32nd Iranian Mathematical Olympiad

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Young Scholars Club

Ministry of Education, I.R.Iran

www.ysc.ac.ir

32^{nd}

Iranian Mathematical Olympiad

Selected Problems with Solutions

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This booklet is prepared by Yeganeh Alimohammadi, Goodarz Mehr and Hesameddin Rajabzadeh.

With special thanks to Amirreza Ahmadzadeh, Ali Golmakani, Amir Ali Moinfar and Morteza Saghafian.

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Iranian Team Members in the 56^{th} IMO
(Chiang Mai - Thailand)

From left to right:

- Ali Daei Nabi
- Farbod Ekbatani
- Ali Sayyadi
- Aria Halavati
- Amin Bahjati
- Mojtaba Zare Bidaki

Contents

Preface

The 32^{nd} Iranian National Mathematical Olympiad consisted of four rounds. The First Round was held on 20 February 2014 all over the country. The exam consisted of 16 short-answer questions and 9 multiple-choice questions that had to be solved in 3 hours. In total, more than 20000 students participated in the exam and more than 1500 of them were validated for the next round.
The Second Round was conducted on 1 and 2 May 2014. In each day, participants were given 3 problems that had to be solved in 4.5 hours. In this round, 48 number of participants were selected to participate in the Third Round.
The examination of the Third Round consisted of four separate exams, and a final exam containing 8 problems, each problem having it's specified time to be solved in. By the end of this round, 10 students were awarded bronze medal, 25 students were awarded silver medal, and the top 13 students were awarded gold medal. The following list represents the names of the gold medalists:

Hossein Baktash
Amin Bahjati
Ali Daei Nabi
Farbod Ekbatani
Hooman Fattahi Moghaddam
Amin Fathpour
Ali Golmakani
Aria Halavati
Amir Hossein Pouya
Sina Saleh
Ali Sayyadi
Arian Taj Mirriahi
Mojtaba Zare Bidaki

The Team Selection Test was conducted on 6 days, each day having 3 problems that had to be solved in 4.5 hours. In the end, the top 6 participants were selected to participate in the 56^{th} IMO as members of the Iranian Team.
In this booklet, we present 6 problems of the Second Round, 8 problems of the final exam of the Third Round, and 18 problems of the Team Selection Test, accompanying their solutions.
It's a pleasure for the authors to offer their grateful appreciation to all the people who have contributed to the conduction of the 32^{nd} Iranian Mathematical Olympiad, including the National Committee of Mathematics Olympiad, problem proposals, problem selection groups, exam preparation groups, coordinators, editors, instructors and all who have shared their knowledge and effort to increase the Mathematics enthusiasm in our country, and assisted in various ways to the conduction of this scientific event.

Problems

Second Round

1 . (Morteza Saghafian) A local supermarket is responsible for the distribution of 100 supply boxes. Each box is ought to contain 10 kilograms of rice and 30 eggs. It is known that a total of 1000 kilograms of rice and 3000 eggs are in these boxes, but in some of them the amount of either item is more or less than the amount required. In each step, supermarket workers can choose two arbitrary boxes and transfer any amount of rice or any number of eggs between them. At least how many steps are required so that, starting from any arbitrary initial condition, after these steps the amount of rice and the number of eggs in all these boxes is equal?

(→ p.25)

2 . (Arian Mohammadi) Square $ABCD$ is given. Points N and P are selected on sides AB and AD, respectively, such that $PN = NC$, and point Q is selected on segment AN such that $\angle NCB = \angle QPN$. Prove that $\angle BCQ = \frac{1}{2}\angle PQA$.

(→ p.25)

3 . (Mohammad Ahmadi) Let x, y and z be nonnegative real numbers. Knowing that $2(xy + yz + zx) = x^2 + y^2 + z^2$, prove

$$\frac{x+y+z}{3} \geq \sqrt[3]{2xyz}.$$

(→ p.25)

4 . (Hadi Khodabandeh) Find all of the solutions of the following equation in natural numbers:

$$n^{n^n} = m^m.$$

(→ p.26)

5 . (Morteza Saghafian) A non-empty set S of positive real numbers is called **powerful** if for any two distinct elements of it like a and b, at least one of the numbers a^b or b^a is an element of S.

a) Present an example of a powerful set having four elements.

b) Prove that a finite powerful set cannot have more than four elements.

(→ p.26)

6 . (Seyyed Reza Hosseini) In the **Majestic Mystery Club (MMC)**, members are divided into several groups, and groupings change by the end of each week in the following manner: in each group, a member is selected as king; all of the kings leave their respective groups and form a new group. If a group has only one member, that member goes to the new group and his former group is deleted. Suppose that MMC has n members and at the beggining all of them form a single group. Prove that there comes a week for which thereafter each group will have at most $1 + \sqrt{2n}$ members.

(→ p.27)

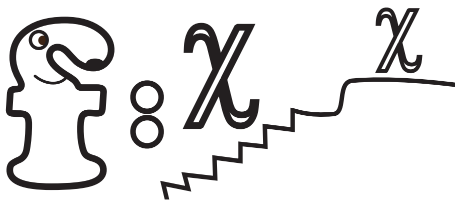
Third Round

1 . (Erfan Salavati, Hesameddin Rajabzadeh and Amir Ali Moinfar)
Increasing Function from What to Where!

In each of the parts (a) through (d), determine whether there exists an increasing bijection between the two sets A and B.

a) $A = \{x \in \mathbb{Q} | x < \sqrt{2}\}$ and $B = \{x \in \mathbb{Q} | x < \sqrt{3}\}$.

b) $A = \mathbb{Q}$ and $B = \mathbb{Q} \cup \{\pi\}$.



For parts (c) and (d), order on points of a plane ($\mathbb{R}^2$) is defined as follows: to compare two ordered pairs, we first consider their first component, if one was greater than the other, we say that ordered pair is the greater one, and if this component was the same in both pairs, the pair with the greater second component is determined to be the greater one (this order is called the bibliographical order). In other words,

$$(a, b) < (c, d) \Leftrightarrow (a < c) \text{ or } (a = c \text{ and } b < d).$$

Using the bibliographical order, we can define an increasing function between subsets of $\mathbb{R}^2$ and other ordered sets (like $\mathbb{R}$). Using the aforementioned information, answer the following parts.

c) $A = \mathbb{R}$ and $B = \mathbb{R}^2$.

d) $A = X \times (X \cup \{0\})$ and $B = (X \cup \{0\}) \times X$, where $X = \{2^{-n} | n \in \mathbb{N}\}$.

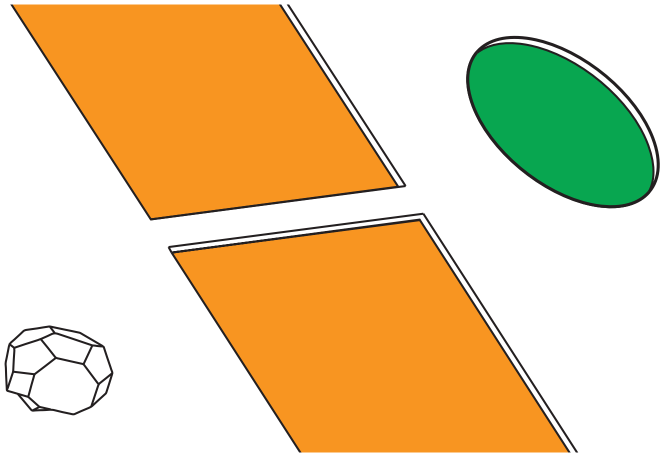
e) Let A and B be two subsets of real numbers such that there exists an increasing surjection from A to B and also an increasing surjection from B to A. Is it always possible to find an increasing bijection between these two sets?

120 minutes ($\rightarrow$ p.29)

2 . (Morteza Saghafian) **Cavity Challenge!**

Cosider a perfectly straight land having a canyon in the form of an infinite strip of width ω. A polyhedron of diameter (maximum distance between any two points on or inside the polyhedron) d is placed on one side of the canyon and a cavity of radius d on the other. We want to roll this polyhedron and drop it inside the cavity in such a way that in any time during its motion, there is at least one contact point between the polyhedron and the ground (unlike the real world, the polyhedron will not fall into the canyon even if it has only one contact point with the ground). To this end, we have constructed a bridge of width $\frac{d}{10}$ over the canyon. Prove that it is possible to drop the polyhedron into the cavity.

60 minutes (→ p.30)



3 . (Mahan Malihi) **Persian Non-residue!**

a) Consider n coprime natural numbers greater than 1 like $d_1, d_2, \ldots, d_n$ and arbitrary natural numbers $r_1, r_2, \ldots, r_n$. Prove that there exists a natural number x, $1 \leq x \leq 3^n$, that satisfies the following system of modular inequalities:

$$\begin{gathered} x \overset{d_1}{\not\equiv} r_1 \\ x \overset{d_2}{\not\equiv} r_2 \\ \vdots \\ x \overset{d_n}{\not\equiv} r_n. \end{gathered}$$

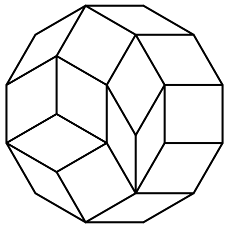
b) For each real number $\epsilon > 0$, prove that there exists a number N such that for each natural number $n > N$ and natural numbers $d_1, d_2, \ldots, d_n$ and $r_1, r_2, \ldots, r_n$, where d_i's $(1 \leq i \leq n)$ are coprime, the above system of inequalities has a solution x that $1 \leq x \leq (2+\epsilon)^n$.

Comment. The assertion stated in part (a) is true even if 3^n is replaced with 2^n.

75 minutes (→ p.31)

4 . (Ali Khezeli) **Fourhombus!**

Let P be a regular $2n$-gon. A Rhombination! of P is defined as drawing a number of rhombi inside P such that they cover the inside area of P, no two overlap, and no vertex of any of the rhombi is placed on the edge of another rhombus or an edge of P. Figure below shows one possible Rhombination of a 12-gon.



a) Prove that the number of rhombi used is a function of n and find it's value. Also, determine the number of vertices and edges in terms of n.

b) Is it true that one may always find an edge such that by omitting all the rhombi that have an edge parrallel to it, P reduces to a polygon with a smaller number of vertices?

c) Is it true that each two Rhombinations can be transformed to each other by applying the following step for a finite number of times?

In each step, we are allowed to omit a vertex that is included in only three rhombi along with the rhombi containing it, and Rhombinate the empty hexagon in a different way.

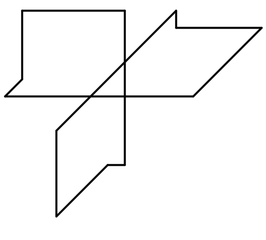
d) If $f(n)$ is the number of possible Rhombinations of a regular $2n$-gon, prove that

$$\prod_{k=1}^{n-1}\left(\binom{k}{2}+1\right) \leq f(n) \leq \prod_{k=1}^{n-1} k^{n-k}.$$

120 minutes (→ p.33)

5 . (Morteza Saghafian) **Eventful Polygons!**

We call a spatial polygon (a polygon in the three domensional space) **Latticelike** if it's edges are parrallel to the coordinate axes.



a) For an arbitrary Latticelike polygon, any two consecutive edges form a right angle that lies in either xy, yz or zx plane. Prove that the number of angles of these three kinds have the same parity.

b) A Latticelike polygon is called an Inscribed Latticlike polygon if there exists a point in space that has equal distances from every vertex of the polygon. Prove that if a hexagon is not planar (i.e. there doesn't exist a plane containig all six vertices), then it's an Inscribed Latticelike polygon.

c) Does there exist a Latticelike 2014-gon with distinct vertices for which there exists a plane that intersects every edge of it in an inner point (i.e. a point other than it's two vertices)?

d) a, b and c are three natural numbers greater than 1. Prove that one can find three points in plane having mutual distance equal to a, b and c, if and only if there exists a Latticelike polygon with a, b and c edges in each of the three coordinate directions.

60 minutes (→ p.35)

6 . (Alireza Fallah) **Polynomial from a Function!**

Polynomial $p(x) \in \mathbb{R}[x]$ is of odd degree m greater than one. Also $f : \mathbb{R} \to \mathbb{Z}$ is a function such that for each real number x, we have $p(f(x)) = f(p(x))$.

a) Prove that the range of function f is a finite set.

b) If f is a non-constant function, prove that the equation $p(x) = x$ has at least two distinct real solutions.

c) Prove that for each natural number $n > 1$, there exist a function f with a range containing exactly n members and a polynomial $p(x)$ that satisfy all the aforementioned conditions.

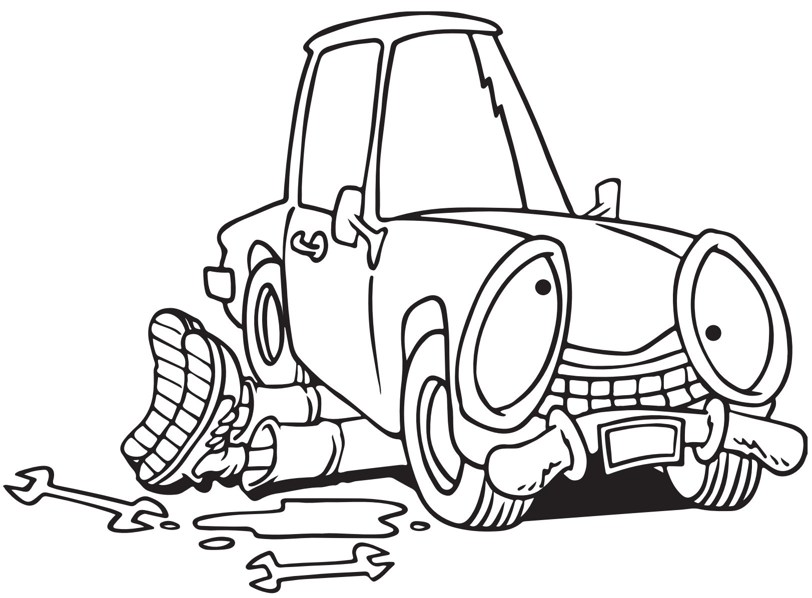
105 minutes (→ p.36)

7 . (Mostafa Eynollahzadeh) **Machine Reparation!**

As you might know, machine M has an input and an output. The input can be any of the English letters (namely, set I) and the output at each moment is one of the colors from the set $C = \{c_1, ..., c_p\}$. At each moment, the machine has a condition which is a member of the finite set S, and entering each letter will change the machine's condition according to some prescribed rule. The machine's output is a surjective function of it's condition. We can only see the machine's output and it's not possible to know the machine's condition directly.

In other words, $M = (S, I, C, t, o)$, where S, I and C are finite sets and $t : S \times I \to S$ and $o : S \to C$ are two functions. t is the rule for condition change and o is a surjective function that determines machine's output according to it's condition. We assume that different conditions of M can be classified by entering a word (i.e. a sequence of letters) and observing the output (i.e. for each two conditions s_i and s_j, one can find a word such that by entering it to the machine, the output for each condition be different).

You may also know that a procedure for M is an algorithm for working with M according to it's outputs that eventually leads to a message. To be more precise, in each step of a procedure, either the user is asked to enter a letter, or a message (that marks the end of the procedure and may have some information for the user) is shown to the user according to the outputs observed thus far. A procedure's length is defined as the maximum number of letters entered during all possible executions.



During different executions of the above procedure, the user may be asked to enter words a, aa or ab, and since these words have a maximum of 2 letters, we say that the procedure is of length 2.

a) Machine M has n different conditions and p output colors. Prove that for each two different conditions of M, there exists a word having at most $n-p$ letters such that by entering it to the machine, two different outputs are observed (after entering each letter, the machine's condition and hence it's output change according to the prescribed rule. So entering a word of length k, gives us a sequence of length $k+1$ of outputs (counting the initial output). However, here only the last output may be considered).

b) Prove that for each machine M with n conditions, it's possible to design a procedure having a length of at most n^2 such that it's execution gives the final condition of the machine to the user (according to it's outputs), starting from any unknown initial condition. (in some way, this can be the same as repairing the machine, because by knowing the machine's condition, we know what input to enter to get the desired output, and losing it's condition may seem as braking it!)
Can you solve this problem for a procedure length of at most $\frac{n^2}{2}$?

120 minutes (→ p.38)

8 . (Morteza Saghafian) K_n **and Nothing More!**

For each nonnegative integer n, polynomial $K_n(x_1, x_2, ..., x_n)$ is defined recursively as follows,

$$K_0 = 1$$
$$K_1(x_1) = x_1$$
$$K_n(x_1, \ldots, x_n) = x_n K_{n-1}(x_1, \ldots, x_{n-1}) + (x_n^2 + x_{n-1}^2) K_{n-2}(x_1, \ldots, x_{n-2}).$$

Prove that $K_n(x_1, x_2, ..., x_{n-1}, x_n) = K_n(x_n, x_{n-1}, ..., x_2, x_1)$.

75 minutes (→ p.39)

Team Selection Test

1 . (Mohammad Ahmadi) Find all polynomials $P(x)$ and $Q(x)$ with rational coefficients such that

$$P(x)^3 + Q(x)^3 = x^{12} + 1.$$

(→ p.41)

2 . (Ali Zamani) Point I_b is the B-excenter of triangle ABC. If we denote by M the midpoint of arc BC of the circumcircle of triangle ABC (the one that does not contain vertex A), and MI_b intersects the circumcircle of triangle ABC at T, prove that $TI_b^2 = TB \times TC$.

(→ p.41)

3 . (Navid Safaei) $b_1 < b_2 < \dots$ is the sequence of all natural numbers that can be written as the sum of squares of two natural numbers. Prove that for infinitely many natural numbers n, $b_{n+1} - b_n = 2015$.

(→ p.42)

4 . (Amirhossein Gorzi) Let n be a natural number. Determine the smallest natural number k such that among any k natural numbers, it is always possible to select an even number of them having a sum dividable by n.

(→ p.43)

5 . (Shayan Dashmiz) Consider an $n \times n$ array of points. For a subset A of the edges of this array (i.e. the lattice edges), we denote by $V(A)$ the set of vertices of A and by $J(A)$ the set of connected components of A. For each natural number l, prove that

$$\frac{l}{2} \leq \min_{|A| \geq l} \left(|V(A)| - |J(A)|\right) \leq \frac{l}{2} + \sqrt{\frac{l}{2}} + 1.$$

(For a set X, by $|X|$ we mean the number of elements of X).

(→ p.44)

6 . (Ali Zamani) Quadrilateral $ABCD$ is both inscribed and circumscribed. Let E be the intersection point of AD and BC, F the intersection point of AB and CD, S the intersection point of AC and BD and O the circumcenter of quadrilateral $ABCD$. E' and F' are selected on AB and AD such that $\angle BEE' = \angle AEE'$ and $\angle AFF' = \angle DFF'$. Let M be the midpoint of arc BAD of the circumcircle of the quadrilateral and X a point collinear with O and E' such that $\frac{XA}{XB} = \frac{EA}{EB}$. Also let Y be a point collinear with O and F' such that $\frac{YA}{YD} = \frac{FA}{FD}$. Prove that the circle with diameter OS, the circumcircle of triangle OAM and the circumcircle of triangle OXY are co-axis.

(→ p.45)

7 . (Mohammad Jafari) For positive real numbers a, b, c and d such that $\sum_{cyc} \frac{1}{ab} = 1$, prove that

$$abcd + 16 \geq 8\sqrt{(a+c)(\frac{1}{a}+\frac{1}{c})} + 8\sqrt{(b+d)(\frac{1}{b}+\frac{1}{d})}.$$

(→ p.45)

8 . (Ali Zamani) Point D is the intersection point of the angle bisector of vertex A with side BC of triangle ABC, and point E is the tangency point of the inscribed circle of triangle ABC with side BC. A_1 is a point on the circumcircle of triangle ABC such that $AA_1 || BC$. If we denote by T the second intersection point of line EA_1 with the circumcircle of triangle AED and by I the incenter of triangle ABC, prove that $IT = IA$.

(→ p.46)

9 . (Hesameddin Rajabzadeh, Tina Torkaman and Ali Khezeli) Find the maximum number of rectangles with sides equal to 1 and 2 and parallel to the coordinate axes such that each two have an area equal to 1 in common.

(→ p.46)

10 . (Mohammad Javad Shabani) Let ABC be an acute-angled triangle. Point Z on the altitude of vertex A and points X and Y on the extensions of the altitudes of vertices B and C are selected such that,

$$\angle AYB = \angle BZC = \angle CXA = 90°.$$

Prove that X, Y and Z are collinear if and only if the length of the tangent from vertex A to the nine-point circle of the triangle is equal to the sum of the lengths of tangents from vertices B and C to this circle.

(→ p.50)

11 . (Mohsen Jamali) Permutation $(a_1, a_2, \ldots, a_n)$ of the set $\{1, 2, \ldots, n\}$ is called **messy** if for any three indices $1 \leq i < j < k \leq n$, $a_i + a_k - 2a_j$ is not dividable by n. Find all natural numbers $n \geq 3$ for which there exists a messy permutation of the set $\{1, 2, \ldots, n\}$.

(→ p.51)

12 . (Mohammad Ahmadi) For positive real numbers a, b and c such that $a+b+c = abc$, prove that

$$\sum_{cyc} \frac{a}{a^2+1} \leq \frac{\sqrt{abc}}{3\sqrt{2}} \sum_{cyc} \frac{\sqrt{a^3+b^3}}{ab+1}.$$

(→ p.52)

13 . (Farhad Seifollahi) From a point A outside circle ω, tangents AS and AT are drawn to the circle. Points X and Y are the midpoints of segments AT and AS, respectively. Tangent XR is drawn from point X to the circle and P and Q are the midpoints of segments XT and XR, respectively. If XY and PQ intersect each other at K, and SX and TK intersect at L, prove that $KRLQ$ is an inscribed quadrilateral.

(→ p.52)

14 . (Morteza Saghafian) Natural numbers a_1, a_2 and a_3 are given. A recursive sequence of integers is defined as follows,

$$a_{n+1} = [a_n, a_{n-1}] - [a_{n-1}, a_{n-2}] \quad n \geq 3.$$

Prove that there exists natural number $k \leq a_3 + 4$ such that $a_k \leq 0$ (note that by the lcm of two integers, we mean their least common positive multiple. Also it may be reminded that lcm can only be defined for non-zero integers, hence the elements of this sequence can be defined only to the point where one becomes zero).

(→ p.53)

15 . (Morteza Saghafian) $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are $2n$ positive numbers. We know that all the a_i's, $1 \leq i \leq n$ are not equal, and that they can be separated into two partitions of equal sum. These two properties hold for the b_i's, $1 \leq i \leq n$, as well. Prove that there exists a simple $2n$-gon with sides parallel to the coordinate axes such that the lengths of it's horizontal edges are equal to a_i's and the lengths of it's vertical edges are equal to b_i's (a simple polygon is one that does not cross itself).

(→ p.54)

16 . (Morteza Saghafian) A puts 5 points on the plane such that no three of them are collinear. B adds a sixth point that is not collinear with any two of the former points. A wants to eventually construct two triangles from the six points such that one can be placed inside another. Can A put the 5 points in such a manner so that he would always be able to construct the desired triangles? (We say that triangle $\triangle_1$ can be placed inside triangle $\triangle_2$ if $\triangle_1$ is congruent to a triangle that is located completely inside $\triangle_2$.)

(→ p.56)

17 . (Morteza Saghafian, Ali Khezeli) For each natural number d, prove that there exists a unique monic polynomial of degree d with the property that $P(1) \neq 0$ such that for a sequence of real numbers $a_1, a_2, \ldots$ that satisfies the following recursive relation,

$$P(n)a_1 + P(n-1)a_2 + ... + P(1)a_n = 0 \quad n > 1,$$

there exists a natural number N such that for any natural number $m > N$, $a_m = 0$.

(→ p.57)

18 . (Ali Zamani) H is the foot of the altitude of vertex A of triangle ABC and H' is the reflection of H with respect to the midpoint of BC. If tangents to the circumcircle of triangle ABC at points B and C intersect each other at X and the perpendicular to XH' at H' intersects lines AB and AC at Y and Z, respectively, prove that $\angle YXB = \angle ZXC$.

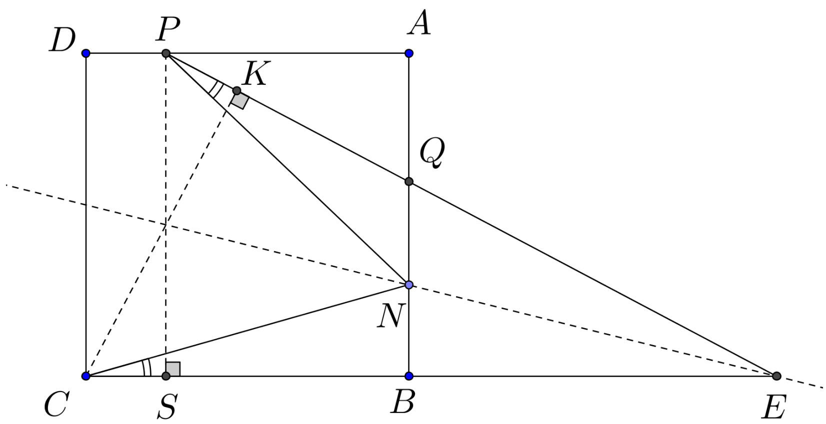
(→ p.58)

Solutions

Second Round

1 . The answer is 99. First consider the initial condition that all of eggs are in just one of boxes. In each step, we can transfer eggs to at most one new box and so we need at least 99 steps. We claim that 99 steps is always enough. For this end, we call a box containing exactly 30 eggs and 10 kilograms of rice a *good box*, and a box which is not good a *bad box*! In each step, consider one of bad boxes containing the most number of eggs and one of bad boxes containing the most amount of rice. If these two boxes were the same, consider another arbitrary bad box (note that if all other boxes were good, this box also must be good and there is nothing to prove). Evidently, we have at least 30 number of eggs and 10 kilograms of rice in these two boxes. So by transfering eggs and rice between them, we can make one of them a good box. Therefore, after 99 steps we have at least 99 good boxes and so the last box is also good and we are done.

2 . Let E be the intersection point of PQ and BC. According to the problem assumption, $PN = NC$ and so $\angle NPC = \angle PCN$. On the other hand, we know $\angle QPN = \angle NCB$. From these we conclude that EPC is an isosceles triangle. Therefore, its altitudes PS and CK have equal length. So $CK = PS = AB = BC$ and therefore right-angled trinagles QBC and QKC are congruent. So QC is the bisector of angles $\angle KCB$ and $\angle KQB$. Hence, $\angle BCQ = \frac{1}{2}\angle KCB$ $(*)$.



On the other hand, since $\angle QBC + \angle QKC = 90° + 90° = 180°$, we get that $QBCK$ is a cyclic quadrilateral which implies $\angle BCK = \angle AQP$. This together with $(*)$ completes the proof.

3 . There is no loss of generality in assuming that $x \geq y \geq z$. We have

$$\begin{aligned}
& x^2 + y^2 + z^2 = 2(xy + yz + zx) \\
\Rightarrow\ & x^2 + x(-2y - 2z) + y^2 + z^2 - 2yz = 0 \\
\Rightarrow\ & x = (y + z) \pm \sqrt{(y + z)^2 - y^2 - z^2 + 2yz} = (y + z) \pm 2\sqrt{yz} = (\sqrt{y} \pm \sqrt{z})^2 \\
\Rightarrow\ & \sqrt{x} = \sqrt{y} \pm \sqrt{z}
\end{aligned}$$

Since $y, z \leq x$ the case $\sqrt{x} = \sqrt{y} - \sqrt{z}$ is not admissibles and so we get $\sqrt{x} = \sqrt{y} + \sqrt{z}$ or equivalently $x = y + z + 2\sqrt{yz}$. After substituting this equality in the statement of the problem, we deduce

$$\frac{y + z + 2\sqrt{yz} + y + z}{3} \geqslant \sqrt[3]{2(y + z + 2\sqrt{yz})yz}$$

If $y = 0$, the assertion is trivial. Therefore, we assume $y \neq 0$. Let we define $t = \frac{z}{y}$. Now we must prove

$$\frac{2t + 2\sqrt{t} + 2}{3} \geqslant \sqrt[3]{2(t + 1 + 2\sqrt{t})t}$$

Which is a consequence of AM-GM inequality.

$$\frac{2t + 2\sqrt{2t} + 2}{3} = \frac{(\sqrt{t} + 1) + (\sqrt{t} + 1) + 2t}{3} \geq \sqrt[3]{2(t + 1 + 2\sqrt{t})t}$$

4 . We start with a lemma.

Lemma 1. *Let n be a positive integer and p, q some positive rational numbers. If $n^p = q$, then q is itself an integer.*

Proof. Suppose $p = \frac{a}{b}$ and $q = \frac{c}{d}$ where $a, b, c, d \in \mathbb{N}$. We have

$$n^p = q \Rightarrow n^{\frac{a}{b}} = \frac{c}{d} \Rightarrow n^a = \left(\frac{c}{d}\right)^b = \frac{c^b}{d^b} \Rightarrow d^b | c^b \Rightarrow d | c \Rightarrow q \in \mathbb{N}$$

□

Now for the main problem, note that if $n = 1$, then $m = n = 1$ and this is a solution for the equation. So we may assume thagt $n > 1$. Let $r = \log_n m$, $(m = n^r)$. We have

$$n^{n^n} = m^m = (n^r)^{n^r} = n^{rn^r}$$

Since $n > 1$, we must have

$$n^n = rn^r \Rightarrow r = n^{n-r}$$

On the other hand, since $n^{n^n} = m^m$, we get $n^n = m \log_n m = r$ or $r = \frac{n^n}{m} \in \mathbb{Q}$. According to the lemma $n - r$ and r playing the role of p and q, respectively, we get r is an integer. Now if $r < n$, then $n^{n-r} \geq n^1 > r = n^{n-r}$, which is impossible. And if $r > n$, then $n^{n-r} < 1 \leq r = n^{n-r}$ which is again impossible.
So we must have $n = r$. Hence, $n = r = n^{n-r} = 1$. This contradicts with the assumption $n > 1$ and consequently, the only solution is $m = n = 1$.

5 . a) $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{16}\}$ is an example of a powerful set with four elements. (Part b shows that this is the unique powerful set with four elements.)

b) First we prove a lemma.

Lemma 1. *A finite powerful set S can not have an element greater than one and an element less than one.*

Proof. Suppose by contradiction that there exist such elements. Now let a be the least element of S and b the least element of S which is greater than 1. By assumption, we must have $a < 1$. Now $a < 1 < b$ and so

- $a^b < a^1 = 1$, but a was the least element of S. Thus, $a^b \notin S$.
- $1 < b^a < b^1 = b$, but b was the least element of S greater than one. Therefore, $b^a \notin S$.

So our assumption leads to a contradiction proving the lemma. $\square$

According to this lemma, if S is a finite powerful set, all of its elements are in $[1, \infty)$ or in $(0, 1]$.
Firstly, suppose that S is a powerful set with $n > 3$ number of elements in $[1, \infty)$ ($S = \{1 = a_1 < a_2 < \cdots < a_n\}$). Note that we can assume that $a_1 = 1$, because if $1 \notin S$, we can add it to S to get a powerful set with more elements. For $i \geq 2$, $a_n^{a_i} > a_n$ and so $a_i^{a_n}$ must be in S. We have

$$a_1 < a_2 < a_2^{a_n} < a_3^{a_n} < \cdots < a_{n-1}^{a_n}$$

So for $2 \leq i \leq n-1$, $a_i^{a_n} = a_{i+1}$. Now for $a_2 < a_{n-1}$ ($n > 3$) we have

$$a_2 < a_2^{a_{n-1}} < a_2^{a_n} = a_3 \Rightarrow a_2^{a_{n-1}} \notin S$$

$$a_{n-1} < a_{n-1}^{a_2} < a_{n-1}^{a_n} = a_n \Rightarrow {a_{n-1}}^{a_2} \notin S$$

but this contradicts, because S was a powerful set.

Now suppose that S is a powerful set with $n > 4$ elements in $(0, 1]$. Let $S = \{a_1 < a_2 < \cdots < a_n = 1\}$. Again we may assume $1 \in S$. Similar to the previous part, for each $1 \leq i \leq n-2$, $a_{n-1} < a_{n-1}^{a_i} < 1$. So $a_n^{a_i} \notin S$, and consequently $a_i^{a_n} \in S$. We have

$$a_1 < a_1^{a_{n-1}} < a_2^{a_{n-1}} < \cdots < a_{n-2}^{a_{n-1}} < 1$$

So we get $a_i^{a_{n-1}} = a_{i+1}$ for each $2 \leq i \leq n-2$. Now if we denote a_{n-1} by a, we get

$$a_{n-2} = a^{\frac{1}{a}}, a_{n-3} = a^{\frac{1}{a^2}}, \ldots$$

Now by looking at a_{n-1} and a_{n-2}, we conclude

$$a_{n-1} = a_{n-2}^{a_{n-1}} < a_{n-1}^{a_{n-2}} < 1 \Rightarrow a_{n-1}^{a_{n-2}} \notin S$$

This implies $a_{n-2}^{a_{n-1}} \in S$. Since $a_{n-2}^{a_{n-1}} > a_{n-2}^{a_n} = a_{n-1}$, we get $a_{n-2}^{a_{n-1}} = a_n$. So

$$a_{n-2}^{a_{n-1}} = \left(a^{1/a^2}\right)^{\left(a^{1/a}\right)} = a \quad \Longrightarrow \quad a^{a^{(1/a)-2}} = a \Rightarrow a^{\frac{1}{a}-2} = 1$$

But $a \neq 1$ and so $a = \frac{1}{2}$. Therefore, $a_{n-1} = \frac{1}{2}$, $a_{n-2} = \frac{1}{4}$ and $a_{n-3} = \frac{1}{16}$. Now since $n > 4$, $a_{n-4} = \frac{1}{256} \in S$ but it is easy to see none of $a_{n-3}^{a_{n-4}}$ and $a_{n-4}^{a_{n-3}}$ is not in S.
So there is no powerful set with more than 4 elements.

6 . First of all note that if we have k groups with number of members $1, 2, \ldots, k$ after a week we have again k groups with $1, 2, \ldots, k$ number of members.

Lemma 1. *If $n = \binom{k}{2}$ for some positive integer k, after $\binom{k}{2}$ weeks, there would be $k-1$ groups such that for each $1 \le i \le k-1$ there is a group with i members.*

Proof. The proof for $n = 2, 3$ is trivial. Suppose that the assertion is true for each integer $1 \le k \le m$ and now we have a group with $\binom{m+1}{2}$ members. By induction after $\binom{m}{2}$ weeks, we have $m-1$ groups with $1, 2, \ldots, m-1$ members. In such situation, the number of members in the first group will be $\binom{m+1}{2} - \binom{m}{2} = m$. Therefore, the assertion is proved. $\square$

Now Suppose that we have $n = \binom{k}{2} + t \le \binom{k+1}{2}$. We want to prove that after some weeks the number of members in the largest group is k. For this reason, we add $\binom{k+1}{2} - \binom{k}{2} - t$ fake members to this group to get a group with $\binom{k+1}{2}$ members. Referring to the lemma, after some weeks, for each $1 \le i \le k$ there always exists a group with i members. We suppose that if there is at least one real member in a group, the fake mambers cannot became king. So the number of real members in each group is at most k. This implies the assertion

$$\binom{k}{2} < n \le \binom{k+1}{2} \Rightarrow k^2 - k + \frac{1}{4} = (k - \frac{1}{2})^2 < 2n \Rightarrow k \le \sqrt{2n} + \frac{1}{2} < \sqrt{2n} + 1$$

Third Round

1 . a) Yes, there exists such function. Firstly, note that if a, b, c and d are four rational numbers the function

$$f(x)=\frac{d-c}{b-a}(x-a)+c$$

defines an increasing bijection between (a,b) and (c,d). Now for the main problem consider two strictly increasing sequences $\{p_n\}_{n\geq 0}$ and $\{q_n\}_{n\geq 0}$ such that $p_0=q_0=0$, $p_n\to\sqrt{2}$ and $q_n\to\sqrt{3}$. Then we can define $f:A\to B$ as follows.

$$f(x)=\begin{cases} x & x\leq 0\\ \frac{q_{i+1}-q_i}{p_{i+1}-p_i}(x-p_i)+q_i & x\in[p_i,p_{i+1}]\end{cases}$$

Obviously, f defines an increasing bijection from A to B.

b) Yes, there exists such function. The sets A and B are both countable, so we can count their elements

$$A=\{a_1,a_2,a_3,\ldots\},\quad B=\{b_1,b_2,b_3,\ldots\}$$

Furthermore, note that both of sets A and B are dense in $\mathbb{R}$, i.e. in every open interval there exist infinitely many elements of A and infinitely many elements of B.
Now we define f inductively as follows. First we define $f(a_1)=b_1$.

(i.) Now suppose that in some step, we have defined f for any $x\in\{a_{i_1}<\ldots<a_{i_k}\}$ such that f is strictly increasing on this domain. Let k be the least positive integer that $f(a_k)$ has not been defined in the previous steps. Assume that a_k lies between a_{i_r} and $a_{i_{r+1}}$ and so $f(a_k)$ must be in $\big(f(a_{i_r}),f(a_{i_{r+1}})\big)$. Choose an aribitrary element of B in this interval say q, such that q is not equal to $f(x)$ for $x\in\{a_{i_1}<\ldots<a_{i_k}\}$ and then define $f(a_k)=q$.

(ii.) Suppose that in some step, l is the least natural number such that a_l is not in the range of function f (which is defined before this step). Let $\{b_{i_1}<b_{i_2}<\ldots<b_{i_k}\}$ be the range of f and $b_l\in(b_{i_r},b_{i_{r+1}})$. Since b_{i_r} and $b_{i_{r+1}}$ are in the range of f, there exist $a_s,a_t\in A$ such that $f(a_s)=b_{i_r}$ and $f(a_t)=b_{i_{r+1}}$. Same as **(i)**, we can choose $a\in A\cap(a_s,a_t)$ such that a is not in the domain of f. Now we can define $f(a)=b_l$.

In each step, by using this two operations we add two numbers to the number of elements of domain and range of f. Furthermore, the least k and l which a_k was not in the domain of f and b_l was not in the range of f will be added to the domain and range, respectively. So if we repeat these two operations to infinity we get

$$\text{Domain of } f=A \text{ and Range of } f=B$$

and so f is surjective. Moreover, since f is strictly increasing in each step, we get an increasing bijection as desired.

c) No, there is not such function. Suppose by contradiction that f is such function. Consider the subset $C=\{0\}\times\mathbb{R}\subseteq B$ and define $D=f^{-1}(C)\in\mathbb{R}$. If $x,y\in C$ and $x<z<y$, then $z\in C$. Since f is assumed to be an increasing bijection, D

has the same property. It is a well-known fact that the subsets of $\mathbb{R}$ satisfying this property are intervels. We claim that D is an open interval. Because if D has a maximum or minimum element, then C must have a maximum or minimum element, too. But C does not have any maximum or minimum element. Let $D = (a, b)$ and $x = (x_1, x_2) = f^{-1}(b)$. Since b is greater than all the elements of D, x must be greater than any element of C and so $x_1 > 0$. But $(\frac{x_1}{2}, 0) \in B$ is greater than any element of C and less than x. Therefore, $f^{-1}(y) \in \mathbb{R}$ is greater than any element of D, but is less than b, which is impossible. This contradiction shows that there is not such function f.

d) No, there is not such function. In an ordered set X, we call that y is the *successor* of x whenever $x < y$ and there is no element between x and y. Furthermore, we call x an *end* if there is no successor for x in X. Notice that the end elements of A are

$$\mathbf{End}(A) = \Big\{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{2}), (\frac{1}{8}, \frac{1}{2}), \dots\Big\}$$

where the end elements of B are

$$\mathbf{End}(B) = \Big\{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{2}), (\frac{1}{8}, \frac{1}{2}), \dots, (0, \frac{1}{2})\Big\}$$

So $(0, \frac{1}{2})$ is the minimum of $\mathbf{End}(A)$. Therefore, if there is some increasing bijection f, $f^{-1}(0, \frac{1}{2})$ must be the minimum of $\mathbf{End}(A)$. But $\mathbf{End}(A)$ has no minimum element. This leads to a contradiction and so there is not such function.

e) Let $Y = \{2^{-n} | n \in \mathbb{N}\} \cup \{0\}$ and define

$$A = Y \cup (Y+1) \cup (Y+2) \cup \cdots, \quad B = A \cup \{-1\}$$

(Note that by $Y + m$ we mean $\{y + m | y \in Y\}$.)
Firstly, we define increasing surjective functions $f : A \to B$ and $g : B \to A$ as follows

$$f(x) = \begin{cases} -1 & x < 1 \\ x - 1 & x \geq 1 \end{cases} \qquad g(x) = \begin{cases} 0 & x = -1 \\ x & x \neq -1 \end{cases}$$

It is easy to check that f and g are both surjective and increasing. Now we want to show that there is not any increasing bijection from A to B. Assume, to the contrary, that h is such function. Since 0 is the minimum element of A, $h(0)$ must be the minimum element of B and so $h(0) = -1$. Note that in B, 0 is the successor of -1. Therefore, $h^{-1}(0)$ must be the successor of 0 in A. But 0 does not have any successor in A. This contradiction shows that there is not such h.

2 . We claim that it is possible to roll the polyhedron in each direction, as far as we want.
Firstly, we draw a line from a point inside the face of polyhedron touching the ground to the destination point. Since the polyhedron is convex, this line intersects one of edges of that face. We can roll the polyhedron around this edge. After this motion, the distance from the destination point and the intersection of polygon and the ground became smaller. We can continue the same process to get the destination point.
To this end, we should prove that there is some positive number ϵ such that after each motion our distance will be decrease at least by ϵ. In the other words, if we denote our distance after n-th step by d_n, the sequence $d_n - d_{n-1}$ has a lower bound or equivalently

cannot tend to zero. If this sequence tends to zero, we deduce that the polyhedron has been rolled around some edge connecting to a single vertex (after some step), infinitely many times.

Lemma 1. *We can roll the polyhedron around one of its vertices at most n times, where n is the number of edges connecting to that vertex.*

Proof. Look at the trace of the line (on the ground) on the polyhedron. If we open the polyhedron, the trace became an straight line. Since a line in the plane intersect each convex polygon at most once, we deduce that during the motions around each vertex each face containing that vertex touches the ground at most once. □

Reffering to the lemma, we infer that $d_n - d_{n-1}$ cannot tend to zero and so we will get our destination point. This shows that if there is no canyon we can get everywhere in the plane.

Now for the case that we have a bridge, we can connect the initial point and the center of cavity by a broken line. So using above method we can roll the polyhedron on this path to drop into the cavity.

3 . a) Without loss of generality, we may suppose that $1 < d_1 < \cdots < d_n$. First, suppose that $d_1 > 2$. For each subset $I = \{i_1 < \cdots < i_k\} \subseteq \{1, 2, \ldots, n\}$, we define d_I to be equal to the product $d_{i_1} d_{i_2} \cdots d_{i_k}$. By the *Chinese Reminder Theorem*, we know that the modular equalities

$$x \overset{d_{i_1}}{\equiv} r_{i_1},\ \ldots, x \overset{d_{i_k}}{\equiv} r_{i_k} \qquad (*)$$

have a unique solution modulo d_I. Let r_I be one of its solutions (If $I = \emptyset$ there is no equation and so all integers are solutions. In this case, we set $r_I = 1$). So $x \equiv r_I \pmod{d_I}$ is equivalent to the system $(*)$.
Now suppose that M is an arbitrary integer. According to the *Inclusion-Exclusion Principle*, the number of solutions of modular inequalities

$$x \overset{d_1}{\not\equiv} r_1,\ \ldots\ , x \overset{d_n}{\not\equiv} r_n$$

which are greater than M is

$$\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} |\{1 \leq x \leq M | x \overset{d_I}{\equiv} r_I\}|.$$

It is easy to see that the number of solutions of each modular equality like $x \overset{d}{\equiv} r$ in $[1, M]$ is $\lfloor \frac{M}{d} \rfloor$ or $\lceil \frac{M}{d} \rceil$. In each case, the difference of the number of solutions and $\frac{M}{d}$ is at most one, so the number of solutions is greater than

$$\begin{aligned} \Big(\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \frac{M}{d_I} \Big) - 2^n &= M\Big(1 - \frac{1}{d_1} - \cdots - \frac{1}{d_n} + \frac{1}{d_1 d_2} + \cdots \Big) - 2^n \\ &= M(1 - \frac{1}{d_1}) \cdots (1 - \frac{1}{d_n}) - 2^n. \end{aligned}$$

Now, if $M = 3^n$ and $d_i \geq 3$ for all i, this number is at least

$$3^n(1-\frac{1}{3})^n - 2^n = 0.$$

And so in this case the number of solutions of inequlities is not zero. For the case $d_1 = 2$, parity of x is determined by the parity of r_1. Thus, we can use one of substitutions $x = 2y$ or $x = 2y - 1$. This substitution changes the other inequalities to some new ones in terms of y

$$y \overset{d_2}{\not\equiv} s_2, \ \ldots \ , y \overset{d_n}{\not\equiv} s_n$$

Since $3 \leq d_2 < \cdots < d_n$, these inequlities have a solution $y_0 \leq 3^{n-1}$. Hence, we can find a solution for the main inequalities (x) not greater than $2 \times 3^{n-1} < 3^n$. So the proof is complete.

b) Similar to the previous part, we assume that $d_1 < d_2 < \cdots < d_n$. Therefore, for each i, we have $d_i \geq i$. Furthermore, suppose that $\epsilon > 0$ has been given. We choose a real number $a \in (\frac{2}{2+\epsilon}, 1)$. Obviously, there is some N_1 such that $1 - \frac{1}{N_1} > a$. On the other hand, since $\frac{(2+\epsilon)a}{2} > 1$, there is N_2 such that

$$\left(\frac{(2+\epsilon)a}{2}\right)^{N_2} > 2^{N_1}.$$

Now we set $N = N_1 + N_2$. We will show that for each $n > N$ the number of solutions of the system of modular inequalities is at most $(2+\epsilon)^n$. According to part (a), we know that the number of solutions of this inequalities in $[1, (2+\epsilon)^n]$ is more than

$$(2+\epsilon)^n(1-\frac{1}{d_1})\cdots(1-\frac{1}{d_n}) - 2^n.$$

But for $n > N$ we have

$$\begin{aligned}
(2+\epsilon)^n(1-\frac{1}{d_1})\cdots(1-\frac{1}{d_n}) &\geq (2+\epsilon)^n(1-\frac{1}{2})\cdots(1-\frac{1}{n+1}) \\
&\geq (2+\epsilon)^n(1-\frac{1}{2})^{N_1}(1-\frac{1}{N_1})^{n-N_1} \\
&> (2+\epsilon)^n 2^{-N_1} a^{n-N_1} \\
&= (2+\epsilon)^n 2^{-N_1} \left(\frac{2}{2+\epsilon}\right)^{n-N_1} \left(\frac{(2+\epsilon)a}{2}\right)^{n-N_1}.
\end{aligned}$$

Note that $\frac{2}{2+\epsilon} < 1$, $\frac{(2+\epsilon)a}{2} > 1$ and $n - N_1 > N_2$. Therefore,

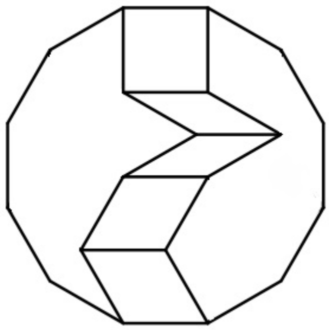
$$\begin{aligned}
(2+\epsilon)^n 2^{-N_1} \left(\frac{2}{2+\epsilon}\right)^{n-N_1} \left(\frac{(2+\epsilon)a}{2}\right)^{n-N_1} &> (2+\epsilon)^n 2^{-N_1} \left(\frac{2}{2+\epsilon}\right)^{n} \left(\frac{(2+\epsilon)a}{2}\right)^{N_2} \\
&> (2+\epsilon)^n 2^{-N_1} \left(\frac{2}{2+\epsilon}\right)^{n} 2^{N_1} \\
&= 2^n.
\end{aligned}$$

And finally, we get

$$(2+\epsilon)^n(1-\frac{1}{d_1})\cdots(1-\frac{1}{d_n}) - 2^n > 0$$

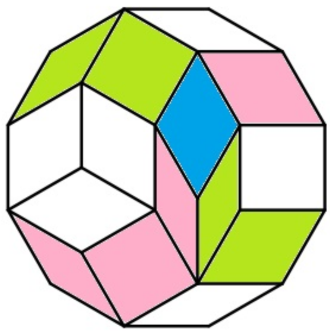
4 . We will solve the same problem for any convex $2n$-gon $P = A_1A_2 \cdots A_{2n}$ such that $A_1A_2 = A_2A_3 = \cdots = A_{2n}A_1$ and $A_iA_{i+1} \parallel A_{n+i}A_{n+i+1}$ for each i. This generalization is important for our induction in part (c).

a) Consider an arbitrary rhombination of P and an arbitrary rhombus of this rhombination. Suppose that two parallel sides of this rhomus are parallel to the x axis. In this case, we can extend this rhombus from up and down, using rhombi having two sides parallel to this. Continuing in this fashion, we will get two sides of P from up and down. Since there is no internal point of sides of P in our rhombination, we infer that each edge in this rhombination has length equal to the sides of P.



Now consider two parallel sides of P, say a and a'. By similar arguments, we can construct some paths consinsting of rhombi with at least one side parallel to these two, connecting these sides. On the other hand, since each path of rhombi can be extended by a unique rhombus in one direction, we conclude that the path connecting a and a' is unique.

Thus, each rhombus is the intersection of two paths of rhombi. We know that the number of such paths is n and so there are $\binom{n}{2}$ rhombi in our rhombination.



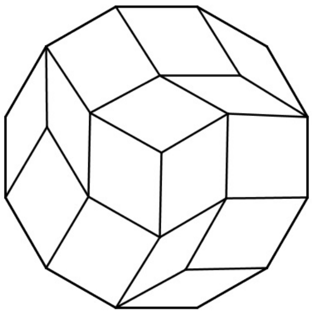
Counting the number of sides of rhombi in two ways determines the number of edges in this rhombination. The number of sides of rhombi is $4\binom{n}{2}$ in one hand. On the other hand, this number is equal to $2n + 2E$, where E is the number of internal edges. So

$$\text{The Number of All Edges} = E + 2n = n + 2\binom{n}{2} = n + (n^2 - n) = n^2$$

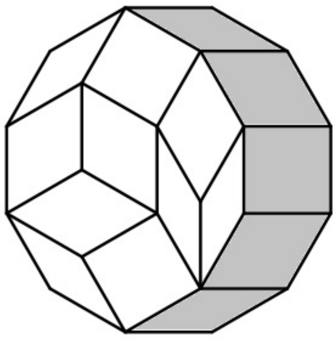
For the number of vertices (which we denote by V), according to the Euler formula for planar graphs we have

$$\left(\binom{n}{2} + 1\right) - n^2 + V = 2 \Rightarrow V = \binom{n+1}{2} + 1$$

b) This is not correct. Just look at the following rhombination.



c) We will prove the assertion by induction on n. The base case $n = 2$ is trivial. Now suppose that the assertion is true for $2(n-1)$-gons. Let $e_1, e_2, \ldots, e_{2n}$ be the sides of P. We call a rhombination *Standard*, whenever the path connecting e_1 to e_{n+1} is adjacent to the sides $e_2, \ldots, e_n$. You can see an standard rhombination in the following shape.



We claim that by applying the step mentioned in the problem statement, we can change each arbitrary rhombination to an standard one. For this end, suppose that we have an arbitrary rhombination of P and let C be the path connecting e_1 to e_{n+1} in this. Assume that C is adjacnet to $e_1, \ldots, e_{i-1}$ and e_{j+1} but not to $e_i, e_{i+1}, \ldots, e_j$ $(2 \le i \le j \le n)$. Note that if C is adjacent to all $e_1, \ldots, e_n$ there is nothing to prove. Otherwise, there is a rhombinated $2(j-i+1)$-gon Q bounded by C and sides $e_i, \ldots, e_j$ (Q may not be convex). Denote the other sides of Q by $e'_i, \ldots, e'_j$ (e_i and e'_i have common vertex). It suffices to show that one of vertices of Q between $e'_i, \ldots, e'_j$ is adjacent to only one rhombus in Q. Therefore, this vertex is adjacent to exactly 3 rhombi in P and if we apply the step on this vertex, the number of rhombi in the right of C became smaller. So we will get an standard rhombination by several use of this trick.
Since the path contaning each e_k intersects C $(i \le k \le j)$, we deduce that each of $e'_i, \ldots, e'_j$ has length and direction same as one of $e_i, \ldots, e_j$. Let M_k be the rhombus containing e'_k in Q. Suppose that e'_k is parallel to e_{g_k} and the other side of M_k is parallel to e_{f_k}. Note that $f_k \neq g_k$ (for each k). Our goal is to find some k such that $f_k = g_{k+1}$. Suppose that there is not such k. We must have $f_i < g_i$, because if not M_i will intersect e_i or lies out of Q. Now suppose that $f_k < g_k$ (induction on $k \ge i$). Since e'_{k+1} is out of M_k, we infer $g_{k+1} \ge f_k$ and because of our assumption $g_{k+1} > f_k$. On the other hand, $f_{k+1} < g_{k+1}$ because if not M_{k+1} is out of Q or may intersects M_k (here we use $f_k \neq g_{k+1}$, because we may have $M_k = M_{k+1}$ and then the conclusion became false). Now the induction is finished and so we have $f_k < g_k$ and $f_k < g_{k+1}$.

Consider the side $e_{g_{k+1}}$ is parallel to e_i. Thus, $g_{k+1} = i$. But we have $f_k \geq i$ and this leads to a contradiction by $g_{k+1} > f_k$.
Finally, we can change each two different rhombinations to an standard one. Then by removing the path connecting e_1 to e_{n+1}, we get two rhombinations in a convex $(2n-2)$-gon P'. Now by induction we deduce that we can transfer two these two rhombinations to a similar one. Therefore, we can transfer each rhombination to another by applying the step several number of times.

d) According to the part (a), the path connecting e_1 to e_{n+1} has $n-1$ rhombi, each one is the intersection of this path and another path connecting two other parallel sides. We have $n-1$ ways for choosing the second direction of the bottom rhombus. $n-2$ ways for choosing the second direction of the next rhombus,...
After removing this path and translating two remaining parts, we get a rhombination in a $2(n-1)$-gon. So if we denote the number of rhombinations of an $2n$-gon by $f(n)$, we have

$$f(n) \leq (n-1)! f(n-1) \Rightarrow f(n) \leq (n-1)! \times (n-2)! \times \cdots \times 2! \times 1! = \prod_{k=1}^{n-1} k^{n-k}$$

Now for the lower bound we define a semipath. By a *semipath* we mean a collection of sides of rhombi connecting to parallel sides of the polygon such that for each direction it has exactly one edge of that direction. Cutting a rhombinated $2(n-1)$-gon from one of its semipaths, then translating one of parts by a vector with length equal to the side of polygon and direction different from sides of $2(n-1)$-gon, and finally, filling the the space between two parts using some rhombi, we get a rhombination for a $2n$-gon. So

$$\Big(\text{ Semipaths in a } 2(n-1)\text{-gon}\Big) \times f(n-1) \leq f(n) \qquad (*)$$

And by induction it is easy to show

$$k + \Big(\text{ Semipaths in a } 2(k-1)\text{-gon}\Big) \leq \Big(\text{ Semipaths in a } 2k\text{-gon }\Big)$$

Therefore, the number of semipaths in a $2k$-gon is at least $1 + \binom{k}{2}$. This and $*$ imply the desired lower bound.

5 . First, we set some notations.

A	Number of sides parallel to the x axis
B	Number of sides parallel to the y axis
C	Number of sides parallel to the z axis
X	Number of angles parallel to the yz plane
Y	Number of angles parallel to the zx plane
Z	Number of angles parallel to the xy plane

a) Consider a side parallel to the x axis. Two vertices of this side are vertices of two angles which are in the xy or zx plane. On the other hand, each angle parallel to the xy plane has a side parallel to the y axis and a side parallel to the x axis. Similar arguments for other directions imply

$$Y + Z = 2A, \quad Z + X = 2B, \quad X + Y = 2C$$

Therefore, X, Y and Z have the same parity.

b) According to the part (a), we have two cases for a lattice hexagon which is not planar.

- $X = Y = Z = 2$.
- $\{X, Y, Z\} = \{0, 2, 4\}$.

In the first case, we can divide the polygon into two rectangles. Now consider the lines l and l' perpendicular to the plane of each rectangle at its center. The plane passing through the centers of these two rectengles and the midpoint of their common side is perpendicular to both of rectangle planes. Therefore, this plane contains lines l and l'. Since l and l' are coplanar, they meet each other at some point O which has equal distance from all vertices of the polygon.
In the second case, according to the equations of the part (a), we have only one side parallel to one of axes. But this is contradiction, because we must have at least two sides parallel to each coordinate axis.

c) There exists such polynomial. We will use $A_1, A_2, \ldots, A_{2014}$ for its vertices. We define

$$A_n = \begin{cases} (k, k, 0) & n = 2k+1, 1 \leq n \leq 1007 \\ (k, k-1, 0) & n = 2k, 1 \leq n \leq 1007 \\ (503-k, 503-k, 1) & n = 1007 + 2k + 1, 1008 \leq n \leq 2014 \\ (503-k, 503-k+1, 1) & n = 1007 + 2k, 1008 \leq n \leq 2014 \end{cases}$$

It is easy to see that the plane $2x - 2y + 2z = 1$ passing through the midpoints of all sides of the polynomial.
d) Referring to the equations in the part (a), we have

$$X + Y + 2Z = 2A + 2B$$

so

$$2C + 2Z = 2A + 2B, C + Z = A + B$$

hence

$$C < A + B$$

By similar arguments, we can show $A < B + C$ and $B < C + A$. Hence, A, B and C are sides of a triangle.

For the converse, suppose that we have a triangle with sides $c \leq b \leq a$. We set $n = \frac{b+c-a}{2}$. First we draw $b + a - c$ sides in the xy plane parallel to the x and y axes, alternatively ($\frac{b+a-c}{2}$ parallel to the x axis and $\frac{b+a-c}{2}$ parallel to the y axis). Next, we $\frac{c+a-b}{2}$ sides parallel to x and z axes. Now we have n number of sides parallel to the y and z axes remained. We can draw them to close the polynomial.

6 . First we claim that $p : \mathcal{R}(f) \to \mathcal{R}(f)$ is a surjective function, where by $\mathcal{R}(f)$ we mean the range of the function f.

$$y \in \mathcal{R}(f) \Rightarrow \exists x_0 \in \mathbb{R};\ f(x_0) = y \Rightarrow p(y) = p(f(x_0)) = f(p(x_0)) \Rightarrow p(y) \in \mathcal{R}(f)$$

So $p : \mathcal{R}(f) \to \mathcal{R}(f)$ is indeed a function. Let $y \in \mathcal{R}(f)$ be an arbitrary element. Thus, there is some $x_0 \in \mathbb{R}$ such that $f(x_0) = y$. Note that since p has odd degree there is $z \in \mathbb{R}$ such that $p(z) = x_0$ and therefore,

$$y = f(x_0) = f(p(z)) = p(f(z))$$

It means that y is the image of $f(z) \in \mathcal{R}(f)$ under p and so $p : \mathcal{R}(f) \to \mathcal{R}(f)$ is surjective.
a) Assume to the contrary that $\mathcal{R}(f)$ is not finite. Since $\deg(P)$ is odd we can find $N > 0$ such that

$$x > N \Rightarrow p(x) > x, \quad x < -N \Rightarrow p(x) < x$$

If $\mathcal{R}(f)$ contains infinitely many positive and infinitely many negative elements, we can choose $x_0, y_0 \in \mathcal{R}(f)$ such that $y_0 < -N < N < x_0$. Note that an element $b \in p^{-1}(a)$, where $a \in \mathcal{R}(f) \cap [y_0, x_0]$ must be in $\mathcal{R}(f) \cap [y_0, x_0]$. Because if $b < y_0$, then $p(b) = a < b < y_0$ and if $x_0 < b$, then $p(b) = a > b > x_0$ which are not possible. On the other hand, $f(x_0) > x_0$ and $f(y_0) < y_0$. Now since $\mathcal{R}(f) \cap [y_0, x_0]$ is a finite set (all its elements are integers), there must be some element which is not covered by elements of $\mathcal{R}(f)$ under p and this contradicts because $p : \mathcal{R}(f) \to \mathcal{R}(f)$ was surjective. The argument in the case which $\mathcal{R}(f)$ contains only finitely many positive or negative integers are similar.

b) Since $\deg(p(x)) > 1$ is an odd number, $p(x) - x$ has odd degree. Therefore, it has at least one real root x_0. Suppose that this polynomial has no real root other than x_0. Thus, $p(x) > x$ for $x > x_0$ and $p(x) < x$ for $x < x_0$.
On the other hand, we have $p(f(x_0)) = f(p(x_0)) = f(x_0)$ and since x_0 was the unique solution of $p(x_0) = x_0$, we get $f(x_0) = x_0$. Suppose that $\mathcal{R}(f) = \{x_{-m} < \cdots < x_{-1} < x_0 < x_1 < \cdots < x_n\}$. If $n > 0$, $p(x_n) > x_n$ and this contradicts because $p(\mathcal{R}(f)) \subseteq \mathcal{R}(f)$. So $n = 0$. Similarly, $m = 0$ and therefore, $|\mathcal{R}(f)| = 1$. But we assumed that $|\mathcal{R}(f)| \geq 2$.

c) Suppose that $z, y_1, \ldots, y_m$ are $n = m+1$ arbitrary distinct integers. Using *Lagrange Interpolation*, we can find a polynomial $p(x) \in \mathbb{R}[x]$ such that

$$p(z) = z, \quad p(y_1) = y_2, \quad p(y_2) = y_3, \quad \ldots, \quad p(y_m) = y_1$$

If it is necessary by changing $x^k(x - z)(x - y_1) \cdots (x - y_m) + p(x)$ with $p(x)$ $(k \in \mathbb{N})$, we can assume that p is a monic polynomial of odd degree. Now we define function f as follows. If there is some integer $k \geq 0$ and $1 \leq i \leq m$ such that $p^k(x) = y_i$, set $f(x) = y_{i-k}$ (Indices are assumed to be modulo m). Note that if $p^{k_1}(x) = y_i$ and $p^{k_2}(x) = y_j$, $y_{i-k_1} = y_{j-k_2}$. Thus, there is no ambiguity in the definition. Otherwise, define $f(x) = z$. We claim that this definition satisfies problem statement.

- If $f(x) = y_i$, there is some integer k such that $p^k(x) = y_{i+k}$. Now $p^k(p(x)) = p(y_{i+k}) = y_{i+k+1}$ and so $f(p(x)) = y_{i+k+1-k} = y_{i+1} = p(y_i) = p(f(x))$.

- If $f(x) = z$, then obviously $f(p(x)) = z$ and so $f(p(x)) = z = p(z) = p(f(x))$.

Therefore, the proof is complete.

7 . Throughout the solution we call two conditions with same output, isochromatic. Furtheremore, we use the following font for input letters

$$I = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots\}$$

Moreover, we mean by a *word* a sequence of input letters (elements of I) and the condition after condition s and entering word w by s_w. For example

$$s_{\mathfrak{a}} = t(s, \mathfrak{a}) \text{ and if } w = \mathfrak{a}\mathfrak{b},\ s_w = t(t(s, \mathfrak{a}), \mathfrak{b}).$$

a) Since $o : S \to C$ is surjective, $n - p \geq 0$. We prove the assertion by induction on $n - p$. The base case $n - p = 0$ is trivial, since we can recognize the situation by looking only at the output and it does not need any input. For the induction step we need the following lemma.

Lemma 1. *If $n > p$, there are two isochromatic situations s, t and some letter $a \in I$ such that t_a and s_a are not isochromatic.*

Proof. Since $p < n$, there are two isochromatic situations like $\tilde{s}$ and $\tilde{t}$. According to the problem condition, there is a word which makes output of $\tilde{s}$ and $\tilde{t}$ different. Let w be the word with least letters satisfying this property. Therefore, by starting with $\tilde{s}$ and $\tilde{t}$ and entering every word with number of letters less than w the outputs are the same, but $\tilde{t}_w$ and $\tilde{s}_w$ are different. Now let a be the last letter of w ($w = \tilde{w}a$). So if we set $s = \tilde{s}_{\tilde{w}}$ and $t = \tilde{t}_{\tilde{w}}$, s and t are isochromatic, but $s_a = \tilde{s}_w$ and $t_a = \tilde{t}_w$ are not isochromatic. □

Now suppose that $n - p > 0$ and the assertion is true for every machine with smaller $n - p$. Reffering to the lemma, there are two isochromatic conditions s and t (for example with color c_1) and letter a such that s_a and t_a are not isochromatic. Set $\tilde{M}$ with conditions, inputs and rules same as M. The only difference is in its output. Consider two elements c' and c'' out of C (outputs of M). Let the output of arbitrary condition u in $\tilde{M}$ equal to the output of u in M, whenever it has a color different from c_1 in M. And if the output in M is c_1, according to the color of u_a in M define it. If the color of u_a is s_a, define it to be c', and otherwise, define it to be c''. Now $\tilde{M}$ is a machine with n conditions and $p + 1$ outputs which satisfies the problem statement according to induction hypothesis.
Consider two arbitrary conditions u and v. We know that there is some word w of length at most $n - p - 1$ such that the output of u_w and v_w are not same in $\tilde{M}$. If one of u_w and v_w has a color different from c' and c'' obviously, they have different colors in M. Otherwise, we can suppose that the colors of u_w and v_w are c' and c'', respectively. By definition of colors in $\tilde{M}$, pairs (u_{wa}, s_a) and (v_{wa}, s_a) are isochromatic. Therefore at least one of words w or wa, which has length at most $n - p$, has an output different from u and v.

b) Suppose that we know the output of the machine is in some subset $S_1 \subseteq S$ with n_1 elements. If $n_1 > 1$, there are two elements $s, t \in S_1$. By part (a), there is a word w of length at most n such that s_w and t_w have different colors. Thus, the output of machine for each initial condition after entering w is mutually different. Suppose that we know the condition is in S_1 and then enter the word w. All the conditions of S_1 which has color c after entering w is a proper subset of S_1, like $T = \{t_1, \dots, t_k\}$.

Hence, the condition after w is one of elements of $T_w = \{(t_1)_w, \dots, (t_k)_w\}$. But the number of elements of T_w is less than S_1.
Using this idea, we can give an algorithm of length at most n^2 such that after entering a word of length n the number of possible conditions decreases and so finally we can recognize the condition.
For the bound $\frac{n^2}{4}$, we use the following lemma.

Lemma 2. *Suppose that S_1 is a subset of S with n_1 elements. Then there exists a word w with length at most $\max(n - n_1 - p + 2, 0)$ such that the output of elements of S_1 after entering this word are all different.*

The case $n = 1, 2$ in the lemma, are part (a) of the problem. For the general case, same as part (a) we can use induction on $n - p$. We must see what will happen if $\max(n - n_1 - p + 2, 0)$.

8 . Consider a $1 \times n$ table with cells labelled by variables $x_1, x_2, \dots, x_n$ from left to right.

x_1	x_2	$\cdots$		x_n

We assign a polynomial in terms of variables $x_1, x_2, \dots, x_n$ to each tilling of this table by 1×1 and 1×2 tiles, as follows. We define the weight of 1×1 tiles to be the variable of its cell in the table and the weight of a 1×2 tile to be the sum of squares of variables of its cells. At the end, we associate to each tilling the product of weights of its tiles. For examples there for $n = 4$, there are only 5 ways of tilling a 1×4 table with mentioned tiles

<table><tr><td colspan="2">x_1 x_2</td><td colspan="2">x_3 x_4</td></tr></table>

<table><tr><td colspan="2">x_1 x_2</td><td>x_3</td><td>x_4</td></tr></table>

<table><tr><td>x_1</td><td>x_2</td><td colspan="2">x_3 x_4</td></tr></table>

<table><tr><td>x_1</td><td colspan="2">x_2 x_3</td><td>x_4</td></tr></table>

<table><tr><td>x_1</td><td>x_2</td><td>x_3</td><td>x_4</td></tr></table>

And the polynomials assigned to these tillings are $(x_1^2 + x_2^2)(x_3^2 + x_4^2)$, $(x_1^2 + x_2^2)x_3x_4$, $x_1x_2(x_3^2 + x_4^2)$, $x_1(x_2^2 + x_3^2)x_4$ and $x_1x_2x_3x_4$.
Now for each integer $n \geq 0$, we define the polynomial P_n to be sum of all polynomials corresponding to the tillings of $1 \times n$ table. For example

$$P_4(x_1, x_2, x_3, x_4) = (x_1^2+x_2^2)(x_3^2+x_4^2)+(x_1^2+x_2^2)x_3x_4+x_1x_2(x_3^2+x_4^2)+x_1(x_2^2+x_3^2)x_4+x_1x_2x_3x_4$$

Note that there is only empty tilling in the case $n = 0$ and there is one trivial tilling when $n = 1$. Therefore, $P_0() = 1, P_1(x_1) = x_1$.
We claim that for each integer $n \geq 0$, $P_n(x_1, \dots, x_n) = K_n(x_1, \dots, x_n)$. For proving the claim, since $P_0 = K_0$ and $P_1 = K_1$, it suffices to show that P_n satisfies the same recursive relation as K_n. This is easy by looking at the last tile of each tilling (the tile covering x_n). We have two cases.

1. The last tile of tilling is a 1×2 tile. The part of P_n corresponding to these tillings is $(x_{n-1}^2 + x_n^2)P_{n-2}(x_1, \ldots, x_{n-2})$, because $x_{n-1}^2 + x_n^2$ is the weight of the last tile and the remaining table after removing this tile has $n-2$ cells.

2. The last tile of tilling is a 1×1 tile. The part of P_n corresponding to these tillings is $x_n P_{n-1}(x_1, \ldots, x_{n-1})$ because x_n is the weight of the last tile and the remaining table after removing this tile has $n-1$ cells.

So the proof of the claim is finished. Finally, note that reflection of each tilling with respect to vertical axis of symmetry of the table is again a tilling. Therefore, if we label the cells by $x_n, x_{n-1}, \ldots, x_1$ rather than $x_1, x_2, \ldots, x_n$, we have again the polynomial $K_n(x_1, x_2, \ldots, x_n)$. So $K_n(x_1, x_2, \ldots, x_n) = K_n(x_n, \ldots, x_2, x_1)$ as desired.

Team Selection Test

1 . We have $x^{12} + 1 = (x^4 + 1)(x^8 - x^4 + 1)$. It is easy to check that these two factors are irreducible in $\mathbb{Z}[x]$ (or equivalently in $\mathbb{Q}[x]$). On the other hand, $P^3 + Q^3 = (P + Q)(P^2 - PQ + Q^2)$. Therefore, our goal is to find P and Q such that $(P+Q)(P^2-PQ+Q^2) = (X^4+1)(x^8-x^4+1)$. Irreducibility of x^4+1 and x^8-x^4+1 imply

- $x^4 + 1$ divides $P + Q$ or $P^2 - PQ + Q^2$.
- $x^8 - x^4 + 1$ divides $P + Q$ or $P^2 - PQ + Q^2$.

So we have four cases

- $P + Q = 1$ and $P^2 - PQ + Q^2 = x^{12} + 1$.

  $$1 - 3PQ = (P + Q)^2 - 3PQ = P^2 - PQ + Q^2 = 1 + x^{12} \Rightarrow 3P(1 - P) = -x^{12}$$

  So zero is the unique root of both P and $1 - P$ which is impossible.

- $P + Q = x^{12} + 1$ and $P^2 - PQ + Q^2 = 1$. Same as the previous part, we get $3PQ = x^{12}(x^{12} + 2)$. So $P(0) = 0$ or $Q(0) = 0$, but at most one of them can be zero because $P + Q = x^{12} + 1$. Because of symmetry we assume that P is divisible by x^{12}. Now since $3PQ = x^{12}(x^{12} + 2)$, we conclude $\deg(Q) \leq 12$. If $\deg(Q) < 12$, then degree of P must be greater than 12 and so we cannot have $P + Q = x^{12} + 1$. Therefore, $\deg(P) = \deg(Q) = 12$. Hence, there must be rational numbers a and b such that $P(x) = ax^{12}$ and $Q(x) = b(x^{12} + 2)$. So $P + Q = (a + b)x^{12} + 2b = x^{12} + 1$. Thus, $a = b = \frac{1}{2}$. This is not possible because $3PQ = 3abx^{12}(x^{12} + 2) = \frac{3}{4}x^{12}(x^{12} + 2) \neq x^{12}(x^{12} + 2)$. Hence, there is no solution in this case.

- $P + Q = x^8 - x^4 + 1$ and $P^2 - PQ + Q^2 = x^4 + 1$. In this case we have $3PQ = x^4(x^{12} - 2x^8 + 3x^4 - 3)$ and by arguments similar to the previous part, we deduce that $\deg(P) = \deg(Q) = 8$. Let p and q be the leading coefficients of P and Q, respectively. We have

  $$\begin{array}{rcll} P + Q & = & x^8 - x^4 + 1 & \Rightarrow \quad p + q = 1 \\ 3PQ & = & x^4(x^{12} - 2x^8 + 3x^4 - 3) & \Rightarrow \quad pq = \frac{1}{3} \end{array}$$

  But $4pq = \frac{4}{3} > 1 = (p + q)^2$ and so p and q are not real numbers. Thus, we do not have any solutions in this case.

- $P+Q = x^4+1$ and $P^2-PQ+Q^2 = x^8-x^4+1$. We get $PQ = x^4$. Note that again we cannot have $P(0) = Q(0) = 0$, because $P(0) + Q(0) = 1$. So we can assume that for example P is divisible by x^4 and so Q is a constant polynomial. Suppose that $P(x) = px^4$ and $Q(x) = q$ $(p, q \in \mathbb{Q})$. Now since $P + Q = px^4 + q = x^4 + 1$ we get $p = q = 1$. This leads to the solution $P(x) = x^4$ and $Q(x) = 1$. $P(x) = 1$ and $Q(x) = x^4$ is another solution.

2 . Firstly, we prove two lammas.

Lemma 1. *In each triangle ABC with altitude AH and circumradius R, we have $AC.AB = 2R.AH$.*

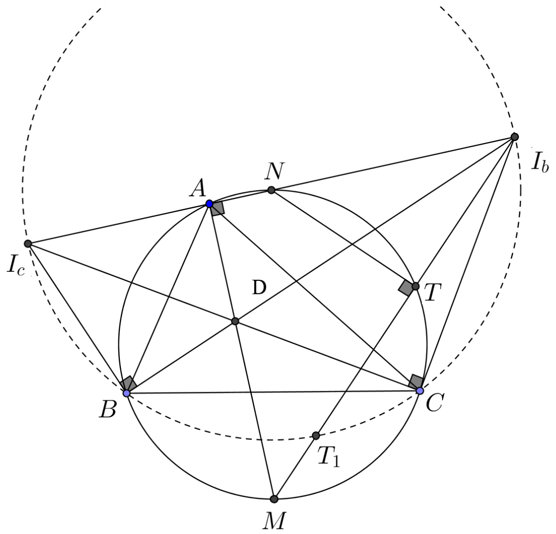
Proof. The proof is easy. $\square$

Lemma 2. *In triangle ABC, let N be the second intersection point of I_bI_c and ω (other than A), where I_b and I_c are excenters of ABC and ω is its circumcircle. Then, the points I_b, I_c, C and B lie on a common circle with center N.*

Proof. Since CI_c and CI_b are internal and external angle bisectors of $\angle C$, we have $\angle I_cCI_b = 90°$. Similarly, $\angle I_bBI_c = 90°$, and so quadrilateral I_cBCI_b is cyclic. Now we have

$$\angle NI_cB = \angle AI_cB = 180° - \angle I_cAB - \angle I_cBA = 90° - \frac{\angle C}{2}$$

Now since $\angle I_cNB = \angle ANB = \angle C$, we conclude that triangle I_cNB is isosceles and so N lies on perpendicular bisector of I_cB. By simialar arguments, N lies on the perpendicular bisector of I_bC, too. Therefore, N is the center this circle. $\square$



We keep using notations in lemma 2. Furthermore, dentoe by Γ the circumcirle of I_bCBI_c and let T_1 be the second intersection point of MI_b and Γ. Since M and N lie on the internal and external bisectors of $\angle A$, respectively, then MN must be a diameter of ω and so $\angle NTM = 90°$. Now T must be the midpoint of I_bT_1, because $NT \perp I_bT_1$ and N is the center of circle passing through T_1 and I_b. So NT is the perpendicular bisector of I_bT_1 and $T_1T = TI_b$. If T' is the foot of perpendicualr from T to line BC(radical axis of ω and Γ) then by *Casey's theorem* for point T and circles ω and Γ we get

$$P_\Gamma^T - P_\omega^T = 2ON.TT' \Rightarrow TI_b^2 = TI_b.TT_1 = 2R.TT' \quad (1)$$

According to the lemma 1 for triangle TBC, we deduce $2R.TT' = TB.TC \quad (2)$. (1) and (2) imply $TI_b^2 = TB.TC$, as desired.

3 . We prove that for any odd integer $m > 0$, there are infinitely many positive integers n, such that $b_{n+1} - b_n = m$. For sake of this reason we will use that following lemma

Lemma 1. *Let a be a positive integer which is not a perfect square. There exist infinitely many primes $p \equiv 3 \pmod 4$ such that a is quadratic non-residue modulo them.*

Proof. let $p_1 p_2 \cdots p_s$ be square free part of integer a, where p_i's are distinct primes. Now choose the quadratic non-residue r_1 modulo p_1 and set $r_2, \dots, r_s$ as quadratic residues modulo $p_2, \dots, p_s$, repectively. Now by *Chinese Reminder Theorem* and *Dirichlet's Theorem* one can find infinitely many prime p satisfying

$$p \equiv r_i \pmod{p_i} \qquad \text{for } i = 1, 2, \dots, s$$

Note that if one of primes name p_1 being equal to 2, then omit the first congruence and just add the criterion $p \equiv 3 \pmod 8$ and we are done. □

Now set $m = 2M + 1$ and consider the sequence $k^2 + M^2, k^2 + M^2 + 1, \dots, k^2 + M^2 + 2M + 1$. In this sequence the first and the last term is represented as sum of two squares. We will prove that there are infinitely many integers k such that only this two terms could be representable as sum of two squares. According to the lemma, there exist primes $p_1, p_2, \dots, p_{2M} \equiv 3 \pmod 4$ such that for each j, $M^2 + j$ is quadratic non-residue modulo p_j. Now we establish the following lemma

Lemma 2. *Let r be a quadratic residue modulo prime P and $(x, P) = 1$. Then there is integer x such that $0 < x < 2P$, $x^2 \equiv r \pmod P$ and $x^2 \not\equiv r \pmod{p^2}$.*

Proof. Let us note that $(x + P)^2 - x^2 = P^2 + 2xP \not\equiv 0 \pmod{P^2}$. Then one of x and $x + P$ satisfies the desired conditions. □

Now by this lemma there exist positive integers $n_1, n_2, \dots, n_{2M}$ such that $0 < n_j < 2p_j$ $(1 \le j \le 2M)$ and we have

$$n_j^2 \equiv -(M^2 + j) \pmod{p_j}, \quad x^2 \not\equiv -(M^2 + j) \pmod{p_j^2}$$

Next, let $k \equiv n_j \pmod{p_j^2}$ for $1 \le j \le 2M$. Then, the numbers $k^2 + M^2 + j$ are all divisible by p_j and not p_j^2. Now, since $p_j \equiv 3 \pmod 4$, $k^2 + M^2 + j$ cannot be written as sum of two squares. So we are done.

4 . We start with a lemma.

Lemma 1. *Let $a_1, a_2, \dots, a_n$ be n even integers. Then is a subsequence $i_1 < i_2 < \cdots < i_k$ of $1, 2, \dots, n$ such that $a_{i_1} + \cdots + a_{i_k}$ is divisible by $2n$.*

Proof. Consider the following integers

$$S_i = \frac{1}{2}(a_1 + a_2 + \cdots + a_i) \quad 1 \le i \le n$$

If there is some i such that $n | S_i$, we are done. So we assume that n divides none of S_i's and since there are n numbers there exist $i < j$ such that $S_i \equiv S_j \pmod n$. This implies

$$n | \frac{1}{2}(a_{i+1} + \cdots + a_j) \Rightarrow 2n | a_{i+1} + \cdots + a_j$$

and the proof is complete. □

Now for the main problem, we have two cases.

- n is an odd number. The answer is $k = 2n$. Firstly, note that for $k < 2n$ if we set $a_1 = a_2 = \cdots = a_k = 1$, we cannot choose an even number of a_i's with sum divisible by n. On the other hand, if $a_1, a_2, \ldots, a_{2n}$ are $2n$ integers, then we define $S_i = a_1 + \cdots + a_{2i}$ for $i = 1, 2, \ldots, n$. By arguments same as the proof of the lemma, there is some i such that $n|S_i$ or there exists $i < j$ such that $n|S_j - S_i$. In both cases we have found an even number of a_i's having sum divisible by n.

- $n = 2m$ is an even number. We claim that $k = n+1 = 2m+1$ is the answer. Note that for $k \leq 2m$, if we set $a_1 = a_2 = \cdots = a_{k-1} = 1$ and $a_k = 0$, it is not possible to select an even number of a_i's having a sum divisible by $2m$. On the other hand, suppose that $a_1, a_2, \ldots, a_{2m+1}$ are $2m+1$ arbitrary integers. Assume that $b_1, b_2, \ldots, b_s$ are the even numbers among a_i's and $c_1, c_2, \ldots, c_r$ are the odd numbers. Since $r+s = 2m+1$, exactly one of r of s is odd and the other is even. We suppose that r is odd and s is even (the other case is similar). Now look at the m numbers $a_1+a_2, a_3+a_4, \ldots, a_{r-2}+a_{r-1}, b_1+b_2, \ldots, b_{s-1}+b_s$. Note that all these sums are even, so reffering to the lemma, we can select some of them with sum divisible by $2m = n$. Since each of these numbers is sum of two member of a_i's we have find an even number of a_i's having a sum divisible by n, as desired.

5 . First we prove the lower bound. Throughout the solution, we denote by v_i and l_i the number of vertices and edges of the i-th connected component, respectively. Each lattice point has 4 adjacent lattice points, and so the degree of each vertex in a connected component is at most 4. Clearly, the degree of the rightmost vertex in each row of a connected component is at most 3 and similar statement holds for the leftmost vertex in each row, and the topmost and the bottommost vertex in each column of a connected component. Thus, we have at least 4 vertices of degree 3 and so

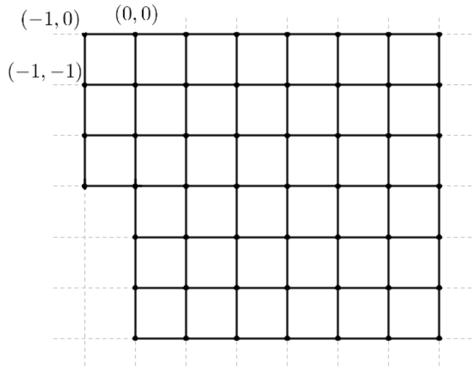
$$2l_i = \text{Sum of Degrees in the } i\text{-th Connected Component} \leq 4v_i - 4$$

Summing these inequlitiese for all connected components yields

$$2|A| \leq 4\sum v_i - 1 \Rightarrow \frac{l}{2} \leq |V(A)| - |J(A)|$$

And this implies the left inequality.
Now we go to the right inequality. We know that there are $(n+1)^2$ vertices and $2n(n+1)$ edges in an $(n+1)\times(n+1)$ array of points. Let n be the greastest positive integer such that $2n(n+1) \leq l$. So we can write $l = 2n(n+1) + k$, where $0 \leq k < 4(n+1)$ is an integer. We will introduce a graph with l edges, satisfying the right inequality. Firstly, consider an $n \times n$ square, which has $2n(n+1)$ edges. If $k > 0$, we start to add some edges to the square. Suppose that the point $(0,0)$ is the leftmost point of the top side of the square. We add the edge connecting $(-1,0)$ to $(0,0)$. Then edges connecting $(-1,-1)$ to $(-1,0)$ and $(0,-1)$. In the same manner, if we need more edges we choose the next edge on the square perimeter (counterclockwise) and add two edges to the graph using that edge. Let t be the number of sides in the square perimeter, which have been used in our process. Thus, we have added $t + 1 + [\frac{k-t}{2}]$ new vertices for constructing k edges. We know $t \leq \left\lceil \frac{k}{2(n+1)} \right\rceil$. Therefore, we have $(n+1)^2 + t + \frac{k-t}{2} + 1$ vertices in one connected component. So we must prove

$$(n+1)^2 + \frac{k+t}{2} < \frac{l}{2} + \sqrt{\frac{l}{2}} + 1 = n(n+1) + \frac{k}{2} + \sqrt{n(n+1) + \frac{k}{2}} + 1$$



Which is in turn equivalent to

$$n^2 + tn + \frac{t^2}{4} < n^2 + n + \frac{k}{2}$$

If $t = 1$, the inequality is obvious. And if $t = 2$, the inequality will be $n + 1 \leq \frac{k}{2}$ which is again true because $2 = t = \lceil \frac{k}{2(n+1)} \rceil$ and so $k \geq 2(n + 1)$. So the assertion is proved.

6 . An inversion with center O and constant r^2 (r is the radius of the circumcircle of quadrilateral), takes the circle with diameter OS to the line EF and the circumcircle of the triangle OAM to the line AM. Furthermore, since XO is the bisector of $\angle AXB$ and $OA = OB$, we get the quadrilateral $AXBO$ is cyclic and so $OE'.OX = r^2$. Simillarly, $OF'.OY = r^2$. Therefore, under this inversion the circumcircle of triangle OXY maps to the line $E'F'$. So it is enough to prove that the line EF, $E'F'$ and AM are concurrent.
Denote by ω, ω_E and ω_F the incircles of quadrilateral $ABCD$, triangle EAB and triangle FAD, respectively. Now E and E' are the external and internal homothetic centers of ω_E and ω, respectively. In the same manner F and F' are the external and internal homothetic centers of ω_F and ω, respectively. Therefore, EF and $E'F'$ meet at the internal homothetic center of ω_F and ω_E, say R. On the other hand, AM is the bisector of $\angle EAB = \angle FAD$ and so is the line of centers of ω_F and ω_E. Thus, R the internal homothetic center of ω_E and ω_F lies on AM. So we have proved that the lines EF, $E'F'$ and AM meet at K, which completes the proof.

7 . First note that $(a + c)(\frac{1}{a} + \frac{1}{c}) = \frac{a}{c} + \frac{c}{a} + 2 = \left(\sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}}\right)^2$ and so

$$\sqrt{(a + c)(\frac{1}{a} + \frac{1}{c})} = \sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}}$$

Similarly $\sqrt{(b + d)(\frac{1}{b} + \frac{1}{d})} = \sqrt{\frac{b}{d}} + \sqrt{\frac{d}{b}}$.
On the other hand, for the left hand side of inequality, because of the hypothesis $\sum_{cyc} \frac{1}{ab} = 1$, we have

$$abcd = (a + c)(b + d)(\frac{1}{a} + \frac{1}{c})(\frac{1}{b} + \frac{1}{d}) = \left(\sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}}\right)^2 \left(\sqrt{\frac{b}{d}} + \sqrt{\frac{d}{b}}\right)^2$$

So we must prove $X^2Y^2 + 16 \geq 8(X + Y)$, where $X = \sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}}$ and $Y = \sqrt{\frac{b}{d}} + \sqrt{\frac{d}{b}}$. Note that by AM-GM inequality $X \geq 2$ and $Y \geq 2$. Therefore, $(X - 1)(Y - 1) \geq 1$ and consequently, $XY \geq X + Y$. This implies the assertion, because $(XY)^2 + 16 \geq 8XY \geq 8(X + Y)$.

8 . Let E_1 be the reflection of E with respect to the midpoint of BC and X the intersection point of AE_1 and EA_1. We claim that $IX \parallel BC$. For this reason, we have (Suppose that R is the radius of circumcircle of ABC and $\angle B \geq \angle C$).

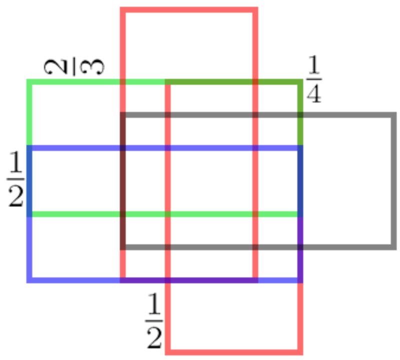
$$\frac{AX}{XE_1} = \frac{AA_1}{EE_1} = \frac{AA_1}{AC - AB}$$
$$\frac{AI}{IE} = \frac{AB}{BD} = \frac{AC}{CD} = \frac{AB + AC}{BC}$$

So referring to the *Thales' Theorem*, we must prove $AA_1.BC = AC^2 - AB^2$.
We have $AA_1 = BC - 2BH = BC - 2AB.\cos(\angle B)$, where H is the foot of perpendicular from A to BC and on the other hand, by *The Law of Cosines* we have $AC^2 - AB^2 = BC^2 - 2AB.BC\cos(\angle B)$. Therefore, the claim is proved.
Now since the quadrilateral $DEAT$ is cyclic and $AA_1 \parallel IX$, we get that the quadrilateral $IATX$ is cyclic. Also, since pairs (E, E_1) and (A, A_1) are symmetric with respect to the perpendicular bisector of the side BC, we have $XE = XE_1$ and so

$$\angle ATI = \angle AXI = \angle XE_1E = \angle XEE_1 = \angle IXE = \angle TAI$$

Thus, $IT = IA$.

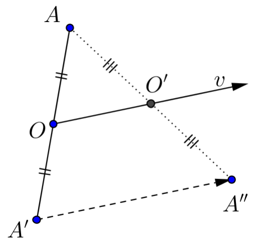
9 . The answer is 5. The picture below shows five 1×2 rectangles mutually intersecting at rectangles with unit area.



It is enough to show that there are no six 1×2 horizontal or vertical rectangles with such a property. Assume that there are 6 such rectangles.
Note that it is obvious that the intersection of two rectangles with parallel sides is again a rectangle. In solution first we show that there are at most three vertical and three horizontal rectangles. At the end we show that there are not three horizontal and three vertical rectangles satisfying the intersection property. For these we prove some lemmas.

Lemma 1. *Composition of a reflection with respect to a point and a translation in the plane, is a reflection, too.*

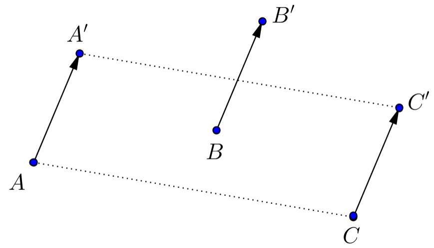
Proof. Let O be the center of the reflection, and $\vec{v}$ be the vector of the traslation. Then by Thales' Theorem it is easy to see the composition is a reflection with respect to point O' such that $OO' = \frac{1}{2}\vec{v}$.



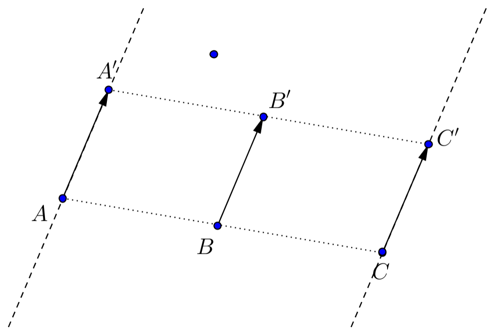
□

Lemma 2. *Let K be a strictly convex shape in the plane. For any real number $r > 0$ and any direction in the plane, there are at most two slices of K with length r parallel to the given direction. (We mean by a slice of K, intersection of a line in the plane with K. Obvoiusly each slice is a line segment.)*

Proof. Assume to the contrary there are three slices of length r parallel to the given direction, say AA', BB' and CC'. Suppose that $\overrightarrow{AA'} = \overrightarrow{BB'} = \overrightarrow{CC'}$ and BB' lies between two others.
If B or B' is outside the parallelogram $AA'C'C$, then the other point lies in the interior of K and therefore the intersection of line containing BB' has length greater than r.



If B lies on AC and B' lies on $A'C'$, then since K is strictly convex there must be a point of K between lines AA' and CC' outside the parallelogram, contradicts with the length r of slice containing BB'.



□

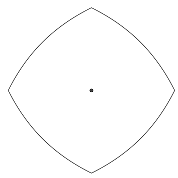
Lemma 3. *There are no four horizontal or four vertical rectangles satisfying the conditions.*

Proof. Assume that there are at least 4 horizontal rectangles. By a horizontal scaling with factor $\frac{1}{2}$ we go to the case that there are 4 unit squares with sides parallel to the axes, such that mutually intersecting at rectangles of area $\frac{1}{2}$.

Every unit square determined uniquely by its center. It is easy to see that two unit squares with centers (x, y) and (x', y') interesect at a rectangle of area $\frac{1}{2}$ iff

$$(1 - |x - x'|)\,(1 - |y - y'|) = \tfrac{1}{2} \quad (*)$$

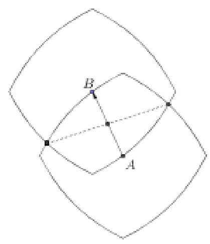
Locus of points (x', y') satisfying $(*)$ (for fixed (x, y)) is boundary of a strictly convex shape in the plane (Look at the shape below). We call such shape an oval with center (x, y).



Suppose that $A = (x_A, y_A)$,$B = (x_B, y_B)$, $C = (x_C, y_C)$ and $D = (x_D, y_D)$ are the centers of these 4 squares such that $y_A \leq y_B \leq y_C \leq y_D$. Note that oval of B is a translation of oval of A with vector $\overrightarrow{AB}$. By lemma 2 these two ovals intersect in at most two points (Note that these ovals are strictly convex because they are intersection of 4 strictly convex shapes). These two points must be C and D. Since A is center of symmetry of oval of A, by lemma 1, these two intersection points are symmetric with respect to the midpoint of AB. Note that $y_B > y_A$, because if $y_A = y_B$, y_C or y_D become less than $y_A = y_B$ contradicts with the minimality of y_A. Now we have

$$y_C \leq \frac{y_C + y_D}{2} = \frac{y_A + y_B}{2} < y_B$$

Contradicts with $y_B \leq y_C$.

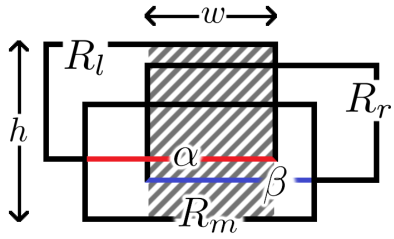


$\square$

Lemma 4. *Every horizontal rectangle intersects every vertical rectangle in a 1×1 square.*

Proof. Intersection of such rectangles is a rectangle with sides at most 1. As intersection of every two rectangles has a unit area, both sides must be unit. $\square$

Using lemma 3, we get that 3 rectangles are vertical (V) and other 3 are horizontal (H). Consider three horizontal rectangles with their centers sorted by x. Fix the middle one and call it R_m. Call left and right rectangles R_l and R_r respectively. Without loss of genrality, we can assume that R_r is not upper than R_l. As in the figure below R_m, R_r intersect in a rectangle with length β and R_m, R_l intersect in a rectangle with length α. Left side of R_r, right side of R_l, top side of R_l and bottom side of R_m construct a rectangle named R_H. Width of R_H is w_H and height of it is h_H. These variables are based on rectangles in (H), we can define the same variables in (V) named R_V, w_V and h_V.



Referring to the lemma 4, every two rectangles, one in (V) and one in (H) intersect in a unit square. So every rectangle in (V) must have a horizontal side in R_H this means $h_V \leq w_H$. Similarly we can show that $h_H \leq w_V$. Without loss of generality, assume that $w_V \leq w_H$. This yields:

$$h_H \leq w_H$$

Now we can calculate all variables in terms of α, β (note that $1 \leq \alpha \leq \beta \leq 2$).
Intersection of R_l, R_r is a rectangle with sides $\alpha + \beta - 2, 1 - \frac{1}{\beta} + \frac{1}{\alpha}$ so:

$$(\alpha + \beta - 2) \times (1 - \frac{1}{\beta} + \frac{1}{\alpha}) = 1(1)$$

We had $h_H \leq w_H$. $h_H = 2 - \frac{1}{\alpha}$ and $w_H = \alpha + \beta - 2$ so:

$$2 - \frac{1}{\alpha} \leq \alpha + \beta - 2(2)$$

Now we change our variables to have simpler results. Assume that:

$$\begin{array}{ll} \alpha = a + 1 & (0 \leq a \leq 1) \\ \beta = b + 1 & (0 \leq b \leq 1) \end{array}$$

From (1) we have:

$$\begin{array}{ll} & (b+a)(1 - \frac{1}{b+1} + \frac{1}{a+1}) = 1 \\ \Rightarrow & b - \frac{b}{b+1} + \frac{b}{a+1} + a - \frac{a}{b+1} + \frac{a}{a+1} = 1 \\ \Rightarrow & b^2 + b - b + \frac{b(b+1)}{a+1} + ab + a - a + \frac{ba}{a+1} + \frac{a}{a+1} = b + 1 \\ \Rightarrow & Q(b) = b^2(1 + \frac{1}{1+a}) + ba - \frac{1}{a+1} = 0 \end{array}$$

From (2) we conclude that the above quadratic polynomial must have a root in the interval $[0, 1]$ satisfying the following condition:

$$2 - \frac{1}{a+1} \leq a + b \Leftrightarrow \underbrace{1 - \frac{a^2}{a+1}}_{k} = 2 - a - \frac{1}{a+1} \leq b$$

$a \in [0, 1]$ so $k \in [0, 1]$. Coefficients of Q are positive so Q is ascending so to prove that b cannot be a root of Q, it's enough to show that $Q(k) > 0$. $Q(k) \geq k^2 + ka - \frac{1}{a+1}$ so we prove that $k^2 + ka - \frac{1}{a+1} \geq 0$.

$$\begin{array}{ll} & k^2 + ka - \frac{1}{a+1} \geq 0 \\ \Leftrightarrow & (1 - \frac{a^2}{a+1})^2 + (1 - \frac{a^2}{a+1})a - \frac{1}{a+1} \geq 0 \\ \Leftrightarrow & (a + 1 - a^2)^2 + a(a+1)(a + 1 - a^2) \geq a + 1 \\ \Leftrightarrow & (a + 1 - a^2)(2a + 1) \geq a + 1 \end{array}$$

Equivalently,

$$2a(a+1) - a^2(2a+1) = -2a^3 + a^2 + 2a = -a(2a^2 - a - 2) \geq 0$$

$a \in [0,1]$ so the last inequality is obvious because $2a^2 - a - 2 = (a^2 - a) + (a^2 - 1) - 1$. Thus we showed that there is no a with those algebraic properties. This means there is no α with that geometric properties and this is a big contradiction because α was a length in our figure. Now we can say there are no six 1×2 rectangles mutually intersecting at rectangles with unit area. This yields there are no $n > 5$ such rectangles.

10 . We denote the feet of altitudes of vertices A, B and C by D, E and F, respectively. Firstly, we prove that if X, Y and Z are collinear then the length of the tangent from A to the nine-point circle of triangle ABC (which we denote by ω) is equal to the sum of the lengths of tangents from vertices B and C to this circle.

$$P_\omega^A = \frac{1}{2}AE.AC = \frac{1}{2}AF.AB = \frac{1}{2}AX^2 = \frac{1}{2}AY^2$$

So the length of the tangent from A to ω is $\frac{\sqrt{2}}{2}AX = \frac{\sqrt{2}}{2}AY$. Similarly, the lengths of tangents from vertices B and C are equal to BY and CX, respectively. If X, Y and Z lie on a common line we have $YZ + ZX = YX$ and $\angle XZC + \angle YZB = 90°$ (1). Since $CX^2 = CE.CA = CD.CB = CZ^2$, we get $CX = CZ$. In the same manner, we infer $AX = AY$ and $BY = BZ$. Using (1) we can deduce $\angle XCZ + \angle ZBY = 180°$. The quadrilateral $FZCB$ is cyclic and so $\angle ABZ = \angle FCZ$. Therefore,

$$\angle YBA + \angle FCX = 90° \Rightarrow \angle YBA + (90° - \angle A) + \angle ACX = 90°$$

Thus, $\angle YAX = 90°$ and so $\angle AXY = 45°$. From these results, we get $\angle ZBY = \angle ZCX = 90°$. Hence, $\sqrt{2}YB + \sqrt{2}CX = \sqrt{2}AX$ and so $YB + CX = AX$, which is what we wanted to show.
For the converse we present the following lemma.

Lemma 1. *Let C_1 and C_2 be two perpendicular circles meeting each other at two points A and B. Then, There are exactly two points like T on the line AB satisfying the following property. If TY and TX are tangents from T to C_1 and C_2, respesctively, suth Y and X are in two different sides of AB. We have A, Y and X are on a common line. Furthermore, these two points are symmetric with respect to the line of centers of C_1 and C_2 and power of them with respect to C_1 and C_2 are both equal to $(R_1 + R_2)^2$, where R_i is the radius of C_i.*

Proof. First note that if the point T has this property, we have

$$\frac{1}{2}\overset{\frown}{AY} = \angle TYA = \angle TYX = \angle TXY = \angle TXA = \frac{1}{2}\overset{\frown}{AX}$$

And since $\angle O_1AO_2 = 90°$ (O_i is the center of C_i), we get

$$\overset{\frown}{AY} = \overset{\frown}{AX} = \frac{1}{2}\overset{\frown}{AY} + \frac{1}{2}\overset{\frown}{AX} = \angle XAO_2 + \angle YAO_1 = 180° - \angle O_1AO_2 = 90°$$

So we must have $\overset{\frown}{AY} = \overset{\frown}{AX} = 90°$.
Now for the existence of such T, consider points Y and X on C_1 and C_2, respectively,

such that $\overset{\frown}{AY} = \overset{\frown}{AX} = 90°$. Note that we can choose such points in a uniqe way in one side of O_1O_2. It is easy to see Y, A and X are collinear. Therefore, if T is the intersection point of the tangent to C_1 at Y and the tangent to C_2 at X, we infer

$$\angle TYX = \overset{\frown}{YA} = \overset{\frown}{XA} = \angle TXY$$

So $TY = TX$. Thus, T lies on the radical axis of C_1 and C_2.
On the other hand, in the hexagon TXO_2AO_1Y we have

$$\angle TYO_1 = \angle YO_1A = \angle O_1AO_2 = \angle AO_2X = \angle O_2XT = 90°$$

Hence, $\angle YTX = 90°$, and obviously $TX = AO_2 + YO_1 = R_2 + R_1$, as desired. $\square$

Now for the main problem note that circle ω_1 with center B and radius BZ is perpendicular to the circle ω_2 with center C and radius CZ, because $\angle BZX = 90°$. Furthermore, since $\angle AYB = \angle AXC = 90°$, AY is tangent to ω_1 at Y and AX is tangent to ω_2 at X. on the other hand $AY = AX = BY + CX$, so according to the lemma the point having this property is unique in one side of BC. Therefore, again by lemma X, Y and Z are collinear.

11 . We claim that there exists a messy permutation of $\{1, 2, \ldots, n\}$ if and only if n is a power of 2.
First suppose that there is some messy permutation $(a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$. There must be some j such that $a_j \equiv 2a_2 - a_1 \pmod{n}$. If $j \geq 3$, then $n | a_j + a_1 - 2a_2$, contradicts because we have assumed that the permutation is messy. If $j = 2$, then $a_2 \equiv a_1 \pmod{n}$ which is not possible. So $j = 1$ and $2a_2 \equiv 2a_1 \pmod{n}$. Now if n is odd, we get $a_2 \equiv a_1 \pmod{n}$, which is again impossible. Therefore, we get there is no messy permutations for odd values of n.
On the other hand, suppose that $(a_1, a_2, \ldots, a_n)$ is a messy permutation and d is a divisor of n. Set $m = \frac{n}{d}$. We remove the numbers in the permutation, which are not divisible by m to get $(a_{i_1}, a_{i_2}, \ldots, a_{i_d})$ where $m | a_{i_j}$ for each j. Now $(\frac{a_{i_1}}{m}, \frac{a_{i_2}}{m}, \ldots, \frac{a_{i_d}}{m})$ is a permutation of $\{1, 2, \ldots, d\}$ modulo d. We claim that this permutation is messy. Assume to the contrary, there are $p < q < r$ among i_k's such that $d | \frac{a_p}{m} + \frac{a_r}{m} - 2\frac{a_q}{m}$. This implies $n = dm | a_p + a_r - 2a_q$, which is impossible because the first permutation was assumed to be messy.
Therefore, if there is a messy permutation for some integer n, n is not divisible by an odd number other than 1 and so n must be a power of 2.
Finally, we construct a messy permutation of $\{1, 2, \ldots, 2^t\}$ by induction on t. Obviously, $\{1, 3, 2, 4\}$ is a messy permutation for $t = 2$. Suppose that $(a_1, a_2, \ldots, a_{2^t})$ is a messy permutation for $n = 2^t$. Now consider the sequence

$$(b_1, \ldots, b_{2^{t+1}}) = (2a_1 + 1, \ldots, 2a_{2^t} + 1, 2a_1, \ldots, 2a_{2^t})$$

This sequence is a permutation for $n = 2^{t+1}$. We claim this permurtion is messy. If $2^{t+1} | b_i + b_k - 2b_j$ for some $i < j < k$, then $k \geq 2^t + 1$ and $i \leq 2^t$ because $(a_1, a_2, \ldots, a_{2^t})$ was assumed to be messy. Now $b_i + b_k - 2b_j$ is an odd number and so can not be divisible by 2^{t+1}. This contradiction shows that $(b_1, \ldots, b_{2^{t+1}})$ is a messy permutation and so the proof is complete.

12 . $a^3+b^3 \geq ab(a+b)$. So we have

$$\frac{\sqrt{a^3+b^3}}{ab+1} \geq \frac{\sqrt{ab(a+b)}}{ab+1}$$

After substitution $a=\frac{1}{x}$, $b=\frac{1}{y}$ and $c=\frac{1}{z}$, it is sufficient to prove that for any positive real numbers x, y and z such that $xy+yz+zx=1$, we have

$$\sum_{cyc}\frac{x}{1+x^2} \leq \frac{1}{3\sqrt{2xyz}}\sum_{cyc}\frac{\sqrt{x+y}}{1+xy} \quad (*)$$

Since $xy+yz+zx=1$, for the left hand side we have

$$\frac{x}{x^2+1} = \frac{x}{x^2+xy+xz+yz} = \frac{x}{(x+y)(x+z)} = \frac{x(y+z)}{(x+y)(y+z)(z+x)}$$

And thus,

$$\sum_{cyc}\frac{x}{1+x^2} = \frac{1}{(x+y)(y+z)(z+x)}\sum_{cyc}(xy+xz) = \frac{2}{(x+y)(y+z)(z+x)}$$

Therefore, $(*)$ is equivalent to

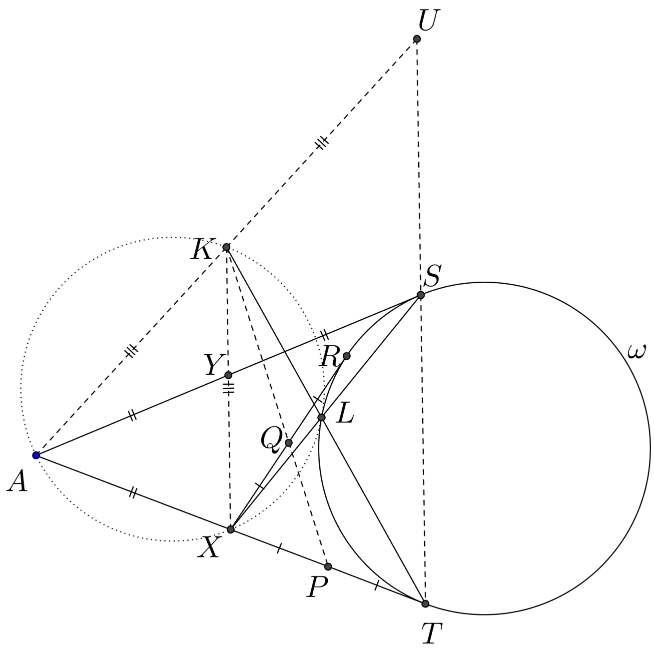
$$\frac{2}{(x+y)(y+z)(z+x)} \leq \frac{1}{3\sqrt{2xyz}}\sum_{cyc}\frac{\sqrt{x+y}}{1+xy}$$

On the other hand, by *Cauchy-Schwarz inequality*, we have $(1+x^2)(1+y^2) \geq (1+xy)^2$ and so

$$\begin{aligned}
\frac{1}{3\sqrt{2xyz}}\sum_{cyc}\frac{\sqrt{x+y}}{1+xy} &\geq \frac{1}{3\sqrt{2xyz}}\sum_{cyc}\frac{\sqrt{x+y}}{\sqrt{(1+x^2)(1+y^2)}} \\
&= \frac{1}{3\sqrt{2xyz}}\sum_{cyc}\frac{\sqrt{x+y}}{\sqrt{(x+y)^2(x+z)(y+z)}} \\
&= \frac{1}{\sqrt{2xyz(x+y)(y+z)(z+x)}} \\
&\geq \frac{2}{(x+y)(y+z)(z+x)}
\end{aligned}$$

The last inequality was true because we have $(x+y)(y+z)(z+z) \geq 8xyz$, which is an immidiate consequence of AM-GM inequality.

13 . If we consider A and X as circles with radius zero, then K is the radical center of A, X and ω. Therefore, K lies on the perpendicular bisector of AX, and so $\angle STA = \angle KXA = \angle KAT$. Let U be the intersection point of AK and TS.



By *Thales's Theorem* K is the midpoint of AU. On the othe hand $\angle ATS = \angle AST$. Thus, triangles AUT and ATS are similar. Since TK and SX are medians of these triangles, we infer that $\angle XSA = \angle STK$. Hence, L lies on ω and also $AXLK$ is a cyclic quadrilateral. Now we have

$$RT \parallel PK \Rightarrow \angle RTK = \angle QKL$$

On the other hand, since XR is tangent to ω, we have $\angle XRL = \angle RTK$.Therefore, $\angle XRL = \angle QKL$ and so four points K, R, L and Q lie on a common circle.

14 . We start with a lemma.

Lemma 1. *For any $n \geq 4$, we have $a_{n-2}|a_n$.*

Proof.

$$a_{n-2}\Big|[a_{n-1}, a_{n-2}], a_{n-2}\Big|[a_{n-2}, a_{n-3}]$$

So a_{n-2} divides $a_n = [a_{n-1}, a_{n-2}] - [a_{n-1}, a_{n-2}]$. □

According to the lemma 1, we know that $a_{n-2}|[a_n, a_{n-1}]$ and so a_{n+1} is divisible by a_{n-2}. We define

$$c_n = \frac{a_{n+1}}{[a_{n-1}, a_{n-2}]} \in \mathbb{Z}$$

If there is some n, such that $a_n \leq 0$, we are done. Therefore, we may suppose that for all positive integers n, $a_n \in \mathbb{N}$. We claim that $c_n = c_{n-1} - 1$, for any $n \geq 5$. If we prove the claim, we get $c_n = c_4 - (n-4)$ and so $c_{c_4+5} = 0$, which implies $a_{c_4+5} = 0$. Note that

$$c_4 + 1 = \frac{a_5}{[a_3, a_2]} + 1 = \frac{[a_4, a_3]}{[a_3, a_2]} \leq \frac{a_4 a_3}{[a_3, a_2]} = a_3\Big(\frac{a_4}{[a_3, a_2]}\Big) \leq a_3$$

Therefore, $c_4 + 5 \leq a_3 + 4$. So proving the claim completes our solution.

$$\begin{aligned}
c_n = c_{n-1} - 1 \quad &\Leftrightarrow \quad \frac{a_{n+1}}{[a_{n-1}, a_{n-2}]} = \frac{a_n}{[a_{n-2}, a_{n-3}]} - 1\\
&\Leftrightarrow \quad \frac{[a_n, a_{n-1}]}{[a_{n-1}, a_{n-2}]} = \frac{[a_{n-1}, a_{n-2}]}{[a_{n-2}, a_{n-3}]} - 1\\
&\Leftrightarrow \quad \frac{[a_n, a_{n-1}]}{[a_{n-1}, a_{n-2}]} = \frac{a_{n+1}}{[a_{n-2}, a_{n-3}]}\\
&\Leftrightarrow \quad \frac{(a_n, a_{n-1})}{a_{n-1}} = \frac{(a_n, a_{n+1})}{[a_{n-1}, a_{n-2}]}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(a_{n+1}, a_n) &= \Big([a_n, a_{n-1}] - [a_{n-1}, a_{n-2}], a_n\Big) = \Big([a_{n-1}, a_{n-2}], a_n\Big) = \Big([a_{n-1}, a_{n-2}], [a_{n-2}, a_{n-3}]\Big)\\
(a_n, a_{n-1}) &= \Big([a_{n-1}, a_{n-2}] - [a_{n-2}, a_{n-3}], a_{n-1}\Big) = \Big([a_{n-2}, a_{n-3}], a_{n-1}\Big)
\end{aligned}$$

So we must prove that

$$\frac{\Big([a_{n-2}, a_{n-3}], a_{n-1}\Big)}{a_{n-1}} = \frac{\Big([a_{n-1}, a_{n-2}], [a_{n-2}, a_{n-3}]\Big)}{[a_{n-1}, a_{n-2}]}$$

By dividing both sides by $[a_{n-2}, a_{n-3}]$, we get that the assertion is equivalent to

$$\Big[[a_{n-2}, a_{n-3}], a_{n-1}\Big] = \Big[[a_{n-1}, a_{n-2}], [a_{n-2}, a_{n-3}]\Big]$$

Which is true. So we are done.

15 . We start with a lemma.

Lemma 1. *Suppose that are given two sequences $a_1 > a_2 > \cdots > a_m$ and $b_1 < b_2 < \cdots < b_m$ of positive real numbers as lengths of segments. We start from the origin and at the step i ($1 \le i \le m$), we go up with a segment of length a_i and then we go right with a segment of length b_i. Suppose that l is the line connecting origin to the endpoint of the last segment. Show that all segments lie in the top of line l.*

Proof. Assume to the contrary that there is a first segment which intersects l, say l'. Obviously, l' is a horizontal segment because if it was vertical, it couldnot be the first segment intersecting l. Suppose that $l' = b_i$. We denote by O and X the origin and the endpoint of the segment b_i, respectively. Since X lies below the line l, we get that the slope of OX is less than the slope of l.

$$\frac{\sum_{j=0}^{i} a_j}{\sum_{j=0}^{i} b_j} = \text{Slope of } OX < \text{Slope of } l = \frac{\sum_{j=0}^{m} a_j}{\sum_{j=0}^{m} b_j}$$

or equivalently

$$\frac{\sum_{j=0}^{i} a_j}{\sum_{j=0}^{m} a_j} < \frac{\sum_{j=0}^{i} b_j}{\sum_{j=0}^{m} b_j} \qquad (*)$$

Note that since b_i's are increasing we have

$$i\sum_{j=i+1}^{m} b_j > (m-i)\sum_{j=0}^{i} b_j \Rightarrow i\sum_{j=0}^{m} b_j > m\sum_{j=0}^{i} b_j$$

This implies that the right hand side of $(*)$ is less than $\frac{i}{m}$. Similar arguments show that the left hand side of $(*)$ is greater than $\frac{i}{m}$ (note that a_i's assumed to be decreasing). This contradiction established the lemma. $\square$

For the main problem suppose that we have divided both a_i's and b_i's into two sets having equal sums.

- $\sum_{i=1}^{k} a_i = \sum_{i=k+1}^{n} a_i$ such that $a_k \leq a_{k-1} \leq \cdots \leq a_2 \leq a_1, a_n \leq a_{n-1} \leq \cdots \leq a_{k+1}$, where $a_1 \geq a_{k+1}$.

- $\sum_{i=1}^{s} b_i = \sum_{i=s+1}^{n} b_i$ such that $b_s \geq b_{s-1} \geq \cdots \geq b_2 \geq b_1, b_n \geq b_{n-1} \geq \cdots \geq b_{s+1}$, where $b_1 \leq b_{s+1}$.

There is no loss of generality in assuming that $k \leq \frac{n}{2} \leq s$. We will use the following algorithm for constructing the polygon.

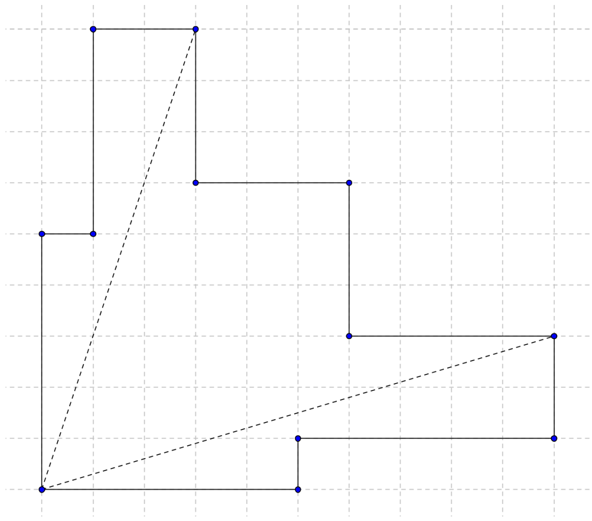
- We start from the origin (C_0 is the origin).

- For $1 \leq i \leq s$, at the i-th step, we start from $C_{2(i-1)}$, then we go up in a segment of length a_i to get C_{2i-1} and then right in a segment of length b_i to get C_{2i}.

- For $s < i \leq k$, at the i-th step, we start from $C_{2(i-1)}$, then we go down in a segment of length a_i to get C_{2i-1} and then left in a segment of length b_i to get C_{2i}.

  Note that since the lengths of a_i's and b_i's are assumed to be monotone, according to the lemma, first $2s$ sides of polygon lie in the top of the segment C_0C_{2s} and the next $2(k-s)$ sides lie in the bottom of $C_{2s}C_{2k}$.

- For $k < i \leq n$, at the i-th step, we start from $C_{2(i-1)}$, then we go down in a segment of length a_i to get C_{2i-1} and then left in a segment of length b_i to get C_{2i}.
  Again because of the lemma, all these sides lie in the bottom of the segment $C_{2k}C_{2n}$.



Because $\sum_{i=1}^{k} a_i = \sum_{i=k+1}^{n} a_i$ and $\sum_{i=1}^{s} b_i = \sum_{i=s+1}^{n} b_i$, we return to the origin at the end of the algorithm and so we get a polygon. Now it suffices to prove that this polygon is simple.
Obviously, in each part of algorithm the segments can not intersect each other. On the other hand, according to the lemma, two segments from two different parts of algorithm can intersect only if the segment connecting the first and the last vertex of each part lie on the same line. But this is possible only if $k = s$ (it means that there is not any side in the second part of algorithm). In this case we have intersection on the connecting line only if there are some $1 \leq t \leq k$ and $k+1 \leq l \leq n$, such that

$$a_1 + \cdots + a_t = a_{l+1} + \cdots + a_n, \quad a_{t+1} + \cdots + a_k = a_{k+1} + \cdots + a_l \tag{1}$$
$$b_1 + \cdots + b_t = b_{l+1} + \cdots + b_n, \quad b_{t+1} + \cdots + b_k = b_{k+1} + \cdots + b_l \tag{2}$$

Since $\frac{a_1+\cdots+a_t}{t} \geq \frac{a_{t+1}+\cdots+a_k}{k-t}$ and $\frac{a_{l+1}+\cdots+a_n}{n-l} \leq \frac{a_{k+1}+\cdots+a_l}{l-k}$, from (1) we get

$$\frac{t}{k-t}(a_{t+1} + \cdots + a_k) \leq a_1 + \cdots + a_t = a_{l+1} + \cdots + a_n \leq \frac{n-l}{l-k}(a_{k+1} + \cdots + a_l) \tag{3}$$

So $\frac{t}{k-t} \leq \frac{n-l}{l-k}$. Similar arguments for b_i's instead of a_i's imply $\frac{n-l}{l-k} \leq \frac{t}{k-t}$. Therefore, $\frac{t}{k-t} = \frac{n-l}{l-k}$. Thus, all inequlities in (3) are equalities. So

$$a_t \leq \frac{a_1 + \cdots + a_t}{t} \leq \frac{a_{t+1} + \cdots + a_k}{k-t} \leq a_{t+1} \leq a_t$$

Therefore, $a_1 = a_2 = \cdots = a_k$ (say this common value a) and similarly, $a_{k+1} = \cdots = a_n$ (say this common value a'). Now since $a_1 + \cdots + a_k = a_{k+1} + \cdots + a_n$, we have $ka = (n-k)a'$. In the same manner, $b_1 = b_2 = \cdots = b_k$ (say this common value b), $b_{k+1} = \cdots = b_n$ (say this common value b') and $kb = (n-k)b'$. But since $a = a_1 \geq a_{k+1} = a'$ and $b = b_1 \leq b_{s+1} = b'$, we have $1 \leq \frac{a}{a'} = \frac{n-k}{k} = \frac{b}{b'} \leq 1$. Hence, $a = a'$ and $b = b'$. It means that all the horizontal segments have equal lengths and all the vertical segments have equal lengths, which contradicts problem conditions.

16 . Firstly, we present an obvious lemma.

Lemma 1. *If for two triangles ABC and $A'B'C'$, we have $AB \leq A'B'$, $AC \leq A'C'$ and $\angle BAC \leq B'A'C'$, then ABC can be placed into $A'B'C'$.*

Let XYZ be a equilateral triangle with center O. We denote the radius of circumcircle of this triangle and the length of its altitudes by R and h, respectively (Clearly, $h = \frac{3}{2}R$). Let P be a point close to O such that $OP < \frac{h-R}{2}$ and angles $\angle OXP$, $\angle OYP$ and $\angle OZP$ are all in the interval $(0, 30°)$. We claim that if A puts the points X, Y, Z, O and P; he would be able to construct the desired triangles.
Now we go to some cases, according to the place of sixth point, say Q.

- If Q lies inside the triangle, then OQP is located completely inside XYZ.

- If Q lies inside the circumcircle of XYZ but outside of the triangle, we have

$$QP < QO + OP \leq R + OP < h < XY, \quad QO \leq R < h < XZ$$

  On the other hand, since Q is outside of XYZ, $OQ \geq h - R > 2OP$ and $QP > OQ - OP > OP$. So OP is the shortest side of triangle OPQ. Therefore, $\angle OQP \leq 60°$. Now since $QP < XY$, $QO < XZ$ and $\angle OQP \leq 60° = \angle YXZ$, according to the lemma, triangle OQP can be placed inside XYZ.

- If Q lies outside the circumcircle of XYZ, triangles QXY, QYZ and QZX cover the triangle XYZ. So one of them, for example QYZ contains O. Therefore, triangle OYZ is located inside QYZ. On the other hand, since OPX can be placed into OYZ (P is in one of congruent triangles OXY, OYZ or OZX), we obtain that OPX can be placed into QYZ.

17 . We claim that the polynomial $P(x) = x(x+1)\cdots(x+d-1)$ of degree d satisfies the problem condition. Note that P is monic and $P(1) = d! \neq 0$. On the other hand, it is easy to verify by induction that if sequence a_n satisfies the recursive relation mentioned in the problem statement, then we have

$$a_m = (-1)^{m-1}\binom{d+1}{m-1}a_1 \quad m > 1$$

So if $m \geq d+2$, then $a_m = 0$. It means that P satisfies all conditions in the problem.
We call a polynomial $P(x)$ which satisfies the conditions in the problem, except it may be not monic *a good polynomial.*
Next, For uniqueness we may need the following lemmas.

Lemma 1. *For every good polynomial $P(x)$, we have $P(0) = 0$.*

Proof. Starting with an initial value a_1, using the recursive reltion mentioned in the probles, we get a sequence a_n, which is zero for large values of n. Let s be the greatest natural number such that $a_s \neq 0$. Now for $n > s$ we have

$$P(n)a_1 + P(n-1)a_2 + \cdots + P(n-s+1)a_s = 0 \quad n > s$$

Define the polynomial $Q(x) = P(x)a_1 + P(x-1)a_2 + \cdots + P(x-s+1)a_s$. Q is zero for infinitely many values of x, consequently it must be always equal to zero. In particular, $Q(s-1) = 0$. Hence,

$$P(s-1)a_1 + P(s-2)a_2 + \cdots + P(1)a_{s-1} + P(0)a_s = 0 \Rightarrow P(0)a_s = 0 \Rightarrow P(0) = 0$$

□

Lemma 2. *Let $P(x)$ be a good polynomial of degree d, then $h(x) = P(x) - P(x-1)$ is a good polynomial of degree $d-1$.*

Proof. Let $\{b_n\}_{n\geq 1}$ be a sequence of real numbers satisfying the recursive relation for polynomial h.

$$h(n)b_1 + h(n-1)b_2 + \cdots + h(1)b_n = 0 \quad n > 1$$

So in terms of P

$$P(n)(b_1 - 0) + P(n-1)(b_2 - b_1) + \cdots + P(1)(b_n - b_{n-1}) = 0$$

Since P is a good polynomial, there is k such that $b_k - b_{k-1} = b_{k+1} - b_k = \cdots = 0$. Thus, the sequence b_n is eventually constant ($b_{k-1} = b_{k+1} = b_{k+2} = \cdots = b$). Furthermore,

$$P(n)(b_1 - 0) + P(n-1)(b_2 - b_1) + \cdots + P(n+1-k)(b_k - b_{k-1}) = 0 \quad (1)$$

We must show that $b = 0$.

$$\begin{aligned} & h(n)b_1 + h(n-1)b_2 + \cdots + h(1)b_n = 0 \quad n > k \\ \Rightarrow \quad & h(n)b_1 + \cdots + h(n+2-k)b_{k-1} + b\Big(h(n+1-k) + \cdots + h(1)\Big) = 0 \\ \Rightarrow \quad & h(n)b_1 + \cdots + h(n+2-k)b_{k-1} + bP(n+1-k) = 0 \qquad (2) \end{aligned}$$

But

$$\begin{aligned} & h(n)b_1 + \cdots + h(n+2-k)b_{k-1} = \\ & \quad P(n)b_1 + P(n-1)(b_2 - b_1) + \cdots + P(n+1-k)(b_k - b_{k-1}) + P(n+2-k)b_{k-1} \end{aligned}$$

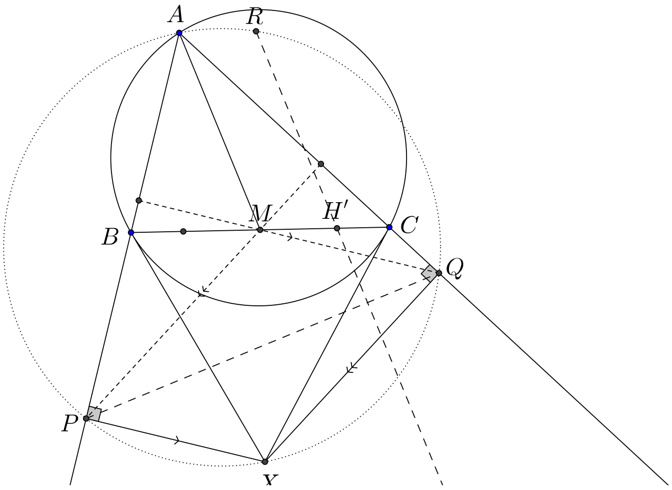
But By (1), this is equal to $P(n+2-k)b_{k-1} = P(n+2-k)b$. Hence, from this and (2), we get $b(P(n+1-k) + P(n+2-k)) = 0$ for all $n > k$. Since P is not zero polynomial, we deduce that $b = 0$ and so $h(x)$ is a good polynomial, as desired. □

Now we use induction on the degree for uniqueness of good monic polynomials. If $\deg(p) = 1$, $P(x) = x + P(0)$ and by lemma 1, $P(x) = x$. Now suppose that we know there is exactly one good monic polynomial of degree $n-1$. Assume that there are two monic good polynomials $P(x)$ and $Q(x)$ of degree n. Now, $\frac{P(x)-P(x-1)}{n}$ and $\frac{Q(x)-Q(x-1)}{n}$ are two good monic polynomials of degree $n-1$. But according to the induction hypothesis, such a polynomial is unique. Therefore, $P(x) - P(x-1) = Q(x) - Q(x-1)$ or equivalently $P(x) - Q(x) = P(x-1) - Q(x-1)$. This implies that $P(x) - Q(x)$ is a constant polynomial. $P(x) - Q(x) = P(0) - Q(0)$. On the other hand, by lemma 1, $P(0) = Q(0) = 0$ and so $P(x) = Q(x)$. Therefore, the proof is complete.

18 . Let P, Q and M be the feet of perpendicular lines from X to AB, AC and BC, respectively. Obviousely, M is the midpoint of BC. We have

$$\angle ZXY = \angle ZXH' + \angle H'XY = \angle AQH' + \angle APH' = \angle PH'Q - \angle A$$

Since we know $\angle BXC = 180° - \angle 2A$, it suffices to prove $\angle PH'Q = 180° - \angle A$. Note that $\angle APM = \angle BXM = 90° - \angle A$ and so $PM \perp AQ$. Similarly, $QM \perp AP$. So M is the orthocenter of triangle APQ, and as a consequence $AM \perp PQ$.



Let R be the foot of the perpendicular line from A to XM. Thus, $ARH'M$ is a parallelogram. Now since $AM \perp PQ$ and $RH' \parallel AM$, we infer $PH' \perp PQ$ (1). On the other hand, since $\angle ARX = \angle APX = \angle AQX$, R lies on the circumcircle of APQ and so $\angle PRQ = \angle A$. Therefore, $RH' = AM = 2r\cos(\angle A) = 2r\cos(\angle PRQ)$ (2), where r is the radius of circle passing through A, R, P and Q. From (1) and (2), we deduce that H' is the orthocenter of triangle PRQ. This implies that $\angle PH'Q = 180° - \angle A$ and this completes the proof.