High School Olympiads

Euler's circle and the problem involved

Reply

takita_louis

#1 Jan 3, 2013, 8:57 pm

ΔABC is an acute triangle with circumcenter O and E is the center of the Euler's circle. $B' \in AB, C' \in AC$ such that E is the midpoint of $B'C'$. Prove that O lies in the Euler line of $\Delta AB'C'$.
Thanks a lot!

Grigoris

#2 Jan 4, 2013, 2:46 am • 1

Dear takita_louis, your problem is very interesting!

I have found a rather complicated proof that involves a lot of ratios. The key point is the application of **Theorem of Analogous Divisions** (which is a result stemming from **Thales Theorem**).
Mr. Kostas Vittas, a Greek Geometer, has mentioned this theorem somewhere on the forum in the past but I am not able to find it. I would be grateful if someone could do so.

I promise to post the complete solution as soon as possible although I am going to be busy the next 2-3 days.

With Best Regards,
Grigoris

Luis González

#3 Jan 4, 2013, 9:38 am • 2

Y, Z are the feet of the altitudes on AC, AB. $H \equiv BY \cap CZ$ is the orthocenter of $\triangle ABC$ and O_A is the circumcenter of $\triangle HBC$, which is clearly the reflection of O across BC. $HAOO_A$ is a parallelogram with diagonal intersection $E \equiv OH \cap AO_A$, i.e. E is midpoint of $\overline{AO_A} \Longrightarrow AB'O_AC'$ is a parallelogram $\Longrightarrow O_AC' \parallel AB \perp CH$ and $O_AB' \parallel AC \perp BH \Longrightarrow O_AC'$ and O_AB' are the perpendicular bisectors of $\overline{HC}$ and $\overline{HB} \Longrightarrow \triangle HBB'$ and $\triangle HCC'$ are isosceles, even similar, since $\angle HBZ = \angle HCY \Longrightarrow \triangle HYC \sim \triangle HZB$ are similar with corresponding cevians HC' and HB'. Hence

$$\frac{\overline{B'B}}{\overline{B'Z}} = \frac{\overline{C'C}}{\overline{C'Y}} \Longrightarrow \frac{\overline{B'B} \cdot \overline{B'A}}{\overline{B'Z} \cdot \overline{B'A}} = \frac{\overline{C'C} \cdot \overline{C'A}}{\overline{C'Y} \cdot \overline{C'A}}.$$

This means that the ratio of powers of B' and C' WRT $\odot(AYHZ) \equiv (H_A)$ and (O) are equal $\Longrightarrow \odot(AB'C') \equiv (O')$ is coaxal with (H_A) and (O), i.e. they meet at A and P. It's well known that P, H and the midpoint M of $\overline{BC}$ are collinear, thus the antipode A' of A WRT (O') is on HM. $OMHH_A$ is clearly a parallelogram with diagonal intersection E, so reflection of A' about the midpoint E of $\overline{B'C'}$ (orthocenter of AB'C') is on $OO'H_A \Longrightarrow OO'$ is Euler line of AB'C'.

Quick Reply

High School Olympiads

Circle passing through a fixed point

Reply

robinson123

#1 Jan 3, 2013, 10:11 pm

Given a circle (C) with center (O) and A, B are 2 fixed points on (C). E lies on AB. C, D are on (C) and CD pass through E. P lies on the ray DA, Q lies on the ray DB such that E is the midpoint of PQ. Prove that the circle passing through C and touch PQ at E also pass through the midpoint of AB

Luis González

#2 Jan 3, 2013, 11:47 pm • 2

Let M be the midpoint of $\overline{AB}.$ We prove that $\odot(CEM)$ is tangent to PQ at $E.$ Polar of E WRT $\mathcal{C}$ cuts AB at $S.$ Inversion with center E and power $\overline{EA} \cdot \overline{EB} = \overline{EC} \cdot \overline{ED}$ carries $\mathcal{C}$ into itself and $\odot(CEM)$ into the line $SD,$ because of $\overline{EA} \cdot \overline{EB} = \overline{EM} \cdot \overline{ES}.$ Since $D(A, B, E, S) \equiv D(P, Q, E, S) = -1$ and E is the midpoint of $\overline{PQ},$ it follows that $DS \parallel PQ \Longrightarrow \odot(CEM)$ is tangent to PQ at $E.$

Grigoris

#3 Jan 4, 2013, 12:11 am • 1

Dear robinson123 and Luis González, lets see an elementary proof:

The parallel line through A to PQ intersects CD at T. Let M be the midpoint of AB.

It is obvious that $MT \parallel BD \implies \frac{EM}{EB} = \frac{ET}{ED} \implies \frac{EM}{ET} = \frac{EB}{ED} \implies \frac{EM}{ET} = \frac{CE}{AE}$.

Consequently, $ACMT$ is cyclic and it follows that $\angle CME = \angle ETA = \angle CEP \implies PE\ tangent.$

With Best Regards,
Grigoris

Quick Reply

High School Olympiads

Concurrent on Euler line

Reply

Source: Own

buratinogigle

#1 Jan 2, 2013, 11:59 am

Let ABC be a triangle with circumcircle (O). Altitudes AD, BE, CF are concurrent at H. X is reflection of H through the line passing through D and perpendicular to OD.

a) Prove that X lies on (O).

b) Similary we have Y, Z. Prove that AX, BY, CZ are concurrent at a point on OH.

See I need some pure geometry :))

pi37

#2 Jan 2, 2013, 2:18 pm • 2

a
b

Luis González

#3 Jan 2, 2013, 10:25 pm • 1

a) Let X_A be the projection of H on the perpendicular to OD at D. DX_A is a tangent of the inconic with foci $O, H,$ whose pedal circle is the 9-point circle $\odot(DEF) \Longrightarrow X_A \in \odot(DEF) \Longrightarrow$ reflection X of H about X_A is on $(O),$ since H is the center of the direct homothety with coefficient 2 that takes $\odot(DEF)$ into (O).

b) Let P and M be the midpoints of $\overline{AH}$ and $\overline{BC},$ i.e. midpoints of the arcs EF and EDF of $\odot(DEF)$. S is the projection of H on EF and PS cuts $\odot(DEF)$ again at $X_A{}'$. Then $PE^2 = PF^2 = PH^2 = PS \cdot PX_A{}' \Longrightarrow$ $\angle DX_A{}'M = \angle DPM = \angle SHP = \angle PX_A{}'H \Longrightarrow \angle HX'_A D = \angle PX_A{}'M = 90^\circ \Longrightarrow X_A \equiv X_A{}'$. So reflections A, H_A, X of H about P, S, X_A are collinear $\Longrightarrow AX \equiv AH_A$ is the A-cevian of the Begonia point X_{24} of H. Similarly, BY and CZ pass through X_{24}, which lies on OH (see the topic Concurrent).

TelvCohl

#4 Oct 10, 2014, 5:27 am • 3

My solution:

Lemma:

Let P, Q be the isogonal conjugate of triangle.
Let X, Y, Z be the projection of P on BC, CA, AB, respectively.
Let X', Y', Z' be the projection of Q on BC, CA, AB respectively.
Let D, E, F be the intersection of (XYZX'Y'Z') and XT, YT, ZT, respectively.
Then X'D, Y'E, Z'F are concurrent if and only if T lie on PQ (In this case X'D, Y'E, Z'F concur at a point lie on PQ)

Proof of the lemma:

Let M be the midpoint of PQ.
Let K be the intersection of Y'Z and YZ'.
Let G be the intersection of ZX and ZX'.
Let R, S be the intersection of (XYZX'Y'Z') and QY', QZ', respectively.

By Pascal theorem (for YZ'SZY'R) we get K lie on PQ.
By Pascal theorem (for YZ'FZY'E) we get the intersection of Y'E and Z'F lie on KT.
We can prove G lie on PQ and the intersection of Z'F and X'D lie on GT similarly,
so T lie on PQ <=> the intersection of (Y'E,Z'F) and (Z'F,X'D) lie on PQ <=>X'D, Y'E, Z'F, PQ are concurrent.

Back to the main problem

Let L be the reflection of H with respect to O.
Let A'B'C' be the anti-complement triangle of triangle ABC.
Let D', E', F' be the intersection of (O) and AD, BE, CF, respectively.
Let P, Q, R be the intersection of (O) and B'C', C'A', A'B', respectively.

(a)

Since DH=DD'=DX,
so angle HXD' = 90,
hence we get OD is the perpendicular bisector of D'X.
ie. X lie on (O)

(b)

Since P, Q, R is the projection of A', B', C' on B'C', C'A', A'B', respectively,
so we get PH//OD (notice that H is the circumcenter of triangle A'B'C'),
hence we get P, H, X are collinear (notice that HX //OD).
Similarly, we can prove Q, H, Y are collinear and R, H, Z are collinear.
ie. PX, QY, RZ concur at H

Since L is the orthocenter of triangle A'B'C',
so H and L is the isogonal conjugate of triangle A'B'C',
By the lemma we get AX, BY, CZ, HL are concurrent.
ie. AX, BY, CZ are concurrent at a point on the Euler line of ABC

Q.E.D

High School Olympiads

Tangents to the incircle

Source: All-Russian MO 1998

v_Enhance
#1 Jan 1, 2013, 2:37 pm

In scalene $\triangle ABC$, the tangent from the foot of the bisector of $\angle A$ to the incircle of $\triangle ABC$, other than the line BC, meets the incircle at point K_a. Points K_b and K_c are analogously defined. Prove that the lines connecting K_a, K_b, K_c with the midpoints of BC, CA, AB, respectively, have a common point on the incircle.

Luis González
#2 Jan 2, 2013, 12:27 am

Posted before. The referred lines concur at the Feuerbach point of ABC.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=419434
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=487067

anantmudgal09
#3 Mar 12, 2016, 2:41 pm

Clearly, the concurrence point is the Feuerbach point since by an easy angle chase, $\triangle K_aK_bK_c \sim \triangle MNP$ for midpoints M, N, P of BC, CA, AB respectively and these are also homothetic.

Thus, these lines concur at the Feuerbach point. :)

jayme
#4 Mar 12, 2016, 4:07 pm

Dear Mathlinkers,
you can see and more on

http://jl.ayme.pagesperso-orange.fr/Docs/la%20droite%20de%20Hamilton.pdf p. 31-33.

Sincerely
Jean-Louis

High School Olympiads

More Mixtillinear Incircle Bisectors

Reply

Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□□

XmL

#1 Dec 30, 2012, 11:53 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

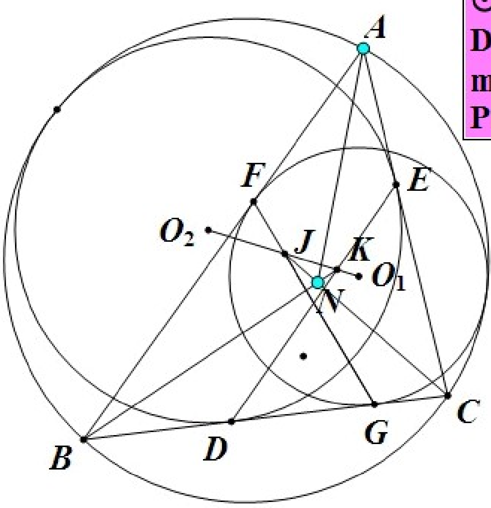
Attachments:

文武光华数学工作室
南京　潘成华

2012　12　20　18:30

已知　△ABC的B-,C-伪内切圆⊙O$_1$,⊙O$_2$切BC分别于G,D,
⊙O$_1$切AB于F,⊙O$_2$切AC于E,DE交O$_1$O$_2$于K,FG交O$_1$O$_2$于J,
直线CK交BJ于N,
求证　AN是∠BAC平分线

$\odot O_1, \odot O_2$ are the B-,C- mixtillinear incircles of $\triangle$ABC. They touch BC at G,D respectively. $\odot O_1$ touches AB at F. $\odot O_2$ touches AC at E, DE, FG meet O_1O_2 at K,J respectively. CK meets BJ at N.
Prove: AN bisects $\angle$BAC



hofamo

#2 Dec 31, 2012, 2:08 pm

(ED,FG)=I.now use Desargues Theorem in triangles IJK,ABC.(FG,AC)=X,(ED,AB)=Y,(O1O2,BC)=Z.we must prove that X,Y,Z are collinear.((O2),(ABC))=M,((O1),(ABC))=N.its easy to now MN,BC,O1O2 are concurrent.now write Menelaus Theorem in ABC and sin law in ABC and Z.

Luis González

#3 Jan 1, 2013, 3:03 am

$I \equiv DE \cap FG$ is the incenter of $\triangle ABC$ and S is the midpoint of the arc BC of the circumcircle (O) of $\triangle ABC,$ not containing $A.$ A-mixtilinear incircle touches $AC, AB, (O)$ at $U, V, P.$ PS, BC, O_1O_2 concur at X (see the 1st two paragraphs of the solution of the problem Mixtilinear Incircles Parallels). Further, $X \in UV$ (see Internally tangent circles and lines and concurrency), thus $X, Y \equiv FG \cap CA$ and $Z \equiv DE \cap AB$ are collinear on the orthopolar of I WRT $\triangle ABC \Longrightarrow \triangle IKJ$ and $\triangle ABC$ are perspective through $\overline{XYZ}.$ By Desargues theorem, AI, BK and CJ concur and the conclusion follows.

Quick Reply

High School Olympiads

Toshio Seimiya Crux Math Problem 1

Reply

Source: Toshio Seimiya, Crux Mathematicorum 2638

thugzmath10

#1 Dec 31, 2012, 8:41 pm

Acute angled triangle ABC with $AB \neq AC$, orthocenter H and centroid G satisfy $\dfrac{1}{[HAB]} + \dfrac{1}{[HAC]} = \dfrac{2}{[HBC]}$, where the brackets denotes area. Prove that $\angle AGH = 90^\circ$.

Luis González

#2 Jan 1, 2013, 1:16 am

$AG \perp HG \Longleftrightarrow A, G$ and infinity point of the orthic axis of $\triangle ABC$ are collinear. Using barycentric coordinates with respect to $\triangle ABC$, we have that $A(1:0:0), G(1:1:1)$ and $P_\infty(S_B - S_C : S_C - S_A : S_A - S_B)$ are collinear.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ S_B - S_C & S_C - S_A & S_A - S_B \end{pmatrix} = 0 \Longleftrightarrow S_B + S_C = 2S_A \ \ (1).$$

$$\frac{[ABC]}{[HAB]} + \frac{[ABC]}{[HAC]} = 2 \cdot \frac{[ABC]}{[HBC]} \Longleftrightarrow S_B + S_C = 2S_A \ \ (2).$$

From (1) and (2) we conclude that $AG \perp HG \Longleftrightarrow \dfrac{1}{[HAB]} + \dfrac{1}{[HAC]} = \dfrac{2}{[HBC]}$.

Quick Reply

High School Olympiads

Prove that BFO is right if AEDC is cyclic

Reply

Source: All-Russian MO 1999

v_Enhance

#1 Dec 31, 2012, 7:09 am

A circle through vertices A and B of triangle ABC meets side BC again at D. A circle through B and C meets side AB at E and the first circle again at F. Prove that if points A, E, D, C lie on a circle with center O then $\angle BFO$ is right.

Luis González

#2 Dec 31, 2012, 9:43 am • 1

AD, CE and BF are pairwise radical axes of $\odot(ABD), \odot(CBE)$ and (O), concurring at their radical center R. Since $\angle ECD = \angle EAD = \angle BFD \Longrightarrow D, R, F, C$ are concyclic. Thus if BR cuts (O) at X, Y, we have $BD \cdot BC = BX \cdot BY = BR \cdot BF$, but since $(X, Y, B, R) = -1$, then F is midpoint of $\overline{XY} \Longrightarrow OF \perp BF$.

P.S. The problem is just a variation of the configuration presented at Circle center O passes through the vertices A and C.

AllonsyAllonso

#3 Oct 16, 2013, 3:52 pm

> *Luis González wrote:*
> we have $BD \cdot BC = BX \cdot BY = BR \cdot BF$, but since $(X, Y, B, R) = -1$, then F is midpoint of $\overline{XY} \Longrightarrow OF \perp BF$.

I don't understand this. Could you explain it?

Quick Reply

High School Olympiads

Incenter lies on common tangent

Reply

Source: All-Russian MO 1999

v_Enhance

#1 Dec 31, 2012, 7:08 am

A triangle ABC is inscribed in a circle S. Let A_C and C_0 be the midpoints of the arcs BC and AB on S, not containing the opposite vertex, respectively. The circle S_1 centered at A_C is tangent to BC, and the circle S_2 centered at C_0 is tangent to AB. Prove that the incenter I of $\triangle ABC$ lies on a common tangent to S_1 and S_2.

Luis González

#2 Dec 31, 2012, 7:59 am

Posted before. That common tangent is a parallel to AC through the incenter I.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=359172
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=361224

Quick Reply

High School Olympiads

Suppose O,K,L are collinear

Reply

Source: All-Russian MO 2000

v_Enhance
#1 Dec 30, 2012, 7:02 pm

A quadrilateral $ABCD$ is circumscribed about a circle ω. The lines AB and CD meet at O. A circle ω_1 is tangent to side BC at K and to the extensions of sides AB and CD, and a circle ω_2 is tangent to side AD at L and to the extensions of sides AB and CD. Suppose that points O, K, L lie on a line. Prove that the midpoints of BC and AD and the center of ω also lie on a line.

Falcon.1
#2 Dec 31, 2012, 2:53 am

First, i'm going to prove the following two lemmas:

Lemma 1: Let P be a point on side YZ of $\triangle XYZ$. Then $\frac{YP}{PZ}=\frac{XY}{XZ}\frac{\sin\angle PXY}{\sin\angle PXZ}$.

Proof: It's clear that $\frac{YP}{PZ}=\frac{(XYP)}{(XZP)}=\frac{XY\cdot XP\sin\angle PXY}{XZ\cdot XP\sin\angle PXZ}$. And we're done. End Lemma 1.

Lemma 2: In triangle XYZ, let T_x be the tangency point of the incircle on side YZ and E_x be the tangency point of the excircle opposite to X on side YZ. Then $YE_x=ZT_x$. That is, the midpoint of T_xE_x is the midpoint of YZ.

Proof: Let T_y and T_z be the tangency points of the incircle on sides XZ and XY respectively. Also, let P_y and P_z be the tangency points of the excircle opposite to X on XZ and XY respectively. By power of a point, $XT_y=XT_z=x$, $YT_x=YT_z=y$ and $ZT_y=ZT_x=z$. Also, $YE_x=YP_z=\ell_y$ and $ZE_x=ZP_y=\ell_z$. It's clear that $XP_y=XP_z$. Then $x+y+\ell_y=x+z+\ell_z$. That is $y+\ell_y=z+\ell_z$. It's also clear that $\ell_y+\ell_z=y+z$. We sum ℓ_y in both sides of our equation and we get $y+2\ell_y=z+\ell_z+\ell_y=2z+y$, so $\ell_y=z$, and we're done. End Lemma 2.

Now, let $\mathcal{C}(\omega)$ be the center of ω, and M,N the midpoints of BC and AD respectively. It's easy to see that $M\mathcal{C}(\omega)\perp BC$ and $N\mathcal{C}(\omega)\perp AD$. So, M,N and $\mathcal{C}(\omega)$ are collinear if and only if $AD\parallel BC$.

Let L' be the tangency point of the incircle of $\triangle OBC$ on side BC. As $ABCD$ is cyclic, it's clear that OL' is the reflection of OL through the angle bisector of $\angle BOC$. So, $\angle BOK=\angle COL'$ and $\angle COK=\angle BOL'$

Due to Lemma 1: $\frac{BK}{CK}=\frac{OB}{OC}\frac{\sin\angle BOK}{\sin\angle COK}$ and $\frac{BL'}{CL'}=\frac{OB}{OC}\frac{\sin\angle BOL'}{\sin\angle COL'}$. By the previous paragraph and joining both equations, we get $\frac{BK}{CK}\frac{BL'}{CL'}=\frac{OB^2}{OC^2}$.

But due to Lemma 2, we have that $BK=CL'$ and $CK=BL'$, so $OB=OC$ and that implies $\triangle OBC$ and $\triangle OAD$ are both isosceles. Thus, $AD\parallel BC$. And we're done. $\blacksquare$

This post has been edited 1 time. Last edited by Falcon.1, Dec 31, 2012, 5:50 am

Luis González
#3 Dec 31, 2012, 4:38 am

Lemma. Incircle (I) and A-excircle (I_a) of $\triangle ABC$ touch BC at $X,Y,$ respectively. If M is the midpoint of $\overline{BC}$, then $MI\parallel AY$ and $MI_a\parallel AX.$

Proof. A is the exsimilicenter of $(I)\sim(I_a)\Longrightarrow AY$ passes through the antipode X' of X WRT $(I).$ Since M is also midpoint of $\overline{XY}$, then MI is X-midline of $\triangle XYX'\Longrightarrow MI\parallel YX'\equiv AY.$ By extraversion, we have $MI_a\parallel AX.\blacksquare$

Back to the problem, let M, N be the midpoints of $\overline{BC}, \overline{AD}$ and J the center of ω. WLOG assume that ω_1 becomes O-excircle of $\triangle OBC$ and ω_2 becomes incircle of $\triangle OAD$. From the previous lemma, we deduce that $JM \parallel OK$ and $JN \parallel OL \Longrightarrow J, M, N$ lie on a line parallel to $\overline{OLK}$.

jred

#4 May 8, 2015, 9:03 pm

> *Falcon.1 wrote:*
>
> ...
>
> Now, let $\mathcal{C}(\omega)$ be the center of ω, and M, N the midpoints of BC and AD respectively. It's easy to see that $M\mathcal{C}(\omega) \perp BC$ and $N\mathcal{C}(\omega) \perp AD$. So, M, N and $\mathcal{C}(\omega)$ are collinear if and only if $AD \parallel BC$.
>
> ...

Actually,$M\mathcal{C}(\omega) \perp BC$ is not true, unless M is the tangency point. Thus your solution is incorrect, and the claim is still true even AD unparallel to BC.

This post has been edited 1 time. Last edited by jred, May 8, 2015, 9:04 pm
Reason: typo

Quick Reply

High School Olympiads

K is the circumcenter of triangle AOC

Reply

Source: All-Russian MO 2000

v_Enhance
#1 Dec 30, 2012, 7:00 pm

Let O be the center of the circumcircle ω of an acute-angle triangle ABC. A circle ω_1 with center K passes through A, O, C and intersects AB at M and BC at N. Point L is symmetric to K with respect to line NM. Prove that $BL \perp AC$.

Luis González
#2 Dec 30, 2012, 10:50 pm

$\angle ONC = \angle OAC = 90^\circ - \angle ABC \Longrightarrow NO \perp AB$, i.e. NO is the perpendicular bisector of $\overline{AB} \Longrightarrow \triangle ANB$ is N-isosceles $\Longrightarrow \angle MKN = 2\angle BAN = 2\angle MBN$. Thus $\triangle MLN$ is L-isosceles with $\angle MLN = \angle MKN = 2\angle MBN \Longrightarrow L$ is the circumcenter of $\triangle BMN$. Since $BO \perp MN$ (MN is antiparallel to AC WRT BA,BC), then BL is the isogonal of BO WRT $\angle ABC \Longrightarrow BL$ is B-altitude of $\triangle ABC$.

sunken rock
#3 Dec 31, 2012, 3:39 am

$\angle AMC = \angle ANC = \angle AOC = 2\angle ABC$, hence triangles ABN, BMC are $M-$ and $N-$ isosceles, so $\angle CAN = \hat{A} - \hat{B} \iff \angle CKN = 2(\hat{A} - \hat{B})$; with $\angle CKM = 2\hat{A}$ we get $\angle MKN = 2\hat{B}$, i.e. L is the circumcenter of ΔBMN. Further, we know that $BOKL$ is a parallelogram, from $\Delta BMN \sim \Delta BCA$.

Best regards,
sunken rock

AnonymousBunny
#4 Nov 7, 2014, 10:50 pm

Solution

Quick Reply

High School Olympiads

Incircle Concurrence

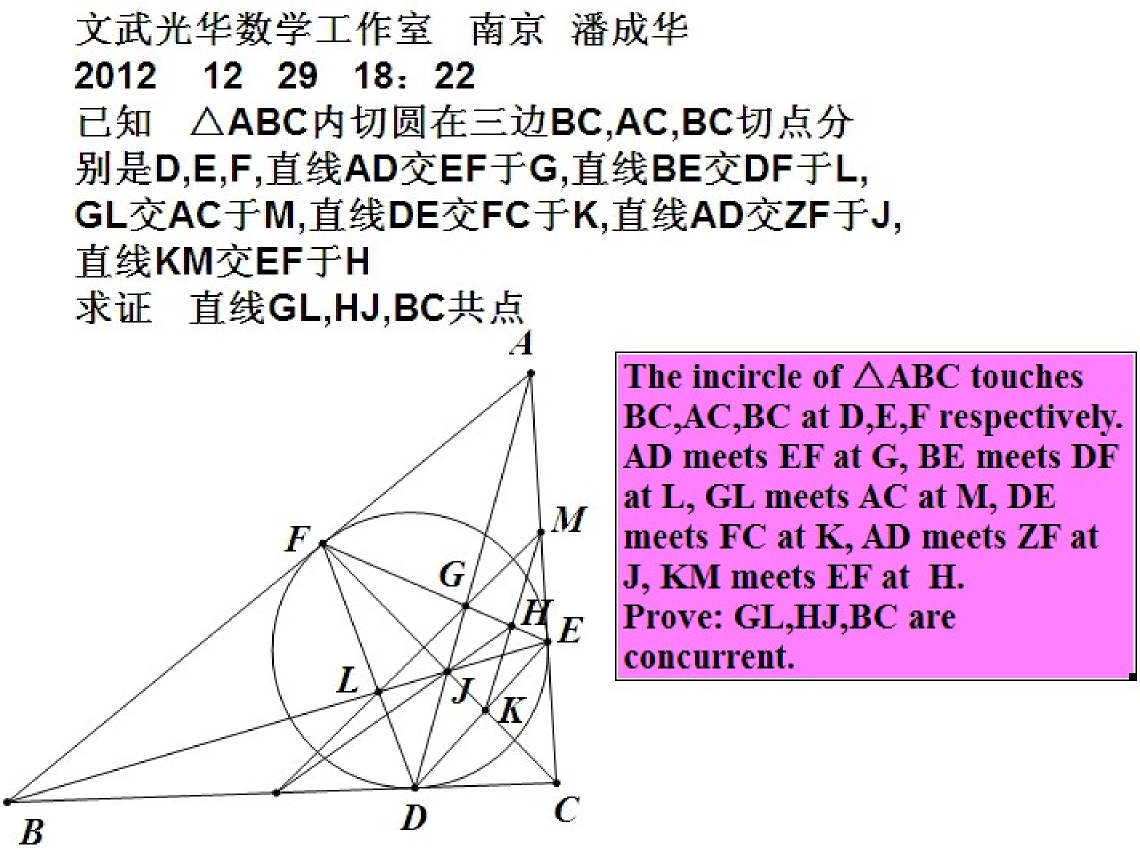
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 30, 2012, 5:33 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:


文武光华数学工作室　南京　潘成华
2012　12　29　18：22
已知　△ABC内切圆在三边BC,AC,BC切点分别是D,E,F,直线AD交EF于G,直线BE交DF于L,GL交AC于M,直线DE交FC于K,直线AD交ZF于J,直线KM交EF于H
求证　直线GL,HJ,BC共点

The incircle of △ABC touches BC,AC,BC at D,E,F respectively. AD meets EF at G, BE meets DF at L, GL meets AC at M, DE meets FC at K, AD meets ZF at J, KM meets EF at H. Prove: GL,HJ,BC are concurrent.

Luis González

#2 Dec 30, 2012, 8:46 am

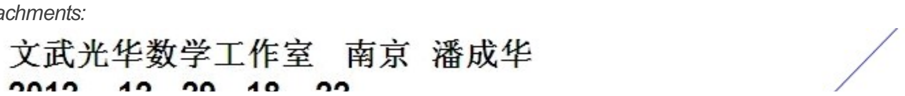
Since J is always inside the incircle ω of $\triangle ABC$, then there exists a central projection sending ω into a circle with center J $\Longrightarrow \triangle ABC$ becomes equilateral with center J. If $P \equiv LG \cap BC$, then obviously $PL = LG = GM$, i.e. $PG = \frac{2}{3}PM$. But $GJ \parallel MH$ (both perpendicular to EF) and $GJ = \frac{1}{3}GD = \frac{1}{3}MK = \frac{2}{3}HM$, thus P, J, H are collinear, i.e. GL, JH, BC concur at P.

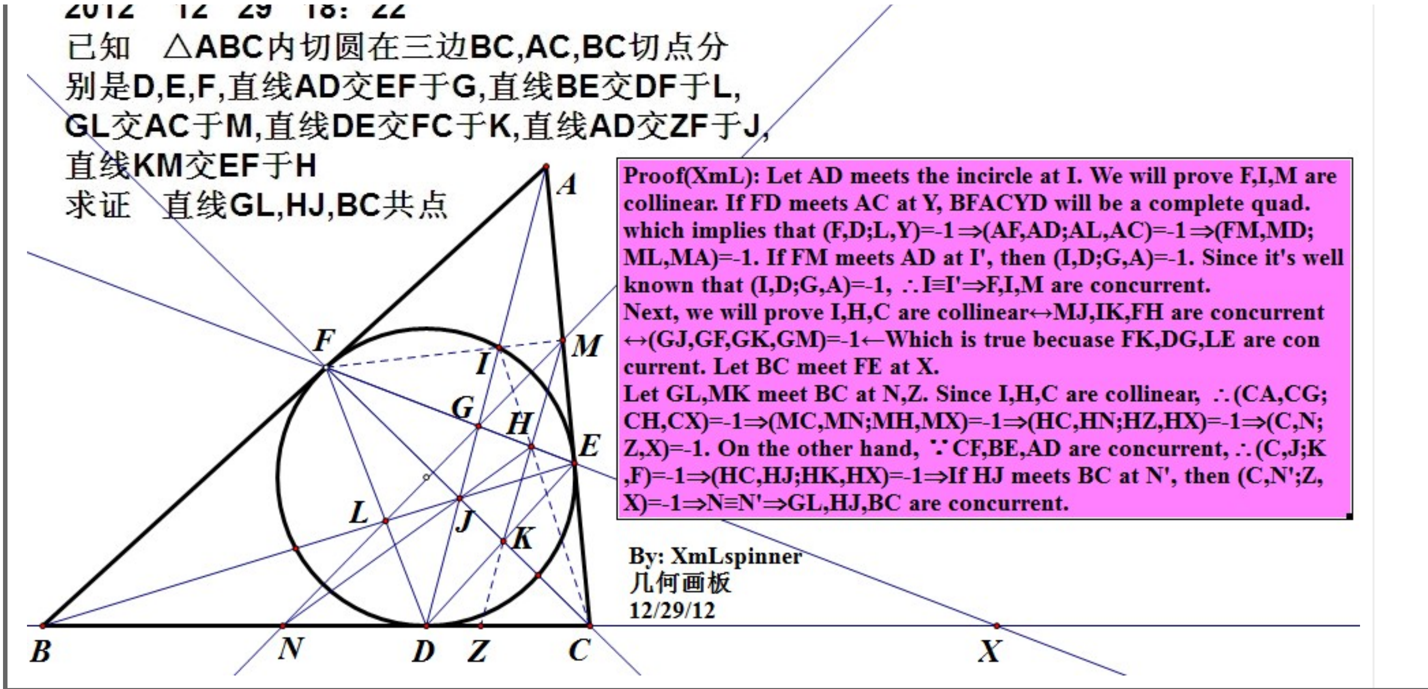
XmL

#3 Dec 30, 2012, 2:03 pm

See Attachment for my solution.
I specifically used cross ratio to show that it's a good tool.

Attachments:


文武光华数学工作室　南京　潘成华


2012 12 29 18: 22
已知 △ABC内切圆在三边BC,AC,BC切点分别是D,E,F,直线AD交EF于G,直线BE交DF于L,GL交AC于M,直线DE交FC于K,直线AD交ZF于J,直线KM交EF于H
求证 直线GL,HJ,BC共点
Proof(XmL): Let AD meets the incircle at I. We will prove F,I,M are collinear. If FD meets AC at Y, BFACYD will be a complete quad. which implies that (F,D;L,Y)=-1⇒(AF,AD;AL,AC)=-1⇒(FM,MD;ML,MA)=-1. If FM meets AD at I', then (I,D;G,A)=-1. Since it's well known that (I,D;G,A)=-1, ∴I≡I'⇒F,I,M are concurrent.
Next, we will prove I,H,C are collinear↔MJ,IK,FH are concurrent↔(GJ,GF,GK,GM)=-1←Which is true becuase FK,DG,LE are con current. Let BC meet FE at X.
Let GL,MK meet BC at N,Z. Since I,H,C are collinear, ∴(CA,CG;CH,CX)=-1⇒(MC,MN;MH,MX)=-1⇒(HC,HN;HZ,HX)=-1⇒(C,N;Z,X)=-1. On the other hand, ∵CF,BE,AD are concurrent, ∴(C,J;K,F)=-1⇒(HC,HJ;HK,HX)=-1⇒If HJ meets BC at N', then (C,N';Z,X)=-1⇒N≡N'⇒GL,HJ,BC are concurrent.
A
F
I
M
G
H
E
L
J
K
B
N
D
Z
C
X
By: XmLspinner
几何画板
12/29/12

Quick Reply

High School Olympiads

A problem with inversely similar triangles

Mr_Cow

#1 Dec 29, 2012, 10:24 pm

$MNPQ$ is a cyclic quadrilateral and $MP \cap NQ = O$. The circles (OMN), (OPQ) intersect at $K \neq O$ and K lies inside $MNPQ$. L is a point such that ΔNPL and ΔQMK are inversely similar. Prove that $NK + PL = NL + PK$.

Thanks a lot!!!

Luis González

#2 Dec 30, 2012, 5:37 am • 1

I believe it should be NPL similar to MQK in that order. This was posted before.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=105470
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=312083

High School Olympiads

Mixtilinear Incircle+Excircle Perpendicular

Reply

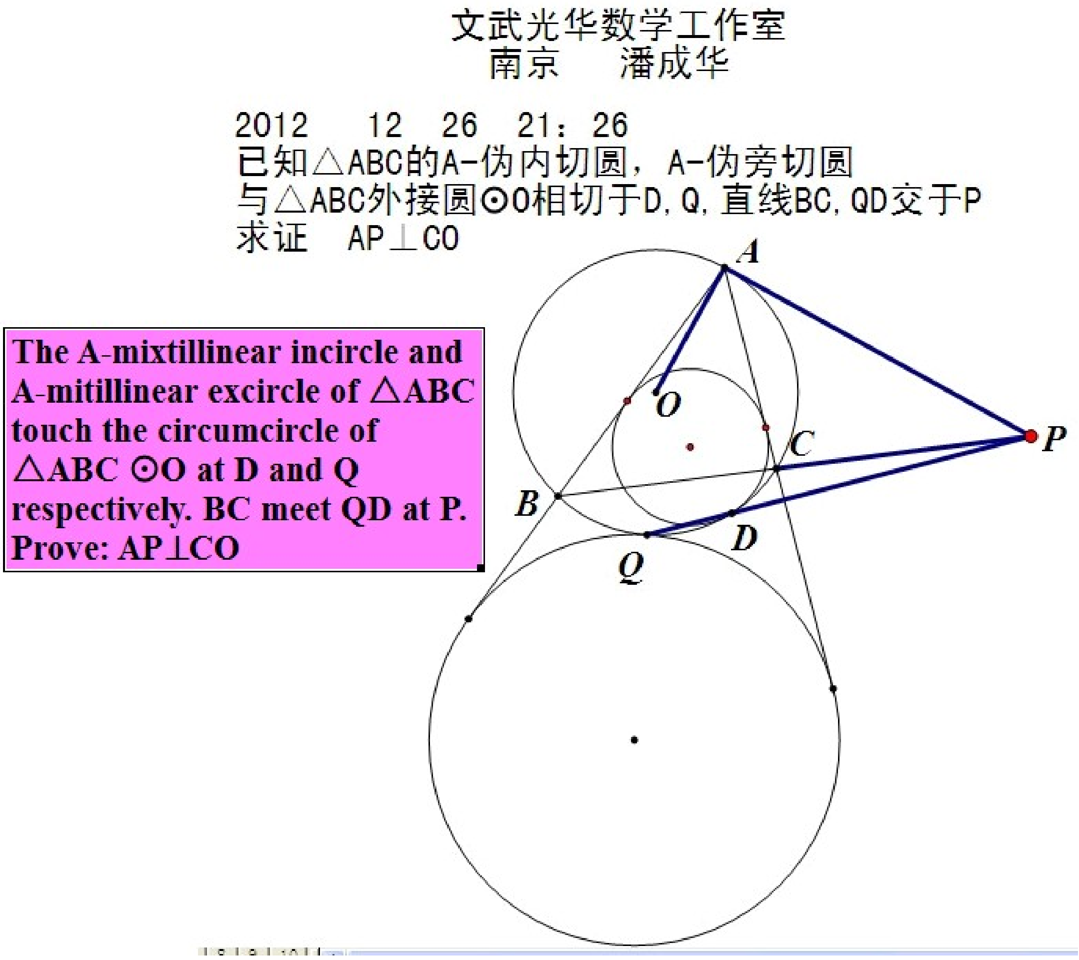
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 27, 2012, 7:51 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



hofamo

#2 Dec 27, 2012, 5:05 pm

sorry I think AO is perpendicular to AP.

XmL

#3 Dec 28, 2012, 8:40 am

> *hofamo wrote:*
> sorry I think AO is perpendicular to AP.

yes you are right! typing mistake sorry

yes you are right! typing mistake sorry~

nsato

#4 Dec 28, 2012, 12:18 pm • 1

Invert the diagram through a circle centered at A, so that the A-mixtilinear incircle goes to itself. Under this inversion, B goes to point B', etc. Then the image of the circumcircle of triangle ABC is $B'C'$. The A-mixtilinear incircle is tangent to $B'C'$ at D'. In other words, the A-mixtilinear incircle is the A-excircle of triangle $AB'C'$.

```
[asy] import geometry; pair excentre(pair A, pair B, pair C) { return((-abs(B - C)*A + abs(C - A)*B + abs(A - B)*C)/(-abs(B - C) + abs(C - A) + abs(A - B))); }; pair mixtilinearcirclecenter(pair A, pair B, pair C) { pair I, D, E; I = incenter(A,B,C); D = extension(A, B, I, rotate(90)*(A - I) + I); E = extension(A, C, I, rotate(90)*(A - I) + I); return(extension(D, rotate(90)*(A - D) + D, E, rotate(90)*(A - E) + E)); }; real mixtilinearcircleradius(pair A, pair B, pair C) { pair I, D, E, O; I = incenter(A,B,C); D = extension(A, B, I, rotate(90)*(A - I) + I); E = extension(A, C, I, rotate(90)*(A - I) + I); return(abs(mixtilinearcirclecenter(A,B,C) - D)); }; path mixtilinearcircle(pair A, pair B, pair C) { return(Circle(mixtilinearcirclecenter(A,B,C),mixtilinearcircleradius(A,B,C))); }; pair mixtilinearcircletangentpoint(pair A, pair B, pair C) { pair[] R; R[0] = abs(circumcenter(A,B,C) - A); R[1] = mixtilinearcircleradius(A,B,C); return((R[0]*mixtilinearcirclecenter(A,B,C) - R[1]*circumcenter(A,B,C))/(R[0] - R[1])); }; pair mixtilinearexcirclecenter(pair A, pair B, pair C) { pair E, P, Q; E = excentre(A,B,C); P = extension(A, B, E, rotate(90)*(A - E) + E); Q = extension(A, C, E, rotate(90)*(A - E) + E); return(extension(P, rotate(90)*(A - P) + P, Q, rotate(90)*(A - Q) + Q)); }; real mixtilinearexcircleradius(pair A, pair B, pair C) { pair E, P, Q, O; E = excentre(A,B,C); P = extension(A, B, E, rotate(90)*(A - E) + E); Q = extension(A, C, E, rotate(90)*(A - E) + E); return(abs(mixtilinearexcirclecenter(A,B,C) - P)); }; path mixtilinearexcircle(pair A, pair B, pair C) { return(Circle(mixtilinearexcirclecenter(A,B,C),mixtilinearexcircleradius(A,B,C))); }; pair mixtilinearexcircletangentpoint(pair A, pair B, pair C) { pair[] R; R[0] = abs(circumcenter(A,B,C) - A); R[1] = mixtilinearexcircleradius(A,B,C); return((R[0]*mixtilinearexcirclecenter(A,B,C) + R[1]*circumcenter(A,B,C))/(R[0] + R[1])); }; unitsize(0.3 cm); pair[] A, B, C, D, O, P, Q; real[] r; A[0] = (2,12); B[0] = (0,0); C[0] = (14,0); D[0] = mixtilinearcircletangentpoint(A[0],B[0],C[0]); Q[0] = mixtilinearexcircletangentpoint(A[0],B[0],C[0]); O[0] = circumcenter(A[0],B[0],C[0]); r[0] = abs(A[0] - O[0]); O[1] = mixtilinearcirclecenter(A[0],B[0],C[0]); r[1] = mixtilinearcircleradius(A[0],B[0],C[0]); O[2] = mixtilinearexcirclecenter(A[0],B[0],C[0]); r[2] = mixtilinearexcircleradius(A[0],B[0],C[0]); P[0] = extension(A[0], A[0] + rotate(90)*(A[0] - O[0]), B[0], C[0]); B[1] = inversion(abs(A[0] - O[1])^2 - r[1]^2, A[0])*(B[0]); C[1] = inversion(abs(A[0] - O[1])^2 - r[1]^2, A[0])*(C[0]); D[1] = inversion(abs(A[0] - O[1])^2 - r[1]^2, A[0])*(D[0]); P[1] = inversion(abs(A[0] - O[1])^2 - r[1]^2, A[0])*(P[0]); Q[1] = inversion(abs(A[0] - O[1])^2 - r[1]^2, A[0])*(Q[0]); draw(incircle(A[0],B[1],C[1])); draw(B[0]--A[0]--C[0]--P[0]--A[0]); draw(circumcircle(A[0],B[0],C[0])); draw(mixtilinearcircle(A[0],B[0],C[0])); draw(B[1]--C[1]); draw(circumcircle(A[0],D[1],Q[1])); draw(O[0]--A[0]); draw(P[0]--Q[0]); draw(circumcircle(A[0],B[1],C[1])); label([aopsnowrap]"$$A$$"[/aopsnowrap], A[0], N); label([aopsnowrap]"$$B$$"[/aopsnowrap], B[0], SW); dot([aopsnowrap]"$$B'$$"[/aopsnowrap], B[1], SW); label([aopsnowrap]"$$C$$"[/aopsnowrap], C[0], SE); dot([aopsnowrap]"$$C'$$"[/aopsnowrap], C[1], E); dot([aopsnowrap]"$$D$$"[/aopsnowrap], D[0], SW); dot([aopsnowrap]"$$D'$$"[/aopsnowrap], D[1], SE); dot([aopsnowrap]"$$O$$"[/aopsnowrap], O[0], S); label([aopsnowrap]"$$P$$"[/aopsnowrap], P[0], W); dot([aopsnowrap]"$$P'$$"[/aopsnowrap], P[1], NW); dot([aopsnowrap]"$$Q$$"[/aopsnowrap], Q[0], S); dot([aopsnowrap]"$$Q'$$"[/aopsnowrap], Q[1], SE); [/asy]
```

The A-mixtilinear excircle is tangent to AB and AC, so the image of the A-mixtilinear circle is also tangent to AB and AC. This image is also tangent to $B'C'$, so the image of the A-mixtilinear excircle is the incircle of triangle $AB'C'$. The incircle of triangle $AB'C'$ is tangent to $B'C'$ at Q'. This tells us that $B'D' = C'Q'$.

Since P is the intersection of BC and DQ, P' lies on both the circumcircle of triangles $AB'C'$ and $AD'Q'$. Since $B'D' = C'Q'$, the circumcenters of triangles $AB'C'$ and $AD'Q'$ both lie on the perpendicular bisector of $B'C'$, so AP', the line joining the intersections of the circumcircles of triangles $AB'C'$ and $AD'Q'$, is parallel to $B'C'$.

Since B' and C' are the inverses of B and C, respectively, $B'C'$ is anti-parallel to BC. By a straight-forward angle chase, $B'C'$ is perpendicular to OA. Therefore, AP' is perpendicular to OA, which means AP is perpendicular to OA. In other words, AP is tangent to the cirucmcircle of triangle ABC at A.

Luis González

#5 Dec 29, 2012, 3:26 am • 2

It suffices to show that $PB : PC = c^2 : b^2$. Let the incircle and A-excircle of $\triangle ABC$ touch BC at X, Y, respectively. It's well known that AX, AY are the isogonals of AQ, QD WRT $\angle BAC$. Hence

$$\frac{PB}{PC} = \frac{QB}{QC}\cdot\frac{DB}{DC} = \frac{\sin\widehat{CAX}}{\sin\widehat{BAX}}\cdot\frac{\sin\widehat{CAY}}{\sin\widehat{BAY}} = \left[\frac{CX}{BX}\cdot\frac{c}{b}\right]\cdot\left[\frac{CY}{BY}\cdot\frac{c}{b}\right] = \frac{c^2}{b^2}.$$

proglote

#6 Dec 30, 2012, 9:24 am • 1

Let τ denote the inversion centered at A with radius $r^2 = AB \cdot AC$, and ϕ the reflection w.r.t. the A-angle bisector. Denote the composition $\tau \circ \phi$ by the A-*reversion*. Note that it maps B to C, C to B, BC to (O), (O) to BC, the incircle to the A-mixtilinear excircle, and the A-excircle to the A-mixtilinear incircle.

Let the incircle, A-excircle touch BC at Q_0, D_0, respectively, and the parallel to BC through A hit (O) at P_0. Clearly $AQ_0D_0P_0$ is symmetric w.r.t. the perpendicular bisector of BC, hence cyclic $\Longrightarrow$ the A-reversion sends Q_0, D_0, P_0 to collinear points, i.e. Q, D, P.

Quick Reply

High School Olympiads

pc=ck

Reply

JRD

#1 Dec 17, 2012, 12:16 am • 1

On circumcircle of triangle ABC , T and K are midpoints of arcs BC and BAC respectively . And E is foot of altitude from C on AB . Point P is on extension of AK such that PE is perpendicular to ET . Prove that PC=CK

Luis González

#2 Dec 17, 2012, 7:50 am • 1

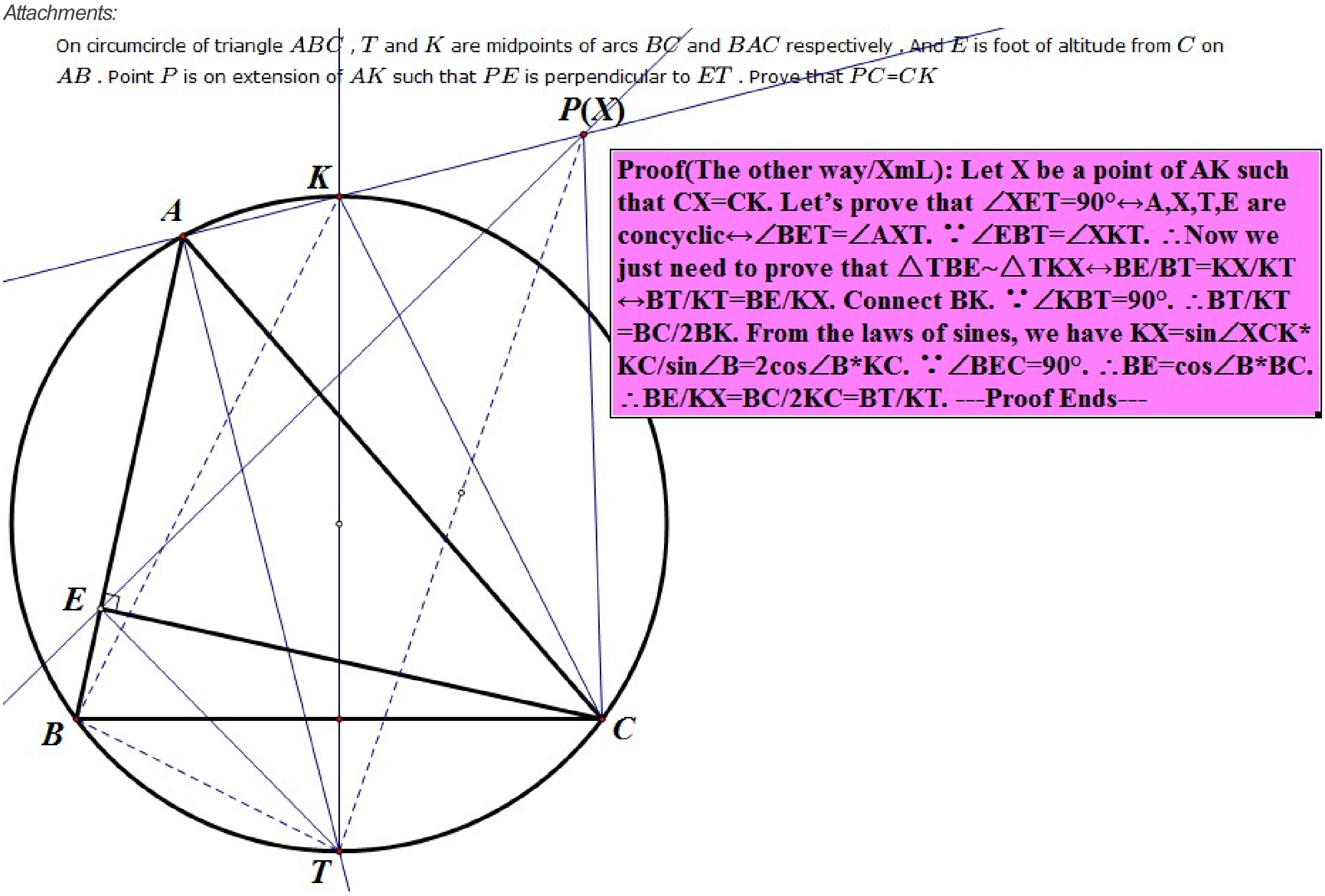
D,F are the feet of the altitudes on BC,CA. $H \equiv AD \cap BF \cap CE$ is the orthocenter. Inversion with center A and power $\overline{AH} \cdot \overline{AD}$ takes the circumcircle (O) of $\triangle ABC$ into $EF \Longrightarrow T' \equiv AT \cap EF$ is the inverse of T and BT' is the inverse of $\odot(AETP) \Longrightarrow P' \equiv BT' \cap AK$ is the inverse of P. Since the pencils $F(D,E,A,B)$ and $A(E,F,T',P')$ are harmonic, we deduce that D,F,P' are collinear $\Longrightarrow H,C,P,A$ are concyclic $\Longrightarrow$ $\angle APC = \angle CHD = \angle ABC = \angle PKC \Longrightarrow \triangle CPK$ is isosceles with $CP = CK$.

XmL

#3 Dec 17, 2012, 11:54 am

See attachment for my proof..
haven't written a proof myself for a while.

Attachments:



nima1376

#4 Apr 28, 2014, 6:18 pm

let T' in AK such that $CK = CP'$ .let point X on AB such that $TB = TC = TX$.
$CEX \sim ACT, \widehat{ECP'} = \widehat{TXE} \Rightarrow CEP' \sim TEX \Rightarrow \widehat{CEP'} = \widehat{TEX}$
done

fmasroor

#5 Apr 29, 2014, 3:18 am

Why are D, F, P' collinear, Luis?

jayme

#6 May 1, 2014, 6:46 pm

Dear Mathlinkers,
1. Q the point of intersection of CE and the perpendicular to AT at T
2. A, E, T, Q and P are concyclic

A question: how can we continue in order to have the result?

Sincerely
Jean-Louis

jayme

#7 May 2, 2014, 5:06 pm

Dear Mathlinlkers,
I continue with

3. the circle passing through A, H (the orthocenter of AVC) and C.
4. we have to prove that this last circle passes through P.

theorem concerning angle between two circle can be fruitfull...

Sincerely
Jean-Louis

jayme

#8 Jan 17, 2015, 8:23 pm

Dear Mathlinkers,

finally revisiting this problem with a new regard, I have found an elementary proof.

1. (O) the circumcircle of ABC
2. ABCK being cyclic, <CKP = <CBA
3. T, E, A, P are concyclic on (1)
4. Q the second point of intersection of PC with (1)
M the point of intersection of BC and KT
(2) the circle with diameter CT ; it goes through M
5. According to the pivot theorem applied to the triangle BEM with (O), (1), (2),
E, Q and M are collinear
6. AEQP being cyclic, <BEQ = <APQ …. then CPK is C-isoceles and we are done.

Sincerely
Jean-Louis

TelvCohl

#9 Jan 18, 2015, 9:17 pm

My solution:

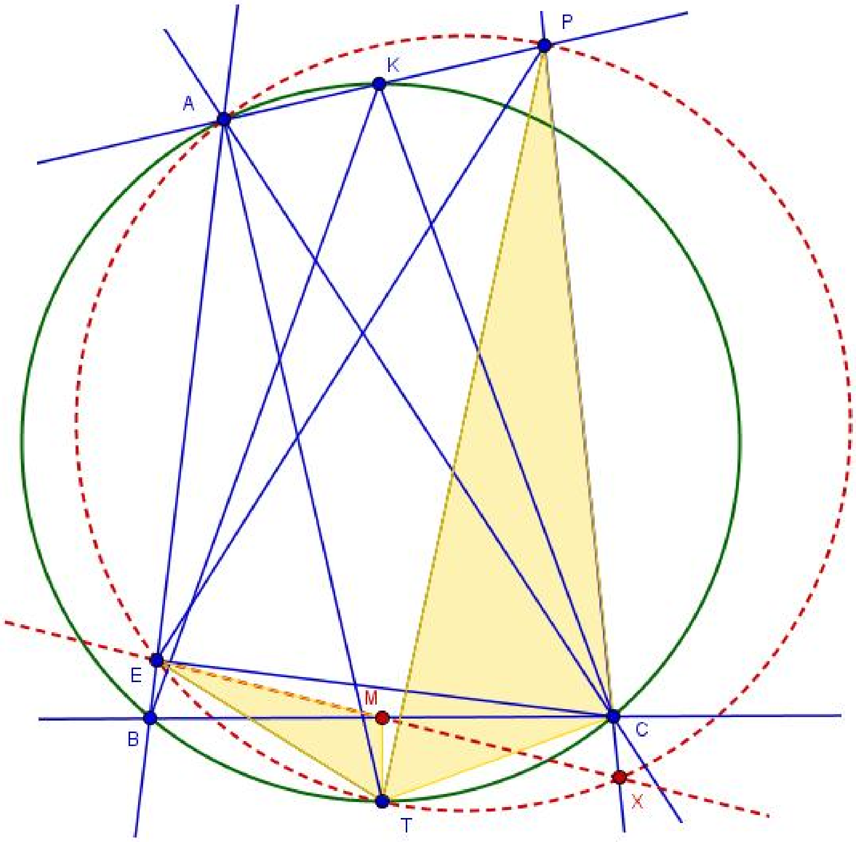
Let M be the midpoint of BC and $X = PC \cap EM$.

From $\angle TEP = \angle TAP = 90^\circ \Longrightarrow A, E, P, T$ are concyclic

From $\angle TEP = \angle TAP = 90^\circ \implies A, E, T, P$ are concyclic ,
so $\angle EPT = \angle BAT = \angle MCT \Longrightarrow Rt\triangle TEP \sim Rt\triangle TMC$,
hence we get $\triangle TEM \sim \triangle TPC$ and $X \in \odot(AEPT)$.
Since $\angle KPC = \angle BEM = \angle CBA = \angle CKP$,
so we get $\triangle CKP$ is an isosceles triangle and $CP = CK$.

Q.E.D

Attachments:



leeky

#10 Mar 24, 2016, 9:14 pm

Invert wrt A with arbitrary radius, denote X' as the image of X in the inversion. In the inverted diagram, $B'T'C'K'$ is collinear in that order and is harmonic, P' lies on AK', $E' \equiv AB' \cap P'T'$(since T, E, A, P lie on a circle with diameter TP). Let $D' \equiv P'T' \cap AC'$, then $(E', D'; T', P') = (B', C'; T', K') = -1$, combining with $\angle AC'E' = 90^\circ \implies \angle AC'P' = \angle AC'T'$. Back to the original diagram, $\angle APC = \angle ATC = \angle B = \angle PKC \implies PC = CK$.

This post has been edited 1 time. Last edited by leeky, Mar 24, 2016, 9:14 pm

Quick Reply

High School Olympiads

Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 16, 2012, 12:23 pm

See Attachment.
These two problems are proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:

文武光华数学工作室

南京　潘成华

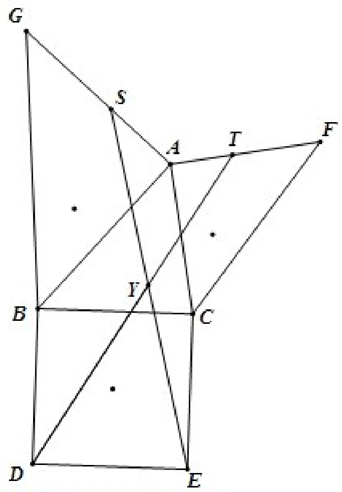
已知　正方形 BCED，等腰直角△ABG，

等腰直角△ACF，∠GAB=∠CAF=90°，

∠ABG,∠ACF 角平分线交 AG,AF 于 S,T

SE 交 DT 于 Y

求证　SE=DT,∠SYT=45°



BCED is a square, △ABG and △ACF are two isosceles right triangles. ∠GAB=∠CAF=90°. The angle bisectors of △ABG and ∠ACF meet AG and AF at S and T. SE mmet DT at Y. Prove SE=DT, ∠SYT=45°

Luis González

#2 Dec 17, 2012, 2:04 am

Let M,N,L be the midpoints of $\overline{BS},\overline{CT},\overline{BE},$ i.e. circumcenters of $\triangle BAS,\triangle CAT,BCED.$ LM and LN are the B-midline and C-midline of $\triangle BES$ and $\triangle CDT.$ So it suffices to show that $\triangle LMN$ is L-isosceles with $\angle MLN=45^\circ.$

$\angle BSA+\angle CTA+\angle BLC=2\cdot 67.5^\circ+90^\circ=180^\circ \Longrightarrow$ circumcircles $(M),(N),(L)$ of $\triangle BAS,\triangle CAT,$ $BCED$ concur at a point $P.$ Hence $\angle BPC=135^\circ$ and $\angle APB=\angle APC=112.5^\circ,$ but since PA,PB,PC are pairwise radical axes of $(M),(N),(L)$ respectively perpendicular to $NM,ML,LN,$ we get $\angle LNM=\angle LMN=67.5^\circ$ $\Longrightarrow \triangle LMN$ is isosceles with $\angle MLN=45^\circ$ and $LM=LN,$ as desired.

jayme

#3 Sep 15, 2013, 7:16 pm

Dear Mathlinkers,
for history...
the result "circumcircles of... concur at a point" has fascinated Samuel L. Greitzer well known for his book "Geometry revisited".
Sometimes this point is named "Greitzer's point"
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Mixtillinear Incircle Collinearity

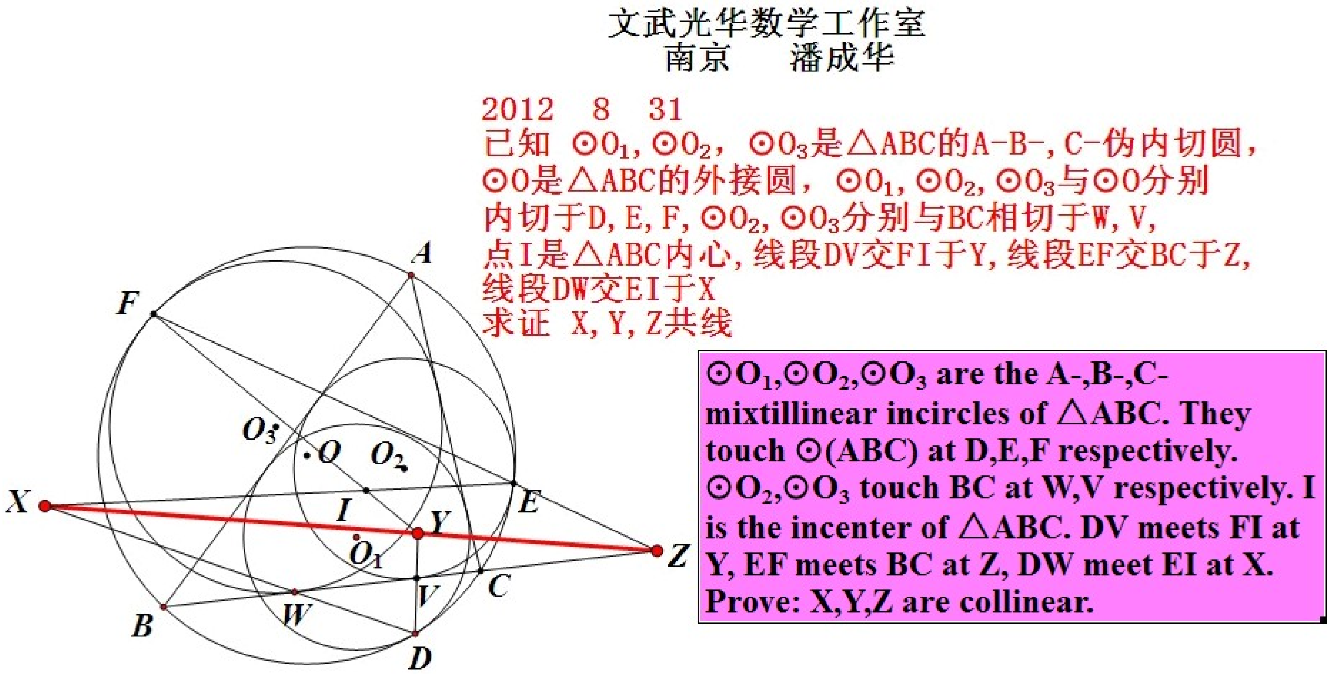
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 15, 2012, 5:57 am

See Attachment.
These two problems are proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Dec 15, 2012, 11:42 am • 1

$M \equiv EV \cap FW$ is the midpoint of the arc BC of the circumcircle (O) of $\triangle ABC$ and $FEVW$ and $MDVW$ are both cyclic (see the first two paragraphs of the solution of the problem Mixtilinear Incircles Parallels). $Z \equiv EF \cap BC \cap DM$ is then radical center of $(O), \odot(FEVW)$ and $\odot(MDVW)$.

Let $P \equiv FV \cap EW$ and K the center of $\odot(FEVW)$. MP is polar of Z WRT $(K) \Longrightarrow MP \perp ZK$ at T and $ZT \cdot ZK = ZD \cdot ZM \Longrightarrow KTDM$ is cyclic $\Longrightarrow KD \perp ZM$. Since ZM is the polar of P WRT (K), then $P \in KD$, i.e. $\angle MDP$ is right $\Longrightarrow DP$ passes through the midpoint of the arc BAC of $(O) \Longrightarrow I \in DP$. Hence, $\triangle IEF$ and $\triangle DWV$ are perspective through P. By Desargues theorem, the intersections $X \equiv IE \cap DW, Y \equiv IF \cap DV$ and $Z \equiv EF \cap WV$ are collinear.

High School Olympiads

Orthic Triangle Excenters Properties

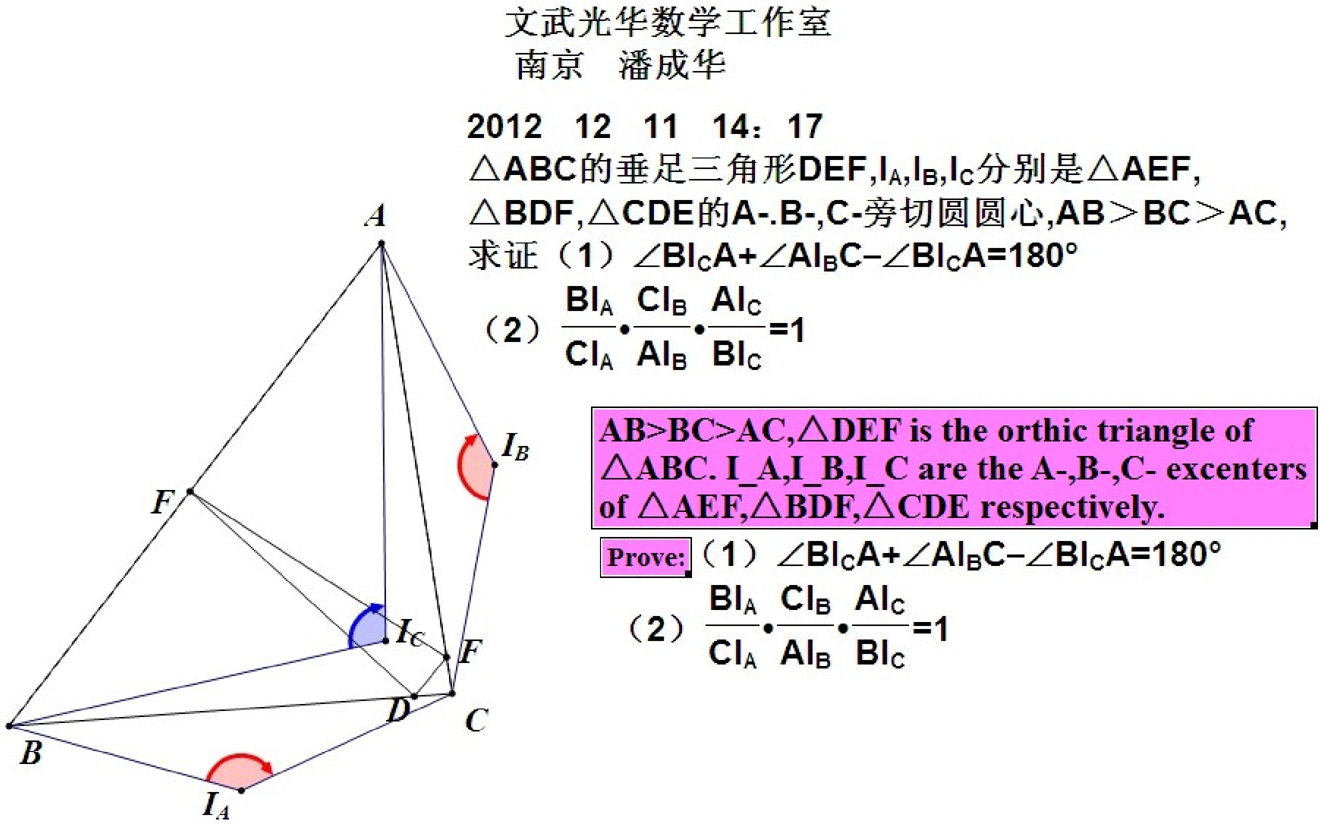
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 13, 2012, 6:34 am

See Attachment.
These two problems are proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Dec 14, 2012, 5:59 am

Let $(J_A),(J_B),(J_C)$ be the excircles of $\triangle ABC$ tangent to BC,CA,AB at X_A,X_B,X_C. $V\equiv J_AX_A\cap J_BX_B\cap J_CX_C$ is the circumcenter of $\triangle J_AJ_BJ_C$. (J_A) touches AC,AB at M,N. Since $\triangle AEF$ and $\triangle ABC$ are similar with corresponding excenters I_A,J_A, we have

$$\frac{AI_A}{AJ_A}=\frac{EF}{BC}=\cos A\Longrightarrow I_A \text{ is orthocenter of } \triangle AMN.$$

Hence, I_A is the reflection of J_A about MN. If C_0,B_0 are the midpoints of $\overline{J_AJ_B},\overline{J_AJ_C}$, then it follows that

$$\frac{J_AI_A}{J_AX_A}=\frac{J_AI_A}{J_AM}=2\sin\frac{A}{2}=2\cos\widehat{J_BJ_AJ_C}=2\cdot\frac{BC}{J_BJ_C}=\frac{BC}{B_0C_0}.$$

Since $\triangle J_ABC\sim\triangle J_AC_0B_0$ are similar with corresponding altitudes J_AX_A and J_AI_A, then we deduce that $\triangle I_ABC\sim\triangle X_AC_0B_0\Longrightarrow$

$\angle BI_AC = \angle C_0X_AB_0 = 180^\circ - (\angle B_0VC_0 - \angle BVC) = \angle I_BI_AI_C + \angle BVC$

$$\angle BI_AC = \angle C_0X_AB_0 = 180^\circ - (\angle B_0VC_0 - \angle BVC) = \angle J_BJ_AJ_C + \angle BVC.$$

Similarly $\angle AI_BC = \angle J_AJ_BJ_C + \angle CVA$ and $\angle BI_CA = \angle AVB - \angle J_BJ_CJ_A$, since the condition $AB > BC > AC$ puts V inside the acute $\triangle ABJ_C$. Thus

$$\angle BI_AC + \angle AI_BC - \angle BI_CA = 180^\circ + \angle BVC + \angle CVA - \angle AVB = 180^\circ.$$

In addition, we have the proportions

$$\frac{CI_A}{BI_A} = \frac{X_AB_0}{X_AC_0} = \frac{BV \cdot \sin \widehat{J_ABC}}{CV \cdot \sin \widehat{J_ACB}} = \frac{VB}{VC} \cdot \frac{J_AJ_C}{J_AJ_B}.$$

Multiplying the cyclic expressions together yields

$$\frac{CI_A}{BI_A} \cdot \frac{AI_B}{CI_B} \cdot \frac{BI_C}{AI_C} = \frac{VB}{VC} \cdot \frac{J_CJ_A}{J_AJ_B} \cdot \frac{VC}{VA} \cdot \frac{J_AJ_B}{J_BJ_C} \cdot \frac{VA}{VB} \cdot \frac{J_BJ_C}{J_CJ_A} = 1.$$

Quick Reply

High School Olympiads

altitudes and circumcircle

Locked

AndrewTom

#1 Dec 13, 2012, 2:25 am

The altitudes through the vertices A, B and C of an acute-angled triangle ABC meet the opposite sides at D, E and F respectively.

The line through D parallel to EF meets the lines AC and AB (extended if necessary) at Q and R, respectively.

The line through E and F meets the line through B and C at P.

Prove that the circumcircle of $\triangle PQR$ passes through the midpoint of BC.

Luis González

#2 Dec 13, 2012, 3:55 am • 1

AndrewTom, you are just reposting old shortlisted problems. Please go through the Olympiad resources or try search before posting.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=219812
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=225253
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=358724

High School Olympiads

Two Angle Bisectors Concurrence Problems

Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

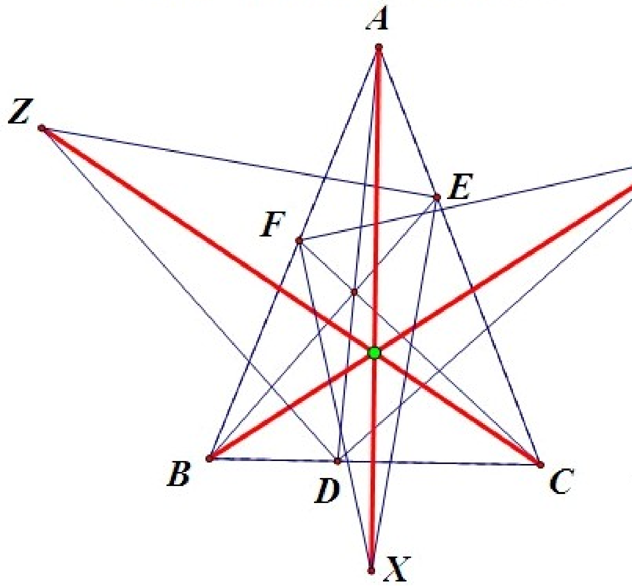
#1 Dec 12, 2012, 8:11 am

See Attachment.
These two problems are proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:

文武光华数学工作室
南京　潘成华

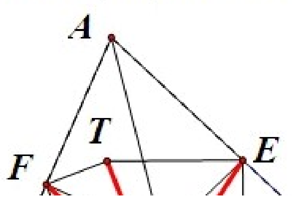
2012　12　12　9：45
已知　直线BE,CF,AD共点，∠AEB,∠ADB角平分线交于Z,∠BEC,∠BFC角平分线交于X,∠ADC,∠AFC角平分线交于Y
求证　直线AX,BY,CZ共点

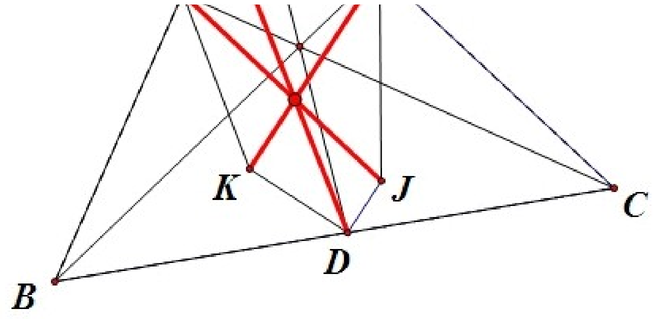


BE,CF,AD are concurrent. The angle bisectors of ∠AEB and ∠ADB meet at Z. The angle bisectors of ∠BEC and ∠BFC meet at X. The angle bisectors of ∠ADC and ∠AFC meet at Y.
Prove: AX,BY,CZ are concurrent.

文武光华数学工作室
南京　潘成华
2012　12　12　11：15
已知　直线BE,CF,AD共点，∠AEB,∠AFC角平分线交于T,∠BFC,∠BDA角平分线交于K,∠BEC,∠ADC角平分线交于J
求证　直线KE,FJ,DT共点

BE,CF,AD are concurrent. The angle bisectors of ∠AEB and ∠AFC meet at T. The angle bisectors of ∠BFC and ∠BDA meet at K. The angle bisectors of ∠BEC and ∠ADC meet at J.
Prove: KE,FJ,DT are concurrent.





Luis González

#2 Dec 13, 2012, 3:25 am

1) Assuming that $P \equiv AD \cap BE \cap CF$ is inside $\triangle ABC,$ then project $\triangle ABC$ with its interior point P into an acute $\triangle ABC$ with orthocenter $P,$ using parallel projection. Pencils $D(K,J,P,B), E(J,T,P,C)$ and $F(T,K,P,A)$ remain harmonic. Since PD,PE,PF are perpendicular to $BC,CA,AB,$ then PD,PE,PF bisect $\angle KDJ, \angle JET, \angle TFK,$ respectively. Since PD,PE,PF are internal bisectors of $\triangle DEF,$ then we deduce that DK,DJ are isogonals WRT $\angle FDE, EJ, ET$ are isogonals WRT $\angle DEF$ and FT,FK are isogonals WRT $\angle EFD.$ By Jacobi's theorem, we conclude that DT,EK,FJ concur at Jacobi's perspector $J^*(\angle A-45^\circ)$ of $\triangle DEF.$

2) As before, assuming that $P \equiv AD \cap BE \cap CF$ is inside $\triangle ABC,$ then project $\triangle ABC$ with its interior point P into an acute $\triangle ABC$ with orthocenter $P,$ using parallel projection. Analogous reasoning shows that AX,BY,CZ concur at the isogonal conjugate of the Jacobi's perspector $J(135^\circ-\angle A)$ of $\triangle ABC.$

Quick Reply

High School Olympiads

acute-angled triangle

Locked

AndrewTom

#1 Dec 13, 2012, 1:08 am

Let ABC be an acute-angled triangle with $BC > CA$. Let O be the circumcentre, H the orthocentre and F the foot of the altitude CH of $\triangle ABC$. Let the perpendicular to OF at F meet the side CA at P.

Prove that $\angle FHP = \angle BAC$.

What happens when $BC \leq CA$, but the triangle is still acute-angled?

Luis González

#2 Dec 13, 2012, 1:28 am • 1

Please use the search function before posting. This is IMO Shortlist 1996 problem G3.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=1133
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=388761

High School Olympiads

Mixtillinear Incircle Parallels

Reply

Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 12, 2012, 6:59 am

See Attachment.
This is a problem proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:

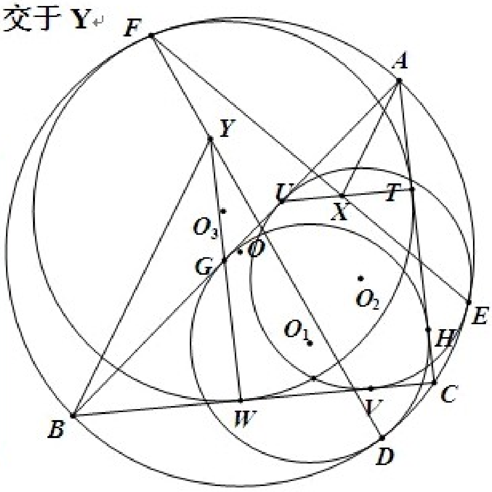
文武光华数学工作室

南京　潘成华

已知 △ABC 的三伪内切圆$\odot O_1,\odot O_2,\odot O_3$,切△ABC 外接圆 O 分别内切于点 D,E,F,三伪内切圆在 AB,BC,AC 边上切点分别是 U,G,W,V,H,T,直线 UT,EF 交于 X,直线 DF,GW 交于 Y

求证：AX//BY

$\odot O\_1$,$\odot O\_2$,$\odot O\_3$ are the three mixtillinear incircles of $\triangle ABC$. They cut the circumcircle of $\triangle ABC$ at D,E,F. Their points of tangency on AB,BC,AC are U,G,W,V,H,T respectively. UT meet EF at X. DF meet GW at Y. Prove: AX//BY.



Luis González

#2 Dec 12, 2012, 11:50 am

Further, we prove that $AX \parallel BY$ is the direction of the isogonal conjugate of the Feuerbach point X_{100} of the antimedial triangle of $\triangle ABC$.

Let EF cut AB, AC at Z, Y, respectively. FG and FZ are isogonals WRT $\angle AFB$ (see On mixtilinear incircles 2), thus by Steiner theorem, we get

$$\frac{BZ}{ZA}=\frac{AG}{GB}\cdot\frac{FB^2}{FA^2}=\frac{b}{s-b}\cdot\left(\frac{a}{b}\cdot\frac{s-b}{s-a}\right)^2=\left(\frac{a}{s-a}\right)^2\cdot\frac{s-b}{b}.$$

Similarly, we have $\frac{CY}{YA}=\left(\frac{a}{s-a}\right)^2\cdot\frac{s-c}{c}$. If AX cuts BC at A', we get then

$$\frac{BA'}{A'C}=\frac{\frac{BU}{UA}-\frac{BZ}{ZA}}{\frac{CY}{YA}-\frac{CT}{TA}}=\frac{\frac{a}{s-a}-\left(\frac{a}{s-a}\right)^2\cdot\frac{s-b}{b}}{\left(\frac{a}{s-a}\right)^2\cdot\frac{s-c}{c}-\frac{a}{s-a}}=\frac{c}{b}\cdot\frac{a-b}{c-a}.$$

This latter expression reveals that $AA'\equiv AX$ is the isogonal of AX_{100} WRT $\angle BAC$. By similar reasoning, BY is the isogonal of BX_{100} WRT $\angle ABC \Longrightarrow AX \parallel BY$.

Quick Reply

High School Olympiads

Orthic Triangle Excenters Concurrence

Reply

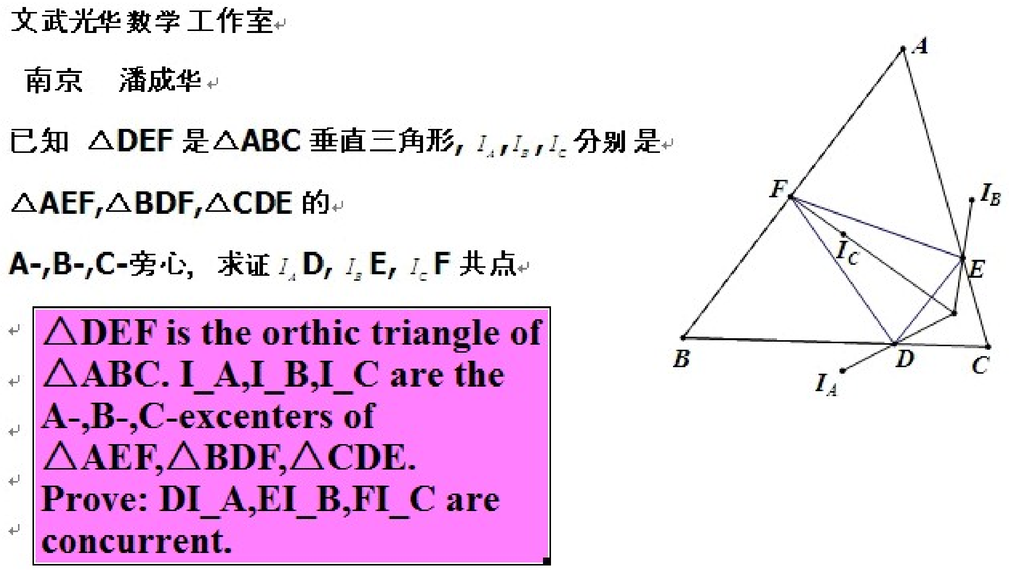
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 11, 2012, 6:55 am

See Attachment.
This is a problem proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Dec 11, 2012, 10:07 am • 1

$\angle EDI_b = \angle BDI_b - \angle BDE = 90^\circ + \frac{1}{2}\angle A - (180^\circ - \angle A) = \frac{3}{2}\angle A - 90^\circ$. Similarly, we have that $\angle FDI_c = \frac{3}{2}\angle A - 90^\circ \Longrightarrow DI_b, DI_c$ are isogonals WRT $\angle EDF$. Analogously, EI_c, EI_a are isogonals WRT $\angle FED$ and FI_a, FI_b are isogonals WRT $\angle DFE$. By Jacobi's theorem, we conclude that lines DI_a, EI_b and FI_c concur at the Jacobi's perspector $J(\frac{3}{2}\angle A - 90^\circ)$ of $\triangle DEF$.

P.S. The concurrency point J is Kimberling center X_{2665} of $\triangle ABC$.

hofamo

#3 Dec 26, 2012, 10:12 pm

Click to reveal hidden text

Quick Reply

High School Olympiads

Mixtilinear Excircles Perpendicular

Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 9, 2012, 5:41 am

See Attachment.
This is a problem proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:

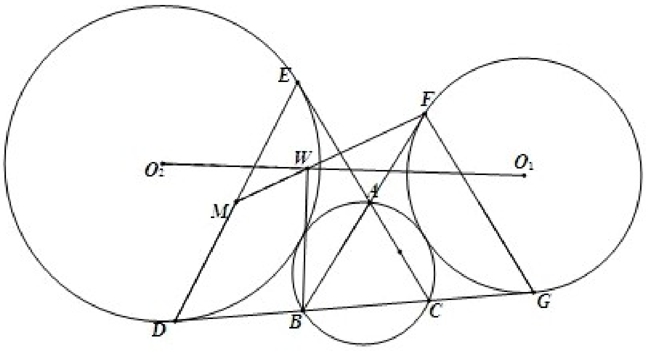
文武光华数学工作室

南京　　潘成华

已知　如图，圆O_1，圆O_2是△ABC 的 B-, C-伪旁切圆，切点分别是 D, E, F, G, 点 M 是 DE 中点，　连接 MF 交O_1O_2于 W

求证：BW⊥O_1O_2

⊙O_1 and ⊙O_2 are the B-,C- mixtillinear excircles of △ABC respectively. The points of tangency are D,E,F,G. M is the midpoint of DE. MF meet O_1O_2 at W. Prove: BW⊥O_1O_2



Luis González

#2 Dec 10, 2012, 12:08 pm • 1

Redefine W as the projection of B on O_1O_2, i.e. the 2nd intersection of the circles with diameters $\overline{BO_1}$ and $\overline{BO_2}$. Then we prove that $W \in FM$.

Let I_b, I_c be the excenters of $\triangle ABC$ againts B, C. It's well-known that $M \equiv I_c$. Obviously G, F are on circle with diameter $\overline{BO_1}$ and D is on circle with diameter $\overline{BO_2}$. Thus if $S \equiv GF \cap DO_2$, then W is Miquel point of $\triangle SDG$ WRT $BFO_2 \Longrightarrow W \in \odot(SFO_2)$. If CO_2 cuts $\odot(SFO_2)$ again at P, then $\angle FPO_2 = \angle FSD = \angle GBI_b = \angle CI_cI_b \Longrightarrow PF \parallel I_bI_c$. So if AC cuts PF at Q, then $\triangle AQF$ is isosceles with legs $AQ = AF$. Thus

$$\frac{CI_c}{I_cP} = \frac{CA}{AQ} = \frac{CA}{AF} \Longrightarrow I_cP = \frac{AF}{AC} \cdot CI_c = \frac{AF}{AC} \cdot \frac{I_cD^2}{I_cO_2} \quad (1)$$

$$\angle FAI_b = \angle CAI_b, \angle AI_bF = \angle ACI_b \Longrightarrow \triangle AFI_b \sim \triangle AI_bC \sim \triangle I_cBD \Longrightarrow$$

$$\frac{AI_b}{AC} = \frac{AF}{AI_b} \Longrightarrow \frac{AF}{AC} = \frac{AI_b{}^2}{AC^2} = \frac{I_cB^2}{I_cD^2} \quad (2)$$

Combining (1) and (2) gives $I_cB^2 = I_cO_2 \cdot I_cP$. Since I_cB is clearly tangent to $\odot(BFG)$, then the latter expression means that I_c has equal power WRT $\odot(BFG)$ and $\odot(SFO_2) \Longrightarrow I_c$ is on their radical axis FW.

TelvCohl

#3 Oct 30, 2014, 10:13 am

My solution:

Let N be the midpoint of FG and $D' \equiv BO_1 \cap DO_2, G' \equiv CO_2 \cap GO_1$.

Easy to see D, D', G, G', M, N are concyclic and $DGG'D'$ is a rectangle .

From **Mannheim theorem** we get M, N are excenter of $\triangle ABC \Longrightarrow B, C, M, N$ are concyclic .

Since $\angle MNB = \angle O_2CD = \angle O_2G'D'$,

so $\triangle MNB \sim \triangle O_2G'D' \Longrightarrow \frac{DG}{BN} = \frac{D'G'}{BN} = \frac{O_2G'}{MN}$ ... (1)

Since $\angle NBF = \angle GBO_1 = \angle G'D'O_1$,

so $\triangle NBF \sim \triangle G'D'O_1 \Longrightarrow \frac{DG}{BN} = \frac{D'G'}{BN} = \frac{G'O_1}{NF}$ ... (2)

Since $\angle O_1G'O_2 = \angle FNM$,

so combine with (1) and (2) we get $\triangle FNM \sim \triangle O_1G'O_2$,

hence $\angle O_1WF = \angle O_1GF \Longrightarrow O_1, G, B, W, F$ are concyclic

so $\angle BWO_1 = 180^\circ - \angle O_1GB = 90^\circ$. i.e. $BW \perp O_1O_2$

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Jul 25, 2015, 10:28 pm

Quick Reply

High School Olympiads

Perspective triangles with mixtilinear incircles

Source: Own

buratinogigle

#1 Nov 22, 2012, 11:14 pm

Let ABC ba triangle with mixtilinear incircles $(O_a),(O_b),(O_c)$. (O_a) cuts BC at A_1,A_2 such that A_1 is between B,A_2. (O_b) cuts CA at B_1,B_2 such that B_1 is between C,B_2. (O_c) cuts AB at C_1,C_2 such that C_1 is between A,C_2. Prove that A_2B_1,B_2C_1,C_2A_1 intersect base a triangle that is perspective with triangle ABC.

Luis González

#2 Dec 9, 2012, 9:40 am

Let (O_a) touch AC,AB at Y,Z and YZ cuts BC at A_0. B_0 and C_0 are defined similarly. By Menelaus' theorem for $\triangle ABC$ cut by A_0YZ, we get

$$\frac{A_0B}{A_0C}=\frac{YA}{AZ}\cdot\frac{BZ}{CY}=\frac{BZ}{CY}=\sqrt{\frac{BA_1\cdot BA_2}{CA_1\cdot CA_2}}.$$

Now, multiplying the cyclic expressions together, keeping in mind that A_0,B_0,C_0 are collinear on the orthopolar of the incenter I, then we deduce by Carnot's theorem that A_1,A_2,B_1,B_2,C_1,C_2 lie on a same conic. Hence by Pascal theorem, intersections $BC\cap B_2C_1, CA\cap C_2A_1$ and $AB\cap A_2B_1$ are collinear and the conclusion follows.

High School Olympiads

Hard concurrent

changpotato
#1 Dec 7, 2012, 5:16 pm

Let ABC be a triangle. P is a point inside ABC. AP, BP, CP cut BC, AC, AB at E, F, D respectively. The angle bisector of $\angle ADC$ cuts the angle bisector of $\angle AFB$ at K.
The angle bisector of $\angle BDC$ cuts the angle bisector of $\angle AEB$ at I. The angle bisector of $\angle AEC$ cuts the angle bisector of $\angle CFB$ at J.Prove that CJ, BI, AK are concurrent.

Luis González
#2 Dec 8, 2012, 9:42 am

Project $\triangle ABC$ with its interior point P into a $\triangle ABC$ with orthocenter $P,$ using parallel projection.

Pencils $D(K,I,P,A), E(I,J,P,B)$ and $F(J,K,P,C)$ remain harmonic. Since $PD \perp AB, PE \perp BC$ and $PF \perp CA,$ it follows that PD,PE,PF bisect $\angle KDI, \angle IEJ, \angle JFK.$ Since PD,PE,PF are internal angle bisectors of the orthic $\triangle DEF,$ then DK,DI are isogonals WRT $\angle FDE.$ EI,EJ are isogonals WRT $\angle DEF$ and FJ,FK are isogonals WRT $\angle EFD \Longrightarrow AK,BI,CJ$ concur at the isogonal conjugate of the Jacobi's perspector $J(45^{\circ}-\angle A)$ of $\triangle ABC.$ For a proof see the topic Similarities (post #6 and the subsequent reply).

changpotato
#3 Dec 8, 2012, 11:05 pm

> Luis González wrote:
> Project $\triangle ABC$ with its interior point P into a $\triangle ABC$ with orthocenter $P,$ using parallel projection.

What is parallel projection ? and Is it possible to make P as an orhtocenter without making some change to other point ?

High School Olympiads

Mixtillinear Incircle Parallels

Reply

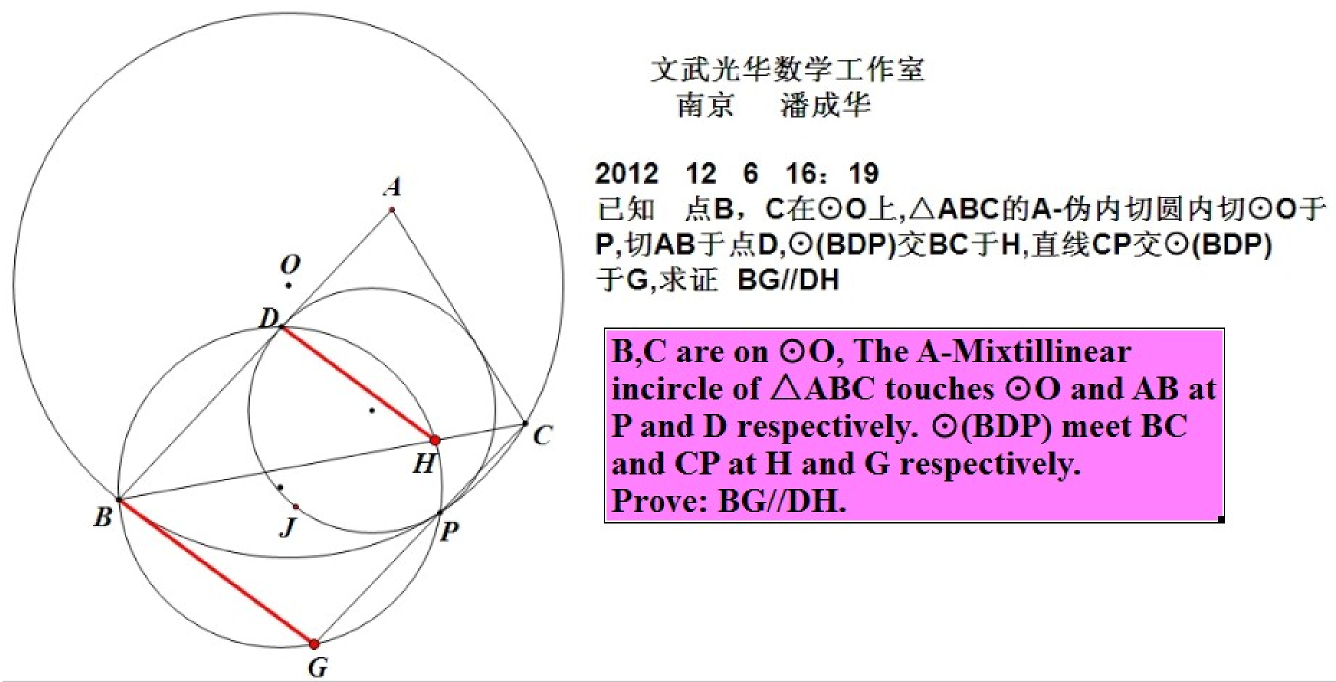
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 7, 2012, 6:21 am

See Attachment.
This is a problem proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Dec 7, 2012, 9:51 am

Circle $\odot(BPD)$ passes through the incenter I of $\triangle ABC$ and PI bisects $\angle BPC$ (see Fairly difficult (Iran 1999), Concyclic points with triangle incenter, incenter of triangle and elsewhere). BI bisects $\angle HBD \Longrightarrow I$ is the midpoint of the arc DH of $\odot(BPD) \Longrightarrow PI$ also bisects $\angle DPH \Longrightarrow PD, PH$ are isogonals WRT $\angle BPC \Longrightarrow \angle BPH = \angle CPD.$ But from cyclic $PHDB$ and $PDBG,$ we have $\angle BPH = \angle ADH, \angle CPD = \angle GBD \Longrightarrow \angle ADH = \angle GBD \Longrightarrow$ $BG \parallel DH.$

Quick Reply

High School Olympiads

Mixtilinear Excircles Concurrence

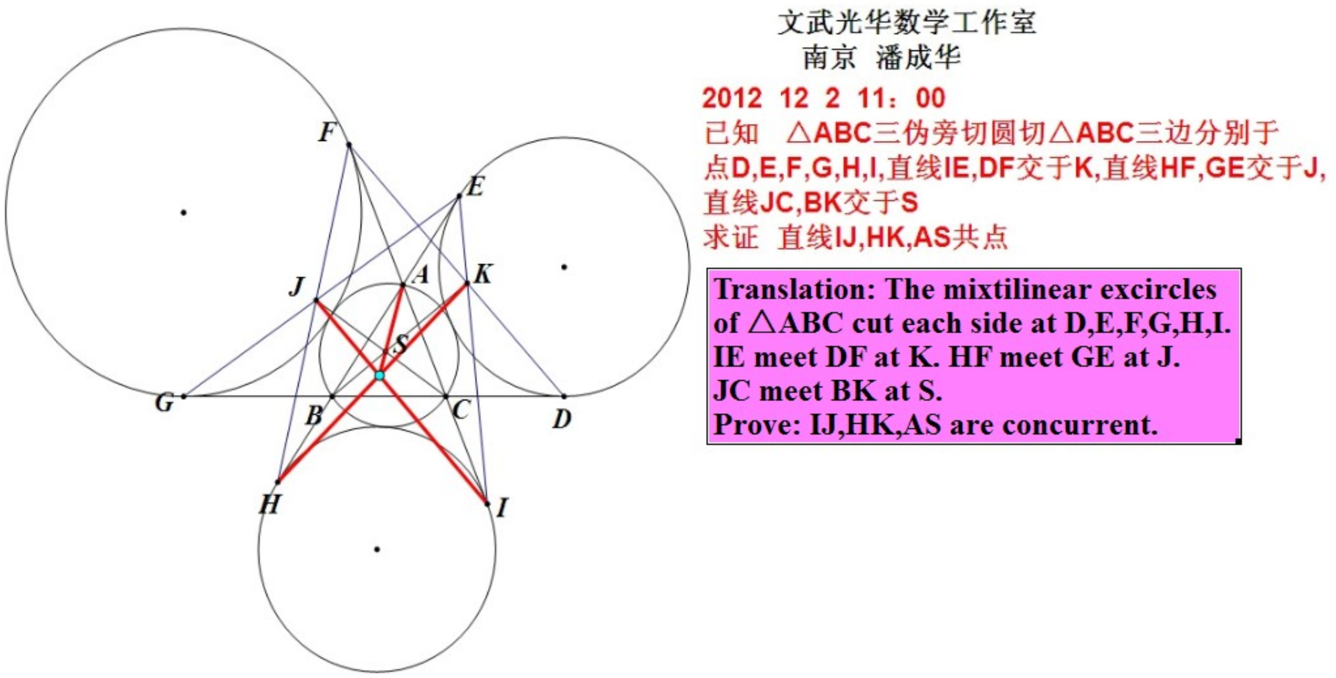
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 2, 2012, 9:33 am

See Attachment.
This is a problem proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Dec 7, 2012, 7:15 am • 1

$$\frac{\overline{IC}}{\overline{IA}}=\frac{s-b}{c}\ ,\ \frac{\overline{FC}}{\overline{FA}}=\frac{a}{s-b}\Longrightarrow\frac{\overline{IC}\cdot\overline{FC}}{\overline{IA}\cdot\overline{FA}}=\frac{a}{c}\Longrightarrow$$

$$\frac{\overline{IC}\cdot\overline{FC}}{\overline{IA}\cdot\overline{FA}}\cdot\frac{\overline{EA}\cdot\overline{HA}}{\overline{EB}\cdot\overline{HB}}\cdot\frac{\overline{GB}\cdot\overline{DB}}{\overline{GC}\cdot\overline{DC}}=\frac{a}{c}\cdot\frac{b}{a}\cdot\frac{c}{b}=1.$$

By Carnot's theorem D,E,F,G,H,I lie on a same conic. Thus by Pascal theorem for $HIEGDF$, the intersections $P\equiv HI\cap GD, K\equiv IE\cap DF$ and $J\equiv EG\cap FH$ are collinear, i.e. JK,BC,HI concur at $P\Longrightarrow\triangle AHI$ and $\triangle SKJ$ are perspective through $PBC\Longrightarrow IJ,HK,AS$ concur at their perspector.

High School Olympiads

A Good Property on Mixtillinear Incircle

Reply

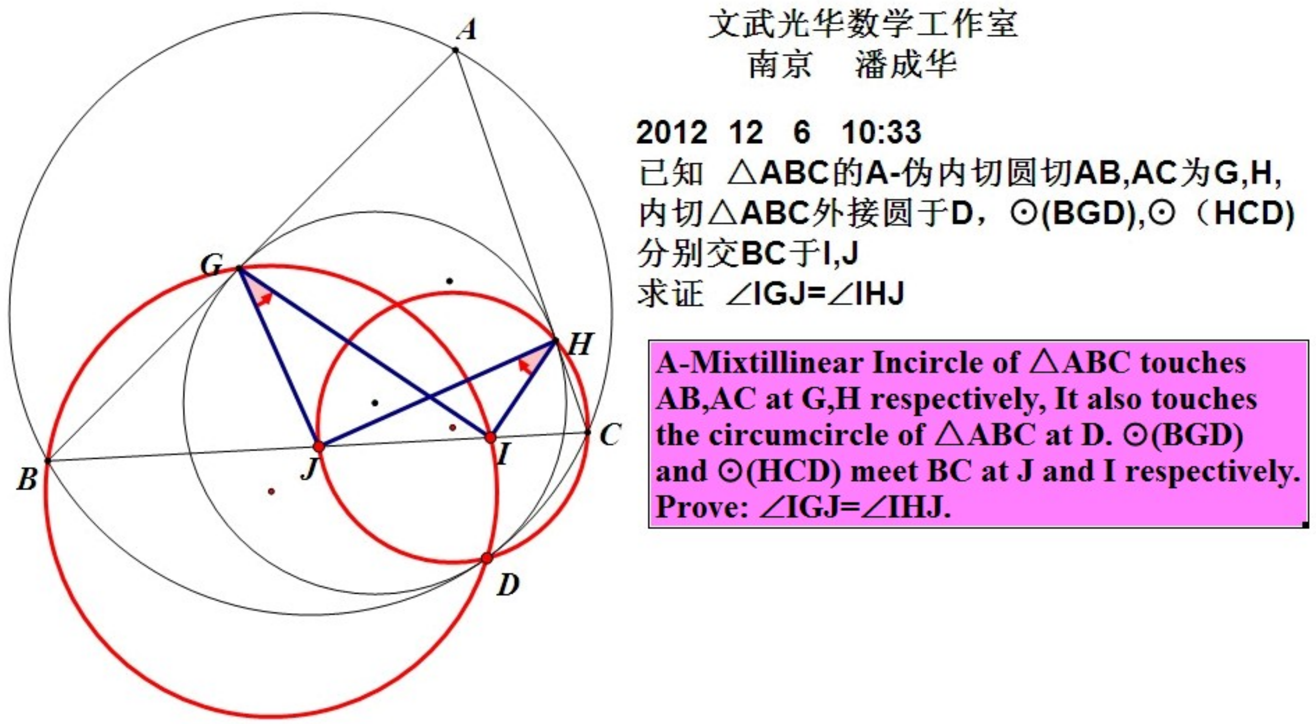
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 6, 2012, 9:09 am

See Attachment.
This is a property discovered(?) by PCHP from WenWuGuangHua Mathematics Workshop in China
The proof is easy btw.

Attachments:



Luis González

#2 Dec 7, 2012, 5:03 am

Circles $\odot(BDG)$ and $\odot(CDH)$ pass through the incenter S of $\triangle ABC$ (see Fairly difficult (Iran 1999), Concyclic points with triangle incenter, incenter of triangle and elsewhere). BS bisects $\angle IBG \Longrightarrow S$ is the midpoint of the arc IG of $\odot(BGI) \Longrightarrow SG=SI$. Similarly, $SH=SJ$, but since I is the midpoint of $\overline{GH}$, then $SG=SH=SI=SJ \Longrightarrow$ quadrilateral $GHIJ$ is cyclic with circumcenter $S \Longrightarrow \angle IGJ=\angle IHJ$.

Quick Reply

High School Olympiads

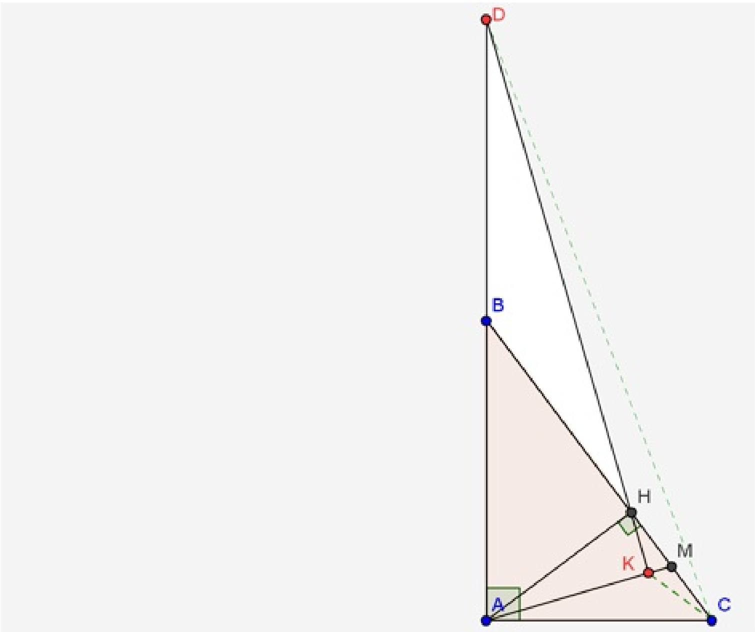
Angle Bisector!!!

khayyam-guilan
#1 Dec 6, 2012, 8:45 pm

See the below figure:

Suppose that $AB = BD$ and $MC = MH$. Show that CM is angle bisector of the triangle KCD.

Attachments:



Luis González
#2 Dec 6, 2012, 9:41 pm • 1

From $\frac{CM}{CA} = \frac{CH}{2CA} = \frac{AH}{2AB} = \frac{AH}{AD}$ and $\angle ACB = \angle BAH$, we deduce that $\triangle AMC$ and $\triangle DHA$ are similar by SAS criterion $\Longrightarrow \angle HMA = \angle AHK \Longrightarrow HK \perp AM \Longrightarrow MC^2 = MH^2 = MK \cdot MA \Longrightarrow \odot(AKC)$ is tangent to $HC \Longrightarrow \angle HCK = \angle MAC = \angle ADH \Longrightarrow BDCK$ is cyclic. Since $BA = BD = BK$, due to $\angle DKA = 90^\circ$, then CMB bisects $\angle KCD$, as desired.

sco0orpiOn
#3 Dec 7, 2012, 1:19 am • 1

let S be the midpoint of the AH.
we know triangle ABH, CAH are simillar and points S, N are Corresponding in these triangles so $\angle BSH = \angle HMA$
assume that BS intersect AH at R so $SRMH$ is cyclic and the lines BS, DK are parallel so $DK \perp AM$
we have $BA^2 = BH \cdot BC \Longrightarrow BD^2 = BH \cdot BC \Longrightarrow \angle BDH = \angle BCD$
so now we need to prove $BDCK$ is cyclic :

$MH^2 = MK \cdot MA \Longrightarrow MC^2 = MK \cdot MA \Longrightarrow \angle MKC = \angle BCA$ and we are done.

High School Olympiads

Mixtilinear Incircles Parallels

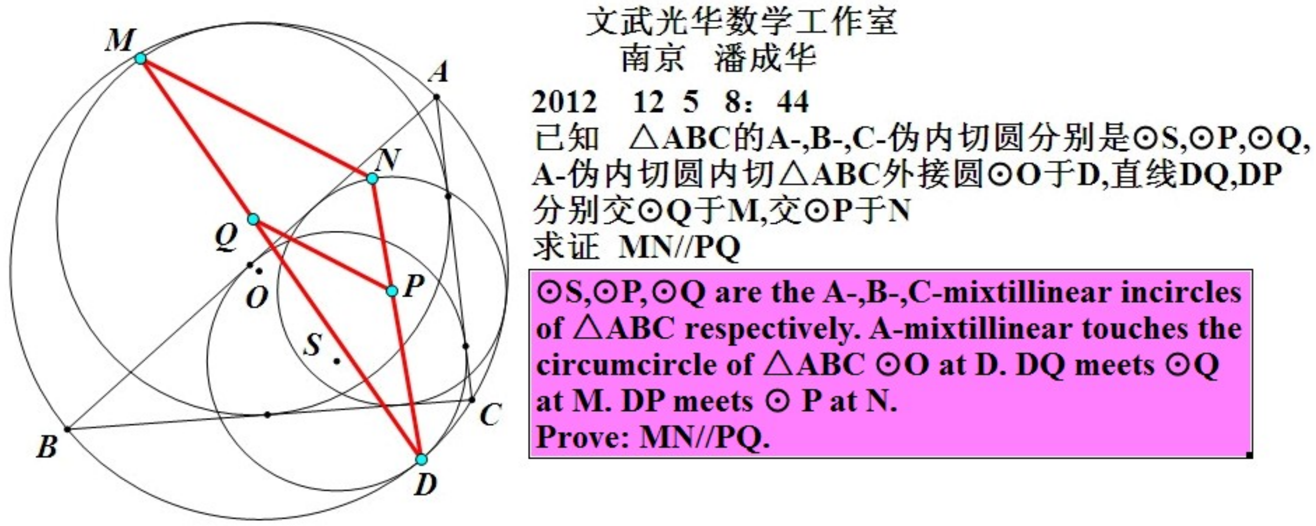
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 5, 2012, 6:59 am

See Attachment.
This is a problem proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Dec 6, 2012, 7:55 am • 1

B-mixtilinear incircle (P,ϱ_b) touches $BC,(O)$ at U,Y and C-mixtilinear incircle (Q,ϱ_c) touches $BC,(O)$ at V,Z. Since YU and ZV bisect $\angle BYC$ and $\angle BZC$, then $S \equiv UY \cap VZ$ is the midpoint of the arc BC of (O). Furthermore, the quadrilateral $UVZY$ is cyclic, due to $SB^2=SV \cdot SZ=SU \cdot SY$.

DU,DZ and DV,DY are pairs of isogonals WRT $\angle BDC$ (see On mixtilinear incircles 2). Thus, $\angle UDV=\angle YDZ=\angle YSZ \equiv \angle USV \Longrightarrow UVSD$ is cyclic $\Longrightarrow SD,YZ$ and $UV \equiv BC$ are pairwise radical axes of (O), $\odot(UVSD)$ and $\odot(UVZY)$, concurring at their radical center K. But since Y and Z are the exsimilicenters of $(O) \sim (P)$ and $(O) \sim (Q)$, then K is the exsimilicenter of $(P) \sim (Q) \Longrightarrow K \in PQ$.

Exsimilicenter K of $(P) \sim (Q)$ is also center of inversion that transforms these circles into each other. So inversion with center K and power $KU \cdot KV=KY \cdot KZ=KD \cdot KS$ takes intersections L,L' of $(P),(Q)$ into themselves and takes D into S. S,L,L' are obviously collinear on the radical axis of $(P),(Q)$, thus D is on circle $\odot(KLL')$, which is the Apollonius circle of $\overline{PQ}$ referent to the ratio $\varrho_b:\varrho_c=KP:KQ \Longrightarrow$

$$\frac{DP}{DQ}=\frac{\varrho_b}{\varrho_c}=\frac{PN}{QM} \Longrightarrow MN \parallel PQ.$$

High School Olympiads

Condition for inscribed similar triangle

Reply

Source: Own?

sunken rock

#1 Dec 5, 2012, 4:30 pm

Let ABC be a triangle, O its circumcenter, B', C' random points on the line segments $\overline{AC}, \overline{AB}$ respectively.
Prove that one can find a point A' on the line segment $\overline{BC}$ so that $\triangle A'B'C' \sim \triangle ABC$ if and only if $AC'OB'$ is cyclic.

Best regards,
sunken rock

Luis González

#2 Dec 5, 2012, 9:14 pm • 1

Assume that $\triangle A'B'C' \sim \triangle ABC.$ Circles $\odot(AB'C'), \odot(BC'A'), \odot(CA'B')$ concur at the Miquel point $M.$ Since $\angle B'MC' = \angle CAB = \angle B'A'C'$ and $\angle C'MA' = \angle ABC = \angle C'B'A' \pmod \pi),$ it follows that M is orthocenter of $\triangle A'B'C' \Longrightarrow \angle B'AM = \angle B'C'M = \angle B'A'M = \angle B'CM \Longrightarrow MA = MC.$ Similarly, we have $MA = MB \Longrightarrow M \equiv O,$ i.e. $A, C,' O, B'$ are concyclic.

Assume that $AC'OB'$ is cyclic. Circles $\odot(OB'C)$ and $\odot(OC'B)$ meet at a point A' on $BC.$ From $OA = OB = OC,$ we get $\angle OC'B' = \angle OAB' = \angle OCB'$ and $\angle OC'A' = \angle OBA' = \angle OCA' \Longrightarrow \angle B'C'A' = \angle BCA.$ Likewise, $\angle A'B'C' = \angle ABC \Longrightarrow \triangle ABC \sim \triangle A'B'C'.$

Quick Reply

High School Olympiads

Concurrent Lines

Headhunter

#1 Dec 4, 2012, 2:10 pm

Hello.

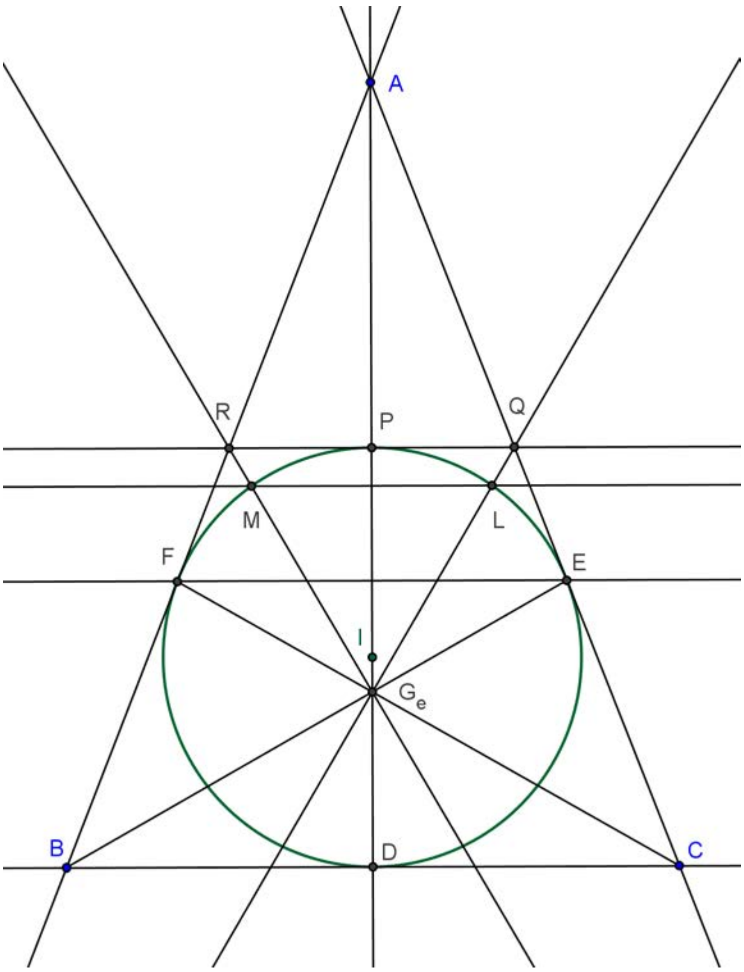
The incircle (I) of $\triangle ABC$ touch BC,CA,AB at D,E,F respectively.
AD cut (I) at P and the tangent at P of (I) cut CA,AB at Q,R
QG_e, RG_e cut (I) at L,M , where G_e is the Gergon point.
show that LM,EF,BC are concurrent.

Luis González

#2 Dec 4, 2012, 8:05 pm • 1

Project a line through the intersection of BC,EF, not cutting (I), to infinity and the incircle (I) into another circle. ABC becomes isosceles with legs AB=AC. The result is now trivial.

Attachments:



Headhunter

#3 Dec 5, 2012, 2:32 am

to Luis González.

Thanks a lot. Is there a proof for the existence of such projection?
Really, I got the same approach but stuck on the existence.

Luis González

#4 Dec 5, 2012, 12:04 pm • 2

Let (O) and ℓ be a circle and a line that have no points in common on plane $\alpha.$ AB is the diameter of (O) perpendicular to $\ell.$ $\mathcal{E}$ is a sphere through $(O).$ T is the vertex of the cone circumscribed in $\mathcal{E}$ along $(O).$ The line connecting T with the pole P of ℓ WRT (O) cuts $\mathcal{E}$ at V (V and T are on different sides of the plane α).

Let $A'B'$ be an antiparallel to AB WRT $VA,VB,$ passing through $T.$ The antiparallel section of the cone $V(O)$ cut by a plane $\beta \perp VAB$ through $A'B'$ is then a circle (T) with center $T,$ since $TA'=TB'$ (V-symmedian VT of VAB bisects all antiparallels to AB WRT VA,VB). So the central projection with center $V,$ projecting α into $\beta,$ sends the circle (O) into the circle (T) and sends the polar ℓ of P WRT (O) into the polar of T WRT $(T),$ which is a line at infinity.

leader

#5 Dec 5, 2012, 4:35 pm • 1

it''s well known that RQ,EF,BC are concurrent at the pole of AD. Let's call that point Z now let ZL cut (I) at M'
by Desargues theorem QEL and RFM' are perspective. since $QE \cap RF = A$ and LE,FM' intersect at the pole of Z which is AD than QL,RM' intersect on AD since $QL \cap AD = G$ than $RM' \cap AD = G$ so $M=M'$ and we have the result.
clearly we could have chosen some other point on AD

Quick Reply

High School Olympiads

Excircles Midpoints Concurrence

Reply

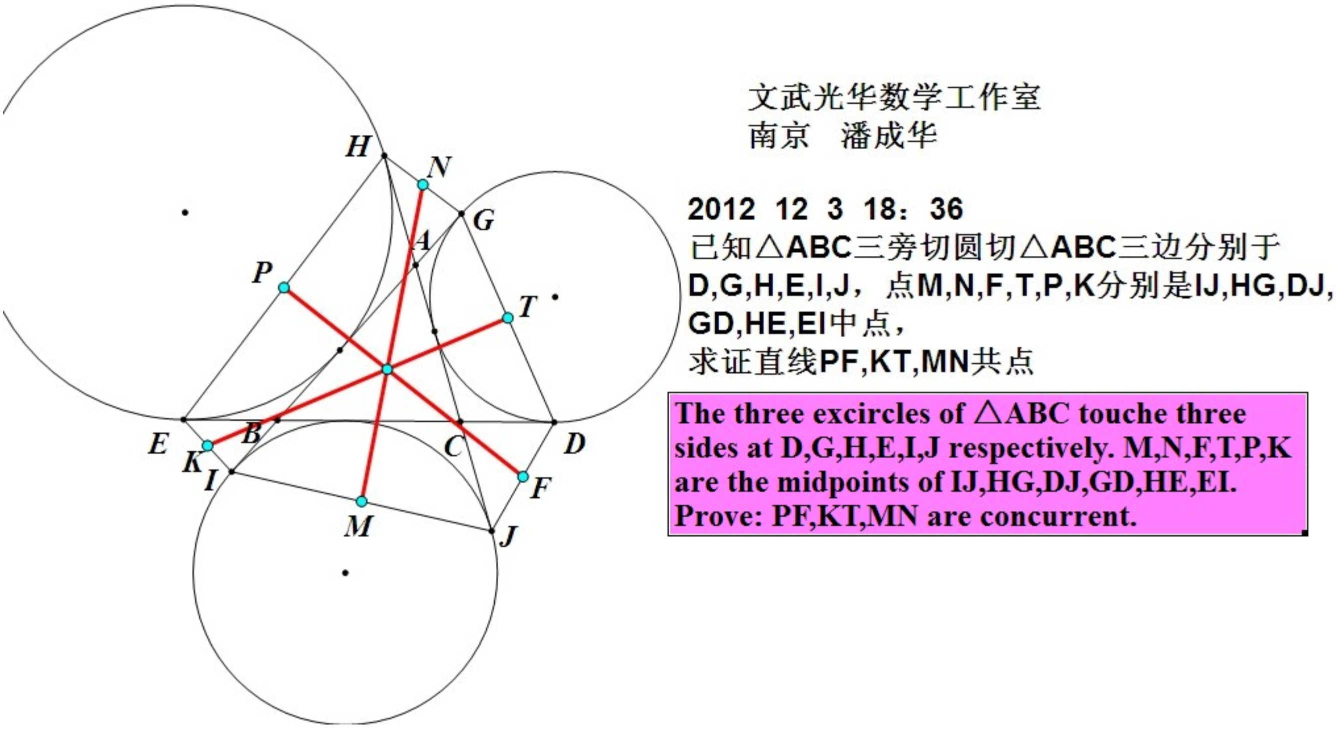
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 4, 2012, 6:09 am

See Attachment.
This is a problem proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Dec 5, 2012, 6:53 am

We use barycentric coordinates WRT $\triangle ABC$ with standard triangle notation.

$$D(0:a-s:s)\ ,\ E(0:s:a-s)\ ,H(s:0:b-s)$$

$$J(b-s:0:s)\ ,\ I(c-s:s:0)\ ,\ G(s:c-s:0)$$

$$\Longrightarrow N(-s(b+c):b(s-c):c(s-b))\ ,\ M(2bc-s(b+c):sb:sc)$$

$$\Longrightarrow MN \equiv s(b-c)x+[2c(s-b)+(b+c)s]y-[2b(s-c)+s(b+c)]z=0$$

By cyclic exchange, the equations of TK and PF are given by

$$TK \equiv -[2c(s-a)+(c+a)s]x+s(c-a)y+[2a(s-c)+s(c+a)]z=0$$

$$PF \equiv [2b(s-a)+(a+b)s]x-[2a(s-b)+(a+b)s]y+s(a-b)z=0$$

Thus, the lines MN, TK and PF concur at a point X with triangle center function

$$f(a,b,c)=2a^3+b^3+c^3+2a^2(b+c)+b^2(c+a)+c^2(a+b)+4abc$$

$f(a,b,c) = 2a + b + c + 2a(b+c) + b(c+a) + c(a+b) + 4abc.$

X is not in the current edition of ETC, but for instance, it lies on the line connecting the centroid $X_2(1:1:1)$ and symmedian point $X_6(a^2:b^2:c^2)$ of $\triangle ABC.$

timon92

#3 Dec 6, 2012, 5:50 am

there's no need of using barycentrics

we can use following lemma: Given hexagon $ABCDEF$. If area of triangle ACE equals to area of BDF, then lines passing through midpoints of opposite sides of $ABCDEF$ concur.

Quick Reply

High School Olympiads

Mixtilinear Incircles Angle Concurrence

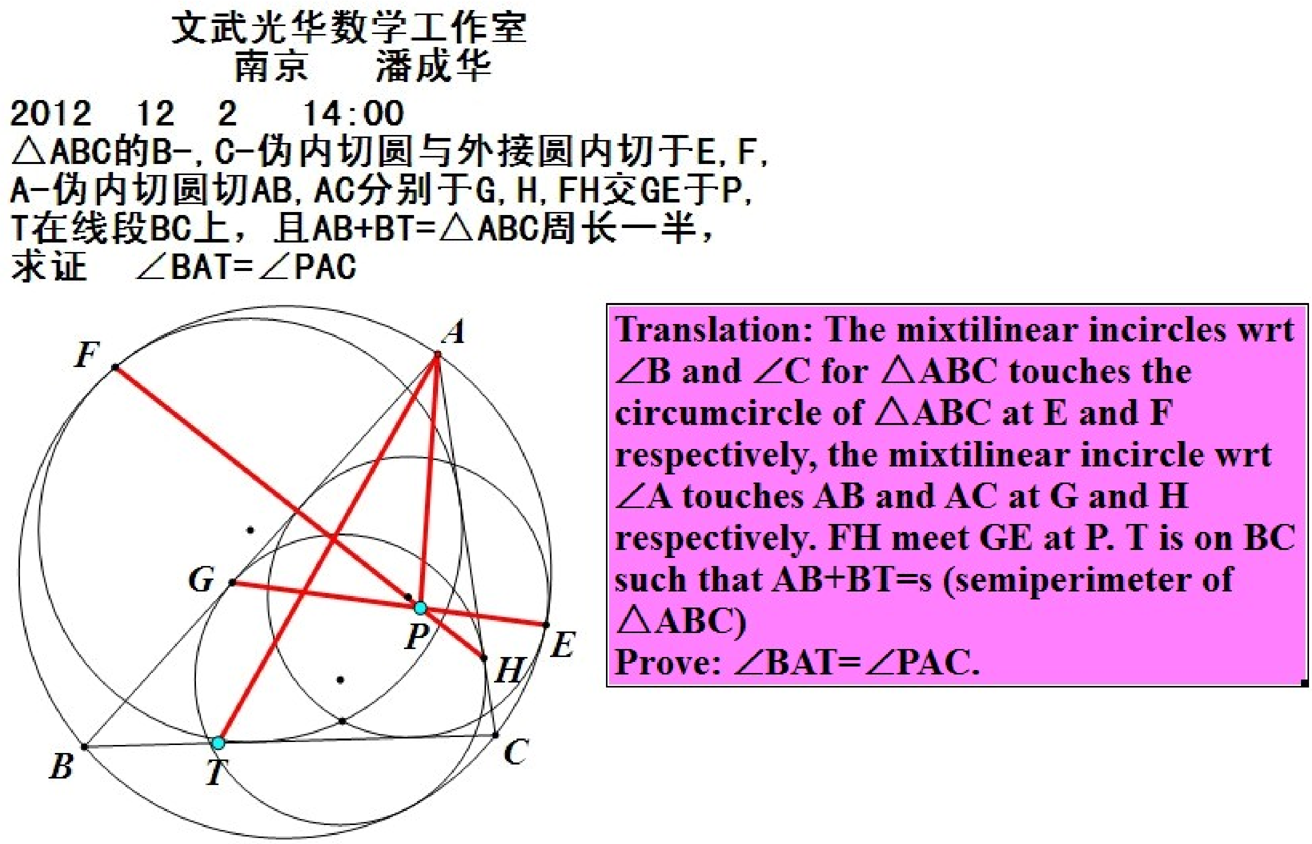
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 2, 2012, 12:35 pm • 1

See Attachment.
This is a problem proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



yunustuncbilek

#2 Dec 2, 2012, 10:29 pm

I think there is no need to define T as the tangent point of the A-excircle. Let A-mixtilinear circle touches the circumscribed circle at K. Since we know that $\triangle BAT \sim \triangle KAC$ which implies, $\angle BAT = \angle KAC$. Defining only K is sufficient. The problem consist of the concurrence of AK, GE and FH.

Luis González

#3 Dec 4, 2012, 5:21 am

Let the A-mixtilinear incircle touch the circumcircle at D. It's well known that AD, BE, CF are the isogonals of the Nagel cevians of $\triangle ABC$, concurring at the isogonal conjugate K of its Nagel point. Thus, it suffices to show that EG, FH and AD concur. If DF cuts AC at U, we get

$$\frac{UC}{UA} = \frac{DC}{DA} \cdot \frac{\sin \widehat{UDC}}{\sin \widehat{UDA}} = \frac{DC}{DA} \cdot \frac{\sin \widehat{FAC}}{\sin \widehat{FCA}} = \frac{DC}{DA} \cdot \frac{FC}{FA}$$

But if S denotes the tangency point of the C-excircle with AB, we have $\triangle FAC \sim \triangle BSC, \triangle DAC \sim \triangle BAT \Longrightarrow$

$$\frac{DC}{DA} = \frac{BT}{BA} = \frac{s-c}{c} \quad , \quad \frac{FC}{FA} = \frac{BC}{BS} = \frac{a}{s-a} \Longrightarrow$$

$$\frac{UC}{UA} = \frac{s-c}{c} \cdot \frac{a}{s-a} = \frac{a(s-c)}{c(s-a)} \Longrightarrow$$

$$F(D,K,A,H) = (U,C,A,H) = \frac{\overline{UC}}{\overline{UA}} \cdot \frac{\overline{HA}}{\overline{HC}} = -\frac{a(s-c)}{c(s-a)} \cdot \frac{c}{s-c} = -\frac{a}{s-a}.$$

Similarly, we have $E(D,K,A,G) = -\dfrac{a}{s-a} \Longrightarrow F(D,K,A,H) = E(D,K,A,G) \Longrightarrow EG$ and FH intersect on AD, as desired.

XmL

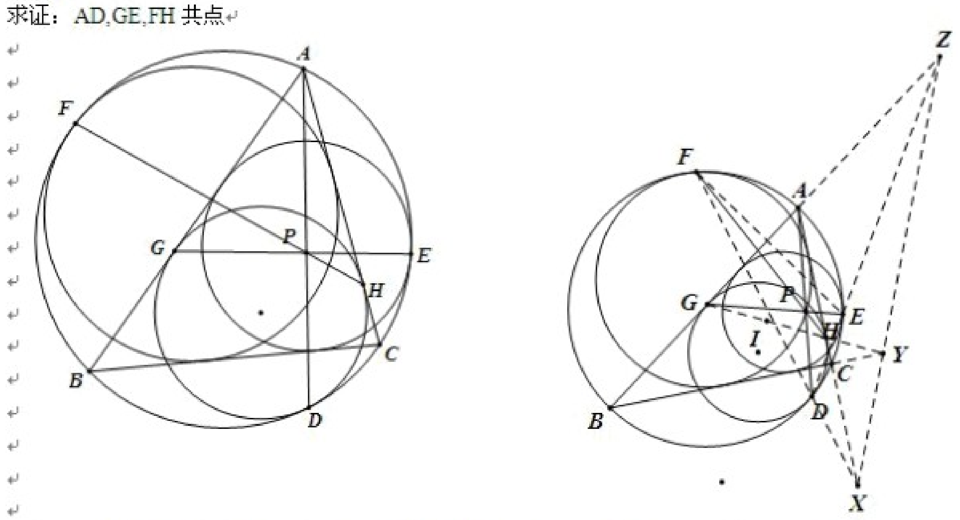
#4 Dec 4, 2012, 6:26 am

See attachment for solution provided by PHCP from WenWuGuangHua Mathematics Workshop In China himself.

Attachments:

已知 如图，△ABC 三伪内切圆与△ABC 外接圆内切于 D,E,F，A-伪内切圆切 AB,AC 分别于 G,H

求证：AD,GE,FH 共点



证明：设 FD 交 AC 于 X,DE 交 AB 于 Z,GH 交 BC 于 Y,易知 X,Y,Z 共线，因此在△AGH 与△DEF 中，根据 Desargue 定理，AD,GE,GH 共线

Proof: Let ED meet AC at X, DE meet AB at Z, GH meet BC at Y. It's well known(might not be for you) that X,Y,Z are collinear. By the Desargue's theorem concerning △AGH and △DEF, we conclude that AD,GE,GH are concurrent.

Quick Reply

High School Olympiads

Mixtilinear Excircles Midpoint Concurrence

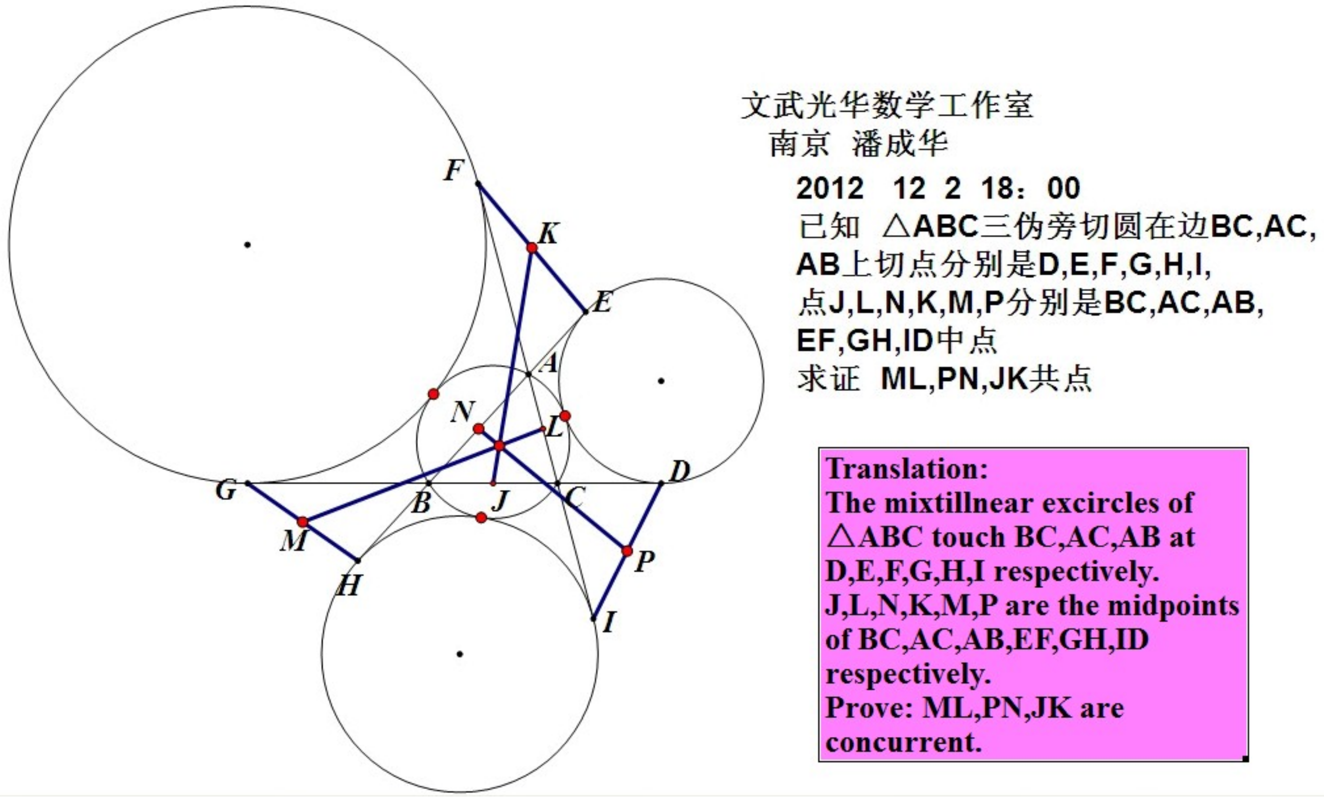
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 3, 2012, 7:28 am

See Attachment.
This is a problem proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Dec 3, 2012, 11:11 am • 1

Let I_a, I_b, I_c be the excenters of $\triangle ABC$ againts A, B, C. It's well known that I_a, I_b, I_c are the midpoints of $\overline{HI}, \overline{DE}, \overline{FG}$. Let $X \equiv EI_b \cap FI_c$. $I_aI_bXI_c$ is clearly a parallelogram and EAI_cX is cyclic, due to $\angle I_bXI_c = \angle I_bI_aI_c = \angle EAI_b \Longrightarrow \overline{I_bE} \cdot \overline{I_bX} = \overline{I_bA} \cdot \overline{I_bI_c} = \overline{I_bC} \cdot \overline{I_bI_a} \Longrightarrow CEXI_a$ is cyclic $\Longrightarrow \angle ECI_b = \angle I_bXI_a = \angle XI_aI_c$. But $\triangle I_aBC \sim \triangle I_aI_bI_c$ with corresponding medians $I_aJ, I_aX \Longrightarrow \angle ECI_b = \angle XI_aI_c = \angle JI_aI_b \Longrightarrow CE \parallel I_aJ$.

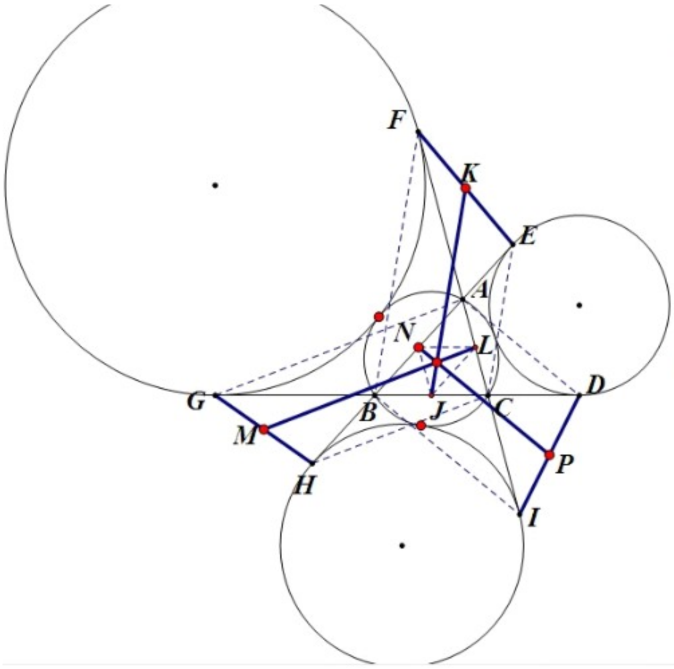
By similar reasoning, $BF \parallel I_aJ$. Thus I_aJ is midline of the trapezoid $BCEF$, meeting EF at its midpoint, i.e. K lies on the symmedian I_aJ of $\triangle I_aI_bI_c$. Likewise, M and P lie on symmedians of $\triangle I_aI_bI_c$ issuing from I_b and $I_c \Longrightarrow ML, PN, JK$ concur at the symmedian point of $\triangle I_aI_bI_c$ (Mittenpunkt of ABC).

XmL

#3 Dec 3, 2012, 12:48 pm

See attachment for solution provided by PHCP from WenWuGuangHua Mathematics Workshop In China himself.

Attachments:



文武光华数学工作室

南京 潘成华

2012 12 2 18：00

已知 △ABC三伪旁切圆在边BC,AC,AB上切点分别是D,E,F,G,H,I,点J,L,N,K,M,P分别是BC,AC,AB,EF,GH,ID中点

求证 ML,PN,JK共点

Proof: BF//CE,NJ//AC,JL//AB, ∴ KJ//BF ⇒ ∠NJK=∠ACE,KJL=∠AEC,∠NLM=∠BCH,∠MLJ=∠AHC,∠JNP=∠IAD,∠LNP=∠ADC,

$$\therefore \frac{\sin\angle NJK}{\sin\angle KJL}\cdot\frac{\sin\angle JLM}{\sin\angle NLM}\cdot\frac{\sin\angle LNP}{\sin\angle JNP}=\frac{\sin\angle ACE}{\sin\angle AEC}\cdot\frac{\sin\angle AHC}{\sin\angle BCH}\cdot\frac{\sin\angle CDA}{\sin\angle DAC}=\frac{AB}{AF}\cdot\frac{BC}{BH}\cdot\frac{CA}{DC}=1$$

High School Olympiads

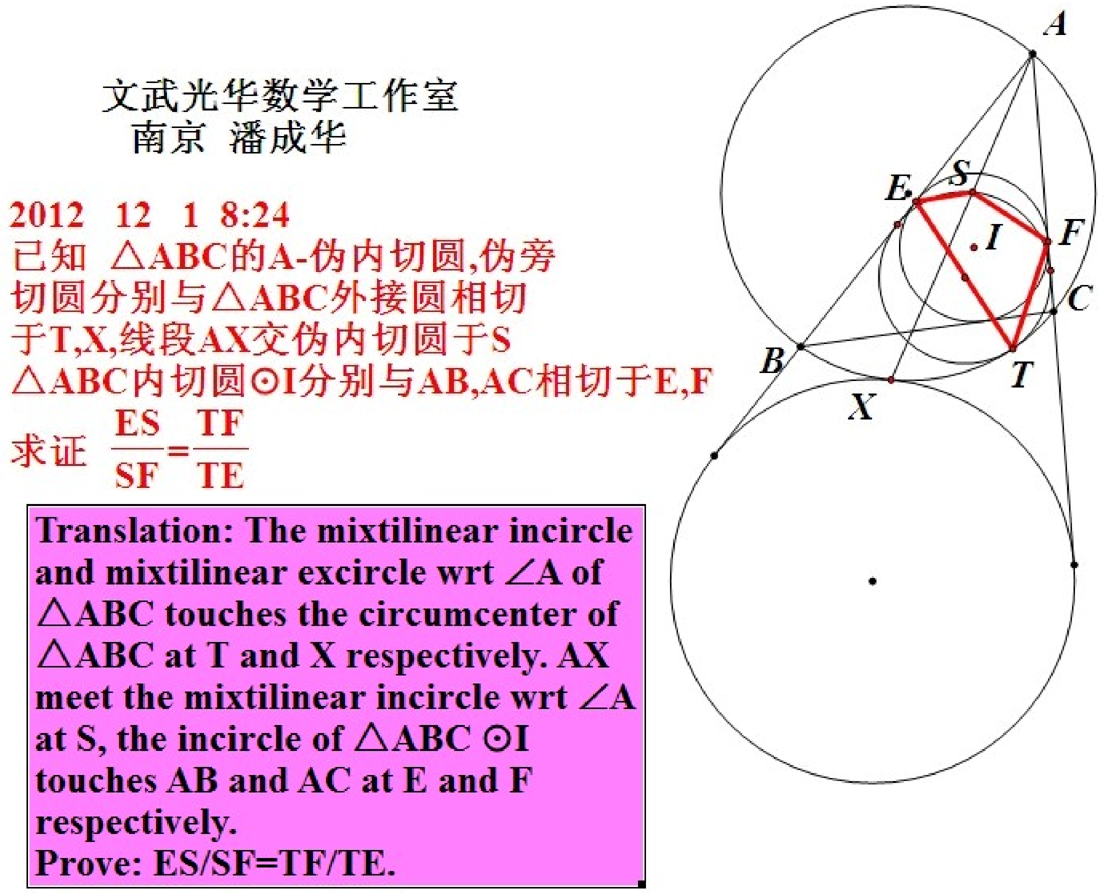
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 2, 2012, 1:33 pm

See Attachment.
This is a problem proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Dec 3, 2012, 7:02 am

(I) touches BC at D and A-mixtilinear incircle touches AB, AC at M, N. AD and AT cut (I) at L, K, $L \neq D$ and K is closer to A. It's well known that AS and AT are the isogonals of the Gergonne cevian AD and Nagel cevian AJ, respectively. Thus, by obvious symmetry $SLFE$ and $JKNM$ are isosceles trapezoids, further $DMJN$ is a parallelogram (MN and DJ bisect each other at I). Thereby

$$\frac{SE}{SF} = \frac{LF}{LE} \ (1) \ , \quad \frac{KN}{KM} = \frac{JM}{JN} = \frac{DN}{DM} \ (2)$$

The inversion with center A, carrying (I) into the A-mixtilinear incircle, takes D, E, F, K into $L, M, N, T \Longrightarrow LEMD$, $LFND$, $KEMT$ and $KFNT$ are all cyclic. Thus

$$\frac{LE}{DM} = \frac{AE}{AD}, \ \frac{LF}{DN} = \frac{AF}{AD} \Longrightarrow \frac{LF}{LE} = \frac{DN}{DM} \ (3)$$

$$\frac{TF}{KN} = \frac{AT}{AN}, \ \frac{TE}{KM} = \frac{AT}{AM} \Longrightarrow \frac{TF}{TE} = \frac{KN}{KM} \ (4)$$

Combining $(1), (2), (3), (4)$ yields $\frac{SE}{SF} = \frac{LF}{LE} = \frac{DN}{DM} = \frac{KN}{KM} = \frac{TF}{TE}$.

High School Olympiads

Cyclic quadrilateral with angle 60 degrees

Reply

Source: Bulgarian Mathematical Olympiad 2001

Lyub4o
#1 Dec 2, 2012, 4:35 pm

$ABCD$ is cyclic and $\angle BAD = 60$. E is the intersecting point of AC and BD and $AE = 3EC$. Prove that the sum of some two sides of the quadrilateral equals the sum of the other two sides.

Luis González
#2 Dec 2, 2012, 10:10 pm • 1

$$AB^2 + AD^2 - 2 \cdot AB \cdot AD \cdot \cos 60^\circ = CB^2 + CD^2 - 2 \cdot CB \cdot CD \cdot \cos 120^\circ$$

$$AB^2 + AD^2 - AB \cdot AD = CB^2 + CD^2 + CB \cdot CD$$

$$(AB - AD)^2 + AB \cdot AD = (CB - CD)^2 + 3 \cdot CB \cdot CD \ \ (1).$$

$$AE = 3 \cdot EC \Longrightarrow [ABD] = 3 \cdot [CBD] \Longrightarrow$$

$$\tfrac{1}{2} AB \cdot AD \cdot \sin 60^\circ = \tfrac{3}{2} CB \cdot CD \cdot \sin 120^\circ \Longrightarrow 3 \cdot CB \cdot CD = AB \cdot AD \ \ (2).$$

$$(1) \cup (2) \Longrightarrow (AB - AD)^2 = (CB - CD)^2 \Longrightarrow |AB - AD| = |CB - CD|.$$

Quick Reply

High School Olympiads

Cyclic quad.

Reply

subham1729

#1 Dec 2, 2012, 10:58 am

Let $ABCD$ be a cyclic quadrilateral with $AB=AD+BC$ prove that bisectors of $\angle ADC$ and $\angle BCD$ meets on AB.
(Without trigonometry :))

Luis González

#2 Dec 2, 2012, 12:31 pm

Take the point P on $\overline{AB}$, such that $BP=BC$. Thus $AP=AB-BC=AD \Longrightarrow \triangle PBC$ and $\triangle PAD$ are both isosceles with apices B and A. $\odot(PCD)$ cuts AB again at Q. Then $\angle PCB=\angle CPB=\angle CDQ$, but $\angle ADC=\pi-\angle CBP=\pi-(\pi-2\angle CPB)=2\angle CPB \Longrightarrow \angle ADQ=\angle CDQ \Longrightarrow DQ$ bisects $\angle ADC$. By similar reasoning, CQ bisects $\angle BCD$. Thus, angle bisectors of $\angle ADC$ and $\angle BCD$ meet at $Q \in AB$.

MillenniumFalcon

#3 May 17, 2015, 6:22 pm

Alternatively, take Q as the intersection of the angle bisectors.
By some angle chasing, note that PQCD is cyclic!
Then by some more angle chasing, angle QPB = 0 (depends on your diagram, but you get what I mean)
Hence Q lies on AB.

Quick Reply

High School Olympiads

Perspective triangles

Source: Own

buratinogigle

#1 Nov 18, 2012, 12:56 am

Let ABC be a triangle with altitudes AD,BE,CF, medians AM,BN,CP and three Feuerbach points F_a,F_b,F_c, (E) is nine points circles. Tangents at F_a,F_b,F_c of (E) intersect base a triangle XYZ.

a) Prove that two triangle XYZ and DEF are perspective.

b) Prove that two triangle XYZ and MNP are perspective.

Luis González

#2 Dec 1, 2012, 11:05 pm • 1

Let $(I_a),(I_b),(I_c)$ be the excircles of $\triangle ABC$ againts $A,B,C.$ $(I_b),(I_c)$ touch BC at $U,V,$ respectively. Label $\delta(K,\ell)$ the distance from a point K to a line $\ell.$ From the external tangency of the 9-point circle and $(I_b),$ we deduce that F_bU is the external bisector of $\angle MF_bD.$ Thus

$$\frac{\delta(D,ZX)}{\delta(M,ZX)}=\frac{DF_b{}^2}{MF_b{}^2}=\frac{UD^2}{UM^2}$$

Similarly, we have the expression $\dfrac{\delta(D,XY)}{\delta(M,XY)}=\dfrac{VD^2}{VM^2} \Longrightarrow$

$$\frac{\delta(D,ZX)}{\delta(D,XY)}\cdot\frac{\delta(M,ZX)}{\delta(M,XY)}=\frac{UD^2}{UM^2}\cdot\frac{VM^2}{VD^2}=\frac{UD^2}{VD^2}=\frac{I_bA^2}{I_cA^2}=\frac{r_b{}^2}{r_c{}^2}$$

Multiplying the cyclic expressions together yields

$$\frac{\delta(D,ZX)}{\delta(D,XY)}\cdot\frac{\delta(E,XY)}{\delta(E,YZ)}\cdot\frac{\delta(F,YZ)}{\delta(F,ZX)}=\frac{\delta(M,ZX)}{\delta(M,XY)}\cdot\frac{\delta(N,XY)}{\delta(N,YZ)}\cdot\frac{\delta(P,YZ)}{\delta(P,ZX)}\ (\star).$$

Since $MU=MV$ and $XF_b=XF_c,$ then M and X have equal powers WRT (I_b) and $(I_c) \Longrightarrow MX$ is radical axis of $(I_b),(I_c).$ Similarly, NY and PZ are radical axes of $(I_c),(I_a)$ and $(I_a),(I_b) \Longrightarrow MX,NY,PZ$ concur at the radical center S of $(I_a),(I_b),(I_c),$ which is nothing but the incenter of $\triangle MNP$ (Spieker point of ABC), i.e. $\triangle ABC$ and $\triangle MNP$ are perspective. Now, the RHS of the expression $(\star)$ equals 1, which implies that DX,EY,FZ are concurrent cevians of $\triangle XYZ,$ i.e. $\triangle ABC$ and $\triangle DEF$ are perspective through a point $T.$

P.S. Interestingly, the perspector T is the Kimberling center X_{1826}, the Zosma transform of the symmedian point of $\triangle ABC.$ This can be verified using barycentric coordinates.

High School Olympiads

EF and O1O2 perpendicular - in trapezoid

Reply

Source: Bulgarian - Spring Mathematics Contest 1997

school5

#1 Dec 1, 2012, 9:34 pm

Point F lies on the base AB of a trapezoid $ABCD$ and is such that $DF = CF$. Let E be the intersecting point of AC and BD and O_1 and O_2 are circumcentres of triangles ADF and FBC respectively. Prove that the straight lines FE and O_1O_2 are orthogonal.

Luis González

#2 Dec 1, 2012, 9:49 pm • 1

Let X, Y be the second intersections of AC, BD with $(O_1), (O_2),$ respectively.
$\angle CYB = \angle CFB = \angle DFA = \angle DXA \Longrightarrow CXYD$ is cyclic $\Longrightarrow \angle BYX = \angle ACD = \angle BAX \Longrightarrow ABXY$ is cyclic. Thus if $\{F, G\} \equiv (O_1) \cap (O_2)$, then AX, BY, FG are pairwise radical axes of $(O_1), (O_2), \odot(ABXY)$ concurring at their radical center $E \Longrightarrow FE$ is radical axis of $(O_1), (O_2)$ perpendicular to its center line O_1O_2.

Quick Reply

High School Olympiads

Prove that: $\angle APM + \angle BPC = 180^o$

Locked

NLT

#1 Dec 1, 2012, 11:03 am • 1

Let ΔABC with $AC = BC$. P is a point inside the triangle such that $\angle PAB = \angle PBC$. M is a midpoint of AB. Prove that: $\angle APM + \angle BPC = 180^o$

NLT

Luis González

#2 Dec 1, 2012, 11:24 am • 1

Posted many times before. Just prove that CP is P-symmedian of PAB.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=151&t=167318
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=286457
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=151&t=502500

High School Olympiads

when R=2r-a

Reply

MBGO
#1 Dec 1, 2012, 3:21 am

ABC a triangle,
A', B', C' the tangency points of $A-$ excircle , $B-$ excircle , $C-$ excircle with BC, CA, AB, respectively, A, B', A', C' lie on a circle.

Prove that $R = 2r_a$, where R is the circumradius of ABC and r_a is the circumradius of $A-$ excircle.

Luis González
#2 Dec 1, 2012, 3:51 am

> MBGO wrote:
> Prove that $R = 2r_a$, where R is the circumradius of ABC and r_a is the circumradius of $A-$ excircle.

The relation is stated incorrectly, it should be $R = \frac{1}{2}r_a$.

Let I_a, I_b, I_c be the excenters of $\triangle ABC$ againts A, B, C. $\triangle ABC$ and its circumcircle (O, R) become orthic triangle and 9-point circle of $\triangle I_aI_bI_c$. Thus, perpendiculars I_aA', I_bB', I_cC' from I_a, I_b, I_c, to BC, CA, AB concur at the circumcenter V of $\triangle I_aI_bI_c$. Since $AC'VB'$ is cyclic, then V and A' coincide $\Longrightarrow 2R = VI_a = r_a$.

Quick Reply

High School Olympiads

Gergonne point

Reply

yaphets

#1 Nov 30, 2012, 9:25 pm

Let ΔABC and let (I) is its incirle, AB and AC touch (I) at C_1, B_1, respectively. M, N are midpoints of BB_1, CC_1, B_2, C_2 are two point on BN and CM such that $MC = MC_2$ and $NB = NB_2$, BB_1 cuts CC_1 at G. Prove that $GB_2 = GC_2$.

Luis González

#2 Nov 30, 2012, 10:40 pm • 2

Let the A-excircle (I_a) touch BC, CA, AB at D, E, F, respectively. BCB_2C_1 is clearly a parallelogram $\Longrightarrow$ $CB_2 = BC_1 = CD$ and $C_1B_2 = BC = C_1F \Longrightarrow C$ and C_1 have equal power WRT (I_a) and the circle (B_2) with zero radius $\Longrightarrow CC_1$ is radical axis of $(I_a), (B_2)$. Analogously, BB_1 is radical axis of $(I_a), (C_2) \Longrightarrow G \equiv BB_1 \cap CC_1$ is radical center of $(I_a), (B_2), (C_2) \Longrightarrow G$ is on radical axis of $(B_2), (C_2)$, i.e. G is on perpendicular bisector of $\overline{B_2C_2}$, or $GB_2 = GC_2$, as desired.

yunxiu

#3 Dec 2, 2012, 11:44 am • 1

2009 IMO Shortlist
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1932935&sid=bd00e742c49e242652d587728c1cb679#p1932935

Quick Reply

High School Olympiads

A conjecture with orthopole

Reply

Source: Droz-Farny

buratinogigle

#1 Nov 12, 2012, 10:10 pm

Let K be orthopole of a line d with respect to triangle ABC. x, y are two perpendicular lines passing through K. x, y cut BC at A_1, A_2, x, y cut CA at B_1, B_2, x, y cut AB at C_1, C_2, x, y cut d at X, Y. Assume that midpoints of A_1A_2, B_1B_2, C_1C_2 are collinear on a line then that line passes though midpoint of XY.

Luis González

#2 Nov 14, 2012, 1:04 pm • 2

Since the midpoints O_A, O_B, O_C of $\overline{A_1A_2}, \overline{B_1B_2}, \overline{C_1C_2}$ are collinear, then the circles $(O_A), (O_B), (O_C)$ with diameters $\overline{A_1A_2}, \overline{B_1B_2}, \overline{C_1C_2}$ are coaxal, i.e. they meet at K and P. Hence, BC, CA, AB, x, y are all tangent to a parabola $\mathcal{P}$ with focus P on the circumcircle (O) of $\triangle ABC$. The directrix p of $\mathcal{P}$ is then the Steiner line of P WRT $\triangle ABC$, which passes through its orthocenter H. Since the tangents x, y from K to $\mathcal{P}$ are perpendicular, then K is on its directrix $p \Longrightarrow p \equiv HK$.

On the other hand, let $\mathcal{P}^*$ be the parabola tangent to BC, CA, AB, d. Its directrix is the Steiner line τ of the fourline BC, CA, AB, d. But the orthopole K of d is on τ, thus $\tau \equiv HK \equiv p \Longrightarrow \mathcal{P} \equiv \mathcal{P}^* \Longrightarrow d$ is also tangent to $\mathcal{P} \Longrightarrow$ circumcircle of $\triangle KXY$ bounded by tangents x, y, d of $\mathcal{P}$ passes through its focus $P \Longrightarrow \odot(KXY)$ is coaxal with $(O_A), (O_B), (O_C) \Longrightarrow$ midpoint of $\overline{XY}$ is on $O_AO_BO_C$.

Quick Reply

High School Olympiads

Another orthocenter property

Reply

Source: mpdb

borislav_mirchev

#1 Nov 12, 2012, 5:14 am

In the acute-angled triangle ABC with orthocenter H and circumcircle k is drawn a line l through H (non-intersecting AB). K and L are the intersection points of k and l (K is from the smaller arc AC, L is from the smaller arc BC). M, N and P are the feets of the perpendiculars from the vertices A, B and C to l, respectively (M, N, P are internal to k). Prove that $PH = |KM - LN|$.

borislav_mirchev

#2 Nov 13, 2012, 4:21 pm

Note that l not intersects the segment AB, but can intersect the line AB.

Luis González

#3 Nov 13, 2012, 11:50 pm • 1

Let O be the circumcenter of $\triangle ABC$. D is the midpoint of $\overline{AB}$ (projection of O on AB), F is the midpoint of $\overline{MN}$ (projection of D on MN) and E is the midpoint of $\overline{KL}$ (projection of O on KL). WLOG assume that KM>LN. From $KM = HK - HM$ and $LN = HL - HN$, we get

$$PH = KM - LN \Longleftrightarrow PH = (HK - HM) - (HL - HN) \Longleftrightarrow$$

$$PH = (HN - HM) + (HK - HL) = 2 \cdot HF - 2 \cdot HE = 2 \cdot EF \ (\star).$$

If Z is the projection of D on OE, then the right $\triangle CHP$ and $\triangle ODZ$ are similar with similarity coefficient $\frac{CH}{OD} = 2 \Longrightarrow PH = 2 \cdot ZD = 2 \cdot EF$. Thus, from $(\star)$, we conclude that $PH = KM - LN$.

Quick Reply

High School Olympiads

Prove that : $KF=r$

vuvanhoacam
#1 Nov 13, 2012, 10:31 am

For square $ABCD$. E runs on BC.$ADEF$ is balance trapezoidal. EF cuts AB at K.
Prove that :

$$KF=r(KBE)$$

Luis González
#2 Nov 13, 2012, 11:21 am

I assume that "balance trapezoidal" means isosceles trapezoid and r(KBE) denotes the inradius of triangle KBE.

$\angle FED=\angle ADE=\angle DEC \Longrightarrow ED$ bisects $\angle KEC$. Since BD bisects $\angle ABC$, then D is B-excenter of $\triangle KBE$ $\Longrightarrow A,C$ are tangency point of the B-excircle of $\triangle KBE$ with $KB,BE \Longrightarrow$ $EF=DA=BA=\frac{1}{2}(KB+BE+EK)$. Therefore $KF=EF-EK=\frac{1}{2}(KB+BE-EK)=r$.

Virgil Nicula
#3 Nov 13, 2012, 9:42 pm

PP. Let the square $ABCD$ and $E\in BC$. Construct the isoseles trapezoid $ADEF$, where

$AF\parallel DE$ and denote $K\in AB\cap EF$. Prove that KF is the length of the inradius for $\triangle BEK$.

Proof, Denote the distance $\delta_d(X)$ of X to the line d . Thus, denote the projection L of D on EF and observe that $ADEF$ - isosceles

trapezoid $\Longrightarrow \underline{\delta_{EK}(D)}=\delta_{EF}(D)=\delta_{AD}(E)=BA=DA=\underline{\delta_{BK}(D)}=CB=\underline{\delta_{BE}(D)}$, i.e. the point D is B-exincenter

of $\triangle BEK$. Thus, $FK=FE-KE=BA-KE=\frac{1}{2}\cdot(BK+KE+EB)-KE=$ $\frac{1}{2}\cdot(BK+BE-KE)=$

$r(\triangle BEK)$. I"ll used the well-known property: the length of the inradius of the A-right triangle ABC is equally to $\frac{b+c-a}{2}$" .

High School Olympiads

My problem

Locked

vuvanhoacam

#1 Nov 13, 2012, 10:38 am

For triange ABC. BE, CF are bisector rays of ABC $(F \in AB; E \in AC)$. BH, CK are altitudes of ABC $(H \in AC, K \in AB)$.$(I), (O)$.
Prove that:

$$I \in HK \Leftrightarrow O \in EF$$

Luis González

#2 Nov 13, 2012, 10:46 am

Posted many times before. It's ISL 1997 Q6 and German Mathematical Competition Q4.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=50582
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=139275
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=219811
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=402242

High School Olympiads

Prove that

Reply

vuvanhoacam

#1 Nov 12, 2012, 11:19 pm

For triangle ABC, AA_0, BB_0, CC_0 are bisector rays of ABC. AA_0 cuts B_0C_0 at A_1, from A_1 the perpendicular of BC cuts BC at A_2. Similar to B, C. Prove that: AA_2, BB_2, CC_2 closing or parallel

Luis González

#2 Nov 13, 2012, 12:33 am

$I \equiv AA_0 \cap BB_0 \cap CC_0$ is the incenter of $\triangle ABC$ and the perpendiculars from I to BC, CA, AB cut AA_2, BB_2, CC_2 at I_1, I_2, I_3, respectively. Since the pencil $A_2(I, A, A_1, A_0)$ is harmonic and $II_1 \parallel A_2A_1$, it follows that A_2A_0 bisects $\overline{II_1}$, i.e. I_1 is the reflection of I about BC. Similarly, I_2 and I_3 are the reflections of I about CA, AB. By Kariya's theorem, $AA_2 \equiv AI_1, BB_2 \equiv BI_2$ and $CC_2 \equiv CI_3$ concur at a point, in fact, the Gray's point X_{79} of $\triangle ABC$.

P.S. This concurrency holds for all points lying on the Neuberg cubic of ABC.

Quick Reply

High School Olympiads

medians

Reply

Source: 0

Alizamani

#1 Nov 4, 2012, 6:11 pm

Let ABC be a triangle. D,E,F are the foot of perpendicular from A,B,C. G is the intersection of medians, and A_1, B_1, C_1 are the feets of the perpendiculars from G to BC, CA, and AB. Prove that intersection of the medians of EDF, $A_1B_1C_1$, and ABC are in a line.

This post has been edited 1 time. Last edited by Alizamani, Nov 20, 2012, 4:55 pm

apgoucher

#2 Nov 4, 2012, 7:28 pm

Let the centroid of DEF be G_1 and the centroid of $A_1B_1C_1$ be G_2.

Let A, B, C be represented by the complex numbers a, b, c of unit modulus, so the circumcentre is 0. By the Euler line properties, we have that the centroid is $\frac{1}{3}(a+b+c)$ and the orthocentre is $a+b+c$.

D is halfway between the orthocentre $a+b+c$ and its reflection $b+c-(a^\star+b^\star+c^\star)bc$, so has the form $D = \frac{1}{2}a+b+c-\frac{1}{2}(a^\star+b^\star+c^\star)bc$, which simplifies to $\frac{1}{2}(a+b+c)-\frac{1}{2}a^\star bc$. Taking one-third of the cyclic sum of D, E, F, we obtain their centroid, which is $G_1 = \frac{1}{2}(a+b+c)-\frac{1}{6}(a^\star bc+b^\star ca+c^\star ab)$.

A_1 is halfway between the centroid $\frac{1}{3}(a+b+c)$ and its reflection $b+c-\frac{1}{3}(a^\star+b^\star+c^\star)bc$ in the edge BC. This gives it the expression $A_1 = \frac{1}{6}a+\frac{2}{3}b+\frac{2}{3}c-\frac{1}{6}(a^\star+b^\star+c^\star)bc$, which simplifies to $\frac{1}{6}a+\frac{1}{2}b+\frac{1}{2}c-\frac{1}{6}a^\star bc$. The centroid of $A_1B_1C_1$ is again given by one-third of the cyclic sum, or $G_2 = \frac{7}{18}(a+b+c)-\frac{1}{18}(a^\star bc+b^\star ca+c^\star ab)$.

So, all we need to do is show that $G = \frac{1}{3}(a+b+c)$, $G_1 = \frac{1}{2}(a+b+c)-\frac{1}{6}(a^\star bc+b^\star ca+c^\star ab)$ and $G_2 = \frac{7}{18}(a+b+c)-\frac{1}{18}(a^\star bc+b^\star ca+c^\star ab)$ are collinear. It is easy to observe that $G_2 = \frac{2}{3}G+\frac{1}{3}G_1$, so the three points are collinear.

The derivation of the formula for reflecting in BC is given as an exercise in chapter 8 of Mathematical Olympiad Dark Arts. AoPS protests to me linking to it for some reason (I have no idea why).
[Mod: Probably because you don't have enough posts. Linked for you (assuming I got the right link).]

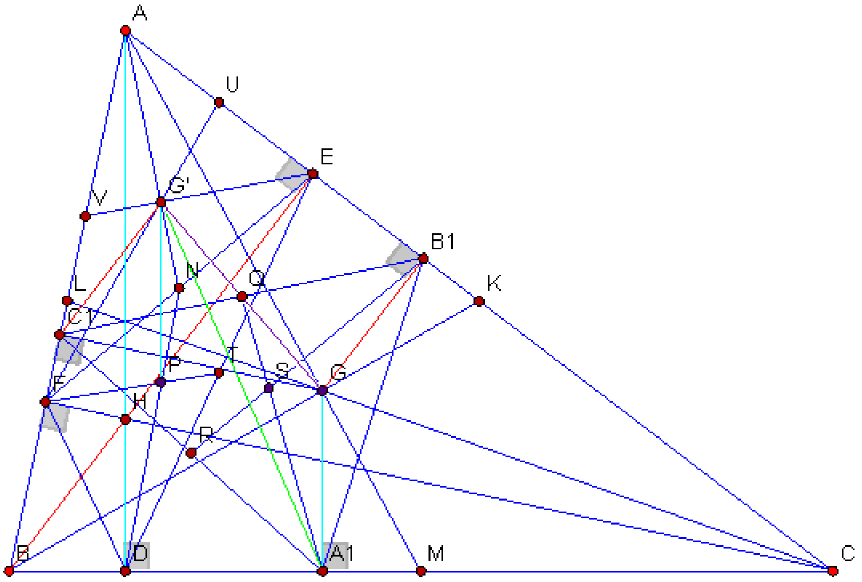
vslmat

#3 Nov 10, 2012, 5:11 pm

Notice that $\Delta AEF \sim \Delta ABC$. Let AM, BK, CL be the medians in ΔABC and AN, EV, FU be the medians in ΔAEF with centroid G'. Then easy to see that $G'P$ $\|$ and $= 1/2AD$, but the same does GA_1, so $G'PA_1G$ is a parallelogram and PG meets $G'A_1$ at the midpoint of $G'A_1$. To prove that P, S, G are collinear is now equivalent to prove that SG meets $G'A_1$ at the midpoint of $G'A_1$.
Now we see that we can prove that C_1GB_1G' is a parallelogram then $G'G$ meets C_1B_1 at Q, the midpoint of them and the centroid S of $A_1B_1C_1$ is also the centroid of $G'A_1G$ and SG meets $G'A_1$ at the midpoint of $G'A_1$. But as GB_1 $\|$ and $= 1/3BE$, it suffices to prove that C_1G' $\|$ and $= 1/3BE$ or $VC_1 = 1/3VB$
But easy to see that $VB = AB - 1/2AF = 1/2AF + BF$ (*)
$VC_1 = VF - 2/3LF = 1/2AF - 2/3LF = 1/2AF - 2/3(1/2AB - BF) =$
$= 1/2AF - 1/3AB + 2/3BF = 1/2AF - 1/3(AF + BF) + 2/3BF =$

$= 1/2AF - 1/3AB + 2/3BF = 1/2AF - 1/3(AF + BF) + 2/3BF =$
$= 1/6AF + 1/3BF$ (**)
(*) and (**) mean the desired.

Attachments:



Alizamani

#4 Nov 12, 2012, 5:04 pm

The general of this problem: l is a line and $\triangle ABC$ is a triangle and P moves on l and if A', B', C' are the feet P on BC, AC, AB. Then the the intersection is medians of $\triangle A'B'C'$ moves on a line.

Luis González

#5 Nov 12, 2012, 11:27 pm

> *Alizamani wrote:*
>
> Let ABC be a triangle. D, E, F are the foot of perpendicular from A, B, C. G is the intersection of medians, and A_1, B_1, C_1 are the feets of the perpendiculars from G to BC, CA, and AB. Prove that intersection of the medians of EDF, $A_1B_1C_1$, and ABC are in a line.

> *Alizamani wrote:*
>
> The general of this problem: l is a line and $\triangle ABC$ is a triangle and P moves on l and if A', B', C' are the feet P on BC, AC, AB. Then the the intersection is medians of $\triangle A'B'C'$ moves on a line.

See the thread 4 centroids on a parallel line to the Brocard axis.

Quick Reply

High School Olympiads

Hard problem

Reply

vuvanhoacam

#1 Nov 12, 2012, 10:19 pm

1. For triangle ABC. (I) is inscribed in the triangle. K is midpoint of AB. Prove that:

$$\widehat{KIB} = 90 \Leftrightarrow b + c = 3a$$

Luis González

#2 Nov 12, 2012, 10:50 pm • 1

Incircle (I) touches AB at Z and the perpendicular to BI at I cuts AB at D.

$$BD = BI \cdot \sec\frac{B}{2} = BZ \cdot \sec^2\frac{B}{2} = (s-b) \cdot \frac{ac}{s(s-b)} = \frac{2ac}{a+b+c}.$$

$$\widehat{KIB} = 90^\circ \Longleftrightarrow BD = BK = \tfrac{1}{2}c \Longleftrightarrow \frac{2ac}{a+b+c} = \frac{c}{2} \Longleftrightarrow b + c = 3a.$$

Virgil Nicula

#3 Nov 13, 2012, 1:44 am • 1

Let $\triangle ABC$ with the incircle $w = C(I, r)$. Denote the midpoint M of $[BC]$.

Prove that $\left\{\begin{array}{l} \mathrm{Case\ 1} \blacktriangleright\ b > c\ :\ IM \perp IB \iff a + b = 3c\ . \\ \mathrm{Case\ 2} \blacktriangleright\ b < c\ :\ IM \perp IC \iff a + c = 3b\ . \end{array}\right.$

Proof 1 (metric). $IB \perp IM \iff IB^2 + IM^2 = MB^2 \iff \frac{ac(s-b)}{s} + r^2 + (\frac{b-c}{2})^2 = (\frac{a}{2})^2 \iff$

$ac(s-b) + sr^2 = s(s-b)(s-c) \iff ac(s-b) + (s-a)(s-b)(s-c) = s(s-b)(s-c) \iff$

$ac(s-b) = a(s-b)(s-c) \iff s = 2c \iff \boxed{a + b = 3c}$.

Proof 2 (metric). Denote $T \in BC \cap w$ and suppose that $b > c$. In this case $\underline{IM \perp IB} \iff TB \cdot TM = IT^2 \iff$

$(s-b)(b-c) = 2r^2 \iff s(s-b)(b-c) = 2(s-a)(s-b)(s-c) \iff s(b-c) = 2(s-a)(s-c) \iff$

$(a+b+c)(b-c) = b^2 - (a-c)^2 \iff \underline{a + b = 3c}$. Prove analogously that in the second case exists the equivalence

$IM \perp IC \iff \boxed{a + b = 3c}$.

Proof 3 (synthetic). Denote $S \in AB \cap w$ and suppose that $b > c$. Observe that $IM \perp IB \iff \triangle BSI \sim \triangle BIM$

$\iff \frac{BS}{BI} = \frac{BI}{BM} \iff BI^2 = BM \cdot BS \iff \frac{ac(s-b)}{s} = \frac{a}{2} \cdot (s-b) \iff s = 2c \iff \boxed{a + b = 3c}$.

Proof 4 (synthetic). Suppose $b > c$. Denote the midpoint N of $[AC]$ and $\{B, S\} = BI \cap w$. Observe

that $\triangle BIM \sim \triangle ANS \iff \frac{BI}{AN} = \frac{BM}{AS} \iff \frac{BI}{AN} = \frac{BM}{IS} \iff AN \cdot BM = IB \cdot IS \iff$

$\frac{ab}{4} = 2Rr \iff abc = 8Rrc \iff 4Rrs = 8Rrc \iff s = 2c \iff \boxed{a + b = 3c}$. See **PP7** from **here**.

This post has been edited 15 times. Last edited by Virgil Nicula, Nov 13, 2012, 7:50 pm

yetti

#4 Nov 13, 2012, 3:17 pm

(I) touches AB at F and $[FZ]$ is diameter of (I). CZ cuts AB at F', the tangency point of C-excircle with AB.
$IK \parallel CZF'$ is midline of $\triangle FZF'$.
$BI \perp IK \iff BI \perp CF' \iff \triangle BCF'$ is B-isosceles $\iff [BF'] = [BC] \iff \frac{1}{2}(b + c - a) = a \iff b + c = 3a$.

Virgil Nicula

#5 Nov 13, 2012, 7:51 pm

Nice proof, **Yetty**. Thanks.

underzero

#6 Nov 13, 2012, 9:49 pm

The incircle touches BC,AC at S,P
BI meets SP at H
if $KIB = 90$ so we have $BI = IH$
thus AI is the median of the triangle ABH
Assume that $A = 2b$,$B = 2a$ so $sin(90 - a - b) = sinb/sina$
thus $(s - a)(s - b)/ab = (s - b/b).(a/s - a)$ so $s - a = a$ and $3a = b + c$
similarly you can prove if $b + c = 3a$ then $KIB = 90$
QED

Quick Reply

High School Olympiads

Area

Reply

Source: Austrian Mathematical Olympiad 1983

MinatoF

#1 Oct 20, 2012, 9:30 pm • 2

Let P be a point in the plane of a triangle ABC. Lines AP, BP, CP respectively meet lines BC, CA, AB at points A', B', C'. Points A'', B'', C'' are symmetric to A, B, C with respect ti A', B', C' ,respectively. Show that

$$S_{A''B''C''} = 3S_{ABC} + 4S_{A'B'C'}$$

yetti

#2 Nov 12, 2012, 5:28 am • 2

Assuming P is inside $\triangle ABC$. Otherwise, signs in the proposed area relation would have to be changed or directed areas used.
Any $\triangle ABC$ with any interior point P can be parallel projected to an acute $\triangle A^*B^*C^*$ with orthocenter P^*, such that $P \mapsto P^*$.
Parallel projection presesrves both area ratios and segment ratios on a line $\Longrightarrow$ it suffices to prove the area relation, when P is orthocenter of acute $\triangle ABC$.
Let P be orthocenter and (O) circumcircle of acute $\triangle ABC$. Let the altitudes AP, BP, CP cut (O) again at X, Y, Z, resp.
$\triangle XYZ \sim \triangle A'B'C'$ are centrally similar with similarity center P and coefficient $2 \Longrightarrow S_{XYZ} = 4S_{A'B'C'}$.
$AA' = A'A''$ and $PA' = A'X \Longrightarrow XA'' = AP$ and similarly, $YB'' = BP, ZC'' = CP$.

$$S_{A''B''C''} = \sum_{\text{cyc}} S_{PB''C''} = \sum_{\text{cyc}} \tfrac{1}{2} \cdot PB'' \cdot PC'' \cdot \sin \widehat{A} = \sum_{\text{cyc}} \tfrac{1}{2} \cdot (PY + YB'') \cdot (PZ + ZC'') \cdot \sin \widehat{A} =$$

$$= \sum_{\text{cyc}} \tfrac{1}{2} \cdot PY \cdot PZ \cdot \sin \widehat{A} + \sum_{\text{cyc}} \tfrac{1}{2} \cdot YB'' \cdot ZC'' \cdot \sin \widehat{A} + \sum_{\text{cyc}} \tfrac{1}{2} \cdot (PY \cdot ZC'' + PZ \cdot YB'') \cdot \sin \widehat{A} =$$

$$= \sum_{\text{cyc}} \tfrac{1}{2} \cdot PY \cdot PZ \cdot \sin \widehat{A} + \sum_{\text{cyc}} \tfrac{1}{2} \cdot BP \cdot CP \cdot \sin \widehat{A} + \sum_{\text{cyc}} (PB' \cdot CP + PC' \cdot PB) \cdot \sin \widehat{A} =$$

$$= \sum_{\text{cyc}} S_{PYZ} + \sum_{\text{cyc}} S_{PBC} + \sum_{\text{cyc}} 2(S_{PCB'} + S_{PBC'}) =$$

$$= S_{XYZ} + S_{ABC} + 2S_{ABC} = 4S_{A'B'C'} + 3S_{ABC}$$

Luis González

#3 Nov 12, 2012, 8:24 am • 1

Let $(u:v:w)$ be the barycentric coordinates of P WRT $\triangle ABC$. Coordinates of its traces on BC, CA, AB are $A'(0:v:w), B'(u:0:w), C'(u:v:0) \Longrightarrow A''(v+w:-2v:-2w)$ and cyclic expressions for B'' and C''. Working with directed areas, we have

$$\frac{S_{A'B'C'}}{S_{ABC}} = \frac{\begin{pmatrix} 0 & v & w \\ u & 0 & w \\ u & v & 0 \end{pmatrix}}{(v+w)(w+u)(u+v)} = \frac{2uvw}{(v+w)(w+u)(u+v)}$$

$$\frac{S_{A''B''C''}}{S_{ABC}} = \frac{\begin{pmatrix} v+w & -2v & -2w \\ -2u & w+u & -2w \\ -2u & -2v & u+v \end{pmatrix}}{-(v+w)(w+u)(u+v)} = \frac{8uvw + 3(v+w)(w+u)(u+v)}{(v+w)(w+u)(u+v)}$$

$$\frac{S_{A''B''C''}}{S_{ABC}} = \frac{8uvw}{(v+w)(w+u)(u+v)} + 3 = 4 \cdot \frac{S_{A'B'C'}}{S_{ABC}} + 3.$$

Quick Reply

High School Olympiads

Concurrency with inversions

Reply

buratinogigle

#1 Nov 12, 2012, 12:24 am • 1

Let ABC be triangle with incircle (I). (I) touches BC, CA, AB at D, E, F, respectively. (ω_1) and (ω_2) are two circles center I. Let M, N, P are in inverse points of D, E, F with respecto circle (ω_1). Let X, Y, Z are in inverse points of A, B, C with respecto circle (ω_2). Prove that MX, NY, PZ are concurrent.

Luis González

#2 Nov 12, 2012, 5:49 am • 2

Two pairs of inverse points B, C and Y, Z are clearly concyclic $\Longrightarrow YZ$ is antiparallel to BC WRT IB, IC. Since IA is the A-circumdiameter of $\triangle IBC$, it follows that $YZ \perp IA \Longrightarrow EF \parallel YZ$. Similarly, $ZX \parallel FD$ and $XY \parallel DE \Longrightarrow \triangle XYZ$ and $\triangle DEF$ are homothetic. $\triangle MNP$ is obviously homothetic to $\triangle DEF$ with center their commom circumcenter I. Thus $\triangle MNP$ and $\triangle XYZ$ are homothetic $\Longrightarrow MX, NY, PZ$ concur at their homothetic center.

Quick Reply

High School Olympiads

Ortocenter

Reply

Aiscrim

#1 Nov 11, 2012, 5:00 pm

In $\triangle ABC$ AD, BE and CF are concurrent cevians in H. Prove that H is the ortocentre of the triangle if and only if $HA \cdot HD = HB \cdot HE = HC \cdot HF$.

Luis González

#2 Nov 12, 2012, 12:24 am

Assume that $HA \cdot HD = HB \cdot HE = HC \cdot HF$. Then $BCEF, CAFD$ and $ABDE$ are all cyclic quadrilaterals $\Longrightarrow$ DE is antiparallel to AB and DF is antiparallel to $AC \Longrightarrow \angle EDC = \angle BAC = \angle FDB \Longrightarrow BC$ bisects $\angle EDF$. But since the line pencil $D(E,F,H,B)$ is harmonic, it follows that $AHD \perp BC \Longrightarrow AD$ is the A-altitude of $\triangle ABC$. Analogously, BE and CF are the B- and C- altitude of $\triangle ABC \Longrightarrow H$ is orthocenter of $\triangle ABC$. The converse is well-known.

ACCCGS8

#3 Nov 12, 2012, 1:49 pm

If H is the orthocentre, the relation holds as $BFEC, AFDC, AEDB$ become cyclic.
Now assume that $HA \cdot HD = HB \cdot HE = HC \cdot CF \implies BFEC, AFDC, AEDB$ cyclic. Then $\angle BEC = \angle BFC = 180 - \angle AFC = 180 - \angle ADC = \angle ADB = \angle AEB \implies \angle BEC = \angle AEB = 90$. Then $\angle BFC = \angle AFC = 90$ Thus BE, CF are altitudes so H is the orthocentre of $\triangle ABC$.

Quick Reply

High School Olympiads

very hard problem

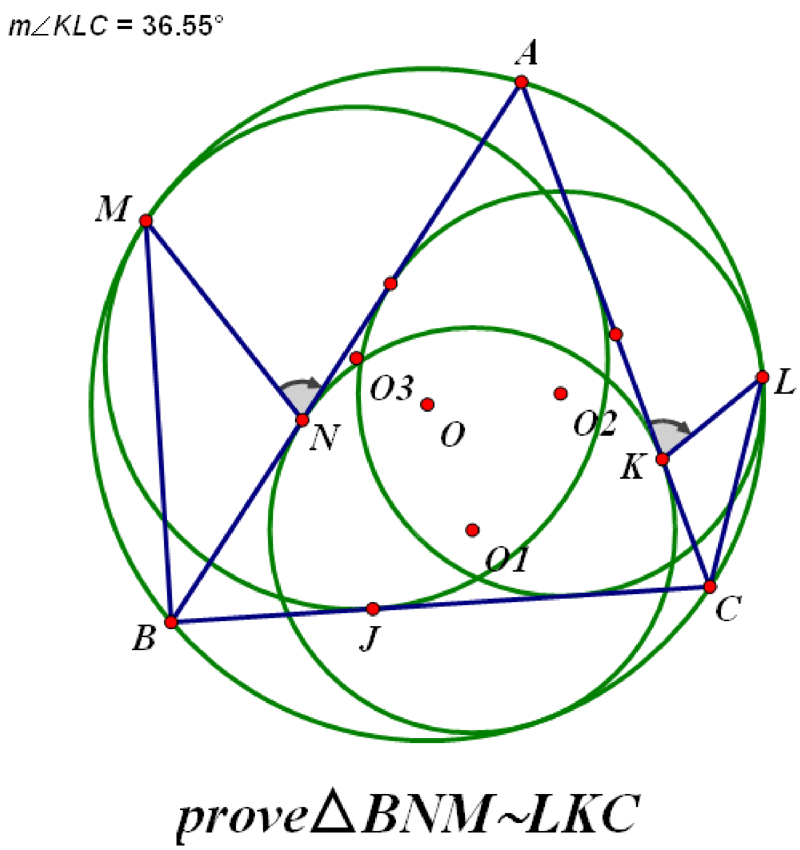
Reply

DANNY123

#1 Nov 11, 2012, 2:47 am

Given triangle ABC and its circumcircle (O). (O_1) is a circle that tangents AB, AC at N, K and circle (O). Circles (O_2) and (O_3) are defined similarly. According to the given diagram, prove that $\triangle BNM \sim \triangle LKC$.

Attachments:



Luis González

#2 Nov 11, 2012, 4:01 am

I have edited the original proposition in order to make it more comprehensible. DANNY123, for God's sake, care about the presentation of your posts!!. Use punctuaction and LaTeX code properly. Also use meaningful subjects: "Very hard problem", "Very easy", etc, do not describe the purpose of the thread.

From the solution of the problem On mixtilinear incircles 2, we know that MN, ML are isogonals WRT $\angle AMB \Longrightarrow \angle BMN = \angle LMA$. Since $\angle MBN = \angle MLA$, it follows that $\triangle BNM \sim \triangle LAM$. Similarly, LK, LM are isogonals WRT $\angle ALC \Longrightarrow \triangle LKC \sim \triangle LAM$. Hence $\triangle BNM \sim \triangle LAM \sim \triangle LKC$.

Quick Reply

High School Olympiads

Fixed point

Reply

buratinogigle

#1 Nov 8, 2012, 10:33 pm • 1

Let (I) be a circle inside circle (O). P, Q are two points on (I). There are two circles (K) and (L) which is passing through P, Q and intouch (O) at M, N, respectively. Prove that MN always passes through a fixed point when P, Q move on (I).

Luis González

#2 Nov 8, 2012, 10:58 pm • 2

Let m, n be the tangents of (O) through M, N and $R \equiv m \cap n. m, n$ and PQ are pairwise radical axes of $(O), (K)$ and (L), concurring at their radical center $R \Longrightarrow RM^2 = RN^2 = RP \cdot RQ$, i.e. R has equal power WRT (O) and $(I) \Longrightarrow R$ moves on the radical axis τ of (O) and $(I) \Longrightarrow$ polar MN of R WRT (O) goes through the fixed pole T of τ WRT (O). Thus, MN passes through a fixed point T of OI.

buratinogigle

#3 Nov 8, 2012, 11:32 pm • 1

Thank you dear Luis for this solution, actually I think this result contains an extension of the problem The three lines AA', BB' and CC' meet on the line IO from Romanian Master Of Mathematics 2012. It is as following

Let ABC be a triangle with circumcircle (O). (I) is a circle inside (O). (ω_{a_1}) and (ω_{a_2}) are two circles passing though B, C and touch (I) at A_1, A_2, respectively. Similarly we have B_1, B_2, C_1, C_2. Prove that A_1A_2, B_1B_2, C_1C_2 are concurrent at a point on OI.

When (I) is incircle we have problem on the post.

Luis González

#4 Nov 11, 2012, 1:57 am • 2

> *buratinogigle wrote:*
> Let ABC be a triangle with circumcircle (O). (I) is a circle inside (O). (ω_{a_1}) and (ω_{a_2}) are two circles passing though B, C and touch (I) at A_1, A_2, respectively. Similarly we have B_1, B_2, C_1, C_2. Prove that A_1A_2, B_1B_2, C_1C_2 are concurrent at a point on OI.

It also follows that the circles $\odot(AA_1A_2), \odot(BB_1B_2)$ and $\odot(CC_1C_2)$ are coaxal. This was submitted at Triad of coaxal circles coming from an arbitrary circle.

Radical axis τ of $(O), (I)$ cuts BC, CA, AB at P_A, P_B, P_C, respectively. We already know that A_1A_2, B_1B_2, C_1C_2 are the polars of P_A, P_B, P_C WRT $(I) \Longrightarrow A_1A_2, B_1B_2, C_1C_2$ concur at the pole T of τ WRT $(I) \Longrightarrow T$ has equal power WRT $\odot(AA_1A_2), \odot(BB_1B_2)$ and $\odot(CC_1C_2)$.

Let $\odot(AA_1A_2), \odot(BB_1B_2), \odot(CC_1C_2)$ cut (O) again at A_0, B_0, C_0, respectively. AA_0, A_1A_2 and τ concur at the radical center R_A of $(O), (I), \odot(AA_1A_2)$. Similarly, $R_B \equiv BB_0 \cap B_1B_2$ and $R_C \equiv CC_0 \cap C_1C_2$ are on τ. Now project τ to infinity and (I) into a circle (T). Polars of P_A, P_B, P_C (at infinite) WRT (T) become the perpendiculars from T to $BC, CA, AB \Longrightarrow AA_0, BB_0, CC_0$ become altitudes of $\triangle ABC$ concurring at its orthocenter. Hence, the original lines AA_0, BB_0, CC_0 also concur at a point $K \Longrightarrow K$ has equal power WRT $\odot(AA_1A_2), \odot(BB_1B_2), \odot(CC_1C_2) \Longrightarrow$ $\odot(AA_1A_2), \odot(BB_1B_2)$ and $\odot(CC_1C_2)$ are coaxal with common radical axis TK.

ACCCGS8

#5 Nov 12, 2012, 1:43 pm • 1

> buratinogigle wrote:
> Let ABC be a triangle with circumcircle (O). (I) is a circle inside (O). (ω_{a_1}) and (ω_{a_2}) are two circles passing though B, C and touch (I) at A_1, A_2, respectively. Similarly we have B_1, B_2, C_1, C_2. Prove that A_1A_2, B_1B_2, C_1C_2 are concurrent at a point on OI.

The above problem quoted is a nice generalisation of http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=497757&p=2797325#p2797325

buratinogigle, I don't quite understand how letting the (I) be the incircle (in the above quoted problem) yields the Romanian Master of Mathematics 2012 problem. In fact, I think letting (I) be the incircle yields the link I provided.

Thanks a lot for posting these beautiful problems - how to you think of them? By experimenting using Geometry software?

buratinogigle

#6 Nov 12, 2012, 10:04 pm • 1

Thank you dear ACCCGS8 for your interrest.

I think in problem Romanian Master of Mathematics 2012 problem, when letting the (I) be the incircle, (I) touch BC at D and touch (ω_A) at X then DX passes through radical center of (ω_A), (ω_B) and (ω_C).

I found these problems sometimes by generalized the onther nice problems and check them by Cabri. I think Cabri is the best dynamic geometry software :).

Quick Reply

High School Olympiads

hard question!!

Reply

safa698

#1 Nov 8, 2012, 3:52 pm • 3

Let ABC be a triangle and O,I,H its circumcenter, incenter and orthocenter, respectively. The incircle touches the sides AB and AC at B' and C' respectively. E is the midpoint of segment AH, and F is the reflection of E with respect to $B'C'$. Prove that O, I, and F are collinear.

Luis González

#2 Nov 9, 2012, 12:47 pm • 3

Generalization. O,I,H are the circumcenter, incenter and orthocenter of $\triangle ABC.$ P is an arbitrary point on AI and Y,Z are the projections of P on $AC,AB.$ Then the reflection of the midpoint E of $\overline{AH}$ about YZ lies on $OP.$

Proof. D is the midpoint of $\overline{BC}$ and L is the reflection of O about $YZ.$ AO cuts YZ at M and LE,LM cut PA at $Q,R,$ respectively. Reflection MR of MO about YZ is parallel to $AH,$ because AO,AH are isogonals WRT $\angle A.$ Further, $AROL$ is an isosceles trapezoid with congruent diagonals $RL=AO=R.$ Hence

$$\frac{AQ}{QR}=\frac{AE}{RL}=\frac{\frac{1}{2}AH}{R}=\frac{OD}{R}=\cos A.$$

This expression implies that Q coincides with the orthocenter of the A-isosceles $\triangle AYZ \Longrightarrow Q$ is reflection of P about YZ $\Longrightarrow$ reflection LQ of OP about YZ passes through $E \Longrightarrow$ reflection of E about YZ lies on $OP.$

dinamometre1123

#3 Nov 9, 2012, 1:51 pm

What do you mean by:

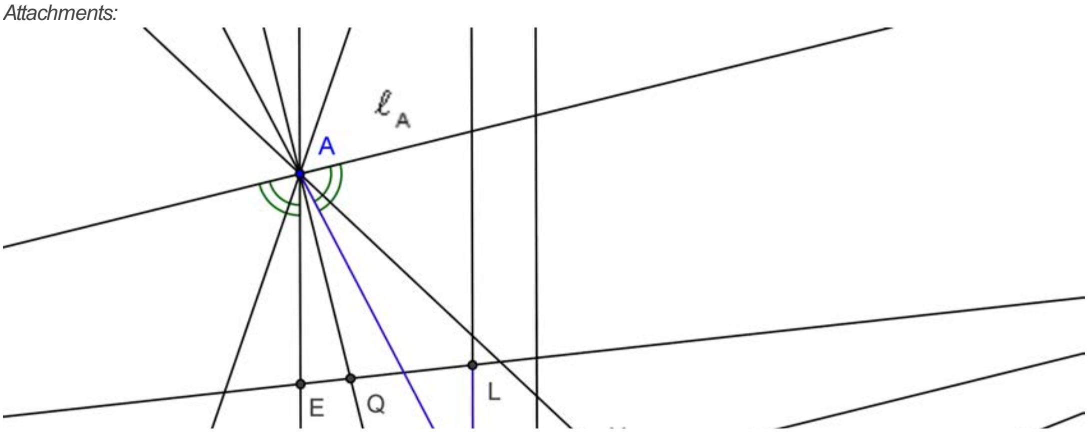
> *Luis González wrote:*
> Reflection MR of MO about YZ is parallel to $AH,$ because AO,AH are isogonals WRT $\angle A.$

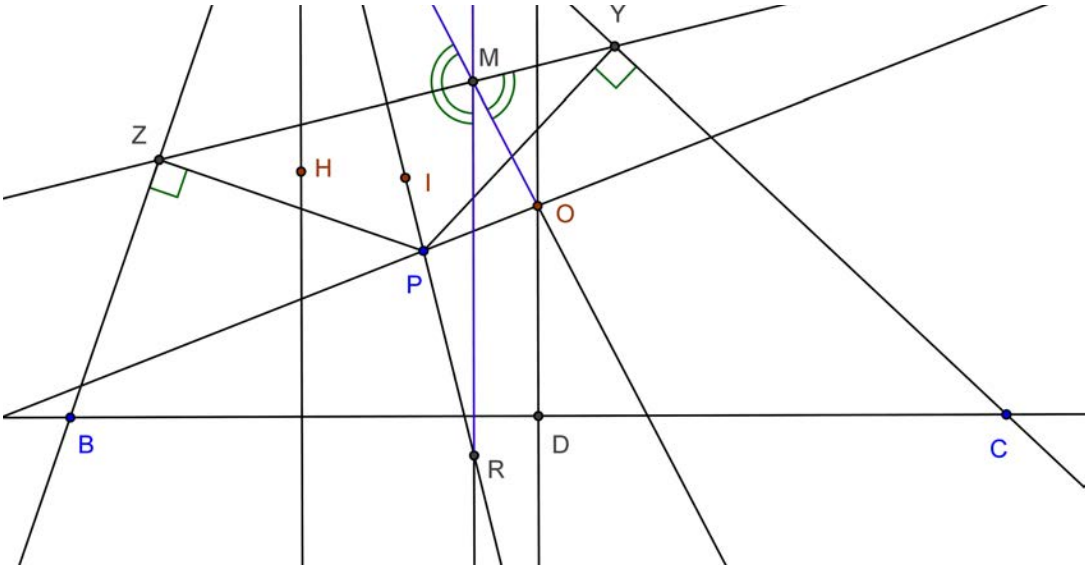
Luis González

#4 Nov 10, 2012, 4:16 am • 1

That means that $\angle BAH=\angle CAO,$ in other words, AH and AO are symmetric about the external bisector $\ell_A \perp AI$ of $\angle BAC.$ Since $YZ \parallel \ell_A,$ then reflection MR of MO about YZ is parallel to $AH.$

Attachments:





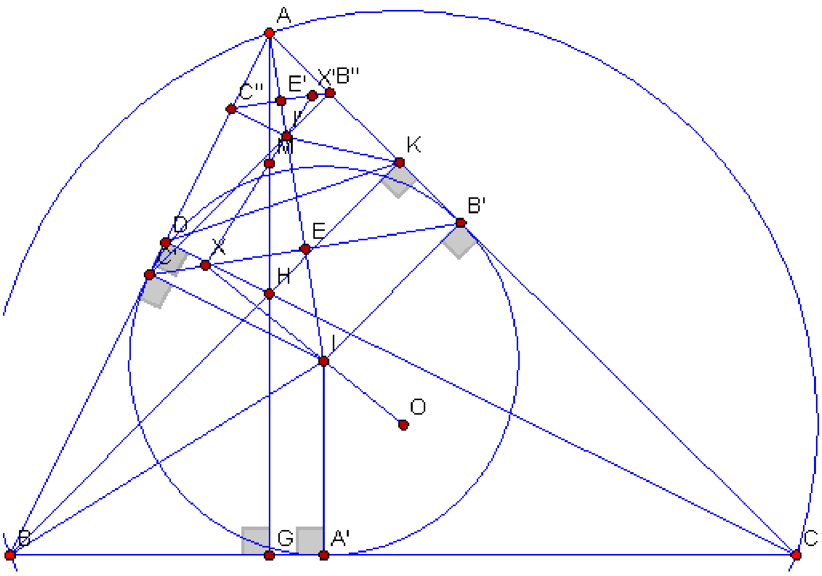
vslmat

#5 Nov 10, 2012, 5:14 pm • 1

Let AG, BK, CD be the altitudes in ΔABC. Denote $\angle BAC$ simply as $\angle A$. Notice that $\Delta ADK \sim ACB$ (the ratio of similarity being d) and point M is in fact the circumcenter of ADK.
Let I' be the incenter of $\Delta ADK, AI$ meet $B'C'$ at E. Easy to see that $AI'/AI = d = AK/AB = cosA$. Therefore, $II'/AI = 1 - cosA = 2sin^2(A/2)$. Also, $EI/AI = IB.sin(A/2)/(IB/sin(A/2)) = sin^2(A/2)$. Thus $II' = 2EI$, or $I'E = IE, I'$ is the reflection of I over $B'C'$.
Let B'', C'' be the projections of I' on AC, AB, respectively. OI meets $B'C'$ at X, MI' meets $B''C''$ at X', AI meets $C''B''$ at E'.
Notice that $E'X'/EX = d = E'I'/EI = E'I'/I'E$, this means that line MX' passes through X. Moreover, $\angle OXB' = \angle C''X'X = \angle X'XB'$ (as $B'C' \parallel B''C''$), line XO is the reflection of line XX' over $B'C'$. Then the reflection of M over $C'B'$ must lie on XO q.e.d.

Note: I apologize for marking the midpoint of AH as M instead of E

Attachments:



tranquanghuy7198

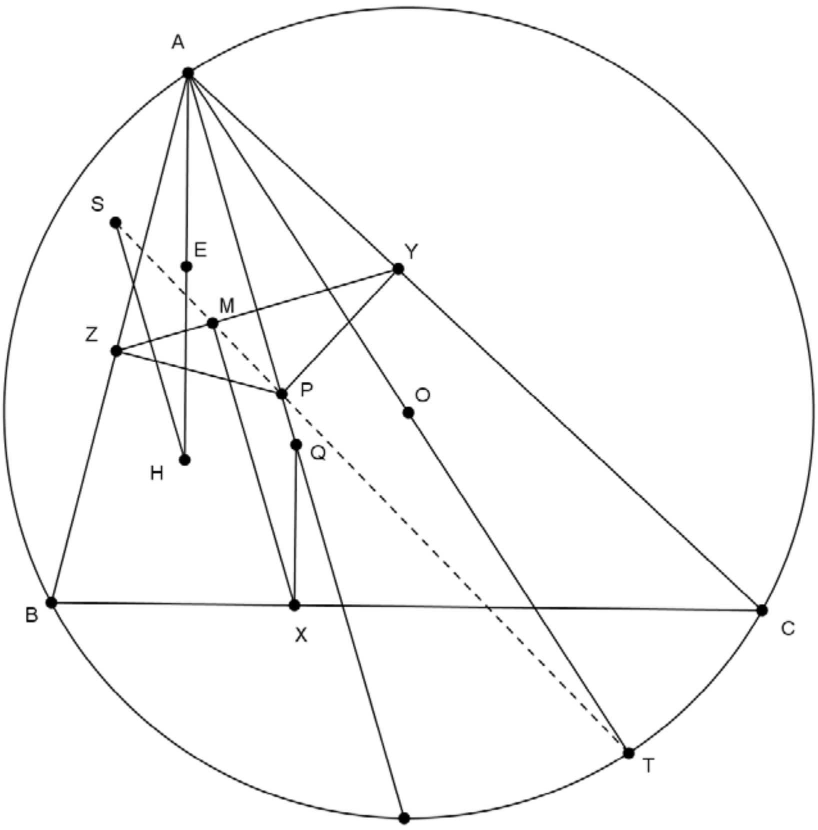
#6 Jun 12, 2015, 6:41 pm

> *Luis Gonzalez wrote:*
> **Generalization.** O, I, H are the circumcenter, incenter and orthocenter of $\triangle ABC. P$ is an arbitrary point on AI and Y, Z are the projections of P on $AC, AB.$ Then the reflection of the midpoint E of $\overline{AH}$ about YZ lies on $OP.$

My solution:

diameter AT and an orthocenter H. P is an arbitrary point on the bisector of $\angle A$. Y, Z are the projections of P on CA, AB, resp. Prove that the reflection S of H WRT YZ lies on TP.

Proof.
Q is the isogonal conjugate of P in $\triangle ABC$ and obviously $\overline{A, P, Q}$. X is the projection of Q on BC
Let $PT \cap YZ = M \Rightarrow XM \perp YZ$ (familiar property)
From here (post#4) we get: MX bisects $\angle PMH$
$\Rightarrow MPT$ and MH are reflective WRT YZ (because $YZ \perp MX$) (1)
On the other hand, according to the subject, MS, MH are reflective WRT YZ (2)
$\Rightarrow \overline{S, M, P, T}$, in other word, $S \in TP$ as desired.
Q.E.D
Attachments:



jayme
#7 Jun 24, 2015, 1:16 pm

Dear Mathlinkers,
have also a look at

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=481938

Sincerely
Jean-Louis

TelvCohl
#9 Sep 1, 2015, 1:04 am

> *Luis González wrote:*
> **Generalization.** O, I, H are the circumcenter, incenter and orthocenter of $\triangle ABC$. P is an arbitrary point on AI and Y, Z are the projections of P on AC, AB. Then the reflection of the midpoint E of $\overline{AH}$ about YZ lies on OP.

Let U, V be the midpoint of CA, AB, resp. Since the reflection Q of P in YZ is the orthocenter of $\triangle AYZ$, so notice E is the orthocenter of $\triangle AUV$ we get QE is the Steiner line of the complete quadrilateral $\{CA, AB, UV, YZ\}$. Since the projection T of A on OP lie on $\odot(AUV), \odot(AYZ)$, so T is the Miquel point of the complete quadrilateral $\{CA, AB, UV, YZ\} \Longrightarrow$ the reflection of T in YZ lie on QE, hence OP, QE are symmetry WRT $YZ \Longrightarrow$ the reflection of E in YZ lie on OP.

jayme

#10 Sep 1, 2015, 4:06 pm

Dear Mathlinkers,
it is
Turkey IMO 2012 Team Camp Exam P4
Sincerely
Jean-Louis

jayme

#11 Sep 1, 2015, 4:10 pm

Dear Mathlinkers,
by considering the symmetric I' of I wrt B'C', I' is the orthocenter of AB'C' and the line I'E goes through the Feuerbach point of ABC.
Now it is easy to conclude....
Sincerely
Jean-Louis

High School Olympiads

Cyclic Hexagon

Reply

Source: Victoria Krakovna

Daniel_Stone

#1 Nov 8, 2012, 6:50 pm

Point A_1 and A_2 that lie inside a circle centered at O are symmetric through point O. Points P_1, P_2, Q_1, Q_2 lie on the circle such that the rays A_1P_1 and A_2P_2 are parallel and in the same direction, and rays A_1Q_1 and A_2Q_2 are also parallel and in the same direction. Prove that lines P_1Q_2, P_2Q_1 and A_1A_2 are concurrent.

Luis González

#2 Nov 8, 2012, 8:58 pm

Let A_2P_2 and A_2Q_2 cut the given circle (O) again at X and Y, respectively. By obvious symmetry $A_1P_1A_2X$ is a parallelogram with diagonal intersection $O \equiv A_1A_2 \cap XP_1$ and $A_1Q_1A_2Y$ is a parallelogram with diagonal intersection $O \equiv A_1A_2 \cap YQ_1$. By Pascal theorem for the cyclic nonconvex hexagon $Q_1P_2XP_1Q_2Y$, the intersections $Q_1P_2 \cap P_1Q_2$, $A_2 \equiv P_2X \cap Q_2Y$ and $O \equiv XP_1 \cap YQ_1$ are collinear $\Longrightarrow P_1Q_2, P_2Q_1$ and A_1A_2 are concurrent.

Quick Reply

High School Olympiads

inequality tetrahedron 109

Reply

Source: Nicusor Zlota

nicusorz

#1 Oct 19, 2012, 3:29 pm

Denote r_a, r_b, r_c, r_d and h_a, h_b, h_c, h_d the radii of exinscribed spheres and the altitudes in tetrahedron, r is the radius of inscribed sphere, then we have inequality
$\frac{r_a+r}{r_a-r}+\frac{r_b+r}{r_b-r}+\frac{r_c+r}{r_c-r}+\frac{r_d+r}{r_d-r}\ge 12$
http://www.infomate.ro

Luis González

#2 Nov 7, 2012, 11:31 pm

This is similar to Inequality tetrahedron ABCD 63. By AM-HM, we obtain

$$\frac{r_a}{r_a-r}+\frac{r_b}{r_b-r}+\frac{r_c}{r_c-r}+\frac{r_d}{r_d-r}\ge\frac{16}{\frac{r_a-r}{r_a}+\frac{r_b-r}{r_b}+\frac{r_c-r}{r_c}+\frac{r_d-r}{r_d}}=$$

$$=\frac{16}{4-r\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}+\frac{1}{r_d}\right)}=\frac{16}{4-r\cdot\frac{2}{r}}=8\Longrightarrow$$

$$\frac{r_a+r}{r_a-r}+\frac{r_b+r}{r_b-r}+\frac{r_c+r}{r_c-r}+\frac{r_d+r}{r_d-r}\ge 12.$$

Quick Reply

High School Olympiads

a little bit harder than the others

Locked

Source: iran (final exam)

sco0orpiOn

#1 Nov 7, 2012, 9:48 pm

Let P be a variable point on arc BC of the circumcircle of ABC. Let I, J be the centers of incircles APB, APC, respectively. Then prove that the circumcircle of PIJ intersect the circumcircle of ABC in a fixed point.

Luis González

#2 Nov 7, 2012, 10:51 pm

The fixed point is the tangency point of the A-mixtilinear incircle with the circumcircle.

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=349634
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=313870
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=393618

High School Olympiads

short and nice solution

Reply

sco0orpiOn
#1 Nov 7, 2012, 2:52 am

Let **ABCDE** be polygon and let **X,Y,Z,T,R** be the midpoint of lines **CD,DE,AE,AB,BC**
prove that if lines **AX,BY,CZ,DT** are concurrent at **S**, then points **E,S,R** are on the same line.

Luis González
#2 Nov 7, 2012, 9:31 am

We use that a point M is on the U-median of $\triangle UVW \Longleftrightarrow [MUV]=[MUW]$. Since S is the common point of the A-median of $\triangle ACD$, B-median of $\triangle BDE$, C-median of $\triangle CEA$ and D-median of $\triangle DAB$, we have then

$[SBE]=[SBD]\ ,\ [SBD]=[SDA]\ ,\ [SDA]=[SCA]\ ,\ [SCA]=[SCE]$

$\Longrightarrow [SBE]=[SCE] \Longrightarrow S$ is on E-median ER of $\triangle EBC$.

Quick Reply

High School Olympiads

nice and not so easy

Reply

sco0orpiOn
#1 Nov 6, 2012, 7:38 pm

Let **L,W** be fixed line an circle respectively and let **P,Q** be fixed points such that **P** is on the line and **Q** is on the circle. Let **M,N** be point on the line **L** such that **PM.PN=constant**. Suppose that **QM,QN** intersect the circle in **X,Y**. Prove that the line **XY** passes through a fixed point

Click to reveal hidden text

Luis González
#2 Nov 7, 2012, 3:27 am • 1

Please, next time use meaningful subjects and use LaTeX code, enclosing formulas between dollar signs, to make your post more readable.

Since $\overline{PM} \cdot \overline{PN} = \text{const}$, then $M \mapsto N$ is an involution on the fixed line $\ell \Longrightarrow QM \mapsto QN$ is also an involution $\Longrightarrow$ $X \mapsto Y$ is a circular involution on the fixed circle ω. Let X', Y' be other pair of homologous points under this involution. The double correspondence implies that the intersections $XX' \cap YY', XY' \cap YX'$ are on the fixed axis τ of the involution $\Longrightarrow$ all lines $XY, X'Y'$ pass through the pole of τ WRT ω.

sco0orpiOn
#3 Nov 7, 2012, 7:12 pm

My own solution:
LEMMA 1: If **ABCD** be four points on a circle. Let P be a point out of circle such that the lines PA, PC are tangent to the circle. Then $(ABCD) = -1 \Longleftrightarrow P, B, D$ collinear

Click to reveal hidden text

LEMMA 2: If Q, A, B, C, D be points on a cirle. If $L - 1$ be an arbitary line. Suppose that X, Y, Z, T be the intersection of lines QA, QB, QC, QD whith L, then $(ABCD) = (XYZT)$.

LEMMA 3: If A, B, C, D be four points on a line respectively. If the midpoint of BD be P
$PB^2 = PC.PA \Longleftrightarrow (ACBD) = -1.$

These lemmas are obvious if you need hint tell me let's get back to the solution
assume that $PM.PN = a^2$ then we choose K, L on the line such that $PK = PL = a$
then assume that QK, QL intersect the circle at B, C and assume that the tangency intersection from B, C to the circle be S.
Then XY passes through S.

Quick Reply

High School Olympiads

inequality tetrahedron 110

Reply

Source: Nicusor Zlota

nicusorz

#1 Oct 19, 2012, 3:33 pm

Denote r_a, r_b, r_c, r_d and h_a, h_b, h_c, h_d the radii of exinscribed spheres and the altitudes in tetrahedron, r is the radius of inscribed sphere, then we have inequality

$$\frac{h_a - r}{h_a + r} + \frac{h_b - r}{h_b + r} + \frac{h_c - r}{h_c + r} + \frac{h_d - r}{h_d + r} \geq \frac{12}{5}$$

http://www.infomate.ro

Luis González

#2 Nov 6, 2012, 5:30 am

This is similar to Inequality tetrahedron ABCD 62. By AM-HM we obtain

$$\frac{h_a}{h_a + r} + \frac{h_b}{h_b + r} + \frac{h_c}{h_c + r} + \frac{h_d}{h_d + r} \geq \frac{16}{\frac{h_a+r}{h_a} + \frac{h_b+r}{h_b} + \frac{h_c+r}{h_c} + \frac{h_d+r}{h_d}} =$$

$$= \frac{16}{4 + r\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} + \frac{1}{h_d}\right)} = \frac{16}{4 + r \cdot \frac{1}{r}} = \frac{16}{5} \Longrightarrow$$

$$\frac{h_a - r}{h_a + r} + \frac{h_b - r}{h_b + r} + \frac{h_c - r}{h_c + r} + \frac{h_d - r}{h_d + r} \geq \frac{12}{5}.$$

Quick Reply

High School Olympiads

Cono Sur Olympiad 2012

Reply

Source: Problem 2

Mualpha7

#1 Nov 4, 2012, 2:49 am • 2

2. In a square $ABCD$, let P be a point in the side CD, different from C and D. In the triangle ABP, the altitudes AQ and BR are drawn, and let S be the intersection point of lines CQ and DR. Show that $\angle ASB = 90$.

Luis González

#2 Nov 4, 2012, 5:14 am • 2

Let AQ, BR cut BC, AD at $E, F,$ respectively. Since $\angle BAE = \angle CBP$ and $AB = BC$, then $\triangle ABE \cong \triangle BCP$ by ASA. Thus, $BE = CP \Longrightarrow CE = DP.$ Likewise, $\triangle BAF \cong \triangle ADP$ gives $DF = CP \Longrightarrow \triangle PCE \cong \triangle FDP$ by SAS $\Longrightarrow \angle EPC = \angle PFD.$ Since $EQPC$ and $FRPD$ are both cyclic, due to their right angles at C, Q and $D, R,$ then $\angle PRD = \angle PFD = \angle EPC = \angle EQC \Longrightarrow ARQS$ is cyclic $\Longrightarrow S$ is on the circle with diameter $AB.$

armpist

#3 Nov 5, 2012, 1:33 am

Dear MLs

M.T.

Attachments:
505386.doc (28kb)

mathreyes

#4 Nov 5, 2012, 3:00 am

If you set $ABCD$ as only a rhombus, the statement remains true!

jayme

#5 Nov 5, 2012, 6:47 pm

Dear Mathlinkers and M.T.
you can see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=362839

or
http://perso.orange.fr/jl.ayme vol. 7 Miniatures géométriques p. 10-12
for an original synthetic proof.

M.T. : your interesting figure lead for a circularly thema which can offer nice results... perhaps a new thema...

Sincerely
Jean-Louis

buratinogigle

#6 Nov 5, 2012, 11:16 pm

More general

Let $ABCD$ be a quadrilateral. AC cuts BD at E. (O) is circumcircle of triangle ECD. P is a point on AB. PC, PD cut (O) again at M, N. Prove that AN intersects BM on (O).

mathreyes

#7 Nov 10, 2012, 10:43 pm • 1

You can prove your generalization by using the converse of Pascal's theorem to hexagon $NDEGMC$ (G is where $\overline{AM}$ cuts $\overline{BN}$)

ElGeek

#8 Feb 3, 2013, 4:18 am • 4

Here the video solution at this problem, by my partner Cristian.

junioragd

#9 Feb 3, 2014, 9:12 pm

There is another approach.Let O be the intersection of the diagonals AC and BD.
Then we have cyclic quadritedrals DPOR and CQOP ,and then easy by angle chasing get <RSQ=<QAR=<QBR=90-<APB,so we are done.

drmzjoseph

#10 Aug 27, 2015, 1:22 pm

This is Serbia TST 2004, you can see at Imomath

Quick Reply

High School Olympiads

Concyclic

Reply

bigbang195

#1 Nov 3, 2012, 3:17 pm

Let points A and B lie on the circle k , and let C be a point inside the circle. Suppose that w is a circle tangent to segments $AC; BC$ and k. Let w touch AC and k at P and Q. Show that the circumcircle of APQ passes through the incenter of ABC.

How do you construction w ?

Luis González

#2 Nov 3, 2012, 9:04 pm • 1

The construction of the circle w is nothing but the Apollonius problem line-line-circle, but there is a slightly simpler approach using Sawayama's lemma. Let AC cut k again at D. Circle w becomes a Thebault circle of the cevian BC of $\triangle ABD \Longrightarrow$ PR passes through the incenter J of $\triangle ABD$. Hence, perpendicular dropped from J to the internal bisector of $\angle ABC$ cuts CA, CB at P, R. w is then the circle tangent to CA, CB through P, R.

As for the concyclicity of A, P, Q and the incenter of $\triangle ABC$, see Fairly difficult (Iran 1999), Concyclic points with triangle incenter, incenter of triangle, and elsewhere.

Quick Reply

High School Olympiads

Touching circles

Reply

Dragonboy

#1 Nov 2, 2012, 8:19 pm

Let ABC be an acute angle triangle with T,S and R being its incircle, circumcircle, and circumradius,respectively. Circle T_a is tangent internally to S at A and tangent externally to T. Circle S_a is tangent internally to S at A and internally to T. Let P_a, Q_a are centers of T_a, S_a respectively. Define P_b, Q_b, P_c, Q_c analogously. Prove that

$$8P_aQ_a.P_bQ_b.P_cQ_c \leq R^3$$

with equality if and only if ABC is equilateral.

[Mod: See USAMO, 2007.]

Luis González

#2 Nov 3, 2012, 12:28 pm • 1

We use standard triangle notation. E,F are the tangency points of the incircle (I) of $\triangle ABC$ with CA, AB. Inversion with center A and power $AE^2=AF^2=(s-a)^2$ takes (I) into itself and carries T_a and S_a into tangents t_a and s_a of (I), which are antiparallel to BC. (I) becomes incircle of $\triangle(AB,AC,t_a)$ and A-excircle of $\triangle(AB,AC,s_a)$, respectively. By inversion property, we have then $\text{dist}(A,t_a) \cdot AP_a = \text{dist}(A,s_a) \cdot AQ_a = \frac{1}{2}(s-a)^2$.

From $\triangle ABC \cong \triangle(AC,AB,t_a) \sim \triangle(AC,AB,s_a)$, we get $\text{dist}(A,t_a)=h_a$ and

$$\frac{\text{dist}(A,s_a)}{h_a}=\frac{r}{r_a}=\frac{s-a}{s} \Longrightarrow \text{dist}(A,s_a)=\frac{h_a \cdot (s-a)}{s}.$$

$$P_aQ_a=AQ_a-AP_a=\frac{1}{2}\left(\frac{s(s-a)}{h_a}-\frac{(s-a)^2}{h_a}\right)=\frac{a(s-a)}{2h_a}.$$

Similarly, we have $P_bQ_b=\frac{b(s-b)}{2h_b}$, $P_cQ_c=\frac{c(s-c)}{2h_c}$.

Thus, the desired inequality is equivalent to

$$\frac{abc(s-a)(s-b)(s-c)}{h_ah_bh_c} \leq R^3 \Longleftrightarrow 8(s-a)(s-b)(s-c) \leq abc.$$

After Ravi's substitution we get $8xyz \leq (y+z)(z+x)(x+y)$. Expanding out into symmetric sums yields $[2,1,0] \geq [1,1,1]$, which is true by Muirhead's theorem, because $[2,1,0] \succ [1,1,1]$.

Quick Reply

High School Olympiads

three circle

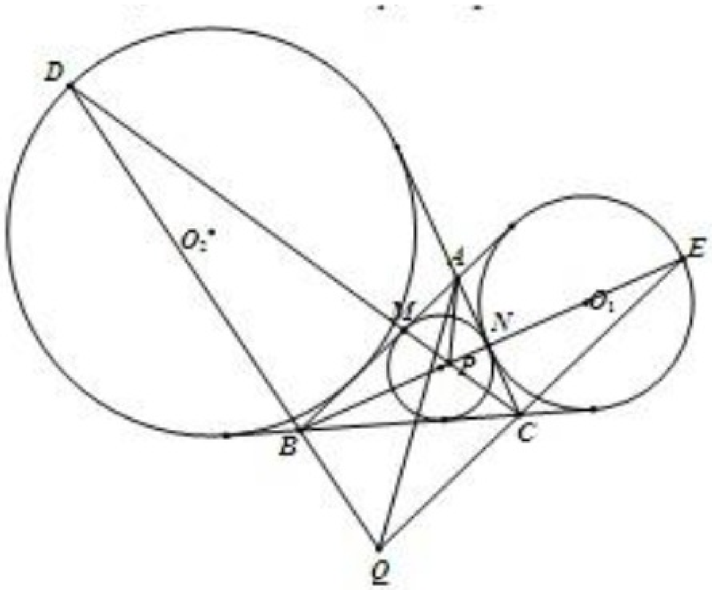
Reply

DANNY123

#1 Oct 27, 2012, 10:19 pm • 1

Triangle $\triangle ABC$ and two external tangent circles, (O_3) is the inscribed circle that touches AB, AC at M, N. Prove that $\angle BAQ = \angle PAC$.

Attachments:



vslmat

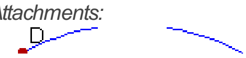
#2 Oct 30, 2012, 11:46 pm • 2

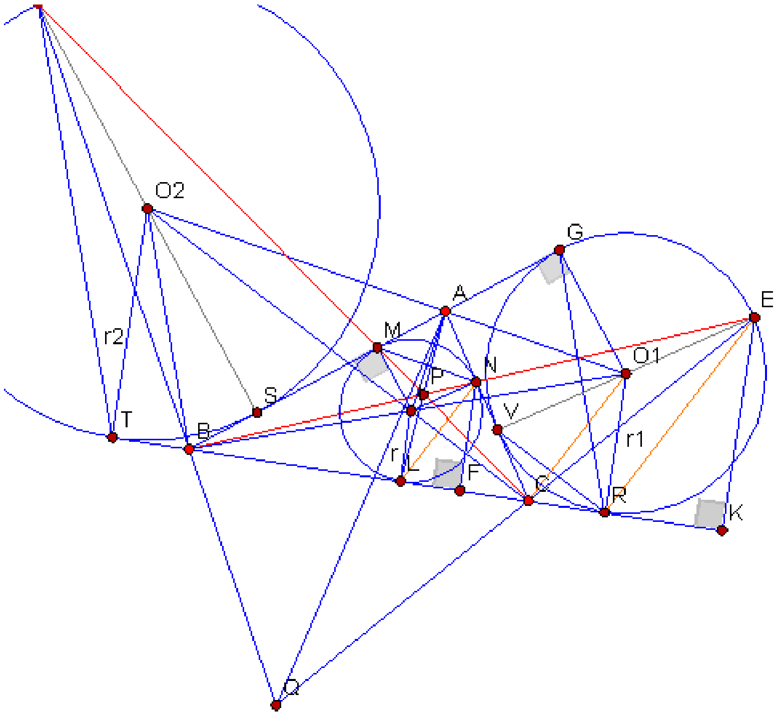
Let the contact triangle be MNL, the excircle with center O_1, radius r_1 is tangent to AB, BC, CA at G, R, V, respectively. The excircle with center O_2, radius r_2 is tangent to AB, BC at S, T, respectively.
It is well known (by using Ceva) that BN, MC, AL concur at P. Let I be the incenter of ABC and the inradius be r.
Notice that $\frac{BI}{BO_1} = \frac{IM}{GO_1} = \frac{IN}{O_1E}$, hence $IN \parallel O_1E$, so $O_1E \perp AC$ and O_1, E, V are collinear. Similarly, D, O_2, S are collinear and $LN \parallel CO_1 \parallel RE$. Denote the interior angles of ABC simply as $\angle A, \angle B, \angle C$.
Let K, F be the projections of E, N on BC. Notice that $\frac{BN}{BE} = \frac{r}{r_1}$. As $CR = CV$, we have:
$\frac{sinRCE}{sinVCE} = \frac{areaRCE}{areaVCE} = \frac{EK}{EV} = \frac{FN.r_1}{r.2r_1} = \frac{NC.sinC}{4r} = cotan(C/2).\frac{2sin(C/2)cos(C/2)}{4} = \frac{cos^2(C/2)}{2}$
Similarly, $\frac{sinDBT}{sinDBS} = \frac{cos^2(B/2)}{2}$
Now denote $\angle PAC = A_1, \angle BAP = A_2$. By sinus rule we have
$\frac{sinA_1}{sinA_2} = \frac{LC}{LB}.\frac{sinC}{sinB} = \frac{tan(B/2)}{tan(C/2)}.\frac{sinC}{sinB} = \frac{cos^2(C/2)}{cos^2(B/2)}$ (*)

Denote $\angle BAQ = A_3, \angle QAC = A_4$. Using sinus rule we have
$\frac{sinA_3}{sinA_4} = \frac{BQ.sinABQ}{CQ.sinACQ} = \frac{BQ}{CQ}.\frac{sinDBS}{sinVCE} = \frac{sinRCE}{sinDBT}.\frac{sinDBS}{sinVCE} = \frac{cos^2(C/2)}{cos^2(B/2)}$ (**)

(*), (**) together with the fact that $A_1 + A_2 = A_3 + A_4 = A$ mean $A_1 = A_3$ (q.e.d) and $A_2 = A_4$.

Attachments:





Luis González

#3 Nov 3, 2012, 12:13 am • 2

Rename $(I_A),(I_B),(I_C)$ the 3 excircles of $\triangle ABC.$ The incircle (I) touches BC at $L.$ $P \equiv AL \cap BN \cap CM$ is Gergonne point of $\triangle ABC.$ $(I_A),(I_B)$ touch BC,CA at $U,V,$ respectively. AL cuts (I_A) at $F,$ such that F is the farthest point from $A.$

Since A is the exsimilicenter of $(I) \sim (I_A),$ then $IL \parallel I_AF,$ i.e. F is the antipode of U WRT $(I_A).$ Similarly, E is the antipode of V WRT $(I_B).$ The right $\triangle CUI_A$ and $\triangle CVI_B$ are obviously similar $\Longrightarrow$ right $\triangle CUF$ and $\triangle CVE$ are also similar $\Longrightarrow$ $\angle BCF=\angle ACE,$ i.e. $CE \equiv CQ$ and CF are isogonals WRT $\angle ACB.$ Likewise, BQ,BF are isogonals WRT $\angle ABC$ $\Longrightarrow Q,F$ are isogonal conjugates WRT $\triangle ABC \Longrightarrow AP \equiv AF$ and AQ are isogonals WRT $\angle BAC.$

Quick Reply

High School Olympiads

Regular hexagon

Reply

Source: JBMO 2005 Shortlist

MinatoF

#1 Oct 18, 2012, 3:36 am • 1

Let $ABCDEF$ be a regular hexagon and $M\in (DE)$, $N\in (CD)$ such that $m(\widehat{AMN})=90^\circ$ and $AN=CM\sqrt{2}$. Find the value of $\frac{DM}{ME}$.

Luis González

#2 Oct 18, 2012, 7:05 am • 1

Since $\widehat{AMN}=\widehat{ACN}=90^\circ$, then AN is a diameter of the circumcircle of $\triangle ACM \Longrightarrow CM=AN\cdot \sin \widehat{MAC} \Longrightarrow$ $\sin \widehat{MAC}=\frac{1}{\sqrt{2}} \Longrightarrow \widehat{MAC}=45^\circ$, since $\widehat{MAC}<60^\circ$. Thus $\widehat{EAM}=60^\circ-45^\circ=15^\circ \Longrightarrow AM$ bisects $\widehat{DAE}$. By angle bisector theorem on $\triangle DAE$, we have then

$$\frac{DM}{ME}=\frac{DA}{AE}=\frac{\frac{2\sqrt{3}}{3}AE}{AE}=\frac{2\sqrt{3}}{3}.$$

Quick Reply

High School Olympiads

4 circles tangent to three sides of quadrilateral

Reply

dorina

#1 Oct 17, 2012, 11:23 pm

Let $ABCD$ be a convex quadrilateral. Analyze 4 circles which are tangent to three sides of this quadrilateral.

1. Prove that that centers of this circles are in one circle.

2. $\frac{AB}{r_1}+\frac{CD}{r_3}=\frac{BC}{r_2}+\frac{AD}{r_4}$

Moderator edit: Please use meaningful subjects and type equations, in LaTeX code, enclosed between dollar signs. Topic edited.

Luis González

#2 Oct 18, 2012, 1:30 am • 1

1) (P,r_1) is tangent to DA,AB,BC, (Q,r_2) is tangent to AB,BC,CD, (R,r_3) is tangent to BC,CD,DA and (S,r_4) is tangent to $CD,DA,AB.$ We assume that all the circles are outside of $ABCD.$ PQ,QR,RS,SP are external bisectors of $\angle ABC, \angle BCD, \angle CDA, \angle DAB.$ Therefore

$\angle PBA=90^{\circ}-\tfrac{1}{2}\angle B \ , \ \angle PAB=90^{\circ}-\tfrac{1}{2}\angle A \Longrightarrow$

$\angle SPQ \equiv \angle APB=180^{\circ}-\angle PBA-\angle PAB=\tfrac{1}{2}(\angle A+\angle B).$

Similarly, we have $\angle QRS=\tfrac{1}{2}(\angle C+\angle D) \Longrightarrow$

$\angle SPQ+\angle QRS=\tfrac{1}{2}(\angle A+\angle B+\angle C+\angle D)=180^{\circ} \Longrightarrow PQRS$ is cyclic.

2) Let X be the tangency point of (P,r_1) with $AB.$

$$\frac{XB}{r_1}=\cot \widehat{PBA}=\tan \frac{B}{2} \ , \ \frac{XA}{r_1}=\cot \widehat{PAB}=\tan \frac{A}{2} \Longrightarrow$$

$$\frac{XA+XB}{r_1}=\frac{AB}{r_1}=\tan \frac{A}{2}+\tan \frac{B}{2}.$$

Cyclically, we have the expressions:

$$\frac{BC}{r_2}=\tan \frac{B}{2}+\tan \frac{C}{2} \ , \ \frac{CD}{r_3}=\tan \frac{C}{2}+\tan \frac{D}{2} \ , \ \frac{DA}{r_4}=\tan \frac{D}{2}+\tan \frac{A}{2}$$

$$\Longrightarrow \frac{AB}{r_1}+\frac{CD}{r_3}=\frac{BC}{r_2}+\frac{DA}{r_4}=\tan \frac{A}{2}+\tan \frac{B}{2}+\tan \frac{C}{2}+\tan \frac{D}{2}.$$

Quick Reply

High School Olympiads

Equilateral triangle

Reply

dorina

#1 Oct 17, 2012, 11:14 pm

Let ABC be an equilateral triangle with side a. M a point in the plane of triangle with the distance d from the center of circumscribed circle. Prove that the area of triangle with sides MA, MB, MC can be expressed with the formula:

$$S=\frac{\sqrt{3}}{12}|a^2-3d^2|$$

Moderator edit: Please use meaningful subjects and type equations, in LaTeX code, enclosed between dollar signs. Topic edited.

Luis González

#2 Oct 18, 2012, 12:12 am • 1

Parallels from M to $AB, BC, CA,$ meet BC, CA, AB at $D, E, F,$ respectively. $MDCE, MEAF$ and $MFBD$ are all isosceles trapezoid with bases $\overline{ME} \parallel \overline{DC}, \overline{MF} \parallel \overline{EA}$ and $\overline{MD} \parallel \overline{FB} \Longrightarrow MA=EF, MB=FD, MC=DE$ $\Longrightarrow$ the sides of $\triangle DEF$ equal $MA, MB, MC.$ Let X, Y, Z be the projections of M on $BC, CA, AB.$ Then the right $\triangle MDX, \triangle MEY, \triangle MFZ$ are similar with $\angle DMX=\angle EMY=\angle FMZ=30^\circ \Longrightarrow \triangle DEF$ and $\triangle XYZ$ are spirally similar with center $M,$ rotational angle 30° and factor $\frac{MD}{MX}=\frac{2}{\sqrt{3}}.$ Thus

$$[DEF]=\left(\frac{2}{\sqrt{3}}\right)^2[XYZ]=\frac{4}{3}\cdot[XYZ].$$

If R denotes the circumradius of $\triangle ABC,$ then by Euler's theorem for the pedal triangle $\triangle XYZ$ of $M,$ we have

$$[XYZ]=[ABC]\cdot\frac{|R^2-d^2|}{4R^2}=\frac{\sqrt{3}}{4}a^2\cdot\frac{|\frac{1}{3}a^2-d^2|}{4\cdot\frac{1}{3}a^2}=\frac{\sqrt{3}}{16}|a^2-3d^2| \Longrightarrow$$

$$[DEF]=\frac{4}{3}\cdot\frac{\sqrt{3}}{16}|a^2-3d^2|=\frac{\sqrt{3}}{12}|a^2-3d^2|.$$

duanby

#3 Oct 19, 2012, 7:54 pm

Let XYZ be the feet of M then XY=sqrt3/2MC then the area of the triangle is 4/3 of the triangle XYZ the according to the Euler theorem the area of XYZ is1/4SABC*(d^2-r^2)

underzero

#4 Oct 20, 2012, 7:23 pm

You can use of analytic geometry for prove your problem.

Quick Reply

High School Olympiads

Prove $|AP|=|AQ|$

Reply

monang1993

#1 Aug 10, 2012, 11:38 am

Given an acute triange ABC. Points M and N in the segment AB and AC such that $|BM|=|CN|$. The line joinig the circumcentre of triangle ABN and ACM intersect AB and AC at P and Q (P,Q are different with M,N). Prove that $|AP|=|AQ|$

Sorry for my bad english

yetti

#2 Aug 10, 2012, 1:34 pm • 1

Circles $\odot(ACM), \odot(ABN)$ meet at A and again at $X \Longrightarrow PQ \perp AX$. From cyclic $ACXM$, $\angle XMB = \angle XCA = \angle XCN$ and from cyclic $ABXN$, $\angle XNC = \angle XBA = \angle XBM$.
Combined with $|BM| = |CN| \Longrightarrow \triangle XMB \cong \triangle XCN$ are congruent by ASA $\Longrightarrow$ their X-altitudes are congruent $\Longrightarrow$ AX internally bisects $\angle CAB \Longrightarrow |AP| = |AQ|$.

sunken rock

#3 Aug 12, 2012, 1:06 pm

Nice solution, thanks!
Additional requirement (easy):

the circumcenters of $\triangle ABC, \triangle AMN, \triangle ABN, \triangle ACM$ are the vertices of a rhombus.

Best regards,
sunken rock

Luis González

#4 Oct 17, 2012, 3:19 am

The P.P was posted before at Line through circumcentres forms an isosceles triangle.

As for sunken rock's remark, let O, O_1, O_2, O_3 be the circumcenters of $\triangle ABC, \triangle ABN, \triangle AMN, \triangle ACM$, respectively. Clearly $OO_1O_2O_3$ is a parallelogram, since OO_1, O_2O_3 are both perpendicular to AB and OO_3, O_1O_2 are both perpendicular to AC. Angle bisector AI of $\angle BAC$ is radical axis of $(O_1), (O_3) \Longrightarrow AI \perp O_1O_3 \Longrightarrow$ $\angle OO_3O_1 = \angle CAI = \frac{1}{2}\angle BAC$. Similarly, $\angle OO_1O_3 = \frac{1}{2}\angle BAC \Longrightarrow \triangle OO_1O_3$ isosceles with apex $O \Longrightarrow$ $OO_1O_2O_3$ is a rhombus.

Quick Reply

High School Olympiads

prove that angle AOC + angle MOB = 180

Reply

minhtue0605

#1 Oct 15, 2012, 4:49 pm

Let be given triangle ABC with $AB = AC$. O is a point inside the triangle such that $\widehat{OBA} = \widehat{OCB}$ and $\widehat{OCA} = \widehat{OBC}$. If M is the midpoint of BC, prove that $\widehat{AOC} + \widehat{MOB} = 180^o$.

Farenhajt

#2 Oct 15, 2012, 9:12 pm

See the attached diagram.

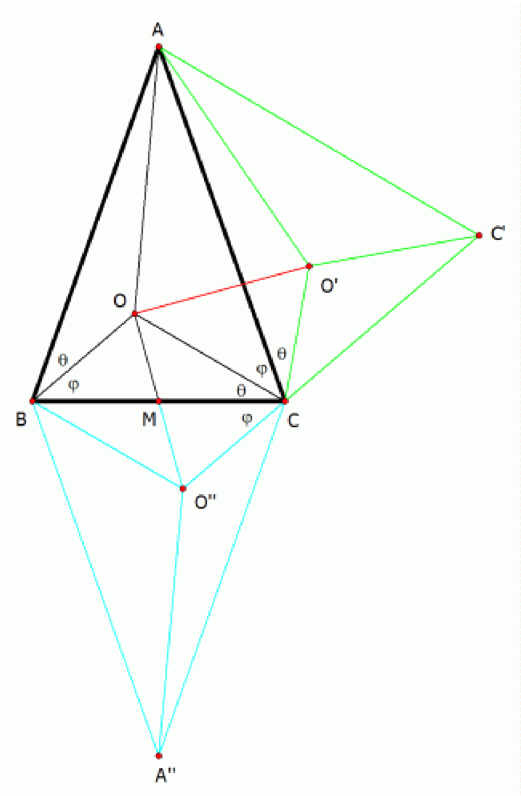
Apply two rotations of $\triangle ABC$ - around A for $\angle BAC$ and around M for 180°. Then
$\angle OAO' = \angle BAC \wedge AO = AO' \implies \triangle AOO' \sim \triangle ABC$, hence
$AO : OO' = AC : CB \iff AO : AC = OO' : CB \quad (1)$

Also, $CO' = CO'' = BO$ and $\angle OCO' = \angle OCO'' = \angle ABC$, hence $\triangle OCO' \cong \triangle OCO'' \implies OO' = OO''$.

Therefore $OO' : CB = OO'' : CB = \dfrac{OO''}{2} : \dfrac{CB}{2} = MO : MB \quad (2)$

Combining (1) and (2) we get $AO : AC = MO : MB$. Interpreting this using the Sine Law in $\triangle AOC$ and $\triangle BOM$, we get $\dfrac{\sin\phi}{\sin\angle AOC} = \dfrac{\sin\phi}{\sin\angle BOM} \iff \sin\angle AOC = \sin\angle BOM$ and the conclusion follows.

Attachments:



Luis González

#3 Oct 16, 2012, 11:30 am

$\angle OBA = \angle OCB$ and $\angle OCA = \angle OBC$ imply that the circumcircle of $\triangle OBC$ is tangent to AB, AC through $B, C \Longrightarrow$ OA is O-symmedian of $\triangle OBC$, which is the isogonal of its O-median OM with respect to $\angle BOC$. If OA cuts BC at P, we have then $\angle MOB = \angle POC = 180^\circ - \angle AOC$.

P.S. See also the topic isosceles triangle for an alternate formulation.

Quick Reply

High School Olympiads

Orthocenter property

Source: mpdb

borislav_mirchev
#1 Oct 15, 2012, 5:51 am • 3

Let aqute-angled triangle ABC with orhtocenter H is inscribed in circle k. Through H is drawn line l. M and N are the intersection points of l with the sides AC and BC. k' is a circle through M, N and C. P is the second intersection point of k and k'. Through P and H is drawn line intersecting k' at the point Q. Prove that $MN \perp CQ$.

This problem is dedicated to Luis González with many thanks for all his beautiful solutions.

Luis González
#2 Oct 16, 2012, 8:25 am • 4

Thanks for your dedication, dear Borislav.

AH, BH cut k again at the reflections X, Y of H across $BC, CA \Longrightarrow S \equiv XN \cap YM$ is the anti-Steiner point of l lying on k. Thus, $\odot(HMY), \odot(HNX)$ and k concur at the Miquel point P^* of HXY WRT $\triangle SMN$. Then $\angle HP^*N = \angle AXS$ and $\angle HP^*M = \angle BYS \Longrightarrow \angle MP^*N = \angle ACB \Longrightarrow P^* \in k'$, i.e. $P \equiv P^*$. Hence $\angle MCQ = \angle MPQ = \angle MYH = 90^\circ - \angle CMN \Longrightarrow CQ \perp MN$, as desired.

borislav_mirchev
#3 Oct 18, 2012, 8:59 pm

The problem is solved in extremely beautiful way here: http://www.math10.com/f/viewtopic.php?f=49&t=10963

armpist
#4 Oct 20, 2012, 1:41 am • 1

Dear Borislav and MLs

Attachments:
borislav.doc (28kb)

borislav_mirchev
#5 Oct 20, 2012, 2:17 am

It is a nice sketch. Can you provide more detailed description? As I understand you introduce two new circles and different additional constructions and use inscribed angles properties to solve the problem.

armpist
#6 Oct 20, 2012, 3:23 am

Dear Borislav,

It's supposed to be a PWW.

Any additional detail (if necessary, as it is in your case)
has to come from viewers like you.

Regards,
M.T.

borislav_mirchev
#7 Oct 20, 2012, 1:17 pm

Even with the hints given on the picture - it is not easy to solve the problem. At least I'm not able to solve it. Indeed - the proof written on the Bulgarian site is totally different. It seems that I'm not the only person that cannot solve the problem using constructions from the picture given.
May you give more details. If someone can understand it - it will be interesting to post more details because I think it is a creative approach to solve the problem.

borislav_mirchev
#8 Oct 26, 2012, 10:35 pm

The following problem is harder than the orthocenter problem and can be solved with idea similar to one of the IMO 2007 shortlist problems.
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=502271
If someone understand the picture without words, please dechipher it.

jayme
#9 Apr 20, 2015, 5:02 pm

Dear Mathlinkers,
for the original problem of Borislav Mirchev

1. X the second point of intersection of AH and (O)
(1) the circumcircle of MNV
(2) the circumcircle of HNX
R the second point of intersection of NB and (2)

2. X being the symmetric of H wrt BC, R is the antipole of N wrt (2).
3. according to the pivot theorem applied to HAM with points X, C, N, (2) goes through P
4. according to the Reim's theorem applied to (1) and (2), CQ // RH

and we are done…

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Perpendicular

Reply

MinatoF
#1 Oct 15, 2012, 11:25 am

Given a triangle ABC . M is midpoint of BC . S,T is points on BM,CM ,respectively that $SM=MT$. P,Q is point on AT,AS ,respectively that $\angle BAS=\angle PST$ and $\angle CAT=\angle QTS$. BQ,CP meet at R . Prove that $RM\perp BC$

Luis González
#2 Oct 16, 2012, 2:52 am • 1

This was posted before in the topic Lost problem with a synthetic solution.

For convenience, interchange $S\equiv B$ and $T\equiv C$. Thus, we prove that $R\equiv TP\cap SQ$ is on the perpendicular bisector of $\overline{BC}$. We use barycentric coordinates WRT $\triangle ABC$. $S(0:1:k)$ and $T(0:k:1)$ for some $k\in\mathbb{R}, S,T\neq B,C$. Equations of the circumcircles $\mathcal{K}_B$ and $\mathcal{K}_C$ of $\triangle ABS$ and $\triangle ACT$ are then

$$\mathcal{K}_B\equiv a^2yz+b^2zx+c^2xy-\frac{a^2z(x+y+z)}{k+1}=0$$

$$\mathcal{K}_C\equiv a^2yz+b^2zx+c^2xy-\frac{a^2y(x+y+z)}{k+1}=0$$

Equations of tangents τ_B,τ_C of $\mathcal{K}_B,\mathcal{K}_C$ at B,C are

$$\tau_B=c^2(k+1)x+ka^2z=0\ ,\ \tau_C\equiv b^2(k+1)+ka^2y=0\Longrightarrow$$

$$P\equiv(a^2k:0:-c^2(k+1)),\ Q\equiv(a^2k:-b^2(k+1):0)\Longrightarrow$$

$$TP\equiv c^2(k+1)x-a^2y+a^2kz=0\ ,\ SQ\equiv b^2(k+1)x+a^2ky-a^2z=0$$

$$R\equiv TP\cap SQ\equiv(a^2(1-k):kb^2+c^2:b^2+kc^2).$$

Eliminating k from the coordinates of R, we get $(b^2-c^2)x+a^2(y-z)=0$, which is presicely the equation of the perpendicular bisector of $\overline{BC}$, i.e. $RM\perp BC$.

Quick Reply

High School Olympiads

Find the ratio of two line segments

Reply

Source: 2012 China Second Round

littletush
#1 Oct 14, 2012, 5:14 pm

Let F be the focus of parabola $y^2=2px(p>0)$, with directrix l and two points A,B on it. Knowing that $\angle AFB=\frac{\pi}{3}$, find the maximal value of $\frac{|MN|}{|AB|}$, where M is the midpoint of AB and N is the projection of M to l.

Luis González
#2 Oct 15, 2012, 9:36 am

Let X,Y be the orthogonal projections of A,B on the directrix $\ell.$ By definition of parabola, we have $AX=FA$ and $BY=FB.$ MN is median of the right trapezoid $AXYB \Longrightarrow MN=\tfrac{1}{2}(AX+BY)=\tfrac{1}{2}(FA+FB).$ Thus

$$\left(\frac{MN}{AB}\right)^2=\frac{\frac{1}{4}(FA+FB)^2}{FA^2+FB^2-2\cdot FA\cdot FB\cdot \cos\frac{\pi}{3}}=\frac{FA^2+FB^2+2\cdot FA\cdot FB}{4(FA^2+FB^2-FA\cdot FB)}.$$

By RMS-GM we have $FA^2+FB^2 \ge 2\cdot FA\cdot FB,$ which can be written in the form

$$FA^2+FB^2+2\cdot FA\cdot FB \le 4(FA^2+FB^2-FA\cdot FB).$$

Therefore $\frac{MN}{AB} \le 1.$ Equality holds when $FA=FB,$ i.e. $\triangle FAB$ is equilateral.

Quick Reply

High School Olympiads

Geometric Inequality (2)

Reply

MinatoF

#1 Oct 10, 2012, 3:00 pm

Let ABC be an acute triangle, γ its circumcircle, O its circumcenter and let a, b, c be the lenghts of BC, CA and AB respectively. Let AQ and PB be the diameters of γ and let Z be the intersection of AP and QC. Prove that $OZ \geq \frac{bc}{a}$

Luis González

#2 Oct 15, 2012, 7:27 am • 1

$OA = OQ = R$ is the circumradius of $\triangle ABC$. M is the midpoint of BC and D is the foot of the A-altitude. Since $\angle PAQ = \angle OBC + \angle OAC = \angle ACB$ and $\angle CQA = \angle CBA$, then $\triangle ABC$ and $\triangle ZQA$ are similar with corresponding medians AM and $ZO \Longrightarrow \frac{OZ}{AM} = \frac{2R}{BC}$. Substituting OZ into the requested inequality, we get

$$\frac{2R \cdot AM}{BC} \geq \frac{AB \cdot AC}{BC} \Longrightarrow AM \geq \frac{AB \cdot AC}{2R} = AD.$$

Which is clearly true. So equality holds when $AD = AM$, i.e. $\triangle ABC$ is A-isosceles.

Quick Reply

High School Olympiads

centroid

Reply

elegant

#1 Oct 7, 2012, 5:02 pm

Let T be the centroid, and O arbitrary point inside ΔABC. If A_1, B_1, C_1 are intersections of OT with BC, CA, and AB, respectively, prove that

$$OA_1 \cdot OB_1 \cdot OC_1 \leq TA_1 \cdot TB_1 \cdot TC_1$$

[Mod edit: LaTeXified]

Luis González

#2 Oct 14, 2012, 4:05 am

OA and TA cut BC at O_1 and its midpoint D. By Menelaus' theorem for $\triangle AOT$, cut by BC, we get

$$\frac{OA_1}{TA_1} = \frac{DA}{TD} \cdot \frac{OO_1}{AO_1} = 3 \cdot \frac{OO_1}{AO_1} = 3 \cdot \frac{[OBC]}{[ABC]} = \frac{3[OBC]}{[OBC] + [OCA] + [OAB]}.$$

Analogously, we have the expressions

$$\frac{OB_1}{TB_1} = \frac{3[OCA]}{[OBC] + [OCA] + [OAB]}, \quad \frac{OC_1}{TC_1} = \frac{3[OAB]}{[OBC] + [OCA] + [OAB]}$$

Hence, the requested inequality is equivalent to

$$\frac{[OBC] + [OCA] + [OAB]}{3} \geq \sqrt[3]{[OBC] \cdot [OCA] \cdot [OAB]}.$$

Which is just AM-GM on the positive quantities $[OBC], [OCA], [OAB]$.

Quick Reply

High School Olympiads

Heights and circles

Reply

Source: mpdb

borislav_mirchev

#1 Sep 9, 2012, 7:13 pm

In the acute-angled triangle ABC - H_A, H_B, H_C are feets of the peprendiculars from the vertices A, B, C to the corresponding sides respectively. A_1 and A_2 are the feets of the perpendiculars from H_A to BH_B and CH_C respectively. B_1 and B_2 are feets of the perpendiculars from H_B to CH_C and AH_A respectively. C_1 and C_2 are the feets of the perpendiculars from H_C to AH_A and BH_B respectively. Prove that

$a)$ If A' and B' are the feets of the perpendiculars from H_A and H_B to AB then points $A', B', A_1, B_2, C_1, C_2$ are concylic.

$b)$ If $k_a(O_A)$ is the circle through A_1, A_2, B_1, C_2; $k_b(O_B)$ is the circle through B_1, B_2, C_1, A_2; $k_c(O_C)$ is the circle through C_1, C_2, A_1, B_2; then circumradii of the triangles $O_AO_BO_C$ and $H_AH_BH_C$ are equal.

Click to reveal hidden text

Luis González

#2 Oct 14, 2012, 1:18 am • 1

ABH_AH_B and $H_AH_BB_2A_1$ are cyclic due to the right angles at H_A, H_B and $A_1, B_2 \Longrightarrow \angle H_AB_2A_1 = \angle H_AH_BB = \angle H_AAB \Longrightarrow AB \parallel A_1B_2$. BH_AA_1A' is cyclic due to the right angles at $A_1, A' \Longrightarrow \angle AA'A_1 = \angle BH_AA_1 = \angle ACB$. Similarly, $\angle BB'B_2 = \angle ACB \Longrightarrow \angle AA'A_1 = \angle BB'B_2 \Longrightarrow A'B'B_2A_1$ is an isosceles trapezoid. $A'H_AC_1H_C$ is cyclic due to the right angles at $A', C_1 \Longrightarrow \angle A'C_1B_2 = \angle AH_CH_A = \angle AB'B_2 \Longrightarrow C_1$ is on the circumcircle of $A'B'B_2A_1$. Likewise, C_2 is on the circumcircle of $A'B'B_2A_1$. Hence, $A', B', A_1, B_2, C_1, C_2$ are on a same circle (O_C).

If H is the orthocenter of $\triangle ABC$, then $(O_A), (O_B), (O_C)$ are the Taylor circles of $\triangle HBC, \triangle HCA, \triangle ABC$, respectively. Thus if A_0, B_0, C_0 denote the midpoints of H_BH_C, H_CH_A, H_AH_B, then O_A, O_B, O_C are the excenter of $\triangle A_0B_0C_0$ opposite to $A_0, B_0, C_0 \Longrightarrow \triangle A_0B_0C_0$ becomes orthic triangle of $\triangle O_AO_BO_C \Longrightarrow$ circumradius of $\triangle O_AO_BO_C$ is twice the circumradius of $\triangle A_0B_0C_0 \Longrightarrow \odot(O_AO_BO_C)$ is congruent to $\odot(H_AH_BH_C)$. Moreover, $\triangle O_AO_BO_C$ is the image of the medial triangle of $\triangle ABC$ under the translation defined by $\overrightarrow{X_3X_{389}}$.

Quick Reply

High School Olympiads

TWO circles

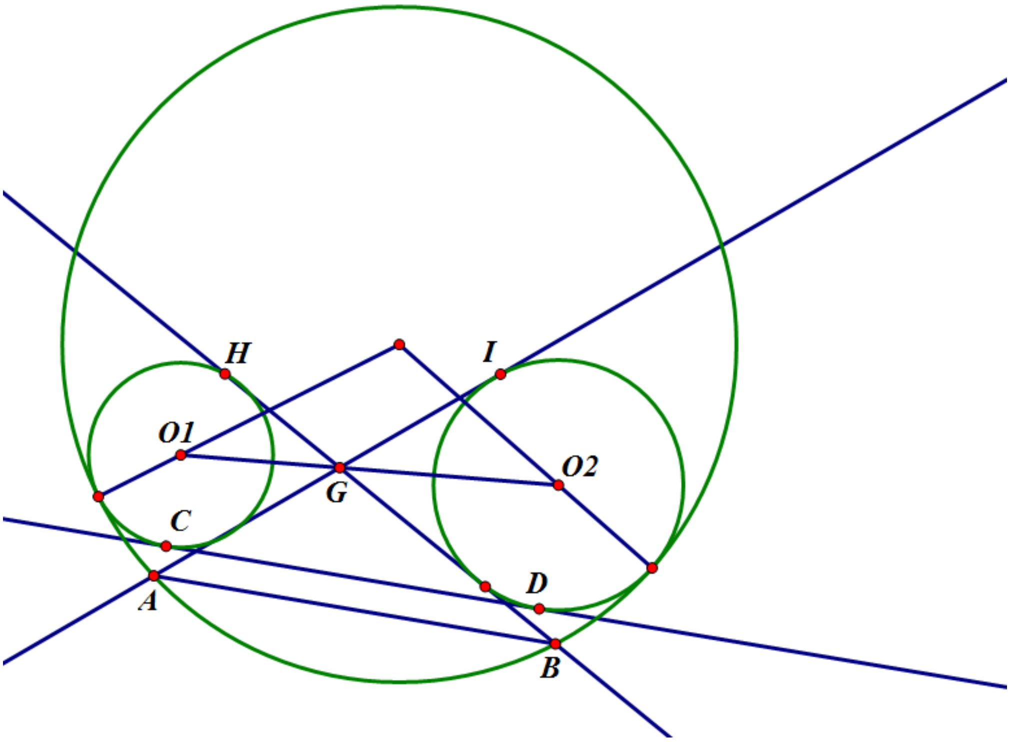
Locked

DANNY123

#1 Oct 13, 2012, 11:09 pm

Two circles, $(O_1), (O_2)$ inscribed in a big circle (O). AI, HB are internal tangent lines and CD is external tangent line. Prove $CD \parallel (AB)$.

Attachments:



Luis González

#2 Oct 13, 2012, 11:31 pm

Discussed many times before, it's the "infamous" Parallel tangent theorem. Thus for further discussions use any of the links below. This topic is locked.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=15945
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=247604
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=310017
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=430441
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=463503

High School Olympiads

Circles, Parallelogram

Source: Solve it

vikan94

#1 Oct 13, 2012, 8:55 pm

A circle is inscribed in $ABCD$, and AB is parallel to CD, $BC = AD$.
The diagonals AC and BD intersect at E. The circles inscribed in $\triangle ABE, \triangle BCE, \triangle CDE, \triangle DAE$ have radius r_1, r_2, r_3, r_4 respectively. Show that: $\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}$

Luis González

#2 Oct 13, 2012, 10:51 pm

These conditions are not necessary for the relation; ABCD can be an arbitrary tangential quadrilateral. See the links below

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=350695
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=373855
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=376589

High School Olympiads

Hard problem: Two similar triangle

Reply

Source: own

physics_1995

#1 Oct 11, 2012, 11:41 am • 2

1. Let ABC be a triangle with orthocenter H.
2. E lies on AB,F lies on AC.
3. D, A are same side with BC such that $\triangle DBC \backsim \triangle HEF$.
4. EF cuts DH at M.
5. G is projection of B on CD.

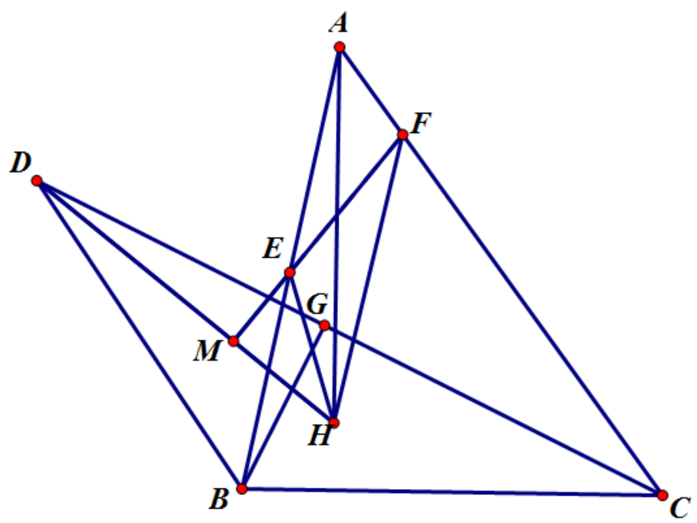
Prove that: $\triangle DMG \backsim \triangle HFA$.

Proposed by lugang_10

Click to reveal hidden text

♥ CQT ♥

Attachments:



Luis González

#2 Oct 13, 2012, 4:24 am • 2

$Y \equiv AC \cap BH$ and $Z \equiv AB \cap CH$ are the feet of the B- and C- altitude. $\odot(AEF)$ cuts AH again at P. Then $\angle PEF = \angle PAC = \angle HBC$. Similarly $\angle PFE = \angle HCB \Longrightarrow \triangle HEF \cup P \sim \triangle DBC \cup H \Longrightarrow$ $\angle HDC = \angle PHF$. So, it suffices to show that $\angle DGM = \angle HAC$.

$\angle DHZ = \angle DHB - \angle ZHB = \angle HPE - \angle EAF = \angle APF = \angle AEF \Longrightarrow HEMZ$ is cylic $\Longrightarrow$ $\angle HME = \angle HZE = 90^\circ$. $HYFM$ and $BCYG$ are cyclic, due to right angles at M, Y and $Y, G \Longrightarrow$ $\angle GYH = \angle GCB = \angle MFH \Longrightarrow G \in MY$. Thus $\angle DGM = \angle CGY = \angle CBY = \angle HAC$, as desired.

Quick Reply

High School Olympiads

Prove BQ = AQ + CQ

Reply

FoolMath

#1 Oct 12, 2012, 3:29 pm

Let ABC be an acute triangle with $\angle A > \angle B > \angle C$. (O) and (I) are circumcirrcle and incircle of ABC, respectively. AI meet (O) at M. N is midpoint of BC. E is a point what symmetry with N. ME meet (O) at Q. Prove that:
$BQ = AQ + CQ$

JustN

#2 Oct 12, 2012, 7:12 pm

With the respect to what E is symmetry with N ?

Luis González

#3 Oct 12, 2012, 9:59 pm

It's a problem from Vietnam Northern Summer Camp of Mathematics, 2011 (P3). Amir submitted it before and the enuntiation had exactly the same problem describing the point N, but livetolove212 posted the problem before in its correct form, so I'm sure E is reflection of I about N.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=323883
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=424438

Quick Reply

High School Olympiads

Geometic Inequality

Reply

MinatoF

#1 Oct 10, 2012, 2:57 pm

Let ABC be a triangle with $\angle A < 60^\circ$. Let X and Y be the points on the sides AB and AC respectively, such that $CA + AX = CB + BX$ snd $BA + AY = BC + CY$. Let P be the point in the plane such that the lines PX and PY are perpendicular to AB and AC respectively. Prove that $\angle BPC < 120^\circ$

Luis González

#2 Oct 12, 2012, 11:29 am • 1

Let I, O be the incenter and circumcenter of $\triangle ABC$. If $\angle A < 60^\circ$, then $2\angle A < 90^\circ + \frac{1}{2}\angle A \Longrightarrow \angle BOC < \angle BIC \Longrightarrow$ I is inside of $\odot(BOC)$.

$(I_a), (I_b), (I_c)$ are the 3 excircles against $A, B, C. CA + AX = CB + BX \Longleftrightarrow X$ is the tangency point of (I_c) with AB. Similary, Y is the tangency point of (I_b) with AC. Since $\triangle ABC$ is orthic triangle of $\triangle I_aI_bI_c$, then intersection P of the perpendiculars from I_b, I_c to CA, AB is the circumcenter of $\triangle I_aI_bI_c \Longrightarrow P$ is reflection of orthocenter I of $\triangle I_aI_bI_c$ about its 9-point center O. Since I is inside of $\odot(BOC)$, then P is outside of it $\Longrightarrow \angle BPC < 2\angle A < 120^\circ$.

Quick Reply

High School Olympiads

Mongolia TST 2011

Reply

MinatoF

#1 Oct 10, 2012, 2:30 pm

Point P,Q and X are chosen on the arc segments AB,BC,CA of the circumcircle ω of triangle ABC respectively. Let P' and Q' be points on the half-lines $[PX)$ and $[QX)$ such that $PB+PA=PP'$ and $QB+QC=QQ'$. Suppose that the circumcircle of triangle $XP'Q'$ meets ω at point R. Prove that R is a constant point that doesn't depend on the location of X on arc CA.

Luis González

#2 Oct 12, 2012, 3:22 am • 1

> MinatoF wrote:
> Let P' and Q' be points on the half-lines $[PX)$ and $[QX)$ such that $PB+PA=PP'$ and $QB+QC=QQ'$.

These conditions are just distracting, we only need two points P',Q' on $PX,QX,$ such that PP' and QQ' are constant. 2nd intersection R of $\odot(XPQ)$ and $\odot(XP'Q')$ is center of spiral similarity that swaps $\overline{PP'}$ and $\overline{QQ'} \Longrightarrow \frac{RP}{RQ}=\frac{PP'}{QQ'}=\text{const}$ $\Longrightarrow R$ is a fixed point on $\omega.$

Assuming that R lies on the arc $PXQ,$ then R is the second intersection of ω with the line joining the midpoint of the arc PBQ and the insimilicenter of the circles (P,PP') and $(Q,QQ').$

Quick Reply

High School Olympiads

Geometric Inequality (3)

Reply

MinatoF

#1 Oct 10, 2012, 3:03 pm

Let M, N and P be points on the sides AB, BC and CA of $\triangle ABC$, respectively. The lines through M, N and P, parallel to BC, CA and AB, respectively, meet at a point T. Prove that

$$[MNP] \le \frac{1}{3}[ABC]$$

Luis González

#2 Oct 11, 2012, 12:26 pm • 1

Project $\triangle ABC$ into an equilateral triangle $\triangle A'B'C'$ through parallel projection. M, N, P go to M', N', P' on $A'B', B'C', C'A'$ and $T'M', T'N', T'P'$ remain parallel to $B'C', C'A', A'B'$. The ratio of the areas of $\triangle ABC$ and $\triangle MNP$ is invariant.

For convenience, get rid of the primes in the projected figure. Let X, Y, Z be the projections of T on BC, CA, AB. The right $\triangle TNX, \triangle TPY$ and $\triangle TMZ$ are similar with $\angle NTX = \angle PTY = \angle MTZ = 30^\circ \Longrightarrow \triangle NPM$ and $\triangle XYZ$ are spirally similar with center T, rotational angle 30° and factor $\frac{TN}{TX} = \frac{2}{\sqrt{3}}$. Thus, if (K, ϱ) denotes the circumcircle of the equilateral $\triangle ABC$, we have

$$[MNP] = \left(\frac{2}{\sqrt{3}}\right)^2 [XYZ] = \frac{4}{3}[XYZ] = \frac{4}{3} \cdot \frac{\varrho^2 - TK^2}{4\varrho^2}[ABC] \Longrightarrow$$

$$\frac{[MNP]}{[ABC]} = \frac{\varrho^2 - TK^2}{3\varrho^2} \le \frac{1}{3}.$$

Tsikaloudakis

#3 Oct 11, 2012, 1:04 pm • 1

Let:

$\upsilon = d(A, BC)$, $ME//BC$, $DN//AC$, $PF//AB$

and

$MH//DN$, $DL//EM$, $AK \perp DL$, $\mathrm{DS} \perp ME$, $\mathrm{M\Sigma} \perp \mathrm{BC}$

and $M\Sigma \geqslant DS$, AK . Then , $\upsilon \leqslant 3\mathrm{M\Sigma}$.

So we have:

$$(MNP) = (TMP) + (TMN) + (TNP) =$$
$$\frac{1}{2}((TDAP) + (TMBF) + (TECN))$$

and

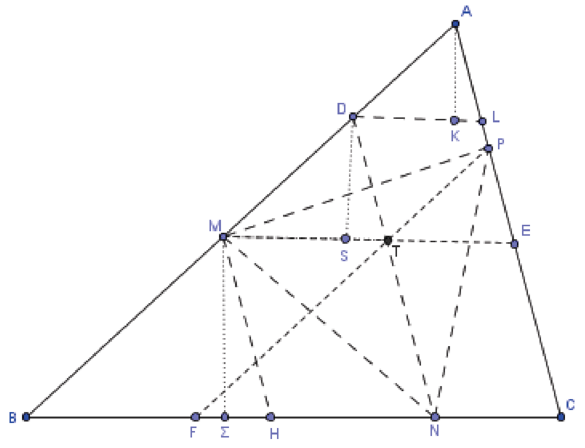
$$(MNP) \leqslant \frac{1}{3}(ABC) \Leftrightarrow \frac{1}{2}[(TDAP) + (TMBF) + (TNCE)] \leqslant \frac{1}{3}(ABC) \Leftrightarrow$$
$$\frac{1}{2}[(ABC) - (DAL) - (TMD) - (BHM)] \leqslant \frac{1}{3}(ABC) \Leftrightarrow$$

$$(ABC) \leqslant 3[(DAL) + (TMD) + (BHM)] \quad (1)$$

(1) is true because:

$$v \leqslant 3M\Sigma \Rightarrow$$
$$(ABC) \leqslant 3\left[(DAL) + (TDM) + BHM\right]$$

Attachments:



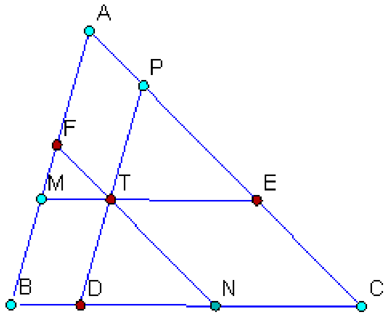
vslmat

#4 Oct 12, 2012, 5:00 pm • 1

Let TP cut BC at point D
As $\frac{[TPN]}{[TDN]} = \frac{TP}{TD}$ and $\frac{[TDN]}{[ABC]} = \frac{TD.TN}{AB.AC}$,we have $\frac{[TPN]}{[ABC]} = \frac{TP.TN}{AB.AC}$
Similarly with $[TMN]$ and $[TMP]$, the given inequality is equivalent to:

$\frac{TP.TN}{AB.AC} + \frac{TP.TM}{AB.BC} + \frac{TM.TN}{BC.AC} \le \frac{1}{3}$ But as $3.LS \le (\frac{TP}{AB} + \frac{TN}{AC} + \frac{TM}{BC})^2$
And it is well known and can be simply proved using similarity of triangles that $\frac{TP}{AB} + \frac{TN}{AC} + \frac{TM}{BC} = 1$ we get $LS \le \frac{1}{3}$ q.e.d.

Attachments:



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High School Olympiads

DM perpendicular BC

Reply

Source: own

physics_1995

#1 Oct 10, 2012, 5:12 pm

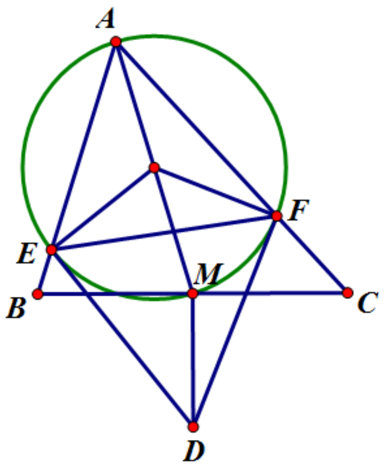
1. Let ABC be a triangle.
2. M is midpoint of BC.
3. (1) is circle with diameter AM.
4. (1) cuts AB, AC at E, F respectively.
5. Tangent at E of (1) cuts tangent at F of (1) at D.

Prove that: $DM \perp BC$.

Click to reveal hidden text

♥ CQT ♥

Attachments:



Luis González

#2 Oct 10, 2012, 9:22 pm • 1

It's a problem from Kolmogorov tournament (2003) and also Kazakhstan BMO TST (2007). Posted many times before.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=16779
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=23765
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=202517
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=253726
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=313997

phuongtheong

#3 Oct 14, 2012, 7:25 am • 2

I have a simple solution of this problem:
G is on (1) such that $AG||BC$. Because M is the midpoint of BC so $A(GMCB) = -1 \Rightarrow EGFM$ is a harmonic quadrilateral. So G, M, D are collinear. And because $AG \perp GM$ so $DM \perp BC$.

Virgil Nicula

#4 Oct 15, 2012, 4:28 pm • 1

PP. Let $\triangle ABC$. Denote the midpoint M of $[BC]$, the point $E \in AB$ so that $ME \perp AB$, the point $F \in AC$ so that

PP. Let $\triangle ABC$. Denote the midpoint M of $[BC]$, the point $E \in AB$ so that $ME \perp AB$, the point $F \in AC$ so that $MF \perp AC$

and the intersection $D \in EE \cap FF$ of the tangent lines at E , F to the circle with the diameter $[AM]$. Prove that $DM \perp BC$.

Proof. Let L be a point for which $LA \parallel BC$ and $LM \perp BC$. Denote $AM = m_a$. Observe that $[MAB] = [MAC] \implies$

$$\boxed{\frac{c}{MF} = \frac{b}{ME}}\ (*) \text{ and } \left\{\begin{array}{c} [ABL] = [AML] = [ACL] \\ \\ \widehat{AEL} \equiv \widehat{AML} \equiv \widehat{AFL} \end{array}\right\| \implies c \cdot LE = m_a h_a = b \cdot LF \stackrel{(*)}{\implies}$$

$MF \cdot LE = ME \cdot LF$, i.e. $LEMF$ is an harmonic cyclic quadrilateral. From an well-known property obtain that

$D \in LM$ - the bisector of the side $[BC]$, i.e. $DB = DC \iff DM \perp BC$. I was inspired by **phuongtheong**. Thank you.

See **PP1** from **here**.

This post has been edited 4 times. Last edited by Virgil Nicula, Nov 16, 2012, 3:00 pm

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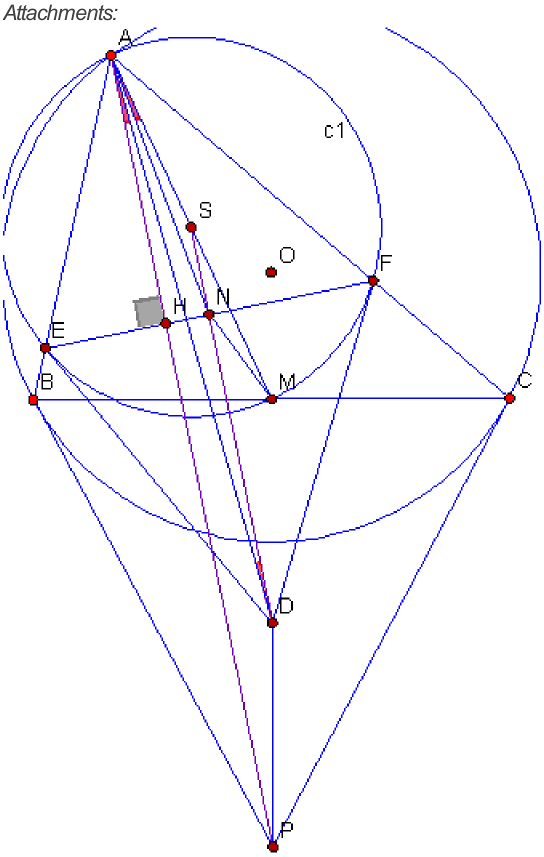
vslmat

#5 Oct 18, 2012, 7:33 pm • 1

Another proof:
Let the tangents at B and C to the circumcirle of ABC meet at P. Let S be the midpoint of AM, i.e. the circumcenter of AEF and N the midpoint of EF.
As AD is the A-symmedian in ΔAEF and AP the A-symmedian in ΔABC, it is easy to show that $\angle PAD = \angle NAM$.
But easy to see that $\angle NAS = \angle ADS$ (both equals $90^\circ + \angle NAF - \angle EAF - \angle AFE$), hence $AP \parallel SD$
Now notice that $\frac{AM}{sin\angle ACB} = \frac{MC}{sin\angle MAC}$ and $\frac{AP}{sin(\angle ABC + \angle BAC)} = \frac{BP}{sin\angle BAP}$ and $\angle MAC = \angle BAP$
thus $\frac{AM}{AP} = \frac{MC}{BP} = cos\angle BAC = cos\angle DSF = \frac{SF}{SD} = \frac{AM}{2.SD}$. This means that $AP = 2.SD$, therefore, P, D, M are collinear and $MD \perp BC$. Moreover, we have shown that D is the midpoint of PM

Attachments:



Quick Reply

High School Olympiads

Constant

Reply

MinatoF

#1 Oct 10, 2012, 2:52 pm

On the plane given a circle (O) and a point A lying outside the circle. Through A, draw the tangents to (O); let B and C be the tangent points. Consider a point P moving on the opposite ray to ray AB and a point Q moving on the opposite ray to ray CA, such that the line PQ is tangents to (O). The line BC intersects the line passing through P, parallel with AC ,in E, and intersects the line passing through Q' parallel to AB, in F. Show that
(1) The line EQ passes through a fixed point, say M;the line FP passes through a fixed point, say N.
(2) The product $PM \cdot QN$ is constant

Luis González

#2 Oct 10, 2012, 8:34 pm • 1

This is problem 2 of Vietnam TST 2011. The line EQ always goes through the intersection M of AB with the perpendicular ℓ to AO through O and analogously, FP always goes through the intersection N of ℓ with AC. The product $PM \cdot QN$ is constant and equal to $OM^2 = ON^2$.

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=404134.

Quick Reply

High School Olympiads

2 Square

MinatoF

#1 Oct 10, 2012, 2:40 pm

Let $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ be two squares in opposite direction (that is, if the vertices A_1, B_1, C_1, D_1 are in clockwise order, then A_2, B_2, C_2, D_2 are ordered counterclockwise) with centers O_1, O_2 respectively. Suppose that D_2, D_1 are respectively in A_1B_1, A_2B_2. Prove that the lines B_1C_1, B_2C_2 and O_1O_2 are concurrent.

Luis González

#2 Oct 10, 2012, 8:09 pm • 1

I believe the problem asks to prove that the lines B_1B_2, C_1C_2 and O_1O_2 are concurrent. I remember solving this problem before, but unfortunately search is not working right now. So I'm writing it out again.

Note that B_1, B_2, D_1, D_2 lie on a same circle ω, because $\angle D_1B_1D_2 = \angle D_2B_2D_1 = 45^\circ$. If $P \equiv A_1C_1 \cap A_2C_2$, from $\angle PA_2D_1 = \angle PA_1D_1 = \angle PA_2D_2 = \angle PA_1D_2 = 45^\circ$, we deduce that P, A_1, D_1, D_2 are concyclic $\Longrightarrow$ $\angle D_1PD_2 = 90^\circ$ and $PD_1 = PD_2$, i.e. $\triangle PD_1D_2$ is isosceles right with apex $P \Longrightarrow P$ is the center of ω.

Let B_1C_1 and B_2C_2 cut ω again at X and Y. Since $\angle XB_2D_2 = \angle YB_1D_1 = 90^\circ$, then XD_2 and YD_1 are diameters of ω intersecting at P. By Pascal theorem for the cyclic hexagon $YD_1B_1XD_2B_2$, the intersections $P \equiv YD_1 \cap XD_2$, $M \equiv D_1B_1 \cap D_2B_2$ and $N \equiv XB_1 \cap YB_2$ are collinear $\Longrightarrow \triangle O_1B_1C_1$ and $\triangle O_2B_2C_2$ are perspective. Hence by Desargues theorem, O_1O_2, B_1B_2, C_1C_2 concur.

Quick Reply

High School Olympiads

Consider the triangles...

Reply

S.E.Louridas
#1 Oct 8, 2012, 12:14 pm

Consider the triangles ABC, PQR. Prove that there is a triangle KLM,
inscribed to triangle ABC (K, L, M are points on different sides) similar to PQR, such that the ratio $\frac{A_{KLM}}{A_{ABC}}$ (*), to be the least.
Can we have a constructive determination?

(*) *A*=area.

Luis González
#2 Oct 10, 2012, 5:18 am

Assume that $K \in BC, L \in CA$ and $M \in AB.$ It suffices to find the $\triangle KLM$ similar to $\triangle PQR$ with minimum area.

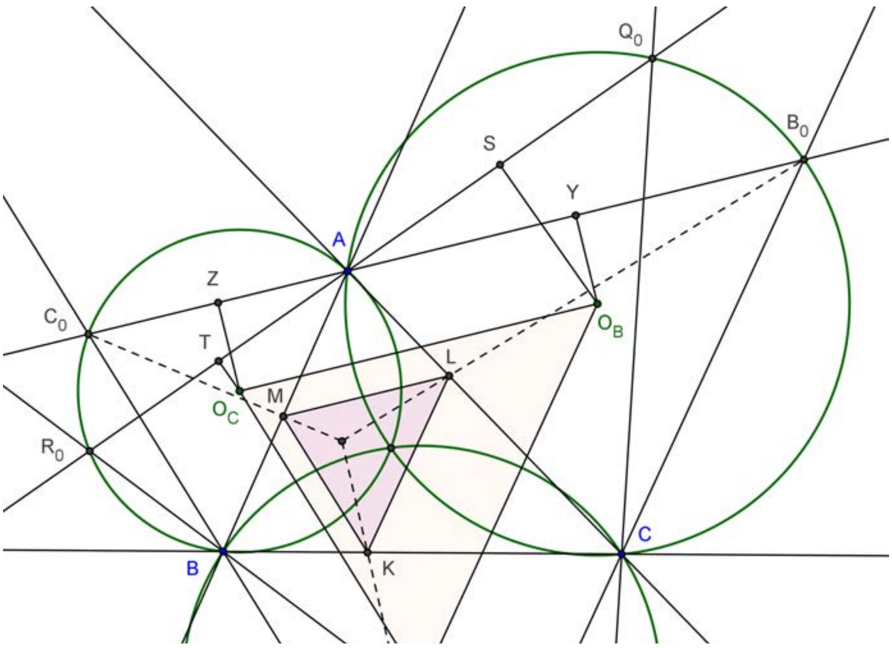
Construct the circles $(O_A), (O_B), (O_C)$ that see BC, CA, AB under the given angles $\angle P, \angle Q, \angle R$, respectively. $P_0 \in (O_A)$ is outside of $\triangle ABC. CP_0$ cuts (O_B) again at Q_0 and AQ_0 cuts (O_C) again at R_0. By Miquel theorem $B \in P_0R_0 \Longrightarrow \triangle P_0Q_0R_0 \sim \triangle PQR$ is circumscribed in $\triangle ABC$. If $\triangle K_0L_0M_0$ is the triangle inscribed in $\triangle ABC$ and homothetic to $\triangle P_0Q_0R_0$, then by Gergonne-Ann theorem, we have $[ABC]^2 = [P_0Q_0R_0] \cdot [K_0L_0M_0]$. Thus, $[K_0L_0M_0]$ is minimum $\Longleftrightarrow [P_0Q_0R_0]$ is maximum.

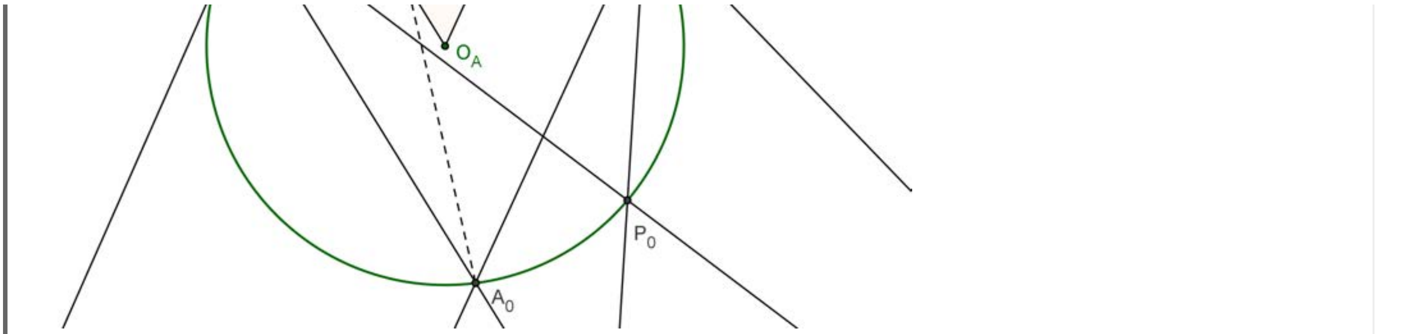
Parallel from A to O_BO_C cuts $(O_B), (O_C)$ again at B_0, C_0. CB_0 cuts (O_A) again at A_0. $\triangle A_0B_0C_0$ is circumscribed in $\triangle ABC$ and is homothetic to $\triangle O_AO_BO_C$. Among all the similar triangles $\triangle P_0Q_0R_0$ (eventually $\triangle P_0Q_0R_0 \equiv \triangle A_0B_0C_0$), it's clear that the $\triangle P_0Q_0R_0$ with maximum area is that with maximum side length Q_0R_0. Let Y, S denote the midpoints of AB_0, AQ_0 and Z, T denote the midpoints of AC_0, AR_0. YZO_CO_B is a rectangle and STO_CO_B is a right trapezoid at S, T.

$B_0C_0 = 2 \cdot YZ = 2 \cdot O_BO_C \geq 2 \cdot ST = Q_0R_0.$

This means that $\triangle A_0B_0C_0$ is the circumscribed triangle with maximum area. Thus, the $\triangle KLM$ with minimum area is that homothetic to $\triangle A_0B_0C_0$.

Attachments:





Quick Reply

High School Olympiads

A relation for a concurrence in a triangle.

Reply

Source: own

Virgil Nicula

#1 Oct 9, 2012, 2:22 pm

PP. Let $\triangle ABC$ and $E \in (AC)$, $F \in (AB)$ so that $\left\{\begin{array}{c} EA = m \cdot EC \\ \\ FA = n \cdot FB \end{array}\right\|$, where $\{m, n\} \subset \mathbb{R}_+^*$, $|m-n| < 1$.

Denote the midpoints M , N of $[BE]$, $[CF]$ respectively. Prove that $EN \cap FM \cap BC \ne \emptyset \iff m^2 + n^2 = mn + 1$.

Luis González

#2 Oct 9, 2012, 11:28 pm

We use barycentric coordinates WRT $\triangle ABC$. From $EC : EA = 1 : m$ and $FB : FA = 1 : n$, we obtain $E(1:0:m)$ and $F(1:n:0) \Longrightarrow M(1:1+m:m)$ and $N(1:n:1+n)$. Equations of lines EN and FM are then

$$EN \equiv mnx + (1+n-m)y - nz = 0$$

$$FM \equiv mnx - my + (1+m-n)z = 0$$

$$P \equiv EN \cap FM \equiv (1+mn-m^2-n^2 : -mn(1+m) : -mn(1+n))$$

$$EN \cap FM \cap BC \ne \emptyset \iff P \in BC \iff 1+mn-m^2-n^2 = 0.$$

sunken rock

#3 Oct 20, 2012, 1:59 am

Let $CE = b'$, $AE = mb'$, $BF = c'$, $AF = nc'$ and $X = EN \cap BC$.

By transversal theorem for $\overline{ENX}$ in $\triangle ABC$: $\frac{AE}{CE} \cdot BF + \frac{BX}{CX} \cdot AF = \frac{FN}{CN} \cdot AB$, or $m + n \cdot \frac{BX}{CX} = n + 1$.

By transversal theorem for $\overline{FMX}$ in the same triangle: $\frac{AF}{BF} \cdot CE + \frac{CX}{BX} \cdot AE = \frac{EM}{MB} \cdot AC$, or $n + m \cdot \frac{CX}{BX} = m + 1$.

The two equations are equivalent to $n \cdot \frac{BX}{CX} = 1 + n - m$ and $m \cdot \frac{CX}{BX} = 1 + m - n$.

Multiplying the last two relations we get the desired $m^2 + n^2 = mn + 1$.

Since the transversal (**Cristea's**) theorem is 'if and only if', our problem keeps this property as well.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Circumcenters on a circle

Reply

Source: own ?

MariusStanean

#1 Oct 8, 2012, 7:34 pm • 1

Let $ABCD$ an inscriptible quadrilateral and M, N be the midpoints of AC, BD respectively. Prove that the circumcenters of the triangles ABM, ABN, CDM, CDN are on a circle.

Luis González

#2 Oct 8, 2012, 11:45 pm • 1

Denote $(O_1),(O_2),(O_3),(O_4)$ the circumcircles of $\triangle MAB, \triangle NAB, \triangle MCD, \triangle NCD,$ respectively. Let $P \equiv AB \cap CD.$ Since $\overline{PA} \cdot \overline{PB} = \overline{PC} \cdot \overline{PD},$ we deduce that PM and PN are radical axes of $(O_1),(O_3)$ and $(O_2),(O_4) \Longrightarrow PM \perp O_1O_3$ and $PN \perp O_2O_4.$ Since $O_1O_2 \perp AB$ and $O_3O_4 \perp CD,$ then $\angle O_1O_2O_4 = \angle BPN$ and $\angle O_1O_3O_4 = \angle CPM.$ But $\triangle PAC$ and $\triangle PDB$ are similar with corresponding medians $PM, PN \Longrightarrow$ $\angle CPM = \angle BPN \Longrightarrow \angle O_1O_2O_4 = \angle O_1O_3O_4 \Longrightarrow O_1,O_2,O_3,O_4$ are concyclic.

Quick Reply

High School Olympiads

Elegant formulation, but is there an elegant solution?

yellowhanoi
#1 Oct 7, 2012, 7:44 pm

Let $ABCD$ be a convex quadrilateral, and M and N the midpoints of the sides CD and AD, respectively. The lines perpendicular to AB passing through M and to BC passing through N intersect at point P. Prove that if P is on the diagonal BD then the diagonals AC and BD are perpendicular.

Luis González
#2 Oct 8, 2012, 1:49 am • 1

Actually, we only need the condition $MN \parallel AC.$

Let H be the orthocenter of $\triangle ABC.$ PM, HC are both perpendicular to $AB \Longrightarrow PM \parallel HC$ and PN, HA are both perpendicular to $CB \Longrightarrow PN \parallel HA.$ $\triangle HAC$ and $\triangle PNM$ having parallel sides are homothetic with center $D \equiv AN \cap CM \Longrightarrow D, P, H$ are collinear $\Longrightarrow P$ is on B-altitude of $\triangle ABC \Longrightarrow BD \equiv BP \perp AC.$

yellowhanoi
#3 Oct 8, 2012, 2:55 am

Your solution is very elegant Mr. Luis Gonzalez, thanks!

auj
#4 Dec 17, 2015, 12:44 am

> *Luis González wrote:*
> Actually, we only need the condition $MN \parallel AC.$
>
> Let H be the orthocenter of $\triangle ABC.$ PM, HC are both perpendicular to $AB \Longrightarrow PM \parallel HC$ and PN, HA are both perpendicular to $CB \Longrightarrow PN \parallel HA.$ $\triangle HAC$ and $\triangle PNM$ having parallel sides are homothetic with center $D \equiv AN \cap CM \Longrightarrow D, P, H$ are collinear $\Longrightarrow P$ is on B-altitude of $\triangle ABC \Longrightarrow BD \equiv BP \perp AC.$

Dear Luis González:
What about Your implication "$\Longrightarrow P$ is on B-altitude of $\triangle ABC$"?
Just check the entire matter for *non-square rectangles* ...
All the best over to Venezuela, auj

High School Olympiads

Nice Geometry

Reply

himanshu786

#1 Oct 7, 2012, 7:30 pm

AB and CD are two perpendicular segments meeting at S. K, L, M, N are the reflections of S on AC, CB, BD, DA. AL cuts the circumcircle of SKL again at F. AM cuts the circumcircle of SMN again at E. Show that $KFEN$ is cyclic.

This post has been edited 1 time. Last edited by himanshu786, Oct 7, 2012, 11:13 pm

Luis González

#2 Oct 7, 2012, 10:04 pm • 1

Let K', L', M', N' be the projections of S on AC, CB, BD, DA. $K'L' \parallel KL$ is S-midline of $\triangle SKL \Longrightarrow \angle SKL = \angle SK'L' = \angle SCB$. Similarly, $\angle SKN = \angle SAD \Longrightarrow \angle NKL = \angle SCB + \angle SAD$. Analogously, $\angle NML = \angle SBC + \angle SDA$. Thus, $\angle NKL + \angle NML = 2\angle BSD = 180^\circ \Longrightarrow KLMN$ is cyclic.

C, D are obviously the centers of $\odot(SKL), \odot(SMN) \Longrightarrow AB$ is common internal tangent of $\odot(SKL)$ and $\odot(SMN)$, i.e. their radical axis. Hence, inversion with center A and radius $AS = AK = AN$ takes $\odot(SKL)$ and $\odot(SMN)$ into themselves and carries concyclic poins K, L, M, N into concyclic points K, F, E, N.

Quick Reply

High School Olympiads

Heights, middles and a circle

Reply

Source: mpdb

borislav_mirchev

#1 Sep 28, 2012, 4:33 am

Let AA_1, BB_1, CC_1 are the heights of the triangle ABC. M, N, P are the middles of the AA_1, BB_1, CC_1, respectively. M_1 and M_2 are the feets of the perpendiculars from M to BA and AC respectively. N_1 and N_2 are the feets of the perpendiculars from N to the sides CB and BA respectively. P_1 and P_2 are the feets of the perpendiculars from P to the sides AC and CB respectively. Prove that $M_1, M_2, N_1, N_2, P_1, P_2$ are concyclic.

Note that the statement is true for all three points M, N, P with the following property: $\frac{AM}{AA_1} = \frac{BN}{BB_1} = \frac{CP}{CC_1}$.

The case $M = A_1, N = B_1, P = C_1$ was proposed for IMO TST in Bulgaria. This circle is known as Taylor's circle.

This post has been edited 1 time. Last edited by borislav_mirchev, Oct 9, 2012, 2:07 am

Luis González

#2 Oct 7, 2012, 6:59 pm • 1

R denotes the circumradius of $\triangle ABC$ and let $\frac{AM}{AA_1} = \frac{BN}{BB_1} = \frac{CP}{CC_1} = k.$

$$M_1M_2 = AM \cdot \sin A = k \cdot AA_1 \cdot \sin A = k \cdot AA_1 \cdot \frac{BC}{2R} = \frac{k}{R} \cdot [ABC].$$

Since the latter expression is symmetric, we deduce that $M_1M_2 = N_1N_2 = P_1P_2$. M_1M_2, N_1N_2 and P_1P_2 are clearly antiparallel to BC, CA and AB, respectively. Thus, $P_1P_2N_1N_2$ is an isosceles trapezoid with legs $P_1P_2 = N_1N_2 \Longrightarrow P_1N_2 \parallel BC$. Similarly, $M_1P_2 \parallel CA$, $N_1M_2 \parallel AB$. Thus $\angle BN_1N_2 = \angle BAC = \angle N_2M_1P_2 \Longrightarrow M_1$ is on the circumcircle of $P_1P_2N_1N_2$. Analogously, M_2 is on the circumcircle of $P_1P_2N_1N_2 \Longrightarrow M_1, M_2, N_1, N_2, P_1, P_2$ are concyclic, as desired.

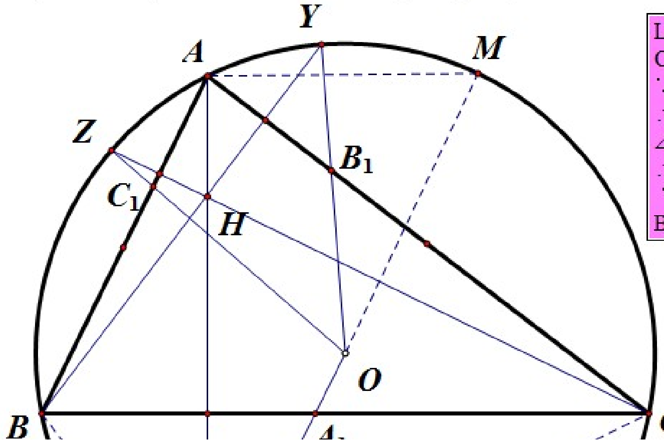
XmL

#3 Oct 8, 2012, 12:42 am

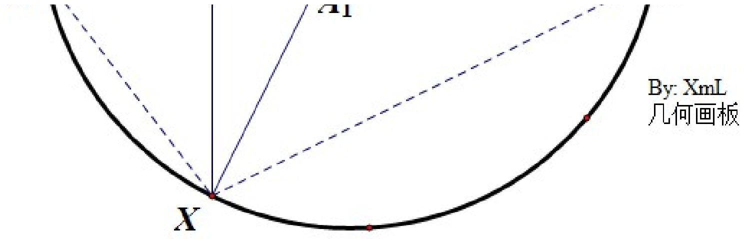
Lol after looking at Luis' solution I realized that I did the wrong part of the problem.
But here's my solution anyway:

Attachments:

Let acute-angled triangle ABC is inscribed in a circle $k(O)$. Heights through A, B, C intersects k at the points H_A, H_B, H_C respectively. A_1 is the intersection point of OH_A and BC, B_1 is the intersection point of OH_B and CA, C_1 is the intersection point of OH_C and AB. Prove that AA_1, BB_1, CC_1 intersects at a common point.



Let's denote Ha,Hb,Hc as X,Y,Z respectively
Connect BX,CX. Extend XO to meet ⊙O at M
∵ ∠MAX=90° ∴AM//BC ∴AC=BM⇒∠ABC=∠BXM
∴∠OXC=180-∠A-∠B=∠ACB. Similarily, we can get:
∠AYO=∠BAC, ∠BYC=∠ACB. ∠OZA=∠BAC, ∠BZO=∠ABC.
∴BA1/CA1=BX/CX*sin∠BXO/sin∠CXO=BX/CX*sin∠ABC/sin∠ACB
∵BX=BZ,ZA=AY,CY=CX ∴BA1/CA1*CB1/AB1*AC1/BC1=1
By Ceva's theorem converse. AA1,BB1,CC1 are concurrent.



Quick Reply

High School Olympiads

prove that angle DAP = angle CAB

Reply

minhtue0605
#1 Oct 6, 2012, 4:27 pm

Let be given parallelogram $ABCD$. P is a point inside triangle ACD. BP intersects DC at L, and DP intersects BC at K. If $BK.BC = DL.DC$, prove that: $\widehat{DAP} = \widehat{CAB}$.

Luis González
#2 Oct 7, 2012, 5:01 am

$BK \cdot BC = DL \cdot DC \Longrightarrow \frac{BK}{DL} = \frac{DC}{BC} = \frac{BA}{DA}$. Since $\angle ABC = \angle ADC$, then $\triangle ABK \sim \triangle ADL$ by SAS criterion $\Longrightarrow \angle BAK = \angle DAL$. Let E be the isogonal conjugate of C WRT $\triangle AKL$. Since KE, KB are isogonals WRT $\angle AKL$, LE, LD are isogonals WRT $\angle ALK$ and AB, AD are isogonals WRT $\angle KAL$, then by Jacobi's theorem, $P \equiv BL \cap DK \cap AE$. Therefore, $AP \equiv AE$, AC are isogonals WRT $\angle KAL \Longrightarrow AP, AC$ are isogonals WRT $\angle BAD$, or $\angle DAP = \angle CAB$.

Virgil Nicula
#3 Oct 7, 2012, 6:45 pm

See **PP11** from **here**.

sdsert
#4 Oct 12, 2012, 9:44 am

DK intersects AB at I, AP intersects DC at H.
Thus, $DL/DH = IB/IA = BK/AD \to DL.DA = BK.DH$
$\to DL.DA.DC = BK.DH.BA \leftrightarrow BK.DA.BC = BK.DH.BA$
$\leftrightarrow DA/DH = BA/BC$.
So $\triangle ADH \sim \triangle ABC$.
Hence, $\angle DAP = \angle BAC$.

sunken rock
#5 Oct 12, 2012, 8:17 pm

Outline of my solution: Extend DK, BL to intersect AB, AD respectively at K', L', and take $\{X\} \in BD \cap AP$, then apply **Ceva** to $\triangle ABD$ and cevians DK', BL', AX, finally getting, with the given relation, $\frac{BX}{DX} = \frac{AB^2}{AD^2}$, so AX is the symmedian, done.

Best regards,
sunken rock

Quick Reply

High School Olympiads

prove that GO is perpendicular to CE

Reply

minhtue0605
#1 Oct 6, 2012, 3:06 pm

Let be given triangle ABC with $AB = AC$. E is the midpoint of AB, and G is the centroid of triangle ACE. If O is the circumcenter of triangle ABC, prove that $GO \perp CE$.

Luis González
#2 Oct 7, 2012, 1:13 am • 1

AO and CE are medians of $\triangle ABC$ intersecting at its centroid M. $F \equiv CG \cap AB$ is the midpoint of $\overline{AE}$. Since $\overline{GC} : \overline{GF} = \overline{EB} : \overline{EF} = -2$ and $\overline{GC} : \overline{GF} = \overline{MC} : \overline{ME} = -2$, it follows that $GE \parallel BC$ and $GM \parallel AB \Longrightarrow OM$ and OE are perpendicular to $BC \parallel GE$ and $AB \parallel GM$, respectively $\Longrightarrow O$ becomes orthocenter of $\triangle GEM \Longrightarrow GO \perp ME \equiv CE$, as desired.

minhtue0605
#3 Oct 7, 2012, 6:40 am

Thanks a lot, great solution

Quick Reply

High School Olympiads

Incircle & Mixtilinear circle

Reply

Manolescu

#1 Oct 5, 2012, 1:17 pm

ΔABC
$\odot O$ the circumcircle
T the tangent point of $A - mixtilinearcircle$ and $\odot O$
D be an arbitary point on $arcBC$ (not contain A)
DE is a tangent line of incircle $\odot I$ which is closer to T intersecting $\odot O$ at another point E
DM is another tangent line of incircle $\odot I$ intersecting BC at M
TM intersects $\odot O$ at another point F

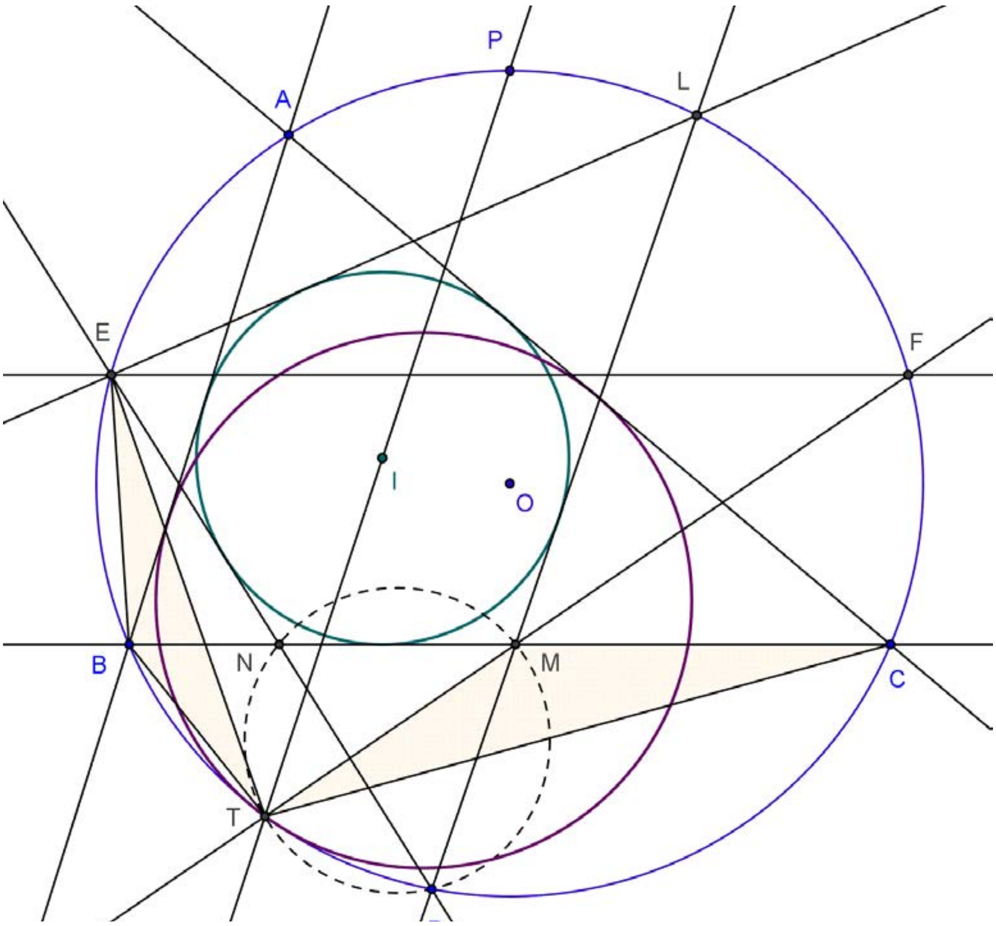
Prove that:
$\angle BAF = \angle CAE$

Luis González

#2 Oct 6, 2012, 9:43 am

DE cuts BC at N and DM cuts (O) again at $L.$ By Poncelet porism, EL is tangent to $(I).$ $\odot(DMN)$ cuts (O) again at the Miquel point T^* of the tangential $MNEL.$ $\angle BCT^* = \angle BET^*$ and $\angle T^*MN = \angle T^*DE = \angle T^*BE \pmod \pi$ yield $\angle MT^*C = \angle E^*TB.$ But since IT^* bisects $\angle MT^*E$ (see the solution of the problem Circumscribed quadrilateral with inversion), it follows that IT^* bisects $\angle BT^*C.$ Thus T^*, I and the midpoint P of the arc BAC of (O) are collinear $\Longrightarrow$ $T \equiv T^*$ is the tangency point of the A-mixtilinear incircle with $(O).$ Consequently, $\angle TMN = \angle TDE = \angle TFE \Longrightarrow$ $EF \parallel BC \Longrightarrow \angle BAF = \angle CAE.$

Attachments:



Manolescu

#3 Oct 6, 2012, 9:45 pm

It's really rare to see your post with a picture Luis
Nice Proof!

Quick Reply

High School Olympiads

Common tangent lines

Reply

Manolescu
#1 Oct 4, 2012, 9:46 pm • 1

$\odot O$ the circumcircle of ΔABC
$\odot I_a$ the $A-excircle$
Two common tangent lines of $\odot O$ and $\odot I_a$
intersects BC at D E respectively (WLOG D is closer to B)

Prove that:
$\angle DAB = \angle EAC$

Luis González
#2 Oct 6, 2012, 12:28 am • 1

I is the incenter of $\triangle ABC. IA, IB, IC$ cut BC, CA, AB at $X, Y, Z. YZ$ cuts (O) at P, Q (assume that Y is between P and Z). According to topic Very hard, the tangents of (O) at P, Q also touch the A-excircle (I_a). Now, project any line through $BC \cap YZ$, not cutting (O), to infinity and (O) into a circle (O'). Denote projected points with primes. $\triangle ABC$ becomes isosceles $\triangle A'B'C'$ with symmetry axis $A'X'$ and convex $BCPQ$ becomes isosceles trapezoid $B'C'P'Q'$. By obvious symmetry, we have

$$(D,B,C,X) \cdot (E,B,C,X) = (D',B',C',X') \cdot (E',B',C',X') = 1 \Longrightarrow$$

$$\frac{DB}{DC} \cdot \frac{EB}{EC} = \frac{XB^2}{XC^2} = \frac{AB^2}{AC^2}.$$

By Steiner theorem, AD, AE are isogonals WRT $\angle BAC$, i.e. $\angle DAB = \angle EAC$.

Quick Reply

High School Olympiads

hard problem: midpoint lies on a line

Reply

Source: own

physics_1995

#1 Sep 30, 2012, 8:35 pm • 1

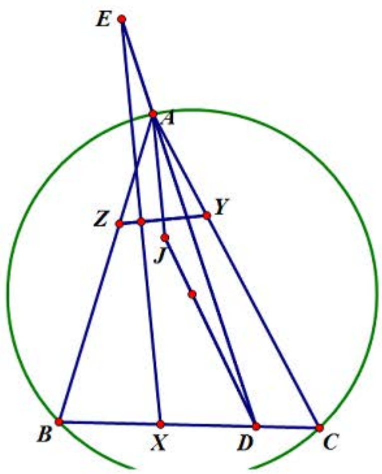
1. Let ABC be a triangle with circumcenter O.
2. J lies on bisector of $\angle BAC$.
3. X, Y, Z is projection of J on BC, CA, AB respectively.
4. JO cuts BC at D.
5. The line through X parallel to AJ cuts AD at E.

Prove that: midpoint of EX lies on YZ.

Click to reveal hidden text

♥ CQT ♥

Attachments:



vslmat

#2 Oct 5, 2012, 5:32 am • 2

Denote the angles of ΔABC simply as $\angle A, \angle B, \angle C$. Let M be the midpoint of BC and AH be the altitude from A, the line going through A parallel to the angle bisector AL cuts AO at G. AO cuts BC at R.

Notice that $\frac{HL}{XL} = \frac{AH}{JX}$ and $\frac{LM}{XL} = \frac{MK}{JX}$, thus $\frac{MH}{XL} = \frac{AH + MK}{JX}$, but as $\frac{XL}{LM} = \frac{JX}{MK}, \frac{LM}{MH} = \frac{MK}{AH + MK}$ /1/

Also notice that $\frac{XL}{LM} = \frac{JX}{MK}$, so $XL = \frac{JX.LM}{MK} = \frac{JX.MH.MK}{(AH + MK).MK}$, or $\frac{XL}{MH} = \frac{JX}{AH + MK}$ /2/

/1/+/2/ give $\frac{MX}{MH} = \frac{JX + MK}{AH + MK}$ (*)

First, we are going to prove that G, E, M are collinear.

$$\frac{EX}{AL} = \frac{XD}{LD} = \frac{\frac{JX}{\tan\angle d}}{\frac{JX}{\tan\angle d} - \frac{JX}{\tan(A/2+C)}} = \frac{\tan(A/2 + C)}{\tan(A/2 + C) - \tan\angle d} =$$

$$= \frac{\tan(A/2 + C)}{\tan(A/2 + C) - \frac{JX - OM}{XM}} = \frac{\tan(A/2 + C).XM}{\tan(A/2 + C).XM - JX + OM} =$$

$$= \frac{\tan(A/2+C).(XL+LM)}{\tan(A/2+C)(XL+LM)-JX+OM} = \frac{JX+MK}{MK+OM}$$

But $\frac{AL}{HG} = \frac{LR}{HR} = \frac{AR}{AR+AH}$, therefore, $\frac{EX}{HG} = \frac{JX+MK}{MK+OM}.\frac{AR}{AR+AH}$

To prove G, E, M collinear we have to show that $\frac{MX}{HM} = \frac{EX}{HG}$, or
$(MK+OM)(AH+AR) = (AH+MK)AR$, or $MK.AH+OM.AH = AH.AR-OM.AR$, or
$AH.OK = AR.(AH-OM)$, or $\frac{AH}{AR} = \frac{AH-OM}{OK}$
As $\frac{AH-OM}{OK} = \frac{1/2AA'}{1/2AA"}$ (see the smaller diagram), the last equation holds, so G, E, M are indeed collinear.

Let the midpoint of HG be S and the projection of X on YZ be T. Now we are going to prove that S, T, M collinear, or
$\frac{XT}{SH} = \frac{XM}{HM} = \frac{JX+MK}{AH+MK}$

As $XT = XY.\sin(A/2+C_1)$ and $\frac{XY}{\sin C} = \frac{JX}{\sin C_1}$ and $\frac{JK}{\sin(A/2+C_1)} = \frac{JC}{\sin B}$, we get

$$XT = \frac{JX.\sin C.JK.\sin B}{\sin C_1.JC} = \sin C.JK.\sin B = \frac{AH}{AC}.JK.\sin B = \frac{AH.JK}{AC/\sin B} =$$

$$= \frac{AH.JK}{AK/\sin(A/2+C)} = \frac{AH.\sin(A/2+C).JK}{AK} = \frac{AH.(JX+MK)}{AK} =$$

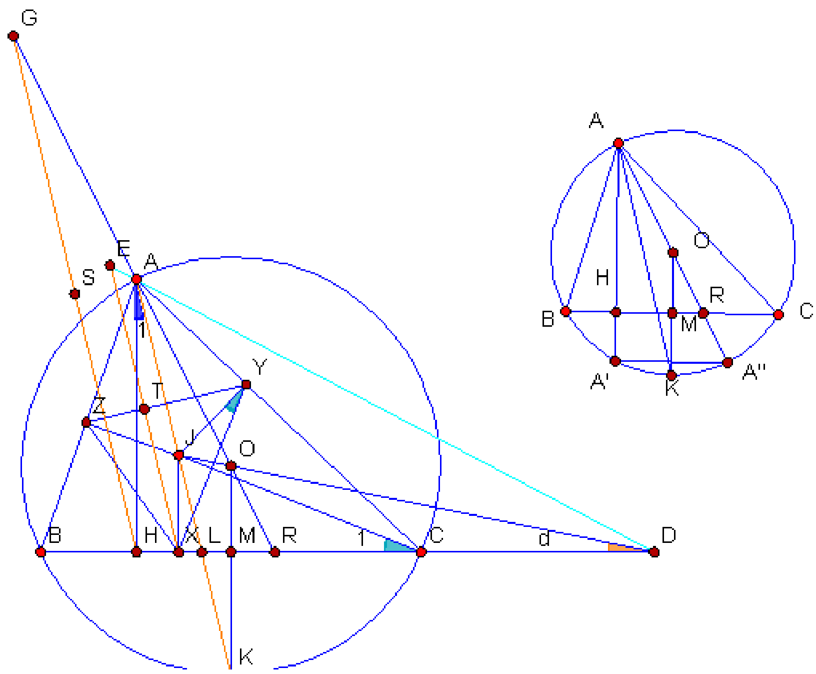
$$= \frac{AH.(JX+MK)}{(AH+MK)/\sin(A/2+C)}$$

Also, $HS = AH.\cos A_1 = AH.\frac{JX}{JL}$

Hence $\XT}{HS} = \frac{(JX+MK).\sin (A/2+C).JL}{(AH+MK).JX} = \frac{JX+MK}{AH+MK}$ (*). Hence S, T, M are indeed collinear.

It follows that T must be the midpoint of XE q.e.d.

Attachments:



This post has been edited 1 time. Last edited by vslmat, Oct 5, 2012, 2:26 pm

Luis González

#3 Oct 5, 2012, 9:38 am • 1

This problem and its generalizations were discussed earlier.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=346956
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=420917
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=425224
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=495103

High School Olympiads

Hard problem: Bisector

Source: own

physics_1995

#1 Oct 2, 2012, 6:45 pm • 1

1. Let ABC be a triangle with circumcircle (O).
2. P is a point on AB.
3. Circle O_1 intouches (O) at I and touches PB, PC at D, E respectively.
4. Circumcircle of $\triangle CEI$ cuts circumcircle of $\triangle BDI$ at G.
5. PG cuts circumcircle of $\triangle CEI$ at S.

Prove that: AS is bisector of $\angle BAC$

Proposed by PHCP

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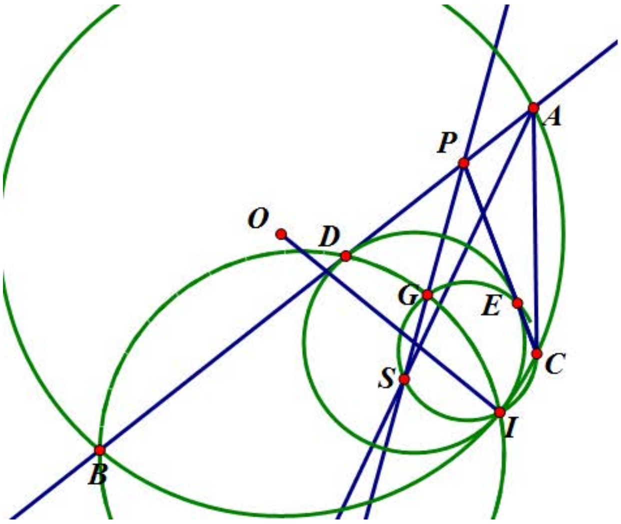
♥ CQT ♥

Attachments:



Luis González

#2 Oct 5, 2012, 8:05 am • 1

Circumcircles of $\triangle CEI$ and $\triangle BDI$ pass through the incenter of $\triangle PBC$ (see Fairly difficult (Iran 1999), Concyclic points with triangle incenter, incenter of triangle and elsewhere) $\Longrightarrow$ G is the incenter of $\triangle PBC \Longrightarrow PG$ is external bisector of $\angle APC$. Using the extraversion of the same result on $\triangle ACP$ and (O_1) touching $(O), \overline{PC}$ and the extension of AP, it follows that $\odot(CEI)$ passes through the A-excenter of $\triangle ACP \Longrightarrow S$ is the A-excenter of $\triangle ACP \Longrightarrow AS$ is internal bisector of $\angle BAC$.

High School Olympiads

Angle bisector

Reply

Manolescu

#1 Sep 26, 2012, 8:34 pm

ΔABC
I the incenter
E the midpoint of BC
AI intersects the circumcircle of ΔABC at F
N the midpoint of EF
M the midpoint of BI
MN intersects BC at D
Prove that:
DM is the angle bisector of $\angle ADB$

Luis González

#2 Sep 26, 2012, 10:59 pm • 1

It's well known that F is circumcenter of $\triangle BIC \Longrightarrow FM$ is perpendicular bisector of BI. Thus, $FEMB$ is cyclic, due to the right angles at E and $M \Longrightarrow \angle EFM = \angle EBI = \angle IBA$ and $\angle EMF = \angle EBF = \angle IAB \Longrightarrow \triangle EMF$ and $\triangle IAB$ are similar with corresponding medians $MN, AM \Longrightarrow \angle EMN = \angle IAM$.

$$\angle AMD = \angle IMA + \angle IME + \angle EMN = \angle EFB + (\angle IMA + \angle IAM) =$$

$$= \frac{\pi}{2} - \frac{\angle ACB}{2} + \frac{\pi}{2} - \frac{\angle BAC}{2} = \frac{\pi}{2} + \frac{\angle ABC}{2}.$$

This means that M is incenter of $\triangle ABD \Longrightarrow DM$ bisects $\angle ADB$.

Quick Reply

High School Olympiads

midpoint and angles

Reply

mikolez

#1 Sep 23, 2012, 11:51 pm

Given acute-angled, scalene triangle ABC. M is a midpoint of the side AC. N, L - projections of the points A, C on CB, AB respectively. $a \perp BM$, $a \cap BM = B$, $LN \cap a = K$. Prove that $\angle AMB = \angle BMK$.

Luis González

#2 Sep 24, 2012, 10:10 am

This was discussed earlier in the topic Prove the angle is equal.

Let $H \equiv AN \cap CL, E \equiv AC \cap NL, F \equiv AC \cap BK$. EH is the polar of B WRT the circle (M) with diameter $\overline{AC} \Longrightarrow EH \perp BM$. Further, $E(C, N, H, B)$ is harmonic $\Longrightarrow E(F, K, H, B)$ is harmonic. Since $EH \parallel FK$, it follows that B is midpoint of $\overline{FK} \Longrightarrow BM$ is perpendicular bisector of $\overline{FK} \Longrightarrow \angle AMB = \angle BMK$.

Quick Reply

High School Olympiads

centroid of DEF is a circumcenter of ABC

Reply

school5
#1 Sep 23, 2012, 2:16 pm

O is a center of circumcircle of triangle $\triangle ABC$. G is its centroid. D, E, F are centers of circles circumscribed to triangles $\triangle GBC, \triangle GCA$ and $\triangle GAB$ respectively. Prove that O is a centroid of triangle $\triangle DEF$.

tobash_co
#2 Sep 23, 2012, 3:50 pm • 1

Let H, I, J be midpoints of BC, CA, AB resp, and K, L, M be midpoints of AG, BG, CG. Note that O, H, D are collinear, and $DH \perp BC$. But $DF \perp BG$ since BG is radical axis of the circles centered at D, F. Thus D, H, L, B are concyclic and so $\angle ODF = \angle GBC$. Similarly $\angle ODE = \angle GCB$. Also clearly E, G, M, K are concyclic, so $\angle HGC = \angle DEF$. Similarly $\angle HGC = \angle DFE$.

From the sine rule we now have

$$\frac{\sin \angle ODF}{\sin \angle ODE} = \frac{\sin \angle GBC}{\sin \angle GCB} = \frac{\sin \angle HGC}{\sin \angle HGB} = \frac{\sin \angle DEF}{\sin \angle DFE}$$

So OD must bisect EF and similarly for the other sides we get that O must be the centroid of $\triangle DEF$, as desired.

Luis González
#3 Sep 24, 2012, 5:45 am

Let $\triangle A_0B_0C_0$ be the antipedal triangle of G WRT $\triangle ABC$. G is centroid of its pedal triangle WRT $\triangle A_0B_0C_0 \Longrightarrow G$ is symmedian point of $\triangle A_0B_0C_0$ (see post #4 at Very interesting collinear problem and elsewhere). Hence, centroid G_0 of $\triangle A_0B_0C_0$ is the isogonal conjugate of G WRT $\triangle A_0B_0C_0 \Longrightarrow$ center of pedal circle (O) of G WRT $\triangle A_0B_0C_0$ is midpoint of $\overline{GG_0}$. Since $\triangle DEF$ and $\triangle A_0B_0C_0$ are homothetic under homothety $(G, \frac{1}{2})$, then the centroid of $\triangle DEF$ is the midpoint O of $\overline{GG_0}$.

MariusBocanu
#4 Sep 26, 2012, 1:11 am

Or, taking the unit circle the circumcircle of $\triangle ABC$ and if we denote by small letter the afix of a point, we have $d = (6abc - a^2b - a^2c - ac^2 - bc^2 - ab^2 - b^2c)\frac{1}{(a-b)(a-c)}$(where c is $3, 4, 6$ or something like that, which i just ignored because our goal was to prove $d + e + f = 0$)

Quick Reply

High School Olympiads

Nine-point circles are concurrent

Reply

nsato
#1 Sep 14, 2012, 8:44 pm

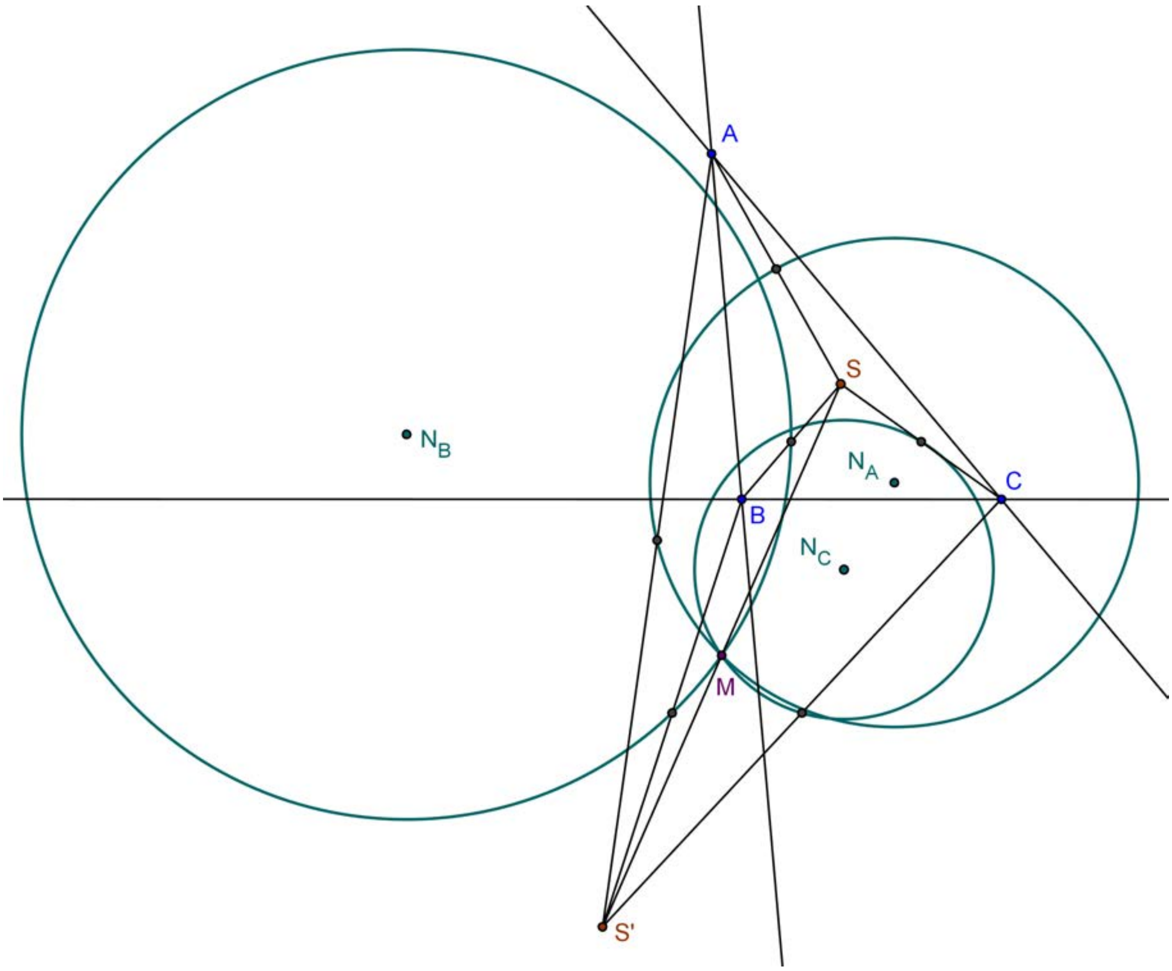
Let S and S' be the isodynamic points of triangle ABC. Prove that the nine-point circles of triangles ASS', BSS', and CSS' are concurrent.

(Inspired by http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=498374.)

Luis González
#2 Sep 24, 2012, 1:27 am

Trivially, they concur at the midpoint M of SS' and have no other common point.

Attachments:



nsato
#3 Sep 25, 2012, 4:50 am

Of course. That was a very silly post on my part.

Quick Reply

High School Olympiads

Incircle Midpoint

Manolescu

#1 Sep 20, 2012, 9:00 pm

Incircle (I) touches ΔABC at DEF resp
DM is perpendicular to EF (intersects at M)
P the midpoint of DM
H the orthocenter of ΔBIC
Prove that:
PH bisects EF

Luis González

#2 Sep 21, 2012, 7:16 am

Let M_B, M_C, N be the midpoints of AC, AB, EF. NP is D-Schawtt line of $\triangle DEF$ passing through its symmedian point $G_e \equiv BE \cap CF$, the Gergonne point of $\triangle ABC$. Thus, it suffices to show that H, G_e, N are collinear. EF, FD, DE cut BC, CA, AB at X, Y, Z. External bisector ℓ_A of $\angle BAC$ cuts $M_B M_C$ at V. Since $BC \parallel M_B M_C$ and $EF \parallel \ell_A$, then $\triangle AVM_B$ and $\triangle EXC$ are homothetic with center lying on AC. But $(A, C, E, Y) = -1$ yields $\frac{YA}{YE} = \frac{YM_B}{YC} \Longrightarrow Y$ is homothetic center of $\triangle AVM_B$ and $\triangle EXC \Longrightarrow Y, X, V$ are collinear $\Longrightarrow$ polars $\ell_A, M_B M_C$ and XYZ of N, H and G_e WRT (I) concur at $V \Longrightarrow N, H, G_e$ are collinear.

vslmat

#3 Sep 21, 2012, 3:55 pm • 1

Another solution:

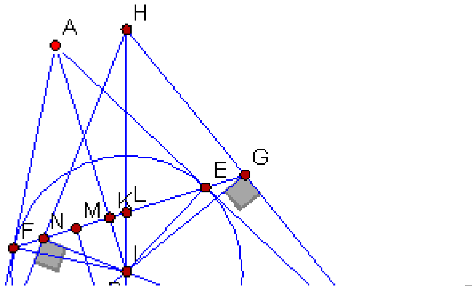
Let AI intersect EF at K, the the midpoint of EF.
Let CI cut EF at N. $\angle NEC = 90^\circ + \angle\frac{BAC}{2}$, hence $\angle CNE = \angle\frac{ABC}{2}$, $NIBF$ is cyclic and $\angle BNC = 90^\circ$, what is a very well known fact. Simillarly, if BI cut EF at G, then $BG \perp HC$.
Notice that to prove H, K, P collinear, it is sufficient to prove $\frac{KI}{PD} = \frac{HI}{HD}$, or $2.\frac{LI}{LD} = \frac{HI}{HD}$
Now for simplicity, let denote $\angle ABC = \angle B, \angle BAC = \angle A, \angle BCA = \angle C$
By sinus theorem in ΔLIG and LDG:

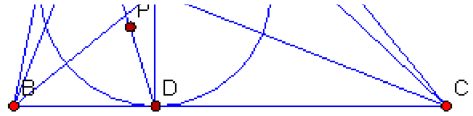
$\frac{LI}{sin(C/2)} = \frac{LG}{sin(90^\circ - B/2)}$ and $\frac{LD}{sinC} = \frac{LG}{sinA/2}$. Hence $\frac{LI}{LD} = \frac{sin(A/2).sin(C/2)}{cos(B/2).sinC}$

In the same way, using sinus theorem in ΔHBI we obtain: $\frac{HI}{HD} = \frac{sin(A/2)}{cos(B/2).cos(C/2)}$

Therefore, $2.\frac{LI}{LD} = \frac{HI}{HD}$ q.e.d.

Attachments:





Manolescu

#4 Sep 22, 2012, 8:49 pm

> *Luis González wrote:*
> NP is D-Schawtt line of $\triangle DEF$ passing through its symmedian point $G_e \equiv BE \cap CF$,

What is D-Schawtt line?
Thank you!

tobash_co

#5 Sep 23, 2012, 12:00 am

Here's another way to finish it off after the well-known fact that N, G, E, F are collinear (using notations from vslmat's solution). Here $N = IC \cap BH$ and $G = IB \cap CH$.

As above let K be midpoint of EF, and now drop perpendicular HJ from H to EF. Now notice that H is part of a complete quadrilateral w.r.t. B, C, G, N and thus since $HD \cap GN = L$, we have (H, L, I, D) harmonic. Projecting it onto EF, since $MD, IK \perp EF$, (M, K, L, J) is also harmonic. Now let $HK \cap DM = P'$, then from the pencil $A(M, K, L, J)$ projected onto DM, we see that since $HJ \cap DM$, harmonic conjugate of P' w.r.t. D, M lies at infinity. Thus $P' \equiv P$ must be the midpoint of DM, and we're done.

Luis González

#6 Sep 23, 2012, 9:42 pm

@Manolescu, the Schwatt lines of ABC are the lines passing through the midpoints of BC,CA,AB and the midpoints of their corresponding altitudes. They pass through the symmedian point K of ABC.

Let M be the midpoint of BC and X, Y, Z the projections of K on BC, CA, AB. AM and AK are isogonals WRT $\angle BAC$ $\Longrightarrow AM \perp ZY$, i.e AM becomes the A-altitude of $\triangle AYZ$. Since K is the antipode of A WRT $\odot(AYZ)$, then its reflection E about the midpoint U of ZY is the orthocenter of $\triangle AYZ \Longrightarrow E \in AM$. It's well-known that K is the centroid of its pedal triangle $\triangle XYZ \Longrightarrow X, K, E$ are collinear and $\overline{KX} = -2 \cdot \overline{KU} = -\overline{KE}$, i.e. K is the midpoint of $\overline{XE}$. If H_A is the foot of the A-altitude, then $EX \parallel AH_A$ implies that MK passes through the midpoint of $\overline{AH_A}$.

vslmat

#7 Sep 27, 2012, 3:28 pm

If wanting to use harmonic division, we can complete the proof after showing the well known facts this way:
Notice that $(H, I; L, D)$ is harmonic, then the bundle $K(HILD)$ cut MD gives us (P', ∝; M, D) - where $P' = KH \cap MD$ - harmonic, then P must be the midpoint of MD, $P' \equiv P$ and H, K, P collinear.

Manolescu

#8 Sep 27, 2012, 7:42 pm

> *Luis González wrote:*
> It's well-known that K is the centroid of its pedal triangle $\triangle XYZ$

Why?
Is it because of anti-parralel?

jayme

#9 Sep 28, 2012, 8:51 pm

Dear Mathlinkers,

you can see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=277108

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Concyclic

Reply

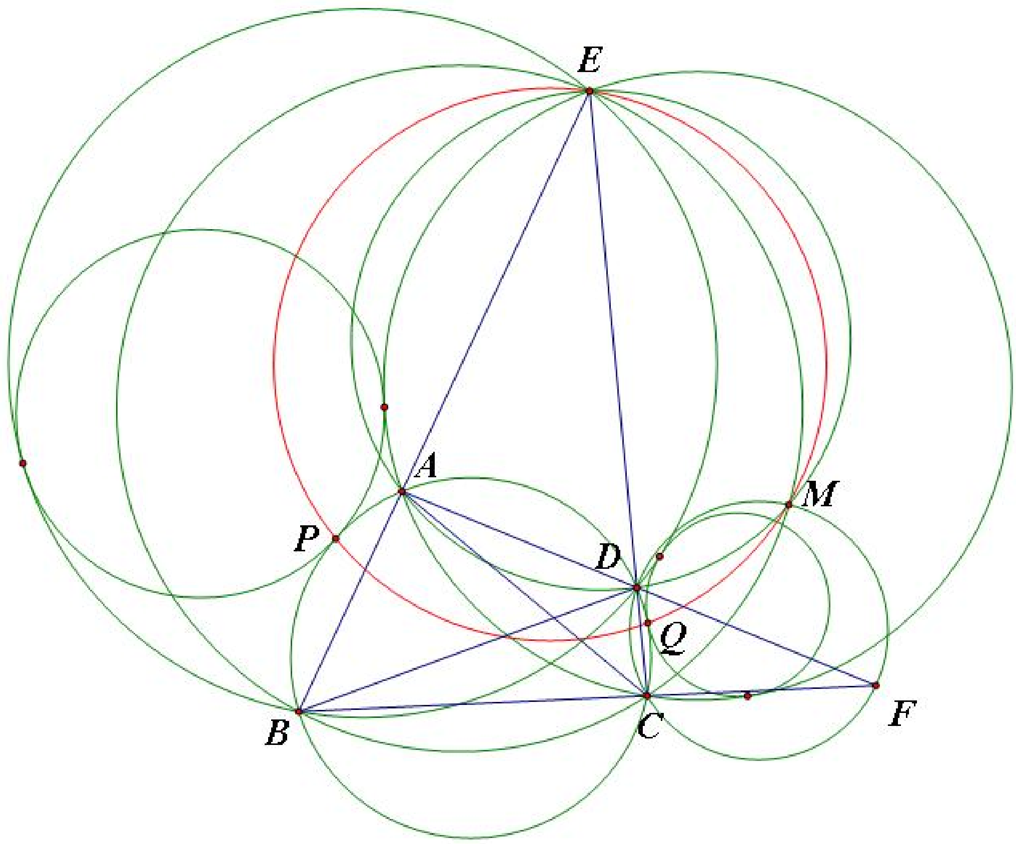
Source: own

phuongtheong

#1 Sep 23, 2012, 9:13 am

I have found this problem to solve another one, but I don't have a solution:
Let a concyclic quadrilateral $ABCD$. E and F is the intersection of AB, CD and AD, BC. (ω_1) and (ω_2) are the circles which is tangent to the circle $(ABCD)$ at P, Q and tangent internally to both of (EDB) and (EAC). M is the Miquel point of complete quadrilateral $ABCDEF$. Prove that M, E, P, Q are on a circle.
This is my figure:

Attachments:



Luis González

#2 Sep 23, 2012, 10:06 am • 1

Let $K \equiv AC \cap BD$. (O) denotes the circumcircle of $ABCD$. Inversion with center E and power equal to the power of E WRT (O), takes (O) into itself and $\odot(EDB), \odot(EAC)$ into the lines AC, BD, respectively. By conformity, the inverse of ω_1 is the circle Ω_1 tangent to rays KA, KB and externally tangent to (O) at the inverse P^* of P. Likewise, the inverse of ω_2 is the circle Ω_2 tangent to rays KC, KD and externally tangent to (O) at the inverse Q^* of Q. Circles $\odot(EAD)$ and $\odot(EBC)$ go to the lines BC and $AD \Longrightarrow F \equiv AD \cap BC$ is the inverse of the Miquel point $M \equiv \odot(EAD) \cap \odot(EBC)$. Hence, it suffices to show that F, P^*, Q^* are collinear.

According to 3 circles with common tangency point, there exists a circle λ_1 externally tangent to (O) at P^* and tangent to lines AD, BC. Analogously, there exists a circle λ_2 externally tangent to (O) at Q^* and tangent to lines AD, BC. Now, P^* is the insimilicenter of $(O) \sim \lambda_1, Q^*$ is the insimilicenter of $(O) \sim \lambda_2$ and F is the exsimilicenter of $\lambda_1 \sim \lambda_2$. Thus, by Monge and d'Alembert theorem, F, P^*, Q^* are collinear, as desired.

Quick Reply

High School Olympiads

Triangle area

Reply

lambosama

#1 Sep 22, 2012, 10:46 pm

Let M be the point outside the $\triangle ABC$. Using a ruler and compass, draw a line go through M and divide the triangle into 2 parts have the same area.

Luis González

#2 Sep 23, 2012, 2:16 am • 1

Assume that M lies inside the angle $\angle ACB.$ d_b and d_c denote the distances fom M to AC and $AB.$ Construct a point P on $\overline{AC},$ such that $AP \cdot d_b$ equals half the area of $\triangle ABC.$ Let Y,Z (unknown) be the intersections of the desired line through M with $AC,AB.$ Then we have

$$AY \cdot d_b - AZ \cdot d_c = AP \cdot d_b \Longrightarrow \frac{d_b}{d_c} = \frac{AZ}{AY-AP} = \frac{AZ}{PY}.$$

Now, the problem is reduced to the construction of the line through M that cuts AC,AB at points $Y,Z,$ such that the ratio $AZ:PY$ is given. See Problem 215b from Hadamard's Elementary Geometry. The same procedure is used to construct the other solution line.

Quick Reply

High School Olympiads

Vector

supama97
#1 Sep 22, 2012, 2:46 pm

Let triangle ABC (I) is an incircle of ABC. Let d pass I and d cuts AC,AB at E,F. Prove that:
$\frac{1}{AE}+\frac{1}{AF}=\frac{a+b+c}{bc}$$(BC=a, CA=b, AB=c)$

Luis González
#2 Sep 23, 2012, 12:10 am

Inversion with center A and arbitrary power takes d into a circle cutting the lines AC, AB, AI again at the inverses E', F', I' of $E,F,I.$ By Ptolemy's theorem for $AF'I'E'$ with $I'F'=I'E'$, we get $AE'+AF'=AI' \cdot \tfrac{E'F'}{I'E'}.$ Since $\angle E'I'F'=\pi-\angle BAC,$ then all the isosceles $\triangle I'E'F'$ are similar. As a result, $\tfrac{E'F'}{I'E'}$ is constant $\Longrightarrow AE'+AF'$ is constant $\Longrightarrow \tfrac{1}{AE}+\tfrac{1}{AF}$ is constant for any $d.$

Let $V \equiv BI \cap AC$ and assume that $d \parallel AC. E$ goes to the infinity point of $AC.$ From $\triangle BFI \sim \triangle BAV,$ we get

$$\frac{AF}{c}=\frac{IV}{BV}=\frac{b}{a+b+c} \Longrightarrow \frac{1}{AE}+\frac{1}{AF}=\frac{a+b+c}{bc}.$$

supama97
#3 Sep 23, 2012, 10:33 am

Why you dont use vector ?. More easy

underzero
#4 Sep 29, 2012, 12:03 am

FE meets BC at D and AI meets BC at M.
We know $AF/(c-AF).BD/DM.a/b+c$=1
so$CM/DM=((b+c)(c-AF)-aAF)/aAF$
now apply menelaus theorem
so we have $AE=AF.c.b.(b+c)/AF.c.(b+c)+ac.AF-b((b+c)(c-AF)-a.AF)$
Thus $1/AE+1/AF=a+b+c/bc$
QED

High School Olympiads

Prove that OM always goes through 1 fixed point

Reply

thiennhan97

#1 Sep 22, 2012, 3:24 pm

Given triangle ABC, M, N, P respectively on the sides BC, AC, AB so that APMN is parallelogram. BN and CP intersect at O. Prove that OM always goes through one fixed point when M moves on BC.

Click to reveal hidden text

Luis González

#2 Sep 22, 2012, 10:49 pm • 1

Actually, D should be vertex of parallelogram $ABDC$ not $ABCD$.

Let D be vertex of parallelogram $ABDC. X \equiv BC \cap AD, Y \equiv AM \cap NP$ and Z are the midpoints of $\overline{BC}, \overline{NP}$ and $\overline{AO}$, respectively. X, Y, Z are collinear on a Newton line of the quadrangle $BCNP \Longrightarrow$ reflections D, M, O of A about X, Y, Z are collinear as well, i.e. OM always goes through the fixed D.

Quick Reply

High School Olympiads

Nice

Reply

supama97

#1 Sep 22, 2012, 2:57 pm

Let triangle ABC and A is a right angle, AM is a median of ABC. (I_1) is an incircle of AMB, (I_2) is an incircle of AMC. Prove that line I_1I_2 bisects $S(ABC)$

auj

#2 Sep 22, 2012, 6:57 pm

> supama97 wrote:
> Let triangle ABC and A is a right angle, AM is a median of ABC. (I_1) is an incircle of AMB, (I_2) is an incircle of AMC. Prove that line I_1I_2 bisects $S(ABC)$

Dear supama:

What does the symbol $S(ABC)$ stand for?

Thanks, auj

yetti

#3 Sep 22, 2012, 7:55 pm

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=128947.

Luis González

#4 Sep 22, 2012, 10:15 pm • 1

See the problem Line bisects the area with circumellipse. After projecting the ellipse into a circle with diameter $\overline{BC}$, we get the same configuration. If A' denotes the antipode of A WRT (O), then OM is obviously perpendicular bisector of BA', i.e. M is midpoint of the arc BA' of $(O) \Longrightarrow AM$ bisects $\angle BAO \Longrightarrow P \equiv AM \cap ON$ is incenter of $\triangle ABO$. Similarly, $Q \equiv AN \cap OM$ is incenter of $\triangle ACO$.

vslmat

#5 Sep 24, 2012, 2:30 pm

What a nice property!

Let the tangent point of O_1 and AB be F, the tangent point of O_2 and AC be G. O_1O_2 cut AB, AC at D and E, respectively. As $AM = BM = CM$, is is easy to show that F, G is the midpoint od AB, resp. AC.

Since BO_1 is the angle bisector of $\angle ABC$, we have: $\frac{S_{FDO_1}}{S_{O_1MO_2}} = \frac{FO_1^2}{O_1M^2} = \frac{BF^2}{BM^2} = \frac{c^2/4}{a^2/4} = \frac{c^2}{a^2}$.

Similarly, $\frac{S_{GEO_2}}{S_{O_1MO_2}} = \frac{GC^2}{CM^2} = \frac{b^2}{a^2}$.

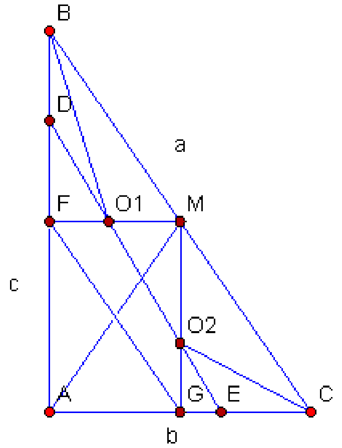
$S_{FDO_1} + S_{GEO_2} = S_{O_1MO_2}$, hence $S_{FDEG} = S_{FMG} = S/4$ and $S_{ADE} = S/2$

O_1O_2 really bisects S_{ABC}

Problem: Given ΔABC with median AM, O_1 is the incenter of ΔBAM and O_2 the incenter of ΔCAM. If O_1O_2 bisects S_{ABC}, then $\angle BAC = 90^\circ$.

Is the statement true?

Attachments:



phuongtheong

#6 Sep 29, 2012, 9:51 pm

Or we can solve this problem as the way below

Denote T and H are the midpoint of AB and AC. P, Q is the intersection of I_1I_2 and AB, AC. E, F is the foot of the altitudes from I_1, I_2 to AM. Because $\Delta I_1TP \sim \Delta I_1MI_2 \sim \Delta QHI_2$ so we have:

$$\frac{S_{I_1TP}+S_{I_2QH}}{S_{MI_1I_2}}=\frac{I_1T^2}{I_1M^2}+\frac{I_2H^2}{I_2M^2}=\frac{I_1E^2}{I_1M^2}+\frac{I_2F^2}{I_2M^2}$$
$$=\sin^2 I_1MA+\sin^2 I_2MA=\sin^2 I_1MA+\cos^2 I_1MA=1$$

So we obtain $S_{MI_1I_2}=S_{I_1PT}+S_{I_2HQ}\Rightarrow S_{APQ}=S_{ATMH}=\dfrac{S_{ABC}}{2}$

We done!

Quick Reply

High School Olympiads

one is equal to sum of two others

Locked

MBGO
#1 Sep 22, 2012, 12:12 pm

Dear all,

$1:$4 distinct points A, B, C, D lie on a circle such that $\measuredangle BCD \neq 90^\circ$.

$2:$Perpendicular bisectors of segments AB, AC intersect the line AD at W, V.

$3:$$CV, BW$ meet at T.

Prove that length of one of the segments AD, BT, CT is equal to the sum of lengths of two others.

##sorry if the problem posted before, I didn't find it by Search Function.##

Luis González
#2 Sep 22, 2012, 12:30 pm

MBGO the problem was posted before. You are either ignoring the source of the problem or you are using the search function with unsuitable keywords. It's IMO 1997 Problem 2, so topic locked.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=32558
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=58299

High School Olympiads

Collinearity in triangle WRT the incircle and circumcircle

Reply

CTK9CQT

#1 Sep 22, 2012, 9:28 am

Let (I) and (O) be incircle and circumcircle of triangle ABC, respectively. (I) touches BC at D. A' is symmetric point of A wrt O. AI and $A'I$ intersect (O) at E and F respectively. Prove that D, E, F are collinear

Luis González

#2 Sep 22, 2012, 10:05 am • 1

Let ED cut (O) again at F'. Since $\angle BF'E = \angle BCE = \angle DBE$, we have $EI^2 = EC^2 = EB^2 = ED \cdot EF' \Longrightarrow$ EI is tangent to $\odot(IDF') \Longrightarrow \angle EID = \angle EF'I$. If H is the orthocenter of $\triangle ABC$, then $\angle EAA' = \angle EAH = \angle EID = \angle EF'I \Longrightarrow F' \in IA' \Longrightarrow F \equiv F'$.

P.S. See also the problem incenter I and touches BC side with D.

CTK9CQT

#3 Sep 22, 2012, 10:14 am

Thank you very much, Luis!

Quick Reply

High School Olympiads

sum of the distances from the orthocenter

Reply

Source: Serbia 2004

elegant
#1 Sep 20, 2012, 7:08 am

Let r be the inradius of an acute triangle. Prove that the sum of the distances from the orthocenter to the sides of the triangle does not exceed 3r.

galaas
#2 Sep 20, 2012, 8:49 am

P*d=P*3r ?

yetti
#3 Sep 21, 2012, 11:46 pm • 1

Assuming $\triangle ABC$ is not equilateral.

External bisectors of angles $\widehat{A}, \widehat{B}, \widehat{C}$ cut BC, CA, AB at X, Y, Z, resp. I is incenter and I_a, I_b, I_c are A-, B-, C-excenters of $\triangle ABC$. $(O), (J)$ are circumcircles of $\triangle ABC, \triangle I_aI_bI_c$, resp.
(O) is 9-point circle of $\triangle I_aI_bI_c$ and I its orthocenter $\Longrightarrow JOI$ is its Euler line. $\diamond BCI_bI_c$ is cyclic because of right $\measuredangle I_cBI_b, \measuredangle I_cCI_b \Longrightarrow \overline{XB}\cdot\overline{XC} = \overline{XI_b}\cdot\overline{XI_c}$.
Similarly, $\overline{YC}\cdot\overline{YA} = \overline{YI_c}\cdot\overline{YI_a}$ and $\overline{ZA}\cdot\overline{ZB} = \overline{ZI_a}\cdot\overline{ZI_b} \Longrightarrow XYZ \perp JOI$ is radical axis of $(J), (O)$.
Circumcircles $(J) \sim (O)$ are centrally similar
with similarity center I and similarity coefficient 2. Since I is inside of $\triangle ABC$, which is inside of $(O) \Longrightarrow (O)$ is inside of (J) and their radical axis is outside of $(O), (J)$.
Let $K \equiv JOI \cap XYZ \Longrightarrow K$ is on the ray $(JOI$ is outside of $(O), (J)$.

Let point $P_0 \in XYZ$ be arbitrary, let D_0, E_0, F_0 be feet of perpendiculars from P_0 to BC, CA, AB and let $\overline{P_0D_0}, \overline{P_0E_0}, \overline{P_0F_0}$ be directed distances of P_0 from BC, CA, AB, resp.
$\overline{P_0D_0} > 0 \Longleftrightarrow A, P_0$ are on the same side of BC and cyclically for $\overline{P_0E_0}, \overline{P_0F_0}$. Let parallel to AX through P_0 cut CA at P_1 and let $\overline{P_1F_1}$ be directed distance of P_1 from $AB \Longrightarrow \overline{P_0E_0} + \overline{P_0F_0} = \overline{P_1F_1}$.
Let parallel to BC through P_0 cut AB at Q and let parallel to AB through P_1 cut BC at S. Let $R \equiv P_0Q \cap P_1S$.
$\triangle P_1RP_0 \sim \triangle ABX$, having parallel sides, are centrally similar with similarity center $Y \Longrightarrow$ Y, R, B are collinear $\Longrightarrow$ diagonal BR of parallelogram $BQRS$ bisects $\measuredangle SBQ \Longrightarrow BQRS$ is a rhombus $\Longrightarrow$ $\overline{P_0D_0} = -\overline{P_1F_1}$ and $\overline{P_0D_0} + \overline{P_0E_0} + \overline{P_0F_0} = 0$.

Let P be arbitrary point and let parallel $p \parallel XYZ$ through P cut the center line JOI at K'. Let $X' \equiv p \cap IX, Y' \equiv p \cap IY, Z' \equiv p \cap IZ$.
Parallels to BC, CA, AB through X', Y', Z', resp., pairwise intersect at $A', B', C' \Longrightarrow \triangle A'B'C' \sim \triangle ABC$ are centrally similar with similarity center I and similarity coefficient $\kappa = \frac{\overline{IK'}}{\overline{IK}}$.
Let D, E, F be feet of perpendiculars from P to BC, CA, AB and let D', E', F' be feet of perpendiculars from P to $B'C', C'A', A'B'$, resp. $\Longrightarrow \overline{D'D} = \overline{E'E} = \overline{F'F} = r(1-\kappa)$.
$A'X', B'Y', C'Z'$ are external bisectors of angles $\widehat{A'}, \widehat{B'}, \widehat{C'}$. Since $P \in X'Y'Z' \Longrightarrow$ sum of its directed distances from $B'C', C'A', A'B'$ is $\overline{PD'} + \overline{PE'} + \overline{PF'} = 0$.
Sum of directed distances of P fom BC, CA, AB is then
$\overline{PD} + \overline{PE} + \overline{PF} = (\overline{PD'} + \overline{D'D}) + (\overline{PE'} + \overline{E'E}) + (\overline{PF'} + \overline{F'F}) = 3r(1-\kappa)$.

Let ϱ be circumradius of $\triangle ABC$. Its 9-point circle $(N, \frac{1}{2}\varrho)$ is internally tangent to its incircle $(I, r) \Longrightarrow$ 9-point center N is on circle $(I, \frac{1}{2}\varrho - r)$.
Let H be orthocenter of $\triangle ABC$. Since $\overline{OH} = 2\overline{ON} \Longrightarrow H$ is on circle $(O', \varrho - 2r) \sim (I, \frac{1}{2}\varrho - r)$ with similarity center O and similarity coefficient $2 \Longrightarrow \overline{OO'} = 2\overline{OI}$.

Let $(O', \varrho - 2r)$ cut the center line OI at L, L', points O, I, O', L' following in this order. Since $[LO'] = \varrho - 2r = \frac{[OI]^2}{\varrho} < [OI] = [IO'] \Longrightarrow L$ is inside of the segment $[IO']$.
If we substitute H for the arbitrary point $P \Longrightarrow$ foot K' of perpendicular from H to the center line OI is on diameter $[LL']$ of $(O', \varrho - 2r)$, which is on the ray $(IK \Longrightarrow \kappa > 0$.
But since $\triangle ABC$ is acute $\Longrightarrow$ its orthocenter H is inside of it, which is inside of its circumcircle $(O) \Longrightarrow K'$ is on the part of $[LL']$ inside of (O), which may be entire $[LL'] \Longrightarrow \kappa < 1$.
Directed distances of interior orthocenter H from BC, CA, AB are all positive and their sum is $[HD] + [HE] + [HF] = \overline{HD} + \overline{HE} + \overline{HF} = 3r(1 - \kappa) < 3r$.

Luis González

#4 Sep 22, 2012, 5:51 am • 2

H is the orthocenter of the acute $\triangle ABC$ and D, E, F are the feet of the altitudes on BC, CA, AB. R, r, ϱ denote the radii of its circumcircle, incircle and incircle of orthic $\triangle DEF$. Using the result of Two inequalities regarding radii (2nd inequality) for $\triangle DEF$ with its excentral $\triangle ABC$, we have $r \ge 2\varrho$. Combining with Euler's inequality $R \ge 2r$, we obtain $r \ge \varrho + \frac{r^2}{R}$. But from topic Sum of altitudes in the acute triangle, we have

$$HD + HE + HF = AD + BE + CF - 2(R + r) = 2r + \varrho + \frac{r^2}{R} \Longrightarrow$$

$$r \ge HD + HE + HF - 2r \Longrightarrow 3r \ge HD + HE + HF.$$

P.S. See also nice problem?!! [d(H; a) + d(H; b) + d(H; c) <= 3r] for more solutions.

Quick Reply

High School Olympiads

Two perpendicular lines and a locus

Source: Iran 3rd round 2012-Geometry exam-P5

goodar2006

#1 Sep 20, 2012, 11:18 am • 1

Two fixed lines l_1 and l_2 are perpendicular to each other at a point Y. Points X and O are on l_2 and both are on one side of line l_1. We draw the circle ω with center O and radius OY. A variable point Z is on line l_1. Line OZ cuts circle ω in P. Parallel to XP from O intersects XZ in S. Find the locus of the point S.

Proposed by Nima Hamidi

This post has been edited 1 time. Last edited by goodar2006, Sep 28, 2012, 7:26 pm

Luis González

#2 Sep 21, 2012, 9:01 pm

We assume that X is between O, Y and P is between $O, Z. V$ is the midpoint of $\overline{XY}$ and $R = OY$ denotes the radius of ω. K is the orthogonal projection of S on ℓ_2. From $\triangle SKX \sim \triangle ZYX$ and $\triangle PXZ \sim \triangle OSZ$, we get

$$\frac{SK^2}{ZY^2} = \frac{SK^2}{PZ(PZ+2R)} = \frac{SX^2}{ZX^2} = \frac{R^2}{PZ^2} \Longrightarrow SK^2 = \frac{PZ+2R}{PZ} \cdot R^2 \quad (1)$$

$$\frac{XK}{XY} = \frac{SX}{ZX} = \frac{R}{PZ} \Longrightarrow \frac{2KV}{XY} = \frac{2(XK + \frac{1}{2}XY)}{XY} = \frac{PZ+2R}{PZ} \quad (2)$$

Combining the expressions (1) and (2), we obtain $SK^2 = KV \cdot \dfrac{2R^2}{XY}$.

This means that the locus of S is a parabola wih focal axis ℓ_2, vertex V the midpoint of $\overline{XY}$ and whose latus rectum length is given by $\frac{2R^2}{XY}$.

High School Olympiads

Incircle and intersecting lines

Reply

Source: mpdb

borislav_mirchev

#1 Sep 21, 2012, 12:54 pm

Let A_0, B_0, C_0 are the tangency points of the incircle of the triangle ABC with the sides BC, CA, AB respectively. A', B', C' are the symmetric points of the A, B, C with respect to A_0, B_0, C_0 respectively. A'', B'', C'' are feets of the perpendiculars from A', B', C' to the sides BC, CA, AB respectively. Prove that $A'A'', B'B'', C'C''$ intersects at a common point.

Luis González

#2 Sep 21, 2012, 2:56 pm • 1

Let H_A be the projection of A on BC. $H \in AH_A$ and I are the orthocenter and incenter of $\triangle ABC$, respectively. IA_0 is then the midparallel of $AH_A \parallel A'A'' \Longrightarrow A'A''$ passes through the reflection X_{944} of H about I. Similarly, $B'B''$ and $C'C''$ pass through X_{944}.

P.S. There are infinitely many points besides the Gergonne point satisfying the desired property. Namely, all the points lying on the Lucas cubic of ABC.

Quick Reply

High School Olympiads

Nice Concyclic(again)

Reply

Manolescu

#1 Sep 20, 2012, 8:55 pm

P is a point outside $\odot O$
PA, PB are tangent lines
PCD intersects $\odot O$ at C, D
such that $BC \perp AD$ and intersects at E
AB, CD intersects at F
the perpendicular bisector of PF intersects BC at G
Prove that:
$PEFG$ concyclic

Luis González

#2 Sep 21, 2012, 1:29 pm

AB is obviously the polar of P WRT $(O) \Longrightarrow$ cross ratio (C, D, F, P) is harmonic. Together with $EC \perp ED,$ it follows that EC, ED bisect $\angle PEF$ internally and externally. Thus, ECB cuts the circumcircle of $\triangle PEF$ again at the midpoint G^* of its arc PF (not containing E) $\Longrightarrow FG^* = PG^* \Longrightarrow G^*$ is on perpendicular bisector of $\overline{PF} \Longrightarrow G \equiv G^* \Longrightarrow PEFG$ is cyclic, as desired.

Quick Reply

High School Olympiads

Nine-point circles are tangent

Reply

buratinogigle
#1 Sep 14, 2012, 5:47 pm • 1

Let ABC be a triangle with two Fermat points F, F'. Prove that nine-point circles of triangles AFF', BFF', CFF' are tangent.

Luis González
#2 Sep 15, 2012, 2:38 am • 4

Generalization. P, Q are antigonal conjugates WRT $\triangle ABC$. Then 9-point circles of $\triangle APQ, \triangle BPQ, \triangle CPQ$ are tangent.

Let $(N_A), (N_B), (N_C)$ denote the 9-point circles of $\triangle APQ, \triangle BPQ, \triangle CPQ$. M is the midpoint of $\overline{PQ}$. Tangent ℓ_B of the circumcircle of $\triangle BPQ$ at B is clearly parallel to the tangent τ_B of (N_B) at M. Similarly, tangent ℓ_C of the circumcircle of $\triangle CPQ$ at C is parallel to the tangent τ_C of (N_C) at M. Simple angle chase (modulo 180°) gives

$$\angle(\ell_B, BC) = \angle PBC - \angle PQB \ , \ \angle(\ell_C, BC) = \angle CQP - \angle BCP.$$

But $\angle CPB = \angle BQC \Longrightarrow \angle PBC + \angle BCP = \angle PQB + \angle CQP \Longrightarrow \angle(\ell_B, BC) = \angle(\ell_C, BC) \Longrightarrow \ell_B \parallel \ell_C$ $\Longrightarrow \tau_B \parallel \tau_C \Longrightarrow \tau_B$ and τ_C coincide $\Longrightarrow (N_B)$ and (N_C) are tangent through M. By similar reasoning, (N_A) is tangent to (N_B) and (N_C) through M.

buratinogigle
#3 Sep 15, 2012, 11:41 pm • 2

Thank you dear Luis for excellent solution, I don't know about antigonal conjugates until your post.

I found an other problem with this configuration

Let P, Q be two antigonal conjugates with respect to triangle ABC. Circumcircle of triangle APQ, BPQ, CPQ cut circumcircle of triangle ABC again at A', B', C', respectively. Prove that AA', BB', CC' are concurrent.

I don't have solution, yet.

A-B-C
#4 Mar 10, 2016, 5:05 pm

Remark. The common tangent line is pendicular to orthotransversal of P.

P, Q are symmetrically through center T of $\mathcal{H}$.
Let G_a be centroid of $\triangle APQ$. G_a is insimilcenter of (APQ) and nine-point circle of $\triangle APQ$.
$\Rightarrow$ Tangent line at T of nine-point circle of $\triangle APQ$ is parallel to tangent line at A of (APQ).
According to lemma 1, tangent line at A of (APQ) is parallel to tangent line at P of $\mathcal{H}$.
According to post #2, tangent line at P of $\mathcal{H}$ is pendicular to orthotransversal of P WRT $\triangle ABC$.

TelvCohl
#5 Mar 10, 2016, 5:54 pm

> Luis González wrote:
> **Generalization.** P, Q are antigonal conjugates WRT $\triangle ABC$. Then 9-point circles of $\triangle APQ, \triangle BPQ, \triangle CPQ$ are

> tangent.

Since the midpoint M of PQ lies on the 9-point circle of $\triangle APQ, \triangle BPQ$ and $\triangle CPQ$, so it suffices to prove the tangent of these three circles passing through M are parallel $\Longleftrightarrow$ the tangent τ_A, τ_B, τ_C of $\odot(APQ), \odot(BPQ), \odot(CPQ)$ passing through A, B, C, respectively are parallel.

Let τ_A cuts the circum-rectangular hyperbola $\mathcal{H}$ of $\triangle ABC$ passing through P, Q again at X. Since the isogonal conjugate of $\mathcal{H}$ WRT $\triangle APQ$ is the perpendicular bisector of PQ, so the A-symmedian of $\triangle APQ$ is tangent to $\mathcal{H}$ at $A \Longrightarrow (A, X; P, Q) = A(A, X; P, Q) = -1$, hence τ_A is parallel to the tangent σ of $\mathcal{H}$ at P (or Q). Similarly, we can prove $\tau_B \parallel \tau_C \parallel \sigma$.

> *buratinogigle wrote:*
> I found an other problem with this configuration
>
> Let P, Q be two antigonal conjugates with respect to triangle ABC. Circumcircle of triangle APQ, BPQ, CPQ cut circumcircle of triangle ABC again at A', B', C', respectively. Prove that AA', BB', CC' are concurrent.

AA', BB', CC' are concurrent at the radical center of $\odot(ABC), \odot(APQ), \odot(BPQ), \odot(CPQ)$.

High School Olympiads

Tangent circles of mixtilinear incircles

Source: Own

buratinogigle

#1 Sep 12, 2012, 12:24 am

Let ABC be triangle inscribed (O). (O_a) is A-mixtilinear incircle of ABC. Circle (ω_a) other than (O) passing through B,C and touches (O_a) at A'. Similarly we have B',C'. Prove that AA',BB',CC' are concurrent.

Luis González

#2 Sep 13, 2012, 2:39 am • 2

(I) is the incircle of $\triangle ABC$ and $(I_A),(I_B),(I_C)$ are its three excircles tangent to BC,CA,AB at X_A,X_B,X_C, respectively. The inversion with center A, that swaps (I) and the A-mixtilinear incircle (O_A), takes (O) into a line τ_A tangent to (I) and antiparallel to BC. τ_A cuts AC,AB at the inverses B_0,C_0 of C,B. (I) becomes then A-excircle of $\triangle AB_0C_0$. By conformity, the inverse of ω_A is the circle Ω_A passing through B_0,C_0 and tangent to (I) through the inverse A_c of A'. Let λ_A be the circle passing through B,C and tangent to (I_A) at U_A. U_B and U_C are defined cyclically. Then $\triangle ABC \cup \lambda_A \sim \triangle AB_0C_0 \cup \Omega_A \Longrightarrow \angle B_0AA_0 = \angle BAU_A$, i.e. $AA_0 \equiv AA'$ and AU_A are isogonals WRT $\angle A$. Likewise, BB',BU_B and CC',CU_C are isogonals WRT $\angle B$ and $\angle C$, respectively.

Let AU_A,BU_B,CU_C cut BC,CA,AB at P_A,P_B,P_C. According to problem Three concurrent lines, AU_A is the U_A-cevian of the isogonal conjugate of the isotomic conjugate of the incenter of $\triangle U_ABC \Longrightarrow$

$$\frac{BP_A}{CP_A} = \frac{{BU_A}^2}{{CU_A}^2} \cdot \frac{BX_A}{CX_A} = \frac{{BX_A}^2}{{CX_A}^2} \cdot \frac{BX_A}{CX_A} = \frac{{BX_A}^3}{{CX_A}^3} = \left(\frac{a+b-c}{c+a-b}\right)^3 \quad (\star).$$

Multiplying the cyclic expressions together, we deduce that AP_A,BP_B,CP_C concur at a point X, due to Ceva's theorem. Hence, AA',BB',CC' concur at the isogonal conjugate X^* of X WRT $\triangle ABC$. From $(\star)$, it follows that the barycentric coordinates of X^* WRT ABC are

$$X^* \left(\frac{a^2}{(b+c-a)^3} : \frac{b^2}{(c+a-b)^3} : \frac{c^2}{(a+b-c)^3}\right).$$

High School Olympiads

triangle intersection problem

Reply

Lomisedu

#1 Sep 10, 2012, 9:21 am

Consider a triangle ABC satisfy the following condition:
Let the angle bisector of angle A intersect side BC at a point D, $\angle ADC = 60^\circ$.
Find a point E on the line AD such that $BD = DE$. Draw the line BE until its intersection with AC at a point F.
Draw the line CE until its intersection with AB at a point G.

show that $GF \perp GD$.

yetti

#2 Sep 11, 2012, 9:11 pm

Let $a = [BC], b = [CA], c = [AB]$ be triangle sides. Let $H \equiv AD \cap GF$ and $K \equiv GF \cap BC$.
(1) Two diagonals AE, GF of complete $\diamondsuit AGEF$ cut its 3rd diagonal BC harmonically at D, K, resp. $\Longrightarrow$ cross ratio $(B, D, C, K) = -1$ is harmonic.
(2) Two diagonals GF, BC of complete $\diamondsuit AGEF$ cut its 3rd diagonal AE harmonically at H, D, resp. $\Longrightarrow$ cross ratio $(A, H, E, D) = -1$ is harmonic.
(3) Two diagonals AE, BC of complete $\diamondsuit AGEF$ cut its 3rd diagonal GF harmonically at H, K, resp. $\Longrightarrow$ cross ratio $(G, H, F, K) = -1$ is harmonic.
From **(1)** $\Longrightarrow AD \perp AK$ internally / externally bisect $\measuredangle CAB$. Since $\measuredangle ADK \equiv \measuredangle ADC = 60^\circ$ in A-right $\triangle ADK \Longrightarrow$ $\overline{DA} = \frac{1}{2}\ \overline{DK} = \dfrac{abc}{c^2 - b^2}$.

From **(2)** $\Longrightarrow \overline{DH} = \overline{DE} + \overline{EH} = \overline{DE} + \overline{EA} \cdot \dfrac{\overline{EH}}{\overline{EH} + \overline{HA}} = \overline{DE} + (\overline{DA} - \overline{DE}) \cdot \dfrac{\overline{DE}}{\overline{DE} + \overline{DA}} = \dfrac{2\ \overline{DE} \cdot \overline{DA}}{\overline{DE} + \overline{DA}} = \dfrac{2\ \overline{BD} \cdot \overline{DA}}{\overline{BD} + \overline{DA}}$.

Substituting $\overline{BD} = \dfrac{ca}{b+c}, \overline{DA} = \dfrac{abc}{c^2 - b^2} \Longrightarrow \overline{DH} = \dfrac{2ab}{b+c} = 2\ \overline{DC} \Longrightarrow \triangle HDC$ with $\measuredangle HDC \equiv \measuredangle ADC = 60^\circ$ is C-right $\Longrightarrow CH \perp CD$.
Let $L \equiv CH \cap AK \Longrightarrow H \equiv CL \cap DA$ is orthocenter of $\triangle DKL$. From **(3)** $\Longrightarrow G \in LD$ is foot of K-altitude KH of $\triangle DKL \Longrightarrow (GF \equiv HK) \perp (LD \equiv GD)$.

Luis González

#3 Sep 12, 2012, 1:06 am

External angle bisector of $\angle BAC$ cuts BC at P. A-Apollonius circle with diameter $\overline{PD}$ cuts AB again at $M \Longrightarrow MD$ bisects $\angle BMC$ internally. Let PM cut AC at $N \Longrightarrow Q \equiv CM \cap BN$ is on AD, since $A(B, C, D, P) = -1$. From cyclic quadrilateral $AMDP$, we get $\angle BMD = \angle APD = 90^\circ - \angle ADC = 30^\circ \Longrightarrow \angle BMQ = 2 \cdot 30^\circ = 60^\circ = \angle ADC$ $\Longrightarrow D \in \odot(BMQ)$ is midpoint of its arc $BQ \Longrightarrow BD = DQ \Longrightarrow Q \equiv E, M \equiv G$ and $N \equiv F \Longrightarrow$ $\angle DGF = \angle DAP = 90^\circ$.

Quick Reply

High School Olympiads

Bisector

Reply

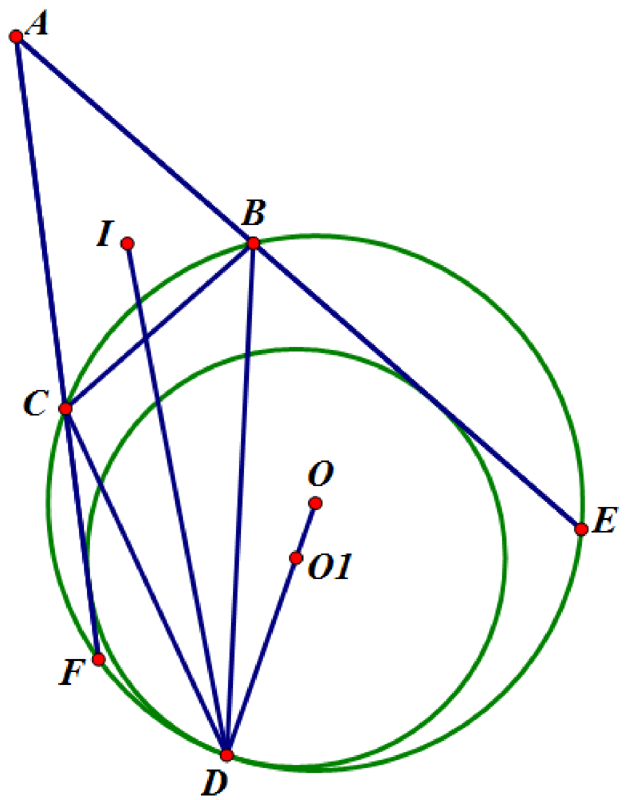
Source: lugang_10

aries

#1 Sep 11, 2012, 7:11 pm

The circle O_1 is in the circle (O) and tangent to (O) at D. A is a point not in (O) or $\in (O)$. l_1 tangent to O_1 cuts (O) at B, E such that B is between A and E, l_2 tangent to (O_1) cuts (O) at C, F such that C is between A and F. I is incenter of $\triangle ABC$. Prove that DI is bisector of $\angle BDC$.
Hope to see a solution without much trigonometry calculation and please don't spam.
Thank you and best regards.

Attachments:



Luis González

#2 Sep 11, 2012, 8:17 pm • 1

I find it amusing how many times this problem appears in the forum over and over again.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=41667
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=6086
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=253207
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=407366

Quick Reply

High School Olympiads

Miquel point and Steiner line

Reply

buratinogigle

#1 Sep 10, 2012, 11:15 pm • 1

Let ABC be a triangle inscribed circle (O) and orthocenter H. P is a point on (O). d is Steiner line of P. M is Miquel point of d with respect to triangle ABC. Prove that M, P, H are collinear iff $OP \perp d$.

Luis González

#2 Sep 11, 2012, 9:18 am • 2

Let d cut AC, AB at E, F, respectively. Circumcircle of $\triangle AEF$ cuts (O) again at the Miquel point M of d WRT $\triangle ABC$. BH cuts (O) again at the reflection Y of H across $AC \Longrightarrow P \in EY$.
$\angle YME = \angle AME - \angle AMY = \angle AFE - \angle ABH = \angle YHE \Longrightarrow M, H, E, Y$ are concyclic.

Assume that M, P, H are collinear $\Longrightarrow d \equiv EH$ and MY are antiparallel WRT $PM, PY \Longrightarrow d$ is perpendicular to the P-circumdiameter PO of $\triangle PMY$. Conversely, assume that $PO \perp d$, i.e. d and MY are antiparallel WRT $PM, PY \Longrightarrow$ M, Y, E and $d \cap PM$ are concyclic $\Longrightarrow H \equiv d \cap PM$, i.e. M, P, H are collinear.

Quick Reply

High School Olympiads

Line bisects the area with circumellipse

buratinogigle

#1 Sep 9, 2012, 7:48 pm

Let ABC be a triangle inscribed an Ellipse (E) with center O is midpoint of BC. M,N lie on (E) such that $OM \parallel AB, ON \parallel AC$ and M,N are different side of A with BC. Let AM cuts ON at P, AN cuts OM at Q. Prove that line PQ bisects the area of triangle ABC.

Luis González

#2 Sep 10, 2012, 12:22 pm • 1

Let PQ cut AC, AB at U, V. Project the ellipse into a circle (O) through parallel projection. It's center O is midpoint of $\overline{BC}$ $\Longrightarrow$ the projected $\triangle ABC$ is right at A and the ratio of areas of $\triangle AUV$ and $\triangle ABC$ is invariant. M and N become midpoints of the arcs ABC and ACB of (O). Let M', N' be the antipodes of M, N WRT (O). By obvious symmetry $P \equiv NN' \cap AM \cap BM' \Longrightarrow APM$ bisects $\angle NAN'$. So if the incirle (I, r) of $\triangle ABC$ touches BC at D, we deduce that

$$\frac{AU}{UC} = \frac{NP}{PN'} = \frac{NA}{AN'} = \frac{CD}{r} = \frac{s-c}{s-a}$$

Thus, U is tangency point of the B-excircle with AC. Likewise, V is tangency point of the C-excircle with AB.

$$\frac{[AUV]}{[ABC]} = \frac{AU \cdot AV}{AB \cdot AC} = \frac{(s-b)(s-c)}{bc} = \frac{1}{2}.$$

buratinogigle

#3 Sep 10, 2012, 11:12 pm

Thank for your nice solution. I found it from the post Line bisects area !

High School Olympiads

B,P,Q,R are collinear

Reply

Source: Indonesia National Science Olympiad D2 P4

nivotko

#1 Sep 5, 2012, 2:52 pm

Given a triangle ABC, let the bisector of $\angle BAC$ meets the side BC and circumcircle of triangle ABC at D and E, respectively. Let M and N be the midpoints of BD and CE, respectively. Circumcircle of triangle ABD meets AN at Q. Circle passing through A that is tangent to BC at D meets line AM and side AC respectively at P and R. Show that the four points B, P, Q, R lie on the same line.

Proposer: Fajar Yuliawan

Luis González

#2 Sep 6, 2012, 8:02 am • 1

Since $\angle ABD = \angle AEC$ and $\angle BAD = \angle EAC$, then $\triangle ABD$ and $\triangle AEC$ are similar with corresponding medians AM and $AN \Longrightarrow \angle DAQ = \angle BAM$. But $MB^2 = MD^2 = MP \cdot MA \Longrightarrow BD$ is tangent of $\odot(ABP) \Longrightarrow$ $\angle BAM = \angle DBP \Longrightarrow \angle DAQ = \angle DBP \Longrightarrow P \in BQ$. Further, $\angle MPD = \angle ADB$ and $\angle MPB = \angle ABD \Longrightarrow$ $\angle BPD = \pi - \angle BAD$, or $\angle DPQ = \angle BAD = \angle DAR \Longrightarrow R \in BPQ$.

Quick Reply

High School Olympiads

Equilateral triangle

Reply

Source: easy, own compilation

sunken rock

#1 Sep 5, 2012, 4:10 am

Let ABC be a equilateral triangle, O its circumcircle; midline DE (D, E- midpoints of $[AB], [AC]$ respectively). The arrow $(ED$ intersects the arc $\overset{\frown}{AB}$ at F, $G = CF \cap AB$. The perpendicular from G onto BO intersects the arc $\overset{\frown}{AB}$ at H.
Prove that $AG = BH$.

Best regards,
sunken rock

Luis González

#2 Sep 6, 2012, 2:49 am • 1

In addition, $AG = BH$ is the golden section of the side length of $\triangle ABC$. By power of D WRT (O, R), we deduce that $FD \cdot (DE + FD) = FD \cdot FE = DE^2 \Longrightarrow \frac{DE}{FD} = \varphi$ is the golden ratio. By Menelaus' theorem for $\triangle ADE$ cut by FGC, we get

$$\frac{AG}{GD} = \frac{CA}{CE} \cdot \frac{FE}{FD} = 2 \cdot \frac{FE}{FD} \Longrightarrow \frac{AB}{AG} = 2 - \frac{FD}{FE} = 2 - \frac{1}{\varphi + 1} = \varphi.$$

Hence, AG is golden section of $AB \Longrightarrow AG = 2 \cdot FD$. Thus

$$\frac{AG}{2AB} = \frac{FD}{BC} = \frac{GD}{BG} = \frac{AG - BG}{2BG} \Longrightarrow AG \cdot BG = AB(AG - BG) \Longrightarrow$$

$$R^2 - OG^2 = AG^2 - BG^2.$$

But $GH \perp OB$ yields $R^2 - OG^2 = BH^2 - BG^2 \Longrightarrow AG = BH = \frac{1}{\varphi}AB.$

sunken rock

#3 Sep 6, 2012, 11:41 pm • 1

The inversion of pole B and power $BG \cdot BA$ sends the circle to the line GH, hence $BH^2 = BG \cdot BA$ (1).
To prove that $AG^2 = BG \cdot BA$ (2), see here.
As usual, the solution of ***Luis*** is gewelry!

Best regards,
sunken rock

Quick Reply

High School Olympiads

concurent

unt

#1 Sep 5, 2012, 10:44 am

Let AA_1, AA_2 be altitude, bisector of $\triangle ABC$.
ω -- circle going through A_1, A_2 and tangent to incircle of $\triangle ABC$ at A_3. Difine similary points B_3, C_3.
Show that lines AA_3, BB_3, CC_3 are concurent.

Luis González

#2 Sep 5, 2012, 11:25 am • 2

Incircle (I) and A-excircle (I_a) are tangent to BC at X and Y, respectively. M is the midpoint of BC and MA_4 is the 2nd tangent from M to (I). $A, A_2, I, I_a,$ are harmonically separeted $\Longrightarrow$ their projections A_1, A_2, X, Y on BC are also harmonically separeted. Since M is also midpoint of XY, then by Newton's theorem, we have $MX^2 = MY^2 = MA_4{}^2 = MA_1 \cdot MA_2 \Longrightarrow \odot(A_1A_2A_4)$ is tangent to (I) at $A_4 \Longrightarrow A_4 \equiv A_3$. $\angle XA_3Y$ is clearly right, thus YA_3 goes through the antipode X' of X WRT $(I) \Longrightarrow AA_3 \equiv YX'$ is the A-Nagel cevian of $\triangle ABC$. Similarly, BB_3 and CC_3 are the B- and C- Nagel cevians of $\triangle ABC$.

High School Olympiads

Intersect on Jerabek hyperbola

Reply

buratinogigle

#1 Sep 3, 2012, 10:09 pm • 1

Let ABC be a triangle inscribed circle (O). P is a point on (O). Prove that Steiner line of P with respect to triangle ABC and orthotransversals of P with respect to triangle ABC intersect on Jerabek hyperbola of triangle ABC.

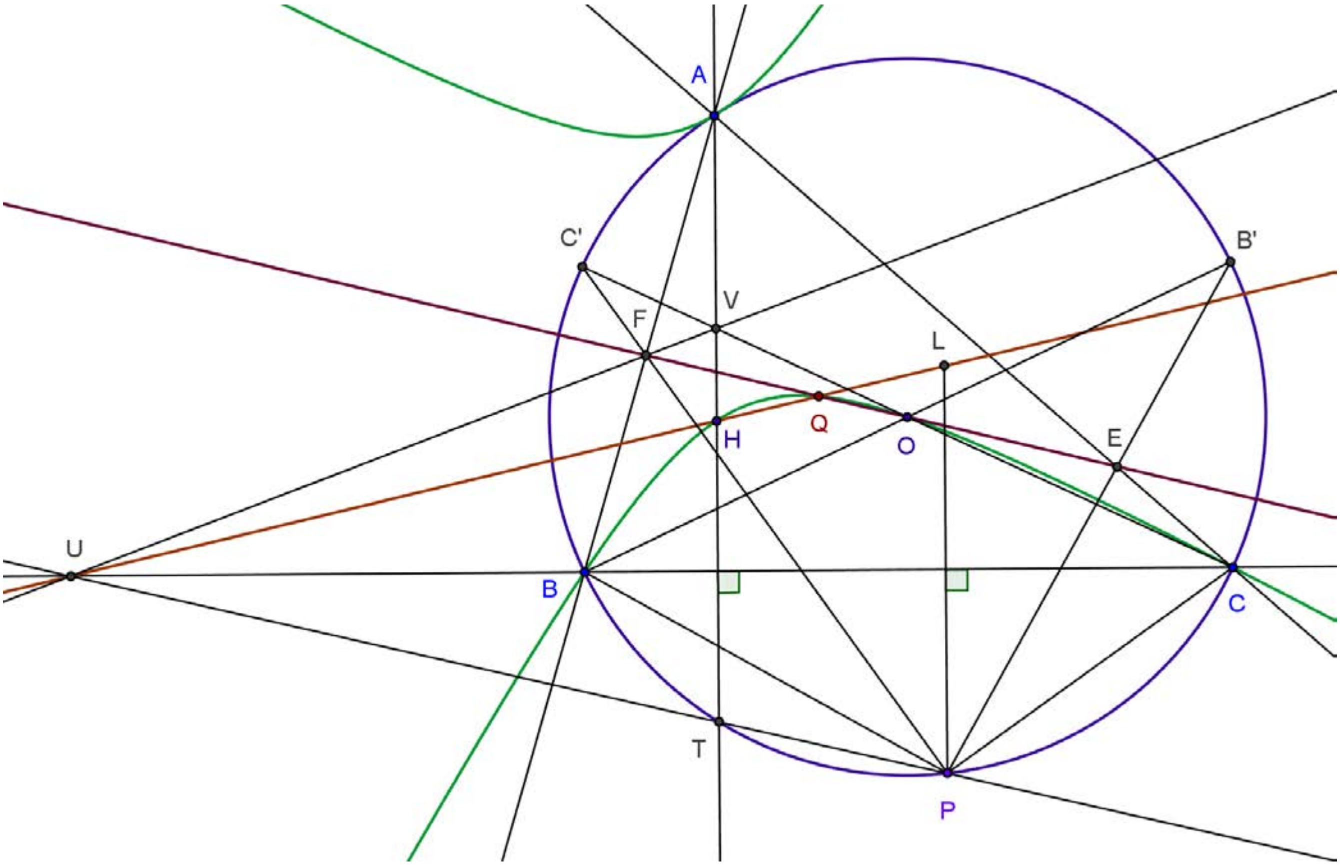
Luis González

#2 Sep 4, 2012, 9:15 am • 3

Perpendicular to PB at P cuts AC at E and (O) again at the antipode B' of B. Perpendicular to PC at P cuts AB at F and (O) again at the antipode C' of $C \Longrightarrow EF$ is orthopolar of P WRT $\triangle ABC$. By Pascal theorem for the hexagon $PB'BACC'$, the intersections $E \equiv PB' \cap AC, O \equiv BB' \cap CC'$ and $F \equiv AB \cap PC'$ are collinear, i.e. $O \in EF$. Let H be the orthocenter of $\triangle ABC$ and L the reflection of P across $BC \Longrightarrow HL$ is Steiner line of P WRT $\triangle ABC$. Since AH cuts (O) again at the reflection T of H about BC, then it follows that PT, HL, BC concur at U.

Let $Q \equiv EF \cap HL$ be the intersection of the orthopolar and Steiner line of P. By Pascal theorem for the hexagon $ATPC'CB$, the intersections $V \equiv AT \cap CC', U \equiv PT \cap BC$ and $F \equiv PC' \cap AB$ are collinear. Thus, in the hexagon $AHQOCB$, the intersections $V \equiv AH \cap CO, U \equiv HQ \cap CB$ and $F \equiv QO \cap AB$ are collinear. By the converse of Pascal theorem, A, B, C, O, H, Q lie on a same conic, i.e. Q is on Jerabek hyperbola of $\triangle ABC$.

Attachments:



Quick Reply

High School Olympiads

Quick one with Triangle ABC (2)

Reply

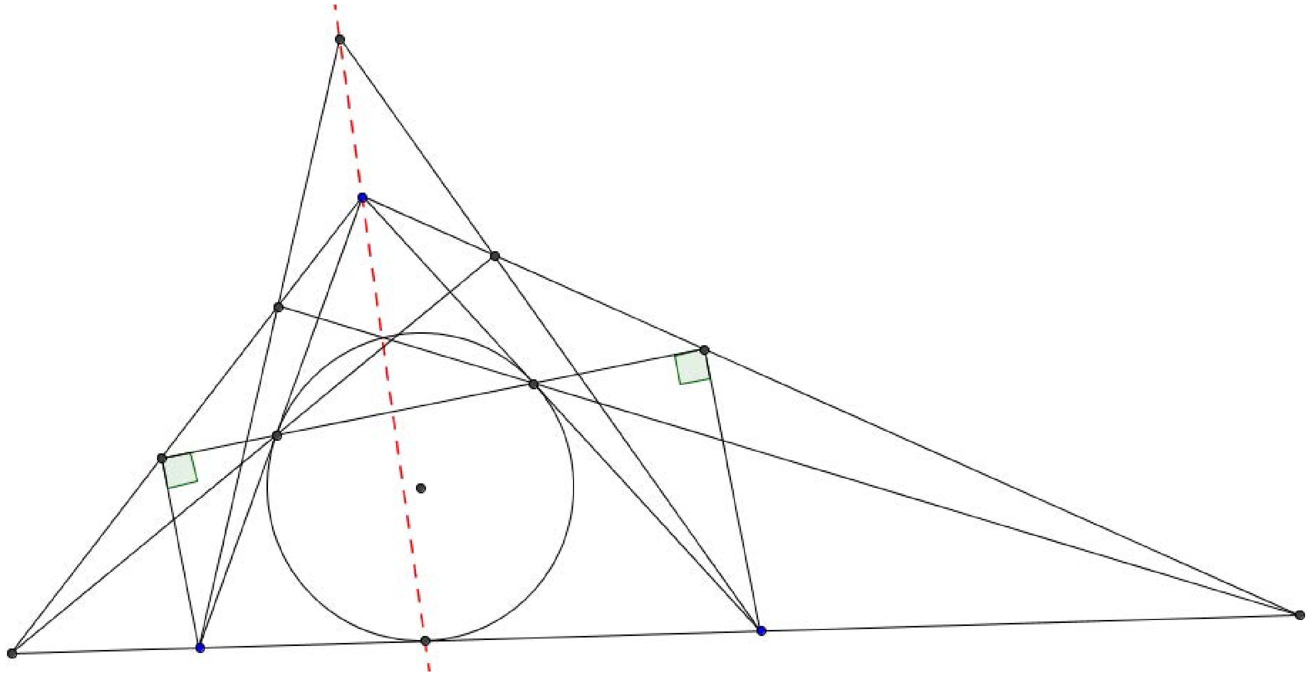
Source: own

applepi2000

#1 Sep 3, 2012, 1:49 am

Let ABC be a triangle with intouch triangle DEF, and let G, H be the feet of the perpendiculars from B, C to EF respectively. Let AG, AH intersect BC at P, Q respectively, and let $PF \cap AQ = X, QE \cap AP = Y$. Finally, let $BY \cap CX = Z$. Prove that A, D, Z are collinear.

Attachments:



Luis González

#2 Sep 4, 2012, 12:53 am • 2

$\triangle DEF$ can be an arbitrary cevian triangle of $\triangle ABC$ and G, H any two distinct points on the line EF. Using the same notations, let $U \equiv PF \cap AC$. Then $(E, F, G, H) = (U, F, P, X) = C(A, F, D, X)$. Similarly, $B(A, E, D, Y) = (E, F, G, H) \Longrightarrow B(A, E, D, Y) = C(A, F, D, X)$. Since the intersection $BE \cap CF$ is on AD, then the intersection $Z \equiv CX \cap BY$ is also on AD, i.e. A, D, Z are collinear.

hyperbolictangent

#3 Sep 4, 2012, 11:22 pm

Could you explain what $C(A, F, D, X)$ means? I assume that (E, F, G, H) is the cross-ratio, but A, F, D, X are not colinear, so I wasn't sure this meant.

applepi2000

#4 Sep 5, 2012, 1:53 am

> *hyperbolictangent wrote:*
>
> Could you explain what $C(A, F, D, X)$ means? I assume that (E, F, G, H) is the cross-ratio, but A, F, D, X are not colinear, so I wasn't sure this meant.

For points A, B, C, D, E, the notation $A(B, C, D, E)$ denotes the cross ratio of the 4 concurrent lines AB, AC, AD, AE. This is well-defined since for any line hitting AB, AC, AD, AE at P, Q, R, S respectively, the cross ratio (P, Q, R, S) is constant. (See here)

Quick Reply

High School Olympiads

Quick one with Triangle ABC

Reply

Source: own

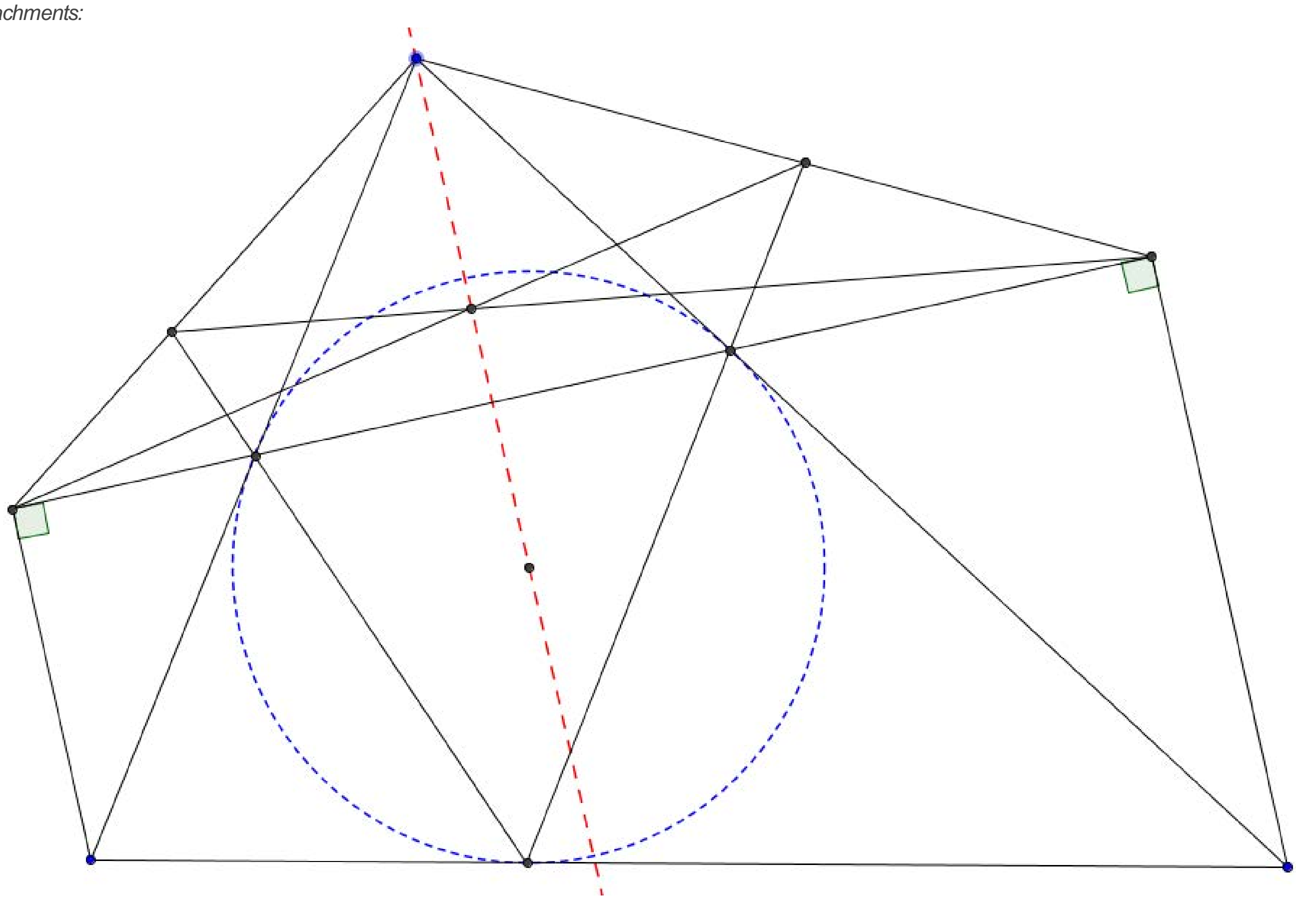
applepi2000

#1 Sep 2, 2012, 6:44 am

Let ABC be a triangle with incenter I and an intouch triangle DEF. The feet of the perpendiculars from B, C to EF are M, N respectively, and $DE \cap AN = X, DF \cap AM = Y, MX \cap NY = Z$. Prove that A, Z, I are collinear.

Edit: Added diagram.

Attachments:



This post has been edited 1 time. Last edited by applepi2000, Sep 2, 2012, 8:50 am

hyperbolictangent

#2 Sep 2, 2012, 8:41 am • 2

Brute Force

I wonder if there is a synthetic approach ...

Luis González

#3 Sep 2, 2012, 10:33 am

AD cuts MN at P and AI cuts BC, MN at $V, Q. QV \parallel MB \parallel NC \Longrightarrow \frac{MQ}{QN} = \frac{BV}{VC} = \frac{c}{b}$. Using that the right triangles $\triangle CNE \sim \triangle BMF$ are clearly similar and that DP is D-symmedian of $\triangle DEF$, we have

$$\frac{MQ}{QN} \cdot \frac{NX}{XA} \cdot \frac{AY}{YM} = \frac{c}{b} \cdot \frac{[DNE]}{[DAE]} \cdot \frac{[DAF]}{[DMF]} = \frac{c}{b} \cdot \frac{[DNE]}{[DMF]} \cdot \frac{[DAF]}{[DAE]} =$$

$$= \frac{c}{b} \cdot \frac{EN}{FM} \cdot \frac{FP}{PE} = \frac{c}{b} \cdot \frac{s-c}{s-b} \cdot \frac{DF^2}{DE^2} = \frac{c}{b} \cdot \frac{s-c}{s-b} \cdot \frac{b(s-b)}{c(s-c)} = 1.$$

By the converse of Ceva's theorem in $\triangle AMN,$ the lines AI, MX, NY concur.

Quick Reply

High School Olympiads

Collinearity in triangle wrt perpendicular bisector

Locked

Narcissus

#1 Sep 2, 2012, 8:48 am

Given a triangle ABC and its circumcircle (O). Let d_A be the perpendicular bisector of OA and X be the intersection of d_A and BC. We define Y, Z similarly. Prove that X, Y, Z are collinear

Luis González

#2 Sep 2, 2012, 8:59 am • 1

Posted at least 4 times before, thus for further discussions use any of the links below.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=1395
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=227144
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=373509
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=462456

High School Olympiads

Sharing same incircle

oneplusone

#1 Aug 19, 2012, 5:18 pm

Let ABC be a non-isosceles triangle with incenter I. The incircle touches the sides of $\triangle ABC$ at A_1, B_1, C_1 (you know where to put them). B_1C_1 intersects the line through I parallel to BC at A_2. Similarly define B_2, C_2. The lines A_1A_2, B_1B_2, C_1C_2 form a triangle $A_3B_3C_3$. Extend A_3A to a point A_4 such that IA bisects $\angle A_3IA_4$. Similarly define B_4, C_4. Prove that $\triangle A_4B_4C_4$ share the same incircle as $\triangle ABC$.

simplependulum

#2 Aug 19, 2012, 7:11 pm

Let's apply pole-polar transformation and we will obtain the same configuration as that from IMO 00 P6.

Luis González

#3 Sep 1, 2012, 2:29 am

O, H are the circumcenter and orthocenter of $\triangle ABC$. $\triangle V_1V_2V_3$ and $\triangle H_1H_2H_3$ are its incentral and orthic triangle. A_2, B_2, C_2 are the poles of AH, BH, CH WRT $(I) \Longrightarrow A_1A_2, B_1B_2, C_1C_2$ are the polars of H_1, H_2, H_3 WRT $(I) \Longrightarrow A_3, B_3, C_3$ are the poles of H_2H_3, H_3H_1, H_1H_2 WRT (I). Since $IA_3 \perp H_2H_3 \perp AO$ and $IA_1 \parallel AH$, then deduce that AI bisects $\angle A_1IA_3$ externally $\Longrightarrow A_4 \in IA_1$. Similarly, $B_4 \in IB_1$ and $C_4 \in IC_1$. So if $P_A \equiv H_2H_3 \cap B_1C_1$, $P_B \equiv H_3H_1 \cap C_1A_2$ and $P_C \equiv H_1H_2 \cap A_1B_1$, then the polars of A_4, B_4, C_4 WRT (I) are the lines ℓ_A, ℓ_B, ℓ_C through P_A, P_B, P_C and perpendicular to IA_1, IB_1, IC_1, respectively.

P_A, P_B, P_C are the A-, B-,C- Pelletier points of $\triangle ABC$, lying on V_2V_3, V_3V_1, V_1V_2. Let V_1X, V_2Y, V_3Z be the 2nd tangents from V_1, V_2, V_3 to (I). V_1V_3 is the polar of $XA_1 \cap ZC_1$ WRT $(I) \Longrightarrow P_B \in XZ$. By symmetry, we have $\angle CB_1X = \angle BC_1A_1 = \angle BA_1C_1 = \angle AB_1Z \Longrightarrow B_1X$ and B_1Z are symmetric WRT $IB_1 \Longrightarrow XZ \perp IB_1$. Consequently, $\ell_B \equiv XZ$. Analogously, $\ell_C \equiv XY \Longrightarrow X \in (I)$ is the pole of B_4C_4 WRT $(I) \Longrightarrow B_4C_4$ is tangent of (I) at X. Likewise, C_4A_4 and A_4B_4 are tangents of (I).

High School Olympiads

Altitudes and Incenter(Own)

Reply

Arab

#1 Aug 31, 2012, 10:20 am

D is the pedal of A on BC of $\triangle ABC$ where $\angle A = 60^\circ$. Let I be the incenter of $\triangle ABC$, M be the midpoint of BC and MI meets AD at N. The circumcircle of $\triangle IDN$ meets AI at P. Prove that, $AD = 2AP$.

Luis González

#2 Aug 31, 2012, 10:49 am • 1

Incircle (I,r) and A-excircle (I_a,r_a) touch BC at X,Y. X' is the antipode of X WRT (I). Since A is the exsimilicenter of $(I) \sim (I_a)$ and their radii $\overline{IX'}$ and $\overline{I_aY}$ are parallel, then $X' \in AY$. Since M is also midpoint of $\overline{XY}$, then IM is X-midline of $\triangle XYX' \Longrightarrow IN \parallel AX' \Longrightarrow ANIX'$ is a parallelogram $\Longrightarrow AN = IX' = r$. By power of point, $AD \cdot AN = AD \cdot r = AI \cdot AP$. When $\angle A = 60^\circ$, then $AI = 2r \Longrightarrow AD \cdot r = 2r \cdot AP \Longrightarrow AD = 2 \cdot AP$.

Arab

#3 Aug 31, 2012, 11:43 am

Let E,F be the intersections of the incircle and AC, AB and we have, $\dfrac{d(E,BC)\cdot d(F,BC)}{d(A,BC)} = \sin^2\dfrac{A}{2}$, where $d(X,YZ)$ means the distance of X and line YZ. On the other hand, $AN = r$.

Quick Reply

High School Olympiads

Right Triangle and a Trapezoid, [KMN] >= 2[AKM]

xeroxia
#1 Aug 30, 2012, 12:59 pm

Let $\triangle ABC$ be a right triangle with $BC > AC > AB$.
Let K and M be arbitrary points on $[AC]$ and $[AB]$, respectively, such that $BCKM$ is a trapezoid.
Let L be the intersection of the diagonals of the trapezoid.
Let N be the foot of perpendicular from L to $[BC]$.
If AN is an angle bisector of $\triangle ABC$, show that $[KMN] \geq 2 \cdot [AKM]$ where $[XYZ]$ denotes the area of $\triangle XYZ$.

Luis González
#2 Aug 31, 2012, 12:23 am

Since $MK \parallel BC$, then L is on the A-median of $\triangle ABC \Longrightarrow AL$ cuts MK at its midpoint S. Perpendicular bisector of MK cuts AN at the midpoint O of the arc MK of $\odot(AMK) \Longrightarrow \triangle OMK$ is right isosceles at O. Let NA, NL cut MK at T, H. Since the pencil formed by AN, AL, AH and the parallel to $BC \parallel MK$ through A is harmonic, then S is the midpoint of $TH \Longrightarrow OS$ is T-midline of $\triangle TNH \Longrightarrow NH = 2 \cdot OS = KM \Longrightarrow [KMN] = \frac{1}{2}KM^2$. If F is the projection of A on KM, the desired inequality is then equivalent to $\frac{1}{2}KM^2 \geq 2 \cdot \frac{1}{2}KM \cdot AF \Longleftrightarrow \frac{1}{2}MK = AS \geq AF$, which is obvious.

xeroxia
#3 Aug 31, 2012, 3:28 am

In fact, it is $[KMN] > 2 \cdot [AKM]$ because the triangle is scalene. I have stated that $BC > AC > AB$ because if it is isosceles the intersection point of the diagonals should always be on AN and $LN \perp BC$.

xeroxia
#4 Aug 31, 2012, 11:33 pm

Let LN cut KM at H. We have $BN/NC = KH/HM$. Let AN cut KM at T. We have $MT/TK = BN/NC$. So we have $MT = KH$.
Let P be the projection of K on BC. $PN = MT = KH$. Let MP cut AN at O. We have $MO = PO$. And since $\triangle MKP$ is right, we have $KO = MO = PO$.
Focus on $\triangle KAM$. O is a point on the angle bisector and $KO = MO$. So either $\angle OMA = \angle OKA$ or $\angle OMA + \angle OKA = 180°$.
The former makes $AK = AM \Rightarrow AB = AC$ which violates $BC > AC > AB$.
So $AKOM$ is cyclic, and $\angle KMO = \angle OAK = 45°$. This makes $\triangle MKP$ an isosceles right triangle.
Since $[KOM] > [AKM]$ because $\triangle KMO$ has greater altitude which equals to circumradius of cyclic $AKOM$.
So $[KMN] = [MPK] = [KOM] + [POK] = 2 \cdot [KOM] > 2 \cdot [AKM]$
(I made much effort not to use $[KMN] = \frac{1}{2}KM^2$, Oh God, I have used it :))

sunken rock
#5 Sep 1, 2012, 2:23 pm

See my 3^{rd} remark **here**, KM is the side of a square inscribed into the right angled triangle ABC, and its side, by calculation is $\frac{KM}{BC} = \frac{bc}{a^2 + bc} \leq \frac{1}{3}$, done. ($a, b, c$ are the sides of the triangle).

Best regards,
sunken rock

High School Olympiads

V midpoint

Reply

Source: PHCP

Manolescu

#1 Aug 22, 2012, 3:38 pm • 1

O is the circumcircle of $\triangle ABC$
J is on the angle bisector of $\angle BAC$
X, Y, Z the pedal of J on BC, CA, AB respectively
JO intersects BC at D
E on AD such that $AJ \parallel EX$
EX intersects YZ at V
Prove that: V is the midpoint of EX

Luis González

#2 Aug 30, 2012, 9:56 am • 2

Redefine E as the reflection of X about YZ and $D \equiv AE \cap BC.$ Then we prove that $D \in OJ.$ M and L are midpoints of $\overline{BC}$ and the arc BC of the circumcircle $(O).$ L_B, L_C are the projections of L on $AC, AB \Longrightarrow L_BML_C$ is Simson line of L WRT $\triangle ABC.$ Since $AZJY$ and AL_CLL_B are homothetic with center $A,$ then it follows that the intersection $F \equiv JX \cap YZ$ is on $AM.$ Hence, all the right $\triangle FXV$ are homothetic with center $M \Longrightarrow V$ and E run on two fixed lines through $M.$ If $H \equiv XE \cap AM,$ then $\frac{EH}{EX}$ is constant. When $J \equiv S \equiv AL \cap BC,$ we deduce that $\frac{EH}{EX} = \frac{MO}{ML}.$ Thus, if S_∞ denotes the infinity point of $AJ \parallel HX,$ we have

$$\frac{\overline{MO}}{\overline{ML}} = \frac{\overline{EH}}{\overline{EX}} = A(E, H, X, S_\infty) = (D, M, X, S) = \frac{\overline{DM}}{\overline{DX}} \cdot \frac{\overline{SX}}{\overline{SM}} = \frac{\overline{DM}}{\overline{DX}} \cdot \frac{\overline{XJ}}{\overline{ML}}$$

$$\Longrightarrow \frac{\overline{MO}}{\overline{XJ}} = \frac{\overline{DM}}{\overline{DX}} \Longrightarrow D \in OJ,$$ as desired.

TelvCohl

#3 Feb 14, 2015, 1:33 pm

My solution:

Redefine E as the reflection of X in YZ, It suffice to prove $AE \cap OJ \in BC$.

Notice that $\angle AYE = \angle AYZ - \angle EYZ = 90^\circ - \frac{1}{2}\angle BAC - \angle ZYX$
$= 90^\circ - \frac{1}{2}\angle BAC - \frac{1}{2}\angle BAC - \angle JYX = \angle OCB - \angle JCX = \angle OCJ$ (Similarly $\angle AZE = \angle OBJ$) ,

so from $XY = CJ \cdot \sin\angle ACB, XZ = BJ \cdot \sin\angle CBA$

$$\Longrightarrow \frac{EY}{EZ} = \frac{CJ}{BJ} \cdot \frac{\sin\angle ACB}{\sin\angle CBA}$$

$$\Longrightarrow \frac{\sin\angle YAE}{\sin\angle ZAE} \cdot \frac{\sin\angle AZE}{\sin\angle AYE} = \frac{\sin\angle COJ}{\sin\angle BOJ} \cdot \frac{\sin\angle OBJ}{\sin\angle OCJ} \cdot \frac{\sin\angle ACB}{\sin\angle CBA}$$

$$\Longrightarrow \frac{\sin\angle YAE}{\sin\angle ZAE} = \frac{\sin\angle COJ}{\sin\angle BOJ} \cdot \frac{\sin\angle ACB}{\sin\angle CBA}$$

$$\Longrightarrow \frac{\sin\angle YAE}{\sin\angle ZAE} \cdot \frac{\cos\angle ACB}{\cos\angle CBA} = \frac{\sin\angle COJ}{\sin\angle BOJ} \cdot \frac{\sin 2\angle ACB}{\sin 2\angle CBA}$$

$$\Longrightarrow \frac{\sin\angle YAE}{\sin\angle ZAE} \cdot \frac{\sin\angle ZAO}{\sin\angle YAO} = \frac{\sin\angle COJ}{\sin\angle BOJ} \cdot \frac{\sin\angle AOB}{\sin\angle COA}$$

$$\Longrightarrow A(Y, Z; E, O) = O(C, B; J, A) \Longrightarrow AE \cap OJ \in BC .$$

Q.E.D

Quick Reply

High School Olympiads

An interesting metric relation with cevians in triangle.

Source: An extension of one characterization for harmonic division.

Virgil Nicula

#1 Aug 30, 2012, 4:06 am

Let $\triangle ABC$ be a triangle. Consider $M\in (BC)$, $N\in (CA)$, $P\in (AB)$ and denote

$R\in AM\cap NP$, $S\in CR\cap AB$. Prove that $1+\frac{MB}{MC}\cdot\frac{NC}{NA}\cdot\frac{PA}{PB}=\frac{PA}{PB}\cdot\frac{SB}{SA}\ (*)$.

Particular case . $\blacktriangleright$ $T\in AM$, where $T\in BN\cap CP$. In this case $(*)$ becomes $\frac{PA}{PB}=2\cdot\frac{SA}{SB}\ (1)$. Is well-known a similar

relation $\frac{TA}{TM}=2\cdot\frac{RA}{RM}\ (2)$. The relations (1) and (2) mean that the divisions (A,S,P,B) and (A,R,T,M) are harmonically.

I was inspired by the proposed problem from **here**.

This post has been edited 9 times. Last edited by Virgil Nicula, Aug 30, 2012, 5:06 am

Luis González

#2 Aug 30, 2012, 4:40 am

Project $\triangle ABC\cup R$ into a triangle $\triangle A^*B^*C^*$ with centroid R^* under central projection. $(MBC)\cdot(NCA)\cdot(PAB)$ and $(PSAB)$ are invariant, thus

$$\frac{PA}{PB}\cdot\frac{SB}{SA}-\frac{MB}{MC}\cdot\frac{NC}{NA}\cdot\frac{PA}{PB}=\frac{P^*A^*}{P^*B^*}\cdot\frac{S^*B^*}{S^*A^*}-\frac{M^*B^*}{M^*C^*}\cdot\frac{N^*C^*}{N^*A^*}\cdot\frac{P^*A^*}{P^*B^*}=$$

$$=\frac{P^*A^*}{P^*B^*}-\frac{N^*C^*}{N^*A^*}\cdot\frac{P^*A^*}{P^*B^*}=\frac{P^*A^*}{P^*B^*}\left(1-\frac{N^*C^*}{N^*A^*}\right)=1$$

Last equality follows from Cristea's theorem for a secant passing through R^*.

High School Olympiads

a line passing through incenter

Reply

mlm95

#1 Aug 30, 2012, 1:01 am

Let ABC be a triangle and I it's incenter. Excircles opposite to B and C touche AC and AB at X_b and X_c respectively. Suppose that the line passing through I and parallel to X_bX_c intersects BC at P. Prove that $\widehat{ATP}=90$ (here T is the intersection of AI with circumcircle of triangle ABC)

Click to reveal hidden text

Luis González

#2 Aug 30, 2012, 1:12 am • 1

Posted before at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=149157
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=298925
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=379785

For a generalization see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=307115

Quick Reply

High School Olympiads

Convex pentagon

Reply

BlackSelena

#1 Aug 28, 2012, 11:24 pm • 1

Give the convex pentagon $ABCDE$ with $\angle B = \angle E = 90^o$. Given that $\angle BAC = \angle EAD$. $BD \cap CE = O$
Prove that $AO \perp EB$.

This post has been edited 1 time. Last edited by BlackSelena, Sep 15, 2014, 6:48 pm

Luis González

#2 Aug 29, 2012, 10:07 am • 5

Let $P \equiv BC \cap DE.$ Then $ABPE$ is cyclic due to the right angles at $B, E \Longrightarrow \angle ABE = \angle APE.$ If Q is the isogonal conjugate of P WRT $\triangle ACD,$ we have $\angle PAE = \angle BAQ \Longrightarrow \angle ABE + \angle BAQ = \angle APE + \angle PAE = 90^\circ,$ i.e. $AQ \perp EB.$ But since $\angle QDC = \angle EDA, \angle QCD = \angle BCA$ and $\angle BAC = \angle EAD,$ then by Jacobi's theorem in ACD, AQ, CE, DB concur at $O \Longrightarrow AO \equiv AQ \perp EB.$

P.S. See also Perpendicularity in a particular pentagon for more solutions.

sunken rock

#3 Aug 29, 2012, 3:56 pm • 3

Alternate proof:

Years ago, my son did show me like this:
Take $N = BC \cap AD, M = DE \cap AC, P$ midpoint of $\overline{CD}, Q$ - midpoint of $\overline{MN}, R$ - midpoint of $\overline{BE}, O = BD \cap CE$.

It is known (and easy to prove) that P, Q, R are collinear and $PQ \perp BE$. We see that $[ABD] = [ECA]$ ($AB \cdot AD \cdot\ sin(\angle BAD) = AC \cdot AE \cdot\ sin(\angle EAC)$) and we want to prove that the locus of points $X|[XND] = [XCM]$ is a parallel through A to PQ.
$2[XND] = \overrightarrow{XA} * \overrightarrow{ND}$ and $2[XCM] = \overrightarrow{XA} * \overrightarrow{CM}$ that is, by substracting the last 2 equalities:
$2([XND] - [XCM]) = \overrightarrow{XA} * (\overrightarrow{ND} - \overrightarrow{CM})$ so, in order for the 2 areas to be equal, we need $\overrightarrow{0} = \overrightarrow{XA} * \overrightarrow{QP} \implies\ AX \parallel PQ$ and we are done. ($2\ \overrightarrow{QP} = \overrightarrow{ND} - \overrightarrow{CM}$).
Note: here $*$ means vector product.

Best regards,
sunken rock

Andrew64

#4 Aug 17, 2013, 9:15 am • 2

> *BlackSelena wrote:*
> Give the convex pentagon $ABCDE$ with $\angle B = \angle E = 90^o$. Know that $\angle BAC = \angle EAD$. $BD \cap CE = O$
> Prove that $AO \perp EB$.

As shown in the figure below.

Let
$BH \perp EC, EG \perp BD, AK \perp BH, AL \perp EG.$

BH intersects AF at P, EG intersects AF at P'

Then

$$\frac{FH}{FG}=\frac{BH}{EG}$$
$$=\frac{BH}{AK}\times\frac{AL}{EG}\times\frac{AK}{AL}$$
$$=\frac{BC}{AB}\times\frac{AE}{ED}\times\frac{AK}{AL}$$
$$=\frac{AK}{AL}$$

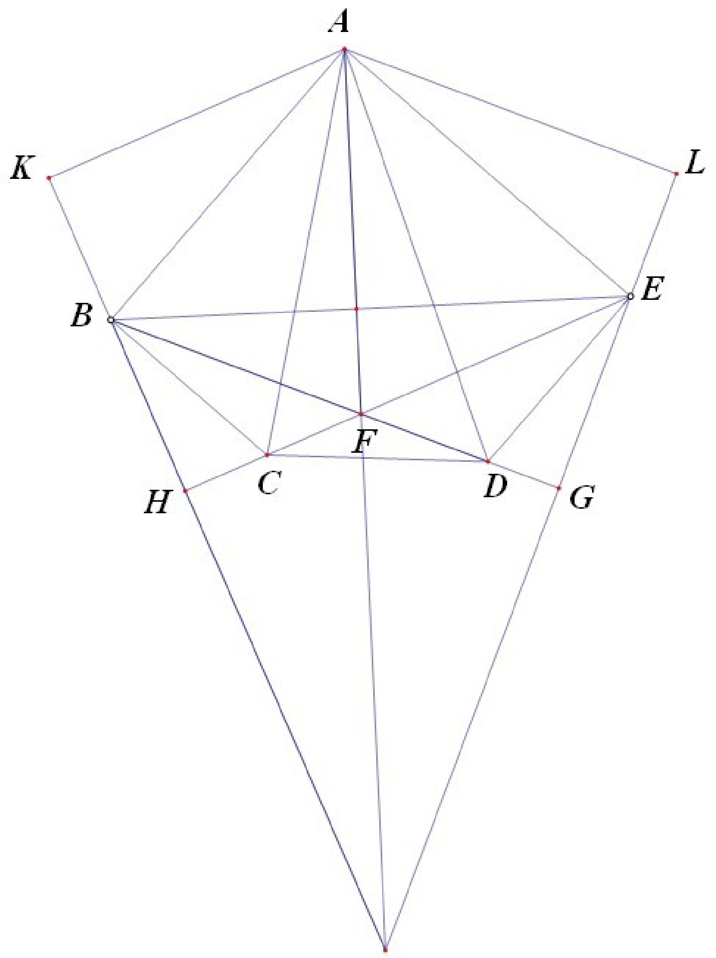
Therefore
$$\frac{AP}{FP}=\frac{AK}{FH}=\frac{AL}{FG}=\frac{AP'}{FP'}$$
P, P' are identical.

So F is the orthocenter of $\triangle BPE$.

Consequently

$AF\perp BE$.

Attachments:



Quick Reply

High School Olympiads

Acute triangle ABC

Reply

Source: Korea Final Round 2011

cadiTM

#1 Aug 28, 2012, 7:38 pm • 1

ABC is an acute triangle. P(different from B, C) is a point on side BC.
H is an orthocenter, and D is a foot of perpendicular from H to AP.
The circumcircle of the triangle ABD and ACD is O_1 and O_2, respectively.
A line l parallel to BC passes D and meet O_1 and O_2 again at X and Y, respectively.
l meets AB at E, and AC at F. Two lines XB and YC intersect at Z.
Prove that $ZE = ZF$ is a necessary and sufficient condition for $BP = CP$.

Luis González

#2 Aug 29, 2012, 6:04 am • 2

$\angle ZXD = \angle BAD, \angle ZYD = \angle CAD \Longrightarrow \angle BZC = \pi - \angle BAC \Longrightarrow Z$ is on circumcircle (O) of $\triangle ABC$. Moreover $\angle ZAC = \angle ZBC = \angle BXD = \angle BAD \Longrightarrow AD, AZ$ are isogonals WRT $\angle BAC$. Let BD, CD cut (O) again at M, N. If MZ cuts AC at F', then $AMF'D$ is cyclic due to $\angle CAD = \angle BAZ = \angle BMZ \Longrightarrow$ $\angle MBC = \angle MAC = \angle MDF' \Longrightarrow DF' \parallel BC \Longrightarrow F \equiv F'$. Likewise, $E \in NZ$ and $ANED$ is cyclic.

Assuming that $ZE = ZF \Longrightarrow \angle MFD = \angle NED \Longrightarrow \angle MAD = \angle NAD$. If AD cuts (O) again at Q, then $QM = QN$, i.e. circumcevian $\triangle QMN$ of D is Q-isosceles $\Longrightarrow D$ is on the A-Apollonius circle (L) of $\triangle ABC \Longrightarrow AD$ is radical axis of (L) and the circle (K) with diameter $\overline{AH}$. This cuts BC at its midpoint, since the circle with diameter $\overline{BC}$ is orthogonal to both (L) and $(K) \Longrightarrow P$ is midpoint of $\overline{BC}$.

duanKHTN

#3 Mar 6, 2013, 11:51 pm

Can you resolved case rest? OK

EDIT: Ah, if $PB = PC$ then BC is common tangent of (O_1) and (O_2). So D and Z opposite each other across BC, so $ZE = ZF$.

rkm0959

#4 Mar 11, 2016, 7:52 am • 1

This took 3 hours last night

Sketch of Proof: We first prove that Z, A, B, C are cyclic. Then, we prove that AP, AZ are isogonal wrt $\angle BAC$.
Finally, we prove that $CD \cap EZ = N$ and $BD \cap FZ = M$ lie on the circumcircle of ABC.
Then, we go into the main proof. The main proof will be in two parts, each part proving one direction.

Step 1: A, B, C, Z are cyclic.
We have

$$\angle BZC = 180 - \angle BXD - \angle CYD = 180 - \angle BAD - \angle CAD = 180 - \angle BAC$$

so we have A, B, C, Z cyclic as desired. ■

Step 2: AP, AZ are isogonal wrt $\angle BAC$.
We have

$$\angle BAD = \angle BXD = \angle ZBC = \angle ZAC$$

which gives us that $AP \equiv AD$ and AZ are isogonal wrt $\angle BAC$. ■

Step 3: $CD \cap EZ = N, ZF \cap BD = M$ lie on the circumcircle of $\triangle BAC$.
We will just prove that $CD \cap EZ$ lie on the circumcircle of $\triangle BAC$. The other part is similar.
We will reword this problem to the following.
Let CD hit the circumcircle of $\triangle BAC$ at N. Prove that N, E, Z are colinear.

We have $\angle AED = \angle ABC = \angle ANC$, so A, N, E, D are cyclic. This gives us

$$\angle DNE = \angle DAE = \angle PAB = \angle ZAC = \angle ZBC = \angle ZNC$$

which is enough to claim that N, E, Z are colinear. ■

We are now ready to go into the main proof.

Sufficient Condition We prove that $ZE = ZF \implies BP = PC$.
First, $\angle ZEF = \angle ZFE$, so $\angle NAD = \angle MAD$. Now, we angle chase

$$\angle XAD = \angle XAB + \angle BAD = \angle XDB + \angle BXD = \angle DBZ = \angle DBC + \angle CBZ = \angle CAM + \angle ZAC = \angle ZAM$$

which gives us $\angle XAZ = \angle DAM$. Similarly we have $\angle YAZ = \angle DAN$, so $\angle XAZ = \angle YAZ$.

I claim that D lies on the A-Apollonius Circle of $\triangle ABC$. It suffices to show that $\dfrac{BD}{CD} = \dfrac{AB}{AC}$.

By Sine law, it suffices to prove

$$\frac{\sin \angle ADB}{\sin \angle BAD} = \frac{\sin \angle ADC}{\sin \angle CAD}$$

and then using $\sin\theta = \sin(180-\theta)$ (in degrees) gives us that it suffices to prove

$$\frac{\sin \angle AXZ}{\sin \angle YXZ} = \frac{\sin \angle AYZ}{\sin \angle XYZ}$$

Now we have

$$\frac{\sin \angle AXZ}{\sin \angle YXZ} = \frac{AZ \cdot \frac{\sin \angle XAZ}{XZ}}{YZ \cdot \frac{\sin \angle XZY}{XY}} = \frac{AZ \cdot XY}{XZ \cdot YZ} \cdot \frac{\sin \angle XAZ}{\sin \angle XZY} = \frac{AZ \cdot XY}{XZ \cdot YZ} \cdot \frac{\sin \angle YAZ}{\sin \angle XZY} = \frac{AZ \cdot \frac{\sin \angle YAZ}{YZ}}{XZ \cdot \frac{\sin \angle XZY}{XY}} = \frac{\sin \angle AYZ}{\sin \angle XYZ}$$

so we are done.

Now AD is the radical axis of the A-Apollonius Circle and the circle with AH as its diameter.
I will prove that the midpoint of BC, M', lie on this radical axis, and this will conclude the proof.

First, the power of the point for the circle with AH as its diameter, we will have

$$(c\sin B - R\cos A)^2 + (\frac{a}{2} - c\cos B)^2 - (R\cos A)^2 = c^2\sin^2 B + c^2\cos^2 B - 2cR\sin B\cos A + \frac{a^2}{4} - ac\cos B = c^2 + \frac{a^2}{4} - c(b\cos A + a\cos B) = \frac{a^2}{4}$$

The power of the point for the A-Apollonius Circle is just $M'B^2 = \dfrac{a^2}{4}$ by a well-known harmonic bundle lemma.

Therefore, M' lie on AD, the radical axis. This gives us that $P \equiv M'$, or $BP = CP$ as required. ■

Necessary Condition We prove that $BP = CP \implies ZE = ZF$.
Take A_0, B_0, C_0 as perpendiculars from A, B, C to BC, CA, AB.
Now

$$AC_0 \cdot AB = AH \cdot AA_0 = AD \cdot AP = AB_0 \cdot AC$$

and now by cyclic quads and easy angle chasing gives us that

$$\angle PDC = \angle PB_0C = \angle ACB$$

and

$$\angle PDB = \angle PC_0B = \angle ABC$$

Now we have

$$\frac{BD}{CD} = \frac{\sin \angle PDC}{\sin \angle PDB} = \frac{\sin \angle ACB}{\sin \angle ABC} = \frac{AB}{AC}$$

$$CD \qquad \sin\angle PDB \qquad \sin\angle ABC \qquad AC$$

Meanwhile, easy angle chasing gives us $\triangle BAZ \sim \triangle PAC$ and $\triangle BAP \sim \triangle ZAC$.
We also have $\frac{AE}{EB} = \frac{AD}{DP} = \frac{AF}{FC}$, so $\triangle AZE \sim \triangle ACD$ and $\triangle AFZ \sim \triangle ADB$.

This gives us that $\frac{EZ}{AZ} = \frac{CD}{ZC} = \frac{BD}{AB} = \frac{FZ}{AZ}$, giving $EZ = FZ$ as required. $\blacksquare$

Quick Reply

High School Olympiads

Bisects

Reply

Source: 2009 Turkey MO Summer Camp

yunustuncbilek

#1 Aug 29, 2012, 1:49 am

Let the circumscribed circle of points A, B, C, D is K. Consider circle K' which is tangent to K, AC, BD at T, E, F respectively. Intersection point of EF and AB is P. Prove that TP bisects ATB.

Luis González

#2 Aug 29, 2012, 2:45 am

This has been posted before with different formulations, e.g.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=397123
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=399496

Quick Reply

High School Olympiads

Simple in structure(Own)

Reply

Arab

#1 Aug 27, 2012, 2:38 pm

Γ is the $Z-$ excircle of $\triangle XYZ$ and Γ tangents XZ, YZ, XY at P, Q, R respectively. ZR meets Γ again at S and T is the midpoint of RS. The angle bisector of $\angle XZY$ meets XY at U. Prove that, $\angle PUQ = \angle XTY + \angle XZY$.

Luis González

#2 Aug 28, 2012, 3:34 am

Let J be the center of the Z-excircle Γ. PQ cuts ZJ at M. PQ, JT, XY concur at the pole L of ZRS WRT Γ. Since $(X, Y, R, L) = -1$ and $TR \perp TL$, then TR bisects $\angle XTY$. RQ cuts ZU at the projection D of X on ZU, i.e. XD is parallel to PQ, meeting ZR at V. Since $R(P, Q, T, L) = -1$, then it follows that $P \in UV$.

$ZPJT$ is cyclic due to right angles at $P, T \Longrightarrow \angle ZTP = \angle ZJP = \angle ZPM = \angle ZXV \Longrightarrow XVTP$ is cyclic $\Longrightarrow$ $\angle XTR = \angle XPV \equiv \angle XPU$. Hence $\angle PUQ = 2\angle PUM = 2(\angle XPU + \angle XZU) = \angle XTY + \angle XZY$.

Arab

#3 Aug 28, 2012, 1:38 pm

Thanks a lot,Luis. :)

Using this we can easily solve the following problem.
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=493348

Quick Reply

High School Olympiads

Concurrent and Perpendicular(Own)

Arab

#1 Aug 27, 2012, 2:33 pm

O is the circumcenter of $\triangle A_1A_2A_3$ and H is the orthocenter. A_1H meets Γ the circumcircle of $\triangle A_1A_2A_3$ at B_1 and C_1 is the midpoint of $\widehat{A_2B_1A_3}$. Similarly define B_2, B_3, C_2, C_3. $B_2C_2 \cap B_3C_3 = X, B_1C_1 \cap B_2C_2 = Y, B_1C_1 \cap B_3C_3 = Z$. Prove that, A_1X, A_2Y, A_3Z are concurent.

Luis González

#2 Aug 27, 2012, 9:18 pm

Generalization. P, Q are two arbitrary points on the plane of $\triangle ABC$ with circumcircle (O). PA, PB, PC cut (O) again at P_1, P_2, P_3 and QA, QB, QC cut (O) again at Q_1, Q_2, Q_3. Let $X \equiv P_2Q_2 \cap P_3Q_3, Y \equiv P_3Q_3 \cap P_1Q_1$ and $Z \equiv P_1Q_1 \cap P_2Q_2$. Then AX, BY, CZ concur.

Proof. Let P_1Q_1, P_2Q_2, P_3Q_3 cut BC, CA, AB at A_1, B_1, C_1, respectively.

$$\frac{A_1B}{A_1C} = \frac{BP_1}{CP_1} \cdot \frac{\sin \widehat{A_1P_1B}}{\sin \widehat{A_1P_1C}} = \frac{\sin \widehat{BAP_1}}{\sin \widehat{CAP_1}} \cdot \frac{\sin \widehat{BAQ_1}}{\sin \widehat{CAQ_1}}$$

Multiply cyclic expressions together, keeping in mind that

$$\frac{\sin \widehat{BAP_1}}{\sin \widehat{CAP_1}} \cdot \frac{\sin \widehat{CBP_2}}{\sin \widehat{ABP_2}} \cdot \frac{\sin \widehat{ACP_3}}{\sin \widehat{BCP_3}} = 1 \ , \ \frac{\sin \widehat{BAQ_1}}{\sin \widehat{CAQ_1}} \cdot \frac{\sin \widehat{CBQ_2}}{\sin \widehat{ABQ_2}} \cdot \frac{\sin \widehat{ACQ_3}}{\sin \widehat{BCQ_3}} = 1$$

$$\Longrightarrow \frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1.$$

Thus, by the converse of Menelaus' theorem, A_1, B_1, C_1 are collinear, i.e. $\triangle ABC$ and $\triangle XYZ$ are perspective through $A_1B_1C_1$. By Desargues theorem, the lines AX, BY and CZ concur.

High School Olympiads

Constructed square over a right triangle

xeroxia
#1 Aug 26, 2012, 11:10 pm

Let ABC be a right triangle with $m(\widehat{A}) = 90^\circ$. Let $KLMN$ be a square such that $N, K \in [BC]$, $M \in [AB]$, and $L \in [AC]$. If AP is an angle bisector of $\triangle ABC$, show that BL and CM intersect at the orthocenter of $\triangle PLM$.

Luis González
#2 Aug 27, 2012, 1:23 am

Center O of $KLMN$ is clearly midpoint of the arc ML of $\odot(AML) \Longrightarrow O \in AP$. Let AP cut ML at T.

$$M(P,T,O,A) = \frac{OP}{OT} \cdot \frac{AT}{AP} = \frac{AT}{AP} = \frac{ML}{BC} = \frac{NB}{NK} \cdot \frac{CK}{CB} = L(B,K,N,C)$$

Since $\angle OMT = \angle NLK = 45^\circ$ and $\angle AMT = \angle CLK$, it follows that $\angle BLK = \angle PMT \Longrightarrow BL \perp PM$. Likewise, $CM \perp PL$ and the conclusion follows.

xeroxia
#3 Aug 30, 2012, 11:45 am

Let AH be the altitude of $\triangle ABC$.
Let BL meet MN and AH at X and Y, respectively.
Let AH meet ML at Z.

$$\frac{AZ}{ZH} = \frac{AL}{LC}, \text{ since } ML \parallel BC \ (1)$$

$$\frac{AZ}{AH} = \frac{ML}{BC} \Rightarrow AH \cdot ML = AZ \cdot BC, \text{ since } ML \parallel BC \ (2)$$

Apply Menelaus at $\triangle AHC$ for the points B, Y, L.

$$\frac{AY}{YH} \cdot \frac{HB}{BC} \cdot \frac{CL}{LA} = 1 \Rightarrow \frac{AY}{YH} = \frac{LA}{CL} \cdot \frac{BC}{BH} \ (3)$$

Merge (1) and (3),

$$\frac{AY}{YH} = \frac{AZ}{ZH} \cdot \frac{BC}{BH}$$

Then merge the above one with (2),

$$\frac{AY}{YH} = \frac{AH \cdot ML}{ZH \cdot BH}$$

Since $ML = ZH$,

$$\frac{AY}{YH} = \frac{AH}{BH}$$

We know $\frac{AY}{YH} = \frac{MX}{XN}$ and $\frac{AH}{BH} = \frac{AC}{AB} = \frac{AL}{AM}$.

So $\frac{MX}{XN} = \frac{AL}{AM}$ (4)

Let AP cut ML at T. Since AP is the angle bisector,

$$\frac{LT}{TM} = \frac{AL}{AM} (5)$$

Merge (4) and (5), we have $MX = TL$ because $MN = ML$. (6)

As **Luis González** stated, the center O of $KLMN$ is on AP because $\angle MAL = \angle MOL = 90^\circ \Rightarrow O \in (AML)$, and $\angle OML = \angle OLM = 45^\circ \Rightarrow O \in AP$.

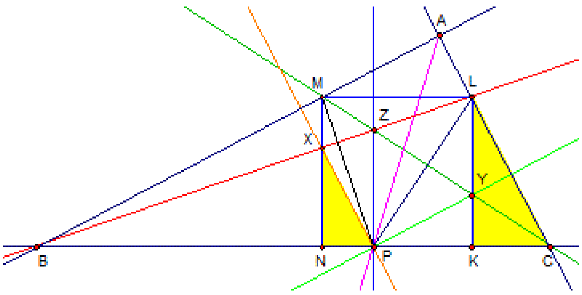
Since MK and LN bisect each other, $TL = NP$. From (6), we have $MX = NP$. So $\angle NMP = \angle MLX \Rightarrow MP \perp BL$. Similarly, $CM \perp LP$. So BL and CM are the altitudes of $\triangle PLM$. Thus they meet at orthocenter of $\triangle PLM$

yetti

#4 Aug 30, 2012, 11:17 pm

Let perpendiculars to AB, CA through P cut MN, KL at X, Y, the orthocenters of $\triangle MBP, \triangle LCP$, resp. Right $\triangle NPX \sim \triangle KCL$ are centrally similar, having parallel sides.
Because of $\frac{BN}{NK} = \frac{[BN]}{[NM]} = \frac{[AB]}{[CA]} = \frac{BP}{PC} \Longrightarrow B$ is their similarity center $\Longrightarrow$ points B, X, L are collinear and $BXL \perp PM$ is both B-altitude of $\triangle MBP$ and L-altitude of $\triangle PLM$.
Likewise, points C, Y, M are collinear and $CYM \perp PL$ is both C-altitude of $\triangle LCP$ and M-altitude of $\triangle PLM$. Combined, $Z \equiv BXL \cap CYM$ is orthocenter of $\triangle PLM$.

Attachments:



xeroxia

#5 Aug 31, 2012, 10:59 pm

Merge bests of **yetti** and **Luis González**.

> *yetti wrote:*
> Let perpendiculars to AB, CA through P cut MN, KL at X, Y, the orthocenters of $\triangle MBP, \triangle LCP$, resp.

> *Luis González wrote:*
> Center O of $KLMN$ is clearly midpoint of the arc ML of $\odot(AML) \Longrightarrow O \in AP$. Let AP cut ML at T.

$\triangle AML \sim \triangle NBM \sim \triangle NXP$, because X is the orthocenter of $\triangle BPM$.
Clearly, $NP = TL$ and $MT = PK$. Since AT is angle bisector of $\triangle AML$, $\frac{AL}{AM} = \frac{TL}{MT} = \frac{NP}{XN}$. So $XN = MT$ and $MX = NP$.
We have $\triangle PNM \cong \triangle XML$ because $PN = MX$,$ML = MN$, and $\angle MNP = \angle XML$. So $\angle NMP = \angle MLX \Rightarrow MP \perp LX$.
Because X is the orthocenter of $\triangle BPM$, XL passes through B. Similarly MY passes through M. So CM and BL are also altitudes of $\triangle PML$. So they meet at the orthocenter of $\triangle MPL$.

sunken rock

#6 Sep 1, 2012, 1:48 pm

Some remarks:
1) Let $BCDE$ the square outside the triangle ABC; L, P, E are collinear, M, P, D collinear as well.
2) AB, PL, CD are concurrent, likewise AC, PM, BE