Hagge circles revisited

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Abstract

In 1907, **Karl Hagge** wrote an article on the construction of circles that always pass through the orthocenter of a given triangle. The purpose of his work is to find an extension of the Wallace-Simson theorem when the generating point is not on the circumcircle. Then they was named Hagge circles. In this paper, we present the new insight into Hagge circles with many corollaries. Furthermore, we also introduce three problems which are similar to Hagge circles.

1 Hagge circles and their corollaries

Problem 1 (The construction of Hagge circles). Given triangle ABC with its circumcircle (O) and orthocenter H. Choose P an arbitrary point in the plane. The cevian lines AP, BP, CP meets (O) again at A_1, B_1, C_1. Denote A_2, B_2, C_2 the reflections of A_1, B_1, C_1 with respect to BC, CA, AB, respectively. Then H, A_1, B_1, C_1 are concyclic. The circle (H, A_1, B_1, C_1) is called P-Hagge circle.

Here we give two proofs for problem 1. Both of them start from the problem which appeared in China Team Selection Test 2006.

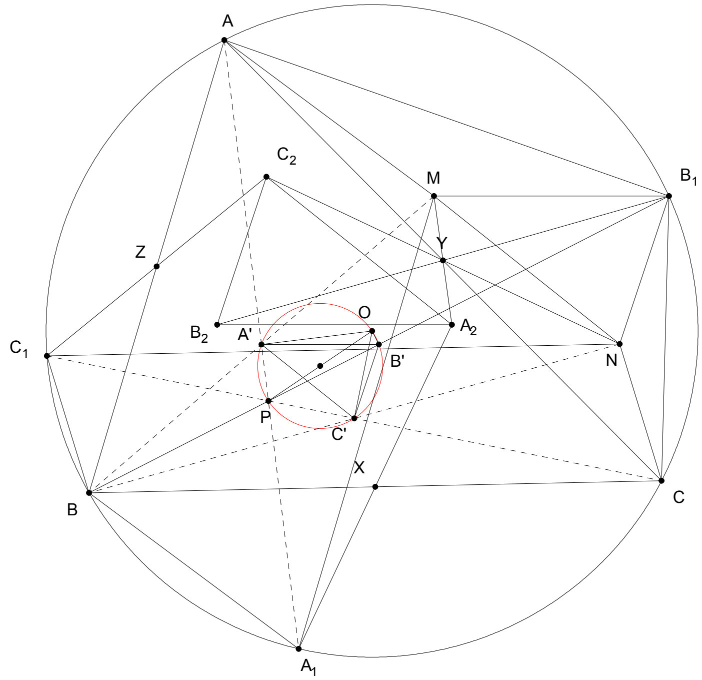
Problem 2 (China TST 2006). Given triangle ABC with its circumcircle (O) and orthocenter H. Let P be an arbitrary point in the plane. The cevian lines AP, BP, CP meets (O) again at A_1, B_1, C_1. Denote A_2, B_2, C_2 the reflections of A_1, B_1, C_1 across the midpoints of BC, CA, AB, respectively. Then H, A_1, B_1, C_1 are concyclic.

Proof.

We introduce a lemma:

Lemma 1. $(B_2C_2, B_2A_2) \equiv (PC, PA)(mod\pi)$

Proof.



Denote M, N the reflections of A_2, C_2 across the midpoint of AC then ΔMB_1N is the reflection of $\Delta A_2B_2C_2$ across the midpoint of AC.

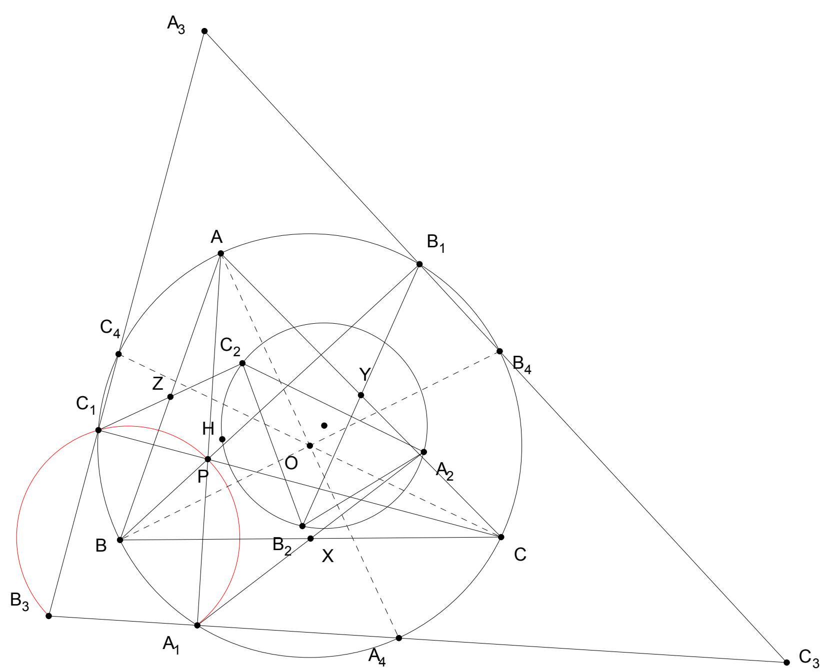
Since $AM \parallel CA_2, AM = CA_2$ and $BA_1 \parallel CA_2, BA_1 = CA_2$ we get AMA_1B is a parallelogram. Similarly, BC_1NC is a parallelogram too.

Let A', B', C' be the midpoints of AA_1, BB_1, CC_1 then A', C' are the midpoints of BM, BN, respectively. The homothety $H_B^{\frac{1}{2}} : M \mapsto A', B_1 \mapsto B', N \mapsto C'$ therefore $\Delta MB_1N \mapsto \Delta A'B'C'$. This means $(B_1M, B_1N) \equiv (B'A', B'C')(mod\pi)$.

On the other side, OA', OB', OC' are perpendicular to AA_1, BB_1, CC_1 hence O, A', B', C', P are concyclic, which follows that:

$(B_2A_2, B_2C_2) \equiv (B_1M, B_1N) \equiv (B'A', B'C') \equiv (PA', PC') \equiv (PA, PC)(mod\pi)$. We are done.

Back to problem 2.



Construct three lines through A_1, B_1, C_1 and perpendicular to AA_1, BB_1, CC_1, respectively. They intersect each other and form triangle $A_3B_3C_3$, intersect (O) again at A_4, B_4, C_4, respectively.

Note that AA_4, BB_4, CC_4 are diameters of (O) therefore A_4, C_4 are the reflections of H across the midpoints of BC, AB. We get $HA_2A_4A_1, HC_2C_4C_1$ are parallelograms, which follows that $HA_2 \parallel A_1A_4$, $HC_2 \parallel C_1C_4$.

But $PC_1B_3A_1$ is a cyclic quadrilateral and applying lemma 1 we have $(HC_2, HA_2) \equiv (B_3C_1, B_3A_1)$ $\equiv (PC_1, PA_1) \equiv (PC, PA) \equiv (B_2C_2, B_2A_2)(mod\pi)$ or H, A_2, B_2, C_2 are concyclic. Our proof is completed.

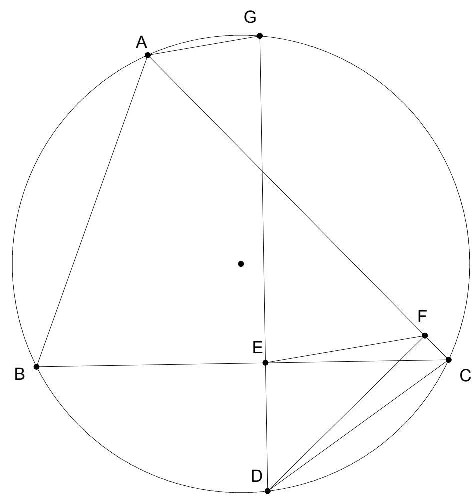
Remark 1: Now come back to problem 1. Let A_3, B_3, C_3 be the reflections of A_2, B_2, C_2 across the midpoints of BC, CA, AB, respectively. Since (BA_2C) is the reflections of (ABC) wrt BC then $A_3 \in (ABC)$. It is easy to see that BC is the midline of the triangle $A_1A_2A_3$ hence $A_1A_3 \parallel BC$. We claim that AA_3, BB_3, CC_3 concur at the isogonal conjugate Q of P wrt ΔABC. Then according to problem 2 we obtain H, A_2, B_2, C_2 are concyclic.

Another proof (Nsato) [9]

We introduce a lemma.

Lemma 2. Given triangle ABC and its circumcircle (O). Let D be an arbitrary point on (O). A line d through D and $d \perp BC.d \cap (O) = \{G\}$ then AG is parallel to the Simson line of point D.

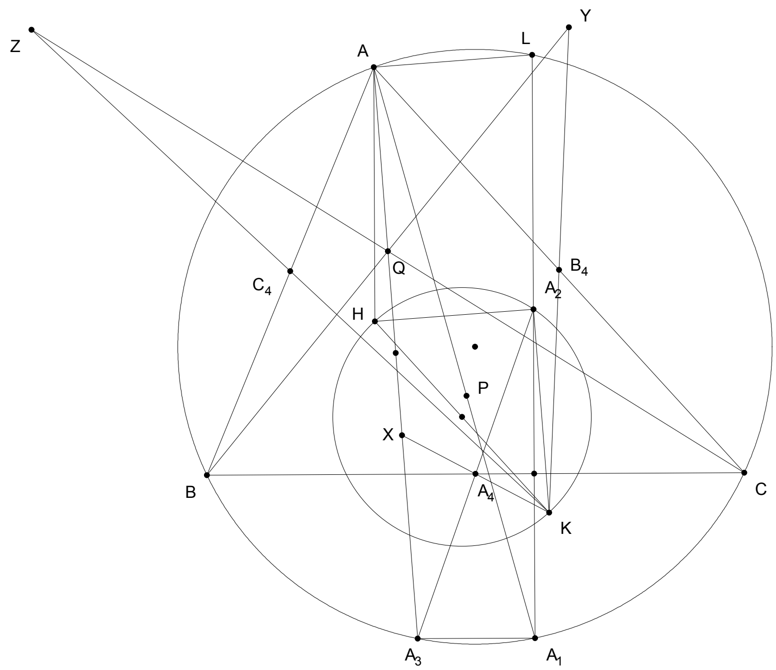
Proof.



Denote $d \cap BC = \{E\}, F$ the projection of D on AC then EF is the Simson line of D.

Since $DEFC$ is a cyclic quadrilateral we get $\angle GAC = \angle GDC = \angle EFA$. Therefore $AG \parallel EF$.

Back to problem 1.



Let Q be the isogonal conjugate of P wrt ΔABC; X, Y, Z be the reflections of A, B, C wrt Q; $A_3B_3C_3$ be the circumcevian triangle of Q; $A_4B_4C_4$ be the median triangle of ΔABC.

The symmetry with center Q takes A to X, B to Y, C to Z then $\Delta ABC \mapsto XYZ$. But $A_4B_4C_4$ is the median triangle of ΔABC hence there exist a point K which is the center of the homothety with ratio $\dfrac{1}{2}$. This transformation maps ΔXYZ to $\Delta A_4B_4C_4$.

On the other hand, let L be the second intersection of A_1A_2 and (O). Applying lemma 2, $AL \parallel HA_2$.

Moreover, LA_1 is perpendicular to BC and A_3A_1 is parallel to BC therefore $A_3A_1 \perp LA_1$, which follows that $AA_3 \perp AL$ or $AA_3 \perp HA_2$.

From remark 1, A_4 is the midpoint of A_2A_3 hence XA_2KA_3 is a parallelogram. This means $KA_2 \parallel AA_3$.

So $KA_2 \perp HA_2$ or A_2 lies on the circle with diamenter KH. Similarly we are done.

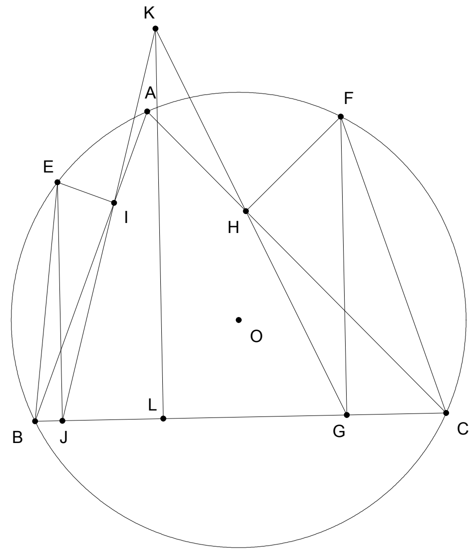
Problem 3. Given triangle ABC with its circumcircle (O) and orthocenter H. Let P be an arbitrary point in the plane. The cevian lines AP, BP, CP meets (O) again at A_1, B_1, C_1. Denote A_2, B_2, C_2 the reflections of A_1, B_1, C_1 with respect to BC, CA, AB, respectively. Then $\Delta A_2B_2C_2 \sim \Delta A_1B_1C_1$.

Proof.

Firstly we introduce a lemma:

Lemma 3. Given triangle ABC and its circumcircle (O). Let E, F be two arbitrary points on (O). Then the angle between the Simson lines of two points E and F is half the measure of the arc EF.

Proof.



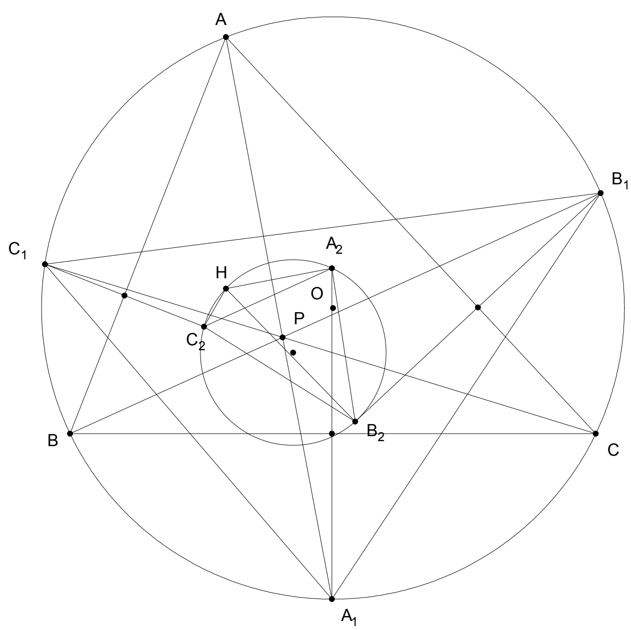
See on the figure, IJ, HG are the Simson lines of two points E and F, respectively. $IJ \cap HG = \{K\}$

Denote L the projection of K on BC. We have $KL \parallel FG$ and $FHGC$ is a cyclic quadrilateral thus $(KG, KL) \equiv (GK, GF) \equiv (CA, CF)(mod\pi)(1)$

Similarly, $(KL, KJ) \equiv (BE, BA)(mod\pi)(2)$

From (1) and (2) we are done.

Back to problem 3.



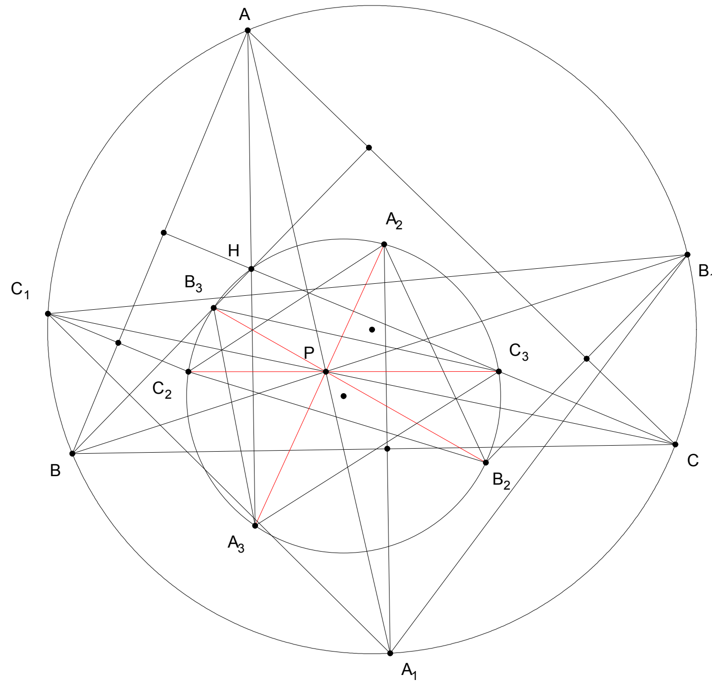
According to Hagge circle, we have H, A_2, B_2, C_2 are concyclic.

Since A_2 is the reflections of A_1 wrt BC we get HA_2 is the Steiner line of A_1, which is parallel to it's Simson line.

Analogously, HB_2 is the Steiner line of B_1. Then applying lemma 3, $(C_2A_2, C_2B_2) \equiv (HA_2, HB_2)$ $\equiv (C_1A_1, C_1B_1)(mod\pi)$. Do similarly with other direct angles we refer to the similarity of two triangles $A_1B_1C_1$ and $A_2B_2C_2$.

Problem 4. Using the same notations as problem 1, let A_3, B_3, C_3 be the second intersections of AH, BH, CH and $(A_2B_2C_2)$, respectively. Then A_2A_3, B_2B_3, C_2C_3 concur at P.

Proof.



$(A_2C_2, A_2A_3) \equiv (HC_2, HA_3) \equiv (HC_2, BC) + (BC, HA) \equiv \pi/2 + (HC_2, BC)(mod\pi)$. Let L, Z be the projections of C_1 on BC, AB, respectively then LZ is the Simson line of C_1. We get $HC_2 \parallel LZ$

Therefore $(HC_2, BC) \equiv (LZ, LC) \equiv (C_1Z, C_1B)(mod\pi)$

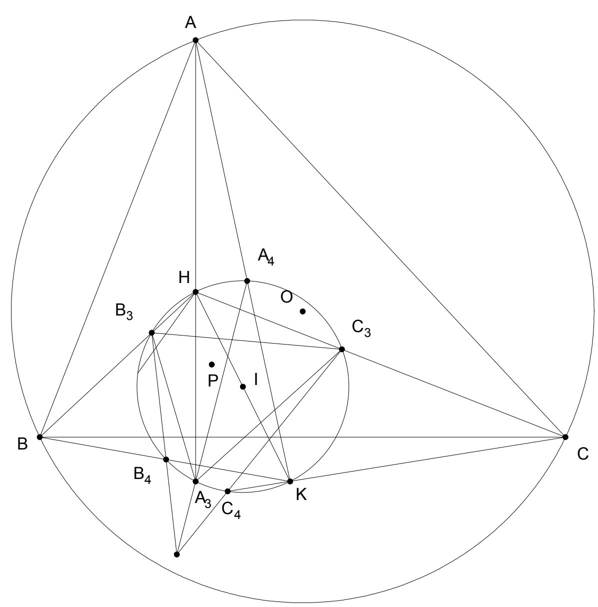
Hence $(A_2C_2, A_2A_3) \equiv \pi/2 + (C_1Z, C_1B) \equiv (BA, BC_1) \equiv (CA, CC_1)(mod\pi)$

Since H, A_3, B_3, C_3 are concyclic then $(C_3B_3, C_3A_3) \equiv (HB_3, HA_3) \equiv (CA, CB)(mod\pi)$. Similarly we get $\Delta A_3B_3C_3 \sim \Delta ABC$. But from problem 3, $\Delta A_2B_2C_2 \sim \Delta A_1B_1C_1$.

Thus there exist a similarity transformation which takes $A_3C_2B_3A_2C_3$ to $AC_1BA_1CB_1$ and it follows that A_2A_3, B_2B_3, C_2C_3 concur at P'. Moreover, $\dfrac{AP}{A_1P} = \dfrac{A_3P'}{A_2P'}$ hence $PP' \parallel AH$. Similarly, $PP' \parallel BH, CH$. This means $P \equiv P'$. Our proof is completed then.

Problem 5. Let I be the circumcenter of triangle $A_2B_2C_2$. K is the reflection of H across I. AK, BK, CK cut (I) again at A_4, B_4, C_4. Then A_3A_4, B_3B_4, C_3C_4 are concurrent.

Proof.



Since $KA_3 \perp AH$ we have $(C_3C_4, C_3A_3) \equiv (KC_4, KA_3) \equiv (CK, CB) (mod \pi)$.

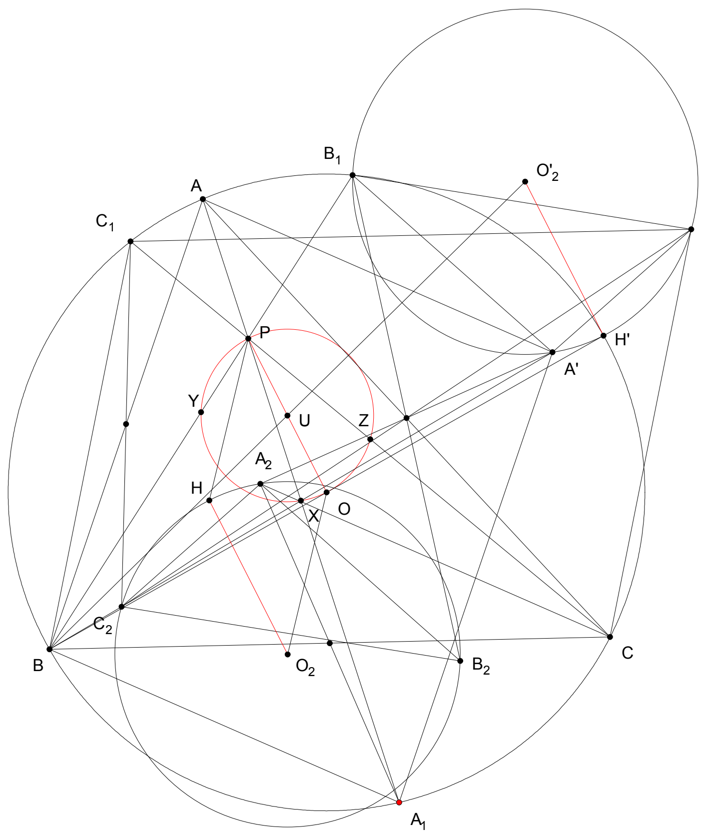
Similarly, $(C_3C_4, C_3B_3) \equiv (CK, CA) (mod \pi)$. Do the same with other angles then applying Ceva-sine theorem for triangle $A_3B_3C_3$ we are done.

Problem 6. Given triangle ABC with its circumcircle (O). Let P and Q be two isogonal conjugate points, O_1, O_2 be the circumcenters of P-Hagge and Q-Hagge circles, respectively. Then $PQ \parallel O_1O_2$ and $PQ = O_1O_2$.

Proof.

Lemma 4. Given triangle ABC with its orthocenter H, circumcenter O and let P be an arbitrary point in the plane. AP, BP, CP intersect (O) again at A_1, B_1, C_1, respectively. Denote A_2, B_2, C_2 the reflections of A_1, B_1, C_1 across the midpoints of BC, CA, AB, respectively, O_2 the center of $(A_2B_2C_2)$. Then OO_2HP is a parallelogram.

Proof.



Denote A', C' the reflections of A_2, C_2 across the midpoint of AC, respectively. Then $\Delta A'B_1C'$ is the reflections of $\Delta A_2B_2C_2$ across the midpoint of AC.

Since $AA' \parallel A_2C, AA' = A_2C, A_2C \parallel BA_1, A_2C = BA_1$ we get $AA'A_1B$ is a parallelogram. (1)

Similarly, $BCC'C_1$ is a parallelogram. (2)

Let X, Y, Z be the midpoints of AA_1, BB_1, CC_1, respectively then X, Y, Z lies on (OP).

But from (1) and (2), X, Z are the midpoints of BA', BC'.

A homothety $H_B^{\frac{1}{2}} : A' \mapsto X, B_1 \mapsto Y, C' \mapsto Z$ therefore $\Delta A'B_1C' \mapsto \Delta XYZ$

A symmetry with center is the midpoint of BC:

$S : B \mapsto B', H \mapsto H'$ the orthocenter of triangle $AB'C$, $O_2 \to O_2'$

$\Rightarrow O_2H \to O_2'H'$

It is easy to show that B, O, H' are collinear. Hence the homothety $H_B^{\frac{1}{2}} : H' \mapsto O, O_2' \mapsto U$ the circumcenter of triangle XYZ.

We conclude that $OP \parallel= O_2'H \parallel= O_2H$. Therefore $HPOO_2$ is a parallelogram. We are done.

Back to our problem.

From the lemma above, $OO_1 \parallel= QH, OO_2 \parallel= PH$ thus $PQ \parallel= O_1O_2$

Our proof is completed.

Remark 2. The radius of P-Hagge circle is equal to the distance of Q and O.

Problem 7. Given triangle ABC, its Nine-point circle (E) and an arbitrary point P on (E). Let Q be the isogonal conjugate point of P wrt ΔABC. Then the center I of Q-Hagge circle lies on (E) too. Moreover, I is the reflection of P across point E.

Proof.

The conclusion follows immediately from remark 2. Note that the center of Q-Hagge circle is always the reflection of P wrt E.

Problem 8. Given triangle ABC and its circumcenter O. P and Q are isogonal conjugate wrt triangle ABC such that O, P, Q are collinear. Then P-Hagge circle and Q-Hagge circle are tangent.

Proof.

This problem is also a corollary of remark 2. We will leave the proof for the readers.

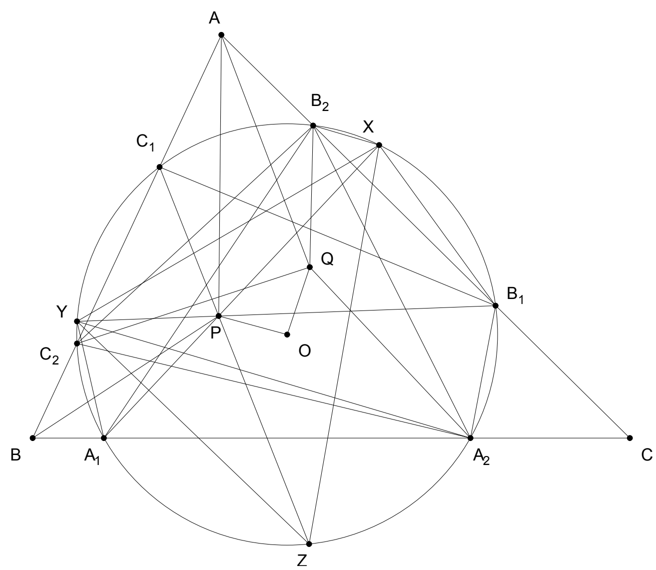
Problem 9. Given triangle ABC and its circumcircle (O, R). A circle which has center O and radius $r < R$ meets BC, CA, AB at $A_1, A_2, B_1, B_2, C_1, C_2$, respectively. Let E, F be the Miquel points of triangle ABC wrt (A_1, B_1, C_1) and (A_2, B_2, C_2) respectively. Then E-Hagge circle and F-Hagge circle are congruent.

Proof.

We will introduce a lemma.

Lemma 5. Given triangle ABC. A circle (O) intersects BC, CA, AB at 6 points $A_1, A_2, B_1, B_2, C_1, C_2$, respectively. Let P, Q be the Miquel points of ΔABC wrt (A_1, B_1, C_1) and (A_2, B_2, C_2). Then P and Q are isogonal conjugate and $OP = OQ$.

Proof.



Let X, Y, Z be the second intersections of A_1P, B_1P, C_1P and (O), respectively.

We have $(ZY, ZX) \equiv (A_1Y, A_1X) \equiv (A_1Y, PY) + (PY, A_1X) \equiv (A_2C, A_2B_1) + (CA_2, CB_1) \equiv (B_1B_2, B_1A_2) (mod \pi)$

This means $XY = A_2B_2$. Similarly, $YZ = B_2C_2, XZ = A_2C_2$. Therefore $\Delta XYZ = \Delta A_2B_2C_2$.

On the other side, $(PX, PY) \equiv (QA_2, QB_2)(mod\pi), (PY, PZ) \equiv (QB_2, QC_2)(mod\pi)$ then there exist a rotation with center O which takes ΔXYZ to $\Delta A_2B_2C_2$ and P to Q. We conclude that $OP = OQ$.

Moreover, $(AP, AB) \equiv (B_1P, B_1C_1) \equiv (ZY, ZP) \equiv (C_2B_2, C_2Q) \equiv (AB_2, AQ)(mod\pi)$.

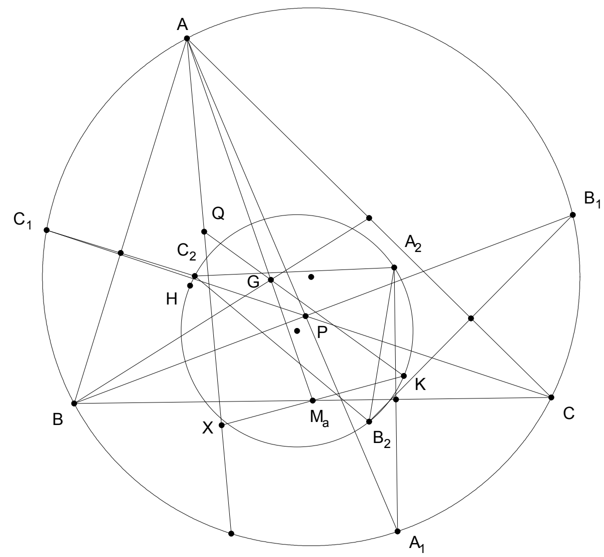
Similarly we get P and Q are isogonal conjugate wrt ΔABC.

Back to our problem.

According to lemma 5, P and Q are isogonal conjugate wrt ΔABC and $OP = OQ$. Then applying remark 2 the conclusion follows.

Problem 10. Given triangle ABC with its orthocenter H. Let P, Q be two isogonal conjugate points wrt ΔABC. Construct the diameter HK of P-Hagge circle. Let G be the centroid of the triangle ABC. Then Q, G, K are collinear and $QG = \dfrac{1}{2}GK$.

Proof.

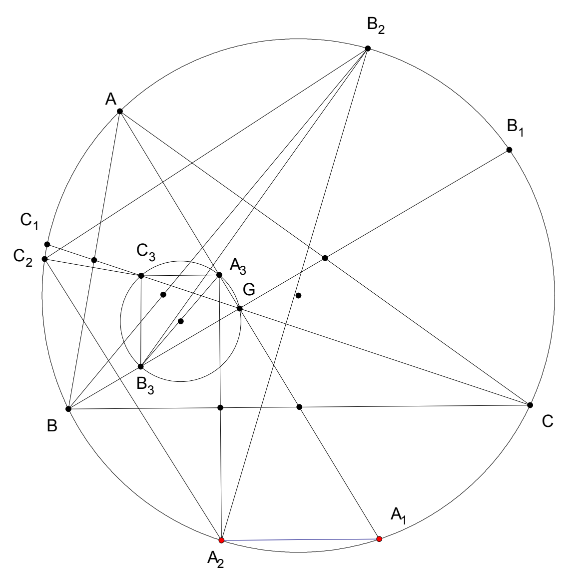


Let X be the reflection of A wrt Q, M_a be the midpoint of BC then from proof 2 of problem 1, we obtain M_a is the midpoint of XK.

Then applying Menelaus theorem for the triangle AXM_a with the line (Q, G, K) we have Q, G, K are collinear. Applying Menelaus theorem again for the triangle QXK with the line (A, G, M_a) then $QG = \dfrac{1}{2}$.

Problem 11. The centroid G lies on L-Hagge circle where L is the Symmedian point.

Proof.



Let $A_1B_1C_1, A_2B_2C_2$ be the circumcevian triangles of G and L, respectively; A_3, B_3, C_3 be the reflections of A_2, B_2, C_2 wrt BC, CA, AB, respectively.

It is easy to see that A_3, B_3, C_3 lie on AG, BG, CG, respectively.

From problem 3, $\Delta A_3B_3C_3 \sim \Delta A_2B_2C_2$. Then $(A_2B_2, A_2C_2) \equiv (A_3C_3, A_3B_3)(mod\pi)$

On the other side, $(GC_3, GB_3) \equiv \dfrac{1}{2}(\overset{\frown}{CB_1} + \overset{\frown}{C_1B}) \equiv \dfrac{1}{2}(\overset{\frown}{B_2A} + \overset{\frown}{AC_2})$

$\equiv (A_2B_2, A_2C_2) \equiv (A_3C_3, A_3B_3)(mod\pi)$.

This means G, A_3, B_3, C_3 are concyclic. We are done.

More results.

1. When $P \equiv I$ the incenter of triangle ABC, P-Hagge circle is called Fuhrmann circle of ΔABC, after the 19th-century German geometer **Wilhelm Fuhrmann**. The line segment connecting the orthocenter H and the Nagel point N is the diameter of Fuhrmann circle. The solution for this result can be found in [8]. Moreover, denote O, I the circumcenter and the incenter of triangle ABC then $HN = 2OI$.

2. When P lies on the circumcircle of triangle ABC, P-Hagge circle becomes the Steiner line of P.

3. When Q lies on the circumcircle of triangle ABC, we have a problem:

Given triangle ABC and an arbitrary point P on its circumcircle (O). Denote A', B', C' the reflections of P across the midpoints of BC, CA, AB. Then H, A', B', C' are concyclic.

4. When P lies at the infinity, problem 1 can be re-written as below:
Given triangle ABC with its circumcircle (O), orthocenter H. Denote A_1, B_1, C_1 points on (O) such that $AA_1 \parallel BB_1 \parallel CC_1, A_2, B_2, C_2$ the reflections of A_1, B_1, C_1 across the lines BC, CA, AB, respectively. Then H, A_2, B_2, C_2 are concylic.

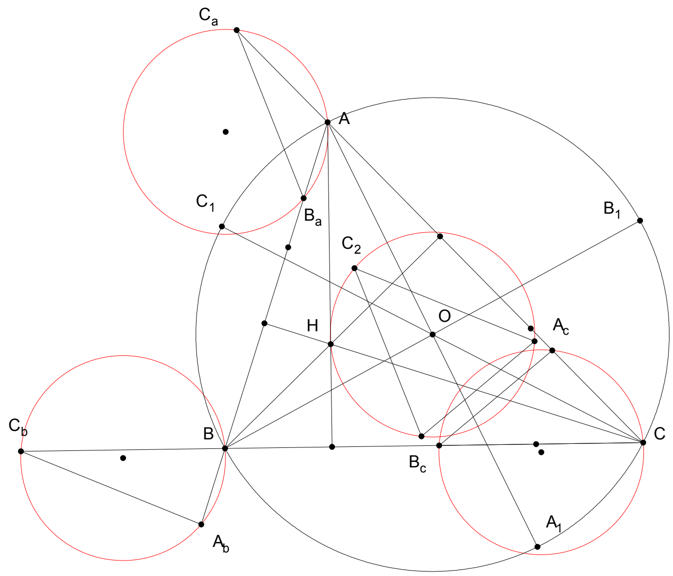
5. When $P \equiv H$ the orthocenter of ΔABC, P-Hagge circle becomes point H.

6. When $P \equiv O$ the circumcenter of ΔABC, the center of P-Hagge circle coincides with O.

2 Results from the triads of congruent circles

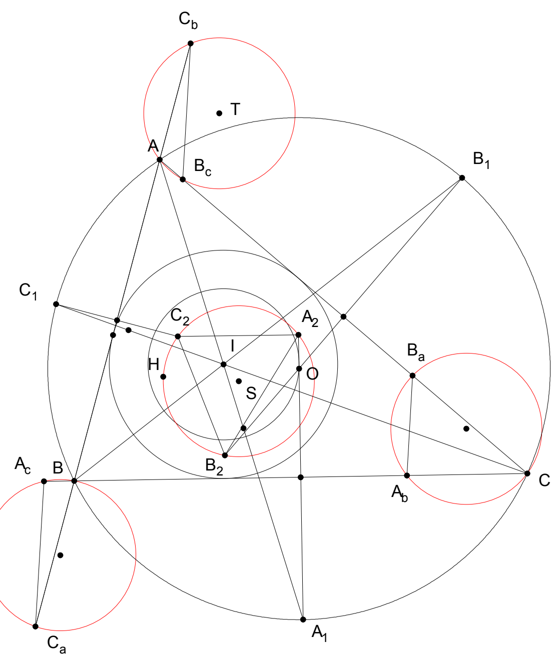
Three year ago, **Quang Tuan Bui**, the Vietnamese geometer, wrote an article about "two triads of congruent circles from reflections" in the magazine Forum Geometricorum. In that note, he constructed two triads of congruent circles through the vertices, one associated with reflections in the altitudes, and the other reflections in the angle bisectors. In 2011, **Tran Quang Hung**, the teacher at High School for Gifted Student, Hanoi University of Science, also constructed another triad which associated with reflections in the lines joining the vertices and the Nine-point center. His work was published in a abook called "Exploration and Creativity". Here we introduce the involvement of Hagge circles in these problems.

Problem 12. Given triangle ABC with its orthocenter H, circumcenter O. Let B_a, C_a be the reflections of B and C across the line AH. Similarly, consider the reflections C_b, A_b of C, A, respectively across the line BH, and A_c, B_c of A, B across the line CH. Then the circumcircles of three triangles $AC_bB_c, BA_cC_a, CA_bB_a$ and O-Hagge circle are congruent.



Note. In [2], Quang Tuan Bui proved that three circles $(AC_bB_c), (BA_cC_a), (CA_bB_a)$ are congruent with the circle (H, HO). By remark 2, HO is the radius of O-Hagge circle. We are done.

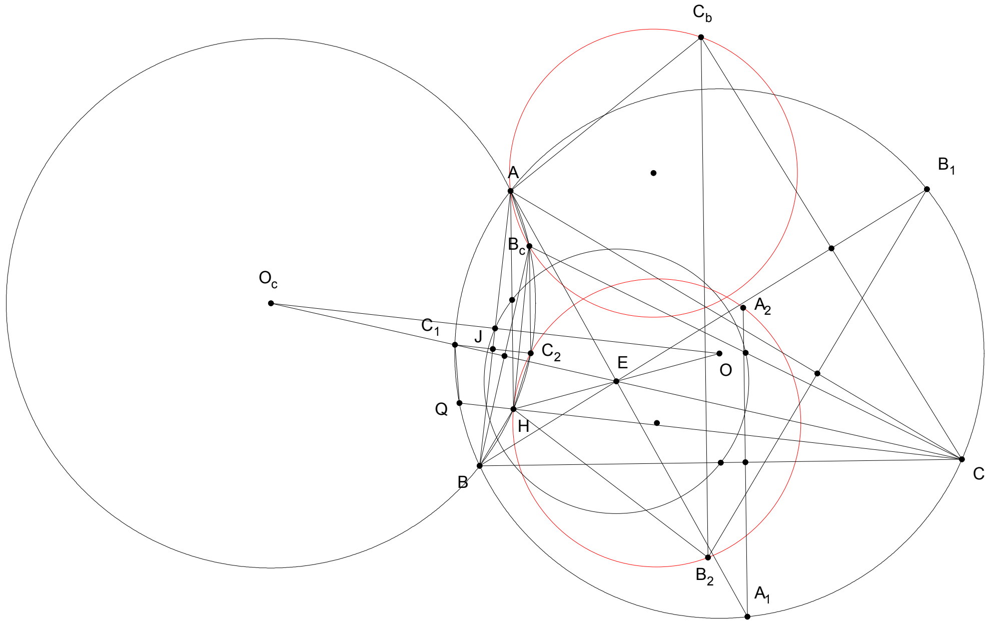
Problem 13. Let I be the incenter of triangle ABC. Consider the reflections of the vertices in the angles bisectors: B_a, C_a of B, C in AI, A_b, C_b of A, C in BI, A_c, B_c of A, B in CI. Then the circumcircles of three triangles $AC_bB_c, BA_cC_a, CA_bB_a$ and Fuhrmann circle are congruent.



Note. In his article, Quang Tuan Bui also proved that three circles $(AC_bB_c), (BA_cC_a), (CA_bB_a)$ are congruent with the circle (I, IO). By remark 2, IO is the radius of Fuhrmann circle and it completes the proof.

Problem 14. Given triangle ABC with its Nine-point center E. Let B_a, C_a be the reflections of B and C across the line AE. Similarly, consider the reflections C_b, A_b of C, A, respectively across the line BE, and A_c, B_c of A, B across the line CE. Then the circumcircles of three triangles $AC_bB_c, BA_cC_a, CA_bB_a$ and E-Hagge circle are congruent.

Proof.



Denote $A_1B_1C_1$ the circumcevian triangle of E. A_2, B_2, C_2 the reflections of A_1, B_1, C_1 across BC, CA, AB, respectively. Let H be the orthocenter of triangle ABC. CH meets the circumcircle (O) of triangle ABC again at Q, (O_c) be the circumcircle of triangle AHB.

Since (O_c) and (O) are symmetric wrt AB, C_2 is the reflection of C_1 wrt AB we obtain $C_2 \in (O_c)$. It is not hard to see that O_c lies on CE. However, B and B_c are symmetric wrt CE we conclude that $B_c \in (O_c)$.

We have $(BA, BB_c) \equiv (CQ, CC_1)(mod \pi)$, which follows from the fact that $CQ \perp AB, BB_c \perp CC_1$. This means the direct angle (BA, BB_c) is half the measure of the arc QC_1 or HC_2. Therefore $AB_c = HC_2$ or AHC_2B_c is a isosceles trapezoid.

Analogously, AHB_2C_b is also a isosceles trapezoid. We claim (AB_cC_b) is the image of (HC_2A_2) across the symmetry whose the axis is the perpendicular bisector of AH. This means two circles (AB_cC_b) and E-Hagge circle are congruent. Similarly we are done.

Remark 3. Other synthetic proofs of the congruent of three circles $(AC_bB_c), (BA_cC_a), (CA_bB_a)$ can be found in [6] or [7]. We have some properties:

1. The radius of three circles $(AC_bB_c), (BA_cC_a), (CA_bB_a)$ is equal to the distance of O and Kosnita point K (the isogonal conjugate of E).

2. The orthocenter of triangle $I_aI_bI_c$ is the reflection of H wrt the center of N-Hagge circle (I_a, I_b, I_c are the circumcenters of three triangles $AC_bB_c, BA_cC_a, CA_bB_a$, respectively).

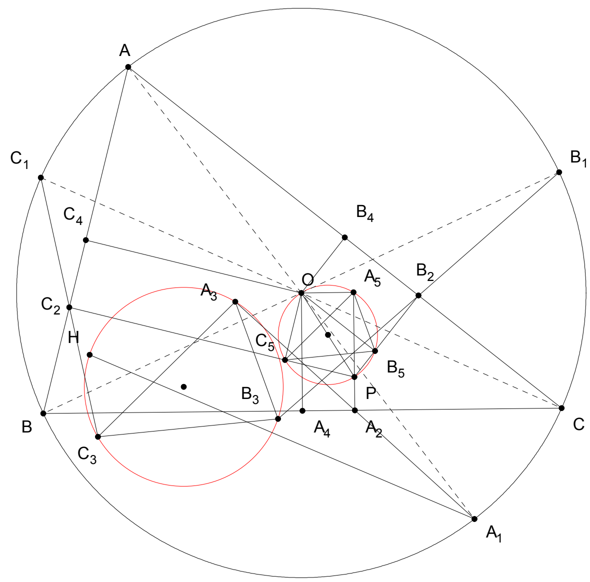
3. N is the center of K-Hagge circle.

3 Near Hagge circles

Finally, we give 3 interesting problems which involve in Hagge circles.

Problem 15. Given triangle ABC, its orthocenter H and its circumcircle (O). Denote A_1, B_1, C_1 the reflections of A, B, C wrt O, respectively, P an arbitrary point in the plane. Let $A_2B_2C_2$ be the pedal triangle of P wrt $\Delta ABC, A_3, B_3, C_3$ be the reflections of A_1, B_1, C_1 wrt A_2, B_2, C_2, respectively. Then 4 points H, A_3, B_3, C_3 are concyclic.

Proof.



Let A_4, B_4, C_4 be the midpoints of BC, CA, AB, respectively then A_4 is the midpoint of A_3H. Similar with B_4, C_4.

We get $HA_3 \parallel= 2A_2A_4, HB_3 \parallel= 2B_2B_4, HC_3 \parallel= 2C_2C_4$.(1)

Construct three rectangles $OA_4A_2A_5, OB_4B_2B_5, OC_4C_2C_5$ hence A_5, B_5, C_5 lie on the circle with diameter OP.(2)

But $OA_5 \parallel= A_4A_2, OB_5 \parallel= B_4B_2, OC_5 \parallel= C_4C_2$(3)

From (1), (2) and (3) we claim H, A_3, B_3, C_3 are concyclic. Moreover, we can show that $\Delta A_3B_3C_3 \sim \Delta ABC$.

Problem 16. Given triangle ABC with its circumcircle (O) and its orthocenter H. Let $H_aH_bH_c$ be the orthic triangle of triangle ABC. P is an arbitrary point on Euler line. AP, BP, CP intersect (O) again at A_1, B_1, C_1. Let A_2, B_2, C_2 be the reflections of A_1, B_1, C_1 wrt H_a, H_b, H_c, respectively. Then H, A_2, B_2, C_2 are concyclic (See [12] or [14]).

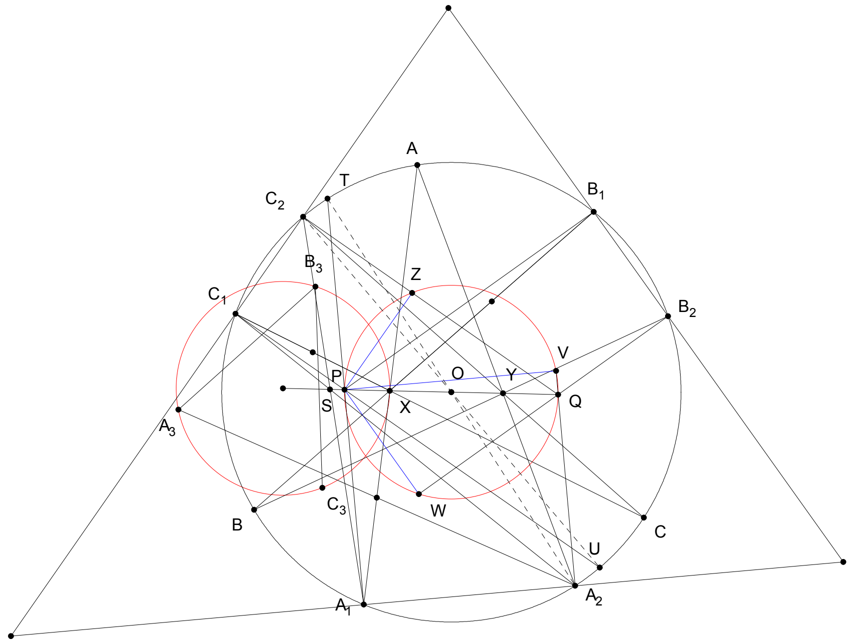
Proof.

We will prove this problem in the general case as below.

Generalization.

Given triangle ABC and its circumcircle (O). Let X, Y be two points inside (O) such that O lies on the line segment XY. AX, BX, CX intersect (O) again at A_1, B_1, C_1, AY, BY, CY intersect (O) again at A_2, B_2, C_2. Let A_3, B_3, C_3 be the reflections of A_2, B_2, C_2 wrt the midpoints of A_1X, B_1Y, C_1Z. Then A_3, B_3, C_3, X lie on a circle (O') and O' lies on XY.

Proof.



Let S be the intersection of A_1C_2 and A_2C_1. Applying Pascal's theorem for 6 points C_1, C_2, A_1, A_2, A, C we get $S \in XY$.

Denote T, U the second intersections of the perpendicular lines through A_1, C_1 to A_1A_2, C_1C_2 with (O), respectively.

Applying Pascal's theorem again for 6 points A_1, A_2, C_1, C_2, T, U We conclude that $P \in XY$. Similarly we obtain the perpendicular line through B_1 to B_1B_2 also passes through P, three lines which pass through A_2, B_2, C_2 and perpendicular to A_1A_2, B_1B_2, C_1C_2, respectively concur at $Q \in XY$.

Construct three rectangles $PA_1A_2V, PB_1B_2W, PZC_2C_1$ then $XA_3 \parallel A_1A_2 \parallel PV$ and $XA_3 = A_1A_2 = PV$. Let L be the midpoint of XP then $LA_3 = LV = \frac{1}{2}A_3V$.

Similarly we get $(A_3B_3C_3)$ is the reflection of (VWZ) wrt L. But P, V, W, Z lie on a circle with diameter PQ so X lies on $(A_3B_3C_3)$. Moreover, O' is the reflection of O wrt L, which follows that $O' \in XY$. We are done.

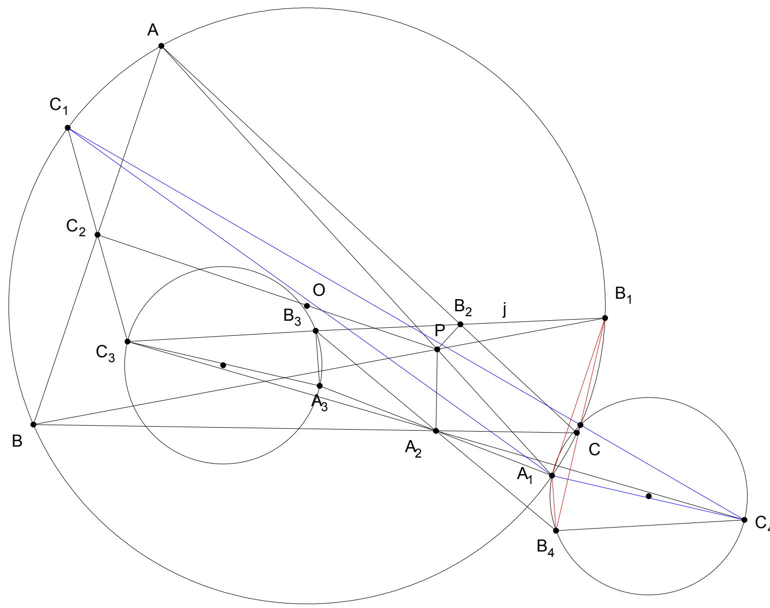
Problem 17. Given triangle ABC and its circumcircle (O), its orthocenter H. Let P be an arbitrary point in the plane. AP, BP, CP intersect (O) again at A_1, B_1, C_1, respectively. Let $A_2B_2C_2$ be the pedal triangle of P wrt $\Delta ABC, A_3, B_3, C_3$ be the reflections of A_1, B_1, C_1 wrt A_2, B_2, C_2, respectively. Then A_3, B_3, C_3, H are concyclic.

Proof.

We introduce two lemmas.

Lemma 6. Three triangles $A_1B_1C_1, A_2B_2C_2, A_3B_3C_3$ are similar.

Proof.



The result $\Delta A_1B_1C_1 \sim \Delta A_2B_2C_2$ is trivival. So we only show that $\Delta A_3B_3C_3 \sim \Delta A_1B_1C_1$.

Let B_4, C_4 be the reflections of B_3, C_3 wrt A_2, respectively. We get $\Delta A_1B_4C_4 \sim \Delta A_3B_3C_3$.(1)

Since A_2C_2 is the midline of triangle $C_1C_3C_4$ then $C_1C_4 \parallel A_2C_2$. Analogously, $B_1B_4 \parallel A_2B_2$. Denote R the intersection of B_1B_4 and C_1C_4. We conclude that $(RB_1, RC_1) \equiv (A_2B_2, A_2C_2) \equiv (A_1B_1, A_1C_1)(mod\pi)$, which follows that $R \in (O)$.

Hence $(C_1A_1, C_1C_4) \equiv (B_1A_1, B_1B_4)(mod\pi)$.

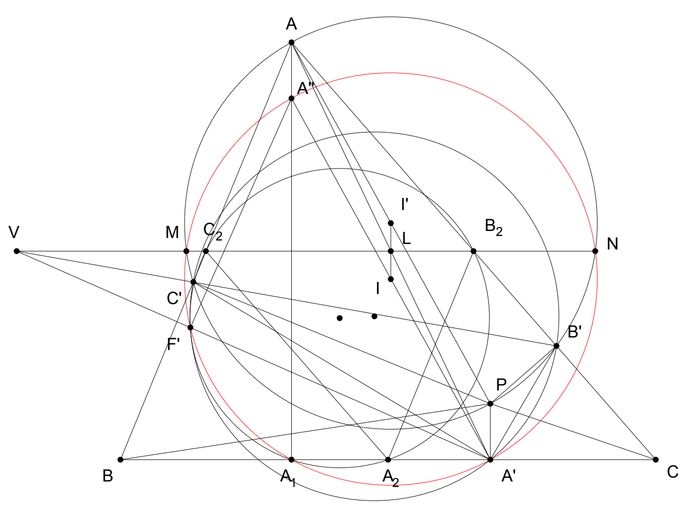
On the other side, $\dfrac{A_1C_1}{A_1B_1} = \dfrac{A_2C_2}{A_2B_2} = \dfrac{C_1C_4}{B_1B_4}$. Therefore $\Delta A_1C_1C_4 \sim \Delta A_1B_1B_4$.

This means $(A_1B_1, A_1B_4) \equiv (A_1C_1, A_1C_4)(mod\pi)$, then $(A_1B_4, A_1C_4) \equiv (A_1B_1, A_1C_1)(mod\pi)$. Similarly we get $\Delta A_1B_1C_1 \sim \Delta A_1B_4C_4$.(2)

From (1) and (2) we are done.

Lemma 7. Given triangle ABC. Let A_1 be the projection of A on BC, A_2, B_2, C_2 be the midpoints of BC, CA, AB, respectively. Let P be an arbitrary point in the plane, $A'B'C'$ be the pedal triangle of P wrt $\Delta ABC, F$ and F' be the intersections of $(A'B'C')$ and $(A_2B_2C_2)$. A line through A' and parallel to AP intersects AA_1 at A''. Then the circle with diameter $A'A''$ passes through one of two point F, F'.

Proof.



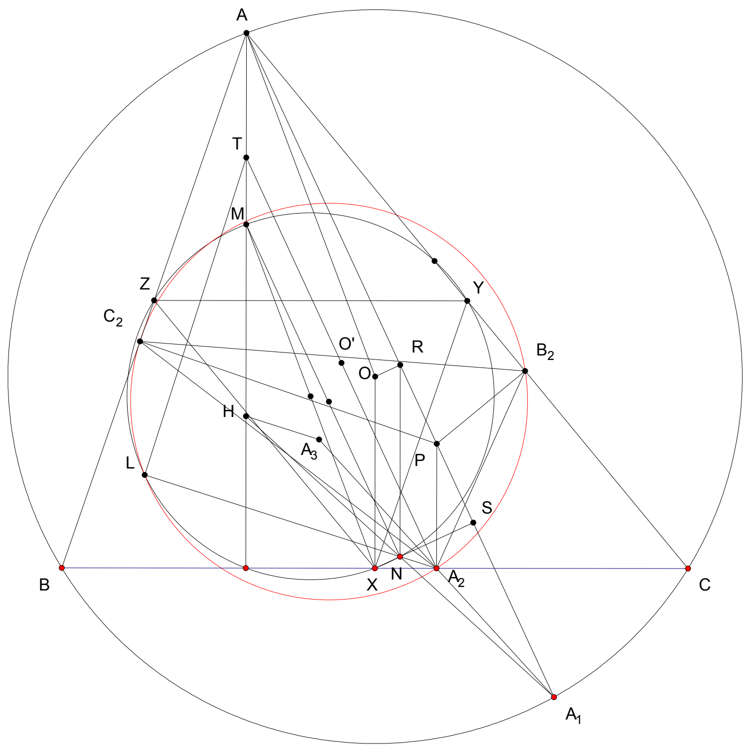
Let V be the intersection of $B'C'$ and B_2C_2. According to the first Fontene's theorem (see [4] or [5]), we obtain A', V, F' are collinear.

On the other side, denote I', L the midpoints of AP, AA', respectively, I the center of $(A'A'')$.

Because $AA"A'P$ is a parallelogram then I, L, I' are collinear and $II' \perp BC$. But L lies on B_2C_2 then I' is the reflection of I wrt B_2C_2. Thus $(I', I'A)$ is the reflection of (I, IA'') wrt B_2C_2, which follows that two circles intersect at two points M and N which lie on B_2C_2.

Since $B', C' \in (I', I'A)$ we get $VF'.VA' = VB'.VC' = VM.VN$ hence F', A', M, N are concyclic or (I, IA'') passes through F'.

Back to our problem.



Let L be the intersection of $(A_2B_2C_2)$ and the Nine-point circle of triangle ABC, X, Y, Z be the midpoints of BC, CA, AB, respectively, N be the second intersection of LA_2 and (XYZ). Let M be the midpoint of AH, K be the projection of A on BC, R be the midpoint of AA_1.

Since $OX \parallel= AM$ then $AO \parallel= XM$.

Denote T the intersection of the lines through A_2 and parallel to AA_1 and AK. Applying lemma 7 we obtain (A_2T) passes through L.

Then $(A_2T, A_2L) \equiv (KT, KL) \equiv (NM, NL)(mod\pi)$, which follows that $MN \parallel A_2T \parallel AP$. Hence $(MX, MN) \equiv (AO, AR)(mod\pi)$.

But $(NM, NX) \equiv (OR, OA) \equiv \pi/2(mod\pi)$ thus $\Delta XMN = \Delta OAR$.

Then $OR \parallel= XN$, we obtain $ORNX$ is a parallelogram. This means $RN \parallel= OX \parallel= \dfrac{1}{2}AH$.

Therefore N is the midpoint of HA_1 or NA_1 is the midline of triangle HA_3A_1. We have $HA_3 \parallel A_2L$. Similarly, $HB_3 \parallel B_2L$.

Then $(HA_3, HB_3) \equiv (LA_2, LB_2) \equiv (C_2A_2, C_2B_2)(mod\pi)$.

Applying lemma 6 we get $(HA_3, HB_3) \equiv (C_3A_3, C_3B_3)(mod\pi)$ or H, A_3, B_3, C_3 are concyclic. Our proof is completed then.

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