Collection of Number Theory Problems: APMO

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Abstract

In this note, I have tried to compile all number theory problems that appeared at APMO (Asian Pacific Mathematical Olympiad) from 1989 to 2015. For now, there are only the problems. Later I may add the solutions as well. I would like to mention beforehand that, the reader may not agree with me on some problems, whether they belong to number theory or not. Probably because, I consider problems regarding combinatorial number theory or some miscellaneous problems belong to number theory in a wider sense. But let's agree not to argue about it. Have fun!

1 Problems

Problem 1 (1989, Problem 2)**.** Prove that

$$5n^2 = 36a^2 + 18b^2 + 6c^2$$

has no integer solutions except $a = b = c = n = 0$.

Problem 2 (1991, Problem 4). A sequence of values in the range $0, 1, 2, ..., k-1$ is defined as follows:

$$a_1 = 1, a_n = a_{n-1} + n \pmod k$$

For which values of k does the sequence assume all possible values?

Problem 3 (1992, Problem 3). Given three distinct positive integers $\frac{n}{2} < a, b, c \leq n$. Prove that, the 8 numbers we get using one multiplication and and addition

$$a + b + c, a + bc, b + ac, c + ab, (a + b)c, (b + c)a, (c + a)b$$

are all distinct. Show that if p is a prime and $n \geq p^2$, then there are $\tau(p-1)$ ways to choose two distinct numbers b, c from

$$\{p + 1, p + 2, ..., n\}$$

so that the 8 numbers derived from p, b, c are not all distinct.

Problem 4 (1992, Problem 5). $a_1, a_2, ..., a_n$ is a sequence of non-zero integers such that the sum of any 7 consecutive terms is positive, whereas the sum of any 11 consecutive terms is negative. What is the largest possible value of n?

Problem 5 (1993, Problem 2). How many different values can be taken by the expression

$$[x] + [2x] + \left[\frac{5x}{3}\right] + [3x] + [4x]$$

for real $x \in [0, 100]$?

Problem 6 (1993, Problem 4). Find all positive integers n for which

$$x^n + (x + 2)^n + (2 - x)^n = 0$$

has an integral solution.

Problem 7 (1994, Problem 3). Find all positive integers n such that

$$n = a^2 + b^2$$

with $\gcd(a, b) = 1$ and every prime less than or equal to $\sqrt{n}$ divides ab.

Problem 8 (1994, Problem 5). Prove that, for any $n > 1$, there is a power of 10 with n digits in base 2 or in base 5 but not both.

Problem 9 (1995, Problem 2). Find the smallest n such that any sequence $a_1, a_2, ..., a_n$ whose values are relatively prime square-free integers between 2 and 1995 must contain a prime. n is square-free if it divisible by no square other than 1.

Problem 10 (1995, Problem 5). $F : \mathbb{Z} \to \{1, 2, ..., n\}$ is a function such that $F(a)$ and $F(b)$ are not equal whenever a and b differ by $5, 7$ or 12. Find the smallest value of n.

Problem 11 (1996, Problem 4). For which n in $[1, 1996]$ is it possible to divide n married couples into exactly 17 groups of single gender, so that the size of any two groups differ by at most 1?

Problem 12 (1997, Problem 2). Find an $n \in [100, 1997]$ such that n divides $2^n + 2$.

Problem 13 (1998, Problem 2). Show that, $(36m + n)(36n + m)$ is never a power of 2.

Problem 14 (1998, Problem 5). What is largest possible positive integer divisible by all positive integers less than its cube root?

Problem 15 (1999, Problem 1). Find the smallest positive integer n such that no arithmetic progression of 1999 real contains just n integers.

Problem 16 (1999, Problem 4). Find all pairs of positive integers (m, n) such that

$$m^2 + 4n \text{ and } n^2 + 4m$$

are perfect squares.

Problem 17 (2000, Problem 1). Compute the sum

$$S = \sum_{i=0}^{101} \frac{x_i^3}{1 - 3x_i + 3x_i^2}$$

where $x_i = \dfrac{i}{101}$.

Problem 18 (2000, Problem 2). Find all permutations $(a_1, a_2, ..., a_9)$ of $1, 2, ..., 9$ such that

$$a_1 + a_2 + a_3 + a_4 = a_4 + a_5 + a_6 + a_7 = a_7 + a_8 + a_9 + a_1$$

and

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = a_4^2 + a_5^2 + a_6^2 + a_7^2 = a_7^2 + a_8^2 + a_9^2 + a_1^2$$

Problem 19 (2000, Problem 4). If $m < n$ are positive integers prove that

$$\frac{n^n}{m^m(n-m)^{n-m}} > \frac{n!}{m!(n-m)!} > \frac{n^n}{m^m(n+1)(n-m)^{n-m}}$$

Problem 20 (2001, Problem 1). For a positive integer n let $S(n)$ be the sum of digits in the decimal representation of n. Any positive integer obtained removing several (at least one) digits from the right-hand end of the decimal representation of n is called a *stump* of n. Let $T(n)$ be the sum of all stumps of n. Prove that $n = S(n) + 9T(n)$.

Problem 21 (2001, Problem 2). Find the largest positive integer N so that the number of integers in the set $\{1, 2, \ldots, N\}$ which are divisible by 3 is equal to the number of integers which are divisible by 5 or 7 (or both).

Problem 22 (2002, Problem 2). Find all positive integers a and b such that

$$\frac{a^2+b}{b^2-a} \quad \text{and} \quad \frac{b^2+a}{a^2-b}$$

are both integers.

Problem 23 (2003, Problem 3). Let $k \geq 14$ be an integer, and let p_k be the largest prime number which is strictly less than k. You may assume that $p_k \geq 3k/4$. Let n be a composite integer. Prove:
(a) if $n = 2p_k$, then n does not divide $(n-k)!$;
(b) if $n > 2p_k$, then n divides $(n-k)!$.

Problem 24 (2004, Problem 1). Determine all finite nonempty sets S of positive integers satisfying

$$\frac{i+j}{(i,j)} \quad \text{is an element of } S \text{ for all } i, j \text{ in } S,$$

where (i, j) is the greatest common divisor of i and j.

Problem 25 (2004, Problem 4)**.** For a real number x, let $\lfloor x \rfloor$ stand for the largest integer that is less than or equal to x. Prove that

$$\left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor$$

is even for every positive integer n.

Problem 26 (2005, Problem 1)**.** Prove that for every irrational real number a, there are irrational real numbers b and b' so that $a+b$ and ab' are both rational while ab and $a+b'$ are both irrational.

Problem 27 (2006, Problem 1)**.** Let n be a positive integer. Find the largest nonnegative real number $f(n)$ (depending on n) with the following property: whenever $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1 + a_2 + \ldots + a_n$ is an integer, there exists some i such that $|a_i - \frac{1}{2}| \geq f(n)$.

Problem 28 (2006, Problem 2)**.** Prove that every positive integer can be written as a finite sum of distinct integral powers of the golden mean $\phi = \dfrac{1+\sqrt{5}}{2}$. Here, an integral power of ϕ is of the form ϕ^i, where i is an integer (not necessarily positive).

Problem 29 (2007, Problem 1)**.** Let S be a set of 9 distinct integers all of whose prime factors are at most 3. Prove that S contains 3 distinct integers such that their product is a perfect cube.

Problem 30 (2008, Problem 4)**.** Consider the function $f : \mathbb{N}_0 \to \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all non-negative integers, defined by the following conditions :

- $f(0) = 0$.
- $f(2n) = 2f(n)$.
- $f(2n+1) = n + f(n)$ for all $n \geq 0$.

i. Determine three sets $L = \{n | f(n) < f(n+1)\}$, $E = \{n | f(n) = f(n+1)\}$ and $G = \{n | f(n) > f(n+1)\}$.

ii. For each $k \geq 0$, find a formula for $a_k = \max\{f(n) : 0 \leq n \leq 2^k\}$ in terms of k.

Problem 31 (2008, Problem 5)**.** Let a, b, c be integers satisfying $0 < a < c-1$ and $1 < b < c$. For each k, $0 \leq k \leq a$, let r_k, $0 \leq r_k < c$, be the remainder of kb when divided by c. Prove that the two sets $\{r_0, r_1, r_2, \ldots, r_a\}$ and $\{0, 1, 2, ..., a\}$ are different.

Problem 32 (2009, Problem 4)**.** Prove that for any positive integer k, there exists an arithmetic sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_k}{b_k}$$

of rational numbers, where a_i, b_i are relatively prime positive integers for each $i = 1, 2, \ldots, k$, such that the positive integers $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ are all distinct.

Problem 33 (2009, Problem 5)**.** Larry and Rob are two robots travelling in one car from Argovia to Zillis. Both robots have control over the steering and steer according to the following algorithm: Larry makes a 90° left turn after every ℓ kilometer driving from start; Rob makes a 90° right turn after every r kilometer driving from start, where ℓ and r are relatively prime positive integers. In the event of both turns occurring simultaneously, the car will keep going without changing direction. Assume that the ground is flat and the car can move in any direction. Let the car start from Argovia facing towards Zillis. For which choices of the pair (ℓ, r) is the car guaranteed to reach Zillis, regardless of how far it is from Argovia?

Problem 34 (2010, Problem 2)**.** For a positive integer k, call an integer a *pure k-th power* if it can be represented as m^k for some integer m. Show that for every positive integer n there exist n distinct positive integers such that their sum is a pure 2009-th power, and their product is a pure 2010-th power.

Problem 35 (2011, Problem 1)**.** Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers $a^2+b+c, b^2+c+a, c^2+a+b$ to be perfect squares.

Problem 36 (2012, Problem 3)**.** Find all positive integer n and prime p with $\dfrac{n^p+1}{p^n+1}$ an integer.

Problem 37 (2013, Problem 2)**.** Determine all positive integers n for which $\dfrac{n^2+1}{[\sqrt{n}]^2+2}$ is an integer. Here, $[x]$ is the greatest integer less than or equal to x for a real number x.

Problem 38 (2013, Problem 3)**.** Consider $2k$ real numbers $a_1, \ldots, a_k, b_1, \ldots, b_k$ and define X_n as:

$$X_n = \sum_{i=1}^{k} [a_i n + b_i]$$

where $[x]$ is the greatest integer less than or equal to x for a real number x. If the sequence $(X_n)_{n\geq 1}$ forms an arithmetic progression, then prove that $\sum_{i=1}^{k} a_i$ is an integer.

Problem 39 (2014, Problem 1)**.** For a positive integer m denote by $S(m)$ and $P(m)$ the sum and product, respectively, of the digits of m. Show that for each positive integer n, there exist positive integers $a_1, a_2, \ldots, a_n$ satisfying the following conditions:

(a) $S(a_1) < S(a_2) < \ldots < S(a_n)$.

(b) $S(a_i) = P(a_{i+1})$.

(c) and $a_{n+1} = a_1$.

Problem 40 (2014, Problem 3)**.** Find all positive integers n such that for any integer k there exists an integer a for which $a^3 + a - k$ is divisible by n.

Problem 41 (2015, Problem 2)**.** Let $S = \{2, 3, 4, \ldots\}$ denote the set of integers that are greater than or equal to 2. Does there exist a function $f : S \to S$ such that $f(ab) = f(a^2)f(b^2)$ for all $a, b \in S$ with $a \neq b$?

Problem 42 (2015, Problem 3)**.** A sequence of real numbers $a_0, a_1, \ldots$ is said to be *good* if the following three conditions hold:

i. The value of a_0 is a positive integer.

ii. For each non-negative integer i we have $a_{i+1} = 2a_i + 1$ or $a_{i+1} = \dfrac{a_i}{a_i + 2}$.

iii. There exists a positive integer k such that $a_k = 2014$.

Find the smallest positive integer n such that there exists a good sequence $a_0, a_1, \ldots$ of real numbers with the property that $a_n = 2014$.

Problem 43 (2015, Problem 5)**.** Determine all sequences $a_0, a_1, a_2, \ldots$ of positive integers with $a_0 \geq 2015$ such that for all integers $n \geq 1$:

- a_{n+2} is divisible by a_n.
- $|s_{n+2} - (n+1)a_n = 1$ where $s_{n+1} = a_n - a_{n-1} + \ldots + (-1)^{n+1}a_0$.

If there is any error or typo, just send me a private message or email me.

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