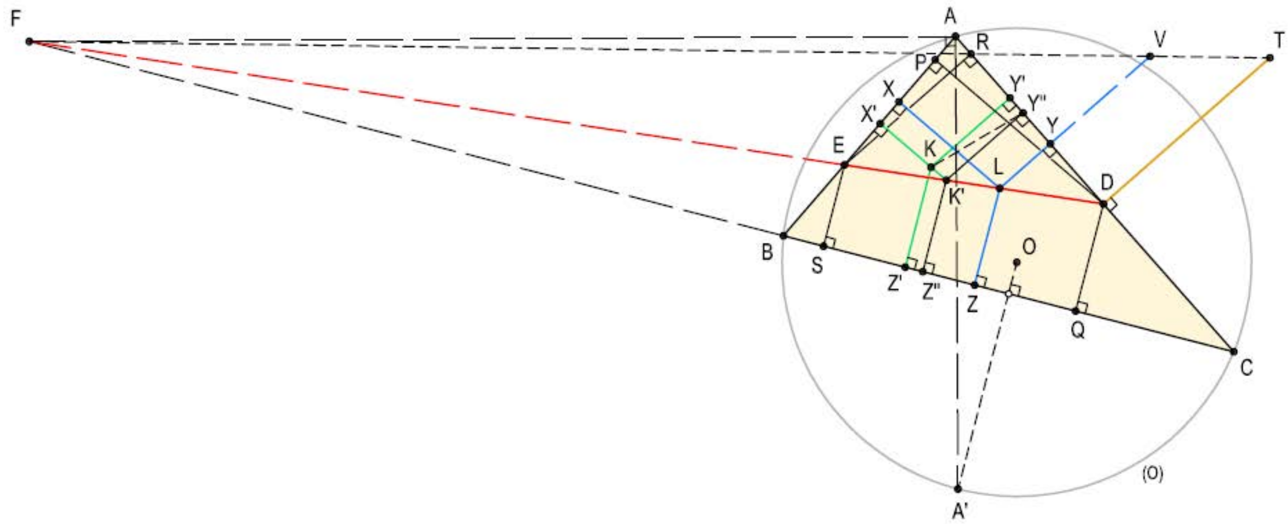
, (1) holds, where X', Y', Z', are the orthogonal projections of K, on $AB,\ AC,\ BC$, respectively.

Click to reveal hidden text

Hence, the locus Λ_k of the point L inwardly to $\triangle ABC$ as the problem states, is the segment $DE,$ connecting the points $D,\ E$ on $AB,\ AC$ respectively, on condition of (1), which are collinear with the point $F,$ as the feet of the external angle bisector of $\angle A,$ on the sideline BC of $\triangle ABC$ and the proof is completed.

Kostas Vittas.

Attachments:



Quick Reply

High School Olympiads

angles

Reply

Kandinsky1
#1 Mar 2, 2011, 2:40 pm

Given triangle ABC with heights AP and BQ. Draw point M on the side AB such that angles AQM and BPM are equal.

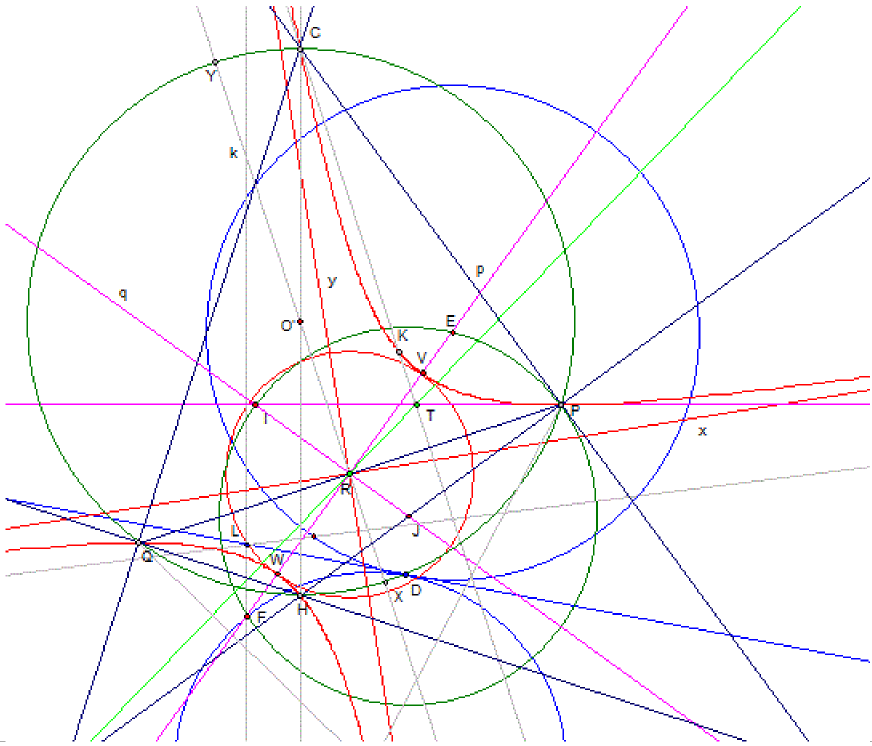
yetti
#2 Mar 2, 2011, 8:38 pm

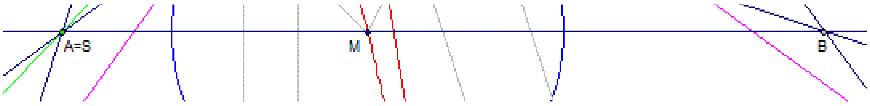
$\angle MQA = \angle MPB \Longrightarrow M$ is isogonal conjugate WRT $\triangle CPQ$ of a point M^* on the perpendicular bisector k of PQ. Isogonal conjugate of k WRT $\triangle CPQ$ is a rectangular hyperbola $\mathcal{K}$ through C, P, Q (and through orthocenter K of $\triangle CPQ$) with center at the midpoint R of PQ. You have to construct intersections $M, M' \in AB \cap \mathcal{K}$.

k cuts circumcircle (O') of $\triangle CPQ$ at X, Y. The hyperbola asymptotes are Simson lines x, y of $\triangle CPQ$ with poles X, Y. If H is diametrically opposite of C on circumcircle $(O') \equiv \odot(CPQ)$, (orthocenter of $\triangle ABC$) then $PKQH$ is parallelogram with diagonal intersection $R \Longrightarrow H \in \mathcal{K}$. Consider degenerate hexagon $PPKCQH$ inscribed in $\mathcal{K}$. By Pascal theorem, $R \equiv PQ \cap KH, S \equiv PH \cap CQ, T \equiv PP \cap CK$ are collinear ($S \equiv A$ for AP, BQ altitudes of $\triangle ABC$). Line RS can be constructed and CK cuts RS at T. Then $PP \equiv PT$ is hyperbola tangent at P. Bisectors p, q of right angles formed by the asymptotes x, y are the hyperbola main axes. Let PT cut q at I. If P, I are on opposite sides of p, then p is the major axis. If not, relabel $p \longleftrightarrow q$ and repeat construction of I. Hyperbola tangent PI bisects $\angle EPF$ between P and the hyperbola foci $E, F \Longrightarrow$ circle (J) through P, I and centered on the minor axis q cuts the major axis p at E, F. Feet of perpendiculars to the hyperbola tangent $PP \equiv PT$ from the foci E, F are on the hyperbola pedal circle (R), which cuts the major axis p at the hyperbola vertices V, W.

Construct circle (E) with center E and radius equal to the hyperbola major axis length VW. Reflect F in AB into F'. Construct circle (M) with center M passing through F, F' (centered on AB) and externally tangent to (E) at D. Then $EM - FM = EM - DM = ED = VW \Longrightarrow M \in \mathcal{K}$. Draw arbitrary circle (U) through F, F' intersecting (E) at N, N'. Radical axis FF' of $(M), (U)$ and radical axis NN' of $(E), (U)$ meet at the radical center L of these 3 circles. Common tangent LD of $(E), (M)$ from L is radical axis of $(E), (M)$ and this tangent can be constructed. Then $(M) \equiv \odot(DFF')$. The other tangent of (E) from L gives another solution; pick the one with M on the segment AB.

Attachments:





Luis González

#3 Mar 2, 2011, 11:16 pm

We redefine the notations as follow: Perpendiculars to the sides AC, BC of $\triangle ABC$ through A, B cut CB, CA at P, Q. $M \in PQ$ such that $\angle MAQ = \angle MBP \Longrightarrow$ points M are isogonal conjugates of the intersections of the perpendicular bisector of AB with the isogonal conjugate of PQ WRT $\triangle ABC$. Using barycentric coordinates WRT $\triangle ABC$, we have:

$$Q(-a^2 : 0 : S_B) \ , \ P(0 : -b^2 : S_A) \implies PQ \equiv b^2 S_B x + a^2 S_A y + a^2 b^2 z = 0.$$

Isogonal conjugate of PQ WRT $\triangle ABC$ is then $\mathcal{E} \equiv S_B yz + S_A zx + c^2 xy = 0$. If PQ does not cut the circumcircle (O), then $\mathcal{E}$ is an ellipse centered at the midpoint $(1 : 1 : 0)$ of AB with focal axis the straight line $c^2(x - y) + (S_B - S_A)z = 0$, i.e. the perpendicular bisector ℓ_c of $AB \Longrightarrow$ desired points M are the isogonal conjugates of the vertices V, V' of $\mathcal{E}$ WRT $\triangle ABC$. The construction of V, V' given the axes c, ℓ_c of $\mathcal{E}$, the length of its minor axis AB and a point C on $\mathcal{E}$ is very straightforward by keeping in mind the rectangular canonical form of the ellipse.

P.S. Note that this alternate construction only works under the given conditions of the problem (AP and BQ perpendicular to CB,CA). yetti's reasoning works in general.

skytin

#4 Mar 7, 2011, 6:17 pm

Its very easy problem from VII GEOMETRICAL OLYMPIAD IN HONOUR OF
I.F.SHARYGIN
THE CORRESPONDENCE ROUND
See it here :
http://www.geometry.ru/olimp/sharygin/2011/zaochn-e.pdf
(Problem 12)

yetti

#5 Mar 7, 2011, 10:13 pm

Construct circle through P, Q tangent to, AB at M. Works for altitudes AP, BQ, but not in general.

Quick Reply

High School Olympiads

Problems about Circles (2)!

Reply

EQSon

#1 Mar 1, 2011, 8:22 pm

Two circles pass through vertex of an angle and an arbitrary point on angle bisector. Prove that pieces of angle's edges which are pent in circles are equal.

Luis González

#2 Mar 1, 2011, 11:38 pm

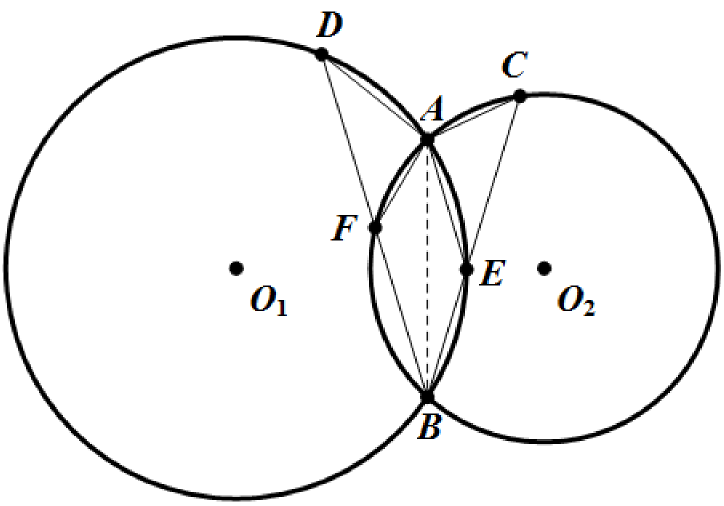
$\triangle ABC$ is scalene and M is a point on its A-angle bisector. Circles $\odot(AMC)$ and $\odot(AMB)$ cut AB, AC again at P, Q respectively. M is the center of the spiral similarity $\mathcal{M}$ taking the oriented segments BP and QC into each other. But since M is the midpoint of the arcs BQ and CP of $\odot(AMB)$ and $\odot(AMC)$, then $MB = MQ$ and $MC = MP \Longrightarrow \mathcal{M}$ is a rotation $\Longrightarrow$ segments BP and CQ are congruent.

Headhunter

#3 Mar 7, 2011, 8:06 am

$\angle AFD = \angle FAB + \angle ABF = \angle FCB + \angle ACF = \angle ACE$
$\angle ADF = \angle AEC$ and $AD = AE$, $AF = AC \implies \triangle AFD \cong \triangle ACE$ $\therefore DF = CE$

Attachments:



Quick Reply

High School Olympiads

hard problem of concurent lines (my own)

Reply

paul1703

#1 Mar 1, 2011, 8:12 pm

let ABC be a triangle with circumceter O and A1 , B1 C1 be the middle of the lines BC CA AB and let A2 B2 C2 be the diametral oposed points of A B C with respect to the circumcenter . let the circumcener of OC1B1 meet the second time the circumecenter of A2OA1 at A3 points B3, C3 defined the same, prove that the lines A1-A3,B1-B3,C1-C3 , are concurent/

Luis González

#2 Mar 1, 2011, 10:36 pm

$\triangle A_0B_0C_0$ is the tangential triangle of $\triangle ABC$. A_0, B_0, C_0 againts A, B, C. Inversion in (O) takes A_1, B_1, C_1 into A_0, B_0, C_0 and A_2, B_2, C_2 into themselves. Circles $\odot(OB_1C_1), \odot(OC_1A_1), \odot(OA_1B_1)$ are taken into the straight lines B_0C_0, C_0A_0, A_0B_0. Circles $\odot(OA_1A_2), \odot(OB_1B_2), \odot(OC_1C_2)$ are taken into the Nagel lines A_0A_2, B_0B_2, C_0C_2 of $\triangle A_0B_0C_0 \Longrightarrow$ their feet $A_4 \equiv A_0A_2 \cap B_0C_0$, $B_4 \equiv B_0B_2 \cap C_0A_0$ and $C_4 \equiv C_0C_2 \cap A_0B_0$ are the inverses of $A_3, B_3, C_3 \Longrightarrow A_1A_3, B_1B_3$ and C_1C_3 are taken into the circles $\odot(OA_0A_4), \odot(OB_0B_4)$ and $\odot(OC_0C_4)$. Lines A_1A_3, B_1B_3, C_1C_3 concur, since $\odot(OA_0A_4), \odot(OB_0B_4), \odot(OC_0C_4)$ are coaxal. For a proof see Coaxal circles and for a further generalization see Coaxial circles.

Quick Reply

High School Olympiads

Problems about Circles (1)!

Reply

EQSon

#1 Mar 1, 2011, 8:11 pm

In ABC triangle, Points A_1, B_1, C_1 are chosen on BC, CA, AB. Imagine that M is an arbitrary point on plane. Line BM passes by A_1BC_1 circumcircle in point B_2. Choose points C_2, A_2 like point B. Prove that points M, A_2, B_2, C_2 are on a common circle.

Luis González

#2 Mar 1, 2011, 9:51 pm

Circles $\odot(AB_1C_1), \odot(BC_1A_1), \odot(CA_1B_1)$ concur at the Miquel point U of $\triangle ABC$ and $\triangle A_1B_1C_1$. Since $\angle A_1UC_2 = \angle BCM$ and $\angle A_1UB_2 = \angle CBM$, it follows that $\angle C_2UB_2 = \angle A_1UC_2 + \angle A_1UB_2 = \angle C_2MB_2 \Longrightarrow$ Points M, U, B_2, C_2 are concyclic. By similar reasoning, we'll have that points M, U, B_2, A_2 are concyclic. Therefore, five points M, U, A_2, B_2, C_2 are concyclic.

jayme

#3 Mar 1, 2011, 11:38 pm

Dear Mathlinkers,
this circle is the Mannheim's circle....
For a synthetic proof
http://perso.orange.fr/jl.ayme vol. 2 Les cercles de Morley, Euler, Mannheim... p. 6.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Romania 2005

swaqar

#1 Feb 28, 2011, 8:55 pm

Given a triangle ABC, let P be a point in the interior of ABC and let AX , BY and CZ concur at P and X , Y and Z lie on the circumcircle Γ of ABC. Let A' , B' and C' be on Γ such that AA' , BB' and CC' are parallel to BC CA and AB respectively. Let $A'X$, $B'Y$ and $C'Z$ meet BC , CA and AB at X' , Y' and Z' respectively. Show that AX' , BY' and CZ' are concurrent .

Luis González

#2 Feb 28, 2011, 10:02 pm

By sine law in $\triangle BXC$ we have $\frac{BX'}{CX'}=\frac{BX}{XC}\cdot\frac{\sin\widehat{BXA'}}{\sin\widehat{CXA'}}=\frac{BX}{XC}\cdot\frac{\sin B}{\sin C}$

Similarly, we have the expressions $\frac{CY'}{AY'}=\frac{CY}{YA}\cdot\frac{\sin C}{\sin A}$, $\frac{AZ'}{BZ'}=\frac{AZ}{ZB}\cdot\frac{\sin A}{\sin B}$

$\frac{BX'}{CX'}\cdot\frac{CY'}{AY'}\cdot\frac{AZ'}{BZ'}=\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}\cdot\frac{\sin B}{\sin C}\cdot\frac{\sin C}{\sin A}\cdot\frac{\sin A}{\sin B}=\frac{BX}{CX}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}$

Since diagonals of the cyclic hexagon $AZBXCY$ concur at P, then $\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=1$

$\Longrightarrow\frac{BX'}{CX'}\cdot\frac{CY'}{AY'}\cdot\frac{AZ'}{BZ'}=1\Longrightarrow$ By Ceva's theorem, AX',BY',CZ' concur.

High School Olympiads

gravity point M such that AMB=2ACB

Source: Albanian Mathematical Olympiad 12 GRADE 2011--Question 5

ridgers

#1 Feb 27, 2011, 12:58 am

The triangle ABC acute with gravity center M is such that $\angle AMB = 2\angle ACB$. Prove that:

(a) $AB^4 = AC^4 + BC^4 - AC^2 \cdot BC^2$,

(b) $\angle ACB \geq 60^o$.

Luis González

#2 Feb 28, 2011, 10:37 am

Let O, H be the circumcenter and ortocenter of $\triangle ABC$. If $\angle AMB = 2\angle C$, then $M \in \odot(OAB)$. Since M lies between O and H, then H is outside $\odot(OAB) \Longrightarrow \angle AHB < \angle AOB \Longrightarrow 180^\circ - \angle C < 2\angle C \Longrightarrow \angle C > 60^\circ$.

Barycentric equation of the circle passing through A, B and $O(a^2S_A : b^2S_B : c^2S_C)$ is

$$\odot(OAB) \equiv a^2yz + b^2zx + c^2xy - \frac{a^2b^2}{2S_C}z(x+y+z) = 0$$

Thus, coordinates $(1:1:1)$ of M satisfy the latter equation

$$2S_C(a^2+b^2+c^2) - 3a^2b^2 = (a^2+b^2-c^2)(a^2+b^2+c^2) - 3a^2b^2 = 0$$

$$a^4 + a^2b^2 + a^2c^2 + a^2b^2 + b^4 + b^2c^2 - a^2c^2 - b^2c^2 - c^4 - 3a^2b^2 = 0$$

$$\Longrightarrow \ c^4 = a^4 + b^4 - a^2b^2$$

yetti

#3 Mar 1, 2011, 7:08 pm

Medians AM, BM cut circumcircle (O) of $\triangle ABC$ again at X, Y. $M \in \odot(AOB) \Longrightarrow MO$ bisects $\angle BMA$ externally $\Longrightarrow ABXY$ is isosceles trapezoid with diagonal intersection M and equal diagonals $AX = BY \Longrightarrow$ power of M to (O) is $\frac{4}{9}m_am_b = AM \cdot BM = R^2 - OM^2 = \frac{1}{9}(a^2+b^2+c^2)$. Squaring:

$$(2b^2+2c^2-a^2)(2c^2+2a^2-b^2) = (a^2+b^2+c^2)^2 \Longleftrightarrow$$

$$4b^2c^2 + 4a^2b^2 - 2b^4 + 4c^4 + 4c^2a^2 - 2b^2c^2 - 2c^2a^2 - 2a^4 + a^2b^2 = a^4 + b^4 + c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2 \Longleftrightarrow$$

$$3c^4 = 3a^4 + 3b^4 - 3a^2b^2$$

Cassius

#4 May 13, 2012, 5:22 pm

(a) Let $A = (0,0)$, $B = (1,0)$, $C = (c,d)$; then $M = \left(\frac{c+1}{3}, \frac{d}{3}\right)$, $O = \left(\frac{1}{2}, \frac{c^2+d^2-c}{2d}\right)$. The condition implies that A, B, M, O are concyclic; the circle passing through A, B, M has equation $(x-\ell)^2 + \left(y - \frac{1}{2}\right)^2 = \ell^2 + \frac{1}{4}$ where $\ell = \frac{c^2+2c+d^2-3d+1}{6(c+1)}$; plugging in the coordinates of O we get a multinomial which has

$c^4 - 2c^3 + 2c^2d^2 + 5c^2 - 2cd^2 - 4c + d^4 + d^2 - 1$ as a factor, and this is equivalent to the thesis (that is, $1 = (c^2 + d^2)^2 + ((c-1)^2 + d^2)^2 - (c^2 + d^2)((c-1)^2 + d^2)$ once the other possibilities are excluded.

(b) It suffices to note that, from the previous point, the triangle with sides AB^2, BC^2, CA^2 has an angle of $60°$.

drmzjoseph

#5 Apr 19, 2015, 12:01 pm

Let L the symmedian point of ABC, and $X \equiv AL \cap BC, Y \equiv BL \cap AC \Rightarrow XLYC$ is cyclical (angle-chasing)

By Steiner's Theorem we obtain $BX = \frac{ac^2}{b^2 + c^2}$ and $AY = \frac{bc^2}{a^2 + c^2}$, we use Lemma $\mathcal{P}_B + \mathcal{P}_A = c^2$ (power with respect to $\odot(CYLX)$)

$\Rightarrow \frac{a^2c^2}{b^2 + c^2} + \frac{b^2c^2}{a^2 + c^2} = c^2 \Rightarrow c^4 = a^4 + b^4 - a^2b^2$

$a^4 + b^4 - a^2b^2 \geq (a^2 + b^2 - ab)^2 = a^4 + b^4 + 3a^2b^2 - 2ab(a^2 + b^2)$

$\iff ab(a^2 + b^2) \geq 2a^2b^2$ it's true, then $c^2 \geq a^2 + b^2 - ab \Rightarrow \angle C \geq 60°$

Quick Reply

High School Olympiads

Ineq-G102 - Geometry

Reply

Ligouras

#1 Feb 26, 2011, 2:02 am

A triangle ABC is given in a plane.
The internal angle bisectors of the angles A, B, C of the triangle intersect the sides BC, CA, AB at D, E, F.
Let P be the point of intersection of the angle bisector of the angle A with the line EF.
The parallel to the side BC through the point P intersects the sides AB and AC in the points S and T.
Prove that

$$\frac{1}{BS}+\frac{1}{CT}\geq\frac{2}{ST}$$

Luis González

#2 Feb 27, 2011, 6:42 am

Line EF is the locus of points whose sum of oriented distances to AB, AC equals the oriented distance to BC. If X, Y, Z denote the orthogonal projections of P onto BC, CA, AB, we have then $PX = PY + PZ \Longrightarrow PX = 2PZ$

From $[ABC] = [AST] + [STCB]$ we obtain

$$PZ\cdot AB+PY\cdot AC+PX\cdot BC=PZ\cdot AS+PY\cdot AT+PX(ST+BC)$$

$$PZ\cdot AB+PZ\cdot AC=PZ\cdot AS+2PZ\cdot ST$$

$$AB+AC=AS+AT+2ST \Longrightarrow BS+CT=2ST$$

By AM-HM we get: $\dfrac{BS+CT}{2}=ST\geq 2\cdot\dfrac{BS\cdot CT}{BS+CT}\Longrightarrow\dfrac{1}{BS}+\dfrac{1}{CT}\geq\dfrac{2}{ST}$

Ligouras

#3 Feb 27, 2011, 2:50 pm

Very nice my friend **LuisGeometra**!!! thanks

Quick Reply

High School Olympiads

APMC 1997

Reply

ShahinBJK
#1 Feb 25, 2011, 11:33 pm

Let P be the intersection of lines l_1 and l_2. Let S_1 and S_2 be two circles externally tangent at P and both tangent to l_1, and let T_1 and T_2 be two circles externally tangent at P and both tangent to l_2.
Let A be the second intersection of S_1 and T_1, B that of S_1 and T_2, C that of S_2 and T_1, and D that of S_2 and T_2. Show that the points A, B, C, D are concyclic if and only if l_1 and l_2 are perpendicular.

Luis González
#2 Feb 26, 2011, 10:16 pm

Inversion with center P and abitrary power takes $\mathcal{S}_1, \mathcal{S}_2$ into the lines s_1, s_2 perpendicular to ℓ_1 and takes $\mathcal{T}_1, \mathcal{T}_2$ into the lines τ_1, τ_2 perpendicular to ℓ_2. Pairwise lines s_1, s_2, τ_1, τ_2 meet at the inverses $A'B', C', D'$ of A, B, C, D. Assume that $\ell_1 \perp \ell_2$, then $A'B'C'D'$ is a rectangle $\Longrightarrow A, B, C, D$ are also concyclic. Conversely, assume that A, B, C, D are concyclic. Since their inverses A', B', C', D' are vertices of a parallelogram, then $A'B'C'D'$ is a rectangle $\Longrightarrow \ell_1 \perp \ell_2$.

ShahinBJK
#3 Feb 26, 2011, 11:38 pm

> luisgeometra wrote:
> Inversion with center P and abitrary power takes $\mathcal{S}_1, \mathcal{S}_2$ into the lines s_1, s_2 perpendicular to ℓ_1 and takes $\mathcal{T}_1, \mathcal{T}_2$ into the lines τ_1, τ_2 perpendicular to ℓ_2. Pairwise lines s_1, s_2, τ_1, τ_2 meet at the inverses $A'B', C', D'$ of A, B, C, D. Assume that $\ell_1 \perp \ell_2$, then $A'B'C'D'$ is a rectangle $\Longrightarrow A, B, C, D$ are also concyclic. Conversely, assume that A, B, C, D are concyclic. Since their inverses A', B', C', D' are vertices of a parallelogram, then $A'B'C'D'$ is a rectangle $\Longrightarrow$ $\ell_1 \perp \ell_2$.

thanks for your solution but can you solve it without inversion? if you can i am waiting your post.

oneplusone
#4 Feb 27, 2011, 7:30 am

Let X, Z be any points on l_2 and in T_1 and T_2 respectively, Y, W are any point on l_1 and lies in S_1, S_2. Then $\angle ABD = \angle ABP + \angle DBP = \angle APX + \angle WPD$. Similarly $\angle ACD = \angle APY + \angle DPZ$. They both add up to 180 iff $l_1 \perp l_2$.

ShahinBJK
#5 Feb 27, 2011, 3:31 pm

> oneplusone wrote:
> Let X, Z be any points on l_2 and in T_1 and T_2 respectively, Y, W are any point on l_1 and lies in S_1, S_2. Then $\angle ABD = \angle ABP + \angle DBP = \angle APX + \angle WPD$. Similarly $\angle ACD = \angle APY + \angle DPZ$. They both add up to 180 iff $l_1 \perp l_2$.

Thanks for your solution I thought that this question is hard

Quick Reply

High School Olympiads

Tangent circles

livetolove212

#1 Dec 2, 2010, 5:45 pm

Problem (own): Given a triangle ABC with its circumcircle (O), its centroid G. Let M_a, M_b, M_c be the midpoints of BC, CA, AB. Let $A_1B_1C_1$ be the triangle formed by the intersections of three lines through M_a, M_b, M_c and perpendicular to AM_a, BM_b, CM_c, respectively. Prove that the pedal circle of G wrt ΔABC is tangent to the 9-point circle of triangle $A_1B_1C_1$.

Luis González

#2 Feb 26, 2011, 9:48 pm

$\triangle G_aG_bG_c$ is the pedal triangle of G WRT $\triangle ABC$. $\odot(G_aG_bG_c)$ cuts BC, CA, AB again at the projections of the symmedian point K of $\triangle ABC$ onto BC, CA, AB. K_bK_c, K_cK_a, K_aK_b cut M_bM_c, M_cM_a, M_aM_b at U, V, W and let D, E, F be the midpoints of B_1C_1, C_1A_1, A_1B_1. Note that $\triangle DEF$ and $\triangle K_aK_bK_c$ are homothetic. Then it suffices to show that D, E, F lie on UK_a, VK_b, WK_c, respectively, since the 1st Fontené theorem will yield that the exsimilicenter $T \equiv DK_a \cap EK_b \cap FK_c$ of $\odot(DEF) \sim \odot(G_aG_bG_c)$ lies on $\odot(G_aG_bG_c) \Longrightarrow \odot(G_aG_bG_c)$ and $\odot(DEF)$ are tangent through T.

We use barycentric coordinates with respect to $\triangle ABC$. Then

$$M_a(0:1:1) \ , \ M_b(1:0:1) \ , \ M_c(1:1:0)$$

$$K_a(0:b^2+S_C:c^2+S_B)$$

$$K_b(a^2+S_C:0:c^2+S_A)$$

$$K_c(a^2+S_B:b^2+S_A:0)$$

$$U \ ((a^2+b^2+c^2)(c^2-b^2):2(c^2-a^2)(b^2+S_A):2(a^2-b^2)(c^2+S_A))$$

$$D \ ((S_C-S_B)^2:S^2+S_C(a^2+4S_A):S^2+S_B(a^2+4S_A))$$

We verify that U, D, K_a are indeed collinear and the conclusion follows.

$$\begin{bmatrix} 0 & b^2+S_C & c^2+S_B \\ (a^2+b^2+c^2)(c^2-b^2) & 2(c^2-a^2)(b^2+S_A) & 2(a^2-b^2)(c^2+S_A) \\ (S_C-S_B)^2 & S^2+S_C(a^2+4S_A) & S^2+S_B(a^2+4S_A) \end{bmatrix} = 0$$

High School Olympiads

Nice problem about concyclic feet of altitudes

Reply

limac

#1 Feb 26, 2011, 5:36 am

Points D and E are chosen on the legs AC and CB of a right triangle $ABC.$ Prove that the feet of the perpendiculars from C to $DE, EA, AB,$ and BD lie on a circle.

Luis González

#2 Feb 26, 2011, 9:27 am • 1

Let M,N,P,Q be the orthogonal projections of C onto the lines $DE,EA,AB,DB.$ Then, B,P,Q,C are concyclic, due to the right angles $\angle CQB=\angle CPB=90^\circ.$ Hence, we have $\angle PQB=\angle PCB=\angle CAP \Longrightarrow A,P,Q,D$ are concyclic. Analogously, B,P,N,E are concyclic. C,M,N,E are concyclic, due to $\angle CNE=\angle CME=90^\circ$ and similarly C,M,Q,D are concyclic. Let $F \equiv CM \cap AB.$ Then N and Q are Miquel points of $\triangle BCF \cup MPE$ and $\triangle ACF \cup MPD \Longrightarrow F \in \odot(MNP)$ and $F \in \odot(MQP).$ Therefore, M,N,P,Q are concyclic.

Quick Reply

High School Olympiads

collinear

mousavi
#1 Feb 25, 2011, 7:22 pm

Let triangle ABC and its circumcircle and I is its incircle center. M and N are midpoints of arcs AB and AC, respectively and D is the midpoint of segment MN. G is point on arc BC. I_1 and I_2 are centers of incircles of triangles AGC and ABG. Circumcircles of triangles I_1I_2G and ABC intersect each other in point P. Prove that points D, I, P are collinear.

Luis González
#2 Feb 25, 2011, 10:07 pm

This configuration has been discussed before. For instance, see Geometry problem.

Redefine P as the intersection of $\overrightarrow{DI}$ with the circumcircle (O) of $\triangle ABC$. Then we shall show that $P \in \odot(GI_1I_2)$. Let L, L' be the midpoints of the arcs BC and CAB of the circumcircle (O). Since I is the orthocenter of its circumcevian triangle $\triangle MNL$, it follows that $L' \equiv DI \cap OL$, thus $AL' \parallel MN$ implies that PA is the P-symmedian of $\triangle PMN \Longrightarrow \frac{AN}{AM} = \frac{PN}{PM}$. Since $NA = NC = NI_1$ and $MA = MB = MI_2$, we have $\frac{NI_1}{MI_2} = \frac{PN}{PM}$. Then $\angle PNG = \angle PMG$ implies that $\triangle PNI_1$ and $\triangle PMI_2$ are similar by SAS criterion $\Longrightarrow \angle PI_1G = \angle PI_2G \Longrightarrow P \in \odot(GI_1I_2)$.

jayme
#3 Feb 26, 2011, 7:15 pm

Dear Mathlinkers,
this nice problem can be see as a result of a study on mixtilinear incircles.
For a synthetic proof, you can see
http://perso.orange.fr/jl.ayme vol. 4 A new mixtilinear incircle adventure I p. 41 than 38.
Use Google Translator.
Sincerely
Jean-Louis

mahanmath
#4 Feb 26, 2011, 7:19 pm

Infact P is the intersection of (O) and A-Appolonius of AMN , which is clearly implies the result .

CTK9CQT
#5 Aug 6, 2014, 7:22 am

@Luis: What is SAS criterion?

High School Olympiads

orthocenter

Reply

Akiyama

#1 Feb 24, 2011, 8:18 pm

Let ABC be a triangle such that it's circumcircle radius is equal to the radius of outer inscribed circle with respect to A.
Suppose that the outer inscribed circle with respect to A touches BC, CA, AB at M, N, L.
Prove that O (Center of circumcircle) is the orthocenter of MNL.

PS. iran tst 2006 day 2-5

Luis González

#2 Feb 24, 2011, 8:39 pm

Lines AM, BN, CL concur at the A-Gergonne point of $\triangle ABC$ and its isogonal conjugate T^* WRT $\triangle ABC$ is the exsimilicenter of (O) and the A-excircle (I_a) (extraversion of the well-known incircle case). Thus if $(O) \cong (I_a)$, then T^* is at infinity $\Longrightarrow T \in (O)$. Hence, LM is the polar of N WRT (O) and NM is the polar of L WRT $(O) \Longrightarrow LM \perp ON$ and $NM \perp OL \Longrightarrow O$ is orthocenter of $\triangle MNL$.

Quick Reply

High School Olympiads

Parallel

Reply

77ant

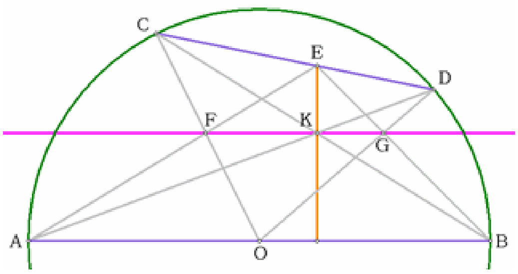
#1 Oct 28, 2009, 8:55 pm

Dear everyone.

Circle (O) has diameter AB and CD is a chord. $AD \cap BC = K$. $E \in CD$ such that $KE \perp AB$. Lines EA, EB cut OC, OD at F, G. Prove that F, K, G are collinear on a parallel to AB.

Thanks for your beautiful answers, in advance.

Attachments:

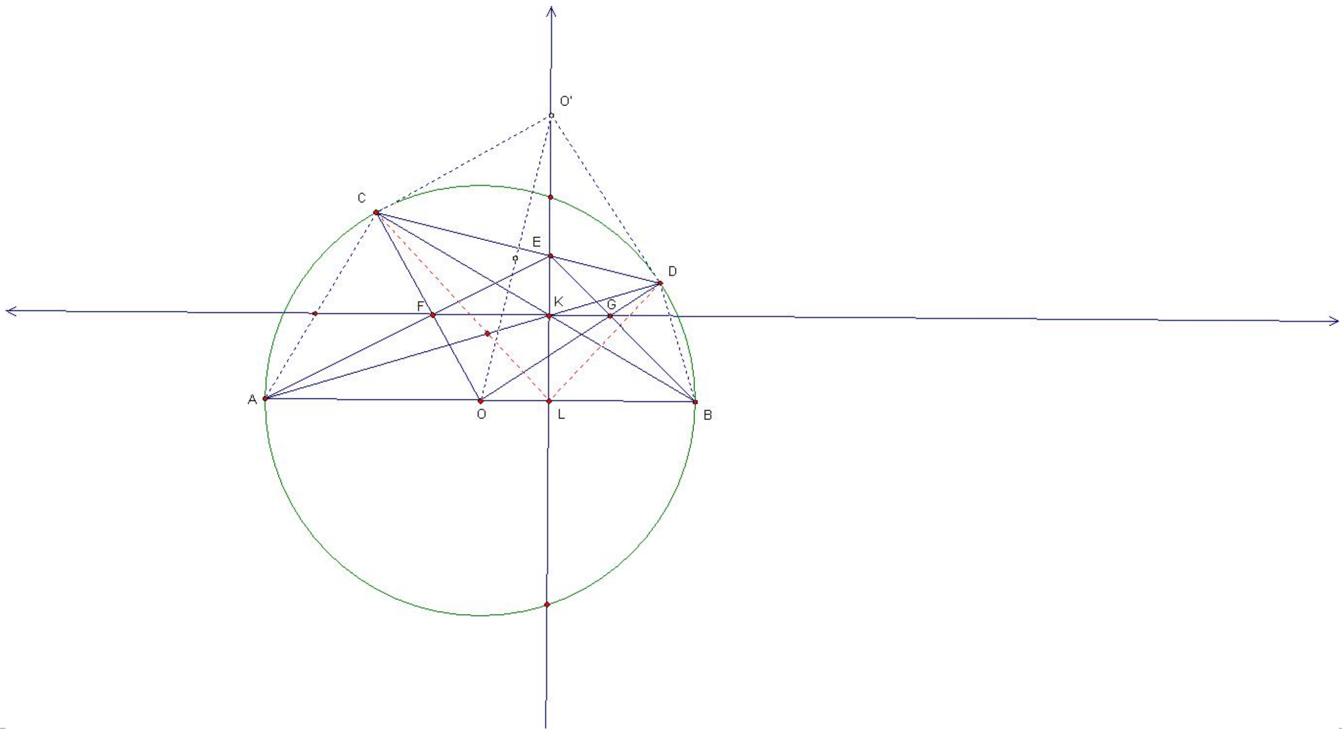


plane geometry

#2 Oct 29, 2009, 6:11 am

HI,dear 77ant:
I am busy recently so I just give hints:
I am sorry
1.denote O' is the circumcenter of CKD then O' is on KE
2.K is the incenter of triangle LCD
3.using sin law in triangleECA triangleEDB

Attachments:



sunken rock

#3 Nov 12, 2009, 12:19 am

Keeping the previous notations (**plane geometry's**), seeing that COLD is cyclic and OO' is a diameter of (COLD), let's note $\angle COD = 2\alpha, \angle AOC = \beta, \angle DOB = \gamma$. If $AB = 2R$, we shall get:

$CD = 2R \cdot sin\alpha;\ OO' = \frac{R}{cos\alpha};\ CL = \frac{R \cdot sin\beta}{cos\alpha};\ LD = \frac{R\sin\gamma}{cos\alpha}$.

With the previous remark (K- incenter of $\triangle CLD$) we shall find $\frac{KE}{KL} = \frac{sin2\alpha}{sin\beta + sin\gamma}$ then $CE = \frac{2R \cdot sin\alpha \cdot sin\beta}{sin\beta + sin\gamma}$.

Further, applying the sine theorem in the triangles $\triangle CFE\ and \triangle OFA$ we find $\frac{KE}{KL} = \frac{EF}{AF}$, hence $KF \parallel AL$.

Seeing that F, K and G are collinear from Pappus, we are done.

Best regards,
sunken rock

Luis González

#4 Feb 24, 2011, 10:58 am

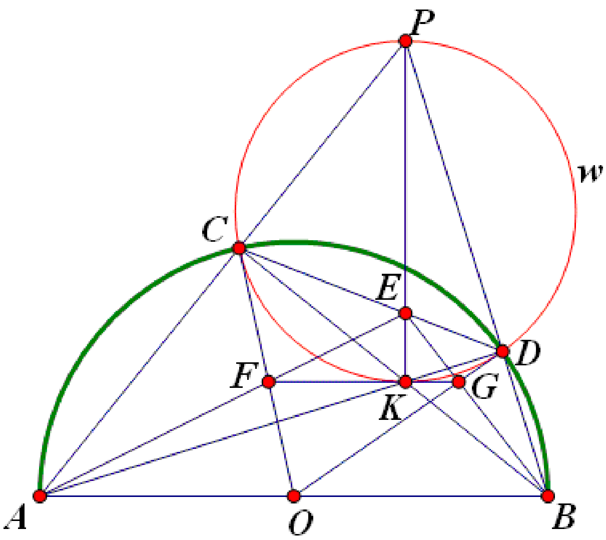
Let $P \equiv AC \cap BD \Longrightarrow K$ is orthocenter of $\triangle PAB$. Thus, circle ω with diameter PK is orthogonal to (O), i.e. OD and OC are tangent to ω. Tangent of ω through K cuts OC and OD at F' and G'. Then B, E, G' lie on the polar of A WRT ω and A, E, F' lie on the polar of B WRT $\omega \Longrightarrow F \equiv F', G \equiv G'$. Thus K, F, G lie on a parallel to AB, as desired.

yunxiu

#5 Feb 24, 2011, 1:37 pm

Another way of finishing Luis' idea
Suppose ω is the circle with diameter PK, than $\angle OCK = \angle OBC = \angle CPK$, so OC is tangents to ω, and so does OD.
Let's see ω, the polar of B and the polar of C pass F, so CKB is the polar of F, hence FK is tangents to ω, and so does GK.
So F, K, G are collinear, and $FG \perp PK$, so $FKG \parallel AB$.

Attachments:



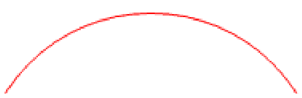
This post has been edited 1 time. Last edited by yunxiu, Feb 25, 2011, 6:37 pm

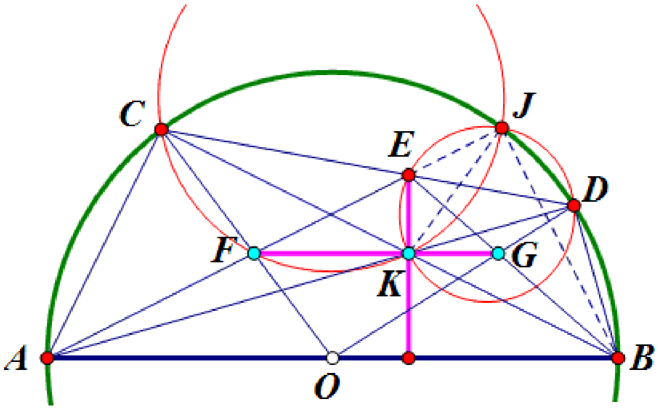
lym

#6 Feb 24, 2011, 4:30 pm

Let AE intersect (O) at the second point J□then easy to see $EKDJ$ is a circle□so $\angle FJK = \angle EDK = \angle FCK$□so $CFKJ$ is a circle .
So Acording to **Reim's Theorem** we get $FK \parallel AB$□and apply **Pappus's Theorem** we have FKG is a line□So $FG \parallel AB$.

Attachments:





vittasko

#7 Feb 24, 2011, 5:18 pm • 2

The problem is also true, in general configuration of an arbitrary circle (O), taken as cords the segments $AB,\ CD.$

So, we denote the points $K \equiv AD \cap BC,\ P \equiv AC \cap BD,\ E \equiv CD \cap PK$ and we will prove that the points $F \equiv AE \cap MC,\ K,\ G \equiv BE \cap MD$, where M is the midpoint of AB, are collinear with $FG \parallel AB.$

$($ As **Luis** said before, the point K is coincided with the orthocenter of the triangle $\triangle PAB$ and then, the line segment $KE \perp AB$, passes through the point P $).$

• Applying the **Pappos theorem**, for the triads of points $A,\ M,\ B$ and $C,\ E,\ D$, on $AB,\ CD$ respectively, we have the collinearity of $F,\ K,\ G$, as well.

Applying again the **Pappos theorem**, for the triads of points $A,\ M,\ B$ and $D,\ E,\ C$, we have also the collinearity of the points $R \equiv AE \cap DM,\ P,\ Q \equiv BE \cap CM.$

We denote the points $S \equiv AB \cap CD,\ T \equiv AB \cap PK$ and from the complete quadrilateral $PCKDAB$, we have that the points $S,\ A,\ T,\ B$, are in harmonic conjugation and so, we conclude that the line segment XY, where $X \equiv PA \cap BE,\ Y \equiv PB \cap AE$, passes through the point S, as the harmonic conjugate of T, with respect to $A,\ B$ $($ consider the complete quadrilateral $PXEYAB$ $).$

Because of the pencil $S.ACXP$ is harmonic now, we have that the pencil $Q.ACXP$ is also harmonic and from $MA = MB$, (1) we conclude that $QR \parallel AB$, (2)

• From (2), applying the **Thales theorem**, $\Longrightarrow \dfrac{MG}{GR} = \dfrac{MB}{QR}$, (3) and $\dfrac{MF}{FQ} = \dfrac{MA}{QR}$, (4)

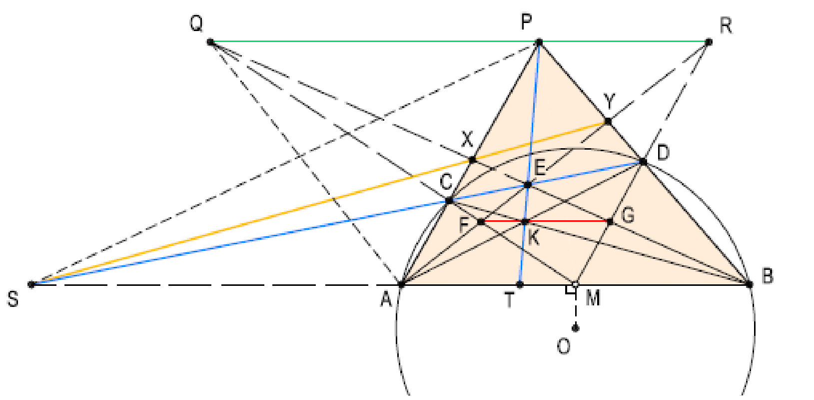
From $(1),\ (3),\ (4) \Longrightarrow \dfrac{MG}{GR} = \dfrac{MF}{FQ} \Longrightarrow FG \parallel QR$, (5)

From $(2),\ (5) \Longrightarrow FG \parallel AB$ and the proof is completed.

Kostas Vittas.

PS. Hence (because of we didn't need any property of the cyclic quadrilateral), we can say the same words about the proof in the more general configuration of an arbitrary quadrilateral $ABDC$, not necessary cyclic.

Attachments:



This post has been edited 2 times. Last edited by vittasko, Feb 24, 2011, 11:11 pm

skytin

#8 Feb 24, 2011, 9:25 pm

Let BD and AC intersects at H
Let line BE intersect (O) at points B and X
BE*EX = CE*ED = KE*EH , so BKXH is cyclic
angle KXB = KHB = BAK = ADO , so DGKX is cyclic
Let circle (GKX) intersect DH at points D and Y
Use Reim's theorem , so GY || to tangent from B to (AB) , so GY || KH
So angle YGK = YDA = 90 = GKH
Like the same angle EKF = 90 . Problem done

Quick Reply

High School Olympiads

Circles And Locus

Reply

Headhunter

#1 Feb 23, 2011, 10:33 pm

Hello.

Given two circles (O), (P) , then a variable circles (X) is tangent to (O), (P) at A, B externally.
A variable circle (Y) is tangent to (O), (P) at C, D internally. A, B, C, D are collinear.
Let l be the radical axis of (X), (Y) and l meet XY at K. Find the locus of K

Luis González

#2 Feb 23, 2011, 11:32 pm • 1

Let U be the exsimilicenter of $(O) \sim (P)$. A is the insimilicenter $(O) \sim (X)$ and B is the insimilicenter of $(P) \sim (X)$. By Monge & d'Alembert theorem, we get that $U \in AB$. Since U is also the center of the positive inversion taking (O), (P) into each other, it follows that U has equal power to (X) and $(Y) \Longrightarrow U \in \ell \ (\star)$. On the other hand, since $OA \parallel PD$ and $OC \parallel PB \Longrightarrow XPYO$ is a parallelogram, thus XY always passes through the midpoint V of OP. Therefore, $\ell \perp XY$ and $(\star)$ imply that the locus of K is the circle with diameter UV.

Quick Reply

High School Olympiads

Geometry

Reply

letrongquang1995

#1 Feb 23, 2011, 8:38 pm

Given two segments AB and CD not in the same plane, find the locus of points M such that AM^2+BM^2=CM^2+DM^2

Luis González

#2 Feb 23, 2011, 9:24 pm

Let E,F be the midpoints of $AB,CD.$ By Apollonius theorem, we have:

$$ME^2=\frac{1}{2}(MA^2+MB^2)-\frac{1}{4}AB^2 \ , \ MF^2=\frac{1}{2}(MC^2+MD^2)-\frac{1}{4}CD^2$$

$$MA^2+MB^2=MC^2+MD^2 \Longrightarrow \ ME^2-MF^2=\frac{1}{4}(CD^2-AB^2)=\text{const}$$

Locus of M is a plane perpendicular to the line connecting the midpoints of $AB,CD.$

Quick Reply

High School Olympiads

Finding the area of a triangle

Reply

alphabeta1729

#1 Feb 21, 2011, 8:32 pm

In $\triangle ABC, X, Y$ are points on the sides AC, BC. If Z is on segment XY such that :

$$\frac{AX}{XC}=\frac{CY}{YB}=\frac{XZ}{ZY}$$

then prove that : $[ABC]=\left([AXZ]^{1/3}+[BYZ]^{1/3}\right)^3$; where $[PQR]$ denotes the area of $\triangle PQR$

Luis González

#2 Feb 21, 2011, 11:37 pm

Let $\frac{AX}{XC}=\frac{CY}{YB}=\frac{XZ}{ZY}=k.$ Then we have:

$$\frac{[AXZ]}{[AXY]}=\frac{[AXY]}{[ACY]}=\frac{[ACY]}{[ABC]}=\frac{k}{1+k}\Longrightarrow\frac{[AXZ]}{[ABC]}=\left(\frac{k}{1+k}\right)^3$$

$$\frac{[BYZ]}{[BYX]}=\frac{[BYX]}{[BCX]}=\frac{[BCX]}{[ABC]}=\frac{1}{1+k}\Longrightarrow\frac{[BYZ]}{[ABC]}=\left(\frac{1}{1+k}\right)^3$$

$$\Longrightarrow\sqrt[3]{[AXZ]}+\sqrt[3]{[BYZ]}=\sqrt[3]{[ABC]}\left(\frac{k}{1+k}+\frac{1}{1+k}\right)=\sqrt[3]{[ABC]}$$

Quick Reply

High School Olympiads

Area of a triangle is a GM of area of two other triangles

Reply

alphabeta1729

#1 Feb 21, 2011, 8:27 pm

In $\triangle ABC$, D, E, F are points on the sides BC, CA, AB respectively. Also, A, B, C are points on sides YZ, ZX, XY of $\triangle XYZ$ for which $EF \parallel YZ$, $FD \parallel ZX$, $DE \parallel XY$.

Then prove that $[ABC]^2 = [DEF] \cdot [XYZ]$; where $[PQR]$ denotes the area of $\triangle PQR$

Luis González

#2 Feb 21, 2011, 9:49 pm

Since $\triangle XYZ$ and $\triangle DEF$ have corresponding parallel sides, XD, YE, ZF concur at their homothetic center H. Let M, N, L be the projections of H onto YZ, ZX, XY.

$[ABC] = [HEAF] + [HFBD] + [HDCE]$, (1)

$[XYZ] = [HYZ] + [HZX] + [HXY] = \frac{HM \cdot YZ + HN \cdot ZX + HL \cdot XY}{2}$, (2)

$[HEAF] = \frac{HM \cdot EF}{2}$, $[HFBD] = \frac{HN \cdot FD}{2}$, $[HDCE] = \frac{HL \cdot DE}{2}$

Combining these three latter expressions with (1) gives

$[ABC] = \frac{HM \cdot EF + HN \cdot FD + HL \cdot DE}{2}$

But $\triangle DEF \sim \triangle XYZ \Longrightarrow \frac{YZ}{EF} = \frac{ZX}{FD} = \frac{XY}{DE} = k$

$[ABC] = \frac{HM \cdot YZ + HN \cdot ZX + HL \cdot XY}{2k}$

Combining this latter expression with (2) yields $[ABC] = \frac{[XYZ]}{k}$

Thus, from $\frac{[XYZ]}{[DEF]} = k^2$, we get that $[ABC] = \sqrt{\frac{[DEF]}{[XYZ]}} \cdot [XYZ]$

$\Longrightarrow [ABC]^2 = [XYZ] \cdot [DEF]$.

Quick Reply

High School Olympiads

Intersection of Four Circles forms a cyclic quadrilateral?

Reply

Yunone

#1 Feb 20, 2011, 7:57 am

Hello, I'm trying to go through the section of Newer Results in Hartshorne's Geometry: Euclid and Beyond.

This particular exercise has been bugging me for a good while:
I have uploaded below a picture of the particular exercise.

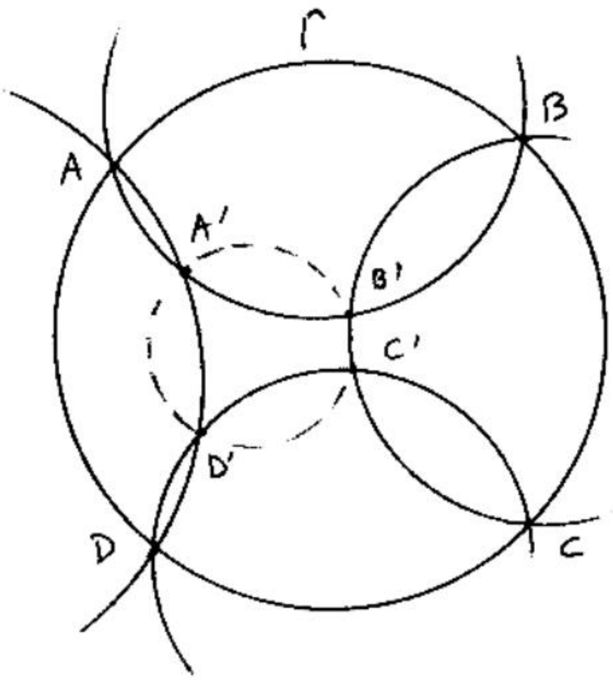
My first approach was to show that the perpendicular bisectors of $A'B'$ and $A'D'$ intersect at the same point as the perpendicular bisectors of $A'B'$ and $B'C'$. I figured since the center of the circle circumscribed around a triangle has its center at the intersection of the three perpendicular bisectors, this would show that A', B', C' and D' would all be on the same circle. However, I didn't see any way to show this.

I also tried extending $A'D'$ and $A'C'$ down into the circle containing D, D' and C with hopes that it $\angle D'A'C' \cong \angle D'B'C'$, but this seemed like a dead end also.

I'm going a little mad, if any one sees a possible solution, I'd be quite grateful. Thanks.

Attachments:

5.17 Let A, B, C, D be four points on a circle Γ. Let four more circles pass through AB, BC, CD, DA, respectively, meeting in further points A', B', C', D'. Show that $A'B'C'D'$ is a cyclic quadrilateral.



yetti

#2 Feb 20, 2011, 8:48 am

You can use inversion centered at one 3-circle intersection point, say B. This takes circles $\odot(AA'B'B), \odot(BB'C'C), \odot(ABCD)$ into lines forming a triangle $\triangle A_iB_i'C_i$ with points $A_i' \in A_iB_i', C_i' \in B_i'C_i, D_i \in C_iA_i$. Circles $\odot(CC'D'D), \odot(DD'A'A)$ go to circles $\odot(C_iC_i'D_i'D_i), \odot(D_iD_i'A_i'A_i)$. Then $\angle B_i'A_i'D_i' + \angle D_i'C_i'B_i' = \angle D_i'D_iA_i + \angle C_iD_iD_i' = \angle C_iD_iA_i = 180^\circ \implies A_i'B_i'C_i'D_i'$ is cyclic (Miquel theorem) $\implies A'B'C'D'$ is also cyclic.

Luis González

#3 Feb 20, 2011, 9:06 am

For the sake of ease when referring angles, let $X \equiv AA' \cap DD', Y \equiv BB' \cap CC'$.

$\angle YB'C' = \angle BCC'$, $\angle YB'A' = \angle BAA'$

$\angle XD'A' = \angle DAA'$,$\angle XD'C' = \angle DCC'$

$\angle A'B'C' + \angle A'D'C' = \angle BAA' + \angle BCC' + \angle DAA' + \angle DCC' = 180^\circ$.

Yunone

#4 Feb 20, 2011, 9:13 am

Thank you both, yetti and luisgeometra!

These are quite helpful solutions that I don't think I would have seen myself any time soon.

jayme

#5 Feb 20, 2011, 1:48 pm

Dear Mathlinkers,
in order to have a general look, you can see
http://perso.orange.fr/jl.ayme vol. 2 Du théorème de Reim au théorème des six cercles p.8-...
Use Google translator
Sincerely
Jean-Louis

soulhunter

#6 Feb 20, 2011, 2:35 pm

hey this is micquel's six circle thoerm

Quick Reply

High School Olympiads

Square Problem (own)

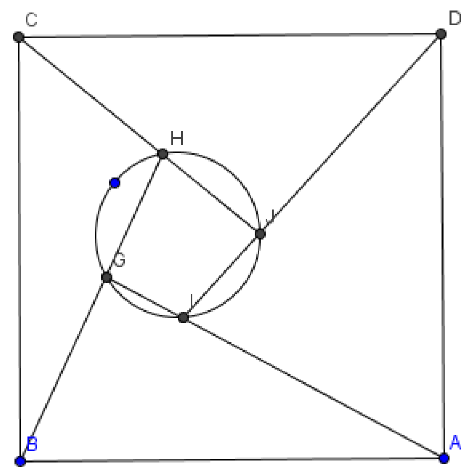
Reply

skytin

#1 Feb 19, 2011, 8:08 pm

Given Square ABCD and circle w
Construct Quadrilateral JIGH were A is on IG , B is on GH , C is on HJ and D is on JI

Attachments:



Luis González

#2 Feb 19, 2011, 10:42 pm

In fact, A,B,C,D can be vertices of any quadrilateral.

Problem: Given a circle ω and 4 points P_1, P_2, P_3, P_4 arranged on its plane, then find a quadrangle $ABCD$ whose consecutives sidelines AB, BC, CD, DA pass through P_1, P_2, P_3, P_4.

Solution: Pick an arbitrary point A_0 on ω. If P_1A_0 cuts ω again at A_1, then the relation between A_0 and A_1 is projective, namely an involution with center P_1. Similarly, let P_2A_1 cut ω again at A_2. Line P_3A_2 cuts ω again at A_3 and P_4A_3 cuts ω again at A' $\Longrightarrow \mathcal{F} : A_0 \mapsto A'$ is a circular projectivity. Thus, the possible vertices A are the double points of $\mathcal{F}$, i.e. repeat the same construction for two more points, say B_0, C_0 to obtain B', C', then three pairs of homologous points A_0, B_0, C_0 and A', B', C' are sufficient to construct the axis f of $\mathcal{F} \Longrightarrow f$ passes through $A_0B' \cap A'B_0$ and $A_0C' \cap A'C_0 \Longrightarrow A \equiv f \cap \omega$.

P.S. Note that this reasoning can be extented exactly in the same way for a cyclic n-gon whose sidelines have to pass through n arranged points. (extension of Castillon's problem).

Quick Reply

High School Olympiads

Please Help... point inside a Paralerogram

Reply

paul1703
#1 Feb 19, 2011, 2:44 am

Let M be a point inside the paralelogram ABCD prove that <MDA=<MBA if and only if <MAD=<MCD

Luis González
#2 Feb 19, 2011, 4:46 am

I think this problem comes from the 2009 Ibero team selection test

Let $X \equiv AM \cap CD$ and $Y \equiv CM \cap AD$. Note that quadrilateral $ACXY$ is cyclic $\Longleftrightarrow \angle DAX = \angle DCY \Longleftrightarrow \angle YXM = \angle MCA$ and $\angle XYM = \angle MAC$. Since $\angle MXD = \angle MAB$ and $\angle MYD = \angle MCB$, it follows that $MYDX$ and $MABC$ are similar $\Longleftrightarrow \angle MBA = \angle MDA$.

yetti
#3 Feb 19, 2011, 5:07 am

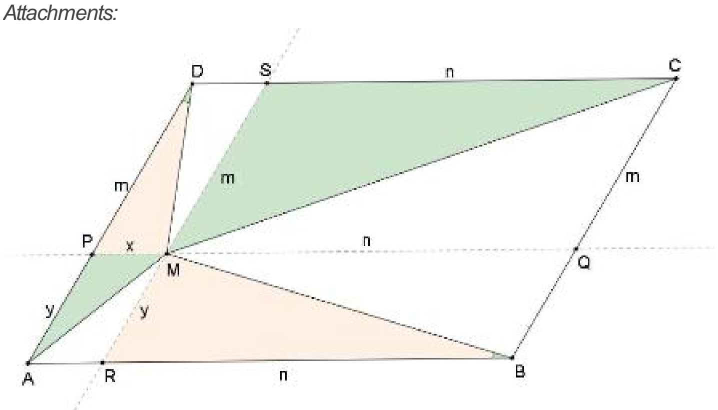
1. A, B, C, D are not orthocerntric $\Longrightarrow \exists$ unique normal hyperbola $\mathcal{H}$ through A, B, C, D. Centerr of $\mathcal{H}$ is $E \equiv AC \cap BD$. This hyperbola is isogonal conjugate of the perpendicular bisector of BD WRT $\triangle ABD$ and isogonal conjugate of the perpendicular bisector of AC WRT $\triangle ACD$. $\angle MDA = \angle MBA \Longleftrightarrow M \in \mathcal{H} \Longleftrightarrow \angle MAD = \angle MCD$.

2. $\angle MDA = \angle MBA \Longrightarrow$ circumcircles of $\triangle ABM, \triangle AMD$ are congruent. Translate M by $\overrightarrow{AD}$ into M'. $\triangle ABM \cong \triangle DCM'$ and $\triangle AMD \cong \triangle M'DM \Longrightarrow$ circumcircles of $\triangle DCM', \triangle M'DM$ are congruent $\Longrightarrow DMCM'$ is cyclic $\Longrightarrow$ circumcircles of $\triangle CMD, \triangle AMD$ are congruent $\Longrightarrow \angle MAD = \angle MCD$. Similarly, $\angle MAD = \angle MCD \Longrightarrow \angle MDA = \angle MBA$.

estoyanovvd
#4 Aug 2, 2011, 10:34 pm • 1

Let $PQ \parallel AB, P \in AD, Q \in BC$ and $RS \parallel AD, R \in AB, S \in CD$. Then $\Delta DMP \sim \Delta BMR \Rightarrow \frac{x}{y} = \frac{m}{n} \Rightarrow \Delta AMP \sim \Delta CMS \Rightarrow \measuredangle PAM = \measuredangle SCM$. Done.

Attachments:



Quick Reply

High School Olympiads

Euler line

Reply

mousavi
#1 Feb 18, 2011, 9:47 pm

Let triangle ABC and H is its orthocenter. Circumcircles of triangles ABH and AHC intersect with segment AC and AB in points M and N. Prove that Euler line of triangle ABC pass from circumcenter of triangle MHN.

mahanmath
#2 Feb 18, 2011, 10:59 pm

Too easy with complex numbers

(Of course I don`t call it a solution)

mousavi
#3 Feb 18, 2011, 11:17 pm

> *mahanmath wrote:*
> Too easy with complex numbers
>
> (Of course I don`t call it a solution)

can you write it?

Luis González
#4 Feb 18, 2011, 11:47 pm

D, E, F are the feet of the altitudes onto BC, CA, AB. Inversion through pole H with power $\overline{HA} \cdot \overline{HD}$ takes the sidelines AC, AB into the circles with diameters HB, HC and the circles $\odot(AHC)$ and $\odot(AHB)$ into the lines DF, DE. Thus, DE, DF cut the circles with diameters HB, HC again at the inverses M', N' of $M, N \Longrightarrow M'N'$ is the inverse of $\odot(HMN)$. Hence, it suffices to show that $M'N'$ is perpendicular to the Euler line OH of $\triangle ABC$, i.e. the diacentral line of $\triangle DEF$. Note that M', N' are the reflections of F, E across EH, FH, i.e. $EF = EM' = FN'$, thus according to this topic (see replies #6 and #12) we get that $M'N' \perp OH$, as desired.

P.S. See also APMO 2010 problem 4 for alternate solutions.

Quick Reply

High School Olympiads

cyclic quadrilaterals

Reply

ridgers

#1 Feb 18, 2011, 10:06 pm

Let A and B be the common points of two circles. A line passing through A intersects the circles at C and D. Let P and Q be the projections of B onto the tangents to the two circles at C and D. Prove that PQ is tangent to the circle of diameter AB.

Luis González

#2 Feb 18, 2011, 10:37 pm

Let $(O_1),(O_2)$ denote the given circles meeting at $A,B.$ Tangents to $(O_1),(O_2)$ through C,D meet at $E.$ Since $\angle BCA=\angle ECD$ and $\angle BDA=\angle EDC,$ it follows that B lies on the circle $\odot(DEC) \Longrightarrow$ Orthogonal projections P,Q,R of B onto the lines EC,ED,CD are collinear on the Simson line of B with respect to $\triangle ECD.$ Since $\angle CRP=\angle CBP$ and $\angle CBP=\angle CBO_1=\angle ABR,$ we deduce that $PR \equiv PQ$ is tangent to the circle with diameter AB through $R.$

Quick Reply

High School Olympiads

Parallelogram

Reply

Swistak

#1 Feb 18, 2011, 1:20 am

Let ABCD be a given parallelogram. E and F are points satisfying $\angle DAE=\angle FAB$ and $E\in DC,\quad F\in BC.$ Let P be the common point of DF and BE. Let l be a line parallel to DC and passing through P. l intersects AC in Q. Prove that PQFB is a cyclic quadrilateral.

Luis González

#2 Feb 18, 2011, 11:32 am

$\frac{RQ}{AB}=\frac{CR}{CB} \ , \ \frac{RP}{AB}=\frac{RF}{CF} \Longrightarrow RQ \cdot RP=AB^2 \cdot \frac{CR}{CB} \cdot \frac{RF}{CF} \ \ (1)$

By Menelaus' theorem for $\triangle BEC$ cut by $\overline{DPF}$, we get $\frac{DE}{DC} \cdot \frac{CF}{FB} \cdot \frac{BP}{PE}=1$

Since $\frac{DE}{FB}=\frac{CB}{AB}$ and $\frac{BP}{PE}=\frac{RB}{CR}$, then we obtain $\frac{AB^2}{CB}=\frac{CF \cdot RB}{CR} \ \ (2)$

From (1) and (2), we get $RQ \cdot RP=RF \cdot RB \Longrightarrow P,Q,F,B$ are concyclic.

Quick Reply

High School Olympiads

in trapezoid ABCD ....

Reply

alirezamath

#1 Feb 17, 2011, 9:49 pm

in trapezoid ABCD lateral side CD is perpendicular to AD & BC(trapezoid`s bases).
a circle with diameter AB meet AD at P. tangent to the circle at point P, meet CD at point M. MQ is another tangent to circle.if BQ meet CD at K.
prove KC=KD

Luis González

#2 Feb 18, 2011, 1:15 am

Define the intersections $U \equiv PB \cap AQ$ and $V \equiv BQ \cap AD$. Then U, V are conjugate points with respect to the circle ω with diameter AB, i.e. circle with diameter UV, intersecting ω at P, Q, is ortogonal to ω, i.e. M is the midpoint of $UV \Longrightarrow \triangle MPV$ is isosceles with apex $M \Longrightarrow DV = DP = BC \Longrightarrow K \equiv CD \cap BQ$ is the midpoint of CD.

mousavi

#3 Feb 18, 2011, 1:50 am

let $\angle ABP = \alpha \Longrightarrow \angle DPM = \angle AQP = \alpha$ and $\angle QBA = x$
$\Longrightarrow \angle APQ = \angle MQA = x$
$\angle DMP = \angle PQM = 90 - \alpha \Longrightarrow MKQP$ is cyclic.
so $\angle MKP = \angle MQP = x + \alpha$ (1)
$\angle KBC = 90 - x - \alpha \Longrightarrow \angle CKB = x + \alpha$ (2)

(1),(2) $\Longrightarrow \angle BKC = \angle PKD \Longrightarrow DK = KC$

Quick Reply

High School Olympiads

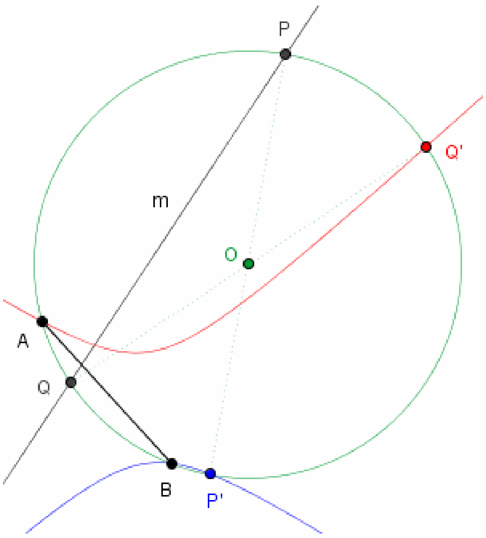
locus of antipode

Reply

77ant
#1 Sep 16, 2010, 1:42 am

Dear everyone
A, B are two fixed points, and m is a fixed line. A variable circle through A, B cut m at P, Q.
Let P', Q' be the antipodal points of P, Q. Find the locus of P', Q'.
May I ask you synthetic solutions (projective as possible) ? My analytic one isn't beautiful.

Attachments:



Luis González
#2 Feb 17, 2011, 11:48 pm

Let H be the orthocenter of $\triangle PAB.$ Since $AHBP'$ is a paralelogram, then it suffices to find the locus $\mathcal{H}$ of $H.$ Define the rectangular reference $A:(0,0), B:(c,0), H:(x,y)$ and P moves along the line $m \equiv py+qx+r=0.$ If E denotes the projection of P onto the line $AB,$ by orthocenter property, we get then

$$\overline{PE}\cdot\overline{HE}=\overline{EA}\cdot\overline{EB} \Longleftrightarrow -\frac{qx+r}{p}\cdot y=x(c-x)$$

$$\Longleftrightarrow \ px^2-ry-pcx-qxy=0$$

In general, locus of H is the hyperbola $\mathcal{H} \equiv px^2-ry-pcx-qxy=0$ passing through $A,B.$ Thus, locus of P',Q' is the hyperbola $\mathcal{H}'$ symmetrical to $\mathcal{H}$ about the midpoint of $AB.$

lym
#3 Feb 19, 2011, 3:36 pm

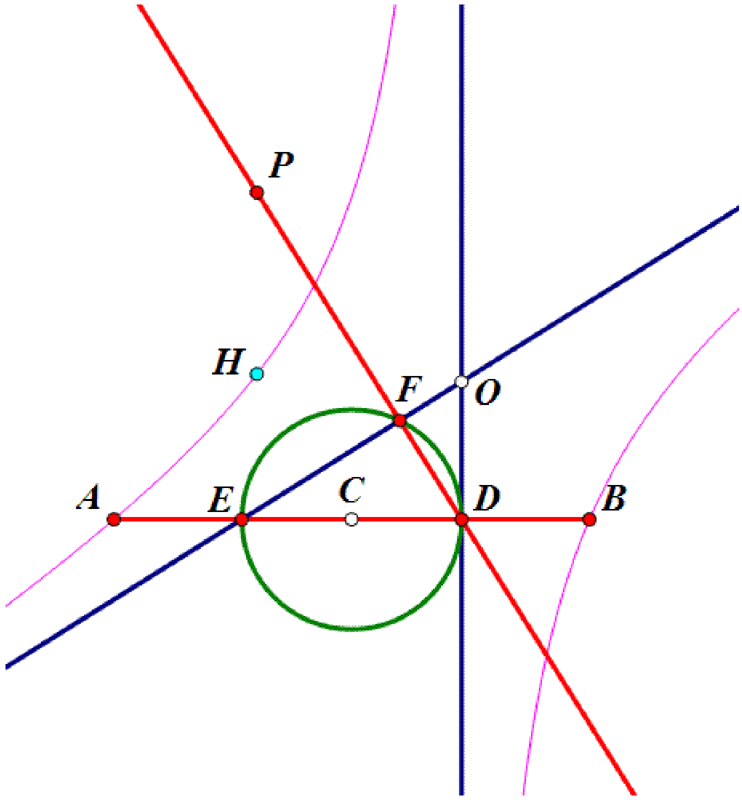
Let PQ intersect AB at D□C the midpoint of AB□$\odot(C,CD)$ cut AB□PD at E□F resp□a perpendicular through D to AB intersect EF at O .

□1□□Let P^* be the reflection of P about D□H□H' are the orthocenters of $\triangle PAB$□$\triangle P^*AB$ resp□then O is the midpoint of HH' .
□2□□Let A'□B' be the reflection of A□B about O resp□then the locus of H pass through A□A'□B□B'□so A□A'□B□B'□H□H' are on a conic $\mathcal{H}$ which's center is O□on the other hand□since H cann't be on OD□OE□so $\mathcal{H}$ must be a hyperbola and OD□OE are it's asymptotes□it means that H is on a fixed hyperbola $\mathcal{H}$ which pass through A□A'□B□B' and it's asymptotes are OD□OE . When $PQ \perp AB$□then $\mathcal{H} \equiv PQ$. When $PQ \parallel AB$□then O become

infinity□$\mathcal{H}$ degenerate parabola .

So□locus of P'□Q' is $\mathcal{H}'$ symmetrical to $\mathcal{H}$ about C .

Attachments:



lym

#4 Feb 19, 2011, 4:16 pm

Sorry□I'v found my analysis is chaos and not correct□can anybody help me to correct it □

Quick Reply

High School Olympiads

Speiker circle

Reply

alphabeta1729

#1 Feb 17, 2011, 9:32 pm

If A', B' and C' are the midpoints of sides BC, CA and AB of $\triangle ABC$ respectively; S is the incenter of $\triangle A'B'C'$ and $AS \cap BC = D$.

Then prove that $\frac{BD}{DC} = \frac{a+b}{a+c}$

Luis González

#2 Feb 17, 2011, 10:55 pm

X, Y are the projections of S on AB, AC. Thus $, SX = \frac{1}{2}(h_c - r), SY = \frac{1}{2}(h_b - r)$

$$\frac{BD}{DC} = \frac{c}{b} \cdot \frac{SX}{SY} = \frac{c \cdot (h_c - r)}{b \cdot (h_b - r)} = \frac{r \cdot (2s - c)}{r \cdot (2s - b)} = \frac{a+b}{a+c}$$

● Alternatively, we can use the barycentrics of S, namely $X_{10} : (b + c : c + a : a + b)$

Quick Reply

High School Olympiads

fixed point

Reply

macrosoft

#1 Feb 17, 2011, 8:48 pm

. Let B and C be points on two given rays from the same point A, such that AB + AC is a constant. Show that there is a point D not same as A such that the circumcircles of the triangle ABC passes through D for all choices of B and subject to constraint above.

Luis González

#2 Feb 17, 2011, 10:30 pm

$AB+AC=k=\text{const}.$ Let X,Y be fixed points of the rays AB,AC such that $AX=AY=\frac{1}{2}k.$ Then we have $\overline{XB}=\overline{YC}=\frac{1}{2}|AC-AB|.$ Circles $\odot(ABC)$ and $\odot(AXY)$ meet at A and the center D of the rotation taking the oriented segments $\overline{XB} \cong \overline{YC}$ into each other $\Longrightarrow DB=DC, DX=DY \Longrightarrow D$ is the midpoint of the arc XY of $\odot(AXY) \Longrightarrow \odot(ABC)$ pass through A and its antipode WRT $\odot(AXY).$

Quick Reply

High School Olympiads

Geometry

Reply

lim phalkun

#1 Feb 14, 2011, 7:02 pm

The diagonals [AC] and [BD] of a cyclic quadrilateral ABCD meet at right in I .
Prove that :
$\frac{1}{IA^2}+\frac{1}{IB^2}+\frac{1}{IC^2}+\frac{1}{ID^2}>=\frac{4}{R^2}$
. Where R is the radius of the circumscribing circle .

Luis González

#2 Feb 16, 2011, 12:08 pm

Let (O,R) be the circumcircle of $ABCD.$ From the power of I to $(O,R),$ we get

$$\frac{1}{IA^2}+\frac{1}{IB^2}+\frac{1}{IC^2}+\frac{1}{ID^2}=\frac{IA^2+IB^2+IC^2+ID^2}{(R^2-IO^2)^2} \Longrightarrow$$

$$\frac{1}{IA^2}+\frac{1}{IB^2}+\frac{1}{IC^2}+\frac{1}{ID^2} \ge \frac{IA^2+IB^2+IC^2+ID^2}{R^4}$$

But we know that $IA^2+IB^2+IC^2+ID^2=4R^2.$ For instance, see this topic

$$\frac{1}{IA^2}+\frac{1}{IB^2}+\frac{1}{IC^2}+\frac{1}{ID^2} \ge \frac{4R^2}{R^4}=\frac{4}{R^2}.$$

Quick Reply

High School Olympiads

Pedal Triangle

Reply

Headhunter
#1 Feb 2, 2011, 12:06 pm

Hello.

Given $\triangle ABC$, find the location of P such that its pedal triangle is an isosceles right-angled triangle.

Luis González
#2 Feb 16, 2011, 6:25 am

Let P_1, P_2, P_3 be the orthogonal projections of P on the sidelines $BC, CA, AB.$ Assume that $\triangle P_1P_2P_3$ is isosceles right with apex $P_1.$ By generalized Simson theorem, $\triangle P_1P_2P_3$ is similar to the triangle $\triangle A'B'C'$ formed by the inverses A', B', C' of A, B, C under any inversion with center P and arbitrary power $k^2.$ Thus, $P_1P_2 = P_1P_3 \Longleftrightarrow A'B' = A'C'.$ By inversion properties, we get

$$\frac{A'B'}{AB} = \frac{k^2}{PA \cdot PB} \ , \ \frac{A'C'}{AC} = \frac{k^2}{PA \cdot PC} \Longrightarrow \frac{AB}{AC} = \frac{PB}{PC}$$

Hence, P lies on the A-Apollonius circle of $\triangle ABC.$ On the other hand, we have that $\angle P_2P_1P_3 = 90^\circ,$ thus from the cyclic quadrilaterals PP_1BP_3 and PP_1CP_2 we deduce that $\angle PBA + \angle PCA = 90^\circ \Longrightarrow \angle BPC = 90^\circ + \angle BAC \pmod \pi).$ In other words, if the perpendicular to AC through C cuts AB at $D,$ then P lies on the circle $\odot(BCD).$ Therefore, A-Apollonius circle of $\triangle ABC$ and $\odot(BCD)$ intersect at two points whose pedal triangles are isosceles right with apex on $BC.$ Repeating the same construction for CA, AB yields at most 6 distinct points whose pedal triangles with respect to $\triangle ABC$ are isosceles right.

Quick Reply

High School Olympiads

Hard geometry problem (own)

Reply

jayme
#1 Jan 8, 2011, 2:03 pm

Dear Mathlinkers,
it is known that
1. the extangent triangle and the orthic triangle of ABC being homothetic, are perspective.
2. The radical triangle determined by the radical axis of the circumcircle with the resp. three excircles of ABC, is also perspective to ABC.

My question :
how can we prove synthetically that the two perspective centers are the same?

Sincerely
Jean-Louis

jayme
#2 Jan 8, 2011, 8:38 pm

Dear Mathlinkers,
no one has a little idea?
Sincerely
Jean-Louis

Luis González
#3 Jan 9, 2011, 11:09 am

Dear Jean-Louis, your configuration can be generalized but I only have a proof with barycentric coordinates:

P is a point in the plane of $\triangle ABC$. ℓ_a is the radical axis of $\odot(PBC)$ and the A-excircle of $\triangle ABC$. ℓ_b, ℓ_c are defined similarly. Then ℓ_a, ℓ_b, ℓ_c bound a triangle perspective with $\triangle ABC$. If P lies on the circumcircle of $\triangle ABC$, the perspector is the Clawson point of $\triangle ABC$, i.e. homothetic center of the orthic triangle and extangents triangle.

k.l.l4ever
#4 Jan 9, 2011, 2:40 pm

Dear Jean Louis,
In my figure which drew by Geometer's Sketchpad,two perspective center you mentioned are not the same.
Can you check your result again,please?
Thank you.

jayme
#5 Jan 9, 2011, 4:06 pm

Dear k.l.l4ever and mathlinkers,
my references :
http://faculty.evansville.edu/ck6/tcenters/recent/clawpt.html
http://forumgeom.fau.edu/FG2010volume10/FG201020.pdf p 205
I have found the same center with my programm...
Check again? What do you think?
Sincerely
Jean-Louis

jayme

#6 Jan 10, 2011, 9:03 pm

Dear Luis and Mathlinkers,
I am working on this situation for a weak and I have found some interest results. But the problem mentioned above resist to my different approach.
Luis has a solution based on trilinear coordinate if I have well understand...
My question is:
can we imagine a problem (algebrically, be built on second grade) which cannot be proved synthetically? What do you think?
Sincerely
Jean-Louis

Luis González

#7 Feb 14, 2011, 4:21 am

> Quote:
> P is a point in the plane of $\triangle ABC.$ ℓ_a is the radical axis of $\odot(PBC)$ and the A-excircle of $\triangle ABC.$ ℓ_b, ℓ_c are defined similarly. Then ℓ_a, ℓ_b, ℓ_c bound a triangle perspective with $\triangle ABC.$ If P lies on the circumcircle of $\triangle ABC,$ the perspector is the Clawson point of $\triangle ABC,$ i.e. homothetic center of the orthic triangle and extangents triangle.

Let $(u:v:w)$ be the barycentric coordinates of P with respect to $\triangle ABC.$ Equations of the circumcircles of $\triangle PBC, \triangle PCA$ and $\triangle PAB$ are given by:

$$\omega_a \equiv a^2yz + b^2zx + c^2xy - x(x+y+z)\frac{\lambda}{u} = 0$$

$$\omega_b \equiv a^2yz + b^2zx + c^2xy - y(x+y+z)\frac{\lambda}{v} = 0$$

$$\omega_c \equiv a^2yz + b^2zx + c^2xy - z(x+y+z)\frac{\lambda}{w} = 0$$

Where $\lambda = \dfrac{a^2vw + b^2wu + c^2uv}{u+v+w}$

Equations of excircles $(I_a), (I_b), (I_c)$ of $\triangle ABC$ are

$$(I_a) \equiv a^2yz + b^2zx + c^2xy - (x+y+z)(s^2x + (s-c)^2y + (s-b)^2z) = 0$$

$$(I_b) \equiv a^2yz + b^2zx + c^2xy - (x+y+z)((s-c)^2x + s^2y + (s-a)^2z) = 0$$

$$(I_c) \equiv a^2yz + b^2zx + c^2xy - (x+y+z)((s-b)^2x + (s-a)^2y + s^2z) = 0$$

Therefore, equations of ℓ_a, ℓ_b, ℓ_c are given by:

$$\ell_a \equiv \left(s^2 - \frac{\lambda}{u}\right)x + (s-c)^2y + (s-b)^2z = 0$$

$$\ell_b \equiv (s-c)^2x + \left(s^2 - \frac{\lambda}{v}\right)y + (s-a)^2z = 0$$

$$\ell_c \equiv (s-b)^2x + (s-a)^2y + \left(s^2 - \frac{\lambda}{w}\right)z = 0$$

Thus, ℓ_a, ℓ_b, ℓ_c bound a triangle perspective with $\triangle ABC$ through a point X with barycentric coordinates

$$X \equiv \left(\frac{1}{(s-b)^2(s-c)^2 - (s-a)^2\left(s^2 - \frac{\lambda}{u}\right)}\right)$$

If P lies on the circumcircle, then $\lambda = 0$ and X becomes the Clawson point of ABC

$$X_{19} \equiv \left(\frac{1}{(s-b)^2(s-c)^2 - s^2(s-a)^2}\right) \equiv (a \cdot \tan A).$$

jayme

#8 Feb 14, 2011, 2:56 pm

Dear Luis,
thank for your approach...
But why we cannot find for the moment a synthetic proof?
Do you have a beginning of esplication?
Sincerely
Jean-Louis

Luis González

#9 Feb 15, 2011, 5:52 am

N is the Nagel point of $\triangle ABC$ and ℓ_a, ℓ_b, ℓ_c cut BC, CA, AB at $X, Y, Z,$ respectively. A-excircle (I_a) touches BC at $D.$ Then $XD^2 = XB \cdot XC \Longrightarrow X$ is the exsimilicenter of the circles with diameters $DB, DC \Longrightarrow X$ lies on the trilinear polar τ of $N^2.$ By similar reasoning, Y, Z lie on $\tau \Longrightarrow \triangle ABC$ and the triangle bounded by ℓ_a, ℓ_b, ℓ_c are perspective through the trilinear polar τ of $N^2.$

P.S. However, I'm still not able to characterize the intersections $\ell_b \cap \ell_c, \ell_c \cap \ell_a$ and $\ell_a \cap \ell_b$ synthetically, in order to show that the perspector is the Clawson point when $P \in (O).$ The synthetic approach might be much tougher than the straightforward proof with barycentrics.

Quick Reply

High School Olympiads

Geometry Tough Problem

Reply

diks94

#1 Feb 14, 2011, 7:38 am

Assume that the circle with centre I touches the sides BC, CA, AB of $[ABC]$ in the points D, E, F respectively, Assume that the lines AI and EF intersect at K,the lines ED and KC at L, and the lines DF and KB at M.Prove That LM is parallel to BC.

$[ABC] \implies$Traingle ABC
Click to reveal hidden text

Luis González

#2 Feb 14, 2011, 11:17 am

I guess you meant $LM \parallel EF,$ instead of $LM \parallel BC.$ ED cuts AB, BK at $U, P.$ BK cuts AC at $V.$ Then $E(U,F,A,B)=E(P,K,V,B)=-1 \Longrightarrow A(P,K,E,F)=-1 \Longrightarrow EF \parallel AP.$ Thus, AP is the A-external bisector. Similarly, lines CK and DF meet at a point Q lying on the A-external bisector. Since $EF \parallel PQ$ and K is the midpoint of $EF,$ it follows that DK is the D-median line of $\triangle DPQ.$ Consequently, we have $LM \parallel PQ \parallel EF.$

diks94

#3 Feb 14, 2011, 9:45 pm

The Question is correct

Quick Reply

High School Olympiads

Geometry Problem (25)

Reply

vladimir92

#1 Feb 14, 2011, 4:17 am

Problem (Own).
Let ABC be a triangle. Call $\triangle A_1B_1C_1$ the cevian triangle of the Nagel point of $\triangle ABC$ and $\triangle A_2B_2C_2$ the triangle which vertices are the midpoints of the altitudes of $\triangle ABC$. Prove that A_1A_2 , B_1B_2 and C_1C_2 are concurent.

Luis González

#2 Feb 14, 2011, 5:11 am

Incircle (I,r) and A-excircle (I_a,r_a) touch BC at X, A_1. IX cuts A_1A_2 at I'. Then

$$\frac{XI'}{A_2D}=\frac{XA_1}{DA_1} \Longrightarrow \frac{h_a}{XI'}=2+\frac{h_a}{r_a} \Longrightarrow XI'=r \Longrightarrow I \equiv I'$$

Similarly, lines B_1B_2 and C_1C_2 pass through the incenter I of $\triangle ABC$.

vladimir92

#3 Feb 15, 2011, 4:16 am

Nice solution, The Idea is to see that those lines concur at the incenter, indeed I have a different solution

Quick Reply

High School Olympiads

Cyprus selection test

Reply

Eukleidis

#1 Feb 13, 2011, 7:56 pm

Let ABCD be a parallelogram and T the intersection point of its diagonals. Let (O,OD) be a circle that is tangent to BD at D, cuts CD at E and AD at Z. If B,E and Z are collinear prove that angles ATD and DOB are equal.

Luis González

#2 Feb 14, 2011, 12:02 am

Since (O) is tangent to BD through D, it follows that $\angle DZB = \angle BDC = \angle ABD \Longrightarrow$ Circle $\odot(ZDB)$ is tangent to AB through $B \Longrightarrow AB^2 = AD \cdot AZ.$ Hence, A and T have equal powers to (O) and the degenerate circle (B), i.e. AT is radical axis of $(O),(B) \Longrightarrow OB \perp AC.$ If $S \equiv OB \cap AC$, then the quadrilateral $DTSO$ is cyclic due to the right angles $\angle ODT$ and $\angle TSO \Longrightarrow \angle DOB = \angle ATD.$

Quick Reply

High School Olympiads

KB=KC in triangle ABC, K is intersection of perpendiculars.

Reply

Goutham
#1 Feb 13, 2011, 11:05 am • 1

In acute triangle ABC, an arbitrary point P is chosen on altitude AH. Points E and F are the midpoints of sides CA and AB respectively. The perpendiculars from E to CP and from F to BP meet at point K. Prove that $KB = KC$.

Luis González
#2 Feb 13, 2011, 11:52 am • 2

From $EC^2 - EP^2 = KC^2 - KP^2$ and $FP^2 - FB^2 = KP^2 - KB^2$, we get

$KB^2 - KC^2 + EC^2 - EP^2 + FP^2 - FB^2 = 0.$

Since $AP \perp EF$, then $FP^2 - EP^2 = FA^2 - EA^2 = FB^2 - EC^2$

$\Longrightarrow KB^2 - KC^2 + FB^2 - EC^2 + EC^2 - FB^2 = 0 \Longrightarrow KB = KC.$

yetti
#3 Feb 13, 2011, 12:26 pm

Let D be midpoint of BC. Parallels to BP, CP through F, E meet at $J \in AH$, on account of central similaity $\triangle AFE \sim \triangle ABC$. Reflections of EF in EJ, FJ meet at Q. Then J, K are incenter and Q-excenter of $\triangle QFE$. Incircle (J) touches EF at $S \equiv AH \cap EF$ and excircle (K) at T, such that $FT = ES \Longrightarrow K$ is on D-altitude of $\triangle DEF$, identical with perpendicular bisector of BC.

anantmudgal09
#4 Dec 7, 2015, 7:11 am

The beauty of this nice problem is just astounding. Here's a sketch of a new idea not in the thread I suppose,

Let the feet of altitudes from E to CP and F to BP be Y, X respectively. Let U, V be the reflections of B, C in Y, X respectively. Now, some power of a point bashing gives B, C, U, V are con-cyclic.

Then, we see that K is the circumcenter of this circle and hence we conclude.

TelvCohl
#5 Dec 8, 2015, 5:39 am • 1

Let D be the midpoint of BC and let K^* be the complement of the orthocenter of $\triangle BPC$ WRT $\triangle ABC$. From $K^*E \perp CP$ and $K^*F \perp BP$ we get $K^* \equiv K$, so DK is perpendicular to $BC \Longrightarrow KB = KC$.

kapilpavase
#6 May 28, 2016, 10:22 am

Lets use dot product. Set O the circumcentre as origin .From $KF \perp BP, EK \perp CP$ we get

$$(\frac{a+c}{2} - k).(c-p) = 0$$

$$(\frac{a+b}{2} - k).(b-p) = 0$$

Subtract second from first and get

$$(\frac{c-b}{2}).(a-p)-k.(c-b)=0$$

Now as $AP \perp CB$ we have $(c-b).(a-p)=0$ so that we must have $k.(c-b)=0$ that is $OK \perp BC$ and we are done.

This post has been edited 4 times. Last edited by kapilpavase, May 28, 2016, 10:26 am

Dukejukem

#7 May 28, 2016, 10:43 am

Let D be the midpoint of $\overline{BC}.$ Since the sidelines of the medial triangle $\triangle DEF$ are parallel to those of $\triangle ABC$, the perpendiculars from B, C, P to DF, DE, EF are concurrent at the orthocenter of $\triangle ABC.$ Therefore, $\triangle BCP$ and $\triangle ABC$ are orthologic. Hence, K is the orthology center of $\triangle DEF$ w.r.t. $\triangle BCP$, so $KD \perp BC.$ Thus, K lies on the perpendicular bisector of $\overline{BC}$, i.e. $KB = KC.$

kapilpavase

#9 May 28, 2016, 10:56 am

Another solution on similar lines of dukejukem..
We see that the condition ensures that P is orthology centre of ABC wrt KEF (perps from A to EF etc concur there). So the other orthology centre of KEF wrt ABC must be O the circumcentre(bcoz perps from E to AC and from F to AB concur there). Hence $OK \perp BC$ and done.

Quick Reply

High School Olympiads

Concurrent Lines

Headhunter

#1 Feb 12, 2011, 11:14 pm

Hello.

Let D,E,F be the feet of three altitudes (AD,BE,CF) of a triangle ABC
Let I_1, O_1 be the incenter, the circumcenter of the triangle AEF
Likewise, we define I_2, O_2 and I_3, O_3
Show that I_1O_1, I_2O_2, I_3O_3 are concurrent at Feuerbach point of the triangle ABC

Luis González

#2 Feb 13, 2011, 2:45 am

Incircle (I,r) touches BC, CA, AB at $X, Y, Z.$ From $\triangle ABC \sim \triangle AEF,$ it follows that $\frac{AI_1}{AI} = \frac{EF}{BC} = \cos A.$ Since AI is the A-circumdiameter of $\triangle AYZ,$ we deduce that I_1 is the reflection of I across $YZ.$

Let (N) be the 9-point circle of $\triangle ABC$ and U the midpoint of $AI.$ A-Garitte circle with diameter XU passes through the Feuerbach point $F_e \equiv (I) \cap (N),$ i.e. F_e, U and the antipode X' of X WRT (I) are collinear. Further, $r^2 = \overline{IU} \cdot \overline{II_1},$ implies that F_e, I, I_1, X' are concyclic. If M denotes the midpoint of $BC,$ then we have $\angle I_1F_eX' = \angle I_1IX' = \angle MF_eX.$ Since $\angle XF_eX'$ is right, then $\angle MF_eI_1$ is also right. Since M is the midpoint of the arc FDE of $(N),$ then F_eI_1 and F_eM bisect $\angle EF_eF \Longrightarrow F_eI_1$ cuts (N) again at the midpoint of the arc EF of $(N),$ i.e. circumcenter O_1 of $\triangle AEF.$ Similarly, diacentral lines of $\triangle BFD$ and $\triangle CDE$ pass through $F_e.$

High School Olympiads

Find the locus of points M in the plane

Amir Hossein

#1 Oct 8, 2010, 2:22 am

In a plane we are given two distinct points A, B and two lines a, b passing through B and A respectively $(a \ni B, b \ni A)$ such that the line AB is equally inclined to a and b. Find the locus of points M in the plane such that the product of distances from M to A and a equals the product of distances from M to B and b (i.e., $MA \cdot MA' = MB \cdot MB'$, where A' and B' are the feet of the perpendiculars from M to a and b respectively).

Luis González

#2 Oct 11, 2010, 10:32 am

Define the rectangular reference $A : (0, 0)\ ,\ b : y = 0\ ,\ B : (p, q)\ ,\ M : (x, y)$

$$MA \cdot MA' = MB \cdot MB' \iff \sqrt{x^2 + y^2} \cdot |q - y| = |y| \cdot \sqrt{(y - q)^2 + (x - p)^2}$$

$$\Longrightarrow\ (x^2 + y^2)(q^2 + y^2 - 2qy) = y^2(y^2 + x^2 + q^2 + p^2 - 2qy - 2px)$$

$$\Longrightarrow\ p^2y^2 - q^2x^2 - 2pxy^2 + 2qyx^2 = 0 \Longrightarrow\ (py - qx)(py + qx - 2xy) = 0$$

Thus, locus of M is the union of the line $AB : py - qx = 0$ and the rectangular hyperbola $\mathcal{K} : py + qx - 2xy = 0$ passing through A, B. Its center coincides with the midpoint D of segment AB and its asymptotes are given by $2y = q, 2x = p$, i.e. the parallel and perpendicular from D to the direction $a \parallel b$.

Luis González

#3 Feb 10, 2011, 9:51 am

Generalization: M is a point on the plane of the scalene triangle $\triangle ABC$. B', C' are the orthogonal projections of M on the sidelines AC and AB. Then the locus of M such that $\overline{MB} \cdot \overline{MB'} = \overline{MC} \cdot \overline{MC'}$ is the union of the circumcircle of $\triangle ABC$ and a rectangular circum-hyperbola.

Let $(x : y : z)$ be the barycentric coordinates of M with respect to $\triangle ABC$. Then

$$\overline{MB} \cdot \overline{MB'} = \overline{MC} \cdot \overline{MC'} \iff$$

$$\frac{(a^2z^2 + 2S_Bzx + c^2x^2)y^2}{b^2} = \frac{(b^2x^2 + 2S_Cxy + a^2y^2)z^2}{c^2} \iff$$

$$(a^2yz + b^2zx + c^2xy)(c^2xy + c^2yz - b^2yz - b^2xz) = 0$$

M either lies on the circumcircle $\mathcal{O} \equiv a^2yz + b^2zx + c^2xy = 0$ or it lies on the circum-hyperbola $\mathcal{H}_a$, isogonal conjugate of the perpendicular bisector of $\overline{BC}$.

High School Olympiads

Prove that expression in □ABC

Reply

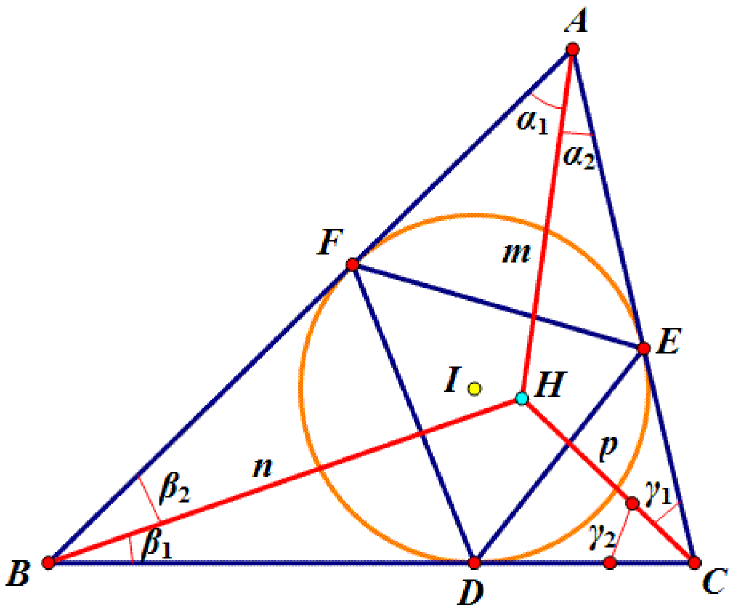
lym

#1 Feb 4, 2011, 5:50 pm

Figure□given Non-equilateral $\triangle ABC$□H is the orthocenter of $\triangle DEF$□$AH=m$□$BH=n$□$CH=p$. Prove that

$\triangle ABC$ is isosceles $\Longleftrightarrow \dfrac{m\cdot(sin\alpha_1-sin\alpha_2)}{sinA}+\dfrac{n\cdot(sin\beta_1-sin\beta_2)}{sinB}+\dfrac{p\cdot(sin\gamma_1-sin\gamma_2)}{sinC}=0$.

Attachments:



lym

#2 Feb 6, 2011, 1:33 am

Is there anybody killed this problem □

If a point H is satisfy that□then H is on a special line ....

Luis González

#3 Feb 9, 2011, 9:31 am

Let $(x:y:z)$ be the barycentric coordinates of $H\equiv X_{65}$ with respect to $\triangle ABC$. Then

$$\frac{m\cdot(\sin\alpha_1-\sin\alpha_2)}{\sin A}+\frac{n\cdot(\sin\beta_1-\sin\beta_2)}{\sin B}+\frac{p\cdot(\sin\gamma_1-\sin\gamma_2)}{\sin C}=0 \iff$$

$$\frac{1}{\sin A}\left(\frac{z}{c}-\frac{y}{b}\right)+\frac{1}{\sin B}\left(\frac{x}{a}-\frac{z}{c}\right)+\frac{1}{\sin C}\left(\frac{y}{b}-\frac{x}{a}\right)=0 \iff$$

$$(b-c)x+(c-a)y+(a-b)z=0$$

Which coincides with the equation of line passing through the incenter $I\ (a:b:c)$ and centroid $G\ (1:1:1)$ of $\triangle ABC$. Since the orthocenter of the intouch triangle $\triangle DEF$ lies on the diacentral line IO, it follows that X_{65} lies on $IG \iff I$ lies on the Euler line of $\triangle ABC \iff$ either $b=c, c=a$ or $a=b$.

Quick Reply

High School Olympiads

AC=MC iff AM bisects angle DAB

Reply

Source: Romanian TST 2002

WakeUp
#1 Feb 5, 2011, 9:31 pm

Let ABC be a triangle such that $AC \neq BC, AB < AC$ and let K be it's circumcircle. The tangent to K at the point A intersects the line BC at the point D. Let K_1 be the circle tangent to K and to the segments $(AD), (BD)$. We denote by M the point where K_1 touches (BD). Show that $AC = MC$ if and only if AM is the bisector of the $\angle DAB$.

Neculai Roman

Luis González
#2 Feb 6, 2011, 8:09 am

If (K_1) touches AD at $N,$ we know that MN passes through the C-excenter $E.$ Since lines DK_1 and AE are parallel, it follows that $ME \perp AE.$ Assume that $\triangle AMC$ is isosceles with apex $C.$ Then $\angle AEB = \angle AMC = 90^\circ - \frac{1}{2}\angle C \Longrightarrow$ quadrilateral $AEMB$ is cyclic $\Longrightarrow \angle ABC = 90^\circ.$ Therefore

$\angle MAC = 90^\circ - \frac{1}{2}\angle C - (90^\circ - \angle C) = \frac{1}{2}\angle C \Longrightarrow AM$ bisects $\angle BAD.$

Conversely, assume that AM bisects $\angle BAD.$ Since $\angle MEB = \frac{1}{2}\angle MAB = \frac{1}{2}\angle C,$ it follows that quadrilateral $AEMB$ is cyclic, thus $\angle AEB = \angle AMC = 90^\circ - \frac{1}{2}\angle C \Longrightarrow \triangle AMC$ is isosceles with apex $C.$

Quick Reply

High School Olympiads

Concurrent 8

buratinogigle

#1 Feb 5, 2011, 8:26 pm

Problem 1 (Well known). Let ABC be a triangle inscribed circle (O). (O_a) touchs AB, AC and touchs (O) internally at A' (A-mixtilinear incircle). Similarly we have B', C'. Prove that AA', BB', CC' are concurrent.

Problem 2 (Maybe well known ?). Let ABC be a triangle inscribed circle (O). (K_a) is circle diameter BC. (O_a) touchs AB, AC and touchs (K_a) externally at A' (O_a is inside triangle ABC). Similarly we have B', C'. Prove that AA', BB', CC' are concurrent.

Problem 3 (Own ?). Let ABC be a triangle inscribed circle (O). K_a is reflection of O through BC. (K_a) is circle center K_a and passes through B, C. (O_a) touchs AB, AC and touchs (K_a) externally at A' (O_a is inside triangle ABC). Similarly we have B', C'. Prove that AA', BB', CC' are concurrent.

Luis González

#2 Feb 5, 2011, 8:46 pm

1) A is the exsimilicenter of the incircle (I) of $\triangle ABC$ and the A-mixtilinear incircle. A' is the exsimilicenter of (O) and (O_a). Hence, by Monge & d'Alembert theorem, it follows that AA' cuts IO at the exsimilicenter X_{56} of $(I), (O)$. Similarly, lines BB', CC' go through X_{56}.

2) See Touching circles and concurrency. AA',BB',CC' concur at the Paasche point of ABC.

lym

#3 Feb 6, 2011, 1:21 am

For 2□Let a line l parallel to BC pass through I cut AB□AC□(I) at B_1□C_1□(P_1, Q_1) . A'' is the intersection of BQ_1 and CP_1□similary B''□C''□then we just prove AA''□BB''□CC'' are concurrent□then use ceva .

For 3□Let $\triangle DEF$ be the tangency triangle of (I) of $\triangle ABC$□then consider the orthocenter H of $\triangle DEF$. Let P_1□Q_1 be the second intersection of EH□FH with (I)□A'' the intersection of BQ_1 and CP_1□similary B''□C''□then we just prove AA''□BB''□CC'' are concurrent□then also use ceva .

Luis González

#4 Feb 6, 2011, 3:25 am

3) D, E, F are the feet of the altitudes on $BC, CA, AB. H \equiv AD \cap BE \cap CF$ is the orthocenter of $\triangle ABC$, thus (K_a) is identical to $\odot(HBC)$. Inversion through pole A with power $\overline{AH} \cdot \overline{AD}$ takes $\odot(HBC)$ into the 9-point circle (N) of $\triangle ABC$ and takes (O_a), tangent to AB, AC and the arc BHC of (K_a), into the A-excircle (I_a) tangent to AB, AC and the arc FDE of (N). $A'' \equiv (N) \cap (I_a)$ is the inverse of A', i.e. $AA' \equiv AA''$. Let (I) be the incircle of $\triangle ABC$. Then A is the exsimilicenter of $(I) \sim (I_a)$ and A'' is the insimilicenter of $(N) \sim (I_a)$. Thus, AA' cuts IN at the insimilicenter X_{12} of $(I) \sim (N)$. Similarly, lines BB' and CC' pass through X_{12}.

jayme

#5 Feb 6, 2011, 11:58 am

Dear Mathlinkers,
for problem 1, in order to go farly,
http://perso.orange.fr/jl.ayme vol. 4 A new mixtilinear incircle adventure I p. 22.
Sincerely
Jean-Louis

buratinogigle

#6 Feb 6, 2011, 12:38 pm

Thanks for your interest, I have another problem

Problem 4. Incircle (I) of triangle ABC touchs BC, CA, AB at D, E, F ,resp. D- mixtilinear incircle of DEF touchs circumcircle (DEF) at A'. Similarly we have B', C'. Prove that AA', BB', CC' are concurrent.

lym

#7 Feb 6, 2011, 5:35 pm

Another very simple proof of Problem 3□

Obviously□AA' pass through the insimilicenter of $(I) \sim (k_a)$□but A is the exsimilicenter of $(N) \sim (K_a)$□so AA' pass through the insimilicenter X_{12} of $(I) \sim (N)$□similarly□BB'□CC' pass though X_{12} .

jayme

#8 Feb 6, 2011, 7:31 pm

Dear Mathlinkers,
for problem 4
1. Use the problem 1
2. Use the Steinbart's theorem
see
http://perso.orange.fr/jl.ayme vol. 3 Les points de Steinbart et de Rabinovitz
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

parallelogram

Reply

tkrass

#1 Feb 4, 2011, 1:12 am

P and Q are points on the sides AB and AD respectively of a parallelogram ABCD. E is a point such that APEQ is a parallelogram too. F is the point of intersection of BQ and DP. Prove that C, E and F are collinear.

sunken rock

#2 Feb 4, 2011, 4:38 pm

Let $\{F\}\in CD\cap BQ,\ \{G\}\in BC\cap DP,\ \{K\}\in BQ\cap DP,\ \{L\}\in CK\cap BD,$ $\{M\}\in PE\cap BD,\ \{N\}\in QE\cap BD$; applying Ceva in $\triangle BCD$ with $BF,\ DG,\ CL$ we get $\frac{BG}{CG}\cdot\frac{CF}{FD}\cdot\frac{DL}{BL}=1(\ *\ )$.

But $\frac{GB}{GC}=\frac{BP}{CD},\ \frac{FC}{FD}=\frac{BC}{DQ}\ (\ 1\ )$.

With (1), from $(*)$ we get $\frac{BL}{DL}=\frac{BP}{CD}\cdot\frac{BC}{DQ}\ (\ 2\ )$.

As $\frac{BP}{CD}=\frac{BP}{AB}=\frac{BM}{BD}\ (\ 3\ )$ and $\frac{BC}{DQ}=\frac{AD}{DQ}=\frac{BD}{DN}\ (\ 4\ )$, hence $(2)\implies\frac{BL}{DL}=\frac{BM}{DN}\implies C,\ E, L$ collinear (from triangles $\triangle BCD$ and $\triangle MEN$ being perspective).

Best regards,
sunken rock

Luis González

#3 Feb 4, 2011, 11:17 pm

Lines EP, EQ cut CD, CB at U, V respectively. By Pappus theorem for lines AB, QV, DC concurring at infinity and AD, PU, BC concurring at infinity, it follows that lines PQ, DB, UV concur at a point $R\Longrightarrow\triangle BCD$ and $\triangle QPE$ are perspective through $\overline{RUV}\Longrightarrow BQ, DP, CE$ concur, i.e. C, E, F are collinear, as desired.

Headhunter

#4 Feb 5, 2011, 9:25 pm • 1

Applying Pappus to $\overline{QDX_\infty}, \overline{PBY_\infty}$ own this problem.

Quick Reply

High School Olympiads

Collinear with excenter

Reply

Vinoth

#1 Feb 2, 2011, 9:07 am

Find a nice and synthetic solution:

In a triangle ABC with $AB > AC$, let ω be a circle tangent to the circumcircle of ABC, the line BC, and the tangent AK to the circumcircle at A. (Here K is the intersection of the tangent to the circumcircle at A with BC.) If ω is tangent to BC and AK at D and E respectively, prove DE passes through the ex-center E_b corresponding to B.

Luis González

#2 Feb 2, 2011, 10:18 am

Actually, the result is still true for any K on the extensions of the segment BC. This is the 'extraversion' of the so called Sawayama's lemma. See the links belows for proofs a more details.

http://forumgeom.fau.edu/FG2003volume3/FG200325.pdf
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=46342
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=37236

Quick Reply

High School Olympiads

Concurrent 3

Reply

Source: In Ellipse

buratinogigle

#1 Sep 26, 2009, 1:22 pm

Let $ABCD$ be convex quadrilateral inscribed a Ellipse with center O. M, N are midpoint of AC, BD. Through A, C draw two line ℓ_a, ℓ_c parallel to ON, Through B, D draw two line ℓ_b, ℓ_d parallel to OM. Let ℓ_a intersect ℓ_b, ℓ_d at Q, R, resp, ℓ_c intersect ℓ_b, ℓ_d at P, S, resp. $AC \cap BD = I$. Let X, Y, Z, T be centroids of triangle IAB, IBC, ICD, IDA. Prove that four lines XQ, ZS, YP, TR are concurrent.

Luis González

#2 Feb 1, 2011, 12:32 pm • 1

Lemma. $ABCD$ is a cyclic quadrilateral and P is the intersection of its diagonals AC, BD. Then Euler lines of $\triangle PAB$, $\triangle PBC, \triangle PCD$ and $\triangle PDA$ concur.

Let O_a, O_b, O_c, O_d and H_a, H_b, H_c, H_d be the circumcenters and orthocenters of $\triangle PDA, \triangle PAB, \triangle PBC, \triangle PCD$, respectively. We will show that H_aO_a, H_bO_b and H_cO_c concur, then the conclusion follows by analogous reasoning. M, N are the midpoints of PA, PB and S, T are the orthogonal projections of A, B onto BD, AC. Then we have

$$\frac{H_bH_c}{TH_c} = \frac{AH_b + CH_c}{CH_c} = \frac{2(NO_b + NO_c)}{CH_c} = \frac{2O_bO_C}{CH_c}$$

From $\triangle PBC \sim \triangle PAD$ we get then $\dfrac{H_bH_c}{O_bO_c} = \dfrac{H_aH_b}{O_aO_b} = 2 \cdot \dfrac{TH_c}{CH_c} = 2 \cdot \dfrac{SH_a}{DH_a}$

Since $H_aH_b \parallel O_cO_b$ and $H_cH_b \parallel O_aO_b$, then $\triangle O_aO_bO_c \sim \triangle H_aH_bH_c \ (\star)$

Define $Q \equiv H_aH_c \cap O_aO_c, U \equiv AH_a \cap MO_a, V \equiv BH_c \cap NO_c$. NO_c, BH_c cut H_aH_c, O_aO_c at E, F, respectively. From $(\star)$ we have $\angle EH_cF = \angle EO_cF \Longrightarrow E, F, O_c, H_c$ are concyclic $\Longrightarrow \angle VFE = \angle VO_cH_c = \angle UO_aH_a \Longrightarrow$ EF is parallel to H_aO_a. As a result, $\triangle UH_aO_a$ and $\triangle VEF$ are homothetic $\Longrightarrow Q, U, V$ are collinear, thus $\triangle H_aH_bH_c$ and $\triangle O_aO_bO_c$ are perspective through the line $\overline{QUV}$. By Desargues theorem, it follows that H_aO_a, H_bO_b and H_cO_c concur.

Back to the problem, project the circum-ellipse of $ABCD$ into a cirle (O') through parallel projection. Label projected points with primes. A, B, C, D go to A', B', C', D' on (O'). Since midpoints go to midpoints under parallel projection, then centroids X, Y, Z, T go to centroids X', Y', Z', T' of $\triangle I'A'B', \triangle I'B'C', \triangle I'C'D', \triangle I'D'A'$. Parallels ℓ_a, ℓ_c to ON go to the perpendiculars from A', C' to $D'B'$ and parallels ℓ_b, ℓ_c to OM go to the perpendiculars fom B', D' to $A'C' \Longrightarrow$ Q', P', S', R' are orthocenters of $\triangle I'A'B', \triangle I'B'C', \triangle I'C'D', \triangle I'D'A' \Longrightarrow X'Q', Y'P', Z'S', T'R'$ are Euler lines of $\triangle I'A'B', \triangle I'B'C', \triangle I'C'D', \triangle I'D'A'$. Now, according to our previous lemma, $X'Q', Y'P', Z'S', T'R'$ concur, thus the primitive lines XQ, YP, ZS, TR concur as well.

buratinogigle

#3 Feb 3, 2011, 8:35 pm

Thank you dear luisgeometra, that's the way which I creat this problem !

jackmath101

#4 Jan 27, 2013, 4:07 pm

About the lemma, I thing we can let M, N, P, Q be the reflections of P about the midpoints the sides AB, BC, CD, DA. And then the Euler line of $\triangle PAB, \triangle PBC, \triangle PCD$ and $\triangle PDA$ concur. Can you prove it ?

Quick Reply

High School Olympiads

Metric Relation&

Reply

Headhunter
#1 Jan 31, 2011, 10:16 am

Hello.

Show that the length of the common chord of the circumcircle and the excircle is $\sqrt{\frac{r^3(4R-r)}{R(R+2r)}}$

R is the circumradius and r is the radius of the excircle.

Luis González
#2 Jan 31, 2011, 8:44 pm

Excircle (U,r) of $\triangle ABC$ touches BC, CA, AB at $X,Y,Z.$ Inversion WRT (U) takes (O) into the 9-point circle of $\triangle XYZ.$ Thus, by inversion properties we obtain:

$$\frac{r}{2R}=\frac{r^2}{|p(U,(O))|}=\frac{r^2}{UO^2-R^2}\Longrightarrow UO=\sqrt{2Rr+R^2}$$

$(U),(O)$ meet at D,E and $M\equiv OU\cap DE\Longrightarrow MD=\frac{1}{2}DE=\frac{1}{2}d.$ By Pythagorean theorem for right triangles $\triangle ODM$ and $\triangle UDM,$ we get:

$$\sqrt{R^2-\frac{d^2}{4}}+\sqrt{r^2-\frac{d^2}{4}}=\sqrt{2Rr+R^2}$$

This expression yields a d-quadratic equation with symmetrical roots $d=\pm\sqrt{\frac{r^3(4R-r)}{R(R+2r)}}$

Quick Reply

High School Olympiads

Nice problem

Reply

jindo

#1 Jan 31, 2011, 11:53 am

Let ABC be a triangle and AX, BY, CZ be the altitudes of its. Prove that the Euler line of triangles AYZ, BZX, CXY have a common point.

Luis González

#2 Jan 31, 2011, 12:24 pm

Posted at least 3 times before. Euler lines and 9-point circles of AYZ, BZX, CXY have a common point that coincides with the center of the Jerabek hyperbola of ABC and the anticomplement of the Feuerbach point of the orthic triangle XYZ. For instance see the topics Name of concurrent point, With the Feuerbach point and Euler line. Further, a nice generalization of the 3 Euler lines concurring on the 9-point circle was posted by gemath in the topic Triangle problem 4.

Quick Reply

High School Olympiads

Metric Relation

Reply

Headhunter
#1 Jan 31, 2011, 10:11 am

Hello.

Show that $a\cdot IA^2+b\cdot IB^2+c\cdot IC^2=4rsR$

Luis González
#2 Jan 31, 2011, 10:38 am

Use Leibniz formula for the incenter $I\equiv(a:b:c)$ and the circumcenter O of ABC.

$BC\cdot R^2+CA\cdot R^2+AB\cdot R^2=2s\cdot IO^2+BC\cdot IA^2+CA\cdot IB^2+AB\cdot IC^2$

$BC\cdot IA^2+CA\cdot IB^2+AB\cdot IC^2=2sR^2-2s(R^2-2Rr)=4Rrs.$

Quick Reply

High School Olympiads

About coaxial circles

Reply

hurricane

#1 Jan 31, 2011, 8:19 am

Let ABC be a triangle and $(O),(I)$ be the circumcircle, incircle of its. (I) touches BC,CA,AB at D,E,F respectively. AO,BO,CO cut (O) at A',B',C' respectively. EF,FD,DE cut IA,IB,IC at A'',B'',C'' respectively. IA',IB',IC' cut $(IB"C"),(IC"A"),(IA"B")$ at X,Y,Z respectively. Prove that $(IA"X),(IB"Y),(IC"Z)$ are coaxial

Sorry, I don't know how to post figure

Luis González

#2 Jan 31, 2011, 10:12 am

IX,IY,IZ cut BC,CA,AB at $U,V,W,$ respectively. Inversion with respect to (I) takes midpoints A'',B'',C'' of $EF,$ FD,DE into $A,B,C.$ Consequently, circles $\odot(IB''C''),\odot(IC''A''),\odot(IA''B'')$ are taken into the sidelines $BC,CA,$ $AB \Longrightarrow U,V,W$ are the inverses of $X,Y,Z \Longrightarrow$ Lines AU,BV,CW are the inverses of $\odot(IXA''),\odot(IYB''),$ $\odot(IZC'').$ According to the topic Concurrent lines, AU,BV,CW concur at $X_{77}.$ Hence $\odot(IXA''),\odot(IYB''),\odot(IZC'')$ meet at I and the inverse of X_{77} in the incircle. This is, $\odot(IXA''),\odot(IYB'')$ and $\odot(IZC'')$ are coaxal with common radical axis the Soddy line of $\triangle ABC,$ i.e. the Brocard axis of the intouch triangle $\triangle DEF.$

Quick Reply

High School Olympiads

Concurrent lines

Reply

novae

#1 Jan 30, 2011, 8:03 pm

Let ABC be a triangle. It's circumcircle (O) and incircle (I). $D;E;F$ be the intersection of $AO;BO;CO$ and (O). $X;Y;Z$ be the intersection of $ID;IE;IF$ and $BC;CA;AB$ respectively. Show that $AX;BY;CZ$ are concurrent
[geogebra]924e68c355a504bcbf4e3f972b3b08d2fee4bc0f[/geogebra]

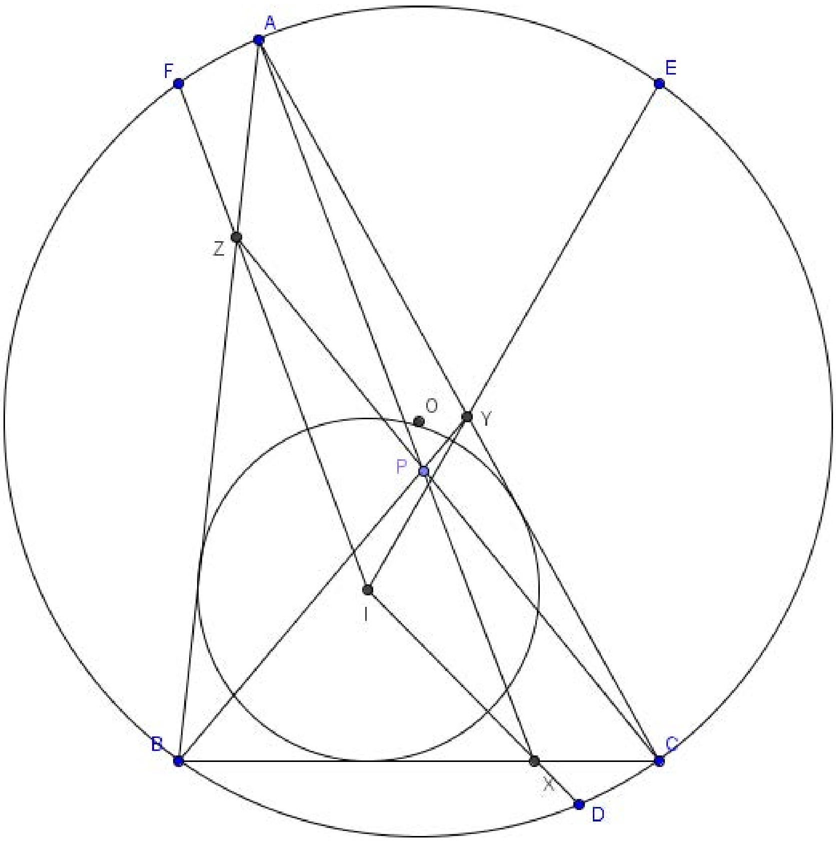
This post has been edited 1 time. Last edited by novae, Jan 30, 2011, 9:22 pm

nguyenhung

#2 Jan 30, 2011, 9:11 pm

Picture of your problem

Attachments:



Luis González

#3 Jan 31, 2011, 12:11 am

We use barycentric coordinates WRT $\triangle ABC.$ Lines AO, BO, CO cut (O) again at

$D\ (-S_BS_C : b^2S_B : c^2S_C)$

$E\ (a^2S_A : -S_CS_A : c^2S_C)$

$F\ (a^2S_A : b^2S_B : -S_AS_B)$

Equation of ID is then $bc(cS_C - bS_B)x - cS_C(S_B + ca)y + bS_B(S_C + ab)z = 0$

$$X \equiv BC \cap ID \equiv \left(0 : \frac{bS_B}{S_B + ca} : \frac{cS_C}{S_C + ab}\right)$$

By cyclic permutations of a, b, c we get the coordinates of Y, Z and then we conclude that AX, BY, CZ concur at

$$X_{77}\left(\frac{aS_A}{S_A + bc} : \frac{bS_B}{S_B + ca} : \frac{cS_C}{S_C + ab}\right),$$

which is the isogonal conjugate of the perspector of the orthic and intangents triangle of $\triangle ABC$.

Quick Reply

High School Olympiads

Cyclic Quadrangle

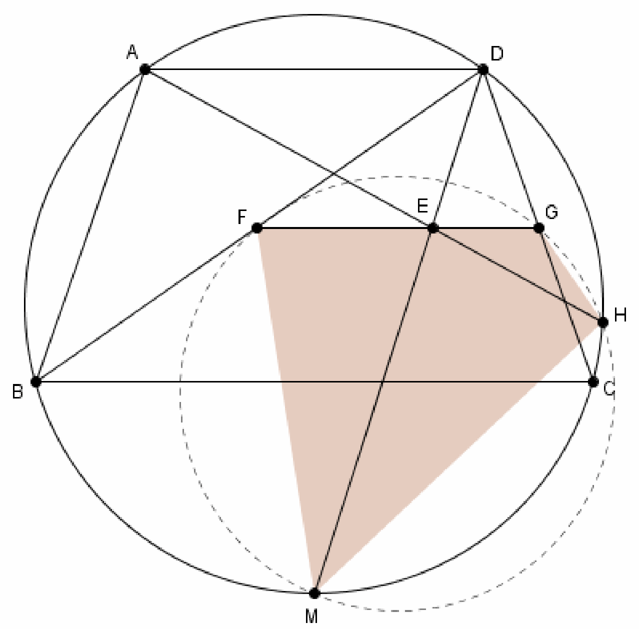
Reply

77ant

#1 Jan 29, 2011, 11:45 pm

Dear everyone
$\square ABCD$ is isosceles trapezoid with its circumcircle (O).
The midpoint of arc $\widehat{BC}$ is M and E move on DM.
Any line through E, parallel to BC, cut BD, CD at F, G respectively.
AE cut (O) at H. Prove that $\square FMHG$ is cyclic.

Attachments:



Luis González

#2 Jan 30, 2011, 8:43 am

Ray CE cuts (O) at U. MU cuts DB at F'. Since $\angle CUM = \angle BDM$, it follows that $DUF'E$ is cyclic $\Longrightarrow$ $\angle DF'E = \angle DUC = \angle DBC \Longrightarrow EF' \parallel BC$, i.e. $F \equiv F'$. Note that E, G, C, H are concyclic due to $\angle DGE = \angle DCB = \angle AHC$. Therefore, $\angle FGH = \angle UCH = \angle FMH \pmod \pi \Longrightarrow FMHG$ is cyclic.

P.S. The inverse exercise can be proved with the same arguments.

Headhunter

#3 Jan 31, 2011, 12:47 pm

From $\square EGHC$ is concyclic and $\triangle DFM \sim \triangle DEC$, we can get it.

Quick Reply

High School Olympiads

Coaxial circles

buratinogigle

#1 Jan 19, 2010, 3:18 pm

Let ABC ba a triangle circumcircle (O). P is an arbitrary point. A', B', C' are the second intersections of PA, PB, PC with circumcircle (O) of triangle ABC. Q is a point on line OP. $A''B''C''$ is pedal triangle of P. Prove that $(A'QA''), (B'QB''), (C'QC'')$ have two common points (they are coaxial circles).

Note that, this is generalization of the problem on the post Concurrent 4.

livetolove212

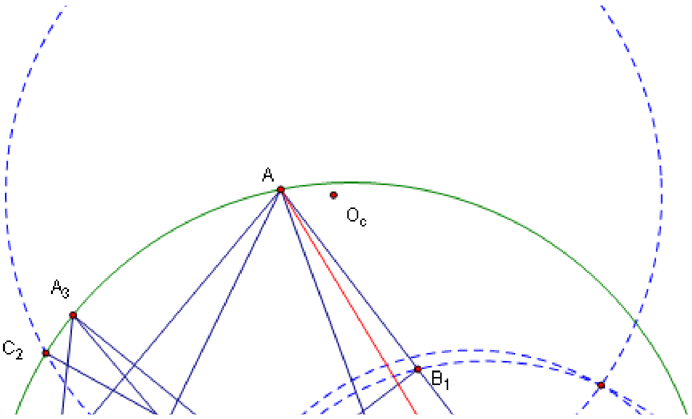
#2 Jan 25, 2011, 6:53 am • 2

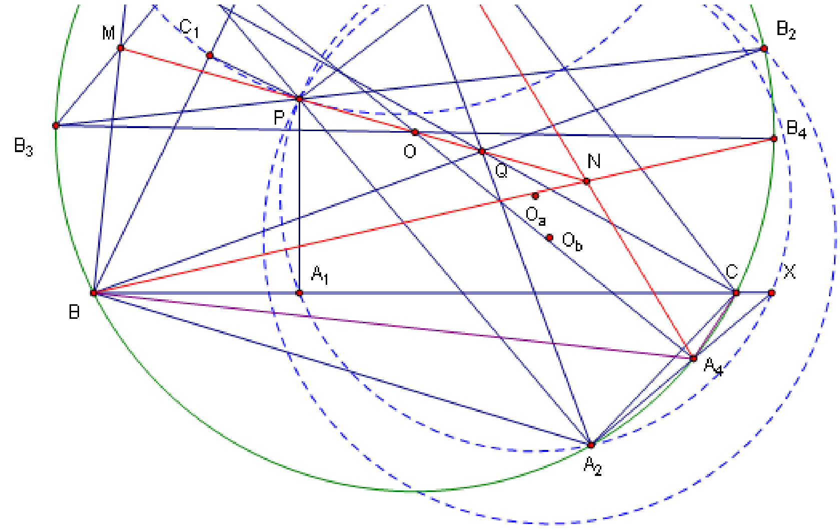
Problem: Let ABC be a triangle with its circumcircle (O). Let P and Q be arbitrary points such that P, O, Q are collinear. $A_1B_1C_1$ is the pedal triangle of P wrt ΔABC, $A_2B_2C_2$ is the circumcevian triangle of Q wrt ΔABC. Show that $(PA_1A_2), (PB_1B_2), (PC_1C_2)$ are coaxal.

Proof:
Denote O_a, O_b, O_c the centers of $(PA_1A_2), (PB_1B_2), (PC_1C_2)$; X the intersection of the line through A_2 and perpendicular to PA_2. Similarly we define Y, Z.
Note that O_a is the midpoint of PX therefore $(O_a), (O_b), (O_c)$ are coaxal iff O_a, O_b, O_c are collinear iff X, Y, Z are collinear. (1)
We have $\frac{XB}{XC} = \frac{S_{XBA_2}}{S_{XCA_2}} = \frac{\sin\angle XA_2B.A_2B}{\sin\angle XA_2C.A_2C} = \frac{\sin\angle XA_2B}{\sin\angle XA_2C}.\frac{\sin\angle BAQ}{\sin\angle CAQ}$
But $\frac{\sin\angle XA_2B}{\sin\angle XA_2C} = \frac{\cos\angle BA_2P}{\cos\angle CA_2P}$.
Let A_3 be the intersection of A_2P and (O), A_4 be the intersection of A_3O and (O). Similarly we define B_3, B_4, C_3, C_4.
We have $\frac{\cos\angle BA_2P}{\cos\angle CA_2P} = \frac{\cos\angle BA_4A_3}{\cos\angle CA_4A_3} = \frac{\sin\angle BA_3A_4}{\sin\angle CA_3A_4} = \frac{\sin\angle BAA_4}{\sin\angle CAA_4}$
Hence $\frac{XB}{XC} = \frac{\sin\angle BAQ}{\sin\angle CAQ}.\frac{\sin\angle BAA_4}{\sin\angle CAA_4}$.
Do the same with $\frac{YC}{YA}, \frac{ZA}{ZB}$ then applying Ceva-sine theorem we obtain $(1) \Leftrightarrow AA_4, BB_4, CC_4$ are concurrent.
On the other side, according to Pascal's theorem for 6 points A, B, A_3, B_3, A_2, B_2 we claim M, P, Q are collinear (M is the intersection of AB_3 and BA_3)
Applying Pascal's theorem again for 6 points A, B, A_3, B_3, A_4, B_4 we get the intersection of AA_4 and BB_4 lies on PQ.
Similarly the intersection of CC_4 and BB_4 lies on PQ. This means AA_4, BB_4, CC_4 are concurrent. We are done.

PS: Another special case:
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=386935

Attachments:





Luis González

#3 Jan 30, 2011, 4:27 am • 3

My solution is similar to Linh's but I went for different arguments. Keeping the same initial notations of Linh's solution, let lines PA_2, PB_2 cut (O) again at D, E and DO, EO cut (O) again at U, V. Define $T \equiv VA_2 \cap UB_2$. By Pascal theorem for nonconvex cyclic hexagons VA_2DUB_2E and AUB_2BVA_2, it follows that T, P, O and R, T, Q are collinear $\Longrightarrow$ $R \equiv AU \cap BV \cap PQ$. If we define the point W similarly as U, V, then AU, BV, CW concur on PQ. Let UA_2, VB_2 and WC_2 cut BC, CA, AB at X, Y, Z. Then we have

$$\frac{XB}{XC} = \frac{A_2B}{A_2C} \cdot \frac{UB}{UC}, \quad \frac{YC}{YA} = \frac{B_2C}{B_2A} \cdot \frac{VC}{VA}, \quad \frac{ZA}{ZB} = \frac{C_2A}{C_2B} \cdot \frac{WA}{WB}$$

Since cyclic hexagons $AWBUCV$ and $AC_2BA_2CB_2$ have concurrent diagonals, we get

$$\frac{UB}{UC} \cdot \frac{VC}{VA} \cdot \frac{WA}{WB} = 1, \quad \frac{A_2B}{A_2C} \cdot \frac{B_2C}{B_2A} \cdot \frac{C_2A}{C_2B} = 1 \Longrightarrow \frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = 1$$

By the converse of Menelaus theorem, the points X, Y, Z are collinear. But since $\angle PA_2U$ is right, then PX is a diameter of the circle $(O_a) \equiv \odot(PA_1A_2) \Longrightarrow O_a$ is the midpoint of PX. Likewise, centers O_b, O_c of $\odot(PB_1B_2)$ and $\odot(PC_1C_2)$ are the midpoints of PY, PZ. If X, Y, Z are collinear, then O_a, O_b, O_c are collinear $\Longrightarrow (O_a), (O_b), (O_c)$ are coaxal.

jayme

#4 Feb 23, 2015, 4:52 pm

Dear Mathlinkers,
this problem can be simply solved with the Jerabek's theorem.

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Two circumscribed quadrilaterals

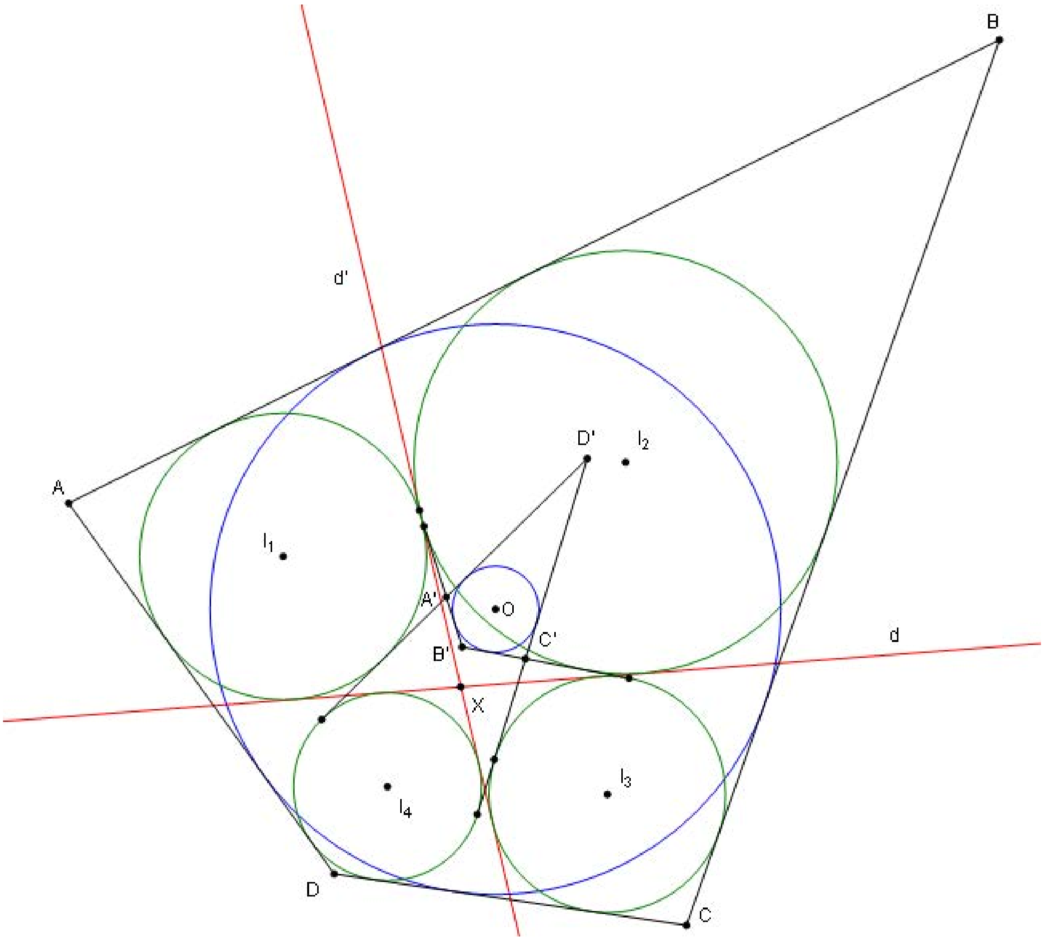
Reply

livetolove212

#1 Jan 28, 2011, 2:36 pm • 1

Problem (own?): Given two lines d and d' in the plane. $d \cap d' = \{X\}$. Four circles $(I_1), (I_2), (I_3), (I_4)$ are tangent to d and d'. The second common external and internal tangent lines of them form two quadrilaterals $ABCD$ and $A'B'C'D'$. Prove that $ABCD$ and $A'B'C'D'$ are the circumscribed quadrilaterals and they have a common incenter.

Attachments:



Luis González

#2 Jan 29, 2011, 12:11 am • 1

The fact that $ABCD$ is tangential follows from a more general configuration of 4 directed circles with 4 directed common tangents touching a single directed circle. See the theorem metioned by yetti in the topic Tangential quadrangle. Directed common tangents d, d', d, d' of $(I_1) \cup (I_2), (I_2) \cup (I_3), (I_3) \cup (I_4), (I_4) \cup (I_1)$ are tangent to the degenerate circle (X), thus directed common tangents AB, BC, CD, DA touch a single directed circle (O).

Let (I_1) touch DA, AB at E, F. Common internal tangents of $(I_1), (I_2)$ and $(I_1), (I_4)$, different from d, d', cut BA, AD at U, V and (I_1) touches d', d at G, H, respectively. Since $EV = XH = XG = FU$, it follows that $AV = AU$, thus due to symmetry, $A' \in AI_1 \Longrightarrow AO$ bisects $\angle B'A'D'$. Similar reasoning yields that $A'B'C'D'$ has incenter O.

Quick Reply

High School Olympiads

Equilateral triangle

Reply

shixian105

#1 Jan 28, 2011, 8:47 am

In an equilateral triangle, let h_1,h_2,h_3 be the distance of a point to the triangle's sides. Prove that all the points which satisfy
$\sqrt{h_1}+\sqrt{h_2}=\sqrt{h_3}$
form the inscribed circle of the triangle.

Luis González

#2 Jan 28, 2011, 11:45 am • 3

The problem can be generalized to any scalene $\triangle ABC.$ Let $(\alpha:\beta:\gamma)$ be the trilinear coordinates of a point P WRT $\triangle ABC.$ Let h_1,h_2,h_3 be the distances from P to $BC,CA,AB.$ Then $\sqrt{h_1}=\sqrt{h_2}+\sqrt{h_3}$ or cyclic permutations $\Longleftrightarrow$ $\sqrt{\alpha}+\sqrt{\beta}=\sqrt{\gamma}$ or cyclic permutations. Squaring any of the former expressions yields the equation $\mathcal{E}$ as

$$\mathcal{E} \equiv \alpha^2+\beta^2+\gamma^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)=0$$

$\mathcal{E}$ is an ellipse tangent to BC,CA,AB through the feet of the angle bisectors $(0:1:1),(1:0:1),(1:1:0).$ It's center is the crosspoint X_{37} of the incenter and centroid. Clearly, when $\triangle ABC$ is equilateral, $\mathcal{E}$ becomes its incirle.

Quick Reply

High School Olympiads

Ineq-Ligouras-G049 - Geometry-(Murray S. Klamkin, 1996)

Reply

Ligouras
#1 Jan 28, 2011, 1:01 am

Let ABC be an acute-angled, non-equilateral triangle.
Prove that there is an interior point P other than the circumcenter such that

$$(PB-CP)a^2+(PC-PA)b^2+(PA-PB)c^2=0$$

mavropnevma
#2 Jan 28, 2011, 1:39 am

For $P \in \triangle ABC$ define $f(P)=(PB-PC)a^2+(PC-PA)b^2+(PA-PB)c^2$. Then
$f(A)=(b-c)(b^2+c^2+bc-a^2)=bc(b-c)(1+2\cos A)$
$f(B)=(c-a)(c^2+a^2+ca-b^2)=ca(c-a)(1+2\cos B)$
$f(C)=(a-b)(a^2+b^2+ab-c^2)=ab(a-b)(1+2\cos C)$
But $1+2\cos A, 1+2\cos B, 1+2\cos C>0$, since $\triangle ABC$ is acute-angled, while $(b-c)+(c-a)+(a-b)=0$, and not all three 0, since $\triangle ABC$ not equilateral, therefore (at least one) is positive, and (at least one) is negative.

It means (at least) one of $f(A), f(B), f(C)$ is positive, and (at least) one of $f(A), f(B), f(C)$ is negative (while clearly $f(O)=0$ for O the circumcentre of $\triangle ABC$). WLOG assume $f(A)<0$ and $f(B)>0$. Since f is clearly continuous, on any arc γ connecting A and B there will lie (at least) a point P_γ such that $f(P_\gamma)=0$.
All we need is take arcs contained in $\triangle ABC$ which avoid O; we find infinitely many such points P.

This post has been edited 1 time. Last edited by mavropnevma, Jan 28, 2011, 12:50 pm

Luis González
#3 Jan 28, 2011, 1:46 am

$F \equiv P$ is the 1st Fermat point of $\triangle ABC$, i.e. $\angle BFC=\angle CFA=\angle AFB=120^\circ$. By cosine law for triangles $\triangle AFC$ and $\triangle AFB$ we obtain

$$b^2=FA^2+FC^2-2\cdot FA\cdot FC\cdot\cos 120^\circ=FA^2+FC^2+FA\cdot FC$$

$$c^2=FA^2+FB^2-2\cdot FA\cdot FB\cdot\cos 120^\circ=FA^2+FB^2+FA\cdot FB$$

$$\Longrightarrow\ (b^2-c^2)\cdot FA=FA\cdot FC^2+FC\cdot FA^2-FA\cdot FB^2-FB\cdot FA^2\ \ (1)$$

By permutating a,b,c and FA,FB,FC cyclically we get the identities:

$$(c^2-a^2)\cdot FB=FA\cdot FB^2+FB\cdot FA^2-FB\cdot FC^2-FC\cdot FB^2\ \ (2)$$

$$(a^2-b^2)\cdot FC=FC\cdot FB^2+FB\cdot FC^2-FC\cdot FA^2-FA\cdot FC^2\ \ (3)$$

Summing up the identities (1), (2) and (3) yields:

$$a^2\cdot(FB-FC)+b^2\cdot(FC-FA)+c^2\cdot(FA-FB)=0$$

So far I'm unable to describe the set of all points P satisfying the desired relation.

Ligouras
#4 Jan 28, 2011, 1:50 am

NICE dear Friends!!!!!!
Thanks

Thanks

mavropnevma

#5 Jan 28, 2011, 5:46 am

Notice that in my solution, if all angles are less than $2\pi/3$, then it works the same.
If say $\angle A \geq 2\pi/3$, then $f(B) < 0$ and $f(C) > 0$, and we can apply the same idea. Also, if $\triangle ABC$ is equilateral, then $f(P) = 0$ for any P (and conversely, if $f(P) = 0$ for all P, then $\triangle ABC$ is equilateral). Thus the thesis is true for all triangles!

Quick Reply

High School Olympiads

Two perspective triangles (own)

Reply

jayme

#1 Jan 27, 2011, 5:02 pm

Dear Mathlinkers,
let ABC be a triangle, XYZ the ortic triangle of ABC, 1a, 1b, 1c the resp. A, B, C-excircles of ABC,
PQR the extangent triangle of ABC.
Wrt A :
D, E the points of intersection of the second external tangent to 1b, 1c wtr AB, AC
D', E' the points of intersection of YZ wtr CD, BE
A* the point of intersection of CE' and BD'.

Cyclically, we have B*, C*.

Prove : A*B*C* is perspective to ABC.

Sincerely
Jean-Louis

Luis González

#2 Jan 28, 2011, 1:00 am

Let CE', BD' cut AB, AC at $U, V.$ By Menelaus theorem for the isosceles $\triangle AEB$ cut by $\overline{YZE'}$ and $\overline{CUE'}$ we get:

$$\frac{YA}{YE}\cdot\frac{EE'}{E'B}\cdot\frac{BZ}{ZA}=1\ ,\ \frac{CA}{CE}\cdot\frac{EE'}{E'B}\cdot\frac{BU}{UA}=1 \Longrightarrow \frac{BU}{UA}=\frac{CE}{CA}\cdot\frac{YA}{YE}\cdot\frac{BZ}{ZA}$$

By similar reasoning, we'll have $\frac{CV}{VA}=\frac{BD}{AB}\cdot\frac{ZA}{ZD}\cdot\frac{CY}{YA}$

If AA^* cuts BC at $A_0,$ then by Ceva's theorem we have:

$$\frac{BA_0}{A_0C}=\frac{VA}{CV}\cdot\frac{BU}{UA}=\left(\frac{AB}{BD}\cdot\frac{ZD}{ZA}\cdot\frac{YA}{CY}\right)\left(\frac{CE}{CA}\cdot\frac{YA}{YE}\cdot\frac{BZ}{ZA}\right)$$

Since $BEDC$ is an isosceles trapezoid with diagonals $BD=CE$ and $YZ \parallel ED,$ we get

$$\frac{BA_0}{A_0C}=\frac{AB}{CA}\cdot\frac{YA}{CY}\cdot\frac{BZ}{ZA}=\frac{AB}{CA}\cdot\frac{BX}{XC}$$

$\Longrightarrow AA^* \equiv AA_0$ is the A-cevian of the Clawson point X_{19} of $\triangle ABC.$ By similar reasoning, we conclude that $\triangle ABC$ and $\triangle A^*B^*C^*$ are perspective through $X_{19}.$

jayme

#3 Jan 28, 2011, 4:19 pm

Dear Luis and Mathlinkers,
thank for you proof. Yes, of course, my problem has something to do with the Clawson's point on which I am working without any success...
My hint is to prove that the line AA* is going through the corresponding vertex of the radical triangle of ABC (determined by the three intersection of the radical axis of the circumcircle of ABC wrt 1a, 1b, 1c).
Dear Luis, do you have a little how to prove this result ?
Very sincerely
Jean-Louis

Quick Reply

High School Olympiads

Nagelian point

Reply

Source: Iranian National Olympiad (3rd Round) 2002

Omid Hatami
#1 Oct 1, 2006, 10:45 pm

H, I, O, N are orthogonal center, incenter, circumcenter, and Nagelian point of triangle ABC. I_a, I_b, I_c are excenters of ABC corresponding vertices A, B, C. S is point that O is midpoint of HS. Prove that centroid of triangles $I_aI_bI_c$ and SIN concide.

arline
#2 Oct 10, 2006, 9:50 pm

vectors and or barycentres

Xixas
#3 Oct 11, 2006, 12:42 am • 1

> *Omid Hatami wrote:*
> H, I, O, N are orthogonal center, incenter, circumcenter, and Nagelian point of triangle ABC. I_a, I_b, I_c are excenters of ABC corresponding vertices A, B, C. S is point that O is midpoint of HS. Prove that centroid of triangles $I_aI_bI_c$ and SIN concide.

A cute little problem! Here is my solution to it.

Define M to be the centroid of ABC, L to be the centroid of $I_aI_bI_c$ and P be such point that O is the midpoint of a line segment IP. Because I is an orthocenter and O - a nine-point center in $I_aI_bI_c$ we infer that P is a circumcenter of $I_aI_bI_c$ and $\vec{IL} = 2\vec{LP}$. Thus note that we are done if we showed that $\vec{SP} = \vec{PN}$. But $\vec{SP} = \vec{IH}$ because of the central symmetry at O. Thus it remains to show that $\vec{PN} = \vec{IH}$. But it is well known that $\vec{NM} = 2\vec{MI}$ and $\vec{HM} = 2\vec{MO}$, thus having considered a homothety centered at M with coefficient $-1/2$ we see that $\vec{HN} = 2\vec{IO} = \vec{IP}$ (the last one follows from the definition of P). We deduce that $HNPI$ is a parallelogram and thus $\vec{PN} = \vec{IH}$. Hence the result.

What's the point of your post, arline (besides showing that you know barycentic coordinates and vectors)? It's absolutely useless.

arline
#4 Oct 11, 2006, 8:34 am

xixas,its straighforward computation using barycantres and or vectors.its true remark.and somewhat general method.

hucht
#5 Oct 14, 2006, 4:12 am

> *arline wrote:*
> vectors and or barycentres

barycentric give us a solution much simpler

Ramchandran

#6 Jan 27, 2011, 9:02 am

Is there an elementary solution without vectors or barycentric co-ordinates?

Thank you.

Luis González

#7 Jan 27, 2011, 11:32 am

Since N, H are the incenter and circumcenter of the antimedial triangle of $\triangle ABC$, it follows that $IO \parallel HN$ and $IO = \frac{1}{2}HN.$ Therefore, reflection B_e of I about O (Bevan point of ABC) is the midpoint of $\overline{NS}$ $(\star).$ B_e is the circumcenter of $\triangle I_aI_bI_c$ since B_e is the concurrency point of the perpendiculars from I_a, I_b, I_c to the sides BC, CA, AB of its orthic triangle $\triangle ABC.$ Thus, the centroid G_0 of $\triangle I_aI_bI_c$ lies on the segment connecting its circumcenter B_e and orthocenter I such that $\overline{G_0B_e} : \overline{G_0I} = -1 : 2.$ Together with $(\star),$ we conclude that G_0 is the centroid of $\triangle SIN$ as well.

Ramchandran

#8 Jan 27, 2011, 12:42 pm

I understand that you find your solution's arguments a little trivial, but can you please help me by elaborating more? :(

Why is the Nagel point the incentre of the anti-medial triangle?
and Why does it being so, and H being the circumcentre result in the parallelism with IO?

and so much more of what you have said..

P.S. I have no doubts that your solution is right, It is still too complicated for me. :(

Rijul saini

#9 Jan 27, 2011, 3:52 pm

> *Ramchandran wrote:*
> I understand that you find your solution's arguments a little trivial, but can you please help me by elaborating more? :(
>
> Why is the Nagel point the incentre of the anti-medial triangle?
> and Why does it being so, and H being the circumcentre result in the parallelism with IO?
>
> and so much more of what you have said..

I find this post a bit unnecessary. Why not try it for yourselves? It could've been a good challenge :) Anyways also, instead of posting it here, you could've easily got the answers in a second by searching AoPS or Mathworld. But well,

> *Ramchandran wrote:*
> Why is the Nagel point the incentre of the anti-medial triangle?

The answer that comes most quickly to me is just a homothety centred about G, the centroid of $\triangle ABC$, and ratio -2. This takes $\triangle ABC$ to its anti-median triangle. Also, by this, it takes the Incentre of $\triangle ABC$ to the Nagel Point of $\triangle ABC$. Since it must take the Incentre of $\triangle ABC$ to the Incentre of its Anti-Medial Triangle, the result follows.

> *Ramchandran wrote:*
> Why does it being so, and H being the circumcentre result in the parallelism with IO?

Think about that homothety discussed above.

> *Ramchandran wrote:*
> And so much more of what you have said..

What else?

Quick Reply

High School Olympiads

Constant Value

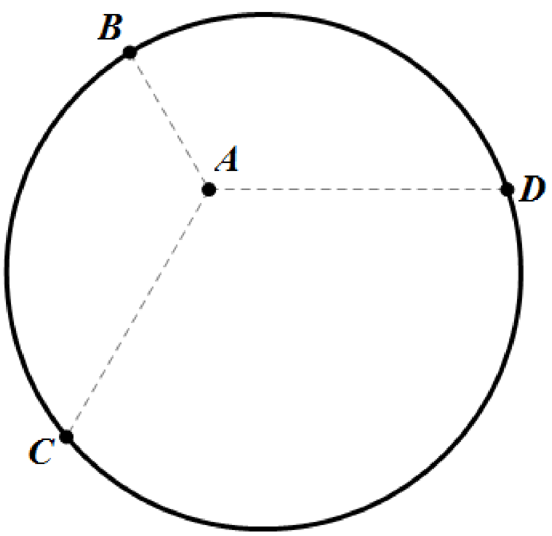
Reply

Headhunter
#1 Jan 26, 2011, 1:46 am

Hello.

Given a point A inside a circle, three points B,C,D move on it satisfying $\angle BAC=\angle CAD=\angle DAB=120^\circ$
Let AB,AC,AD be x,y,z respectively. Show that $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$ is constant.

Attachments:



nsato
#2 Jan 26, 2011, 3:37 am

> *Headhunter wrote:*
> Hello.
>
> Given a point A inside a circle, three points B,C,D move on it satisfying $\angle BAC=\angle CAD=\angle DAB=120^\circ$
> Let AB,AC,AD be x,y,z respectively. Show that $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$ is constant.

This doesn't seem right. If A approaches B, for example, then x approaches 0, and your expression goes to infinity.

Headhunter
#3 Jan 26, 2011, 4:03 am

to nsato.

The point A is fixed.

Redeem

Redeem

#4 Jan 26, 2011, 5:35 am

> Headhunter wrote:
>
> to nsato.
>
> The point A is fixed.

In which case, wouldnt B, C, and D be fixed as well? (aside from equivalent permutations, and rotations of the circle)

Headhunter

#5 Jan 26, 2011, 6:12 am

to All.

Draw a circle. then the circle is fixed.
Draw a point A inside the circle. then the point A is fixed.
Three points B,C,D move on the circle satisfying the condition that $\angle BAC=\angle CAD=\angle DAB=120^{\circ}$

Luis González

#6 Jan 26, 2011, 10:58 am

The relation as stated is not constant for an arbitrary position of A. Let BA, CA, DA cut the fixed circle (O,R) again at P,Q,R. A step of the solution to problem About Fermat point yields $AB+AC+AD=AP+AQ+AR$, hence:

$$(AB+AC+AD)\left(\frac{1}{AB}+\frac{1}{AC}+\frac{1}{AD}\right)=\frac{(AB+AC+AD)^2}{R^2-OA^2}$$

$AB+AC+AD$ cannot be expressed as function of AO,R exclusively. See area 2

SomePig

#7 Jan 26, 2011, 11:28 am

[geogebra]d74349c434c109f09cf90d9543b9c81a94f875db[/geogebra]
Move B around the circle.

atomicwedgie

#9 Jan 26, 2011, 4:07 pm

The claim is true if A is the center of the circle, or the lines drawn from A are rotated by some integer multiple of $\pi/3$. If A is not the center of the circle, then the expression does not remain constant for any position of B.

77ant

#10 Jan 29, 2011, 1:21 am

I think that Headhunter's statement is wrong,
and have seen a problem similar to it in Russian Math Olympiad.

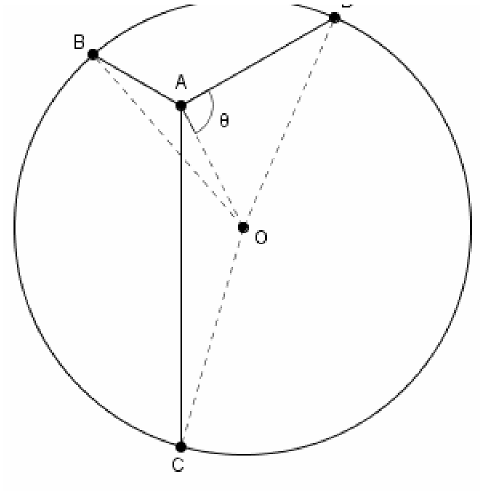
$DO=r, OA=a$
x',y',z' are supplementary segments of x,y,z

$r^2-a^2=z^2-2az\cos\theta$
$r^2-a^2=x^2-2ax\cos(\theta+120^{\circ})$
$r^2-a^2=y^2-2ay\cos(\theta+240^{\circ})$

$$\therefore\ (r^2-a^2)\left(\frac{1}{z}+\frac{1}{x}+\frac{1}{y}\right)=z+x+y \implies x'+y'+z'=x+y+z$$

Just, $\dfrac{x+y+z}{\frac{1}{z}+\frac{1}{x}+\frac{1}{y}}$ is constant.

Attachments:



Quick Reply

High School Olympiads

concurrent,,,,

Reply

jaydoubleuel

#1 Jan 25, 2011, 8:05 pm

given an acute triangle $\triangle ABC$ and it's nine-point circle w
$A', B', C'$are poles of BC, CA, AB wrt w
prove that $AA', BB', CC'$are concurrent

Luis González

#2 Jan 25, 2011, 8:30 pm

A' is the intersection of the tangent of ω at the midpoint of BC and the perpendicular dropped from N to BC. Its anticomplement A_0 is then the intersection of the tangent τ_a of the circumcircle (O) at A with the perpendicular bisector ℓ_a of BC. Using barycentric coordinates with respect to $\triangle ABC$, we obtain:

$$\ell_a \equiv (b^2 - c^2)x + a^2y - a^2z = 0 \ , \ \tau_a \equiv c^2y + b^2z = 0$$

$$A_0 \equiv \ell_a \cap \tau_a \equiv (a^2(b^2 + c^2) : b^2(c^2 - b^2) : c^2(b^2 - c^2))$$

Thus $A' : (2b^2c^2 - b^4 - c^4 : a^2b^2 + a^2c^2 + b^2c^2 - c^4 : a^2b^2 + a^2c^2 + b^2c^2 - b^4)$

Coordinates of B', C' are found by cyclic permutation of a, b, c. Thus AA',BB',CC' concur at

$$\left(\frac{1}{a^2b^2 + a^2c^2 + b^2c^2 - a^4} : \frac{1}{a^2b^2 + a^2c^2 + b^2c^2 - b^4} : \frac{1}{a^2b^2 + a^2c^2 + b^2c^2 - c^4}\right)$$

Which is the isotomic conjugate of X_{1078} $(a^2b^2 + a^2c^2 + b^2c^2 - a^4)$.

Quick Reply

High School Olympiads

Two parallels (own)

jayme

#1 Jan 24, 2011, 7:55 pm

Dear Mathlinkers,
ABC a triangle, 1b, 1c the resp. B, C-excircles of ABC, Ib, Ic the resp. centers of 1b, 1c,
U, V the points of contact of BC resp. with 1b, 1c
and A* the midpoint of UV.
Prove : A*Ib // AV.
Sincerely
Jean-Louis

Luis González

#2 Jan 25, 2011, 12:20 am

This is a quite simple problem dear Jean-Louis.

Since A is the insimilicenter of $(I_b) \sim (I_c)$, it follows that VA pass through the antipode U' of U WRT (I_b). Midpoint M of BC is also midpoint of UV, due to $MU = MV = \frac{1}{2}(b+c)$. Hence, MI_b is the U-midline of $\triangle UVU' \Longrightarrow MI_b \parallel AV$.

High School Olympiads

Triangle

Reply

rrmm

#1 Jan 24, 2011, 10:14 pm

If the AA1, BB1 nad CC1 height of the triangle ABC, A2, B2 and C2 the points at which real AA1, BB1 and CC1 circumcircle of the triangle ABC, prove |AA2|/|AA1| + |BB2|/|BB1| + |CC2|/|CC1| = 4 !

This post has been edited 1 time. Last edited by rrmm, Jan 25, 2011, 12:55 am

Luis González

#2 Jan 24, 2011, 11:47 pm

$H \equiv AA_1 \cap BB_1 \cap CC_1$ is the orthocenter of $\triangle ABC$ and A_2, B_2, C_2 are the reflections of H about BC, CA, AB, since $\angle BCA_2 = \angle BAA_1 = \angle BCH$, etc.

$$\frac{\overline{AA_2}}{\overline{AA_1}} = \frac{\overline{AA_1} + \overline{HA_1}}{\overline{AA_1}} = \frac{\overline{HA_1}}{\overline{AA_1}} + 1 = \frac{|\triangle BHC|}{|\triangle ABC|} + 1$$

Similarly, we have the expressions

$$\frac{\overline{BB_2}}{\overline{BB_1}} = \frac{|\triangle CHA|}{|\triangle ABC|} + 1 \ \ , \ \ \frac{\overline{CC_2}}{\overline{CC_1}} = \frac{|\triangle AHB|}{|\triangle ABC|} + 1$$

$$\frac{\overline{AA_2}}{\overline{AA_1}} + \frac{\overline{BB_2}}{\overline{BB_1}} + \frac{\overline{CC_2}}{\overline{CC_1}} = 3 + \frac{|\triangle BHC| + |\triangle CHA| + |\triangle AHB|}{|\triangle ABC|} = 4.$$

Subrata

#3 Jan 25, 2011, 12:38 pm

Please ewplain why A2 is the reflection of H about BC

rrmm

#4 Jan 25, 2011, 2:54 pm

Thank You very much!

Raja Oktovin

#5 Jan 25, 2011, 5:20 pm

> *Subrata wrote:*
> Please ewplain why A2 is the reflection of H about BC

you can just angle chase like $\angle BCA_2 = \angle BAA_2 = \angle C_1AA_1 = \angle C_1CA_1 = \angle HCB$.

Quick Reply

High School Olympiads

With the radical triangle (Own)

Reply

jayme

#1 Jan 22, 2011, 12:36 pm

Dear Mathlinkers,
let ABC be a triangle, (O) the circumcircle of ABC, 1a, 1b, 1c the resp. A, B, C-excircles of ABC, and UVW the radical triangle (determined by the intersection of the radical axis of (O) wrt 1a, 1b, 1c).
Let PQR the extangent triangle of ABC,
V' the points of contact of BC with 1b,
V", W" the points of contact of QR wrt 1b, 1c, and X the midpoint of V"W".
Prove : the parallel to UV through V' goes through X.
Sincerely
Jean-Louis

jayme

#2 Jan 22, 2011, 9:16 pm

Dear Mathlinkers,
no a small idea?
Sincerely
Jean-Louis

Luis González

#3 Jan 24, 2011, 9:42 am

(O,R) is circumcirle of $\triangle ABC.$ The excircles $(I_b),(I_c)$ touch BC at X,Y and their common external tangent (different from BC) touches $(I_b),(I_c)$ at $P,Q.$ M,L are midpoints of BC,PQ and let D,T be the orthogonal projections of A onto $BC,PQ.$ Since PQ is antiparallel to $BC,$ then $O \in AT.$ Assume that $XL \perp OI_b.$ Then

$$I_bL^2-OL^2=I_bX^2-OX^2 \Longrightarrow \ OL^2=I_bM^2-I_bX^2+OX^2=MX^2+OX^2$$

By Pythagorean theorem for OTL, we get $OL^2=OT^2+TL^2=OT^2+DM^2 \Longrightarrow$

$$OT^2=MX^2-DM^2+OX^2=2(OM^2+MX^2)-OM^2-DM^2$$

$$OT^2=OM^2+2MX^2-DM^2=R^2-\tfrac{1}{4}a^2+\tfrac{1}{2}(b+c)^2-({m_a}^2-{h_a}^2)$$

$$OT^2=R^2+{h_a}^2+2R\cdot h_a=(R+h_a)^2 \Longrightarrow OT=R+h_a$$

Which is a true identity $\Longrightarrow XL$ is parallel to the radical axis of $(I_b),(O).$

jayme

#4 Jan 25, 2011, 7:43 pm

Dear Luis and Mathlinkers,
thank very much for your proof based on a caracterization of an orthodiagonel quadrilateral.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Rectangle in □ABC (oWn)

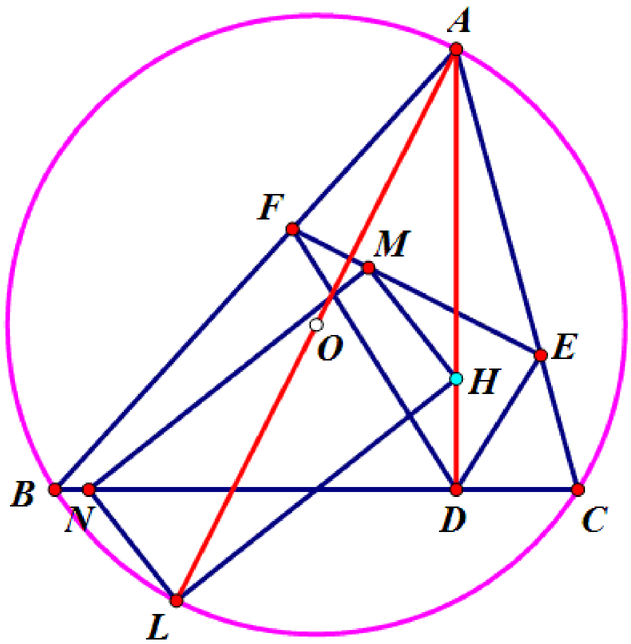
Reply

lym
#1 Jan 23, 2011, 8:42 pm

Figure□(O)□H and $\triangle DEF$ resp. are the circumcircle□orthocenter and orthic triangle of $\triangle ABC$□L is opposite point of A wrt (O) . Prove that

(1)□Exist M and N where are on EF and BC such that $HMNL$ si an rectangle .
(2)□Call ω_a the $\odot HMNL$□similar delimit ω_b□ω_c□then ω_a□ω_b and ω_c are coaxal .

Attachments:



Luis González
#2 Jan 24, 2011, 12:39 am

AL cuts EF at T. Then $TECL$ is cyclic on account of $\angle LTE=\angle LCE=90^\circ$. Since $HDCE$ is also cyclic we have then $AT\cdot AL=AE\cdot AC=AH\cdot AD\Longrightarrow THDL$ is cyclic and let its circumcircle cut EF,BC at M,N. Then $\angle MNL=\angle MTL=90^\circ$ and $\angle NLH=\angle NDH=90^\circ\Longrightarrow HMNL$ is a rectangle, as desired. $\odot(DHT)\equiv\omega_a$. Then according to the topic Coaxal circles, we deduce that circles $\omega_a,\omega_b,\omega_c$ are coaxal.

lym
#3 Jan 24, 2011, 12:55 am

Nice solution□dear luis .
My method is complicated□yours is easy and clever .

Quick Reply

High School Olympiads

Locus of centroid and orthorcenter

Reply

vulalach
#1 Jan 23, 2011, 9:49 am

Let acute-angle xOy and point P interior xOy. A circle pass through P and O intersects Ox and Oy at M and N, respectively. Find locus of centroid and orthorcenter of triangle OMN.

Immanuel Bonfils
#2 Jan 23, 2011, 10:10 am

Does both are variable, point P and acute angle?

vulalach
#3 Jan 23, 2011, 10:39 am

sorry, xOy and point P are fix. a circle is variable.

Luis González
#4 Jan 23, 2011, 12:51 pm

Since $\angle MOP = \angle MNP$ and $\angle NOP = \angle MNP$ are fixed, it follows that triangles $\triangle MNP$ are all similar. If L is the midpoint of $\overline{MN}$, then $\angle NPL = \varphi$ and ratio $\frac{PL}{PN}$ are constant $\Longrightarrow$ Locus of L is the image ℓ of the line OX under the spiral similarity with center P, ratio $\frac{PL}{PN}$ and rotational angle φ clockwise. Therefore, locus of centroids G of $\triangle OMN$ is the homothetic line of ℓ under the homothety with center O and cofficient $\frac{2}{3}$.

Let P_1, P_2 be the reflections of P about $OX, OY \Longrightarrow P_1P_2$ is the Steiner line of P with respect to all $\triangle OMN$. Thus, locus of orthocenters H of $\triangle OMN$ is the line P_1P_2.

Quick Reply

High School Olympiads

Prove that PNA = AMB - [UKRMO 2009 Grade 9]

Reply

Amir Hossein

#1 Jan 23, 2011, 7:44 am • 1

In triangle ABC points M, N are midpoints of BC, CA respectively. Point P is inside ABC such that $\angle BAP = \angle PCA = \angle MAC$. Prove that $\angle PNA = \angle AMB$.

Luis González

#2 Jan 23, 2011, 9:18 am

(O) is circumcircle of $\triangle ABC$ and tangents through A, B to (O) cut the tangent through C at Q, R. A-symmedian AR cuts (O) again at D and P' is the midpoint of the chord AD, i.e. projection of O on AD. Thus, P' lies on the circumcircle of the cyclic quadrilateral $OCRB$. Let $\odot(OBC)$ cut AC again at S. Then $\triangle ASB$ is isosceles with apex S and SR bisects $\angle BSA$ externally $\Longrightarrow SR \parallel AB \Longrightarrow \angle BAP = \angle PRS = \angle PCA \Longrightarrow P \equiv P'$. Now, since NP is the A-midline of $\triangle ADC$, it follows that $\angle PNA = \angle DCA = \angle AMB$.

FantasyLover

#3 Jan 23, 2011, 11:41 am

Solution

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dgreenb801

#4 Jan 24, 2011, 9:10 am • 2

Let D be such that $ABDC$ is a parallelogram, then A,M, and D are collinear. We have $\angle BAM = \angle PAC$ and $\angle BDA = \angle DAC = \angle ACP$, so $\triangle ABD \sim \triangle APC$. But M is the midpoint of AD and N is the midpoint of AC, so $\triangle ABM \sim \triangle APN$. So $\angle PNA = \angle AMB$.

---

mathuz

#5 Apr 24, 2013, 6:02 pm

Let points D, B lie opposite sides from the line AC and $\triangle ABC$ similar to $\triangle DCA$.
That $ADCP$ is cyclic and D, N, P are collinear. From
angles DNC and AMB are equal then $\angle ANP$ equal to $\angle AMB$.
Also solution the problem Grade 11, P:3 by the method is very easy.

---

sunken rock

#6 Apr 25, 2013, 4:56 am

Obviously, P lies onto $A-$symmedian of ΔABC, and let AP intersect the circle $\odot(ABC)$ at D. Clearly, $ABDC$ is a harmonic quadrilateral and $\widehat{AMB} = \widehat{BMD} = \widehat{ACD}$. Consequently we need to prove $\angle ANP = \angle ACD$, i.e. P is midpoint of AD, but this is true, since $\angle CPD = \angle CAP$.

Best regards,
sunken rock

Quick Reply

High School Olympiads

A problem

Reply

refrigerator

#1 Jan 23, 2011, 12:36 am

Let (O) is circumcircle of ΔABC. A line l pass A intersects (O) at M and circumcircle BCH (H_ orthocenter of ABC) at N inwardly (O). Midperpendicular of MN intersection BC at P. Prove NP passes through a fixed point

Luis González

#2 Jan 23, 2011, 2:33 am

Since $(O) \cong \odot(HBC)$ are symmetrical about $BC,$ it follows that reflections L,T of H,N about BC lie on $(O).$ Thus, P becomes circumcenter of $\triangle NMT.$ Ray TN cuts (O) at $S.$ From the isosceles trapezoids $HNTL$ and $ASTL$ with circumcircle $(O),$ we obtain $\angle NMT=\angle SCL=\angle STL=\angle HNT \Longrightarrow HN$ is tangent to $\odot(MNT)$ through $N,$ i.e. $\angle NHP$ is right. Thus, NP passes through the antipode of H WRT $\odot(HBC),$ i.e. the reflection of A about the midpoint of $BC.$

Quick Reply

High School Olympiads

Geometry

Reply

refrigerator

#1 Jan 23, 2011, 12:24 am

Let ΔABC and G inside it. AG, BG, CG intersections BC, CA, AB at D, E, F. Let M,N,P are the midpoints of BC, CA, AB. Let H, I, K are the midpoints of EF, DF, DE. Prove: MH, NI, PK concurrent

Luis González

#2 Jan 23, 2011, 1:11 am

Let X, Y, Z be the midpoints of GA, GB, GC. X, H, M are collinear on the Newton line of the complete quadrangle $AFGE$. Likewise, Y, I, N and Z, K, P are collinear on the Newton lines of $BDGF$ and $CEGD$, respectively. Since $\triangle XYZ$ is homothetic to $\triangle ABC$ under the homothety with center G and factor $\frac{1}{2}$, it follows that $\triangle XYZ$ and the medial $\triangle MNP$ are centrally congruent $\Longrightarrow$ lines $MX \equiv MH, NY \equiv NI$ and $PZ \equiv PK$ concur at their symmetry center.

Quick Reply

High School Olympiads

prove

Reply

informationed
#1 Jan 21, 2011, 11:42 pm

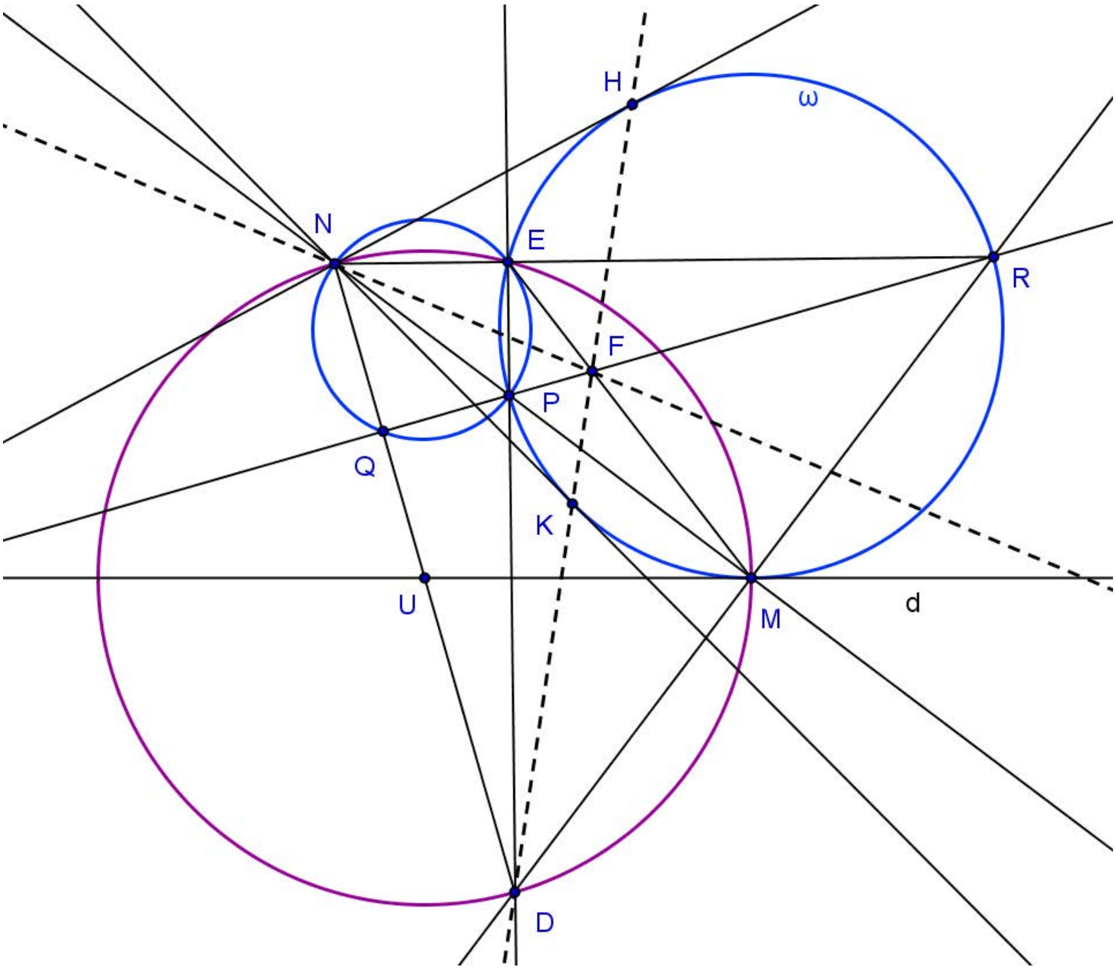
Let MN not perpendicular d, M lie on d. The circle ω variable touching d at M. NH, NK touches ω at H,K. Prove HK passes through a fixed point

Luis González
#2 Jan 22, 2011, 10:20 pm

Let (U) be the fixed circle centered at d and passing through $M, N.$ Perpendicular d' to NM at M cuts (U) again at the fixed $D.$ Variable circle ω cuts lines NM and d' again at $P, R.$ Let $Q \equiv RP \cap DN.$ Then the circles $\omega, (U)$ and $\odot(PNQ)$ concur at the Miquel point E of $\triangle DMN \cup \overline{PQR}.$ But P is ortocenter of $\triangle DNR,$ due to $\angle PRD = \angle PND \Longrightarrow N, E, R$ are collinear.

Let $F \equiv EM \cap PR.$ Since line pencil $N(M, R, F, D)$ is harmonic, it follows that NF is the polar of D WRT $\omega.$ Therefore, the polar HK of N WRT ω pass through the fixed point $D.$

Attachments:



Quick Reply

High School Olympiads

A lemma

Reply

hurricane
#1 Jan 22, 2011, 8:54 am

Let AD, BE, CF be the altitudes of an acute triangle ABC. Let M, N be respectively the points of itersection of DE and CF and of DF and BE. Denote by $B'; C'$ respestively ,the symmetric points of the vertices B, C through the sides AC, AB. Denote by O' be the circumcenter of circle $(AB'C')$. Prove that $AO' \perp MN$

diks94
#2 Jan 22, 2011, 9:32 am

Can u pls post it's figure

hurricane
#3 Jan 22, 2011, 9:50 am

> diks94 wrote:
> Can u pls post it's figure

I don know how to post figure. I think this problem is understandable

diks94
#4 Jan 22, 2011, 9:59 am

I Hope U Are Trying To Sa That The Symmetric point of B with Respect to AC is it's mirror image i hope

Luis González
#5 Jan 22, 2011, 10:09 am

$H \equiv AD \cap BE \cap CF$ is the orthocenter of $\triangle ABC$. From the cyclic quadrilaterals $DHEC$ and $DHFB$, we have that $M \equiv DE \cap CH$ and $N \equiv DF \cap BH$ have equal powers to $(L) \equiv \odot(HBC)$ and 9-point circle $(K) \equiv \odot(DEF) \Longrightarrow$ MN is radical axis of $(K), (L) \Longrightarrow KL \perp MN$. But A is the exsimilicenter of $(K) \sim (L) \Longrightarrow A, L, K$ are collinear, i.e. $AK \perp MN$. Now, according to this topic, the circumcenter O' of $\triangle AB'C'$ lies on $AK \Longrightarrow AO' \perp MN$.

diks94
#6 Jan 22, 2011, 10:27 am

Let AD be along the y axis and Bc be along the X axis
D be (0,0)
Then Equation Of AC is

$$2bx = ay - ab$$

..1
Equation Of AB is

$$ab - 2bx = ay$$

...2
Equation of BE is Perpendicular to AC
Any Line perpendicular to given line can be written as Bx - ay + $ =0

Where Equation of original line is ax+ by +c =0
After Doing This We can Find the Equation OF MN Whch Comes Out To be parallel to x axis
After This We Find The Circumcentre of Triangle AB'C'
Which lies on the Y axis
Therefore OA Perpendicular to MN

jayme
#7 Jan 22, 2011, 4:17 pm

Dear Mathlinkers,
the problem can be more clear if in my point of view, we don't forget that the orthocenter is the incenter of the orthic triangle.
Starting with a triangle ABC, we consider the incenter I and the I-cevian triangle A'B'C' of ABC.
Now it is known according some message on Mathlinks
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=299372
that for example B'C' is orthogonal to the line joining the circum center of ABC to the A-excenter of ABC.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Circles through the Schroder point

Source: 0

Luis González

#1 May 6, 2009, 3:23 am • 1

Incircle (I, r) of $\triangle ABC$ is tangent to BC, CA, AB at X_a, X_b, X_c. Perpendiculars from I to BC, CA, AB meet external bisectors of angles $\angle A, \angle B, \angle C$ at A', B', C', respectively. Circles $\odot(IBB'), \odot(ICC')$ meet at I, Y_a and define cyclically Y_b, Y_c. Let M, N, L be the midpoints of $\overline{X_bX_c}, \overline{X_cX_a}, \overline{X_aX_b}$.

1) Show that $\odot(AIX_a), \odot(BIX_b), \odot(CIX_c), \odot(MIY_a), \odot(NIY_b), \odot(LIY_c)$ meet at the Schroder point X_{1155} of $\triangle ABC$, i.e. the inverse of the insimilicenter of $(I) \sim (O)$ in the circumcircle.

2) Let R_a, R_b, R_c be the radii of $\odot(IAA'), \odot(IBB'), \odot(ICC')$ and Ψ denotes the sum of the distances from the orthocenter of $\triangle X_aX_bX_c$ to the sides of $\triangle ABC$. Show that

$$\frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c} = \frac{\Psi}{r^2}$$

admire9898

#2 Sep 16, 2009, 9:28 pm

Thanks! 😁

jayme

#3 Sep 16, 2009, 10:42 pm

Dear Luis and Mathlinkers,
nice result.
I think that you know the great article written by Dary Grinberg
http://de.geocities.com/darij_grinberg/Schroeder/Schroeder.html
Sincerely
Jean-Louis

Luis González

#4 Sep 17, 2009, 6:20 am

Dear Jean Louis, Darij's article is indeed a great material. The concurrency of the circles $\odot(AIX_a), \odot(BIX_b)$ and $\odot(CIX_c)$ at the Schröder point S is well-known. Basically, I'm pointing out the concurrency of the circles $\odot(MIY_a), \odot(NIY_b)$ and $\odot(LIY_c)$ at the inverse of the insimilicenter of $(I) \sim (O)$ in the circumcircle.

jayme

#5 Sep 17, 2009, 9:34 am

Dear Luis,
sorry, I have read to quickly your message.
Your result is very interesting.
How do you prove it?
Sincerely
Jean-Louis

Luis González

#6 Jan 22, 2011, 4:33 am • 2

Sorry for my late reply dear Jean-Louis, I stopped thinking about Schröder point for a while.

A-mixtilinear incircle ω_a touches AB, AC and the circumcirle (O) at $P, Q, U.$ It is well-known that the A-excenter I_a is the midpoint of $PQ.$ Inversion through pole A with power $AI_a{}^2$ swaps (I_a) and ω_a and takes (O) into a straight line τ tangent to (I_a) through the inverse U' of $U.$ τ cuts AB, AC at the inverses B', C' of $B, C.$ Since $B'C'$ is antiparallel to $BC,$ then there exists the composition of the axial symmetry accross the angle bisector of $\angle BAC$ and a positive homothety with center A taking $\triangle AC'B'$ with incircle ω_a into $\triangle ABC$ with incircle $(I) \Longrightarrow AX_a$ and AUU' are isogonals WRT $\angle BAC.$ Since U is the insimilicenter of $(O) \sim \omega_a$ and A is the exsimilicenter of $(I) \sim \omega_a,$ it follows that $AU \cap IO$ is the insimilicenter X_{55} of (I) and $(O).$ By similar reasoning, we conclude that X_{55} is the isogonal conjugate of the Gergonne point G_e of $\triangle ABC$ $(\star).$

Inversion WRT (I) takes X_a, X_b, X_c into themselves and M, N, L into $A, B, C,$ respectively. Thus, $\odot(AIX_a), \odot(BIX_b),$ $\odot(CIX_c)$ are taken into median lines X_aM, X_bN, X_cN of the intouch triangle $\triangle X_aX_bX_c \Longrightarrow \odot(AIX_a), \odot(BIX_b),$ $\odot(CIX_c)$ meet at I and the inverse of the centroid V of $\triangle X_aX_bX_c.$ Since IA' is a diameter of $\odot(IAA'),$ it follows that the inverse line ℓ_a of $\odot(IAA')$ is the parallel to BC passing through $M,$ i.e the tangent at M of the 9-point cirle $\odot(MNL)$ of $\triangle X_aX_bX_c.$ Likewise, inverse lines ℓ_b, ℓ_c of $\odot(IBB'), \odot(ICC')$ are tangent to $\odot(MNL)$ through $N, L.$ Therefore, pairwise lines ℓ_a, ℓ_b, ℓ_c meet at the inverses Z_a, Z_b, Z_c of $Y_a, Y_b, Y_c \Longrightarrow AZ_a, BZ_b, CZ_c$ are the inverses of $\odot(MIY_a),$ $\odot(NIY_b), \odot(LIY_c).$ But since $\triangle ABC \sim \triangle Z_aZ_bZ_c$ are homothetic, it follows that AZ_a, BZ_b, CZ_c concur at the insimilicenter of their incircles $(I) \sim \odot(MNL),$ i.e. the centroid V of $\triangle X_aX_bX_c.$ Consequently, $\odot(MIY_a), \odot(NIY_b),$ $\odot(LIY_c)$ also pass through the inverse of V in the incircle $(I).$ Now, according to Concurrent 2 (PC point again) (see the remark between bold lines) the circles $\odot(AIX_a), \odot(BIX_b), \odot(CIX_c)$ meet at I and the inverse of the isogonal conjugate of the Gergonne point G_e in the circumcircle $(O).$ Together with $(\star),$ we conclude that $\odot(AIX_a), \odot(BIX_b), \odot(CIX_c),$ $\odot(MIY_a), \odot(NIY_b), \odot(LIY_c)$ concur at the Schröder point of $\triangle ABC.$

Quick Reply

High School Olympiads

concyclic points (oWn)

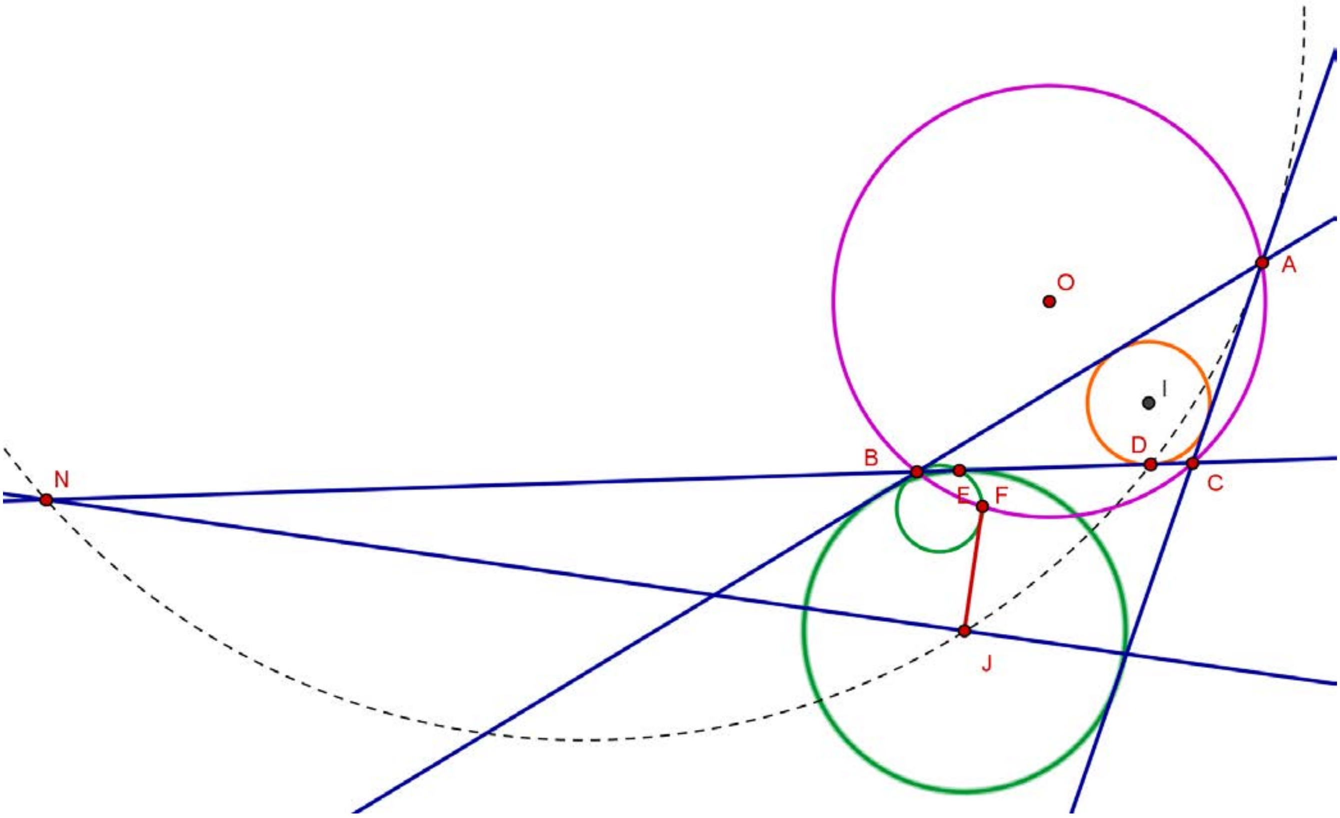
Reply

lym

#1 Jan 20, 2011, 9:46 pm

The incircle (I) and A-excircle (J) of $\triangle ABC$ respectively touch BC at D and E□a circle $\mathcal{W}$ pass through E and tangent to AB at B intersect $O(ABC)$ at F . A line perpendicular to JF at J intersect BC at N . Prove that A□D□J□N are concyclic .

Attachments:

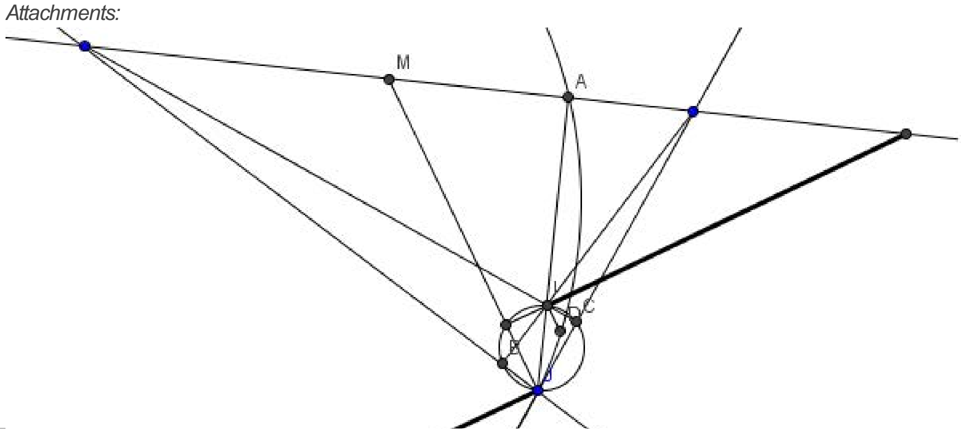


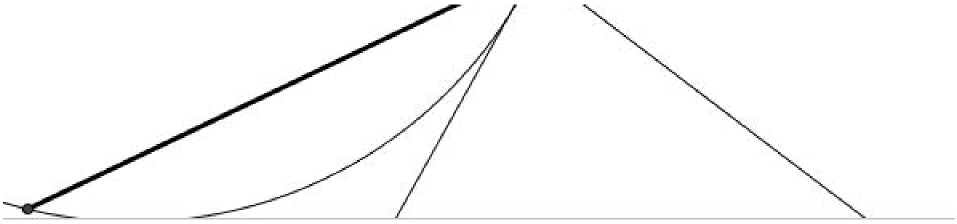
skytin

#2 Jan 20, 2011, 11:07 pm

1 . Prove that JF goes thru midpoint of arc BC (largest)
2 . construct other excenters

Attachments:





Luis González

#3 Jan 21, 2011, 3:39 am

A-mixtilinear excircle ω_a touches $AB, AC, (O)$ at P, Q, T respectively. It's well-known that J is the midpoint of $PQ.$ Inversion through pole A with power AJ^2 takes ω_a into the A-excircle (J) and the circumcircle (O) into a line τ tangent to (J) through the inverse T' of $T.$ τ cuts AB, AC at the inverses B', C' of $B, C.$ Since $B'C'$ is antiparallel to $BC,$ then there exists the composition of the axial symmetry about the angle bisector of $\angle BAC$ and a direct homothety centered at A taking $\triangle AB'C'$ with incircle ω_a into $\triangle ABC$ with incircle $(I) \Longrightarrow$ rays AD, AT are isogonal WRT $\angle BAC.$

Let AD cut (O) again at $K \Longrightarrow BTKC$ is an isosceles trapezoid with legs $TB = KC.$ Since $DC = EB,$ then due to obvious symmetry, it follows that $\angle ETB = \angle DKC = \angle ABC \Longrightarrow \odot(TEB)$ is tangent to AB though $B \Longrightarrow T \equiv F.$ Now, FJ, FA become F-median and F-symmedian of $\triangle PFQ$ and FP, FQ bisect $\angle AFB$ and $\angle AFC$ externally, due to the external tangency of $(O), \omega_a.$ Thus, we deduce that $\angle JFB = \angle JFC \Longrightarrow JF$ bisects $\angle BFC,$ i.e. JF passes through the midpoint M of the arc BAC of $(O).$ If L is the midpoint of the arc BC of $(O),$ we obtain

$\angle JND = \angle EFJ = \angle FML = FAL = \angle JAD \Longrightarrow A, D, J, N$ are concyclic.

jayme

#4 Jan 21, 2011, 5:39 pm

Dear Mathlinkers,
a simple observation.
By a A-extraversion of the situation involves in
http://perso.orange.fr/jl.ayme vol. 4 A new mixtilinear incircle adventure I p. 31,
F is the point of contact of the A-mixilinear excircle of ABC ;
in consequence, JF goes through the midpoint of the arc BC which contains A.
Sincerely
Jean-Louis

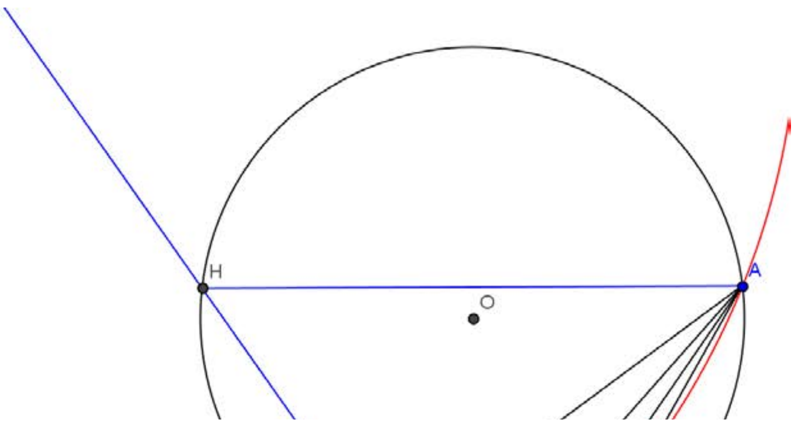
THVSH

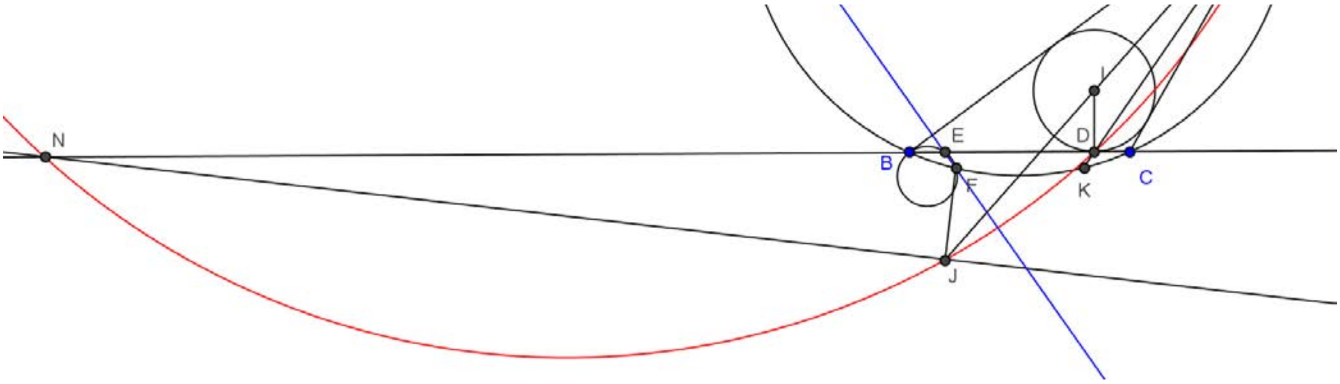
#5 May 10, 2015, 3:56 pm • 2

My solution:

Let $EF \cap (O) = H; K \in (O)$ such that $FK \parallel BC$
(BEF) is tangent to AB, so $\angle HFB \equiv \angle EFB = \angle ABC \Longrightarrow AH \parallel BC$
Now, since the reflection wrt the perpendicular bisector BC, we get: A, D, K are collinear.
$\Longrightarrow AD, AF$ are isogonal conjugates in $\angle BAC$.
$\Longrightarrow \triangle ACD \sim \triangle AFB \Longrightarrow AD.AF = AB.AC$
We also have $\triangle AIB \sim \triangle ACJ \Longrightarrow AI.AJ = AB.AC$
$\Longrightarrow AD.AF = AI.AJ \Longrightarrow \triangle AFJ \sim \triangle AID \Longrightarrow \angle AJF = \angle ADI$
$\Longrightarrow \angle AJN = 90^\circ + \angle AJF = 90^\circ + \angle ADI = \angle ADN$
$\Longrightarrow A, D, J, N$ are concyclic. Q.E.D

Attachments:





![avatar]

buratinogigle

#6 May 10, 2015, 9:42 pm • 1

Nice dear THVSH, base on your solution, I can proposed an interesting extension as following

Let ABC be a triangle inscribed circle (O) and P,Q are two isogonal conjugate point. AP cuts (O) again at D. E is a point on (O). EQ cuts (O) again at F. G lies on BC such that $PG \parallel DF$. APG cuts BC again at H. AP cuts (PBC) again at K. Prove that $HK \parallel DE$.

My solution is the same as An extension of a property from mixtilinear incircle post #6

Quick Reply

High School Olympiads

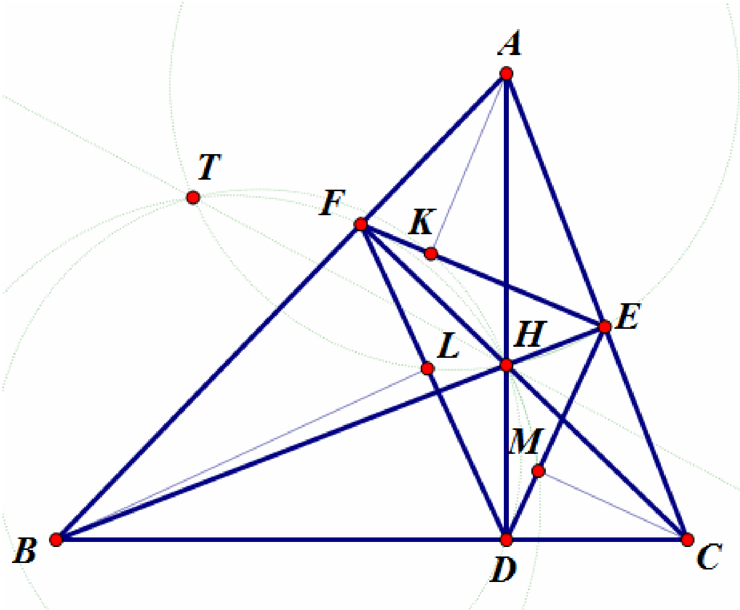
Coaxal Circles

Reply

77ant
#1 Jan 17, 2011, 11:57 am • 1

Dear everyone
For $\triangle ABC$ with its orthocenter H, $\triangle DEF$ is its orthic triangle.
AK, BL, CM are perpendicular to EF, FD, DE respectively.
Prove that three circumcircles of $\triangle KHD$, $\triangle LHE$, $\triangle FHM$ are coaxal.

Attachments:



vulalach
#2 Jan 19, 2011, 9:04 am

Denote DE = a, DF = b, EF = c. We calculate KF, DL, EM in a, b, c.
Next we can prove that DK, EL and FM are concurrent at I and IK.ID = IE.IL = IF.IM. @

Luis González
#3 Jan 19, 2011, 9:31 am • 1

> *vulalach wrote:*
> Denote DE = a, DF = b, EF = c. We calculate KF, DL, EM in a, b, c.
> Next we can prove that DK, EL and FM are concurrent at I and IK.ID = IE.IL = IF.IM. @

This is a complete nonsense. What makes you think that Nagel point "I" of the scalene DEF satisfies IK.ID = IE.IL = IF.IM ?

vulalach
#4 Jan 19, 2011, 11:32 am

Sorry, i'm wrong.

jayme

#5 Jan 19, 2011, 12:25 pm

Dear Mathlinkers,
in order to have a better look on this nice problem
let ABC be a triangle, I the incenter and D'E'F' the Nagel triangle of ABC.
We have to prove that the circles (AID'), (BIE') and (CIF') are coaxial.
The second point of intersection is the so called by Darij Grinberg "The Nagel-Schröder point".
The proof is not so easy...
Sincerely
Jean-Louis

Luis González

#6 Jan 20, 2011, 12:10 am • 2

Perpendicular to HK through K cuts BC at P_0 and we define R_0, Q_0 on CA, AB similarly. Then circumcenters O_a, O_b, O_c of $\triangle DHK, \triangle EHL, \triangle FHM$ are the midpoints of HP_0, HQ_0, HR_0. Thus it suffices to show that P_0, Q_0, R_0 are collinear, since $(O_a), (O_b), (O_c)$ are coaxal $\Longleftrightarrow O_a, O_b, O_c$ are collinear $\Longleftrightarrow P_0, Q_0, R_0$ are collinear. Let HP_0 cut AC, AB at Q, R. Then since $FKHR$ and $QKHE$ are both cyclic, it follows that $\angle ARQ = \angle FHK$ and $\angle AQR = \angle EHK$ (mod 180 degrees). Therefore

$$\frac{AR}{AQ} = \frac{\sin \widehat{EHK}}{\sin \widehat{FHK}} = \frac{KE}{KF} \cdot \frac{HF}{HE} \quad (1)$$

Since $ARHQ$ is cyclic and AK, AH are isogonals WRT $\angle BAC$, we get

$$AR \cdot AQ = AK \cdot AH = AF \cdot AE \ (2)$$

From (1) and (2) we get $AR^2 = \dfrac{AF \cdot AE \cdot KE \cdot HF}{KF \cdot HE} = AB^2 \cdot \dfrac{\cos^2 B \cdot \cos^2 A}{\cos^2 C}$

$$\Longrightarrow RA = AB \cdot \frac{\cos B \cdot \cos A}{\cos C} \Longrightarrow RB = AB \left(\frac{\cos C - \cos B \cos A}{\cos C} \right)$$

Similarly, we'll have the expressions

$$QA = AC \cdot \frac{\cos C \cdot \cos A}{\cos B} \quad , \quad QC = AC \left(\frac{\cos B - \cos C \cos A}{\cos B} \right)$$

By Menelaus' theorem for $\triangle ABC$ cut by transversal $\overline{P_0QR}$, we get then

$$\frac{P_0B}{P_0C} = \frac{QA}{QC} \cdot \frac{RB}{RA} = \frac{\cos C(\cos C - \cos A \cos B)}{\cos B(\cos B - \cos A \cos C)} \quad (3)$$

By cyclic exchange, we get the expressions

$$\frac{Q_0C}{Q_0A} = \frac{\cos A(\cos A - \cos B \cos C)}{\cos C(\cos C - \cos A \cos B)} \quad (4)$$

$$\frac{R_0A}{R_0B} = \frac{\cos B(\cos B - \cos A \cos C)}{\cos A(\cos A - \cos B \cos C)} \quad (5)$$

Multiplying (3), (4) and (5) together yields: $\dfrac{P_0B}{P_0C} \cdot \dfrac{Q_0C}{Q_0A} \cdot \dfrac{R_0A}{R_0B} = 1$

By the converse of the Menelaus' theorem, we conclude that P_0, Q_0, C_0 are collinear $\Longrightarrow O_a, O_b, O_c$ are collinear $\Longrightarrow$ $(O_a), (O_b), (O_c)$ are coaxal and the proof is completed.

livetolove212

#7 Jan 24, 2011, 10:22 am

Dear Mathlinkers,
This problem is only a special case of the problem at here.
It's also base on the generalization of Steinbart's theorem.

livetolove212

#8 Jan 24, 2011, 3:44 pm • 2

First I will change the names of points in this problem:

" Given a triangle ABC with its orthocenter H, circumcenter O. Let $A_1B_1C_1$ be the orthic triangle of ABC, X, Y, Z be the intersections of AO and B_1C_1, BO and A_1C_1, CO and A_1B_1. Show that $(HA_1X), (HB_1Y), (HC_1Z)$ are coaxal".

Proof:

Let O_a, O_b, O_c be the center of $(HA_1X), (HB_1Y), (HC_1Z)$; P be the intersection of AO and (O). We have $\angle AB_1C_1 = \angle ABC = \angle APC$ then PXB_1C is a cyclic quadrilateral.

We get $AH.AA_1 = AB_1.AC = AX.AP$, which follows that $P \in (HXA_1)$.

Let A_3 be the intersection of the line through P and perpendicular to HP with BC. Similarly we define B_3, C_3. Note that $\angle HA_1A_3 = \angle HPA_3 = 90^o$ thus O_a is the midpoint of HA_3.

$(O_a), (O_b), (O_c)$ are coaxal iff O_a, O_b, O_c are collinear, iff A_3, B_3, C_3 are collinear.

Let A_2, B_2, C_2 be the midpoints of BC, CA, AB; A_4 be the intersections of the lines through H and perpendicular to HA_2 and BC. Similarly we define B_4, C_4.

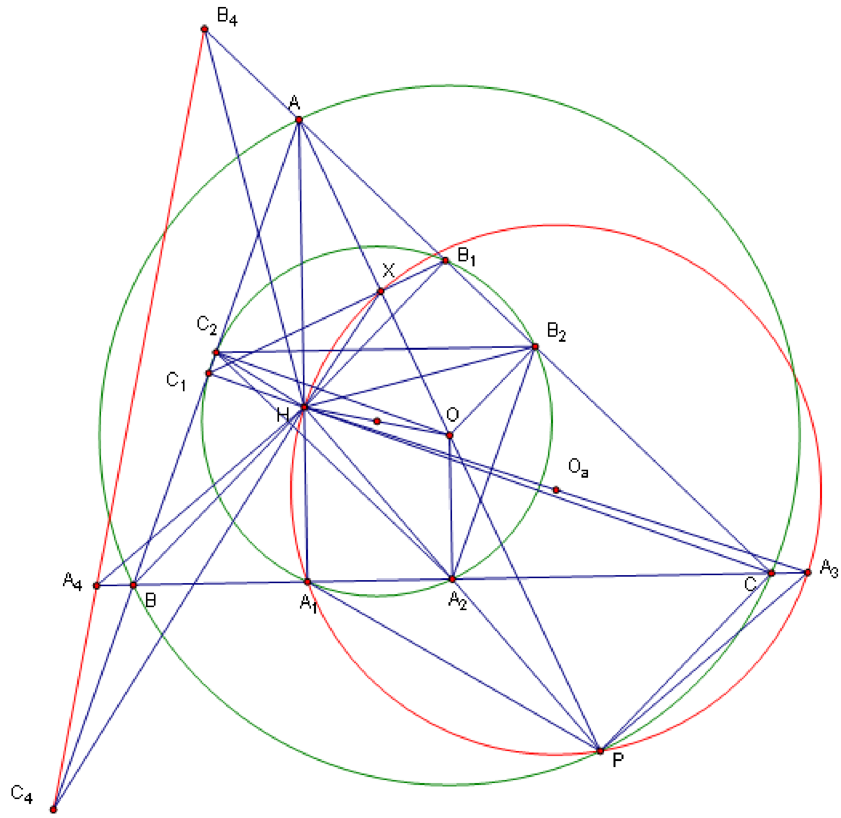
Since H is the reflection of P wrt A_2 then A_3, B_3, C_3 are collinear iff A_4, B_4, C_4 are collinear.

We have $\frac{A_4B}{A_4C} = \frac{S_{A_4BH}}{S_{A_4CH}} = \frac{BH.\sin\angle A_4HB}{CH.\sin\angle A_4HC} = \frac{BH}{CH}.\frac{\cos BHA_2}{\cos CHA_2}$

Similarly we get $\frac{A_4B}{A_4C}.\frac{B_4C}{B_4A}.\frac{C_4A}{C_4B} = \frac{BH}{CH}.\frac{CH}{AH}.\frac{AH}{BH}.\frac{\cos\angle BHA_2}{\cos\angle BHC_2}.\frac{\cos\angle B_2HC}{\cos\angle A_2HC}.\frac{\cos\angle C_2HA}{\cos\angle B_2HA}$

Note that $\frac{\cos\angle B_2HC}{\cos\angle A_2HC} = \frac{\sin\angle HA_2B_2}{\sin\angle HB_2A_2} = \frac{HB_2}{HA_2}$. Do the same with $\frac{\cos\angle BHA_2}{\cos\angle BHC_2}$ and $\frac{\cos\angle C_2HA}{\cos\angle B_2HA}$ we are done.

Attachments:



Quick Reply

High School Olympiads

b+c=3a (Nagel Point)

Reply

Headhunter
#1 Jan 19, 2011, 7:42 am

Hello.

If the Nagel point of a triangle lies on its incircle,
Show that the sum of two of the sides of the triangle equals to three times the third side.

Luis González
#2 Jan 19, 2011, 9:07 am

A-excircle and B-excircle touch BC, CA at $X, Y. N \equiv AX \cap BY$ is the Nagel point of $\triangle ABC.$ Incircle (I, r) touches BC, CA, AB at P, Q, R and let P_0, Q_0, R_0 be the antipodes of P, Q, R WRT $(I).$ Then $N \in (I) \Longleftrightarrow$ either $N \equiv P_0,$ $N \equiv Q_0$ or $N \equiv R_0.$ Assume that $N \equiv P_0.$ Then the distance δ_a from N to BC equals $2r.$

By Menelaus' theorem for $\triangle AXC$ cut by the transversal $\overline{BNY}$, we get

$$\frac{AN}{NX} = \frac{a}{s-a} \Longrightarrow \frac{AX}{NX} = \frac{h_a}{\delta_a} = \frac{s}{s-a} \Longrightarrow 2r \cdot s = h_a(s-a)$$

$$\Longrightarrow a \cdot h_a = h_a(s-a) \Longrightarrow b + c = 3a.$$

Quick Reply

High School Olympiads

Concurrency

Reply

juancarlos

#1 Oct 21, 2006, 12:01 am

Let AD, BE, CF be the internal bisectors of ABC triangle (D, E, F on the sides).DE, EF, FD cut AB, BC, CA at Z, X, Y respectively. Draw the tangent XU to the circumcircle of ABC, U on the arc that not contain A, draw the tangents YV, ZW similarly. Prove that AU, BV, CW concur.

yetti

#2 Oct 23, 2006, 2:07 am

By Ceva/Menelaus theorems, X, Y, Z are feet of the external bisectors of $\angle A,\ \angle B,\ \angle C$, which are collinear. XYZ is the radical axis of the circumcircle (O) of $\triangle ABC$ and of the circumcircle (P) of its excentral triangle $\triangle I_aI_bI_c$. For example $X \equiv BC \cap I_bI_c$, quadrilateral BCI_BI_c is cyclic on account of the right angles $\angle I_bCI_c,\ \angle I_cCI_b$ (B, C are altitude feet of the excentral triangle), so that $XB \cdot XC = XI_b \cdot XI_c$, X is on the radical axis of (P), (O), similarly Y, Z. The excentral triangle is always acute, for example $\angle I_cI_aI_b = \frac{\angle B + \angle C}{2} = 90^\circ - \frac{\angle A}{2} < 90^\circ$, etc. Thus its circumcircle (P) and its 9-point circle (O) never intersect, hence the radical axis XYZ never intersects (O). Project the circle (O) into a circle (O') and the line XYZ to infinity. Tangents to (O') from X', Y', Z' (now at infinity) are parallel to B'C', C'A', A'B', tangency points U', V', W' are midpoints of the arcs B'C', C'A', A'B' of (O') opposite to A', B', C', respectively, A'U', B'V', C'W' are the internal bisectors of $\angle A', \angle B',\ \angle C'$ intersecting at the incenter I' of the projected $\triangle A'B'C'$, hence the original cevians AU, BV, CW are also concurrent.

vittasko

#3 Nov 26, 2006, 3:46 pm

This problem is also true in general configuration of an arbitrary line, which intersects the sidelines $AB, BC, CA,$ at points so be it, $Z, X, Y,$ respectively.

A simple elementary proof of the general problem is as follows: (In my drawing $AB = 6.0, BC = 8.0, AC = 7.2,$ $BZ = 4.8, CX = 6.8$).

Because of the points $Z, X, Y,$ are collinear, by Menelaus's theorem, we have that $\frac{XC}{XB} \cdot \frac{ZB}{ZA} \cdot \frac{YA}{YC} = 1$, (1)

From similar triangles $\triangle UBX \sim \triangle UCX \Longrightarrow \frac{XC}{XU} = \frac{UC}{UB} = \frac{XU}{XB} \Longrightarrow \frac{XC}{XB} = \frac{(UC)^2}{(UB)^2}$, (2) and similarly $\frac{ZB}{ZA} = \frac{(WB)^2}{(WA)^2}$, (3) and $\frac{YA}{YC} = \frac{(VA)^2}{(VC)^2}$, (4)

From $(1), (2), (3), (4), \Longrightarrow (UC)\cdot(VA)\cdot(WB) = (UB)\cdot(WA)\cdot(VC)$, (5)

From (5), we conclude that the diagonals $AU, BV, CW,$ of the inscribed hexagon $AWBUCV,$ are concurrent at one point, and the proof is completed.

Kostas Vittas.

PS. It is well known that in an inscribed hexagon, it's diagonals are concurrent at one point, if and only if, the product of it's three non-adjoined side segments, is equal with the one, of the other three it's side segments (easy to prove).

Luis González

#4 Jan 19, 2011, 3:26 am

vittasko wrote:

> This problem is also true in general configuration of an arbitrary line, which intersects the sidelines $AB, BC, CA,$ at points so be it, $Z, X, Y,$ respectively.

Let A_0, B_0, C_0 be the vertices of the tangential triangle of $\triangle ABC \Longrightarrow A_0U, B_0V, C_0W$ are the polars of X, Y, Z with respect to the circumcircle (O). Since X, Y, Z are collinear, then A_0U, B_0V, C_0W concur at the pole K of $\overline{XYZ}$ with respect to (O). By Steinbart theorem for $\triangle A_0B_0C_0$, if $K \equiv A_0U \cap B_0V \cap C_0W$, it follows that AU, BV, CW concur at a point P_0. Furter, if P is the trilinear pole of $\overline{XYZ}$, then $P_0 \equiv I \cdot \sqrt{P}$, i.e. AU, BV, CW concur at the product of the incenter I of $\triangle ABC$ and the barycentric square root of P.

jayme

#5 Jan 19, 2011, 9:46 pm

Dear Mathlinkers,
1. X, Y, Z are collinear because ABC and DEF are perspective
2. the three tangents determine a triangle noted A'B'C' perspective with ABC ;
in consequence, AA', BB' and CC' are concurrent
3. According to the Steinbart's theorem (see http://perso.orange.fr/jl.ayme vol. 3 les points de Steinart et de Rabinowitz), and we are done.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Equal Segments

Reply

77ant

#1 Jan 17, 2011, 10:59 am • 1

Dear everyone
In triangle ABC, the internal bisector of $\angle A$ meets BC at D, and the external bisectors of $\angle B$ and $\angle C$ meet AC and AB at E and F respectively. Suppose that three lines perpendicular to BC, AC, AB at D, E, F, respectively, are concurrent. Prove that $AB = AC$

Luis González

#2 Jan 17, 2011, 12:14 pm

In general, the cevian triangle of $P(x:y:z)$ is also a pedal triangle, if and only if

$$(S_AS_B + S_AS_C + S_BS_C)(S_A(y^2-z^2)x + S_B(z^2-x^2)y + S_C(x^2-y^2)z) = 0$$

Particularly, cevian triangle of the A-excenter $I_a(-a:b:c)$ is a pedal triangle

$$\Longleftrightarrow \ aS_A(b^2-c^2) - bS_B(c^2-a^2) - cS_C(a^2-b^2) = 0$$

$$\Longleftrightarrow \ (a+b)(a+c)(b-c)(b+c-a)^2 = 0 \ \Longleftrightarrow b = c$$

Quick Reply

High School Olympiads

Parallelogram and Circle

Reply

Kunihiko_Chikaya

#1 Jan 17, 2011, 8:25 am

Given a parallelogram $OABC$. A circle passing through the points $O,\ B$ intersects with the extented lines of $OA,\ OC$ at $P,\ Q$ respectively.

(1) Prove that $\overline{OA}\cdot\overline{OP}+\overline{OC}\cdot\overline{OQ}=\overline{OB}^2$.

(2) For acute angles $\alpha,\ \beta$, let $\angle POB=\alpha,\ \angle BOQ=\beta$ and denote $S,\ T$ the areas of triangles $OPB,\ OQB$ respectively. If $OA=a,\ OB=b$, then
Prove that $a^2S+b^2T=\frac{c^4\sin\alpha\cdot\sin\beta}{2\sin(\alpha+\beta)}$.

This post has been edited 1 time. Last edited by Kunihiko_Chikaya, Jan 19, 2011, 6:32 am

Luis González

#2 Jan 17, 2011, 8:46 am

Problem (1) was posted before. It's from Baltic way 2001.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=378175
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=324722

Quick Reply

High School Olympiads

Tangents to the circumcircle of ABC

Reply

WakeUp
#1 Jan 16, 2011, 10:51 pm

The tangents at A and B to the circumcircle of the acute triangle ABC intersect the tangent at C at the points D and E, respectively. The line AE intersects BC at P and the line BD intersects AC at R. Let Q and S be the midpoints of the segments AP and BR respectively. Prove that $\angle ABQ = \angle BAS$.

dgreenb801
#2 Jan 17, 2011, 5:03 am

Where is Q? And what is the point of having point S if you could just say $\angle BAP$ instead of $\angle BAS$?

WakeUp
#3 Jan 17, 2011, 6:14 am

> dgreenb801 wrote:
> Where is Q? And what is the point of having point S if you could just say $\angle BAP$ instead of $\angle BAS$?

Sorry about this, the mistake is not my own as this is the exact wording of the problem. However http://www.imomath.com/ believes that Q is the midpoint of AP and S is the midpoint of BR. I've just done a quick sketch and this seems right, so i've edited the problem.

Luis González
#4 Jan 17, 2011, 6:31 am • 1

Oh my Lord, it's this problem again !.

http://www.artofproblemsolving.com/viewtopic.php?t=19806
http://www.artofproblemsolving.com/viewtopic.php?t=207440
http://www.artofproblemsolving.com/viewtopic.php?f=46&t=275380

cwein3
#5 Aug 29, 2011, 2:31 pm • 1

Very nice problem

It is well known that AP and BR are symmedians of the triangle ABC. See here: http://web.mit.edu/yufeiz/www/olympiad/geolemmas.pdf Let the midpoint of BC be X and the midpoint of AC be Y. Let AX intersect the circumcircle again at Z and BY intersect the circumcircle at W.

Note that $\triangle BAP \sim \triangle ZAC$, and $\triangle ARB \sim \triangle WCB$, from trivial angle chasing using the fact that AP and BR are symmedians. Then we must have $\angle AZR = \angle ABQ$ and $\angle XWB = \angle SAB$ by similar triangles. But since $\angle XRB = \angle WBA = \angle WZA$, $WRXZ$ is cyclic, so $\angle SAB = \angle XWB = \angle AZR = \angle ABQ$, as desired.

Quick Reply

High School Olympiads

Parabola

Reply

Headhunter

#1 Jan 12, 2011, 12:54 am

Hello.

For a parabola α with its focus F and a point A outside α, let two lines tangent to α (passing through A) touch α at P, Q. Show that $\angle AFQ = \angle AFP$ with no use of coordinates system as possible.

SomePig

#2 Jan 15, 2011, 7:09 pm

Solution

Luis González

#3 Jan 15, 2011, 11:51 pm

The result is true for any nondegenerate conic section.

Proposition. $\mathcal{K}$ is a nondegenerate conic with foci F, F'. A is a point outside $\mathcal{K}$ and tangents from A to $\mathcal{K}$ touch $\mathcal{K}$ through P, Q. Then $\angle AFQ = \angle AFP$. Likewise, $\angle AF'Q = \angle AF'P$.

Proof. We know that rays AF, AF' are isogonals WRT $\angle PAQ$. If M, N are the reflections of F on tangents AQ, AP, it follows that $Q \in MF'$, $P \in NF'$ and $MN \perp AF'$. Therefore, $\triangle AMN$ is isosceles with apex A and F' lies on its symmetry axis $\Longrightarrow \angle AMQ = \angle ANP \Longrightarrow \angle AFQ = \angle AFP$. Similarly, we'll have $\angle AF'Q = \angle AF'P$.

Headhunter

#4 Jan 16, 2011, 2:12 am

to All.

Thank you for your beautiful solutions.

Quick Reply

High School Olympiads

Vietnam NMO 1990_4

Reply

April
#1 Oct 26, 2008, 11:06 am

A triangle ABC is given in the plane. Let M be a point inside the triangle and A', B', C' be its projections on the sides BC, CA, AB, respectively. Find the locus of M for which $MA \cdot MA' = MB \cdot MB' = MC \cdot MC'$.

Vo Duc Dien
#2 Jan 15, 2011, 9:56 am

It's just one single dot.

Vo Duc Dien

Luis González
#3 Jan 15, 2011, 11:20 am

Wonder why the problem restricts point M in the interior of ABC ?. The locus of M would be the union of the circumcircle of $\triangle ABC$ and its orthocenter. Let $(x:y:z)$ be the barycentric coordinates of M with respect to $\triangle ABC$.

For instance, condition $MB \cdot MB' = MC \cdot MC'$ implies alone

$$\frac{(a^2z^2+2S_Bzx+c^2x^2)y^2}{b^2}=\frac{(b^2x^2+2S_Cxy+a^2y^2)z^2}{c^2} \Longleftrightarrow$$

$$(a^2yz+b^2zx+c^2xy)(c^2xy+c^2yz-b^2yz-b^2xz)=0$$

Thus, M either lies on the circumcircle $\mathcal{O} \equiv a^2yz+b^2zx+c^2xy=0$ or it lies on the rectangular circum-hyperbola $\mathcal{H}_a$, isogonal conjugate of the perpendicular bisector of $\overline{BC}$. Hence, cyclic conditions yield that locus of M is the union of $\mathcal{O}$ and the orthocenter of ABC.

Quick Reply

High School Olympiads

Two parallels (own)

jayme
#1 Jan 14, 2011, 3:56 pm

Dear Mathlinkers,
ABC a triangle, (O) the circumcircle of ABC, A'B'C' the orthic triangle of ABC, H the orthocenter of ABC,
D the second point of intersection of AA' with (O), E the second point of intersection of DB' with (O), B* the point of intersection of BE and A'C'.
Prove that B*H is parallel to B'C'.
Sincerely
Jean-Louis

Luis González
#2 Jan 15, 2011, 4:42 am • 1

$F \equiv B'C' \cap AA'$, $P \equiv BC \cap B'C'$ and U is the midpoint of $B'C'$. Since cross ratio (B', C', F, P) is harmonic, it follows that $\overline{PF} \cdot \overline{PU} = \overline{PB'} \cdot \overline{PC'} = \overline{PB} \cdot \overline{PC} \Longrightarrow B, C, U, F$ are concyclic $\Longrightarrow \angle UBC = \angle B'FC$, but quadrilateral $FDCB'$ is cyclic on account of $\angle AB'C' = \angle ABC = \angle ADC$, thus $\angle EDC = \angle UBC \Longrightarrow E, U, B$ are collinear. Let the parallel from H to $B'C'$ cut $A'C'$, AB at B_0, K, respectively. Since H is incenter of $\triangle A'B'C'$, then $\triangle C'HB_0$ is isosceles with apex $B_0 \Longrightarrow B_0$ is the midpoint of $HK \Longrightarrow BB_0$ is identical to the B-median line of $\triangle BB'C' \Longrightarrow EB$ and $A'C'$ intersect at $B_0 \Longrightarrow B_0 \equiv B^* \Longrightarrow HB^* \parallel B'C'$.

jayme
#3 Mar 31, 2015, 1:29 pm

Dear Mathlinkers,

1. X the second point of intersection of the parallel to B'C' through B with (O)
2. Y the second point of intersection of BB' with (O)
3. the pencil (B ; A, Y, E, X) is harmonic
4. the pencil (C' ; A, H, A', B') is harmonic
5. these pencils aving a common ray (AB) and two correspondant rays parallel C'B' and BX, we are done…

Sincerely
Jean-Louis

TelvCohl
#4 Mar 31, 2015, 5:26 pm

My solution:

Let $T = BH \cap C'A'$.

From A nice midpoint we get BE cut $B'C'$ at its midpoint M ,

so from Menelaus theorem (for $\triangle C'B'T$ and line $\overline{BB^*M}$) we get $\frac{CB^*}{TB^*} = \frac{BB'}{BT} = \frac{HB'}{HT} \Longrightarrow B^*H \parallel B'C'$.

Q.E.D

High School Math

Maximum Area of a Quadrilateral

Reply

edwinsampang
#1 Jan 11, 2011, 1:09 pm

ABCD is a quadrilateral with sides
AB=3 cm, BC=4 cm, CD=6cm, and DA=5 cm
What is its maximum area?

castigioni
#2 Jan 11, 2011, 4:41 pm

The maximum area is found when the quadrilateral is cyclic and the area is 6*10^.5

luie1168e
#3 Jan 14, 2011, 8:18 am

> *castigioni wrote:*
> The maximum area is found when the quadrilateral is cyclic and the area is 6*10^.5

How do you prove that?

Luis González
#4 Jan 14, 2011, 10:09 am

a,b,c,d denote the fixed side lengths of a convex quadrilateral $ABCD$ with semiperimeter $p=\frac{1}{2}(a+b+c+d)$. Due to Brahmagupta, we know that:

$$[ABCD]=\sqrt{(p-a)(p-b)(p-c)(p-d)-abcd\cdot\cos^2\frac{A+C}{2}}$$

$[ABCD]$ is maximum $\Longleftrightarrow \cos^2\frac{A+C}{2}=0 \Longleftrightarrow A+C=180^\circ.$

Quick Reply

High School Olympiads

H' is on a fixed line (Mathematical Reflections 2006)

jestrada

#1 Jan 13, 2011, 8:32 pm

In the triangle ABC we pick points D, E on sides AC, AB respectively such that $CD = BE$. Let F be the intersection of BD and CE. let G be the midpoint of AF. Let H be the 2nd point of intersection of the circumcircles of triangles ABD, ACE. Let H' be the reflection of H through G. Prove that H' is on a fixed line.

Luis González

#2 Jan 14, 2011, 8:46 am

Let $\triangle PQR$ be the antimedial triangle of $\triangle ABC$. P, Q, R against A, B, C. M is the midpoint of BC and K is the reflection of F about M. Then $BF \parallel CK$ and $CF \parallel BK$ imply that $|\triangle KCD| = |\triangle KBE| = \frac{1}{2}|BFCK|$. Since $BE = CD$, then it follows that K is equidistant from $BE, CD \Longrightarrow K$ lies on the A-angle bisector of $\triangle ABC$, hence F lies on the P-angle bisector of $\triangle PQR$.

On the other hand, $\odot(ABD)$ and $\odot(ACE)$ meet at A and the center of the rotation taking the oriented segments BE, DC into each other $\Longrightarrow HE = HC \Longrightarrow AH$ bisects $\angle EAC$, thus H moves on the A-angle bisector of $\triangle ABC$. Now, since $AHFH'$ is a parallelogram, it follows that $FH' \parallel AHK \Longrightarrow H' \in PF$, i.e. Locus of H' is the P-angle bisector of the antimedial triangle $\triangle PQR$.

jestrada

#3 Jan 21, 2011, 9:10 pm

Nice solution, Luis
When you say "the antimedial triangle of ABC" you mean the triangle with A, B, C as its midpoints?

Here is my solution.

Since $ABHD, ACHE$ are cyclic, we have $\angle ABH = \angle HDC$ and $\angle ACH = \angle HEB$. Combine this with $BE = CD$ and we have $\triangle HBE = \triangle HDC$ and therefore $HB = HD$ so $\angle HBD = \angle HDB$ and $\angle BAH = \angle HAD$, so H is on the inner bisector of $\angle BAC$.
WLOG let $AB < AC$ (If $AB = AC$ then clearly F is also on the bisector of $\angle BAC$, so H' is on this bisector too). Let M be on the prolongation of AB through A such that $BM = AC$. Let $N \in AC$ such that $CN = AB$. I will prove F, M, N are collineal.
We have $\dfrac{DC}{AC} \times \dfrac{AE}{EB} \times \dfrac{BF}{FD} = 1$ (Menelaus Theorem)
Since $BE = CD$, we have $\dfrac{BF}{FD} = \dfrac{AC}{AE} = \dfrac{BM}{DN}$, since $AE = AB - BE = CN - CD = CN$
Also $AM = BM - AB = AC - CN = AN$
This implies $\dfrac{AM}{BM} \times \dfrac{BF}{FD} \times \dfrac{DN}{AN} = 1$ so F, M, N are collineal (Reciprocal of Menelaus)

Since $AM = AN$, $\angle AMN = \angle ANM = 1/2\angle BAC = \angle BAH$ so $AH||MF$. Since $AHFH'$ is a parallelogram, H' lies on the parallel through F to AH, i.e H' lies on MN, but MN is a fixed line (q.e.d).

High School Olympiads

Nice tangent (oWn)

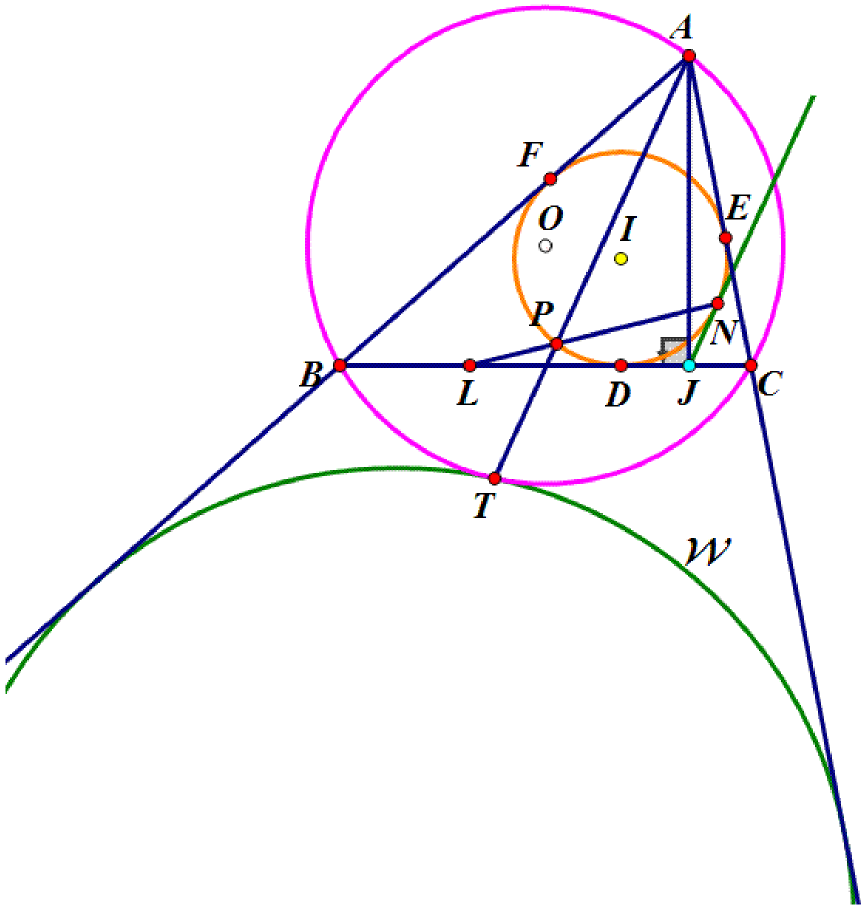
Reply

lym

#1 Jan 13, 2011, 9:04 pm

Figure□the incircle $\odot I$ tangent BC□AC□AB at D□E□F□$\overline{BL}=-\overline{CD}$□$J$ is the foot of A-altitude .
A circle ω tangent AB and AC tangent (O) at T□AT intersect $\widehat{EDF}$ at P□LP intersect $\odot I$ at N .
Prove that JN is tangent of $\odot I$.

Attachments:



Luis González

#2 Jan 14, 2011, 12:24 am

A is the exsimilicenter of $(I) \sim \mathcal{W}$ and T is the insimilicenter of $(O) \sim \mathcal{W}$. By Monge & d'alembert theorem, it follows that $IO \cap AT$ is the insimilicenter of $(O) \sim (I) \Longrightarrow$ Radii OA, IP are parallel $\Longrightarrow$ Tangent of (I) through P passes through $V \equiv AI \cap BC$. On the other hand, $LB = DC$ implies that L is the tangency point of the A-excircle (I_a) with BC. Since cross ratio (A, V, I, I_a) is harmonic, so is its orthogonal projection on BC, namely $(J, V, D, L) = -1, (\star)$. Let the tangent from J to (I) (different from BC) touch (I) at N_0 and $Q \equiv JN_0 \cap PV$. (I) becomes Q-excircle of $\triangle QVJ$. Then by degenerate Brianchon theorem, it follows that QD, JP, VN_0 concur $\Longrightarrow PN_0$ cuts BC at the harmonic conjugate of D WRT (V, J). Thus, from $(\star)$ we deduce that $L \equiv PN_0 \cap BC \Longrightarrow N_0 \equiv N$.

Quick Reply

High School Olympiads

Hard Geometry Problem With Cevian Triangles

Reply

TheIronChancellor
#1 Jan 13, 2011, 1:33 am

Let ABC triangle and also DEF, D'E'F' the cevian triangles for any two points respectively on the inside (D,E,F just between B,D' and A,E' and A,F' respectively).
Prove that the points Q=DE∩D'E', R=DF∩D'F', S=EF∩E'F' belong in the same straight line passing through A.

jayme
#2 Jan 13, 2011, 11:32 am • 1

Dear Iron and mathlinkers,
why not send a figure in order to write more scientifically your nice problem?
Sincerely
Jean-Louis

Luis González
#3 Jan 13, 2011, 11:37 am • 2

The last intersection should be $S \equiv EF' \cap FE'$, instead of $S \equiv EF \cap E'F'$.

Let $\triangle DEF$ and $\triangle D'E'F'$ be the cevian triangles of P, P'. Let $U \equiv FD \cap AC$ and $V \equiv D'E' \cap AB. UF'$ cuts sideline BC at T. Then we have $(TCB) = (F'AB) \cdot (UCA) = (VAB) \cdot (ECA) \Longrightarrow V, E, T$ are collinear $\Longrightarrow \triangle UED$ and $\triangle F'VD'$ are perspective through $T \Longrightarrow$ Intersections $A \equiv EU \cap VF'$, $P \equiv DF \cap D'F'$ and $Q \equiv DE \cap D'E'$ are collinear $(\star)$. Since D, E, F, D', E', F' lie on a same conic, namely the bicevian conic of P, P', by Pascal theorem for nonconvex hexagon $DEF'D'E'F$, the intersections $Q \equiv DE \cap D'E'$, $S \equiv EF' \cap FE'$ and $R \equiv F'D' \cap FD$ are collinear. Together with $(\star)$, A, Q, R, S are collinear.

jayme
#4 Jan 13, 2011, 5:23 pm

Dear Mathlinkers,
this problem appears as a generalization of the Schroeter's points.
For more details and developpements, you can see
http://perso.orange.fr/jl.ayme vol. 2 les deux points de Schroeter.
Sincerely
Jean-Louis

TheIronChancellor
#5 Jan 13, 2011, 7:33 pm

Luisgeometra you are the best in geometry i think that if I can give you any problem you will solve it !!!

Thanks to all of you helped me with that beautiful problem.

vittasko
#6 Jan 15, 2011, 7:42 pm

This problem has been posted **Here** lately and it is inspired from the problem **Pretty Hard**, posted by **libra_gold**.

● Based on the General problem, posted in the topic **Pelletier's triangle** of **Jean-Louis Ayme**, we have that the points
$Q \equiv DE \cap D'F'$ and $R \equiv DF \cap D'E'$ and A are collinear and also that the points Q, X, R, A where

$Q \equiv DE \cap D'E'$ and $R \equiv DF \cap D'F'$ and A, are collinear and also that the points $Q,\ X,\ R,\ A$, where $X \equiv BC \cap AQ$, are in harmonic conjugation (see please in the link, the corresponded harmonic conjugation of the points $Q,\ X,\ P,\ C$, from the complete quadrilateral $SQPRBC$).

So, the line segment UV, where $U \equiv EQ \cap D'F',\ V \equiv D'E' \cap DF$, passes through the point A, as the harmonic conjugate of X, with respect to $Q,\ R$.

Because of now, the collinearity of the points $A,\ U,\ V$, based on the **Pascal theorem**, we conclude that the non-convex hexagon $DEE'D'F'F$ is inscribed in a conic so be it (c) (consider them as the points of intersection, $U \equiv DE \cap D'F'$ and $A \equiv EE' \cap F'F$ and $V \equiv E'D' \cap FD$).

We consider also, the non-convex hexagon $DEF'D'E'F$ inscribed in (C) and applying again the **Pascal theorem**, we have that the points $Q \equiv DE \cap D'E'$ and $S \equiv EF' \cap E'F$ and $R \equiv F'D' \cap FD$, are collinear.

Hence, because of the line segments $QRA,\ QRS$, have two common points, we conclude that they are coincided and the proof is completed.

Kostas Vittas.

Attachments:
t=386252.pdf (5kb)

Quick Reply

High School Olympiads

property of parabola

Reply

77ant

#1 Jan 13, 2011, 1:55 am

Dear everyone
For a parabola η with its axis y, let the line tangent to η at its vertex O be x
A point P move on x, and a line passing through P cut η at A, B. It cut y at C
Prove that $PA.PB = PC^2$. I think it's so cool.

Luis González

#2 Jan 13, 2011, 6:07 am

Let F, G be the reflections of C about O, P and X_∞, Y_∞ denote the infinity points of $x, y,$ respectively. Perpendicular to y through C cuts η at P, Q. Then $(P, Q, C, X_\infty) = -1$ and $(C, F, O, Y_\infty) = -1$ implies that FG is the polar of C WRT η $\Longrightarrow$ Cross ratio (A, B, C, G) is harmonic, thus by Newton's theorem $PG^2 = PC^2 = PA \cdot PB$.

Quick Reply

Sorry, your request for community topics is taking too long. Please try again in a bit.

High School Olympiads

Acute-angled triangle ABC

Reply

Alfons_Nasir
#1 Jan 13, 2011, 12:25 am

An acute-angled triangle ABC is inscribed in a circle with center O. A point P is taken on the shorter $arcAB$. The perpendicular from P to BO intersects AB at S and BC at T . Likewise, the perpendicular from P to AO intersects AB at Q and AC at R.

(a) Prove that the triangle PQS is isosceles.
(b) Show that $PQ^2 = QR * ST$

Luis González
#2 Jan 13, 2011, 2:32 am

$\triangle PQS$ is obviously isosceles, due to $\angle PSQ = \angle PQS = \angle BCA$. Let AO cuts (O) again at D. Then $\angle ADP = \angle RPA = \angle ABP$ implies that $\triangle AQP \sim \triangle PSB$, thus

$$\frac{AQ}{PS} = \frac{PQ}{BS} \implies PQ^2 = AQ \cdot BS \ (\star)$$

On the other hand, since QR and ST are antiparallel WRT BC, CA, we have

$$\frac{QR}{BC} = \frac{AQ}{AC}, \frac{ST}{AC} = \frac{BS}{BC} \Longrightarrow QR \cdot ST = \frac{AQ \cdot BC}{AC} \cdot \frac{BS \cdot AC}{BC} = AQ \cdot BS.$$

Together with $(\star)$, we obtain $PQ^2 = QR \cdot ST$, as desired.

sunken rock
#3 Apr 24, 2011, 12:52 am

Working on the same way, we see $\Delta ARQ \sim \Delta TBS$ and $\Delta AQP \sim \Delta PSB$, which solves the problem.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Parallel of AC at B meets external bisector at D

Reply

Source: Baltic Way 1998

WakeUp
#1 Jan 12, 2011, 12:32 am

Given triangle ABC with $AB < AC$. The line passing through B and parallel to AC meets the external angle bisector of $\angle BAC$ at D. The line passing through C and parallel to AB meets this bisector at E. Point F lies on the side AC and satisfies the equality $FC = AB$. Prove that $DF = FE$.

Luis González
#2 Jan 12, 2011, 8:20 am

Let parallels from B, C to AC, AB intersect at P. Then $ABPC$ is a parallelogram and $\triangle PDE$ is isosceles, due to $\angle ADB = \angle AEC = 90^\circ - \frac{1}{2}\angle BAC$. Internal bisector of $\angle BAC$ cut $\overline{PB}$ at Q. Then $\angle CAQ = \angle AQB = \angle BAQ$ implies that $\triangle BAQ$ is isosceles with apex $B \Longrightarrow BA = BQ = CF \Longrightarrow PF \parallel AQ \Longrightarrow PF$ bisects $\angle BPC$ internally, i.e. PF is the P-angle bisector of the isosceles $\triangle PDE \Longrightarrow FD = FE$.

IstekOlympiadTeam
#3 Nov 3, 2015, 12:45 am

$\angle PAB = \angle APB$ Then $AB = PB = FC$ sice $PB \parallel FC$ and this implyes that $PFCB$ is parallelogram. Then $DF = BC$ and we must show that $BC = FE$. Whis obviously true because $\triangle BAC = \triangle FCE$

EDIT: 222 post 🙂

This post has been edited 1 time. Last edited by IstekOlympiadTeam, Nov 3, 2015, 12:47 am

MouN
#4 Nov 3, 2015, 1:23 am

Let M be the midpoint of the arc $\widehat{BAC}$ and G the intersection of the circumcircle of $\triangle ABC$ and BD. Since $AB = FC$, $MB = MC$ and $\angle ABM = \angle FCM$ the triangles ABM and FCM are congruent. Hence $\angle MEC = 180^\circ - MAB = 180^\circ - MFC$ so $CFME$ is cyclic. Then $\angle FMC = \angle AMB = \angle ACB = \angle GBC = \angle GMC$ so M, F and G are collinear, and so $\angle ADG = \angle MAC = \angle MGB$. Together with $DG \perp AF$ this means $ADGF$ is an isosceles trapetz and hence cyclic. Thus $\angle FDA = \angle MGA = \angle MCA = \angle MEF$ so $DF = FE$.

Quick Reply

High School Olympiads

Diagonals AO, BP and DN are concurrent

Reply

WakeUp

#1 Jan 12, 2011, 4:06 am

Through a point O on the diagonal BD of a parallelogram $ABCD$, segments MN parallel to AB, and PQ parallel to AD, are drawn, with M on AD, and Q on AB. Prove that diagonals AO, BP, DN (extended if necessary) will be concurrent.

Luis González

#2 Jan 12, 2011, 7:22 am

$ABCD \sim MOPD$ are homothetic through D and $ABCD \sim QBNO$ are homothetic through $B \Longrightarrow$ Midpoints E, F of segments MP, QN lie on diagonal DB and $QN \parallel MP.$ Let $G \equiv MQ \cap PN.$ Then GO is the G-median of $\triangle GMP \Longrightarrow G \in BD.$ Therefore, $\triangle ABD$ and $\triangle OPN$ are perspective through $\overline{GBD}.$ By Desargues theorem, it follows that lines AO, BP, DN concur, as desired.

sunken rock

#3 Jan 12, 2011, 10:13 pm

It seems that we can use Ceva for $\triangle NOP$ as well.

Best regards,
sunken rock

Ariel S

#4 Jan 13, 2011, 12:51 am

I have found a rather straightforward solution using analytic geometry. One can intersect two of the lines and then prove that the interesction belongs to the third one as well.

Anyways, it's just another of the many possible solutions.

Best wishes,
Ariel S

Quick Reply

High School Olympiads

Perpendicular

Reply

Headhunter
#1 Jan 12, 2011, 1:00 am

Hello.

For a trapezoid $ABCD$, where AD is parallel to BC, AB meet CD at E
Let two orthocenters of $\triangle ABF$, $\triangle CDF$ be P, Q
Show that PQ is perpendicular to EF

Luis González
#2 Jan 12, 2011, 6:44 am

F is not previously defined, but I'm guessing that $F \equiv AC \cap BD.$

Let U, V be the orthogonal projections of A, D onto BD, AC and X, Y the orthogonal projections of B, C onto $AC, BD.$ Let N, M be the midpoints of $AD, BC.$ Then U, V and X, Y lie on the circles $(N), (M)$ with diameters AD, BC, respectively. Since $PU \cdot PA = PX \cdot PB$ and $QV \cdot QD = QY \cdot QC$, it follows that P, Q have equal powers to $(N), (M) \Longrightarrow PQ$ the radical axis of $(N), (M) \Longrightarrow PQ \perp MN.$ But $AD \parallel BC$, then M, N, F, E are collinear $\Longrightarrow PQ \perp EF$, as desired.

jayme
#3 Jan 12, 2011, 9:32 pm

Dear Mathlinkers,
this problem can be resolved by considering the Steiner's line and the Gauss line of the complete quadrilateral determined by the triangle ABF and the menelian CDE.
For a synthetic and original proof of the perpendicularity you can see
http://perso.orange.fr/jl.ayme vol. 4 la droite de Gauss et la droite de Steiner
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

The point D on the side BC for the right triangle ABC

Reply

Source: Baltic Way 1998

WakeUp

#1 Jan 12, 2011, 12:08 am

In a triangle ABC, $\angle BAC = 90^\circ$. Point D lies on the side BC and satisfies $\angle BDA = 2\angle BAD$. Prove that

$$\frac{2}{AD} = \frac{1}{BD} + \frac{1}{CD}$$

Luis González

#2 Jan 12, 2011, 1:56 am

$\overrightarrow{AD}$ cuts circumcircle (O) at P. Then $\angle BOP = 2\angle BCP = 2\angle BAD = \angle BDA \Longrightarrow \triangle ODP$ is isoceles with apex P $\Longrightarrow DP = OP = \frac{1}{2}BC$. Then, from the power of D with respect to (O), we obtain

$$AD \cdot (BD + CD) = 2 \cdot BD \cdot CD \Longrightarrow \frac{2}{AD} = \frac{1}{BD} + \frac{1}{CD}$$

Quick Reply

High School Olympiads

A question

Reply

borislav_mirchev
#1 Dec 31, 2010, 1:22 am

It is given a cyclic quadrilateral ABCD. O is it circumcenter and P is the intersection point of its diagonals. Do you know some formula for the lenght of the segment OP?

BigSams
#2 Dec 31, 2010, 5:02 am • 1

Well, if you had the coordinates of the four vertices, you could determine the equations of the diagonals and find the coorinates of the intersection.
The forumula for coordinates of the circumcenter are given at
http://en.wikipedia.org/wiki/Circumscribed_circle#Cartesian_coordinates (just choose any three of the four vertices), which would take pretty skillful algebraic manipulation to derive on your own.
Then use distance formula.
I dunno about a non-analytic method though. This is pretty ugly I must admit.

borislav_mirchev
#3 Dec 31, 2010, 2:12 pm

I ask the question because of Euler formula exists for the OI distance and in the triangle almost all center distances can be expressed in therms of 3 sides but in the quadrilateral ... who knows.

Titanium
#4 Jan 6, 2011, 6:24 pm • 1

According to E. W. Hobson, "A treatise on plane and advanced trigonometry" p. 207, the distance OP in a cyclic quadrilateral is given by

$$OP=\frac{R}{(ab+cd)(ad+bc)}\sqrt{(ac+bd)(ac(b^2-d^2)^2+bd(a^2-c^2)^2)}$$

where R is the circumradius. This is given by Parameshvara's formula

$$R=\frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$$

where s is the semiperimeter. Thus OP is expressed in terms of the sides a,b,c,d.

borislav_mirchev
#5 Jan 10, 2011, 11:00 pm

Wow! These are very interesting results. If you have time you can scan the proofs and post them or write them in mathlinks. Do you know a book where it is written something about that problem:
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=369459
? It is not solved in mathlinks for too long time. It is the only problem posted by me and not solved.

Luis González

#6 Jan 11, 2011, 6:49 am • 1

Remark: If cyclic quadrilateral $ABCD$ with circumcirce (O,R) has also an incircle (I,r), then $OP=f(R,r)$ due to Poncelet porism. Further, from the topic Bicentric quadrilateral 4, we deduce that length OP is given by

$$OP=\frac{2R^2\sqrt{R^2+r^2-r\sqrt{r^2+4R^2}}}{2R^2+r^2-r\sqrt{r^2+4R^2}}$$

borislav_mirchev

#7 Jan 11, 2011, 12:57 pm

Thank you! You guessed my further thoughts. My final question is:

How about bicentric quadrilateral diagonals/sides? Can they be expressed in therms of R and r?
(Somewhere i saw a formula for r in therms of a, b, c, d for circumscribed quadrilateral)

Titanium

#8 Jan 12, 2011, 3:14 am • 1

There is no proof of the formula for OP given in Hobson's book, since it was not a theorem but a problem I cited (without any solution).
Instead I will give my solution: Inserting the formula for R and simplifying gives

$$OP^2=\frac{(ac+bd)^2(ac(b^2-d^2)^2+bd(a^2-c^2)^2)}{16K^2(ab+cd)(ad+bc)}$$

where

$$K=\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

according to Brahmagupta's formula. I derive this formula. Let M be the midpoint of diagonal AC where

$$AC=\sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$$

(this formula is well known). By the Pythagorean theorem $OP^2=PM^2+OM^2$ and $OM^2=R^2-MC^2$. Thus

$$OP^2=(PC-MC)^2+R^2-MC^2=PC^2-PC\cdot AC+R^2$$

(since $2MC=AC$) where

$$PC=bc\sqrt{\frac{ac+bd}{(ab+cd)(ad+bc)}}$$

according to "Advanced Trigonometry" by Durell and Robson (problem 14 at p. 26). Hence we get

$$OP^2=\frac{b^2c^2(ac+bd)}{(ab+cd)(ad+bc)}-\frac{bc(ac+bd)}{ab+cd}+\frac{(ab+cd)(ac+bd)(ad+bc)}{16K^2}=\frac{(ac+bd)}{16K^2(ab+cd)(ad+bc)}\left(16b^2c^2K^2-16bcK^2(ad+bc)+(ab+cd)^2(ad+bc)^2\right)=\frac{(ac+bd)^2\left(ac(b^2-d^2)^2+bd(a^2-c^2)^2\right)}{16K^2(ab+cd)(ad+bc)}$$

and we are done.

Attachments:

OP.pdf (17kb)

Titanium

#9 Jan 12, 2011, 4:00 am • 1

A few additional comments

1. By the way, a proof of the formulas for diagonal AC and circumradius R can be found at
http://forumgeom.fau.edu/FG2007volume7/FG200720.ps

2. I don't have any solution to your other problem with four inraii, nor have I ever seen it in a book. It is an interesting problem, hope someone else can solve it!

3. About the diagonals in a bicentric quadrilateral, I know a formula with radii R, r and the product of the diagonals pq. It is

$$\frac{pq}{4r^2} - \frac{4R^2}{pq} = 1$$

It is given as problem 5 on p. 164 in Paul Yiu's "Notes on Euclidean Geometry". You can find it here:
http://math.fau.edu/Yiu/EuclideanGeometryNotes.pdf

4.

> *Quote:*
> Somewhere i saw a formula for r in therms of a, b, c, d for circumscribed quadrilateral

Note: In that formula a, b, c, d are not the sides but the four distances from the vertices to the points where the incircle is tangent to the sides (see Lemma 2 in http://forumgeom.fau.edu/FG2008volume8/FG200814.ps). It is not possible to find a formula for the inradius r in a circumscribed quadrilateral (i.e. a tangential quadrilateral) in terms of the sides, since the sides alone do not determine a circumscribed quadrilateral. (But they do it in a cyclic quadrilateral, hence we have Parameshvara's formula.)

borislav_mirchev

#10 Jan 12, 2011, 7:58 pm

Thank you for the all valuable resources you shared. It is a very impressive collection and answered lots of my questions. I downloaded hobson's book. It is a very good book about trigonometry.

Quick Reply

High School Olympiads

Concurrent Tangent

Reply

Source: All-Russian MO 1999

Headhunter
#1 Jan 11, 2011, 4:04 am

The incircle of $\triangle ABC$ touch AB,BC,CA at K,L,M. The common external tangents to the incircles of $\triangle AMK$,$\triangle BKL$,$\triangle CLM$, distinct from the sides of $\triangle ABC$, are drawn. Show that these three lines are concurrent.

Luis González
#2 Jan 11, 2011, 5:17 am

Let $(I_1), (I_2), (I_3)$ be the incircles of $\triangle AMK, \triangle BKL$ and $\triangle CLM.$ It's easy to see that I_1, I_2, I_3 are the midpoints of the arcs MK, KL, LM of the incircle (I), e.g. $\angle KI_1M = 90^\circ + \frac{1}{2}\angle BAC = 180^\circ - \angle KLM \Longrightarrow I_1KLM$ is cyclic. Therefore, $I_0 \equiv LI_1 \cap MI_2 \cap KI_3$ is the incenter of $\triangle KLM,$ which becomes orthocenter of its circumcevian triangle $\triangle I_1I_2I_3 \Longrightarrow L, M, K$ are the reflections of I_0 across $I_2I_3, I_3I_1, I_1I_2.$ As a result, reflections of the sidelines BC, CA, AB across $I_2I_3, I_3I_1, I_1I_2,$ (common external tangents of $(I_1), (I_2), (I_3)$ different from BC, CA, AB) concur at $I_0.$

Headhunter
#3 Jan 12, 2011, 12:55 am

Thanks for your really nice solution.

Quick Reply

High School Olympiads

Hard Geometry problem again (own)

jayme

#1 Jan 10, 2011, 1:56 pm

Dear Mathlinkers,
this is a conjecture:
the Spieker's point of ABC is the center of perspective of the medial triangle of ABC with the radical triangle of ABC.
The radical triangle of ABC is determined by the intersection of the radical axes of the circumcircle of ABC with the resp. excircles of ABC.
Sincerely
Jean-Louis

Luis González

#2 Jan 10, 2011, 11:57 pm • 3

Dear Jean-Louis, your conjecture is true and very easy to prove. It follows from the fact that the Spieker point S of $\triangle ABC$ is the radical center of its 3 excircles $(I_a),(I_b),(I_c).$

Let M,N,L be the midpoints of BC,CA,AB and let $(I_b),(I_c)$ touch the sideline BC at $U,V.$ Since $MU=MV=\frac{1}{2}(b+c),$ it follows that M has equal power WRT $(I_b),(I_c) \Longrightarrow$ Radical axis τ_a of $(I_b),(I_c)$ goes through M orthogonally to $I_bI_c,$ i.e. τ_a is the A-angle bisector of the medial triangle. Likewise, radical axes τ_b,τ_c of $(I_c),(I_a)$ and $(I_a),(I_b)$ are the B- and C- angle bisectors of the medial triangle $\Longrightarrow$ Incenter of the medial triangle (Spieker point) is the radical center of $(I_a),(I_b),(I_c).$ Now, pairwise radical axes of (O) with $(I_a),(I_b),(I_c)$ meet on radical axes SM,SN,SL and the conclusion follows.

High School Olympiads

interesting

Reply

aspava

#1 Jan 10, 2011, 12:48 am

Given three circles A, B, and C tangent to each other at D, E, and F, respectively. Lines DE and FE intersect circle A at G and H, respectively.
Prove that the points H, A, and G are collinear.

Luis González

#2 Jan 10, 2011, 3:20 am

A more general result:

Three circles $(A),(B),(C)$ are externally tangent to each other at $D,E,F,$ such that D is the tangency point of $(B),(C)$ and (A) touches $(B),(C)$ at $E,F.$ U is an arbitrary point on $(A).$ Ray UE cuts (B) at $X,$ ray XD cuts (C) at Y and ray YF cuts (A) at $V.$ Then A,U,V are collinear. (Yaglom, Transformations géométriques II)

Proof: Let $\ell_U,\ell_X,\ell_Y,\ell_V$ be the tangents of $(A),(B),(C),(A)$ through $U,X,Y,V,$ respectively. Since E,D,F are insimilicenters of $(A) \sim (B),(B) \sim (C)$ and $(C) \sim (A),$ it follows that $\ell_U \parallel \ell_X, \ell_X \parallel \ell_Y$ and $\ell_Y \parallel \ell_V \Longrightarrow \ell_U \parallel \ell_V \Longrightarrow$ Points U,V are diametrically opposed in the circle $(A).$

jayme

#3 Jan 10, 2011, 6:30 pm

Dear Mathlinkers,
this is the Boutin's theorem.
A synthetic proof can be seen on
http://perso.orange.fr/jl.ayme vol. 1 A propos du théorème de Boutin
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

nice problem

Reply

aspava

#1 Jan 10, 2011, 12:39 am

Draw squares ABDE and BCFG on sides AB and BC of a triangle ABC. Prove the following:
i) CD and AG are congruent and perpendicular

ii) CD, AG, and EF are concurrent in H

iii) EF is bisector of angle GHC.

Luis González

#2 Jan 10, 2011, 2:46 am

Triangles $\triangle BCD$ and $\triangle BGA$ are congruent by SAS criterion, due to $BD=BA$, $BG=BC$ and $\angle DBC=\angle ABG.$ Hence $CD=AG$ and $\angle BDC=\angle BAG,$ which implies that B,D,A,H are conclyclic $\Longrightarrow \angle DHA=90^\circ.$ H lies on the circumcircles of the squares $BDEA$ and $BGFC.$ Since arcs $DE=AE$ are equal, then HE bisects $\angle DHA.$ Likewise, HF bisects $\angle GHC \Longrightarrow CD,AG,EF$ concur and EF bisects $\angle (CD,AG).$

jayme

#3 Jan 10, 2011, 6:33 pm

Dear Mathlinkers,
for more results and synthetic proofs including the present result, you can see
http://perso.orange.fr/jl.ayme vol. 5 La figure de Vecten
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

nice one

aspava

#1 Jan 10, 2011, 1:10 am

A triangle cuts off from the circumcircle three circular segments.

prove that:

2klm = Rr2,

where R is the circumradius, r the inradius, and k, l, and m are the three segment altitudes.

Luis González

#2 Jan 10, 2011, 2:08 am

M,N,L are midpoints of $\overline{BC},\overline{CA},\overline{AB}$ and P,Q,R are midpoints of the arcs BC,CA,AB of its circumcircle (O), i.e. AP,BQ,CR are internal angle bisector. From right triangles $\triangle CPM,\triangle AQN$ and $\triangle BRL$, we get

$$\tan\frac{A}{2}=\frac{2\cdot MP}{BC} \ , \ \tan\frac{B}{2}=\frac{2\cdot NQ}{CA} \ , \ \tan\frac{C}{2}=\frac{2\cdot LR}{AB}$$

$$\Longrightarrow MP\cdot NQ\cdot LR=\frac{1}{8}\tan\frac{A}{2}\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}\cdot BC\cdot CA\cdot AB$$

Now, we use the identities

$$|\triangle ABC|=\frac{BC\cdot CA\cdot AB}{4R} \ \ , \ \ r^2=\tan\frac{A}{2}\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}\cdot|\triangle ABC|$$

$$\Longrightarrow 2\cdot MP\cdot NQ\cdot LR=R\cdot r^2.$$

Site Support

Bug Reports

Reply

rrusczyk
#1 Apr 19, 2010, 11:45 pm • 6

With any new site release, there are bound to be some bugs. If you find something that does not work properly, please post it here.

Please note: Please confine your comments to bugs. **We will be cleaning out this thread periodically, deleting the posts as we log their contents.** This helps us keep track of what we've read and we haven't. To report a desired feature or a missing feature post here. Only use this list for things that are broken. Posts not directly related to bugs will be deleted or moved.

Overall post ratings not always updating automatically when rating a post.

See posts below for known issues. FTW issues are listed in FTW support forum.

BOGTRO
#2 May 17, 2010, 9:58 am • 1

Another bug: immediatly after commenting on a blog post, it will show a blank wall of comment with the text "page n+1 of n". For example, in PhireKaLK's blog, in the developing post for his game, after making a comment, I got a blank wall saying "page 61 of 60". Win?

phiReKaLk6781
#3 Jun 15, 2010, 10:53 am

I'm not sure if this is intentional, and there certainly is an element of benefit from knowing where a post used to be, but a thread (I found from this one, ironically) still displays the red dot for a user having posted even after the post has been deleted.

Admin response: Per design. This is intentional.

levans
#4 Jun 15, 2010, 10:48 pm

Thanks to v_enhance for his detailed report!

- list item 1
- list item 2

1. list item 1
2. list item 2

a. list item 1
b. list item 2

Not sure if we want to use green diamonds over black squares in forum.

levans
#5 Jun 16, 2010, 3:23 am

JBL wrote:

Not all posts on which I have posted have the dot indicating that I've posted there. In particular, many older posts lack it. For example, at the present moment, this post (in this forum) which was recently bumped up does not indicate that I've posted in it.

Per phpBB3 design, only the last 6 months are tracked.

$LaTeX$

#6 Jun 17, 2010, 5:50 am

In rrusczyk's post, when he mentions what icons you need to use. The second bolded (not a word?) text.

> rusczyk wrote:
>
> The Classroom Math forum is for middle school students to discuss the math from their classes.
>
> **This forum should only be used by middle school students and teachers.** (Others are free to read, but please leave the discussion to middle school students.)
>
> Middle schoolers should use this board to answer challenge questions posted by board moderators, administrators, and other users. They can also use this board to ask questions they have about their math studies.
>
> **When you make a new topic in this forum, you will have the ability to place a Message Icon next to your post. Please use the icon when you are posting a question you need help with and use when you are posting a question you are challenging others to answer.**
>
> Older students should leave the challenge questions for the younger students - answer the challenge problems that are appropriate for your age and experience level only! You are free to help out students who have questions.
>
> We expect the math in this forum to be topics you might find in a middle school classroom. Typical classes in middle school classrooms include Pre-Algebra, Algebra 1, and very basic Geometry. If you wish to discuss contest problems, use The Basics for beginning contest problems and use MATHCOUNTS for harder middle school contest problems.
>
> If you are uncertain which forum to place your question in, always place it in the more advanced forum of those you are considering.
>
> Please refrain from titling a post as "easy" or "hard." If it is in the right forum, it is *appropriate* regardless.
>
> Please do not post messages that are meant to "teach" others about forum etiquette. Messages such as "do not revive old topics" should not be placed on the forums.
>
> If you feel someone is abusing the forum, please contact a moderator or administrator.

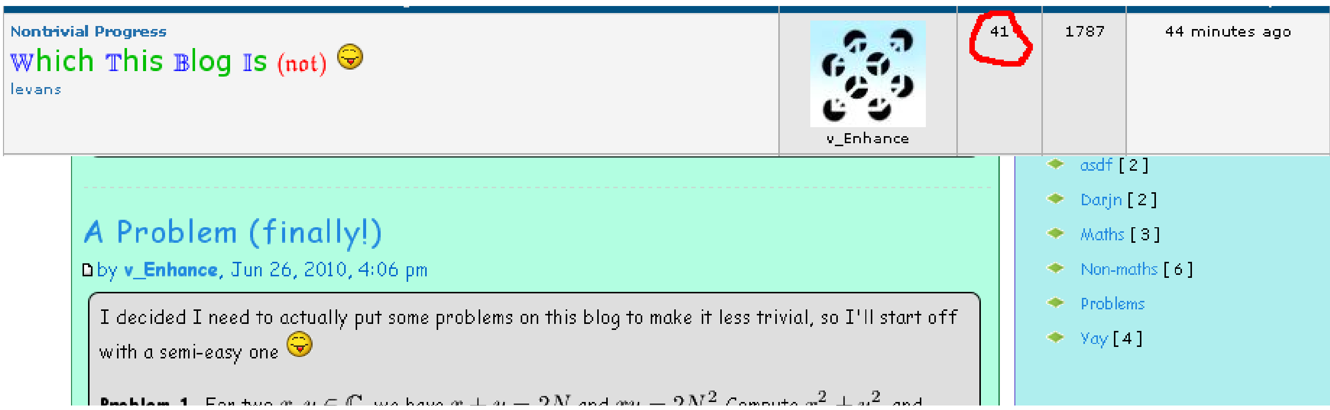
the

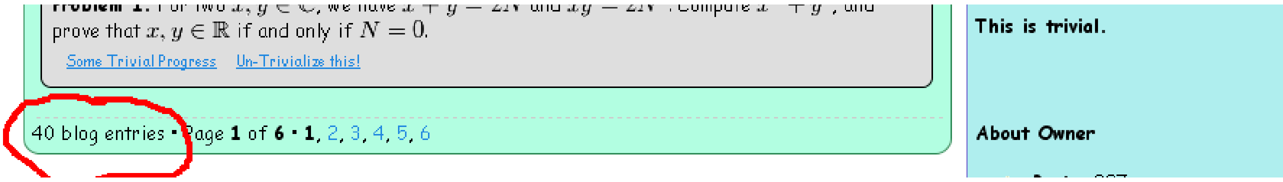
phiReKaLk6781

#7 Jun 30, 2010, 8:44 am

The number of entries displayed for a blog on the list of blogs page and the count displayed at the bottom of the page for a specific blog do not seem to always match. Here's an example with v_Enhance's blog (Mewto55555's blog has the two counts differing by three):

Attachments:





exmath89
#8 Jun 30, 2010, 10:15 am • 1

I am not sure if this is a bug, but when you log in, if you don't click anything for about 4-5 seconds, it automatically logs you out. Why does that happen?

Admin response: Because you are clicking the login link from the logout page. After you login, it redirects you to the last page you were on, which in this case, was the logout page.

QuantumTiger
#9 Dec 28, 2010, 2:46 am

I don't have a link for this, but in some cases if you post and immediately after click the New Posts button, it will show a new post by you with the "unread" icon. Is this supposed to happen?

levans
#10 Dec 28, 2010, 4:45 am

Doesn't sound out of the ordinary.

Luis González
#11 Jan 9, 2011, 10:53 pm

Clicking on [moderator control panel] at the bottom of the forum is leading to the page: *This topic has been deleted*. The link is not working in any of the 5 sub fora of the Geometry forum. No idea, if other moderators have noticed this bug in their forums.

P.S. Link to the moderator control panel works fine in the sidebar, but not with wider forum option.

QuantumTiger
#12 Jan 9, 2011, 10:59 pm

This doesn't occur for me. Which forum do you moderate?

Dojo
#13 Jan 10, 2011, 4:10 am

Same for AoPS Blogs.

Quick Reply

High School Olympiads

Search Of Locus

Headhunter
#1 Jan 2, 2011, 11:12 pm

Hello.

There are two fixed points B, C on a fixed line l.
A point A move on the line m parallel to l
Find the locus of the orthocenter of $\triangle ABC$ with no use of coordinates system.

Luis González
#2 Jan 9, 2011, 9:32 am

I'm aware that your asking for solutions without coordinate bashing, but the solution with rectangular coordinates is so easy that it's worth showing it. Let $B \equiv (0,0), C \equiv (L,0)$ and A moves on the horizontal line $m \equiv y = h$. Let $H \equiv (x,y)$ be the orthocenter of $\triangle ABC$ and $D \equiv (x,0)$ the foot of the A-altitude.

$$\overline{AD} \cdot \overline{HD} = \overline{DB} \cdot \overline{DC} \Longrightarrow h \cdot y = x \cdot (L-x) \Longrightarrow \left(x - \frac{L}{2}\right)^2 = -h\left(y - \frac{L^2}{4h}\right)$$

Locus of H is parabola $\mathcal{K}$ passing through B, C whose focal axis is the perpendicular bisector τ of $\overline{BC}$ and its latus rectum measures h. If M denotes the midpoint of $\overline{BC}$ and τ cuts m at N, then the midpoint of $\overline{MN}$ is the vertex of $\mathcal{K}$.

High School Olympiads

Prove P is the orthocenter of AQR.

Reply

mathwizarddude
#1 Dec 26, 2010, 1:31 pm

In triangle ABC with pedal triangle DEF, the perpendicular bisector to PD intersects sides AB and AC at Q and R respectively, and let P be the reflection of D across QR. Prove that P is the orthocenter of triangle AQR.

oneplusone
#2 Dec 27, 2010, 10:21 am

> *mathwizarddude wrote:*
> In triangle ABC with pedal triangle DEF, the perpendicular bisector to PD intersects sides AB and AC at Q and R respectively, and let P be the reflection of D across QR. Prove that P is the orthocenter of triangle AQR.

What is the first P, in the perpendicular bisector of PD?

mathwizarddude
#3 Dec 28, 2010, 1:05 am

P is the intersection of AD and EF. Sorry for being vague.

Luis González
#4 Jan 7, 2011, 5:48 am

I assume that $\triangle DEF$ is the orthic triangle. Let H be the orthocenter of $\triangle ABC$ and M the midpoint of PD. Since $(P,D,H,A)=-1,$ it follows that $\overline{AP}\cdot\overline{AD}=\overline{AH}\cdot\overline{AM},$ but $FHMQ$ is cyclic due to $\angle HFQ=\angle HMQ=90^{\circ}$ $\Longrightarrow \overline{AH}\cdot\overline{AM}=\overline{AF}\cdot\overline{AQ}.$ Thus, $\overline{AF}\cdot\overline{AQ}=\overline{AP}\cdot\overline{AD},$ i.e. $PFQD$ is cyclic $\Longrightarrow \angle FQP=\angle FDP=\angle FBH$ $\Longrightarrow QP \parallel BH \perp AR$ and the conclusion follows.

Quick Reply

High School Olympiads

Fixed Point

Reply

Headhunter
#1 Jan 7, 2011, 1:35 am

Hello.

A, C, B are collinear in the order, where C is not the midpoint of AB
A point P move on the line through C, perpendicular to AB
Let two feet of the perpendicular lines from C onto AP, BP be Q, R respectively.
Show that QR pass through a fixed point.

Luis González
#2 Jan 7, 2011, 2:21 am

Let M,N be the orthogonal projections of A,B onto lines $PB,PA.$ Line MN cuts AB at $K,$ such that the cross ratio (A,B,C,K) is harmonic $\Longrightarrow K$ is fixed on the line $AB.$ Let PC cut MN,RQ at $U,V,$ respectively. Then according to the topic Probl 5 (XX OMM), V is the midpoint of $\overline{UC}.$ Since $RQ \parallel MNK$ (both antiparallel to AB WRT PA,PB) it follows that RQ passes through the midpoint of $\overline{CK}.$

Quick Reply

High School Olympiads

old problem

MJ GEO

#1 Dec 18, 2009, 9:27 pm

$(h_a)^2=2(R)^2$ that h_a is altitude from A and R is circumradius.h_a is intersect BC at H.HE,HF is perpendicular to AB,AC (E is on AB and F is on AC) if O be the circumcenter of ABC prove that E,F,O are lies on a line

Luis González

#2 Jan 6, 2011, 7:03 am

Since $\angle AEF=\angle AHF=\angle ACB,$ it follows that $EFCB$ is cyclic, i.e $\triangle AFE$ and $\triangle ABC$ are similar $\Longrightarrow EF$ is antiparallel to $BC,$ i.e. $AO \perp EF.$ If $P \equiv AO \cap EF,$ then E,O,F are collinear $\Longleftrightarrow O \equiv P$

$$\Longleftrightarrow \frac{R}{AH}=\frac{EF}{BC}=\frac{AF}{AB}=\frac{AH^2}{AC \cdot AB}=\frac{AH}{2R} \Longleftrightarrow AH=\sqrt{2}R.$$

sunken rock

#3 Jan 6, 2011, 12:13 pm

The inversion of pole A and power AH^2 transforms the circle $\odot(ABC)$ into its diameter perpendicular to OA and, if this diameter intersect AB and AC at E' and F' respectively, we shall get $AE'\cdot AB=AF'\cdot AC=AH^2$, i.e. $HE'\perp AB$ and $HF'\perp AC$, or $E'\equiv E$ and $F'\equiv F$, done.

Best regards,
sunken rock

Luis González

#4 Jan 6, 2011, 12:32 pm

> sunken rock wrote:
> $HE'\perp AB$ and $HF'\perp AC$, or $E'\equiv E$ and $F'\equiv F$, done.

How does this lead directly to E,O,F are collinear $\Longleftrightarrow AH=\sqrt{2}R$?

My reasoning: Line EF cuts (O) at double points U,V under the inversion $\Longrightarrow AH=AU=AV$ is the radius of the inversion. Then $AU=AV=AH=AO\sqrt{2}$

sunken rock

#5 Jan 6, 2011, 2:52 pm

To: luisgeometra
1) The problem does not ask 'iff'
2) $AU=AO\sqrt{2}$ iff UV is a diameter of the circle, i.e. E,O,F collinear.

With best regards,
Stan Fulger,
Master Master

High School Olympiads

very nice about two-center-joining line

Reply

mr2

#1 Jan 5, 2011, 2:24 pm • 1

Let a triangle ABC. (O) is its circumcircle. The incircle (I) touches the sides BC, CA, AB at D, E, F respectively. The line DE cuts (O) at points M, N. The line DF cuts (O) at points P, Q. Let (O_1) be the circle which passes through F, M, N; (O_2) be the circle which passes through E, P, Q.

a, Prove that the O_1O_2 is parallel to EF.

b, Prove that O_1O_2 passes through the midpoint of BC.

Luis González

#2 Jan 5, 2011, 8:51 pm

Result a) follows from the fact that the radical axis of circles (O_1) and (O_2) is the D-altitude of the intouch triangle DEF, hence EF is parallel to the center line O_1O_2. See the second part of the problem Radical center and line IO, but proposition b) is certainly not true in general, could you please check it?.

mr2

#3 Jan 6, 2011, 10:21 am

Yes, it seems easy to know D lies the radical axis, but hard to show another point which has the same radical power wrt $(O_1), (O_2)$, on the D-altitude of the intouch triangle

I wait that point ... add on a proof, of course ... not only a consequence of the result.

Luis González

#4 Jan 6, 2011, 12:12 pm

Let A_0, B_0, C_0 be the midpoints of BC, CA, AB. Line ED cuts AB at R. From $(A, B, F, R) = -1$, we have $RA \cdot RB = RF \cdot RC_0$, but $RA \cdot RB = RM \cdot RN \Longrightarrow RM \cdot RN = RF \cdot RC_0 \Longrightarrow C_0 \in (O_1)$ and similarly $B_0 \in (O_2)$. O_1 is the the intersection of the perpendicular bisectors of FC_0 and MN. Thus if $S \equiv OO_1 \cap IF$, then O_1 is the midpoint of OS. Likewise, if $T \equiv OO_2 \cap IE$, then O_2 is the midpoint of OT. Let H_0 denote the orthocenter of $\triangle DEF$. Since $EH_0 \parallel TO, FH_0 \parallel OS$ and lines ET, FS, OH_0 concur at I, it follows that $\triangle OST \sim \triangle H_0EF$ are centrally similar through I. Thus, the remaining pair of sidelines are parallel $\Longrightarrow EF \parallel ST \parallel O_1O_2$.

mr2

#5 Jan 6, 2011, 4:26 pm

Nothing to say ... simply, it's a masterpiece...

Quick Reply

High School Olympiads

tangent problem 1##

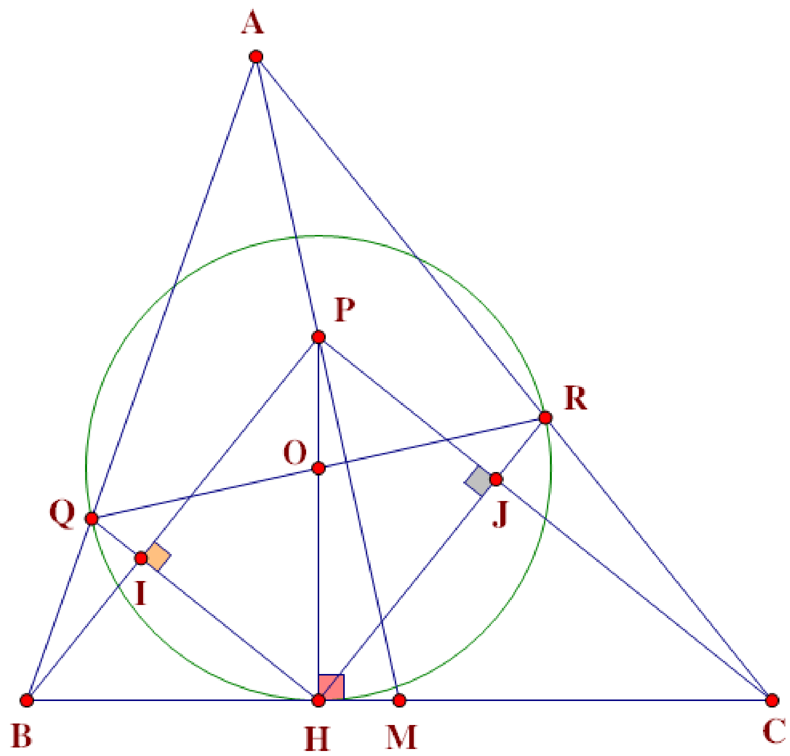
Reply

thanhnam2902

#1 Jan 5, 2011, 8:11 pm

Let ABC is a acute triangle. Let M is midpoint of BC, let P is a point lie on AM satisfy $MB = MP$. Let H is perpendicular foot of point P on BC. Let (d_1) is a line pass to point H and (d_1) perpendicular with PB, let (d_1) meet AB at point Q. Let (d_2) is a line pass to point H and (d_2) perpendicular with PC, let (d_2) meet AC at point R. Prove that BC tangent with the circumcircle of QRH triangle at point H.

Attachments:



Immanuel Bonfils

#2 Jan 6, 2011, 1:19 am

As $BP^2 + CP^2 = CB^2$, triangle BCP is rectangle in P, and so triangle HRQ in H (catheti respectively orthogonal).

So triangle HQR is inscribed in the semicircle QHR.

Luis González

#3 Jan 6, 2011, 2:09 am

Immanuel Bonfils your assertion does not imply that circle (QHR) is tangent to BC at H.

Let the tangents of the circumcircle $\odot(ABC)$ through B, C meet at T and let P^* be the isogonal conjugate of P WRT $\triangle ABC$, lying on the A-symmedian AT. Since $\angle BPC = 90^\circ$, then it follows that $\angle BP^*C = 90^\circ + \angle BAC \Longrightarrow P^* \in \odot(T, TB)$. Since $\angle BAT = \angle CAM$ and $\angle ABT = \pi - \angle ACB$, we have:

$$\frac{TP^*}{TA} = \frac{TB}{TA} = \frac{\sin \widehat{BAT}}{\sin \widehat{ABT}} = \frac{\sin \widehat{CAM}}{\sin \widehat{ACB}} = \frac{MC}{MA} = \frac{MP}{MA},$$

which means that $TM \parallel PP^* \Longrightarrow PP^* \perp BC \Longrightarrow$ pedal circle ω of P, P^* WRT $\triangle ABC$ is tangent to BC through H. Further, R, Q coincide with the orthogonal projections of P^* on $AC, AB \Longrightarrow \odot(HQR) \equiv \omega$.

Quick Reply

High School Olympiads

Reply

Amir Hossein

#1 Jan 2, 2011, 3:07 am

Let squares be constructed on the sides BC, CA, AB of a triangle ABC, all to the outside of the triangle, and let A_1, B_1, C_1 be their centers. Starting from the triangle $A_1B_1C_1$ one analogously obtains a triangle $A_2B_2C_2$. If S, S_1, S_2 denote the areas of triangles $ABC, A_1B_1C_1, A_2B_2C_2$, respectively, prove that $S = 8S_1 - 4S_2$.

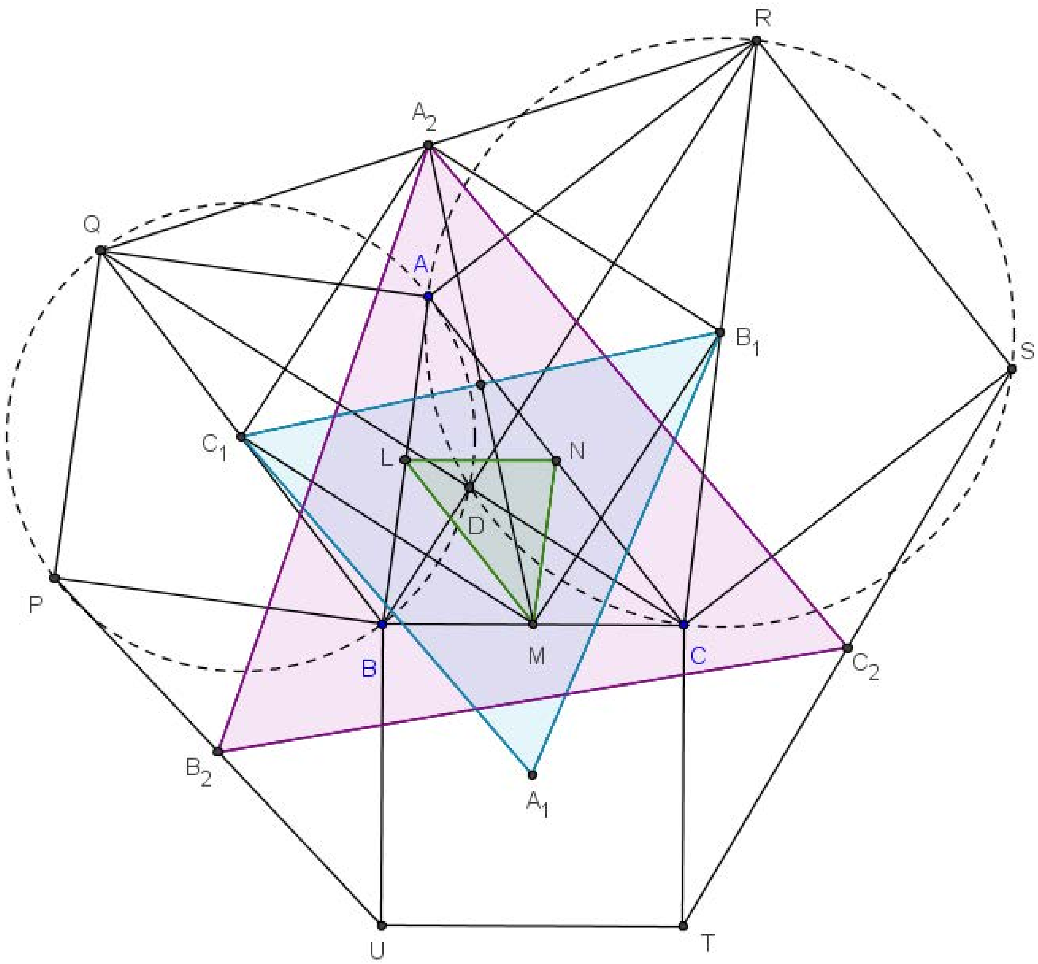
Luis González

#2 Jan 5, 2011, 8:52 am

Let $ABPQ$ and $ARSC$ be the consecutive vertices of the squares constructed outwardly on BA, AC. Thus, $\triangle QAC$ and $\triangle BAR$ are congruent by SAS criterion, due to $AQ = AB, AC = AR$ and $\angle QAC = \angle BAR = 90^\circ + \angle BAC \Longrightarrow \angle ABR = \angle AQC$. If $D \equiv QC \cap BR$, then it follows that Q, A, D, B are concylic $\Longrightarrow \angle QDR = 90^\circ, (\star)$. Let M, N, L be the midpoints of BC, CA, AB. Since MC_1 and MB_1 become B- and C- midlines of $\triangle BCQ$ and $\triangle CBR$, from $(\star)$ we deduce that $\triangle MC_1B_1$ is isosceles right at M. In exactly the same way, we deduce that the midpoint of RQ and B_1, C_1 form another isosceles right triangle $\Longrightarrow A_2$ is the midpoint of QR, i.e. M is the reflection of A_2 about B_1C_1. Likewise, N, L are the reflections of B_2, C_2 about $C_1A_1, A_1B_1 \Longrightarrow \triangle A_2B_2C_2$ and the medial $\triangle MNL$ of $\triangle ABC$ become the outer and inner Vecten triangles of $\triangle A_1B_1C_1$. Now, we know that the area of any triangle is the arithmetic mean between the areas of its outer and inner Vecten triangles. Therefore

$$2|\triangle A_1B_1C_1| = |\triangle A_2B_2C_2| + |\triangle MNL| \Longrightarrow 2S_1 = S_2 + \frac{1}{4}S.$$

Attachments:



Quick Reply

High School Olympiads

nice relation of fermet point

emptyfunction
#1 Jan 3, 2011, 7:39 pm

let T be torcelis point of triangle ABC, whose sides are a,b,c then prove the following relation
AT(b^2-c^2)+BT(c^2-a^2)+CT(a^2-b^2) =0

Luis González
#2 Jan 4, 2011, 10:27 am

By cosine law for $\triangle ATC$ and $\triangle ATB$ with $\angle ATC = \angle ATB = 120^\circ$, we get:

$$b^2 = TA^2 + TC^2 - 2 \cdot TA \cdot TC \cdot \cos 120^\circ = TA^2 + TC^2 + TA \cdot TC$$

$$c^2 = TA^2 + TB^2 - 2 \cdot TA \cdot TB \cdot \cos 120^\circ = TA^2 + TB^2 + TA \cdot TB$$

$$\Longrightarrow (b^2 - c^2) \cdot TA = TA \cdot TC^2 + TC \cdot TA^2 - TA \cdot TB^2 - TB \cdot TA^2 \ \ (1)$$

By cyclic permutation of a,b,c and TA,TB,TC we get the identities:

$$(c^2 - a^2) \cdot TB = TA \cdot TB^2 + TB \cdot TA^2 - TB \cdot TC^2 - TC \cdot TB^2 \ \ (2)$$

$$(a^2 - b^2) \cdot TC = TC \cdot TB^2 + TB \cdot TC^2 - TC \cdot TA^2 - TA \cdot TC^2 \ \ (3)$$

Summing up the identities (1), (2) and (3) yields:

$$(b^2 - c^2) \cdot TA + (c^2 - a^2) \cdot TB + (a^2 - b^2) \cdot TC = 0$$

Luis González
#3 Jan 5, 2011, 2:46 am

There's a beautiful consequence coming from the proposed relation above.

F, E are the 1st Fermat point and Euler reflection point of the acute triangle $\triangle ABC$. Then with appropiate choice of signs we have the relation:

$$\pm \ EA \cdot FA \cdot BC \ \pm \ EB \cdot FB \cdot CA \ \pm \ EC \cdot FC \cdot AB = 0$$

High School Olympiads

Half-equilateral triangles are constructed [ILL 1974]

Reply

WakeUp

#1 Jan 3, 2011, 6:13 am

Outside an arbitrary triangle ABC, triangles ADB and BCE are constructed such that $\angle ADB = \angle BEC = 90^\circ$ and $\angle DAB = \angle EBC = 30^\circ$. On the segment AC the point F with $AF = 3FC$ is chosen. Prove that $\angle DFE = 90^\circ$ and $\angle FDE = 30^\circ$.

Luis González

#2 Jan 3, 2011, 7:30 am

Let P be the orthogonal projection of D onto AB. Since $\overline{PB} : \overline{PA} = -1 : 3$, it follows by Thales theorem that $PF \parallel BC$. Therefore, $\angle DPF = 90^\circ + \angle ABC$, but on the other hand $\angle DBE = 90^\circ + \angle ABC$ and $\frac{DB}{DP} = \frac{BE}{PF}$ imply that $\triangle DBE \sim \triangle DPF$ by SAS criterion. Thereby, $\frac{DB}{DE} = \frac{DP}{DF}$ and $\angle EDF = \angle BDP$ imply that $\triangle DEF \sim \triangle DBP$ by SAS $\Longrightarrow \triangle DEF$ is right at F and $\angle FDE = 30^\circ$.

Quick Reply

High School Olympiads

Lines from excentres are concurrent [ILL 1974]

Reply

WakeUp

#1 Jan 3, 2011, 4:25 am

Let K_a, K_b, K_c with centres O_a, O_b, O_c be the excircles of a triangle ABC, touching the interiors of the sides BC, CA, AB at points T_a, T_b, T_c respectively.
Prove that the lines O_aT_a, O_bT_b, O_cT_c are concurrent in a point P for which $PO_a = PO_b = PO_c = 2R$ holds, where R denotes the circumradius of ABC. Also prove that the circumcentre O of ABC is the midpoint of the segment PI, where I is the incentre of ABC.

Luis González

#2 Jan 3, 2011, 5:16 am

This is such a well-known result. Incircle (I) touches BC at X and let M be the midpoint of BC. Since $\overline{MX} = -\overline{MT_a}$, it follows that line O_aT_a parallel to IX, OM passes through the reflection P of I about O. Likewise, O_bT_b, O_cT_c pass through P. Since $\triangle ABC$ and its circumcircle (O) become orthic triangle and 9-point circle of the excentral $\triangle O_aO_bO_c$, it follows that P, as the concurrency point of the perpendiculars dropped from O_a, O_b, O_c to BC, CA, AB, is the circumcenter of $\triangle O_aO_bO_c \Longrightarrow PO_a = PO_b = PO_c = 2R.$

Quick Reply

High School Olympiads

The product is equal two 2 - [IMO LongList 1971]

Reply

Amir Hossein

#1 Jan 2, 2011, 4:08 am

Let M be the circumcenter of a triangle ABC. The line through M perpendicular to CM meets the lines CA and CB at Q and P, respectively. Prove that

$$\frac{\overline{CP}}{\overline{CM}} \cdot \frac{\overline{CQ}}{\overline{CM}} \cdot \frac{\overline{AB}}{\overline{PQ}} = 2.$$

Luis González

#2 Jan 2, 2011, 5:29 am

H is the foot of the C-altitude. $\angle CPM = \angle CAB$ implies that $\triangle ABC \sim \triangle PQC$

$$\Longrightarrow \frac{AB}{PQ} = \frac{CB}{CQ}, \ \frac{CP}{CM} = \frac{CA}{CH} \Longrightarrow \frac{CP}{CM} \cdot \frac{CQ}{CM} \cdot \frac{AB}{PQ} = \frac{CA \cdot CB}{CH \cdot CM} = 2$$

sunken rock

#3 Apr 23, 2011, 10:13 pm

By using inversion, it works as well: PQ is a diameter, perpendicular to CM, i.e. the inversion $I(C, CM \cdot 2 \cdot CM)$ sends the circle to the line PQ, hence $\frac{AB}{PQ} = \frac{k}{CP \cdot CQ}$, where $k = 2 \cdot CM^2$.

Best regards,
sunken rock

Quick Reply

High School Olympiads

A pentagram problem

Reply

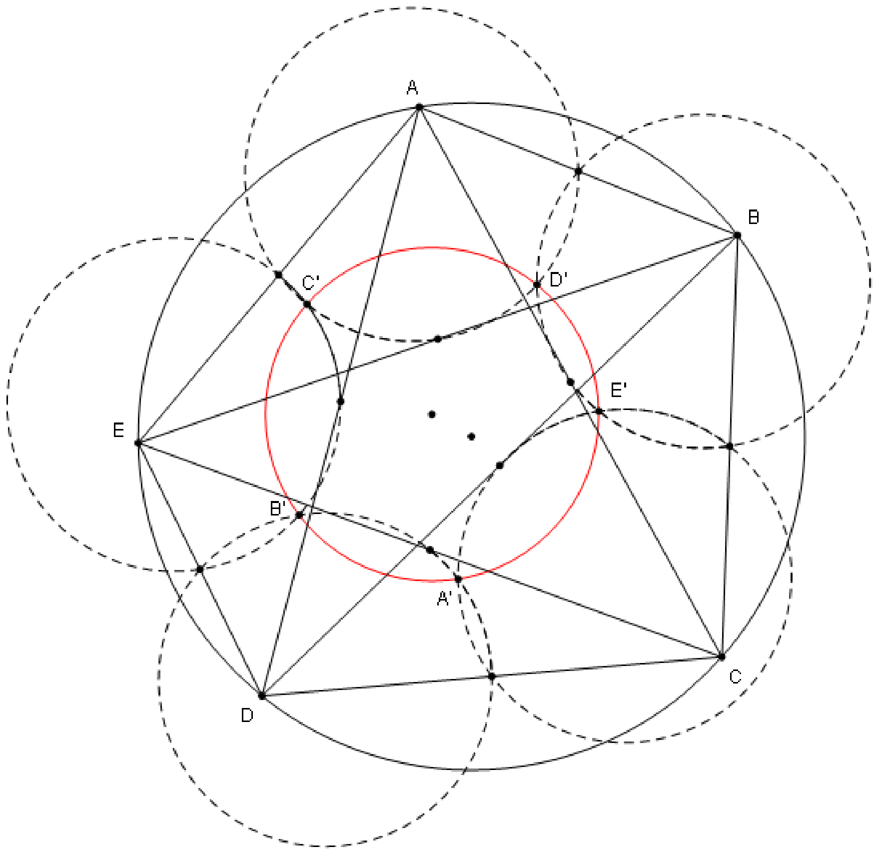
livetolove212

#1 Jan 1, 2011, 8:18 pm

Problem (own):
Given a cyclic pentagon $ABCDE$. Let A', B', C', D', E' be the second intersections of the 9-point circles of triangles ABC, BCD, CDE, DEA, EAB. Prove that A', B', C', D', E' are concyclic.

Happy new year!

Attachments:



Luis González

#2 Jan 2, 2011, 12:51 am

Let M,N,P,Q be the midpoints of BA, BC, BE, BD. Then M,N,P,Q lie on the image (U) of the circumcircle (O) of $ABCD$ through the homothety $(B, \frac{1}{2})$. D' is Poncelet point of quadrangle $ABCE \Longrightarrow \odot(PND')$ is 9-point circle of $\triangle BCE$. Since $\odot(MPD') \cong \odot(PND') \cong \odot(NMD')$, it follows that D' is the orthocenter of $\triangle MNP \Longrightarrow$ $PD' \perp MN$. By similar reasoning, we'll have $QE' \perp MN \Longrightarrow PD' \parallel QE'$. But since $\triangle MNP$ and $\triangle MNQ$ have the same circumcircle (U), then $PD' = QE'$ is twice the distance from U to MN. Therefore $PQE'D'$ is a parallelogram $\Longrightarrow$ $D'E' \parallel PQ \parallel DE$ and $D'E' = \frac{1}{2}DE$. Similarly, we'll get $EA \parallel E'A'$ and $E'A' = \frac{1}{2}EA \Longrightarrow \triangle D'E'A'$ and $\triangle DEA$ are homothetic with coefficient $-\frac{1}{2} \Longrightarrow A'D' \parallel AD$ and $A'D' = \frac{1}{2}AD'$. By similar reasoning, we conclude that $ABCDE$ and $A'B'C'D'E'$ are homothetic $\Longrightarrow$ Pentagon $A'B'C'D'E'$ is also cyclic.

Quick Reply

High School Olympiads

two geometry problems

Reply

kevsiho

#1 Dec 31, 2010, 8:33 am

1. triangle ABC/ Point P is on the arc BCof Circumcirle(doesn't contain A).
two tangent line to incircle of ABC : PD, PE.
PD and BC meet at point M
PE and BC meet at point N
then, the circumcircle of PMN pass the constant point(-maybe the intersection point with circumcircle of ABCand mixtilinear circle(A))

2. triangle ABC/ BD and CE are bisectors (D, E lie on the AC, AB)
DE and the circumcircle of ABC meet at point M, N
I is the incenter of ABC/ then the radius of circumcircle of MIN is 2R(R:radius of circumcircle of ABC)

Luis González

#2 Dec 31, 2010, 10:22 am

> Quote:
> 2. In triangle ABC BD and CE are bisectors (D, E lie on the AC, AB). DE and the circumcircle of ABC meet at points M, N. I is the incenter of ABC, then the radius of circumcircle of MIN is 2R (R:radius of circumcircle of ABC)

Let I_a, I_b, I_c be the excenters of $\triangle ABC$ against A, B, C. $\triangle ABC$ and its circumcircle (O, R) become orthic triangle and 9-point circle of $\triangle I_aI_bI_c \Longrightarrow$ radius of the congruent circles $\odot(I_aI_bI_c)$ and $\odot(II_bI_c)$ is $2R$. Now, it suffices to prove that M, N lie on $\odot(II_bI_c)$. Since BC, I_bI_c and $MN \equiv DE$ concur at P such that $A(B, C, I, P) = -1$ and quadrilateral BCI_bI_c is cyclic on account of the right angles $\angle I_cBI_b = \angle I_bCI_c = 90^\circ$, it follows that $PB \cdot PC = PI_b \cdot PI_c \Longrightarrow$ PDE is the radical axis of $(O), \odot(II_bI_c) \Longrightarrow M, N \equiv (O) \cap \odot(II_bI_c)$.

> Quote:
> 1. In triangle ABC, point P is on the arc BC of its circumcirle that doesn't contain A. Two tangent line PD,PE to incircle of ABC are drawn. PD and BC meet at point M, PE and BC meet at point N then, the circumcircle of PMN pass through a constant point (maybe the intersection point with circumcircle of ABC and mixtilinear incircle(A))

See http://www.artofproblemsolving.com/Forum/viewtopic.php?t=186117.

Quick Reply

High School Olympiads

Nice(own)

Reply

TRAN THAI HUNG

#1 Dec 29, 2010, 11:19 pm

Let (O_1) intersect (O_2) at M,N. The common tangent of $(O_1), (O_2)$ closer to M touches (O_1) at A and (O_2) at B. The common tangent of $(O_1), (O_2)$ closer to N touches (O_1) at C and (O_2) at D. Prove that: $\frac{MN^2}{AC.BD} = \cos^2 \angle ANB$

This post has been edited 1 time. Last edited by TRAN THAI HUNG, Dec 31, 2010, 5:54 pm

Luis González

#2 Dec 31, 2010, 1:21 am • 2

Radical axis MN of $(O_1), (O_2)$ bisects $\overline{AB}$ through $U.$ From $MN = UN - UM$ we get

$MN^2 = UN^2 + UM^2 - 2 \cdot UN \cdot UM = UN^2 + (UN - MN)^2 - \frac{1}{2}AB^2$

$\Longrightarrow MN = \frac{UN^2 - \frac{1}{4}AB^2}{UN} \ \ (1)$

NU is the N-median of $\triangle NAB \Longrightarrow UN^2 = \frac{1}{2}(NA^2 + NB^2) - \frac{1}{4}AB^2 \ \ (2)$

Combining the expressions (1) and (2) yields

$MN = \frac{NA^2 + NB^2 - AB^2}{2 \cdot UN} = \frac{NA \cdot NB \cdot \cos \widehat{ANB}}{UN} \ \ (3)$

Let the tangents of $\odot(NAB)$ at A, B intersect at $T,$ i.e. NT is the N-symmedian of $\triangle NAB.$ Let S be the second intersection of NT with $\odot(NAB).$ Then $P \equiv MS \cap AB$ is the midpoint of $\overline{MS},$ due to $\angle SAB = \angle SNB = \angle ANM = \angle MAB.$ If L denotes the midpoint of $MN,$ we have then

$NS = 2 \cdot LP = \frac{NA \cdot NB}{UN} \ \ (4)$

$H \equiv O_1O_2 \cap AB \cap CD$ is the exsimilicenter of $(O_1) \sim (O_2)$ and HM becomes the M-circumdiameter of $\triangle PML \Longrightarrow PL = HM \cdot \sin \widehat{UHL}.$ But on the other hand

$2 \cdot \sin \widehat{UHL} = \frac{AC}{HA} \ \ , \ \ 2 \cdot \sin \widehat{UHL} = \frac{BD}{HB} \ \Longrightarrow$

$4 \sin^2 \widehat{UHL} = \frac{AC \cdot BD}{HA \cdot HB} = \frac{AC \cdot BD}{HM^2} \Longrightarrow 4 \cdot PL^2 = AC \cdot BD$

Combining the latter expression with (4) gives $: \ AC \cdot BD = \left(\frac{NA \cdot NB}{UN}\right)^2 \ \ (5)$

Now, from (3) and (5) we get $: \ \frac{MN^2}{AC \cdot BD} = \cos^2 \widehat{ANB}.$

Quick Reply

High School Olympiads

Inscribe a triangle - [Iran Second Round 1985]

Reply

Amir Hossein

#1 Dec 29, 2010, 3:12 pm

Inscribe in the triangle ABC a triangle with minimum perimeter.

Luis González

#2 Dec 29, 2010, 8:25 pm

This is known as Fagnano's problem

We shall assume that $\triangle ABC$ is acute with orthocenter H and circumcenter $O. X, Y, Z$ are the feet of the altitudes onto $BC, CA, AB.$ If $\triangle DEF$ is an arbitrary triangle such that D, E, F lie on BC, CA, AB respectively, we have to prove

$$DE + EF + FD \geq XY + YZ + ZX.$$

In the quadrangles $AEOF$, $BFOD$ and $CDOE$ we have the following inequalities:

$$[AEOF] \leq \frac{R \cdot EF}{2} \ , \ [BFOD] \leq \frac{R \cdot FD}{2} \ , \ [CDOE] \leq \frac{R \cdot DE}{2}$$

$$\Longrightarrow [\triangle ABC] \leq \frac{R \cdot (DE + EF + FD)}{2}$$

Since O, H are isogonal conjugates, we get $OA \perp YZ, \ OB \perp ZX, \ OC \perp XY$

$$[AYHZ] = \frac{R \cdot YZ}{2} \ , \ [BZHX] = \frac{R \cdot ZX}{2} \ , \ [CXHY] = \frac{R \cdot XY}{2}$$

$$\Longrightarrow [\triangle ABC] = \frac{R \cdot (XY + YZ + ZX)}{2}$$

Therefore $DE + EF + FD \geq XY + YZ + ZX$, as desired.

Quick Reply

High School Olympiads

AD and BC meet MN at P and Q

Reply

Source: Baltic Way 2005

WakeUp

#1 Dec 28, 2010, 10:39 pm

Let $ABCD$ be a convex quadrilateral such that $BC=AD$. Let M and N be the midpoints of AB and CD, respectively. The lines AD and BC meet the line MN at P and Q, respectively. Prove that $CQ=DP$.

Luis González

#2 Dec 29, 2010, 12:02 am

Let R,S be the midpoints of the diagonals $AC,BD.$ Since $BC=AD,$ then $NR=NS \Longrightarrow$ Parallelogram $NRMS$ is a rhombus $\Longrightarrow NM$ bisects $\angle RNS$ internally. Let $U \equiv AD \cap BC.$ Since $\angle CUD$ and $\angle RNS$ have corresponding parallel sides, it follows that their angle bisectors are parallel as well. By Menelaus' theorem for $\triangle UCD$ cut by $\overline{QPN},$ keeping in mind that $UP=UQ$ and $CN=ND,$ we get

$$\frac{QU}{QC} \cdot \frac{CN}{ND} \cdot \frac{DP}{PU}=1 \Longrightarrow QC=DP.$$

jgnr

#3 Dec 29, 2010, 12:19 pm

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=377297 then use sine law.

sunken rock

#4 Jan 8, 2011, 8:59 pm

It is easy to see that MN is parallel to the bisector of the angle $(\widehat{BC, AD})$, hence the triangles $\triangle MBQ$ and $\triangle AMP$ are pseudo-similar and our problem is solved then.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Line through circumcentres forms an isosceles triangle

Reply

Source: Baltic Way 2005

WakeUp
#1 Dec 28, 2010, 10:33 pm

Let the points D and E lie on the sides BC and AC, respectively, of the triangle ABC, satisfying $BD = AE$. The line joining the circumcentres of the triangles ADC and BEC meets the lines AC and BC at K and L, respectively. Prove that $KC = LC$.

Luis González
#2 Dec 28, 2010, 11:29 pm

Let circles $\odot(BEC)$ and $\odot(ADC)$ intersect at C, P. Then P is the center of the rotation taking the oriented segments $\overline{BD}$ and $\overline{EA}$ into each other $\Longrightarrow PB = PE \Longrightarrow CP$ bisects $\angle ECB$ internally, i.e. radical axis of $\odot(BEC)$ and $\odot(ADC)$ is the internal bisector of $\angle ACB \Longrightarrow CP$ is the perpendicular bisector of $\overline{KL}$ and the conclusion follows.

littletush
#3 Nov 10, 2011, 10:24 am

we consider it in xOy system,with x-axis the bisector of $\angle ACB$.
let $D(d, kd), E(e, -ke), A(e+c, -k(e+c)), B(d+c), k(d+c))$
it suffices to prove that $x_S = x_T$
and it's easy to prove:
$$x_S = x_T = \frac{(k^2+1)(d+e+c)}{4}.$$

sunken rock
#4 Nov 17, 2011, 3:38 am

My contribution:

The circumcenters of triangles $\triangle ABC, \triangle ACD, \triangle BCE, \triangle CDE$ are the vertices of a rhombus!

Best regards,
sunken rock

Quick Reply

High School Olympiads

Fermat's isogonal conjugate

Reply

oneplusone

#1 Dec 27, 2010, 10:31 am

Let ABC be an acute triangle. Let F be a point in the triangle such that $\angle AFB=\angle BFC=\angle CFA$. Let E be the isogonal conjugate of F, i.e. $\angle ABE=\angle CBF,\angle BCE=\angle ACF$. Prove that

$$AE\cdot BC=BE\cdot AC=CE\cdot AB$$

Luis González

#2 Dec 27, 2010, 11:03 am

It's true for both Fermat points. Isogonal conjugate of F with respect to $\triangle ABC$ is one of the two common points of the three Apollonian circles of $\triangle ABC,$ since their pedal triangles are equilateral. Thus E satisfies

$$\frac{EB}{EC}=\frac{AB}{CA} \ , \ \frac{EC}{EA}=\frac{BC}{AB} \Longrightarrow EA \cdot BC=EB \cdot CA=EC \cdot AB$$

jayme

#3 Dec 27, 2010, 3:44 pm

Dear Mathlinkers,
for an interesting figure and some proofs,
http://perso.orange.fr/jl.ayme vol. 2 La fascinante figure de Cundy
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Common point

Reply

Swistak

#1 Dec 26, 2010, 8:25 pm

D and E are points outside triangle ABC. Let X, Y be the points that $X \in CD \quad Y \in CE \quad \angle CAX + \angle CBY = 180$. Prove that all XY lines have common point.

Luis González

#2 Dec 27, 2010, 1:26 am

Let $P \equiv AX \cap BY$. Then $\angle CAX + \angle CBY = \pi$ implies that P, A, C, B are concyclic on the fixed circle $\mathcal{M}$. Let $\mathcal{M}$ cut lines CD, CE again at U, V. By Pascal theorem for hexagon $CUBPAV$, the intersections $X \equiv CU \cap PA$, $M \equiv UB \cap VA$ and $Y \equiv CV \cap PB$ are collinear $\Longrightarrow XY$ goes through the fixed $M \equiv UB \cap VA$.

Quick Reply

High School Olympiads

a^2+b^2+c^2-4S sqrt(3)

Reply

trbst
#1 Dec 25, 2010, 10:47 pm

Consider a triangle ABC with side-lenghts a, b, c and area (surface) S .

Is there any distance between two remarkable points equal to : $a^2+b^2+c^2-4S\sqrt{3}$? .

Luis González
#2 Dec 25, 2010, 11:52 pm • 1

I guess you're trying to show Weitzenbock's inequality by using distances between triangle centers. Then the distance between the 1st Fermat point F and centroid G works.

$$FG^2=\frac{1}{18}(a^2+b^2+c^2-4\sqrt{3}[\triangle ABC])$$

sunken rock
#3 Jan 9, 2011, 12:51 am

See this one as well, although the points are not 'remarkable': http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=270559

Best regards,
sunken rock

Thalesmaster
#4 Mar 1, 2011, 4:40 am

> *luisgeometra wrote:*
> I guess you're trying to show Weitzenbock's inequality by using distances between triangle centers, then the distance between the 1st Fermat point F and centroid G works.
>
> $$FG^2=\frac{1}{18}(a^2+b^2+c^2-4\sqrt{3}[\triangle ABC])$$

Hence, we obtain the inequality : $a^2+b^2+c^2\geq 4\sqrt{3}[\triangle ABC])$
Which is easy to prove using Heron's Formula and the well known inequality $abc\geq (a+b-c)(c+a-b)(b+c-a)$:
$4\sqrt{3}[\triangle ABC]=\sqrt{3(a+b+c)(a+b-c)(c+a-b)(b+c-a)}$
$\leq \sqrt{3abc(a+b+c)}\leq ab+bc+ca\leq a^2+b^2+c^2$
But this inequality is not as strong as the Hadwiger-Finsler one :
$a^2+b^2+c^2\geq (a-b)^2+(b-c)^2+(c-a)^2+4\sqrt{3}[\triangle ABC]$

Quick Reply

High School Olympiads

Midpoint is a Fixed Point - [Iran Second Round 1986]

Reply

Amir Hossein

#1 Dec 24, 2010, 5:13 pm • 2

O is a point in the plane. Let O' be an arbitrary point on the axis Ox of the plane and let M be an arbitrary point. Rotate M, 90° clockwise around O to get the point M' and rotate M, 90° anticlockwise around O' to get the point M''. Prove that the midpoint of the segment MM'' is a fixed point.

Luis González

#2 Dec 24, 2010, 11:23 pm • 2

General problem: Let A, B two fixed points and P a variable point in the plane. Notation $\mathcal{R}_U(\theta)$ stands for the rotation with center U and oriented angle $\theta.$ Let Q be the image of A under $\mathcal{R}_A(\alpha)$ and P' is the image of Q under $\mathcal{R}_B(\beta).$ Assume that both are directed clockwise. Composition $\mathcal{L} \equiv \mathcal{R}_A(\alpha) \circ \mathcal{R}_B(\beta)$ is a certain rotation. $\triangle UBA$ is constructed in the upper half-plane such that $\angle UAB = \frac{1}{2}\alpha, \angle UBA = \frac{1}{2}\beta$ and let U' be the reflection of U across $AB.$ Then $\mathcal{R}_A(\alpha) : U \mapsto U'$ and $\mathcal{R}_B(\beta) : U' \mapsto U \Longrightarrow U$ is double under $\mathcal{L}$, i.e. U is the center of $\mathcal{L}.$ For $A \equiv P \equiv Q$ we obtain the isosceles $\triangle BPP'$ with apex B and U lies on its symmetry axis $\Longrightarrow \angle PUP' = \alpha + \beta.$ As a result, we have $\mathcal{L} \equiv \mathcal{R}_U(\alpha + \beta).$

Particularly, for rotational angles $\alpha = \beta = \frac{\pi}{2}, \mathcal{R}_U(\pi)$ becomes a central symmetry about $U \Longrightarrow$ Midpoint of $\overline{PP'}$ is the fixed $O,$ i.e. apex of the isosceles right $\triangle UBC$ constructed in the upper half-plane.

Quick Reply

High School Olympiads

Turkey NMO 1999, P-5, nice geometric inequality on altitudes

Reply

$f(e)$

efoski1687

#1 Dec 24, 2010, 2:45 am

In an acute triangle $\triangle ABC$ with circumradius R, altitudes $\overline{AD}, \overline{BE}, \overline{CF}$ have lengths h_1, h_2, h_3, respectively. If t_1, t_2, t_3 are lengths of the tangents from A, B, C, respectively, to the circumcircle of triangle $\triangle DEF$, prove that
$\sum_{i=1}^{3}\left(\frac{t_i}{\sqrt{h_i}}\right)^2 \leq \frac{3}{2}R$.

Luis González

#2 Dec 24, 2010, 5:53 am

Powers P_A, P_B, P_C of A, B, C WRT 9-point circle $\odot(DEF)$ are

$$P_A = \tfrac{1}{2}AB \cdot AC \cdot \cos A \ , \ P_B = \tfrac{1}{2}BC \cdot BA \cdot \cos B \ , \ P_C = \tfrac{1}{2}CA \cdot CB \cdot \cos C$$

Lengths of altitudes AD, CE, CF are given by

$$AD = \frac{AB \cdot AC}{2R} \ \ , \ \ BE = \frac{BC \cdot BA}{2R} \ \ , \ \ CF = \frac{CA \cdot CB}{2R}$$

Substituting the latter expressions into the proposed inequality yields

$\cos A + \cos B + \cos C \le \frac{3}{2}$, which follows from Jensen's inequality for $\cos\theta$, $(0, \frac{\pi}{2})$

Quick Reply

High School Olympiads

Turkey NMO 1999, P-2,a relation on a circle

Reply

$f(e)$

efoski1687

#1 Dec 24, 2010, 2:52 am

Problem-2:
Given a circle with center O, the two tangent lines from a point S outside the circle touch the circle at points P and Q. Line SO intersects the circle at A and B, with B closer to S. Let X be an interior point of minor arc PB, and let line OS intersect lines QX and PX at C and D, respectively. Prove that
$\frac{1}{|AC|}+\frac{1}{|AD|}=\frac{2}{|AB|}$.

Luis González

#2 Dec 24, 2010, 4:32 am

Arcs BP, BQ are equal $\Longrightarrow XB, XA$ bisect $\angle CXD$ internally and externally $\Longrightarrow$

$$\Longrightarrow (C,D,B,A)=-1 \Longrightarrow \frac{BC}{AC}=\frac{BD}{AD} \Longrightarrow \frac{AB}{AC}-2=\frac{BD}{AD}-1=-\frac{AB}{AD}$$

$$\Longrightarrow \frac{2}{AB}=\frac{1}{AC}+\frac{1}{AD}$$

sunken rock

#3 Apr 24, 2011, 1:03 am

Or $\frac{BC}{BD}=\frac{AC}{AD} \leftrightarrow \frac{AB-AC}{AD-AB}=\frac{AC}{AD} \leftrightarrow AB\cdot(AC+AD)=2\cdot AC\cdot AD \leftrightarrow$ to relation to prove.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Incentre, circumcentre and midpoints of AC,BC are concyclic

Reply

Source: Baltic Way 1999

WakeUp
#1 Dec 24, 2010, 12:33 am

In a triangle ABC it is given that $2AB=AC+BC$. Prove that the incentre of $\triangle ABC$, the circumcentre of $\triangle ABC$, and the midpoints of AC and BC are concyclic.

Luis González
#2 Dec 24, 2010, 3:03 am • 1

D,E are the midpoints of $BC,CA.$ Then $I\in\odot(ODE)\Longleftrightarrow\angle CIO=90^\circ$

$$\Longleftrightarrow IO^2+IC^2=OC^2\Longleftrightarrow R^2-2Rr+IC^2=R^2$$

$$\Longleftrightarrow\frac{ab(s-c)}{s}=2Rr\Longleftrightarrow 2ab(s-c)=abc\Longleftrightarrow a+b=2c$$

sunken rock
#3 Apr 24, 2011, 1:16 am • 1

Remark: If, additionally, F is the foot of the internal angle bisector of $\angle ACB$ on AB, see that $AF=AE$ and $BF=BD$, hence $IE=ID$ and, easily, $CEID$ is cyclic, but $CEOD$ is cyclic as well...

Observation: I is the midpoint of CP, where P is the second intersection of the internal angle bisector of $\angle ACB$ with the circle $\odot O$.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Nice Geometric

Reply

vntbqpqh234

#1 Dec 22, 2010, 9:52 pm

Let ABC be a triangle, H is orthocenter, O is circumcenter, and R its circumradius.
Let D,E,F be the reflection of point A across the line BC,B across the line AC,C across the line BA,respective. Prove that D,E,F are collinear if and only if $OH=2R$

Luis González

#2 Dec 23, 2010, 1:24 am • 1

Lemma: Let G,N be the centroid and 9-point center of $\triangle ABC.$ Reflection triangle $\triangle DEF$ of $\triangle ABC$ is homothetic to the pedal triangle $\triangle A_0B_0C_0$ of N WRT $\triangle ABC$ under the homothety with center G and coefficient $4.$

Let P be the foot of the A-altitude, M be the midpoint of BC and $T \equiv AP \cap (O),$ different from $A.$ It is well-known that $\overline{OM}=\frac{1}{2}\overline{AH}$ and $\overline{HP}=\frac{1}{2}\overline{HT}.$ Since NA_0 is the median of the right trapezoid $HOMP,$ we have

$$\overline{NA_0}=\frac{\overline{OM}+\overline{HP}}{2}=\frac{\overline{AH}+\overline{HT}}{4}=\frac{\overline{HD}}{4} \Longrightarrow \frac{\overline{HD}}{\overline{NA_0}}=4.$$

Because of $\frac{\overline{GH}}{\overline{GN}}=4 \Longrightarrow G,A_0,D$ are collinear $\Longrightarrow \frac{\overline{GD}}{\overline{GA_0}}=4.$

Similarly $\frac{\overline{GE}}{\overline{GB_0}}=\frac{\overline{GF}}{\overline{GC_0}}=4 \Longrightarrow \triangle A_0B_0C_0 \sim \triangle DEF$ are homothetic under $(G,4).$

As a result, D,E,F are collinear $\Longleftrightarrow A_0,B_0,C_0$ are collinear. Then $\overline{A_0B_0C_0}$ becomes a Simson line with pole N WRT $\triangle ABC \Longleftrightarrow N \in (O) \Longleftrightarrow ON=R=\frac{1}{2}OH.$

jayme

#3 Dec 23, 2010, 4:31 pm

Dear Mathlinkers,
this problem has been proposed as problem 5 at the Concours Général France in 1999.
Perhaps it comes from another origin?
Any idea?
Sincerely
Jean-Louis

Luis González

#4 Dec 23, 2010, 9:34 pm

Dear Jean Louis, according to Mathworld this result comes from Bottema, O. Hoofdstukken uit de elementaire meetkunde, 2nd ed. Utrecht, Netherlands: Epsilon, pp. 83-87. 1987. The problem also appeared as G5 of IMO shorlist 1998.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=18080
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=2791

Quick Reply

High School Olympiads

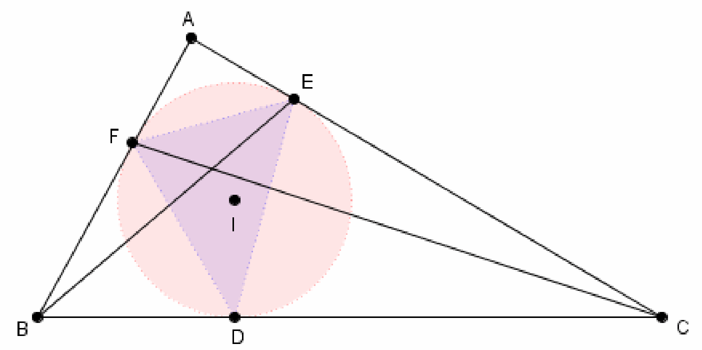
area of triangle

Reply

77ant
#1 Dec 21, 2010, 12:27 pm

Dear everyone
For a triangle ABC with its incircle (I), let it touch BC, CA, AB at D, E, F respectively.
$\triangle BEF = 10, \triangle CEF = 20$. Find the area of $\triangle DEF$

Attachments:



Luis González
#2 Dec 21, 2010, 11:44 pm

$\triangle XYZ$ is tangential triangle of $\triangle ABC$. Barycentric coordinates of Y, Z are given by

$$Y\ (a^2 : -b^2 : c^2)\ \ ,\ \ Z\ (a^2 : b^2 : -c^2) \implies$$

$$\frac{|\triangle YBC|}{|\triangle ABC|} = \frac{1}{2S_B}\begin{pmatrix} a^2 & -b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{a^2}{2S_B}$$

$$\frac{|\triangle ZBC|}{|\triangle ABC|} = \frac{1}{2S_C}\begin{pmatrix} a^2 & b^2 & -c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{a^2}{2S_C}$$

$$\implies \frac{1}{|\triangle YBC|} + \frac{1}{|\triangle ZBC|} = \frac{2(S_B + S_C)}{a^2} \cdot \frac{1}{|\triangle ABC|} = \frac{2}{|\triangle ABC|}$$

Use the latter relation for $\triangle DEF$ and its tangential triangle $\triangle ABC$.

oneplusone
#3 Dec 23, 2010, 12:47 pm

Let EF intersect CB at G. Then since $[CEF] = 2[BEF]$, we have $GB = BC$. Also since $GBDC$ is harmonic, we have $\frac{CD}{DB} = \frac{CG}{GB} = 2$. Thus $[DEF] = \frac{2}{3}[BEF] + \frac{1}{3}[CEF] = \frac{40}{3}$.

Quick Reply

High School Olympiads

Nice Geometry Problem

Reply

hvaz

#1 Dec 20, 2010, 6:12 am

Let ABC be a non-isosceles triangle, A_0 and B_0 be the midpoints of sides BC and AC, A_1 and B_1 be the feet of the altitudes from A and B, respectively. Let H be the orthocenter of triangle ABC, O its circuncenter and C' = $A_1B_1 \cap A_0B_0$. Prove that CC' is perpendicular to OH.

Luis González

#2 Dec 20, 2010, 6:32 am

The result is valid for any pair of isogonal conjugates WRT $\triangle ABC.$

Theorem: P,Q are two isogonal conjugates WRT $\triangle ABC.$ $\triangle P_1P_2P_3$ and $\triangle Q_1Q_2Q_3$ are the pedal triangles of P,Q WRT $\triangle ABC.$ If $M \equiv P_1P_2 \cap Q_1Q_2,$ then $CM \perp PQ.$

It's known that P_1,P_2,P_3,Q_1,Q_2,Q_3 lie on a same circle $\mathcal{M}.$ Let U,V be the midpoints of $CP,CQ,$ i.e. U,V are the centers of the circumcircles of $\triangle CP_1P_2$ and $\triangle CQ_1Q_2.$ From power of M WRT $\mathcal{M},$ it follows that $\overline{MP_1} \cdot \overline{MP_2} = \overline{MQ_1} \cdot \overline{MQ_2} \Longrightarrow M$ has equal power WRT $\odot(CP_1P_2)$ and $\odot(CQ_1Q_2) \Longrightarrow CM$ is radical axis of $\odot(CP_1P_2)$ and $\odot(CQ_1Q_2) \Longrightarrow CM \perp UV,$ but obviously $UV \parallel PQ.$

Quick Reply

High School Olympiads

Quite Hard

Source: Unknown source

Flakky

#1 Mar 18, 2009, 3:17 am

Let (k) be a circle with diameter AB. Points C, D, E, F (in that order) $\in$ semicircle $\bar{k}$. If $\angle AMC = \angle FMB$ and $\angle AMD = \angle EMB$ prove that:
a)$PM \perp AB$ where $P = CD \cap EF$
b)$QM \perp AB$ where $Q = CE \cap DF$
EDIT: M is an arbitrary point on AB.

This post has been edited 1 time. Last edited by Flakky, Mar 19, 2009, 3:11 am

Heebeen, Yang

#2 Mar 18, 2009, 5:56 am

M maybe center of circle?
But if it is, it's too obvious.

Luis González

#3 Mar 18, 2009, 6:00 am

If M is the center of the circle, the result is indeed trivial. I believe M is an arbitrary point on $\overline{AB}$.

Let $R \equiv AC \cap BF$ and $S \equiv AD \cap BE$. Since M is the unique point of AB such that AB bisects angles $\angle CMF$ and $\angle DME$ externally, it follows that $\triangle CFM$ and $\triangle DEM$ become orthic triangles of $\triangle RAB$ and $\triangle SAB$, respectively $\Longrightarrow$ $N \equiv CF \cap DE$ is the harmonic conjugate of M WRT (A, B), i.e. RS is polar of N WRT the circle k with diameter AB. But indeed the intersections $P \equiv CD \cap EF$ and $Q \equiv CE \cap DF$ lie on the polar of N WRT $k \Longrightarrow P, Q, R, S$ are collinear.

Flakky

#4 Mar 19, 2009, 1:04 am

I'm truly sorry. M is an arbitrary point on AB.

Luis González

#5 Dec 18, 2010, 6:58 am

I found a slighty different approach:

Let X be the intersection of DE with the perpendicular τ to AB through M and let $N \equiv DE \cap AB.$ Since MX, MN bisect $\angle DME$, then $(D, E, X, N) = -1 \Longrightarrow \tau$ is the polar of N WRT the circle with diameter $\overline{AB}.$ Similarly CF passes through N and then in the complete quadrilateral $CDEF$, it follows that $P \equiv CD \cap EF$ and $Q \equiv CE \cap DF$ lie on τ.

Zhero

#6 Dec 18, 2010, 12:48 pm

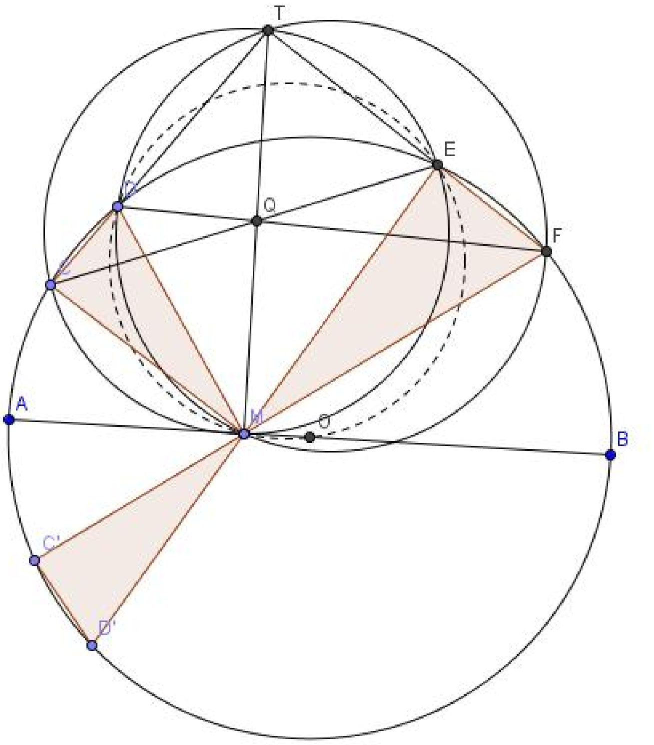
Let O be the center of (k). Let the circumcircle of OMF hit (k) at C'. We have $\angle FMO = \angle OC'F = \angle C'FO = \angle C'MA$, whence $C' = C$, so $CFOM$ is cyclic. Similarly, we find that $DEOM$ is cyclic. Let the tangents to (k) at C and F meet at U, and let the tangents at D and E meet at V. $\angle UCO = \angle UFO = 90^\circ$, sp $CMOFU$ is cyclic; hence, $\angle UMO = \angle UCO = 90^\circ$, so $UM \perp AB$. Similarly, $VM \perp AB$. Finally, Pascal's theorem on $CCEFFD$ and $EEFDDC$ shows that P, Q, U, V are collinear, so we are done.

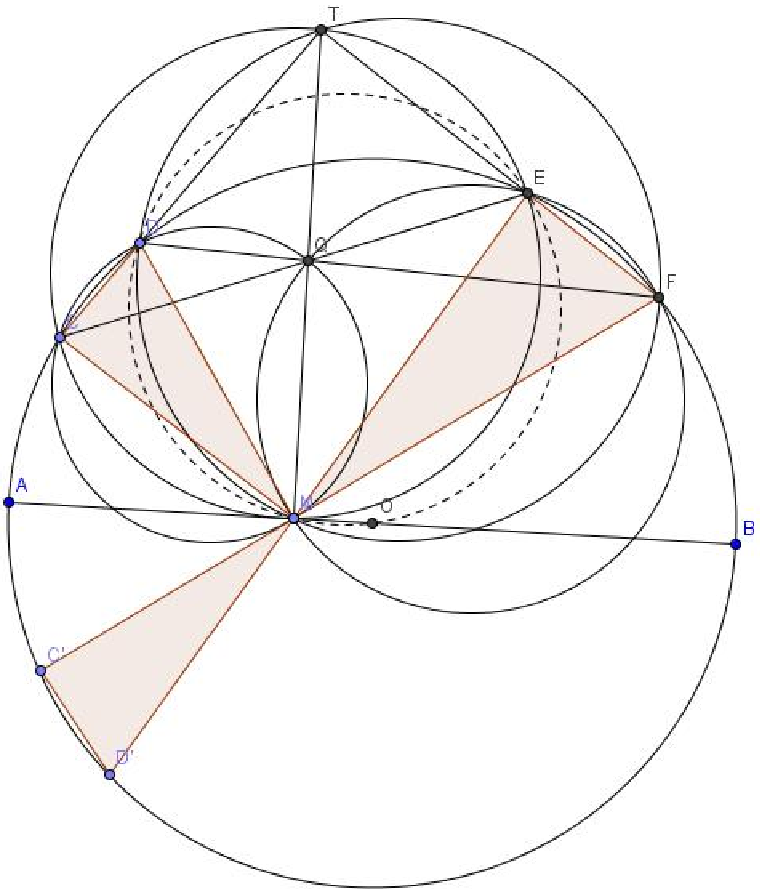
skytin

#7 Dec 18, 2010, 9:10 pm

Let make simmetry of points C D wrt AB and get points C' D' . Easy to see that M is on C'F and D'E and triangle MEF ~ MC'D' = MCD , so MEF ~ MCD . Let EF intersect (FMD) at point T , aesy to see that angle EFM = MCD , so T is on CD . So like the same T is on (MCE) . Let O is center of (k) well known that O is on (MDE) , so angle EMD = EOD = 2*TFD , so MT is angle bissector of EMD , so AB is perpendicular to MT and T = P and Q is on MT (Q = radical center of ((k) (MCP) (MDP)) . done

Attachments:





Quick Reply

High School Olympiads

incircle very interesting problem

Reply

Source: if you solve please post in a short time (i couldn't solve)

Ulanbek_Kyzylorda KTL

#1 Mar 21, 2009, 9:21 am

A circumscribed triangle ABC is also inscribed. M is midpoint of AC and N is midpoint of arc AC (that consists point B).Prove that <BNI=<IMA(I is center of inscribed circle). (please solve)

Virgil Nicula

#3 Mar 24, 2009, 8:03 am

> Ulanbek_Kyzylorda KTL wrote:
>
> Let ABC be a triangle ($b \ne c$) for which denote the circumcircle $w = C(O,R)$, the incircle $C(I,r)$, the midpoint M of $[BC]$ and the point $N \in w$ so that $MN \perp BC$ and BC doesn't separate A, N. Prove that $\widehat{ANI} \equiv \left\{\begin{array}{ccc} \widehat{IMC} & \mathrm{if} & b<c \\ \widehat{IMB} & \mathrm{if} & b>c\end{array}\right.$.

Proof I. Suppose $b<c$. Denote the diameter $[NS]$ of circle w, $T\in BC$, $IT\perp IA$ and $R\in NI\cap ST$.

Prove easily that $R\in w$ and the quadrilateral $MITS$ is cyclically. Thus, $IT\parallel NA$ and $\widehat{IMC}\equiv\widehat{ISR}\equiv\widehat{ANI}$.

Proof II. Suppose $b<c$. Denote $D\in BC$ for which $ID\perp BC$. Observe that $MD=\frac{c-b}{2}$

and $\tan\widehat{IMC}=\tan\widehat{IMD}=\frac{ID}{MD}=\frac{2r}{c-b}\Longrightarrow\boxed{\tan\widehat{IMC}=\frac{2r}{c-b}}$.

Since $AN\perp AI$ obtain $\tan\widehat{ANI}=\frac{AI}{AN}=\frac{b+c}{2p}\cdot\frac{2bc\cdot\cos\frac{A}{2}}{b+c}\cdot\frac{1}{2R\sin\frac{C-B}{2}}=$

$\frac{bc\cdot\cos\frac{A}{2}\sin\frac{A}{2}}{2pR\sin\frac{C-B}{2}\cos\frac{B+C}{2}}=\frac{bc\sin A}{2pR(\sin C-\sin B)}=\frac{2pr}{p(c-b)}\Longrightarrow\boxed{\tan\widehat{ANI}=\frac{2r}{c-b}}$.

In conclusion, if $b<c$, then $\tan\widehat{ANI}=\tan\widehat{IMC}=\frac{2r}{c-b}$, i.e. $\widehat{ANI}\equiv\widehat{IMC}$.

You can prove analogously in the case $b>c$ that $\tan\widehat{ANI}=\tan\widehat{IMB}=\frac{2r}{b-c}$, i.e. $\widehat{ANI}\equiv\widehat{IMB}$.

This post has been edited 1 time. Last edited by Luis González, Mar 12, 2015, 10:41 am

Luis González

#4 Dec 16, 2010, 2:59 am

Ray AI cuts circumcircle (O) of $\triangle ABC$ at the midpoint P of the arc AC not containing B. Inversion through pole P with radius $PA=PC=PI$ takes M into N and I is double $\Longrightarrow$ Circle $\odot(IMN)$ is tangent to PI through $I\Longrightarrow$ $\angle INM=\angle MIP$. Which implies that $\angle BNI=\angle IMA$.

nsato

#5 Dec 16, 2010, 9:35 am

See this post:
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=32163

Quick Reply

High School Olympiads

Angle Bisectors

Reply

Stephen
#1 Oct 20, 2010, 9:31 am

In triangle ABC, let D the intersection point of segment AC and the bisector of □ABC.

And E the intersection point of segment AB and the bisector of □ACB.

Let X be the intersection point of line DE and arc $\widehat{AB}$ of the circumcircle of triangle ABC.

Prove that $\frac{1}{BX} = \frac{1}{AX} + \frac{1}{CX}$.

Luis González
#2 Oct 20, 2010, 8:47 pm • 1

Line DE is the geometric locus of the points whose sum of the oriented distances to AC and AB equals the oriented distance to BC, since the trilinear equation of line DE with respect to $\triangle ABC$ is given by $\beta + \gamma - \alpha = 0$. Therefore, if P, Q, R are the orthogonal projections of X on CB, BA, AC, we have that $XP = XR - XQ$.

Since $\angle XBP = \angle XAR$ and $\angle XCB = \angle XAB$, we have $\triangle XBP \sim \triangle XAR$ and $\triangle XCP \sim \triangle XAQ$

$$\Longrightarrow \frac{BX}{AX} = \frac{XP}{XR} \ , \ \frac{CX}{AX} = \frac{XP}{XQ}$$

$$\Longrightarrow \frac{1}{BX} - \frac{1}{CX} = \frac{XR - XQ}{AX \cdot XP} = \frac{XP}{AX \cdot XP} = \frac{1}{AX}.$$

madhusudan kale
#3 Dec 14, 2010, 12:46 am

Luis, prove your first statement.

Luis González
#4 Dec 14, 2010, 9:37 pm

Trilinear coordinates of D and E are given by $(1:0:1)$ and $(1:1:0)$, respectively

$$DE : \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ \alpha & \beta & \gamma \end{pmatrix} = 0 \implies DE : \beta + \gamma - \alpha = 0$$

Virgil Nicula
#5 Dec 17, 2010, 9:52 pm

See **here** the first proposed problem (**PP1**).

Quick Reply

High School Olympiads

N-secting a,b,c gives a rational expression

Reply

Source: IberoAmerican 1988 Q4

WakeUp

#1 Dec 14, 2010, 4:36 am

$\triangle ABC$ is a triangle with sides a, b, c. Each side of $\triangle ABC$ is divided in n equal segments. Let S be the sum of the squares of the distances from each vertex to each of the points of division on its opposite side. Show that $\frac{S}{a^2+b^2+c^2}$ is a rational number.

kaszubki

#2 Dec 14, 2010, 5:57 am

hint

Luis González

#3 Dec 14, 2010, 6:47 am • 1

Assume that $\overline{BC}$ is divided into n equal segments by $n-1$ points $P_1, P_2, P_3, ..., P_k$. By Stewart theorem, we obtain the general expression

$$(AP_k)^2 = \left(\frac{k}{n}\right) b^2 + \left(1-\frac{k}{n}\right) c^2 - \frac{k}{n}\left(1-\frac{k}{n}\right) a^2 \ , \ \ k = 1, 2, 3, 4, 5, ..., n$$

$$(AP_k)^2 = c^2 + \left(\frac{a \cdot k}{n}\right)^2 - \frac{k}{n}(a^2+c^2-b^2)$$

Sum S_A of the square of the $n-1$ cevians AP_k is then

$$S_A = nc^2 + \frac{a^2}{n^2} \cdot \sum_{k=1}^{n} k^2 - \left(\frac{a^2+c^2-b^2}{n}\right) \cdot \sum_{k=1}^{n} k - b^2$$

Note that $k = n$ gives $AP_n \equiv AC$, thus in the latter summation we have subtracted b^2, since it does not count. Then, using sum formulas yields

$$S_A = nc^2 + \frac{a^2}{n^2} \cdot \left[\frac{n(n+1)(2n+1)}{6}\right] - \left(\frac{a^2+c^2-b^2}{n}\right) \cdot \left[\frac{n(n+1)}{2}\right] - b^2$$

$$S_A = \frac{(n-1)(b^2+c^2)}{2} - \frac{(n^2-1)a^2}{6n}$$

By cyclic permutation of elements, we obtain the expressions:

$$S_B = \frac{(n-1)(a^2+c^2)}{2} - \frac{(n^2-1)b^2}{6n}$$

$$S_C = \frac{(n-1)(b^2+a^2)}{2} - \frac{(n^2-1)c^2}{6n}$$

$$S = S_A + S_B + S_C = \left[(n-1) - \frac{(n-1)(n+1)}{6n}\right] \cdot (a^2+b^2+c^2)$$

$$\Longrightarrow \ \frac{S}{a^2+b^2+c^2} = \frac{(n-1)(5n-1)}{6n}.$$

Quick Reply

High School Olympiads

Six Simson lines

Reply

nsato

#1 Dec 14, 2010, 1:33 am • 1

Let A, B, C, P, Q, and R be six concyclic points. Show that if the Simson lines of P, Q, and R with respect to triangle ABC are concurrent, then the Simson lines of A, B, and C with respect to triangle PQR are concurrent. Furthermore, show that the points of concurrence are the same.

Luis González

#2 Dec 14, 2010, 4:13 am • 3

Let A_0, B_0, C_0 be the orthogonal projections of A, B, C onto PR, RQ, PR and A_1, B_1, C_1 the orthogonal projections of A, B, C onto PQ, PQ, QR, respectively. Perpendiculars from A_0, C_0 to BC, AB meet at the orthopole O_Q of PR WRT $\triangle ABC$ and perpendiculars from B_0, C_1 to AC, BA meet at the orthopole O_P of QR WRT $\triangle ABC$. Since Simson lines p, q, r with poles P, Q, R WRT $\triangle ABC$ meet at the orthopoles of the lines connecting their poles, it follows that $U \equiv p \cap q \cap r \Longleftrightarrow U \equiv O_P \equiv O_Q$. Which implies that $C_1 \in UC_0$ and $A_1 \in UA_0$. Hence, A_0A_1, B_1B_0 and C_1C_0, meeting at U, are Simson lines of A, B, C WRT $\triangle PQR$.

Ramchandran

#3 Jan 23, 2011, 2:42 pm

Hi, I am not really a geometry buff, so please tell me , if this is right -

solution

Quick Reply

Spain

16th IBERO - URUGUAY 2001.

Reply

carlosbr

#1 Mar 26, 2006, 8:50 pm

$16th$ Iberoamerican Olympiad

Minas, URUGUAY. [2001]

Edited by djimenez

Carlos Bravo

Attachments:
2001.pdf (31kb)

Luis González

#2 Jan 31, 2010, 10:44 am

> Quote:
> **Problema 2.** El incirculo (I) de $\triangle ABC$ es tangente a BC, CA, AB en X, Y, Z. Las rectas BI, CI cortan a la recta YZ en P, Q. Mostrar que $\triangle ABC$ es isósceles si y solo si $XP = XQ$.

Es bien sabido que la paralela media a AB, la recta YZ y la bisectriz BI de $\angle ABC$ concurren en P. Así mismo la paralela media a AC, la recta YZ y la bisectriz CI de $\angle ACB$ concurren en Q. Una sencilla demostración puede apreciarse en el tópico *Another unlikely concurrency* en el sitio http://pagesperso-orange.fr/jl.ayme/.

Consecuentemente los ángulos $\angle BPC$ y $\angle CQB$ son rectos, es decir que los cuadriláteros $PQBC$, $PIXC$ y $QIXB$ son cíclicos. Así se tienen las siguentes relaciones angulares:

$$\angle QPX = \angle QPI + \angle IPX = \angle ICX + \angle ICX = \angle ACB$$

$$\angle PQX = \angle PQI + \angle IQX = \angle IBX + \angle IBX = \angle ABC$$

Por lo tanto $\triangle ABC \sim \triangle XQP$ y evidentemente $\triangle ABC$ es isósceles si y solo si $\triangle XQP$ es isósceles.

Luis González

#3 Dec 13, 2010, 4:16 am

> Quote:
> **Problema 6.** Probar que es imposible cubrir un cuadrado de lado 1 con 5 cuadrados iguales de lado < 1/2.

Considérese los siguientes 9 puntos en el plano cartesiano $(0,0), (\frac{1}{2},0), (1,0), (0,\frac{1}{2}), (\frac{1}{2},\frac{1}{2}), (1,\frac{1}{2}), (0,1), (\frac{1}{2},1), (1,1)$. Cualquier cuadrado de lado menor que $\frac{1}{2}$ solo podrá cubrir uno de de estos puntos. Asi, ello es cierto para 8 cuadrados.

Quick Reply

Spain

Examenes OMCC 2006

Reply

wx69yz
#1 Aug 5, 2006, 1:57 am

Aca les dejo los examenes de la OMCC 2006 realizada en Panama del 30 de julio al 5 de agosto.

Attachments:
Prueba_dia_1.pdf (578kb)
Prueba_dia_2.pdf (560kb)

Dave Vather
#2 Aug 5, 2006, 11:51 am

¿Alguien tiene los resultados por países?

conejita
#3 Aug 6, 2006, 3:56 am

Los resultados se encuentran en la pagina oficial de la VIII centro, los resultados de la delegacion mexicana fueron:

Oro:
Paul y Jose Daniel

Plata:
Andres Emilsson

Vamos Mexico!!!!

M4RI0
#4 Aug 7, 2006, 10:34 am

Mis soluciones para los tres primeros.

Attachments:
Sol01.pdf (636kb)

conejita
#5 Aug 7, 2006, 7:38 pm

Los problemas de la OMCC 2006 ya me salieron, pero con excepcion del 6, la vdd ya he hecho varias cosas, y todas me resultan contradictorias. Me gustaria saber si alguien lo ha hecho, para k me de cuenta de n k ando fallando.
Si kieren k escriba lo k he hecho asta aora, k me digan y con gusto lo pongo!!

Y no mas como comentario, creo k esta OMCC no fue dificil!!

Espero sus soluciones

Tony2006
#6 Aug 11, 2006, 7:06 am

EXELENTE PRIMER PERFECTO EN OMCC FELICIDADES DANIEL!!!!

José
#7 Aug 11, 2006, 11:12 pm

> conejita wrote:
> Si kieren k escriba lo k he hecho asta aora, k me digan y con gusto lo pongo!!

Los que podrías hacer es abrir 6 topics, cada uno con un problema, y ahi compartir la solución

mhuarancca
#8 Aug 26, 2006, 8:26 am

seria bueno q alguien ponga la soluciones , especialmente para el 5 y la 6 en mi caso ,

Luis González
#9 Mar 17, 2010, 11:11 pm

> Quote:
> Sean Γ y Γ' dos circunferencias de igual radio con centros O y O', respectivamente. Γ y Γ' se cortan en dos puntos y A es uno de ellos. Se escoge un punto B cualquiera en Γ. Sea C el otro punto de corte de la recta AB con Γ' y D un punto en Γ' tal que $OBDO'$ es un paralelogramo. Demuestre que la longitud de CD es constante, es decir, no depende de la elección de B.

Sea $E \equiv BD \cap \Gamma$, diferente de B. Basta observar que como $\Gamma \cong \Gamma'$, entonces por evidente simetría si $BD \parallel OO'$, los arcos AE y AD en Γ y Γ' son congruentes $\Longrightarrow \angle ABE = \angle ACD$. Así el triángulo $\triangle BDC$ es isósceles con ápice $D \Longrightarrow CD = BD = OO' = \text{const}$.

Luis González
#10 Mar 19, 2010, 3:22 am

> Quote:
> Sea $ABCD$ un cuadrilátero convexo. Sea I el punto de intersección de las diagonales AC y BD. Sean E, H, F y G puntos sobre los segmentos AB, BC, CD y DA, respectivamente, tales que EF y GH se cortan en I. Sea M el punto de intersección de EG y AC y N el punto de intersección de HF y AC. Demuestre que $\frac{AM}{IM} \cdot \frac{IN}{CN} = \frac{IA}{IC}$.

Note que los triángulos $\triangle AEG$ y $\triangle CFH$ son perspectivos a través del perspector I, entonces por teorema de Desargues las intersecciones $X \equiv AD \cap BC, Y \equiv AB \cap DC$ y $P \equiv EG \cap HF$ son colineales. Considérese $Q \equiv AC \cap XY$. Luego por teorema de Menelao en los triángulos $\triangle MQP$ y $\triangle NQP$ cortados por las rectas AB y CD, tenemos

$$\frac{YQ}{YP} \cdot \frac{PE}{EM} \cdot \frac{MA}{QA} = 1 \ , \ \frac{YQ}{YP} \cdot \frac{PF}{NF} \cdot \frac{NC}{QC} = 1$$

$$\Longrightarrow \frac{MA}{NC} \cdot \frac{PE}{EM} \cdot \frac{NF}{PF} = \frac{QA}{QC} \ (*)$$

Pero por teorema de Menelao en $\triangle MNP$ cortado por $\overline{FIE}$, se tiene

$\frac{PE}{EM} \cdot \frac{NF}{PF} = \frac{IN}{IM}$. Combinando con $(*)$ resulta

$\frac{MA}{NC} \cdot \frac{IN}{IM} = \frac{QA}{QC}$. Luego como $(A, C, I, Q) = -1$, se sigue que

$$\frac{AM}{IM} \cdot \frac{IN}{CN} = \frac{QA}{QC} = \frac{IA}{IC}.$$

Luis González
#11 Dec 12, 2010, 11:21 pm • 1

El problema anterior puede ser generalizado de la siguiente forma

Proposición: $ABCD$ es un cuadrilátero convexo y J es un punto en la diagonal AC. Una recta por J corta a AB, CD en P, Q y otra recta por J corta a BC, DA en R, S. AC corta a PS, RQ en M, N. Entonces $\frac{IA}{IC} = \frac{AM}{JM} \cdot \frac{JN}{CN}$.

Proyéctese la recta a través de $AD \cap BC$ y $AB \cap DC$ al infinito. Denotando los puntos projectados con subíndice cero, $ABCD$ con intercepto I de diagonales se transforma en un paralelogramo $A_0B_0C_0D_0$ con centro I_0 y claramente $\triangle A_0P_0S_0$ y $\triangle C_0Q_0R_0$ son homotéticos a través de J_0. Como $I_0A_0 = I_0C_0$ y $\frac{J_0M_0}{J_0N_0} = \frac{A_0M_0}{C_0N_0}$, se sigue que

$$\left(\frac{I_0A_0}{I_0C_0} \cdot \frac{M_0C_0}{M_0A_0}\right) \cdot \left(\frac{J_0M_0}{J_0N_0} \cdot \frac{C_0N_0}{C_0M_0}\right) = (A_0, C_0, I_0, M_0) \cdot (M_0, N_0, J_0, C_0) = 1$$

Así, $(A, C, I, M) \cdot (M, N, J, C) = 1 \Longrightarrow \frac{IA}{IC} \cdot \frac{CN}{JN} \cdot \frac{JM}{MA} = 1.$

Quick Reply

High School Olympiads

Cevian triangle of ABC is similar to ABC

Source: CentroAmerican 2004

WakeUp

#1 Dec 11, 2010, 12:29 am • 1

ABC is a triangle, and E and F are points on the segments BC and CA respectively, such that $\frac{CE}{CB}+\frac{CF}{CA}=1$ and $\angle CEF=\angle CAB$. Suppose that M is the midpoint of EF and G is the point of intersection between CM and AB. Prove that triangle FEG is similar to triangle ABC.

Luis González

#2 Dec 11, 2010, 3:00 am • 1

Let the parallel to CA through E cut AB at G'. From $\triangle BEG' \sim \triangle BCA$ we get

$$\frac{CE}{CB}=\frac{AG'}{AB}=1-\frac{CF}{CA} \Longrightarrow \frac{CF}{CA}=\frac{AB-AG'}{BA}=\frac{BG'}{BA} \Longrightarrow FG' \parallel CB$$

Thus, $CEG'F$ is a parallelogram $\Longrightarrow CM$ passes through $G' \Longrightarrow G \equiv G'$. $\angle EGF=\angle BCA$, $\angle CEF=\angle EFG=\angle CAB$ imply that $\triangle ABC \sim \triangle FEG$.

panamath

#3 Feb 7, 2014, 12:09 pm

Since $\triangle CFE \sim \triangle ABC \implies CE \cdot CB = CF \cdot CA$ (1). From hypotesis we have that $CE \cdot CA = CB \cdot FA$ (2). Multiplying (1) and (2) we get $CE^2 = CF \cdot CA$ (3). Now we have that CG is symmedian, so $\frac{AG}{GB}=\frac{AC^2}{BC^2}=\frac{CE^2}{CF^2}$ (4).
Finally from (3) aqnd (4) we get $\frac{FA}{CF}=\frac{AG}{GB} \implies FG \parallel BC$ analogously
$GE \parallel AC \implies \triangle FGE \sim \triangle CFE \sim \triangle ABC$

JackXD

#4 Dec 20, 2015, 10:12 am

Note that $\frac{CE}{CB}+\frac{CF}{AC}=1 \implies \frac{CE}{CB}=\frac{AF}{AC}=k \implies$ parallels through F and E to BC and AC meet at a common point G'.Now let the C-median of $\triangle CEF$ meet AB at G.As $AFEB$ is cyclic it follows that CG is the C-symmedian of $\triangle ABC \implies \frac{AG}{BG}=\frac{b^2}{a^2}$.Also $CE*CB=CF*CA \implies ka^2=(1-k)b^2 \implies \frac{k}{1-k}=\frac{b^2}{a^2}$.Therefore $\frac{AG'}{BG'}=\frac{CE}{BE}=\frac{k}{1-k}=\frac{b^2}{a^2}=\frac{AG}{BG} \implies G \equiv G'$ and the result immmediately follows.

High School Olympiads

77ant

#1 Dec 8, 2010, 11:43 pm • 1

Dear everyone
There is an acute triangle ABC with its centroid, circumcenter G, O
OG is parallel to BC. Let D, E, F be the midpoints of BC, CA, AB respectively.
The length of the circumradius of the triangle ABC is equal to 2.
Prove that $OD = OE \cdot OF$ with no trig, as possible.

This post has been edited 1 time. Last edited by 77ant, Dec 9, 2010, 12:45 am

Luis González

#2 Dec 9, 2010, 12:08 am

Let H be the orthocenter of $\triangle ABC$. Since O is the orthocenter of the medial triangle $\triangle DEF$, then it is enough to show that $HA = HB \cdot HC$. Let P the foot of the A-altitude. Then $HGO \parallel BC \Longleftrightarrow \frac{HA}{HP} = \frac{GA}{GD} = 2 \Longleftrightarrow HA = 2HP \ (\star)$. But because of $\angle BHC = \angle BAC \pmod \pi$, circumcircles of $\triangle ABC$ and $\triangle BHC$ are congruent $\Longrightarrow$ $HB \cdot HC = 2R \cdot HP$. Together with $(\star)$, it yields $HA = HB \cdot HC$.

77ant

#3 Dec 9, 2010, 12:19 am

Thanks so much. I seldom understand why $HB \cdot HC = 2R \cdot HP$
Sorry. I got it just right now.

High School Olympiads

Locus of points symmetric to a fixed point about a chord.

Reply

Goutham

#1 Dec 7, 2010, 2:49 pm

A fixed point A inside a circle is given. Consider all chords XY of the circle such that $\angle XAY$ is a right angle, and for all such chords construct the point M symmetric to A with respect to XY . Find the locus of points M.

Luis González

#2 Dec 7, 2010, 10:15 pm

Let (O,R) be the given circle. Lines AX, AY cut (O) again at Z,W and let N be the midpoint of $XY.$ Then $\angle XAN=\angle AXN=\angle AWZ \Longrightarrow NA \perp ZW.$ Likewise, if L denotes the midpoint of $ZW,$ we get $LA \perp XY.$ As a result, quadrilateral $OLAN$ is a parallelogram $\Longrightarrow$ segments AO,NL bisect each other at U and the orthogonal projection of A on XY lies on the circle with diameter $\overline{NL}.$

By median theorem in $\triangle ALN,$ we obtain $UN^2=\frac{1}{2}(AN^2+AL^2)-\frac{1}{4}AO^2$

But by Pythagorean theorem for right triangles $\triangle OZL$ and $\triangle OYN,$ we have

$$AN^2=R^2-\tfrac{1}{4}ZW^2 \ , \ AL^2=R^2-\tfrac{1}{4}XY^2$$

$$AN^2+AL^2=2R^2-(AN^2+AL^2) \ \Longrightarrow \ AN^2+AL^2=R^2$$

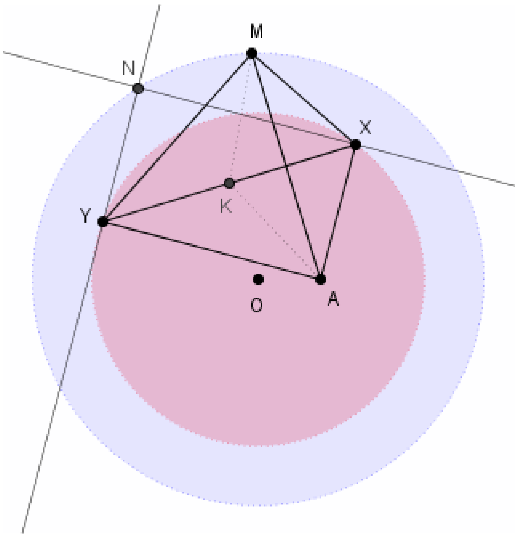
Thus, locus of M is a circle with center O and radius $OM=2\sqrt{\frac{1}{2}R^2-\frac{1}{4}AO^2}.$

77ant

#3 Dec 8, 2010, 12:27 am

Make rectangle AXNY, XYNM is isosceles trapezoid, and OK is perpendicular bisector of NM.
ON=OM, OM^2+OA^2=OX^2+OY^2 . Thus OM=constant, where its locus is a circle.

Attachments:



Vo Duc Dien

#4 Oct 23, 2012, 3:38 am

Let XA and XM intersect the circle again at W and Z, respectively. Just prove that AW = MZ, then MZ×MX = AW×AX = constant and done.

Quick Reply

High School Olympiads

Parabola's Property

Reply

Headhunter
#1 Dec 6, 2010, 9:36 pm • 1

Hello.

There is a parabola α with its axis β , vertex A. Let l be the line passing through A (perpendicular to β)
An arbitrary point B is on l . An arbitrary line passing through B cut α at two points C, D and β at E
Show that $BE^2 = BC \cdot BD$ without the use of coordinates system.

Luis González
#2 Dec 7, 2010, 8:22 am • 2

Let F, G be the reflections of E about A, B and U_∞, V_∞ denote the infinity points of lines β, ℓ respectively. Perpendicular to β through E cuts α at P, Q. Then $(P, Q, E, V_\infty) = -1$ and $(E, F, A, U_\infty) = -1$ imply that FG is the polar of E WRT α $\Longrightarrow$ cross ratio (C, D, E, G) is harmonic, hence by Newton's theorem $BG^2 = BE^2 = BC \cdot BD$.

Headhunter
#3 Dec 10, 2010, 1:43 pm

to luisgeometria.

Many thanks.

Quick Reply

High School Olympiads

Plane passes through a fixed point [Iran Second Round 1989]

Reply

Amir Hossein

#1 Dec 6, 2010, 3:16 pm • 1

A sphere S with center O and radius R is given. Let P be a fixed point on this sphere. Points A, B, C move on the sphere S such that we have $\angle APB = \angle BPC = \angle CPA = 90^\circ$. Prove that the plane of triangle ABC passes through a fixed point.

Luis González

#2 Dec 7, 2010, 4:39 am

Ortogonal projections O_A, O_B, O_C of O onto the faces PBC, PCA, PAB of the right trihedron $P(ABC)$ are clearly the midpoints of BC, CA, AB. If M, N, L denote the midpoints of PA, PB, PC, then $MO_BOO_CPLO_AN$ is a cuboid $\Longrightarrow$ Diagonals LO_C and OP bisect each other at Q and plane $ABC \equiv O_AO_BO_C$ passes through the midpoint D of OL. Let the trace of ABC on the plane $OLPO_C$ cut OP at E. Since OQ and O_CD are medians of $\triangle OLO_C, E$ is centroid of $\triangle OLO_C$. Thus, $\overline{EO} : \overline{EP} = -1 : 2 \Longrightarrow E$ is fixed on the plane ABC.

Quick Reply

High School Olympiads

Pelletier's triangle

jayme
#1 Dec 5, 2010, 2:37 pm

Dear Mathlinkers,
prove that the axis of perspectivity betwenn the Pelletier's triangle of ABC and the triangle ABC is tangent to the incircle of ABC.
Sincerely
Jean-Louis

vittasko
#2 Dec 6, 2010, 1:00 pm

The side-segments of the **Pelletier triangle** pass through the vertices of the given triangle $\triangle ABC$ as well.

This result is also true for every pair of **Cevian triangles** (instead of the pair of the orthic and the intouch triangles here) and it is easy to prove applying the **Pappos theorem**.

So, let $\triangle DEF,\ \triangle XYZ$ be, the orthic and intouch tiangles of $\triangle ABC$ respectively and we denote the points $P\equiv EF\cap YZ$ and $Q\equiv DF\cap XZ$ and $R\equiv DE\cap XY.$

The line segments $PQ,\ PR,\ QR,$ pass through the points $C,\ B,\ A,$ respectively (in my drawing, $BC > AC > AB$).

Because of the triangles $\triangle ABC,\ \triangle PQR$ are perspective at point so be it $K\equiv AP\cap BQ\cap CR,$ based on the **Desarques theorem**, we have that the points $S\equiv AB\cap PQ$ and $T\equiv BC\cap QR$ and $V\equiv AC\cap PR$ are collinear.

This line segment, is the axis of perspectivity as the problem states, of the triangles $\triangle ABC,\ \triangle PQR$ and because of the point $Q\equiv AT\cap CS$ as the point of intersection of the diagonals of the quadrilateral $ASTC,$ lies on the line segment XZ which connecting the tangent points of the circle (I) to $AS,\ CT,$ applying the **Newton theorem**, we conclude that ST tangents to (I) and the proof is completed.

Kostas Vittas.

Attachments:
t=380947.pdf (9kb)

jayme
#3 Dec 6, 2010, 5:12 pm

Dear Mathlinkers,
from whom comes this result?
Sincerely
Jean-Louis

jayme
#4 Dec 6, 2010, 5:32 pm

Dear Kostas and Mathlinkers,
nice proof...
mine is based on my article "les deux points de Schroeter"
http://perso.orange.fr/jl.ayme vol. 2 p. 15-21
so that I can specifie the nature of the point of contact.
I will remastered this article next.
Sincerely
Jean-Louis

Luis González

#5 Dec 6, 2010, 8:43 pm

The problem can be generalized as follow:

P and Q are two arbitrary points on the plane of $\triangle ABC.$ $\triangle P_aP_bP_c$ and $\triangle Q_aQ_bQ_c$ are the cevian triangles of P and Q WRT $\triangle ABC,$ respectively. $A' \equiv P_bP_c \cap Q_bQ_c$ and B', C' are defined cyclically. Then $\triangle ABC, \triangle A'B'C'$ are perspective and the perspective axis is tangent to the two inconics of $\triangle ABC$ with perspectors P and $Q.$

vittasko

#6 Dec 7, 2010, 9:50 pm

> *vittasko wrote:*
>
> The side-segments of the **Pelletier triangle** pass through the vertices of the given triangle $\triangle ABC$ as well. This result is also true for every pair of **Cevian triangles** (instead of the pair of the orthic and the intouch triangles here) and it is easy to prove applying the **Pappos theorem**.

GENERAL PROBLEM. - A triangle $\triangle ABC$ is given and let $\triangle DEF,\ \triangle D'E'F'$ be, the cevian triangles of $\triangle ABC,$ with respect two arbitrary points $K,\ K'$ respectively, inwardly to it. Prove that the triangles $\triangle ABC,\ \triangle PQR$ are perspective, where $P \equiv EF \cap E'F',\ Q \equiv DF \cap D'F',\ R \equiv DE \cap D'E'$ and also that the sidelines of $\triangle PQR,$ pass through the vertices of $\triangle ABC$ (one per one).

PROOF. - Let be the points $P \equiv EF \cap E'F',\ Q \equiv DF \cap CP,\ R \equiv DE \cap BP$ and we denote the points $X \equiv AB \cap CP,\ Y \equiv AC \cap BP.$

We consider the triads of points $B,\ D,\ C$ and $E,\ P,\ F,$ on the line segments $BC,\ EF$ respectively and based on the **Pappos theorem**, we have that the points $Q,\ A \equiv BF \cap CE,\ R,$ are collinear.

From the complete quadrilateral $AEKFBC$ and because of the harmonic pencil $E.AFKD$ intersected from the line segment $BR,$ we have that the points $B,\ P,\ Y,\ R,$ are in harmonic conjugation.

Similarly, because of the harmonic pencil $F.AEKD$ intersected from the line segment $CQ,$ we have that the points $C,\ P,\ X,\ Q,$ are also in harmonic conjugation.

So, we conclude that the line segment $QR,$ passes through the point $Z \equiv BC \cap XY.$

• We denote the points $Q' \equiv F'D' \cap CP,\ R' \equiv D'E' \cap BP$ and by the same way as before, we can say that the points $Q',\ A,\ R'$ are collinear and also that the line segment $Q'R',$ passes through the point $Z \equiv BC \cap XY.$

Hence, we conclude that $Q'R' \equiv QR \Longrightarrow Q' \equiv Q$ and $R' \equiv R$ and then, we have that the sidelines of $\triangle PQR,$ pass through the vertices of $\triangle ABC,$ one per one.

Because of now, the collinearity of $X \equiv AB \cap PQ,\ Y \equiv AC \cap PR$ and $Z \equiv BC \cap QR,$ based on the **Desarques theorem**, we conclude that the triangles $\triangle ABC,\ \triangle PQR$ are perspective and the proof of the general problem is completed.

• The **Pelletier triangle** as the proposed problem states, is a particular case of the above general problem, considering the orthic triangle and the intouch triangle, as the cevian ones of $\triangle ABC,$ with respect to its ortocenter and incenter respectively.

Kostas Vittas.

Attachments:

t=380947(a).pdf (8kb)

Quick Reply

High School Olympiads

Point in triangle, looks simple enough ...

Reply

Merlinaeus
#1 Dec 6, 2010, 2:29 am

... but I can't do it.

Let P be an internal point of triangle ABC and let α, β, γ be defined by

$\alpha = BPC - BAC$

$\beta = CPA - CBA$

$\gamma = APB - ACB$

Prove that
$PA * sin(BAC)/sin(\alpha) = PB * sin(CBA)/sin(\beta) = PC * sin(ACB)/sin(\gamma)$

Merlin

Luis González
#2 Dec 6, 2010, 3:04 am

Lines AP, BP, CP cut circumcircle (O) of $\triangle ABC$ again at A', B', C'. Then it's easy to figure out that $\angle PBC' = \angle BPC - \angle BAC = \alpha$. Similarly, we have $\angle PAB' = \gamma$ and $\angle PCA' = \beta$.

By sine law in triangles $\triangle PBC', \triangle PAB'$ and $\triangle PCA'$, we obtain

$$\frac{PB}{PC'} = \frac{\sin A}{\sin \alpha}, \ \frac{PA}{PB'} = \frac{\sin C}{\sin \gamma}, \ \frac{PC}{PA'} = \frac{\sin B}{\sin \beta}$$

But from power of P WRT (O), we have $PA \cdot PA' = PB \cdot PB' = PC \cdot PC'$

Hence, combining with the three previous expressions yields

$$PA \cdot \frac{\sin A}{\sin \alpha} = PB \cdot \frac{\sin B}{\sin \beta} = PC \cdot \frac{\sin C}{\sin \gamma}$$

Quick Reply

High School Olympiads

The line t through the centroid of triangle ABC

Reply

Source: Baltic Way 2006

WakeUp

#1 Dec 5, 2010, 1:35 am

Let the medians of the triangle ABC intersect at point M. A line t through M intersects the circumcircle of ABC at X and Y so that A and C lie on the same side of t. Prove that $BX \cdot BY = AX \cdot AY + CX \cdot CY$.

dgreenb801

#2 Dec 5, 2010, 2:51 am

Let t meet AB and BC at P and Q, respectively.
Let PQ meet the line AC at Z.
We have to show
$BX \cdot BY = AX \cdot AY + CX \cdot CY$
$BX \cdot BY \cdot sinXBY = AX \cdot AY \cdot sinXAY + CX \cdot CY \cdot sinXCY$
$2[BXY] = 2[AXY] + 2[CXY]$
$1 = \frac{[AXY]}{[BXY]} + \frac{[CXY]}{[BXY]}$
$1 = \frac{AP}{PB} + \frac{CQ}{QB}$

By Menelaus,
$\frac{ZT}{ZA} \cdot \frac{AP}{PB} \cdot 2 = 1$ and
$\frac{ZC}{ZT} \cdot \frac{1}{2} \cdot \frac{BQ}{QC} = 1$
Thus we find $\frac{AP}{PB} = \frac{ZA}{2ZT}$ and $\frac{CQ}{QB} = \frac{ZC}{2ZT}$
So $\frac{AP}{PB} + \frac{CQ}{QB} = \frac{ZA}{2ZT} + \frac{ZC}{2ZT}$, and we have to show this equals 1, or $ZA + ZC = 2ZT$.
But $ZA + ZC = (ZA - AT) + (ZC + CT) = 2ZT$, which completes the proof.

This post has been edited 1 time. Last edited by dgreenb801, Dec 6, 2010, 5:04 am

Luis González

#3 Dec 5, 2010, 3:17 am

Let N be the midpoint of AC and A', B', C', N' the orthogonal projections of A, B, C, N onto line t. Segment NN' becomes the median of the trapezoid $ACC'A'$, thus $NN' = \frac{1}{2}(AA' + CC')$. But from $\triangle MBB' \sim \triangle MNN'$ we get

$$\frac{BB'}{NN'} = \frac{BM}{NM} = 2 \Longrightarrow \ BB' = AA' + CC' \ (\star)$$

Let R be the circumradius of $\triangle ABC$. Using that the altitudes and circumdiameters issuing from the same vertices are isogonals, we obtain

$$BX \cdot BY = 2R \cdot BB' \ , \ AX \cdot AY = 2R \cdot AA' \ , \ CX \cdot CY = 2R \cdot CC'$$

Combining these latter expressions with $(\star)$ yields

$$\frac{BX \cdot BY}{2R} = \frac{AX \cdot AY}{2R} + \frac{CX \cdot CY}{2R} \Longrightarrow BX \cdot BY = AX \cdot AY + CX \cdot CY.$$

littletush

#4 Nov 10, 2011, 10:19 am

let XY intersect AB, BC at S, T respectively.
then $\frac{BX*BY}{CX*CY}=\frac{BT}{TC}, \frac{BX*BY}{AX*AY}=\frac{BS}{SA}$
it suffices to prove $\frac{AS}{SB}+\frac{CT}{TB}=1$
which is trivial

---

duanKHTN

#5 Feb 27, 2013, 9:51 pm

> *dgreenb801 wrote:*
> Let t meet AB and BC at P and Q, respectively.
> Let PQ meet the line AC at Z.
> We have to show
> $BX \cdot BY = AX \cdot AY + CX \cdot CY$
> $BX \cdot BY \cdot sinXBY = AX \cdot AY \cdot sinXAY + CX \cdot CY \cdot sinXCY$
> $2[BXY] = 2[AXY] + 2[CXY]$
> $1 = \frac{[AXY]}{[BXY]} + \frac{[CXY]}{[BXY]}$
> $1 = \frac{AP}{PB} + \frac{CQ}{QB}$
>
> By Menelaus,
> $\frac{ZT}{ZA} \cdot \frac{AP}{PB} \cdot 2 = 1$ and
> $\frac{ZC}{ZT} \cdot \frac{1}{2} \cdot \frac{BQ}{QC} = 1$
> Thus we find $\frac{AP}{PB} = \frac{ZA}{2ZT}$ and $\frac{CQ}{QB} = \frac{ZC}{2ZT}$
> So $\frac{AP}{PB} + \frac{CQ}{QB} = \frac{ZA}{2ZT} + \frac{ZC}{2ZT}$, and we have to show this equals 1, or $ZA + ZC = 2ZT$.
> But $ZA + ZC = (ZA - AT) + (ZC + CT) = 2ZT$, which completes the proof.

There is one other way to solve the problem
Let AM meet the line BC at K
By Menelaus for triangle ABK
easy to see that to prove

Quick Reply

High School Olympiads

AE,EF,FC make a triangle with an angle 60

Reply

sororak
#1 Dec 4, 2010, 2:49 am

Let $ABCD$ be a square. We choose points M,N on segments AB,BC, respectively, such that $BM=CN$. Let E,F be the meet points of segments DM,DN with AC, respectively. Prove that segments AE,EF,FC are the lengths of the sides of a triangle with an angle 60°.

Luis González
#2 Dec 4, 2010, 3:38 am

Let $O \equiv DB \cap AC$. By Menelaus' theorem for $\triangle BCO$ cut by DFN, we get

$$\frac{DO}{DB}\cdot\frac{BN}{NC}\cdot\frac{CF}{FO}=1 \Longrightarrow \frac{BN}{AB-BN}\cdot\frac{CF}{OC-CF}=2$$

Substituting $OC=\frac{\sqrt{2}}{2}AB$ into the latter one yields: $\frac{BN}{AB}=\frac{\sqrt{2}AB-2CF}{\sqrt{2}AB-CF}$

Similarly, we'll have the expression $\frac{BM}{AB}=\frac{\sqrt{2}AB-2AE}{\sqrt{2}AB-AE}$

$$BM+BN=AB \Longrightarrow \frac{\sqrt{2}AB-2CF}{\sqrt{2}AB-CF}+\frac{\sqrt{2}AB-2AE}{\sqrt{2}AB-AE}=1$$

$$\Longrightarrow 2AB^2-2\sqrt{2}\cdot AB(AE+CF)+3\cdot AE\cdot CF=0$$

Substituting $AB=\frac{\sqrt{2}}{2}AC=\frac{\sqrt{2}}{2}(AE+EF+CF)$ gives

$$(AE+EF+CF)^2-2(AE+EF+CF)(AE+CF)+3\cdot AE\cdot CF=0$$

$$\Longrightarrow EF^2=AE^2+CF^2-AE\cdot CF$$

Hence, lengths EF,AE,CF form a triangle with angle 60° against EF.

oneplusone
#3 Dec 5, 2010, 7:25 am

$$\frac{AE}{EF+FC}+\frac{CF}{FE+EA}=\frac{AM}{DC}+\frac{CN}{AD}=1$$

Manipulating gives

$$EF^2=AE^2+CF^2-AE\cdot CF$$

Quick Reply

High School Olympiads

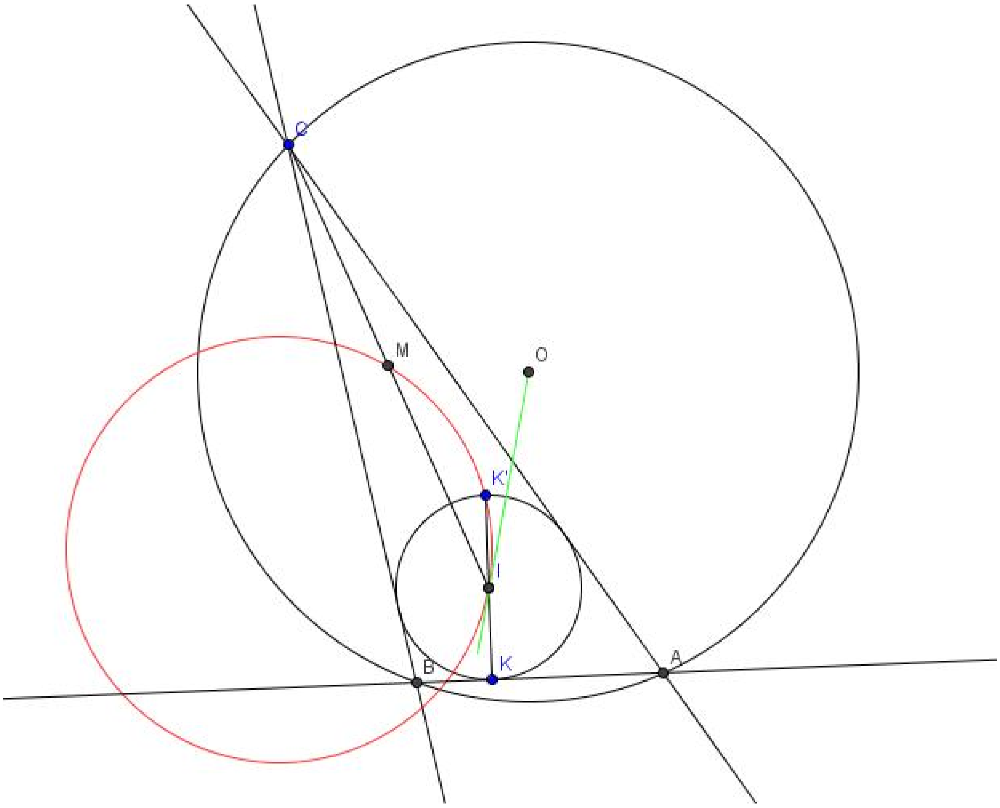
Line OI

Reply

skytin
#1 Dec 4, 2010, 1:19 am

Incircle in triangle ABC tangent to AB at point K , and has incenter at point I . Point K' is on line KI and KI = IK' . M is midpoint of IC , O is circumcenter of ABC , prove that IO is tangent to (K'IM)

Attachments:



Luis González
#2 Dec 4, 2010, 2:18 am • 1

Let (I) touch AC, CB at $L, N.$ Inversion WRT (I) takes Euler line IO of $\triangle NLK$ into itself and takes midpoint of IA, IB, IC into the reflections D, E, F of I about $LK, KN, NL.$ Since antipodes K', N', L' of K, N, L WRT (I) are double, it follows that circle $\odot(IMK')$ is taken into the line $FK'.$ Hence, $\odot(IMK')$ is tangent to $IO \Longleftrightarrow IO \parallel FK'.$ Indeed, congruent $\triangle N'L'K' \cong \triangle DEF$ with parallel sidelines are homologous under a translation with direction $DN' \parallel EL' \parallel FK'.$ But since Euler lines of $\triangle DEF, \triangle N'L'K'$ and $\triangle NLK$ coincide, then $FK' \parallel IO.$

lym
#3 Dec 4, 2010, 10:24 am

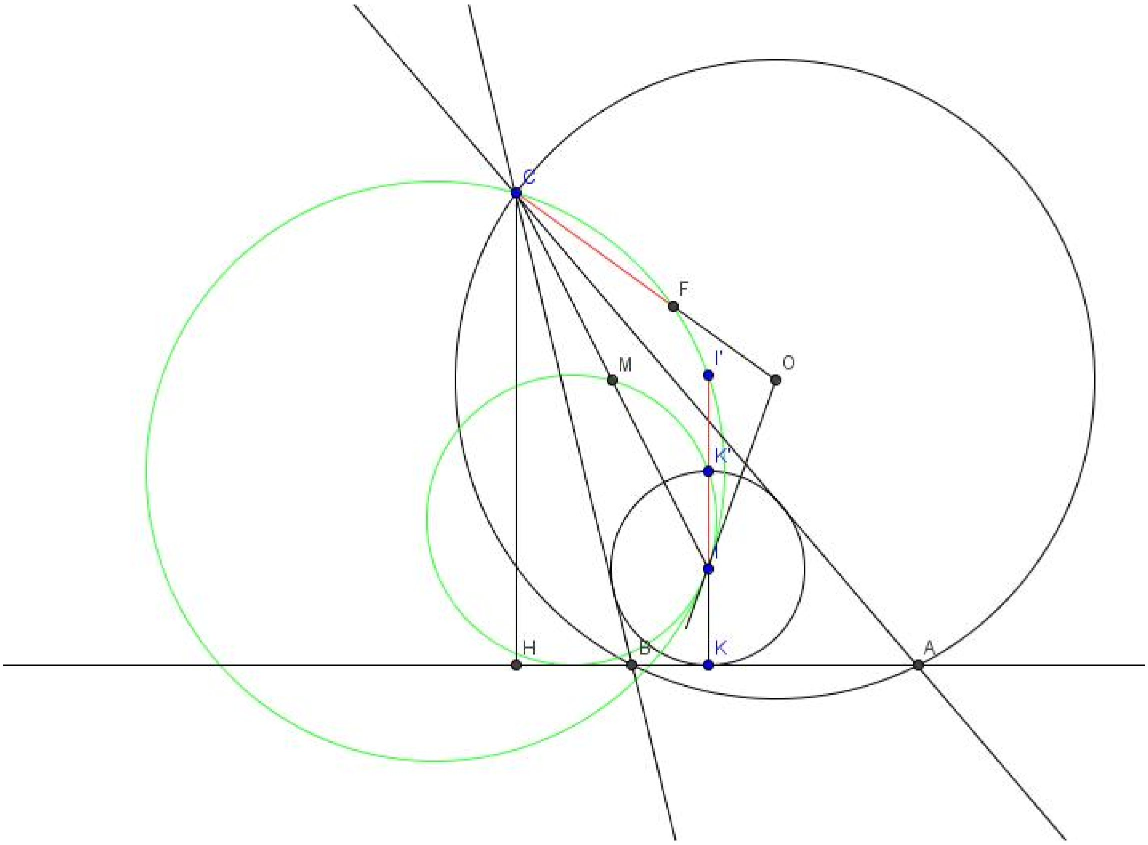
Let H be the orthocenter of inner tangented triangle then we just need to prove $\triangle KIH \sim \triangle IMK'$. It follows $r^2 = KH \cdot IM$

skytin
#4 Dec 4, 2010, 7:01 pm • 1

Solution :

Let I' is reflection of point I wrt K' . Easy to see after Homotety that (I'IC) is tangent to (K'IM) . Let CH is height from C on BA . Well known that angle HCI = ICO , so angle I'IC = ICO . Let (I'IC) intersect CO at point C and F . Easy to see that CF = II' = 2r , so power of point O wrt (I'IC) = OF*OC = (R - 2r)*R = OI^2 (Euler formula) , so IO is tangent to (I'IC) , so IO is tangent to (K'IM) . done

Attachments:



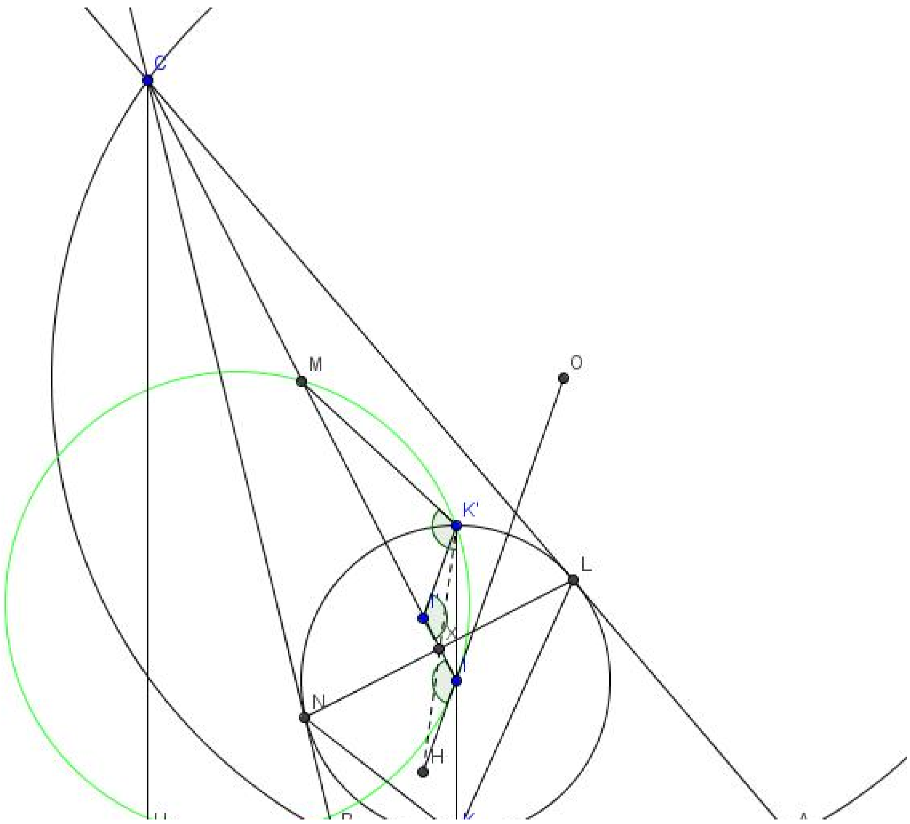
skytin

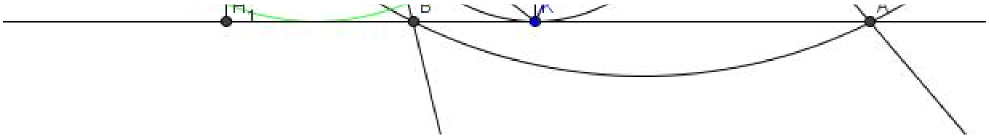
#5 Dec 4, 2010, 7:17 pm

Second solution .

Let H is orthocenter of triangle KNL and X is midpoint of NL . Easy to see that X is midpoint of HK' . Let I' is reflection of point I wrt X . Easy to see that II'*IM = IX*IC = IN^2 = IK'^2 , so angle MK'I = II'K' = I'IH , so IO is tangent to K'IM . done

Attachments:





mathVNpro

#6 Dec 6, 2010, 2:38 am • 1

> Quote:
>
> Incircle in triangle ABC tangent to AB at point K , and has incenter at point I . Point K' is on line KI and KI = IK' . M is midpoint of IC , O is circumcenter of ABC , prove that IO is tangent to (K'IM)

Let r be the radii of incricle (I) and H be the orthocenter of the tagency triangle of $\triangle ABC$. Since the *9-point circle* is the image of (O) through the inversion $\mathcal{I}(I, k = r^2)$. Therefore, O also lies on the *Euler line* of the tangency triangle of $\triangle ABC \Longrightarrow H \in OI$. Now, $\mathcal{I}(I, k)$ also maps K' to itself, and M to M'- where M' is the reflection of I across B' and C'- where B' and C' are the tangency points of (I) with CA and AB respectively. Thus, $\mathcal{I}(I, k) : (IK'M) \mapsto K'M'$. However, note that both (I, M') and (H, K') are two pairs of points which are symmentry wrt to the midpoint of $B'C' \Longrightarrow IHM'K'$ is parallogram. Or in other words, $IH \parallel K'M'$, which implies the result of the problem. $\square$

Quick Reply