc}\frac{xz-xy}{\frac{x^2z-z}{x^2+1}\frac{x^2y-x}{(x^2+1)}}=-\frac{1}{2x}\sum_{cyc}\frac{z-y}{\frac{xz-1}{x^2+1}\frac{xy-1}{x^2+1}}=-\frac{1}{2x}\sum_{cyc}\frac{(x^2+1)^2(z-y)}{(xz-1)(xy-1)}=-\frac{1}{2xyz(xz-1)(xy-1)(yz-1)}\sum_{cyc}yz(yz-1)(z-y)(x^2+1)^2.$$

Let us now determine

$$\sum_{cyc}yz(yz-1)(z-y)(x^2+1)^2=\sum_{cyc}yz(yz-1)(z-y)(x^4+2x^2+1)=\sum_{cyc}yz(yz-1)(x^4z-x^4y+2x^2z-2x^2y+z-y)=\sum_{cyc}\text{[illegible]}$$

Therefore,

$$\pm i\left(\frac{a}{BP\cdot PC}+\frac{b}{CN\cdot NA}+\frac{c}{AM\cdot MB}\right)=-\frac{1}{2xyz(xz-1)(xy-1)(yz-1)}\sum_{cyc}yz(yz-1)(z-y)(x^2+1)^2=-\frac{\sum_{cyc}yz^2-y^2z}{2xyz}=\frac{(z-y)(y-x)(x-z)}{2xyz}.$$

Let us now calculate the right-hand side of the equation.

$a=|\dfrac{2xz}{x+z}-\dfrac{2xy}{x+y}|$

The angle of $\dfrac{2xz}{x+z}-\dfrac{2xy}{x+y}$ is like the angle of ix

Therefore $\pm ia=\dfrac{\frac{2xz}{x+z}-\frac{2xy}{x+y}}{x}=\dfrac{2z}{x+z}-\dfrac{2y}{x+y}=2\dfrac{xz-xy}{(x+z)(x+y)}=2x\dfrac{z-y}{(x+z)(x+y)}.$

Therefore,

$$\pm i\frac{(a+b+c)^2}{abc}=\frac{\left(\sum_{cyc}2x\frac{z-y}{(x+z)(x+y)}\right)^2}{\frac{2xyz(z-y)(y-x)(x-z)}{(x+z)^2(x+y)^2(y+z)^2}}=\frac{\left(\sum_{cyc}2x\frac{(z-y)(z+y)}{(z+y)(x+z)(x+y)}\right)^2}{\frac{2xyz(z-y)(y-x)(x-z)}{(x+z)^2(x+y)^2(y+z)^2}}=\frac{\left(\sum_{cyc}2x(z-y)(z+y)\right)^2}{2xyz(z-y)(y-x)(x-z)}=\frac{\left(\sum_{cyc}2xz^2-2xy^2\right)^2}{2xyz(z-y)(y-x)(x-z)}=\frac{(z-y)(y-x)(x-z)}{2xyz}.$$

We can see that both sides are equal up to the sign, so we need to check it for one point, and by continuity it will follow that the sign is the same in all other cases.

If ABC is an equilateral triangle and d is parallel to BC, then $BP=PC=\infty$, and therefore the first term is 0.

All other terms are clearly positive. Therefore the sign is always +.

Quick Reply

High School Olympiads

easy geometry

Reply

andria

#1 Apr 16, 2015, 5:26 pm

A variable line L passes throw the incenter of a given triangle ABC and intersects BC, AC, AB at points M, N, P prove that value of S is independent of choice of L:

$S=\frac{AB}{PA.PB}+\frac{AC}{NA.NC}+\frac{BC}{MB.MC}$

Luis González

#2 Apr 29, 2015, 6:48 am

Powers $PA\cdot PB, NC\cdot NA, MB\cdot MC$ should be signed, thus assuming WLOG that B is between $C, M,$ we'll take $-MB\cdot MC$ in the expression.

Let $D\equiv AI\cap BC$ and let the parallel from I to BC cut AC, AB at $Y, Z.$ We have:

$$\frac{MB}{IZ}=\frac{MP}{IP} \ , \ \frac{MC}{IY}=\frac{MN}{IN} \Longrightarrow \frac{MB\cdot MC}{IY\cdot IZ}=\frac{MP\cdot MN}{IP\cdot IN}.$$

But $\frac{IZ}{c}=\frac{BZ}{c}=\frac{ID}{AD}=\frac{a}{a+b+c}$ and similarly $\frac{IY}{b}=\frac{a}{a+b+c} \Longrightarrow$

$$\frac{a}{MB\cdot MC}=\frac{(a+b+c)^2}{abc}\cdot \frac{IP\cdot IN}{MP\cdot MN}$$

Thus, adding the cyclic expressions together gives

$$-\frac{a}{MB\cdot MC}+\frac{b}{NC\cdot NA}+\frac{c}{PA\cdot PB}=\frac{(a+b+c)^2}{abc}\left(-\frac{IP\cdot IN}{MP\cdot MN}+\frac{IP\cdot IM}{NP\cdot NM}+\frac{IN\cdot IM}{PN\cdot PM}\right)$$

Simple algebraic manipulation shows that $\tfrac{IP\cdot IM}{NP\cdot NM}+\tfrac{IN\cdot IM}{PN\cdot PM}=1+\tfrac{IP\cdot IN}{MP\cdot MN}.$ Hence

$$-\frac{a}{MB\cdot MC}+\frac{b}{NC\cdot NA}+\frac{c}{PA\cdot PB}=\frac{(a+b+c)^2}{abc}=\text{const}.$$

Luis González

#3 Apr 29, 2015, 7:06 am

Apparently this was an issue of Mathematics and Youth Magazine. See also Problem in geometry (no solutions) and Incenter (solution with complex numbers).

Quick Reply

High School Olympiads

geometry

Reply

andria

#1 Apr 28, 2015, 6:14 pm

$ABCD$ is a cyclic quadrilateral. $AC \cap BD = F, AD \cap BC = E$ let M, N midpoints of AB, CD prove that:
$\frac{MN}{EF} = \frac{1}{2}|\frac{CD}{AB} - \frac{AB}{CD}|$.

TelvCohl

#2 Apr 28, 2015, 7:23 pm

My solution:

Let $\mathbf{R}(\ell)$ be the reflection with respect to axis ℓ .
Let $\mathbf{H}(O, t)$ ($t \in \mathbb{R}$) be the homothety with center O and factor t .
Let ℓ_E, ℓ_F be the bisector of $\angle(BC, AD), \angle(AC, BD)$, respectively .
Let E' be the image of E under $\mathbf{H}(F, \frac{AB}{CD}) \circ \mathbf{R}(\ell_F)$ and F' be the image of F under $\mathbf{H}(E, \frac{CD}{AB}) \circ \mathbf{R}(\ell_E)$.

WLOG $CD > AB$

Since $D \mapsto A$ under $\mathbf{H}(F, \frac{AB}{CD}) \circ \mathbf{R}(\ell_F)$,
so $\triangle FDE \sim \triangle FAE' \Longrightarrow \angle E'AF = \angle EDF = 180^\circ - \angle ACB \Longrightarrow AE' \parallel EB$.
Similarly, we can prove $BE' \parallel EA \Longrightarrow AEBE'$ is a parallelogram $\Longrightarrow M$ is the midpoint of EE' .

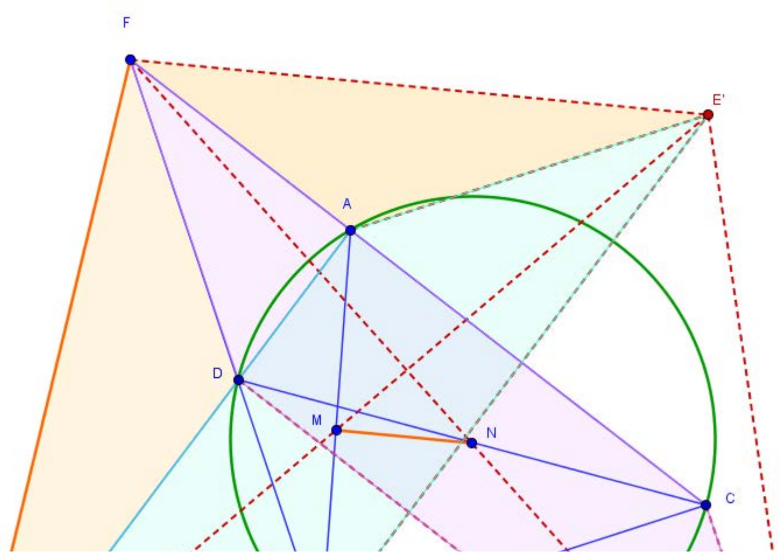
By similar discussion we get $CFDF'$ is a parallelogram and N is the midpoint of FF' .

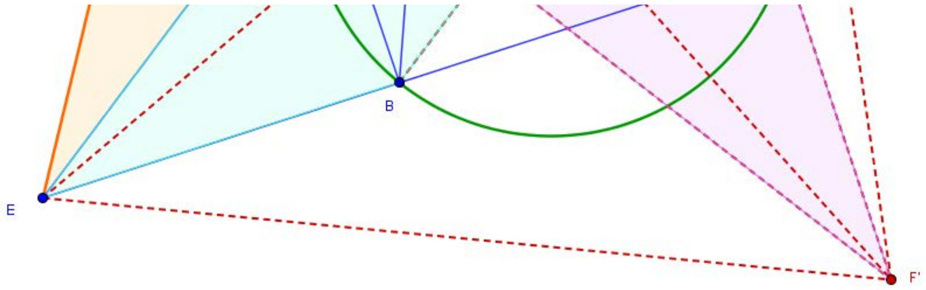
Since $\{EF, EF'\}, \{FE.FE'\}$ are symmetry WRT ℓ_E, ℓ_F, respectively ,
so from $\ell_E \perp \ell_F$ (well-known) we get $EF' \parallel FE' \Longrightarrow FEF'E'$ is a trapezoid ,

hence we get $MN = \frac{1}{2} \cdot (EF' - FE') = \frac{1}{2} \cdot EF \cdot \left(\frac{CD}{AB} - \frac{AB}{CD}\right) \Longrightarrow \frac{MN}{EF} = \frac{1}{2} \cdot \left(\frac{CD}{AB} - \frac{AB}{CD}\right)$.

Q.E.D

Attachments:





Luis González

#3 Apr 28, 2015, 10:09 pm

Posted before at [Synthetic solution desired [2 MN/EF = |AC/BD - BD/AC|] and involving absolute value. Note that the relation holds for any cyclic ABCD (convex or self-intersecting).

TelvCohl

#5 Jul 26, 2015, 1:48 pm • 1

Another solution :

Let Y, X $(X \in MN)$ be the midpoint of DF, EF, respectively .

WLOG $CD > AB$

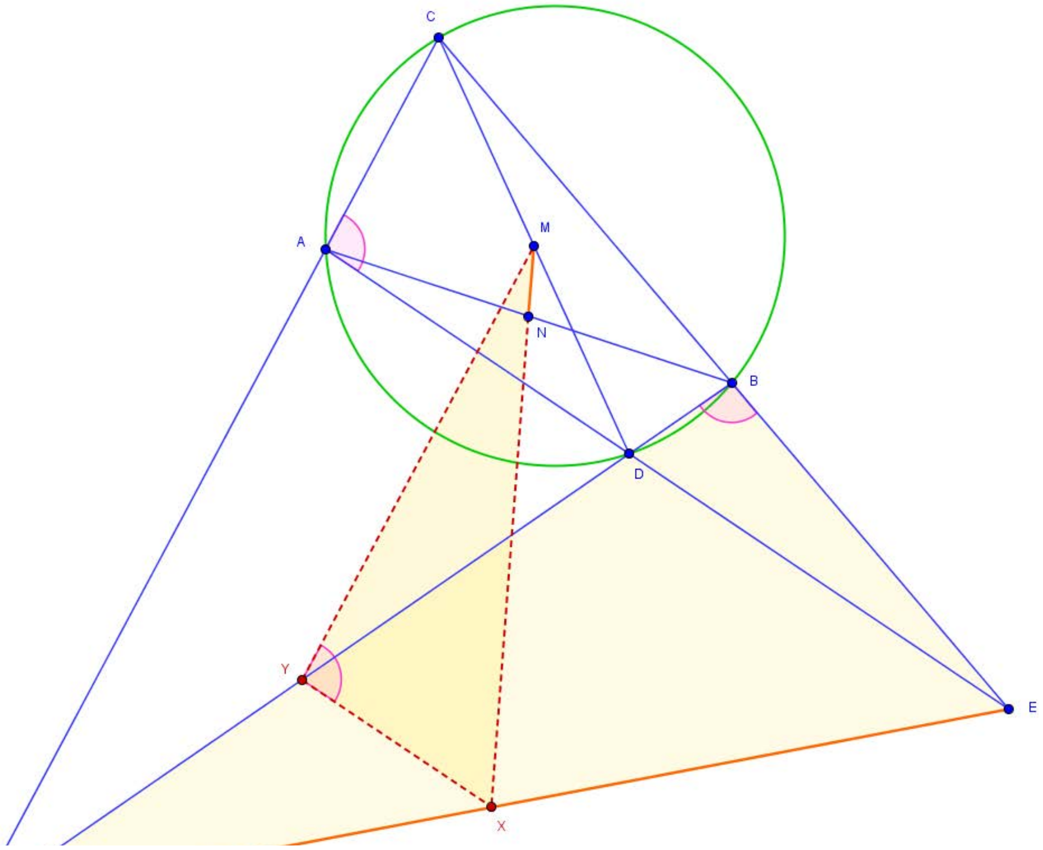
From $\triangle EAB \sim \triangle ECD$ and $\triangle FAB \sim \triangle FDC \Longrightarrow \frac{YM}{YX} = \frac{FC}{DE} = \frac{BF}{BE}$,

so combine $\angle XYM = \angle DAC = \angle EBF \Longrightarrow \triangle MXY \sim \triangle FEB \Longrightarrow \frac{XM}{EF} = \frac{YM}{BF} = \frac{1}{2}\frac{FC}{BF} = \frac{1}{2}\frac{CD}{AB}$. $(\star)$

Similarly, we can prove $\frac{XN}{EF} = \frac{1}{2}\frac{AB}{CD}$, so combine $(\star)$ we conclude that $\frac{MN}{EF} = \frac{XM}{EF} - \frac{XN}{EF} = \frac{1}{2}\left(\frac{CD}{AB} - \frac{AB}{CD}\right)$.

Q.E.D

Attachments:



Quick Reply

Spain

OMCC 2012, problema 3

Reply

Jutaro

#1 Jul 3, 2012, 7:40 am

Sean a, b, c números reales que satisfacen $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}=1$ y $ab+bc+ca>0$. Demostrar que

$$a+b+c-\frac{abc}{ab+bc+ca}\geq 4.$$

Anthony Erb (Puerto Rico)

Luis González

#2 Apr 28, 2015, 8:56 am

Con manipulación algebraica se tiene:

$$(a+b+c)(ab+bc+ca)-abc=(a+b)(b+c)(c+a)$$

La hipótesis del problema se puede reexpresar

$$(a+b)(b+c)(c+a)=(a+b)(b+c)+(b+c)(c+a)+(c+a)(a+b)$$

Ahora, más manipulación algebraica muestra que:

$$(a+b)(b+c)+(b+c)(c+a)+(c+a)(a+b)=a^2+b^2+c^2+3(ab+bc+ca)$$

Teniendo en cuenta la desigualdad $a^2+b^2+c^2\geq ab+bc+ca$, se tendrá

$$(a+b+c)(ab+bc+ca)-abc\geq 4(ab+bc+ca)$$

$$a+b+c-\frac{abc}{ab+bc+ca}\geq 4.$$

Quick Reply

High School Olympiads

Nice perpendicular

Reply

Source: PHCP

qthh213
#1 Apr 28, 2015, 12:07 am

let ABC be a triangle and M is the midpoint of BC. D is a point in $\triangle ABC$ and $\angle DBA = \angle DCA . DE \perp AC$ at E and $DF \perp AB$ at $F . P$ is a point on EF. $BP \cap ME$ at Q.Prove that $AQ \perp DP$.

tkhalid
#2 Apr 28, 2015, 4:56 am

Is this true for any point P on EF?

Luis González
#3 Apr 28, 2015, 5:38 am • 1

More general, let $D \equiv X$ and Y be two isogonal conjugates WRT $\triangle ABC$ and M is the projection of Y on BC.

If $S \equiv BX \cap ME,$ then easy angle chase reveals that $\angle BSM = \angle XAC \Longrightarrow S \in \odot(AFXE).$ Now if $U \equiv AQ \cap XP,$ then the intersections $P \equiv EF \cap UX, B \equiv AU \cap XS$ and $Q \equiv ES \cap AU$ are collinear. Thus by the converse of Pascal theorem on $EFAUXS,$ we have $U \in \odot(AFXE) \Longrightarrow \angle AUX = \angle AFX = 90^\circ$ or $AQ \perp XP.$

Quick Reply

High School Olympiads

P lies on Euler line iff P lies on Euler line

Reply

Source: Own

buratinogigle

#1 Apr 28, 2015, 1:05 am

Let ABC be a triangle inscribed circle (O). And (K) is Hagge circle of point P. H is orthocenter of triangle ABC. HA, HB, HC cut (K) again at X, Y, Z, resp. Prove that P lies on Euler line of ABC iff P lies on Euler line of XYZ.

Luis González

#2 Apr 28, 2015, 2:59 am

See Some properties of Hagge circle (Theorem 4). This follows from $\triangle ABC \cup P \sim \triangle XYZ \cup P$, so P is on Euler line of $\triangle ABC \Longleftrightarrow P$ is on Euler line of $\triangle XYZ$.

Quick Reply

High School Olympiads

Some properties of Hagge circle

TelvCohl
#1 Feb 11, 2015, 7:21 pm • 8

Some properties of **Hagge circle**

Theorem 1:

Let H be the orthocenter of $\triangle ABC$.
Let $\triangle A_1B_1C_1$ be the circumcevian triangle of P WRT $\triangle ABC$.
Let A_2, B_2, C_2 be the reflection of A_1, B_1, C_1 in BC, CA, AB, respectively .

Then A_2, B_2, C_2, H are concyclic .

Proof:

Let P^* be the isogonal conjugate of P WRT $\triangle ABC$.
Let $\triangle A_1^*B_1^*C_1^*$ be the circumcevian triangle of P^* WRT $\triangle ABC$.
Let B_2^*, C_2^* be the reflection of B_2, C_2 in the midpoint D of BC, respectively .
Let O be the circumcenter of $\triangle ABC$ and A^* be the antipode of A in $\odot(ABC)$.

Easy to see A_2, B_2, C_2 is the reflection of A_1^*, B_1^*, C_1^* in the midpoint of BC, CA, AB, respectively .

Since $CB_2AB_1^*, CB_2BB_2^*$ are parallelogram ,
so $ABB_2^*B_1^*$ is a parallelogram $\Longrightarrow$ the midpoint of AB_2^* is the projection of O on BP^* .
Similarly we can get the midpoint of AC_2^* is the projection of O on CP^* ,
so the midpoint of $AA_1^*, AB_2^*, AC_2^*, AA^*$ lie on a circle with diameter OP^* ,
hence after doing homothety $\mathbf{H}(A, 2)$ we get A_1^*, B_2^*, C_2^*, A^* are concyclic .

Notice that A_1^*, B_2^*, C_2^*, A^* is the reflection of A_2, B_2, C_2, H in D , respectively ,
so we get A_2, B_2, C_2, H are concyclic . $\blacksquare$

The circle in **Theorem 1** is called the Hagge circle $\mathcal{H}_P$ of P WRT $\triangle ABC$.

Theorem 2:

The center T of $\mathcal{H}_P$ is the reflection of P^* in the 9-point center N of $\triangle ABC$.

Proof:

Let Q be the image of P^* under homothety $\mathbf{H}(A, 2)$.

From the proof of **Theorem 1** we get the reflection of Q in D is the antipode of H in $\mathcal{H}_P$,
so from $HT = \frac{1}{2}QA^* = P^*O$ and $HT \parallel QA^* \parallel P^*O \Longrightarrow P^*OTH$ is a parallelogram ,
hence we get T is the reflection of P^* in the midpoint N of OH . $\blacksquare$

Theorem 3:

Let $A_3 = \mathcal{H}_P \cap AH, B_3 = \mathcal{H}_P \cap BH, C_3 = \mathcal{H}_P \cap CH$.

Then $P \in A_2A_3, P \in B_2B_3, P \in C_2C_3$.

Proof:

Lemma:

Let S, S^* be the isogonal conjugate of $\triangle ABC$.
Let $X = AS \cap \odot(ABC), X^* = AS^* \cap \odot(ABC), V = AS^* \cap BC$.

Then $\frac{AS}{SX} = \frac{S^*V}{VX^*}$.

Proof of the lemma:

Since $\angle VCX^* = \angle BAX^* = \angle XAC$,
so we get $\triangle VCX^* \sim \triangle CAX$ . ... (1)
Since $\angle X^*S^*C = \angle S^*AC + \angle S^*CA = \angle BCX + \angle SCB = \angle SCX$,
so we get $\triangle CX^*S^* \sim \triangle SXC$ . ... (2)

From $(1), (2) \Longrightarrow X^*V \cdot XA = X^*C \cdot XC = X^*S^* \cdot XS$. i.e. $\frac{AS}{SX} = \frac{S^*V}{VX^*}$

Back to the main proof:

Let H^* be the antipode of H in $\mathcal{H}_P$.

From symmetry it suffices to prove $P \in A_2A_3$.

Since $OP^* \parallel HH^*$ and $HH^* = 2OP^*$,
so H^* is the anti-complement of P^* WRT $\triangle ABC$,
hence from $H^*A_3 \parallel BC$ we get $AA_3 = 2 \cdot \text{dist}(P^*, BC)$,

so combine with the lemma $\Longrightarrow \frac{AP}{PA_1} = \frac{\text{dist}(P^*, BC)}{\text{dist}(A_1^*, BC)} = \frac{\text{dist}(P^*, BC)}{\text{dist}(A_1, BC)} = \frac{AA_3}{A_1A_2} \Longrightarrow P \in A_2A_3$. ■

Theorem 4:

$\triangle ABC \cup \triangle A_1B_1C_1 \cup P \sim \triangle A_3B_3C_3 \cup \triangle A_2B_2C_2 \cup P$.

Proof:

Let A_4, B_4, C_4 be the midpoint of AA_1^*, AB_2^*, AC_2^*, respectively .

From the proof of **Theorem 1** we get A_4, B_4, C_4 is the projection of O on AP^*, BP^*, CP^*, respectively .

Since $\angle A_4B_4C_4 = \angle A_4P^*C_4 = \angle(AP^*, CP^*) = \angle BAP + \angle BCP = \angle A_1B_1C_1$ (Similarly $\angle B_4C_4A_4 = \angle B_1C_1A_1$) ,
so we get $\triangle A_1B_1C_1 \sim \triangle A_4B_4C_4 \Longrightarrow \triangle A_1B_1C_1 \sim \triangle A_1^*B_2^*C_2^* \sim \triangle A_2B_2C_2$.

Since $\angle C_2A_2A_3 = \angle C_2HA = \angle(OC_4, HA) = \angle P^*CB = \angle ACP = \angle C_1A_1A$,
so we get $\triangle A_1B_1C_1 \cup A \sim \triangle A_2B_2C_2 \cup A_3$ (Similar discussion for (B, B_3) and (C, C_3)) ,
hence combine with **Theorem 3** $\Longrightarrow \triangle ABC \cup \triangle A_1B_1C_1 \cup P \sim \triangle A_3B_3C_3 \cup \triangle A_2B_2C_2 \cup P$. ■

Some topics related to **Hagge circle**:
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=444395
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=320075
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=318484
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=50&t=329272

jayme
#2 Feb 11, 2015, 8:35 pm

Dear Mathlinkers,

thank to remind to us the Hagge circle which appear as a generalization of the Fuhrmann's circle.
For me it appears also like a Mannheim's circle...

see: http://jl.ayme.pagesperso-orange.fr/Docs/Le%20cercle%20de%20Hagge.pdf
where you can find Theorem 1, 2, 3 and perhaps 4? with some examples and extension...

Sincerely
Jean-Louis

Jean-Louis

mhq

#3 Aug 4, 2015, 4:37 pm

> TelvCohl wrote:
>
> Some properties of **Hagge circle** :)
>
> **Theorem 1**:
>
> Let H be the orthocenter of $\triangle ABC$.
> Let $\triangle A_1B_1C_1$ be the circumcevian triangle of P WRT $\triangle ABC$.
> Let A_2, B_2, C_2 be the reflection of A_1, B_1, C_1 in BC, CA, AB, respectively .
>
> Then A_2, B_2, C_2, H are concyclic .
>
> **Proof**:
>
> Let P^* be the isogonal conjugate of P WRT $\triangle ABC$.
> Let $\triangle A_1^*B_1^*C_1^*$ be the circumcevian triangle of P^* WRT $\triangle ABC$.
> Let B_2^*, C_2^* be the reflection of B_2, C_2 in the midpoint D of BC, respectively .
> Let O be the circumcenter of $\triangle ABC$ and A^* be the antipode of A in $\odot(ABC)$.
>
> Easy to see A_2, B_2, C_2 is the reflection of A_1^*, B_1^*, C_1^* in the midpoint of BC, CA, AB, respectively .
>
> Since $CB_2AB_1^*, CB_2BB_2^*$ are parallelogram ,
> so $ABB_2^*B_1^*$ is a parallelogram $\Longrightarrow$ the midpoint of AB_2^* is the projection of O on BP^* .
> Similarly we can get the midpoint of AC_2^* is the projection of O on CP^* ,
> so the midpoint of $AA_1^*, AB_2^*, AC_2^*, AA^*$ lie on a circle with diameter OP^* ,
> hence after doing homothety $\mathbf{H}(A, 2)$ we get A_1^*, B_2^*, C_2^*, A^* are concyclic .
>
> Notice that A_1^*, B_2^*, C_2^*, A^* is the reflection of A_2, B_2, C_2, H in D , respectively ,
> so we get A_2, B_2, C_2, H are concyclic . $\blacksquare$
>
> The circle in **Theorem 1** is called the Hagge circle $\mathcal{H}_P$ of P WRT $\triangle ABC$.
>
> ---
>
> **Theorem 2**:
>
> The center T of $\mathcal{H}_P$ is the reflection of P^* in the 9-point center N of $\triangle ABC$.
>
> **Proof**:
>
> Let Q be the image of P^* under homothety $\mathbf{H}(A, 2)$.
>
> From the proof of **Theorem 1** we get the reflection of Q in D is the antipode of H in $\mathcal{H}_P$,
> so from $HT = \frac{1}{2}QA^* = P^*O$ and $HT \parallel QA^* \parallel P^*O \Longrightarrow P^*OTH$ is a parallelogram ,
> hence we get T is the reflection of P^* in the midpoint N of OH . $\blacksquare$
>
> ---
>
> **Theorem 3**:
>
> Let $A_3 = \mathcal{H}_P \cap AH, B_3 = \mathcal{H}_P \cap BH, C_3 = \mathcal{H}_P \cap CH$.
>
> Then $P \in A_2A_3, P \in B_2B_3, P \in C_2C_3$.
>
> **Proof**:
>
> **Lemma**:
>
> Let S, S^* be the isogonal conjugate of $\triangle ABC$.
> Let $X = AS \cap \odot(ABC), X^* = AS^* \cap \odot(ABC), V = AS^* \cap BC$.
>
> Then $\dfrac{AS}{SX} = \dfrac{S^*V}{VX^*}$.
>
> **Proof of the lemma**:
>
> Since $\angle VCX^* = \angle BAX^* = \angle XAC$

Since $\angle VCX^* = \angle BAX^* = \angle XAC$,
so we get $\triangle VCX^* \sim \triangle CAX$ . ... (1)
Since $\angle X^*S^*C = \angle S^*AC + \angle S^*CA = \angle BCX + \angle SCB = \angle SCX$,
so we get $\triangle CX^*S^* \sim \triangle SXC$ . ... (2)

From $(1), (2) \Longrightarrow X^*V \cdot XA = X^*C \cdot XC = X^*S^* \cdot XS$. i.e. $\frac{AS}{SX} = \frac{S^*V}{VX^*}$

Back to the main proof:

Let H^* be the antipode of H in $\mathcal{H}_P$.

From symmetry it suffices to prove $P \in A_2A_3$.

Since $OP^* \parallel HH^*$ and $HH^* = 2OP^*$,
so H^* is the anti-complement of P^* WRT $\triangle ABC$,
hence from $H^*A_3 \parallel BC$ we get $AA_3 = 2 \cdot \text{dist}(P^*, BC)$,

so combine with the lemma $\Longrightarrow \frac{AP}{PA_1} = \frac{\text{dist}(P^*, BC)}{\text{dist}(A_1^*, BC)} = \frac{\text{dist}(P^*, BC)}{\text{dist}(A_1, BC)} = \frac{AA_3}{A_1A_2} \Longrightarrow P \in A_2A_3$. ■

Theorem 4:

$\triangle ABC \cup \triangle A_1B_1C_1 \cup P \sim \triangle A_3B_3C_3 \cup \triangle A_2B_2C_2 \cup P$.

Proof:

Let A_4, B_4, C_4 be the midpoint of AA_1^*, AB_2^*, AC_2^*, respectively .

From the proof of **Theorem 1** we get A_4, B_4, C_4 is the projection of O on AP^*, BP^*, CP^*, respectively .

Since $\angle A_4B_4C_4 = \angle A_4P^*C_4 = \angle(AP^*, CP^*) = \angle BAP + \angle BCP = \angle A_1B_1C_1$ (Similarly $\angle B_4C_4A_4 = \angle B_1C_1A_1$) ,
so we get $\triangle A_1B_1C_1 \sim \triangle A_4B_4C_4 \Longrightarrow \triangle A_1B_1C_1 \sim \triangle A_1^*B_2^*C_2^* \sim \triangle A_2B_2C_2$.

Since $\angle C_2A_2A_3 = \angle C_2HA = \angle(OC_4, HA) = \angle P^*CB = \angle ACP = \angle C_1A_1A$,
so we get $\triangle A_1B_1C_1 \cup A \sim \triangle A_2B_2C_2 \cup A_3$ (Similar discussion for (B, B_3) and (C, C_3)) ,
hence combine with **Theorem 3** $\Longrightarrow \triangle ABC \cup \triangle A_1B_1C_1 \cup P \sim \triangle A_3B_3C_3 \cup \triangle A_2B_2C_2 \cup P$. ■

Some topics related to **Hagge circle**:
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=444395
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=320075
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=318484
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=50&t=329272

Could you please help me to find any information about anticompliment?

Quick Reply

High School Olympiads

Generalization of Schiffer point

Reply

Source: Own

buratinogigle

#1 Apr 28, 2015, 12:55 am • 1

Let ABC be a triangle inscribed circle (O) and P, Q are two isogonal conjugate points. Let DEF, XYZ be circumcevian triangle of P, Q, resp. Let U, V, W are isogonal conjugate of X, Y, Z with respect to triangle PBC, PCA, PCA, resp. Prove that DU, EV, FW are concurrent.

Luis González

#2 Apr 28, 2015, 2:13 am

Let PA, PB, PC cut EF, FD, DE at D', E', F', resp. $\widehat{PBU} = \widehat{XBC} = \widehat{PED} \Longrightarrow UB \parallel DE$ and similarly $UC \parallel DF$. Thus if $M \equiv DE \cap UC, N \equiv DF \cap UB$, then $DMUN$ is parallelogram, i.e. DU is the D-median of $\triangle DMN$. Hence

$$\frac{DF'}{DM} = \frac{FF'}{FC}, \ \frac{DE'}{DN} = \frac{EE'}{EB} \Longrightarrow \frac{DM}{DN} = \frac{EE'}{EB} \cdot \frac{FC}{FF'} \cdot \frac{DF'}{DE'} \Longrightarrow$$

$$\frac{DM}{DN} = \left[\frac{DE'}{DB} \cdot \frac{\sin \widehat{EDF}}{\sin \widehat{EDB}}\right] \cdot \left[\frac{DC}{DF'} \cdot \frac{\sin \widehat{FDC}}{\sin \widehat{EDF}}\right] \cdot \frac{DF'}{DE'} = \frac{DC}{DB} \cdot \frac{CF}{BE} \Longrightarrow$$

$$\frac{\sin \widehat{FDU}}{\sin \widehat{EDU}} = \frac{DM}{DN} = \frac{DC}{DB} \cdot \frac{CF}{BE}.$$

Hence, multiplying the cyclic expressions together, we conclude by trig Ceva that DU, EV, FW concur.

Quick Reply

High School Olympiads

Circumcenter lies on circumcircle

Reply

Source: Own

livetolove212

#1 Apr 26, 2015, 12:03 pm

Given a triangle ABC with circumcircle (O), incenter I. Let l_1, l_2 be two arbitrary lines through I. l_1, l_2 intersect (BIC) at A_1, A_2. Similarly we define B_1, B_2, C_1, C_2. Let XYZ be the triangle formed by the intersections of A_1A_2, B_1B_2, C_1C_2. Prove that the circumcenter of XYZ lies on (ABC).

Note that when$l_1 \perp l_2$, A_1A_2, B_1B_2, C_1C_2 concur at a point on (O). Then take an inversion center I we get Droz-Farny line.

SalaF

#2 Apr 26, 2015, 4:15 pm • 1

Very nice problem. Here is my solution:

Let D, E, F be the centers of $\odot(IBC), \odot(ICA), \odot(IAB)$. Clearly they are the midpoints of the arcs $\widehat{BC}, \widehat{CA}, \widehat{AB}$ of $\odot(ABC)$.
Let P be the second intersection of $\odot(BC_1A_1), \odot(CB_1A_1)$: we have
$\angle(BP, PC) = \angle(BP, PA_1) + \angle(PA_1, PC) = \angle(C_1B, \ell_1) + \angle(\ell_1, B_1C) = \angle(AB, AI) + \angle(AI, AC) = \angle(AB, AC)$
so that $P \in \odot(ABC)$. We immediately conclude that also $P \in \odot(AB_1C_1)$. Moreover
$\angle(PB, PA_1) = \angle(C_1B, C_1I) = \angle(AB, AD) = \angle(PB, PD) \implies P, A_1, D$ are collinear. Similarly, $P \in EB_1, FC_1$.
Furthermore, $\angle(YC_1, YA_1) = \angle(C_2C_1, \ell_2) + \angle(\ell_2, A_2A_1) = \angle(BC_1, BI) + \angle(BI, BA_1) = \angle(BC_1, BA_1)$ and $Y \in \odot(BA_1C_1)$. We also get analogous results for Y, Z.
Analogously we can define Q to be the common point of
$\odot(ABC), \odot(AB_2C_2X), \odot(BA_2C_2Y), \odot(CA_2B_2Z), DA_2, EB_2, FC_2$.
Now we have
$\angle(PZ, QZ) = \angle(PZ, ZA_1) + \angle(ZY, ZX) + \angle(ZB_1, QZ) = \angle(B_1P, \ell_1) + \angle(PA_1, PB_1) + \angle(\ell_2, QA_2)$
(keeping in mind that $Z \in \odot(PA_1B_1)) = \angle(\ell_2, \ell_1) + \angle(PA_1, QA_2)$. But $PA_1 \cap QA_1 = D$ so that
$\angle(PA_1, QA_2) = \angle(DA_1, DA_2) = 2\angle(IA_1, IA_2) = 2\angle(\ell_1, \ell_2)$. This implies
$\angle(PZ, QZ) = \angle(\ell_2, \ell_1) + 2\angle(\ell_1, \ell_2) = \angle(\ell_1, \ell_2)$. This is independent of Z and also happens for
$X, Y \implies P, Q, X, Y, Z$ are concyclic. Finally let U be the circumcenter of XYZ: we have
$\angle(UP, UQ) = 2\angle(ZP, ZQ) = 2\angle(\ell_1, \ell_2) = \angle(DP, DQ)$ and we conclude that $U \in \odot(DPQ) \equiv \odot(ABC)$ as we wanted.

TelvCohl

#4 Apr 26, 2015, 6:52 pm • 1

My solution:

Let $M_1 = \odot(ABC) \cap \odot(AB_1C_1), M_2 = \odot(ABC) \cap \odot(AB_2C_2)$.

Since C is the Miquel point of the complete quadrilateral $\{\ell_1, \ell_2, A_1A_2, B_1B_2\}$,
so we get $C \in \odot(A_1B_1Z)$ and $C \in \odot(A_2B_2Z)$.
Similarly, we can prove $B \in \odot(C_1A_1Y), B \in \odot(C_2A_2Y), A \in \odot(B_1C_1X), A \in \odot(B_2C_2X)$.

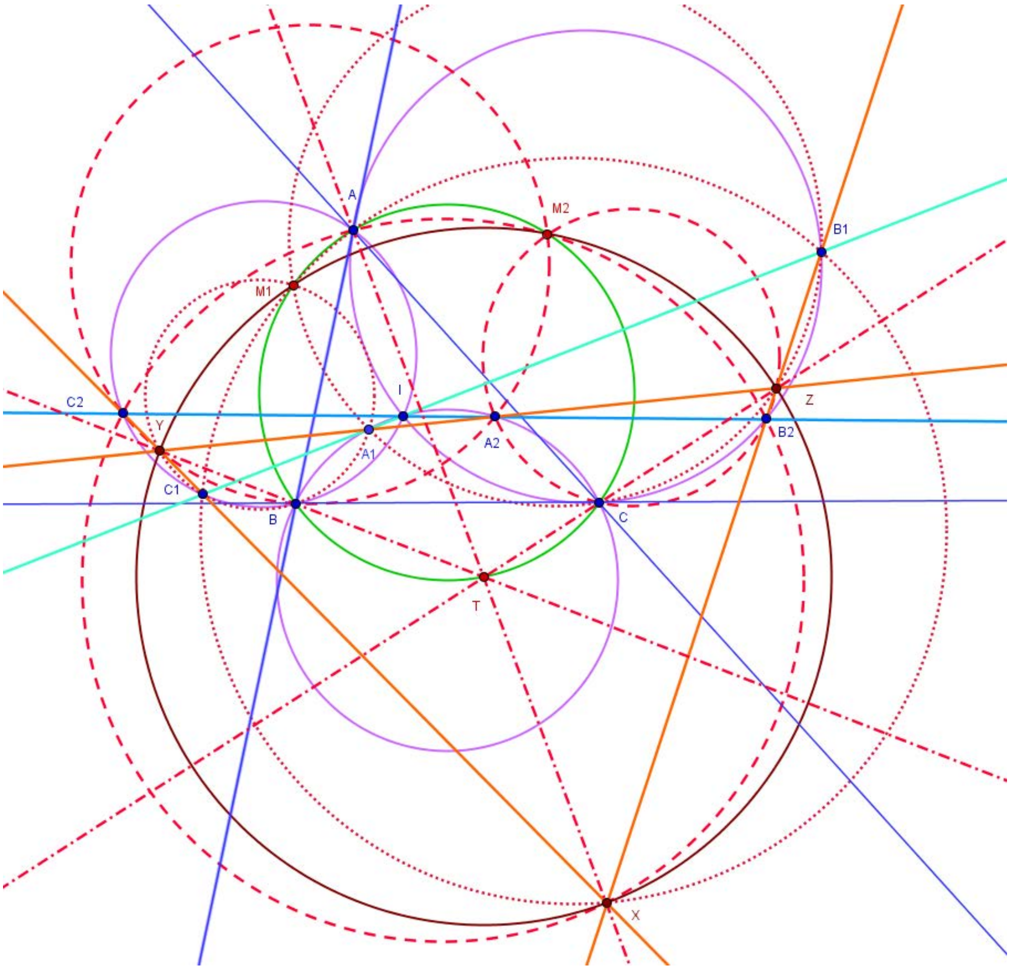
Since $\angle CM_2B_2 = \angle AM_2B_2 - \angle AM_2C = \angle CBA - \angle B_2C_2A = \frac{1}{2}\angle CBA$,
so from $\angle CA_2B_2 = \angle CBI$ we get $M_2 \in \odot(CA_2B_2)$ (Similarly, $M_2 \in \odot(BC_2A_2)$) ,
hence M_2 is the Miquel point of the complete quadrilateral $\{A_1A_2, B_1B_2, C_1C_2, \ell_2\} \implies M_2 \in \odot(XYZ)$.
Similarly, we can prove M_1 is the Miquel point of complete quadrilateral $\{A_1A_2, B_1B_2, C_1C_2, \ell_1\}$ and $M_1 \in \odot(XYZ)$.

Since $\angle BYZ + \angle CZY = \angle BM_2A_2 + \angle CM_2A_2 = \angle BAC$,
so we get $CTB = 180^\circ - \angle BAC$ where $T = BY \cap CZ \implies T \in \odot(ABC)$.
Similarly, we can prove $CZ \cap AX \in \odot(ABC) \implies AX, BY, CZ$ are concurrent at $T \in \odot(ABC)$.

Since $\angle ZXY = 180^\circ - \angle C_2AB_2 = 180^\circ - \angle C_2IB - \angle BAC - \angle CIB_2 = 90^\circ - \frac{1}{2}\angle BAC$,
so $\angle YTZ = 2\angle YXZ$ (Similarly, $\angle ZTX = 2\angle ZYX, \angle XTY = 2\angle XZY$) $\Longrightarrow T$ is the circumcenter of $\triangle XYZ$.

Q.E.D

Attachments:



Luis González

#5 Apr 27, 2015, 11:40 pm • 2

A_1A_2 cuts B_1B_2, C_1C_2 at Y, Z resp and $J \equiv BZ \cap CY, D \equiv CB_1 \cap BC_1$. Since $\widehat{CA_1A_2} = \widehat{CIA_2} = \widehat{CB_1B_2} \Longrightarrow$ A_1YB_1C is cyclic $\Longrightarrow \widehat{CYA_1} = \widehat{CB_1I} = \frac{1}{2}\widehat{A}$ and similarly $\widehat{BZA_2} = \frac{1}{2}\widehat{A} \Longrightarrow \triangle JYZ$ is J-isosceles with apex angle $180^\circ - \widehat{A} \Longrightarrow J \in (O)$.

In the same way we have $D \in (O)$. Thus since $\widehat{XB_2I} = \widehat{DCI}$ and $\widehat{XC_2I} = \widehat{DBI} \Longrightarrow$
$\widehat{XB_2I} + \widehat{XC_2I} = 360^\circ - \widehat{BIC} - \widehat{BDC} = 360^\circ - (90^\circ + \frac{1}{2}\widehat{A}) - (180^\circ - \widehat{A}) = 90^\circ + \frac{1}{2}\widehat{A} \Longrightarrow$
$\widehat{YXZ} = 90^\circ - \frac{1}{2}\widehat{A} = \frac{1}{2}\widehat{YJZ}$, which means that J is the circumcenter of $\triangle XYZ$.

Quick Reply

High School Olympiads

Simple geometry

Source: 2015 Flanders Mathematical Olympiad, Problem 2

gnej

#1 Apr 27, 2015, 9:12 pm

Consider two points Y and X in a plane and a variable point P which is not on XY. Let the parallel line to YP through X intersect the internal angle bisector of $\angle XYP$ in A, and let the parallel line to XP through Y intersect the internal angle bisector of $\angle YXP$ in B. Let AB intersect XP and YP in S and T respectively. Show that the product $|XS| * |YT|$ does not depend on the position of P.

Luis González

#2 Apr 27, 2015, 10:21 pm

Since $\angle XYA = \angle PYA = \angle YAX \Longrightarrow \triangle XAY$ is isosceles wih legs $XY = XA$ and similarly $\triangle YBX$ is isosceles with $XY = YB$. Now if $Q \equiv AX \cap BY$, we have then

$$\frac{XS}{QB} = \frac{AX}{AQ} = \frac{XY}{AQ}, \ \frac{YT}{AQ} = \frac{BY}{QB} = \frac{XY}{QB} \Longrightarrow XS \cdot YT = XY^2.$$

sunken rock

#3 Apr 29, 2015, 1:35 pm

There is no actually need of Q: simply, $\triangle AXS \sim \triangle TYB$, as having parallel sides, wherefrom $\frac{AX}{TY} = \frac{SX}{BY} \iff SX \cdot TY = AX \cdot BY = XY^2$ (taking into account the above proven $AX = XY = BY$).

Best regards,
sunken rock

High School Olympiads

collinear

Reply

Source: 2015 Taiwan TST Round 3 Quiz 1 P2

wanwan4343
#1 Apr 27, 2015, 9:23 am

Let O be the circumcircle of the triangle ABC. Two circles O_1, O_2 are tangent to each of the circle O and the rays $\overrightarrow{AB}$, $\overrightarrow{AC}$, with O_1 interior to O, O_2 exterior to O. The common tangent of O, O_1 and the common tangent of O, O_2 intersect at the point X. Let M be the midpoint of the arc BC (not containing the point A) on the circle O, and the segment $\overline{AA'}$ be the diameter of O. Prove that X, M, and A' are collinear.

Luis González
#3 Apr 27, 2015, 9:57 am • 1

Let P, Q be the tangency points of (O_1) and (O_2) with (O). It's well-known that AP and AQ go through the exsimilicenter U and insimilicenter V of (O) and the incircle (I) (for this use Monge & d'Alembert theorem for $(I), (O), (O_1)$ and similarly for $(I), (O), (O_2)$). Since $(O, I, U, V) = -1$, then the pencil $A(A', M, P, Q)$ is harmonic $\Longrightarrow$ quadrilateral $PMQA'$ is harmonic $\Longrightarrow X \in MA'$.

TelvCohl
#4 Apr 27, 2015, 10:11 am

My solution:

Let I, I_a be the Incenter, A-excenter of $\triangle ABC$, respectively .
Let $T = AI \cap BC$ and H be the projection of A on BC .
Let Y, Z be the tangent point of $\odot(I), \odot(I_a)$ with BC, respectively .
Let Y^*, Z^* be the tangent point of $\odot(O_2), \odot(O_1)$ with $\odot(O)$, respectively .

Since $\{AY, AY^*\}, \{AZ, AZ^*\}$ are isogonal conjugate of $\angle A$ (well-known) ,
so from $(A, T; I, I_a) = -1 \Longrightarrow A(H, T; Y, Z) = -1 \Longrightarrow A(A', M; Y^*, Z^*) = -1$,
hence we get $A'Y^*MZ^*$ is a harmonic quadrilateral $\Longrightarrow X, M$ and A' are collinear .

Q.E.D

YaWNeeT
#5 Apr 27, 2015, 12:01 pm

X is the radical center of three circles,that is,to prove $A'M$ is radical axis of O_1, O_2,and it is not hard by using mannheim theorem and chicken claw theorem.

polya78
#6 Aug 3, 2015, 11:30 pm

Let I, J be incenter and A-excenter of $\triangle ABC$. Let U, V be the points of tangency of O_1, O_2 and $\overrightarrow{AB}$. Let f be the inversion followed by reflection about AM such that $f(B) = B, f(C) = C$. Then f brings O_1, O_2 into the A-excircle and incircle of $\triangle ABC$ respectively, and also $f(I) = J$, so we have that $IU, JV \perp AM$. Since M is the midpoint of IJ, it follows that if W is the midpoint of UV, $MW \perp AM$. W and its corresponding point on $\overrightarrow{AC}$ are on the radical axis of O_1, O_2. Since $A'M \perp AM$, it follows that M, A', X are all on this radical axis.

Quick Reply

High School Olympiads

Kiepert hyperbola Problem 1

Reply

Source: Own

VUThanhTung

#1 Apr 27, 2015, 12:23 am

Let k be the Kiepert hyperbola of a triangle ABC. Let K_1, K_2, K_3 be 3 points on k. Prove that the orthocenter of $K_1K_2K_3$ also lie on k.

Luis González

#2 Apr 27, 2015, 12:27 am • 1

It isn't a special property of the Kiepert hyperbola. In general, a hyperbola through 3 points X,Y,Z is rectangular if and oly if it passes through the orthocenter of XYZ.

TelvCohl

#3 Apr 27, 2015, 3:22 am • 1

You can see here for the proof

Quick Reply

High School Olympiads

Common tangent passes through antipodal point

Reply

Source: Own

buratinogigle

#1 Apr 27, 2015, 12:13 am

Let ABC be a triangle inscribed circle (O) with altitude AH. Circle (K) is tangent to segment HA, HB and (O). Circle (L) is tangent to segment HA, HC and (O). Prove that the common internally tangent other than AH of (K) and (L) passes through symmetric point of A through O.

Reference

http://www.artofproblemsolving.com/community/c6h88823

http://artofproblemsolving.com/community/c6h210518p1159893

http://www.artofproblemsolving.com/community/c6t48f6h1082284_abackbc

buratinogigle

#2 Apr 27, 2015, 12:17 am

More general problem

Let ABC be a triangle inscribed circle (O) and D is a point on segment BC. Circle (K) is tangent to segment DA, DB and (O). Circle (L) is tangent to segment DA, DC and (O). AD cuts (O) again at E. F lies on (O) such that $EF \parallel BC$. Prove that the common internally tangent other than AD of (K) and (L) passes through F.

Luis González

#3 Apr 27, 2015, 12:22 am • 1

> *buratinogigle wrote:*
> More general problem
>
> Let ABC be a triangle inscribed circle (O) and D is a point on segment BC. Circle (K) is tangent to segment DA, DB and (O). Circle (L) is tangent to segment DA, DC and (O). AD cuts (O) again at E. F lies on (O) such that $EF \parallel BC$. Prove that the common internally tangent other than AD of (K) and (L) passes through F.

This is nothing but another formulation of the "Parallel tangent theorem". See http://www.artofproblemsolving.com/community/c6h15945, http://www.artofproblemsolving.com/community/c6h430441 and elsewhere.

Quick Reply

High School Olympiads

Find the maximum value of AB+BC+CA

Reply

thanhnam2902

#1 Apr 26, 2015, 6:22 am

Let (O) is a circle, let A is the fixed point and A lie on (O). Let B, C are change points and B, C lie on (O). Find the location of B, C for $AB + BC + CA$ is maximum.

Luis González

#2 Apr 26, 2015, 9:55 pm

If R denotes the radius of (O), we have $BC = 2R \cdot \sin \widehat{A}, CA = 2R \cdot \sin \widehat{B}$ and $AB = 2R \cdot \sin \widehat{C} \Longrightarrow$ $BC + CA + AB = 2R \cdot (\sin \widehat{A} + \sin \widehat{B} + \sin \widehat{C})$. Since the sine function is concave in the interval $(0, \pi)$, then by Jensen inequality we get

$$\sin \widehat{A} + \sin \widehat{B} + \sin \widehat{C} \le 3 \sin \left(\frac{\widehat{A} + \widehat{B} + \widehat{C}}{3} \right) = 3 \sin 60^\circ = \frac{3\sqrt{3}}{2} \Longrightarrow$$

Therefore $BC + CA + AB \le 3\sqrt{3}R$ with equality iff $\widehat{A} = \widehat{B} = \widehat{C}$, i.e. iff $\triangle ABC$ is equilateral. The construction of B and C is now straightforward.

Quick Reply

High School Olympiads

Own invention (maybe)

drmzjoseph

#1 Apr 26, 2015, 9:12 pm

Let D, E and F be points on the sides BC, CA and AB of a triangle ABC respectively. If B and C are conjugate points with respect to $\odot(AFE)$, If C and A are conjugate points with respect to $\odot(FBD)$. Prove that A and B are conjugate points with respect to $\odot(DEC)$

Luis González

#2 Apr 26, 2015, 9:52 pm • 1

B, C are conjugate points WRT $\odot(AFE) \Longleftrightarrow P \equiv BE \cap CF$ is on $\odot(AEF)$ and likewise C, A are conjugate points WRT $\odot(FBD) \Longleftrightarrow Q \equiv AD \cap CF$ is on $\odot(FBD)$. Hence $\angle ADB = \angle AFC = \angle BEC \Longrightarrow R \equiv AD \cap BE$ is on $\odot(DEC) \Longrightarrow A, B$ are conjugate points WRT $\odot(DEC)$.

TelvCohl

#5 Apr 26, 2015, 9:57 pm • 1

My solution:

Let H be the orthocenter of $\triangle ABC$.
Let A', B', C' be the midpoint of BC, CA, AB, respectively .
Let X, Y, Z be the projection of H on AA', BB', CC', respectively .

Since $\odot(BC) \perp \odot(AEF), \odot(BC) \perp \odot(AH)$,
so AA' is the radical axis of $\{\odot(AEF), \odot(AH)\} \Longrightarrow X \in \odot(AEF)$.

Similarly, we can prove $Y \in \odot(BFD)$,
so from Mannheim theorem we get $Z \in \odot(CDE) \Longrightarrow \odot(CDE) \perp \odot(AB)$,
hence we conclude that A, B are conjugate points WRT $\odot(CDE)$.

Q.E.D

High School Olympiads

tangent to the circle

Reply

LeVietAn

#1 Apr 26, 2015, 9:37 am

Let ABC be a triangle with circumcircle (O). AB, AC cuts $\odot(OBC)$ again at A_b, A_c . BC, BA cuts $\odot(OCA)$ again at B_c, B_a. CA, CB cuts $\odot(OAB)$ again at C_a, C_b. Show that the circumcircle of the triangle determined by the lines A_bA_c, B_cB_a and C_aC_b is tangent to the circle (O).

Luis González

#2 Apr 26, 2015, 9:56 am • 1

Note that A_b, A_c are just the intersections of the perpendicular bisectors of AC, AB with AB, AC, resp. Now see the problem The circle tangents to the the circumcircle.

Quick Reply

High School Olympiads

The circle tagents to the the circumcircle

yumeidesu
#1 Jan 6, 2014, 9:05 am

Let ABC be a triangle. Suppose that perpendicular bisector of AB cuts AC at A_1, perpendicular bisector of AC cuts AB at A_2. Similarly define B_1, B_2, C_1, C_2. Three lines A_1A_2, B_1B_2, C_1C_2 cut each other define a triangle DEF. Prove that (ABC) tangent to (DEF).

Luis González
#2 Jan 6, 2014, 9:51 am • 2

Let (O) denote the circumcircle of $\triangle ABC$ and $\triangle XYZ$ its tangential triangle; X, Y, Z againts A, B, C, respectively.

In any triangle, it's well known that the internal bisector issuing from one vertex, the polar of another vertex WRT its incircle and the midline referent to the third concur. Hence, $A_1 \equiv ZO \cap CA$ and $A_2 \equiv YO \cap AB$ lie on the X-midline of $\triangle XYZ \Longrightarrow$ A_1A_2 is X-midline of $\triangle XYZ.$ Similarly, B_1B_2 and C_1C_2 are the Y- and Z- midline of $\triangle XYZ \Longrightarrow \triangle DEF$ is medial triangle of $\triangle XYZ.$ Hence, by Feuerbach theorem (O) and $\odot(DEF)$ are tangent.

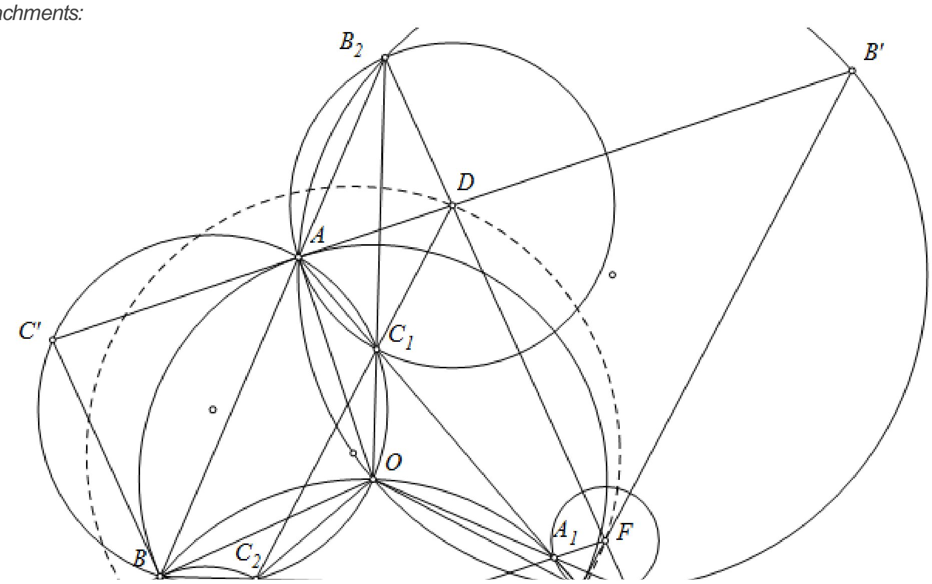
jayme
#3 Jan 6, 2014, 8:05 pm

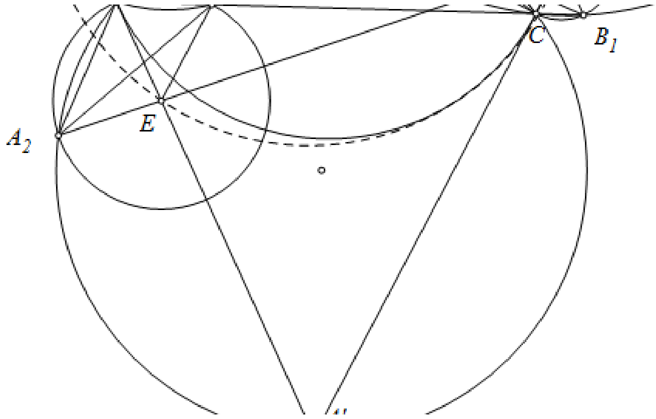
very nice proof Dear Luis
Jean-Louis

livetolove212
#4 Apr 26, 2015, 11:13 am • 1

We know that $A_1, A_2 \in (BOC), B_1, B_2 \in (AOC), C_1, C_2 \in (AOB).$
$\angle DC_2B_1 = \angle BAC = \angle DB_1C_2$ then $\angle OC_1C_2 = 90^\circ - \angle C_1C_2C = 90^\circ - \angle C_2B_1B_2 = \angle C_1B_2D$. We get $DB_2 = DC_1$. Moreover, $\angle B_2DC_1 = 2\angle DC_2B_1 = 360^\circ - 2\angle B_2AC_1$ then D is the circumcenter of triangle $B_2AC_1.$
This means $\angle DAC_1 = \angle DC_1A = \angle ABC$ or DA is the tangent of $(O).$ Let $A'B'C'$ be the triangle formed by the intersections of $AD, BE, CF.$ We get (O) is the incircle of $A'B'C'.$ On the other side,
$A_1A_2 \parallel AD, B_1B_2 \parallel BE, C_1C_2 \parallel CF$ then DEF is the median triangle of $A'B'C'$. Therefore (DEF) is tangent to (O) at Feuerbach point of triangle $A'B'C'.$

Attachments:





This post has been edited 1 time. Last edited by livetolove212, Apr 26, 2015, 11:58 am
Reason: abc

Quick Reply

High School Olympiads

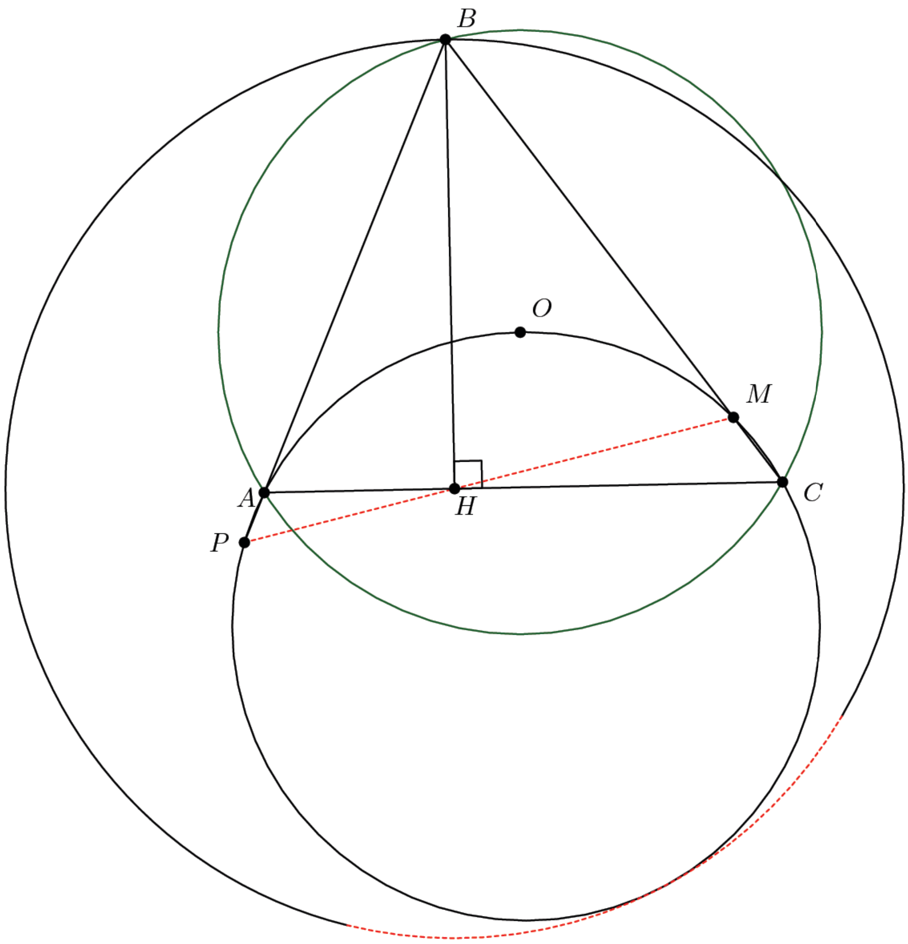
Own invention

Reply

drmzjoseph

#1 Apr 26, 2015, 7:19 am

Let ABC be an acute-angled triangle with circumcenter O, Let H be foot of the altitude from B and let ω be the circle of radius BH and center H. If $\odot(AOC)$ cut again to AB and BC at P and M respectively. Prove that P, H and M are collinear $\Longleftrightarrow$ ω and $\odot(AOC)$ are tangents.



This post has been edited 1 time. Last edited by drmzjoseph, Apr 29, 2015, 3:28 am
Reason: Typo

tkhalid

#3 Apr 26, 2015, 9:18 am

Part 1

Luis González

#4 Apr 26, 2015, 9:45 am • 1

Assume that ω and $\odot(AOC)$ are tangent. Inversion with center B and power $BM\cdot BC=BP\cdot BA$ fixes $\odot(AOC)$ and carries AC into $(K)\equiv\odot(BMP)$. Hence, by conformity ω goes to the perpendicular to BH at K tangent to $\odot(AOC)\Longrightarrow$ $OK\perp BH\Longrightarrow BK$ is radical axis of $\odot(ABC),(K)\Longrightarrow BK,PM,AC$ are pairwise radical axes of $\odot(ABC)$, $\odot(AOC),(K)$ concurring at its radical center H, i.e. P,H,M are collinear.

Conversely, if P, H, M are collinear $\Longrightarrow BH$ is radical axis of $(K), \odot(ABC) \Longrightarrow OK \perp BH.$ Thus, under the referred inversion we have that ω and $\odot(AOC)$ are tangent.

TelvCohl

#5 Apr 26, 2015, 11:45 am • 1

Here is another approach :

Let ℓ be the bisector of $\angle B$.
Let N, T be the 9-point center, orthocenter of $\triangle ABC$, respectively .
Let Ψ be the composition of Inversion $\mathbf{I}(B, \sqrt{\frac{1}{2}BA \cdot BC})$ and reflection $\mathbf{R}(\ell)$.

Easy to see $\Psi(A), \Psi(C)$ is the midpoint of BC, BA, respectively .

Since $O \longleftrightarrow H$ under Ψ ,
so $\Psi(\odot(H, HB))$ is the perpendicular bisector τ of BO .
Since $\Psi(\odot(AOC))$ is the 9-point circle of $\triangle ABC$,
so $\Psi(P), \Psi(M)$ is the projection of A, C on BC, BA, respectively .

Now the original problem become **:** τ is tangent to $\odot(N) \iff O \in \odot(B\Psi(P)\Psi(M)) \equiv \odot(BT)$.
i.e. we need to prove that $\text{dist}(N, \tau) = \frac{1}{2}\text{R} \iff \angle BOT = 90^\circ$ and this is almost trivial :) .

Q.E.D

tkhalid

#7 Apr 26, 2015, 11:26 pm

Wait did my comment get deleted? If so may I ask why?

tkhalid

#8 Apr 27, 2015, 12:01 am

Extension

1. Prove $BO \perp PM$
2. Quadrilateral $BPYM$ is a kite (Y is the tangency point of the two circles)
3. If H is the orthocenter of $\triangle ABC$, and O_1 the center of $\odot(AOC)$, then O_1H is parallel to BO.

Can someone prove or disprove 2 and 3 because I am not sure about them.

drmzjoseph

#9 Apr 27, 2015, 6:03 am

> *tkhalid wrote:*
> **Extension**
>
> 1. Prove $BO \perp PM$
> 2. Quadrilateral $BPYM$ is a kite (Y is the tangency point of the two circles)
> 3. If H is the orthocenter of $\triangle ABC$, and O_1 the center of $\odot(AOC)$, then O_1H is parallel to BO.
>
> Can someone prove or disprove 2 and 3 because I am not sure about them.

1. $\angle OBC + \angle BAC = 90^\circ$ further $\angle BAC = \angle PMB \Rightarrow BO \perp PM$
2 Let X be the symmetrical of B WRT $PM \Rightarrow X \in \omega(HB = HX)$ and $X \in \odot(AOC)$ because $\angle PXM = \angle ABC = \angle MAB \Rightarrow X \equiv Y$
3. It's not true $BO \parallel O_1H \iff \angle ABC = 60^\circ$ Because BOO_1H is a parallelogram i.e. $OO_1 = BH \iff O_1$ is the symmetrical of O WRT $AC \iff \angle AOC = 120^\circ$

This post has been edited 1 time. Last edited by drmzjoseph, Apr 27, 2015, 6:03 am

tkhalid

#10 Apr 27, 2015, 6:36 am

Thank you drmzjoseph. I enjoyed your problem too. 🙂

livetolove212

#11 May 8, 2015, 3:14 pm

Assume that P, H, M are collinear. We have O is the orthocenter of triangle BPM; BH, BO are isogonal wrt $\angle ABC$. Let I, J be the centers of $(BPM), (BOC)$ then B, I, H are collinear. Let T be the reflection of B wrt PM then $T \in (BOC)$. Since I and J are symmetric wrt PM we get H, J, T are collinear and $HT = HA$. Therefore ω is tangent to (BOC). Conversely, if ω or (H, HA) is tangent to (BOC). Let H' be the intersection of AH and PM. From the fact above, $(H', H'A)$ is tangent to (BOC). Since H', H, A are collinear, we get $H' \equiv H$ or H, P, M are collinear.

Quick Reply

Spain

OMC 2011 Problema 6

Jutaro

#1 Jun 26, 2011, 9:52 pm

Sea ABC un triángulo acutángulo y sean D, E y F los pies de las alturas desde A, B y C, respectivamente. Sean Y y Z los pies de las perpendiculares desde B y C sobre FD y DE, respectivamente. Sea F_1 la reflexión de F con respecto a E y sea E_1 la reflexión de E con respecto a F. Si $3EF = FD + DE$, demostrar que $\angle BZF_1 = \angle CYE_1$.

Luis González

#2 Apr 26, 2015, 4:00 am

Como $\triangle ABC$ es el triángulo excentral de $\triangle DEF$, entonces el ortocentro H de $\triangle ABC$ es el incentro de $\triangle DEF$ y Y, Z son los puntos de contacto de sus excírculos referentes a E y F. Ahora sabiendo que $EZ = FY = \frac{1}{2}(FD + DE - EF)$, de la condición del problema se desprende que $EF = FY = EZ$.

Del $\triangle EFZ$ isósceles en E, resulta entonces $\angle FZD = 90^\circ + \frac{1}{2}\angle DEF = \angle DHF \Longrightarrow Z \in \odot(BDHF) \Longrightarrow$ $\angle BZD = \angle BFD = \angle ACB \Longrightarrow \angle BZE = 180^\circ - \angle ACB$. Ahora, como F_1 pasa a ser el simétrico de Z en AC, se tiene $\angle EZF_1 = 90^\circ - \angle DEC = 90^\circ - \angle ABC$. Por ende $\angle BZF_1 = \angle BZE + \angle EZF_1 = 270^\circ - \angle ACB - \angle ABC$. Analogamente se obtendrá $\angle CYE_1 = 270^\circ - \angle ACB - \angle ABC$ y se sigue la conclusión.

High School Olympiads

Passes through the circumcentre

Reply

Source: Own

LeVietAn
#1 Apr 25, 2015, 11:21 am

Problem: The circle inscribed a triangle ABC meets the sides BC, CA, AB at D, E, F, respectively. Points H, K, E are on EF, FA, AE such that DH, DK, DL are perpendicular to EF, FA, AE respectively. Let M the midpoint of KL. Prove that HM is through the center circumcircle of a triangle HBC.

tkhalid
#2 Apr 25, 2015, 11:30 am

I think you meant points H, K, and L. Also you should say are on lines EF, FA, and AE in my opinion, just for clarification.

LeVietAn
#3 Apr 25, 2015, 11:31 am

yes, thanks tkhalid :))

TelvCohl
#4 Apr 25, 2015, 1:01 pm

My solution:

Let I be the incenter of $\triangle ABC$.
Let T be the center of $\odot(HBC)$ and N be the midpoint of EF .
Let $G = AD \cap \odot(I)$ ($G \neq D$) and D' be the reflection of D in AI .

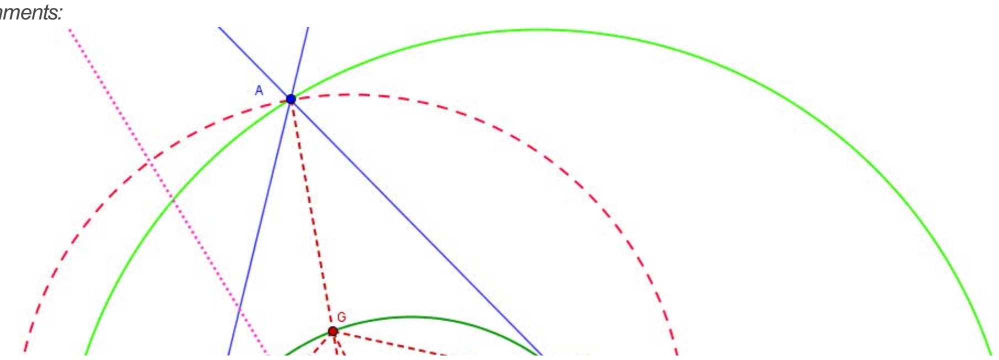
From $(EF, HD; HB, HC) = -1$ and $DH \perp EF \Longrightarrow HD$ is the bisector of $\angle BHC$,
so combine with $\angle AEF = \angle AFE$ we get $\angle FHB = \angle EHC$ and $\angle HBA = \angle HCA$.

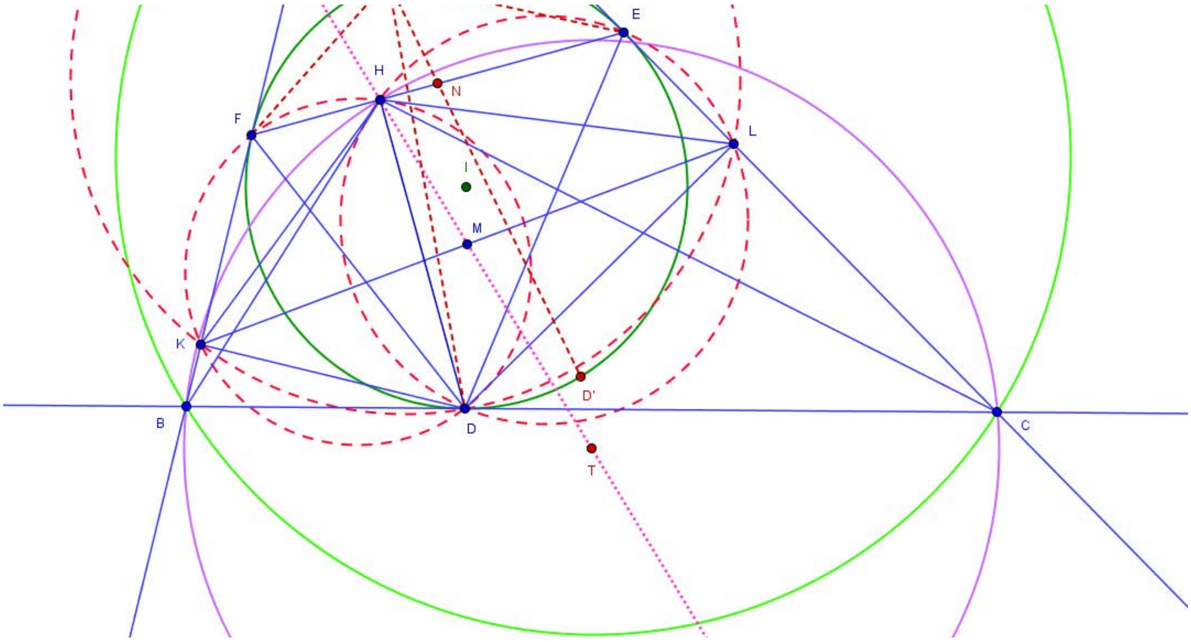
Since DG is D-symmedian of $\triangle DEF$,
so from symmetry we get D', N, G are collinear (or notice that $DEGF$ is harmonic quadrilateral) .
Since $\angle HKL = \angle HKD - \angle DKL = \angle DFE - \angle DAL = \angle GFE$ (similarly, $\angle HLK = \angle GEF$) ,
so we get $\triangle HKL \sim \triangle GFE \Longrightarrow \triangle HKL \cup M \sim \triangle GFE \cup N \Longrightarrow \angle KHM = \angle FGD' = \angle DFE$. $(\star)$

Since $\angle KHT = \angle KHB + \angle BHT = \angle HKA - \angle HBA + 90^\circ - \angle HCB$
$= \angle HDF - \angle HCA + 90^\circ - \angle HCB = 90^\circ - \angle DFE + 90^\circ - \angle ACB = \angle DFE$,
so combine with $(\star)$ we get H, M, T are collinear .

Q.E.D

Attachments:





Luis González

#5 Apr 26, 2015, 12:13 am

Since $H(B,C,D,E)=-1$ and $DH\perp EF,$ then HD,HE bisect $\angle BHC.$ Thus if T is the projection of H on $BC,$ it suffices to show that HM is the reflection of HT on $HD.$

From cyclic $DHEL$ and $DHFK,$ we get $\widehat{DLH}=\widehat{DEH}=\widehat{DFK}=\widehat{DHK}\Longrightarrow\odot(LHD)$ touches HK and likewise $\odot(KHD)$ touches $HL\Longrightarrow D$ is midpoint of the H-symmedian chord of $\triangle HKL$ cut by its circumcircle $\Longrightarrow$ $\widehat{MHL}=\widehat{DHK}=\widehat{DEF}\Longrightarrow\widehat{DHM}=|\widehat{DHL}-\widehat{MHL}|=|\widehat{DFE}-\widehat{DEF}|=\widehat{HDI}=\widehat{DHT}\Longrightarrow HM$ is the reflection of HT on $HD.$

Quick Reply

High School Olympiads

Incenter, circumcenter and excenter

Locked

Source: maybe famous but cannot find it

gobathegreat
#1 Apr 25, 2015, 1:53 pm

Let I and O be incenter and circumcenter of triangle ABC, respectively. Excircle k_A is tangent to lines AB, BC and CA at K, M and N, respectively. Point P is midpoint of KM and it belongs to circumcircle of ABC. Prove that points I, O and N are collinear.

Kaito
#2 Apr 25, 2015, 3:46 pm

"Point P is midpoint of KM and it belongs to circumcircle of ABC" What did this mean while the question is only about I,O,N

gobathegreat
#3 Apr 25, 2015, 6:14 pm

Of course it matters. This is just an ordinary triangle if P does not lie on the circumcircle. But as it lies on the circumcircle you need to show that if P lies on circumcircle then it implies that I O and N are collinear

Kaito
#4 Apr 25, 2015, 7:10 pm

> *gobathegreat wrote:*
> Of course it matters. This is just an ordinary triangle if P does not lie on the circumcircle. But as it lies on the circumcircle you need to show that if P lies on circumcircle then it implies that I O and N are collinear

Sorry, I have just glanced the problem.

Kaito
#5 Apr 25, 2015, 7:22 pm

> *gobathegreat wrote:*
> Let I and O be incenter and circumcenter of triangle ABC, respectively. Excircle k_A is tangent to lines AB, BC and CA at K, M and N, respectively. Point P is midpoint of KM and it belongs to circumcircle of ABC. Prove that points I, O and N are collinear.

And, I think you meant N is on BC

Luis González
#6 Apr 25, 2015, 9:51 pm

Indeed, there is a typo in the problem; the points M and N are switched. This was posted before at Points O,N, I lie on a line [Russia 2003].

High School Olympiads

Points O,N, I lie on a line [Russia 2003]

Reply

Amir Hossein

#1 Nov 4, 2010, 7:07 pm

In a triangle ABC, O is the circumcenter and I the incenter. The excircle ω_a touches rays AB, AC and side BC at K, M, N, respectively. Prove that if the midpoint P of KM lies on the circumcircle of $\triangle ABC$, then points O, N, I lie on a line.

Luis González

#2 Nov 4, 2010, 11:48 pm

Assume that P lies on the circumcircle (O, R) of $\triangle ABC$. Hence, P coincides with the midpoint of II_a. From the right triangle $\triangle AMI_a$ with hypotenuse AI_a and altitude MP, we have

$$\overline{I_aM}^2 = r_a{}^2 = \overline{I_aP} \cdot \overline{I_aA} = p(I_a, (O)) = 2R \cdot r_a \Longrightarrow \ r_a = 2R$$

Since $OP \parallel NI_a$, it follows that O, N, I are collinear, such that $\overline{OI} = -\overline{ON}$.

yetti

#3 Nov 5, 2010, 4:19 am

(I) is incircle and $(I_a), (I_b), (I_c)$ are excircles of $\triangle ABC$. Inversion in (I_a) takes A, B, C to midpoints of MK, KN, NM and circumcircle (O) of $\triangle ABC$ to 9-point circle of $\triangle NMK$. Midpoint P of MK is in $(O) \Longrightarrow (O)$ goes to itself and it is 9-point circle of $\triangle NMK \Longrightarrow$ centrally similar $\triangle NMK \sim \triangle II_cI_b$ have the same 9-point circle $(O) \Longrightarrow$ they are centrally congruent with similarity center O and coefficient $-1 \Longrightarrow \overline{ON} = -\overline{OI}$.

Quick Reply

High School Math

HKIMO Prelim 2002 P12

Reply

YanYau

#1 Apr 15, 2015, 9:00 pm • 1

In trapezium $ABCD$, $BC\perp AB$, $BC\perp CD$, and $AC\perp BD$. Given $AB=\sqrt{11}$ and $AD=\sqrt{1001}$. Find BC

Luis González

#2 Apr 25, 2015, 3:46 am

For convenience denote $BC=x, CD=y.$ If P is the projection of A on $CD,$ then by Pythagorean theorem, we get $AD^2=PA^2+PD^2=x^2+(y-AB)^2 \Longrightarrow 1001=x^2+(y-\sqrt{11})^2 \Longrightarrow x^2+y^2-2\sqrt{11}y-990.$ But from $AC\perp BD,$ we get $y^2-x^2=AD^2-AB^2=990 \Longrightarrow y^2-\sqrt{11}y-990=0.$ Solving this quadratic equation for $y>0$ gives $y=10\sqrt{11} \Longrightarrow x^2=(10\sqrt{11})^2-990 \Longrightarrow x=\sqrt{110}.$

Quick Reply

High School Olympiads

Schwatt's lines

Reply

jayme
#1 Jan 15, 2015, 5:42 pm

Dear Mathlinkers,

recently, the Schwatt's line has been mentioned.

I propose to indicate the different proofs and perhaps have a new regard with a new proof.

I recall : the A-Schwatt's line of a triangle ABC goes through the midpoints of BC and the A-altitude, and also through the Lemoine's point K of ABC.

Sincerely
Jean-Louis

TelvCohl
#2 Jan 15, 2015, 10:26 pm

My solution:

Let $\triangle A'B'C'$ be the tangential triangle of $\triangle ABC$.
Let M be the midpoint of BC and H be the projection of A on BC .

It's well known that $K \equiv AA' \cap BB' \cap CC'$,
so $M(A', K; H, A) = -1 \Longrightarrow MK$ pass through the midpoint of AH .
i.e. K lie on $A-$ Schwatt line of $\triangle ABC$

Q.E.D

Luis González
#3 Jan 15, 2015, 10:36 pm

It's a particular case of the following well-known result:

If $\triangle A'B'C'$ is cevian triangle of arbitrary point P WRT $\triangle ABC$, then $\triangle ABC$ and the medial triangle of $\triangle A'B'C'$ are perspective through the isotomcomplement P^* of P. Therefore, if M and A'' are the midpoints of BC and AA', respectively, we have $P^* \in MA''$.

jayme
#4 Jan 16, 2015, 1:53 pm

Dear Mathlinkers,
my proof consequently of a twin theorem....

1. ABC a triangle
2. A', B' the feet of the A, B-altitude
3. M, I, E, D the midpoints of AA', BC, CA, AB
4. K the Lemoine's point
5. U, V the points of intersection of IE and A'B', A'B' and DE.

According to "Another unlikely concurrence" (Crux Mathematicorum, 8 (2003) 511-513
or http://jl.ayme.pagesperso-orange.fr/Docs/An%20Another%20Unlikely%20Concurrence.pdf)
K is the point of intersection of AU and BV

R is the point of intersection of AU and BV.

According to Pappus, (KIM) is the pappusian's line of the hexagon AUEVBA'A.

Sincerely
Jean-Louis

sunken rock

#5 Jan 17, 2015, 2:03 pm

To be noted that any rectangle inscribed into that triangle having 2 vertices on subject side (and one on each of other sides) have the center on this line as well.

Best regards,
sunken rock

TelvCohl

#6 Jan 20, 2015, 9:15 am

More general:

Let $\triangle P_aP_bP_c, \triangle Q_aQ_bQ_c$ be the cevian triangle of P, Q, respectively .
Let $AQ \cap P_bP_c = P_a^*, BQ \cap P_cP_a = P_b^*, CQ \cap P_aP_b = P_c^*$.
Let $AP \cap Q_bQ_c = Q_a^*, BP \cap Q_cQ_a = Q_b^*, CP \cap Q_aQ_b = Q_c^*$.

Then $P_aP_a^*, P_bP_b^*, P_cP_c^*, Q_aQ_a^*, Q_bQ_b^*, Q_cQ_c^*$ are concurrent at the cross point of $\{P, Q\}$.

Luis González

#7 Jan 20, 2015, 11:31 am • 1

> TelvCohl wrote:
> More general:
>
> Let $\triangle P_aP_bP_c, \triangle Q_aQ_bQ_c$ be the cevian triangle of P, Q, respectively.
> Let $AQ \cap P_bP_c = P_a^*, BQ \cap P_cP_a = P_b^*, CQ \cap P_aP_b = P_c^*$.
> Let $AP \cap Q_bQ_c = Q_a^*, BP \cap Q_cQ_a = Q_b^*, CP \cap Q_aQ_b = Q_c^*$.
>
> Then $P_aP_a^*, P_bP_b^*, P_cP_c^*, Q_aQ_a^*, Q_bQ_b^*, Q_cQ_c^*$ are concurrent at the cross point of $\{P, Q\}$.

Let $\mathcal{C}$ be the unique conic through A, B, C, P, Q. Since P_bP_c is the polar of P_a WRT $\mathcal{C}$, then P_a and $P_a{}^*$ are conjugate points WRT $\mathcal{C} \Longrightarrow P_aP_a{}^*$ and the sideline PQ of $\triangle APQ$ are conjugate lines WRT its circumconic $\mathcal{C} \Longrightarrow P_aP_a{}^*$ goes through the pole R of PQ WRT $\mathcal{C}$ (Seydewitz-Staudt theorem). In the same way, the lines $P_bP_b{}^*, P_cP_c{}^*$ and $Q_aQ_a{}^*, Q_bQ_b{}^*, Q_cQ_c{}^*$ go through R; the cross point of P, Q WRT $\triangle ABC$.

TelvCohl

#8 Jan 20, 2015, 12:10 pm

> TelvCohl wrote:
> More general:
>
> Let $\triangle P_aP_bP_c, \triangle Q_aQ_bQ_c$ be the cevian triangle of P, Q, respectively .
> Let $AQ \cap P_bP_c = P_a^*, BQ \cap P_cP_a = P_b^*, CQ \cap P_aP_b = P_c^*$.
> Let $AP \cap Q_bQ_c = Q_a^*, BP \cap Q_cQ_a = Q_b^*, CP \cap Q_aQ_b = Q_c^*$.
>
> Then $P_aP_a^*, P_bP_b^*, P_cP_c^*, Q_aQ_a^*, Q_bQ_b^*, Q_cQ_c^*$ are concurrent at the cross point of $\{P, Q\}$.

Another proof :

Let $A_1 = BP \cap CQ, A_2 = BQ \cap CP$ and define B_1, B_2, C_1, C_2 similarly .

From Cevian nest theorem we get $P_aP_a^*, P_bP_b^*, P_cP_c^*$ are concurrent at T .
From Desargue theorem (consider $\triangle A_1P_bP_c^*$ and $\triangle A_2P_b^*P_c$) we get $T \in A_1A_2$.
Similarly, we can prove $T \in B_1B_2$ and $T \in C_1C_2 \Longrightarrow T = A_1A_2 \cap B_1B_2 \cap C_1C_2$

Similarly, we can prove $T' \in B_1B_2$ and $T' \in C_1C_2 \Longrightarrow T' = A_1A_2 \cap B_1B_2 \cap C_1C_2$.

From Cevian nest theorem we get $Q_aQ_a^*, Q_bQ_b^*, Q_cQ_c^*$ are concurrent at T'.
From Desargue theorem (consider $\triangle A_1Q_b^*Q_c$ and $\triangle A_2Q_bQ_c^*$) we get $T' \in A_1A_2$.
Similarly, we can prove $T' \in B_1B_2$ and $T' \in C_1C_2 \Longrightarrow T' \equiv A_1A_2 \cap B_1B_2 \cap C_1C_2 \equiv T$.
i.e. $P_aP_a^*, P_bP_b^*, P_cP_c^*, Q_aQ_a^*, Q_bQ_b^*, Q_cQ_c^*$ are concurrent (cross point of $\{P, Q\}$)

Remark :

Let $\mathcal{C}$ be the circumconic of $\triangle ABC$ passing through P, Q.

From my proof we get the cross point of $\{P, Q\}$ is the intersection of A_1A_2, B_1B_2, C_1C_2.

Since A_1A_2 is the polar of $PQ \cap BC$ WRT $\mathcal{C}$,
so A_1A_2 pass through the pole Z of PQ WRT $\mathcal{C}$.
Similarly, we can prove $Z \in B_1B_2$ and $Z \in C_1C_2 \Longrightarrow Z$ is the cross point of $\{P, Q\}$.

This post has been edited 1 time. Last edited by TelvCohl, Jan 20, 2015, 11:04 pm

jayme
#9 Jan 20, 2015, 1:29 pm

Dear Mathlinkers,
you can see also

http://jl.ayme.pagesperso-orange.fr/Docs/The%20cross-cevian%20point.pdf

with references and more.

Sincerely
Jean-Louis

IDMasterz
#10 Jan 20, 2015, 6:01 pm

> *TelvCohl wrote:*
>
> > *TelvCohl wrote:*
> > More general:
> >
> > Let $\triangle P_aP_bP_c, \triangle Q_aQ_bQ_c$ be the cevian triangle of P, Q, respectively .
> > Let $AQ \cap P_bP_c = P_a^*, BQ \cap P_cP_a = P_b^*, CQ \cap P_aP_b = P_c^*$.
> > Let $AP \cap Q_bQ_c = Q_a^*, BP \cap Q_cQ_a = Q_b^*, CP \cap Q_aQ_b = Q_c^*$.
> >
> > Then $P_aP_a^*, P_bP_b^*, P_cP_c^*, Q_aQ_a^*, Q_bQ_b^*, Q_cQ_c^*$ are concurrent at the cross point of $\{P, Q\}$.
>
> **Another proof :**
>
> Let $A_1 = BP \cap CQ, A_2 = BQ \cap CP$ and define B_1, B_2, C_1, C_2 similarly .
>
> From Cevian nest theorem we get $P_aP_a^*, P_bP_b^*, P_cP_c^*$ are concurrent at T.
> From Desargue theorem (consider $\triangle A_1P_bP_c^*$ and $\triangle A_2P_b^*P_c$) we get $T \in A_1A_2$.
> Similarly, we can prove $T \in B_1B_2$ and $T \in C_1C_2 \Longrightarrow T \equiv A_1A_2 \cap B_1B_2 \cap C_1C_2$.
>
> From Cevian nest theorem we get $Q_aQ_a^*, Q_bQ_b^*, Q_cQ_c^*$ are concurrent at T'.
> From Desargue theorem (consider $\triangle A_1Q_b^*Q_c$ and $\triangle A_2Q_bQ_c^*$) we get $T' \in A_1A_2$.
> Similarly, we can prove $T' \in B_1B_2$ and $T' \in C_1C_2 \Longrightarrow T \equiv T'$.
> i.e. $P_aP_a^*, P_bP_b^*, P_cP_c^*, Q_aQ_a^*, Q_bQ_b^*, Q_cQ_c^*$ are concurrent (cross point of $\{P, Q\}$

I think maybe we can simplify the idea by taking the conic through $ABCPQ$ to a circle with P, Q being antipodal points. I believe the concurrence is then the point at infinity perpendicular to the line PQ.

High School Olympiads

inverse wrt Brocard circle

Leonhard Euler
#1 Mar 8, 2009, 8:33 am

Let ABC be a triangle, G the centroid, O the circumcenter, K the symmedian point, and E the Euler reflection point. Prove that G and E are inverse wrt Brocard circle, i.e. circle with diameter OK.

Leonhard Euler
#2 Mar 8, 2009, 9:46 am

E is kimberling center X_{110}, the intersection of reflection of Euler line in the sides of triangle ABC.

Luis González
#3 Mar 8, 2009, 10:55 am

The collinearity of G, X_{110} and X_{182} can be proved easily using barycentric coordinates

$$\begin{bmatrix} 1 & 1 & 1 \\ a^2(c^2-a^2)(a^2-b^2) & b^2(b^2-c^2)(a^2-b^2) & c^2(c^2-a^2)(b^2-c^2) \\ \sin A\cos(A-\omega) & \sin B\cos(B-\omega) & \sin C\cos(C-\omega) \end{bmatrix} = 0,$$

where ω is the Brocard angle of $\triangle ABC$.

Leonhard Euler
#4 Mar 8, 2009, 11:50 am

I found this problem by solving http://xtec.net/~qcastell/ttw/ttweng/resultats/r432.html. There are only problems with no solution in http://xtec.net/~qcastell/ttw/ttweng/portada.html. Isn't it? Then what is meaning of "not solved"?

yetti
#5 Mar 11, 2009, 8:39 am

Just a start, proving that inversion of E in Brocard circle is on Euler line.

Let (J) be the Brocard circle with diameter OK and center $J \equiv X_{182}$. Let $(X), (Y), (Z)$ be the A-, B-, C-Apollonius circles of $\triangle ABC$ centered at $X \in BC, Y \in CA, Z \in AB$. The A-, B-, C-symmedians are radical axes of $(X), (Y), (Z)$ and (O) and these circles are all perpendicular to $(O) \Longrightarrow X, Y, Z$ are poles of the symmedians AK, BK, CK and $XYZ \perp OK$ is polar of the symmedian point K. Consequently, the line $j \equiv XYZ$ is inversion of the Brocard circle (J) in the circumcircle (O). Denote $\mathbf{R}_{(O)}, \mathbf{R}_{(J)}, \mathbf{R}_j$ reflections in $(O), (J), j$. Then $\mathbf{R}_{(J)} = \mathbf{R}_{(O)} \circ \mathbf{R}_j \circ \mathbf{R}_{(O)}^{-1} = \mathbf{R}_{(O)} \circ \mathbf{R}_j \circ \mathbf{R}_{(O)}$. Complete quadrilateral $\mathcal{Q}$ formed by the lines (BC, CA, AB, XYZ) has 3 diagonals are AX, BY, CZ, let L, M, N be their midpoints. Circles $(L), (M), (N)$ with diameters AX, BY, CZ are all tangent to OA, OB, OC at A, B, C, therefore powers of O to $(L), (M), (N)$ are equal. They pass through vertices of $\triangle ABC$ and are centered on its midlines, therefore they pass through its altitude feet, and powers of the orthocenter H to $(L), (M), (N)$ are all equal to half the power of H to $(O) \Longrightarrow$ Euler line OH is their common radical axis. Their centers form Newton line LMN of this complete quadrilateral and $OH \perp LMN$ is its orthocentric (Steiner) line. It follows that 4 circumcircles $(O) \equiv \odot(ABC), \odot(AYZ) \odot (BZX), \odot(CXY)$ of the 4 triangles of this complete quadrilateral concur at E, pole of the Euler line OH of $\triangle ABC$ taken as Steiner line. Consequently, reflection E_j of E in XYZ is on OH. Initial reflection $\mathbf{R}_{(O)}$ keeps $E \in (O)$ in place, subsequent reflection $\mathbf{R}_j$ takes it into $E_j \in OH$, and final reflection $\mathbf{R}_{(O)}$ keeps it on OH. This means the reflection $\mathbf{R}_{(J)}$ puts E on OH.

It remains to show that EG cuts OK at its midpoint J.

This post has been edited 1 time. Last edited by yetti, Mar 20, 2009, 8:21 am

yetti

#6 Mar 20, 2009, 9:56 pm

Let EJ cuts OH at G' Let F_a, F_b, F_c, F_j be feet of perpendiculars from E to BC, CA, AB, XYZ and $E_a, E_b, E_c, E_j \in OH$ reflections of E in $BC, CA, AB, XYZ.$ E_j is reflection of G' in $(O).$ $F_aF_bF_cF_j$ is Simson line with pole E WRT all 4 triangles of the complete quadrilateral $(BC, CA, AB, XYZ).$ From this Simson line and cyclic $EF_jXF_a, EF_jYF_b, EF_jZF_c,$ the angles $\angle EXF_a = \angle EYF_b = \angle EZF_c$ are equal and isosceles $\triangle EE_aX \sim \triangle EE_bY \sim \triangle EE_cZ$ similar $\Longrightarrow$ spiral similarity $\mathbf{S}_E$ with center $E,$ rotation angle $\phi = \angle EE_aX = \angle EE_bY = \angle EE_cZ$ and coefficient $k = \frac{EX}{EE_a} = \frac{EY}{EE_b} = \frac{EZ}{EE_c}$ takes the Euler line $OH \equiv E_aE_bE_c$ into the line $XYZ.$ Consequently, perpendicular bisectors g, o of EG', EO cut XYZ at the images $P = \mathbf{S}_E(G'), Q = \mathbf{S}_E(O).$ As XYZ is perpendicular bisector of $EE_j,$ circumcircle (Q) of $\triangle EOE_j$ is centered at $Q.$ (This is Miquel circle of the complete quadrilateral $(BC, CA, AB, XYZ),$ in this case centered on one of its lines.) From the reflection $\mathbf{R}_{(O)}, \angle(o, g) = \angle OEG' = \angle OE_jE = \frac{1}{2}\angle EQO = \frac{1}{2}\angle EPG' \Longrightarrow EP \parallel o.$ Since $o \perp EO, EP$ is tangent of (O) at $E.$ Circle (P) with center P and radius $PE = PG'$ is centered on XYZ and perpendicular to $(O),$ just like the 3 Apollonius circles $(X), (Y), (Z)$ of $\triangle ABC,$ therefore (P) is coaxal with the Apollonius circles and passes through their 2 intersection $D_1, D_2,$ the 2 isodynamic points of $\triangle ABC.$ This means that (P) is Parry circle of $\triangle ABC$ and $G' \equiv G$ its centroid.

Click to reveal hidden text

yetti

#7 Mar 23, 2009, 9:47 pm

Synthetic proof without using Parry circle (P):

O_a, O_b, O_c are circumcenters of $\triangle AYZ, \triangle BZX, \triangle CXY$ intersecting at $E \in (O).$ $\angle EO_aA = 2\angle EF_jA = \angle EXE_a$ $\Longrightarrow$ isosceles $\triangle EO_aA \sim \triangle EXE_a$ are similar, $O_a = \mathbf{S}_E(A)$ and similarly, $O_b = \mathbf{S}_E(B), O_c = \mathbf{S}_E(C) \Longrightarrow$ spiral similarity $\mathbf{S}_E$ takes $\triangle ABC$ with circumcircle (O) into $\triangle O_aO_bO_c$ with circumcircle $(Q).$ Circumcenter O of the remaining $\triangle ABC$ of the complete quadrilateral (BC, CA, AB, XYZ) is also on $(Q).$ Since $O \in E_aE_bE_c, Q \in XYZ.$

A-symmedian cuts BC, OX, XYZ at $K_a, X_a, L_a.$ Cross ratios $(B, C, X, K_a) = (Y, Z, X, L_a) = -1$ are harmonic $\Longrightarrow$ if M_a is midpoint of $XL_a,$ then $\overline{M_aY} \cdot \overline{M_aZ} = M_aX^2.$ Circle (M_a) with diameter XL_a goes through $X, X_a \Longrightarrow$ $(M_a) \perp (O)$ and power of M_a to (O) is equal to $M_aX^2.$ It follows that $M_a \in XYZ$ is on radical axis EA of $(O), (O_a).$

Let $\triangle A_jB_jC_j$ with circumcircle (O_j) be reflection of $\triangle ABC$ with circumcircle (O) in $j \equiv XYZ.$ Let EX cut (O) again at $N_a.$ N_a is reflection of E is perpendicular to EX through $O,$ which is parallel to the perpendicular bisector O_bO_c of $EX.$ From $\overline{M_aE} \cdot \overline{M_aA} = M_aX^2, \angle AOO_j = \angle XAM_a = \angle M_aEX = \angle AEN_a = \frac{1}{2}\angle AON_a, N_a$ is also reflection of A in $OO_j.$ Reflection of the line EP in the perpendicular bisector O_bO_c of EX goes through X and is perpendicular to the reflection of the segment OE in $O_bO_c,$ i.e., also perpendicular to the segment $ON_a.$ But reflection O_jA_j of OA in XYZ is parallel to ON_a and so the reflection of EP in $O_bO_c,$ going through $X,$ is perpendicular to $O_jA_j,$ hence it is identical with reflection A_jX of AX in $XYZ.$ Similarly, reflections of EP in O_cO_a, O_aO_b are identical with reflections B_jY, C_jZ of BY, CZ in $XYZ.$ These reflections are all tangents of $(O_j).$

Let R be Parry reflection point of $\triangle ABC,$ i.e., reflection of O in $E.$ Since reflections of EP in O_bO_c, O_cO_a, O_aO_b are tangents of $(O_j),$ reflections of a parallel $s \parallel EP$ through R in O_bO_c, O_cO_a, O_aO_b concur at $O_j \in (Q).$ This means that s is Steiner line of $\triangle O_aO_bO_c$ with the pole O_j and it cuts its Euler line XYZ at its orthocenter $S.$ If M is midpoint of $OE,$ then $MQ \parallel EP \parallel RS$ and $\frac{\overline{QP}}{\overline{PS}} = \frac{\overline{ME}}{\overline{ER}} = \frac{1}{2}.$ Since Q, S are circumcenter and orthocenter of $\triangle O_aO_bO_c, P$ is its centroid. But P is image of G' is spiral similarity $\mathbf{S}_E$ taking $\triangle ABC$ into $\triangle O_aO_bO_c,$ therefore $G' \equiv G$ is centroid of $\triangle ABC.$

Luis González

#8 Mar 27, 2009, 12:16 am

I come back with a more concrete proof. In order to get that $E \equiv X_{110}$ is the inverse of G in the Brocard circle, we need to show that EG passes through the midpoint X_{182} of $\overline{OK}$ (already proved) and that the polar of G WRT $\mathcal{B}$ passes through E (to be proved).

Equation of Brocard circle $\mathcal{B}$ is given by

$$\mathcal{B} \equiv a^2yz + b^2xz + c^2xy - \frac{a^2b^2c^2}{a^2+b^2+c^2}(x+y+z)\left(\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z}\right) = 0$$

Hence polar of $G(1:1:1)$ WRT $\mathcal{B}$ is given by

Hence, polar τ of $G\ (1:1:1)$ WRT B is given by:

$\tau \equiv (b^2-c^2)^2x+(c^2-a^2)^2y+(a^2-b^2)^2z=0$

Indeed, coordinates of X_{110} satisfy the equation of τ and the conslusion follows.

TelvCohl

#9 Feb 4, 2015, 1:25 am • 1

My solution:

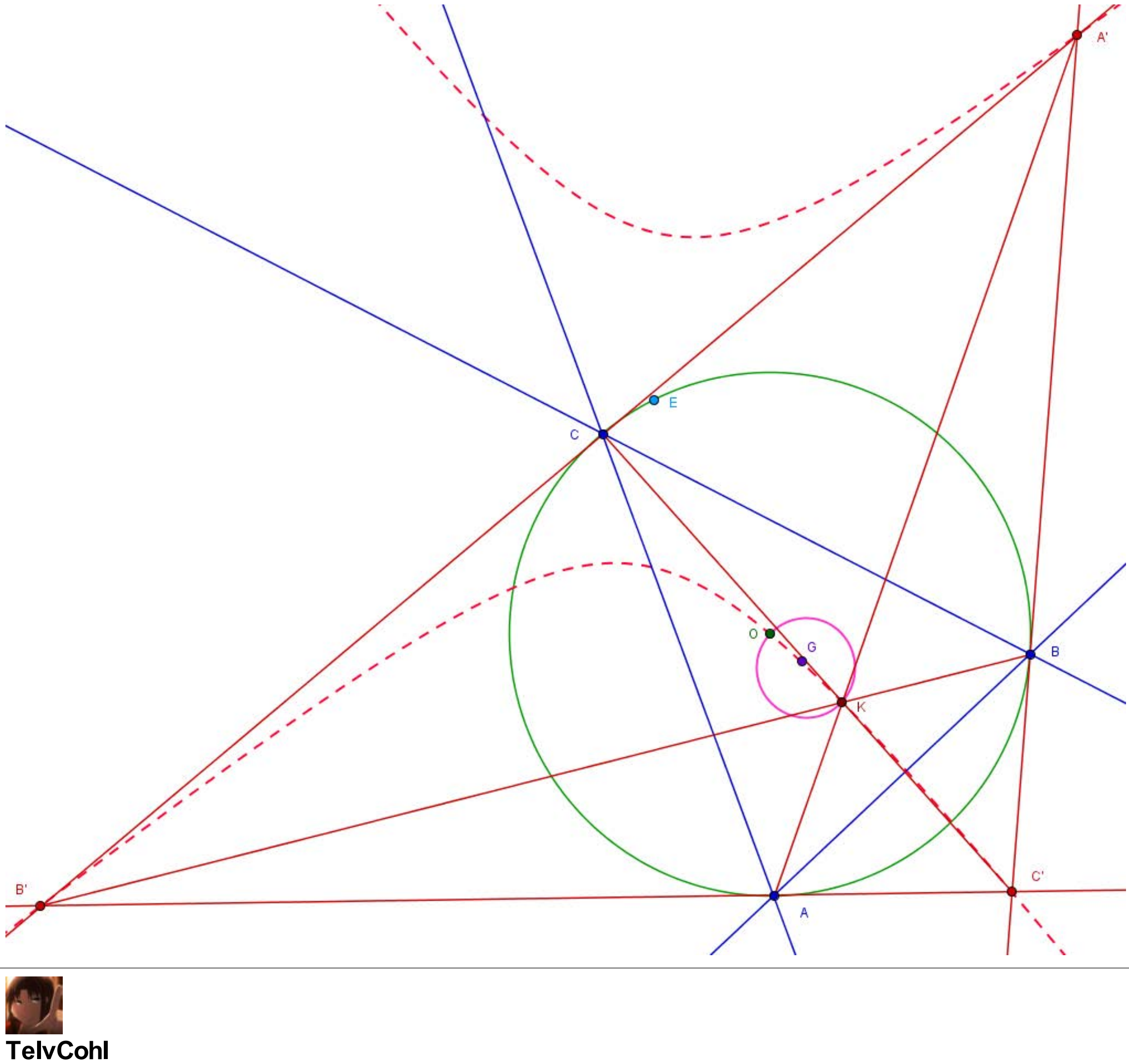
Let $\triangle A^*B^*C^*$ be the tangential triangle of $\triangle ABC$.

Since O, K, E is the incenter, Gergonne point, Feuerbach point of $\triangle A^*B^*C^*$, respectively ,
so we get E is the center of the Feuerbach hyperbola $\mathcal{F}$ of $\triangle A^*B^*C^*$ and $O, K \in \mathcal{F}$.

Since G is the cross point of $\{O, K\}$ WRT $\triangle A^*B^*C^*$,
so from the generalization discussed in the topic Schwatt's lines we get OK is the polar of G WRT $\mathcal{F}$,
hence from the theorem mentioned in the topic Rectangular circumhyperbola and circle (posts # 2 and # 4) $\Longrightarrow$ G and E are inverse WRT $\odot(OK)$ (Brocard circle) .

Q.E.D

Attachments:



TelvCohl

#10 May 3, 2016, 10:17 pm

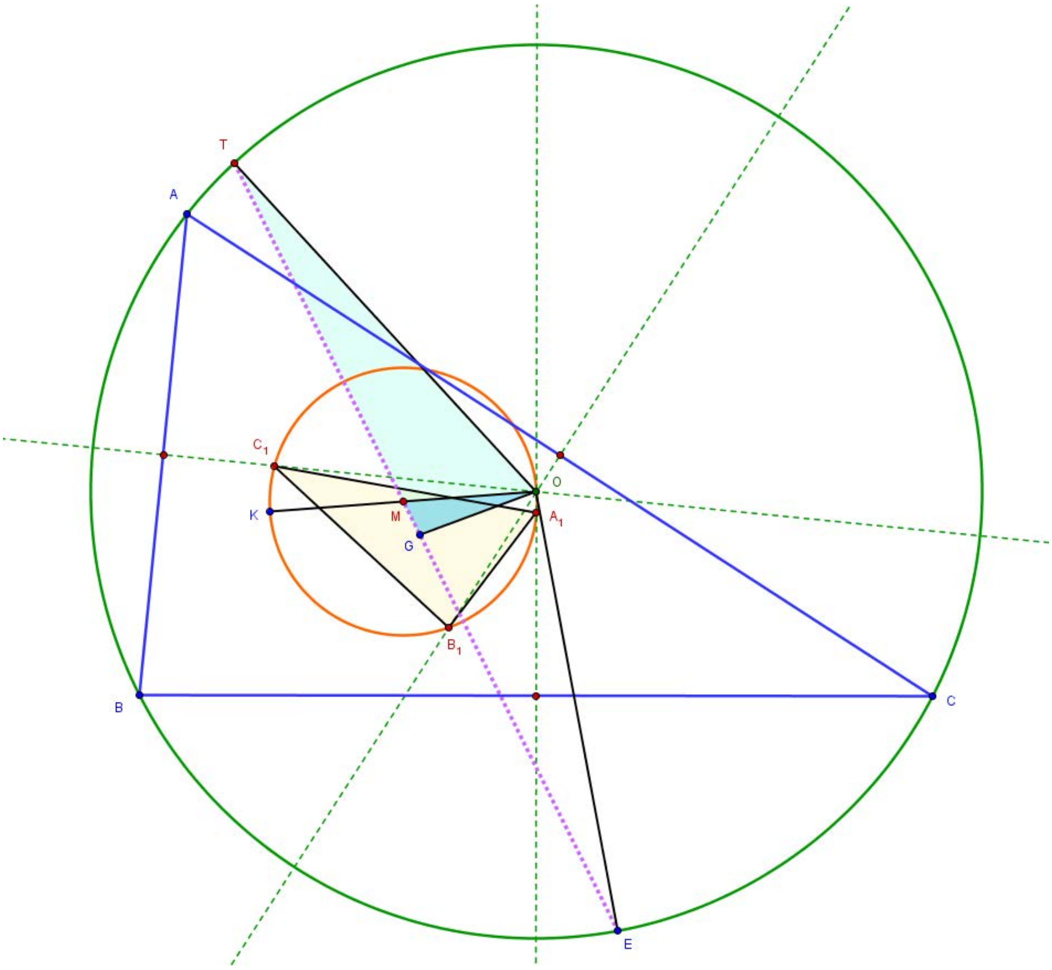
Let $\triangle A_1B_1C_1$ be the first Brocard triangle of $\triangle ABC$ and let T be the Tarry point of $\triangle ABC$. It's well-known that $\triangle ABC \overset{-}{\sim} \triangle A_1B_1C_1$ and G, O is the Centroid, Tarry point of $\triangle A_1B_1C_1$, respectively, so if M is the midpoint of OK, then we get

$$\triangle ABC \cup O \cup G \cup T \overset{-}{\sim} \triangle A_1B_1C_1 \cup M \cup G \cup O \Longrightarrow \triangle OGT \overset{-}{\sim} \triangle MGO,$$

hence $\measuredangle OGT = \measuredangle OGM \Longrightarrow G, M, T$ are collinear. Notice that the steiner line of T WRT $\triangle ABC$ is perpendicular to OK, so from $\measuredangle GTO = \measuredangle MOG = \measuredangle(OK, OG)$ we get the second intersection of TG with the circumcircle of $\triangle ABC$ is the anti-steiner point E of OG WRT $\triangle ABC$. Finally, from $\measuredangle OEM = \measuredangle GTO = \measuredangle MOG$ we get $MO^2 = MG \cdot ME$.

Attachments:

Attachments:



Quick Reply

High School Olympiads

Nice and hard geometry

Locked

junior2001

#1 Apr 25, 2015, 12:37 am

Let (I_c) and (I_a) excircles of ABC.The sircle S passes through B and tangent to (I_c) and (I_a) and intersect AC at M,N.Prove that $\angle ABM = \angle NBC$.

Luis González

#2 Apr 25, 2015, 1:17 am

Discussed before at http://www.artofproblemsolving.com/community/c6h585827.

High School Olympiads

St.Peterburg, P6 Grade 11, 2013

Reply

mathuz

#1 Apr 17, 2014, 10:50 am

Let (I_b), (I_c) are excircles of a triangle ABC. Given a circle ω passes through A and externally tangents to the circles (I_b) and (I_c) such that it intersects with BC at points M, N.
Prove that $\angle BAM = \angle CAN$.

A. Smirnov

This post has been edited 1 time. Last edited by mathuz, Apr 20, 2014, 8:00 pm

Luis González

#2 Apr 17, 2014, 11:26 am • 1

Insimilicenter A of $(I_b) \sim (I_c)$ is also center of negative inversion that transforms them into each other. By conformity, ω is transformed into the common external tangent τ_A of $(I_b), (I_c)$ different from BC. The center of ω is on the perpendicular form A to τ_A, but τ_A is antiparallel to BC WRT $AB, AC \Longrightarrow$ it is on the A-circumdiameter of $\triangle ABC \Longrightarrow \omega$ is tangent to the circumcircle (O) of $\triangle ABC \Longrightarrow A$ is exsimilicenter of $\omega \sim (O)$. Hence, if AM, AN cut (O) again at M', N', we have $M'N' \parallel MN \equiv BC \Longrightarrow$ arcs BM', CN' are equal $\Longrightarrow \angle BAM = \angle CAN$.

hal9v4ik

#3 Apr 17, 2014, 12:53 pm • 1

Grade 6? I am speechless.

dizzy

#4 Apr 17, 2014, 1:32 pm • 1

haha, most probably it is Grade 9 :)

nima1376

#5 Apr 17, 2014, 4:11 pm • 1

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=524295&hilit=Saint+Peterburg

Quick Reply

High School Olympiads

isogonal conjugate of prasolov point

Reply

r1234

#1 Dec 22, 2011, 1:17 pm • 1

Hi, here I saw a beautiful property of prasolov point. But I don't have a synthetic proof of it.

Show the Prasolov point of $\triangle ABC$ is the isogonal conjugate of the perspector of $\triangle ABC$ and the orthic triangle of the orthic triangle of $\triangle ABC$.

Luis González

#2 Dec 24, 2011, 1:43 pm • 2

D, E, F are the projections of A, B, C on BC, CA, AB and X, Y, Z are the projections of D, E, F on EF, FD, DE. $H \equiv AD \cap BE \cap CF$ is the orthocenter of $\triangle ABC$. (O, R) and $(N, \frac{R}{2})$ are the circumcircle and 9 point circle of $\triangle ABC$. D', E', F' are the antipodes of D, E, F WRT (N). Let M and U be the midpoints of BC and AH. From $\triangle BFD \sim \triangle ECD$ we have $DF \cdot DE = DB \cdot DC = AD \cdot HD$. But $DF \cdot DE = R \cdot XD$, thus $AD \cdot HD = XD \cdot R$. Since $UDMD'$ is a rectangle and $AH = 2UH = 2OM$, we have $HD = OD'$. Hence, $AD \cdot OD' = XD \cdot R$, i.e. $\frac{DA}{DX} = \frac{OA}{OD'} \Longrightarrow \triangle XDA \sim \triangle D'OA$ by SAS, due to $\angle XDA = \angle D'OA$. As a result, $\angle HAX = \angle OAD'$, i.e. AX, AD' are isogonals WRT $\angle A$. Similarly, BY, BE' and CZ, CF' are isogonals WRT $\angle B$ and $\angle C$. $P_r \equiv AD' \cap BE' \cap CF'$ is the Prasolov point of $\triangle ABC$, hence AX, BY, CZ concur at the isogonal conjugate of P_r.

TelvCohl

#3 Apr 24, 2015, 10:09 am

My solution:

Let $\triangle DEF$ be the orthic triangle of $\triangle ABC$.
Let $\triangle XYZ$ be the orthic triangle of $\triangle DEF$.
Let A', B', C' be the midpoint of BC, CA, AB, respectively .
Let D', E', F' be the antipode of D, E, F in $\odot(DEF)$, respectively .
Let D_b, D_c be the projection of D on AC, AB, respectively . (define E_c, E_a, F_a, F_b similarly)
Let O be the circumcenter of $\triangle ABC$ and $P \equiv AD' \cap BE' \cap CF'$ be the Prasolov point of $\triangle ABC$.

Since D_bD_c is the Simson line of D WRT $Rt\triangle ABE$,
so from Steiner theorem $\Longrightarrow$ D_bD_c pass through the midpoint of DE .
Similarly, we can prove D_bD_c pass through the midpoint of $DF \Longrightarrow D_bD_c$ is D-midline of $\triangle DEF$.

Since D', X is the reflection of O, D in $B'C', D_cD_b$, respectively ,
so from $\triangle AB'C' \cup O \sim \triangle AD_cD_b \cup D \Longrightarrow \triangle AB'C' \cup O \cup D' \sim \triangle AD_cD_b \cup D \cup X$,
hence AX is the isogonal conjugate of $AD' \equiv AP$ WRT $\angle BAC$.

Similarly, BY, CZ is the isogonal conjugate of BP, CP WRT $\angle CBA, \angle ACB$, respectively ,
so the perspector of $\triangle ABC$ and $\triangle XYZ$ is the isogoanl conjugate of P WRT $\triangle ABC$.

Q.E.D

Luis González

#4 Apr 24, 2015, 10:40 am

A generalization was posted afterwards. See http://www.artofproblemsolving.com/community/c6h455311.

Quick Reply

High School Olympiads

Isogonal ray

Reply

buratinogigle
#1 Dec 31, 2011, 8:51 pm

Let ABC be triangle. A circle (K) passing through B,C cuts CA,AB at E,F. BE cuts CF at G. AG cuts BC at H. L is projection of H on EF. M is midpoint of BC. MK cuts circumcircle (KEF) again at N. Prove that $\angle LAB=\angle NAC$.

See more, isogonal conjugate of prasolov point.

Luis González
#2 Jan 1, 2012, 2:45 am • 3

Let $\odot(KEF)$ cut $\overline{AC},\overline{AB},\overline{AG}$ at U,V,P respectively. According to the topic An extension of Nine-point circle, $\triangle PUV$ and $\triangle GCB$ are homothetic with center $A.$ Further, $PNUV$ is an isosceles trapezoid with $PN \parallel UV \parallel BC.$ On the other hand, let T be the reflection of G about $EF.$ Since $L(G,T,E,H)=-1$ and $L(G,A,E,H)=-1,$ then $T \in AL.$ Hence, from $\triangle NUV \cong \triangle PVU \sim \triangle GFE \cong \triangle TFE$ and $\triangle AEF \sim \triangle AVU,$ it follows that $AETF \sim AVNU \Longrightarrow$ $\angle TAF=\angle NAU,$ i.e. $\angle LAB=\angle NAC.$

Quick Reply

High School Olympiads

hard geometry

Locked

junior2001

#1 Apr 24, 2015, 9:43 am

$ABCD$ be the parallelogramm, line l passes through B and perpendicular to CD.Two circles with common chord CD and tangent to l at P, Q,respectively.Let M be the midpoint of AB.Prove that $\angle DMP = \angle DMQ$.

Luis González

#2 Apr 24, 2015, 10:34 am

Please give your topics meaningful subjects and proofread before posting. The correct statement states that l is perpendicular to BC and not CD. See the topic An interesting construction with parallelogram.

High School Olympiads

tangent circle

Locked

junior2001

#1 Apr 24, 2015, 9:35 am

Let the circle passes through the vertex A and C of $\triangle ABC$, intersect the sides AB and BC at the points Y and X,respectively.Segments AX and CY intersect at O.Let M is midpoint of AC and N is midpoint of XY.Prove that BO tangent to (MON).

Luis González

#2 Apr 24, 2015, 10:13 am

Posted many times before. It's IMO Shortlist 2009 - Problem G4.

High School Olympiads

IMO Shortlist 2009 - Problem G4

Reply

April

#1 Jul 5, 2010, 5:49 pm • 5

Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

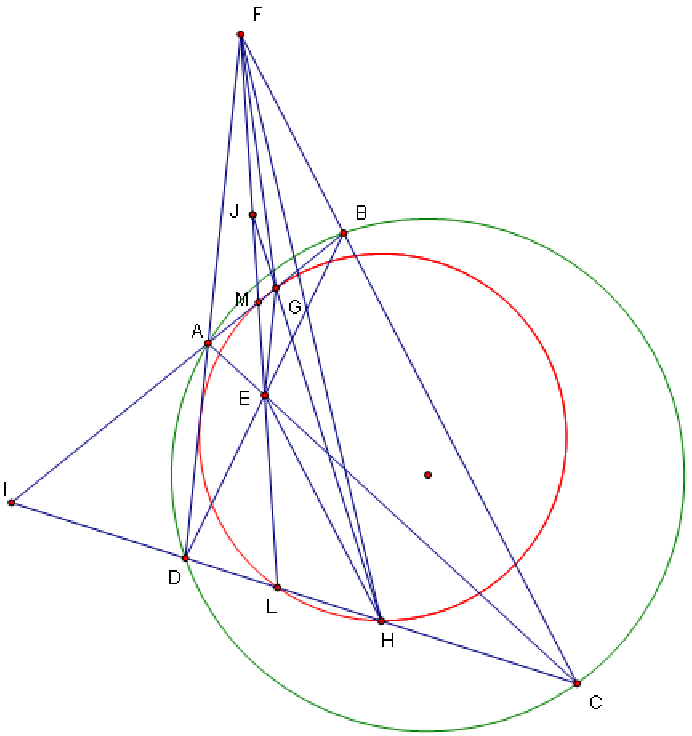
Proposed by David Monk, United Kingdom

livetolove212

#2 Jul 5, 2010, 8:03 pm • 10

Solution:
Let L, M be the intersection of EF and DC and $AB.GH$ cuts EF at J.
Applying Gauss's line we get J is the midpoint of EF.
Since $(FEML) = -1$ then $JE^2 = JM.JL.(1)$
On the other side, $(ILDC) = -1$ and H is the midpoint of DC then $IL.IH = ID.IC$. Similarly, $IM.IG = IA.IB$
So $IM.IG = IL.IH$, which follows that $MGHL$ is a cyclic quadrilateral.
Hence $JM.JL = JG.JH(2)$
From (1) and (2) we obtain $JE^2 = JG.JH$. We are done.

Attachments:



daniel73

#3 Jul 9, 2010, 3:07 pm • 1

Consider the transformation that takes AB into CD after a reflection on the internal bisector of $\angle AFB$ and a dilation of factor $\frac{FC}{FA} = \frac{FD}{FB}$ (equality holds because $ABCD$ is cyclic, hence $FA \cdot FD = FB \cdot FC$ is the power of F with respect to its circumcircle). This transformation takes G into H, and E into E' such that $CE'D$ is similar to AEB. But AEB is also similar to DEC, or $CE'D$ and DEC are equal, and $CEDE'$ is a parallelogram. Similarly, E is the result of applying the transformation to E'' such that $AEBE''$ is a parallelogram. Now, E', E'' are on the symmetric to FE with respect to the internal bisector of $\angle AFB$, or P, E', E'' are collinear. Note also that G, H are the respective midpoints of EE', EE'', or $GH \parallel E'E''$. Now, the transformation preserves angles, or $\angle FEG = \angle FE'H = \angle E''E'E = \angle GHE$, and by semiinscribed angles, the conclusion follows.

MJ GEO

#4 Jul 9, 2010, 9:31 pm

> *April wrote:*
>
> Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

It was posted before here.

Best regards,
Majid.

BlackMax

#5 Jul 14, 2010, 12:15 am

As a matter of fact, exactly this problem (though with another designaions) had been proposed at an olympiad in St. Petersburg (10th grade, problem 7) several months before the shortlist was formed (14.12.2008). Can anyone comment on this coincidence, and how come that the problem from quite a well-known competition managed to pass all checks and had real chances to be posed to those who had been solving it before?

Translation of the official wording from Russian:
A circle passing through points A and C of a triangle ABC intersects its sides AB and BC at points Y and X respectively. The segments AX and CY intersect at a point O. Let M and N be the midponits of the segments AC and XY. Prove that the line BO is tangent to the circumcircle of MON.

daniel73

#6 Jul 14, 2010, 7:06 am • 5

> *BlackMax wrote:*
>
> As a matter of fact, exactly this problem (though with another designaions) had been proposed at an olympiad in St. Petersburg (10th grade, problem 7) several months before the shortlist was formed (14.12.2008). Can anyone comment on this coincidence, and how come that the problem from quite a well-known competition managed to pass all checks and had real chances to be posed to those who had been solving it before?

I was in the Problem Selection Committee for the 2008 IMO in Madrid. We looked everywhere for precedents in competitions, magazines or online site (such as this one) of the longlisted problems. We scratched quite a few problems due to the fact that, either the exact problem, or a subtle variation which introduced no concept variation, had already been proposed/published/solved. Imagine our distress and shame when it was pointed out that still a couple of shortlisted problems (I don't remember exactly whether 2 or 3) had in fact been proposed in some competition; at least, these problems had not been given a wide exposure to the public and, if I remember correctly, no online source for the material could be quoted, just the booklet from the competition; nonetheless we learnt the hard way that it is close to impossible to check everything... there is just too much material out there (which is great for students preparing themselves for the competition, but not for submission of original problems). The only thing that can be done is, when you are on a PSC or Jury, do your best in order to find the problems that, through happenstance or just plain bad will (I will always prefere to believe in the first case if I do not have enough evidence against - innocent until proven guilty), slipped into the longlist/shortlist. And of course, encourage fair play with your own example...

There is something else that also distresses me quite a bit: shortlists are supposed to be kept secret until the following year's IMO, but we find here and there these problems breaking out into the open. I think shortlist problems are a very valuable material for student training/selection (heck, I use it!), but whenever such a problem is used, it should be made sure that the students who are exposed to them, know very well that it is "top-secret" stuff and should not be discussed or commented outside of the team... And just for the sake of extra security, I only work with students on the previous year IMO shortlist about 1-2 weeks before the following IMO...

mgao

#7 Aug 17, 2010, 1:54 am

> livetolove212 wrote:
> $(FEML) = -1$

How do we know that M and L are conjugate points? Is this just a well-known fact?

$\frac{d}{dt}\text{Fail} > t$

dnkywin

#8 Nov 16, 2010, 5:09 am • 2

Sorry to revive this topic, but I suddenly felt the urge to post the solution to this problem that I came up with during a test at MOP.

WLOG let E, F be on opposite sides of AB, and let E be closer to BC than to AD. Let line EF intersect lines AB and CD at J and K, respectively.

Now, if we can prove that $\frac{\sin\angle GHE}{\sin\angle HGE} = \frac{\sin\angle GEJ}{\sin\angle HEK}$, then we are done, since for fixed $0 < \alpha < \pi$, $\frac{\sin x}{\sin(\alpha - x)}$ is increasing, so the above would imply that $\angle GHE = \angle GEJ$, or that EF is tangent to the circumcircle of GEH.

Thus we will now try to prove that

$$\frac{\sin\angle GHE}{\sin\angle HGE} = \frac{\sin\angle GEJ}{\sin\angle HEK}$$

We know that ΔFAB is similar to ΔFCD, ΔFAC is similar to ΔFBD, and ΔAED is similar to ΔBEC. Also, By Ceva's theorem,

$$\frac{GJ}{JB} = \frac{1}{2}\left(\frac{AJ}{JB} - 1\right) = \frac{1}{2}\left(\frac{CF\cdot DA}{BC\cdot FD} - 1\right) = \frac{CF\cdot DA - BC\dot{F}D}{2BC\cdot FD}$$

Similarly,

$$\frac{HK}{KC} = \frac{AD\cdot BF - FA\cdot CB}{2FA\cdot CB}$$

Now we have:

$$\begin{aligned}
1 &= \left(\frac{FB}{FD}\right)^2\left(\frac{FD}{FB}\right)^2 && (1)\\
&= \frac{FB^2 - FA^2}{FD^2 - FC^2}\cdot\frac{FB}{FD}\cdot\left(\frac{FD}{FB}\right)^3 && (2)\\
&= \frac{FA(FD-FA) - FB(FC-FB)}{FD(FD-FA) - FC(FC-FB)}\cdot\frac{FB}{FD}\cdot\left(\frac{CD}{AB}\right)^3 && (3)\\
&= \frac{FA\cdot DA - BC\cdot FB}{AD\cdot AF - FC\cdot CB}\cdot\frac{FB}{FD}\cdot\left(\frac{CD}{AB}\right)^3 && (4)\\
&= \frac{\frac{CD}{AB}(FA\cdot DA - BC\cdot FB)}{\frac{AB}{CD}(AD\cdot AF - FC\cdot CB)}\cdot\frac{FB}{FD}\cdot\frac{CD}{AB} && (5)\\
&= \frac{CF\cdot DA - BC\cdot FD}{AD\cdot BF - FA\cdot CB}\cdot\frac{FA}{FD}\cdot\frac{FB}{FA}\cdot\frac{CD}{AB} && (6)\\
&= \frac{\frac{CF\cdot DA - BC\cdot FD}{2BC\cdot FB}}{\frac{AD\cdot BF - FA\cdot CB}{2FA\cdot BC}}\cdot\frac{FB}{FA}\cdot\frac{CD}{AB} && (7)\\
&= \frac{\frac{JG}{BJ}}{\frac{KH}{CK}}\cdot\frac{FB}{FA}\cdot\frac{CD}{AB} && (8)\\
&= \frac{BE}{CE}\left(\frac{CD}{AB}\right)^2\cdot\frac{\frac{JG}{BJ}}{\frac{KH}{CK}}\cdot\frac{\frac{FB\sin\angle FBE}{EF}}{\frac{FA\sin\angle FAE}{EF}} && (9)\\
&= \frac{BE}{CE}\left(\frac{CD}{AB}\right)^2\cdot\frac{\frac{JG}{BJ}}{\frac{KH}{CK}}\cdot\frac{\sin\angle JEB}{\sin\angle AEJ} && (10)\\
&= \left(\frac{CD}{AB}\right)^2\cdot\frac{JG}{KH}\cdot\frac{\frac{BE\sin\angle JEB}{BJ}}{\frac{CE\sin\angle KEC}{CK}} && (11)\\
&= \left(\frac{CD}{AB}\right)^2\cdot\frac{JG}{KH}\cdot\frac{\sin\angle BJE}{\sin\angle CKE} && (12)\\
&= \left(\frac{CD}{AB}\right)^2\cdot\frac{JG}{KH}\cdot\frac{\sin\angle GJE}{\sin\angle HKE} && (13)
\end{aligned}$$

$$= \left(\frac{CD}{AB} \cdot \frac{GE}{HE}\right) \cdot \frac{HE}{GE} \cdot \frac{\frac{}{GE}}{\frac{KH \sin\angle HKE}{HE}} \qquad (14)$$

$$= \frac{\frac{\sin\angle GEJ}{\sin\angle HEK}}{\frac{\sin\angle GHE}{\sin\angle HGE}} \qquad (15)$$

so we are done.

Notes: steps 9,10,11,12,14, and 15 use the law of sines; step 9 follows from the fact that ΔAED is similar to ΔBEC; step 15 followes from the fact that HE and GE are the corresponding medians in similar triangles ΔDEC and ΔAEB.

sunken rock

#9 Jan 4, 2011, 2:25 am

Take the relation (4) from http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=202546&hilit=Ptolemy and apply it to the quadrilateral $DECF$ and get $\frac{GJ}{HJ} = \left(\frac{GE}{HE}\right)^2$, i.e. FE is tangent to circle (EGH). Here J is the midpoint of EF and, as per Gauss line theorem G, E and J are collinear.

Best regards,
sunken rock

Zhero

#10 Jan 9, 2011, 11:52 am

> *mgao wrote:*
> How do we know that M and L are conjugate points? Is this just a well-known fact?

Let IE hit BC at X. $-1 = (F, X; B, C) = I(F, X; B, C) = I(F, E; M, L) = (F, E; M, L)$.

sunken rock

#11 Jan 15, 2011, 11:27 am • 2

The subject problem has been proposed at a Romanian contest as well; its outstanding official solution can be read here: http://forum.gil.ro/viewtopic.php?f=25&t=255&p=400#p400 .
I hope none of you will face problems in understanding; if there is the case, I might translate it.

Best regards,
sunken rock

daniel73

#12 Jan 15, 2011, 2:02 pm

> *sunken rock wrote:*
> The subject problem has been proposed at a Romanian contest as well; its outstanding official solution can be read here: http://forum.gil.ro/viewtopic.php?f=25&t=255&p=400#p400 .
> I hope none of you will face problems in understanding; if there is the case, I might translate it.
>
> Best regards,
> sunken rock

Very beautiful solution indeed, thanks for the link!

ConcaveCircle

#13 Jan 27, 2011, 5:10 am

This is my solution, I rather like it.
Let O be the circumcenter of $ABCD$. Let the intersection of AB and CD be X.

Lemma: The polar of X with respect to the circumcircle of $ABCD$ is FE.
Proof: Consider G' and H', the poles of AB and CD respectively, which lie on the polar of X. By Pascal's Theorem on $AACBBD$ and $CCADDB$, we have that E, F, G', H' are collinear.

We invert around O. Let the images of $A, B, C, D, E, F, G, H, X$ be A', B'...and so on, respectively. We want to show that the circumcircles of XOE' and $E'G'H'$ are tangent, since inversion preserve tangency.

Let the foot of the perpendicular from E' to XO be Z. Let the pole of $E'Z$ be Y. Let $E'Y$ intersect CD at K, whose image after inversion about O is K'.
The circumcenter K' of $E'G'H'$ is the intersection of CD and the perpendicular bisector of $E'H'$. Let Y' be the intersection of EH' and ZO.
$EKHO$ and $E'ZK'O$ are cyclic because $\angle OEK, \angle OHK, \angle E'ZO, \angle E'K'O$ are right, and thus,
$\angle EYO = \angle EKO + \angle YOK = \angle EHO + \angle ZE'Y' = \angle OE'Y' + \angle ZE'Y' = \angle EE'Z = \angle EY'O$
Thus $Y' = Y$, and so E', Y, K' collinear $\Rightarrow E', Z, K$ collinear.
Since Z, K are the circumcenters of XOE' and $E'G'H'$, and E' is a common point, E' is their point of tangency, and we are done.

RaleD
#14 Jun 6, 2011, 12:18 am

New points: O center of the circle $(ABCD)$, X, Y intersections of tangents in C, D and A, B on $(ABCD)$, N, M intersections of XY with CD and AB, P interesection of CD and AB, T intersection XY and OP.
It is known that XY passes through E and that XY is polar of P with respect to $(ABCD)$.
We will show that $HNMG$ is cyclic. This is enough because then
$\angle HEN = \angle ENC - \angle EHN = \angle HGM - \angle EGM = \angle HGE$.
Now we have $\angle OGM = \frac{\pi}{2} = \angle OTM$ so $OTMG$ is cyclic. Similar $OTNH$ is cyclic. So we have
$PM \cdot PG = PT \cdot PO = PN \cdot PH$; $PM \cdot PG = PN \cdot PH$ and we have $HNMG$ is cyclic.

swaqar
#15 Sep 14, 2011, 3:35 am

I was just wondering about this problem and tried to solve it with spiral symmetries; it revealed many facts about spiral centres. Here is the solution ..

Consider the spiral symmetry with centre S which sends A to D , B to C . $GE \cap CD = H'$ and $HE \cap AB = G'$, $AB \cap CD = I$ then.

It is well-known that S lies on polar $e = IF$ of E and is the inverse of E in circumcircle of $ABCD$. Clearly, $S \in$ the circle γ passing through E , B and A.

$\mathit{Lemma\ 1.}$ The tangent t to the circle γ at S and the line l parallel though CD, passing through E intersect at AB.
$\mathit{Proof:}$ The line l is tangent to the circumcircle ω of $\triangle AEB$. Let $l \cap t = U$. Line AB is the radical axis of circles γ and ω_1 so it's enough to show that $US = UE$.
Then, $\angle ESU = ESI - \angle USI = \angle FSU - 90 = \angle FSA + \angle ASU - 90 =$
$\angle CBA + \angle AFS - 90 = \angle IDA + AFS - 90 = \angle DFI + \angle IDF - 90 = 90 - \angle FID$.
We use oriented angles to avoid case work.
Also, $\angle UES = \angle IVS = 180 - \angle VSI - \angle SIV = 90 - \angle SIB = 90 - \angle FID$
and this proves the lemma.

$\mathit{Lemma\ 2.}$ Lines SG' and FE intersect at E' , E' on the circle γ .
$\mathit{Proof:}$ Let $FE \cap \gamma = E'$. The bundle $(FI, FE; FA, FB)$ is harmonic and so is the quadrilateral $SE'AB$. For triangle BEA, lines EG' and EU are respectively, the symmedian and tangent to ω at E, so the range $(U, G'; A, B)$ is harmonic and so is the bundle $(SU, SG'; SA, SB)$ and so intersecting this bundle with γ gives a harmonic quadrilateral $SG''AB$ and so $G'' = E'$ which proves the lemma.

We now prove the main result. Let $H'G'$ intersect CB in X . Since the initial spiral symmetry also sends G' to H', the four points X , B , G' and S lie on a circle, so $\angle G'SB = G'XB$. As, $SE'BF$ is cyclic,
$\angle G'SB = \angle E'SB = \angle E'FB = \angle EFB$.

So, $\angle GX'B = \angle EFB$ which implies that the antiparallel to GH in triangle HEG is parallel to EF which completes the proof.

sjaelee
#16 Sep 10, 2012, 7:24 am

Let the point H' such that CFDH' is a parallelogram, and G' be a point such that FBG'A is a parallelogram. Extende FE to intersect CD at J and AB at I. Note that quadrilaterals ECFD and EBG'A are similar since ▲BFA~▲CFD, with F corresponding to G'.

Since EF correspands to EG', EH' corresponds to EF, and □FEH'=□G'EF (from similar quadrilaterals), points E, H', G' are collinear. Let EG' intersect CD at K and AB at L. ▲EKD~▲EIB (corresponding parts of quadrilaterals), thus JKLI is cyclic. Since HG || H'G', JHGI is cyclic. Then, □HGA=□FJH and □FGB=□JHF since FH and FG are medians of similar triangles BFA and CFD.

We have □HGA+□FGB+□FGH=180=□FJH+□JHF+□JFH. Using the above angle equalities, □JFH= □FGH. Thus EF is tangent to the circumcircle of FGH.

thecmd999

#17 Dec 21, 2013, 2:04 am • 1

Nice problem.

Quick

IDMasterz

#18 Dec 21, 2013, 9:20 am • 1

Note Newton Gauss line, so M be the midpoint of EF, and $M \in GH$. Also, let O be circumcentre of $ABCD$ and AB, CD meet at I where it follows $OGHI$ are concyclic, or GH, EF are antiparallel, or $EF \cap AB = X, EF \cap CD = Y, G, H$ are concyclic. But $ME^2 = MX \cdot MY = MG \cdot MH$, so the problem just got destroyed.

Blitzkrieg97

#19 Apr 13, 2014, 12:56 am

> *sunken rock wrote:*
> The subject problem has been proposed at a Romanian contest as well; its outstanding official solution can be read here: http://forum.gil.ro/viewtopic.php?f=25&t=255&p=400#p400 .
> I hope none of you will face problems in understanding; if there is the case, I might translate it.
>
> Best regards,
> sunken rock

can you tell me why M is midpoint of AO? the rest is clear

sunken rock

#20 Apr 13, 2014, 3:31 am • 1

@Blitzkrieg97: $AEODBC$ is a complete quadrilateral ($AEOB$ the quad, and B, C the intersections of opposite sides), hence AO, DE, BC are its diagonals, which have their midpoints collinear (Newton-Gauss line).

Best regards,
sunken rock

junioragd

#21 Aug 7, 2014, 12:56 am

During to similarites and angle chasing we easy get <FGE+<FHE=180.Also,from similar triangles(I won't wrote this part because it is obvious) we get GE/GF=HE/HF.Now,we will to prove that
EF is the tangent to both circumcircles GHE and FHG.Now,this is a well known,but I will write the solution: Invert about G and let X' be the picture of X under this inversion.Now,we see that reduces to prove that E'H'=F'H',but this is easy using the fact GE/GF=HE/HF,so we are finished.

utkarshgupta

#22 Apr 25, 2016, 5:55 pm • 3

Solution :

We will work in the Cartesian plane.
$X = (x_1, y_1)$ will mean that the x,y coordinates of the point X are x_1, y_1 respectively.
The slope of a line l will be denoted by m_l.

The slope of a line ℓ will be denoted by m_ℓ.

Without loss of Generality,

$$E = (0, 0)$$

,

$$A = (1, 0)$$

,

$$C = (-pq, 0)$$

Let $m_{BD} = \tan\theta$

So set

$$B = (p\cos\theta, p\sin\theta)$$

Using B, D lie on the opposite sides of AC, and $EB \cdot ED = EA \cdot EC$.

$$D = (-q\cos\theta, -q\cos\theta)$$

Now easy calculations yield
$F = AD \cap BC$

$$F = (\frac{q(p + q + \cos\theta + pq\cos\theta)}{q^2 - 1}, \frac{q\sin\theta(1 + pq)}{q^2 - 1})$$

$$G = (\frac{1 + p\cos\theta}{2}, \frac{p\sin\theta}{2})$$

$$H = (\frac{-pq - q\cos\theta}{,} \frac{-q\sin\theta}{2})$$

Observe that the given problem is equivalent to proving directed angles, $\angle FEG = \angle EHG$

That is $\dfrac{m_{GH} - m_{EH}}{1 + m_{GH}m_{EH}} = \dfrac{m_{GE} - m_{EF}}{1 + m_{GE}m_{EF}}$

Now this is just easy calculations (they are actually easy as most of the things cancel out).

This post has been edited 1 time. Last edited by utkarshgupta, Apr 25, 2016, 5:57 pm

AMN300

#23 May 22, 2016, 3:42 am

We do not use Cartesian coordinates.

solution

Quick Reply

High School Olympiads

Isogonal conjugates points

Reply

Source: A collolary of a problem on Mathematic and Youth Magazine

A-B-C

#1 Apr 23, 2015, 8:44 pm

Given 4 points A, B, C, D
Let P be an arbitary point such that P does not lie on these circles: $(BCD), (CDA), (DAB), (ABC)$
A', B', C', D' are the isogonal conjugates of P with respect to $\triangle BCD, \triangle CDA, \triangle DAB, \triangle ABC$
Prove that A', B', C', D' are concyclic if and only if A, B, C, D are concyclic.

Luis González

#2 Apr 23, 2015, 9:12 pm • 1

Recalling that the midpoint between two isogonal conjugates is the center of their pedal circle, then this follows from 4 centers of the pedal circles are concyclic. All steps are reversible, so the converse is proved in the same way.

Quick Reply

High School Olympiads

4 centers of the pedal circles are concyclic

Source: own

livetolove212

#1 Aug 7, 2011, 10:11 am

Given a cyclic quadrilateral $ABCD$. Let X be an arbitrary point in the plane such that X is not lie on $(ABCD)$. Prove that the centers of the pedal circles of X wrt ABC, BCD, CDA, DAB are concyclic.

Luis González

#2 Aug 7, 2011, 10:57 pm

We also prove that those circles concur for any quadrilateral $ABCD$. Let P, Q, R, S, U, V be the projections of X onto AB, BC, CD, DA, AC, BD. $(O_1), (O_2), (O_3), (O_4)$ denote the pedal circles of X WRT $\triangle ABC, \triangle BCD, \triangle CDA, \triangle DAB$. Let T be the 2nd intersection of $(O_1), (O_4)$. Using oriented angles $\pmod \pi$, we have

$$\angle(TV, TQ) = \angle(TP, TQ) + \angle(TV, TP) = \angle(UP, UQ) + \angle(SV, SP) \ \ (1)$$

$$\angle(UP, UQ) = \angle(UC, UQ) + \angle(UP, UA) = \angle(RC, RQ) + \angle(SP, SA) \ \ (2)$$

$$\angle(SV, SP) = \angle(SV, SA) - \angle(SP, SA) = \angle(RV, RD) - \angle(SP, SA) \ \ (3)$$

Substituting $\angle(UP, UQ)$ and $\angle(SV, SP)$ from (2) and (3) into (1) yields

$\angle(TV, TQ) = \angle(RC, RQ) + \angle(RV, RD) = \angle(RV, RQ) \Longrightarrow T \in (O_2)$. Likewise, we'll have $T \in (O_3)$. $\square$

Now, we assume that $ABCD$ is cyclic. From $O_1O_4 \perp TP$ and $O_1O_2 \perp TQ$, we get

$$\angle(O_1O_4, O_1O_2) = \angle(TP, TQ) = \angle(UP, UQ) \ , \ \angle(O_3O_4, O_3O_2) = \angle(US, UR)$$

But $\angle(UQ, UP) = \angle(CB, CD) + \angle(AD, AB) - \angle(US, UR) \Longrightarrow \angle(UP, UQ) = \angle(US, UR)$. Therefore, $\angle(O_1O_4, O_1O_2) = \angle(O_3O_4, O_3O_2) \Longrightarrow O_1, O_2, O_3, O_4$ are concyclic.

High School Math

Prove or Disprove

KudouShinichi

#1 Apr 23, 2015, 2:14 pm

Inspired from ISL 10 G2

Let P be a point interior or exterior of triangle $\triangle ABC$ (with $CA \neq CB$) but not on circumcircle of $\triangle ABC$. The lines AP, BP and CP meet again its circumcircle Γ at K, L, respectively M. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.

(Prove or disprove)

Luis González

#2 Apr 23, 2015, 8:50 pm

From $\triangle PBC \sim \triangle PML$ and $\triangle PAC \sim \triangle PMK,$ we get $\frac{PB}{PM} = \frac{CB}{ML}$ and $\frac{PA}{PM} = \frac{CA}{MK} \Longrightarrow \frac{PA}{PB} = \frac{CA}{CB} \cdot \frac{ML}{MK}.$ Consequently, $MK = ML \Longleftrightarrow \frac{PA}{PB} = \frac{CA}{CB} \Longleftrightarrow P$ is on the C-Apollonius circle $\odot(S, SC)$ of $\triangle ABC,$ i.e. $SC = CP.$

KudouShinichi

#3 Apr 24, 2015, 12:44 am

$\triangle PBC \sim \triangle PML$ was of very great use thanks...... Sir Luis Gonzales

High School Math

Geometry problem

Reply

Higgsboson79

#1 Apr 22, 2015, 8:16 am

Let $ABCD$ be a cyclic quadrilateral, and let E and F be points on rays BA and DC, respectively, such that $BE=BC$ and $DF=DA$. Prove that line BD bisects segment EF.

Luis González

#2 Apr 22, 2015, 11:59 am • 1

Let $P \equiv AB \cap CD$ and $M \equiv BD \cap EF$. Apply Menelaus' theorem to $\triangle PEF$ cut by BD, keeping in mind that $\triangle PBC \sim \triangle PDA$, i.e. $\frac{BP}{BC}=\frac{DP}{DA}$.

$$\frac{BE}{BP}\cdot\frac{PD}{DF}\cdot\frac{FM}{ME}=1 \Longrightarrow \frac{FM}{ME}=\frac{BP}{BE}\cdot\frac{FD}{DP}=\frac{BP}{BC}\cdot\frac{DA}{DP}=1 \Longrightarrow FM=ME.$$

sunken rock

#3 Apr 22, 2015, 6:20 pm • 1

Excellent solution!
Remark: generalization
If $\frac{BE}{BC}=\frac{DF}{AD}=k\neq 1$, the property still holds, similar proof.

Best regards,
sunken rock

sunken rock

#4 Apr 23, 2015, 3:33 am • 1

Let $P\in AB\cap CD, M\in EF\cap BD$.
Applying sine law in $\triangle BME \implies \frac{BE}{sin\widehat{BME}}=\frac{ME}{sin\widehat{ABD}}$ (1).
applying sine law in $\triangle DMF \implies \frac{DF}{sin\widehat{DMF}}=\frac{MF}{sin\widehat{BDC}}$ (2).
Dividing side by side relation $(1):(2) \implies \frac{BE}{DF}=\frac{ME}{MF}\cdot\frac{sin\widehat{BDC}}{sin\widehat{ABD}}$ (3).
$\triangle BCP\sim\triangle DAP \implies \frac{BC}{AD}=\frac{BP}{DP}$ (4).
sine law in $\triangle BDP \implies \frac{BP}{sin\widehat{BDC}}=\frac{DP}{sin\widehat{ABD}}$ (5).
From (3), with (4) and (5), keeping in mind $BE=BC, DF=AD$ we get the desired $ME=MF$.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Angle related to Fermat points and Isodynamic points

Reply

Source: Own

TelvCohl

#1 Mar 13, 2015, 12:06 am • 1

Let ℓ_1 be a line connecting 1st Fermat point F_1 and 2nd Isodynamic point S_2 of $\triangle ABC$.
Let ℓ_2 be a line connecting 2nd Fermat point F_2 and 1st Isodynamic point S_1 of $\triangle ABC$.
Let ℓ_3 be the reflection of ℓ_1 in BC and $X = \ell_2 \cap \ell_3$.

Prove that $\angle BXC = 60^\circ$.

buratinogigle

#2 Mar 13, 2015, 9:47 am • 2

Nice dear Telv. I see this problem is true for two antigonal conjugate points as following

Let ABC be a triangle and P_1, P_2 are two antigonal conjugate points. Q_1, Q_2 are isogonal conjugate of P_1, P_2. Reflection of line P_2Q_1 thourgh BC cuts line P_1Q_2 at X then $\angle BXC = \angle BP_1C$.

Luis González

#3 Mar 13, 2015, 11:20 am • 3

> *buratinogigle wrote:*
> Let ABC be a triangle and P_1, P_2 are two antigonal conjugate points. Q_1, Q_2 are isogonal conjugate of P_1, P_2. Reflection of line P_2Q_1 through BC cuts line P_1Q_2 at X then $\angle BXC = \angle BP_1C$.

Lemma: If $\{X, X^*\}$ and $\{Y, Y^*\}$ are pairs of isogonal conjugates WRT $\triangle ABC$, then the intersections $XY \cap X^*Y^*$ and $XY^* \cap YX^*$ are also isogonal conjugates WRT $\triangle ABC$.

The proof can be found in the book Geometry of conics by A.V. Akopyan and A.A. Zaslavsky (page 90).

Back to the problem, let $\mathcal{H}$ be the rectangular hyperbola through A, B, C, P_1, P_2. Isogonals Q_1, Q_2 of antigonal conjugates P_1, P_2 lie then on the isogonal line τ of $\mathcal{H}$ WRT $\triangle ABC$. From the previous lemma, we deduce that $P_1P_2 \cap Q_1Q_2 \equiv \tau$ is the isogonal of $R \equiv P_1Q_2 \cap P_2Q_1$ WRT $\triangle ABC \Longrightarrow R \in \mathcal{H}$. But $\mathcal{H}$ is precisely the isogonal of the perpendicular bisector of $\overline{P_1P_2}$ WRT either $\triangle AP_1P_2, \triangle BP_1P_2$ or $\triangle CP_1P_2 \Longrightarrow \angle RP_1C = \angle RP_2C \Longrightarrow$
$\angle P_2RP_1 = \angle P_1CP_2 + \angle CP_1R + \angle CP_2R = \angle P_1CP_2 + 2 \cdot \angle CP_2R$ $(\star)$.

If $U \in RP_2 \cap BC$ and $V \in UX$ is the reflection of P_2 on BC, we get $\angle P_1XV \equiv \angle P_1XU = \angle P_1RP_2 - 2 \cdot \angle P_2UC$ and $\angle P_1CV = \angle P_1CP_2 + 2 \cdot \angle P_2CU \Longrightarrow \angle P_1XV - \angle P_1CV = \angle P_2RP_1 - 2 \cdot \angle CP_2R - \angle P_1CP_2$. Combined with $(\star)$, we get then $\angle P_1XV = \angle P_1CV \Longrightarrow C, P_1, X, V$ are concyclic and in the same way B, P_1, X, V are concyclic $\Longrightarrow B, C, X, P_1$ are concyclic $\Longrightarrow \angle BXC = \angle BP_1C \pmod{\pi}$.

Quick Reply

High School Olympiads

Collinear

Reply

oneplusone
#1 Apr 16, 2011, 2:20 pm

Let AD, BE be angle bisectors of $\triangle ABC$, with D on BC and E on AC. Let AD intersect BE at I. Let X be any point and Y is its isogonal conjugate wrt $\triangle ABC$. Let AX, BX intersect BC, AC at M, N. Prove that D, Y, E are collinear if and only if M, I, N are collinear.

Luis González
#2 Apr 16, 2011, 9:47 pm • 2

This is a nice generalization of the problem discussed in bisectors and altitudes. We use barycentric coordinates WRT $\triangle ABC$. Thus, let $(u:v:w)$ be the coordinates of X.

$M\ (0:v:w)\ ,\ N\ (u:0:w)$ and $I\ (a:b:c)$ are collinear $\Longleftrightarrow$

$$\begin{pmatrix} 0 & v & w \\ u & 0 & w \\ a & b & c \end{pmatrix} = 0 \Longleftrightarrow avw + buw - cuv = 0 \ \ (1)$$

$D\ (0:b:c)\ ,\ E\ (a:0:c)$ and $Y\left(\frac{a^2}{u}:\frac{b^2}{v}:\frac{c^2}{w}\right)$ are collinear $\Longleftrightarrow$

$$\begin{pmatrix} 0 & b & c \\ a & 0 & c \\ \frac{a^2}{u} & \frac{b^2}{v} & \frac{c^2}{w} \end{pmatrix} = 0 \Longleftrightarrow \frac{abc(avw + buw - cuv)}{uvw} = 0 \ \ (2)$$

From (1) and (2) we conclude that M, I, N are collinear $\Longleftrightarrow D, E, Y$ are collinear.

oneplusone
#3 Apr 17, 2011, 7:48 am • 2

This is my solution.
First suppose M, I, N are collinear. Let J, K be excenters opposite A, B. Let Y' be intersection of JN and KM. Then by Pappus, E, Y', D are collinear. Let T be the intersection of NJ and BE. Then since $AIDJ$ is harmonic, $NTY'J$ is harmonic. Since $\angle TBJ = 90$, we have BE is the angle bisector of $\angle NBY'$. Smilarly AD is the angle bisector of $\angle MAY'$. Thus X, Y' are isogonal conjugates, so $Y \equiv Y'$, so E, Y, D are collinear.

Now for the other direction. Let NI intersects BC at M', and BN intersects AM' at X'. Then the isogonal conjugate of X', Y' is on DE. But since Y' lies on BY and D, Y, E are collinear, we must have $Y \equiv Y'$, and everything else follows.

TelvCohl
#4 Apr 19, 2015, 6:49 pm • 3

Generalization :

Let $\{P, P'\}, \{Q, Q'\}$ be two pairs of isogonal conjugate of $\triangle ABC$.
Let $\triangle P_AP_BP_C, \triangle Q_AQ_BQ_C$ be the cevian triangle of P, Q WRT $\triangle ABC$, respectively .

Then $Q' \in P_BP_C \Longleftrightarrow P' \in Q_BQ_C$.

Proof :

By symmetry, it suffices to prove $Q' \in P_BP_C \Longrightarrow P' \in Q_BQ_C$.

Let $\triangle P_aP_bP_c, \triangle P'_aP'_bP'_c$ be the circumcevian triangle of P, P' WRT $\triangle ABC$, respectively .
Let $\triangle Q_aQ_bQ_c, \triangle Q'_aQ'_bQ'_c$ be the circumcevian triangle of Q, Q' WRT $\triangle ABC$, respectively .
Let X be the intersection of $P_BQ'_b$ and $P_CQ'_c$.

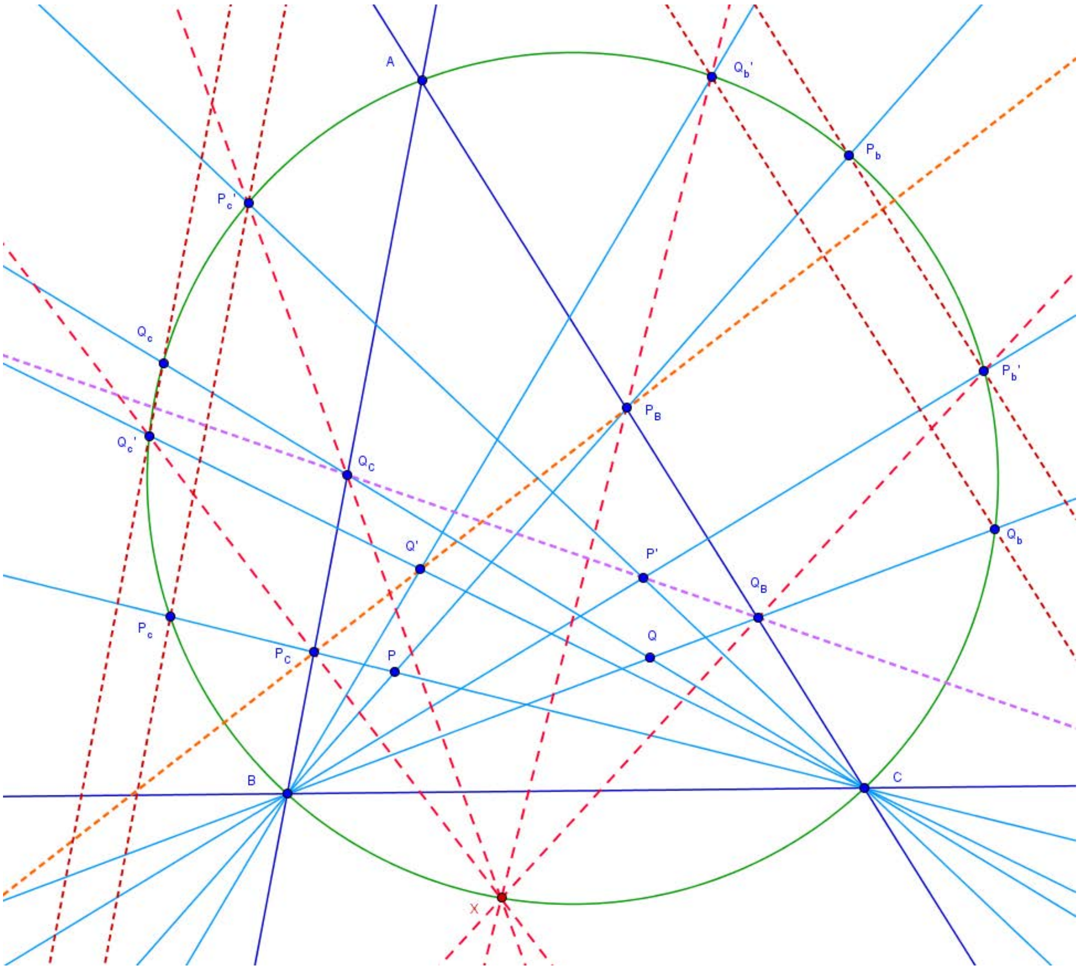
From Pascal theorem (for $XQ'_cCABQ'_b$) we get $X \in \odot(ABC)$.

From Pascal theroem (for $XP'_cP_cCQ_cQ'_c$) we get $X \in Q_CP'_c$.
From Pascal theroem (for $XP'_bP_bBQ_bQ'_b$) we get $X \in Q_BP'_b$.

From Pascal theroem (for $XP'_bBACP'_c$) we get $P' \in Q_BQ_C$.

Done 🙂

Attachments:



Luis González

#5 Apr 22, 2015, 4:38 am • 1

Another proof to the generalization mentioned by Telv:

Let $P'B, P'C$ cut CA, AB at Y, Z and let $Q'B, Q'C$ cut CA, AB at E, F. Assume that $Q' \in P_BP_C \Longrightarrow (A, B, P_C, F) = (A, E, P_B, C)$. But $(A, B, P_C, F) = (B, A, Z, Q_C)$ and $(A, E, P_B, C) = (C, Q_B, Y, A) \Longrightarrow (B, A, Z, Q_C) = (C, Q_B, Y, A)$ or $(B, A, Z, Q_C) = (Y, A, C, Q_B) \Longrightarrow P' \in Q_BQ_C$. The converse is proved in the same way.

P.S. The problem is still true if we replace $\{P, P'\}$ and $\{Q, Q'\}$ by two pairs of isotomic conjugates.The proof is exactly the same as the previous one.

jayme

#6 May 21, 2015, 6:17 pm

Dear Mathlinkers,
you can see

http://jl.ayme.pagesperso-orange.fr/Docs/Deux%20couples%20de%20points%20isogonaux.pdf

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

BMO1 2007/08 Question 5 Geomertry Problem

Locked

Source: BMO1 2007/08

MadChickenMan

#1 Apr 21, 2015, 10:48 pm

$5.$ Let P be an internal point of triangle ABC. The line through P parallel to AB meets BC at L, the line through P parallel to BC meets CA at M, and the line through P parallel to CA meets AB at N. Prove that

$$\frac{BL}{LC} \times \frac{CM}{MA} \times \frac{AN}{NB} \leq \frac{1}{8}$$

and locate the position of P in triangle ABC when equality holds.

Can anyone come up with a full solution? I've got the first part, but I'm not so sure about the second part and I'm interested to see what answer people come up with.

Luis González

#2 Apr 21, 2015, 10:53 pm

Posted before at http://www.artofproblemsolving.com/community/c6h620698.

High School Olympiads

triangle problem

Source: BMO 2007

cr1

#1 Jan 12, 2015, 6:34 am

Let P be internal point of triangle ABC . The line through P parallel to AB meets BC at L , The line through P parallel to BC meets CA at M, The line through P parallel to CA meets AB at N. Prove that

$$\frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN}{NB}\leq\frac{1}{8}$$

Luis González

#2 Jan 14, 2015, 9:54 am

Let u,v,w denote the areas of $\triangle PBC, \triangle PCA, \triangle PAB,$ respectively. If PC cuts AB at C', we have

$$\frac{CL}{LB}=\frac{CP}{PC'}=\frac{CC'-PC'}{PC'}=\frac{CC'}{PC'}-1=\frac{u+v+w}{w}-1=\frac{u+v}{w}.$$

Similarly, we have $\frac{CM}{MA}=\frac{u}{v+w}$ and $\frac{AN}{NB}=\frac{v}{w+u}$.

Substituting these expressions into the object inequality, we obtain $8\cdot u\cdot v\cdot w\leq(u+v)(v+w)(w+u)$, which simply follows by multiplying $2\sqrt{u\cdot v}\leq u+v$ (AM-GM) and the cyclic expressions. Equality holds iff $u=v=w$, i.e. iff P is the centroid of $\triangle ABC$.

P.S. For more solutions see the topic A Geometric Inequality.

High School Olympiads

Nice theorem

Reply

Source: van khea

vankhea

#1 Mar 26, 2015, 9:07 pm

Let D, E, F be points on the side $[BC], [CA], [AB]$ respectively.
Let $P = EF \cap AD, Q = FD \cap BE, R = DE \cap CF$. Let $X = AB \cap QR, Y = AD \cap QR, Z = AC \cap QR$.
Prove that: $\frac{BX}{XA}.\frac{AF}{FB} + \frac{DY}{YA}.\frac{AP}{PD} + \frac{CZ}{ZA}.\frac{AE}{EC} = 1$

Luis González

#2 Apr 21, 2015, 10:47 pm

Consider a homology sending the line QR to infinity. Since the cross ratios in the object expression remain invariant, it suffices to prove in the new figure that $\frac{AF}{FB} + \frac{AP}{PD} + \frac{AE}{EC} = 1.$

Let $U \equiv DF \cap AC$ and $V \equiv DE \cap AB.$ Since $CF \parallel DE$ and $BE \parallel DF,$ we get $\frac{AF}{FB} = \frac{AU}{UE} = \frac{[AFD]}{[DEF]}$ and $\frac{AE}{EC} = \frac{AV}{VF} = \frac{[ADE]}{[DEF]} \Longrightarrow$

$$\frac{AF}{FB} + \frac{AP}{PD} + \frac{AE}{EC} = \frac{[AFD]}{[DEF]} + \frac{[ADE]}{[DEF]} + \frac{[AEF]}{[DEF]} = \frac{[DEF]}{[DEF]} = 1.$$

vankhea

#3 Apr 26, 2015, 7:29 am

Thanks you Teacher
Here is my solution
Theorem 1
Let D, E, F be points on the sides BC, CA, AB and let O be intersection of AD and EF. Then we get
$\frac{DO}{OA}.BC = \frac{BF}{FA}.DC + \frac{CE}{EA}.BD.$
Theorem 2 (Mihaileanu's theorem)
Let C' and B' be points on the sides AB and AC and let O be intersection of BB' and CC'. The line (l) pass through point O cut AB at M and cut AC at N. Then we get: $\frac{BM}{MA}.\frac{AC'}{C'B} + \frac{BN}{NA}.\frac{AB'}{B'C} = 1$

Solution of my problem
Let AD met CF and BE at K and L.
By using theorem 1 we have
$\frac{DY}{YA} = \frac{BX}{XA}.\frac{DC}{BC} + \frac{CZ}{ZA}.\frac{BD}{BC}; (1)$
By using theorem 2 in triangle ΔABD we get:
$\frac{BX}{XA}.\frac{AF}{FB} + \frac{DY}{YA}.\frac{AL}{LD} = 1; (a)$
By using Theorem 2 in triangle ΔACD we have:
$\frac{CZ}{ZA}.\frac{AE}{EC} + \frac{DY}{YA}.\frac{AK}{KD} = 1; (b)$
Adding the equation above we get
$\frac{BX}{XA}.\frac{AF}{FB} + \frac{CZ}{ZA}.\frac{AE}{EC} + \frac{DY}{YA}(\frac{AK}{KD} + \frac{AL}{LD}) = 2$
$\Rightarrow \frac{BX}{XA}.\frac{AF}{FB} + \frac{CZ}{ZA}.\frac{AE}{EC} = 2 - \frac{DY}{YA}(\frac{AK}{KD} + \frac{AL}{LD}); (2)$
From the equation (a) and (b) we have
$\frac{BX}{} = (1 - \frac{DY}{}.\frac{AL}{}).\frac{FB}{}$

$$\frac{CZ}{ZA} = (1 - \frac{DY}{YA}.\frac{AK}{KD}).\frac{EC}{AE}$$
Apply to (1) we get:
$$\frac{DY}{YA} = (1 - \frac{DY}{YA}.\frac{AL}{LD}).\frac{FB}{AF}.\frac{DC}{BC} + (1 - \frac{DY}{YA}.\frac{AK}{KD}).\frac{EC}{AE}.\frac{BD}{BC}$$
By using Menelaus's theorem in triangle ΔABD with F, K, C collinear and in triangle ΔACD with E, L, B collinear we get:
$$\frac{BF}{FA}.\frac{DC}{BC} = \frac{KD}{AK}; \frac{CE}{EA}\frac{BD}{BC} = \frac{LD}{AL}$$
$$\Rightarrow \frac{DY}{YA} = (1 - \frac{DY}{YA}.\frac{AL}{LD}).\frac{KD}{AK} + (1 - \frac{DY}{YA}.\frac{AK}{KD}).\frac{LD}{AL}$$
$$\Rightarrow \frac{DY}{YA} = \frac{\frac{KD}{AK} + \frac{LD}{AL}}{1 + \frac{AL}{LD}.\frac{KD}{AK} + \frac{AK}{KD}.\frac{LD}{AL}}; (c)$$
Apply to the equation (2) we get
$$\Rightarrow \frac{BX}{XA}.\frac{AF}{FB} + \frac{CZ}{ZA}.\frac{AE}{EC} = \frac{\frac{AL}{LD}.\frac{KD}{AK} + \frac{AK}{KD}.\frac{LD}{AL}}{1 + \frac{AL}{LD}.\frac{KD}{AK} + \frac{AK}{KD}.\frac{LD}{AL}}; (3)$$
By using theorem 1 we have
$$\frac{DP}{PA} = \frac{BF}{FA}.\frac{DC}{BC} + \frac{CE}{EA}.\frac{BD}{BC} = \frac{KD}{AK} + \frac{LD}{AL}$$
$$\Rightarrow \frac{AP}{PD} = \frac{1}{\frac{KD}{AK} + \frac{LD}{AL}}; (d)$$
From the equation (c) and (d) we get:
$$\frac{DY}{YA}.\frac{AP}{PD} = \frac{1}{1 + \frac{AL}{LD}.\frac{KD}{AK} + \frac{AK}{KD}.\frac{LD}{AL}}; (4)$$
From (3) and (4) we get the result.

Quick Reply

High School Olympiads

Need help with hard geometry problem

Locked

mihaisar

#1 Apr 21, 2015, 1:04 pm

It probably makes use of homothety but I wasn't able to find it.

Given a triangle ABC find on the sides AC and BC points X and Y such that AX=XY=YB.

Luis González

#2 Apr 21, 2015, 10:16 pm

Discussed earlier at http://www.artofproblemsolving.com/community/c6h1079974.

High School Olympiads

Nice geometry

Reply

mamavuabo
#1 Apr 20, 2015, 9:27 pm

How can we contruct M, N respectively on the sides AB, AC of the triangle ABC such that BM=MN=NC?

Luis González
#2 Apr 20, 2015, 9:46 pm • 1

Denote $P \equiv BN \cap CM$. Easy angle chase reveals that $\angle BPC = 90^\circ + \tfrac{1}{2}\angle BAC = \angle BIC$, where I is the incenter of $\triangle ABC$. On the other hand, if D is the point forming the parallelogram $ABDC$, then the locus of the points P fulfilling $BM = CN$ alone is the parallel ℓ from D to AI. Thus P is found as the intersection of the circular arc BIC with ℓ. BP, CP cut AC, AB at N, M.

mamavuabo
#3 Apr 20, 2015, 10:00 pm

Thank you very much, Luis.Very nice solution!

mamavuabo
#4 Apr 20, 2015, 10:12 pm

But wait, why the locus of P is the parallel from D to AI?

Luis González
#5 Apr 20, 2015, 10:53 pm • 1

Let M be the midpoint of BC and let D and Q be the reflections of A and P about M, resp. Since $BPCQ$ is parallelogram $\Longrightarrow [BMQ] = [BPQ] = [CPQ] = [CNQ] \Longrightarrow \tfrac{1}{2}\text{dist}(Q, BM) \cdot BM = \tfrac{1}{2}\text{dist}(Q, CN) \cdot CN$. Since $BM = CN$ $\Longrightarrow \text{dist}(Q, BM) = \text{dist}(Q, CN) \Longrightarrow Q$ is equidistant from $AB, AC \Longrightarrow Q$ is on internal bisector AI of $\angle BAC \Longrightarrow P$ is on internal bisector of $\angle BDC$, i.e. the parallel to AI through D.

Quick Reply

High School Olympiads

Q is centroid of KLN

Reply

Source: Own

buratinogigle

#1 Sep 16, 2013, 11:23 pm

Let ABC be a triangle and P,Q,R are three collinear points. PA,PB,PC cut BC,CA,AB at D,E,F, respectively. QA,QB,QC cut RD,RE,RF at X,Y,Z, respectively. K,L,N are the points on PX,PY,PZ, respectively, such that $QK\parallel PA,QL\parallel PB,QN\parallel PC$. Prove that Q is centroid of triangle KLN.

THVSH

#2 Apr 10, 2015, 5:39 pm • 2

My solution

Let $EF\cap BC=U\longrightarrow (B,C;D,U)=-1$.
The line passing through B and parallel to PU intersects PC at V.
Let l be the line passing through Q and parallel to PU.
We have:
$\frac{QL}{QN}=\frac{QL}{PB}.\frac{PC}{QN}.\frac{PB}{PC}=\frac{YQ}{YB}.\frac{ZC}{ZQ}.\frac{PB}{PC}=(\frac{RQ}{RP}.\frac{EP}{EB}).(\frac{RP}{RQ}.\frac{FC}{FP}).\frac{PB}{PC}=\frac{EP}{EB}.\frac{FC}{FP}.\frac{PB}{PC}=\frac{DC}{DB}.\frac{PB}{PC}=\frac{UC}{UB}.\frac{PB}{PC}=\frac{PC}{PV}.\frac{PB}{PC}=\frac{PB}{PV}$
Since $QL\parallel PB;QN\parallel PC$, we get $\triangle QLN\sim\triangle PBV\Longrightarrow NL\parallel BV\parallel PU\parallel l$
So $(QL,QN;QK,l)=P(B,C;D,U)=-1$. Since $l\parallel NL$, we get: QK passes through the midpoint of NL.
Similarly, QL passes through the midpoint of NK, QN passes through the midpoint of KL.
Therefore, Q is the centroid of $\triangle KLN$ Q.E.D

Luis González

#3 Apr 21, 2015, 11:49 am • 1

Fix the line PQ and animate R. Clearly the series Y,Z are projective and therefore the series L,N are also projective. When $R\equiv Q$, then $N\equiv L\equiv Q$ and when $R\equiv P$, then L,N go to infinity $\Longrightarrow L\mapsto N$ is a perspectivity with fixed center at infinity, i.e. all lines NL are parallel.

Consider the case when $R\in EF$. Let $S\equiv EF\cap BC$ and PS cuts QB,QC,QN,QL at Y',Z',N',L'. Then $(N,Q,N',\infty)=(Z,Q,Z',C)=(Y,Q,Y',B)=(L,Q,L',\infty)\Longrightarrow\frac{NQ}{NN'}=\frac{LQ}{LL'}\Longrightarrow NL\parallel PS$. Now as the pencil formed by parallels from Q to PB,PC,PA,PS is harmonic, we deduce that QK cuts NL at its midpoint $\Longrightarrow KQ$ is the K-median of $\triangle KLN$ and similarly LQ,NQ are their other medians $\Longrightarrow Q$ is centroid of $\triangle KLN$.

tranquanghuy7198

#4 May 22, 2015, 3:36 pm

This is the solution using vectors:

Set: $P=\frac{\Sigma\alpha.A}{\Sigma\alpha},\frac{\overline{QR}}{\overline{RP}}=k$

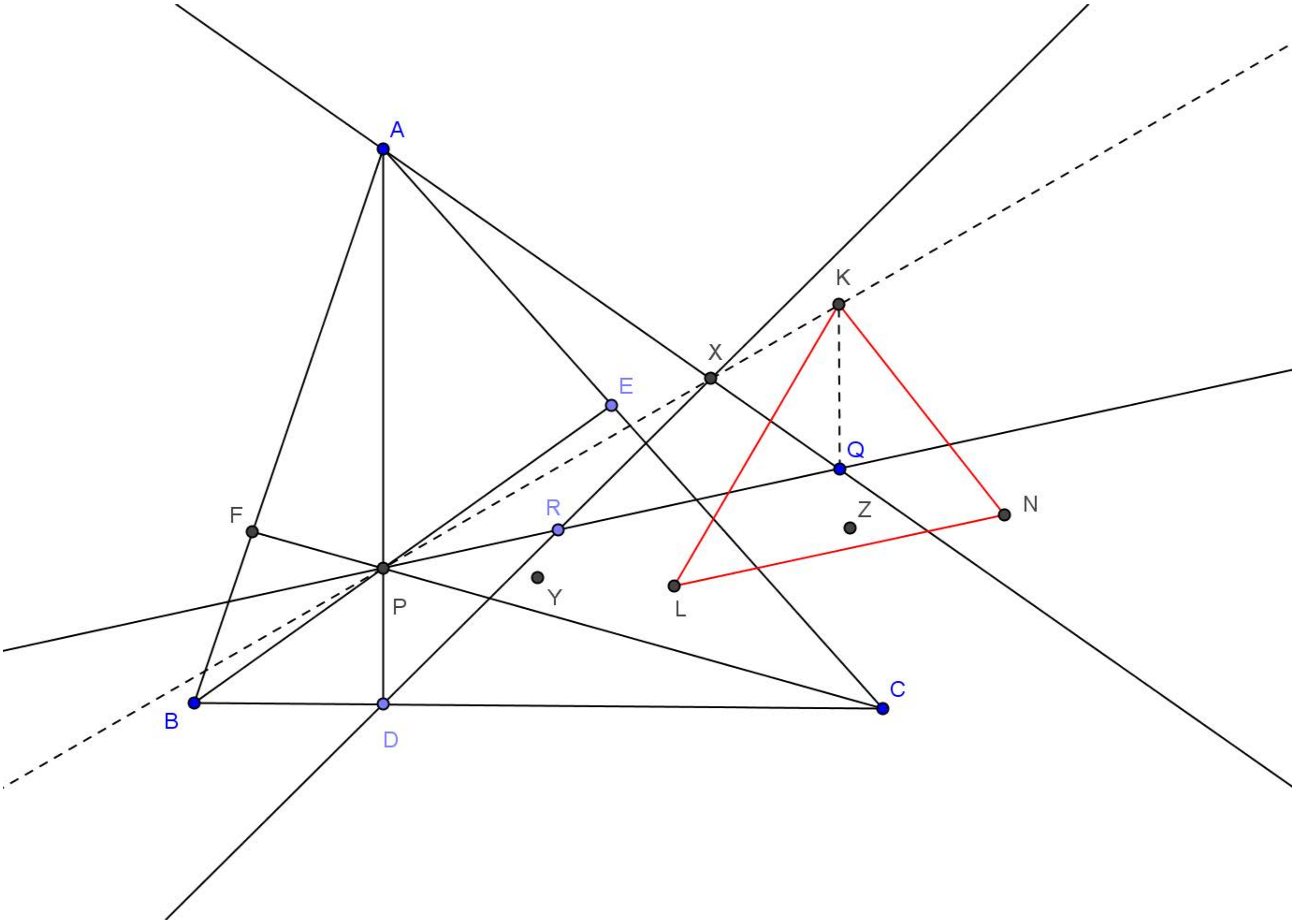
Notice that: $\frac{\overline{QK}}{\overline{AP}}=\frac{\overline{XQ}}{\overline{XA}}=\frac{\overline{QR}}{\overline{RP}}.\frac{\overline{PD}}{\overline{DA}}$(Menelaus) $=k.\left(-\frac{\alpha}{\alpha+\beta+\gamma}\right)$

$\Rightarrow\overrightarrow{QK}=k.\left(-\frac{\alpha}{\alpha+\beta+\gamma}\right).\overrightarrow{AP}$

$\Rightarrow\Sigma\overrightarrow{QK}=k.\Sigma\left(-\frac{\alpha}{\alpha+\beta+\gamma}\right).\overrightarrow{AP}=-\frac{k}{\alpha+\beta+\gamma}.(\Sigma\alpha.\overrightarrow{AP})=\overrightarrow{0}$

$\Rightarrow Q$ is the centroid of $\triangle KLN$
Q.E.D.

Attachments:



This post has been edited 1 time. Last edited by tranquanghuy7198, May 22, 2015, 3:41 pm

Quick Reply

High School Olympiads

hard for me

Reply

Leon000

#1 Apr 20, 2015, 7:53 pm

There is a triangle ABC but not an isosceles triangle. Let H be an orthocenter of this triangle. Prove that we can locate two points P,Q that satisfy both of these two conditions:

□) PA + QA = PB + QB = PC + QC

□)AH, BH, CH bisect the angle PAQ, PBQ, PCQ, respectively.

Of course 'XY' represents Line segment XY !

Luis González

#2 Apr 20, 2015, 9:15 pm

From (i), P,Q are the foci of an ellipse $\mathcal{S}$ passing through A,B,C and from (ii), it follows that the perpendiculars to AH,BH,CH at A,B,C are tangents of $\mathcal{S} \Longrightarrow \mathcal{S}$ is the Steiner circum-ellipse of $\triangle ABC$ (Steiner in-ellipse of its antimedial triangle) and P,Q are its foci (Bickart points).

To construct P,Q with ruler and compass alone you may use the general construction given at http://www.artofproblemsolving.com/community/c6h359945.

Quick Reply

High School Olympiads

Construction of conic

Fermat's Little Turtle

#1 Aug 3, 2010, 7:48 am

Is there a website explaining how, or if someone could explain how, to construct the conic through 4 points knowing it has a tangent at one of them (tangent given).
Similarly through 3 points, 2 of which have a known tangent.

Thanks~!~

Luis González

#2 Aug 3, 2010, 9:07 am

Any non-degenerate conic section is unambiguously defined by 5 elements (points, incident tangents). Five points, four points and a tangent through one of them, three points and two tangents through two of them, five tangents, four tangents and a contact point or three tangents and the contact points on two of them. We cannot draw the curve using ruler-compass, but we can define it by constructing as many elements as we want (center, asymptotes, symmetry axes, etc). For instance, let us consider the case where it's given four points M, N, P, Q and a tangent m through $M.$

We construct the homologous P', Q', N' of P, Q, N in a circumference γ tangent to m through $M,$ which is the center of the homology mapping the wanted conic $\mathcal{H}$ into $\gamma.$ Three pairs of homologous points $(P, P'), (Q, Q')$ and (N, N') are sufficient to construct the axis e and limit line ℓ of the homology. Construct the pole O' of ℓ WRT γ and the conjugate points K, K' on ℓ WRT γ such that $MK \perp MK'.$ This is, construct $U \equiv \ell \cap m$ and the circle (U) with radius UM cuts ℓ at $K, K' \Longrightarrow MK, MK'$ are parallel to the axes of $\mathcal{H} \Longrightarrow O'K, O'K'$ cut γ at A', B' and C', D', i.e. images of the vertices A, B and C, D of $\mathcal{H}.$ Thus, it remains to construct the homologous points A, B, C, D of A', B', C', D' under the referred homology to get the axes AB, CD and the center $O \equiv AB \cap CD$ of $\mathcal{H}.$ Keep in mind that, if the limite line ℓ cuts γ (say at two points), then the wanted conic $\mathcal{H}$ is a hyperbola and we might consider the construction of its asymptotes.

High School Olympiads

Soddy line and Gergonne point

Reply

Geftus

#1 Apr 20, 2015, 6:59 pm

How to prove that Gergonne point lie on Soddy line?
Especially synthetic proof is required

TelvCohl

#2 Apr 20, 2015, 7:39 pm • 1

My solution :

Let $\mathcal{P}(T, \odot)$ be the power of the point T WRT circle $\odot$.
Let I, G_e be the Incenter, Gergonne point of $\triangle ABC$, respectively .
Let D, E, F be the tangent point of $\odot(I)$ with BC, CA, AB, respectively .
Let S_1, S_2 be the inner Soddy point, outer Soddy point of $\triangle ABC$, respectively .
Let $\odot(A)$ be a circle with center A and pass through E, F (define $\odot(B)$ and $\odot(C)$ similarly) .
Let $\odot(S_1)$ be a circle with center S_1 and tangent to $\odot(A), \odot(B), \odot(C)$ at A_1, B_1, C_1, respectively .
Let $\odot(S_2)$ be a circle with center S_2 and tangent to $\odot(A), \odot(B), \odot(C)$ at A_2, B_2, C_2, respectively .
Let A_3, B_3, C_3 be the exsimilicenter of $\odot(B) \sim \odot(C)$, $\odot(C) \sim \odot(A)$, $\odot(A) \sim \odot(B)$, respectively .

From D'Alembert theorem $\Longrightarrow$ EF, BC, B_1C_1, B_2C_2 are concurrent at A_3 ,
so from
$A_3B_1 \cdot A_3C_1 = A_3B_2 \cdot A_3C_2 = A_3D^2 = A_3E \cdot A_3F \Longrightarrow \mathcal{P}(A_3, \odot(I)) = \mathcal{P}(A_3, \odot(S_1)) = \mathcal{P}(A_3, \odot(S_2))$.

Similarly, we get $\mathcal{P}(B_3, \odot(I)) = \mathcal{P}(B_3, \odot(S_1)) = \mathcal{P}(B_3, \odot(S_2))$ and
$\mathcal{P}(C_3, \odot(I)) = \mathcal{P}(C_3, \odot(S_1)) = \mathcal{P}(C_3, \odot(S_2))$,
so $\odot(I), \odot(S_1), \odot(S_2)$ are coaxial with common radical axis $\overline{A_3B_3C_3} \Longrightarrow I, S_1, S_2$ are collinear and $\overline{IS_1S_2} \perp \overline{A_3B_3C_3}$.
$(\star)$

On the other hand, from Sondat theorem (for $\triangle ABC$ and $\triangle DEF$) we get $\overline{IG_e} \perp \overline{A_3B_3C_3}$,
so combine with $(\star)$ we get I, G_e, S_1, S_2 are collinear . i.e. G_e lie on the Soddy line S_1S_2 of $\triangle ABC$

Q.E.D

jayme

#3 Apr 20, 2015, 8:13 pm

Dear Mathlinkers,
I don't know how you define the Soddy line....

see for example

http://jl.ayme.pagesperso-orange.fr/Docs/La%20droite%20de%20Soddy.pdf

Sincerely
Jean-Louis

Luis González

#4 Apr 20, 2015, 8:45 pm • 1

Discussed before at http://www.artofproblemsolving.com/community/c6h399665.

Quick Reply

High School Olympiads

On the line through Incentre and Gergonne Point

Rijul saini
#1 Mar 31, 2011, 6:19 pm

Let $\triangle ABC$ be a triangle with ω and I as incircle and incenter, respectively. Circle ω touches the sides AB, BC, CA at points C_1, A_1, B_1, respectively. Segments AA_1 and BB_1 meet at point G. Circle ω_A is centered at A with radius AB_1. Circles ω_B and ω_C are defined analogously. Circles $\omega_A, \omega_B, \omega_C$ are externally tangent to circle ω_1. Circles $\omega_A, \omega_B, \omega_C$ are internally tangent to circle ω_2. Let O_1 and O_2 be the centers of ω_1 and ω_2, respectively. Lines A_1B_1 and AB meet at C_2, and lines A_1C_1 and AC meet at B_2. Prove that points I, G, O_1, O_2 lie on a line that is perpendicular to line B_2C_2.

BigSams
#2 Mar 31, 2011, 7:37 pm

Correct me if I am wrong, but this problem sounds suspiciously like Mathematical Reflections O191 at http://awesomemath.org/wp-content/uploads/reflections/2011_2/MR_2_2011_Problems.pdf ... to which solutions are still being accepted...

Rijul saini
#3 Mar 31, 2011, 7:43 pm

> *BigSams wrote:*
> Correct me if I am wrong, but this problem sounds suspiciously like Mathematical Reflections O191 at http://awesomemath.org/wp-content/uploads/reflections/2011_2/MR_2_2011_Problems.pdf ... to which solutions are still being accepted...

Well, if you would have tried this problem, you wouldn't have said that.
Anyways, if you're really interested in the source, it was given in a test to the Black Group in the MOP.

BigSams
#4 Mar 31, 2011, 7:46 pm

Sorry about that , just sounded like certain parts were equivalent statements.

Luis González
#5 Apr 1, 2011, 12:21 am • 6

Firstly, it is well known that the line connecting the incenter I and Gergonne point G_e of $\triangle ABC$ (Soddy line) is perpendicular to the trilinear polar $\ell \equiv \overline{A_2B_2C_2}$ of G_e. Note that AA_1, BB_1, CC_1 are the polars of A_2B_2, C_2 WRT (I), i.e. ℓ is the polar of G_e WRT $(I) \Longrightarrow IG_e \perp \ell$.

Let X, Y, Z be the tangency points of ω_1 with $\omega_A, \omega_B, \omega_C$ and X', Y', Z' the tangency point of ω_2 with $\omega_A, \omega_B, \omega_C$. A_2 is the exsimilicenter of $\omega_B \sim \omega_C$ and Y, Z are the insimilicenters of $\omega_B \sim \omega_1$ and $\omega_C \sim \omega_1$. Thus, by Monge & d'Alembert theorem, it follows that Y, Z, A_2 are collinear. Since Y, Z are also inverse points under the inversion with center A_2 taking ω_B and ω_C into each other, we have $\overline{A_2Y} \cdot \overline{A_2Z} = \overline{A_2C_1} \cdot \overline{A_2B_1}$, i.e. A_2 has equal power to ω_A and $\omega_1 \Longrightarrow A_2X$ is tangent to ω_1, ω_A. Likewise, A_2X' is tangent to ω_A, ω_2. Since (I) is the radical circle of $\omega_A, \omega_B, \omega_C$, the inversion in (I) takes these circles into themselves and by conformity, ω_2 is the inverse of $\omega_1 \Longrightarrow I \equiv XX' \cap O_1O_2$. Note that $U \equiv AA_1 \cap B_1C_1$ is the pole of AA_2 WRT (I), thus $U \in XX' \Longrightarrow$ cross ratio (X, X', U, I) is harmonic $\Longrightarrow$ pencil $A_2(X, X', U, I)$ is harmonic $\Longrightarrow$ pencil $A(O_1, O_2, I, G_e)$ formed by the perpendiculars from A to A_2X, A_2X', B_1C_1 and A_2I is also harmonic. Therefore, by analogous reasoning we conclude that I, G_e, O_1, O_2 are collinear and harmonically separated.

P.S. ω_1, ω_2 are known as Soddy circles of ABC. Their centers O_1, O_2 are X_{175}, X_{176} in ETC.

This post has been edited 1 time. Last edited by Luis González, Mar 8, 2015, 1:55 am

Rijul saini

#6 Apr 1, 2011, 7:20 pm • 4

Really nice solution, Luis. Didn't notice Y, Z, A_2 were collinear while working on it .

Anyways, here's my solution.
First, we prove that IG is perpendicular to the line $A_2B_2C_2$.
This is achieved by showing that the line $A_2B_2C_2$ is the polar of IG wrt ω.

Now, we prove that O_1, I, O_2 are collinear. For this, invert about A_1. This takes ω_B, ω_C to two parallel lines ω'_B, ω'_C respectively. ω'_A is a circle tangent to both these lines, and not passing through A_1. B'_1, C'_1 are the points at which it is tangent to those lines. Further, ω'_1, ω'_2 are the two circles tangent to the parallel lines and ω'_A. These also don't pass through A_1. And finally, ω is the line through B'_1, C'_1, I' is the reflection of A_1 across ω, and O'_1, O'_2 are the inverses of A_1 wrt ω'_1, ω'_2 respectively.

We have to prove O'_1, I', A_1, O'_2 are concyclic. For this, let I_1 be the inverses of I' wrt ω'_1. We have that I', A_1, O'_2, I_1 form an isosceles trapezoid, and are hence concyclic. Also, if X, r is the centre, and radius of ω_1, then $XI_1 \cdot X'_I = XO'_1 \cdot XA_1$, and thus I_1, I', A_1, O'_1 are concyclic. And thus, O'_1, I', A_1, O'_2 are concyclic.

We now show that the four points O_1, I, O_2, G are collinear.
Now, since B', C' are the reflections of A_1 across the lines ω'_B, ω'_C. Therefore, to prove that we only need to prove that the circles $\odot(B'A_1B'_1), \odot(C'A_1C'_1), \odot(O'_1I'A_1O'_2)$ have a point in common other than A_1. Therefore, it suffices to prove that they are coaxial. And since they already have a point in common, which is A_1, it suffices to show that their centres are collinear.
Now, the centre of $\odot(B'A_1B'_1)$, say M is the intersection of the perpendicular bisector of $A_1B'_1$ with the line ω'_B,the centre of $\odot(C'A_1C'_1)$, say N is the intersection of the perpendicular bisector of $A_1C'_1$ with the line ω'_C, and the centre of $\odot(O'_1I'A_1O'_2)$ lies on ω'. Therefore, we now need to prove that if P is the intersection of the lines MN and ω', then the circle with centre P and radius PA_1 is orthogonal to ω'_1. Thus we are reduced to proving that $PX^2 = PA_1^2 + r^2$. Now, if T is the centre of ω_A, then $PX^2 = PT^2 + XT^2 = PT^2 + (2r)^2$. Combining this with the fact that $B_1C_1 = 2r$, we are reduced to proving that $PA_1^2 = PT^2 + \frac{3}{4}B_1C_1^2$. But this was accomplished in this thread, even though I don't have a synthetic solution.
This completes it, and we are through.

Aside: If anyone worked through my solution, what does the point I_1, when inverted back, look like?

Quick Reply

High School Olympiads

brocard points in harmonic quadrilateral

linqaszayi

#1 Apr 19, 2015, 9:09 pm

given a harmonic quadrilateral ABCD. prove that there exist points P,Q such that $\angle PAB = \angle PBC = \angle PCD = \angle PDA$, $\angle QBA = \angle QCB = \angle QDC = \angle QAD$.
let AP and BQ meets at M_1,BP and CQ meets at M_2,CP and DQ meets atM_3,DP and AQ meets at M_4.prove that M_1, M_2, M_3, M_4 lies on the circle with diameter OK,where K is the point AC and BD meets.

TelvCohl

#2 Apr 19, 2015, 11:09 pm • 3

I think you forgot to mention O is the center of $\odot(ABCD)$

My solution:

Let $\mathcal{C}_1$ be a circle through D and tangent to AB at A .
Let $\mathcal{C}_2$ be a circle through A and tangent to BC at B .
Let $\mathcal{C}_3$ be a circle through B and tangent to CD at C .
Let $\mathcal{C}_4$ be a circle through C and tangent to DA at D .
Let $\mathcal{C}_1'$ be a circle through A and tangent to CD at D .
Let $\mathcal{C}_2'$ be a circle through D and tangent to BC at C .
Let $\mathcal{C}_3'$ be a circle through C and tangent to AB at B .
Let $\mathcal{C}_4'$ be a circle through B and tangent to DA at A .
Let M, N be the midpoint of AC, BD, respectively .

It's well-known that M, N are the isogonal conjugate of harmonic quadrilateral $ABCD$.

Since $\angle BAM = \angle BDC = \angle ADM$,
so $\odot(ADM)$ is tangent to AB at $A \Longrightarrow M \in \mathcal{C}_1$.
Similarly, we can prove $N \in \mathcal{C}_2, M \in \mathcal{C}_3, N \in \mathcal{C}_4, N \in \mathcal{C}_1', M \in \mathcal{C}_2', N \in \mathcal{C}_3', M \in \mathcal{C}_4'$.

Let P be the intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$ ($P \neq A$) .

Since $\angle BPM = \angle ADM + 180^\circ - \angle APB = \angle BDC + 180^\circ - \angle ANB$
$= \angle BDC + 180^\circ - \angle NAD - \angle ADB = \angle BDC + 180^\circ - \angle BAC - \angle ADB = 180^\circ - \angle MCB$,
so B, C, P, M are concyclic $\Longrightarrow P \in \mathcal{C}_3$ (similarly, we can prove $P \in \mathcal{C}_4$) ,
hence $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ are concurrent at P and $\angle PAB = \angle PBC = \angle PCD = \angle PDA$.

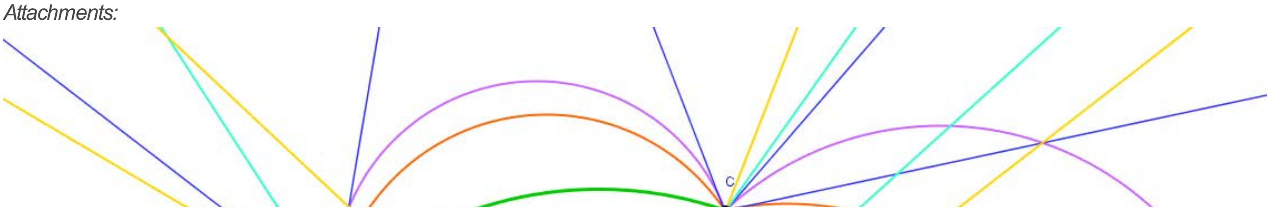
Similarly, we can prove $\mathcal{C}_1', \mathcal{C}_2', \mathcal{C}_3', \mathcal{C}_4'$ are concurrent at Q and $\angle QBA = \angle QCB = \angle QDC = \angle QAD$.

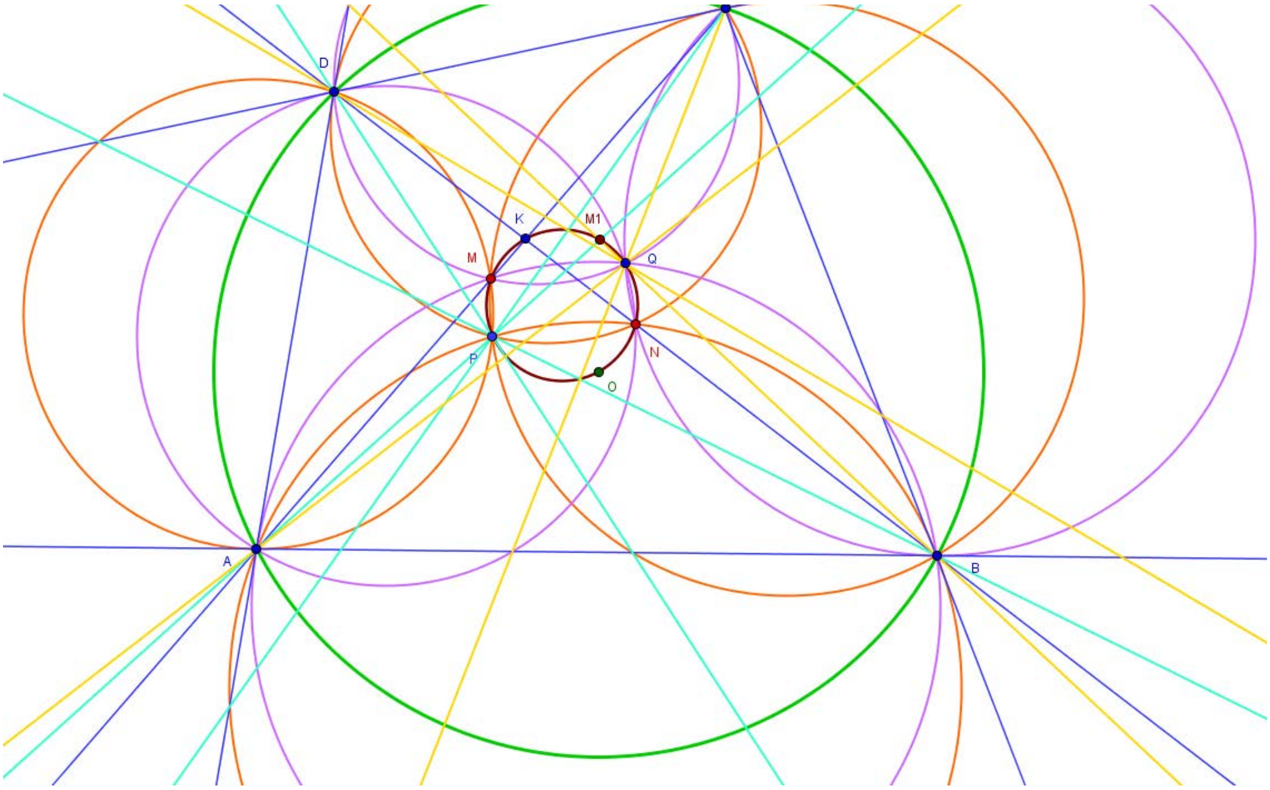
From $\angle NPM = \angle NBA + \angle ADM = \angle DCA + \angle KDC = 180^\circ - \angle MKN \Longrightarrow P \in \odot(OK)$.
Similarly, we can prove $Q \in \odot(OK) \Longrightarrow O, K, P, Q, M, N$ all lie on a circle with diameter OK .

From $\angle NQB = \angle KBA = \angle NPM_1 \Longrightarrow M_1 \in \odot(OK)$.
Similarly, we can prove M_2, M_3, M_4 all lie on $\odot(OK)$.

Q.E.D

Attachments:





Luis González

#3 Apr 20, 2015, 1:43 am

Let M, N be the midpoints of AC, BD. For any $\triangle ABC$, the circle passing through A and tangent to BC at B passes through the midpoint N of the B-symmedian chord BD. Hence if P is the 2nd intersection of $\odot(ABN)$ and $\odot(BCM)$, we have $\angle PNB = \angle PAB = \angle PBC = \angle PDC \Longrightarrow P \in \odot(CDN) \Longrightarrow \odot(CDP)$ touches DA and by symilar reasoning $\odot(DAP)$ touches $AB \Longrightarrow \angle PAB = \angle PBC = \angle PCD = \angle PDA = \omega$ and similarly there is a point Q verifying $\angle QBA = \angle QCB = \angle QDC = \angle QAD = \omega$ (isogonal conjugate of P WRT $ABCD$).

Since $\angle PMC = \angle PBC = \angle PCD = \angle PNK$, then $P \in \odot(KMN)$ and similarly $Q \in \odot(KMN)$, but $OMKN$ is cyclic on account of the right angles at $M, N \Longrightarrow P, Q$ lie on circle Ω with diameter OK and moreover $\angle KOP = \angle KOQ = \omega$. Thus since $\triangle ABM_1$ is isosceles with base angle $\angle M_1AB = \angle M_1BA = \omega$, then $\angle PM_1B = 2\omega = \angle POQ \Longrightarrow M_1 \in \Omega$ and likewise $M_2, M_3, M_4 \in \Omega$.

TelvCohl

#4 Apr 20, 2015, 2:30 am • 2

Remark : We can prove the converse of the first part :

Exist a point P in cyclic quafrilateral $ABCD$ such that
$\angle PAB = \angle PBC = \angle PCD = \angle PDA \Longrightarrow AB \cdot CD = AD \cdot BC$.

Proof :

Let K be a point such that $\triangle KBA$ and $\triangle PCD$ are directed similar .

From $\angle KAP + \angle PBK = \angle PDC + \angle BAP + \angle DCP + \angle PBA = 180^\circ \Longrightarrow K \in \odot(PAB)$.
From $\angle ABK = \angle DCP = \angle BAP \Longrightarrow AP \parallel KB \Longrightarrow APBK$ is an isosceles trapezoid $\Longrightarrow AB = PK$. $(\star)$

From Ptolemy theorem and $(\star)$ we get $AB^2 = PA \cdot KB + PB \cdot KA$,
so combine with $\triangle KBA \sim \triangle PCD \Longrightarrow AB \cdot CD = PA \cdot PC + PB \cdot PD$.
Similarly, we can prove $AD \cdot BC = PA \cdot PC + PB \cdot PD \Longrightarrow AB \cdot CD = AD \cdot BC$.

Done

Luis González

#5 Apr 20, 2015, 4:51 am

Yup, the converse is also true. Here is another proof.

Invert the figure with center B. Then A, D, C go to collinear points A', D', C' and the inverse P' of P fulfills $P'A' \parallel BC'$, BA' touches $\odot(P'D'A')$ and $P'C'$ touches $\odot(BD'C')$.

If $P'D'$ cuts BC' at X, we have $\angle BA'D' = \angle A'P'D' = \angle BXD' \Longrightarrow X \in \odot(BA'D')$. Thus $\angle XA'D' = \angle D'BC' = \angle P'C'D' \Longrightarrow XA' \parallel P'C' \Longrightarrow XA'P'C'$ is parallelogram $\Longrightarrow BD' \equiv BD$ is the B-median of $\triangle BA'C' \Longrightarrow BD$ is the B-symmedian of $\triangle BAC \Longrightarrow ABCD$ is harmonic.

Quick Reply

High School Olympiads

collinear points,KOMAL 2014

Reply

dan121

#1 Apr 20, 2015, 3:42 am

The angle bisector drawn from right-angled vertex C of triangle ABC intersects the circumscribed circle at point P, and the angle bisector from vertex A intersects the circumscribed circle at point Q. K is the intersection of line segments PQ and AB. The centre of the inscribed circle is O, and its point of tangency on side AC is E. Prove that points E, O and K are collinear.

Luis González

#2 Apr 20, 2015, 4:05 am

Let R be the midpoint of the arc CA of $\odot(ABC)$. Since $RO = RC, QO = QC$ (well-known for any triangle) and $EC = EO$, then E, R, Q are collinear on the perpendicular bisector of CO. Now by Pascal theorem for $CPQRBA$, the intersections $O \equiv CP \cap RB, K \equiv PQ \cap BA$ and $E \equiv QR \cap AC$ are collinear.

Quick Reply

High School Olympiads

Coilenear and concurrent

Reply

ThE-dArK-lOrD

#1 Apr 19, 2015, 11:19 pm

Given triangle ABC and its circumcircle Y.
Choose arbitrary point Z in triangle ABC.
D is the intersection of AZ and Y.
Difine similary E and F.
G is the intersection of AZ and BC.
Difine similary H and I.
J is the second intersection of AB and circumcircle of BGD.
K is the second intersection of AC and circumcircle of CGD.
Difine similary L,M N,O.
P is the midpoint of JK.
Difine similary Q and R
A_1 is the point such that BCA_1 is equilateral triangle
and A_1 is on the other side of A respect to BC.
Difine similary B_1 and C_1.
A_2 is the point such that BCA_2 is equilateral triangle
and A_2 is on the same side of A respect to BC.
Difine similary B_2 and C_2.
Prove that
(1) AP, BQ and CR concurrent at point W
(2) AA_i, BB_i and CC_i concurrent at point T_1 and T_2 for $i=1,2$
(3) T_1,T_2 and W colinear.

ThE-dArK-lOrD

#3 Apr 19, 2015, 11:29 pm

And prove that W is the fixed point of triangle ABC.

Luis González

#4 Apr 19, 2015, 11:50 pm

$AK\cdot AC=AG\cdot AD=AJ\cdot AB \Longrightarrow JK$ is antiparallel to BC WRT $AB,AC \Longrightarrow AP$ is the A-symmedian of $\triangle ABC$ and similarly BQ and CR are the B- and C- symmedian $\Longrightarrow W$ is the symmedian point of $\triangle ABC$. Now, it's known that the symmedian point is on the line joining the 2 Fermat points T_1,T_2 of $\triangle ABC$.

Quick Reply

High School Olympiads

A lemma on Kiepert triangles

Reply

Source: Telv Cohl, Luis González

VUThanhTung
#1 Apr 19, 2015, 7:48 am

Let $A_\alpha B_\alpha C_\alpha$, $A_\beta B_\beta C_\beta$ be two Kiepert triangles of $\triangle ABC$. Prove that if $\alpha + \beta = 0$ then $AB_\beta A_\alpha C_\beta$ is a parallelogram.

This post has been edited 2 times. Last edited by VUThanhTung, Apr 19, 2015, 9:32 pm

Luis González
#2 Apr 19, 2015, 8:16 am • 1

More general: construct $\triangle ZAB, \triangle YCA$ inwardly (or outwardly) and $\triangle XBC$ outwardly (or inwardly), such that $\triangle ZAB \sim \triangle XBC \sim \triangle YCA$, then $AYXZ$ is a parallelogram. This has been discussed many times before, for instance see prove that two triangles share their centroid and elsewhere.

TelvCohl
#3 Apr 19, 2015, 1:55 pm • 1

Actually, I used this lemma to solve another your poblem (post #6) .
For the proof of this lemma you can see here (post #5) .

VUThanhTung
#4 Apr 19, 2015, 9:32 pm

Thank you both, I made a correction on the source

Quick Reply

High School Olympiads

Twice the Angle

Locked

minimario

#1 Apr 19, 2015, 5:41 am

Triangle ABC is such that $AB = AC$. Let D be a point on side BC such that $BD = 2DC$. Point P lies on segment AD and satisfies $\angle ABP = \angle PAC$. Prove that $\angle BAC = 2\angle DPC$.

Luis González

#2 Apr 19, 2015, 5:51 am

Posted before at http://www.artofproblemsolving.com/community/c6h483125.

High School Olympiads

A geometry problem

Reply

math-o-fun
#1 Jun 9, 2012, 10:35 pm

Let ABC be an isosceles triangle with $AB = AC$. Let D be a point on the segment BC such that $BD = 2DC$. Let P be a point on the segment AD such that $\angle BAC = \angle BPD$. Prove that $\angle BAC = 2\angle DPC$.

BBAI
#2 Jun 10, 2012, 12:33 am • 1

Soln:
Extend AD to X such that$BP = PX$.
So $\triangle BPX$ is similar to $\triangle ABC.\implies X$ lies on $\odot ABC$.
As $BD = 2DC.$,so $BX = 2XC$.
Let a perpendicular is drawn from $P$to BX.at Y.
So $XY = XC$.
$\triangle XYC$ is isosceles $.\implies \triangle PYC$ is isosceles.
So $\angle CPD$ =$\angle XPY$ =$\frac{\angle BAC}{2}$.
So done.

subham1729
#3 Jun 10, 2012, 12:38 am

It was a problem of Indian Imotc 2012 practise test
I did there by trigonmetry.

BBAI
#4 Jun 10, 2012, 1:49 am

Everyone knows you solve geometry by Sine rule and coordinate. COOL!

sunken rock
#5 Jun 10, 2012, 1:59 am

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=316752 as well!

Best regards,
sunken rock

Goutham
#6 Jun 10, 2012, 10:01 am

Take K the midpoint of BP. Simple sine rule gives $BP = 2AP = BK$ and prove that $ABK \cong CAP$ and angle chase gives the result.

Quick Reply

High School Olympiads

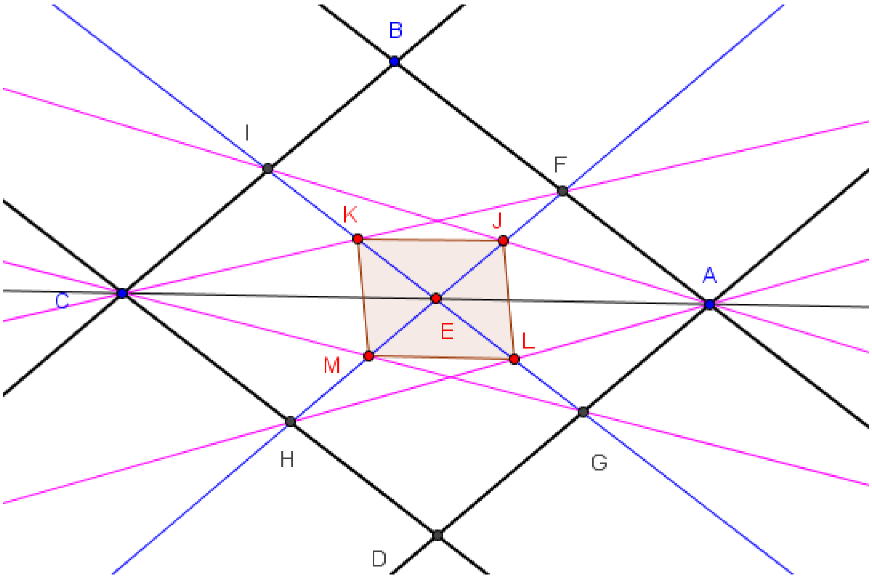
Square in a square

daothanhoai

#1 Apr 18, 2015, 5:48 pm

Let $ABCD$ be a parallelogram. $IG//AB//CD$, $FH//BC//AD$. Denote as the figure attachment, then show that $KJLM$ be a parallelogram.
Special case:
if $ABCD$ be a rectangle then show that $KJLM$ be a rhombus
If $ABCD$ be a square then show that $KJLM$ be a square,

Attachments:



Luis González

#2 Apr 18, 2015, 10:40 pm

This is a particular case of the following configuration:

$ABCD$ is arbitrary quadrilateral. $X \equiv AB \cap CD, Y \equiv BC \cap DA. E$ is arbitrary point on $AC. EX$ cuts BC, DA at I, G and EY cuts AB, CD at $F, H. J \equiv AI \cap FH, K \equiv CF \cap IG, M \equiv CG \cap FH, L \equiv AH \cap IG.$ Then KM, LJ, BD, XY concur and JK, ML, AC, XY concur.

Solution: Let $P \equiv XY \cap BD.$ Consider the involutive homology that fixes the the line pencil $P,$ the line AC and that carries $B \mapsto D.$ Thus since $EX \mapsto EY,$ it follows that $I \mapsto H$ and $F \mapsto G \Longrightarrow CF \mapsto CG, AI \mapsto AH \Longrightarrow K \mapsto M$ and $J \mapsto L \Longrightarrow KM, LJ, BD, XY$ concur at P and JK, ML, AC, XY concur.

Quick Reply

High School Olympiads

Geometry

Reply

MarryLysine

#1 Apr 18, 2015, 5:26 pm

Let ABC be an isosceles triangle with the vertex at A. Let E and F be the point on AB and BC such that EF//AC. Let M be the midpoint of CE and N be the intersection of the perpendicular bisectors of the triangle BEF.

Prove that: $\widehat{MAN} = \frac{1}{2}\widehat{BAC}$

TelvCohl

#2 Apr 18, 2015, 7:05 pm • 1

My solution:

Let O be the circumcenter of $\triangle ABC$.
Let T, S be the midpoint of BC, CA, respectively .

Easy to see $M \in TS$ and $N \in BO$.

Since $OA = OB, SA = ST$,
so from $\angle OAB = \angle SAT \Longrightarrow \triangle OAB \sim \triangle SAT$.
Since $\triangle BEF \cup N$ and $\triangle BAC \cup O$ are homothety (with center B),
so $BN : NO = BE : EA = TM : MS \Longrightarrow \triangle OAB \cup N \sim \triangle SAT \cup M$,
hence we get $\angle BAN = \angle TAM, \angle NAO = \angle MAS \Longrightarrow \angle NAM = \angle BAT = \frac{1}{2}\angle BAC$.

Q.E.D

Luis González

#3 Apr 18, 2015, 9:26 pm • 1

Let $D \in AC$ such that $ED \parallel BC. X, Y$ are the midpoints of $DE, EF.$ Clearly $EXMY$ is parallelogram and $(AX \parallel NE) \perp DE.$ Thus from $\triangle AEX \sim \triangle ENY \Longrightarrow \frac{AE}{AX} = \frac{EN}{EY} = \frac{EN}{XM}$ and since $\angle AXM = \angle AEN,$ then $\triangle AXM \sim \triangle AEN$ by SAS $\Longrightarrow \angle XAM = \angle EAN \Longrightarrow \angle MAN = \angle BAX = \frac{1}{2}\angle BAC.$

Quick Reply

High School Olympiads

Concentric Circles

Reply

BartSimpsons

#1 Apr 18, 2015, 1:07 pm

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two concentric circles with $\mathcal{C}_2$ inside $\mathcal{C}_1$. Let A and C be points on $\mathcal{C}_1$ such that AC is tangent to $\mathcal{C}_2$ at B. Let D be the midpoint of AB. A line passing through A meets $\mathcal{C}_2$ at E and F. Let the perpendicular bisectors of ED and CF meet at M. Then show that M lies on AC if and only if $\angle DFC = 90^\circ$.

Luis González

#3 Apr 18, 2015, 1:27 pm

$AE \cdot AF = AB^2 = \frac{1}{4}AC^2 = AD \cdot AC \Longrightarrow EDCF$ is cyclic with circumcircle (M). Hence $M \in AC \Longleftrightarrow DC$ is diameter of $(M) \Longleftrightarrow \angle DFC = 90^\circ$.

Quick Reply

High School Math

Easy lemma - Conjugate points

Reply

drmzjoseph

#1 Apr 18, 2015, 11:39 am

Denote $\mathcal{P}_X$ the power of X with respect to ω(an circle). Prove that $\mathcal{P}_A+\mathcal{P}_B=AB^2 \iff A$ and B are conjugate points with respect to ω

Luis González

#2 Apr 18, 2015, 1:07 pm • 1

As it's not possible that both A and B lie inside of ω, then WLOG we assume that A is outside of ω. AM, AN are the tangents from A to ω. $\mathcal{P}_A+\mathcal{P}_B=AB^2 \Longleftrightarrow \mathcal{P}_B=AB^2-AM^2 \Longleftrightarrow B$ has equal power WRT ω and $\odot(A,AM) \Longleftrightarrow B$ is on their radical axis MN; the polar of A WRT $\omega \Longleftrightarrow A$ and B are conjugate points WRT ω.

suli

#3 Apr 19, 2015, 4:56 am

I forgot what conjugate points were

drmzjoseph

#4 Apr 19, 2015, 6:03 am

> suli wrote:
>
> I forgot what conjugate points were

A and B are conjugate points with respect to $\omega \iff A$ lies on polar of B with respect to ω

Or

A and B are conjugate points with respect to $\omega \iff$ the circle of diameter AB is orthogonal to ω

suli

#5 Apr 19, 2015, 8:21 am

Thanks for telling me that I need to get a book of projective and inversive geometry ASAP.

drmzjoseph

#6 Apr 19, 2015, 8:26 am • 1

I have the book, in PDF of "Geometry of Conics" (Arseniy Akopyan) maybe you are interested (Any user, too)

drmzjoseph

#7 May 1, 2015, 10:41 am

My solution:

Denote r the radius of ω and O the center. If $p=\overrightarrow{OP}$ denotes the vector from O.

A and B are conjugate points with respect to $\omega \iff$ the circle of diameter AB is orthogonal to ω i.e.

$$(\frac{a+b}{2})^2=(\frac{a-b}{2})^2+r^2$$

$$\iff ab=r^2 \iff a^2-r^2+b^2-r^2=(a-b)^2 \iff \mathcal{P}_A+\mathcal{P}_B=AB^2$$

Quick Reply

High School Olympiads

EF, GH, BC are concurrent

Source: Inspire from buratinogigle

THVSH

#1 Apr 18, 2015, 10:23 am

Let ABC be a triangle. P, Q are isogonal conjugate in $\triangle ABC$. $\triangle DEF$ is the pedal triangle of P wrt $\triangle ABC$. X is the projection of Q on BC. $AP \cap (ABC) = G$. GG' is the diameter of (ABC). $G'X \cap (ABC) = H$ (other than G'). Prove that EF, GH, BC are concurrent

Luis González

#2 Apr 18, 2015, 11:55 am • 2

Let GH cut BC at T and let AQ cut $\odot(ABC)$ again at $L.$ If we fix the ray AG and animate $P,$ clearly the series P, Q, E, F, X are all homographic, thus the pencils $GH \barwedge G'X$ and EF (perpendicular to the fixed direction AQ) are homographic. Hence all we need to prove is that they induce the same homography on $BC,$ i.e. that EF, GH, BC concur for at least 3 positions of $P.$

When $P \equiv G,$ then Q goes to infinity $\Longrightarrow GH \perp BC \Longrightarrow T \equiv D$ and $\overline{DEF}$ is Simson line of G WRT $\triangle ABC \Longrightarrow$ the concurrency holds. When P is at infinity, then E, F go to infinity and $Q \equiv H \equiv L \Longrightarrow GH \parallel BC \Longrightarrow$ the concurrency holds.

Finally consider the case when $P \equiv A.$ Here $Q \in BC$ and the line EF is the perpendicular to AQ at $A.$ If $M \equiv GG' \cap BC,$ then $\angle G'MQ = \angle G'GL = \angle G'AQ \Longrightarrow AG'QM$ is cyclic $\Longrightarrow \angle AMG' = \angle AQG'$ and since $\angle AGG' = \angle ALG',$ then $\angle GAM = \angle HG'L = \angle HGL = \angle GTM \Longrightarrow ATGM$ is cyclic $\Longrightarrow \angle TAG = \angle TMG = \angle QAG' \Longrightarrow$ $\angle TAQ = \angle GAG' = 90^\circ \Longrightarrow$ the concurrency holds.

TelvCohl

#4 Apr 18, 2015, 12:37 pm • 1

My solution:

Let R be the projection of X on EF and $T = EF \cap BC$.

It's well-known that D, E, F, X are concyclic .

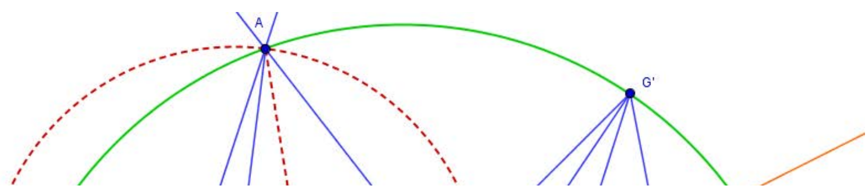
Since $\angle CBG' = 90^\circ - \angle PAC = \angle EPA = \angle EFA$,
so combine with $\angle FAE = \angle BG'C$ we get $\triangle FAE \sim \triangle BG'C$.
Since $\angle XFE = \angle CDE = 90^\circ - \angle ACP = 90^\circ - \angle QCB$ (similarly, $\angle XEF = 90^\circ - \angle QBC$) ,

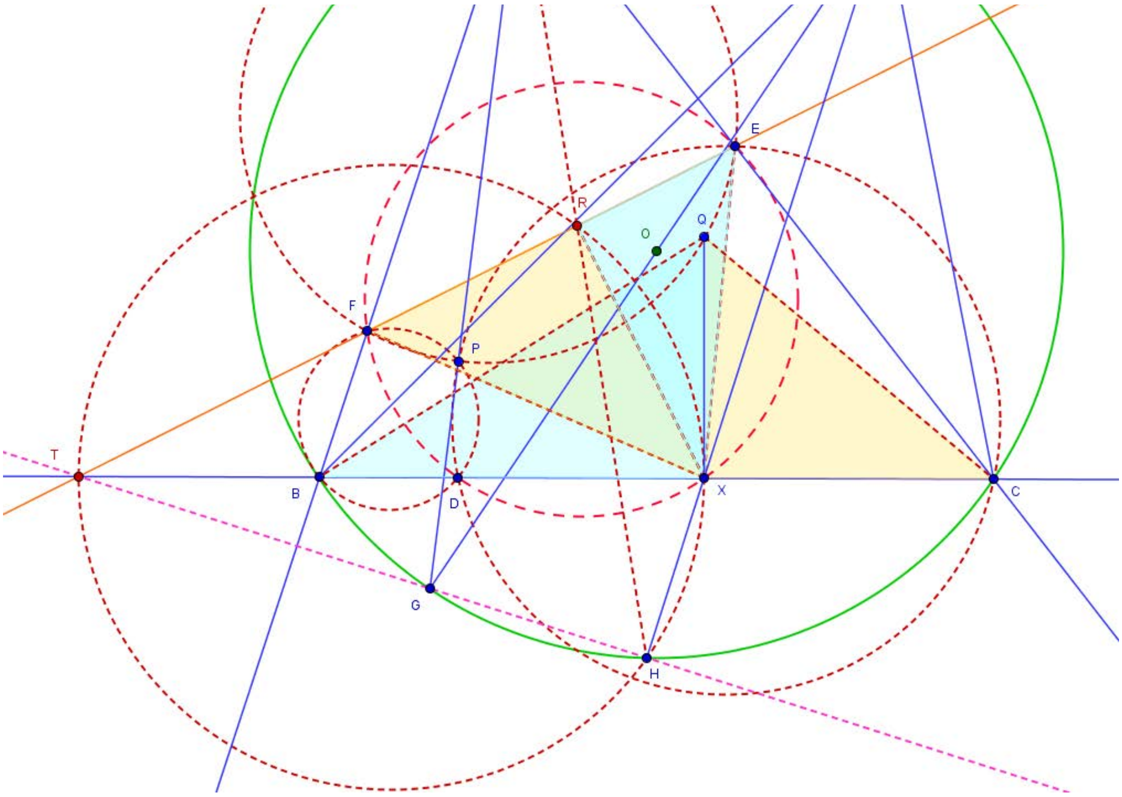
so $\frac{FR}{ER} = \frac{\tan \angle REX}{\tan \angle RFX} = \frac{\cot \angle QBC}{\cot \angle QCB} = \frac{BX}{CX} \Longrightarrow \triangle FAE \cup R \sim \triangle BG'C \cup X \Longrightarrow A, R, H$ are collinear .

Since $\angle XTR = \angle(BC, FE) = \angle(G'X, AR) = \angle XHR$,
so we get X, H, R, T are concyclic $\Longrightarrow \angle XHT = \angle XRE = 90^\circ$,
hence combine with $\angle XHG = 90^\circ \Longrightarrow G, H, T$ are collinear . i.e. EF, BC, GH are concurrent at T

Q.E.D

Attachments:





Quick Reply

High School Olympiads

Problems with circumcevian triangle

Source: Own

buratinogigle

#1 Apr 17, 2015, 3:02 pm • 2

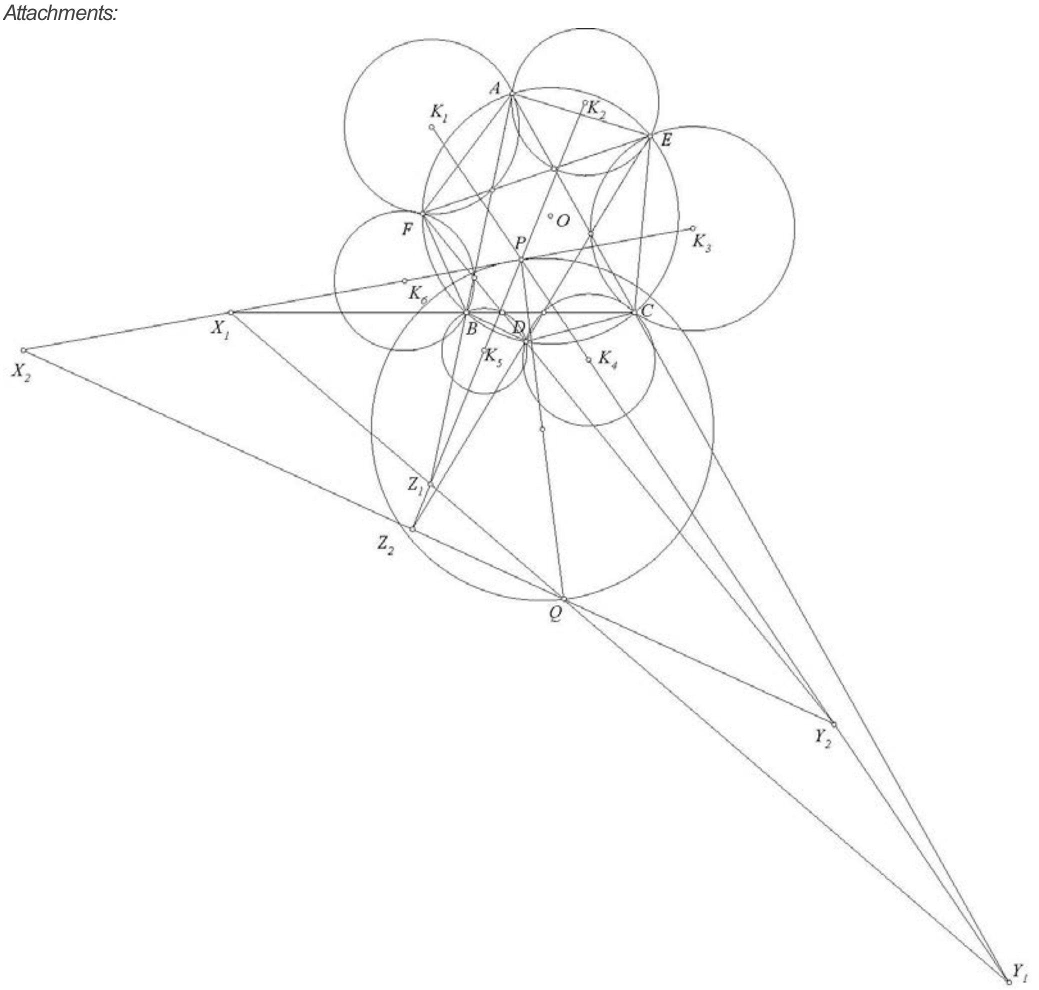
Triangle DEF is circumcevian triangle of a point P with respect to triangle ABC.

a) If K_3K_6 cuts BC at X_1. K_1K_4 cuts CA at Y_1. K_2K_5 cuts AB at Z_1. Then X_1, Y_1, Z_1 are collinear on line ℓ_1.

b) If K_3K_6 cuts EF at X_2. K_1K_4 cuts FD at Y_2. K_2K_5 cuts DE at Z_2. Then X_2, Y_2, Z_2 are collinear on line ℓ_2.

c) Let ℓ_1 cuts ℓ_2 at Q. Prove that circle with diameter PQ are orthogonal to circle (O).

Attachments:



Luis González

#2 Apr 18, 2015, 6:52 am • 1

Solutions to problems a) and b).

Let $U \equiv DE \cap CA$ and $V \equiv DF \cap AB$. By Pascal theorem for $BEDFCA$, we get $P \in UV$. Now if UV cuts (K_3) again at S, we have $\angle USC = \angle UEC = \angle VFP \Longrightarrow F, V, S, C$ are concyclic $\Longrightarrow$ $PB \cdot PE = PC \cdot PF = PV \cdot PS \Longrightarrow P$ is center of inversion interchanging (K_3) and $(K_6) \Longrightarrow P \in K_3K_6$. Therefore

$$\frac{X_1B}{X_1C} = \frac{PB}{PC} \cdot \frac{\sin \angle BPK_6}{\sin \angle FPK_6} = \frac{PB}{PC} \cdot \frac{\text{dist}(K_6, PB)}{\text{dist}(K_6, PF)} = \frac{PB}{PC} \cdot \frac{\sin \angle PBK_6}{\sin \angle PFK_6}.$$

But $\angle PBK_6 = |90^\circ - \angle BFD + \angle ABP| = |90^\circ - \angle(DE, AB)|$ and similarly $\angle PFK_6 = |90^\circ - \angle(FD, CA)|$ $\Longrightarrow$

$$\frac{X_1B}{X_1C} = \frac{PB}{PC} \cdot \frac{\cos \angle(DE, AB)}{\cos \angle(FD, CA)}.$$

Now, multiplying the cyclic expressions together, we conclude that X_1, Y_1, Z_1 are collinear by Menelaus' theorem and in the same way X_2, Y_2, Z_2 are collinear.

Luis González

#3 Apr 18, 2015, 7:03 am • 1

Solution to problem c).

This is merely projective. Consider a homology taking (O) into a conic with center the image of $P.$ Thus $\triangle DEF$ and $\triangle ABC$ become symmetric WRT $P \Longrightarrow \ell_1$ and ℓ_2 are also symmetric about $P \Longrightarrow \ell_1 \parallel \ell_2.$ Hence in the primitive figure, $Q \equiv \ell_1 \cap \ell_2$ is on the polar of P WRT $(O) \Longrightarrow P, Q$ are conjugate points WRT $(O) \Longrightarrow$ circle with diameter $\overline{PQ}$ is orthogonal to $(O).$

Quick Reply

High School Olympiads

Ratio between two Areas

Source: Fall 2005 Tournament of Towns Senior O-Level #4

bluecarneal

#1 Mar 30, 2015, 11:29 pm

On all three sides of a right triangle ABC external squares are constructed; their centers denoted by D, E, F. Show that the ratio of the area of triangle DEF to the area of triangle ABC is:

a) *(2 points)* greater than 1;
b) *(2 points)* at least 2.

Luis González

#2 Apr 18, 2015, 3:54 am

According to http://www.artofproblemsolving.com/community/c6h422667, we have in general $[DEF]=\frac{1}{8}(a^2+b^2+c^2)+[ABC]$. When $\angle BAC=90^\circ$, we have

$$\frac{[DEF]}{[ABC]}=\frac{a^2+b^2+c^2}{8[ABC]}+1=\frac{2(b^2+c^2)}{8\cdot\frac{1}{2}bc}=\frac{b^2+c^2}{2bc}+1.$$

Thus, it immediately follows that $\frac{[DEF]}{[ABC]}>1$ and $\frac{[DEF]}{[ABC]}\geq 2$ is equivalent to $b^2+c^2\geq 2bc$, which is clearly true.

High School Olympiads

Triangle

Reply

ngocduy
#1 Aug 6, 2011, 8:50 am

Give a triangle ABC, AB=c, BC=a, CA=b. Make right triangles BCA', CAB', ABC' outside ABC which A'B=A'C, B'A=B'C, C'A=C'B.
a) Prove that we can make a triangle from AA', BB', CC'.
b) Find the area of that triangle

Luis González
#2 Aug 6, 2011, 10:11 am

Using Van Aubel's theorem for the degenerate quadrilateral AABC, we get that AA' and $B'C'$ are equal and perpendicular. Likewise, $BB', C'A'$ are equal and perpendicular and $CC', A'B'$ are equal and perpendicular. Thus, the lengths AA', BB', CC' coincide with the side lengths of $\triangle A'B'C'$. By cosine law for $\triangle AB'C'$, we get

$$B'C'^2 = AC'^2 + AB'^2 - 2 \cdot AB' \cdot AC' \cdot \cos(90^\circ + A)$$

Substituting $AB' = \frac{\sqrt{2}}{2}b$ and $AC' = \frac{\sqrt{2}}{2}c$ into the latter expression gives

$$B'C'^2 = \frac{1}{2}(b^2 + c^2) + bc \sin A$$

Since the diagonals of $AB'A'C'$ are equal and perpendicular, then its area is given by

$$[AB'A'C'] = \frac{B'C'^2}{2} = \frac{b^2 + c^2}{4} + \frac{bc \sin A}{2} = \frac{b^2 + c^2}{4} + [ABC]$$

$\triangle A'B'C'$ is equivalent to the difference between $AB'A'C'$ and $\triangle AB'C'$, thus

$$[A'B'C'] = \frac{b^2 + c^2}{4} + [ABC] - \frac{AB' \cdot AC'}{2} \cdot \sin(90^\circ + A) \implies$$

$$[A'B'C'] = \frac{b^2 + c^2}{4} + [ABC] - \frac{bc}{4} \cdot \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 + b^2 + c^2}{8} + [ABC].$$

ngocduy
#3 Aug 6, 2011, 10:32 pm

That's right. This is a line problem , start from many easy theorem.

Quick Reply

High School Olympiads

Inverses of the traces of a point

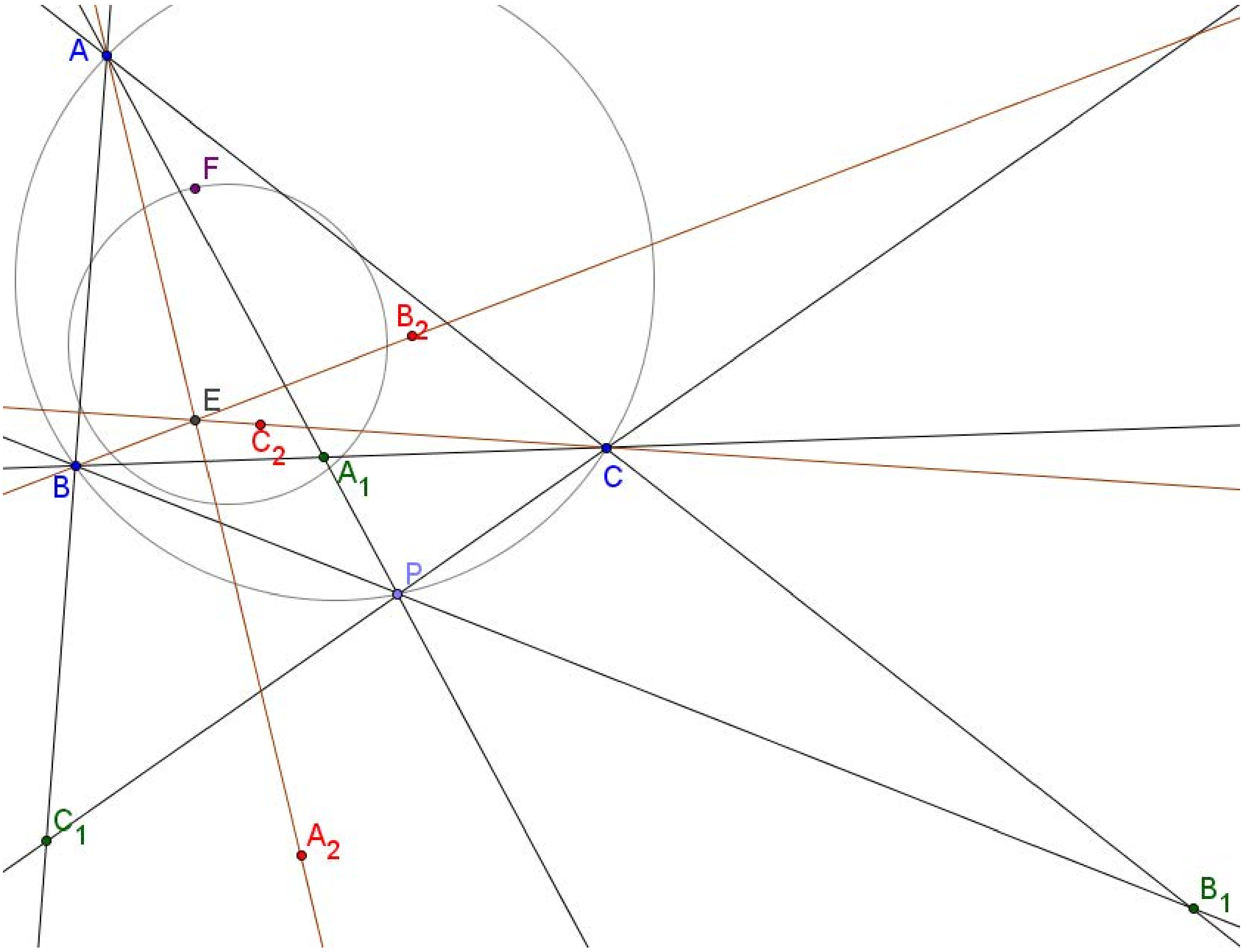
Reply

Cezar

#1 Apr 16, 2015, 10:42 pm • 1

Let P be on $\odot ABC$ and its traces are A_1, B_1, C_1.
Let the inverses of A_1, B_1, C_1 wrt $\odot ABC$ be A_2, B_2, C_2 .
Prove that CC_2, AA_2, BB_2 are concurrent .
Let $CC_2 \cup AA_2 \cup BB_2 = E$. Prove that the isogonal conjugate of E belongs to the nine-point circle.

Attachments:



This post has been edited 1 time. Last edited by Cezar, Apr 17, 2015, 1:45 pm
Reason: ..

TelvCohl

#2 Apr 16, 2015, 11:24 pm • 1

My solution:

Let Q (at infinity) be the isogonal conjugate of P WRT $\triangle ABC$.
Let $\triangle A_3B_3C_3$ be the circumcevian triangle of Q WRT $\triangle ABC$.
Let A_4, B_4, C_4 be the reflection of A_3, B_3, C_3 in BC, CA, AB, respectively .
Let Ψ be the composition of Inversion $\mathbf{I}(A, \sqrt{AB \cdot AC})$ and Reflection $\mathbf{R}(\ell)$ (ℓ is the bisector of $\angle A$).

Since $B \longleftrightarrow C, BC \longleftrightarrow \odot(ABC)$ under Ψ ,
so $A_1 \longleftrightarrow A_3$ under $\Psi \Longrightarrow A_2 \longleftrightarrow A_4$ under Ψ

so $A_1 \longleftrightarrow A_3$ under $\Psi \Longrightarrow A_2 \longleftrightarrow A_4$ under Ψ,
hence AA_4 is the isogonal conjugate of AA_2 WRT $\angle BAC$.

Since A_4 and P are symmetry WRT the midpoint of BC ,
so AA_4 pass through the complement R of P WRT $\triangle ABC$.

Similarly we can prove $R \in BB_4$ and $R \in CC_4$,
so AA_4, BB_4, CC_4 are concurrent at the complement of P WRT $\triangle ABC$.
(obviously R (complement of P) lie on the 9-point circle of $\triangle ABC$)

Since AA_4, BB_4, CC_4 are concurrent at R ,
so AA_2, BB_2, CC_2 are concurrent at the isogonal conjugate of R WRT $\triangle ABC$.

Q.E.D

Luis González

#3 Apr 17, 2015, 3:04 am • 1

Since B_1C_1 is the polar of A_1 WRT $\odot(ABC),$ then A_2 is the projection of A_1 on $B_1C_1,$ i.e. the Miquel point of the cyclic $ABPC \Longrightarrow A_2$ is the 2nd intersection of $\odot(ACC_1)$ and $\odot(ABB_1).$ Now according to Three concurrent radical axes (synthetic solutions at posts #5 and #6), AA_2 goes through the isogonal of the complement of P and similarly BB_2 and CC_2 $\Longrightarrow E$ is the isogonal of the complement of $P.$ Thus, the isogonal of E lies on the complement of $\odot(ABC),$ i.e. the 9-point circle of $\triangle ABC.$

Cezar

#4 Apr 17, 2015, 3:57 am

TelvCohl could you please explain why

> *Quote:*
> $\Psi \Longrightarrow A_2 \longleftrightarrow A_4$ under Ψ ,

TelvCohl

#5 Apr 17, 2015, 2:34 pm

> *Cezar wrote:*
> TelvCohl could you please explain why
>
> > *Quote:*
> > $\Longrightarrow A_2 \longleftrightarrow A_4$ under Ψ ,

Since A_2 is the image of A_1 under the Inversion WRT $\odot(ABC)$,
so $\Psi(A_2)$ is the image of $\Psi(A_1) \equiv A_3$ under the Inversion WRT $\Psi((\odot(ABC)) \equiv BC$.
i.e. $\Psi(A_2)$ is the reflection of A_3 in $BC \Longrightarrow A_2 \longleftrightarrow A_4$ under Ψ :)

Quick Reply

High School Olympiads

Three concurrent radical axes

Source: 0

Luis González

#1 Jun 14, 2009, 12:03 am • 1

P is an arbitrary point on the plane of $\triangle ABC$ and let $\triangle A'B'C'$ be the cevian triangle of P WRT $\triangle ABC$. The circles $\odot(ABB')$ and $\odot(ACC')$ meet at A, X. Similarly, define the points Y and Z WRT B and C. Prove that the lines AX, BY, CZ concur at the isogonal conjugate of the complement of P WRT $\triangle ABC$.

jayme

#2 Jun 14, 2009, 11:10 am

Dear Luis,
nice result.
See for historical note
Stevanovic M. R., Symmedian as radical axis, Message Hyacinthos:
http://tech.groups.yahoo.com/group/Hyacinthos/message/10931
http://tech.groups.yahoo.com/group/Hyacinthos/message/10905
http://tech.groups.yahoo.com/group/Hyacinthos/message/10904
Sincerely
Jean-Louis

Luis González

#3 Jun 14, 2009, 10:47 pm

Thanks for the links Jean-Louis, I found this result very interesting and with lots of applications. There is also a couple of additional things that I would like to highlight.

Let $P\ (u:v:w)\ ,\ A'\ (0:v:w)\ ,\ B'(u:0:w)\ ,\ C'(u:v:0)$

Barycentric equations of circles $\odot(ABB')$ and $\odot(ACC')$ are then:

$$a^2yz+b^2xz+c^2xy-\frac{ub^2z}{u+w}(x+y+z)=0$$

$$a^2yz+b^2xz+c^2xy-\frac{uc^2y}{u+v}(x+y+z)=0$$

Thus, their radical axis AX has equation $AX\equiv\frac{c^2y}{u+v}-\frac{b^2z}{u+w}=0$

Similarly, equations of BY, CZ are given by:

$$BY\equiv\frac{c^2x}{u+v}-\frac{a^2z}{w+v}=0\ ,\ CZ\equiv\frac{a^2y}{v+w}-\frac{b^2x}{u+w}=0$$

Therefore, AX, BY, CZ concur at $J\left(\frac{a^2}{v+w}:\frac{b^2}{u+w}:\frac{c^2}{u+v}\right)$,

which is the isogonal conjugate of the complement of P

Additionally, if we consider the three centers of the spiral similarities that takes the vectors $\overrightarrow{BP}, \overrightarrow{CP}, \overrightarrow{AP}$ into the vectors $\overrightarrow{CA}, \overrightarrow{AB}, \overrightarrow{BC}$, then the triangle formed by these three centers is perspective with $\triangle ABC$ and the perspector is J.

Further, the incenter of this triangle is the first Stevanovic point X_{1130} of $\triangle ABC$.

jayme

#4 Jun 15, 2009, 7:01 pm

Dear Luis,
sorry, I have enconter no name for that point.
Sincerely
Jean-Louis

TelvCohl

#5 Feb 19, 2015, 9:54 pm

My solution:

Let ℓ_A be the bisector of $\angle BAC$.
Let $Q \in AC, R \in AB$ such that $BQ \parallel CP, CR \parallel BP$ and $T = BQ \cap CR$.
Let Ψ be the composition of inversion $\mathbf{I}(A, \sqrt{AB \cdot AC})$ and reflection $\mathbf{R}(\ell_A)$.

Since $AB \cdot AC = AB' \cdot AR$,
so R is the image of B' under Ψ .
Similarly we can prove Q is the image of C' under Ψ ,
so BQ, CR is the image of $\odot(ACC'), \odot(ABB')$ under Ψ, respectively ,
hence $T \equiv BQ \cap CR$ is the image of $X \equiv \odot(ACC') \cap \odot(ABB')$ under Ψ . ... $(\star)$

Since AT pass through the complement P^* of P WRT $\triangle ABC$,
so from $(\star)$ we get AX passes through the isogonal conjugate V of P^* WRT $\triangle ABC$.

Similarly we can prove BY, CZ pass through V ,
so we conclude that AX, BY, CZ are concurrent at V .

Q.E.D

This post has been edited 3 times. Last edited by TelvCohl, Apr 29, 2016, 12:17 am

Luis González

#7 Apr 17, 2015, 2:44 am

Here is my proof without using barycentric coordinates:

Let D be the point forming the parallalelogram $BPCD \Longrightarrow AD$ is the A-cevian of the complement of P. Since $[DCB'] = [DCP] = [DBP] = [DBC'] \Longrightarrow \text{dist}(D, AC) : \text{dist}(D, AB) = BC' : CB'$. But since X is center of the spiral similarity that swaps $\overline{CB'}$ and $\overline{C'B}$, we have $\text{dist}(X, AB) : \text{dist}(X, AC) = BC' : CB' \Longrightarrow$ $\text{dist}(D, AC) : \text{dist}(D, AB) = \text{dist}(X, AB) : \text{dist}(X, AC) \Longrightarrow AD$ and AX are isogonals WRT $\angle BAC \Longrightarrow AX$ goes through the isogonal of the complement of P and similarly BY and CZ.

TelvCohl

#8 Apr 17, 2015, 10:15 am

Another proof:

Let P_A, P_B, P_C be the reflection of P in the midpoint of BC, CA, AB, respectively .

Since X is the center of spiral similarity that maps $CP \mapsto AB$,
so $\triangle XCP \sim \triangle XAB \Longrightarrow XC : BP_A = XC : PC = XA : AB$ . ... $(\star)$
Since $\angle P_ABA = \angle PCB + \angle CBP + \angle PBA = \angle B'XA + \angle CXB' = \angle CXA$,
so combine with $(\star)$ we get $\triangle CXA \sim \triangle P_ABA \Longrightarrow \angle BAP_A = \angle XAC$.

Similarly we can prove $\angle CBP_B = \angle YBA, \angle ACP_C = \angle ZCB$,
so AX, BY, CZ are concurrent at the isogonal conjugate of the complement of P WRT $\triangle ABC$.

Q.E.D

andria

#11 Apr 17, 2015, 12:56 pm

Another solution:let S reflection of P in the midpoint of BC then note that $\angle PXB = \angle CC'A = \angle CXA, \angle XBP = \angle XC'B = \angle XCA \longrightarrow \triangle PXB \sim \triangle CXA$ so $\frac{AC}{PB} = \frac{AX}{BX}$ $\longrightarrow \frac{AC}{CS} = \frac{AX}{BX}$ also $\angle ACS = \angle AB'B = \angle AXB$ so $\triangle ACS \sim \triangle AXB \longrightarrow \angle BAX = \angle CAS$ DONE

pi37

#12 Apr 28, 2016, 11:58 pm

Let D be the reflection of P across M, the midpoint of BC, and let E be the intersection of CX and BD. Since $PC \parallel BD$,

$$\angle XEB = \angle XCP = \angle XB'P = \angle XAB$$

so $AB'XBE$ is cyclic. Now $B'B \parallel DC$, so by Reim's theorem $ACDE$ is cyclic. Then

$$\angle XAB = \angle XEB = \angle CED = \angle CAD$$

as desired.

Quick Reply

High School Olympiads

geometry

Reply

andria

#1 Mar 31, 2015, 3:51 pm

Two circles W_1 and W_2 intersect each other at A, B let the tangent lines from A, B to W_1 and W_2 intersect at P let the second tangent lines from P to W_1 and W_2 meet them at C, D prove that:
$\angle APD = \angle BPC$

Luis González

#3 Apr 16, 2015, 4:25 am • 1

Label $W_1 \equiv (O_1, r_1), W_2 \equiv (O_2, r_2)$ and let I and J be their insimilicenter and exsimilicenter, respectively. Circle $\omega \equiv \odot(AIBJ)$ with diameter $\overline{IJ}$ is the Apollonius circle of $\overline{O_1O_2}$ for $\frac{r_1}{r_2}$.

If $Q \equiv AO_1 \cap BO_2$, then $PABQ$ is cyclic due to $\angle PAQ = \angle PBQ = 90^\circ$ and I becomes incenter of $\triangle AQO_2 \Longrightarrow QI$ bisects $\angle O_1QO_2 \Longrightarrow Q \in \omega \Longrightarrow P \in \omega \Longrightarrow \frac{PO_1}{PO_2} = \frac{r_1}{r_2} = \frac{O_1C}{O_2D} \Longrightarrow PAO_1C \sim \triangle PDO_2B \Longrightarrow \angle APC = \angle BPD$
$\Longrightarrow \angle APD = \angle BPC.$

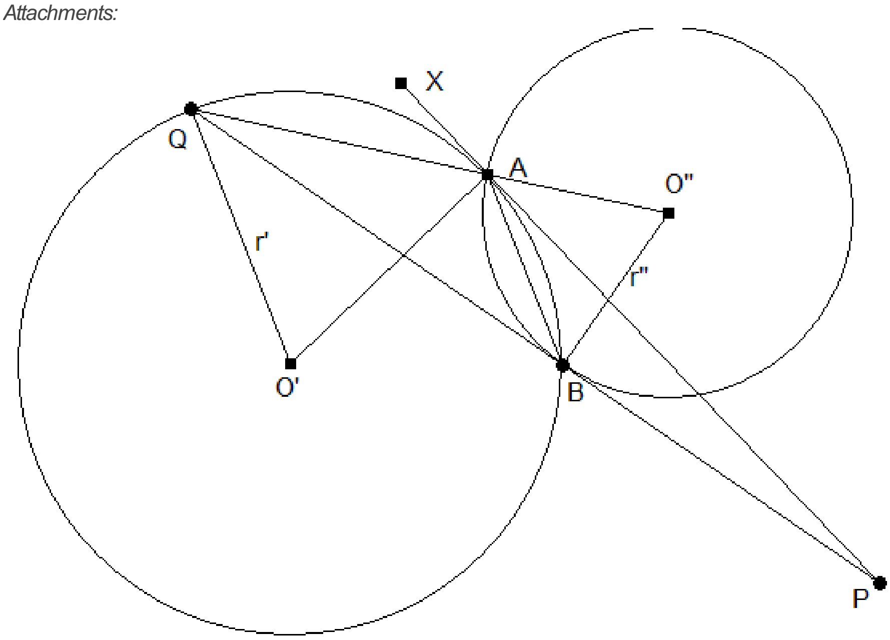
sunken rock

#4 Apr 18, 2015, 4:52 am

Let BP intersect the second time the circle W_1 at Q (see drawing). We need to show that, from P both circles are seen under equal angles, i.e. $\frac{r'}{r''} = \frac{AP}{PB}$ ($*$). Since $\triangle AQP \sim \triangle BAP \implies \frac{AP}{BP} = \frac{AQ}{AB}$ (1) and $\angle ABQ = \angle XAQ$ (2).
Lastly, we need that $\triangle AQO' \sim \triangle ABO''$ which is true, since with (2) we get
$\angle QAO' = \angle 90^\circ - \angle XAQ = 90^\circ - \angle ABQ = \angle ABO''$, hence $\frac{r'}{r''} = \frac{AQ}{AB} = \frac{AP}{BP}$, done.

Best regards,
sunken rock

Attachments:



Quick Reply

High School Olympiads

easy geometry

Locked

andria

#1 Apr 15, 2015, 11:22 pm

Let W be an arbitrary circle in the plane of a given triangle ABC let O_A, O'_A be the circles throw B, C which are tangent to W at A_1, A_2 similarly define B_1, B_2, C_1, C_2 prove that circumcircles of triangles $AA_1A_2, BB_1B_2, CC_1C_2$ are coaxal.

Luis González

#2 Apr 15, 2015, 11:40 pm

Discussed before at http://www.artofproblemsolving.com/community/c6h595152.

High School Olympiads

Beautiful Geometry

ThE-dArK-lOrD
#1 Apr 15, 2015, 3:30 pm

Given triangle ABC.
Let I is its incenter.
D,E,F be the foot perpendicular from I to AB,BC,CA.
X,Y,Z is the intersection of AI,BI,CI with BC,CA,AB respectively.
P is the intersection of DE and XY.
Q is the intersection of EF and YZ.
R is the intersection of DF and XZ.
Proof that A lie on PR,B lie on PQ,C lie on PQ.
And proof that I is the orthocenter of triangle PQR

TelvCohl
#2 Apr 15, 2015, 4:15 pm • 2

I think you mean D,E,F is the projection of I on BC,CA,AB, respectively .

My solution:

Let $G=AD \cap BE \cap CF$ be the Gergonne point of $\triangle ABC$.
Let $\mathcal{H}$ be the circumconic of $\triangle ABC$ passing through I,G and $T=DE \cap QR$.

Since DE, XY is the polar of F, Z WRT $\mathcal{H}$, respectively ,
so $P \equiv DE \cap XY$ is the pole of $FZ \equiv AB$ WRT $\mathcal{H} \Longrightarrow PA, PB$ are the tangents of $\mathcal{H}$.
Similarly we can prove QB,QC,RC,RA are the tangents of $\mathcal{H} \Longrightarrow A \in RP, B \in PQ, C \in QR$.

Since DE is the polar of C WRT $\odot(I)$,
so C lie on the polar τ of P WRT $\odot(I)$.
Since $Q(E,D;P,T)=(EF \cap BC,D;B,C)=-1$,
so $T \in \tau \Longrightarrow CT \equiv QR$ is the polar of P WRT $\odot(I) \Longrightarrow IP \perp QR$.
Similarly we can prove $IQ \perp RP, IR \perp PQ \Longrightarrow I$ is the orthocenter of $\triangle PQR$.

Q.E.D

ThE-dArK-lOrD
#4 Apr 15, 2015, 5:08 pm

Sorry i don't know this circumconic. Can you explain to me please ?

Luis González
#8 Apr 15, 2015, 9:52 pm

For the first problem, see http://www.artofproblemsolving.com/community/c6h553528 (post #4).

High School Olympiads

Intersection of Three Lines

Source: Spring 2006 Tournament of Towns Junior A-Level #3

bluecarneal

#1 Apr 15, 2015, 6:52 pm

On sides AB and BC of an acute triangle ABC two congruent rectangles $ABMN$ and $LBCK$ are constructed (outside of the triangle), so that $AB = LB$. Prove that straight lines AL, CM and NK intersect at the same point.

(5 points)

Luis González

#2 Apr 15, 2015, 8:24 pm

Since $\angle ABL = \angle CBM = \angle ABC + 90^\circ$, then the B-isosceles $\triangle BAL$ and $\triangle BMC$ are similar $\Longrightarrow$ $\angle BMC = \angle BAL$ and $\angle BLA = \angle BCM \Longrightarrow P \equiv CM \cap AL$ is the 2nd intersection of $\odot(ABMN)$ and $\odot(LBCK)$. Now $\angle NPM = \angle NBM = \angle CBK = \angle CPK \Longrightarrow N, P, K$ are collinear.

High School Olympiads

a geometry problem , maybe old and easy

Reply

Source: 2015 Taiwan TST Round 1 Quiz2 P2

YaWNeeT
#1 Apr 14, 2015, 9:53 pm

Given any triangle ABC. Let O_1 be it's circumcircle, O_2 be it's nine point circle, O_3 is a circle with orthocenter of ABC, H, and centroid G, be it's diameter. Prove that: O_1, O_2, O_3 share axis. (i.e. chose any two of them, their axis will be the same one, if ABC is an obtuse triangle, the three circle share two points.)

This post has been edited 3 times. Last edited by Luis González, Apr 15, 2015, 12:52 am

Luis González
#2 Apr 15, 2015, 12:46 am • 1

Let D, E, F be the feet of the altitudes on BC, CA, AB. M is the midpoint of BC and EF, FD, DE, cut BC, CA, AB at P, Q, R. Since $(B, C, D, P) = -1 \Longrightarrow PB \cdot PC = PM \cdot PD \Longrightarrow P$ has equal power WRT (O_1) and 9-point circle $\odot(DEF) \equiv (O_2)$ and similarly Q and $R \Longrightarrow \overline{PQR}$ is radical axis of $(O_1), (O_2)$.

Since PH is the polar of A WRT the circle (M) with diameter BC, then PH is perpendicular to AM at T, the 2nd intersection of $\odot(AEHF)$ and (O_3). Hence $PH \cdot PT = PE \cdot PF \Longrightarrow P$ has equal power WRT (O_2) and (O_3) and similarly $Q, R \Longrightarrow \overline{PQR}$ is radical axis of $(O_2), (O_3)$. The conclusion follows.

TelvCohl
#3 Apr 15, 2015, 6:27 am

In general, given tow circles $\odot(O_1, R_1)$ and $\odot(O_2, R_2)$.
Let P, Q be the insimilicenter, exsimilicenter of $\odot(O_1) \sim \odot(O_2)$, respectively .

Then $\odot(O_1), \odot(O_2), \odot(O_3) \equiv \odot(PQ)$ are coaxial .

Proof :

Since $\odot(O_3)$ is the locus of the point T such that $TO_1 : TO_2 = R_1 : R_2$,

so for any point T on $\odot(O_3)$ we have $\frac{{TO_1}^2 - {R_1}^2}{{TO_2}^2 - {R_2}^2} = \left(\frac{R_1}{R_2}\right)^2 = \text{Const} \Longrightarrow \odot(O_1), \odot(O_2), \odot(O_3)$ are coaxial .

Done

MathPanda1
#4 Nov 16, 2015, 12:46 am

Does this work?
Note that the centres of the circles are collinear on some line l. Then, their axis' are perpendicular to l and they are concurrent by the Radical Axis Theorem. Thus, they must coincide i.e. They have the same axis.

MathPanda1
#5 Nov 16, 2015, 7:00 pm

Does the solution above work? It seems a little short for the problem...

EulerMacaroni
#6 Nov 16, 2015, 8:09 pm

> MathPanda1 wrote:
> Does the solution above work? It seems a little short for the problem...

All you did was prove that the radical axii are parallel

MathPanda1
#7 Nov 16, 2015, 10:47 pm

> aops wrote:
> Theorem 3: (Radical Axis Concurrence Theorem) The three pairwise radical axes of three circles concur at a point, called the radical center.

See http://www.artofproblemsolving.com/wiki/index.php/Radical_axis
Is parallel radical axii a possibility?

EulerMacaroni
#8 Nov 16, 2015, 11:00 pm

> MathPanda1 wrote:
>> aops wrote:
>> Theorem 3: (Radical Axis Concurrence Theorem) The three pairwise radical axes of three circles concur at a point, called the radical center.
>
> See http://www.artofproblemsolving.com/wiki/index.php/Radical_axis
> Is parallel radical axii a possibility?

Yes, but just because they're parallel doesn't mean that they're the same line...

MathPanda1
#9 Nov 17, 2015, 3:31 am

Well, don't they have to meet? The only way parallel lines meet is if they coincide. Unless point of infinity exists...

EulerMacaroni
#10 Nov 17, 2015, 5:50 am

> MathPanda1 wrote:
> Well, don't they have to meet? The only way parallel lines meet is if they coincide. Unless point of infinity exists...

Parallel radical axii coincide at the point at infinity; a valid way to do this problem would be to show that there exists some point with equal power wrt to all three circles, which would force the conclusion.

MathPanda1
#11 Nov 17, 2015, 8:58 am

Thank you for pointing out the errors in my proof, EulerMacaroni. I really appreciate it!

Quick Reply

High School Olympiads

easy geometry

Reply

andria

#1 Apr 14, 2015, 12:46 pm

$ABCD$ is a cyclic quadrilateral $AB \cap CD = E, BC \cap AD = F$ and $\odot(ABCD) \cap \odot(\triangle AEF) = P$ prove that PC bisects EF.

TelvCohl

#2 Apr 14, 2015, 1:35 pm

My solution:

Let $B^* = PE \cap BC, D^* = PF \cap CD$.

From $\angle PBA = \angle PDA, \angle PEA = \angle PFA \Longrightarrow \triangle PBE \sim \triangle PDF$,
so combine with $\angle PBB^* = \angle PDD^*$ we get $\triangle PBE \cup B^* \sim \triangle PDF \cup D^*$,
hence $PB^* : PE = PD^* : PF \Longrightarrow B^*D^* \parallel EF$ . ... $(\star)$

Since $(PE, PF; PC, B^*D^*) = -1$,
so combine with $(\star)$ we get PC pass through the midpoint of EF .

Q.E.D

Luis González

#3 Apr 15, 2015, 12:17 am

Let CP cut $\odot(CEF)$ again at $Q. \widehat{QEF} = \widehat{PCB} = \widehat{PAE} = \widehat{PFE} \Longrightarrow EQ \parallel FP$ and similarly we have $FQ \parallel EP$ $\Longrightarrow EPFQ$ is parallelogram $\Longrightarrow PC$ bisects $\overline{EF}$.

Quick Reply

High School Olympiads

Equal segments

Reply

buratinogigle

#1 Jul 30, 2011, 5:13 pm

Let ABC be a triangle. P, P' are two isogonal conjugate point with respsect to ABC. K, K' are projection of P and P' on line BC, resp. AH is altitude of ABC. $A_1, A_2 \in AH$ such that $AA_1 = PK, AA_2 = P'K'$. $P'A_1, PA_2$ intersect line BC at U, V. Prove that $BU = CV$.

Note that. It is the generalization of the following problem

Let (I, r) be incircle of triangle ABC. AH is altitude, $K \in AH$ such that $AK = r$, M is midpoint of BC. Prove that M, I, K are collinear.

Luis González

#2 Apr 14, 2015, 9:53 am • 2

Let AP, AP' cut $\odot(ABC)$ again at M, M' and we redefine U, V as the projections of M, M' on BC. Since $MM' \parallel BC$, then by symmetry we have $BU = CV$. We'll prove that $U \in P'A_1$ and $V \in PA_2$.

If $D \equiv PM' \cap BC$ and $E \equiv DP' \cap MM'$, we have $DP' \parallel AP$ (this was discussed at the problems Isogonal points and parallelism and Concurrent with PQ line).Therefore $\frac{AA_2}{MV} = \frac{P'K'}{M'U} = \frac{P'D}{DE} = \frac{AP}{PM} \Longrightarrow V \in PA_2$ and in the same way we get $U \in P'A_1$, as desired.

Quick Reply

High School Olympiads

Concurrent with PQ line

Source: Own

buratinogigle

#1 Oct 23, 2014, 11:20 am

Let ABC be a triangle with circumcircle (O) and P,Q are two isogonal conjugate point. M,N are projection of P,Q on BC. AP,AQ cut (O) agian at S,T. Prove that SM,TN and PQ are concurrent.

TelvCohl

#2 Oct 23, 2014, 3:01 pm • 3

My solution:

Lemma 1:

Let D,E be two points on the bisector of $\angle ABC$ satisfy $\angle BAD=\angle EAC$.
Let $G=AD \cap (ABC)$ and $F=EG \cap BC$.

Then $AE//DF$

Proof of lemma 1:

Let X be a point on the bisector of $\angle ABC$ satisfy $AX//BG$.
Redefine F as the point on BC satisfy $AE//DF$, so we only have to prove E,F,G are collinear .

Since $\angle AXE=\angle XBG=\angle XBC+\angle CBG=\angle ABE+\angle BAE=\angle AEX$,
so $\triangle AEX$ is a isosceles triangle . ie. $AX=AE$
Since $\angle ABX=\angle FBD$ and $\angle AXB=\angle AEX=\angle FDB$,
so we get $\triangle BAX \sim \triangle BFD$,
hence $\frac{GD}{GA}=\frac{BD}{BX}=\frac{DF}{XA}=\frac{DF}{AE}$. ie. E,F,G are collinear

Lemma 2:

P,P' are isogonal conjugate of $\triangle ABC$.
Let $N=AP \cap (ABC)$, $M=P'N \cap BC$.

Then $PM//AP'$

Proof of lemma 2:

Let D,E be the intersection of AP,AP' and the bisector of $\angle ABC$.
Let F,I,G be the intersection of NE,AP,AP' and BC .
Let $H=BP \cap AP'$.

From lemma 1 we get $DF//AP'$.
Since $(I,F;M,G)=(A,E;P',G)=(G,E;H,A)=(I,D;P,A)$,
so we get FD,MP,GA are concurrent. ie. $FD//MP//P'A$

Lemma 3 :

P,P' are isogonal conjugate of $\triangle ABC$.
Let $X=AP \cap (ABC)$ and Y be the projection of P on BC .
Let $Z=XY \cap (ABC)$.

Then Z lie on (AP')

Proof of lemma 3:

Let $Q = AP' \cap (ABC)$.
Let $R = P'X \cap BC$ and V be a point on (AP') satisfy $AV \perp BC$.
Redefine $Z = (AP') \cap (ABC)$, so we only have to prove X, Y, Z are collinear .

Easy to see $P'V//XQ//BC$.
From lemma 2 we get $PR//AP'$.
From Reim theorem we get V, X, Z are collinear . ... (1)
Since $\triangle AP'V$ and $\triangle PRY$ are homothety ,
so we get V, X, Y are collinear . ... (2)
Combine (1) and (2) we get X, Y, Z are collinear .

Back to the main problem:

Let $P' = (AP) \cap (ABC), Q' = (AQ) \cap (ABC)$ and A' be the antipodal point of A'.

Easy to see P', P, A' are collinear and Q, Q', A' are collinear .

From lemma 3 we get S, M, Q' are collinear and T, N, P' are collinear ,
so from Pascal theorem (for $ASQ'A'P'T$) we get $P, Q, SM \cap TN$ are collinear .

Q.E.D

Luis González
#3 Oct 24, 2014, 4:47 am • 1

Fix the isogonals AS, AT. Parallel through Q to AS cuts BC at U. The applications $P \mapsto Q$ and $Q \mapsto U$ are clearly homographic $\Longrightarrow P \barwedge U$. When $P \equiv A$, we have $Q \equiv U \equiv AT \cap BC$, when $Q \equiv A$, we have $P \equiv U \equiv BC \cap AS$ and when $P \equiv S, Q$ goes to the infinite point of AT and U goes then to the infinite point A_∞ of $BC \parallel ST$. Consequently, the superposed pencils TP, TU are identical, i.e. $U \in TP$ and similarly if $V \equiv QS \cap BC$, then $PV \parallel AQ \Longrightarrow$ pencils $P(A_\infty, V, M, A)$ and $Q(A_\infty, A, N, U)$ are similar, thus projective $\Longrightarrow S(T, V, M, A)$ and $T(S, A, N, U)$ are projective with double ray $ST \Longrightarrow P \equiv SA \cap TU, Q \equiv SV \cap TA$ and $SM \cap TN$ are collinear.

buratinogigle
#4 Dec 27, 2014, 2:05 pm • 2

Problem. Let ABC be a triangle inscribed circle (O) with P, Q are isogonal conjugate points. AP, AQ cut (O) again at M, N. S, T are projections of P, Q on BC. Prove that SM, TN and PQ are concurrent.

Solution by my pupil **Phan Anh Quan.**

Let MQ, NP, AM, AN cut BC at E, F, G, H, resp. Because P, Q are isogonal conjugate, by angle charsing we have $\triangle CHN \sim \triangle ACM$ và $\triangle CPM \sim \triangle QCN$ (a.a) deduce $HN.AM = CM.CN = QN.PM$ deduce $\dfrac{MP}{MA} = \dfrac{NH}{NQ} = \dfrac{ME}{MQ}$ thus $PE \parallel AQ$.

Similarly, we have $QF \parallel AG$. From this $\triangle PGE \sim \triangle QFH$ and they have the altitudes PS, QT. Let MS cuts NT at R then $M(PQ, RN) = (GE, S) = (FH, T) = N(PQ, RM)$. Therefore P, Q, R are collinear.

Note that, the part of this solution solve the problem Isogonal points and parallelism by the nice and short way.

Attachments:
Figure2608.pdf (11kb)

XmL
#5 Dec 31, 2014, 12:19 am • 3

Define $(APC) \cap SC = C', BC \cap AP, AQ = Y, X, TN, SM$ intersect the A-altitude at P', Q' resp.
By some angle chasing, we have $ASC'P \sim QTCX \Rightarrow \dfrac{AP}{PS} = \dfrac{QX}{XT} = \dfrac{QN}{d(N, ST)} = \dfrac{AP}{d(P', ST)}$($d(A, BC)$ means the distance from A to BC)$\Rightarrow PP' \parallel BC$. Likewise we can obtain $QQ' \parallel BC$. Hence by ratio equalities created by the parallel lines, we have $P'(P, Q'; T, Q) = Q'(P, P'; S, Q) \Rightarrow P'T, Q'S, PQ$ concur.

IDMasterz

#6 Jan 1, 2015, 12:08 am • 1

my solution to this problem from memory was:

Let $\odot AQC$ meet $\odot ABC$ at T. Then, as P moves along AP, then $M \mapsto T$ is a projectivity (You can see this by considering perpendicular pencil created by circumcentre). Now just take the three points (they are pretty easy to choose), and then finish with pascal theorem :)

Arab

#7 Jan 2, 2015, 4:23 am • 2

See Corollary 3.1.

This post has been edited 1 time. Last edited by Arab, Jul 22, 2015, 5:12 pm

Quick Reply

High School Olympiads

Coaxal circles in incenter/excenter configuration

Reply

Source: own

jlammy
#1 Apr 14, 2015, 12:55 am

• Let ABC be a triangle, whose incircle touches BC, CA, AB at P, Q, R respectively. Denote D as the intersection of the lines PQ and AB, and E as the intersection of the lines RP and CA. If I is the incenter of $\triangle ABC$, prove that the circumcircles of triangles PQE, PRD and PIA are coaxal.

• Let ABC be a triangle, whose excircle (opposite A) touches BC, CA, AB at P, Q, R respectively. Denote D as the intersection of the lines PQ and AB, and E as the intersection of the lines RP and CA. If I_a is the excenter of $\triangle ABC$, opposite A, prove that the circumcircles of triangles PQE, PRD and PI_aA are coaxal.

Luis González
#2 Apr 14, 2015, 3:34 am

Inversion WRT (I) takes A into the midpoint M of QR and takes D, E into the projections D', E' of I on CR, BQ (because CR, BQ are the polars of D, E WRT (I)). Hence $\odot(PIA)$ goes to the line PM and $\odot(PQE), \odot(PRD)$ go to $\odot(PQE'), \odot(PRD')$. Thus, it's enough to show that PM is radical axis of $\odot(PQE'), \odot(PRD')$.

Since $E' \in \odot(IPBR)$, then $\angle PE'B = \angle PRB = \angle PQR \Longrightarrow QR$ is tangent of $\odot(PQE')$ and similarly QR is tangent of $\odot(PRD') \Longrightarrow QR$ is external common tangent of $\odot(PQE')$ and $\odot(PRD') \Longrightarrow PM$ is their radical axis, as desired.

The 2nd problem is proved in exactly the same way, i.e. it follows by extraversion.

TelvCohl
#3 Apr 14, 2015, 8:30 am

Another solution:

Let ω_Q, ω_R be the circle through P and tangent to QR at Q, R, respectively .
Let ℓ be the bisector of $\angle QPR$ and T be the intersection of $\odot(I)$ with $P-$ median of $\triangle PQR$.
Let Ψ be the composition of Inversion $\mathcal{I}(P, \sqrt{PQ \cdot PR})$ and Reflection $\mathcal{R}(\ell)$ (i.e. $\Psi = \mathcal{I}(P, \sqrt{PQ \cdot PR}) \circ \mathcal{R}(\ell)$) .
Let $A^* \equiv \omega_Q \cap \omega_R$ be the reflection of T in the midpoint of QR and I^* be the reflection of P in QR .

Easy to see $Q \longleftrightarrow R, QR \longleftrightarrow \odot(PQR), I \longleftrightarrow I^*$ under Ψ ,

From $RD \longleftrightarrow \omega_Q, QE \longleftrightarrow \omega_R$ under $\Psi \Longrightarrow \Psi(A) = A^*$,
so $\Psi(D) = D^*, \Psi(E) = E^*$ is $\omega_Q \cap PR, \omega_R \cap PQ$, respectively .
Notice that A^*, Q, R, I^* are concyclic at the reflection of $\odot(I)$ in QR ,
so from $\triangle I^*QR \cong \triangle PQR \Longrightarrow I^*A^*$ is I^*- symmedian of $\triangle I^*QR$ . ... $(\star)$

From $\angle RQD^* = \angle QPR = \angle QRE^* \Longrightarrow QD^*, RE^*$ are the tangents of $\odot(I^*QR)$,
so combine with $(\star) \Longrightarrow I^*A^*, QD^*, RE^*$ are concurrent $\Longrightarrow \odot(PIA), \odot(PDR), \odot(PEQ)$ are coaxial .

Q.E.D

jayme
#4 Apr 14, 2015, 5:12 pm

Dear Mathlinkers,
thanl jlammy for this nice problem...
I research a proof without inversion...

Sincerely
Jean-Louis

TelvCohl

#8 Apr 14, 2015, 7:33 pm

Another solution (without inversion) :

Let r be the radius of $\odot(I)$.
Let $\mathcal{P}(P, \odot)$ be the power of a point P WRT circle $\odot$.
Let O_D, O_E be the center of $\odot(PRD), \odot(PQE)$, respectively .

Easy to see $O_D \in BI, O_E \in CI$.

From $\angle BIP = \angle RQP, \angle IO_DP = \angle BDP \Longrightarrow \triangle DQR \sim \triangle O_DIP$,

so $\frac{\mathcal{P}(I, \odot(O_D))}{\mathcal{P}(A, \odot(O_D))} = \frac{IO_D{}^2 - PO_D{}^2}{DA \cdot AR} = \frac{QD^2 - RD^2}{DA} \cdot \frac{QR}{\mathrm{r} \cdot AR} = \frac{-QD \cdot QP}{DA} \cdot \frac{QR}{\mathrm{r} \cdot AR} = \frac{-2 \cdot QR \cdot \sin \angle A}{AR}$.

Similarly we can prove $\frac{\mathcal{P}(I, \odot(O_E))}{\mathcal{P}(A, \odot(O_E))} = \frac{-2 \cdot QR \cdot \sin \angle A}{AQ}$,

so we get $\frac{\mathcal{P}(I, \odot(O_D))}{\mathcal{P}(A, \odot(O_D))} = \frac{\mathcal{P}(I, \odot(O_E))}{\mathcal{P}(A, \odot(O_E))} \Longrightarrow \frac{\mathcal{P}(I, \odot(O_D))}{\mathcal{P}(I, \odot(O_E))} = \frac{\mathcal{P}(A, \odot(O_D))}{\mathcal{P}(A, \odot(O_E))}$.

i.e. $\odot(PIA), \odot(PRD), \odot(PQE)$ are coaxial .

Q.E.D

jayme

#9 Apr 15, 2015, 8:09 pm

Dear jlammy and Mathlinkers,

jlammy how are the main ideas of your nice proof ?

Sincerely
Jean-Louis

liberator

#10 Apr 16, 2015, 1:09 am • 2

For drawability of the diagram, we solve the second version (although the proof of the first is similar).

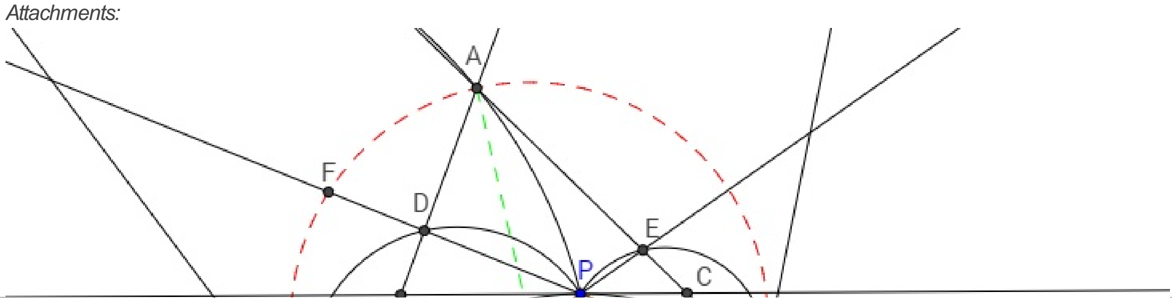
Let X denote the second intersection of circles (PQE) and (PRD). It suffices to show that X is on (PI_aA).

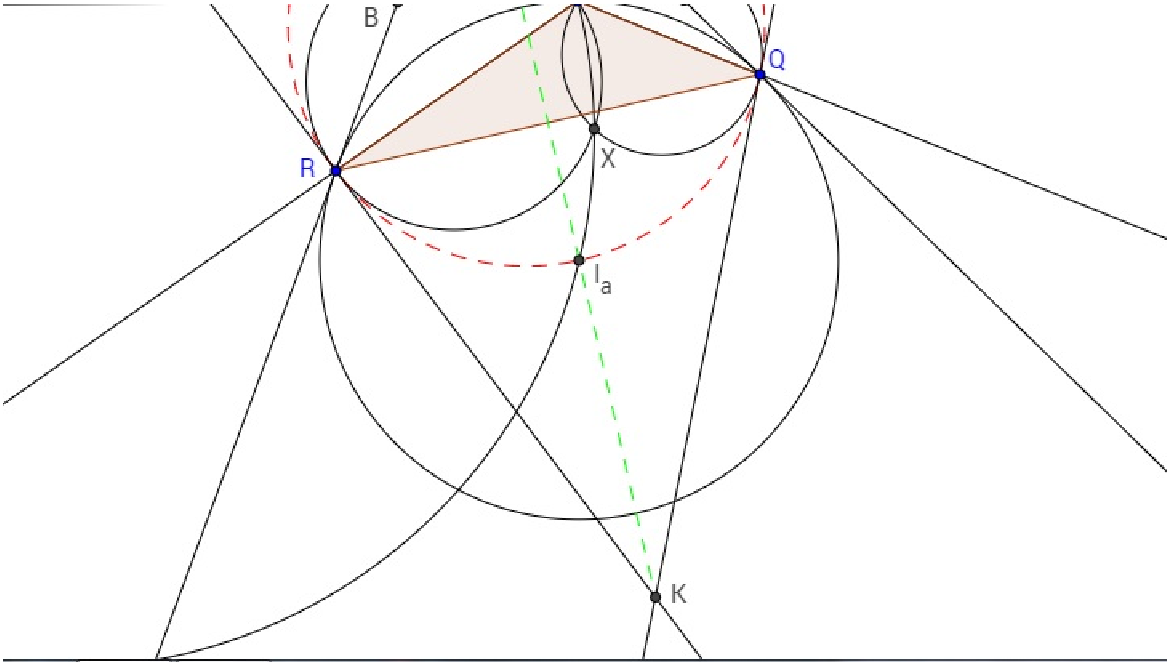
First, observe that AQI_aR is cyclic, from perpendicularity of tangents.

We claim that (AQI_aR) is tangent to (PQE) at Q and (PRD) at R. Let PQ meet (AQI_aR) again at F. By Reim's theorem on (PQR) and $FAQR)$, we have $AF \parallel PR \iff AF \parallel PE$. Then by Reim's theorem again, on (FAQ) and (PEQ), the two circles are tangent at Q. Similarly, (AQI_aR) is tangent to (PRD) at R, which establishes our claim.

Now, let K be the radical center of the three circles $(PQE), (PRD), (AQR)$. Being the intersection of the two tangents to (AQR) at Q and R, K is on AI_a, the perpendicular bisector of $\overline{QR}$. Hence $KI_a \cdot KA = KQ^2 = KX \cdot KP$, so X is on (PI_aA), as required.

Attachments:





jayme

#11 Apr 16, 2015, 11:48 am

Dear Liberator and Mathlinkers,
this jlammy nice problem and your elegant proof give a real regard to the figure in particular to the first problem...
Sincerely
Jean-Louis

jayme

#12 Apr 16, 2015, 12:08 pm

Dear Liberator and Mathlinkers,
the question of the tangency of two circles at Q can be solved by considering the pivot theorem which can be proved with the Reim's theorem...
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Easy Geometry

Reply

mihajlon

#1 Apr 14, 2015, 1:21 am

In acute $\triangle ABC$ $(AB \neq AC)$ with $\angle BAC = \alpha$, E is euler's center and P is on segment AE. If $\angle ABP = \angle ACP = x$, prove that $x = 90^{\circ} - 2\alpha$.
(Serbian TST 2013)

This post has been edited 1 time. Last edited by mihajlon, Apr 14, 2015, 3:14 am

Luis González

#2 Apr 14, 2015, 2:05 am

Let O be the circumcenter of $\triangle ABC$ and let K be the circumcenter of $\triangle OBC$. It's well-known that AE and AK are isogonals WRT $\angle BAC$. Therefore, the isogonal conjugate of P WRT $\triangle ABC$, which must lie on the perpendicular bisector of BC, is none other than $K \Longrightarrow \angle ABP = \angle CBK = 90^{\circ} - \angle BOC = 90^{\circ} - 2\alpha$.

TelvCohl

#3 Apr 14, 2015, 8:49 am

For more solution you can see 2013 Serbia Additional TST Problem 2

Quick Reply

High School Olympiads

Isogonal conjugate in a trapezium

Reply

tranquanghuy7198
#1 Apr 13, 2015, 1:24 pm

Let $ABCD$ be a trapezium with bases $AB \parallel CD$. AC intersects BD at P. The circumcircles of $\triangle PAD$ and $\triangle PBC$ intersect at P and O. Points M, N are constructed such that $AM \parallel CN$ and $DM \parallel BN$. MN intersects AC, BD at K, L, respectively. S is isogonal conjugate of O in $\triangle PKL$. Prove that:
$\triangle OAC \sim \triangle ODB \sim \triangle SMN$.

Luis González
#2 Apr 14, 2015, 1:42 am • 1

The condition $AB \parallel CD$ is unnecessary; this holds for any $ABCD.$ Clearly we have $\triangle OAC \sim \triangle ODB$ and we redefine S as the point verifying that $\triangle SMN$ is inversely similar to $\triangle OAC \sim \triangle ODB.$ We prove then that O, S are isogonal conjugates WRT $\triangle PKL.$

Indeed, from $DM \parallel BN \Longrightarrow \frac{BL}{LD} = \frac{NL}{LM} \Longrightarrow \triangle ODB \cup L \sim \triangle SMN \cup L \Longrightarrow \angle OLB = \angle SLN \Longrightarrow LO, LS$ are isogonals WRT $\angle PLK$ and similarly KO, KS are isogonals WRT $\angle PKL \Longrightarrow O, S$ are isogonal conjugates WRT $\triangle PKL,$ as desired.

Quick Reply

High School Olympiads

Concurrency in harmonic quadrilateral (1000 th post !!!!)

Reply

mahanmath

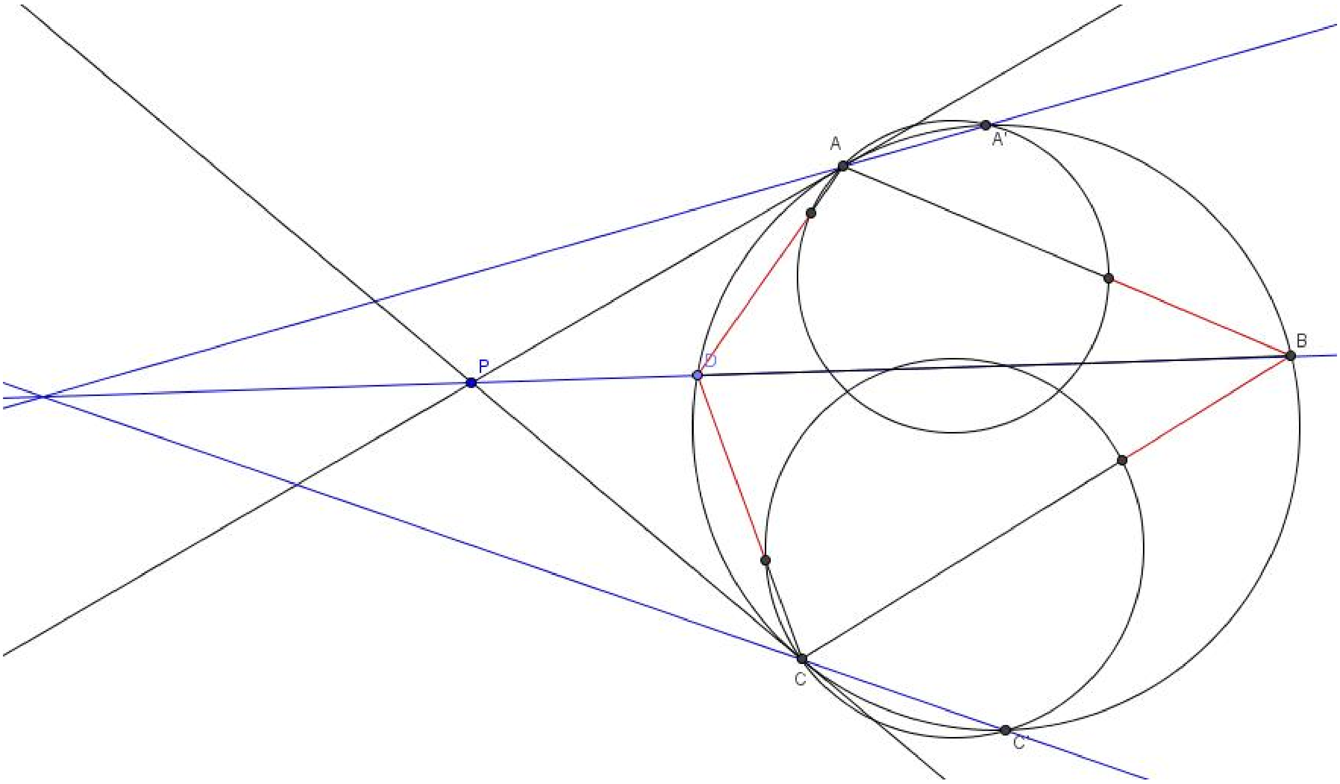
#1 Nov 5, 2011, 5:17 am • 2

It's my 1000^{th} post .....

The following problem is mine , hope you enjoy it !

PA, PC are tangent to circle ω . An arbitrary line pass through P and cuts ω at D, B .
X_1, X_2 lie on DA, DC and Y_1, Y_2 lie on BA, BC respectively such that $DX_1 = DX_2$ and $BY_1 = BY_2$.
$(AX_1Y_1) \cap \omega = A, A'$ and $(CX_2Y_2) \cap \omega = C, C'$ .Prove that AA', CC', BD are Concurrent

Attachments:



Luis González

#2 Nov 5, 2011, 6:37 am • 2

If X_3, Y_3 are the reflections of X_1, Y_1 about the midpoints of AD, AB, and X_4, Y_4 are the reflections of X_2, Y_2 about the midpoints of CD, CB, then AA' and CC' are tangent to $\odot(AX_3Y_3)$ and $\odot(CX_4Y_4)$, respectively (see the hidden lemma). So if AA', CC' cut BD at U, V, we deduce that

$$\frac{UD}{UB} = \frac{AD}{AB} \cdot \frac{AX_3}{AY_3} = \frac{AD}{AB} \cdot \frac{DX_1}{BY_1} \quad , \quad \frac{VD}{VB} = \frac{CD}{CB} \cdot \frac{CX_4}{CY_4} = \frac{CD}{CB} \cdot \frac{DX_2}{BY_2}$$

Since $\frac{AD}{AB} = \frac{CD}{CB}$, it follows that $U \equiv V$, i.e. AA', CC', BD concur.

Click to reveal hidden text

Love_Math1994

#3 Nov 5, 2011, 9:49 am • 1

Congrat mahanmath.U r so active .

It is my proof:

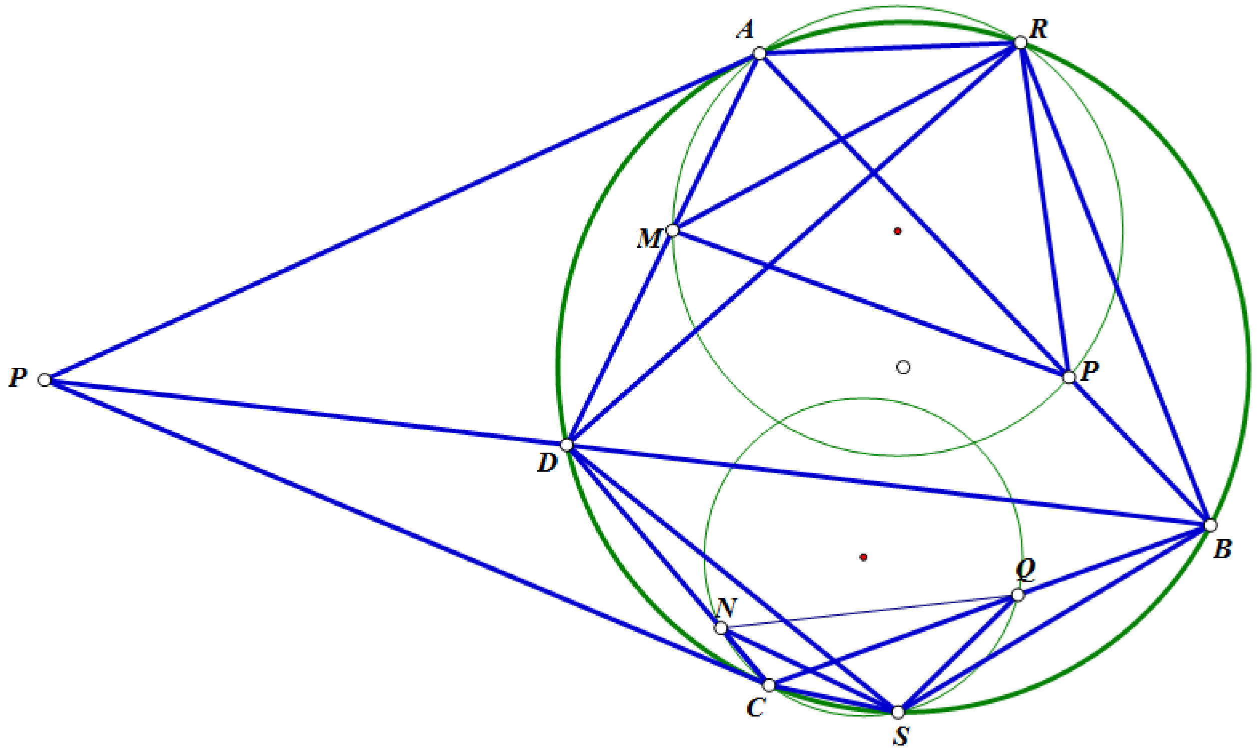
Easy to see that AR,CS and BD is concur if and only if DRBS is harmonic.(Use menelaos theorem and similar triangle to see they concur iff DR.BS=RB.SD)

But we have $\triangle RMD \sim \triangle RPB(a.a)$ and $\triangle DNS \sim \triangle BQS$ so $RB/RD = PB/DM = BQ/DN = BS/DS$.

GED.

Hope it right .

Attachments:



mahanmath

#4 Nov 5, 2011, 5:55 pm

It suffice to prove that B, D lie one a circle which is coaxal with $(AX_1Y_1), (CX_2Y_2)$, but the ratio of power of B wrt $(AX_1Y_1), (CX_2Y_2)$ is equal to the ratio of power of D wrt $(AX_1Y_1), (CX_2Y_2)$ ....FINISH !

(See this topic)

goldeneagle

#5 Nov 6, 2011, 4:36 am

> *mahanmath wrote:*
>
> It suffice to prove that B, D lie one a circle which is coaxal with $(AX_1Y_1), (CX_2Y_2)$, but the ratio of power of B wrt $(AX_1Y_1), (CX_2Y_2)$ is equal to the ratio of power of D wrt $(AX_1Y_1), (CX_2Y_2)$ ....FINISH !
>
> (See this topic)

exactly mine!!

Quick Reply

High School Olympiads

Parallel

Reply

77ant

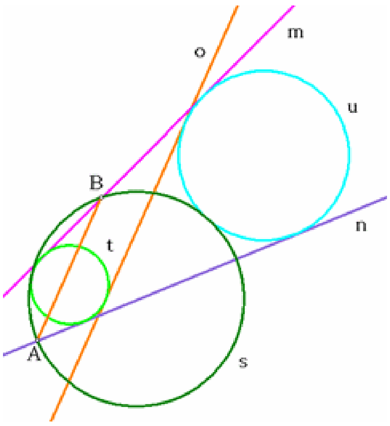
#1 Nov 4, 2009, 10:11 pm

Dear everyone.

Please think of the following.
There are three circles s, t, u so that t, u are tangent to s. Three lines m, n, o are common tangents of t and u. Then AB is parallel to the line o.

Thank you for reading. :)

Attachments:



Luis González

#2 Apr 13, 2015, 5:01 am

I'm surprised this problem went unnoticed as it is nothing but an extraversion of the so called "Parallel tangent theorem". See the threads http://www.artofproblemsolving.com/community/c6h15945, http://www.artofproblemsolving.com/community/c6h430441 for proofs.

Quick Reply

High School Olympiads

Bisectors and Angle Ratios

Reply

Source: Fall 2005 Tournament of Towns Senior A-Level #5

bluecarneal

#1 Mar 31, 2015, 7:18 am

In triangle ABC bisectors AA_1, BB_1 and CC_1 are drawn. Given $\angle A : \angle B : \angle C = 4 : 2 : 1$, prove that $A_1B_1 = A_1C_1$.

(7 points)

Luis González

#2 Apr 13, 2015, 1:59 am

Let D, E denote the midpoints of the arcs BC, CA of the circumcircle $\odot(ABC)$ and $J \equiv DB_1 \cap CC_1$.

$\widehat{A} : \widehat{B} : \widehat{C} = 4 : 2 : 1 \Longrightarrow \widehat{A} = 2 \cdot \widehat{B}$ and $\widehat{B} = 2 \cdot \widehat{C} \Longrightarrow \triangle BCB_1$ is isosceles with apex $B_1 \Longrightarrow BCEA$ is isosceles trapezoid with bases AE, BC, thus by symmetry, we have $\widehat{JDA_1} = \frac{1}{2}\widehat{ABE} = \widehat{JCA_1} \Longrightarrow DCJA_1$ is cyclic $\Longrightarrow$ $\widehat{DJA_1} = \widehat{DCB} = \widehat{CAD} \Longrightarrow AA_1JB_1$ is cyclic $\Longrightarrow \widehat{AA_1J} = \widehat{DCJ} = \frac{1}{2}\widehat{C} + \frac{1}{2}\widehat{A} = \frac{1}{2}\widehat{C} + \widehat{B} = \widehat{AC_1J} \Longrightarrow$ AJA_1C_1 is cyclic $\Longrightarrow AB_1A_1C_1$ is cyclic and since AA_1 bisects $\angle B_1AC_1$, then $A_1B_1 = A_1C_1$.

Quick Reply