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The Associated Harmonic Quadrilateral

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Abstract. In this article we study a natural association of a harmonic quadrilateral to every non-parallelogrammic quadrilateral. In addition we investigate the corresponding association in the case of cyclic quadrilaterals and the reconstruction of the quadrilateral from its harmonic associated one. Finally, we associate to a generic quadrilateral a cyclic one.

1. Harmonic quadrilaterals

Harmonic quadrilaterals, introduced by Tucker and studied by Neuberg ([1, p.206], [6]) can be defined in various equivalent ways. A simple one is to draw the tangents FA, FC to a circle κ from a point F (can be at infinity) and draw also an additional secant FBD to the circle (see Figure 1(I)). Another definition starts

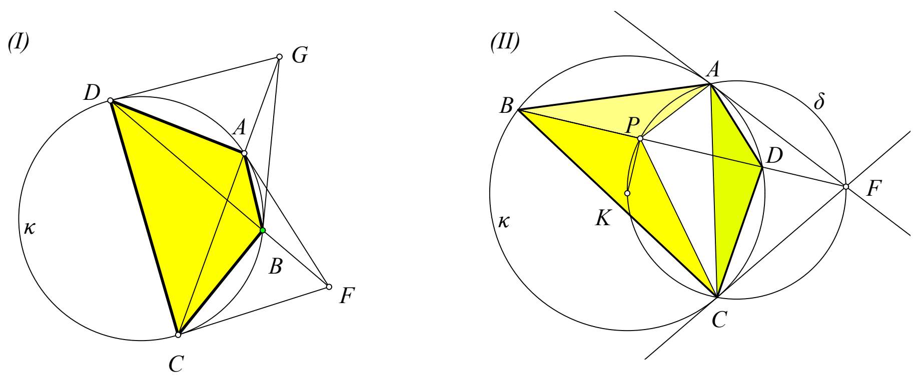

Figure 1. Definition and a basic property

with an arbitrary triangle ABD and its circumcircle κ and defines C as the intersection of κ with the symmedian from A. These convex quadrilaterals have several interesting properties exposed in textbooks and articles ([5, p.100,p.306], [8]). One of them, used in the sequel, is their characterization as convex cyclic quadrilaterals, for which the products of opposite side-lengths are equal $|AB||CD| = |BC||DA|$ or, equivalently, the ratios of adjacent side-lengths are equal $\frac{|AB|}{|AD|} = \frac{|CB|}{|CD|}$. Another property, also used below, deals with a dissection of the quadrilateral in similar triangles (see Figure 1(II)), which I formulate as a lemma without a proof.

Lemma 1. *Let $ABCD$ be a harmonic quadrilateral and P be the projection of its circumcenter K onto the diagonal BD. Then triangles ADC, APB and BPC are similar. Furthermore, the tangents of its circumcircle at points A and C intersect*

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at a point F of the diagonal BD and the circumcircle δ of ACF passing through K and P.

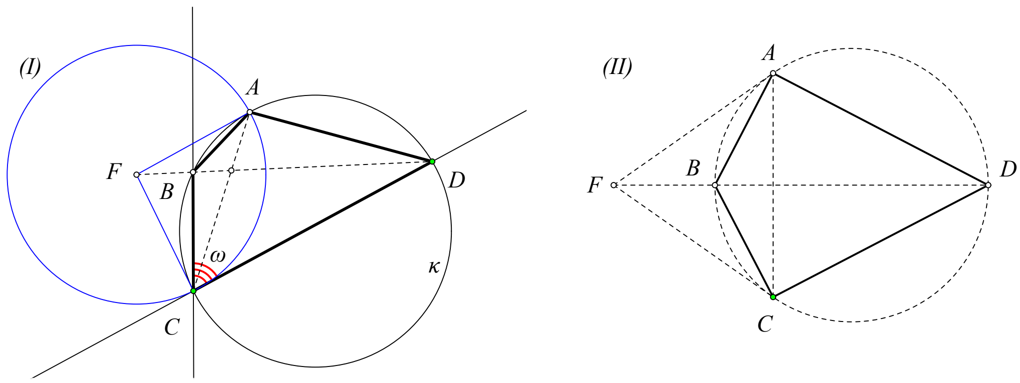

Figure 2. Determination by ω and $r = \frac{|AB|}{|AD|}$ Kite

Note that, up to similarity, a harmonic quadrilateral is uniquely determined by its angle $\omega = \angle BCD$ and the ratio $r = \frac{|AB|}{|AD|} = \frac{|CB|}{|CD|}$ (see Figure $2(I)$). In fact, fixing the circle κ and taking an inscribed angle of measure ω, the angle-sides determine a chord BD of length depending only on κ and ω. Then, points A, C on both sides of BD are determined by intersecting κ with the Apollonius circle ([2, p.15]), dividing BD in the given ratio r.

A special class of harmonic quadrilaterals, comprising the squares, is the one of *kites*, which are symmetric with respect to one of their diagonals (see Figure $2(II)$). Excluding this special case, for all other harmonic quadrilaterals there is a kind of symmetry with respect to the two diagonals, having the consequence, that in all properties, including one of the diagonals, it is irrelevant which one of the two is actually chosen.

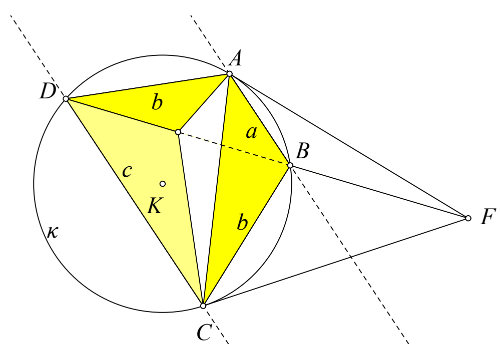

Figure 3. Harmonic trapezia

Another class of special harmonic quadrilaterals is the one of *harmonic trapezia*, comprising all equilateral trapezia with side lengths satisfying $ac = b^2$ (see Figure

3). This, up to similarity, is also a one-parameter family of harmonic quadrilaterals. Given the circle $\kappa(r)$, each harmonic trapezium, inscribed in κ, is determined by the ratio $k = \frac{a}{b} < 1$ of the small parallel to the non-parallel side-length. A short calculation shows that to each such trapezium corresponds a special triangle ABD with data

$$a = k'r, \qquad b = \frac{k'}{k}r, \qquad \cos B = \frac{1-k^2}{2k},$$

where $k' = \sqrt{\frac{4k^2-(1-k^2)^2}{2}}$.

2. The associated harmonic quadrilateral

In the sequel we restrict ourselves to non-parallelogrammic convex quadrilaterals. For every such quadrilateral $p = ABCD$ there is a harmonic quadrilateral q, naturally associated to p. The next theorem shows how to construct it.

Theorem 2. *The two centers Z_1, Z_2 of the similarities f_1, f_2, mapping respectively $f_1(A) = C, f_1(B) = D, f_2(B) = D, f_2(C) = A$, of a non-parallelogramic quadrilateral $p = ABCD$, together with the midpoints M, N of its diagonals AC, BD, are the vertices of a harmonic quadrilateral $q = NZ_1MZ_2$, whose circumcircle κ passes through the intersection point E of the diagonals.*

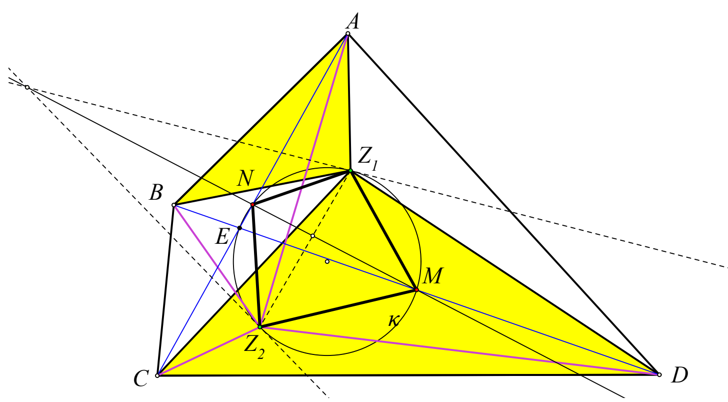

Figure 4. The harmonic quadrilateral associated to a quadrilateral

In fact, let κ be the circle passing through the midpoints M, N of the diagonals and also passing through their intersection point E. Some special cases in which point E is on line MN are handled below. Point Z_1 is the center of similarity f_1 ([3, p.72], [11, II, p.43]) mapping the triangle ABZ_1 correspondingly onto CDZ_1 (see Figure 4). Analogously, point Z_2 is the center of the similarity mapping the triangle BCZ_2 onto DAZ_2. It follows easily, that the triangles based on the diagonals, ACZ_1 and BDZ_1, are also similar, their similarity ratio being equal to those of their medians from Z_1, as well as their corresponding bases coinciding with the diagonals $\lambda = \frac{|Z_1N|}{|Z_1M|} = \frac{|AC|}{|BD|}$. This implies also that the angles formed by corresponding medians of the two similar triangles are equal, i.e., ANZ_1 and

EMZ_1 are equal angles. This implies that Z_1 is on κ. Analogously is seen that Z_2 is also on κ and that the ratio $\frac{|Z_2N|}{|Z_2M|} = \lambda$. Thus,

$$\frac{|Z_1N|}{|Z_1M|} = \frac{|Z_2N|}{|Z_2M|},$$

which means that the cyclic quadrilateral Z_1MZ_2N is harmonic.

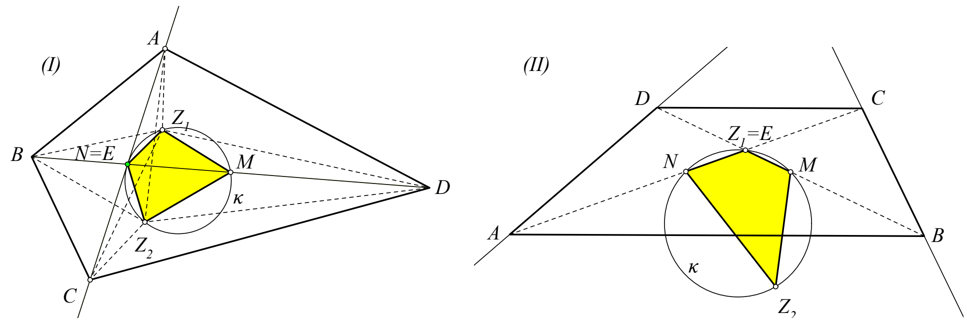

Figure 5. Point E coinciding with N Point Z_1 coinciding with E

In the case one of the midpoints of the diagonals coincides with their intersection point ($N = E$) the circle κ passes through the midpoints M, N of the diagonals and is tangent to the diagonal (AC), whose midpoint coincides with E (see Figure $5(I)$). Another particular class is the one of trapezia, characterized by the fact that one of the similarity centers (Z_1) coincides with the intersection E of the diagonals (see Figure $5(II)$).

3. The inverse construction

Fixing a harmonic quadrilateral q and selecting two opposite vertices Z_1, Z_2 of it, we can easily construct all convex quadrilaterals p having the given one as their associated. This reconstruction is based on the following lemma.

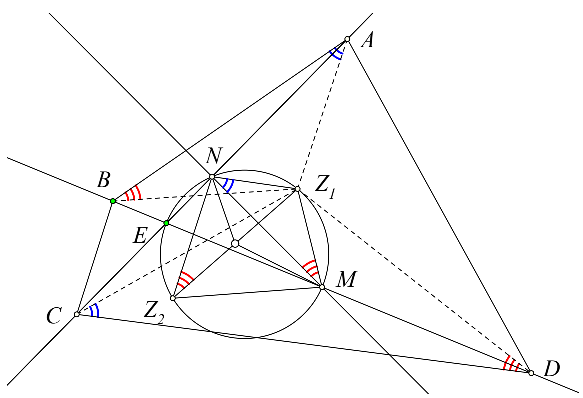

Figure 6. Generating the quadrilateral from its associated harmonic one

Lemma 3. *Let $p = ABCD$ be a convex quadrilateral with associated harmonic one $q = NZ_1MZ_2$, such that Z_1 is the similarity center of triangles ABZ_1, CDZ_1. Then triangle NMZ_1 is also similar to the above triangles.*

In fact, by the Theorem 2, triangles ACZ_1, BDZ_1 are also similar, and N, being the midpoint of side AC, maps, by the similarity sending ACZ_1 to BDZ_1, to the corresponding midpoint M of CD (see Figure 6). This implies that triangles Z_1AN, Z_1BM are also similar, hence $\frac{|Z_1N|}{|Z_1M|} = \frac{|Z_1A|}{|Z_1B|}$. Since the rotation angle, involved in the similarity mapping ACZ_1 to BDZ_1, is the angle AZ_1B, this angle will be also equal to angle NZ_1M, thereby proving the similarity of triangles ABZ_1 and NMZ_1.

Lemma 3 implies that all quadrangles $p = ABCD$, having the given quadrangle $q = NZ_1MZ_2$ as their associated harmonic, are parameterized by the similarities f with center at Z_1. For, each such similarity produces a triangle $ABZ_1 = f(NMZ_1)$ and defines through it the two vertices A, B. The other two vertices C, D of the quadrilateral p are found by taking, correspondingly, the symmetrics of A, B with respect to N and M. Note, that, by reversing the argument in Lemma 3, the diagonals AC, BD of the resulting quadrilateral intersect at a point E of the circumcircle of the harmonic quadrilateral. Hence their angle is the same with angle NZ_1M. Also the ratio of the diagonals of $ABCD$ is equal to the ratio $\frac{|Z_1N|}{|Z_1M|}$, thus it is determined by the harmonic quadrilateral $q = NZ_1MZ_2$. We have proved the following theorem.

Theorem 4. *Given a harmonic quadrilateral $q = NZ_1MZ_2$, there is a double infinity of quadrilaterals $p = ABCD$ having q as their harmonic associate with similarity centers at Z_1 and Z_2 and midpoints of diagonals at M and N. All these quadrilaterals have their diagonals intersecting at the same angle NZ_1M, the same ratio $\frac{|AC|}{|BD|} = \frac{|Z_2N|}{|Z_2M|}$ and their Newton lines coinciding with MN. Each of these quadrilaterals is characterized by a similarity f with center at Z_1, mapping $f(Z_1NM) = Z_1AB$.*

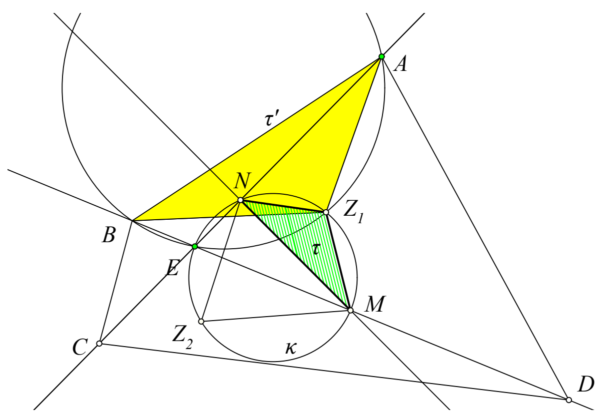

Figure 7. Alternative generation of $ABCD$ from the harmonic quadrilateral

An alternative way to generate all quadrilaterals with given harmonic associate $q = NZ_1MZ_2$ and similarity centers at Z_1, Z_2, is to use a point E on the circumcircle κ of q, draw lines EM, EN, and consider their intersections A, C with the circles passing through E and Z_1. Equivalently, construct all triangles Z_1AB similar to Z_1NM and having the vertex A on line EN. Then the other vertex B moves on line EN ([11, II, p.68]) and C, D are again, respectively, the symmetrics of A, B with respect to N and M. A fourth method is described in §7.

4. Two related similar quadrilaterals

In order to prove some additional properties of our configuration, the following lemma is needed, which, though elementary in character, I could not locate a proof of it in the literature. For the completeness of the exposition I outline a short proof of it.

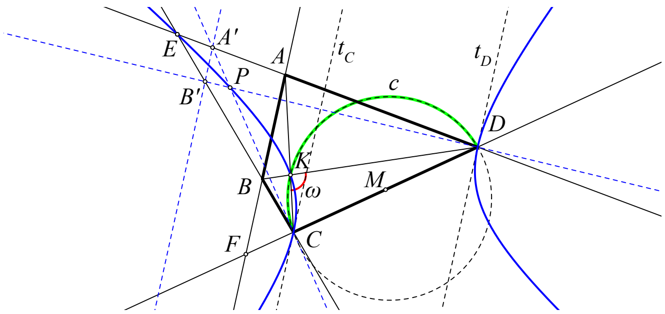

Figure 8. Quadrilateral from angles and angle of diagonals

Lemma 5. *Two quadrilaterals having equal corresponding angles and equal angles between diagonals are similar.*

In fact, let $ABCD$ be a quadrilateral with given angles and the angle ω between its diagonals. The two triangles ECD, FAD, where E, F are the intersection points of opposite sides, have known angles and are constructible up to similarity. Thus, we can fix triangle ECD and move a line parallel to AF intersecting the sides EC, ED correspondingly at B', A'. The quadrilateral with the required data must have the angle formed at the intersection point $P = (A'C, B'D)$ equal to ω. This position K for P is found as follows (see Figure 8). As $A'B'$ moves parallel to itself it creates a homographic correspondence $B' \mapsto A'$ between the points of the lines EC and ED and induces a corresponding homography between the pencils of lines at C and D. Then, according to the Chasles-Steiner theorem, the intersection point P of corresponding rays CA', DB' describes a conic ([9, p.109]). It is easily seen that this conic is a hyperbola passing through the vertices of triangle ECD, whose tangents at C, D are parallel to $A'B'$ and its center is the midpoint M of CD. The intersection point K of the conic with a circular arc c of points viewing CD under the angle ω determines the quadrilateral with the required properties and shows that it is unique, up to similarity.

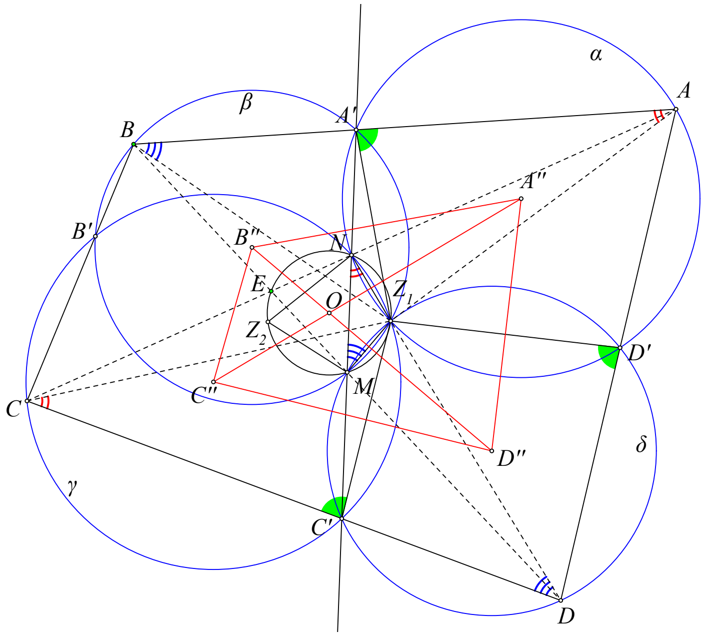

Figure 9. Four circles intersecting on the sides

Theorem 6. (1) *The circles* $\alpha = (Z_1NA), \beta = (Z_1MB)$ *pass through the intersection point* A' *of the Newton line with side* AB*. Analogously, the circles* $\gamma = (Z_1NC), \delta = (Z_1MD)$ *pass through the intersection point* C' *of the Newton line with side* CD*.*

(2) *Circles* β *and* γ *intersect at a point* B' *of* BC*. Analogously circles* α *and* δ *intersect at a point* D' *of* AD*.*

(3) *The centers* A'', B'', C'', D'' *of corresponding circles* $\alpha, \beta, \gamma, \delta$ *build a quadrilateral* $A''B''C''D''$ *similar to* $ABCD$*, whose diagonals pass through* O*.*

(4) *Analogous to the above properties hold by replacing* Z_1 *with* Z_2 *and defining* A', B', C', D' *and circles* $\alpha, \beta, \gamma, \delta$ *properly.*

In fact, (1) and (2) result by a simple angle chasing argument (see Figure 9). (3) follows from the Lemma 5 and the fact that $A''B''C''D''$ has the same angles with $ABCD$ and also the same angle of diagonals, which intersect at O. (4) is proved by the same arguments.

5. The case of cyclic quadrilaterals

The location of the similarity centers Z_1, Z_2 in the case of a cyclic quadrilateral is, in most cases, immediate according to the following.

Theorem 7. *In the case of a cyclic quadrilateral* $p = ABCD$*, whose opposite sides intersect at points* F, G*, the similarity centers* Z_1, Z_2 *are the intersections of the circumcircle* κ *of the associated harmonic quadrilateral with the circle* μ *on diameter* FG*.*

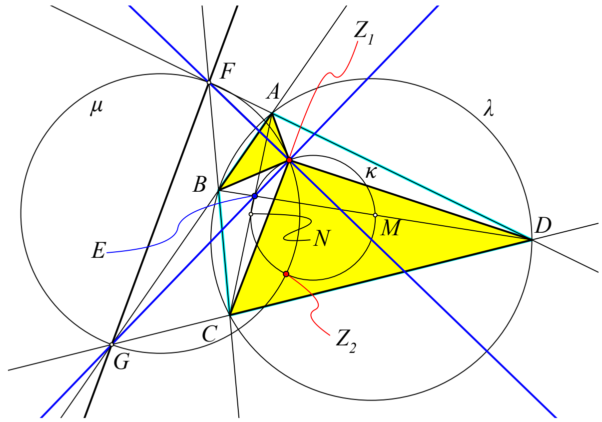

Figure 10. The case of cyclic $ABCD$

In [7] it is proved that a quadrilateral p is cyclic if and only if the circle μ, with diameter FG, is orthogonal to the corresponding circle $\kappa = (MNE)$. Thus, in this case there are indeed two intersection points Z_1, Z_2 on κ (see Figure 10). There is also proved, that in this case line FG is the polar of E and coincides with the radical axis of the pencil of circles generated by κ and the circumcircle λ of $ABCD$. Since angle FZ_1G is a right one and points (B, C, N, F) make a harmonic division, the two lines Z_1G, Z_1F are the bisectors of the angle BZ_1C as well as of angle AZ_1D. Thus, angles AZ_1B and CZ_1D are equal and angles AZ_1C, BZ_1D are also equal. Since G is on the radical axis of κ and λ the quadrilateral CDZ_1E is cyclic, hence the angles ECZ_1 and EDZ_1 are equal. This implies that triangles AZ_1C and BZ_1C are similar and from this follows that triangles AZ_1B, CZ_1D are also similar. This identifies point Z_1 with the center of similarity transforming AB to CD. Analogously is proved the corresponding property for the other intersection point Z_2.

Next theorem explores the possibility to determine a generic cyclic quadrilateral $p = ABCD$ on the basis of its associated harmonic one.

Theorem 8. *A convex cyclic quadrilateral p, whose opposite sides intersect, is uniquely determined from its associated harmonic quadrilateral q and the location of the intersection E of the diagonals of p on the circumcircle κ of q. Point E can be taken arbitrarily on the arc defined by Z_1Z_2, which is less than half the circumference of κ. All cyclic quadrilaterals resulting by such a choice of E have the angle between their diagonals equal to $\angle Z_1MZ_2$ or its complementary and the ratio of diagonal-lengths equal to $\frac{|Z_1M|}{|Z_1N|} = \frac{|Z_2M|}{|Z_2N|}$.*

The first statement follows easily from two facts. The first is that, according to Theorem 7, the circle μ on diameter FG, where F, G are the intersections of opposite sides of $p = ABCD$, is orthogonal to the circumcircle κ of q and its center is at the intersection P of tangents to κ, respectively at Z_1 and Z_2 or the pole of Z_1Z_2 with respect to κ. Hence this circle is constructible from the data of the harmonic

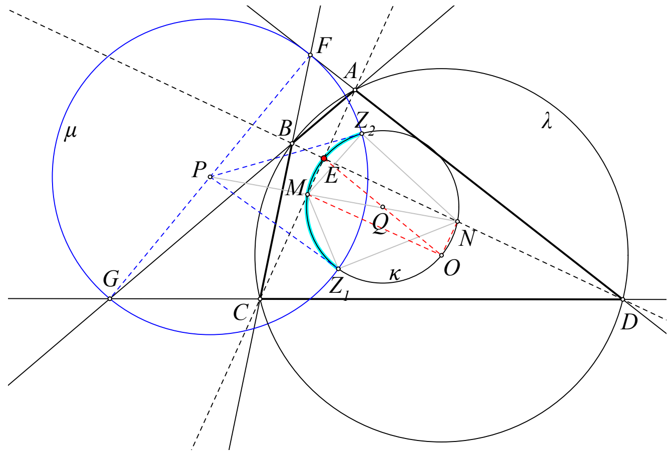

Figure 11. Constructing the cyclic $ABCD$ from its associated harmonic

quadrilateral $q = MZ_1NZ_2$. The second fact, proved in the aforementioned reference, is that the circumcircle λ of the quadrilateral p is orthogonal to μ and its center is the diametral point O of E with respect to circle κ. This implies that λ can be constructed as the circle, which is orthogonal to μ and has its center at O. Having this circle, we obtain the vertices of the quadrilateral p by intersecting it with lines EM and EN. The other statements follow from fundamental properties of the harmonic quadrilateral, such as, for example, the fact, that M, N are separated by Z_1, Z_2 and that generic cyclic convex quadrilaterals have the intersection point E always in the arc Z_1Z_2, which is less than half the circumference of κ.

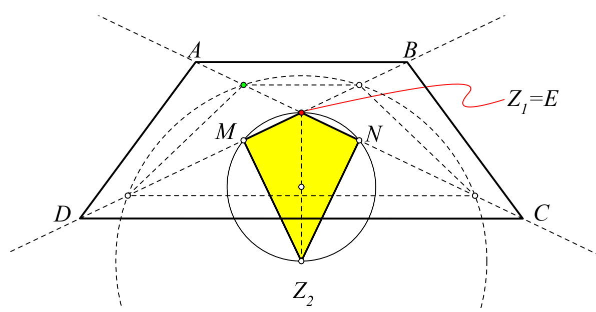

Figure 12. Associated harmonic quadrilateral of an isosceles trapezium

Having excluded from the beginning the parallelogrammic quadrilaterals, which have both pairs of opposite sides intersecting at infinity, the case of cyclic quadrilaterals, not included in both theorems, is the one of equilateral trapezia, having one pair of sides intersecting at infinity. In this case the harmonic associated is found easily, having the similarity centers coinciding correspondingly with the intersection point $E = Z_1$ of the diagonals and the circumcenter $O = Z_2$ (see Figure 12). Theorem 8 is not valid in this case, since, then, there are infinite many cyclic quadrilaterals with the same harmonic associate. In fact, in this case, every circle centered at $Z_2 = O$, with radius $r > |Z_1Z_2|$ defines, through its intersections with lines Z_1M, Z_1N, an equilateral trapezium having the given $q = NZ_1MZ_2$ for harmonic associated. Two other cases, in which the intersection point E of the diagonals of $p = ABCD$ coincides with a particular point, are the quadrilaterals having $E = N$, i.e., coinciding with the midpoint of one diagonal (see Figure 13),

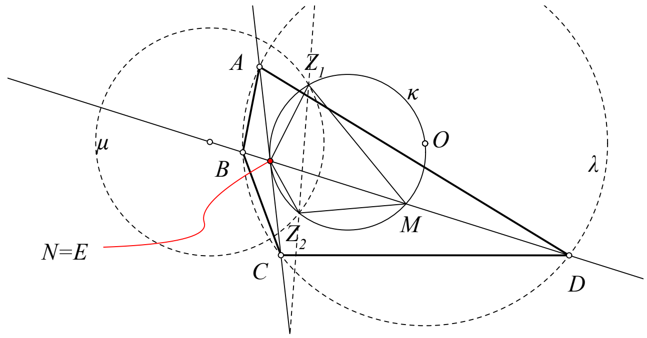

Figure 13. The case $E = N$

and the quadrilaterals $p = ABCD$, which are also themselves harmonic. In this

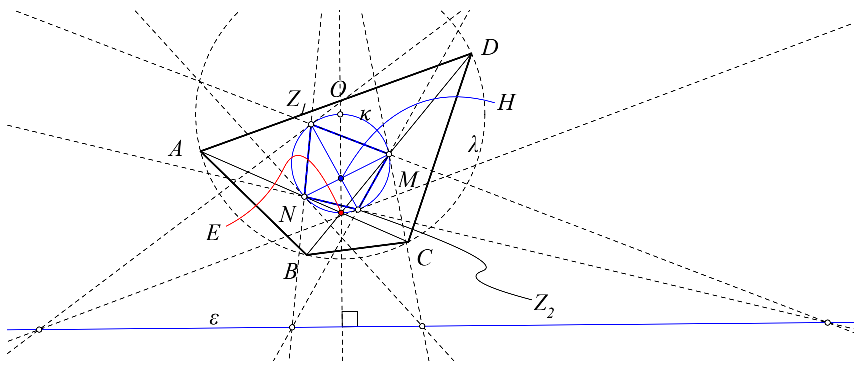

Figure 14. The case $p = ABCD$ is also harmonic

case E is on the diameter of the circumcircle κ of q, which contains the intersection point H of the diagonals of q. Then the polar ε of H with respect to κ coincides with the radical axis of the circle κ and the circumcircle λ of p (see Figure 14).

6. The two lemniscates

Fixing the harmonic quadrilateral $q = NZ_1MZ_2$, as seen in the previous section, all cyclic quadrilaterals p, having q as their associated, are parameterized by a point E varying on an arc Z_1Z_2 of the circumcircle κ of q. The following theorem shows that the vertices of the resulting quadrilaterals $p = ABCD$ vary on two lemniscates of Bernoulli ([10, p.13], [4, p.110]).

Theorem 9. *The vertices of all convex cyclic quadrilaterals $p = ABCD$, having the same harmonic associated quadrilateral $q = NZ_1MZ_2$ are on two Bernoulli lemniscates with nodes, respectively, at M and N. Each pair of opposite vertices lies on the same lemniscate and is symmetric with respect to the corresponding node.*

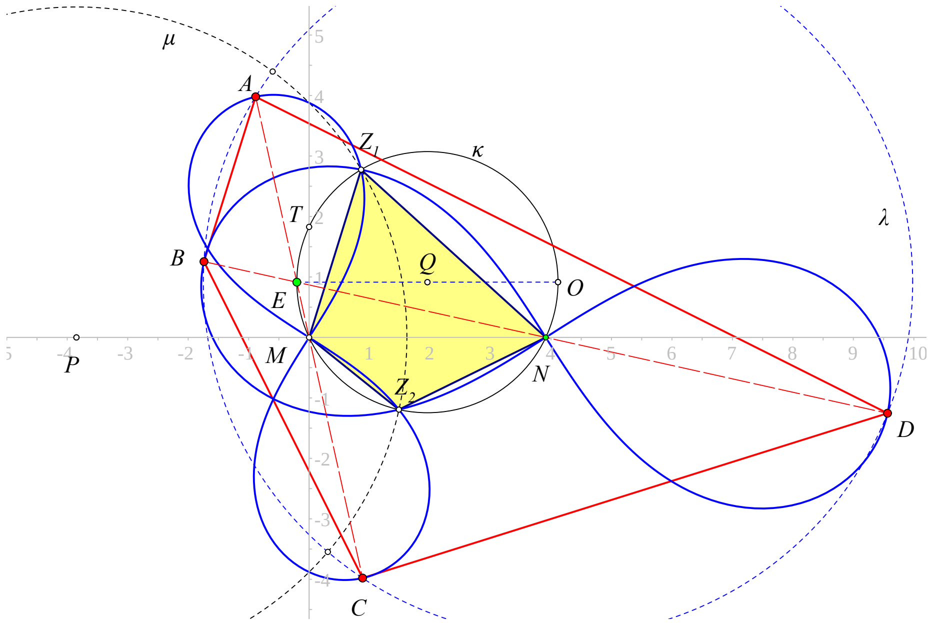

Figure 15. Geometric locus of vertices of $ABCD$ with given harmonic associated

The proof of the theorem follows from a simple calculation, using cartesian coordinates centered at the vertex M of the given harmonic quadrilateral $q = NZ_1MZ_2$. Vertex N is set at $(n, 0)$ and the circumcircle κ of q intersects the y-axis at $(0, t)$. Point $P(p, 0)$ is the center of the circle μ, which passes through Z_1, Z_2 and is orthogonal to κ. The equations can be set in dependence of the parameters n, p and t by following the recipe of reconstruction of p from q, described in Theorem 8. For a variable point $E(u, v)$ on κ, the intersection points of line EM and the circle λ, centered at the diametral O of the point E and orthogonal to μ, are found by eliminating (u, v) from the three equations representing the circle λ, the line ME and the circle κ. These are correspondingly:

$$\begin{aligned} x^2 + y^2 - 2x(n-u) - 2y(t-v) - 2pu + pn &= 0, \\ vx - uy &= 0, \\ u^2 + v^2 - nu - tv &= 0. \end{aligned}$$

Eliminating (u, v) from these equations, leads to an equation of the 8-th degree, which splits into the two quadratics $(x-p)^2 = 0$, $(x-n)^2 + (y-t)^2 - (n^2 + t^2) + np = 0$ and the equation of the fourth degree

$$(x^2 + y^2)^2 + np(y^2 - x^2) - 2ptxy = 0,$$

for the coordinates (x, y) of the points A and C. The first equation represents the line $x = p$ not satisfied by the points A, C. The second represents the circle λ obtained when $E = M$ and satisfied by A, C only when AC is tangent to κ at M. Finally the last equation, by inverting on the unit circle, leads to

$$np(y^2 - x^2) - 2ptxy + 1 = 0,$$

representing a rectangular hyperbola centered at the origin. By the well known property of Bernoulli's lemniscates to be the inverses of such hyperbolas ([4, p.110]), this proves the theorem for the pair of opposite vertices A and C. For the other pair of opposite vertices, B and D, an analogous calculation, leads to a corresponding system of three equations

$$\begin{aligned} x^2 + y^2 - 2x(n-u) - 2y(t-v) - 2pu + pn &= 0, \\ vx + (n-u)y - nv &= 0, \\ u^2 + v^2 - nu - tv &= 0. \end{aligned}$$

Here again, elimination of (u, v), transfer of the origin at N, and inversion on the unit circle centered at N, leads, through the factorization of an equation of th 8-th degree, to the equation of the rectangular hyperbola

$$(n^2 - np)(y^2 - x^2) + 2t(n-p)xy + 1 = 0.$$

This, using the aforementioned property of Bernoulli's lemniscate, proves the theorem for the vertices B and D.

Remarks. (1) Using, for convenience, the corresponding equations of the rectangular hyperbolas, one can easily compute the symmetry axes of the lemniscates and

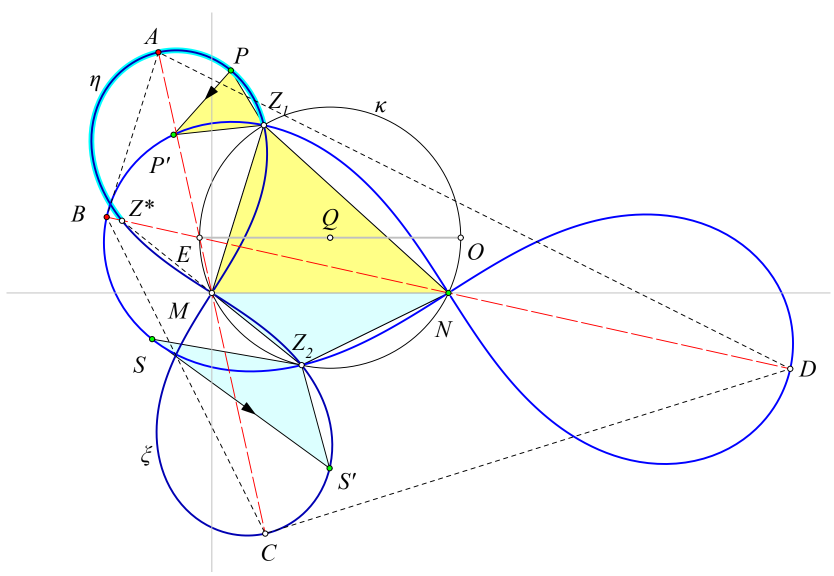

Figure 16. The similarities of the two lemniscates

see that they are obtained, respectively, by lines AC, BD, when their intersection

E is such that EO is parallel to line MN (see Figure 16). A simple computation shows also that the two lemniscates are similar with respect to two similarities. The first one $P' = f_1(P)$ has its center at Z_1, its oriented rotation-angle equals $\angle MZ_1N$ and its ratio is $r_1 = \frac{|Z_1M|}{|Z_1N|}$. The second similarity $S' = f_2(S)$ has its center at Z_2, its oriented rotation-angle equals $\angle NZ_2M$ and its ratio i s $r_2 = \frac{|Z_2N|}{|Z_2M|} = r_1^{-1}$.

(2) Fixing a certain lemniscate ξ, one can use the above results to give a parametrization of all cyclic quadrilaterals, up to similarity, by three points Z_1, Z_2, P properly chosen on the lemniscate. In fact, select first two points Z_1, Z_2, each on a different loop and on the same side of the axis AC of ξ (see Figure 16). This, together with the node M of ξ creates a triangle Z_1MZ_2 with the angle at M greater than a right one. This triangle defines also a unique point N, such that $q = NZ_1MZ_2$ is a harmonic quadrilateral. Excepting the squares, all other harmonic quadrilaterals, up to similarity, are obtained in this way. Having q, one can define the similarity f_1 of the previous remark. Then, every point A on the arc $\eta = Z_1Z^*$, where Z^* the symmetric of Z_2 with respect to M, defines a cyclic quadrilateral $p = ABCD$.

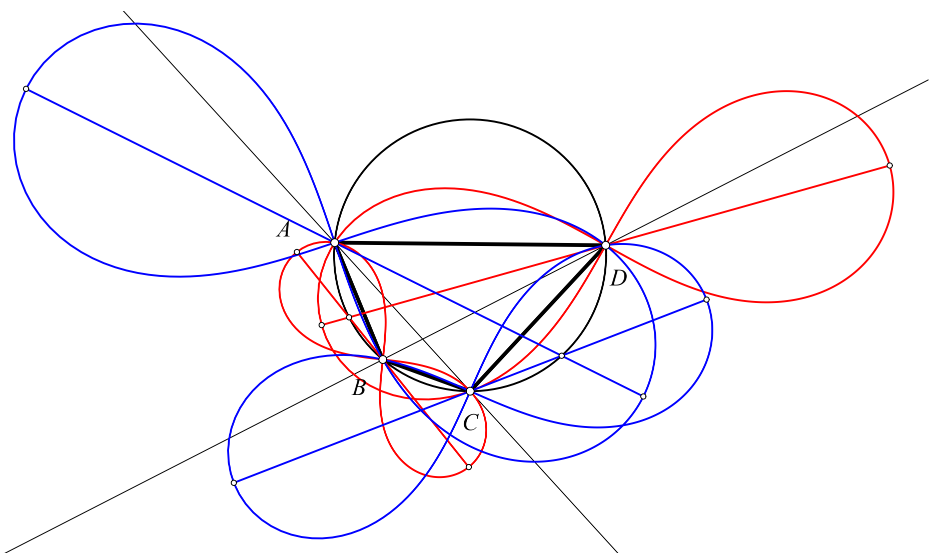

Figure 17. The four lemniscates

Point $B = f_1(A)$, point C is the symmetric of A with respect to M and point D is the symmetric of B with respect to N.

(3) The symbol $q = NZ_1MZ_2$ for the harmonic quadrilateral sets a certain order on its vertices. In the resulting construction of the cyclic quadrilateral $p = ABCD$ it is assumed that Z_1, Z_2 play the role of the similarity centers and M, N are the midpoints of the diagonals. Interchanging these roles, changes also the related cyclic quadrilaterals. Thus, giving q without an ordering for its vertices, produces two families of cyclic quadrilaterals, depending on how we interpret its two pairs of opposite vertices. Figure 17 shows the two pairs of lemniscates

corresponding to the two interpretations of the opposite vertices of the harmonic quadrilateral $q = ABCD$. All cyclic quadrilaterals having q for their associated harmonic, have their vertices on these lemniscates.

7. The associated cyclic quadrilateral

Starting with an arbitrary convex quadrilateral $p = ABCD$ with intersections of opposite sides F and G, we can, through the intermediate construction of its associated harmonic, pass to a natural *associated cyclic* quadrilateral $p' = A'B'C'D'$. In fact, consider the associated harmonic $q = Z_1NZ_2M$ of p and from this, con-

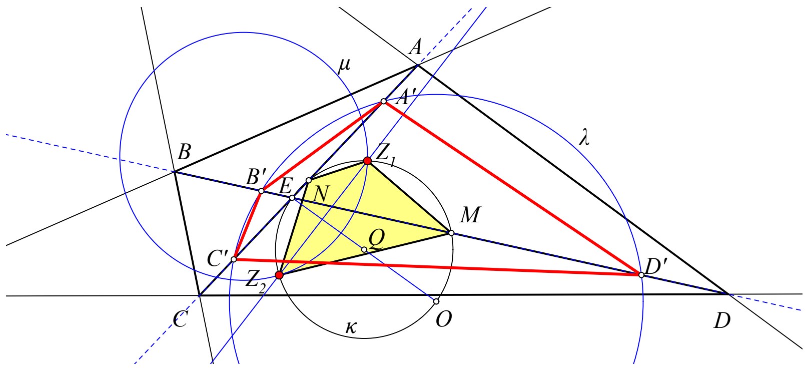

Figure 18. Quadrilateral $p = ABCD$ and its associated cyclic $p' = A'B'C'D'$

struct, following the recipe of Theorem 8, the corresponding cyclic $p' = A'B'C'D'$ (see Figure 18). From its definition, p' has the same harmonic associated q with p. Further it is easy to see that $|AA'| = |CC'|, |BB'| = |DD'|$ and the ratio $\frac{|AA'|}{|BB'|} = \frac{|AC|}{|BD|}$(see Figure 18). If one of the intersection points F, G of the opposite sides is at infinity then p is a trapezium and the corresponding harmonic quadrilateral has one of the similarity centers (Z_1) coinciding with the intersection E of its diagonals. Excluding this case, the procedure described above can be reversed. Starting from the convex cyclic quadrilateral $p' = A'B'C'D'$ and taking on its diagonals segments

$$|AA'| = |CC'|, |BB'| = |DD'| \qquad \text{in ratio} \qquad \frac{|AA'|}{|BB'|} = \frac{|A'C'|}{|B'D'|},$$

we obtain quadrilaterals $p = ABCD$ with the same associated harmonic quadrilateral. This gives an alternative construction of the one exposed in §3. In the excluded case of trapezia $p = ABCD$, the result is different and the procedure must be slightly modified. In fact, in this case there is no proper associated cyclic quadrilateral, the corresponding construction leading to a degenerate cyclic quadrilateral, which coincides with a triangle $Z_1C''D''$ (see Figure 19). In this case the quadrilaterals $p' = A'B'C'D'$, having the same associated harmonic quadrilateral $q = NZ_1MZ_2$ with p are also trapezia and are obtained by taking an arbitrary point A' on Z_1N, on the other halfline than N and projecting it parallel to MN

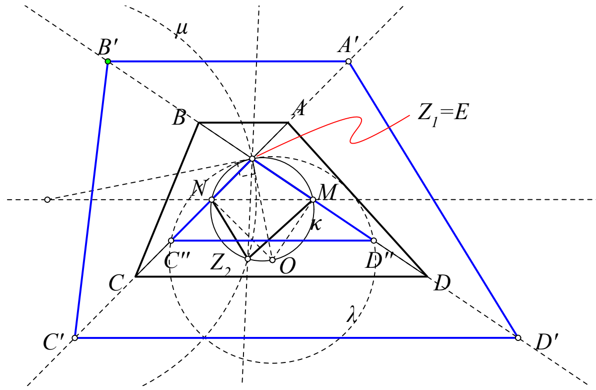

Figure 19. For trapezia the associated cyclic degenerates to a triangle

onto B' on Z_1M. Then taking, respectively, the symmetrics, C', D' with respect to N and M.

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