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INEQUALITIES OF KARAMATA, SCHUR AND MUIRHEAD, AND SOME APPLICATIONS

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Abstract. Three classical general inequalities—those of Karamata, Schur and Muirhead—are proved in this article. They can be used in proving other inequalities, particularly those appearing as problems in mathematical competitions, including International Mathematical Olympiads. Some problems of this kind are given as examples. Several related inequalities—those of Petrović, Steffensen and Szegö—are treated, as well.

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1. Introduction

Three classical inequalities are proved in this article. They can be used in proving several other inequalities, particularly those appearing as problems in mathematical competitions, including International Mathematical Olympiads. Some problems of this kind are given as examples.

The article is adapted according to our book [7], intended for preparation of students for mathematical competitions. Its (shortened) Serbian version was published in Nastava matematike, L, 4 (2005), 22–31.

We start by recalling some well-known notions which will be used in the sequel.

A function $f\colon (a,b) \to \mathbf{R}$ is said to be *convex* if for each two points $x_1, x_2 \in (a,b)$ and each two nonnegative real numbers λ_1, λ_2 satisfying $\lambda_1 + \lambda_2 = 1$, the following inequality is valid

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leqslant \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

The function f is *concave* if the function $-f$ is convex, i.e., if the opposite inequality

$$f(\lambda_1 x_1 + \lambda_2 x_2) \geqslant \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

always holds. If, in the previous inequalities (assuming $x_1 \neq x_2$), the equality takes place only in the case when $\lambda_1 = 0$ or $\lambda_2 = 0$, then the function f is said to be *strictly convex* (resp. *strictly concave*).

It can be easily checked that the function $f\colon (a,b) \to \mathbf{R}$ is convex (strictly convex) if and only if the inequality

$$\frac{f(x) - f(x_1)}{x - x_1} \leqslant \frac{f(x_2) - f(x)}{x_2 - x} \tag{1}$$

(resp. $\dfrac{f(x)-f(x_1)}{x-x_1} < \dfrac{f(x_2)-f(x)}{x_2-x}$) holds for arbitrary points x_1, x_2, x from (a,b), such that $x_1 < x < x_2$,. An analogous criterion is valid for concave (strictly concave) functions.

Let $f\colon (a,b) \to \mathbf{R}$ and $x, y \in (a,b)$. The quotient

$$\Delta_f(x,y) = \frac{f(y)-f(x)}{y-x}$$

is called the *divided difference* of the function f at the points x, y. It is clear that the divided difference is a symmetric function in x, y, i.e., $\Delta_f(x,y) = \Delta_f(y,x)$.

LEMMA 1. *A function $f\colon (a,b) \to \mathbf{R}$ is convex (strictly convex) if and only if its divided difference $\Delta_f(x,y)$ is increasing (strictly increasing) in both variables. An analogous assertion is valid for concave (strictly concave) functions.*

Proof. Let the function f be convex and let $x_1, x_2 \in (a,b)$ so that $x_1 < x_2$. Choose an arbitrary $x \in (a,b)$ such that, for instance, $x_1 < x < x_2$ (for other values $x \in (a,b)$ the proof is similar). Applying the previous assertion, the convexity of the function f implies that the inequality (1) is valid. In other words,

$$\Delta_f(x_1,x) = \Delta_f(x,x_1) \leqslant \Delta_f(x_2,x), \tag{2}$$

which means that the function Δ_f is increasing in its first argument. As far as it is symmetric, it is increasing in its second argument, as well.

Conversely, if Δ_f is increasing in both arguments, then for $x_1 < x < x_2$ the inequality (2) holds, which implies (1), and so the function f is convex on (a,b). ■

2. Majorization relation for finite sequences and Karamata's inequality

Let us introduce a majorization relation for finite sequences of real numbers.

DEFINITION 1. Let $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ be two (finite) sequences of real numbers. We say that the sequence a *majorizes* the sequence b and we write

$$a \succ b \qquad \text{or} \qquad b \prec a,$$

if, after a possible renumeration, the terms of the sequences a and b satisfy the following three conditions:

1° $a_1 \geqslant a_2 \geqslant \dots \geqslant a_n$ and $b_1 \geqslant b_2 \geqslant \dots \geqslant b_n$;

2° $a_1 + a_2 + \dots + a_k \geqslant b_1 + b_2 + \dots + b_k$, for each k, $1 \leqslant k \leqslant n-1$;

3° $a_1 + a_2 + \dots + a_n = b_1 + b_2 + \dots + b_n$.

The first condition is obviously no restriction, since we can always rearrange the sequence. The second condition is essential.

Clearly, $a \succ a$ holds for an arbitrary sequence a.

EXAMPLE 1. (a) If $a = (a_i)_{i=1}^n$ is an arbitrary sequence of nonnegative numbers, having the sum equal to 1, then

$$(1, 0, \dots, 0) \succ (a_1, a_2, \dots, a_n) \succ \Big(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\Big).$$

(b) The sequences $(4, 4, 1)$ and $(5, 2, 2)$ are incomparable in the sense of the relation $\succ$, i.e., non of the two majorizes the other one. $\triangle$

The following inequality appears in literature connected with various names—I. Schur [15], G. H. Hardy, J. I. Littlewood, G. Polya [3], H. Weyl [20], and J. Karamata [8]. Following articles [5], [9] and [13], we shall call it *Karamata's inequality.*

THEOREM 1. *Let $a = (a_i)_{i=1}^n$ and $b = (b_i)_{i=1}^n$ be (finite) sequences of real numbers from an interval (α, β). If the sequence a majorizes b, $a \succ b$, and if $f\colon (\alpha, \beta) \to \mathbf{R}$ is a convex function, then the inequality*

$$\sum_{i=1}^n f(a_i) \geqslant \sum_{i=1}^n f(b_i)$$

holds.

First proof. In this proof we use Abel's transformation [2].

Denote by c_i the divided difference of the function f at the points a_i, b_i,

$$c_i = \Delta_f(a_i, b_i) = \frac{f(b_i) - f(a_i)}{b_i - a_i}.$$

Since the function f is convex, the condition 1°, by Lemma 1, implies that the sequence (c_i) is decreasing.

Further, denote

$$A_k = \sum_{i=1}^k a_i, \quad B_k = \sum_{i=1}^k b_i, \quad (k = 1, \dots, n); \qquad A_0 = B_0 = 0.$$

The assumption 3° implies that $A_n = B_n$. Now, we have

$$\begin{aligned}
\sum_{i=1}^n f(a_i) - \sum_{i=1}^n f(b_i) &= \sum_{i=1}^n (f(a_i) - f(b_i)) = \sum_{i=1}^n c_i(a_i - b_i) \\
&= \sum_{i=1}^n c_i(A_i - A_{i-1} - B_i + B_{i-1}) = \sum_{i=1}^n c_i(A_i - B_i) - \sum_{i=1}^n c_i(A_{i-1} - B_{i-1}) \\
&= \sum_{i=1}^{n-1} c_i(A_i - B_i) - \sum_{i=0}^{n-1} c_{i+1}(A_i - B_i) = \sum_{i=1}^{n-1} (c_i - c_{i+1})(A_i - B_i).
\end{aligned}$$

As mentioned before, $c_i \geqslant c_{i+1}$, and by assumption 2° it is $A_i \geqslant B_i$ for $i = 1, 2, \dots, n-1$. Hence, the last sum, and so also the difference

$$\sum_{i=1}^n f(a_i) - \sum_{i=1}^n f(b_i)$$

is nonnegative, which was to be proved.

Second proof. In this proof we use Stieltjes' integral [9]. We shall need the following lemma.

LEMMA 2. [16] *Let* $\psi_1, \psi_2\colon [\alpha,\beta] \to \mathbf{R}$ *be two integrable functions, such that* $\psi_1 \succ \psi_2$*, in the sense that*

$$\int_\alpha^x \psi_1\,dt \geqslant \int_\alpha^x \psi_2\,dt \quad \textit{for} \quad x \in [\alpha,\beta) \quad \textit{and} \quad \int_\alpha^\beta \psi_1\,dt = \int_\alpha^\beta \psi_2\,dt.$$

Further, let $\varphi\colon [\alpha,\beta] \to \mathbf{R}$ *be an increasing (integrable) function. Then*

$$\int_\alpha^\beta \varphi\psi_1\,dx \leqslant \int_\alpha^\beta \varphi\psi_2\,dx.$$

Proof. Put $\psi(x) = \psi_1(x) - \psi_2(x)$ and $g(x) = \int_\alpha^x \psi(t)\,dt$. Then, by the hypothesis, $g(x) \geqslant 0$ for $x \in [\alpha,\beta]$ and $g(\alpha) = g(\beta) = 0$. Using integration by parts in the Stieltjes integral, we get

$$\begin{aligned}\int_\alpha^\beta \varphi(t)\psi(t)\,dt &= \int_\alpha^\beta \varphi(t)\,dg(t) = \varphi(t)g(t)\Big|_\alpha^\beta - \int_\alpha^\beta g(t)\,d\varphi(t)\\ &= -\int_\alpha^\beta g(t)\,d\varphi(t) \leqslant 0. \quad \blacksquare\end{aligned}$$

Proof of the Theorem. The given function f, being convex, is continuous and it can be represented in the form $f(x) = \int_\alpha^x \varphi\,dt$ for an increasing function φ. Introduce functions $A(x)$ and $B(x)$ by

$$A(x) = \sum_{i=1}^n m\{[\alpha,x] \cap [\alpha,a_i]\}, \qquad B(x) = \sum_{i=1}^n m\{[\alpha,x] \cap [\alpha,b_i]\},$$

where mS denotes the measure of the set S. It is easy to see that

$$A(x) \leqslant B(x), \qquad A(a_1) = B(a_1)$$

and that $A'(x)$ and $B'(x)$ exist everywhere except in a finite set of points. Applying Lemma 2, we conclude that

$$\int_\alpha^{a_1} f\,dA(x) \geqslant \int_\alpha^{a_1} f\,dB(x). \tag{3}$$

But,

$$\begin{aligned}\int_\alpha^{a_1} f\,dA(x) &= n\int_\alpha^{a_n} \varphi\,dx + (n-1)\int_{a_n}^{a_{n-1}} \varphi\,dx + \cdots + \int_{a_2}^{a_1} \varphi\,dx\\ &= f(a_1) + f(a_2) + \cdots + f(a_n),\end{aligned}$$

and the similar relation holds for the integral on the right-hand side of (3). This proves Karamata's inequality. ■

NOTE 1. The condition that Karamata's inequality holds for every convex function f on (α, β) is not only necessary, but also sufficient for the relation $a \succ b$. The proof can be found in [4] or [11].

NOTE 2. If the function f is strictly convex, it can be easily checked that the equality in Karamata's inequality is obtained if and only if the sequences (a_i) and (b_i) coincide.

NOTE 3. Jensen's inequality [6] in the form

$$f\Big(\frac{x_1 + x_2 + \cdots + x_n}{n}\Big) \leqslant \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n},$$

for a Jensen-convex function f, is obtained as a special case of Karamata's inequality, by putting $b_1 = b_2 = \cdots = b_n = \dfrac{a_1 + \cdots + a_n}{n}$. The general form of Jensen's inequality follows from the weighted form of Karamata's inequality [2]:

$$\sum_{i=1}^{n} \lambda_i f(a_i) \geqslant \sum_{i=1}^{n} \lambda_i f(b_i)$$

if $\lambda_i \in \mathbf{R}^+$ and (a_i) and (b_i) satisfy condition 1° of Definition 1, $\sum\limits_{i=1}^{k} \lambda_i a_i \geqslant \sum\limits_{i=1}^{k} \lambda_i b_i$ for $k = 1, 2, \dots, n-1$ and $\sum\limits_{i=1}^{n} \lambda_i a_i = \sum\limits_{i=1}^{n} \lambda_i b_i$.

Lemma 2 can also be used [9] in proving Steffensen's inequality [17]:

COROLLARY 1. *Let* $f, g\colon [0, a] \to \mathbf{R}$, $0 \leqslant g(x) \leqslant 1$, f *be decreasing on* $[0, a]$, *and let* $F(x) = \int_0^x f\,dt$. *Then*

$$\int_0^a fg\,dx \leqslant F\Big(\int_0^a g\,dx\Big).$$

Proof. If we denote $c = \int_0^a g\,dx$, then $0 < c \leqslant a$. Let $\tilde{g}(x) = \begin{cases} 1, & x \in [0, c], \\ 0, & x \in (c, a] \end{cases}$. Then, it is easy to check that $\tilde{g} \succ g$ (in the sense of Lemma 2), and so, applying this Lemma, we obtain Steffensen's inequality in the form

$$\int_0^c f\,dx = \int_0^a f\tilde{g}\,dx \geqslant \int_0^a fg\,dx. \quad \blacksquare$$

EXAMPLE 2. [5] Prove that for arbitrary positive numbers a, b and c the inequality

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leqslant \frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c}$$

holds.

Solution. Suppose that the numbers a, b and c are such that $a \geqslant b \geqslant c$, i.e., the sequence (a, b, c) is decreasing (this can be done without loss of generality). Then we have $(2a, 2b, 2c) \succ (a+b, b+c, c+a)$, and so, applying Karamata's inequality to the function $f(x) = \dfrac{1}{x}$, which is convex on the interval $(0, +\infty)$, we obtain the desired inequality. △

EXAMPLE 3. [13] Prove that the inequality

$$\cos(2x_1 - x_2) + \cos(2x_2 - x_3) + \cdots + \cos(2x_n - x_1) \leqslant \cos x_1 + \cos x_2 + \cdots + \cos x_n$$

holds for arbitrary numbers x_1, x_2, ... , x_n from the interval $[-\pi/6, \pi/6]$.

Solution. The numbers $2x_i - x_{i+1}$, $i = 1, 2, \ldots, n$ $(x_{n+1} = x_1)$, as well as the given numbers x_i, belong to the interval $[-\pi/2, \pi/2]$. The function $f(x) = \cos x$ is concave on this interval, and so the Karamata's inequality holds with the opposite sign. Thus, it is sufficient to prove that the sequences $a = (2x_1 - x_2, 2x_2 - x_3, \ldots, 2x_n - x_1)$ and $b = (x_1, x_2, \ldots, x_n)$, when arranged to be decreasing, satisfy the conditions of Theorem 1.

Let indices m_1, ... , m_n and k_1, ... , k_n be chosen so that

$$\{m_1, \ldots, m_n\} = \{k_1, \ldots, k_n\} = \{1, \ldots, n\},$$

$$2x_{m_1} - x_{m_1+1} \geqslant 2x_{m_2} - x_{m_2+1} \geqslant \cdots \geqslant 2x_{m_n} - x_{m_n+1}, \tag{4}$$

$$x_{k_1} \geqslant x_{k_2} \geqslant \cdots \geqslant x_{k_n}. \tag{5}$$

Then

$$2x_{m_1} - x_{m_1+1} \geqslant 2x_{k_1} - x_{k_1+1} \geqslant x_{k_1}$$

(the first inequality holds because $2x_{m_1} - x_{m_1+1}$ is, by the choice of the numbers m_i, the greatest of the numbers of the form $2x_{m_i} - x_{m_i+1}$; the second one follows by the choice of the numbers k_i). By similar reasons,

$$(2x_{m_1} - x_{m_1+1}) + (2x_{m_2} - x_{m_2+1}) \geqslant (2x_{k_1} - x_{k_1+1}) + (2x_{k_2} - x_{k_2+1}) \geqslant x_{k_1} + x_{k_2},$$

and, generally, the sum of the first l terms of sequence (4) is not less than the sum of the first l terms of sequence (5), for $l = 1, \ldots, n-1$. For $l = 1$, obviously, the equality is obtained, and so all the conditions for applying the Karamata's inequality are fulfilled. △

The following inequality of M. Petrović [14] comes close to these ideas (the proof is taken from [10]).

THEOREM 2. *Let $f\colon [0, +\infty) \to \mathbf{R}$ be a convex function, and $(x_i)_{i=1}^n$, be a sequence of positive numbers. Then the inequality*

$$f(x_1) + \cdots + f(x_n) \leqslant f(x_1 + \cdots + x_n) + (n-1)f(0)$$

holds.

Proof. Denote $s = \sum_{i=1}^{n} x_i$ and $\lambda_i = \dfrac{x_i}{s}$. Then $\sum_{i=1}^{n} \lambda_i = 1$ and

$$x_i = (1-\lambda_i) \cdot 0 + \lambda_i s, \quad i = 1, 2, \dots, n.$$

The convexity of the function f implies that

$$f(x_i) \leqslant (1-\lambda_i) f(0) + \lambda_i f(s) \quad i = 1, \dots, n,$$

and hence, summing up, we obtain

$$\sum_{i=1}^{n} f(x_i) \leqslant (n-1) f(0) + f(s). \quad \blacksquare$$

The following variations on Petrović's inequality are also proved in [10].

Corollary 2. (a) *Let $f\colon [a,b] \to \mathbf{R}$ be a convex function, $0 \leqslant a < b$ and let $(x_i)_{i=1}^{n}$ be a sequence of positive numbers, such that $x_1 + \dots + x_n \leqslant b - a$. Then the inequality*

$$f(a+x_1) + \dots + f(a+x_n) \leqslant f(a+x_1+\dots+x_n) + (n-1) f(a)$$

holds.

(b) *Let $f\colon [0,b_1] \to \mathbf{R}$ be a convex function, and $b_1 \geqslant b_2 \geqslant b_3 \geqslant 0$. Then the following holds:*

$$f(b_1 - b_2 + b_3) \leqslant f(b_1) - f(b_2) + f(b_3). \tag{6}$$

Proof. (a) Apply Petrović's inequality to the (convex) function $\varphi\colon [0, b-a] \to \mathbf{R}$, given by $\varphi(x) = f(a+x)$.

(b) In the inequality (a) put $n = 2$, $b_1 = a + x_1 + x_2$, $b_2 = a + x_1$ and $b_3 = a$. $\blacksquare$

An easy consequence is now an inequality of G. Szegö [18].

Corollary 3. *Let $f\colon [0,b_1] \to \mathbf{R}$ be a convex function and $b_1 \geqslant b_2 \geqslant \dots \geqslant b_{2n+1} \geqslant 0$. Then the inequality*

$$f(b_1 - b_2 + \dots + b_{2n+1}) \leqslant f(b_1) - f(b_2) + \dots + f(b_{2n+1})$$

holds.

Proof. For $n = 1$, the inequality reduces to (6). Assuming that it holds for $n-1$, where $n \geqslant 1$, let us prove that it holds also for n.

Note that $b' = b_1 - b_2 + \dots + b_{2n-1} \geqslant b_{2n-1} \geqslant b_{2n} \geqslant b_{2n+1}$. Thus, using (6), we obtain that

$$\begin{aligned} f(b' - b_{2n} + b_{2n+1}) &\leqslant f(b') - f(b_{2n}) + f(b_{2n+1}) \\ &\leqslant f(b_1) - f(b_2) + \dots + f(b_{2n-1}) - f(b_{2n}) + f(b_{2n+1}). \quad \blacksquare \end{aligned}$$

3. Inequalities of Schur and Muirhead

Definition 2. Let $F(x_1, x_2, \dots, x_n)$ be a function in n nonnegative real variables. Define $\sum^! F(x_1, x_2, \dots, x_n)$ as the sum of $n!$ summands, obtained from the expression $F(x_1, x_2, \dots, x_n)$ as all possible permutations of the sequence $x = (x_i)_{i=1}^n$.

Particularly, if for some sequence of nonnegative exponents $a = (a_i)_{i=1}^n$, the function F is of the form $F(x_1, x_2, \dots, x_n) = x_1^{a_1} \cdot x_2^{a_2} \cdot \dots \cdot x_n^{a_n}$, then, instead of $\sum^! F(x_1, x_2, \dots, x_n)$, we shall write also

$$T[a_1, a_2, \dots, a_n](x_1, x_2, \dots, x_n),$$

or just $T[a_1, a_2, \dots, a_n]$ if it is clear which sequence x is used.

Example 4. $T[1, 0, \dots, 0] = (n-1)! \cdot (x_1 + x_2 + \dots + x_n)$, and $T[\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}] = n! \cdot \sqrt[n]{x_1 x_2 \cdots x_n}$. Using this terminology, the Arithmetic-Geometric Mean Inequality can be written as

$$T[1, 0, \dots, 0] \geqslant T\left[\frac{1}{n}, \dots, \frac{1}{n}\right]. \quad \triangle$$

Let us prove now *Schur's inequality*.

Theorem 3. *The inequality*

$$T[a + 2b, 0, 0] + T[a, b, b] \geqslant 2T[a + b, b, 0]$$

holds for arbitrary positive numbers a and b.

The previous inequality means that

$$T[a + 2b, 0, 0](x, y, z) + T[a, b, b](x, y, z) \geqslant 2T[a + b, b, 0](x, y, z)$$

holds for an arbitrary sequence (x, y, z) of positive numbers.

Proof. Let (x, y, z) be a sequence of positive numbers. Using elementary transformations, we obtain

$$\begin{aligned} &\frac{1}{2}T[a + 2b, 0, 0] + \frac{1}{2}T[a, b, b] - T[a + b, b, 0] \\ &\qquad = x^a(x^b - y^b)(x^b - z^b) + y^a(y^b - x^b)(y^b - z^b) + z^a(z^b - x^b)(z^b - y^b). \end{aligned}$$

Assume, without loss of generality, that $x \geqslant y \geqslant z$. Then only the second summand in the last expression is negative. It is sufficient to prove that

$$x^a(x^b - y^b)(x^b - z^b) + y^a(y^b - x^b)(y^b - z^b) \geqslant 0,$$

i.e., $(x^b - y^b)(x^a(x^b - z^b) - y^a(y^b - z^b)) \geqslant 0$. The last inequality is equivalent to $x^{a+b} - y^{a+b} - z^b(x^a - y^a) \geqslant 0$. However,

$$x^{a+b} - y^{a+b} - z^b(x^a - y^a) \geqslant x^{a+b} - y^{a+b} - y^b(x^a - y^a) = x^a(x^b - y^b) \geqslant 0,$$

which proves the theorem. ∎

Corollary 4. *If x, y and z are nonnegative real numbers, and $r \geqslant 0$, then the inequality*

$$x^r(x - y)(x - z) + y^r(y - z)(y - x) + z^r(z - x)(z - y) \geqslant 0$$

holds.

Proof. Transforming, the given inequality can be written as

$$T[r+2,0,0]+T[r,1,1]\geqslant 2T[r+1,1,0]$$

which is a special case of Schur's inequality for $a=r$, $b=1$. ■

EXAMPLE 5. Putting $a=b=1$ in Schur's inequality (or $r=1$ in Corollary 4), we obtain

$$x^3+y^3+z^3+3xyz\geqslant x^2y+xy^2+y^2z+yz^2+z^2x+zx^2. \quad \triangle$$

For arbitrary sequences a and b, $T[a]$ may be incomparable with $T[b]$, in the sense that it is not true that either $T[a](x)\leqslant T[b](x)$ or $T[a](x)\geqslant T[b](x)$ holds for arbitrary values of the variable sequence $x=(x_i)$. It appears that a necessary and sufficient condition for these two expressions to be comparable is the condition that one of the sequences a and b majorizes the other one. More precisely, the following *Muirhead's theorem* [12] holds.

THEOREM 4. *The expression $T[a]$ is comparable with the expression $T[b]$ for all positive sequences x, if and only if one of the sequences a and b majorizes the other one in the sense of relation $\prec$. If $a\prec b$ then*

$$T[a]\leqslant T[b].$$

The equality holds if and only if the sequences a and b are identical, or all the x_i's are equal.

Proof. We prove the necessity of the condition first. Taking the sequence x to be constant, with all the terms equal to c, we obtain that

$$c^{\sum a_i}\leqslant c^{\sum b_i}.$$

This can hold for arbitrary large, as well as for arbitrary small values of c, only if the condition 3° of Definition 1 is satisfied. Put now $x_1=\cdots=x_k=c$ and $x_{k+1}=\cdots=x_n=1$. Comparing the highest powers of c in the expressions $T[a]$ and $T[b]$, and taking into account that $T[a]\leqslant T[b]$ has to be valid for c arbitrary large, we conclude that $a_1+\cdots+a_k\leqslant b_1+\cdots+b_k$, $1\leqslant k<n$.

Let us prove now the sufficiency of the condition. The assertion will be a consequence of the following two lemmas. But first, let us define a linear operation L which can be applied to sequences b of exponents.

Let b_k and b_l be two distinct terms of the sequence b, such that $b_k>b_l$. We can write

$$b_k=\rho+\tau,\quad b_l=\rho-\tau\qquad(0<\tau\leqslant\rho).$$

Now, if $0\leqslant\sigma<\tau\leqslant\rho$, define the sequence $a=L(b)$ as follows

$$\begin{aligned} a_k&=\rho+\sigma=\frac{\tau+\sigma}{2\tau}\,b_k+\frac{\tau-\sigma}{2\tau}\,b_l,\\ a_l&=\rho-\sigma=\frac{\tau-\sigma}{2\tau}\,b_k+\frac{\tau+\sigma}{2\tau}\,b_l,\\ a_\nu&=b_\nu,\quad(\nu\neq k,\nu\neq l). \end{aligned}$$

The given definition does not require that either of the two sequences b and a be decreasing.

LEMMA 3. *If $a = L(b)$, then $T[a] \leqslant T[b]$, while the equality holds only if the sequence x is constant.*

Proof. We can rearrange the sequence so that $k = 1$ and $l = 2$. Then

$$\begin{aligned} T[b] - T[a] &= \sum\nolimits^{!} x_3^{b_3} \cdots x_n^{b_n} (x_1^{\rho+\tau} x_2^{\rho-\tau} + x_1^{\rho-\tau} x_2^{\rho+\tau} - x_1^{\rho+\sigma} x_2^{\rho-\sigma} - x_1^{\rho-\sigma} x_2^{\rho+\sigma}) \\ &= \sum\nolimits^{!} (x_1 x_2)^{\rho-\tau} x_3^{b_3} \cdots x_n^{b_n} (x_1^{\tau+\sigma} - x_2^{\tau+\sigma})(x_1^{\tau-\sigma} - x_2^{\tau-\sigma}) \geqslant 0. \end{aligned}$$

The equality holds only if all the x_i's are equal to each other. ■

LEMMA 4. *If $a \prec b$, but a differs from b, then a can be obtained from b applying the transformation L finitely many times.*

Proof. Denote by m the number of differences $b_\nu - a_\nu$ which are not equal to zero. m is an integer, and we shall prove that we can apply the transformation L in such a way, that after each application the number m strictly decreases (which means that the procedure will stop after finitely many steps). Since $\sum(b_\nu - a_\nu) = 0$, and not all of the differences are equal to zero, both positive and negative differences exist, but the first of them is positive. We can choose k and l in such a way that

$$a_k < b_k, \quad a_{k+1} = b_{k+1}, \quad \dots, \quad a_{l-1} = b_{l-1}, \quad a_l > b_l$$

hold ($b_l - a_l$ is the first among the negative differences, and $b_k - a_k$ is the last among the positive differences preceding it). Let $b_k = \rho + \tau$ and $b_l = \rho - \tau$ and define σ by

$$\sigma = \max\{|a_k - \rho|, |a_l - \rho|\}.$$

At least one of the following two equalities holds:

$$a_l - \rho = -\sigma, \qquad a_k - \rho = \sigma,$$

since $a_k > a_l$. It is also $\sigma < \tau$, since $a_k < b_k$ and $a_l > b_l$. Let

$$c_k = \rho + \sigma, \qquad c_l = \rho - \sigma, \qquad c_\nu = b_\nu \quad (\nu \neq k, \nu \neq l).$$

We shall consider now the sequence $c = (c_i)$ instead of the sequence b. The number m has decreased at least by 1. It can be easily checked that the sequence c is decreasing and that it majorizes a.

Applying this procedure several times, the sequence a can be obtained, which proves Lemma 4, and so also the theorem as a whole. ■

EXAMPLE 6. The Arithmetic-Geometric Mean Inequality is now a trivial consequence of Muirhead's inequality (see Example 4). △

EXAMPLE 7. (Yugoslav Federal Competition 1991) Prove that

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \leqslant \frac{1}{abc}$$

holds for arbitrary positive numbers a, b and c.

Solution. Multiplying both sides of the inequality by

$$abc(a^3+b^3+abc)(b^3+c^3+abc)(c^3+a^3+abc),$$

we obtain that the given inequality is equivalent to

$$\frac{3}{2}T[4,4,1]+2T[5,2,2]+\frac{1}{2}T[7,1,1]+\frac{1}{2}T[3,3,3]\leqslant$$
$$\leqslant\frac{1}{2}T[3,3,3]+T[6,3,0]+\frac{3}{2}T[4,4,1]+\frac{1}{2}T[7,1,1]+T[5,2,2]$$

which holds, since from Muirhead's theorem it follows that $T[5,2,2]\leqslant T[6,3,0]$. △

Example 8. (International Mathematical Olympiad 1995) Let a,b and c be positive real numbers such that $abc=1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geqslant\frac{3}{2}.$$

Solution. In order to apply Muirehead's theorem, the expressions have to be homogeneous. So, divide the right-hand side by $(abc)^{4/3}=1$ and multiply both sides by $a^3b^3c^3(a+b)(b+c)(c+a)(abc)^{4/3}$. The inequality becomes equivalent with

$$2T\left[\frac{16}{3},\frac{13}{3},\frac{7}{3}\right]+T\left[\frac{16}{3},\frac{16}{3},\frac{4}{3}\right]+T\left[\frac{13}{3},\frac{13}{3},\frac{10}{3}\right]\geqslant 3T[5,4,3]+T[4,4,4].$$

The last inequality can be obtained by summing up the following three inequalities which follow directly from Muirhead's theorem:

$$2T\left[\frac{16}{3},\frac{13}{3},\frac{7}{3}\right]\geqslant 2T[5,4,3],$$
$$T\left[\frac{16}{3},\frac{16}{3},\frac{4}{3}\right]\geqslant T[5,4,3],$$
$$T\left[\frac{13}{3},\frac{13}{3},\frac{10}{3}\right]\geqslant T[4,4,4].$$

Equality holds if and only if $a=b=c=1$. △

4. Problems

1. (Yugoslav International Selection Test 1969) Real numbers a_i, b_i $(i=1,2,\ldots,n)$ are given, such that

$$a_1\geqslant a_2\geqslant\ldots\geqslant a_n>0,$$
$$b_1\geqslant a_1,$$
$$b_1b_2\geqslant a_1a_2,$$
$$\ldots\ldots\ldots$$
$$b_1b_2\cdots b_n\geqslant a_1a_2\cdots a_n.$$

Prove that $b_1+b_2+\cdots+b_n\geqslant a_1+a_2+\cdots+a_n$.

Hint. Apply the following variant of Karamata's inequality [19] to the function $f(x) = e^x$.

Let $(a_i)_{i=1}^n$ *and* $(b_i)_{i=1}^n$ *be two sequences of real numbers satisfying:*

1° $a_1 \geqslant a_2 \geqslant \dots \geqslant a_n$, $b_1 \geqslant b_2 \geqslant \dots \geqslant b_n$;

2° $\sum\limits_{i=1}^{k} a_i \geqslant \sum\limits_{i=1}^{k} b_i$ *for all* $k \in \{1, 2, \dots, n\}$.

If $f\colon \mathbf{R} \to \mathbf{R}$ *is an* **increasing** *convex function, then the inequality*

$$\sum_{i=1}^{n} f(a_i) \geqslant \sum_{i=1}^{n} f(b_i)$$

holds.

2. [13] Prove that the inequality

$$\frac{a_1^3}{a_2} + \frac{a_2^3}{a_3} + \dots + \frac{a_n^3}{a_1} \geqslant a_1^2 + a_2^2 + \dots + a_n^2$$

holds for arbitrary positive numbers a_1, a_2, ... , a_n.

Hint. Similarly as in Example 3, applying Theorem 1 to the convex function $f(x) = e^x$, prove that the inequality

$$e^{3x_1 - x_2} + e^{3x_2 - x_3} + \dots + e^{3x_n - x_1} \geqslant e^{2x_1} + e^{2x_2} + \dots + e^{2x_n}$$

holds, and substitute $x_i = \log a_i$, $i = 1, \dots, n$.

3. Prove that the inequality

$$(a + b - c)(b + c - a)(c + a - b) \leqslant abc$$

holds for arbitrary positive numbers a, b, c.

Hint. Apply Corollary 4 of Schur's inequality $T[3, 0, 0] + T[1, 1, 1] \geqslant 2T[2, 1, 0]$ or Karamata's inequality to the concave function $f(x) = \log x$.

4. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leqslant 1.$$

Hint. Using the condition $abc = 1$, the given inequality can be written in a homogeneous form

$$\frac{1}{a+b+(abc)^{1/3}} + \frac{1}{b+c+(abc)^{1/3}} + \frac{1}{c+a+(abc)^{1/3}} \leqslant \frac{1}{(abc)^{1/3}}.$$

Substituting $a = x^3$, $b = y^3$, $z = c^3$, it becomes the inequality from Example 7.

5. If a, b and c are positive real numbers, prove that the inequality

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \geqslant 3\frac{ab + bc + ca}{a + b + c}$$

holds.

Hint. Combine the following four consequences of Muirhead's inequality:

$$1.\quad T[9,2,0] \geqslant T[7,4,0], \qquad 2.\quad T[10,1,0] \geqslant T[7,4,0],$$
$$3.\quad T[6,5,0] \geqslant T[6,4,1], \qquad 4.\quad T[6,3,2] \geqslant T[4,4,3],$$

with the consequence $T[4,2,2] + T[8,0,0] \geqslant 2T[6,2,0]$ of Schur's inequality, which can be, multiplying by abc written as

$$5.\quad T[5,3,3] + T[9,1,1] \geqslant 2T[7,3,1].$$

6. (Shortlisted problem for IMO'98) Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{a^3}{(1+b)(1+c)} + \frac{b^3}{(1+c)(1+a)} + \frac{c^3}{(1+a)(1+b)} \geqslant \frac{3}{4}.$$

7. (International Mathematical Olympiad 1984) Let x, y and z be nonnegative real numbers satisfying the equality $x + y + z = 1$. Prove the inequality

$$0 \leqslant xy + yz + zx - 2xyz \leqslant \frac{7}{27}.$$

Hint. The inequality on left-hand side is easy to prove. The right-hand one is equivalent to

$$12T[2,1,0] \leqslant 7T[3,0,0] + 5T[1,1,1].$$

This inequality is true since it can be obtained by summing up the inequalities $2T[2,1,0] \leqslant 2T[3,0,0]$ and $10T[2,1,0] \leqslant 5T[3,0,0]+5T[1,1,1]$ (the first one follows from Muirhead's theorem, and the second one is Schur's inequality for $a = b = 1$).

8. (International Mathematical Olympiad 1999) Let n be a fixed integer, $n \geqslant 2$.

(a) Determine the minimal constant C such that the inequality

$$\sum_{1 \leqslant i < j \leqslant n} x_i x_j (x_i^2 + x_j^2) \leqslant C \Big(\sum_{1 \leqslant i \leqslant n} x_i \Big)^4$$

is valid for all real numbers $x_1, x_2, \dots, x_n \geqslant 0$.

(b) For the constant C found in (a) determine when the equality is obtained.

Solution. As far as the given inequality is homogeneous, we can assume that $x_1 + x_2 + \cdots + x_n = 1$. In this case the inequality can be written as

$$x_1^3(1 - x_1) + x_2^3(1 - x_2) + \cdots + x_n^3(1 - x_n) \leqslant C.$$

The function $f(x) = x^3(1-x)$ is increasing and convex on the segment $[0, 1/2]$. Let x_1 be the greatest of the given numbers. Then the numbers $x_2, x_3, \dots, x_n$ are not greater than $1/2$. If $x_1 \in [0, 1/2]$ as well, then from $(x_1, x_2, \dots, x_n) \prec (\frac{1}{2}, \frac{1}{2}, 0, \dots)$ using Theorem 1, we obtain that

$$f(x_1) + f(x_2) \cdots + f(x_n) \leqslant f\Big(\frac{1}{2}\Big) + f\Big(\frac{1}{2}\Big) + (n-2)f(0) = \frac{1}{8}.$$

If, to the contrary, $x_1 > 1/2$, then it is $1 - x_1 < 1/2$ and we have that $(x_2, x_3, \dots, x_n) \prec (1 - x_1, 0, \dots, 0)$. Applying Karamata's inequality once more, we obtain that

$$f(x_1) + f(x_2) \cdots + f(x_n) \leqslant f(x_1) + f(1 - x_1) + (n-2)f(0) = f(x_1) + f(1 - x_1).$$

(Alternatively, in this case Petrović's inequality can be applied to obtain the same result.) It is easy to prove that the function $g(x) = f(x) + f(1 - x)$ has the maximum on the segment $[0, 1]$ equal to $g(1/2) = 1/8$. Thus, in this case also, $f(x_1) + f(x_2) + \dots + f(x_n) \leqslant 1/8$ follows.

Equality holds, e.g., for $x_1 = x_2 = 1/2$, which proves that $C = 1/8$.

9. (International Mathematical Olympiad 2000) Let x, y, z be positive real numbers, such that $xyz = 1$. Prove that

$$\left(x - 1 + \frac{1}{y}\right)\left(y - 1 + \frac{1}{z}\right)\left(z - 1 + \frac{1}{x}\right) \leqslant 1.$$

Solution. The condition $xyz = 1$ implies that the numbers x, y and z can be written in the form $x = \dfrac{a}{b}$, $y = \dfrac{b}{c}$ and $z = \dfrac{c}{a}$ for some $a, b, c > 0$. Using the new variables, the given inequality reduces to our Problem 3.

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