Number Theory Problems From IMO

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Abstract

It's been on my planning list for a long time, to make this document. Finally I was able to compile all the number theory problems that appeared at the IMO so far. However, this time I am only compiling the problems. On a later version, I may write the solutions as well. Each problem has the year and chronological number of its appearance with the problem number, though some may be missing. Finally, I have put the problems which I think belongs to the class of number theory. Some may not agree with me about some problems.

1 Problems

Problem 1 (1959, Problem 1)**.** Prove that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible for every $n \in \mathbb{N}$.

Proposed by-Poland

Problem 2 (1960, Problem 1)**.** Find all the three-digit numbers for which one obtains, when dividing the number by 11, the sum of the squares of the digits of the initial number.

Proposed by-Bulgaria

Problem 3 (1962, Problem 1)**.** Find the smallest natural number n with the following properties:

(1) In decimal representation it ends with 6.

(2) If we move this digit to the front of the number, we get a number 4 times larger.

Proposed by-Poland

Problem 4 (1964, Problem 1)**.** (a).Find all natural numbers n such that 7 divides $2^n - 1$.
(b). Prove that for all natural numbers n the number $2^n + 1$ is not divisible by 7.

Proposed by-Czechoslovakia

Problem 5 (1967, Problem 3)**.** Let k, m, and n be positive integers such that $m + k + 1$ is a prime number greater than $n + 1$. Write $c_s = s(s + 1)$. Prove that the product

$$(c_{m+1} - c_k)(c_{m+2} - c_k) \cdots (c_{m+n} - c_k)$$

is divisible by $c_1 c_2 \cdots c_n$.

Proposed by-United Kingdom

Problem 6 (1968, Problem 2)**.** Find all positive integers x for which

$$p(x) = x^2 - 10x - 22$$

where $p(x)$ denotes the product of the digits of x.

Proposed by-Czechoslovakia

Problem 7 (1968, Problem 6)**.** Let $[x]$ denote the integer part of x, i.e., the greatest integer not exceeding x for positive x. If n is a positive integer, express as a simple function of n the sum

$$\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \ldots + \left[\frac{n+2^i}{2^{i+1}}\right]$$

Proposed by-United Kingdom

Problem 8 (1969, Problem 1)**.** Prove that there exist infinitely many natural numbers a with the following property: the number

$$z = n^4 + a$$

is composite for any natural n.

Proposed by-Germany, DR

Problem 9 (1970, Problem 2)**.** Let a and b be the bases of two number systems and let

$$A_n = \overline{x_1 x_2 \ldots x_n}_{(a)}, \qquad A_{n+1} = \overline{x_0 x_1 x_2 \ldots x_n}_{(a)}$$

$$B_n = \overline{x_1 x_2 \ldots x_n}_{(b)}, \qquad B_{n+1} = \overline{x_0 x_1 x_2 \ldots x_n}_{(b)}$$

be numbers in the number systems with respective bases a and b, so that $x_0, x_1, x_2, ..., x_n$ denote digits in the number system with base a as well as in the number system with base b. Suppose that neither x_0 nor x_1 is zero. Prove that $a > b$ if and only if

$$\frac{A_n}{A_{n+1}} > \frac{B_n}{B_{n+1}}$$

Proposed by-Romania

Problem 10 (1970, Problem 4)**.** For what natural numbers n can the product of some of the numbers $n, n+1, n+2, n+3, n+4, n+5$ be equal to the product of the remaining ones?

Proposed by-Czechoslovakia

Problem 11 (1971, Problem 3)**.** Prove that the sequence $2^n - 3, (n > 1)$ contains a sub-sequence of numbers relatively prime in pairs.

Proposed by-Poland

Problem 12 (IMO 1972, Problem 1)**.** A set of 10 positive integers is given such that the decimal expansion of each of them has two digits. Prove that there are two disjoint subsets of the set with equal sums of their elements.

Proposed by-Soviet Union

Problem 13 (1972, Problem 3)**.** Let m and n be nonnegative integers. Prove that

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer.

Proposed by-United Kingdom

Problem 14 (1974, Problem 3)**.** Prove that there does not exist a natural number n for which the number

$$5 \nmid \sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$$

Proposed by-Romania

Problem 15 (1974, Problem 6)**.** Let $P(x)$ be a polynomial with integer coefficients. If $n(P)$ is the number of (distinct) integers k such that $P^2(k) = 1$, prove that $n(P) \deg(P) 2$, where $\deg(P)$ denotes the degree of the polynomial P.

Proposed by-Sweden

Problem 16 (1975, Problem 2)**.** Let $a_1, a_2, a_3, ...$ be any infinite increasing sequence of positive integers. (For every integer $i > 0, a_{i+1} > a_i$.) Prove that there are infinitely many m for which positive integers x, y, h, k can be found such that $0 < h < k < m$ and

$$a_m = xa_h + ya_k$$

Proposed by-United Kingdom

Problem 17 (1975, Problem 4)**.** Let A be the sum of the digits of the number 4444^{4444} and B the sum of the digits of the number A. Find the sum of the digits of the number B.

Proposed by-Soviet Union

Problem 18 (1976, Problem 2)**.** Let $P_1(x) = x^2 - 2$ and

$$P_j(x) = P_1(P_{j-1}(x)), j = 2, 3, ...$$

Show that for arbitrary n, the roots of the equation

$$P_n(x) = x$$

are real and different from one another.

Proposed by-Finland

Problem 19 (1976, Problem 4)**.** Find the largest number obtainable as the product of positive integers whose sum is 1976.

Proposed by-USA

Problem 20 (1977, Problem 2)**.** In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

proposed by-Vietnam

Problem 21 (1977, Problem 3)**.** Let n be a given integer greater than 2, and let V_n be the set of integers $1 + kn$, where $k = 1, 2, ...$ A number $m \in V_n$ is called indecomposable in V_n if there do not exist numbers $p, q \in V_n$ such that $pq = m$. Prove that there exists a number $r \in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Expressions that differ only in order of the elements of V_n will be considered the same.)

Proposed by-Netherlands

Problem 22 (1977, Problem 5)**.** Let a and b be natural numbers and let q and r be the quotient and remainder respectively when $a^2 + b^2$ is divided by $a + b$. Determine the numbers a and b if

$$q^2 + r = 1977$$

Proposed by-Germany, FR

Problem 23 (1977, Problem 6)**.** Let $f : \mathbb{N} \to \mathbb{N}$ be a function that satisfies the inequality

$$f(n+1) > f(f(n))$$

for all $n \in \mathbb{N}$. Prove that $f(n) = n$ for all natural numbers n.

Proposed by-Bulgaria

Problem 24 (1978, Problem 1)**.** Let $n > m \geq 1$ be natural numbers such that the groups of the last three digits in the decimal representation of $1978m, 1978n$ coincide. Find the ordered pair (m, n) of such m, n for which $m + n$ is minimal.

Proposed by-Cuba

Problem 25 (1978, Problem 3)**.** Let $f\{(n)\}$ be a strictly increasing sequence of positive integers: $0 < f(1) < f(2) < f(3) < \dots$. Of the positive integers not belonging to the sequence, the nth in order of magnitude is $f(f(n)) + 1$. Determine $f(240)$.

Proposed by-United Kingdom

Problem 26 (1978, Problem 5)**.** Let $\varphi : \{1, 2, 3, ...\} \to \{1, 2, 3, ...\}$ be injective. Prove that for all n,

$$\sum_{k=1}^{n} \frac{\varphi(k)}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}$$

Proposed by-France

Problem 27 (1979, Problem 1)**.** Given that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1318} + \frac{1}{1319} = \frac{p}{q}$$

for co-prime p, q. Prove that, $1979|p$.

Proposed by-Germany, FR

Problem 28 (1981, Problem 3)**.** Determine the maximum value of $m^2 + n^2$ where m and n are integers satisfying

$$m, n \in \{1, 2, ..., 1981 \quad \text{and} \quad (n^2 - mn - m^2)^2 = 1$$

Proposed by-Netherlands

Problem 29 (1981, Problem 6)**.** Assume that $f(x, y)$ defined for all positive integers x and y, and that the following equations are satisfied:

$$\begin{aligned} f(0, y) &= y + 1, \\ f(x+1, 0) &= f(x, 1), \\ f(x+1, y+1) &= f(x, f(x+1, y)) \end{aligned}$$

Determine $f(4, 1981)$.

Proposed by-Finland

Problem 30 (1982, Problem 1)**.** The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m, n,

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1$$

$$f(2) = 0, f(3) > 0 \text{ and } f(9999) = 3333$$

Determine $f(1982)$.

Proposed by-United Kingdom

Problem 31 (1982, Problem 4)**.** Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers (x, y), then it has at least three such solutions. Show that the equation has no solution in integers when $n = 2891$.

Proposed by-United Kingdom

Problem 32 (1983, Problem 3)**.** Let a, b, c be positive integers s.t. $(a, b) = (b, c) = (c, a) = 1$. Show that $2abc - ab - bc - ca$ is the largest integer not representable as

$$bcx + cay + abz$$

for non-negative integer x, y, z.

Proposed by-Germany, FR

Problem 33 (1983, Problem 5)**.** Prove or disprove the following statement: In the set $\{1, 2, 3, ..., 10^5\}$ a subset of 1983 elements can be found that does not contain any three consecutive terms of an arithmetic progression.

Proposed by-Poland

Problem 34 (1984, Problem 2)**.** Find two positive integers a, b such that none of the numbers $a, b, a + b$ is divisible by 7 but $(a + b)^7 - a^7 - b^7$ is.

Proposed by-Netherlands

Problem 35 (1984, Problem 6)**.** Let a, b, c, d be odd positive integers such that $a < b < c < d, ad = bc$, and

$$a + d = 2^k, b + c = 2^m$$

for some positive integer k, m. Prove that, $a = 1$.

Proposed by-Poland

Problem 36 (1985, Problem 3)**.** The weight $w(p)$ of a polynomial p,

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

with integer coefficients a_i defined as the number of its odd coefficients. For $i = 0, 1, 2, ...$ let

$$q_i(x) = (1 + x)^i$$

Prove that for any finite sequence $0 \leq i_1, i_2, ..., i_n$,

$$w(q_{i_1} + q_{i_2} + \ldots + q_{i_n}) \geq w(q_{i_1})$$

Proposed by-Netherlands

Problem 37 (1985, Problem 4)**.** Given a set M of 1985 positive integers, none of which has a prime divisor larger than 26, prove that M has four distinct elements whose geometric mean is an integer.

Proposed by-Mongolia

Problem 38 (1986, Problem 1). The set $S = \{2, 5, 13\}$ has the property that for every $a, b \in S, a \neq b$, the number $ab - 1$ is a perfect square. Show that for every positive integer d not in S, the set $S \cup \{d\}$ does not have the above property.

Proposed by-Germany, FR

Problem 39 (1987, Problem 1). Let S be a set of n elements. We denote the number of all permutations of S that have exactly k fixed points by $p_n(k)$. Prove that

$$\sum_k p_n(k) = n!$$

Proposed by-Germany, FR

Problem 40 (1987, Problem 4). Does there exist a function $f : N \to N$, such that

$$f(f(n)) = n + 1987$$

for every natural number n?

Proposed by-Soviet Union

Problem 41 (1987, Problem 5). Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.

Problem 42 (1987, Problem 6). Let n be an integer greater than or equal to 2. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{\frac{n}{3}}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$.

Proposed by-Soviet Union

Problem 43 (1988, Problem 3). A function f defined on the positive integers (and taking positive integer values) is given by

$$\begin{aligned} f(1) = 1, &\qquad f(3) = 3 \\ f(2n) &= f(n) \\ f(4n+1) &= 2f(2n+1) - f(n) \\ f(4n+3) &= 3f(2n+1) - 2f(n) \end{aligned}$$

for all positive integers n. Determine with proof the number of positive integers n less than or equal to 1988 for which $f(n) = n$.

Proposed by-United Kingdom

Problem 44 (1988, Problem 6). Let a and b be two positive integers so that

$$k = \frac{a^2 + b^2}{ab + 1} \in \mathbb{N}$$

Prove that k is a square.

Proposed by-Germany, FR

Problem 45 (1989, Problem 5). For which positive integers n does there exist a positive integer N such that none of the integers $1+N, 2+N, ..., n+N$ is the power of a prime number?

Proposed by-Sweden

Problem 46 (1990, Problem 3). Find all positive integers n such that

$$n^2 | 2^n + 1$$

Proposed by-Romania

Problem 47 (1990, Problem 5). Two players A and B play a game in which they choose numbers alternately according to the following rule: At the beginning, an initial natural number $n_0 > 1$ is given. Knowing n_{2k}, player A may choose any $n_{2k+1} \in \mathbb{N}$, such that

$$n_{2k} \leq n_{2k+1} \leq n_{2k}^2$$

Then player B chooses a number $n_{2k+2} \in \mathbb{N}$ such that

$$\frac{n_{2k+1}}{n_{2k+2}} = p^r$$

where $p \in \mathbb{P}, r \in \mathbb{N}$.
It is stipulated that player A wins the game if he (she) succeeds in choosing the number 1990, and player B wins if he (she) succeeds in choosing 1. For which natural numbers n_0 can player A manage to win the game, for which n_0 can player B manage to win, and for which n_0 can players A and B each force a tie?

Proposed by-Germany, FR

Problem 48 (1991, Problem 2)**.** Let $n > 6$ and let $a_1 < a_2 < ... < a_k$ be all natural numbers that are less than n and relatively prime to n. Show that if $a_1, a_2, ..., a_k$ is an arithmetic progression, then n is a prime number or a natural power of two.

Proposed by-Romania

Problem 49 (1992, Problem 1)**.** Find all integer triples (p, q, r) such that $1 < p < q < r$ and $(p-1)(q-1)(r-1)$ is a divisor of $(pqr-1)$.

Proposed by-New Zealand

Problem 50 (1992, Problem 6)**.** For each positive integer n, denote by $s(n)$ the greatest integer such that for all positive integers $k \leq s(n), n^2$ can be expressed as a sum of squares of k positive integers.

(i) Prove that $s(n) \leq n^2 - 14$ for all n.

(ii) Find a number n s.t. $s(n) = n^2 - 14$.

(iii) Prove that there are infinitely many n s.t. $s(n) = n^2 - 14$.

Proposed by-United Kingdom

Problem 51 (1993, Problem 1)**.** Let $n > 1$ be an integer and $f(x) = x^n + 5x^{n-1} + 3$. Prove that there do not exist polynomials $g(x), h(x)$, each having integer coefficients and degree at least one, such that

$$f(x) = g(x)h(x)$$

Proposed by-Ireland

Problem 52 (1993, Problem 5)**.** Determine whether there exists a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ with the following properties:

$$\begin{aligned} f(1) &= 2 \\ f(f(n)) &= f(n) + n \end{aligned}$$

Proposed by-Germany

Problem 53 (1994, Problem 1)**.** Let m and n be positive integers. The set $A = \{a_1, a_2, ..., a_m\}$ is a subset of $1, 2, ..., n$. Whenever $a_i + a_j \leq n, 1 \leq i \leq j \leq m, a_i + a_j$ also belongs to A. Prove that

$$\frac{a_1 + a_2 + \ldots + a_m}{m} \geq \frac{n+1}{2}$$

Proposed by-France

Problem 54 (1994, Problem 3)**.** For any positive integer k, A_k is the subset of

$$\{k+1, k+2, ..., 2k\}$$

consisting of elements whose digit in base 2 contain exactly three 1's. Let $f(k)$ denote the number of elements of A_k.

(a) Prove that for any positive integer $m, f(k) = m$ has at least one solution.

(b) Determine all positive integers m for which $f(k) = m$ has a unique solution.

Proposed by-Romania

Problem 55 (1994, Problem 4)**.** Determine all pairs (m, n) of positive integers such that

$$\frac{n^3+1}{mn+1} \in \mathbb{N}$$

Proposed by-Australia

Problem 56 (1994, Problem 6)**.** Find a set A of positive integers such that for any infinite set P of prime numbers, there exist positive integers $m \in A$ and $n \notin A$, both the product of the same number (at least two) of distinct elements of P.

Proposed by-Finland

Problem 57 (1995, Problem 6)**.** Let p be an odd prime. Find the number of p-element subsets A of $1, 2, ..., 2p$ such that the sum of all elements of A is divisible by p.

Proposed by-Poland

Problem 58 (1996, Problem 3)**.** Let $\mathbb{N}_0$ denote the set of non-negative integers. Find all $f : \mathbb{N}_0 \to \mathbb{N}_0$ s.t.

$$f(m + f(n)) = f(f(m)) + f(n) \quad \text{for all } n, m \in \mathbb{N}_0$$

Proposed by-Romania

Problem 59 (1996, Problem 4)**.** The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

Proposed by-Russia

Problem 60 (1996, Problem 6)**.** Let p, q, n be three positive integers s.t. $p + q < n$. Let $(x_0, x_1, ..., x_n)$ be a $n + 1$-tuple of integers satisfying

(a) $x_0 = x_n = 0$

(b) For each i with $1 \leq i \leq n$, $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$. Show that there exists a pair (i, j) of distinct indices with $(i, j) \neq (0, n)$ such that $x_i = x_j$.

Proposed by-France

Problem 61 (1997, Problem 5)**.** Find all pairs of integers x, y s.t.

$$x^{y^2} = y^x$$

Proposed by-Czechoslovakia

Problem 62 (1997, Problem 6)**.** For a positive integer n, let $f(n)$ denote the number of ways to represent n as a sum of non-negative integer powers of 2. Representations that differ only in the ordering in their summation are not considered to be distinct(For example, $f(4) = 4$). Prove the inequality

$$2^{\frac{n^2}{4}} \leq f(n) \leq 2^{\frac{n^2}{2}}$$

Proposed by-Lithuania

Problem 63 (1998, Problem 3)**.** For any $n \in \mathbb{N}$, $\tau(n)$ denotes the number of positive divisors of n. Determine all positive integers m s.t.

$$m = \frac{\tau(k^2)}{\tau(k)}$$

has a solution.

Proposed by-Belarus

Problem 64 (1998, Problem 4)**.** Determine all pairs (x, y) of positive integers such that

$$xy^2 + y + 7 | x^2y + x + y$$

Proposed by-United Kingdom

Problem 65 (1998, Problem 6)**.** Determine the least possible value of $f(1998)$, where f is a function from the set $\mathbb{N}$ of positive integers into itself such that for all $m, n, \in \mathbb{N}$,

$$f(n^2 f(m)) = m \left(f(n)^2\right)$$

Proposed by-Bulgaria

Problem 66 (1999, Problem 4)**.** Find all pairs of positive integers (x, p) such that $x \leq 2p, p \in \mathbb{P}$, and

$$x^{p-1} | (p-1)^x + 1$$

Proposed by-Taiwan

Problem 67 (2000, Problem 5)**.** Does there exist a positive integer n such that n has exactly 2000 prime divisors and $n|2^n + 1$.

Proposed by-Russia

Problem 68 (2001, Problem 4)**.** Let n be an odd positive integer greater than 1 and $c_1, c_2, ..., c_n$ be integers. For each permutation $a = (a_1, a_2, ..., a_n)$ of $(1, 2, ..., n)$ define

$$S(a) = \sum_{i=1}^{n} c_i a_i$$

Prove that, there exists permutations a, b s.t. $n! | S(a) - S(b)$.

Proposed by-Canada

Problem 69 (2001, Problem 6)**.** Let $a > b > c > d$ be positive integers and suppose

$$ac + bd = (b + d + a - c)b + d - a + c$$

Prove that, $ad + bc$ isn't a prime.

Proposed by-Bulgaria

Problem 70 (2002, Problem 3)**.** Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$a^n + a^2 - 1|a^m + a - 1$$

Proposed by-Romania

Problem 71 (2002, Problem 4)**.** Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < ... < d_k = n$. Prove that

$$d_1d_2 + d_2d_3 + \quad + d_{k1}d_k$$

is always less than n^2, and determine when it is a divisor of n^2.

Proposed by-Romania

Problem 72 (2003, Problem 2)**.** Determine all pairs (a, b) of positive integers such that

$$\frac{a^2}{2ab^2 - b^3 + 1} \in \mathbb{N}$$

Proposed by-Bulgaria

Problem 73 (2003, Problem 6)**.** Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number n^pp is not divisible by q.

Proposed by-France

Problem 74 (2004, Problem 6)**.** We call a positive integer *alternate* if its decimal digits are alternately odd and even. Find all positive integers n such that n has an alternate multiple.

Proposed by-Iran

Problem 75 (2005, Problem 2). Let $a_1, a_2, ...$ be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for each positive integer n , the numbers $a_1, a_2, ..., a_n$ gives different remainder upon division by n. Prove that each integer occurs exactly once in the sequence.

Proposed by-Holland

Problem 76 (2006, Problem 4). Consider the sequence $a_1, a_2, ...$ defined by

$$a_n = 2^n + 3^n + 6^n - 1$$

Determine all positive integers that are relatively prime to every term of the sequence.

Proposed by-Poland

Problem 77 (2006, Problem 4). Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2$$

Problem 78 (2006, Problem 5). Let $P(x)$ be a polynomial in x with $\deg(P) = n > 1$ and $k \in \mathbb{N}$.

$$Q(x) = P(P(...(P(x))...))$$

where P occurs k times. Prove that there are at most n integers t such that

$$Q(t) = t$$

Proposed by-Dan Schwarz, Romania

Problem 79 (2007, Problem 5). Let a and b be positive integers s.t.

$$4ab - 1 | (4a^2 - 1)^2$$

Prove that, $a = b$.

Proposed by-Kevin Buzzard and Edward Crane, United Kingdom

Problem 80 (2008, Problem 3)**.** Prove that there are infinite n s.t. $n^2 + 1$ has a prime factor greater than $2n + \sqrt{2n}$.

Problem 81 (2009, Problem 1)**.** Let n be a positive integer and let $a_1, ..., a_k$ ($k \geq 2$) be distinct integers in the set $\{1, 2, ..., n\}$ s.t.

$$n | a_i(a_{i+1} - 1)$$

for $i = 1, 2, ..., k - 1$. Prove that $n | a_k(a_1 - 1)$.

Proposed by - Ross Atkins, Australia

Problem 82 (2009, Problem 3)**.** Let $s_1, s_2, ...$ be a strictly increasing sequence of positive integers. If

$$s_{s_1}, s_{s_2}, ..., s_{s_n} \quad \text{and } s_{s_1+1}, s_{s_2+1}, ..., s_{s_n+1}$$

are both arithmetic sequences, then show that, $s_1, s_2, ...$ it-self is an arithmetic sequence.

Proposed by - Gabriel Carroll, United States of America

Problem 83 (2009, Problem 5)**.** Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b + f(a) - 1)$$

A triangle is non-degenerate if its vertices are not collinear.

Proposed by - France

Problem 84 (2010, Problem 1)**.** Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than a.

Proposed by - Pierre Bornsztein, France

Problem 85 (2010, Problem 3)**.** Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that $(g(m) + n)(g(n) + m)$ is a perfect square for all $m, n \in \mathbb{N}$.

Proposed by-Gabriel Carroll, USA

Problem 86 (2011, Problem 1)**.** Given any set

$$A = \{a_1, a_2, a_3, a_4\}$$

of four distinct positive integers, we denote the sum $s_A = a_1 + a_2 + a_3 + a_4$. Let n_A be the number of pairs (i, j) with $1 \leq i \leq j \leq 4$ s.t. $a_i + a_j | s_A$. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

Proposed by-Fernando Campos, Mexico

Problem 87 (2011, Problem 5)**.** Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n,

$$f(m - n) | f(m) - f(n)$$

Prove that, for all m, n with $f(m) \leq f(n)$, $f(m) | f(n)$.

Proposed by-Mahyar Sefidgaran, Iran

Problem 88 (2012, Probem 4)**.** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, and c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

Here $\mathbb{Z}$ denotes the set of integers.

Problem 89 (2012, Problem 6)**.** Find all positive integers n for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1$$

Proposed by - Dušan Djukić, Serbia

Problem 90 (2013, Problem 1)**.** Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, ..., m_k$ (not necessarily different) such that

$$1 + \frac{2^k - 1}{n} = (1 + \frac{1}{m_1})(1 + \frac{1}{m_2})...(1 + \frac{1}{m_k})$$

Proposed by - Japan

Problem 91 (2014, Problem 1)**.** Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers, Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + \cdots + a_n}{n} \leq a_{n+1}.$$

Proposed by - Austria

Problem 92 (2014, Problem 5)**.** For each positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

Proposed by - Luxembourg

Problem 93 (2015, Problem 2)**.** Find all postive integers (a, b, c) such that

$$ab - c, \quad bc - a, \quad ca - b$$

are all powers of 2.

Proposed by - Serbia