High School Olympiads

unsolved olympiad geometry extra hard problem plz solve.....

Locked

sarthak7
#1 Apr 12, 2015, 10:37 pm

Let ABC be a triangle. Angle B=120 and the angle bisectors of A, B, and C meet at O. The angle bisectors are AD, BE and CF. Prove that angle DEF=90.

Luis González
#3 Apr 12, 2015, 11:22 pm

Please use the search before posting.This is a rather old problem

http://www.artofproblemsolving.com/community/c6h1336
http://www.artofproblemsolving.com/community/c6h114998
http://www.artofproblemsolving.com/community/c6h623032

High School Olympiads

Tough problem - Angle bisectors (< RPQ = 90°)

Reply

Arne
#1 Oct 22, 2003, 8:35 pm

Let ABC be a triangle. Let P, Q, R be the points of intersection of the angle bisectors of its angles $\angle A$, $\angle B$, $\angle C$ with the sides $[BC]$, $[CA]$, $[AB]$ respectively. Prove that $\measuredangle RPQ = 90^\circ$ if and only if $\measuredangle CAB = 120^\circ$.

Valentin Vornicu
#2 Oct 24, 2003, 11:23 pm

i have not completed the calculations, but a simple algebraic aproach seems to kill the problem.

we express everything in terms of a,b,c the side lengths of the triangle ABC.

for easier computations we can denote x=BP, y=CQ, z=AR.

then we need to prove that

a² = b²+c² + bc \sqrt 3 <=>
RQ² = RP² + PQ ² .

But from Cosine law in triangles ARQ, BRP and CPQ we obtain the expressions of RQ², RP² and PQ² as a function of x,y,z,a,b,c.

We also know that x= ac / (b+c) , y = ba / (c+a) and z= cb / (a+b) and we have obtained a relation only in a,b,c (no \sqrt -s involved) and we just group, cancel, and factorize the terms to obtain the LHS.

grobber
#3 Oct 25, 2003, 12:46 am

I've tried desperately to find a geometric approach, but I couldn't come up with anything really different from Valentin's approach. The calculations look too ugly to me, though. I don't think I've got the patience to finish them...

sprmnt21
#4 Oct 30, 2003, 10:26 pm

Let's considere the equilateral triangle BCD (D is opposite to A whit respect to BC).
Given that BACD is cyclic, is easy to see that D is on the line AP. It results that BPD and APC are similar. Then BP/BD=AP/AC from which we can write BP/AP=BD/AC=BC/AC=BR/AR then PR is the bisector of angle(BPA).

the rest follows ...

Arne
#5 Oct 31, 2003, 1:50 am

Wow, what a beautiful solution !

Arne
#6 Nov 1, 2003, 1:07 am

Well ...

This obviously proves <A = 120° => <RPQ = 90°.

Probably I'm missing something but how do you prove <RPQ = 90° => <A = 120° ?

Arne

#7 Nov 2, 2003, 3:33 pm

So the wonderful geometric proof above works only for

<CAB = 120° => <RPQ = 90°

and not for

<RPQ = 90° => <CAB = 120° ?

Is there nothing similar that can be done for

<RPQ = 90° => <CAB = 120° ?

bugzpodder

#8 Nov 2, 2003, 9:38 pm

extend PR and PQ to meet a line parallel to BC and passes thru A at R' and P' respectively

R'AR and BPR, AP'Q and PQC are similar

so we have:
R'A=BP*AR/RB=BP*AC/BC
AP'=PC*AQ/QC=PC*AB/BC

notice that AB/BP=AC/PC
BP=AB*PC/AC
sub that into R'A we get
R'A=AB*PC/BC which is equal to AP'
it follows that A is the midpoint of R'P' since R'PP' is right triangle A is the center of the circumcircle R'P'P and it follows that AP=AR'=PP' and APR'=AR'P=R'PB, AP'P=APP'=P'PC
hence APB is bisected by RP and APC is bisected by QP
continue as sprmnt21

Arne

#9 Nov 3, 2003, 1:57 am

Thanks.

kueh

#10 Feb 2, 2005, 3:16 am

Trigo shows it one way nicely, with an interesting lemma that the intersection of CR and PQ is the incentre of APC, and the intersection of BQ and PR is the incentre of APB.

This is because: $\frac{AQ}{CQ} = \frac{AB}{BC} = \frac{AP}{PC}$. Thus by angle bisector thm, it is the incentre. (It shows that it is 90 only one way though... because it proves a stronger theorem.)

juancarlos

#11 May 17, 2005, 8:55 pm

We have R, Q are excenters of triangles APC, ABP so PR, PQ are bisectors of $\angle BPA$ and $\angle APC$, hence $\angle RPQ = 90^0$.

kueh

#12 May 17, 2005, 10:30 pm

-shrugs- that was exactly the same as my solution!

ddziabenko

#13 May 24, 2005, 4:51 am

Let ABC be a triangle. Let P, Q, R be the intersections of the angle bisectors of <A, <B, <C with [BC], [CA], [AB] respectively. Prove that <RPQ = 90° if and only if <CAB = 120°.

My solution is as follows:
Drop perpendiculars QD, QE and QF on AB (possibly extended), AP and BC respectively as well as perpendiculars RD_1, RE_1 and RF_1 on AC, AP and BC respectively. Note that $RD_1 = RF_1$ and $QD = QF$. We also have $\angle RAD_1 = \angle QAD$ and $\angle BAP = \angle CAP$, so that triangles $ADQ \sim ARD_1$ and $AQE \sim ARE_1$ from which, since triangles ADQ and AQE share the side AQ and triangles ARD_1 and ARE_1 share the side AR, we get $\frac{DQ}{QE} = \frac{RD_1}{RE_1}$. Hence we have $\frac{QF}{QE} = \frac{RF_1}{RE_1}$, if this ratio exceeds 1 then (from the triangles QPF, QPE and PRE_1, PRF_1) we would get $\angle RPF_1 > \angle RPE_1$ and $\angle QPF > \angle QPE$ so that $\frac{\angle PC}{2} = \frac{\pi}{2} > \angle QPE + \angle RPE_1 = \angle RPQ$. If the ratio is less than 1, from the same triangles we would get $\angle RPQ = \angle QPE + \angle RPE_1 > \frac{\angle BPC}{2} = \frac{\pi}{2}$. Hence $\angle RPQ = \frac{\pi}{2}$ only if $QF = QD = QE$ which can happen only if $\angle BAC$ is obtuse and AQ bisects $\angle PAD$, so that $\angle BAP = \angle PAC = \angle CAD$. Since those angles add up to $180°$ each of them is $60°$, hence $\angle CAB = 120°$

Quick Reply

High School Olympiads

foot of angle bisector

Reply

Source: foot of angle bisector and angle 120°

Bob75
#1 Oct 18, 2006, 3:50 pm

A, B and C are the feet of the interior bisectors of a triangle EFG.
Show that the triangle ABC is a right triangle iff one of the angles of EFG is 120°.

cuenca
#2 Oct 19, 2006, 7:22 am

$Lemma$: For a triangle ABC ($BC > AC$) let the interior angle bisectors AD and CF and the exterior angle bisector BE, then D, E and F are collinear. $Prove$ By Menelaus, occur iff and only if $\frac{BD.CE.AF}{DC.EA.FB} = 1$, by angle bisector theorem we have $\frac{AB.BC.AC}{AC.AB.BC} = 1$, which is true.
Suppose <GEF=120, the angle bisectors are EA, FB and GC, and I, the incenter. In the triangle EAF, EG is an exterior angle bisector and FI is an interior angle bisector, so by the lemma AC is an interior angle bisector,so <EAC=<CAF, similarly, we have: <EAB=<BAG, and we are done.

OHO
#3 Oct 19, 2006, 3:17 pm

What's about the other side?

yetti
#4 Oct 20, 2006, 9:41 am

Let I be the incenter of arbitrary $\triangle EFG$ and $I_E,\ I_F,\ I_G$ excenters opposite to E, F, G. The circumcircle (O) of $\triangle EFG$ is the common 9-point circle of $\triangle II_EI_F,\ \triangle II_EI_G$ with orthocenters $I_G,\ I_F,$ hence their circumcircles (P), (Q) are congruent. $PI_G,\ QI_F$ are the Euler lines of these 2 triangles cutting each other at their common midpoint O, hence PQI_GI_F is a parallelogram. Let $A \in FG,\ B \in GE,\ C \in EF$ be feet of internal angle bisectors of $\triangle EFG.$ Quadrilaterals $IFI_EG,\ IEI_FG,\ IEI_GF$ with diagonal intersections A, B, C are all cyclic, having right angles at the common vertices E, F, G, hence $AF \cdot AG = AI \cdot AI_E, BG \cdot BE = BI \cdot BI_F, CF \cdot CE = CI \cdot CI_G.$ This means AB is radical axis of (O), (P) and AC is radical axis of (O), (Q), so that $AB \perp OP,\ AC \perp OQ.$ The center line PQ of the congruent circles (P), (Q) is the perpendicular bisector of their common chord $II_E,$ cutting it at its midpoint M. (O) also cuts II_E at its midpoint M, being the 9-point circle of $\triangle I_EI_FI_G$ with ortocenter I. Since $II_E \equiv EA$ bisects $\angle E,$ M is the midpoint of the arc FG of (O) opposite to E. From the circumcircle (P) of $\triangle II_EI_F$ and from the cyclic quadrilateral $IEI_FG,$

$$\angle IPQ \equiv \angle IPM = \frac{\angle IPI_E}{2} = \angle II_FI_E \equiv \angle II_FG = \angle IEG = \frac{\angle E}{2}$$

and similarly $\angle IQP = \frac{\angle E}{2}.$ Since $\angle MFG = \angle MGF = \frac{\angle E}{2}$ as well, the isosceles $\triangle FMG \sim \triangle PIQ$ are similar, having equal base angles.

$\angle GEF = 120^\circ \iff \angle FMG = 60^\circ \iff \triangle FMG$ is equilateral $\iff \triangle PIQ$ is equilateral $\iff PQ = PI = PI_F \iff PQI_GI_F$ is a rhombus $\iff OP \perp OQ \iff AB \perp AC.$

April
#5 Oct 20, 2006, 7:01 pm

> *Bob75 wrote:*
> A', B' and C' are the feet of the interior bisectors of a triangle ABC.
> Prove that: $\angle BAC = 120^\circ \iff \angle B'A'C' = 90^\circ$.

Assume that A lies on ray Bx.
$\star \angle BAC = 120^\circ$
$\implies \angle xAC = \angle A'AC = 60^\circ$
$\implies A'B'$ is the external bisector of $\angle AA'B$
Similarly $A'C'$ is the external bisector of $\angle AA'C$
$\implies \angle B'A'C' = 90^\circ$
$\star \angle B'A'C' = 90^\circ$
Let $I = AA' \cap BB' \cap CC', K = BB' \cap C'A'$
We have:
$(BIKB') = -1$
$\implies A'(BIKB') = -1$
And $B'A' \perp A'K$
$\implies \angle AA'B' = \angle CA'B'$
$\implies B'$ is the center of B-excircle of triangle ABC.
$\implies \angle A'AC = \angle xAC = \angle A'AB$
$\implies \angle BAC = 120^\circ$.

OHO
#6 Oct 22, 2006, 12:35 am

Could you tell me what $(BIKB') = -1$ and $A'(BIKB') = -1$ means?

This post has been edited 1 time. Last edited by OHO, Oct 23, 2006, 5:06 pm

April
#7 Oct 23, 2006, 10:24 am

> *OHO wrote:*
> Could you tell me what this $(BIKB') = -1$ and $A'(BIKB') = -1$ means?

It's harmonious division and harmonically. Do you know it?

OHO
#8 Oct 23, 2006, 4:15 pm

> *April wrote:*
>> *OHO wrote:*
>> Could you tell me what this $(BIKB') = -1$ and $A'(BIKB') = -1$ means?
>
> It's harmonious division and harmonically. Do you know it?

No.

April
#9 Oct 25, 2006, 9:43 am

OK. I have another solution:
$\star \angle B'A'C' = 90^\circ$
Let Δ be the line through A and parallel $BC. A_1B_1 \cap \Delta = B_2, A_1C_1 \cap \Delta = C_2$.
We have:
$\frac{AB_2}{CA_1} = \frac{AB_1}{CB_1} = \frac{AB}{CB}\frac{AC_2}{BA_1} = \frac{AC_1}{BC_1} = \frac{AC}{BC} \implies \frac{AB_2}{CA_1}.\frac{BA_1}{AC_2} = \frac{AB}{BC}.\frac{AC}{BC} \implies \frac{AB_2}{AC_2}.\frac{BA_1}{CA_1} = \frac{AB}{AC} \implies \frac{AB_2}{AC_2}.\frac{AB}{AC} = \frac{AB}{AC} \implies \frac{AB_2}{AC_2} = 1$
$\Delta A_1B_2C_2$ have $\angle B_2A_1C_2 = 90^\circ, AB_2 = AC_2$
$\implies AA_1 = AB_2 = AC_2 \implies \angle AA_1B_2 = \angle AB_2A_1 = \angle CA_1B_2 \implies B_1$ is the excenter of triangle $ABA_1 \implies \angle xAC = \angle CAA_1 = \angle A_1AB \implies \angle BAC = 120^\circ$.

OHO

#10 Oct 26, 2006, 1:26 am

Very nice. Thank you for this solution.

In the first line, $\frac{AB}{BC}\cdot\frac{AC}{BC}$ should be $\frac{AB}{BC}\cdot\frac{BC}{AC}$

In your first solution, I understand that $BK\cdot IB' = BB'\cdot KI$. Am I right? and could you prove it for me please?

April

#11 Oct 27, 2006, 10:44 am

Yes, you are right. You can prove it by applying Menelau's theorem for triangle BIC with the line $\overline{A'KC'}$ and $\overline{PB'C'}(P = B'C'\cap BC)$

Note that: $\frac{PB}{PC} = \frac{A'B}{A'C}$.

And you can find some problems and properties of a harmonious division at http://www.mathlinks.ro/Forum/viewtopic.php?highlight=division+harmonious&t=46146

OHO

#12 Oct 27, 2006, 4:18 pm

Thanks.

Konigsberg

#13 Mar 5, 2014, 8:07 am

this is quickly baryable. I will post my solution sometime soon.

Konigsberg

#14 Mar 5, 2014, 5:11 pm

> *April wrote:*
>
> > *Bob75 wrote:*
> >
> > A', B' and C' are the feet of the interior bisectors of a triangle ABC.
> > Prove that: $\angle BAC = 120^\circ \iff \angle B'A'C' = 90^\circ$.
>
> Assume that A lies on ray Bx.
> $\star\angle BAC = 120^\circ$
> $\implies \angle xAC = \angle A'AC = 60^\circ$
> $\implies A'B'$ is the external bisector of $\angle AA'B$
> Similarly $A'C'$ is the external bisector of $\angle AA'C$
> $\implies \angle B'A'C' = 90^\circ$
> $\star\angle B'A'C' = 90^\circ$
> Let $I = AA'\cap BB'\cap CC', K = BB'\cap C'A'$
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> $(BIKB') = -1$
> $\implies A'(BIKB') = -1$
> And $B'A'\perp A'K$
> $\implies \angle AA'B' = \angle CA'B'$
> $\implies B'$ is the center of B-excircle of triangle ABC.
> $\implies \angle A'AC = \angle xAC = \angle A'AB$
> $\implies \angle BAC = 120^\circ$.

Nice application of Harmonics! However, is there any angle chasing solution (no harmonic divisions, length bashing, etc.)

Thanks

Quick Reply

High School Olympiads

Angle bisector

Reply

Source: INMOTC 2015

geniusramanujan
#1 Jan 29, 2015, 11:45 pm

In triangle ABC, angle BAC is 120 degrees. Let the angle bisectors of angles A, B, C meet the opposite sides in D, E, F respectively. Prove that the circle on diameter EF passes through D.

TelvCohl
#2 Jan 29, 2015, 11:57 pm • 1

My solution:

Since AE is the external bisector of $\angle BAD$,
so E is the $B-$ excenter of $\triangle ABD \Longrightarrow \angle CDE = \angle EDA$ . ... (1)
Similarly, we can prove F is the $C-$ excenter of $\triangle ACD$ and $\angle ADF = \angle FDB$ . ... (2)

From (1) and (2) $\Longrightarrow \angle EDF = 90^\circ \Longrightarrow D \in \odot(EF)$.

Q.E.D

geniusramanujan
#3 Jan 30, 2015, 12:51 pm

ok
great
thanks
wonderful solution
i never thought of excentres

Quick Reply

High School Olympiads

collineaR

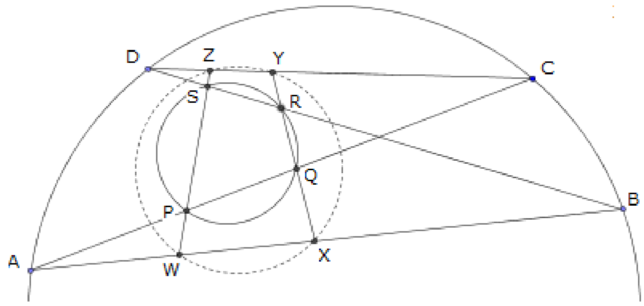
Reply

ricarlos

#1 Apr 12, 2015, 8:15 am • 1

Let $ABCD$ be a cyclic quadrilatheral. Let P, Q, R, S be points on AC and BD so that $PQRS$ is cyclic.
Line PS intersect AB and CD at W and Z,
line QR intersect AB and CD at X and Y.
Prove that $WXYZ$ is cyclic and the circumcenter of $ABCD$, $WXYZ$ and $PQRS$ are collinear.

Attachments:



Luis González

#3 Apr 12, 2015, 8:57 am • 3

Since $\angle APW=\angle DRY$ and $\angle PAW=\angle RDY \Longrightarrow \angle AWP=\angle DYR \Longrightarrow WXYZ$ is cyclic. Now, if RQ and SP cut $\odot(ABCD)$ at $\{R',Q'\}$ and $\{S',P'\}$, resp, then by Desargues involution theorem for $ABCD$ cut by RQ,SP, there exist points $U \in RQ$ and $V \in SP$, such that $UR \cdot RQ=UX \cdot UY=UR' \cdot UQ'$ and $VS \cdot VP=VW \cdot VZ=VS' \cdot VP'$ (center of the involutions), thus $\odot(ABCD),\odot(PQRS),\odot(WXYZ)$ are coaxal with common radical center $UV \Longrightarrow$ their centers are collinear.

Quick Reply

High School Olympiads

Radical axis of pedal triangles

Reply

61plus

#1 Mar 29, 2015, 9:47 pm

Let ABC be a triangle and D a fixed point, line l through D parallel to BC, E a variable point on line l and E' the reflection of E across D. Show that the radical axis of the circumcircle of the pedal triangles of E, E' to ABC passes through a fixed point.

TelvCohl

#3 Mar 29, 2015, 11:43 pm • 1

This problem is the special case of **Lemoine theorem :**

Let S be the orthopole of line ℓ WRT $\triangle ABC$.
Let P be a point moves on ℓ and $\triangle XYZ$ be the pedal triangle of P WRT $\triangle ABC$.

Then the power $\mathcal{P}(S, \odot(XYZ))$ of S WRT $\odot(XYZ)$ is constant

---

Proof :

Let D be the projection of A on ℓ .
Let $R = AD \cap \odot(ABC)$ and H be the projection of R on BC .
Let $B' \in AB$ such that $DB' \parallel RB$ and $C' \in AC$ such that $DC' \parallel RC$.
Let D' be the reflection of D in A-midline of $\triangle ABC$ and $X' \in PX$ be a point such that $AX' \parallel BC$.
Let $W = YZ \cap B'C', S^* = XW \cap DD'$ and K be the second intersection of $\odot(DD'XX') \cap \odot(XYZ)$.

Easy to see A, D, P, X', Y, Z lie on a circle with diameter AP .

Since $\triangle RBC$ and $\triangle DB'C'$ are homothetic with center A ,
so $D \in \odot(AB'C') \Longrightarrow \angle WYD = \angle ZAD = \angle BAR = \angle BCR = \angle WC'D$,
hence we get C', D, W, Y are concyclic $\Longrightarrow \angle DWC' = \angle DYC' = \angle DX'A \Longrightarrow D, W, X'$ are collinear .

From homothety we get $\overline{SD} = \overline{RH}$.

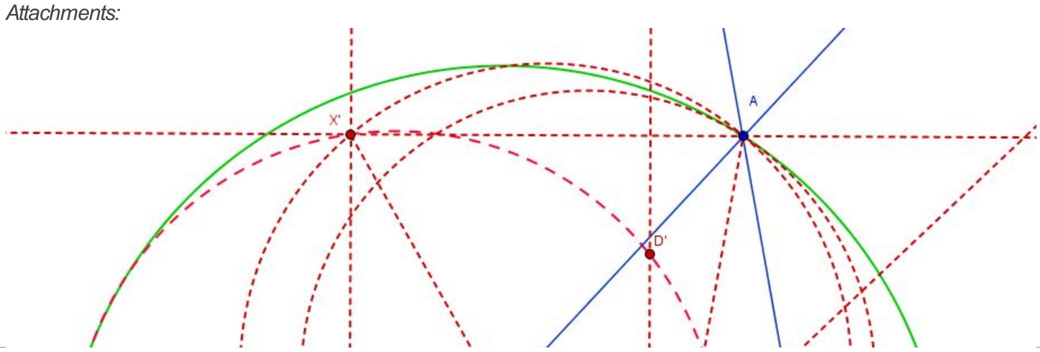
Since $\frac{\overline{S^*D}}{\overline{XX'}} = \frac{\overline{DW}}{\overline{WX'}} = \frac{\text{dist}(D, B'C')}{\text{dist}(A, B'C')} = \frac{\text{dist}(R, BC)}{\text{dist}(A, BC)} = \frac{\overline{RH}}{\overline{XX'}}$,

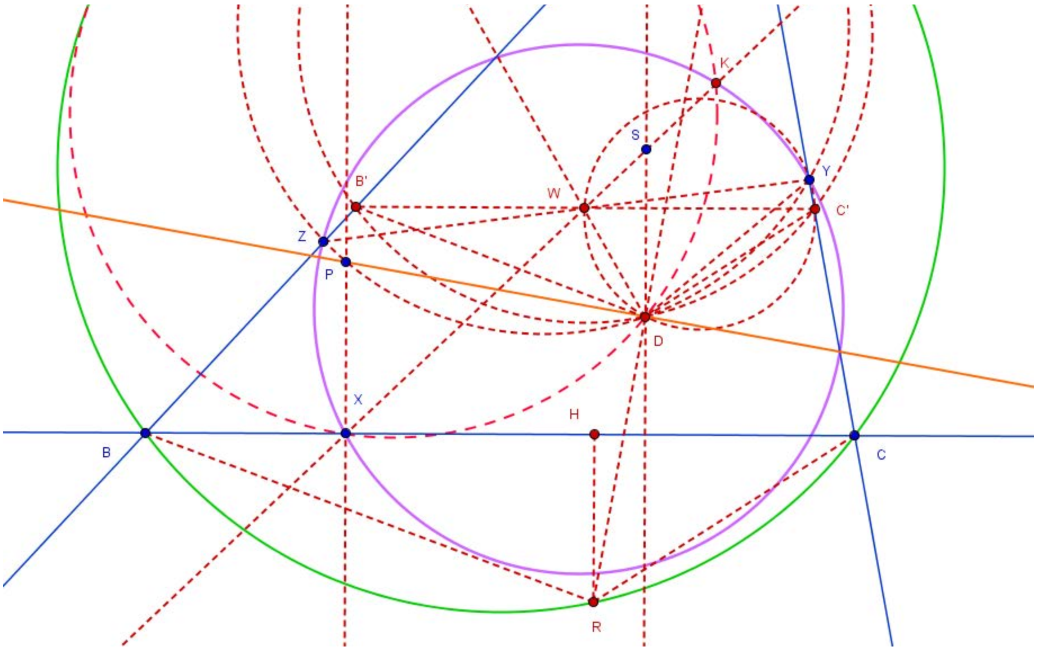
so we get $\overline{S^*D} = \overline{RH} \Longrightarrow S^* \equiv S$. i.e. S, W, X are collinear

Since W is the radical center of $\{\odot(AP), \odot(XYZ), \odot(DD'XX')\}$,
so W lie on the radical axis XK of $\{\odot(XYZ), \odot(DD'XX')\} \Longrightarrow S, K, W, X$ are collinear ,
hence we get $\mathcal{P}(S, \odot(XYZ)) = XS \cdot SK = DS \cdot SD' = \text{Const}$.

Q.E.D

Attachments:





Luis González

#4 Apr 12, 2015, 6:38 am

The Lemoine theorem was also discussed at the thread radical axis and Simpson's lines are concurrent.

High School Olympiads

radical axis and Simpson's lines are concurrent

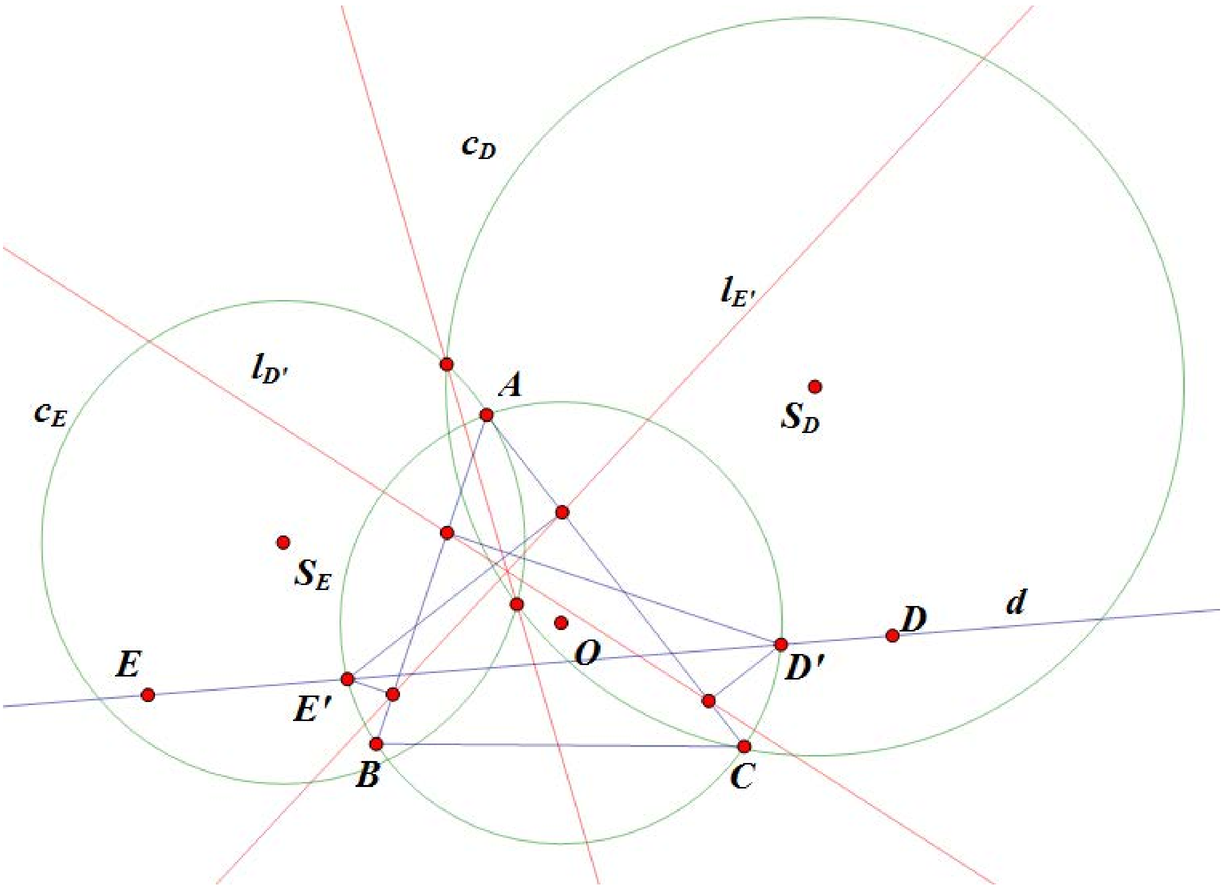
Reply

perfectstrong

#1 May 30, 2013, 6:03 pm

Given triangle ABC with its circumcircle (O). Let D, E is two point in the same plane. The line DE cuts (O) at two distinct points which are D', E'. Denote by $(c_D), (c_E)$ the pedal circle of D, E with respect to $\triangle\ ABC$, respectively.
$l_{D'}, l_{E'}$ denote the Simpson's line of D', E' with respect to $\triangle\ ABC$, respectively.
Prove or disprove that the radical axis of $(c_D), (c_E)$ and $l_{D'}, l_{E'}$ are concurrent.

Attachments:



Luis González

#2 Jun 21, 2013, 8:56 pm • 3

Let X, Y, Z be the projections of D on BC, CA, AB. AO cuts DE at T and M, N are the projections of T on AC, AB. The orthogonal projection K of A on DE is clearly the 2nd intersection of the circles $\odot(AYZ)$ and $\odot(AMN)$ with diameters AD, AT, i.e. K is Miquel point of the quadrangle bounded by AC, AB, YZ, MN. If $U \equiv MN \cap YZ$, then K, U, N, Z are concyclic. Thus if KU cuts $\odot(AYZ)$ again at L, we have $\angle ZAL = \angle ZKL = \angle ANU = \angle ABC \Longrightarrow AL \parallel BC$, i.e. L is the 2nd intersection of DX with $\odot(AYZ)$.

Since $AL \parallel MN \parallel BC$, then as D runs on the fixed $D'E'$, the ratio of the speeds of U and X is constant, namely $KU : KL = \text{dist}(K, MN) : \text{dist}(K, AL) \Longrightarrow$ all lines XU pass through a fixed point. When D coincides with D' or E' then $\overline{XYZ}$ become their Simson lines passing through the orthopole R of $D'E'$ (for a proof see the lemma in the solution of the problem Six orthopoles lie on a circle), hence $R \in XU$.

If RX cuts (c_D) again at F, we have $UX \cdot UF = UY \cdot UZ = UL \cdot UK \Longrightarrow X, L, F, K$ are concyclic. $\odot(XLK)$ clearly forms a pencil with common radical axis RK, since they go through K and its reflection K' on the A-midline of $\triangle ABC$.

Hence, power of R WRT (c_D) is then $RX \cdot RF = RK \cdot RK' = \text{const}$. Analogously, the power of R WRT (c_E) is $RK \cdot RK' \Longrightarrow R$ is on the radical axis of $(c_D), (c_E)$.

perfectstrong

#3 Jun 22, 2013, 7:57 am

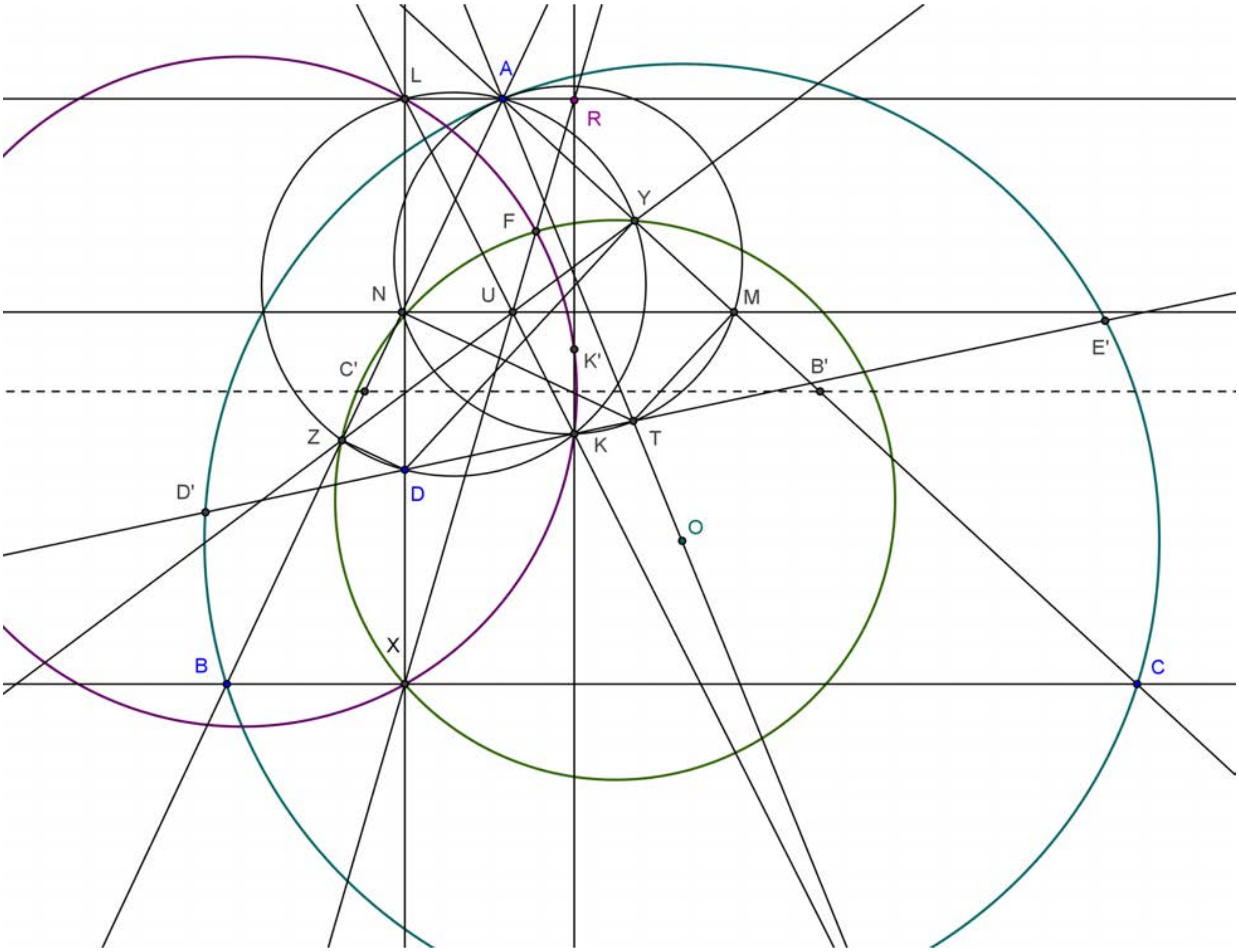
Great Solution But could you explain this sentence: "$\odot(XLK)$ clearly forms a pencil with common radical axis RK, since they go through K and its reflection K' on the A-midline of $\triangle ABC$." ?

Luis González

#4 Jun 22, 2013, 10:22 am • 1

It simply means that $\odot(XLK)$ goes through the fixed points K, K'. Note that its center is on the perpendicular bisector of $\overline{XL}$, which clearly is the line passing through the midpoints B', C' of AC, AB. Hence by symmetry, $\odot(XLK)$ passes through the reflection of K' of K about $B'C'$.

Attachments:



Quick Reply

High School Olympiads

Circles passing through a point

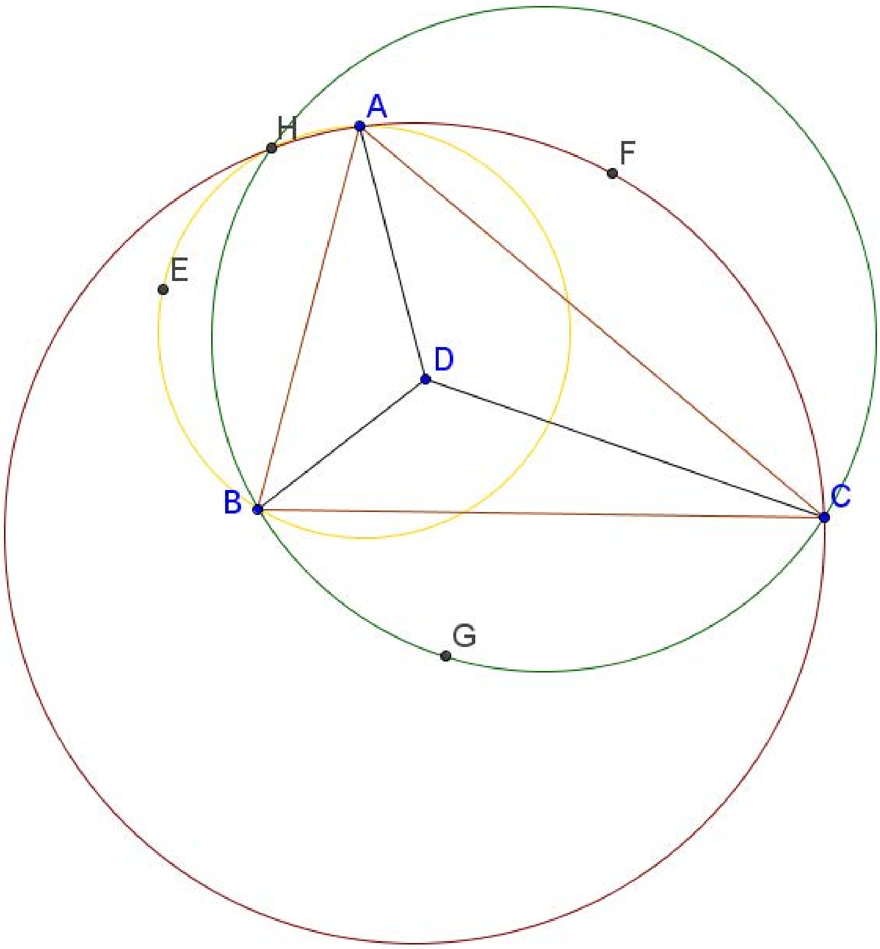
Reply

Cezar

#1 Apr 12, 2015, 5:18 am

Let a point D and $\triangle ABC$.
Let points G, E, F be the isogonal points of points A, C, B wrt to triangles $\triangle BDC, \triangle ABD, \triangle ADC$.
Prove that the the circumcircles of the triangles $\triangle AEB, \triangle CGB, \triangle AFC,$ have a common point.

Attachments:



Luis González

#2 Apr 12, 2015, 5:48 am

Let H be the 2nd intersection of $\odot(AEB)$ and $\odot(BGC)$. Using the diagram above, we get
$\angle BHC = \angle GBC + \angle GCB = \angle DBA + \angle DCA$ and similarly $\angle AHB = \angle AEB = 180^\circ - \angle DBC - \angle DAC$
$\Longrightarrow \angle AHC = \angle AHB - \angle BHC = 180^\circ - \angle ABC - \angle DAC - \angle DCA \Longrightarrow$
$\angle AHC = \angle DAB + \angle DCB = \angle FAC + \angle FCA \Longrightarrow H \in \odot(AFC).$

TelvCohl

#3 Apr 12, 2015, 11:46 am

My solution:

Let Q be the isogonal conjugate of D WRT $\triangle ABC$.

Since $\angle CGB = 180^\circ - \angle ACD - \angle ABD = 180^\circ - (\angle BDC - \angle BAC) = \angle BQC$,
so G lie on the reflection of $\odot(QBC)$ in $BC \Longrightarrow \odot(GBC)$ pass through the antigonal conjugate T of Q WRT $\triangle ABC$.
Similarly we can prove $T \in \odot(FCA)$ and $T \in \odot(EAB) \Longrightarrow \odot(EAB), \odot(FCA), \odot(GBC)$ are concurrent at T .

Q.E.D

Quick Reply

High School Olympiads

Proving parallel segments

Reply

NEWDORMANTUSER

#1 Apr 12, 2015, 1:28 am

Let XYZ be a triangle, and let ω be its circumcircle. So, let the tangent to ω from Y and line through Z parallel to XY meet in X_1. Cyclically define Y_1 (as the intersection of the tangent to ω from Z and the line through X parallel to YZ). Next we define X_2 as the intersection of the tangent to ω from Z and the line through Y parallel to ZX. Cyclically define Y_2. Then prove that

$$X_1X_2 \parallel Y_1Y_2.$$

TelvCohl

#2 Apr 12, 2015, 1:54 am • 1

My solution:

From my proof at previous problem we get $\triangle ZX_1X \sim \triangle ZYY_1, \triangle ZX_2X \sim \triangle ZYY_2$,

so $ZX_1 \cdot ZY_1 = ZX \cdot ZY = ZX_2 \cdot ZY_2 \Longrightarrow \frac{ZX_1}{ZY_2} = \frac{ZX_2}{ZY_1} \Longrightarrow X_1X_2 \parallel Y_1Y_2$.

Q.E.D

Luis González

#3 Apr 12, 2015, 2:39 am • 1

Let $U \equiv YX_1 \cap XY_2, V \equiv XY_1 \cap YX_2$ and $P \equiv UZ \cap XY$. ZP is the Z-symmedian of $\triangle XYZ$ and since Y_1X_2 is antiparallel to XY WRT $VX, VY,$ then V-median VZ of $\triangle VXY$ becomes V-symmedian of $\triangle VY_1X_2$. Hence

$$\frac{ZY_1}{ZX_2} = \frac{VY^2}{VX^2} = \frac{XZ^2}{ZY^2} = \frac{PX}{PY} = \frac{ZY_2}{ZX_1} \Longrightarrow X_1X_2 \parallel Y_1Y_2.$$

Quick Reply

High School Olympiads

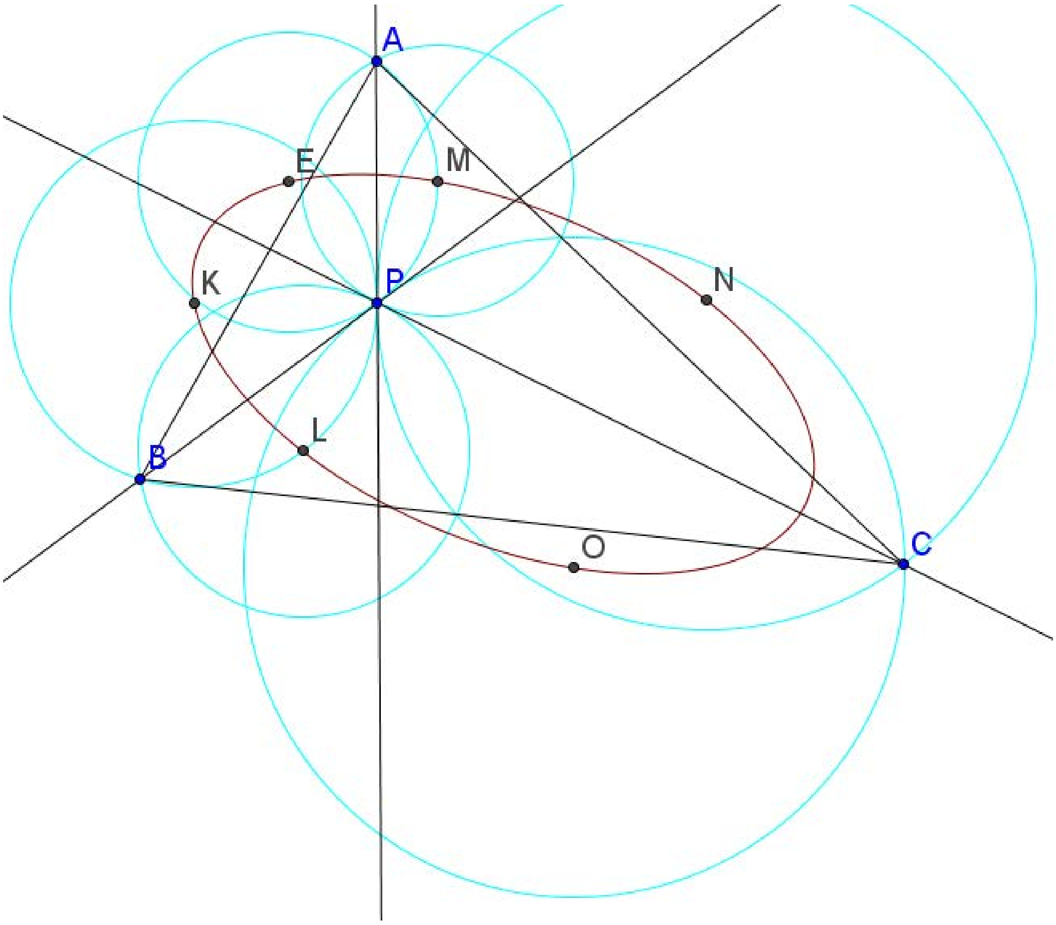
Tangent circles

izaya-kun

#1 Apr 11, 2015, 8:23 pm

Let a point P and $\triangle ABC$.
Let ω_{AB} be the circle that is tangent to BP at point P and passes through point A.
We also have the circles ω_{BA}, ω_{CB},ω_{BC} .....
1)Prove that the centers of those six circles lie on a conic.
2)Special case if point P is the centroid of $\triangle ABC$ then all those centers lie on a circle.

Attachments:



TelvCohl

#2 Apr 11, 2015, 9:28 pm

My solution:

1.

Let A_B be the center of ω_{AB} and define A_C, B_A, B_C, C_A, C_B similarly .
Let D, E, F be the circumcenter of $\triangle BCP, \triangle CAP, \triangle ABP$, respectively .

Easy to see $A_B, A_C \in EF$ and $B_C, B_A \in FD$ and $C_A, C_B \in DE$.

From $B_AC_A \perp AP \Longrightarrow B_AC_A \parallel EF$,
Similarly we can prove $C_BA_B \parallel FD, A_CB_C \parallel DE$,

so $\frac{FA_C}{EA_C} \cdot \frac{FA_B}{EA_B} \cdot \frac{DB_A}{FB_A} \cdot \frac{DB_C}{FB_C} \cdot \frac{EC_B}{DC_B} \cdot \frac{EC_A}{DC_A} = 1$,

hence we get $A_B, A_C, B_C, B_A, C_A, C_B$ lie on a conic (Carnot theorem) .

Done

For **2.** you can see Dao's Six Point Circle

Luis González

#3 Apr 11, 2015, 11:41 pm • 1

Let $\triangle XYZ$ be the antipedal triangle of P WRT $\triangle ABC$ (X, Y, Z againts A, B, C). The perpendicular to AP at P cuts XY, XZ at M', K', the perpendicular to BP at P cuts YX, YZ at O', E' and the perpendicular to CP at P cuts ZY, ZX at M', L'. Hexagon $M'N'O'L'K'E'$ is then image of hexagon $MNOLKE$ under homothety $(P, 2)$.

1) Intersections $E'M' \cap K'N', M'L' \cap N'O', L'K' \cap O'E'$, all at infinity, are collinear, thus by Pascal theorem $E'M'L'K'N'O'$ is inscribed in a conic $\mathcal{C}' \Longrightarrow EMLKNO$ is inscribed in a conic $\mathcal{C}$ image of $\mathcal{C}'$ under homothety $(P, \frac{1}{2})$.

2) If P is the centroid of $\triangle ABC$, pedal triangle of P WRT $\triangle XYZ$, then P is symmedian point of $\triangle XYZ \Longrightarrow \mathcal{C}'$ becomes the 1st Lemoine circle of $\triangle XYZ \Longrightarrow \mathcal{C}$ is the circle image of $\mathcal{C}'$ under homothety $(P, \frac{1}{2})$.

Quick Reply

High School Olympiads

M, I, N are collinear

Source: own

THVSH

#1 Apr 10, 2015, 9:33 pm • 2

Let ABC be a triangle with incircle (I). The tangent of (I) and parallel to BC intersects CA, AB at E, F, respectively. J, K are reflection of I in the line CA, AB, respectively. $BJ \cap CK = M; FJ \cap EK = N$.
Prove that M, I, N are collinear

Luis González

#2 Apr 10, 2015, 11:31 pm • 3

Let (I) touch CA, AB at $R, S. IR, IS$ cut AB, AC at U, V and JK cuts AB, AC at Z, Y. Since $AKIJ$ is a kite, then $JK \perp AI \Longrightarrow Z, Y$ become orthocenters of $\triangle AIK, \triangle AIJ$. Thus $\widehat{AIZ} = 90^\circ - \widehat{A} \Longrightarrow$ $\widehat{FIZ} = \widehat{AIZ} - \widehat{AIF} = 90^\circ - \widehat{A} - \tfrac{1}{2}\widehat{C} = \tfrac{1}{2}(\widehat{B} - \widehat{A}) = \widehat{CIV}$ and likewise $\widehat{YIE} = \widehat{UIB}$. Now since $\widehat{UIZ} = \widehat{VIY}$ (by symmetry) and $\widehat{CIE} = \widehat{BIF} = 90^\circ$, it follows that $(F, Z, B, U) = (C, V, E, Y)$ or $(B, U, F, Z) = (C, V, E, Y)$ $\Longrightarrow J(B, U, F, Z) = K(C, V, E, Y) \Longrightarrow M \equiv JB \cap KC, I \equiv JU \cap KV$ and $N \equiv JF \cap KE$ are collinear.

Quick Reply

High School Olympiads

MN bisects AC

Locked

dothef1

#1 Apr 10, 2015, 9:25 pm

Let M and N denote the projections of the orthocenter of a triangle ABC on the internal and the external bisectors of the angle B.
Prove that MN bisects AC.

Luis González

#2 Apr 10, 2015, 9:56 pm

Old problem. See the topics projections of orthocenter on bisectors, Collinear and elsewhere.

High School Olympiads

projections of orthocenter on bisectors

Reply

jbmorgan

#1 Feb 6, 2009, 9:41 am

Let ABC be a triangle with orthocenter H. Let the projections of H on the internal and external bisectors of A be P and Q respectively. Let M be the midpoint of BC.

Prove that P, Q, M are collinear.

April

#2 Feb 6, 2009, 10:51 am

It's true for any point H inside triangle ABC such that $\angle HBA = \angle HCA$.

Let E, F the projections of H on AB, AC, respectively. Hexagon $AEPHFQ$ is cyclic, therefore, we can easily prove that $\angle PEF = \angle PFE$ and $\angle QEF = \angle QFE$. Hence, we conclude that P, Q lie on the perpendicular bisector of EF (1)

On the other hand, denote by K, L the middle points of BH, CH, respectively. We have $\triangle MKE = \triangle FLM$, i.e. $ME = MF$ (2)

Combining (1) and (2), the result follows.

jayme

#3 Feb 6, 2009, 5:25 pm

Dear jbmorgan and Mathlinkers,
the center of the Euler's circle of ABC is also on this line.
Nice generalization of April
Sincerely
Jean-Louis

Mashimaru

#4 Apr 23, 2009, 2:15 pm

> jayme wrote:
> Dear jbmorgan and Mathlinkers,
> the center of the Euler's circle of ABC is also on this line.
> Nice generalization of April
> Sincerely
> Jean-Louis

I don't think so. If $\overline{PQM}$ passes through center E of Euler's circle of $\triangle ABC$ then for every point H satisfies $\widehat{HBA} = \widehat{HCA}$ we must have H lies on ME, thus the locus of H is a line (or even a part of a line. But it is wrong, the locus of H is a part of a hypebol curve.

jayme

#5 Apr 26, 2009, 8:08 pm

Dear Mashimaru and Mathlinkers,
this result comes from Droz-Farny

Please, see again your figure and toys with it...
Sincerely
Jean-Louis

Virgil Nicula
#6 Apr 27, 2009, 5:26 am

Very nice the **April**'s extension and its proof !

I remained without words. Thank you ! The **April**'s proof is a combination of the following lemmas :

> Quote:
> **Lemma 1.** Let $AQHP$ be a rectangle inscribed in the circle $w=C(O)$. Consider the points $\{E,F\}\subset w$ so that
>
> the line AP separates E , F and $\widehat{PAE}\equiv\widehat{PAF}$. Then Q , P belong to the bisector of the segment $[EF]$.
>
> Proof.

> Quote:
> **Lemma 2 (well-known).** Let PBC be a triangle. Denote the midpoint M of the side $[BC]$. Construct outside of $\triangle\ PBC$
>
> two P-right triangles PBE , PCF so that $\widehat{PBE}\equiv\widehat{PCF}$. Then the point M belongs to the bisector of the segment $[EF]$.

sunken rock
#7 Nov 3, 2009, 3:03 am

Another "smart proof" for extension:
CH intersects AB at R, BH – AC at S. The midpoints of diagonals AH, RS and BC of the complete quadrilateral ARHSBC are collinear (Gauss line), but BCSR is cyclic ($\angle ABH=\angle ACH$), hence the line connecting the midpoints of 2 opposite sides is the perpendicular bisector of the line connecting the feet of the perpendiculars from the intersection of its diagonals (H) to the other opposite sides (EF); from above see that PQ is the perpendicular bisector of EF.

Best regards,
sunken rock

randomusername
#8 Mar 26, 2016, 3:37 pm

Consider the homothety centered at H sending the nine-point circle of $\triangle ABC$ to the circle $\omega=\odot(ABC)$. Then it is well-known that X and M (X is the midpoint of HA) map to A and A', resp., where O, the center of ω is on AA'. Thus, $AA'||XM$. It is also known that AX and AA' are isogonals w.r.t. $\angle A$, and thus if $Z=XM\cap AP$, then $\angle XAZ=\angle ZAA'=\angle AZX$, whence $XA=XZ$. It follows from Thales that $\angle AZH=90^\circ$, so $Z=P$. Also, $APHQ$ is a rectangle, and line XM passes through its center and vertex P, so this line also passes through Q. $\blacksquare$

Ankoganit
#9 Mar 26, 2016, 3:53 pm

> jbmorgan wrote:
> Let ABC be a triangle with orthocenter H. Let the projections of H on the internal and external bisectors of A be P and Q respectively. Let M be the midpoint of BC.
>
> Prove that P,Q,M are collinear.

Also see Passes through midpoint and MN bisects AC.

Quick Reply

High School Olympiads

Collinear

Reply

77ant

#1 Jun 14, 2010, 9:46 am

Dear mathlinkers. Here one problem is. I hope it is not bad. Thanks.
For an acute triangle **ABC**, let **M**, **H**, **I**, and **Ib** be the midpoint of **BC**, the orthocenter, the incenter, and the **B**-excenter. Draw two perpendiculars from **H** to **AI**, **AIb** and let **D**, **E** be the feet of two perpendiculars. Then prove that **M**, **D**, **E** are collinear.

Luis González

#2 Jun 14, 2010, 11:18 am

Let F be the midpoint of $AH.$ Since $AEHD$ is a rectangle, then ED passes through $F.$ If U is the reflection of H across $AI_b,$ then AU,AH are isogonals with respect to $\angle BAC \Longrightarrow AU$ passes through the circumcenter O of $\triangle ABC.$ EF becomes the H-midline of $\triangle HUO \Longrightarrow ED$ passes through the midpoint N of $OH,$ i.e. 9 point center of $\triangle ABC.$ Since MF is a diameter of the 9-point circle $(N),$ we conclude that E,D,M,N are collinear.

dothef1

#3 Apr 12, 2015, 1:41 am

Can someone post a projective solution to this one ?

jayme

#4 Apr 12, 2015, 12:50 pm

Dear Mathlinkers,
this well known result on AoPS comes from Droz-Farny in 1894.

Sincerely
Jean-Louis

sunken rock

#5 Apr 12, 2015, 6:41 pm

$ADHE$ is a rectangle, so DE passes through F, the midpoint of AH. Let BN, CP be altitudes of $\triangle ABC, N, P$ on $AC, , AB$ respectively. MF is the **Newton-Gauss** line of the complete quad $APHNCB$, so it passes through the midpoint of NP $(\ 1\ )$. Let DH istersect AC, AB at Q, R respectively. ARQ is isosceles, since the angle bisector AD is altitude as well, so D is midpoint of RQ $(\ 2\ )$.
RQ is the bisector of angles H of the similar triangles BHP, CHN, hence $\frac{PR}{RB}=\frac{NQ}{CQ}$ $(\ 3\ )$.
By a well-known property, for a quadrilateral $BCNP$, the point D, the midpoint of the segment RQ, with R, Q - dividing the opposite sides CN, BP at the same ratio (3) - lies onto the line joining the midpoints of the other opposite sides, BC, PN, thus we are done.

Best regards,
sunken rock

jayme

#6 Apr 12, 2015, 7:54 pm

Dear Mathlinkers,
just a link

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=569545

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Isogonal conjugates

Reply

Source: Romania TST 2015 Day 1 Problem 1

ComplexPhi

#1 Apr 9, 2015, 11:19 pm

Let ABC be a triangle, let O be its circumcenter, let A' be the orthogonal projection of A on the line BC, and let X be a point on the open ray AA' emanating from A. The internal bisectrix of the angle BAC meets the circumcircle of ABC again at D. Let M be the midpoint of the segment DX. The line through O and parallel to the line AD meets the line DX at N. Prove that the angles BAM and CAN are equal.

This post has been edited 1 time. Last edited by ComplexPhi, Jun 4, 2015, 11:59 pm

TelvCohl

#2 Apr 10, 2015, 12:44 am

My solution:

Let H_∞, I_∞ be the infinity point on AA', AD, respectively .
Let $T = AA' \cap ON$ and S, R be the midpoint of AO, AD, respectively .

Easy to see M, S, R are collinear .

Since $\angle ATO = \angle XAD = \angle DAO = \angle AOT$,
so we get $AT = AO = DO \Longrightarrow ATOD$ is a parallelogram $\Longrightarrow T, S, D$ are collinear .

Notice that $\{AA', AO\}$ are isogonal conjugate of $\angle BAC$,
so from $A(M, O; D, A') = D(M, S; R, H_\infty) = (N, T; I_\infty, O) = A(N, A'; D, O) \Longrightarrow \angle BAM = \angle CAN$.

Q.E.D

Luis González

#3 Apr 10, 2015, 3:00 am

If AM cuts OD at P, then $AXPD$ is obviously parallellogram.Thus, from $\triangle PDX \cong \triangle AXD \sim \triangle ODN$, we get $\frac{AX}{XP} = \frac{AX}{AD} = \frac{OD}{ON} = \frac{AO}{ON}$. But clearly $\angle AXP = \angle AON \Longrightarrow \triangle AXP \sim \triangle AON$ by SAS $\Longrightarrow \angle XAM = \angle OAN \Longrightarrow AM, AN$ are isogonals WRT AO, AX, which in turn are isogonals WRT $\angle BAC \Longrightarrow \angle BAM = \angle CAN$.

Quick Reply

High School Olympiads

Nice Geometry

Reply

Source: vankhea

vankhea

#1 Apr 9, 2015, 9:11 pm

ABC inscribe in an ellipse (E).
K be any point inside of ABC.
The rays AK, BK, CK cuts BC, CA, AB at A', B', C' and cuts ellipse (E) at A'', B'', C'' respectively.
Prove that:
$$\frac{AK}{KA''}.\frac{A''A'}{A'A}+\frac{BK}{KB''}.\frac{B''B'}{B'B}+\frac{CK}{KC''}.\frac{C''C'}{C'C}=1$$

Luis González

#2 Apr 10, 2015, 1:36 am • 1

There exists a homography taking the given ellipse $\mathcal{E}$ with its interior point K into a circle with center K. Cross ratios remain invariant, thus it suffices to prove the relation for an acute $\triangle ABC$ with circumcircle $\odot(ABC) \equiv (K)$. If H is its orthocenter, then clearly $\triangle HBC \cong \triangle A''CB$ and cyclically $\Longrightarrow$

$$(K, A, A'', A') = \frac{AK}{KA''} \cdot \frac{A''A'}{A'A} = \frac{A''A'}{A'A} = \frac{[A''BC]}{[ABC]} = \frac{[HBC]}{[ABC]}.$$

Adding the cyclic expressions together yields

$$\frac{AK}{KA''} \cdot \frac{A''A'}{A'A} + \frac{BK}{KB''} \cdot \frac{B''B'}{B'B} + \frac{CK}{KC''} \cdot \frac{C''C'}{C'C} = \frac{[HBC]+[HCA]+[HAB]}{[ABC]} = 1.$$

Quick Reply

High School Olympiads

midpoints of sides lie on center locus

Reply

mineiraojose
#1 Apr 9, 2015, 8:07 pm

Let A,B,C,D be four points and consider the locus of the centers of the pencil of conics formed by A,B,C,D. Prove that the midpoints M,N,P,Q of the sides AB,BC,CD,DA lie on the locus.

TelvCohl
#2 Apr 9, 2015, 8:26 pm • 1

My solution:

Let $\mathcal{C}$ be a conic passing through A,B,C,D and T be the center of $\mathcal{C}$.
Let $M_\infty, N_\infty, P_\infty, Q_\infty$ be the infinity point on AB,BC,CD,DA, respectively .

Since the direction of $\{TM,M_\infty\},\{TN,N_\infty\},\{TP,P_\infty\},\{TQ,Q_\infty\}$ are conjugate WRT $\mathcal{C}$,
so $T(M,N;P,Q)=(M_\infty,N_\infty;P_\infty,Q_\infty)=\text{Const} \Longrightarrow$ the locus of T is a conic through M,N,P,Q .

Q.E.D

P.S
The locus of the center of a pencil of conics is called the 9-point conic of these four points
(Easy to see that the 9-point conic of $\{A,B,C,D\}$ also pass through $AB \cap CD, AC \cap BD, AD \cap BC$.)

Luis González
#3 Apr 10, 2015, 12:42 am • 1

Discussed before at http://www.artofproblemsolving.com/community/c6h351779. See post #7 for a synthetic approach.

Quick Reply

High School Olympiads

About Nine-point Conic

Reply

bbblow

#1 Jun 6, 2010, 11:01 pm • 1

Given A, B, C, D, locus of center of conics passing through $ABCD$ is the nine-point conic for $ABCD$.

*Nine-point conic for $ABCD$ is a conic which passes through:
midpoints of AB, AC, AD, BC, BD, CD
cevian traces of D, which can also be thought as $AB \cap CD, AC \cap BD, BC \cap DA$.

A mere conjecture, but GSP tells no lie. (Usually)
A thesis as simple as this is probably a result that was proven previously. Can anyone enlighten me?

Luis González

#2 Jun 7, 2010, 4:58 am • 3

Let us use barycentric coordinates with respect to $\triangle ABC$. Pencil $\mathcal{H}$ of conics through A, B, C and $D \equiv (u : v : w)$ has fundamental equation:

$$\mathcal{H} \equiv BC \cdot DA + \delta CA \cdot DB = 0 \Longrightarrow \delta y(wx - uz) + x(vz - wy) = 0$$

The centers O of $\mathcal{H}$ (poles of the line at infinity x+y+z=0) have then coordinates:

$$O \equiv \left(\frac{\delta(u+w) + v - w}{vw(\delta - 1)} : \frac{\delta(u-w) + w + v}{uw\delta(\delta - 1)} : \frac{\delta(w+u) - w - v}{uv\delta} \right)$$

Eliminating δ from the latter coordinates gives the locus described by O as:

$$vwx^2 + uwy^2 + uvz^2 - (v+w)uyz - (u+w)vxz - (u+v)wxy = 0$$

This is a conic passing through the midpoints $(0:1:1), (1:0:1), (1:1:0)$ of BC, CA, AB and the traces $(0:v:w)$, $(u:0:w)$ and $(u:v:0)$ of the D-cevians on BC, CA, AB, i.e the 9-point conic of $ABCD$.

This post has been edited 1 time. Last edited by Luis González, Jun 7, 2010, 8:02 am

bbblow

#3 Jun 7, 2010, 7:38 am

Thank you very much.

May I know how $\mathcal{H} \equiv BC \cdot DA + \delta CA \cdot DB = 0$ was derived?

Luis González

#4 Jun 7, 2010, 9:20 am • 3

Let us consider homogeneus coordinates in the projective plane, which embrace the barycentric coordinates.

Given two conics $\mathcal{H}_1$ and $\mathcal{H}_2$ with equations $f_1((x^0, x^1, x^2)) = {}^tXA_1X = 0$ and $f_2((x^0, x^1, x^2)) = {}^tXA_2X = 0$, respectively, we define a pencil of conic as the family/set of conics whose points satisfy $\alpha \cdot {}^tXA_1X + \beta \cdot {}^tXA_2X = 0$ for $(\alpha,\beta)\not\eq=0.$ Obviously, by the definition $\mathcal{H}_1$ and $\mathcal{H}_2$ belong to the pencil, thus it is sufficient to assign $(1, 0)$ and $(0, 1)$ to α, β, respectively. Setting $\delta=\frac{\beta}{\alpha} \ , (\alpha \not\eq= 0),$ we rewrite the equation of the pencil as ${}^tX(A_1 + \delta A_2)X = 0$, recalling that $\delta = \infty$ represents the conic ${}^tXA_2X = 0$.

A pencil of conics contains at most three degenerate conics, unless that all conics are degenerate and also any pencil of conics has either four distintc or identical points. According to the configuration of these points, we basically have 5 types of pencils; from 4 distinct points to 4 identical points. We are interested in the first one. Let P, Q, R, S be four distinct points $RS \equiv p$, $PQ \equiv q, QS \equiv r$ and $PR \equiv s$. In such case, we have three degenerate conics into pairs of lines, namely $PQ \cup RS$, $QR \cup SP$ and $PR \cup QS$. If we take two of these conics as the fundamental conics $r = 0, s = 0, p = 0$ and $q = 0$, then the pencil of conics passing through P, Q, R, S is given by $r \cdot s + \delta p \cdot q = 0$.

bbblow

#5 Jun 8, 2010, 12:54 pm

Though I don't really understand that well, still thanks. I'll read it up when I feel like doing so =)

pldx1

#6 Jun 12, 2010, 8:13 pm

Hello,
An example is as follows. Take a triangle ABC and its orthocenter H. Then the "curve" obtained as $BC \cup AH$ is a degenerated rectangular hyperbola (RH). In the same vein, $CA \cup BH$ and $AB \cup CH$ are RH either. All the other conics through A, B, C, H are regular RH. And the locus of their centers is the Euler celebrated nine point conic.

Best regards, Pierre.

Luis González

#7 Nov 15, 2012, 8:45 am • 1

Rewriting the proof without calculations:

Let K, L, M, N be the midpoints of AB, BC, CD, DA. Let O be the center of a conic $\mathcal{H}$ of the pencil through A, B, C, D. The poles of OK, OL, OM, ON WRT $\mathcal{H}$ are the intersections $K_\infty, L_\infty, M_\infty, N_\infty$ of AB, BC, CD, DA with the line at infinity. Cross ratio of the line pencil $O(K, L, M, N)$ equals the cross ratio $(K_\infty, L_\infty, M_\infty, N_\infty)$ of their poles $\Longrightarrow$ $O(K, L, M, N) = \text{const} \Longrightarrow$ locus of O is a conic through K, L, M, N.

The centers of the degenerate conics $AB \cup CD, BC \cup DA$ and $AC \cap BD$ are nothing but $AB \cap CD, BC \cap DA$ and $AC \cap BD$, respectively. Hence, $\mathcal{H}$ is the 9-point conic of $ABCD$.

Quick Reply

High School Olympiads

isogonal conjugate of two Fermat points

Reply

daothanhoai

#1 Oct 16, 2013, 3:18 pm

Dear Mathlinkers!

Let a triangle ABC, let first end secon Fermat points is F_1, F_2 and D_1, D_2(respectively) are isogonal conjugate of two Fermat points. Then:

F_1, D_2 and the centroid are collinear

F_2, D_1 and the centroid are collinear

D_1, D_2 and centre of circumcircle are collinear

Pedal circle of F_1, F_2 and Nine points circle are concurrent at midpoint of F_1F_2

Best regard

Dao Thanh Oai

Luis González

#2 Mar 22, 2015, 10:36 pm

It's known that D_1, D_2 lie on Brocard axis, i.e D_1D_2 passes through the circumcenter and the centroid is the intersection $F_1D_2 \cap F_2D_1$ (see problem 2 at http://www.artofproblemsolving.com/community/c6h1064217).

Since D_1, D_2 are inverse points in the circumcircle, their isogonals F_1, F_2 are antigonal conjugates, i.e. midpoint of F_1F_2 is the center X_{115} of the Kiepert hyperbola; Poncelet point of $ABCF_1$ and $ABCF_2$. Thus pedal circles of F_1, F_2 and the 9-point circle of $\triangle ABC$ concur at X_{115}.

Quick Reply

High School Olympiads

Properties of Brokard and Isodynamic points

Reply

izaya-kun

#1 Mar 18, 2015, 7:33 pm

1)Let Br_1 and Br_2 be the two brokard points of $\triangle ABC$.
Let I_1 and I_2 be the two isodynamic points of $\triangle ABC$.
Prove $\angle LBr_1I_1 = \angle LBr_1I_2 = \angle LBr_2I_1 = \angle LBr_2I_2 = 60^\circ$, where L is the lemoine point of $\triangle ABC$.
2) Let F_1 and F_2 be the two fermat points of $\triangle ABC$.
Prove F_1I_2 and F_2I_1 intersect at the centroid of $\triangle ABC$.

Luis González

#3 Mar 19, 2015, 4:50 am • 2

1) For convenience rename K the symmedian point and Ω_1, Ω_2 the Brocard points. ω denotes the Brocard angle of $\triangle ABC$, $O \in KI_1I_2$ is the circumcenter, H is its orthocenter, M is the midpoint of BC and X is outside of $\triangle ABC$, such that $\triangle XBC$ is equilateral.

$$\frac{I_1O}{I_1K} = \frac{R}{AK} \cdot \frac{\sin \widehat{OAI_1}}{\sin \widehat{KAI_1}} = \frac{R}{AK} \cdot \frac{\sin \widehat{HAF_1}}{\sin \widehat{MAF_1}} = \frac{R}{AK} \cdot \frac{\sin \widehat{MXA}}{\sin \widehat{MAX}} =$$

$$= \frac{R}{AK} \cdot \frac{AM}{MX} = \frac{2R}{AK} \cdot \frac{AM}{\sqrt{3}a}.$$

But $AK = \dfrac{bc}{a^2+b^2+c^2} \cdot \sqrt{2(b^2+c^2)-a^2} = \dfrac{2bc}{a^2+b^2+c^2} \cdot AM \Longrightarrow$

$$\frac{I_1O}{I_1K} = \frac{2R}{\sqrt{3}a} \cdot \frac{a^2+b^2+c^2}{2bc} = \frac{a^2+b^2+c^2}{4\sqrt{3}[ABC]} = \frac{\cot \omega}{\sqrt{3}} \quad (1).$$

From previous problem http://www.artofproblemsolving.com/community/c6h1064160, we know that $\triangle OK\Omega_1$ is right with hypotenuse OK. Hence if $J_1 \in \overline{OK}$, such that $\angle K\Omega_1J_1 = 60^\circ$, we have

$$\frac{J_1O}{J_1K} = \frac{O\Omega_1}{K\Omega_1} \cdot \frac{\sin 30^\circ}{\sin 60^\circ} = \frac{\cot \omega}{\sqrt{3}} \quad (2).$$

From (1) and (2), it follows that $I_1 \equiv J_1 \Longrightarrow \angle K\Omega_1I_1 = 60^\circ$. In the same way $\angle K\Omega_1I_2 = 60^\circ$ and the same holds for Ω_2 as it is the reflection of Ω_1 on OK.

Luis González

#5 Mar 19, 2015, 5:17 am • 2

2) Since F_1 and F_2 are Kiepert perspectors $K(60^\circ)$ and $K(-60^\circ)$, then F_1F_2 hits the Brocard axis I_1I_2 at the symmedian point K. In fact, in general, we have $K \in K(\theta)K(-\theta)$. For a proof see Property of Kariya point (remark at post #3 and post #4). Now, from the lemma mentioned in the topic Angle related to Fermat points and Isodynamic points (see post #3), it follows that $F_1I_2 \cap I_1F_2$ is the isogonal conjugate of $K \equiv F_1F_2 \cap I_1I_2$ WRT $\triangle ABC \Longrightarrow F_1I_2 \cap I_1F_2$ is the centroid of $\triangle ABC$.

Quick Reply

High School Math

geometry

Reply

andria

#1 Mar 21, 2015, 7:15 pm

Incircle of triangle ABC touches AB, BC,AC at F,D and E. Assume that bisector of angle BAC meet the incircle at P. PB and PC intersect the lines FD and ED at T and S prove that AP is perpendicular to TS.

Submathematics

#2 Mar 21, 2015, 10:29 pm

since $FB = FD$

$\frac{FT}{TD} = \frac{Sin(\angle FBT)}{Sin(\angle DBT})$

Also,

$\frac{Sin(\angle FBT)}{Sin(\angle DBT}) = \frac{AP \times BE}{PE \times AB}$

SO, $\frac{FT}{TD} = \frac{AP \times BE}{PE \times AB}$

Similarly,

$\frac{SE}{SD} = \frac{AP \times EC}{PE \times AC}$

by the angle bisector theorem,

$\frac{AB}{AC} = \frac{BE}{EC}$

so,

$\frac{SE}{SD} = \frac{FT}{TD}$

SO,

TS is parallel to FE

Since,

$AE = AF$

AP IS perpendicular to FE

SO,

AP IS PERPENDICULAR TO TS

Luis González

#3 Mar 22, 2015, 3:51 am

Since PT and PS are the P-symmedians of $\triangle PDF$ and $\triangle PDE$, we get

$$\frac{DT}{TF} = \frac{DP^2}{PF^2} = \frac{DP^2}{PE^2} = \frac{DS}{DE} \Longrightarrow (ST \parallel EF) \perp AP.$$

Quick Reply

High School Math

geometry

Reply

andria

#1 Mar 21, 2015, 6:54 pm

Assume that two circles W and W' intersect at A and B, P is an arbitrary point on a line AB such that it is outside of W and W'. Q is a point on W and outside of W' such that PQ is tangent to W. R is a point on PQ such that PR=PQ (P is between Q and R) the tangents from R to W' meet W' at T and S. Prove that T,S and Q are collinear.

Luis González

#2 Mar 22, 2015, 12:52 am

Label $(O),(O)'$ the given circles. Since $PA \cdot PB = PQ^2 = PR^2$, then the circle (P) with diameter $\overline{QR}$ is orthogonal to $(O)'$. Hence if $O'R$ cuts $(O)'$ at X, Y and cuts (P) again at M (projection of Q on RO'), we get $O'X^2 = O'Y^2 = O'M \cdot O'R \Longrightarrow (X, Y, M, R) = -1 \Longrightarrow QM$ is the polar of R WRT $(O)' \Longrightarrow QM \equiv ST$, i.e. T, S, Q are collinear.

le_truong_son

#3 Mar 22, 2015, 5:12 pm

I don't understand why $O'X^2 = O'Y^2 = O'M \cdot O'R$ .M is the projection of Q on RO' but not sure that $O'X^2 = O'Y^2 = O'M \cdot O'R$

Quick Reply

High School Olympiads

Generalization of 2014 Taiwan TST2 Quiz2 P1

Reply

Source: Own

TelvCohl

#1 Mar 18, 2015, 12:05 am • 2

Let I, O be the incenter, circumcenter of $\triangle ABC$, respectively .
Let D be the tangent point of $\odot(I)$ with BC and D^* be the antipode of D in $\odot(I)$.
Let ℓ be the tangent of $\odot(I)$ passing through D^* and E, F be two points on ℓ .
Let $X = IE \cap \odot(AEF), Y = IF \cap \odot(AEF)$ and R be the radius of $\odot(O)$.

Prove that $\operatorname{dist}(O, XY) = |\mathrm{R} \cdot \cos \angle EIF|$

P.S.

buratinogigle

#2 Mar 19, 2015, 10:26 pm • 1

Nice generalization dear Telv!

Combine with my generalization in this post Prove that A, X, O, Y are concyclic and yours, I propose the problem

Let ABC be a triangle and P is a point on bisector of $\angle BAC$. D, E, F are projections of P on BC, CA, AB. PD cuts (DEF) again at G. d is the line passing through G and parallel to BC. O is circumcenter of ABC. Q is isogonal conjugate of P. M, N are two points on line d. QM, PN cut circumcircle of triangle AMN agian at X, Y then $\operatorname{dist}(O, XY) = |\mathrm{R} \cdot \cos(QM, PN)|$.

Arab

#3 Mar 20, 2015, 8:00 am • 2

Hello, TelvCohl!

This property was also discovered in 2009 by Ye Zhonghao.

For reference (in simplified Chinese), see A Challenging Problem (#44).

This post has been edited 1 time. Last edited by Arab, Mar 20, 2015, 8:01 am

Luis González

#5 Mar 21, 2015, 4:18 am • 1

Let I_a denote the A-excenter of $\triangle ABC$ and let P be the midpoint of the arc BC of (O). AI cuts BC, ℓ, XY at V, L, S respectively and cuts $\odot(EFXY)$ again at Z. K is the circumcenter of $\triangle IXY$, lying on perpendicular ID from I to EF (antiparallel to XY WRT IX, IY). Using the result of the problem http://www.artofproblemsolving.com/community/c6h1063007 in the cyclic $EFXY$ cut by the line IA, we get

$$\frac{1}{IL} - \frac{1}{IA} = \frac{1}{I_aV} - \frac{1}{I_aA} = \frac{1}{IP} = \frac{1}{IS} - \frac{1}{IZ}.$$

But $IZ \cdot IA = 2 \cdot IK \cdot ID^* = 2r \cdot IK \Longrightarrow \dfrac{1}{IP} = \dfrac{1}{IS} - \dfrac{IA}{2r} \cdot \dfrac{1}{IK},$

which means that $S \mapsto K$ is a perspectivity between IA and ID. Recall that a general perspectivity would be $k_1 \cdot \frac{1}{IS} + k_2 \cdot \frac{1}{IK} = \text{const}$ for $k_1, k_2 \in \mathbb{R}$. Hence all lines SK go through a fixed point. When K is at infinity, then $S \equiv P$ $\Longrightarrow SK$ becomes perpendicular bisector of BC. When $K \equiv D$, we have then

$$\frac{1}{IP}=\frac{1}{IS}-\frac{IA}{2r^2}\Longrightarrow\frac{SP}{IS}=\frac{IP-IS}{IS}=\frac{IA\cdot IP}{2r^2}=\frac{2Rr}{2r^2}=\frac{R}{r}=\frac{OP}{IK},$$

which implies that O,S,K are collinear $\Longrightarrow O$ is the fixed point. Hence

$$\frac{\text{dist}(O,XY)}{\text{dist}(K,XY)}=\frac{\text{dist}(O,XY)}{|IK\cdot\cos\widehat{EIF}|}=\frac{SO}{SK}=\frac{R}{IK}$$

$$\Longrightarrow \text{dist}(O,XY)=R\cdot|\cos\widehat{EIF}|.$$

P.S. In exactly the same way, we can approach the generalization mentioned by buratinogigle.

Quick Reply

High School Olympiads

Prove the equality

Reply

HHT23

#1 Mar 14, 2015, 9:57 pm

Given convex quadrilateral $ABCD$ inscribed circle (O). Let I be the intersection of AC and BC. Line Δ passes through I, intersects the segments AB, CD at M, N and intersects (O) at P, Q, respectively (M, N lie on the segments IQ, IP, respectively). Prove that

$$\frac{1}{IM}+\frac{1}{IP}=\frac{1}{IN}+\frac{1}{IQ}$$

Luis González

#3 Mar 14, 2015, 10:21 pm • 1

By Desargues involution theorem, Δ cuts the opposite sidelines of $ABCD$ and its circumcircle (O) at pairs of points in involution $\Longrightarrow (P,M,I,N)=(Q,N,I,M) \Longrightarrow$

$$\frac{PM}{PI}\cdot\frac{NI}{NM}=\frac{QN}{QI}\cdot\frac{MI}{MN} \Longrightarrow \frac{MP}{IM\cdot IP}=\frac{QN}{IN\cdot IQ} \Longrightarrow$$

$$\frac{IM+IP}{IM\cdot IP}=\frac{IN+IQ}{IN\cdot IQ} \Longrightarrow \frac{1}{IM}+\frac{1}{IP}=\frac{1}{IN}+\frac{1}{IQ}.$$

P.S. Note that the relation still holds for any $ABCD$ and a conic through A, B, C, D.

HHT23

#4 Mar 14, 2015, 10:32 pm

Can you tell me what Desargues involution theorem is and how to prove it? Thank you.

TelvCohl

#5 Mar 14, 2015, 10:50 pm • 1

> HHT23 wrote:
>
> Given convex quadrilateral $ABCD$ inscribed circle (O). Let I be the intersection of AC and BD. Line Δ passes through I, intersects the segments AB, CD at M, N and intersects (O) at P, Q, respectively (M, N lie on the segments IQ, IP, respectively). Prove that
>
> $$\frac{1}{IM}+\frac{1}{IP}=\frac{1}{IN}+\frac{1}{IQ}$$

Typo corrected :)

My solution:

Let $X=\Delta \cap \odot(ICD), Y=\Delta \cap \odot(IAB)$.
Let Ψ be the inversion with center I which swaps P, Q .
Let A^*, D^* be the reflection of A, D in the perpendicular bisector ℓ of PQ, respectively .

Since $\angle AA^*D=180^\circ-\angle DCA=180^\circ-\angle DXY$,

so combine with $AA^* \parallel XY$ we get A^*, D, X are collinear .
Similarly we can prove $Y \in D^*A \Longrightarrow X, Y$ are symmetry WRT $\ell \Longrightarrow QX = PY$.

Since X, Y, P, Q is the image of M, N, Q, P under Ψ, respectively ,

so from $IQ + IX = QX = PY = IP + IY \Longrightarrow \frac{1}{IP} + \frac{1}{IM} = \frac{1}{IQ} + \frac{1}{IN}$.

Q.E.D

P.S. You can find some information about Desargue involution theorem at here

Quick Reply

High School Olympiads

Circumcircle of AMN is tangent to omega

Reply

Source: 2015 China TST 2 Problem 3

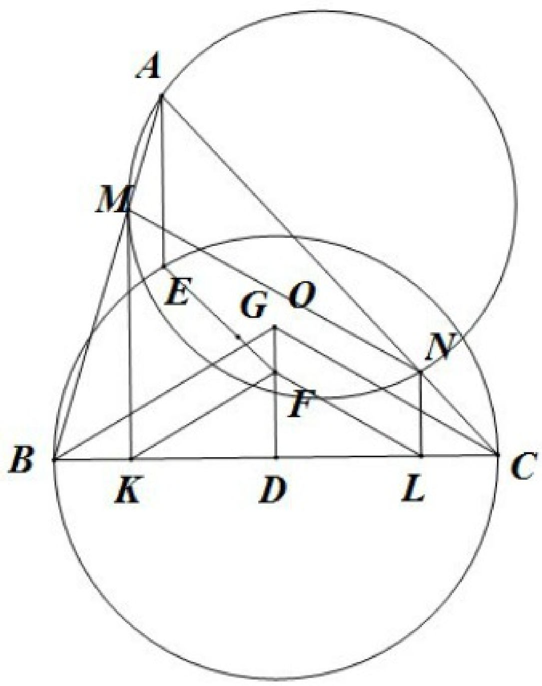
TelvCohl

#1 Mar 19, 2015, 3:38 pm • 2

Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G.
Let D be the midpoint of BC and $E \in \odot(BC)$ be a point inside $\triangle ABC$ such that $AE \perp BC$.
Let $F = EG \cap OD$ and K, L be the point lie on BC such that $FK \parallel OB, FL \parallel OC$.
Let $M \in AB$ be a point such that $MK \perp BC$ and $N \in AC$ be a point such that $NL \perp BC$.
Let ω be a circle tangent to OB, OC at B, C, respectively .

Prove that $\odot(AMN)$ is tangent to ω

Attachments:



TelvCohl

#2 Mar 19, 2015, 5:11 pm • 3

My solution:

Lemma :

Let $P, Q \in BC$ be the point such that $BP = CQ$.
Let $E \in AC, F \in AB$ be the point such that $EP \perp BC, FQ \perp BC$.
Let O be the circumcenter of $\triangle ABC$ and $T = \odot(AEF) \cap \odot(O)$ ($T \neq A$) .

Then $A(B, C; O, T) = -1$

Proof of the lemma :

Let $\mathcal{P}(P, \odot)$ be the power of a point P WRT circle $\odot$.
Let D be the midpoint of BC and $Y = OD \cap AC, Z = OD \cap AB$.

Since $\frac{\mathcal{P}(E, \odot(O))}{\mathcal{P}(E, \odot(AYZ))} = \frac{EC}{EY} = \frac{PC}{PD} = \frac{QB}{QD} = \frac{FB}{FZ} = \frac{\mathcal{P}(F, \odot(O))}{\mathcal{P}(F, \odot(AYZ))}$,

so we get $\odot(O), \odot(AEF), \odot(AYZ)$ are coaxial . i.e. A, Y, Z, T are concyclic

From $\angle OAY = \angle OZA$ we get $\triangle OAY \sim \triangle OZA \Longrightarrow OT^2 = OA^2 = OY \cdot OZ$,
so OA, OT is tangent to $\odot(AYZ) \Longrightarrow AYTZ$ is a harmonic quadrilateral $\Longrightarrow A(B, C; O, T) = -1$.

Back to the main problem :

Let H be the orthocenter of $\triangle ABC$ and R be the center of ω .
Let P be the antipode of A in $\odot(O)$ and $I = OD \cap AC, J = OD \cap AB$.
Let A^*, B^*, C^* be the projection of A, B, C on BC, AC, AB, respectively .
Let $E^* = AE \cap \odot(BC), T = \odot(AMN) \cap \odot(O), \{S, S^*\} = AO \cap \omega$ ($AS < AS^*$) .
Let Ψ be the composition of Inversion $\mathbf{I}(A, \sqrt{AB \cdot AC})$ and Reflection $\mathbf{R}(\ell)$ where ℓ is the bisector of $\angle BAC$.

Easy to see RB, RC is the tangent of $\odot(O)$.

Since AR, AD are isogonal conjugate of $\angle BAC$,
so $\odot(BC) \longleftrightarrow \omega$ under Ψ (notice that $B \longleftrightarrow C$ under Ψ) ,
hence we get $E \longleftrightarrow S^*, E^* \longleftrightarrow S$ under $\Psi \Longrightarrow AE \cdot AS^* = AE^* \cdot AS$. i.e. $SE \parallel S^*E^*$ $(\star)$

From $OA^2 = OP^2 = OS \cdot OS^*$ we get $(A, P; S, S^*) = -1$ . ... (1)
From $AH \cdot AA^* = AB \cdot AC^* = AC \cdot AB^* = AE \cdot AE^*$ we get $(A, H; E, E^*) = -1$ . ... (2)
Combine with $(1), (2)$ and $(\star)$ we get $PH \parallel SE \parallel S^*E^*$.

From the proof of the lemma we get $T \in \odot(AIJ)$ and OA is the tangent of $\odot(AIJ)$,
so from $BM : MJ = CN : NI = OF : FD = HE : EA = PS : SA \Longrightarrow S \in \odot(AMNT)$.

From the lemma we get $(B, C; Q, U) = -1$ where $Q = AT \cap BC, U = AO \cap BC$,
so we get QS is tangent to ω at S (notice that SBS^*C is a harmonic quadrilateral) ,
hence from $QS^2 = QB \cdot QC = QT \cdot QA \Longrightarrow QS$ is tangent to $\odot(ATS) \equiv \odot(AMN)$ at S ,
so we get ω and $\odot(AMN)$ are tangent to each other at S .

Q.E.D

buratinogigle
#3 Mar 19, 2015, 5:19 pm • 2

But this problem is mine , please the link
http://forumgeom.fau.edu/FG2014volume14/FG201425.pdf

Luis González
#4 Mar 20, 2015, 6:34 am • 1

Let H be the orthocenter of $\triangle ABC$ and let U be the antipode of A in $(O) \equiv \odot(ABC)$. If BU, CU cut AC, AB at Y, Z, then U is orthocenter of $\triangle AYZ \Longrightarrow \omega$ is the circle with diameter YZ. If OD cuts AB, AC, at P, Q, then clearly $\overline{PBZ} \sim \overline{QCY} \Longrightarrow (O), \odot(AYZ)$ and $\odot(APQ)$ meet at A and the center R of the spiral similarity that swaps $\overline{BZ}$ and $\overline{CY}$. Furthermore, (O) and $\odot(APQ)$ are orthogonal, due to $\angle APQ = 90^\circ - \angle ABC = \angle OAQ$.

AR, BC, YZ are pairwise radical axes of $\omega, (O), \odot(AYZ)$ concurring at their radical center T and AO is the polar of T WRT ω. Thus if AO cuts ω at S (inside of $\triangle ABC$), then TS is tangent of $\omega \Longrightarrow TS^2 = TB \cdot TC = TA \cdot TR \Longrightarrow$ $\odot(ARS)$ is tangent to ω at S. If $\odot(ARS)$ cuts $\overline{AB}, \overline{AC}$ at M', N', then the ratio of the powers of M', N', S WRT $\odot(APQ)$ and (O) are equal $\Longrightarrow$

$$\frac{M'A \cdot M'P}{M'A \cdot M'B} = \frac{N'A \cdot N'Q}{N'A \cdot N'C} = \frac{SA^2}{SA \cdot SU} \Longrightarrow \frac{PM'}{M'B} = \frac{QN'}{N'C} = \frac{AS}{SU} \quad (1).$$

But on the other hand, keeping in mind that $\triangle ABC \cup E \sim \triangle AYZ \cup S$, we have

$$\frac{AS}{SU} = \frac{AE}{EH} = \frac{DF}{FO} = \frac{DL}{LC} = \frac{DK}{KB} = \frac{PM}{MB} = \frac{QN}{NC} \quad (2).$$

From (1) and (2), we get $M \equiv M', N \equiv N' \Longrightarrow \odot(AMN)$ touches ω at S.

Wolstenholme

#5 Mar 26, 2015, 6:02 am

Here is a length bash solution I found - originally I tried to find various lengths as to use Casey's Theorem, however when I noticed that $MK + NL = MN$ the solution fell into place.

For the remainder of the proof, let $a = BC, b = CA, c = AB,$ and R be the circumradius of $\triangle ABC.$

Lemma 1: $MK + NL = MN.$

Proof: Since A, G, D are collinear and $OD \parallel AE$ we find that $\frac{AE}{FD} = \frac{AG}{GD} = 2.$

Now let X projection from A onto $BC.$ Looking at the Power of X with respect to the circle with diameter $BC,$ we find that $EX = \sqrt{BX * CX}.$ Now it is easy to compute that $AX = b\sin C$ and $BX = c\cos B$ and $CX = b\cos C$ so we find that:

$$FD = \frac{AE}{2} = \frac{AX - EX}{2} = \frac{b\sin C - \sqrt{bc\cos B\cos C}}{2}$$

Now it is easy to compute that $OD = R\cos A.$ Therefore we have that $\frac{DK}{FD} = \frac{DB}{OD}$ so we find that:

$$CL = BK = DB - DK = DB - \frac{DB\cdot FD}{OD} = \frac{a}{2} - \frac{a\cdot FD}{2OD} = \frac{a}{2} - \frac{ab\sin C - a\sqrt{bc\cos B\cos C}}{4R\cos A} \qquad (1)$$

Now note that $MK = BK\tan B$ and $NL = BK\tan C$ so $MK + NL = BK(\tan B + \tan C) = \frac{BK\sin A}{\cos B\cos C}.$ Moreover note that $AM = c - MB = c - \frac{BK}{\cos B}$ and similarly $AN = b - \frac{BK}{\cos C}.$ Therefore by the Law of Cosines on $\triangle AMN$ we find that

$$MN^2 = \left(c - \frac{BK}{\cos B}\right)^2 + \left(b - \frac{BK}{\cos C}\right)^2 - 2\left(c - \frac{BK}{\cos B}\right)\left(b - \frac{BK}{\cos C}\right)\cos A =$$

$$a^2 + \left(\frac{2b\cos A\cos C + 2c\cos A\cos C - 2b\cos B - 2c\cos C}{\cos B\cos C}\right)BK + \left(\frac{\cos^2 B + \cos^2 C - 2\cos A\cos B\cos C}{\cos^2 B\cos^2 C}\right)BK^2$$

So we want BK to be the positive solution to:

$$a^2 + \left(\frac{2b\cos A\cos C + 2c\cos A\cos C - 2b\cos B - 2c\cos C}{\cos B\cos C}\right)x + \left(\frac{\cos^2 B + \cos^2 C - 2\cos A\cos B\cos C}{\cos^2 B\cos^2 C}\right)x^2 = \left(\frac{1 - \cos^2 A}{\cos^2 B\cos^2 C}\right)x^2$$

Using the facts that $a\cos A + b\cos B - c\cos C = 2c\cos A\cos B$ (and cyclic) and $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$ we have that this quadratic simplifies to:

$$(4\cos A)\,x^2 + (4a\cos B\cos C)\,x - a^2\cos B\cos C = 0$$

Which by the quadratic formula and the well-known identities $AX = 2R(\cos A + \cos B\cos C)$ has solution:

$$\frac{-a\cos B\cos C + a\sqrt{\cos B\cos C(\cos A + \cos B\cos C)}}{2\cos A} =$$

$$\frac{a(A2R\cos A - AX) + a\sqrt{2R\cos B\cos C\cdot AX}}{4R\cos A} = BK$$

So the lemma is proven

So the lemma is proven.

Now, returning to the original problem, let Y be the point on MN such that $MY = MK$ and $NY = NL$. Let the rotation centered at M that takes K to Y also take B to the point Z. Note that the rotation centered at N that takes L to Y also takes C to Z.

Lemma 2: Quadrilateral $AMZN$ is cyclic.

Proof:

Note that $\angle MZN = \angle MZY + \angle NZY = \angle B + \angle C = 180 - \angle A$ so the lemma is proven.

Lemma 3: $Z \in \omega$.

Proof:

$\angle BZC = 360 - \angle BZM - \angle MZN - \angle CZN = 360 - \left(\frac{180 - \angle BMZ}{2}\right) - (180 - \angle A) - \left(\frac{180 - \angle CNZ}{2}\right) = \angle A + \frac{\angle ANZ + \angle AMZ}{2} = \angle A + 90$
so the lemma is proven.

Lemmas 2 and 3 indicate that Z is one of the intersections between ω and the circumcircle of $\triangle AMN$. Letting R be the center of the circumcircle of $\triangle AMN$ and S be the center of ω it suffices to show that R, Z, S are collinear. But $\angle RZM + \angle MZB + \angle BZS = (90 - \angle ZNM) + \angle MBZ + (90 - \angle ZCB) = \angle C + (\angle B - \angle ZBC) + (90 - \angle ZCB) = \angle B + \angle C + \angle A = 180$ so we are done.

Phew!

Phun_TZ

#6 Apr 8, 2015, 4:12 am

Just wanna point out that verifying $MK + NL = KL$ can be less computational. It seems to me that once you've found the relation, it's fairly easy to proceed to what we seek.AThe main reason is that one may find it intuitive to rotate MK around M and NL around N until they meet on MN, and see what the intersection point gets you from there. However, speculating it first-handed is not so obvious, at least not to me anyway.
Now we shall prove it!
One can see that $(MK - NL)^2 + KL^2 = MN^2$, due to Pythagoras' Theorem.So it suffices to show that

$$KL^2 = 4MK * NL.$$

Clearly, $BK = LC$ and $KD = DL$. So for our convenience, let'es denote them by x and y respectively. Now we have $MK = x * tanB$ and $NL = x * tanC$, and obviously $KL = 2y$.
Hence, our goal has now become showing that

$$y^2 = x^2 * tan(B)tan(C).$$

Let X be the foot of the alitude from A on BC. Then it's easy to see that $BX * XC = EX^2$.Now one can easily obtain that $BX = AX/tan(B), XC = AX/tan(C)$.Therefore, $\frac{AX^2}{tanB * tanC} = EX^2$, which implies $tanB * tanC = (AX/EX)^2$.
Again, The result we seek has now mutated to proving

$$AX : EX = y : x.$$

This seems simple enough to prove synthetically.I ,however, still have to rely on some more trigonometrical notations, although it's still short and concise.
Since $FD = AE/2$, we have that $y = DL = \frac{1}{2} * AE * tan(A)$,and then $x = \frac{1}{2} * (a - AE * tanA)$.Moreover, we have $EX = AX - AE$.Combining all these, we obtain the ultimate relation of what we have to prove

$$\frac{AX}{AX - AE} = \frac{AE * tan(A)}{a - AE * tan(A)}.$$

Now Multiplying the denominators up, rearranging them a bit, we have that

$$AX * a = AE * tanA * (2AX - AE).$$

Observe that $AX * a = AC * ABsinA = 2 * [ABC]$ thus

Observe that $AX * a = AC * ABsinA = 2 * [ABC]$, thus
$AX * a = AC * ABcosA * tanA = AC * AH_b * tan(A)$,where H_b is obviously the foot of altitude from B on CA.Hence, we finally obtain

$$AC * AH_b = AE * (2h_a - AE).$$

However, this is merely just the power of A wrt. the circle passing through B, C,centred at D.

This post has been edited 2 times. Last edited by Phun_TZ, Apr 8, 2015, 4:23 am

drmzjoseph

#8 Apr 12, 2015, 11:49 am • 5

Let H orthocenter of $\triangle ABC$, and $Y \equiv BH \cap AC, X \equiv OD \cap AC$
Now $\frac{AE}{EH} = \frac{DF}{FO} = \frac{DL}{LC} = \frac{XN}{NC}$, then $\triangle XDC \cup N \sim \triangle AYH \cup E \Rightarrow \angle HYE = \angle NDC = \angle ECB$ analogously $\angle MDB = \angle EBC \Rightarrow MDN = 90°$, if $P \equiv MN \cap OD \Rightarrow MP = NP = PD$ (Because $KD = DL$) Then $\angle DNM = \angle MDB$
Let Z the miquel point of D, N, M WRT $\triangle ABC$ (i.e. $Z \equiv \odot(BMD) \cap \odot(MAN) \cap \odot(NDC)$)
$\angle BZC = \angle BMD + \angle CND = 90° + \angle BAC$, easy see that $Z \in \omega$,
$\angle BZM = \angle MND = \angle MNZ + \angle DNZ = \angle MNZ + \angle ZCB$
Notice $Z \in \omega, Z \in \odot(AMN)$ and $\angle BZM = \angle MNZ + \angle ZCB \Rightarrow \omega$ is tangent to $\odot(AMN)$

sceptre

#9 May 26, 2015, 2:09 pm

> *Luis González wrote:*
>
> Let H be the orthocenter of $\triangle ABC$ and let U be the antipode of A in $(O) \equiv \odot(ABC)$. If BU, CU cut AC, AB at Y, Z, then U is orthocenter of $\triangle AYZ \Longrightarrow \omega$ is the circle with diameter YZ. If OD cuts AB, AC, at P, Q, then clearly $\overline{PBZ} \sim \overline{QCY} \Longrightarrow (O), \odot(AYZ)$ and $\odot(APQ)$ meet at A and the center R of the spiral similarity that swaps $\overline{BZ}$ and $\overline{CY}$. Furthermore, (O) and $\odot(APQ)$ are orthogonal, due to $\angle APQ = 90° - \angle ABC = \angle OAQ$.
>
> AR, BC, YZ are pairwise radical axes of $\omega, (O), \odot(AYZ)$ concurring at their radical center T and AO is the polar of T WRT ω. Thus if AO cuts ω at S (inside of $\triangle ABC$), then TS is tangent of $\omega \Longrightarrow TS^2 = TB \cdot TC = TA \cdot TR$ $\Longrightarrow \odot(ARS)$ is tangent to ω at S. If $\odot(ARS)$ cuts AB, AC at M', N', then the ratio of the powers of M', N', S WRT $\odot(APQ)$ and (O) are equal $\Longrightarrow$
>
> $$\frac{M'A \cdot M'P}{M'A \cdot M'B} = \frac{N'A \cdot N'Q}{N'A \cdot N'C} = \frac{SA^2}{SA \cdot SU} \Longrightarrow \frac{PM'}{M'B} = \frac{QN'}{N'C} = \frac{AS}{SU} \quad (1).$$
>
> But on the other hand, keeping in mind that $\triangle ABC \cup E \sim \triangle AYZ \cup S$, we have
>
> $$\frac{AS}{SU} = \frac{AE}{EH} = \frac{DF}{FO} = \frac{DL}{LC} = \frac{DK}{KB} = \frac{PM}{MB} = \frac{QN}{NC} \quad (2).$$
>
> From (1) and (2), we get $M \equiv M', N \equiv N' \Longrightarrow \odot(AMN)$ touches ω at S.

Why the ratio of power of M',N',S (APQ),(O) equal

Quick Reply

High School Olympiads

Prove perpendicular

Locked

thanhnam2902

#1 Mar 19, 2015, 8:16 am

Let (O) is the circumscribed circle of ABC triangle. The tangent of the circle (O) at A meet BC at point D. Let DO meet AB, AC at E, F respectively. let M, N are the midpoints of AB, AC respectively. Prove that EN, EM, AO are concurent.

Luis González

#2 Mar 19, 2015, 10:15 am

Posted before at http://www.artofproblemsolving.com/community/c6h603937

High School Olympiads

The lines are concurrence

yumeidesu

#1 Aug 26, 2014, 2:43 pm

Let ABC be a triangle inscribed in circle (O) and $BA=BC.$ The tangent line of (O) at A cut the BC at D. Line DO cut AB, AC at E, F respectively. The midpoint of AB is M, the midpoint of AC is N. Prove that AO, EN, MF are concurrent.

Luis González

#2 Aug 29, 2014, 5:13 am • 1

The object lines concur for any $\triangle ABC;$ condition $BA=BC$ is unnecessary.

Let L be the projection of A on $OD,$ lying on circle $\odot(OMAN)$ with diameter $\overline{OA}.$ AL is polar of D WRT $(O) \Longrightarrow$ $A(M,N,L,D)=O(M,N,L,A)=-1.$ Hence if $U \equiv MN \cap OA$ and $P \equiv EF \cap MN,$ then $O(M,N,L,A)=-1$ yields $(M,N,P,U)=-1,$ therefore in the complete quadrangle $EFMN,$ we get that EN, FM and $AU \equiv AO$ concur.

High School Olympiads

Property of Kariya point

Source: Well Known

TelvCohl
#1 Nov 2, 2014, 5:45 am

Let I be the incenter of $\triangle ABC$.
Let D, E, F be the tangent point of (I) with BC, CA, AB, respectively .
Let X, X' be two point on ID satisfy $IX = IX'$ (X, D are on the same side of I) .
Let Y, Y' be two point on IE satisfy $IY = IY'$ (Y, E are on the same side of I) .
Let Z, Z' be two point on IF satisfy $IZ = IZ'$ (Z, F are on the same side of I).
Let $T = AX \cap BY \cap CZ, T' = AX' \cap BY' \cap CZ'$.

Prove that TT' pass through the orthocenter of $\triangle DEF$ ($X(65)$ in **ETC**)

Luis González
#2 Nov 2, 2014, 11:29 am • 3

As the points X, Y, Z vary, they induce an affine homography between $ID, IE, IF \Longrightarrow T \equiv BY \cap CZ$ is on a fixed conic $\mathcal{F}$ through B, C. Evaluating the cases when $\{I \equiv X \equiv Y \equiv Z\}, \{X \equiv D, Y \equiv E, Z \equiv F\}$ and when X, Y, Z go to infinity, we deduce that $\mathcal{F}$ passes through B, C, I, the orthocenter H and the Gergonne point of $\triangle ABC$, i.e. the Feuerbach circum-hyperbola of $\triangle ABC$ and similarly X' is on $\mathcal{F}$. Since $AX \mapsto AX'$ is an involution $\Longrightarrow T \mapsto T'$ is an involutive homography fixing $\mathcal{F} \Longrightarrow TT'$ go through the pole P of the involution. As the fixed points are I and H, then P is the pole of IH WRT $\mathcal{F}$. So it suffices to show that P coincides with the orthocenter of $\triangle DEF$.

Let $T \equiv X_{65}$ be the orthocenter of $\triangle DEF$ and let DT, ET, FT cut (I, r) again at U, V, W (reflections of T on EF,FD,DE). $(N, \frac{1}{2}R)$ is 9-point circle of $\triangle ABC$ tangent to (I) at the Feuerbach point Fe; Poncelet point of $ABCI$ and therefore center of $\mathcal{F}$. If L, J are the midpoints of HA, HI, we clearly have $IU \parallel NL \Longrightarrow LU$ goes through exsimilicenter Fe of $(I) \sim (N)$ and as $\triangle UVW \cup T$ is homothetic to $\triangle ABC \cup I$, then

$$\frac{TU}{IA} = \frac{r}{R} \Longrightarrow \frac{TU}{JL} = \frac{r}{\frac{1}{2}R} = \frac{FeU}{FeL}.$$

Since $JL \parallel TU$ (both perpendicular to EF), then it follows that Fe, T, J are collinear $\Longrightarrow FeTJ$ and IH have conjugate directions WRT $\mathcal{F}$. Since OIT is tangent of $\mathcal{F}$ at I (fixed point of the isogonal conjugation), then we deduce that T is the pole of IH WRT $\mathcal{F} \Longrightarrow P \equiv T$.

TelvCohl
#3 Jan 21, 2015, 12:47 am

Here is another way to prove the pole of IH WRT Feuerbach hyperbola $\mathcal{F}$ is orthocenter T of the intouch triangle :

According to the generalization at Schwatt's lines , we only have to prove T is the cross point of $\{I, H\}$ WRT $\triangle ABC$.

Theorem:

Let I be the incenter of $\triangle ABC$.
Let D, E, F be the projection of A, B, C on BC, CA, AB, respectively .
Let $D^* = AI \cap EF, E^* = BI \cap FD, F^* = CI \cap DE$ and T be the orthocenter of the intouch triangle of $\triangle ABC$.

Then DD^*, EE^*, FF^* are concurrent at T .

Proof:

Let $\triangle XYZ$ be the intouch triangle of $\triangle ABC$

Let $\triangle XYZ$ be the intouch triangle of $\triangle ABC$.
Let I^* be the reflection of I in YZ and $S = AI \cap BC, P = XT \cap AD$.

Easy to see I^* is the orthocenter of $\triangle AYZ$.

From $II^* = XT, II^* \parallel XT \Longrightarrow I^*IXT$ is a parallelogram ,
so combine with $\triangle AEF \cup D^* \sim \triangle ABC \cup S \Longrightarrow \frac{AD^*}{AS} = \frac{EF}{BC} = \cos \angle BAC = \frac{AI^*}{AI} = \frac{PT}{PX}$.
i.e. $T \in DD^*$

Similarly, we can prove $T \in EE^*$ and $T \in FF^*$,so DD^*, EE^*, FF^* are concurrent at T .

Done :)

Remark:

Use same idea we can prove following property of Kiepert point **:**

Let D be a point out of $\triangle ABC$ such that $\angle DBC = \angle DCB = \theta$.
Let D^* be a point in $\triangle ABC$ such that $\angle D^*BC = \angle D^*CB = \theta$.
Define E, E^* for CA and F, F^* for AB similarly .
Let $K_\theta = AD \cap BE \cap CF$ and $K_{-\theta} = AD^* \cap BE^* \cap CF^*$.

Then $K_\theta K_{-\theta}$ pass through the Symmedian point K of $\triangle ABC$.

Since $K_\theta K_{-\theta}$ pass through the pole of GH WRT the Kiepert hyperbola $\mathcal{K}$ of $\triangle ABC$ (G, H is the centroid, orthocenter of $\triangle ABC$, respectively .) , so we only have to prove K is the cross point of $\{G, H\}$ WRT $\triangle ABC$ which followed from the fact that K is the isotomcomplement of H WRT $\triangle ABC$:) .

This post has been edited 1 time. Last edited by TelvCohl, Aug 27, 2015, 1:06 pm

Luis González

#4 Jan 21, 2015, 2:01 am

> *TelvCohl wrote:*
> Use same idea we can prove following property of Kiepert point :
>
> Let D be a point out of $\triangle ABC$ such that $\angle DBC = \angle DCB = \theta$.
> Let D^* be a point in $\triangle ABC$ such that $\angle D^*BC = \angle D^*CB = \theta$.
> Define E, E^* for CA and F, F^* for AB similarly.
> Let $K_\theta = AD \cap BE \cap CF$ and $K_{-\theta} = AD^* \cap BE^* \cap CF^*$.
>
> Then $K_\theta K_{-\theta}$ pass through the Symmedian point K of $\triangle ABC$.

Remark: there is also a nice way to prove this without resorting to the Kiepert hyperbola. Letting $\triangle A'B'C'$ be the tangential triangle of $\triangle ABC$, just note that $\angle ACE^* = \angle ACE = \angle ABF = \angle ABF^*$ and $\angle ACB' = \angle ABC$, $\angle ABC' = \angle ACB \Longrightarrow C(E, E^*, B, B') = B(F, F^*, C, C') \Longrightarrow$ $B(E, E^*, C, B') = C(E, E^*, B, B') = B(F, F^*, C, C') = C(F, F^*, B, C') \Longrightarrow K_\theta \equiv BE \cap CF$, $K_{-\theta} \equiv BE^* \cap CF^*$ and $K \equiv BB' \cap CC'$ are collinear.

jayme

#5 May 7, 2016, 2:18 pm

Dear Mathlinkers,
I come back to this initial problem...and research a proof without conic of homography...is this possible?

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

A cute geometry problem

Reply

izaya-kun

#1 Mar 18, 2015, 8:13 am

Let Br_1 and Br_2 be the two Brokard points of $\triangle ABC$.
Let point O be the circumcenter of $\triangle ABC$ and point L the lemoine point of $\triangle ABC$.
Prove that the points Br_1, Br_2, O and L all lie on a circle.

IDMasterz

#2 Mar 18, 2015, 9:06 am

this is very well-known, and can be proved simply using projective transformations.

Luis González

#3 Mar 18, 2015, 9:42 am

Indeed, it's a well-known result and I think Darij submitted a nice proof somewhere in the forum. Here is my solution that I found some time ago. Just for convenience we'll denote K the symmedian (Lemoine) point of $\triangle ABC$ and Ω_1, Ω_2 its 1st and 2nd Brocard points.

Let $\triangle XYZ$ be the tangential triangle of $\triangle ABC$ and let B' and C' be the 2nd intersections of BY and CZ with the circumcircle (O). V and W are the midpoints of BB' and CC' (projections of O on BB' and CC'). Since V is on circle with diameter OY, it follows that $\angle VYC \equiv \angle BYC = \angle VAC \Longrightarrow$ $\angle BAV = \angle BAC - \angle BYC = \angle BXC - \angle BYC = \angle YBC \Longrightarrow \odot(BAV)$ touches BC and similarly $\odot(BWC)$ touches AC. Hence $\{B, \Omega_1\} \equiv \odot(BAV) \cap \odot(BWC) \Longrightarrow$ $\angle V\Omega_1 W = \angle WCB + \angle VAB = \angle WCB + \angle VBC = \angle VKW \Longrightarrow O, K, V, W, \Omega_1$ are concyclic, i.e. Ω_1 is on Brocard circle with diameter $\overline{OK}$ and by similar reasoning Ω_2 is also on Brocard circle.

Furthermore, $K\Omega_1 O\Omega_2$ is cyclic kite because $\angle VO\Omega_1 = \angle KV\Omega_1 = \angle BA\Omega_1 = \omega$, where ω is the Brocard angle. Thus $\angle KO\Omega_1 = \angle KO\Omega_2 = \omega \Longrightarrow \Omega_1, \Omega_2$ are symmetric WRT OK.

TelvCohl

#5 Mar 18, 2015, 4:31 pm • 1

This circle is known as Brocard circle ,
the proof base on Second Lemoine circle can be found in **Advanced Euclidean Geometry, Roger A. Johnson** .

Here is another approach :

For convenience I change the notation $Br_1, Br_2 \longrightarrow T_1, T_2$.

Let ω be the Brocard angle of $\triangle ABC$.
Let A^*, B^*, C^* be the antipode of A, B, C in $\odot(O)$, respectively .
Let $D = BA^* \cap CB^*, E = CB^* \cap AC^*, F = AC^* \cap BA^*$.

From $Rt\triangle AA^*C \sim Rt\triangle BDC \Longrightarrow \triangle CAB \sim \triangle CA^*D \sim \triangle BB^*D$,
so from $BO = OB^*$ we get DO is D-symmedian of $\triangle DEF$.

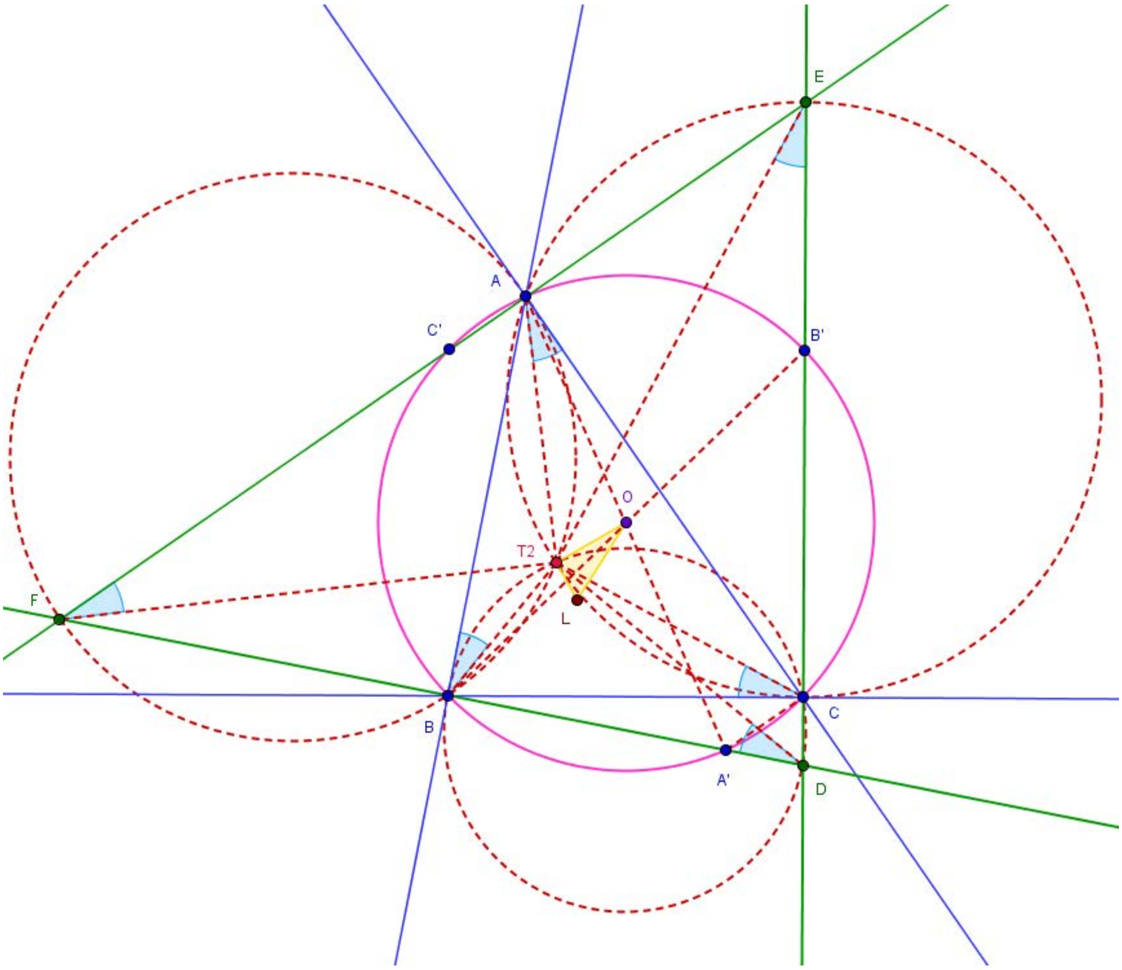
Similarly we can prove EO is E-symmedian of $\triangle DEF \Longrightarrow O$ is the Symmedian point of $\triangle DEF$.

Since T_2 lie on $\odot(ABF), \odot(BCD), \odot(CAE)$,
so from $\triangle ABC \sim \triangle FDE \Longrightarrow T_2$ is 2nd Brocard point of $\triangle DEF$,
hence we get $\triangle T_2FA \sim \triangle T_2OK \Longrightarrow \angle LT_2O = 90^\circ, \angle T_2OL = \omega$.

Similarly we can prove $\angle LT_1O = 90^\circ$, $\angle T_1OL = \omega$,
so we get T_1, T_2, O, L lie on a circle with diameter OL and $\angle T_1OL = \angle T_2OL = \omega$.

Q.E.D

Attachments:



Quick Reply

High School Math

Find Locus

Reply

Headhunter

#1 Oct 13, 2010, 1:39 pm

Hello.

From any point P on the base of a triangle, perpendiculars PX and PY are drawn to the sides.
Find the locus of the middle point of XY

Luis González

#2 Oct 25, 2010, 2:14 am

Let us define the cartesian reference where sideline BC coincides with y-axis and $P(0,\varrho)$ is a varible point on it. Sidelines AB, AC are given by the equations $ax+by+c=0$ and $dx+ey+f=0.$ Thereby, coordinates of the orthogonal projections $P(0,\varrho)$ onto lines AB, AC are given by

$$X\left(\frac{-ac-ab\varrho}{a^2+b^2},\frac{a^2\varrho-bc}{a^2+b^2}\right) \ \ , \ Y\left(\frac{-df-de\varrho}{d^2+e^2},\frac{d^2\varrho-ef}{d^2+e^2}\right)$$

Hence, coordinates of the midpoint $M(\overline{x},\overline{y})$ of $\overline{XY}$ are given by:

$$\overline{x}=-\frac{\varrho}{2}\left(\frac{ab}{a^2+b^2}+\frac{de}{d^2+e^2}\right)-\frac{1}{2}\left(\frac{ac}{a^2+b^2}+\frac{df}{d^2+e^2}\right)$$

$$\overline{y}=\frac{\varrho}{2}\left(\frac{a^2}{a^2+b^2}+\frac{d^2}{d^2+e^2}\right)-\frac{1}{2}\left(\frac{bc}{a^2+b^2}+\frac{ef}{d^2+e^2}\right).$$

This represents a parametric equation of a line $\ell.$ Hence, locus of midpoints M of $\overline{XY}$ is the line ℓ passing through the midpoints of the altitudes issuing from $B, C.$

Luis González

#4 Mar 18, 2015, 4:31 am

For more solutions, see http://www.artofproblemsolving.com/community/c6h303987

Quick Reply

High School Olympiads

Locus of the midpoint of EF

Source: proposed by my friend

shoki
#1 Oct 3, 2009, 8:36 pm

Let ΔABC be a triangle and P be a point on BC. Let E, F be the projections of P on AC, AB, respectively ($E \in AC, F \in AB$). Find the locus of the midpoint of EF.

Mathias_DK
#2 Oct 3, 2009, 8:54 pm

It's going to be a line, and you can find it by putting $P = B$ and $P = C$, and then drawing a line between the two different midpoints of EF.

TelvCohl
#3 Nov 24, 2014, 8:21 am • 2

My solution:

Let $S(\triangle)$ be the area of $\triangle$.
Let D be the midpoint of EF .
Let Y, Z be the projection of B, C on CA, AB, respectively .

Since
$$S(\triangle DYC) + S(\triangle DBZ) = \frac{1}{2} \cdot [S(\triangle FYC) + S(\triangle EBZ)]$$
$$= \frac{PC}{BC} \cdot S(\triangle BYC) + \frac{BP}{BC} \cdot S(\triangle ZYC) + \frac{PC}{BC} \cdot S(\triangle YBZ) + \frac{BP}{BC} \cdot S(\triangle CBZ)$$
$$= \frac{1}{2} \cdot S(BCYZ),$$

so D lie on the Newton line of complete quadrilateral $\{BC, CY, YZ, ZB\}$,
hence the locus of the midpoint of EF is a line passing through the midpoint of BY, CZ .

Luis González
#4 Nov 24, 2014, 8:56 am • 3

Circle $\odot(PEAF)$ with diameter $\overline{PA}$ cuts BC again at the foot D of the A-altitude. Angles $\angle DEF = \angle DAB$ and $\angle DFE = \angle DAC$ are constant $\Longrightarrow$ all $\triangle DEF$ are similar. Thus if M is the midpoint of $\overline{EF}$, then $\angle FDM$ is constant and $\frac{DF}{DM} = k$ is constant $\Longrightarrow$ locus of M is the line ℓ image of AB under spiral similarity with center D, rotational angle $\angle(DM, DF)$ and coefficient k. Making $P \equiv B$ and $P \equiv C$, then ℓ is just the line through the midpoints of the altitudes issuing from B and C.

High School Olympiads

Prove AP is perpendicular to HQ

Reply

Source: Own

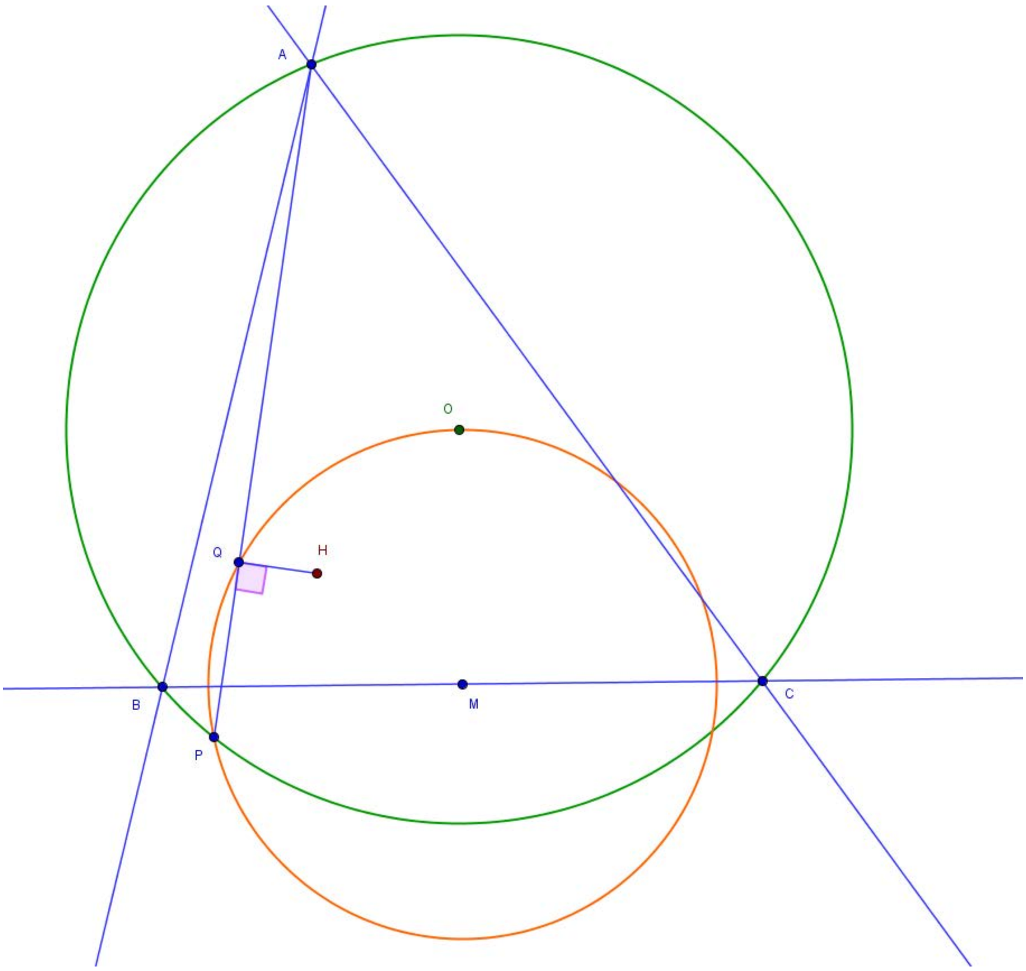
TelvCohl

#1 Mar 16, 2015, 6:14 pm • 1

Let M be the midpoint of BC .
Let O, H be the circumcenter, orthocenter of $\triangle ABC$, respectively .
Let $P = \odot(O) \cap \odot(M, MO)$ and $Q = AP \cap \odot(M, MO)$ $(Q \neq P)$.

Prove that $AP \perp HQ$.

Attachments:



IDMasterz

#2 Mar 16, 2015, 6:52 pm • 1

Second intersection of (M, MO) with (O) be P'. If $AP' \cap (M, MO) = Q'$, then $QQ' \perp AO$. Since $\angle OPQ = \angle PAO$, if $AO \cap (M, MO) = T$ then $TQ = QA \implies T$ is the reflection of A over QQ'. So, O is the orthocentre of AQQ'. Since $PP' \parallel BC$, we have AH contains the circumcentre O' of $\odot AQQ'$. But, $AO' = MO = \dfrac{AH}{2} \implies H$ is the anti-pode of A in AQQ'.

mineiraojose

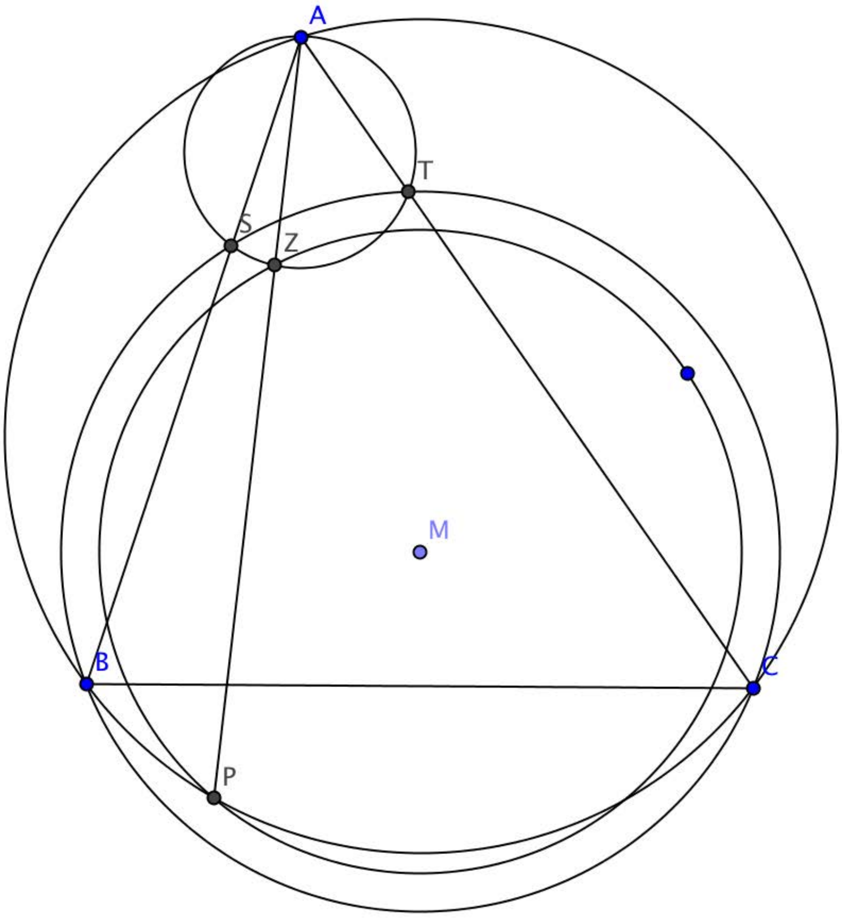
mineiraojose

#4 Mar 18, 2015, 2:01 am • 1

Here is a generalization of the problem :

Let M be a variable point on the perpendicular bisector of BC of $\triangle ABC$ with circumcircle (O). Denote γ a circle with center M and arbitrary radius that intersects (O) at P. Let the circle (M, MB) intersect AB, AC at S and T. Let $AP \cap \gamma = Z$. Prove that Z lies on AST.

Attachments:



TelvCohl

#5 Mar 18, 2015, 3:25 am • 1

> *mineiraojose wrote:*
>
> Here is a generalization of the problem :
>
> Let M be a variable point on the perpendicular bisector of BC of $\triangle ABC$ with circumcircle (O). Denote γ a circle with center M and arbitrary radius that intersects (O) at P. Let the circle (M, MB) intersect AB, AC at S and T. Let $AP \cap \gamma = Z$. Prove that Z lies on AST.

Thank you for your nice generalization , here is my solution:

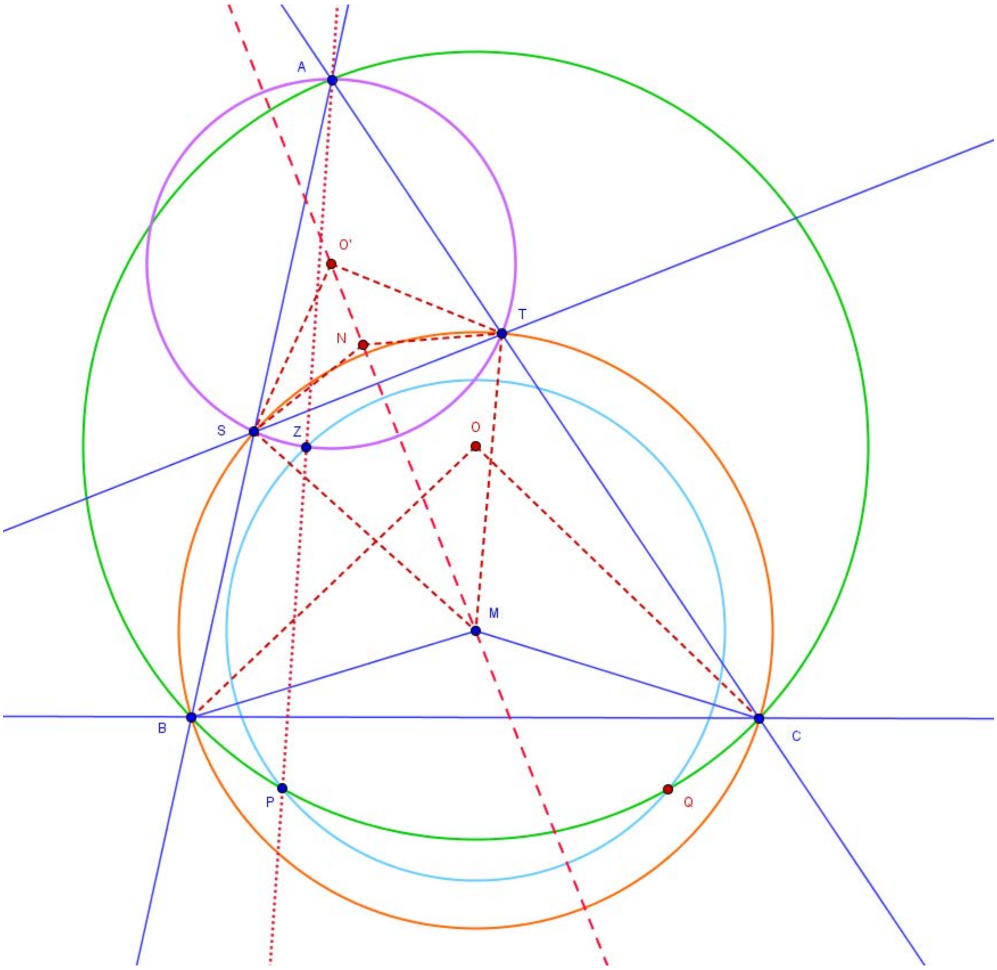
Let Q be intersection of $\odot(ABC)$ with γ ($Q \neq P$) .
Let N be a point such that $\triangle ABC \cup M \sim \triangle ATS \cup N$.
Let O, O^* be the circumcenter of $\triangle ABC, \triangle ATS$, respectively .

Redefine Z as the intersection of $\odot(AST)$ and γ . It suffices to prove A, Z, P are collinear .

Since $\angle TMS = 180^\circ - \angle MST - \angle STM = 180^\circ - (360^\circ - 2\angle MBA - 2\angle MCA - \angle MBC - \angle MCB)$
$= 180^\circ - (2\angle BAC + \angle MBC + \angle MCB) = \angle BMC - \angle BOC = \angle OBM + \angle OCM = \angle O^*TN + \angle O^*SN$
,
so M is the image of N under the inversion WRT $\odot(AST) \Longrightarrow MQ : NZ = MZ : NZ = MS : NS = MC : NS$,
hence we get $\triangle ATS \cup N \cup Z \sim \triangle ABC \cup M \cup Q \Longrightarrow \angle SAZ = \angle CAQ = \angle BAP$. i.e. A, Z, P are collinear

Q.E.D

Attachments:



Luis González

#7 Mar 18, 2015, 3:41 am • 1

Another proof to mineiraojose's generalization:

Let Z^* be the second intersection of AP with $\odot(AST)$ and let D, E, F be the midpoints of PZ^*, BS, CT, resp. A, D, E, F lie on midcircle of (O) and $\odot(AST)$, but $AEMF$ is cyclic due to the right angles at $E, F \Longrightarrow \angle ADM = \angle AEM = 90^\circ \Longrightarrow MD$ is perpendicular bisector of $PZ^* \Longrightarrow MP = MZ^* \Longrightarrow Z \equiv Z^*$.

Quick Reply

High School Olympiads

20 concyclic points

Reply

Source:

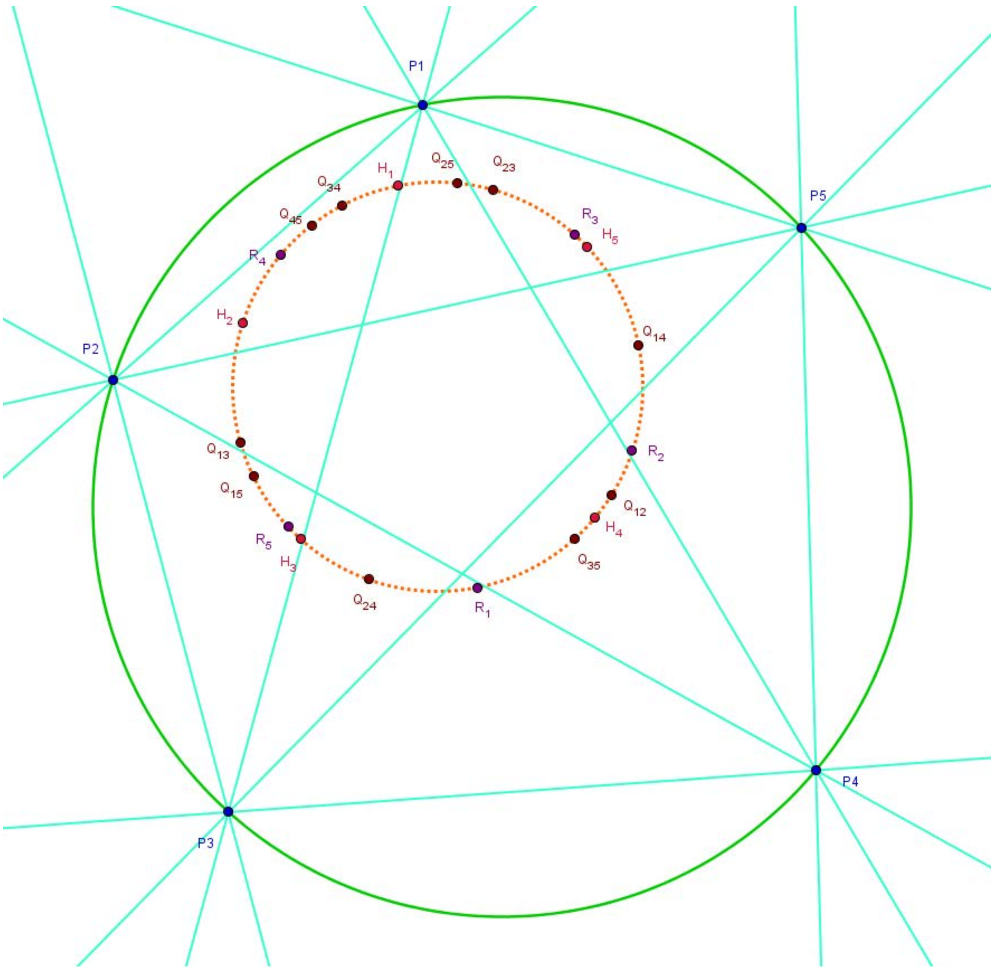
TelvCohl

#1 Mar 17, 2015, 5:47 am • 1

Given five points P_1, P_2, P_3, P_4, P_5 lie on a circle .
Let R_i be the Poncelet point of $\{P_j, P_k, P_m, P_n\}$ ($\{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$) .
Let Q_{ij} be the orthopole of P_iP_j WRT $\triangle P_kP_mP_n$ ($i < j, \{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$) .
Let H_i be the midpoint of the orthocenter of $\triangle P_iP_jP_k$ and $\triangle P_iP_mP_n$ ($\{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$) .

Prove that
$R_1, R_2, R_3, R_4, R_5, H_1, H_2, H_3, H_4, H_5,$
$Q_{12}, Q_{13}, Q_{14}, Q_{15}, Q_{23}, Q_{24}, Q_{25}, Q_{34}, Q_{35}, Q_{45}$
are concyclic

Attachments:



Luis González

#2 Mar 17, 2015, 10:37 am • 3

Let T_{34} denote the orthocenter of $\triangle P_1P_2P_5$. Since R_3 and R_4 are the anticenters of cyclic $P_1P_2P_4P_5$ and $P_1P_2P_3P_5$, then R_3, R_4 are the midpoints of $P_4T_{34}, P_3T_{34} \Longrightarrow R_3R_4 = \frac{1}{2}P_3P_4$ and $\overrightarrow{R_3R_4} \parallel \overrightarrow{P_3P_4}$. Same holds for the remaining pairs of corresponding vertices, thus $R_1R_2R_3R_4R_5$ is image of $P_1P_2P_3P_4P_5$ under a homothety with coefficient $-\frac{1}{2} \Longrightarrow$
$R_1R_2R_3R_4R_5$ is cyclic with a circumcircle ω

$R_1R_2R_3R_4R_5$ is cyclic with a circumcircle ω.

Orthopole Q_{34} of P_3P_4 WRT $\triangle P_1P_2P_5$ is the intersection of its Simson lines with poles P_3, P_4 and the oriented angle between them is half the measure of the arc $P_3P_4 \Longrightarrow \angle(Q_{34}R_3, Q_{34}R_4) = \angle(P_1P_3, P_1P_4) = \angle(R_1R_3, R_1R_4) \Longrightarrow Q_{34} \in \omega$. In the same way, the other 9 orthopoles lie on ω.

Let T_{23} and T_{45} denote the orthocenters of $\triangle P_1P_4P_5$ and $\triangle P_1P_2P_3$, resp. Again R_3, R_4 are the midpoints of P_2T_{23}, P_5T_{45}, therefore $(H_1R_3 \parallel P_2T_{45}) \perp P_1P_3$ and $(H_1R_4 \parallel P_5T_{23}) \perp P_1P_4 \Longrightarrow$
$\angle(H_1R_3, H_1R_4) = (P_1P_3, P_1P_4) = \angle(R_1R_3, R_1R_4) \Longrightarrow H_1 \in \omega$. In the same way H_2, H_3, H_4, H_5 lie on ω.

Hence we conclude that $R_1, R_2, R_3, R_4, R_5, H_1, H_2, H_3, H_4, H_5, Q_{12}, Q_{13}, Q_{14}, Q_{15}, Q_{23}, Q_{24}, Q_{25}, Q_{34}, Q_{35}, Q_{45}$ lie on a same circle ω.

Quick Reply

High School Olympiads

Kazakhstan National Olympiad 2015 Final Round, 11grade, P2

Reply

Source: Kazakhstan National Olympiad 2015 Final Round, 11grade, P2

zhumazhenis

#1 Mar 14, 2015, 7:24 pm

Given convex quadrilateral $ABCD$. K and M are the midpoints of BC and AD respectively. Segments AK and BM intersect at the point N, and the segments KD and CM at the point L. And quadrilateral $KLMN$ is inscribed. Let the circumscribed circles of triangles BNK and AMN second time intersect at the point Q, and circumscribed circles of triangles KLC and DML at the point P. Prove that the areas of quadrilaterals of $KLMN$ and $KPMQ$ are equal.

Luis González

#3 Mar 16, 2015, 9:29 am • 1

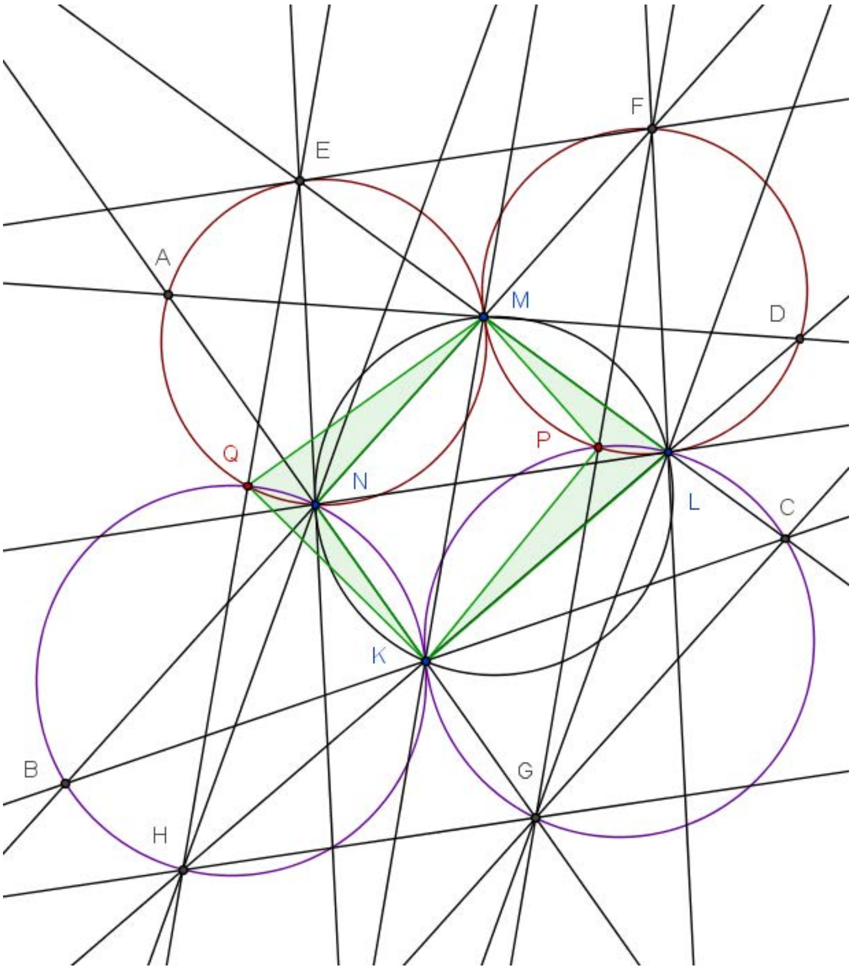
Let ML, MN cut $\odot(AMN), \odot(DML)$ again at E, F and let KN, KL cut $\odot(KLC), \odot(BNK)$ again at G, H. $\angle EMN = \angle NKL = \angle LCG \Longrightarrow BM \parallel CG \Longrightarrow K$ is also midpoint of $GN \Longrightarrow NLGH$ is parallelogram. Similarly $NLFE$ is parallelogram $\Longrightarrow EFGH$ is parallelogram. By Miquel theorem in $\triangle LEH$, it follows that $Q \in EH$ and similarly $P \in FG$. Thus since $HQ \parallel GP, HN \parallel GL \Longrightarrow \angle NHQ = \angle LGP \Longrightarrow NQ = LP$, due to $\odot(BNK) \cong \odot(KLC)$.

Since K, M are midpoints of $NG, NF \Longrightarrow KM \parallel FG \parallel EH$. But $\angle FPL = \angle FML = \angle EMN = \angle EQN \pmod{\pi}$, which means that LP and NQ form the same angle θ with KM. Therefore

$$[PMLK] = \tfrac{1}{2}LP \cdot KM \cdot \sin\theta = \tfrac{1}{2}NQ \cdot KM \cdot \sin\theta = [QMNK],$$

which implies that $[KLMN] = [KPMQ]$, as desired.

Attachments:



Quick Reply

High School Olympiads

Five concurrent lines in a rectangular

Reply

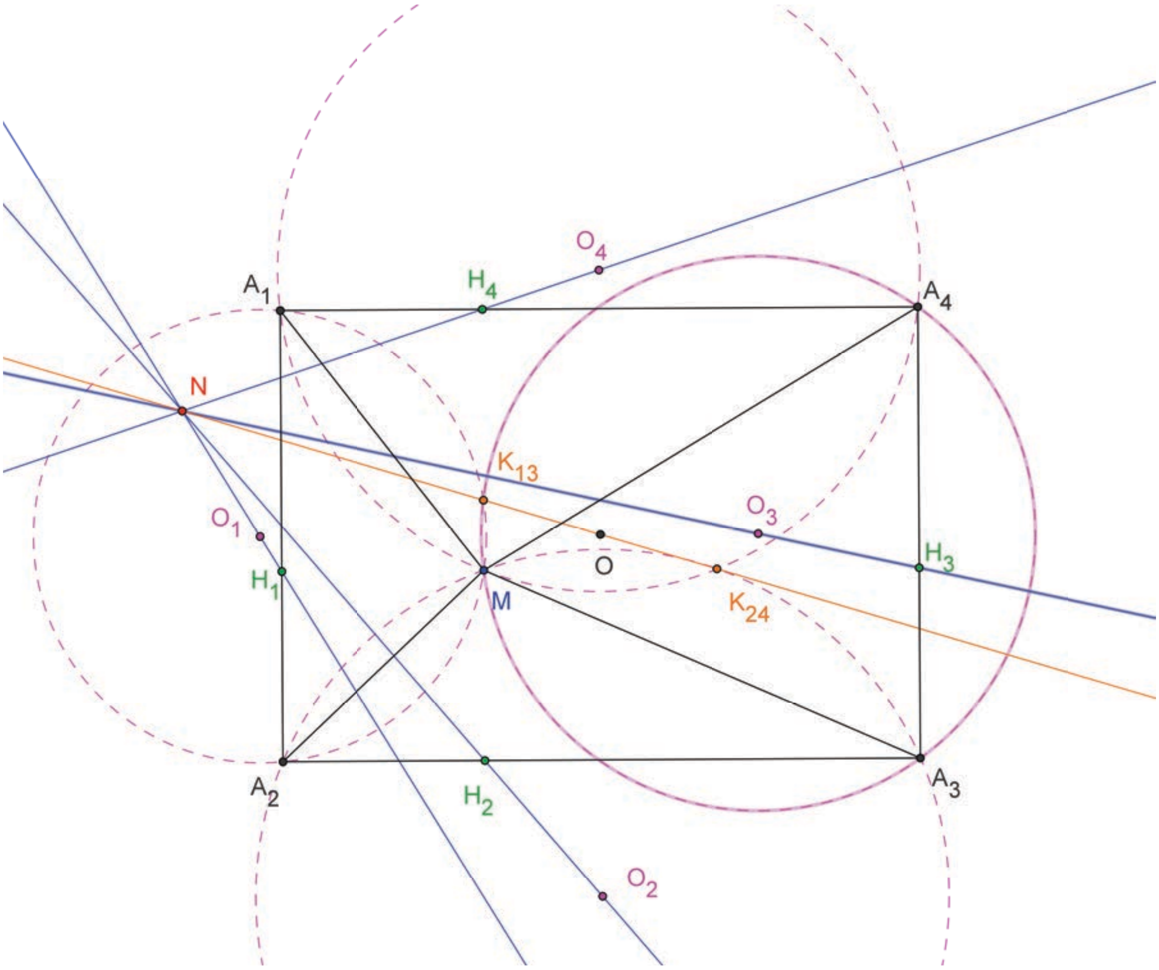
Source: Own

VUThanhTung

#1 Mar 15, 2015, 7:00 am

Given one point M and a rectangular $A_1A_2A_3A_4$ with center O.
H_1, H_2, H_3, H_4 are projections of M on $A_1A_2, A_2A_3, A_3A_4, A_3, A_4A_1$.
O_1, O_2, O_3, O_4 are centers of $(MA_1A_2), (MA_2A_3), (MA_3A_4), (MA_3, A_4A_1)$.
$(O_1) \cap (O_3) = \{M, K_{13}\}$; $(O_2) \cap (O_4) = \{M, K_{24}\}$; $(MA_1A_3) \cap (MA_2A_4) = K, M$
Then:
1. K, O, M are collinear, K_{13}, K_{24}, O are collinear.
2. $O_1H_1, O_2H_2, O_3H_3, O_4H_4, K_{13}K_{24}$ are concurrent.

Attachments:



Luis González

#3 Mar 15, 2015, 12:40 pm • 1

1) Let (O) be the circumcircle of $A_1A_2A_3A_4$. Thus A_1A_3, A_2A_4 are pairwise radical axes of $(O), \odot(MA_1A_3)$, $\odot(MA_2A_4)$ concurring at their radical center $O \Longrightarrow O \in KM$. By symmetry K_{13} and K_{24} are the reflections of M on the perpendicular bisectors of A_1A_2 and A_1A_4, thus their intersection O is the midpoint of $K_{13}K_{24}$, i.e. $O \in K_{13}K_{24}$.

2) O_3O_4, MA_4, H_3H_4 concur at the midpoint X_4 of MA_4 and OO_4 meets H_4K_{24} at its midpoint U. Thus since $UX_4 \parallel H_3K_{24}$, then $\triangle OO_3O_4$ and $\triangle K_{24}H_3H_4$ are perspective through UX_4, thus by Desargues theorem $P \equiv O_3H_3 \cap O_4H_4 \cap OK_{24}K_{13}$. Likewise O_1H_1 and O_2H_2 pass through P.

Quick Reply

High School Math

rhomb-prove

Reply

Pirkuliyev Rovsen

#1 Mar 14, 2015, 1:21 am

Through the vertex C with a rhombus $ABCD$ held line that intersects the segments AB and BD at M and K, respectively. Circle passing through the points A, M, and K, crosses the lines BD and AD at points P and L, respectively. Prove that orthocenter triangle CKP coincides with the center of the circumscribed circle of triangle CML.

Luis González

#2 Mar 15, 2015, 4:05 am • 2

Let H be the second intersection of $\odot(AMK)$ with $AC.$ Since $\angle KCH=\angle KAH=\angle KPH=\angle KMH,$ it follows that $PH \perp CK$ (because of $CH \perp PK$), thus H is orthocenter of $\triangle CKP.$ Furthermore, $\triangle HMC$ is isosceles with legs $HC=HM,$ but since AH bisects $\angle MAL,$ then H is midpoint of the arc MKL of $\odot(AMK) \Longrightarrow HL=HM \Longrightarrow HC=HM=HL \Longrightarrow H$ is circumcenter of $\triangle CML.$

sunken rock

#3 Mar 25, 2015, 12:44 am

Since $KPLM$ is cyclic, CH belongs to a diameter of $\odot(CLM)$. By symmetry, $\angle CPK=\angle KPA$ and $\angle KPA=\angle KMB$, so $BCPM$ is cyclic; BP being bisector of $\angle ABC$, we get $CP=CM$, hence PH is perpendicular bisector of CM, thus H is indeed the circumcenter of $\triangle CLM$.

Best regards,
sunken rock

Quick Reply

High School Olympiads

line pass the fixed point

Reply

phuong
#1 Mar 14, 2015, 7:48 pm

Let ABC ba an acute triagle inscribed circle (O), with B,C are the fixed point on (O), A is movable point on (O). Two squares $ABDE$ and $ACGF$ are located outside the triangle. AF cuts BD at M, AE cuts CG at N. The circles (DMF) and (GNE) cut each other at P and Q. Prove that PQ always pass through the fixed point wherever A runs around the circle (O).

TelvCohl
#3 Mar 14, 2015, 8:44 pm • 1

My solution:

Let $Q^*=DE \cap FG$ and M^*,N^* be the midpoint of AM, AN, respectively .
Let T be a point out of the $\triangle ABC$ such that $\angle TBC=\angle TCB=\angle BAC$.

Since $Q^* \in \odot (DMF)$ and $Q^* \in \odot (GNE)$,
so Q^* is one of the intersection of $\odot (DMF)$ and $\odot (GNE) \Longrightarrow Q^* \equiv Q$.
From $Rt \triangle ABM \sim Rt \triangle ACN \Longrightarrow AE:AF=AB:AC=AM:AN$,
so $AE \cdot AN=AF \cdot AM$. i.e. A lie on the radical axis PQ of $\{ \odot (DMF), \odot (GNE) \}$

Since $\angle APM=\angle QPM=90^{\circ}, \angle APN=\angle QPN=90^{\circ}$,
so we get M,P,N are collinear and $AP \perp MN \Longrightarrow AP \perp M^*N^*$,
hence from my post at here (post #4 lemma) we get T lie on $AP \equiv PQ$.
i.e. PQ pass through a fixed point T when A varies on $\odot (O)$

Q.E.D

Luis González
#4 Mar 14, 2015, 11:11 pm

See the topic 2 squares on acute ABC, angle ACS=angle BCP. PQ is the A-symmedian of $\triangle ABC$, so the fixed point is the intersection of the tangents of (O) at B,C.

Quick Reply

High School Olympiads

2 squares on acute ABC, angle ACS=angle BCP

Source: ARMO 2013, 9th grade, p7

Sayan

#1 May 20, 2013, 6:40 am

Squares $CAKL$ and $CBMN$ are constructed on the sides of acute-angled triangle ABC, outside of the triangle. Line CN intersects line segment AK at X, while line CL intersects line segment BM at Y. Point P, lying inside triangle ABC, is an intersection of the circumcircles of triangles KXN and LYM. Point S is the midpoint of AB. Prove that angle $\angle ACS = \angle BCP$.

Luis González

#2 May 20, 2013, 8:12 am • 2

Let $D \equiv MN \cap KL$, which is clearly the 2nd intersection of $\odot(KXN)$ and $\odot(LYM)$. Since $\angle ACX = \angle BCY = 90^\circ - \angle ACB$, then $CAKL \cup X \sim CBMN \cup Y \Longrightarrow \triangle LCX \sim \triangle NCY \Longrightarrow XYNL$ is cyclic $\Longrightarrow CX \cdot CN = CY \cdot CL \Longrightarrow C$ is on radical axis DP of $\odot(KXN), \odot(LYM)$.

If U is the reflection of A about C, then $\triangle CBU \cong \triangle CNL$ by SAS, because $CB = CN, CU = CA = CL$ and $\angle BCU = \angle NCL = 180^\circ - \angle ACB$. Therefore, $\angle CUB = \angle CLN$, but since $LC \perp CU$, then $CS \parallel BU \perp NL$ $\Longrightarrow CS$ and $CD \equiv CP$ are then the altitude and circumdiameter of $\triangle CNL$ issuing from $C \Longrightarrow$ they are isogonals WRT CN, CL, which in turn are isogonals WRT $CA, CB \Longrightarrow CS, CP$ are isogonals WRT $\angle ACB$.

sunken rock

#3 May 20, 2013, 12:30 pm • 1

As above, since $KL \perp KX, MN \perp NX, KXND$ is cyclic, and so is $LYMD, D \in KL \cap MN$.
Clearly $\Delta CAX \sim \Delta CBY \implies CY \cdot AC = CX \cdot BC$, but $AC = CL, BC = CN$, hence $CX \cdot CN = CY \cdot CL$ and CD is the radical axis of the two circles. As CL is distance from D to AC and CN distance from D to BC we infer that D belongs to the $C-$symmedian of ΔABC and we are done (*any point of symmedian has distances to the adjacent sides proportional to the lengths of the sides*).

Best regards,
sunken rock

NewAlbionAcademy

#4 May 26, 2013, 3:59 am

Solution

junioragd

#5 Jul 21, 2014, 10:24 pm

Let S be the intersection point of KL and MN.Now,first we have that CX.CN=CL.CY(By similariti of triangles CAX and CBY) so we have that XYNL is a cyclic.Now,from angle chase we have that KXNS and YMSL are cyclic,so from the radical axis theorem applyed on this cyclic quadritedrals we have that SP,XN and LY concur at one point,so we obtain that S,C and P are colinear.Now,it is easy to see that LCNS is a cyclic,so after some little angle chase it remains to prove that CS is orthogonal to LN.Now,just reflect B with respect to C to B',and now triangles ACB' and LCN are congruent,and CS//AB',so we are finished

thecmd999

#6 Sep 24, 2014, 2:04 pm

Solution

jlammy

#7 Jan 1, 2015, 2:37 am

This question is a consequence of a result which appears in **jayme**'s paper **here** on page 88. It appears to have first mentioned in **[1]**.

Here is my proof.

Lemma (Mosnat, [2]) *CS is the C-altitude of $\triangle LCN$, the B-flank.*

Proof Let $T = SC \cap LN$. Rotate $\triangle LCN$ 90° about C so that

$$\begin{cases} L \to A \\ N \to N' \\ T \to T'. \end{cases}$$

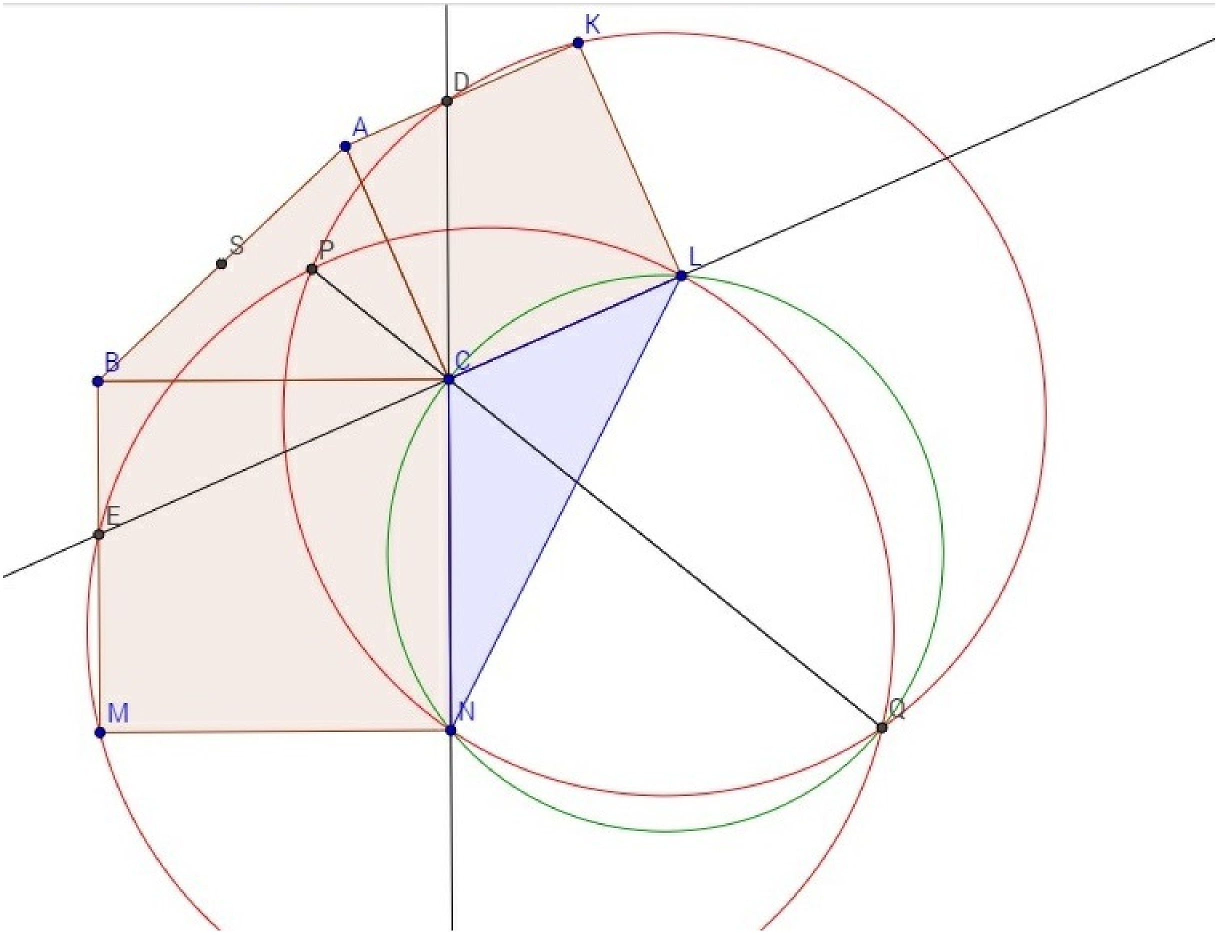
From midpoints, $SC \parallel AN'$, so as $\angle SCT' = 90^\circ$, we have $CT' \perp AN'$, and the result follows. $\square$

We return to the main proof. Let $Q = MN \cap KL$. As before, we have $Q \in (LYM), Q \in (KXN)$, and $C \in \overline{PQ}$. It suffices to show that CS, CP are isogonal: i.e. the circumcenter O of $\triangle LCN$ lies on CP. But this is easy, since $\overline{CQ}$ is a diameter of (LCN), and we are done.

References

[1] E. W. Grebe , "Das geradlinige Dreieck ... Seiten Fallen kann", *Grünerts Archiv 9* (1847)
[2] E. Mosnat, *Problèmes de géométrie analytique*, Vuibert et Nony (1892)

Attachments:



utkarshgupta

#8 Dec 11, 2015, 9:10 pm

We have to prove that CP is the symmedian and we are done.
First observe that C lies on the radical axis of $\odot LMY$ and $\odot KXN$
Let $\odot LMY \cap \odot KXN = P, Q$

It is easy to see that $Q = MN \cap KL$
Thus $C \in PQ$

Now since $KL || AC$,
distance of Q from $AC = KA = AC$,
similarly, distance of Q from BC is BC
Thus, Q lies on the symmedian that is CP is the symmedian.

va2010
#9 Jan 18, 2016, 6:45 am

Hmm, I think this is a record for one of the most unmotivated solutions I've ever posted. Sorry for that.

Extend LK to intersect NM at Q. Observe that $QLYM$ and $QNXK$ are cyclic. Indeed, for $QLYM$ this is just a consequence of the fact that

$$\angle MQK = 180 - \angle NCL = \angle ACB = 90 - \angle YCB = \angle CYB = 180 - \angle MYL$$

, and similar for $QNXK$. So the problem has now been changed to proving that (QLM) and (QKN) intersect on the C-symmedian. Now, the intersection point of these is just the center of spiral similarity which takes MN to LK, which means that it is the center of spiral similarity which takes BC to AC. This is well-known to lie on the symmedian.

This point P has many other interesting properties. For example, it is the fixed point in 2008 USA TST #7 as well as the fixed point in 2008 USAMO #2. The property that it lies on the symmedian can be found here:
http://yufeizhao.com/olympiad/three_geometry_lemmas_sol.pdf

Quick Reply

High School Olympiads

Kazakhstan National Olympiad 2015 Final Round, 9grade, P3

Source: Kazakhstan National Olympiad 2015 Final Round, 9grade, P3

zhumazhenis

#1 Mar 14, 2015, 7:58 pm

Given right trangle with $\angle C = 90^\circ$. Inscribed and escribed circles of ABC are tangent to side BC at points A_1 and A_2. Similarly, we define the points B_1 and B_2. Prove that the segments A_1B_2 and B_1A_2 intersect at an altitude drawn from vertex C of the triangle ABC.

Luis González

#2 Mar 14, 2015, 9:53 pm

B-excircle touches AB at Y. D is the foot of the C-altitude and A_1B_2 cuts CD at X. Using standard triangle notation, we get

$$\frac{YA}{YB} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_2}{B_2A} = \frac{s-c}{s} \cdot \frac{s-b}{s-c} \cdot \frac{s-a}{s-c} = \frac{(s-a)\cdot(s-b)}{s\cdot(s-c)} = 1.$$

Thus, by Menelaus' theorem A_1, B_2, Y are collinear $\Longrightarrow$ $\widehat{CXA_1} = \widehat{AB_2Y} + \widehat{ACD} = \tfrac{1}{2}\widehat{A} + 90^\circ - \widehat{A} = 90^\circ - \tfrac{1}{2}\widehat{A} = 90^\circ - \tfrac{1}{2}\widehat{XCA_1} \Longrightarrow \triangle CA_1X$ is C-isosceles $\Longrightarrow$ $CX = CA_1 = s - c$. Similarly B_1A_2 will cut CD at the same point X.

High School Olympiads

Reflection of P lie on BC iff Gamma pass through O

Reply

Source: 2015 China TST Day 1 Problem 1

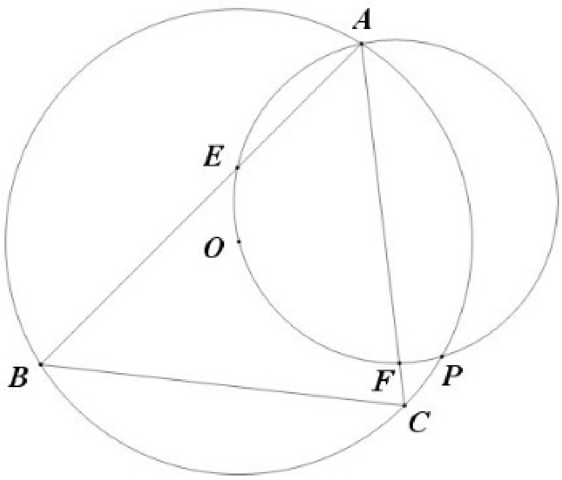
TelvCohl

#1 Mar 13, 2015, 11:15 pm

Let Γ be a circle passing though the vertex A of $\triangle ABC$.
Let Γ cut segment $AC,$ segment AB at $F, E,$ respectively .
Let P be the intersection of Γ and $\odot(ABC)$ $(P \neq A)$.

Prove that the reflection of P in EF lies on BC $\Longleftrightarrow$ Γ pass through the circumcenter O of $\triangle ABC$

Attachments:

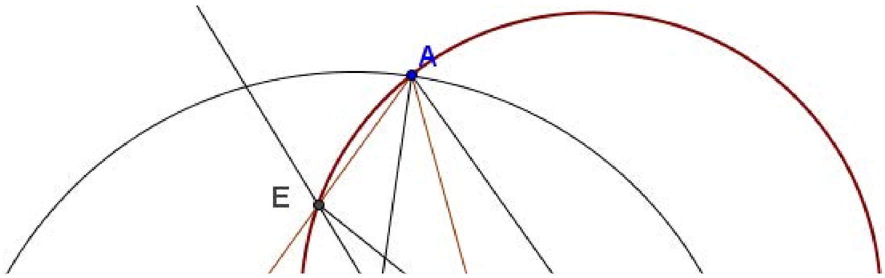


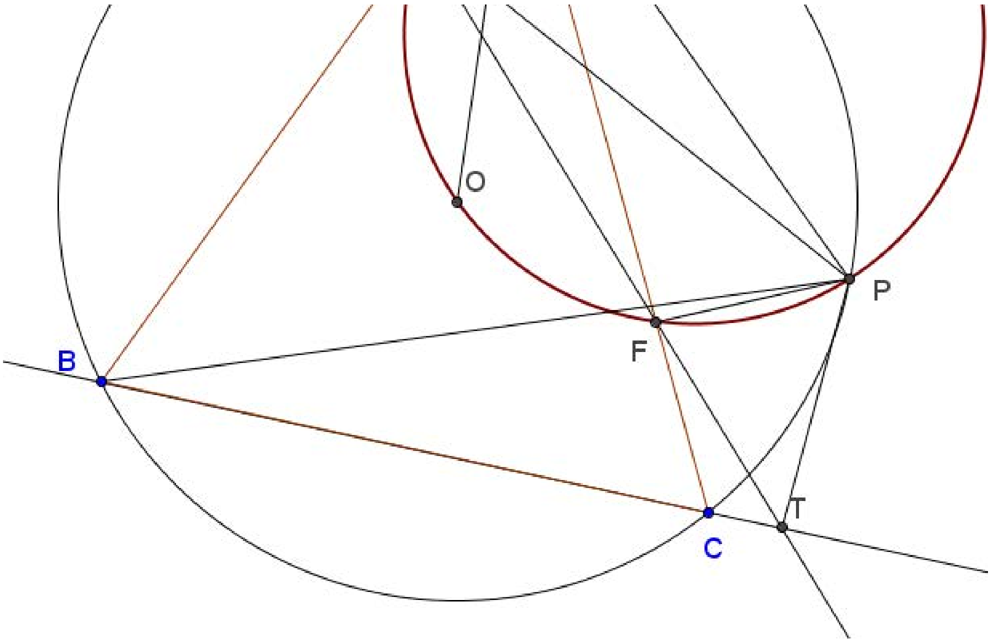
izaya-kun

#2 Mar 14, 2015, 12:19 am

First case
The circle passes through O.
Let the line that passes through EF intersect the ray BC at point T. So we need to prove $\angle ETP = \angle ETB$.
1)$\angle PAF = \angle PEF = \angle PBC$, so the quadrilateral $EPTB$ is cyclic. So we need to prove $EP = EB$.
2)$2\angle ABP = \angle AOP = \angle AEP = \angle ABP + \angle EPB$ so $\angle ABP = \angle EPB$. And we find $EP = EB$, as desired.
Second case
Again let the line that passes through EF intersect the ray BC at point T. The same way we find that $EPTB$ is cyclic.
Obviously we must have $\angle ETP = \angle ETB$. So $EP = EB$.
2)$\angle AEP = 2\angle ABP = \angle AOP$ so point O belongs to the circle that passes through points A, E, F, P. And we are done.
.

Attachments:





Luis González

#4 Mar 14, 2015, 12:33 am

Let $D \equiv EF \cap BC.$ As P is the Miquel point of EF WRT $\triangle ABC,$ then it lies on $\odot(BED)$ and $\odot(CFD).$ If the reflection of P on EF is on $BC,$ then DEF bisects $\angle PDB \Longrightarrow EB = EP$ and $FC = FP \Longrightarrow OE$ and OF are perpendicular bisectors of $\overline{PB}$ and $\overline{PC} \Longrightarrow OE \perp PB, OF \perp PC \Longrightarrow \angle(OE, OF) = \angle(PB, PC) = \angle(AE, AF) \Longrightarrow O \in \Gamma.$

The converse is proved with similar arguments. Letting K denote the center of $\Gamma,$ we get $\angle(PA, PB) = \angle(CA, CB) = \angle(OK, OE),$ but since $OK \perp PA,$ then $OE \perp PB \Longrightarrow EB = EP$ and likewise $FC = FP.$ The result follows.

Tsarik

#5 Mar 14, 2015, 1:49 am

Let $\angle ABP = \alpha, \angle PBC = x$ and $\angle AEP = y$. Notice that $\angle FEP = \angle PEF = \angle PBC = x$. Let T be a point on BC such that $\angle TEF = \angle FEP$. Notice that $ET = EP$ implies that T is the reflection of P in EF and on BC. Using sine theorem in $\triangle BET$ and $\triangle BEP$ we get that $ET = \frac{sin(x+\alpha)EB}{sin(x+y-\alpha)} = \frac{sin(x+\alpha)}{sin(x+y-\alpha)}\frac{sin(y-\alpha)}{sin(\alpha)}EP$. Thus $EP = ET$ iff $\frac{sin(x+\alpha)}{sin(\alpha)} = \frac{sin(x+y-\alpha)}{sin(y-\alpha)}$ or $ctg(\alpha) = ctg(y-\alpha)$ or $y = 2\alpha$, but as $\angle AOP = 2\alpha$ this means that $O \in \Gamma$.

MathPanda1

#6 Mar 14, 2015, 2:57 am

When will the other problems of the China TST be posted? Thanks a lot!

Dukejukem

#7 Mar 14, 2015, 8:06 am

Here is a quick way to prove both directions simultaneously:

That this solution is far easier to find if one is familiar with the Miquel point and spiral similarity. However, neither of these concepts are strictly necessary, and the following solution relies solely on angle chasing: Define $X \equiv EF \cap BC.$ Then using the two cyclic quads in the figure, we find that

$$\measuredangle PEX = \measuredangle PEF = \measuredangle PAF = \measuredangle PAC = \measuredangle PBC = \measuredangle PBX$$

where the angles are directed. Then since $\measuredangle PEX = \measuredangle PBX$, it follows that the points P, E, B, X are concyclic. Now, let P' be the reflection of P in $EF.$ Then $P' \in BC$ if and only if $\measuredangle EXP' = \measuredangle EXB.$ Since reflection preserves angles, this is equivalent to $\measuredangle PXE = \measuredangle EXB.$ Because P, E, B, X are concyclic, angle chasing yields

$$\measuredangle PXE = \measuredangle EXB \iff \measuredangle PBE = \measuredangle EPB$$

$$= -\measuredangle PBE - \measuredangle BEP$$
$$\iff 2\measuredangle PBE = -\measuredangle BEP$$
$$\iff 2\measuredangle PBA = -\measuredangle AEP$$
$$\iff \measuredangle POA = \measuredangle PEA.$$

Then since $\measuredangle POA = \measuredangle PEA$ if and only if P, A, E, O are concyclic, we're done. $\square$

Wolstenholme
#8 Mar 24, 2015, 7:11 am

Case 1: O lies on the circumcircle of $\triangle AEF$.

One immediately notices the Miquel configuration, so it is natural to construct the point $T = EF \cap BC.$ It now suffices to show that $\angle PTE = \angle BTE.$ Now by Miquel's Theorem we have that points B, E, P, T are concyclic - thus, it suffices to show that $BE = PE.$ But $\angle BEP = 180 - \angle AEP = 180 - \angle AOP = 180 - 2\angle EBP$ so $\triangle BEP$ is isosceles as desired.

Case 2: The reflection of P over EF lies on $BC.$

Every step in Case 1 is reversible, so we're done.

dibyo_99
#9 Mar 25, 2015, 10:26 pm

P is the Miquel point of EF wrt $\triangle ABC$. Let $EF \cap BC = X$. Then, $BEPX$ and $CFPX$ are cyclic quadrilaterals. For the 'if' part, we have

$$\angle PXF = \angle PBE = \frac{1}{2}\angle POA = \frac{1}{2}\angle PFA = \frac{1}{2}\angle PXC$$

So, EF bisects $\angle PXC$. Hence, the reflection of P on EF lies on BC. For the 'only if' part,

$$\angle PXF = \frac{1}{2}\angle PXC \implies \angle PBA = \frac{1}{2}\angle PFA \implies \angle POA = \angle PFA$$

So, $AOFP$ is cyclic.

Konigsberg
#10 Jul 2, 2015, 3:15 pm

Did anyone try $\sqrt{bc}$ inversion on this one? It seems cool because the two circles in the problem would map to lines and the circumcenter basically maps to the reflection of A across BC. However, the problem is that EF becomes a circle, while BC becomes a line, and I don't know what's the analogue of reflecting when you have a circle.

Konigsberg
#11 Jul 4, 2015, 9:41 am

Does anyone have a solution with $\sqrt{bc}$ inversion?

utkarshgupta
#12 Dec 13, 2015, 10:36 pm

Too much related to ARO 2001

suli
#13 Dec 14, 2015, 12:48 am • 1

Let's try a two-column proof!

Reflection of P lies on EF
iff: if $OF \cap BC = G$, then OF bisects $\angle PGC$ (Definition of reflection)
iff: $2\angle FGP = \angle BGP$ (Definition of angle bisector)
$FPGC$ is cyclic (P is Miquel point of $BEFC$)
$2\angle FGP = 2\angle FCP$ (Definition of cyclic quadrilateral)

$2\angle FGP = 2\angle FCP$ (Definition of cyclic quadrilateral)
$2\angle FCP = 2\angle ABP$ ($ABCP$ is cyclic; def. cyclic quadrilateral)
$2\angle ABP = \angle AOP$ (Def. circumcenter)
$2\angle FGP = \angle AOP$ (Transitive property)
$\angle BGP = \angle AFP$ (Def. cyclic quadrilateral)
iff: $\angle AOP = \angle AFP$ (Transitive property)
iff: $AOFP$ is cyclic (Def. cyclic quadrilateral)
iff: Γ passes through O (Def. cyclic quadrilateral)

(Wait, did I do this correctly?)

This post has been edited 1 time. Last edited by suli, Dec 14, 2015, 12:48 am

djmathman

#14 Dec 14, 2015, 1:08 am • 3

I'm sorry, but that would be a non-perfect example of a two column proof. Perhaps you might want to work on your proof writing skills?

Attached is an example of clearly A+ work. This is one direction of the proof; the other direction is analogous.

Statements	Reasons
1.	1. Given
2. Let $OF \cap BC = G$	2. Labeling new points
3. OF bisects $\angle PGC$	3. Definition of reflection
4. $2\angle FGP = \angle BGP$	4. Definition of angle bisector
5. P is Miquel point of $BEFC$	5. P is center of spiral similarity sending $\overline{BC}$ to $\overline{EF}$
6. $FPGC$ is cyclic	6. Definition of miquel point
7. $\angle FGP = \angle FCP$	7. Definition of cyclic quadrilateral
8. $2\angle FGP = 2\angle FCP$	8. If $a = b$, then $2a = 2b$
9. $ABCP$ is cyclic	9. P lies on circumcircle of $\triangle ABC$
10. $\angle FCP = \angle ABP$	10. See 7
11. $2\angle FCP = 2\angle ABP$	11. See 8
12. $2\angle ABP = \angle AOP$	12. Definition of circumcenter
13. $2\angle FCP = \angle AOP$	13. Transitive Property from step 10
14. $2\angle FGP = \angle AOP$	14. Transitive Property from step 8
15. $\angle BGP = \angle AFP$	15. Definition of cyclic quadrilateral
16. $2\angle FGP = \angle AFP$	16. Transitive property from step 4
17. $2\angle FCP = \angle AFP$	17. Transitive property from step 7
18. $\angle AOP = \angle AFP$	18. Transitive property from step 13
19. $AOFP$ is cyclic	19. Definition of cyclic quadrilateral
20. Γ passes through O	20. Definition of cyclic quadrilateral

(Interestingly enough, some of the two-column proofs I wrote for my Geometry Honors class were about this long as well....)

EDIT @below: Wait actually in our class my teacher allowed us to leave the first statement blank because it was easier to do that than to write out all the information given in the problem statement

but w/e

This post has been edited 2 times. Last edited by djmathman, Dec 14, 2015, 1:27 am

suli

#15 Dec 14, 2015, 1:16 am • 1

Not A+ work: you lose 100/20 = 5 points for leaving statement 1 blank. Here is A+ work:

EDIT @above: What does **Common Core** say about the first statement?

Statements	Reasons
1. Reflection of P lies on BC	1. Given
2. Let $OF \cap BC = G$	2. Labeling new points
3. OF bisects $\angle PGC$	3. Definition of reflection
4. $2\angle FGP = \angle BGP$	4. Definition of angle bisector
5. P is Miquel point of $BEFC$	5. P is center of spiral similarity

	sending $\overline{BC}$ to $\overline{EF}$
6. $FPGC$ is cyclic	6. Definition of miquel point
7. $\angle FGP = \angle FCP$	7. Definition of cyclic quadrilateral
8. $2\angle FGP = 2\angle FCP$	8. If $a = b$, then $2a = 2b$
9. $ABCP$ is cyclic	9. P lies on circumcircle of $\triangle ABC$
10. $\angle FCP = \angle ABP$	10. See 7
11. $2\angle FCP = 2\angle ABP$	11. See 8
12. $2\angle ABP = \angle AOP$	12. Definition of circumcenter
13. $2\angle FCP = \angle AOP$	13. Transitive Property from step 10
14. $2\angle FGP = \angle AOP$	14. Transitive Property from step 8
15. $\angle BGP = \angle AFP$	15. Definition of cyclic quadrilateral
16. $2\angle FGP = \angle AFP$	16. Transitive property from step 4
17. $2\angle FCP = \angle AFP$	17. Transitive property from step 7
18. $\angle AOP = \angle AFP$	18. Transitive property from step 13
19. $AOFP$ is cyclic	19. Definition of cyclic quadrilateral
20. Γ passes through O	20. Definition of cyclic quadrilateral

This post has been edited 2 times. Last edited by suli, Dec 14, 2015, 1:40 am

mssmath

#16 Dec 14, 2015, 1:33 am

Both of those fail as they fail to do both direction simultaneously. You get a 50.

Quick Reply

High School Olympiads

Squares Constructed on Sides of Quadrilateral

Reply

Source: ILL 1970 - Problem 7.

BigSams

#1 May 24, 2011, 8:52 am • 1

Let $ABCD$ be an arbitrary quadrilateral. Squares with centers M_1, M_2, M_3, M_4 are constructed on AB, BC, CD, DA respectively, all outwards or all inwards. Prove that $M_1M_3 = M_2M_4$ and $M_1M_3 \perp M_2M_4$.

Luis González

#2 May 24, 2011, 9:41 am

This is known as Van Aubel's theorem. The synthetic proof is very figure-dependent, so we assume that ABCD is convex and the squares are constructed outside ABCD. The remaining configurations can be treated with analogous arguments.

Let E be the midpoint of $\overline{AC}$. Then according to the topic Easy geometry, we get that $\triangle EM_1M_2$ and $\triangle EM_3M_4$ are isosceles right triangles with commom apex E. Since $\angle M_1EM_3 = \angle M_2EM_4 = 90^\circ + \angle M_1EM_4$, then it follows that $\triangle M_1EM_3$ and $\triangle M_2EM_4$ are congruent by SAS $\Longrightarrow \overline{M_1M_3} \cong \overline{M_2M_4}$ and $\angle EM_4M_2 = \angle EM_3M_1$ imply that $M_1M_3 \perp M_2M_4$.

Quick Reply

High School Olympiads

Miquel's points problem

Source: Own

VUThanhTung

#1 Mar 12, 2015, 7:23 pm • 1

Part A:
Consider a triangle ABC and 6 points $A_1, A_2 \in BC; B_1, B_2 \in CA; C_1, C_2 \in AB$. Let us define the circles $(a_1) = (AB_1C_1), (a_2) = (AB_2C_2); (b_1) = (BC_1A_1); (b_2) = (BC_1A_1); (c_1) = (CA_1B_1); (c_2) = (CA_1B_1)$.
Let $(a_1) \cap (b_1) \cap (c_1) = P_1$; $(a_2) \cap (b_2) \cap (c_2) = P_2$; $(a_1) \cap (a_2) = \{A, A'\}$; $(b_1) \cap (b_2) = \{B, B'\}$; $(c_1) \cap (c_2) = \{C, C'\}$. Then

1. 3 lines AA', BB', CC' are concurrent at a point X.
2. 6 points A', B', C', P_1, P_2, X lies on a circle (ω).

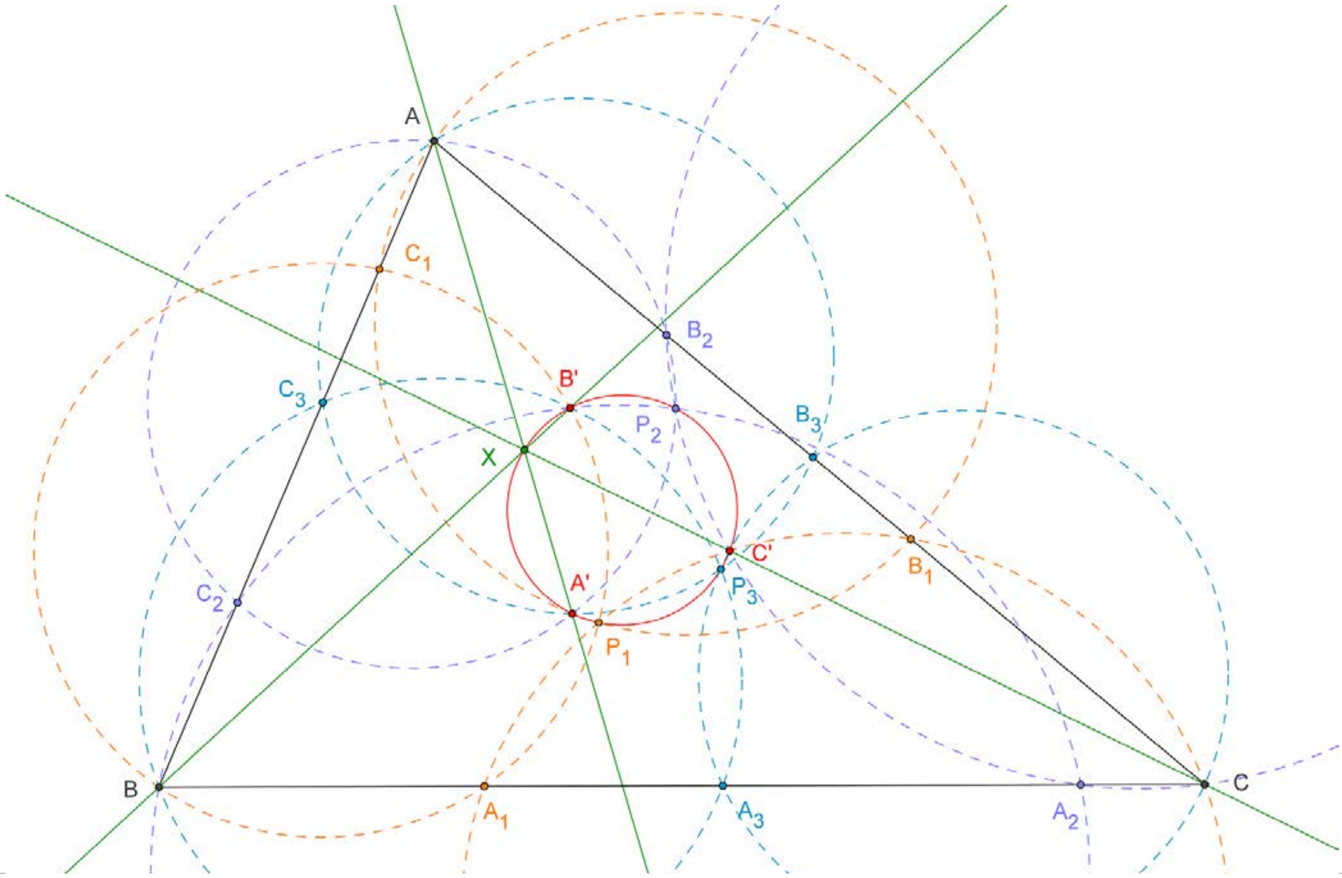
Part B:

Let $k \in R$, $A_3 \in A_1A_2; B_3 \in B_1B_2; C_3 \in C_1C_2$ such that $\dfrac{\overline{A_3A_1}}{\overline{A_3A_2}} = \dfrac{\overline{B_3B_1}}{\overline{B_3B_2}} = \dfrac{\overline{C_3C_1}}{\overline{C_3C_2}} = k$. Let $(a_3) = (AB_3C_3); (b_3) = (BC_3A_3); (c_3) = (CA_3B_3)$ and $(a_3) \cap (b_3) \cap (c_3) = P_3$. Then

1. $A' \in (a_3); B' \in (b_3); C' \in (c_3)$.
2. $P_3 \in (\omega)$ (the locus of P_3 is (ω) when k varies).
3. when $k \to +1$ or -1 then $P_3 \to X$.

The problem can be easily generalized by replacing the triangle ABC by a polygon.

Attachments:



This post has been edited 4 times. Last edited by VUThanhTung, Mar 12, 2015, 7:29 pm

TelvCohl

#2 Mar 12, 2015, 9:33 pm • 1

My solution:

Lemma:

Let $D \in BC, E \in CA, F \in AB$ and M be the Miquel point of $\{D, E, F\}$ WRT $\triangle ABC$.
Let P be a point and $A^* = AP \cap \odot(AEF), B^* = BP \cap \odot(BFD), C^* = CP \cap \odot(CDE)$.

Then M, P, A^*, B^*, C^* are concyclic .

Proof:

From $\angle MB^*P = \angle MDB = \angle MC^*P \Longrightarrow M, P, B^*, C^*$ are concyclic .
Similarly we can prove M, P, C^*, A^* are concyclic $\Longrightarrow M, P, A^*, B^*, C^*$ are concyclic .

Back to the main problem:

Part A

Since $\angle B_2B_1A' = \angle C_2C_1A', \angle B_1B_2A' = \angle C_1C_2A'$,

so $\triangle A'B_1B_2 \sim \triangle A'C_1C_2 \Longrightarrow \frac{\text{dist}(A', AB)}{\text{dist}(A', AC)} = \frac{C_1C_2}{B_1B_2}$.

Similarly we can prove $\frac{\text{dist}(B', BC)}{\text{dist}(B', BA)} = \frac{A_1A_2}{C_1C_2}$ and $\frac{\text{dist}(C', CA)}{\text{dist}(C', CB)} = \frac{B_1B_2}{A_1A_2}$,

so $\frac{\text{dist}(A', AB)}{\text{dist}(A', AC)} \cdot \frac{\text{dist}(B', BC)}{\text{dist}(B', BA)} \cdot \frac{\text{dist}(C', CA)}{\text{dist}(C', CB)} = 1 \Longrightarrow AA', BB', CC'$ are concurrent at X .

From the lemma we get A', B', C', X, P_1 are concyclic and A', B', C', X, P_2 are concyclic ,
so A', B', C', X, P_1, P_2 lie on a circle (ω) .

Part B

Since $\triangle A'B_1B_2 \cup B_3 \sim \triangle A'C_1C_2 \cup C_3$,
so A', B_3, C_3, A are concyclic (Similar discussion for B' and C') .

From the lemma and **Part B** 1. $\Longrightarrow P_3 \in \odot(A'B'C') \equiv (\omega)$.

Q.E.D

VUThanhTung

#3 Mar 12, 2015, 9:59 pm

Thank you, the lemma is interesting. I notice that : let O be the center of the circle $(MPA^*B^*C^*)$ then:
(a) if P moves on a line then the locus of O is a line;
(b) if P moves on a circle then the locus of O is a circle.

TelvCohl

#4 Mar 12, 2015, 10:44 pm

> *VUThanhTung wrote:*
> Thank you, the lemma is interesting. I notice that : let O be the center of the circle $(MPA^*B^*C^*)$ then:
> (a) if P moves on a line then the locus of O is a line;
> (b) if P moves on a circle then the locus of O is a circle.

Notice that all isosceles triangle $\triangle POM$ are similar, so the locus of O is similar to the locus of P :) .

Luis González

#5 Mar 13, 2015, 12:09 am • 1

The concurrency was discussed before at Circumferences and concurrency and the concyclicity follows from Mannheim's theorem, which is exactly the lemma mentioned by Telv.

Quick Reply

High School Olympiads

Circumferences and concurrency

Reply

bozzio

#1 May 29, 2013, 11:47 pm

ABC is a triangle. Take D,E on AC (D closer to A), F,G on BC (F closer to C), H,I on AB (H closer to B). The circumcircles of CEG and CFD meet at X, the circumcircles of AEI and AHD meet at Z and the circumcircles of BGI and BFH meet at Y. Show that AZ,BY and CX are concurrent.

Luis González

#2 May 30, 2013, 12:42 am

Let $\delta(P,\ell)$ denote the distance from the point P to the line ℓ. Clearly Z is the center of the spiral similarity that swaps $\overline{HI}$ and $\overline{DE} \Longrightarrow \triangle ZHI$ and $\triangle ZDE$ are directly similar $\Longrightarrow \delta(Z,CA) : \delta(Z,AB) = ED : IH$. Multiplying the cyclic expressions together gives

$$\frac{\delta(Z,CA)}{\delta(Z,AB)} \cdot \frac{\delta(Y,AB)}{\delta(Y,BC)} \cdot \frac{\delta(X,BC)}{\delta(X,CA)} = \frac{ED}{IH} \cdot \frac{IH}{GF} \cdot \frac{GF}{ED} = 1.$$

By the converse of Ceva's theorem, we conclude that AZ, BY, CX concur.

Quick Reply

High School Olympiads

Passes Through Incenter

Reply

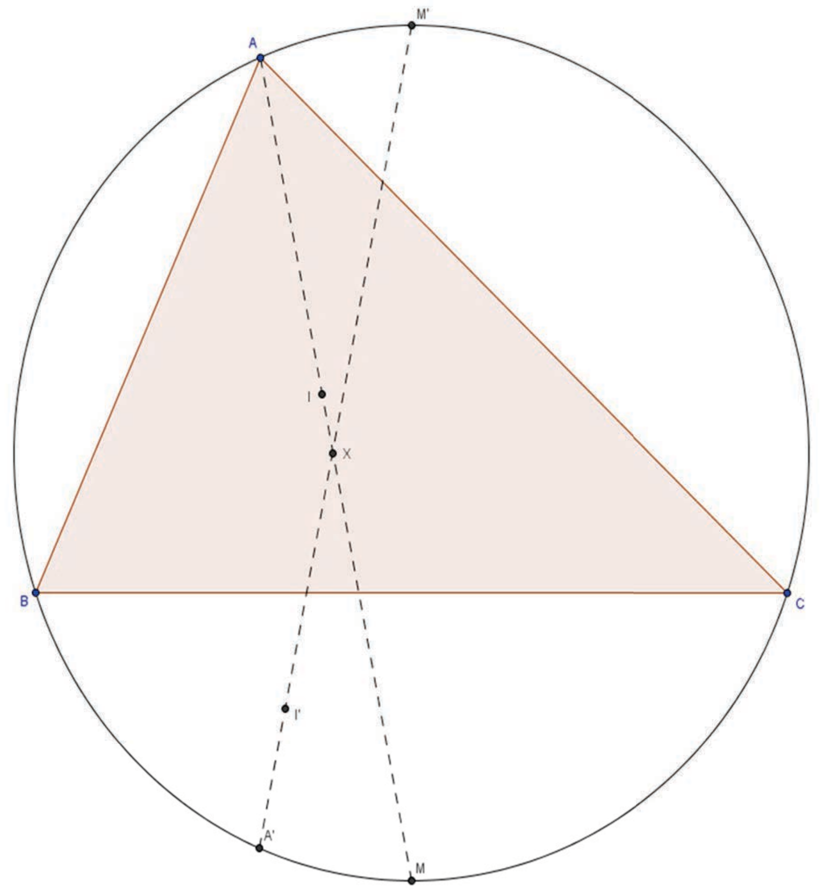
Jettywang828

#1 Mar 12, 2015, 7:51 am

We have fixed segment BC and variable point A.
Take diameter of circumcircle of triangle ABC parallel to $\overline{BC}$, then reflect A over it to A'. Find the locus of points A such that the angle bisector of $\angle BA'C$ passes through the incenter of triangle ABC.

my progress

Attachments:



tastymath75025

#2 Mar 12, 2015, 8:54 am

Does this help? Let K be the point where the incircle touches BC. Then $IO||BC \implies HK||AO$.

(tournament of towns 2003)

Another way to proceed may be to note that $IO||BC \implies D(I, BC) = D(O, BC) \implies r = R\cos A$.

(Note this gives us a weak restriction on A since Euler tells us $R \geq 2r$ so $2\cos A \leq 1$ and $A \geq 60$.

Also we note A is acute or else the circumcenter is outside ABC. So $90 > A \geq 60$.)

Luis González

#3 Mar 12, 2015, 10:16 am

By problem condition, A' is the tangency point of the A-mixtilinear incircle of $\triangle ABC$ with its circumcircle. According to topic http://www.artofproblemsolving.com/community/c6h352772, this happens if and only if $AB = AC$ or $\cos B + \cos C = 1$. So the locus of A is the union of the perpendicular bisector τ of $\overline{BC}$ and a quartic $\mathcal{Q}$.

Define the rectangular reference $A \equiv (x, y)$ for $\{BC \equiv y = 0, \tau \equiv x = 0\}$ and WLOG $B \equiv (-1, 0), C \equiv (1, 0)$. Using $\cos B + \cos C = 1$, this quartic has implicit equation:

$$\mathcal{Q} \equiv \frac{1}{\sqrt{1 + \frac{y^2}{(x+1)^2}}} + \frac{1}{\sqrt{1 + \frac{y^2}{(x-1)^2}}} = 1.$$

It does not seem special, does it ?. For a Wolfram Alpha graph see here.

tastymath75025

#4 Mar 12, 2015, 11:49 pm

Another way to get Luis Gonzales's result is to note our equation rearranges as $\frac{r}{R} = \cos A$ from my post but it is well-known that $\frac{r}{R} = \cos A + \cos B + \cos C - 1$, giving us $\cos B + \cos C = 1$.

Quick Reply

High School Olympiads

if and only if cosB+cosC=1 (own)

Reply

seifi-seifi
#1 Jun 14, 2010, 6:46 pm

ABC be a triangle.circle ω is tangent to AB and AC and circumcircle of ABC.

ω is tangent to circumcircle of ABC at G. prove that $AG \perp BC$ if and only if

$cosB + cosC = 1$ or $AB = AC$

This post has been edited 1 time. Last edited by seifi-seifi, Jun 14, 2010, 10:20 pm

pldx1
#2 Jun 14, 2010, 8:33 pm • 1

Or if AB=AC.

frenchy
#3 Jun 14, 2010, 8:34 pm • 1

> pldx1 wrote:
> Or if AB=AC.

Maybe it is obviously but why?
EDIT:Sorry for stupid question.I thought that pldx1 deduce from $cosB + cosC = 1$ the fact that $AB = AC$

This post has been edited 1 time. Last edited by frenchy, Jun 15, 2010, 4:34 am

seifi-seifi
#4 Jun 14, 2010, 10:22 pm

> pldx1 wrote:
> Or if AB=AC.

thanks dear pldx1. i edited my post.

Luis González
#5 Jun 14, 2010, 11:49 pm

It's well known that AG is the isogonal ray of the A-Nagel cevian. For instance, see here, i.e. AG passes through X_{56}. By the problem conditions, the A-altitude τ_a is identical to the A-cevian η_a of X_{56}. Using barycentric coordinates WRT $\triangle ABC$, we have:

$$X_4 \equiv \left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right) \ , \ X_{56} \equiv (a(1-\cos A) : b(1-\cos B) : c(1-\cos C))$$

$$\tau_a \equiv \frac{y}{z} = \frac{S_C}{S_B} = \frac{b\cos C}{c\cos B} \ , \ \eta_a \equiv \frac{y}{z} = \frac{b(1-\cos B)}{c(1-\cos C)}$$

$$\quad \frac{\cos C}{} \quad \frac{1-\cos B}{}$$

$r_a \equiv \eta_a \iff \frac{\quad}{\cos B} = \frac{\quad}{1-\cos C} \iff (\cos B - \cos C)(\cos B + \cos C - 1) = 0$

Either $\cos B = \cos C \iff AB = AC$ or $\cos B + \cos C = 1$

seifi-seifi

#6 Jun 15, 2010, 7:05 am

thanks dear luis for your solution.

another problem(own):

in triangle ABC , A_excircle touch BC at M and O hs circumcircle of ABC.

prove that A, M, O are collinear if and only if $cosB + cosC = 1$

Virgil Nicula

#8 Jun 15, 2010, 8:46 pm

Very very nice problem !

> *seifi-seifi wrote:*
> Let ABC be a triangle with $AB \neq AC$. The circle ω is tangent to AB and AC and circumcircle of ABC.
>
> The circle ω is tangent to circumcircle of ABC at G. Prove that $AG \perp BC \iff \cos B + \cos C = 1$.

Proof (metrical !). Denote $w = C(S, \rho)$ and $T \in AB \cap w$. Is well-known or prove easly that $IT \perp IA$, $ST \perp AB$, $AT = \frac{bc}{s}$

and $\rho = \frac{r}{\cos^2 \frac{A}{2}} = \frac{bcr}{s(s-a)}$, where $2s = a + b + c$. Therefore, $\boxed{AG \perp BC} \iff$ the ray $[AS$ is the bisector of $\widehat{GAO}$ $\iff$

$m(\widehat{AGS}) = 2 \cdot m(\widehat{GAS}) \overset{(*)}{\iff} AS^2 = SG \cdot (SG + AG) \iff AS^2 - \rho^2 = \rho \cdot GA \iff AT^2 = \rho \cdot GA \iff$

$\frac{b^2c^2}{s^2} = \frac{bcr}{s(s-a)} \cdot AG \iff AG = \frac{bc(s-a)}{sr} \iff 2R\cos(B-C) = \frac{bc(s-a)}{sr} \iff 2R\cos(B-C) = \frac{2(s-a)}{\sin A}$ $\iff$

$2(s-a) = 2R\sin A\cos(B-C) \iff 2(s-a) = R(\sin 2B + \sin 2C) \iff 2(s-a) = b \cdot \cos B + c \cdot \cos C \iff$

$b + c - a = (b+c)(\cos B + \cos C) - a \iff b + c = (b+c)(\cos B + \cos C) \iff \boxed{\cos B + \cos C = 1}$.

==

$(*)$ Here I used the well-known property : "In a triangle ABC exists the equivalence $B = 2C \iff b = c(c+a) \quad \vee \quad b = c$.

seifi-seifi

#9 Jun 16, 2010, 2:50 pm

thanks all. now prove this problem :

1-prove that $AIH = 90$ if and only if $cosB + cosC = 1$(H=orthocenter and I=incenter)

MJ GEO

#10 Jun 16, 2010, 8:22 pm

Very nice

I will prove some other things.Note that some of them are from seifi-seifi in Iranian forum with my own solutions.

Its well known that $cosA + cosB + cosC = 1 + \frac{r}{R} \implies cosA = \frac{r}{R}$

Let H be the orthocenter of triangle ABC.Then $AH = 2RcosA \Longrightarrow AH = 2r$
Let I be the incenter of triangle ABC and E, L be the intersections of A_altitude with BC and circumcircle
And let M, N the midpoints of the arc BC and side BC and O is circumcenter.we have:
we know that $EL = EH.h_a - r = EL + r = EH + r = h_a - 2r + r = h_a - r$ so $AI = IL$
Its easy to show that $HAI = \frac{B-C}{2} \Longrightarrow LIM = B - C = LOM \Longrightarrow IOML$ is cyclic.but
$LMI = B - C + 90 + C - B = 90$ so $IO \parallel BC$.
We know that $MI^2 = ME.MA$ and from above $\frac{MN}{NO} = \frac{ME}{MI}$.From this we can conclude that $IN \parallel AO$ and so A, O and Nagel point are collinear.
$HAI = IAO = NIM = IMN$ so $IM = IN$.
And now i will prove that $AIH = 90$.we must prove that $AH.AD = AI.AD$:
$\frac{AI}{ID} = \frac{b+c}{a} \Longrightarrow AI.AD = AD^2.\frac{b+c}{2p}$ but its well known that $AD = \frac{2}{b+c}.\sqrt{bcp(p-a}.(2p = a+b+c)$ so we must prove that $rh_a = bc(p-a)$.
but we have $\frac{r}{p-a} = tan\frac{A}{2}$ and $bc = 2Rh_a$.so we must prove that $tan\frac{A}{2} = 2R$ wich is left to you.

MJ GEO
#11 Jun 16, 2010, 8:45 pm

Here is things that i proved in last post:(names like above)
1)$AI = IL \Longleftrightarrow cosB + cosC = 1$
2)$OI \parallel BC \Longleftrightarrow cosB + cosC = 1$
3)$AH = 2r \Longleftrightarrow cosB + cosC = 1$
4)$cosA = \frac{r}{R} \Longleftrightarrow cosB + cosC = 1$
5)$MN = NI \Longleftrightarrow cosB + cosC = 1$
6)$AIH = 90 \Longleftrightarrow cosB + cosC = 1$
7)A, O, X_a are collinear $\Longleftrightarrow cosB + cosC = 1$($X_a$ is where BC touches A_excircle)
8)$tan\frac{A}{2} = 2R \Longleftrightarrow cosB + cosC = 1$
9)$1 - cosA = a \Longleftrightarrow cosB + cosC = 1$

Virgil Nicula
#12 Jun 17, 2010, 4:46 am

> *MJ GEO wrote:*
> $IA \perp IH \iff \cos B + \cos C = 1 \iff X_a \in OA$, where X_a is a point where the A-excircle touches the side $[BC]$.

Proof.

► $\boxed{IA \perp IH} \iff IA = AH\cos\frac{B-C}{2} \iff 2R\cos A\cos\frac{B-C}{2} = IA \iff$

$2R\cos A\cos\frac{B-C}{2}\cos\frac{B+C}{2} = IA\sin\frac{A}{2} \iff R\cos A(\cos B + \cos C) = r \iff$

$\cos A(\cos B + \cos c) = \frac{r}{R} \iff \cos A(\cos B + \cos c) = \cos A + (\cos B + \cos C) - 1 \iff$

$(\cos B + \cos C)(1 - \cos A) = (1 - \cos A) \iff \boxed{\cos B + \cos C = 1}$ because $\cos A \ne 1$.

► Denote $D \in AO \cap BC$. Observe that $DC = \frac{ab\cdot\cos B}{b\cdot\cos B + c\cdot\cos C} =$
$\frac{ab\cdot\cos B}{(b+c)(\cos B + \cos C) - (b\cdot\cos C + c\cdot\cos B)} \Longrightarrow$

$DC = \frac{ab\cdot\cos B}{(b+c)(\cos B + \cos C) - a}$. Therefore, $\boxed{\cos B + \cos C = 1} \iff DC = \frac{ab(1-\cos C)}{2(s-a)} \iff$

$DC = \frac{ab}{\ldots}\sin^2\frac{C}{\ldots} \iff DC = \frac{ab}{\ldots}\cdot\frac{(s-a)(s-b)}{\ldots} \iff DC$

$DC = \frac{\quad}{s-a} \cdot \sin \frac{\quad}{2} \iff DC = \frac{\quad}{s-a} \cdot \frac{\quad}{ab} \iff DC =$
$s-b \iff D \equiv X_a \iff \boxed{X_a \in OA}$.

pldx1
#13 Jun 17, 2010, 1:53 pm

Hello,

We use Γ and O to denote the ABC-circumcircle and its center ; $K,\ L,\ M$ to denote the contacts of circle ω with resp. $AB,\ AC,\ \Gamma$ (sparing G to denote X(2), the centroid, that will probably occur somewhere in the future).

The key point is that $AB,\ AC,\ \Gamma$ are three cycles through the same point. Therefore, the number of cycles ω that are in contact with all them three is FOUR, instead of the usual eight. The net result is that FOUR possibilities are open, and must be discussed.

Let us restart from the beginning. The center D of ω is located on one of the bisectors of angle $(AB,\ AC)$. Using barycentrics, we have:

$$D = u : v : w$$

where u is to be determined, $v = b$ and either $w = +c$ or $w = -c$. Projection of D on AB is

$$K = uc^2 + S_b w : vc^2 + S_a w : 0$$

and circle centered at D through K can be computed. Its radical axis with the circumcircle is:

$$x\,(a+b-c)^2\,(c+a-b)^2 + 4\,(cu - S_b)^2\,y + 4\,(ub + S_c)^2\,z = 0$$

Solving in z and substituting into Γ lead to a second degree equation in y. In order to have a contact, the discriminant must vanish. This leads to a second degree equation in u, that can be factored and solved easily. We obtain:

$$\begin{array}{rcl} D_1 & = & (1+\cos A - \cos C - \cos B)\,a : 2\,b : 2\,c \\ D_2 & = & -(1+\cos A + \cos B + \cos C)\,a : 2\,b : 2\,c \\ D_3 & = & (1-\cos A - \cos B + \cos C)\,a : -2\,b : 2\,c \\ D_4 & = & a\,(-1+\cos A - \cos B + \cos C) : -2\,b, 2\,c \end{array}$$

where labels $1, 2$ are related with $w = +c$ and labels $3, 4$ with $w = -c$. The contacts points have, respectively, the following barycentrics:

$$\begin{array}{rcl} M_1 & = & a\,(a+c-b)\,(a+b-c) : -2\,(a+b-c)\,b^2 : -2\,(a+c-b)\,c^2 \\ M_2 & = & a\,(a+c-b)\,(a+b-c) : -2\,(a+c-b)\,b^2 : -2\,(a+b-c)\,c^2 \\ M_3 & = & a\,(a+b+c)\,(a-b-c) : -2\,(a-b-c)\,b^2 : -2\,(a+b+c)\,c^2 \\ M_4 & = & a\,(a+b+c)\,(a-b-c) : -2\,(a+b+c)\,b^2 : -2\,(a-b-c)\,c^2 \end{array}$$

As expected, we can jump from one result to another by using the *transformation continue* of EH Lemoine, i.e. from incircle to V-excircle by the substitution $v \mapsto -v$ (V is any of the three vertices A, B, C and v the corresponding sidelength).

To discuss the orthogonality of BC and AM_j, the well-known formula that gives the oriented angle ϑ between lines $[p, q, r]$ and $[u, v, w]$:

$$\cot \vartheta = \frac{(wq+vr)\,S_a + (wp+ur)\,S_b + (vp+uq)\,S_c - pua^2 - qvb^2 - wrc^2}{2S\,((q-r)\,u + (r-p)\,v + (p-q)\,w)}$$

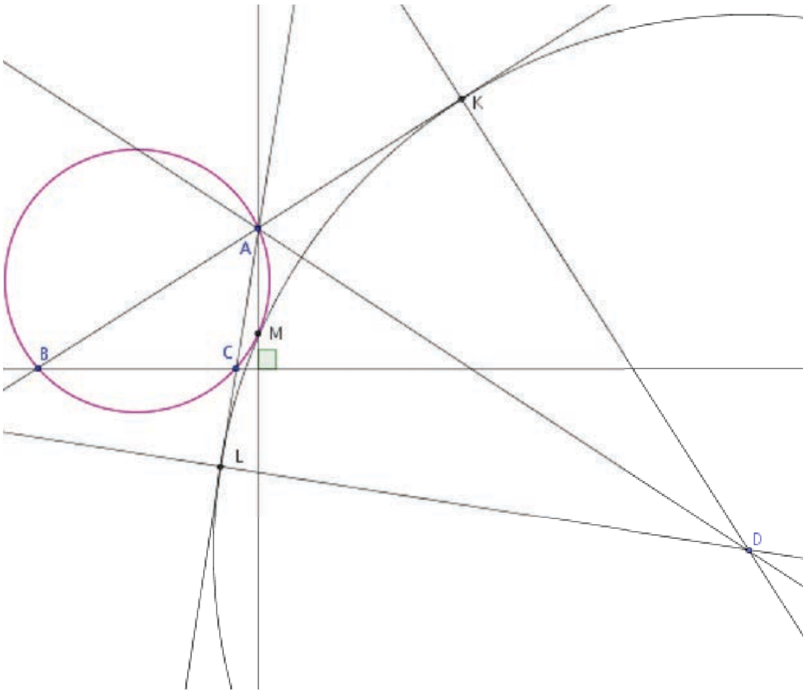
is not required since $AM \perp BC$ results clearly into $y\,S_b = z\,S_c$ where $M = x : y : z$. Some algebra leads to the conditions:

$$\begin{array}{rcl} \mathcal{C}_1 & = & (b-c)\,(a+b+c)\,(1-\cos B - \cos C) \\ \mathcal{C}_2 & = & (b-c)\,(b+c-a)\,(1+\cos B + \cos C) \\ \mathcal{C}_3 & = & (b+c)\,(a+c-b)\,(1-\cos B + \cos C) \\ \mathcal{C}_4 & = & (b+c)\,(a+b-c)\,(1+\cos B - \cos C) \end{array}$$

Therefore, if we take $C = 1.7278759594743862812$ and $B = .5669079168750871613$, we will satisfy condition $\mathcal{C}_3$ but not condition $\mathcal{C}_1$. And nevertheless, we obtain $AM \perp BC$, as it can be checked on the attached Figure.

Best regards, Pierre.

Attachments:



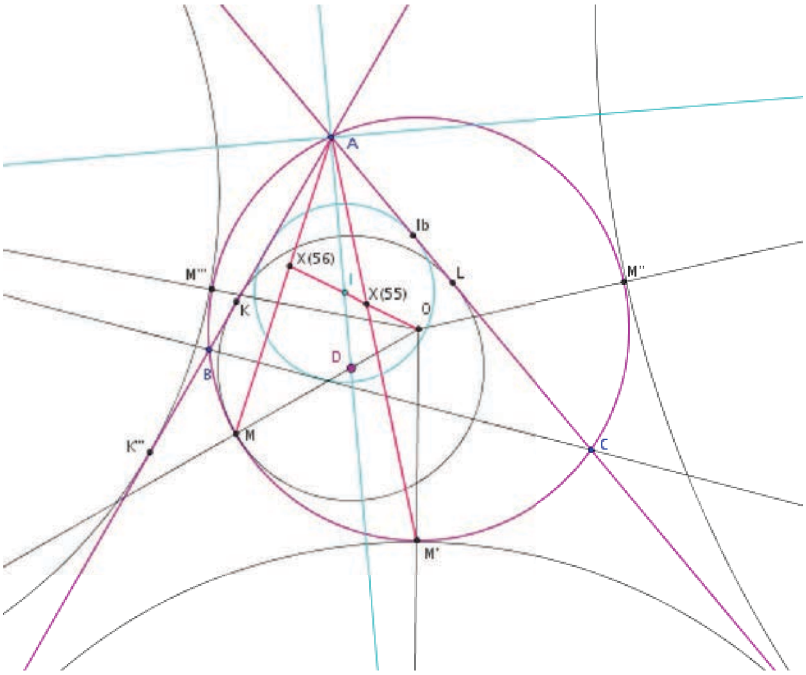
pldx1

#14 Jun 17, 2010, 3:59 pm

Hello,
An additional result. Rename the former point $M = M_1$ as Ma and construct points $Mb,\ Mc$ accordingly. As stated by Luis, triangle $Ma,\ Mb,\ Mc$ is perspective with ABC at X(56), the in-similicenter of the inscribed and the circumscribed circles.
Rename point $M' = M_2$ as $M'a$ and construct $M'b,\ M'c$ accordingly. Then triangle $M'a,\ M'b,\ M'c$ is perspective with ABC at X(55), the ex-similicenter of the inscribed and the circumscribed circles. This is easily obtained from the barycentrics, and can be checked in the attached figure (red lines).
The other two don't lead to perspective triangles.

Best regards, Pierre.

Attachments:



Virgil Nicula

#15 Jun 17, 2010, 7:40 pm

Remark. We can use an well-known property : "If M is the midpoint of $[BC]$ and $V \in MI \cap AH$, then $AV = r$ " . Thus $\cos B + \cos C = 1 \Longleftrightarrow R \cdot \cos A = r \Longleftrightarrow AV = OM \Longleftrightarrow$ the quadrilateral $AVMO$ is a parallelogram $\Longleftrightarrow$ $IM \parallel AO \Longleftrightarrow IM = AO \Longleftrightarrow$ the **Nagel**'s point $N \in AO$.

armpist

#16 Jun 17, 2010, 8:51 pm

Hello MLs

see http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=25730

Historical note:

One of my solution is exactly like Darij's.
I dedicate this solution to all-out struggle for peace coming to the Middle East area.

Mr. T.

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seifi-seifi

#17 Jun 26, 2010, 7:15 pm

a new problem(own)
in triangle ABC , M is midpoint of BC and I is incenter of ABC. MI intersect incircle at T.
prove that $AT\perp BC$ if and only if $cosB+cosC=1$

MJ GEO

#18 Jun 27, 2010, 12:50 am

> seifi-seifi wrote:
> a new problem(own)
> in triangle ABC , M is midpoint of BC and I is incenter of ABC. MI intersect incircle at T.
> prove that $AT\perp BC$ if and only if $cosB+cosC=1$

Assume that incircle touch BC in D and let S be the second intersection of DI with incircle.As above points A,O,S,X_a are collinear.But $\angle IAO=\angle AIS=\frac{B-C}{2}\Longrightarrow AS=r$
But its well known that $MI\parallel AX_a\Longrightarrow AT\parallel SI$ and we are done. :)

---

seifi-seifi

#19 Jun 29, 2010, 12:00 am

new problems :

1- in triangle ABC , M is midpoint of BC and I is incenter of ABC. prove that $\angle BMI-\angle CMI=\frac{3}{2}(\angle B-\angle C)$ if and only if $\angle B=\angle C$ or $cosB+cosC=1$

2- let E be tangent point of A_excircle with BC. AE intersect incircle at T. prove that $AT=r$ if and only if $cosB+cosC=1$.
(r is radus of incircle.)

MJ GEO

#20 Jun 29, 2010, 4:21 am

> seifi-seifi wrote:
> new problems :
>
> 1- in triangle ABC , M is midpoint of BC and I is incenter of ABC. prove that $\angle BMI-\angle CMI=\frac{3}{2}(\angle B-\angle C)$ if and only if $\angle B=\angle C$ or $cosB+cosC=1$

Are you sure?...We know that $MI \parallel AX_a$,and X_a, A, O are collinear...its easy to see that $\angle BMI - \angle CMI = 2(\angle B - \angle C)$.

> seifi-seifi wrote:
>
> 2- let E be tangent point of A_excircle with BC. AE intersect incircle at T. prove that $AT = r$ if and only if $cosB + cosC = 1$.
> (r is radus of incircle.)

But I proved it in my last post...$AS = r$...

Best regards,
Majid.

seifi-seifi

#21 Jul 1, 2010, 12:51 am

oh yes im sorry.

now with R, r construct a triangle such that $cosB + cosC = 1$.

Quick Reply

High School Olympiads

Similar Quaderilaterals

Reply

Source: Handout

AdithyaBhaskar

#1 Feb 23, 2015, 11:46 am

In a triangle ABC, construct altitudes AD, BE, CF and let H be the orthocenter of $\triangle ABC$. Let O_1, O_2, O_3 be the incenters of triangles EHF, FHD, DHE respectively. Prove that the lines $AO_1; BO_2; CO_3$ are concurrent.
Hint : Use similar quadrilaterals.

Solution:
Click to reveal hidden text

This post has been edited 1 time. Last edited by math_explorer, Jul 21, 2015, 11:11 am
Reason: typo fix frompost #2 incorporated, other spelling

TelvCohl

#2 Feb 23, 2015, 12:50 pm

> AdithyaBhaskar wrote:
> In a triangle ABC, construct altitudes AD, BE, CF and let H be the orthocenter of $\triangle ABC$. Let O_1, O_2, O_3 be the incenters of triangles EHF, FHD, DHE respectively. Prove that the lines $AO_1; BO_2; CO_3$ are concurrent.

Typo corrected

My solution:

Let A^*, B^*, C^* be the antipode of A, B, C in $\odot(ABC)$, respectively .
Let I_a, I_b, I_c be the incenter of $\triangle A^*BC, \triangle B^*CA, \triangle C^*AB$, respectively .

Since $\triangle AEF \cup H \sim \triangle ABC \cup A^*$,
so AI_a is the isogonal conjugate of AO_1 WRT $\angle BAC$.
Similarly BI_b, CI_c is the isogonal conjugate of BO_2, CO_3 WRT $\angle CBA, \angle ACB$, respectively .

Since $\angle B^*AC = \angle C^*AB, \angle C^*BA = \angle A^*BC, \angle A^*CB = \angle B^*CA$,
so from Jacobi theorem we get AI_a, BI_b, CI_c are concurrent at a point Q ,
hence AO_1, BO_2, CO_3 are concurrent at the isogonal conjugate of Q WRT $\triangle ABC$.

Q.E.D

MariusBocanu

#3 Feb 23, 2015, 2:32 pm

My solution is not nice, but at least it's easy. We have that $\frac{O_1F}{sin(\widehat{FAO_1})} = \frac{AO_1}{sin(\widehat{AFO_1})}$ and $\frac{O_1E}{sin(\widehat{O_1AE})} = \frac{O_1A}{sin(\widehat{AEO_1})}$, so we have that $\frac{sin(\widehat{FAO_1})}{sin(\widehat{EAO_1})} = \frac{sin(45-\frac{B}{2}) \cdot sin(45+\frac{C}{2})}{sin(45-\frac{C}{2}) \cdot sin(45+\frac{B}{2})}$. Multiply all these equalities, and we are done.

Long live geometry bash!

Luis González

#4 Mar 10, 2015, 11:23 am

Posted before at http://www.artofproblemsolving.com/community/c6h317996.

Quick Reply

High School Olympiads

geometry

Reply

nguyenqn1998

#1 Mar 9, 2015, 10:15 pm

Let ABC be a triangle, let $Bx \perp BA$ (Bx is on half plane without C) and $Cy \perp CA$ (Cy is on half plane without B).

Let E,F be points on lines Bx,Cy respectively so that $\frac{BE}{CF} = \frac{AB}{AC}$. BF intersect CE at point H. Prove that $AH \perp BC$

Luis González

#2 Mar 9, 2015, 10:29 pm

$\tfrac{AB}{AC} = \tfrac{BE}{CF}$ means that the right triangles $\triangle ABE$ and $\triangle ACF$ are similar by SAS $\Longrightarrow \angle BAE = \angle CAF$. Hence, if A_{∞} denotes the point at infinity of $\perp BC$, then by Jacobi's theorem, it follows that $H \equiv BF \cap CE \cap AA_{\infty} \Longrightarrow AH \perp BC$.

Quick Reply

High School Olympiads

Geometry Inequality

Reply

Nimplesy

#1 Mar 9, 2015, 5:49 pm

Let ABC with $AB=c$, $BC=a$ and $CA=b$. Let d_a,d_b,d_c be the length of the inner angle bisector of A,B,C respectively. Prove that:

$$\frac{1}{d_a}+\frac{1}{d_b}+\frac{1}{d_c}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

ShadowThunder

#2 Mar 9, 2015, 9:01 pm

With EB//AC and E∈ AC, we have:
$\frac{b}{b+c}=\frac{d_a}{EB}<\frac{d_a}{2c}$ so $\frac{1}{d_a}>\frac{1}{2}.(\frac{1}{b}+\frac{1}{c})$ (1)
The similar: $\frac{1}{d_b}>\frac{1}{2}.(\frac{1}{a}+\frac{1}{c})$ (2)
$\frac{1}{d_c}>\frac{1}{2}.(\frac{1}{b}+\frac{1}{a})$ (3)
We conclude the argument by combining (1), (2) and (3)
Q.E.D

PROF65

#3 Mar 9, 2015, 9:15 pm • 1

let D,D' be the intersections of the A-angle bisector with BC and the circumcircle we know that $AD.AD'=bc$ then $\frac{1}{a.AD.AD'}=\frac{1}{abc}\implies\frac{1}{AD}=\frac{AD'.a}{abc}\implies\frac{1}{AD}=\frac{D'B.c+D'C.b}{abc}$ but $D'B=D'C\geq\frac{a}{2}$ thus $\frac{1}{d_a}\geq\frac{\frac{a(b+c)}{2}}{abc}$
similarly $\frac{1}{d_b}\geq\frac{\frac{b(a+c)}{2}}{abc}$and $\frac{1}{d_c}\geq\frac{\frac{c(b+a)}{2}}{abc}$ therefore sum up and conclude

Luis González

#5 Mar 9, 2015, 10:16 pm

Using the well-known identity $\frac{\cos\frac{A}{2}}{d_a}=\frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}\right)$, keeping in mind that $\cos\frac{A}{2}<1$ for a non-degenerate $\triangle ABC\Longrightarrow$ $\frac{2}{d_a}>\frac{1}{b}+\frac{1}{c}$. Adding the 3 inequalities together yields the result.

YESMAths

#6 Mar 9, 2015, 10:28 pm

> Nimplesy wrote:
> Let ABC with $AB=c$, $BC=a$ and $CA=b$. Let d_a,d_b,d_c be the length of the inner angle bisector of A,B,C respectively. Prove that:
>
> $$\frac{1}{d_a}+\frac{1}{d_b}+\frac{1}{d_c}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

It is well-known that

$$d_a = \frac{2bc}{b+c}\cos\frac{A}{2}$$

and other analogous equalities hold.
Then we get that

$$\frac{1}{d_a} = \frac{a(b+c)}{2abc}\sec\frac{A}{2}$$

And since $\sec\frac{A}{2} > 1$, we get

$$\sum_{cyc}\frac{1}{d_a} > \sum_{cyc}\frac{a(b+c)}{2abc} = \sum_{cyc}\frac{1}{a}$$

proving the statement. :)

Nimplesy

#7 Mar 10, 2015, 6:50 pm

any prove for this identity $\frac{\cos\frac{A}{2}}{d_a} = \frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}\right)$

PROF65

#8 Mar 12, 2015, 12:04 am

> *ShadowThunder wrote:*
>
> With EB//AC and E∈ AC, we have:
> $\frac{b}{b+c} = \frac{d_a}{EB} < \frac{d_a}{2c}$ so $\frac{1}{d_a} > \frac{1}{2}.(\frac{1}{b}+\frac{1}{c})$ (1)
> The similar: $\frac{1}{d_b} > \frac{1}{2}.(\frac{1}{a}+\frac{1}{c})$ (2)
> $\frac{1}{d_c} > \frac{1}{2}.(\frac{1}{b}+\frac{1}{a})$ (3)
> We conclude the argument by combining (1), (2) and (3)
> Q.E.D

I think you mean EB is parallel to the angle bisector

> *Nimplesy wrote:*
>
> any prove for this identity $\frac{\cos\frac{A}{2}}{d_a} = \frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}\right)$

use the begining of $ShadowThun$ and express EB in terms of c and $\cos\frac{A}{2}$

Quick Reply

High School Olympiads

Projective axis

Reply

Seventh

#1 Mar 9, 2015, 8:26 pm

Is given the straight lines r and s, such that there are a projective transformation (keeping the cross ratio) that takes r to s, where the point $A_1 \in r$ is taken to the point $B_1 \in s$. Let $C_{i,j} = A_iB_j \cap A_jB_i$. Prove that all the points $C_{i,j}$ are collinear.

Luis González

#2 Mar 9, 2015, 9:28 pm

Since the pencils $A_i(B_i, B_j, \cdots)$ and $B_i(A_i, A_j, \cdots)$ are projective with common ray A_iB_i, it immediately follows that all C_{ij} lie on a same line τ, the projective axis of these pencils. Now to prove that τ is independent of A_i, B_i, let $M \equiv r \cap s$. Then using the projective axis τ, the image of M in these two series are none other than the intersections X, Y of τ with $r, s \Longrightarrow$ $\tau \equiv XY$ is independent of the points A_i, B_i chosen. The conclusion follows.

TelvCohl

#3 Mar 9, 2015, 9:36 pm

In general ,
for a homography Φ preserve conic $\mathcal{C}$ map $A_k \mapsto B_k$,
all points $C_{i,j} = A_iB_j \cap A_jB_i$ lie on a line (the homography axis of Φ) .
🙂

Quick Reply

High School Math

Angle Bisector

Reply

yimingz89

#1 Mar 5, 2015, 5:13 am • 1

In acute triangle ABC, l is the bisector of $\angle BAC$. A line through the midpoint of BC parallel to l meets AC and AB at E and F respectively. Given that $AE=1, EF=\sqrt{3}, AB=21$, find the sum of all possible values of BC.

Answer

Luis González

#2 Mar 8, 2015, 8:04 am • 1

Let D be the midpoint of $\overline{BC}$ and let P be the midpoint of the arc BAC of the circumcircle $\odot(ABC)$, lying on external bisector of $\angle BAC$. Simson line of P WRT $\triangle ABC$ goes through D parallel to the internal bisector of $\angle BAC \Longrightarrow E$ and F are the orthogonal projections of P on AC, AB. Thus $EF=\sqrt{3}=AP\cdot \sin A$ and by Archimede's Midpoint Theorem, it follows that $AE=\frac{1}{2}|AB-AC|=1$. By Ptolemy's theorem for cyclic $ABCP$, keeping in mind that $PB=PC$, we get $AP\cdot BC=PB\cdot |AB-AC|=2\cdot PB\cdot AE \Longrightarrow$

$$\frac{1}{AP}=\frac{\sin A}{\sqrt{3}}=\frac{\frac{1}{2}BC}{PB}=\frac{DB}{PB}=\sin\frac{A}{2}\Longrightarrow$$

$$\frac{\sin\frac{A}{2}}{\sin A}=\frac{1}{\sqrt{3}}\Longrightarrow \cos\frac{A}{2}=\frac{\sqrt{3}}{2}\Longrightarrow A=60^\circ.$$

We clearly have two cases. For $AC>AB$, then $\{AC=23, AB=21\}$ and for $AB>AC$, then $\{AB=21, AC=19\}$. Calculating BC, using cosine theorem yields

$$BC=\sqrt{23^2+21^2-2\cdot 23\cdot 21\cdot\cos 60^\circ}=\sqrt{487} \text{ or}$$

$$BC=\sqrt{21^2+19^2-2\cdot 21\cdot 19\cdot\cos 60^\circ}=\sqrt{403}.$$

yimingz89

#3 Mar 8, 2015, 9:43 am

Thanks for the solution, but I still have a few questions.
How do you deduce that E and F are the projections? I know the Simson line theorem states that the feet of the projections from a point on the circumference to the 3 sides of the triangle are collinear, but how do you apply that here? Finally, how do you get $EF=\sqrt{3}=AP\cdot\sin A$, because angle A (or the exterior angle A) are not part of a right triangle, as FE is the a diagonal of the cyclic quadrilateral.

This post has been edited 2 times. Last edited by yimingz89, Apr 23, 2015, 10:59 pm

Quick Reply

High School Olympiads

Euler Line and Equilateral Triangles

Reply

Isogonics

#1 Mar 7, 2015, 7:18 pm

Let $\triangle ABC$ be a triangle and l be its euler line. Let P be a point on BC that AP is perpendicular to l.
Let $\triangle BCS, \triangle BCT$ be two equilateral triangle. Let Q be circumcenter of $\triangle AST$.
Prove: $BP = CQ$.

Luis González

#2 Mar 8, 2015, 2:53 am • 2

WLOG we assume that $AC > AB$. If B', C' lie on CA, AB, such that $\odot(ABB')$ and $\odot(ACC')$ touch BC, then $P \equiv BC \cap B'C'$ (for a proof see AF perpendicular to Euler line). Therefore $a^2 = c \cdot BC' = b \cdot CB' \Longrightarrow AC' = \frac{|a^2-c^2|}{c}$, $AB' = \frac{|b^2-a^2|}{b}$. Hence by Menelaus' theorem for $\triangle ABC$ cut by $\overline{PB'C'}$, we obtain

$$\frac{\overline{PB}}{\overline{PC}} = \frac{\overline{C'B}}{\overline{C'A}} \cdot \frac{\overline{B'A}}{\overline{B'C}} = -\frac{\frac{a^2}{c}}{\frac{a^2-c^2}{c}} \cdot \frac{\frac{b^2-a^2}{b}}{\frac{a^2}{b}} = \frac{b^2-a^2}{c^2-a^2} \quad (1).$$

On the other hand, from the problem Squares of some lengths equal another (see post #2), we get

$$\frac{\overline{QC}}{\overline{QB}} = \frac{\cot A - \cot B}{\cot A - \cot C} = \frac{b^2-a^2}{c^2-a^2} \quad (2).$$

From (1) and (2) we get $\frac{\overline{PB}}{\overline{PC}} = \frac{\overline{QC}}{\overline{QB}}$, which means that P, Q are isotomics WRT BC, i.e. $BP = CQ$, as desired.

TelvCohl

#4 Mar 8, 2015, 3:55 pm • 2

Thanks for **Luis** for point me out this link, I did not know this link before :) .

Here is another way to finish the proof :

For any line $\mathcal{L}$, denote $\mathcal{L}_\infty$ be the infinity point on $\mathcal{L}$.
Let O, H be the circumcenter, orthocenter of $\triangle ABC$, respectively .
Let ℓ_B^*, ℓ_C^* be the perpendicular bisector of AC, AB, respectively .
Let ℓ_B, ℓ_C be the line perpendicular to BC passing through B, C, respectively .
Let $B^* = \ell_B \cap \ell_B^*, C^* = \ell_C \cap \ell_C^*$ and $Y \in \ell_B, Z \in \ell_C$ such that $QY \perp AB, QZ \perp AC$

From the problem Squares of some lengths equal another (post # 2) we get $Q \in B^*C^*$.
From Pappus theorem (for $Q - B - C$ and $\ell_{B\infty} \equiv \ell_{C\infty} - \ell^*_{C\infty} - \ell^*_{B\infty}$) we get $H \in YZ. ...(1)$
From Pappus theorem (for $Q - B^* - C^*$ and $\ell_{B\infty} \equiv \ell_{C\infty} - \ell^*_{C\infty} - \ell^*_{B\infty}$) we get $O \in YZ. ...(2)$

From $(1), (2)$ we get YZ coincide with the Euler line l of $\triangle ABC$,
so from the problem An extension of a problem of perpendicularity we get $BP = CQ$.

Q.E.D

TelvCohl

#5 Feb 19, 2016, 1:24 am • 1

Let A_1 be the reflection of A in BC. Let $O, J \in A_1S$ be the circumcenter, 1st isodynamic point of $\triangle ABC$, respectively. Since AT, AJ are isogonal conjugate WRT $\angle A$, so we get $\measuredangle OSJ = \measuredangle AA_1S = \measuredangle TAA_1 = \measuredangle OAJ \Longrightarrow A, J, O, S$ are concyclic, hence $\measuredangle A_1AQ = 90^\circ - \measuredangle ASA_1 = 90^\circ - \measuredangle AOJ$ ... $(\star)$.

Let $D \equiv AA_1 \cap \odot(O), V \equiv AQ \cap \odot(O)$. Since the steiner line of D WRT $\triangle ABC$ is perpendicular to AO, so from $(\star) \Longrightarrow$ the steiner line of V WRT $\triangle ABC$ is parallel to the Brocard axis OJ of $\triangle ABC$, hence V is the steiner point of $\triangle ABC$ (X_{99} in ETC) which is the isotomic conjugate of the point at infinity with direction $\perp l \Longrightarrow BP = CQ$.

Quick Reply

High School Olympiads

Squares of some lengths equal another

Reply

Rijul saini

#1 Mar 27, 2011, 2:54 pm

Let M, N be the points in the plane of $\triangle ABC$ satisfying $MB \perp BC,\ AM = MC,\ NC \perp BC,\ AN = NB$. Let P be the intersection of the lines BC and MN.

Prove that $AP^2 = PD^2 + \frac{3}{4}BC^2$, where D is the midpoint of BC.

yetti

#2 Mar 30, 2011, 1:19 pm • 1

(O, R) is circumcircle of $\triangle ABC$. Parallel to BC through O cuts MB, NC at M', N'.

$\frac{\overline{PB}}{\overline{PC}} = \frac{\overline{MB}}{\overline{NC}} = \frac{\overline{M'B} - \overline{M'M}}{\overline{N'C} - \overline{N'N}} = \frac{\cot A - \cot C}{\cot A - \cot B}.$

$\triangle ZBC$ is equilateral, Z on the same side of BC as $A.\ \omega = \angle ZOA = \angle B - \angle C.$ With coordinate origin O and x-axis parallel to $\overline{BC}$, perpendicular bisector k of AZ has equation

$$y - \tfrac{R}{2}(\cos\omega - \cos A + \sqrt{3}\sin A) = \frac{\sin\omega}{\cos\omega + \cos A - \sqrt{3}\sin A}(x + \tfrac{R}{2}\sin\omega).$$

Substitute $y = -R\cos A$ to get x-coordinate $\overline{DK}$ of its intersection K with BC:

$$\overline{DK} = -R \cdot \frac{\cos 2A + \cos\omega\cos A}{\sin\omega}.$$

As $-\overline{DB} = \overline{DC} = R\sin A$, it follows that

$$\frac{\overline{KB}}{\overline{KC}} = \frac{\cos 2A + \cos(A-\omega)}{\cos 2A + \cos(A+\omega)} = \frac{\cos 2A - \cos 2C}{\cos 2A - \cos 2B} = \frac{\sin(A-C)\sin B}{\sin(A-B)\sin C} = \frac{\cot A - \cot C}{\cot A - \cot B}.$$

As a result, $K \equiv P$ are identical, $AP = ZP$ and $AP^2 = ZP^2 = PD^2 + ZD^2 = PD^2 + \frac{3}{4}BC^2$.

Rijul saini

#3 Apr 1, 2011, 7:20 pm • 1

Toss those points on the coordinate plane. Without Loss of generality, take

$$A = (2a, 2b),\ B = (-2, 0),\ C = (2, 0),\ D = (0, 0)$$

The midpoint of AC, i.e. $T = (a+1, b)$. Similarly, the midpoint of AB, i.e. $U = (a-1, b)$.

Slope of $AC\ = \frac{2b}{2a-2} = \frac{b}{a-1}$.

This implies that the slope of any perpendicular to AC, in particular, the perpendicular bisector of $AC\ = \frac{-1}{b/(a-1)} = \frac{1-a}{b}$.

Combining this with the fact that the perpendicular bisector of AC passes through T, and it's nothing but the line TM, we get that the equation of

$$TM : y = \frac{1-a}{b}(x - (a+1)) + b$$

And therefore, since x coordinate of M is already known to be -2, we get the y coordinate of M as

$$\frac{1-a}{b}(-1-(a-1))+b=\frac{(a-1)(a+3)}{b}+b$$

Therefore,

$$M=\left(-2,\frac{(a-1)(a+3)}{b}+b\right)$$

Similarly,

$$N=\left(2,\frac{(a+1)(a-3)}{b}+b\right)$$

Therefore, now, setting $y=0$ in the equation of MN and performing an easy calculation, we get that the x coordinate of $P=\frac{a^2+b^2-3}{a}$.

Now,

$$\begin{aligned}PA^2-PD^2&=\left(\frac{a^2+b^2-3}{a}-2a\right)^2+4b^2-\left(\frac{a^2+b^2-3}{a}\right)^2\\&=4a^2+4b^2-4a\cdot\frac{a^2+b^2-3}{a}\\&=12\end{aligned}$$

And since $BC=4$, we get that $PA^2=PD^2+\frac{3}{4}BC^2$, which is exactly what we needed to prove.

drmzjoseph

#4 Mar 18, 2015, 2:57 pm • 1

Let B_1 and C_1 the points symmetric of B and C WRT C and B respectively.
$\Rightarrow M$ and N are circumcenters of $\triangle ACC_1$ and $\triangle ABB_1$ respectively.
Let X and Y two points such that $\triangle BCX$ and $\triangle BCY$ are equilaterals.

$D\in XY \Rightarrow YD.XD=C_1D.CD=B_1D.BD=\frac{3}{4}BC^2 \Rightarrow D$ have equal power in $\odot(ACC_1), \odot(ABB_1)$ and $\odot(AXY)$

$\Rightarrow \odot(ACC_1), \odot(ABB_1)$ and $\odot(AXY)$ are coaxiales circles (radical axis AD). i.e. the center of $\odot(AXY)$ lies on MN and the center of $\odot(AXY)$ lies on perpendicular bisector of XY (Line BC)

$\Rightarrow P$ is the center of $\odot(AXY) \Rightarrow Pot_{\odot(AXY)}(D)=PD^2-AP^2=-XD.YD=-\frac{3}{4}BC^2$

Q.E.D

This post has been edited 1 time. Last edited by drmzjoseph, Mar 18, 2015, 6:04 pm
Reason: Nothing without importance

Quick Reply

High School Olympiads

show that IMH'O are concyclic

Reply

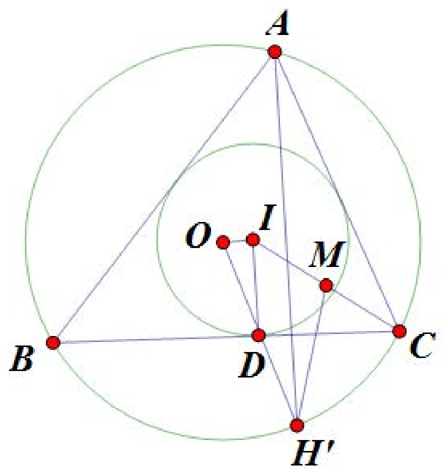
Australia

#1 Mar 7, 2015, 9:50 pm

Let ΔABC be a triangle with circumcentre O,and the incenter is I, inscribed circle of ΔABC touch BC at D,if M is midpoint with IC,and $\odot O \bigcap OD = H', OI//BC$

show that
I, M, H', O are concyclic.

Attachments:



Luis González

#2 Mar 7, 2015, 10:36 pm

See http://www.artofproblemsolving.com/community/c6h341355. Bear in mind that $OI \parallel BC \Longleftrightarrow AO \parallel HD$, where H is the orthocenter of $\triangle ABC$.

Quick Reply

High School Olympiads

Cyclic

Reply

77ant

#1 Mar 28, 2010, 2:38 pm

Hi, everyone.

Please read the following.

For an acute triangle ABC ($\angle B < \angle C$), let I, O, H be its incenter, circumcenter, orthocenter respectively. Its incircle touch BC at D.
$AO//HD$, $OD \cap AH = E$, F=midpoint of CI. Prove that E, F, I, O are cyclic.

It is from 2010 **KMO** final.
Thanks. :)

Luis González

#2 Aug 14, 2010, 5:40 am • 1

Let M, T be the midpoints of BC and AH. Then $AT = OM \Longrightarrow ATMO$ is a parallelogram $\Longrightarrow DH \parallel MT \parallel OA$. Hence, OA meets BC at the reflection of D about $M \Longrightarrow AO$ coincides with the A-Nagel cevian $\Longrightarrow AO$ passes through the antipode D_0 of D in the incircle (I). From the parallelogram $AHDD_0$, we have then, $DD_0 = AH = 2OM \Longrightarrow OM = ID$, i.e. $IO \parallel BC$ $(*)$. Since OA, BC, OE bound an isosceles triangle, we have $\angle(DH, DC) = \angle(OA, BC) = \angle(DC, DE)$, which implies that E is the reflection of H across $BC \Longrightarrow E \equiv OD \cap AH$ lies on the circumcircle (O). Ray AI cuts (O) at the midpoint V of the arc BC and ray AO cuts (O) and BC at L, U. Then $\angle AVE = \angle ALE = \angle AUM = \angle ODM$. Together with $(*)$, it follows that $\angle ODM = \angle DOI \Longrightarrow \angle EOI = \angle EVI$. Thereby, O, I, E, V are concyclic on a circle with diameter IV, but V is the circumcenter of $\triangle BIC$ (well-known), so $VF \perp IC \Longrightarrow F \in \odot(OIE)$.

leader

#3 May 21, 2013, 3:43 pm

let AO cut BC and circle ABC again at L, R since M is the midpoint of HR by Tales $LM = DM$ so L is where the A-excircle touches BC so if DI meets the incircle again at P $A-P-L$ and IM is the midline in DPL so $IM||PO$ now $PIMO$ is a parallelogram so $OM = IP = ID$ and $OI||BC$. Since AI meets circle ABC again at T the circumcenter of BIC $\angle TFI = \angle TOI = 90$ now $\angle OEA = \angle EOM = \angle LOM = \angle OAE$ so $OE = OA$ and IO is the perpendicular bisector of AE let TO meet circle ABC again at S since $A-I-T$ are collinear by symmetry wrt OI $E-I-S$ are collinear and $\angle IET = \angle SET = 90$ meaning $OIFET$ is cyclic.

leeky

#4 Mar 21, 2015, 1:02 pm

After knowing $OD \cap AH = E \in \odot(ABC)$, note $FI = FD = FC$, then $\angle IFD = \angle C$. Also $\angle DEC = \angle DEA + \angle AEC = \angle OAH + \angle B = \angle B + \angle A - 2(90^\circ - \angle C)$. It is easy to check that $\angle IFD = \angle DEC$, thus F, D, E, C concyclic.
Then $\angle IOE = \angle CDE = \angle CFE$, thus E, F, I, O concyclic.

This post has been edited 1 time. Last edited by leeky, Nov 9, 2015, 8:27 pm
Reason: typo

Quick Reply

High School Olympiads

Concurrency of circles and line

Reply

Peres123
#1 Mar 7, 2015, 8:57 pm

Let ABC be a triangle. Circle Γ, which passes through A, is tangent to the circumcircle of ABC at A and tangent to side BC at D. Let AD intersects the circumcircle again at Q. Let O be the circumcenter of ABC. If AO bisects angle DAC, show that the circle with center Q and radius QB, circle Γ, and the line which is perpendicular to AD and passes through B, these three are concurrent

Luis González
#2 Mar 7, 2015, 10:06 pm

Let X,Y be the 2nd intersections of AC,AB with $\Gamma.$ Since A is the exsimilicenter of $(O) \equiv \odot(ABC)$ and $(U) \equiv \Gamma \Longrightarrow BC \parallel XY,$ thus the arcs DX,DY of Γ are equal $\Longrightarrow AD \equiv AQ$ bisects $\angle BAC.$

Let P be the foot of the A-altitude. Since AP,AO are isogonals WRT $\angle BAC,$ then AP bisects $\angle BAD \Longrightarrow \triangle ABD$ is A-isosceles and since AUO bisects $\angle DAC,$ then $\triangle ADX$ is A-isosceles $\Longrightarrow AB=AD=AX \Longrightarrow X$ coincides with the reflection of B on $AD \Longrightarrow BX \perp AD$ and $QC=QB=QX.$ The conclusion follows.

Peres123
#3 Mar 12, 2015, 8:03 pm

i have understand about $96/100$ but how to prove this part $QC=QB=QX$

Quick Reply

High School Olympiads

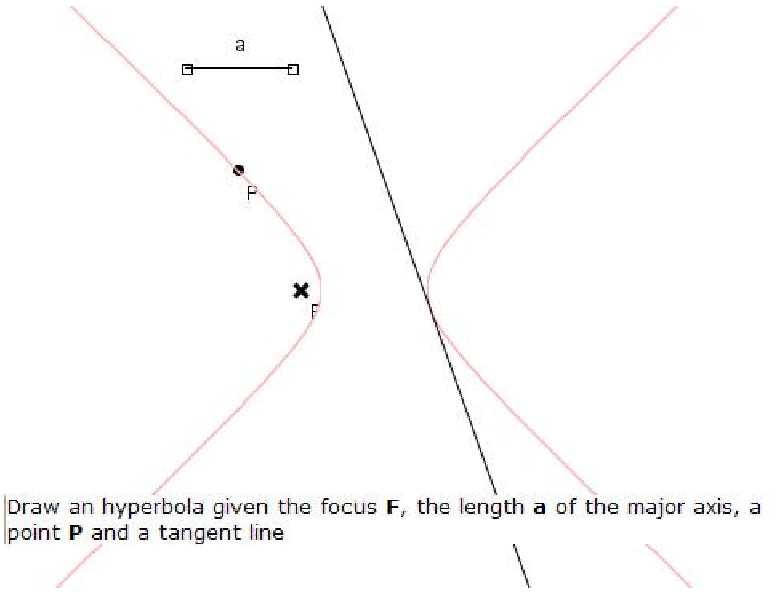
5 points of hyperbola

Reply

jrrbc

#1 Jul 12, 2009, 1:16 pm

Attachments:



Luis González

#6 Mar 7, 2015, 10:28 am

Denote F' the another focus. Since $PF'-PF=a,$ then construct circle $\mathcal{C}$ with center P and radius $PF+a \Longrightarrow F'$ lies on $\mathcal{C}.$ Construct orthogonal projection Q of F on the given tangent. The center O of the hyperbola must lie on the circle $\mathcal{K}$ with center Q and radius $\tfrac{1}{2}a,$ but the reflection F' of F about O lies on $\mathcal{C}.$ Hence, construct the image $\mathcal{K}'$ of $\mathcal{K}$ under the homothety with center F and coefficient 2. $\mathcal{K}'$ and $\mathcal{C}$ meet at the focus F' (consider the one according to the sketch). The object hyperbola is now defined.

Quick Reply

High School Olympiads

Triangle problem 10

Source: Not of me

gemath

#1 Jul 11, 2007, 9:12 pm

Given triangle ABC (non right) with altitude AA', BB', CC'. Let D, E, Fbe excenters of $\angle B'AC', \angle C'A'B, \angle B'A'C$ resp.A-exradius of ABC touch BC, CA, AB at M, N, P. Prove that circumcenter of DEF is orthocenter of MNP.
Compare with my problem Triangle problem 7.

TelvCohl

#2 Mar 7, 2015, 7:19 am

My solution:

Let I_a be the $A-$ excenter of $\triangle ABC$.
Let G, T be the orthocenter, 9-point center of $\triangle MNP$, respectively .

Since $\triangle ABC \cup I_a \sim \triangle AB'C' \cup D \sim \triangle A'B'C \cup F$,
so $\triangle I_aBC \sim \triangle DB'C' \sim \triangle FB'C \Longrightarrow \triangle B'C'C \sim \triangle B'DF$,

hence we get $DF = C'C \cdot \frac{B'D}{B'C'} = C'C \cdot \frac{BI_a}{BC} = BI_a \cdot \sin \angle CBA = MP$.

From $\angle(C'C, DF) = \angle C'B'D = \angle I_aBC = PBI_a = \angle(C'C, MP) \Longrightarrow DF \parallel MP$.
Similarly we can prove $DE = MN$ and $DE \parallel MN \Longrightarrow \triangle DEF$ and $\triangle MNP$ are homothetic and congruent .

Since D is the reflection of I_a in NP (well-known) ,
so T is the midpoint of $DM \Longrightarrow \triangle DEF$ is the image of $\triangle MNP$ under homothety $\mathbf{H}(T, -1)$,
hence we get G is the circumcenter of $\triangle DEF$ ($\because$ G is the image of I_a under homothety $\mathbf{H}(T, -1)$) .

Q.E.D

Luis González

#3 Mar 7, 2015, 8:08 am

Let I_a be the A-excenter of $\triangle ABC$. Since $\triangle AB'C' \cup D \sim \triangle ABC \cup I_a \Longrightarrow \frac{AD}{AI_a} = \frac{B'C'}{BC} = \cos \widehat{A} = \cos \widehat{PAN} \Longrightarrow$ D is orthocenter of A-isosceles $\triangle ANP \Longrightarrow D$ is reflection of I_a on NP and by similar reasoning, E, F are the reflections of I_a on $PM, MN \Longrightarrow \triangle DEF$ is image of the medial triangle of $\triangle MNP$ under dilatation with center I_a and coefficient 2.
The conclusion follows.

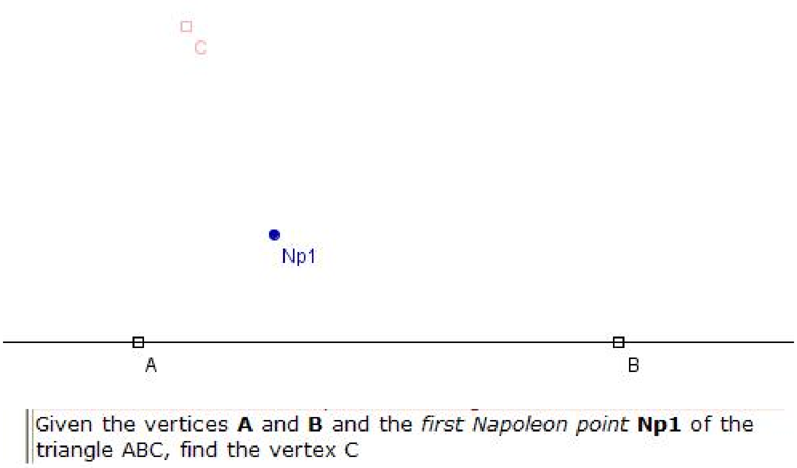
High School Olympiads

A Np1 B

Reply

jrrbc
#1 Aug 10, 2009, 6:37 am

Attachments:



Luis González
#2 Mar 7, 2015, 4:11 am

Let A', B', C' denote the apices of the equilateral triangles $\triangle A'BC, \triangle B'CA, \triangle C'AB$ erected outwardly. Their centers are $X, Y, Z,$ respectively, thus $Np \equiv AX \cap BY \cap CZ$ is the 1st Napoleon point of $\triangle ABC$.

The point Z is constructible and C is then on ZNp, such that $\frac{BX}{BC} = \frac{\sqrt{3}}{3}$ and $\angle XBC = 30^\circ$. Hence, C is found as the intersection of ZNp with the line image of ANp under the spiral similarity with center B, coefficient $\frac{\sqrt{3}}{3}$ and rotational angle 30° counterclockwise.

Quick Reply

High School Olympiads

collinearity

Moron
#1 Mar 6, 2015, 5:01 pm

Given a non-isosceles triangle ABC with incircle k with center S. k touches the side BC, CA, AB at P, Q, R respectively. The line QR and line BC intersect at M. A circle which passes through B and C touches k at N. The circumcircle of triangle MNP intersects AP at L. Prove that S, L, M are collinear.

TelvCohl
#2 Mar 6, 2015, 9:06 pm

My solution:

Let T be the midpoint of MP .

From $(M, P; B, C) = -1 \Longrightarrow TP^2 = TB \cdot TC$,
so T is the radical center of $\{\odot(S), \odot(BNC), \odot(ABC)\} \Longrightarrow NT$ is the tangent of $\odot(S)$,
hence we get $TN = TP = TM \Longrightarrow MP$ is the diameter of $\odot(MNP) \Longrightarrow ML \perp AP$ . ... $(\star)$

Since M lie on the polar QR of A WRT $\odot(S)$,
so AP is the polar of M WRT $\odot(S) \Longrightarrow MS \perp AP$,
hence combine with $(\star)$ we get S, L, M are collinear .

Q.E.D

P.S. For more property about this configuration you can see :

http://www.artofproblemsolving.com/community/c6h17323
http://www.artofproblemsolving.com/community/c6h332584

gavrilos
#3 Mar 6, 2015, 9:38 pm

Hello!Tevl Cohl was way faster than me.My solution is almost the same.Let me note that another approach for the relation $ML \perp AP$ is to use the lemma that NP bisects $\angle BNC$ which,since M, B, P, Q are a harmonic series,gives that $MN \perp NP$ as internal and external bisectors of the angle $\angle BNC$.

Luis González
#4 Mar 6, 2015, 9:49 pm

Posted before at http://www.artofproblemsolving.com/community/c6h318871

High School Olympiads

Kazakhstan olympiad 2009

Reply

Source: Presendential Kazakhstan olympiad 1day 3 problem

Ulanbek_Kyzylorda KTL
#1 Dec 16, 2009, 5:40 pm

Triangle ABC has incircle w centered as S that touches the sides BC,CA and AB at P,Q and R respectively. AB isn't equal AC, the lines QR and BC intersects at point M, the circle that passes through points B and C touches the circle w at point N, circumcircle of triangle MNP intersects with line AP at L (L isn't equal to P). Then prove that S,L and M lie on the same line

Luis González
#2 Dec 16, 2009, 8:43 pm • 1

Perpendicular bisector of $\overline{BC}$ cuts the circle $\odot(BNC)$ at E lying on the same side of A with respect to $BC.$ Since NE and NP are the external and internal bisector of $\angle BNC,$ it follows that EN cuts BC at the harmonic conjugate of P with respect to $BC,$ in other words, EN goes through $M \Longrightarrow \angle MNP=90^\circ.$ Then $\angle MLP=\angle MNP=90^\circ.$ Since LP is the polar of M with respect to the incircle $(S),$ then PL goes through $A.$

Ulanbek_Kyzylorda KTL
#3 Dec 16, 2009, 9:14 pm

can you explain me polarity please

Ulanbek_Kyzylorda KTL
#4 Dec 17, 2009, 11:42 am

who has got another solution

nikolapavlovic
#5 Mar 13, 2016, 10:41 pm

Let the circle that touches k and passes thru BC be ω
The homothety centered at N taking k to ω.It takes P to the midpoint of arc BC thus NP is the angle bisector of $\angle CNB.AP,BQ,CR$ are concurrent thus $(B,C;P,M) \implies \odot MNP$ is the Apollonius circle of $\triangle CNB \implies \angle MNP=\frac{\pi}{2}$.All we need to prove now is :

$$AP\perp MS$$

M is on the polar of A wrt k and P wrt k thus AP is the polar of M wrt k so $MS\perp AP$ and we are done.

This post has been edited 2 times. Last edited by nikolapavlovic, Mar 14, 2016, 12:40 am

Quick Reply

High School Olympiads

Unique points and collinearity

Locked

Moron

#1 Mar 6, 2015, 4:51 pm

Given acute triangle ABC with circumcenter O and the center of nine-point circle N. Point N_1 are given such that $\angle NAB = \angle N_1AC$ and $\angle NBC = \angle N_1BA$. Perpendicular bisector of segment OA intersects the line BC at A_1. Analogously define B_1 and C_1. Show that all three points A_1, B_1, C_1 are collinear at a line that is perpendicular to ON_1.

TelvCohl

#2 Mar 6, 2015, 8:15 pm

My solution:

Let ℓ_A, ℓ_B, ℓ_C be the perpendicular bisector of OA, OB, OC, respectively .
Let $\triangle A_2B_2C_2$ be a triangle with sides ℓ_A, ℓ_B, ℓ_C $(A_2 = \ell_B \cap \ell_C, B_2 = \ell_C \cap \ell_A, C_2 = \ell_A \cap \ell_B)$.

Since N_1 is the Kosnita point of $\triangle ABC$,
so N_1 lie on AA_2, BB_2, CC_2 (well-known) .

From Desargue theorem (for $\triangle ABC$ and $\triangle A_2B_2C_2$) we get A_1, B_1, C_1 are collinear .
From Sondat's theorem (for $\triangle ABC$ and $\triangle A_2B_2C_2$) we get ON_1 is perpendicular to their perspectrix $\overline{A_1B_1C_1}$.

Q.E.D

Luis González

#5 Mar 6, 2015, 9:33 pm

Posted at least 3 times before. See
http://www.artofproblemsolving.com/community/c6h1395
http://www.artofproblemsolving.com/community/c6h373509
http://www.artofproblemsolving.com/community/c6h462456

High School Olympiads

Iranian challenge [Kosnita point property]

Source: Iran 2001

Arne
#1 Oct 31, 2003, 3:32 am

Suppose that a triangle ABC has circumcenter O, and let N be the center of the nine-point circle of the triangle ABC.
Choose a point N' such that $\angle N'BA = \angle NBC$ and $\angle N'AB = \angle NAC$.
Suppose the perpendicular bisector of the segment OA meets BC in A'.
Define two points B' and C' similarly.
(a) Prove that the points A', B', C' lie on a line l.
(b) Prove that l is perpendicular to ON'.

grobber
#2 Oct 31, 2003, 3:48 am

Hmmm... I wouldn't say that yet. I get the feeling (b) is going to take a LONG time.

Arne
#3 Oct 31, 2003, 3:52 am

Yeah, the official solution ... it is really weird and long.

But maybe it might be a good one for you ... your favourite technique, you know ?

grobber
#4 Oct 31, 2003, 5:56 am

Ok, I do have a solution for (a), but I used a theorem which i'm not going to prove here (you can look it up on the Net; the proof isn't all that hard, it's just a basic application of Ceva's thm):

KARIYA'S THEOREM: Let I be the incenter of ABC and let D, E, F the projections of I on BC, CA, AB respectively. Take A1 on (ID, B1 on (IE and C1 on (IF s.t. IA1=IB1=IC1. then AA1, BB1, CC1 are concurrent.

Now for (a): Let A1 be the intersection between the perpendicular bisectors of OB and OC. By the same method we obtain pts B1 and C1. Then O is the incenter of A1B1C1 and by using Kariya's thm for triangle A1B1C1 we get AA1, BB1, CC1 concurrent. Then, by using Desargue's thm for triangles ABC and A1B1C1 we get the fact that the intersection pts of the pairs of lines (BC,B1C1), (CA, C1A1), (AB, A1B1) are collinear, which is exactly what we wanted.

This isn't exactly what I understand by a nice solution, but it's still a solution (I'm really not happy with it, it has no "poetry").
The bad thing is that I really don't know how to use any of this to prove (b). Maybe we need a different type of soln for (a), one which would help us approach (b)...

grobber
#5 Nov 1, 2003, 12:48 am

I've discovered a really interesting thing (by using a dynamic geometry program - Euklides):

I've defined A1, B1, C1 in my previous post. I've also proved that AA1, BB1, CC1 are concurrent. Well, guess what their concurrency point is. That's right, it's N' (unfortunately I don't have a soln for this, I've just observed it). This is the only thing I have to prove in order to have a complete soln (it's not obvious from what I've said so far; the rest of the solution involves an inversion; I'll post it when or if I can prove the thing Ive observed).

Man! I feel like I'm so close, yet so far...

grobber

#6 Nov 1, 2003, 3:12 pm • 1

I've managed to prove that thing I mentioned in my previous post, but let's assume it known for now (it's a computational proof and I don't know if I'll post it; I'll post the main idea though).

So we assume we know that N' is on AA1, BB1, CC1. Let T, U, V be the midpts of OA, OB, OC respectively, and X, Y, Z be the midpts of BC, CA, AB respectively. Now consider the inversion of pole O and power R^2/2. The circumcircles of XOT, YOU, ZOV have a second common point O' because they have a common point O and their centers are collinear (the line is obtained from I by a homothety of center O and ration 1/2). What we must prove is that N' is on OO'. By the inversion I've mentioned the circles XOT, YOU, ZOV turn into the lines AA1, BB1, CC1 respectively. These lines intersect at N', which must then be the image of O' (the other common point of the 3 circles which have the 3 lines concurrent at N' as images), so O, N', O' are collinear, which is what we want to prove.

The idea to prove that AA1, BB1, CC1 are concurrent at N':

Let's denote by (b,c)(X) the ratio between the distance from X to the side b and the distance from X to the side c. It would be enough to prove that (b,c)(N') = (b,c)(A1), which means that N' is on AA1 (it's obvious that we would use an analogous method to prove N' to be on BB1, CC1). Since N' and N are isogonal points (this means that thing abt the angles) we get (b,c)(N')=1/(b,c)(N). (b,c)(N) isn't at all hard to calculate, it's just boring (we get it in terms of cos and sin of A, B, C). The same could be said abt (b,c)(A1). To calculate this one I took the intersection point between the perpendicular at B to OB and the one at C to OC (call it K) for which we can easily calculate the distances to the sides b and c. A1 is the midpt of OK, so we can calculate d(A1,b) (for example) as (d(O,b)+d(K,b))/2 etc.

I hope it's clear enough.

darij grinberg

#7 Jul 2, 2004, 4:04 pm

> *Arne wrote:*
> Suppose that a triangle ABC has circumcenter O, and let N be the center of the nine-point circle of the triangle ABC. Choose a point N' such that $\measuredangle N'BA = \measuredangle NBC$ and $\measuredangle N'AB = \measuredangle NAC$.

This means that the point N' is simply the isogonal conjugate of the nine-point center N of triangle ABC. This point N' is called the **Kosnita point** of triangle ABC; its main property of this point is the following: If X, Y, Z are the circumcenters of triangles BOC, COA, AOB, then the lines AX, BY, CZ concur at N'.

> *Arne wrote:*
> Suppose the perpendicular bisector of the segment OA meets BC in A'.
> Define two points B' and C' similarly.
> **(a)** Prove that the points A', B', C' lie on a line l.
> **(b)** Prove that l is perpendicular to ON'.

The solution is immediate using the Sondat theorem. Indeed, consider the circumcenters X, Y, Z of triangles BOC, COA, AOB. Then, since the circumcenters Y and Z of triangles COA and AOB both lie on the perpendicular bisector of the segment OA, the line YZ is the perpendicular bisector of the segment OA, and hence this line meets the line BC at the point A'. Similarly, the lines ZX and XY meet CA and AB at the points B' and C'.

Now, since the line YZ is the perpendicular bisector of the segment OA, we have $YZ \perp OA$, and similarly $ZX \perp OB$ and $XY \perp OC$, so that the perpendiculars from the vertices A, B, C of triangle ABC to the sidelines YZ, ZX, XY of triangle XYZ concur at the point O.

On the other hand, the circumcenters O and X of triangles ABC and BOC both lie on the perpendicular bisector of the segment BC. Hence, $BC \perp OX$, and again we have by analogy $CA \perp OY$ and $AB \perp OZ$. Thus, the perpendiculars from the vertices X, Y, Z of triangle XYZ to the sidelines BC, CA, AB of triangle ABC concur at the point O, too.

Therefore, the triangles ABC and XYZ are orthologic, and both of their orthologic centers coincide with O. Now, since the lines AX, BY and CZ concur at the point N', it follows from the Sondat theorem that the intersections of the lines YZ, ZX, XY with the lines BC, CA, AB - i. e. the points A', B', C' - lie on one line l, and the points O, O and N' lie on one line perpendicular to l. In other words, the line ON' is perpendicular to l. This solves the problem.

But I don't know any elegant proof for the Sondat theorem. Do you know any good proof for your problem, so I could see whether it generalizes?

Darij

This post has been edited 1 time. Last edited by darij grinberg, Mar 22, 2007, 7:05 pm

sunchips

#8 Feb 24, 2007, 12:35 am

I know this is probably too much to ask, but does a simpler solution exist? cuz this question is also listed in a hmwrk problem at MOP 2002, as a "medium hard" problem, and by the looks of it, it should be VERY hard.....

see *http://www.unl.edu/amc/a-activities/a6-mosp/a6-1-mosparchives/2002-ma/hwpages/02mosphw.pdf*

treegoner

#9 Feb 25, 2007, 10:51 am

See http://forumgeom.fau.edu/FG2005volume5/FG200502index.html

Quick Reply

High School Olympiads

circumcircle and center of nine-point circle

Reply

bah_luckyboy
#1 Feb 7, 2012, 10:56 am

Let O and N be the circumcenter and center of nine-point circle of ΔABC, respectively. Let N_1 be a point such that $\angle N_1BA = \angle NBC$ and $\angle N_1AB = \angle NAC$. Let the perpendicular bisector of OA meets BC at A_1. Define B_1 and C_1 on the same way. Prove that
a. A_1, B_1, C_1 are collinear.
b. Let their line be l. Prove that $l \perp ON_1$

r1234
#2 Feb 7, 2012, 3:41 pm • 1

For **part a**, let A' be the midpoint of OA.Similarly define B', C'.Let D, E, F be the feet of the A,B,C altitudes.H is the orthocenter of ABC.Consider the circles $AA'A_1D, BB'B_1E, CC'C_1F$. Note that all these circles are coaxal with OH.Hence their centres i.e the midpoints of AA_1, BB_1, CC_1 are collinear.Hence by Gaussian line theorem A_1, B_1, C_1 are collinear.

For **part b**, N_1 is the isogonal conjugate of the nine point center N, i.e it is the Kosnita point of ABC.Now AA_1 passes through the centre of $\odot BOC$.Let O_1 be the reflection of O wrt BC.Similarly define O_2, O_3.Let l_1 be the line perpendicular to OO_1 through O_1.Similarly define l_2, l_3.Let the lines p, q, r be tangents through A, B, C to $\odot ABC$.Let $A_2 = p \cap l_1$.Similarly define B_2, C_2.
Note that A_2 is the pole of AN_1 wrt $\odot ABC$. Note that $A_1B_1C_1$ is mapped to the line $A_2B_2C_2$ with a homotehty centred at O with ratio 2.So $A_2B_2C_2$ is the polar of N_1.So $ON_1 \perp A_2B_2C_2$.
Since $A_2B_2C_2 \parallel l$, we conclude that $ON_1 \perp l$.

This post has been edited 1 time. Last edited by r1234, Feb 8, 2012, 2:55 pm

nsato
#3 Feb 7, 2012, 9:46 pm

See also
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=1395

bah_luckyboy
#4 Feb 8, 2012, 5:30 am

> *r1234 wrote:*
> Note that all these circles are coaxal with OH.Hence their centres i.e the midpoints of AA_1, BB_1, CC_1 are collinear.Hence by Gaussian line theorem A_1, B_1, C_1 are collinear.

Thank you so much, but what is Gaussian line theorem about? I've been searching on internet and only found that theorem was all about 3-dimensional space, not 2-dimensional.. I'm very pleased if you give me some reference to that theorem. Best regards

Learner94
#5 Feb 8, 2012, 3:44 pm

See here http://www.cut-the-knot.org/Curriculum/Geometry/Quadri.shtml

Quick Reply

High School Olympiads

Find r_{n} in function of r

pedro22

#1 Dec 6, 2010, 9:26 am

The circumference $O(2r)$ has diameter AB. The circumference $C(r)$ is tangent to $O(2r)$ in D and AB in O. The circumference $O_1(r_1)$ is inscribed in the curvilinear triangle ABD and there is a chain of circles tangent $O_i(r_i)$ $(i=2,3,...)$ where $O_i(r_i)$ is tangent to $O(2r)$ in $C(r)$ and $O_{i-1}(r_{i-1})$ for each i. Find r_n in function of r.

Luis González

#4 Mar 6, 2015, 6:44 am

I believe the correct proposition refers the curvilinear triangle OBD and not ABD. For convenience, let r denote the radius of the circle with diameter $\overline{AB}$ and let the circle inscribed in the curvilinear triangle ODB have indices $O_0(r_0)$. This touches AB at H.

$$OH^2+{r_0}^2={OO_0}^2 \ , \ OH^2=2r_0 \cdot r \ , \ OO_0=r-r_0$$

$$\Longrightarrow 2r_0 \cdot r+{r_0}^2=(r-r_0)^2 \Longrightarrow \ r_0=\tfrac{1}{4}r.$$

Inversion with center D and power r^2 takes $C(\frac{r}{2})$ into the line AB and takes $O(r)$ into the perpendicular bisector ℓ of $OD \Longrightarrow$ (O_0) is fixed. Hence by conformity, the chain of circles $O_i(r_i)$ is taken into another chain (K_i) tangent to $AB, \ell \Longrightarrow$ $(K_i) \cong (O_0)$. By inversion properties, we have $\dfrac{r_n}{r_0}=\dfrac{r^2}{\mathcal{K}_n}$ $(\star)$, where $\mathcal{K}_n$ denotes the power of D WRT (K_n).

Let the midparallel τ of ℓ, AB cut DO at N. Clearly all centers K_i lie on τ. Thus

$$NK_n=NO_0+n \cdot 2r_0=OH+n \cdot 2r_0=\frac{\sqrt{2}}{2}r+\frac{n}{2}r.$$

By Pythagorean theorem for the right triangle $\triangle DNK_n$, we get

$$DK_n{}^2=DN^2+NK_n{}^2=\left(\frac{3}{4}r\right)^2+\left(\frac{\sqrt{2}}{2}r+\frac{n}{2}r\right)^2=r^2\left(\frac{17}{16}+\frac{n^2}{4}+\frac{\sqrt{2}}{2}n\right)$$

$$\mathcal{K}_n=DK_n{}^2-{r_0}^2=r^2\left(\frac{17}{16}+\frac{n^2}{4}+\frac{\sqrt{2}}{2}n\right)-\frac{r^2}{16}$$

$$\Longrightarrow \mathcal{K}_n=r^2\left(\frac{n^2}{4}+\frac{\sqrt{2}}{2}n+1\right)$$

Substituting the value of $\mathcal{K}_n$ into $(\star)$ and rearranging original conditions yields

$$r_n=\frac{2r}{(n-1)^2+2\sqrt{2}(n-1)+4} \ \ , \ \ n=1,2,3,4,5,...$$

Quick Reply

High School Olympiads

Hard geometry

Reply

lenhathoang1998

#1 Mar 5, 2015, 9:32 pm

Let ABC be a triangle with incirle (I) touch BC, CA, AB at D, E, F, respectively. X, Y, Z are midpoints of BC, CA, AB. EF intersects BC, YZ at G, Q. Let M is midpoint of GD. MQ intersects (I) at U, V. $BV \cap CU = S$, $BU \cap CV = T$. Prove that: $TX \perp MS$

Luis González

#2 Mar 5, 2015, 9:53 pm

According to Mongolian test 2, problem 2, U and V lie on the circle (X) with diameter $\overline{BC}$. Therefore, it follows that MS is the polar of T WRT $(X) \Longrightarrow TX \perp MS$.

Quick Reply

High School Olympiads

Mongolian test2, problem 2

Reply

baysa

#1 May 22, 2011, 7:42 pm

Incircle of triangle ABC touch sides BC, CA, AB at points D, E, F. The line EF intersects with BC at point P, with the midline parallel to side BC at point Q. Median QM of triangle DQP intersects the incircle at R. Prove that $\angle BRC = 90$.

Luis González

#2 May 22, 2011, 11:30 pm • 2

In any triangle, it's well known that the polar of a vertex with respect to the incircle, the midline referent to a second vertex and the inner angle bisector issuing from the third concur. This has been posted before several times, for instance see CSMO 2008 Problem 6 Day 2 or Collinearity. If I denotes the incenter of $\triangle ABC$ and X, Y, Z the midpoints of BC, CA, AB, then FE, BI, XY concur at U and FE, CI, XZ concut at V. Thus, angles $\angle BUC$ and $\angle BVC$ are right, i.e. U, V lie on the circle (X) with diameter $\overline{BC}$. From the parallels $ZF \parallel YU$ and $ZV \parallel YE$, we obtain

$$\frac{QF}{QU} = \frac{QZ}{QY} = \frac{QV}{QE} \Longrightarrow QV \cdot QU = QF \cdot QE$$

Thus, Q has equal power to (I) and (X). On the other hand, since cross ratio (B, C, D, P) is harmonic, then by Newton theorem we have $MD^2 = MB \cdot MC \Longrightarrow M$ has equal power to (I) and (X). Therefore, QM is the radical axis of $(I), (X)$. If QM cuts (I) at a point R, then obviously $\angle BRC = 90^\circ$.

Quick Reply

High School Olympiads

hard to be drawn

Locked

Nimplesy

#1 Mar 5, 2015, 7:37 pm

Given two triangles ABC and PQR with the following conditions:
(i) P is the midpoint of BC and A is the midpoint of QR.
(ii) QR is the internal bisector of BAC and BC is the internal bisector of QPR.
Prove that $BA+AC=QP+PR$. Figures must be attached within the solution.

Luis González

#2 Mar 5, 2015, 8:59 pm

Posted before at
http://www.artofproblemsolving.com/community/c6h488793
http://www.artofproblemsolving.com/community/c6h350292

High School Olympiads

P is the mid-point of BC and A is the midpoint of QR

Source: India tst 2002 p13

Sayan

#1 Jul 13, 2012, 11:09 am

Let ABC and PQR be two triangles such that

(a) P is the mid-point of BC and A is the midpoint of QR.

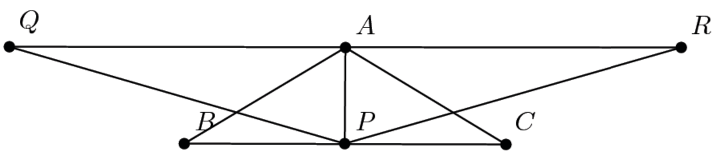
(b) QR bisects $\angle BAC$ and BC bisects $\angle QPR$

Prove that $AB + AC = PQ + PR$.

applepi2000

#2 Jul 14, 2012, 5:01 am

This seems very false, just consider when $BC||QR$ like so:



I think this problem need the condition $BC \not\parallel QR$ to be true. Edit:
Ah, right. Thanks.

This post has been edited 1 time. Last edited by applepi2000, Jul 14, 2012, 5:26 am

negativebplusorminus

#3 Jul 14, 2012, 5:08 am

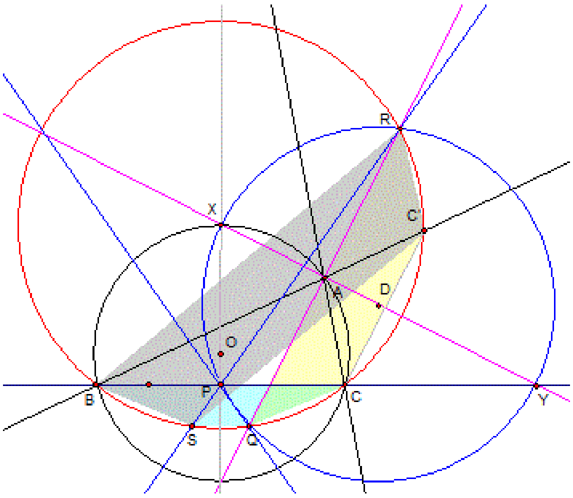
Or perhaps by "bisect", the problem meant an internal angle bisector, rather than an external one.

yetti

#4 Jul 14, 2012, 5:08 am

Let $(O), (D)$ be circumcircles of $\triangle ABC, \triangle PQR,$ resp. Let perpendicular bisectors of $|BC|, |QR|$ meet at X and let $Y \equiv BC \cap XA \Longrightarrow X \in (O) \cap (D)$ and $Y \in (D)$. Let $C' \in AB$ be reflection of C in $XA \Longrightarrow |AC'| = |AC|$ and perpendicular bisectors XP, XA of $|BC|, |CC'|$ meet at circumcenter X of $\triangle BCC'$. $\triangle AXB \sim \triangle ACY$ are similar, having equal angles $\Longrightarrow \overline{AB} \cdot \overline{AC'} = \overline{AB} \cdot \overline{AC} = \overline{AX} \cdot \overline{AY} = \overline{AQ} \cdot \overline{AR} \Longrightarrow BQC'R$ is cyclic. Isosceles trapezoid $QCC'R$ is also cyclic $\Longrightarrow BQCC'R$ is cyclic with circumcircle (X) of $\triangle BCC'$. Let PR cut (X) again at S. Since P is midpoint of $|BC|$ and BC bisects $\angle RPQ \Longrightarrow S$ is reflection of Q in perpendicular bisector XP of $|BC| \Longrightarrow |PS| = |PQ|$ and $BSQC$ is isosceles trapezoid $\Longrightarrow BSC'R$ is also isosceles trapezoid and $|AB| + |AC| = |AB| + |AC'| = |BC'| = |SR| = |PS| + |PR| = |PQ| + |PR|$.

Attachments:



Quick Reply

High School Olympiads

A hard problem (From Japan 2001)

Reply

Source: Japanese MO Finals 2001

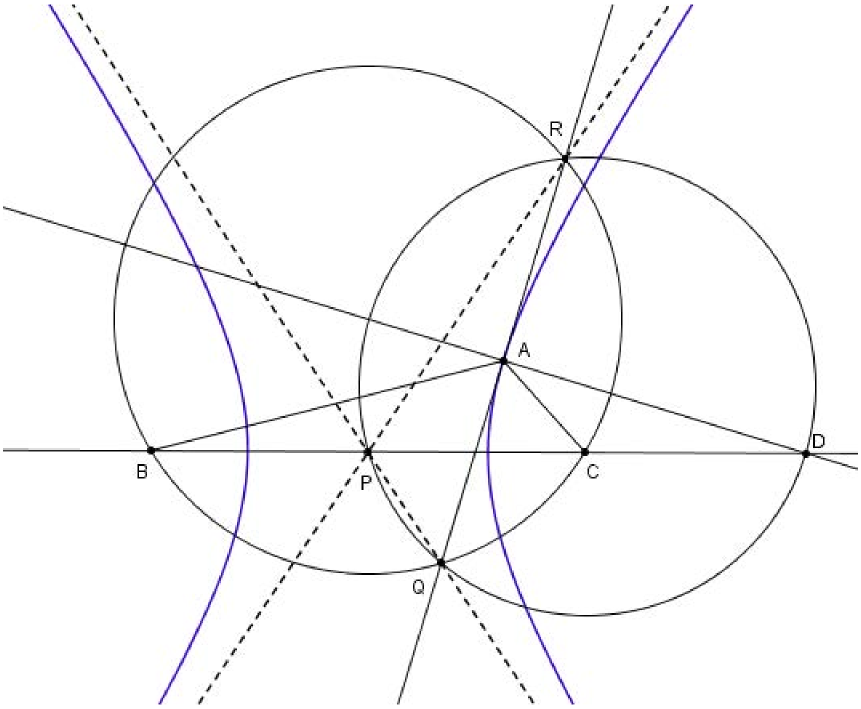
litongyang
#1 May 24, 2010, 7:11 pm

Suppose that ABC and PQR are triangles such that A, P are the midpoints of QR, BC respectively, and QR, BC are the bisectors of $\angle BAC, \angle QPR$. Prove that $AB + AC = PQ + PR$.

Luis González
#2 May 24, 2010, 9:11 pm

See the problem On the hyperbola tangent for an equivalent formulation.

Attachments:



litongyang
#3 May 25, 2010, 5:10 pm

Thank you very much.

Quick Reply

High School Olympiads

some hard problems

Source: me

vinoth_90_2004

#1 Jun 20, 2004, 9:55 am

Let X be a circle, Y a point outside the circle and Z a point on the circle (Y, Z fixed). Consider the pencil of lines through Y cutting the circle in variable points M and N. Find the locus (as M and N vary) of (i) the orthocentre (ii) symmedian point of MNZ.

(error fixed now)

This post has been edited 1 time. Last edited by vinoth_90_2004, Jun 20, 2004, 11:55 am

grobber

#2 Jun 20, 2004, 12:42 pm

The locus of the orthocenter is a disk centered at Z, of radius twice that of circle (X): Assume M is also fixed. In this case the locus of the orthocenter is a circle the same radius as (X), passing throuh Z and M. The locus we're looking for is obtained by rotating this circle around Z, thus covering a disk of radius twice that of the circle (X).

In the second case, the locus seems to be the disk (X). This is because according to my sketch, the locus of the symmedian pt when Z, M are fixed is an ellipse situated inside the circle (X), tangent to this circle in M, Z. I didn't try to prove this yet, but I will. Anyway, as we move M around the circle, this ellipse will cover the entire disk and only the disk, I think we should eliminate the point Z from the locus, but this is merely a detail. It's not a symmedian point of any non-degenerate triangle ZMN, but it's a limit point. Actually, the same goes for all the points of the circle (X).

Of course, the proof of the fact that the locus of the symmedian pt is an ellipse will have to use some projective stuff, I just don't know what yet.

grobber

#3 Jun 20, 2004, 12:52 pm

Ok, I've got it:

Let z, m, n be the lines tangent to (X) in Z, M, N respectively. It's well-known that the lines formed by $(Z, m \cap n),\ (M, z \cap n), (N, m \cap z)$ are the symmedians. When M, Z are fixed, so are m, z. n varies being tangent to (X). This means that it determines a homographic map from m to z, the map being $m \cap n \to z \cap n$. This means that the transformation from the pencil through M to the pencil through Z mapping the symmedian through M to the symmedian through Z is projective, so the locus of the intersection of two corresponding lines through this transformation must be a conic (these are all well-known properties of projective transformations) passing through M, Z.

Now all we need to show in order for the entire thing to be over is that the symmedian pt can't lie outside the circumcircle, and this is really easy because the symmedian point is always inside the triangle, and the triangles ZMN always lie inside (X).

darij grinberg

#4 Jun 20, 2004, 1:01 pm

> *grobber wrote:*
>
> The locus of the orthocenter is a disk centered at Z, of radius twice that of circle (X): Assume M is also fixed. In this case the locus of the orthocenter is a circle the same radius as (X), passing throuh Z and M. The locus we're looking for is obtained by rotating this circle around Z, thus covering a disk of radius twice that of the circle (X).

I believe this is a misunderstanding. The line MN must pass through Y, hence the sought locus is a curve (it is, indeed, a circular

I believe this is a misunderstanding. The line MN must pass through Y, hence the sought locus is a curve (it is, indeed, a circular arc, if my dynamic geometry drawing is correct, but I have no proof).

Darij

grobber
#5 Jun 20, 2004, 1:25 pm

> *darij grinberg wrote:*
>
> > *grobber wrote:*
> > The locus of the orthocenter is a disk centered at Z, of radius twice that of circle (X): Assume M is also fixed. In this case the locus of the orthocenter is a circle the same radius as (X), passing throuh Z and M. The locus we're looking for is obtained by rotating this circle around Z, thus covering a disk of radius twice that of the circle (X).
>
> I believe this is a misunderstanding. The line MN must pass through Y, hence the sought locus is a curve (it is, indeed, a circular arc, if my dynamic geometry drawing is correct, but I have no proof).
>
> Darij

Yes, it is a misunderstanding. For some reason, I was looking for the locus of those pts when Z is fixed and M, N just go around the circle. I don't know why I did that, since what Vinoth wanted is pretty clear.

grobber
#6 Jun 20, 2004, 2:28 pm

Ok, let me make up for that mistake. I only have the answer to the first problem, though.

Let X be the center of the circle (X) and let X' be the center of the circumcircle of the triangle formed by M, N and the orthocenter H of MNZ. It's easy to show that X' is on a circle centered at Y. Now we prove that the vector $\vec{X'H}$ is constant, thus obtaining the result that H moves on a circle obtained from the locus of X' by a translation.

In order to show that $\vec{X'H}$ is constant it's enough to observe that $\vec{X'H}=\vec{XZ}$, which is constant because X, Z are fixed.

We've shown that H is on that circle, but not the entire circle works. However, determining the exact arc is easy. We simply consider its limit points reached by H when MN becomes tangent to (X).

[Tex question: How do you make wider arrows over vectors?]

grobber
#7 Jun 20, 2004, 2:39 pm

As for the second problem, the locus seems to be a conic, so it definitely won't be as easy to find. I don't even know if it's always the same sort of conic (sometimes it looks like an arc of an ellipse tangent to (X) in Z and sometimes it looks like an arc of a parabola with the same property.

darij grinberg
#8 Jun 20, 2004, 4:12 pm

Grobber, thanks for the solution of the first problem! I had a similar idea: The locus for (i) is (a part of) the circle with center T and radius TZ, where X is the center of the circle (X), and T is the point where the parallel to XZ through Y meets the parallel to XY through Z. You will easily see that this is equivalent to what you said.

[Concerning the TeX question: I have always been using $\overrightarrow{XZ}$.]

Darij

grobber

#9 Jun 25, 2004, 1:21 pm

Here's an idea for the second problem:

Assume the variable line through Y cuts (X) in M, N. The tangents in M, N to (X) intersect at point T the locus of which is a line (it's the pole of MN, which always passes through Y, so T must be on the polar of Y; maybe the locus of T isn't the entire line, but that's not really important :)). Let's try to find the locus of $S = MN \cap ZT$. This point is the fourth harmonious conjugate of Z, T', T, where $T' = ZT \cap (X)$. The transformation which maps a point P of the plane to the fourth harmonious conjugate of $Z, P, ZP \cap \ell$ (ℓ is a line) is projective, so it turns conics into conics (this can be shown by linear algebra, using the matrices of the conic and the projective transformation). We apply this for $\ell =$the polar of Y and the conic we want to transform being (X). This turns the points of (X) into the points of the locus of S, which is thus a conic (call it (S)).

We now apply the same result to the same ℓ, but this time trying to transform the conic (S). We obtain the locus of K (the symmedian of ZMN), which must then be a conic too.

I think it's not that hard to show that because in these cases Z is on the conics we are transforming, the resulting conics will all be tangent to each other in Z.

The locus of K is an arc of a conic tangent to (X) in Z. Moreover, it's the arc which is situated inside (X).

grobber

#10 Jun 25, 2004, 1:37 pm

I said that by a projective transformation a conic turns into another conic, I don't know if this is true for transformations which don't have a non-singular matrix, but I think our transformation has a non-singular matrix.

You might have noticed that I'm not really sure about all of this, so I apologuize just in case it's wrong :D.

[Edit: here's is something that's very useful]

TelvCohl

#11 Mar 4, 2015, 5:07 pm

My solution for **(ii)**:

Let K be the symmedian point of $\triangle MNZ$.
Let the polar of Y WRT $\mathcal{X}$ cut $\mathcal{X}$ at Y_1 and Y_2 .
Let T be the pole of MN WRT $\mathcal{X}$ and $S = ZT \cap MN$.

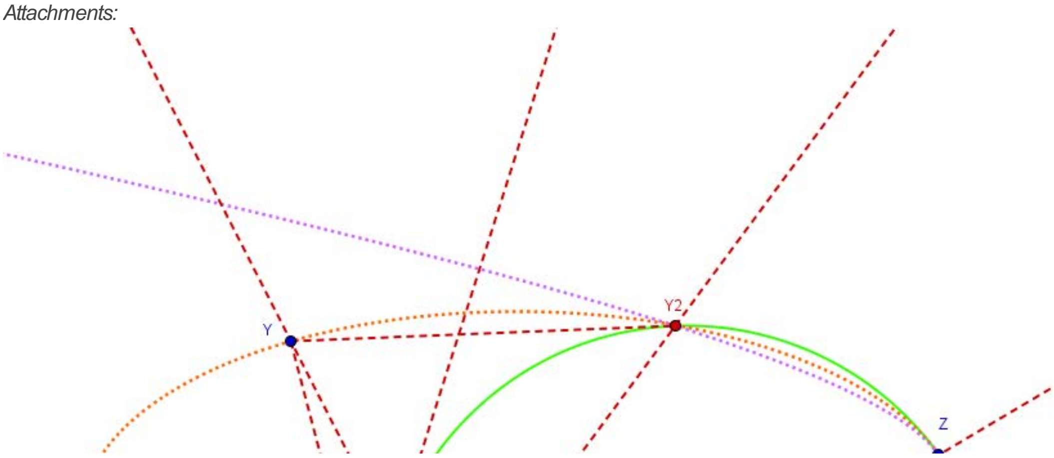
It's well-known that $K \in ZT$.

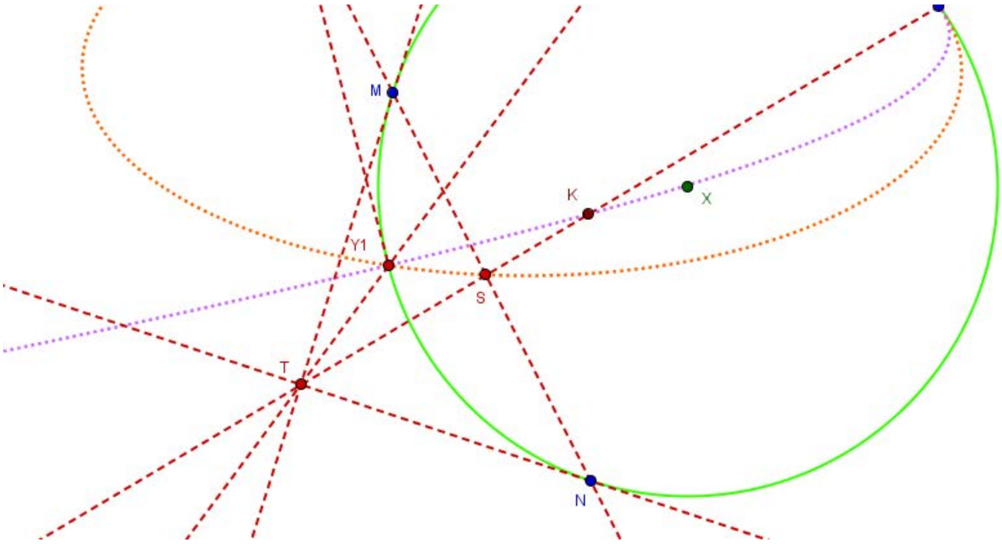
Since Y lie on the polar MN of T WRT $\mathcal{X}$,
so T lie on the polar Y_1Y_2 of Y WRT $\mathcal{X}$ . ... $(\star)$
From $Y(Y_1, Y_2; T, S) = -1 \Longrightarrow$ pencil $YT \mapsto$ pencil YS is homography ,
so combine with $(\star)$ we get pencil $ZT \mapsto$ pencil YS is homography $\Longrightarrow$ S lie on a conic $\mathcal{C}_1$.
Consider the case $M \equiv N \equiv Y_1$ and the case $M \equiv N \equiv Y_2 \Longrightarrow \mathcal{C}_1$ pass through Y, Y_1, Y_2, Z .

Since $(T, K; S, Z) = -1$,
so pencil $Y_1S \mapsto$ pencil Y_1K is homography $\Longrightarrow$ pencil $Y_1K \mapsto$ pencil ZT is homography ,
hence $K \equiv Y_1K \cap ZT$ lie on a conic $\mathcal{C}_2$ (pass through Y_1, Y_2, Z) $\Longrightarrow$ the locus of K is the part of $\mathcal{C}_2$ inside $\mathcal{X}$.

Q.E.D

Attachments:





Luis González

#12 Mar 4, 2015, 9:38 pm

Another solution for (ii):

Label (O) the fixed circle. Tangents of (O) at M, N meet at E and ZE cuts (O) again at F. K denotes the symmedian point of $\triangle ZMN.$

Cross ratio (Z,K,F,E) equals that of equilateral triangle $\triangle ZMN$ with center K and circumcircle $(O),$ as there is a homology transforming $\triangle ZMN$ into an equilateral triangle with circumcircle the image of the circumcircle of $\triangle ZMN.$ Hence $(Z,K,F,E)=\tfrac{1}{4} \Longrightarrow$ the application $K \mapsto F$ is therefore homologic, fixing the polar of Y WRT (O) and the line pencil through $Z \in (O) \Longrightarrow$ locus of K is a conic $\mathcal{C}$ tangent to (O) at $Z.$

Quick Reply

High School Olympiads

Good problem on circumcenter of an isosceles triangle

Reply

Grinver

#1 Oct 19, 2010, 9:51 am

Let ABC be an isosceles triangle in which angle C measures 120°. A line that passes through O, which is the circumcenter of triangle ABC, cuts AB, BC, and CA at X, Y, and Z, respectively. Show that

$$\frac{1}{OX} = \frac{1}{OY} + \frac{1}{OZ}$$

Luis González

#2 Oct 20, 2010, 1:09 am

This relation is true for $X \in \overline{AB}$. When X lies on the extensions of $\overline{AB}$, then appropriate choice of signs has to be considered. Assuming $X \in \overline{BC}$, $AOBC$ is a rhombus formed by equilateral triangles $\triangle AOC$ and $\triangle BOC$, i.e. $OA = OB = OC = CA = CB = L$. From $\triangle OAX \sim \triangle YBX$ and $\triangle OBX \sim \triangle ZAX$, we obtain

$$\frac{OY}{OX} = \frac{YB+L}{L}, \ \frac{OZ}{OX} = \frac{AZ+L}{L} \Longrightarrow \frac{OX}{OY} + \frac{OX}{OZ} = \frac{L}{YB+L} + \frac{L}{AZ+L} \quad (1)$$

But from the similar triangles $\triangle BOY \sim \triangle AZO$ we get

$$\frac{YB}{L} = \frac{L}{AZ} \Longrightarrow \frac{1}{YB+L} + \frac{1}{AZ+L} = \frac{1}{L} \quad (2)$$

Combining (1) and (2) yields : $\dfrac{1}{OY} + \dfrac{1}{OZ} = \dfrac{1}{OX}$.

Grinver

#3 Oct 20, 2010, 11:50 am

Where did you get OY/OX from and its result?

Luis González

#5 Mar 4, 2015, 9:37 am

Another solution:

As before, $ACBO$ is a rhombus, thus reflection X' of O on X is clearly on the parallel from C to AB. Therefore $C(B, A, O, X') = -1 \Longrightarrow (Y, Z, O, X') = -1 \Longrightarrow \dfrac{1}{OX} = \dfrac{2}{OX'} = \dfrac{1}{OY} + \dfrac{1}{OZ}$.

sunken rock

#6 Mar 9, 2015, 2:17 am • 1

$\triangle OBY \sim \triangle ZAO$, so $\dfrac{BY}{OB} = \dfrac{AO}{AZ}$ (1), AB is bisector of angles $\angle OAZ$ and $\angle OBC$, hence $\dfrac{XY}{OX} = \dfrac{BY}{OB} = \dfrac{AO}{AZ} = \dfrac{OX}{XZ}$, i.e. $\dfrac{XY}{OX} = \dfrac{OX}{XZ} \implies OX^2 = XY \cdot XZ$, or $OX^2 = (OY - OX)(OZ - OX) \iff OY \cdot OZ = OX \cdot OY + OX \cdot OZ$ (2), the relation (2) being equivalent to the relation to prove.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Touching circles and length ratio

Reply

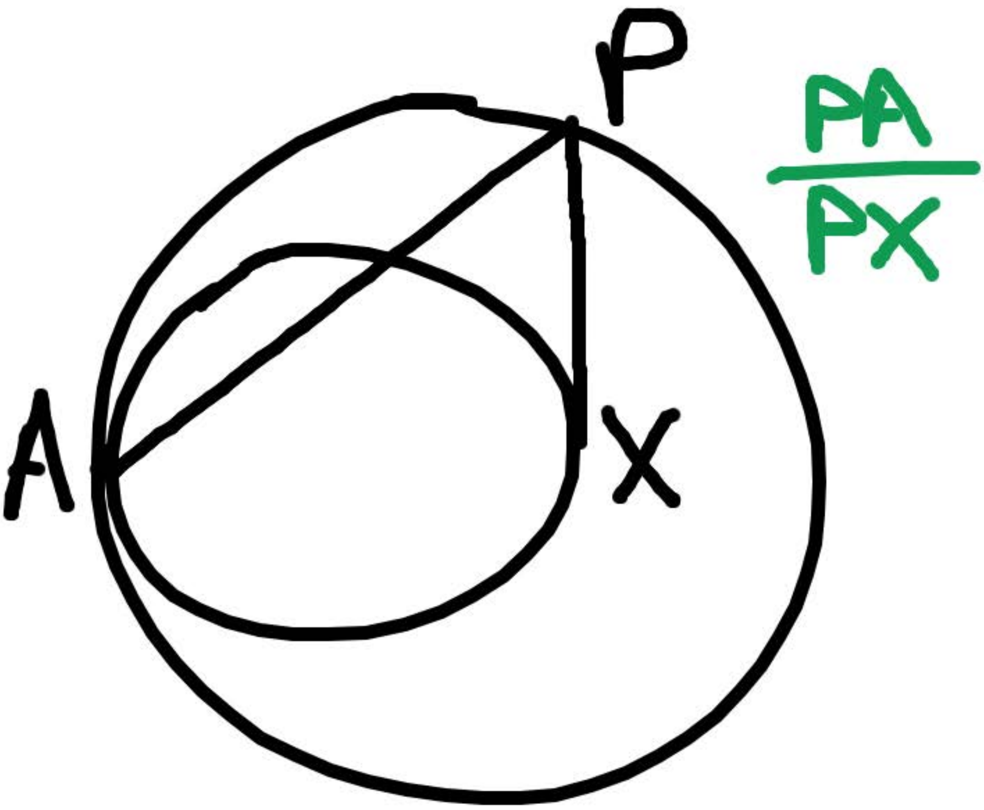
Michael888

#1 Mar 3, 2015, 11:04 pm

Given two circles of different radius touching internally at point A, take any point P on the larger circle (where $P \neq A$), and let the tangent from P to the smaller circle touch it at X. Prove that $\frac{PA}{PX}$ is constant. See the (badly drawn) diagram for clarification.

Coordinate geometry works fine, but is there a nice Euclidean solution to this? Thanks.

Attachments:



Luis González

#3 Mar 3, 2015, 11:23 pm

It's a particular case of the **Casey's chord theorem** when one of the object circles degenerates into a point. See Casey's theorem and its applications (Theorem 1).

Michael888

#4 Mar 3, 2015, 11:29 pm

That's amazing! Thank you; the pdf is excellent.

Quick Reply

High School Olympiads

Casey's theorem and its applications

Reply

Luis González
#1 Jul 2, 2011, 10:20 am • 38

The following article presents a proof of the generalized Ptolemy's theorem, also known as Casey's theorem, and its applications in the resolution of difficult geometry problems. First and second sections contain its direct proof and some applications. Third section contains some proposed problems.

Attachments:
Casey's theorem and its applications.pdf (252kb)

Amir Hossein
#2 Jul 2, 2011, 10:22 am • 1

Very nice! Thanks!

Goutham
#3 Jul 2, 2011, 3:27 pm • 1

Thank you for the file.

Dr Sonnhard Graubner
#4 Jul 2, 2011, 3:28 pm • 3

hello, nice work!
Sonnhard.

jatin
#5 Jul 2, 2011, 3:55 pm • 1

Excellent article, Luis!
Keep up the good work, and thanks a lot for the file

goodar2006
#6 Jul 2, 2011, 6:30 pm • 6

very nice luis. Thanks a lot.

RaleD
#7 Jul 11, 2011, 9:04 pm

Thanks for this nice file; using theorem 1 (didn't know it before) + some cosine law, we can easily solve imo 1999 #5.

Ligouras
#8 Aug 1, 2011, 10:09 pm

Very nice! Thanks!!!!

Leon
#9 Aug 3, 2011, 1:39 pm
Thanks a lot for your excellent article !

Leon

geotopo
#10 Nov 17, 2011, 7:46 pm
Thanks for this nice file

phuongtheong
#11 Dec 17, 2011, 11:51 am • 1
Thanks a lot!

joybangla
#12 May 22, 2013, 2:48 pm • 1
that was a really good article!

War-Hammer
#13 May 26, 2013, 11:59 pm
Hi ;

Thanks. Nice and useful.

Best Regard

romario
#14 Jan 26, 2014, 12:27 am
Thanks a lot

Quick Reply

High School Olympiads

Prove that the radical axis is parallel to the side

Reply

Source: Practice Problem 6

NEWDORMANTUSER

#1 Mar 2, 2015, 10:37 pm

Let XYZ be a triangle. Let P, Q, R be respective points on YZ, ZX, XY such that $XQ \cdot QZ = XR \cdot RY$. Let the circumcircles of XYZ and XRQ intersect in $W \neq X$. Then, prove that $XW \parallel QR$.

TelvCohl

#2 Mar 2, 2015, 11:05 pm

My solution:

Let O, O^* be the circumcenter of $\triangle XYZ, \triangle XQR$, respectively .
Let QR cut $\odot(XYZ)$ at Q^*, R^* (Q lie on the segment Q^*R and R lie on the segment R^*Q) .

Since
$XR \cdot RY = R^*R \cdot RQ^* = R^*R \cdot RQ + R^*R \cdot QQ^*$ ... (1)
$XQ \cdot QZ = Q^*Q \cdot QR^* = Q^*Q \cdot QR + Q^*Q \cdot RR^*$ ... (2)
so from (1) and (2) we get $QQ^* = RR^* \Longrightarrow OQ = OR$,
hence OO^* is the perpendicular bisector of $QR \Longrightarrow XW \parallel QR$ ($\because XW \perp OO^*$) .

Q.E.D

Luis González

#3 Mar 2, 2015, 11:23 pm

Let (O, ϱ) and (O', ϱ') denote the circumcircles of $\triangle XYZ$ and $\triangle XQR$. Then $\varrho^2 - OR^2 = XR \cdot RY = XQ \cdot QZ = \varrho^2 - OQ^2 \Longrightarrow OR = OQ \Longrightarrow OO'$ is perpendicular bisector of both $\overline{QR}$ and $\overline{XW} \Longrightarrow XW \parallel QR.$

Quick Reply

High School Olympiads

easy geometry

Reply

luciano

#1 Mar 2, 2015, 3:16 am

Let ABCD be a trapezoid,AB || CD, P the symmetry of A at BD and Q the symmetry of B at AC.Prove that:
a) <APC=<BQD

Luis González

#2 Mar 2, 2015, 4:18 am

Let X,Y be the projections of A,B on BD,AC and let M,N be the midpoints of $AC,BD.$ Since $MN \parallel AB$ and $AXYB$ is cyclic due to the right angles at $X,Y,$ then $\angle BNM=\angle XBA=\angle XYA \Longrightarrow XNMY$ is cyclic $\Longrightarrow$ $\angle NXM=\angle MYN \Longrightarrow \angle AXM=\angle BYN.$ But since $XM \parallel PC, YN \parallel QD \Longrightarrow$ $\angle APC=\angle AXM=\angle BYN=\angle BQD.$

TelvCohl

#3 Mar 2, 2015, 4:37 am

My solution:

Let P^* be the projection of A on BD and Q^* be the projection of B on AC .

Since A, B, P^*, Q^* are concyclic at a circle with diameter AB ,
so from Reim theorem we get C, D, P^*, Q^* are concyclic $\Longrightarrow \angle ACP^*=\angle BDQ^*$,
hence combine with $\angle CAP^*=\angle DBQ^* \Longrightarrow \triangle ACP^* \cup P \sim \triangle BDQ^* \cup Q \Longrightarrow \angle APC=\angle BQD$.

Q.E.D

Quick Reply

High School Olympiads

Property of PC-point

TelvCohl

#1 Feb 27, 2015, 9:08 am

Let A^*, B^*, C^* be the antipode of A, B, C in $\odot(ABC)$, respectively .
Let P be a point and $D = PA^* \cap BC, E = PB^* \cap CA, F = PC^* \cap AB$.

Prove that AD, BE, CF are concurrent $\Longleftrightarrow$ P lie on Darboux cubic

Luis González

#2 Feb 27, 2015, 11:03 pm • 1

Let $\triangle XYZ$ be the antipedal triangle of P WRT $\triangle ABC$ (X, Y, Z againts A, B, C). $\widehat{ACY} = \widehat{PCA^*} = 90^\circ - \widehat{PCA}$ and similarly $\widehat{ABZ} = \widehat{PBA^*}$. Hence, we have

$$\frac{DB}{DC} = \frac{[PBA^*]}{[PCA^*]} = \frac{PB \cdot BA^* \cdot \sin \widehat{PBA^*}}{PC \cdot CA^* \cdot \sin \widehat{PCA^*}} = \frac{PB}{PC} \cdot \frac{\cos C}{\cos B} \cdot \frac{\sin \widehat{ABZ}}{\sin \widehat{ACY}} =$$

$$= \frac{PB}{PC} \cdot \frac{\cos C}{\cos B} \cdot \frac{AC}{AB} \cdot \frac{AZ}{AY} \cdot \frac{\sin \widehat{AZB}}{\sin \widehat{AYC}} = \frac{AC}{AB} \cdot \frac{\cos C}{\cos B} \cdot \frac{PB}{PC} \cdot \frac{AZ}{AY} \cdot \frac{XY}{XZ}.$$

Thus, multypliying the cyclic expressions together, we deduce by Ceva's theorem that AD, BE, CF concur $\Longleftrightarrow$ AX, BY, CZ concur, i.e. if and only if $\triangle ABC$ and the antipedal triangle of P are perspective $\Longleftrightarrow$ P is on Darboux cubic of $\triangle ABC$.

High School Olympiads

equilateral triangle-prove

Reply

Pirkuliyev Rovsen

#1 Feb 24, 2015, 9:41 pm

Inside the an equilateral triangle ABC take the point P. From a point P on the sides AB, BC and CA perpendiculars PX,PY,PZ. It is known that $PY=\sqrt{PX\cdot PZ}$. Prove that $\angle BPC=120^\circ$.

TelvCohl

#2 Feb 24, 2015, 10:09 pm • 1

My solution:

Since $\angle XPY=\angle YPZ=120^\circ$ and $PX \cdot PZ=PY^2$,
so we get $\triangle XPY \sim \triangle YPZ \Longrightarrow \angle PXY+\angle PZY=60^\circ$,
hence $\angle PBC+\angle PCB=\angle PXY+\angle PZY=60^\circ \Longrightarrow \angle BPC=120^\circ$.

Q.E.D

Luis González

#3 Feb 27, 2015, 9:44 am

More general: $\triangle ABC$ is scalene and P is a point on its plane. X,Y,Z are the projections of P on $BC,CA,AB.$ Hence locus of $P,$ such that $PX=\sqrt{PY \cdot PZ}$ is the conic $\mathcal{C}$ through $B,C,$ the incenter I and tangent to $AB,AC.$ $\mathcal{C}$ also contains the A-excenter.

The condition $PX^2=PY \cdot PZ$ can be expressed as $\frac{PX}{PZ}=\frac{PY}{PX} \Longrightarrow$

$$\frac{\sin \widehat{PBC}}{\sin \widehat{PBA}} \cdot \frac{\sin \widehat{IBA}}{\sin \widehat{IBC}}=\frac{\sin \widehat{PCA}}{\sin \widehat{PCB}} \cdot \frac{\sin \widehat{ICB}}{\sin \widehat{ICA}} \Longrightarrow B(C,A,P,I)=C(A,B,P,I),$$

which means that P is on the conic $\mathcal{C}$ through B,C,I and tangent to $AB,AC.$ It also contains the A-excenter of $\triangle ABC$ as it clearly fulfills the locus condition.

Quick Reply

High School Olympiads

Reflections of vertices of triangle

Reply

Source: Hyacinthos #23136

rodinos

#1 Feb 27, 2015, 3:21 am

Let ABC be a triangle and P a point.

Denote:

Ab, Ac = the reflections of A in PB, PC, resp.

Ea = the Euler line of PAbAc
(= perp. bisector of AbAc, since PAbAc is isosceles)

La = the reflection of Ea in BC
A' = BC /\ Ea

Similarly.................

1. La, Lb, Lc are concurrent.
(the point of concurrence is the isogonal conjugate of P)

2. ABC, A'B'C' are perspective.

APH

Luis González

#2 Feb 27, 2015, 3:51 am

1. Let Q denote the isogonal conjugate of P WRT $\triangle ABC$ and let X be the reflection of Q on BC. Since $\angle PCA = \angle QCB = \angle XCB$ and $\angle PBA = \angle QBC = \angle XBC$, it follows that A, X are isogonal conjugates WRT $\triangle PBC \Longrightarrow PX \perp A_bA_c \Longrightarrow PX$ is perpendicular bisector of A_bA_c and the conclusion follows.

2. Note that the inconic $\mathcal{C}$ with foci P, Q is tangent to BC at A' and similarly it touches CA, AB at B', C'. Hence $\triangle ABC$ and $\triangle A'B'C'$ are perspective through the perspector of $\mathcal{C}$ (degenerate case of Brianchon's theorem).

Quick Reply

High School Olympiads

Bisects segments

Reply

Source: HSGS senior Olympiad 2014

buratinogigle

#1 Feb 19, 2015, 1:08 am

Let ABC be an acute triangle with $AB < AC$. (O) is circumcircle of triangle ABC. AD, BE, CF are altitude. Circle (ω) with center A radius AD cuts (O) at M, N.

a) Prove that MN bisects DE, DF.

b) Let EF cut BC at G. DP are diamter of (ω). PG cuts (ω) again at Q. Prove that (O) bisects segment DQ.

TelvCohl

#2 Feb 19, 2015, 6:35 am • 1

My solution:

Let $R = \odot(AD) \cap \odot(ABC)$ and $S = AR \cap BC$.
Let Y, Z be the projection of D on AC, AB, respectively .

Since $AY \cdot AC = AZ \cdot AB = AD^2 = AM^2 = AN^2$,
so $C \longleftrightarrow Y, B \longleftrightarrow Z, \odot(ABC) \longleftrightarrow MN$ under the inversion $\mathcal{I}(A, AD^2)$,
hence we get M, N, Y, Z are collinear .

Since MN is the Simson line of D WRT $\triangle ABE$,
so from Steiner theorem we get MN pass through the midpoint of DE .
Similarly we can prove MN pass through the midpoint of $DF \Longrightarrow MN$ is $D-$ midline of $\triangle DEF$.

Since $SB \cdot SC = SR \cdot SA = SD^2$,
so S lie on the radical axis MN of $\odot(A, AD)$ and $\odot(ABC)$.
Since $\odot(A, AD), G, P$ is the image of $\odot(AD), S, A$ under homothety $\mathbf{H}(D, 2)$, respectively ,
so $Q \equiv PG \cap \odot(A, AD)$ is the image of $R \equiv AS \cap \odot(AD)$ under $\mathbf{H}(D, 2)$. i.e. the midpoint R of DQ lie on $\odot(O)$

Q.E.D

buratinogigle

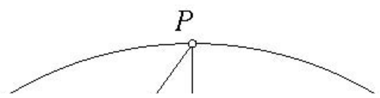
#3 Feb 19, 2015, 8:44 am • 1

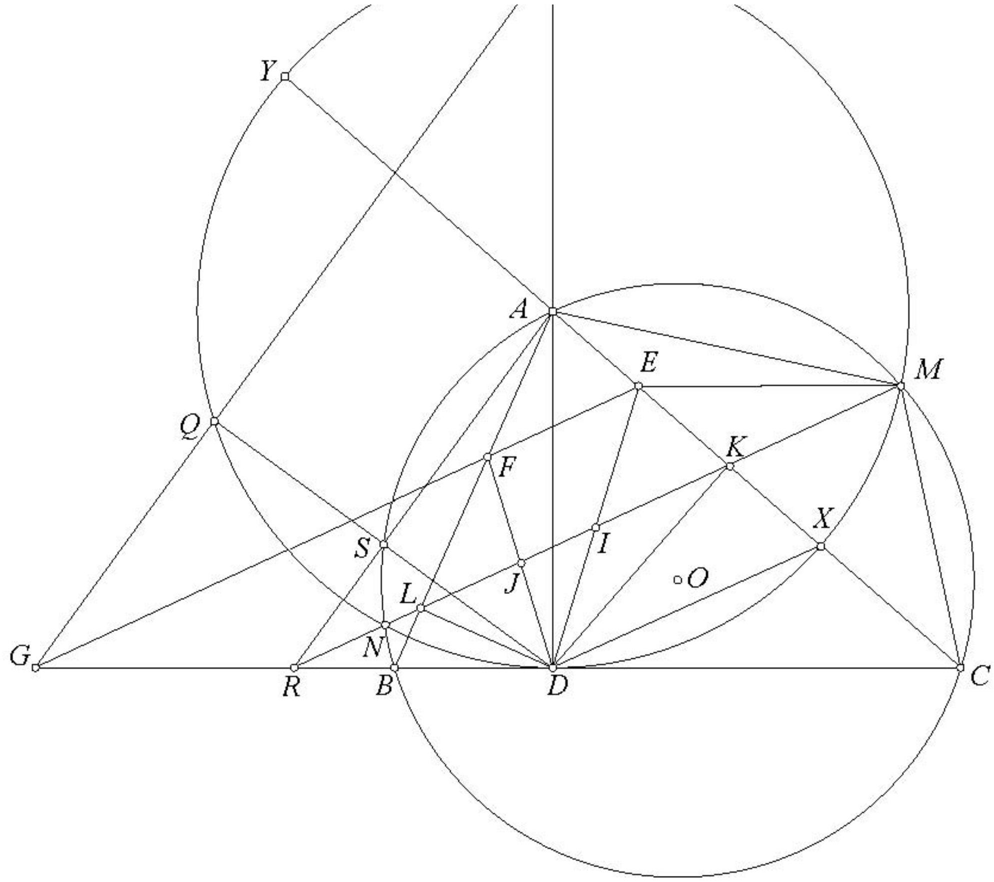
Thank you dear Telv Cohl for your interset, here is my solution

a) Let MN cuts DE, DF, AC, AB at I, J, K, L, resp. (ω) cuts AC at X, Y. We have $\overline{KX}.\overline{KY} = \overline{KM}.\overline{KN} = \overline{KA}.\overline{KC}$ deduce $(KC, XY) = -1$, thus $AD^2 = AX^2 = AY^2 = \overline{AK}.\overline{AC}$. So $DK \perp AC$. Similarly, $DL \perp AB$. Thus, $AKDL$ is cyclic, so $\angle LKF = \angle LAD = 90^\circ - \angle ABD = 90^\circ - \angle DEC = \angle IDK$. From this IDK is isoceles, DKE is right deduce I is midpoint of DE. Similarly, J is midpoint of DF.

b) Let MN cuts BC at R. AR cuts QD at S. Follow a) R is midpoint of GD. A is midpoint of DP deduce S is midpoint of QD. Because $DQ \perp PG$ therefore $DS \perp SA$. From this, S, K, L lie on circle diameter AD. We see $(BC, GD) = -1$ and R is midpoint of DG then $\overline{RS}.\overline{RA} = \overline{RK}.\overline{RL} = RD^2 = RG^2 = \overline{RB}.\overline{RC}$. From this, $ASBC$ is cyclich S deduce (O). We are done.

Attachments:





Luis González

#4 Feb 26, 2015, 9:23 pm

Problem a) was posted before at a triangle ABC.

b) Let U be the midpoint of $\overline{DG}$ and AU cuts (O) again at $R.$ Since $(B,C,D,G)=-1 \Longrightarrow$ $UD^2=UB \cdot UC=UA \cdot UR \Longrightarrow \angle ARD=\angle ADU=90^{\circ}.$ Thus since $\angle DQP=90^{\circ}$ and AU is D-midline of $\triangle DPG,$ it follows that D,Q,R are collinear, R being midpoint of $\overline{DQ}.$

Quick Reply

High School Olympiads

Two perspective triangle

Reply

Source: own

jayme

#1 Feb 25, 2015, 5:21 pm

Dear Mathlinkers,

1. ABC an acute triangle
2. (O) the circumcircle of ABC
3. A'B'C' the O-circumcevian triangle of ABC
4. (I) the incircle of ABC
5. DEF the contact triangle of ABC
6. A"B"C" the I-pedal triangle wrt DEF
7. X the second point of intersection of IA' with the circle with diameter ID and circularly
8. A+ the point of intersection near A of the circle (IA"X) with (I) and circularly.

Prove : A+B+C+ and ABC are perspective.

Sincerely
Jean-Louis

TelvCohl

#2 Feb 25, 2015, 11:09 pm

My solution:

Let $D^* = IA' \cap BC, E^* = IB' \cap CA, F^* = IC' \cap AB$.
Let $A^- = EF \cap DX, B^- = FD \cap EY, C^- = DE \cap FZ$.

Since

$$\frac{BD^*}{D^*C} = \frac{[IBA']}{[ICA']} = \frac{BI \cdot BA' \cdot \cos\frac{1}{2}\angle CBA}{CI \cdot CA' \cdot \cos\frac{1}{2}\angle ACB} \text{ ... } (1)$$

$$\frac{CE^*}{E^*A} = \frac{[ICB']}{[IAB']} = \frac{CI \cdot CB' \cdot \cos\frac{1}{2}\angle ACB}{AI \cdot AB' \cdot \cos\frac{1}{2}\angle BAC} \text{ ... } (2)$$

$$\frac{AF^*}{F^*B} = \frac{[IAC']}{[IBC']} = \frac{AI \cdot AC' \cdot \cos\frac{1}{2}\angle BAC}{BI \cdot BC' \cdot \cos\frac{1}{2}\angle CBA} \text{ ... } (3)$$

so from $(1), (2), (3)$ we get $\frac{BD^*}{D^*C} \cdot \frac{CE^*}{E^*A} \cdot \frac{AF^*}{F^*B} = 1 \Longrightarrow AD^*, BE^*, CF^*$ are concurrent ,

hence their pole WRT $\odot(I)$ are collinear . i.e. A^-, B^-, C^- are collinear . ... $(\star)$

Since the center of $\odot(IA''X), \odot(IB''Y), \odot(IC''Z)$ is the midpoint of IA^-, IB^-, IC^-, respectively ,
so from $(\star)$ we get the center of $\odot(IA''X), \odot(IB''Y), \odot(IC''Z)$ are collinear $\Longrightarrow \odot(IA''X), \odot(IB''Y), \odot(IC''Z)$ are coaxial . ... (Ψ)

Since $AA'' \cdot AI = AE^2 = AF^2$,
so A lie on the radical axis of $\{\odot(I), \odot(IA''X)\} \Longrightarrow AA^+$ is the radical axis of $\{\odot(I), \odot(IA''X)\}$.
Similarly we can prove BB^+, CC^+ is the radical axis of $\{\odot(I), \odot(IB''Y)\}, \{\odot(I), \odot(IC''Z)\}$, respectively ,
so combine with (Ψ) we get AA^+, BB^+, CC^+ are concurrent at the radical center of
$\{\odot(I), \odot(IA''X), \odot(IB''Y), \odot(IC''Z)\}$.

Q.E.D

Luis González

#3 Feb 26, 2015, 8:45 am

Inversion WRT (I) takes A'' and X into A and $U \equiv BC \cap IA'$, resp, and intersections A^+, A^- of $(I) \cap \odot(IXA'')$ are double, thus $\odot(IXA'')$ is transformed into the line $AU \equiv AA^+$. But according to Concurrent lines, $AU \equiv AA^+$ passes through X_{77} of $\triangle ABC$ and similarly for BB^+ and $CC^+ \Longrightarrow \triangle ABC$ and $\triangle A^+B^+C^+$ are perspective with perspector X_{77} on Soddy line of $\triangle ABC$.

Quick Reply

High School Olympiads

An interesting result about circumscribed quadrilateral

Source: own

tranquanghuy7198

#1 Jan 29, 2015, 8:52 am • 1

Let ABCD be a circumscribed quadrilateral. Its incircle (I) touches AB, BC, CD, DA at X, Y, Z, T, respectively. XY, ZT, AC concur at E. XZ, YT, AC concur at F. Prove that the line which passes through I and is parallel to the line joining the centers of the circumcircles of triangles AYZ and CXT bisects EF.

Luis González

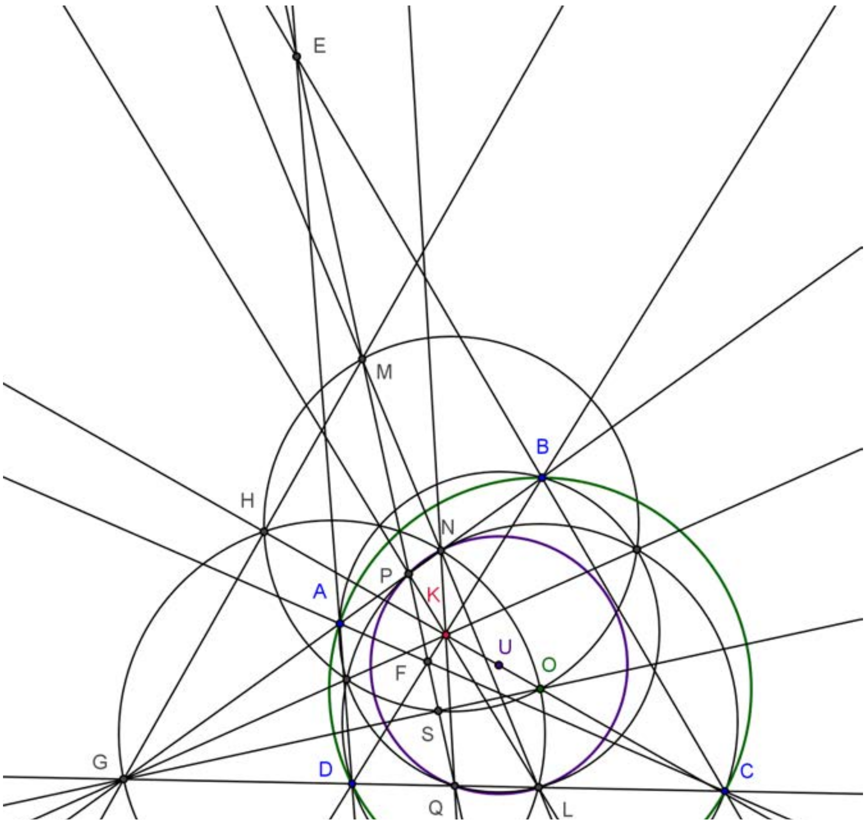
#2 Feb 8, 2015, 5:22 am • 4

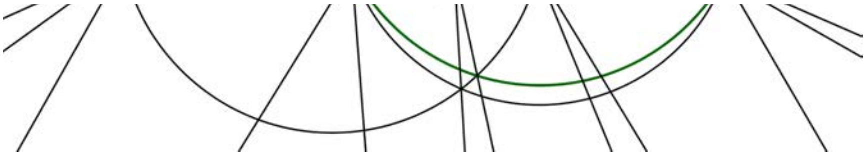
Lemma: $ABCD$ is cyclic with circumcircle (O). $E \equiv AD \cap BC$ and $F \equiv AC \cap BD$. M, N, L are the midpoints of EF, AB, CD, on Newton line of $ABCD$. Then $\odot(LAB), \odot(NCD)$ and the circle Γ with diameter $\overline{OM}$ are coaxal.

Proof: Let $G \equiv AB \cap CD$ and EF cuts AB, CD at P, Q. Since $(A, B, P, G) = -1$ and $(D, C, Q, G) = -1 \Longrightarrow GP \cdot GN = GA \cdot GB = GC \cdot GD = GQ \cdot GL$, thus $P \in \odot(NCD), Q \in \odot(LAB), PNLQ$ is cyclic with circumcircle (U) and G is on radical axis of $(O), (U)$. Since E, F are conjugate points WRT both (O) and (U), then the circle with diameter $\overline{EF}$ is orthogonal to both $(O), (U) \Longrightarrow M$ is on radical axis of $(U), (O)$. So GM is radical axis of $(U), (O) \Longrightarrow OU \perp GM$, but since $K \equiv PL \cap QN$ is pole of GM WRT (U), then $UK \perp GM \Longrightarrow OK \perp GM$ at H.

From $(D, C, Q, G) = -1$, we get $QC \cdot QD = QL \cdot QG \Longrightarrow Q$ has equal power WRT $\odot(NCD)$ and $\odot(OHG) \Longrightarrow NQ$ and OH are radical axis of $\odot(NCD), \odot(OHG)$ and $\odot(OHG), \odot(OHM) \equiv \Gamma$ meeting at their radical center $K \Longrightarrow K$ is on radical axis of $\Gamma, \odot(NCD)$. If $S \equiv OG \cap EF$, then S is on Γ, as $EF \perp OG$ (EF is the polar of G WRT (O)), thus from cyclic $OSPN$, we get $GS \cdot GO = GP \cdot GN \Longrightarrow G$ is on radical axis of $\Gamma, \odot(NCD)$. As a result, GK is radical axis of $\Gamma, \odot(NCD)$. Analogously, GK is radical axis of $\Gamma, \odot(LAB)$, thus $\Gamma, \odot(LAB), \odot(NCD)$ are coaxal with common radical axis GK. $\blacksquare$

Attachments:





Luis González

#3 Feb 8, 2015, 5:22 am • 2

Back to the main problem. The inversion WRT (I) fixes X, Y, Z, T and takes A, C into the midpoints A', B' of $TX, YZ,$ respectively, thus $\odot(AYZ), \odot(CXT)$ go to $\odot(A'YZ), \odot(C'XT)$. If M is the midpoint of $EF,$ then the polar m of M WRT (I) is taken into the circle with diameter $MI.$ Using the previous lemma for $XYZT,$ then $\odot(A'YZ), \odot(C'XT)$ and the circle with diameter MI are coaxal, thus their inverses concur, i.e. m is the radical axis of $\odot(AYZ), \odot(CXT) \Longrightarrow IM \perp m \Longrightarrow IM$ is parallel to the line connecting the centers of $\odot(AYZ), \odot(CXT)$ and the conclusion follows.

Quick Reply

High School Olympiads

Property of Salmon circle

Reply

Source: Well Known

TelvCohl
#1 Feb 5, 2015, 2:51 pm • 1

Let P be a point on BC and N be the 9-point center of $\triangle ABC$.
Let O_1, O_2 be the circumcenter of $\triangle ABP, \triangle ACP$, respectively .
Let O^* be the center of $\odot(AO_1O_2)$ and H be the projection of O^* on BC .

Prove that $AP \parallel NH$

jayme
#2 Feb 5, 2015, 4:28 pm

Dear Mathlinkers,
what is the Salmon circle? Do you have a reference?
Thank in advance
Sincerely
Jean-Louis

rodinos
#3 Feb 5, 2015, 6:37 pm

The name "Salmon Circle" comes, I think, from the 1st Salmon Theorem.
http://mathworld.wolfram.com/SalmonsTheorem.html

aph

toto1234567890
#4 Feb 5, 2015, 8:37 pm

Well, I think I have seen it somewhere but I can't remember where.

rodinos
#5 Feb 5, 2015, 11:12 pm

Salmon Circle is, I think, the circle I know as Miquel circle:

Let (a,b,c,d) be a quadrilateral and Oa, Ob, Oc, Od the circumcenters of the component triangles (b,c,d), (c,d,a), (d,a,b), (a,b,c), resp.
Oa, Ob, Oc, Od are concyclic.

Luis González
#6 Feb 5, 2015, 11:35 pm • 1

Since $\angle AO_1B = 2\angle APB = \angle AO_2C$, then the isosceles $\triangle O_1AB$ and $\triangle O_2AC$ are spirally similar $\Longrightarrow \triangle ABC$ and $\triangle AO_1O_2$ are spirally similar with center A. Hence $\angle(AO_1, AO^*)$ is constant and $\frac{AO_1}{AO^*}$ is constant $\Longrightarrow O^*$ runs on the line image of the perpendicular bisector of AB under spiral similarity with center A, rotational angle $\angle(AO_1, AO^*)$ and coefficient $\frac{AO_1}{AO^*}$. Thus, it follows that the series O^*, O_1 are similar, but the series O_1, P and O^*, H are clearly similar, thus the series P, H are similar. Therefore, it suffices to prove that $NH \parallel AP$ holds somewhere.

When P coincides with the foot of the A-altitude, then O_1, O_2 become midpoints of $AB, AC,$ hence O^*N is perpendicular bisector of $O_1O_2 \Longrightarrow (AP \parallel NH) \perp BC.$ Consequently, $NH \parallel AP$ will hold for any $P,$ as desired.

P.S. Apparently this is a problem from China IMO team training (2006). See problem geometry.

TelvCohl

#7 Feb 6, 2015, 2:37 pm • 1

Sorry for re-post . This problem appeared in Math club in Taiwan as homework without resource,
I tried to search some words such as " Salmon circle " but I did not find it. Anyway, I think it's an interesting problem and here is my proof :

Let $Q = PO_1 \cap \odot(O_1), R = PO_2 \cap \odot(O_2), G = O^*H \cap QR$.

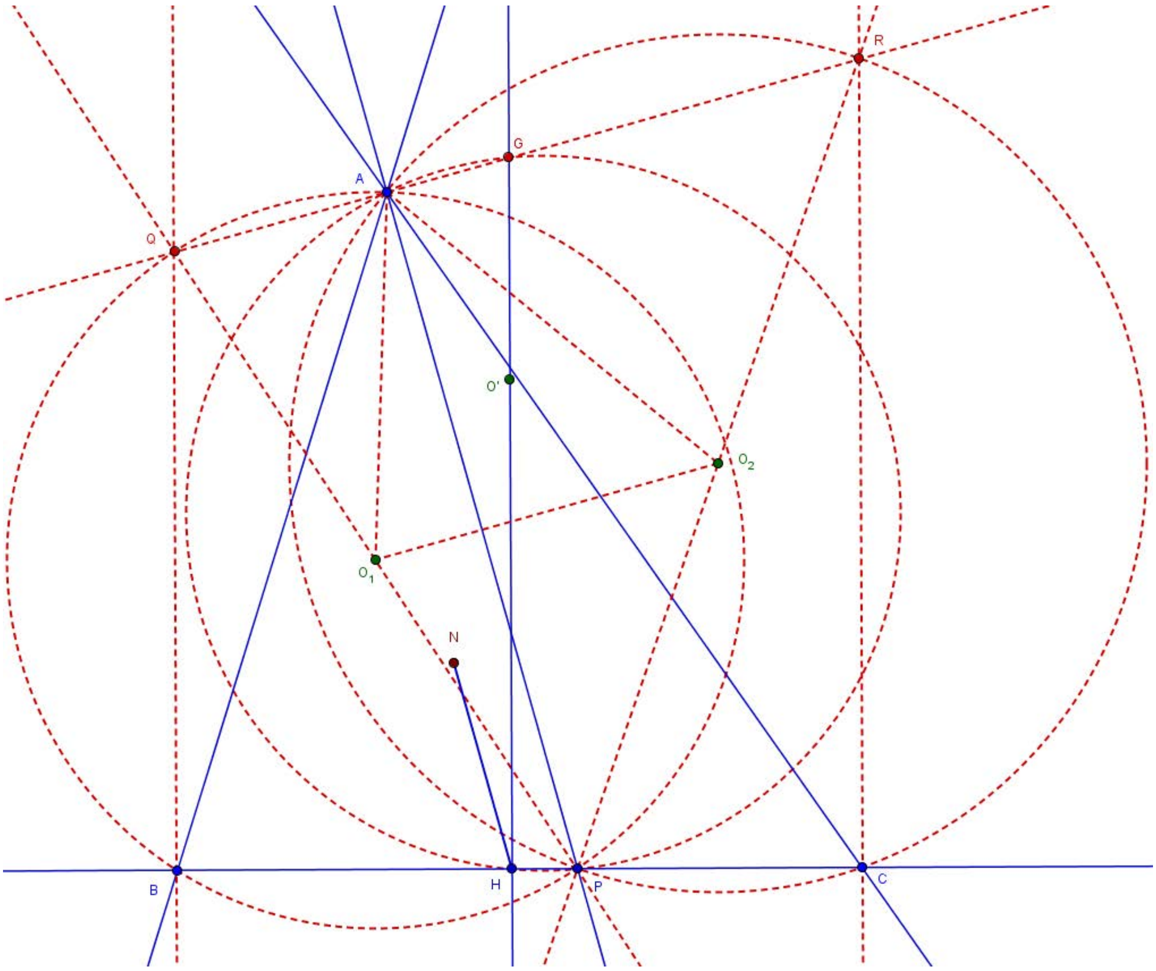
Easy to see $\triangle AO_1O_2 \sim \triangle ABC$.

Since $\angle QAP = \angle RAP = 90^\circ$,
so A is the projection of P on QR .
Since $\angle PQR = \angle ABC, \angle QRP = \angle ACB$,
so combine with $\frac{BH}{HC} = \frac{QG}{GR} \Longrightarrow \triangle ABC \cup N \cup H \sim \triangle PQR \cup O^* \cup G$.

Since A, G, H, P lie on a circle (with diameter GP) ,
so we get $\angle NHB = \angle O^*GQ = \angle APB \Longrightarrow AP \parallel NH$.

Q.E.D

Attachments:



buratinogigle

#8 Feb 7, 2015, 6:36 pm

Hi Telv you can see source of this problem in http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=282210

I have a general problem

Let ABC be a triangle with circumcenter O and P is a point on OA. E, F are projection of P on CA, AB. Q is a point on EF. The line passing through Q and perpendicular to AQ cuts PE, PF at M, N. K is circumcenter of triangle AMN. D is projection of K on BC. R is isogonal conjugate of P. Perpendicular bisector of EF cuts OR at G. Prove that $DG \parallel AQ$.

Luis González

#9 Feb 8, 2015, 2:37 am • 1

> buratinogigle wrote:
>
> I have a general problem
>
> Let ABC be a triangle with circumcenter O and P is a point on OA. E, F are projection of P on CA, AB. Q is a point on EF. The line passing through Q and perpendicular to AQ cuts PE, PF at M, N. K is circumcenter of triangle AMN. D is projection of K on BC. R is isogonal conjugate of P. Perpendicular bisector of EF cuts OR at G. Prove that $DG \parallel AQ$.

Clearly $EF \parallel BC$ and R is on the A-altitude of $\triangle ABC$. From cyclic $AQEM$ and $AQNF$, we get $\angle AMN = \angle AEF = \angle ACB$ and $\angle ANM = \angle AFE = \angle ABC \Longrightarrow \triangle ANM \sim \triangle ABC$. Hence as Q varies on EF, the series Q, K are similar, but the series K, D are obviously similar, hence the series Q, D are similar. Thus it's enough to show that $AQ \parallel GD$ holds somewhere and this is trival when $Q \equiv AR \cap EF$, in which $(AQ \parallel KGD) \perp BC$, as desired.

Quick Reply

High School Olympiads

triangle and line problem

Locked

cr1

#1 Feb 7, 2015, 1:51 am

Line l divides triangle into 2 parts of same area and perimeter. Prove that l passes through incenter of triangle.

Luis González

#2 Feb 7, 2015, 5:34 am

Posted before, so for further discussions use any of the links below.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=133939
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=150&t=43649
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=151&t=396636

High School Olympiads

intersect on Euler line

Reply

titanian

#1 Feb 4, 2015, 11:03 pm

Let ABC be a triangle with circumcenter O. D, E, F are the midpoints of BC, CA, AB, respectively. AO cut (O) at M, AD cut (O) at D_1, MD cut (O) at D_2
Similarly, We have E_1, E_2, F_1, F_2
Prove that D_1D_2, E_1E_2, F_1F_2 intersect on Euler line of triangle ABC.

Luis González

#2 Feb 4, 2015, 11:32 pm • 2

Let H be the orthocenter of $\triangle ABC$. Since $(BH \parallel CM) \perp AC$ and $(CH \parallel BM) \perp AB$, then $HBMC$ is parallelogram $\Longrightarrow H \in DM$. If OG cuts (O) at U, V, then by Desargues involution theorem for AMD_1D_2 cut by OG, it follows that $X \equiv OG \cap D_1D_2$ is the image of O under the unique involution defined by the pairs of points U, V and H, G, i.e. the Gob's point X_{25} of $\triangle ABC$. Similarly, the lines E_1E_2 and F_1F_2 hit the Euler line OG at X_{25}.

TelvCohl

#3 Feb 5, 2015, 6:14 am • 1

My solution:

Let D_3, E_3, F_3 be the pole of D_1D_2, E_1E_2, F_1F_2, respectively .
Let N be the 9-point center of $\triangle ABC$ and $T = MD_1 \cap AD_2$.

From Pascal theorem (for $MD_1D_1AD_2D_2$) we get $D_3 \in TD$,
so notice that D is the orthocenter of $\triangle TAM \Longrightarrow D_3D \perp AM$. i.e. $D_3D \perp ND$
Since $\angle D_3DD_1 = 90^\circ - \angle D_1AM = \angle AMD_1 = \angle DD_1D_3$,
so we get $D_3D = D_3D_1 = D_3D_2 \Longrightarrow D_3D^2 = D_3D_1{}^2 = D_3D_2{}^2$.
i.e. D_3 lie on the radical axis $\mathcal{R}$ of $\odot(O)$ and $\odot(N)$

Similarly, we can prove E_3, F_3 lie on $\mathcal{R}$,
so D_1D_2, E_1E_2, F_1F_2 are concurrent at the pole P of $\mathcal{R}$ WRT $\odot(O)$,
hence from $OP \perp \mathcal{R}$ and $ON \perp \mathcal{R} \Longrightarrow P \in ON$.
i.e. D_1D_2, E_1E_2, F_1F_2 are concurrent at a point on the Euler line of $\triangle ABC$

Q.E.D

P.S. The pole P of the orthic axis $\mathcal{R}$ of $\triangle ABC$ WRT $\odot(O)$ is X_{25} in ETC :)

Quick Reply

High School Olympiads

Radical circle and radical centers

Reply

Source: own

proglote

#1 Mar 13, 2013, 7:13 am

Let $\Gamma_A, \Gamma_B, \Gamma_C$ be non-coaxal circles in the plane centered at O_A, O_B, O_C and with radical circle Γ. Let R_A denote the radical center of $\Gamma, \Gamma_B, \Gamma_C$, and similarly define R_B, R_C. Prove that O_AR_A, O_BR_B, O_CR_C are concurrent.

My solution is synthetic, hope to see other nice solutions :)

Luis González

#2 Mar 13, 2013, 8:06 am • 1

Since Γ is orthogonal to $\Gamma_A, \Gamma_B, \Gamma_C$, then O_A, O_B, O_C are the poles of R_BR_C, R_CR_A, R_AR_B WRT Γ, i.e. $\triangle O_AO_BO_C$ is the polar triangle of Γ WRT $\triangle R_AR_BR_C$.

http://mathworld.wolfram.com/ChaslessPolarTriangleTheorem.html
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=310396
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=439414

TelvCohl

#3 Feb 3, 2015, 1:22 pm

Generalization:

Let $\Gamma_A, \Gamma_B, \Gamma_C$ be non-coaxial circles with center O_A, O_B, O_C, respectively .
Let P be the radical center of $\{\Gamma_A, \Gamma_B, \Gamma_C\}$ and Γ be a circle with center P .
Let R_A be the radical center of $\{\Gamma, \Gamma_B, \Gamma_C\}$ and define R_B, R_C similarly .

Then O_AR_A, O_BR_B, O_CR_C are concurrent .

Proof:

Let $\mathcal{P}(T, \odot)$ be the power of the point T WRT circle $\odot$.
Let $A = R_BR_C \cap O_BO_C, B = R_CR_A \cap O_CO_A, C = R_AR_B \cap O_AO_B$.

Since $\frac{AO_B}{AO_C} \cdot \frac{BO_C}{BO_A} \cdot \frac{CO_A}{CO_B}$

$$= \frac{\text{dist}(O_B, R_BR_C)}{\text{dist}(O_C, R_BR_C)} \cdot \frac{\text{dist}(O_C, R_CR_A)}{\text{dist}(O_A, R_CR_A)} \cdot \frac{\text{dist}(O_A, R_AR_B)}{\text{dist}(O_B, R_AR_B)}$$

$$= \frac{|\mathcal{P}(O_B, \Gamma_A) - \mathcal{P}(O_B, \Gamma)|}{|\mathcal{P}(O_C, \Gamma_A) - \mathcal{P}(O_C, \Gamma)|} \cdot \frac{|\mathcal{P}(O_C, \Gamma_B) - \mathcal{P}(O_C, \Gamma)|}{|\mathcal{P}(O_A, \Gamma_B) - \mathcal{P}(O_A, \Gamma)|} \cdot \frac{|\mathcal{P}(O_A, \Gamma_C) - \mathcal{P}(O_A, \Gamma)|}{|\mathcal{P}(O_B, \Gamma_C) - \mathcal{P}(O_B, \Gamma)|}$$

$$= \frac{|\mathcal{P}(O_B, \Gamma_A) - \mathcal{P}(O_B, \Gamma)|}{|\mathcal{P}(O_A, \Gamma_B) - \mathcal{P}(O_A, \Gamma)|} \cdot \frac{|\mathcal{P}(O_C, \Gamma_B) - \mathcal{P}(O_C, \Gamma)|}{|\mathcal{P}(O_B, \Gamma_C) - \mathcal{P}(O_B, \Gamma)|} \cdot \frac{|\mathcal{P}(O_A, \Gamma_C) - \mathcal{P}(O_A, \Gamma)|}{|\mathcal{P}(O_C, \Gamma_A) - \mathcal{P}(O_C, \Gamma)|}$$

$= 1$,

so from Menelaus theorem we get A, B, C are collinear ,
hence from Desargue theorem $\Longrightarrow$ O_AR_A, O_BR_B, O_CR_C are concurrent .

Q.E.D

buratinogigle

#4 Feb 3, 2015, 4:19 pm

> *TelvCohl wrote:*
>
> **Generalization**:
>
> Let $\Gamma_A, \Gamma_B, \Gamma_C$ be non-coaxial circles with center O_A, O_B, O_C, respectively .
> Let P be the radical center of $\{\Gamma_A, \Gamma_B, \Gamma_C\}$ and Γ be a circle with center P .
> Let R_A be the radical center of $\{\Gamma, \Gamma_B, \Gamma_C\}$ and define R_B, R_C similarly .
>
> Then O_AR_A, O_BR_B, O_CR_C are concurrent .

Very nice dear Telv. I have remark.

If O_AR_A, O_BR_B, O_CR_C are concurrent at Q. Let A', B', C' be pole of O_BO_C, O_CO_A, O_AO_B wrt Γ. Then O_AA', O_BB', O_CC' are conrruent at R. We have O_A, O_B, O_C, P, Q, R lie on a rectangular hyperbola!

Luis González

#5 Feb 4, 2015, 1:55 am • 1

> *TelvCohl wrote:*
>
> **Generalization**:
>
> Let $\Gamma_A, \Gamma_B, \Gamma_C$ be non-coaxial circles with center O_A, O_B, O_C, respectively.
> Let P be the radical center of $\{\Gamma_A, \Gamma_B, \Gamma_C\}$ and Γ be a circle with center P.
> Let R_A be the radical center of $\{\Gamma, \Gamma_B, \Gamma_C\}$ and define R_B, R_C similarly.
>
> Then O_AR_A, O_BR_B, O_CR_C are concurrent.

R_BR_C is radical axis of $\Gamma, \Gamma_A \Longrightarrow PO_A \perp R_BR_C$ and similarly $PO_B \perp R_CR_A$ and $PO_C \perp R_AR_B$. PR_A is radical axis of $\Gamma_B, \Gamma_C \Longrightarrow PR_A \perp O_BO_C$ and similarly $PR_B \perp O_CO_A, PR_C \perp O_AO_B$. Hence $\triangle O_AO_BO_C$ and $\triangle R_AR_BR_C$ are bilogic (common orthology center P), thus they are also perspective, i.e. O_AR_A, O_BR_B, O_CR_C are concurrent.

> *buratinogigle wrote:*
>
> If O_AR_A, O_BR_B, O_CR_C are concurrent at Q. Let A', B', C' be pole of O_BO_C, O_CO_A, O_AO_B wrt Γ. Then O_AA', O_BB', O_CC' are conrruent at R. We have O_A, O_B, O_C, P, Q, R lie on a rectangular hyperbola!

In general, it's known that if $\triangle O_AO_BO_C$ and $\triangle R_AR_BR_C$ are perspective and orthologic, then the conic through O_A, O_B, O_C, the perspector Q and the orthology center P of $\triangle R_AR_BR_C$ WRT $\triangle O_AO_BO_C$ is a rectangular hyperbola $\mathcal{H}$. Now, if ϱ denotes the radius of Γ and P_A, P_B, P_C denote the projections of P on O_BO_C, O_CO_A, O_AO_B, respectively, then $\varrho^2 = PP_A \cdot PA' = PP_B \cdot PB' = PP_C \cdot PC'$. Thus when Γ varies, with its center P fixed, the proyective pencils $O_AR \equiv O_AA', O_BR \equiv O_BB', O_CR \equiv O_CC'$ generate none other that $\mathcal{H} \Longrightarrow R \in \mathcal{H}$.

Quick Reply

High School Olympiads

Two Circles

Headhunter

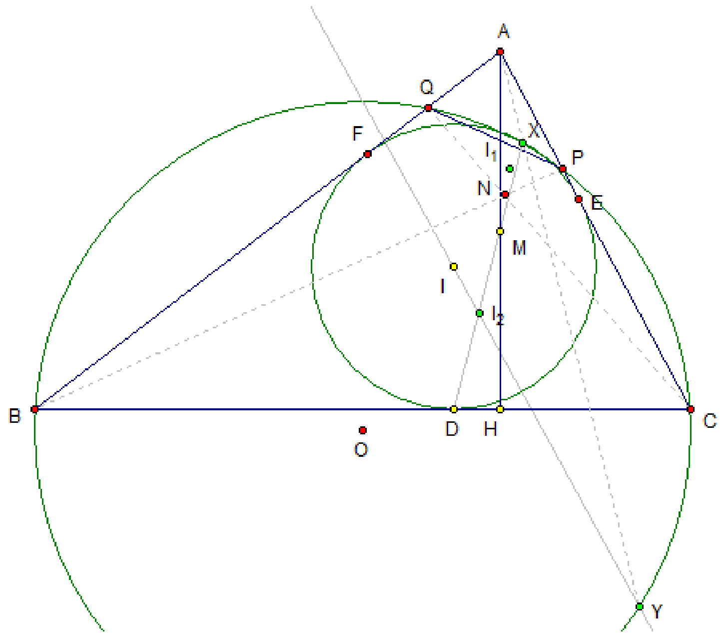
#1 Jun 27, 2012, 7:54 am • 2

Hello.

A circle (O) through B,C is tangent to the incircle (I) of $\triangle ABC$ at X.
(O) cut CA,AB at P,Q and BP meet CQ at N. AH is the altitude of $\triangle ABC$.
M is the midpoint of AH. I_1, I_2 are the incenters of $\triangle PNQ$, $\triangle CNB$.
AX meet (O) at Y (different from X).
show the following.

(1) D,I_2,M,X are collinear, where it's well known that D,M,X are collinear.
(2) I,I_2,Y are collinear.
(3) X,I_1,I_2,Y are cyclic.
(4) when a tangent to (I) at X cut BC at K, a circle (K, KX) pass I_1.

Attachments:



Luis González

#2 Jun 30, 2012, 11:14 am • 3

Let ω be the circle tangent to AB, AC and internally tangent to (O) at Y^*. Then A, X, Y^* are the exsimilicenters of $(I) \sim \omega$, $(I) \sim (O)$ and $\omega \sim (O)$, respectively $\Longrightarrow A, X, Y^*$ are collinear, i.e. $Y \equiv Y^*$. Let ω_1, ω_2 be the circles tangent to BP, CQ, such that ω_1 is internally tangent to (O) through its arc PQ and ω_2 is internally tangent to (O) through its arc BC. According to 3 circles with common tangency point, we have $X \in \omega_1$ and $Y \in \omega_2$. Thus, from incenter of triangle we claim that YI_1 bisects $\angle PYQ$, XI_1 bisects $\angle PXQ$, both YI, YI_2 bisect $\angle BYC$, yielding the collinearity of Y, I, I_2, and XI_2 bisects $\angle BXC$. But XD bisects $\angle BXC$ internally, due to the internal tangency of (I) and (O), hence X, D, I_2 are collinear.

YI_1, YI_2 cut (O) again at the midpoints E, F of its arcs PXQ, CXB, respectively and XI_1, XI_2 cut (O) again at the midpoints U, V of its arcs PYQ, CYB, respectively. UE, VF are then perpendicular bisectors of $\overline{PQ}, \overline{BC} \Longrightarrow EFUV$ is a rectangle with center $O \Longrightarrow \angle I_1XI_2 \equiv \angle UXV = \angle UFV = \angle FVE = \angle FYE \equiv \angle I_1YI_2$. Hence, X, Y, I_1, I_2 are concyclic. Furthermore, the circle through these points is orthogonal to $(O), (I), \omega, \omega_1$ and ω_2.$\square$

> *Headhunter wrote:*

(4) when a tangent to (I) at X cut BC at K, a circle (K, KX) pass I_1.

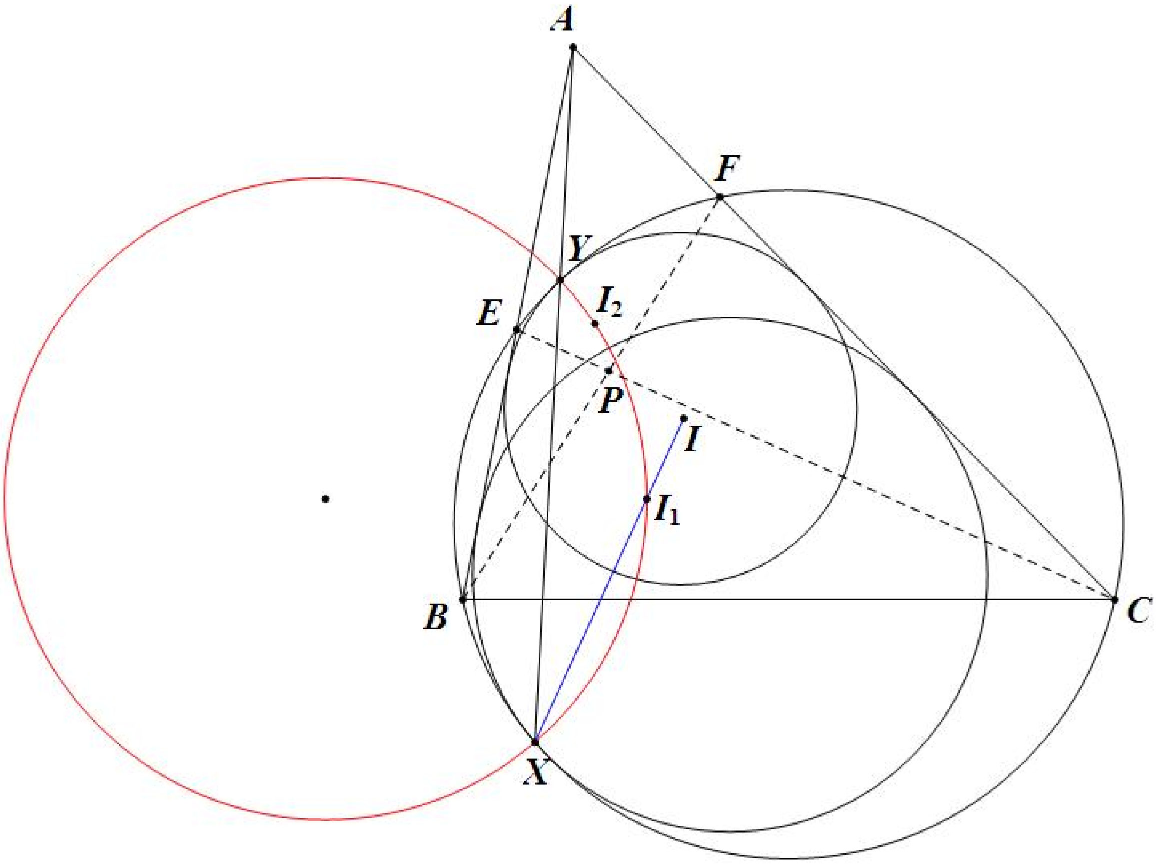
This assertion is not true.

phuongtheong

#3 Jun 30, 2012, 5:12 pm • 2

Generalization of (2) and (3)
Given ABC is a triangle and (ω) is a circle pass through B, C and cut the sides AB, AC at E, F, respectively. EC intersecs FB at P. I_1, I_2 are incenter of triangle PBC and PEF. (ω_1) and (ω_2) are two circle that tangent to the sides AB, AC and internally tangent to BC at X and EF at Y as fingure. I is the incenter of triangles ABC. Prove that:
1. X, I, I_1 are collinear.
2. X, Y, I_1, I_2 are concylic.
Click to reveal hidden text
Anyone have an another generalization?
Thanks you.

Attachments:



buratinogigle

#4 Jul 1, 2012, 5:32 pm • 3

Good idea dear Phuong, I don't have solution but I have another problem base on your idea.

Let $ABCD$ be cyclic quadrilateral with circumcircle (O). AC cuts BD at I. E, F, G, H are incenters of triangles IAB, IBC, ICD, IDA, respectively. $(K), (L), (M), (N)$ are circles tangent to lines $IA, IB; IB, IC; IC, ID; ID, IA$ and tangent to (O) internally at X, Y, Z, T, respectively.

a) Prove that X, E, G, Z and Y, F, H, T are concyclic on (O_1) and (O_2), respectively.

b) Prove that O lies on radical axis of (O_1) and (O_2).

Attachments:

Figure848.pdf (9kb)

Luis González

#5 Jul 1, 2012, 10:18 pm • 4

I remarked in my previous post that the circle through X, Y, I_1, I_2 is orthogonal to (O).

ZE and ZG cut (O) again at the midpoints P, Q of its arcs AB and DBC. XE and XG cut (O) again at the midpoints U, V of its arcs BCA and CD. Thus, UP and QV are perpendicular bisectors of $\overline{AB}$ and $\overline{CD}$, meeting at $O \Longrightarrow PQUV$ is a rectangle.

$\angle EXG \equiv \angle UXV = \angle UQV = \angle PVQ = \angle PZQ \equiv \angle EZG$

$\angle XZE \equiv \angle XZP = \angle XUP \equiv XUO = \angle UXO \equiv \angle OXE$

Thus, X, E, G, Z lie on a same circle (O_1) and OX is tangent to (O_1), i.e. (O) and (O_1) are orthogonal. Likewise, Y, F, H, T lie on a same circle (O_2) orthogonal to (O). Hence, (O) is orthogonal to both $(O_1), (O_2) \Longrightarrow O$ is on their radical axis.

TelvCohl

#6 Feb 2, 2015, 1:10 pm • 1

> *phuongtheong wrote:*
> Generalization of (2) and (3)
> Given ABC is a triangle and (ω) is a circle pass through B, C and cut the sides AB, AC at E, F, respectively. EC intersecs FB at P. I_1, I_2 are incenter of triangle PBC and PEF. (ω_1) and (ω_2) are two circle that tangent to the sides AB, AC and internally tangent to BC at X and EF at Y as fingure. I is the incenter of triangles ABC. Prove that:
> 1. X, I, I_1 are collinear.
> 2. X, Y, I_1, I_2 are concylic.

My solution:

Let U, V be the midpoint of arc CF, arc BE, respectively .
Let B', C' be the tangent point of ω_1 with AB, AC, respectively .
Let E', F' be the tangent point of ω_2 with AB, AC, respectively .

1.
From 3 circles with common tangency point we get there exist a circle ω_3 tangent to BF, CD and internally tangent to ω at X , so from the problem incenter of triangle we get XI, XI_1 are the bisector of $\angle CXB$. i.e. X, I, I_1 are collinear

2.
Easy to see $V \in XB', U \in XC'$.

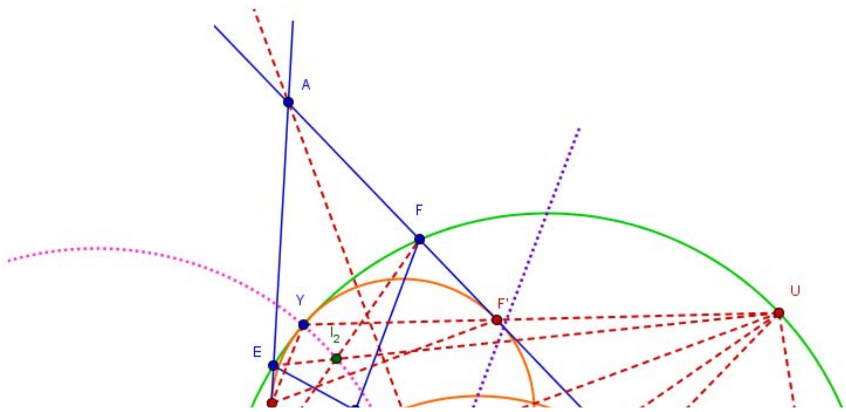
Since $\triangle VBB' \sim \triangle VXB, \triangle UCC' \sim \triangle UXC$,
so we get $VI_1^2 = VI_2^2 = VB^2 = VB' \cdot VX$ ... (1) and $UI_1^2 = UI_2^2 = UC^2 = UC' \cdot UX$ ... (2)

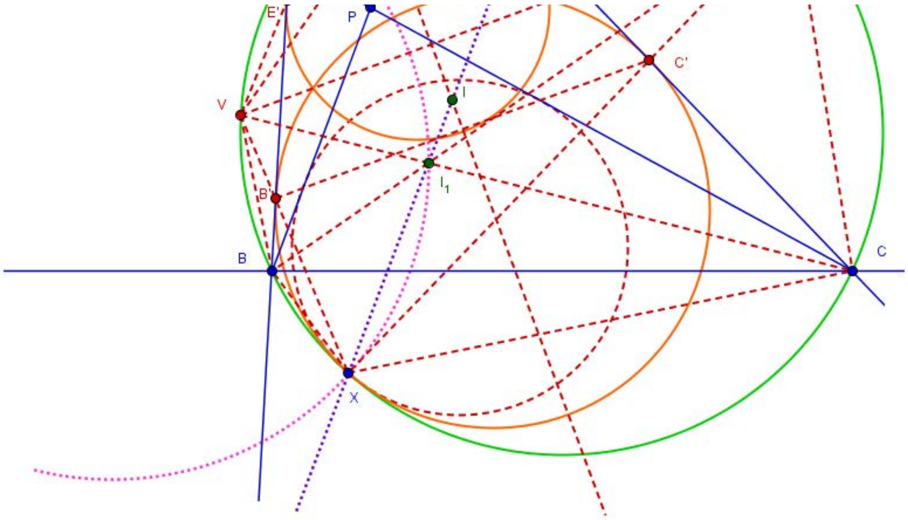
From $AI \perp UV$ (well-known) $\Longrightarrow B'C' \parallel UV$,
so combine with $(1), (2) \Longrightarrow UX : VX = UI_1 : VI_1 = UI_2 : VI_2$.
Similarly, we can prove $UY : VY = UI_1 : VI_1 = UI_2 : VI_2 \Longrightarrow X, Y, I_1, I_2$ are concyclic (Apollonius's Theorem) .

Q.E.D

Remark:
Since X, Y, I_1, I_2 lie on the Apollonius circle of $\{U, V\}$, so $\odot(XYI_1I_2)$ is orthogonal to ω (and $\omega_1, \omega_2, \omega_3$) .

Attachments:





Luis González

#7 Feb 2, 2015, 10:54 pm • 1

> phuongtheong wrote:
> Generalization of (2) and (3)
> Given ABC is a triangle and (ω) is a circle pass through B, C and cut the sides AB, AC at E, F respectively. EC intersecs FB at P. I_1, I_2 are incenter of triangle PBC and PEF. (ω_1) and (ω_2) are two circle that tangent to the sides AB, AC and internally tangent to BC at X and EF at Y as fingure. I is the incenter of triangles ABC. Prove that:
> 1. X, I, I_1 are collinear.
> 2. X, Y, I_1, I_2 are concylic.

Just for the record, my proof given in the 1st reply is valid for this generalization. No tangency between the circle (I) and BC was used for solving problems (2) and (3).

Quick Reply

High School Olympiads

Smarandache – Pătraşcu theorem

Reply

Source: internet

AdithyaBhaskar

#1 Feb 2, 2015, 9:40 am

Prove the Smarandache – Pătraşcu theorem:
Consider triangle ABC and the inscribed triangle $A_1B_1C_1$, ortho-homological, Q,Q_1 their centers of orthology, P the homology center and d their homology axes. If $A_2B_2C_2$ is the podar triangle of Q_1, P_1 is the homology center of triangles ABC and $A_2B_2C_2$, and d_1 their homology axes, then the points P, Q, Q_1, P_1 are collinear and the lines d and d_1 are parallel.

Luis González

#2 Feb 2, 2015, 10:26 am

I'm not familiar with the term "ortho-homological", I guess it means that Q is a PC point, i.e. $\triangle A_1B_1C_1$ is the pedal triangle of Q and cevian triangle of P. If yes, then this simply follows by Sondat's theorem. Q, Q_1, P are collinear and this line is perpendicular to the homology axis d. If Q is a PC-point, then so is its isogonal conjugate Q_1, thus again by Sondat's theorem Q, Q_1, P_1 are collinear and $d_2 \perp QQ_1$. Hence P, P_1, Q, Q_1 are collinear and $(d \parallel d_1) \perp QQ_1$.

Quick Reply

High School Olympiads

Prove the equality of angles

Reply

Pinionrzek

#1 Jan 31, 2015, 11:53 pm

There are given 5 points A, B, C, D, E on the plane such that A, B, C, D lie on one circle and $ABDE$ is parallelogram. Let F and G be points of interesections AB and CD, AC and BD respectively. Show that $\angle GFA = \angle ECD$.

TelvCohl

#2 Feb 1, 2015, 12:41 am • 1

My solution:

Let M, N, P, Q be the midpoint of AD, BC, BD, FG, respectively .

It's well-known that M, N, Q are collinear (Newton line) and easy to see $PM = \frac{1}{2}GA, PQ = \frac{1}{2}DF$. $(\star)$

Since $PQ \parallel DF, MQ \parallel EC$,
so we get $\angle ECD = \angle MQP$ . ... (1)

Since $\frac{FB}{FD} = \frac{\sin \angle CDB}{\sin \angle DBF} = \frac{\sin \angle GAB}{\sin \angle ABG} = \frac{BG}{AG}$,

so combine with $(\star)$ and $\angle QPM = \angle FCA = \angle FBG \Longrightarrow \triangle FBG \sim \triangle QPM$ . ... (2)

From (1) and (2) we get $\angle ECD = \angle GFA$.

Q.E.D

Luis González

#3 Feb 1, 2015, 3:56 am

Let $(O) \equiv \odot(ABCD), H \equiv AD \cap BC$ and let M, N, P, Q be the midpoints of DA, BC, BA, BD, resp. Thus $EC \parallel MN, ED \parallel MQ \Longrightarrow \widehat{ECD} = \widehat{MNQ} = \widehat{ONQ} - \widehat{ONM}$. But from cyclic $ONBQ$ and $ONHM$, we get $\widehat{ONQ} = \widehat{OBD} = 90^\circ - \widehat{HCD}$ and $\widehat{ONM} = \widehat{OHD} \Longrightarrow$
$\widehat{MNQ} = 90^\circ - \widehat{HCD} - \widehat{OHD} = 90^\circ - \widehat{HAB} - \widehat{OHD} = \widehat{POH}$. But since $OP \perp FA$ and $OH \perp FG$ (FG is polar of H WRT (O)), we have $\widehat{GFA} = \widehat{POH} = \widehat{MNQ} = \widehat{ECD}$.

Quick Reply

High School Olympiads

Circumscribed quadrilateral

Reply

Source: Own

buratinogigle

#1 Jan 23, 2015, 4:27 pm

Let $ABCD$ be circumscribed quadrilateral circle (I). (I) touches DC,DA at Z,T. Tagent of (I) cuts DA,DC at P,Q. AB cuts CD at E. AD cuts BC at F. Prove that $\frac{QC^2}{QE^2}\cdot\frac{PA^2}{PF^2}=\frac{ZC}{ZE}\cdot\frac{DC}{DE}\cdot\frac{TA}{TF}\cdot\frac{DA}{DF}$.

Luis González

#2 Jan 31, 2015, 9:08 pm • 1

It can be written as $(Q,C,E,D)\cdot(P,A,F,T)=(Q,E,C,Z)\cdot(P,F,A,D)$, thus it is invariant under any homology. Considering the homology sending EF to infinity and taking (I) to another circle (this does exist), it reduces to prove that $QC\cdot PA=DC\cdot TA$ for a rhombus $ABCD$ with incircle (I).

Let PQ cut BC,BA at X,Y and let S be the tangency point of (I) with BA. Since $\angle IXY=\angle IXC$ and $\angle XIY=\angle XCI=90^\circ-\tfrac{1}{2}\angle ABC\Longrightarrow\triangle IXY\sim\triangle CXI\Longrightarrow\frac{CI}{CX}=\frac{IY}{IX}$ and in the same way $\frac{AI}{AY}=\frac{IX}{IY}\Longrightarrow$ $CX\cdot AY=AI^2=AS\cdot AB=TA\cdot DC$. But from $\triangle CQX\sim\triangle AYP$, we have $QC\cdot PA=CX\cdot AY\Longrightarrow$ $QC\cdot PA=TA\cdot DC$, as desired.

Quick Reply

High School Olympiads

Locus

Reply

Source: 10.8 Final Round of Sharygin geometry Olympiad 2013

Nguyenhuyhoang

#1 Aug 9, 2013, 1:36 pm

Two fixed circles are given on the plane, one of them lies inside the other one. From a point C moving arbitrarily on the external circle, draw two chords CA, CB of the larger circle such that they tangent to the smalaler one. Find the locus of the incenter of triangle ABC.

pi37

#2 Aug 10, 2013, 11:04 am

Let CA, CB be tangent to the smaller circle at X, Y. By a well known lemma, since the smaller circle is the C- mixtillinear incircle of ABC, its incenter is the midpoint of XY, or the inverse of C about the smaller circle. But then the locus of the incenters is simply the inverse of the bigger circle about the smaller one.
EDIT: Misread and assumed that the two circles were tangent.

hqdhftw

#3 Aug 11, 2013, 9:21 am

Assume that the smaller circle is $(I;r)$ and the external circle is $(O;R)$, the incenter of $\triangle CAB$ be J. We shall prove that $IJ.IC = const$.
Let $CI \cap (O) = P$, hence $PJ = PA = PB$. We assume that J lies between I and P (the other case can be proved similarly). Let CA meets (I) at Q.
We have $IP.IC = R^2 - OI^2$ (1)
$\frac{IJ}{IP} = 1 - \frac{PI}{PJ} = 1 - \frac{PI}{PA}$. (2)
On the other hand, $\frac{IP}{PA} = \frac{IP}{2R.sinACP} = \frac{R^2 - OI^2}{2R.IC.sinACP} = \frac{R^2 - OI^2}{2R.IQ} = \frac{R^2 - OI^2}{2Rr} = const.$ (3)
From (1), (2) and (3), we can calculate the value $IJ.IC = \frac{(2Rr + OI^2 - R^2)(R^2 - OI^2)}{2Rr} = k$, so J lies on the circle which is the image of (O) through the inversion center I, radius k.

Luis González

#4 Jan 31, 2015, 3:54 am

Label (I, r) and (O, R) the given circles, this latter being the external one. J is the incenter of $\triangle ABC$ and CJ cuts (O) again at D. Keeping in mind that $DA = DB = DJ$ (well-known), we get

$$IC \cdot IJ = IC \cdot (ID - JD) = IC \cdot ID - IC \cdot DA =$$

$$= IC \cdot ID - \frac{r}{\sin \widehat{ACD}} \cdot 2R \cdot \sin \widehat{ACD} = IC \cdot ID - 2R \cdot r = \text{const} = -\varrho^2.$$

Thus, locus of J is the inverse circle of (O) under inversion with center I and power $-\varrho^2$.

Quick Reply

High School Olympiads

ptolemy

Reply

calhanSPheiro2

#1 Jan 30, 2015, 4:24 pm

The trianggle ABC is inscribed (O). BE,CF is bisector of angle B and C (E is in AC,F is in AB). EF intersect (O) at M,N (E is between F and N). Prove that $\frac{1}{BM}+\frac{1}{CN}=\frac{1}{AM}+\frac{1}{AN}+\frac{1}{BN}+\frac{1}{CM}$

Luis González

#2 Jan 30, 2015, 9:43 pm

calhanSPheiro2, please next time use $\LaTeX$ when typing formulae. You just need to enclose them between dollar signs. Uploading them as images from a photobucket is not a good idea as they might eventually disappear. As for the problem, it was posted before at http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=340280.

Quick Reply

High School Olympiads

Feuerbach point

Locked

Source: Hyacinthos #23079

rodinos

#1 Jan 30, 2015, 1:38 am

Let ABC be a triangle

Denote:

Oa, Ob, Oc = the circumcenters of IBC, ICA, IAB, resp.

Aa, Ab, Ac = the reflections of Oa in AI, BI, CI, resp.
(since Oa lies on AI we have Aa = Oa).

(N1) = the NPC of AaAbAc. Similarly (N2), (N3).

(N1), (N2), (N3) concur at the Feuerbach point of ABC.

References if known ?

Proof

APH

Luis González

#2 Jan 30, 2015, 2:09 am • 1

Isn't this "trivial" after we noticed that the 9-point circle of $\triangle O_aA_bA_c$ is nothing but the 9-point circle of $\triangle IBC$?. They concur at the Poncelet point of $ABCI$, i.e. the intersection of the pedal circle of I (incircle of $\triangle ABC$) and the 9-point circle of $\triangle ABC$, the Feuerbach point of $\triangle ABC$.

High School Olympiads

A property of a line through circumcenter

Reply

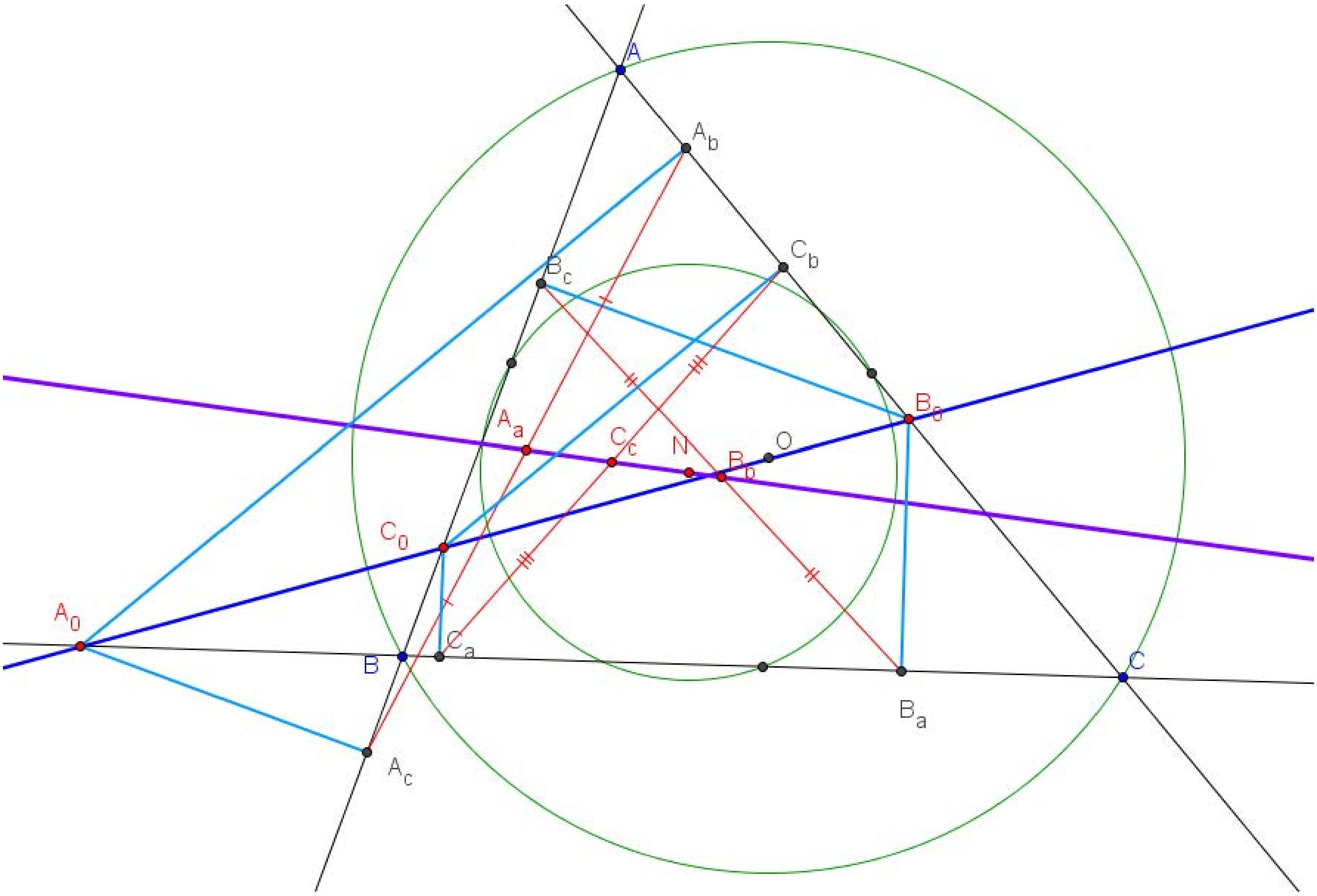
daothanhoai

#1 Jan 29, 2015, 8:05 am

Let ABC be a triangle, let a line L through circumcenter and the line meets BC, CA, AB at A_0, B_0, C_0. Let A_b, C_b be projection of A_c on AC and AB respectively. Let A_a be the midpoint of A_bA_c define B_b, C_c cyclically. show that A_b, C_b, C_c and Nine point center of ABC are collinear.

Note that: If the line L doesn't pass through circumcenter then A_a, B_b, C_c also are collinear

Attachments:



TelvCohl

#2 Jan 29, 2015, 2:22 pm

My solution:

Lemma

Given $\triangle ABC$ and a line ℓ .
Let P_1, P_2, P_3 be three points on ℓ .
Let $\triangle D_1E_1F_1$ be the pedal triangle of P_1 WRT $\triangle ABC$.
Let G_1 be the centroid of $\triangle D_1E_1F_1$ and R_1 be a point on ray P_1G_1 such that $P_1G_1 : G_1R_1 = 2 : 1$.
Define $\triangle D_2E_2F_2, G_2, R_2$ for P_2 and $\triangle D_3E_3F_3, G_3, R_3$ for P_3 similarly .

Then R_1, R_2, R_3 are collinear .

Proof:

Let M_1, M_2, M_3 be the midpoint of E_1F_1, E_2F_2, E_3F_3, respectively .

Since $E_1E_2 : E_2E_3 = P_1P_2 : P_2P_3 = F_1F_2 : F_2F_3$,
so from ERIQ lemma $\Longrightarrow M_1, M_2, M_3$ are collinear and $M_1M_2 : M_2M_3 = P_1P_2 : P_2P_3$.

Since $D_1G_1 : G_1M_1 = D_2G_2 : G_2M_2 = D_3G_3 : G_3M_3 = 2 : 1$,
so from ERIQ lemma $\Longrightarrow G_1, G_2, G_3$ are collinear and $G_1G_2 : G_2G_3 = P_1P_2 : P_2P_3$.

Since $P_1G_1 : G_1R_1 = P_2G_2 : G_2R_2 = P_3G_3 : G_3R_3 = 2 : 1$,
so from ERIQ lemma $\Longrightarrow R_1, R_2, R_3$ are collinear and $R_1R_2 : R_2R_3 = P_1P_2 : P_2P_3$.

Back to the main problem:

Let P be a moving point on L .
Let P_a, P_b, P_c be the projection of P on BC, CA, AB, respectively .
Let G be the centroid of $\triangle P_aP_bP_c$ and R on the ray PG such that $PG : GR = 2 : 1$.

When P coincide with $A_0 \Longrightarrow R \equiv A_a$ . ... (1)
When P coincide with $B_0 \Longrightarrow R \equiv B_b$ . ... (2)
When P coincide with $C_0 \Longrightarrow R \equiv C_c$ . ... (3)

From (1), (2), (3) and the lemma we get A_a, B_b, C_c are collinear .

If L pass through the circumcenter O of $\triangle ABC$:
Consider the case $P \equiv O \Longrightarrow R$ coincide with the 9-point center N of $\triangle ABC$,
so from the lemma we get A_a, B_b, C_c, N are collinear

Q.E.D

Luis González

#3 Jan 29, 2015, 11:09 pm

In general, the applications $B_0 \mapsto B_b$ and $C_0 \mapsto C_c$ are affine homographies, thus when ℓ varies, B_b and C_c describe two line ℓ_b and ℓ_c. Since $B_0 \mapsto C_0$ is a perspectivity, then $B_b \mapsto C_c$ is also a perspectivity between ℓ_b, ℓ_c, because when $B_0 \equiv C_0 \equiv A$, then $B_b \equiv C_c$ is the midpoint of the A-altitude $\Longrightarrow B_bC_c$ goes through a fixed point.

Assume the case when $C_0 \equiv B_c$ is the midpoint of AB. From cyclic BCB_0O and $BB_aB_0C_0$, we get $\angle COB_0 = \angle B_aBB_0 = \angle B_aC_0B_0 \Longrightarrow C_0B_bB_a \parallel CO \Longrightarrow C_0B_b$ goes through the 9-point center N of $\triangle ABC$. Further, the pencil formed by C_0C_a, C_0C_b, C_0B_a and the parallel from C_0 to C_aC_b and the pencil formed by CB, CA, the tangent of (O) at C and the C-symmedian are projective, as they have corresponding perpendicular rays, thus C_0B_a passes trough the midpoint C_c of $C_aC_b \Longrightarrow N \in B_bC_c$ and similarly we'll have $N \in B_bC_c$ when B_0 is the midpoint of AC. Analogously $C_cA_a \in N \Longrightarrow A_a, B_b, C_c, N$ are collinear for all ℓ.

Quick Reply

High School Olympiads

collinear

Reply

Source: Korea Final Round 2008 2nd Day#5

johnkwon0328

#1 Jan 28, 2015, 6:06 pm

Quadrilateral $ABCD$ is inscribed in a circle O.
Let $AB\cap CD=E$ and $P\in BC, EP\perp BC, R\in AD, ER\perp AD, EP\cap AD=Q, ER\cap BC=S$
Let K be the midpoint of QS

Prove that E,K,O are collinear.

Luis González

#2 Jan 28, 2015, 10:01 pm

Let $F\equiv BC\cap AD, G\equiv AC\cap BD$ and $X\equiv PR\cap QS.$ Clearly $\triangle EBC$ and $\triangle EDA$ are similar with corresponing E-altitudes EP,ER and their isogonals $ES,EQ,$ hence it follows that $\triangle EPS\sim\triangle ERQ\Longrightarrow PQRS$ is cyclic with circumcircle $(K).$ Now, from the complete $ABCD$ and $PQRS,$ we get $F(A,B,G,E)=F(Q,P,X,E)=-1\Longrightarrow$ FGX is the polar of E WRT $(K)\Longrightarrow FG\perp EK,$ but since FG is the polar of E WRT $(O),$ we have $FG\perp EO\Longrightarrow$ E,K,O are collinear.

For other solutions see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=228708
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=358553

Mikasa

#3 Feb 2, 2015, 12:52 pm

Let $AD\cap BC=X, AC\cap BD=N, QS\cap PR=M$. Now, it is a well known fact that, $(XE,XM;XR,XS)=-1$ and $(XE,XN;XD,XC)=-1$. But XR and XD are the same line, and the same goes for XS and XC. So XM and XN are the same line too. This means X,M,N are collinear.

Note that K is the center of $\odot PQRS$. Now by Brokard's theorem, $XM\perp EK$ and $XN\perp EO$. But X,M,N are collinear. So, E,K,O must be collinear too.

rkm0959

#4 Jun 9, 2015, 7:03 pm

Define O_1 as the circumcenter of $\triangle AED$ and O_2 as the circumcenter of $\triangle EBC$
Since $\angle BEP=90-\angle PBE=90-\angle EDA=\angle DER$, we can get that O_1 lies on EQ and O_2 lies on ES.
$$\triangle EAD\sim\triangle ECB \implies \frac{EO_1}{EQ}=\frac{EO_2}{ES}\implies \triangle O_1EO_2\sim\triangle QES\implies O_1O_2\parallel QS$$
Letting M the midpoint of O_1O_2, points E,K,M are collinear.
It suffices to show that points E,M,O are collinear.
$OO_1\perp AD, EO_2\perp AD \implies OO_1\parallel EO_2$
$OO_2\perp BC, EO_1\perp BC \implies OO_2\parallel EO_1$
Therefore $\square OO_1EO_2$ is a parallelogram, and points E,M,O are collinear, as desired. $\blacksquare$

This post has been edited 1 time. Last edited by rkm0959, Jun 9, 2015, 7:04 pm

mathmatecS

#5 Jun 9, 2015, 7:32 pm

rkm0959 wrote:

Define O_1 as the circumcenter of $\triangle AED$ and O_2 as the circumcenter of $\triangle EBC$
Since $\angle BEP = 90 - \angle PBE = 90 - \angle EDA = \angle DER$, we can get that O_1 lies on EQ and O_2 lies on ES.
$$\triangle EAD \sim \triangle ECB \implies \frac{EO_1}{EQ} = \frac{EO_2}{ES} \implies \triangle O_1EO_2 \sim \triangle QES \implies O_1O_2 \parallel QS$$
Letting M the midpoint of O_1O_2, points E, K, M are collinear.
It suffices to show that points E, M, O are collinear.
$OO_1 \perp AD, EO_2 \perp AD \implies OO_1 \parallel EO_2$
$OO_2 \perp BC, EO_1 \perp BC \implies OO_2 \parallel EO_1$
Therefore $\square OO_1EO_2$ is a parallelogram, and points E, M, O are collinear, as desired. $\blacksquare$

Because AD is radical axis of O_1 and $O \implies OO_1, OO_2$ is perpendicularity with AD and BC.

This post has been edited 2 times. Last edited by mathmatecS, Jun 9, 2015, 7:32 pm

Quick Reply

High School Olympiads

Perpendicularity in a cyclic quadrilater

Reply

Source: Maybe own

StanleyST
#1 Jan 28, 2015, 3:18 am

Given a cyclic quadrilater $ABCD$, let the diagonals AC and BD meet at E. The midpoints of AB and CD are G and H, respectively. Let $G' = EG \cap C_{\triangle EDC}$, $H' = HE \cap C_{\triangle AEB}$, $F = CD \cap AB$ and denote by O the circumcenter of the quadrilater $ABCD$. Prove that the line $H'G'$ is perpendicular on the line FO.($C_{\triangle ABC}$ denotes the circumcircle of $\triangle ABC$)

Luis González
#2 Jan 28, 2015, 4:06 am

$\triangle EAB$ and $\triangle EDC$ are similar with corresponding E-medians EG, EH and corresponding E-symmedians EH', EG', since $\widehat{AEH'} = \widehat{CEH} = \widehat{BEG} = \widehat{DEG'}$. Hence if $P \equiv EH \cap AB$ and $Q \equiv EG \cap CD$, we have $\frac{EG}{EH} = \frac{EP}{EQ} = \frac{EH'}{EG'}$ $\Longrightarrow GHQP, GHG'H'$ are cyclic $\Longrightarrow PQ \parallel H'G'$ are antiparallel to GH WRT $FG, FH \Longrightarrow$ they are perpendicular to the F-circumdiameter FO of $\triangle FGH$, i.e. $H'G' \perp FO.$

TelvCohl
#3 Jan 28, 2015, 11:58 am

My solution:

Since $\triangle EAB \cup G \sim \triangle EDC \cup H$,
so EH', EG' is the $E-$ symmedian of $\triangle EAB, \triangle EDC$, respectively ,
hence we get $\triangle EAB \cup G \cup H' \sim \triangle EDC \cup H \cup G'$ and $EAH'B, EDG'C$ are harmonic quadrilateral .

From $\angle EGA = \angle EHD = \angle G'HD$
$\Longrightarrow \angle G'HO + \angle OGG' = \angle FHO + \angle OGF = 180^\circ \Longrightarrow G' \in \odot(OF)$.
Similarly, we can prove $H' \in \odot(OF) \Longrightarrow O, G, H, G', H', F$ lie on a circle with diameter OF ,
so from $\angle FGG' = \angle FHH' \Longrightarrow F$ is the midpoint of arc $G'H'$ in $\odot(OF) \Longrightarrow OF \perp G'H'$.

Q.E.D

Quick Reply

High School Olympiads

Circumcenter runs on a fixed circle

Reply

BlackSelena
#1 Jan 25, 2015, 8:48 pm • 1

Let circles (I) and (O) are fixed, (I) is in (O). Point X runs on (I). Tangent of (I) at X cut (O) at A, B. Prove that when X moves, the circumcenter of triangle IAB runs on a fixed circle.

sasanineq
#2 Jan 25, 2015, 11:21 pm • 1

use repercussion obviously blackselena!

BlackSelena
#3 Jan 26, 2015, 10:41 am

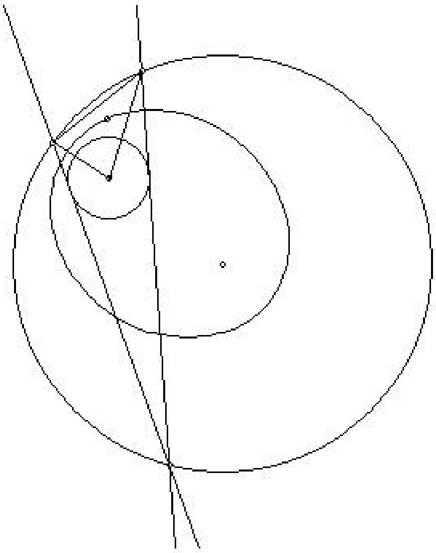
> *sasanineq wrote:*
> use repercussion obviously blackselena!

Sorry but what is 'repercussion' ?
Thanks for your attention anyway.

buratinogigle
#4 Jan 26, 2015, 12:27 pm

I think this assertion is not true. The locus is 2 degreee curve, only. It is not a circle. Please see the figure.

Attachments:



Luis González
#5 Jan 27, 2015, 9:14 am • 1

Dear buratinogigle, I think you missunderstood the statement. Letting R, r denote the radii of $(O), (I)$, we prove that the locus of the circumcenter K of $\triangle IAB$ is a circle with center O and radius $\frac{R^2-OI^2}{2r}$.

Let M be the midpoint of $\overline{AB}$. In the $\triangle IAB$, we have

$$AB^2 = IA^2 + IB^2 - 2 \cdot IA \cdot IB \cdot \cos \widehat{AIB} = IA^2 + IB^2 + 4 \cdot KM \cdot r.$$

Substituting $IA^2 = r^2 + XA^2, IB^2 = r^2 + XB^2$ and $AB^2 = XA^2 + XB^2 + 2 \cdot XA \cdot XB$ into the latter expression yields $XA \cdot XB = r^2 + 2 \cdot KM \cdot r$. But on the other hand, we have

$$XA \cdot XB = MA^2 - MX^2 = MA^2 - [OI^2 - (OM - r)^2] =$$

$$= MA^2 + OM^2 - OI^2 + r^2 - 2 \cdot OM \cdot r = R^2 + r^2 - OI^2 - 2 \cdot OM \cdot r \Longrightarrow$$

$$R^2 + r^2 - OI^2 - 2 \cdot OM \cdot r = r^2 + 2 \cdot KM \cdot r \Longrightarrow$$

$$2r \cdot (KM + OM) = R^2 - OI^2 \Longrightarrow OK = KM + OM = \frac{R^2 - OI^2}{2r} = \text{const}.$$

buratinogigle

#6 Jan 27, 2015, 12:37 pm

Thank BlackSelena and Luis much for nice problem and nice solution. I only have remark if (IBA) cuts (I) at PQ the reflection of I through PQ lies on fixed circle, too.

TelvCohl

#8 May 20, 2016, 1:16 am • 1

Let T be the circumcenter of $\triangle AIB$ and let R, r be the radius of $\odot(O), \odot(I)$, respectively. From cosine rule for $\triangle AOT$ and $\triangle IOT$ we get $R^2 = AT^2 + OT^2 - 2 \cdot AT \cdot OT \cdot \cos \angle ATO$ and $IO^2 = IT^2 + OT^2 - 2 \cdot IT \cdot OT \cdot \cos \angle ITO$, so we conclude that

$$R^2 - IO^2 = 2 \cdot OT \, (IT \cdot \cos \angle ITO - AT \cdot \cos \angle ATO) = 2r \cdot OT \Longrightarrow OT = \frac{R^2 - IO^2}{2r}.$$

More general result : Given two fixed circles $\odot(O), \odot(I)$ ($\odot(I)$ lies inside $\odot(O)$.) and a fixed point J. Let a tangent of $\odot(I)$ cuts $\odot(O)$ at A, B. Then the circumcenter T of $\triangle ABJ$ moves on a conic when AB varies on $\odot(O)$.

Proof : I'll only prove the case when J lies inside $\odot(O)$ (other cases can be proved similarly.). Let R, r be the radius of $\odot(O)$, $\odot(I)$, respectively and let S be the point such that $OT \stackrel{\parallel}{=} JS$. After proceeding as my proof above we get

$$\frac{R^2 - JO^2}{2} = OT \cdot \text{dist}(J, AB) = JS \cdot \text{dist}(J, AB),$$

so AB is the polar of S WRT the circle with center J and radius $\sqrt{\frac{R^2 - JO^2}{2}}$. Let τ be the polar of I WRT $\odot(J)$. From Salmon's theorem we get $\frac{JS}{\text{dist}(S, \tau)} = \frac{IJ}{r}$, so S moves on a conic with focus J and directrix τ when AB varies on $\odot(O)$, hence T moves on a conic when AB varies on $\odot(O)$.

Quick Reply

High School Olympiads

tangeancy

Reply

Source: Morocco MO 2015

Legend-crush

#1 Jan 27, 2015, 12:04 am

Let ΔABC be a triangle and O its circumcenter. Let T be the intersection of circle (COB) and the circle through A and C and tangeant to AB . K is the intersection of lines TO and BC.
Prove that: AK is tangeant to (ABC)

TelvCohl

#2 Jan 27, 2015, 12:34 am • 1

My solution:

Let ℓ be the bisector of $\angle BAC$.
Let A', B', C' be the midpoint of BC, CA, AB, respectively .
Let H, X be the projection of A on BC, OA', respectively .

Let Ψ be the composition of inversion $\mathcal{I}(A, \sqrt{\frac{AB \cdot AC}{2}})$ and reflection $\mathcal{R}(\ell)$.

Since $\Psi(B) = B', \Psi(C) = C', \Psi(O) = H$,
so $\Psi(\odot(OBC))$ is the 9-point circle $\odot(HB'C')$ of $\triangle ABC$.
From AB is the tangent of $\odot(ATC)$ we get $\Psi(\odot(ATC))$ is the line $A'C'$,
so $\Psi(T) \equiv A'C' \cap \odot(HB'C') = A' \Longrightarrow \Psi(\overline{OT}) = \odot(AHA')$,
hence we get $\Psi(K) \equiv \odot(AHA') \cap \odot(AB'C') = X$.

Since $AX \parallel BC$,
so AK is the tangent of $\odot(ABC)$.

Q.E.D

Luis González

#3 Jan 27, 2015, 2:38 am • 1

Let M be the midpoint of the arc BC of $\odot(BOC)$. $\angle MBO = \angle MCO = 90^\circ \Longrightarrow MB, MC$ are tangents of (O). If AM cuts $\odot(OBC)$ again at T', then $\angle CT'M = \angle COM = \angle BAC \Longrightarrow AB$ touches $\odot(ACT') \Longrightarrow T \equiv T' \Longrightarrow OTK \perp AM \Longrightarrow K$ is the pole of AM WRT $(O) \Longrightarrow AK$ is tangent to (O).

Mikasa

#4 Feb 2, 2015, 10:31 pm

Let the tangent to $\odot ABC$ at A intersect the line BC at K'. WLOG assume that B lies between K' and C'. Now let T' be the inverse of K' wrt $\odot ABC$. This means K', T', O are collinear. Now the polar of A wrt $\odot ABC$ is AK', which passes through K'. So the polar of K' wrt $\odot ABC$ passes through A. This implies that AT' is the polar of K' wrt $\odot ABC$. So $OK' \perp AT'$. Let $AT' \cap \odot ABC = D$. Then $AT' = T'D$ and $K'D$ is also tangent to $\odot ABC$. Now it is a well-known result that BC is the B-symmedian of $\triangle ABD$ and CB is the C-symmedian of $\triangle ACD$. So, $\angle BAD = \angle BCD = \angle ACT'$, which implies that $\odot ACT'$ touches AB. Also, $\angle K'T'B = 90^\circ - \angle BT'D$. But it is also well-known that $\triangle BT'D \sim \triangle BAC$. So $\angle BT'D = \angle BAC \Rightarrow \angle K'T'B = 90^\circ - \angle BAC = \angle OCB \Rightarrow T', O, C, B$ are con-cyclic. So, $T' \equiv T$. Then, $K' \equiv K$. So KA is tangent to $\odot ABC$ at A.

Sketshup

#5 Feb 3, 2015, 10:55 pm

One can also prove by angle chasing that $(OTA), (ABC)$ are tangents. We have $\angle OTC = \angle OBC = \frac{\pi}{2} - \angle A$. We also have $\angle ATC = \pi - \angle A$. Thus $\angle ATO = \frac{\pi}{2}$. This implies that AO is a diameter of (OTA). If O' is the center of OTA, O', O, A are collinear. Hence $(OTA), (ABC)$ are internally tangents.

From now on we have 2 options:
1) One may invert the diagram by considering the inversion with respect to circle (ABC). (OBC) inverts to line BC and $(OTBC)$ is concyclic. Thus T inverts to the intersection of OT and BC, or, T inverts to K. Thus $(ABC), (OTA)$ are internally tangents, imply that $(ABC), (KA)$ are tangents.

2)OT is the radical axis of $(OBC), (OTA)$. BC is the radical axis of $(OBC), (ABC)$. Hence, OT, BC and the radical axis of $(OTA), (ABC)$ are concurrent. Given the tangency, this last radical axis is the tangent of (ABC) at A. And because the concurrency point is K, we have AK is tangent to ABC.

jayme

#6 Feb 4, 2015, 2:32 pm

Dear Mathlinkers,

1. X the second point of intersection of AB with (BOC)
2. Y the second point of intersection of TX with the circle through A and C and tangeant to AB.
3. according to the pivot theorem applied to these three circles, AY is tangent to (ABC) at A.
4. we have to prove AY goes through K which seems to be not difficult.

Sincerely
Jean-Louis

jayme

#7 Feb 4, 2015, 4:47 pm

Dear Mathlinkers,
I come back with my approach...
I am waiting a proof of 4.

Sincerely
Jean-Louis

jayme

#8 Feb 5, 2015, 7:42 pm

Dear Mathlinkers,
I have proved that AT is perpendicular to OK with the Reim's theorem applied two time and finished the the power of K.
Sincerely
Jean-Louis

Seventh

#9 Feb 5, 2015, 9:59 pm

Consider $V = AT \cap BC$ and $S = AT \cap \odot(ABC)$. Chasing some angles (to get it easy, chase the angles $\angle ABT, \angle BAC, \angle ACB$), we get $AT \perp TO$ and $\angle BTV = \angle CTV$, so $(K, V; B, C) = -1$. So, it is enough to prove that $(A, S; B, C) = -1$. Well, notice that CC, BB (wrt $\odot(ABC)$) and AT are concurrent in the midpoint M of the arc $\widehat{\text{BC}}$ of $\odot(OBC)$ which doesn't contains O, since $\angle OBM = \angle OCM = \angle OTM = 90^o$. So, by the construction of a harmonic quadrilateral, follows that $ABSC$ is harmonic, as desired.

jayme

#10 Feb 6, 2015, 5:24 pm

Dear Mathlinkers,
Finally, this problem is based on two result

1. AT is perpendicular to OK (with the Reim's theorem applied two times)
2. the three chords theorem (by considering the circle with diameter AK).

Sincerely
Jean-Louis

sunken rock

#11 Feb 8, 2015, 1:00 am

An easier way to approach it (my opinion). Let AN be a diameter of $\odot(ATC)$, O, B, C, T are concyclic and $\angle CTK = \angle OBC = 90^\circ - \angle BAC = \angle CAN$, so clearly $ATCN$ is cyclic, i.e. $O - T - N - K$ are collinear and $AT \perp OK$, hence AT passes through M, OM being a diameter of the circle $\odot BOC$.
BM, CM are tangent to the circle O, so $A - T - M$ are collinear and TK is the external bisector of $\angle BTC$ and $\frac{KC}{KB} = \frac{TC}{BT}$ (1).
Easy angle chase gives $\angle BAT = \angle ACT, \angle CAT = \angle TBA$, hence $\triangle ABT \sim \triangle CAT \implies \frac{BT}{AT} = \frac{AT}{CT} = \frac{AB}{AC}$, or $\frac{BT}{AT} \cdot \frac{AT}{CT} = \frac{AB^2}{AC^2}$. With (1) we get $\frac{KC}{KB} = \frac{AC^2}{AB^2}$, or AK is, indeed, tangent to the circle $\odot(ABC)$.

Best regards,
sunken rock

Dukejukem

#12 Feb 8, 2015, 11:32 pm

Call Γ the circumcircle of $\triangle ABC$ and ω the circle passing through $B, O, C.$ Define $X \in \omega$ so that $\overline{OX}$ is a diameter of ω, and let AX meet ω for a second time at $T'.$ Then note that since OX bisects $\angle BOC$, we have $\measuredangle BAC = \measuredangle XOC$, where the angles are directed. Therefore,

$$\begin{aligned}\measuredangle BAT' &= \measuredangle BAC - \measuredangle T'AC \\ &= \measuredangle XOC - \measuredangle T'AC \\ &= \measuredangle XT'C - \measuredangle T'AC \\ &= \measuredangle AT'C - \measuredangle T'AC \\ &= -\measuredangle CT'A - \measuredangle T'AC \\ &= \measuredangle ACT'.\end{aligned}$$

Then since $\measuredangle BAT' = \measuredangle ACT'$, it folllows by the "Tangent-Angle Theorem" that AB is tangent to $\odot(AT'C).$ Therefore, we conclude that $T' = T.$

Finally, notice that since $\overline{OX}$ is a diameter of ω, we have $BO \perp BX$ and $CO \perp CX.$ Therefore, X is the intersection of the tangents to Γ at B and $C.$ Then since $K \in BC$, it follows that K lies on the polar of X w.r.t. $\Gamma.$ By La Hire's Theorem, X lies on the polar of $K.$ Because AX is a line passing through X that is perpendicular to KO (since $TX \perp TO$), we deduce that AX is the polar of K w.r.t. $\Gamma.$ Hence KA is tangent to Γ, and we're done. $\square$

Wolowizard

#13 Feb 9, 2015, 2:53 am • 1

Here is mine solution , with a little bit different approach:
Let F intersection of tangent from A with BC. Now let$Q$ be intersection of $FTandcirclearoundCOB$.
$COQBcyclic \implies FQ * FO = FB * FC$ (we assumed that $F - B - C$ and its trivial that it is$F - T - O$. We also have
$AF * AF = FB * FC$ (AF is tangent) from this we get
$AF * AF = FQ * FB \implies \angle ATF = \angle OAF = 90 also \angle OTC = 90 - A \implies \angle ATC = 180 - A$ from where its trivial that Q is on circle that is tangent to $ABandcontainsA, C \implies Q = T \implies F = K$.
Done.

dothef1

#14 May 10, 2015, 3:02 am

Basic angle chasing yields that (OTA) and (BAC) are tangent at A
Since K is the radical center of $(TOBC)$, (ABC) and (ATO)
we get that (AK) is the radical axis of (ABC) and (ATO)
And so (AK) is the tangent to both circles (ABC) and (ATO) .

Quick Reply

High School Olympiads

Around Polar triangle

Reply

daothanhoai

#1 Jan 26, 2015, 8:19 am

Let ABC be a triangle, let (O) be a circle, let $A_0B_0C_0$ be a triangle bounded by three polar line: A to (O), B to (O), C to (O). The line AA_0 meets (O) again at A' and A''; define B', B'' and C', C'' cyclically(see the figure below).
Let $A_1B_1C_1$ be a triangle bounded by three tangent line of (O) at A', B', C'. Let A_2, B_2, C_2 be a triangle bounded by three tangent line of (O) at A'', B'', C''.

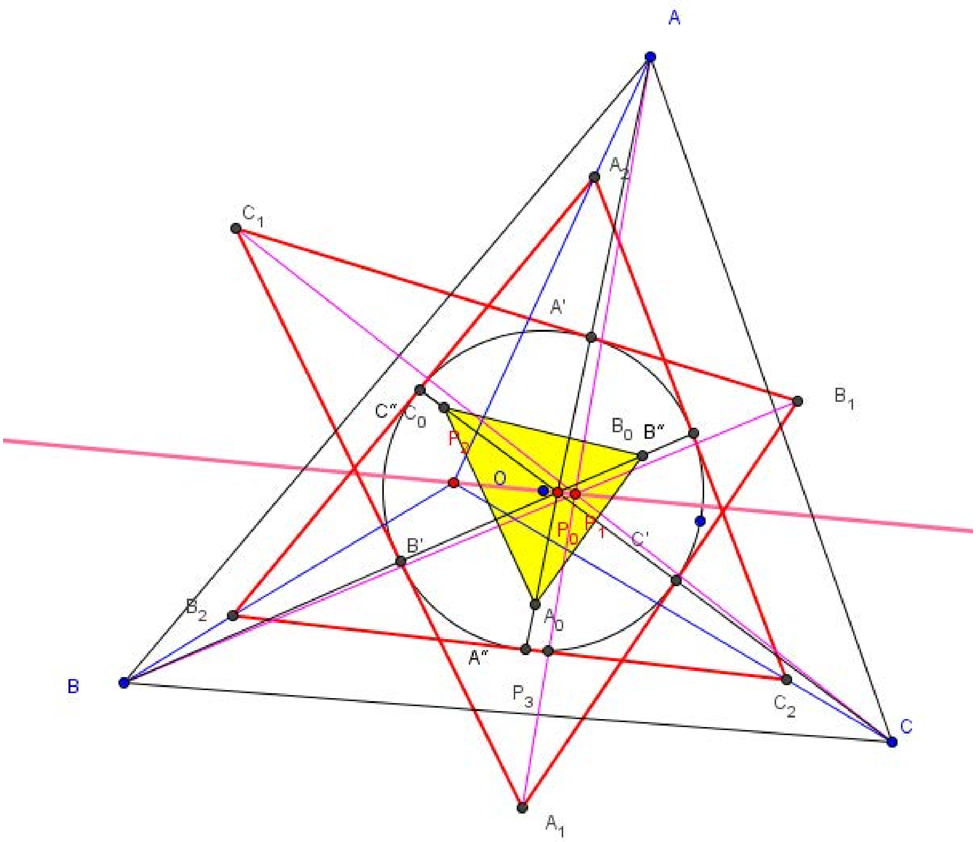
1- Two triangle $A_0B_0C_0$ and ABC are perpective (well-known result); the perpector is P_0

2- Two triangle $A_1B_1C_1$ and ABC are perpective; the perpector is P_1

3- Two triangle $A_2B_2C_2$ and ABC are perpective; the perpector is P_2

4- Three points P_0, P_1, P_2 are collinear.

Attachments:



Luis González

#2 Jan 26, 2015, 11:26 am

$BC, B_0C_0, B_1C_1, B_2C_2$ concur at the pole X of AA_0 WRT (O) and similarly $CA, C_0A_0, C_1A_1, C_2A_2$ and $AB, A_0B_0, A_1B_1, A_2B_2$ concur at the poles Y and Z of BB_0 and CC_0 WRT $(O) \Longrightarrow \overline{XYZ}$ is the polar of P_0 WRT (O) $\Longrightarrow$ by Desargues theorem $\triangle ABC, \triangle A_0B_0C_0, \triangle A_1B_1C_1, \triangle A_2B_2C_2$ are in perspective through the same perspectrix $\overline{XYZ}$. Thus, by the 3 homologies theorem, we deduce that the perspectors P_0, P_1, P_2 are collinear.

Quick Reply

High School Olympiads

Two Equilateral

Reply

Source: Mathematical Excalibur

KudouShinichi
#1 Jan 26, 2015, 1:52 am

Let D be on side BC of equilateral triangle ABC. Let P and Q be the incenters of $\triangle ABD$ and $\triangle ACD$ respectively. Let E be the point so that $\triangle EPQ$ is equilateral and D , E are on opposite sides of PQ .Prove that lines BC and DE are perpendicular.

Luis González
#2 Jan 26, 2015, 3:31 am

More general, let $\triangle ABC$ be A-isosceles and let $\triangle EPQ$ be E-isosceles such that $\angle PEQ = \angle ABC.$ Then BC and DE are perpendicular.

Since $\angle(QD, QA) = 90^\circ - \frac{1}{2}\angle(CA, CB) = \angle(QE, QP) \Longrightarrow QE$ and QA are isogonals WRT $\angle DQP$ and similarly we get that PE, PA are isogonals WRT $\angle DPQ \Longrightarrow A, E$ are isogonal conjugates WRT $\triangle DPQ \Longrightarrow \angle EDP = \angle ADQ$ $\Longrightarrow \angle EDC = \angle PDQ = 90^\circ$, i.e. $DE \perp BC.$

ATimo
#3 Jan 26, 2015, 1:28 pm • 2

Let us solve this:
In triangle ABC , AD is the bisector of A. Let P and Q be the incenters of ABD and ACD respectively.Let X be a point such that PX=QX and PXQ=A/2. X and A are on opposite sides of PQ. Prove that the lines XD and BC are perpendicular.

sasanineq
#4 Jan 26, 2015, 1:58 pm • 2

we must prove anglePDA=angleQDE it`s trivial by the lemma:

LEMMA:if sin(a)/sin(K-A)=sin(B)/sin(K-B) then we have A=B or K=180

sunken rock
#5 Jan 26, 2015, 3:08 pm

For the original problem, see this link as well (just different ... wording):
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=272602

Best regards,
sunken rock

Quick Reply

High School Olympiads

Inequality!

Reply

Source: Korea 1998 Problem 2

ComplexPhi

#1 Jan 25, 2015, 2:27 am

Let D,E,F be points on the sides BC,CA,AB respectively of a triangle ABC. Lines AD,BE,CF intersect the circumcircle of ABC again at P,Q,R, respectively.Show that:

$$\frac{AD}{PD}+\frac{BE}{QE}+\frac{CF}{RF}\geq 9$$

and find the cases of equality.

Luis González

#2 Jan 25, 2015, 7:31 am

Posted before at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=405665
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=157905

For a better bound see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=159968

Quick Reply

High School Olympiads

Fixed point, bisect and perpendicular

Source: HSGS class 10 test (Own)

buratinogigle

#1 Jan 24, 2015, 7:29 am • 1

Let ABC be a triangle wihe M,N lie on segment CA,AB such that $MN\parallel BC$. P is on MN. E is the point such that $EP\perp AC$ and $EC\perp BC$. F is the point such that $FP\perp AB$ and $FB\perp BC$.

a) Prove that line EF is always passes through fixed point when P move.

b) Q lies on BC such that $AQ\perp EF$. Prove that perpendicular bisector of BC bisects PQ.

c) Let EM cuts FN at L. AQ cuts MN at R. Prove that $RL\perp BC$.

Reference link

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=471881

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=471928

shinichiman

#2 Jan 24, 2015, 4:36 pm • 1

a) Let F_1,F_2 lie on line BF such that $F_1N\perp AB, F_2M\perp AC$, E_1,E_2 lie on line CE such that $E_1N\perp AC, E_2M\perp AC$. E_1F_1 intersects E_2F_2 at K then K is fixed point.
Since $P\in MN$ so $\dfrac{FF_1}{FF_2}=\dfrac{EE_1}{EE_2}=\dfrac{PN}{PM}$, we obtain E,F,K are collinear.
b) Draw $PH\perp BC$ with $H\in BC$. The problem is equivalent to $BH=QC$. Since $AQ\perp EF, AC\perp EP$ so $\angle QAC=\angle PEF$. Similarly, we get $\angle PFE=\angle QAB$. We have

$$\frac{QC}{QB}=\frac{AC\sin\angle QAC}{AB\sin\angle QAB}=\frac{AC}{AB}\cdot\frac{\sin\angle PEF}{\sin\angle PFE}=\frac{AC}{AB}\cdot\frac{PF}{PE}$$

We also have

$$\begin{aligned}\frac{HB}{HC}\cdot\frac{PE}{PF}&=\frac{HB}{PF}\cdot\frac{PE}{HC}=\frac{\sin\angle PFB}{\sin\angle PEC}\\&=\frac{\sin\angle ABC}{\sin\angle ACB}=\frac{AC}{AB}\end{aligned}$$

Thus, $\dfrac{HB}{HB}=\dfrac{QC}{QB}$ or $BH=QC$.
c) Let $LR'\perp MN$ at R'. Hence, $\angle NLR'=\angle PBC$ and $\angle MLR'=\angle PCB$. We have

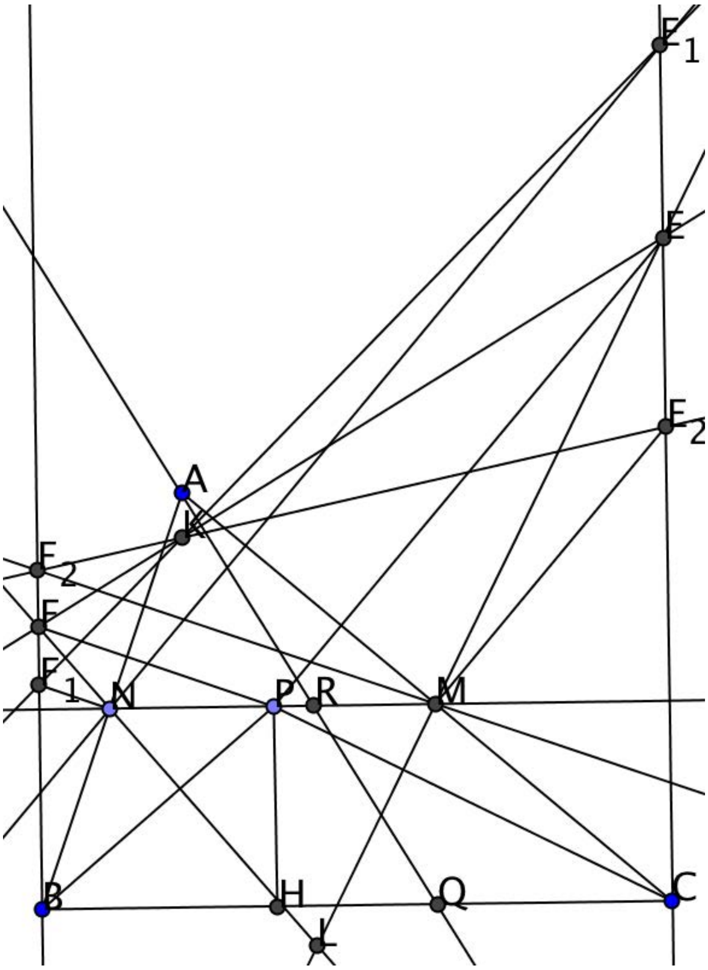
$$\frac{R'N}{R'M}=\frac{LN\sin\angle R'LN}{LM\sin\angle R'LM}=\frac{LN}{LM}\cdot\frac{PC}{PB}.$$

We also have $\angle HPC=\angle LMN, \angle HPB=\angle LNM$ so

$$\frac{HC}{HB}=\frac{PC}{PB}\cdot\frac{\sin\angle HPC}{\sin\angle HPB}=\frac{PC}{PB}\cdot\frac{LN}{LM}.$$

Thus, $\frac{R'N}{R'M}=\frac{HC}{HB}=\frac{QB}{QC}=\frac{RN}{RM}$. Therefore, $R\equiv R'$ or $LR\perp BC$.

Attachments:



buratinogigle

#3 Jan 24, 2015, 5:30 pm • 1

Thank you shinichiman for your interest, here is my solution

a) Let AD be altitude of ABC. MN cuts CE, BF at S, T. The line passes through S perpendicular to AC cuts EF at I. We have $\triangle SPE\sim\triangle DAC$ and $\triangle TPF\sim\triangle DAB$. From this, $\frac{IE}{IF}=\frac{ES}{FG}=\frac{ES}{PS}.\frac{PS}{FG}=\frac{ES}{PS}.\frac{TP}{TF}=\frac{CD}{AD}.\frac{AD}{DB}=\frac{DC}{DB}$. This means I lies on AD deduce I is intersection of AD and SG. So I is fiexed.

b) H is projection of P on BC. We will prove that $QB=HC$ deduce perpendicular bisector of BC bisects PQ, indeeed. From $\triangle SPE\sim\triangle DAC$ and $\triangle TPF\sim\triangle DAB$, we have $\frac{PE}{PF}=\frac{PE}{PS}.\frac{PS}{PT}.\frac{PT}{PF}=\frac{AC}{AD}.\frac{HC}{HB}.\frac{AD}{AB}=\frac{AC}{AB}.\frac{HB}{HC}$.

Let K be on AC such that $BK\parallel AQ$. We have, $\triangle ABK\sim\triangle PFE$ deduce $\frac{QB}{QC}=\frac{BQ}{AK}.\frac{AK}{AB}.\frac{AB}{QC}=\frac{QC}{AC}.\frac{PE}{PF}.\frac{AB}{QC}=\frac{AB}{AC}.\frac{AC}{AB}.\frac{HB}{HC}=\frac{HB}{HC}$. This mean $QB=HC$.

c) We must prove that EM, FN and the line passing through R and perpendicular to MN are concurrent. We must prove $\frac{ES}{FT}=\frac{MS}{MR}.\frac{NR}{NT}$. We have, $\triangle SME\sim\triangle SCP$ and $\triangle TNF\sim\triangle TBP$. From this, $\frac{ES}{FT}=\frac{ES}{MS}.\frac{MS}{NT}.\frac{NT}{FT}=\frac{MS}{NT}.\frac{PS}{SC}.\frac{BT}{PT}=\frac{MS}{NT}.\frac{PS}{PT}=\frac{MS}{MR}.\frac{HB}{HC}=\frac{MS}{MR}.\frac{RN}{NT}$.

We are done.

Attachments:

Figure2616.pdf (13kb)

TelvCohl

#4 Jan 24, 2015, 10:49 pm • 3

My solution:

Let $X = EF \cap MN, T = EF \cap AQ$.
Let $E_1 = MN \cap CE, F_1 = MN \cap BF, E_2 = FP \cap CE, F_2 = EP \cap BF$.
Let H, Y, Z be the projection of A, E_1, F_1 on BC, AB, AC, respectively .
Let $G = E_1Y \cap F_1Z$ and $A_\infty, E_\infty, F_\infty$ be the infinity point on AH, E_1Y, F_1Z, respectively .

Easy to see $\triangle ABC \cup H \sim \triangle PE_2E \cup E_1 \sim \triangle PFF_2 \cup F_1$.

From Pappus theorem (for $E_1 - P - F_1$ and $F_\infty - A_\infty - E_\infty$) we get $G \in EF$,
so EF pass through a fixed point G which is lie on AH ($\because G$ is the orthopole of MN WRT $\triangle ABC$) .

Since $(B, C; H, Q) = (Y, Z; G, T) = (E_1, F_1; \infty, X)$,

so we get $\frac{BH}{CH} \cdot \frac{CQ}{BQ} = \frac{F_1X}{E_1X}$ . ... $(\star)$

From $(\star) \Longrightarrow \frac{CQ}{BQ} = \frac{F_1X}{E_1X} \cdot \frac{CH}{BH} = \frac{FF_1}{EE_1} \cdot \frac{F_1F_2}{F_1F} = \frac{F_1F_2}{EE_1} = \frac{PF_1}{PE_1}$,

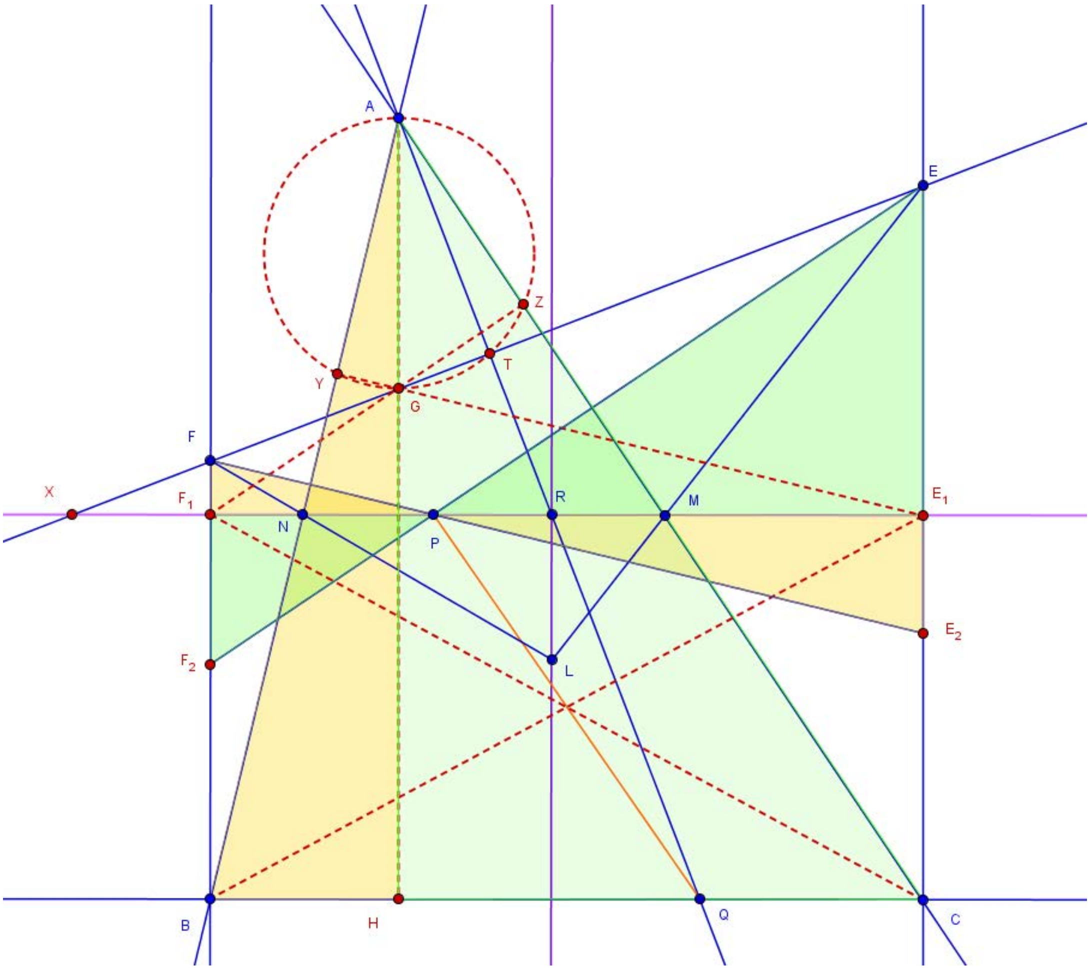
so we get the midpoint of PQ is the center of rectangle BCE_1F_1 ,
hence the perpendicular bisector of BC pass through the midpoint of PQ .

Let ℓ be the perpendicular from R to BC and $L_1 = FN \cap \ell, L_2 = EM \cap \ell$.

From $(\star) \Longrightarrow \frac{RL_1}{RL_2} = \frac{RN}{RM} \cdot \frac{FF_1}{EE_1} \cdot \frac{ME_1}{NF_1} = \frac{BQ}{CQ} \cdot \frac{F_1X}{E_1X} \cdot \frac{CH}{BH} = 1 \Longrightarrow L_1 \equiv L_2 \equiv L$.

Q.E.D

Attachments:



Luis González

#5 Jan 25, 2015, 5:13 am • 1

a) Denote U, V the intersections of MN with $FB, EC.$ When P varies on $MN,$ the series P, E, F are clearly similar inducing a proyectivity between $CV, BU,$ but when P is at inifinity, then $E \equiv F$ is the point at infinity of $BU \parallel CV \Longrightarrow E \mapsto F$ is a perspectivity, i.e. EF goes through a fixed point $X,$ which is then the concurrency point of the perpendiculars from A, U, V to $MN, AM, AN.$

b) If S is the projection of P on $BC,$ it suffices to prove that S, Q are isotomic points WRT $BC.$ Since the pencils AR and XE are projective, it follows that $P \mapsto R$ is a proyectivity on $MN \Longrightarrow S \mapsto Q$ is a proyectivity on $BC,$ so all we need to show is that it is identical with the projectivity taking S to its isotomic WRT $BC.$ Clearly B, C swap and $S \equiv Q$ is the point at infinity of BC when P is at infinity. When $EF \parallel MN, EFUV$ is a rectangle and $ARQ \perp MN$ and since $(UX \parallel EP) \perp AC \Longrightarrow$ by symmetry $SC = PV = XF = QB \Longrightarrow Q, S$ are isotomic points WRT $BC.$ Thus we conclude it holds for all points $P.$

c) Since the pencils EM, FN are projective, it follows that $L \barwedge E \barwedge P \barwedge R.$ Clearly when P is at infinity, then L is the point at infinity of $BU \parallel CV$ and when P coincides with U and $V,$ then $L \equiv R \equiv M$ and $L \equiv R \equiv N,$ resp. Thus, it suffices to show that $RL \parallel BF \parallel CE$ holds for another $P.$ Indeed when $EF \parallel MN, \triangle ABC$ and $\triangle LEF$ are perspective through $MN \Longrightarrow$ by Desargues theorem BF, CE, AL concur, i.e. $AL \parallel BF \parallel CE,$ as desired.

Quick Reply

High School Olympiads

Three circles with a common point!

Source: Korea 1998 Problem 5

ComplexPhi
#1 Jan 25, 2015, 2:29 am

Let I be the incenter of triangle ABC, O_1 a circle through B tangent to CI, and O_2 a circle through C tangent to BI. Prove that O_1,O_2 and the circumcircle of ABC have a common point.

Luis González
#2 Jan 25, 2015, 3:32 am

Additional conditions are missing here, as there are infinetely many circles through B tangent to CI and there are infinetely many circles through C tangent to BI. Probably the correct version refers (O_1) as the circle through B tangent to CI at I and the same for (O_2). If yes, the problem is quite easy.

Let P be the 2nd intersection of (O_1) with $\odot(ABC)$. $\widehat{BPI} = 180^\circ - \widehat{BIC} = \frac{1}{2}(180^\circ - \widehat{BAC}) = \frac{1}{2}\widehat{BPC} \Longrightarrow PI$ bisects $\widehat{BPC}$, i.e. P is the second intersection of $\odot(ABC)$ with the line though I and the midpoint of the arc BAC and similarly (O_2) goes through P.

jayme
#3 Jan 25, 2015, 5:07 pm

Dear Mathlinkers,
according the correction of Luis, we can have an enterely synthetic proof with angle by considering the circle (BIC) : a converse of the pivot theorem, then Reim's theorem for (O1) and siilarly for (O2).

Sincerely
Jean-Louis

High School Olympiads

□XAY = □XYM!

ATimo
#1 Jan 24, 2015, 12:49 am • 2

Let ABC be a triangle. Let M be the midpoint of BC. Let ω be the circle with center M and radius BC/2. This circle is intersecting AB and AC in E and F. Let P be the intersecting point of AM and EF. Let the point X be on the arc of EF (from ω) XP is intersecting ω for the second time in Y. Prove that $\angle XAY = \angle XYM$.

Luis González
#2 Jan 24, 2015, 1:48 am

Obviously E,F are the projections of C,B on AB,AC. Let $H \equiv BF \cap CE$ be the orthocenter of $\triangle ABC$. Since $PX \cdot PY = PE \cdot PF$, then P is on radical axis of $\odot(AXY)$ and $\odot(AEHF) \Longrightarrow AP$ is their radical axis $\Longrightarrow$ they meet again at $D \in AM$ (projection of H on AM). Therefore HD is the polar of A WRT $\omega \Longrightarrow A, D$ are conjugate points WRT ω $\Longrightarrow \odot(AXDY)$ is orthogonal to $\omega \Longrightarrow MY$ is tangent to $\odot(AXY) \Longrightarrow \angle XAY = \angle XYM$.

TelvCohl
#3 Jan 24, 2015, 2:58 am • 1

My solution:

Let $H = BF \cap CE$ and T be the projection of H on AM .

Easy to see H is the orthocenter oF $\triangle ABC$.

Since H lie on the polar of A WRT ω ,
so TH is the polar of A WRT $\omega \Longrightarrow MT \cdot MA = MX^2 = MY^2$ . ... (1)

Since A, E, F, H, T are concyclic ,
so $AP \cdot PT = EP \cdot PF = XP \cdot PY \Longrightarrow A, T, X, Y$ are concyclic . ... (2)

From (1) and $(2) \Longrightarrow MX, MY$ are the tangent of $\odot(ATXY) \Longrightarrow \angle XAY = \angle XYM$.

Q.E.D

godofgeometry
#4 Jan 24, 2015, 1:08 pm • 3

ATimo very interesting question
draw the circle AEF. suppose that the intersection point of AM and the circle is k. it is easy to prove that ME is tangent to this circle. ME^2=MK*MA and ME=MY so MY^2=MK*MA then KYM=KAY so only we need to prove that AXKY is cyclic.
AP*PK=EP*PF=YP*PX so AXKY is cyclic.

High School Olympiads

Inradius of Midpoint of Arcs Triangle

Source: Own (kind of solved)

tastymath75025

#1 Jan 23, 2015, 10:22 am

Try to do it synthetically please

Let ABC be a triangle with circumcircle w. Let D, E, F be the midpoints of minor arcs BC, CA, AB. Show that the inradius of ABC is less than or equal to the inradius of DEF.

The problem has already been solved with other methods such as

Let r_1, r_2 be the inradii and note $1+\frac{r_1}{R} = \cos A + \cos B + \cos C$ while $1+\frac{r_2}{R} = \cos\frac{A+B}{2} + \cos\frac{B+C}{2} + \cos\frac{C+A}{2}$.

However, it would be nice to see a synthetic proof or mostly synthetic at least.

(sorry if this has been posted before)

Luis González

#2 Jan 23, 2015, 9:49 pm • 1

Let $\triangle I_aI_bI_c$ be the excentral triangle of $\triangle ABC$. Circumcircle (O,R) of $\triangle ABC$ becomes 9-point circle of $\triangle I_aI_bI_c \Longrightarrow D, E, F$ are the midpoints of II_a, II_b, II_c and the circumradius of $\triangle I_aI_bI_c$ is $2R$. Thus if ϱ is the inradius of $\triangle I_aI_bI_c$, it suffices to prove that $\varrho \geq 2r$, where r is the inradius of $\triangle ABC$.

Let r_a, r_b, r_c denote the exradii of $\triangle ABC$. If X and X' are the projections of I and I_a on BC, we have $II_a \geq IX + I_aX' = r + r_a$ and similarly $II_b \geq r + r_b$ and $II_c \geq r + r_c \Longrightarrow II_a + II_b + II_c \geq 3r + r_a + r_b + r_c$. But in the acute $\triangle I_aI_bI_c$, the sum of the distances from its orthocenter I to its vertices equals twice the sum of its circumradius and inradius $\Longrightarrow 2(2R+\varrho) \geq 3r + r_a + r_b + r_c$. Finally substituting $r_a + r_b + r_c = 4R + r$ yields $\varrho \geq 2r$, as desired.

High School Olympiads

A very hard problem(Maybe known)

Locked

Source: ???

toto1234567890

#1 Jan 23, 2015, 6:02 pm

There is a triangle ABC. A point D is on the bisector of $\angle BAC$. And O is the circumcenter of $\triangle ABC$.
X,Y,Z are projections of D on BC,CA,AB . And OD meets BC at E.
The symmetry point of X on YZ is W .

Prove that A,W,E are collinear.

Luis González

#2 Jan 23, 2015, 6:46 pm

Posted before at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=346956
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=420917
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=495103

In addition O,E and the orthocenter of X*YZ are also colinear, where X* is the second intersection of BC with (XYZ).
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=615866

High School Olympiads

Six point lie on a circle-A generalization Nine point circle

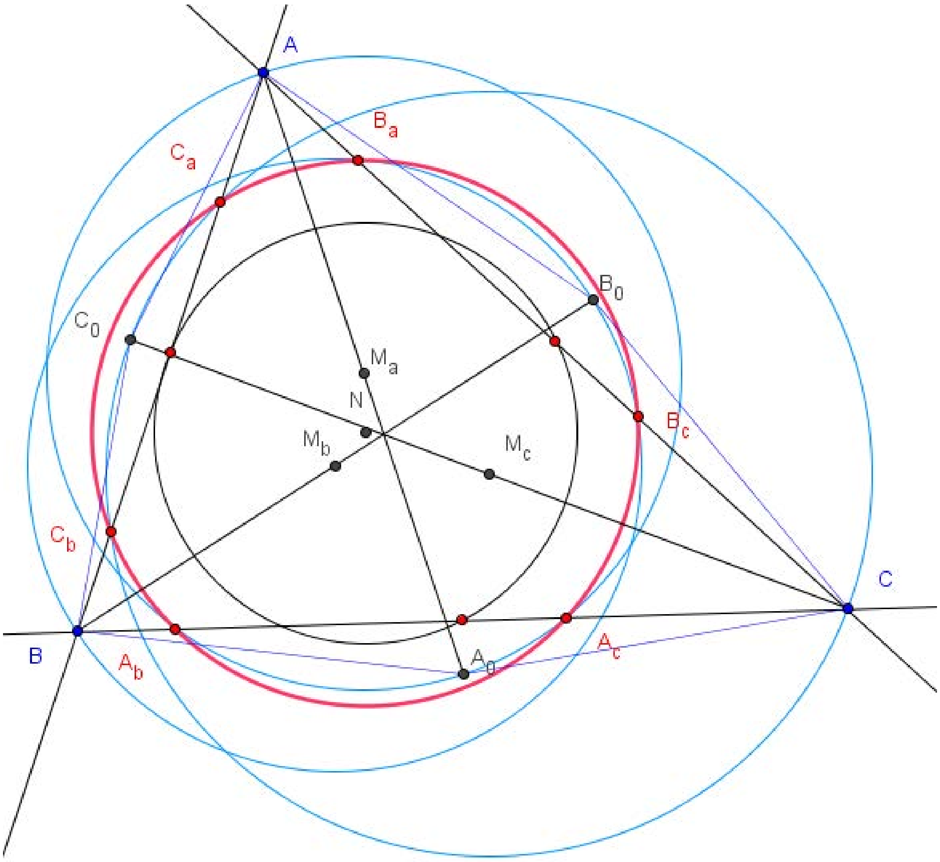
Source: The result is well-know?

daothanhoai

#1 Jan 22, 2015, 7:37 am • 1

Let ABC be a triangle, let $A_0B_0C_0$ be Kiepert triangle of ABC. The circle with diameter AA_0 meets BC at A_bA_c. Define B_a, B_c, C_a, C_b cyclically. Show that six point $A_bA_c, B_a, B_c, C_a, C_b$ lie on a circle. When the Kiepert triangle is median triangle of ABC the circle is Nine point circle. The result is well-known?

Attachments:



TelvCohl

#2 Jan 22, 2015, 10:03 am • 2

My solution:

Let A_B, A_C be the projection of A_0 on AB, AC, respectively .
Let B_C, B_A be the projection of B_0 on BC, BA, respectively .
Let C_A, C_B be the projection of C_0 on CA, CB, respectively .

Easy to see $B_A \in \odot(BB_0)$ and $C_A \in \odot(CC_0)$.
From $Rt\triangle AB_0B_A \sim Rt\triangle AC_0C_A \Longrightarrow AB_A : AC_A = AB_0 : AC_0 = AC : AB$,
so we get $AB_c \cdot AB_a = AB \cdot AB_A = AC \cdot AC_A = AC_a \cdot AC_b \Longrightarrow B_c, B_a, C_a, C_b$ are concyclic .
Similarly, we can prove C_a, C_b, A_b, A_c are concyclic and A_b, A_c, B_c, B_a are concyclic ,
so from Davis theorem we get $A_b, A_c, B_c, B_a, C_a, C_b$ are concyclic .

Q.E.D

buratinogigle

#3 Jan 22, 2015, 10:34 am

Nice problem, I just have remark AA_0, BB_0, CC_0 are concurrent at P on Kiepert Hyperbola, and radical center H of circles diameter AA_0, BB_0, CC_0 lie on Kiepert Hyperbola, too. We get center of circle passing though $A_b, A_c, B_a, B_c, C_a, C_b$ lie on line PH.

TelvCohl

#4 Jan 22, 2015, 1:15 pm • 2

> buratinogigle wrote:
>
> Nice problem, I just have remark AA_0, BB_0, CC_0 are concurrent at P on Kiepert Hyperbola, and radical center H of circles diameter AA_0, BB_0, CC_0 lie on Kiepert Hyperbola, too. We get center of circle passing though $A_b, A_c, B_a, B_c, C_a, C_b$ lie on line PH.

My solution:

Lemma:

Let D be a point out of $\triangle ABC$ satisfy $\angle DBC = \angle DCB = \theta$.
Let E be a point out of $\triangle ABC$ satisfy $\angle EAC = \angle ECA = 90^\circ - \theta$.
Let F be a point out of $\triangle ABC$ satisfy $\angle FAB = \angle FBA = 90^\circ - \theta$.

Then $AD \perp EF$.

Proof of the lemma:

Let $B' \in AF, C' \in AE$ satisfy $AB = AB', AC = AC'$ and $T = BC' \cap CB'$.

Easy to see $\triangle ABB' \cup F \sim \triangle ACC' \cup E \Longrightarrow EF \parallel B'C'$.

From $\triangle AB'C \sim \triangle ABC' \Longrightarrow \angle BTC = 180^\circ - (90^\circ - \theta) = 90^\circ + \theta$,
so combine with $\angle DBC = \angle DCB = \theta$ we get D is the circumcenter of $\triangle BTC$,
hence from 2006 USA TST Problem 6 we get $AD \perp B'C'$. i.e. $AD \perp EF$

Back to the main problem:

Let D, E, F be the midpoint of BC, CA, AB, respectively .
Let A_1 be a point satisfy $\angle A_1BA_0 = \angle A_1CA_0 = 90^\circ$.
Let B_1 be a point satisfy $\angle B_1CB_0 = \angle B_1AB_0 = 90^\circ$.
Let C_1 be a point satisfy $\angle C_1AC_0 = \angle C_1BC_0 = 90^\circ$.
Let O_a, O_b, O_c be the midpoint of AA_0, BB_0, CC_0, respectively .
Let B_0', C_0' be the reflection of B_0, C_0 in CA, AB, respectively .

From my proof for the original problem we get A lie on the radical axis of $\{\odot(O_b), \odot(O_c)\}$. $(\star)$
Easy to see $\triangle ABC \cup B_0' \cup C_0' \sim \triangle DEF \cup O_b \cup O_c \Longrightarrow O_bO_c \parallel B_0'C_0'$,
so combine with the lemma and $(\star)$ we get $AA_1 \perp B_0'C_0' \Longrightarrow AA_1 \perp O_bO_c \Longrightarrow AA_1$ is the radical axis of $\{\odot(O_b), \odot(O_c)\}$.

Similarly, we can prove BB_1, CC_1 is the radical axis of $\{\odot(O_c), \odot(O_a)\}, \{\odot(O_a), \odot(O_b)\}$, respectively ,
so we get $H \equiv AA_1 \cap BB_1 \cap CC_1$ lie on the Kiepert hyperbola of $\triangle ABC$.

From Sondat theorem (for $\triangle ABC$ and $\triangle O_aO_bO_c$) we get the center of $\odot(A_bA_cB_cB_aC_aC_b)$ lie on PH .

Q.E.D

Remark:

(1)

If we denote $\phi = \angle A_0BC = \angle A_0CB = \angle B_0CA = \angle B_0AC = \angle C_0AB = \angle C_0BA$,
then P is the Kiepert perspector K_ϕ and the radical center H of $\{\odot(O_a), \odot(O_b), \odot(O_c)\}$ is the Kiepert perspector $K_{90+\phi}$.

(2)

The center of $\odot(A_bA_cB_cB_aC_aC_b)$ is actually the nine point center of $\triangle ABC$,
so we get the property of Kiepert point : $K_\phi K_{90+\phi}$ pass through the nine point center of $\triangle ABC$.
(You can find more properties about Kiepert point at Collinearity with Lemoine point of a triangle.(post # 5))

(3)

From the lemma we get the following property about Kiepert triangle :
The pedal triangle of the isogonal conjugate of $K_{90-\phi}$ WRT $\triangle ABC$ and the Kiepert triangle with angle ϕ are homothetic .
(Moreover, the homothety center of these two triangles is the Symmedian point of $\triangle ABC$!)

daothanhoai

#5 Jan 22, 2015, 1:48 pm

1-Center of the New circle $(A_bA_cB_aB_cC_aC_b)$ is Nine point circle;

2-Intersection of three circles which diameter AA0,BB0,CC0 at A*,B*,C*(outer) and A",B",C"(inner). Show that ABC perpective with two triangle A*B*C* and A"B"C"; the perpector lie on Kiepert hyperbola

Luis González

#6 Jan 23, 2015, 6:00 am • 1

If one goes for the center and radius of the circle the result is quite nice. If θ denotes the Kiepert angle and R and S denote the circumradius and area of $\triangle ABC$, respectively, we prove that these 6 points lie on a circle with center the 9-point center N and radius $\varrho = \sqrt{\tfrac{1}{4}R^2 + S\cdot\tan\theta}.$

Let O, H denote the circumcenter and orthocenter of $\triangle ABC. X$ is the projection of A on BC, M is the midpoint of BC and L is the midpoint of $XM.$ If AX cuts the circle with diameter AA_0 again at $U,$ then by symmetry $A_bUA_0A_c$ is an isosceles trapezoid. In the cyclic AA_bUA_c with perpendicular diagonals, we have

$$A_bA_c{}^2 = (XA_b + XA_c)^2 = XA_b{}^2 + XA_c{}^2 + 2\cdot XA_b\cdot XA_c =$$

$$= AA_0{}^2 - (AX^2 + XU^2) + 2\cdot AX\cdot XU = AA_0{}^2 - (AX - XU)^2 =$$

$$= (AX + XU)^2 + XM^2 - (AX - XU)^2 = XM^2 + 4\cdot AX\cdot XU \Longrightarrow$$

$$NA_b{}^2 = NL^2 + LA_b{}^2 = NL^2 + \tfrac{1}{4}XM^2 + AX\cdot XU.$$

Substituting $XM^2 = OH^2 - (HX - OM)^2, NL = \tfrac{1}{2}(OM + HX)$ and $OM = \tfrac{1}{2}HA$ into the latter expression and factoring yields

$$NA_b{}^2 = \tfrac{1}{2}HA\cdot HX + \tfrac{1}{4}OH^2 + AX\cdot XU.$$

Now, substituting $HA\cdot HX = \tfrac{1}{2}(R^2 - OH^2)$ and $XU = MA_0 = BM\cdot\tan\theta$ into the latter expression, we obtain $NA_b{}^2 = \tfrac{1}{4}R^2 + S\cdot\tan\theta,$ which is obviously a symmetric expression. Thus we conclude the 6 described points lie on a circle with center N and radius $\varrho = \sqrt{\tfrac{1}{4}R^2 + S\cdot\tan\theta}.$

Quick Reply

High School Olympiads

points family

Reply

PROF65

#1 Jan 22, 2015, 11:33 pm

Let $C(O)$ a circle ,K a point outside it ;l,k variable lines passing through K and intersecting C at $M,P;N,Q$ resp. s.t. $K,M,P;K,N,Q$ are in this order. the circles of OMN and OPQ intersect at O and $T_{l,k}$. what is the locus of $T_{l,k}$

Luis González

#2 Jan 22, 2015, 11:54 pm

MN,PQ and OT are pairwise radical axes of $(O),\odot(OMN),\odot(OPQ)$ concurring at its radical center $L.$ Thus if OT cuts (O) at $U,V,$ we have $LU \cdot LV=LM \cdot LN=LO \cdot LT.$ Since O is midpoint of $UV,$ it follows that $(U,V,T,L)=-1 \Longrightarrow KT$ is the polar of L WRT $(O) \Longrightarrow LOT \perp KT \Longrightarrow T$ is on circle with diameter $\overline{KO}.$

Quick Reply

High School Olympiads

AK is perpendicular to MN

Reply

THVSH
#1 Jan 22, 2015, 3:39 pm

Let ABC be a triangle and K is the Kosnita point of ABC. BE, CF are the altitudes of the triangle ABC. M, N lie on the segment BF, CE, $FM = \frac{1}{3}BF, EN = \frac{1}{3}CE$. Prove that $AK \perp MN$

Sardor
#2 Jan 22, 2015, 4:06 pm

What is Kosnita point?

TelvCohl
#3 Jan 22, 2015, 4:17 pm • 2

Dear **Sardor**, Kosnita point is the isogonal conjugate of the nine point center :) .

My solution:

Let T be the midpoint of BC .
Let G, N be the centroid, nine point center of $\triangle ABC$, respectively .
Let A', B', C' be the reflection of A, B, C in BC, CA, AB, respectively .

Since $BM : MF = 2 : 1$ and $CF = FC'$,
so M is the centroid of $\triangle BCC' \Longrightarrow M \in C'T$ and $TM : MC' = 1 : 2$.
Similarly, we can prove $N \in B'T$ and $TN : NB' = 1 : 2 \Longrightarrow MN \parallel B'C$. $(\star)$
Since $\triangle A'B'C'$ is the image of the pedal triangle of N WRT $\triangle ABC$ under homothety $\mathcal{H}(G, 4)$ (well-known) ,
so we get $AK \perp B'C' \Longrightarrow AK \perp MN$ (from $(\star)$) .

Q.E.D

jayme
#4 Jan 22, 2015, 4:18 pm

Dear Mathlinkers,

see a figure and more on

http://jl.ayme.pagesperso-orange.fr/Docs/Le%20point%20de%20Kosnitza.pdf P. 8

Sincerely
Jean-Louis

PROF65
#5 Jan 22, 2015, 9:04 pm • 1

I will use two results which are easy to prove
1) let ABC triangle M, F two points in the side AB ;N, E two points in the side AC s.t.
$M = Bar\{(B, k)(F, l)\}, N = Bar\{(C, k)(E, l)\}$ so if the centers of ABC, AEF, AMN areO, O', O'' resp. then $O'' = Bar\{(O, k)(O', l)\}$. (Bar: barycenter)
2) the isogonal of the diametrical cevian is the orthogonal cevian ie the altitude.

Applying the result 1 to the problem yield $O'O'' = \frac{1}{3}O'O$ where O, O', O'' centers of ABC, AEF, AMN resp. but O' is the midpoint of AH (H orthocenter of ABC) thus AO'' pass through the ;idpoint of HO which is the ninepoint center therefore by result 2 AK the isogonal of AO'' is orthogonal to MN

Luis González

#6 Jan 22, 2015, 10:08 pm • 2

Let $H \equiv BE \cap CF$ and let N bet 9-point center of $\triangle ABC,$ which is also 9-point center of $\triangle HBC.$ If L is the centroid of $\triangle HBC,$ then $(LM \parallel CF) \perp AB$ and $(LN \parallel BE) \perp AC.$ Thus it's enough to prove that $N \in AL.$ Indeed AL is Euler line of $\triangle HBC$ passing through its 9-point center $N.$

yunxiu

#7 Jan 23, 2015, 7:25 am • 1

Let J is the nine-point center of ΔABC, $JX \bot AB$, $JY \bot AC$.

Denote $AE = a$,$CE = b$,then $AN = a + \frac{1}{3}b$.

Because Y is the midpoint of EP, so $AY = \frac{1}{2}\left(a + \frac{1}{2}(a + b) \right) = \frac{3}{4}a + \frac{1}{4}b = \frac{3}{4}AN$.

From $\frac{AY}{AN} = \frac{3}{4} = \frac{AX}{AM}$, we have $XY \parallel MN$, hence $AK \bot MN$.

High School Olympiads

Mapping of P on the pedal circle of P

Reply

Source: Hyacinthos #23044

rodinos

#1 Jan 22, 2015, 1:37 am

Let ABC be a triangle, P a point and A'B'C' the pedal triangle of P

Denote:

I1 = the reflection of I in AP
Ia = the reflection of I1 in AI

I2 = the reflection of I in BP
Ib = the reflection of I1 in BI

I3 = the reflection of I in CP
Ic = the reflection of I1 in CI

1. The circumcenter of IaIbIc is the isogonal conjugate of P.

2. The parallels through A', B', C' of the perpendicular
bisectors of IbIc, IcIa, IaIb, resp. concur on the pedal circle of P.
(ie circumcircle of A'B'C').

aph

Luis González

#2 Jan 22, 2015, 2:48 am • 1

1) Composition of axial simmetries on AP, AI is a rotation with center A and rotational angle $2\angle(AP, AI) \Longrightarrow \triangle AII_a$ is A-isosceles with $\angle IAI_a = 2 \cdot \angle PAI \Longrightarrow$ perpendicular bisector of II_a is the reflection of AP on AI, i.e. the isogonal of AP passing through the isogonal conjugate Q of P. Likewise BQ is perpendicular bisector of $II_b \Longrightarrow Q$ is circumcenter of $\triangle II_aI_b$ $\Longrightarrow$ perpendicular bisector of I_aI_b goes through Q and the same holds for the others, thus Q is circumcenter of $\triangle I_aI_bI_c$ and in addition $I \in \odot(I_aI_bI_c)$.

2) Since $(II_a \parallel B'C') \perp AQ$ and $(II_b \parallel A'C') \perp BQ \Longrightarrow \angle I_aI_cI_b = \angle I_aII_b = \angle A'C'B'$ and similarly for the others. Thus we deduce that $\triangle A'B'C'$ and $\triangle I_aI_bI_c$ are inversely similar, consequently they are orthologic with orthology centers lying on their corresponding circumcircles.

rodinos

#3 Jan 22, 2015, 3:22 am

A'B'C', IaIbIc are orthologic and parallelogic.

rodinos

#4 Jan 22, 2015, 9:18 am

Analytic studies by

Angel Montesdeoca
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/23047

Cesar Lozada
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/23048

Quick Reply

High School Olympiads

Two Thebault circle equal incircle

Reply

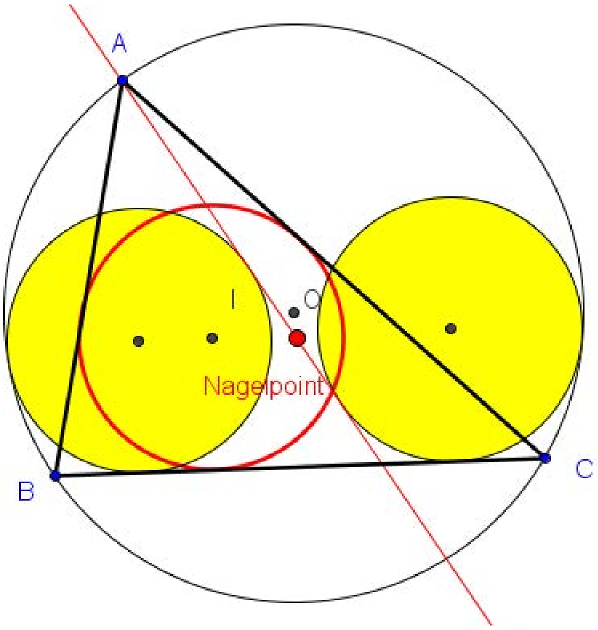
Source: Francisco Javier and I found this

daothanhoai

#1 Jan 21, 2015, 11:00 pm

Show that: Two Thebault circle respect to the line AN_a has radii = r , r = radii of incircle and N_a is Nagel point of ABC.(But I don't know it is new or old)

Attachments:



This post has been edited 1 time. Last edited by daothanhoai, Jan 21, 2015, 11:26 pm

Luis González

#2 Jan 21, 2015, 11:17 pm • 1

Old problem. Further, two Thebault circles of a cevian AA' of ABC are congruent if and only if AA' is the A-Nagel cevian. The configuration also appeared in a Bulgarian NMO (2010).

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=201858
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=349730

Quick Reply

High School Olympiads

Nice collinearity!

Reply

Source: Romania TST 2014 Day 1 Problem 1

ComplexPhi

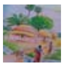
#1 Jan 21, 2015, 9:53 pm

Let ABC be a triangle, let A', B', C' be the orthogonal projections of the vertices A ,B ,C on the lines BC, CA and AB, respectively, and let X be a point on the line AA'.Let γ_B be the circle through B and X, centred on the line BC, and let γ_C be the circle through C and X, centred on the line BC.The circle γ_B meets the lines AB and BB' again at M and M', respectively, and the circle γ_C meets the lines AC and CC' again at N and N', respectively.Show that the points M, M', N and N' are collinear.

Luis González

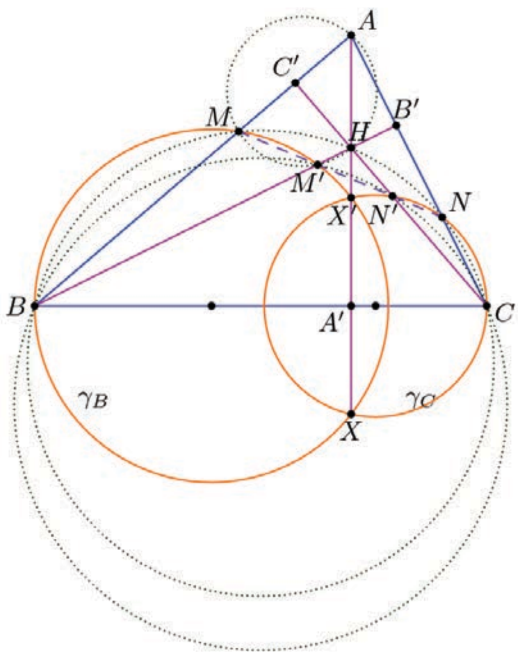
#2 Jan 21, 2015, 10:31 pm

Clearly AA' is the radical axis of $\gamma_B,\gamma_C,$ thus it follows that $BCNM$ and $BCM'N'$ are cyclic $\Longrightarrow$ $\angle BM'N'=\angle BCC'=\angle BB'C \Longrightarrow MN,M'N'$ are antiparallel to BC WRT $AB,AC,$ i.e. $MN \parallel M'N' \parallel B'C'.$ But if γ_B cuts BC again at $Y,$ we have $(YM' \parallel CA) \perp BB' \Longrightarrow \angle AMM'=\angle BYM'=\angle BCA=\angle AC'B' \Longrightarrow$ $MM' \parallel B'C'.$ Consequently M,M',N,N' are collinear on a parallel to $B'C'.$



Ankoganit

#3 May 21, 2016, 11:34 am



Suppose a reflection about BC takes X to X'; clearly $X'\in AA'$. Now since the centers of γ_B and γ_C are on BC, this reflection keeps them fixed. Thus they pass through X' as well. So by power of point,

$$AM\cdot AB=AX\cdot AX'=AN\cdot AC\implies BMNC\text{ is cyclic.}$$

On the other hand,

$$HM'\cdot HB=HX\cdot HX'=HN'\cdot HC\implies BM'N'C\text{ is cyclic.}$$

Now we have

$$\begin{aligned} BM' \cdot BH =& (BH - HM') \cdot HB \\ =& BH^2 - HM' \cdot HB = BH^2 - HX' \cdot HX \\ =& BH^2 - (HA - X'A')(HA + XA') = BH^2 - HA^2 + XA'^2 \qquad [\text{Since } X'A' = XA'] \\ =& BA'^2 + XA'^2 \\ =& BA^2 - AA'^2 + XA'^2 = BA^2 - (AA' + XA')(AA' - X'A') \\ =& BA^2 - AX \cdot AX' \\ =& BA^2 - AM \cdot AB = BA \cdot (BA - AM) \\ =& BA \cdot BM. \end{aligned}$$

Therefore $BM' \cdot BH = BM \cdot BA \implies AMM'H$ is cyclic. Therefore

$$\angle MM'B = \angle BAH = \angle HCB = \angle HM'N'.$$

This implies M, M', N' are collinear. Similarly, we can prove N, M', N' are collinear, so that M, M', N, N' are collinear, as desired. ■

This post has been edited 3 times. Last edited by Ankoganit, May 21, 2016, 11:49 am

WizardMath

#4 May 23, 2016, 1:26 pm

This configuration is like 2013 G1

Quick Reply

High School Olympiads

Tripole lies on OI line

Reply

Source: Own

buratinogigle
#1 Jan 18, 2015, 5:28 pm

Let ABC be triangle. A-excicle (I_a) touches BC, CA, AB at A_a, A_b, A_c. $B-$excicle (I_b) touches CA, AB, BC at B_b, B_c, B_a. C-excicle (I_c) touches AB, BC, CA at C_c, C_a, C_b. Let B_bC_c cuts B_cC_b at D. Similarly we have E, F. Prove that D, E, F are collinear on line d and tripole of d with respect to triangle $A_aB_bC_c$ lies on OI line of triangle ABC.

Sardor
#2 Jan 18, 2015, 6:06 pm • 1

I think this problem very nice !
But i dont understand, what is tripole of the line ?

TelvCohl
#3 Jan 18, 2015, 6:19 pm • 1

My solution:

Let $A' = AI_a \cap B_bC_c, B' = BI_b \cap C_cA_a, C' = CI_c \cap A_aB_b$.
Let $B_e \equiv I_aA_a \cap I_bB_b \cap I_cC_c$ be the Bevan point of $\triangle ABC$.

From symmetry we get $D \in I_bI_c, E \in I_cI_a, F \in I_aI_b$.

From AD is the external bisector of $\angle C_cAB_b \Longrightarrow \frac{C_cD}{B_bD} = \frac{C_cA}{B_bA}$.

Similarly, we can prove $\frac{A_aE}{C_cE} = \frac{A_aB}{C_cB}$ and $\frac{B_bF}{A_aF} = \frac{B_bC}{A_aC}$,

so we get $\frac{C_cD}{B_bD} \cdot \frac{A_aE}{C_cE} \cdot \frac{B_bF}{A_aF} = \frac{C_cA}{B_bA} \cdot \frac{A_aB}{C_cB} \cdot \frac{B_bC}{A_aC} = 1 \Longrightarrow D, E, F$ are collinear at d .

From AA' is the internal bisector of $\angle C_cAB_b \Longrightarrow (D, A'; C_c, B_b) = -1$.
Similarly, we can prove $(E, B'; A_a, C_c) = -1$ and $(F, C'; B_b, A_a) = -1$,
so the trilinear pole of of d WRT $\triangle A_aB_bC_c$ is the perspector of $\triangle A_aB_bC_c$ and $\triangle A'B'C'$.

Since d is the common perspective axis of any two triangles of $\{\triangle I_aI_bI_c, \triangle A_aB_bC_c, \triangle A'B'C'\}$,
so the perspector of $\{\triangle I_aI_bI_c, \triangle A_aB_bC_c\}, \{\triangle A_aB_bC_c, \triangle A'B'C'\}, \{\triangle I_aI_bI_c, \triangle A'B'C'\}$ are collinear ,
hence the trilinear pole of d WRT $\triangle A_aB_bC_c$ lie on $B_eI \equiv OI$.

Q.E.D

buratinogigle
#4 Jan 18, 2015, 9:19 pm • 1

Thank Telv for nice solution.

@Sardor If DEF is cevian triangle of P wrt ABC then EF, FD, DE cut BC, CA, AB at three collinear points on line d. P is called tripole of d and d is called tripolar of P.

buratinogigle

#5 Jan 19, 2015, 11:12 am • 1

I have seen some other properties from this configuration.

Easily seen B_b, C_c, B_c, C_b lie on circle (ω_a). Similarly we have $(\omega_b), (\omega_c)$. (ω_b) cuts (ω_c) again at X. Similarly we have Y, Z.

a) Prove that AX, BY, CZ are concurrent at circumcenter O of ABC.

b) Prove that circumcenter of triangle XYZ lies on OI line of ABC.

c) Let L be radical center of $(\omega_a), (\omega_b), (\omega_c)$. H is orthocenter of ABC. Prove that HL is paralllel to OI line of ABC.

Sardor

#6 Jan 19, 2015, 4:23 pm • 1

Thank you @buratinogigle !

Luis González

#7 Jan 20, 2015, 9:32 pm • 3

There is more to say, this tripole is Kimberling center X_{3057} of $\triangle ABC$, intersection of OI with the line connecting the Spieker point and the Feuerbach point.

Let Fe, Na, S denote the Feuerbach point, Nagel point and Spieker point (midpoint of $\overline{INa}$) of $\triangle ABC$, respectively. Since AD is obviously the external bisector of $\angle BAC \equiv \angle C_cAB_b$, then AI cuts B_bC_c at the harmonic conjugate D' of D WRT $\{B_b, C_c\}$ and likewise BI, CI cut C_cA_a, A_aB_b at the harmonic conjugates E', F' of E, F WRT $\{C_c, A_a\}, \{A_a, B_b\}$. According to the generalization discussed in the topic Schwatt's lines, $X \equiv A_aD' \cap B_bE' \cap C_cF'$ is the pole of INa WRT the conic $\mathcal{F}$ through A, B, C, I, Na, the Feuerbach hyperbola of $\triangle ABC$ with center Fe. Hence the tangent OI of $\mathcal{F}$ at I goes through X and since the lines FeS and IN_a have conjugate directions WRT $\mathcal{F}$, it follows that $X \in FeS \Longrightarrow X \equiv OI \cap FeS \equiv X_{3057}$.

Quick Reply

High School Olympiads

You can solve it!

Locked

ATimo

#1 Jan 20, 2015, 12:15 pm

Let ABC be a triangle. Suppose D is on BC such that AD bisects \angle BAC. Suppose M is on AB such that \angle MDA = \angle ABC, and N is on AC such that \angle NDA = \angle ACB. If AD and MN intersect on P, prove that AD^3 = AB \cdot AC \cdot AP.

Luis González

#2 Jan 20, 2015, 12:25 pm

First off, give your topics meaningful subjects and use LaTeX to make your post more comprehensive, just type formulae between dolar signs. Secondly, use the search before posting contest problems, this is Indonesian Mathematical Olympiad 2014 Day 2 (p6). See the topic http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=605055.

High School Olympiads

very hard, help me

Reply

calhanSPheiro2
#1 Jan 19, 2015, 8:19 pm

A quadrilateral ABCD is inscribed into a circle ω with center O. Let M and N be the midpoints of segments AB and CD respectively. Let Ω be the circumcircle of triangle OMN.Let P and Q be the common points of ω and Ω, and X and Y the second common points of Ω with the circumcircles of triangles CDM and ABN. Prove PQ||XY

.

ATimo
#2 Jan 19, 2015, 10:20 pm • 1

Guidance:
Suppose that AB and CD are intersecting in P.we must prove that OP is Perpendicular to XY. use that the main axises of three circles are Intersecting in one point.(it is important to find the main axis of the circles CDM and ABN) and use also the power of intersecting points to solve the problem.

calhanSPheiro2
#3 Jan 19, 2015, 11:25 pm

can you present the solution clearly? ATimo

YESMAths
#4 Jan 19, 2015, 11:30 pm • 1

> ATimo wrote:
> (it is important to find the main axis of the circles CDM and ABN)

Usually they are called the radical axes, than main axes of the circles. I hope you mean that only.

> calhanSPheiro2 wrote:
> can you present the solution clearly? ATimo

You may first try to work on the guidelines provided by ATimo. If there is a dead end, then you may ask for help.

TelvCohl
#5 Jan 19, 2015, 11:59 pm • 1

My solution:

Let $E=AB \cap CD, F=DA \cap BC, M'=PQ \cap \odot AB, N'=PQ \cap CD$.

Since OE is the diameter of $\odot(OMN)$,
so EP, EQ are the tangents of $\omega \Longrightarrow F, P, Q$ are collinear at the polar of E WRT ω .
From $(E, M'; A, B)=-1 \Longrightarrow EM' \cdot EM=EA \cdot EB=EC \cdot ED \Longrightarrow M' \in \odot(CDM)$.
Similarly, we can prove $N' \in \odot(ABN)$.

Consider the inversion Φ with center E and factor $EP^2 = EQ^2$.

Since $C \longleftrightarrow D, M \longleftrightarrow M'$,
so the image X' of X under Φ is the intersection of PQ, EX and $\odot(CDM)$.
Similarly, we can prove the image Y' of Y under Φ is the intersection of PQ, EY and $\odot(ABN)$.

Since $M'E \cdot MM' = M'P \cdot M'Q = M'A \cdot M'B = M'N' \cdot M'Y'$,
so we get E, M, N', Y' are concyclic .
Similarly, we can prove E, N, M', X' are concyclic ,
so $\angle X'Y'E = \angle N'MM' = \angle N'NM' = \angle EX'Y'$ (notice M, N, M', N' are concyclic) ,
hence we get $EX' = EY' \Longrightarrow XY \parallel X'Y' \equiv PQ$ (notice X, Y, X', Y' are concyclic) .

Q.E.D

Luis González

#6 Jan 20, 2015, 4:21 am

Let $K \equiv AB \cap CD. \odot(MCD), \odot(NAB)$ cut AB, CD again at $U, V \Longrightarrow$
$KM \cdot KU = KC \cdot KD = KA \cdot KB = KV \cdot KN \Longrightarrow UMNV$ is cyclic and $(A, B, U, K) = -1 \Longrightarrow$
$UM \cdot UK = UA \cdot UB \Longrightarrow U$ is on radical axis NY of $\Omega, \odot(NAB) \Longrightarrow U \equiv PQ \cap AB \cap NY$ is radical center of Ω, $\omega, \odot(NAB)$ and likewise $V \equiv PQ \cap CD \cap MX \Longrightarrow \angle NYX = \angle NMV = \angle NUV \Longrightarrow XY \parallel UV \equiv PQ.$

Quick Reply

High School Olympiads

Geometry concurrent

Source: Kazakhstan NMO 2015 (second round) P3

rightways

#1 Jan 19, 2015, 10:56 pm

A rectangle is said to be $inscribed$ in a triangle if all its vertices lie on the sides of the triangle. Prove that the locus of the centers (the meeting points of the diagonals) of all inscribed in an acute-angled triangle rectangles are three concurrent unclosed segments.

Luis González

#2 Jan 20, 2015, 2:31 am

The proposition is missing the three sides of the triangle, as it is possible to inscribe the rectangle in such a way that two opposite vertices lie on a side, otherwise the center lies on the line connecting the midpoint of a side with the midpoint of the corresponding altitude. These are the 3 Schwatt lines of ABC concurring at its symmedian point. See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=598678.

PROF65

#3 Jan 21, 2015, 3:49 am

i think, if we mean by the side the strict meaning which is the segment delimited by two vertices the solution is just the three segments concurrent at the symmedian point ;but if we mean by it the line ,it s ok that the solution include certainly the sides

IDMasterz

#4 Jan 21, 2015, 7:24 pm

Consider the set of rectangles $TUVW$ with $VW \in BC$ and $T \in AB, U \in AC$. We must have $TU \parallel BC \equiv VW$, and that if V is the projection of T on BC, $V \mapsto T \mapsto U$ preserves ratio. Hence, there is a spiral similarity mapping $V \mapsto U$ from the set $BC \mapsto AC$. Hence, the midpoint of VU moves along a line i.e. the centres of the rectangles move along a line.

When $T = A = U$, the centre is the midpoint of AD, where D is the foot of altitude from A to BC. When $T = B, U = C$ then the centre is the midpoint of BC. So the locus is the A-Schwatt line. Similarly, we conclude they move on the Schwatt lines.

Let the tangents at B, C to $\odot ABC$ meet at P, point at infintiy perpendicular to BC be ℓ_A and symmedian point of ABC be L. If Q, R are midpoints of BC and AD, then

$$Q(P, L; D, A) \equiv Q(\ell_A, L; D, A) = -1 = Q(\ell_A, R; D, A)$$

$\implies L \in QR$. Hence, the concurrence point of the locus lines is the symmedian point.

High School Olympiads

A property of the in-circle of a quadrilateral

Reply

BSJL

#1 Jan 19, 2015, 4:57 pm

Suppose there is a point O inside the quadrilateral $ABCD$ such that $d(O,AB)=d(O,BC)=d(O,CD)=d(O,DA)$. What's more, we have P,O,R and Q,O,S are collinear where P,Q,R,S are the mid-point of the sides AB,BC,CD,DA, respectively. Prove: $OA\cdot OC=OB\cdot OD$.

EDIT: $d(X,l)$ means the distance from the point X to the line l.

Luis González

#2 Jan 20, 2015, 12:41 am

Let $U\equiv AB\cap DC$ and the perpendicular to UO at O cuts CD,AB at $M,N.$ WLOG we assume that (O) becomes incircle of $\triangle UBC$ and U-excircle of $\triangle UAD.$ Since $\angle ODM=\angle ODA$ and $\angle DOA=\angle DMO=90^\circ-\frac{1}{2}\angle AUD$ $\Longrightarrow \triangle ODA\sim\triangle MDO \Longrightarrow \frac{OA}{OD}=\frac{OM}{MD}.$ By similar reasoning we get $\frac{MC}{OM}=\frac{OC}{OB}\Longrightarrow \frac{MC}{MD}=\frac{OA\cdot OC}{OB\cdot OD}$ and similarly we obtain $\frac{NA}{NB}=\frac{OA\cdot OC}{OB\cdot OC}\Longrightarrow$

$$\frac{NA}{NB}=\frac{MC}{MD}=\frac{OA\cdot OC}{OB\cdot OD}.$$

Since $PQRS$ is clearly a parallelogram, then O is the midpoint of $PR.$ Thus, either $AB\parallel CD$ or the lines PR and NM coincide, hence from the latter expression we get $OA\cdot OC=OB\cdot OD.$

Quick Reply