Courage

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> Another good thing to remember when picking your target is that the problems that are most simply stated are often the most difficult ... tackling simply-stated problems often involves generating an entire framework in which to attack the problem.
>
> — Richard Rusczyk, USAMO strategy session in WOOT

§1 Lecture Notes

In my opinion, the scariest problems aren't the one with lots of intimidating notation but those two-line problems[1] you see stuck in the #6 slot. The distinct characteristic of the problems in this unit is they give you seemingly nothing to work with.

Advice:

- **Don't be scared**, hence the name of this lecture. Sometimes the problems are genuinely hard, but sometimes they are not, and you can't tell by looking.
- **Be more willing to do more elaborate setups**. For simply stated problems, you should be more willing to set up frameworks (think back to "free"). Getting footholds helps you at least make progress, especially if the problem doesn't let you look at small cases.
- **It's okay to have long solutions**. It turns out that it's hard to have a problem satisfying all three constraints:
  - The problem statement is simple and natural,
  - The solution is also short, but
  - The problem is not trivial.

  So it's not uncommon to have simply stated problems in the #3 slot turn out to have fairly long details.

(Idiosyncratic note: a lot of your final problems will fit this category, too.)

[1] I don't mean two-line just in the sense they are short: most inequalities and functional equations are one line, for example, but the displayed equation has enough complexity that you can at least start. In these problems, the statement is both short and simple, almost mysteriously.

§1.1 Examples

Example 1.1 (HMMT 2016 C9)

Let $V = \{1, \dots, 8\}$. How many permutations $\sigma \colon V \to V$ are automorphisms of some tree?

Walkthrough. This is sort of a good example because it looks somewhat scary at first, but can be solved by calmly making the necessary observations and setting up the right framework.

(a) Prove that σ works as long as it has a fixed point.

(b) Show that whether or not σ works depends only on the *cycle structure* of σ (corresponding to a partition of 8).

(c) Make a list of all possible cycle structures which do not have a 1 (i.e. a fixed point). I think there should be seven of them.

(d) Among these seven, show that exactly three of them work (and determine which ones they are).

(e) Use complementary counting to extract the final answer 30212.

Example 1.2 (China TST 2018/2/2)

Let $p(n)$ denote the partition function. Find all positive integers n so that $p(n) + p(n+4) = p(n+2) + p(n+3)$.

Walkthrough. We begin by generating functions. Let $N = n+4$, so we want $p(N) + p(N-4) - p(N-1) - p(N-2) = 0$.

(a) Relate the given quantity in the problem to the generating function

$$\frac{1 - X - X^2 + X^4}{(1-X)(1-X^2)(1-X^3)\dots} = \frac{\frac{1-X-X^2+X^4}{(1-X)(1-X^2)}}{(1-X^3)(1-X^4)(1-X^5)\dots}$$

(b) Show that the generating function of (a) becomes

$$\frac{1 - X^3 - X^5 - X^7 - X^9 - \dots}{(1-X^3)(1-X^4)(1-X^5)\dots}.$$

This gives us the idea that most of the coefficients should be eventually negative. So let's think about this instead.

A partition of n is called *coarse* if all the parts are at least three. For example, here are all the coarse partitions of the first few numbers.

$$\begin{aligned}
&3 \\
&4 \\
&5 \\
6 &= 3+3 \\
7 &= 3+4 \\
8 &= 3+5 = 4+4 \\
9 &= 3+6 = 4+5 = 3+3+3 \\
10 &= 3+7 = 4+6 = 5+5 = 3+3+4 \\
11 &= 3+8 = 4+7 = 5+6 = 3+3+5 = 3+4+4 \\
12 &= 3+9 = 4+8 = 5+7 = 6+6 = 3+3+6 = 3+4+5 \\
&= 4+4+4 = 3+3+3+3 \\
13 &= 3+10 = 4+9 = 5+8 = 6+7 = 3+3+7 \\
&= 3+4+6 = 3+5+5 = 4+4+5 = 3+3+3+4 \\
14 &= 3+11 = 4+10 = 5+9 = 6+8 = 7+7 \\
&= 3+3+8 = 3+4+7 = 3+5+6 = 4+4+6 = 4+5+5 \\
&= 3+3+3+5 = 3+3+4+4.
\end{aligned}$$

Let $q(n)$ denote the number of coarse partitions of n. By convention $q(0) = 1$.

(c) Use (b) to conclude that we seek N such that $q(N) = q(N-3) + q(N-5) + q(N-7) + \dots$.

(d) Identify three N for which the equality in (c) holds.

We'll now prove these are the only ones.

(e) Show directly that for all odd N, we have

$$q(N) \le q(N-3) + q(N-5) + q(N-7) + \dots.$$

(f) Prove that if $N \ge 11$ then there exists a coarse partition of N whose terms are not all the same. Conclude that for odd $N \ge 11$ we cannot have equality in (e).

When N is even, we run into the issue that N may have no odd parts, so the proof in (e) breaks down. However, one should sense that we are close, and should be able to force it through.

(g) Show that if $N \in \{6, 8, 10, 12\}$, we have

$$q(N) > q(N-3) + q(N-5) + q(N-7) + \dots.$$

Now assume $N \ge 14$ is even. Let's deal with the troublesome permutations. Suppose

$$\pi \colon N = a_1 + \dots + a_k$$

is a coarse partition *whose parts are all even*, and with $k > 1$. We define

$$f(\pi) = (a_1 - 1) + a_2 + \dots + (a_k + 1).$$

(h) Show that f is injective, and has at least two distinct terms.

(i) Repeating the idea of (e), use the map f to prove that $q(N) \le 1 + q(N-3) + q(N-5) + \dots$; the extra 1 term comes from the trivial partition $N = N$ not being processed by f.

(j) For even $N \ge 14$, give two examples of partitions with at least two distinct terms which are not of the form $f(\pi)$ for any π as described.

(k) Conclude that $q(N) \le -1 + q(N-3) + q(N-5) + \dots$ for even $N \ge 14$, finishing the problem.

Not an easy problem.

Example 1.3 (USA TST 2004/3)
What is the largest n for which one can draw a convex n-gon whose vertices lie in a 2004×2004 unit lattice?

Walkthrough. As usual, we're going to adopt the point of view that a polygon is a collection of n vectors with sum zero. Each vector can be written as $v_i = c_i \langle a_i, b_i \rangle$ where $\gcd(a_i, b_i) = 1$ and $c_i \in \mathbb{Z}_{>0}$. Here the integers a_i and b_i need not be positive or nonzero, but they should not both be zero (and if $a_i = 0$ then $b_i = \pm 1$).

(a) Show that $\sum c_i |b_i| \le 2 \cdot 2003$ over all the vectors i. (Consider the highest and lowest vertices of the polygon.)

(b) Prove that $4 \cdot 2003 \ge \sum_i |a_i| + |b_i|$.

(c) Show that any pair (a, b) of relatively prime positive integers, there are at most four vectors v_i with $|a_i| = a$ and $|b_i| = b$.

(d) Deduce from (c) that for any $\ell \ge 1$, at most $4\varphi(\ell)$ vectors have $T(v_i) = \ell$.

(e) Evaluate $\sum_{\ell=1}^{21} \ell\varphi(\ell)$ and $\sum_{\ell=1}^{21} \varphi(\ell)$.

(f) Use (b), (d), (e) in tandem to prove that $n \le 561$.

(g) Why did we pick the constant 21 in (e)?

Thus we have an upper bound of 561. We will achieve it:

(h) Show that we can fit a unique lattice 560-gon in a square of length 1997 (i.e. a 1998×1998 lattice).

(i) Remove two vectors and add three more to obtain a 561-gon fitting in a square of length 2003 (i.e. a 2004×2004 lattice).

§2 Practice problems

Instructions: Solve [72♣]. If you have time, solve [90♣]. Problems with red weights are mandatory.

> Is there an endless supply of foolish heroes with death wishes? Honestly, where do you all come from?
>
> Mal Keshar in *Descent into Darkness*, from *The Battle for Wesnoth*

[9♣] **Problem 1** (HMMT 2010)**.** Let S be a convex subset of $\mathbb{R}^2$ which has finite area a. Prove that either $a = 0$ or S is bounded.

[9♣] **Problem 2** (USAMO 2006)**.** Find all positive integers n for which there exist $k \geq 2$ positive rational numbers $a_1, \dots, a_k$ satisfying $a_1 + a_2 + \cdots + a_k = a_1 a_2 \dots a_k = n$.

[9♣] **Problem 3** (TSTST 2011)**.** Let ABC be a triangle. Its excircles touch sides BC, CA, AB at D, E, F. Prove that the perimeter of triangle ABC is at most twice that of triangle DEF.

[9♣] **Problem 4** (HMMT 2017)**.** Let $w = w_1 w_2 \dots w_n$ be a word of length n. Show that w has at most n *distinct* contiguous palindromic substrings.

[9♣] **Problem 5** (HMMT 2016)**.** Fix positive integers $r > s$, and let $\mathcal{F}$ be an infinite family of sets, each of size r, no two of which share fewer than s elements. Prove that there exists a set of size $r - 1$ that shares at least s elements with each set in $\mathcal{F}$.

[9♣] **Required Problem 6** (IMO 2008/3)**.** Prove that there are infinitely many positive integers n such that $n^2 + 1$ has a prime factor greater than $2n + \sqrt{2n}$.

[9♣] **Problem 7** (Shortlist 2014)**.** Let P be a chessboard polygon, and assume P may be tiled with S-tetrominoes. Prove that if P is tiled with S-tetrominoes and Z-tetrominoes, then the number of Z-tetrominoes used is even.

[9♣] **Required Problem 8** (Iran TST 2008/3)**.** Let T be a tree with n edges. Let Q_n denote the n-dimensional hypercube graph (with 2^n vertices and $n \cdot 2^{n-1}$ edges). Prove that the edges of Q_n can be partitioned into graphs isomorphic to T.

[9♣] **Problem 9** (Math Prize 2019/4)**.** Let n and d be positive integers such that $d \geq n$ and d divides $1+2+\cdots+n$. Show that the set $\{1, \dots, n\}$ can be partitioned into disjoint subsets such that the sum of the numbers in each subset equals d.

[9♣] **Problem 10** (USAMO 2004/3)**.** For what real values of $k > 0$ is it possible to dissect a $1 \times k$ rectangle into two similar but noncongruent polygons?

[9♣] **Problem 11** (TSTST 2015/3)**.** Let P be the set of all primes, and let M be a non-empty subset of P. Suppose that for any non-empty subset $\{p_1, p_2, \dots, p_k\}$ of M, all prime factors of $p_1 p_2 \cdots p_k + 1$ are also in M. Prove that $M = P$.

[9♣] **Problem 12** (ELMO Shortlist 2011)**.** Prove that a convex pentagon $A_1A_2A_3A_4A_5$ has exactly one pair $\{P, Q\}$ of isogonal conjugates in its interior (meaning $\angle PA_iA_{i-1} = \angle A_{i+1}A_iQ$ for $1 \leq i \leq 5$, indices taken mod 5).

[9♣] **Problem 13** (USAMO 2007/3)**.** Let S be a set containing $n^2 + n - 1$ elements. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

[9♣] **Problem 14** (Shortlist 2015)**.** Let S be a nonempty set of positive integers. We say that a positive integer n is *clean* if it has a unique representation as a sum of an odd number of distinct elements from S. Prove that there exist infinitely many positive integers that are not clean.

[1♣] **Mini Survey.** At the end of your submission, answer the following questions.

(a) About how many hours did the problem set take?

(b) Name any problems that stood out (e.g. especially nice, instructive, boring, or unusually easy/hard for its placement).

Any other thoughts are welcome too. Examples: suggestions for new problems to add, things I could explain better in the notes, overall difficulty or usefulness of the unit.

§3 Solutions to the walkthroughs

§3.1 Solution 1.1, HMMT 2016 C9

The answer is 30212. The proof is based on the cycle type of σ (unsurprising since if σ works so does its entire conjugacy class!).

Note that the sum of degrees of T is 14. The ones that work are:

- If σ has a fixed point, then take a star, so it works.
- $2+6$: double broom with two ends with three hairs at each end ($\to - \leftarrow$)
- $2+2+2+2$: take a path
- $2+2+4$: $>---<$.

The ones that don't work are:

- $3+5$: suppose ab is an edge where a has order 3, b has order 5. Then $a\sigma^3(b)$ is an edge, etc, so a neighbors with all order-five vertices. This gives us a $K_{3,5}$ which is clearly not a subgraph of a tree.

  The number of permutations with this cycle type is $\binom{8}{3} \cdot 2! \cdot 4! = 2688$.
- $2+3+3$: similar argument. The number of permutations with this cycle type is $\binom{8}{2} \cdot \frac{1}{2} \cdot \binom{6}{3} \cdot (2!)^2 = 1120$.
- $4+4$: sum of degrees is not divisible by 4. The number of permutations with this cycle type is $\frac{1}{2} \cdot \binom{8}{4} \cdot (3!)^2 = 1260$.
- 8: sum of degrees not divisible by 8. The number of permutations with this cycle type is $7! = 5040$.

Hence the answer $8! - (2688+1120+1260+5040) = 30212$.

§3.2 Solution 1.2, China TST 2018/2/2

Solution with Brice Huang, Kevin Sun, and Ashwin Sah: The answers are $n = 1, 3, 5$. We instead show that $N = 5, 7, 9$ are the only solutions to

$$p(N) + p(N-4) = p(N-1) + p(N-2).$$

(which is equivalent via $N = n+4$).

Terminology A partition of n is called *coarse* if all the parts are at least three. It is *nontrivial* if it has more than one part. Let $q(n)$ denote the number of coarse partitions of n.

Given a coarse partition π, we let $t(\pi)$ denote the number of distinct odd parts it has. We say a coarse partition π is *even* if $t(\pi) = 0$ (meaning all parts of π are even), and *odd* otherwise.

Examples For concreteness, the coarse partitions of the first few values of n are:

$$\begin{aligned}
&3 \\
&4 \\
&5 \\
6 &= 3+3 \\
7 &= 3+4 \\
8 &= 3+5 = 4+4 \\
9 &= 3+6 = 4+5 = 3+3+3 \\
10 &= 3+7 = 4+6 = 5+5 = 3+3+4 \\
11 &= 3+8 = 4+7 = 5+6 = 3+3+5 = 3+4+4 \\
12 &= 3+9 = 4+8 = 5+7 = 6+6 = 3+3+6 = 3+4+5 \\
&= 4+4+4 = 3+3+3+3 \\
13 &= 3+10 = 4+9 = 5+8 = 6+7 = 3+3+7 \\
&= 3+4+6 = 3+5+5 = 4+4+5 = 3+3+3+4 \\
14 &= 3+11 = 4+10 = 5+9 = 6+8 = 7+7 \\
&= 3+3+8 = 3+4+7 = 3+5+6 = 4+4+6 = 4+5+5 \\
&= 3+3+3+5 = 3+3+4+4
\end{aligned}$$

which gives $q(6) = q(7) = 2$, $q(8) = 3$, $q(9) = 4$, $q(10) = 5$, $q(11) = 6$, $q(12) = 9$, $q(13) = 10$, $q(14) = 13$. By convention, $q(0) = 1$.

Reduction to coarse partitions The motivation for defining $q(n)$ comes from the following generating functions computation.

Claim — For all $N \ge 5$,

$$\begin{aligned}
&p(N) + p(N-4) - p(N-1) - p(N-2) \\
&= q(N) - q(N-3) - q(N-5) - q(N-7) - q(N-9) - \dots.
\end{aligned}$$

Proof. This proof is by generating functions. First, write:

$$\begin{aligned}
\frac{1-X-X^2+X^4}{(1-X)(1-X^2)(1-X^3)\dots} &= \frac{\frac{1-X-X^2+X^4}{(1-X)(1-X^2)}}{(1-X^3)(1-X^4)(1-X^5)\dots} \\
&= \frac{\frac{1-X^2-X^3}{1-X^2}}{(1-X^3)(1-X^4)(1-X^5)\dots} \\
&= \frac{1-\frac{X^3}{1-X^2}}{(1-X^3)(1-X^4)(1-X^5)\dots} \\
&= \frac{1-X^3-X^5-X^7-X^9-\dots}{(1-X^3)(1-X^4)(1-X^5)\dots}.
\end{aligned}$$

Since $\prod_{k\ge1}(1-X^k)^{-1} = \sum_N p(N)X^N$, and $\prod_{k\ge3}(1-X^k)^{-1} = \sum_N q(N)X^N$, this implies the desired result. $\square$

Estimates First, we completely resolve the case when N is odd.

Claim — For all odd $N \ge 5$,

$$q(N) \le q(N-3) + q(N-5) + q(N-7) + \dots$$

with equality if and only if $N = 5$, $N = 7$, $N = 9$.

Proof. The main observation is that $q(N-3)$ counts the number of coarse partitions of N containing at least one 3, et cetera. As N is odd, $t(\pi) \ge 1$ for all coarse partitions π of N. Then we have

$$q(N) = \sum_{\pi} 1 \le \sum_{\pi} t(\pi) = q(N-3) + q(N-5) + \dots$$

where the sum is over coarse partitions of N as desired.

Equality holds if and only if $t(\pi) = 1$ for every π. This occurs when $N = 5, 7, 9$ but not for $N \ge 11$ in light of the coarse partition $3 + 5 + (N-8)$. □

The even case will require some more care.

Claim — For $N \in \{6, 8, 10, 12\}$ we have

$$q(N) - 1 = q(N-3) + q(N-5) + q(N-7) + \dots$$

and for all even $N \ge 14$ we have

$$q(N) < q(N-3) + q(N-5) + q(N-7) + \dots.$$

Proof. The proof is similar to before, but this time we must take care of even partitions.

Consider the following operation f: given an even coarse partition $\pi = a_1 + a_2 + \dots + a_k$ with $k \ge 2$ (here $4 \le a_1 \le \dots \le a_k$), we consider the partition

$$f(\pi) = (a_1 - 1) + a_2 + \dots + (a_k + 1).$$

This is an odd coarse partition with $t(f(\pi)) = 2$. Observe that f is injective, and let S denote its image.

Thus

$$\begin{aligned} q(N) &= 1 + \sum_{\text{odd } \pi} 1 + \sum_{\text{even nontriv } \pi} 1 = 1 + \sum_{\text{odd } \pi} 1 + \sum_{\text{odd } \pi \in S} 1 \\ &= 1 + \sum_{\text{odd } \pi \notin S} 1 + \sum_{\text{odd } \pi \in S} 2 \\ &\le 1 + \sum_{\text{odd } \pi \notin S} t(\pi) + \sum_{\text{odd } \pi \in S} t(\pi) = 1 + \sum_{\text{odd } \pi} t(\pi) \\ &= 1 + q(N-3) + q(N-5) + \dots. \end{aligned}$$

One can then observe that for $N \in \{6, 8, 10, 12\}$, all the estimates are sharp, in the sense that $t(\pi) \le 2$ with equality when $\pi \in S$ for odd coarse partitions of N.

As for $N \ge 14$, notice that $\pi_1 = 3 + 3 + 3 + (N-9)$ and $\pi_2 = 3 + 5 + (N-8)$ are two partitions of N with $t(\pi_1) = t(\pi_2) = 2$ but which are not in S, thus giving us $q(N) \le -2 + (1 + q(N-3) + \dots)$ as claimed. □

Remark. An alternative approach is to write

$$\frac{1 - X - X^2 + X^4}{(1-X)(1-X^2)(1-X^3)\dots} = \frac{1 - X^5 - X^7 - X^8 - X^9 - \dots}{(1-X^4)(1-X^5)\dots}.$$

which makes the casework much more manageable, since a $-X^k$ will appear for every $k \ge 7$.

§3.3 Solution 1.3, USA TST 2004/3

The answer is 561.

We first prove that this is an upper bound. Orient the polygon counterclockwise, so each side of the polygon can be thought of as a nonzero vector $v_i = c_i \langle a_i, b_i \rangle$ where $\gcd(a_i, b_i) = 1$ and $c_i \in \mathbb{Z}_{>0}$. Here the integers a_i and b_i need not be positive or nonzero, but they should not both be zero (and if $a_i = 0$ then $b_i = \pm 1$).

We define the *reduced taxicab length* of such a vector by $T(v_i) = |a_i| + |b_i|$. (Our construction will have $c_i = 1$ for all i, so this is essentially synonymous with the taxicab length.)

By considering the highest and lowest vertices of the polygon, we find that the sum of absolute values of the vertical components of all n vectors is at most $2003 \cdot 2$, (2003 upwards and 2003 downwards). Similarly, the sum of absolute values of horizontal components of all n vectors is at most $2003 \cdot 2$. This gives an upper bound on the sum of the taxicab lengths.

On the other hand, for any pair (a, b) of relatively prime positive integers there are at most four vectors v_i with $|a_i| = a$, $|b_i| = b$ (corresponding to choices of $\pm$ signs). From this we can conclude for a given $\ell \ge 1$, there are at most $4\varphi(\ell)$ vectors with $T(v_i) = \ell$.

Compute

$$\begin{aligned} \varphi(1) + 2\varphi(2) + \dots + 21\varphi(21) &= 1997 \\ \varphi(1) + \varphi(2) + \dots + \varphi(21) &= 140. \end{aligned}$$

Thus we compute

$$\begin{aligned} 4 \cdot 2003 &\ge \sum_{i=1}^{n} T(v_i) \\ &\ge 1 \cdot 4\varphi(1) + 2 \cdot 4\varphi(2) + 3 \cdot 4\varphi(3) + \dots + 21 \cdot 4\varphi(21) + 22 \cdot (n - 560) \\ &= 4 \cdot 1997 + 22(n - 560). \end{aligned}$$

Thus $n \le 561$ as desired.

To give a construction, one might note that taking the 560 relatively prime vectors of length ≤ 21 and arranging them suitably gives a convex 560-gon which fits in a square of length 1997. Now remove $(8, 13)$, $(1, -20)$; add $(-3, 19)$, $(15, -7)$, $(-3, -19)$; many other choices possible. (Note that the inequality has exactly 2 in "wiggle room", so the choices are fairly constrained.)

Remark. Legend has it this problem was solved by only one student on the exam.