High School Olympiads

Circumcentre, Incentre, Intersection of Diagonals Collinear

Locked

Source: Czech-Polish-Slovak 2005 Q2

djb86

#1 Apr 28, 2013, 12:25 am • 1

A convex quadrilateral $ABCD$ is inscribed in a circle with center O and circumscribed to a circle with center I. Its diagonals meet at P. Prove that points O, I and P lie on a line.

Luis González

#2 Apr 28, 2013, 12:35 am

Use the search before posting contest problems

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=31693
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=226758&
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=302934

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High School Olympiads

Five points on a circle define a rectangular triangle

Locked

Source: Czech-Polish-Slovak 2006 Q1

djb86

#1 Apr 27, 2013, 11:20 pm

Five points A, B, C, D, E lie in this order on a circle of radius r and satisfy $AC = BD = CE = r$. Prove that the orthocentres of triangles ACD, BCD, BCE form a rectangular triangle.

Luis González

#2 Apr 28, 2013, 12:28 am

Use the search before posting contest problems

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High School Olympiads

Two triangles have the same Euler circle

Reply

Source: Romania TST 2013 Test 2 Problem 2

Drytime

#1 Apr 26, 2013, 9:44 pm

The vertices of two acute-angled triangles lie on the same circle. The Euler circle (nine-point circle) of one of the triangles passes through the midpoints of two sides of the other triangle. Prove that the triangles have the same Euler circle.

EDIT by pohoatza (in concordance with Luis' PS): Alternate/initial version

This post has been edited 3 times. Last edited by bluecameal, Mar 13, 2016, 10:33 pm

Luis González

#2 Apr 27, 2013, 2:31 am • 2

Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles with the same circumcircle $(O,R).$ $(N,\frac{R}{2})$ is 9 point circle of $\triangle ABC$ and assume that midpoints E',F' of $\overline{C'A'},\overline{A'B'}$ are on $(N).$ There are only two distinct circles with radius $\frac{1}{2}R$ passing through E',F', namely, $\odot(A'E'F),$ because $\triangle A'E'F'$ is image of $\triangle A'B'C'$ under homothety $(A',\frac{1}{2}),$ and the 9-point circle of $\triangle A'B'C' \Longrightarrow (N)$ coincides with the 9-point circle of $\triangle A'B'C'.$

P.S. $\triangle ABC, \triangle A'B'C'$ is a poristic family with fixed circumcircle (O) and MacBeath inconic (conic with focus O and pedal circle (N)).

mathbuzz

#3 Apr 27, 2013, 12:59 pm • 1

suppose both the triangles have the unit circle centered at origin as their circumcircle.
suppose a,b,c be the vertices of the first triangle and that of the 2nd one are d,e,f
suppose the nine point circle of the first triangles contains the midpoints of 2 sides DE,DF
then the equation of the nine point circle of the 1st triangle is $\mid z-(a+b+c)/2=1/2$
then $\mid d+e-a-b-c\mid=1$ and $\mid d+f-a-b-c\mid=1.$ suppose $a+b+c=d+e+f+p$. then $\mid f+p\mid=1$ and $\mid e+p\mid=1.$ so , $p=0$ or ,$p=-e-f$ .note that , $p=-e-f$ implies $a+b+c=d$, ,which implies that , the orthocentre of ΔABC is D ,which is impossible. so , $p=0$,so orthocentre of both the triangles is the same. so , the nine point centre and the radius of the nine point circle of both triangles is the same. hence done

iarnab_kundu

#4 Apr 27, 2013, 3:30 pm

> *mathbuzz wrote:*
> then $\mid f+p\mid=1$
> and $\mid e+p\mid=1.$ so , $p=0$ or ,$p=-e-f$.

I'm really sorry, but I don't understand why.

Aiscrim

#5 Apr 24, 2014, 12:20 am

Perform an inversion that leaves circle (O) invariant. The hypothesis reduces to the following: there are given two triangles, which share the same incircle and 5 out of 6 vertices lie on the same circle. By Poncelet porism, the sixth vertice will also lie on the circle, which (by 'undoing' the inversion) proves that the third midpoint of the second triangle also lie on the 9 point circle, meaning that the two triangle share the same Euler circle.

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High School Olympiads

Similar Isosceles Triangles Parallels

Reply

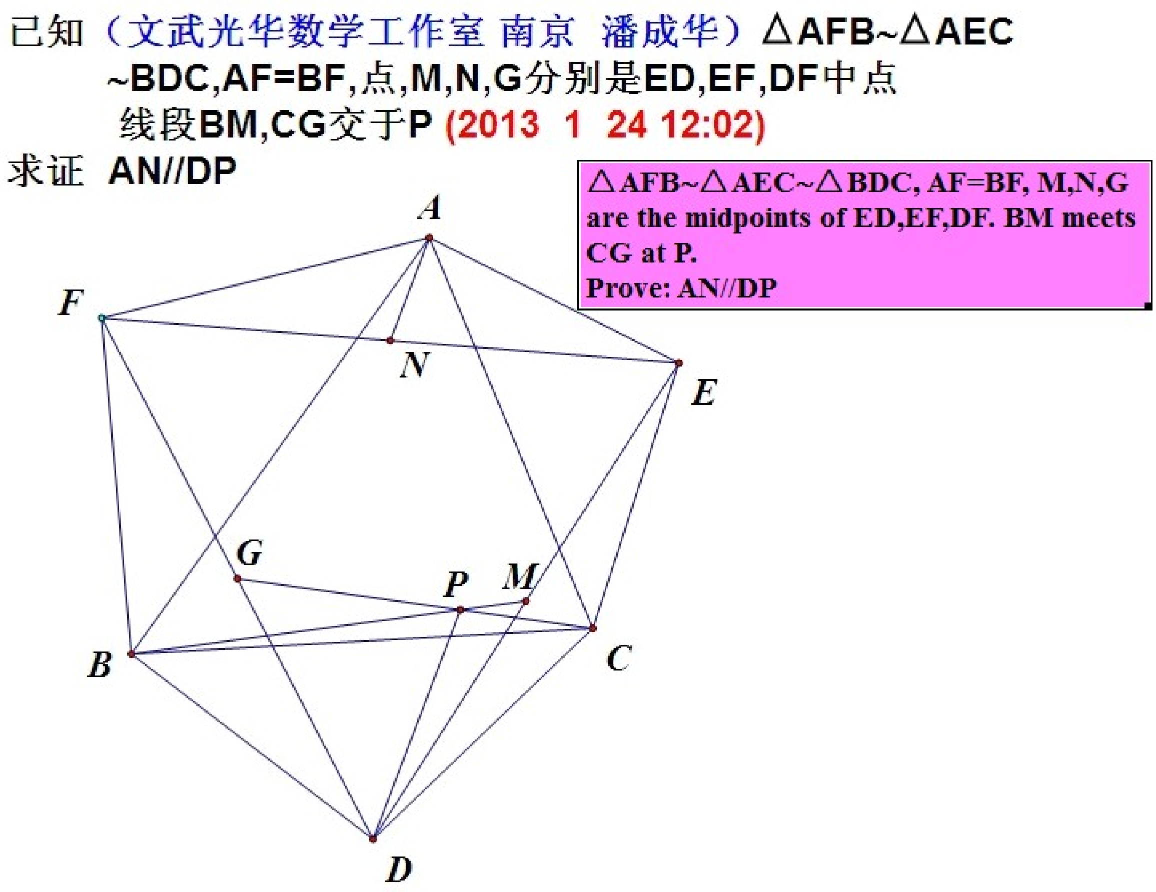
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□□

XmL

#1 Jan 25, 2013, 10:12 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Apr 25, 2013, 8:57 am • 1

MG is D-midline of $\triangle DEF$, meeting DN at its midpoint Y. X is the midpoint of $\overline{BC}$ and U the reflection of D about X. Since $\angle ABC = \angle FBU$ and $\frac{AB}{BC} = \frac{FB}{DB} = \frac{FB}{BU}$, then $\triangle ABC \sim \triangle FBU$ by SAS $\Longrightarrow \frac{FU}{AC} = \frac{FB}{AB} = \frac{AE}{AC} \Longrightarrow FU = AE$. Similarly $EU = AF$, thus $AEUF$ is a parallelogram $\Longrightarrow U \in AN$.

Since $\angle BAE = \angle CAF$ and $\frac{AB}{AC} = \frac{AF}{AE}$, i.e. $AB \cdot AE = AC \cdot AF$, it follows that $[BAE] = [CAF]$. By Kiepert theorem, AD, BE, CF concur at a point K, hence

$$\frac{[BAE]}{[BED]} = \frac{AK}{KD} = \frac{[CAF]}{[CFD]} \Longrightarrow [BED] = [CFD] \Longrightarrow [BMD] = [CGD] \Longrightarrow$$

$$[BPD] + [MPD] = [CPD] + [GPD] \Longrightarrow$$

$$[MPD] - [GPD] = [CPD] - [BPD] \Longrightarrow 2[YDP] = 2[XDP] \Longrightarrow DP \parallel XY.$$

But XY is the D-midline of $\triangle DNU \Longrightarrow DP \parallel XY \parallel UN \equiv AN$.

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High School Olympiads

cyclic quadrilateral

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AndrewTom

#1 Apr 24, 2013, 2:20 am

The sides of the cyclic quadrilateral $ABCD$ satisfy $AD + BC = AB$. Prove that the lines bisecting ADC and BCD meet on AB.

Luis González

#2 Apr 24, 2013, 2:49 am • 1

Posted before. It's IMO 1985 (Q1) and also IBMO 1994 (Q2).

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=16634
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High School Olympiads

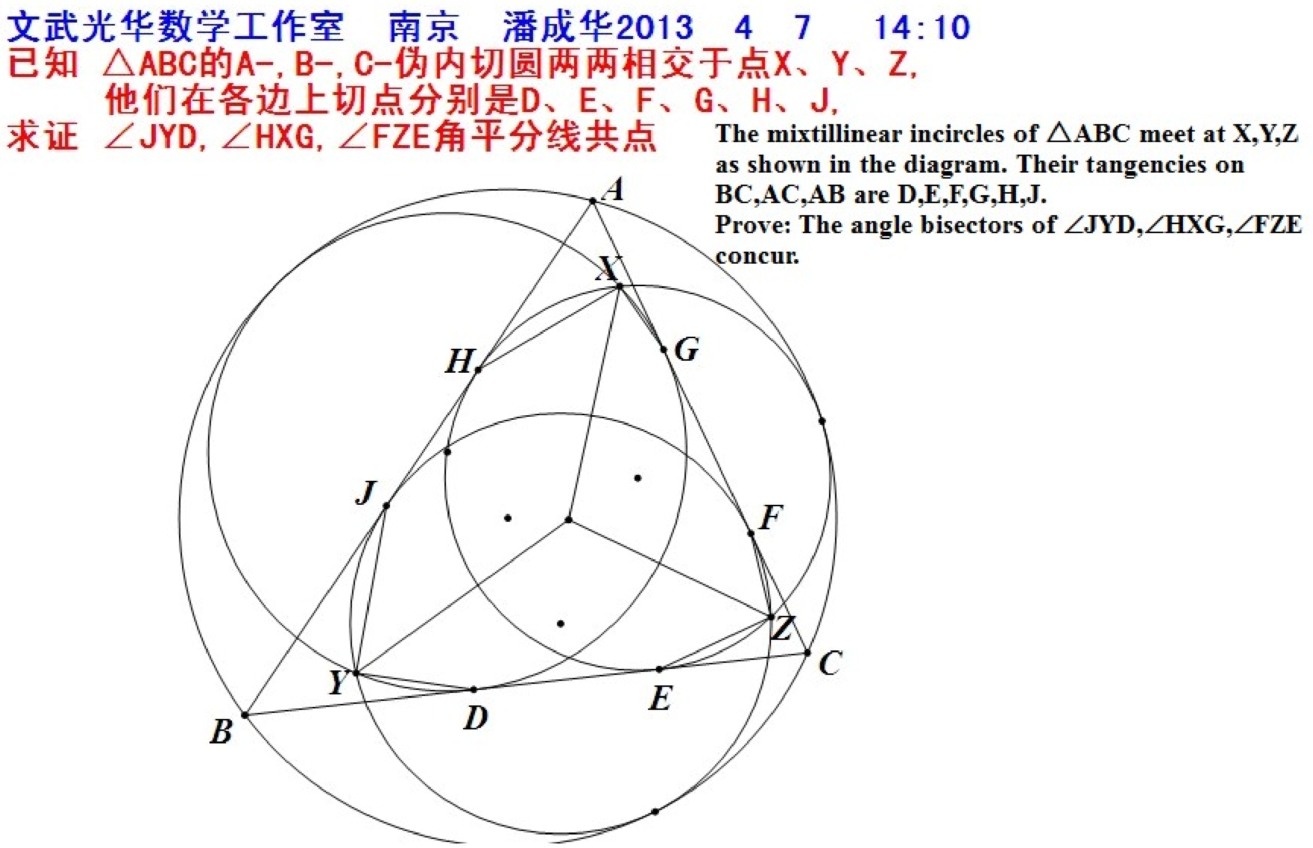
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Apr 9, 2013, 8:27 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Apr 24, 2013, 12:25 am • 2

It's well-known that the incenter I of $\triangle ABC$ is the common midpoint of $\overline{FJ}, \overline{EH}$ and $\overline{DG}.$ Let M be the midpoint of the arc BC of the circumcircle $(O).$ If the B- and C- mixtilinear incircles touch (O) at $V,W,$ then VW,BC,UM concur at their exsimilicenter $S,$ which is also center of inversion that transforms then into each other (see the 1st two paragraphs of the solution of the problem Mixtilinear Incircles Parallels). Further, $S \in JF$ (see Concurrent). Since $JF \perp IM$ is tangent of the circumcircle (M) of $\triangle BIC,$ we get then $SI^2=SB \cdot SC=SV \cdot SW=SX^2 \Longrightarrow X$ lies on I-Apollonius circle (S,SI) of $\triangle BIC.$ Now, keeping in mind that ID,IE are isogonals WRT $\angle BIC,$ due to $\angle BIE=\angle CID=90^\circ,$ we deduce then

$$\frac{XB^2}{XC^2}=\frac{IB^2}{IC^2}=\frac{BD}{DC} \cdot \frac{BE}{EC},$$

which means that XD,XE are isogonals WRT $\angle BXC,$ i.e. $\angle BXE=\angle CXD.$ Since XI,XB are the median and symmedian of $\triangle XEH$ issuing from $X \Longrightarrow \angle BXE=\angle IXH.$ Similarly XI,XC are the median and symmedian of $\triangle XDG$ issuing from $X \Longrightarrow \angle CXD=\angle IXG.$ Therefore $\angle IXG=\angle IXH,$ i.e. IX bisects $\angle HXG$ internally. Analogously, internal bisectors of $\angle JYD$ and $\angle FZE$ pass through $I.$

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High School Olympiads

Problems for Pre-Olympiad

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Jayjayniboon

#1 Apr 22, 2013, 2:00 pm

1. Let ABC be a triangle, with P and Q arbitrary points on CA, AB respectively. Let PQ meet the circumcircle of ABC at X and Y. Prove that the midpoints of BP, CQ, PQ and XY are concyclic.

2. The incircle ω of $\triangle ABC$ touches sides AB and BC at F and D, respectively. Segments AD and CF meet ω at H and K, respectively. Prove that $\frac{FD \times HK}{FH \times DK} = 3$.

3. In a right angled-triangle ABC, $A\hat{C}B = 90^\circ$. Its incircle O meets BC, AC, AB at D, E, F respectively. AD cuts O at P. If $B\hat{P}C = 90^\circ$, prove $AE + AP = PD$

4. Circles C_1 and C_2 are externally tangent at M, and radius of C_2 is greater than radius of C_1. A is a point on C_2 which does not lie on the line joining the centers of the circles. Let B and C be points on C_1 such that AB and AC are tangent to C_1. Lines BM and CM intersect C_2 again at E and F, respectively. Let D be the intersection of tangent to C_2 at A and line EF. Show that the locus of D as A varies is straight line.

5. In a cyclic quadrilateral $ABCD$, let E be the intersection of AD and BC (so that C is between B and E), and F be the intersection of AC and BD. Let M be the midpoint of side CD, and $N \neq M$ be a point on the circumcircle of triangle ABM such that $\frac{AM}{MB} = \frac{AN}{NB}$. Show that E, F, N are collinear.

6. Let ABC be a triangle with $\angle A < 60^\circ$. Let X and Y be the points on the sides AB and AC respectively, such that $CA + AX = CB + BX$ snd $BA + AY = BC + CY$. Let P be the point in the plane such that the lines PX and PY are perpendicular to AB and AC respectively. Prove that $\angle BPC < 120^\circ$

7. Let ABC be an acute-angled triangle with $|AB| < |AC|$, altitudes AD, BE, CF and orthocentre H. Let P be the intersection of BC and EF, M be the midpoint of BC and Q be the intersection of the circumcircle of MBF and MCE.
(a) Prove that $\angle PQM = 90^\circ$.
(b) Conclude that P, H, Q are collinear.
(c) Let ω be the circle passing through B, C, E, F. What are the images each of the points $A, B, C, D, E, F, P, Q, M$ the midpoints of AB, BC, CA and the midpoints of AH, BH, CH under the inversion about ω?

8. Let ABC be a triangle with circumcircle ω. Let C_A be the circle tangent to AB and AC and internally tangent to ω, touching ω at A'. Define B', C' analogously. Prove that AA', BB', CC' are concurrent.

Luis González

#2 Apr 23, 2013, 2:15 am • 1

Please, use the search before posting. All these problems were already discussed.

1) http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=487476
2) http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=380153
3) http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=361852
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7) http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=4919
8) http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=202363

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High School Olympiads

Mixtillinear Incircle Concurrence

Reply

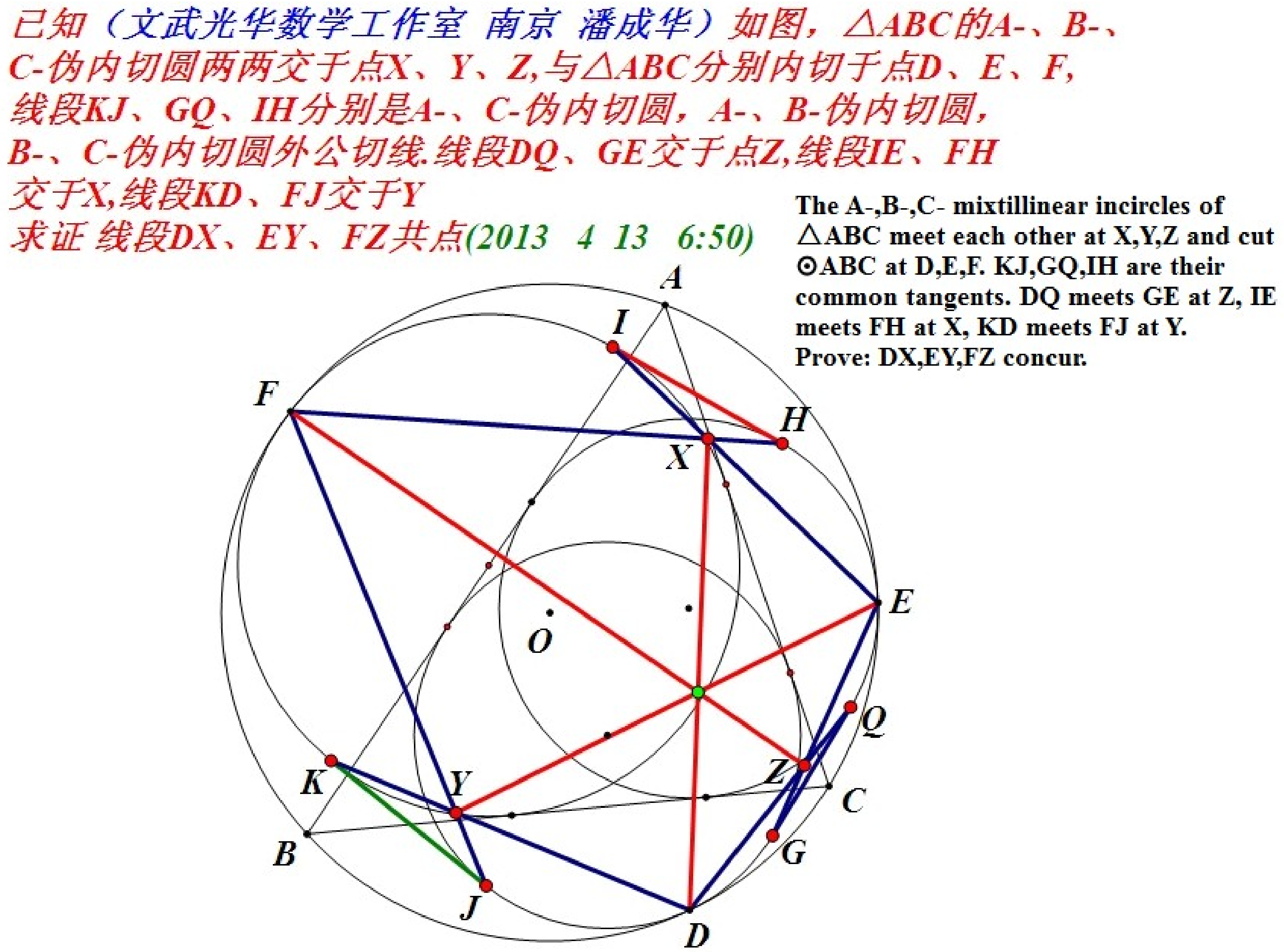
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□□

XmL

#1 Apr 14, 2013, 12:06 pm

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Apr 22, 2013, 10:59 am • 1

Label $\omega_A, \omega_B, \omega_C$ the mixtilinear incircles of $\triangle ABC$. ω_B, ω_C touch BC at M, N. From the internal tangency of the circumcirle (O) with ω_B, ω_C, we deduce that FN, EM bisect $\angle BFC, \angle BEC \Longrightarrow A_1 \equiv FN \cap EM$ is the midpoint of the arc BC of (O). Moreover, $A_1B^2 = A_1M \cdot A_1E = A_1N \cdot A_1F \Longrightarrow EFNM$ is cyclic and A_1 is on radical axis UV of ω_B, ω_C, where $\{U, V\} \equiv \omega_B \cap \omega_C$. For the same reason, $A_2 \equiv UV \cap EH \cap FI$ is on (O).

DE, DN and DM, DF are pairs of isogonals WRT $\angle BDC$ (see On mixtilinear incircles 2), hence it follows that $\angle MDN = \angle EDF = \angle EA_1F \equiv \angle MA_1N \Longrightarrow DMNA_1$ is cyclic. Therefore, DA_1, EF and $MN \equiv BC$ concur at the radical center A_c of $(O), \odot(EFNM)$ and $\odot(DMNA_1)$. But E, F are the exsimilicenters of $(O) \sim \omega_B$ and $(O) \sim \omega_C$, so A_c is nothing but the exsimilicenter of $\omega_B \sim \omega_C$, which is also center of the inversion that transforms these circles into each other.

In this inversion, D goes to A_1 and U, V are double points $\Longrightarrow A_0, U, V, D$ lie on the inverse circle of UV. But if A_2D cuts IH at L, we have $A_2U \cdot A_2V = A_2D \cdot A_2L \Longrightarrow L \in \odot(A_0UVD) \Longrightarrow L$ is the inverse of the midpoint $W \equiv UV \cap IH$ of $\overline{IH} \Longrightarrow A_0H \cdot A_0I = A_0L \cdot A_0W \Longrightarrow (H, I, L, A_0) = -1 \Longrightarrow$ intersection X of the diagonals EI, FH of the complete quadrilateral $EFIH$ is on DA_2, in other words $D(E, F, X, A_0) = -1$. Similarly, if FD, DE cut CA, AB at B_0, C_0 we have $E(F, D, Y, B_0) = -1$ and $F(D, E, Z, C_0) = -1$. Since A_0, B_0, C_0 are collinear on a homothety axis τ of $\omega_A, \omega_B, \omega_C$, then it follows that DX, EY, FZ concur at the trilinear pole of τ WRT $\triangle DEF$.

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High School Olympiads

Ratio between parallel segments in an isosceles triangle

Source: Nordic MO 2011 Q2

djb86

#1 Apr 21, 2013, 10:13 pm

In a triangle ABC assume $AB = AC$, and let D and E be points on the extension of segment BA beyond A and on the segment BC, respectively, such that the lines CD and AE are parallel. Prove $CD \geq \frac{4h}{BC}CE$, where h is the height from A in triangle ABC. When does equality hold?

Luis González

#2 Apr 22, 2013, 5:12 am

$AE \geq h$ and $BC = BE + CE \geq 2\sqrt{BE \cdot CE} \Longrightarrow BC^2 \geq 4 \cdot BE \cdot CE$

$$\Longrightarrow \frac{4h}{BC} \cdot CE \leq \frac{AE}{BE} \cdot BC = \frac{CD}{BC} \cdot BC = CD.$$

Equality obviously holds when $AE = h, BE = CE$, i.e. E is midpoint of $\overline{BC}$.

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High School Olympiads

Right-angled trapezoid ABCD

Source: Sharygin First Round 2013, Problem 14

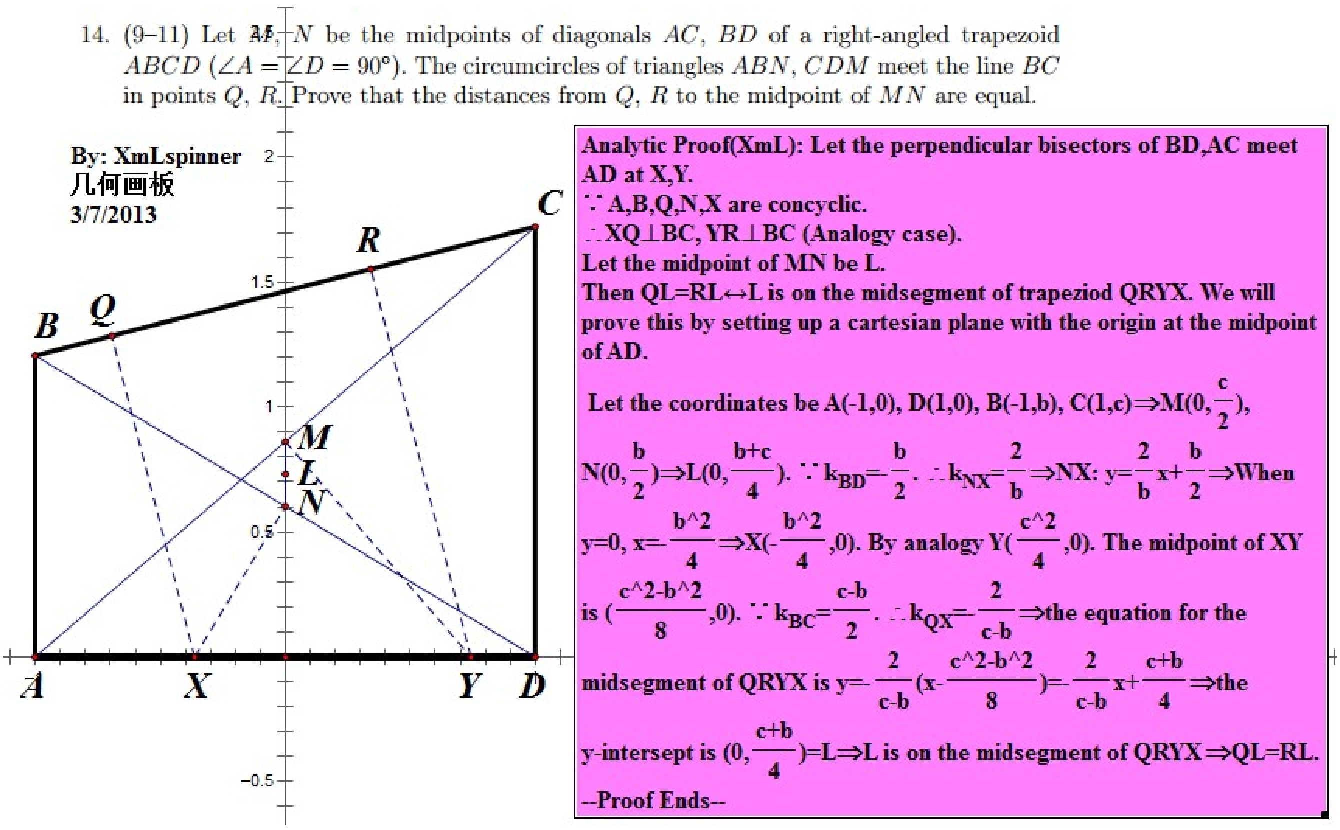
LarrySnake
#1 Apr 5, 2013, 1:07 am

Let M, N be the midpoints of diagonals AC, BD of a right-angled trapezoid $ABCD$ ($\measuredangle A = \measuredangle D = 90^\circ$).
The circumcircles of triangles ABN, CDM meet the line BC in the points Q, R.
Prove that the distances from Q, R to the midpoint of MN are equal.

XmL
#2 Apr 5, 2013, 1:40 am

See attachment for my solution

Attachments:



This post has been edited 1 time. Last edited by XmL, Apr 7, 2013, 10:32 pm

yetti
#3 Apr 7, 2013, 3:43 pm

Let $[AB] = 2a$, $[BC] = 2b$, $[CD] = 2c$, $[DA] = 2d$ and WLOG, $a < c$. Let (O_1, r_1), (O_2, r_2) be circumcircles of N-, M-isosceles $\triangle ABN, \triangle CDM$.
Putting coordinate origin at midpoint G of $[MN]$, centroid of trapezoid $ABCD$, and positive x-axis along ray $(AD$, line BC and perpendicular to BC through G have equations:
$y = \dfrac{c-a}{d} \cdot x + \frac{1}{2}(c+a)$ and $y = -\dfrac{d}{c-a} \cdot x$, respectively. x-coordinate of foot K of perpendicular from G to BC is then
$x_K = -\dfrac{d}{2} \cdot \dfrac{c^2-a^2}{d^2+(c-a)^2} = -\dfrac{d}{2} \cdot \dfrac{c^2-a^2}{b^2}$.

Circumcircle $(O_1, r_1) \equiv \odot(ABN)$ has equation $(x + r_1)^2 + \left(y + \frac{1}{2}(c - a)\right)^2 = r_1^2$. Substituting equation of BC for y yields quadratic equation for x-coordinates of A, Q:

$x^2 + 2r_1 \cdot x + \left(\dfrac{c - a}{d} \cdot x + c\right)^2 = 0 \implies x^2 + (\ldots) \cdot x + \dfrac{c^2 d^2}{b^2} = 0$. Since x-coordinate $x_A = -d$ of A is known, x-coordinate of Q is the other root, or $x_Q = -\dfrac{dc^2}{b^2}$.

x-coordinate of R is obtained similarly, or just by replacing $c \longleftrightarrow a$ and $-d \longleftrightarrow +d$ in formula for $x_Q \implies x_R = +\dfrac{da^2}{b^2}$.

x-coordinate of midpoint of $[QR]$ is then $\frac{1}{2}(x_Q + x_R) = -\dfrac{d}{2} \cdot \dfrac{c^2 - a^2}{b^2} = x_K \implies K$ is midpoint of $[QR] \implies [GQ] = [GR]$.

mathreyes

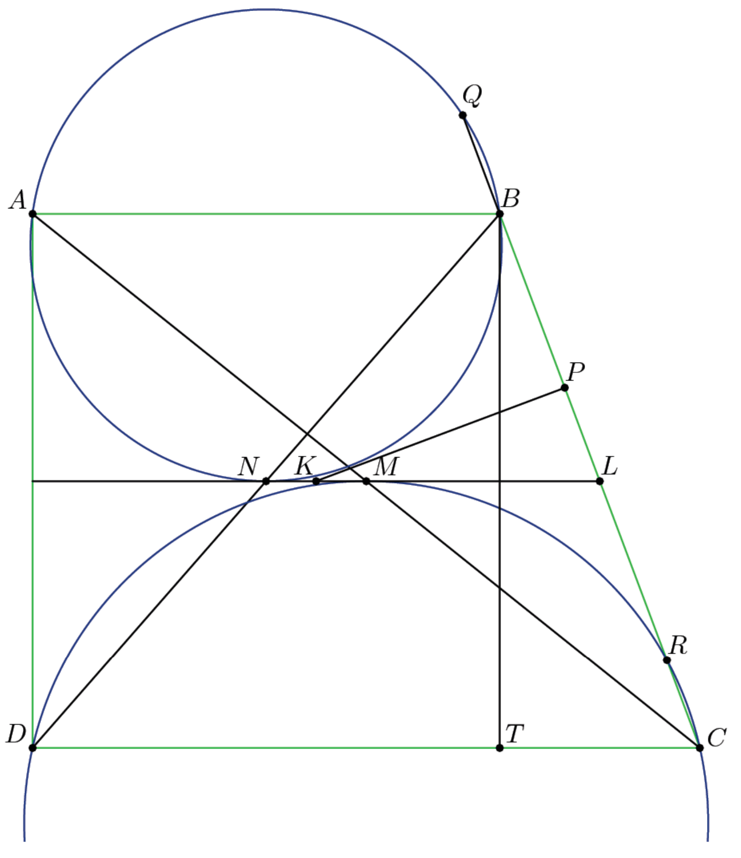
#4 Apr 7, 2013, 8:20 pm

There is a more geometrical approach once you note that midpoint of $\overline{MN}$ is in fact mid point of $\overline{UV}$ where U is midpont of $\overline{AD}$ and V is midpoint of $\overline{BC}$.

v_Enhance

#5 Apr 7, 2013, 10:21 pm

Okay, so I guess it's safe to discuss.



Let $AB = 2x$, $CD = 2y$, and assume without loss of generality that $x < y$. Let L be the midpoint of BC and denote $BC = 2\ell$. Let P be the midpoint of QR. Let T be the foot of B on DC.

Since N is the midpoint of the hypotenuse of $\triangle ABD$, it follows that $AN = BN$. Since $MN \parallel AB$, we see that MN is tangent to (ABN). Similarly, it is tangent to (BCM).

Noting that $LM = AB$ via $\triangle ABC$, we obtain

$$LR \cdot LC = LM^2 = (AB)^2 = x^2 \implies LR = \frac{x^2}{\ell}$$

Similarly, $LQ = \frac{y^2}{\ell}$. Then,

$$PL = \frac{LQ - LR}{2} = \frac{y^2 - x^2}{2\ell} \text{ and } KL = \frac{ML + NL}{2} = x + y.$$

But then, we find that

$$\frac{KL}{PL} = \frac{\frac{y^2 - x^2}{2\ell}}{x + y} = \frac{y - x}{2\ell} = \frac{TC}{BC}$$

Combined with $\angle KLP = \angle BCT$, we find that $\triangle KLP \sim \triangle BCT$. Therefore, $\angle KPL = \angle BTC = 90\dg$. But P is the midpoint of QR, so $KQ = KR$.

applepi2000

#6 Apr 8, 2013, 7:52 pm

Such a good solution (sketch):

Let E, F be the feet of the perpendiculars from N, M to BC respectively. We will show that $QE = FR$ from which the result follows immediately. We have:

$$\frac{EQ}{FR} = (\frac{EQ}{QN})(\frac{QN}{BN})(\frac{BN}{BD})(\frac{BD}{AD})(\frac{AD}{AC})(\frac{AC}{CM})(\frac{CM}{CR})(\frac{RM}{RF})$$

$$= (\cos EQN)(\frac{\sin QBN}{\sin BQN})(\frac{1}{2})(\frac{BD}{AC})(2)(\frac{\sin CRM}{\sin MCR})(\frac{1}{\cos MRF})$$

$$= (\cos ABD)(\frac{\sin CBD}{\sin ABD})(\frac{BD}{AC})(\frac{\sin ACD}{\sin BCA})(\frac{1}{\cos ACD})$$

By Law of sines, we have $\sin CBD = \frac{(CD)(AD)}{(BC)(BD)}$ and $\sin BCA = \frac{(BA)(AD)}{(BC)(CA)}$. Then then plugging this in and simplifying gives:

$$\frac{EQ}{FR} = \frac{(AB)(AC)^2(AD)^2(BC)(BD)^2(CD)}{(AB)(AC)^2(AD)^2(BC)(BD)^2(CD)} = 1$$

So $EQ = FR$ and the result immediately follows.

Luis González

#7 Apr 22, 2013, 12:51 am

Line MN is midparallel of $AB \parallel CD$, cutting $\overline{BC}, \overline{AD}$ at their midpoints U, V, respectively. Circles $\odot(ABN), \odot(CDM)$ cut AD again at G, H, respectively. QG, RH meet UV at Y, Z, respectively. $\angle BQG$ and $\angle CRH$ are obviously right, i.e. $QRHG$ is a right trapezoid.

$\angle ABN = \angle BAN = \angle ANV \Longrightarrow UV$ is tangent to $\odot(ABN) \Longrightarrow YN^2 = YG \cdot YQ = YV \cdot YU$. Hence, if N' is the reflection of N about Y, it follows that $(U, V, N, N') = -1$. Likewise, if M' is the reflection of M about Z, we have $(U, V, M, M') = -1$. Since M, N are symmetric about the common midpoint E of $\overline{MN}, \overline{UV}$, then we deduce that M', N' are symmetric about $E \Longrightarrow Y, Z$ are symmetric about $E \Longrightarrow E$ is on midparallel of $QG \parallel RH$, in other words, E is on perpendicular bisector of $\overline{QR}$, and the conclusion follows.

leader

#8 Apr 22, 2013, 1:59 am

Y, L, X are midpoints of AB, MN, DC and E, F, G, H, I are projections of A, D, M, N, L on BC. since wwe need that I is the midpoint of RQ and I is already the midpoint of HG we only need $QH = GR$.
since $\angle GRM = \angle XDM$ and $\angle MXD = \angle MGR = 90$ $MGR \sim XDM$ so $RG = DX * MG/MX$. similarly $QH = AY * NH/YN$ since $YN = AD/2 = XM$ we only need $XD * MG = AY * NH$ multiplying by 4 both sides

we only need $DC/AB = DF/AE$ swhich is true since $DCF \sim ABE(\angle ABE = \angle DCF$ and $\angle AEB = \angle DFC = 90)$

Ashutoshmaths

#9 Jul 28, 2014, 3:23 pm • 1

solution

Quick Reply

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High School Olympiads

Geometry Problem

Locked

NLT

#1 Apr 21, 2013, 10:57 pm

Let ΔABC with incircle (I). A circle contact AC, AB and circumscribed circle of ΔABC at P,Q,R respectively . T is intersection of AI and circumscribed circle of ΔABC.Prove The intersection of TR, PQ an BC is a point.

Luis González

#2 Apr 21, 2013, 11:18 pm

This problem and its generalization were already discussed.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=151&t=145105
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=397123
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=399496

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High School Olympiads

Property of Isogonal Conjugates

Reply

ged3.14
#1 Apr 20, 2013, 7:56 pm

Let P be a point in triangle ABC, and let P' be its isogonal conjugate. Prove that $AP \cdot \sin BPC = AP' \cdot \sin BP'C$.

Luis González
#2 Apr 21, 2013, 10:09 pm

D, E, F are the orthogonal projections of P on BC, CA, AB and X, Y, Z are the orthogonal projections of P' on $BC, CA, AB. CP' \perp DE$ and $BP' \perp DF \Longrightarrow \angle(P'B, P'C) = \angle(DF, DE) \Longrightarrow$ $\sin\angle(P'B, P'C) = \sin\angle(DF, DE)$. Similarly $\sin\angle(PB, PC) = \sin\angle(XZ, XY)$.

It's well known that D, E, F, X, Y, Z lie on a same circle (let ϱ be the radius of this circle), hence $\triangle AEF \sim \triangle AZY$. But since the right $\triangle PAF$ and $\triangle P'AY$ are clearly similar, then $AEPF \sim AZP'Y \Longrightarrow$

$$\frac{AP}{AP'} = \frac{EF}{YZ} = \frac{2\varrho \cdot \sin\angle(DF, DE)}{2\varrho \cdot \sin\angle(XZ, XY)} = \frac{\sin\angle(P'B, P'C)}{\sin\angle(PB, PC)}.$$

Quick Reply

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High School Olympiads

tangent line of a ellipse

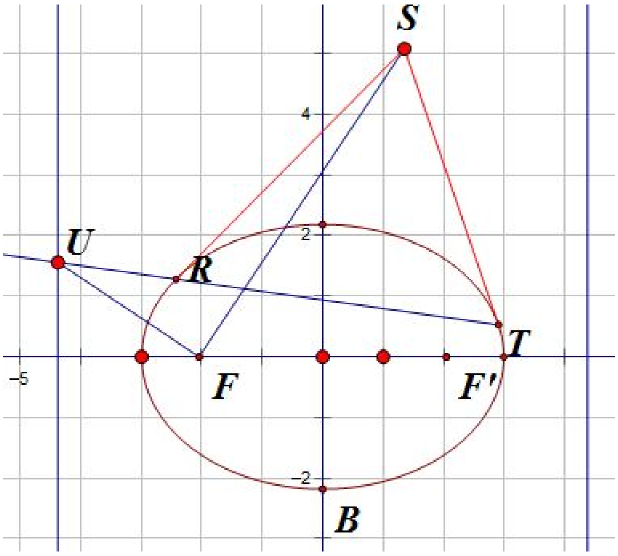
Reply

DANNY123

#1 Apr 20, 2013, 8:37 am

given a ellipse E, the elliptic focus F,and a outside point S, SR and ST are two tangent lines, RT meets the directrix line of ellipse at U. Prove that $< SFU$=90

Attachments:



Luis González

#2 Apr 20, 2013, 9:42 am • 1

Let M, N be the reflections of F on ST, SR. By reflective property, RN and TM pass through F'. Moreover, SF, SF' are isogonals WRT $\angle RST$, hence $\triangle SMN$ becomes isosceles with legs $SF = SM = SN$ and symmetry axis $SF' \perp MN$ $\Longrightarrow \angle SFT = \angle SMF' = \angle SNR = \angle SFR \Longrightarrow FS$ bisects $\angle RFT$ $(\star)$.

Directrix f corresponding to F is polar of F WRT $\mathcal{E}$. Since RT is polar of S WRT $\mathcal{E}$, then $U \equiv f \cap RT$ is pole of SF WRT $\mathcal{E}$ $\Longrightarrow S(R, T, F, U) = -1 \Longrightarrow F(R, T, S, U) = -1$. Together with $(\star)$, FU also bisects $\angle RFT \Longrightarrow FS \perp FU$.

DANNY123

#3 Apr 20, 2013, 9:44 pm

You are right, but how to prove the reflexive property?

Luis González

#4 Apr 21, 2013, 7:32 am • 1

The reflective property of conics is well-known, several proofs can be found on textbooks and internet. Here is one:

We assume the case where our conic is an ellipse $\mathcal{E}$ with foci F, F'. The rest of the cases are treated analogously. τ is a tangent of $\mathcal{E}$ through a point P. We prove that τ bisects $\angle FPF''$ externally.

Let Q be an arbitrary point on τ. M is a point on $\mathcal{E}$ inside $\triangle QFF'$. FM cuts QF' at U. By triangle inequality $QF + QU \geq MF + MU$ and $MU + UF' \geq MF' \Longrightarrow QF + QU + MU + UF' \geq MF + MU + MF' \Longrightarrow$ $QF + QF' \geq MF + MF' = PF + PF' \Longrightarrow P$ is the unique point on ℓ that minimizes the sum $QF + QF'$. But if X is the reflection of F on τ and XF' cuts τ at P^*, we have $QF + QF' = QX + QF' \geq XF' = P^*F + P^*F' \Longrightarrow$ $P \equiv P^* \Longrightarrow \tau$ bisects $\angle FPF'$ externally, as desired.

Quick Reply

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Nice geometry

Reply

Jayjayniboon

#1 Apr 19, 2013, 4:26 pm

1) $ABCD$ is quadrilateral that have incenter I. Points E, F are on BI, DI respectively. Suppose that $2\angle EAF = \angle BAD$. Prove $2\angle FCE = \angle DCA$

This post has been edited 2 times. Last edited by Jayjayniboon, Jul 27, 2013, 10:43 am

Luis González

#2 Apr 20, 2013, 2:47 am • 2

Please do not triple post, use meaningful subjects and search before posting.

1) http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=466300
2) http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=333140
3) http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=46029

Jayjayniboon

#3 Apr 20, 2013, 6:33 pm

Thanks you!

Quick Reply

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perpendicular to bisector

Reply

Source: All-Russian Olympiad 1996, Grade 11, Second Day, Problem 6

micliva

#1 Apr 19, 2013, 3:04 pm

In isosceles triangle ABC ($AB = BC$) one draws the angle bisector CD. The perpendicular to CD through the center of the circumcircle of ABC intersects BC at E. The parallel to CD through E meets AB at F. Show that BE = FD.

M. Sonkin

Luis González

#2 Apr 20, 2013, 12:07 am

Let M, N be the midpoints of $BC, CA.$ $I \equiv BN \cap CD$ is the incenter of $\triangle ABC.$ Since $OE \perp CI,$ the intersection $P \equiv OE \cap CI$ is on circle $\odot(ONCM)$ with diameter $\overline{OC}.$ Moreover P is midpoint of its arc $MON,$ because CP bisects $\angle MCN \Longrightarrow PO$ bisects $\angle MON \Longrightarrow \angle EOM = 90^\circ - \frac{1}{2}\angle MON = \frac{1}{2}\angle BCA \Longrightarrow \angle OEM = \angle CIN \Longrightarrow$ $OECI$ is cyclic $\Longrightarrow \angle OIE = \angle OCE = \angle OBC = \angle OBA \Longrightarrow \triangle BEI$ is E-isosceles and $IE \parallel AB \Longrightarrow IEFD$ is a parallelogram $\Longrightarrow BE = EI = FD.$

shivangjindal

#3 Feb 1, 2014, 7:11 pm

Let $\angle C = 2\alpha$. Let $OB \cap CD = M$. Then $\angle BCM = \alpha$, And $\angle BOE = 180 - (90 + \alpha + 90 - 2\alpha) = \alpha$. Thus quadrilateral $MOCE$ cylic.

Thus
$\angle EMO = 180 - \angle OCE = 180 - \angle OBE \implies \angle BME = \angle MBE \implies ME = BE.$

Since BO bisects $\angle B \implies ABM = \angle BME \implies DF \parallel EM$ Thus $DFEM$ is parellogram. Thus we are done! :)

Quick Reply

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A bisector in the triangle formed by the incircle

Reply

Source: Mediterranean MO 1998 Q3

djb86
#1 Apr 19, 2013, 1:31 pm

In a triangle ABC, I is the incenter and D, E, F are the points of tangency of the incircle with BC, CA, AB, respectively. The bisector of angle BIC meets BC at M, and the line AM intersects EF at P. Prove that DP bisects the angle FDE.

Luis González
#2 Apr 19, 2013, 10:34 pm

Posted several times before at
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=24699
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=110139
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=337049
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=331763 (P437)

jayme
#3 Apr 21, 2013, 4:01 pm

Dear Mathlinkers,
I saw this problem for the first time in
GM, 3 (1991) 113.
Did someone have a photocopy of this problem in GM?
Sincerely
Jean-Louis

Quick Reply

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High School Olympiads

Line BK is tangent to circle (ABD)

Reply

dorina

#1 Apr 19, 2013, 2:29 am

Two circles intersect at point A and B. A line passing through B cuts the first circle at C and the second circle at D.Tangents of the first circle at C and tangents of the second circle at D intersect at *M*. The parallel line with CM, which passes through the intersection of AM and CD cuts AC at *K*. Prove that BK is tangent with the second circle.

Moderator edit: Problem subject edited.

Luis González

#2 Apr 19, 2013, 4:21 am

$\angle MCD = \angle BAC$ and $\angle MDC = \angle BAD \Longrightarrow \angle CAD = \angle MCD + \angle MDC \Longrightarrow \angle CAD = \pi - \angle CMD \Longrightarrow$ $ACMD$ is cyclic $\Longrightarrow \angle ADB = \angle AMC$. But $\angle ABD = \angle ACM \Longrightarrow \triangle ABD \sim \triangle ACM \Longrightarrow \angle DAB = \angle CAP$.

On the other hand, if $P \equiv AM \cap CD$, then $\angle KPC = \angle MCB = \angle KAB \Longrightarrow ABPK$ is cyclic $\Longrightarrow$ $\angle CAP = \angle KBP$, hence $\angle DAB = \angle KBP \Longrightarrow KB$ is tangent to $\odot(ABD)$, as desired.

Quick Reply

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High School Olympiads

prove

Locked

dorina

#1 Apr 19, 2013, 2:34 am

A circle with center O passes through vertices A and C of triangle ABC and cuts the segment AB and BC at points K and N, respectively. The circle circumscribing triangle ABC and KBN intersect at exactly two different points B and M.Prove that OM is perpendicular to MB.

Luis González

#2 Apr 19, 2013, 3:25 am

Firstly, please, give your posts meaningful subjects. Secondly, use the search function before posting contest problems. This IMO 1985 Problem 5.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=549
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=60787
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=84104
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=251993

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High School Olympiads

Romania National Olympiad ,2013,problem 3

ionbursuc

#1 Apr 16, 2013, 10:40 pm

Given P a point m inside a triangle acute-angled ABC and DEF intersections of lines with that AP, BP, CP with$[BC]$, $[CA]$,respective $[AB]$
a) Show that the area of the triangle DEF is at most a quarter of the area of the triangle ABC
b) Show that the radius of the circle inscribed in the triangle DEF is at most a quarter of the radius of the circle circumscribed on triangle $4ABC$.

Luis González

#2 Apr 19, 2013, 12:32 am • 4

Denote ϱ and $p(\triangle DEF)$ the inradius and semiperimeter of $\triangle DEF.$ R denotes the circumradius of $\triangle ABC$ and X,Y,Z the feet of the altitudes on $BC,CA,AB.$ If $(u:v:w)$ represent the barycentric coordinates of P WRT $\triangle ABC,$ then the area of its cevian triangle $\triangle DEF$ is given by

$$\frac{[DEF]}{[ABC]}=\frac{2uvw}{(v+w)(w+u)(u+v)}$$

Since P is inside $\triangle ABC,$ then $u,v,w>0.$ Thus by AM-GM we have

$$v+w \ge 2\sqrt{vw}\ ,\ \ w+u \ge 2\sqrt{wu}\ ,\ \ u+v \ge 2\sqrt{uv} \Longrightarrow$$

$$(v+w)(w+u)(u+v) \ge 8uvw \Longrightarrow \frac{1}{4} \ge \frac{2uvw}{(v+w)(w+u)(u+v)}=\frac{[DEF]}{[ABC]}.$$

Since $\triangle ABC$ is acute, then its orthic triangle $\triangle XYZ$ is the inscribed triangle with the least perimeter (celebrated Fagnano's problem), for a proof see problem 4 in the thread Geometry problems. Hence $p(\triangle XYZ) \le p(\triangle DEF)$ together with the previous $\varrho \cdot p(\triangle DEF)=[DEF] \le \frac{1}{4}[ABC]$ gives

$$\varrho \le \frac{[ABC]}{4 \cdot p(\triangle XYZ)}=\frac{R}{4}.$$

mamangava12345678

#3 Apr 20, 2013, 5:15 pm

Where can I find the results of this National Olympiad?

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High School Olympiads

Prove that ID and AC are parallel

Reply

Source: All-Russian Olympiad 1996, Grade 9, Second Day, Problem 6

micliva

#1 Apr 18, 2013, 8:21 pm

In the isosceles triangle ABC $(AC = BC)$ point O is the circumcenter, I the incenter, and D lies on BC so that lines OD and BI are perpendicular. Prove that ID and AC are parallel.

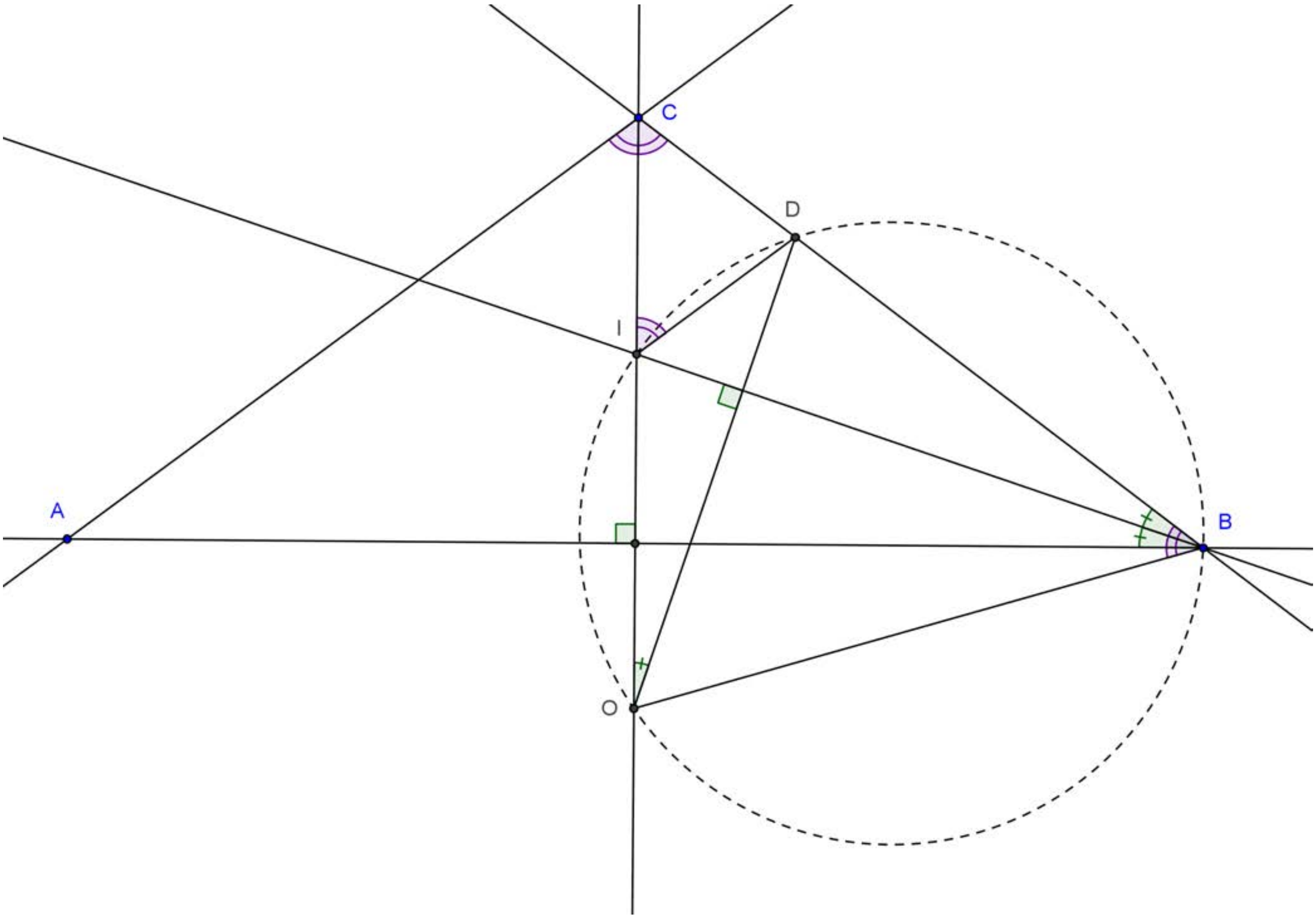
M. Sonkin

Luis González

#2 Apr 18, 2013, 9:01 pm

See the diagram below for a proof without words

Attachments:



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High School Olympiads

a point lies on the euler line

Reply

Lawasu
#1 Apr 18, 2013, 4:14 pm

Let ABC be a triangle inscribed into a circle ω and let M be the reflection of C across AB. Tangent lines to ω in A and B meets in point C'. The line MC' and the tangent to ω in C meets in point K. Prove that K lies on the Euler line of triangle ABC.

Luis González
#2 Apr 18, 2013, 8:05 pm

Equivalent problem
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=291602

Generalizations
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=346956
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=420917
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=495103

XmL
#3 Apr 19, 2013, 5:28 am

Let H, O be the orthocenter and circumcenter of $\triangle ABC \rightarrow HO$ is the euler line. Let $HO \cap CK = K'$. We will prove that K, M, C' are collinear. Since $HM \perp AB, OC' \perp AB$, so $HM \parallel OC'$, which means that K, M, C' are collinear $\leftrightarrow \frac{KH}{KO} = \frac{HM}{OC'}$.
Let X be the midpoint of AB, $CM \cap AB, \odot ABC = Z, Y$. Since $HZ = YZ$, so $HM = HZ + MZ = CZ + YZ = CY$. Since $\frac{KH}{KO} = \frac{[CHK]}{[KCO]} = \frac{sin\angle KCH}{sin\angle KCO} * \frac{CH}{CO} = \frac{sin\angle CBY}{1} * \frac{CH}{R}$, also $OC' = \frac{R^2}{OX}$. So now we just have to prove $sin\angle CBY * \frac{CH}{R} = \frac{CY * OX}{R^2} \leftrightarrow \frac{sin\angle CBY}{CY} * \frac{CH}{OX} = \frac{1}{R}$. Since $\frac{CH}{OX} = 2$ and by the law of sines $\frac{sin\angle CBY}{CY} = \frac{1}{2R}$, So now the equality above becomes $\frac{1}{2R} * 2 = \frac{1}{R}$, which is obviously true. $\square$

duanby
#4 Apr 21, 2013, 8:00 am

My solution:
Let X be the meet point of OH and tangent at A
It's sufficient to prove X,M,C'are collinear
Let Y be the antipodal point of A,AH meet ωat D,N be the foot of H on AO
then sufficient to prove

$$\frac{OC}{HM} = \frac{OX}{HX}$$

that is

$$\frac{2R^2}{AH * AD} = \frac{OX}{HX}$$

that is

$$AH * AD = AN * AY$$

And we have

$$\angle YDH = \angle YNH = 90$$

so done

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High School Olympiads

Prove that I,J,P,H are concyclic

Reply

Source: Iran TST 2013: TST 1, Day 1, Problem 1

mlm95

#1 Apr 17, 2013, 11:44 pm • 1

In acute-angled triangle ABC, let H be the foot of perpendicular from A to BC and also suppose that J and I are excenters oposite to the side AH in triangles ABH and ACH. If P is the point that incircle touches BC, prove that I, J, P, H are concyclic.

Math-lover123

#2 Apr 18, 2013, 12:07 am

Do you mean J,I are excenters of excircles inscribed in angles ABH and ACH ?

Luis González

#3 Apr 18, 2013, 12:22 am

@Math-lover123, no tricky wording here. J is simply the B-excenter of ABH and I is simply the C-excenter of ACH.

The concyclicity is true for any H on BC, not necessarily the foot of the A-altitude as the problem states. See two problems about cyclic quadrilateral (problem 1), incenters and cyclic, etc. The proof is analogous for excircles.

leader

#4 Apr 18, 2013, 12:23 am

nice problem
let X, S be feet from I, J to BC then.
$XP = BX - BP = ((BH + BA + AH) - (BC + BA - CA))/2 = (AH - CH + AC)/2 = HS = SJ$
and similarly $SP = XI$ and since $\angle IXP = \angle JSP = 90$ then $JSP \cong IXP$ so $\angle IPX = \angle SJP = 90 - \angle SPJ$ so $\angle IPJ = 90 = 2 * 45 = \angle IHJ$ therefore $IHPJ$ is cyclic.

This post has been edited 1 time. Last edited by leader, May 4, 2013, 2:14 am

subham1729

#5 Apr 18, 2013, 12:36 am

First of all we've $\frac{Cos(A-B)/2 - CosC/2}{Cos(A-B)/2 - CosA/2} + \frac{CosB}{Cos(B-C)/2} 2SinB/2 = 2CosC/2$. Now note $\frac{S-c}{CI} = \frac{Cos(A-B)/2 - CosC/2}{Cos(A-B)/2 - CosA/2}$ and $\frac{CosB}{Cos(B-C)/2} 2SinB/2 = \frac{CosB/2}{Sin(B+A/2)} \times \frac{BH}{AH} = \frac{Cos(C/2+\alpha)}{Sin(\alpha)}$ now also we had $\frac{Cos(C/2-\theta)}{Sin(\theta)} = \frac{S-c}{CI}$. Where $\angle HIA = \theta, \angle PJC = \alpha$. Now so we've $\frac{Cos(C/2+\alpha)}{Sin(\alpha)} + \frac{Cos(C/2-\theta)}{Sin(\theta)} = 2CosC/2$ and that implies $\frac{Cos(C/2+\alpha)}{Sin(\alpha)} = \frac{Cos(C/2+\theta)}{Sin(\theta)}$ now so we've $Cot(\alpha) = Cot(\theta)$ and certainly that implies $\theta = \alpha$ and so done.

Goutham

#6 Apr 19, 2013, 11:15 pm

In fact, H can be any point on BC. For this, if we denote I', J' as the projections of I, J onto BC, then we have to prove that $II'P \sim PJJ'$. Moreover, assume without loss of generality that $BH < BP$. Then the problem is equivalent to proving that(after a few simplifications)

$$4xybc \sin B \sin C = ((c+t)^2 - x^2)((b+t)^2 - y^2)$$

After using cosine rule and simplifying, we have to prove

$$bc \sin B \sin C = t^2(1+\cos\alpha)(1-\cos\alpha)$$

where $\angle AHB = \cos\alpha$. This is obvious by sine rule in triangles AHB, AHC.

Previously, I had copied the question wrong and got the following result:
In triangle ABC, H is any point on BC and the A-excircle touches BC at P. Also, I, J are the H-excentres of triangle ABH, ACH respectively. Then $IHPJ$ is cyclic.

goodar2006
#7 Apr 21, 2013, 7:52 am

Proposed by Mehdi E'tesami Fard

seledeur
#8 Aug 26, 2013, 12:38 am

> *leader wrote:*
> nice problem
> let X, S be feet from I, J to BC then.
> $XP = BX - BP = ((BH + BA + AH) - (BC + BA - CA))/2 = (AH - CH + AC)/2 = HS = SJ$
> and similarly $SP = XI$ and since $\angle IXP = \angle JSP = 90$ then $JSP \cong IXP$ so
> $\angle IPX = \angle SJP = 90 - \angle SPJ$ so $\angle IPJ = 90 = 2 * 45 = \angle IHJ$ therefore $IHPJ$ is cyclic.

Please, why $XP = BX - BP$? Please cite the theorems you used, thank you .

tuvie
#9 Aug 26, 2013, 12:52 am • 1

It is not a theorem, but I'll explain it for you
Since B, P, X lie on BC on this order, i.e, are collinear, it follows that $BX = BP + PX$ and the result follows.

seledeur
#10 Aug 26, 2013, 12:59 am

Oh yeah I copied wrong .. But why $XP = BX - BP = ((BH + BA + AH) - (BC + BA - CA))/2$?

Mathematicalx
#11 Nov 6, 2013, 1:53 pm

Dear tuvie,
Please explain the steps which seledur wrote.Because i did not understand too. Thanks...

IDMasterz
#12 Nov 6, 2013, 6:19 pm

I still cant believe this, I spent an hour solving the wrong problem! If you are confused, the incircle tangency P is actually the excircle tangency... You have to be kidding me.

Here is a quick solution for any point:

Suppose X, Y are the tangent points of $(J), (I)$ with BC. We get

$$PX = BP + BX = \frac{AB + AF - BF - AB + AC + BC}{2} = \frac{AC + AF + FC}{2} = FY.$$

Since $FJX \sim FIY \implies \frac{JX}{FY} = \frac{FX}{IY} \implies \frac{JX}{PX} = \frac{PY}{IY} \implies DJX \sim DIY$ so done.

That was a really really bad typo seriously.

gearss

#13 Nov 11, 2013, 9:04 am

can someone tell me what is excenter?

MichiPanaitescu

#14 Nov 12, 2013, 1:32 pm

@gearss:
Look at the first image from:
http://en.wikipedia.org/wiki/Incircle_and_excircles_of_a_triangle

jayme

#15 Nov 14, 2013, 4:30 pm

Dear Mathlinkers,
for more precision, you can see
http://perso.orange.fr/jl.ayme vol.5 Le theoreme de Feuerbach-Ayme p.10-13
Sincerely
Jean-Louis

shivangjindal

#16 Jan 6, 2014, 11:47 am

Let I', J' be projection of I and J onto line BC. Then , by properties of $Ex-circle$. We see that, $BJ' = \frac{AB+BH+AH}{2}$ and $BP = \frac{AB+BC-AC}{2} \implies PJ' = \frac{AH+AC-HC}{2}$. Now, $II' = \frac{AH \cdot HC}{AC+CH-AH}$. Now using Pythagoras theorem, one see that $PJ' = II'$. Similarly, $PI' = JJ'$. Thus we have $\triangle PII' \cong PJJ'$ Thus $\angle IPJ = 90^\circ$ and easy to see that $\angle IHJ = 90^\circ$. So $IHPJ$ is a cyclic quadrilateral. and II' is its diameter. So we are done! $\square$.

Quick Reply

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Reflection across midpoint

Reply

Source: Moldova TST 2013

mikeshadow

#1 Apr 17, 2013, 6:07 pm

Consider the triangle $\triangle ABC$ with $AB \neq AC$. Let point O be the circumcenter of $\triangle ABC$. Let the angle bisector of $\angle BAC$ intersect BC at point D. Let E be the reflection of point D across the midpoint of the segment BC. The lines perpendicular to BC in points D, E intersect the lines AO, AD at the points X, Y respectively. Prove that the quadrilateral B, X, C, Y is cyclic.

Luis González

#2 Apr 17, 2013, 11:29 pm

M is the midpoint of $\overline{BC}$ and N, L the midpoints of the arcs BC and BAC of the circumcircle (O, R). MN is then D-midline of $\triangle DEY$. From $\triangle DMN \sim \triangle LAN$ and $\triangle ADX \sim \triangle ANO$, we get

$$\frac{2R}{AN} = \frac{DN}{MN}, \ \frac{XD}{R} = \frac{AD}{AN} \Longrightarrow 2 \cdot MN \cdot XD = AD \cdot DN \Longrightarrow$$

$$YE \cdot XD = AD \cdot DN = DB \cdot CD = CE \cdot CD \Longrightarrow \frac{YE}{CD} = \frac{CE}{XD}.$$

The latter ratio means that the right ΔYEC and $\triangle CDX$ are similar by SAS criterion $\Longrightarrow \angle YCE = \angle CXD \Longrightarrow$ $\angle YCX = 90^\circ$. Similarly, $\angle YBX = 90^\circ \Longrightarrow BXCY$ is cyclic.

XmL

#3 Apr 18, 2013, 5:37 am

Let L, M be the midpoints of BC and arc$BC \Rightarrow OLM$ is the perpendicular bisector of BC. Since $XD \perp BC$. So $\angle ADX = \angle AMO = \angle MAO$. Let K be on XD such that $DEYK$ is a rectangle. Since $DL = LE$, so $DM = MY \Rightarrow \angle DKM = \angle XDA = \angle XAD \Rightarrow A, X, M, K$ are concyclic
$\Rightarrow XD * DK = AD * DM = BD * DC \Rightarrow X, B, K, C$ are concyclic, which implies that X, B, K, Y, C are concyclic.
$\square$

First time writing in Latex form!

jayme

#4 Apr 18, 2013, 1:46 pm

Dear Mathlinkers,
after your angle chasing, we can involve the three chords theorem and we are done...
Sincerely
Jean-Louis

MellowMelon

#5 Jun 30, 2013, 8:53 pm

ISL 2012 G4

Quick Reply

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High School Olympiads

Three elipses

Reply

Source: (mpdb)

borislav_mirchev

#1 Apr 7, 2013, 1:23 pm

Let quadrilateral $ABCD$ is circumscribed around an ellipse. Its tangency point with the sides are K, L, M, N. Prove that A, C, K, L, M, N lie on an ellipse and B, D, K, L, M, N lie on an ellipse. What can you say about main lines of these ellipses?

Luis González

#2 Apr 17, 2013, 4:57 am • 1

Label K, L, M, N the tangency points of the conic with AB, BC, CD, DA, respectively. By degenerate Brianchon theorem (or Newton theorem), the lines AC, BD, KM, LN concur at P, i.e. intersections $B \equiv AK \cap CL, P \equiv KM \cap LN$ and $D \equiv MC \cap NA$ are collinear, thus by the converse of Pascal theorem A, C, K, L, M, N lie on a same conic. Similarly, B, D, K, L, M, N lie on a same conic.

As for the last question, I'm not aware of any special property in a pencil through 4 general points K,L,M,N. But if these points are concyclic, the axes are parallel.

borislav_mirchev

#3 Apr 17, 2013, 2:33 pm

For the last question it seems that if $ABCD$ is circumscribed around a circle - they are perpendicular.

Quick Reply

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High School Olympiads

Complete quadrilateral properties

Reply

Source: (mpdb)

borislav_mirchev

#1 Apr 16, 2013, 3:45 am

1. Let $ABCD$ is a convex quadrilateral. E is the intersection point of AB and CD. F is the intersection point of AD and BC. Through B, D, E, F are drawn four parallel lines intersecting AC at the points B_1, D_1, E_1, F_1. Prove that:
$\frac{1}{BB_1}+\frac{1}{FF_1}=\frac{1}{DD_1}+\frac{1}{EE_1}$.

2. http://gogeometry.com/problem/p750-complete-quadrilateral-diagonal-parallel-metric-relations-high-school-college.htm

Luis González

#2 Apr 16, 2013, 12:26 pm • 1

1) The proposed relation does not hold in general.

2) Let AC cut BD, EF at $P, Q,$ respectively. By Gergonne-Euler theorem for $\triangle AEF$ and its interior point $C,$ keeping in mind that (A,C,P,Q) is harmonic, we get

$$\frac{CQ}{AQ}+\frac{CB}{FB}+\frac{CD}{ED}=\frac{CP}{PA}+\frac{CB}{FB}+\frac{CD}{ED}=1.$$

Substituting the ratios $\frac{c}{a}=\frac{CP}{PA}, \ \frac{c}{e}=\frac{CD}{ED}, \ \frac{c}{f}=\frac{CB}{FB} \Longrightarrow \frac{1}{a}+\frac{1}{e}+\frac{1}{f}=\frac{1}{c}.$

borislav_mirchev

#3 Apr 16, 2013, 2:52 pm

For 1) I'm not sure you are correct. Can you give a counter-example?

djb86

#4 May 5, 2013, 10:12 pm • 1

According to my calculations, (1) is correct. I do not have the energy to type a full solution, but here is an outline:

Outline

sunken rock

#5 May 10, 2013, 6:12 pm

For 1), counter-example: take $ABCD$ a kite, and $D_1\equiv B_1$, you get $EE_1=FF_1, BB_1\ne DD_1$.

Best regards,
sunken rock

Quick Reply

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Concurrent lines and colinear points of concurrence

Source: Own

Papangu

#1 Apr 15, 2013, 10:00 am

Let ABC be scalene triangle. Let D, E and F be the points of contact between the incircle and the sides BC, CA and AB, respectively. Let I be the center of the incircle and let Q, R and S be the reflection of D, E and F with relation to AI, BI and CI, respectively. Let M, N and P be the midpoints of BC, CA and AB respectively. Prove that:

i) AM, BN and CP are concurrent.
ii) MQ, NR and PS are concurrent in a point that belong to the incircle.
iii) AQ, BR and CS are concurrent.
iv) The above three points of concurrence are collinear.

Luis González

#2 Apr 15, 2013, 10:57 am

AM,BN,CP concur at the centroid G of $\triangle ABC.$ AQ,BR,CS are the isogonals of the Gergonne cevians, concurring at the isogonal conjugate U of the Gergonne point $AD \cap BE \cap CF,$ which is the insimilicenter of the incircle (I) and circumcircle $(O).$ MQ,NR,PS concur at the Feuerbach point $F_e \equiv \odot(MNP) \cap (I)$ (see Lines concurrent on incircle, Two Homothetic Triangles Related to the Feuerbach point, Concurrency and elsewhere).

G is insimilicenter of $\odot(MNP) \sim (O)$, F_e is exsimilicenter of $\odot(MNP) \sim (I)$ and U is the insimilicenter of $(I) \sim (O)$. By Monge and d'Alembert theorem G, F_e and U are collinear.

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Results in projective geometry

Source: geogebra

peregrinefalcon88

#1 Apr 15, 2013, 2:20 am

Consider a point P in triangle ABC. Let $\triangle A_1B_1C_1$ be the cevian triangle of P with respect to $\triangle ABC$. Let B_1C_1 intersect BC at A_2 and define B_2 and C_2 similarly. It is known that A_2, B_2, and C_2 are collinear. Let A_3 be the midpoint of A_1A_2 and define B_3 and C_3 similarly. It should also be well known that A_3, B_3, and C_3 are collinear. Let circle O_A be the circle centered at A_3 that passes through A_1 and A_2, and define O_B and O_C similarly. It can be shown that these 3 circles have a single radical axis L, which always passes through the circumcenter of $\triangle ABC$. What I am unable to prove, is that L always passes through the orthocenter of $\triangle A_1B_1C_1$, which i am certain is true from geogebra.

Luis González

#2 Apr 15, 2013, 2:55 am • 1

Since (B, C, A_1, A_2) is harmonic, it follows that $\overline{A_3A_1}^2 = \overline{A_3B} \cdot \overline{A_3C} \Longrightarrow$ circle $\mathcal{O}_A$ with diameter $\overline{A_1A_2}$ is orthogonal to the circumcircle (O) of $\triangle ABC$. Similarly, $\mathcal{O}_B$ and $\mathcal{O}_C$ are orthogonal to (O). Since A_3, B_3, C_3 are collinear on the Newton line of the complete quadrangle bounded by BC, CA, AB and the trilinear polar $A_2B_2C_2$ of P, then $\mathcal{O}_A, \mathcal{O}_B$ and $\mathcal{O}_C$ are coaxal with radical axis passing through O.

Let A_4, B_4, C_4 be the projections of A_1, B_1, C_1 on B_1C_1, C_1A_1, A_1B_1. They clearly lie on $\mathcal{O}_A, \mathcal{O}_B, \mathcal{O}_C$, respectively. $T \equiv A_1A_4 \cap B_1B_4 \cap C_1C_4$ is orthocenter of $\triangle A_1B_1C_1$. By orthocenter property $\overline{TA_1} \cdot \overline{TA_4} = \overline{TB_1} \cdot \overline{TB_4} = \overline{TC_1} \cdot \overline{TC_4} \Longrightarrow T$ has equal power WRT $\mathcal{O}_A, \mathcal{O}_B, \mathcal{O}_C \Longrightarrow T$ is on their radical axis.

capu

#3 Apr 15, 2013, 5:00 am

Take the circumcenter of the triangle with vertices B1, C1 and the foot of the perpendicular from C1 to B1A1. Call the circle, Omega. trivially, the orthocenter of A1B1C1 is on the radical axis of O_B and Omega, as well as the radical axis of O_C and Omega. Therefore it's on the radical axis of O_B and O_C and since the center os OB, OC and OA are collinear, the orthocenter is on the radical axis of all 3 circles.

Quick Reply

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colinear

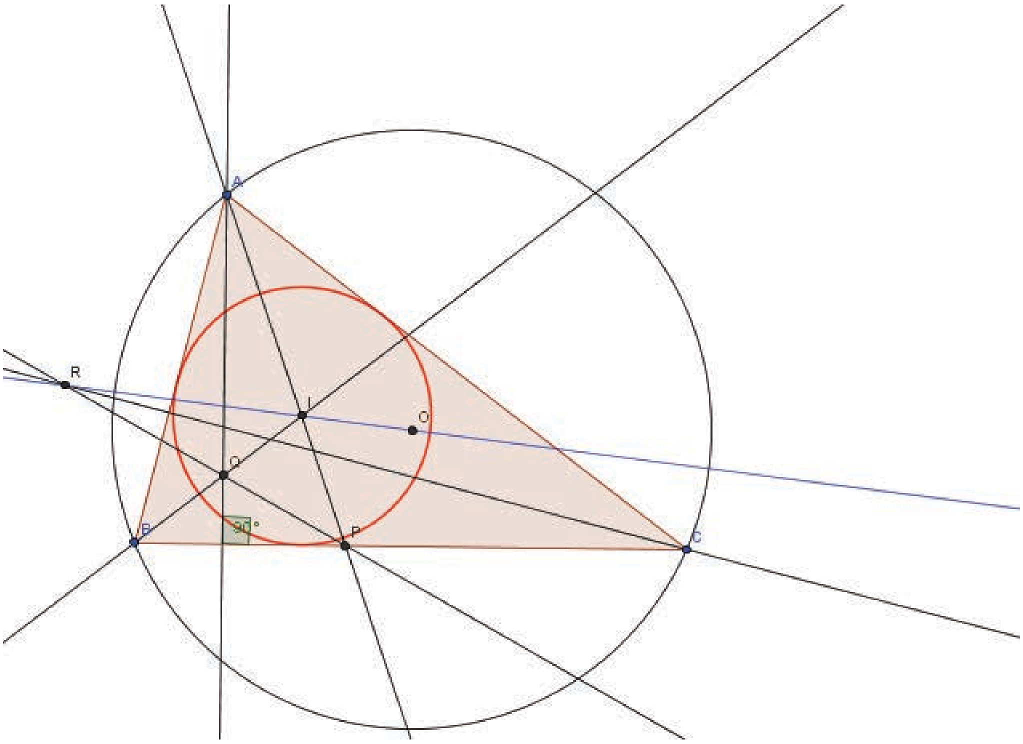
Reply

a00012025

#1 Apr 9, 2013, 9:11 pm

Let ABC triangle with incenter I and circumcenter O,and $P=AI\cap BC$,Q lies on BI such that $AQ\perp BC$, R lies on PQ such that $CR\perp AB$.Prove that O,I and R are colinear

Attachments:



Luis González

#2 Apr 14, 2013, 11:17 am • 2

The problem can be generalized as follows

Theorem: I is incenter of $\triangle ABC$ and P is an arbitrary point on its plane. P^* is isogonal conjugate of P WRT $\triangle ABC$. AI cuts BC at D and BI cuts AP^* at Q. Then IP, QD, CP^* concur.

Dual theorem: G is centroid of $\triangle ABC$ and P is an arbitrary point on its plane. P^* is isotomic conjugate of P WRT $\triangle ABC$. AG cuts BC at D and BG cuts AP^* at Q. Then GP, QD, CP^* concur.

P.S. The proof is rather straightforward using barycentric coordinates.

jayme

#3 May 2, 2013, 4:54 pm

Dear Mathlinkers,
from where comes this problem?
Sincerely
Jean-Louis

TelvCohl

#4 Dec 8, 2015, 10:49 am

> *Luis González wrote:*
>
> **Theorem:** I is incenter of $\triangle ABC$ and P is an arbitrary point on its plane. P^* is isogonal conjugate of P WRT $\triangle ABC.$ AI cuts BC at D and BI cuts AP^* at $Q.$ Then IP, QD, CP^* concur.

Since IP is tangent to the circumconic of $\triangle ABC$ passing through I, P^* at I, so from Pascal theorem (for BCP^*AII) we get $IP \cap CP^*, AI \cap BC \equiv D, AP^* \cap BI \equiv Q$ are collinear. i.e. IP, QD, CP^* are concurrent

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An extension of NMO - Bulgary, 1997.

Reply

Source: 0

Virgil Nicula

#1 Apr 9, 2013, 8:41 pm

PP. Let $\triangle ABC$ with the circumcircle w and $M\in (AC)$, $N\in (AB)$, $D\in MN\cap w$.

Prove that $\boxed{\left|\frac {b}{DB}\cdot\frac {NB}{NA}-\frac {c}{DC}\cdot\frac {MC}{MA}\right|=\frac {a}{DA}}$.

Remark. If $I\in BM\cap CN$ is incenter of $\triangle ABC$, then obtain **P.P.** from **NMO**, **Bulgary 1997**. Study another particular cases.

This post has been edited 2 times. Last edited by Virgil Nicula, Sep 17, 2013, 9:10 pm

Luis González

#2 Apr 13, 2013, 9:57 am

This is equivalent to the generalization mentioned in the topic 1/bx = 1/ax + 1/cx. Letting (p,q,r) be the trilinear coordinates of $BM\cap BN$ WRT ABC, we get

$$\frac{1}{p\cdot DA}=\left|\frac{1}{q\cdot DB}-\frac{1}{r\cdot DB}\right|.$$

Using the relations $\frac{p}{q}=\frac{NB}{NA}\cdot\frac{b}{a}$, $\frac{p}{r}=\frac{MC}{MA}\cdot\frac{c}{a}$, we obtain the desired expression.

Virgil Nicula

#3 Sep 17, 2013, 8:22 pm

See **PP1** from **here** and an easy its extension.

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nice problem

Reply

shekast-istadegi

#1 Apr 11, 2013, 6:59 pm

Let K be a point on perpendicular bisector of BC of acute triangle ABC and K is inside ABC. prove that $\sin(\widehat{KAB}+\widehat{KBC})/\sin(\widehat{KAC}+\widehat{KCB})=AC/AB$.

proposed by J_Shirani

Luis González

#2 Apr 13, 2013, 8:32 am

AK cuts $\odot(ABC)$ again at $P \Longrightarrow \widehat{KCP}=\widehat{PCB}+\widehat{KCB}=\widehat{KAB}+\widehat{KBC}$. Similarly, we have $\widehat{KBP}=\widehat{KAC}+\widehat{KCB}$. Hence

$$\frac{[KCP]}{[KBP]}=\frac{KC\cdot PC\cdot \sin \widehat{KCP}}{KB\cdot PB\cdot \sin\widehat{KBP}}=\frac{PC}{PB}\cdot\frac{\sin(\widehat{KAB}+\widehat{KBC})}{\sin(\widehat{KAC}+\widehat{KCB})} \Longrightarrow$$

$$\frac{\sin(\widehat{KAB}+\widehat{KBC})}{\sin(\widehat{KAC}+\widehat{KCB})}=\frac{PB}{PC}\cdot\frac{PK\cdot PC\cdot\sin\widehat{APC}}{PK\cdot PB\cdot\sin\widehat{APB}}=\frac{\sin\widehat{ABC}}{\sin\widehat{ACB}}=\frac{AC}{AB}.$$

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A result of cevian triangles

Source: My own problem

peregrinefalcon88

#1 Apr 13, 2013, 3:40 am

Given a point P in triangle ABC, define the "cevian conjugate" P' as follows. Let AP, BP, and CP intersect BC, CA, and AB at A_1, B_1, and C_1. Draw lines l_a, l_b, and l_c paralel to B_1C_1, C_1A_1 and A_1B_1 that pass through A, B, and C respectively. Let l_a and l_b intersect at C_2, and define A_2, and B_2 similarly. Let M_A be the point of BC, and label the midpoints of CA and AB similarly. It is well known (i think) that A_2M_A, B_2M_B, and C_2M_C concur at a point that we will call P'. Now, given points R and S in triangle ABC, prove that RS is parallel to $R'S'$.

I will post a solution to this soon, but I was curious what other people thought of the problem in terms of difficulty and potential generalizations.

$[hide="'hint'"]Ipurposelymadetheproblemseemabitobscurebyfindingaseparateresult, usingthatresult, anddeletingtheassociatedlinesfromthediagram.[hide="'hintcontinued'"]Addinthecentroid[/hide][/hide]$

edit 1: My 'well known' claim can be explored in the following 2 links.

http://www.cut-the-knot.org/Curriculum/Geometry/CevaCradle.shtml

http://www.cut-the-knot.org/Curriculum/Geometry/CevaNest.shtml

does anyone know is cevian nest/cradle are the official terms for these ideas?

edit 2: My solution is now attached to this post, its just the solution below written in more detail.

Attachments:

ComplementsThroughCevianTriangles.pdf (255kb)

This post has been edited 1 time. Last edited by peregrinefalcon88, Apr 14, 2013, 9:45 pm

Luis González

#2 Apr 13, 2013, 6:59 am • 2

Your "cevian conjugate" is nothing but the complement of P WRT ABC.

The pencil formed by A_2B, A_2C, A_2M_A and the parallel to BC through A_2 is clearly harmonic $\Longrightarrow$ the pencil formed by the parallels from A_1 to A_2B, A_2C, A_2M_A, BC is also harmonic, but $A_1(C_1, B_1, P, B) = -1 \Longrightarrow PA_1 \parallel A_2M_A$. Similarly, $PB_1 \parallel B_2M_B$ and $PC_1 \parallel C_2M_C \Longrightarrow A_2M_A, B_2M_B, C_2M_C$ concur at the complement P' of P WRT $\triangle ABC$, i.e. P' is on the line connecting P with the centroid G of $\triangle ABC$, such that $\overline{GP'} : \overline{GP} = -1 : 2$.

The second part of the problem is then trivial by the definition of complements.

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Tangent Circles+Centers Concyclic

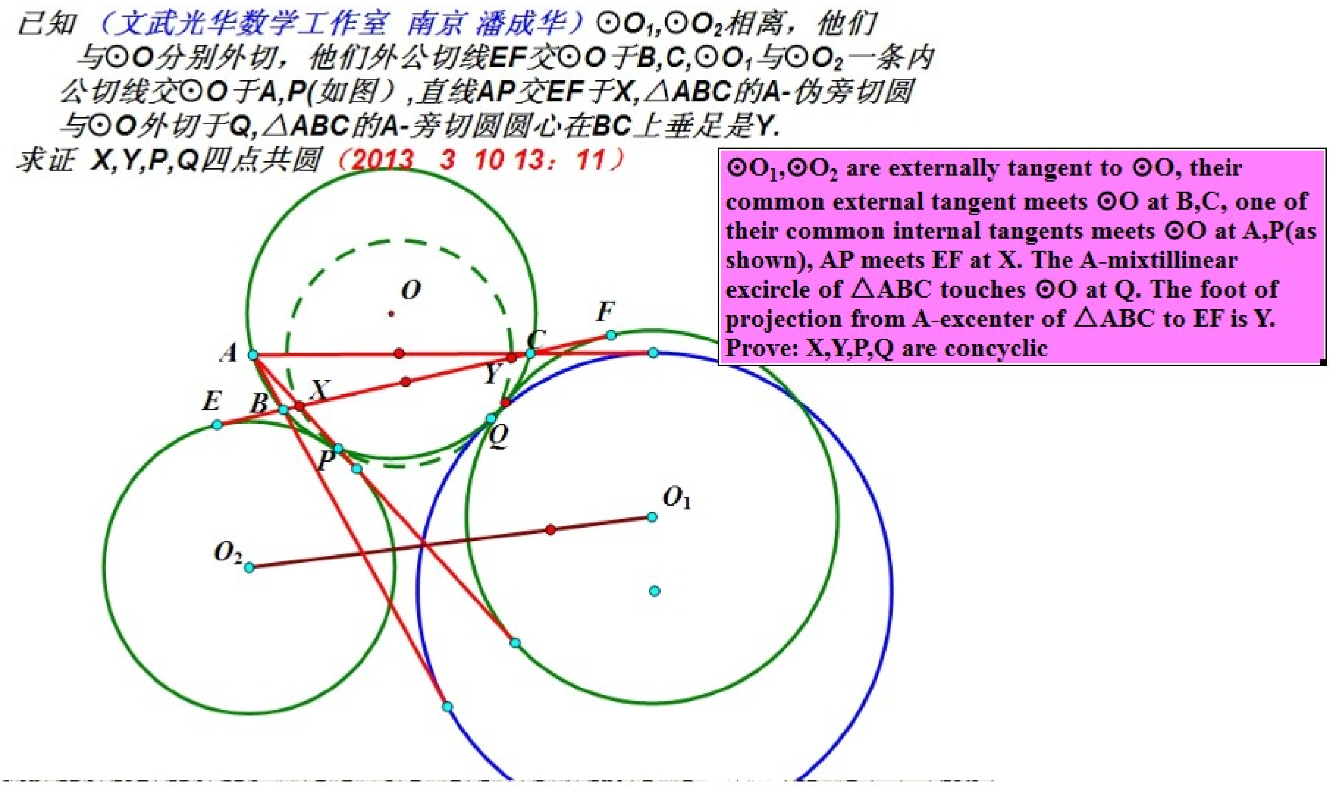
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Mar 29, 2013, 12:04 pm

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Apr 4, 2013, 8:33 am • 1

Take away the circles $(O_1), (O_2)$, they are just distracting. Let X be an arbitrary point on the side BC of $\triangle ABC$. AX cuts the circumcircle (O) of $\triangle ABC$ again at P. A-excircle touches BC at Y and the A-mixtilinear excircle touches (O) at Q. Then P, Q, X, Y are concyclic.

Let Q^* be the 2nd intersection of $\odot(PXY)$ with (O) and YQ^* cuts (O) again at M. By Reim's theorem, $AM \parallel BC \Longrightarrow M$ is fixed $\Longrightarrow Q^*$ is fixed, i.e. $\odot(PXY)$ goes through a fixed point of (O). Now, move X to the tangency point of the incircle (I) with BC. It's known that AX, AQ are isogonals WRT $\angle BAC \Longrightarrow PQ \parallel BC$, and since X, Y are symmetric WRT the midpoint of $\overline{BC}$, then by obvious symmetry $PQYX$ is an isosceles trapezoid $\Longrightarrow Q \in \odot(PXY) \Longrightarrow Q^* \equiv Q$.

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Circumscribed quadrilateral

Reply

Source: (mpdb)

borislav_mirchev

#1 Apr 3, 2013, 8:41 pm • 1

Let $ABCD$ is a circumscribed quadrilateral. E is the intersection point of AB and CD. F is the intersection point of AD and BC. Prove that incircles of the triangles AEF and CEF have a common tangency point with EF.

Luis González

#2 Apr 4, 2013, 12:15 am • 2

Let the incircle of $ABCD$ touch AB, BC, CD, DA at $P, Q, R, S,$ respectively.

$$FC - CE = FQ + CQ - CR - ER = FS + CQ - CQ - EP =$$

$$= FS - EP = FA + AS - EA - AP = FA + AS - EA - AS = FA - EA.$$

But if the incircles $(U), (V)$ of $\triangle AEF, \triangle CEF$ touch EF at $X, Y,$ we have $FX = \frac{1}{2}(FE + FA - EA)$ and $FY = \frac{1}{2}(FE + FC - CE) \Longrightarrow FX = FY,$ i.e. $X \equiv Y \Longrightarrow (U), (V)$ touch EF at the same point.

Quick Reply

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perpendicular bisector of BH

Reply

hEatLove

#1 Apr 2, 2013, 9:54 pm

Let H be orthocenter of triangle ABC and perpendicular bisector of segment BH intersects AB at E and D be the foot of altitude of C to AB . Prove that KD is perpendicular to EO where O is circumcenter and K is midpoint of BH.

Luis González

#2 Apr 2, 2013, 10:37 pm • 2

Let the perpendicular bisector of $\overline{BH}$ cut BC at P. Orthogonal projections of H on BE, BP, EP and orthogonal projections of O on BE, BP (midpoints of BA,BC) and EP are on 9-point circle (N) of $\triangle ABC \Longrightarrow O, H$ have the same pedal circle (N) WRT $\triangle BEP \Longrightarrow O, H$ are isogonal conjugates WRT $\triangle BEP \Longrightarrow EO, EH$ are isogonal lines WRT $\angle BEP \Longrightarrow$ EO is perpendicular to the line DK connecting the projections of H on BE, EP.

hEatLove

#3 Apr 3, 2013, 4:23 pm

CAN YOU EXPLAIN THIS PART? Orthogonal projections of H on BE, BP, EP and orthogonal projections of O on BE, BP (midpoints of BA,BC) and EP are on 9-point circle (N) of $\triangle ABC$

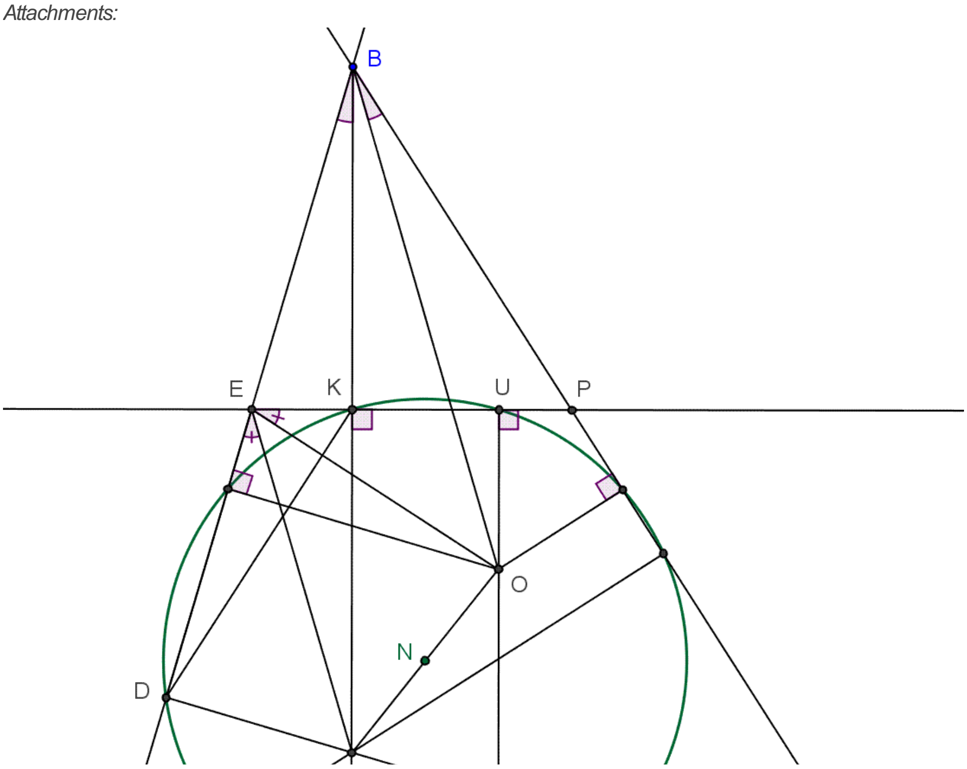
I don't understand why projection of O to EP is on that circle

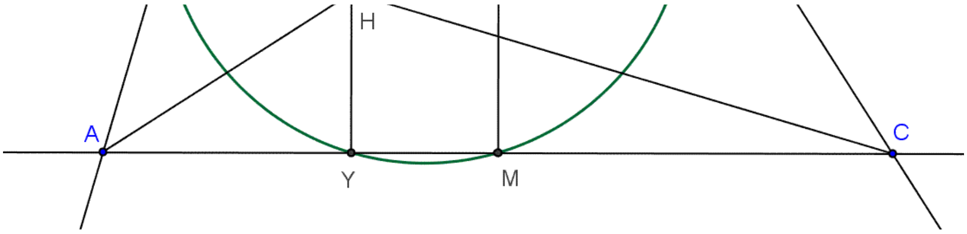
Luis González

#4 Apr 3, 2013, 11:25 pm

Let M be the midpoint of AC and Y the foot of the B-altitude. If U is the projection of O on EP, then obviously KYMU is a rectangle with circumcircle (N).

Attachments:





leader

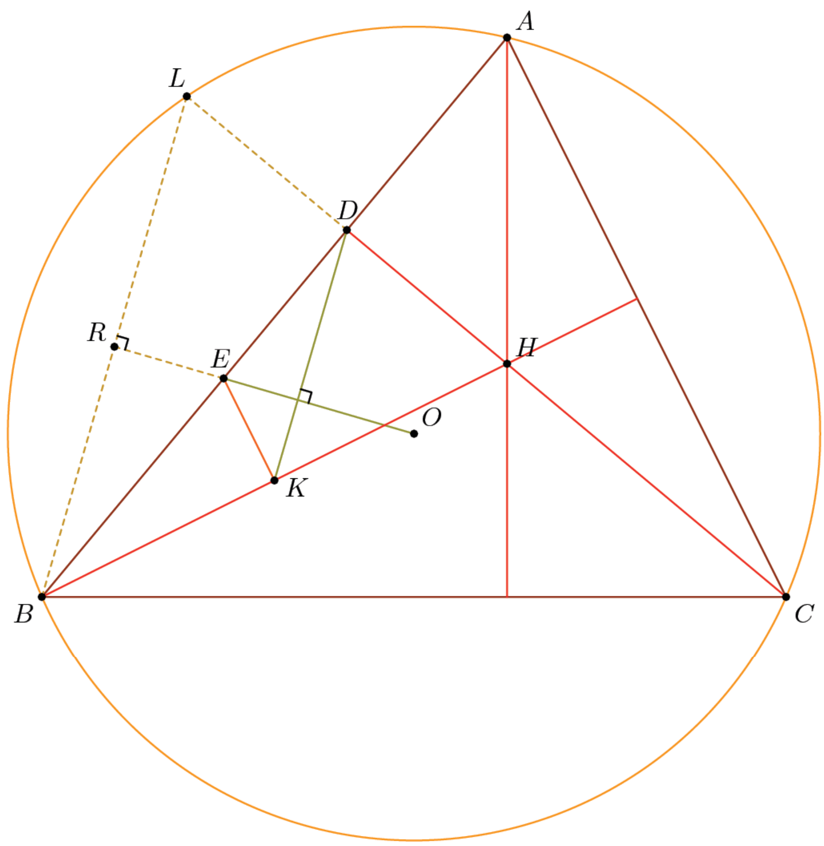
#5 Apr 3, 2013, 11:58 pm

let P be the midpoint of OH than D, K belong to the Euler's circle of ABC centered at P so $DP = KP$ and $(DO^2 + DH^2)/2 + HP * HO = DP^2 = KP^2 = (KH^2 + KO^2)/2 + HP * HO$ so $DO^2 - OK^2 = HK^2 - DH^2 = EH^2 - EK^2 - (EH^2 - ED^2) = ED^2 - EK^2$ so $OD^2 - OK^2 = ED^2 - EK^2$ therefore $EO \perp DK$

v_Enhance

#6 Apr 4, 2013, 5:40 am • 1

Reflect H over AB to L on (ABC). Then $DK \parallel LB$, and E is the circumcenter of $\triangle LBH$. So EO is the perpendicular bisector of LB, done!



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AO/OD - BC/AT=4

Reply

Source: Turkey TST 2013 - Day 1 - P3

xeroxia

#1 Apr 2, 2013, 10:00 pm • 1

Let O be the circumcenter and I be the incenter of an acute triangle ABC with $m(\widehat{B}) \neq m(\widehat{C})$. Let D, E, F be the midpoints of the sides $[BC]$, $[CA]$, $[AB]$, respectively. Let T be the foot of perpendicular from I to $[AB]$. Let P be the circumcenter of the triangle DEF and Q be the midpoint of $[OI]$. If A, P, Q are collinear, prove that

$$\frac{|AO|}{|OD|} - \frac{|BC|}{|AT|} = 4.$$

Luis González

#2 Apr 3, 2013, 12:42 am • 2

Let H be the orthocenter of $\triangle ABC.$ P is midpoint of $\overline{OH}.$ AI cuts the circumcircle (O) of $\triangle ABC$ again at midpoint M of its arc $BC.$ Since $AH=2 \cdot OD,$ then the reflection S of O about BC forms the parallelogram $AHSO \Longrightarrow AS, OH$ bisect each other at $P.$ Hence if A,P,Q are collinear, then A,Q,S are collinear. By Menelaus theorem for $\triangle OMI,$ cut by $\overline{AQS},$ we get

$$\frac{AI}{AM} \cdot \frac{MS}{SO} \cdot \frac{OQ}{QI} = 1 \Longrightarrow \frac{AM}{AI} = \frac{MS}{SO} = \frac{AO-2 \cdot OD}{2 \cdot OD} \Longrightarrow$$

$$\frac{AO}{OD} = 2 \cdot \frac{AM}{AI} + 2 = \left(1 + \frac{p}{p-a}\right) + 2 = 4 + \frac{a}{p-a} = 4 + \frac{BC}{AT}.$$

mathuz

#3 May 18, 2013, 5:02 am • 1

hello;
thank you, Luis Gonzalez.
(**the beautiful solution**).

But i have an other solution,
it is very nice too.

Let H - orthocenter, AA_1, BB_1, CC_1 - altitudes, r, R - inradius and circumradius of the triangle ABC.

Idea: A, P, Q are collinear if and only if

$$\frac{HB_1+OE}{HC_1+OF} = \frac{OE+r}{OF+r}$$

We have very nice two lemmas:
(1) $HA_1 \cdot R = 2d_b \cdot d_c$ and ...
(2) $d_a+d_b+d_c=r+R$.

So, since its, we see not difficult

$$\frac{AO}{OD} - \frac{BC}{AT} = 4.$$

Quick Reply

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KM.LN=BM.CN

Source: Turkey TST 2013 - Day 2 - P2

xeroxia
#1 Apr 2, 2013, 10:04 pm

Let the incircle of the triangle ABC touch $[BC]$ at D and I be the incenter of the triangle. Let T be midpoint of $[ID]$. Let the perpendicular from I to AD meet AB and AC at K and L, respectively. Let the perpendicular from T to AD meet AB and AC at M and N, respectively. Show that $|KM|\cdot|LN|=|BM|\cdot|CN|$.

Luis González
#2 Apr 2, 2013, 11:16 pm

Perpendicular KL from I to AD meets tangent BC of (I) through D at the pole P of AD WRT $(I) \Longrightarrow$ cross ratio (B,C,D,P) is harmonic. MN is clearly D-midline of $\triangle DIP$, meeting DP at its midpoint U, hence $UP^2=UD^2=UB\cdot UC$, or $\frac{PU}{UB}=\frac{UC}{PU}$. But by Thales theorem $\frac{UC}{UP}=\frac{CN}{LN}$ and $\frac{PU}{UB}=\frac{KM}{BM} \Longrightarrow \frac{KM}{BM}=\frac{CN}{LN}$.

xeroxia
#3 Apr 3, 2013, 12:10 am • 1

Let I be the incenter of $\triangle ABC$. AI meet BC at N. The incircle touches BC at D. Parallel through I to BC meet AD at P.
$PI=\dfrac{(u-a)(b-c)}{a+b+c}$, where u is semiperimeter.

Proof:

$CD=u-c$ and $NC=\dfrac{ab}{b+c} \Longrightarrow DN=\dfrac{(u-a)(b-c)}{b+c}$.

$$\frac{AI}{AN}=\frac{AC}{AC+CN}=\frac{b}{b+\dfrac{ab}{b+c}}=\frac{b+c}{a+b+c}.$$

$$\frac{PI}{DN}=\frac{AI}{AN}\Rightarrow PI=DN\cdot\frac{AI}{AN}=\frac{(u-a)(b-c)}{a+b+c}. \blacksquare$$

Let's go back to the problem.

Let KL meet AD at R, and MN meet AD at S.
Let X be the foot of perpendicular from B to AD.
Let Y be the foot of perpendicular from C to AD.

$\dfrac{KM}{BM}=\dfrac{RS}{SX}$ and $\dfrac{LN}{CN}=\dfrac{RS}{SY}$.

So $\dfrac{KM\cdot LN}{BM\cdot CN}=1 \iff RS^2=SX\cdot SY$

Let $\angle XBD=\theta$. So $\angle ADI=\angle DCY=\theta$.
We have $XD=(u-b)\sin\theta$ and $DY=(u-c)\sin\theta$.
$SX=SD-XD=RS-XD$ and $SY=SD+DY=RS+DY$.
$SX\cdot SY=(RS-XD)(RS+DY)=RS^2+RS(DY-XD)-XD\cdot DY$.
We will show that $RS(DY-XD)=XD\cdot DY$.
$RS=\dfrac{r}{2}\cdot\cos\theta$ where $ID=r$.

So $\iff \frac{r}{2}\cdot\cos\theta(b-c)\sin\theta=(u-b)(u-c)\sin^2\theta$

$\iff r=\frac{2(u-b)(u-c)}{(b-c)}\cdot\tan\theta$

$\iff u(u-a)r=\frac{2u(u-a)(u-b)(u-c)}{2(b-c)}\cdot\tan\theta$

$\iff u(u-a)r=\frac{2u^2r^2}{(b-c)}\cdot\tan\theta$

$\iff (u-a)(b-c)=2ur\cdot\tan\theta$

$\iff \frac{(u-a)(b-c)}{2u}=r\cdot\tan\theta$

We know $PI=\frac{(u-a)(b-c)}{a+b+c}$ and we also know that $PI=r\tan\theta$.

So the last statement is true. So we have $RS(DY-XD)=XD\cdot DY.\blacksquare$

This post has been edited 1 time. Last edited by xeroxia, Apr 3, 2013, 1:27 pm

---

MMEEvN

#4 Apr 3, 2013, 12:54 pm

> *Luis González wrote:*
> $UP^2=UD^2=UB\cdot UC,$

I don't undestand this part .Can someone explain this more.

---

specialone

#5 Apr 3, 2013, 3:43 pm

bariz kolay soru = obviously easy question

---

fystic

#6 Apr 6, 2013, 8:03 pm

> *MMEEvN wrote:*
>
> > *Luis González wrote:*
> > $UP^2=UD^2=UB\cdot UC,$
>
> I don't undestand this part .Can someone explain this more.

from

> *Luis González wrote:*
> cross ratio (B,C,D,P) is harmonic.

we have $BD*PC=PB*DC$ so $(UP-UB)(UC+UP)=(UP+UB)(UC-UP)$ by opening the brackets the desired conclusion results.

:)

Quick Reply

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Construction problem

Reply

Vutra
#1 Apr 2, 2013, 1:38 am

If we have $a = BC, b = AC$ and $\beta - \gamma = \angle ABC - \angle ACB$, can we construct triangle ABC?

yetti
#2 Apr 2, 2013, 5:08 am • 1

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=44096&p=283190#p283190. In this post,

> *yetti wrote:*
> If the sides $a = BC, b = CA$ and the angular difference $\angle B - \angle C$ are given, the problem cannot be solved with compass and straightedge, because it leads to finding the intersection of a known circle with a conchoid of Nicomedes.

Luis González
#3 Apr 2, 2013, 7:28 am • 1

It also leads to finding the intersections of a circle and a hyperbola, which, in general, is not approachable using compass and straightedge.

Fix the points B, C. A will run on the locus that satisfy that $\angle ABC - \angle ACB$ equals the constant $\beta - \gamma$. Fix a X on the locus, $\angle ABC - \angle ACB = \angle XBC - \angle XCB \Longrightarrow \angle ABC - \angle XBC = \angle ABX = \angle ACB - \angle XCB = \angle ACX$ $\Longrightarrow$ isogonal conjugate of A WRT $\triangle XBC$ is on perpendicular bisector ℓ of $\overline{BC} \Longrightarrow$ locus of A is the isogonal of ℓ WRT $\triangle XBC$, i.e. a rectangular hyperbola $\mathcal{H}$ through B, C. Intersection A of $\mathcal{H}$ and the circle (C, b) clearly cannot be constructed with compass and straightedge alone.

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k=AA1=BB1=CC1

Reply

bdjck

#1 Mar 31, 2013, 6:35 pm

In a triagle ABC, there is P inside ABC such that
AP, BP, CP meet BC, CA, AB respectively at A_1, B_1, C_1.
If $k = AA_1 = BB_1 = CC_1 < min(AB, BC, CA)$, show that
there exists a point Q inside ABC such that
AQ, BQ, CQ meet BC, CA, AB respectively at A_2, B_2, C_2
and $l = AA_2 = BB_2 = CC_2 < min(AB, BC, CA)$.

of course P is different from Q.

Luis González

#2 Apr 2, 2013, 5:33 am

P and Q coincide with the foci of the Steiner circumellipse of ABC, i.e. the ellipse that goes through A,B,C and whose center is the centroid of ABC.

http://www.xtec.cat/~qcastell/ttw/ttweng/resultats/r163.html
http://tech.groups.yahoo.com/group/Hyacinthos/message/237
http://tech.groups.yahoo.com/group/Hyacinthos/message/7811
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=406505

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High School Olympiads

Tangent line, tangent circles, reflection point

Reply

hvaz

#1 Apr 2, 2013, 1:39 am

Let S to be a circle and A, B points on S. M is the midpoint of arc AB and P is a point on segment AB which is not its midpoint. Circle S_1 is tangent to S on point X, and to segments AP and MP. Circle S_2 is tangent to segment BP on L, to segment MP and to S. If MP intersects with S on point Q, and Z is the intersection of QL and S, prove that $MX = MZ$.

Luis González

#2 Apr 2, 2013, 3:39 am • 1

It is enough to show that QM bisects $\angle XQL,$ in other words, QX,QL are isogonals WRT $\angle AQB.$ Let Y be the tangency point of $\mathcal{S}_2$ with $\mathcal{S}.$ From the internal tangencies, we deduce that XN,YL bisect $\angle AXB,\angle AYB.$

Let U,V be the incenters of $\triangle PAQ,\triangle PBQ \Longrightarrow I \equiv AU \cap BV \cap CP$ is the incenter of $\triangle QBC.UV,AB,XY$ concur at K (see Five concurrent lines or Ordinary and Thebault incircles). Since $P(U,V,I,K)$ is harmonic, then it follows that (A,B,P,K) is harmonic $\Longrightarrow$

$$\frac{QA}{QB}=-\frac{PA}{PB}=\frac{KA}{KB}=\frac{XA}{XB} \cdot \frac{YA}{YB}=\frac{XA}{XB} \cdot \frac{LA}{LB} \Longrightarrow \frac{QA}{QB}=\frac{XA}{XB} \cdot \frac{LA}{LB}.$$

The latter relation implies that QX,QL are isogonals WRT $\angle AQB,$ as desired.

hvaz

#3 Apr 2, 2013, 7:34 am

Man, you are like the human dictionary of Geometry. Nice solution!

PS: I really believe nobody untrained in geometry may enter your house haha!

Quick Reply

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High School Olympiads

Circumcentres of 6 circles are concyclic

Reply

Source: Chinese TST 1 2013 Day 2 Q2

iarnab_kundu

#1 Apr 1, 2013, 12:26 pm

Let P be a given point inside the triangle ABC. Suppose L,M,N are the midpoints of BC,CA,AB respectively and

$$PL:PM:PN=BC:CA:AB.$$

The extensions of AP,BP,CP meet the circumcircle of ABC at D,E,F respectively. Prove that the circumcentres of APF,APE,BPF,BPD,CPD,CPE are concyclic.

Luis González

#2 Apr 2, 2013, 1:56 am

Let A_1,A_2,B_1,B_2,C_1,C_2 denote the circumcenters of $\triangle PBF,\triangle PCE,\triangle PCD,\triangle PAF,\triangle PAE,\triangle PBD,$ respectively. Perpendiculars to AD,BE,CF through its endpoints A,B,C,D,E,F clearly bound a hexagon whose opposite sides are parallel and whose diagonals bisect each other at $O,$ because O is equidistant from its opposite sides. Vertices of this hexagon are nothing but the images of A_1,A_2,B_1,B_2,C_1,C_2 under homothety with center P and coeffient 2 $\Longrightarrow$ segments A_1A_2,B_1B_2,C_1C_2,OP bisect each other at $K.$ This is true for arbitrary $P.$

Now, it suffices to prove that $A_1A_2=B_1B_2=C_1C_2.$ By Stewart theorem for the median PK of $\triangle PA_1A_2,$ we obtain

$$PK^2=\frac{PA_1{}^2+PA_2{}^2}{2}-\frac{A_1A_2{}^2}{4}.$$

But the circumradii PA_1 and PA_2 verify $\dfrac{PB}{PA_1}=\dfrac{PC}{PA_2}=2\sin A=\dfrac{BC}{R}.$

$$PK^2=\frac{R^2}{2BC^2}(PB^2+PC^2)-\frac{A_1A_2{}^2}{4}=\frac{R^2}{2BC^2}\left(2PL^2+\frac{BC^2}{2}\right)-\frac{A_1A_2{}^2}{4}$$

$$\Longrightarrow A_1A_2{}^2=4R^2\left(\frac{PL^2}{BC^2}+\frac{1}{4}\right)-4PK^2 \ \ (1)$$

By similar reasoning, we get the expressions:

$$B_1B_2{}^2=4R^2\left(\frac{PM^2}{CA^2}+\frac{1}{4}\right)-4PK^2 \ \ (2)$$

$$C_1C_2{}^2=4R^2\left(\frac{PN^2}{AB^2}+\frac{1}{4}\right)-4PK^2 \ \ (3)$$

Since $\frac{PL}{BC}=\frac{PM}{CA}=\frac{PN}{AB},$ then from $(1),(2),(3),$ we get $A_1A_2=B_1B_2=C_1C_2 \Longrightarrow A_1,A_2,B_1,B_2,C_1,C_2$ lie on a circle with center K the midpoint of $OP.$

Remark: Actually there are two points P that satisfy the problem conditions, namely the Isologic points of the medial triangle MNL, which lie on the Euler line of MNL,ABC.

duanby

#3 Apr 2, 2013, 6:45 pm

my solution:

more complatated way

Click to reveal hidden text

polya78

#4 May 8, 2013, 11:12 pm

Let O be the circumcenter of $\triangle ABC$, Z be the circumcenter of $\triangle AFP$, and R, Y the midpoints of OP, AO, v the circumradius of $\triangle ABC$, and $\theta = PM/AC = PL/BC = PN/AB$.

Then $\triangle AZP \sim \triangle AOC$, so $\triangle AZO \sim \triangle APC$. Thus $ZY/v = PM/AC = \theta$. Also, using the median formula, we have $ZR^2 + OR^2 = (ZP^2 + OZ^2)/2 = (AZ^2 + OZ^2)/2 = ZY^2 + OY^2$, so that $ZR^2 = (v\theta)^2 + (v/2)^2 - OR^2$. So all of the circumcenters in question lie on a circle with center R.

Sorry about the diagram. I mislabeled the midpoints K, L, M instead of L, M, N.

Attachments:

china 2013.pdf (419kb)

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High School Olympiads

Circumcircles and Reflections

Reply

Source: 2013 CMO #5

JSGandora

#1 Apr 1, 2013, 2:47 am • 4

Let O denote the circumcentre of an acute-angled triangle ABC. Let point P on side AB be such that $\angle BOP = \angle ABC$, and let point Q on side AC be such that $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to the circumcircle of triangle APQ.

Luis González

#2 Apr 1, 2013, 5:13 am • 1

If AO cuts BC at R, we have $\triangle BOP \sim \triangle ABR$ and $\triangle COQ \sim \triangle ACR$, which yields $BP \cdot BA = AR \cdot OB$ and $CQ \cdot CA = AR \cdot OC \Longrightarrow BP \cdot BA = CQ \cdot CA$, i.e. B, C have equal powers WRT $(K) \equiv \odot(APQ) \Longrightarrow KB = KC$, thus by symmetry (K) cuts the circumcircle (O) of $\triangle ABC$ again at the reflection S of A across the perpendicular bisector of $\overline{BC}$, i.e. $AS \parallel BC$.

WLOG assume that $\angle ABC > \angle ACB$. Since S is also center of the spiral similarity that takes $\overline{PQ}$ into $\overline{BC}$, then $\triangle SPQ \sim \triangle SBC \cong \triangle ACB \Longrightarrow \angle SPQ = \angle ACB \Longrightarrow$ $\angle APS = \angle APQ - \angle SPQ = \angle AOQ - \angle ACB = 2(\angle ABC - \angle ACB)$. But $\angle ABS = \angle ABC - \angle ACB \Longrightarrow$ $\angle APS = 2\angle ABS \Longrightarrow \triangle BPS$ is isosceles with legs $PB = PS$. But if PQ cuts BC at M, then B, P, S, M are concyclic (S is Miquel point of PQ WRT ABC) $\Longrightarrow MP$ bisects $\angle BMS$ and $\angle(AS, AP) = \angle(BC, BA) = \angle(SM, SP)$ $\Longrightarrow MS$ is tangent to (K). Hence, reflection MS of BC across PQ touches $\odot(APQ)$.

subham1729

#3 Apr 1, 2013, 5:20 am • 1

I'm showing method, calculations are easy.Suppose extended BC meets extended PQ at M. Take $BP = x, CQ = y$.Now just by menalaus we have MB in terms of x, y, b, c. Now note x, y are also fixed and dependable on sides of the triangle.Let $\angle PMB = \theta$.Now by cosine rule on APQ we get PQ again by menulas we get MP.Now letting center $\odot(APQ)$ as O' radius R' and $\angle O'MP = \alpha$ we have $\frac{R'}{Sin\alpha} = \frac{MQ}{Cos(A-\alpha)}$, so we get $tan\alpha$. Now also $\frac{x}{Sin\theta} = \frac{MP}{SinB}$ from here we get $Sin\theta$. Using those we'll get $\frac{Sin(\theta-\alpha)}{Sin\alpha} = CosA$ and that implies $\angle QPQ' = \theta$ where Q' is tangent to $\odot(APQ)$.

yetti

#4 Apr 1, 2013, 10:49 am

WLOG, $\widehat C > \widehat B$. Reflection of $(OB$ in OP cuts (O) at Z. $\measuredangle ZOB = 2\widehat B = \measuredangle AOC \Longrightarrow$ O-isosceles $\triangle ZOB \cong \triangle AOC$ $\Longrightarrow ZA \parallel BC$. Likewise, reflection of $(OC$ in OQ cuts (O) at Z. Perpendicular bisectors OP, OQ of $[BZ], [CZ]$ cut them at their midpoints M, N. From M-right $\triangle BPM \Longrightarrow \measuredangle APO = \frac{\pi}{2} - (\widehat C - \widehat B)$ and from N-right $\triangle CQN \Longrightarrow$ $\measuredangle OQA = \frac{\pi}{2} + (\widehat C - \widehat B) \Longrightarrow APOQ$ is cyclic with circumcircle (K). From O-isosceles $\triangle AZO \Longrightarrow$ $\measuredangle AZO = \frac{\pi}{2} - (\widehat C - \widehat B) = \measuredangle APO \Longrightarrow Z \in (K)$. Let $Y \equiv OZ \cap BC$. From $\triangle CYZ \Longrightarrow$ $\measuredangle BYZ = \frac{\pi}{2} - (\widehat C - \widehat B) = \measuredangle BPM \Longrightarrow P$ is circumcenter of $\triangle BYZ$. Likewise, Q is circumcenter of $\triangle CYZ \Longrightarrow$ PQ is perpendicular bisector of $[YZ]$. Let PQ cut BC, ZA at X, T, resp. $\Longrightarrow XZ$ is reflection of BC in PQ and $XYTZ$ is rhombus $\Longrightarrow \measuredangle OZX = \measuredangle AZO = \measuredangle OAZ \Longrightarrow XZ$ is tangent of $(K) \equiv \odot(AZPOQ)$ at Z.

leader

#5 Apr 1, 2013, 11:08 pm

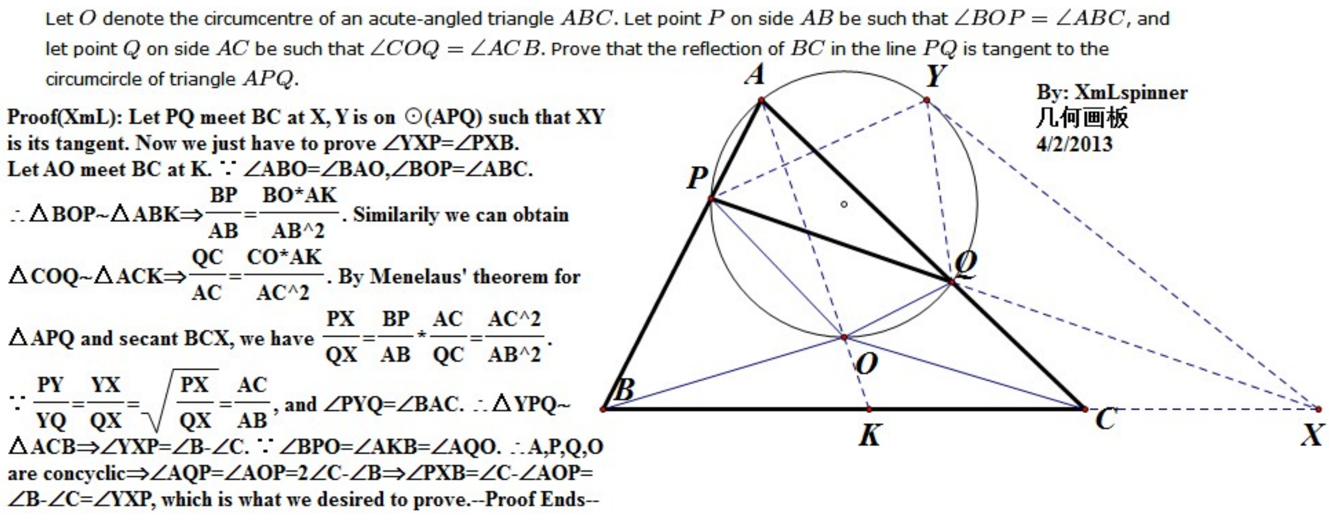
let A' be the reflection of A in PQ. then by symmetry wrt PQ it's enough to prove that circle $A'PQ$ touches BC. let circle COQ intersect BC again at R. first
$\angle APO = 90 - \angle ACB + \angle ABC = 180 - (90 + \angle ACB - \angle ABC) = 180 - \angle AQO$ so $AQOP$ is cyclic and now $\angle ROP = 360 - (180 - \angle BAC + 180 - \angle ACB) = 180 - \angle CBA$ so $BPOR$ is cyclic and
$\angle PRQ = \angle PRO + \angle QRO = 90 - \angle ABC + 90 - \angle ACB = \angle BAC = \angle PA'Q$ so R is on the circle $PA'Q$
similarly $\angle PQR = \angle ABC$ so $\angle PRB = \angle POB = \angle ABC = \angle PQR$ so BC touches circle RPQ which is actually circle $A'PQ$

XmL

#6 Apr 3, 2013, 5:28 am

See attachment for my semi-computational proof for this problem

Attachments:



Let O denote the circumcentre of an acute-angled triangle ABC. Let point P on side AB be such that $\angle BOP = \angle ABC$, and let point Q on side AC be such that $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to the circumcircle of triangle APQ.

Proof(XmL): Let PQ meet BC at X, Y is on ⊙(APQ) such that XY is its tangent. Now we just have to prove ∠YXP=∠PXB. Let AO meet BC at K. ∵ ∠ABO=∠BAO,∠BOP=∠ABC. ∴ΔBOP~ΔABK⇒$\frac{BP}{AB}=\frac{BO*AK}{AB^2}$. Similarily we can obtain ΔCOQ~ΔACK⇒$\frac{QC}{AC}=\frac{CO*AK}{AC^2}$. By Menelaus' theorem for ΔAPQ and secant BCX, we have $\frac{PX}{QX}=\frac{BP}{AB}*\frac{AC}{QC}=\frac{AC^2}{AB^2}$. ∵ $\frac{PY}{YQ}=\frac{YX}{QX}=\sqrt{\frac{PX}{QX}}=\frac{AC}{AB}$, and ∠PYQ=∠BAC. ∴ΔYPQ~ΔACB⇒∠YXP=∠B-∠C. ∵ ∠BPO=∠AKB=∠AQO. ∴A,P,Q,O are concyclic⇒∠AQP=∠AOP=2∠C-∠B⇒∠PXB=∠C-∠AOP=∠B-∠C=∠YXP, which is what we desired to prove.--Proof Ends--

Particle

#7 Apr 3, 2013, 8:10 pm • 3

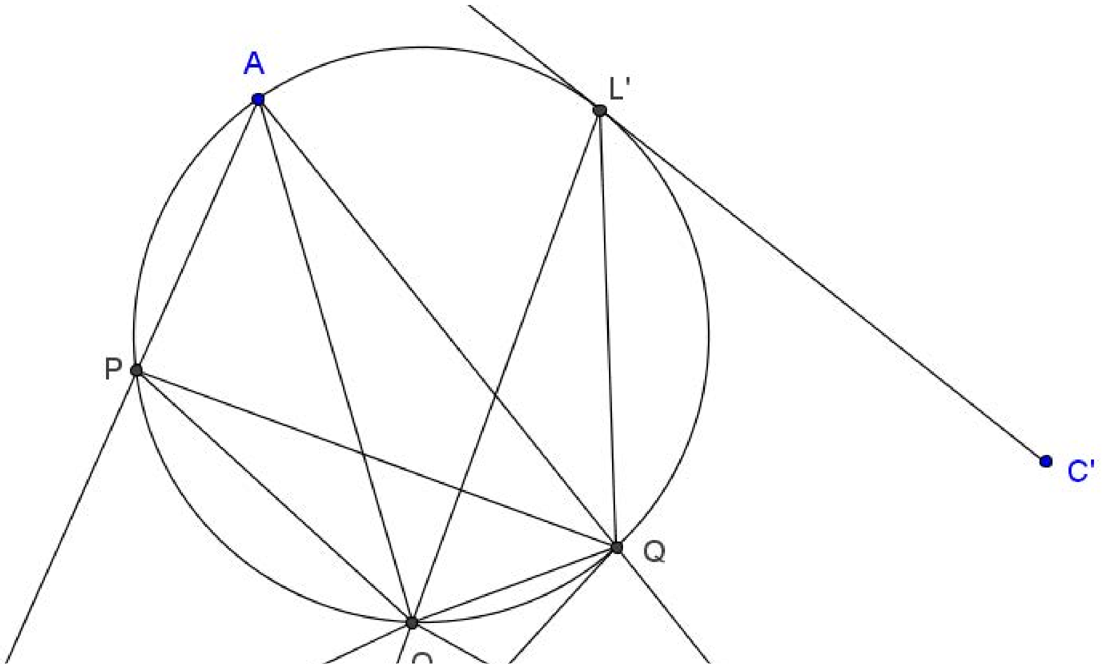
$\angle OPA = \angle OQC = \angle B + \pi/2 - \angle C$. So A, P, Q, O concyclic. Suppose the line passing through O and perpendicular to PQ intersect $\bigcirc APQ$ again at L' and BC at L. We claim the reflection of BC across PQ touches $\bigcirc APQ$ at L'.

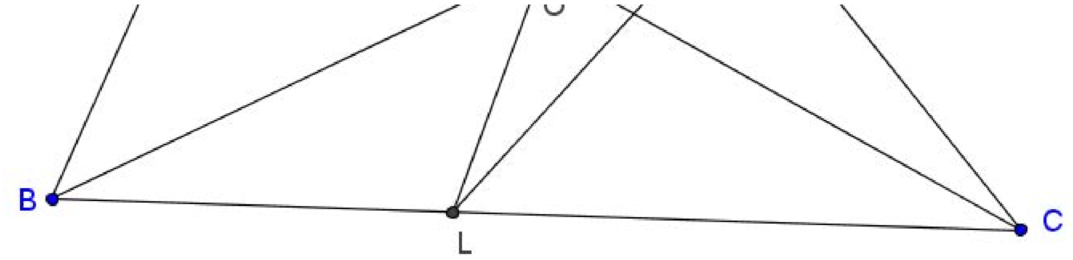
WLOG we may assume $\angle B \geq \angle C$.
$\angle QOL' = \pi/2 - \angle PQO = \angle C \implies O, Q, C, L$ concyclic. So $\angle OL'Q = \angle OAQ = \angle OCQ = \angle OLQ$. Since $LL' \perp PQ$ we find that L' is the reflection of L across PQ. Suppose after *that* reflection C maps to C'.

$$\angle C'L'Q = \angle CLQ = \angle COQ = \angle ACB = \angle QOL'$$

So $C'L'$ is the tangent of $\bigcirc APQ$ as desired.

Attachments:





JuanOrtiz

#8 May 27, 2015, 9:53 am

First prove APOQ are concyclic, then prove that if X is the intersection of the 2 circles then AX || BC. Then notice XQCD is cyclic (D is PQ cut BC) to prove that DX is tangent to the small circle. Finally, notice that <ACX = <QXC to finish.

This post has been edited 1 time. Last edited by JuanOrtiz, May 27, 2015, 9:53 am

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High School Olympiads

Intersections of Circumcircles and Hypotenuse

Reply

Source: 2013 CMO #3

JSGandora
#1 Apr 1, 2013, 2:45 am

Let G be the centroid of a right-angled triangle ABC with $\angle BCA = 90^\circ$. Let P be the point on ray AG such that $\angle CPA = \angle CAB$, and let Q be the point on ray BG such that $\angle CQB = \angle ABC$. Prove that the circumcircles of triangles AQG and BPG meet at a point on side AB.

Luis González
#2 Apr 1, 2013, 3:30 am • 3

Let M be the midpoint of $\overline{AB}$. $\angle CPA = \angle CAB = \angle ACM \equiv \angle ACG \Longrightarrow AC$ is tangent to $\odot(PCG) \Longrightarrow AC^2 = AG \cdot AP$. Thus if K denotes the orthogonal projection of C on AB, we have $AK \cdot AB = AC^2 = AG \cdot AP \Longrightarrow B, K, G, P$ are concyclic, i.e. circumcircle of $\triangle BPG$ goes through K. Similarly, the circumcircle of $\triangle AQG$ goes through $K \Longrightarrow$ circumcircles of $\triangle AQG, \triangle BPG$ meet on AB.

exmath89
#3 Apr 1, 2013, 6:47 am

Solution

supercomputer
#4 Apr 14, 2014, 10:19 am

what is the motivation to project C onto AB?

Konigsberg
#5 Dec 22, 2014, 7:17 pm

We are trying to find the common point. Maybe we can guess that it's the point, and then work from there?

IDMasterz
#6 Dec 22, 2014, 7:52 pm

Well, I can safely say visualising it in my head, the way I did it was to think of inversion: B, under inversion through A with pole AC^2, goes to the foot of the altitude from C, so $AK \cdot AB = AC^2 = AG \cdot AP \implies K, G, P, B$ are concyclic (having notation from Luis). And inversion is natural. From here, one should note G can be replaced by any point on the line CM, where M is the midpoint of AB (or circumcentre of ABC).

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High School Olympiads

Quadrilateral with B=108 and C=153

Source: Antalya Math Olympiad - 2013 - 1st Round

xeroxia

#1 Mar 31, 2013, 3:04 pm

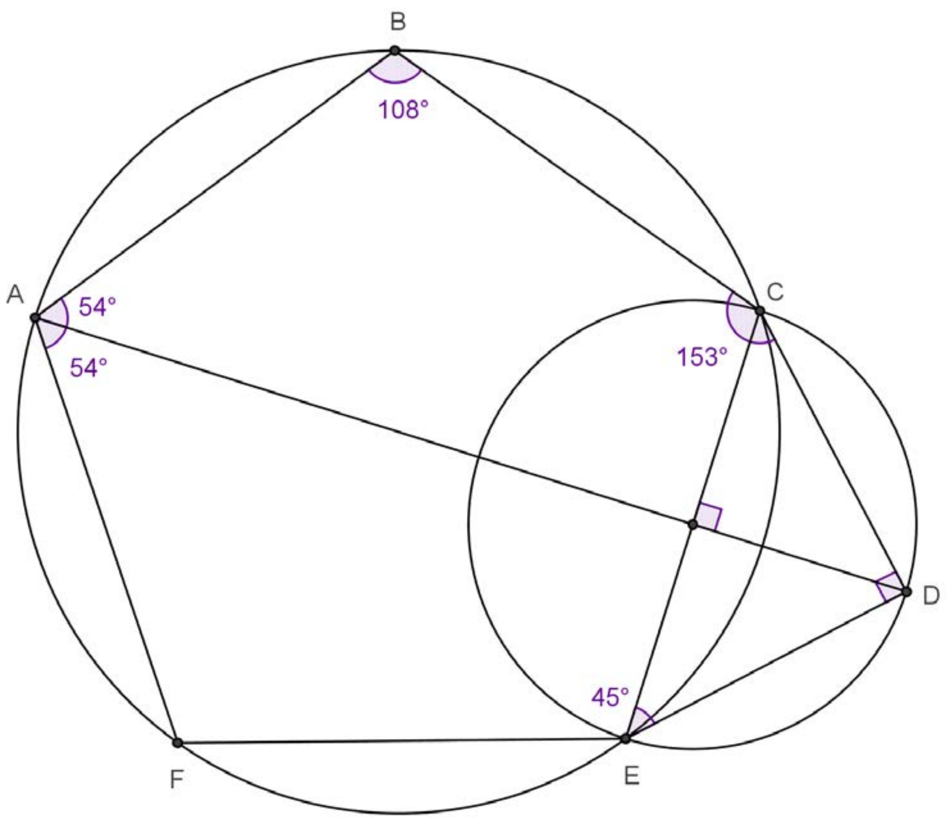
Let $ABCD$ be a quadrilateral with $AB=BC=CD\sqrt{2}$, $\angle ABC=108^\circ$, and $\angle BCD=153^\circ$. What is $\angle BAD$?

Luis González

#2 Mar 31, 2013, 10:45 pm • 3

See the diagram below for a proof without words.

Attachments:



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High School Olympiads

Cyclic quadrilateral ABCD

Reply

Source: own?

vankhea

#1 Mar 29, 2013, 1:42 am

Let $ABCD$ be an arbitrary concyclic quadrilateral.
Let X, Y, Z, T be midpoints of AB, BC, CD, DA respectively, The rays OX, OY, OZ, OT meets the circumcircle of quadrilateral $ABCD$ at E, F, G, H repsectively.
Prove that $EG \perp FH$.

This post has been edited 1 time. Last edited by v_Enhance, Apr 7, 2013, 10:49 pm
Reason: Edited accordingly.

yetti

#2 Mar 31, 2013, 1:28 am

Perhaps you mean that the rays OX, OY, OZ, OT meet circumcircle of the cyclic quadrilateral $ABCD$ at E, F, G, H, respectively :

vankhea

#3 Mar 31, 2013, 3:17 am

> *yetti wrote:*
> Perhaps you mean that the rays OX, OY, OZ, OT meet circumcircle of the cyclic quadrilateral $ABCD$ at E, F, G, H, respectively :

yes i mean like that.
sorry for my bad English.

yetti

#4 Mar 31, 2013, 4:34 am • 1

Let (O) be circumcircle of $ABCD$. Tangents $e \parallel AB, f \parallel BC, g \parallel CD, h \parallel DA$ of (O) at E, F, G, H pairwise meet at $P \equiv e \cap f, Q \equiv f \cap g, R \equiv g \cap h, S \equiv h \cap e$.
Quadrilateral $PQRS$ cyclic, having equal angles as cyclic quadrilateral $ABCD$, and also tangential with incircle (O). Let (K) be circumcircle of $PQRS$.
Inversion in (O) takes P, Q, R, S into midpoints P', Q', R', S' of EF, FG, GH, HE, and circumcircle (K) of $PQRS$ into circumcircle (K') of $P'Q'R'S'$.
But $P'Q' \parallel EG \parallel S'R'$ as midlines of $\triangle EFG, \triangle EHG$ and likewise, $Q'R' \parallel FH \parallel P'S' \Longrightarrow P'Q'R'S'$ is cyclic parallelogram, i.e., rectangle $\Longrightarrow$
$P'Q' \perp Q'R'$ are perpendicular $\Longrightarrow$ their parallels $EG \perp FH$ are also perpendicular.

Luis González

#5 Mar 31, 2013, 8:52 am

E, F, G, H are obviously midpoints of the arcs AB, BC, CD, DA, respectively.

$$\angle(EG, FH) = \tfrac{1}{2}(\widehat{EF} + \widehat{GH}) = \tfrac{1}{2}(\widehat{EB} + \widehat{FB} + \widehat{GD} + \widehat{HD}) =$$

$$= \tfrac{1}{4}(\widehat{AB} + \widehat{BC} + \widehat{CD} + \widehat{DA}) = \tfrac{1}{4}(360^\circ) = 90^\circ.$$

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High School Olympiads

Two Tangents, Angle Bisector and Concurrency

Reply

Source: own

thugzmath10
#1 Mar 27, 2013, 5:47 pm

Let ABC be an acute triangle with circumcircle ω. The tangents of ω at B and C intersect at X. The internal bisector of $\angle ABC$ intersects BC at D and ω at the second point E. The line through X perpendicular to the internal bisector of $\angle ABC$ meets BC at Y. The circumcircle of triangle AXY meets ω again at M. MX and MY intersect ω again at F and G, respectively. Prove that FB, GC and XY are concurrent.

Math-lover123
#2 Mar 31, 2013, 1:57 am

> *thugzmath10 wrote:*
> Let ABC be an acute triangle with circumcircle ω. The tangents of ω at B and C intersect at X. The internal bisector of $\angle ABC$ intersects BC at D and ω at the second point E. The line through X perpendicular to the internal bisector of $\angle ABC$ meets BC at Y. The circumcircle of triangle AXY meets ω again at M. MX and MY intersect ω again at F and G, respectively. Prove that FB, GC and XY are concurrent.

There must be some mistake!!

Luis González
#3 Mar 31, 2013, 2:32 am

I think the poster refers the internal bisector of $\angle BAC$ instead, but even those definitions of Y and M are not necessary. Y and M can be any points on BC and the circumcircle ω of $\triangle ABC$, respectively.

Quadrilateral $MCFB$ is harmonic $\Longrightarrow$ pencil $A(B,C,F,M)$ is harmonic $\Longrightarrow$ pencil $G(B,C,F,Y)$ is harmonic. If $P \equiv GF \cap BC$, then (B,C,P,Y) is harmonic $\Longrightarrow$ XY is the polar of P WRT $\omega \Longrightarrow$ the sidelines FB, GC of the complete quadrangle $BFCG$ meet on XY.

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High School Olympiads

Geometry

Reply

maths_lover5

#1 Mar 31, 2013, 1:49 am

We have a triangle ABC where $AB+BC=2AC$. The circle with center O inscribed in the triangle touches AB in M point and BC in N point. P is the circumcenter of the triangle. Prove that $OMNP$ is cyclic.

Luis González

#2 Mar 31, 2013, 2:08 am

It follows from the fact that $BP \perp OP \Longleftrightarrow AB+BC=2\cdot AC.$

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=82999
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=383434
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=417792

TheBernuli

#3 Mar 31, 2013, 2:47 am

$OMPNB$ is cyclic, too.

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High School Olympiads

Concurrent

Reply

Source: Luis Gonzales

TrungK40PBC

#1 Mar 29, 2013, 3:44 pm

Problem. Let ABC be a triangle with incenter I and let X,Y,Z be points lying on the internal angle bisectors AI,BI,CI. Furthermore, let M,N,P be the midpoints of the sides BC,CA,AB, and let D,E,F be the tangency points of the incircle of ABC with these sides. Prove that if MX,NY,PZ are parallel, then DX,EY,FZ are concurrent on the incircle of ABC.

Luis González

#2 Mar 31, 2013, 12:29 am

The problem and its general version were discussed in the forum already.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=472689
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=294728
http://forumgeom.fau.edu/FG2006volume6/FG200629.pdf

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High School Olympiads

Two equal angles

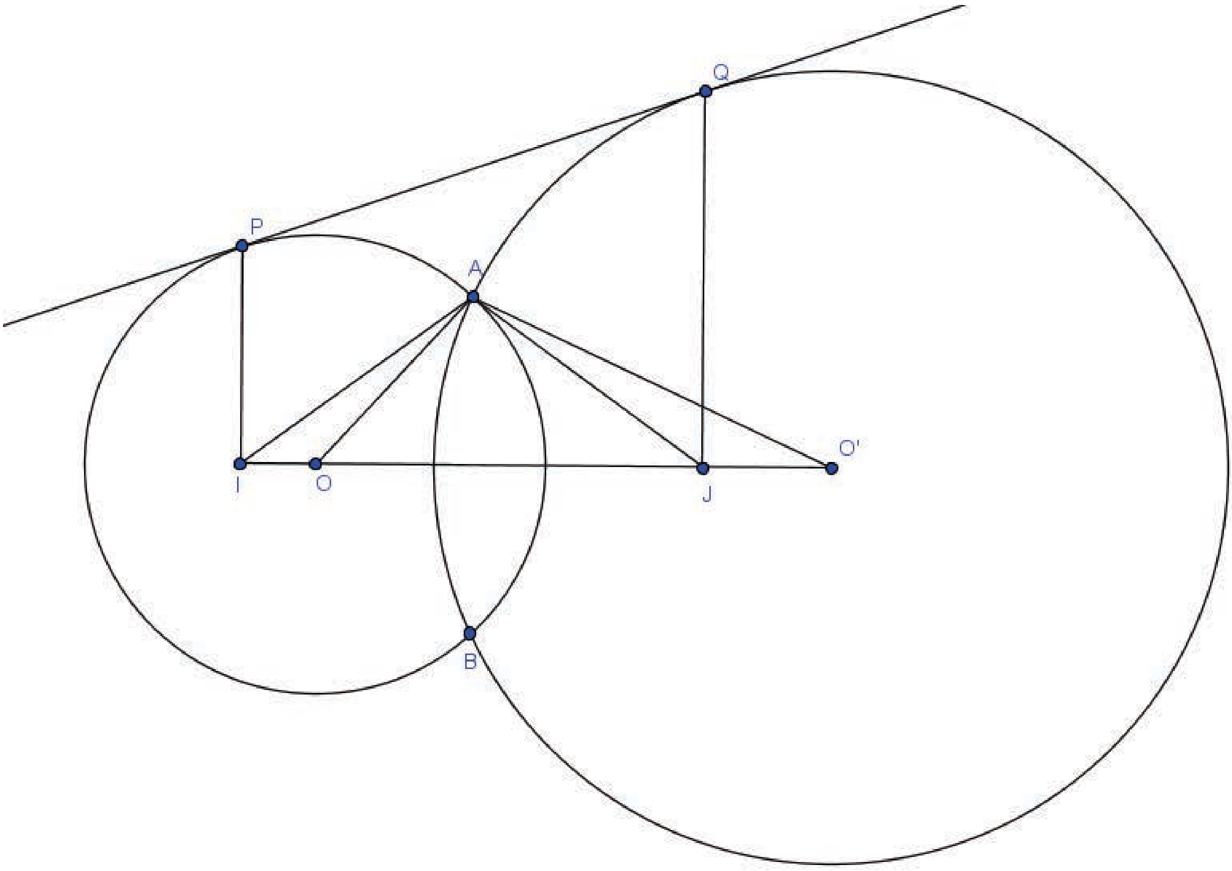
Reply

CTK9CQT

#1 Mar 30, 2013, 8:04 pm

Let two circles (O) and (O') intersect at A and B. PQ is their common ex-tangent. Let I, J are feets of P, Q on OO'. Prove that $\angle IAJ = \angle OAO'$

Attachments:



Luis González

#2 Mar 30, 2013, 10:29 pm

$U \equiv PQ \cap OO'$ is the exsimilicenter of $(O) \sim (O)'$, which is also center of inversion that transforms these circles into each other $\Longrightarrow UA^2 = UP \cdot UQ$, but $OPQJ$ is cyclic due to the right angles at P,J $\Longrightarrow UP \cdot UQ = UO \cdot UJ = UA^2 \Longrightarrow$ UA is tangent to $\odot(AOJ) \Longrightarrow \angle UAO = \angle AJO$ $(\star)$.

$AB \perp OO'$ is radical axis of $(O), (O)'$, bisecting their common tangent $\overline{PQ} \Longrightarrow AB$ is perpendicular bisector of $\overline{IJ}$, i.e. $\triangle AIJ$ is A-isosceles. If V is insimilicenter of $(O) \sim (O')$, then $AV \perp AU$ bisect $\angle OAO'$, since $\frac{AO}{AO'} = -\frac{VO}{VO'} = \frac{UO}{UO'}$.
From $(\star)$, we get $90^\circ - \angle UAO = \frac{1}{2}\angle OAO' = 90^\circ - \angle AJO = \frac{1}{2}\angle IAJ \Longrightarrow \angle OAO' = \angle IAJ.$

Particle

#3 Mar 31, 2013, 5:26 pm

(I considered all angles modulo π)

Define,
$U = PQ \cap OO'$; $\quad I_1 = IA \cap (O)$; $\quad O_1 = OA \cap (O)$; $\quad J_1 = JA \cap (O')$; $\quad O_1' = O_1'A \cap (O')$
AO_1 is a diameter of (O). So $AB \perp O_1B$. Similarly $AB \perp O_1'B$. Therefore O_1, B, O_1' are collinear.

Notice that $\triangle PIQ \sim \triangle UIP$. So $\dfrac{PI}{IO} = \dfrac{UI}{IP} \implies PI^2 = UI \cdot IO$. Hence
$UI \cdot IO = PI^2 = PO^2 - IO^2 = AI \cdot II_1 \implies U, I_1, O, A$ are concyclic. $(\star)$

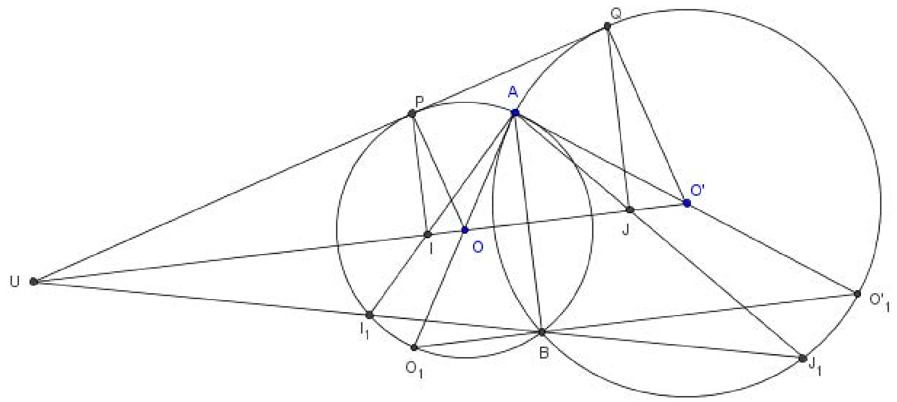
O and O' are midpoints of AO_1 and AO_1', respectively. So $OO'||O_1O_1'$. Therefore
$\angle O'OA = \angle O_1'O_1A = \angle BO_1A = \angle BI_1A \quad (\star\star)$

Combining $(\star)$ and $(\star\star)$, we derive $BI_1 \cap OO' = U$. Similarly we can show $BJ_1 \cap OO' = U$. So I_1, B, J_1 are collinear.

So a spiral similarity takes $\triangle I_1AJ_1$ to $\triangle O_1AO_1'$. Now obviously we will have $\angle IAJ = \angle OAO'$

Comment: This problem is from 1983IMO.

Attachments:



sunken rock

#4 Mar 31, 2013, 6:42 pm

Let $X = OO' \cap PQ$; from right-angled ΔPOX we get $PO^2 = OI \cdot OX$, but $PO = AO$, so AO is tangent to the circle $\odot(AIX)$ and $\angle IAO = \angle AXO$.
Similarly, $\angle JAO' = \angle AXO$, and we are done.

Best regards,
sunken rock

jayme

#5 Mar 31, 2013, 7:11 pm

Dear Mathlinkers,
this problem comes from
24-ième O.I.M. France (1983)
Sincerely
Jean-Louis

Quick Reply

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High School Olympiads

hard question about quadrangular

Reply

mathematics123321
#1 Mar 30, 2013, 2:07 am

Quadrangular $ABCD$,$AB=CD$, $\measuredangle ABC=150^{\circ}$, $\measuredangle BCD=90^{\circ}$. Calculate the angle between line BC and the line that passing through the middle of the edges BC and AD.

Luis González
#2 Mar 30, 2013, 3:06 am

Let M,N,U,V denote the midpoints of $\overline{BC},\overline{AD},\overline{BD},\overline{AC}.$ The parallelogram $UMVN$ is a rhombus because $UM=VN=\tfrac{1}{2}CD=\tfrac{1}{2}AB=VM=VN \Longrightarrow MN$ is internal bisector of $\angle UMV,$ which is then parallel to the internal bisector of the angle formed by $AB,CD.$ If $P \equiv AB \cap DC,$ then $\triangle BCP$ is right at C and $\angle PBC=30^{\circ} \Longrightarrow$ the angle between BC and the internal bisector of $\angle BPC$ is obviously $60^{\circ} \Longrightarrow \angle (BC,MN)=60^{\circ}.$

mathematics123321
#3 Mar 30, 2013, 3:31 pm

I didn't understand the sulution, can someone draw the sulution or explain more?

leader
#4 Mar 30, 2013, 4:01 pm

this was on the tournament of towns.
here is a different approach. let M,N be midpoints of BC,AD draw point E such that E,D are on the same side of BC and $\angle CBE=90$ and $BE=AB$ call X the midpoint of DE first of $BE=AB=CD$ and clearly $BE||CD$ so $BEDC$ is a parallelogram giving $MX||BE$ and $MX=BE=BA$ but notice that $\angle EBA=\angle CBA-\angle CBE=150-90=60$ and since $AB=BE$ we have $\triangle ABE$ is equilateral so $AE=AB$ and $\angle AEB=60$ but NX is the midline of $\triangle ADE$ so $NX||AE$ and $NX=AE/2=AB/2=MX/2$ but since $\angle MXN=\angle BEA=60$ we have that MXN is right with $\angle NMX=30$ finally $\angle NMC=\angle XMC=\angle NMX=90+30=120$.

Quick Reply

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Excircles and equal areas

Reply

Source: (mpdb)

borislav_mirchev

#1 Mar 28, 2013, 5:08 pm

Let k_a, k_b, k_c are the excircles of the triangle ABC. A_1 and B_2 are tangency points of k_c with AC and BC. B_1 and C_2 are the tangency points of k_a with the sides AB and AC. C_1 and A_2 are the tangency points of k_b with the sides BC and AB. Prove that $S_{A_1B_1C_1} = S_{A_2B_2C_2}$.

Luis González

#2 Mar 29, 2013, 6:46 am • 1

This is a particular case of the following result: P_1, P_2, P_3 are three arbitrary points on the sidelines BC, CA, AB of $\triangle ABC$. Q_1, Q_2, Q_3 are the reflections of P_1, P_2, P_3 about the midpoints of BC, CA, AB, respectively. Then $\triangle P_1P_2P_3$ and $\triangle Q_1Q_2Q_3$ have equal areas. (Johnson 1929).

For a proof, see the topic collinear-sharigin and elsewhere.

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Prove an area of triangle

Source: own

vankhea

#1 Mar 8, 2013, 10:36 pm

Let D,E,F be the points on the sides BC,CA,AB of triangle ABC respectively such that
$DE\perp CA, EF\perp AB, FD\perp BC$. Let $M=AD\cap BE, N=BE\cap CF, P=CF\cap AD$.
Prove that $[BMD]+[CNE]+[APF]=[MNP]$
Noted: $[MNP]$ be area of triangle MNP.

Luis González

#2 Mar 29, 2013, 12:55 am • 1

Let the F-altitude FY of $\triangle DEF$ (parallel to AC through F) cut BC at $Z.$ Perpendiculars to EF through E and D (parallels to AB through E and D) cut BC,CA at $U,V,$ respectively. Since $\triangle VDC \sim \triangle ABC \sim \triangle EFD,$ we get

$$\frac{DZ}{ZC}=\frac{DY}{YE}=\frac{CE}{EV}=\frac{CU}{UD} \Longrightarrow DU=CZ \Longrightarrow BC=BD+CU+CZ \Longrightarrow$$

$$\frac{BD}{BC}+\frac{CU}{BC}+\frac{CZ}{BC}=\frac{BD}{BC}+\frac{CE}{CA}+\frac{AF}{AB}=1.$$

Now, the conclusion follows using the result of the problem Area of triangle.

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Area of triangle

Reply

Source: vankhea

vankhea

#1 Mar 16, 2013, 8:17 am

Let D, E, F be the points on the sindes BC, CA, AB of triangle ΔABC so that $\frac{BD}{BC}+\frac{CE}{CA}+\frac{AF}{AB}=1$. Let $M=AD\cap BE, N=BE\cap CF, P=CF\cap AD$.
Prove that $[MNP]=[MDB]+[NEC]+[PFA]$
Click to reveal hidden text

Luis González

#2 Mar 28, 2013, 10:58 am • 1

$$\frac{BD}{BC}+\frac{CE}{CA}+\frac{AF}{AB}=\frac{[ABD]}{[ABC]}+\frac{[BCE]}{[ABC]}+\frac{[CAF]}{[ABC]}=1 \Longrightarrow$$

$$[ABD]+[BCE]=[ABC]-[CAF]=[BFC]=[BNF]+[BCN] \Longrightarrow$$

$$[ABD]+[BCE]-[BCN]=[ABD]+[NEC]=[BNF] \Longrightarrow$$

$$[BMPF]+[MNP]=[ABD]+[NEC] \Longrightarrow$$

$$[MNP]=[ABD]-[BMPF]+[NEC]=[MDB]+[NEC]+[PFA].$$

Quick Reply

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Extremely beautiful problem

Reply

Source: (mpdb)

borislav_mirchev

#1 Mar 27, 2013, 7:26 pm

Let k_1, k_2, k_3 are the excircles tangent to the sides AB, BC, CA, respectively. A_1, B_1 and C_1 are the tangency points of k_1 with the lines formed by the triangle sides and the segment AB, respectively. M_1 and M_2 are the middles of the segments A_1C_1 and B_1C_1. M_3 and M_4 are defined in simillar manner. M_5 and M_6, too. Prove that M_1, M_2, M_3, M_4, M_5, M_6 lie on an ellipse. Is this statement remains true if k_1, k_2, k_3 were ellipses instead of circles?

Luis González

#2 Mar 28, 2013, 2:44 am • 1

Obviously M_4, M_5 lie on the external bisector of $\angle ACB.$ Since M_1M_2 is the C_1-midline of $\triangle A_1B_1C_1,$ then $M_1M_2 \parallel A_1B_1 \parallel M_4M_5.$ Likewise, $M_2M_3 \parallel M_5M_6$ and $M_3M_4 \parallel M_6M_1,$ i.e. intersections $M_1M_2 \cap M_4M_5,$ $M_2M_3 \cap M_5M_6$ and $M_3M_4 \cap M_6M_1$ are on the line at infinity, thus by the converse of Pascal theorem, the hexagon $M_1M_2M_3M_4M_5M_6$ is inscribed in a conic.

As for your last question, no, it is not true in general. Draw your sketch for 3 arbitrary cevian triangles.

borislav_mirchev

#3 Mar 28, 2013, 3:10 am

Just a remark. When M_1, ..., M_6 are the middles of the arcs instead of the segments this statement is also true.

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Square and perpendicular lines

Reply

barcelona

#1 Mar 27, 2013, 7:54 pm

It is given the square $ABCD$ with side 1. The points M, N, P, Q are on the sides AB, BC, CD, DA such that $MP \perp NQ$. Find $QA + AM + NC + CP$.

Luis González

#2 Mar 27, 2013, 11:26 pm

Let $O \equiv MP \cap NQ$ and U, V are the orthogonal projections of P, N on AB, AD, respectively. $AMOQ$ is cyclic due to the right angles at $A, O \Longrightarrow \angle PMB = \angle NQA$. Since $PU = NV = AB$, then the right $\triangle PMU$ and $\triangle NQV$ are congruent $\Longrightarrow MU = QV$. But $MU = DP - AM$ and $QV = QA - NB \Longrightarrow DP - AM = QA - NB \Longrightarrow$ $QA + AM = DP + NB = 2 \cdot AB - NC - CP \Longrightarrow QA + AM + NC + CP = 2 \cdot AB.$

Tsikaloudakis

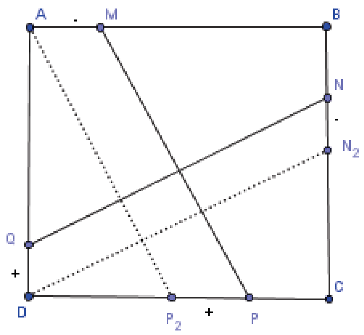
#3 Mar 28, 2013, 1:41 am

$$\left.\begin{array}{l} AP_2//MP \\ DN_2//QN \end{array}\right\} \Rightarrow \left.\begin{array}{l} AM = P_2P \\ NN_2 = DQ \\ DP_2 = CN_2 \end{array}\right\} \Rightarrow$$

$$QA + AM + NC + CP =$$

$$= (QA + QD) + (DP_2 + P_2P + PC) = AD + DC$$

Attachments:



sunken rock

#4 Mar 28, 2013, 12:45 pm

Same thing if drawing parallels through the center of the square.

Remark: study the problem for a rectangle.

Best regards,
sunken rock

Quick Reply

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A B point on circle G

Locked

hEatLove

#1 Mar 27, 2013, 7:23 pm

A, B are points on circle ω , C be the point inside the circle. Suppose that the circle is tangent to AC at P , ω at Q , BC at ... . Show that the circumcircle of triangle APQ pass through the incenter of triangle ABC

Luis González

#2 Mar 27, 2013, 7:45 pm • 1

This has been discussed many times before. For further discussions use the links below.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=6086
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=41667
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=253207
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=407366

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Tangents+Tangents

Reply

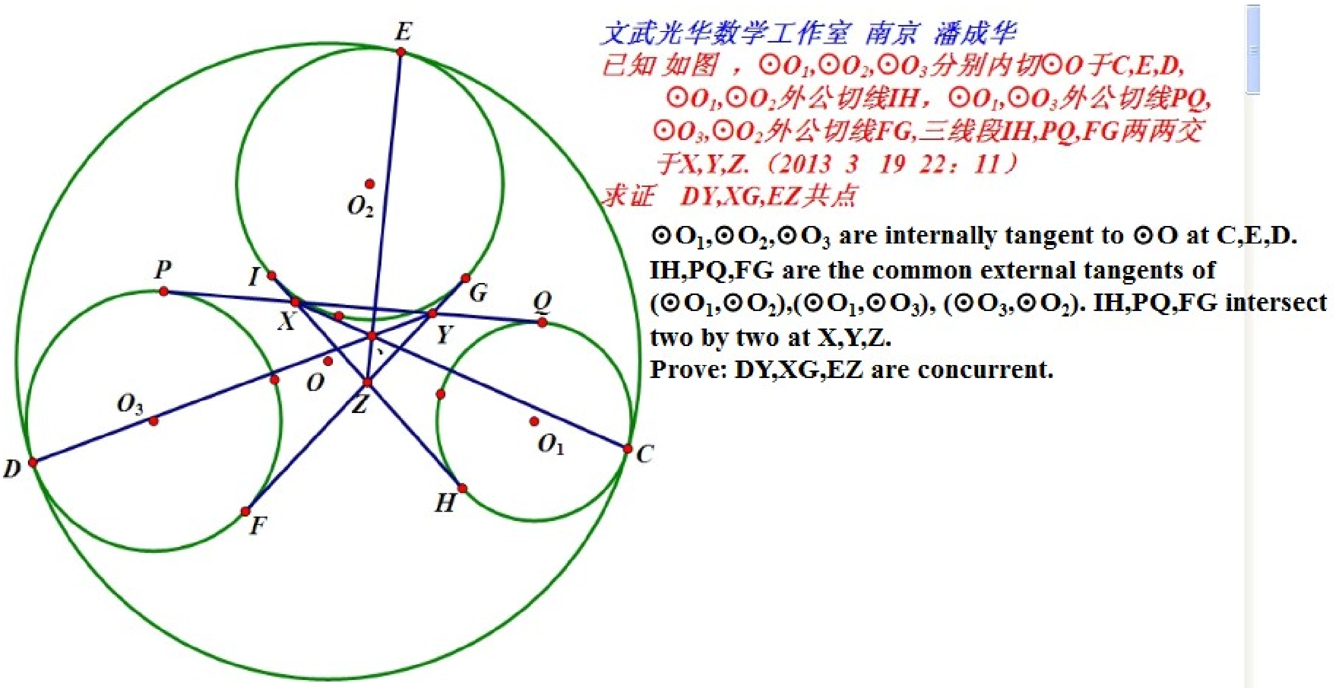
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Mar 25, 2013, 10:07 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Mar 27, 2013, 12:36 am • 1

Let (J) be the incircle of $\triangle XYZ. X$ is the exsimilicenter of $(J) \sim (O_1)$ and C is the exsimilicenter of $(O_1) \sim (O)$, thus by Monge & d'Alembert theorem, XC passes through the exsimilicenter U of $(J) \sim (O)$. Similarly, YD and ZE pass through U, i.e. XC, YD, ZE concur at U.

P.S. When $(O_1), (O_2), (O_3)$ coincide with the excircles of $\triangle XYZ$, then the concurrency point U is the Apollonius point X_{181} of $\triangle XYZ$.

Quick Reply

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Prove or disprove the geometry property

Reply

Source: Own

Nguyenhuyhoang

#1 Mar 22, 2013, 9:51 pm • 1

An extension of IMO Shortlist 2005 G1:
Using the same notation as here: http://www.artofproblemsolving.com/Forum/viewtopic.php?p=439274#p439274
Prove that the orthocenter of the intriangle of triangle ABC lies on KL.
The IMOSL 2005 G1 is a really interesting problem and it can be extend further through many ways.

Luis González

#2 Mar 26, 2013, 1:57 am

Let the incircle (I) touch AB at $F.U,V$ denote the midpoints of $FE, FD.$ Inversion WRT (I) takes L,K into themselves and carries U,V into $A,B.$ Since I,A,B,L,K are concyclic, then U,V,L,K are collinear, i.e. LK is F-midline of $\triangle DEF,$ cutting the F-altitude FZ of $\triangle DEF$ at its midpoint $P.$ Since $EDKL$ is a rectangle with center $I,$ the distance from I to ED is twice the distance $|PZ|=|FP| \Longrightarrow P$ coincides with the orthocenter of $\triangle DEF.$

Quick Reply

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Prove IN is fixed line

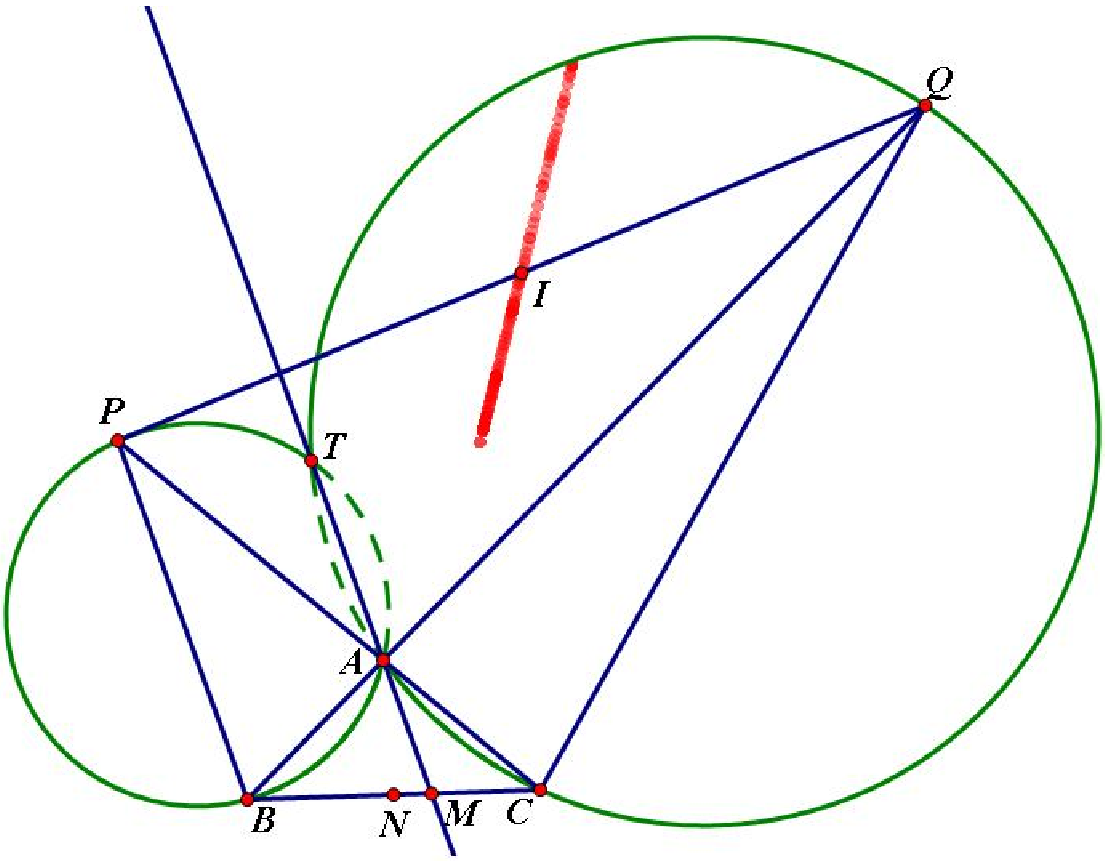
coriander

#1 Mar 25, 2013, 9:41 pm

Let triangle ABC. M is a fixed point on the segment BC. A moving ponit T is on ray MA but not on segment MA. $(ABT) \cap TC = P, (ACT) \cap TB = Q$. I is midpont of PQ. N is midpoint of BC.
Prove that line passes I and N is a fixed line

Thanks

Attachments:



Luis González

#2 Mar 26, 2013, 12:32 am

It should be $(ABT) \cap AC = P, (ACT) \cap AB = Q$ as your figure suggests.

Note that the second intersection T of $\odot(ABP)$ and $\odot(ACQ)$ is the center of the spiral similarity that swaps $\overline{PC}$ and $\overline{BQ}$, thus it follows that the series P, Q with base lines AC, AB are similar, hence the midpoint I of $\overline{PQ}$ runs on a fixed line that passes through N, because $I \equiv N$ when $T \equiv A$. Even all points $I \in PQ$ verifying $\overline{IP} : \overline{IQ} = \text{const}$ describe a fixed line.

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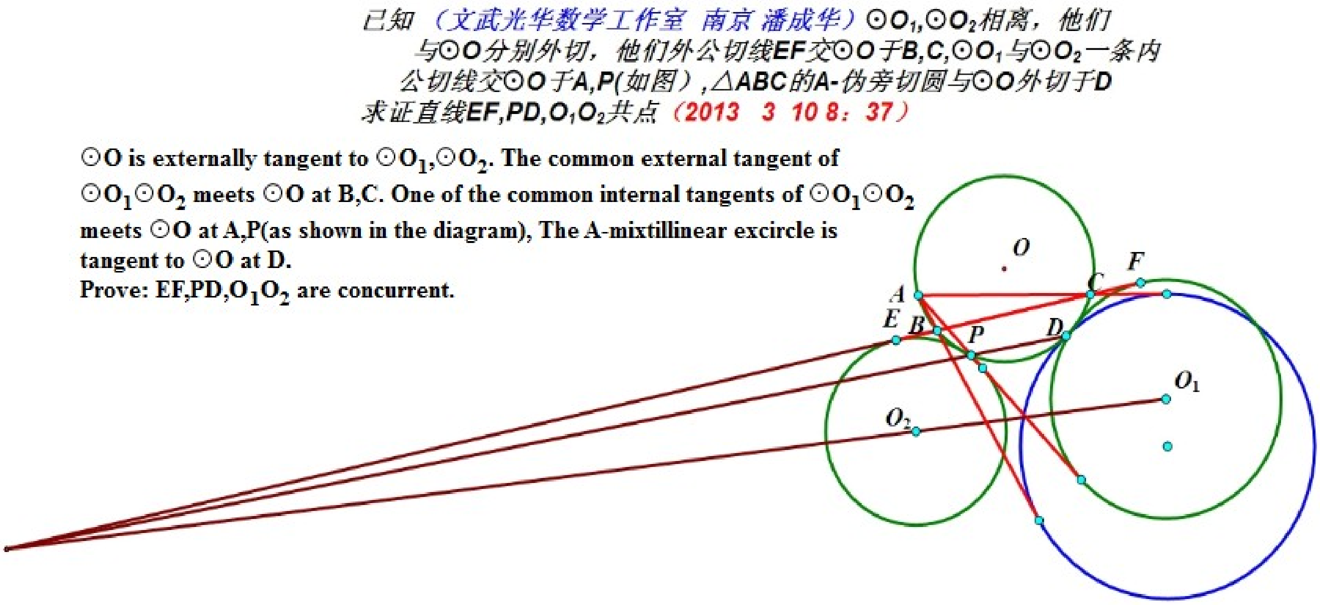
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Mar 12, 2013, 7:42 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Mar 13, 2013, 4:06 am • 1

Let U,V be the tangency points of (O) with $(O_1),(O_2)$ and PA cuts BC at M. Since U,V are the insimilicenters of $(O)\sim(O_1)$ and $(O)\sim(O_2)$, it follows that EF,UV and O_1O_2 concur at the exsimilicenter S of $(O_1)\sim(O_2)$. Thus if we assume that PD cuts BC at T, then we have to show that $S\equiv T$. From the external tangencies of $(O),(O_1)$ and $(O),(O_2)$, we deduce that UE and VF bisect $\angle BUC$ and $\angle BVC$ externally. Moreover if (O_1) touches AP at E', then by Sawayama's lemma EE' passes through the C-excenter J of $\triangle PBC$. Hence, easy angle chase gives $\angle BJE=90^\circ-\frac{1}{2}\angle BPM$ and $\angle CJE=90^\circ+\frac{1}{2}\angle CPM \Longrightarrow$

$$\frac{UB}{UC}=\frac{EB}{EC}=\frac{JB}{JC}\cdot\frac{\sin\widehat{BJE}}{\sin\widehat{CJE}}=\frac{\sin\frac{\widehat{BCP}}{2}}{\cos\frac{\widehat{CBP}}{2}}\cdot\frac{\cos\frac{\widehat{BPM}}{2}}{\cos\frac{\widehat{CPM}}{2}}.$$

Similarly, we have $\dfrac{VC}{VB}=\dfrac{\sin\frac{\widehat{CBP}}{2}}{\cos\frac{\widehat{BCP}}{2}}\cdot\dfrac{\cos\frac{\widehat{CPM}}{2}}{\cos\frac{\widehat{BPM}}{2}}\Longrightarrow$

$$\frac{SB}{SC}=\frac{UB}{UC}\cdot\frac{VB}{VC}=\frac{\sin\frac{\widehat{BCP}}{2}}{\cos\frac{\widehat{CBP}}{2}}\cdot\frac{\cos\frac{\widehat{BCP}}{2}}{\sin\frac{\widehat{CBP}}{2}}\cdot\frac{\cos\frac{\widehat{BPM}}{2}}{\cos\frac{\widehat{CPM}}{2}}\cdot\frac{\cos\frac{\widehat{BPM}}{2}}{\cos\frac{\widehat{CPM}}{2}}=$$

$$=\frac{\sin\widehat{BCP}}{\sin\widehat{CBP}}\cdot\frac{\cos^2\frac{\widehat{BPM}}{2}}{\cos^2\frac{\widehat{CPM}}{2}}=\frac{PB}{PC}\cdot\frac{\cos^2\frac{\widehat{ACB}}{2}}{\cos^2\frac{\widehat{ABC}}{2}}=\frac{PB}{PC}\cdot\frac{DB}{DC}=\frac{TB}{TC}.$$

Hence, $\dfrac{SB}{SC}=\dfrac{TB}{TC}$ means that S and T coincide, as desired.

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High School Olympiads

Geometry interested

cristianobalobalo

#1 Mar 10, 2013, 8:19 pm

The circles (O) and (O') meet at point A, B. Point C is fixed on (O) and point D is fixed on (O'). A moving point P is on the opposite ray of ray BA. The circumcircles of triangle PBC, PBD intersect BD, BC at second point E, F respectively. Prove that the midpoint of line segment EF is always on a fixed straight line.

Luis González

#2 Mar 11, 2013, 12:19 am • 1

Let $(O),(O)'$ cut BD,BC again at $U,V.$ Perform an inversion with center B and arbitrary power, we denote inverse points with primes. A,C,D,P go to $A',C',D',P' \Longrightarrow U' \equiv A'C' \cap BD', V' \equiv D'A' \cap BC', E' \equiv P'C' \cap BD'$ and $F' \equiv P'D' \cap BC'.$ From the complete quadrangles $C'D'U'V'$ and $C'D'F'E',$ we deduce that $C'D',U'V',E'F'$ concur at the harmonic conjugate H' of $C'D' \cap BA'$ WRT $C'D' \Longrightarrow$ their inverses $\odot(BCD),\odot(BUV)$ and $\odot(BEF)$ concur at $B,H.$

H is then the center of the spiral similarity that takes $\overline{CD} \mapsto \overline{VU} \mapsto \overline{FE}.$ Hence, midpoints M,N,L of CD,UV,EF are homologous under such spiral similarity $\Longrightarrow \triangle HMD \sim \triangle HNU \sim \triangle HLE \Longrightarrow M,N,L$ lie on the image of BD under the spiral similarity with center $B,$ rotational angle $\angle(HD,HM)$ and coefficient $\frac{HM}{HD},$ i.e. L moves on fixed line MN.

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collinear

Reply

Source: MMO

unt

#1 Mar 10, 2013, 7:58 pm

Let I be incenter of $\triangle ABC$. A_1 is a middle of arc (BC), which does not contain point A, A_2 is a middle of arc (ABC). Perpendicular, droped from point A_1 on line A_2I, intesect line BC at a point A'. B', C' difine in similar way.

Show that
a) A', B', C' are collinear;

b) that line is perpendicular to OI, where O is a circumcenter of ABC.

jayme

#2 Mar 10, 2013, 8:41 pm

Dear Mathlinkers,

1. Note A* the point of intersection of the perpendicular in question, and circularly B*, C*
2. the triangle ABC and A*B*C* are perspective and we are done... for the first question.

See for more Http://perso.orange.fr/jl.ayme vol. 4 a New mixtilinear incircle adventure I and II.

Sincerely
Jean-Louis

Luis González

#3 Mar 10, 2013, 8:54 pm

It's well known that the 2nd intersection P of IA_2 with (O) is the tangency point of the A-mixtilinear incircle ω_A with (O). If ω_A touches AC, AB at Y, Z, then YZ, BC, PA_1 concur (see Internally tangent circles and lines and concurrency) $\Longrightarrow A'$ is on the perpendicular to AI at I. If (I) touches BC, CA, AB at D, E, F, then A' is clearly the pole of the D-altitude of $\triangle DEF$ WRT (I). Similarly, B' and C' are the poles of the E- and F- altitudes WRT $(I) \Longrightarrow A', B', C'$ lie on the polar of the orthocenter T of $\triangle DEF$ WRT (I), which lies on Euler line OI of $\triangle DEF \Longrightarrow \overline{A'B'C'} \perp TI \equiv OI$.

unt

#4 Mar 11, 2013, 5:54 pm

A', B', C' is just radical axe of generate circle (I) and circumcenter of ABC.

Quick Reply

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Excircles Parallel

Reply

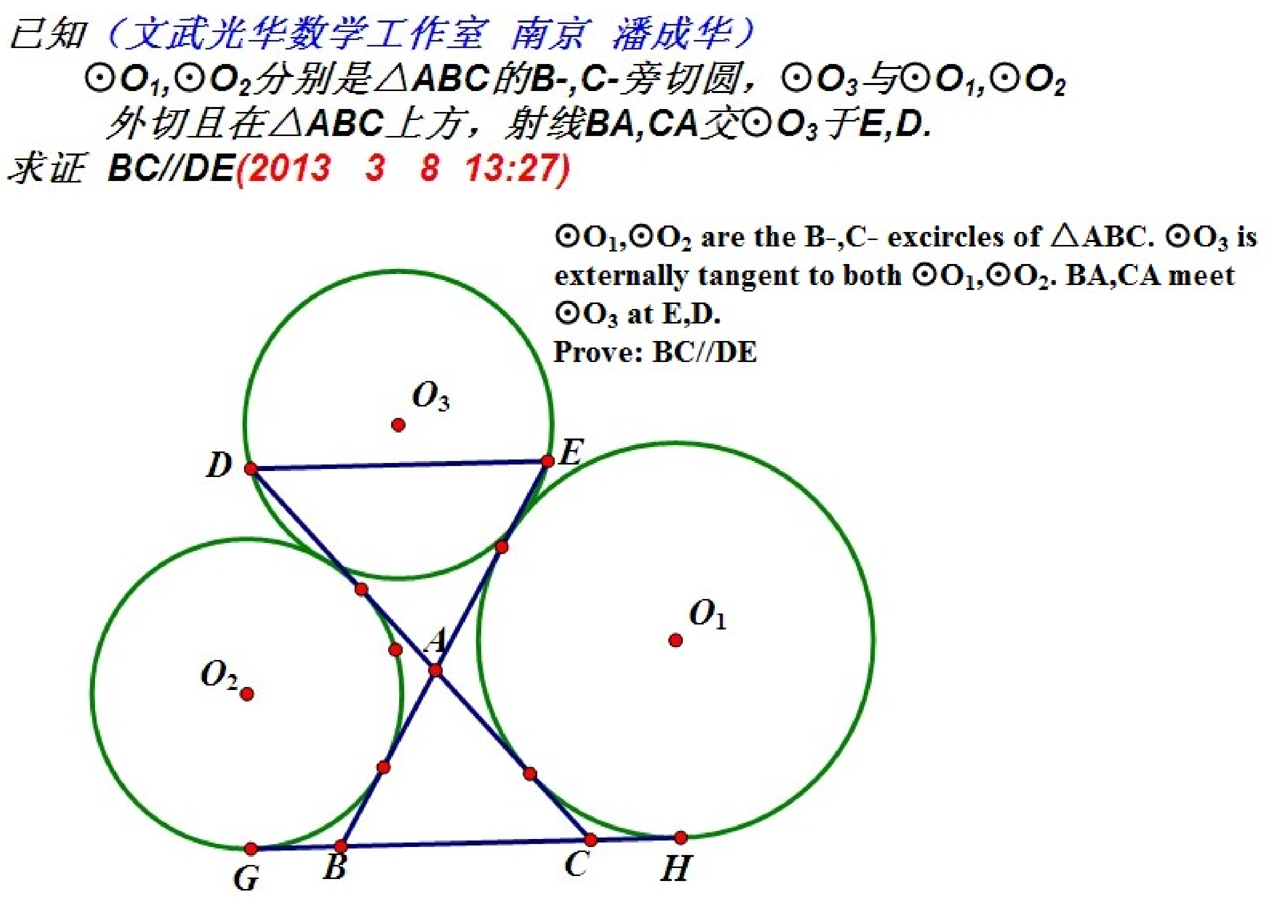
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Mar 10, 2013, 5:57 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China
Note: I know that this is a special case of a particular problem, but I would appreciate it if a proof is given to this.

Attachments:



Luis González

#2 Mar 10, 2013, 7:05 am

This is the "Parallel tangent theorem" posted many times before.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=15945
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=247604
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=310017
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=430441
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=463503

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High School Olympiads

B and C excircle equal angles

Source: own

Math-lover123

#1 Mar 8, 2013, 7:36 pm • 1

Triangle ABC is given which is not isosceless.
Let the points of contact of B excircle with the side AC and C excircle with side AB are D and E respectively.
Intersection of lines DE and BC is F.
Point of contact of the incircle of triangle ABC and the side BC is L and midpoint of arc BC not containing A of circumcircle of triangle ABC is K.
Prove that $\angle DFB = \angle AKL$.
Waiting for some synthetic solution.

Luis González

#2 Mar 9, 2013, 11:29 pm • 1

Incircle (I) touches AC, AB at Y, Z. AK cuts BC, DE at V, M, KL cuts the circumcircle (O) of $\triangle ABC$ again at T and AT cuts ED at U. From the problem incenter I and touches BC side with D, it follows that T is the second intersection of $\odot(AYZ)$ with (O).

Denote by (O_1) and (O_2) the circumcircles of $\triangle AYZ$ and $\triangle ADE$. Since the midpoints of AC, AB are also midpoints of DY, EZ, then the circle (O_0) with diameter $\overline{OA}$ is midcircle of $(O_1), (O_2)$. Hence if AT cuts (O_0) again at H, then $\overline{OO_1H}$ is perpendicular bisector of $\overline{AT}$. But since AO_1OO_2 is a parallelogram with diagonal intersection O_0, then $O_2A \perp TA$, i.e. TA is tangent to $(O_2) \Longrightarrow (U, UA)$ is A-Apollonius circle of $\triangle ADE$ passing through $M \Longrightarrow \triangle UAM$ is U-isosceles. Since $ATLV$ is cyclic, due to $KB^2 = KL \cdot KT = KV \cdot KA$, then $\angle KMF = \angle UAM = \angle KLF \Longrightarrow FMLK$ is cyclic $\Longrightarrow \angle DFB = \angle AKL$.

TelvCohl

#3 Nov 13, 2014, 11:14 pm

My solution:

Let I be the incenter of $\triangle ABC$.
Let Y, Z be the tangent point of (I) with AC, AB, respectively .
Let M be the midpoint of arc BAC and $X = KL \cap (ABC)$.

It's well-known $\triangle KIL \sim \triangle KXI$ and $X \in (AI)$ (see incenter I and touches BC side with D).

Since X is the Miquel point of complete quadrilateral $\{BC, CA, AB, YZ\}$,
so we get $\triangle XYC \sim \triangle XZB$ and $\frac{XY}{XZ} = \frac{YC}{ZB} = \frac{AD}{AE}$.
hence combine with $\angle ZXY = \angle EAD$ we get $\triangle XYZ \sim \triangle ADE$ . ... $(*)$

Since $MB = MC, BE = CD, \angle EBM = \angle DCM$,
so we get $\triangle MBE$ and $\triangle MCD$ are concurrent .
ie. $\triangle MDE \sim \triangle MCB \sim \triangle AYZ$ (all isosceles triangle)
Combine with $(*)$ we get $\triangle MDE \cap A \sim \triangle AYZ \cap X$.

Since $\angle XIA = \angle XYA = \angle ADM, \angle DCM = \frac{\angle ACB - \angle CBA}{2} = \angle KIL = \angle IXK$,
so we get $\triangle KIX \sim \triangle MDC \sim \triangle MEB$,

hence combine with $\triangle MDE \sim \triangle MCB$ we get $\angle BFD = \angle CMB = \angle LKA$.

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Aug 21, 2015, 1:26 am

IDMasterz

#4 Nov 14, 2014, 11:41 am

This follows because, if the incircle touches AC, AB at M, N respectively then $\odot AMN \cap \odot ABC \in KL$.

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High School Olympiads

Prove BCP is equilateral triangle

Reply

sunken rock

#1 Mar 9, 2013, 2:39 am

Let $M \in [BC], N \in [CD]$ of the rectangle $ABCD$ so that ΔAMN is equilateral.
If P is midpoint of $[AN]$, then ΔBCP is an equilateral triangle.

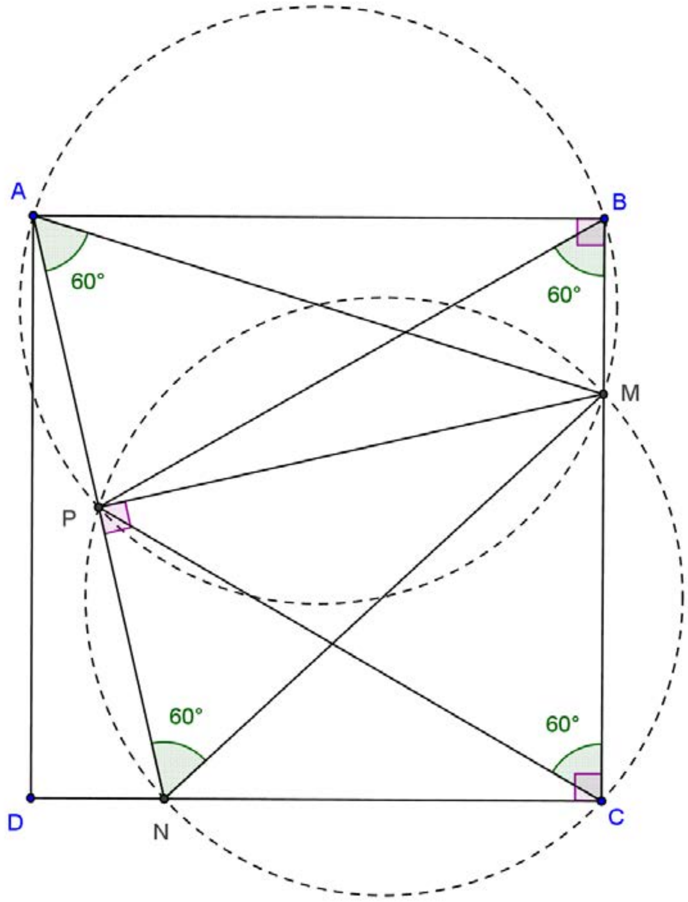
Best regards,
sunken rock

Luis González

#2 Mar 9, 2013, 3:05 am

See the diagram below for a proof without words

Attachments:



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Similar Isosceles Triangles Area 4

Reply

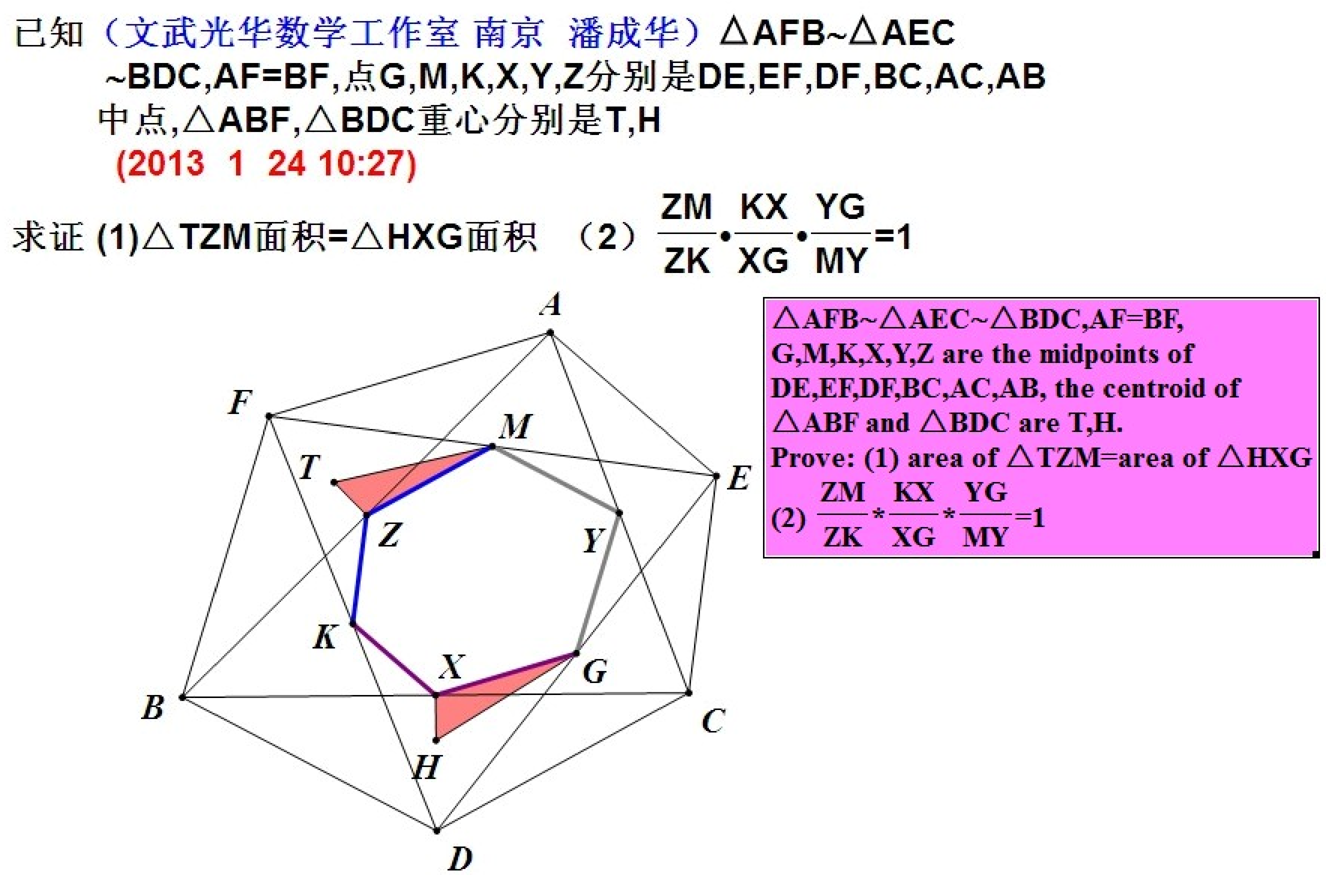
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Jan 31, 2013, 9:50 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Mar 7, 2013, 12:46 am

1) By Kiepert theorem, AD, BE, CF concur at P. From $\triangle ACE \sim \triangle ABF$, we get $AB \cdot AE = AC \cdot AF$, but $\angle BAE = \angle CAF \Longrightarrow [BAE] = [CAF]$. Since $[BAE] : [EBD] = AP : PD = [CAF] : [FCD]$, then $[EBD] = [FCD]$. Analogously, $[DAF] = [EBF]$ and $[FCE] = [DAE] \Longrightarrow$

$$[EBD] + [DEF] - [FCE] - [FCD] = [EBF] + [DEF] - [DAE] - [DAF]$$

$$\Longrightarrow [EBD] - [ECD] = [EBF] - [EAF] \Longrightarrow [EXD] = [EZF].$$

Since $[HXG] = \frac{1}{6}[EXD]$ and $[TZM] = \frac{1}{6}[EZF]$, we get $[HXG] = [TZM]$.

Luis González

#3 Mar 7, 2013, 7:31 am

#3 Mar 7, 2013, 7:31 am

2) Since $\triangle ABC$ and $\triangle DEF$ have the same centroid L (see the problem prove that two triangles share their centroid), then it follows that $\triangle XKG$ and $\triangle AEF$ are homothetic under homothety $(L, -\frac{1}{2}) \Longrightarrow \frac{XK}{XG} = \frac{AE}{AF}$. Multiplying the cyclic expressions together gives

$$\frac{XK}{XG} \cdot \frac{YG}{YM} \cdot \frac{ZM}{ZK} = \frac{AE}{AF} \cdot \frac{BF}{BD} \cdot \frac{CD}{CE} = \frac{AE}{AF} \cdot \frac{AF}{CD} \cdot \frac{CD}{AE} = 1.$$

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High School Olympiads

The line pass through Euler point

Reply

yumeidesu

#1 Mar 5, 2013, 3:04 am • 2

Let ABC be an acute triangle and B', C' are the symmetric point of B, C to AC, AB, respectively. Assume that $B'C, BC'$ intersects at A_0. Let D, E, F are the feet of perpendicular lines of A_0 to BC, CA, AB and I is the center of circle (DEF). Prove that AI pass through Euler point of ABC.

Luis González

#2 Mar 5, 2013, 4:37 am • 1

From cyclic $FBDA_0$, we get $\angle FDA_0 = \angle FBA_0 = \angle ABC = \angle FA_0D \Longrightarrow \triangle FDA_0$ is F-isosceles. Similarly, $\triangle EDA_0$ is E-isosceles $\Longrightarrow EF$ is perpendicular bisector of $\overline{DA_0}$. Let P be the foot of the A-altitude and Y, Z the midpoints of AC, AB. U is the antipode of P WRT 9 point circle $(N) \equiv \odot(PYZ)$. Trivial angle chase gives $\angle UZY = \angle UPY = 90^\circ - \angle ABC = \angle DFE \Longrightarrow UZ \parallel DF$. Likewise, $UY \parallel DE \Longrightarrow \triangle UYZ$ and $\triangle DEF$ are homothetic with center $A \Longrightarrow A$ is also homothetic center of their circumcircles (N) and $(I) \Longrightarrow A, N, I$ are collinear.

Nguyenhuyhoang

#3 Mar 5, 2013, 10:37 pm • 1

We can also prove that the Euler line of ABC parallel to A_0I.
This problem is really cute :)

TelvCohl

#4 Dec 3, 2014, 11:31 am

My solution:

Let A_1 be the reflection of A in BC .
Let X be the projection of A on BC .
Let O, H be the circumcenter, orthocenter of $\triangle ABC$, respectively .

Since $\angle A_1BC = \angle FBA_0, \angle BCA_1 = \angle A_0CE$,
so A_1 is the isogonal conjugate of A_0 WRT $\triangle ABC$,
hence combine with $\angle CA_0B = 180^\circ - 2\angle BAC$ we get $A_0 \equiv AO \cap (BOC)$.

Since $\triangle BAX \sim \triangle BA_0F$,
so $\triangle BAA_0 \sim \triangle BXF$,
hence we get $\angle XFA = \angle OA_0B = \angle OCB = 90^\circ - \angle BAC$.
ie. $XF \perp AC$
Similarly, we can prove $XE \perp AB$,
so we get XEA_0F is a parallelogram .

Since $\frac{AA_1}{AH} = \frac{2AX}{AH} = \frac{2AE}{AC} = \frac{AA_0}{AO}$,
so we get $OH \parallel A_0A_1$.
Since I is the midpoint of A_0A_1 ,
so we get AI pass through the midpoint of OH .
ie. AI pass through the nine point center of $\triangle ABC$

Q.E.D

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Geometry hard

Reply

yamatunga

#1 Mar 4, 2013, 12:05 pm

Let sharp triangle ABC inscribed circle $(O;R)$ and H is orthocenter of triangle ABC. circle $(E;r)$ exposure HB, HC and exposure in with circle $(O;R)$.
Prove that: midpoint of HE is center of the circle inscribed the triangle HBC

Luis González

#2 Mar 4, 2013, 10:58 pm • 1

CH cuts the circumcircle of $\triangle ABC$ again at the reflection Z of H about AB. (E) is a Thebault circle of the cevian BH of $\triangle ZBC$. Thus if M,N denote the tangency points of (E) with $\overline{HB},\overline{HC}$, then by Sawayama's lemma, MN passes through the incenter V of $\triangle ZBC$. If J is the incenter of $\triangle BHC$, then simple angle chase gives
$\angle BJV = 90^\circ - \frac{1}{2}\angle BHC = \angle HMN = \angle BMV \Longrightarrow BJMV$ is cyclic $\Longrightarrow$
$\angle JMB = \angle BVC = 90^\circ + \frac{1}{2}\angle BZC = 90^\circ + \frac{1}{2}\angle BHZ = 180^\circ - \angle JHM \Longrightarrow \angle JMH = \angle JHM \Longrightarrow$
$\triangle JHM$ is J-isosceles $\Longrightarrow J$ coincides with the circumcenter of the H-isosceles $\triangle HMN \Longrightarrow J$ is midpoint of $\overline{HE}$.

realz

#3 Mar 6, 2013, 5:43 am • 1

We have that $\triangle ABC$ and $\triangle HBC$ have the same Euler circle so from Feuerbach theorem we have that the incicrle of $\triangle HBC$ is tangent to Euler circle of $\triangle ABC$. It is well known that homothety in H with coefficent 2 maps the Euler circle in to the circumcircle , so the image of the incircle of $\triangle HBC$ is a circle which is tangent to HB,HC and the circum circle $\Longrightarrow$ its image is $(E,r) \Longrightarrow$ the incenter is the middle of HE.

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Romanian Master of Mathematics 2013 - p3

Reply

Source: Romanian Master of Mathematics

Swistak

#1 Mar 3, 2013, 3:42 am

Let $ABCD$ be a quadrilateral inscribed in a circle ω. The lines AB and CD meet at P, the lines AD and BC meet at Q, and the diagonals AC and BD meet at R. Let M be the midpoint of the segment PQ, and let K be the common point of the segment MR and the circle ω. Prove that the circumcircle of the triangle KPQ and ω are tangent to one another.

My solution

Luis González

#2 Mar 4, 2013, 9:41 am

If Y is the 2nd intersection of ω and $\odot(CPQ)$, then M, A, Y are collinear (see the solution of the problem Three collinear points). YR cuts ω again at Z and $S \equiv CY \cap AZ$, thus MS is the polar of R WRT $\omega \Longrightarrow S \in PQ$. Since MR is the polar of S WRT ω, then SK is tangent to ω. Therefore $SP \cdot SQ = SY \cdot SC = SK^2 \Longrightarrow SK$ is tangent to $\odot(KPQ) \Longrightarrow \odot(KPQ)$ and ω are tangent.

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tangent to the incircle

Locked

JRD

#1 Mar 4, 2013, 4:03 am

let ABC be a triangle and H be foot of altitude through A . point L is the midpoint of AH . incircle of $\triangle ABC$ touch BC at D . and T is the intersection point of incircle and DL . prove that circumcircle of triangle BTC is tangent to the incircle .

Luis González

#2 Mar 4, 2013, 4:09 am

Use the search function before posting. This is problem 7 of IMO Shortlist 2002.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=14741
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=17323
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=205790
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=265444

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Concurrence of lines determined by midpoints

Reply

Source: Turkey MOWP 2013

yunustuncbilek

#1 Mar 4, 2013, 12:25 am

Let A_1, B_1, C_1 are points on BC, AC, AB, respectively, such that AA_1, BB_1, CC_1 intersect. D, D', E, E', F, F' are the midpoints of $BC, B_1C_1, AC, A_1C_1, AB, A_1B_1$. Prove that DD', EE', FF' intersect at a point collinear with the gravity center of ABC and the intersection of AA_1, BB_1, CC_1.

Luis González

#2 Mar 4, 2013, 12:51 am • 1

Denote $P \equiv AA_1 \cap BB_1 \cap CC_1$ and A_2, B_2, C_2 the midpoints of PA, PB, PC. D, D', A_2 are collinear on the Newton line of the complete quadrangle BCB_1C_1. Likewise, EE' and FF' pass through B_1 and C_1. Obviously, $\triangle A_2B_2C_2 \sim \triangle ABC$ are homothetic under homothety $(P, \frac{1}{2})$ and $\triangle ABC \sim \triangle DEF$ are homothetic under homothety $(G, -2)$, where G denotes the centroid of $\triangle ABC$. Hence, DD', EE' and FF' concur at the center K of the composition $(K, -1) \equiv (P, \frac{1}{2}) \circ (G, -2)$, whose center is then on PG.

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Nice Tangent Circles Perpendicular Problem

Reply

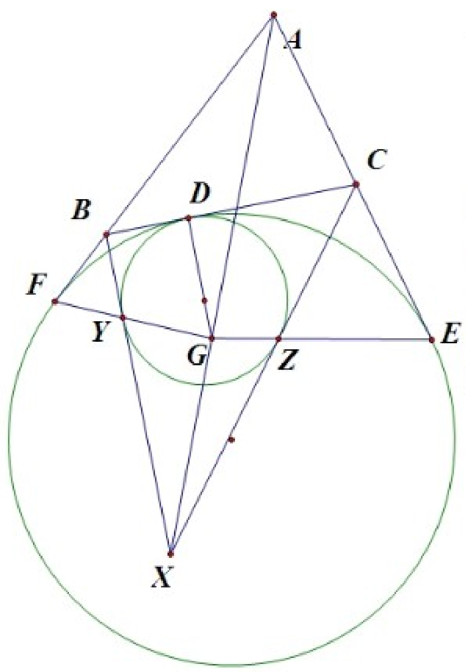
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Mar 3, 2013, 5:54 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



已知 （文武光华数学工作室 南京 潘成华）△ABC的A-旁切圆与边BC与AC,AB延长线分别内切于D,E,F,点X是∠BAC内一点，△XBC的内切圆也在D与BC相切并与XB,XC分别相切于Y,Z.直线EY,FZ交于G.
（2013 2 27 10：35）
求证 DG⊥BC

The A-excircle of △ABC touches BC,AC,AB at D,E,F. X is a point inside ∠BAC such that The incircle of △XBC touches BC,XB,XC at D,Y,Z. EY meets FZ at G.
Prove: DG⊥BC

Luis González

#2 Mar 3, 2013, 6:29 am

This is an extraversion of a configuration discussed for the incircle. AX, FY and EZ concur at the insimilicenter G of the incircle of XBC and the A-excircle of ABC.

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http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=207396

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High School Olympiads

Circumcircle,incenter and perpendicular

Locked

Source: Unknown

MMEEvN

#1 Mar 2, 2013, 11:08 pm

Let I be the incenter and AD be a diameter of circumcircle of a triangle ABC.If the point E on ray BA and the point F on the ray CA satisfy the condition $BE=CF=\frac{AB+BC+CA}{2}$. Show that the lines EF and DI are perpendicular

Luis González

#2 Mar 2, 2013, 11:23 pm

Use the search function before posting.

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High School Olympiads

Beautiful conditions - show that point lies on excircle

Reply

Source: Burii

timon92

#1 Feb 28, 2013, 1:30 am

Given triangle ABC in which $\sphericalangle BAC = \frac{\pi}{3}, \sphericalangle ACB = \frac{\pi}{4}$ and $\sphericalangle CBA = \frac{5\pi}{12}$. Incircle touches BC at point D. Point X lies inside angle BAC and outside triangle and satisfies following conditions:

$$BD \cdot BX = CD \cdot CX$$

$$\tan \sphericalangle CXB = -\frac{1086}{1081} + \frac{1404\sqrt{2}}{1081} - \frac{1054\sqrt{3}}{1081} + \frac{634\sqrt{6}}{1081}$$

Show that X lies on A-excircle.

timon92

#2 Mar 1, 2013, 6:02 am

I have edited the value of tangent of angle CXB because at first it was computed wrongly (thanks to Luis for pointing it out).

Luis González

#3 Mar 1, 2013, 8:51 am

Label (I, r) and (I_a, r_a) the incircle and A-excircle. (I_a) touches BC, CA, AB at E, Y, Z and YZ cuts BC at F (harmonic conjugate of E WRT B,C). $\frac{BX}{CX} = \frac{CD}{BD} = \frac{BE}{CE} \Longrightarrow X$ is on Apollonius circle ω_A with diameter $\overline{EF}$. If U is the 2nd intersection of (I_a) and $\omega_{A:}$ then it suffices to show that $\tan \widehat{BUC}$ equals the value given in the 2nd condition. Hence we'll have $X \equiv U$.

Since UE bisects $\angle BUC$, then (I_a) is internally tangent to $\odot(BCU)$. In this configuration, it's well-known that U, E, I are collinear, thus perpendicular bisector of BC cuts IE at the midpoint P of the arc BC of $\odot(UBC)$. If M is the midpoint of BC (also midpoint of ED), then MP is E-midline of $\triangle IDE \Longrightarrow PM = \frac{1}{2}ID = \frac{1}{2}r$.

$$\tan \frac{\widehat{BUC}}{2} = \tan \widehat{PCM} = \frac{PM}{MC} = \frac{r}{a} \Longrightarrow$$

$$\tan \widehat{BUC} = \frac{2 \cdot \tan \frac{\widehat{BUC}}{2}}{1 - \tan^2 \frac{\widehat{BUC}}{2}} = \frac{2 \cdot \frac{r}{a}}{1 - \frac{r^2}{a^2}} = \frac{2ar}{a^2 - r^2} \ (\star).$$

For a $\triangle ABC$ with the given conditions, the inradius r in terms of $BC = a$ is given by

$$r = \frac{\sqrt{2} + \sqrt{6} - 2}{4\sqrt{3}} \cdot a \Longrightarrow \tan \frac{\widehat{BUC}}{2} = \frac{\sqrt{2} + \sqrt{6} - 2}{4\sqrt{3}}.$$

Substituting this value into the expression $(\star)$, we get

$$\tan \widehat{BUC} = \frac{2 \cdot \frac{\sqrt{2}+\sqrt{6}-2}{4\sqrt{3}}}{1 - \left(\frac{\sqrt{2}+\sqrt{6}-2}{4\sqrt{3}}\right)^2} = -\frac{1086}{1081} + \frac{1404\sqrt{2}}{1081} - \frac{1054\sqrt{3}}{1081} + \frac{634\sqrt{6}}{1081}.$$

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High School Olympiads

Concurrent Euler lines

Reply

Source: Own

buratinogigle
#1 Feb 21, 2013, 11:07 pm

Problem1. Let ABC be a triangle with circumcenter O. P is a point. Let O_a, O_b, O_c be circumcenter of triangles PBC, PCA, PAB, respectively. Prove that Euler lines of triangles $OO_bO_c, OO_cO_a, OO_aO_b$ are concurrent iff AO_a, BO_b, CO_c are concurrent.

Problem 2. Let ABC be equilateral triangle with center O. P is a point. Let O_a, O_b, O_c be circumcenter of triangles PBC, PCA, PAB, respectively. Prove that Euler lines of triangles $OO_bO_c, OO_cO_a, OO_aO_b$ are concurrent.

Luis González
#2 Feb 26, 2013, 10:05 pm • 1

> *buratinogigle wrote:*
> **Problem1.** Let ABC be a triangle with circumcenter O. P is a point. Let O_a, O_b, O_c be circumcenter of triangles PBC, PCA, PAB, respectively. Prove that Euler lines of triangles $OO_bO_c, OO_cO_a, OO_aO_b$ are concurrent iff AO_a, BO_b, CO_c are concurrent.

We assume that P does not lie either on the circumcircle (O) or the line at infinite. Let $\triangle A_0B_0C_0$ be the antipedal triangle of P WRT $\triangle ABC$ and Q the isogonal conjugate of P WRT $\triangle A_0B_0C_0$ (reflection of P on O). Hence, in the homothety with center P and and factor 2, O goes to Q, A, B, C go to the reflections A_1, B_1, C_1 of P on B_0C_0, C_0A_0, A_0B_0 and O_a, O_b, O_c go to $A_0, B_0, C_0 \Longrightarrow$ Euler lines ℓ_a, ℓ_b, ℓ_c of $\triangle OO_bO_c, \triangle OO_cO_a, \triangle OO_aO_b$ go to Euler lines τ_a, τ_b, τ_c of $\triangle QB_0C_0$, $\triangle QC_0A_0, \triangle QA_0B_0$.

Hence, ℓ_a, ℓ_b, ℓ_c concur $\Longleftrightarrow \tau_a, \tau_b, \tau_c$ concur $\Longleftrightarrow Q$ is on Neuberg cubic $\mathcal{N}$ of $\triangle A_0B_0C_0 \Longleftrightarrow P$ is on Neuberg cubic $\mathcal{N}$ of $\triangle A_0B_0C_0$ ($\mathcal{N}$ is self-isogonal) $\Longleftrightarrow A_1A_0, B_1B_0, C_1C_0$ concur $\Longleftrightarrow AO_a, BO_b, CO_c$ concur.

Luis González
#3 Feb 26, 2013, 11:09 pm • 1

> *buratinogigle wrote:*
> **Problem 2.** Let ABC be equilateral triangle with center O. P is a point. Let O_a, O_b, O_c be circumcenter of triangles PBC, PCA, PAB, respectively. Prove that Euler lines of triangles $OO_bO_c, OO_cO_a, OO_aO_b$ are concurrent.

As the previous solution, let $\triangle A_0B_0C_0$ be the antipedal triangle of P WRT $\triangle ABC$ and Q the isogonal conjugate of P WRT $\triangle A_0B_0C_0$. Pedal triangle $\triangle ABC$ of P WRT $\triangle A_0B_0C_0$ is equilateral $\Longrightarrow P$ is an isodynamic point of $\triangle A_0B_0C_0 \Longrightarrow Q$ is a Fermat point of $\triangle A_0B_0C_0 \Longrightarrow$ Euler lines of $\triangle QB_0C_0, \triangle QC_0A_0, \triangle QA_0B_0$ concur at the centroid G_0 of $\triangle A_0B_0C_0$ (see Fermat point and Euler lines). Therefore, Euler lines of $\triangle OO_bO_c, \triangle OO_cO_a, \triangle OO_aO_b$ concur at the midpoint of PG_0.

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Inscribed and circumscribed quadrilateral

Reply

hEatLove

#1 Feb 24, 2013, 4:49 pm

$ABCD$ be the inscribed and circumscribed quadrilateral . Line pass through these 2 circles be l
a). Prove that line l pass through intersection point of diagonals of $ABCD$
b). P be the intersection of external bisectors of angle A , C and Q be the intersection of external bisectors of angle B , D . Prove at l is perpendicular with PQ

Luis González

#2 Feb 26, 2013, 1:03 am

Let $(I),(O)$ be the incircle and circumcircle of $ABCD.$ The collinearity of I,O and the intersection $AC \cap BD$ of diagonals is well-known (see collinearity in bicentric quardilateral and elsewhere).

Let (I) touch AB,BC,CD,DA at $K,M,N,L.$ IA,IB,IC,ID cut LK,KM,MN,NL at their midpoints $A',B',C',D'.$ External bisectors of $\angle A,\angle B,\angle C,\angle D$ are then the polars of A',B',C',D' WRT $(I) \Longrightarrow P,Q$ are the poles of $A'C'$ and $B'D'$ WRT $(I) \Longrightarrow S \equiv A'C' \cap B'D'$ is the pole of PQ WRT $(I) \Longrightarrow IS \perp PQ.$ The inversion WRT (I) takes the vertices of the parallelogram $A'B'C'D'$ into the vertices of the cyclic $ABCD \Longrightarrow A'B'C'D'$ is a rectangle with center S lying on $OI.$ Hence, $OIS \perp PQ.$

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Nice Orthogonal and Concurrent(Own)

Reply

Arab

#1 Feb 24, 2013, 11:17 am

Let I be the incenter of $\triangle ABC$ and $AI \cap BC = A_1$,similarly define B_1, C_1.A_2 lies on BC such that $IA_1 = IA_2$,similarly define B_2, C_2.
(1).Prove that,$\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are orthogonal.
(2).Let A_3 be the intersection of AI and Γ the circumcircle of $\triangle ABC$,similarly define B_3, C_3.
ℓ_1 passes through A_3 and is perpendicular to B_2C_2,similarly define ℓ_2, ℓ_3.Prove that,ℓ_1, ℓ_2, ℓ_3 are concurrent.

Luis González

#2 Feb 25, 2013, 11:25 am • 2

1) Incircle (I) touches BC, CA, AB at $D, E, F \Longrightarrow A_2, B_2, C_2$ are reflections of A_1, B_1, C_1 WRT $D, E, F,$ respectively. If T is the orthocenter of $\triangle DEF,$ then DT is the midparallel of IA_1 and the perpendicular from A_2 to $EF \Longrightarrow$ perpendicular from A_2 to EF goes through the reflection J of I about $T.$ Similarly perpendiculars from B_2 and C_2 to FD and DE pass through $J \Longrightarrow \triangle DEF$ and $\triangle A_2B_2C_2$ are orthologic through $J \Longrightarrow$ perpendiculars from D, E, F to B_2C_2, C_2A_2, A_2B_2 concur at their second orthology center $K.$ So if L denotes the orthocenter of $\triangle A_2B_2C_2,$ then the perpendiculars from A_1, B_1, C_1 to B_2C_2, C_2A_2, A_2B_2 concur at the reflection of L about $K,$ i.e. $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are orthologic.

2) $\triangle A_3B_3C_3$ is clearly homothetic to $\triangle DEF,$ thus it is also orthologic to $\triangle A_2B_2C_2 \Longrightarrow \ell_1, \ell_2$ and ℓ_3 concur at one orthology center.

Arab

#3 Feb 25, 2013, 7:31 pm

Yeah.These two is actually very easy.

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High School Olympiads

Intersect on circle and collinear

Source: Own

buratinogigle

#1 Feb 9, 2013, 1:16 am • 2

Let ABC be a triangle with circumcenter O. E,F are on CA,AB, respectively such that $EF\parallel BC$. (K) is circumcircle of triangle AEF. Assume that (K) cuts BC at M,N such that M is between B and N. Let S,T be circumcenter of triangle ABM and ACM, respectively. Prove that ET and FS intersect at a point P on (K) and P,O,N are collinear.

Click to reveal hidden text

Luis González

#2 Feb 24, 2013, 3:29 am • 2

WLOG assume that $\angle AMB$ is acute, thus S is inside $\triangle ABC$ and T is outside $\triangle ABC$. Since $\angle ASB=\angle ATC=2\angle AMB$, then the isosceles $\triangle ASB$ and $\triangle ATC$ are spirally similar with center $A \Longrightarrow \triangle ABC$ and $\triangle AST$ are also spirally similar with center A. Moreover $FA:FB=EA:EC$ means that SF and TE are corresponding cevians of $\triangle ASB$ and $\triangle ATC$, thus if $P \equiv ET \cap FS$, then $\angle PTA=\angle PSA \Longrightarrow A,S,T,P$ are concyclic $\Longrightarrow \angle TPS=\angle TAS=\angle EAF \Longrightarrow P \in (K)$.

Since OT,OS are perpendicular bisectors of CA,AB, then $\angle SOT=\angle BAC \pmod{\pi} \Longrightarrow O \in \odot(AST)$. Hence $\angle EPO=\angle TAO=\angle BAM$, but $\angle BAM \equiv \angle FAM=\angle EAN=\angle EPN \Longrightarrow \angle EPO=\angle EPN \Longrightarrow$ $O \in PN$.

disneyalice710

#3 Aug 14, 2014, 10:09 pm • 1

Hic my solution is the same as Luis González!
I dont know I am happy or a little sad!

disneyalice710

#4 Aug 15, 2014, 4:16 pm • 2

Let see:
a) $\angle BTA==2\angle AMC$
then,$\triangle ATBissimilarto\triangle ASC$
more over, AF:AB=$AE:AC$
then, ATF is similar to ASE(cgc)
so, $\angle TFA=\angle SEAor\angle PFA=\angle PEA$, which implies APFE is cyclic.
b) We have$\angle EPN=\angle EMN=\angle FAM$, more over, $\angle BAO=\angle MAS=90-\angle ACB$
then, $\angle BAM=\angle SAO=\angle SPO$(APOS is cyclic_===), so$\angle SPO=\angle SPN$, which impliesP,O,N are collinear

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High School Olympiads

Prove that I is incenter

Reply

Source: Own

buratinogigle

#1 Feb 10, 2013, 9:58 pm

Let ABC be a triangle with bisector AD, D is in BC. M is on CA such that $DM \parallel AB$. Bisector of $\angle ABM$ cuts AD at E. Circumcircle of triangle ABE cuts BM again at F. AF cuts BE at I. CI cuts BM at K. Prove that I is the incenter of triangle KAB.

See more IMO Shortlist 2011, G6.

Luis González

#2 Feb 22, 2013, 11:29 pm • 1

E is clearly incenter of $\triangle ABM$. $\angle AFB = \angle AEB = 90^\circ + \frac{1}{2}\angle AMB \Longrightarrow \triangle MAF$ is M-isosceles $\Longrightarrow ME$ is perpendicular to AF at Q and $MA = MF = MD \Longrightarrow \triangle MDF$ is M-isosceles, thus the triangle bounded by BM, DF and AB is also isosceles $\Longrightarrow BE$ is perpendicular to DF at N.

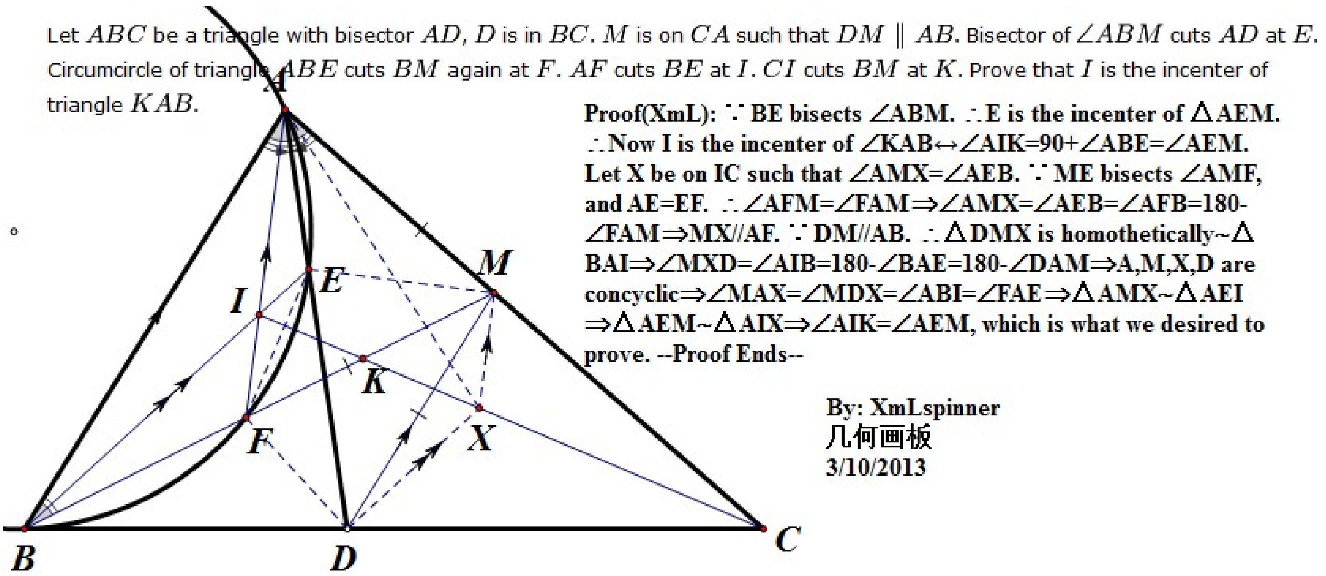
$\angle MDA = \angle MAD = \angle BAE = \angle MFE \Longrightarrow MEFD$ is cyclic $\Longrightarrow \angle BFN = \angle AEQ$. From cyclic $EFNQ$ (due to right angles at Q,N) we get $\angle IFN = \angle IEQ$. If the parallel from E to $AB \parallel DM$ cuts $\odot(AEB)$ again at S, then $\angle BES = \angle EBA = \angle EFI$. Hence, we conclude that the pencils $E(A,B,Q,S)$ and $F(B,I,N,E)$ are congruent. If MQ cuts BC, DF at P, R, then $R(B,I,N,E) = E(A,B,Q,S) = M(A,B,Q,D) = R(B,C,D,P) \Longrightarrow R, I, C$ are collinear. Since I is orthocenter of $\triangle FER$, then $CI \perp FE \Longrightarrow \angle AIK = 90^\circ + \angle AFE = 90^\circ + \angle ABI \Longrightarrow I$ is incenter of $\triangle KAB$.

XmL

#3 Mar 11, 2013, 10:57 am • 1

See attachment for my solution

Attachments:



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High School Olympiads

o in convex quadrilateral

Locked

sahadian

#1 Feb 21, 2013, 1:45 am

Let $ABCD$ be a convex quadrilateral. Lines AB and CD meet at P, and lines AD and BC meet at Q. Let O be a point in the interior of $ABCD$ such that $\angle BOP = \angle DOQ$. Prove that
$\angle AOB + \angle COD = 180$.

Luis González

#2 Feb 21, 2013, 2:14 am

Use the search function before posting.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=15103
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=85920
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=213544

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High School Olympiads

inequality 212

Reply

Source: Nicusor Zlota,CTATV

nicusorz

#1 Feb 19, 2013, 7:51 pm

In any triangle ABCwe have the following inequalities :
$\sum a^4 \geq 16r^2\sum r_a^2$

Luis González

#2 Feb 20, 2013, 6:22 am

Using Ravi's substitution $x=p-a, y=p-b$ and $z=p-c$, we obtain

$$r^2 \cdot {r_a}^2 = r^4 \cdot \frac{p^2}{(p-a)^2} = \frac{(xyz)^2}{(x+y+z)^2} \cdot \frac{(x+y+z)^2}{x^2} = y^2z^2.$$

Therefore, the proposed inequality is equivalent to

$$(y+z)^4+(z+x)^4+(x+y)^4 \geq 16(y^2z^2+z^2x^2+x^2y^2),$$

which is obviously true, since by AM-GM we have $(y+z)^4 \geq 16y^2z^2$ and cyclically.

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An easy one

Reply

seby97

#1 Feb 17, 2013, 9:07 pm

Let ABCD be a rhombus and O its center. Let N onto [AB], $MN||BC, M \in [CD], NE||AC, E \in MB, NF||BD,$ $F \in [AM]$, S the midpoint of AB and G be the intersection of EF and MS. Prove:

a) $\frac{EB}{EM} + \frac{FA}{FM} = 1$

b) G is the centroid of AMB

c) O,G,N collinear and 2OG=GN.

Luis González

#2 Feb 20, 2013, 4:23 am

From $NE \parallel AC, NM \parallel AD$, we get $\angle BNE = \angle BAO = \angle DAO = \angle MNE \Longrightarrow NE, NF$ bisect $\angle(AB, MN)$. By angle bisector theorem, we obtain then

$$\frac{EB}{EM} = \frac{NB}{NM}, \frac{FA}{FM} = \frac{NA}{NM} \Longrightarrow \frac{EB}{EM} + \frac{FA}{FM} = \frac{NB + NA}{NM} = \frac{AB}{NM} = 1.$$

By Cristea's theorem for $\triangle MAB$, cut by transversal EF, we get then

$$AS \cdot \frac{EB}{EM} + BS \cdot \frac{FA}{FM} = AB \cdot \frac{GS}{GM} \Longrightarrow$$

$$\frac{AB}{2}\left(\frac{EB}{EM} + \frac{FA}{FM}\right) = AB \cdot \frac{GS}{GM} \Longrightarrow GM = 2 \cdot GS \Longrightarrow G$$ is centroid of AMB.

$MN \parallel AD \parallel OS$ and $MN = DA = 2 \cdot SO$ implies that O is the complement of N WRT $\triangle MAB \Longrightarrow O, G, N$ are collinear and $GN = 2 \cdot OG$.

sunken rock

#3 Feb 21, 2013, 11:18 pm

Alternate solution for a):

Let $E' = CD \cap NE, F' = CD \cap NF$. Then $ME' = MF' = CD$ (1). Now $\frac{BE}{ME} = \frac{BN}{ME'}, \frac{AF}{MF} = \frac{AN}{MF'}$, so $\frac{AF}{MF} + \frac{BE}{ME} = \frac{BN + AN}{CD} = 1$, and this is a necessary and sufficient condition for the centroid of ΔABM to lie onto EF, and point $a)$ is solved.

Best regards,
sunken rock

hero12

#4 Feb 27, 2013, 7:22 pm

This is from Local Mathematical Olympiad,Suceava(Romania),2012!

Quick Reply

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High School Olympiads

A good geometry problem

Locked

TONCAS

#1 Feb 19, 2013, 10:38 pm

Let $\square ABCD$ be a square. A point P belongs to interior of $\square ABCD$ such that $PA : PB : PC = 1 : 2 : 3$. Find $\angle APB$.

Luis González

#2 Feb 20, 2013, 12:56 am

Posted before, so for further discussions use any of the links below.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=151&t=59825
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=150&t=502397
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=150&t=506726

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interesting geometry!!!

---Fermat---

#1 Feb 17, 2013, 2:23 pm

Let KL and KN be the tangents from a point K to a circle k. Point M is arbitrarily taken on the extension of KN past N, and P is the second intersection point of k with the circumcircle of triangle KLM. The point Q is the foot of the perpendicular from N to ML. Prove that □MPQ = 2□KML

---Fermat---

#2 Feb 17, 2013, 7:09 pm

> ---Fermat--- wrote:
>
> Let KL and KN be the tangents from a point K to a circle k. Point M is arbitrarily taken on the extension of KN past N, and P is the second intersection point of k with the circumcircle of triangle KLM. The point Q is the foot of the perpendicular from N to ML. Prove that □MPQ = 2□KML

Hint: Click to reveal hidden text

yetti

#3 Feb 19, 2013, 9:57 pm

Label the circle tangent to KL, KN at L, N as (J). Let $[LM]=k, [MK]=l, =[KL]=m$ and $p=\frac{1}{2}(k+l+m)$.
Let (O) be circumcircle of $\triangle KLM$. Let LM cut (J) again at R.
Power of M to (J) is $[MN]^2=[LM]\cdot[RM]$, where $[MN]=[MK]-[NK]=[MK]-[KL]=l-m\implies$ $[RM]=\frac{[MN]^2}{[LM]}=\frac{(l-m)^2}{k}$.
Let PR cut KM at S and (O) again at T. By Reim for circles $(O),(J)$ intersecting at $L,P\implies KT\parallel LM$. By Ptolemy for isosceles cyclic trapezoid $KLMT\implies[KT]=\frac{[KM]\cdot[TL]-[KL]\cdot[TM]}{[LM]}=\frac{l^2-m^2}{k}$.
From central similarity of $\triangle MSR\sim\triangle KST$ with similarity center S similarity coefficient $-\frac{[RM]}{[KT]}=-\frac{l-m}{l+m}=-\frac{[MS]}{[SK]}$ and from $[MS]+[SK]=[MK]=l\implies[MS]=\frac{1}{2}(l-m),[SK]=\frac{1}{2}(l+m)$.
Since $[MN]=l-m\implies S$ is midpoint of $[MN]\implies[QS]=[MS]=[SN]\implies\triangle SQM$ is S-isosceles $\implies$ $\measuredangle SPM=\measuredangle TPM=\measuredangle TLM=\measuredangle LMK=\measuredangle QMS=\measuredangle SQM\implies$ $PQSM$ is cyclic. $[QS]=[MS]$ then $\implies PS$ bisects $\measuredangle QPM\implies\measuredangle QPM=2\measuredangle SPM=2\measuredangle LMK$.

Luis González

#4 Feb 19, 2013, 11:23 pm • 1

ML cuts k again at S. $\angle SPL=\angle MLK=\angle MPK\Longrightarrow\angle MPS=\angle LPK=\angle LMK\Longrightarrow\odot(PMS)$ is tangent to MK. Hence PS is radical axis of k and $\odot(PMS)$, cutting their common tangent $\overline{MN}$ (hypotenuse of MQN) at its midpoint $D\Longrightarrow\triangle MDQ$ is D-isosceles $\Longrightarrow\angle DQM=\angle DMQ=\angle DPM\Longrightarrow PQDM$ is cyclic. Hence $DM=DQ$ implies that PD bisects $\angle MPQ\Longrightarrow\angle MPQ=2\angle MPS=2\angle KML$.

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Mixtillinear Excircles Collinearity

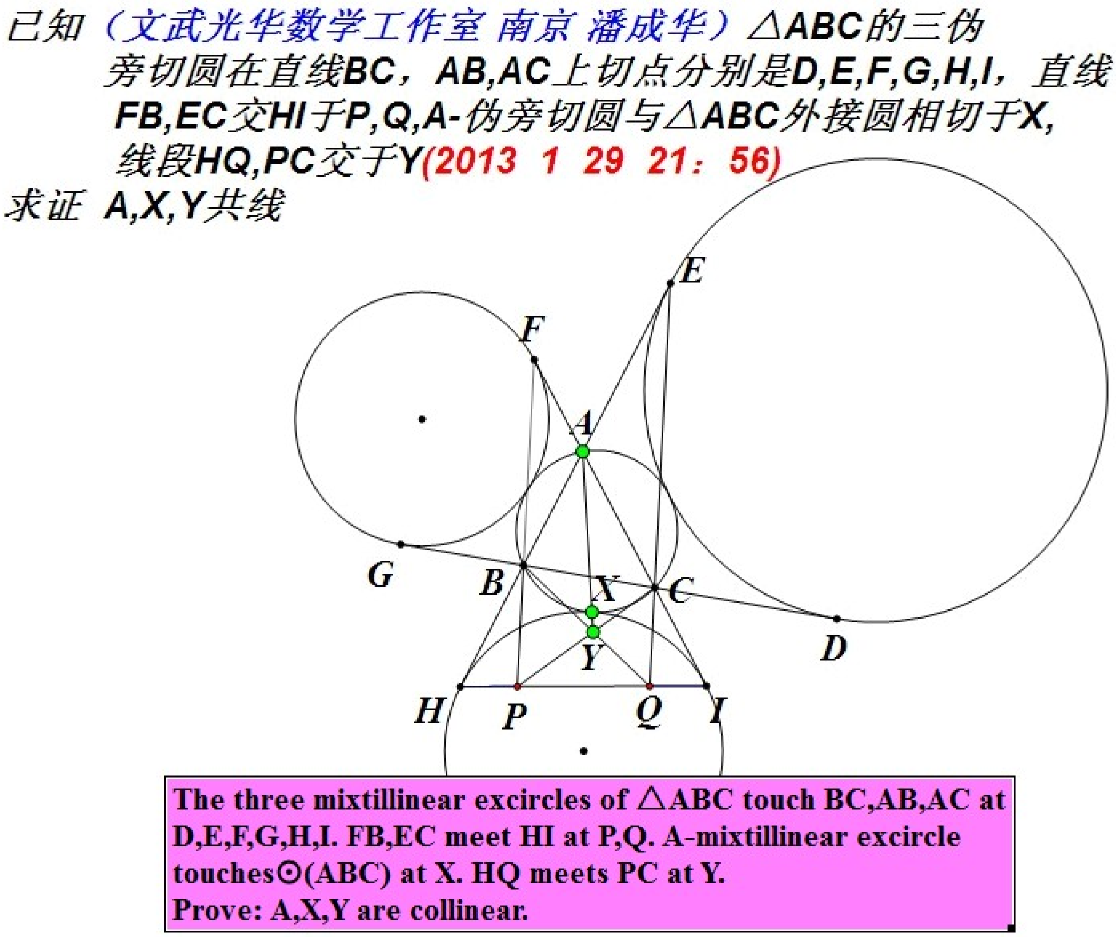
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Feb 12, 2013, 9:28 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Feb 19, 2013, 4:33 am

Let AY, BQ, CP cut BC, CA, AB at M, N, L, respectively. From $(C, A, N, I) = (E, A, B, H)$, we get

$$\frac{NC}{NA}\cdot\frac{IA}{IC}=\frac{BE}{BA}\cdot\frac{HA}{HE}\Longrightarrow\frac{NC}{NA}=\frac{BE}{BA}\cdot\frac{IC}{HE}=\frac{BE}{BA}\cdot\frac{IC}{BH+BE}.$$

Using $BE=\frac{ac}{p-b}$, $HA=\frac{bc}{p-a}$, $BH=\frac{c(p-c)}{p-a}$, $IC=\frac{b(p-b)}{p-a}$, we get

$$\frac{NC}{NA}=\frac{ab(p-b)}{c[a(p-a)+(p-b)(p-c)]}.$$

Similarly, we have $\frac{LB}{LA} = \frac{ac(p-c)}{b[a(p-a)+(p-b)(p-c)]}$.

By Ceva's theorem for concurrent $AM, BN, CL,$ we have $\frac{BM}{MC} = \frac{NA}{NC} \cdot \frac{LB}{LA}$. Hence

$\frac{BM}{MC} = \frac{c[a(p-a)+(p-b)(p-c)]}{ab(p-b)} \cdot \frac{ac(p-c)}{b[a(p-a)+(p-b)(p-c)]} = \frac{c^2}{b^2} \cdot \frac{p-c}{p-b}$.

By Steiner theorem, AYM is the isogonal of the A-Gergonne cevian of $\triangle ABC \Longrightarrow AX \equiv AY.$

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Concurrency in a special case

Reply

mathreyes
#1 Feb 18, 2013, 8:08 pm

Let ΔABC our triangle and take randomly $P \in \overline{AB}$.
$X \in \overline{AC}$ and $Y \in \overline{BC}$ are such that $\overline{PX}$ and $\overline{PY}$ bisect $\angle APC$ and $\angle BPC$ respectively.
Show that ΔABC and the medial triangle of ΔPXY are in point perspective.

Luis González
#2 Feb 18, 2013, 9:20 pm • 1

PX cuts BC at M and BX cuts CP at Q. Since PX, PY bisect $\angle BPC$, then the cross ratio (B, C, Y, M) is harmonic. But from the complete quadrangle $BCXP$, the pencil $A(B, C, Q, M)$ is harmonic $\Longrightarrow Y \in AQ \Longrightarrow \triangle PXY$ is cevian triangle of Q WRT $\triangle ABC$. Now by Cevian Nest Theorem, $\triangle ABC$ and the medial triangle of $\triangle PXY$ are perspective through the cevian quotient G^*/Q of $\triangle PXY$, where G^* denotes the centroid of $\triangle PXY$.

mathreyes
#3 Feb 18, 2013, 11:57 pm

> Luis González wrote:
> ...$\Longrightarrow \triangle PXY$ is cevian triangle of Q WRT $\triangle ABC$....

This is a straightforward result of using the angle bisector theorem and ceva's theorem. (this can be usefull for those (including ME xD) who are just babys on harmonic ratios.)

> Luis González wrote:
> Now by Cevian Nest Theorem, $\triangle ABC$ and the medial triangle of $\triangle PXY$ are perspective through the cevian quotient G^*/Q of $\triangle PXY$, where G^* denotes the centroid of $\triangle PXY$.

This is JUST I asked to prove! but thanks xD I didn't know about this result :)

Definitely I learn more by reading your solutions than reading a book :)

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Reflection And Collinear

Reply

Headhunter

#1 Feb 17, 2013, 11:21 pm • 1

Hello.

For $\triangle ABC$, let two perpendicular bisectors of AB , BC be m , n
Reflect m , n in BC , AB respectively and let the intersection of the reflections be X
Reflect m , n in AC and let the intersection of the reflections be Y
show that B , X , Y are collinear.

Luis González

#2 Feb 18, 2013, 4:09 am • 1

Y is obviously the reflection of the circumcenter O of ABC about AC. If A',C' denote the reflections of A,C about CB,AB, then the reflections of m,n about BC,AB become perpendicular bisectors of BA' and BC', thus X is circumcenter of BA'C'. Now the problem is exactly the same as P3 of Tuymaada 2009 Senior League, Second Day.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=289760

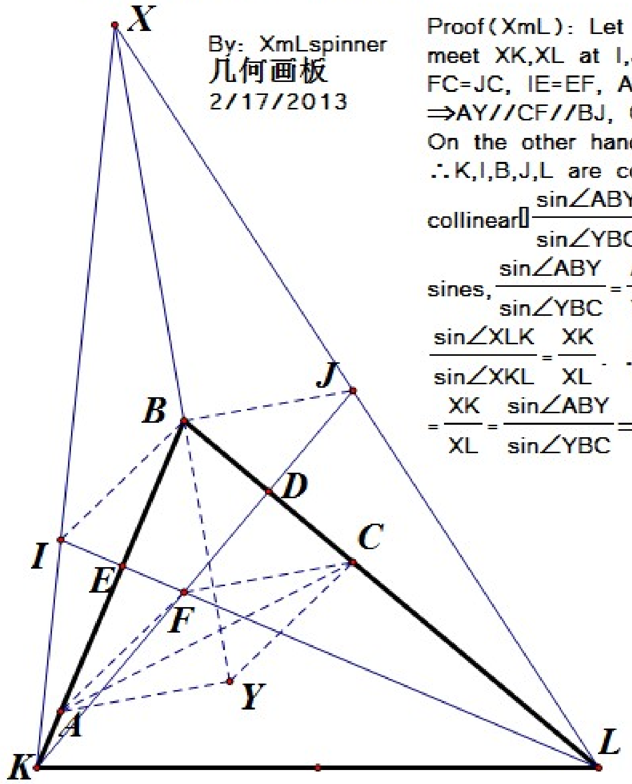
XmL

#3 Feb 18, 2013, 4:21 am • 1

See Attachment for an alternate trig solution. Long time no problem solving....

Attachments:

For $\triangle ABC$, let two perpendicular bisectors of AB , BC be m , n
Reflect m , n in BC , AB respectively and let the intersection of the reflections be X
Reflect m , n in AC and let the intersection of the reflections be Y
show that B , X , Y are collinear.



Proof(XmL): Let m,n meet BC,AB at L,K resp. m,n meet XK,XL at I,J resp. m,n meet at F. ∵BD=DC, FC=JC, IE=EF, AE=BE. ∴IB=AF=AY=CY=CF=BJ ⇒AY//CF//BJ, CY//AF//IB.
On the other hand, ∵∠KBL=∠DFL=∠KJL=∠KIL. ∴K,I,B,J,L are concyclic. Connect XB,BY. B,X,Y are collinear $\Leftrightarrow \frac{\sin\angle ABY}{\sin\angle YBC}=\frac{\sin\angle XBK}{\sin\angle XBL}$. By the law of sines, $\frac{\sin\angle ABY}{\sin\angle YBC}=\frac{AY}{YC}*\frac{\sin\angle BAY}{\sin\angle BCY}*\frac{BY}{BY}=\frac{\sin\angle KBJ}{\sin\angle IBL}=\frac{\sin\angle XLK}{\sin\angle XKL}=\frac{XK}{XL}$. ∴ $\frac{\sin\angle XBK}{\sin\angle XBL}=\frac{XK}{XL}*\frac{\sin\angle XLB}{\sin\angle XKB}*\frac{XB}{XB}=\frac{XK}{XL}=\frac{\sin\angle ABY}{\sin\angle YBC}$ ⇒B,X,Y are collinear. -Proof Ends-

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An extension of a problem

Reply

Source: own

MariusStanean

#1 Feb 18, 2013, 12:43 am

In triangle ABC let D be the feet of altitude from A and $E \in (AD)$, such that $\frac{DE}{AE}=\frac{BD}{CD}$. Let F be the feet of perpendicular from D to the line BE. Prove that $AF \perp CF$.

Luis González

#2 Feb 18, 2013, 3:03 am • 1

Redefine F as the second intersection of the circles with diameters $\overline{AC}$ and $\overline{BD}$. Then we prove that BF cuts AD at a point E satisfying $\frac{DE}{AE}=\frac{BD}{CD}$.

Since $\angle BFD$ is right, then BF cuts the circle (M) with diameter $\overline{AC}$, again at the antipode P of D, i.e. the reflection P of D on $M \Longrightarrow ADCP$ is a rectangle $\Longrightarrow \overline{AP}$ and $\overline{CD}$ are congruent and parallel. Hence, from $\triangle AEP \sim \triangle DEB$, we obtain $\frac{DE}{AE}=\frac{BD}{AP}=\frac{BD}{CD}$, as desired.

MMEEvN

#3 Feb 18, 2013, 9:11 pm

$\frac{DC}{BD}=\frac{AE}{ED} \Longrightarrow \frac{BC}{AD}=\frac{BD}{DE}$

Hence $\frac{BC}{AD}=\frac{BD}{DE}=\frac{BF}{FD}$ Since $\angle FBD=\angle FDE$ we have that two triangles FDA and FBC are similar $\Longrightarrow \angle FAD=\angle FCD \Longrightarrow AFDC$ is cyclic from which the result follows immediately

sunken rock

#4 Feb 19, 2013, 12:11 am • 1

The given relation is equivalent to $\frac{DE}{BD}=\frac{AE}{CD}$, but from $\Delta BDE\sim\Delta DFE$ we get $\frac{DE}{BD}=\frac{EF}{FD}$, consequently $\frac{EF}{FD}=\frac{AE}{CD}$ (1). With $\angle AEF=\angle CDF$ we see $\Delta AEF\sim\Delta CDF$. Since $AE\perp CD, EF\perp FD$, there is a spiral similarity which maps the two triangles after a 90° rotation about F, followed by a homothecy. Consequently $AF\perp CF$, but this proof appeared before on this site.

Best regards,
sunken rock

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Property for 2 Secants of a Parabola

Reply

Source: 2013 Waseda University entrance exam/Science and Technology

Kunihiko_Chikaya

#1 Feb 17, 2013, 12:37 am • 1

Given a parabola $C : y^2 = 4px\ (p > 0)$ with focus $F(p,\ 0)$. Let two lines $l_1,\ l_2$ passing through F intersect orthogonaly each other,
C intersects with l_1 at two points $P_1,\ P_2$ and C intersects with l_2 at two points $Q_1,\ Q_2$. Answer the following questions.

(1) Set the equation of l_1 as $x = ay + p$ and let the coordinates of $P_1,\ P_2$ as $(x_1,\ y_1),\ (x_2,\ y_2)$, respectively. Express $y_1 + y_2,\ y_1y_2$ in terms of $a,\ p$.

(2) Show that $\frac{1}{P_1P_2} + \frac{1}{Q_1Q_2}$ is constant regardless of way of taking $l_1,\ l_2$.

Luis González

#2 Feb 17, 2013, 8:02 am • 1

> kunny wrote:
> (2) Show that $\frac{1}{P_1P_2} + \frac{1}{Q_1Q_2}$ is constant regardless of way of taking $l_1,\ l_2$.

Valid for any conic, so for the sake of ease we approach the case when $\mathcal{C}$ is an ellipse with focus F. The proof is similar when $\mathcal{C}$ is a hyperbola, except that the reciprocals of the lengths of the chords P_1P_2 and Q_1Q_2 have to be taken with appropriate sign, that is, (+) for chord outside the hyperbola and (-) for chord inside the hyperbola.

It's known that the polar transform of $\mathcal{C}$ WRT any circle (F, k) centered at its focus is a circle (O, ϱ). Hence, polars p_1, p_2, q_1, q_2 of P_1, P_2, Q_1, Q_2 WRT (F, k) bound a square with incircle (O, ϱ). They cut FP_1, FP_2, FQ_1, FQ_2 at the inverses X_1, X_2, Y_1, Y_2 of P_1, P_2, Q_1, Q_2 under inversion (F, k^2). By inversion properties we get then

$$k^2\left(\frac{1}{P_1P_2} + \frac{1}{Q_1Q_2}\right) = \frac{FX_1 \cdot FX_2}{X_1X_2} + \frac{FY_1 \cdot FY_2}{Y_1Y_2} = \frac{FX_1 \cdot FX_2 + FY_1 \cdot FY_2}{2\varrho}.$$

If M, N denote the projections of O on X_1X_2, Y_1Y_2, then we have $FX_1 \cdot FX_2 = \varrho^2 - ON^2$ and $FY_1 \cdot FY_2 = \varrho^2 - OM^2$. Hence

$$\frac{1}{P_1P_2} + \frac{1}{Q_1Q_2} = \frac{2\varrho^2 - (OM^2 + ON^2)}{2\varrho \cdot k^2} = \frac{2\varrho^2 - OF^2}{2\varrho \cdot k^2} = \text{const.}$$

Headhunter

#3 Feb 17, 2013, 10:56 pm • 3

Hello.

The following is a synthetic proof.

Let the feet of the perpendicular lines from P_1 , P_2 , Q_1 , Q_2 onto the directrix M , L , N , O
It's well known that the angle bisector of $\angle MP_1F$ is the tangent line at P_1
$\triangle MAP_1 \equiv \triangle FAP_1 \Longrightarrow FA = MA$, $\angle AFP_1 = 90^\circ$
Let the tangent at P_2 cut the parabola at A' and likewise, $\angle A'FP_2 = 90^\circ \Longrightarrow A' = A$
From the above, $\angle P_1AP_2 = 90^\circ$ and $AM = AF = AL \Longrightarrow \triangle MAP_1 \sim \triangle P_2LA$
Hence, $P_1M \cdot P_2L = AF^2$. Likewise, $Q_1N \cdot Q_2O = BF^2$

It's well known that $\dfrac{1}{P_1F}+\dfrac{1}{P_2F}=\dfrac{1}{p}$ and $\dfrac{1}{Q_1F}+\dfrac{1}{Q_2F}=\dfrac{1}{p}$

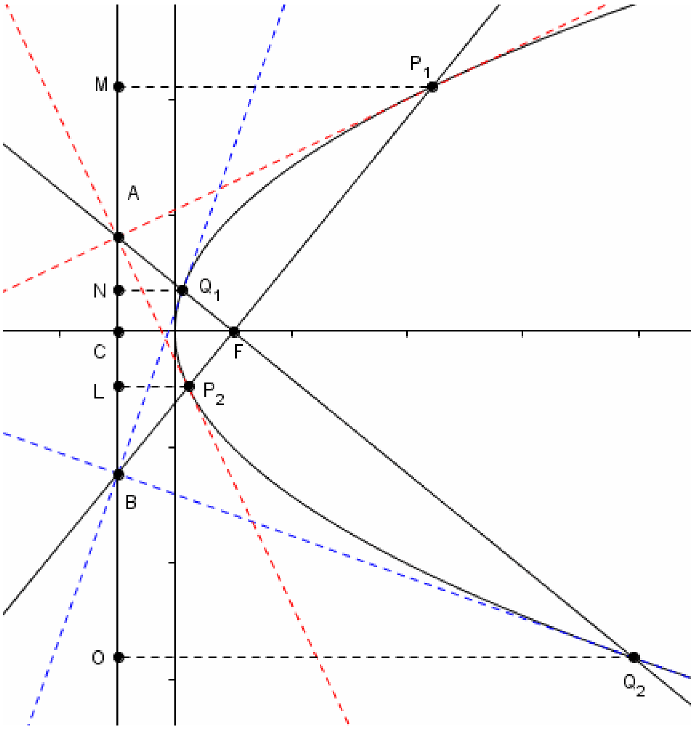
$$\implies \frac{1}{P_1F}+\frac{1}{P_2F}=\frac{P_1F+P_2F}{P_1F\cdot P_2F}=\frac{P_1P_2}{P_1F\cdot P_2F}=\frac{1}{p}\implies \frac{1}{P_1P_2}=\frac{p}{P_1F\cdot P_2F}$$

Likewise, $\dfrac{1}{Q_1Q_2}=\dfrac{p}{Q_1F\cdot Q_2F}$

$$\frac{1}{P_1P_2}+\frac{1}{Q_1Q_2}=p\left(\frac{1}{P_1F\cdot P_2F}+\frac{1}{Q_1F\cdot Q_2F}\right)=p\left(\frac{1}{AF^2}+\frac{1}{BF^2}\right)$$

$$=p\cdot\frac{AB^2}{AF^2\cdot BF^2}=\frac{p}{CF^2}=\frac{1}{4p}=constant$$

Attachments:



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Euler-Poncelet point

Reply

Source: own

jayme

#1 Feb 16, 2013, 6:43 pm • 1

Dear Mathlinkers,

1. ABCD a cyclic convex quadrilateral
2. (O) the circumcircle of ABCD
3. P, Q the resp. points of intersection of AD and BC, AB and CD
4. Lp the P-isogonal line of PO wrt the triangle PAB
5. Lq the Q-isogonal line of QO wrt the triangle QBC.
6. N the point of intersection of Lp and Lq.

Prouve : N is the Poncelet's point of ABCD.

Sincerely
Jean-Louis

Luis González

#2 Feb 17, 2013, 3:10 am • 1

It's known that when a quadrilateral is cyclic, its Poncelet point coincides with its anticenter. Define N as the anticenter of the cyclic $ABCD.$ Let U,V be the midpoints of BC,DA (projections of O on BC,DA). Since $UN \perp DA$ and $VN \perp BC,$ then N is the orthocenter of $\triangle PUV \Longrightarrow PN \perp UV \Longrightarrow PN,PO$ are isogonals WRT $\angle APB.$ Analogously, QN,QO are isogonals WRT $\angle AQD.$

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Nice property of OI line.

Reply

kaszubki

#1 Feb 16, 2013, 1:14 am

Let O be circumcenter, I be incenter and F be Feuerbach point of triangle ABC. Suppose that lines OI and AB intersect each other in point P. Let P_A and P_B be projections of point P onto lines AC and BC, respectively.
Prove that P, P_A, P_B, F lie on one circle.

Luis González

#2 Feb 16, 2013, 1:43 am • 1

In general, if a line τ passing through O cuts AB at P, then the circle $\odot(PP_AP_B)$ with diameter $\overline{CP}$ goes through the orthopole of τ WRT $\triangle ABC$ (see the highligthed lemma in the topic I count on it as a hard problem). When $\tau \equiv OI$, its orthopole is the Feuerbach point F.

Now, for a proof of the latter assertion, you may see the general theorem at Two Yango's problem (2nd post).

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Concurrency on OP

Reply

Source: Own

buratinogigle

#1 Feb 11, 2013, 10:06 pm

Let ABC be a triangle with circumcircle (O). P is a point such that if Q is isogonal conjugate of P with respect to triangle ABC then O,P,Q are collinear. D,E,F are projections of P on BC,CA,AB, respectively. QA,QB,QC cut (O) again at K,L,N, respectively. X,Y,Z are on (O) such that $XK\perp BC, YL\perp CA, ZN\perp AB$. Prove that DX,EY,FZ are oncurrent at a point on OP.

Click to reveal hidden text

Luis González

#2 Feb 15, 2013, 10:29 pm • 1

PA,PB,PC cut (O) again at U,V,W. It's known that P,Q,O are collinear (P is on McKay cubic of ABC) $\Longleftrightarrow \triangle DEF$ and $\triangle UVW$ are homothetic. Since $UK \parallel BC$, then $\angle UKX$ is right $\Longrightarrow UX$ is a diameter of (O). Similarly, VY and WZ are diameters of $(O) \Longrightarrow \triangle XYZ \cong \triangle UVW$ are symmetric about $O \Longrightarrow \triangle XYZ$ and $\triangle DEF$ are homothetic $\Longrightarrow$ DX,EY,FZ concur at their insimilicenter S, which is then the insimilicenter of their circumcircles $\odot(DEF) \sim (O) \Longrightarrow S$ is on their center line. Since the center of $\odot(DEF)$ is the midpoint of $\overline{PQ}$, then $S \in OP$.

Quick Reply

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High School Olympiads

Four concurrent circles

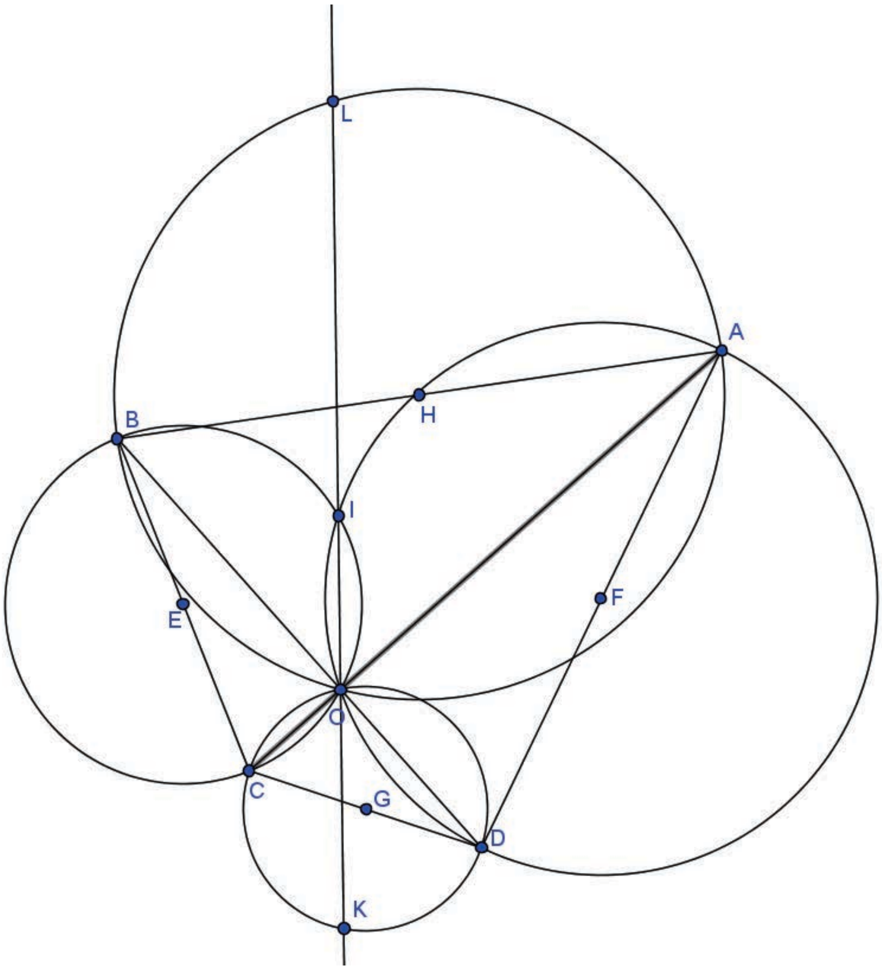
Reply

Narcissus

#1 Feb 15, 2013, 11:45 am

Let $ABCD$ is a quadrilaterial with two perpendicular diagonals: $AC \perp BD$ and O is the intersection. We denote (XY) is circle with diameter XY. (AD) cuts (BC) at I (#O). OI cuts (AB) and (CD) at L, K respectively. Prove that: $IL = IK$.
We have the figure below!

Attachments:



Luis González

#2 Feb 15, 2013, 12:08 pm

Just let ABCD be an arbitrary quadrilateral and its diagonals AC and BD meet at O. Circles (OAD) and (OBC) meet again at I. OI cuts the circles (OAB) and (OCD) again at L,K, respectively. Then IL=-IK.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=319862

Narcissus

#3 Feb 15, 2013, 3:16 pm

> *Luis González wrote:*

Just let ABCD be an arbitrary quadrilateral and its diagonals AC and BD meet at O. Circles (OAD) and (OBC) meet again at I. OI cuts the circles (OAB) and (OCD) again at L,K, respectively. Then IL=-IK.
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=319862

I don't know what would happen if the two diagonals are not perpendicular to each other. But if they are, my teacher said that he had a synthetic soln not using inversion. I would try to solve it again, and if I could, I would post my soln later.

CTK9CQT

#4 Feb 16, 2013, 10:14 am • 1

Triangle IAL and DAB are similar, here we have: $\frac{IL}{IA}=\frac{DB}{DA} (1)$

Triangle ICK and BCD are similar, here we have: $\frac{IK}{IC}=\frac{BD}{BC} (2)$

Triangle IAD and ICB are similar, here we have: $\frac{IA}{DA}=\frac{IC}{BC} (3)$

From $(1), (2), (3)$ we have QED

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High School Olympiads

Korea Third Round (FKMO) 2012 #4

Reply

Source: FKMO 2012

syk0526
#1 Mar 25, 2012, 11:36 am • 1

Let ABC be an acute triangle. Let H be the foot of perpendicular from A to BC. D, E are the points on AB, AC and let F, G be the foot of perpendicular from D, E to BC. Assume that $DG \cap EF$ is on AH. Let P be the foot of perpendicular from E to DH. Prove that $\angle APE = \angle CPE$.

Luis González
#2 Mar 25, 2012, 12:30 pm

Let $R \equiv DE \cap BC$ and $Q \equiv DH \cap AC$. Clearly F, G, H, R are harmonically separeted, thus the pencil $A(D, E, H, R)$ is harmonic, i.e. $(B, C, H, R) = -1 \Longrightarrow D(A, C, Q, E) = -1$. Hence, $PE \perp PQ$ implies that PE, PQ bisect $\angle APC$ internally and externally, i.e. $\angle APE = \angle CPE$.

sunken rock
#3 Mar 25, 2012, 3:23 pm

With Thales few times we get AH, BE, CD concurrent and, if $\{Q\} \in DH \cap AC$, clearly $\frac{QA}{QC} = \frac{AE}{CE}$ and, $PE \perp PQ$ yields the required angle equality.

Best regards,
sunken rock

ACCCGS8
#4 Aug 31, 2012, 12:06 pm

Let $AH = a, BH = b, CH = c, BF = bx, GC = cy$.
$\frac{FH}{HG} = \frac{DY}{YG} = \frac{DF}{EG} \implies \frac{b(1-x)}{c(1-y)} = \frac{x}{y} \implies \frac{1-x}{x}\frac{b}{c}\frac{y}{1-y} = 1$ so AH, BE, CD concurrent by converse of Ceva. If DH meets AC at Y then Y, A, E, C are harmonic so we are done.

hyperspace.rulz
#5 Feb 15, 2013, 6:59 am

> *Luis González wrote:*
> Clearly F, G, H, R are harmonically separeted

I can see that this is true (by noting that AH, DF, EG concur at the point at infinity, since all three lines are parallel), but how exactly would I explain this in an exam situation?

Luis González
#6 Feb 15, 2013, 11:35 am

hyperspace.rulz, in the projective plane, ordinary points and points at infinity have the same treatment. The projective plane consists of the standard Euclidean plane, together with a set of points called points at infinity, one for each collection of parallel lines. Point at infinity equals direction in the plane, so any two lines always have one common point, parallels are no exception.

lines. Point at infinity equals direction in the plane, so any two lines always have one common point, parallels are no exception.

Suppose we have a triangle XGF, D and E lie on XF and XG, resp. ED cuts FG at R, EF cuts DG at K and XK cuts FG at H. By well-known property of the complete quadrangle, the cross ratio (F,G,H,R) is harmonic. When X is a point at infinite (DF // EG // KH), the cross ratio (F,G,H,R) is still harmonic.

hyperspace.rulz

#7 Feb 16, 2013, 11:05 am

Note that the proof this well-known property of the complete quadrangle uses Ceva's and Menelaus' Theorems, can these still be used when dealing with points at infinity?

rkm0959

#8 Mar 11, 2016, 7:17 am

The main claim is that AH, BE, CD are concurrent. We prove this by Ceva.
Let $DF = a$, $EG = b$, $\frac{FH}{DF} = \frac{HG}{EG} = x$, $\frac{BF}{DF} = y$, and $\frac{CG}{GE} = z$.
The above claim is equivalent to

$$\frac{BD}{DA} \cdot \frac{AE}{CE} \cdot \frac{CH}{BH} = 1 \iff \frac{y}{x} \cdot \frac{x}{z} \cdot \frac{b(x+z)}{a(x+y)} \iff \frac{by(x+z)}{az(x+y)} = 1$$

Meanwhile, we have $AH = \frac{a(y+x)}{y} = \frac{b(z+x)}{z}$, so $by(x+z) = az(x+y)$ as required.

Now take $X = DH \cap AC$ and $Y = DE \cap BC$. (Y, B, H, C) is harmonic, so taking perspectivity at D we have (E, A, X, C) harmonic. Combining this with $EP \perp DH$ gives us the desired conclusion.

This post has been edited 1 time. Last edited by rkm0959, Mar 11, 2016, 7:17 am

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High School Olympiads

Right triangle and angle bisector.

Reply

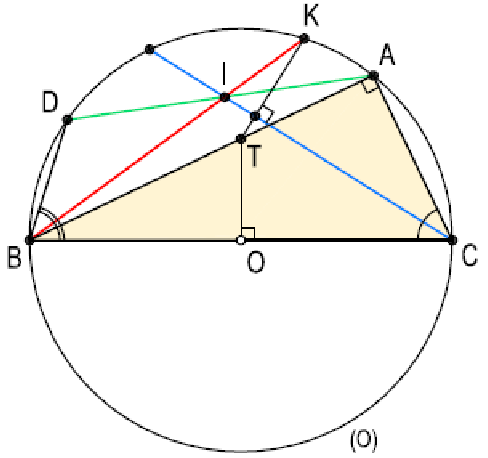
vittasko

#1 Feb 1, 2013, 3:58 am • 1

A right triangle $\triangle ABC$ with $\angle A = 90^o$ is given and let T be, the point of intersection of AB, from the midperpendicular of the segment BC. We draw the line through the point T and perpendicular to the angle bisector of $\angle C$, which intersects the circumcircle (O) of the given triangle $\triangle ABC$, at one point so be it K. The line BK intersects the angle bisector of $\angle C$ at point I and we denote the point $D \equiv (O) \cap AI$. Prove that the line BK bisects the angle $\angle DBC$.

Kostas Vittas.

Attachments:



yetti

#2 Feb 15, 2013, 6:34 am

Let $X \equiv BD \cap CA$. Let $BC = x, CX = b, XB = c, p = \frac{1}{2}(x+b+c)$ and $\widehat{X} = \measuredangle AXD \equiv \measuredangle CXB$, $\widehat{B} = \measuredangle DBC \equiv \measuredangle XBC, \widehat{C} = \measuredangle BCA \equiv \measuredangle BCX$.
Let (P, R) be circumcircle and (I_x, r_x) X-excircle of $\triangle XBC$. BA, CD are B-, C-altitudes of this triangle $\Longrightarrow$ $\triangle XAD \sim \triangle XBC$ with similarity coefficient $\cos \widehat{X}$.
Let XI cut BC at J and (P) again at Y. Let (I_x) touch XB at W, so that $XW = p$, and let N be foot of perpendicular from Y to XC. By Archimedes theorem, $XN = \frac{1}{2}(b+c)$.

BK bisects $\widehat{B} = \angle DBC \Longleftrightarrow$ incenter I of $\triangle XBC$ is on $AD \Longleftrightarrow \dfrac{b+c}{x+b+c} = \dfrac{XI}{XJ} = \cos \widehat{X} \Longleftrightarrow$
$XN = \frac{1}{2}(b+c) = p\cos \widehat{X} = XW \cos \widehat{X} \Longleftrightarrow$ points W, Y, N are collinear $\Longleftrightarrow$
$\triangle YWI_x$ with $\measuredangle WI_xY = \measuredangle WYI_x = \frac{\pi}{2} - \frac{\widehat{X}}{2}$ is W-isosceles $\Longleftrightarrow$ W-, P-isosceles $\triangle WI_xY \cong \triangle PBY$ with $\measuredangle YWI_x = \widehat{X} = \measuredangle YPB$ and $YB = YI_x$ are congruent $\Longleftrightarrow r_x = WI_x = PB = R \Longleftrightarrow$
$1 = \dfrac{r_x}{R} = 1 - \cos \widehat{X} + \cos \widehat{B} + \cos \widehat{C} = 2\cos^2 \frac{\widehat{B}+\widehat{C}}{2} + 2\cos \frac{\widehat{B}+\widehat{C}}{2} \cos \frac{\widehat{B}-\widehat{C}}{2} = 4 \sin \frac{\widehat{X}}{2} \cos \frac{\widehat{B}}{2} \cos \frac{\widehat{C}}{2} \Longleftrightarrow$
$2\cos \frac{\widehat{B}}{2} \sin \widehat{C} = \dfrac{\sin \frac{\widehat{C}}{2}}{\sin \frac{\widehat{X}}{2}}$.
On the other hand, $BK = x \cos \frac{\widehat{B}}{2}$ and $BT = \dfrac{x}{2 \sin \widehat{C}}$. By sine theorem for $\triangle BTK \Longleftrightarrow$
$\dfrac{\sin \widehat{BTK}}{\dots} = \dfrac{BK}{\dots} = 2\cos \frac{\widehat{B}}{2} \sin \widehat{C} = \dfrac{\sin \frac{\widehat{C}}{2}}{\dots} = \dfrac{\sin(\pi - \frac{\widehat{C}}{2})}{\dots}.$

$\sin TKB \quad BT \quad 2 \quad \sin\frac{X}{2} \quad \sin\frac{X}{2}$

Since in addition, $\measuredangle BTK + \measuredangle TKB = \pi - \measuredangle KBT = \pi - \frac{\widehat{B}}{2} + \frac{\pi}{2} - \widehat{C} = \pi - \frac{\widehat{C}}{2} + \frac{\widehat{X}}{2}$, this is $\Longleftrightarrow \measuredangle BTK = \pi - \frac{\widehat{C}}{2}$ and $\measuredangle TKB = \frac{\widehat{X}}{2} \Longleftrightarrow TK \perp CI$.

Lemma

Luis González

#3 Feb 15, 2013, 9:59 am • 1

We don't need CI to be the internal bisector of <C, that's probably a distraction.

KO, KT cut IA, IC at $P, Q. Q$ is obviously on circumcircle ω of the cyclic $OTAC$. Intersections of opposite sidelines of the hexagon $PATQCO$ are collinear, i.e. $I \equiv PA \cap QC, B \equiv AT \cap CO$ and $K \equiv TQ \cap OP$ are collinear $\Longrightarrow P \in \omega$ (converse of Pascal theorem). From cyclic $APOC, \angle(OC, OP) = \angle(AC, AD) = \angle(BC, BD) \Longrightarrow OPK \parallel BD \Longrightarrow \angle DBK = \angle OKB = \angle OBK \Longrightarrow BK$ bisects $\angle DBC$.

armpist

#4 Feb 15, 2013, 9:39 pm

Dear MLs and Jean-Louis,

Now that we finally have two circles intersecting at A and C,

do you think THE GREAT REIM theorem should be employed to prove

parallelism ?

M.T.

jayme

#5 Feb 16, 2013, 6:47 pm

Dear M.T. and Mathlinkers,
yes the Reim's theorem allows to avoid angles... and the conclusion follows without angles also.
But all the ways to prouve a result allow to each other to have a fruitfull point of view.

Sincerely
Jean-Louis

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High School Olympiads

Line passing two circumcenters bisects AD

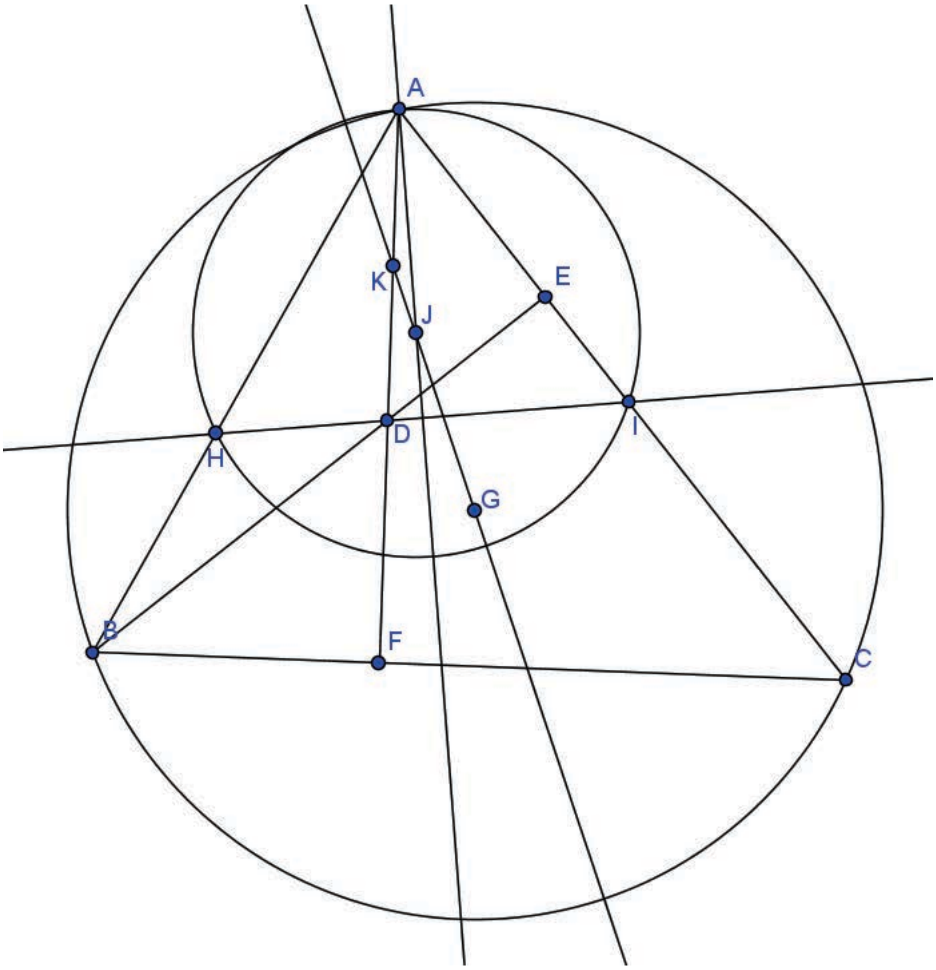
Reply

Narcissus

#1 Feb 14, 2013, 4:16 pm

Let ABC is a triangle inscribed in circle (G) and let D be its orthocenter. The line passing through D and perpendicular to A's in-bisector meets AB, AC at H, I, respectively. Let J be the circumcenter of triangle AHI. Prove that GJ bisects the segment AD (id, GJ passes through the midpoint K of AD).
We have the figure below!

Attachments:



Luis González

#2 Feb 14, 2013, 8:39 pm

This configuration has been discussed many times before. It just follows from the fact that the circles (G), (AIH) and (ADE) are coaxal.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=89098
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=320620
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=409570

MMEEvN

#3 Feb 14, 2013, 9:49 pm

Let the perpendicular from G to BC meet the side BC at F

$\frac{GF}{cosA} = \frac{BC/2}{sinA}$ Hence $GF = \frac{BC}{2sinA}(cosA) = \frac{c}{2sinC}(cosA)$

Using trigonometry yields $AD = \frac{c}{sinC}(cosA)$ Hence $AD = 2GF$

let the intersection of HI and the angular bisector of $\angle BAC$ be Y and the intersection of DF and AY be X Let $\angle DAY = x$ It is easy to observe that $\angle YAG = x$

Hence by trigonometry $AX = \frac{2.AD.AL}{AD + AL}(cosx)$($L$ is the intersection of AG again with the circumcircle and AX is the angle bisector)

$AX = \frac{4RcosA}{cosA + 1}(cosx)$($R$ is the circumradis) By applying law of sines to triangle AYD we have $AY = AD.cosx = 2RcosAcosx$ and applying the same law to triangle AHY yields $AH = \frac{AY}{cos\frac{A}{2}} = \frac{2RcosAcosx}{cos\frac{A}{2}}$

Observe that $\frac{AH}{AY} = \frac{1}{cos\frac{A}{2}}$ similarly we can calculate $\frac{AX}{AH}$ by using the expressions obtained for AX and AH which yields $\frac{AH}{AY} = \frac{AX}{AH}$ Hence $\angle AHX = 90$

similarly $\angle AIX = 90$ Hence X lies on the circumcircle of triangle AHI. $JA = JX$ Since $AD = 2GH \implies GK||DF$ by the converse midpoint theorem GK moves through the center of AX which is the circumceter of triangle AHI

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High School Olympiads

in a equilateral triangle

Reply

Stephen

#1 Feb 14, 2013, 4:46 pm

Triangle XYZ is equilateral, and circle w is its circumcircle.

P is a point in the interior of XYZ, and XP, YP, ZP intersects with w at A, B, C.

D, E, F are incenters of triangles PBC, PCA, PAB.

Prove that AD, BE, CF are concurrent.

Luis González

#2 Feb 14, 2013, 8:18 pm

In addition, the concurrency point is on the line connecting P with the incenter of ABC.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=318684
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=356459

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High School Olympiads

ratio of lengths are equal

Reply

Stephen

#1 Feb 13, 2013, 3:18 pm

In acute triangle ABC, point D, E, F are in the exterior of ABC, satisfying the condtion:

$$\angle DBC = \angle DCB = \angle EAC = \angle ECA = \angle FAB = \angle FBA = \angle BAC$$

M is the midpoint of BC, P is intersection of DE and AC, Q is intersection of AB and DF.

Prove that $MP : MQ = AB : AC$.

Luis González

#2 Feb 14, 2013, 6:03 am

Let N, L be the midpoints of CA, AB. $\triangle ABC, \triangle MNL$ and $\triangle DEF$ have common centroid G (see the problem prove that two triangles share their centroid).

$$\frac{PE}{PD} = \frac{CE}{CD} \cdot \frac{\sin \widehat{ECA}}{\sin \widehat{DCA}} = \frac{AC}{BC} \cdot \frac{\sin \widehat{A}}{\sin(\widehat{A} + \widehat{C})} = \frac{AC}{BC} \cdot \frac{\sin \widehat{A}}{\sin \widehat{B}} = 1.$$

Hence, P is the midpoint of DE. Similarly, Q is the midpoint of $DF \Longrightarrow G \equiv EQ \cap FP \cap AM \Longrightarrow$ pentagons $AECBF$ and $MQLNP$ are homothetic under the homothety with center G and coefficient $-\frac{1}{2} \Longrightarrow \triangle MPN$ and $\triangle MQL$ are also similar isosceles triangles $\Longrightarrow MP : MQ = MN : ML = AB : AC.$

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inside the incircle

Reply

Stephen

#1 Feb 13, 2013, 3:47 pm

In triangle ABC, D, E, F are the intersection points of incircle and BC, CA, AB.

G is the second intersection point of incircle and AD.

K, L are points on the incircle satisfying $FG = FK, EG = EL$.

Prove that $\frac{DL}{DK} = \frac{EG^2}{FG^2}$.

Luis González

#2 Feb 14, 2013, 1:41 am • 1

AD cuts EF at P. GPD is G-symmedian of $\triangle GEF$. Since E, F are the midpoints of the arc GL, GK of the incircle, then $GL \parallel AC$ and $GK \parallel AB$, i.e. $\angle DGL = \angle PAE$ and $\angle DGK = \angle PAF$. Hence

$$\frac{DL}{DK} = \frac{\sin \widehat{DGL}}{\sin \widehat{DGK}} = \frac{\sin \widehat{PAE}}{\sin \widehat{PAF}} = \frac{PE}{PF} \cdot \frac{AF}{AE} = \frac{PE}{PF} = \frac{EG^2}{FG^2}.$$

sunken rock

#3 Feb 18, 2013, 10:37 pm • 1

$FG = FK \iff \angle FDK = \angle FDG$; with $\angle AFG = \angle FDG$ we get $\angle FDK = \angle AFG$ (1).
From the cyclic $FKDG$ we get $\angle FKD = \angle FGA$ (2).
From $(1) \wedge (2) \implies \Delta DKF \sim \Delta FGA \iff \frac{AG}{FK} = \frac{FG}{KD} \iff AG \cdot KD = FG^2$ (3); similarly, $AG \cdot DL = GE^2$ (4). From $(3) \wedge (4)$, dividing side by side, we get the required relation.

Best regards,
sunken rock

MMEEvN

#4 Feb 19, 2013, 2:40 pm

It is easy to observe that $KG||AF$ and $GL||AE \implies \angle DGK = \angle DAF$ and $\angle DGL = \angle DAE$

Apply the trigonometric form of Ceva's theorem to $\Delta AEF \implies \frac{sin\widehat{FAG}}{sin\widehat{GAE}} \cdot \frac{sin\widehat{AEG}}{sin\widehat{GEF}} \cdot \frac{sin\widehat{GFE}}{sin\widehat{GFA}} = 1$

Since $\angle GEF = \angle GFA$ and $\angle GFE = \angle GEA$ we have
$$\frac{GF^2}{GE^2} = \frac{sin(\widehat{GEF})^2}{sin(\widehat{GFE})^2} = \frac{sin\widehat{FAG}}{sin\widehat{GAE}} = \frac{sin\widehat{KGD}}{sin\widehat{LGD}} = \frac{sin\widehat{KLD}}{sin\widehat{DKL}} = \frac{DK}{DL}$$

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High School Olympiads

euler line perpendicular

Stephen

#1 Feb 13, 2013, 4:22 pm

O and H are the circumcenter and orthocenter of acute triangle ABC.

M, N are the midpoints of BC, AB.

P is the intersection point of BC and the line that go through N and perpendicular to HN.

Q is the intersection point of AB and the line that go through M and perpendicular to HM.

Prove that OH and PQ are perpendicular

Luis González

#2 Feb 13, 2013, 11:42 pm

$X \equiv AH \cap BC$ and $Z \equiv CH \cap BA$ are the feet of the A- and C- altitudes. P is the 2nd intersection of $\odot(HNX)$ with BC and Q is the 2nd intersection of $\odot(HMZ)$ with BA. If R is the 2nd intersection of $\odot(HNX)$ and $\odot(HMZ)$, then $\angle PRH = \angle QRH = 90^\circ \Longrightarrow PQ$ is perpendicular to HR at R. Hence it suffices to prove that $O \in HR$.

Let $E \equiv MN \cap XZ$. Since $MXNZ$ is clearly cyclic (9-point circle), we have $EM \cdot EN = EX \cdot EZ \Longrightarrow E$ has equal power WRT the circles with diameters $\overline{AH}$ and $\overline{AO} \Longrightarrow BE$ is radical axis of these circles, which is then perpendicular to its center line, the B-midline of $\triangle BHO \Longrightarrow BE \perp OH$. Hence NX, MZ and OH concur at the pole V of BE WRT the 9-point circle $\odot(MXNZ)$, which is precisely the radical center of $\odot(MXNZ), \odot(HNX)$ and $\odot(HMZ) \Longrightarrow OH$ is radical axis of $\odot(HNX)$ and $\odot(HMZ) \Longrightarrow O \in HR$, as desired.

P.S. The result still holds if O,H are replaced by any two isogonal conjugates and M,N the projections of O on BC,BA. The proof is exactly the same.

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incenter and circles

Reply

Stephen
#1 Feb 13, 2013, 3:27 pm

I is the incenter of triangle ABC. Circle w go through B, C and circumscribes with incircle on point D.

E is the second intersection point of AD and w. Prove that line EI go through the midpoint of w's arc BC.

(The arc BC we are saying includes the point D.)

MMEEvN
#2 Feb 13, 2013, 3:52 pm

> *Stephen wrote:*
> . Circle w go through B, C and circumscribes with incircle on point D.

I don't understand this part .Can you please explain more

Stephen
#3 Feb 13, 2013, 4:26 pm

Sorry for not selecting good words to describe the problem.

Circle w passes through B, C and its tangent to the incircle on point D.

Now you can understand?

Luis González
#4 Feb 13, 2013, 10:19 pm

Let (J) be the circle tangent to AB, AC and internally tangent to w at E^*, this is different from (I). A, D and E^* are the exsimilicenters of $(I) \sim (J), (I) \sim w$ and $w \sim (J)$. By Monge and d'Alembert theorem, A, D, E^* are collinear $\Longrightarrow$ $E \equiv E^*$. Now EI bisects $\angle BEC$, i.e. EI passes through the midpoint of the arc BDC of w. For a proof see Fairly difficult (Iran 1999), Concyclic points with triangle incenter, incenter of triangle and elsewhere.

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Distance from a Point to its Simson Line

Reply

1/0 = 42

admin25

#1 Feb 13, 2013, 6:24 am

Given a point N on the circumcircle of triangle ABC. Show that the distance from N to its Simson Line with respect to ABC is

$$\frac{(NA)(NB)(NC)}{4R^2},$$

where R is the radius of the circumcircle.

Luis González

#2 Feb 13, 2013, 8:38 am • 1

Let Q, R be the projections of N on $AC, AB. QR$ is Simson line of N WRT $\triangle ABC. X$ is the projection of N on $QR.$ We use that in any triangle the product of its circumdiameter and an altitude equals the product of its adjacents sides.

$$NX \cdot NA = NR \cdot NQ \ \ , \ \ NR = \frac{NA \cdot NB}{2R} \ \ , \ \ NQ = \frac{NA \cdot NC}{2R} \Longrightarrow$$

$$NX \cdot NA = \frac{NA^2 \cdot NB \cdot NC}{4R^2} = \frac{NA \cdot NB \cdot NC}{4R^2}.$$

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Radical axis of incircles

Reply

tuanh208

#1 Feb 13, 2013, 12:00 am

Let ABC be an acute triangle and a point P inside s.t $PB+BA=PC+CA$. Let M be the midpoint of BC. Prove that M lies on radical axis of incircles of triangles ABP and ACP.

Luis González

#2 Feb 13, 2013, 1:03 am

Already posted at Midpoint lies on the radical axis of two incircle.

Related problems:
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=340482
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=340735
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=389721

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Inscribe conic

Reply

Source: maybe application of carnot's theorem

Liebig

#1 Feb 12, 2013, 11:40 am

Let points $A_1, A_2, B_1, B_2, C_1, C_2$ lie on the sides BC, CA, AB of ΔABC respectively, and these six points lie on a conic. Suppose $AA_1, AA_2, BB_1, BB_2, CC_1, CC_2$ form a convex hexagon T. Prove that T has an inscribe conic.

Luis González

#2 Feb 12, 2013, 12:46 pm

Let $\mathcal{C}$ be the conic tangent to the 5 lines AA_1, AA_2, BB_1, BB_2 and CC_1. Second tangent from C to $\mathcal{C}$ cuts AB at C_3. Using the projective version of the problem Humdinger (a conic can be projected into a circle through a homology), then A_1, A_2, B_1, B_2, C_1 and C_3 lie on a same conic $\Longrightarrow C_2 \equiv C_3 \Longrightarrow CC_2$ touches $\mathcal{C}$, thus the hexagon T is circumscribed in the conic $\mathcal{C}$.

Quick Reply

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High School Olympiads

triangle (Prove the equality)

Pirkuliyev Rovsen

#1 Feb 8, 2013, 1:08 pm

In triangle ABC the bisector conducted CL, and triangles CAL CBL and inscribed circle, which relate the line AB at points M and N respectively. Prove the equality $\frac{2}{CL}=\frac{1}{LN}-\frac{1}{AM}$.

Azerbaijan Land of the Fire

yetti

#2 Feb 11, 2013, 4:15 pm • 1

> Pirkuliyev Rovsen wrote:
>
> In $\triangle ABC$, $L\in AB$ such that CL is the internal bisector of $\widehat{C}$. Inscribed circles of $\triangle CAL, \triangle CBL$ touch the line AB at points M and N, respectively.
> Prove the equality $\frac{2}{CL}=\frac{1}{LN}-\frac{1}{AM}$.

Let $BC=a, CA=b, AB=c$ and $p=\frac{1}{2}(a+b+c)$.

$$AM=\tfrac{1}{2}(CA+AL-CL)=\tfrac{1}{2}\left(b+\frac{bc}{a+b}-CL\right)=\frac{pb}{a+b}-\tfrac{1}{2}CL$$

$$BN=\tfrac{1}{2}(BC+BL-CL)=\tfrac{1}{2}\left(a+\frac{ca}{a+b}-CL\right)=\frac{pa}{a+b}-\tfrac{1}{2}CL$$

$$LN=BL-BN=\frac{ca}{a+b}-\frac{pa}{a+b}+\tfrac{1}{2}CL\implies AM-LN=p-\frac{ca}{a+b}-CL$$

$$\frac{2}{CL}=\frac{1}{LN}-\frac{1}{AM}\iff AM\cdot LN=\tfrac{1}{2}CL(AM-LN)\iff$$
$$\left(\frac{pb}{a+b}-\tfrac{1}{2}CL\right)\left(\frac{ca}{a+b}-\frac{pa}{a+b}+\tfrac{1}{2}CL\right)=\tfrac{1}{2}CL\left(p-\frac{ca}{a+b}-CL\right)\iff$$
$$\frac{abp(p-c)}{(a+b)^2}=\tfrac{1}{4}CL^2\iff\frac{\sqrt{4abp(p-c)}}{a+b}=CL,$$ which is well known fromula for CL.

Luis González

#3 Feb 12, 2013, 2:49 am • 1

L-excircle (U) of $\triangle CAL$ and B-excircle (V) of $\triangle CBL$ touch CL at Y,Z, respectively. $CZ=LN$ and $CY=AM$. External bisectors CU and CV of $\angle ACL$ and $\angle BCL$ are obviously isogonals WRT $\angle ACB$, i.e. bisectors of $\angle ACB$ also bisect $\angle UCV$. Hence if the external bisector of $\angle ACB$ cuts UVL at D, then U,V,D,L are harmonically separeted $\Longrightarrow$ their projections Y,Z,C,L on CL are also harmonically separated $\Longrightarrow$ they verify the relation

$$\frac{2}{CL}=\frac{1}{CZ}-\frac{1}{CY}=\frac{1}{LN}-\frac{1}{AM}.$$

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High School Olympiads

Concurrency with incircle

Reply

Source: Own

buratinogigle

#1 Feb 11, 2013, 11:17 pm

Let ABC be a triangle with incircle (I) touches BC, CA, AB at D, E, F, respectively. (A) is circle center A and passing through E, F. (K_a) is the circle tangent externally to (A) at X and tangent to DE, DF. Similarly, we have Y, Z. Prove that DX, EY, FZ are concurrent.

Luis González

#2 Feb 12, 2013, 12:12 am • 1

Parallel from A to BC cuts DE, DF at E', F'. $\angle AE'E = \angle CDE = \angle CED \Longrightarrow AE = AE'$. Similarly, $AF = AF'$ $\Longrightarrow \overline{E'F'}$ is a diameter of $(A) \Longrightarrow DX$ becomes the D-cevian of the Paasche point of $\triangle DE'F'$ (see the solution of the problem touching circles and concurrency). Since $\triangle DEF \sim \triangle DF'E'$, then DX is the isogonal of the D-cevian of the Paasche point of $\triangle DEF$ WRT $\angle EDF$. Hence we conclude that DX, EY, FZ concur at the isogonal conjugate of the Paasche point of $\triangle DEF$, i.e X_{1124} of $\triangle DEF$.

Quick Reply

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High School Olympiads

2013 Japan Mathematical Olympiad Finals Problem 4

Reply

Kunihiko_Chikaya
#1 Feb 11, 2013, 10:25 pm • 1

Given an acute-angled triangle ABC, let H be the orthocenter. A cirlcle passing through the points $B,\ C$ and a cirlcle with a diameter AH intersect at two distinct points $X,\ Y$. Let D be the foot of the perpendicular drawn from A to line BC, and let K be the foot of the perpendicular drawn from D to line XY. Show that $\angle BKD=\angle CKD$.

This post has been edited 1 time. Last edited by Kunihiko_Chikaya, Feb 11, 2013, 10:54 pm

Luis González
#2 Feb 11, 2013, 10:37 pm • 3

Let $E \equiv BH \cap CA$ and $F \equiv CH \cap AB$ be the feet of the B- and C- altitudes. XY, EF and BC are pairwise radical axes of the circle $\odot(AEF)$ with diameter $\overline{AH}$, the circle $\odot(BCEF)$ with diameter $\overline{BC}$ and the circle $\odot(BCXY) \Longrightarrow$ XY, EF, BC concur at their radical center R. From the complete quadrangle $BCEF$, the cross ratio (B,C,D,R) is harmonic. Since $DK \perp RK$, then it follows that KD, KR bisect $\angle BKD$ internally and externally.

buratinogigle
#3 Feb 11, 2013, 10:54 pm • 1

Nice problem here is a generalization.

Let ABC be a triangle and P is point such that $AP\perp BC$. E,F are projection of P on CA,AB, respectively. BE cuts CF at H. AH cuts BC at D. (K) is circle diameter AP. (L) is a circle passing through B,C. d is radical axis of $(K),(L)$. N is projection of D on d. Prove that $\angle BND=\angle CND$.

MariusBocanu
#4 Feb 14, 2013, 12:29 pm • 1

> *buratinogigle wrote:*
> Nice problem here is a generalization.
>
> Let ABC be a triangle and P is point such that $AP\perp BC$. E,F are projection of P on CA,AB, respectively. BE cuts CF at H. AH cuts BC at D. (K) is circle diameter AP. (L) is a circle passing through B,C. d is radical axis of $(K),(L)$. N is projection of D on d. Prove that $\angle BND=\angle CND$.

Let $AP\cap BC=\{U\}$. $BUPF, CUPE, AFPE$ are cyclic, so $\widehat{CBA}=\widehat{UBF}=\widehat{APF}=\widehat{AEF}$, so $FECB$ is cyclic, now, all you have to do is to apply the same argument as in the initial problem.

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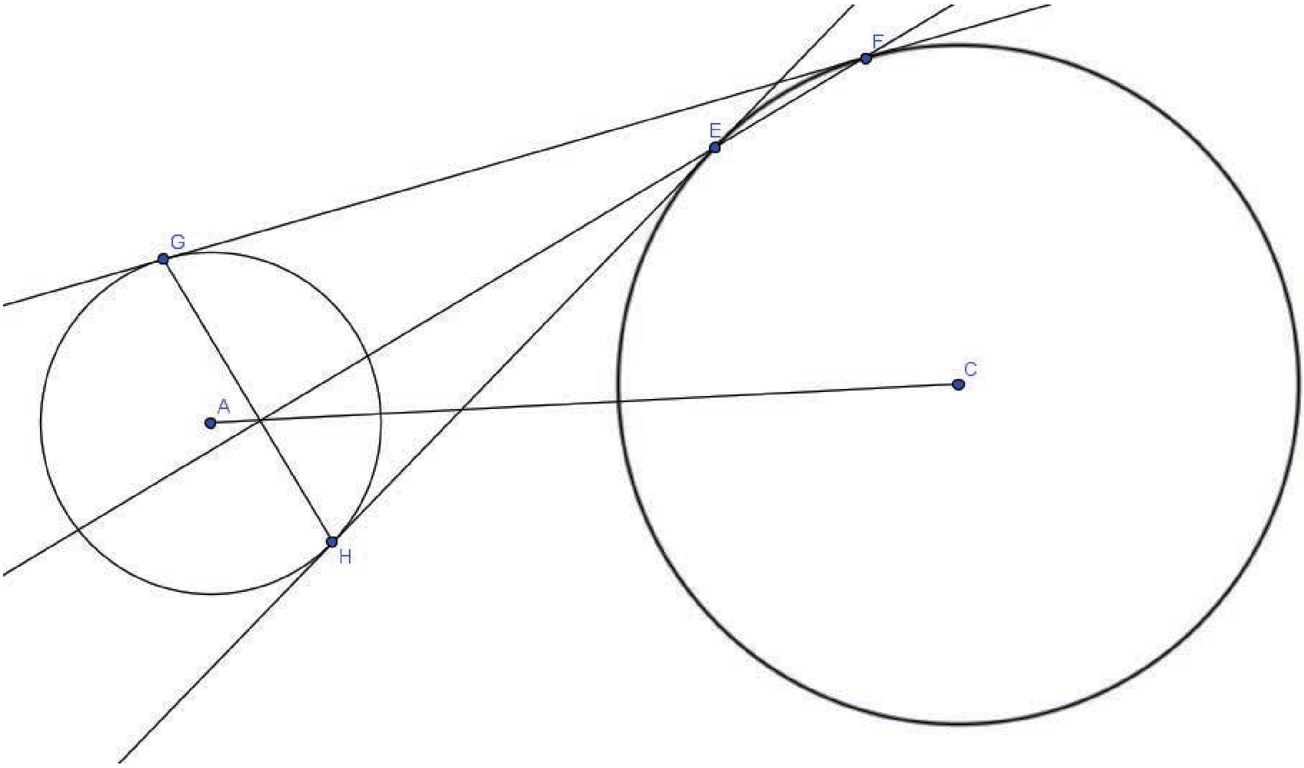
A familiar Property

Locked

Narcissus
#1 Feb 11, 2013, 7:10 pm

I know it it a familiar property, but I cannot prove it well:
Two circle (A) and (C) is outside each other. Their in-tangent touches (A) and (C) at H, E, respectively. Their ex-tangent touches $(A), (C)$ at G, F, respectively. Prove that GH, EF, AC are concurrent.

Attachments:



Luis González
#2 Feb 11, 2013, 9:42 pm

Search before posting. This is a problem from Polish MO 2009 2nd round.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=223789
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=266894
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=151&t=488965

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High School Olympiads

nice one

Locked

Source: tournament of the towns 1992

JRD

#1 Feb 11, 2013, 1:57 pm

let ABC be a triangle and let bisector of angle A meet circumcircle of $\triangle ABC$ at point D . I and N are incenter and midpoint of BC respectively . point P is on extension of IN such that $NP=NI$ and M is the intersection point of DP and circumcircle of $\triangle ABC$. prove that one of segments AM, BM, CM is equal to sum of the others.

Luis González

#2 Feb 11, 2013, 9:10 pm

Please give your posts meaningful subjects. As for the problem, it was posted before

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=323883
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=424438

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High School Olympiads

A triad of circles tangent to the 9-point circle

Reply

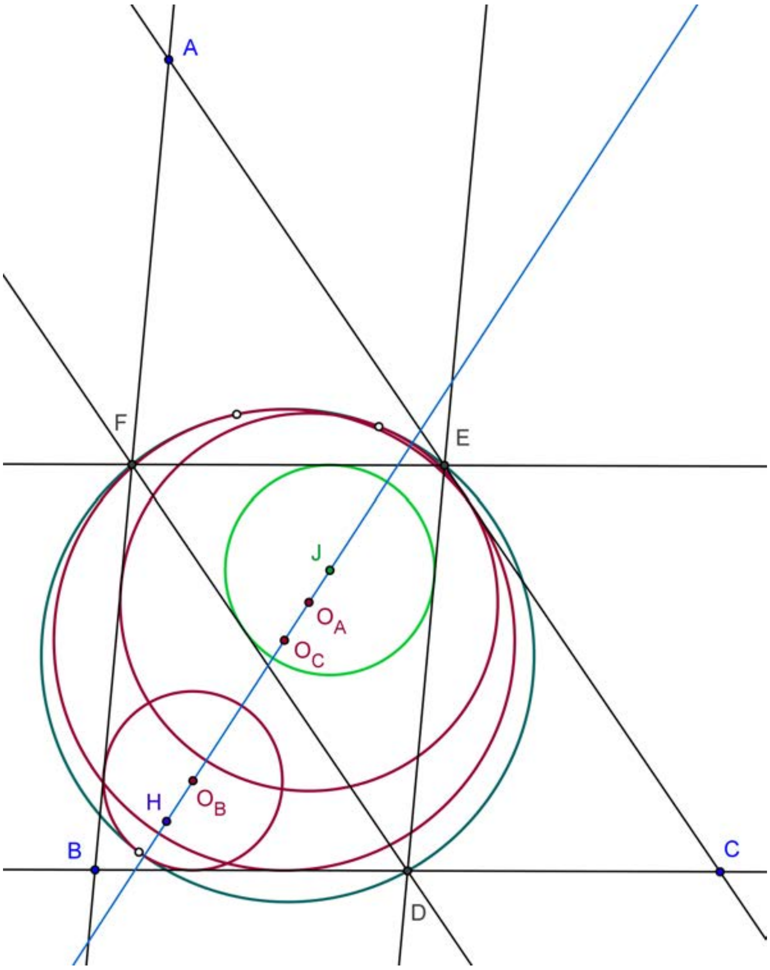
Source: Nikolaos Dergiades and Alexei Myakishev

Luis González

#1 Feb 10, 2013, 1:54 pm • 2

$\triangle ABC$ is acute with orthocenter $H.$ D,E,F are the midpoints of BC,CA,AB and J is the incenter of $\triangle DEF.$ $(O_A),$ (O_B) and (O_C) are the circles tangent to two sides of $\triangle ABC$ and tangent internally to the 9-point circle $\odot(DEF)$ (all different from the incircle of ABC). Show that their centers O_A,O_B and O_C fall on the line $HJ.$

Attachments:



hofamo

#2 May 31, 2013, 3:32 pm

tangent point of (Oa),(Ob),(Oc),(I) to 9-point circle are Xa,Xb,Xc,F respectively.its easy to know that A,Xa,F are collinear.foot of perpendiculars from H,J,I,Oa to BC,AC,AB are (Ha,Hb,Hc),(Ja,Jb,Jc),(Ia,Ib,Ic),(Oaa,Oab,Oac) respectively.we know AXa.AF=AE.AHb and AXa.AF=AIb.AOab or AE.AHb=AIb.AOab . now we have AOab=AOac=b.c.cos(A)/2(P-a) .now if we prove OabHb/HbJb=OacHc/HcJc we prove problem because we prove Oa,H,J are collinear.its not very long and its true. Click to reveal hidden text

duanby

#3 Jun 8, 2013, 6:52 pm • 1

My solution:
let Y,Z be the foot of B,C and S,T be the foot of O[a] on AB,AC
Then we need to prove that $\frac{ZS}{YT}=\frac{ZP}{ZQ}$
Here,P,Q be the tangent of the extangent circle of C,B on AB,AC
I have two method
the first one is more directly
YE=$\frac{c^2-a^2}{2b}$,ZF=$\frac{b^2-a^2}{2c}$
use Casey theorem,we get
$ZS*EF=(ZF-ZS)*EZ+ZF*ET$ that is $\ZS*(EF+EA)=ZF*AT$
$YT*EF=(TE-YT)*YF+EY*FS$ that is $\YT*(EF+AF)=FZ*AS$
so $\frac{ZS}{YT}=\frac{(a+c)*FZ}{(a+b)*YE}=\frac{(b-a)*b}{(c-a)*c}$
another is
Let R be the tangent of O[A],Fe be the Ferabhe point
then$\frac{ZS}{YT}=\frac{ZR}{YR}=\frac{FFe*AZ}{EFe*AY}=\frac{FF'*AZ}{EE'*AY}=\frac{(b-a)*b}{(c-a)*c}$
where F',E' be the tangent of the incircle on AB,AC

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High School Olympiads

Perpendicular wrt Incircle

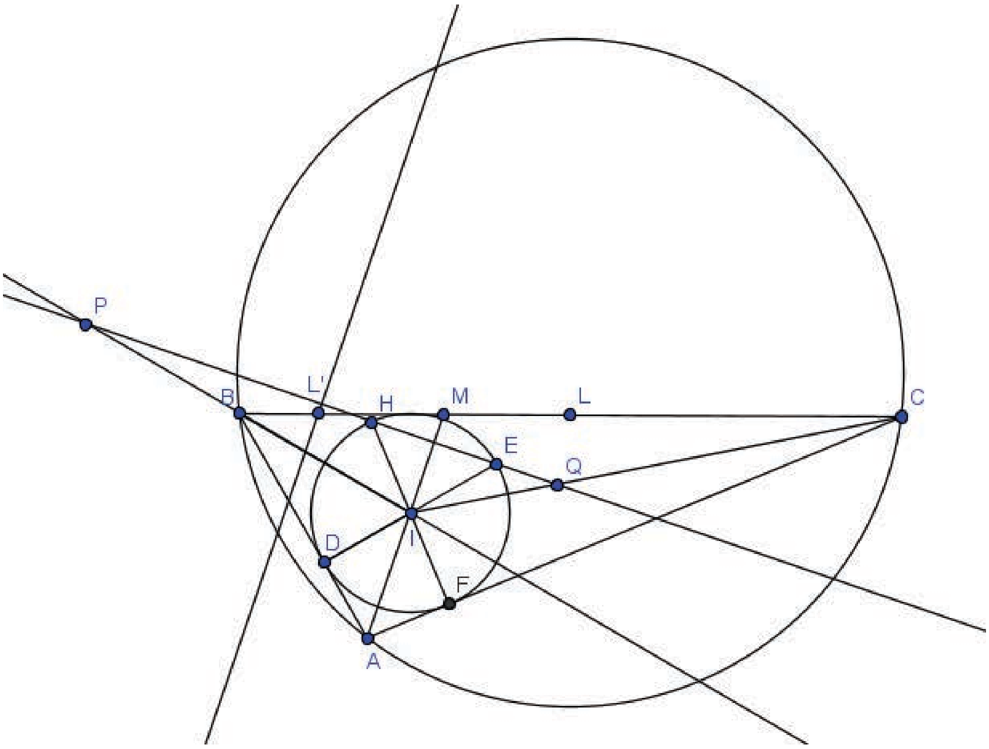
Source: From Vietnam NMO 2013

CTK9CQT

#1 Feb 10, 2013, 9:30 am

Let ABC is a triangle and (I) is its incircle. (I) touches AB, AC at D, F respectively. ID, IF cuts (I) again at E, H. EH cuts IB, IC at P, Q, respectively. AI meets BC at M and L is midpoint of BC. Let L' be symmetric point of L through M. Prove that L' lies on the perpendicular bisector of PQ
We have the figure below!

Attachments:



Luis González

#2 Feb 10, 2013, 12:50 pm

If BI, CI cut DF at Y, Z, then it's clear that P, Q are the reflections of Y, Z about I. Furthermore, simple angle chase gives $\angle BIZ = \angle ADF = 90^\circ - \frac{1}{2}\angle A \Longrightarrow B, I, D, Z$ are concyclic $\Longrightarrow \angle BZI = 90^\circ$. Thus if K is the midpoint of CZ, then LK is perpendicular bisector of CZ.

Let N be the midpoint of the arc BC of the circumcircle $\odot(ABC)$ (circumcenter of BIC). $PQ \parallel DF$ is then perpendicular to the I-circumdiameter IN of $\triangle BIC \Longrightarrow P, Q, B, C$ lie on a same circle ω. If V, S denote the midpoints of IC, CQ, we get $VK = \frac{1}{2}(CZ - CI) = \frac{1}{2}IQ = \frac{1}{2}(CI - CQ) = VS$, i.e. V is midpoint of KS. Hence NV becomes midparallel of LK and the perpendicular bisector of $CQ \Longrightarrow$ perpendicular bisector of CQ meets LN at the reflection J of L about N, which is the center of ω. Therefore JL' becomes L-midline of $\triangle LJL' \Longrightarrow JL' \perp PQ$ is perpendicular bisector of PQ.

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Reflection point lies on a line

Reply

buratinogigle

#1 Feb 10, 2013, 12:35 am

Let (O) be a circle and E, F are two points inside (O). $(K), (L)$ are two circles passing though E, F and tangent internally to (O) at A, D, respectively. AE, AF cut (O) again at B, C, respectively. BF cuts CE at G. Prove that reflection of A though EF lies on line DG.

See more IMO Shortlist 2011, G4.

Luis González

#2 Feb 10, 2013, 2:55 am • 2

Since A is the exsimilicenter of $(O) \sim (K)$, it follows that $BC \parallel EF$. If CE, BF cut (O) again at E', F', then $\angle GEF = \angle GCB = \angle E'F'F \Longrightarrow E, F, E', F'$ are concyclic $\Longrightarrow EF, E'F'$ and the tangent of (O) at A concur at the radical center R of $(O), (K)$ and $\odot(EFE'F')$. If M denotes the reflection of A on EF, then $\odot(MEF)$ and (K) are symmetric about $EF \Longrightarrow RM$ is tangent to $\odot(MEF) \Longrightarrow RM^2 = RE' \cdot RF' \Longrightarrow \odot(ME'F')$ is internally tangent to $\odot(MEF)$.

Reflection X of A about the midpoint of EF is obviously on $\odot(MEF)$. Insimilicenter G of $\triangle ABC \sim \triangle XFE$ is also insimilicenter of their circumcircles $(O) \sim \odot(MEF) \Longrightarrow G$ is also center of inversion that takes $\odot(MEF)$ into (O). In this inversion, E, F go to E', F' and M goes to the intersection D' of (O) with the ray GM. Since $\odot(ME'F')$ is internally tangent to (MEF), then by conformity $\odot(D'EF)$ is internally tangent to $(O) \Longrightarrow \odot(D'EF)$ and (L) coincide $\Longrightarrow D \equiv D'$.

TelvCohl

#3 Oct 10, 2014, 5:29 pm • 2

My solution:

Let E', F' be the intersection of (O) and DE, DF , respectively.
Let T be the intersection of BF' and CE' and M be the midpoint of BC .
Let A' be the intersection of (O) and DG .

Since A, D is the exsimilicenter of $((K) \sim (O)), ((L) \sim (O))$, respectively,
so we get $EF \parallel BC \parallel E'F'$.
By Pascal theorem (for $DE'CABF'$) we get E, T, F are collinear.
Since $E'F'CB$ is an isosceles trapezoid,
so T is the projection of M on EF .
Since $EF \parallel BC \parallel E'F'$,
so by Desargue theorem (for $\triangle CEE'$ and $\triangle BFF'$) we get D, G, T are collinear.
Easy to see A, G, M are collinear.
Since $\angle ETM = 90^\circ$ and $(TA, TG; TE, TM) = -1$,
so we get $\angle ATE = \angle A'TF$. ie. $AA' \parallel BC$
hence the reflection of A with respect to EF lie on DG .

Q.E.D

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A surprisedly beautiful Lemma

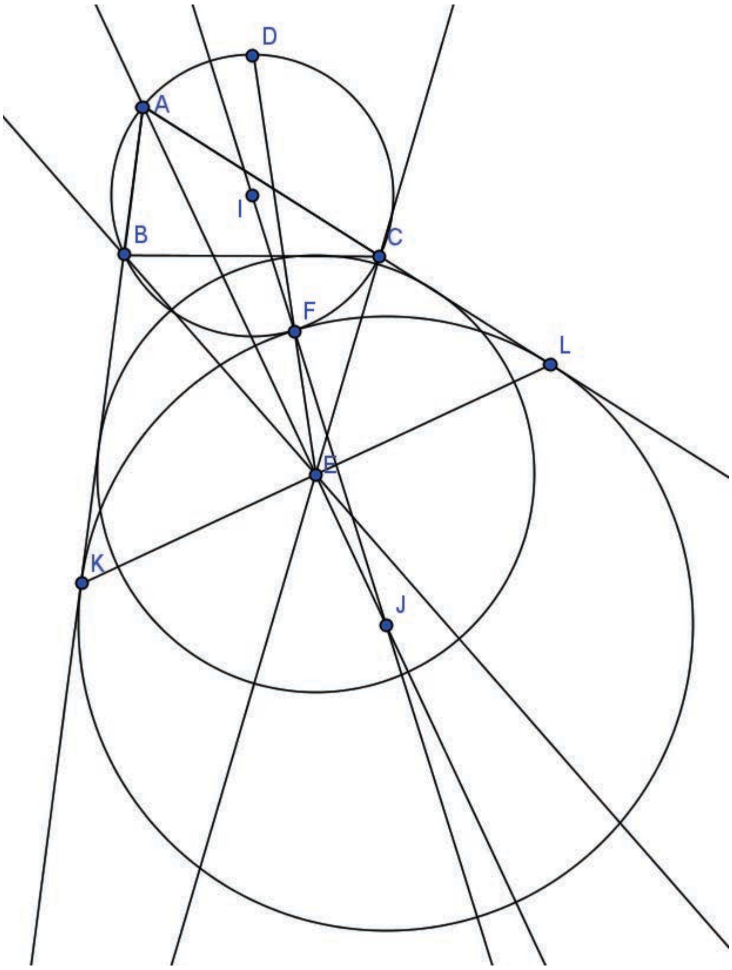
Locked

Narcissus

#1 Feb 9, 2013, 8:12 pm • 1

ABC is a triangle inscribed in (I). Circle (J) touches rays AB, AC and (I) at K, L, F respectively. D is the midpoint of arc BC containing A. E is the intersection of DF and KL. Prove that E is the A-excenter of triangle ABC

Attachments:



Luis González

#2 Feb 9, 2013, 10:22 pm • 1

This is just an extraversion of the mixtilinear incircle configuration. A-excenter E is the midpoint of LK (Sawayama's lemma) and FE bisects <BFC (Protassov theorem)

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=6086
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=31739
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=41667
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=407366

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Concurrency with excircles.

mathreyes

#1 Dec 18, 2012, 11:53 pm

Let ΔABC a triangle and trace the excircles $\mathcal{C}_A$, $\mathcal{C}_B$ y $\mathcal{C}_C$.
$\mathcal{C}_A$ touches $\overleftrightarrow{AB}$ in $X_{\mathcal{C}_A,\overleftrightarrow{AB}}$ and define in similar fashion points $X_{\mathcal{C}_A,\overleftrightarrow{AC}}$, $X_{\mathcal{C}_B,\overleftrightarrow{BA}}$, $X_{\mathcal{C}_B,\overleftrightarrow{BC}}$, $X_{\mathcal{C}_C,\overleftrightarrow{CA}}$ and $X_{\mathcal{C}_C,\overleftrightarrow{CB}}$.

Supose:

$\overline{X_{\mathcal{C}_A,\overleftrightarrow{AB}}, X_{\mathcal{C}_B,\overleftrightarrow{BC}}}$ cuts $\overline{X_{\mathcal{C}_A,\overleftrightarrow{AC}}, X_{\mathcal{C}_C,\overleftrightarrow{BC}}}$ in P.
$\overline{X_{\mathcal{C}_B,\overleftrightarrow{AB}}, X_{\mathcal{C}_C,\overleftrightarrow{BC}}}$ cuts $\overline{X_{\mathcal{C}_A,\overleftrightarrow{AB}}, X_{\mathcal{C}_C,\overleftrightarrow{AC}}}$ in Q.
$\overline{X_{\mathcal{C}_B,\overleftrightarrow{BC}}, X_{\mathcal{C}_C,\overleftrightarrow{AC}}}$ cuts $\overline{X_{\mathcal{C}_A,\overleftrightarrow{AC}}, X_{\mathcal{C}_B,\overleftrightarrow{AB}}}$ in R.
$\overline{X_{\mathcal{C}_B,\overleftrightarrow{AB}}, X_{\mathcal{C}_A,\overleftrightarrow{AC}}}$ cuts $\overline{X_{\mathcal{C}_A,\overleftrightarrow{AB}}, X_{\mathcal{C}_B,\overleftrightarrow{BC}}}$ in S.
$\overline{X_{\mathcal{C}_A,\overleftrightarrow{AC}}, X_{\mathcal{C}_C,\overleftrightarrow{BC}}}$ cuts $\overline{X_{\mathcal{C}_C,\overleftrightarrow{AC}}, X_{\mathcal{C}_A,\overleftrightarrow{AB}}}$ in T.
$\overline{X_{\mathcal{C}_B,\overleftrightarrow{AB}}, X_{\mathcal{C}_C,\overleftrightarrow{BC}}}$ cuts $\overline{X_{\mathcal{C}_C,\overleftrightarrow{AC}}, X_{\mathcal{C}_B,\overleftrightarrow{BC}}}$ in U.

Prove that $\overline{UP}$, $\overline{QS}$ and $\overline{RT}$ concur.

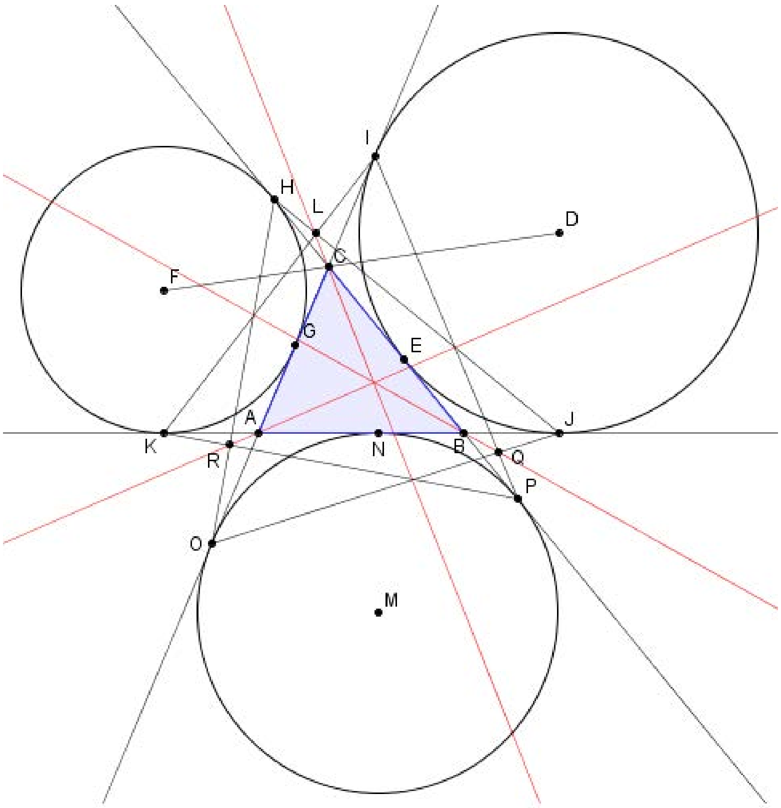
mathreyes

#2 Feb 9, 2013, 8:42 am

I will pull up this topic leaving here the graphic situation in order to make it fully understandable for you.

I have changed the names because they are just horrific in my previous post.

Attachments:

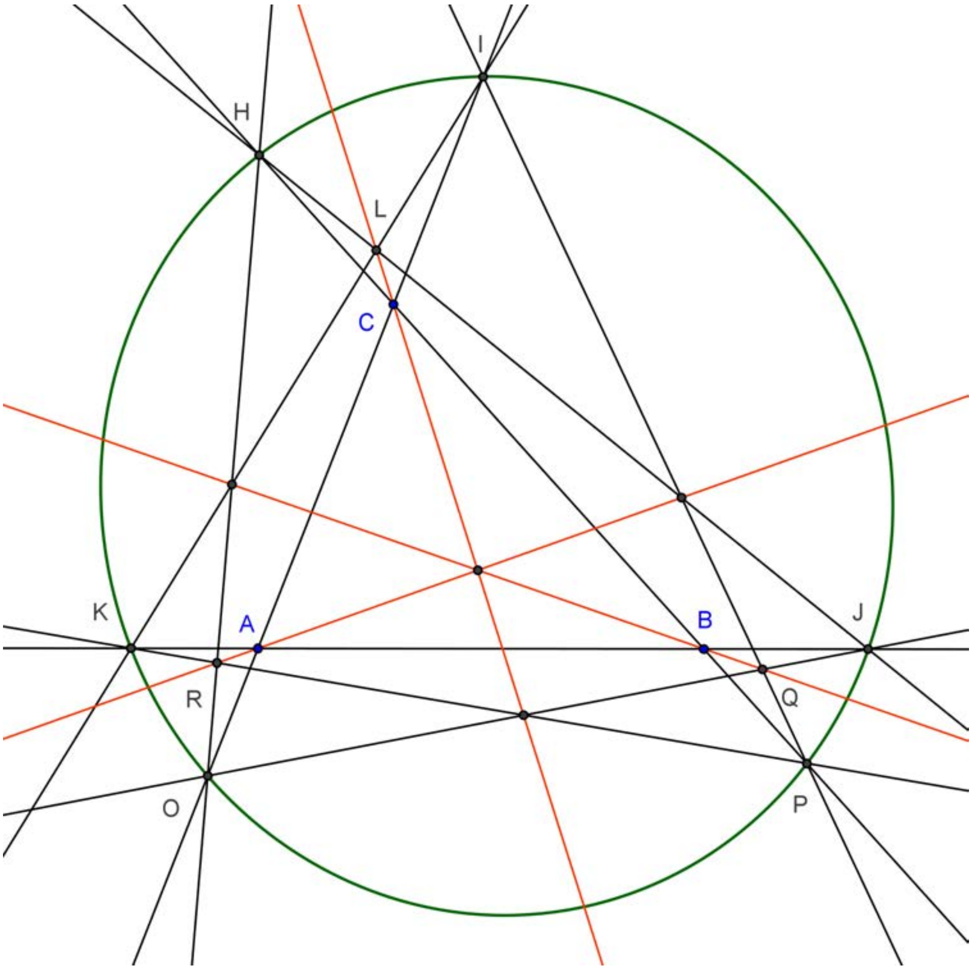


Luis González

#3 Feb 9, 2013, 9:32 am

Annoying problem just like Mixtilinear Excircles Concurrence. After proving that H,I,J,P,O,K are on a same conic (use Carnot's theorem), the problem becomes merely projective, i.e. Pascal-Desargues madness.

Attachments:



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Mixtilinear Excircles Perpendicular 2

Reply

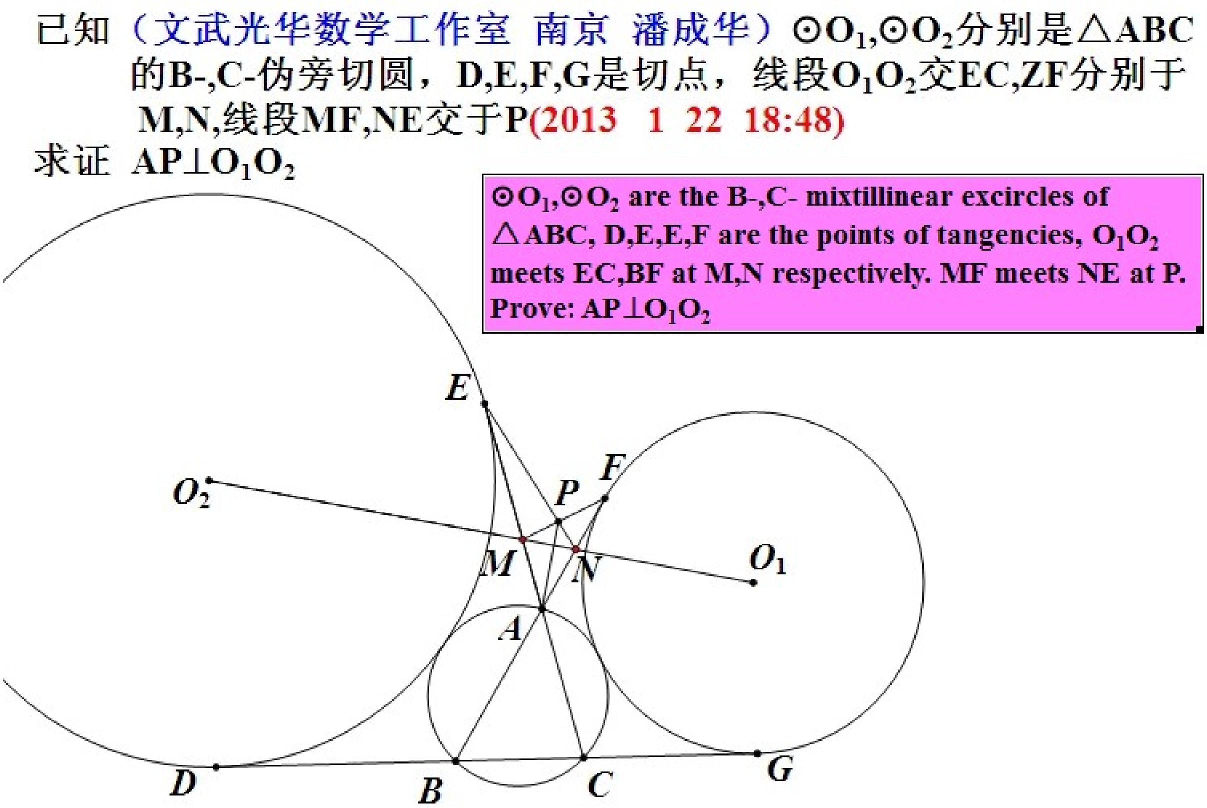
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Jan 24, 2013, 8:11 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Feb 9, 2013, 6:27 am • 1

Let the incircle (I,r) of $\triangle ABC$ touch AB at Z. From $\triangle BZI \sim \triangle BFO_1$, we get

$$\frac{O_1F}{r}=\frac{BF}{p-b} \Longrightarrow \frac{O_1F}{AF}=\frac{r}{p-b}\cdot\frac{BF}{AF}=\frac{r}{p-b}\cdot\frac{a}{p-c}=\frac{a\cdot r}{(p-b)(p-c)}.$$

Similarly, we have $\dfrac{O_2E}{AE}=\dfrac{a\cdot r}{(p-b)(p-c)} \Longrightarrow \dfrac{O_1F}{AF}=\dfrac{O_2E}{AE}$.

Therefore, the right triangles $\triangle AFO_1$ and $\triangle AEO_2$ are similar by SAS $\Longrightarrow \angle F\overline{AO_1}=\angle \overline{EAO_2}$. If Q is the projection of A on O_1O_2, then $AQFO_1$ and $AQEO_2$ are inscribed in the circles with diameters $\overline{AO_1}$ and $\overline{AO_2} \Longrightarrow$ $\angle FQO_1=\angle FAO_1=\angle EAO_2=\angle EQO_2 \Longrightarrow O_1O_2$ and AQ bisect $\angle EQF$ externally and internally. So if $U\equiv EF\cap O_1O_2$, then the pencil $A(E,F,Q,U)$ is harmonic, but from the complete quadrangle $EMNF$, the pencil $A(E,F,P,U)$ is harmonic $\Longrightarrow AP\equiv AQ\perp O_1O_2$.

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Mixtilinear Excircles Parallels

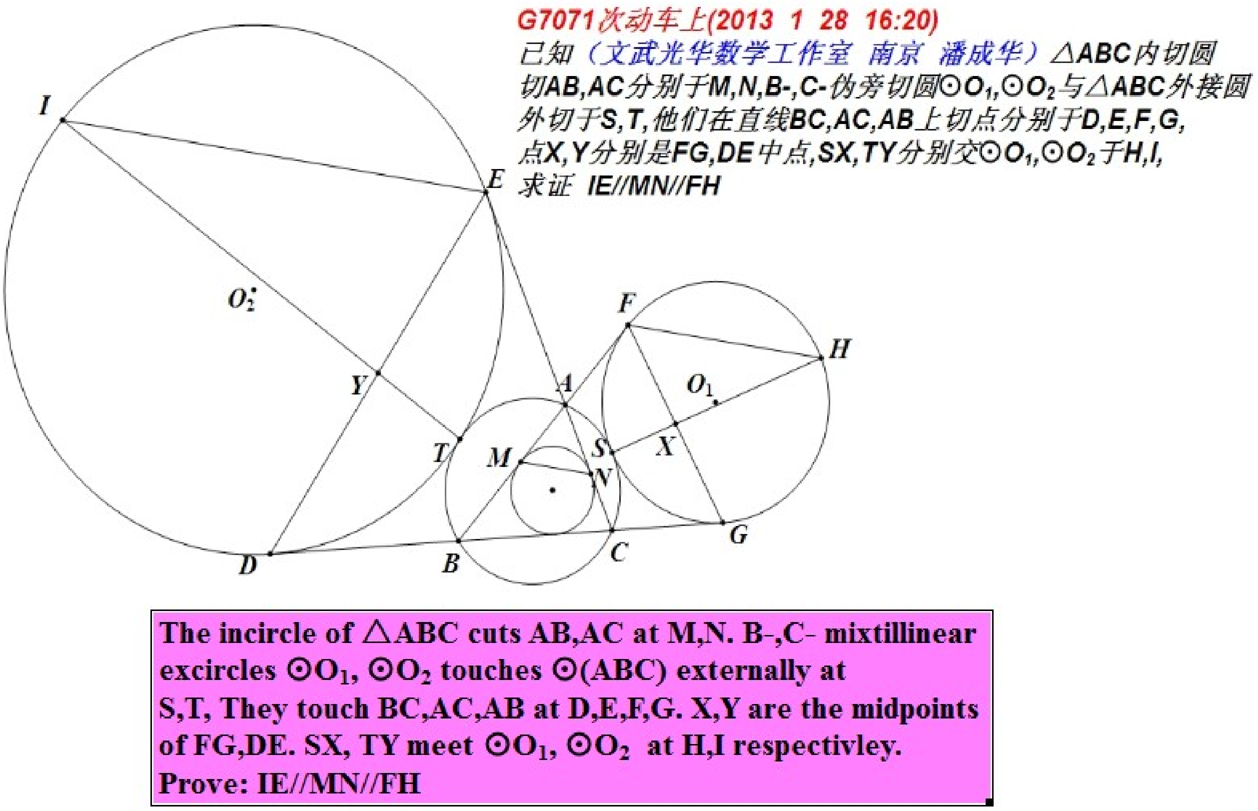
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Feb 2, 2013, 8:56 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Feb 8, 2013, 6:30 am

Y is the C-excenter of $\triangle ABC$ (Sawayama's lemma) and B, D, Y, T are concyclic (for a proof see the topic incenter of triangle and elsewhere). Moreover from the external tangency of (O_2) and the circumcircle (O), we deduce that TD bisects $\angle BTC$ externally $\Longrightarrow DT$ passes through the midpoint K of the arc BAC of $(O) \Longrightarrow$ $KC^2 = KB^2 = KY^2 = KT \cdot KD \Longrightarrow KY$ is tangent to $\odot(DYT) \Longrightarrow \angle AYT = \angle YDT = \angle EIT \Longrightarrow$ $IE \parallel AY \parallel MN$. Analogously $FH \parallel MN$.

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Nice Problem That Leads to A Nice Conclusion

Reply

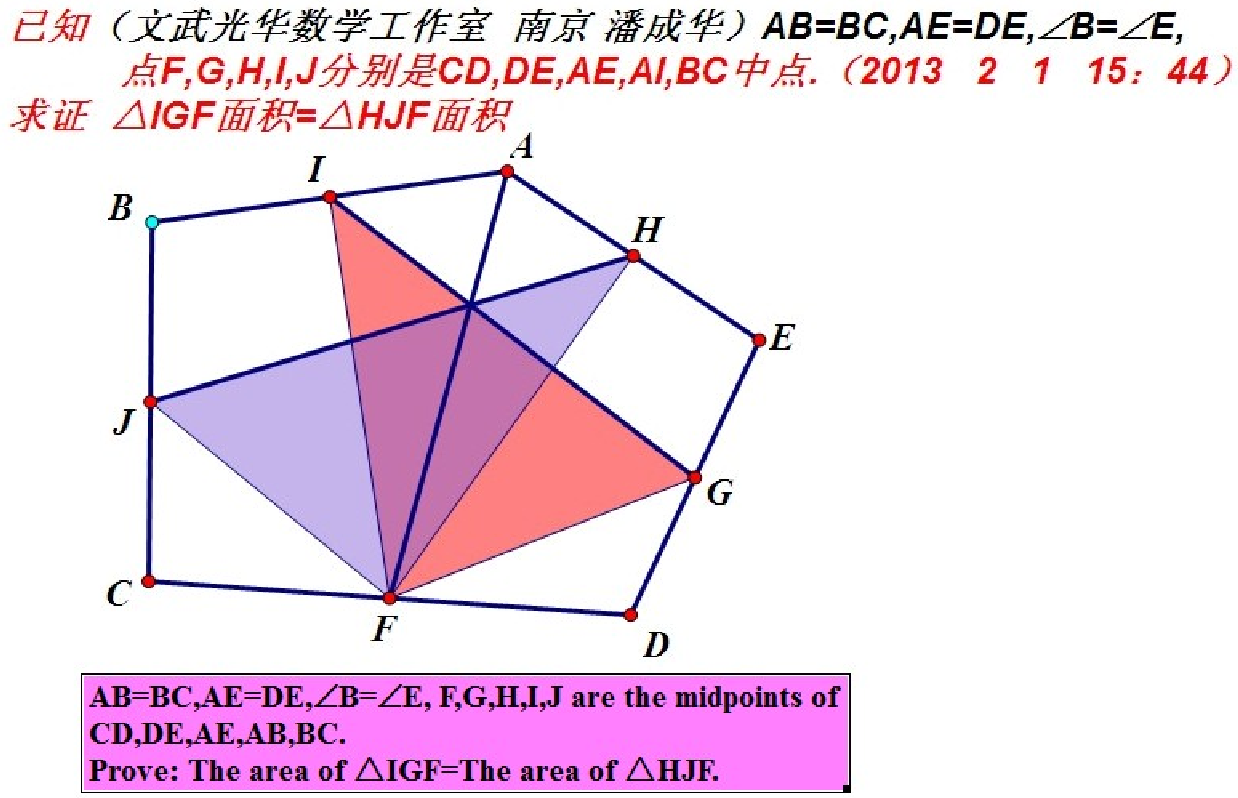
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Feb 3, 2013, 7:38 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Feb 8, 2013, 5:19 am

Since $JF \parallel BD$ and $JF = \frac{1}{2}BD$, we deduce that the area of the quadrilateral $BHDF$ with diagonals BD and FH is twice the area of $\triangle HJF$. Let S denote the area of the pentagon $ABCDE$. Then

$$2[HJF] = [BHDF] = S - \tfrac{1}{2}[ADE] - \tfrac{1}{2}[BDC] - \tfrac{1}{2}[ABE] \Longrightarrow$$

$$4[HJF] = 2S - (S - [ABD] + [ABE]) = S + [ABD] - [ABE].$$

By similar reasoning, we have $4[IGF] = S + [AEC] - [ABE]$. So it suffices to prove that $[ABD] = [AEC]$. Indeed, from $\triangle ABC \sim \triangle AED$, we get $AB \cdot AD = AE \cdot AC$, and since $\angle BAD = \angle EAC$, we deduce that $[ABD] = [AEC]$, as desired.

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Another Nice Problem

Reply

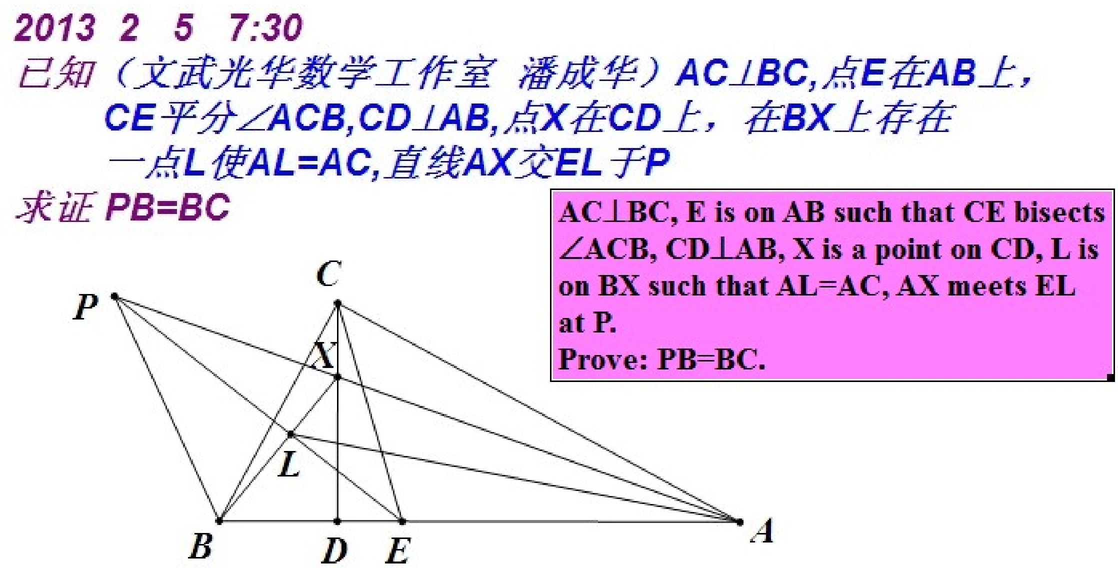
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Feb 5, 2013, 7:06 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Feb 8, 2013, 12:15 am

This is just an alternate formulation of IMO 2012 Problem 5.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=834&t=488511
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=489045

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Parallels

Reply

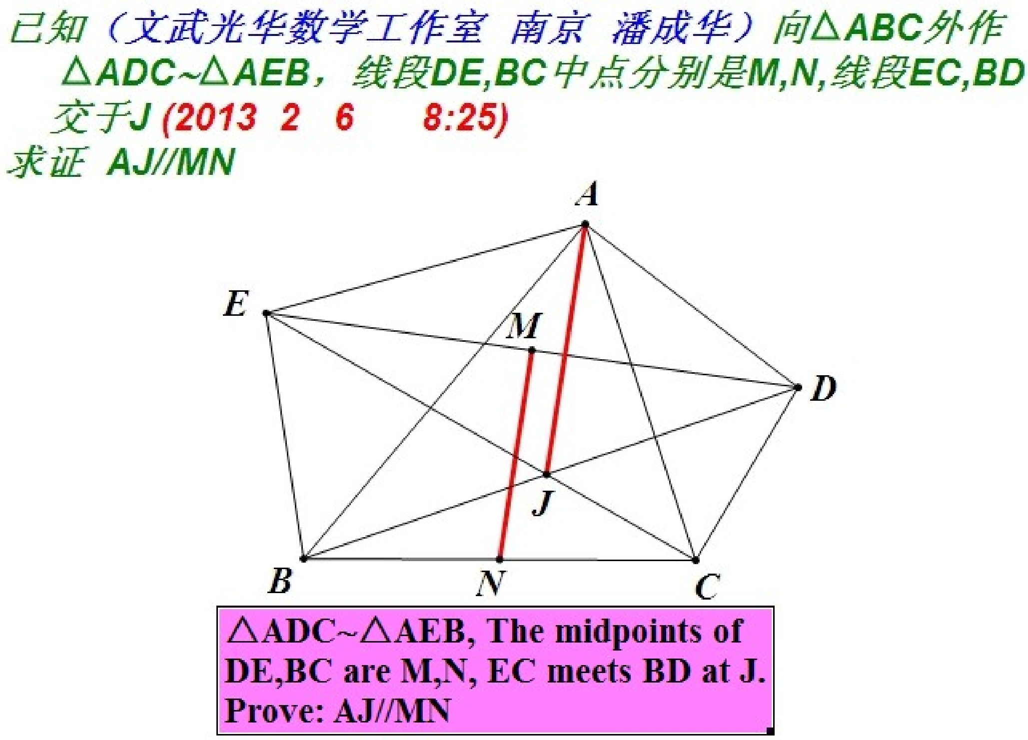
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Feb 6, 2013, 7:29 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Feb 7, 2013, 3:09 am

Construct point F outside $\triangle ABC$, such that $\angle FBC = \angle FCB = \angle EBA$, thus by Jacobi's theorem $J \equiv AF \cap BD \cap CE$. Further $H \equiv CD \cap BE$ is then the isogonal conjugate of F WRT $\triangle ABC$, thus if Y, Z are the projections of H on AC, AB, then $AF \equiv AJ \perp YZ$. Hence, it suffices to show that $MN \perp YZ$.

Let CD, BE cut AB, AC at U, V, respectively. $\triangle BAV$ and $\triangle CAU$ are clearly similar with corresponding cevians $AE, AD \Longrightarrow \overline{EB} : \overline{EV} = \overline{DC} : \overline{DU} \Longrightarrow D, E$ are homologous points under the spiral similarity with center P that swaps $\overline{BV}$ and $\overline{CU} \Longrightarrow \triangle PBC, \triangle PVU, \triangle PED$ are all directly similar with corresponding medians $PN, PL, PM \Longrightarrow$ M, N, L lie on the image of CU under the spiral similarity with center P, rotational angle $\angle(PC, PN)$ and coefficient $\frac{PN}{PC} \Longrightarrow$ LMN is Newton line of $ABHC$, meeting $\overline{AH}$ at its midpoint K (circumcenter of AYZ). Now, using the result of the problem Power of point 2 for $\triangle ABC$ and the point H satisfying $\angle ABH = \angle ACH$, we get $NY = NZ \Longrightarrow MNK$ is perpendicular bisector of $\overline{YZ} \Longrightarrow MN \perp YZ$, as desired.

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High School Olympiads

Point in cyclic cuadrilateral with angle condition

Locked

Source: Czech - Polish - Slovak Match 2004

hatchguy

#1 Feb 5, 2013, 3:58 pm

A point P in the interior of a cyclic quadrilateral $ABCD$ satisfies $\angle BPC = \angle BAP + \angle PDC$. Denote by E, F and G the feet of the perpendiculars from P to the lines AB, AD and DC, respectively. Show that the triangles FEG and PBC are similar.

Luis González

#2 Feb 6, 2013, 10:33 am • 1

Posted several times before. It's also UK FST2 2006 (Q2) and a problem from Peru TST for IBMO 2005.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=49710
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=316491
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=397093
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=475315

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High School Olympiads

AB^2-AC^2=AB.BC and A=30, B=?

xeroxia
#1 Feb 6, 2013, 2:03 am

Let ABC be a triangle with $AB^2-AC^2=AB\cdot BC$. If $m(\widehat{A})=30^\circ$, what is $m(\widehat{B})$?

yetti
#2 Feb 6, 2013, 4:57 am

By cosine theorem, $2\cdot AB\cdot BC\cdot\cos\widehat{B}-BC^2=AB^2-AC^2=AB\cdot BC\implies 2\cos\widehat{B}-1=\frac{BC}{AB}$.

By sine theorem, $\frac{BC}{AB}=\frac{\sin\widehat{A}}{\sin\widehat{C}}=\frac{1}{2\sin(150^\circ-\widehat{B})}$.

Combined, $4\sin(150^\circ-\widehat{B})\cos\widehat{B}-1=2\sin(150^\circ-\widehat{B})\iff$

$\sin(150^\circ)+\sin(150^\circ-2\widehat{B})-\frac{1}{2}=\sin(150^\circ-\widehat{B})\iff$

$\sin(150^\circ-2\widehat{B})=\sin(150^\circ-\widehat{B})\iff$

$150^\circ-2\widehat{B}=150^\circ-\widehat{B}$ or $150^\circ-2\widehat{B}=180^\circ-(150^\circ-\widehat{B})\iff$

$\widehat{B}=2\widehat{B}$ or $3\widehat{B}=120^\circ\iff$

$\widehat{B}=0^\circ$ or $\widehat{B}=40^\circ$.

Luis González
#3 Feb 6, 2013, 5:50 am

Let D,E,F be the midpoints of BC,CA,AB and let P be the orthogonal projection of A on BC.
$AB^2-AC^2=PB^2-PC^2=2\cdot PD\cdot BC\Longrightarrow PD=\frac{1}{2}AB=DE\Longrightarrow\triangle DEP$ is D-isosceles, but since $DPEF$ is an isosceles trapezoid with bases $EF\parallel DP$, we have $\angle EPD=\angle FDC=180^\circ-\angle C$. Hence $\angle EDP+2\angle EPD=180^\circ\Longrightarrow\angle B+2(180^\circ-\angle C)=180^\circ\Longrightarrow\angle B+2(30^\circ+\angle B)=180^\circ\Longrightarrow\angle B=40^\circ$.

xeroxia
#4 Feb 6, 2013, 12:43 pm

Clearly, $AB-\frac{AC^2}{AB}=BC<AB$.

Take D on $[BA]$ such that $BC=BD$. $AD=AB-BC$.

$AB^2-AC^2=BC\cdot AB\Rightarrow AB^2-AB\cdot BC=AC^2$

$\Rightarrow AB(AB-BC)=AB\cdot AD=AC^2\Rightarrow\angle ABC=\angle DCA$.

$\angle BCD=90^\circ-\angle ABC/2\Rightarrow\angle ACB=90^\circ+\angle ABC/2$.

$\angle BAC+\angle BCA+\angle ABC=180^\circ\Rightarrow 30^\circ+\angle ABC+90^\circ+\angle ABC/2=180^\circ$

$\Rightarrow\angle ABC=40^\circ.\blacksquare$

Note: 30° is not a magic number. If exterior angle of B is twice of exterior angle of C, we have $AB^2-AC^2=BC\cdot AB$.

Virgil Nicula

#5 Feb 8, 2013, 1:51 am

$$\left\{\begin{array}{ccc} b^2 & = & a^2+c^2-2ac\cdot\cos B\\ c^2-b^2 & = & ac\end{array}\right\|\bigoplus\implies\left\{\begin{array}{c} a=c(2\cos B-1)\\ C=150^\circ-B\end{array}\right\|\implies$$

$$\frac{1}{2}=\sin(30^\circ+B)(2\cos B-1)\iff\frac{1}{2}=\sin(30^\circ+2B)+\frac{1}{2}-\sin(30^\circ+B)\iff$$

$$\sin(30^\circ+2B)=\sin(30^\circ+B)\iff(30^\circ+2B)+(30^\circ+B)=180^\circ\iff\boxed{B=40^\circ}.$$

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High School Olympiads

Three concurrent lines [Brocard axes of AB'C', BC'A', CA'B']

Reply

skywalker

#1 Mar 3, 2007, 5:19 pm

Let $A'B'C'$ be orthic triangle of ABC prove that Brocard axes of three triangle $AB'C'$, $BC'A'$, $CA'B'$ are concurrent at a point.

Hint

Luis González

#2 Feb 5, 2013, 2:36 am

Let O,K,H be the circumcenter, symmedian point and orthocenter of $\triangle ABC. O_A,O_B,O_C$ are the midpoints of HA,HB,HC (circumcenters of AB'C', BC'A', CA'B') and D,E,F are the midpoints of $BC,CA,AB. X,Y,Z$ are the projections of K on $BC,CA,AB. K_A,K_B,K_C$ are the symmedian points of $\triangle AB'C',\triangle BC'A',\triangle CA'B'$, respectively $\Longrightarrow O_AK_A,O_BK_B,O_CK_C$ are Brocard axes of $\triangle AB'C',\triangle BC'A',\triangle CA'B'$, respectively.

Let the tangents of (O) at B,C meet at A_0. Then $\triangle ABC \sim \triangle AB'C'$ are similar with corresponding symmedians AKA_0 and $AK_AD \Longrightarrow AK_A:DK_A=AK:A_0K \Longrightarrow KK_A \parallel A_0D \perp BC \Longrightarrow K_A \equiv AD \cap KX$. Since XK passes through the midpoint of YZ (well-known), then the intersection of the line passing through K and the midpoint of YZ with the A-altitude AD of $\triangle AYZ$ is nothing but the orthocenter K_A of $\triangle AYZ$.

Since O_A is the orthocenter of $\triangle AEF$, then it follows that O_AK_A is the Steiner line of the four-line AB,AC,EF,YZ. Hence, Miquel point A_1 of this four-line, which is the 2nd intersection A_1 of the circles $\odot(AEF)$ and $\odot(AYZ)$ (projection of A on OK) is then the pole of O_AK_A WRT $\triangle AEF$ and $\triangle AYZ \Longrightarrow O_AK_A$ goes through the reflection of A_1 on EF, which is nothing but the anti-Steiner point of OK WRT $\triangle DEF$, in other words the orthopole of OK WRT $\triangle ABC$. So we conclude that O_AK_A,O_BK_B,O_CK_C concur at the orthopole of OK WRT $\triangle ABC$, which clearly falls on the 9-point circle $\odot(DEF)$.

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High School Olympiads

Equal angles

Reply

Source: Tsitsifas book- In greek

stergiu

#1 Jan 16, 2013, 4:00 am

Let a circumscribed quadrilateral $ABCD$. The circle which is tangent to AB at point B and also to line CD has center K.The circle which is tangent to AD at point D and also

to line BC has center L.

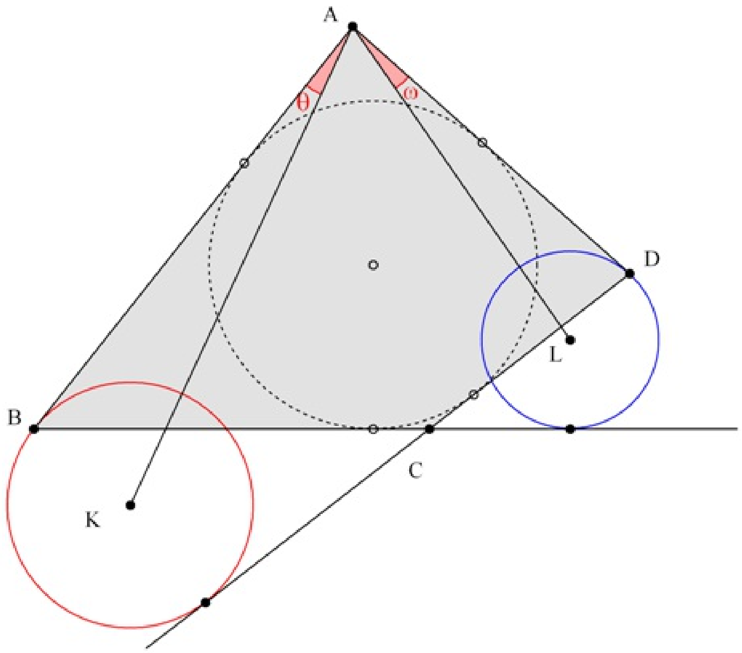
Prove that $\angle KAB = \angle LAD$

Babis

(Thanks to Nick Fragakis for making the image)

* edit

Attachments:



This post has been edited 1 time. Last edited by stergiu, Feb 7, 2013, 6:58 pm

Luis González

#2 Feb 3, 2013, 10:47 am

Incircle (I,r) of $ABCD$ touches AB,AD at $M,N.P \equiv AB \cap CD$ and $Q \equiv AD \cap BC.BD,PQ$ and MN clearly concur at the pole R of AC WRT (I,r), thus by Menelaus' theorem for $\triangle APQ$ cut by $\overline{RBD}$ and $\overline{RPQ}$, we get

$$\frac{RP}{RQ}=\frac{AD}{QD}\cdot\frac{PB}{AB}=\frac{AN}{QN}\cdot\frac{PM}{AM} \Longrightarrow \frac{AB}{AD}=\frac{PB}{PM}\cdot\frac{QN}{QD} \ \ (1).$$

From $\triangle PBK \sim \triangle PMI$ and $\triangle QDL \sim \triangle QNI$, we obtain

$$\frac{KB}{r}=\frac{PB}{PM} \ , \ \frac{LD}{r}=\frac{QD}{QN} \Longrightarrow \frac{KB}{LD}=\frac{PB}{PM}\cdot\frac{QN}{QD} \ \ (2).$$

$(1) \cup (2)$ gives $\dfrac{KB}{LD}=\dfrac{AB}{AD} \Longrightarrow \triangle ABK \sim \triangle ADL \Longrightarrow \angle KAB=\angle LAD.$

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High School Olympiads

Excircle and Circumcircle.

Reply

Aandabaccha93

#1 Feb 2, 2013, 4:07 pm

In $\triangle ABC$ Let Incentre I, Excentre opposite to A is A_A. $P, Q \in BC$ such that $AP = AQ = s$ where s is the semi perimeter. $A_AF \perp BC$. Let H be the circumcentre of $\triangle APQ$. HA_A intersect the excircle at S. Prove that A, F, S are collinear.

Luis González

#2 Feb 2, 2013, 11:18 pm • 1

It follows from the fact that circle (APQ) is internally tangent to the A-excircle at S. See

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=34337
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=221941

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High School Olympiads

symmertric on midpoints and the equality of length

Stephen

#1 Feb 1, 2013, 4:39 pm

O is the circumcenter of triangle of ABC. A_1, A_2 are the points on BC which are symmertric to the midpoint of segment BC. Similarly, B_1, B_2 are the points on CA which are symmertric to the midpoint of segment CA, and C_1, C_2 are the points on AB which are symmertric to the midpoint of segment AB. Z_1 is the cross point of the circumcircles of A_1BC_1, A_1CB_1 which is not A_1, and Z_2 is the cross point of the circumcircles of A_2BC_2, A_2CB_2 which is not A_2. Prove that $OZ_1 = OZ_2$.

Luis González

#2 Feb 2, 2013, 10:31 pm

This configuration was discussed before. Note that Z_1 and Z_2 are the Miquel points of A_1B_1C_1 and A_2B_2C_2 WRT ABC. See the topics

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=298400
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=363074

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High School Olympiads

A metrical relation in a triangle.

Reply

Virgil Nicula
#1 Jan 31, 2013, 11:03 pm • 1

PP. Let ABC be a triangle and $D\in (BC)$. The incircles of $\triangle ABD$ and $\triangle ACD$ touch BC at M , N respectively. Prove that the ray $[AD$ is the A-bisector of $\triangle ABC\iff \frac {1}{MB}+\frac {1}{MD}=\frac {1}{ND}+\frac {1}{NC}$.

Luis González
#2 Feb 2, 2013, 1:06 am • 1

Let the incircle (I) of $\triangle ABC$ touch BC at X and h_a denotes the length of the A-altitude. Then

$$\frac{1}{BX}+\frac{1}{CX}=\frac{1}{s-b}+\frac{1}{s-c}=\frac{a}{(s-b)(s-c)}=\frac{2\cot \frac{A}{2}}{h_a}.$$

Using the latter relation on triangles $\triangle ABD$ and $\triangle ACD$ with common A-altitude h_a, we get that

$$\frac{1}{MB}+\frac{1}{MD}=\frac{1}{ND}+\frac{1}{NC} \Longleftrightarrow \cot \frac{\widehat{BAD}}{2}=\cot \frac{\widehat{CAD}}{2} \Longleftrightarrow \widehat{BAD}=\widehat{CAD}.$$

Virgil Nicula
#3 Feb 2, 2013, 4:00 pm

Nice proof ! Thank you. See **PP9 & PP10** from **here**.

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High School Olympiads

circumcenters are collinear

Reply

Stephen
#1 Feb 1, 2013, 11:40 am

In triangle ABC, I is the incenter. A_1 is the cross point of AI and BC.

l_a is the line which go through I and perpendicular the AI.

A_2 is the cross point of l_a and BC. Similarly define B_1, B_2, C_1, C_2.

Prove that the circumcenters of triangles $AA_1A_2, BB_1B_2, CC_1C_2$ are collinear.

Luis González
#2 Feb 1, 2013, 11:32 pm

Lemma. $\triangle ABC$ is scalene with circumcenter O and orthocenter H. D, E, F are the midpoints of BC, CA, AB. D', E', F' are the projections of A, B, C on OD, OE, OF and O_A, O_B, O_C are the projections of O on HA, HB, HC. Then the circles $\odot(O_ADD'), \odot(O_BEE')$ and $\odot(O_CFF')$ are coaxal.

Proof. The center S_A of $\odot(O_ADD')$ is obviously on the perpendicular bisector of $\overline{DD'}$, i.e. the A-midline EF. Thus $\odot(O_ADD')$ passes through the reflection H_A of O_A about EF, which is the midpoint of $\overline{H_A}$. Similarly, the centers S_B, S_C of $\odot(O_BEE'), \odot(O_CFF')$ are on FD, DE and they pass through midpoints H_B, H_C of $\overline{HB}, \overline{HC}$.

DH_A, EH_B, FH_C are circumdiameters of the 9-point circle (N) of $\triangle ABC$, meeting at $N \Longrightarrow N$ has equal power WRT $\odot(O_ADD'), \odot(O_BEE'), \odot(O_CFF')$ and their centers are collinear, since $\overline{S_AS_BS_C}$ is the orthopolar of N WRT $\triangle DEF$. Consequently, $\odot(O_ADD'), \odot(O_BEE')$ and $\odot(O_CFF')$ are coaxal.

Back to the problem, let A_0, B_0, C_0 be the tangency points of the incircle (I) of $\triangle ABC$ with BC, CA, AB. H_0 is the orthocenter of $\triangle A_0B_0C_0$. A_2, B_2, C_2 are clearly the poles of A_0H_0, B_0H_0, C_0H_0 WRT (I). Hence, inversion WRT (I) takes A_2, B_2, C_2 into the projections I_1, I_2, I_3 of I on A_0H_0, B_0H_0, C_0H_0. This also takes A, B, C into the midpoints D_1, E_1, F_1 of B_0C_0, C_0A_0, A_0B_0 and A_1, B_1, C_1 into the projections D_2, E_2, F_2 of A_0, B_0, C_0 on ID_1, IE_1, IF_1.

Using the previous lemma for $\triangle A_0B_0C_0$, the circles $\odot(I_1D_1D_2), \odot(I_2E_1E_2)$ and $\odot(I_3F_1F_2)$ are coaxal, thus their inverses $\odot(AA_1A_2), \odot(BB_1B_2)$ and $\odot(CC_1C_2)$, under the referred inversion, are also coaxal $\Longrightarrow$ their centers are collinear.

nsato
#3 Mar 29, 2013, 8:56 pm

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=315751.

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High School Olympiads

Incircle Tangency

Reply

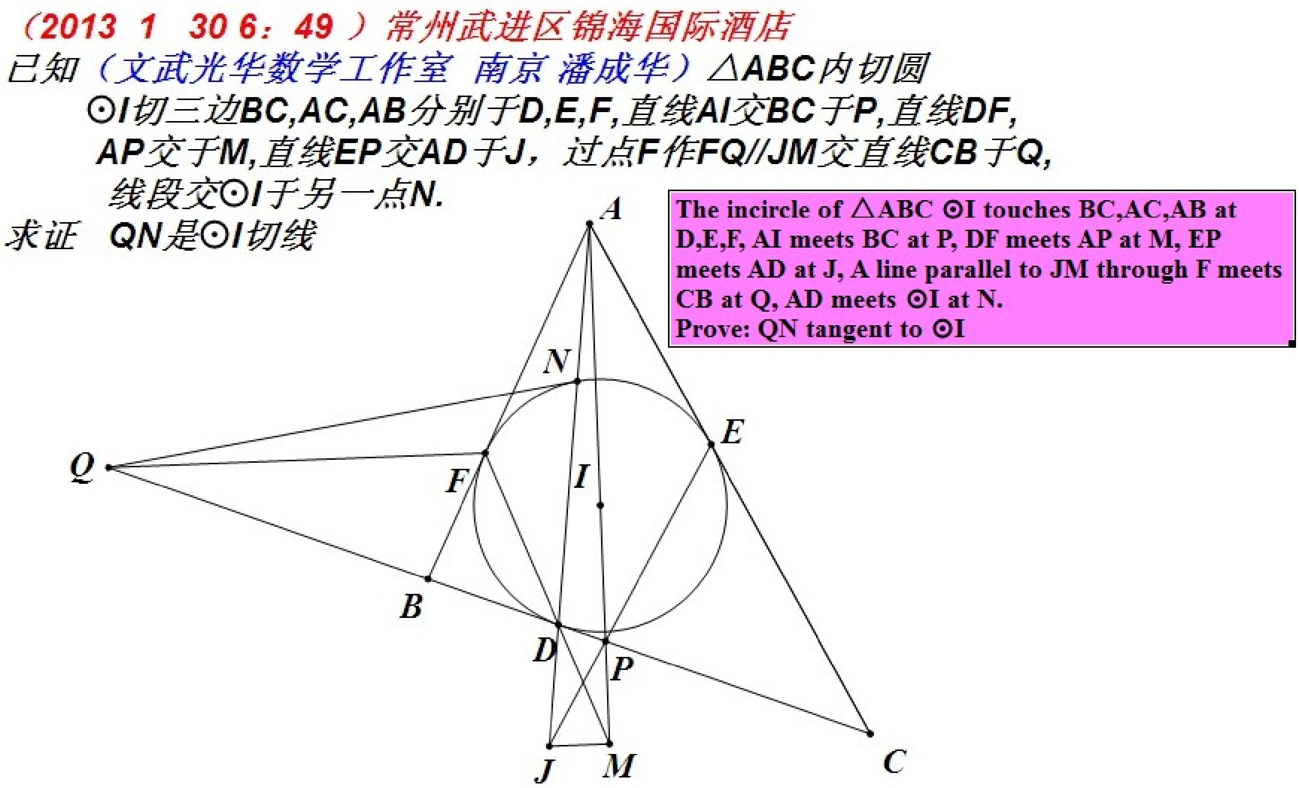
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Jan 30, 2013, 11:55 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Jan 31, 2013, 4:09 am • 1

Since $\angle CIM = 90^\circ - \frac{1}{2}\angle ABC = \angle BDF = \angle CDM$, then C, I, D, M are concyclic $\Longrightarrow$ $\angle CMI = \angle CDI = 90^\circ$, i.e. $CM \perp AI$. CM cuts AD at J^* and MDF cuts AC at S. From the complete quadrilateral $ADMC$, it follows that $J^*(A, C, P, S) = -1$, but $(A, C, E, S) = -1 \Longrightarrow E \in PJ^* \Longrightarrow J \equiv J^*$. Hence $CMJ \parallel EF$ $\Longrightarrow Q \in EF$. Since AD is the polar of Q WRT (I), it follows that QN is tangent to (I).

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High School Olympiads

Nice problem about Lemoine point and Lemoine circle

Locked

Source: Pham Hy Hieu, silver medalist at IMO 2009

Nguyenhuyhoang

#1 Jan 30, 2013, 1:59 pm

Given $\triangle ABC$ circumscribed in (O), symmedian AA_1, draw parallelogram $AC_aA_1B_a$ such that C_a, B_a are on AB, AC respectively. Prove that:
a) BCB_aC_a is cyclic(I believe there are at least two solutions for this one)
b) Let (O_a) be the circumcircle of BCB_aC_a, prove that AO_a passes through OL, where L is the Lemoine point of $\triangle ABC$. (I have a solution using geometry transformation, but I'm expecting a solution without using transformation).

Luis González

#2 Jan 30, 2013, 9:56 pm • 1

For proposition b), I believe it should be that AOa passes through the midpoint of OL. This is a problem from China TST 2005 and also Italian Prelmo 2012 (recently posted).

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=558640
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=482207

High School Olympiads

Bicentric quadrilateral

Source: Me

juancarlos

#1 May 4, 2005, 4:46 am

Another well known property of bicentric quadrilateral?:
Let $ABCD$ bicentric quadrilateral, with I incenter, M midpoint of BD and P cut point of AC and BD. Prove that $\angle BIP = \angle MID$.

yetti

#2 May 8, 2005, 4:02 am

Let the opposite sides AB, CD and BC, DA of the bicentric quadrilateral $ABCD$ intersect at points X, Y, respectively. WLOG, assume that the quadrilateral incircle (I) is also the incircle of the triangles $\triangle ABX, \triangle BCY$. Since the quadrilateral $ABCD$ is tangential, the line MN connecting the midpoints M, N of the diagonals BD, AC (Newton's line) passes through its incenter I. This can be demonstrated in the following way: AM, CM are medians of the triangles $\triangle DAB, \triangle BCD$ and BN, DN are medians of the triangles $\triangle ABC, \triangle CDA$, cutting their areas in half. Hence,

$$|\triangle ABM| + |\triangle CDM| = |\triangle DAM| + |\triangle BCM| = \frac{|ABCD|}{2}$$

$$|\triangle ABN| + |\triangle CDN| = |\triangle DAN| + |\triangle BCN| = \frac{|ABCD|}{2}$$

Since the quadrilateral $ABCD$ is tangential, $AB + CD = BC + DA$ and the altitudes of the triangles $\triangle ABI, \triangle BCI, \triangle CDI, \triangle DAI$ from their common vertex I are all equal to the quadrilateral inradius r. Hence, sums of the following triangle areas are also equal:

$$|\triangle ABI| + |\triangle CDI| = \frac{r}{2}(AB + CD) =$$

$$= \frac{r}{2}(BC + DA) = |\triangle BCI| + |\triangle DAI| = \frac{|ABCD|}{2}$$

Since the locus of common vertices V of 2 triangles $\triangle ABV, \triangle CDV$ with the fixed bases AB, CD, such that the sum of their areas is constant, is a straight line and since the points M, N, I are all on this line, the line MN passes through the incenter I.

Let K, L be the intersections of the perpendicular bisector of the diagonal BD with the minor and major arcs BD of the circumcircle (O), i.e., the midpoints of these arcs. This perpendicular bisector passes through the midpoint M of this diagonal and through the quadrilateral circumcenter O. The bisectors AI, CI of the angles $\angle A = \angle DAB, \angle C = \angle BCD$ pass through the quadrilateral incenter I and they also cut the minor and major arcs BD at their midpoints K, L, respectively. The angles $\triangle ACL = \angle AKL$ spanning the same arc AL of the circumcircle (O) are equal. The diagonal intersection $P \equiv AC \cap BD$, the incenter I and the circumcenter O of a bicentric quadrilateral are collinear (numerous proofs of this fact have been posted, for example, see Concurrency problem; related to Poncelet's theorem). The quadrilateral $OMIN$ is cyclic, because the angles $\angle OMI = \angle ONI = 90^o$ are both right. Hence, the triangles $\triangle OMI \sim \triangle NIP$ are similar by SAS, because the angles $\angle OIM = \angle NIP$ are vertical and the power of the incenter I with respect to the circumcircle of this quadrilateral is equal to $OI \cdot PI = MI \cdot NI$, i.e., $\frac{OI}{MI} = \frac{CI}{PI}$. Consequently, their external angles $\angle KMI = \angle CPI$ are equal. As a result, the triangles $\triangle KMI \sim \triangle CPI$ are similar and their remaining angles $\angle KIM = \angle CIP$ are also equal.

$$\angle BIM = \angle AIM - \angle AIB = (180^o - \angle KIM) - \left[180^o - \left(\frac{\angle A + \angle B}{2}\right)\right] =$$

$$= \frac{\angle A + \angle B}{2} - \angle KIM$$

$= \frac{\quad}{2} - \angle KIM$

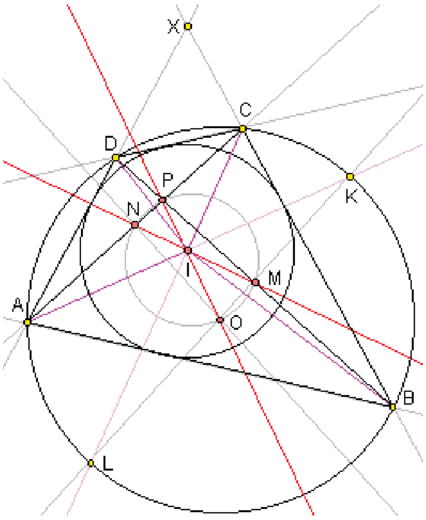
$$\angle DIP = \angle CID - \angle CIP = 180^o - \left(\frac{\angle C + \angle D}{2}\right) - \angle CIP =$$

$$= \frac{\angle A + \angle B}{2} - \angle KIM$$

$$\angle BIP = \angle BID - \angle BIM = \angle BID - \angle DIP = \angle DIM$$

In exactly the same way, it can be shown that the angles $\angle AIN = \angle CIP$ and $\angle AIP = \angle CIN$ are also equal. It also immediately follows from the fact that the angles $\angle CIP = \angle KIM$ are equal and the angles $\angle AIN = \angle KIM$ are vertical.

Attachments:



Luis González

#3 Jan 30, 2013, 2:27 am

Denote by O the circumcenter of $ABCD.$ Simple angle chase yields

$$\angle ODI = \angle ODA - \angle IDA = 90^\circ - \angle ABD - \tfrac{1}{2}\angle ADC =$$

$$= \tfrac{1}{2}(\angle ADC + \angle ABC) - \angle ABD - \tfrac{1}{2}\angle ADC = \tfrac{1}{2}\angle ABC - \angle ABD = \angle DBI.$$

Thus, OD is tangent to $\odot(IBD).$ Similarly OB is tangent to $\odot(IBD) \Longrightarrow IO$ is I-symmedian of $\triangle BID.$ But it's well-known that O, I, P are collinear (see collinearity in bicentric quardilateral and elsewhere), hence IP is I-symmedian of $\triangle IBC$ $\Longrightarrow IP, IM$ are isogonals WRT $\angle BID$ and the conclusion follows.

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High School Olympiads

Similar Isosceles Triangles Area 2

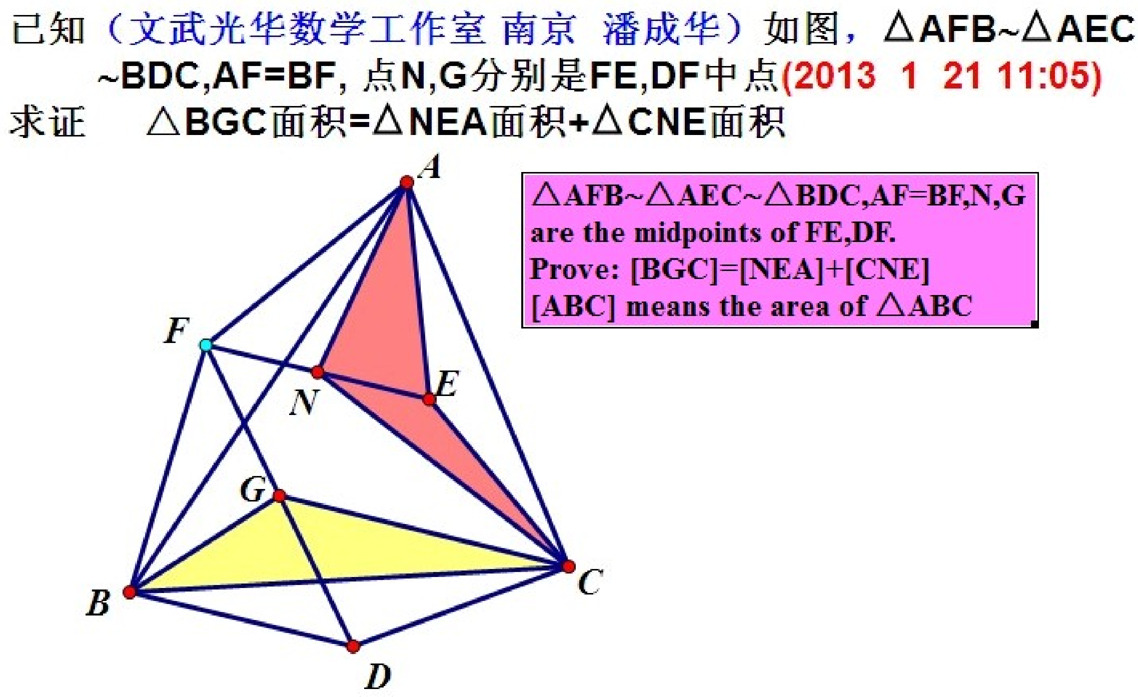
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Jan 21, 2013, 11:37 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Jan 29, 2013, 9:51 am

Let H be the reflection of E about $AC \Longrightarrow AECH$ is a rhombus. $\triangle BAH$ having a pair of equal and parallel sides to the diagonals of the concave quadrilateral $AEBC$ is equivalent to it $\Longrightarrow [BAH]=[AEBC]$. From $\triangle AHC \sim \triangle AFB$, we get $AH \cdot AB = AC \cdot AF$ and since $\angle BAH = \angle CAF = \angle BAC + \angle FAB$, we deduce that $[BAH]=[CAF]$. Hence, $[ACF]=[AEBC]=[AEC]+[EBC] \Longrightarrow [AFE]+[CFE]=[EBC]$. But we know that $EFBD$ is a parallelogram, thus G is midpoint of $\overline{EB} \Longrightarrow [NEA]+[CNE]=[BGC]$.

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High School Olympiads

MYM Olympiad Problem 2 (400th Post)

Reply

Source: Math and Youth Magazine 418

thugzmath10

#1 Jan 28, 2013, 7:44 pm

Let ABC be a triangle with circumcenter O and incenter I. BC touches the incircle at D. The circle with diameter AI meets the circumcircle of ABC again at M and the line through A parallel to BC at N. Prove that MO bisects DN.

Luis González

#2 Jan 29, 2013, 12:56 am • 1

P, Q are the midpoints of the arcs BC and BAC of (O). PD cuts (O) again at M'. Since $PB = PC = PI$, we have $PI^2 = PB^2 = PD \cdot PM' \Longrightarrow PI$ is tangent to $\odot(DIM') \Longrightarrow \angle IM'D = \angle DIP = \angle APQ = \angle AM'Q \Longrightarrow \angle AM'I = \angle PM'Q = 90^\circ \Longrightarrow M \equiv M'$, i.e. M, D, P are collinear.

$\angle AMN = \angle AIN = 180^\circ - \angle AID = \frac{1}{2}|\angle B - \angle C| = \angle ABQ = \angle AMQ \Longrightarrow M, N, Q$ are collinear. Since $\overline{DIN}$ and $\overline{POQ}$ are parallel (both perpendicular to BC) and MO is M-median of $\triangle MPQ$, then MO is also the M-median of $\triangle MDN$, i.e. MO bisects $\overline{DN}$.

Quick Reply

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Similar Isosceles Triangles Area 3

Reply

Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

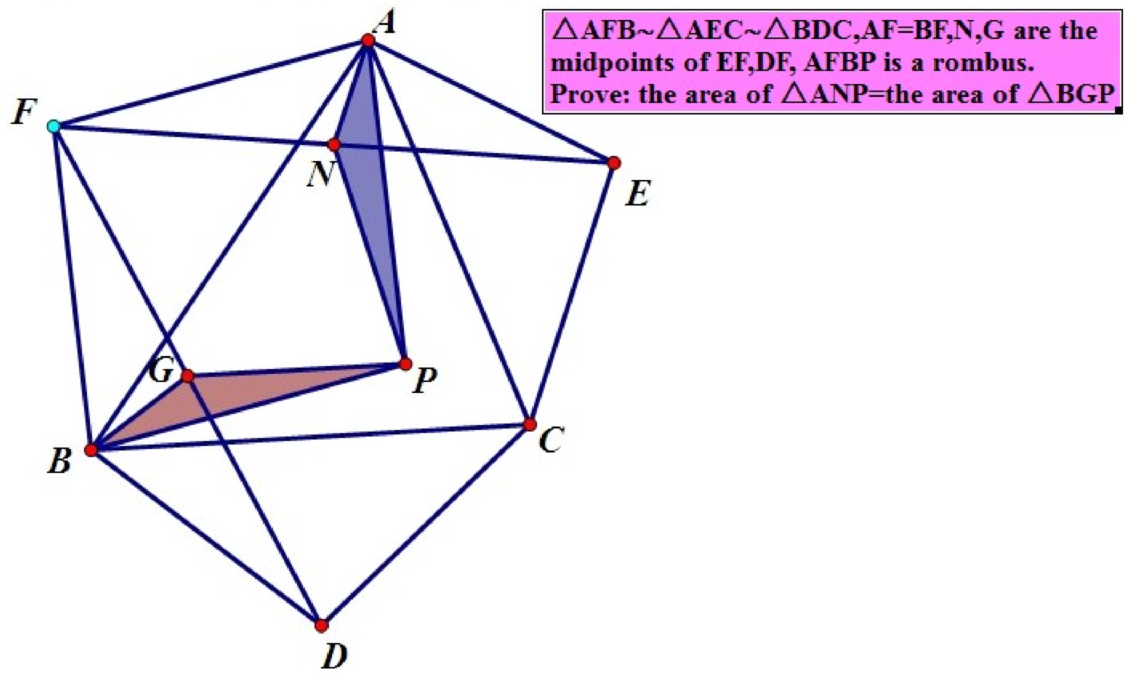
XmL

#1 Jan 28, 2013, 9:45 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:

已知（文武光华数学工作室 南京 潘成华）△AFB~△AEC
~△BDC,AF=BF,点,N,G分别是EF,DF中点,四边
形AFBP是菱形 (2013 1 17 20:50)
求证 △ANP面积=△BGP面积



Luis González

#2 Jan 28, 2013, 11:53 am

We already know that $CDPE$ is a parallelogram (see the solution of the problem Similar Isosceles Triangles Median Concurrence), i.e. $PE = DC = DB$ and $PD = EC = EA$. Since $PB = PA$, then it follows that $\triangle PEA \cong \triangle BDP \Longrightarrow [PEA] = [BDP]$. Simple area chase gives $2[ANP] = [AFP] - [PEA]$ and $2[BGP] = [BFP] - [BDP]$, but obviously $[AFP] = [BFP]$, thus $[ANP] = [BGP]$, as desired.

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Prove that it touches the circumcircle

Reply

chxris

#1 Jan 27, 2013, 1:25 am

In ΔABC let D, E, F be the midpoint of the sides BC, CA, AB respectively and let G, H, I be the points where the incircle of ΔABC touches the sides BC, CA, AB respectively. Now consider circle Γ_1 with center D and passes through G, Γ_2 with center E and passes through H and Γ_3 with center F and passes through I. Prove that if Γ_1 touches the circumcircle of ΔABC at a point on the arc BC that doesn't contain A, then Γ_2 or Γ_3 will touch the circumcircle too.

Luis González

#2 Jan 27, 2013, 5:13 am • 1

Assume that Γ_1 touches the circumcircle (O) of $\triangle ABC$ at M, which is nothing but the midpoint of the arc BC. Thus

$$DG = \tfrac{1}{2}|b-c| = MD = \tfrac{1}{2}a \cdot \tan\frac{A}{2} \Longrightarrow$$

$$\tan^2\frac{A}{2} = \frac{(b-c)^2}{a^2} \Longrightarrow \frac{(a+c-b)(a+b-c)}{(a+b+c)(b+c-a)} = \frac{(b-c)^2}{a^2} \Longrightarrow$$

$$(b-c)^2(a+b+c)(b+c-a) - a^2(a+c-b)(a+b-c) = 0 \Longrightarrow$$

$$(a^2+b^2-c^2)(a^2+c^2-b^2) = 0.$$

Thus, either $a^2+b^2=c^2 \Longrightarrow \triangle ABC$ is right with cathethi $CB, CA \Longrightarrow \Gamma_2$ also touches (O), or $a^2+c^2=b^2 \Longrightarrow \triangle ABC$ is right with catheti $BC, BA \Longrightarrow \Gamma_3$ also touches (O).

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Three collinear points

Reply

Source: (mpdb)

borislav_mirchev

#1 Jan 25, 2013, 5:16 am

Let $ABCD$ is an inscribed quadrilateral in a circle k. P is the intersection point of the diagonals. E is the intersection point of AB and CD. F is the intersection point of AD and BC. k_1 is the circumcircle of the triangle AEF. k_2 is the circumcircle of the triangle CEF. M is the intersection point of k and k_1. N is the intersection point of k and k_2. Prove that M, N and P are collinear.

Luis González

#2 Jan 27, 2013, 2:28 am • 1

Let $G \equiv EF \cap BD.$ Since N is the Miquel point of the complete quadrangle $EFBD,$ it follows that N, D, E, G are concyclic $\Longrightarrow \angle END = \angle EGD \Longrightarrow \angle ENA = \angle END + \angle AND = \angle EGD + \angle EBD = \angle FEB \Longrightarrow EF$ is tangent to $\odot(ANE).$ Analogously, $\odot(ANF)$ is tangent to $EF \Longrightarrow NA$ is radical axis of $\odot(ANE)$ and $\odot(ANF),$ meeting its common tangent $\overline{EF}$ at its midpoint $K.$ By similar reasoning, CM passes through $K.$

Let NP cut k again at M' and CM' cuts AN at $K'.$ From the cyclic complete quadrilateral $ANCM',$ it follows that K' is on the polar EF of P WRT $k \Longrightarrow K \equiv K'$ and $M \equiv M' \Longrightarrow M, N, P$ are collinear.

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Bisecting the diagonal and equal segments

Reply

Source: (mpdb)

borislav_mirchev

#1 Jan 25, 2013, 2:03 am

Let $ABCD$ is an inscribed quadrilateral. E is the intersection point of AB and CD. F is the intersection point of AD and BC. E' and F' are the feets of the perpendiculars from E and F to AD and AB, respectively. E'' and F'' are the feets of the perpendiculars from E and F to CB and CD, respectively. Prove that $E'F'$ and $E''F''$ are passing through the middle of the diagonal BD and $E'F' = E''F''$.

Luis González

#2 Jan 26, 2013, 9:37 am • 1

Obviously E', F', E'', F'' lie on the circle ω with diameter $\overline{EF}$. $\triangle FAB$ and $\triangle FCD$ are similar with corresponding altitudes FF' and $FF'' \Longrightarrow \angle BFF' = \angle DFF'' \Longrightarrow$ arcs $F'E''$ and $E'F''$ of ω are congruent $\Longrightarrow$ cyclic $E'E''F'F''$ is an isosceles trapezoid with $E'E'' \parallel F'F'' \Longrightarrow E'F' = E''F''$. Let $E'F'$ cut BD at P, thus by Menelaus' theorem for $\triangle DAB$ cut by $\overline{E'PF'}$, we get

$$\frac{PD}{BP} = \frac{E'D}{E'A} \cdot \frac{AF'}{F'B} = \frac{\cot \widehat{EDE'}}{\cot \widehat{EAE'}} \cdot \frac{\cot \widehat{FAF'}}{\cot \widehat{FBF'}}.$$

Since $\widehat{EAE'} = \widehat{FAF'}$ and $\widehat{EDE'} = \widehat{FBF'}$, then $PD = BP$, i.e. $E'F'$ passes through the midpoint P of $\overline{BD}$. Likewise, $P \in E''F''$.

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Cyclic quadrilaterals

Locked

Source: BMO tst 2009-italy

bozzio

#1 Jan 10, 2013, 12:58 am

ABCD is a convex quadrilateral. P and Q are two points in ABCD such that ADQP and BCQP are cyclic. E is a point on the segment PQ such that:
$\angle PBE = \angle QCE$
$\angle PAE = \angle QDE$
Show that ABCD is cyclic.

Luis González

#2 Jan 10, 2013, 1:42 am • 1

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http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=287865
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=356615

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High School Olympiads

Locus of second intersection of circumcircles

Reply

xeroxia
#1 Jan 9, 2013, 9:51 pm

A circle with center O, and two points A and B are given.
Let C be an arbitrary point on the circle.
Let D be the intersection of AB and OC.
Show that the locus of the second intersection of (ADC) and (ODB) is a line when $|OC| = |AB|$, and a circle, otherwise.

Luis González
#2 Jan 9, 2013, 11:19 pm

Inversion in the circle (O) carries A, B, C into themselves and the line AB is taken into the circle $\odot(OAB)$. OD cuts $\odot(OAB)$ again at the inverse D' of D, hence BD' and $\odot(CAD')$ are the inverses of $\odot(OBD)$ and $\odot(CAD)$, meeting again at the inverse P' of P. $\angle ABD' = \angle AOD' = 2\angle ACD' = 2\angle AP'D' \Longrightarrow \triangle ABP'$ is isosceles with legs $BA = BP' \Longrightarrow P'$ runs on the circle $(B, BA) \Longrightarrow P$ runs on the inverse circle ω of (B, BA). Now, ω is a line $\Longleftrightarrow$ $O \in (B, BA) \Longleftrightarrow AB = OB = OC$.

xeroxia
#3 Jan 11, 2013, 2:33 pm

Let the second intersection be P.
Since P, O, D, B are on same circle, $\angle PBA = \angle POC$.
Since P, A, C, D are on same circle, $\angle PAD = \angle PCO$.
So $\triangle PAB \sim \triangle PCO$ by $A.A$.

Then we have $\dfrac{BA}{OC} = \dfrac{PB}{PO}$. So $\dfrac{PB}{PO}$ is constant because AB and OC are constant.

Since, O, B, $\dfrac{PB}{PO}$ are constant, then the locus of P is the Apollonius circle of B and O with the ratio $k = \dfrac{PB}{PO}$.
Since a line is a circle with infinite radius, if $\dfrac{BA}{OC} = \dfrac{PB}{PO} = 1$, the Apollonius circle will be a line.
It is in fact, if $BA = OC$, then $PB = OP$, then P is on the perpendicular bisector of $[OB]$.

Quick Reply

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Bevan point

Source: Equal angles

juancarlos

#1 Nov 30, 2006, 6:24 pm

From the excenter I_a drawn the tangent I_aT to the circumcircle (O) of ABC. If I_aP cut (O) at S, Q where P is the Bevan point of ABC and K is the touch point of the excircle (I_a) with BC, prove that $< KTS =< PTQ$

ciprian

#2 Dec 2, 2006, 10:21 pm

Maybe I can't make the translation right but what do you mean by' the Beaven point of the triangle ABC'?

perfect_radio

#3 Dec 3, 2006, 3:38 am

http://mathworld.wolfram.com/BevanPoint.html

yetti

#4 Dec 18, 2006, 9:05 am

Let $R,\ r,\ r_a,\ s,\ \triangle$ be the circumradius, inradius, A-exradius, and semiperimeter, and area of the triangle $\triangle ABC.$ The Bevan point P of this triangle is the circumcenter of its excentral triangle $\triangle I_aI_bI_c,$ (O) is the 9-point circle of the excentral triangle, hence $I_aP = 2R.$ Since $I_aK \perp BC,\ \ I_aA \perp I_bI_c$ are isogonals of $I_aI_b,\ \ I_aI_c$ in the excentral triangle and I_aA its altitude, $P \in I_aK.$ In order to show that TP, TK are isogonals of TQ, TS in the triangle $\triangle TQS,$ we have to show that

$$(?)\quad \frac{KQ}{KS} \cdot \frac{PQ}{PS} = \frac{TQ^2}{TS^2}$$

Since I_aT is a tangent from I_a to (O), a circle centered at I_a and with radius I_aT is perpendicular to (O), hence it is the T-Apollonius circle of the triangle $\triangle TQS$ and consequently, $\frac{TQ^2}{TS^2} = \frac{I_aQ}{I_aS}.$ Thus we have to show that

$$(?)\quad \frac{KQ}{KS} \cdot \frac{PQ}{PS} = \frac{I_aQ}{I_aS}$$

Assume the points I_a, S, K, Q follow on the line I_aK in this order and $KQ > 0,\ \ KS < 0.$ Then $PQ = I_aQ - I_aP = I_aQ - 2R > 0, PS = I_aS - I_aP = I_aS - 2R < 0,$ which allows to elliminate PQ, PS:

$$(?)\quad \frac{KQ}{KS} \cdot \frac{I_aQ - 2R}{I_aS - 2R} = \frac{I_aQ}{I_aS}$$

Power of I_a to (O) is $I_aS \cdot I_aQ = I_aO^2 - R^2 = 2r_aR,$ which allows to elliminate I_aS :

$$(?)\quad \frac{KQ}{KS} \cdot \frac{I_aQ^2 - 2R \cdot I_aQ}{2r_aR - 2R \cdot I_aQ} = \frac{I_aQ^2}{2r_aR}$$

Reducing by $\frac{I_aQ}{2R},$

$$(?)\quad \frac{KQ}{KS} \cdot \frac{I_aQ - 2R}{r_a - I_aQ} = \frac{I_aQ}{r_a}$$

Power of K to (O) is $KQ\cdot KS=KB\cdot KC=-(s-b)(s-c)$, which allows to elliminate KS :

$$(?)\quad \frac{KQ^2}{(s-b)(s-c)}\cdot\frac{I_aQ-2R}{I_aQ-r_a}=\frac{I_aQ}{r_a}$$

Finally, $I_aQ=I_aK+KQ=r_a+KQ$, which allows to elliminate I_aQ :

$$(?)\quad \frac{KQ\cdot(KQ+r_a-2R)}{(s-b)(s-c)}=\frac{KQ+r_a}{r_a}$$

Thus we have to show that $y_1=KQ>0,\ y_2=KS<0$ are the positive and negative roots of the quadratic equation

$$y^2-[2R-(r_a-r)]\ y-(s-b)(s-c)=0$$

where we substituted

$$\frac{(s-b)(s-c)}{r_a}=\frac{(s-a)(s-b)(s-c)}{\triangle}=\frac{\triangle}{s}=r$$

Simplifiying,

$$2R-(r_a-r)=2R-a\tan\frac{A}{2}=2R\left(1-\sin A\ \frac{1-\cos A}{\sin A}\right)=2R\cos A$$

$$y^2-2R\cos A\cdot y-(s-b)(s-c)=0$$

But it is clear that $KQ\cdot KS=-(s-b)(s-c)$ from the power of K to (O) and $KQ+KS=2R\cos A$ (where $KQ>0,\ KS<0$) just from reflecting the line BC in the triangle circumcenter O.

This post has been edited 1 time. Last edited by yetti, Dec 19, 2006, 5:25 am

Virgil Nicula

#5 Dec 19, 2006, 2:49 am

Thank you, Yetti, very nice, congratulations !

yetti

#6 Dec 19, 2006, 2:15 pm

Thanks, but I could have made it simpler. Multiplying out the 3rd equation in question,

$$(?)\quad \frac{KQ}{KS}\cdot\frac{I_aQ-2R}{I_aS-2R}=\frac{I_aQ}{I_aS}$$

$$(?)\quad KQ\cdot(I_aQ-2R)\cdot I_aS=KS\cdot(I_aS-2R)\cdot I_aQ$$

Power of I_a to (O) is $I_aS\cdot I_aQ=I_aO^2-R^2=2Rr_a$. Substituting this,

$$(?)\quad KQ\cdot(2Rr_a-2R\cdot I_aS)=KS\cdot(2Rr_a-2R\cdot I_aQ)$$

Reducing by 2R,

$$(?)\quad KQ\cdot(r_a-I_aS)=KS\cdot(r_a-I_aQ)$$

Finally, substituting $I_aQ=I_aK+KQ=r_a+KQ, I_aS=I_aK+KS=r_a+KS$, where $KQ>0,\ KS<0$,

$$(?)\quad KQ\cdot KS=KS\cdot KQ$$

which is an obvious identity. I did not even need the power of K to (O).

Luis González

#7 Jan 9, 2013, 9:13 am • 1

$\triangle ABC$ and its circumcircle (O,R) become orthic triangle and 9-point circle of its excentral triangle $\triangle I_aI_bI_c \Longrightarrow$ circumcenter of $\triangle I_aI_bI_c$ (Bevan point P of ABC) is on perpendicular I_aK from I_a to BC and $PI_a=2R$. Power of I_a WRT

(O,R) is $I_aT^2 = 2R \cdot r_a = I_aK \cdot I_aP \Longrightarrow \odot(TPK)$ is internally tangent to (O) at $T \Longrightarrow T$ is exsimilicenter of $(O) \sim \odot(TPK)$. Thus if TP, TK cut (O) again at P', K', we have $PK \parallel P'K'$, i.e. $SQK'P'$ is isosceles trapezoid with circumcircle $(O) \Longrightarrow$ arcs SP' and QK' are equal $\Longrightarrow \angle STP' = \angle QTK'$, i.e. TP, TK are isogonals WRT $\angle STQ \Longrightarrow \angle KTS = \angle PTQ$.

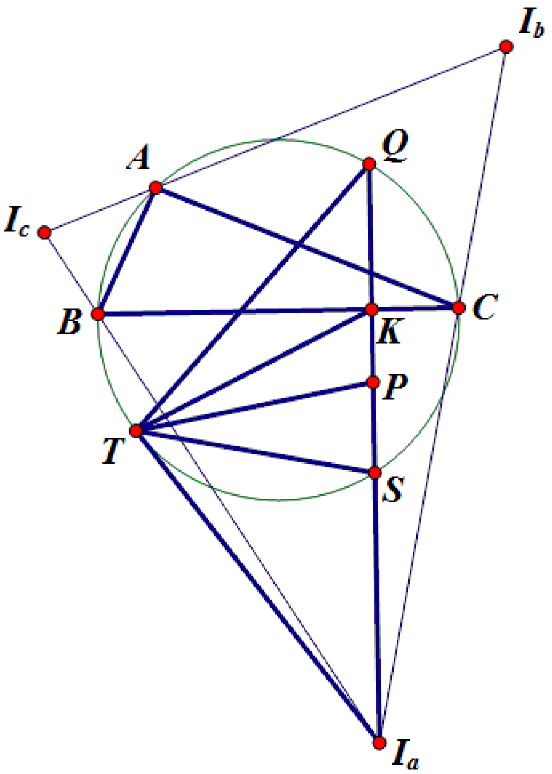
yunxiu

#8 Jan 9, 2013, 6:23 pm

> *Luis González wrote:*
>
> $\triangle ABC$ and its circumcircle (O,R) become orthic triangle and 9-point circle of its excentral triangle $\triangle I_aI_bI_c \Longrightarrow$ circumcenter of $\triangle I_aI_bI_c$ (Bevan point P of ABC) is on perpendicular I_aK from I_a to BC and $PI_a = 2R$. Power of I_a WRT (O,R) is $I_aT^2 = 2R \cdot r_a = I_aK \cdot I_aP \Longrightarrow \odot(TPK)$ is internally tangent to (O) at T.

So $\angle PKT = \angle I_aTP$,$\angle SQT = \angle I_aTS$, hence $\angle QTK = \angle I_aTP - \angle I_aTS = \angle PTS$, and we have $\angle KTS = \angle PTQ$.

Attachments:



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convex pentagon

Reply

Source: own

syncmaster

#1 Jan 9, 2013, 3:26 am

In convex pentagon ABCDE, AB=BC and <BCD=<EAB=90. Inside ABCDE there is a point X, such that AX is perpendicular to BE and CX is perpendicular to BD. Prove that BX is also perpendicular to DE

Luis González

#2 Jan 9, 2013, 5:12 am

Let $P \equiv AD \cap CE.$ The right triangles $\triangle BAP$ and $\triangle BCP$ are obviously symmetric about $PB \Longrightarrow BP \perp AC.$ Thus, perpendiculars from B,D,E to AC,BA,BC concur at $P \Longrightarrow \triangle BDE$ and $\triangle BCA$ are orthologic and P is one orthology center $\Longrightarrow$ perpendiculars from A,B,C to BD,DE,BE concur at the other orthology center $X,$ i.e. BX is perpendicular to $DE.$

JRD

#3 Jan 10, 2013, 8:44 pm

let $AX \cap BE = L$ and $CX \cap BD = T \implies L,T,B,X$ are one a same circle so by angle chasing it's easy to see that it's enough to prove that L,T,D,E are one a same circle and we know $BL.BE = AB^2 = BC^2 = BT.BD$ so done

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an interesting problem

Locked

Source: own

syncmaster
#1 Jan 9, 2013, 3:21 am

Inside an acute triangle ABC there is a point P, such that <PAC=<PBC. Let L and N be foots of perpendiculars from poin P to BC and AC respectively. D is the midpoint of AB. Prove that DL=DN

Luis González
#2 Jan 9, 2013, 4:41 am

Posted at least 4 times before, hence thread locked.

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geometry hard

thanhthuan

#1 Dec 21, 2012, 7:36 am

Given triangel ABC with (O) be the circumcircle. Let P be an arbitary in the plane.AP,BP,CP intersect (O) again at A_1, B_1, C_1 .The circles with deameters AP,BP,CP intersect (O) again at A_2, B_2, C_2 . prove that four line $A_1A_2, B_1B_2, C_1C_2, OP$ are concurrent.

Thank you very much !!!

apgoucher

#2 Dec 28, 2012, 6:56 pm

Invert about P.

A_2A (the inverse of the circle on diameter AP) is perpendicular to line APA_1. Hence, A_2 is diametrically opposite A_1 by the converse of Thales' theorem. Denote the circumcentre of $ABCA_1B_1C_1A_2B_2C_2$ by Q. Let P' be the point on QP such that $QP.QP' = R^2$ and Q lies between P and P', where R is the radius of the circumcircle obtained. By power of a point, A_1PA_2P' are concyclic.

Inverting about P again to restore the original configuration, all of the lines A_1A_2, B_1B_2 and C_1C_2 concur at P', which trivially lies on the line OPQ.

Luis González

#3 Jan 8, 2013, 10:38 pm

Obviously, PA_2, PB_2, PC_2 cut (O) again at the antipodes A_0, B_0, C_0 of A, B, C WRT (O). By Pascal theorem for the cyclic hexagon $AA_0A_2BB_0B_2$, the intersections $O \equiv AA_0 \cap BB_0, P \equiv A_0A_2 \cap B_0B_2$ and $R \equiv A_2B \cap B_2A$ are collinear. By Pascal theorem for the cyclic hexagon $BA_2A_1AB_2B_1$, the intersections $R \equiv BA_2 \cap AB_2$, $Q \equiv A_2A_1 \cap B_2B_1$ and $P \equiv A_1A \cap B_1B$ are collinear. Hence, O, P, Q, R are collinear $\Longrightarrow$ $Q \equiv OP \cap A_1A_2 \cap B_1B_2$. Similarly the line C_1C_2 goes through Q.

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cyclic quadrilateral & perpendicular lines

Locked

dan23

#1 Jan 7, 2013, 10:25 pm

ABCD is cyclic quadrilateral. (AC,BD)=E. Draw two perpendicular lines from E to AB&CD . H & F are the foots respectively. (EH□AB & EF□CD). M is midpoint of AD. Prove that :MF=MH.

Luis González

#2 Jan 7, 2013, 10:41 pm • 1

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Concurrence of 4 Euler lines [Schiffler point]

Reply

Bismarck
#1 Aug 29, 2006, 5:57 am

Let I be incenter of $\triangle ABC$.

a) Prove that Euler lines of triangles $\triangle ABC, \triangle BCI, \triangle CAI, \triangle ABI$ concur.

b) Prove that the isogonal conjugate of this point of concurrency (wrt $\triangle ABC$) is orthocenter of $\triangle DEF$ where D, E, F are points where incircle touches $\triangle ABC$.

cuenca
#2 Aug 31, 2006, 4:09 am

Look here: http://forumgeom.fau.edu/FG2003volume3/FG200312.pdf#search=%22Schiffler%20point%22

livetolove212
#3 Jul 24, 2011, 3:00 pm

For part a.
This proof is in the **Mathematics and Youth** Magazine 11/2007 by **Le Duc Thinh**.
Let O be the circumcircle of triangle ABC, G be the centroid of triangle ABC, M be the midpoint of BC, G' be the centroid of triangle $BIC. AI$ intersects BC at D, intersects (O) at J. (I) touches BC at K, JG' intersects OG at S, intersects AM at E.
It's easy to see that J is the circumcenter of triangle BIC. Then $J'G$ is the Euler line of triangle BIC.
Applying Menelaus's theorem for triangle GOM with line (S, E, J) we have:
$$\frac{SG}{SO}.\frac{JO}{JM}.\frac{EM}{EG} = 1.$$
Then $\frac{SG}{SO} = \frac{JM}{JO}.\frac{EG}{EM} = \frac{JM}{R}.\frac{EG}{EM}$
Applying Menelaus's theorem for triangle IAM with line (J, G', E) we have :
$$\frac{JI}{JA}.\frac{EA}{EM}.\frac{G'M}{G'I} = 1.$$
Since G' is the centroid of triangle BIC we get:
$$\frac{EA}{EM} = 2.\frac{JA}{JI} = 2.\frac{JA}{JB} = 2.\frac{JB}{JD} = 2.\frac{JI}{JD}.$$
So $\frac{EG}{EM} = \frac{GM}{EM} - 1 = \frac{1}{3}.\frac{AM}{EM} - 1$
$$= \frac{1}{3}.(\frac{EA}{EM} + 1) - 1 = \frac{1}{3}.\frac{EA}{EM} - \frac{2}{3}$$
$$= \frac{2}{3}.\frac{JI}{JD} - \frac{2}{3} = \frac{2}{3}.\frac{ID}{JD} = \frac{2}{3}.\frac{IK}{JM} = \frac{2}{3}.\frac{r}{JM}$$
Therefore $\frac{SG}{SO} = \frac{JM}{r}.\frac{EG}{EM} = \frac{JM}{R}.\frac{2}{3}.\frac{r}{JM} = \frac{2}{3}.\frac{r}{R}$
Similarly, the Euler line of CIA, AIB intersect OG at S. We are done.

Another synthetic proof. See here:
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=278562
In this proof, you note that BO_b, CO_c, IO concur at I then 3 Euler lines of triangles AIB, AIC and ABC are concurrent.

varun.tinkle
#4 Jul 28, 2011, 11:41 am • 1

A beautiful solution in fact but my question is who the hell could have thought of such an elmentary solution man

A beautiful solution in fact but my question is who the hell could have thought of such an elmentary solution man......

jayme

#5 Jul 28, 2011, 4:04 pm

Dear Mathlinkers,
I think that a more simple proof is possible. But all the proof are cery interesting because it shows some particular aspect of the general problem.
Sincerely
Jean-Louis

jayme

#6 Aug 1, 2011, 9:32 pm

Dear Mathlinkers,
I have found a nice proof of a) (next on my web site)
but can some give a hint of b) or a reference?
Sincerely
Jean-Louis

RSM

#7 Aug 3, 2011, 12:10 am

Here is a property of Schiffler point which is just a different form of the previous property I posted.
Schiffler point of a triangle ABC is the Kosnita point of $A'B'C'$, where A', B', C' are the midpoints of the arcs BC, CA, AB on $\odot ABC$.
My proof for Schiffler point reveals this directly.

Luis González

#8 Aug 3, 2011, 1:12 am • 1

> *RSM wrote:*
> Schiffler point of a triangle ABC is the Kosnita point of $A'B'C'$, where A', B', C' are the midpoints of the arcs BC, CA, AB on $\odot ABC$.

Let I and I_a, I_b, I_c be the incenter and three excenters of $\triangle ABC$ against A, B, C. Clearly, the reflection of I about the midpoint M_a of BC is the orthocenter H_1 of $\triangle I_aBC$. Thus, the reflection of the centroid of $\triangle IBC$ about M_a is the centroid G_1 of $\triangle BCH_1 \Longrightarrow I_aG_1$ the image of the Euler line of $\triangle IBC$ under the homothety $(I, 2)$.

Let G_a be the centroid of $\triangle II_bI_c$. Then $\triangle I_aI_bI_c \cup G_a \sim \triangle I_aBC \cup G_1 \Longrightarrow \angle I_bI_aG_a = \angle BI_aG_1 \Longrightarrow I_aG_a, I_aG_1$ are isogonal lines WRT $\angle I_bI_aI_c$. I_aG_a passes through the complement of the complement of I WRT $\triangle I_aI_bI_c$, which is the 9-point center O of $\triangle I_aI_bI_c$. Hence, I_aG_1 passes through the isogonal conjugate of O WRT $\triangle I_aI_bI_c$, i.e. the Kosnita point K^* of $\triangle I_aI_bI_c \Longrightarrow$ Euler line of $\triangle IBC$ passes through the midpoint of IK^*, i.e. the Kosnita point K' of $\triangle A'B'C'$. Likewise, Euler lines of $\triangle ICA, \triangle IAB$ pass through K'.

jayme

#9 Aug 4, 2011, 3:23 pm

Dear Mathlinkers,
an article concerning the “Schiffler point and a result by L. and T. Emelyanov” has been put on my website
with a new approach.
http://perso.orange.fr/jl.ayme vol. 9
You can use Google translator
Sincerely
Jean-Louis

Luis González

#10 Jan 7, 2013, 9:43 pm • 2

> *Bismarck wrote:*
> b) Prove that the isogonal conjugate of this point of concurrency (wrt $\triangle ABC$) is orthocenter of $\triangle DEF$ where D, E, F

b) Prove that the isogonal conjugate of this point of concurrency (wrt $\triangle ABC$) is orthocenter of $\triangle DEF$ where D, E, F are points where incircle touches $\triangle ABC$.

Rename X_0, Y_0, Z_0 the tangency points of the incircle (I, r) with BC, AC, AB. A-excircle (I_a, r_a) touches BC, CA, AB at X_1, Y_1, Z_1. G is the centroid and (O, R) is the circumcircle. M and D are the midpoints of $\overline{BC}$ and the arc BC of (O). OI_a cuts BC at A_1 and AA_1 cuts OG, OD at S, U. By Menelaus' theorem for $\triangle ODI_a$ cut by $\overline{AUA_1}$, we get

$$\frac{UD}{UO} = \frac{AD}{AI_a} \cdot \frac{I_aA_1}{A_1O} = \frac{AD}{AI_a} \cdot \frac{r_a}{OM} \Longrightarrow$$

$$\frac{UM}{OU} = \frac{1}{R}\left[\left(\frac{UD}{OU} + 1\right) OM - R\right] = \frac{1}{R}\left[\left(\frac{AD}{AI_a} \cdot \frac{r_a}{OM} + 1\right) OM - R\right]$$

Let P be the projection of D on AB. Then $\dfrac{AD}{AI_a} = \dfrac{DP}{r_a} \Longrightarrow$

$$\frac{UM}{UO} = \frac{1}{R}\left[\left(\frac{DP}{OM} + 1\right) OM - R\right] = \frac{DP + OM - R}{R} = \frac{DP - DM}{R}.$$

But $DP = \frac{1}{2}(r + r_a)$ and $DM = \frac{1}{2}(r_a - r) \Longrightarrow DP - DM = r.$

$\Longrightarrow \frac{UM}{UO} = \frac{r}{R}$. Hence, by Menelaus' theorem for $\triangle OGM$ cut by $\overline{ASU}$, we obtain

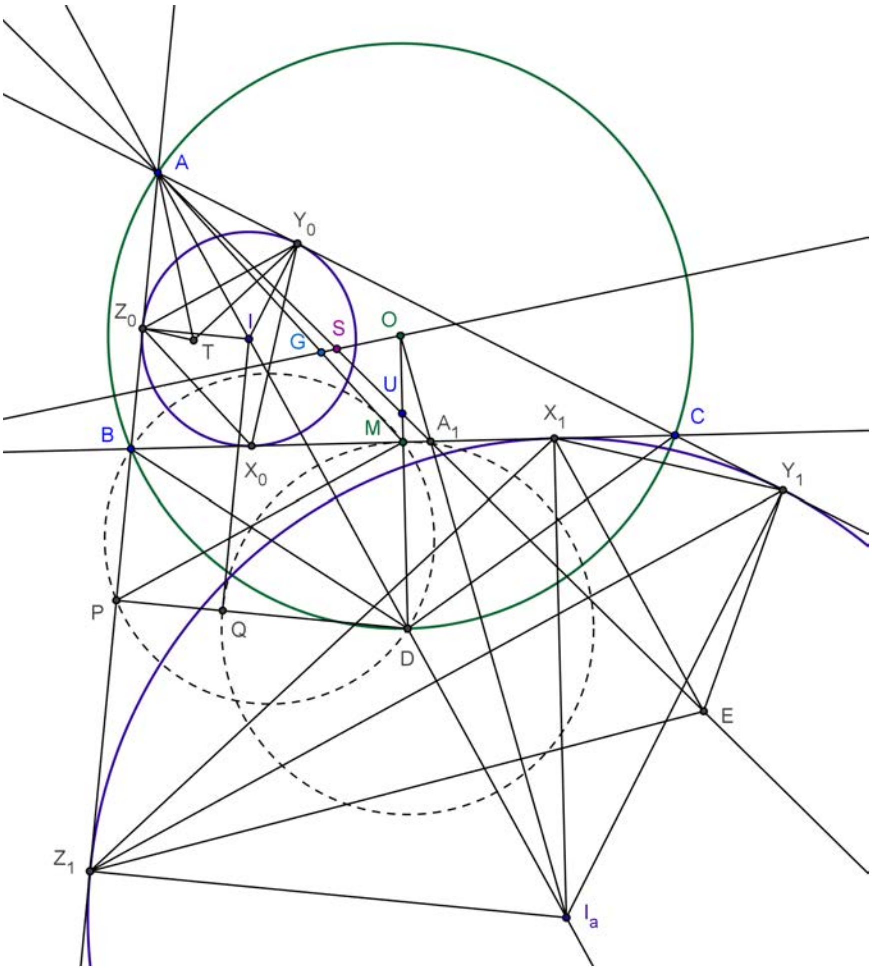
$$\frac{SG}{SO} = \frac{AG}{AM} \cdot \frac{UM}{OU} = \frac{2}{3} \cdot \frac{r}{R} \Longrightarrow S \text{ is Schiffler point of } \triangle ABC.$$

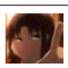
Let E be the reflection of X_1 about Y_1Z_1. Using the result of the problem V midpoint for I_a and its pedal triangle $\triangle X_1Y_1Z_1$, we get that ASA_1 goes through E. If T is the orthocenter of $\triangle X_0Y_0Z_0$, we have then

$$\angle TY_0Z_0 = \angle IY_0X_0 = \angle ICB = \angle CX_1Y_1 = \angle X_1Z_1Y_1 = \angle EZ_1Y_1.$$

Likewise, $\angle TZ_0Y_0 = \angle EY_1Z_1$. Thus, $AZ_0TY_0 \sim AY_1EZ_1 \Longrightarrow \angle Z_0AT = \angle Y_1AE$, i.e. AT and $AE \equiv AS$ are isogonals WRT $\angle BAC$. Similarly, BT, BS are isogonals WRT $\angle CBA \Longrightarrow S, T$ are isogonal conjugates WRT $\triangle ABC$.

Attachments:





TelvCohl
#11 Nov 27, 2014, 9:00 pm

> *Bismarck wrote:*
> b) Prove that the isogonal conjugate of this point of concurrency (wrt $\triangle ABC$) is orthocenter of $\triangle DEF$ where D, E, F are points where incircle touches $\triangle ABC$.

My solution **:**

Let H be the orthocenter of $\triangle DEF$.
Let I_a, I_b, I_c be three excenters of $\triangle ABC$.
Let O, G, be the circumcenter, centroid of $\triangle ABC$, respectively .
Let X, Y, Z be the tangent point of (I_a) with BC, CA, AB, respectively .
Let X' be the reflection of X in YZ and M be the midpoint of BC .
Let $V = ID \cap EF, V' = I_aX \cap YZ, P = XX' \cap YZ, N = DH \cap EF$.
Let K, T, S be the intersection of AX' and I_aX, OM, OG, respectively .

It's well known that A, V, V', M are collinear ,
so $\triangle DNV$ and $\triangle XPV'$ are homothetic with center M ,
hence combine with $DN \parallel XP$ we get P, M, N are collinear and $XP = DN$.

Easy to see $\triangle DEF \cap N \sim \triangle I_aI_bI_c \cap A$ . ... $(\star)$

From here we get $Q = AX' \cap BC \in OI_a$,
so from $(\star)$ we get $\frac{OT}{TM} = \frac{I_aK}{KX} = \frac{I_aA}{2DN} = \frac{R}{r}$,
hence from Menelaus theorem (for $\triangle OGM$ and $\overline{AST}$) we get $\frac{GS}{SO} = \frac{2r}{3R}$.
i.e. S is the Schiffler point of $\triangle ABC$

Since $AEHF \sim AZX'Y$,
so we get $\{AS, AH\}$ are isogonal conjugate of $\angle BAC$.
Similarly, $\{BS, BH\}, \{CS, CH\}$ are isogonal conjugate of $\angle CBA, \angle ACB$, respectively ,
so we conclude that the orthocenter H of $\triangle DEF$ is the isogonal conjugate of S WRT $\triangle ABC$.

Q.E.D

Attachments:

This post has been edited 1 time. Last edited by TelvCohl, Mar 26, 2016, 9:42 pm

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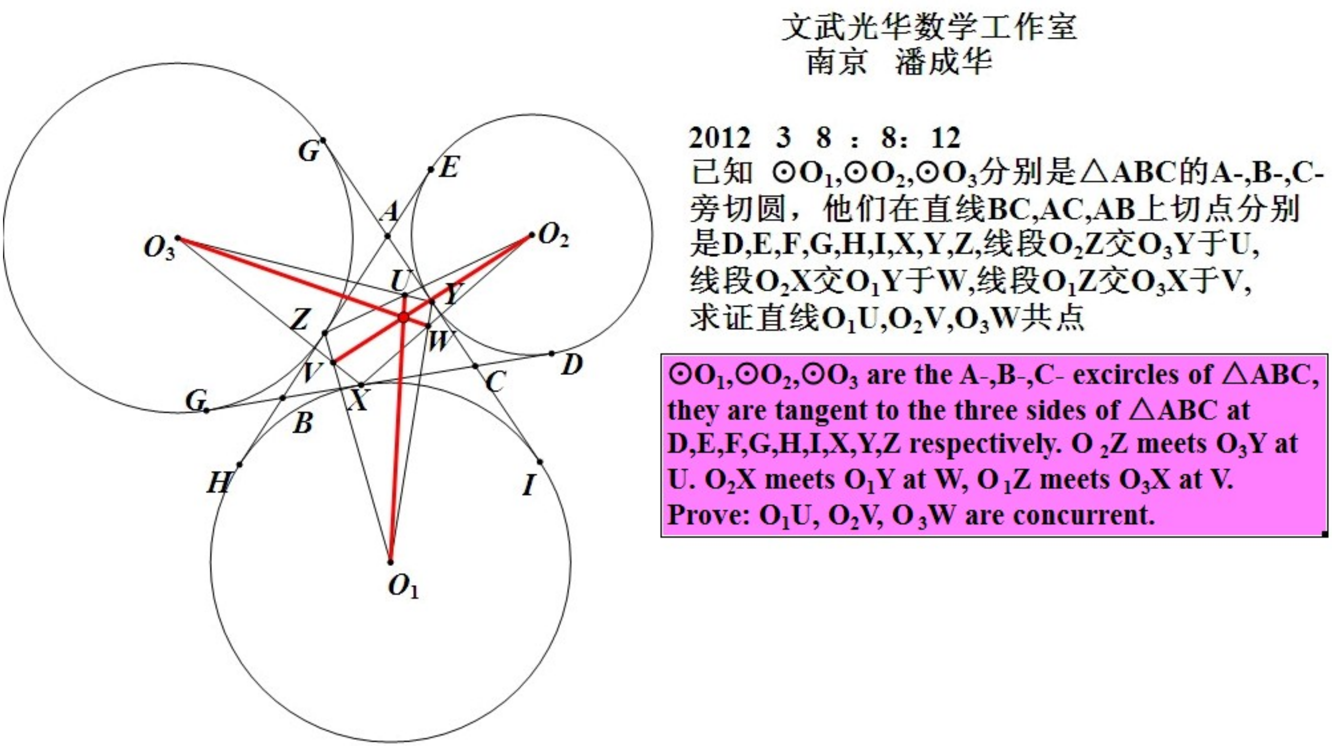
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Jan 7, 2013, 7:28 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Jan 7, 2013, 11:14 am

The problem is just a particular case of the following projective configuration:

P and Q are two points on the plane of $\triangle ABC$. PA,PB,PC cut BC,CA,AB at P_1,P_2,P_3 and QA,QB,QC cut P_2P_3,P_3P_1,P_1P_2 at M_1,M_2,M_3, respectively. $A_1 \equiv BM_3 \cap CM_2, B_1 \equiv CM_1 \cap AM_3$ and $C_1 \equiv AM_2 \cap BM_1$. Then AA_1,BB_1 and CC_1 concur.

P.S. The proof is rather easy using Ceva and Menelaus theorem.

hofamo

#3 Jan 7, 2013, 1:21 pm

use ceva Theorem in sin mode in triangle O1O2O3 and points U,W,V.(3 times)((O1O2,AB)=X`,(O1O3,AC)=Y`,(O2O3,BC)=Z`))
(use sin law in ceva)
now use ceva in O1O2O3 and lines O1U,O2V,O3W.

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High School Olympiads

Parallel lines in the external circles

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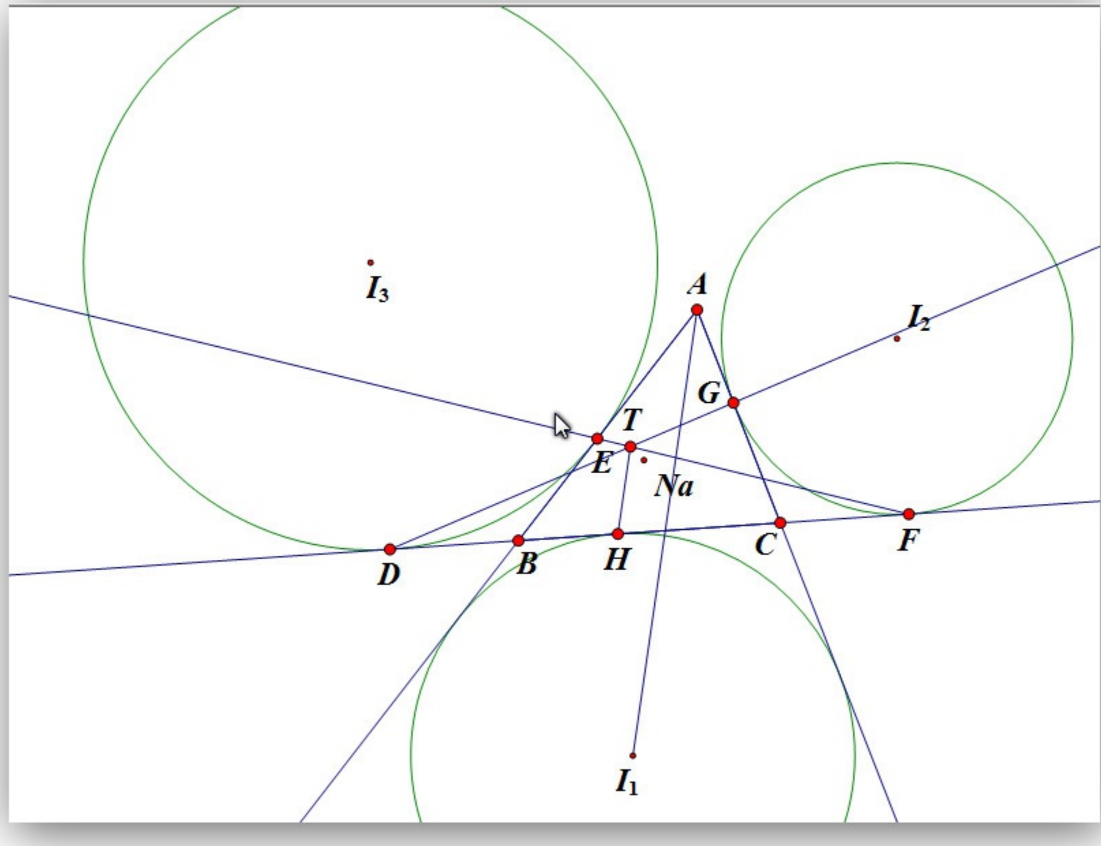
DANNY123

#1 Jan 5, 2013, 11:04 am

$(I_1), (I_2), (I_3)$ are excircles of triangle ABC. (I_1) touches the side BC at H. (I_3) touches BC and AB at D, E. (I_2) touches BC and AC at F, G. Let EF meets DG at T. Prove that TH parallels to AI_1.

Attachments:



Luis González

#2 Jan 6, 2013, 1:08 am • 1

$V \equiv AI_1 \cap BC$ is the foot of the internal bisector of $\angle BAC$. (I_2) and (I_3) touch AB, AC at M, N and MG, NE cut BC at K, J. DG and FE cut the parallel from H to $AI_1 \parallel MG \parallel NE$ at Y, Z, respectively. We prove that $Y \equiv Z$.

By Menelaus' theorem for $\triangle ABC$ cut by $\overline{MGK}$, we get

$$\frac{BK}{KC} = \frac{MB}{MA} \cdot \frac{GA}{CG} = \frac{MB}{CG} = \frac{p}{p-a}.$$

Similarly, $\dfrac{CJ}{JB} = \dfrac{p}{p-a} \Longrightarrow J, K$ are symmetric about the midpoint of BC $(\star)$.

On the other hand, from $\triangle DHY \sim \triangle DKG$ and $\triangle CGK \sim \triangle CAV$, we get

$$\frac{HY}{GK} = \frac{DH}{DK} = \frac{b}{DK}, \ \frac{GK}{AV} = \frac{CG}{AC} = \frac{p-a}{b} \Longrightarrow HY = \frac{(p-a) \cdot AV}{DK}.$$

By similar reasoning, $HZ = \dfrac{(p-a) \cdot AV}{FJ}$. Since $DK = FJ$, due to $(\star)$, then $HY = HZ \Longrightarrow Y \equiv Z$, as desired.

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