2) AE, PL, CD are concurrent, likewise AC, PM, BE
3) Alternate construction of the square $KLMN$ inside a right-angled triangle $\triangle ABC$, (out of the classical one, where $AD \cap BC = K$, a.s.o.):
draw the bisector AP and median AF; the perpendicular at P onto BC intersects the median at Z. $BZ \cap AC = L$, a.s.o.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Like harmonic division

Reply

Source: Mongolia TST 2011

Darkmage48706

#1 Aug 26, 2012, 11:57 pm

Incircle of triangle $\triangle ABC$ touch sides BC, CA, AB at points D, E, F respectively. The line EF intersects with line BC at P and with the middle line parallel to side BC at Q. Median QM of triangle DQP intersects the incircle at R. Prove that $\angle PRC = 90$.

Luis González

#2 Aug 27, 2012, 12:43 am

I believe it should be $\angle BRC = 90$, since I solved this problem before but for some reason the search function is not friendly today, so I'm just giving the outline of my proof.

Let X, Y, Z be the midpoints of BC, CA, AB. Clearly, it suffices to show that MQ is radical axis of the incircle (I) and the circle (X) with diameter $\overline{BC}$. It's known that XY, BI, EF concur at $S \in (X)$ and similarly XZ, CI, EF concur at $T \in (X)$. From here we obtain $QE \cdot QF = QS \cdot QT$ and from $(B, C, D, P) = -1$, we get $MD^2 = MB \cdot MC \Longrightarrow Q, M$ have equal powers WRT $(I), (X)$ and the conclusion follows.

Quick Reply

High School Olympiads

Concyclic points

Source: Romania TST 1 2010, Problem 4

mavropnevma

#1 Aug 25, 2012, 5:31 pm • 2

Two circles in the plane, γ_1 and γ_2, meet at points M and N. Let A be a point on γ_1, and let D be a point on γ_2. The lines AM and AN meet again γ_2 at points B and C, respectively, and the lines DM and DN meet again γ_1 at points E and F, respectively. Assume the order M, N, F, A, E is circular around γ_1, and the segments AB and DE are congruent. Prove that the points A, F, C and D lie on a circle whose centre does not depend on the position of the points A and D on the respective circles, subject to the assumptions above.

***

Luis González

#2 Aug 26, 2012, 12:35 pm

Since $\angle MAN=\angle MEN$ and $\angle MBN=\angle MDN,$ it follows that $\triangle NAB$ and $\triangle NED$ are congruent by ASA $\Longrightarrow$ $\angle ANB=\angle END$ and $NA=NE, NB=ND \Longrightarrow$ isosceles $\triangle NAE$ and $\triangle NDB$ with common apex N are similar. Thus, $\angle DBC=\angle AND=\angle AEF$ implies that the cyclic quadrangles $NAEF$ and $NDBC$ are similar $\Longrightarrow$ $\angle FAN=\angle CDN \Longrightarrow A,F,C,D$ are concyclic.

Let the perpendicular to MN at M cut γ_1 and γ_2 again at $X,Y.$ According to problem Concyclic, the center of the circumcircle of $AFCD$ is the midpoint of $\overline{XY}.$

Potla

#3 Aug 26, 2012, 8:01 pm

I solved the second part in a different manner, although it leads to the same thing. :)

Note that if O is the centre of $\odot(AFCD)\equiv\gamma$, then $\angle(OO_1,O_1N)=\angle(AE,AF)$ because OO_1 is perpendicular to the radical axis of γ_1 and γ. So, $\angle OO_1N=\angle EAF=\angle END$ and similarly $\angle OO_2N=\angle ANB.$ Since these angles are equal, and also $\angle ANB=\pi-\angle NAB-\angle NBA=\pi-\frac{1}{2}(\angle MO_1N+\angle MO_2N)\equiv\alpha$; therefore the position of O is independent of our selection of A and $D.$ To construct O, we can just construct equal angles of α on NO_1 and $NO_2,$ one clockwise and the other counterclockwise to meet at the required circumcentre.

[asy] import graph; size(15cm, 15cm); real labelscalefactor = 0.55; pen dps = linewidth(0.7) + fontsize(10); defaultpen(dps); pen dotstyle = black; real xmin = -16.03, xmax = 27.44, ymin = -10.67, ymax = 16.09; pen qqwuqq = rgb(0,0.39,0); pen ffqqff = rgb(1,0,1); pen dcrutc = rgb(0.86,0.08,0.24); draw(circle((-3.02,0), 5.01), red); draw(circle((6,0), 7.21), red); draw((-7.67,1.85)--(xmax, -0.76*xmax-4), qqwuqq); /* ray */ draw((13.21,0.29)--(xmin, 0.32*xmin-4), gray); /* ray */ draw(circle((2.98,4.01), 10.88), linetype("4 4") + ffqqff); draw((-3.51,4.99)--(13.21,0.29), blue); draw((9.05,6.54)--(-7.67,1.85), blue); draw((0,-4)--(-3.51,4.99), qqwuqq); draw((0,-4)--(9.05,6.54), gray); draw((0,4)--(0,-4), dcrutc); /* dots and labels */ dot((-3.02,0),dotstyle); label([aopsnowrap]"$$O_1$$"[/aopsnowrap], (-3,0.4), NE * labelscalefactor); dot((0,4),dotstyle); label([aopsnowrap]"$$M$$"[/aopsnowrap], (-0.25,4.5), NE * labelscalefactor); dot((6,0),dotstyle); label([aopsnowrap]"$$O_2$$"[/aopsnowrap], (6.25,0.4), NE * labelscalefactor); dot((0,-4),dotstyle); label([aopsnowrap]"$$N$$"[/aopsnowrap], (-0.35,-5.29), NE * labelscalefactor); dot((-7.67,1.85),dotstyle); label([aopsnowrap]"$$A$$"[/aopsnowrap], (-8.67,2), NE * labelscalefactor); dot((13.21,0.29),dotstyle); label([aopsnowrap]"$$D$$"[/aopsnowrap], (13.49,-0.11), NE * labelscalefactor); dot((9.05,6.54),dotstyle); label([aopsnowrap]"$$B$$"[/aopsnowrap], (9,7), NE * labelscalefactor); dot((3.73,-6.84),dotstyle); label([aopsnowrap]"$$C$$"[/aopsnowrap], (3.31,-8.3), NE * labelscalefactor); dot((-3.51,4.99),dotstyle); label([aopsnowrap]"$$E$$"[/aopsnowrap], (-3.22,5.4), NE * labelscalefactor); dot((-3.11,-5.01),dotstyle); label([aopsnowrap]"$$F$$"[/aopsnowrap], (-3.67,-6.13), NE * labelscalefactor); dot((2.98,4.01),dotstyle); label([aopsnowrap]"$$O$$"[/aopsnowrap], (3.31,4.12), NE * labelscalefactor); clip((xmin,ymin)--(xmin,ymax)--(xmax,ymax)--(xmax,ymin)--cycle); [/asy]

$\square$

Aiscrim

#4 Mar 23, 2014, 4:42 pm

It is kind of obvious from angle-chasing that $AFCD$ is cyclic.

Let $\Omega_1 \in \gamma_1,\ \Omega_2 \in \gamma_2$ such that $N\Omega_1$ is tangent to circle γ_2 and $N\Omega_2$ is tangent to circle γ_1.

Let Q be the midpoint of AF and $\{R\} = NQ \cap CD,\ \{S\} = NQ \cap \gamma_2$. As $\triangle NAF \sim \triangle NDC$, and NQ is median in $\triangle NAF$, it follows easily that NR is symmedian in triangle $\triangle NCD$, so the quadrilateral $NCSD$ is harmonic.

Intersecting the fascicle (NC, ND, NS, NN) with the line AF, we see that we get $N(A, F, Q, NN \cap AF) = -1 \Leftrightarrow N\Omega_1 \parallel AF$, so $AFN\Omega_1$ is an isosceles trapezoid, i.e. AF and $N\Omega_1$ share the same perpendicular bisector.

Analogously, CD and $N\Omega_2$ share the same perpendicular bisector, so the center of $AFCD$ is the circumcenter of triangle $\triangle N\Omega_1\Omega_2$, which is, indeed, a fixed point.

sunken rock

#5 Mar 24, 2014, 1:17 am

Since $DE = AB$, a rotation of $\angle AME$ centered at N will map the triangles NED, NAB, so $\triangle DNB \sim \triangle ENA \therefore \angle NEA = \angle DBN\ (\ 1\ )$; from cyclic $AENF, NBDC$, with (1) we get $\angle AFN = \angle NCD$, i.e. $FADC$ is cyclic.

Let O_1, O_2, O be the centres of the circles $(AMN), (BMN), (AFCD)$. Easy angle chase shows $AE \parallel CD, AF \parallel BD$; since $NO_1 \perp AE, NO_2 \perp BD \implies NO_1 \parallel OO_2, NO_2 \parallel OO_1$ and O is the symmetrical of N w.r.t. midpoint of O_1O_2, a fixed point.

Best regards,
sunken rock

Sardor

#6 Dec 21, 2014, 1:26 am

a) Easy by spiral similarity
b) We have by Proizvolov's problem (from Sharygin's book), we have OO_1NO_2 is a parallelogramm, hence O is fixed point.

High School Olympiads

Brocard point

Source: IMO Longlist 1969

huyvietnam

#1 Aug 25, 2012, 4:36 pm

Let M be the point inside the right-angled triangle $ABC(\widehat{C}=90^\circ)$, such that $\widehat{MAB}=\widehat{MBC}=\widehat{MCA}=\varphi$. Let ψ be the acute angle between the medians of AC and BC. Prove that $\frac{\sin(\varphi+\psi)}{\sin(\varphi-\psi)}=5$.

Luis González

#2 Aug 26, 2012, 5:50 am

RHS should be -5 as LHS is defined. Rename $\angle MAB=\angle MBC=\angle MCA=\omega$ the Brocard angle of $\triangle ABC$ and let G be the centroid of $\triangle ABC$. Tangent of the Brocard angle can be found using the fact that a median and a symmedian issuing from two vertices of any triangle intersect on a Brocard ray issuing from the third vertex.

$$\tan\omega=\frac{4[\triangle ABC]}{AB^2+BC^2+CA^2}=\frac{4[\triangle ABC]}{AB^2+AB^2}=\frac{2[\triangle ABC]}{AB^2}$$

$$\cot\Psi=-\frac{GA^2+GB^2-AB^2}{4[\triangle GAB]}=-\frac{3(\frac{2}{3}AB^2-GC^2-AB^2)}{4[\triangle ABC]}=\frac{AB^2}{3[\triangle ABC]}$$

$$\frac{\sin(\omega+\Psi)}{\sin(\omega-\Psi)}=\frac{\sin\omega\cos\Psi+\cos\omega\sin\Psi}{\sin\omega\cos\Psi-\cos\omega\sin\Psi}=\frac{\tan\omega+\tan\Psi}{\tan\omega-\tan\Psi}=\frac{2+3}{2-3}=-5.$$

High School Olympiads

Concurrent lines

Reply

Source: Romania TST 2 2010, Problem 3

mavropnevma
#1 Aug 25, 2012, 5:44 pm • 2

Let γ_1 and γ_2 be two circles tangent at point T, and let ℓ_1 and ℓ_2 be two lines through T. The lines ℓ_1 and ℓ_2 meet again γ_1 at points A and B, respectively, and γ_2 at points A_1 and B_1, respectively. Let further X be a point in the complement of $\gamma_1 \cup \gamma_2 \cup \ell_1 \cup \ell_2$. The circles ATX and BTX meet again γ_2 at points A_2 and B_2, respectively. Prove that the lines TX, A_1B_2 and A_2B_1 are concurrent.

***

Luis González
#2 Aug 26, 2012, 5:01 am

Inverting the figure with center T, we get the following equivalent problem: γ_1 and γ_2 are two parallel lines. T, X are two arbitrary points on its plane. A, B are distinct points on γ_1. TA, TB cut γ_2 at A_1, B_1 and XA, XB cut γ_2 at A_2, B_2. Then we prove that TX is the radical axis of $\odot(TA_1B_2)$ and $\odot(TB_1A_2)$.

XA cuts $\odot(TB_1A_2)$ again at D and XB cuts $\odot(TA_1B_2)$ again at E. $\angle TDA_2 = \angle TB_1A_2 = \angle TBA$ and $\angle TEB = \angle TA_1B_2 = \angle TAB \Longrightarrow D$ and E lie on $\odot(TAB) \Longrightarrow \angle XDE = \angle XBA = \angle XB_2A_2 \Longrightarrow$ D, E, A_2, B_2 are concyclic $\Longrightarrow XE \cdot XB_2 = XD \cdot XA_2$, i.e. X has equal power WRT $\odot(TA_1B_2)$ and $\odot(TB_1A_2)$ $\Longrightarrow TX$ is their radical axis.

ACCCGS8
#3 Aug 28, 2012, 4:45 pm

I performed the same inversion as in the above post. Let XT meet A_2B_2 at P. We want to prove that TP is the radical axis of the circumcircles of A_2B_1T and B_2A_1T. Clearly T lies on the radical axis. So we want to prove P has equal powers w.r.t. both circumcircles i.e. $PB_1 \cdot PA_2 = PA_1 \cdot PB_2 \Leftrightarrow \frac{PA_2}{PA_1} = \frac{PB_2}{PB_1} \Leftrightarrow \frac{A_1A_2}{PA_1} = \frac{B_1B_2}{PB_1} \Leftrightarrow \frac{A_2A}{AX} \cdot \frac{XT}{TP} = \frac{B_2B}{BX} \cdot \frac{XT}{TP}$.
The equality was reached via Menelaus. Now the last statement holds because AB is parallel to A_2B_2.

junior2001
#4 Dec 25, 2014, 12:04 am

any solution without inversion ?

IDMasterz
#5 Dec 26, 2014, 11:50 pm

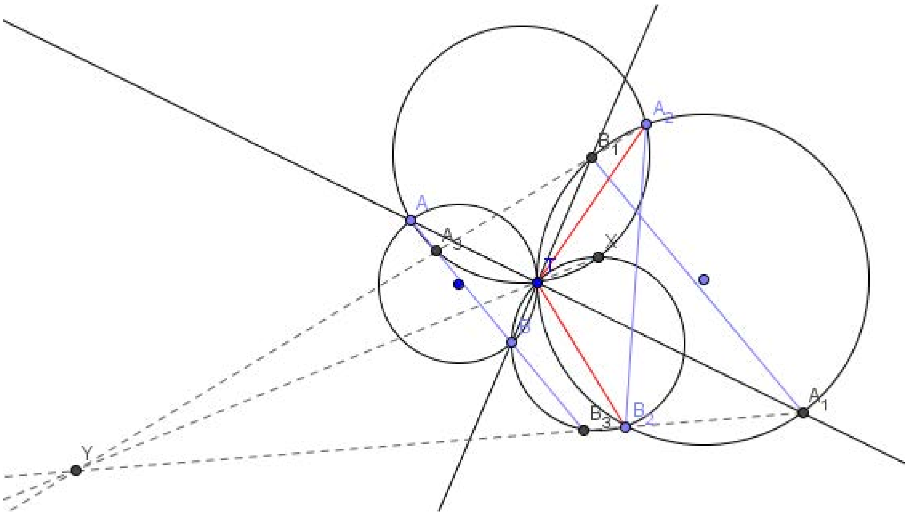
^^Why, inversion is just similar triangles :)

Panoz93
#6 Jul 28, 2015, 2:13 am

very easy for Romanian tst
Consider Y the intersection of lines A_1B_2 and A_2B_1. $Y \in TX$ is equivalent to proving that Y belongs to the radical axis of the circumcircles of ATX and BTX .

Consider A_3, B_3 the intersection of lines A_1B_2 and A_2B_1 with the circumcircles of BTX and ATX respectively.
Furthermore we need to prove $YA_2 * YA_3 = YB_3 * YB_2$.
Then $\angle BB_3B_2 = \angle B_1TB_2 = \angle B_1A_1B_2 \Leftrightarrow B_1A_1||BB_3$
Similarly $B_1A_1||AA_3$. Hence A, B, A_3, B_3 lie all in a line parallel to B_1A_1.
Therefore $\angle A_3B_3A_1 = 180 - \angle B_2A_1B_1 = 180 - \angle B_2A_2B_1$ and A_3, B_3, A_1, B_1 are conclyclic as we wanted.

Attachments:



This post has been edited 2 times. Last edited by Panoz93, Jul 28, 2015, 2:15 am
Reason: typo

Quick Reply

High School Olympiads

Two circles, and two lines meeting on a circle

Reply

Source: Romania TST 5 2010, Problem 2

mavropnevma

#1 Aug 25, 2012, 6:15 pm • 2

Let ℓ be a line, and let γ and γ' be two circles. The line ℓ meets γ at points A and B, and γ' at points A' and B'. The tangents to γ at A and B meet at point C, and the tangents to γ' at A' and B' meet at point C'. The lines ℓ and CC' meet at point P. Let λ be a variable line through P and let X be one of the points where λ meets γ, and X' be one of the points where λ meets γ'. Prove that the point of intersection of the lines CX and $C'X'$ lies on a fixed circle.

Gazeta Matematica

Luis González

#2 Aug 26, 2012, 2:54 am • 3

Let $Q \equiv CX \cap C'X'$. CX cuts ℓ at M and γ again at E. $C'X'$ cuts ℓ at N and γ' again at F. Since ℓ is polar of C and C' WRT γ and γ', then the pencils $P(C,M,E,X)$ and $P(C',N,F,X')$ are harmonic $\Longrightarrow P,E,F$ are collinear. Hence, by Menelaus' theorem for $\triangle QCC'$ cut by XX' and EF, we get

$$\frac{QX}{QX'}=\frac{PC'}{PC}\cdot\frac{CX}{C'X'}\ ,\ \frac{QE}{QF}=\frac{PC'}{PC}\cdot\frac{CE}{C'F}\Longrightarrow$$

$$\frac{QX\cdot QE}{QX'\cdot QF}=\left(\frac{PC'}{PC}\right)^2\cdot\frac{CX\cdot CE}{C'X'\cdot C'F}=\left(\frac{PC'\cdot CA}{PC\cdot C'A'}\right)^2=\text{const}.$$

Thus, the ratio of the powers of Q WRT γ and γ' is constant $\Longrightarrow$ locus of Q is a circle ω coxal with γ and γ'. This circle ω then passes through the intersections $CA\cap C'A'$, $CB\cap C'B'$, $CA\cap C'B'$ and $CB\cap C'A'$.

simplependulum

#3 Aug 27, 2012, 8:55 am • 1

> Quote:
> Thus, the ratio of the powers of Q WRT γ and γ' is constant $\Longrightarrow$ locus of Q is a circle ω coxal with γ and γ'.

I have learnt this theorem very recently but is there any name given to it?

Quick Reply

High School Olympiads

Line passing through a fixed point

Reply

Source: Romania TST 4 (All Geometry) 2010, Problem 1

mavropnevma

#1 Aug 25, 2012, 6:02 pm • 3

Let P be a point in the plane and let γ be a circle which does not contain P. Two distinct variable lines ℓ and ℓ' through P meet the circle γ at points X and Y, and X' and Y', respectively. Let M and N be the antipodes of P in the circles PXX' and PYY', respectively. Prove that the line MN passes through a fixed point.

Mihai Chis

Luis González

#2 Aug 25, 2012, 11:20 pm • 2

Let O, U, V be the centers of γ, $\odot(PXX')$ and $\odot(PYY')$, respectively. $OU \perp XX'$ and $OV \perp YY'$ are the perpendicular bisectors of $\overline{XX'}$ and $\overline{YY'}$. Since XX' and YY' are antiparallel WRT PX, PX', it follows that $PU \perp YY'$. Similarly, $PV \perp XX'$. Hence $PU \parallel OV$ and $PV \parallel OU \Longrightarrow PUOV$ is a parallelogram $\Longrightarrow$ line UV passes through the midpoint of $\overline{PO}$. Then, the image MN of UV under the homothety $(P, 2)$ goes through O.

ACCCGS8

#3 Aug 26, 2012, 1:18 pm

Assume without loss of generality that P, X, Y and P, X', Y' lie in that order.
Let W be the midpoint of XY, U be the foot of the perpendicular from M to YN, T be the point on MN such that $TW \perp XY$ and S be the intersection of MU and WT. Note that $XM, YN \perp XY$.
$XW = YW \implies MS = US \implies MT = NT$ as $XMUY$ is a rectangle and Midpoint Theorem. Similarly, if Z is the midpoint of $X'Y'$ and V is the point on MN such that $ZV \perp X'Y'$, $MV = NV$. Thus $T = V$ so the perpendicular bisectors of XY and $X'Y'$ concur on MN so MN passes through O, the circumcentre of γ. Done.

Jul

#4 Nov 8, 2014, 10:10 pm

My solution :
Let $\{P, T\} = (PXX') \cap (PYY')$. Let $J = XX' \cap YY'$. Easily, we get P, T, J are conlinear.
From the problem 5, IMO 1985 :
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=366594&sid=3824115ccf232b168958a0149e7b5313#p366594

We have $OT \perp PJ$. But $MT \perp PJ, NT \perp PJ$. Hence, M, N, O are conlinear.

TelvCohl

#5 Nov 8, 2014, 10:50 pm • 1

My solution:

Let L be the polar of P WRT γ .
Let Z be the projection of P on L and O be the center of γ.
Let P_x, P_y be the projection of P on XX', YY', respectively .

From $XX' \cap YY' \in L$ we get (PP_xP_y) pass through Z .
Invert with center P and factor $PX \cdot PY = PX' \cdot PY'$ we get MN passes through O .

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Jul 19, 2015, 7:14 am

Quick Reply

High School Olympiads

Three circumcircles having two meeting points

Reply

Source: Romania TST 6 2010, Problem 2

mavropnevma

#1 Aug 25, 2012, 6:25 pm • 2

Let ABC be a scalene triangle, let I be its incentre, and let A_1, B_1 and C_1 be the points of contact of the excircles with the sides BC, CA and AB, respectively. Prove that the circumcircles of the triangles AIA_1, BIB_1 and CIC_1 have a common point different from I.

Cezar Lupu & Vlad Matei

Luis González

#2 Aug 25, 2012, 10:42 pm • 1

Posted before at:
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=386935
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=489896

For a generalization see:
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=325959

v_Enhance

#3 Mar 17, 2013, 1:18 pm • 1

> *Quote:*
> Hi, can you solve Romania TST 2010 Day 6 Q2 with barycentrics?

Yup. Without inversion, too!

Let $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$ and define a, b, c in the usual fashion. Then, we get

$$A_1 = (0 : s - b : s - c)$$

and its cyclic variants, as well as $I = (a : b : c)$.

Let us calculate $\omega_A = (AIA_1)$ and its cyclic variants. Upon using the generic circle form $-a^2yz - b^2zx - c^2xy + (x + y + z)(ux + vy + wz)$ we find $u = 0$ and the system

$$abc = vb + wc$$
$$a(s - b)(s - c) = v(s - b) + w(s - c)$$

Solving, we find that $v = \dfrac{ac(s - c)(2b - s)}{s(b - c)}$ and $w = \dfrac{ab(s - b)(2c - s)}{s(c - b)}$. Symmetrically, so we obtain the equations:

$$\omega_A : 0 = -a^2yz - b^2zx - c^2xy + (x + y + z)\left(\frac{ac(s - c)(2b - s)}{s(b - c)}y + \frac{ab(s - b)(2c - s)}{s(c - b)}z\right)$$
$$\omega_B : 0 = -a^2yz - b^2zx - c^2xy + (x + y + z)\left(\frac{ba(s - a)(2c - s)}{s(c - a)}z + \frac{bc(s - c)(2a - s)}{s(a - c)}x\right)$$
$$\omega_C : 0 = -a^2yz - b^2zx - c^2xy + (x + y + z)\left(\frac{cb(s - b)(2a - s)}{s(a - b)}x + \frac{ca(s - a)(2b - s)}{s(b - a)}y\right)$$

Now the *radical axis* ℓ_{AB} of ω_A and ω_B is given by subtracting the RHS's:

$$\frac{bc(s-c)(2a-s)}{s(c-a)}x+\frac{ac(s-c)(2b-s)}{s(b-c)}y+\left(\frac{ab(s-b)(2c-s)}{s(c-b)}-\frac{ba(s-a)(2c-s)}{s(c-a)}\right)z$$

We can compute

$$\begin{aligned}\frac{ab(s-b)(2c-s)}{s(c-b)}-\frac{ba(s-a)(2c-s)}{s(c-a)} &= \frac{ab(2c-s)}{s}\left(\frac{s-b}{c-b}-\frac{s-a}{c-a}\right)\\ &= \frac{ab(2c-s)}{s}\frac{(s-b)(c-a)-(s-a)(c-b)}{(c-b)(c-a)}\\ &= \frac{ab(2c-s)}{s}\frac{-(a-b)(s-c)}{(c-b)(c-a)}\end{aligned}$$

So, the equation of the radical axis is just

$$\ell_{AB}:0=\frac{ca(s-c)(2a-s)}{s(c-a)}x+\frac{ca(s-c)(2b-s)}{s(b-c)}y+\frac{ab(s-c)(2c-s)(a-b)}{(b-c)(c-a)}z$$

or, after multiplying through by some common factors:

$$\ell_{AB}:0=ca(2a-s)(b-c)x+ab(2b-s)(c-a)y+bc(2c-s)(a-b)z$$

which is symmetric, so the equations for ℓ_{BC} and ℓ_{CA} are the same, and hence the conclusion holds.

Quick Reply

High School Olympiads

Tangents to the Euler circle

Locked

Source: Romania TST 4 (All Geometry) 2010, Problem 2

mavropnevma

#1 Aug 25, 2012, 6:06 pm • 1

Let ABC be a scalene triangle. The tangents at the perpendicular foot dropped from A on the line BC and the midpoint of the side BC to the nine-point circle meet at the point A'\,; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent.

Gazeta Matematica

Luis González

#2 Aug 25, 2012, 10:27 pm

Seriously, How many more times are we going to see this problem in the forum?

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=310396
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=439414
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=455292
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=486970
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=488722
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=484304

High School Olympiads

Euler line

Source: My teacher

qua96

#1 Aug 23, 2012, 7:05 pm

Problem: Let ABC be a triagle with orthocentre H and circumcircle (O); M is midpoint of BC. Line MH intersect (O) again at E. The line though A parallel BC intersect (O) at D; DE met OH at T. Prove that: TA touches (O).

novae

#2 Aug 24, 2012, 6:45 am

> *qua96 wrote:*
> Problem: Let ABC be a triagle with orthocentre H and circumcircle (O); M is midpoint of BC. Line MH intersect (O) again at E. The line though A parallel BC intersect (O) at D; DE met OH at T. Prove that: TA touches (O).

Restate as follow : Let ABC be a triangle with orthocentre H and circumcircle (O); M is midpoint of BC. **Ray** MH intersects (O) again at E. The line though A parallel BC intersects (O) at D. Tangent at A to (O) intersects OH at T. Prove that T, D, E are collinear.

[geogebra]43147f69ff0995289a17a85061795d29ac7bdb54[/geogebra]

We have a well-known result that $MH \perp AE$. Therefore, $(AD, AT) \equiv (AH, AO) \pmod{\pi}$.
Let D' be the intersection of TE and (O) which differs from E. Then $(ED', EA) \equiv (AD, AT) \pmod{\pi}$.
To prove T, D, E are collinear, we need to prove $(ED, EA) \equiv (AH, AO) \pmod{\pi}$, which is obviously true by angle chasing.

Luis González

#3 Aug 24, 2012, 11:07 am

As already mentioned, it's well-known that $MH \perp AE$, i.e. EH cuts (O) again at the antipode P of A WRT (O). If AH cuts (O) again at Q, then clearly $ADPQ$ is a rectangle with center O. By Pascal theorem for the degenerate cyclic hexagon $AAQDEP$, the intersections $T^* \equiv AA \cap ED, H \equiv AQ \cap EP$ and $O \equiv QD \cap PA$ are collinear $\Longrightarrow T \equiv T^* \Longrightarrow TA$ touches (O).

High School Olympiads

Concurrency

Reply

Source: Own

buratinogigle

#1 Jul 21, 2012, 12:11 am

Let ABC be a triangle with circumcircle (O) and P, Q are two isogonal conjugate points. A_1, B_1, C_1 are midponts of BC, CA, AB, resp. PA, PB, PC cut (O) again at A_2, B_2, C_2, resp. A_2A_1, B_2B_1, C_2C_2 cut (O) again at A_3, B_3, C_3. A_3Q, B_3Q, C_3Q cut BC, CA, AB at A_4, B_4, C_4. Prove that AA_4, BB_4, CC_4 are concurrent.

Luis González

#2 Aug 9, 2012, 5:39 am • 2

AQ, BQ, CQ cut BC, CA, AB at Q_1, Q_2, Q_3. $X \equiv Q_2Q_3 \cap BC, Y \equiv Q_3Q_1 \cap CA, Z \equiv Q_1Q_2 \cap AB$. AQ_1 and AX cuts (O) again at D, E, respectively. Tangent of (O) at E cuts BC at T. $\angle CED = \angle CAD, \angle BED = \angle BAD$, $\angle BET = \angle BCE = \angle BAX \Longrightarrow$ pencils $E(B, C, D, T) \sim A(B, C, D, X)$ are harmonic $\Longrightarrow$ quadrangle $EBDC$ is harmonic $\Longrightarrow ED$ is the E-symmedian of $\triangle EBC \Longrightarrow ED, EA_1$ are isogonals WRT $\angle BEC$. Consequently, EA_1 passes through $A_2 \Longrightarrow A_3 \equiv E$. Similarly, $B_3 \in BY$ and $C_3 \in CZ$. If $A_0 \equiv BB_3 \cap CC_3, B_0 \equiv CC_3 \cap AA_3$ and $C_0 \equiv AA_3 \cap BB_3$, then $\triangle A_0B_0C_0$ is anticevian triangle of Q WRT $\triangle ABC$ and A_0A_3, B_0B_3, C_0C_3 concur at the cyclocevian conjugate R of Q WRT $\triangle A_0C_0C_0$. Now, proving the concurrency of AA_4, BB_4, CC_4 is merely projective.

Project XYZ to infinite. Q and $\triangle A_0B_0C_0$ become centroid and antimedial triangle of $\triangle ABC$. Thus $\overline{QA} : \overline{QA_0} = \overline{QQ_1} : \overline{QA} = \overline{QA_4} : \overline{QA_3} = -1 : 2 \Longrightarrow AA_4$ is the complement of A_0A_3 WRT $\triangle ABC$. Similarly, BB_4 and CC_4 are the complements of B_0B_3 and C_0C_3 WRT $\triangle ABC$. Hence, AA_4, BB_4 and CC_4 concur at the complement of R WRT $\triangle ABC$.

Quick Reply

High School Olympiads

Circumscribed quadrilateral with inversion

Reply

Source: Own

buratinogigle

#1 Jul 22, 2012, 1:19 am • 3

Let $ABCD$ be circumscribed quadrilateral and M is its Miquel point. (M) is a circle center M. Let A', B', C', D' invert A, B, C, D through (M), respectively. Prove that $A'B'C'D'$ is circumscribed quadrilateral.

Luis González

#2 Aug 8, 2012, 11:39 pm • 4

Let $P \equiv AD \cap BC$. $\odot(PAB)$ and $\odot(PCD)$ meet at P and the Miquel point M of $ABCD$. Incircle (I) of $ABCD$ touches AB, BC, CD, DA at W, X, Y, Z. Inversion with respect to (I) takes P, A, B, C, D into the midpoints of XZ, ZW, WX, XY, YZ, respectively. Thus, the inverse images of $\odot(PAB)$ and $\odot(PCD)$ are the 9-point circles of $\triangle WXZ$ and $\triangle XZY$, which meet at the midpoint of $\overline{XZ}$ and the Poncelet point of $WXYZ$, i.e. the anticenter U of the cyclic $WXYZ$. So, if E is the midpoint of $\overline{YZ}$, then $EU \perp XW$, i.e. $EU \parallel IB$. Therefore, the inverse $\odot(IDM)$ of EU is tangent to IB at $I \Longrightarrow \angle MDI = \angle BIM$. In the same way, $\angle MBI = \angle DIM \Longrightarrow \triangle MID \sim \triangle MBI$. Analogously, $\triangle MIA \sim \triangle MCI$.

Thus, MI bisects both $\angle BMD, \angle AMC$ and $MI^2 = MB \cdot MD = MA \cdot MC$. Hence, reflections A_0, B_0, C_0, D_0 of A, B, C, D about MI lie on MC, MD, MA, MB, respectively and $MA \cdot MC_0 = MB \cdot MD_0 = MC \cdot MA_0 = MD \cdot MB_0 = MI^2 \Longrightarrow A_0, B_0, C_0, D_0$ are the inverses of C, D, A, B WRT (M, MI). It's known that the inverses of a same figure under inversions with the same center are all centrally similar, thus $A'B'C'D' \sim C_0D_0A_0B_0 \Longrightarrow A'B'C'D'$ is also tangential with an incircle.

phuongtheong

#3 Oct 14, 2012, 3:39 pm • 2

Proof.

We have:

$$\Delta MBC \sim \Delta MAD \Rightarrow \frac{MB}{MA} = \frac{MC}{MD} = \frac{BC}{AD}$$

$$\Delta MCD \sim \Delta MBA \Rightarrow \frac{MC}{MB} = \frac{MD}{MA} = \frac{BC}{AD}$$

Because A', B', C', D' invert A, B, C, D through M so:

$$A'B' = k.\frac{AB}{MA.MB}; B'C' = k.\frac{BC}{MB.MC}$$

$$C'D' = k.\frac{CD}{MC.MD}; D'A' = k.\frac{DA}{MD.MA}$$

$$A'B' + C'D' = k\left(\frac{AB}{MA.MB} + \frac{CD}{MC.MD}\right)$$

$$= k.\frac{DC}{MD.MC}\left(1 + \frac{DC}{AB}\right) (\text{Because } \frac{AB^2}{MA.MB} = \frac{DC^2}{MD.MC})$$

Similarly, $B'C' + D'A' = k.\frac{BC}{MB.MC}\left(1 + \frac{BC}{AD}\right)$

We also have: $\frac{DC}{MD} = \frac{AB}{MA}$ and $\frac{BC}{MB} = \frac{AD}{MA}$

Thus,

$$A'B' + C'D' = k.\frac{AB}{MA.MC}\left(1 + \frac{DC}{AB}\right) = \frac{AB + DC}{MA.MC}$$

$$B'C' + D'A' = k.\frac{AD}{\ldots}\left(1 + \frac{BC}{\ldots}\right) = \frac{AD + BC}{\ldots}$$

$B'C'+D'A'=k.\frac{\quad}{MA.MC}\left(1+\frac{\quad}{AD}\right)=\frac{\quad}{MA.MC}$

$ABCD$ is a circumscribed quadrilateral that means $AB+DC=AD+BC$.
And we obtain $A'B'+C'D'=B'C'+D'A'$. We done!

livetolove212

#4 Jul 31, 2014, 9:25 am

Let $AB\cap CD=\{E\}$, $AD\cap BC=\{F\}$, M be the Miquel point of complete quadrilateral $ABCDEF$.
Construct two tangents of (I) through M, intersect (EBC) at H and G. According to Poncelet's porism, (I) is the incircle of MHG.
Let the line through I and perpendicular to MI cuts MH, MG at X, Y, respectively, ω be the circle tangent to MH, MG at X, Y. By Sawayama theorem, ω is the M-mixtilinear circle of triangle MHG or ω is tangent to (EBC). Similarly ω is also tangent to $(EAD), (FAB), (FCD)$.
Let ω' be the image of ω through $\mathcal{I}_M^k$, we conclude that ω' is inscribed in $A'B'C'D'$.

livetolove212

#5 Jul 31, 2014, 9:39 am

> *Luis González wrote:*
>
> Let $P\equiv AD\cap BC.$ $\odot(PAB)$ and $\odot(PCD)$ meet at P and the Miquel point M of $ABCD.$ Incircle (I) of $ABCD$ touches AB, BC, CD, DA at $W, X, Y, Z.$ Inversion with respect to (I) takes P, A, B, C, D into the midpoints of $XZ, ZW, WX, XY, YZ,$ respectively. Thus, the inverse images of $\odot(PAB)$ and $\odot(PCD)$ are the 9-point circles of $\triangle WXZ$ and $\triangle XZY,$ which meet at the midpoint of $\overline{XZ}$ and the Poncelet point of $WXYZ,$ i.e. the anticenter U of the cyclic $WXYZ.$ So, if E is the midpoint of $\overline{YZ},$ then $EU\perp XW,$ i.e. $EU\parallel IB.$ Therefore, the inverse $\odot(IDM)$ of EU is tangent to IB at $I\Longrightarrow\angle MDI=\angle BIM.$ In the same way, $\angle MBI=\angle DIM\Longrightarrow$ $\triangle MID\sim\triangle MBI.$ Analogously, $\triangle MIA\sim\triangle MCI.$
>
> Thus, MI bisects both $\angle BMD, \angle AMC$ and $MI^2=MB\cdot MD=MA\cdot MC.$ Hence, reflections A_0, B_0, C_0, D_0 of A, B, C, D about MI lie on $MC, MD, MA, MB,$ respectively and $MA\cdot MC_0=MB\cdot MD_0=MC\cdot MA_0=MD\cdot MB_0=MI^2\Longrightarrow A_0, B_0, C_0, D_0$ are the inverses of C, D, A, B WRT $(M, MI).$ It's known that the inverses of a same figure under inversions with the same center are all centrally similar, thus $A'B'C'D'\sim C_0D_0A_0B_0\Longrightarrow A'B'C'D'$ is also tangential with an incircle.

We can prove the existence of circle which is tangent to $(MAB), (MBC), (MCD), (MDA)$ by the similar way as Luis's.
Note that $\mathcal{I}_I^{r^2}$ maps $(MAB), (MBC), (MCD), (MDA)$ to the 4 Euler circles of ZWX, WXY, XYZ, YZW, which have the same radius of $r/2$ and concurrent at Euler-Poncelet point J of $XYZW$. Therefore (J, r) is tangent to the 4 Euler circles.

Quick Reply

High School Olympiads

Many circles

Reply

Source: mpdb

borislav_mirchev

#1 Aug 5, 2012, 7:56 pm

Acute-angled triange ABC is inscribed in a circle k. D is the feet of the altitude from the vertex C to AB. With diameters AD and BD respectively are constructed circles k_1 and k_2. E is the intersection point of k and k_1 and F is the intersection point of k and k_2. Externally from the triangle ABC are drawn tangents CA_1 and CB_1 from C to k_1 and k_2 respectively. CE intersects k_1 at the point A_2. CF intersects k_2 at the point B_2. Prove that:
a) AA_1BB_1 is concyclic.
b) $A_1A_2B_1B_2$ is concyclic.
c) A_1B_1EF is concyclic.
d) A_2B_2EF is concyclic.

Luis González

#2 Aug 7, 2012, 12:19 pm • 1

Let P,Q be the 2nd intersections of CA,CB with k_1,k_2 (projections of D on CA,CB). Inversion with center C and power CD^2 takes k_1,k_2 into themselves and swaps k and $PQ \Longrightarrow A_2,B_2,E,F$ are concyclic and P,Q,A_2,B_2 are collinear. Since $\odot(C,CD)$ and $\odot(CPDQ)$ are orthogonal to both k_1 and k_2, we deduce that A_1,B_1 and P,Q are pairs of inverse points under the inversion that takes k_1 into k_2. So, in this inversion, A goes to B, A_2 goes to B_2 and E goes to F. As a result, $ABB_1A_1, A_1B_1B_2A_2, A_1B_1FE$ are cyclic.

borislav_mirchev

#3 Aug 7, 2012, 12:51 pm

Can it be solved without using inversion? I suppose - inscribed angles/quadrilaterals and power of point may solve the problem.

Quick Reply

High School Olympiads

perspective

Reply

Source: well-known most probably

r1234

#1 Aug 6, 2012, 11:14 pm

In a triangle $\triangle ABC$, P is a point inside it. P' is its isogonal conjugate w.r.t $\triangle ABC$. $\triangle A_1B_1C_1$ is the pedal triangle of P. Let Q be an arbitrary point on the line PP'. Draw the circumcevian triangle of Q w.r.t $\triangle A_1B_1C_1$. Let it be $\triangle A_2B_2C_2$. Then $\triangle ABC$ and $\triangle A_2B_2C_2$ are perspective.

Luis González

#2 Aug 7, 2012, 8:09 am • 1

This configuration has been already discussed:

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=366219
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=389717
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=431409

Quick Reply

High School Olympiads

4AM=3AC in regular pyramid VABCD

Reply

Source: Romanian MO 2010 Grade 8

WakeUp

#1 Aug 6, 2012, 6:41 pm

Let $VABCD$ be a regular pyramid, having the square base $ABCD$. Suppose that on the line AC lies a point M such that $VM=MB$ and $(VMB)\perp(VAB)$. Prove that $4AM=3AC$.

Mircea Fianu

Luis González

#2 Aug 7, 2012, 12:39 am

Let O be the center of the square $ABCD. VO, BO$ are both perpendicular to AC, i.e. plane VOB is perpendicular to AC $\Longrightarrow$ skew lines VB and AC are orthogonal, thus there exists a plane through AC perpendicular to VB at P. Since $MB=MV$, then P is midpoint of $\overline{BV} \Longrightarrow AV=AB \Longrightarrow \triangle VAB, \triangle VBC, \triangle VCD, \triangle VDA$ are all equilateral with side length L. Let $N\in\overline{AC}$, such that $AN=\frac{3}{4}AC$ (N is midpoint of OC). By Pythagorean theorem, we have

$$NB^2=OB^2+ON^2=\left(\frac{\sqrt{2}}{2}L\right)^2+\left(\frac{\sqrt{2}}{4}L\right)^2=\frac{5}{8}L^2$$

$PN^2=NB^2-PB^2=\frac{_5^}{^8}L^2-\frac{_1}{^4}L^2=\frac{_3}{^8}L^2 \Longrightarrow$

$$PA^2+PN^2=\left(\frac{\sqrt{3}}{2}L\right)^2+\frac{3}{8}L^2=\frac{9}{8}L^2=\left(\frac{3}{4}AC\right)^2=AN^2.$$

Hence $\angle APN=90^\circ \Longrightarrow$ planes VNB and VAB are perpendicular $\Longrightarrow N\equiv M$.

sunken rock

#3 Aug 7, 2012, 2:17 am • 1

Little bit shorter proof: reshaping **Luis'**, with less calculation:

Triangle $\triangle APC$ is isosceles, with: $AC=l\sqrt{2}, AP=PC=l\frac{\sqrt{3}}{2}, PO=\frac{l}{2}$. If $\triangle APM$ is right-angled at P, then, by altitude theorem, $PO^2=AO\cdot OM$, wherefrom we get $OM=l\frac{\sqrt{2}}{4} \iff 3\cdot AC=4\cdot AM$.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Quadrilateral construction

Reply

delegat

#1 Aug 5, 2012, 9:14 pm

Construct circumscribed and cyclic (both at same time) quadrilateral $ABCD$ if R (circumradius), length of diagonal AC and angle between diagonals are given.

Luis González

#2 Aug 6, 2012, 5:25 am

Let $(O),(I)$ be the circumcircle and incircle of $ABCD.$ Tangents of (O) at A and C meet at $U.$ Simple angle chase gives $\angle UAI=\frac{1}{2}\angle DAB+\angle ABD, \angle UCI=\frac{1}{2}\angle DCB+\angle CBD \Longrightarrow$

$$\angle UAI+\angle UCI=2\pi-\angle AIC-\angle AUC=\frac{1}{2}(\angle DAB+\angle DCB)+\angle ABC$$

$$2\pi-\angle AIC-\angle AUC=\frac{1}{2}\pi+\frac{1}{2}\pi-\frac{1}{2}\angle AUC \Longrightarrow \angle AIC+\frac{1}{2}\angle AUC=\pi.$$

Thus I lies on the small arc AC of the circle (U) with center U and radius $UA=UC.$ Let $E \equiv AD \cap IU.$ Again, by simple angle chase we get

$$\angle AEI=\angle AUI+\angle UAD=2\angle ACI+\angle ACD=$$

$$=2\left(\frac{1}{2}\angle BCD-\angle ACD\right)+\angle ACD=\angle BCA=\angle BDA \Longrightarrow UI \parallel BD.$$

Construction: Draw circumcircle (O,R) and construct a chord $\overline{AC}.$ Tangents of (O) at $A,C,$ mee at $U.$ The line through U with the given direction of BD cuts the small arc AC of the circle (U,UA) at the incenter I of the desired $ABCD.$ IO cuts AC at the diagonal intersection F of $ABCD$ (see collinearity in bicentric quardilateral and elsewhere). So, the line through F with the given direction of BD cuts (O) at $B,D,$ completing the quadrangle.

Quick Reply

High School Olympiads

spieker point

Reply

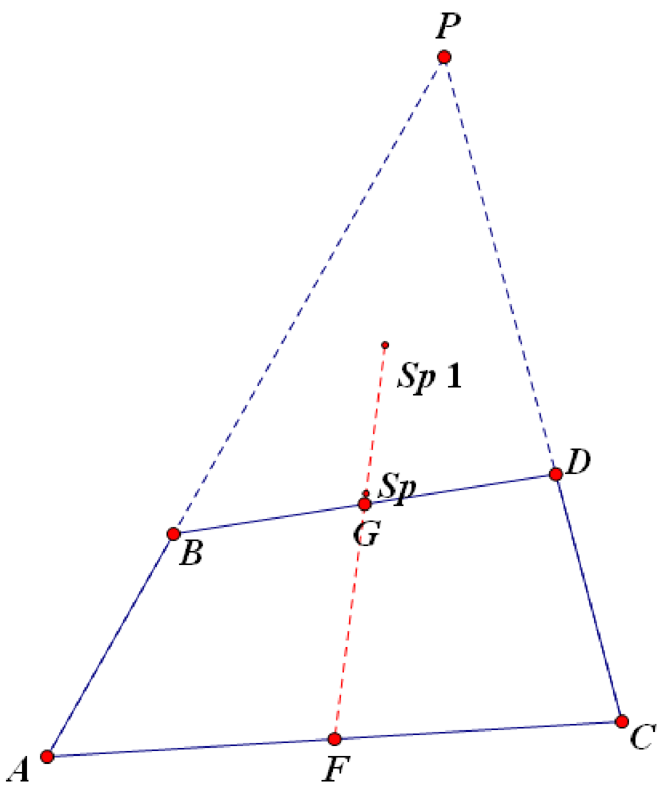
DANNY123

#1 Aug 5, 2012, 8:43 pm

$AB = CD. AB \cap CD = P. F, G$ are midpoints of $AC, BD.$ Spieker point of $\triangle PBD$ and $\triangle PAC$ are denoted as $Sp_1, Sp.$ Then F, G, Sp_1, Sp are collinear.

Moderator: $\LaTeX$ fixed.

Attachments:



Luis González

#2 Aug 6, 2012, 12:13 am

DANNY123, please use $\LaTeX$ properly and make sure your posts are readable. As for the problem, it suffices to show that FG is parallel to the angle bisector of <BPD. See

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=116208
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=214627
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=377297

Quick Reply

High School Olympiads

fixed point and circle

neversaynever

#1 Aug 4, 2012, 7:15 am

Given a fixed point P outside the fixed circle S , $A, B \in \odot S$ and the length of AB is given.
M si the midpoint of AB, l pass through B and parallel to PM meet $\odot S$ at B, C.
Prove that line AC pass through a fixed point.

Luis González

#2 Aug 5, 2012, 1:40 am • 1

Let τ be the perpendicular to SP at $P. Q \equiv AC \cap \tau. PM \parallel BC$ is A-midline of $\triangle ABC,$ meeting AC at its midpoint N (projection of S on AC). Quadrangles $SNQP$ and $SANM$ are inscribed in the circles with diameters $\overline{SQ}$ and $\overline{SN},$ respectively $\Longrightarrow \angle SQP=\angle SNM=\angle SAM.$ Since AB is constant, then $\angle SAM$ is constant $\Longrightarrow \angle SQP$ is constant $\Longrightarrow Q$ is fixed, i.e. AC passes through a fixed point Q of $\tau.$

High School Olympiads

Circumscribed and inscribed quadrilateral

Reply

hyperspace.rulz
#1 Aug 4, 2012, 6:13 pm

Given three points A, B, C, construct with straightedge and compass a point D such that the points A, B, C, D (not necessarily in that order) form a quadrilateral with both a circumcircle and an incircle.

Luis González
#2 Aug 4, 2012, 10:01 pm

WLOG assume that AC is a diagonal of the desired quadrangle and $BC > BA$. (O) is the circumcircle of $\triangle ABC$. Thus, it suffices to find the point D on the arc AC of (O), such that $DC - DA = BC - BA$. Let $P \in \overline{DC}$, such that $CP = BC - BA = DC - DA$. Then $\triangle DAP$ is D-isosceles $\Longrightarrow \angle APC = 90^\circ + \frac{1}{2}\angle ADC$. Hence, pick an arbitrary D' on the arc AC of (O) and $P' \in \overline{CD'}$, such that $D'P' = D'A$. Circle with center C and radius $BC - BA$ cuts the circular arc $AP'C$ at P. CP cuts (O) again at the desired D.

P.S. See also Construct a bicentric quadrilateral and IMO 1962 Day 2, Problem 5.

Quick Reply

High School Olympiads

Distance betven H and I

Reply

Number1
#1 Aug 3, 2012, 2:00 am

Let ρ be inradius for $A'B'C'$ which is pedal triangle for H with respect to triangle ABC.
Let R and r be circumradius and inradius for ABC.

I ... incenter for ABC
H ...orthocenter for ABC

Prove (elementary!!!): $HI^2 = 2r^2 - 2R\rho$

Luis González
#2 Aug 3, 2012, 2:55 am • 1

Let O and N be the circumcenter and 9-point center of $\triangle ABC$. Assuming that $\triangle ABC$ is acute, then H is incencer of $\triangle A'B'C'$. IN is I-median of $\triangle IOH \Longrightarrow$

$$IN^2 = \frac{1}{2}(IH^2 + IO^2) - \frac{1}{4}OH^2 \Longrightarrow IH^2 = 2(IN^2 + HN^2) - IO^2$$

Substituting $IN = \frac{1}{2}R - r$, since (I, r) and $(N, \frac{1}{2}R)$ are internally tangent, and IO and HN using Euler's formula for $\triangle ABC$ and $\triangle A'B'C'$, we get

$$IH^2 = 2\left(\frac{R}{2} - r\right)^2 + 2\left(\frac{R^2}{4} - R\rho\right) - (R^2 - 2Rr) = 2r^2 - 2R\rho.$$

Number1
#3 Aug 3, 2012, 3:30 am

> Luis González wrote:
> $IN = \frac{1}{2}R - r$, since (I, r) and $(N, \frac{1}{2}R)$ are internally tangent...

Do you think this is elementary?

Quick Reply

High School Olympiads

AP, BC and OH are concurrent iff AH=HN

Reply

Source: French TST 2012

WakeUp

#1 Aug 2, 2012, 7:07 pm

Let ABC be an acute-angled triangle with $AB \neq AC$. Let Γ be the circumcircle, H the orthocentre and O the centre of Γ. M is the midpoint of BC. The line AM meets Γ again at N and the circle with diameter AM crosses Γ again at P. Prove that the lines AP, BC, OH are concurrent if and only if $AH = HN$.

subham1729

#2 Aug 2, 2012, 7:54 pm • 1

My solution is based on long calculation. So I'm giving outline:
Let the intersection point be D. We'll easily get BD in terms of a, b, c.
So we'll also get AD^2 using cosine rule.
We've $PD^2 \cdot AD^2 = CD^2 \cdot BD^2$. Hence we'll get PD^2.
Also we've $AP^2 - PD^2 = AM^2 - MD^2$. Hence we've AP^2
Using $AD = AP + PD$ we'll get a equation in terms of a, b, c.
From here it's easy to show the iff part.

Luis González

#3 Aug 2, 2012, 8:45 pm

Let K be the antipode of A WRT Γ. Since KC, BH are both perpendicular to AC and KB, CH are both perpendicular to AB, then $HCKB$ is parallelogram $\Longrightarrow K$ is the reflection of H about M. Thus, $\angle APM = 90^\circ$ implies that M, H, P are collinear. If $D \equiv AP \cap BC$, then H is also the orthocenter of $\triangle ADM$. Consequently, $OH \in D \Longleftrightarrow OH \perp AM \Longleftrightarrow$ OH is perpendicular bisector of $\overline{AN} \Longleftrightarrow AH = HN$.

r1234

#4 Aug 3, 2012, 12:10 pm • 1

In other way....
Let (M) be the circle with diameter BC.Now let Γ be the circle with diameter AM.Then AP is the radical axis of (O) and Γ. Again BC is the radicla axis of (O) and (M). Now note that H has the sam e power w.r.t Γ and (M). So we just need to prove that O lies on the radical axis of Γ and (M).But centre of Γ lies on AM.Let $OH \cap BC = K$. Then KOH is the radical axis of Γ and (M). So $OH \perp AN \implies AH = HN$. The only if part can be proven in the same way.

bah_luckyboy

#5 Aug 3, 2012, 7:33 pm

The case when $AB < AC$ is similar with $AB > AC$, then suppose that $AB < AC$. Let HM intersect Γ at Q and R, with Q on the arc AB and R on the arc BC. Then we claim that $\angle AQR = 90$. To prove this, suppose that HM intersect AO at point R'. Note that $AH||OM$, and $AH = 2OM$. Then we get that $\Delta R'MO$ is similar with $\Delta R'AH$ with the ratio of similitude 2. Then $R'A = 2OA$ and R', O, A are collinear $\rightarrow R'A$ is the diameter of Γ, which implies that R' lies on Γ, and finally $R' = R$ (since H, M, R' are collinear too). Then RA is the diameter of Γ, and our claim was proven.

We see that $\angle AQR = 90 \rightarrow \angle AQM = 90$. Then Q lies on a circle with diameter AM, and finally the circle with diameter AM cuts Γ at A and Q, which directly implies that $Q = P$. Now let AP intersect BC at point X. Consider the triangle XAM. Note that $AH \perp BC \rightarrow AH \perp XM$, and $MH \perp AP \rightarrow MH \perp AX$. Then H is the orthocentre of ΔXAM, and we get that $XH \perp AM$.

Note that $AH = HN$ is equivalent with H lies on the perpendicular bisector of AN. Since O is obviously lies on the perpendicular bisector of AN, we get that the condition is equivalent with $HO \perp AN \Leftrightarrow HO \perp AM$. We have known that $XH \perp AM$. If $HO \perp AM$, then X, H, O are collinear $\Leftrightarrow AP, BC, OH$ meet at a common point. Then we have that AP, BC, OH are concurrent iff $AH = HN$ (Q.E.D)

Quick Reply

High School Olympiads

Simple but beautiful(Own)

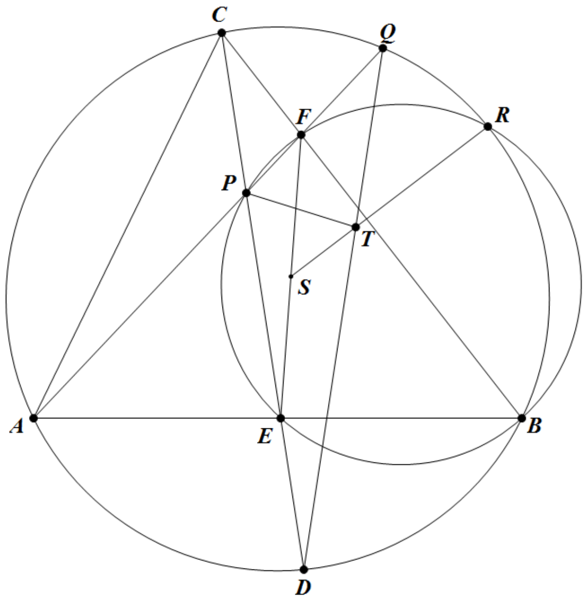
Reply

Arab

#1 Aug 1, 2012, 5:26 pm

Γ is the circumcircle of $\triangle ABC$ and $AB = BC$. E, F lies on AB, BC respectively and AF meets CE at P, such that B, E, F, P are concyclic, whose circumcircle is ω. AF, CE meet Γ at Q, D respectively and S is the midpoint of EF. Let R be the second intersection of Γ and ω and RS meets DQ at T. Prove that, PT is the bisector of $\angle DPQ$.

Attachments:



Luis González

#2 Aug 2, 2012, 12:02 am • 1

M is the midpoint of $\overline{AC}$. MP cuts ω again at R^* and ray PM cuts Γ at L. $\angle ALC = \pi - \angle ABC = \angle APC$ implies that $APCL$ is a parallelogram. Since A and C are conjugate points WRT ω, it follows that the circle (M) with diameter $\overline{AC}$ is orthogonal to $\omega \Longrightarrow MC^2 = MP \cdot MR^* \Longrightarrow \angle CR^*L = \angle PCM = \angle CAL \Longrightarrow R^* \in \Gamma$, i.e. $R \equiv R^*$.

R is the center of the spiral similarity that takes $\overline{EF}$ into $\overline{AC}$ and their midpoints S, M are obviously homologous $\Longrightarrow \triangle RSF \sim \triangle RMC$. But $\angle TRF = \angle PRC = \angle DCA = \angle DQA \Longrightarrow T, R, Q, F$ are concyclic $\Longrightarrow \angle RFT = \angle RQT = \angle RCP \Longrightarrow FT, CP$ are corresponding cevians of $\triangle RSF \sim \triangle RMC$, which means that $\triangle TEF$ and $\triangle PAC$ are spirally similar with center R. Therefore, if AP cuts Γ again at K, then $TEPF$ and $PAKC$ are spirally similar with center R. This is true regardless of $\triangle ABC$ being B-isosceles. When $AB = BC$, then KPB bisects $\angle AKC \Longrightarrow PT$ bisects $\angle EPF$.

iborol

#3 Aug 2, 2012, 12:55 am

Arab, which program did you use to draw the picture?

Arab

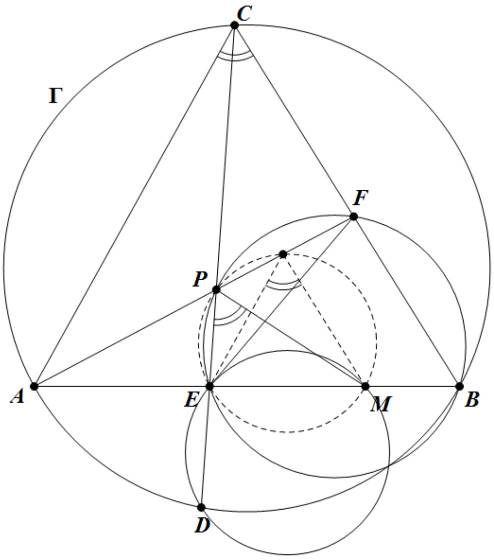
#4 Aug 2, 2012, 10:10 am

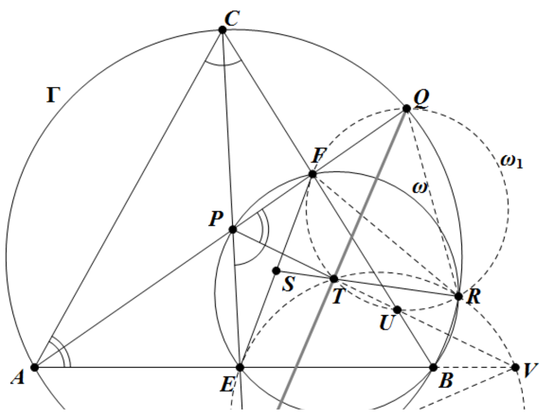
I usually use The Geometer's Sketchpad to draw pictures. :)

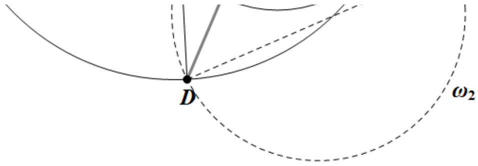
My solution uses the conclusion of http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=490433.

We should notice the following facts:
(1).ω_1 that passes Q,F,R is tangent to EF.
ω_2 that passes E,D,R is tangent to EF.
(2).Let $U=\omega_1\cap BC$ and $V=\omega_2\cap AB$,we have $\angle UPF=\angle BAC$ and $\angle VPE=\angle ACB$.
(3).Note that,$\angle BAC+\angle ABC=\angle UPF+\angle UPE=180^\circ-\angle ABC$,so $\angle UPE=\angle VPE$,by which we get,U,V,V are collinear.
(4).Let T be the second intersection of ω_1 and ω_2,then Q,T,D are collinear and T lies on UV.
(5).Since TR is the radical axis of ω_1 and ω_2,we get TR bisects EF,which means RS,UV,DQ are concurrent.

Attachments:







Arab

#5 Aug 3, 2012, 2:46 pm

Now we get another nice conclusion:Let $U = PT \cap BC$ and $V = PT \cap AB$,where $\triangle ABC$ is arbitrary.Prove that,$\frac{CU}{UB} = \frac{AB}{BV}$.

Quick Reply

High School Olympiads

A problem about circumscribed

Reply

dien9c

#1 Jul 31, 2012, 3:21 pm

A quadrilateral $ABCD$ is inscribed in a circle such that the circle of diameter CD intersects the line segments AC, AD, BC, BD respectively at A_1, A_2, B_1, B_2, and the circle of diameter AB meets the line segments CA, CB, DA, DB respectively at C_1, C_2, D_1, D_2. Prove that there exits a circle that is tangent to the four lines $A_1A_2, B_1B_2, C_1C_2, D_1D_2$.

Luis González

#2 Aug 1, 2012, 2:46 am

The result is still true for two arbitrary circles $(M_1), (M_2)$, one passing through C, D and the another passing through A, B. Label $P \equiv A_1A_2 \cap B_1B_2, Q \equiv B_1B_2 \cap C_1C_2, R \equiv C_1C_2 \cap D_1D_2, S \equiv D_1D_2 \equiv A_1A_2$. Using oriented angles (mod 180 deg), we get $\angle AD_1C_2 = \angle ABC = \angle ADC$ and $\angle AC_1D_2 = \angle ABD = \angle ACD \Longrightarrow D_1C_2 \parallel D_2C_1 \parallel DC \Longrightarrow$ cyclic $C_1C_2D_1D_2$ is an isosceles trapezoid with $C_1C_2 = D_1D_2$, i.e $\triangle RD_1C_2$ is R-isosceles $\Longrightarrow$ $RQ - RS = SD_1 - QC_2$. Similarly, $PQ - PS = SA_2 - QB_1$.

$\angle BC_2Q = \angle BAC = \angle BDC = \angle BB_1Q \Longrightarrow \triangle QB_1C_2$ is Q-isosceles, i.e. $QB_1 = QC_2$. Likewise, $SA_2 = SD_1$. Consequently, $PQ - PS = RQ - RS \Longrightarrow PQRS$ is tangential whose incenter I is the intersection of the perpendiculars dropped from M_1, M_2 to AB, CD, respectively. When (M_1) and (M_2) have diameters $\overline{CD}$ and $\overline{AB}$, then I is the Anticenter of $ABCD$.

Quick Reply

High School Olympiads

interior point and area

Locked

Babai

#1 Jul 30, 2012, 8:22 pm

P is an interior point of ABC.Suppose H_A, H_B, H_C be the orthocentres of PBC, PCA, PAB respectively.Prove that $[H_AH_BH_C] = [ABC]$.

Luis González

#2 Jul 30, 2012, 10:22 pm

Posted before at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=5840
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=299022
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=472685

For a general configuration see the highlighted theorem at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=105181
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=402956

High School Olympiads

Collinear Points

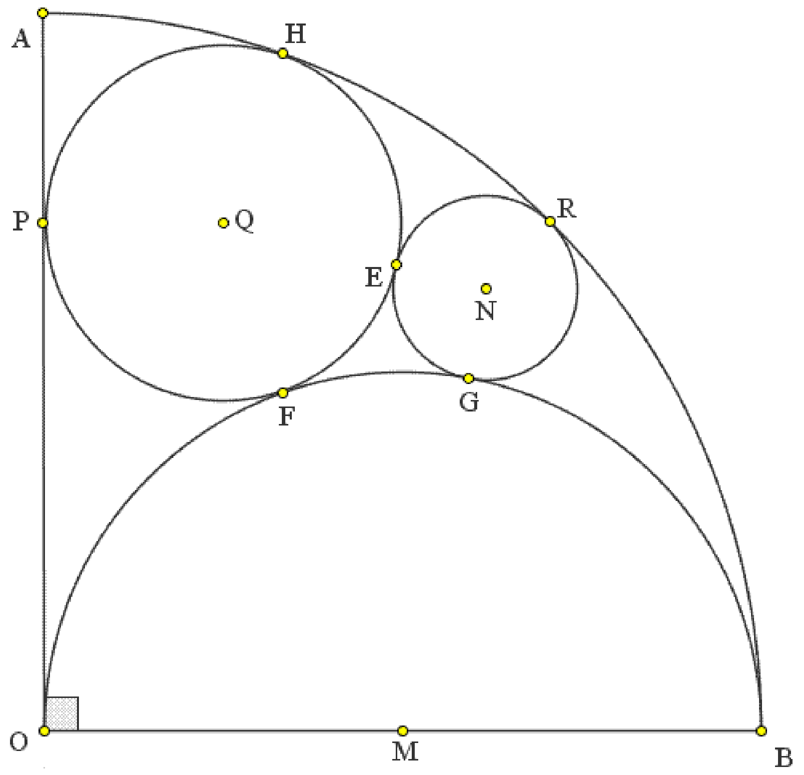
Reply

Speed2001

#1 Jul 29, 2012, 6:54 am

As shown in the figure. Prove that P, Q, R are collinear.

Attachments:



ACCCGS8

#2 Jul 29, 2012, 5:18 pm

Let r be the radius of the circle centred at M and let s be the radius of the circle centred at Q.

$$\frac{s}{2r-s} = \sin\angle POQ = \cos\angle QOM = \frac{(2r-s)^2 + r^2 - (r+s)^2}{2r(2r-s)} =$$

$$= \frac{4r^2 - 4rs + s^2 + r^2 - r^2 - s^2 - 2rs}{2r(2r-s)} = \frac{4r^2 - 6rs}{2r(2r-s)} = \frac{2r-3s}{2r-s}$$

$$\implies s = 2r - 3s \implies s = \frac{r}{2}.$$

So $PO = \sqrt{(2r-s)^2 - s^2} = \sqrt{4r^2 - 4rs} = \sqrt{4r^2 - 2r^2}$.

Now let t be the radius of the circle centred at N. Using the theory of Soddy Circles,

$$-2r = \frac{(r)(\frac{r}{2})(t)}{\frac{r^2}{2} + rt + \frac{rt}{2} - 2\sqrt{(\frac{r^2t}{2})(\frac{3r}{2}+t)}} \implies -r^3 - 3r^2t + 2r\sqrt{r^2t(3r+2t)} =$$

$$= \frac{r^2t}{2} \implies 2\sqrt{(t)(3r+2t)} = r + \frac{7t}{2} \implies \frac{17t^2}{4} - 5rt + r^2 = 0$$

$$\implies t = \frac{10r - 4\sqrt{2}r}{17}.$$

Let X be the foot of the perpendicular from R to OB.

$$\frac{OX}{2r} = \cos\angle NOM = \frac{(2r-t)^2 + r^2 - (r+t)^2}{2r(2r-t)} = \frac{4r^2 - 6rt}{2r(2r-t} = \frac{2r-3t}{2r-t} =$$

$$= \frac{2r - \frac{30r - 12\sqrt{2}r}{17}}{2r - \frac{10r-4\sqrt{2}r}{17}} = \frac{1+3\sqrt{2}}{6+\sqrt{2}} = \frac{1}{\sqrt{2}} \implies OX = \sqrt{2}r$$

$\implies RX = \sqrt{4r^2 - OX^2} = \sqrt{4r^2 - 2r^2} = PO \implies P, Q, R$ collinear (as $PQ \perp PO$).

Speed2001

#3 Jul 29, 2012, 10:01 pm

Thanks **ACCCGS8** for your nice solution.

Luis González

#4 Jul 30, 2012, 12:41 am • 1

Inversion with center B and power $R^2 = \overline{BO}^2 = \overline{BF} \cdot \overline{BP}$ takes (Q) into itself and takes (M) and (O) into OA and the perpendicular ℓ to BO at $M,$ respectively. By conformity, ℓ is then tangent to (Q) at $H^* \Longrightarrow PH^* = OM = \frac{1}{2}R.$ Therefore, $OP^2 = R \cdot \frac{1}{2}R \Longrightarrow OP = \frac{1}{2}\sqrt{2}R.$ By conformity the inverse of (N) is the circle (S) tangent to AO and ℓ at the inverse R^* of $R.$ Since $(S) \cong (Q),$ then $R^*H^* = PH^*.$ Thus if ℓ cuts the arc BFO of (M) at $U,$ it follows that $UR^* = MH^* = OP = UB,$ i.e. $\triangle UBR^*$ is U-isosceles $\Longrightarrow \angle ABR = \frac{1}{2}\angle BUM = \frac{1}{2}(45^\circ) \Longrightarrow R$ is midpoint of the arc AB of $(O).$ As a result, P is the projection of R on $OA,$ i.e. P, Q, R are collinear.

Quick Reply

High School Olympiads

BC/2

Reply

farzinacid

#1 Jul 28, 2012, 10:13 pm

in triangle ABC , suppose D , E , F are on sides BC , AC , AB respectively . such that <DEF = <B and <DFE = <C .
suppose X and Y are perpendicular feet from E and F to BC. prove : XY = BC/2.

Luis González

#2 Jul 28, 2012, 11:51 pm

O is the Miquel point of DEF WRT $\triangle ABC.$ Since $\angle DOE=\angle DCE=\angle EFD$ and $\angle FOD=\angle FBD=\angle DEF \pmod\pi),$ it follows that O is the orthocenter of $\triangle DEF.$ Thus, $\angle ECO=\angle EDO=\angle EFO=\angle EAO \Longrightarrow OA=OC.$ Likewise, $OA=OB \Longrightarrow O$ is the circumcenter of $\triangle ABC.$ Now, let Y^* be a point between B and $X,$ such that $XY^*=\tfrac{1}{2}BC.$ Normal to BC through Y^* cuts AB at $F^*.$ According to the problem fix point, A,E,O,F^* are concyclic $\Longrightarrow F \equiv F^*$ and $Y \equiv Y^*.$

sunken rock

#3 Aug 1, 2012, 2:03 am

Construction of points D, E, F: **rotate the triangle ABC of angle α about the circumcenter O, get $A'B'C'$ the new triangle; $D\in B'C'\cap BC$, a.s.o.**

Take then M, N the midpoints of AB, AC respectively and see that $\widehat{MN, EF}=\frac{\alpha}{2}$. With equal circumradii of triangles $\triangle AOA', \triangle BOB', \triangle COC'$, each being $r'=\frac{R}{2\cdot cos\frac{\alpha}{2}}$, by easy trig. calculations we get the required equality then.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Crazy problem

oneplusone

#1 Jul 26, 2012, 9:32 pm

Let $ABCD$ be a convex quadrilateral. The incircles of $\triangle ABC$ and $\triangle ADC$ touch the side AC at E, F. Suppose that $AE < AF$. Let

$$P, Q, R, S, M, W, X, Y, Z$$

be points on (ext=extension)

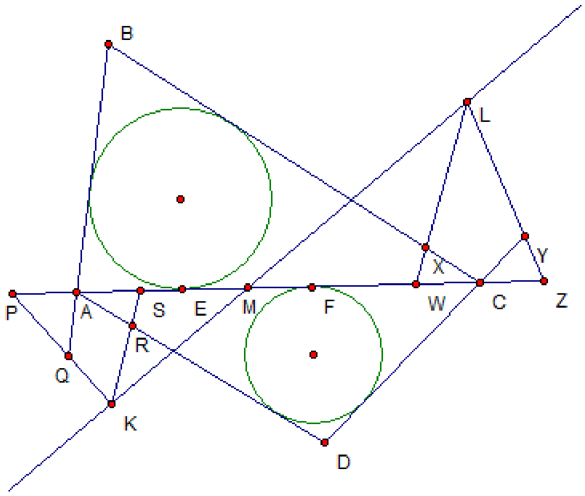
$$\text{ext of } CA, \text{ext of } BA, AD, AC, EF, AC, BC, \text{ext of } DC, \text{ext of } AC$$

such that

$$AP = AQ = AR = AS = EM = MF = CW = CX = CY = CZ$$

Let PQ intersect SR at K and WX intersect ZY at L. Prove that K, L, M are collinear.

Attachments:



leader

#2 Jul 26, 2012, 11:29 pm

from the looks of it KL is the radical axis of the 2 incircles

Luis González

#3 Jul 28, 2012, 11:37 am

Incircles $(I), (J)$ of $\triangle ABC, \triangle ADC$ touch CB, CD at E', F', respectively. If $H \equiv EE' \cap FF'$, then clearly $\triangle HEF \cong \triangle LWZ$ are congruent with corresponding parallel sides $\Longrightarrow$ their corresponding medians HM and LC are equal and parallel, i.e. $HLCM$ is a parallogram $\Longrightarrow$ LM passes through the midpoint T of $\overline{CH}$. Let $U \equiv EE' \cap CI$ and $V \equiv FF' \cap CJ$ be the orthogonal projections of C on HE and HF. Since $IE^2 = IU \cdot IC$ and $JF^2 = JV \cdot JC$, it follows that the circle (T) with diameter $\overline{CH}$ is orthogonal to both (I) and $(J) \Longrightarrow T$ is on radical axis of $(I), (J)$, i.e. LTM is radical axis of $(I), (J)$. By similar reasoning, K is on radical axis of $(I), (J)$.

High School Olympiads

Locus of circumcenter

xeroxia

#1 Jul 27, 2012, 6:50 pm

A circle with diameter AB is given. Let C be an arbitrary point on the diameter AB. Let D be the center of the circle tangent to the given circle and tangent to AB at C. The circle with diameter BC meet BD at P. Let O be the circumcenter of $\triangle APC$. Find the locus of O when C varies.

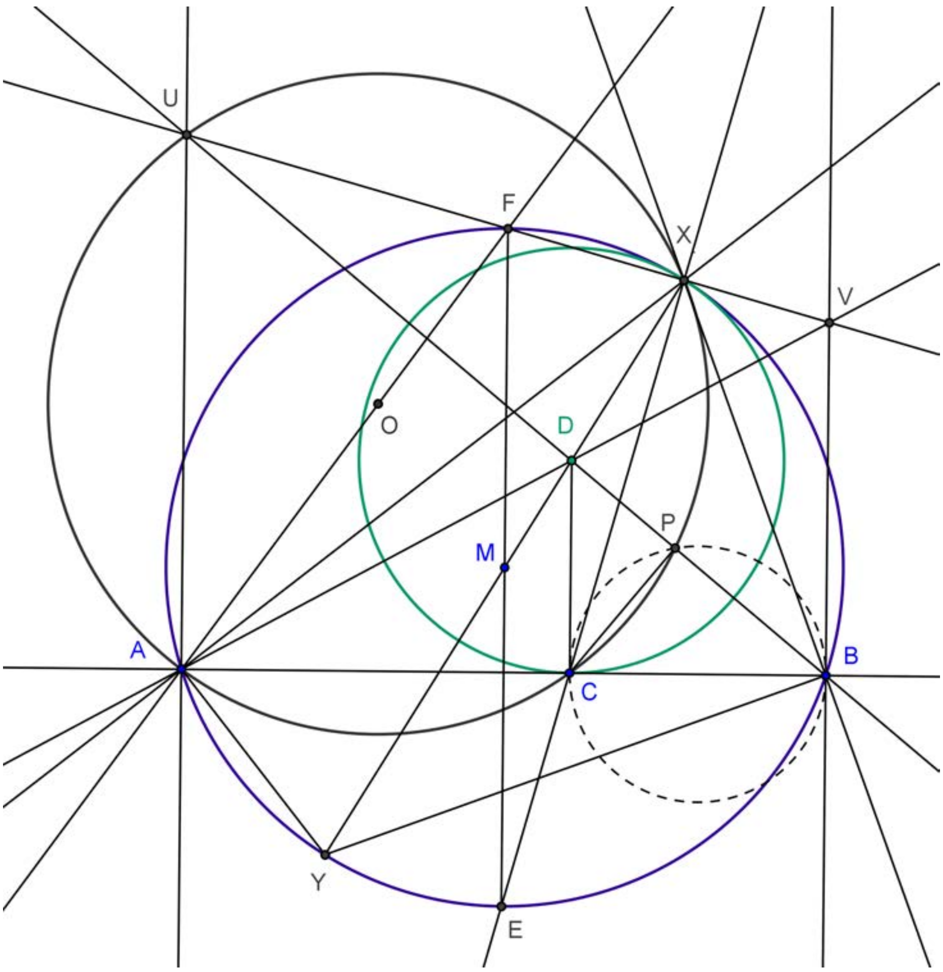
Luis González

#2 Jul 28, 2012, 1:33 am • 1

$\overline{AB}$ can be an arbitrary chord of the given circle (M) and $C \in \overline{AB}$.

X is the tangency point of (D) with (M). E, F are the midpoints of the arcs AB and AXB. XCE and XF bisect $\angle AXB$ internally and externally. Let Y be the antipode of X WRT (M). Normals to AB through A, B cut XF at U, V. Since $\angle UXA = \angle BXV, \angle UAB = \angle XAY = 90^\circ$ and $\angle VBA = \angle XBY = 90^\circ$, by Jacobi's theorem, it follows that XM, AV, BU concur at a point D^*. But $\frac{AU}{BV} = \frac{D^*U}{D^*B} = \frac{CA}{CB} \Longrightarrow CD^* \perp AB \Longrightarrow D \equiv D^*$. Since $CP \perp UDB$, then P is on the circumcircle of the cyclic $AUXC$. Now, locus of the circumcenter O of $\triangle ACX$ is clearly $\overline{AF}$.

Attachments:



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vslmat

#3 Aug 20, 2012, 9:03 pm

Another solution:

Let C_1, the circle with center D that is tangent to AB at C and C_0, the circle with diameter AB, be tangent to the latter at F.
FC cuts C_0 again at H, as both ΔCFD and ΔHFO are isosceles, $HO \parallel CD$, so $HO \perp AB$.
Let HO cut C_0 again at K and FK cut t, the tangent to C_0 at A, at L.
Since $CALF$ is cyclic, $\angle LCA = \angle AFL = 45^\circ$
Notice that $\angle HBA = 45^\circ$. Now let CL cut C_2, the circle with diameter BC and center G, at N.
$\angle BNC = 90^\circ, \angle NCB = 45^\circ$, so $\angle NBC = 45^\circ$, hence N is on BH and $NG \perp AB$.
Let M be the midpoint of CL, since ΔCAL is isosceles right-angled, $MA \perp CL$ and $\angle CAM = 45^\circ$, what tells us that K, M, A are collinear.
Quadrilateral $NGMA$ is cyclic, hence $GC.CA = NC.CM$.
Since M is the midpoint of CL, triangle FMC is isoscales, and as $DC = DF, \angle DFM = \angle DCM = 45^\circ = \angle NBO$ and $\angle DMF = \frac{1}{2}.\angle CMF = \angle MLF = \angle NHC = \angle NOB$
Hence $\Delta NBO \sim \Delta DFM$, and

$$\frac{BN}{BO} = \frac{FD}{FM}$$

But $\frac{BN}{BO} = \frac{NC}{r}$; and $\frac{FD}{FM} = \frac{r_1}{CM}$, so

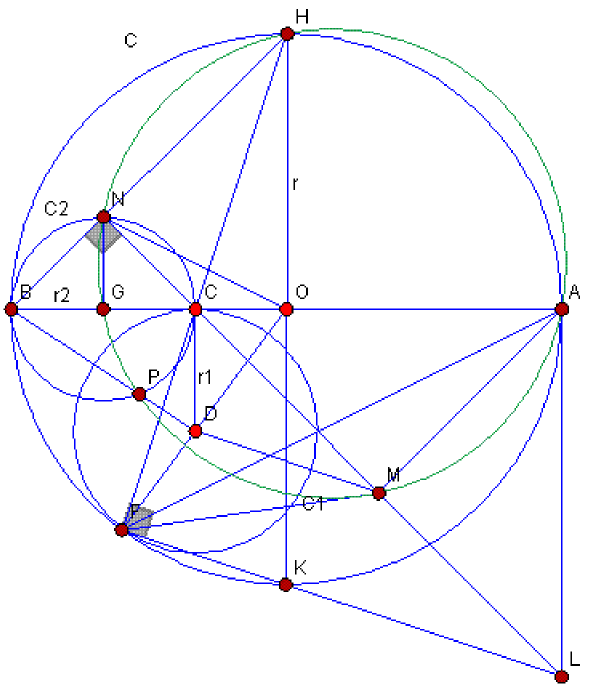
$\frac{NC}{r} = \frac{r_1}{CM}$, or $r.r_1 = NC.CM = GC.CA = r_2.AC$

This is equivalent to (notice that $AC = AL$):

$\frac{r_1}{AC} = \frac{r_2}{r}$, or $\frac{CD}{AL} = \frac{BC}{BA}$.

This means that B, D, L are collinear. It follows that M is the circumcenter of $PCAL$ and we know that M is on AK.
The locus of the circumcenter of PCA is the line AK.

Attachments:



Quick Reply

High School Olympiads

Tangents to incircle and excircles

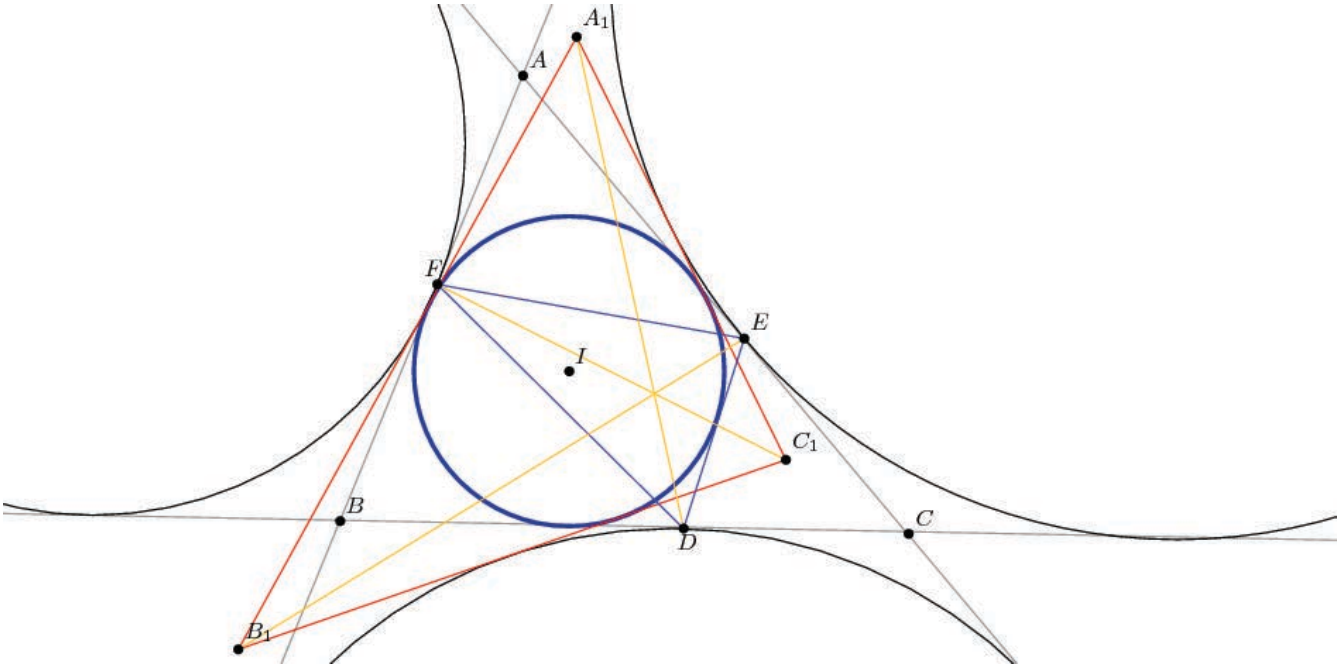
Source: own

proglote

#1 Jul 22, 2012, 1:04 am • 1

Let ABC be a triangle with incircle (I) and excircles $(I_A), (I_B), (I_C)$. Let ℓ_A denote the common tangent to (I) and (I_A) different from BC, and define ℓ_B, ℓ_C similarly. Let $A_1 = \ell_B \cap \ell_C$, and similarly define B_1, C_1. Suppose that $(I_A), (I_B), (I_C)$ are tangent to BC, CA, AB at D, E, F. Prove that $\triangle DEF$ and $\triangle A_1B_1C_1$ are perspective.

Attachments:



Luis González

#2 Jul 27, 2012, 12:29 pm • 4

Incircle (I) touches BC, CA, AB at X, Y, Z. $P_A \equiv \ell_A \cap EF, P_B \equiv \ell_B \cap FD, P_C \equiv \ell_C \cap DE$. ℓ_A cuts AC, AB at U, V. $\triangle ABC \cong \triangle AUV$ are clearly symmetric WRT AI. Thus, by Menelaus' theorem for $\triangle AEF$ cut by ℓ_A, we get

$$\frac{P_AE}{P_AF} = \frac{VA}{FV} \cdot \frac{UE}{UA} = \frac{|CA|}{|AB|} \cdot \frac{|UA - AE|}{|VA - AF|} = \frac{|CA|}{|AB|} \cdot \frac{|AB - CY|}{|CA - BX|}.$$

Similarly, $\dfrac{P_BF}{P_BD} = \dfrac{|AB|}{|BC|} \cdot \dfrac{|BC - AZ|}{|AB - CY|}$, $\dfrac{P_CD}{P_CE} = \dfrac{|BC|}{|CA|} \cdot \dfrac{|CA - BX|}{|BC - AZ|}$

$$\Longrightarrow \frac{P_AE}{P_AF} \cdot \frac{P_BF}{P_BD} \cdot \frac{P_CD}{P_CE} = 1$$

By Menelaus' theorem, we deduce that P_A, P_B, P_C are collinear, i.e. $\triangle DEF$ and $\triangle A_1B_1C_1$ are perspective through $\overline{P_AP_BP_C}$, i.e. DA_1, EB_1 and FC_1 concur.

High School Olympiads

Both cylic and circumscribed quadrilateral

Reply

Source: Tuymaada 2012, Problem 6, Day 2, Seniors

mavropnevma

#1 Jul 22, 2012, 3:40 am • 1

Quadrilateral $ABCD$ is both cyclic and circumscribed. Its incircle touches its sides AB and CD at points X and Y, respectively. The perpendiculars to AB and CD drawn at A and D, respectively, meet at point U; those drawn at X and Y meet at point V, and finally, those drawn at B and C meet at point W. Prove that points U, V and W are collinear.

Proposed by A. Golovanov

Luis González

#2 Jul 22, 2012, 11:02 am • 2

Let UV cut the perpendicular to CD through C at $W'.$ $P \equiv AD \cap BC.$ WLOG assume that the incircle (V) of $ABCD$ is P-excircle of $\triangle PAB.$ Incircle (J) of $\triangle PAB$ touches AB at $L.$ Since AB, DC are antiparallel WRT $PA, PB,$ then $\triangle PAB \sim \triangle PCD$ are similar with incircles $(J),(V).$ Thus

$$\frac{UV}{VW'}=\frac{DY}{YC}=\frac{BL}{LA}=\frac{AX}{XB} \Longrightarrow AU \parallel XV \parallel BW' \Longrightarrow W \equiv W'.$$

subham1729

#3 Jul 22, 2012, 5:35 pm • 1

Hint :
Take in radius=r, keep $\angle B, \angle C$ fixed .We've $XV=r$
Expressing AU, BV in terms of r and by some trigonometric fixed ratio we are done.

antimonyarsenide

#4 Jul 23, 2012, 12:23 am

We could also get $AX/XB=DY/YC$ by letting M,N be the incircle tangency points with AD, BC respectively and noting that by some basic angle chasing, $\triangle BXV \sim \triangle VMD$, $\triangle BNV \sim \triangle VYD$, so $BXVN \sim VMDY$ and similarly $AXVM \sim VNCY$, so $AX*CY=VM*VN=r^2=BX*YD$.
Note that this also shows BC, XY, AD concur I messed around a lot with this kind of idea (except with $P=AB \cap CD$) and also noticed stuff like $\angle WVP = \angle UVP$. I didn't think of the phantom point construction and equal ratios $\iff$ parallelism idea
Nice problem though.

Aiscrim

#5 Jun 14, 2014, 10:39 pm • 1

Oh, Dr Trigo, you're such a nice guy

navi_09220114

#6 Dec 7, 2015, 7:46 pm

It is easy to see AX/XB=DY/YC. We prove this fact first. Let F be intersection of AB and CD, reflect segment BC about the angle bisector of BFC to get segment C'B' (B' lies on AB, C' lies on DC), then since AD and BC are antiparallel, then B'C' is parallel to AD, and B'C' is tangent to the incircle of ABCD, say at K. Let the F-excircle of FB'C' tangent to B'C' at K', then if incircle of ABCD tangent to AD at M and tangent to BC at N, then by homothety F, N, K' colinear. So we get:

AX/XB=AN/BM=AN/C'K=AN/B'K'=DN/C'K=DN/CM=DY/YC.

Now consider the Miquel point P of the quadrilateral ABCD, then F is the center of spiral similarity that maps circumcircle PAD to circumcircle PBC. Since AX/XB=DY/YC, then it also map circumcircle PAD to circumcircle PXY, so PFXY is cyclic. So it is obvious by now, since APU=APV=APW=90. So P, U, V, W are colinear.

Quick Reply

High School Olympiads

Good one

Reply

Source: Own

ACCCGS8

#1 Jul 20, 2012, 10:30 am

Let D, E, F be the midpoints of BC, CA, AB of triangle ABC. D_1 and D_2 lie on EF, E_1 and E_2 lie on FD, and F_1 and F_2 lie on DE such that $AD_1 = AD_2 = BE_1 = BE_2 = CF_1 = CF_2$. Prove that the points D_1, D_2, E_1, E_2, F_1, F_2 are concyclic.

Luis González

#2 Jul 21, 2012, 5:54 pm • 1

Let $AD_1 = AD_2 = BE_1 = BE_2 = CF_1 = CF_2 = k$. P is the foot of the A-altitude. EF cuts AP at its midpoint U. H is the orthocenter of $\triangle ABC$. From $UD_1 \perp AH$, we get
$k^2 - {HD_1}^2 = UA^2 - UH^2 = HA \cdot (UA - UH) = HA \cdot HP$. The RHS is the power of the inversion with center H that takes the circumcircle of $\triangle ABC$ into its 9- point circle. Thus, we deduce that
$HD_1 = HD_2 = HE_1 = HE_2 = HF_1 = HF_2 \Longrightarrow D_1, D_2, E_1, E_2, F_1, F_2$ lie on a circle centered at H.

leader

#3 Jul 21, 2012, 7:13 pm

circles $k_1(A, AD_1)$ and $k_2(B, BE_1)$ are congruent and since $FA = FB$ than F is on the radical axis of these 2 circles so $FE_1 * FE_2 = FD_1 * FD_2$ so D_1, D_2, E_2, E_1 are con-cyclic similarly E_2, E_1, F_1, F_2 are con-cyclic and F_1, F_2, D_1, D_2 are con-cyclic but the pairwise radical axis of these circles form triangle DEF so these circles are actually one circle.

Quick Reply

High School Olympiads

Equivalent to IMO 2012 Q5

Reply

malcolm

#1 Jul 15, 2012, 8:11 am • 1

Let ω_1, ω_2 be orthogonal circles with centres O_1, O_2. Suppose $X \in \omega_1, Y \in \omega_2$ and $O_1Y \cap O_2X$ lies on the radical axis of the circles. Show XY passes through a centre of similitude of the circles.

Luis González

#2 Jul 18, 2012, 3:26 am • 3

Let $P \equiv O_1Y \cap O_2X.$ O_1Y and O_2X cut ω_2 and ω_1 again at $N, M.$ Since P has equal power WRT ω_1 and $\omega_2,$ we have $PX \cdot PM = PY \cdot PN \Longrightarrow X, Y, M, N$ lie on a same circle $\omega.$ $\omega_1 \perp \omega_2$ implies that $O_2Y^2 = O_2M \cdot O_2X \Longrightarrow O_2Y$ is tangent to $\omega.$ Similarly, O_1X is tangent to $\omega.$ If $S \equiv O_1X \cap O_2Y,$ then $SX = SY,$ i.e. circle (S) with center S and radius SX is either externally tangent or internally tangent to both ω_1 and $\omega_2.$ Since X, Y are the insimilicenters/exsimilicenters of $\omega_1 \sim (S)$ and $\omega_2 \sim (S),$ by Monge & d'Alembert theorem, it follows that XY passes through the exsimilicenter of $\omega_1 \sim \omega_2.$

r1234

#3 Jul 20, 2012, 10:20 pm • 1

Well....Here another equivalent version of IMO problem 5, which I used while solving it....this may be clumsy but yet......may be nice....

In a triangle ABC, D is the foot of altitude. Now let H be the orthocenter of $\triangle ABC$. Let $BH \cap \odot ADC = B'$ and $CH \cap \odot ABC = C'$. Then $AB' = AC'$. This is trivial.

Now the main fact is to get the equivalence in these two problems....so I am writing it down here....

(Notations according to IMO P5)..

If we invert the figure wrt D with arbitrary power then the inverted problem becomes as follows...(I'll write X' as the inverse of X)

The lines BK is tangent to the circle KAD and BL is tangent to $\odot LCD$. These two circles goes to the lines $B'K'$ and $A'L'$ respectively such that K' lies on the circumference of $\odot X'DA'$ and L' lies on the circumference of $\odot X'DB'$ and moreover $B'K'$ is tangent to the circle $\odot A'DK'$ and $A'L'$ is tangent to the circle $\odot B'DL'$. Now clearly $\angle K'AD = \angle B'K'D = \angle B'X'D \Longrightarrow A'K'$ passes through the orthocenter of $\triangle X'A'B'$. Similarly $B'L'$ passes through the orthocenter of $\triangle X'A'B'$. Now in the main problem we needed to prove that M lies on the radical axis of the circles $\odot KAD$ and $\odot LCD$. So in the inverted image we need to prove that the radical axis of $\odot DL'B'$ and $\odot DA'K'$ passes through the intersection point of $A'L'$ and $B'K'$ and the point D. That is, we just need to show that $TL' = TK'$ where $T = B'K' \cap A'L'$. So haven't we got back to the equivalent problem?? 😁

Quick Reply

High School Olympiads

Orthocenter, equal angles

Reply

spinx7

#1 Jul 16, 2012, 1:37 am

Let ABC be a triangle and let points O and H be the center of the circumcirle and the orthocenter, respectively. The line perpendicular to AH which crosses its midpoint intersects AB and AC in points D and E respectively. Show that $\angle DOA = \angle EOA$.

yetti

#2 Jul 16, 2012, 7:14 am

AO cuts circumcircle (O) of $\triangle ABC$ again at X. Altitudes BH, CH cut (O) again at Y, Z, resp.
By Pascal theorem for cyclic hexagon $AZBXCY \Longrightarrow$ intersections
$D' \equiv AB \cap XZ, E' \equiv AC \cap XY, H \equiv BY \cap CZ$ are collinear.
D' is on perpendicular bisector AB of $|HZ| \Longrightarrow \triangle D'HZ$ is D'-isosceles.
$\angle HZD' = \angle CZX = \angle CAX = \frac{\pi}{2} - \angle ABC \Longrightarrow$
$\angle AD'H = \frac{1}{2}\angle ZD'H = \angle ABC \Longrightarrow D'H \parallel BC$. Similarly, $E'H \parallel BC$, which already follows from the fact that points D', H, E' are collinear.
$\angle D'XA = \angle ZXA = \angle ZCA = \frac{\pi}{2} - \angle CAB = \angle YBA = \angle YXA = \angle E'XA$.
$\triangle DOE \sim \triangle D'XE'$ are centrally similar with similarity center A and similarity coefficient $\frac{1}{2} \Longrightarrow \angle DOA = \angle EOA$.

Luis González

#3 Jul 17, 2012, 1:07 am

$Q \equiv BH \cap AC$ and $R \equiv CH \cap AB$ are the feet of the B- and C- altitudes. $M \equiv AH \cap ED$ is the midpoint of $\overline{AH}$. Pedal circle $\odot(MQR)$ of H WRT $\triangle ADE$ is Euler circle of $\triangle ABC$ passing through midpoints B', C' of AC, AB. Thus, perpendiculars to AE, AD through B', C' meet at the isogonal conjugate of H WRT $\triangle ADE$, i.e. H, O are isogonal conjugates WRT $\triangle ADE \Longrightarrow \angle(EO, EA) = \angle(ED, EH) = \angle(EA, ED) \Longrightarrow AE$ bisects $\angle OED$. Likewise, AD bisects $\angle ODE \Longrightarrow A$ is either O-excenter or incenter of $\triangle OED \Longrightarrow OA$ bisects $\angle DOE$.

phuongtheong

#4 Jul 17, 2012, 9:25 am

My solution to this nice problem :)
Denote K is the intersection of AO and circle (ABC). P and Q is the intersection of lines CH and BH and (ABC). The line through H and parallel to BC cut AB, AC at N, M respectively.
Because P is the relection of H through AB so we have
$\angle NPA = \angle NHA = 90^0 \Rightarrow \angle NPH = \angle NAH = \angle OAC = \angle KAO$.
Thus P, N, K are collinear.
Similarly K, M, Q are collinear.
It's easy to see $AP = AH = AQ$ so $\angle PKA = \angle QAK$.
Remember O, D, E is the midpoint of AK, AN, AM respectively. So we have $\angle AOD = \angle AOE$.
We are done!

TelvCohl

#5 Jan 31, 2015, 3:11 pm

My solution:

From $\angle DAH = \angle AHD = \angle OAC = \angle ACO \Longrightarrow \triangle ADH \sim \triangle AOC$,
so we get $\triangle ADO \sim \triangle AHC \Longrightarrow \angle AOD = \angle ACH = 90^\circ - \angle BAC$.
Similarly, we can prove $\angle AOE = 90^\circ - \angle BAC \Longrightarrow \angle AOD = \angle AOE$.

Q.E.D

Quick Reply

High School Olympiads

passing through fixed point

Reply

Source: teacher's homework

CTK9CQT

#1 Jul 9, 2012, 9:46 am

Given a circle (O), a point M on the circle (M is fixed) and a line d outside the circle. Let H is projection point of O on d. Let A is an arbitrary point on d and B is the symmetric point of A through H. MA and MB intersect (O) at P and Q respectively.
Prove that the line PQ passes through a fixed point when A moves on d
(I think that the fixed point is the intersection of PQ and d)
sorry for my poor english!

Luis González

#2 Jul 12, 2012, 7:54 am

$N \in (O)$ is the reflection of M across OH and (O) cuts BN again at F. Obviously, we have $MN \parallel PF \parallel AB$. By Pascal theorem for the degenerate cyclic hexagon $MNNFPQ$, the intersections $MN \cap FP$ (at infinity), $V \equiv NN \cap PQ$ and $B \equiv NF \cap MQ$ are collinear $\Longrightarrow V \in d$. Therefore, all lines PQ pass through the fixed intersection of d with the tangent of (O) at N.

sunken rock

#3 Jul 12, 2012, 5:49 pm

It seems it is similar to ***this one***.

Best regards,
sunken rock

Quick Reply

High School Olympiads

harmonic quadrilateral

Reply

Source: one has proposed it on ML without proving

CTK9CQT

#1 Jul 11, 2012, 9:25 am

Let $ABCD$ be a quadrilateral circumscribed about a circle (O) . Let P be any point on AC. Let the segments PA, PB, PC, PD meet (O) at A', B', C', D' respectively. Prove that $A'B'C'D'$ is a harmonic quadrilateral.

Luis González

#2 Jul 11, 2012, 11:51 am

Let (O) touch AB, AD at $U, V. E \equiv AC \cap BD. UV$ cuts BD at the pole K of AC WRT (O). Thus if KB' cuts (O) again at D'', we have $P(E, K, B', D'') = -1$. But $A(E, K, U, V) = -1$ yields $(E, K, B, D) = -1 \Longrightarrow$ $P(E, K, B', D) = -1$, thus $D' \equiv D''$, i.e. $B'D'$ passes through K. Hence, tangents of (O) at B', D' meet on the polar $AC \equiv A'C'$ of K WRT $(O) \Longrightarrow$ quadrilateral $A'B'C'D'$ is harmonic.

r1234

#3 Jul 12, 2012, 5:53 pm

Or in another way.......

Let $AB \cap CD = X, AD \cap BD = Y$. Perform a projective transformation that takes X to infinity and the circle (O) to another circle (O_1). The quadrilateral $A_1B_1C_1D_1$ is circumscribed about (O_1).(Here we denote the image of P by P_1). So $A_1B_1 \parallel C_1D_1$. Now again perform a projective transformation with the singular line through Y_1 parallel to A_1B_1 that projects (O_1) to another circle (O_2).Now the quadrilateral $A_2B_2C_2D_2$ is a square. P_2 is a point on A_2C_2.Clearly the quadrilateral $A_2'B_2'C_2'D_2'$ is harmonic.As projective transformation preserves cross ratio, we conclude that $A'B'C'D'$ is also harmonic.

Quick Reply

High School Olympiads

A nice problem with Feuerbach point

Reply

jayme
#1 Jul 9, 2012, 3:48 pm

Dear Mathlinkers,
1. ABC a triangle
2. O, I the resp. centers of the circumcircle, incircle of ABC
3. Fe the Feuerbach point of ABC
4. DEF the contact triangle of ABC
5. (1) the Miquel circle of ABC with line OI

Prouve : (1) goes through Fe

Sincerely
Jean-Louis

Luis González
#2 Jul 10, 2012, 4:47 am

I think you mean that (1) is the Miquel circle of DEF WRT OI, otherwise, the result isn't true as stated. Since Fe is the anti-Steiner point of OI WRT DEF, then the problem is a consequence of the following lemma:

Lemma. If ℓ is an arbitrary line through the orthocenter H of $\triangle ABC$, then the Miquel circle of $\triangle ABC$ WRT ℓ passes through the anti-Steiner point of ℓ WRT $\triangle ABC$.

Proof. ℓ cuts CA, AB at E, F. BH cuts the circumcircle (O) of $\triangle ABC$ again at the reflection Y of H about CA. YE cuts (O) again at the anti-Steiner point S of ℓ WRT $\triangle ABC$. Circumcircle (O_1) of $\triangle AEF$ cuts (O) again at the Miquel point M of $\triangle ABC$ WRT ℓ, thus $\odot(MOO_1)$ is the Miquel circle of ℓ. Angle chase (mod pi) gives

$$\angle OSM = \frac{\pi}{2} - \angle MYS = \frac{\pi}{2} - \angle MAF - \angle EHY =$$

$$= \frac{\pi}{2} - \angle MAF - \left(\frac{\pi}{2} - \angle AEF\right) = \angle AEM = \angle OO_1M$$

Which implies that M, S, O, O_1 are concyclic, as desired.

jayme
#3 Jul 10, 2012, 1:22 pm

Dear Luis and Mathlinkers,
yes you are right, I meant that (1) is the Miquel circle of DEF WRT OI.
Do you have an idea from where comes this problem?
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Given Triangle And Locus

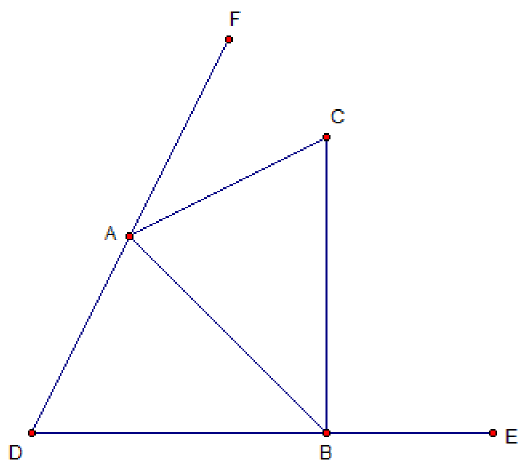
Headhunter
#1 Jul 9, 2012, 8:10 am

Hello.

Given $\triangle ABC$, $\angle FDE$
A, B move on DF, DE respectively.
show that the locus of C is an ellipse.

Please Never Use Coordinates System.

Attachments:



Luis González
#2 Jul 9, 2012, 10:54 am • 1

Let (K) be the circumcircle of $\triangle ABE$. KC cuts (K) at U. Assume that U is between K, C. Since AB and $\angle AEB$ are constant, then the radius $KA = KB$ of (K) is constant, i.e. $ACBK$ and $\triangle AUB$ are rigid $\Longrightarrow \angle ABU = \angle AEU$ is fixed. Thus, $\ell \equiv EU$ is fixed. Let X be the projection of C on ℓ and CX cuts EK at Q. We have $EQ = KE + KC = \varrho = \text{const}$ and $\frac{CX}{QX} = \frac{UC}{EQ} = k = \text{const} < 1$. Hence, Pythagorean theorem for $\triangle EXQ$ gives

$$EX^2 + QX^2 = EX^2 + \frac{CX^2}{k^2} = \varrho^2 \Longrightarrow \frac{EX^2}{\varrho^2} + \frac{CX^2}{\varrho^2 k^2} = 1$$

Therefore, locus of C is an ellipse with center E, pedal circle (E, ϱ) and major axis along the line ℓ.

High School Olympiads

Locus Related To Rectangle

Reply

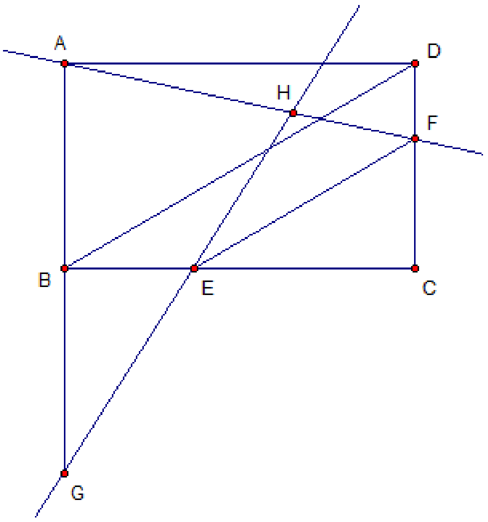
Headhunter
#1 Jul 9, 2012, 8:00 am

Hello.

$ABCD$ is rectangle and EF is parallel to BD.
G is on AB such that $AB=BG$.
GE cut AF at H.
show that the locus of H is an ellipse.

Please Never Use Coordinates System.

Attachments:



Luis González
#2 Jul 9, 2012, 8:33 am • 1

Project the rectangle $ABCD$ into a square $ABC'D'$ under parallel projection (affinity). E,F go to E',F' on $BC',C'D'$, such that $E'F' \parallel BD'$. $H' \equiv AF' \cap GE'$. Since $AD'=AB$ and $D'F'=BE'$, we have $\triangle AD'F' \cong \triangle ABE' \Longrightarrow \angle D'AH' = \angle BAE' = \angle AGH' \Longrightarrow \angle AH'G = 90^\circ \Longrightarrow$ locus of H' is the circle with center B and radius $BA=BG=BC'$. Thus, locus of H is an ellipse through A,C,G with axes AG, BC.

Quick Reply

High School Olympiads

a pole-polar exercise

Locked

CTK9CQT

#1 Jul 7, 2012, 8:19 pm

my teacher give me this problem when we are exploring about pole-polar, but I have no idea about a solution using pole-polar:
Given a triangle ABC and a point O. A line passing O perpendicular to OA intersects BC at M. Define P, Q similarly. Prove that M, P, Q are collinear.
You can do it using diffenrent tools!. Thanks!

Luis González

#2 Jul 7, 2012, 10:23 pm • 2

Well-known and posted many times before. The line MPQ is the orthopolar of O WRT ABC.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=1036
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=51844
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=273212
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=313257
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=346252

High School Olympiads

Tangent to incircles

Source: Polish IMO preparation camp

Swistak

#1 Jul 5, 2012, 2:25 am

Let ABC be a given triangle and D be a variable point in inner of ABC such that $\angle ABD = \angle ACD$. Prove that all possible second inner tangents to incircles of triangles ABD and ACD (other than AD) pass through a fixed point.

Luis González

#2 Jul 7, 2012, 10:32 am • 2

The fixed point if the midpoint of BC. This follows from the following general lemma:

Lemma. $ABCD$ is a quadrilateral. (I_1,r_1) and (I_2,r_2) are the incircles of $\triangle ABC$ and $\triangle ADC$. The internal common tangent ℓ of (I_1) and (I_2), other than AC, cuts BD at P. Then we have $\frac{BP}{CP} = \frac{\cot \widehat{ABI_1}}{\cot \widehat{ADI_2}}$.

Proof. Let ℓ cut CB, CD, CA at M, N, S, respectively. (I_1) touches CA, CB at E, Y and (I_2) touches CA, CD at F, Z. By Newton's theorem for the tangential quadrilateral $ADNS$, the lines AN, DS and FZ concur at a point K. Since $\angle SFK = \angle NZK$, it follows that $\triangle KFS$ and $\triangle KZD$ are pseudo-similar $\Longrightarrow \frac{DK}{KS} = \frac{DZ}{SF}$. Thus, by Menelaus' theorem for $\triangle DSC$, cut by $\overline{ANK}$, we get

$$\frac{CN}{ND} = \frac{AC}{AS} \cdot \frac{KS}{DK} = \frac{AC}{AS} \cdot \frac{SF}{DZ}$$

Similarly, using the tangential quadrilateral $ABMS$, we get $\frac{MB}{CM} = \frac{AS}{AC} \cdot \frac{BY}{SE}$

Thus, by Menelaus' theorem for $\triangle BCD$ cut by ℓ, we get

$$\frac{BP}{CP} = \frac{MB}{CM} \cdot \frac{CN}{ND} = \frac{SF}{SE} \cdot \frac{BY}{DZ} = \frac{r_2}{r_1} \cdot \frac{BY}{DZ} = \frac{\cot \widehat{ABI_1}}{\cot \widehat{ADI_2}}.$$

TelvCohl

#3 Sep 7, 2015, 1:16 am • 1

> Luis González wrote:
> **Lemma.** $ABCD$ is a quadrilateral. (I_1,r_1) and (I_2,r_2) are the incircles of $\triangle ABC$ and $\triangle ADC$. The internal common tangent ℓ of (I_1) and (I_2), other than AC, cuts BD at P. Then we have $\frac{BP}{DP} = \frac{\cot \widehat{ABI_1}}{\cot \widehat{ADI_2}}$.

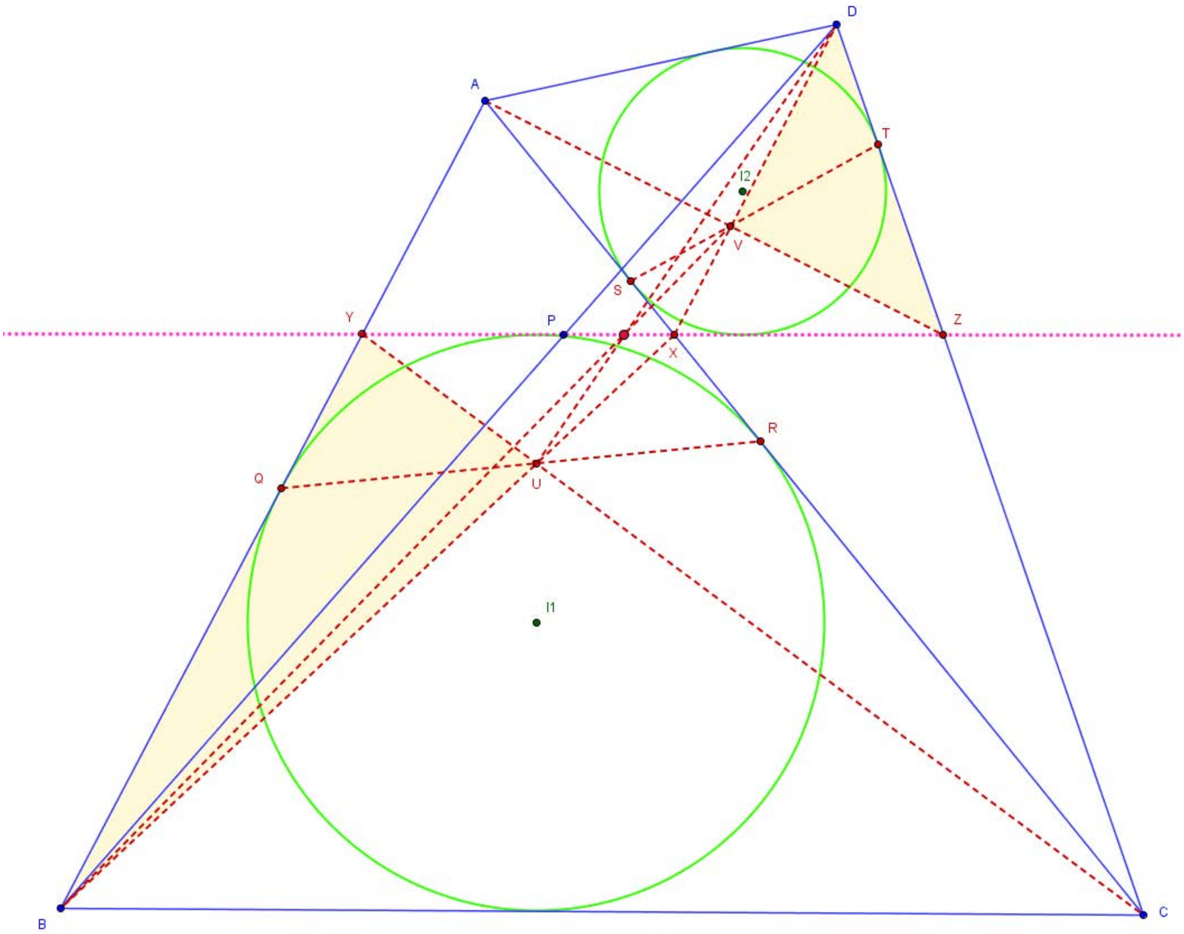
Inspired by Luis's proof, I found the proof as following :) :

Let $Q \equiv \odot(I_1) \cap AB, R \equiv \odot(I_1) \cap AC, S \equiv \odot(I_2) \cap AC, T \equiv \odot(I_2) \cap DC, X \equiv \ell \cap AC, Y \equiv \ell \cap AB, Z \equiv \ell \cap CD.$

From Newton theorem we know BX, CY, QR are concurrent at U and DX, AZ, ST are concurrent at V. Since $A \equiv YB \cap ZV, C \equiv UY \cap DZ$ and $X \equiv BU \cap VD$ are collinear, so from Desargue theorem we get XP, BV, DU are concurrent. $(\star)$ Since $\measuredangle UQB = \measuredangle XRU$, so $\triangle UBQ$ and $\triangle UXR$ are pseudo-similar $\Longrightarrow \frac{BU}{XU} = \frac{BQ}{XR}$. Similarly, we can prove $\frac{DV}{XV} = \frac{DT}{XS}$, so combine with $(\star)$ and from Ceva's theorem (for $\triangle XBD$) we conclude that

$$\frac{DY}{DP} = \frac{DU}{XU}\cdot\frac{XV}{DV} = \frac{DQ}{XR}\cdot\frac{XS}{DT} = \frac{DQ}{DT}\cdot\frac{XS}{XR} = \frac{DQ}{DT}\cdot\frac{r_2}{r_1} = \frac{\cot\widehat{ADI_1}}{\cot\widehat{ADI_2}}.$$

Attachments:



Quick Reply

High School Olympiads

Prove $O;L$ and Q are collinear.

Reply

hoangquan

#1 Jul 6, 2012, 11:26 am

Problem

Let ABC be a triangle inscribed in a circle (O).The tangents to the circle at B and C intersect at P and the tangent at A intersects BC at L. We construct the parallelogram $ANBM$ where $M;N$ are points on BC and BP respectively. Denote by Q the intersection of AP and MN. Show that $O;L$ and Q are collinear.

Luis González

#2 Jul 6, 2012, 12:09 pm • 2

$F \equiv AB \cap MN$ is the midpoint of AB and $MN.$ Let AP cut BC at D and (O) again at $E.$ BC is the polar of P WRT (O) $\Longrightarrow$ pencil $B(A,E,D,P)$ is harmonic, i.e. pencil $B(F,E,M,N)$ is harmonic. Since F is the midpoint of $MN,$ then $BE \parallel MN \equiv FQ \Longrightarrow Q$ is the midpoint of $AE,$ i.e. OQ is the perpendicular bisector of $AE.$ But since AE is the polar of L WRT $(O),$ we have $OL \perp AE \Longrightarrow O,Q,L$ are collinear.

hoangquan

#3 Jul 6, 2012, 7:31 pm

Dear Luis González
Can you use butterfly theorem prove this problem.?

r1234

#4 Jul 6, 2012, 8:50 pm

Note that $\triangle ANB \sim \triangle ABC$. Let $MN \cap AB = X$, then X is the midpoint of AB. Now $\angle ANB = \angle BAC \implies \angle XNA = \angle TAB$ (T is the midpoint of BC) $\implies \angle XMB = \angle BEA$ where $E = AP \cap \odot ABC$. Hence $XE \parallel BC \implies Q$ is the midpoint of AE. Hence $OQ \perp AP$. But since L is he pole of AP wrt (O), we get $OL \perp AP$.Hence O, L, Q are collinear.

Quick Reply

High School Olympiads

Prove that $A;R; S;L$ are colinear

Reply

Source: Konstantinos Pappelis

hoangquan

#1 Jul 5, 2012, 5:53 am

Can you use **buttery theorem** prove :

Problem:

Let $ABCD$ be a quadrilateral inscribed in a circle centered at O and call M the intersection of AC and BD. Let P be a point on segment BC such that PM is perpendicular to MO. Let DP meet the circle at S. Let Q be a point of the circle such that DQ is perpendicular to OM. Denote by R the intersection of the bisectors of angles ABS and AQS and by L the intersection of the tangents through B and Q. Prove that $A; R; S; L$ are colinear.

Luis González

#2 Jul 5, 2012, 8:21 pm

Let MP cut AD at F. By Butterfly theorem, M is the midpoint of $\overline{PF}$. Since $DQ \parallel PF$, the pencil $D(P, F, M, Q)$ is harmonic, i.e. $D(S, A, B, Q)$ is harmonic. But $\angle ADB = \angle AQB, \angle BDS = \angle BQS$ and $\angle SDQ = \angle SQL \Longrightarrow Q(S, A, B, L) \sim D(S, A, B, Q)$ is also harmonic $\Longrightarrow$ the quadrilateral $ABSQ$ is harmonic. Hence, it follows that $L \in AS$ and $BA \cdot QS = QA \cdot BS$, i.e. $\frac{BA}{BS} = \frac{QA}{QS}$. Thus, by angle bisector theorem, the angle bisectors of $\angle ABS$ and $\angle AQS$ meet at a point R on the line ASL.

r1234

#3 Jul 7, 2012, 10:43 am

Let $AB \cap CD = X, AD \cap BC = Y$. Then XY is the polar of M wrt (O).So $OM \perp XY \Longrightarrow MP \parallel XY \parallel DQ$. Now let $BD \cap XY = K$. Then $(BD; MK) = -1$. Let $DQ \cap BC = T$. Then since $XY \parallel DQ \parallel MP$, we get $(BT; PY) = -1$. Hence $(DA, DS; DQ, DB) = -1 \Longrightarrow AQSB$ is harmonic. Hence A, S, R, L are collinear.

Quick Reply

High School Olympiads

Two lines meeting on circumcircle

Reply

Source: ELMO Shortlist 2010, G4; also ELMO #6

Zhero
#1 Jul 5, 2012, 7:49 am

Let ABC be a triangle with circumcircle ω, incenter I, and A-excenter I_A. Let the incircle and the A-excircle hit BC at D and E, respectively, and let M be the midpoint of arc BC without A. Consider the circle tangent to BC at D and arc BAC at T. If TI intersects ω again at S, prove that SI_A and ME meet on ω.

Amol Aggarwal.

Luis González
#2 Jul 5, 2012, 10:38 am • 2

Denote by Ω the circle internally tangent to ω at T and tangent to BC at $D. TB, TC$ cut Ω again at B', C'. Since T is exsimilicenter of $\Omega \sim \omega$, then $BC \parallel B'C' \Longrightarrow$ arcs DB' and DC' of Ω are equal, i.e. TD bisects $\angle BTC \Longrightarrow M \in TD$. ME cuts ω again at F and FI_A cuts ω again at S^*. It's well known that I, I_A lie on the circle (M) with center M and radius $MB = MC$, thus inversion WRT (M) takes D, E into T, F and I, I_A into themselves $\Longrightarrow \odot(IDT)$ and $\odot(I_AEF)$ are orthogonal to $(M) \Longrightarrow \angle MFS^* = \angle EI_AM = \angle DIM = \angle MTS \Longrightarrow S \equiv S^*$. Lines ME and SI_A intersect at $F \in \omega$.

TelvCohl
#3 Dec 17, 2014, 9:51 pm • 1

My solution:

Let $T' = ME \cap (ABC)$.

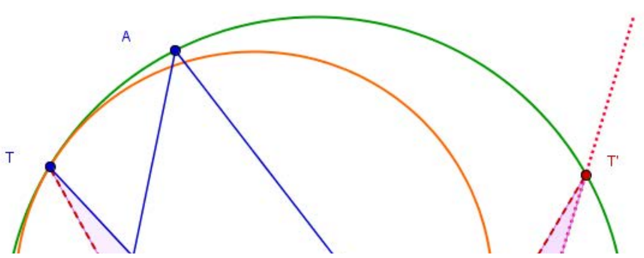
From homothety we get T, D, M are collinear .

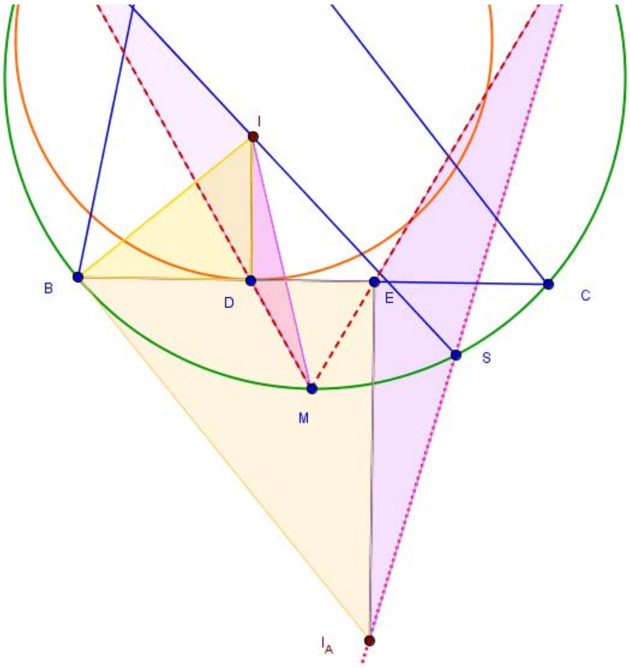
Since $BD = CE$,
so T' is the reflection of T in the bisector of BC .
Since $\triangle MBD \sim \triangle MTB$,
so we get $MI^2 = MB^2 = MD \cdot MT$.
ie. $\triangle MID \sim \triangle MTI$

Since $\text{Rt}\triangle BDI \sim \text{Rt}\triangle I_AEB$,
so we get $DM \cdot ET' = DM \cdot DT = DB \cdot DC = DB \cdot EB = DI \cdot EI_A$,
hence combine with $\angle I_AET' = \angle MDI$ we get $\triangle I_AET' \sim \triangle MDI$,
so from $\angle ET'I_A = \angle DIM = \angle ITM = \angle ET'S$ we get T', S, I_A are collinear .
ie. $ME \cap I_AS \in (ABC)$

Q.E.D

Attachments:





Sardor

#4 Dec 17, 2014, 11:37 pm

Telv Cohl thank you for nice proof..My solution similar to Luiz Gonzalez's solution(but without invesion).

Seventh

#5 Dec 18, 2014, 2:58 am

Other solution:

Claim 1: Are given two circle internally tangent at a point T, AB a chord of the bigger circle tangent to the smaller circle in a point C. So TC is the internal bisector of $\angle ATB$.
Prove: A simple homothety with center T carring one circle in the other.

Applying **Claim 1** in the problem, we have that T, D, E are collinear. Also is well known that $CE = BD$, and since $MB = MC$, follows that $MD = ME$. Denote $ME \cap \omega = T'$, and since $MD = ME$, and the tangent to ω at M is parallel to DE, follows that $MT' = MT$.

Now we gonna prove that T', S and I_A are collinear.

Note that $(A, M; I, I_A) = (A, M; T, SI_A \cap \omega)$, where $(A, M; I, I_A) = \dfrac{AI}{AI_A} = \dfrac{r}{r_A}$, since $MI = MI_A = MB = MC$. But also note that $(A, M; T, T') = \dfrac{AT}{AT'}$, since $MT = MT'$, where $\dfrac{AT}{AT'} = \dfrac{\text{sen}(\angle TMA)}{\text{sen}(\angle AMT')} = \dfrac{DR}{RE}$, since $MD = ME$, where $R = IM \cap DE$. But $\triangle IDR$ and $\triangle I_AER$ are similar, thus $\dfrac{DR}{RE} = \dfrac{r}{r_A}$.

Therefore, $(A, M; T, SI_A \cap \omega) = (A, M; T, T')$, in other words $T' = SI_A \cap \omega$, as desired.

anantmudgal09

#6 Sep 28, 2015, 5:12 am

Similar to Telv Cohl's solution.
Assume without loss of generality, $AB < AC$.(The case $AB = AC$ being trivial.)

Let ω denote the circumcircle of $\triangle ABC$ and let ME meet ω again at X.
Now, let AM meet BC at Y.
Clearly, $MI^2 = MY.MA = ME.MX = MD.MT$ and so A, Y, E, X are concyclic.
Now applying $\sqrt{bc}$ inversion we get that $X \longrightarrow X'$
where X' is the point of intersection of BC with T_AM where T_A is the touch-point of the A-mixitilinear in-circle with ω.
Now it is well known that X' lies on the mixitilinear touch chord of the A-mixitilinear in-circle and so we conclude that

Now it is well-known that X lies on the mixtilinear touch-chord of the A-mixtilinear in-circle and so we conclude that $\angle AXI_A = 90$. Now let AI_A meet ω again at S'. Then clearly, S' is the antipodal point of A w.r.t ω.
So it suffices to showing that T, I, S' are col-linear.
This is equivalent to the fact that $\angle MTI = \angle MAS'$.
But, $\angle MTI = \angle MID = \dfrac{\angle B - \angle C}{2} = \angle MAS'$.
Hence the conclusion follows.

This post has been edited 1 time. Last edited by anantmudgal09, Sep 28, 2015, 5:13 am

EulerMacaroni

#7 Sep 28, 2015, 6:27 am • 1

I think this is the simplest solution yet...

Invert about M with radius MB. The problem becomes to show that the circumcircles of (MDI) and (MEI_a) intersect on BC. Let K be the intersection of (MEI_a) with BC. Then since $I_aE \perp EK$, $\angle I_aEK = \angle I_aMK = \angle AMK = \angle AMS = 90^\circ$, where S is the intersection of MK with ω, so S is the antipode of A in ω. Now, let K' be the intersection of (MDI) with BC, and S' be the intersection of MK' with ω. Then $\angle IDK' = \angle IMK' = \angle AMS' = 90^\circ$, so $S \equiv S'$, hence $K \equiv K'$ and we're done.

infiniteturtle

#8 Yesterday at 10:15 AM

Every problem needs a nice BARY-BASH! The following computation is very instructive in the ways of bashing.

Let $A = (1,0,0), B = (0,1,0), C = (0,0,1)$. Then

$$D = (0 : s-c : s-b), E = (0 : s-b : s-c), I = (a : b : c), I_a = (-a : b : c), M = (-a^2 : b^2+bc : c^2+bc).$$

All fine and good, but now how do we proceed? We need to think *very carefully* about how to make this barybash feasible. In particular, the definition of T is nearly impossible to use as is. We need the synthetic observation that T, D, M are collinear by homothety. This makes T a workable point.

So now, we are going to intersect a bunch of lines with the circumcircle. The only way this is remotely possible is by using Vieta. Indeed, this bash will show just how ridiculously overpowered Vieta is...

Let $X = ME \cap \omega$. Our solution path is as follows: compute $T = MD \cap \omega$ and X by intersecting each line with the circle. Then we'll find $S = XI_a \cap \omega$ by intersecting this line with the circle, and finally check that S lies on XI.

Let's get started! First, the equation of MD is

$$\begin{vmatrix} x & y & z \\ -a^2 & b^2+bc & c^2+bc \\ 0 & s-c & s-b \end{vmatrix} = 0 \iff$$

$$\iff x(b+c)(b(s-b)-c(s-c)) + y(a^2)(s-b) - z(a^2)(s-c) = 0$$

$$\iff x(c^2-b^2)(s-a) + y(a^2)(s-b) - z(a^2)(s-c) = 0$$

$$\iff z = x \cdot \frac{(c^2-b^2)(s-a)}{a^2(s-c)} + y \cdot \frac{(s-b)}{(s-c)}.$$

The circumcircle has equation $a^2yz + b^2zx + c^2xy = 0$, so now we'll plug z into the equation:

$$y^2 \cdot \frac{a^2(s-b)}{(s-c)} + x^2 \cdot \frac{b^2(c^2-b^2)(s-a)}{a^2(s-c)} + Qxy = 0,$$

for some function Q of a, b, c. Note that Vieta allows us to not care what Q is! Now if we normalize to $x = 1$ and observe that one solution for y is when M is the considered point, i.e. $y = \dfrac{-b(b+c)}{a^2}$, we have

$$\left(\frac{-b(b+c)}{a^2}\right)\left(\frac{a^2(s-b)}{(s-c)}\right) y = \frac{(c^2-b^2)(s-a)}{a^2(s-c)}.$$

This reduces to

$$y = \frac{b(b-c)(s-a)}{a^2(s-b)}.$$

Now since $x = 1$ we can easily calculate z as

$$z = \frac{c(c-b)(s-a)}{a^2(s-c)},$$

which is acceptable since it has to agree (symmetrically) with y. Now we re-normalize to the cleaner form

$$T = \left(a^2(s-b)(s-c) : b(b-c)(s-a)(s-c) : c(c-b)(s-a)(s-b)\right).$$

The computation of X is virtually identical; we find

$$X = \left(a^2(s-b)(s-c) : b(c-b)s(s-b) : c(b-c)s(s-c)\right).$$

(Note to reader: We can also synthetically observe that T, X have the same x-coordinate [when normalized, although in the above forms they still do.] This could be used to find X. The time expenditure of setting up such a bash instead of copying the one we already have for T probably makes it counter-productive.)
Now we need to compute the equations of line IT, I_aX. First, line IT has equation

$$\begin{vmatrix} x & y & z \\ a & b & c \\ a^2(s-b)(s-c) & b(b-c)(s-a)(s-c) & c(c-b)(s-a)(s-b) \end{vmatrix} = 0$$

$$\iff x(bc)(s-a)(c-b)a + y(ac)(s-b)(a(s-c)+(b-c)(s-a)) - z(ab)(s-c)(a(s-b)+(c-b)(s-a)) = 0.$$

Now, we divide through by a and, more importantly, observe that

$$a(s-c)+(b-c)(s-a) = S_c, a(s-b)+(c-b)(s-a) = S_b.$$

Thus line IT has equation

$$x(bc)(s-a)(c-b) + y(c)(s-b)S_c - z(b)(s-c)S_b = 0.$$

Again computing I_aX is done in nearly the same way; somewhat surprisingly, we again find a nice simplification, this time

$$a(s-b)+(b-c)s = S_c, a(s-c)+(c-b)s = S_b,$$

which allows us to reduce the equation of I_aX to

$$x(bcs)(b-c) + y(c)(s-c)S_c - z(b)(s-b)S_b = 0.$$

Now we will compute S as the intersection of I_aX and ω. Perhaps it seems unintuitive to not intersect the two lines with each other; however, there are good reasons for intersecting with the circle:
(1) There isn't a terribly easy way to intersect two lines,
(2) we have already seen how powerful Vieta is, and
(3) if S turns out to be nasty, it will be much easier to check if it lies on a line rather than if it lies on a circle.
(The choice to use I_aX here instead of IT is essentially arbitrary; their equations are very similar.)

So let's do this! From the equation for I_aX we obtain

$$z = x \cdot \frac{cs(b-c)}{(s-b)S_b} + y \cdot \frac{c(s-c)S_c}{b(s-b)S_b}.$$

Plugging this into the circumcircle equation we find

$$y^2 \cdot \left(\frac{a^2c(s-c)S_c}{b(s-b)S_b}\right) + x^2 \cdot \left(\frac{b^2cs(b-c)}{(s-b)S_b}\right) + Q \cdot xy = 0,$$

for some function Q of a, b, c. Normalize to $x = 1$, then one solution of this is

$$y = \frac{b(c-b)s(s-b)}{a^2(s-b)(s-c)}.$$

Therefore the one we care about satisfies

$$\frac{b(c-b)s(s-b)}{a^2(s-b)(s-c)} \cdot \frac{a^2c(s-c)S_c}{b(s-b)S_b} \cdot y = \frac{b^2cs(b-c)}{(s-b)S_b},$$

which miraculously simplifies to $y = \frac{-b^2}{S_c}$. Back substituting, we find $z = \frac{-c^2}{S_b}$, which is good since y and z are symmetrical.

Re-normalizing, we find the cleaner form

$$S = (-S_bS_c : b^2S_b : c^2S_c).$$

We're almost there! We just need to check that this lies on IT, whose equation we already have. Plugging in, we want

$$-S_bS_cbc(s-a)(c-b) + b^2c(s-b)S_bS_c - c^2b(s-c)S_bS_c = 0.$$

We can cancel the 6th degree bcS_bS_c from everything, leaving only

$$-(s-a)(c-b) + b(s-b) - c(s-c) = 0.$$

Now factoring $(b-c)$ from this, it becomes

$$(b-c)(2s-a-b-c) = 0,$$

which is clearly true. Hooray!

Most important thing to note from this: Vieta is incredible, making intersecting a line with a circle very very easy (look at how ridiculously easily we compute S.)

Also! I have seen this point S before when bary bashing another problem... bash configuration recognition?

FWIW: This only took about 1.5 pages to solve, and is probably doable in an hour or so (I did it while watching a movie, so it took a while)

Quick Reply

High School Olympiads

Midpoints of segments are concyclic

Reply

Source: ELMO Shortlist 2010, G6

Zhero

#1 Jul 5, 2012, 7:52 am

Let ABC be a triangle with circumcircle Ω. X and Y are points on Ω such that XY meets AB and AC at D and E, respectively. Show that the midpoints of XY, BE, CD, and DE are concyclic.

Carl Lian.

Luis González

#2 Jul 5, 2012, 9:29 am

The infamous generalization of IMO 2009 Problem 2

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=291269
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=372184
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=439734

simplependulum

#3 Jul 6, 2012, 9:32 am

I didn't go through all the links but i think it's a new solution .
Although it is not beautiful compared with those extraordinary solutions , you will see that it's a nice use of **Ptolemy theorem** .

Let M, N, P be the midpoints of BE, CD, DE respectively , O the circumcentre of triangle ABC , and the circumcircle of triangle MNP intersect XY the second time at Q , we want to show that $OQ \perp XY$ or equivalently $OD^2 - OE^2 = QD^2 - QE^2$ or $AE \cdot CE - AD \cdot BD = 2PQ \cdot DE$ ($D - P - Q - E$) .
Since $\Delta QMN \sim \Delta AED$ and $BD = 2MP$, $CE = 2NP$, we have
$AE \cdot CE - AD \cdot BD = 2(\frac{DE}{MN})(MQ \cdot NP - NQ \cdot MP) = 2(\frac{DE}{MN})(MN \cdot PQ) = 2PQ \cdot DE$.

applepi2000

#4 Jul 13, 2012, 3:31 am • 1

Here's a direct solution using the converse of Pascal's:

Let the midpoints of XY, DE, BE, CD be M_1, M_2, M_3, M_4 respectively. Let $AM_1 \cap \Omega = A_1, BM_1 \cap \Omega = BM_1, CM_1 \cap \Omega = C_1$. Let $C_1A_1 \cap XY = C_2, B_1A_a \cap XY = B_2, CB_2 \cap BC_2 = A_2$. Note that since B_2, C_2, M are collinear, the converse of Pascal's on $A_1C_1CA_2BB_1$ implies A_2 lies on Ω. Thus:

$$\angle BAC = \angle BA_2C = \pi - \angle C_2BC - \angle B_2CB = \angle BC_2E + \angle CB_2D - \pi$$

Note that by the Butterfly Theorem wrt M_1, we have $DM_1 = M_1B_2, EM_1 = M_1C_2$. Thus, from SAS similarity we see that $M_1M_3 || C_2B, M_1M_2 || B_2C$. Thus:

$$\angle BAC = \angle BC_2E + \angle CB_2D - \pi = \angle M_3M_1E + \angle M_4M_1D - \pi = \angle M_4M_1M_3$$

Also since $M_2M_3 || AB, M_2M_4 || AC$ we have:

$$\angle BAC = \angle M_4M_2M_3$$

So thus $\angle M_4M_2M_3 = \angle M_4M_1M_3 \implies M_1, M_2, M_3, M_4$ are concyclic as required.

Quick Reply

High School Olympiads

120 angle

Reply

farzinacid

#1 Jul 5, 2012, 4:59 am

In triangle $\triangle ABC$ we have $AC = AB + AM$ and AM is median. AM cuts incircle of $\triangle ABC$ in X and Y. Prove : $\angle XIY = 120^\circ$ (I is incenter of ABC).

Luis González

#2 Jul 5, 2012, 5:28 am

$AC = AB + AM \Longrightarrow AC > AB \Longrightarrow I$ lies inside $\triangle ABM$. Let P be the projection of I on the median $\overline{AM}$. If r denotes the inradius of $\triangle ABC$, we have

$$[\triangle ABM] = \frac{1}{2}[\triangle ABC] = \frac{1}{2}IP \cdot AM + \frac{1}{2}r \cdot BM + \frac{1}{2}r \cdot AB$$

$$[\triangle ABM] = \frac{1}{2}(IP \cdot AM + \frac{1}{2}r \cdot BC + r \cdot AB)$$

$$\frac{1}{2}r(AB + AC + BC) = IP \cdot AM + \frac{1}{2}r \cdot BC + r \cdot AB \Longrightarrow$$

$$\frac{1}{2}r(AC - AB) = IP \cdot AM \Longrightarrow IP = \frac{1}{2}r \Longrightarrow \angle XIP = 60^\circ \Longrightarrow \angle XIY = 120^\circ.$$

Quick Reply

High School Olympiads

Inequality tetrahedron ABCD -63

Reply

Source: Nicusor Zlota,CTATV

nicusorz

#1 Jun 20, 2012, 1:48 pm

Denote r_a, r_b, r_c, r_d and h_a, h_b, h_c, h_d the radii of exinscribed spheres and the altitudes in tetrahedron $ABCD$, r is the radius of inscribed sphere, then we have inequality

$$\frac{r_a-r}{r_a+r}+\frac{r_b-r}{r_b+r}+\frac{r_c-r}{r_c+r}+\frac{r_d-r}{r_d+r}\geq\frac{4}{3}$$

http://www.infomate.ro

Luis González

#2 Jul 4, 2012, 12:02 pm

Let S_A, S_B, S_C, S_D denote the areas of the faces against A, B, C, D, respectively. Let V denote the volume of the tetrahedron $ABCD$. Then we have

$$\frac{1}{r_a}=\frac{S_B+S_C+S_D-S_A}{3V}\ ,\ \frac{1}{r_b}=\frac{S_C+S_A+S_D-S_B}{3V}$$

$$\frac{1}{r_c}=\frac{S_A+S_B+S_D-S_C}{3V}\ ,\ \frac{1}{r_d}=\frac{S_A+S_B+S_C-S_D}{3V}$$

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}+\frac{1}{r_d}=\frac{2(S_A+S_B+S_C+S_D)}{3V}=\frac{2}{r}$$

Therefore, by AM-HM we obtain

$$\frac{r_a}{r_a+r}+\frac{r_b}{r_b+r}+\frac{r_c}{r_c+r}+\frac{r_d}{r_d+r}\geq\frac{16}{\frac{r_a+r}{r_a}+\frac{r_b+r}{r_b}+\frac{r_c+r}{r_c}+\frac{r_d+r}{r_d}}=$$

$$=\frac{16}{4+r\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}+\frac{1}{r_d}\right)}=\frac{16}{4+r\cdot\frac{2}{r}}=\frac{8}{3}\Longrightarrow$$

$$\frac{r_a-r}{r_a+r}+\frac{r_b-r}{r_b+r}+\frac{r_c-r}{r_c+r}+\frac{r_d-r}{r_d+r}\geq\frac{4}{3}.$$

Quick Reply

High School Olympiads

Inequality tetrahedron ABCD - 62

Source: Nicusor Zlota,CTATV

nicusorz

#1 Jun 20, 2012, 1:43 pm

Denote r_a, r_b, r_c, r_d and h_a, h_b, h_c, h_d the radii of exinscribed spheres and the altitudes in tetrahedron $ABCD$, r is the radius of inscribed sphere, then we have inequality

$$\frac{h_a+r}{h_a-r}+\frac{h_b+r}{h_b-r}+\frac{h_c+r}{h_c-r}+\frac{h_d+r}{h_d-r}\ge\frac{20}{3}$$

http://www.infomate.ro

Luis González

#2 Jul 4, 2012, 12:00 pm

Let S_A, S_B, S_C, S_D denote the areas of the faces against $A, B, C, D,$ respectively. Let V denote the volume of the tetrahedron $ABCD.$ Then we have

$$\frac{1}{h_a}=\frac{S_A}{3V}\ ,\ \frac{1}{h_b}=\frac{S_B}{3V}\ ,\ \frac{1}{h_c}=\frac{S_C}{3V}\ ,\ \frac{1}{h_d}=\frac{S_D}{3V}\Longrightarrow$$

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}+\frac{1}{h_d}=\frac{S_A+S_B+S_C+S_D}{3V}=\frac{1}{r}$$

Therefore, by AM-HM we obtain

$$\frac{h_a}{h_a-r}+\frac{h_b}{h_b-r}+\frac{h_c}{h_c-r}+\frac{h_d}{h_d-r}\ge\frac{16}{\frac{h_a-r}{h_a}+\frac{h_b-r}{h_b}+\frac{h_c-r}{h_c}+\frac{h_d-r}{h_d}}=$$

$$=\frac{16}{4-r\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}+\frac{1}{h_d}\right)}=\frac{16}{4-r\cdot\frac{1}{r}}=\frac{16}{3}\Longrightarrow$$

$$\frac{h_a+r}{h_a-r}+\frac{h_b+r}{h_b-r}+\frac{h_c+r}{h_c-r}+\frac{h_d+r}{h_d-r}\ge\frac{20}{3}.$$

High School Olympiads

Unique pair of points satisfying angle condition in pentagon

Reply

Source: ELMO Shortlist 2011, G4

math154

#1 Jul 3, 2012, 9:51 am • 2

Prove that for any convex pentagon $A_1A_2A_3A_4A_5$, there exists a unique pair of points $\{P, Q\}$ (possibly with $P = Q$) such that $\measuredangle PA_iA_{i-1} = \measuredangle A_{i+1}A_iQ$ for $1 \le i \le 5$, where indices are taken $\pmod 5$ and angles are directed $\pmod \pi$.

Calvin Deng.

This post has been edited 2 times. Last edited by math154, Jul 4, 2012, 5:20 am

nsato

#2 Jul 4, 2012, 4:44 am • 1

This problem looks interesting, but I don't quite understand it. Can anyone clarify what it means?

Luis González

#3 Jul 4, 2012, 9:02 am • 1

Assume that there exists two points P, Q, such that PA_1, QA_1 are isogonals WRT $\angle A_5A_1A_2$, PA_2, QA_2 are isogonals WRT $\angle A_1A_2A_3$, etc. Let P_1, P_2, P_3, P_4, P_5 be the projections of P on $A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_1$ and Q_1, Q_2, Q_3, Q_4, Q_5 the projections of Q on $A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_1$. By well known property of isogonals, P_1, Q_1, P_2, Q_2 lie on a circle centered at the midpoint M of $\overline{PQ}$. Similarly, P_2, Q_2, Q_3, P_3 lie on a circle centered at M, etc $\Longrightarrow$ P_1, P_2, P_3, P_4, P_5 lie on a same circle (M). Hence, $A_1A_2 \perp PP_1, A_2A_3 \perp PP_2, A_3A_4 \perp PP_3, A_4A_5 \perp PP_4$ and $A_5A_1 \perp PP_5$ are tangents of the conic $\mathcal{K}$ with focus P, Q and pedal circle (M). Since $\mathcal{K}$ is unambiguosly defined by the five tangents $A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_1$, then the pair P, Q is unique.

ACCCGS8

#4 Jul 10, 2012, 11:40 am • 1

Call a pair of points P, Q that satisfy the conditions of the problem **pentagonal isogonal conjugates.**

Firstly we will show if pentagonal isogonal conjugates P, Q exist, then there exists an ellipse with foci P, Q that is tangent to all five sides of the pentagon. Let the reflection of P in side A_iA_{i+1}, for all $1 \le i \le 5$, be P_i. By a well known result of isogonals, $QP_i = QP_{i+1}$ for all $1 \le i \le 5$, so $QP_1 = QP_2 = QP_3 = QP_4 = QP_5$. Let QP_i intersect A_iA_{i+1} at R_i for all $1 \le i \le 5$. Then $PR_i + QR_i = P_iR_i + QR_i = QP_i$, and PR_i and QR_i are symmetric with respect to A_iA_{i+1}, so there exists an ellipse with foci P, Q that is tangent to all five sides of the pentagon.

Now for any ellipse tangent to the sides of the pentagon, its foci are pentagonal isongonal conjugates. Since there exists exactly one ellipse tangent to all five sides of the pentagon, we are done.

Quick Reply

High School Olympiads

Three mutually tangent circles

Reply

Source: ELMO Shortlist 2011, G2

math154

#1 Jul 3, 2012, 9:50 am

Let $\omega, \omega_1, \omega_2$ be three mutually tangent circles such that ω_1, ω_2 are externally tangent at P, ω_1, ω are internally tangent at A, and ω, ω_2 are internally tangent at B. Let O, O_1, O_2 be the centers of $\omega, \omega_1, \omega_2$, respectively. Given that X is the foot of the perpendicular from P to AB, prove that $\angle O_1XP = \angle O_2XP$.

David Yang.

Luis González

#2 Jul 3, 2012, 11:45 am

ω can be either internally tangent or externally tangent to ω_1 and ω_2, both cases give the same result. Let Q be the exsimilicenter of $\omega_1 \sim \omega_2$. A is the insimilicenter/exsimilicenter of $\omega \sim \omega_1$ and B is the insimilicenter/exsimilicenter of $\omega \sim \omega_2$. By Monge and d'Alembert theorem, A, B, Q are collinear. Since P, Q are harmonically separeted from O_1, O_2 and $PX \perp QX$, it follows that XP, AB bisect $\angle O_1XO_2$ or $\angle O_1XP = \angle O_2XP$.

ACCCGS8

#3 Jul 5, 2012, 4:29 pm

Here's my trigonometric solution to this problem.
Let $\angle APO_1 = \angle PAO_1 = x$ and $\angle BPO_2 = \angle PBO_2 = y$.
Standard angle-chasing yields $\angle BAP = y$ and $\angle ABP = x$.
Scale the diagram so that $XP = 1$.
From right-angled triangle AXP, $AX = \frac{1}{tany}$ and $AP = \frac{1}{siny}$. The Sine Rule in triangle AO_1P gives
$PO_1 = AO_1 = \frac{1}{2cosxsiny}$.
Similarly, $PO_2 = \frac{1}{2cosysinx}$.
Thus $\frac{PO_1^2}{PO_2^2} = \frac{(tanx)^2}{(tany)^2}$.

Applying the Cosine Rule in triangle XAO_1 gives
$O_1X^2 = \frac{1}{(tany)^2} + \frac{1}{4(cosx)^2(siny)^2} - \frac{cos(x+y)}{cosxsinytany}$.
Similarly, $O_2X^2 = \frac{1}{(tanx)^2} + \frac{1}{4(cosy)^2(sinx)^2} - \frac{cos(x+y)}{cosysinxtanx}$.
Thus $\frac{O_1X^2}{O_2X^2} = \frac{\frac{1}{(tany)^2} + \frac{1}{4(cosx)^2(siny)^2} - \frac{cos(x+y)}{cosxsinytany}}{\frac{1}{(tanx)^2} + \frac{1}{4(cosy)^2(sinx)^2} - \frac{cos(x+y)}{cosysinxtanx}}$

$= \frac{(cosy)^2(sinx)^2(1+4sinxsinycosxcosy)}{(cosy)^2(sinx)^2(1+4sinxsinycosxcosy)}$

$= \frac{(tanx)^2}{(tany)^2}$

$= \frac{PO_1^2}{PO_2^2}$

The desired result then follows from the Angle Bisector Theorem.

mathocean97

#4 Dec 4, 2012, 10:58 am

You can also invert through P.

Let the images be the same point (instead of A', keep it as A for simplicity).

The new picture becomes two parallel lines (ω_1 and ω_2) with the circle ω tangent to both lines at points A, B. The centers O_1 and O_2 map to the reflections of P across ω_1 and ω_2 respectively.

Also, the perpendiculars to AP and BP and A, B respectively meet at X. We now must prove that $\angle XO_1P = \angle XO_2P$, equivalent to $XO_1 = XO_2$.

This is probably obvious, but I haven't found a good synthetic solution.

So instead, let the lines ω_1 and ω_2 be the lines $y = 1$ and $y = -1$.

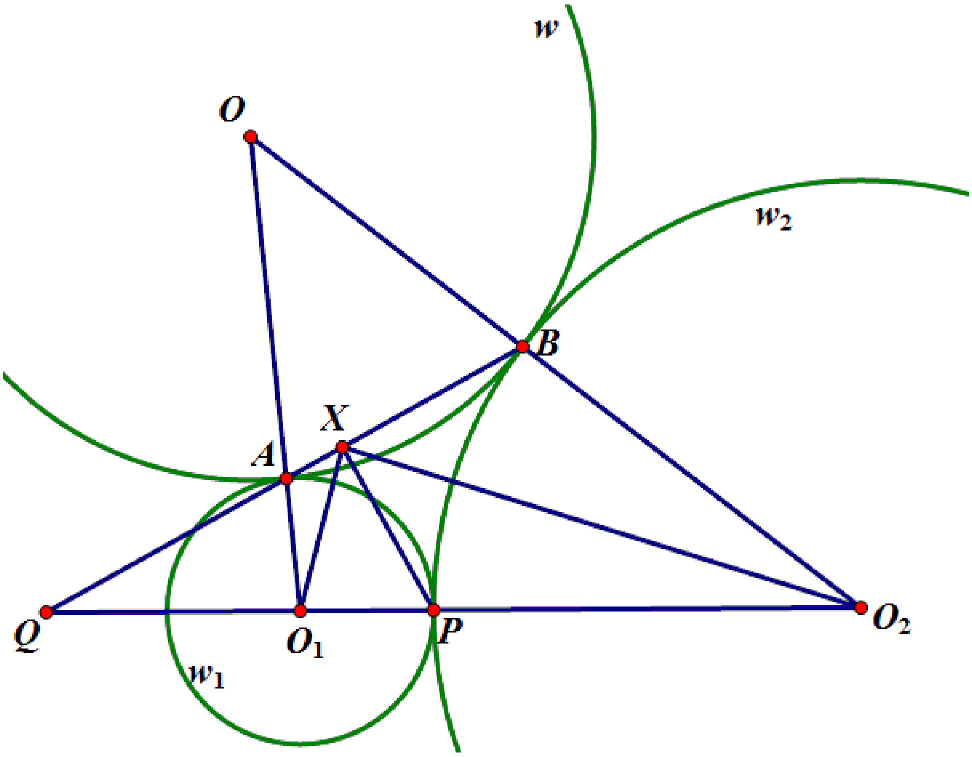
Also, let A be point (0,1), and B is (0,-1). Let P be point (p_1, p_2). We can now easily compute that $O_1 = (p_1, 2 - p_2)$, and $O_2 = (p_1, -2 - p_2)$, and $X = (\frac{p_2^2 - 1}{p_1}, -p_2)$. Now its obvious that $XO_1 = XO_2$, so we're done.

yunxiu

#5 Dec 6, 2012, 5:45 pm

Denotes $AB \cap O_1O_2 = Q$, by Menelaus theorem $\frac{O_2B}{BO}\frac{OA}{AO_1}\frac{O_1Q}{QO_2} = 1$, so $\frac{O_1Q}{QO_2} = \frac{AO_1}{O_2B} = \frac{O_1P}{PO_2}$, hence $X(PQ;O_1O_2) = (PQ;O_1O_2) = -1$.
Because $PX \perp QX$, we have $\angle O_1XP = \angle O_2XP$.

Attachments:



↰ Quick Reply

High School Olympiads

Harmonic (Own)

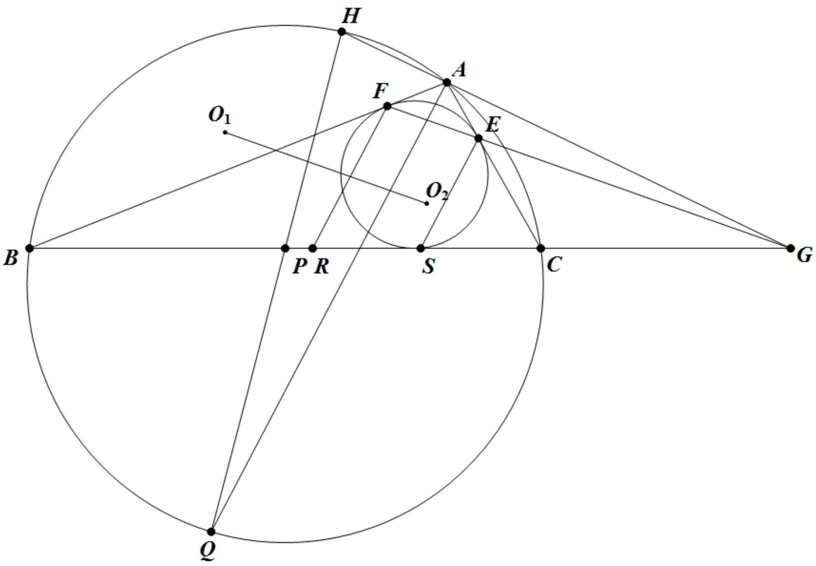
Reply

Arab

#1 Jun 29, 2012, 4:58 pm

The inscribed circle of $\triangle ABC$ tagents AB, AC at F, E respectively and EF meets BC at G. AG meets Γ the circumcircle of $\triangle ABC$ again at H, such that A lies within segment GH. PH meets Γ at Q, where P is the midpoint of BC. R, S lies on BC such that $RF \parallel AQ \parallel SE$. Let O_1, O_2 be the circumcenter of $\triangle ABS$ and $\triangle ACR$. Prove that $O_1O_2 \parallel EF$.

Attachments:



Luis González

#2 Jul 3, 2012, 10:33 am

Incircle (I) touches BC at D and AD cuts Γ again at M. Tangent of Γ at H cuts BC at K. Since $\angle BAM = \angle BHM$, $\angle CAM = \angle CHM$ and $\angle CAG = \angle CHK$, then the pencils $A(B, C, D, G) \sim H(B, C, M, K)$ are harmonic. Therefore, quadrilateral $HBMC$ is harmonic $\Longrightarrow HM$ is H-symmedian of $\triangle HBC \Longrightarrow HM, HP$ are isogonals WRT $\angle BHC \Longrightarrow QM \parallel BC \Longrightarrow AQ$ is the isogonal of AD WRT $\angle BAC$. If $T \equiv AQ \cap BC$, then by Steiner theorem, we deduce that

$$\frac{BA^2}{CA^2} = \frac{BT}{CT} \cdot \frac{BD}{CD} = \frac{BT}{CT} \cdot \frac{BF}{CE}$$

But from $ES \parallel FR \parallel AT$ we obtain $\frac{CS}{\ } = \frac{CE}{\ }$, $\frac{BT}{\ } = \frac{BA}{\ } \Longrightarrow$

$CT \quad CA \quad BR \quad BF$

$$\frac{BA^2}{CA^2} = \frac{CE}{BF} \cdot \frac{BR}{CS} \cdot \frac{BA}{CA} \cdot \frac{BF}{CE} \Longrightarrow \frac{BR}{CS} = \frac{BA}{CA} \ (\star).$$

Let $(O_1), (O_2)$ cut AC, AB again at $Y, Z.$ Then $CS \cdot CB = CY \cdot CA$ and $BR \cdot BC = BZ \cdot BA.$ Together with $(\star),$ we get $CY = BZ.$ As a result, the 2nd intersection U of (O_1) and (O_2) is the center of the rotation that takes $\overline{CY}$ into $\overline{ZB}.$ Thus, $UY = UB \Longrightarrow U$ is the midpoint of the arc BSY of $(O_1) \Longrightarrow AU$ bisects $\angle BAY,$ i.e. AI is radical axis of $(O_1), (O_2) \Longrightarrow O_1O_2 \parallel EF.$

Quick Reply

High School Olympiads

polar problem

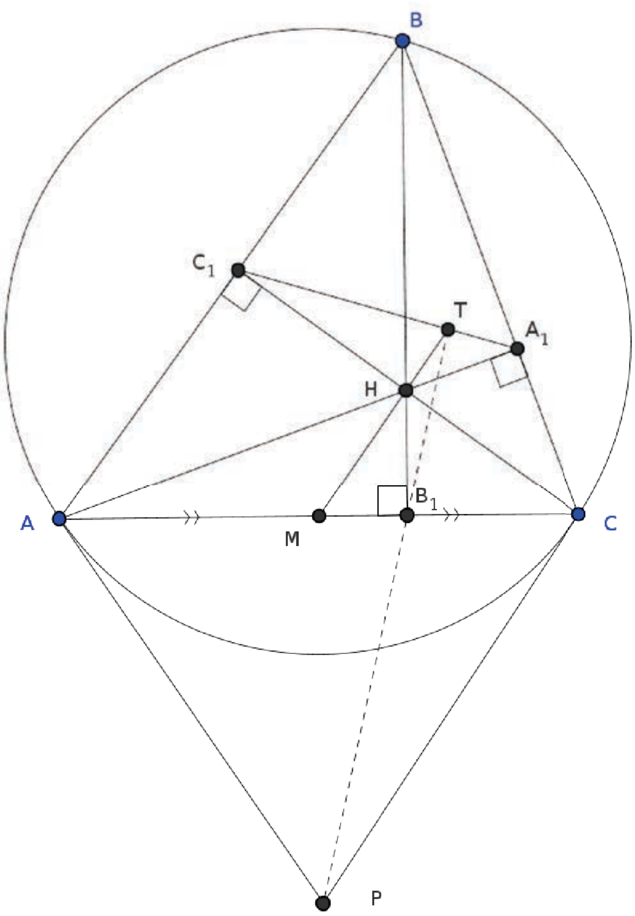
Reply

unt

#1 Jul 3, 2012, 3:50 am

Show P, B_1, T are collinear (see diagram).

Attachments:



Luis González

#2 Jul 3, 2012, 5:35 am

BP cuts the circumcircle (O) of $\triangle ABC$ again at F and E is the antipode of B WRT (O). Then EA, CH are both perpendicular to BA and EC, AH are both perpendicular to $BC \Longrightarrow HAEC$ is a parallelogram $\Longrightarrow H, M, E$ are collinear. Hence if ray MH cuts (O) at Q, then $\angle BQH = 90^\circ \Longrightarrow Q$ is the 2nd intersection of $\odot(BMB_1)$ and $\odot(BA_1C_1)$. Thus, keeping in mind that BE, BB_1 and BM, BF are isogonals WRT $\angle ABC$, we have $\angle EQB_1 = \angle MBB_1 = \angle EBF \Longrightarrow B_1 \in QF$.

Since MA_1, MC_1 are tangents of $\odot(BA_1C_1)$, then M is pole of A_1C_1 WRT $\odot(BA_1C_1) \Longrightarrow B_1(H, Q, T, M) = -1$. But P is pole of AC WRT $(O) \Longrightarrow B_1(B, F, P, M) = -1$. Consequently B_1, T, P are collinear.

comboishard

#3 Jul 3, 2012, 7:09 am • 1

HB_1 TH

Since $HB_1||MP$, it suffices to show that $\frac{HB_1}{MP} = \frac{TH}{MH}$.

From $\angle A_1C_1M = \angle CBB_1 + \angle C_1CM = \angle A$ and similarly $\angle C_1A_1M = \angle A$, we find $\triangle A_1C_1M \sim \triangle CAP$ with ratio $\frac{A_1C_1}{CA}$.

Since A_1CAC_1 is cyclic, we find $\triangle A_1HC_1 \sim \triangle CHA$ with ratio $\frac{A_1C_1}{CA}$.

Now $\frac{TH}{MH} = \frac{[A_1HC_1]}{[A_1MC_1]} = \frac{[CHA]}{[CPA]} = \frac{HB_1}{MP}$, as desired.

Quick Reply

High School Olympiads

Three lines concur

Reply

Source: ELMO Shortlist 2012, G2

math154

#1 Jul 2, 2012, 9:12 am

In triangle ABC, P is a point on altitude AD. Q, R are the feet of the perpendiculars from P to AB, AC, and QP, RP meet BC at S and T respectively. the circumcircles of BQS and CRT meet QR at X, Y.

a) Prove SX, TY, AD are concurrent at a point Z.

b) Prove Z is on QR iff $Z = H$, where H is the orthocenter of ABC.

Ray Li.

Luis González

#2 Jul 2, 2012, 1:35 pm

Since QR is antiparallel to BC WRT AB, AC, we have $AR \cdot AC = AQ \cdot AB \Longrightarrow A$ has equal power WRT the circles $\odot(BQS)$ and $\odot(CRT)$ with diameters $\overline{BS}$ and $\overline{CT} \Longrightarrow AD \perp BC$ is radical axis of $\odot(BQS)$ and $\odot(CRT)$. Since $\angle ACB = \angle AQX = \angle XSB$, then $XS \parallel AC$. Similarly $YT \parallel AB$, thus $XYTS$ is cyclic $\Longrightarrow SX, TY, AD$ are then pairwise radical axes of $\odot(XYTS), \odot(BQS)$ and $\odot(CRT) \Longrightarrow SX, TY, AD$ concur at their radical center Z.

Assume that $Z \in QR$, i.e. $X \equiv Y \equiv Z$. Then $BZ \perp ZS \parallel AC, CZ \perp ZT \parallel AB$, i.e. Z is orthocenter of $\triangle ABC$.

Dukejukem

#3 Dec 23, 2014, 12:03 am

Outline: We first show that AD is the radical axis of $\odot(BQXS)$ and $\odot(CRYT)$. Afterwards, we prove that the points S, X, Y, T are concyclic. It follows that $\odot(BQXS)$ and $\odot(CRYT)$ intersect twice on AD, and by the Radical Axis Theorem, we finish the first part. For the second part, we show that $Z \in QR \iff BZ \perp AC,\ CZ \perp AB$, which is true iff $Z = H$.

Proof. First, we claim that line AD is the radical axis of $\odot(BQXS)$ and $\odot(CRYT)$. First, remark that since $\angle BQS = \angle CRT = 90^\circ$, it follows that BS and CT are diameters of $\odot(BQXS)$ and $\odot(CRYT)$, respectively. Therefore, the line joining the centers of these circles is the line BC. Hence, the radical axis of these two circles is a line perpendicular to BC. Therefore, if we can show that some point on the line AD has equal power with respect to $\odot(BQXS)$ and $(CRYT)$, we will have shown the desired fact.

We claim that point P has equal power with respect to both circles. By Power of a Point, it suffices to show that $PQ \cdot PS = PR \cdot PT$. By invoking Power of a Point again, this equality is equivalent to quadrilateral $QRST$ being cyclic. Indeed this follows from the cyclic quadrilateral $ARPQ$: We have that

$$\measuredangle RTS = \measuredangle PTD = 90^\circ - \measuredangle DPT = 90^\circ - \measuredangle APR$$

$$= 90^\circ - \measuredangle AQR = \measuredangle RQS.$$

Therefore, $\measuredangle RTS = \measuredangle RQS$, so the points Q, R, S, T are concyclic. Thus, we conclude that AD is the radical axis of $\odot(BQXS)$ and $(CRYT)$. $\blacksquare$

Now, we claim that the points S, X, Y, T are concyclic. To show this, we first prove that $SX \parallel AC$. For these lines to be parallel, it suffices to show that $\measuredangle QRC = \measuredangle QXS$. This follows from cyclic quadrilaterals $BQRC$ and $BQSX$, since

$$\measuredangle QRC = \measuredangle QXS \iff \measuredangle QBC = \measuredangle QBS$$

which is true. Thus, we have that $SX \parallel AC$, and similarly, $TY \parallel AB$. By using these pairs of parallel lines, we have that

$$\measuredangle YXS = \measuredangle QRC = \measuredangle QBC = \measuredangle YTS.$$

Therefore, $\measuredangle YXS = \measuredangle YTS$, which implies that the points S, X, Y, T are concyclic. $\blacksquare$

Now, since AD is the radical axis of $\odot(BQSX)$ and $\odot(CRYT)$, it follows that these two circles intersect twice on the line AD. Call these intersections points U, V. Then, by applying the radical axis theorem to the following three pairs of concyclic points $U, V, T, Y; \quad U, V, S, X; \quad S, X, T, Y$ we deduce that the lines SX, TY, AD concur. This concludes the proof of the first part. $\blacksquare$

For the second part, note that since $X, Y \in QR$, the lines SX, TY, AD concur on the line $QR \iff X = Y = Z$. Let us call this triple intersection point H'. We show that $H' = H$, the orthocenter of $\triangle ABC$.

Recall that the lines SX and AC are parallel. Therefore, since $BX \perp XS$, it follows that $BX \perp AC$. Hence, $BH' \perp AC$. Similarly, we find that $CH' \perp AB$. These two relations hold iff H' is the orthocenter of $\triangle ABC$. This concludes the proof of the second part. $\blacksquare$.

TelvCohl

#4 Dec 23, 2014, 11:48 am • 2

My solution:

Since QR is anti-parallel to BC WRT $\angle BAC$,
so we get B, C, Q, R are concyclic and $AB \cdot AQ = AC \cdot AR$ . ... $(\star)$
Since the center of $\odot(BQS), \odot(CRT)$ is the midpoint of BS, CT ,
so combine with $(\star)$ we get AD is the radical axis of $\{\odot(BQS), \odot(CRT)\}$.

From Reim theorem we get $XS \parallel AC$ and $YT \parallel AB$,
so we get X, Y, T, S are concyclic ,
hence SX, TY, AD are concurrent at the radical center of $\{\odot(BQS), \odot(CRT), \odot(XYTS)\}$.

Since P lie on the radical axis AD of $\{\odot(BQS), \odot(CRT)\}$,
so we get $PT \cdot PR = PS \cdot PQ$. ie. Q, R, S, T are concyclic
From Reim theorem we get $BX \parallel TR$ and $CY \parallel SQ$,
so $Z \in QR \Longleftrightarrow X \equiv Y \equiv Z \Longleftrightarrow BZ \perp AC, CZ \perp AB \Longleftrightarrow Z \equiv H$.

Q.E.D

Quick Reply

High School Olympiads

Parallel lines and angle congruences

Reply

Source: ELMO Shortlist 2012, G5; also ELMO #5

math154

#1 Jul 2, 2012, 9:14 am • 2

Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $PD \parallel AE$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.

Calvin Deng.

Luis González

#2 Jul 2, 2012, 11:49 am • 3

We use barycentrics WRT $\triangle ABC$. $D(0:1:k), E(0:k:1)$ for some k. Infinity point of the line AE is $A_{\infty}(k+1:-k:-1)$. Equation of line passing through D and A_{∞} is then $\ell_A \equiv (k-1)x+ky-z=0.$

Isogonal $\lambda_A \equiv kc^2y-b^2z=0$ of AE WRT $\angle BAC$ cuts ℓ_A at

$P(k(c^2-b^2):(k-1)b^2:k(k-1)c^2)$

Eliminating k and setting $(x:y:z) \to \left(\frac{a^2}{x}:\frac{b^2}{y}:\frac{c^2}{z}\right)$ gives the equation of the perpendicular bisector of $\overline{BC}$, namely $\tau_A \equiv (c^2-b^2)x+a^2(z-y)=0 \Longrightarrow$ Isogonal conjugate of P lies on $\tau_A \Longrightarrow \angle PBA = \angle PCA$.

simplependulum

#3 Jul 2, 2012, 2:31 pm • 2

Let F be a point in the plane such that $ABFC$ is parallelogram .
Then using $BD = CE$, we have $DF||AE$ or DF passes through P .
Let h_1, h_2, h_3, h_4 be the distances of P from lines AB, BF, FC, CA respectively . From $\angle BAP = \angle CAE = \angle BFD$ and $\angle BAC = \angle BFC$, we have $h_1 : h_4 = h_2 : h_3$, which implies $h_1 : h_2 = h_4 : h_3$. Since $\angle ABF = \angle ACF$, we have $\angle ABP = \angle ACP$ as desired .

apluscactus

#4 Jul 3, 2012, 1:45 am • 2

Sort, but what's ELMO?

And can you tell me if the following scheme of solution is right :

There is a unique point P on a secant AQ (when it's different of the angle bissector of an isosceles triangle) of a triangle ABC such that angle ABP=angle PCA
So we can reconstruct the exercise : let E be on BC with angle EAC=x. Let Q be on BC with angle QAB =x. Let P be thé unique point on AQ such that angle PBA =angle PCA= y. Let D be on BC such that PD is parallel to AE. We havé to show that BD=CE.
Let thé intersection of PC and AR be R. By angle Chasing and Law of sines we have
BD = BP sin(BAC - x + y)/sin(ACB + x)
And CE = CR sin (x+y) /sin (ACB+x)

But BPA is similar to CRA so
BP / CR= AP/AR =sin(x+y) / sin (BAC-x+y)

Hence the result....

Is this right ?

apluscactus
#5 Jul 3, 2012, 2:01 am

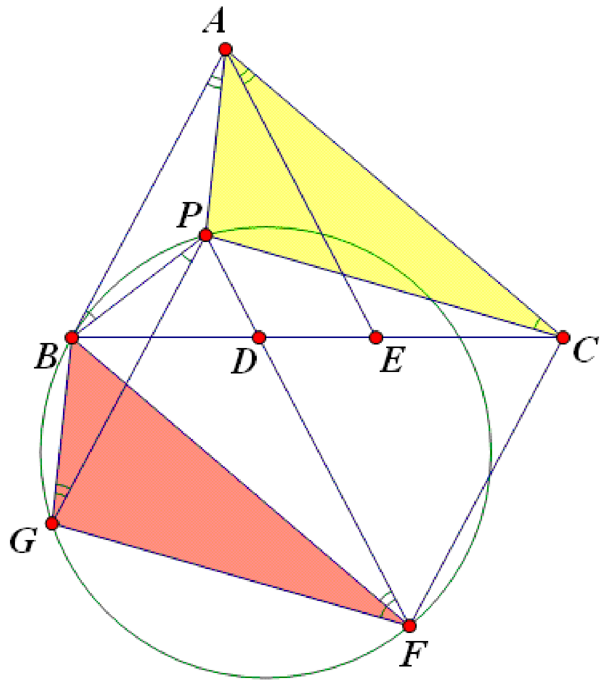
> apluscactus wrote:
> Sorry, but what's ELMO ?

I 've now seen the description of thé compétition...

yunxiu
#6 Jul 4, 2012, 8:26 pm • 7

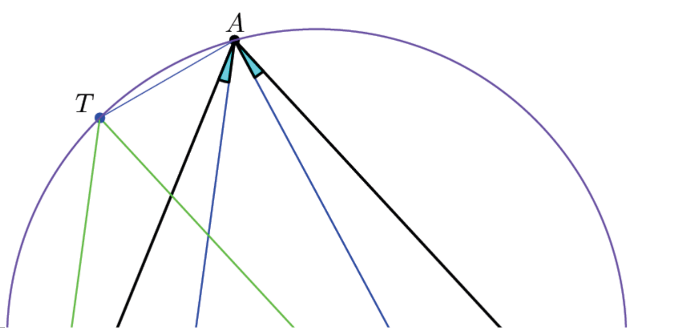
Let $ABFC$ and $ABGP$ are parallelograms. Because $\Delta AEC \cong \Delta FDB$,
$\angle BFP = \angle BFD = \angle CAE = \angle BAP = \angle BGP$, so $BGFP$ are concyclic. Since $\Delta APC \cong \Delta BGF$, we have $\angle ABP = \angle BPG = \angle BFG = \angle ACP$.

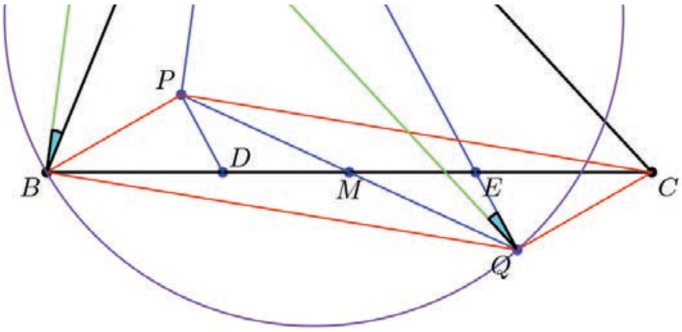
Attachments:



prime04
#7 Jul 6, 2012, 6:54 pm • 2

Let $\angle PAB = \angle EAC = \theta$.
Let M be the midpoint of BC and Q be a point such that $BPCQ$ is a parallelogram.
Then $MD = ME$ and $MP = MQ \implies PDQE$ is a parallelogram $\implies PD \parallel QE \implies A, E, Q$ are collinear.
Now, choose T such that $APBT$ is a parallelogram. Observe that $ACQT$ is also a parallelogram.
Hence, $\angle PAB = \angle ABT = \theta = \angle QAC = \angle AQT \implies ATBQ$ is cyclic $\implies BQT = \angle BAT$.
Bu then $\angle BQT = \angle PCA$ and $\angle BAT = \angle PBA$. Hence $\angle PBA = \angle PCA$ $\square$





math154

#8 Feb 6, 2013, 11:14 pm • 1

Since nobody has posted this yet, here's why I think Luis González's and apluscactus's approaches work so well:

We want to show that P, Q are isogonal conjugates, where Q is the intersection of AE and the perpendicular bisector ℓ of BC. Let Q' be the isogonal conjugate of Q, and suppose the in-ellipse of $\triangle ABC$ with foci Q, Q' touches BC, CA, AB at X, Y, Z, respectively. Observe that X is just the point along BC that minimizes $QX + Q'X$, or equivalently, the point on BC collinear with Q' and the reflection of Q over BC.

First note that if P, Q were isogonal conjugates, we would have $\angle PDB = \angle QED = \angle QDE$, so D would equal X. (This is not strictly necessary but it helps motivate what follows.)

Since $Q \in \ell$, we have $\angle CQE = \angle BQD$. By this property of ellipses, $\angle BQX = \angle CQE$, so $D = X$. But then $\angle Q'DB = \angle QDC = \angle QEB$, so $Q'D \parallel QE$ implies $P = Q'$, as desired.

(Alternatively, define $D' \in BC$ such that $Q'D' \parallel AE$; then $\angle AQ'B + \angle D'Q'C = 180^\circ$, so $D' = X$, and then $\angle QD'E = \angle Q'D'B = \angle QED'$ finishes the proof.)

Of course, a lot of this can be phrased without ellipses, but this is just a "higher-level" perspective.

sjaelee

#9 Jun 9, 2013, 8:21 am • 1

Let $AP \cap BC = M$, $X = PC \cap AE$, and P' the point on AM such that $\triangle BP'A \sim \triangle CXA$. Let D' be the point on BC such that $P'D'||AE$. Note that $\angle AXP' = \angle D'P'X = \angle BP'M$ from the above information. $\angle BP'M = \angle D'P'C$ implies that $\angle BP'D' = \angle MP'C$.

We will first prove that $\dfrac{AC}{AB} = \dfrac{BD'}{PD'} \cdot \dfrac{P'M}{BM}$.

By the law of sines, $\dfrac{BD}{PD} = \dfrac{\sin\angle BP'D}{\sin\angle P'BD'}$ and $\dfrac{PM}{BM} = \dfrac{\sin\angle P'BD'}{\sin\angle BP'M}$. As $\angle BP'D' = \angle MP'C$ and $\angle BP'M = \angle D'P'C$, we have

$$\frac{BD}{PD}\cdot\frac{PM}{BM} = \frac{\sin\angle MP'C}{\sin\angle P'BD'}\cdot\frac{\sin\angle P'BD'}{\sin\angle CP'D'} = \frac{\sin\angle MP'C}{\sin\angle D'P'C} = \frac{\sin\angle XP'A}{\sin\angle P'XA} = \frac{AX}{AP'} = \frac{AC}{AB}$$

The last steps come from the similar triangles and parallel lines.

Then using the law of sines on triangles BAM and ACE with $P'D'||AE$

$$\frac{CE}{BM} = \frac{AC}{AB}\cdot\frac{AE}{AM} = \frac{AC}{AB}\cdot\frac{P'D'}{P'M} = \frac{BD'}{PD'}\cdot\frac{P'M}{BM}\cdot\frac{P'D'}{P'M} = \frac{BD'}{BM}$$

Therefore $CM = BD'$. But then $D = D'$ and $P = P'$, and we have the desired equality.

junioragd

juniorag

#10 Nov 20, 2014, 12:52 am • 1

Here is my approach:Let R be the intersection point of BP and AE and S be the intersection point of AE and CP.Now,from Thales theorem and using $BD=CE$ we get $BP/PR=CS/PS$ and since angles SAC and PAB are equal,it is obvious that ABR is similar to APC,so we are finished.

MillenniumFalcon

#11 Jun 26, 2015, 2:48 pm

I tried drawing a line through E,P such that AP is the symmedian, but couldn't do it.

theflowerking

#12 Jun 26, 2015, 9:12 pm

The problem can be solved by reflecting! (Let $R(\triangle)$ denote the circumradius of a triangle).

(I did NOT use trig bash, the sines are only used for technicalities).
Let M be the midpoint of BC and P', A' the reflections of P, A across M. Notice $AE||PD||P'D$ so AEP' are collinear. Similarly $A'DP$ are collinear. Let P'' be the reflection of P across the midpoint of AC. Then $P'A', AP, CP''$ are equal and parallel segments and so

$$\angle CP''P=\angle CA'P'=\angle BAP=\angle CAP',$$

so P'', C, P', A form a cyclic quadrilateral. Thus, $R(AP'C)=R(AP''C)=R(APC)$.

However,

$$R(ABP)=R(A'P'C)=\frac{P'C}{2\sin(\angle P'A'C)}=\frac{P'C}{2\sin(\angle P'AC)}=R(P'AC)=R(APC),$$

which implies

$$\frac{AP}{\sin(\angle ABP)}=\frac{AP}{\sin(\angle ACP)}.$$

So $\angle ABP=\angle ACP$ or $180-\angle ACP$. However, the second choice would imply $APBC$ is cyclic, but it isn't even convex! Therefore, the first choice is correct, so $\boxed{\angle ABP=\angle ACP}$.

(originally posted at: http://artofproblemsolving.com/community/c6h1106437_equiangular_line).

This post has been edited 1 time. Last edited by theflowerking, Jun 26, 2015, 9:12 pm

anantmudgal09

#13 Aug 22, 2015, 2:40 am • 1

An easy synthetic solution. Note that CE/CD =CE/BE=$[QCA]/[QAB]$=CQ/CP=$[QCA]/[PCA]$ this implies $[QAB]$ =$[PCA]$.This implies $AP.AC.SinCAP$ =$AQ.AB.SinBAQ$. Since $\angle CAP$ =$\angle BAQ$ we have $\triangle APB$ similar to $\triangle AQC$.Done.(here Q is the intersection of AE and CP.)

EulerMacaroni

#14 Sep 8, 2015, 7:31 am

Let A_1' be the point such that $BA_1'P$ is a parallelogram and let A' be the point such that $ABA'C$ is a parallelogram. First we claim that $BPA_1'A'$ is cyclic. This is pretty easy angle chasing; note that $\angle BA_1'P=\angle BAP$ and $\angle BA'P=\angle BA'D=180^\circ-\angle A'BD-\angle BDA'=180^\circ-\angle C-\angle AEC=\angle CAE$ so by isogonals, we get the cyclicity. Now, $\angle PBA=\angle BPA_1'$, and since $\triangle BA_1'A'\sim\triangle APC$, $\angle PCA=\angle A_1'A'B=\angle A_1'PB$ and we're done.

Dukejukem

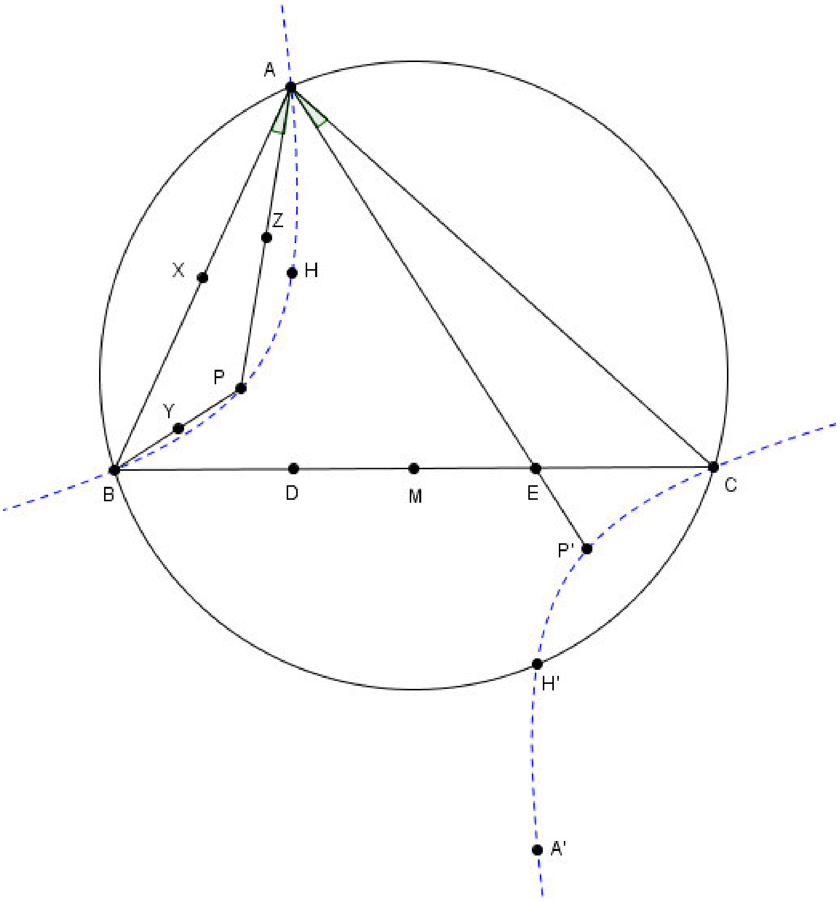
#15 Sep 29, 2015, 5:42 am • 1

Lemma: Let B, C, O be three fixed points in the plane. Then the locus of points A for which O lies on the nine-point circle of $\triangle ABC$ is the rectangular hyperbola $\mathcal{C}$ with center O passing through B, C.

Proof: Let H be the orthocenter of $\triangle ABC$ and let D be the reflection of H in O. Since H is the center of positive homothety that sends the nine-point circle of $\triangle ABC$ to the circumcircle, we deduce that $D \in \odot(ABC)$. Therefore, it is well-known that the midpoint O of $\overline{HD}$ is the center of the rectangular hyperbola passing through A, B, C, D. Hence, $A \in \mathcal{C}$, as desired. $\blacksquare$

Now, let τ be the perpendicular bisector of $\overline{BC}$, let M be the midpoint of $\overline{BC}$, let H be the orthocenter of $\triangle ABC$ and let A', P', H' be the reflections of A, P, H in M. We will show that M lies on the nine-point circle of $\triangle APB$.

Note that $DPEP'$ is a parallelogram because its diagonals bisect one another. Hence, $DP \parallel EP' \implies A, E, P'$ are collinear. Therefore, $\measuredangle BAP = \measuredangle P'AC = \measuredangle PA'B$, where the last step follows from the homothety $\mathcal{H}(M, -1)$. Thus, if X, Y, Z are the midpoints of $\overline{AB}, \overline{BP}, \overline{PA}$, the homothety $\mathcal{H}\left(A, \frac{1}{2}\right)$ implies that $\measuredangle XAZ = \measuredangle ZMX$. But recalling that $AXYZ$ is a parallelogram, it follows that $\measuredangle ZYX = \measuredangle ZMX$, implying that M lies on the nine-point circle of $\triangle APB$.



Thus by the lemma, we deduce that P lies on the rectangular hyperbola $\mathcal{C}$ with center M passing through A, B. By symmetry in M, we see that $C \in \mathcal{C}$, and therefore $\mathcal{C}$ is a circumhyperbola of $\triangle ABC$. Then since $\mathcal{C}$ is rectangular, it follows that $H \in \mathcal{C}$, and by symmetry in M, we deduce $H' \in \mathcal{C}$. But it is well-known that H' is the antipode of A WRT $\odot(ABC)$. Therefore, the fourth intersection H' of $\mathcal{C}$ with $\odot(ABC)$ is the isogonal conjugate of the point at infinity on τ. Hence, $\mathcal{C}$ is the isogonal conjugate of τ. Thus, if Q is the isogonal conjugate of P, we have
$Q \in \tau \implies \angle QBC = \angle QCB \implies \angle PBA = \angle PCA$ as desired. $\square$

This post has been edited 6 times. Last edited by Dukejukem, Sep 29, 2015, 6:02 am

Quick Reply

High School Olympiads

Two circles concur on a line

Reply

Source: ELMO Shortlist 2012, G1; also ELMO #1

math154

#1 Jul 2, 2012, 9:10 am • 2

In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.

Ray Li.

Luis González

#2 Jul 2, 2012, 9:56 am • 2

M is the midpoint of $\overline{BC}$ and H is the orthocenter of $\triangle ABC$. Since the circle with diameter $\overline{BC}$ is orthogonal to the circle ω through A, E, F, H, then ME, MF are tangents of ω. Let $Q \equiv EF \cap BC$. Since $(B, C, D, Q) = -1$, we have $MB^2 = ME^2 = MF^2 = MD \cdot MQ \Longrightarrow \odot(DEQ)$ and $\odot(DFQ)$ are tangent to ω at $E, F \Longrightarrow \odot(DEQ)$ and $\odot(DFQ)$ coincide with ω_1 and $\omega_2 \Longrightarrow P \equiv Q$.

WakeUp

#3 Jul 2, 2012, 10:53 pm

Let O be the centre of ω, on which the orthocentre H clearly lies diametrically opposite to A. An angle chase shows that O, E, D, F are concyclic: $\angle EOF = 2\angle EAF = 2A$ and of course $\angle FDE = 180^\circ - 2A$.

Invert with respect to ω (we now know D' lies on EF).

The problem becomes:

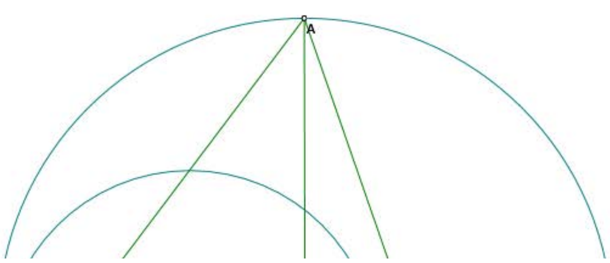
In the triangle AEF with circumcentre O and circumcircle ω, the diameter AH meets EF at D'. The circle tangent to ω at E and passing through D' and the circle tangent to ω at F passing through D' intersect at D' and P. Prove that $OP \perp PD'$.

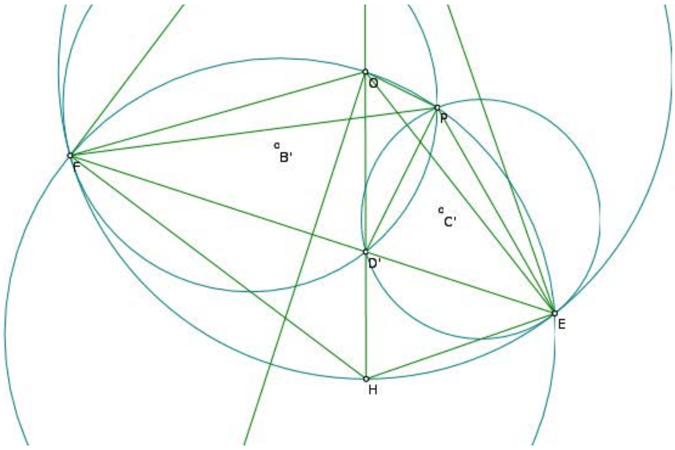
Note that we can actually ignore points B' and C', who also lie on the circle with diameter OD' along with P, so we've only really inverted D, ω_1, ω_2...

Anyway:

Consider the tangent ℓ to ω and $(OD'E)$ at E. By considering the angle ED' makes with this tangent, we get, by the alternate segment theorem that this angle is both equal to $\angle D'PE$ and $\angle FAE$. Hence $\angle D'PE = \angle FAE$ and similarly we get that $\angle FPD' = \angle FAE$. Therefore $\angle FPE = \angle FPD' + \angle D'PE = 2\angle FAE = \angle FOE$ so $EPOF$ is cyclic. Now consider the diameter OO' of (OEF). Since $OF = OE$, O' is the midpoint of the arc EF not containing O. Now, since $\angle FPD' = \angle EPD'$, we have that PD' is the bisector of $\angle FPE$ and so PD' passes through O'. Hence $\angle OPD' = \angle OPO' = 90^\circ$ and the result follows.

Attachments:





r1234

#4 Jul 4, 2012, 5:05 pm

First inverting the figure wrt H with power $HA.HD$ we get the following problem

In $\triangle ABC$, two circles ω_1, ω_2 pass through A and touch the line BC at B, C respectively. Show that the intersection point of these two circles lies on $\odot AEF$. Now inverting this figure again wrt A with power $AH.AD$ we get the following problem:

In $\triangle ABC$, two tangents are drawn at the points E, F to the circle $\odot AEF$, then these two tangent lines intersect on BC, and this is obvious. They meet on at the midpoint of BC.

Bigwood

#5 Jul 7, 2012, 7:18 am

Let H be the orthocenter and M be midpoint of BC, and second intersection point of ω_1, ω_2 be Q. $\angle AEH = \angle AFH = \frac{\pi}{2}$ implies H is on ω. Note that B, F, E, C lie on the circle with diameter BC.
Then $\angle HEM = \angle BEM = \angle EBM = \angle EAD = \angle EAH$, which means ME tangents ω. Similarly so does MF.
Then we can conclude that radical axis of $(\omega, \omega_1), (\omega, \omega_2), (\omega_1, \omega_2)$ is EM, FM, DQ. thus M, D, Q is collinear, which is equivalent to the claim.

polya78

#6 Apr 11, 2013, 11:46 pm

The radical axes of the three circles are DQ and the tangents to w at E, F. But those two tangents meet at M, the midpoint of BC, because M and the center of w are antipodal on the nine-point circle.

yugrey

#7 Apr 15, 2013, 4:03 am

My solution is more or less the same as the last couple; if you let $PB = PC$ on BC, showing that P is the intersection of the tangents to the circle at E and F is straightforward, and then PD, the same line as BC, must be the radical axis so if the circles meet again they must meet on BC.

This has one slight flaw, but that flaw is actually inherent in the problem statement. If $P = D$, then $AB = AC$ and the two circles w_1 and w_2 are tangent, so they don't meet on BC. If the problem were reworded to "If w_1 and w_2 meet at a point P other than D, show that point is on BC" or if it was stated that the triangle was not isosceles, this would have been avoided.

leader

#8 Apr 21, 2013, 7:58 pm

let AD be the diameter of circle ABC and let DH meet circle ABC again at R. now let AR meet BC at K.since $\angle HRA = 90 = \angle HFA = \angle HEA$ R is on ω. since $\angle CRH = \angle CAD = 90\angle CDA = 90 - \angle CBA = \angle BCH$ circle CHR touches BC. Similarly circle BHR touches BC. Consider inversion of pole A and power $AE * AC$. it pictures ω into BC so R into K. it pictures H, E, F into D, C, B so ω touches circles EDK and FDK therefore $K = P$ so P is on BC.

giratina150

#9 Apr 24, 2013, 10:18 am

solution

thecmd999

#10 May 7, 2013, 3:56 am

Nice and quick :)

Solution

anantmudgal09

#11 Jun 5, 2015, 3:02 am

Let EF meet BC at X. Now it is some easy angle chase that X is the desired point.

steppewolf

#12 Jan 29, 2016, 3:02 am

First of all, it's easy to see that $AEHF$ is cyclic.
Let M be the pole of EF wrt the circumcircle of $\triangle AEF$. Let S be the point of intersection of the lines EF and AH.
Because $AEHF$ is cyclic, the polar of S is the line BC, so M lies on BC because S lies on the polar of M. Now if circle ω_1 intersects BC at P, and circle ω_2 intersects BC at Q, by power of point we have

$$MD*MP=MF^2=ME^2=MD*MQ$$

So $MP=MQ$, which means that $P=Q$;

Quick Reply

High School Olympiads

Difficult Isosceles Traingle Problem

Reply

Source: me

Ichserious

#1 Jun 28, 2012, 12:34 am

ABC is an isosceles triangle with $AB = AC$. D is a point on BC, and E, F are two points between A, D such that $\angle BEC = \frac{3}{2}\angle BAC$ and $\angle BFC = 180^\circ - \frac{1}{2}\angle BAC$. Let G be the reflection of A with respect to CE. Prove that $BG = GF$.

This post has been edited 1 time. Last edited by Ichserious, Jul 1, 2012, 1:16 am

xeroxia

#2 Jun 29, 2012, 2:47 pm

sketchpad says they are not equal

xeroxia

#3 Jun 29, 2012, 3:10 pm

When $\triangle ABC$ is equilateral, we get $GF = BF$.
This form of the problem can be a separate question.

Also first impressions say $GF = BF \Leftrightarrow AB = AC = BC$. If it will be proved, it can be a good problem.

xeroxia

#4 Jun 29, 2012, 5:22 pm

I will prove $AB = AC = BC \Rightarrow GF = BF$.

Obviously $\angle BEC = 90^\circ$
Let $CE \cap AG = \{H\}$ and we have $\angle CHA = 90^\circ$ by axis of symmetry.

So $BE \parallel AG$. And it is obvious G is a a point on circle $(C, |AC|)$. So $\angle AGB = 150^\circ$. $\angle GBE = 30^\circ$ because $BE \parallel AG$.
HE is the altitude of right-angled trapezoid and it is $HE = \frac{GB}{2}$.

Let's copy-paste $\triangle GAB$ to outer side of AC such that $\triangle GAB \cong \triangle G'CA$.
Now we have $\angle G'AH = 90^\circ$. $HAG'C$ is right-angled trapezoid. Let H' be the feet of perpendicular that is drawn from G' to CH. $HAG'H'$ is a rectangle. $HH' = AG' = GB \Rightarrow HE = EH'$.
Let $G'H'$ cut AE at F'. Because $HE = EH'$ and $G'H' \parallel AH$, we have $AG = H'F' = G'H'$. And that value is $G'H' = H'F' = \frac{AG}{2} = \frac{G'C}{2}$. Thus $\triangle CG'F'$ is isosceles. It is not only isosceles, it is equilateral because of $G'F' = G'C = AG$.

$\angle ACG' = \angle GAB = \frac{\angle GAC}{2}$. Since $\triangle G'F'C$ is equilateral, $\angle GCF' = \angle ACG'$, and then CF is the bisector of GCB.
We have $\angle GCF' = \angle F'CB = \angle GAB$. And we have $GA = CF'$, and $AB = AG = CB$.
So we have $\triangle GAB \cong \triangle F'CG \cong \triangle F'CB$.
This implies $GB = GF' = BF'$ and $\angle BGA = \angle GF'C = \angle BF'C = 150^\circ$.
There is only one point on AD such that $\angle BFC = 180^\circ - \frac{60^\circ}{2} = 150^\circ$. This implies $F = F'$.

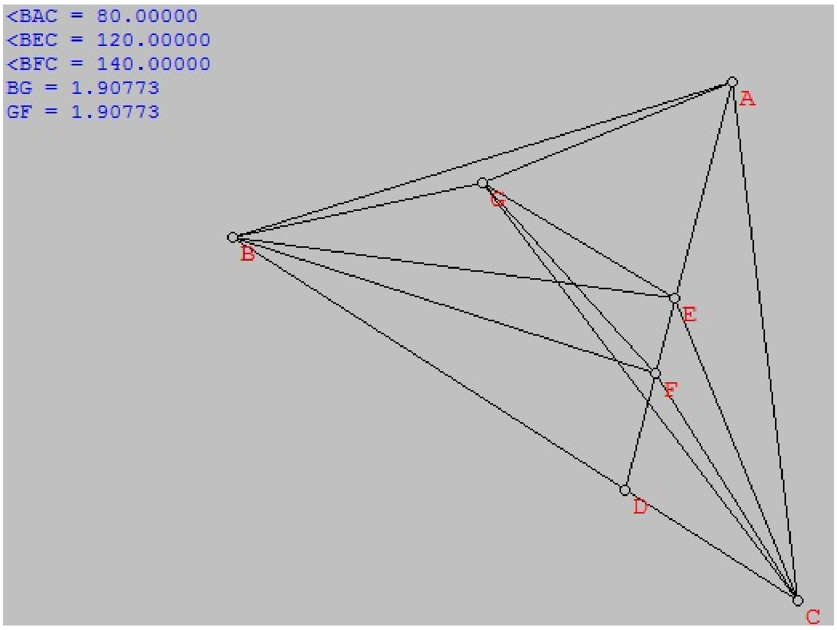
Ichserious

#5 Jun 30, 2012, 10:36 am

> *xeroxia wrote:*
> sketchpad says they are not equal

Are you sure? But my diagram looks perfectly well.

Attachments:



xeroxia

#6 Jun 30, 2012, 2:59 pm

> *Ichserious wrote:*
> Are you sure? But my diagram looks perfectly well.

You stated $BF = GF$ not $BG = GF$.

xeroxia

#7 Jun 30, 2012, 5:28 pm

I will prove $BG = GF$.

Let $\angle BAC = 2\theta$. So $\angle BEC = 3\theta, \angle BFC = 180^\circ - \theta$
$\angle FBC + \angle FCB = \angle ABE + ACE = \theta$ **(..1..)**.

First I will prove $\angle ECA = \angle FBC$.

Let E' be a point such that $\angle ACE' = \angle FBC$. So we have $\angle E'CB = \angle ABF$ because $AB = AC$.
From trigonometric form of Ceva Theorem $\dfrac{\sin\angle E'CA}{\sin\angle E'CB} \cdot \dfrac{\sin\angle E'BC}{\sin\angle E'BA} = \dfrac{\sin\angle FCA}{\sin\angle FCB} \cdot \dfrac{\sin\angle FBC}{\sin\angle FBA}$.

We will get $\angle E'BA = \angle FCB$. Then we have $\angle BE'C = \angle BEC$ from **(..1..)**.

So $\angle ECA = \angle FBC$.

Let's call $\angle FBC = \angle ECA = \angle ECG = \alpha$ and then $\angle FCB = \angle ABE = \theta - \alpha$.
By simple angle arithmetic $\angle FCG = 90^\circ - 2\theta - \alpha$.
Again, by simple angle arithmetic $\angle GAB = 90^\circ - 2\theta - \alpha$.
Look at the triangles $\triangle GAB$ and $\triangle FCG$. We have $CG = AB$, and $\angle GAB = \angle FCG$. If we have $FC = AG$, then these triangles will be equivalent. So this will imply $GF = GB$.
So we will prove the lemma $FC = AG$. **(..2..)**

$AC = \dfrac{AH}{\sin\alpha}$. From Sine Law, $BC = \dfrac{AC}{\sin(90^\circ - \theta)} \cdot \sin 2\theta = \dfrac{2AH\sin\theta}{\sin\alpha}$. **(..3..)**

Now from Sine Law at $\triangle BFC$, $\dfrac{BC}{\sin\theta} = \dfrac{FC}{\sin\alpha}$. **(..4..)**

Merge **(..3..)** and **(..4..)**, then we will get $FC = 2\cdot AH = AG$.
We already know $AB = GC$ and $\angle GAB = \angle GCF$. So we have $\triangle GAB \cong \triangle FCG$. This implies $GF = GB$.

---

xeroxia
#8 Jun 30, 2012, 6:34 pm

> *Ichserious wrote:*
> ABC is an isosceles triangle with $AB = AC$. D is a point on BC, and E, F are two points between A, D such that $\angle BEC = \dfrac{3}{2}\angle BAC$ and $\angle BFC = 180^\circ - \dfrac{1}{2}\angle BAC$. Let G be the reflection of A with respect to CE. Prove that $GF = BF$.

In fact, it should be $BG = GF$.
In my previous post, I have proved it.

Let's change Ichserious's problem:

ABC is an isosceles triangle with $AB = AC$. D is a point on BC, and E, F are two points between A, D such that $\angle BEC = \dfrac{3}{2}\angle BAC$ and $\angle BFC = 180^\circ - \dfrac{1}{2}\angle BAC$. Let G be the reflection of A with respect to CE. Prove that $AB = AC = BC \Leftrightarrow GF = BF$.

Solution

Ichserious
#9 Jul 1, 2012, 1:17 am

> *xeroxia wrote:*
> You stated $BF = GF$ not $BG = GF$.

Sorry about that! I have edited it.

---

yetti
#10 Jul 2, 2012, 6:39 am

Label $AB = AC = r.$ Circles $(B, r), (C, r)$ meet at A and again at the reflection A' of A in $BC.$ Label $BC = 2a$ and $AA' = 2h \Longrightarrow r^2 = h^2 + a^2.$
$\angle BFC = \pi - \frac{1}{2}\widehat{A} \Longrightarrow F$ is on minor arc $\overset{\frown}{BC}$ of circle $(A', r).$ Label $\angle AA'F = \phi.$

Perpendicular bisectors p_B, p_C of the segments $|BF|, |CF|$ cut the circles $(B), (C)$ at A' and again at $G, H,$ resp. By default, $|BG| = |FG|$ and $|CH| = |FH|.$
Perpendicular bisectors q_B, q_C of the segments $|AH|, |AG|$ through $B, C,$ resp., meet at circumcenter E of $\triangle AGH.$ By default, G, H are reflections of A in the lines $BE, CE.$

$\triangle A'CG, \triangle A'BH$ are C-, B-isosceles, resp., and $\angle CA'G + \angle BA'H = \angle BA'C + GA'H = \frac{3}{2}\widehat{A} \Longrightarrow$
$\angle GCB + \angle HBC = 2\pi - 3\widehat{A} - (\pi - \widehat{A}) = \pi - 2\widehat{A} \Longrightarrow \angle GCA + \angle HBA = \pi - \widehat{A} - (\pi - 2\widehat{A}) = \widehat{A} \Longrightarrow$
$\angle BEC = \pi - (\angle GCB + \angle HBC) - \frac{1}{2}(\angle GCA + \angle HBA) = 2\widehat{A} - \frac{1}{2}\widehat{A} = \frac{3}{2}\widehat{A}.$
All we have to do is to show that points A, E, F are collinear or $E \in AF.$

Put coordinate origin at the midpoint of $|BC|$ and positive x-axis along the ray $(BC \Longrightarrow$
$A = (0, h), B = (-a, 0), C = (a, 0), A' = (0, -h), F = (r\sin\phi, r\cos\phi - h).$

Circles $(B),(C)$ have equations

$$\begin{array}{llll}(B): & (x+a)^2+y^2=r^2 & \Longrightarrow & x^2+2ax+y^2=h^2, \\ (C): & (x-a)^2+y^2=r^2 & \Longrightarrow & x^2-2ax+y^2=h^2.\end{array}$$

Perpendicular bisectors $p_B \equiv A'G, p_C \equiv A'H$ of $|BF|, |CF|$ have equations

$$p_B: \quad y+h=\frac{r\cos\phi+h}{r\sin\phi-a}\cdot x=mx,$$

$$p_C: \quad y+h=\frac{r\cos\phi+h}{r\sin\phi+a}\cdot x=nx.$$

They meet the circles $(C),(B)$ at A' and again at $G\in p_B\cap(C), H\in p_C\cap(B)$, resp.
Solving the 2 sets of equations, one root $A'=(0,-h)$ of each set being known,

$$G=(\frac{2(hm+a)}{m^2+1},\ \frac{h(m^2-1)+2am}{m^2+1}),\ H=(\frac{2(hn-a)}{n^2+1},\ \frac{h(n^2-1)-2an}{n^2+1}).$$

This leads to equations of perpendicular bisectors $q_B\equiv BE, q_C\equiv CE$ of the segments $|AH|,|AG|$, resp:

$$q_B: \quad y=\frac{hn-a}{h+an}\cdot(x+a),$$

$$q_C: \quad y=\frac{hm+a}{h-am}\cdot(x-a).$$

Solving the 2 equations yields coordinates of the circumcenter $E\equiv q_B\cap q_C$ of $\triangle AGH$:

$$E=(\frac{ar^2(m+n)}{(h^2-a^2)(m-n)+2ha(mn+1)},\ h-\frac{hr^2(m-n)+2ar^2}{(h^2-a^2)(m-n)+2ha(mn+1)})$$

Substituting back for

$$m+n=\frac{r\cos\phi+h}{r\sin\phi-a}+\frac{r\cos\phi+h}{r\sin\phi+a}=2r\sin\phi\cdot\frac{r\cos\phi+h}{r^2\sin^2-a^2}=\frac{2r\sin\phi}{h-r\cos\phi},$$

$$m-n=\frac{r\cos\phi+h}{r\sin\phi-a}-\frac{r\cos\phi+h}{r\sin\phi+a}=2a\cdot\frac{r\cos\phi+h}{r^2\sin^2-a^2}=\frac{2a}{h-r\cos\phi},$$

$$mn+1=\frac{r\cos\phi+h}{r\sin\phi-a}\cdot\frac{r\cos\phi+h}{r\sin\phi+a}+1=2h\cdot\frac{r\cos\phi+h}{r^2\sin^2-a^2}=\frac{2h}{h-r\cos\phi},$$

$$E=(r\sin\phi\cdot\frac{r^2}{4h^2-r^2},\ h+(r\cos\phi-2h)\cdot\frac{r^2}{4h^2-r^2})$$

Equation of the line AF is

$$AF: \quad y-h=\frac{r\cos\phi-2h}{r\sin\phi}\cdot x.$$

Substituting coordinates of E into this equation leads to a trivial identity:

$$h+(r\cos\phi-2h)\cdot\frac{r^2}{4h^2-r^2}-h=\frac{r\cos\phi-2h}{r\sin\phi}\cdot r\sin\phi\cdot\frac{r^2}{4h^2-r^2},$$

which means that the circumcenter E of $\triangle AGH$ is on the line $AF.\square$

Luis González

#11 Jul 2, 2012, 8:33 am

O is the circumcenter of $\triangle ABC$ and P is the reflection of A about $BC. BO, CO$ cut AC, AB at $Y, Z.$ By the given conditions, E lies on the circumcircle ω of the isosceles trapezoid $BCYZ$ and F lies on the circle (P) with center P and radius $PB=PC. AP$ cuts ω again at K, N (K is between A and N). Since $(K,N,O,A)=-1$ and $\angle KEN=90^\circ$, then EN bisects both $\angle FEO$ and $\angle BEC \Longrightarrow EO, EF$ are isogonals WRT $\angle BEC.$ But
$\angle BEC+\angle BOC=\pi-\frac{1}{2}\angle BAC+2\angle BAC=\pi+\angle BEC \Longrightarrow O, F$ are isogonal conjugates WRT $\triangle BEC \Longrightarrow$

BF, BO are isogonals WRT $\angle EBC$. Thus if S, L are the projections of A on CE, BC, we have

$$\angle FBC = \angle OBE \equiv \angle YBE = \angle ECA \equiv \angle SCA = \angle SLA \Longrightarrow LS \perp BF$$

Therefore, the line connecting the reflections of A about L, S is also perpendicular to $BF \Longrightarrow GP$ is the perpendicular bisector of $\overline{BF}$, i.e. $BG = GF$.

Invader_2011

#12 Jul 8, 2012, 7:17 pm

Another synthetic approach:

Let the perpendicular from F to BC intersect the circle centered at A with radius AB at P.

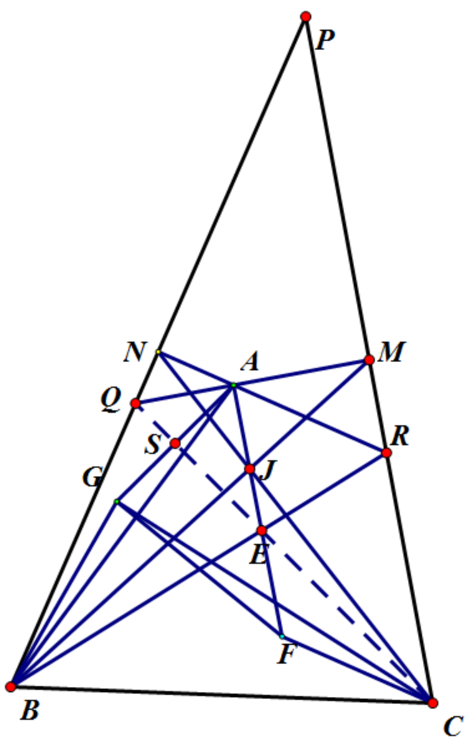
Then A is the circumcenter of ΔPBC. So $\angle BFC = 180^\circ - \frac{1}{2}\angle BAC = 180^\circ - \angle BPC$. So B, F, H, C are concyclic, where H is the orthocenter of ΔPBC. Since $PF \perp BC$, and the intersection point of line PF and the arc BHC is unique, we have $F = H$.

Let the midpoints of PB, PC, be N, M, respectively. Let lines MA, NA, intersect PB, PC, at Q, R, respectively. Let BR, CQ intersect at E'. Then by Pascal's theorem on $NRBMQC$ (or Pappus' Theorem), we have that A, J, E' are collinear, where J is the centroid of ΔPBC. Thus E' is on the Euler Line of ΔPBC, or line AJF.

Since $PR = RB$, $PQ = QC$, we have $\angle BE'C = \angle BQC + \angle PBR = 3\angle P = \frac{3}{2}\angle BAC$. Similar to above we have $E' = E$.

Since $PM = MC$ and $PQ = QC$, we obtain that QM bisects $\angle PQC$. Let $AG \perp BQ$ at S, then $AN \perp PB \Rightarrow AG = 2AS = 2AN$. Also it is a well known fact that $CF = 2AN$, so $CF = AG$. By angle chasing we get that $\angle GAB = \angle FCG$, with $CG = CA = BA$ we obtain that $\Delta AGB \cong \Delta CFG \Rightarrow GF = GB$.

Attachments:



Quick Reply

High School Olympiads

Four nine-point circles

buratinogigle

#1 Jul 2, 2012, 12:04 am

Let ABC be a triangle with circumcenter O. P, Q are two isogonal conjugate with respect to triangle such that P, Q, O are collinear. Prove that four nine-point circles of triangles ABC, APQ, BPQ, CPQ have a same point.

See more Jerabek point.

Click to reveal hidden text

Luis González

#2 Jul 2, 2012, 1:27 am

H is the orthocenter of $\triangle ABC$. Isogonal conjugation with respect to $\triangle ABC$ takes P, Q into each other and O into H. The isogonal of PQ is then the rectangular hyperbola $\mathcal{H}$ through A, B, C, P, Q, H. Hence, the center K of $\mathcal{H}$ lies on 9-point circles of $\triangle ABC, \triangle APQ, \triangle BPQ, \triangle CPQ$, i.e. 9-point circles of $\triangle ABC, \triangle APQ, \triangle BPQ, \triangle CPQ$ are concurrent at K, being the last three coaxal with radical axis the line connecting K with the midpoint of $\overline{PQ}$.

High School Olympiads

Nice geometry

Locked

Source: http://forum.gil.ro/viewtopic.php?f=25&t=3568

drEdrE

#1 Jul 1, 2012, 11:30 pm

Let ABC be a triangle such that $\angle C=90^\circ$, and N be the altitude foot from C.Let Γ be a circle tangent to BN, CN, and to the circumcircle of ABC. $D=\Gamma\cap BN$, prove that CD bisects $\angle BCN$.

Luis González

#2 Jul 1, 2012, 11:43 pm • 1

Posted many times before, so for further discussions use any of the links below.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=17446
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=51954
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=123139
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=146656
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=416223

High School Olympiads

Concurrence

Reply

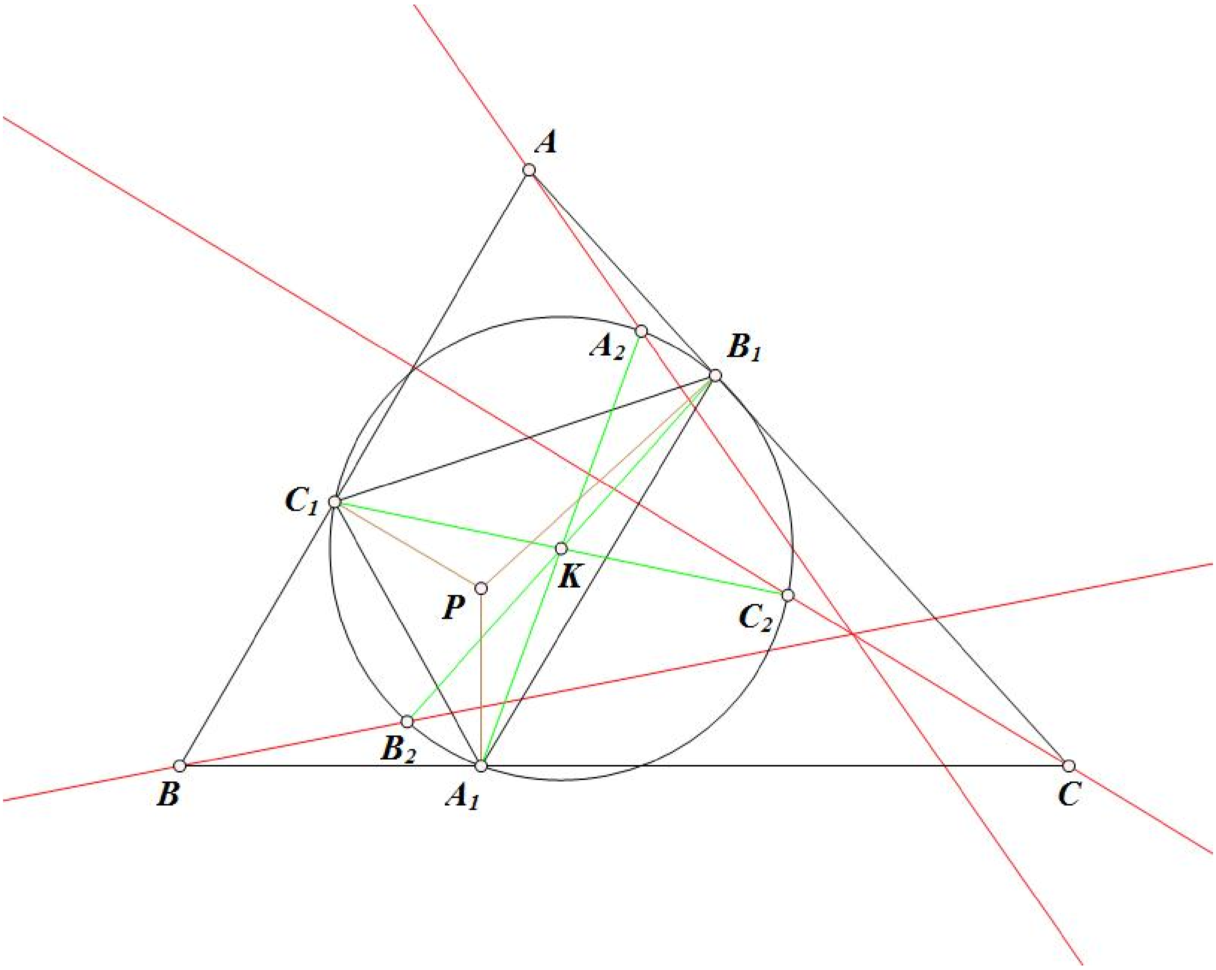
Butterfly

#1 Jul 1, 2012, 6:12 am

Let ABC be a triangle, P be an arbitrary point, the pedal triangle of P with respect to ABC be $A_1B_1C_1$, the antipodal triangle of $A_1B_1C_1$ with respect to the circumcircle of $A_1B_1C_1$ be $A_2B_2C_2$. Then AA_2, BB_2, CC_2 are concurrent.

P.S: This is my accidental discovery, but I don't know it has been already known or not.

Attachments:



Luis González

#2 Jul 1, 2012, 7:43 am

This is a particular case of the following configurations:

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=366219
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=389717
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=431409

Quick Reply

High School Olympiads

two squares

KittyOK

#1 Jun 30, 2012, 9:43 pm

Let $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ be two squares in opposite direction (that is, if the vertices A_1, B_1, C_1, D_1 are in clockwise order, then A_2, B_2, C_2, D_2 are ordered counterclockwise) with centers O_1, O_2 respectively. Suppose that D_2, D_1 are respectively in A_1B_1, A_2B_2. Prove that the lines B_1B_2, C_1C_2, O_1O_2 are concurrent.

Luis González

#2 Jul 1, 2012, 7:22 am

B_1, B_2, D_1, D_2 lie on a same circle ω, since $\angle D_1B_1D_2 = \angle D_2B_2D_1 = 45^\circ$. If $P \equiv A_1C_1 \cap A_2C_2$, from $\angle PA_2D_1 = \angle PA_1D_1 = \angle PA_2D_2 = \angle PA_1D_2 = 45^\circ$, we deduce that P, A_1, D_1, D_2 are concyclic $\Longrightarrow$ $\angle D_1PD_2 = 90^\circ$ and $PD_1 = PD_2$, i.e. $\triangle PD_1D_2$ is isosceles right with apex $P \Longrightarrow P$ is the center of ω.

Let B_1C_1 and B_2C_2 cut ω again at X and Y. $\angle XB_2D_2$ and $\angle YB_1D_1$ are clearly right $\Longrightarrow XD_2$ and YD_1 are diameters of ω meeting at P. By Pascal theorem for the cyclic hexagon $YD_1B_1XD_2B_2$, the intersections $P \equiv YD_1 \cap XD_2$, $M \equiv D_1B_1 \cap D_2B_2$ and $N \equiv XB_1 \cap YB_2$ are collinear $\Longrightarrow \triangle O_1B_1C_1$ and $\triangle O_2B_2C_2$ are perspective through $\overline{NPM}$ $\Longrightarrow O_1O_2, B_1B_2, C_1C_2$ concur.

High School Olympiads

gamma1,2,3 are semicircles show that AB=DE

Reply

Source: India tst 2002 p1

Sayan

#1 Jun 30, 2012, 5:23 pm • 1

Let A, B and C be three points on a line with B between A and C. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be semicircles, all on the same side of AC and with AC, AB, BC as diameters, respectively. Let l be the line perpendicular to AC through B. Let Γ be the circle which is tangent to the line l, tangent to Γ_1 internally, and tangent to Γ_3 externally. Let D be the point of contact of Γ and Γ_3. The diameter of Γ through D meets l in E. Show that $AB = DE$.

Luis González

#2 Jun 30, 2012, 10:37 pm

Inversion with center A and power $\overline{AB} \cdot \overline{AC}$ carries Γ_3 into itself and swaps the line ℓ and Γ_1. Then by conformity, it follows that Γ is double $\Longrightarrow \Gamma$ is orthogonal to the inversion circle. Thus if O_3 denotes the center of Γ_3, then $AD \perp EO_3$. Since $EDBA$ is cyclic, due to $\angle EDA = \angle EBA = 90^\circ$, and $\triangle BDO_3$ is isosceles with legs $DO_3 = BO_3$, then $EDBA$ is an isosceles trapezoid with legs $AB = DE$.

prime04

#3 Jul 1, 2012, 2:10 pm

Let Y be the center of Γ_3 and the circle Γ touch l at G and Γ_1 at F.
The homothety centered at F which takes Γ to Γ_1 maps G to A and so F, G, A are collinear.
Moreover, $FGBC$ is cyclic $\Longrightarrow AG \cdot AF = AB \cdot AC \Longrightarrow A$ lies on the radical axis of the circles $\Gamma, \Gamma_3 \Longrightarrow AD$ is the radical axis of $\Gamma, \Gamma_3 \Longrightarrow \angle ADE = 90^\circ$ and so $ABDE$ is cyclic. But then $YB = YD$ forces $YA = YE$ and so
$AB = DE$ $\square$

[asy] real r=1.2; pen main = linewidth(0.67) + fontsize(10); defaultpen(main); import graph; size(200); unitsize(1cm); import olympiad; import math; pair A,B,C,D,E,G,Z,Y; path a, b, c, d, l, s; A=(0,0); B=(5.28*r,0); C=(9*r,0); a=Arc((A+B)*0.5,0.5*B.x,0,180); b=Arc((B+C)*0.5,(C.x-B.x)*0.5,0,180); c=Arc((A+C)*0.5,0.5*C.x,0,180); draw(b); draw(c); dot(A,4bp+black); dot(B,4bp+black); dot(C,4bp+black); draw(A--B--C--cycle); draw(B--(B+8.38*dir(90))--cycle); Z=(A+(B+C)*0.5)*0.5; s=Arc(Z,Z.x,0,180); D=intersectionpoint(s,b); E=extension(B,B+2*dir(90),(B+C)*0.5,D); dot(E,4bp+blue); draw(E--C--cycle); dot(D,4bp+blue); Y=(B+C)*0.5; pair[] F=intersectionpoints(E--C--cycle,c); dot(F[0],4bp+blue); G=extension(A,F[0],B,B+2*dir(90)); dot(G,4bp+blue); draw(circumcircle(G,D,F[0])); draw(A--G--F[0]--cycle); draw(A--E--cycle); dot(Y,4bp+blue); draw(A--D--E--cycle); draw(D--Y--cycle); label("A",A,SW); label("B",B,S); label("C",C,SE); label("D",D,NE); label("E",E,NW); label("F",F[0],NE); label("G",G,NW); label("Y",Y,S); draw(B--D--cycle, black+linewidth(0.5)); markscalefactor=0.031; draw(rightanglemark(E,B,A),linewidth(0.35)); draw(rightanglemark(E,D,A),linewidth(0.35)); draw(rightanglemark(C,F,A),linewidth(0.35)); [/asy]

This post has been edited 4 times. Last edited by prime04, Jul 1, 2012, 5:57 pm

MBGO

#4 Jul 1, 2012, 2:20 pm

it's obviouse that L, K, A are collinear;which K, L are the tangency points of Γ with l, Γ_1 respectively;since LK intersect Γ at the midpoint of the arc,left to the line l from the circle.
$\angle EBC = \angle ALC = 90$. hence $AK.AL = AB.AC$ and A lies on the radical axis of Γ_2, Γ ; i.e. AD is Tangent to them.
so the rest of proof is similar to what Luis Gonzalez said.

Click to reveal hidden text
Click to reveal hidden text

With Regards.

Quick Reply

High School Olympiads

D, E, I are collinear <=> P, Q, O are collinear

Locked

Source: Iran Third Round MO 1997, Exam 3, P5

Amir Hossein

#1 Jun 30, 2012, 4:34 pm

In an acute triangle ABC let AD and BE be altitudes, and AP and BQ be bisectors. Let I and O be centers of incircle and circumcircle, respectively. Prove that the points D, E, and I are collinear if and only if the points P, Q, and O are collinear.

Luis González

#2 Jun 30, 2012, 9:29 pm • 1

Posted before at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=50582
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=139275
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=219811

For a generalization see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=402242

Spain

8th IBERO - MEXICO 1993.

Reply

carlosbr

#1 Mar 26, 2006, 8:15 pm

8^{th} Iberoamerican Olympiad
Ciudad de Mexico, MEXICO. [1993]

Edited by djimenez

Carlos Bravo

Attachments:
1993.pdf (30kb)

Luis González

#2 Dec 28, 2009, 6:16 am

> Quote:
> **Problema 1**. Un número es capicúa, si cuando éste es escrito en notación decimal, se puede leer igual de izquierda a derecha como viceversa. Sea $x_1 < x_2 < \cdots < x_i < x_{i+1}, \cdots$ la secuencia de todos los números capicúas. Para cada i, definamos $y_i = x_{i+1} - x_i$. Cuántos primos diferentes contienen el conjunto $\{y_1, y_2, \ldots\}$?.

$0 < 1 < \cdots < 9 < 11 < \cdots < 99 < \cdots n \Longrightarrow$ Los posibles valores de y son $1, 2, 11$. Si x_i tiene n digitos $(n \geq 3)$, entonces x_{i+1} termina con los mismos dígitos que x_i, i.e éste es divisible por 10, ó no. En tal caso, los dos capicúas consecutivos pueden ser de la forma $x_i = a9 \cdots 9a, x_{i+1} = b0 \cdots 0b$ con $b = a + 1$ y $y_i = 11$. Si x_{i+1} tiene un dígito mas que x_i, entonces éstos tienen la forma $x_i = 9a \cdots a9, x_{i+1} = 1b \cdots b1$ con $y_i = 2$. Por ende, el conjunto contiene dos primos diferentes.

Luis González

#3 Dec 26, 2011, 8:19 pm

> Quote:
> **Problema 2.** Probar que para cada polígono convexo con área menor o igual a 1, existe un paralelogramo con área 2 que contiene el polígono.

Probaremos que existe un paralelogramo que contiene al polígono y cuya área es a lo mas dos véces el área de éste. Consideremos un polígono con 4 o mas lados, puesto que el caso del triángulo es trivial. Sean A, B, C, D los vértices del polígono que forman el cuadrilátero con mayor área. Sea $PQRS$ el paralelogramo formado por las paralelas de B, D a AC y las paralelas de A, C a BD. El área de $PQRS$ es en efecto dos veces el área de $ABCD$ y contiene al polígono, ya que por lo menos si hay otro vértice E fuera de $PQRS$ de modo que A y E yacen en el mismo lado de BD, entonces $|\triangle EBD| > |\triangle ABD| \Longrightarrow |EBCD| > |ABCD|$, que contradice la hipotesis inicial.

Luis González

#4 Jun 16, 2012, 6:16 am

> Quote:
> **Problema 3.** Sea $\mathbb{N}^* = \{1, 2, \ldots\}$. Hallar todas las funciones $f : \mathbb{N}^* \to \mathbb{N}^*$, tales que si $x < y$, entonces $f(x) < f(y)$. Además $f\left(yf(x)\right) = x^2 f(xy)$ para todo $x, y \in \mathbb{N}^*$.

Note que la primera condición, implica que f es inyectiva. Si $y=1$, entonces $f(f(x))=x^2f(x)$. Si $y=f(z)$, entonces $f(f(x)f(z))=x^2f(xf(z))=x^2z^2f(xz)=f(f(xz))$. Pero como f es inyectiva, entonces $f(xz)=f(x)f(z)$. Supongamos que $f(k)>k^2$, para algún k. Entonces, $f(f(k))>f(k^2)=f(k\cdot k)=f(k)^2$. Pero $f(f(k))=k^2f(k)$, así $k^2>f(k)$, que es contradicción. Analogamente, supongamos que $f(k)<k^2$. Entonces $k^2f(k)=f(f(k))<f(k^2)=f(k)^2$, así $k^2<f(k)$, que es contradicción. Por tanto, se debe tener $f(k)=k^2$.

Luis González

#5 Jun 19, 2012, 3:00 am

> *carlosbr wrote:*
>
> **Problema 4.** Sea $\triangle ABC$ un triangulo equilatero, Γ es su incírculo y sean D y E dos puntos en AB y AC tales que DE es tangente a Γ. Probar que $\frac{AD}{DB}+\frac{AE}{EC}=1$.

Sea $P\equiv BE\cap CD$, $Q\equiv AP\cap BC$ y M,N los puntos de tangencia de Γ con AB,AC. Como el cuadrilátero $DECB$ es tangencial, entonces por teorema de Newton se tiene $P\equiv MN\cap BE\cap CD$ y en vista que $\triangle ABC$ es equilátero, MN es la paralela media de $BC\Longrightarrow P$ es punto medio de AQ. Luego aplicando el teorema de Van Aubel al triángulo $\triangle DEQ$ ceviano del punto P resulta $\frac{\overline{AD}}{\overline{DB}}+\frac{\overline{AE}}{\overline{EC}}=\frac{\overline{AP}}{\overline{PQ}}=1$.

Luis González

#6 Jun 30, 2012, 5:19 am

> *Quote:*
>
> **Problema 5.** Sean P y Q dos puntos distintos en el plano. Denotemos por $m(PQ)$ la mediatriz de $\overline{PQ}$. Sea S un subconjunto finito del plano, con mas de un elemento, el cual satisface que: Si P y Q están en S, entonces $m(PQ)$ intersecta a S. Si $\overline{P_1Q_1},\overline{P_2Q_2},\overline{P_3Q_3}$ son tres segmentos diferentes, tal que sus extremos están en S, entonces, si existe un punto en $m(P_1Q_1),m(P_2Q_2)$ y $m(P_3Q_3)$, éste no pertencece a S. Halle el número de puntos que S puede tener.

Note que hay $\frac{1}{2}n(n-1)$ pares de puntos y cada uno de éstos tiene un punto de S sobre su mediatriz. Pero cada punto de S está a lo mas sobre dos mediatrices, así $2n\geq\frac{1}{2}n(n-1)\Longrightarrow n\leq 5$. Es facil ver que el triángulo equilátero y el pentágono regular muestran que los casos $n=3,5$ son posibles. El caso $n=2$, queda evidentemente descartado. Ahora, si asumimos que $n=4$, hay 6 pares de puntos, entonces al menos un punto de S debe estar sobre dos mediatrices. Sin perdida de generalidad, pongamos que A está sobre las mediatrices de $\overline{BC}$ y $\overline{BD}$, pero también está sobre la mediatriz de $\overline{CD}$ (A es circuncentro de BCD), lo cual es contradicción.

Quick Reply

High School Olympiads

Prove that A_2B_2=2AB

Reply

LlynX

#1 Jun 20, 2012, 11:28 pm

Let A_1, B_1 be two points on the base AB of an isosceles triangle ABC, with $\angle C > 60^\circ$, such that $\angle A_1CB_1 = \angle ABC$. A circle externally tangent to the circumcircle of $\triangle A_1B_1C$ is tangent to the rays CA and CB at the points A_2 and B_2, respectively. Prove that $A_2B_2 = 2AB$.

Luis González

#2 Jun 21, 2012, 12:56 am

Let H be the tangency point of the referred circles, i.e. H is the center of inverse homothety of $\odot(CA_1B_1)$ and $\odot(HA_2B_2)$. Since $\angle A_2HB_2 = 180^\circ - \angle CA_2B_2 = \angle ABB_2 = \angle A_1HB_1$, it follows that $\overline{A_1B_1} \parallel \overline{B_2A_2}$ are correspoding chords under the referred homothety $\Longrightarrow H \equiv A_1B_2 \cap A_2B_1$. Since $\angle HA_2C = \angle HB_2A_2 = \angle HA_1B_1 = \angle HCB_1$, then $B_1C^2 = B_1H \cdot B_1A_2 \Longrightarrow B_1$ has equal power WRT $\odot(HA_2B_2)$ and (C) with zero radius. Likewise, A_1 has equal power WRT $\odot(HA_2B_2), (C) \Longrightarrow A_1B_1 \equiv AB$ is the radical axis of (C) and $\odot(HA_2B_2) \Longrightarrow AB$ is C-midline of $\triangle CA_2B_2$.

yetti

#3 Jun 22, 2012, 4:58 am

See also
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=23118,
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=198482,
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=203188,
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=481364.

Quick Reply

High School Olympiads

tetrahedron and circumsphere

Reply

Source: 36-th Vietnamese Mathematical Olympiad 1998

$\mathbb{Q}(\zeta_p)$

N.T.TUAN

#1 Feb 17, 2007, 1:11 pm • 1

Let be given a tetrahedron whose circumcenter is O. Draw diameters AA_1, BB_1, CC_1, DD_1 of the circumsphere of $ABCD$. Let A_0, B_0, C_0, D_0 be the centroids of triangle BCD, CDA, DAB, ABC. Prove that $A_0A_1, B_0B_1, C_0C_1, D_0D_1$ are concurrent at a point, say, F. Prove that the line through F and a midpoint of a side of $ABCD$ is perpendicular to the opposite side.

Luis González

#2 Jun 20, 2012, 10:25 am

$G \equiv AA_0 \cap BB_0 \cap CC_0 \cap DD_0$ is the centroid of the tetrahedron $ABCD$. Let $F \equiv A_0A_1 \cap OG$. By Menelaus' theorem for $\triangle AOG$ cut by A_0A_1, keeping in mind that $\overline{A_1O} : \overline{A_1A} = 1 : 2$ and $\overline{A_0G} : \overline{A_0A} = 1 : 4$, we get $\overline{FG} : \overline{FO} = 1 : 2$, i.e. F is the reflection of O about G. Likewise, B_0B_1, C_0C_1 and D_0D_1 pass through F.

Let M, N be the midpoints of the edges BC, AD. Since G is the midpoint of the bimedian $\overline{MN}$, then it follows that $ONFM$ is a parallelogram with diagonal intersection $G \Longrightarrow MF \parallel ON \perp AD$, i.e. skew lines FM and AD are orthogonal. Analogously, the lines joining the midpoints of CD, DB, AB, AC, AD with F are orthogonal to the edges AB, AC, CD, DB, BC, respectively.

Quick Reply

High School Olympiads

very good

minhphuc.v

#1 Jun 17, 2012, 6:15 pm

let $A;B;C;D \in (\alpha)$ and S is not into (α). $AB//CD; AB=CD$; intersection of AC and DB is O . Assume $p;q;u;v$ are distances from O to $(SAB);(SBC);(SCD);(SDA)$ respectively. And (SAC) make with (SBD) 90^o.

Prove that $\frac{1}{p^2}+\frac{1}{u^2}=\frac{1}{q^2}+\frac{1}{v^2}$

Luis González

#2 Jun 18, 2012, 12:04 pm • 1

Let δ,λ denote the planes $SBD, SAC,$ respectively. $ABCD$ is obviously a parallelogram with diagonal intersection $O \Longrightarrow$ $[OAB]=[OBC]=[OCD]=[ODA]=S_0, [SOB]=[SOD]=S_1$ and $[SOA]=[SOC]=S_2.$ By cosine theorem for the tetrahedron $SOAB,$ we get

$$[SAB]^2={S_0}^2+{S_1}^2+{S_2}^2-2[S_0S_1\cos \widehat{\alpha \delta}+S_0S_2 \cos \widehat{\alpha \lambda}+S_1S_2\cos 90^{\circ}]$$

$$[SAB]^2={S_0}^2+{S_1}^2+{S_2}^2-2S_0[S_1\cos \widehat{\alpha \delta}+S_2 \cos \widehat{\alpha \lambda}]$$

By cosine theorem for the tetrahedron $SOCD,$ we get

$$[SCD]^2={S_0}^2+{S_1}^2+{S_2}^2-2[-S_0S_1\cos \widehat{\alpha \delta}-S_0S_2 \cos \widehat{\alpha \lambda}+S_1S_2\cos 90^{\circ}]$$

$$[SCD]^2={S_0}^2+{S_1}^2+{S_2}^2+2S_0[S_1\cos \widehat{\alpha \delta}+S_2 \cos \widehat{\alpha \lambda}]$$

$$\Longrightarrow [SAB]^2+[SCD]^2=2({S_0}^2+{S_1}^2+{S_2}^2)$$

Analogously, we get $[SBC]^2+[SDA]^2=2({S_0}^2+{S_1}^2+{S_2}^2)$

$$\Longrightarrow [SAB]^2+[SCD]^2=[SBC]^2+[SDA]^2$$

Since $[SOAB]=[SOBC]=[SOCD]=[SODA]=\tfrac{1}{4}[SABCD],$ then we have

$$\frac{9}{16}\left(\frac{[SABCD]^2}{p^2}+\frac{[SABCD]^2}{u^2}\right)=\frac{9}{16}\left(\frac{[SABCD]^2}{q^2}+\frac{[SABCD]^2}{v^2}\right)$$

$$\Longrightarrow \frac{1}{p^2}+\frac{1}{u^2}=\frac{1}{q^2}+\frac{1}{v^2}$$

High School Olympiads

perpendicular

Reply

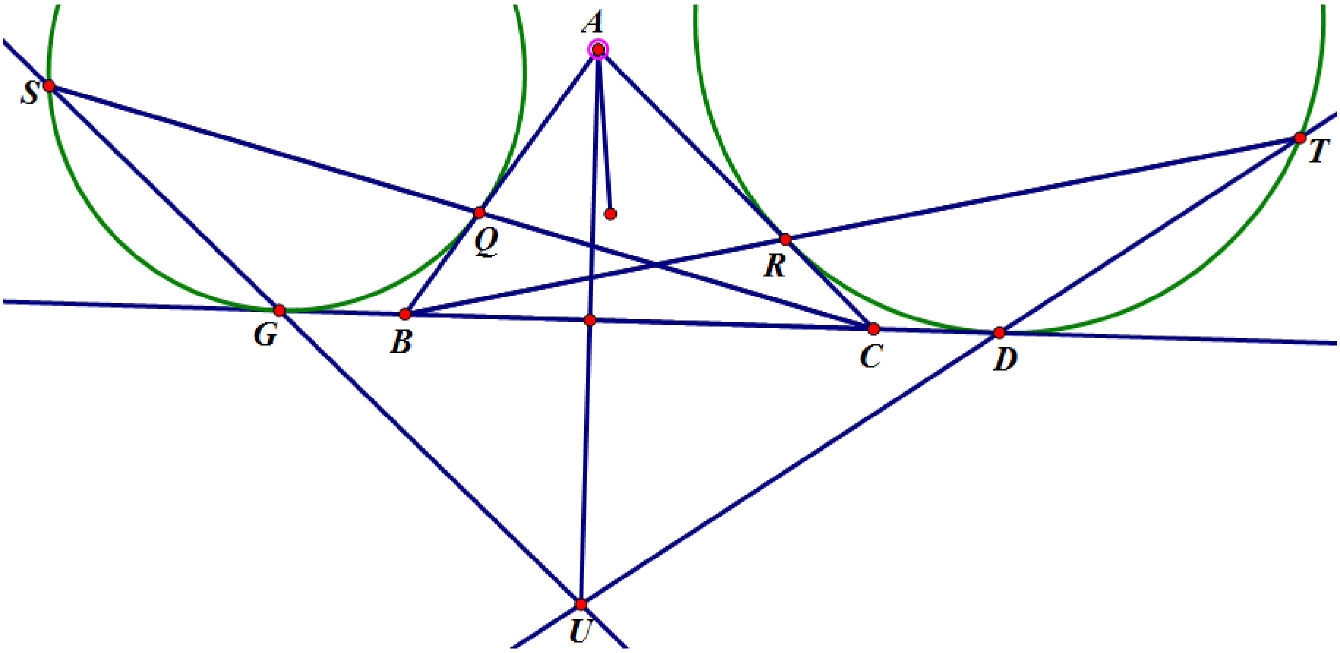
Source: a chinese friend

we_were_in_love_tara

#1 Jun 17, 2012, 6:18 pm

let ABC be a triangle. Let D,R be the points of tangency of the B-excircle to BC,AC respectively. Let G,Q be the points of tangency of C-excircle to BC,AB respectively. BR cuts B-excircle again at T, CQ cuts C-excircle again at S. SG cuts TD at U. Prove $AU \perp BC$

Attachments:



Luis González

#2 Jun 18, 2012, 2:16 am • 1

Let P be the foot of the A-altitude. $E \equiv DR \cap GQ$ lies on AP (See Inradius and altitude, Ex-circle and perpendicular segments, two excircles, Intersection on an altitude and elsewhere). Let the B-excircle (I_b) and C-excircle (I_c) touch AB, AC at M, N, respectively. By degenerate Brianchon theorem, AD, CM, BR concur $\Longrightarrow D(M,R,B,A)=-1 \Longrightarrow$ $F \equiv MD \cap AP$ is harmonic conjugate of E WRT A,P. Likewise, GN meets AP at F. Since MD is the polar of B WRT (I_b), then $D(R,T,M,B)=-1 \Longrightarrow (E,U,F,P)=-1$, where $U \equiv AP \cap DT$. Similarly, SG cuts AP at the harmonic conjugate U of E WRT F,P.

Quick Reply

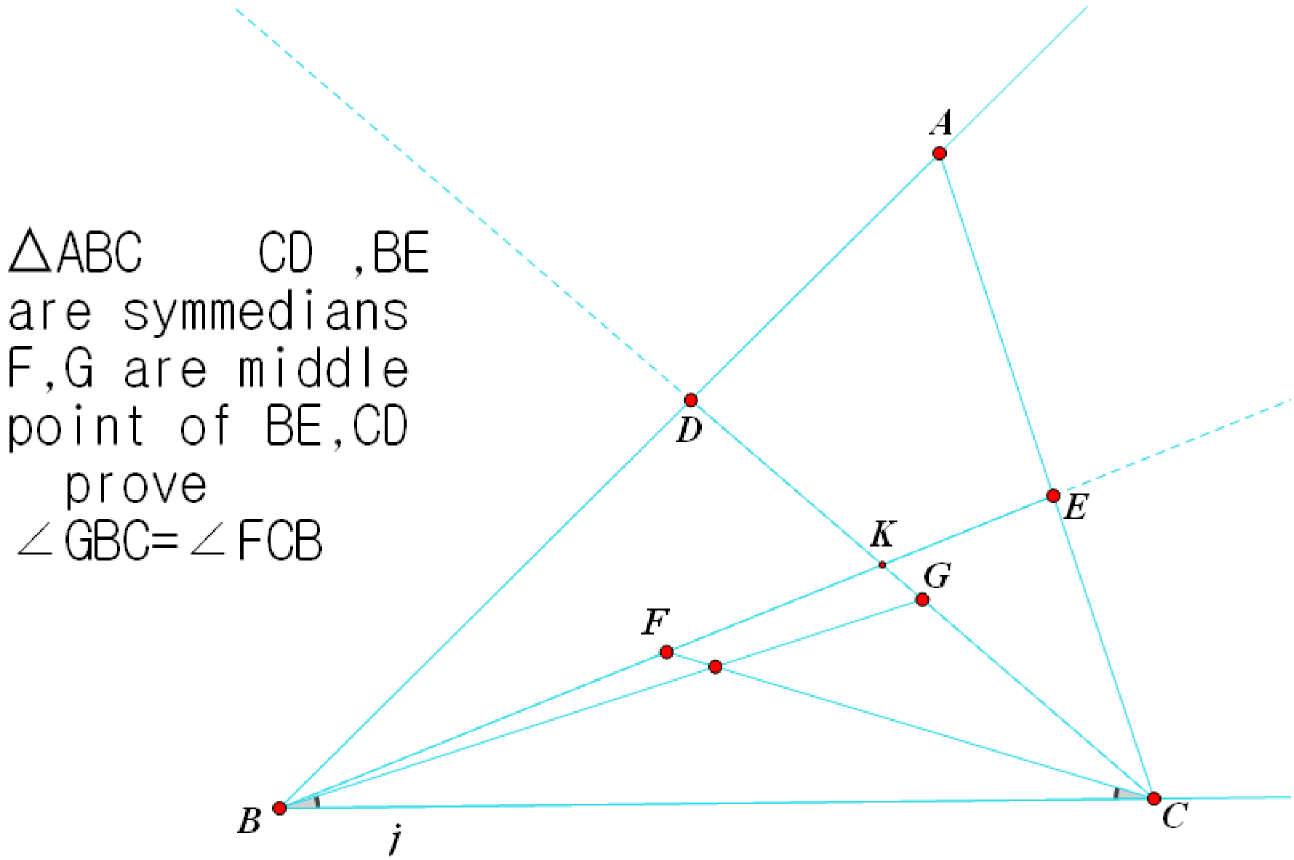
High School Olympiads

symmedians

DANNY123
#1 Jun 17, 2012, 5:47 am

prove two angles equal

Attachments:



Headhunter
#2 Jun 17, 2012, 6:56 am

here

Luis González
#3 Jun 17, 2012, 7:21 am • 1

Generalization. P is a point on the plane of $\triangle ABC$. PC, PB cut AB, AC at D, E. M, N are the midpoints of CD, BE. Then $\angle MBC = \angle NCB \Longleftrightarrow P$ lies on the A-Schwatt line of $\triangle ABC$, i.e. the line passing through the symmedian point X_6 of $\triangle ABC$ and midpoints of BC and the A-altitude.

Proof: Let $Q \equiv BM \cap CN$. Then $\angle MBC = \angle NCB \Longleftrightarrow Q$ is on perpendicular bisector of $\overline{BC}$. Let $(u : v : w)$ be the barycentric coordinates of P WRT $\triangle ABC$. Then $E(u : 0 : w)$ and $D(u : v : 0) \Longrightarrow$

$$M(u : v : u + v) \Longrightarrow BM \equiv (u + v)x - uz = 0$$

$$N(u : u + w : w) \Longrightarrow CN \equiv (u + w)x - uy = 0$$

$Q \equiv BM \cap CN \equiv (u : u + w : u + v)$

Q is on perpendicular bisector $\tau_A \equiv (b^2 - c^2)x + a^2y - a^2z = 0$ of $\overline{BC} \Longleftrightarrow$

$(b^2 - c^2)u + a^2(u + w) - a^2(u + v) = 0 \Longleftrightarrow (b^2 - c^2)u + a^2w - a^2v = 0 \Longleftrightarrow$

P is on the A-Schwatt line through midpoint $(0 : 1 : 1)$ of $\overline{BC}$ and $X_6(a^2 : b^2 : c^2)$.

RSM

#4 Jun 17, 2012, 12:44 pm • 2

Here is a synthetic solution for the generalisation(I will use Luis's notations):-

Suppose, $Q = CN \cap BM, R = MN \cap PQ, S = MN \cap BC, T = PQ \cap BC$.
Note that, $(MN;RS) = -1 \implies (A'M, A'N; A'R, BC) = -1$ where A' is the midpoint of BC. So $A'R$ passes through A. $(PQ;RT) = -1 \implies (A'P, A'Q; A'A, BC) = -1$. Suppose, $AX \perp BC$ with X on BC. X' be the midpoint of AX. So $(A'P, A'Q; A'A, BC) = -1 \implies (AX, YZ) = -1$ where $Y = A'Q \cap AX$ and $Z = A'P \cap AX$. So Y is at infinity iff Z is midpoint of AX. So done

buratinogigle

#5 Jun 17, 2012, 11:26 pm

More general problem

Let ABC be a triangle and D is a point on BC. M is a point on AD. The Line passing through M and parallel to BC cuts CA, AB at E, F, respectively. The line passing through E and parallel to AB cuts the line passing through F and parallel to CA at P. N is a point on line PM. NB cuts PF at K, NC cuts PE at L, CK cuts BL at Q. Prove that $PQ \parallel AD$.

Quick Reply

High School Olympiads

Quite difficult Problem

Reply

minhtue0605

#1 Jun 16, 2012, 8:29 am

Let be given triangle ABC with orthocenter H and circumcenter O. E, F are points on AB, AC respectively such that quadrilateral AEHF is a parallelogram. Show that: OE = OF.

Luis González

#2 Jun 16, 2012, 8:44 am • 3

This holds for all the points H satisfying $\angle HCA = \angle HBA.$ Since $\angle HFC = \angle HEB,$ then $\triangle HFC$ and $\triangle HEB$ are similar $\Longrightarrow \frac{HF}{HE} = \frac{EA}{FA} = \frac{FC}{EB}.$ If R denotes the circumradius of $\triangle ABC,$ we have then

$$EB \cdot EA = OE^2 - R^2 = FC \cdot FA = OF^2 - R^2 \Longrightarrow OE = OF.$$

Quick Reply

High School Olympiads

Angle OLK = Angle OLM

Reply

Source: Baltic Way 1993

WakeUp
#1 Jun 15, 2012, 8:31 pm • 1

In the triangle ABC, $|AB|=15, |BC|=12, |AC|=13$. Let the median AM and bisector BK intersect at point O, where $M\in BC, K\in AC$. Let $OL\perp AB, L\in AB$. Prove that $\angle OLK=\angle OLM$.

Luis González
#2 Jun 16, 2012, 2:16 am • 1

Let KM and CO cut AB at P and L^*. $\angle OLK=\angle OLK$, i.e. LO, AB bisect $\angle MLK \Longleftrightarrow L(M,K,O,P)=-1$ $\Longleftrightarrow C(B,A,L^*,P)=-1 \Longleftrightarrow L\equiv L^* \Longleftrightarrow CO$ is C-altitude of $\triangle ABC$. By Ceva's theorem, we get

$$\frac{AL}{LB}\cdot\frac{BM}{MC}\cdot\frac{CK}{KA}=1\Longleftrightarrow\frac{AL}{LB}=\frac{CL\cdot\cot A}{CL\cdot\cot B}=\frac{KA}{CK}=\frac{c}{a}\Longleftrightarrow$$

$$a(b^2+c^2-a^2)-c(a^2+c^2-b^2)=0\Longleftrightarrow b=\sqrt{\frac{a^3+c^3+ca^2-ac^2}{a+c}}$$

subham1729
#3 Jun 16, 2012, 5:30 pm • 1

My ugly solution

Let the line OC intersect AB in point P. As AM is a median, we have |AP|
|PB| = |AK| |KC| (this obviously holds if |AB| = |AC| and the equality is preserved under uniform compression of the plane along BK). Applying the sine theorem to the triangles ABK and BCK we obtain
$|AP|/|PB=|AK/KC|=|AB/BC|=5/4$
As |AP| + |PB| = |AB| = 15, we have |AP| = 25/3
and |PB| = 20/3 Thus $|AC|^2-|BC|^2=25=|AP|^2-|BP|^2$ and $|AC|^2-|AP|^2=|BC|^2|BP|^2$. Applying now the cosine theorem to the triangles APC and BPC we get cos\APC = cos\BPC, i.e., P = L. As above, we can use a compression of the plane to show that KP k BC and therefore \OPK = \OCB. As |BM| = |MC| and \BPC = 90 we have \OCB = \OPM. Combining these equalities, we get \OLK = \OPK = \OCB = \OPM = \OLM.

Quick Reply

High School Olympiads

An interesting "slicing" problem.

Source: Own.

Virgil Nicula

#1 Jun 14, 2012, 9:36 pm

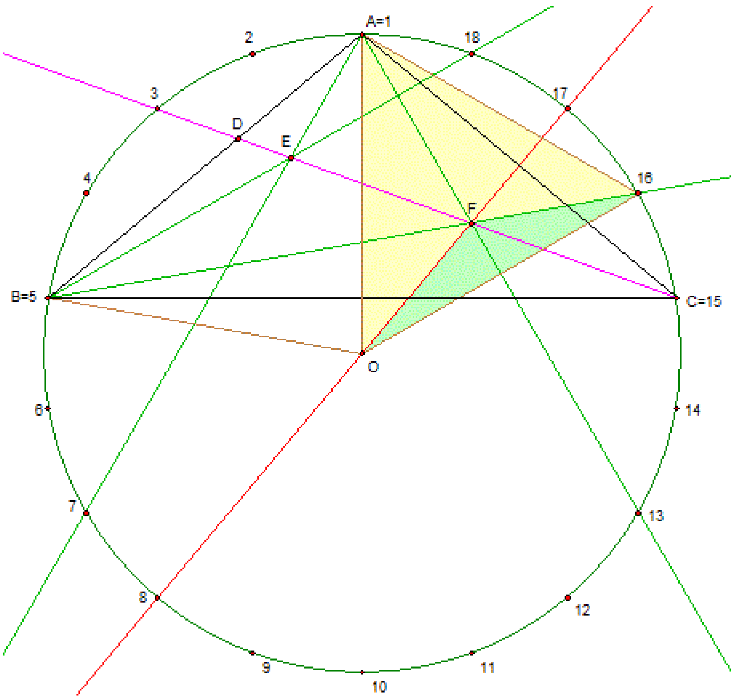
Let ABC be an A-isosceles triangle with $A = 100^\circ$. Denote $D \in (AB)$ so that $\widehat{DCA} \equiv \widehat{DCB}$ and $\{E, F\} \subset (CD)$ so that $m\left(\widehat{EBD}\right) = m\left(\widehat{FBC}\right) = 10^\circ$. Prove that $m\left(\widehat{EAF}\right) = 60^\circ$.

yetti

#2 Jun 15, 2012, 10:19 pm • 1

Take regular 18-gon $P_1P_2...P_{18}$ inscribed in a circumcircle (O), with $\triangle ABC \equiv \triangle P_1P_5P_{15}$. Internal bisector $CD \equiv P_{15}D$ of $\angle BCA \equiv \angle P_5P_{15}P_1$ cuts (O) again at P_3. Undefine point F.
Let $F \equiv BP_{16} \cap P_8P_{17} \equiv P_5P_{16} \cap P_8P_{17}$, where P_8P_{17} is a diameter of (O). $\triangle AOP_{16} \equiv \triangle P_1OP_{16}$ is equilateral and $\triangle FOP_{16}$ is F-isosceles, with base angles at vertices O, P_{16} equal to $20^\circ \Longrightarrow$
F is on perpendicular bisector $AP_{13} \equiv P_1P_{13}$ of $OP_{16} \Longrightarrow F$ is also on reflection $CP_3 \equiv P_{15}P_3$ of $AP_{13} \equiv P_1P_{13}$ in the diameter P_8OP_{17}, satisfiying the problem conditions.
Since E, F are isogonal conjugates WRT $\triangle ABC \equiv \triangle P_1P_5P_{15} \Longrightarrow \angle EAB = \angle CAF = \angle P_{15}P_1P_{13} = 20^\circ$ and $\angle FAE = \angle CAB - (\angle CAF + \angle EAB) = 100^\circ - 2 \cdot 20^\circ = 60^\circ$.

Attachments:



Luis González

#3 Jun 16, 2012, 12:27 am • 1

$I \in CD$ is the incenter of $\triangle ABC. \angle ABI = 20^\circ \Longrightarrow BE$ bisects $\angle DBI.$ Since $\angle DIB = 40^\circ$ and $\angle AID = 70^\circ$, then IA is external bisector of $\angle DIB \Longrightarrow U \equiv BE \cap IA$ is B-excenter of $\triangle BDI \Longrightarrow \angle DUE = 20^\circ.$ But since $\angle UEI = \angle DAI = 50^\circ$, then $ADEU$ is cyclic $\Longrightarrow \angle DAE = \angle DUE = 20^\circ. E, F$ are obviously isogonal conjugates WRT $\triangle ABC \Longrightarrow \angle CAF = \angle BAE = 20^\circ \Longrightarrow \angle EAF = 100^\circ - 2 \cdot 20^\circ = 60^\circ.$

sunken rock

#4 Jun 19, 2012, 4:01 pm

Excellent solution, **Luis**!
We can also judge this way: as explained, E, F are isogonal conjugates w.r.t. $\triangle ABC$ so we need to prove $\angle CAF = 20^\circ$.
Take F' on $(CD$ so that $AF' = CF'$, and show that $\angle CBF' = 10^\circ$.
Take F'', the reflection of F' in the perpendicular bisector of BC, see easily that $\triangle AF'F''$ is equilateral, hence $CF' = F'F'' = BF''$ and, with $BCF'F''$ isosceles trapezoid, easily BF' is the internal angle bisector of $\angle F'BF'' = 20^\circ$, consequently $F' \equiv F$ and we are done.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Using Butterfly theorem

hoangquan
#1 Jun 7, 2012, 5:15 am

For ΔABC be a triangle with incircle (I) and circumcircle (O), M is the middle of the segment BC. AM cut (I) in the points K, L. Lines parallel with BC passing through K, L cut (I) in the points X, Y, AX cut BC in the point P, AY cut (O) in the point D. DM cut (O) in the point E. AM cut (O) in the point F. Prove that $\angle FPE = 180^\circ$.

PS: Can you use Butterfly theorem.?

Luis González
#2 Jun 7, 2012, 8:43 am

Let lines EF and AD intersect BC at P^* and Q, respectively. Since OM is perpendicular to P^*Q, then by Butterfly theorem for the chords AD, EF of (O), we have that M is the midpoint $\overline{P^*Q}$. But according to the topic Geometry (1), M is the midpoint of $\overline{PQ} \Longrightarrow P \equiv P^*$, i.e. P, E, F are collinear, as desired.

r1234
#3 Jun 10, 2012, 8:19 pm • 1

Lemma:-(I) is the incircle of $\triangle ABC.M$ is the midpoint of BC.The line AM cuts (I) at K, L.Lines through K, L parallel to BC meet (I) at X, Y.Then the rays AX, AY are isotomic wrt BC.

Proof:-Let ℓ be the line through A parallel to BC.Now consider the projective transformation that takes ℓ to infinity and the circle (I) to another circle, say ω.Now by A' we mean the image of point A under this transformation.Since BC, KX, LY are parallel to the singular line ℓ, so their images $B'C', K'X', L'Y'$ are also parallel to each other.Now note that under this transformation $A'B', A'C'$ becomes parallel since A' lies on the line at infinity.Now $B'C', A'B', C'A'$ touch the circle ω.Now the image of AX is the line through X' parallel to $A'B'$ and similarly image of AY is the line $A'Y'$ which is parallel to $A'C'$.Since $X'K' \parallel Y'L'$ we conclude that the points $A'X' \cap B'C', A'Y' \cap B'C'$ are isotomic wrt $B'C'$.Since BC is parallel to the singular line, we conclude that AX, AY are isotomic wrt BC.

Back to the main proof:- Let $FE \cap BC = P'$, and $AY \cap BC = Q$. Then by Butterfly theorem, M is the midpoint of $P'Q$.But, as proved in lemma, M is the midpoint of PQ.So $P \equiv P'$.Hence F, P, E are collinear.

jayme
#4 Sep 18, 2013, 7:41 pm

Dear Mathlinkers,
for your lemma, you can see also
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=80872
Sincerely
Jean-Louis

High School Olympiads

Perpendicular Lines

thugzmath10

#1 May 24, 2012, 9:43 pm

The incircle of triangle ABC touches BC and CA at points D and E respectively. Let K be the reflection of D across the midpoint of BC. The line through K perpendicular to BC meets DE at L and N is the midpoint of KL. Prove that $BN \perp AK$.

yunxiu

#2 May 25, 2012, 8:44 am • 1

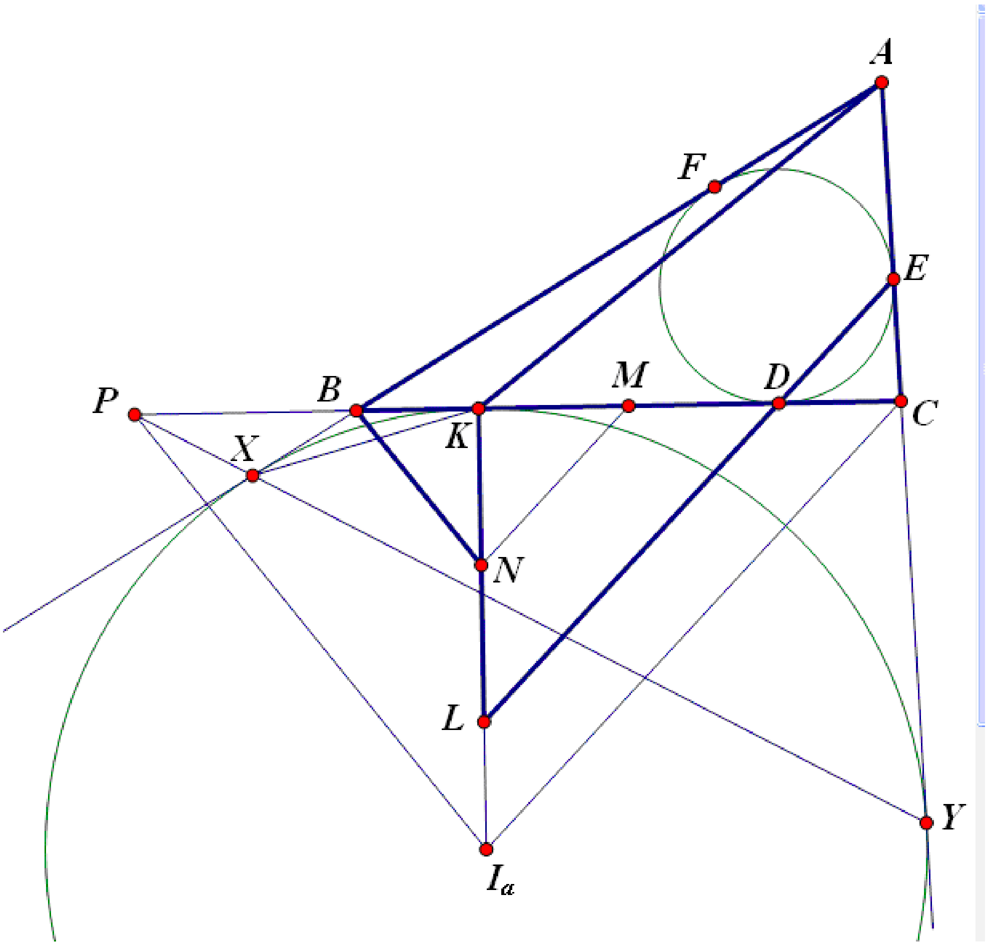
Denote A-excircle (I_a) touches AB, AC at X, Y, $XY \cap BC = P$, M is the midpoint of BC, then P is the pole of AK about (I_a), so $PI_a \perp AK$. We suppose $AB \geqslant AC$.

Because $(BC;KP) = -1$, $\frac{BP}{CP} = \frac{BK}{CK}$, so $\frac{BC}{CP} = \frac{CP-BP}{CP} = \frac{CK-BK}{CK} = \frac{KD}{CK}$.

Because $\angle CDE = \angle BCI_a$, $DE \parallel CI_a$, $MN \parallel DL$, so $MN \parallel CI_a$ and we have

$\frac{BM}{CP} = \frac{1}{2}\frac{KD}{CK} = \frac{KM}{CK} = \frac{MN}{CI_a}$, so $\Delta NBM \sim \Delta I_a PC$, hence $BN \parallel PI_a$, and we have $BN \perp AK$.

Attachments:



Luis González

#3 Jun 7, 2012, 5:35 am • 1

Alternate proposition: In a triangle $\triangle ABC,$ let D, E the tangency points of the incircle (I) with $BC, AC.$ Let K be the reflection of D about the midpoint of $BC.$ Perpendicular to BC at K cuts DE at L and perpendicular from B to AK cuts LK at $N.$ Then N is midpoint of $\overline{LK}.$

Proof: Let M be the midpoint of $\overline{BC}$ and $\overline{DK}.$ Since MI is the M-Nagel ray of the medial triangle of $\triangle ABC,$ we have $MI \parallel AK.$ Therefore $MI \perp BN,$ which implies that $\angle MID = \angle NBK.$ From the similar triangles $\triangle BKN \sim \triangle IDM$ and $\triangle LKD \sim \triangle CDI,$ we have

$$\frac{NK}{BK} = \frac{DM}{ID}\ ,\ \frac{LK}{CD} = \frac{DK}{ID}$$

Since $BK = CD$ and $DM = \frac{1}{2}DK,$ we get $NK = \frac{1}{2}LK \Longrightarrow N$ is midpoint of $\overline{LK}.$

High School Olympiads

Find the distance from the center of O_k to the base

Reply

mathwizarddude
#1 Jun 6, 2012, 8:07 am

Let AB be both the base and diameter of semicircle C_1 and BC be both the base and diameter of semicircle C_2 such that the two semicircles are externally tangent at B. Additionally $AB \geq BC$. Let C_3 be a semicircle with base and diameter AC such that C_1 and C_2 are internally tangent to C_3 at A and C, respectively. Let O_1 be a circle internally tangent to C_3 and externally tangent to C_1 and C_2. For $k \geq 1$, let O_{k+1} be a circle internally tangent to C_3 and externally tangent to C_1 and O_k. Find the distance from the center of O_k to base AC (in terms of AB, BC, k).

yetti
#2 Jun 7, 2012, 5:02 am

> mathwizarddude wrote:
> Let AB be both the base and diameter of semicircle C_1 and BC be both the base and diameter of semicircle C_2 such that the two semicircles are externally tangent at B. Additionally $AB \geq BC$. Let C_3 be a semicircle with base and diameter AC such that C_1 and C_2 are internally tangent to C_3 at A and C, respectively. Let O_1 be a circle internally tangent to $C_\color[rgb]{1,0,0}1$ and externally tangent to $C_\color[rgb]{1,0,0}2$ and $C_\color[rgb]{1,0,0}3$. For $k \geq 1$, let O_{k+1} be a circle internally tangent to C_3 and externally tangent to C_1 and O_k. Find the distance from the center of O_k to base AC (in terms of AB, BC, k).

Typos fixed.

Let C_1, C_2, C_3 be centers and $r_1 = \frac{1}{2}AB, r_2 = \frac{1}{2}BC, r_3 = \frac{1}{2}AC$ radii of the circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, respectively.
Inversion with center A and power AB^2 takes the circles $\mathcal{C}_1, \mathcal{C}_3$ into lines $d_1, d_3 \perp AB$. Line d_1 is tangent of $\mathcal{C}_1$ at B.
Line d_3 cuts the center line AB at D, such that $AD = \dfrac{AB^2}{AC}$ and $BD = \dfrac{AB^2}{AC} - AB$.
Circle $\mathcal{C}_2$ goes to a circle $\mathcal{D}_2$ tangent to the parallel lines $d_1 \parallel d_3$ at B, D, with center D_2 and radius $s_2 = \frac{1}{2}BD = \dfrac{r_1^2}{r_3} - r_1$.
Circle $\mathcal{O}_1$, internally tangent to $\mathcal{C}_1$ and externally tangent to $\mathcal{C}_2, \mathcal{C}_3$ goes to a circle $\mathcal{P}_1 \cong \mathcal{D}_2$ tangent to the parallel lines $d_1 \parallel d_3$ and externally tangent to the circle $\mathcal{D}_2$.
Circle $\mathcal{O}_{k+1}$, internally tangent to $\mathcal{C}_1$ and externally tangent to $\mathcal{O}_k, \mathcal{C}_3$ goes to a circle $\mathcal{P}_{k+1} \cong \mathcal{P}_k$ tangent to the parallel lines $d_1 \parallel d_3$ and externally tangent to the circle $\mathcal{P}_k$.
Let O_k, P_k be centers of the circles $\mathcal{O}_k, \mathcal{P}_k,\ k = 1, 2, ...$ Let F_k be foot of perpendicular from O_k to AB.
Inversion center A is also the external similarity center of the circles $\mathcal{O}_k \sim \mathcal{P}_k$, with similarity coefficient α equal to the power of inversion $AB^2 = 4r_1^2$, divided by power of the inversion center to the inverted circle $\mathcal{P}_k$.
Power $p(A, \mathcal{P}_k)$ of the inversion center A to the inverted circle $\mathcal{P}_k$ and the similarity oefficient α are equal to

$$p(A, \mathcal{P}_k) = AD_2^2 + P_kD_2^2 - s_2^2 = \left(AB + \tfrac{1}{2}BD\right)^2 + (k \cdot BD)^2 - \left(\tfrac{1}{2}BD\right)^2 = (2r_1 + s_2)^2 + 4k^2s_2^2 - s_2^2 = 4(r_1^2 + r_1s_2 + k^2s_2^2)$$

$$\alpha = \frac{AB^2}{p(A, \mathcal{P}_k)} = \frac{r_1^2}{r_1^2 + r_1s_2 + k^2s_2^2} = \frac{O_kF_k}{P_kD_2} = \frac{\text{dist}(O_k, AB)}{2ks_2}$$

As a result,

$$\text{dist}(O_k, AB) = \frac{2ks_2r_1^2}{r_1^2 + r_1s_2 + k^2s_2^2} = \frac{2kr_1r_3(r_1 - r_3)}{r_1r_3 + k^2(r_1 - r_3)^2} = \frac{k \cdot AB \cdot AC \cdot BC}{AB \cdot AC + k^2 \cdot BC^2}$$

Luis González

#3 Jun 7, 2012, 5:27 am • 1

@Vladimir, I don't see any typo in the original proposition. I think that you are assuming that C1 and C2 are internally tangent at B (?).

Inversion with center A and power $\overline{AB} \cdot \overline{AC}$ takes (C_2) into itself and takes (C_1) and (C_3) into the perpendiculars c_1 and c_3 to AC through C, B. Thus, by conformity (O_k) is transformed into a chain of congruent circles (S_k) externally tangent to each other and tangent to c_1, c_3. A is then exsimilicenter of $(O_k) \sim (S_k)$, i.e. A, O_k, S_k are collinear. Let $AB = a$ and $BC = b$. Locus of O_k is clearly an ellipse $\mathcal{E}$ with major axis $a + \frac{1}{2}b$ and foci C_1, C_3. Thus if we define the rectangular reference (x, y) where $A(0,0)$ and $B(a, 0)$, then the equations of $\mathcal{E}$ and the line $\ell \equiv AS_k$ are

$$\mathcal{E} \equiv \frac{(x - \frac{1}{2}a - \frac{1}{4}b)^2}{(\frac{1}{2}a + \frac{1}{4}b)^2} + \frac{y^2}{\frac{1}{4}(a^2 + ab)} = 1 \ , \ \ell \equiv \frac{y}{x} = \frac{bk}{a + \frac{1}{2}b}$$

ℓ cuts $\mathcal{E}$ at $A(0,0)$ and $O_k \left[\dfrac{a(b^2 + 2a^2 + 3ab)}{2(a^2 + ab + b^2k^2)}, \dfrac{ab(a+b)k}{a^2 + ab + b^2k^2}\right] \Longrightarrow$

$$\text{dist}(O_k, AC) = \frac{ab(a+b)k}{a^2 + ab + b^2k^2}$$

yetti

#4 Jun 7, 2012, 6:07 am

You are right Luis, the 1st problem condition specifies that the circle $\mathcal{C}_3$ is largest and points A, B, C follow in this order. I asssumed that circle $\mathcal{C}_1$ is largest and points A, C, B follow in this order.

If the point C is moved over B, the result $\text{dist}(O_k, AB) = \text{dist}(O_k, AC) = \dfrac{k \cdot AB \cdot AC \cdot BC}{AB \cdot AC + k^2 \cdot BC^2}$ is the same, being symmetrical WRT AB, AC.

Quick Reply

High School Olympiads

Intersect on circle

Reply

buratinogigle

#1 Jun 6, 2012, 12:14 am

Let ABC be a triangle with circumcircle (O) and a point P. PA, PB, PC cuts (O) again at A_1, B_1, C_1. A_2, B_2, C_2 is pedal triangle of P with respect to triangle ABC. H is orthocenter of triangle ABC. A_3, B_3, C_3 are symmetric of H thourgh A_2, B_2, C_2 respectively. Prove that A_1A_3, B_1B_3, C_1C_3 are concurrent at point T lies on (O).

Luis González

#2 Jun 6, 2012, 10:21 am • 2

M_a, M_b, M_c are midpoints of BC, CA, AB. Midpoint A_4 of $\overline{HA_1}$ clearly is on 9-point circle $(N) \equiv \odot(M_aM_bM_c)$, since homothety $(H, 2)$ takes (N) into (O). Let D be the 2nd intersection of PA_2 with $\odot(AB_2C_2)$ and S the 2nd intersection of $\odot(AM_bM_c)$ with $\odot(AB_2C_2)$. DS cuts $\odot(AM_bM_c)$ again at A_5. According to this thread (see the highlighted theorem) we have that DS is the reflection of A_2U WRT M_bM_c, where $U \equiv (N) \cap \odot(A_2B_2C_2)$ is the anti-Steiner point of OP WRT $\triangle M_aM_bM_c$, i.e. orthopole of OP WRT $\triangle ABC$. 2nd intersection A_6 of UA_2 with (N) is then reflection of A_5 about M_bM_c. Thus if AH cuts $\odot(AM_bM_c)$ and (N) again at R and F (midpoint of AH), we have

$$\angle HFA_6 = \angle ARA_5 = \angle ASA_5 \equiv \angle ASD = \angle APD = \angle HAP,$$

which means that $FA_6 \parallel AP$, but FA_4 is the H-midline of $\triangle HAA_1 \Longrightarrow A_4 \equiv A_6 \Longrightarrow U \in A_2A_4$. Therefore, image A_1A_3 of A_4A_2 under homothety $(H, 2)$ passes through the reflection T of H about U. Likewise, B_1B_3 and C_1C_3 pass through the reflection $T \in (O)$ of H about the orthopole of OP.

TelvCohl

#3 Oct 11, 2014, 12:15 pm • 2

My solution:

Let D be the intersection of AH and BC .
Let M, X be the midpoint of BC, AH , respectively.
Let O be the circumcenter of triangle ABC and K be the orthopole of OP .
Let A_4 be a point on AH satisfy $A_2A_4 \parallel AP$ and A_5 be the intersection of KA_2 and the nine point circle of ABC .
Let S be the projection of O on AP .

Since (A_2A_4) is the reflection of (AP) with respect to the A-midline,
so (A_2A_4) pass through K .
Since $\angle A_5XD = \angle A_2KD = \angle A_2A_4D$,
so $A_5X \parallel A_2A_4 \parallel AP$.
Since $XM \overset{\parallel}{=} AO$ and $\angle ASO = \angle XA_5M = 90$,
so ASO and XA_5M are congruent and $OS \overset{\parallel}{=} MA_5$,
hence we get OSA_5M is a parallelogram and $A_5S \overset{\parallel}{=} \frac{1}{2}HA$,
so A_5 is the midpoint of A_1H and $A_1A_3 \overset{\parallel}{=} 2 \cdot A_5A_2$,
hence A_1A_3 pass through the reflection of H with respect to K which is lie on (ABC) .

We can prove B_1B_3, C_1C_3 pass through the reflection of H with respect to K similarly,
so we get A_1A_3, B_1B_3, C_1C_3 are concurrent at a point lie on the circumcircle of triangle ABC .

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Aug 21, 2015, 3:15 am

Quick Reply

High School Olympiads

Concurrent lines in pentagon

Reply

Source: 0

delegat

#1 Jun 6, 2012, 12:33 am • 1

Given is a pentagon $ABCDE$ such that $\angle A = \angle B = 90$ and there exists a circle with diameter AB touching sides CD and DE at points M and N. Prove that lines CE, AM and BN are concurrent.

This post has been edited 1 time. Last edited by delegat, Jun 6, 2012, 1:10 am

Luis González

#2 Jun 6, 2012, 2:06 am • 1

Let $P \equiv AN \cap BM$. Since $\angle AMB = \angle BNA = 90^\circ$, then $H \equiv AM \cap BN$ is the orthocenter of $\triangle PAB$. Since the circle $\odot(PMHN)$ with diameter $\overline{PH}$ is clearly orthogonal to the circle with diameter $\overline{AB}$, then it follows that $D \equiv CM \cap EN$ is midpoint of $\overline{PH}$. As a result, if HA, HB cut BC, EA at U, V, then C, E are midpoints of $\overline{BU}$ and $\overline{AV}$, since $PH \parallel BU \parallel AV$. Then H, C, E are collinear (CHE is commom H-median of BHU and AHV).

prime04

#3 Jun 6, 2012, 6:16 pm • 1

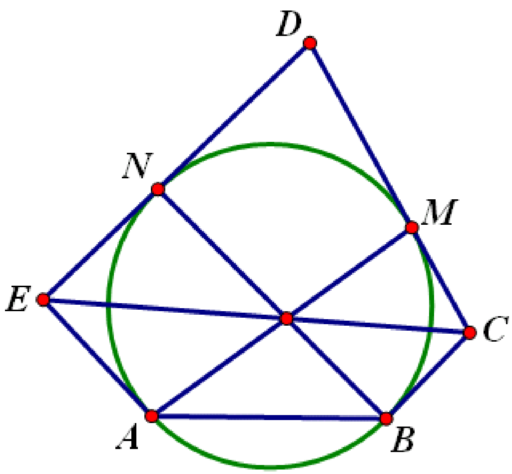
Let ω be the circle with diameter AB. Let $AN \cap BM = P$. Now, BM, AN are respectively the polars of C, E wrt ω $\implies CE$ is the polar of P (wrt ω). But then in cyclic $ABNM$, $H = AM \cap BN$ also lies on the polar of P and therefore E, H, C are collinear $\square$

yunxiu

#4 Jun 11, 2012, 6:51 am

Given is a pentagon $ABCDE$ such that BC, AE tangent a circle ω at B, A, and CD, DE touching ω at points M, N. Prove that lines AM, BN, CE are concurrent.

Attachments:



yunxiu

#5 Jun 11, 2012, 12:13 pm

It is the special case of Brianchon's theorem or Newton 's theorem.

Quick Reply

High School Olympiads

nice concurrence

Reply

prime04
#1 Jun 4, 2012, 2:19 am • 1

Let M be the midpoint of side BC of $\triangle ABC$. BE and CF are altitudes and H is the orthocentre. Suppose MH meets $\odot ABC$ in K (K lies in the arc containing A). KE meets AB in Q. Prove that MQ, AH and EF are concurrent.

This post has been edited 1 time. Last edited by prime04, Jun 5, 2012, 10:32 pm

Luis González
#2 Jun 4, 2012, 3:10 am • 3

Let P be the reflection of H about M. Since $HBPC$ is a parallelogram, then $PB \parallel CH \perp AB \Longrightarrow P$ is the antipode of A WRT the circumcircle (O) of $\triangle ABC \Longrightarrow \angle AKH = 90^\circ \Longrightarrow K$ lies on the circumcirle ω of $AEHF$. Hence AK, EF, BC are pairwise radical axes of $(O), \omega$ and the circle with diameter $\overline{BC}$, which concur at their radical center R. Since ME, MF are tangents of ω and $(F, E, D, R) = -1$, it follows that MD is the polar of R WRT ω. Diagonals AF, KE of the cyclic $AKFE$ clearly meet on the polar of $R \equiv AK \cap EF$ WRT its circumcircle ω. Thus MQ, AH and EF concur at D, as desired.

Luis González
#3 Jun 5, 2012, 6:42 am • 3

Generalization: Circle (U) through B, C cuts sides AC, AB of $\triangle ABC$ at E, F. $H \equiv BE \cap CF$. UH cuts BC at M and the circumcirle of $\triangle ABC$ at K (K lies on the arc BAC). KE cuts AB at Q. Then MQ, AH and EF concur.

Let $R \equiv EF \cap BC$. Then AH, AR are the polars of R, H WRT $(U) \Longrightarrow AH$ is perpendicular to UR at L and AR is perpendicular to UH at $K^* \Longrightarrow \overline{RA} \cdot \overline{RK^*} = \overline{RU} \cdot \overline{RL} = \overline{RB} \cdot \overline{RC} = \overline{RE} \cdot \overline{RF} \Longrightarrow A, B, C, K^*$ are concyclic, i.e. $K \equiv K^*$. Further, K lies on the circumcircle ω of $\triangle AEF$.

On the other hand, $F \in \odot(RBK)$, since F is Miquel point of EF WRT $\triangle ABC$. Thus, inversion with center A and power $\overline{AK} \cdot \overline{AR} = \overline{AF} \cdot \overline{AB}$ takes circle $\odot(KMR)$ with diameter $\overline{MR}$ into itself and carries ω into BC. Since $BC \perp \odot(KMR)$, then $\omega \perp \odot(KMR) \Longrightarrow R, M$ are conjugate points WRT ω. If $D \equiv AH \cap EF$, then MD is the polar of R WRT ω because of $(E, F, D, R) = -1$. Diagonals AF, KE of the cyclic $AKFE$ clearly meet on the polar of $R \equiv AK \cap EF$ WRT its circumcircle ω, thus MQ, AH and EF concur at D.

r1234
#4 Jun 5, 2012, 9:56 am

> prime04 wrote:
> Let M be the midpoint of side BC of $\triangle ABC$. BE and CF are altitudes and H is the orthcentre. Suppose MH meets $\odot ABC$ in K (K lies in the arc containing A). KE meets AB in Q. Prove that MQ, AH and EF are concurrent.

I have not read Luis's solution.....so my solution may be just same as him...

First of all we know that the reflection of H wrt M lies on the circumcircle of $\triangle ABC$ and its the point where the line AO meets $\odot ABC$ for the 2nd time. (O is the circumcenter of $\triangle ABC$). So from this we conclude that $AK \perp HK$, so A, K, H, E, F are cyclic.AK, EF, BC concur due to radical axis theorem on the cirlces $\odot ABC, (M, MB), (AEHF)$. Let $EF \cap BC = X$. And $AH \cap EF = X'$. Then $(X, E, X', F) = -1$.Now let $MQ \cap EF = X''$. Note that ME, MF are tangents to $(AEHF)$.So M is the polar of EF wrt $(AEHF)$.So, MQ is the polar of X wrt $(AEHF)$. Hence $(X, E, X'', F) = -1$. So we conclude that $X' \equiv X''$.So AH, MQ, EF concur.

jayme

#5 Apr 16, 2015, 5:48 pm

Dear Mathlinkers,

1. (A*) the circle with diameter AH
2. Tf the tangent to (A*) at F goes through M
3. by Pascal theorem in a degenerated case, we are done...

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Angle trisection and and incenter

Reply

Source: Iberoamerican SL G4

RobRoobiks

#1 Jun 5, 2012, 12:13 am

Let ABC be a triangle such that is D on side BC and $\angle DAC$ is twice $\angle BAD$ Let w be the incircle of triangle ADC with center I. The circumcircle of triangle AIB intersects w at X and Y. Let P be the intersection of XY and AI, and M the perpendicular of I at AB. Show that $4AP \cdot PI = MI^2$

yetti

#2 Jun 5, 2012, 3:59 am

Let (Z) be circumcircle of $\triangle AIB$. Let incircle (I) of $\triangle ADC$ touch AD, AC at F, E.
Let external bisector of $\angle CAD$ cut the center line IZ at Q. Let J (point at infinity) be direction of $AQ \perp AI$.
IM, IZ are isogonals WRT $\angle AIB \Longrightarrow \triangle MBI \sim \triangle AQI$ by ASA $\Longrightarrow \angle AQI = \angle MBI = \angle ABI \Longrightarrow Q \in (Z)$ $\Longrightarrow$
Q is pole of XY WRT (I). Since A, J are poles of EF, AI, respectively, WRT (I) and Q, A, J are collinear, their polars XY, EF, AI are concurrent at P, the midpoint of EF.

$$MI^2 = \left(AI \cdot \sin \widehat{IAB}\right)^2 = \left(2 \cdot AI \cdot \sin \widehat{IAD} \cdot \cos \widehat{IAD}\right)^2 =$$
$$\left(2 \cdot IF \cdot \cos \widehat{IAD}\right)^2 = (2 \cdot PF)^2 = 4 \cdot EP \cdot PF = 4 \cdot XP \cdot PY = 4 \cdot AP \cdot PI$$

Luis González

#3 Jun 5, 2012, 4:05 am

Any circle through A, I cutting (I) at X, Y does the job. Since AI bisects both $\angle XAY$ and $\angle DAC$, then the external bisector $\ell \perp AI$ of $\angle DAC$ also bisects $\angle XAY$ externally. If $E \equiv XY \cap \ell$, then $(P, X, Y, E) = -1 \Longrightarrow \ell$ is polar of P WRT (I). If (I) touches AC, AD at U, V, then $P \equiv XY \cap UV \cap AI \Longrightarrow$

$$AP \cdot PI = UP^2 = \tfrac{1}{4}UV^2 = \tfrac{1}{4}AI^2 \sin^2 \widehat{UAV} = \tfrac{1}{4}MI^2.$$

Quick Reply

Spain

13rd IBERO - REPUBLICA DOMINICANA 1998.

Reply

carlosbr

#1 Mar 26, 2006, 8:41 pm

$13rd$ Iberoamerican Olympiad

Puerto Plata, REPUBLICA DOMINICANA. [1998]

Edited by djimenez

Carlos Bravo

Attachments:

1998.pdf (26kb)

Luis González

#2 Dec 26, 2009, 12:25 am

> *Quote:*
> **Problema 1.** Hay 98 puntos en un círculo. Dos jugadores juegan alternadamente de la siguiente manera: Cada jugador une dos puntos que no estan ya unidos. El juego termina cuando al menos, cada punto ha sido unido a otro. El ganador es el último jugador a jugar. ¿Quién de los dos jugadores (primero o segundo) tiene la estrategia ganadora?

Asumamos que hay n puntos. El primero en jugar, de modo que n-2 puntos tienen al menos un segmento, pierde, ya que el otro jugador simplemente une los últimos dos puntos y el juego termina. Pero hay N=(n-3)(n-4)/2 jugadas posibles entre los primeros n-3 puntos para obtener un segmento. Para n=1 ó 2 mod 4, N es impar y para n=0 ó 3, N es par. Así, el primer jugador en turno gana para n=1 ó 2 mod 4 (y en particular para el caso de 98 puntos) y el segundo jugador gana para n=0 ó 3 mod 4.

Luis González

#3 May 5, 2012, 3:03 am

> *Quote:*
> **Problema 2.** En un triángulo $\triangle ABC$, sean D, E, F los puntos de tangencia de su incirculo (I) con AC, AB, BC. BD corta a (I) nuevamente en P y $M \equiv EP \cap BC$. Probar que M es punto medio de BF si y solo si $AB = AC$.

Supongamos que $\triangle ABC$ es isósceles con $AB = AC$. Como el ángulo $\angle EPD$ está inscrito en (I), resulta

$$\angle EPD = \angle EFD = \frac{\pi - \alpha}{2} \Longrightarrow \angle EBM = \angle BPM = \frac{\pi - \alpha}{2}$$

Por otro lado, como $\angle BEM = \angle EDP = \angle PBM$, ya que $ED \parallel BC$, resulta $\triangle BEM \sim \triangle PBM$

$$\Longrightarrow \frac{BM}{ME} = \frac{MP}{BM}, \ BM^2 = MP \cdot ME$$

Pero precisamente $MP \cdot ME$ es la potencia de M con respecto a (I), entonces $BM^2 = MP \cdot ME = MF^2 \Longrightarrow BM = MF$. Si $BM = MF$, se tiene $\triangle BEM \sim \triangle PBM$ y luego $\angle BPM = \angle ABC$ implica que $AB = AC$.

Luis González

#4 May 7, 2012, 9:13 am

> *Quote:*
>
> **Problema 3.** Hallar el número natural mínimo n con la siguiente propiedad: Entre cualquier colección de distintos n en el conjunto $\{1, 2, ..., 999\}$, es posible tomar cuatro números a, b, c, d diferentes, tales que $a + 2b + 3c = d$.

Consideremos el conjunto $S = \{166, 167, ..., 999\}$. El valor mínimo de $a + 2b + 3c$ para a, b, c diferentes en S es $168 + 2 \cdot 167 + 3 \cdot 166 = 1000$. Así, no se pueden encontrar a, b, c, d diferentes en S con $a + 2b + 3c = d$. Por ello, el mínimo n debe ser mayor que 834. Ahora, supongamos que S es cualquier subconjunto de 835 elementos que satisfacen la condición. Tomemos sus elementos como $m = a_1 < a_2 < ... < a_{835} = M$. Claramente, $M \geq m + 834 \geq 835$, así $-3m \geq 3 \cdot 834 - 3M$ y por ello $M - 3m \geq 2502 - 2M \geq 2502 - 2 \cdot 999 = 504$. Sea $k = M - 3m$. Entonces, hay al menos 167 parejas disjuntas (a, b) tomadas de $\{1, 2, ..., 999\}$ con $a + 2b = k$, a saber: $(k - 2, 1), (k - 4, 2)$, $(k - 6, 3), ..., (k - 334, 167)$. Note que en el caso extremo $k = 504$, ello es $(170, 167)$.

Al menos, un número de cada par debe ser M o m, o no pertenece a S, o de otro modo tendríamos $a + 2b + 3m = M$ para distintos a, b, m, M en S. Ninguno puede ser M y a lo mas uno de ellos puede ser m, así se tiene al menos 166 números que no están en S. Esto significa que S contiene a lo mas 833 números, lo cual es una contradicción. Por ello S no puede tener 835 elementos, ni puede tener mas de 835 elementos.

Luis González

#5 May 7, 2012, 10:58 pm

> *Quote:*
>
> **Problema 4.** Hay representantes de n paises diferentes sentados alrededor de una mesa circular $(n \geq 2)$, de modo que si dos representantes son del mismo país, entonces sus vecinos de la derecha no son del mismo país. Hallar, para cada n, el número máximo de personas que se pueden sentar en la mesa.

Obviamente no pueden haber mas de n^2 personas. Representemos a alguien de un país i por i. Entonces para $n = 2$ el arreglo 1122 funciona. Supongamos que se tiene un arreglo para n. Entonces cada $11, 22, ..., nn$ deben ocurrir solo una vez en el arreglo. Reemplazemos 11 por $1(n+1)11, 22$ por $2(n+1)22, ...,$ y $(n-1)(n-1)$ por $(n-1)(n+1)(n-1)(n-1)$. Finalmente reemplazemos nn por $n(n+1)(n+1)nn$. Es facil probar que ahora se tiene un arreglo para $n + 1$. Hemos añadido un representante adicional por cada uno de los paises 1 a n y $n + 1$ representantes para el país $n + 1$, así hemos obtenido en efecto $(n+1)^2$ personas en total. Hemos también dado a cada representante de cada país 1 a n un vecino del pais $n + 1$ en su lado derecho y hemos dado a los $(n + 1)$ representantes del país $(n + 1)$, vecinos (a su derecha) de cada uno de los otros países.

Luis González

#6 May 8, 2012, 11:59 pm

> *Quote:*
>
> **Problema 5.** Hallar el máximo valor posible de n, tal que existen puntos $P_1, P_2, P_3, \ldots, P_n$ en el plano y números reales $r_1, r_2, \ldots, r_n$, tal que la distancia entre dos puntos diferentes cualesquiera P_i y P_j es $r_i + r_j$.

Para cada i, j, k, l, note que $P_iP_j + P_kP_l = P_iP_k + P_jP_l = P_iP_l + P_jP_k$. Esto implica claramente que, o se tienen algunos 3 puntos alineados, en cuyo caso éstos son todos los puntos en nuestro conjunto $(n \leq 3)$, o no hay 3 puntos alineados y por tanto no se forma ningún cuadrilátero convexo. Por teorema de Erdos-Szekeres, entre 5 puntos cualesquiera en el plano, no alineados 3 de ellos, se puede encontrar vértices de un cuadrilátero convexo. Esto significa que $n \leq 4$. Note que es posible tener $n = 4$, a saber, tómese 4 círculos tangentes externamente dos a dos cuyos centros son P_1, P_2, P_3, P_4 y tómerse r_i como sus correspondientes radios (Círculos de Soddy).

Luis González

#7 Jun 4, 2012, 10:26 pm

> *Quote:*
>
> **Problema 6.** Sea λ una raiz positiva de la ecuación $x^2 - 1998x - 1 = 0$. Se define la secuencia $x_0, x_1, x_2, \ldots, x_n, \ldots$ para $x_0 = 1,\ x_{n+1} = \lfloor \lambda x_n \rfloor, n = 1, 2 \ldots$ Hallar el residuo de la división de x_{1998} por 1998.

Se tiene $\dfrac{x_{n+1}}{\lambda} < x_n < \dfrac{x_{n+1}+1}{\lambda} \Longrightarrow \left[\dfrac{x_{n+1}}{\lambda}\right] = x_n - 1.$

Como $\lambda = 1998 + \dfrac{1}{\lambda}, x_{n+1} = [\lambda x_n] = 1998x_n + \left[\dfrac{x_n}{\lambda}\right] = 1998x_n + x_{n-1} - 1$, entonces resulta

$x_{n+1} \equiv x_{n-1} - 1 \pmod{1998}$. Por consiguiente $x_{1998} \equiv 1000 \pmod{1998}$.

Quick Reply

High School Olympiads

Concyclic 3

Reply

buratinogigle

#1 Jun 3, 2012, 9:55 pm • 1

Let $ABCD$ be a parallelogram. (O) is circumcircle of triangle ABC. P is a point on BC. K is circumcenter of triangle PAB. L is in AB such that $KL\perp BC$. CL cuts (O) again at M. Prove that M,P,C,D are concyclic.

Note that, it is generalization of nice problem form Russian olympiad K, H, C and D are concyclic.

Luis González

#2 Jun 4, 2012, 6:19 am • 1

Let $\odot(PCD)$ cut AD again at E. Then $\angle DEP=\angle DCP=\angle ABP \pmod\pi \Longrightarrow E\in\odot(PAB)$. Further, $EAPB$ is an isosceles trapezoid with circumcircle (K), thus by symmetry $L\equiv AB\cap EP$. PE is radical axis of $(K),\odot(PCD)$ and AB is radical axis of $(K),\odot(ABC)\Longrightarrow L$ is radical center of $(K),\odot(PCD),\odot(ABC)\Longrightarrow LC$ is radical axis of $\odot(ABC),\odot(PCD)\Longrightarrow M\in\odot(PCD)$.

Quick Reply

High School Olympiads

Prove that MB = MG

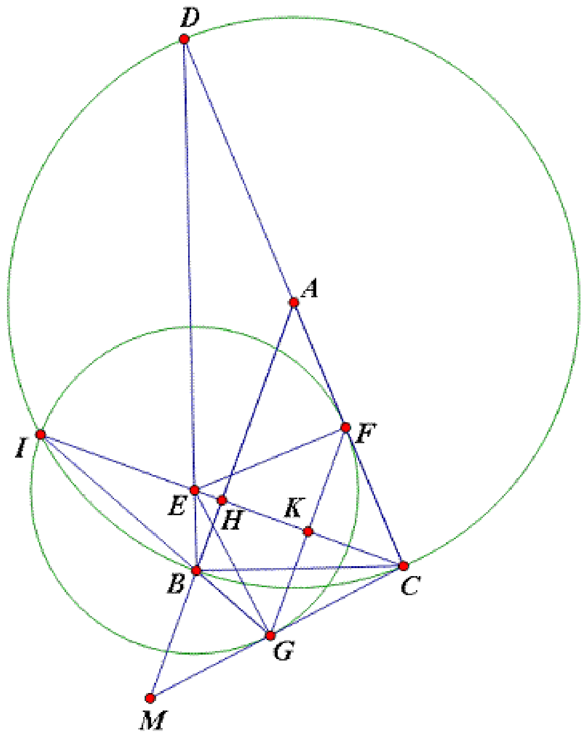
Reply

thanhnam2902

#1 Jun 2, 2012, 6:44 pm

Let BCD is a right triangle with $\angle CBD = 90^0$. Let A is the midpoint of CD and $\angle BAC < 60^0$. Let E is a point lie on BD satisfy $CE \perp AB$ at point H. Let F is the perpendicular foot of point E on CD. Let G is a point lie on the circle $(E; EF)$ satisfy $CE \perp FG$. Let CG meet AB at point M. Prove that $MB = MG$.

Attachments:



Luis González

#2 Jun 3, 2012, 2:05 am • 1

Let CE cut $\odot(BDC)$ again at $I. \angle IDB = \angle ECB = \angle ABD = \angle ADB \Longrightarrow I$ is reflection of F about BD, i.e. I lies on the circle (E, EF). Let FI cut BA, BD at $N, L. \triangle BCE \sim \triangle FDE$ are clearly similar with corresponding altitudes $BH, FL \Longrightarrow \frac{EH}{HC} = \frac{EL}{LD} \Longrightarrow HL \parallel CD \Longrightarrow \angle ELH = \angle ENH = \angle EDC = \angle EBH \Longrightarrow N$ is reflection of B about CE. Since G is obviously the reflection of F about CE and CG is tangent to (E, EF), we have that B, G, I are collinear and $\angle MGI = \angle GFI = \angle NBI \Longrightarrow \triangle MBG$ is isosceles with $MB = MG$.

dgreenb801

#3 Jun 5, 2012, 10:44 am

We see $EGBCF$ is cyclic with diameter EC, let its circumcircle intersect AB at N, then the pairs (A, M), (N, B), (F, G) are symmetric with respect to EC, so to show $MB = MG$ is the same as showing $AN = AF$. But this is obvious by power of a point on A, since $AN \cdot AB = AF \cdot AC$.

Quick Reply

High School Olympiads

On mixtilinear incircles 2

Reply

Fang-jh
#1 Jun 2, 2012, 3:34 pm • 1

Given a triangle ABC, let its B-,C-mixtilinear incircles touch its circumcircle at E, F, respectively. Let its A-mixtilinear incircle touch the sides AB at D. prove that the bisector of $\angle DFE$ passes through the incenter of triangle ABC.

Luis González
#2 Jun 2, 2012, 10:50 pm • 1

B-mixtilinear incircle and C-excircle of $\triangle ABC$ touch AB at P, Q, respectively. CF, CQ are isogonals WRT $\angle ACB$ (see hard and very nice and elsewhere). Therefore we have

$$\frac{FB}{AQ}=\frac{a}{CQ}, \ \frac{BQ}{FA}=\frac{CQ}{b} \Longrightarrow \frac{FB}{FA}=\frac{a}{b}\cdot\frac{AQ}{BQ}=\frac{a}{b}\cdot\frac{s-b}{s-a}$$

On the other hand, we know that $\frac{BD}{DA}=\frac{s-b}{b}, \ \frac{BP}{PA}=\frac{a}{s-a}$

Now, keeping in mind that EP bisects $\angle AEB$, due to the internal tangency of $\odot(ABC)$ with the B-mixtilinear incircle, by angle bisector theorem, we deduce then

$$\frac{BD}{DA}\cdot\frac{BE}{EA}=\frac{BD}{DA}\cdot\frac{BP}{PA}=\frac{s-b}{b}\cdot\frac{a}{s-a}=\frac{FB}{FA}.$$

This latter relation reveals that FE, FD are isogonals WRT $\angle AFB$. But the internal angle bisector of $\angle AFB$ passes through the the incenter I of $\triangle ABC$ (see incenter of triangle), thus FI bisects $\angle DFE$, as desired.

TelvCohl
#3 Oct 23, 2014, 9:45 pm • 1

My solution:

Let X be the tangent point of B -Mixtilinear circle with AB and M be the midpoint of arc AB .

From homothety we get E, X, M are collinear .

Since M, F, D, X are concyclic (well-known) ,
so $\angle ADF=180^\circ-\angle FME=180^\circ-\angle FAE$.
i.e. $\triangle FDB \sim \triangle FAE$ and $\angle BFD=\angle EFA$ ... $(\star)$
Since the bisector of $\angle BFA$ passes through the incenter of $\triangle ABC$,
so combine with $(\star)$ we get the bisector of $\angle DFE$ passes through the incenter of $\triangle ABC$.

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Jun 19, 2015, 6:48 am

Quick Reply

High School Olympiads

Circles involving incenters

Reply

Source: Costa Rican Final TST 2012, problem 3

hatchguy

#1 Jun 2, 2012, 7:14 am • 1

Let P be a variable point in arc BC of the circumcircle of ABC. Let I_1 and I_2 be the incenters of APB and APC, respectively.

a) Show that the circumcircle of PI_1I_2 passes through a fixed point.

b) Show that the circle with diameter I_1I_2 passes through a fixed point.

c) Show that the midpoint of I_1I_2 lies on a fixed circle.

Luis González

#2 Jun 2, 2012, 7:37 am • 2

These are exactly the questions of Iran Pre-Preparation Course Examination 1997, E2, P2, which have appeared separately on different post many times. e.g.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=24483
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=313870
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=355595

Quick Reply

High School Olympiads

Tedrahedron ABCD

Reply

Source: IMO Shortlist 1988, Problem 6, Czech Republic 3, Problem 8 of ILL

orl

#1 Oct 22, 2005, 8:11 pm

In a given tedrahedron $ABCD$ let K and L be the centres of edges AB and CD respectively. Prove that every plane that contains the line KL divides the tedrahedron into two parts of equal volume.

tc1729

#2 Mar 4, 2012, 7:01 am

Let p be the plane through AB parallel to CD. Any plane p' parallel to p which intersects the segment KL will intersect the tetrahedron in a parallelogram with center on the line KL. Any plane q through KL will pass through the center of the parallelogram and hence divide it into two equal parts. So suppose q divides the tetrahedron into two parts T and T'. We have established that the area of the intersection of p' and T is equal to the area of the intersection of p' and T'. It follows from Cavalieri's principle that T and T' have equal volume.

(To see that $PQRS$ is a parallelogram, note that PQ is parallel to CD, and RS is parallel to CD, so PQ and RS are parallel. Similarly QR and PS are parallel to AB and hence to each other.)

Luis González

#3 Jun 2, 2012, 2:56 am

Arbitrary plane through K,L cuts AD,BC at $X,Y.$ It's clear that the locus of the points that bisect the segments with endpoints on $BC,AD,$ is the midplane λ parallel to those passing through BC,AD and parallel to each other, thus $KL \in \lambda \Longrightarrow KL$ bisects $\overline{XY} \Longrightarrow X,Y$ are equidistant from the plane $CDK.$ Since $[DLK]=[CLK],$ then tetrahedra $[DLKX]=[CLKY]$ are equivalent. If V_1 and V_2 denote the volumen of the pentahedra $LXKYBD$ and $LXKYCA,$ we have then

$$V_1=[DKBYL]+[CLKY]=[DKBC]=\tfrac{1}{2}[ABCD]$$

Similarly, $V_2=\tfrac{1}{2}[ABCD] \Longrightarrow V_1=V_2=\tfrac{1}{2}[ABCD]$

Quick Reply

High School Olympiads

Prove relation in special triangle

Reply

sunken rock

#1 Jun 2, 2012, 1:46 am

Given the triangle $\triangle ABC$ with $m(\angle A)=48^\circ, m(\angle C)=18^\circ$, prove the relation:
$AC\cdot(AC-AB)=BC^2$.
In connection with **this problem**.

Note: synthetic solution, please!

Best regards,
sunken rock

Luis González

#2 Jun 2, 2012, 2:41 am

Take point D on segment $\overline{AC}$, such that $AB=AD$. Angle chase reveals that $\angle ABC=114^\circ$ and $\angle ABD=66^\circ \Longrightarrow$ $\angle CBD=114^\circ-66^\circ=48^\circ \Longrightarrow BC$ is tangent to $\odot(ABD)$. Thus $BC^2=CD\cdot AC=(AC-AB)\cdot AC$.

P.S. The property is valid for all $\triangle ABC$ with $\angle C+\frac{3}{2}\angle A=90^\circ$.

Quick Reply

High School Olympiads

euler line

Locked

Source: Italian Prelmo 2012

ctumeo

#1 Jun 2, 2012, 12:23 am

Let ABC be a triangle, and P an internal point so that the angles APB, BPC, CPA are all equals 120°.
Prove that the Euler lines of triangles APB, BPC, CPA are concurrent (i.e. they meet at the same point).
(My English is poor)

I hope for a simple and clear solution.
Thanks

Luis González

#2 Jun 2, 2012, 12:50 am • 2

Posted many times before. They concur at the centroid of ABC.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=6188
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=47955
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=178003
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=211789
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=354576

High School Olympiads

prove that :BC=CD

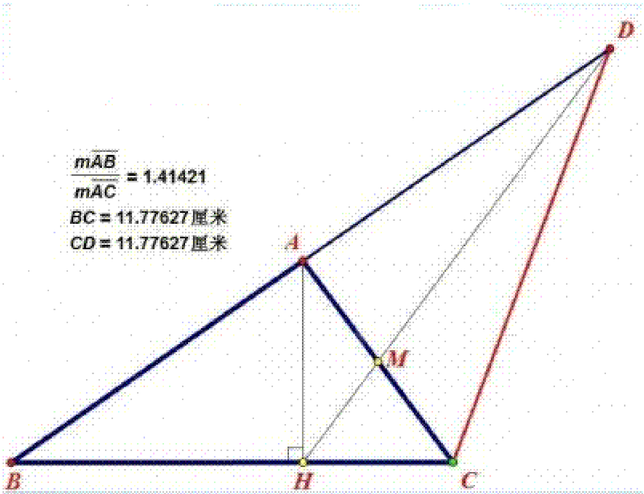
Reply

tian_275461

#1 May 31, 2012, 11:13 pm

Let $\triangle ABC, AB=\sqrt{2}AC, AH\perp BC$ and $AM=MC.$
Prove that $BC=CD.$

Attachments:



sunken rock

#2 May 31, 2012, 11:36 pm

It seems you missed $AC\bot AB$, in which case is a straight forward application of **Menelaos**!

Best regards,
sunken rock

tian_275461

#3 Jun 1, 2012, 12:21 am

> *sunken rock wrote:*
> It seems you missed $AC\bot AB$, in which case is a straight forward application of **Menelaos**!

Thank you , This problem don't have $AC\bot AB$.

yetti

#4 Jun 1, 2012, 9:21 am

$BH^2+HA^2=BA^2=2\cdot AC^2=2\cdot (HC^2+HA^2)\Longrightarrow HA^2=BH^2-2\cdot HC^2.$

Coordinate origin at H, positive x-axis along the ray $(HC.$

Line $BA:\quad y=\frac{\overline{HA}}{\overline{BH}}\cdot (x+\overline{BH}),$

Line $HM:\quad y=\overline{HA}\ \ x$

Line $HM: \quad y = \frac{\quad}{\overline{HC}} \cdot x,$

Point $D \equiv BA \cap HM: \quad x_D = \frac{\overline{BH} \cdot \overline{HC}}{\overline{BH} - \overline{HC}}, \; y_D = \frac{\overline{BH} \cdot \overline{HA}}{\overline{BH} - \overline{HC}}.$

$CD^2 = (x_D - x_C)^2 + (y_D - y_C)^2 =$

$= \frac{\left[\overline{BH} \cdot \overline{HC} - \overline{HC} \cdot (\overline{BH} - \overline{HC})\right]^2 + (BH \cdot HA)^2}{(\overline{BH} - \overline{HC})^2} = \frac{HC^4 + BH^2 \cdot HA^2}{(\overline{BH} - \overline{HC})^2} =$

$= \frac{HC^4 + BH^4 - 2 \cdot BH^2 \cdot HC^2}{(\overline{BH} - \overline{HC})^2} = \frac{(BH^2 - HC^2)^2}{(\overline{BH} - \overline{HC})^2} = (\overline{BH} + \overline{HC})^2 = BC^2.$

Luis González

#5 Jun 1, 2012, 10:22 am

Let K be the foot of the C-altitude and D^* the reflection of B about K. Thus $BC = CD \Longleftrightarrow D \equiv D^*$. We use barycentric coordinates WRT $\triangle ABC$ with Conway's notation. $H(0 : S_C : S_B), M(1 : 0 : 1)$. Reflection of $B(0 : 1 : 0)$ about $K(S_B : S_A : 0)$ is $D^*(2S_B : S_A - S_B : 0)$

$HM \equiv S_C(x - z) + S_B y = 0$ cuts $AB \equiv z = 0$ at $D(S_B : -S_C : 0)$

Hence, $D \equiv D^* \Longleftrightarrow -2S_C = S_A - S_B \Longleftrightarrow 2b^2 = c^2 \Longleftrightarrow c = \sqrt{2}b.$

Quick Reply

High School Olympiads

Bosnia and Herzegovina TST 2012 Problem 5

Reply

teps

#1 May 20, 2012, 7:22 pm

Given is a triangle $\triangle ABC$ and points M and K on lines AB and CB such that $AM=AC=CK$. Prove that the length of the radius of the circumcircle of triangle $\triangle BKM$ is equal to the lenght OI, where O and I are centers of the circumcircle and the incircle of $\triangle ABC$, respectively. Also prove that $OI \perp MK$.

Luis González

#2 May 20, 2012, 9:00 pm • 1

Posted many times before. It's also P2 of Turkey National Olympiad 2002.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=14655
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=52563
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=30539
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http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=182301
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=332588
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=395162

novae

#3 Jul 26, 2012, 8:49 pm • 1

> teps wrote:
>
> Given is a triangle $\triangle ABC$ and points M and K on lines AB and CB such that $AM=AC=CK$. Prove that the length of the radius of the circumcircle of triangle $\triangle BKM$ is equal to the lenght OI, where O and I are centers of the circumcircle and the incircle of $\triangle ABC$, respectively. Also prove that $OI \perp MK$.

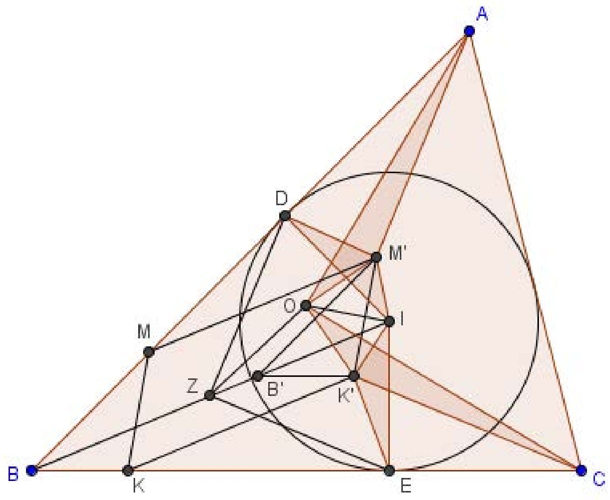
Let M' be intersection of the perpendicular bisector of AM with the line passes through M and parallel to BI, K' is defined similarly; D, E be orthogonal projections of I on BA, BC respectively. Let O^* be the circumcenter of $\triangle M'IK'$.
================================
Firstly, we will prove that $\triangle IM'D = \triangle IK'E$.
Since $\triangle AMM' = \triangle KCK', AD = KE, MD = CE$ and $ID \perp AB, IE \perp BC$, it's easy to verify that $DM' = EK', \widehat{IDM'} = \widehat{IEK'}$. Thus $\triangle IM'D = \triangle IK'E$.
================================
Secondly, we will prove that $O^* \equiv O$.
Let Z be the midpoint of BI. Since O^* is the center of $(IM'K')$, we obtain that $\triangle O^*M'K' \sim \triangle ZDE$. Note that $AM' \parallel ZD$ and $CK' \parallel ZE$, we have

$$\begin{aligned}(M'O^*, M'A) &\equiv (M'O^*, K'O^*) + (K'O^*, K'C) + (K'C, M'A)\\ &\equiv (O^*M', O^*K') + (K'O^*, K'C) + (ZE, ZD)\\ &\equiv (K'O^*, K'C) \pmod{\pi}.\end{aligned}$$

Hence $\triangle O^*M'A = \triangle O^*K'C$ (because $M'A = K'C$). Then $\triangle O^*AC \sim \triangle O^*M'K' \sim \triangle ZDE \sim \triangle OAC$. Thus $O^* \equiv O$.
================================
Now, let B' be intersection of (O^*) with BI (B' differs from I). Hence $B'M' \parallel BM, B'K' \parallel BK$. Moreover, $M'K' \parallel MK$ and $M'K' = MK$ (since $MKM'K'$ is a parallelogram). Therefore, $R_{(BMK)} = R_{(B'M'K')} = OI$.
From part 1 and 2, we deduce that OI is the perpendicular bisector of $M'K'$. Thus $OI \perp M'K'$, equivalently, $OI \perp MK$.

(Q.E.D.)

Attachments:



subham1729

#4 Aug 6, 2012, 10:36 am • 1

For the first part we've to show $MK^2=\frac{(R-2r)b^2}{4R}$
It's easy because using cosine rule we'll get MK^2 easily.
For second part, We've to show $OKTD$ cyclic.
Let $\angle IOD=x$.
So $\frac{b-c}{2OI}=Sinx$
Also we've $2OI=\frac{b-c}{\angle BKM}$.... from BKM
So we get $\angle BKM=x$ hence done.

Quick Reply

High School Olympiads

Isogonal conjugates, MP perpendicular to CF

Reply

Source: Serbia Additional TST 2012, Problem 3

Djile
#1 May 19, 2012, 8:18 pm • 1

Let P and Q be points inside triangle ABC satisfying $\angle PAC = \angle QAB$ and $\angle PBC = \angle QBA$.

a) Prove that feet of perpendiculars from P and Q on the sides of triangle ABC are concyclic.

b) Let D and E be feet of perpendiculars from P on the lines BC and AC and F foot of perpendicular from Q on AB. Let M be intersection point of DE and AB. Prove that $MP \perp CF$.

Luis González
#2 May 19, 2012, 9:04 pm • 8

a) See Nine-point circle generalization, isogonal, pedal circle theorem and elsewhere.

b) Let K be the orthogonal projection of P on AB. Let the circumcircles of $\triangle PKF$ and $PDCE$ meet again at N. DE, PN, FK are pairwise radical axes of $\odot(PDCE), \odot(PKF)$ and $\odot(DEFK)$ concurring at their radical center M, i.e. $N \in PM. \angle PNC = \angle PEC = 90^\circ$, i.e. $MP \perp CF$.

crazyfehmy
#3 May 25, 2012, 4:02 am

The part b) was the last question in BMO2012 Shortlist.

Swistak
#4 May 26, 2012, 9:53 pm

Here's solution to b) using vectors.
Let G be the foot of the perpendicular to AB from P and H foot of the perpendicular to AC from Q.
$CQ \perp DE$ since P ang Q are isogonal conjugate.
Let me write XY instead of $\vec{XY}$, because writing second ones many times is exhausting.
$MP \cdot CF = MP \cdot PF - MP \cdot PC$
$MP \cdot PF = MP \cdot (PQ + QF) = MP \cdot PQ + (MG + GP) \cdot QF = MP \cdot PQ + GP \cdot QF$
$-MP \cdot PC = -(ME + EP) \cdot PC = MY(CQ + QP) - EP \cdot PC = ME \cdot QP - EP \cdot (PE + EC) = ME \cdot QP + PE^2$
$MP \cdot CF = MP \cdot PF - MP \cdot PC = MP \cdot PQ + GP \cdot QF + ME \cdot QP + PE^2 = (MP - ME) \cdot PQ + PE^2 + GP \cdot QF = EP(PQ + EP) + GP \cdot QF = EP \cdot EQ - PG \cdot QF = EP \cdot HQ - PG \cdot QF = 0$
since $AEPG$ and $AFQH$ are similar.

KOSNITA
#5 Jul 16, 2012, 9:26 pm

The thing is to notice that P and Q are isogonal conjugates of each other.
Then the proofs are corollaries of the property.

Quick Reply

High School Olympiads

Parallelogram, segment ratio

Reply

sunken rock

#1 May 17, 2012, 7:15 pm

$ABCD$ is a parallelogram, with $AB=AC, E$ is the midpoint of AB, F the second intersection of the median CE with the circumcircle of $\triangle ABC$, its center being O, and H is the midpoint of BC.
From 9th grade problem of Final MO 2012, Russia, $FHCD$ is cyclic, let's O_1 be its circumcenter.
Prove that the centroid G of $\triangle ACE$ lies on the segment OO_1, dividing it into the ratio: $\frac{OG}{GO_1}=\frac{1}{2}$.

Best regards,
sunken rock

Luis González

#2 May 19, 2012, 12:11 am

First, we prove that $FHCD$ is indeed cyclic. Let K be the midpoint of $\overline{AC}$. From $\triangle AEF \sim \triangle CEB \cong \triangle BKC$ we deduce that $\frac{AE}{CE}=\frac{EH}{KD}=\frac{EF}{EB}=\frac{EF}{KH}$. But since $\angle HEF=\angle DKH$, then $\triangle HEF \sim \triangle DKH \Longrightarrow$ $\angle EHF=\angle KDH \Longrightarrow \angle DHF=\angle DCA+\angle DHK+\angle KDH=\angle DCA+\angle BKH=\angle DCF.$

Let L be the midpoint of $\overline{AE}$. AH cuts EK at M and the circumcircles of $FHCD$ and $\triangle ABC$ again at P and $Q \Longrightarrow$ $O_1 \in CP$. Clearly, $\triangle BEM$ and $\triangle DCH$ are homothetic with homothetic coefficient $-\frac{1}{2}$, thus their circumdiameters EQ, CP issuing from E, C are parallel, i.e. $LO \parallel EQ \parallel CO_1$ and $LO=\frac{1}{2}EQ=\frac{1}{2}CO_1 \Longrightarrow O$ is the complement of O_1 WRT $\triangle ACE$, i.e. $G \in OO_1$ such that $\overline{GO}:\overline{GO_1}:-1:2.$

sunken rock

#3 May 21, 2012, 9:20 pm

Remarks:
For the fact that $FHCD$ is cyclic, in general case ($AB \neq AC$) and H the projection of A onto BC, see my proof **here**. Over there $KHCD$ is required to be proved cyclic;

@luis: it seems that, applying the power of E w.r.t $\odot(ABC)$ solves little bit faster:
$EF\cdot CE=AE^2 \iff EF\cdot KD=EH\cdot KH$, a.s.o.

I shall post later my proof of the problem.

Best regards,
sunken rock

sunken rock

#4 May 23, 2012, 2:11 pm

Let $K=AC\cap BD, L$ antipodal of A in $\odot(ABC), M$ the midpoint of CD. K, O_1 lie on the perpendicular bisector of CH. Since $O_1M\perp CD$, by symmetry $O_1E\perp HE$, but $HE \parallel AC$; also $OK\perp AC \implies EO_1 \parallel OK \parallel CL$. From $EK=CH$ and $\angle KEO_1=\angle HAC=\angle HCL \implies \triangle EKO_1=\triangle CHL \implies EO_1=CL$, but $CL=2\cdot OK$, hence if $G'\in OO_1\cap EK$, then $\frac{G'K}{EG'}=\frac{1}{2}=\frac{G'O}{G'O'}$, or $G'\equiv G$, done.

Remark: the second intersection $FD\cap\odot(ABC)$ lies on the L-median of $\triangle ADL$.

Best regards,
sunken rock

Quick Reply

High School Olympiads

A new central line

Reply

Source: own

jayme
#1 May 9, 2012, 3:02 pm

Dear Mathlinkers,

1. ABC a triangle,
2. P a point
3. (1a) the circumcircle of the triangle PBC
4. A' the second point of intersection of AP with (1a)
5. A'' the A'-de Longchamps point of the triangle A'BC
6. A* the point of intersection of PA'' and BC, and circularly B*, C*.

Conjecture : A*, B*, C* are collinear .

Note : A'' is the point of contact of the A-mixtilinear incircle of A'BC.

Sincerely
Jean-Louis

Luis González
#2 May 10, 2012, 7:10 am

Using the concyclicity of P, A'', B'', C'', which holds for any three points equally characterized on the circumcircles of $\triangle BCA', \triangle CAB', \triangle ABC'$, it follows that A^*, B^*, C^* lie on the the radical axis of $\odot(A''B''C'')$ and the circumcircle of $\triangle ABC$. Now, we prove an adittional result on this configuration, which gives us a different characterization of the referred points.

Proposition. PA, PB, PC cut BC, CA, AB at P_1, P_2, P_3. Then A^* is the exsimilicenter of the incircles of $\triangle BPP_1, \triangle CPP_1$. Similarly, B^* is the exsimilicenter of the incircles of $\triangle CPP_2, \triangle APP_2$ and C^* is the exsimilicenter of the incircles of $\triangle APP_3, \triangle BPP_3$.

$$\frac{A^*B}{A^*C} = \frac{BA''}{CA''} \cdot \frac{\sin \widehat{BA''A^*}}{\sin \widehat{CA''A^*}} = \frac{BA''}{CA''} \cdot \frac{\sin \widehat{PCB}}{\sin \widehat{PBC}} = \frac{BA''}{CA''} \cdot \frac{PB}{PC} \ (1)$$

A'-excircle of $\triangle A'BC$ touches BC at E. Since $A'A'', A'E$ are isogonals WRT $\angle BA'C$ (well-known), it follows that

$$\frac{BA''}{CA''} = \frac{BA'}{CA'} \cdot \frac{CE}{BE} = \frac{\cot \frac{1}{2}\widehat{CBA'}}{\cot \frac{1}{2}\widehat{BCA'}} \cdot \frac{BA'}{CA'} = \frac{\tan \frac{1}{2}\widehat{BPP_1}}{\tan \frac{1}{2}\widehat{CPP_1}} \cdot \frac{BA'}{CA'} \ (2)$$

Combining the expressions (1) and (2) gives

$$\frac{A^*B}{A^*C} = \frac{\tan \frac{1}{2}\widehat{BPP_1}}{\tan \frac{1}{2}\widehat{CPP_1}} \cdot \frac{BA'}{CA'} \cdot \frac{PB}{PC} = \frac{\tan \frac{1}{2}\widehat{BPP_1}}{\tan \frac{1}{2}\widehat{CPP_1}} \cdot \frac{BP_1}{CP_1}$$

This latter expression not only proves that A^*, B^*, C^* are collinear, by multiplying the cyclic expressions together, but it also follows that the line connecting the incenters of $\triangle BPP_1$ and $\triangle CPP_1$ passes through A^*, and similarly for B^* and C^*, which proves our claim.

High School Olympiads

locus of P

Reply

Babai

#1 May 10, 2012, 12:35 am

If pedal triangle of P is perspective with main triangle find the locus of P.

Luis González

#2 May 10, 2012, 1:14 am

Locus is a circular self-isogonal cubic with pivot X(20)= De Longchamps point of ABC.

http://bernard.gibert.pagesperso-orange.fr/Exemples/k004.html
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=334909
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=290020

Quick Reply

High School Olympiads

collinear points

Locked

anca1997

#1 May 9, 2012, 9:06 pm

Let be ABC a triangle with $A, B, C < \pi/2$ and the points A_1, B_1 such as AA_1 and BB_1 are the heights of the triangle $A_1,\ B_1 \in AC$ and AA' and BB' are the bisector of the angles of the triangle and $A' \in BC,\ B' \in AC$. Show that $I \in A_1B_1 \Leftrightarrow O \in A'B'$

Luis González

#2 May 9, 2012, 9:51 pm

Posted before at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=50582
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=139275
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=219811

For a generalization see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=402242

High School Olympiads

Nice indentity

Reply

gemath

#1 Jul 26, 2007, 10:18 pm

Given triangle ABC with centroid G, orthocenter H and first and second Brocard points is Ω_1, Ω_2 prove that

$$G\Omega_1^2 + \Omega_1 H^2 + G\Omega_2^2 + \Omega_2 H^2 = 2GH^2$$

Luis González

#2 May 9, 2012, 12:27 pm

If U is the midpoint of $\overline{GH}$, then it suffices to show that $\Omega_1 U^2 + \Omega_2 U^2 = \frac{1}{2} GH^2$. We resort to barycentric coordinates with respect to $\triangle ABC$. Equation of orthocentroidal circle $\mathcal{G}$ on diameter $\overline{GH}$ is given by

$$\mathcal{G} \equiv a^2yz + b^2xz + c^2xy - \tfrac{2}{3}(x+y+z)(S_Ax + S_By + S_Cz) = 0.$$

Powers p_1, p_2 of $\Omega_1\left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right)$ and $\Omega_2\left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}\right)$ WRT $\mathcal{G}$ are then

$$p_1 = \frac{\frac{2}{3}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{S_A}{b^2}+\frac{S_B}{c^2}+\frac{S_C}{a^2}\right) - \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^2}$$

$$p_2 = \frac{\frac{2}{3}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{S_A}{c^2}+\frac{S_B}{a^2}+\frac{S_C}{b^2}\right) - \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^2}$$

Keeping in mind that $S_B + S_C = a^2, S_C + S_A = b^2$ and $S_A + S_B = c^2$, we get

$$p_1 + p_2 = \frac{1}{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^2}\left[3 \cdot \frac{2}{3}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) - 2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\right] = 0$$

$$\Longrightarrow \Omega_1 U^2 - \tfrac{1}{4}GH^2 + \Omega_2 U^2 - \tfrac{1}{4}GH^2 = 0 \Longrightarrow \Omega_1 U^2 + \Omega_2 U^2 = \tfrac{1}{2}GH^2.$$

Quick Reply

High School Olympiads

circles inscribed in the triangles

Reply

aktyw19

#1 May 7, 2012, 11:35 am

On the side BC of triangle ABC lies the point D. Points E and F are centres of circles inscribed in the triangles ABD and ACD. Prove that if the points B,C,E,F lie on a circle, then $\frac{AD+BD}{AD+CD}=\frac{AB}{AC}$

Luis González

#2 May 8, 2012, 4:46 am • 1

EF cuts AD,AC,AB at K,M,N and DF,DE cut AC,AB at $P,Q.$ If $BEFC$ is cyclic, then $\angle IEF=\angle ICB=\angle ICM \Longrightarrow EIMC$ is cyclic $\Longrightarrow \angle EMC=\angle BIC.$ Similarly, $\angle FNB=\angle BIC \Longrightarrow \triangle AMN$ is isosceles with legs $AM=AN.$ Hence, by angle bisector theorem, we get

$$\frac{AK}{AM}=\frac{FK}{FM} \ , \ \frac{AK}{AN}=\frac{EK}{EN} \Longrightarrow \frac{FM}{FK}=\frac{EN}{EK}$$

Thus, by Menelaus' theorem for $\triangle MAK, \overline{DFP}$ and $\triangle NAK,\overline{DEQ},$ we get

$$\frac{AQ}{AN} \cdot \frac{EN}{EK}=\frac{DA}{DK}=\frac{AP}{PM} \cdot \frac{FM}{FK} \Longrightarrow \frac{AP}{PM}=\frac{AQ}{QN} \Longrightarrow PQ \parallel EF \Longrightarrow$$

$$\frac{DE}{EQ}=\frac{DF}{FP} \Longrightarrow \frac{AD+BD}{AB}=\frac{AD+CD}{AC}.$$

hoangquan

#3 Nov 19, 2012, 4:13 pm

> Luis González wrote:
>
> EF cuts AD,AC,AB at K,M,N and DF,DE cut AC,AB at $P,Q.$ If $BEFC$ is cyclic, then $\angle IEF=\angle ICB=\angle ICM \Longrightarrow EIMC$ is cyclic $\Longrightarrow \angle EMC=\angle BIC.$ Similarly, $\angle FNB=\angle BIC \Longrightarrow \triangle AMN$ is isosceles with legs $AM=AN.$ Hence, by angle bisector theorem, we get
>
> $$\frac{AK}{AM}=\frac{FK}{FM} \ , \ \frac{AK}{AN}=\frac{EK}{EN} \Longrightarrow \frac{FM}{FK}=\frac{EN}{EK}$$
>
> Thus, by Menelaus' theorem for $\triangle MAK, \overline{DFP}$ and $\triangle NAK,\overline{DEQ},$ we get
>
> $$\frac{AQ}{AN} \cdot \frac{EN}{EK}=\frac{DA}{DK}=\frac{AP}{PM} \cdot \frac{FM}{FK} \Longrightarrow \frac{AP}{PM}=\frac{AQ}{QN} \Longrightarrow PQ \parallel EF \Longrightarrow$$
>
> $$\frac{DE}{EQ}=\frac{DF}{FP} \Longrightarrow \frac{AD+BD}{AB}=\frac{AD+CD}{AC}.$$

Point I is ?

sunken rock

#4 Nov 19, 2012, 10:23 pm • 2

Quite obvious, the incenter of ΔABC!

Best regards,
sunken rock

Quick Reply

High School Olympiads

Altitude feet & parallel lines

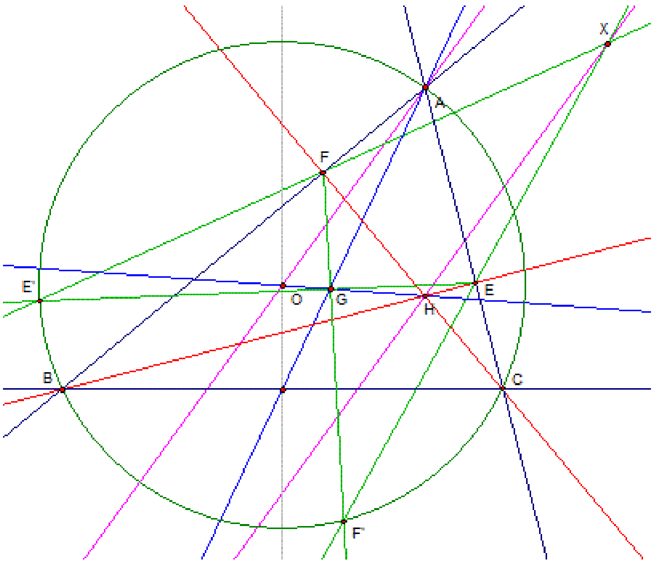
Reply

yetti

#1 May 7, 2012, 1:43 am • 1

$(O), H, G$ are circumcircle, orthocenter and centroid of $\triangle ABC$. $E \in CA, F \in AB$ are feet of the B-, C-altitudes BH, CH.
Rays $(EG, (FG$ cut (O) at E', F'. Let $X \equiv E'F \cap F'E$. Prove that $XH \parallel AO$.

Attachments:



Luis González

#2 May 7, 2012, 6:52 am

Since G is the center of the homothety with ratio $-\frac{1}{2}$ that takes (O) into the 9-point circle of $\triangle ABC$, then it follows that $\overline{GF} : \overline{GF'} = -1 : 2$ and $\overline{GE} : \overline{GE'} = -1 : 2 \Longrightarrow G$ is centroid of $\triangle XE'F'$. Further, $BE' \parallel AC$ and $CF' \parallel AB \Longrightarrow ACBE', ABF'C$ are isosceles trapezoid with the referred bases $\Longrightarrow AE' = BC = AF' \Longrightarrow \triangle AE'F'$ is A-isosceles. If L is midpoint of $\overline{E'F'}$, then AOL is perpendicular bisector of $\overline{E'F'}$. Thus $\overline{GO} : \overline{GH} = \overline{GL} : \overline{GX} : -1 : 2$ yields $XH \parallel AOL$.

Quick Reply

High School Olympiads

a strange metric characterization of a point

Reply

Source: own

pohoatza

#1 Jun 2, 2007, 6:45 pm • 2

Let ABC be a triangle and denote by D, E, F the foot of the internal angle bisectors such that $D \in (BC)$, $E \in (CA)$, $F \in (AB)$. $(I_a, r_a), (I_b, r_b), (I_c, r_c)$ are their three excircles. Prove that there exists a point M on the incircle of the triangle ABC, such that:

$$MD\cdot\frac{I_aI}{I_aD}\cdot\sqrt{R+2r_a}\pm ME\cdot\frac{I_bI}{I_bE}\cdot\sqrt{R+2r_b}\pm MF\cdot\frac{I_cI}{I_cF}\cdot\sqrt{R+2r_c}=0$$

Luis González

#2 May 6, 2012, 10:30 pm • 1

Let $(I,r),(O,R)$ be the incircle and circumcircle of $\triangle ABC$. Since IAI_bC and IAI_cB are both cyclic, we have $\overline{EI}\cdot\overline{EI_b}=\overline{EA}\cdot\overline{EC}$ and $\overline{FI}\cdot\overline{FI_c}=\overline{FA}\cdot\overline{FB}$, i.e. E, F have equal powers WRT the circumcircles $(U),(O)$ of $\triangle II_bI_c$ and $\triangle ABC \Longrightarrow EF \perp OU$ is the radical axis of $(O),(U)$. Since I_a is the exsimilicenter of $(O)\sim(U)$, then $OI_a \perp EF$. Since (O) cuts I_aI_c again at the midpoint of its arc ABC, it follows that $OQ \perp AC$ is the perpendicular bisector of AC, thus $\angle QOI_a=\angle AEF$. By sine law for $\triangle BOI_a$ and $\triangle QOI_a$, we obtain

$$\frac{\sin\widehat{OI_aB}}{\sin\widehat{OBI_a}}=\frac{\sin\widehat{OI_aB}}{\sin\widehat{BEA}}=\frac{R}{OI_a}\ ,\ \frac{\sin\widehat{QOI_a}}{\sin\widehat{OI_aB}}=\frac{QI_a}{R}=\frac{I_bI_c}{2R}\Longrightarrow$$

$$\frac{\sin\widehat{QOI_a}}{\sin\widehat{BEA}}=\frac{\sin\widehat{AEF}}{\sin\widehat{BEA}}=\frac{I_aI_c}{2OI_a}\Longrightarrow EF=\frac{2\sin A}{\sin\widehat{BEA}}\cdot\frac{AF\cdot OI_a}{I_aI_c}\Longrightarrow$$

$$EF=\frac{2BE}{bc}\cdot\frac{bc}{a+b}\cdot OI_a\cdot\sin\frac{B}{2}=\frac{4ac}{(a+b)(a+c)}\cdot OI_a\cdot\cos\frac{B}{2}\sin\frac{B}{2}=$$

$$=\frac{abc}{(a+b)(a+c)}\cdot\frac{OI_a}{R}=\frac{abc}{(a+b)(a+c)}\sqrt{\frac{R+2r_a}{R}}\quad(1)$$

By cyclic exchange of elements, we get the expressions for FD, DE

$$FD=\frac{abc}{(b+c)(b+a)}\sqrt{\frac{R+2r_b}{R}}\quad(2)\ ,\ DE=\frac{abc}{(c+a)(c+b)}\sqrt{\frac{R+2r_c}{R}}\quad(3)$$

Let M be an arbitrary point on the circumcircle of $\triangle DEF$, different from D, E, F. By Ptolemy's theorem for $DEFM$ we get $MD\cdot EF\pm ME\cdot FD\pm MF\cdot DE=0$. Substituting (1), (2) and (3) into the latter equation and simplifying terms gives

$$MD\cdot(b+c)\sqrt{R+2r_a}\pm ME\cdot(c+a)\sqrt{R+2r_b}\pm MF\cdot(a+b)\sqrt{R+2r_c}=0$$

But $\dfrac{I_aI}{I_aD}=\dfrac{r+r_a}{r_a}=\dfrac{b+c}{s}\ ,\ \dfrac{I_bI}{I_bE}=\dfrac{r+r_b}{r_b}=\dfrac{c+a}{s}\ ,\ \dfrac{I_cI}{I_cF}=\dfrac{r+r_c}{r_c}=\dfrac{a+b}{s}$

$$MD\cdot\frac{I_aI}{I_aD}\sqrt{R+2r_a}\pm ME\cdot\frac{I_bI}{I_bE}\sqrt{R+2r_b}\pm MF\cdot\frac{I_cI}{I_cF}\sqrt{R+2r_c}=0$$

Thus, $M\in(I)$ is either the Feuerbach point X_{11} or the 8th Stevanovic point X_{3024}.

Quick Reply

High School Olympiads

Tangent and Euler line

nsato

#1 May 5, 2012, 9:30 pm

Given a triangle, prove that the common tangent of the incircle and nine-point circle is parallel to the Euler line if and only if one of the angles of the triangle is equal to 60°.

Luis González

#2 May 5, 2012, 11:21 pm • 1

Let I, O, H, N be the incenter, circumcenter, orthocenter and 9-point center of $\triangle ABC$. R, r denote the radii of the incircle and circumcircle of $\triangle ABC$ and s denote its semiperimeter. Since (I, r) and $(N, \frac{R}{2})$ are internally tangent, then their common tangent is parallel to $ONH \Longleftrightarrow IN \perp OH$, i.e. IN is perpendicular bisector of $\overline{OH} \Longleftrightarrow IO = IH$. Using the identities $OI^2 = R^2 - 2Rr$ and $IH^2 = 4R^2 + 4Rr + 3r^2 - s^2$, we get

$$IN \perp OH \Longleftrightarrow 4R^2 + 4Rr + 3r^2 - s^2 = R^2 - 2Rr \Longleftrightarrow 3(R + r)^2 = s^2$$

Now, we use the well-known identities

$$R + r = R(\cos A + \cos B + \cos C) \ , \ s = R(\sin A + \sin B + \sin C) \Longrightarrow$$

$$IN \perp OH \Longleftrightarrow \sin A + \sin B + \sin C = \sqrt{3}(\cos A + \cos B + \cos C) \Longleftrightarrow$$

$$\sin(A - 60^\circ) + \sin(B - 60^\circ) + \sin(C - 60^\circ) = 0 \Longleftrightarrow$$

$$\sin(\frac{A}{2} - 30^\circ) \sin(\frac{B}{2} - 30^\circ) \sin(\frac{C}{2} - 30^\circ) = 0 \Longleftrightarrow$$

Either $\angle A = 60^\circ, \angle B = 60^\circ$ or $\angle C = 60^\circ$.

RSM

#3 May 6, 2012, 2:25 am • 2

My solution is almost same as Luis'. but the ending is different. We have $OI = IH$. Now AO, AH are isogonals. So either $AOIH$ is cyclic or $AO = AH$. Similar for B, C. But all of $AOIH, BOIH, COIH$ can't be cyclic. So WLOG, $AO = AH \implies \angle BAC = 60$.

High School Olympiads

Inequality for angle

Reply

Source: Romania TST 3 2009, Problem 1

Drytime
#1 May 4, 2012, 8:57 pm

Let $ABCD$ be a circumscribed quadrilateral such that $AD > \max\{AB, BC, CD\}$, M be the common point of AB and CD and N be the common point of AC and BD. Show that

$$90^\circ < m(\angle AND) < 90^\circ + \frac{1}{2}m(\angle AMD).$$

Fixed, thank you Luis.

This post has been edited 1 time. Last edited by Drytime, May 6, 2012, 12:11 am

Luis González
#2 May 5, 2012, 8:46 pm

I think the correct version refers ABCD as a circumscribed quadrilateral. In addition, the exact problem apparently appeared in a Latvian TST (Problem 3). See the links below.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=276813
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=405637

Quick Reply

High School Olympiads

Simson line tangent to Euler circle

Locked

Source: Romania TST 3 2009, Problem 2

Drytime

#1 May 4, 2012, 8:59 pm

Prove that the circumcircle of a triangle contains exactly 3 points whose Simson lines are tangent to the triangle's Euler circle and these points are the vertices of an equilateral triangle.

Luis González

#2 May 5, 2012, 10:07 am

Already posted. If you know the source of the problem then check first the Olympiad resources before posting.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=25115
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=107065
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=244550
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=277289

High School Olympiads

Cyclic quadrilateral

Reply

Source: mpdb

borislav_mirchev

#1 May 4, 2012, 2:16 am

Let $ABCD$ is a quadrilateral inscribed in a circle k. The line l is a line through the middles of its diagonals. M and N are the intersection points of k and l. P is the intersection point of the lines AD and BC. Prove that $\angle APM = \angle BPN$.

Luis González

#2 May 4, 2012, 11:33 am • 1

More generally, replace midpoints of AC, BD by points $U \in AC, V \in BD$, such that $\overline{UA} : \overline{UC} = \overline{VB} : \overline{VD}$. UV cuts circumcircle of $ABCD$ at M, N and AD, BC at E, F, respectively. $\triangle PAC$ and $\triangle PBD$ are similar with corresponding P-cevians $PU, PV \Longrightarrow PU, PV$ are isogonals WRT $\angle EPF \Longrightarrow \odot(PUV)$ and $\odot(PEF)$ are tangent through P. Now, if I is the center of the involution $(M, E, U) \mapsto (N, F, V)$, then $\overline{IM} \cdot \overline{IN} = \overline{IU} \cdot \overline{IV} = IP^2 \Longrightarrow \odot(PMN)$ and $\odot(PEF)$ are tangent at $P \Longrightarrow PM, PN$ are isogonals WRT $\angle APB$.

RSM

#3 May 4, 2012, 4:21 pm

Suppose, $Q = AB \cap CD$. L be the midpoint of PQ. Note that, MN passes through L. Suppose, $P_w(X)$ denotes the power of any point X wrt $\odot ABCD$. Clearly, $2P_w(L) = P_w(P) + P_w(Q) - \frac{PQ^2}{2}$. But we know that, $P_w(P) + P_w(Q) = PQ^2$. So $P_w(L) = LP^2$. So PM and PN are isogonal conjugates wrt $\angle(PQ, MN)$. So PM and PN are isogonal conjugates wrt $\angle DPC$. So done.

Quick Reply

Spain

10th IBERO - CHILE 1995.

Reply

carlosbr
#1 Mar 26, 2006, 8:27 pm

10^{th} Iberoamerican Olympiad
Region V, CHILE. [1995]

Edited by djimenez

Carlos Bravo

Attachments:
1995.pdf (45kb)

Luis González
#2 Jan 14, 2010, 7:54 am

> Quote:
> **Problema 1.** Hallar todos los valores posibles de la suma de los dígitos de todos los cuadrados perfectos.

$11\cdots 1122\cdots 225$ (n veces 1, n+1 veces 2) es un cuadrado perfecto y la suma de sus dígitos es por tanto $n+2(n+1)+5=3n+7.$ También de $(10^n-1)^2=100^n-2\cdot 10^n+1=99\cdots 9800\cdots 001$ (n-1 veces 9, n-1 veces 0), sabemos que la suma de los dígitos es $9(n-1)+8+1=9n.$ Así $f(n)=0,1,4,7 \mod 9.$

Luis González
#3 Jan 20, 2010, 4:17 am

> Quote:
> **Problema 2.** Sea n un entero positivo mayor que 1. Halle todos los valores posibles de números reales $x_1,\ x_2,\ldots,\ x_{n+1}\geq 1,$ tales que se cumplen las condiciones:
>
> $$x_1^{\frac{1}{2}}+x_2^{\frac{1}{3}}+\cdots+x_n^{\frac{1}{n+1}}=nx_{n+1}^{\frac{1}{2}}\ \ ,\ \ \frac{x_1+x_2+\cdots+x_n}{n}=x_{n+1}$$

Por desigualdad de Cauchy-Schwartz se tiene $\left(\sum {x_i}^{\frac{1}{2}}\right)^2\leq\sum 1\sum x_i,$ con igualdad si y solo si todos los x_i son iguales. Es decir, si colocamos $x_{n+1}=\frac{1}{n}(x_1+x_2+...+x_n),$ entonces $\sum {x_i}^{\frac{1}{2}}\leq n{x_{n+1}}^{\frac{1}{2}},$ pero como $x_i\geq 1,$ tenemos que $\sum {x_n}^{\frac{1}{n+1}}\leq\sum {x_i}^{\frac{1}{2}},$ con igualdad si y solo si $x_2=x_3=...=x_n.$ Así, $\sum {x_n}^{\frac{1}{n+1}}\leq\sum {x_{n+1}}^{\frac{1}{2}},$ con igualdad si y solo si $x_i=1.$

Luis González
#4 Jan 2, 2012, 10:13 am

> Quote:
> **Problema 3.** Sean r,s dos rectas ortogonales no coplanares y AB su perpendicular común de modo que $A\in r$ y $B\in s.$ Considerése la esféra de diametro AB y dos puntos variables $M\in r$ y $N\in s$ tal que MN es tangente a la esféra en $T.$ Hallar el lugar geométrico de $T.$

Denótese α y β los planos determinados por AB, s y AB, r y sean X, Y las proyecciones ortogonales de T en α y $\beta.$ Como X, Y están en las proyecciones ortogonales de MN sobre α y $\beta,$ respectivamente, se tiene $\triangle TXN \sim \triangle MAN$ y $\triangle TYM \sim \triangle NBM$ de las cuales se desprenden $\frac{TX}{MA} = \frac{NT}{MN}, \frac{TY}{NB} = \frac{MT}{MN}.$ Como $NT = NB$ y $MA = MT,$ entonces deducimos que $TX = TY.$

Quiere decir que T yace en alguno de los dos planos bisectores γ y γ' del diedro $\alpha\beta.$ El lugar geométrico de T consiste entonces en dos circunferencias resultantes de la intersección de la esféra con γ y $\gamma'.$

Luis González

#5 Apr 26, 2012, 3:07 am

> *Quote:*
>
> **Problema 5.** La circunferencia inscrita (I) del triángulo $\triangle ABC$ es tangente a BC, CA, AB en $D, E, F.$ Suponga que AD se corta nuevamente con (I) en su punto medio $X.$ Si los rayos XB y XC cortan a (I) en $Y, Z,$ mostrar que $EY = FZ.$

Los rayos XB y XC son las X-simedianas de los triángulos $\triangle XFD$ y $\triangle XED.$ Entonces si M y N son los puntos medios de FD y $ED,$ se tiene $\angle FXB = \angle MXD$ y $\angle EXC = \angle NXD.$ Como $MX \parallel BA$ y $NX \parallel CA,$ se sigue que $\angle BAD = \angle FXB$ y $\angle CAD = \angle EXC.$ Por lo tanto, por ser BF y CE tangentes al incírculo (I) resulta $\angle BAD = \angle FXB = \angle BFY, \angle CAD = \angle EXC = \angle CEZ \Longrightarrow AD \parallel FY \parallel EZ.$ Así, $FEZY$ es un trapecio isósceles $\Longrightarrow EY = FZ.$

Quick Reply

High School Olympiads

Nice collinear

Reply

Source: own

jayme

#1 Apr 24, 2012, 5:19 pm

Dear Mathlinkers,

1. ABC a triangle
2. I the incenter sur ABC
3. A'B'C' the I-cevian triangle of ABC
4. A*B*C* the tangentiel triangle of ABC
5. Ia the A-excenter of ABC
6. X the point of intersection of the parallel to A'C' through B with the parallel to A'B' through C.

Prouve : X, Ia and A* are collinear.

Sincerely
Jean-Louis

Luis González

#2 Apr 25, 2012, 8:21 am

(O,R) is circumcircle of $\triangle ABC$ and $\triangle I_aI_bI_c$ is its excentral triangle. P is the midpoint of the arc BAC of (O,R), i.e. midpoint of $\overline{I_bI_c}$. It's well-known that $OI_c \perp A'B' \parallel XC \Longrightarrow \angle POI_c = \angle BCX$ and $\angle I_aCX = \angle OI_cC$. Since $\angle I_aCA^* = \angle ICO$, then it follows that

$$C(X,A^*,I_a,B) = \frac{OC}{OI_c} \cdot \frac{\sin A}{\sin \widehat{POI_c}} = \frac{2R}{I_bI_c} \cdot \frac{\sin A}{\sin \widehat{AA'B}} = \frac{\sin \frac{A}{2}}{\sin \widehat{AA'B}} = -\frac{a}{b+c}.$$

Similarly, $B(X,A^*,I_a,C) = -\dfrac{a}{b+c}$, thus $C(X,A^*,I_a,B) = B(X,A^*,I_a,C) \Longrightarrow X, I_a, A^*$ are collinear.

RSM

#3 Apr 26, 2012, 3:34 am

Suppose, $I_aI_bI_c$ is the excentrix triangle of ABC. M be the midpoint of ABC, note that, $(A'B', A'C'; A'I, BC) = -1 \implies XM \parallel AI$. Under inversion wrt M with power MB^2, X goes to the nine-point center(call it N) of I_aBC and $A*$ goes to the reflection of O(call it O') on BC where O is the circumcenter of ABC. So $O'N$ is anti-parallel to $A*X$ wrt $\angle BI_aC$. Suppose, O_a is the circumcenter of I_aBC and O'_a is the reflection of O_a on BC. U be the reflection of O'_a on O'. So $O'N \parallel I_aU$ and under inversion wrt O_a with power $O_aI_a^2$, U goes to $A*$, so $A*I_a$ is anti-parallel to I_aU wrt $\angle BI_aC$. So $I_a, X, A*$ are collinear.

Quick Reply

Spain

2nd IBERO - URUGUAY 1987.

Reply

carlosbr

#1 Mar 26, 2006, 10:20 am

2^{nd} Iberoamerican Olympiad

Salto y Paysandú, URUGUAY. [1987]

Edited by djimenez

Carlos Bravo

Attachments:
1987.pdf (26kb)

Luis González

#2 Jan 29, 2010, 9:52 pm

> *Quote:*
>
> **Problema 1.** Hallar las funciones $f(x)$ tales que $f(x)^2 f\left(\frac{1-x}{1+x}\right)=64x\ ,x\neq 0,\pm 1.$

Sea $x=\frac{1-y}{1+y}$, luego $\frac{1-x}{1+x}=y\Longrightarrow f\left(\frac{1-y}{1+y}\right)^2\cdot f(y)=64\cdot\frac{1-y}{1+y}\Longrightarrow$

$f\left(\frac{1-x}{1+x}\right)^2\cdot f(x)=64\cdot\frac{1-x}{1+x}$. Pero $f(x)^4 f\left(\frac{1-x}{1+x}\right)^2=64^2\cdot x^2\Longrightarrow$

$f(x)^3=\frac{64(1+x)x^2}{1-x}\Longrightarrow f(x)=4\sqrt[3]{\frac{(1+x)x^2}{1-x}}$.

Luis González

#3 Jan 29, 2010, 10:28 pm

> *Quote:*
>
> **Problema 2.** En un triángulo $\triangle ABC, M$ y N son los punos medios de $AC,AB,$ respectivamente y P es el punto de intersección de BM y $CN.$ Probar que si es posible inscribir una circunferencia en el cuadrilátero $ANPM,$ entonces $\triangle ABC$ es isósceles.

$ANPM$ es circunscriptible en una circunferencia $\Longleftrightarrow AM+PN=AN+PM$

$\Longrightarrow AB+\frac{2}{3}BM=AC+\frac{2}{3}CN$

Si L es el punto medio de BC y Q el simétrico de P con respecto a $L,$ se tiene

$ AB - AC = \frac {_2}{^3}(CN - BM) = QB - QC^$

Es decir que Q yace en el mismo brazo de la hiperbola $\mathcal{H}$ con focos B,C que pasa por $A.$ Por la simetría de la hiperbola con respecto a su centro $L,$ toda recta que pase por L y la corte en A ha de cortarla por segunda vez en su otro brazo. Entonces es contradictorio que LA corte dos veces el mismo brazo de $\mathcal{H}$, a menos que esta degenere en la mediatriz de $BC\Longrightarrow\triangle ABC$

contradictorio que EA corte dos veces el mismo brazo de $\mathcal{H}$, a menos que ésta degenere en la mediatriz de $BC \Longrightarrow \triangle ABC$ debe ser isósceles con ápice A.

Luis González
#4 Jan 4, 2012, 12:07 pm

> *Quote:*
> **Problema 3.** Si x, y, z son entero positivos y $1 + x + \sqrt{3}y = (2 + \sqrt{3})^{2z-1}$, pruebe que x es cuadrado perfecto.

Si $(2+\sqrt{3})^{2z-1} = (x+1) + y\sqrt{3}$, entonces $(2-\sqrt{3})^{2z-1} = (x+1) - y\sqrt{3} \Longrightarrow$

$$x + 1 = \frac{1}{2}(2+\sqrt{3})^{2z-1} + (2-\sqrt{3})^{2z-1}) = \cosh((2z-1)2a).$$

Donde $a > 0$ y $\cosh(2a) = 2$. Luego $x = 2\sinh((2z-1)a)^2$, así hay que probar que $\sqrt{2}\sinh((2p-1)a)$ es entero.

Se tiene $\cosh(2a) = 2\ ,\ \sinh(2a) = \sqrt{3}\ ,\ \sinh(a) = \dfrac{1}{\sqrt{2}}\ ,\ \cosh(a) = \sqrt{\dfrac{3}{2}}$

Sean a_p, b_p números reales reales, tales que $\sinh((2p-1)a) = \dfrac{a_p}{\sqrt{2}}$ y $\cosh((2p-1)a) = b_p\sqrt{\dfrac{3}{2}}$. Luego

$$\sinh(2p+1)a) = \sinh((2p-1)a)\cosh(2a) + \cosh((2p-1)a)\sinh(2a) \Longrightarrow a_{p+1} = 2a_p + 3b_p$$

$$\cosh(2p+1)a) = \cosh((2p-1)a)\cosh(2a) + \sinh((2p-1)a)\sinh(2a) \Longrightarrow b_{p+1} = a_p + 2b_p$$

Como se tiene $a_1 = 1$ y $b_1 = 1$, obtenemos que $a_p, b_p \in \mathbb{N}$. Por consiguiente, $x = 2\sinh((2z-1)a)^2 = a_z^2$ es cuadrado perfecto.

Luis González
#5 Jan 8, 2012, 6:47 am

> *Quote:*
> **Problema 4.** Se define una sucesión p_n de la siguiente forma: $p_1 = 2$ y para todo n mayor o igual que 2, p_n es el mayor divisor primo de la expresión $p_1p_2p_3\ldots p_{n-1} + 1$. Pruebe que p_n es diferente de 5.

$p_1 = 2$ y $p_2 = 3$. Para todo $n > 2$, $p_1p_2p_3\ldots p_{n-1} + 1$ no es múltiplo de 2 ni es múltiplo de 3. Si el mayor divisor primo de $p_1p_2p_3\ldots p_{n-1} + 1$ es 5, la expresión anterior es una potencia de 5. Por ende $p_1p_2p_3\ldots p_{n-1} + 1 = 5^k - 1$. Como $5^k - 1$ es multiplo de 4 para todo k entero, resulta $p_1p_2p_3\ldots p_{n-1} + 1$ multiplo de 4. Esto es absurdo, ya que $p_1 = 2$ y $p_1p_2p_3\ldots p_{n-1}$ es impar.

Luis González
#6 Apr 24, 2012, 8:40 am

> *Quote:*
> **Problema 5.** Sean r, s, t raices de la ecuación $x(x-2)(3x-7) = 2$. Mostrar que r, s, t son reales positivas y determine el valor de $\arctan r + \arctan s + \arctan t$.

Escribimos $f(x) = x(x-2)(3x-7) - 2 = 3x^3 - 13x^2 + 14x - 2$. Entonces $f(0) = -2$, $f(1) = 2$, ésto indica que hay una raíz entre 0 y 1. $f(2) = -2$, entonces hay otra raíz entre 1 y 2. $f(3) = 4$, así que la tercera raíz está entre 2 y 3. $f(x) = 0$ tiene tres raíces, así que son todas soluciones reales y positivas. Por otro lado, se tiene la identidad

$$\tan(x+y+z) = \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - (\tan x \tan y + \tan y \tan z + \tan z \tan x)}$$

Luego sustitutendo $x = \arctan r,\ y = \arctan s,\ z = \arctan t$, se tiene

$$\tan(x+y+z) = \frac{r+s+t-rst}{1-(rs+st+tr)} = \frac{\frac{13}{3} - \frac{2}{3}}{1 - \frac{14}{3}} = -1 \Longrightarrow x+y+z = -\frac{\pi}{4} + k\pi$$

$\Longrightarrow x+y+z=\arctan r+\arctan s+\arctan t=\frac{3\pi}{4}.$

Luis González

#7 Apr 25, 2012, 12:57 am

> Quote:
>
> **Problema 6.** $ABCD$ es un cuadrilátero convexo y P,Q son dos puntos en AD, BC tales que $\frac{AP}{PD}=\frac{AB}{CD}=\frac{BQ}{QC}$. Probar que PQ forma igual ángulo con AB y CD.

Las paralelas por P a DC, AB cortan a AC, BD, respectivamente en X, Y. Entonces $\frac{BQ}{QC}=\frac{AP}{PD}=\frac{AX}{XC}\Longrightarrow QX \parallel PY \parallel AB$ y analogamente $QY \parallel PX \parallel DC$. Además $\frac{PX}{PY}\cdot\frac{AB}{DC}=\frac{BQ}{CQ}\Longrightarrow PX=PY$, así el paralelogramo $PXQY$ es un rombo $\Longrightarrow PQ$ es bisectriz de $\angle XPY$, paralela pues a la bisectriz del ángulo $\angle(AB,CD)$ y por ende forma ángulos iguales con AB y CD.

Quick Reply

Spain

15th IBERO - VENEZUELA 2000.

Reply

carlosbr
#1 Mar 26, 2006, 8:46 pm

$15th$ Iberoamerican Olympiad
Merida, VENEZUELA. [2000]

Edited by djimenez

Carlos Bravo

Attachments:
2000.pdf (27kb)

Luis González
#2 Jan 31, 2010, 4:26 am • 1

> *Quote:*
> **Problema 1**. Un polígono regular de n lados $(n \geq 3)$ tiene sus vértices numerados del 1 a n. Se trazan todas las diagonales del polígono. Mostrar que si n es impar, entonces es posible asignar a cada lado y a cada diagonal un número entero entre 1 y n, tal que el número asignado a cada lado o diagonal es diferente al número asignado a cualquiera de los vértices de sus puntos extremos y que además, para cada vértice, todos los lados y diagonales que concurren en él tienen diferente número asignado.

Desígnese cada lado con puntos extremos $i, j \in \overline{1, n-1}$ con el número en $\overline{1, n}$ congruente a $i + j \pmod n$. Ahora, para cada $i \in \overline{1, n-1}$ todos los números en $\overline{1, n}$ aparecen en los lados adyacentes a i excepto por uno, el cual es diferente de n (de otro modo, se tendría que $2i \equiv 0 \pmod n \Longrightarrow i \equiv 0 \pmod n$), lo cual es contradicción. La implicación es pues una consecuencia del hecho que n es impar). Asígnese el lado in con este número (asignación buscada).

Luis González
#3 Apr 12, 2012, 9:10 am • 1

> *Quote:*
> **Problema 2.** Sean $\mathcal{S}_1$ y $\mathcal{S}_2$ dos circunferencias con centros O_1 y O_2, respectivamente secantes en M, N. La recta t es la tangente común de $\mathcal{S}_1$ y $\mathcal{S}_2$ mas cerca de M. t toca a $\mathcal{S}_1$ y $\mathcal{S}_2$ en A, B. C es un punto tal que BC es un diametro de $\mathcal{S}_2$ y D es la intersección de la perpendicular a AM desde B con O_1O_2. Probar que C, D, M están alineados.

Primeramente demostremos el siguiente resultado

Teorema: En el triángulo $\triangle ABC$, sea A' el antipodal de A en su circuncírculo (O). La recta tangente λ a (O) por A' corta a BC en P y la recta OP corta a AC, AB en D, E. Entonces O es punto medio de DE.

Basta considerar el antipodal C' de C en (O) y $E' \equiv A'C' \cap AB$. Luego por el teorema de Pascal en el hexagono no convexo y degenerado $A'A'ABCC'$, se tiene que las intersecciones $P \equiv BC \cap \lambda, O \equiv AA' \cap CC'$ y $E' \equiv A'C' \cap AB$ están alineadas $\Longrightarrow E \equiv E'$. Entonces como $\angle C'A'C$ es recto, resulta $A'E \parallel AC \Longrightarrow AEA'D$ es un paralelogramo $\Longrightarrow$ O es punto medio de ED.

Ahora, en la figura problema considérese $T \equiv AB \cap O_1O_2$. Por ser T centro de la homotecia positiva que transforma $\mathcal{S}_1$ en $\mathcal{S}_2$, entonces TM corta por segunda vez a $\mathcal{S}_2$ en el homologo M' de M, y siendo P el antipodal de A en el círculo $\mathcal{S}_1$, resulta $BD \parallel MP \parallel M'C$. Consecuentemente, si $F \equiv CM' \cap O_1O_2$, entonces D ha de ser el simétrico de F con respecto a

$BD \parallel MP \parallel M'C$. Consecuentemente, si $E = CM' \cap O_1O_2$, entonces D ha de ser el simétrico de E con respecto a O_2. Por otro lado, si $ D' \equiv CM \cap O_1O__2,$ aplicando el resultado previo a la configuración de $\triangle CMM'$ con la tangente t por el antipodal B de C, resulta que D' es el simétrico de E con respecto a $O_2 \Longrightarrow D' \equiv D'$, por lo tanto C, D, M están alineados.

Luis González

#4 Apr 22, 2012, 11:29 pm • 1

> *Quote:*
> Problema 3. Hallar todas las soluciones de $(m+1)^a = m^b + 1$, para enteros mayores a 1.

Tomando la ecuación $\mod (m+1)$, tenemos $(-1)^b = -1 \Longrightarrow b$ es impar. Entonces, podemos dividir el lado derecho por $m+1$ para obtener $m^{b-1} - m^{b-2} + ... - m + 1$. Esta tiene un número impar de terminos. Si m es impar, entonces cada termino es impar y por ende el total es impar, pero $(m+1)^{a-1}$ es par (ver que a>1), lo cual es contradicción, así m es par. Tenemos $m^b = (m+1)^a - 1$. Luego, expandiendo el lado derecho de la ecuación usando el teorema del binomio y usando $b > 1$, vemos que m debe dividir a. Así, a es también par. Denotemos $a = 2A, m = 2M$. Factorizamos $(m+1)^a - 1$ como $[(m+1)^A + 1] \cdot [(m+1)^A - 1]$. Los dos factores tiene diferencia 2, entonces su MCD divide 2, pero ambos factores son pares, así su MCD es exactamente 2.

Si $M = 1$ o es una potencia de 2, entonces el menor factor $3^A - 1$ debe ser 2, así $A = 1$ y resulta $3^A + 1 = 4 \Longrightarrow (2M)^b = 8 \Longrightarrow M = 1$ y $b = 3$. Así se tiene la solución $(m, a, b) = (2, 2, 3)$. Si M no es una potencia de 2, entonces $M^b > 2^b$, así debemos tener el factor mayor $2M^b$ y factor menor $2^b - 1$. Pero el factor mayor es ahora mayor a 2^{b+1}, por tanto la diferencia entre los factores es al menos $3 \cdot 2^{b-1} > 2$. Contradicción.

Luis González

#5 Apr 23, 2012, 11:55 pm • 1

> *Quote:*
> **Problema 4.** Algunos términos son borrados de una progresión aritmética infinita $1, x, y, ...$ de números reales para producir una progresión geométrica infinita $1, a, b, ...$ Hallar todos los valores posibles de a.

Si a es negativo, entonces los términos de la PG son alternamente positivos y negativos, mientras todos los términos en la PA desde un cierto punto son positivos o todos los términos en la PA desde un cierto punto son negativos. Por ende, a no puede ser negativo. Si a es cero, entonces todos los términos de la PG son cero, excepto el primero, pero a lo mas un término de la PA es cero, así a tampoco puede ser cero. Es decir, que la PA debe tener infinitos términos positivos y por tanto $x \geq 1$.

Sea $d = x - 1$, así todos los términos de la PA tienen la forma $1 + nd$ para n natural. Supongamos que $a = 1 + md$, $a^2 = 1 + nd \Longrightarrow (1+md)^2 = 1 + nd \Longrightarrow d = (n - 2m)m^{-2}$, el cual es irracional. Así, a es irracional. Supongamos que $a = b/c$, donde b y c son naturales primos relativos y $c > 1$. Luego el denominador del enésimo término de la PG es c^n, el cual se vuelve arbitrariamente grande a medida que n incrementa. Pero si $d = h/k$, entonces todos los términos de la PA tienen denominador a lo mas k. Entonces, no se puede tener $c > 1$. Por tanto a debe ser entero positivo. Por otro lado, es facil ver que cualquier entero positivo funciona. Tómerse $x = 2$, luego la PA incluye todos los enteros positivos y por tanto incluye cualquier PG con términos enteros positivos.

Quick Reply

High School Olympiads

Two perpendiculars

Reply

jayme

#1 Apr 23, 2012, 7:14 pm

Dear Mathlinkers,

1. ABC a triangle
2. I the incenter of ABC
3. A'B'C' the I-cevian triangle of ABC
4. (O) the circumcircle of ABC
5. Pa' the perpendicular to BC through I
6. L the point of interction of Pa' and AO

Prouve : A'L is perpendicular to B'C'

Sincerely
Jean-Louis

Luis González

#2 Apr 23, 2012, 10:41 pm

Let $\triangle I_aI_bI_c$ be the excentral triangle of $\triangle ABC.$ Since IAI_bC and IAI_cB are both cyclic, we have $\overline{B'I} \cdot \overline{B'I_b} = \overline{B'A} \cdot \overline{B'C}$ and $\overline{C'I} \cdot \overline{C'I_c} = \overline{C'A} \cdot \overline{C'B},$ i.e. B',C' have equal powers WRT the circumcircles $(U),(O)$ of $\triangle II_bI_c$ and $\triangle ABC \Longrightarrow B'C' \perp OU$ is the radical axis of $(O),(U).$ Since I_a is the exsimilicenter of $(O) \sim (U),$ then $OI_a \perp B'C'.$ Now, if M is the midpoint of the arc BC of $(O),$ we have $MO \parallel IL \Longrightarrow \triangle ILA'$ and $\triangle MOI_a$ are homothetic with center $A \Longrightarrow A'L \parallel OI_a \perp B'C'.$

Quick Reply

High School Olympiads

three concurrent lines

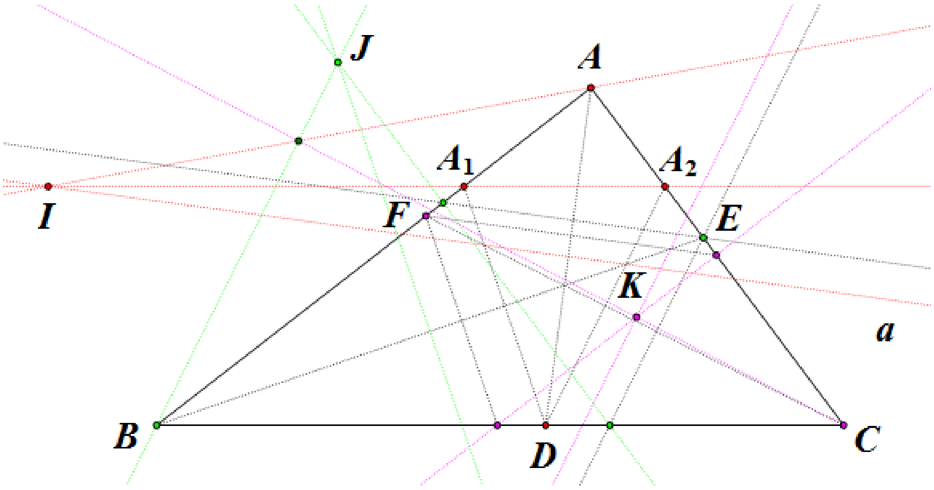
Reply

77ant

#1 Apr 23, 2012, 12:56 pm

Hi, everyone
$\overline{AD}, \overline{BE}, \overline{CF}$ are angle bisectors. Draw the perpendiculars from D to $\overline{BE}, \overline{CF}$
Let them cut $\overline{AB}, \overline{CA}$ at A_1, A_2. The perpendicular bisector of $\overline{AD}$ cut $\overleftrightarrow{A_1A_2}$ at I
Likewise, let's define J, K with respect to B, C. Prove that AI, BJ, CK are concurrent.
Thanks in advance. So sorry if this would be known.

Attachments:



RSM

#2 Apr 23, 2012, 1:23 pm

Suppose, $I_aI_bI_c$ is the excentric triangle of ABC. Note that, AA_1DA_2 is homothetic to ABI_aC. So if the bisector of AI_a intersects BC at X_a, then A, I, X_a are collinear. Also if $A'B'C'$ is the medial triangle of $I_aI_bI_c$, then note that, X_a lies on $B'C'$ and similar for X_b, X_c. So by First Foneten's theorem, AX_a, BX_b, CX_c are concurrent on $\odot ABC$. So done.

Luis González

#3 Apr 23, 2012, 8:33 pm

Furthermore, this concurrency point P is X_{100} of $\triangle ABC$, i.e. the Feuerbach point of the antimedial triangle of $\triangle ABC$.
According to two Yango's problem (post #2), P is anti-Steiner point of the Euler line of $\triangle I_aI_bI_c$ WRT $\triangle A'B'C'$. Now, according to With the Feuerbach's point, P is the Feuerbach point of the antimedial triangle of $\triangle ABC$.

Quick Reply

High School Olympiads

Lemoine point

Reply

Diehard
#1 Apr 23, 2012, 10:11 am

Construct a triangle given, in position, two vertices and the Lemoine point.

Luis González
#2 Apr 23, 2012, 12:34 pm

Label $\triangle ABC$ the wanted triangle with symmedian point $K. B, C, K$ are known. D, E, F are orthogonal projections of K on $BC, CA, AB,$ (D is known). K is centroid of its pedal triangle $\triangle DEF$ (well-known) $\Longrightarrow$ midpoint M of $\overline{EF}$ is constructible, i.e. $M \in DK,$ such that $\overline{KM} : \overline{KD} = -1 : 2.$ Since E, F vary along the circles ω_C, ω_B with diameters $\overline{KC}, \overline{KB},$ then F, F^* are the intersections of ω_B with the reflection of ω_C about $M.$ Likewise, E, E^* are the intersections of ω_C with the refletion of ω_B about $M \Longrightarrow A \equiv BF \cap CE$ and $A^* \equiv BF^* \cap CE^*. \triangle ABC$ and $\triangle A^*BC$ are the possible solutions.

Quick Reply

Spain

11st IBERO - COSTA RICA 1996.

Reply

carlosbr

#1 Mar 26, 2006, 8:31 pm

11^{st} Iberoamerican Olympiad

Limon, COSTA RICA. [1996]

Edited by djimenez

Carlos Bravo

Attachments:

1996.pdf (28kb)

Luis González

#2 Dec 19, 2009, 12:41 am

> Quote:
>
> **Problema 1.** Sea n un número natural. Un cubo de lado n puede ser dividido en 1996 cubos cuya longitud de lados son también números naturales. Hallar el valor mínimo posible de n.

Véase que $n^3 \geq 1996,$ lo que significa $n \geq 13.$ Ahora solo resta conseguir una configuración para $n = 3.$ Si usamos 1984 cubos de lado $1,$ 11 de lado 2 y 1 de lado $5,$ entonces se tendrán 1996 cubos que forman un cubo de lado $13.$

Luis González

#3 Mar 25, 2012, 3:48 am

> Quote:
>
> **Problema 2.** Sea un triángulo $ABC,$ D el punto medio de BC y M el punto medio de $AD.$ La recta BM intersecta a AC en $N.$ Mostrar que AB es tangente al circuncirculo de $\triangle NBC$ si y solo si $\frac{BM}{MN} = \frac{BC^2}{BN^2}$

Es bien sabido que la recta BM divide a AC en dos partes siendo una el doble de la otra $\Longrightarrow CN = 2AN.$ Así, al aplicar teorema de Menelao en el triángulo $\triangle ADC$ cortado por la transversal $\overline{BMN}$ se obtiene $\frac{BM}{MN} = 3.$

Haciendo uso del Teorema de Stewart en la ceviana BN del $\triangle ABC$ resulta

$$BN \cdot BN^2 = AN \cdot BC^2 + CN \cdot BA^2 - AN \cdot CN \cdot AC$$

$$\Longrightarrow BN^2 = \tfrac{1}{3}BC^2 + \tfrac{2}{3}BA^2 - \tfrac{2}{9}AC^2 \Longrightarrow \frac{BC^2}{BN^2} = \frac{BC^2}{\frac{1}{3}BC^2 + \frac{2}{3}BA^2 - \frac{2}{9}AC^2}$$

$$\frac{BM}{MN} = \frac{BC^2}{BN^2} \Longleftrightarrow \frac{BC^2}{BN^2} = \frac{BC^2}{\frac{1}{3}BC^2 + \frac{2}{3}BA^2 - \frac{2}{9}AC^2} = 3$$

$\Longleftrightarrow BA^2 = \frac{1}{3}AC^2 = AN \cdot AC \Longleftrightarrow AB$ es tangente al círculo $\odot(NBC).$

Luis González

#4 Apr 11, 2012, 9:41 pm

> Quote:
>
> **Problema 3.** Se tiene una cuadrícula de k^2-k+1 filas y k^2-k+1 columnas, donde $k=p+1$ y p es primo. Para cada primo p, dar un método para poner los números 0 y 1, un número en cada cuadrado, tal que en cada fila hayan exactamente k ceros, en cada columna hayan exactamente k ceros y que no hayan rectángulos con lados paralelos a los de la cuadrícula con ceros en cada cuatro vértices.

Note que $k^2-k+1=(p+1)^2-(p+1)+1=p^2+p+1$, así la cuadrícula tiene en realidad p^2+p+1 filas y p^2+p+1 columnas. Ahora, consideramos el plano proyectivo $\mathbb{P}^2(\mathbb{Z}_p)$ sobre $\mathbb{Z}_p$. Este plano tiene p^2+p+1 puntos y p^2+p+1 rectas. Podemos definir una biyección entre p^2+p+1 filas de la cuadrícula y los p^2+p+1 puntos del plano proyectivo y una biyección entre las p^2+p+1 columnas de la cuadrícula y las p^2+p+1 rectas del plano proyectivo. Cualquier cuadrado de la cuadrícula corresponde a un par de una fila y una columna de la cuadrícula, y así a un par (P,ℓ) con P siendo un punto y ℓ una recta de plano proyectivo. Escribamos un "0" en el cuadrado si el punto P yace sobre la linea ℓ, y un "1" de lo contrario. Entonces, es facil ver que las condiciones del problema son satisfacidas, a saber:

1) En cada fila, hay exactamente $k=p+1$ cuadrados con "0", ya que cada punto en $\mathbb{P}^2(\mathbb{Z}_p)$ yace exactamente en $p+1$ diferentes rectas.

2) En cada columna, hay exactamente $k=p+1$ cuadrados con "0", ya que cada recta en $\mathbb{P}^2(\mathbb{Z}_p)$ pasa por exactamente $p+1$ puntos diferentes.

3) No hay rectángulos con lados paralelos a los de la cuadrícula con ceros en cada uno de sus cuatro vértices; de hecho, si tal rectángulo existiese, entonces sus dos lados horizontales (filas) corresponderían a dos distintos puntos P,Q, sus dos lados verticales (columnas) corresponderían a dos rectas diferentes ℓ and ℓ', y como los cuatro vértices del rectángulo tienen 0 escrito en ellos, entonces se tendría $P\in\ell, Q\in\ell, P\in\ell', Q\in\ell'$. Así, las dos rectas diferentes ℓ,ℓ' tendrían dos puntos en común, lo cual es imposible.

Luis González

#5 Apr 21, 2012, 12:43 am

> Quote:
>
> **Problema 4.** Dado un número natural $n\geq 2$, consideremos todas las fracciones de la forma $\frac{1}{ab}$, donde a y b son naturales primos relativos tales que $a<b\leq n$ y $a+b>n$. Probar que para cada n, la suma de todas estas fracciones es $\frac{1}{2}$.

Aplicamos inducción sobre n. Es obvio para $n=3$, ya que los únicos pares son $(1,3)$ y $(2,3)$ y $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$. Ahora supongamos que es cierto para n. A medida que nos movemos a $n+1$, estamos introduciendo los nuevos pares $(a,n+1)$ con un primo relativo a $n+1$ y perdemos los pares $(a,n+1-a)$ con un primo relativo a $n+1-a$ y por ende a $n+1$. Así, para cada par de primos relativos a $n+1$ y $\frac{1}{2}(n+1)$, se gana $(a,n+1)$ y $(n+1-a,n+1)$ y pierde $(a,n+1-a)$. Pero

$$\frac{1}{a(n+1)}+\frac{1}{(n+1-a)(n+1)}=\frac{n+1-a+a}{a(n+1-a)(n+1)}=\frac{1}{a(n+1-a)}.$$

Luis González

#6 Apr 22, 2012, 9:58 am

> Quote:
>
> **Problema 6.** Hay n puntos diferentes $A_1,\ldots,A_n$ en el plano y a cada punto A_i se le asigna un número real λ_i distinto de cero, de tal modo que $(\overline{A_iA_j})^2=\lambda_i+\lambda_j$ para todo i,j con $i\neq j$. Probar que
>
> 1) $n\leq 4$.
>
> 2) Si $n=4$, entonces $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4}=0.$

Asumamos que $n\geq 4$. Entonces

$$A_iA_j{}^2+A_kA_l{}^2=(\lambda_i+\lambda_k)+(\lambda_j+\lambda_l)=A_iA_k{}^2+A_jA_l{}^2$$

Lo que implica que $A_iA_l\perp A_kA_j$. Similarmente, se tiene $A_jA_l\perp A_iA_k, A_kA_l\perp A_iA_j \Longrightarrow A_l$ es el ortocentro de

$\triangle A_iA_jA_k$. Siendo éste único para todo triángulo, entonces deducimos que $n \le 4$. Ahora, sea $\triangle ABC$ con ortocentro H y circunradio R. Luego $\lambda_A + \lambda_B = c^2, \lambda_A + \lambda_C = b^2 \Longrightarrow \lambda_B - \lambda_C = c^2 - b^2$. Pero $\lambda_B + \lambda_C = a^2$, así

$$\lambda_B = \tfrac{1}{2}(c^2 + a^2 - b^2) = ca\cos B$$

$$\lambda_C = \tfrac{1}{2}(a^2 + b^2 - c^2) = ab\cos C$$

$$\lambda_A = \tfrac{1}{2}(b^2 + c^2 - a^2) = bc\cos A$$

$$\frac{1}{\lambda_A} + \frac{1}{\lambda_B} + \frac{1}{\lambda_C} + \frac{1}{\lambda_H} = 0 \Longleftrightarrow$$

$$\Longleftrightarrow \sum_{\text{cyclic}} \frac{1}{ab\cos C} = -\frac{1}{AH \cdot BH\cos(180^\circ - C)} = \frac{1}{AH \cdot BH\cos C}$$

$$\Longleftrightarrow \sum_{\text{cyclic}} \frac{1}{4R^2 \sin A \sin B \cos C} = \frac{1}{4R^2 \cos A \cos B \cos C}$$

$$\Longleftrightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Ahora, es bien sabido que esta última identidad es válida para todo $\triangle ABC$.

Quick Reply

High School Olympiads

cyclic quadrilateral in the triangle: angle bisectors

Reply

ivanbart-15

#1 Apr 22, 2012, 1:28 am

Given is triangle $\triangle ABC$. Bisectors of its angles A, B and C meet the opposite sides in A_1, B_1 and C_1 respectively. If $AA_1B_1C_1$ is cyclic quadrilateral, prove that :

$$\frac{BC}{AB+AC}=\frac{AB}{AC+BC}+\frac{AC}{AB+BC}$$

Luis González

#2 Apr 22, 2012, 2:48 am

We use barycentric coordinates WRT ABC. The general equation of a circle is given by

$$a^2yz+b^2zx+c^2xy-(x+y+z)(\delta_A x+\delta_B y+\delta_C z)=0$$

Substituting $A_1(0:b:c),\ B_1(a:0:c),\ C_1(a:b:0)$ into the latter equation gives

$$b\cdot\delta_B+c\cdot\delta_C=\frac{bca^2}{b+c}\ ,\ c\cdot\delta_C+a\cdot\delta_A=\frac{cab^2}{c+a}\ ,\ a\cdot\delta_A+b\cdot\delta_B=\frac{abc^2}{a+b}$$

Solving these equations, we obtain

$$\delta_A=\frac{bc}{2}\left(\frac{b}{c+a}+\frac{c}{a+b}-\frac{a}{b+c}\right)$$

$$\delta_B=\frac{ca}{2}\left(\frac{c}{a+b}+\frac{a}{b+c}-\frac{b}{c+a}\right)$$

$$\delta_C=\frac{ab}{2}\left(\frac{a}{b+c}+\frac{b}{c+a}-\frac{c}{a+b}\right)$$

$$B(0:1:0)\in\odot(A_1B_1C_1)\Longleftrightarrow\delta_B=0\Longleftrightarrow\frac{c}{a+b}+\frac{a}{b+c}-\frac{b}{c+a}=0.$$

ivanbart-15

#3 Apr 22, 2012, 3:39 am

thank you luisgeometra for your solution, but I'm not very familiar with barycentric coordinates.
I was thinking about applying Ptolomy's theorem, but couldn't express A_1C_1, B_1C_1 and A_1B_1 in terms of sidelengths...
anybody has an idea?

Sayan

#4 Apr 22, 2012, 8:54 am

> ivanbart-15 wrote:
> express A_1C_1, B_1C_1 and A_1B_1 in terms of sidelengths... anybody has an idea?

the length of the incentral triangle sides has a horrible expression. See here

Quick Reply

Spain

14th IBERO - CUBA 1999.

Reply

carlosbr

#1 Mar 26, 2006, 8:43 pm

$14th$ Iberoamerican Olympiad
La Habana, CUBA. [1999]

Edited by djimenez

Carlos Bravo

Attachments:
1999.pdf (34kb)

Luis González

#2 Jan 13, 2010, 10:36 am

> Quote:
> **Problema 1.** Hallar todos los enteros positivos menores que 1000, tal que el cubo de la suma de sus dígitos es igual al cuadrado de tal entero.

Solo son 1 y 27. Es facil ver que es imposible para los enteros en el rango 100-999. Si $1 \le n \le 9,$ vemos que solo 1 satisface la condición. Para $10 \le n \le 99,$ tomamos en cuenta la igualdad $(a+b)^3=(10a+b)^2 \Longrightarrow a+b=k^2, 10a+b=k^3$ $\Longrightarrow$ solo $k=3$ y $a=2, b=7.$

Luis González

#3 Jan 14, 2010, 12:14 am

> Quote:
> **Problema 2.** Dadas dos circunferencias $\mathcal{C}_1$ y $\mathcal{C}_2,$ demostrar que existen infinitas circunferencias que bisecan a ambas. Hallar el lugar geométrico de sus centros.

Sean (O_1,r_1) y (O_2,r_2) el centro y radio de $\mathcal{C}_1,\mathcal{C}_2$ y O el centro de una circunferencia con radio r que biseca a ambas cortando a $\mathcal{C}_1$ en A,B y a $\mathcal{C}_2$ en $C,D.$ Basta aplicar el teorema de Pitagoras en los triángulos rectángulos $\triangle OO_1A$ y $\triangle OO_2C$ ya que es evidente que OO_1 y OO_2 son mediatrices de las cuerdas AB y $CD.$ Así

$$OO_1{}^2=r^2-r_1{}^2 \ , \ OO_2{}^2=r^2-r_2{}^2 \ \Longrightarrow OO_1{}^2-OO_2{}^2=r_2{}^2-r_1{}^2=\text{const}$$

El lugar de O es un recta perpendicular a O_1O_2 simétrica del eje radical de $\mathcal{C}_1$ y $\mathcal{C}_2$ con respecto a la mediatriz de $O_1O_2.$

Luis González

#4 Dec 12, 2010, 11:03 pm

> Quote:
> **Problema 3.** Dados puntos $P_1,P_2,...,P_n$ sobre una recta, se construye un círculo con diámetro $\overline{P_iP_j}$ para cada par i,j y se colorea el círculo con uno de los k colores. Para cada $k,$ hallar todos los n para el cual podemos encontrar siempre dos círculos del mismo color con una tangente exterior común.

Hay $n-1$ círculos with diámetro P_iP_{i+1} y claramente, cada par tiene una tangente común. Si $n-1>k,$ entonces dos de ellos deben tener el mismo color. Si $n-1\leq k,$ entonces coloreamos todos los círculos con diámetro $\overline{P_iP_j}$ y $i<j$ con color $i.$ Luego si dos círculos tiene el mismo color, entonces ambos tienen una tangente en uno de los puntos. Por ende, uno yace dentro del otro y no tendrá una tangent común exterior. Así, los n buscados son aquellos mayores que $k+1.$

Luis González

#5 Apr 21, 2012, 9:45 pm

> Quote:
> **Problema 4.** Sea B un número entero mayor que 10 tal que todos sus dígitos pertenecen al conjunto $\{1,3,7,9\}.$ Probar que B tiene un factor primo mayor o igual a $11.$

Como ninguno de los dígitos debe ser par, entonces 2 no divide a $B.$ También, ninguno de ellos puede ser 0 ó $5.$ Sabemos que 5 no divide a $B.$ Supongamos que la declaración es falsa, así existen $n\in\mathbb{N}$ tal que todos sus dígitos pertenecen al conjunto $\{1,3,7,9\}$ y $n=3^a7^b$ para algun número no negativo, siendo a y b enteros. Ahora, nótese que $3^07^0=1,$ así los dígitos unidad pertenecen a $\{1,3,7,9\}$ y los dígitos decena son pares.

Usando inducción, si $n=100x+10y+z,$ donde $x\geq 0,$ y es par y $z\in\{1,3,7,9\},$ entonces $3n$ y $7n$ tendrán un dígito decena par y $1,3,7,$ ó 9 como dígitos unidad. Entonces, por inducción, resulta que todo $a,b\geq 0$ tal que 3^a7^b tendrá un digito decena par, lo cual contradice el hecho de que ninguno de los dígitos es par.

Luis González

#6 Apr 21, 2012, 11:10 pm

> Quote:
> **Problema 5.** $\triangle ABC$ está incrito en un circunferencia de centro $O.$ Las alturas del tríangulo son AD,BE y $CF.$ La recta EF corta a la circunferencia (O) en $P,Q.$
> a) Mostrar que OA es perpendicular a $PQ.$
> b) Si M es punto medio de $BC,$ probar que $AP^2=2AD\cdot OM.$

a) Sea $T\equiv AO\cap EF.$ Como EF es antiparalela a $BC,$ se tiene $\angle AEF=\angle ABC$ y por otro lado sabemos que $\angle OAC=\angle DAB=90^\circ-\angle B.$ Ésto produce la semejanza $\triangle ATE\sim\triangle ADB\Longrightarrow AT\perp EF.$

b) Sea $H\equiv AD\cap BE\cap CF$ el ortocentro de $\triangle ABC$ y R el antipodal de A en $(O).$ Como $\triangle FHE\sim\triangle BHC\cong\triangle CRB$ y $\triangle AEF\sim\triangle ABC,$ se deduce que los cuadriláteros $AEHF$ y $ABRC$ son semejantes $\Longrightarrow AT\cdot AR=AH\cdot AD.$ Como es sabido que AH vale el doble de $OM,$ resulta entonces $AT\cdot AR=2AD\cdot OM.$ Pero por teorema del cateto en el $\triangle APR$ rectángulo en $P,$ se tiene $AP^2=AT\cdot AR\Longrightarrow AP^2=2AD\cdot OM.$

Quick Reply

High School Olympiads

On triangles with vertices on the internal angle bisectors

Reply

Source: generalization of Math. Reflections issue 2, S228 (b)

pohoatza

#1 Apr 1, 2012, 1:41 am • 1

Let ABC be a triangle with incenter I and let X, Y, Z be points lying on the internal angle bisectors AI, BI, CI. Furthermore, let M, N, P be the midpoints of the sides BC, CA, AB, and let D, E, F be the tangency points of the incircle of ABC with these sides. Prove that if MX, NY, PZ are parallel, then DX, EY, FZ are concurrent on the incircle of ABC.

Also, can anyone find some kind of converse? (notice the connection with Daneels' result from http://forumgeom.fau.edu/FG2006volume6/FG200629.pdf).

Luis González

#2 Apr 21, 2012, 11:53 am • 1

The general problem was discussed at On triangles with vertices on the internal angle bisectors with a synthetic proof. Now, we resort to computations with barycentric coordinates WRT ABC. Let $P \equiv MX \cap NY \cap PZ$ and $Q \equiv DX \cap EY \cap FZ$. If $Q \equiv (x:y:z)$, then $P \equiv (s-a)x[(s-b)y+(s-c)z-(s-a)x]$. Hence, $MX \parallel NY \parallel PZ$, i.e. P is at infinity $\Longleftrightarrow$

$$\sum_{\text{cyclic}} (s-a)x[(s-b)y+(s-c)z-(s-a)x]=0 \Longleftrightarrow$$

$$\sum_{\text{cyclic}} (s-a)^2x^2+2(s-b)(s-c)yz=0 \Longleftrightarrow Q \in \odot(DEF).$$

> *pohoatza wrote:*
> Also, can anyone find some kind of converse? (notice the connection with Daneels' result from http://forumgeom.fau.edu/FG2006volume6/FG200629.pdf).

The converse is immediate, $MX \parallel NY \parallel PZ$, i.e. P is at infinity $\Longleftrightarrow Q \in \odot(DEF)$. If instead $DX \parallel EY \parallel FZ$, then $P \equiv MX \cap NY \cap PZ$ lies on the 9-point conic of the Nagel point of $\triangle ABC$. Letting $P \equiv (x:y:z)$, then

$$Q\left(\frac{x(y+z-x)}{s-a} : \frac{y(z+x-y)}{s-b} : \frac{z(x+y-z)}{s-c}\right)$$

Thus, $DX \parallel EY \parallel FZ$, i.e. Q is at infinity $\Longleftrightarrow$

$$\frac{x(y+z-x)}{s-a}+\frac{y(z+x-y)}{s-b}+\frac{z(x+y-z)}{s-c}=0,$$

which is the equation of the 9-point conic of the Nagel point $(s-a:s-b:s-c)$.

Quick Reply

High School Olympiads

Macedonia National Olympiad 2012 - Problem 4

Reply

StefanS
#1 Apr 7, 2012, 9:14 pm

A fixed circle k and collinear points E, F and G are given such that the points E and G lie outside the circle k and F lies inside the circle k. Prove that, if $ABCD$ is an arbitrary quadrilateral inscribed in the circle k such that the points E, F and G lie on lines AB, AD and DC respectively, then the side BC passes through a fixed point collinear with E, F and G, independent of the quadrilateral $ABCD$.

This post has been edited 1 time. Last edited by StefanS, Apr 8, 2012, 6:10 pm

MBGO
#2 Apr 8, 2012, 1:53 am • 1

Dear Friend....

Would you explain your question? what this question want? and you didn't suppose that points E,F,G are collinear....did you?

With Regards.

StefanS
#3 Apr 8, 2012, 6:11 pm

Thank you very much for correcting me and sorry for the mistake (I posted the problem right after the competition when I was in a hurry...!) I edited it and I hope it makes sense now!

PS. I'll try and post the solution as soon as possible.

This post has been edited 1 time. Last edited by StefanS, Apr 14, 2012, 5:56 pm

Nevergiveupbtw
#4 Apr 11, 2012, 10:58 pm

there is a solution by pascal theorem, but it is not beautiful

Luis González
#5 Apr 12, 2012, 3:07 am • 2

E, F, G can be three collinear points in general position (not necessarily E,G outside k as the problem states). Let $\ell \equiv \overline{EFG}$ cut k at P, P' and BC at F'. Projecting the quadruplet $PACP'$ from the vertices B, D of the complete quadrangle $ABCD$, we have $B(PACP') \barwedge D(PACP')$. Cutting both pencils with ℓ and permutating P, P' and E, F' yields $(PFGP') \barwedge (PEF'P') \barwedge (P'F'EP)$. Thus, in the projectivity $\mathcal{I} : (PFGP') \mapsto (P'F'EP)$, $P \mapsto P'$, $F \mapsto F'$, $G \mapsto E$, $P' \mapsto P$, the points P, P' are double $\Longrightarrow \mathcal{I}$ is an involution. Hence, BC always passes through the fixed image F' of F under $\mathcal{I}$.

MBGO
#6 Apr 13, 2012, 11:42 am • 1

> *Nevergiveupbtw wrote:*
> there is a solution by pascal theorem, but it is not beautiful

Would youexplain it somehow? we have to make any hexagon here?

Would youexplain it somehow? we have to make any hexagon here?

And is there any elementry solution for this?

With Regards.

StefanS

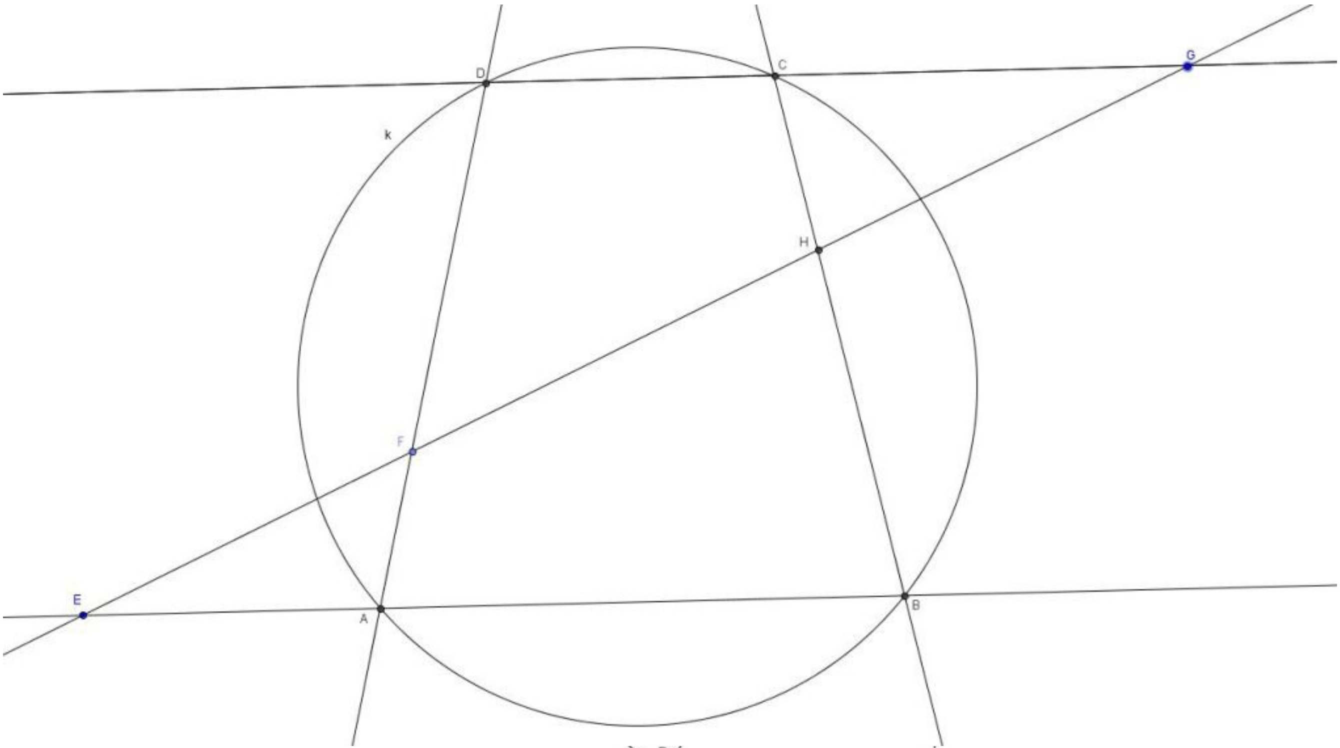
#7 Apr 13, 2012, 6:47 pm • 3

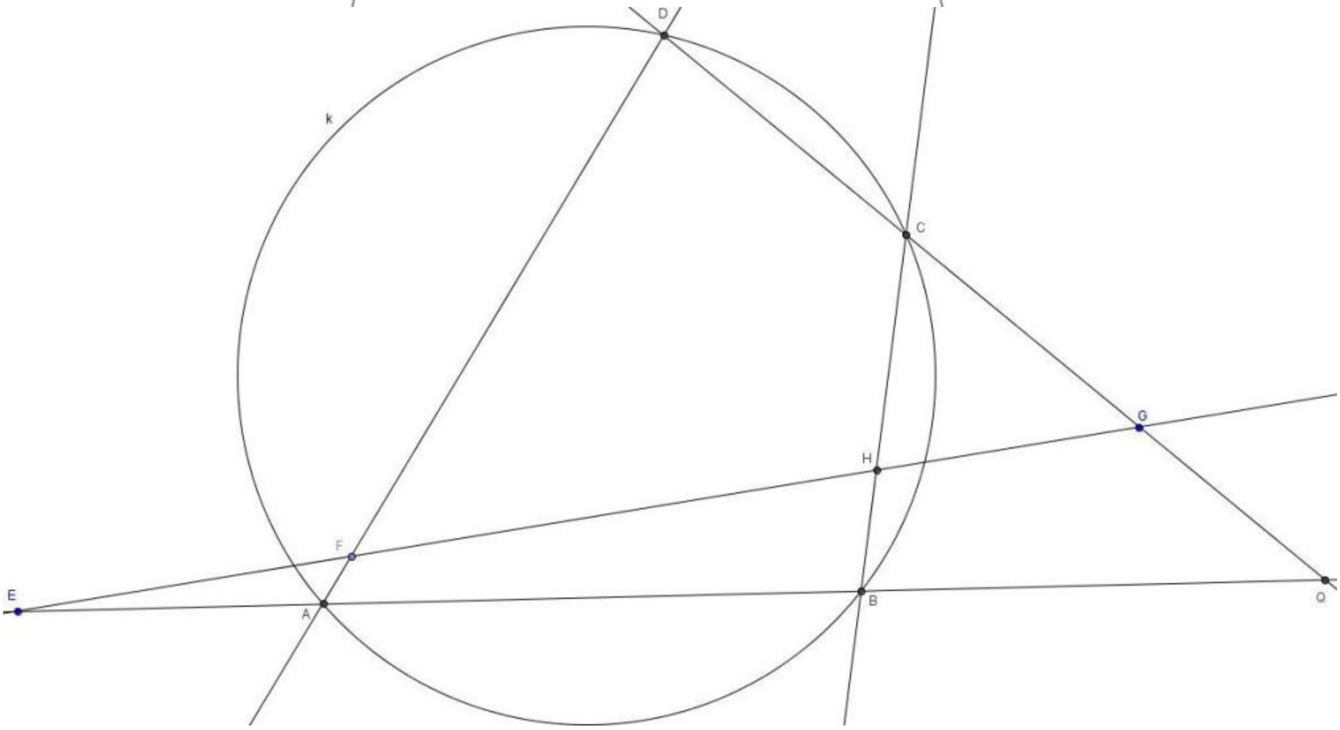
> *MahanBabol wrote:*
> And is there any elementry solution for this?

There is! 🙂

[My Solution]

Attachments:





r1234

#8 Apr 14, 2012, 4:33 pm

I also got a solution like Stefan S..... first projecting the circle k to another circle k' so that the image of F is the centre of k'.Then the line EFG becomes the line passing through the centre of the new circle...... now by some Menelaus the result follows immediately..

StefanS

#9 Apr 14, 2012, 6:00 pm

> *r1234 wrote:*
> ... first projecting the circle k to another circle k' ...

You say you 'project' the circle... Do you use projective geometry? Does Luisgeometra use it too?

Nevergiveupbtw

#10 May 8, 2012, 4:27 pm

> *MahanBabol wrote:*
> Would you explain it somehow? we have to make any hexagon here?
>
> And is there any elementry solution for this?
>
> With Regards.

Oh, I'm sorry, I haven't visited this forum for a long time.
We consider two quadrilateral sastifying hypothesis ABCD an A'B'C'D'; ABCD is fixed. So BC intersect EG at a fixed point.
We will show that BC intersect B'C' at K then K is on EG. BD intersect B'D' at H.
Apply pascal theorem for six point A,B,D,A',B',D' then E,F, H are collinear;
Apply pascal theorem for six point C,B,D,C',B',D' then K,G, H are collinear;
So we have B'C' always pass through fixed point K- done

Quick Reply

High School Olympiads

Romania TST 2010/6

Reply

Maxim Bogdan
#1 Apr 11, 2012, 9:29 pm

Let ABC be a triangle with incentre I, and A_1, B_1, C_1 the contact points of the excirles with BC, CA, AB. Prove that the circumcircles of AIA_1, BIB_1, CIC_1 have a common point different from I.

jayme
#2 Apr 11, 2012, 10:08 pm • 1

Dear Mathlinkers,
we are in the Nagel-Schröder point situation... Darij Grinberg has write a very important article about this subject :

http://www.cip.ifi.lmu.de/~grinberg/Schroeder/Schroeder.html

Sincerely
Jean-Louis

Luis González
#3 Apr 11, 2012, 10:50 pm • 1

See http://tech.groups.yahoo.com/group/Hyacinthos/message/6544. This point was listed X(1339) in ETC afterwards. Also, this problem was posted in Aops with a different fashion, see the topic Coaxal circles.

P.S. For a generalization see the topic Coaxial circles.

Quick Reply

High School Olympiads

Tetrahedron ABCD

Reply

Source: Romanian TST 1992 - Day 1 - Problem 3

AndrewROM

#1 Apr 9, 2012, 9:14 pm

Let $ABCD$ be a tetrahedron; B', C', D' be the midpoints of the edges AB, AC, AD; G_A, G_B, G_C, G_D be the barycentres of the triangles BCD, ACD, ABD, ABC, and G be the barycentre of the tetrahedron. Show that A, G, G_B, G_C, G_D are all on a sphere if and only if A, G, B', C', D' are also on a sphere.

Dan Brânzei

Luis González

#2 Apr 11, 2012, 5:01 am

Let M, N, L be the midpoints of the edges BC, CD, DB. Denote $\Omega \equiv \odot(BCD)$ and $\omega \equiv \odot(MNL)$. Medians AG_A, BG_B, CG_C, DG_D concur at G such that $\overline{GG_A} : \overline{GA} = -1 : 3$, etc. Bimedians NB', LC', MD' bisect each other at G. Assume that A, G, G_B, G_C, G_D lie on a same sphere $\mathcal{S}_1$. If $P \in AG$ such that $\overline{GA} : \overline{GP} = -1 : 3$, then B, C, D, G, P lie on the homothetic sphere of $\mathcal{S}_1$ under homothety $(G, -3)$. Thus $p(G_A, \Omega) = \overline{G_AP} \cdot \overline{G_AG} = -\frac{1}{3}GA \cdot \frac{8}{3}GA = -\frac{8}{9}GA^2$.

Assume that A, B', C', D', G lie on the same sphere $\mathcal{S}_2$. If Q is the reflection of A about G, then Q, N, L, M, G lie on the reflection of $\mathcal{S}_2$ about G. Therefore, $p(G_A, \omega) = \overline{G_AG} \cdot \overline{G_AQ} = -\frac{1}{3}GA \cdot \frac{2}{3}GA = -\frac{2}{9}GA^2$. But clearly $p(G_A, \omega) = \frac{1}{4}p(G_A, \Omega)$, thus $p(G_A, \Omega) = -\frac{8}{9}GA^2$. Consequently, A, G, G_B, G_C, G_D lie on a same sphere $\Longleftrightarrow$ A, B', C', D', G lie on a same sphere.

Quick Reply

High School Olympiads

non-equilateral triangle

Source: Mathley 7 problem: Michel Bataille

mathley

#1 Apr 9, 2012, 11:36 pm

A non-equilateral triangle ABC is inscribed in a circle Γ with centre O, radius R and its incircle has centre I and touches BC, CA, AB at D, E, F, respectively. A circle with centre I and radius ρ intersects the rays $[ID), [IE), [IF)$ at A', B', C'.

[0.1cm] Show that the orthocentre K of $\Delta A'B'C'$ is on the line OI and that $\frac{IK}{IO} = \frac{\rho}{R}$.

Luis González

#2 Apr 10, 2012, 3:17 am

IA, IB, IC cut the circumcircle (O, R) of $\triangle ABC$ again at the midpoints X, Y, Z of its arcs BC, CA, AB. $\triangle A'B'C'$ with circumcirle (I, ρ) and $\triangle XYZ$ with circumcircle (O, R) are clearly homothetic with homothetic center $J \in OI$. Since I is orthocenter of $\triangle XYZ$ (well-known), then K, I, J, O are collinear. Furthermore

$$\frac{JI}{JO} = \frac{JK}{JI} = \frac{\rho}{R} \Longrightarrow \frac{JO + IO}{JO} = \frac{JI + IK}{JI} \Longrightarrow \frac{IK}{IO} = \frac{JI}{JO} = \frac{\rho}{R}.$$

High School Olympiads

Prove that JM, KN and LP are concurrent

Source: Romanian TST 2011

WakeUp

#1 Apr 9, 2012, 9:21 pm

The incircle of a triangle ABC touches the sides BC, CA, AB at points D, E, F, respectively. Let X be a point on the incircle, different from the points D, E, F. The lines XD and EF, XE and FD, XF and DE meet at points J, K, L, respectively. Let further M, N, P be points on the sides BC, CA, AB, respectively, such that the lines AM, BN, CP are concurrent. Prove that the lines JM, KN and LP are concurrent.

Dinu Serbanescu

Luis González

#2 Apr 10, 2012, 1:51 am • 1

WLOG assume that J is between E, F. $\triangle JKL$ is obviously autopolar WRT the incircle (I) of $\triangle ABC$ and $A \in LK$, because its polar EF WRT (I) passes through J. Analogously, $B \in LJ$ and $C \in JK$. Since KL does not cut (I), then project KL to infinity and (I) into a circle, whose center is then J. So $\triangle DEF$ becomes isosceles right at D and $BCEF$ is a rectangle. Let $Y \equiv AM \cap BN \cap CP, T \equiv KN \cap LP$ and $M' \equiv TJ \cap BC$. Thus, it suffices to show that $M \equiv M'$. Let TP, TN cut BC at U, V. Then isosceles right $\triangle TUV$ and $\triangle JBC$ are homothetic through $M' \Longrightarrow$

$$\frac{\overline{M'B}}{\overline{M'C}} = \frac{\overline{M'U}}{\overline{M'V}} = \frac{\overline{BU}}{\overline{CV}} = \frac{\overline{PB}}{\overline{NC}} \Longrightarrow YM' \parallel PB \parallel NC \Longrightarrow M \equiv M'.$$

r1234

#3 Apr 10, 2012, 11:49 pm

Lemma:- In a triangle $\triangle ABC$, P is a point inside it.Let $\triangle P_aP_bP_c$ be the cevian triangle of P.Consider a point Q on the circumference of $\odot P_aP_bP_c$.Let $X = P_bP_c \cap P_aQ$.Analogously define Y, Z.Then AX, BY, CZ are concurrent.

Proof:- Let us consider a projective transformation that takes the circle $\odot P_aP_bP_c$ to the circle $P_a'P_b'P_c'$ and the tri-liner polar of P to infinity(It can be proven that there exists such a projective transformation).Let $A'B'C'$ be the image of ABC.Clearly P' becomes the centroid of $\triangle A'B'C'$.Now the result of the lemma is trivially true for the medial triangle.Hence the lemma is proven.

Main Proof:- Now we just need to prove that poles of JM, NK, LP wrt the incircle are collinear.Note that J is the pole of KL.Similarly K is the pole of JL and L is the pole of JK.Now clearly polars of M, N, P pass through D, E, F respectively.Since AM, BN, CP are concurrent, clearly $\triangle A_1B_1C_1$ is perspective with $\triangle DEF$ where $\triangle A_1B_1C_1$ is the triangle formed by the polars of M, N, P.So $\triangle DEF$ is a cevian triangle of $\triangle A_1B_1C_1$.Now using the above lemma we conclude that JM, NK, LP are concurrent.

TelvCohl

#4 Nov 7, 2014, 7:37 am

My solution:

Let I be the incenter of $\triangle ABC$.

From Cevian nest theorem we get AJ, BK, CL are concurrent at Z .
Since KL is the polar of J WRT (I) ,
so K, A, L are collinear .
Similarly, we can prove $B \in LJ$ and $C \in JK$.
Since $\triangle ABC$ is the Z -cevian triangle of $\triangle JKL$,

so from Cevian nest theorem we get JM, KN, LP are concurrent .

Q.E.D

junior2001

#5 Nov 7, 2014, 8:52 am

What's cevian-nest theorem?

jayme

#6 Nov 7, 2014, 10:38 am • 1

Dear Mathlinkers,
see for example
http://jl.ayme.pagesperso-orange.fr/ vol. 3 the cevian nests theorem
or on search on mathlinks
Sincerely
Jean-Louis

IDMasterz

#7 Nov 7, 2014, 7:19 pm

cevian nest theorem, twice

Quick Reply

High School Olympiads

concylic quadrilateral

Reply

Source: Mathley 7 problem

mathley

#1 Apr 9, 2012, 8:51 am

Let $ABCD$ be a concyclic quadrilateral. Suppose that E is the intersection of AB and CD, F is the intersection of AD and CB; I is the intersection of AC and BD. The circumcircles (FAB), (FCD) meet FI at K, L. Prove that $EK = EL$.

Luis González

#2 Apr 9, 2012, 11:02 am • 1

(O) is the circumcircle of $ABCD$ and M,N are the midpoints of $BC,AD.$ Since FI is the polar of E WRT (O), then FI is perpendicular to OE at $P \Longrightarrow O,P,M,N,A$ lie on the circle Ω with diameter $\overline{OE}.$ Further, powers of M,N WRT $\odot(FAB)$ and $\odot(FCD)$ are in the same ratio, namely -1 $\Longrightarrow \Omega$ is coaxal with $\odot(FAB)$ and $\odot(FCD)$ and the ratio of the powers of P WRT $\odot(FAB)$ and $\odot(FCD)$ is -1. Hence $\overline{PK} \cdot \overline{PF} = -\overline{PL} \cdot \overline{PF} \Longrightarrow \overline{PK} = -\overline{PL}$, i.e. P is midpoint of $\overline{KL} \Longrightarrow EP$ is perpendicular bisector of $\overline{KL} \Longrightarrow EK = EL.$

Quick Reply

High School Olympiads

Cevian quotient

Source: nice

skywalker
#1 Mar 3, 2007, 5:22 pm

Prove that for all M lie on the Feuerbach hyperbola of triangle ABC with incenter I and circumcenter O then the cevian quotient M/I lie on line IO

Luis González
#2 Apr 8, 2012, 3:46 am

We prove a stronger result: M/I lies on OI if and only if M lies on the Feurebach hyperbola of ABC. Using barycentric coordinates WRT ABC, we have

$$O(a^2S_A : b^2S_B : c^2S_C)\ ,\ I(a:b:c)\ ,\ M(x:y:z) \implies$$

$$M/I\left(a\left(\frac{b}{y}+\frac{c}{z}-\frac{a}{x}\right) : b\left(\frac{c}{z}+\frac{a}{x}-\frac{b}{y}\right) : c\left(\frac{a}{x}+\frac{b}{y}-\frac{c}{z}\right)\right)$$

Hence, $O, I, M/I$ are collinear if and only if

$$\begin{bmatrix} a & b & c \\ a^2S_A & b^2S_B & c^2S_C \\ a\left(\frac{b}{y}+\frac{c}{z}-\frac{a}{x}\right) & b\left(\frac{c}{z}+\frac{a}{x}-\frac{b}{y}\right) & c\left(\frac{a}{x}+\frac{b}{y}-\frac{c}{z}\right) \end{bmatrix} = 0$$

$$\iff a(bS_B - cS_C)yz + b(cS_C - aS_A)zx + c(aS_A - bS_B)xy = 0,$$

which is the equation of the Feuerbach hyperbola of ABC, i.e. isogonal conjugate of IO.

RSM
#3 Apr 8, 2012, 4:16 am • 1

Lemma:-

Given a triangle, ABC and a point P, the locus of all points Q so that the ceva quotient Q/P lies on a fixed line is a hyperbola passing through A, B, C.

Proof:-

If we take P to the centroid of ABC by a projective transformation, then Q/P becomes the anti-complement of the isotomic point of Q wrt ABC. So if it moves on a line, then Q moves on a conic passing through A, B, C since isotomic conjugate of a circumconic is a line.

Back to Main Proof:-

In this problem, clearly, $I/I = I$ and if G_e is the Georgenne point of ABC, then clearly, G_e/I is the center of homothety of intouch triangle and excentric triangle of of ABC, which lies on OI. So the locus of Q is the conic passing through A, B, C, I, G_e which is the Feuerbach hyperbola.

This post has been edited 1 time. Last edited by RSM, Apr 8, 2012, 11:04 am

Luis González

#4 Apr 8, 2012, 5:06 am

Nice proof, but there are some assertions that are not necessarily true.

> RSM wrote:
>
> Note that, the ceva quotient Q/P and P/Q are the same. If we take P to the centroid of ABC by a projective transformation, then Q/P becomes the complement of the isotomic point of Q wrt ABC. So if it moves on a line, then Q moves on a hyperbola passing through A, B, C since isotomic conjugate of a hyperbola is a line.

In general, Q/P and P/Q are not the same point and the isotomic conjugate of a line is not necessarily a hyperbola. This depends on whether the line meets the Steiner circumellipse of the reference triangle at two, one or any point.

> RSM wrote:
>
> $I/I = I$ and if G_e is the Georgenne point of ABC, then clearly, G_e/I is the centroid of the intouch triangle of ABC, which lies on OI.

Ge/I is not the centroid of the intouch triangle but the Isogonal Mittenpunkt, i.e. exsimilicenter of the incircle and the circumcircle of the excentral triangle.

RSM

#5 Apr 8, 2012, 11:07 am

Thank you for pointing out the mistakes. I have edited the proof.

Actually I messed up with the definitions. I took the following definition as the definition of ceva-quotient:-

If AP, BP, CP intersects cevian triangle of Q at A', B', C', then perspective center of the cevian triangle of Q and $A'B'C'$ is Q/P.

In this definition P/Q and Q/P are the same. But I searched for the definition in google just and saw that its different.

Quick Reply

High School Olympiads

Line joining isogonal conjugate points passes through O

goodar2006

#1 Apr 6, 2012, 1:08 pm • 1

Let P be a point in triangle ABC such that $\angle PAB+\angle PBC+\angle PCA=\frac{\pi}{2}$. If P' is the isogonal conjugate of P, then prove that PP' passes through O.

Luis González

#2 Apr 6, 2012, 10:49 pm • 4

From Another variant of a conjecture we know that the pedal triangle $\triangle XYZ$ and circumcevian triangle $\triangle A'B'C'$ of P are homothetic through a point H. Perpendiculars from A,B,C to $XY \parallel B'C', ZX \parallel C'A'$ and $XY \parallel A'B'$ concur at the isogonal conjugate P' of P. Thus $\triangle A'B'C'$ and $\triangle ABC$ are orthologic through P', Q and perspective through P, therefore by Sondat's theorem, Q,P,P' are collinear. Since $QA' \parallel PX, QB' \parallel PY, QC' \parallel PZ$, then Q,P,H are collinear, but it's well known that the isogonal conjugate P' of P WRT $\triangle ABC$ is the reflection of P about the circumcenter of $\triangle XYZ$. Thus $\overline{HPP'}$ passes through the circumcenter O of $\triangle A'B'C'$.

P.S. Locus of P is then a self-isogonal cubic with pivot O, namely the M'Cay cubic.

High School Olympiads

Another variant of a conjecture

Reply

goodar2006

#1 Apr 6, 2012, 10:18 pm

Let P be a point in triangle ABC such that $\angle PAB + \angle PBC + \angle PCA = \frac{\pi}{2}$. Prove that it's pedal triangle and it's circumcevian triangle are homothetic.

Luis González

#2 Apr 6, 2012, 10:39 pm • 1

PA, PB, PC cut the circumcircle of $\triangle ABC$ again at A', B', C' and X, Y, Z are the orthogonal projections of P on BC, CA, AB, respectively. $B'C'$ cuts AC at U. Using directed angles (mod 180), we get

$$\angle AUC' = \angle CAB' + \angle AB'C' = \angle PBC + \angle PCA$$

$$\angle AYZ = \frac{\pi}{2} - \angle PYZ = \frac{\pi}{2} - \angle PAB$$

Since $\angle PAB + \angle PBC + \angle PCA = \frac{\pi}{2}$, it follows that $\angle AYZ = \angle AUC' \Longrightarrow B'C' \parallel YZ$. Analogously, $C'A' \parallel ZX$ and $A'B' \parallel XY \Longrightarrow \triangle A'B'C' \sim \triangle XYZ$ are homothetic.

Quick Reply

High School Olympiads

equal angle

Reply

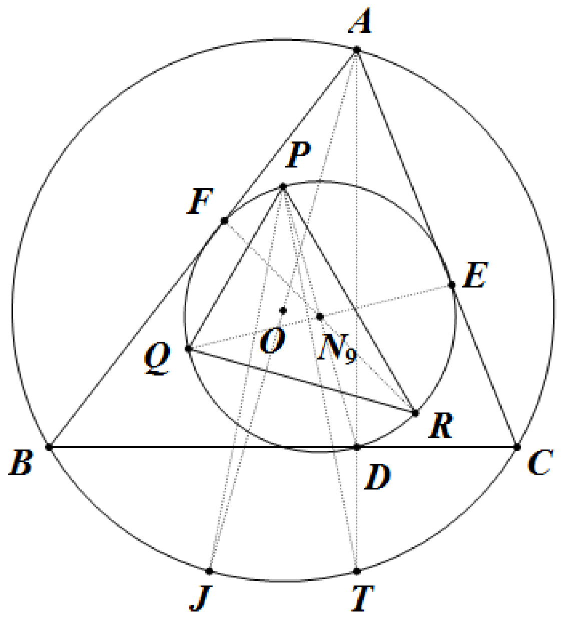
77ant

#1 Apr 4, 2012, 11:17 pm

Hi, everyone.

For $\triangle ABC$ with its circumcenter O, let $\triangle DEF$ be its orthic triangle, which D, E, F are on BC,CA, AB.
P, Q, R are the antipodal points of D, E, F with respect to N_9 of $\triangle ABC$, which is 9-point-circle's center.
J is the antipodal point of A with respect to O and AD meet (O) at T
Show that PJ, PT are isogonal conjugate with respect to $\angle QPR$

Attachments:



Luis González

#2 Apr 6, 2012, 1:50 am • 2

AD cuts (N_9) again at U and PJ at $L.$ Centroid G of $\triangle ABC$ is the center of the homothety with factor $-\frac{1}{2}$ that takes (O) into $(N_9).$ U, P are clearly the images of J, T under this homothety, thus $G \equiv UJ \cap PT$ and $\overline{GU} : \overline{GJ} = \overline{GP} : \overline{GT} = -1 : 2 \Longrightarrow G$ is centroid of the right $\triangle LTJ \Longrightarrow \triangle TPJ$ is isosceles with symmetry axis $\overline{PO}$, i.e. PO bisects $\angle TPJ. DA, DE, DF$ are the reflections of PO, PQ, PR about N_9, being DA angle bisector of $\angle EDF$, thus PO bisects $\angle QPR.$ Hence PJ, PT are isogonals WRT $\angle QPR.$

77ant

#3 Apr 6, 2012, 9:39 am

Thank you for sincere interest and reply, *luisgeometra*.

Quick Reply

High School Olympiads

A circumradius equation

Source: Turkish TST 2012 Problem 4

crazyfehmy

#1 Mar 27, 2012, 3:45 am • 1

In a triangle ABC, incircle touches the sides BC, CA, AB at D, E, F, respectively. A circle ω passing through A and tangent to line BC at D intersects the line segments BF and CE at K and L, respectively. The line passing through E and parallel to DL intersects the line passing through F and parallel to DK at P. If R_1, R_2, R_3, R_4 denotes the circumradius of the triangles AFD, AED, FPD, EPD, respectively, prove that $R_1R_4 = R_2R_3$.

This post has been edited 1 time. Last edited by crazyfehmy, Feb 10, 2013, 8:51 pm

Luis González

#2 Mar 28, 2012, 5:46 am • 3

Clearly $P \in \odot(AEF)$. Let S be the 2nd intersection of $\odot(AEF)$ and ω, i.e. center of the spiral similarity that takes $\overline{EL}$ into $\overline{FK}$. By Reim's theorem for $\odot(AEF)$ and ω with common chord AS, we get that S, P, D are collinear. Since $\odot(DEF)$ and ω are internally tangent at D, it follows that DE, DF bisect $\angle ADL$ and $\angle ADK$. Thus, by angle bisector theorem, we deduce that

$$\frac{DL}{DA} = \frac{EL}{EA}, \ \frac{DK}{DA} = \frac{FK}{FA} \Longrightarrow \frac{DL}{DK} = \frac{EL}{FK} = \frac{SE}{SF} = \frac{\sin \widehat{SPE}}{\sin \widehat{SPF}} \Longrightarrow$$

$$\frac{DL}{DK} = \frac{\sin \widehat{DAL}}{\sin \widehat{DAK}} = \frac{\sin \widehat{SPE}}{\sin \widehat{SPF}} = \frac{DE}{2R_4} \cdot \frac{2R_3}{DF} = \frac{R_3}{R_4} \cdot \frac{2R_2 \cdot \sin \widehat{DAL}}{2R_1 \cdot \sin \widehat{DAK}} \Longrightarrow$$

$$R_1 \cdot R_4 = R_2 \cdot R_3.$$

Ali-mes

#3 Jan 29, 2013, 4:09 am • 1

According to the law of sines, it is sufficient to prove that: $\sin(\angle AFD).\sin(\angle PED) = \sin(\angle AED).\sin(\angle PFD)$.

We have: $\angle PED = \angle EDL$ and according to this one http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1967516&sid=2ea9f7046fedc0f5bf3328b258f7a07f#p1967516, $\angle EDL = \angle EDA$, so $\angle PED = \angle EDA$. And similarly, $\angle PFD = \angle ADF$.

$$\frac{\sin(\angle PED)}{\sin(\angle AED)} = \frac{\sin(\angle EDA)}{\sin(\angle AED)} = \frac{AE}{AD} = \frac{AF}{AD} = \frac{\sin(\angle ADF)}{\sin(\angle AFD)} = \frac{\sin(\angle PFD)}{\sin(\angle AFD)}.$$ QED

High School Olympiads

very nice problem

Reply

Pedram-Safaei

#1 Mar 27, 2012, 7:46 pm

$ABCD$ is a cyclic convex quadrilateral. Find locus of point P inside $ABCD$ such that $\measuredangle BPC = \measuredangle BAP + \measuredangle CDP$.

Luis González

#2 Mar 28, 2012, 1:32 am

Let τ be the tangent of $\odot(PAB)$ at $P. \angle(\tau, PB) = \angle BAP.$ Since $\angle BPC = \angle BAP + \angle CDP,$ then $\angle(PC, \tau) = \angle CDP,$ i.e. τ is tangent to $\odot(PCD) \Longrightarrow \odot(PAB)$ and $\odot(PCD)$ are externally tangent at $P. AB, CD, \tau$ are then radical axes of $\odot(ABCD), \odot(PAB)$ and $\odot(PCD)$ concurring at their radical center $E \Longrightarrow P$ lies then on the circle centered at E and orthogonal to $\odot(ABCD).$

Quick Reply

High School Olympiads

Euler line

Locked

Pedram-Safaei
#1 Mar 26, 2012, 4:43 pm

suppose that $ABCD$ is a cyclic convex quadrilateral and P is intersection point of it's diaagonals.prove that Euler lines of triangles APB, BPC, CPD, DPA are parallel or concurent.

Luis González
#2 Mar 26, 2012, 8:38 pm • 1

Posted before at
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=107997
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=187520
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=204979

For a generalization see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=302860

High School Olympiads

Parallel to OI

Source: inspired by another problem

jayme
#1 Mar 25, 2012, 7:37 pm

Dear Mathlinkers,
1. ABC a triangle
2. (I) the incircle of ABC
3. U the midpoint of AI
4. X the symmetric of U wrt I
5. DEF the contact triangle of ABC
6. O the center of the circumcircle of ABC.

Prouve : the symmetric of DX wrt the D-inner bissector of DEF is parallel to OI

Sincerely
Jean-Louis

Luis González
#2 Mar 26, 2012, 3:58 am

There's more to say about this configuration. The line DX passes through the antipode of the Feuerbach point F_e of $\triangle ABC$ WRT (I). Let D' be the antipode of D WRT (I). According to Intersect on circle (post # 4), UD' cuts (I) again at F_e. Thus, reflection XD of UD' about I passes through the antipode V of F_e WRT (I). D-altitude of $\triangle DEF$ cuts (I) again at the reflection of the orthocenter of $\triangle DEF$ about EF. Since F_e is anti-Steiner point of OI WRT $\triangle DEF$, then PF_e is reflection of OI about EF. Thus by simple angle chase we get

$$\angle(DP, OI) = \angle(PF_e, PD) = \angle(VF_e, VD) = \angle(DV, DI).$$

This clearly implies that the isogonal of DXV WRT $\angle EDF$ is parallel to OI.

r1234
#3 Mar 28, 2012, 8:28 pm

Let us restate this problem as follows:-
In a triangle ABC, tangents at B, C to $\odot ABC$ intersect at T.P is the midpoint of OT(O is the circumcenter), P' is the reflection of P wrt O.Then the isogonal line of AP' wrt $\angle BAC$ is parallel to the euler line of that triangle.

Proof:- Let H be the orthocenter of $\triangle ABC$.Now clearly AP passes through the Kosnita point K i.e the isogonal conjugate of the nine point center N of $\triangle ABC$.So AN, AK are also isogonal wrt $\angle OAH$.Now since $OP' \parallel AH$, so $A(O, K, H, P') = -1$.Let P'' be the isogonal conjugate of P'.then clearly $A(O, N, H, P'') = -1$.But since N is the midpoint of OH, we conclude that $AP'' \parallel OH$.Hence proven.

High School Olympiads

very hard geoproblem

Source: Kazakhstan 2012

math707
#1 Mar 23, 2012, 9:17 pm

Let PQ- tangent line to incircle of triangle ABC(P on AB, Q on AC). M and N-points on AB and AC respectively, where $AP=MB$ and $AQ=CN$. Prove that all lines MN passes through a fixed point.(I solved this problem via complex numbers, may be somebody will find a simple solution)

jayme
#2 Mar 23, 2012, 9:56 pm • 1

Dear Mathlinkers,
the fix point is the Nagel point of ABC... this is only a conjecture

Sincerely
Jean-Louis

sunken rock
#3 Mar 24, 2012, 2:38 am

Hint:
It seems that the **Transversal Theorem (Cristea's theorem)** may work here, so we need to prove:
$(s-b)\cdot\frac{AP}{PB}+(s-c)\cdot\frac{AQ}{CQ}=s-a$, where a, b, c, s are triangle's sides BC, CA, AB and its semiperimeter.

If we prove the above relation, it means that, indeed, MN passes through the Nagel point of $\triangle ABC$.

Best regards,
sunken rock

pohoatza
#4 Mar 24, 2012, 6:48 am

Cristea's theorem after the Romanian Soccer player right?

Luis González
#5 Mar 24, 2012, 8:26 am

Lemma. P,K are two points on the plane of $\triangle ABC$. BP, CP cut AC, AB at E, F and BK, CK cut AC, AB at Y, Z. Y_0, Z_0 are the reflections of Y, Z about the midpoints of AC, AB, respectively. P^{-1} is the isotomic conjugate of P WRT $\triangle ABC$. Then $K \in EF \Longleftrightarrow P^{-1} \in Y_0Z_0$.

Proof. We use barycentric coordinates WRT $\triangle ABC$. Thus

$$K(u:v:w) \Longrightarrow \ Y_0\left(\frac{1}{u}:0:\frac{1}{w}\right) \ , \ Z_0\left(\frac{1}{u}:\frac{1}{v}:0\right)$$

$$P(x:y:z) \Longrightarrow \ P^{-1}\left(\frac{1}{x}:\frac{1}{y}:\frac{1}{z}\right)$$

$$\begin{pmatrix} u & v & w \\ x & 0 & z \\ x & y & 0 \end{pmatrix} = 0 \iff vxz + wxy - uyz = 0 \iff K \in EF$$

$$\begin{pmatrix} x^{-1} & x^{-1} & z^{-1} \\ u^{-1} & 0 & w^{-1} \\ u^{-1} & v^{-1} & 0 \end{pmatrix} = 0 \iff \frac{vxz + wxy - uyz}{uvwxyz} = 0 \iff P^{-1} \in Y_0Z_0$$

Since these conditions are identical, we get that $K \in EF \iff P^{-1} \in Y_0Z_0$. $\square$

Back to the problem, let the incircle (I) of $\triangle ABC$ touch AC, AB at E, F. By Newton's theorem for the tangential quadrilateral $PQCB$, the lines CP, BQ, EF concur at K. Using the previous lemma, if $K \in EF$, then MN passes through the isotomic conjugate of the Gergonne point $BE \cap CF$, i.e. the Nagel point of $\triangle ABC$.

RSM

#6 Mar 24, 2012, 12:28 pm

> *luisgeometra wrote:*
> **Lemma.** P, K are two points on the plane of $\triangle ABC. BP, CP$ cut AC, AB at E, F and BK, CK cut AC, AB at $Y, Z. Y_0, Z_0$ are the reflections of Y, Z about the midpoints of AC, AB, respectively. P^{-1} is the isotomic conjugate of P WRT $\triangle ABC$. Then $K \in EF \iff P^{-1} \in Y_0Z_0$.

Here is a synthetic proof of this lemma:-
Suppose, $B_1 = BP^{-1} \cap AC$ and $C_1 = CP^{-1} \cap AB$.
If K lies on EF, then $(KB, KC; KA, KE) = (BZ; AF) = (YC; AE) \implies (AZ_0; BC_1) = (Y_0A; CB_1)$. So BB_1, CC_1, Y_0Z_0 are concurrent. So Y_0Z_0 passes through P^{-1}.

Quick Reply

High School Olympiads

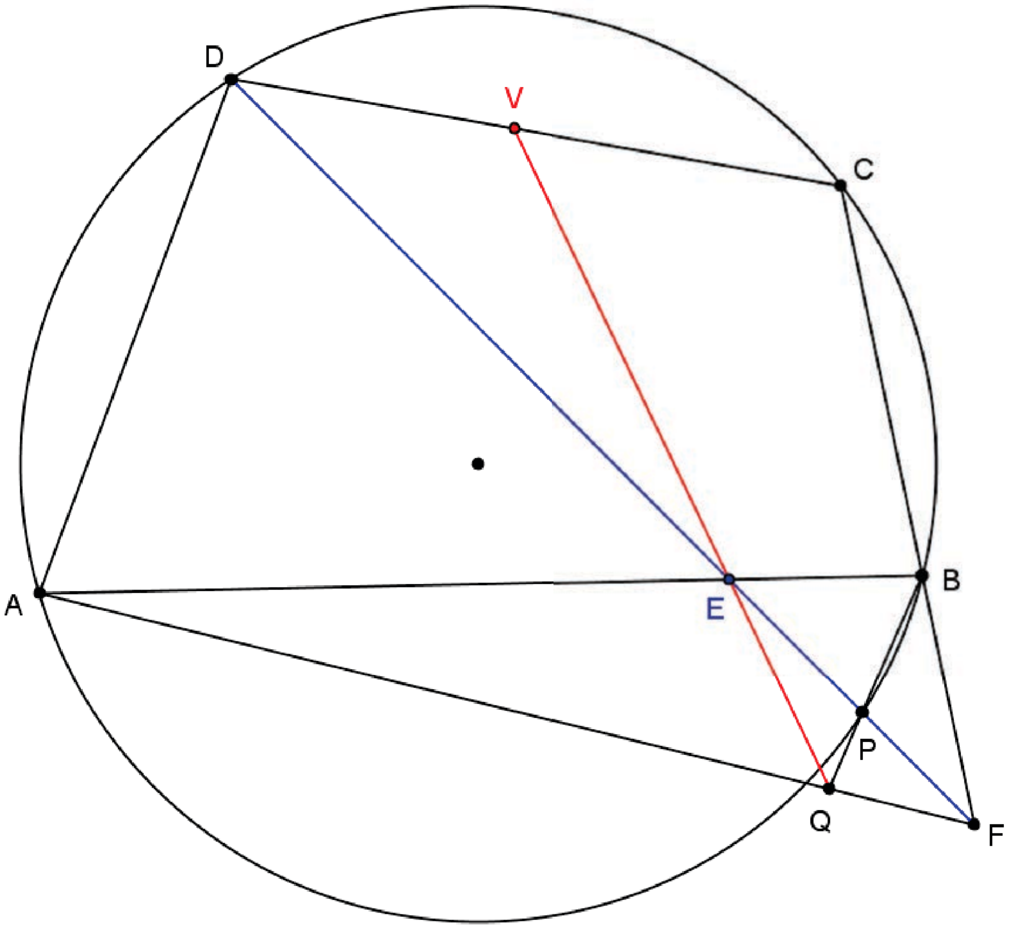
V is a fixed point (own)

Petry

#1 Mar 23, 2012, 1:30 am • 4

Let $ABCD$ be a quadrilateral inscribed in the circle Γ and E is a variable point on AB.
$\{F\} = DE \cap BC$, , $\{Q\} = BP \cap AF$ and $\{V\} = QE \cap CD$.
Prove that V is a fixed point.

Attachments:



vittasko

#2 Mar 23, 2012, 8:46 pm • 1

Thank you for a nice problem, dear **Petrisor**.

● Let be the points $X \equiv CD \cap AF$ and $Y \equiv AD \cap BC$ and $T \equiv XY \cap PB$.

Let D' be the **antidiametric** (= **diametric opposite**) point of D and we denote the points $Z \equiv BC \cap DD'$ and $K \equiv CD \cap AZ$ and $S \equiv KY \cap BD'$.

The line segment CD intersects the pencil $Y.CDKX$ and so, we have
$(Y.CDKX) = (C, D, K, X) = (A.CDKX) = (C, Y, Z, F) = (D.XAD'P)$, (1)

● We see now, that the angles formed by the homologous rays if the pencils $D.XAD'P,\ B.CAD'P$ are equal (

$\angle XDA = \angle CBA$ and $\angle ADD' = \angle ABD'$ and $\angle D'DP = \angle D'BP$) and so, we conclude that they have equal **double ratios** (= **cross ratios**).

That is we have $(D.XAD'P) = (B.CAD'P) \quad ,(2)$

From $(1),\ (2) \Longrightarrow (Y.CDKX) = (B.CAD'P) \quad ,(3)$

From (3) and because of the pencils $Y.CDKX,\ B.CAD'P$ have BY as their common ray, we conclude that the points $A \equiv YD \cap BA$ and $S \equiv YK \cap BD'$ and $T \equiv YX \cap BP$, are collinear.

That is, the point $T \equiv XY \cap BP$ lies on the line segment AS and we denote the point $V \equiv CD \cap AS.$

We will prove now that the line segment QE, where $Q,\ E$ are the points as the problem states, passes through the point V, considering as a fixed point on $CD.$

$($ begining from the point D', we define the points $Z \equiv BC \cap DD'$ and $K \equiv CD \cap AZ$ and $S \equiv KY \cap BD'$ and $V \equiv CD \cap AS$ $).$

● Because of now the collinearity of the points $X \equiv DV \cap FQ$ and $Y \equiv AD \cap BF$ and $T \equiv AV \cap BQ$, based on the **Desarques theorem**, we conclude that the triangles $\triangle\ ADV,\ \triangle\ BFQ$ are perspective and so, the line segments $AB,\ DF,\ VQ$ are concurrent at one point.

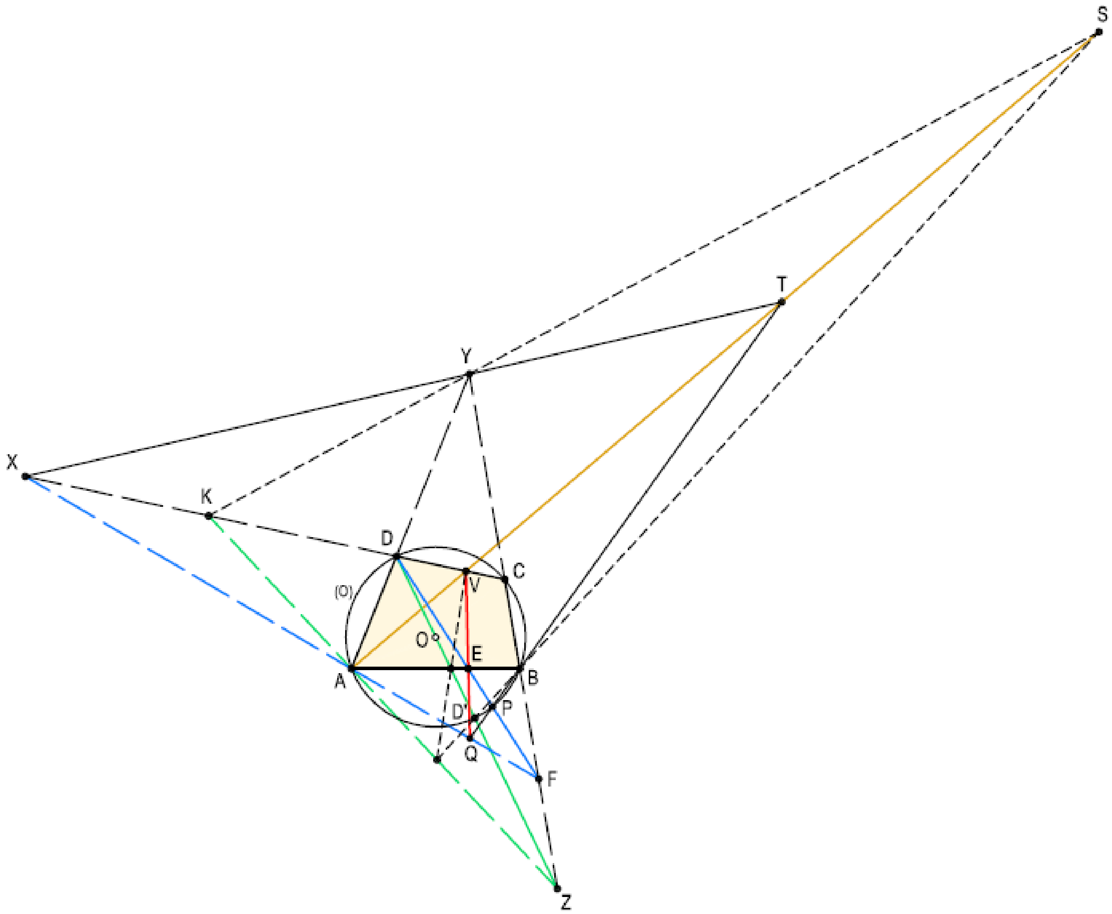
That is, the line segment VQ passes through the point $E \equiv AB \cap DF$ and the proof is completed.

● This proof is dedicated to **Kostas Dortsios**

Kostas Vittas.

PS. It is clear that we can say the same words for every point D' on the circumcircle of the given cyclic quadrilateral $ABCD$, not necessary as the antidiametric point of D.

Attachments:



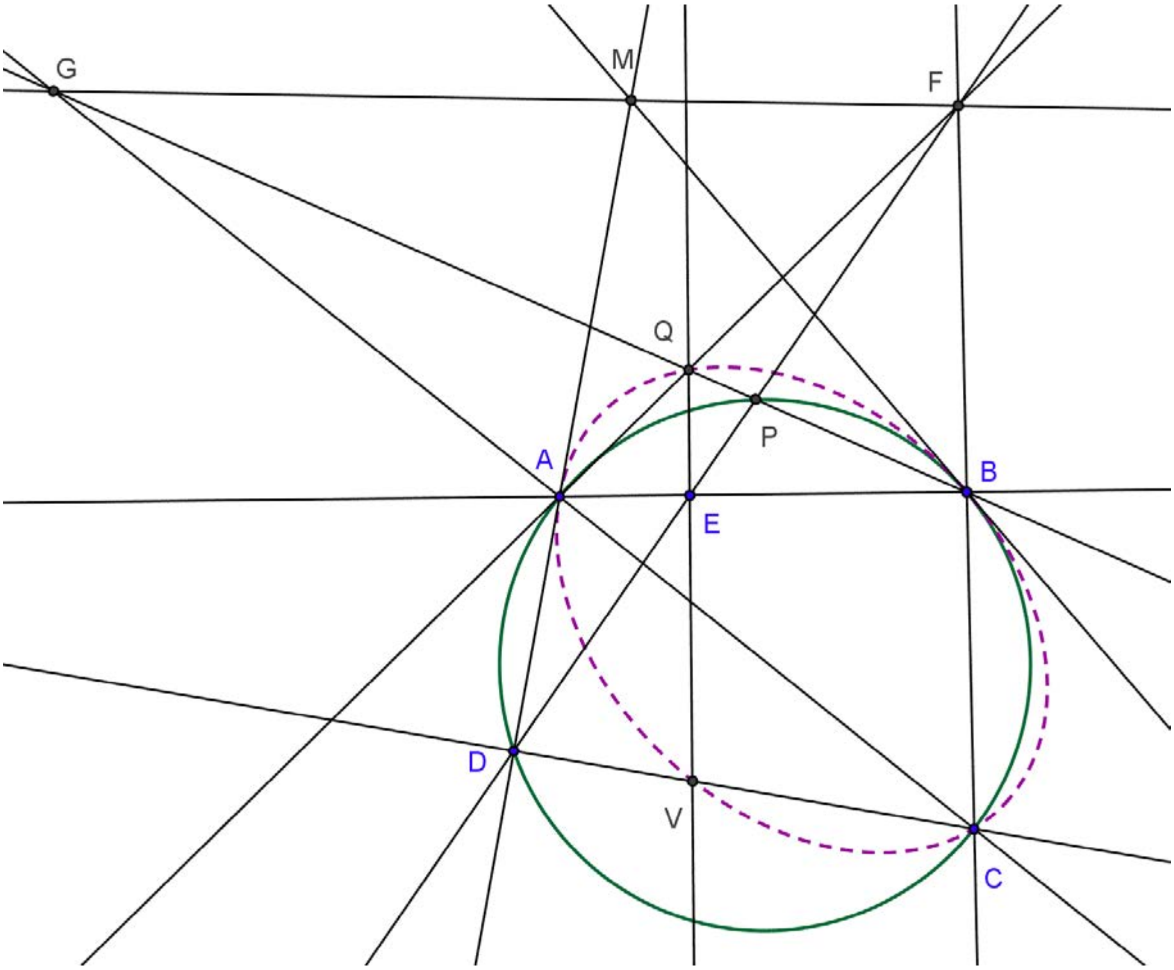
This post has been edited 4 times. Last edited by vittasko, Mar 24, 2012, 3:00 pm

Luis González

#3 Mar 24, 2012, 1:07 am • 6

Tangent of Γ at B cuts AD at M. Let $\mathcal{K}$ be the conic passing through A, B, C and tangent to MA, MB. By Pascal theorem for the degenetare cyclic hexagon $BBCADP$, the intersections $M \equiv BB \cap AD, F \equiv BC \cap DP, G \equiv CA \cap PB$ are collinear. Thus, by Pascal theorem for the degenerate hexagon $AAQBBC$, we get that $Q \in \mathcal{K}$. By Pascal theorem for $AAQVCB$, we get that $V \in \mathcal{K}$. So the fixed V is the 2nd intersection of $\mathcal{K}$ with CD.

Attachments:



vittasko

#4 Mar 24, 2012, 2:07 am

Thank you for a very beautiful approach dear **Luis**.

Best regards, Kostas Vittas.

vittasko

#5 Apr 8, 2012, 4:15 pm

An interesting result mentioned by a friend of mine **Kostas Rekoumis** Here, is that the constant point V as the problem states, lies on the **A**-symmedian of the triangle $\triangle\ ABD.$

If we consider as constant the triangle $\triangle\ ABD$ and the points $C,\ P,$ as movable ones on its circumcircle (O) then, the point $V\equiv CD\cap QE,$ where $E\equiv AB\cap DP$ and $F\equiv BC\cap DP$ and $Q\equiv AF\cap BP,$ always lies on the **A**-symmedian of the $\triangle\ ABD.$

Thank you again dear **Petrisor**, for a nice problem.

Best regards, Kostas Vittas.

Petry

#6 Aug 17, 2014, 7:52 pm

My solution:
Let R be the circumradius of $ABCD$.
$\{S\} = VQ \cap BC$

$$\frac{EF}{ED} = \frac{FB \cdot sin\angle EBF}{BD \cdot sin\angle EBD} = \frac{FB \cdot sin\frac{\widehat{AC}}{2}}{BD \cdot sin\frac{\widehat{AD}}{2}} = \frac{FB \cdot \frac{AC}{2R}}{BD \cdot \frac{AD}{2R}} = \frac{FB \cdot AC}{BD \cdot AD} \text{(*)}$$

$$\frac{EA}{EB} = \frac{AD \cdot sin\angle EDA}{BD \cdot sin\angle EDB} = \frac{AD \cdot sin\frac{\widehat{PA}}{2}}{BD \cdot sin\frac{\widehat{PB}}{2}} = \frac{AD \cdot \frac{PA}{2R}}{BD \cdot \frac{PB}{2R}} = \frac{AD \cdot PA}{BD \cdot PB}$$

$$\frac{FQ}{AQ} = \frac{FB \cdot sin\angle FBQ}{AB \cdot sin\angle ABQ} = \frac{FB \cdot sin\frac{\widehat{PC}}{2}}{AB \cdot sin\frac{\widehat{PA}}{2}} = \frac{FB \cdot \frac{PC}{2R}}{AB \cdot \frac{PA}{2R}} = \frac{FB \cdot PC}{AB \cdot PA}$$

So, $\frac{SF}{SB} = \frac{EA}{EB} \cdot \frac{FQ}{AQ} = \frac{AD \cdot PA}{BD \cdot PB} \cdot \frac{FB \cdot PC}{AB \cdot PA} \Rightarrow$

$$\Rightarrow \frac{SF}{SB} = \frac{AD \cdot FB \cdot PC}{AB \cdot BD \cdot PB} \Rightarrow$$

$$\Rightarrow \frac{SF}{FB} = \frac{AD \cdot FB \cdot PC}{AB \cdot BD \cdot PB - AD \cdot FB \cdot PC} \Rightarrow$$

$$\Rightarrow SF = \frac{AD \cdot FB^2 \cdot PC}{AB \cdot BD \cdot PB - AD \cdot FB \cdot PC}$$

$SC = SF + FC =$

$$= \frac{AD \cdot FB^2 \cdot PC + AB \cdot BD \cdot PB \cdot FC - AD \cdot FB \cdot PC \cdot FC}{AB \cdot BD \cdot PB - AD \cdot FB \cdot PC} =$$

$$= \frac{AB \cdot BD \cdot PB \cdot FC - AD \cdot FB \cdot PC \cdot BC}{AB \cdot BD \cdot PB - AD \cdot FB \cdot PC}$$

So, $\frac{SC}{SF} = \frac{AB \cdot BD \cdot PB \cdot FC - AD \cdot FB \cdot PC \cdot BC}{AD \cdot FB^2 \cdot PC}$ (**)

(*), (**) $\Rightarrow$

$$\Rightarrow \frac{VC}{VD} = \frac{EF}{ED} \cdot \frac{SC}{SF} =$$

$$= \frac{FB \cdot AC}{BD \cdot AD} \cdot \frac{AB \cdot BD \cdot PB \cdot FC - AD \cdot FB \cdot PC \cdot BC}{AD \cdot FB^2 \cdot PC} =$$

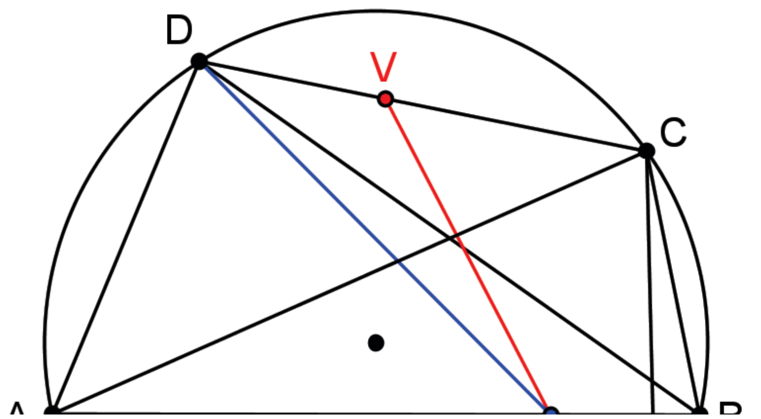
$$= \frac{AB \cdot AC}{AD^2} \cdot \frac{PB \cdot FC}{PC \cdot FB} - \frac{AC \cdot BC}{AD \cdot BD} = \frac{AC}{AD} \cdot \left(\frac{AB \cdot PB \cdot FC}{AD \cdot FB \cdot PC} - \frac{BC}{BD}\right) =$$

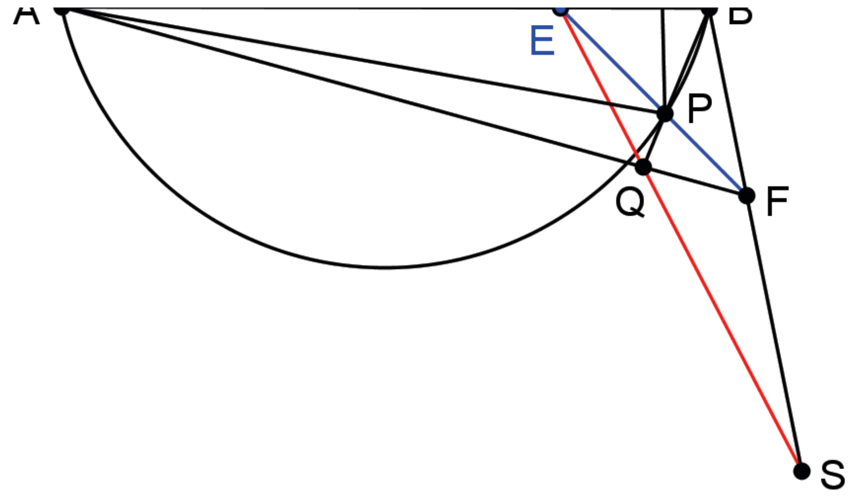
$$= \frac{AC}{AD} \cdot \left(\frac{AB \cdot PB \cdot FD}{AD \cdot FB \cdot BD} - \frac{BC}{BD}\right) = \frac{AC}{AD} \cdot \left(\frac{AB \cdot FB \cdot CD}{AD \cdot FB \cdot BD} - \frac{BC}{BD}\right) \Rightarrow$$

$$\Rightarrow \frac{VC}{VD} = \frac{AC}{AD \cdot BD} \cdot \left(\frac{AB \cdot CD}{AD} - BC\right)$$

So, V is a fixed point.

Attachments:





Petry

#7 Aug 17, 2014, 7:58 pm

Let's prove that the fixed point V lies on the A-symmedian of ΔABD.

AW is A-symmedian of ΔABD, $W \in BD \Rightarrow \dfrac{WB}{WD} = \left(\dfrac{AB}{AD}\right)^2$

$\{V'\} = AW \cap CD$ and $\{A, T\} = AW \cap \Gamma$.

$$\frac{WB}{WD} = \frac{AB \cdot sin\angle WAB}{AD \cdot sin\angle WAD} = \frac{AB \cdot sin\frac{\widehat{TB}}{2}}{AD \cdot sin\frac{\widehat{TD}}{2}} = \frac{AB \cdot \frac{TB}{2R}}{AD \cdot \frac{TD}{2R}} = \frac{AB \cdot TB}{AD \cdot TD}$$

So, $\dfrac{AB \cdot TB}{AD \cdot TD} = \left(\dfrac{AB}{AD}\right)^2 \Rightarrow \dfrac{TB}{TD} = \dfrac{AB}{AD}$ (1)

$$\frac{V'C}{V'D} = \frac{TC \cdot sin\angle V'TC}{TD \cdot sin\angle V'TD} = \frac{TC \cdot sin\frac{\widehat{AC}}{2}}{TD \cdot sin\frac{\widehat{AD}}{2}} = \frac{TC \cdot \frac{AC}{2R}}{TD \cdot \frac{AD}{2R}} = \frac{TC \cdot AC}{TD \cdot AD} \text{ (2)}$$

$BCTD$ is inscribed quadrilateral $\Rightarrow TB \cdot CD = TC \cdot BD + TD \cdot BC \Rightarrow$
$\Rightarrow TC \cdot BD = TB \cdot CD - TD \cdot BC$ (3)

(1), (3) $\Rightarrow \dfrac{TC}{TD} = \dfrac{TC \cdot BD}{TD \cdot BD} = \dfrac{TB \cdot CD - TD \cdot BC}{TD \cdot BD} =$

$$= \frac{TB}{TD} \cdot \frac{CD}{BD} - \frac{BC}{BD} = \frac{AB}{AD} \cdot \frac{CD}{BD} - \frac{BC}{BD} \Rightarrow$$

$$\Rightarrow \frac{TC}{TD} = \frac{1}{BD} \cdot \left(\frac{AB \cdot CD}{AD} - BC\right) \text{ (4)}$$

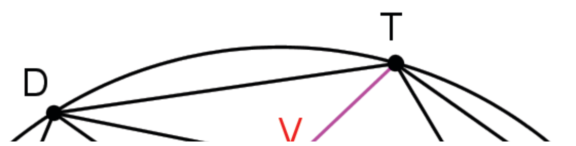
(2), (4) $\Rightarrow \dfrac{V'C}{V'D} = \dfrac{AC}{AD \cdot BD} \cdot \left(\dfrac{AB \cdot CD}{AD} - BC\right)$

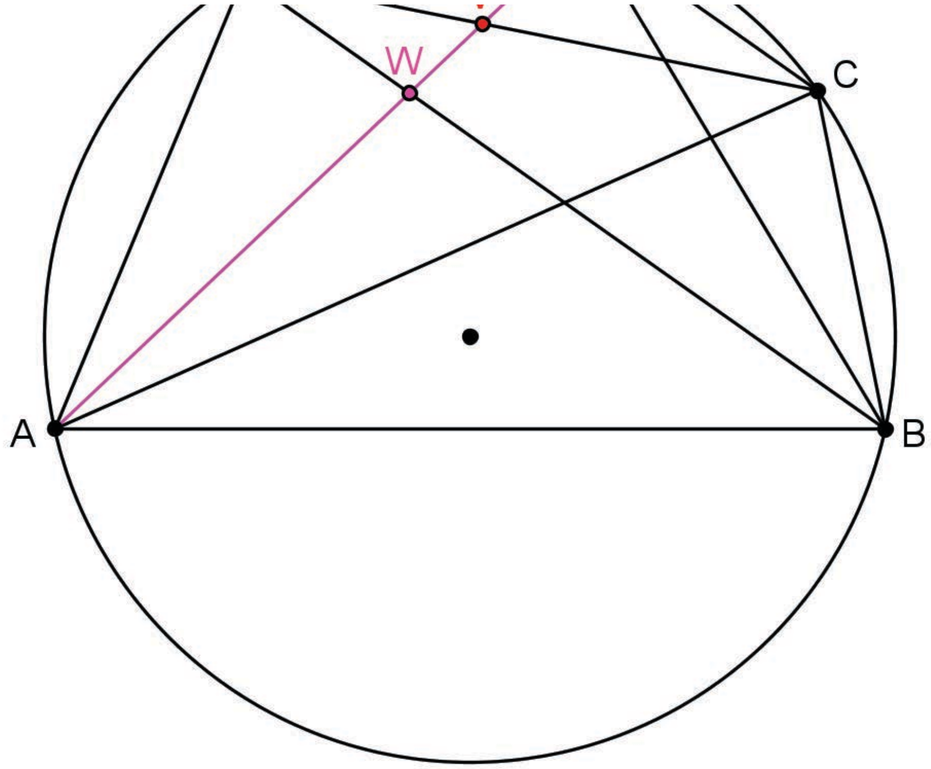
So, $\dfrac{V'C}{V'D} = \dfrac{VC}{VD} \Rightarrow V' = V$.

So, the fixed point V lies on the A-symmedian of ΔABD.

Best regards,
Petrisor Neagoe

Attachments:





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