Cyclic Polygon

1. Suppose $ABCD$ is a quadrilateral with $AB = a,\ BC = b,\ CD = c,\ DA = d$. Find a formula for the area S of $ABCD$. (Note: The area depends on the angles at A, B, C and D.)

2. Let $a_i > 0$, $i = 1, \cdots, k$, $k \geq 3$. Prove that the following conditions are equivalent:

   a) $2\max\{a_i : 1 \leq i \leq k\} < \sum_{i=1}^{k} a_i$.

   b) There exists a non-degenerate polygon with sides a_i, $i = 1, \cdots, k$.

   c) There exists a convex polygon with sides a_i, $i = 1, \cdots, k$.

   d) There exists a cyclic convex polygon with sides a_i, $i = 1, \cdots, k$.

   Moreover, if any of the conditions hold, then, up to permutation of the vertices, there is only one polygon in d), which maximizes the area of a polygon with sides a_i, $i = 1, \cdots, k$.

3. Let x be the diameter of the circle. Then the center of the circle is inside (or on) P iff the equation $S(x) = \sum_{i=1}^{k} \sin^{-1}\left(\dfrac{a_i}{x}\right) = \pi$ has a solution for $x \geq a_1$. Since $S'(x) = \sum_{i=1}^{k} -\dfrac{a_i}{x\sqrt{x^2 - a_i^2}} < 0$ and $S(x) \to 0$ as $x \to \infty$, the solution exists iff $S(a_1) \geq \pi$ iff $\sum_{i=2}^{k} \sin^{-1}\left(\dfrac{a_i}{a_1}\right) \geq \dfrac{\pi}{2}$.

4. Suppose $a_i > 0$, $i = 1, \cdots, k$, $k \geq 3$, satisfy the conditions in 2., find an equation connecting the diameter of the circle given in d) and $a_i > 0$, $i = 1, \cdots, k$.

5. Suppose a_i, $i = 1, \cdots, k-1$ are fixed. prove that

   a) Among all k-sided polygon with $k-1$ sides equal to a_i, $i = 1, \cdots, k-1$, there is a unique choice of a_k and a polygon of sides a_i, $i = 1, \cdots, k$ that maximizes the area.

   b) The polygon in a) is cyclic with a_k as a diameter.

   c) The value of a_k is given by the unique solution of the equation $\sum_{i=1}^{k-1} \sin^{-1}\left(\dfrac{a_i}{x}\right) = \dfrac{\pi}{2}$.

   d) The value of a_k does not depend on the order of a_i, $i = 1, \cdots, k-1$.

   e) For $a_1 = 3$, $a_2 = 4$, $a_3 = 12$, find a_4 such that the area of the cyclic polygon with sides a_i, $i = 1, \cdots, 4$ is a maximum among all quadrilateral with 3 sides equal to 3, 4 and 12.

Solution

1. Suppose $ABCD$ is a quadrilateral with $AB = a,\ BC = b,\ CD = c,\ DA = d$. Then the area S of $ABCD$ satisfies

$$S^2 = \frac{1}{4}\left(a^2b^2 + c^2d^2 - \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2\right) - \frac{1}{2}abcd\cos(B+D)$$

Proof Let $AC = x$. Then, by Cosine Law, we have

$$a^2 + b^2 - 2ab\cos B = x^2 = c^2 + d^2 - 2cd\cos D$$

$$\Rightarrow \quad ab\cos B - cd\cos D = \frac{(a^2+b^2)-(c^2+d^2)}{2}$$

$$\Rightarrow \quad (ab\cos B - cd\cos D)^2 = \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2$$

$$\Rightarrow \quad (ab\cos B)^2 + (cd\cos D)^2 = \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2 + 2abcd\cos B\cos D$$

Then

$$\begin{aligned}
S^2 &= \left(\frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D\right)^2 \\
&= \frac{1}{4}\big((ab\sin B)^2 + (cd\sin D)^2 - 2abcd\sin B\sin D\big) \\
&= \frac{1}{4}\big((ab)^2(1-\cos^2 B) + (cd)^2(1-\cos^2 D) + 2abcd\sin B\sin D\big) \\
&= \frac{1}{4}\left((ab)^2 + (cd)^2 - \left(\left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2 + 2abcd\cos B\cos D\right) + 2abcd\sin B\sin D\right) \\
&= \frac{1}{4}\left(a^2b^2 + c^2d^2 - \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2\right) - \frac{1}{2}abcd(\cos B\cos D - \sin B\sin D) \\
&= \frac{1}{4}\left(a^2b^2 + c^2d^2 - \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2\right) - \frac{1}{2}abcd\cos(B+D)
\end{aligned}$$

Note: The angles B and D are measured counterclockwise and inside the quadrilateral. The result holds for both convex and concave quadrilaterals.

2. For fixed a, b, c and d, the quadrilateral attains maximum area when $B + D = 180°$, i.e. $ABCD$ is a cyclic quadrilateral.

3. Suppose points $A_1, A_2, \cdots, A_k,\ n \geq 3$, (respectively, $A'_1, A'_2, \cdots, A'_k$) are vertices of a cyclic polygon arranged in the counterclockwise direction on a circle with center C (respectively, C') and radius R (respectively, R') such that $|A_iA_{i+1}| = |A'_iA'_{i+1}|$ for $i = 1, \cdots k-1$ and $R > R'$. Then we have

   a. $\angle A_iA_{i+1}A_{i+2} > \angle A'_iA'_{i+1}A'_{i+2}$ for $i = 1, \cdots k-2$.

   b. $\angle A_kA_1A_2 < \angle A'_kA'_1A'_2$.

   c. $\angle A_{k-1}A_kA_1 > \angle A'_{k-1}A'_kA'_1$.

   d. $|A_kA_1| > |A'_kA'_1|$.

Proof. Let $|A_kA_1| = a_1,\ |A'_kA'_1| = a'_1$ and $|A_iA_{i+1}| = a_{i+1} = |A'_iA'_{i+1}|$ for $i = 1, \cdots n-1$.

a. For $i = 1, \cdots k-2$,

$$\begin{aligned}
\angle A_iA_{i+1}A_{i+2} &= \angle A_iA_{i+1}C + \angle CA_{i+1}A_{i+2} \\
&= \frac{\pi}{2} - \sin^{-1}\left(\frac{a_{i+1}}{2R}\right) + \frac{\pi}{2} - \sin^{-1}\left(\frac{a_{i+2}}{2R}\right) \\
&> \frac{\pi}{2} - \sin^{-1}\left(\frac{a_{i+1}}{2R'}\right) + \frac{\pi}{2} - \sin^{-1}\left(\frac{a_{i+2}}{2R'}\right) \\
&= \angle A_iA_{i+1}C + \angle CA_{i+1}A_{i+2} \\
&= \angle A'_iA'_{i+1}A'_{i+2}
\end{aligned}$$

b.

$$\begin{aligned}
\angle A_kA_1A_2 &= \frac{1}{2}\angle A_kCA_2 \\
&= \frac{1}{2}\left(2\sum_{i=3}^{k} \sin^{-1}\left(\frac{a_i}{2R}\right)\right) \\
&< \textstyle\sum_{i=3}^{k} \sin^{-1}\left(\frac{a_i}{2R'}\right) \\
&= \angle A'_kA'_1A'_2
\end{aligned}$$

c. Similar to b.

d. By induction on k. For $k = 3$, the result is obvious.
Suppose $k > 3$. Apply induction assumption on the triangles $A_kA_1A_2$ and $A'_kA'_1A'_2$ and cyclic polygons $A_2, \cdots A_k$ and $A'_2, \cdots A'_k$ we have
$\angle A_2A_1A_k < \angle A'_2A'_1A'_k$, $\angle A_3A_2A_k < \angle A'_3A'_2A'_k$ and $\angle A_3A_2A_1 > \angle A'_3A'_2A'_1$. So we have

$$\angle A_kA_2A_1 = \angle A_3A_2A_1 - \angle A_3A_2A_k > \angle A'_3A'_2A'_1 - \angle A'_3A'_2A'_k = \angle A'_kA'_2A'_1\,.$$

Since $|A_2A_k| > |A'_2A'_k|$, we have

$$a_1 = \frac{|A_2A_k|}{\sin \angle A_2A_1A_k} \sin \angle A_kA_2A_1 > \frac{|A'_2A'_k|}{\sin \angle A'_2A'_1A'_k} \sin \angle A'_kA'_2A'_1 = a'_2\,.$$

Let $a_i > 0$, $i = 1, \cdots, k$, $k \geq 3$.

4. The following conditions are equivalent:

   a) $2\max\{a_i : 1 \leq i \leq k\} < \sum_{i=1}^k a_i$.

   b) There exists a non-degenerate polygon with sides a_i, $i = 1, \cdots, k$.

   c) There exists a convex polygon with sides a_i, $i = 1, \cdots, k$.

   d) There exists a cyclic convex polygon with sides a_i, $i = 1, \cdots, k$.

   Moreover, up to permutation of the vertices, there is only one polygon in d) maximizes the area of a polygon with sides a_i, $i = 1, \cdots, k$.

   Proof: Clearly, $d) \Rightarrow c) \Rightarrow b) \Rightarrow a)$. Suppose a) holds. Without loss of generality, we may assume that $a_1 = \max\{a_i : 1 \leq i \leq k\}$. Choose $2 \leq j < k-1$ such that $a_1+a_2+\cdots+a_{j-1} < a_j+a_{j+1}+\cdots+a_k$ and $a_1 + (a_2 + \cdots + a_{j-1} + a_j) \geq (a_{j+1} + \cdots + a_k)$. Then we have

$$\begin{aligned} a_1 &< (a_2 + \cdots + a_{j-1} + a_j) + (a_{j+1} + \cdots + a_k) && (1)\\ a_2 + \cdots + a_{j-1} + a_j &\leq a_1 + a_2 + \cdots + a_{j-1} < a_j + a_{j+1} + \cdots + a_k \leq a_1 + (a_{j+1} + \cdots + a_k) && (2)\\ a_{j+1} + \cdots + a_k &\leq a_1 + (a_2 + \cdots + a_{j-1} + a_j) && (3) \end{aligned}$$

   Therefore, we can form a triangle of sides a_1, $(a_2 + \cdots + a_{j-1} + a_j)$, $(a_{j+1} + \cdots + a_k)$. This triangle is non-degenerate unless $a_2 + \cdots + a_{j-1} + a_j = a_1 + a_{j+1} + \cdots + a_k$ in (2), which implies that $a_1 = a_j$ and $a_2 + \cdots + a_{j-1} = a_{j+1} + \cdots + a_k$. In this case, we can form a rectangle with sides a_1, $(a_2 + \cdots + a_{j-1})$, a_j, $(a_{j+1} + \cdots + a_k)$. Hence, in all cases, we have a non-degenerate polygon of sides a_i, $i = 1, \cdots, k$.

   Let P be a polygon that maximizes the area. We are going to prove by induction on k that P is a cyclic polygon.

   Suppose the vertices of P are $A_1,\ A_2,\ \cdots A_k$ with $\overline{A_iA_{i+1}} = a_i$ for $i = 1, \cdots, k$, ($A_{k+1} = A_1$).

   If $k = 3$, the result is obvious.

   For $k = 4$, the result follows from 2.

   Suppose the result holds for polygons with fewer than k sides, $k > 4$, then $A_1,\ A_2,\ \cdots A_{k-1}$ are cyclic and $A_2,\ \cdots A_k$ are cyclic. Since $k > 4$, $A_1,\ A_2,\ \cdots A_k$ lie on the circle determined by $A_2,\ \cdots A_{k-1}$. Therefore, P is cyclic.

   Uniqueness of the polygon follows from the result in 3.

5. Let x be the diameter of the circle. Then the center of the circle is inside (or on) P iff the equation $S(x) = \sum_{i=1}^k \sin^{-1}\left(\frac{a_i}{x}\right) = \pi$ has a solution for $x \geq a_1$. Since $S'(x) = \sum_{i=1}^k -\frac{a_i}{x\sqrt{x^2 - a_i^2}} < 0$ and $S(x) \to 0$ as $x \to \infty$, the solution exists iff $S(a_1) \geq \pi$ iff $\sum_{i=2}^k \sin^{-1}\left(\frac{a_i}{a_1}\right) \geq \frac{\pi}{2}$.

6. Suppose a_i, $i = 1, \cdots, k-1$ are fixed. Then

a) Among all k-sided polygon with $k-1$ sides equal to a_i, $i = 1, \cdots, k-1$, there is a unique choice of a_k and a polygon of sides a_i, $i = 1, \cdots, k$ that maximizes the area.

b) The polygon in a) is cyclic with a_k as a diameter.

c) The value of a_k is given by the unique solution of the equation $\sum_{i=1}^{k-1} \sin^{-1}\left(\frac{a_i}{x}\right) = \frac{\pi}{2}$.

d) The value of a_k does not depend on the order of a_i, $i = 1, \cdots, k-1$.

Proof: Let P be a polygon that maximizes the area. We are going to prove by induction on k that P is a cyclic polygon, with diameter a_k.

Suppose the vertices of P are $A_1,\ A_2,\ \cdots A_k$ with $\overline{A_iA_{i+1}} = a_i$ for $i = 1, \cdots, k$, with $A_{k+1} = A_1$. For $k = 3$, the result is obvious.

Suppose the result holds for polygons with number of sides less than k, with $k > 3$. Then $A_1,\ A_3, \cdots A_k$ is cyclic with diameter $\overline{A_1A_k}$ and $A_1,\ A_2,\ A_k$ is cyclic with diameter $\overline{A_1A_k}$. Therefore, $A_1,\ A_2, \cdots A_k$ is cyclic with diameter $\overline{A_1A_k}$. If $x = a_k$, we have $S(x) = \sum_{i=1}^{k-1} \sin^{-1}\left(\frac{a_i}{x}\right) = \frac{\pi}{2}$. Clearly, $S(x)$ is defined for $x \geq a = \max\{a_i : 1 \leq i \leq k-1\}$ and $S(a) > \frac{\pi}{2}$, $S(x)$ is decreasing and $S(x) \to 0$ as $x \to \infty$. Therefore, $S(x) = \frac{\pi}{2}$ for a unique x and this solution does not depend on the order of a_i.

e) For $a_1 = 3$, $a_2 = 4$, $a_3 = 12$, a_4 is the solution of (See 3. Remark)

$$1 - \frac{3*4}{\sqrt{(x^2-3^2)(x^2-4^2)}} - \frac{4*12}{\sqrt{(x^2-4^2)(x^2-12^2)}} - \frac{3*12}{\sqrt{(x^2-3^2)(x^2-12^2)}} = 0$$

$$\Rightarrow \quad x \approx 13.78039394882347$$

7. **Remark** Let $\theta_j = \sin^{-1} t_j$. Then $\sin\theta_j = t_j$ and $\cos\theta_j = \sqrt{1-t_j^2}$. Let $\theta = \sum_{j=1}^{k}\theta_j$. Then,

$$\begin{aligned}
\cos\theta + \mathbf{i}\sin\theta =\ & \prod_{j=1}^{k}\left(\cos\theta_j + \mathbf{i}\sin\theta_j\right) \\
=\ & \sum_{m=0}^{\left[\frac{k}{2}\right]}(-1)^m \left(\sum_{1\le i_1<i_2<\cdots i_{2m}\le k} \prod_{r=1}^{2m}\sin\theta_{i_r} \prod_{\substack{s \\ i_s\ne i_1,i_2,\cdots i_{2m}}} \cos\theta_{i_s}\right) \\
& +\mathbf{i}\sum_{m=0}^{\left[\frac{k-1}{2}\right]}(-1)^m \left(\sum_{1\le i_1<i_2<\cdots i_{2m+1}\le k} \prod_{r=1}^{2m+1}\sin\theta_{i_r} \prod_{\substack{s \\ i_s\ne i_1,i_2,\cdots i_{2m+1}}} \cos\theta_{i_s}\right) \\
=\ & \sum_{m=0}^{\left[\frac{k}{2}\right]}(-1)^m \left(\sum_{1\le i_1<i_2<\cdots i_{2m}\le k} \prod_{r=1}^{2m} t_{i_r} \prod_{\substack{s \\ i_s\ne i_1,i_2,\cdots i_{2m}}} \sqrt{1-t_{i_s}^2}\right) \\
& +\mathbf{i}\sum_{m=0}^{\left[\frac{k-1}{2}\right]}(-1)^m \left(\sum_{1\le i_1<i_2<\cdots i_{2m+1}\le k} \prod_{r=1}^{2m+1} t_{i_r} \prod_{\substack{s \\ i_s\ne i_1,i_2,\cdots i_{2m+1}}} \sqrt{1-t_{i_s}^2}\right)
\end{aligned}$$

Therefore, we have

1. $\sum_{i=1}^{k}\sin^{-1}\left(\frac{a_i}{x}\right) = \pi$

$$\Leftrightarrow \sum_{m=0}^{\left[\frac{k}{2}\right]}(-1)^m \left(\sum_{1\le i_1<i_2<\cdots i_{2m}\le k} \prod_{r=1}^{2m}\left(\frac{a_{i_r}}{x}\right) \prod_{\substack{s \\ i_s\ne i_1,i_2,\cdots i_{2m}}} \sqrt{1-\left(\frac{a_{i_s}}{x}\right)^2}\right) = -1$$

$$\Leftrightarrow \prod_{i=1}^{k}\frac{x}{\sqrt{x^2-a_i^2}} + \sum_{m=0}^{\left[\frac{k}{2}\right]}(-1)^m \left(\sum_{1\le i_1<i_2<\cdots i_{2m}\le k}\prod_{r=1}^{2m}\frac{a_{i_r}}{\sqrt{x^2-a_{i_r}^2}}\right) = 0$$

2. $\sum_{i=2}^{k}\sin^{-1}\left(\frac{a_i}{x}\right) = \sin^{-1}\left(\frac{a_1}{x}\right)$

$$\Leftrightarrow \sum_{m=0}^{\left[\frac{k-1}{2}\right]}(-1)^m \left(\sum_{2\le i_1<i_2<\cdots i_{2m}\le k} \prod_{r=1}^{2m}\left(\frac{a_{i_r}}{x}\right) \prod_{\substack{s \\ i_s\ne i_1,i_2,\cdots i_{2m}}} \sqrt{1-\left(\frac{a_{i_s}}{x}\right)^2}\right) = \sqrt{1-\left(\frac{a_1}{x}\right)^2}$$

$$\Leftrightarrow \sum_{m=0}^{\left[\frac{k-1}{2}\right]}(-1)^m \left(\sum_{2\le i_1<i_2<\cdots i_{2m}\le k}\prod_{r=1}^{2m}\frac{a_{i_r}}{\sqrt{x^2-a_{i_r}^2}}\right) = \frac{\sqrt{x^2-a_1^2}}{\prod_{i=2}^{k}\sqrt{x^2-a_i^2}}$$

3. $\sum_{i=1}^{k-1}\sin^{-1}\left(\frac{a_i}{x}\right) = \frac{\pi}{2}$

$$\Leftrightarrow \sum_{m=0}^{\left[\frac{k-1}{2}\right]}(-1)^m \left(\sum_{1\le i_1<i_2<\cdots i_{2m}\le k-1} \prod_{r=1}^{2m}\left(\frac{a_{i_r}}{x}\right) \prod_{\substack{s \\ i_s\ne i_1,i_2,\cdots i_{2m}}} \sqrt{1-\left(\frac{a_{i_s}}{x}\right)^2}\right) = 0$$

$$\Leftrightarrow \sum_{m=0}^{\left[\frac{k-1}{2}\right]}(-1)^m \left(\sum_{1\le i_1<i_2<\cdots i_{2m}\le k-1}\prod_{r=1}^{2m}\frac{a_{i_r}}{\sqrt{x^2-a_{i_r}^2}}\right) = 0$$