High School Olympiads

Circle through Nagels point and A excenter

Reply

Source: Own

Math-lover123

#1 Oct 30, 2013, 12:15 am

In triangle ABC I_a,I_b,I_c,N are A-,B-,C-excenter and Nagels point respectively.
H_b,H_c are respectively orthocenters of triangles ACI_b and ABI_c respectively.
Lines H_bN and I_aI_c intersect at D.
Lines H_cN and I_aI_b intersect at E.
Prove that N,I_a,D,E are concyclic.

Luis González

#2 Nov 3, 2013, 1:14 am • 1

Let I be the incenter of $\triangle ABC$ and P,Q,R the midpoints of $BC,CA,AB.$ Midpoint S of $\overline{IN}$ is the Spieker point of $\triangle ABC,$ i.e. incenter of $\triangle PQR$ (well-known). Since $AI \parallel CH_b$ (both perpendicular to I_bI_c) and $CI \parallel AH_b$ (both perpendicular to I_aI_b), then $AICH_b$ is a parallelogram $\Longrightarrow Q$ is midpoint of $\overline{IH_b}.$ Similarly, R is midpoint of $\overline{IH_c} \Longrightarrow SQ \parallel NH_b$ and $SR \parallel NH_c \Longrightarrow \angle DNE=\angle QSR=\angle BIC=\pi-\angle BI_aC \Longrightarrow N,I_a,D,E$ are concyclic.

Quick Reply

High School Olympiads

pedal triangle

Reply

Source: own

vankhea

#1 Nov 1, 2013, 6:56 pm

Let O be circumcenter of triangle $\triangle ABC$. Let P be a point in its plane and $\triangle XYZ$ its pedal triangle WRT $\triangle ABC$. Prove that: $\sin BOC\cdot PA^2+\sin COA\cdot PB^2+\sin AOB\cdot PC^2=4|[ABC]-2[XYZ]|$.
With $[ABC]$ be the area of triangle $\triangle ABC$ and $[XYZ]$ be the area of $\triangle XYZ$

Luis González

#2 Nov 3, 2013, 12:11 am

If $\omega\equiv(O,R)$ denotes the circumcircle of $\triangle ABC$, then by Lagrange theorem for P and $O\equiv([OBC]:[OCA]:[OCA])$, we get:

$$[OBC]\cdot PA^2+[OCA]\cdot PB^2+[OAB]\cdot PC^2=[ABC]\cdot(PO^2+p(O,\omega))$$

Substituting $p(O,\omega)=R^2$ and $[OBC]=\frac{1}{2}R^2\cdot\sin\widehat{BOC}$ and cyclic expressions

$$\sin\widehat{BOC}\cdot PA^2+\sin\widehat{COA}\cdot PB^2+\sin\widehat{AOB}\cdot PC^2=2[ABC]\cdot\frac{PO^2+R^2}{R^2}$$

But, from Euler's theorem

$$\frac{[XYZ]}{[ABC]}=\frac{|R^2-PO^2|}{4R^2}\Longrightarrow\frac{|[ABC]-2[XYZ]|}{[ABC]}=\frac{PO^2+R^2}{2R^2}\Longrightarrow$$

$$\sin\widehat{BOC}\cdot PA^2+\sin\widehat{COA}\cdot PB^2+\sin\widehat{AOB}\cdot PC^2=4|[ABC]-2[XYZ]|.$$

Quick Reply

High School Olympiads

About Mixtilinear Circles

Reply

CTK9CQT

#1 Nov 2, 2013, 11:48 am

Let ABC be a triangle and (O) be its circumcircle. $(O_A),(O_B),(O_C)$ be the A,B,C-mixtilinear circles, respectively. $(O_A),(O_B),(O_C)$ touches (O) at P_A,P_B,P_C, respectively. $(O_B),(O_C)$ touches BC at H_B,H_C in that order. Prove that $\triangle P_AH_BP_B$ and $\triangle P_AH_CP_C$ are similar.

Luis González

#2 Nov 2, 2013, 12:14 pm • 1

From the internal tangencies of (O) with $(O_B),(O_C),$ we deduce that P_BH_B and P_CH_C bisect $\angle BP_BC$ and $\angle BP_CC,$ respectively $\Longrightarrow M \equiv P_BH_B \cap P_CH_C$ is midpoint of the arc BC of $(O).$ On the other hand, from the problem On mixtilinear incircles 2, P_AH_B,P_AP_C and P_AH_C,P_AP_B are pairs of isogonals WRT $\angle BP_AC,$ hence $\angle H_BP_AH_C=\angle P_CP_AP_B=\angle H_CMH_B \Longrightarrow MP_AH_CH_B$ is cyclic $\Longrightarrow \angle P_AH_BM=\angle P_AH_CM \Longrightarrow$ $\angle P_AH_BP_B=\angle P_AH_CP_C.$ Since $\angle P_AP_BM=\angle P_AP_CM,$ then $\triangle P_AH_BP_B \sim \triangle P_AH_CP_C.$

Quick Reply

High School Olympiads

Angles between two pairs of circles are equal

Source: Own

Math-lover123

#1 Nov 1, 2013, 12:43 am

In triangle ABC (I) is the incircle.
(I) touches sides of BC,CA,AB of triangle ABC at points D,E,F respectively.
Line EF intersects circumcircle of triangle ABC ate two points K,L,such that $K-F-E-L$.
Prove that angle between the circumcircles of triangles BDF and IDK equals to the angle between circumcircles of triangles CDE and ILD.

Luis González

#2 Nov 2, 2013, 1:59 am

Inversion WRT (I) takes A,B,C into the midpoints P,Q,R of EF,FD,DE and EF,FD,DE into the circles $\odot(IEF)$, $\odot(IFD)$, $\odot(IDE)$. Thus K,L go to the intersections X,Y of $\odot(IEF)$ with $\odot(PQR)$, respectively $\Longrightarrow \odot(IDK)$ and $\odot(IDL)$ go to the lines DX,DY. By conformity the angle between $\odot(BDF)\equiv\odot(IDF)$ and $\odot(IDK)$ is then equal to the angle between DF,DX. Similarly, the angle between $\odot(CDE)$ and $\odot(IDL)$ equals the angle between DE,DY. Hence, it suffices to show that DX,DY are isogonals WRT $\triangle DEF$.

Let $U\in\odot(PQR)$ be the foot of the D-altitude of $\triangle DEF$. Since $DE\cdot DQ=DF\cdot DR=DI\cdot DU$, then $\odot(IEF)$ is image of $\odot(PQR)$ under inversion $(D,DI\cdot DU)$ followed by symmetry across the angle bisector of $\angle EDF$, and vice versa. Thus X,Y swap places $\Longrightarrow DX,DY$ are symmetric across the bisector of $\angle EDF$, i.e. DX,DY are isogonals WRT $\triangle DEF$, as desired.

Quick Reply

High School Olympiads

Perpendicular Lines Involving Incenter

Reply

$f(e)$

efoski1687

#1 Oct 26, 2013, 1:00 am

Let ABC be a triangle with incenter I. Points M and N are the midpoints of side AB and AC, respectively. Points D and E lie on lines AB and AC, respectively, such that $BD = CE = BC$. Line l_1 pass through D and is perpendicular to line IM. Line l_2 passes through E and is perpendicular to line IN. Let P be the intersection of lines l_1 and l_2.Prove that AP is perpendicular to BC.

Luis González

#2 Nov 1, 2013, 10:31 pm • 1

If O is the circumcenter of $\triangle ABC$, then $OI \perp DE$ (see am=ab=bn, Nice but not easy, An interesting property of a triangle and elsewhere). In other words, perpendiculars from I, M, N to DE, AD, AE concur at $O \Longrightarrow \triangle IMN$ and $\triangle AED$ are orthologic through O. Their second orthology center is then $P \Longrightarrow AP \perp MN \Longrightarrow AP \perp BC$, as desired.

P.S. For a proof using Carnot's theorem see hard perpendicularity problem.

Quick Reply

High School Olympiads

same centroid

Source: geometry vill

vanu1996
#1 Oct 30, 2013, 6:20 pm

Given $\triangle ABC$ equilateral triangles $\triangle BCA', \triangle CAB', \triangle ABC'$ are outwardly constructed on the sides. Show that $\triangle ABC, \triangle A'B'C'$ have the same centroid. (Thirdtimelucky)

Luis González
#2 Nov 1, 2013, 9:41 am

It still holds for similar triangles A'BC~B'CA~C'AB constructed outwardly or inwardly. See the topics

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=486504
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=315336

High School Olympiads

Four medians are concurrent

Reply

sunken rock

#1 Oct 30, 2013, 10:52 pm

The diagonals AC, BD of the convex quadrilateral $ABCD$ meet at O and the bisectors (internal & external) of $\angle AOB$ meet the segments AB, BC, CD, DA at M, N, P, Q respectively.
Prove that the medians from A in $\triangle AMQ$, from B in $\triangle BMN$, from C in $\triangle CNP$ and from D in $\triangle DPQ$ are concurrent.

Best regards,
sunken rock

Luis González

#2 Nov 1, 2013, 8:40 am

Since the pencil $O(M,N,B,A) \equiv O(P,Q,D,A)$ is harmonic, it follows that MN, PQ and OA concur at the harmonic conjugate X of O WRT $A, C \Longrightarrow B(M,N,O,X)$ and $D(Q,P,O,X)$ are harmonic. Hence if $\mathcal{K}$ is the conic tangent to AB, BC, CD at M, N, P, then BO is the polar of X WRT $\mathcal{K} \Longrightarrow DQ$ is tangent to $\mathcal{K}$, i.e. there is a conic $\mathcal{K}$ tangent to AB, BC, CD, DA at M, N, P, Q.

Let K denote the center of $\mathcal{K}$ and U the midpoint of $\overline{MQ}$. Note that the polars of A, U, K WRT $\mathcal{K}$ concur at the intersection of QM with the line at infinity, thus A, U, K are collinear, i.e. A-median of $\triangle AMQ$ goes through K. Analogously, medians of $\triangle BMN, \triangle CNP$ and $\triangle DPQ$, issuing from B, C, and D, go through K.

sunken rock

#3 Nov 1, 2013, 12:34 pm

If S, T are the midpoints of AC, BD respectively, then $K \in ST$ and $\frac{KS}{KT} = \frac{AC}{BD}$!

Best regards,
sunken rock

sunken rock

#4 Apr 1, 2014, 12:32 am

> *sunken rock wrote:*
> If S, T are the midpoints of AC, BD respectively, then $K \in ST$ and $\frac{KS}{KT} = \frac{AC}{BD}$!

I shall post the proof my son provided to this statement:
Let S, T be midpoints of $\overline{AC}, \overline{BD}$, L midpoint of $\overline{MQ}$ and E, U, F intersections of AK with PM, BD, QN respectively.
Applying **Menelaos** to $\triangle OST$ with the transversal $\overline{AUK}$: $\frac{KT}{SK} \cdot \frac{SA}{OA} \cdot \frac{OU}{UT} = 1$, replacing $\frac{KT}{SK} = \frac{DT}{AS}$ (as constructed K) and $\frac{OU}{OA} = \frac{FU}{AF}$ (from angle bisector theorem - OF = external bisector of $\triangle OAU$) we get $\frac{DT}{UT} = \frac{AF}{UF} \Longrightarrow FT \parallel AD$ (1). In a similar way we get $ET \parallel AB$ (2).
Now apply the **converse of Menelaos** to $\triangle QMO$ to get that EF passes through L:
$\frac{OF}{FQ} \cdot \frac{QL}{LM} \cdot \frac{ME}{ES} = 1$ ($*$). Substitute first and last ratio with equivalent ones following from (1) and (2):
$\frac{TO}{TD} \cdot \frac{QI}{LM} \cdot \frac{BT}{OT} = 1($ $**$), but $BT = TD$ and from $(**)$ we get $QL = LM$.

In a similar way each median passes through K.

Best regards,
sunken rock

vittasko

#5 Apr 1, 2014, 11:08 pm

Let $A',\ B',\ C',\ D'$ be the midpoints of the segments $QM,\ MN,\ NP,\ PQ,$ respectively.

It is enough to prove that three of the medians as the problem states, are concurrent at one point. So, we will prove that the lines $AA',\ CC,\ DD'$ are concurrent at one point so be it $K.$

Let be the point $T\equiv AC\cap MN$ and because of $OM,\ ON$ as the angle bisectors of the angles $\angle AOB,\ \angle BOC$ respectively, we have that the points $A,\ O,\ C,\ T$ are in harmonic conjugation (easy to prove applying the **Menelaos theorem** in the triangle $\triangle\ ABC$ with transversal $MNT,$ where $OM,\ ON$ are the angle bisectors of $\angle AOB,\ \angle BOC$ respectively).

Similarly, because of $OQ,\ OP$ as the angle bisectors of $\angle AOD,\ \angle COD$ respectively, we conclude that the line PQ passes through the point $T,$ as the harmonic conjugate of $O,$ with respect to $A,\ C.$

● Let be the points $R\equiv AD\cap MP$ and $S\equiv CD\cap NQ.$

The points $R,\ A,\ Q,\ D$ are in harmonic conjugation because of the lines $OQ,\ OR$ bisect (internaly and externaly) the angle $\angle AOD$ of the triangle $\triangle\ OAD.$

Similarly, the points $S,\ C,\ P,\ D$ are also in harmonic conjugation.

So, we have the equality of **Double ratios** (= **cross ratios**) $(R,A,Q,D) = (S,C,P,D)\quad ,(1)$

From (1) we conclude that the lines $RS,\ AC,\ PQ$ are concurrent at one point and then, the line RS passes through the point $T\equiv AC\cap PQ.$

● Let be the points $X\equiv AD\cap A'D',\ Y\equiv CD\cap C'D',$ as the midpoints of the segments $RQ,\ SP$ respectively, as well.

We consider the complete quadrilateral $TNOPMQ$ and based on the **Gauss-Newton theorem**, we conclude that the line $A'C'$ connecting the midpoints of its diagonals $MQ,\ NP,$ passes through the midpoint so be it $Z,$ of the third diagonal $OT.$

It is easy to show now, that the points $X,\ Y,\ Z$ are collinear applying again the **Gauss-Newton theorem** considering the complete quadrilateral $TSOPRQ.$

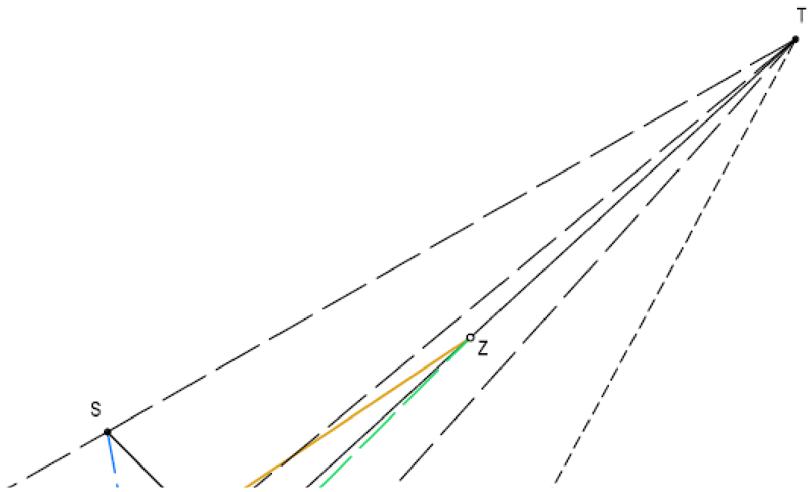
But, because of the collinearity of the points $X\equiv AD\cap A'D'$ and $Y\equiv CD\cap C'D'$ and $Z\equiv AC\cap A'C'$ we can say, based on the **Desarques theorem**, that the triangles $\triangle\ ACD,\ \triangle\ A'C'D'$ are perspective and then, the lines $AA',\ CC'\ DD',$ are concurrent at one point so be it $K.$

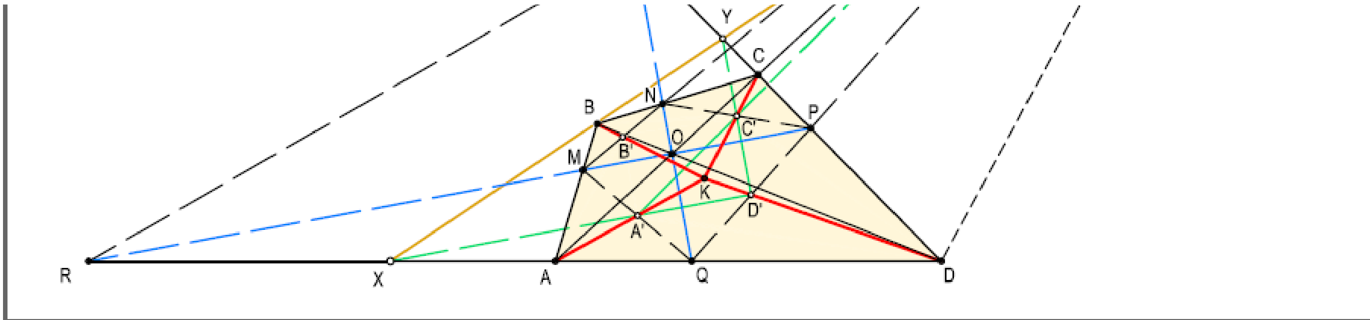
By the same way we can also prove that the lines $AA',\ BB',\ CC'$ are concurrent at one point and so, all the medians as the problem states, are concurrent at point K and the proof is completed.

● This proof is dedicated to **Antonis Zitridis.**

Kostas Vittas.

Attachments:





Quick Reply

High School Olympiads

prove the equality

Reply

DonaldLove
#1 Oct 15, 2013, 10:27 pm

Consider triangle ABC, circumscribe circle (O), BC>CA>AB. (I) is the incribed circle of ABC. AI meets (O) at K. OK meets BC at M. N on IM such that IM=MN. KN meets (O) at L. Prove that LB=LA+LC.

Luis González
#2 Oct 15, 2013, 10:58 pm

Posted before. L is the Feuerbach point of the antimedial triangle of ABC.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=323883
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=424438
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=24959

Quick Reply

High School Olympiads

Cevian triangle locus

Reply

fmasroor

#1 Oct 14, 2013, 5:13 am

Given a point X, triangle ABC, and its cevian triangle A'B'C', what is the locus of points such that their cevian triangle (wrt. ABC) is of equal area to A'B'C'? equal perimeter?

Luis González

#2 Oct 15, 2013, 9:07 am

Let P be a point and $\triangle A_0B_0C_0$ its cevian triangle WRT $\triangle ABC$. We are interested in the locus of P such that $[A_0B_0C_0]$ is constant, i.e. $[A_0B_0C_0]$ is a constant fraction $\frac{1}{k}$ of $[ABC]$. If $(x:y:z)$ are the barycentric coordinates of P WRT $\triangle ABC$, then we have

$$\frac{2xyz}{(x+y)(y+z)(z+x)}=\frac{[A_0B_0C_0]}{[ABC]}=\frac{1}{k}\Longrightarrow$$

$$\mathcal{Q}\equiv x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz(1-k)=0.$$

This is certainly a cubic through $A\equiv(1:0:0), B\equiv(0:1:0), C\equiv(0:0:1)$ and the infinite points of BC, CA, AB, namely $(0:1:-1),(1:0:-1),(1:-1:0)$, i.e. a non-pivotal circumcubic with 3 asymptotes parallel to BC, CA, AB.

Since the cevian triangle of the isotomic conjugate P^+ of P is equivalent to $\triangle A_0B_0C_0$, then $P^+\in\mathcal{Q}$. Also, note that the equation of $\mathcal{Q}$ is invariant under the mappings $(x:y:z)\mapsto(z:x:y)$ and $(x:y:z)\mapsto(y:z:x)$, thus $\mathcal{Q}$ contains the 2 bicentric pairs of P and P^+.

Now, to derive the barycentric equation of the locus of P such that its cevian triangle $\triangle A_0B_0C_0$ has constant perimeter p_0, we use the formula of the distance between two points. The result seems a rather ugly high degree curve:

$$\sum_{\text{cyclic}}\frac{\sqrt{S_Ax^2(y-z)^2+S_By^2(x+z)^2+S_Cz^2(x+y)^2}}{(x+z)(x+y)}=p_0.$$

Quick Reply

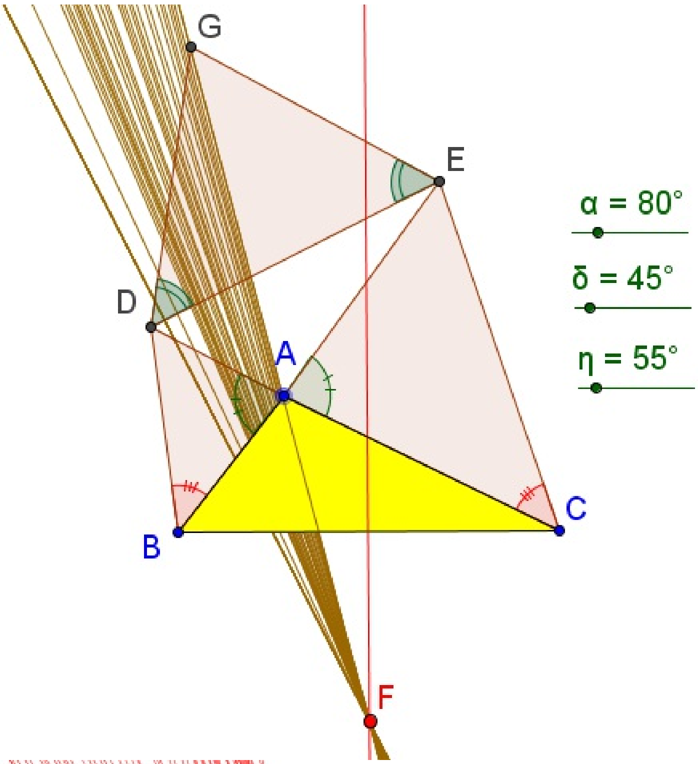
High School Olympiads

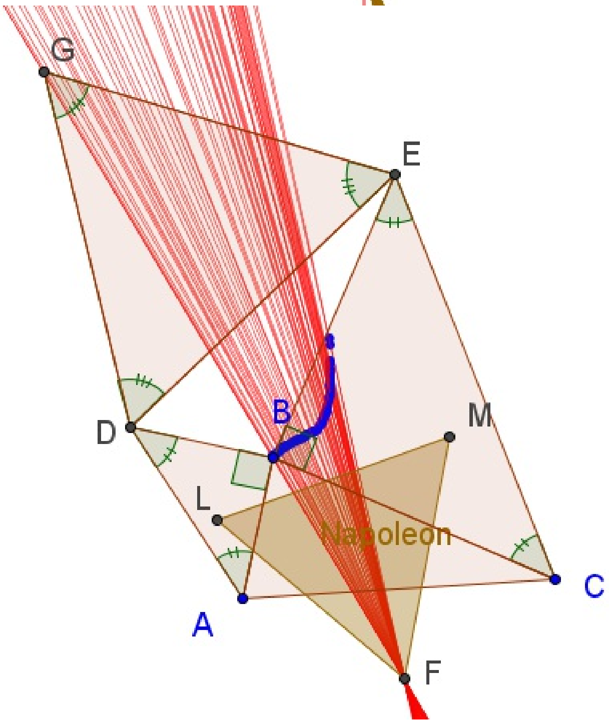
Relative a generalization Bottema theorem

daothanhoai

#1 Oct 12, 2013, 5:55 pm

Let a triangle ABC. Construct two similar triangle ABD and ACE with $\angle BAD = \angle CAE = const$. Construct Isosceles triangle DEG with $\angle DEG = \angle EDG = const$. GA intersects perpendicular bisector of BC at F. Prove that when A move on the plane then F is fixed point. Perpendicular bisector of DE intersects perpendicular bisector of BC at N.





Prove that when A move on the plane then:

1) N is a fixed point.

2) F is a fixed point.

Luis González

#2 Oct 14, 2013, 9:31 am • 1

1) Let $(X),(Y)$ be the circumcircles of $\triangle ABD, \triangle ACE$ meeting at $A, P.$ Redefine N as midpoint of the arc $BPC.$

Since $\angle(PB,PA)=\angle ADB=\angle AEC=\angle(PA,PC),$ it follows that PA,PN bisect $\angle BPC \Longrightarrow \angle APN$ is right $\Longrightarrow PN$ cuts $(X),(Y)$ a the antipodes U,V of $A.$ Moreover, we have $\angle BNC=\angle BPC=2\angle ADB,$ which means that the isosceles $\triangle NBC$ is directly similar with the isosceles $\triangle XBA$ and $\triangle YCA \Longrightarrow \triangle YNC \sim \triangle ABC$ and $\triangle XBN \sim \triangle ABC$ are directly similar, but $NB=NC \Longrightarrow \triangle XBN \cong \triangle YNC \Longrightarrow XA=XB=YN$ and $YA=YC=XN \Longrightarrow AXNY$ is a parallelogram $\Longrightarrow N$ is midpoint of $\overline{UV} \Longrightarrow B,C,P$ and the midpoint of $\overline{UV}$ are concyclic. In the same way, in the configuration of $\triangle ADE$ and the similar $\triangle ABD \sim \triangle ACE,$ we have that E,D,P and the midpoint N of $\overline{UV}$ are concyclic, being N midpoint of the arc $DPE.$ Since $\angle BPC=2\angle ADB$ is constant, then $\odot(PBC)$ is fixed $\Longrightarrow N$ is fixed, i.e. perpendicular bisector of $\overline{DE}$ meets perpendicular bisector ℓ of $\overline{BC}$ at fixed $N.$

Luis González

#3 Oct 14, 2013, 9:32 am • 1

2) Construct I-isosceles $\triangle IBC$ outside $\triangle ABC,$ such that $\angle BIC=2\angle BAD.$ If S,T are the projections of D,E on $AB,AC,$ then $\triangle ADB \sim \triangle AEC$ gives $\frac{AS}{SB}=\frac{AT}{TC} \Longrightarrow ST \parallel BC \Longrightarrow$ parallels to IB,IC through $S,T,$ respectively, meet at $J \in AI,$ because $\triangle JST \sim \triangle IBC$ are homothetic with center $A.$

Let $\odot(JST)$ cut AI again at $H.$ $\angle AHS=\angle JTS=\angle ICB=\angle ADS \Longrightarrow AHDS$ is cyclic $\Longrightarrow \angle DHA=\angle DSA=90^\circ.$ Similarly $\angle EHA=90^\circ \Longrightarrow HA,DE$ bisect $\angle SHT.$ If M,L,K are the midpoints of $ED,DA,AE,$ then $ML=KA=KT$ and $MK=LA=LS.$ But from the parallelogram $MLAK$ and the similar triangles $\triangle LAS \sim \triangle KAT$ outside it, we deduce that $\angle MLS=\angle MKT \Longrightarrow \triangle MLS \cong \triangle TKM$ are congruent by SAS $\Longrightarrow MS=MT \Longrightarrow M$ is midpoint of the arc $SHT.$ Hence, homothety with center $A,$ sending $\triangle JST$ to $\triangle IBC,$ will take M to the antipode W of I in the circle $\odot(IBC) \Longrightarrow AM$ cuts ℓ at fixed $W.$

By Menelaus' theorem for $\triangle WMN$ cut by $\overline{GAF},$ we obtain

$$\frac{\overline{FN}}{\overline{FW}}=\frac{\overline{GN}}{\overline{GM}}\cdot\frac{\overline{AM}}{\overline{AW}}=\frac{\overline{GN}}{\overline{GM}}\cdot\frac{\overline{IN}}{\overline{IW}}.$$

Since I,W are fixed $\Longrightarrow \frac{\overline{IN}}{\overline{IW}}$ is constant and since all $NEGD$ are similar $\Longrightarrow \frac{\overline{GN}}{\overline{GM}}$ is constant. Consequently, $\frac{\overline{FN}}{\overline{FW}}$ is constant $\Longrightarrow F$ is fixed on $\ell.$

daothanhoai

#4 Oct 14, 2013, 7:00 pm

Dear Luis González and Mathlinkers

Why all $NEGD$ are similar??

Thank you very much!

Best regard

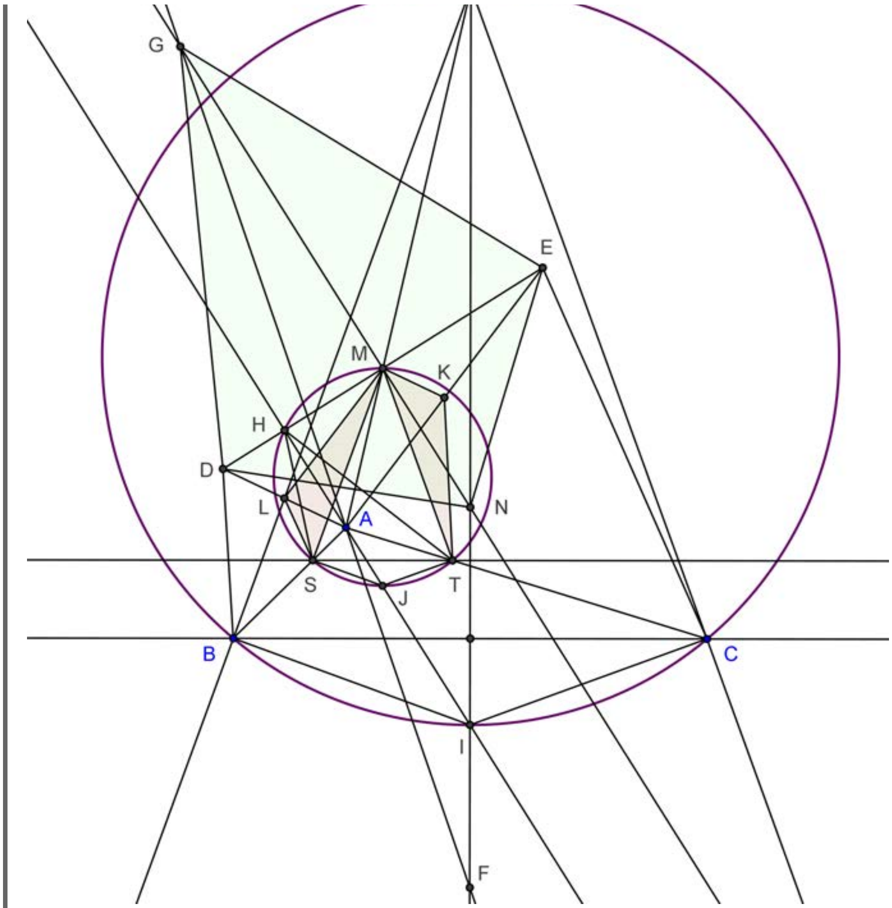
Dao Thanh Oai

Luis González

#5 Oct 14, 2013, 11:20 pm • 1

From 1) N is midpoint of the arc $DPE,$ thus $\angle DNE=\angle DPE=2\angle ABD$ is constant $\Longrightarrow \triangle NDE$ are all similar N-isosceles. Since $\triangle GDE$ are all similar by hypothesis, then $NEGD$ are all similar kites.

Attachments:



Quick Reply

High School Olympiads

prove that

Reply

nvthe_cht

#1 Oct 12, 2013, 6:05 pm

Given a cyclic convex quadrilateral $ABCD$ and be inscribed circle, let F be the intersection of AC and BD, let E be the intersection of BC and AD. Let M, N be the midpoints of AB, CD, Prove that:

$$\frac{EF}{MN} = \frac{1}{2}.\left|\frac{AB}{CD} - \frac{CD}{AB}\right|$$

Luis González

#2 Oct 13, 2013, 9:42 am

The expression holds for any 4 concyclic points, hence the quadrilateral ABCD can be either simple or complex. See the following topics:

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=64864
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=350179

Quick Reply

High School Olympiads

3 points on a line

Reply

lambosama
#1 Oct 12, 2013, 11:41 pm

Give the quadrilateral $ABCD$ inscribed (O). AC cut BD at M. N is the point inside the quadrilateral such that $\angle NAB + \angle NCB = \angle NBA + \angle NDA = 90^\circ$
Prove that: O, M, N are on a (straight) line :)
($ABCD$ is quad cylic just in case you don't understand what i'm saying)

This post has been edited 1 time. Last edited by lambosama, Oct 13, 2013, 7:56 am

Luis González
#2 Oct 13, 2013, 8:01 am • 2

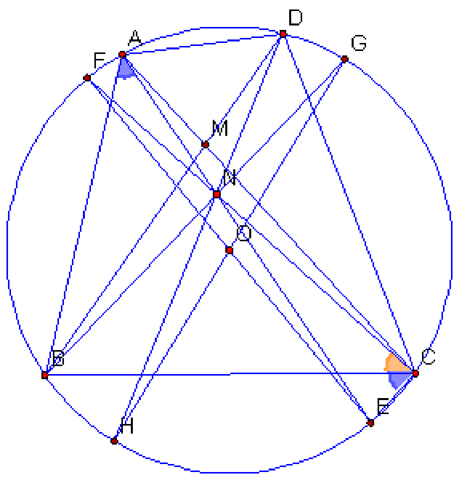
Tangents of (O) at B, D meet at X and tangents of (O) at A, C meet at Y. XY is then the polar of $M \equiv AC \cap BD$ WRT $(O) \Longrightarrow MO \perp XY$. Since (O) is orthogonal to the circles (X) and (Y), with radii XB and YA, then MO is their radical axis.

Since $\angle XBA = \angle ACB$ and $\angle XDA = \angle ACD$, it follows that $\angle NBX + \angle NDX = 90^\circ + \angle BCD \Longrightarrow$ $\angle BND = 270^\circ - \angle BCD - \angle BXD$, but since $\angle BCD = \angle XBD = 90^\circ - \frac{1}{2}\angle BXD \Longrightarrow$ $\angle BND = 180^\circ - \frac{1}{2}\angle BXD$, which means that N is on circle (X). Similarly, N is on $(Y) \Longrightarrow N$ is on their radical axis OM, i.e. O, N, M are collinear.

vslmat
#3 Oct 14, 2013, 4:09 pm

Another solution:
Let AN cut the circumcirle again at E. As $\angle NAB + \angle NCB = \angle NCE = 90^\circ$, NC must cut OE at a point F on the circle. Similarly, if BN meets the circumcircle again at G then DN must cut OG at a point H on the circle.
By Pascal in $ACFBDH : M \equiv AC \cap BD; N \equiv CF \cap DH; X \equiv AH \cap FB$ collinear.
By Pascal in $FEAHGB : O \equiv FE \cap GH; N \equiv EA \cap GB; X \equiv FB \cap AH$ collinear.
Then four point M, N, X, O are collinear.

Attachments:



Quick Reply

High School Olympiads

concyclic points

DonaldLove
#1 Oct 9, 2013, 11:06 pm

consider a non iscoceles, non right triangle ABC, center of circumscribe circle is O, orthocenter H . a line through A parallel to OH meets (O) at K. a line through K parallel to HA meets (O) at L. a line through L parallel to OA meets OH at E. prove that B,C,O,E are concyclic points

Luis González
#2 Oct 10, 2013, 1:24 am • 1

Redefine E as the 2nd intersection of OH with $\odot(BOC)$. We prove that $OA \parallel EL$.

Let the A-altitude cut (O) again at D (reflection of H on BC). Then $AKLD$ is an isosceles trapezoid $\Longrightarrow$ $\angle ADL = \angle DAK = \angle DHO \Longrightarrow DL$ is reflection of OH on BC. Therefore, reflection M of O on BC is on DL and OH, BC, DL concur at $P \Longrightarrow PD \cdot PL = PB \cdot PC = PE \cdot PO \Longrightarrow EOLD$ is cyclic $\Longrightarrow$ $\angle OEL = \angle ODL \equiv \angle ODM$. But from the isosceles trapezoid $OHDM$ and the parallelogram $AHMO$, we have $\angle ODM = \angle OHM = \angle HOA \Longrightarrow \angle HOA = \angle OEL \Longrightarrow OA \parallel EL$, as desired.

kingmathvn
#3 Oct 12, 2013, 5:02 pm • 1

Thanks Luis González.
Are you a teacher? and Do you like geometry?

jayme
#4 Oct 12, 2013, 10:25 pm

Dear Mathlinkers,
after a short investigation, this problem
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=557976
can be usefull with an idea of Luis to resolve the above problem.
Sincerely
Jean-Louis

High School Olympiads

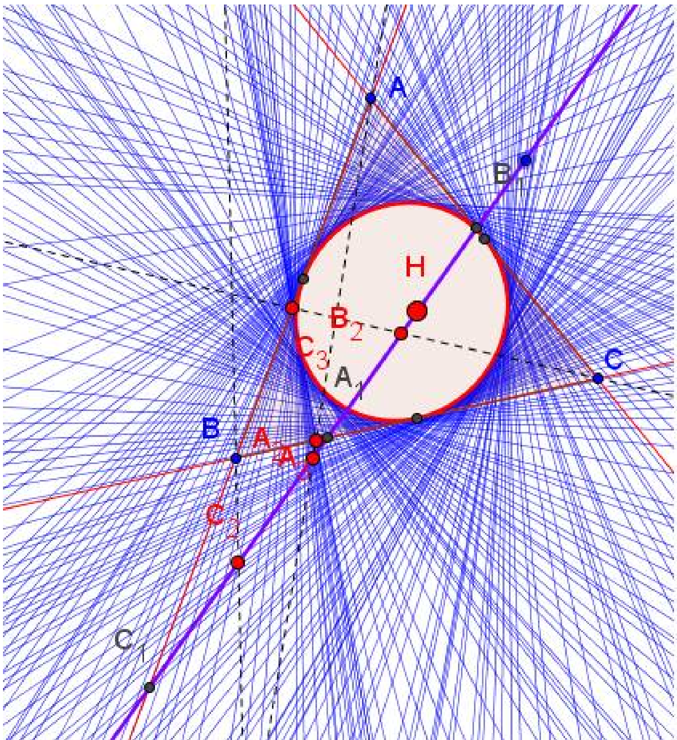
A new line tangent steiner ellipse

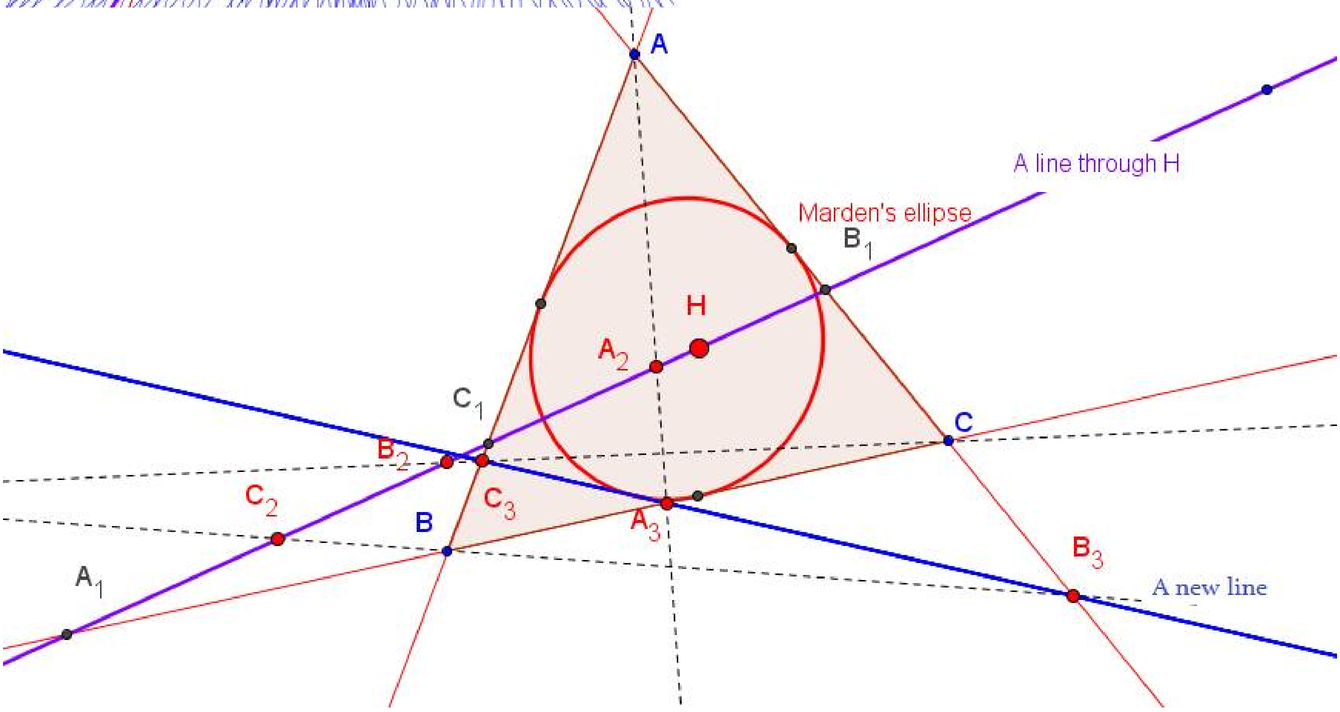
daothanhoai

#1 Sep 24, 2013, 10:15 am

Let a triangle ABC and A_1, B_1, C_1 are on BC, CA, AB and A_1, B_1, C_1 are collinear. A_2, B_2, C_2 are midpoint of B_1C_1, C_1A_1, A_1B_1 (respectively). AA_2, BB_2, CC_2 interect BC, CA, AB at A_3, B_3, C_3 respectively. Heve A_3, B_3, C_3 are collinear. if A_1, B_1, C_1 through orthocenter => A_3B_3 tangent with Steiner's ellipse

Attachments:





Luis González

#2 Oct 9, 2013, 5:45 am • 1

The line $A_3B_3C_3$ is always tangent to the Steiner inellipse of $\triangle ABC$, so $A_1B_1C_1$ does not necessarily have to pass through the orthocenter of $\triangle ABC$.

Let D, E, F be the midpoints of BC, CA, AB, respectively. $P \equiv AA_3 \cap CC_3$ and $Q \equiv BP \cap CA$. By Menalaus' theorem for $\triangle CB_1C_2, \overline{APA_2}$ and $\triangle ABC, \overline{A_1B_1C_1}$, we get the expressions

$$\frac{AB_1}{AC} \cdot \frac{CB}{BA_1} \cdot \frac{A_1C_1}{C_1B_1} = 1 \ , \ \frac{AB_1}{AC} \cdot \frac{CP}{PC_2} \cdot \frac{C_2A_2}{A_2B_1} = 1.$$

But $\frac{C_2A_2}{A_2B_1} = \frac{A_1C_1}{C_1B_1} \Longrightarrow \frac{CP}{PC_2} = \frac{CB}{BA_1} \Longrightarrow BP \parallel A_1B_1C_1.$

Hence, C-median CC_2 of $\triangle CA_1B_1$ bisects the parallel $\overline{BQ}$ to $\overline{A_1B_1}$, i.e. P is midpoint of $\overline{BQ} \Longrightarrow P$ is on the B-midline DF. Thus, by the converse of Brianchon theorem, there is a conic inscribed in the quadrangle CA_3C_3A touching CA_3 and AC_3 at D and $F \Longrightarrow$ it coincides with the Steiner inellipse $\mathcal{S}$ of $\triangle ABC$. Likewise, A_3B_3 is tangent to $\mathcal{S} \Longrightarrow A_3, B_3, C_3$ lie on a tangent of $\mathcal{S}$.

Quick Reply

High School Olympiads

hard problem

Shayanhas
#1 Oct 8, 2013, 2:18 am

Let ABC be triangle and M,N be the midpoint of arc AB,AC and BN,CM cut AC,AB at X,Y.
E,F lie on BC (E,F are not lie on the segment BC) such that $AC = CE$ and $AB = BF$.
O_1 and O_2 are center of circumcircle CNE and BMF and BO_1 cut CO_2 at D.
prove that $AD \perp XY$

Luis González
#2 Oct 8, 2013, 3:25 am • 1

Please, next time give your threads meaningful descriptions. Subjects such as "hard problem", "nice and hard problem", etc. do not describe the problem and makes searching more difficult. As for the proposition, it was posted before at Old and nice Geometry with two solutions.

sco0orpiOn
#3 Oct 9, 2013, 9:56 pm • 1

@shayanhas: salam shayan ,chekhabara omidvaram ke vase tim khoob bekhooni ,in soal ba enkas makoose vakili mitereke :)

there is a short solution for this problem with inversion,
first do an inversion on point B, $(BA.BC)$ and reflect points from the internal bisector of B (the line BN) now we have :

$BY.BN = BA.BC$ so after the invert and reflecting the point Y goes to N ($Y \longleftrightarrow N$)

we also have that $BX.BE = \dfrac{a.c}{a+b}.(a+b) = a.c = BA.BC \implies X \longleftrightarrow E$

and also it is obvious that $C \longleftrightarrow A$ so we can see that $CNE \longleftrightarrow AXY$ now let O be the circumcenter of triangle AXY so we have that $\angle ABO = \angle O_1BC$
as the same way we can do this inversion on point C and see that $\angle O_2CB = \angle OCA$ so we get that the point O and D are isogonal conjugate points in triangle ABC so we easily have that $\angle BAO = \angle CAD$ so it is obvious that AD is perpendicular to XY and we are done

High School Olympiads

Problem of conic and three circles!

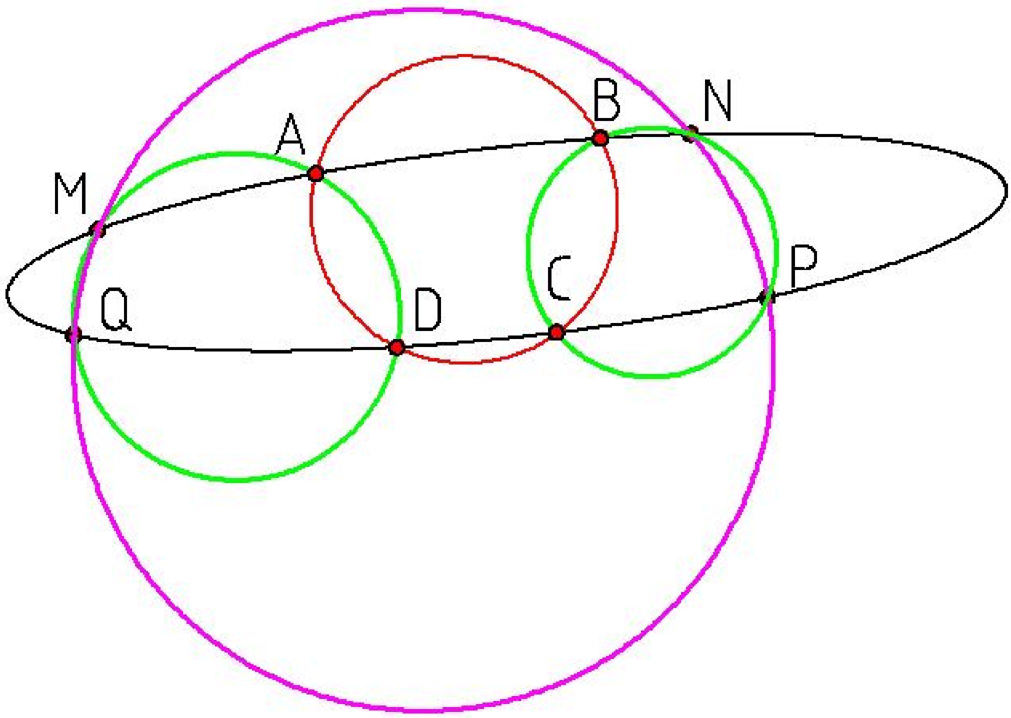
Reply

daothanhoai

#1 Sep 30, 2013, 7:52 am

Let A, B, C, D are concyclic; B, N, P, C are concyclic; M, A, D, Q are concyclic. Let A, B, C, D, M, N, P, Q are on conic. Prove that: M, N, P, Q are concyclic.

Attachments:



Luis González

#2 Oct 8, 2013, 1:35 am • 2

Label $\mathcal{C}$ the given conic with axes ℓ_1, ℓ_2. Arbitrary circle ω through A, D cuts $\mathcal{C}$ again at $X, Y. U, V$ are points on $\mathcal{C}$, such that the tangents of $\mathcal{C}$ at U, V are parallel to XY, AD, respectively. Let I be the intersection of these tangents and $J \equiv AD \cap XY$. By generalized power of point, we have

$$\frac{IV^2}{IU^2} = \frac{JA \cdot JD}{JX \cdot JY} = 1 \Longrightarrow IU = IV.$$

Hence, I is either on ℓ_1 or $\ell_2 \Longrightarrow AD$ and XY are equal inclined to ℓ_1, ℓ_2. Thus when ω varies, keeping A, D fixed, all lines XY go through a fixed direction. As a result, we deduce that $AD \parallel NP$ and $BC \parallel MQ$. If $\odot(MNP)$ cuts $\mathcal{C}$ again at Q^*, then $BC \parallel MQ^* \Longrightarrow Q \equiv Q^* \Longrightarrow M, N, P, Q$ are concyclic.

TelvCohl

#3 Nov 2, 2014, 5:31 am

> *Luis González wrote:*
> Thus when ω varies, keeping A, D fixed, all lines XY go through a fixed direction.

See also here (post #3 lemma) :)

This post has been edited 1 time. Last edited by TelvCohl, Aug 21, 2015, 2:04 am

IDMasterz

#4 Nov 2, 2014, 1:10 pm • 1

By three conics theorem $AD \parallel NP$ so the result follows again by corollary.

Quick Reply

High School Olympiads

nice and hard problem

Shayanhas
#1 Oct 5, 2013, 2:01 am

ω is circumcircle of triangle ABC. ω_a is a circle such that ω_a is tangent to AB,AC,ω ($M_A = (\omega \cap \omega_a)$). The parallel from A to the tangent at M_A of ω cuts BC at A_1. B_1,C_1 are defined similarly. Prove that A_1,B_1,C_1 are collinear.

Luis González
#2 Oct 6, 2013, 2:17 am • 2

If ω_a touches AC, AB at $E, F,$ we have $AE = AF = \frac{bc}{p}, BF = \frac{c(p-b)}{p}$ and $CE = \frac{b(p-c)}{p}$. Hence, the barycentric equation of ω_a WRT $\triangle ABC$ is given by

$$\omega_a \equiv a^2yz + b^2zx + c^2xy - \frac{x+y+z}{p^2}\left(b^2c^2x + c^2(p-b)^2y + b^2(p-c)^2z\right) = 0$$

Equation of radical axis τ_A of ω_a and $\omega \equiv a^2yz + b^2zx + c^2xy = 0$ (their common tangent) is then $\tau_A \equiv b^2c^2x + c^2(p-b)^2y + b^2(p-c)^2z = 0,$ which has infinity point $\left(c^2(p-b)^2 - b^2(p-c)^2 : pb^2(a+b-3c) : -pc^2(c+a-3b)\right) \Longrightarrow$ equation of the parallel from A to τ_A is $c^2(a+c-3b)y - b^2(a+b-3c)z = 0,$ intersecting BC at $A_1 \equiv \left(0 : b^2(a+b-3c) : -c^2(c+a-3b)\right)$.

By cyclic permutations we get the coordinates of B_1, C_1 and we conclude that A_1, B_1, C_1 are collinear on the line

$$\tau \equiv \frac{a^2}{b+c-3a}\cdot x + \frac{b^2}{c+a-3b}\cdot y + \frac{c^2}{a+b-3c}\cdot z = 0,$$

which is the trilinear polar of X_{3445}, i.e. the isogonal conjugate of the Nagel point of the antimedial triangle of $\triangle ABC$.

Luis González
#3 Oct 6, 2013, 9:01 am • 7

OK, we'll prove synthetically that A_1, B_1, C_1 lie on the trilinear polar of the isogonal conjugate of the Nagel point of the antimedial triangle.

Let D, E, F be the midpoints of BC, CA, AB. Incircle (I) touches BC, CA at X, Y and the A-excircle, B-excircle touch BC, CA at P, Q. $N \equiv AP \cap BQ$ is the Nagel point of $\triangle ABC$.

AP cuts (I) at the antipode T of X and S. Since $\angle TSX$ is right, then S is on circle with center D and radius $DX = DP$ $\Longrightarrow DS$ is tangent to (I). Let DS cut AC at U and V is the reflection of U on E. By Newton's theorem for the tangential $ABDU$, the lines AD, BU, XY concur $\Longrightarrow (B, X, D, C) = (U, Y, A, C)$, but $(B, X, D, C) = (C, P, D, B)$ and $(U, Y, A, C) = (V, Q, C, A) = (C, A, V, Q) \Longrightarrow (C, P, D, B) = (C, A, V, Q) \Longrightarrow N \in DV$. Since $DF \parallel CA \equiv UV$, then it follows that the pencil $D(E, F, V, U) \equiv D(E, F, N, S)$ is harmonic.

On the other hand, it's known that AM_A and $AP \equiv AS$ are isogonals, thus the tangent τ_A of ω_a at M_A and DS are antiparallel WRT $AB, AC \Longrightarrow$ isogonal ℓ_A of $AA_1 \parallel \tau_A$ is then parallel to SD. As a result, if N_0 is the Nagel point of the antimedial triangle of $\triangle ABC$, then the pencil AB, AC, AN_0, ℓ_A formed by parallels through A to DE, DF, DN, DS is also harmonic $\Longrightarrow AA_1$ is the harmonic conjugate of the isogonal of AN_0 WRT $AB, AC \Longrightarrow A_1$ is on the trilinear polar τ of the isogonal of N_0. Similarly, B_1 and C_1 fall on the line τ.

High School Olympiads

U, V, E, F lie on a circle

ptk_1411

#1 Oct 4, 2013, 6:45 pm • 1

A, B are arbitrary points on circle (O), their tangents intersect at S. An arbitrary line through S intersect (O) at C, D. Let H be the midpoint of CD, and OH intersects (O) at U, V. E, F respectively lie on SA, SB such that $HE \perp SA, HF \perp SB$. Prove that U, V, E, F lie on a circle.

This post has been edited 1 time. Last edited by ptk_1411, Oct 5, 2013, 10:06 pm

Luis González

#2 Oct 4, 2013, 11:59 pm • 2

Clearly, A, B, and H lie on the circle with diameter $\overline{OS}$, thus EF is Simson line of H WRT $\triangle SAB$, intersecting AB at the projection G of H. If OH cuts AB, EF at M, X, then $\angle XMG = \angle OSH = \angle OBH = \angle BHF = \angle XGM \Longrightarrow X$ is midpoint of the hypotenuse $\overline{HM}$ of the right $\triangle HGM$. Since M is the pole of CD WRT (O), then $(U, V, H, M) = -1 \Longrightarrow XM^2 = XH^2 = XU \cdot XV$. But XH is tangent of $\odot(XEF)$ at H, thus $XH^2 = XE \cdot XF \Longrightarrow XU \cdot XV = XE \cdot XF \Longrightarrow U, V, E, F$ are concyclic.

High School Olympiads

Problem proposed by Omri Solan.

Reply

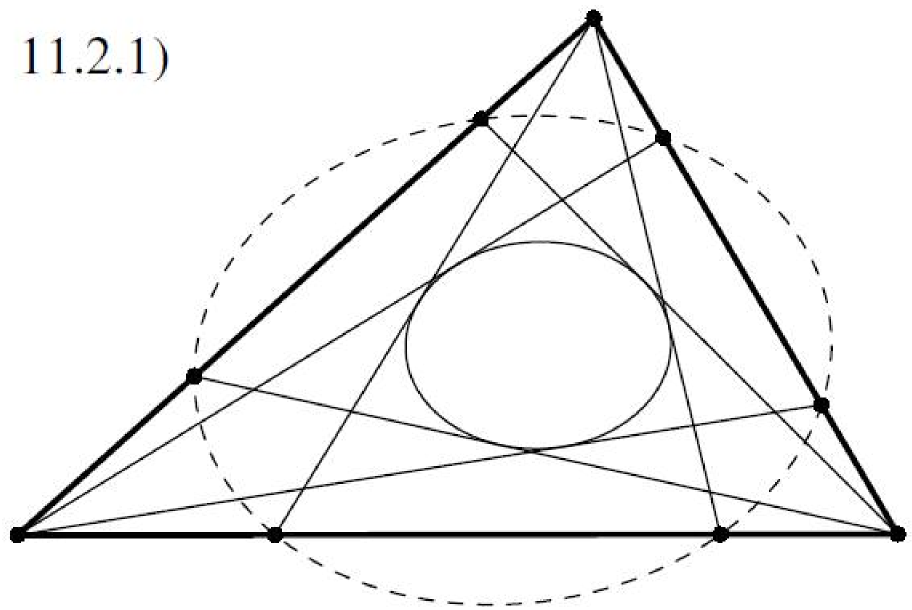
Source: Omri Solan.

mohohoho

#1 Oct 4, 2013, 9:48 pm

Let a circle be inside a triangle such as the circle is not touching any side of the triangle.
Draw pairs of tangent lines coming from the vertices such as intersect the sides of the triangle.
Then, the six points on the side so formed lie on a conic.

Attachments:



Luis González

#2 Oct 4, 2013, 10:09 pm • 1

The problem and its variation have been posted before. See

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=362831
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=520514

Quick Reply

High School Olympiads

Nice collinear problem!

Reply

Source: Proposed by Ali Zouelm

fandogh
#1 Sep 30, 2013, 1:39 pm

Consider triangle ABC, altitudes AA_1, BB_1, CC_1. The incircle is tangent to AB,BC,CA in C_2, A_2, B_2 and I is the incenter. Prove that three points $(AA_1, BB_2), (BI, CC_1), (CC_2, AI)$ are collinear.

Luis González
#2 Oct 4, 2013, 7:01 am

Let $X \equiv AA_1 \cap BB_2, Y \equiv BI \cap CC_1, Z \equiv CC_2 \cap AI$. H, O, G_e are the orthocenter, circumcenter and Gergonne point of $\triangle ABC$. Since the isogonal conjugate of G_e WRT $\triangle ABC$ is the insimilicenter of $(I) \sim (O)$ (well-known), then G_e is on the isogonal of OI WRT $\triangle ABC$, i.e. A, B, C, H, I, G_e lie on a same conic. Now, by Pascal theorem for the hexagon BG_eCHAI, the intersections $X \equiv BG_e \cap HA, Y \equiv CH \cap IB$ and $Z \equiv G_eC \cap AI$ are collinear.

Lyub4o
#3 Nov 28, 2014, 8:32 pm

> *Luis González wrote:*
> Then G_e is on the isogonal of OI WRT $\triangle ABC$, i.e. A, B, C, H, I, G_e lie on a same conic.

Can somebody explain this passage once more, please?

TelvCohl
#4 Nov 28, 2014, 9:41 pm

> *Lyub4o wrote:*
>> *Luis González wrote:*
>> Then G_e is on the isogonal of OI WRT $\triangle ABC$, i.e. A, B, C, H, I, G_e lie on a same conic.
>
> Can somebody explain this passage once more, please?

Since the isogonal conjugate of G_e is the insimilicenter of $(O) \sim (I)$ which is trivial lie on OI ,
so G_e lie on the isogonal of OI WRT $\triangle ABC$ which is the Feuerbach hyperbola of $\triangle ABC$. :)

Quick Reply

High School Olympiads

IK and DL intersect on circumcircle of ABC

Reply

Source: Own

Math-lover123
#1 Sep 17, 2013, 10:42 pm

In a non-isosceles triangle ABC,I is the incenter.
Its incircle touches sides BC,CA and AB at D,E,F respectively.
Projection of point D on EF is K.
L is a foot of perpendicular through I on A-altitude of ABC.
Prove that the lines IK and DL intersect on the circumcircle of triangle ABC.

Luis González
#2 Sep 17, 2013, 11:34 pm

Let M be the midpoint of the arc BC of $\odot(ABC)$. MD goes through the 2nd intersection P of $\odot(ABC) \cap \odot(AEF)$ and MI is tangent to $\odot(PID)$ (see the previous problem Equal angles). Hence $\angle IAL = \angle MID = \angle IPD \Longrightarrow L \in MDP$. Let PI cut EF at Q. Since I is midpoint of the arc EF of $\odot(AEF)$, we have $ID^2 = IE^2 = IQ \cdot IP \Longrightarrow ID$ is tangent to $\odot(PQD) \Longrightarrow \angle IDQ = \angle IPD = \angle DIM \Longrightarrow DQ \parallel MI \perp EF \Longrightarrow Q \equiv K \Longrightarrow IK$ and DL meet at $P \in \odot(ABC)$.

jayme
#3 Sep 18, 2013, 5:16 pm

Der Mathlinkers,
An outline of my proof taking to reference of AoPS
1. A, E, F, I, L are concyclic
2. P is the second point of intersection of this circle and the circumcircle
3. According http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=499885 PI goes through K
4. W is the second points of intersection of AI with the circumcircle
5. according to a convers of Reims theorem, P, L, W are collinear
6. According http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=215418 PL goes through D
and we are done
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Eulers line tangent to circmucircle of OSI_a

Reply

Source: Own

Math-lover123

#1 Sep 16, 2013, 9:40 pm • 1

In acute-angled triangle ABC, $\angle BAC=60^\circ$.
O,H,N,I_a are circumcenter, orthocenter, Nagel point and A-excenter of ABC.
Internal angle bisector of $\angle BAC$ intersects line HN at point S.
Prove that Eulers line of triangle ABC is tangent to the circumcircle of triangle OSI_a.

Luis González

#2 Sep 16, 2013, 10:40 pm • 1

Since $AH=2\cdot OA\cdot\cos\widehat{A}=2\cdot OA\cdot\cos 60^\circ=OA,$ then $\triangle AHO$ is A-isosceles $\Longrightarrow$ internal bisector ASI_a of $\angle OAH$ is perpendicular bisector of $\overline{OH}.$ Since H and N are the circumcenter and incenter of the antimedial triangle of $\triangle ABC,$ then it follows that HN is parallel to $OI,$ where I is the incenter of $\triangle ABC.$ But $\angle BIC=\angle BOC=120^\circ \Longrightarrow O$ is on circle $\odot(BIC)$ with diameter $\overline{II_a} \Longrightarrow \angle IOI_a$ is right $\Longrightarrow OI_a \perp OI \parallel HN.$ Hence, $\angle SI_aO=\angle OHS=\angle HOS$ $\Longrightarrow OH$ is tangent to the circumcircle of $\triangle OSI_a.$

Math-lover123

#3 Sep 16, 2013, 11:23 pm

Nice remark: Eulers line of triangle ABC is also tangent to the circumcircle of triangle HSI_a!

Quick Reply

High School Olympiads

geometry

Locked

Source: APMO 2010

tdv

#1 Sep 16, 2013, 9:44 pm

Let ABC be an acute angled triangle satisfying the conditions AB > BC and AC > BC. Denote by O and H the circumcentre and orthocentre, respectively, of the triangle ABC. Suppose that the circumcircle of the triangle AHC intersects the line AB at M different from A, and the circumcircle of the trinangle AHB intersects the line AC at N different from A. Prove that the circumcentre of the triangle MNH lies on the line OH.

Luis González

#2 Sep 16, 2013, 9:58 pm • 1

Use the search before posting contest problems.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=337222
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=348349
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=392264
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=503545

High School Olympiads

Collinear points

Reply

Source: Isogonal conjugate in projevtive plane

buratinogigle
#1 Sep 14, 2013, 6:58 pm

Let ABC be a triangle inscirbed conic $\mathcal{C}$ and P,Q are two points. Let PA,QA cut $\mathcal{C}$ at A_1,A_2, PB,QB cut $\mathcal{C}$ at B_1,B_2, PC,QC cut $\mathcal{C}$ at C_1,C_2, respectively. Let A_1A_2,B_1B_2,C_1C_2 cut BC,CA,AB at A_3,B_3,C_3, respectively. Prove that A_3,B_3,C_3 are collinear.

Luis González
#2 Sep 14, 2013, 10:17 pm • 1

This has been posted before for a circle. Of course, it is then valid for any conic.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=274286
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=495890

buratinogigle
#3 Sep 14, 2013, 11:24 pm

Thank Luis, I know it was known before. I think it is a theorem of projective geometry, is there projective solution for this ?

Here is a projective version of six points pedal circle

Let ABC be a triangle inscirbed conic $\mathcal{C}$ and P,Q are two points. Let PA,QA cut $\mathcal{C}$ at $A_1,A_2;PB,QB$ cut $\mathcal{C}$ at $B_1,B_2;PC,QC$ cut $\mathcal{C}$ at C_1,C_2, respectively. Let A_1A_2,B_1B_2,C_1C_2 cut BC,CA,AB at A_3,B_3,C_3, respectively.

a) Prove that A_3,B_3,C_3 are collinear on line ℓ.

b) Let A_4,A_5 are the points such that A_4 lies on line BC and (BC,A_3A_4) is harmonic division, and A_5 lies on line A_1A_2 and that (A_1A_2,A_3A_5) is harmonic division. B_4,B_5,C_4,C_5 are defined similarly. Prove that the lines A_4A_5,B_4B_5,C_4C_5 are concurrent.

c) Let A_4A_5,B_4B_5,C_4C_5 cuts ℓ at A_6,B_6,C_6, respectively. Let PA_6,QA_6 cut BC at $A_7,A_8;PB_6,QC_6$ cut CA at $B_7,B_8;PC_6,QC_6$ cut AB at C_7,C_8;. Prove that six points A_7,A_8,B_7,B_8,C_7,C_8 lie on a conic.

Quick Reply

High School Olympiads

Equal angles

Reply

Source: Own

Math-lover123

#1 Sep 14, 2013, 3:33 am

In triangle ABC ($AC > AB$) I and I_a are incenter and A-excenter,respectively.
Circle with diameter AI intersects circumcircle of ABC at two points A and D.
E is foot of A-altitude in ABC.
Prove that $\angle I_aEC = \angle DAI$.

Luis González

#2 Sep 14, 2013, 4:11 am

Let M, N be the midpoints of the arcs BC, BAC of the circumcircle (O). M is obviously the midpoint of $\overline{II_a}$. Incircle (I) touches BC at X and MX cuts (O) again at D'. Inversion with center M and radius $MB = MC = MI$ takes BC into $(O) \Longrightarrow MI^2 = MX \cdot MD' \Longrightarrow \angle MD'I = \angle XIM = \angle AMN = \angle AD'N \Longrightarrow \angle AD'I = \angle MD'N = 90^\circ \Longrightarrow D \equiv D'$. Hence if $V \equiv AM \cap BC$, we get $\frac{VX}{VE} = \frac{VI}{VA} = \frac{VM}{VI_a} \Longrightarrow EI_a \parallel DXM$. But since $ADXV$ is cyclic, due to $MX \cdot MD = MV \cdot MA = MI^2$, then $\angle I_aEC = \angle MXV = \angle DAV \equiv \angle DAI$.

War-Hammer

#3 Sep 25, 2013, 10:23 pm

Let $\omega_1, \omega_2, \omega_3$ be the circumcircle of $\triangle ABC, \triangle AID, \triangle BIC$. It is obvious that ω_2, ω_3 are tangent to each other $\Longrightarrow AD, II, BC$ are concurrent at $K \Longrightarrow \angle AIK = \angle AEK = 90^\circ \Longrightarrow \angle DAI = \angle KAI = \angle IEC$. Let $AI \cap BC = F$, it is well know $(A, I, F, I_a) = -1$ and from $\angle FEA = 90^\circ$ we get $\angle DAI = \angle KAF = \angle IEF = \angle I_aEC$.

Quick Reply

High School Olympiads

Two parallel chords and a locus problem

Source: Turkey TST 1989 - P3

matematikolimpiyati

#1 Sep 11, 2013, 8:46 pm

Let C_1 and C_2 be given circles. Let A_1 on C_1 and A_2 on C_2 be fixed points. If chord A_1P_1 of C_1 is parallel to chord A_2P_2 of C_2, find the locus of the midpoint of P_1P_2.

Luis González

#2 Sep 13, 2013, 12:02 pm

Let O_1, O_2 be the centers of $\mathcal{C}_1, \mathcal{C}_2$. Let M, N be the midpoints of $\overline{P_1P_2}, \overline{A_1A_2}$ and X_1, X_2 the midpoints of $\overline{A_1P_1}, \overline{A_2P_2}$, i.e. the projections of O_1, O_2 on A_1P_1, A_2P_2, respectively. Let K be the midpoint of $\overline{MN}$ and the perpendicular bisector ℓ of $\overline{MN}$ cuts A_1P_2, A_2P_2 at Y_1, Y_2, respectively. Since MN is midparallel of $A_1P_1 \parallel A_2P_2$, then K is midpoint of $\overline{X_1X_2}$ and $\overline{Y_1Y_2} \Longrightarrow X_1Y_1 = X_2Y_2$, which means that O_1 and O_2 are equidistant from $\ell \Longrightarrow \ell$ goes through the midpoint O of $\overline{O_1O_2}$ $\Longrightarrow$ locus of M is the circle with center O and radius ON.

High School Olympiads

nice geo

Reply

DonaldLove

#1 Sep 12, 2013, 11:51 am

consider (O) with diameter AB. C, D are points on (O) such that CD is not the diameter and is perpendicular to ABat E. N is on (O). CN intersects AE at M. (I, r) is tangent to MN, MD and inside tangent to (O). prove that $\frac{1}{r} = \frac{1}{MA} + \frac{1}{ME}$

Luis González

#2 Sep 13, 2013, 3:06 am

Let (I) be the incircle of $\triangle CDN$ tangent to CD at X. (J, r) touches (O) internally and the segments $\overline{MD}, \overline{MN}$ at P, Q. By Sawayama's lemma, PQ goes through I, but $PQ \parallel OMA \parallel IX \Longrightarrow PQ \equiv IX$.

In the $\triangle CDN$, it's known that midpoint B of the arc CD is the circumcenter of $\triangle CID$. Thus if $\odot(CID)$ cuts CN again at K, then the projection U of B on CN is the midpoint of $\overline{CK} \Longrightarrow CU = \frac{1}{2}|NC - NK| = \frac{1}{2}|NC - ND| = EX$. If $S \equiv BU \cap CE$, then $\angle USE = \angle BMN$ and $\angle DCB = \angle CNB$ give $\triangle SBC \sim \triangle MBN \Longrightarrow$

$$\frac{EX}{EB} = \frac{UC}{EB} = \frac{SC}{SB} = \frac{MN}{MB} = \frac{MA}{MC} \Longrightarrow \frac{MA}{AE} = \frac{MC \cdot EX}{AE \cdot EB} = \frac{MC \cdot EX}{EC^2}.$$

But $\frac{MQ}{MC} = \frac{EX}{EC} \Longrightarrow \frac{AE}{MA} = \frac{EC}{MQ} = \frac{ME}{JQ} = \frac{ME}{r} \Longrightarrow$

$$\frac{1}{r} = \frac{AE}{MA \cdot ME} = \frac{MA + ME}{MA \cdot ME} = \frac{1}{MA} + \frac{1}{ME}.$$

P.S. See also the topic a sangaku problem for more solutions.

Quick Reply

High School Olympiads

Collinear with Circumcenter and orthocenter

Reply

Davidinci

#1 Sep 12, 2013, 11:31 pm

Let ABC a triangle with orthocenter H and circumcenter O. Let D and E the interceptions of the circles with diameter AH and BH with the circumcenter (Distintics from A and B) respectively. And let F the interception of the lines AE and BD. Prove that F, H and O are collinear.

Luis González

#2 Sep 13, 2013, 12:19 am

This still holds for any point H on the plane of ABC.

Let DH and EH cut (O) again at X and Y, respectively. Since $\angle ADH \equiv \angle ADX$ and $\angle BEH \equiv \angle BEY$ are right, then AX and BY are diameters of (O) intersection at O. Now, by Pascal theorem for the cyclic hexagon $AXDBYE$, the intersections $O \equiv AX \cap BY, H \equiv XD \cap YE$ and $F \equiv DB \cap EA$ are collinear.

Quick Reply

High School Olympiads

Mixtilinear incircle - concurent lines

Reply

sunken rock

#1 Sep 11, 2013, 9:43 pm

The $A-$mixtilinear incircle of ΔABC touches AB, AC and circumcircle at X, Y, Z respectively. Let $P \in BY \cap CX$. Prove that AP, BC, ZI are concurrent, I being the incenter of ΔABC.

Best regards,
sunken rock

Luis González

#2 Sep 11, 2013, 10:36 pm

If M denotes the midpoint of the arc BC of $\odot(ABC)$, then according to the problem Concurrent, the lines BC, YX and ZM concur at S. If $D \equiv AP \cap BC$, from the complete quadrilateral $BXYC$, it follows that the pecil $A(X, Y, P, S)$ is harmonic $\Longrightarrow (B, C, D, S)$ is harmonic. Since ZM bisects $\angle BZC$ externally, then we deduce that ZD bisects $\angle BZC$ internally $\Longrightarrow I \in ZD$, i.e. AP, BC, ZI concur.

Quick Reply

High School Olympiads

Ellipse - Prove

Reply

luiseduardo

#1 Aug 31, 2013, 10:21 am

Prove that any diameter of an ellipse is the geometric mean between the parallel focal chord and major axis.

yetti

#2 Sep 1, 2013, 4:17 pm • 1

Let $[PQ], [RS]$ be ellipse major and minor axes, respectively, E, F its focal points, and (O) its pedal circle on diameter $[PQ]$.
Let $[AB]$ be arbitrary ellipse diameter, i.e., chord through its center O.
Let $[CD]$ be ellipse chord through E, parallel to the chord $[AB]$, such that A, C are on the same side of PQ. Let perpendiculars through A, B, C, D cut circle (O) at A', B', C', D', so that A', C' are on the same side of PQ as A, C.
Let X be foot of perpendicular from A to PQ and let M be foot of perpendicular from O to $C'D'$, identical with midpoint of the circle chord $[C'D'] \Longrightarrow [C'M]^2 = [OC']^2 - [OM]^2 = [OP]^2 - ([OE]^2 - [ME]^2 = [OR]^2 + [ME]^2$.
On the other hand, $[AO]^2 = [OX]^2 + [AX]^2 = [OX]^2 + [A'X]^2 \cdot \dfrac{[OR]^2}{[OP]^2} =$
$[OX]^2 + ([OA']^2 - [OX]^2) \cdot \dfrac{[OR]^2}{[OP]^2} = [OR]^2 + [OX]^2 \cdot \dfrac{[OE]^2}{[OP]^2}$. Since M- and X-right $\triangle EMO \sim \triangle OXA'$ are similar, having equal angles $\Longrightarrow$
$\dfrac{[ME]}{[OE]} = \dfrac{[OX]}{[OA']} = \dfrac{[OX]}{[OP]}$. Combined, $[AO]^2 = [OR]^2 + [ME]^2 \Longrightarrow [C'D'] = 2[C'M] = 2[AO] = [AB]$. In conclusion, $\triangle ECC' \sim \triangle EDD' \sim \triangle OAA' \sim \triangle OBB'$ are similar, having equal angles $\Longrightarrow$
$\dfrac{[CD]}{[AB]} = \dfrac{[CD]}{[C'D']} = \dfrac{[AB]}{[A'B']} = \dfrac{[AB]}{[PQ]} \Longrightarrow [AB] = \sqrt{[CD] \cdot [PQ]}$.

Luis González

#3 Sep 11, 2013, 10:22 am

Let $\mathcal{E}$ be the given ellipse with center O and foci E, F. AB is arbitrary diameter and XY is a chord through F parallel to AB. Projection P of E on the tangent of $\mathcal{E}$ at X is on its pedal circle (O, ϱ). Let M, N be the midpoints of $\overline{XY}, \overline{XE}$. Since XP bisects $\angle EXF$, then $\angle(PN, PX) = \angle(XP, XN) = \angle(XM, XP) \Longrightarrow PN \parallel XY \Longrightarrow PN$ is E-midline of $\triangle XEF$, passing through the midpoint O of $\overline{EF} \Longrightarrow P \in AB$. Thus, polar p of P WRT $\mathcal{E}$ goes through X and the conjugate direction of the diameter $AB \Longrightarrow p \parallel OM$. If $L \equiv p \cap AB$, then $(A, B, P, L) = -1 \Longrightarrow$
$OA^2 = OB^2 = OL \cdot OP = MX \cdot OP \Longrightarrow AB^2 = XY \cdot 2\varrho$.

Quick Reply

High School Olympiads

Mixtilinear Excircles Isogonals

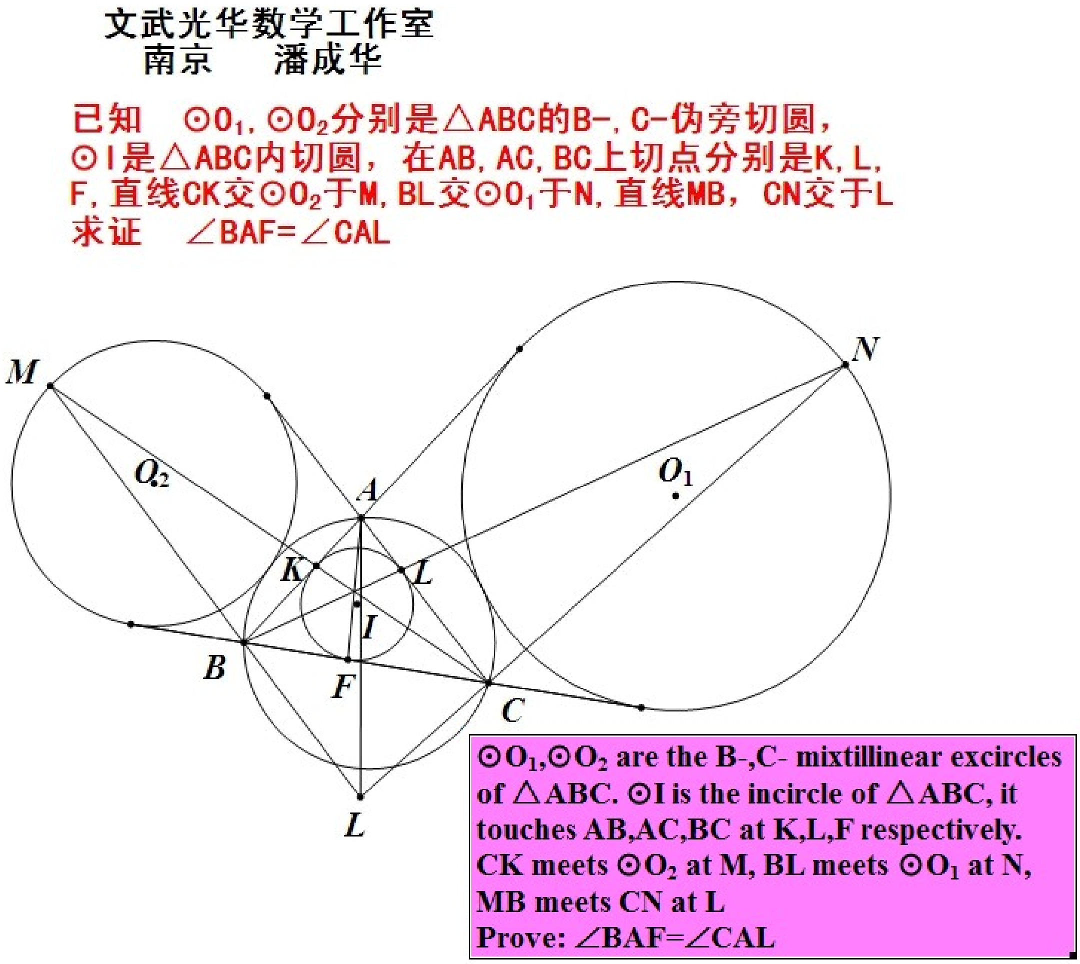
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 24, 2012, 9:24 am

See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Sep 10, 2013, 9:06 am

Let (O_3) be the A-mixtilinear excircle and AF cuts (O_3) at $P,$ such that it is the farthest intersection from $A,$ i.e. P is the image of F under the homothety with center A that carries (I) into (O_3) $\Longrightarrow$ $IF \parallel O_3P \perp BC.$ Similarly, $O_1N \perp CA$ and $O_2M \perp AB.$ Hence, if $(O_1),(O_2)$ touch AB,AC at $Z,Y,$ we have $\angle NO_1Z=\angle BAY=\angle MO_2Y$ $\Longrightarrow$ isosceles $\triangle O_1ZN$ and $\triangle O_2YM$ are similar. But $\triangle AZO_1 \sim \triangle AYO_2$ (for a proof check the solution of Mixtilinear Excircles Perpendicular 2), thus $AYMO_2 \sim AZNO_1 \Longrightarrow \angle NAZ=\angle MAY,$ i.e. AN,AM are isogonals WRT $\angle BAC.$

By similar reasoning BP,BM are isogonals WRT $\angle ABC$ and CP,CN are isogonals WRT $\angle ACB \Longrightarrow$ $L \equiv BM \cap CN$ is the isogonal conjugate of P WRT $\angle ABC \Longrightarrow AL$ and $AF \equiv AP$ are isogonals WRT $\angle BAC,$ or $\angle BAF=\angle CAL,$ as desired.

61plus

#3 Sep 10, 2013, 9:48 pm

By trigo:
Let CN, BM meet at L'.
From Ceva's, $\frac{\sin\angle BAL'}{\sin\angle CAL'}\cdot\frac{\sin\angle ACN}{\sin\angle BCN}\cdot\frac{\sin\angle CBM}{\sin\angle ABM}=1$.
Using $\sin\angle CBM=\frac{CM}{BM}\cdot\sin\angle BKM$,
$\sin\angle ABM=\frac{MK}{BM}\cdot\frac{AC}{AK}\cdot\sin\angle ACK$ and similarly for $\sin\angle ACN, \sin\angle BCN$, and the angle ceva's for AF, BL, CK,
we get $\frac{\sin\angle BAL'}{\sin\angle CAL'}=\frac{CM\cdot LN\cdot AB}{MK\cdot BN\cdot AC}\cdot\frac{\sin\angle CAF}{\sin\angle BAF}$. Hence problem becomes $\frac{CM\cdot LN\cdot AB}{MK\cdot BN\cdot AC}=1$.
Lemma: Let circle O_1 be tangent to AC, BC at X, Y. Then XY passes through C-excenter.
Proof: Let O_1 be tangent to circumcircle at D, and let XD, YD cut circumcircle again at X', Y'. By homothety X', Y' are midpoints of arcAC, BC containing B, A respectively. Hence BX', AY' meet at the C-excenter. Using pascal's on $X'ADBY'C$ we have X, Y, C-excenter collinear.
From the lemma, and noting that C is center of homothety from incircle to circleO_1, we have $\frac{CM}{MK}=\frac{\frac{a+b+c}{2}}{\frac{a+b+c}{2}-\frac{a+b-c}{2}}=\frac{a+b+c}{c}$ and similarly $\frac{LN}{BN}=\frac{b}{a+b+c}$. Substitute back we are done.

DVDthe1st

#4 Sep 30, 2013, 10:13 am

Let circle O_3 be the A-mixtilinear excircle.
Let P be the further intersection of AF and circle O_3

We assume the following lemma: $\angle MAB=\angle NAC$.

Then we note that P and L are isogonal conjugates so we are done.

It remains to show the lemma, which is slightly more difficult to show. Does anyone have a short proof?

Quick Reply

High School Olympiads

Geometry

Locked

Source: Venkatachala Challenge and thrill

sayantan1999

#1 Sep 10, 2013, 12:59 am

S is a given circle and O is a given point. If a variable chord AB subtends a right angle at O, find the locus of the midpoint of AB.

Luis González

#2 Sep 10, 2013, 3:22 am

Posted many times before. It's a problem from APMO 1995.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=141464
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=78745
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=365385
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=383739

High School Olympiads

Miquel point of quadrilaterals

Reply

Source: Own

buratinogigle

#1 Sep 8, 2013, 5:55 pm

Let $ABCD$ be cyclic quadrilateral with Miquel point M. AC cuts BD at E. X,Y,Z,T are projection of E on AB,BC,CD,DA, respectively. Let N be Miquel point of quadlrilateral $XYZT$. Prove that N is midpoint of EM.

Luis González

#2 Sep 9, 2013, 6:31 am • 1

If P,Q,R,S are the reflections of E on $AB,BC,CD,DA,$ then it suffices to prove that $PQRS$ and $ABCD$ have the same Miquel point $M.$ Let $U \equiv AB \cap DC$ and $V \equiv BC \cap DA.$ Since $ABCD$ is cyclic, then its Miquel point M is the projection of E on UV (well-known).

Let EY cut AD at $K.$ Since the pencil $M(E,U,Y,K)$ is harmonic and $EM \perp MU,$ it follows that MU bisects $\angle KMY$ $\Longrightarrow \angle KMU = \angle YMV = \angle YEV = \angle KQV \Longrightarrow MKQV$ is cyclic $\Longrightarrow \angle MQE = \angle MVK = \angle MBA$ and since $\angle MEQ = \pi - \angle MVB = \angle MAB,$ then $\triangle MEQ \sim \triangle MAB.$ Likewise, we obtain $\triangle MSE \sim \triangle MAB \Longrightarrow$ $\triangle MEQ \sim \triangle MSE$ and similarly $\triangle MER \sim \triangle MPE,$ which gives that ME bisects both $\angle PMR, \angle QMS,$ i.e. $\angle PMQ = \angle SMR$ and

$$\frac{MR}{ME} = \frac{ME}{MP} \ , \ \frac{ME}{MS} = \frac{MQ}{ME} \Longrightarrow \frac{MR}{MS} = \frac{MQ}{MP}.$$

This means that $\triangle MPQ$ and $\triangle MSR$ are directly similar by SAS $\Longrightarrow M$ is the center of the spiral similarity that takes $\overline{PQ}$ into $\overline{SR},$ hence it is the Miquel point of $PQRS,$ as desired.

djmathman

#3 Sep 9, 2013, 7:25 am

Wait, how exactly does a quadrilateral have a defined unique Miguel point? I know that for a triangle ABC, the Miguel point is based on both that triangle and the locations of points X, Y, Z on the sides of the triangle (so that, say, $M=(AXY)\cap(BYZ)\cap(CZA)$). So how is it possible for the quadrilateral version to be based solely on the location of points A, B, C, and D?

mathocean97

#4 Sep 9, 2013, 7:33 am • 1

Well, if we consider the complete quadrilateral where $AB \cap CD = E$ and $AD \cap BC = F$, then by simple angle chasing one can show that $\odot EAD, \odot EBC, \odot FAB, \odot FCD$ all pass through a common point, and that is considered to be the Miquel Point of the quadrilateral.

Quick Reply

High School Olympiads

Perpendicular Circles Like TST Problem

Source: Iran Third Round 2013 - Geometry Exam - Problem 3

War-Hammer

#1 Sep 8, 2013, 6:43 am • 1

Suppose line ℓ and four points A, B, C, D lies on ℓ. Suppose that circles ω_1, ω_2 passes through A, B and circles ω'_1, ω'_2 passes through C, D. If $\omega_1 \perp \omega'_1$ and $\omega_2 \perp \omega'_2$ then prove that lines $O_1O'_2, O_2O'_1, \ell$ are concurrent where O_1, O_2, O'_1, O'_2 are center of $\omega_1, \omega_2, \omega'_1, \omega'_2$.

Luis González

#2 Sep 8, 2013, 9:56 am • 1

Inversion with center B and power $\overline{BC} \cdot \overline{BD}$ takes $\omega_1{'}, \omega_2{'}$ into themselves and carries ω_1, ω_2 into lines passing through the inverse P of A and perpendicular to $\omega_1{'}, \omega_2{'}$, due to conformity. In other words, $PO_1{'}$ and $PO_2{'}$ are perpendicular to BO_1 and BO_2. Now, if M, N are the intersections of $O_1O_2, O_1{'}O_2{'}$ with ℓ and L_∞ is the infinity point of $\perp \ell$, then the pencils $B(O_1, O_2, M, L_\infty)$ and $P(O_1{'}, O_2{'}, L_\infty, N)$ with corresponding perpendicular rays have equal cross ratios $\Longrightarrow$ $(O_1, O_2, M, L_\infty) = (O_1{'}, O_2{'}, L_\infty, N) \Longrightarrow MO_1 : MO_2 = NO_2{'} : NO_1{'}$. Since $O_1O_2 \parallel O_2{'}O_1{'}$, then we deduce that $O_1O_2{'}, O_2O_1{'}$ and $MN \equiv \ell$ concur.

mathlink

#3 Dec 18, 2013, 2:35 pm

We don't need to use L_∞. Here is a simplier way to end González's solution:
Denote $\angle O'_1PN = \alpha, \angle O'_2PN = \beta$ and $X = O'_1P \cap O_2B, Y = O'_2P \cap O_1B$ then

$$\frac{O'_1N}{O'_2N} = \frac{\tan\beta}{\tan\alpha} = \frac{\tan XPB}{\tan YPB} = \frac{\cot XBP}{\cot YBP} = \frac{\tan O_1BM}{\tan O_2BM} = \frac{O_1M}{O_2N}$$

Done

High School Olympiads

Collinear Point On Line With Distance R/2 To Circumcenter

Source: Iran Third Round 2013 - Geometry Exam - Problem 5

War-Hammer
#1 Sep 8, 2013, 7:35 am • 2

Let ABC be triangle with circumcircle (O). Let AO cut (O) again at A'. Perpendicular bisector of OA' cut BC at P_A. P_B, P_C define similarly. Prove that :

I) Point P_A, P_B, P_C are collinear.

II) Prove that the distance of O from this line is equal to $\frac{R}{2}$ where R is the radius of the circumcircle.

Luis González
#2 Sep 8, 2013, 7:43 am

This is a particular case of a problem recently discussed. See

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=535534
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=532462

mathuz
#3 Sep 17, 2013, 6:39 pm

by perspectivity, please help me!

We have that a triangle $A_1B_1C_1$ such that construct by the lines - perpendicular bisector of the OA', OB', OC',
here A_1 is intersection point of the perpendicular bisector OB' and OC' and similar that ...

For (i) we need prove that ABC and $A_1B_1C_1$ have a line perspectivity. So we prove that they have a center perspectivity.

But, i haven't proof.
Please help me, in this idea!

War-Hammer
#4 Sep 25, 2013, 1:25 am

But how about second part ???

TelvCohl
#5 Oct 21, 2014, 5:41 pm • 2

My solution:

Let H be the orthocenter of $\triangle ABC$.
Let D be the intersection of AH and (ABC) .
Let K be the Anti-steiner point of OH (so-called Kiepert focus) of $\triangle ABC$.
Let O_a, O_b, O_c be the reflection of O in BC, CA, AB , respectively .

Easy to see P_A, P_B, P_C is the center of $(OO_aA'), (OO_bB'), (OO_cC')$.
Since D is the reflection of H in BC ,

so O_a, D, K are collinear .
Since $\angle OO_aK = \angle ADK = \angle OA'K$,
so K, O_a, A', O are concyclic . ie. K lie on (OO_aA') .
Similarly, we can prove (OO_bB') and (OO_cC') pass through K ,
so we get $(OO_aA'), (OO_bB'), (OO_cC')$ are coaxial. ie. P_A, P_B, P_C are collinear at l

Since l is the perpendicular bisector of OK ,
so the distance between O and l is $\frac{R}{2}$.

Q.E.D

Quick Reply

High School Olympiads

b+c=2a Make Cyclic Quadrilateral

Reply

Source: Iran Third Round 2013 - Geometry Exam - Problem 2

War-Hammer
#1 Sep 8, 2013, 4:39 am

Let ABC be a triangle with circumcircle (O). Let M,N be the midpoint of arc AB,AC which does not contain C,B and let M',N' be the point of tangency of incircle of $\triangle ABC$ with AB,AC. Suppose that X,Y are foot of perpendicular of A to MM',NN'. If I is the incenter of $\triangle ABC$ then prove that quadrilateral $AXIY$ is cyclic if and only if $b+c=2a$.

Luis González
#2 Sep 8, 2013, 5:29 am • 1

From the parallel radii $\overline{OM} \parallel \overline{IM'}$ and $\overline{ON} \parallel \overline{IN'}$, it follows that $V \equiv MM' \cap NN'$ is the exsimilicenter of $(I) \sim (O)$ $\Longrightarrow V \in OI$. Hence, A,X,Y,I are concyclic $\Longleftrightarrow \angle AIV = 90^\circ \Longleftrightarrow \angle AIO = 90^\circ$. It suffices to prove that $\angle AIO = 90^\circ \Longleftrightarrow b+c=2a$, which has been posted before. For instance, see Incentre, circumcentre and midpoints of AC,BC are concyclic and elsewhere.

sayantanchakraborty
#3 Apr 7, 2014, 1:52 pm

What do you mean by exsimilicenter???From where do I get it??

wiseman
#4 Sep 20, 2014, 12:43 am

> *sayantanchakraborty wrote:*
> What do you mean by exsimilicenter???From where do I get it??

I think this can help.

IDMasterz
#5 Sep 21, 2014, 3:51 pm

Tangency and midpoints of arcs... If $MM' \cap NN' = E$, then E is the external centre of similitude of (I) and (O). If (E) is the circle with diametre AE then $I \in (E) \implies \angle AIE \equiv \angle AIO = \frac{\pi}{2}$. Now, let the midpoint of arc CAB be X, P midpoint of arc BC, I_A be the A-excentre. As $OI \parallel AX$, and $OX \cap AI = P$, we have

$$AI = IM \implies 3 = \frac{AI_A}{AI} = \frac{s}{s-a} \implies \frac{a}{s-a} = 2 \implies 2a = b+c.$$

Quick Reply

High School Olympiads

Concyclic points

Locked

vyfukas

#1 Sep 7, 2013, 11:58 pm

Let X be a circle, AC and BD two chords intersecting at the point P. Other circle is tangent to the chords AC, BD and circle at points E, F and T. J is incenter of triangle APD. Prove that points J, F, D and T are concyclic.

Luis González

#2 Sep 8, 2013, 1:50 am • 1

Rather old and posted many times before. Thread locked.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=6086
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=41667
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=253207
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=407366

High School Olympiads

Good Geometry

Locked

Source: My ever active mind(of course joking)

mathdebam

#1 Sep 7, 2013, 10:06 pm

Let ABC be a triangle in which the the incentre is I and the excircle opposite to A touches BC at D. The line ID (when extended) meets the excircle oppossite to A at P. Prove that PD bisects the angle BPC.

Luis González

#2 Sep 8, 2013, 1:12 am

This is basically the extravesion of IMO ShortList 2002, geometry problem 7 for an excircle. See also

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=455279
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=501770
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=547044

High School Olympiads

Theorem (known or not)

Reply

Source: own?

jayme

#1 Aug 27, 2013, 11:23 am • 1

Dear Mathlinkers,
what do you think about this result...?

1. (O) a circle
2. A,B,C three points of (O)
3. P a point inner (O) for the commodity of the figure
4. A'B'C' the P-circumtriangle of ABC
5. Q a point inner (O) for the commodity of the figure
6. A*B*C* the Q-circumtriangle of ABC
7. A+B+C+ the Q-circumtriangle of A'B'C'

Prove : A+B+C+ and A*B*C* are perspective with center R
and P, Q, R are collinear.

Sincerely
Jean-Louis

Luis González

#2 Sep 7, 2013, 11:38 pm • 1

No incidence conditions are necessary, because the problem is merely projective.

By Pascal theorem for the hexagon $A^*B'B^+B^*A'A^+$, the intersections $S \equiv A^*B' \cap B^*A'$, $Q \equiv B'B^+ \cap A'A^+$ and $R \equiv B^+B^* \cap A^+A^*$ are collinear. By Pascal theorem for the hexagon $A^*B'BB^*A'A$, the intersections $S \equiv A^*B' \cap B^*A'$, $P \equiv B'B \cap A'A$ and $Q \equiv BB^* \cap AA^*$ are collinear. Therefore, P, Q, R, are collinear, i.e. A^*A^+ and B^*B^+ meet on $R \in PQ$. Similarly, C^*C^+ goes through R and the conclusion follows.

Quick Reply

High School Olympiads

Equal Angles With Midpoint Of AH

Reply

Source: Iran Third Round 2013 - Geometry Exam - Problem 4

War-Hammer

#1 Sep 6, 2013, 6:44 pm • 1

In a triangle ABC with circumcircle (O) suppose that A-altitude cut (O) at D. Let altitude of B, C cut AC, AB at E, F. H is orthocenter and T is midpoint of AH. Parallel line with EF passes through T cut AB, AC at X, Y. Prove that $\angle XDF = \angle YDE$.

Luis González

#2 Sep 7, 2013, 3:01 am • 5

$\angle AXY = \angle AFE = \angle ACB = \angle ADB \Longrightarrow B, X, T, D$ are concyclic. Further, $\angle TEH = \angle THE = \angle BHD = \angle BDH \Longrightarrow D, B, T, E$ are concyclic, thus D, B, X, T, E are concyclic $\Longrightarrow$ $\angle EDX = \angle ETY$. By similar reasoning, we have $\angle FDY = \angle FTX$. But $\triangle TEF$ is obviously T-isosceles and $XY \parallel EF$ bisects $\angle ETF \Longrightarrow \angle ETY = \angle FTX \Longrightarrow \angle EDX = \angle FDY \Longrightarrow \angle XDF = \angle YDE$.

mathuz

#3 Sep 17, 2013, 1:36 pm • 2

@Luis Gonzalez, nice solution!
But, i have another solution.
Let l be perpendicular line to AD at the point D,
$CH \cap l = P$, $BH \cap l = Q$, $AB \cap l = M$, $AC \cap l = N$ and P is projection of the point X on the lind AD.
We have that

$$\frac{MX}{MA} = \frac{DP}{AD} = 1 - \frac{\cos A \sin B}{2 \sin C}.$$

$\triangle DMA$ similar to $\triangle DPH$ and let a point X' lie on PH and $\angle XDX' = 90^\circ$ and $X'DFX$ is cyclic.
So

$$X'F = \frac{2R \sin A \cos(B - C)}{\sin C}$$

and similar that we have Y' lie on the QH, $Y'DEY$ is cyclic and

$$Y'E = \frac{2R \sin A \cos(B - C)}{\sin B}.$$

Hence $\frac{FX'}{EY'} = \frac{AC}{AB} = \frac{AX}{AY} = \frac{FX}{EY} \Rightarrow \triangle X'FX$ is similar to $\triangle Y'EY$ and $\angle FX'X = \angle EY'Y \Rightarrow$ $\angle FDX = \angle EDY$.

War-Hammer

#4 Sep 17, 2013, 3:08 pm • 1

Nice , it's 5'th solution for this nice problem

halloffame

#5 Apr 24, 2016, 7:00 am

Let EF intersect AD at M. We have $BDMF$ and $BDTX$ cyclic so we easily get $\angle FDX = \angle TBM$. By analogy we get $\angle EDY = \angle MCT$ so we need to prove $\angle MBT = \angle MCT$ or in other word that M is the orthocenter of $\triangle BTC$ or $RM \cdot RT = RB \cdot RC = RH \cdot RA$ where R is the intersection of AD and BC. This is well know to be true because R, H, M, A are harmonic so we are finished.

Quick Reply

High School Olympiads

Owm Problem. Concurrency.

Reply

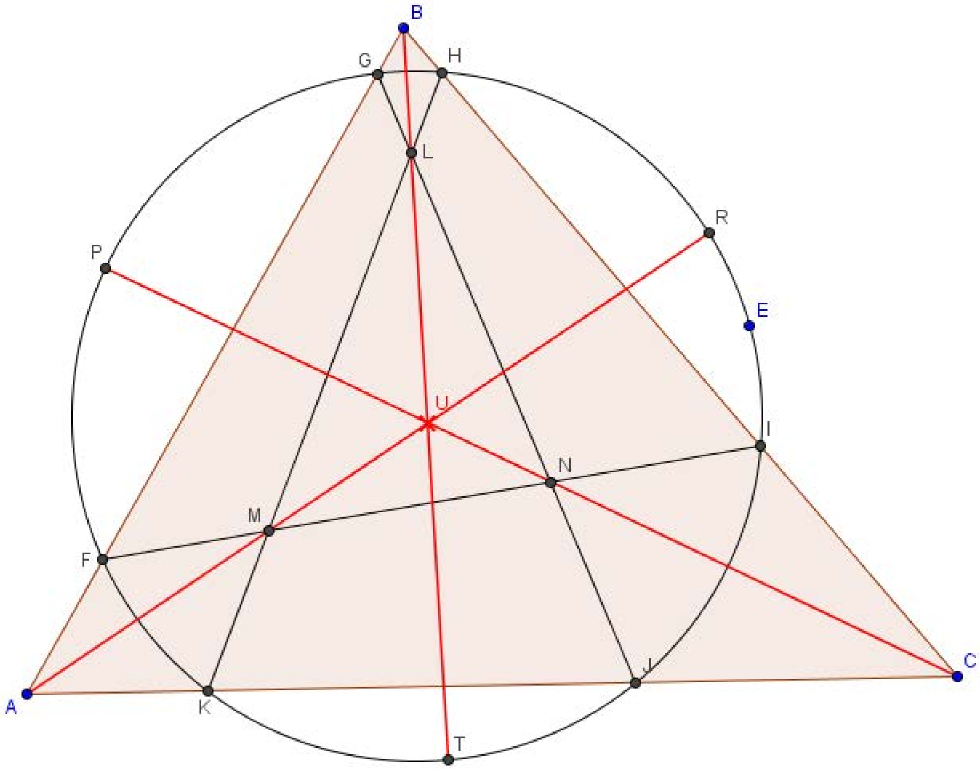
mohohoho

#1 Sep 6, 2013, 10:00 pm

Let ABC be any triangle, now draw a circle such
that intersect twice sides AB, BC, AC at G, F; H, I;
K, J, respectively. Now draw lines GJ, HK, FI such they
form triangle MNL.

Prove that lines AM, BL, CN concur.

Attachments:



Luis González

#2 Sep 6, 2013, 10:25 pm • 1

You already submitted this problems at Own Problem # 5. Concurrency. Please do not double post, otherwise your posts will be locked.

By Pascal theorem for the cyclic hexagon $FIHKJG,$ the intersections $Y \equiv FI \cap KJ, X \equiv IH \cap JG$ and $Z \equiv HK \cap GF$ are collinear, thus $\triangle ABC$ and $\triangle MLN$ bounded by the lines GJ, IF, KH are perspective through $\overline{XYZ}.$ By Desargues theorem, AM, BL, CN concur.

vslmat

#3 Sep 27, 2013, 8:19 pm • 1

Posted before
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=511097

mohohoho

#4 Sep 28, 2013, 8:00 pm

Oh, I didn't know. Thank you, vslmat.

Quick Reply

High School Olympiads

another isogonal conjugate problem

Reply

ams1215

#1 Sep 5, 2013, 11:35 am • 1

Consider triangle ABC, circumscribed circle (O), orthocenter H. AH,BH,CH consecutively intersect (O) at A1,B1,C1. A random point P, not on BC,CA,AB,(O). AP,BP,CP consecutively intersect (O) at A1,B1,C1. X,Y,Z are consecutively the intersections of BC,CA,AB and A1A2,B1B2,C1C2. Prove that X,Y,Z are collinear and the line contains these 3 points is perpendicular to QH (Q is the isogonal conjugate of P to triangle ABC)

Luis González

#2 Sep 6, 2013, 10:04 am • 1

> *ams1215 wrote:*
> Consider triangle ABC, circumscribed circle (O), orthocenter H. AH,BH,CH consecutively intersect (O) at A1,B1,C1. A random point P, not on BC,CA,AB,(O). AP,BP,CP consecutively intersect (O) at A2,B2,C2. X,Y,Z are consecutively the intersections of BC,CA,AB and A1A2,B1B2,C1C2. Prove that X,Y,Z are collinear and the line contains these 3 points is perpendicular to QH (Q is the isogonal conjugate of P to triangle ABC)

Typo corrected in red color. See the problem P'H perpendicular A'B'C' (P' isogonal conjugate of P).

Quick Reply

High School Olympiads

Keep expanding squares outside

Reply

Source: Indonesian Mathematical Olympiad 2013 Problem 7

chaotic_iak
#1 Sep 5, 2013, 7:07 pm

Let $ABCD$ be a parallelogram. Construct squares $ABC_1D_1, BCD_2A_2, CDA_3B_3, DAB_4C_4$ on the outer side of the parallelogram. Construct a square having B_4D_1 as one of its sides and it is on the outer side of AB_4D_1 and call its center O_A. Similarly do it for C_1A_2, D_2B_3, A_3C_4 to obtain O_B, O_C, O_D. Prove that $AO_A = BO_B = CO_C = DO_D$.

Luis González
#2 Sep 6, 2013, 9:29 am

Denote P, Q, R, S the centers of the squares $ABC_1D_1, BCD_2A_2, CDA_3B_3, DAB_4C_4$, respectively. Since $BP = CR, BQ = CQ$ and $\angle PBQ = \angle RCQ = 90^\circ + \angle ABC$, it follows that $\triangle BPQ \cong \triangle CRQ$ by SAS $\Longrightarrow$ $PQ = QR$ and $\angle PQR = \angle BQC = 90^\circ$, i.e $\triangle PQR$ is isosceles right at Q. Likewise, $\triangle RSP$ is isosceles right at S, thus quadrilateral $PQRS$ is a square with side length $PQ = QR = RS = SP = m$.

By Van Aubel's theorem (see Squares Constructed on Sides of Quadrilateral) for the degenerate quadrilateral A_2C_1BB, we deduce that $\overline{BO_B}$ and $\overline{PQ}$ are congruent and perpendicular $\Longrightarrow BO_B = m$. Similarly, AO_A, CO_C, DO_D equal m.

Quick Reply

High School Olympiads

Circumcircle intersections everywhere

Reply

Source: Indonesian Mathematical Olympiad 2013 Problem 2

chaotic_iak

#1 Sep 5, 2013, 6:55 pm

Let ABC be an acute triangle and ω be its circumcircle. The bisector of $\angle BAC$ intersects ω at [another point] M. Let P be a point on AM and inside $\triangle ABC$. Lines passing P that are parallel to AB and AC intersects BC on E, F respectively. Lines ME, MF intersects ω at points K, L respectively. Prove that AM, BL, CK are concurrent.

Luis González

#2 Sep 6, 2013, 8:44 am • 1

If MF, ME cut AC, AB at U, V, then clearly $\triangle PEF$ and $\triangle AVU$ are homothetic with center $M \Longrightarrow UV \parallel EF \equiv BC$. Let UV cut AM at D. Since $MB^2 = MC^2 = ME \cdot MK = MF \cdot ML$, then $EKLF$ is cyclic $\Longrightarrow$ $\angle MAL = \angle MKL = \angle MFE = \angle MUV \Longrightarrow ALUD$ is cyclic $\Longrightarrow \angle MLD = \angle CAM = \angle MLB \Longrightarrow D \in BL$. Similarly, $D \in CK \Longrightarrow AM, BL, CK$ concur at D.

mathuz

#3 Sep 6, 2013, 2:21 pm

my solution is by cheva sinus!

Quick Reply

High School Olympiads

Isogonal conjugate and perpendicularity

Source: created by treegoner for Mathematical Reflection

treegoner
#1 Aug 8, 2006, 9:52 am • 3

Since the deadline for submitting solutions to Mathematical Reflection third issue 's problem column has passed, I can post this problem for discussion.

Given ABC be a triangle, and P be a point inside the triangle. Let A', B', C' be the intersections of AP, BP, CP with the triangle's sides. Through P, we draw a perpendicular to PA that intersects BC at A_1. Define B_1, C_1 analogously. Let P^* be the isogonal conjugate of P with respect to $A'B'C'$. Show that A_1, B_1, C_1 lie on a line that is perpendicular to PP^*.

In fact, one can prove a stronger statement: if $A''B''C''$ be the pedal triangle of P with respect to $A'B'C'$, then the line A_1, B_1, C_1 is the polar of the point P with respect to the circumcircle of triangle $A''B''C''$. :)

Luis González
#2 Aug 21, 2013, 6:14 am • 6

We prove the stronger version of the problem:

$\triangle A'B'C'$ is the cevian triangle of P WRT $\triangle ABC. X, Y, Z$ are the projections of P on $B'C', C'A', A'B'$. Perpendiculars to PA, PB, PC at P cut BC, CA, AB, at A_1, B_1, C_1, respectively. It's known that these points are collinear (orthopolar τ of P WRT ABC). Then τ is the polar of P WRT $\odot(XYZ)$.

Proof: Let the perpendicular to PA at P cut $A'B'$ and $A'C'$ at $N, M. U \equiv MN \cap YZ$ and A_2 is the reflection of P about $U. \angle MNZ = \angle ZPA' = \angle ZYA' \Longrightarrow MNZY$ is cyclic and since UP is clearly tangent to $\odot(PYA'Z)$, then we have $UA_2{}^2 = UP^2 = UY \cdot UZ = UM \cdot UN \Longrightarrow$ circle (U) with diameter $\overline{PA_2}$ is orthogonal to both $\odot(XYZ)$ and $\odot(MNZY) \Longrightarrow P, A_2$ are conjugate points WRT $\odot(XYZ) \Longrightarrow A_2$ is on the polar of P WRT $\odot(XYZ)$. It also follows that $(P, A_2, M, N) = -1 \Longrightarrow A'(P, A_2, C', B') = -1$, but $A'(P, A_1, C', B') - 1 \Longrightarrow A'A_1 \equiv A'A_2 \equiv BC \Longrightarrow A_1 \equiv A_2$, hence A_1 lies on the polar of P WRT $\odot(XYZ)$. Analogously, B_1 and C_1 lie on the polar of P WRT $\odot(XYZ) \Longrightarrow \tau \equiv A_1B_1C_1$ is the polar of P WRT $\odot(XYZ)$.

jayme
#3 Aug 21, 2013, 12:22 pm

Dear Mathlinkers,
the reference in question
Nguyen K. L., O13, Mathematical Reflections 3 (2006) ; http://reflections.awesomemath.org/
Sincerely
Jean-Louis

High School Olympiads

Double elements of three involutions

Reply

Source: created by treegoner

treegoner

#1 Jul 25, 2006, 9:01 pm

Let (C) be a conic and ABC is a triangle such that the sides of BC, CA, AB intersect the conic at M_1 and M_2, N_1 and N_2, P_1 and P_2. Suppose there exist the double elements X_1 and X_2 of the involution formed by (B, C) and (M_1, M_2) and define Y_1, Y_2, Z_1, Z_2 similarly. Show that the six points $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ are the vertices of a complete quadrangle.

Number1

#2 Aug 16, 2006, 12:41 am

What are double elements?

treegoner

#3 Aug 16, 2006, 8:31 pm

Suppose an involutions formed by conjugate pairs of points $(A, A'), (B, B'), \ldots$.. Double elements M and M', if exist, are self conjugate points $(M, M), (M', M')$. I hope it is clear. :)

Luis González

#4 Aug 20, 2013, 12:15 am

It's known that double points in an involution are harmonic conjugates WRT two pairs of homologous points, so WLOG assume that X_1, Y_1, Z_1 are on $\overline{BC}, \overline{CA}, \overline{AB}$, respectively. X_2, Y_2, Z_2 are then on the extensions of BC, CA, AB. By property of the involution $B \mapsto C, M_1 \mapsto M_2, X_1 \mapsto X_1$, we obtain

$$\frac{\overline{BM_1} \cdot \overline{BM_2}}{\overline{CM_1} \cdot \overline{CM_2}} = \frac{\overline{X_1B} \cdot \overline{X_1B}}{\overline{X_1C} \cdot \overline{X_1C}} = \frac{\overline{X_1B}^2}{\overline{X_1C}^2}$$

Similarly, we have the expressions

$$\frac{\overline{CN_1} \cdot \overline{CN_2}}{\overline{AN_1} \cdot \overline{AN_2}} = \frac{\overline{Y_1C}^2}{\overline{Y_1A}^2}, \ \frac{\overline{AP_1} \cdot \overline{AP_2}}{\overline{BP_1} \cdot \overline{BP_2}} = \frac{\overline{Z_1A}^2}{\overline{Z_1C}^2} \Longrightarrow$$

$$\frac{\overline{X_1B}^2}{\overline{X_1C}^2} \cdot \frac{\overline{Y_1C}^2}{\overline{Y_1A}^2} \cdot \frac{\overline{Z_1A}^2}{\overline{Z_1C}^2} = \frac{\overline{BM_1} \cdot \overline{BM_2}}{\overline{CM_1} \cdot \overline{CM_2}} \cdot \frac{\overline{CN_1} \cdot \overline{CN_2}}{\overline{AN_1} \cdot \overline{AN_2}} \cdot \frac{\overline{AP_1} \cdot \overline{AP_2}}{\overline{BP_1} \cdot \overline{BP_2}}.$$

By Carnot's theorem for the conic $\mathcal{C}$ intersecting the sidelines of $\triangle ABC$ at 6 points, the RHS of the latter expression equals 1 $\Longrightarrow$

$$\frac{\overline{X_1B}}{\overline{X_1C}} \cdot \frac{\overline{Y_1C}}{\overline{Y_1A}} \cdot \frac{\overline{Z_1A}}{\overline{Z_1C}} = -1.$$

Hence, by Ceva's theorem AX_1, BY_1, CZ_1 concur at a point $P \Longrightarrow X_2, Y_2, Z_2$ lie on the trilinear polar of P WRT $\triangle ABC$ $\Longrightarrow X_2 \in Y_1Z_1, Y_2 \in Z_1X_1$ and $Z_2 \in X_1Y_1$, i.e. $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ are the vertices of the complete quadrilateral bounded by the sidelines of the cevian triangle of P and its trilinear polar.

Quick Reply

High School Olympiads

An extension from "right" to "any" triangle.

Reply

Source: The source is the particular case - right triangle.

Virgil Nicula

#1 Aug 15, 2013, 10:55 pm • 1

PP. Let a nonacute $\triangle ABC$, the midpoint M of $[BC]$, the tangent points $E\in CA$ and $F\in AB$ of its incircle $w=C(I,r)$ and $X\in MI$, $AX\perp BC$. Define $P\in EX\cap w$ and $Q\in FX\cap w$. Prove that $\frac{\delta_{BC}(P)}{\delta_{BC}(Q)}=\frac{b(s-c)}{c(s-b)}\cdot\frac{2R-h_c}{2R-h_b}$.

Particular case. If $AB\perp AC$, then $\delta_{BC}(P)=\delta_{BC}(Q)$, where $\delta_d(M)$ is the distance from M to the line d .

Luis González

#2 Aug 16, 2013, 4:22 am • 3

Let the incircle (I) and the A-excircle (I_a) touch BC at $D,K.$ Let U be the antipode of D WRT $(I).$ Since A is the exsimilicenter of $(I)\sim(I_a),$ then A,U,K are collinear. M is also midpoint of $DK,$ hence MIX is D-midline of $\triangle DKU$ $\Longrightarrow IX\parallel AU\Longrightarrow AXIU$ is a parallelogram $\Longrightarrow AX=IU=r.$ Furthermore $AXDI$ is also a parallelogram $\Longrightarrow$ $DX\parallel IA\perp EF\Longrightarrow\angle XDE=90^\circ-\angle DEF=\frac{1}{2}\angle ABC.$ Similarly $\angle XDF=\frac{1}{2}\angle ACB.$

Let DX cut (I) again at $R.$ From $\triangle XDQ\sim\triangle XFR$ and $\triangle XDP\sim\triangle XER,$ we get

$$\frac{DP}{RE}=\frac{DX}{EX}\ ,\ \frac{DQ}{RF}=\frac{DX}{FX}\Longrightarrow$$

$$\frac{DP^2}{DQ^2}=\frac{RE^2}{RF^2}\cdot\frac{FX^2}{EX^2}=\frac{\sin^2\widehat{XDE}}{\sin^2\widehat{XDF}}\cdot\frac{FX^2}{EX^2}=\frac{\sin^2\frac{\widehat{B}}{2}}{\sin^2\frac{\widehat{C}}{2}}\cdot\frac{FX^2}{EX^2}$$

But $DP^2=2r\cdot\delta_{BC}(P)$ and $DQ^2=2r\cdot\delta_{BC}(P)\Longrightarrow$

$$\frac{\delta_{BC}(P)}{\delta_{BC}(Q)}=\frac{\sin^2\frac{\widehat{B}}{2}}{\sin^2\frac{\widehat{C}}{2}}\cdot\frac{FX^2}{EX^2}=\frac{b(s-c)}{c(s-c)}\cdot\frac{FX^2}{EX^2}\quad(\star).$$

By cosine law in $\triangle AXF,$ we get $FX^2=r^2+(s-a)^2-2r\cdot(s-a)\cdot\sin\widehat{B}\Longrightarrow$

$$FX^2=r(s-a)\left(\frac{s-a}{r}+\frac{r}{(s-a)}-2\sin\widehat{B}\right)=$$
$$=r(s-a)\left(\cot\frac{\widehat{A}}{2}+\tan\frac{\widehat{A}}{2}-2\sin\widehat{B}\right)=2r(s-a)(\csc\widehat{A}-\sin\widehat{B})=$$
$$=2r(s-a)\left(\frac{2R}{a}-\frac{h_c}{a}\right)=\frac{2r(s-a)}{a}\cdot(2R-h_c).$$

Similarly, we have $EX^2=\frac{2r(s-a)}{a}\cdot(2R-h_b).$ Now, substituting these 2 latter expressions into $(\star),$ we get

$$\frac{\delta_{BC}(P)}{\delta_{BC}(Q)}=\frac{b(s-c)}{c(s-c)}\cdot\frac{2R-h_c}{2R-h_b}.$$

Virgil Nicula

#3 Aug 16, 2013, 4:41 am • 1

Very nice ! **Thank you**. See **PP10** from **here**.

buratinogigle

#4 Aug 19, 2013, 1:40 pm • 1

An other extension

Let ABC be a triangle and incircle (I) touches CA, AB at E, F, respectively. M is the midpoint of BC. N is a point on MI such that N is inside triangle ABC and $AN = AE = AF$. EN, FN cut (I) again at P, Q, respectively. Prove that $AN \perp PQ$.

Luis González

#5 Aug 19, 2013, 10:43 pm

> *buratinogigle wrote:*
> An other extension
>
> Let ABC be a triangle and incircle (I) touches CA, AB at E, F, respectively. M is the midpoint of BC. N is a point on MI such that N is inside triangle ABC and $AN = AE = AF$. EN, FN cut (I) again at P, Q, respectively. Prove that $AN \perp PQ$.

But any point N on the arc EF of the circle (A, AE) verifies $AN \perp PQ$. Using that NA is the N-circumdiameter of $\triangle NEF$ and the cyclic quadrilateral $EFPQ$, we get $\angle PQN = \angle FEN = 90^\circ - \angle FNA \Longrightarrow AN \perp PQ$. In addition $I \in PQ$, which follows from the fact that (I) and (A, AE) are orthogonal.

Quick Reply

High School Olympiads

tangents and angle bisector

Source: 2013 cwmi q7

61plus

#1 Aug 19, 2013, 8:12 am

Let PA, PB be tangents to a circle centered at O, and C a point on the minor arc AB. The perpendicular from C to PC intersects internal angle bisectors of AOC, BOC at D, E. Show that $CD = CE$

Arab

#2 Aug 19, 2013, 9:17 am

Without loss of generality,we may assume that D is inside $\triangle PAB$ and E outside $\triangle PAB$.
Since $\angle AOD = \angle COD, \angle BOE = \angle COE$ and $OA = OB = OC$,we obtain $\triangle AOD \cong \triangle COD$ and $\triangle BOE \cong \triangle COE$.
Hence $CD = AD, CE = BE, \angle OAD = \angle OCD, \angle OBE = \angle OCE$.
Moreover,$\angle PAD = 90^\circ - \angle OAD = 90^\circ - \angle OCD = 90^\circ - (180^\circ - \angle OCE) = 90^\circ - (180^\circ - \angle OBE) = \angle OBE - 90^\circ = \angle PBE$.
Note that $PA^2 + AD^2 - PD^2 = PA^2 + CD^2 - PD^2 = PA^2 - PC^2$.
Analogously,$PB^2 + BE^2 - PE^2 = PB^2 + CE^2 - PE^2 = PB^2 - PC^2$.
Combining $\cos \angle PAD = \cos \angle PBE$,we get that $2PA \cdot AD = 2PB \cdot BE$,which follows that $AD = BE$.Consequently,$CD = CE$,as desired.

$Q.E.D.$

Luis González

#3 Aug 19, 2013, 10:32 am • 3

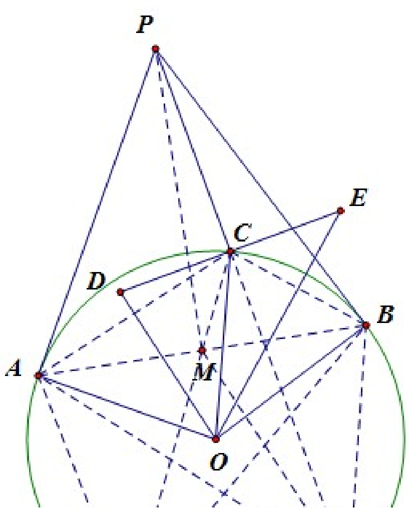
Let γ denote the tangent of (O) at C. Since CP is C-symmedian of $\triangle ABC$, the pencil CA, CB, CP, γ is harmonic. OD, OE are perpendicular bisectors of $\overline{CA}, \overline{CB} \Longrightarrow$ pencil OD, OE, τ, OC formed by perpendiculars from O to CA, CB, CP, γ is also harmonic $\Longrightarrow D, E, C$ and $\tau \cap DE$ (at infinity) are harmonically separeted $\Longrightarrow \overline{CD} = -\overline{CE}$.

XmL

#4 Aug 20, 2013, 1:52 am • 1

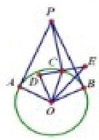
See attachment for my solution

Attachments:

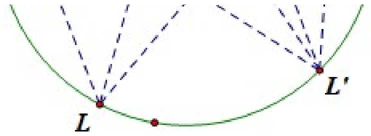


By: XmL
8/18/2013

如图，PA, PB 为圆 O 的切线，点 C 在劣弧 $\overset{\frown}{AB}$ 上（不含点 A, B），过点 C 作 PC 的垂线 l，与 $\angle AOC$ 的平分线交于点 D，与 $\angle BOC$ 的平分线交于点 E，求证：$CD = CE$。



First of all, let M be the midpoint of AB, hence it's well known that CP,CM are isogonals with respect to $\angle$ACB(here we used the fact that C is on the circle). Since CB$\perp$OE,AC$\perp$DO(here we used the angle bisector condition), and PC$\perp$DE. Thus $\angle$CEO=180-$\angle$PCB =$\angle$ACM,$\angle$CDO=$\angle$MCB. Now let CM meet $\odot$O at another point L. Hence $\angle$ABL=$\angle$DEO,$\angle$BAL=$\angle$CDO$\Rightarrow\triangle$ALB~$\triangle$DOE. Since $\angle$COE=$\angle$CLB, hence M,C are two corresponding points in



$\triangle ALB, \triangle DOE \Rightarrow C$ is the midpoint of DE. Q.E.D

The key to this problem is keeping the condition-C is on ⊙O) in mind and noticing that $\angle CEO, \angle CDO$ are angle chase-able, which eventually leads to constructing a similar triangle.

Arab

#5 Aug 20, 2013, 12:50 pm • 1

@XmL
Here is a similar way to deal with the segments based on your solution.

$\angle CAM = \frac{1}{2}\angle COB = \angle COE$,which follows that $\triangle AMC \sim \triangle OCE$.Hence $\frac{CM}{CE} = \frac{AM}{OC}$.
Analogously,$\triangle BMC \sim \triangle OCD$ and $\frac{CM}{CD} = \frac{BM}{OC}$.Consequently,since $AM = BM$,we obtain $CD = CE$.

monsterrr

#6 Aug 21, 2013, 4:12 pm

> *liuyj8526 wrote:*
> Let PA, PB be tangents to a circle centered at O, and C a point on the minor arc AB. The perpendicular from C to PC intersects internal angle bisectors of AOC, BOC at D, E. Show that $CD = CE$

Actually this question is from China Western Mathematical Olympiad, which was renamed as China Western Mathematical Invitational Cmpetition (CWMI) since 2012...

Quick Reply

High School Olympiads

pure geometry

Reply

Pirkuliyev Rovsen

#1 Aug 18, 2013, 6:30 pm

Inside the isosceles right triangle ABC ($\angle C = 90^\circ$) point M is taken such that $AM = 2$, $\angle AMB = 120^\circ$, $\angle AMC = 105^\circ$. Find BM, CM.

Answer: $BM = \sqrt{3}$, $CM = \sqrt{\frac{1}{2}}$.

Luis González

#2 Aug 19, 2013, 3:25 am • 1

Let BM cut $\odot(ABC)$ again at P. Hence $\angle CPA = 135^\circ = \angle CMB, \angle PAC = \angle MBC$ and $CA = CB$ give $\triangle PAC \cong \triangle MBC \Longrightarrow PC = MC \Longrightarrow \angle CPB = \angle CMP = 45^\circ \Longrightarrow \triangle CMP$ is isosceles right at C. In addition, $\triangle PAM$ is obviously right at P with $\angle PMA = 180^\circ - \angle AMB = 60^\circ$. Hence

$$BM = AP = \tfrac{\sqrt{3}}{2} \cdot AM = \sqrt{3}\ , \ MP = \frac{AM}{2} = 1 \Longrightarrow CM = \tfrac{\sqrt{2}}{2} \cdot MP = \sqrt{\tfrac{1}{2}}.$$

Quick Reply

High School Olympiads

Excenters and Extouch

Source: 2013 CWMI

61plus

#1 Aug 18, 2013, 10:08 am

Let ABC be a triangle, and B_1, C_1 be its excenters opposite B, C. B_2, C_2 are reflections of B_1, C_1 across midpoints of AC, AB. Let D be the extouch at BC. Show that AD is perpendicular to B_2C_2

Luis González

#2 Aug 18, 2013, 10:25 am

Discussed before. AD is radical axis of the reflections of the B-excircle and C-excircle on the midpoints of AC and AB.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=35317
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=142767 (b)

leeky

#3 Mar 26, 2016, 9:35 pm

Doable without the excircles. Let A_1 be the excenter opposite A, $B_2C_2 \cap BC \equiv X$, $B_1C_1 \cap BC \equiv E$. Note ABC is the orthic triangle of $A_1B_1C_1$, and AC_1BC_2, AB_1CB_2 are both parallelograms. Thus $\triangle BXC_2 \sim \triangle CXB_2 \implies \frac{CX}{CB_2} = \frac{CX+BX}{CB_2+BC_2} = \frac{BC}{B_1C_1} = \frac{A_1D}{A_1A}$(the last equality due to the two lengths being the two altitudes of $\triangle A_1BC \sim \triangle A_1B_1C_1$). Note the cyclic quadrilateral $EADA_1$(with diameter EA_1) $\implies \angle AA_1D = \angle AEB = \angle BCB_2 \implies \triangle CXB_2 \sim \triangle A_1DA$. Since $CX \perp A_1D, AA_1 \perp B_2C$, the two triangles are a 90° rotation of each other $\implies AD \perp B_2X \equiv B_2C_2$.

High School Olympiads

the value of $\widehat{MSN}$ is constant.

Reply

ams1215

#1 Aug 17, 2013, 7:36 pm

Consider the circle (O) and a point S outside (O). SA, SB is tangent to (O). SO, AB intersects at H. δ is the perpendicular bisector of SH. Consider (I) such that $S \in (I)$ and $(O), (I)$ are outside tangent. (I) intersects δ at M, N. Prove that the value of $\widehat{MSN}$ is constant.

Luis González

#2 Aug 17, 2013, 9:26 pm

Note that δ passes through the midpoints of $SA, SB \Longrightarrow \delta$ is the radical axis of (O) and (S) with zero radius. Hence, inversion with center M and power MS^2 takes (O) into itself and carries (I) externally tangent to (O) into the tangent SB of $(O) \Longrightarrow$ MB passes through the tangency point K of $(O), (I)$. Similarly, NA passes through K.
$\angle MKN = \angle BKA = \pi - \angle SBA \Longrightarrow \angle MSN = \pi - \angle MKN = \angle SBA.$

P.S. See also Prove that A_2B_2=2AB for an equivalent problem.

Burii

#3 Aug 17, 2013, 9:38 pm

Click to reveal hidden text

Quick Reply

High School Olympiads

Locus of midpoints of isogonal conjugates

Reply

Goutham

#1 Sep 21, 2010, 11:53 pm

In triangle ABC, we take cevians AM and AN such that AM passes through the circumcentre of the triangle and AN passes through the orthocentre of the triangle. We take points X, Y on the cevians AM, AN such that $\angle ABX = \angle YBN$. Find the locus of the midpoint of XY.

Luis González

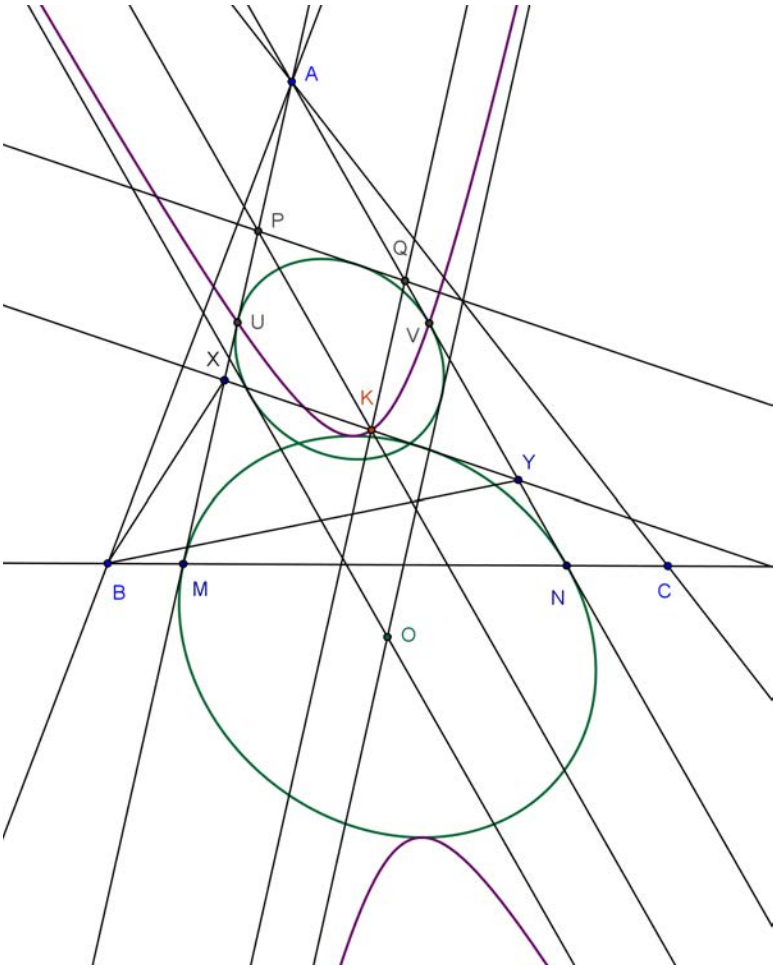
#2 Aug 17, 2013, 1:40 am • 1

Let AM,AN be two arbitrary isogonals, not necessarily the altitude and circumdiameter issuing from A. We prove that the locus of the midpoint K of XY is a hyperbola with asymptotes parallel to AM,AN.

Since $\angle(BX, BA) = \angle(BC, BY)$, then clearly $BX \mapsto BY$ is a proyectivity $\Longrightarrow$ the series X and Y with base lines AM and AN are proyective and not perspective $\Longrightarrow XY$ envelopes a conic $\mathcal{K}$ tangent to AM, AN at M, N, the images of A in the respective series. Thus, if P, Q are the midpoints of AX, AY, then PQ envelopes the conic $\mathcal{K}^*$ homothetic to $\mathcal{K}$ under the homothety with center A and factor $\frac{1}{2}$ (this is obviously tangent to AM,AN through their midpoints U,V) $\Longrightarrow$ series P, Q with base lines AM, AN are proyective. Hence if M_∞ and N_∞ are the infinite points of $AM \parallel KQ$ and $AN \parallel KP$, the pencils $M_\infty Q$ and $N_\infty P$ are proyective $\Longrightarrow K \equiv M_\infty Q \cap N_\infty P$ moves on a conic through M_∞ and N_∞, i.e. a hyperbola $\mathcal{H}$ with asymptotes parallel to AM and AN.

This hyperbola $\mathcal{H}$ goes through U, V, because when $X \equiv M$ or $Y \equiv N$, then K coincides with U or V and it is concentric and bitangent to $\mathcal{K}$, their common tangents XY being parallel to BC.

Attachments:



Quick Reply

High School Olympiads

convex pentagon ABCDE

Reply

Pirkuliyev Rovsen
#1 Aug 16, 2013, 11:30 pm

In the convex pentagon $ABCDE$, $\angle ABC=\angle AED=90^{\circ}$ and $AB\cdot ED=BC\cdot AE$. Let F be the intersection point of CE and BD. Prove that $AF\perp BE$.

Luis González
#2 Aug 16, 2013, 11:52 pm • 2

$\angle ABC=\angle AED=90^{\circ}$ and $\frac{AB}{BC}=\frac{AE}{ED} \Longrightarrow \triangle ABC \sim \triangle AED \Longrightarrow \angle BAC=\angle EAD.$ Now, see the problems

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=151&t=496141
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=79051

Andrew64
#3 Aug 17, 2013, 8:55 am • 1

> *Pirkuliyev Rovsen wrote:*
> In the convex pentagon $ABCDE$, $\angle ABC=\angle AED=90^{\circ}$ and $AB\cdot ED=BC\cdot AE$. Let F be the intersection point of CE and BD. Prove that $AF\perp BE$.

As shown in the figure below.

Let
$BH\perp EC$, $EG\perp BD$, $AK\perp BH$, $AL\perp EG$.

BH intersects AF at P, EG intersects AF at P'

Then

$$\frac{FH}{FG}=\frac{BH}{EG}$$
$$=\frac{BH}{AK}\times\frac{AL}{EG}\times\frac{AK}{AL}$$
$$=\frac{BC}{AB}\times\frac{AE}{ED}\times\frac{AK}{AL}$$
$$=\frac{AK}{AL}$$

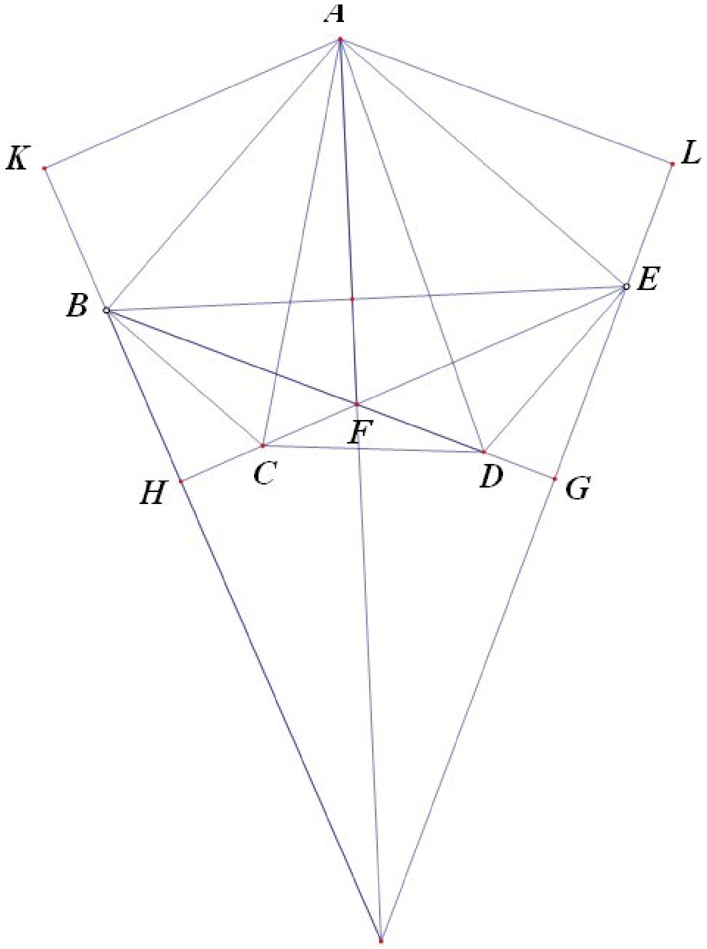
Therefore
$$\frac{AP}{FP}=\frac{AK}{FH}=\frac{AL}{FG}=\frac{AP'}{FP'}$$
P, P' are identical.

So F is the orthocenter of $\triangle BPE$.

Consequently

$AF\perp BE$.

Attachments:



Quick Reply

High School Olympiads

P,Q are on the circle with diameter AB

Source: unknown

bcp123

#1 Aug 16, 2013, 2:30 am

Let ABC be an acute-angled triangle with altitudes AA_1,BB_1,CC_1 and orthocenter H. Take a point K on minor arc AB_1 of the circle with diameter AB, such that $\angle HKB = \angle C_1KB$. Let $BK \cap CC_1 = L$. Let the circle passing through L with center C cuts AA_1 at M and let the circle passing through M with center B cuts CC_1 at P and Q. Prove that P and Q are on the circle with diameter AB.

Luis González

#2 Aug 16, 2013, 8:31 am • 1

Let $S \equiv KA \cap CC_1$ and $\odot(LKS)$ cuts the circle (F) with diameter $\overline{AB}$ again at N. Then S is orthocenter of $\triangle ABL \Longrightarrow S \in BN$. $BSKC_1$ is cyclic due to its right angle at $K, C_1 \Longrightarrow \angle HKL = \angle BKC_1 = \angle NSL = \angle NKL \Longrightarrow H \in KN$. By Pascal theorem for $KKNAB_1B$, the intersections $KK \cap AB_1, H \equiv KN \cap BB_1$ and $L \equiv NA \cap BK$ are collinear $\Longrightarrow CK$ is tangent to $(F) \Longrightarrow C$ is the center of $\odot(KLN)$ orthogonal to (F).

From the complete cyclic $KSNL$, we deduce that AH is the polar of B WRT $\odot(KSL) \Longrightarrow BM$ is tangent to $\odot(KLN)$ $\Longrightarrow \odot(KLN)$ is orthogonal to both (F) and $(B, BM) \Longrightarrow$ perpendicular CC_1 from its center C to $FB \equiv AB$ is the radical axis of (F) and $(B, BM) \Longrightarrow \{P, Q\} \equiv CC_1 \cap (B, BM)$ lie on (F).

High School Olympiads

Cyclic Hexagon

Reply

NewAlbionAcademy

#1 Aug 15, 2013, 8:40 am

Given cyclic hexagon $ABCDEF$, we have that quadrilaterals $ABDF$, $ABCD$, $ADEF$ are harmonic. Prove that quadrilateral $ACDE$ is also harmonic.

Luis González

#2 Aug 15, 2013, 9:13 am • 1

Denote ω the hexagon circumcircle. b,d,f are tangents of ω at $B,D,F.X\equiv f\cap b,Y\equiv b\cap d$ and $Z\equiv d\cap f.$ If $ABDF,ABCD$ and $ADEF$ are harmonic, then $X\in AD,Y\in AC$ and $Z\in AE.$ Now, project any line through $BF\cap YZ,$ not cutting $\omega,$ to infinity and ω into another circle ω'. The projected $\triangle X'Y'Z'$ becomes X'-isosceles and certainly the cyclic $A'C'D'E'$ is a kite $\Longrightarrow$ tangents of ω' at C',E' meet on $X'A'D'$ $\Longrightarrow$ $A'C'D'E'$ is harmonic $\Longrightarrow$ $ACDE$ is harmonic.

Quick Reply

High School Olympiads

About the conclusion of Feuerbach point

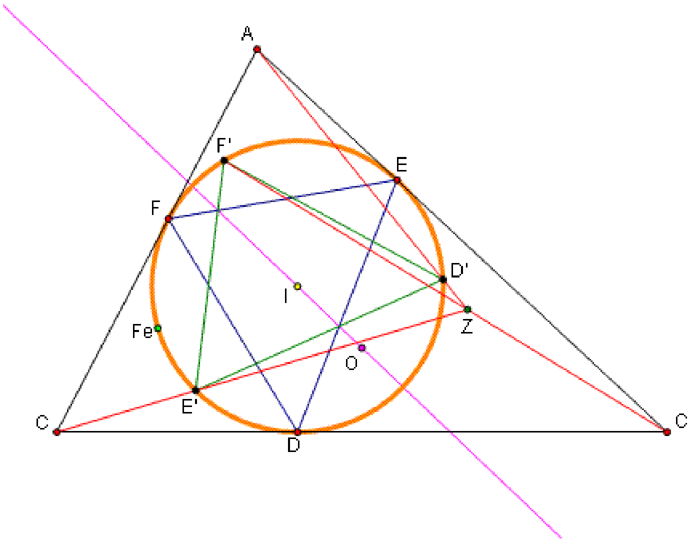
Source: 0

lym

#1 Apr 5, 2009, 2:22 pm

Let O, I, Fe be circumcenter, incenter and Feuerbach point of $\triangle ABC$. The incircle (I) touches BC, CA, AB at D, E, F. D', E', F' are the reflections of D, E, F on OI. Prove: AD', BE', CF' concur at the isogonal conjugate Z of Fe.

Attachments:



Luis González

#2 Aug 15, 2013, 8:08 am

It's well-known than F_e is the anti-Steiner point of OI WRT $\triangle DEF$, i.e. OI is the Steiner line of F_e WRT $\triangle DEF$, hence if Q, R are the projections of F_e on DF, DE, then $QR \parallel OI$. Thus, perpendicular DD' from D to QR is the isogonal of DF_e WRT $\angle EDF$, or $\angle EDD' = \angle FDF_e \Longrightarrow EF \parallel F_eD' \Longrightarrow EFF_eD'$ is an isosceles trapezoid $\Longrightarrow D'$ is the reflection of F_e across the perpendicular bisector AI of $\overline{EF} \Longrightarrow AF_e, AD'$ are isogonals WRT $\angle BAC \Longrightarrow Z \in AD'$. Analogously, BE' and CF' pass through Z.

jayme

#3 Aug 15, 2013, 3:20 pm

Dear Mathlinkers,
for the concurrence, I made a link with a particular case of the Steinbart's theorem.
You can see
http://perso.orange.fr/jl.ayme vol. 3 Les points de Steinbart et de Rabinowitz
Sincerely
Jean-Louis

High School Olympiads

Maybe easy

Reply

bcp123

#1 Aug 14, 2013, 10:04 pm

Let ABC be a triangle with circumcircle $\odot O$. The line ℓ is tangent to $\odot O$ at A. Let $BO\cap \ell=E,CO\cap \ell=F$ and $EC\cap FB=O_A$. Define O_B and O_C similarly. Prove that AO_A,BO_B and CO_C are concurrent.

Luis González

#2 Aug 14, 2013, 11:11 pm • 2

Let H be the orthocenter of $\triangle ABC.$ Parallel τ from A to BC cuts BH,CH at $Y,Z.$ Since ℓ,τ are isogonals WRT $\angle BAC$ and BO,BH are isogonals WRT $\angle ABC,$ it follows that E and Y are isogonal conjugates WRT $\triangle ABC \Longrightarrow CE,CY$ are isogonals WRT $\angle ACB.$ Likewise, BF,BZ are isogonals WRT $\angle ABC \Longrightarrow O_A \equiv BF \cap CE$ is the isogonal conjugate of $X \equiv BZ \cap CY \Longrightarrow AO_A,AX$ are isogonals WRT $\angle BAC.$

Let $D \equiv AH \cap BC$ be the foot of the A-altitude and $U \equiv AX \cap BC.$ From $YZ \parallel BC,$ we get $\frac{UB}{UC}=\frac{AZ}{AY}=\frac{DC}{DB} \Longrightarrow$ U,D are symmetric about the midpoint of $\overline{BC} \Longrightarrow AU \equiv AX$ is the A-cevian of the isotomic conjugate of $H,$ i.e. the retrocenter R of $\triangle ABC \Longrightarrow AO_A$ is the A-cevian of the isogonal conjugate of $R,$ the Gob's point X_{25} of $\triangle ABC.$ Analogously, BO_B and CO_C goes through $X_{25}.$

Quick Reply

High School Olympiads

Collinear with Fe

Reply

Source: a composition

jayme
#1 Aug 14, 2013, 4:44 pm

Dear Mathlinkers,

1. ABC a triangle,
2. DEF the contact triangle of ABC
3. A' the midpoint of BC
4. Q, R the feet of the perpendicular to the A-in bissector of ABC through B, C
5. X the center of the circle going through A', Q, R
6. Fe the Feuerbach's point of ABC.

Prove : D, X and Fe are collinear.

Sincerely
Jean-Louis

Luis González
#2 Aug 14, 2013, 9:14 pm

Let L denote the foot of the A-altitude of $\triangle ABC.$ According to the problem Four points lie on a circle, the circle $\odot(A'QR)$ goes through L and its center X is the midpoint of the arc LA' of 9-point circle $(N).$ Let F_eL, F_eA' cut the incircle (I) again at $Y, Z.$ Since F_e is the exsimilicenter of $(I) \sim (N),$ then $YZ \parallel LA' \Longrightarrow D$ is midpoint of the arc YZ of $(I) \Longrightarrow F_eD$ is internal bisector of $\angle LF_eA'$, passing through the midpoint X of the arc LA', i.e. D, X, F_e are collinear.

jayme
#3 Aug 14, 2013, 9:30 pm

Dear Luis and Mathlinkers,
a proof of the circle and his center can be seen on
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=52070
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

2013 China girls' Mathematical Olympiad problem 7

Reply

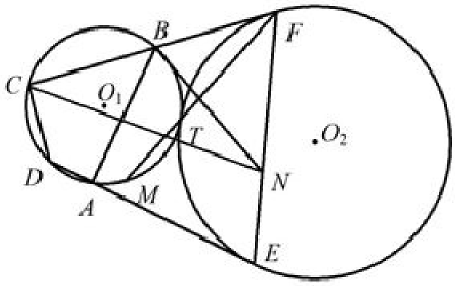
s372102

#1 Aug 13, 2013, 9:18 pm

As shown in the figure, $\odot O_1$ and $\odot O_2$ touches each other externally at a point T, quadrilateral $ABCD$ is inscribed in $\odot O_1$, and the lines DA, CB are tangent to $\odot O_2$ at points E and F respectively. Line BN bisects $\angle ABF$ and meets segment EF at N. Line FT meets the arc $\widehat{AT}$ (not passing through the point B) at another point M different from A. Prove that M is the circumcenter of $\triangle BCN$.

Attachments:

7. As shown in the figure, $\odot O_1$ and $\odot O_2$ touches each other externally at a point T, quadrilateral $ABCD$ is inscribed in $\odot O_1$, and the lines DA, CB are tangent to $\odot O_2$ at points E and F respectively. Line BN bisects $\angle ABF$ and meets segment EF at N. Line FT meets the arc $\widehat{AT}$ (not passing through the point B) at another point M different from A. Prove that M is the circumcenter of $\triangle BCN$.



Luis González

#2 Aug 14, 2013, 12:19 am • 2

Let TB, TC cut $\odot O_2$ again at B', C'. Since T is the exsimilicenter of $\odot O_1 \sim \odot O_2$, then $B'C' \parallel BCF \Longrightarrow F$ is midpoint of the arc $B'TC'$ of $\odot O_2 \Longrightarrow TF \equiv TM$ bisects $\angle B'TC' \equiv \angle BTC$ externally $\Longrightarrow M$ is midpoint of the arc BTC of $\odot O_1 \Longrightarrow AM$ is external bisector of $\angle BAC$ and $MB = MC$.

Note that $\odot O_2$ is a Thebault circle of the cevian AD of $\triangle ABC$ externally tangent to its circumcircle $\odot O_1$. By Sawayama's lemma EF passes through its C-excenter $\Longrightarrow N$ is C-excenter of $\triangle ABC \Longrightarrow N \in AM$.
$\angle BNA = 90^\circ - \frac{1}{2}\angle ACB = 90^\circ - \frac{1}{2}\angle BMN \Longrightarrow \triangle BMN$ is M-isosceles, i.e. $MB = MN$. Hence $MB = MC = MN \Longrightarrow M$ is circumcenter of $\triangle BCN$.

Andrew64

#3 Aug 17, 2013, 2:24 pm

> s372102 wrote:
> As shown in the figure, $\odot O_1$ and $\odot O_2$ touches each other externally at a point T, quadrilateral $ABCD$ is inscribed in $\odot O_1$, and the lines DA, CB are tangent to $\odot O_2$ at points E and F respectively. Line BN bisects $\angle ABF$ and meets segment EF at N. Line FT meets the arc $\widehat{AT}$ (not passing through the point B) at another point M different from A. Prove that M is the circumcenter of $\triangle BCN$.

As shown in the figure.

Let N be the intersection of AM and EF.

It's fairly obvious

$\angle FET = \angle TFK = \angle TBM = \angle TCM = \angle TAM$

So we have

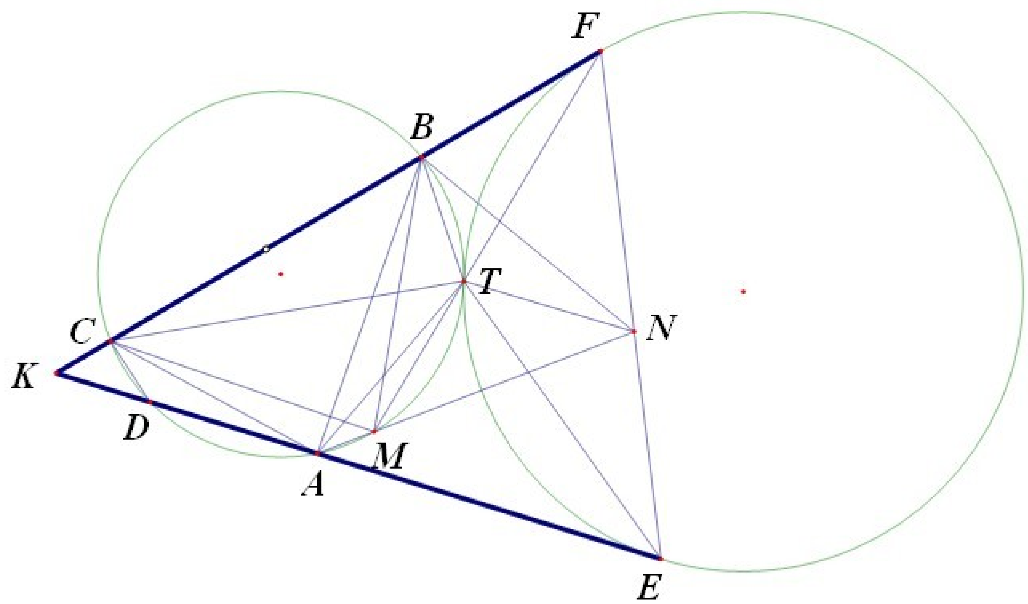
$\angle CBM = \angle CTM = \angle BTF = \angle BCM$
$MB^2 = MB \times MF$, and A, E, N, T are concyclic.

So
$MC = MB$, and
$\angle MNT = \angle TEA = \angle MFN$

Thus
$MN^2 = MT \times MF$,
Consequently
$MB = MC = MN$.

$\angle ABN = \angle ABM + \angle MBN$
$= \angle ABM + \frac{180 - \angle BMN}{2}$
$= \angle ABM + 90 - \frac{\angle BCA}{2}$
$= \angle ABM + 90 - \frac{\angle BCM}{2} - \frac{\angle ACM}{2}$
$= \frac{\angle CMB}{2} + \frac{\angle BCM}{2} + \frac{\angle ACM}{2}$
$= \frac{\angle CAB}{2} + \frac{\angle BCA}{2}$
$= \frac{\angle ABF}{2}$

Namely AN is the bisector of $\angle ABF$.
Attachments:



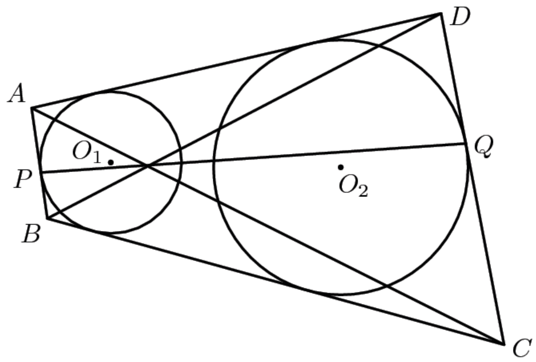
High School Olympiads

2013 China girls' Mathematical Olympiad problem 2

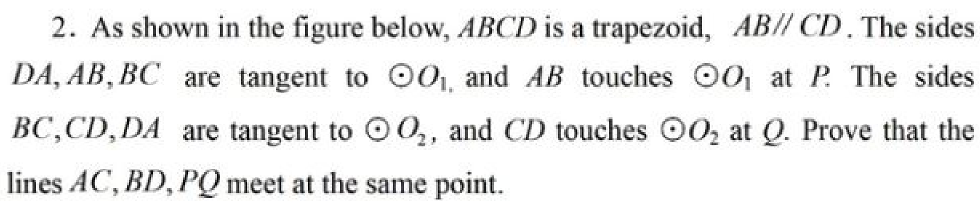
s372102

#1 Aug 13, 2013, 9:11 pm

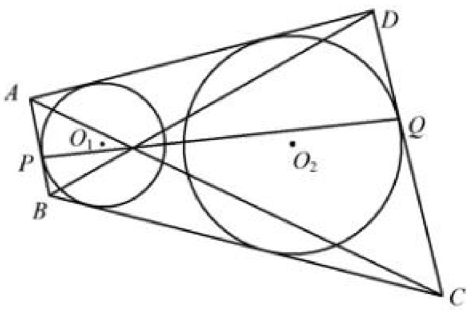
As shown in the figure below, $ABCD$ is a trapezoid, $AB \parallel CD$. The sides DA, AB, BC are tangent to $\odot O_1$ and AB touches $\odot O_1$ at P. The sides BC, CD, DA are tangent to $\odot O_2$, and CD touches $\odot O_2$ at Q. Prove that the lines AC, BD, PQ meet at the same point.



Attachments:



2. As shown in the figure below, $ABCD$ is a trapezoid, $AB // CD$. The sides DA, AB, BC are tangent to $\odot O_1$, and AB touches $\odot O_1$ at P. The sides BC, CD, DA are tangent to $\odot O_2$, and CD touches $\odot O_2$ at Q. Prove that the lines AC, BD, PQ meet at the same point.



Luis González

#2 Aug 13, 2013, 11:15 pm • 2

Let $M \equiv BC \cap AD$ and $N \equiv AC \cap BD$. Let $\odot O_3$ be the incircle of $\triangle MAB$ touching AB at R. Clearly $\triangle MBA$ and $\triangle MCD$ are homothethic with incircles $\odot O_3, \odot O_2 \Longrightarrow \frac{QC}{QD} = \frac{RB}{RA}$, but since P, R are symmetric about the midpoint of $\overline{AB}$ ($\odot O_1$ is the M-excircle of MAB), it follows that $\frac{QC}{QD} = \frac{PA}{PB} \Longrightarrow \triangle NAB$ and $\triangle NCD$ are homothetic with corresponding cevians $NP, NQ \Longrightarrow N \in PQ$.

Andrew64

#3 Aug 17, 2013, 11:19 am

> s372102 wrote:
>
> As shown in the figure below, $ABCD$ is a trapezoid, $AB \parallel CD$. The sides DA, AB, BC are tangent to $\odot O_1$ and AB touches $\odot O_1$ at P. The sides BC, CD, DA are tangent to $\odot O_2$, and CD touches $\odot O_2$ at Q. Prove that the lines AC, BD, PQ meet at the same point.

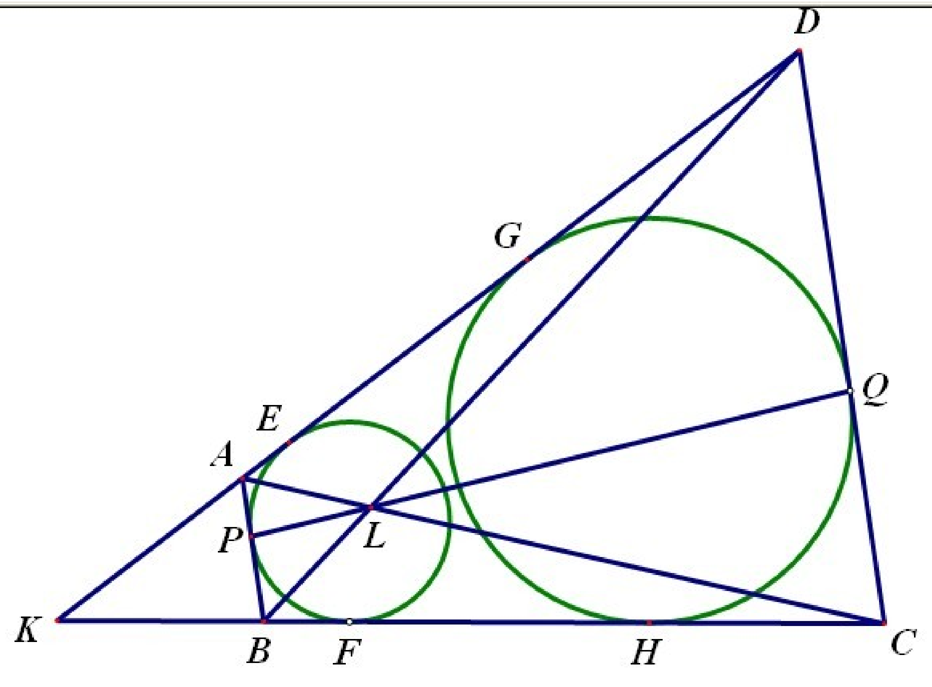
As shown in the figure below.

L is the intersection of AC and BD.

$$\frac{AL}{LC}=\frac{AB}{CD}$$
$$=\frac{KA}{KD}$$
$$=\frac{KB}{KC}$$
$$=\frac{KE-AE}{KE+EG+GD}$$
$$=\frac{KF-BF}{KF+FH+HC}$$
$$=\frac{(KE-AE)-(KF-BF)}{(KE+EG+GD)-(KF+FH+HC)}$$
$$=\frac{BF-AE}{GD-HC}$$
$$=\frac{AB+(BF-AE)}{AC+(GD-HC)}$$
$$=\frac{BP}{DQ}$$

Therefore AC, BD, PQ meet at the same point L.

Attachments:



mojyla222

#4 Aug 23, 2013, 1:15 am

another problem:
lines l and h are tangent from B , D to circlesO_2 , O_1.
prove that l and h are paralel.

Quick Reply

High School Olympiads

Collinear and equal segment

Reply

Source: Own

buratinogigle

#1 Jul 25, 2013, 3:35 pm

Let ABC be a triangle and P is a point such that $PA\perp BC$. PA,PB,PC cut BC,CA,AB at D,E,F, respectively. Q is isogonal conjugate of P with respect to triangle ABC. X,Y,Z are projetions of Q on lines EF,FD,DE, respectively. K is circumcenter of triangle DEF. L,M,N are projection of Q on lines BC,CA,AB. R is on circumcircle of triangle KMN such that $KR\parallel BC$.

a) Prove that D,R,Q are collinear.

b) Prove that $KL=DR$.

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=545403

Luis González

#2 Aug 12, 2013, 1:51 am • 1

Clearly, Y,Z,L lie on the circle ω with diameter $\overline{QD}$. This cuts AD again at the projection U of Q on AD. By well-known Blanchet theorem, DP,BC bisect $\angle EDF\equiv\angle ZDY\Longrightarrow U$ and L are the midpoints of the arc YZ and YDZ of $\omega\Longrightarrow$ UL is perpendicular bisector of $\overline{YZ}$, which passes through K and meets the diameter $\overline{QD}$ of ω at is center $\Longrightarrow QUDL$ is a rectangle $\Longrightarrow DQ$ is reflection of LK across the perpendicular bisector of $\overline{DL}$.

Since Q is on the A-circumdiameter of $\triangle ABC$, then obviously the line connecting the projections M,N of Q on CA,AB is parallel to BC. Thus, the cyclic $DLMN$ is an isosceles trapezoid with bases $\overline{MN}\parallel\overline{DL}$. By symmetry, R is the reflection of K across the perpendicular bisector of $\overline{DL}\Longrightarrow R\in DQ$. Again, by symmetry $KRLD$ is an isosceles trapezoid with congruent diagonals (or legs) $\overline{KL}\cong\overline{DR}$.

Quick Reply

High School Olympiads

Collinear points

Source: Own

buratinogigle

#1 Jul 25, 2013, 2:01 pm

Let ABC be a triangle and (K) is a circle passing through B, C. (K) cuts CA, AB again at E, F, respectively. BE cuts CF at H. d is the line passing though K and perpendicular to AH.

a) P is a point on d. PM, PL are diameters of circumcircle of triangles PBF, PCE. Prove that M, L, H are collinear.

b) Q is a point circumcircle ω of triangle KBC. KN is diameter of ω. NB, NC cut circumcircle of triangle QBF, QCE again at S, T, respectively. Prove that S, T, H are collinear.

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=951&p=3155141#p3155141

Luis González

#2 Aug 11, 2013, 12:25 pm • 1

a) Let D be the projection of K on AH and X the 2nd intersection of $\odot(PBF)$ and $\odot(PCE)$. BF, CE, PX are pairwise radical axes of $(K), \odot(PBF), \odot(PCE)$ concurring at their radical center A. Since $KD \equiv d$ is clearly the polar of A WRT (K), then we deduce that $AH \cdot AD = AE \cdot AC = AX \cdot AP \Longrightarrow P, D, H, X$ are concyclic $\Longrightarrow$ $\angle PXH = \angle PDH = 90^\circ \Longrightarrow HX$ cuts $\odot(PBF)$ and $\odot(PCE)$ again at the antipodes of P, i.e. M, L, H are collinear.

b) Since the pencil $D(B, C, A, K)$ is harmonic and $DA \perp DK$, it follows that DA, DK bisect $\angle BDC$. Hence, K is midpoint of the arc BDC of $\odot(BDC) \Longrightarrow \odot(BDC) \equiv \omega \Longrightarrow N$ is the 2nd intersection of AD with ω.

If R is the 2nd intersection of $\odot(QBF)$ and $\odot(QCE)$, then BF, CE, QR are pairwise radical axes of $(K), \odot(QBF)$, $\odot(QCE)$ concurring at their radical center A. Thus, inversion with center A and power $AE \cdot AC$ takes B, C, Q, D into $F, E, R, H \Longrightarrow EHFR$ is cyclic. Therefore $\angle(RF, RH) = \angle(EF, EH) = \angle(BF, BS) = \angle(RF, RS) \Longrightarrow$ $S \in RH$. By similar reasoning, $T \in RH$, hence S, T, H are collinear.

Quick Reply

High School Olympiads

constant point

lambosama

#1 Aug 9, 2013, 11:14 pm

Give the $\triangle ABC$, M is the point inside the triangle such as $\angle AMB - \angle C = \angle AMC - \angle B$. Prove that the line go through 2 incirle center of 2 triangle AMB, ACM also go through a constant point .
p/s: please show me how to locate M :(

Luis González

#2 Aug 10, 2013, 12:47 am

Let MA, MB, MC cut the circumcircle of $\triangle ABC$ again at P, Q, R.
$\angle PBQ = \angle AMB - \angle APB = \angle AMB - \angle ACB$. Similarly, we have $\angle PCR = \angle AMC - \angle ABC$, hence $\angle PBQ = \angle PCR \Longrightarrow PQ = PR$. Now, from $\triangle MAB \sim \triangle MQP$ and $\triangle MAC \sim \triangle MRP$, we get

$$\frac{AB}{PQ} = \frac{MB}{MP}, \frac{AC}{PR} = \frac{MC}{MP} \Longrightarrow \frac{MB}{MC} = \frac{AB}{AC}.$$

This means that M is a point on the A-Apollonius circle of $\triangle ABC$, i.e. the circle whose diameter is $\overline{DE}$, where D, E are the feet of the internal and external bisector of $\angle BAC$. According to the problem Apollonius circle, the line connecting the incenters of $\triangle AMC$ and $\triangle AMB$ always passes through E.

High School Olympiads

Locus of intersection of polars

Source: own

proglote

#1 Jul 30, 2012, 9:43 pm

Let $(A), (B)$ be two circles in a plane, ℓ an arbitrary line in this plane and $P \in \ell$. P^* is the intersection of the polar lines of P w.r.t. (A) and (B).

Find the locus of P^* as P varies when:

i) ℓ is parallel to the radical axis of (A) and (B);
ii) ℓ is not parallel to the radical axis of (A) and (B).

Luis González

#2 Aug 2, 2012, 10:08 am • 1

i) Polars of P WRT (A) and (B) are perpendicular to PA, PB at D, E. If R_a, R_b denote the radii of $(A), (B)$, we have ${R_a}^2 = AD \cdot AP$ and ${R_b}^2 = BE \cdot BP \Longrightarrow$ circle (K) with diameter $\overline{PP^*}$ is orthogonal to both (A) and $(B) \Longrightarrow$ locus of K is the radical axis τ of $(A), (B)$. Since $\ell \parallel \tau$, then the locus of P^* is the reflection of ℓ about τ.

ii) We assume that (A) and (B) are disjoint with different radii. ℓ is neither parallel nor perpendicular to AB. Using the same arguments as before, the circle (K) with diameter $\overline{PP^*}$ is orthogonal to both (A) and $(B) \Longrightarrow K$ is on radical axis τ of $(A), (B)$ and (K) cuts AB at the limiting points U, V of $(A), (B)$. Let λ be the perpendicular to ℓ through $M \equiv \ell \cap AB$ and λ' the perpendicular to AB through the reflection of M about τ. Let X, Y be the projections of P, P^* on AB. Since midpoint K of $\overline{PP^*}$ is on τ, then X, Y are symmetric about $\tau \Longrightarrow \text{dist}(P^*, \lambda') = MX$. If Q is the projection of P^* on ℓ (2nd intersection of (K) with ℓ), we have

$$MQ \cdot MX = \frac{MU \cdot MV}{MP} \cdot MX = MU \cdot MV \cdot \cos \angle(\ell, AB) \Longrightarrow$$

$$\text{dist}(P^*, \lambda) \cdot \text{dist}(P^*, \lambda') = MU \cdot MV \cdot \cos \angle(\ell, AB) = \text{const}.$$

Thus, locus of P^* is the hyperbola passing through U, V, the poles of ℓ WRT $(A), (B)$ and whose asymptotes are λ, λ'.

proglote

#3 Aug 3, 2012, 11:56 pm

For the case when $\ell \parallel AB$, with analytic methods we find that

$$PX \cdot P^*Y = \text{dist}(P, \tau)^2 + \mathcal{P}(T),$$

where $T \equiv \tau \cap AB$ and $\mathcal{P}(T)$ is the power of T w.r.t. any of the two given circles. Hence, the locus is a parabola symmetric w.r.t. τ. However, I still don't have a synthetic proof of this result.

We also have the following: Suppose (A) and (B) intersect at two points, T_1 and T_2. Then PP^* is tangent to the ellipse with foci T_1, T_2 through X.

Luis González

#4 Aug 4, 2012, 2:17 am • 1

When $\ell \parallel AB$, the conic $\mathcal{H}$ only has one point at infinity, namely, $\perp AB \Longrightarrow \mathcal{H}$ is a parabola with focal axis perpendicular to AB. Let (K) cut ℓ again at R. R^* is the antipode of R WRT (K). R, R^* are then conjugate points WRT (A) and $(B) \Longrightarrow R^*$ is on the polars of R WRT $(A), (B)$, thus $R^* \in \mathcal{H}$. Since PRP^*R^* is a rectangle with $PR \parallel P^*R^* \parallel AB$, then the midparallel of $PR^* \parallel RP^*$ is the focal axis of $\mathcal{H} \Longrightarrow \tau$ is focal axis of $\mathcal{H}$.

As for the second conjecture, PX is obviously tangent to the ellipse $\mathcal{E}$ with foci T_1, T_2 that passes through X. Let (K) cut τ at E, F. Since $(K) \perp (A)$, the cross ratio (E, F, T_1, T_2) is harmonic. Together with $EP \perp FP$, it follows that PE bisects $\angle T_1PT_2$. But $\angle EPX = \angle KEP = \angle KPE \Longrightarrow PE$ bisect $\angle XPP^*$. Hence, PT_1, PT_2 are isogonals WRT $\angle XPP^*$ $\Longrightarrow PP^*$ is the second tangent from P to $\mathcal{E}$.

Luis González

#5 Aug 9, 2013, 10:45 pm • 2

The problem can be generalized as follows:

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two conics in a plane and ℓ an arbitrary line in this plane. P is a variable point on ℓ. Then the locus of the intersection P^* of the polars of P WRT $\mathcal{C}_1$ and $\mathcal{C}_2$ is a conic that passes through the poles of ℓ WRT $\mathcal{C}_1, \mathcal{C}_2$.

Label p_1, p_2 the polars of P WRT $\mathcal{C}_1, \mathcal{C}_2$. They pass through the fixed poles K_1 and K_2 of ℓ WRT $\mathcal{C}_1$ and $\mathcal{C}_2$. p_1, p_2 cut ℓ at P_1, P_2, respectively. Since the polar of P_1 WRT $\mathcal{C}_1$ passes through P, then $P_1 \mapsto P$ is an involution on ℓ. Similarly $P \mapsto P_2$ is an involution on ℓ, thus $P_1 \mapsto P_2$ is a projectivity $\Longrightarrow$ pencils K_1P_1 and K_2P_2 are projective $\Longrightarrow P^* \equiv K_1P_1 \cap K_2P_2$ runs on a projective conic $\mathcal{K}$ that passes through K_1, K_2. Note that when P is at infinity then p_1, p_2 pass through the centers O_1, O_2 of $\mathcal{C}_1, \mathcal{C}_2$, hence $\mathcal{K}$ also goes through the intersection $K_1O_1 \cap K_2O_2$.

Quick Reply

High School Olympiads

Collinear with Feuerbach point

Reply

jayme
#1 Aug 9, 2013, 7:20 pm

Dear Mathlinkers,

1. ABC a triangle
2. H the orthocenter of ABC,
3. A* the midpoint of AH,
4. PQR the orthic triangle of ABC,
5. Fe the Feuerbach's point of ABC,
6. 1a the incircle of the triangle AQR
7. A' the center of 1a.

Prove : A', A* and Fe are collinear.

Sincerely
Jean-Louis

Luis González
#2 Aug 9, 2013, 9:18 pm

Let the incircle (I,r) touch BC, CA, AB at X, Y, Z. From $\triangle ABC \sim \triangle AQR$, it follows that $\frac{AA'}{AI} = \frac{QR}{BC} = \cos A$. Since AI is the A-circumdiameter of $\triangle AYZ$, we deduce that A' is the reflection of I across YZ.

Let (N) be the 9-point circle of $\triangle ABC$ and U the midpoint of AI. According to Intersect on circle, F_e, U and the antipode X' of X WRT (I) are collinear. Since $r^2 = \overline{IU} \cdot \overline{IIA'}$, then the inversion WRT (I) takes the line XUF_e into the circle $\odot(XIF_e) \Longrightarrow A' \in \odot(XIF_e)$. If M denotes the midpoint of BC, we have $\angle A'F_eX' = \angle A'IX' = \frac{1}{2}\angle MF_eP = \angle MF_eX$. But since $\angle XF_eX'$ is right, then $\angle MF_eA'$ is also right. Since M is the midpoint of the arc QPR of (N), then F_eA' and F_eM bisect $\angle QF_eR \Longrightarrow F_eA'$ cuts (N) again at the midpoint A^* of the arc QR of (N), in other words, A', A^*, F_e are collinear.

Quick Reply

High School Olympiads

Geometry Problem

Locked

War-Hammer

#1 Jul 25, 2013, 3:01 am

Let ω_1,ω_2 with center O_1,O_2 be two circles intersect at P,Q. Let AB,CD be common tangent of ω_1,ω_2 with $A,C\in\omega_1, B,D\in\omega_2$. Suppose that M,N are midpoint of segments AC,BD. Prove that $\angle MPO_1=\angle NPO_2$.

Luis González

#2 Jul 25, 2013, 3:46 am

Please, give your posts meaningful subjects and use the search before posting. This is IMO 1983, Day 1, Problem 2.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=60799
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=527344

High School Olympiads

Two problems

Source: Own

buratinogigle

#1 Jul 24, 2013, 9:47 pm

Problem 1. Let ABC be a triangle with orthocenter H and incenter I. M is midpoint of BC. N is symmetric point of I through M. D,E,F are projection of N on lines BC,CH,HB. Prove that circumcenter of triangle DEF lies on circumcircle of triangle HBC.

Problem 2. Let ABC be a triangle with altitudes AD,BE,CF and circumcenter O. OA,OB,OC cut EF,FD,DE at X,Y,Z, respectively. K is circumcenter of triangle XYZ. Let M,N be midpoints of CA,AB, respectively. P is a point on circumcircle of triangle KMN such that $KP\parallel BC$.

a) Prove that D,P,O are collinear.

b) Let L be midpoint of BC. Prove that $KL=DP$.

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=951&p=3153962#p3153962

Luis González

#2 Jul 24, 2013, 11:56 pm • 1

P.1) Let I_a be the A-excenter of $\triangle ABC$ and D the midpoint of the arc BC of its circumcircle. Let $K\in\odot(BHC)$ be the reflection of D on BC. If L denotes the isogonal conjugate of N WRT $\triangle HBC$, then $\angle LCH=\angle NCB=\angle IBC\Longrightarrow$ $\angle LCB=\frac{1}{2}\angle ABC+90^\circ-\angle ABC=90^\circ-\frac{1}{2}\angle ABC=\angle I_aBC\Longrightarrow CL\parallel BI_a$. Similarly, $BL\parallel CI_a\Longrightarrow$ BI_aCL is a parallelogram $\Longrightarrow L$ is reflection of I_a on M. Since D is midpoint of $\overline{II_a}$, then by symmetry K is midpoint of $\overline{NL}$. K is precisely the center of the pedal circle $\odot(DEF)$ of N,L WRT $\triangle HBC$.

Luis González

#3 Jul 25, 2013, 2:15 am • 1

P.2) Clearly Y,Z,L lie on the circle (S) with diameter $\overline{OD}$. Since OL bisects $\angle BOC\equiv\angle YOZ$, then L is the midpoint of the arc $YDZ\Longrightarrow L$ is equidistant from $Y,Z\Longrightarrow KL$ is perpendicular bisector of YZ, meeting the arc YOZ at its midpoint U and the diameter $\overline{OD}$ at $S\Longrightarrow OUDL$ is a rectangle $\Longrightarrow OD$ is reflection of UL across the perpendicular bisector of $\overline{DL}$. Since $MNDL$ is an isosceles trapezoid with bases $DL\parallel MN$, then we deduce that P is just the reflection of K on the perpendicular bisector of $\overline{DL},\overline{MN}\Longrightarrow P\in OD$. By obvious symmetry $KPLD$ is an isosceles trapezoid with congruent diagonals $\overline{KL}\cong\overline{DP}$.

Quick Reply

High School Olympiads

Equal angles in a rhombus

Reply

Source: Tuymaada 2013, Day 1, Problem 2 Seniors

mavropnevma
#1 Jul 24, 2013, 8:58 pm • 4

Points X and Y inside the rhombus $ABCD$ are such that Y is inside the convex quadrilateral $BXDC$ and $2\angle XBY = 2\angle XDY = \angle ABC$. Prove that the lines AX and CY are parallel.

S. Berlov

Luis González
#2 Jul 24, 2013, 10:47 pm

$\angle XBY = \frac{1}{2}\angle ABC = \angle DBC \Longrightarrow \angle ABX = \angle DBY$. Similarly, we have $\angle ADX = \angle BDY$. If Z denotes the isogonal conjugate of X WRT $\triangle ABD$, we have $\angle DBZ = \angle ABX = \angle DBY$ and $\angle BDZ = \angle ADX = \angle BDY \Longrightarrow$ quadrilateral $BYDZ$ is a kite $\Longrightarrow Z$ is reflection of Y across BD. Thus, by obvious symmetry $ACYZ$ is an isosceles trapezoid with legs $AZ, CY \Longrightarrow \angle ACY = \angle CAZ = \angle CAX \Longrightarrow AX \parallel CY$.

AndreiAndronache
#3 Jul 24, 2013, 10:50 pm

We have $\angle ABX \equiv \angle YBD$ and analogues. By trigonometric Ceva in $\triangle ABX$ and $\triangle CBD \Rightarrow \frac{\sin \angle BAX}{\sin \angle DAX} = \frac{\sin \angle DCY}{\sin \angle BCY}$, so $\angle BAX \equiv \angle DCY \Rightarrow AX || CY \square$

Quick Reply

High School Olympiads

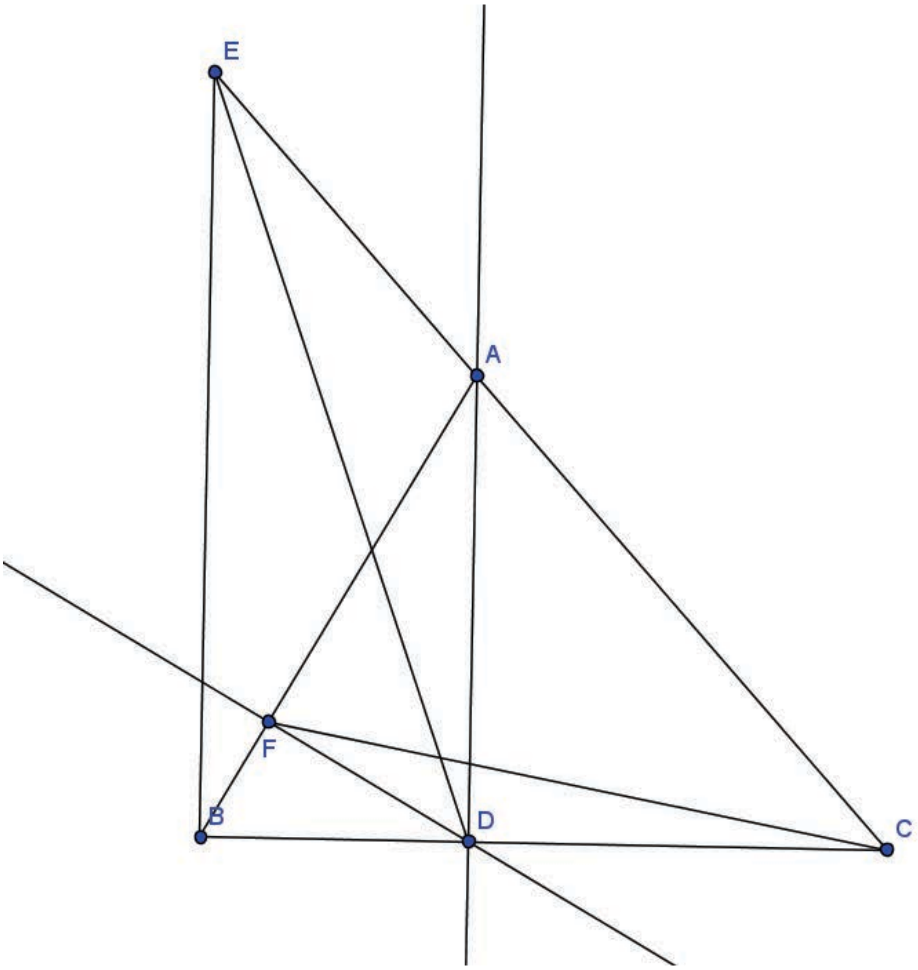
A small problem about Equal Angle

Reply

Narcissus
#1 Jul 24, 2013, 3:11 pm

Let ABC is an acute triangle and D,F are feet of A,D onto BC,AB, respectively. The line passing through B and perpendicular to BC cuts AC at E. Prove that $\widehat{BED}=\widehat{CFD}$

Attachments:



Luis González
#2 Jul 24, 2013, 9:19 pm • 1

1st solution: EB,AD are clearly tangent to the circle $\odot(FBD)$ with diameter $\overline{BD}$. Let ED cut $\odot(FBD)$ again at P. By Pascal theorem for the degerate cyclic hexagon $BBFPDD$, the intersections $E\equiv BB\cap PD, A\equiv BF\cap DD$ and $FP\cap BD$ are collinear $\Longrightarrow F,P,C$ are collinear $\Longrightarrow \angle CFD\equiv\angle PFD=\angle EDA=\angle BED.$

2nd solution: Let $M\equiv AB\cap DE$ and Q the projection of M on BD. BP, MQ, DF are altitudes of $\triangle MBD$ cutting at its orthocenter.

$$\frac{CD}{CB}=\frac{AD}{EB}=\frac{MD}{EM}=\frac{QD}{BQ}\Longrightarrow (B,D,Q,C)=-1\Longrightarrow C\in FP.$$

Xml

xmL

#3 Jul 25, 2013, 12:06 am • 1

Let G be outside of segment AD such that $\angle GFA=\angle CFD$. Since $\angle FDC=180-\angle FDB=180-\angle BAD=\angle FAG$, thus by AA similarity we have $\triangle FAG\sim\triangle FDC$ hence $\frac{EA}{AC}=\frac{BD}{DC}=\frac{AD}{GA}\Rightarrow ED\parallel GA\Rightarrow\angle BED=\angle EDG=\angle DGC=\angle CFD$ (G,C,D,F are concyclic). Q.E.D

sunken rock

#4 Jul 25, 2013, 6:54 am • 1

Let the perpendicular from A to CF intersect FD at M; it has been proven on the site that $\frac{MD}{MF}=\frac{BD}{CD}\ (\ 1\ )$. Let the perpendicular at F to CF intersect AD at N; $(FDCN)$ is cyclic, hence $\angle CND=\angle CFD$. Now all we need is $DE\parallel CN$, but this is true since: from (1) we get $\frac{AN}{AD}=\frac{CD}{BD}$, but $\frac{CD}{BD}=\frac{AC}{AE}$, and our claim is now proven ($\Delta CAN\sim\Delta EAD\implies CN\parallel DE$).

Best regards,
sunken rock

Quick Reply

High School Olympiads

express in terms of 'a'

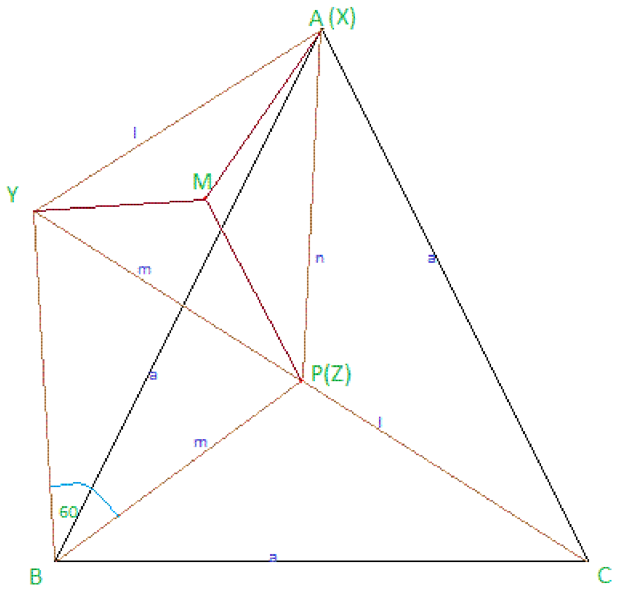
Reply

panther000

#1 Jul 23, 2013, 9:22 pm

Let, ABC be an equilateral triangle with side length a. It can be proved that for any point P inside the triangle, PA,PB,PC forms a triangle. Let, for some point P, this formed triangle is XYZ. Let M be the fermat point of triangle XYZ. Express MX+MY+MZ in terms of 'a'

Attachments:



mavropnevma

#2 Jul 23, 2013, 9:38 pm • 1

> *panther000 wrote:*
> It can be proved that for any point P inside the (equilateral) triangle ABC, (the segments) PA, PB, PC form a triangle.

That is **Pompeiu's theorem**, also valid when P is outside $\triangle ABC$ (when P lies on the circumcircle of $\triangle ABC$, that triangle is degenerated, since one length is equal to the sum of the other two).

Luis González

#3 Jul 24, 2013, 1:01 am

1st solution: Let the parallels through P to AB, BC, CA cut BC, CA, AB at D, E, F, respectively. $\angle ECD = \angle PDC = 60^\circ \Longrightarrow$ trapezoid $PDCE$ is isosceles $\Longrightarrow PC = DE$. Similarly, $PA = EF$ and $PB = FD$ $\Longrightarrow \triangle DEF \cong \triangle XYZ$ is Pompeiu triangle of P. Since $\angle DPE = \angle EPF = \angle FPD = 120^\circ \Longrightarrow P$ is 1st Fermat point of $\triangle DEF$. If PF cuts BC at G, then $\triangle PDG$ is equilateral and $PECG$ is a parallelogram, hence $PF = BD$, $PD = DG$ and $PE = CG \Longrightarrow PD + PE + PF = DG + GC + BD = BC = a \Longrightarrow MX + MY + MZ = a$.

2nd solution: Let Q be the image of P under rotation $(B, 60^\circ)$ counterclockwise. Then $\triangle BPQ$ is equilateral and $\triangle PBC \cong \triangle QBA \Longrightarrow QA = PC \Longrightarrow \triangle PAQ$ is Pompeiu triangle of P. Let $\odot(PBQ)$ cut AB again at M. $\angle AMP = 180^\circ - \angle PQB = \angle PMQ = 120^\circ \Longrightarrow M$ is 1st Fermat point of $\triangle PAQ$. By Ptolemy's theorem for cyclic $PBQM$, we obtain $MP + MQ = MB \Longrightarrow MA + MP + MQ = AB = a \Longrightarrow MX + MY + MZ = a$.

Quick Reply

High School Olympiads

Prove that. (DEF) and (ABC) touch each other

Reply

qua96
#1 Jul 23, 2013, 9:59 pm

Pro. Let´s triangle ABC, (I) is it´s incircle, (I) touch BC at M, AI cut (ABC) at N, NM cut (ABC) at D.(I) cut (BIC) at E, F. Prove that. (DEF) and (ABC) touch each other.

War-Hammer
#2 Jul 23, 2013, 10:15 pm

What is the source of problem ???

qua96
#3 Jul 23, 2013, 11:13 pm

I get it by GSP :-j

Luis González
#4 Jul 23, 2013, 11:38 pm

It's well-known that N is the circumcenter of $\triangle BIC$. Inversion WRT $(N) \equiv \odot(BIC)$ swaps $\odot(ABC)$ and $BC \Longrightarrow M$ goes to D and since E, F are double, then $\odot(DEF)$ is the inverse of (I). Since (I) is tangent to BC, then by conformity $\odot(DEF)$ is tangent to $\odot(ABC)$.

Remark: $\odot(DEF)$ goes through the similitude centers of (I) and $\odot(BIC)$. See ARMO - 2013, Grade 11, day2, p:4.

jayme
#5 Aug 1, 2013, 5:50 pm

Dear Mathlinkers,
we can also use a reduction ad absurdum, considering the circle tangent resp. to circumcircle at D and to BC at M, and conclued with an application of the power of a point to the circles...
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Not like a G6

Source: ELMO Shortlist 2013: Problem G6, by Victor Wang

v_Enhance
#1 Jul 23, 2013, 7:32 am

Let $ABCDEF$ be a non-degenerate cyclic hexagon with no two opposite sides parallel, and define $X = AB \cap DE$, $Y = BC \cap EF$, and $Z = CD \cap FA$. Prove that

$$\frac{XY}{XZ} = \frac{BE}{AD} \frac{\sin|\angle B - \angle E|}{\sin|\angle A - \angle D|}.$$

Proposed by Victor Wang

Luis González
#2 Jul 23, 2013, 1:35 pm • 2

Denote $K \equiv BC \cap FA$. Clearly $\widehat{AZD} = |\widehat{B} - \widehat{E}|$ and $\widehat{BYE} = |\widehat{A} - \widehat{D}|$. Hence, by sine law for $\triangle ZAD$ and $\triangle BYE$, we get:

$$\frac{\sin|\widehat{B} - \widehat{E}|}{AD} = \frac{\sin\widehat{ADC}}{AZ} = \frac{\sin\widehat{ABK}}{AZ}, \frac{\sin|\widehat{A} - \widehat{D}|}{BE} = \frac{\sin\widehat{FEB}}{BY} = \frac{\sin\widehat{BAK}}{BY}$$

$$\Longrightarrow \frac{\sin|\widehat{B} - \widehat{E}|}{\sin|\widehat{A} - \widehat{D}|} \cdot \frac{BE}{AD} = \frac{\sin\widehat{ABK}}{\sin\widehat{BAK}} \cdot \frac{BY}{AZ} = \frac{AK}{BK} \cdot \frac{BY}{AZ} \ (1).$$

By Pascal theorem, X, Y, Z are collinear, hence by Menelaus' theorem for $\triangle KYZ$ cut by the line $\overline{ABX}$, we get

$$\frac{XY}{XZ} \cdot \frac{AZ}{AK} \cdot \frac{BK}{BY} = 1 \Longrightarrow \frac{XY}{XZ} = \frac{AK}{BK} \cdot \frac{BY}{AZ} \ (2).$$

Combining (1) and (2) gives $\dfrac{XY}{XZ} = \dfrac{\sin|\widehat{B} - \widehat{E}|}{\sin|\widehat{A} - \widehat{D}|} \cdot \dfrac{BE}{AD}$.

leminscate
#3 Jun 23, 2014, 5:45 pm

Let (*) be the statement we want to prove.
Complex bash with $(ABCDEF)$ being the unit circle. (The calculations can be done within 45 minutes with the key being to **factorise expressions along the way**).

Let H be the point on $(ABCDEF)$ such that $CH||AF$. Compute $h = \dfrac{af}{c}$. By easy angle chasing we get $\angle DAH = |\angle B - \angle E|$ so by the Extended Sine Law, $DH = 2\sin(\angle B - \angle E)$. Now, note that if u, v are points on the unit circle, $|u-v|^2 = (u-v)(\frac{1}{u} - \frac{1}{v}) = -\dfrac{(u-v)^2}{uv}$ (1).

Hence $DH^2 = \dfrac{(d - \frac{af}{c})^2}{\frac{daf}{c}} = \dfrac{(cd-af)^2}{cdaf}$. So $\sin^2(\angle B - \angle E) = \dfrac{(cd-af)^2}{4cdaf}$. Similarly, $\sin^2(\angle A - \angle D) = \dfrac{(cb-ef)^2}{4cbef}$. From (1) we can easily obtain BE^2, AD^2, so putting this all together gives:

$(RHS(*))^2 = \dfrac{(cd-af)^2(b-e)^2}{(cb-ef)^2(a-d)^2}$. (2)

By Pascal's Theorem, X, Y, Z are collinear. Now by Menelaus' Theorem on $\triangle YFZ$ and transversal CAX, we get

By Pascal's Theorem, X, Y, Z are collinear. Now by Menelaus' Theorem on $\triangle YFZ$ and transversal GAX, we get
$\frac{XZ}{XY}\frac{YG}{GF}\frac{FA}{AZ} = 1 \implies (LHS(*))^2 = \frac{YG^2}{GF^2}\frac{FA^2}{AZ^2}$. (3)
$ \ [\begin{align*} YG^2 &= \left\lvert \frac{ab(e+f)-ef(a+b)}{ab-ef} - \frac{bc(e+f)-ef(b+c)}{bc-ef} \right\rvert ^2\\ &= \left\lvert \frac{ef(a-c)(b-e)(b-f)}{(ab-ef)(bc-ef)} \right\rvert ^2\\ &= -\frac{(a-c)^2(b-e)^2(b-f)^2ef}{(ab-ef)^2(bc-ef)^2}. \end{align*}$
$ \ [\begin{align*} GF^2 &= \left\lvert \frac{ab(e+f)-ef(a+b)}{ab-ef} - f \right\rvert ^2\\ &= \left\lvert \frac{e(f-a)(f-b)}{ab-ef} \right\rvert ^2\\ &= -\frac{(f-a)^2(f-b)^2e}{f(ab-ef)^2}. \end{align*}$
Similarly, $AZ^2 = -\frac{(a-c)^2(a-d)^2f}{a(cd-af)^2}$.
And from (1) $AF^2 = -\frac{(f-a)^2}{fa}$.
Now substituting this all into (3), we get an orgeastic feast of cancellation and obtain
$(LHS(*))^2 = \frac{(cd-af)^2(b-e)^2}{(cb-ef)^2(a-d)^2}$. (4)

So from (2) and (4) $(RHS(*))^2 = (LHS(*))^2$. Since $0 \le |\angle B - \angle E|, |\angle A - \angle D| \le \pi$, $RHS \ge 0$. Clearly $LHS \ge 0$ as well, so we can square root both sides safely to get the desired result.

Quick Reply

High School Olympiads

SD is tangent to (QDM)

Source: ELMO Shortlist 2013: Problem G13, by Ray Li

v_Enhance
#1 Jul 23, 2013, 7:32 am • 1

In $\triangle ABC$, $AB < AC$. D and P are the feet of the internal and external angle bisectors of $\angle BAC$, respectively. M is the midpoint of segment BC, and ω is the circumcircle of $\triangle APD$. Suppose Q is on the minor arc AD of ω such that MQ is tangent to ω. QB meets ω again at R, and the line through R perpendicular to BC meets PQ at S. Prove SD is tangent to the circumcircle of $\triangle QDM$.

Proposed by Ray Li

Luis González
#2 Jul 23, 2013, 12:12 pm • 2

Since the cross ratio (B, C, D, P) is harmonic, then $MB^2 = MC^2 = MD \cdot MP \Longrightarrow$ circle (M) with diameter $\overline{BC}$ is orthogonal to $(K) \equiv \omega \Longrightarrow Q \in (M)$. Inversion with center B and power $BD \cdot BP$ takes (K) into itself and carries (M) into a line orthogonal to (K), due to conformity. R is the inverse of Q, hence RK is perpendicular bisector of $\overline{DP} \Longrightarrow$ $\triangle SDP$ is S-isosceles $\Longrightarrow \angle SDP = \angle SPD \equiv \angle QPD = \angle MQD \Longrightarrow SD$ is tangent to $\odot(QDM)$.

sunken rock
#3 Jul 24, 2013, 4:14 pm • 2

Other solution, thanks to the same brilliant idea of **Luis**:

Q belongs to **Apollonius** circle of ΔABC, hence QD, QP are bisectors of $\angle BQC$ which, being onto the circle of diameter BC, is a right angle, consequently $\angle BQD = 45^\circ$ and R is the midpoint of the arc DP of the circle (ADP) and $\angle SDP = \angle SPD = \angle DQM$, done.

Best regards,
sunken rock

JuanOrtiz
#4 Aug 18, 2013, 12:17 am

The Solution

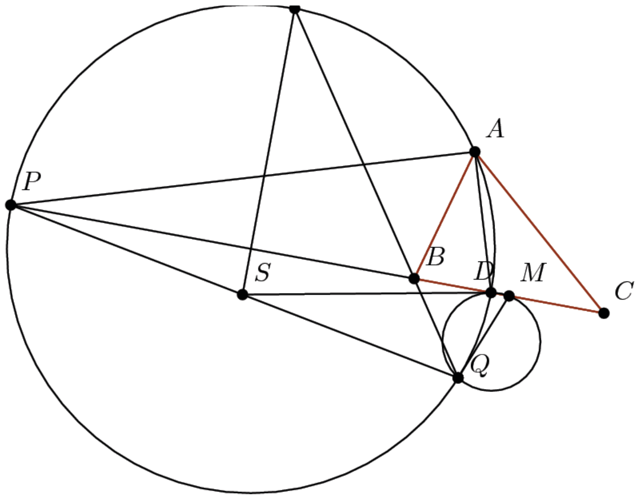
Pretty standard problem.

thecmd999
#5 Apr 30, 2014, 9:53 am

Solution

AnonymousBunny
#6 Jul 6, 2014, 10:23 pm

Wow, G13! To be honest I found the G4 more challenging than this.



Let ω denote the circumcircle of $\triangle APD.$ By the angle bisector theorem, $\frac{BD}{DC}=\frac{AB}{AC}=\frac{BP}{PC}$, so ω is the Appolonius circle of B and C with ratio $\frac{AB}{AC}$. Since Q lies on this circle too, $\frac{QB}{QC}=\frac{BD}{DC}$, implying QD bisects $\angle BQC$ internally. Since $\angle PAD=90^{\circ}$ and quadrilateral $APQD$ is cyclic, $\angle PQD=90^{\circ}$, so QP bisects $\angle BQC$ externally (this follows from the fact that the internal and external angle bisectors are perpendicular). Also, note that since $DA\perp AP, AD$ is a diameter of $\omega.$

Now, note that $PB\times DC=PC\times BD,$ which implies (P,D,B,C) is harmonic. Hence, $MC^2=MB^2=MD\cdot MP=MQ^2,$ where we have used the fact that $MQ^2=MD\times MP$ which follows from PoP. Hence, $MB=MC=MQ,$ which implies that M is the circumcenter of $\triangle BCQ.$ Since M is the midpoint of $BC,$ this forces $\triangle BQC$ to be right angled at $Q.$ By our previous observations, we have that $\angle BQM=\angle QMC=\angle BQP=45^{\circ}.$

Now, $\angle RPD=\angle PQR=45^{\circ}=\angle RQD=\angle RDP,$ so $\triangle RPD$ is isoceles with $RP=RD.$ Since AD is a diameter of ω and R lies on the perpendicular bisector of PD, RS must bisect $PD,$ implying RS is the perpendicular bisector of $PD.$ Hence, $SP=SD$ and $\angle SDP=\angle SPD.$ Since MQ is tangent to $\omega, \angle MQD=\angle QPD.$ It follows that $\angle SDP=\angle MQD,$ completing the proof. $\blacksquare$

This post has been edited 1 time. Last edited by AnonymousBunny, Jul 7, 2014, 10:55 pm

IDMasterz
#7 Jul 7, 2014, 8:37 pm

Obviously, the circle with centre M through B, C is orthogonal to the circle APD, so $\angle BQC=\frac{\pi}{2}$, thus QB bisects $\angle PQD$. So, SDP is isosceles, or $\angle SDP=\angle SPD=\angle MQD$...

sayantanchakraborty
#8 Dec 18, 2014, 9:02 pm

Yes its just an easy cross ratio exercise.

AP,AD,AB,AC being a harmonic pencil implies $(P,D;B,C)=-1 \implies MD\cdot MP=MB^2=MQ^2=MC^2 \implies QB\perp QC$.Also Q being a point on the Apollonius circle means $\angle BQD=\angle DQC=45^{\circ} \implies PR=RM \implies \angle DQM=\angle QPD=\angle SMP$ which implies the result.

Quick Reply

High School Olympiads

Prove that angles AGH and BGH are equal

Source: ELMO Shortlist 2013: Problem G8, by David Stoner

v_Enhance

#1 Jul 23, 2013, 7:32 am

Let ABC be a triangle, and let D, A, B, E be points on line AB, in that order, such that $AC = AD$ and $BE = BC$. Let ω_1, ω_2 be the circumcircles of $\triangle ABC$ and $\triangle CDE$, respectively, which meet at a point $F \neq C$. If the tangent to ω_2 at F cuts ω_1 again at G, and the foot of the altitude from G to FC is H, prove that $\angle AGH = \angle BGH$.

Proposed by David Stoner

exmath89

#2 Jul 23, 2013, 10:53 am

Solution

Luis González

#3 Jul 23, 2013, 11:38 am • 1

Let M be the midpoint of the arc AB of ω_1 and redefine H as the projection of M on CF. It suffices to show that M, H, G are collinear. Label α, β, γ the angles of $\triangle ABC$ at A, B, C.

$\angle CMH = 90^\circ - (\angle MCE - \angle FCE) = 90^\circ - \frac{1}{2}\beta - \frac{1}{2}\gamma - \angle FCE \Longrightarrow \angle CMH = \frac{1}{2}\alpha - \angle FCE$. Let $P \equiv CD \cap EF$. From cyclic $CDEF$, we have $\angle PFC = \angle FEC + \angle FCE = \angle CDA = \frac{1}{2}\alpha$, but since $\angle GFC = \angle FEC \Longrightarrow \angle GFC = \frac{1}{2}\alpha - \angle FCE = \angle CMH \Longrightarrow G \in MH$, as desired.

thecmd999

#4 Apr 27, 2014, 8:53 am

This is straightforward angle chasing, so I won't post my solution. Here are some remarks.

Let O_2 be the center of ω_2. Since the perpendicular bisectors of AD and AE meet at O_2 and are the external angle bisectors of $\angle B$ and $\angle C$, it follows that O_2 is the A-excenter of triangle ABC. In particular, line OO_2 is the perpendicular bisector of segment AF, so F can be reinterpreted as the reflection of A over line I_aO. In fact, I think this makes a harder restatement of the problem.

leminscate

#5 Jun 24, 2014, 9:49 am

Let O be the circumcentre of $\triangle ABC$ and M be the midpoint of arc AB not containing C. Note that since $\triangle DAC$ and $\triangle EBC$ are isosceles, the external angle bisectors of $\angle A, \angle B$ are the perpendicular bisectors of DC, EC, respectively. Hence the circumcentre of $\triangle CDE$ is I_c, the excentre of $\triangle ABC$ opposite C.

Now it is easy to complex with unit circle (ABC) and a^2, b^2, c^2 being the vertices A, B, C. $m = -ab, i_c = -ab + bc + ca$.
$\[\begin{align*}CF \perp I_c O \implies -cf &= -\frac{i_c}{\overline{i_c}}\\ &= -\frac{abc(-ab+bc+ca)}{a+b-c} \end{align*}$

$$\implies f = \frac{ab(-ab+bc+ca)}{c(a+b-c}.$$

So $f - i_c = (-ab+bc+ca)\dfrac{ab - c(a+b-c)}{c(a+b-c)} = \dfrac{(-ab+bc+ca)(c-a)(c-b)}{c(a+b-c)}$.

$$\overline{(f-i_c)} = \frac{(a+b-c)(c-a)(c-b)}{(-ab+bc+ca)abc}.$$

So $\dfrac{f-i_c}{\overline{(f-i_c)}} = \dfrac{(-ab+bc+ca)^2ab}{(a+b-c)^2}$.

This last quantity equals fg since $I_cF \perp FG$ so $g = \dfrac{c(-ab+bc+ca)}{a+b-c}$ from which we observe that $mg = -c^2f$. So $MG \perp CF$ and we are done.

tobash_co

#6 Jun 25, 2014, 4:40 pm

Let I be the incenter of $\triangle ABC$, M be the midpoint of arc AB of ω_1, and $K = AB \cap MC$. Since $AI \parallel DC$, $BI \parallel EC$, $\triangle AIB, \triangle DCE$ are homothetic with center K. It is well-known that the circumcenters of $\triangle AIB, \triangle DCE$ are M, O respectively, so this homothety also sends M to O, thus K, M, O are collinear. But C, K, M are collinear, thus we actually have C, M, O collinear. Now we have

$$\angle FCM + \angle GMC = \angle OFC + \angle CFG = 90^\circ \Rightarrow CF \perp GM$$

Thus G, H, M are collinear. But GM is the bisector of $\angle AGB$, so in fact GH bisects $\angle AGB$, and thus $\angle AGH = \angle BGH$.

SmartClown

#7 Nov 15, 2015, 5:51 am

First note that $\angle AFD = \angle AFB - \angle DFC = \dfrac{\angle B}{2}$. Let $\angle GFC = \phi$. From that we have $\angle CDF = \phi$ so $\angle ADF = \dfrac{\angle A}{2} - \phi$. From this we easily get $\angle FCB = \angle FAB = 90 - \dfrac{\angle C}{2} - \phi$. Now let M be the midpoint of arc AB containing C we have that $\angle BAM = 90 - \dfrac{\angle C}{2}$ which implies $\angle FCM = \phi = \angle GFC$ so $CFGM$ is isosceles trapezoid so $MG||CF$ so $GH \perp MG$ so $\angle AGH = \angle BGH$ so we are finished.

Quick Reply

High School Olympiads

Circumcircles of AEF, BFD, CDE

Reply

Source: ELMO Shortlist 2013: Problem G2, by Michael Kural

v_Enhance

#1 Jul 23, 2013, 7:31 am

Let ABC be a scalene triangle with circumcircle Γ, and let D,E,F be the points where its incircle meets BC, AC, AB respectively. Let the circumcircles of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ meet Γ a second time at X, Y, Z respectively. Prove that the perpendiculars from A, B, C to AX, BY, CZ respectively are concurrent.

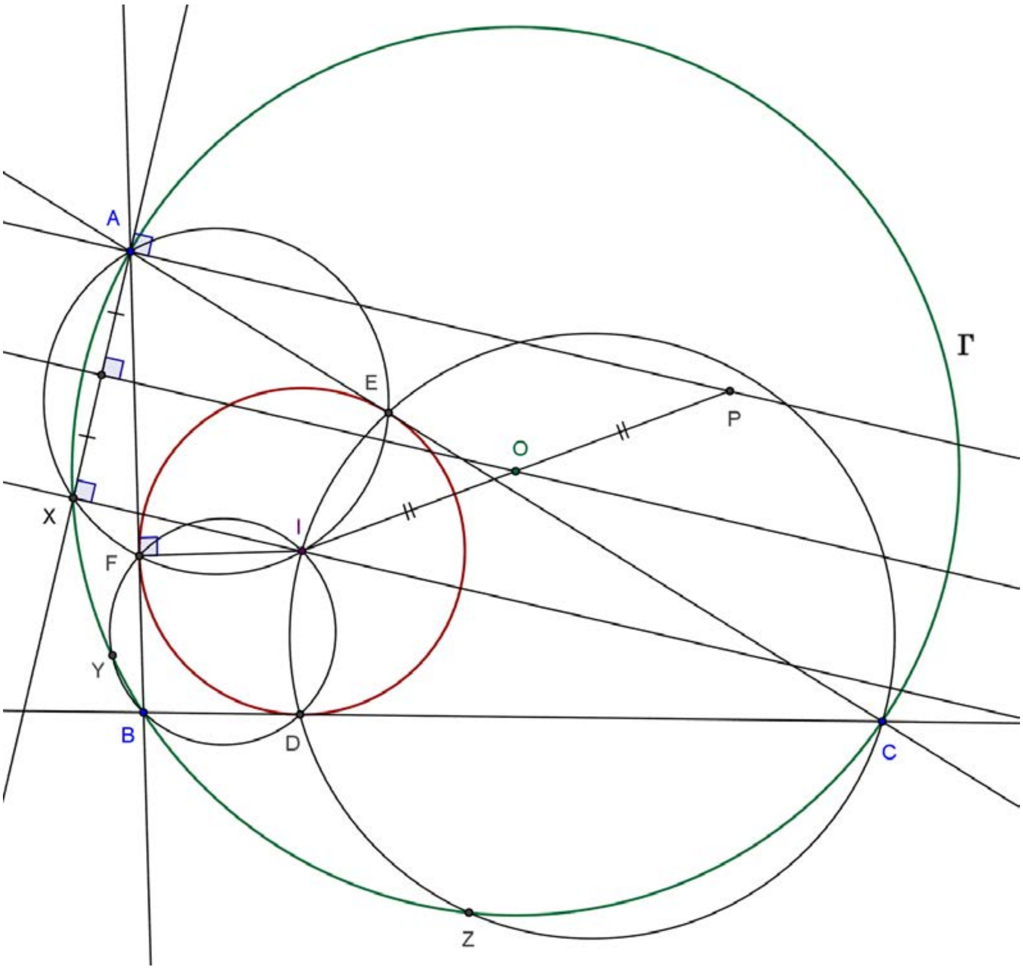
Proposed by Michael Kural

Luis González

#2 Jul 23, 2013, 10:05 am • 1

See the diagram for a proof without words.

Attachments:



djmathman

#3 Dec 18, 2013, 4:09 am

Probably not the intended solution, but a solution nevertheless.

Let $M = AX \cap BY$, $N = BY \cap CZ$, $P = CZ \cap AX$. Note that AX is the radical axis of (AEF) and (ABC) and that BY is the radical axis of (BED) and (ABC), so M lies on the radical axis of (AEF) and (BED). However, since $AE \perp EI$ and $AF \perp FI$, where I is the incenter, $AEIF$ is a cyclic quadrilateral, so $M \in EF$ and $ME \perp AD$. Similarly, $ND \perp BC$ and $PF \perp CA$. Therefore

$$\begin{aligned} AM^2 + BN^2 + CP^2 &= (AE^2 + EM^2) + (BD^2 + DN^2) + (CF^2 + FP^2) \\ &= (BE^2 + EM^2) + (CD^2 + DN^2) + (AF^2 + FP^2) = MB^2 + NC^2 + PA^2, \end{aligned}$$

so by Carnot's Theorem the perpendiculars to MN, NP, PM from B, C, A respectively are concurrent as desired. ■

jayme

#4 Dec 18, 2013, 2:04 pm

Dear Mathlinkers,
after the nice and silent proof of Luis, who want to give the geometric nature of P?
Sincerely
Jean-Louis

thecmd999

#5 Apr 23, 2014, 4:03 am

Solution

jayme

#6 May 20, 2014, 1:58 pm

Dear Mathlinkers,
P is the Bevan's point of ABC...
see
http://perso.orange.fr/jl.ayme vol. 3, Cinq théorèmes de von Nagel p. 24-25
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Given orthogonal circles, show FGOB is cyclic

Reply

Source: ELMO Shortlist 2013: Problem G5, by Eric Chen

v_Enhance
#1 Jul 23, 2013, 7:31 am

Let ω_1 and ω_2 be two orthogonal circles, and let the center of ω_1 be O. Diameter AB of ω_1 is selected so that B lies strictly inside ω_2. The two circles tangent to ω_2, passing through O and A, touch ω_2 at F and G. Prove that $FGOB$ is cyclic.

Proposed by Eric Chen

Luis González
#2 Jul 23, 2013, 9:27 am

Let AB cut ω_2 at $X, Y.$ Since $\omega_1 \perp \omega_2 \Longrightarrow OA^2 = OB^2 = OX \cdot OY \Longrightarrow (X, Y, A, B) = -1 \Longrightarrow B$ is on the polar of A WRT $\omega_2.$ Inversion WRT ω_1 takes ω_2 into itself and carries the circles passing through O, A tangent to ω_2 into two lines tangent to ω_2 at the inverses F^*, G^* of $F, G.$ These lines will meet at the double point $A \Longrightarrow F^*G^*$ is the polar of A WRT ω_2 $\Longrightarrow B, F^*, G^*$ are collinear. Since B is double, then B, F, G, O lie on the inverse circle of $F^*G^*.$

thecmd999
#3 Apr 25, 2014, 1:46 am

Solution

AnonymousBunny
#4 Jul 6, 2014, 1:19 am

Nice and easy. :)

Click to reveal hidden text

junior2001
#5 Dec 26, 2014, 2:21 pm

any solution without inversion ?

EulerMacaroni
#6 Sep 3, 2015, 6:41 am

Slightly different finish

Quick Reply

High School Olympiads

find the size of angle AME

Source: China south east mathematical Olympiad 2012 day2 problem 7

jred
#1 Jul 17, 2013, 3:20 pm

In $\triangle ABC$, point D lies on side AC such that $\angle ABD = \angle C$. Point E lies on side AB such that $BE = DE$. M is the midpoint of segment CD. Point H is the foot of the perpendicular from A to DE. Given $AH = 2-\sqrt{3}$ and $AB = 1$, find the size of $\angle AME$.

emreorhan44
#2 Jul 17, 2013, 10:19 pm

$\tan 15^\circ = 2-\sqrt{3}$, may be it can help you.

Luis González
#3 Jul 22, 2013, 9:28 am

Since $\angle ABD = \angle EDB = \angle ACB$, then EB, ED are tangent to the circumcircle (K) of $\triangle BCD$. Let $\odot(KME)$ cut AC again at F. $\angle FEK = \angle FMK = 90^\circ \Longrightarrow EF \parallel BD \perp EK \Longrightarrow \frac{EF}{BD} = \frac{AE}{AB}$. Moreover $\frac{AH}{AE} = \sin \widehat{AED} = \sin \widehat{BKD} = \frac{BD}{EK}$. Hence

$$\tan \widehat{AME} = \tan \widehat{FKE} = \frac{EF}{EK} = \frac{BD}{EK} \cdot \frac{AE}{AB} = \frac{AH}{AE} \cdot \frac{AE}{AB} = \frac{AH}{AB} = 2-\sqrt{3}$$

$$\Longrightarrow \widehat{AME} = \arctan(2-\sqrt{3}) = 15^\circ.$$

P.S. See the topic the degree of AME for another solution.

High School Olympiads

Geo-problem(excellent)

Reply

Pirkuliyev Rovsen

#1 Jul 21, 2013, 8:07 pm

Points M and N on the AC side of the triangle ABC such that $\angle ABN > \angle ABM = \angle CBN$. Prove that the common interior tangent to the inscribed circle of triangle ABM and CBN intersect on the bisector of the angle ABC.

bcp123

#2 Jul 21, 2013, 8:39 pm • 1

Let incenter of ABM is I_1 and incribed circle's radius is r_1.
Let incenter of CBN is I_2 and incribed circle's radius is r_2.
Let interior tangents intersect at K.
By similarity, we have $\frac{r_1}{r_2} = \frac{I_1K}{I_2K}$.
Let $I_1T \perp BM$ and $I_2L \perp BN$. $\triangle BI_1T \sim \triangle BI_2L$ since $\angle I_1BM = \angle I_2BN$.
Therefore, $\frac{r_1}{r_2} = \frac{I_1B}{I_2B} = \frac{I_1K}{I_2K}$ which implies that $\angle I_1BK = \angle I_2BK$ since I_1, I_2, K are collinear. Then,
$\angle ABK = \angle CBK$ since $\angle ABI_1 = \angle CBI_2$.
$Q.E.D$

This post has been edited 2 times. Last edited by bcp123, Jul 22, 2013, 12:26 am

Luis González

#3 Jul 22, 2013, 12:12 am • 1

Denote by $(I_1), (I_2)$ the incircles of $\triangle ABM, \triangle CBN$. $I \equiv AI_1 \cap CI_2$ is the incenter of $\triangle ABC$. $D \equiv BI \cap AC$ and E are the feet of the internal and external bisector of $\angle ABC$. EI_1 cuts BD, CI at F, J_2. Since $I(A, C, D, E) = -1$, then $(I_1, J_2, F, E) = -1$. Together with $BE \perp BF$, it follows that BF bisects $\angle I_1BJ_2 \Longrightarrow \angle FBI_1 = \angle FBJ_2 \Longrightarrow \angle MBI_1 = \angle NBJ_2 \Longrightarrow I_2 \equiv J_2 \Longrightarrow (I_1, I_2, F, E) = -1 \Longrightarrow F$ is the insimilicenter of $(I_1) \sim (I_2)$ and the conclusion follows.

Quick Reply

High School Olympiads

Metric relation in a hexagon

Source: Tuymaada 2013, Day 1, Problem 2 Juniors

mavropnevma
#1 Jul 21, 2013, 4:02 am • 2

$ABCDEF$ is a convex hexagon, such that in it $AC \parallel DF$, $BD \parallel AE$ and $CE \parallel BF$. Prove that

$$AB^2 + CD^2 + EF^2 = BC^2 + DE^2 + AF^2.$$

N. Sedrakyan

Luis González
#2 Jul 21, 2013, 4:37 am

Perpendiculars from B, D and F to $DF \parallel AC, FB \parallel CE$ and $BD \parallel EA$ obviously concur at the orthocenter H of $\triangle BDF$. Then $AB^2 - BC^2 = HA^2 - HC^2, CD^2 - DE^2 = HC^2 - HE^2$ and $EF^2 - FA^2 = HE^2 - HA^2$. Adding these expressions together gives

$$AB^2 - BC^2 + CD^2 - DE^2 + EF^2 - FA^2 = 0 \Longrightarrow$$

$$AB^2 + CD^2 + EF^2 = BC^2 + DE^2 + AF^2.$$

mavropnevma
#3 Jul 21, 2013, 11:04 am • 1

This vectorial solution captures the same phenomenon as the more synthetic one above.

Associate position vectors to the vertices of the hexagon (with v being the position vector associated to a point V). Then we can take advantage of the writing $XY^2 = \|y - x\|^2 = \langle y - x, y - x \rangle = \|x\|^2 + \|y\|^2 - 2\langle y, x \rangle$. If we write these relations for the sides of the hexagon, the required equality comes to

$$\langle b, a \rangle + \langle d, c \rangle + \langle f, e \rangle = \langle b, c \rangle + \langle d, e \rangle + \langle f, a \rangle,$$

that is

$$\langle b, a - c \rangle + \langle d, c - e \rangle + \langle f, e - a \rangle = 0.$$

But if we take the origin on the perpendicular from B on AC, then $b \perp a - c$, and so $\langle b, a - c \rangle = 0$. Similar consequences will hold for the other two terms. Therefore, if we take the origin at the orthocentre of the triangle BDF, all three terms will be null, and so the required equality will be proved.

We can in fact proceed without considering the orthocentre of $\triangle BDF$, by leaving the origin at an arbitrary point. The equivalent relation obtained may also be written (using $\langle b, a - c \rangle = \langle b, (a - e) + (e - c) \rangle$ as

$$\langle b - f, a - e \rangle + \langle b - d, e - c \rangle = 0.$$

But triangles BDF and EAC are similar, in a similarity ratio ρ, and then $\dfrac{BD}{AE} = \dfrac{FB}{EC} = \rho$, whence $b - d = \rho(a - e)$ and $b - f = -\rho(e - c)$, therefore

$$\langle b - f, a - e \rangle + \langle b - d, e - c \rangle = -\rho\langle e - c, a - e \rangle + \rho\langle a - e, e - c \rangle = 0.$$

One more proof, if needed, of the strength of these dot-product manipulations.

mathuz

#4 Jul 26, 2013, 6:05 pm

it's easy from the Carnot's theorem!

Quick Reply

High School Olympiads

parallel

mathuz
#1 Jul 21, 2013, 12:09 am

In a triangle ABC, k_a, k_b, k_c are excircles, opposite of the A, B, C, respectively. The circle k_a touches to the lines AB and AC at points A_1 and A_2. Points B_1, B_2, C_1, C_2 are defined similar of this sort and $A_1A_2 \cap B_1B_2 = R(.)$,
$A_1A_2 \cap C_1C_2 = Q(.)$, $B_1B_2 \cap C_1C_2 = P(.)$.
H' is orthocenter of triangle PQR.
Let $k(I)$ is incircle of the triangle ABC, $H(.)$ and $O(.)$ are orthocenter and circumcenter of the triangle ABC. Prove that $OI \parallel HH'$.

Luis González
#2 Jul 21, 2013, 1:45 am

Let D be the foot of the A-altitude. Assume that B_1B_2 and C_1C_2 cut AD at two distinct points P_1, P_2. By Menelaus' theorem for $\triangle ABD, \overline{P_1B_1B_2}$ and $\triangle ACD, \overline{P_2C_1C_2}$, we get

$$\frac{P_1A}{P_1D} = \frac{DB_1}{BB_1} \cdot \frac{BB_2}{AB_2} = \frac{DB_1}{AB_2}, \ \frac{P_2A}{P_2D} = \frac{DC_2}{CC_2} \cdot \frac{CC_1}{AC_1} = \frac{DC_2}{AC_1}.$$

But if I_b, I_c denote the centers of k_b, k_c, we have $\frac{AC_1}{AB_2} = \frac{AI_c}{AI_b} = \frac{DC_2}{DB_1}$

$$\Longrightarrow \frac{P_1A}{P_1D} = \frac{P_2A}{P_2D} \Longrightarrow P \equiv P_1 \equiv P_2,$$ i.e. P lies on AH.

Similarly, Q and R lie on BH and CH. If X, Y, Z are the tangency points of $k(I)$ with BC, CA, AB, then clearly $\triangle XYZ$ and $\triangle PQR$ are homothetic with circumcenters I, H, since $PH \parallel IX, QH \parallel IY, RH \parallel IZ$. Hence, their Euler lines OI and HH' are parallel.

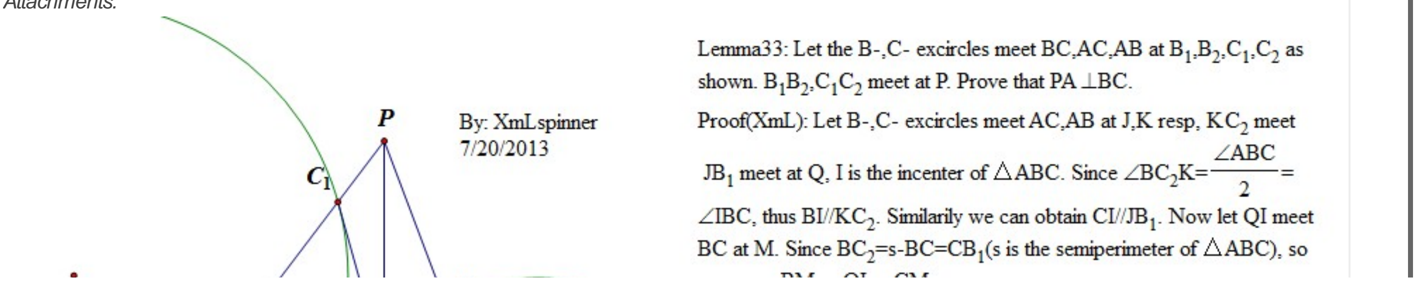
mathuz
#3 Jul 21, 2013, 2:10 am

thank you.
nice idea:
$H(.)$ is circumcenter of the triangle PQR. :)

XmL
#4 Jul 21, 2013, 10:59 am

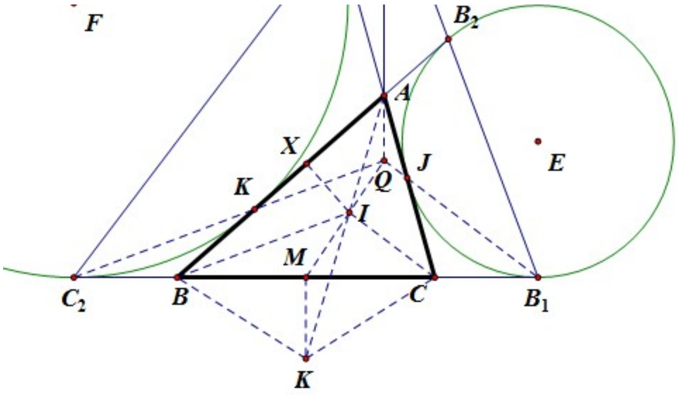
You could also consider constructing the three excenters to form another homothetic triangle with PQR.
Here's another way to prove the lemma that Luis has proven in the attachment below, which is no simpler than Luis's but I thought I would share it anyway~

Attachments:



Lemma33: Let the B-,C- excircles meet BC,AC,AB at B_1,B_2,C_1,C_2 as shown. B_1B_2,C_1C_2 meet at P. Prove that PA $\perp$BC.

Proof(XmL): Let B-,C- excircles meet AC,AB at J,K resp, KC_2 meet JB_1 meet at Q, I is the incenter of $\triangle$ABC. Since $\angle BC_2K = \frac{\angle ABC}{2} = \angle IBC$, thus BI//$KC_2$. Similarily we can obtain CI//JB_1. Now let QI meet BC at M. Since BC_2=s-BC=CB_1(s is the semiperimeter of $\triangle$ABC), so



we have $\frac{BM}{BC_2}=\frac{QI}{IM}=\frac{CM}{CB_1}\Rightarrow BM=CM$. Since $KQ\perp B_1B_2, JQ\perp C_2C_1$, thus Q is the orthocenter of $\triangle PB_1C_2\Rightarrow PQ\perp BC$. So now $PA\perp BC\leftrightarrow AQ\perp BC$. Let K be the midpoint of arcBC of $\odot ABC$, thus K is on AI and $MK\perp BC\Rightarrow$now it suffices to prove $\frac{AI}{KI}=\frac{QI}{MI}=\frac{BC_2}{BM}\leftrightarrow\frac{BK}{BM}=\frac{AI}{BK}$ (well known KI=CK=BK). Now let X be the pedal from I to AB. Thus $AX=s-BC=BK\Rightarrow$now we just have to prove $\frac{BK}{BM}=\frac{AI}{AX}$ which is prove since $\triangle AXI\sim\triangle BMK$. Q.E.D

Quick Reply

High School Olympiads

INAMO 2011 day 1 problem 3

Reply

septian

#1 Jul 20, 2013, 10:09 am

Let ABC be acute-angled triangle. Line ℓ_a passing through point A and perpendicular to AB, line ℓ_b passing through point B and perpendicular to BC and line ℓ_c passing through point C and perpendicular to AC. Suppose $\ell_a \cap \ell_b = D, \ell_b \cap \ell_c = E$ and $\ell_a \cap \ell_c = F$. Prove that $[DEF] \geq 3[ABC]$

lagger11

#2 Jul 20, 2013, 7:22 pm

Notice that $\triangle DEF \sim \triangle BCA$. Hence, $\frac{[DEF]}{[BCA]} = (\frac{DF}{AB})^2$. Now, WLOG $A(0,0), B(b,0), C(a,b)$ with $a,b,c > 0$ with $a < b$. Determine $m_{BC} = \frac{c}{a-b}, m_{CA} = \frac{c}{a}$. Then $m_{l_b} = -\frac{1}{m_{BC}} = \frac{b-a}{c} \Rightarrow y_{l_b} = \frac{b-a}{c}(x-b)$ and $m_{l_c} = -\frac{1}{m_{CA}} = -\frac{a}{c} \Rightarrow y_{l_c} = -\frac{a}{c}(x-a)+c$. Easily we can get $D(0, -\frac{b(b-a)}{c})$ and $F(0, \frac{a^2}{c}+c)$. then $DF = \frac{a^2}{c} + \frac{b(b-a)}{c}$. Since $\frac{(a-\frac{b}{2})^2 + (c-\frac{\sqrt{3}b}{2})^2}{bc} \geq 0$ then $\frac{[DEF]}{[ABC]} = (\frac{DF}{AB})^2 = (\frac{(a-\frac{b}{2})^2 + (c-\frac{b\sqrt{3}}{2})^2}{bc} + \sqrt{3})^2 \geq 3$ equivalent to $[DEF] \geq 3[ABC]$. $Q.E.D$

Luis González

#3 Jul 20, 2013, 11:34 pm • 1

Since $\angle CFA = \angle BAC, \angle ADB = \angle CBA$ and $\angle BEC = \angle ACB$, it follows that $\triangle ABC \sim \triangle FDE$ and $\odot(ADB)$, $\odot(BEC), \odot(CFA)$ are tangent to BC, CA, AB, respectively $\Longrightarrow \odot(ADB), \odot(BEC), \odot(CFA)$ concur at the 1st Brocard point P of $\triangle ABC$, which is then the center of the spiral similarity that transforms $\triangle ABC$ into $\triangle FDE$. The rotational angle is $\angle APF = \angle BPD = \angle CPE = 90^\circ$ and the coefficient is then $\frac{PF}{PA} = \cot \widehat{PFA} = \cot \widehat{PCA} = \cot \omega$ (ω is the Brocard angle of ABC). Hence

$$\sqrt{\frac{[DEF]}{[ABC]}} = \frac{PF}{PA} = \cot \omega = \cot A + \cot B + \cot C \geq 3 \cot \frac{A+B+C}{3} = \sqrt{3}.$$

Quick Reply

High School Olympiads

semicircle problem

Reply

Source: China south east mathematical Olympiad 2007 problem 2

jred

#1 Jul 12, 2013, 9:47 am

AB is the diameter of semicircle O. C,D are two arbitrary points on semicircle O. Point P lies on line CD such that line PB is tangent to semicircle O at B. Line PO intersects line CA, AD at point E, F respectively. Prove that OE=OF.

Luis González

#2 Jul 12, 2013, 11:44 am

Let G be the 2nd intersection of DO with (O) (reflection of D about O). By Pascal theorem for the degenerate cyclic hexagon $DCABBG$, the intersections $P \equiv DC \cap BB, E \equiv CA \cap GB$ and $O \equiv AB \cap GD$ are collinear. Since $\overline{ADF} \parallel \overline{BGE}$ and $\overline{OA} = -\overline{OB}$, then $AFBE$ is parallelogram with diagonal intersection O, i.e. O is midpoint of $\overline{EF}$.

jred

#3 Jul 19, 2013, 2:53 pm

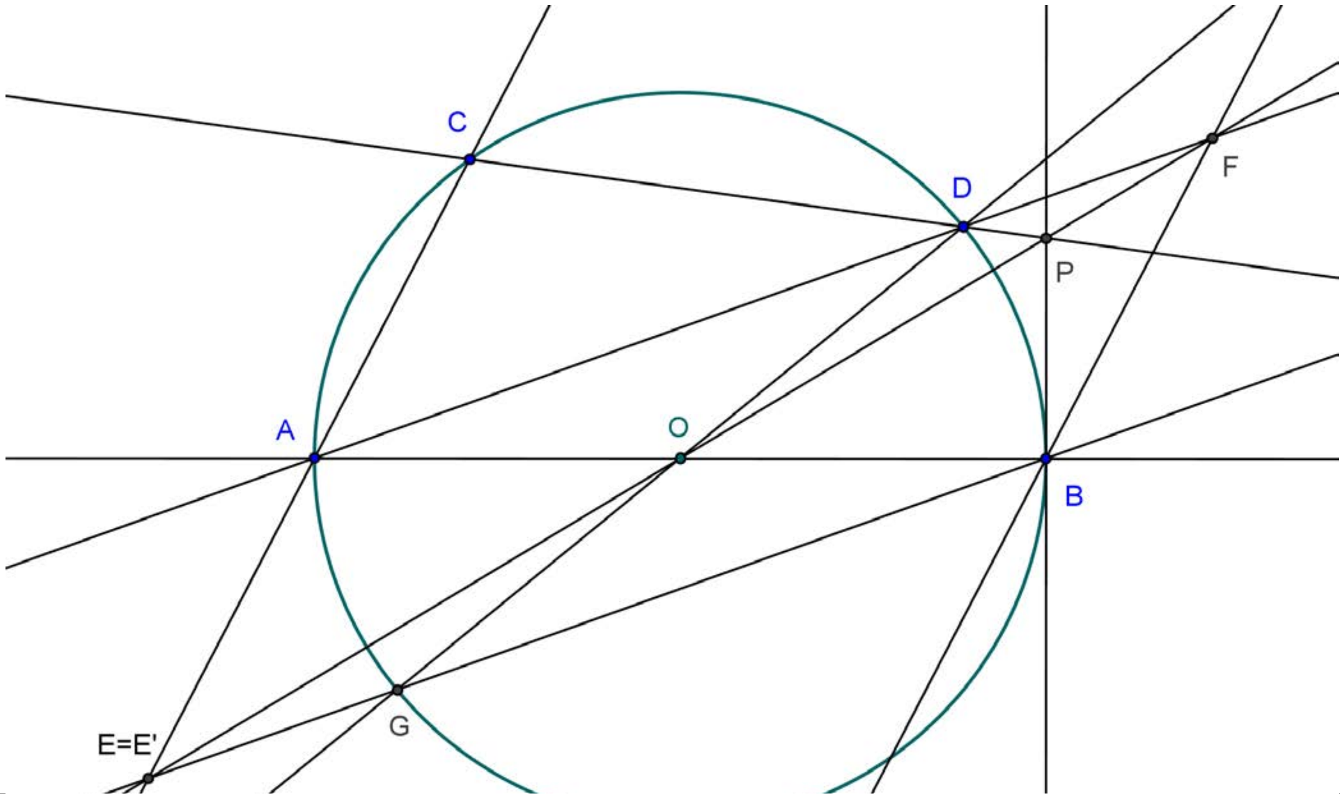
According to the statement, E lies on line PO, so why do you use pascal theorem to prove P,E,O being collinear?
otherwise, i don't understand how do you get $\overline{ADF} \parallel \overline{BGE}$?

Luis González

#4 Jul 19, 2013, 10:33 pm

I used Pascal theorem to prove that the points B, G, E are collinear. It is a matter of denotation, intersections $P \equiv CD \cap BB, E' \equiv CA \cap GB$ and $O \equiv AB \cap GD$ are collinear $\Longrightarrow E \equiv E'$.

Attachments:



jred

#5 Jul 20, 2013, 8:17 am

> *Luis González wrote:*
>
> I used Pascal theorem to prove that the points B, G, E are collinear. It is a matter of denotation, intersections $P \equiv CD \cap BB, E' \equiv CA \cap GB$ and $O \equiv AB \cap GD$ are collinear $\Longrightarrow E \equiv E'.$

i got it now, thanks!

Quick Reply

High School Olympiads

the intangents of (P) and (I) is parallel to BC.

Reply

tangkhaihanh
#1 Jul 1, 2013, 2:55 pm

A circle (O) passing through B, C, tangent to the circles (I) (internally) and (P) (externally), exists if and only if one of the intangents of (P) and (I) is parallel to BC.

oneplusone
#2 Jul 7, 2013, 9:00 pm

Umm... what is O, I, P?

hal9v4ik
#3 Jul 10, 2013, 8:16 pm

It is cirles with centres O, I, P

VincentTandya
#4 Jul 11, 2013, 12:06 pm

Sorry if I 'misread' this problem, but what does the problem ask?

Arab
#5 Jul 18, 2013, 12:04 pm

We wonder what is P.

VincentTandya
#6 Jul 18, 2013, 4:14 pm

Do you have any picture or some sort of that? You don't tell anything about B and C.

@Arab, I think P is a center of a circle, called circle P.

Arab
#7 Jul 18, 2013, 4:35 pm

@VincentTandya, Yes it is. But where is it ?

VincentTandya
#8 Jul 18, 2013, 4:46 pm

I don't know. Obviously outside circle O. The problem should have an image.

oneplusone
#9 Jul 19, 2013, 3:53 pm

I think I discovered the problem.

ABC is a triangle, $(I), (P)$ are circles both tangent to the segments AB, AC with (P) closer to A. Then there exists a circle (O) passing through B, C, internally tangent to (I) and externally tangent to (P) iff one of the internal tangents of $(I), (P)$ is parallel to BC.

Very interesting question, haven't proved it though.

Luis González

#10 Jul 19, 2013, 9:53 pm

> *tangkhaihanh wrote:*
> A circle (O) passing through B, C, tangent to the circles (I) (internally) and (P) (externally), exists if and only if one of the intangents of (P) and (I) is parallel to BC.

> *oneplusone wrote:*
> ABC is a triangle, $(I), (P)$ are circles both tangent to the segments AB, AC with (P) closer to A. Then there exists a circle (O) passing through B, C, internally tangent to (I) and externally tangent to (P) iff one of the internal tangents of $(I), (P)$ is parallel to BC.

This is the extraversion of the "Parallel tangent theorem". See

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=15945
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=430441
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=310017

P.S. See theorem 1 in the article J. Marshall Unger, A new proof of a "hard but important" Sangaku problem. Forum Geometricorum, 10 (2010) 7--13.

Quick Reply

High School Olympiads

Lemoine point

Reply

Source: Own?

Nguyenhuyhoang

#1 Jul 19, 2013, 12:58 am

Given triangle ABC inscribed in (O), AA_1, BB_1, CC_1 are its altitudes. AO intersects (OBC) at A', define similarly for B', C'. Prove that $(AA_1A'), (BB_1B'), (CC_1C')$ have a common radical axis d and d passes through the Lemoine point L of ABC.

Click to reveal hidden text

Luis González

#2 Jul 19, 2013, 2:04 am

Let LA, LB, LC cut (O) again at L_A, L_B, L_C. AO cuts (O) again at P and PL_A cuts BC at D. Pencil formed by AB, AC, AL and the tangent τ_A of (O) at A is clearly harmonic, thus the pencil $PB, PC, PL_A \equiv PD, PA$ formed by perpendiculars from P to AB, AC, AL, τ_A is also harmonic. Since $A'O$ bisects $\angle BA'C$ (O is midpoint of the arc BC of (BA'C)), then it follows that $A'D$ bisects $\angle BA'C$ externally $\Longrightarrow \angle AA'D = 90^\circ \Longrightarrow A, A_1, A', D, L_A$ are concyclic, i.e. $L_A \in (AA'A_1)$. Similarly, $L_B \in \odot(BB'B_1)$ and $L_C \in (CC'C_1)$.

If H is orthocenter of $\triangle ABC$, we have $HA \cdot HA_1 = HB \cdot HB_1 = HC \cdot HC_1$ and since $LA \cdot LL_A = LB \cdot LL_B = LC \cdot LL_C$, then H and L have equal power WRT $\odot(AA'A_1), \odot(BB'B_1)$ and $\odot(CC'C_1)$ $\Longrightarrow HL$ is their common radical axis.

leader

#3 Jul 19, 2013, 3:59 am

> *Luis González wrote:*
> Let LA, LB, LC cut (O) again at L_A, L_B, L_C. AO cuts (O) again at P

M, H are midpoint BC and orthocenter of ABC. Clearly the composition of symmetry wrt the internal bisector of $\angle BAC$ and inversion with center A and square radius $AB * AC$ It takes A', L_A, A_1 to H, M, P respectively which are collinear so AA_1L_AA' is cyclic.

I can't believe i couldn't see that H was the other point on d so i proved that the centers of those circles are collinear which i also easy using Menelaus theorem on the medial triangle of ABC.

Nguyenhuyhoang

#4 Jul 19, 2013, 10:07 am

@Leader:Yes, that was tricky, at first I thought that OL was the radical axis.
My idea is pretty much the same as two above posts but to prove L_A, P, D are collinear, we can take the midpoint M of BC.
Since $PM//A_1A'$, we have $AMPD$ is cyclic, or we can take T is the intersection of AP and BC, and we have the identity $TM.TD = TO.TA' = TB.TC = TA.TP$.
As we have $AMPD$ is cyclic, we have $\widehat{AMB} = \widehat{ACL_A}$, and we now can easily prove that ABL_AC is harmonic, done!

This post has been edited 1 time. Last edited by Nguyenhuyhoang, Oct 24, 2013, 10:54 pm

hqdhftw

#5 Jul 19, 2013, 1:36 pm

Let the tangents of B, C intersect at K. It is obvious that $BCDL_A$, $L_ADA'K$ and $BCA'K$ are concyclic, hence their pairwise radical axises are $BC, DL_A, A'K$ must concurrent. Done.

Quick Reply

High School Olympiads

poles and polar concurrency

Source: myself

fmasroor
#1 Jul 17, 2013, 6:59 pm

Triangle ABC, incircle tangent to BC at D, CA at E, and AB at F. Let A' be the pole of EF wrt the circumcircle of ABC, similarly for B' and C'. Prove that AA', BB', and CC' are concurrent.

Luis González
#2 Jul 17, 2013, 8:18 pm

We only need $\triangle DEF$ as a cevian triangle.

Let d, e, f be the polars of D, E, F WRT the circumcircle of $\triangle ABC$, or even any conic through A, B, C. $A' \equiv e \cap f$, $B' \equiv f \cap d, C' \equiv d \cap E$. d cuts BC at D_0, such that $(B, C, D, D_0) = -1$. Similarly, e, f cut CA, AB at E_0, F_0, such that $(C, A, E, E_0) = -1$ and $(A, B, F, F_0) = -1 \Longrightarrow D_0, \underline{E_0, F_0}$ lie on the trilinear polar of $AD \cap BE \cap CF$ WRT $\triangle ABC \Longrightarrow \triangle ABC$ and $\triangle A'B'C'$ are perspective through $\overline{D_0E_0F_0}$. Hence, by Desargues theorem, AA', BB', CC' are concurrent.

fmasroor
#3 Jul 17, 2013, 10:27 pm

Sorry for all of these questions, but I have only shortly been exposed to poles and polars, etc. Can you explain what (B, C, D, D0)=-1 means? Harmonic division? What does it mean to be perspective through a line?

Luis González
#4 Jul 18, 2013, 9:46 pm • 1

In general, (B, C, D, D_0) stands for the cross ratio of the 4 points B, C, D, D_0, i.e. $(B, C, D, D_0) = \frac{DB}{DC} \cdot \frac{D_0C}{D_0B}$. These points are harmonically separeted when their cross ratio equals $(B, C, D, D_0) = -1$, i.e. $\frac{DB}{DC} = -\frac{D_0B}{D_0C}$. Now, given a fixed point P and a fixed circle ω, a line through P cuts ω at X, Y. The locus of the points Q, such that $(X, Y, P, Q) = -1$ is a line, the polar of P WRT ω.

Two triangles $\triangle ABC$ and $\triangle XYZ$ are said to be perspective through a line, if the intersections $BC \cap YZ, CA \cap ZX$ and $AB \cap XY$ are collinear (the line through these points is the perspectrix of ABC and XYZ). Likewise, two triangles $\triangle ABC$ and $\triangle XYZ$ are said to be perspective through a point, if AX, BY, CZ concur (the concurrency point is the perspector of ABC and XYZ). Desargues theorem states that: Two triangles are perspective through a point $\Longleftrightarrow$ they are perspective through a line.

High School Olympiads

Two interesting locus with distances

Source: KöMaL

mahanmath

#1 Jun 30, 2013, 4:32 am

a) D is an interior point of the acute-angled triangle ABC, such that
$AB.BC.CA = DA.AB.BD + DB.BC.CD + DC.CA.AD$

Find the locus of the points D.

b) E is an interior point of the acute-angled triangle ABC, such that $AB.BC.CA = AE^2.BC + BE^2.CA + CE^2.AB$

Find the locus of the points E.

Luis González

#2 Jul 16, 2013, 1:36 am

a) It is a single point, the orthocenter H of $\triangle ABC.$ See the topic Inequality.

b) It is a single point, the incenter I of $\triangle ABC.$ By Leibniz theorem for the points E and $I \equiv (a:b:c),$ we get

$a \cdot EA^2 + b \cdot EB^2 + c \cdot EC^2 = (a+b+c) \cdot EI^2 + a \cdot IA^2 + b \cdot IB^2 + c \cdot IC^2.$

But $a \cdot IA^2 + b \cdot IB^2 + c \cdot IC^2 = abc$ (well-known triangle identity) and by locus condition, the LHS equals $abc \Longrightarrow$
$EI = 0 \Longrightarrow E \equiv I.$

High School Olympiads

Nice perpendicular

Locked

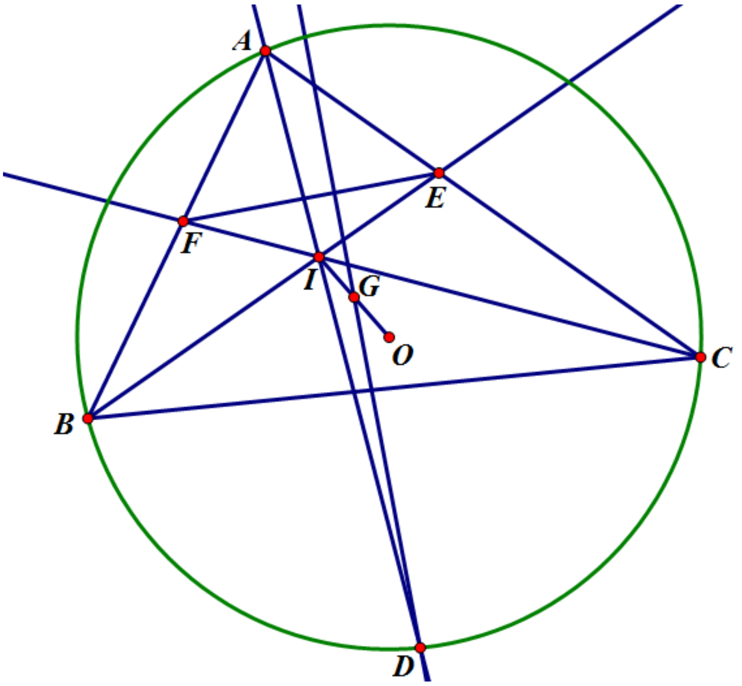
Source: A chinese forum

CQT_95

#1 Jul 15, 2013, 2:14 pm

Dear mathlinkers
1. Let ABC be a triangle and I, O are incenter,circumcenter of triangle,respectively.
2. $BI \cap AC \equiv E, CI \cap AB \equiv F$.
3. G is midpoint of OI.
Prove that $DG \perp EF$ with $D \equiv AI\cap$ circumcircle of triangle ABC.
Best regards.
♥ v`♥

Attachments:



Luis González

#2 Jul 15, 2013, 9:26 pm

It's amazing how this problem an its variations are posted over and over again. Just note that the A-excenter is the reflection of I on D.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=511134
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=509916
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=452634
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=434837
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=400448
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=299372

High School Olympiads

Concyclic about mixtlinear incircle

Reply

Source: Russian problem book

Spats

#1 Jul 15, 2013, 3:59 pm

Let ABC be a triangle and let M be the point of contact of the $A-$ mixtlinear incircle with the circumcircle of ABC. Also, let N be a point on the arc BC of the circumcircle pf ABC which does not contain A. Let U,V be the points on the side BC such that NU and NV are tangent to the incircle of ABC.
Show that the points M,N,U,V are concyclic.

Luis González

#2 Jul 15, 2013, 8:47 pm

Posted at least 3 times before

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=501116
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=384385
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=186117

Quick Reply

High School Olympiads

four points lie on a circle

Reply

Source: China south east mathematical olympiad 2008 day1 problem 3

jred

#1 Jul 14, 2013, 9:22 pm

In $\triangle ABC$, side $BC > AB$. Point D lies on side AC such that $\angle ABD = \angle CBD$. Points Q, P lie on line BD such that $AQ \perp BD$ and $CP \perp BD$. M, E are the midpoints of side AC and BC respectively. Circle O is the circumcircle of $\triangle PQM$ intersecting side AC at H. Prove that O, H, E, M lie on a circle.

Luis González

#2 Jul 14, 2013, 10:03 pm • 1

From the right $\triangle BPC$, we have $\angle PEC = 2\angle PBC = \angle ABC \Longrightarrow PE \parallel AB \Longrightarrow PE$ is C-midline of $\triangle ABC \Longrightarrow M \in PE$. Similarly Q is on the line passing through midpoints M, F of AC, AB. $\angle QHM = \angle QPM = \angle ABD \Longrightarrow BAHQ$ is cyclic, i.e. $\angle BHA = \angle BQA = 90^\circ$, i.e. $\odot(HEM)$ is 9-point circle (N) of $\triangle ABC$. O is then intersection of the perpendicular bisector of $\overline{EF}$ and the perpendicular bisector of $\overline{PQ}$, which bisects $\angle FME$ externally, because $\triangle QMP$ is obviously M-isosceles $\Longrightarrow O$ is midpoint of the arc HM of (N).

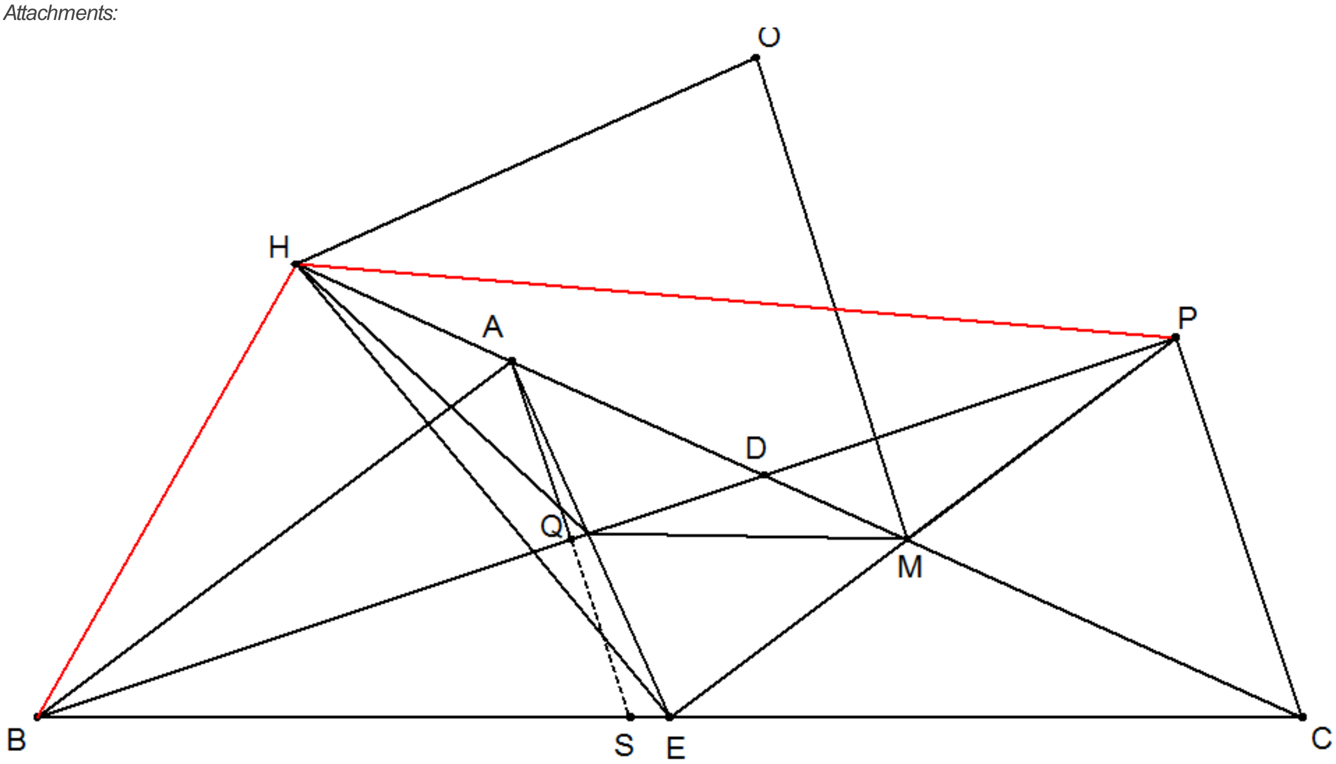
acupofmath

#3 Aug 16, 2013, 9:40 pm

first it's obvious $\triangle AQB = \triangle BQS$ so $QM \| BC$ (thales) now like this we have $EP \| AB$
let $\angle ABC = 2\alpha$ and $\angle ACB = \gamma$
now we can write : $\angle PHM = \angle PQM = \angle PBC \Rightarrow \angle PBC = \angle PHC \Rightarrow$
$PHBC$ is cyclic.
$\Rightarrow \angle CHB = \angle CPB = 90^0 \Rightarrow \angle CBH = 90^0 - \gamma$
but in a right triangle if AM is the median of $BC(\angle A = 90^0)$ then $2AM = BC$
so : $HE = EB \Rightarrow \angle BEH = 180^0 - (90^0 - \gamma + 90^0 - \gamma)$
$\Rightarrow \angle BEH = 2\gamma (*)$
but $PEC = 2\alpha (**)$
and $\angle HOM = 2\widehat{QH} + 2\widehat{QM} = 2(\angle MHQ + \angle HMQ) = 2(\gamma + \angle QPM)$
$= 2(\gamma + \angle PBA) = 2\alpha + 2\gamma \Rightarrow \angle HOM = 2\alpha + 2\gamma (***)$

$(*), (**), (***) \Rightarrow \angle MEH + \angle MOH = 180^0 \Rightarrow$ it's done!!

Attachments:



Quick Reply

High School Olympiads

Finding Locus

Reply

77ant

#1 Jan 25, 2010, 7:06 pm

Dear everyone.

For the following problem which I got from my little and young angel, I got only one and very long solution 3 days ago.
But I long for beautiful, purely geometric and short solutions. May I ask you a favor? I will wait for your advice.
And I wonder if this problem is well known. Thank you for reading my post.

Please read the following.

Draw two parallel lines (parallel to AB, CA respectively) through an arbitrary point P on BC and let the intersection points be D, E respectively.
For the midpoint M of BC, let the line parallel to AM (passing through P) meet DE at K. Find the locus of the point K. :)

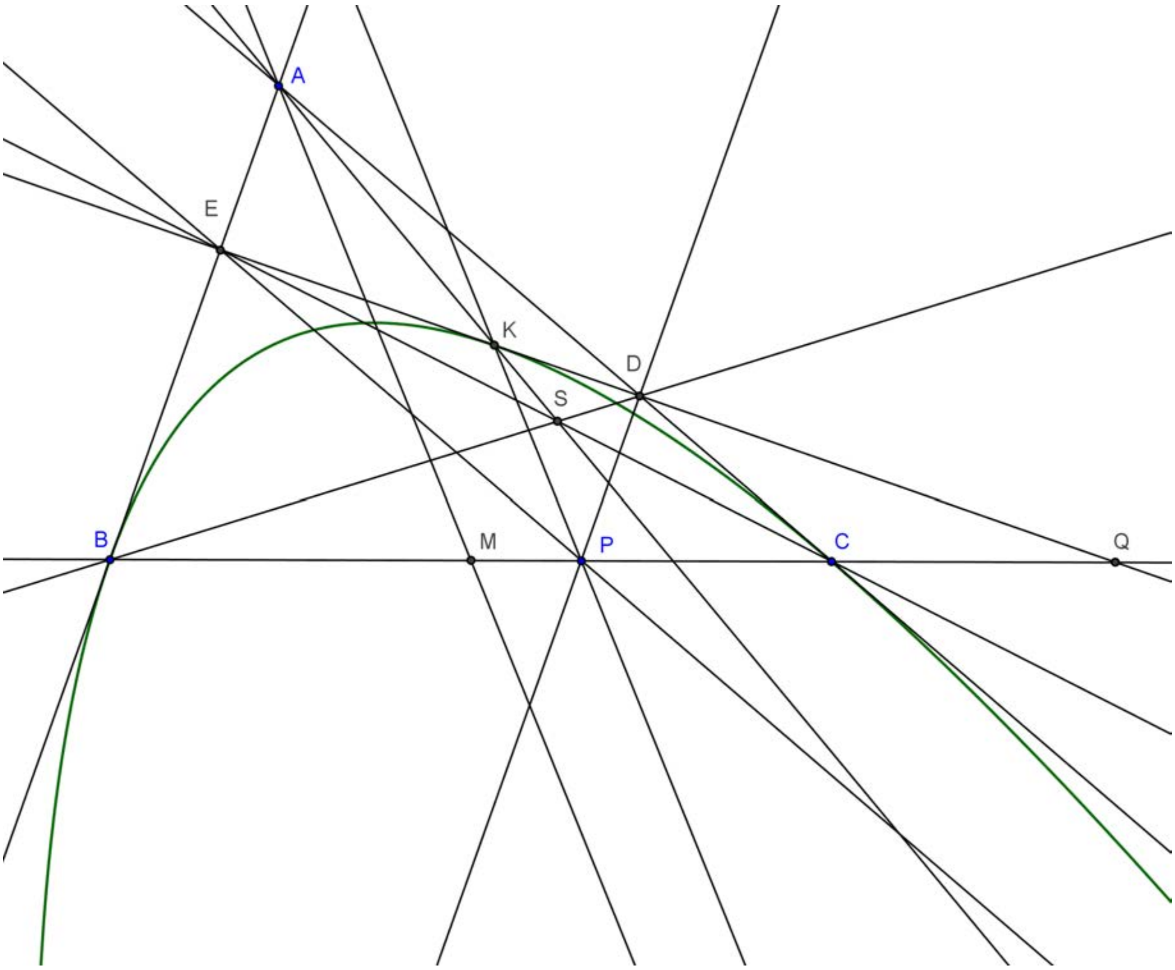
Luis González

#2 Jul 14, 2013, 3:03 am • 1

Since $\frac{BE}{EA}=\frac{BP}{PC}=\frac{AD}{DC}$, then the series E and D with base lines BA, AC are projective and clearly not perspective $\Longrightarrow$ lines DE envelope a projective conic $\mathcal{P}$ tangent to AB, AC at B, C (images of the common point A in the respective series). Since these series are clearly similar, then $\mathcal{P}$ is a parabola (the A- Artzt parabola of ABC).

Let K^* be the tangency point of DE with $\mathcal{P}$. By degenerate Brianchon theorem, the lines AK^*, BD, CE concur at S, thus the pencil $A(B,C,S,Q)$ is harmonic, where $Q \equiv BC \cap DE \Longrightarrow (E,D,K^*,Q)=-1 \Longrightarrow P(E,D,K^*,C)=-1 \Longrightarrow$ pencil formed by parallels through A to PE, PD, PK^*, BC is also harmonic $\Longrightarrow PK^* \parallel AM \Longrightarrow K \equiv K^* \Longrightarrow K$ runs on $\mathcal{P}$.

Attachments:



Quick Reply

High School Olympiads

Find Locus

Reply

Headhunter

#1 Aug 12, 2010, 1:09 am

Hello.

For a conic with its center C, let CP and CQ be its arbitrary conjugate diameters.
From an any point O (fixed), draw the perpendicular m to CP and let the intersection of m and CQ be R.
Find the locus of R.

Luis González

#2 Jul 13, 2013, 9:59 am

Let P_{∞} and Q_{∞} be the intersections of CP, CQ with the line at infinity. CP_{∞} is the polar of Q_{∞} WRT the conic $\mathcal{C}$ and CQ_{∞} is the polar of P_{∞} WRT $\mathcal{C},$ hence $P_{\infty} \mapsto Q_{\infty}$ is nothing but an involution on the line at infinity $\Longrightarrow CP \mapsto CQ$ is an involution. Hence, if M, N are the projections of O on CP, CQ (2nd intersections of CP,CQ with the circle ω with diameter OC), then $CM \mapsto CN$ is an involution on ω defining a fixed pole $K,$ i.e. all lines MN pass through a fixed point $K.$ Therefore, $M \mapsto N$ is a projectivity $\Longrightarrow$ pencils OM and CN are projective $\Longrightarrow$ intersection R of homologous rays OM, CN runs on a conic $\mathcal{H}$ through O, C which is tangent to KO, KC (images of the double rays OC,CO).

When M coincides with the second intersections of ω with the axes u, v of $\mathcal{C},$ then R goes to the intersections of u, v with the line at infinity. Those directions are orthogonal, so $\mathcal{H}$ is an equilateral hyperbola with asymptotes parallel to the axes of $\mathcal{C}.$

Quick Reply

High School Olympiads

Many circles

Reply

Source: Own

JuanOrtiz

#1 Jun 30, 2013, 4:56 am

Say Γ is a circle such that AB and AC are tangent to it, with $B, C \in \Gamma$. Take an arbitrary point P on the minor arc BC of Γ. Let $BP \cap AC = X, CP \cap AB = Y$. Take Ω the circumcircle of PYX, let $\Omega \cap \Gamma = \{Z, P\}$ and $\Omega \cap AP = D$.

Prove that: the circumcircles of BYZ, CXZ and ADZ are coaxial.

Potla

#2 Jun 30, 2013, 7:28 pm

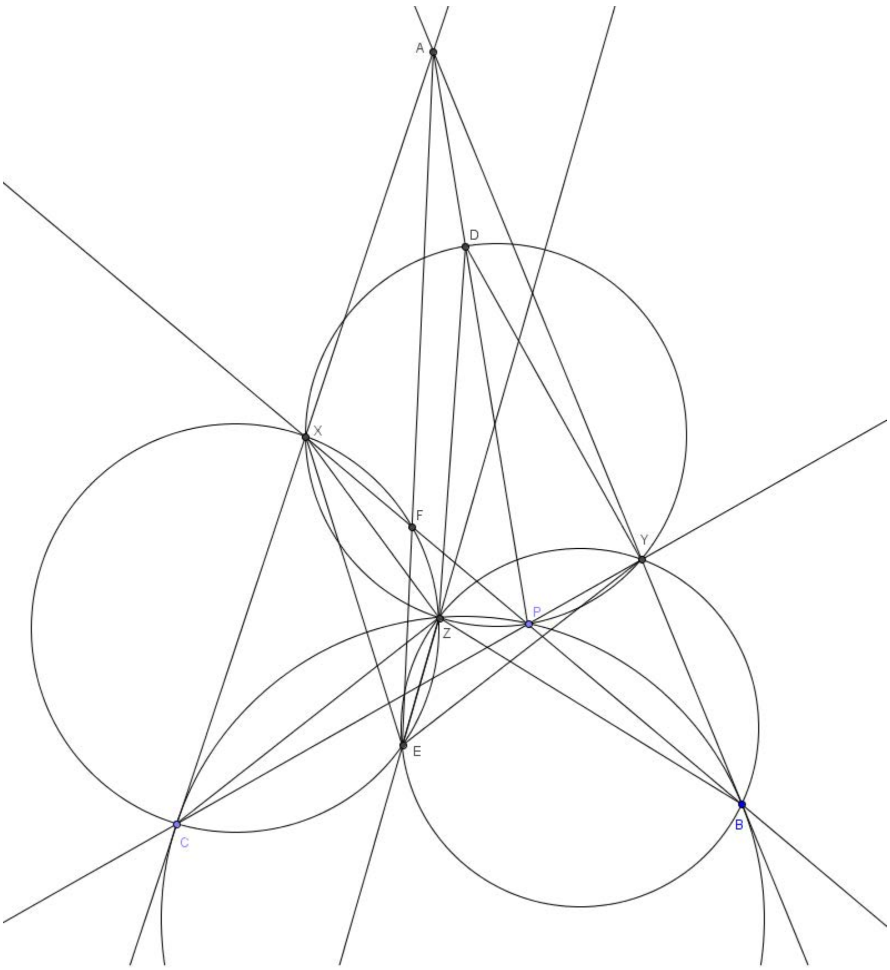
Thanks to IDMasterz for the diagram.

Incomplete Solution

Edit:

In the diagram, we let $F = EA \cap BX$, but I have not shown that $F \in \odot(CEX)$; I missed it out totally and posted on impulse. I will add a solution as soon as I can.

Attachments:



oneplusone

#3 Jun 30, 2013, 7:38 pm • 2

Darn my solution is so long your post appeared before I can finish typing.
But in your solution how do you know point F exists?

My solution:

Let E be the Miquel point of $AYBPXC$, i.e. the intersection of the circumcircles of triangles BYP, CXP, BAX, CAY. Let X', Y' be reflections of X, Y about AO, where O is the circumcenter of Γ. Let the circumcircles of $\triangle CZX, \triangle BZY$ intersect at F. Note that $BYPE, CXPE, YPZX, YZFB, XZFC$ are cyclic, so by Miquel's 6 circle theorem, $YEFX$ is cyclic ($\angle YEX = \angle YEP + \angle PEX = \angle YBP + \angle PCX = \angle YBZ + \angle ZCX = \angle YFX$). At the same time $\angle YEX = \angle YBP + \angle PCX = \angle ABC = \angle AX'X = \angle YY'Y$. So we have $XX'YEFY'$ cyclic, and it's circumcircle is symmetric about AO.
$\angle BEP + \angle PEC = \angle PYA + \angle PXA = \angle ZYA + \angle ZXA = \angle ZFB + \angle ZFC$, so we have $BEFC$ is cyclic. Therefore E, F are symmetric about AO (or $E \equiv F$ but that is clearly not true). So now we have $\angle PDZ = \angle PXZ = \angle AXZ - \angle AXB = \angle ZFC - \angle AEB = \angle ZFC - \angle AFC = \angle ZFA$ therefore $ADZF$ is cyclic and we are done.

JuanOrtiz

#4 Jun 30, 2013, 8:44 pm

Here is a solution I found. It definitely is uglier than the ones posted above, but it works. I renamed the points and the circles from the problem because I got confused.

Problem

Solution

edit

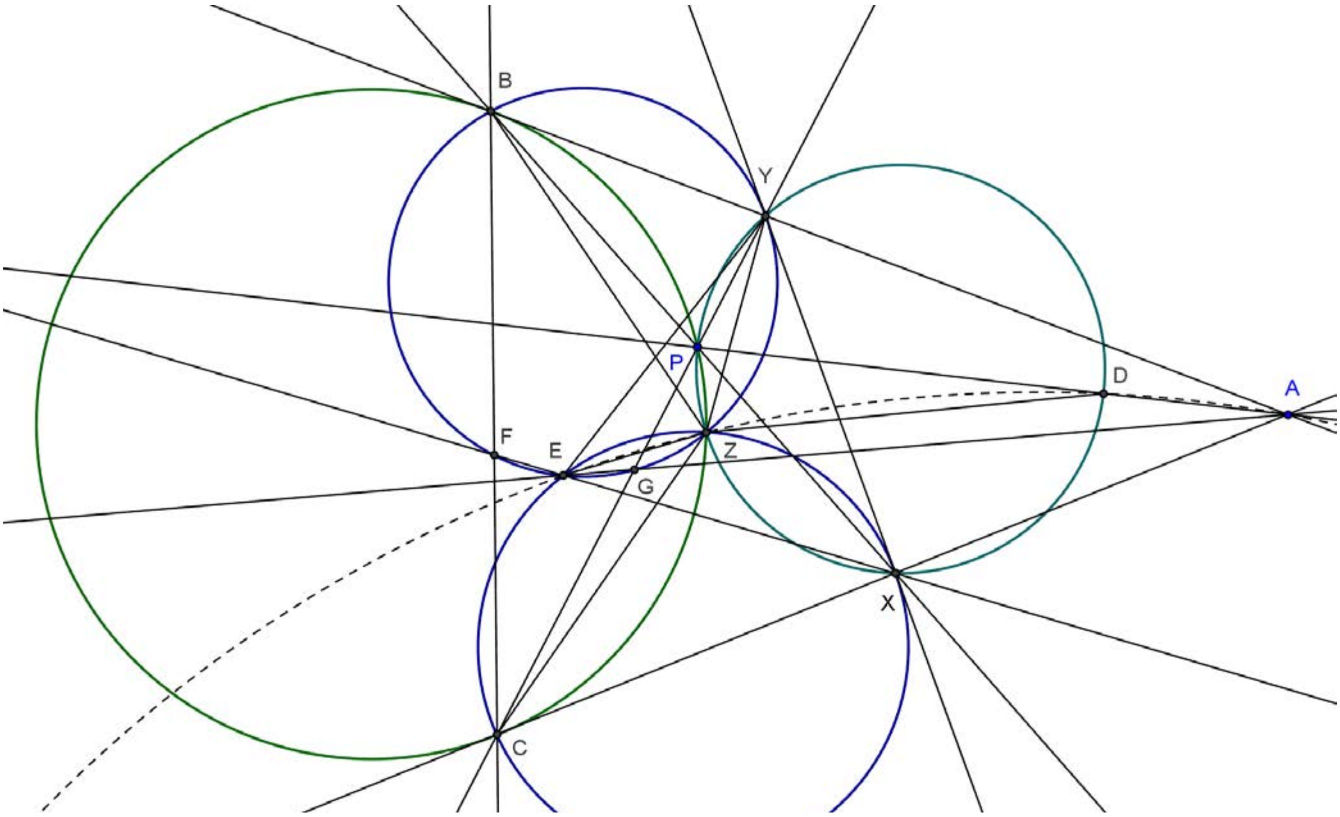
Luis González

#5 Jul 13, 2013, 5:56 am • 2

Let $\{E, Z\} \equiv \odot(CXZ) \cap \odot(BYZ)$. $\angle ZYX = \angle ZPX = \angle BCZ = \angle YBZ \Longrightarrow XY$ is tangent to $\odot(BYZ)$. Similarly, XY is tangent to $\odot(CXZ)$. Hence $\angle YEZ = \angle XYZ, \angle XEZ = \angle YXZ$. If $F \equiv XE \cap BC$, we have $\angle FEY = \pi - \angle XYZ - \angle YXZ = \angle YZX = \angle YPX = \angle BPC = \angle ABC \Longrightarrow F \in \odot(BYZ)$.

Let $G \equiv YC \cap AE$. The intersections of opposite sidelines of the hexagon $BYYGEF$ are collinear, namely, $X \equiv YY \cap EF, C \equiv YG \cap FB$ and $A \equiv GE \cap BY$. By the converse of Pascal theorem, we have $G \in \odot(BYZ)$. Hence $\angle ZEA \equiv \angle ZEG = \angle ZYP = \angle ZDP \Longrightarrow A, E, Z, D$ are concyclic.

Attachments:



Quick Reply

High School Olympiads

right-triangle problem

Reply

Source: China south east mathematical olympiad 2006 day1 problem 2

jred

#1 Jul 4, 2013, 8:10 pm

In $\triangle ABC$, $\angle ABC = 90^\circ$. Points D, G lie on side AC. Points E, F lie on segment BD, such that $AE \perp BD$ and $GF \perp BD$. Show that if $BE = EF$, then $\angle ABG = \angle DFC$.

This post has been edited 1 time. Last edited by Amir Hossein, Jul 11, 2013, 9:33 pm
Reason: Fixed LaTeX, thanks War-Hammer.

Luis González

#2 Jul 13, 2013, 12:36 am

Let GF cut BC at P. Circle (K) with diameter $\overline{BP}$ is tangent to AB at B, hence reflection AF of AB on AE is tangent to (K) through F. Let Q be the 2nd intersection of BG with (K). By Pascal theorem for $BBQFFP$, the intersections $A \equiv BB \cap FF, G \equiv BQ \cap FP$ and $QF \cap PB$ are collinear $\Longrightarrow C \in QF \Longrightarrow \angle DFC \equiv \angle BFQ = \angle ABG$.

TelvCohl

#3 Jan 15, 2015, 9:17 am • 1

My solution:

Let $X \in AB$ such that $CX \parallel GB$ and $Y = AE \cap BG, M = AE \cap CX$.

From $\frac{XM}{MC} = \frac{BY}{YG} = \frac{BE}{EF} = 1 \Longrightarrow MF = MB = MC = MX$

$\Longrightarrow B, C, F, X$ are concyclic $\Longrightarrow \angle CFD = \angle CXB = \angle GBA$.

Q.E.D

Quick Reply

High School Olympiads

right-angle triangle ABC

Reply

Source: China south east mathematical Olympiad 2007 problem 6

jred
#1 Jul 12, 2013, 9:37 am

In right-angle triangle ABC, $\angle C = 90^\circ$, Point D is the midpoint of side AB. Points M and C lie on the same side of AB such that $MB \perp AB$, line MD intersects side AC at N, line MC intersects side AB at E. Show that $\angle DBN = \angle BCE$.

Luis González
#2 Jul 12, 2013, 10:35 am

Let $F \equiv AC \cap MB.$ Circle (O) with diameter $\overline{BF}$ is obviously orthogonal to the circumcircle of $\triangle ABC \Longrightarrow DB$ and DC are tangents of $(O).$ Let G be the second intersection of CM with (O) and $H \equiv FG \cap BC.$ From the complete cyclic quadrilateral $CFGB,$ it follows that $M, FC \cap BG$ and D are on the polar of H WRT $(O) \Longrightarrow N \equiv FC \cap BG \Longrightarrow \angle DBN \equiv \angle EBG = \angle BCE.$

sunken rock
#3 Jul 12, 2013, 5:34 pm

Let the parallel through N to AB intersect CE, BM, BC at P, R, Q respectively and call $S \in BN \cap CE$.
Since D is midpoint of AB and $NP \parallel AB$ it follows that the pencil $N(A, D, B, P)$ is harmonic, and so are $B(C, M, S, P)$ and $B(Q, R, N, P)$ $(\ 1\ )$.
Obviously, F is the orthocenter of ΔBNQ, and from (1) we infer $Q - F - S$ collinear, hence $FS \perp BN$ and $CBSF$ is cyclic, done.

Best regards,
sunken rock

Quick Reply

High School Olympiads

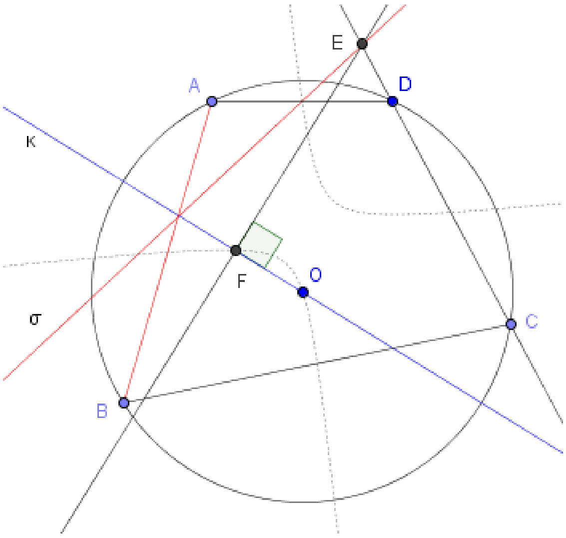
locus problem

77ant

#1 Sep 14, 2010, 4:24 am

Dear everyone
For a cyclic quadrilateral $ABCD$ with its circumcircle (O), let κ be an arbitrary line through O.
Reflect AB in κ, and call it σ. Let σ meet CD at E. Draw the perpendicular line from E to κ, and let it meet κ at F.
What is the locus of F ? I guess it's an equilateral hyperbola. Is this locus well known ? Thanks.

Attachments:



Luis González

#2 Jun 27, 2013, 12:38 am

Let $P \equiv \kappa \cap AB, M \equiv EF \cap AD, N \equiv EF \cap BC$ and $G \equiv EF \cap AB.$ Since PF bisects $\angle EPG$ and $PF \perp EG,$ then $\triangle PEG$ is P-isosceles with symmetry axis $\kappa \Longrightarrow F$ is midpoint of $\overline{EG},$ thus by Butterfly theorem F is also midpoint of $\overline{MN}.$ If A',B',C' are the reflections of A,B,C about $F,$ then M,B',C' lie on the reflection of BC on $F.$ Hence, intersections of the opposite sidelines of the hexagon $ADCC'B'A'$ are collinear, namely $M \equiv AD \cap C'B',$ $E \equiv DC \cap B'A'$ and $F \equiv CC' \cap A'A.$ Thus, by the converse of Pascal theorem A,D,C,A',B',C' lie on a same conic $\mathcal{K}$ whose center is $F \Longrightarrow$ reflection B of B' about F is on $\mathcal{K} \Longrightarrow \mathcal{K}$ forms a pencil through $A,B,C,D.$ Locus of its center F is then the 9-point conic $\mathcal{H}$ of $ABCD.$ For a proof see About Nine-point Conic (post #7 for a synthetic proof).

$\mathcal{H}$ goes through O ((O) is a conic of the pencil) and $X \equiv AB \cap CD, Y \equiv BC \cap DA, Z \equiv AC \cap BD.$ Since O is orthocenter of $\triangle XYZ$ (Brokart theorem), then it follows that $\mathcal{H}$ is an equilateral hyperbola.

High School Olympiads

area of triangle

Reply

aktyw19

#1 Jun 24, 2013, 11:09 pm

Let ABC be an acute triangle with all its angles greater than 45^o. Let DEF be the orthic triangle of ABC, and let MNP be the orthic triangle of DEF. Prove that $S(DEF)^2 \geq S(ABC) \cdot S(MNP)$, where $S(XYZ)$ denotes the area of triangle XYZ.

Luis González

#2 Jun 25, 2013, 7:44 am

Let O,G,H,I be the circumcenter, centroid, orthocenter and incenter of $\triangle ABC. N,T$ are circumcenter and orthocenter of $\triangle DEF.$ Incircle (H) of $\triangle DEF$ touches EF,FD,DE at $X,Y,Z.$ Since $\triangle XYZ$ is homothetic to the excentral $\triangle ABC$ of $\triangle DEF,$ then by Gergonne-Ann theorem we have $[DEF]^2=[XYZ] \cdot [ABC],$ hence our inequality is equivalent to $[XYZ] \ge [MNP].$

$\triangle XYZ$ and $\triangle MNP$ are pedal triangles of H and T WRT $\triangle DEF,$ thus by Euler theorem, $[XYZ] \ge [MNP] \Longleftrightarrow NT \ge NH.$ So we need to prove that in any $\triangle ABC,$ we have $OH \ge OI,$ which follows from the fact the incenter is on the disk with diameter $\overline{GH}$ (see GIH>90 and elsewhere).

Quick Reply

High School Olympiads

Parabola

Reply

a00012025
#1 May 9, 2013, 3:45 pm

Let $\Gamma : y^2 = 4cx$, $A(c, 2c)$, $P(5c, -2c)$ .A line passing through P intersects Γ at B and C , prove that $\angle BAC = 90°$.

Dr Sonnhard Graubner
#2 May 9, 2013, 4:48 pm

hello, after some calculations we obtain
$\overline{BA}^2 + \overline{AC}^2 = \frac{16c^2(5m^2 + 2m + 1)(1 + m^2)}{m^4} = \overline{BC}^2$
Sonnhard.

pvskand
#3 Jun 6, 2013, 7:37 pm

Without any loss of generality we can consider the parabola to be $y^2 = 4x$
Then the points are $A(1, 2), P(5, -2)$
Let the line passing through P be L and let L intersect the Γ at $B(a^2, 2a)$ and $C(b^2, 2b)$
Now on finding $BA^2 + AC^2$ we get
$= a^4 + b^4 + 2(a^2 + b^2) - 8(a + b) + 10$ ....(1)
Now we use the condition that
Slope of PB=Slope of PC we get
$a + b + ab = -5$
$\Rightarrow a + b + = -5 - ab$..$(2)$
Now substituting $a + b = -5 - ab$ in (1) we have
$a^4 + b^4 + 4(a^2 + b^2) + 8ab + 50$ ....(3)
On squaring equation (2)
$a^2 + b^2 - 8ab = 25 + (ab)^2$
$\Rightarrow a^2 + b^2 - 8ab - (ab)^2 = 25$
$\Rightarrow 2a^2 + 2b^2 - 16ab - 2(ab)^2 = 50$...................................(4)
Substituting (4) in (3) we get
$a^4 + b^4 + 4(a^2 + b^2) - 2(ab)^2 - 8ab = BC^2$
Hence $\triangle BAC$ is a right angled triangle at A.

$Q.E.D$

yetti
#4 Jun 24, 2013, 10:10 pm

Let $A = [ka^2,\ a],\ B = [kb^2,\ b],\ C = [kd^2,\ d]$ be 3 points on parabola $x = ky^2$. Slopes of lines BC, CA, AB are then
$m = \frac{1}{k(b+d)},\ n = \frac{1}{k(d+a)},\ p = \frac{1}{k(a+b)}$.
Assuming $CA \perp AB \implies \frac{1}{k(d+a)} = n = -\frac{1}{p} = -k(a+b) \implies d = -a - \frac{1}{k^2(a+b)}$ and
$m = \frac{k(b+a)}{k^2(b^2-a^2)-1}$. Equation of line BC is then $y - b = m(x - kb^2)$ or

$$(1) \quad F(x, y, b) = (k^2b^2 - kx)(y + a) - (k^2a^2 + 1 - kx)(y - b) = 0.$$

Assuming $A = [ka^2,\ a]$ is fixed, this is equation of a family of lines depending on parameter b. Equation of their envelope (if any) is obtained by eliminating this parameter from equations $F(x, y, b) = 0$ and any) is obtained by eliminating this parameter from equations $F(x, y, b) = 0$ and

$$(2) \quad \frac{\partial F(x, y, b)}{\partial b} = 2bk^2(y + a) + (k^2a^2 + 1 - kx) = 0.$$

Substituting (2) into $(1) \implies (x - 2by + kb^2)(y + a) = 0$. Since $x - 2by + kb^2 = 0$ is equation of parabola tangent at $B = [kb^2,\ b]$ (not part of any envelope of parabola secants BC) $\implies y + a = 0$.
Substituting this into $(1) \implies$ the envelope consists of one fixed point $P = \left[\frac{k^2a^2+1}{k},\ -a\right]$. Using the given values $k = \frac{1}{4c}$ and $a = 2c \implies$ this fixed point is $P = [5c,\ -2c]$.

This post has been edited 1 time. Last edited by yetti, Mar 11, 2015, 11:43 am

Luis González

#5 Jun 24, 2013, 11:31 pm • 1

If B, C vary on Γ, such that $\angle BAC = 90^\circ$, then $AB \mapsto AC$ is an involutive homography defining a fixed point P on the normal τ to Γ through A. This is true for any conic (see Line passes through fixed point on normal).

Let A' be the reflection of A on focal axis ℓ. Q is the projection of A on ℓ. When $B \equiv A'$, then C is at infinity $\Longrightarrow PA' \parallel \ell$. If τ cuts ℓ at S, then $|QS| = 2p$ is subnormal length of Γ (distance from focus to directrix). Since QS is A-midline of $\triangle AA'P$, then $PA' = 2 \cdot SQ = 4p$. So if the figure is defined in a rectangular system, such that $\Gamma : y^2 = 4p \cdot x$ and $A \equiv (x_0, y_0) \in \Gamma$, then $P \equiv (4p + x_0, -y_0)$.

Quick Reply

High School Olympiads

Jacobi's Theorem

Locked

trident

#1 Jun 24, 2013, 8:25 am

Can someone provide a (preferably synthetic) proof for Jacobi's Theorem?

The theorem states that in $\triangle ABC$ with points X, Y, Z in the plane, if $\angle ZAB = \angle YAC$, $\angle ZBA = \angle XBC$, and $\angle XCB = \angle YCA$, then AX, BY, and CZ are concurrent.

Luis González

#2 Jun 24, 2013, 8:49 am

Use the search. Several proofs have been posted in the forum.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=30114
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=50&t=154396
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=404816
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=50&t=335519

IDMasterz

#3 Jun 24, 2013, 4:14 pm

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=531740

jayme

#4 Jun 25, 2013, 2:54 pm

Dear Mathlinkers,
also you can see a complete synthetic proof and a lot of applications

http://perso.orange.fr/jl.ayme vol. 5 Le theoreme de Jacobi

Sincerely
Jean-louis

High School Olympiads

prove angles equal

Source: sharygin geometry olympiad

kgpian
#1 Jun 23, 2013, 11:56 pm

A circle with center I touches sides AB,BC,CA of triangle ABC in points C',A',B' . Lines AI,CI,B'I meet A'C' in points X,Y,Z respectively. Prove that <YB'Z=<XB'Z.
BTW can anybody post or give a source of some nice angle chasing problems.Thanks in advance

Luis González
#2 Jun 24, 2013, 8:11 am

Proof 1: Let $P \equiv A'C' \cap AC$. Since AA', BB', CC' concur, then $I(A, C, B', P)$ is harmonic $\Longrightarrow (X, Y, Z, P)$ is harmonic. Together with $B'Z \perp B'P$, we deduce that $B'Z, B'P$ bisect $\angle XB'Y$, or $\angle YB'Z = \angle XB'Z$.

Proof 2 (angle chase): $\angle CA'X \equiv \angle BA'C' = 90^\circ - \frac{1}{2}\angle ABC = \angle CIX \Longrightarrow ICXA'$ is cyclic $\Longrightarrow$ $\angle IXC = \angle IA'C = 90^\circ$. Similarly, $\angle IYA = 90^\circ \Longrightarrow X, Y$ lie on the circle with diameter $\overline{AC}$. Now, from cyclic $ACXY$, $AYIB', CXIB'$, we get $\angle YB'Z = \angle YAX = \angle YCX = \angle XB'Z$.

High School Olympiads

MMC 2012 problem 4

Math-lover123

#1 Jun 23, 2013, 9:51 pm

Let O be the circumcenter, R be the circumradius, and k be the circumcircle of a triangle ABC .
Let k_1 be a circle tangent to the rays AB and AC, and also internally tangent to k.
Let k_2 be a circle tangent to the rays AB and AC , and also externally tangent to k. Let A_1 and A_2 denote the respective centers of k_1 and k_2.
Prove that:
$(OA_1+OA_2)^2-A_1A_2^2=4R^2.$

Luis González

#2 Jun 24, 2013, 12:52 am • 1

Let (I,r) and (I_a,r_a) denote the incircle and A-excircle of $\triangle ABC.$ Let ϱ_1 and ϱ_2 denote the radii of $k_1,k_2.$ If k_1,k_2 touch AC at $D,E,$ then it's well-known that $ID \perp AI$ and $I_aE \perp AI_a$ $(\star).$ Substituting $OA_1=R-\varrho_1, OA_2=R+\varrho_2$ and $A_1A_2{}^2=DE^2+(\varrho_2-\varrho_1)^2$ into the required expression gives

$$4R^2+(\varrho_2-\varrho_1)^2+4R(\varrho_2-\varrho_1)-DE^2-(\varrho_2-\varrho_1)^2=4R^2 \Longrightarrow$$

$$DE^2=4R\cdot(\varrho_2-\varrho_1).$$

But $(\star)$ yields $\varrho_1=r\cdot \sec^2 \frac{A}{2}\ ,\ \varrho_2=r_a\cdot \sec^2 \frac{A}{2}\ ,\ DE=II_a\cdot \sec \frac{A}{2}$

$$\Longrightarrow II_a{}^2\cdot \sec^2 \frac{A}{2}=4R\cdot \sec^2 \frac{A}{2}(r_a-r) \Longrightarrow II_a{}^2=4R\cdot(r_a-r),$$

which is a known triangle identity. Hence the proposed relation is proved.

Quick Reply

High School Olympiads

intersection on the Euler's line

Reply

Source: own

jayme

#1 Jun 22, 2013, 8:50 pm

Dear Mathlinkers,

1. ABC a triangle
2. (E) the Euler's line of ABC
3. P a point on (E)
4. A'B'C' the P-cevian triangle of ABC
5. A"B"C" the median triangle of ABC
6. A* the point of intersection of B'C' and B"C"
7. U the point of intersection of CA* and AA'

Prove that C"U and BC meet on (E)

Sincerely
Jean-Louis

Luis González

#2 Jun 23, 2013, 5:15 am

Generalization: P, Q are two arbitrary points on the plane of $\triangle ABC$. BP, CP cut CA, AB at P_2, P_3 and BQ, CQ cut CA, AB at Q_2, Q_3. $A^* \equiv P_2P_3 \cap Q_2Q_3$ and $U \equiv CA^* \cap PA$. Then PQ, BC, UQ_3 concur.

Let $\mathcal{K}$ be the conic through A, B, C, P, Q. If P_1, Q_1 are the intersections of AP, AQ with BC, then P_1 and Q_1 are the poles of P_2P_3 and Q_2Q_3 WRT $\mathcal{K} \Longrightarrow A^* \equiv P_2P_3 \cap Q_2Q_3$ is the pole of $P_1Q_1 \equiv BC$ WRT $\mathcal{K} \Longrightarrow CA^*$ is tangent of $\mathcal{K}$. Now, by Pascal theorem for the degenerate hexagon $PQCCBA$, the intersections $PQ \cap BC$, $Q_3 \equiv QC \cap BA$ and $U \equiv A^*C \cap AP$ are collinear, in other words, PQ, BC, UQ_3 concur, as desired.

Quick Reply

High School Olympiads

Orthic Triangle Collinearity

Reply

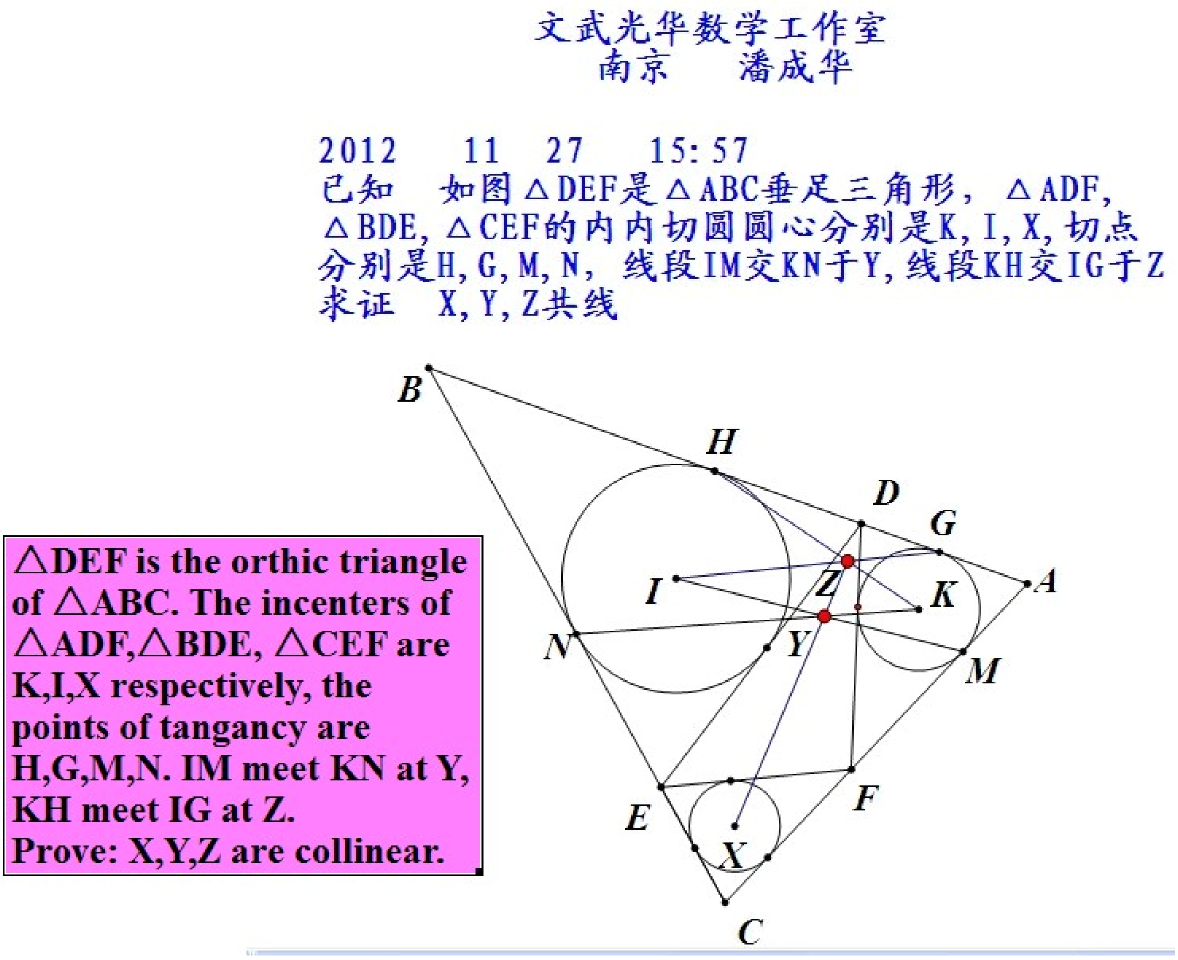
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 5, 2012, 11:58 am

See Attachment.
This is a problem proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



yetti

#2 Jun 19, 2013, 1:15 pm

Problem: $\triangle DEF$ is the orthic triangle of $\triangle ABC$, where $D \in AB, E \in BC, F \in CA$. The incenters of $\triangle AFD, \triangle BDE, \triangle CEF$ are K, I, X, respectively.
The points of tangency of (K) are $G \in AB, M \in AC$ and the points of tangency of (I) are $H \in AB, N \in BC$. IM meets KN at Y and KH meets IG at Z.
Prove: X, Y, Z are collinear.

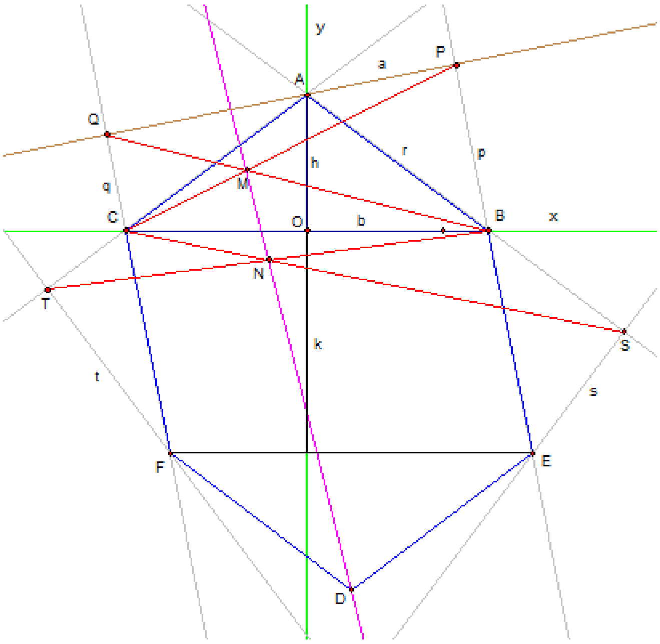
Proof: Let $p = \frac{1}{2}(AB + BC + CA)$ be semiperimeter of reference $\triangle ABC$. Let KG, KM cut AB, AC at U, W. From similarity $\triangle AMG \sim \triangle ABC$ with similarity coefficient $\cos \widehat{A} \implies$
$AU = \dfrac{AM}{\cos \widehat{A}} = p - BC$ and $AW = \dfrac{AG}{\cos \widehat{A}} = p - BC$. Likewise, let IH, IN cut AB, BC at $U, V \implies$
$BU = BV = p - AC$. It follows that $\triangle UVW$ is contact triangle of reference $\triangle ABC$.

its incircle (J,r) touching AB,BC,CA at U,V,W and that K,I,X are orthocenters of isosceles $\triangle AWU, \triangle BUV, \triangle CVW$, respectively. See also **here**. Let (O_A,R_A) be circumcircle of isosceles $\triangle AWU \implies$ $J \in (O_A)$ and $WK = UK = 2R_A \sin \frac{\widehat{A}}{2} = r = WJ = UJ \implies JWKU$ is rhombus. Likewise, $JUIV, JVXW$ are rhombi $\implies K,I,X$ are reflections of J in WU, UV, VW, respectively,
and $KUIVXW$ is equilateral hexagon with parallel opposite sides. By the following lemma, points X and $Y \equiv IM \cap KN, Z \equiv KH \cap IG$ are collinear.

Lemma: Let $CABEDF$ be equilateral hexagon with parallel opposite sides. Let $a \perp (EB \parallel FC)$ be line through A and let $P \equiv a \cap EB, Q \equiv a \cap FC$.
Let S,T be feet of perpendiculars from E,F to AB,AC, respectively. Let $M \equiv CP \cap BQ$ and $N \equiv CS \cap BT$. Points M,N,D are then collinear.

[i]Coordinate proof of the lemma:[/i]

Attachments:



Luis González

#3 Jun 22, 2013, 3:31 am

Let the incircle (J) touch BC,CA,AB at A_0,B_0,C_0. K,I,X are the orthocenters of $\triangle AB_0C_0, \triangle BC_0A_0, \triangle CA_0B_0$ (see P1 at tangent and orthocenter), i.e. K,I,X are reflections of J on $B_0C_0, C_0A_0, A_0B_0 \Longrightarrow \overline{XK}$ and $\overline{A_0C_0}$ are congruent and parallel. Hence, if $U \equiv XH \cap KN$, we have $\frac{XU}{UH} = \frac{XK}{HN} = \frac{C_0A_0}{HN} = \frac{BC_0}{BH} = \frac{JC_0}{IH}$.

Similarly, if XG cuts IM at V, we have $\frac{XV}{VG} = \frac{JC_0}{KG}$. Hence if UV cuts AB at P, then by Menelaus' theorem for $\triangle XGH, \overline{PUV}$, we obtain

$$\frac{PH}{PG} = \frac{HU}{UX} \cdot \frac{XV}{VG} = \frac{IH}{JC_0} \cdot \frac{JC_0}{KG} = \frac{IH}{KG}.$$

Since $\overline{IH} \parallel \overline{KG}$, the latter relation reveals that $P \in IK$. Therefore $\triangle KHU$ and $\triangle IGV$ are perspective through P. By Desargues theorem, intersections $X \equiv UH \cap VG, Y \equiv KU \cap IV$ and $Z \equiv KH \cap IG$ are collinear.

Quick Reply

High School Olympiads

EASY PROBLEMS

Locked

signature

#1 Jun 21, 2013, 1:45 pm

Let A,B,C,P,Q,and R be six concyclic points.Show that if the simson lines of P,Q,and R with respect to triangle ABC are concurrent,then the simson lines of A,B, and C with respect to triangle PQR are concurrent.Furthermore,show that the points of concurrence are the same.

Luis González

#2 Jun 21, 2013, 8:50 pm

Please give your posts meaningful subjects and use the search before posting. Why is it so difficult to you?, you only have to use appropriate keywords. This was already discussed in the topic http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=382210, hence thread locked.

High School Olympiads

Quadrilateral with perpendicular bisectors

Source: Internet page

EnriqueDelaTorre
#1 Jun 17, 2013, 3:21 am

ABCD is a quadrilateral such that BD is perpendicular to AC. M is the midpoint of AC and BD passes through M. We have E,F,G,H on AB,BC,CD, DA, respectively, such that E, M and G are collinear and so are F, M and H. EH intersects AC at X, FG intersects AC at Y. Prove MX=MY.

tobash_co
#2 Jun 17, 2013, 7:48 pm

Reflect the triangle BCD across BD so that C maps to A and G, F are mapped to points on AD, AB respectively. From now on we say F, G for the points after reflection. Note that

$$\frac{AG}{GD}\cdot\frac{DM}{MB}\cdot\frac{BE}{EA}=\frac{AM}{MD}\cdot\frac{\sin\angle AMG}{\sin\angle GMD}\cdot\frac{DM}{MB}\cdot\frac{MB}{MA}\cdot\frac{\sin\angle BME}{\sin\angle AME}=1$$

So by Ceva's theorem, BG, DE intersect on AM. Let $EG\cap BD=S$. Then S is the harmonic conjugate of M w.r.t. B, D. Analogously HF passes through S as well. Since AM hits EG, FH at the harmonic conjugates of S w.r.t. E, G and F, H respectively, by considering the complete quadrilateral formed by E, F, G, H, A, S, it follows that FG, EH intersect on AM as well. So X is the reflection of Y w.r.t. BD, which means that $MX=MY$.

Luis González
#3 Jun 17, 2013, 11:11 pm

Let $P\equiv AB\cap CD$ and $Q\equiv BC\cap DA$. For any complete quadrangle $ABCD$, the intersection $K\equiv EH\cap FG$ is on PQ. Indeed, projecting PQ to infinity, $ABCD$ becomes parallelogram with diagonal intersection M and by obvious symmetry M will be midpoint of EG, FH as well $\Longrightarrow EFGH$ is parallelogram $\Longrightarrow EH\cap FG$ is at infinity, i.e. $K\equiv PQ\cap EH\cap FG$. Since the pencil $P(A,D,M,Q)$ is harmonic, then $K(E,G,M,Q)\equiv K(X,Y,M,Q)$ is harmonic. When BD is perpendicular bisector of $\overline{AC}$, we obviously have $PQ\parallel AC$, i.e. $KQ\parallel XY\Longrightarrow \overline{MX}=-\overline{MY}$.

sunken rock
#4 Jun 17, 2013, 11:50 pm

Let K be symmetrical of A about BD, $J=EM\cap DK, L=HF\cap BK$.
By **Desargues** for $\triangle CGF, \triangle KJL, B, D$ and intersection of GF and JL are collinear, hence the latter is Y. As JL is the symmetrical of EH w.r.t. M, we are done.

Best regards,
sunken rock

theflowerking
#5 Jun 18, 2013, 3:04 am

Isn't just $K=C$, or is K the reflection of A by the midpoint of BC?

MMEEvN
#6 Jun 18, 2013, 8:28 pm

It can be done using co-ordinate geometry.Take M to be the origin and let BD be the y-axis and AC be the x-axis and put $A(-1,0)C(1,0)B(0,a), D(0,-b)$...

sunken rock

#7 Jun 18, 2013, 9:48 pm

> *theflowerking wrote:*
> Isn't just $K=C$, or is K the reflection of A by the midpoint of BC?

For this particular case, isn't it the same??

Best regards,
sunken rock

theflowerking

#8 Jun 19, 2013, 4:48 am

Ehm, no... BD is the perpendicular bisector of AC, not the other way around. I don't really see how your proof is true. If you could explain, I'd be greateful. Thanks

High School Olympiads

Intersect on perpendicular bisector

Reply

Source: Own

buratinogigle

#1 May 31, 2013, 10:12 pm

Let ABC be a triangle. (K) is a circle passing through B, C. (K) cut CA, AB again at E, F, respectively. Circle (L) touches CA, AB and touches (K) internally at T such that T and A are not same side with respect to BC. (I) is B-mixtilinear incircle of triangle EBC and (J) is C-mixtilinear incircle of triangle FCB. $(I), (J)$ touch BC at M, N, respectively. Prove that circumcircle of triangle TMN and (K) intersect on perpendicular bisector of BC.

Attachments:

Figure1062.pdf (7kb)

Luis González

#2 Jun 17, 2013, 3:03 am • 2

Let U, V denote the incenters of $\triangle BCE, \triangle BCF$, respectively. Let P, Q be the tangency points of $(I), (J)$ with BE, CF, respectively and Y, Z the tangency points of (L) with AC, AB. By Sawayama's lemma, we deduce that $U \in PM, V \in QN$ and $\{U, V\} \in YZ$. Let $G \equiv QN \cap AC$. Simple angle chase gives

$$\angle GVY = \angle AGV - \angle AYZ = (90^\circ - \tfrac{1}{2}\angle FCB + \angle ACB) - 90^\circ + \tfrac{1}{2}\angle BAC =$$

$$= 90^\circ + \tfrac{1}{2}\angle ACB - \tfrac{1}{2}\angle ABC - \tfrac{1}{2}\angle FCB = 90^\circ + \tfrac{1}{2}\angle FCE - \tfrac{1}{2}\angle ABC =$$

$$= 90^\circ + \tfrac{1}{2}\angle FBE - \tfrac{1}{2}\angle ABC = 90^\circ - \tfrac{1}{2}\angle EBC = \angle UMN,$$

which means that $UVNM$ is cyclic. If S denotes the midpoint of the arc BC of (K) (not containing E,F), then according to Internally tangent circles and lines and concurrency, ST, YZ, BC concur at H. Since $BCUV$ is clearly cyclic with circumcenter S, then we have $HT \cdot HS = HB \cdot HC = HU \cdot HV = HM \cdot HN \Longrightarrow MNTS$ is cyclic.

alet

#3 Jun 21, 2013, 4:11 am

buratinogigle: What program did you use for the figure?

buratinogigle

#4 Jun 21, 2013, 5:08 pm • 1

I used Cabri II. See http://www.cabri.com/

Quick Reply

High School Olympiads

not easy

Locked

signature

#1 Jun 15, 2013, 10:14 pm

Let ABCD be a triangular pyramid such that no face of the pyramid is right triangle and the orthocenters of triangles ABC,ABD,and ACD are collinear. Prove that the center of the sphere circumscribed about the pyramid belongs to the plene passing through the midpoints of AB,AC,and AD.

Luis González

#2 Jun 16, 2013, 12:10 am

Firstly, please give your post meaningful subjects. Subjects such as "not esy", "very easy", "nice problem", etc, do not describe the purpose of the proposed problem and makes search more difficult. Secondly, use the search function before posting contest problems, this is All Russian Olympiad 2009 - Problem 11.3. For solutions see center of the sphere circumscribed to a pyramid.

High School Olympiads

let ABC

Locked

signature

#1 Jun 15, 2013, 7:45 pm

In a scaiene triangle ABC,H and M are the orthocenter an centroid respectively consider the formed by the lines through A,B and C peependiculer to AM,BM and CM respectively.Prove that the centroid of this triangle lines on the line MH.

Luis González

#2 Jun 15, 2013, 11:35 pm

Use the search before posting contest problems. This is P3 (grade 9) from ARMO 2008.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=209616
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=258869
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=390332
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=490118

High School Olympiads

nice Gepmetry

Locked

signature

#1 Jun 15, 2013, 7:56 pm

Points K and L are taken on the arcs AB and BC respectively of a triangle ABC such that KL is parallel to AC.Prove that the incenters of triangles ABK and CBL are equidistance from the midpoint of arc ABC.

[Mod: See Russia, 2006.]

Luis González

#2 Jun 15, 2013, 11:10 pm

U,V denote the incenters of $\triangle ABK, \triangle CBL$ and M denote the midpoint of the arc ABC of the circumcircle (O). BU,BV cut (O) again at the midpoints D,E of the arcs $AK,CL.$ By obvious symmetry, M is also midpoint of the arc $DBE,$ i.e $MD=ME$ and $DK=EL.$ Since D,E are the circumcenters of $\triangle AKU, \triangle CLV$ (well-known), then $DU=DK=EL=EV.$ Since $\angle BDM=\angle BEM,$ then $\triangle MDU \cong \triangle MEV$ by SAS criterion $\Longrightarrow MU=MV.$

High School Olympiads

Circles touching incircle

Source: Kolmogorov's Cup; Komal March 2012

Spats

#1 May 27, 2013, 11:42 am

The incircle of triangle ABC is k. The circle k_A touches k and the segments AB and AC at at A', A_b and A_c respectively. The circles k_B, k_C and the points B', C' are defined analogously. The second intersection point of the circles $A'B'A_b$ and $A'C'A_c$, other than A', is K. The line $A'K$ meets k at R, other than A'. Prove that R lies on the radical axis of the circles k_B and k_C.

Luis González

#2 May 29, 2013, 2:43 am

k touches BC at D, k_B touches AB, BC at M, N and k_C touches BC, CA at L, P. Since B' is the insimilicenter of $k \sim k_B$, it follows that NB' cuts k again at the antipode X of D WRT k. Similarly, LC' passes through X. Then $XD^2 = XL \cdot XC' = XN \cdot XB' \Longrightarrow X$ is on the radical axis of k_B, k_C and $B'C'LN$ is cyclic. Analogously $C'A'A_cP$ and $A'B'MA_b$ are cyclic.

Let $B'C'$ cut $\odot(A'B'A_b)$ and $\odot(A'C'A_c)$ again at S, T, resp. Simple angle chase gives $\angle B'SA_b = \angle B'MA_b = \angle A'B'C' \Longrightarrow A'B'SA_b$ is an isosceles trapezoid with $B'S = A'A_b$. Similarly, we have $C'T = A'A_c$, but obviously $A'A_b = A'A_c \Longrightarrow B'S = C'T \Longrightarrow \overline{B'C'}$ and $\overline{ST}$ have the same midpoint $J \Longrightarrow J$ has equal power WRT $\odot(A'B'A_b)$ and $\odot(A'C'A_c) \Longrightarrow J$ is on their radical axis $A'K$.

Since $\angle XC'B' = \angle B'NL = \angle A'B'C'$, then $A'B'C'X$ is an isosceles trapezoid with $A'X \parallel B'C'$. Hence, pencil $A'(B', C', J, X) \equiv A'(B', C', R, X)$ is harmonic $\Longrightarrow XB'RC'$ is harmonic $\Longrightarrow XR$ goes through the intersection U of the tangents of k at B', C'. Since UB', UC' also touch k_B, k_C and $UB' = UC'$, then U has equal power WRT $k_B, k_C \Longrightarrow UX$ is radical axis of k_B, k_C, cutting k again at R.

High School Olympiads

Three collinear points

Reply

Source: own conjecture

jayme

#1 May 27, 2013, 6:27 pm • 1

Dear Mathlinkers,

1. ABC a triangle,
2. A'B'C' the medial triangle of ABC
3. DEF the contact triangle of ABC
4. A* the point of intersection of B'C' and EF
5. I, Na the resp. incenter, Nagel point of ABC
6. Y the point of intersection of INa and A'C'

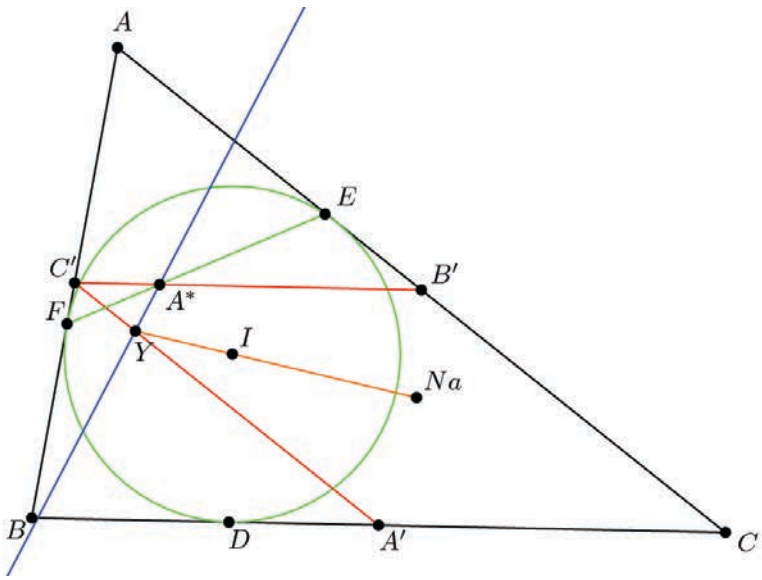
Conjecture : A*Y goes through B

Sincerely
Jean-Louis

robinpark

#2 May 27, 2013, 8:18 pm • 1

Your conjecture doesn't seem to be true:



jayme

#3 May 27, 2013, 9:29 pm • 1

Dear robinpark and Mathlinkers,
can you check again your figure because after toying and verificating my figure it seems OK.
Perhaps my program is old???
Sincerely
Jean-Louis

sunken rock

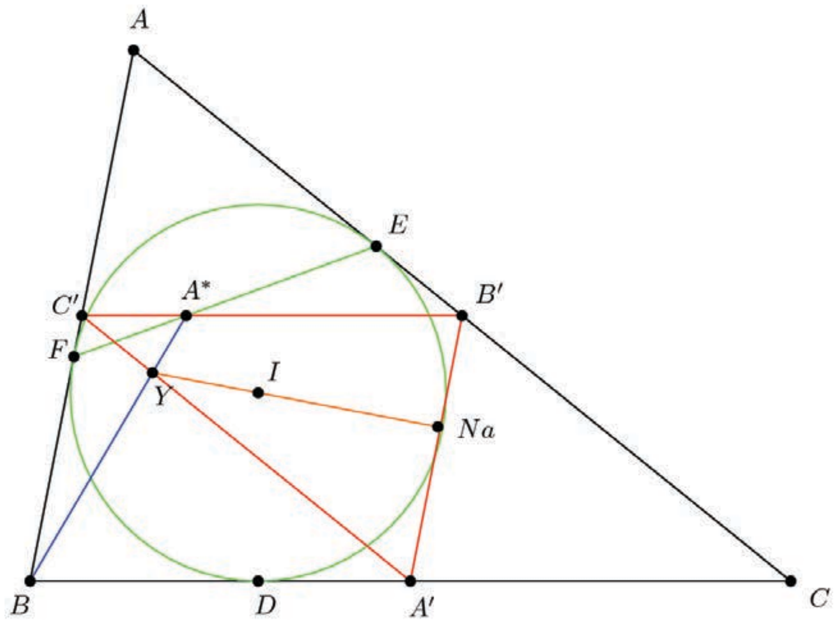
#4 May 27, 2013, 11:50 pm

My program shows as true for any position!

Best regards,
sunken rock

robinpark

#5 May 28, 2013, 7:55 am • 1



We use barycentric coordinates. Let $A = (1,0,0), B = (0,1,0), C = (0,0,1)$, and let $BC = a, CA = b, AB = c$.

Clearly we have $A' = (0:1:1), B' = (1:0:1), C' = (1:1:0)$. We also have $D = (0 : s-c : s-b), E = (s-c : 0 : s-a), F = (s-b : s-a : 0)$ using side ratios, where s is the semiperimeter of $\triangle ABC$. Thus line $B'C'$ has equation $x - y - z = 0$ and EF has equation $(s-a)x - (s-b)y - (s-c)z = 0$. So $A^* = (c-b, c-a, a-b)$.

It is known that $I = (a:b:c)$ and $Na = (b+c-a : c+a-b : a+b-c)$. Thus the line INa has equation $(b-c)x + (c-a)y + (a-b)z = 0$. Since line $A'C'$ has equation $x - y + z = 0$, we have that $Y = (c-b : a-2b+c : a-b)$. Note that

$$\begin{vmatrix} c-b & c-a & a-b \\ c-b & a-2b+c & a-b \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} c-b & c-a & a-b \\ 0 & 2a-2b & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0$$

by row reduction. Hence A^*, Y, B are collinear, as desired.

Luis González

#6 May 28, 2013, 9:40 am • 1

It's known that in any triangle, the polar of one vertex WRT the incircle, the internal angle bisector issuing from a second vertex and the midline referent to the third vertex concur. Thus $EF, BI, A'B'$ concur at U and $EF, CI, A'B'$ concur at V. By Pappus theorem for hexagon $BUA'VCA^*$, the intersections $I \equiv BU \cap VC, Z \equiv UA' \cap CA^*$ and $Y^* \equiv A'V \cap A^*B$ are collinear, but by Pappus theorem for hexagon $BB'A'C'CA^*$, the intersections $G \equiv BB' \cap C'C, Z \equiv B'A' \cap CA^*$ and $Y^* \equiv A'C' \cap A^*B$ are collinear $\Longrightarrow I, G, Z, Y^*$ are collinear $\Longrightarrow Y \equiv Y^* \Longrightarrow IG \equiv INa, BA^*$ and $A'C'$ concur at Y.

Quick Reply

High School Olympiads

Toying again with Feuerbach point

Reply

jayme
#1 May 26, 2013, 3:57 pm

Dear Mathlinkers,
1. ABC a triangle
2. A'B'C' the medial triangle of ABC
3. I, G the resp. incenter, centroid of ABC
4. B* the midpoint of BI
5. Fe the Feuerbach point of ABC.

Prove that A'C', IG and B*Fe are collinear.

Sincerely
Jean-Louis

Luis González
#2 May 27, 2013, 7:33 am • 1

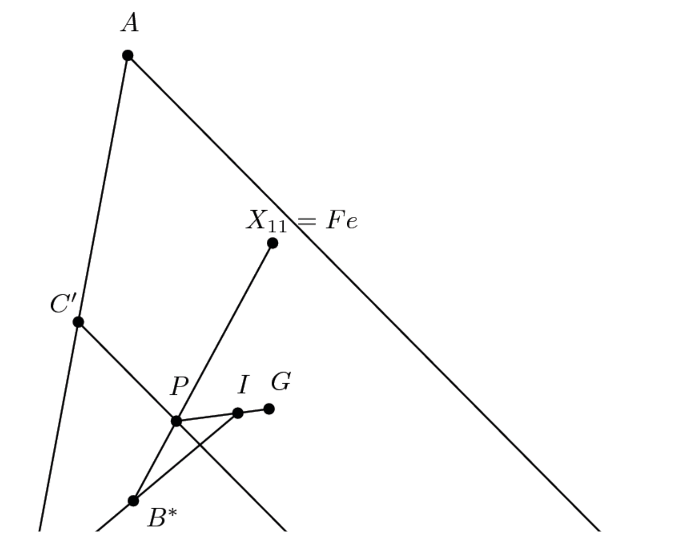
Let E, E' be the tangency points of the incircle (I) and B-excircle (I_b) with AC. It's well known that BE' passes through the antipode Y of E WRT (I) and that I is the Nagel point of the medial $\triangle A'B'C'$, i.e. Nagel point N of $\triangle ABC$ is the anticomplement of $I \Longrightarrow N \equiv IG \cap BE'$. It's also known that the midpoint $U \equiv BE' \cap A'C'$ of $\overline{BE'}$ is also midpoint of $\overline{NY}$ (see USAMO 2001 problem 2), hence if BI cuts $AC, A'C'$ at S, V, we have

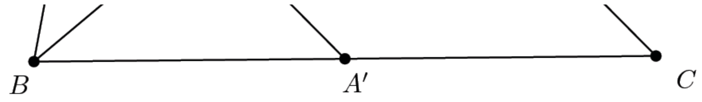
$$\frac{\overline{BN}}{\overline{BY}} = \frac{\overline{BN}}{\overline{NE'}} = \frac{\overline{SI}}{\overline{IV}} = -2 \cdot \frac{\overline{VB^*}}{\overline{VI}} = -\frac{\overline{BI}}{\overline{BB^*}} \cdot \frac{\overline{VB^*}}{\overline{VI}} \Longrightarrow$$

$$\frac{\overline{UY}}{\overline{UN}} \cdot \frac{\overline{BN}}{\overline{BY}} = \frac{\overline{BI}}{\overline{BB^*}} \cdot \frac{\overline{VB^*}}{\overline{VI}} \Longrightarrow (U, Y, N, B) = (V, B^*, I, B).$$

The latter expression reveals that $A'C' \equiv UV, IG \equiv IN$ and YB^* concur. On the other hand, the Feuerbach point of $\triangle ABC$ is on YB^* (see post #4 at Intersect on circle), hence the proof is completed.

robinpark
#3 May 27, 2013, 9:20 am • 1





We use barycentric coordinates. Let $A=(1,0,0), B=(0,1,0), C=(0,0,1)$, and let $BC=a, CA=b, AB=c$.

Clearly, $A'=(0:1:1)$ and $C'=(1:1:0)$. We also have $I=(a:b:c)$ and $G=(1:1:1)$. Since $B=(0:a+b+c:0)$, we have $B^*=(a:a+2b+c:c)$. It is known that the Feuerbach Point $X_{11}=Fe$ has coordinates

$$X_{11}=((b+c-a)(b-c)^2:(c+a-b)(c-a)^2:(a+b-c)(a-b)^2).$$

Now line $A'C'$ has equation $x-y+z=0$, and line IG has equation $(b-c)x+(c-a)y+(a-b)z=0$. Their intersection P then has coordinates $(b-c:-a+2b-c:b-a)$. It remains to prove that P, B^*, X_{11} are collinear; that is,

$$\begin{vmatrix} b-c & -a+2b-c & b-a \\ a & a+2b+c & c \\ (b+c-a)(b-c)^2 & (c+a-b)(c-a)^2 & (a+b-c)(a-b)^2 \end{vmatrix}=0$$

which is not difficult to check.

jayme

#4 May 31, 2013, 4:41 pm

Dear Luis and Mathlinkers,
very nice ideas dear Luis and thank a lot.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Collinear Points

Locked

War-Hammer

#1 May 26, 2013, 10:51 pm

Let ABC be a triangle inscribe in a circle Γ . Suppose that ω is the ex-circle of vertex A . Let bisector of B, C intersect AC, AB at E, F and let the common tangent of Γ, Ω touch circle Γ at X, Y . Prove that X, E, F, Y are collinear.

Luis González

#2 May 26, 2013, 11:30 pm

Posted at least 4 times before, so for further discussions use the links below. Topic locked.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=385175
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=438325
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=412847
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=516894

High School Olympiads

$RT//DE$ $\rightarrow$ R,I,T are collinear

Reply

bah_luckyboy
#1 May 26, 2013, 7:10 pm

Let D be the midpoint of the arc BC of the circumcircle of triangle ABC which does not contain A. X is on the minor arc BD; E is the midpoint of the arc AX. S is on the arc AC that does not contain B; SD meets BC at the point R and SE meets AX at the point T. Prove that, if RT is parallel with DE, then R, I, T are collinear with I is the incenter of the triangle ABC

Luis González
#2 May 26, 2013, 9:36 pm • 1

Let τ be the tangent of $\odot(ABC)$ at S. $\angle(\tau, SD) = \angle SED = \angle STR \Longrightarrow \tau$ is also tangent of $\odot(STR) \Longrightarrow \odot(STR)$ is internally tangent to $\odot(ABC)$ at S. Assume that $\odot(STR)$ cuts BC again at R^*. From the internal tangency, it follows that SR, SR^* are isogonals WRT $\angle BSC$, but $SR \equiv SD$ bisects $\angle BSC \Longrightarrow R \equiv R^* \Longrightarrow BC$ touches $\odot(STR)$. In the same way, using that STE bisects $\angle ASX$, we get that AX touches $\odot(STR)$. Hence $\odot(STR)$ is a Thebault circle of the cevian AX of $\triangle ABC$. By Sawayama's lemma RT passes through the incenter I.

mathuz
#3 May 26, 2013, 10:07 pm

it's chinese olympiad!

mathuz
#4 May 26, 2013, 10:33 pm

thank you, Luis Gonzalez.
I am glad. I search classical solution of the problem, few days.

leader
#5 May 27, 2013, 12:18 am

If $J = DA \cap RT$ $\angle AJT = \angle ADE = \angle ASE = \angle AST$ so $SJTA$ is cyclic.
$\angle DSJ = \angle DSX + \angle XSJ = \angle DAX + \angle XSE - \angle JST = \angle ASE + \angle JAT - \angle JST = \angle AJT = \angle DJR$
with $\angle JDR = \angle SDJ$ $\triangle DJR \sim \triangle DJS$ so $DJ^2 = DR \cdot DS$ but $\angle DCR = \angle DSB = \angle CSD$ so
$\triangle DCR \sim \triangle DCS$ and $DB^2 = DC^2 = DR \cdot DS = DJ^2$ so $DC = DB = DJ$ and
$\angle JBC = \dfrac{1}{2}\angle CDJ = \dfrac{1}{2}\angle CBA$ so $J = I$

nima-amini
#6 Jan 15, 2014, 7:53 pm

it is iran third round 2001 !!!!

Quick Reply

High School Olympiads

Playing with the Feuerbach point

Reply

jayme
#1 May 25, 2013, 8:11 pm

Dear Mathlinkers,
1. ABC a triangle,
2. (I) the incircle of ABC
3. DEF the contact triangle of ABC
4. A'B'C' the médian triangle of ABC
5. A*, C* the points of intersection resp. of B'C' and EF, A'B' and DE,
6. C+ the point of intersection of A'B' and CA*,
7. Fe the Feuerbach's point of ABC.

Prove that the triangle FeC+C* is Fe-rectangular.

Sincerely
Jean-Louis

Luis González
#2 May 25, 2013, 11:14 pm

Let the Feuerbach tangent cut AB and $A'B'$ at $Z, M,$ respectively. According to the problem Tangent at Fe, M is the midpoint of $\overline{C^*C^+}.$ By 1st Fontené theorem FC^* passes through the Feuerbach point $F_e.$ Then since $FZ \parallel MC^*,$ it follows that $\triangle MC^*F_e \sim \triangle ZFF_e \Longrightarrow \triangle MC^*F_e$ is M-isosceles $\Longrightarrow MC^* = MF_e = MC^* \Longrightarrow \angle C^*F_eC^+$ is right.

Quick Reply

High School Olympiads

tangent

Reply

huyhoang

#1 Feb 24, 2012, 6:53 pm

Let O_1AB and O_2AB are two equilateral triangles which O_1 and O_2 lie in different side of AB. Construct two circles $(O_1; O_1A)$ and (O_2, O_2A). Let T be a point on (O_1) and draw two tangents of O_2TX, TY and this two lines intersect (O_1) at Y', X' respectively. Prove that $X'Y'$ is tangent to O_2.

huyhoang

#2 Mar 12, 2012, 12:17 pm

at least someone could show me the idea?

Luis González

#3 May 25, 2013, 4:42 am

By Poncelet's porism, the triangles $\triangle TX'Y'$ form a family of poristic triangles with common circumcircle (O_1, O_1A) and T-excircle $(O_2, O_2A) \Longleftrightarrow O_1O_2{}^2 = O_1A^2 + 2 \cdot O_1A \cdot O_2A$. In other words, $X'Y'$ touches (O_2, O_2A) if and only if the latter relation holds. Indeed

$$O_1O_2{}^2 = \left(\sqrt{3} \cdot O_1A\right)^2 = 3 \cdot O_1A^2 = O_1A^2 + 2 \cdot O_1A \cdot O_2A.$$

Quick Reply

High School Olympiads

Prove A_1,B_1,C_1 is collinear

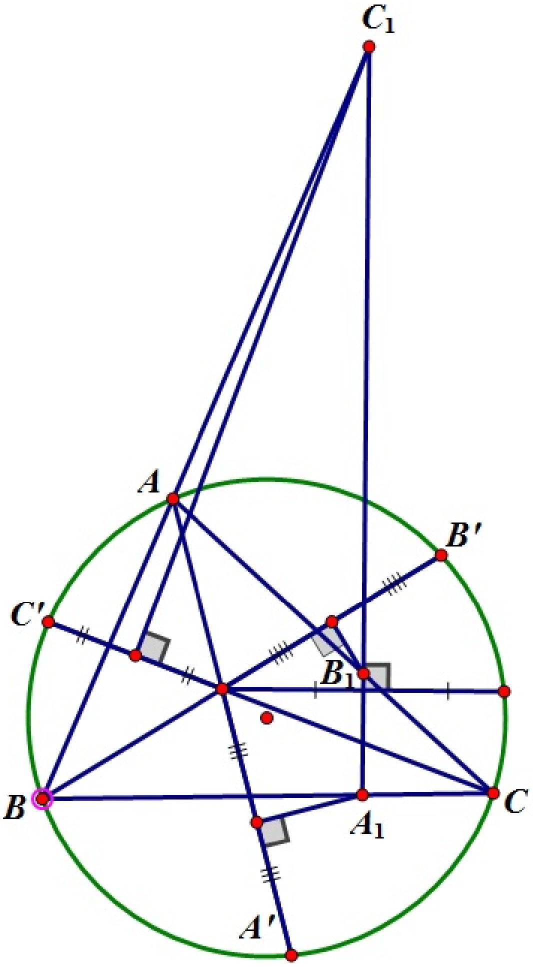
Reply

tangkhaihanh

#1 May 24, 2013, 3:37 pm

Let ABC be a triangle inscribled inside circle (O) . M is a point inside the triangle ABC $(M \notin BC, CA, AB)$ AM, BM, CM meets (O) again at A', B', C' respectively. Midperpendicular of MA', MB', MC' meet BC, CA, AB at A_1, B_1, C_1 respectively.
1.Prove that A_1, B_1, C_1 is collinear
2.M' is symmetric to M through $A_1B_1C_1$. Prove that $M' \in (O)$

Attachments:



Luis González

#2 May 24, 2013, 10:27 pm • 5

H is the orthocenter of $\triangle ABC$ and $D, E, F \in (O)$ its reflections about BC, CA, AB. M_1, M_2, M_3 are the reflections of M on BC, CA, AB. A_1, B_1, C_1 are then the circumcenters of $\triangle MA'M_1, \triangle MB'M_2, \triangle MC'M_3$. DM_1, EM_2, FM_3 are reflections of HM on BC, CA, AB, concurring at the anti-Steiner point $S \in (O)$ of HM. Then, simple angle chase (mod 180º) gives $\angle DSA' = \angle DAA' = \angle M_1MA' \Longrightarrow S \in \odot(MA'M_1)$. Likewise, $\odot(MB'M_2)$ and $\odot(MC'M_3)$ pass through $S \Longrightarrow \odot(MA'M_1), \odot(MB'M_2)$ and $\odot(MC'M_3)$ are coaxal with radical axis $MS \Longrightarrow A_1, B_1, C_1$ are collinear on the perpendicular bisector of $\overline{MS}$ and $M' \equiv S$.

hgdhftw

#3 May 25, 2013, 12:51 am • 1

I have another solution.
Let $O_a; O_b; O_c$ be the circumcenters of $\triangle MB'C'; \triangle MA'C'; \triangle MA'B'$ respectively. Therefore $A_1 = O_bO_c \cap BC; B_1 = O_aO_c \cap AC; C_1 = O_aO_b \cap AB$. By the Desargues theorem, A_1, B_1, C_1 are collinear is equivalent to AO_a, BO_b, CO_c are concurrent. We shall prove that AO_a, BO_b, CO_c concurs at a point lies on (O).
Let N be the isogonal conjugate of M. Let AN, BN, CN meets (O) at A", B", C", $B'A, C'A$ meets (O_a) at B_a and C_a; $B'C$ and $A'C$ meets (O_c) at B_c and A_c. By some simple angle chasing we can prove that there exists spiral similarity V_1: $(O)- > (O_a); BB" - > B'B_a; CC" - > C'C_a; N- > A$.
and spiral similarity V_2: $(O)- > (O_c); BB" - > B'B_c; AA" - > A'A_c; N- > C$. Thus we have: $\angle ONB" = \angle B'AO_a = \angle B'CO_c$, so AO_a intersects CO_c at a point K lies on (O). By similar proof we can prove that K also lies on BO_b. Hence AO_a, BO_b, CO_c are concurrent at a point on (O).
Next, let KM meets (O) at S'. We shall prove that $S' \equiv S$.
Let R be the reflection of M on AC. By some angle chasing we can prove $\angle B'MR = \angle OBN$, and $MR.BO = BN.B'M = MA.A'C$, therefore $\triangle MRB' \sim \triangle BNO$. Hence, $(ON; BB") = (B'R; MR) = (S'K; S'B') = (S'M; S'B')$. Therefore S', R, B', M are concyclic in a circle with center B_1. Thus $S' \equiv S$ and is the reflect of M on A_1B_1.

leader

#4 May 26, 2013, 2:43 am

Obviously there exist P, Q on lines CB, CA such that $MP/PB = MQ/QA$ and $\angle PMQ = \angle ACB$(by moving P on line CB) now let circle CPQ and lines AM, BM meet circle ABC again at X, R, S it's not hard to prove that the symmetric point to M (X') coincides with X by similarity of $X'QA$ and $X'PB$. now it's also not hard by angle chase to prove that R is on $k(P, PM)$ giving $P = A_1$ and similarly $Q = B_1$. now we get the same point X by considering sides $CA, AB(X = (Q, QM) \cap \odot ABC$. now getting that A_1, B_1, C_1 are on the perp bisector of MX. and of course X is on circle ABC.

Quick Reply

High School Olympiads

4 Tangent circles

Reply

xeroxia

#1 May 20, 2013, 12:16 am

Let C_1 and C_2 be two internally tangent circles with radii R and r ($R > r$), respectively. Let ℓ be a chord of C_1 such that ℓ is tangent to C_2 at its midpoint. Let C_3 be a circle tangent to C_1, C_2 and ℓ. Let C_4 be the circle tangent to C_1, C_3, and ℓ. Show that radius of C_4 is equal to $\boxed{\dfrac{(R-r)^2}{(R+r)^2}\cdot r}$.

(C_2, C_3, C_4 lie on same side of ℓ)

Luis González

#2 May 24, 2013, 8:17 am • 1

Label $C_1 \equiv (O_1, R), C_2 \equiv (O_2, r), C_3 \equiv (O_3, r_3)$ and $C_4 \equiv (O_4, r_4)$. Let A denote the tangency point of C_1, C_2 and B, C the intersections of ℓ with C_1. Since C_2 touches BC at its midpoint M, then by obvious symmetry AMO_1 is perpendicular bisector of $\overline{BC}$. Let C_3, C_4 touch BC at P, Q, respectively. F is the reflection of O_2 about A and τ is the perpendicular to AO_1 through F. If PO_3 cuts τ at H, we have $O_3H = PH - r_3 = MF - r_3 = R - r_3 = O_3O_1 \Longrightarrow O_3$ is on the parabola $\mathcal{P}$ with focus O_1 and directrix τ. This is then the locus of the centers of the circles inscribed in the the circular segment determined by arc BAC). The vertex of $\mathcal{P}$ is clearly O_2.

If K is the projection of O_3 on the focal axis AO_1, then by parabola property, $|O_3K|$ is geometric mean between $|KO_2|$ and the latus rectum length $4|O_1O_2| = 4(R-r)$.

$$|O_3K|^2 = |PM|^2 = 4(R-r)\cdot|KO_2| = 4(R-r)\cdot(r-r_3) \Longrightarrow$$

$$(2\sqrt{r\cdot r_3})^2 = 4(R-r)\cdot(r-r_3) \Longrightarrow r_3 = \frac{r(R-r)}{R}.$$

Since O_4 is also on the parabola $\mathcal{P}$, we get similarly

$$|QM|^2 = 4(R-r)\cdot(r-r_4) \Longrightarrow |PM+PQ|^2 = 4(R-r)\cdot(r-r_4) \Longrightarrow$$

$$(2\sqrt{r_3\cdot r} + 2\sqrt{r_4\cdot r_3})^2 = 4(R-r)\cdot(r-r_4) \Longrightarrow$$

$$\left(r\sqrt{\frac{R-r}{R}} + \sqrt{\frac{r_4\cdot r(R-r)}{R}}\right)^2 = (R-r)\cdot(r-r_4) \Longrightarrow$$

$$(r + \sqrt{r_4\cdot r})^2 - R(r-r_4) = 0.$$

Solving the latter quadratic equation for r_4 gives $r_4 = r\cdot\dfrac{(R-r)^2}{(R+r)^2}$.

Quick Reply

High School Olympiads

Tangent at Fe

jayme
#1 May 23, 2013, 4:32 pm

Dear Mathlinkers,
1. ABC a triangle,
2. (I) the incircle of ABC
3. DEF the contact triangle of ABC
4. A'B'C' the median triangle of ABC
5. A*, C* the points of intersection of B'C' and EF, A'B' and DE,
6. C+ the point of intersection of A'B' and CA*
7. Fe the Feuerbach point of ABC

Prove that the tangent to (I) at Fe goes through the midpoint of C+C*.

Sincerely
Jean-Louis

Luis González
#2 May 23, 2013, 11:29 pm • 2

Let $\mathcal{H}$ be the circumconic through the centroid and the Gergonne point of $\triangle ABC$. $A'C'$ and DF are the polars of B' and E WRT $\mathcal{H} \Longrightarrow B^* \equiv A'C' \cap DF$ is the pole of AC WRT $\mathcal{H} \Longrightarrow AB^*$ is tangent of $\mathcal{H}$. Similarly, C^* is pole of AB WRT $\mathcal{H}$ $\Longrightarrow AC^*$ is tangent of $\mathcal{H} \Longrightarrow A, B^*, C^*$ are collinear. Likewise, C, B^*, A^* are collinear. This is actually true for any two cevian triangles.

By 1st Fontené theorem EB^* passes through F_e. Thus if the tangent τ of (I) at F_e cuts BC, AB at X, Z, then by Newton theorem $B^* \equiv AX \cap CZ \cap DF \cap EF_e \Longrightarrow C^*$ is then the pole of CZ WRT $(I) \Longrightarrow C(E, D, B^*, C^*) = -1 \Longrightarrow$ $(A, X, B^*, C^*) = -1 \Longrightarrow Z(A, X, C^+, C^*) = -1$. Since $ZA \parallel A'B' \equiv C^*C^+$, then it follows that $XZ \equiv \tau$ passes through the midpoint of $\overline{C^*C^+}$.

Quick Reply

High School Olympiads

Prove that the circumcircle of APQ passes through I

Locked

a00012025

#1 May 23, 2013, 10:02 pm

Let points A and B lie on the circle Γ , and let C be a point inside the circle.Suppose that ω is a circle tangent to segments AC,BC and Γ. Let ω touch AC and Γ at P and Q.I is the incenter of ABC. Prove that the circumcircle of APQ passes through I

Luis González

#2 May 23, 2013, 10:17 pm • 2

Fairly tricky problem but posted many times before. It's from an Iranian math competition (1999).

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=6086
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=41667
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=253207
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=407366

High School Olympiads

Trapezoid

Reply

bozzio

#1 May 2, 2013, 1:09 am

A trapezoid ABCD with AB //CD has AB=3CD. The circle with center in C and radius CS (S is the midpoint of AC) and the circle with center D and radius DT (T is the midpoint of BD) meet in two points, P and Q. M is the midpoint of AB. Show that P,Q,M are collinear.

Luis González

#2 May 22, 2013, 8:37 pm

Let $K \equiv AD \cap BC$. From $\triangle KDC \sim \triangle KAB$, we get $KD = \frac{1}{3}KA$ and $KC = \frac{1}{3}KB$. By Stewart theorem for the cevians MD and BD of $\triangle KAM$ and $\triangle KAB$, we obtain the relations

$$MD^2 = \frac{1}{12}AB^2 + \frac{2}{3}KM^2 - \frac{2}{9}KA^2 \ , \ DT^2 = \frac{1}{12}AB^2 + \frac{1}{6}KB^2 - \frac{1}{18}KA^2$$

$$\Longrightarrow MD^2 - DT^2 = \frac{2}{3}KM^2 - \frac{1}{6}(KA^2 + KB^2).$$

By similar reasoning, we get $MC^2 - CS^2 = \frac{2}{3}KM^2 - \frac{1}{6}(KA^2 + KB^2)$. Hence $MD^2 - DT^2 = MC^2 - CS^2 \Longrightarrow$ M has equal power WRT the circles (D, DT) and $(C, CS) \Longrightarrow M$ is on their radical axis PQ.

sunken rock

#3 May 23, 2013, 5:35 pm

I shall prove this way the formula proved above : by median theorem for medians CM, DM respectively, keeping into account $AC = 2CS, BD = 2DT$ we find that we need to prove $2(AD^2 - BC^2) = AC^2 - BD^2$, but this is true from median theorem for medians $DS = CT$ (from given $AB = 3CD \implies ST \stackrel{\parallel}{=} CD$).

Best regards,
sunken rock

Quick Reply

High School Olympiads

parallelism and concurrency

Reply

Source: ITAMO 2013 problem 5

bozzio

#1 May 11, 2013, 11:08 pm

ABC is an isosceles triangle with $AB = AC$ and the angle in A is less than 60°. Let D be a point on AC such that $\angle DBC = \angle BAC$. E is the intersection between the perpendicular bisector of BD and the line parallel to BC passing through A. F is a point on the line AC such that $FA = 2AC$ (A is between F and C).
Show that EB and AC are parallel and that the perpendicular from F to AB, the perpendicular from E to AC and BD are concurrent.

Vo Duc Dien

#2 May 20, 2013, 4:14 am

Problem needs to be modified to say that:

Show that .... and that the perpendicular from F to AB, the perpendicular from E to AC and the extension of BD are concurrent.

Luis González

#3 May 22, 2013, 1:07 am

Proposition edited according to Vo Duc Dien's observation.

Let the perpendicular bisector of $\overline{BD}$ cut AC at P. Since $\angle DBC = \angle BAC$, then $\angle BDC = \angle BCD \Longrightarrow \angle BPD = \angle BAC$, i.e. B is on perpendicular bisector of $\overline{AP}$ and since $\angle BAE = \angle ABC = 90^\circ - \angle APE$, then B coincides with the circumcenter of $\triangle PAE \Longrightarrow BE = BA = CA \Longrightarrow \triangle ABC \cong \triangle BAE$, i.e. $AEBC$ is a parallelogram $\Longrightarrow EB \parallel AC$.

If M is the midpoint of $\overline{AF}$, then $MA = AC = AB = EB \Longrightarrow MABE$ is a rhombus $\Longrightarrow ME = MA = MF \Longrightarrow FE \perp AE$. Hence if the perpendicular from F to AB cuts BD at G, we have $\angle EDB = \angle EAB = \angle EFG \Longrightarrow DEFG$ is cyclic $\Longrightarrow \angle FEG = \angle FDG = \angle EAF \Longrightarrow EG \perp AC$, i.e. perpendicular from E to AC, perpendicular from F to AB and BD concur at G.

Quick Reply

High School Olympiads

Cyclic $ABCD$

Reply

War-Hammer
#1 May 19, 2013, 1:06 am

Let ω_1, ω_2 are two circle which intersect at A, E . Suppose that l is line which is tangent to ω_2 at E and suppose that P is the arbitrary point on l . Let PD, PB are tangent to ω_2, ω_1 . If AE intersect PB at T , and C is point on PB such that $BT = TC$ then prove that $ABCD$ is cyclic quadrilateral .

Luis González
#2 May 21, 2013, 5:23 am

Lemma: ω is a fixed circle and E, A are two fixed points on its plane, such that $A \in \omega$. A variable circle through A, E cuts ω again at R. Then the intersection X of the tangents of $\odot(EAR)$ at E, R runs on a fixed line.

Let EA, ER cut ω again at M, N. Inversion with center E and power $\overline{EA} \cdot \overline{EM}$ transforms ω into itself and carries $\odot(EAR)$ into $MN \Longrightarrow$ tangent of $\odot(EAR)$ at R goes to the circle passing through E and tangent to MN at N. This cuts EX again at the inverse Y of X, but since $\angle XER = \angle EAR = \angle ENM \Longrightarrow MN \parallel EY \Longrightarrow \triangle NEY$ is N-isosceles. Hence if U is the midpoint of $\overline{EY}$, then NU is perpendicular to MN, cutting ω again at the fixed antipode L of $M \Longrightarrow$ locus of U is the circle γ with diameter $\overline{EL} \Longrightarrow$ locus of Y is the circle γ^* homothetic to γ under homothety $(E, 2)$. This circle clearly passes through E, hence the locus of X will be the inverse line of γ^*. $\blacksquare$

In the problem, let $R \in \odot(ABC) \equiv \omega$ and tangents of $\odot(EAR)$ at E, R meet at X. As R varies X moves on a fixed line. When R coincides with B or C, the circles $\odot(ABE)$ and $\odot(ACE)$ are tangent to BC because $TB^2 = TC^2 = TE \cdot TA$ $\Longrightarrow X$ lies on BC in both cases $\Longrightarrow BC$ is the locus of X. Therefore, when $X \equiv P$, then $R \equiv D \Longrightarrow A, B, C, D$ are concyclic.

leader
#3 May 22, 2013, 12:08 am

call circle ABC k. let k cut ω_2 again at K and meet EK again at K'. now let L, M by symmetric points to E wrt T and BC respectively. by power of T to ω_2 $TB * TC = TA * TE = TA * TL$ so $L \in k$ but now since $BMLC$ is an isosceles trapezoid $M \in k$ and $ML||BC$ now let X, Y be midpoints of EK, EM now $XY||MK$ but since $\angle K'LE = \angle EKA = \angle LEP$ then $EP||LK'$ now since $\angle EXY = 180 - \angle K'XY = \angle K'KM = 180 - (180 - \angle K'LM) = 180 - \angle EPS = \angle EPY$ so $EXPY$ is cyclic but we know that Y is the foot of E to BC so $\angle EXP = 90 = \angle EYP$ so PX is the perpendicular bisector of KE and K is the second intersection of ω_2 and circle $c(P, PE)$ so $K = D$ and $ABCD$ is cyclic.

This post has been edited 1 time. Last edited by leader, May 23, 2013, 4:01 am

War-Hammer
#4 May 22, 2013, 12:16 pm

Hi ;

Thanks from "Leader and Luis González" , I find 3rd solution with Pascal Theorem .

Best Regard

War-Hammer
#5 May 22, 2013, 3:35 pm

> *leader wrote:*

> call circle ABC k. let k cut ω_1 again at K

Hi leader ;

A, B are on ω_1 !!!

How the cirumcircle of $\triangle ABC$ cut ω_1 again !!!

And can you tell what are L, M ???

Thanks

Best Regard

leader

#6 May 23, 2013, 4:02 am

> *War-Hammer wrote:*
>
> Hi leader ;
>
> A, B are on ω_1 !!!
>
> How the cirumcircle of $\triangle ABC$ cut ω_1 again !!!
>
> And can you tell what are L, M ???
>
> Thanks
>
> Best Regard

you are right it's ω_2
well you take point E reflect it over T to get L and if you reflect E over line BC you get M.

Quick Reply

High School Olympiads

IO Euler line of triangle PQR

Reply

Source: Bosnia and Herzegovina TST 2013 problem 6

Math-lover123

#1 May 20, 2013, 11:39 pm

In triangle ABC, I is the incenter. We have chosen points P, Q, R on segments IA, IB, IC respectively such that $IP \cdot IA = IQ \cdot IB = IR \cdot IC$.
Prove that the points I and O belong to Euler line of triangle PQR where O is circumcenter of ABC.

This post has been edited 2 times. Last edited by Math-lover123, Jun 19, 2013, 6:56 pm

Luis González

#2 May 21, 2013, 1:14 am

Circumcircle (J) of $\triangle PQR$ is obviously the image of $(O) \equiv \odot(ABC)$ under the inversion with center I and power $IP \cdot IA = IQ \cdot IB = IR \cdot IC$. Thus, by inversion property I is also a center of similitude of $(O) \sim (J) \Longrightarrow I, J, O$ are collinear. Since B, C, Q, R are concyclic, then QR is antiparallel to BC WRT $IB, IC \Longrightarrow QR$ is perpendicular to the I-circumdiameter IA of $\triangle BIC$. Likewise, IB, IC are perpendicular to $RP, PQ \Longrightarrow I$ is orthocenter of $\triangle PQR \Longrightarrow IJ$ is Euler line of $\triangle PQR$, passing through O.

War-Hammer

#3 May 21, 2013, 1:20 am

Hi ;

Nice one .

Click to reveal hidden text

Best Regard

This post has been edited 1 time. Last edited by War-Hammer, May 21, 2013, 10:28 am

mathuz

#4 May 21, 2013, 9:43 am

Let incircle of the triangle ABC, touches sides BC, CA, AB at points D, E, F, respectively and
$M \in ID, N \in IE, L \in IF \Rightarrow$
$IF \cdot IL = IE \cdot IN = ID \cdot IM$.
So, we see not difficult
M, R, N - collinear;
N, P, L - collinear;
L, Q, M - collinear;
and $MR = NR$, $NP = LP$, $LQ = MQ$.
Therefore, I - circumcenter of the triangle MNL, and orthocenter of the triangle PQR.
(*)

Let IA, IB, IC intersect circumcircle of the triangle ABC at points A_1, B_1, C_1. Then, we have $\triangle A_1B_1C_1$ and $\triangle PQR$ - gomotetic similar. (center I)
So, points I, O and O' (circumcenter of the $\triangle PQR$) are collinear.(**)
Since (*) and (**), we get OI - Euler's line of the triangle PQR.

sunken rock

#5 May 21, 2013, 1:34 pm

Remark: the centers of the circles $\odot ABPQ, \odot BCQR, \odot CARP$ lie on a circle centered O!

Best regards,
sunken rock

Math-lover123

#6 May 21, 2013, 2:06 pm

Very nice to see many different solutions to this nice problem

jayme

#7 May 17, 2014, 8:09 pm

Dear Mathlinkers,
see also
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=565405
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Orthocentre of ADE is the midpoint of BC

Source: ARMO 2013, 9th grade, p2

Sayan

#1 May 20, 2013, 6:28 am • 2

Acute-angled triangle ABC is inscribed into circle Ω. Lines tangent to Ω at B and C intersect at P. Points D and E are on AB and AC such that PD and PE are perpendicular to AB and AC respectively. Prove that the orthocentre of triangle ADE is the midpoint of BC.

Luis González

#2 May 20, 2013, 6:50 am

Denote by M the midpoint of $\overline{BC}$. AP is obviously the A-circumdiameter of $\triangle ADE$ and since AM, AP are isogonals WRT $\angle BAC$, then it follows that AM is the A-altitude of $\triangle ADE$. From cyclic quadrilaterals $PMCE$ and $PMBD$, we get $\angle PME = \angle PCE = \angle ABC = \angle DPM \Longrightarrow ME \parallel PD$, or $ME \perp AD \Longrightarrow EM$ is E-altitude of $\triangle ADE$, hence M is the orthocenter of $\triangle ADE$.

dibyo_99

#3 May 24, 2013, 3:35 pm • 4

I used Complex Number Bashing.

WLOG, assume that $\triangle ABC$ to be circumcentred at the origin.

Then $p = \dfrac{2bc}{b+c}$.

$\implies d = a + b + \dfrac{2b(c-a)}{b+c}$ and $e = a + c + \dfrac{2c(b-a)}{b+c}$.

Let $m = \dfrac{b+c}{2}$.

To show that M is the intersection of D and E altitudes, we need to equivalently prove that M lies on the line through $D \perp AE$ and also on the line through $E \perp AD$.

To do this we need to show that $\dfrac{m-d}{a-e} = -\dfrac{\overline{m}-\overline{d}}{\overline{a}-\overline{e}}$

And, $\dfrac{m-e}{a-d} = -\dfrac{\overline{m}-\overline{e}}{\overline{a}-\overline{d}}$

This is just a series of not too bad calculations.

NewAlbionAcademy

#4 May 26, 2013, 1:12 am

Solution

mathbuzz

#5 Jun 4, 2013, 11:19 am • 1

let the circumcircle of ABC be the unit circle centred at origin.then suppose X is the midpoint of BC
then $x = \frac{b+c}{2}$,$p = \frac{2bc}{b+c}$, $d = (a + b + p - ab\overline{p})/2$ and
$e = (a + c - ac\overline{p})/2$. then $\mu_{AE} = \mu_{AC} = -ac$
also , after a little bit calculation , we get that , $\mu_{DX} = ac$. Click to reveal hidden text
hence DX is perpendicular to AC.
similarly , EX is perpendicular to AD . hence done.

vanu1996
#6 Jan 16, 2014, 6:25 pm • 1

Let DX and EY be the altitudes of ADE,and DX, EY meet BC at F, G,now $DX \parallel PE$,so $\angle DPE + \angle PDX = 180$,hence $\angle PDX = \angle PBC$,so $PDBF$ is cyclic,hence F is the midpoint of BC ,similarly we can prove G is the midpoint of BC,so $F = G$,hence done.

sunken rock
#7 Jan 16, 2014, 10:26 pm

Let M, N ne midpoints of $BC, AP.N$ is the circumcenter of $ADPE, \angle PDE = \angle PAE$ (1). We claim that M, N are ızogonal conjugate w.r.t. $\triangle ADE$.
AM, AP are ızogonals, since AP is symmedian.
From rıght-angled triangle $ADP, \angle ADN = \angle DAN$ (2), while from cyclic $ADPE \implies \angle PDM = \angle PBM = \angle BAC$ (3). From $(1), (2) \wedge (3) \implies \angle EDM = \angle AND$ and, indeed, M, N are ızogonal conjugate points w.r.t. $\triangle ADE$, as claimed; N being the circumcenter, it follows that M is the orthocenter.

Best regards,
sunken rock

Mikasa
#8 May 19, 2014, 11:30 am

Let M be the midpoint of BC. Let AM, DM intersect DE, AE at K, L respectively. Here we use a lemma.

Lemma: Let the tangents at B, C to the circumcircle of $\triangle ABC$ meet at D. Then AD is the A-symmedian of $\triangle ABC$.
Proof: Let the reflection of AD across the angle bisector of $\angle BAC$ meet BC at M'. Then,
$$\frac{BM'}{M'C} = \frac{AM' \times \frac{\sin \angle BAM'}{\sin \angle ABC}}{AM' \times \frac{\sin \angle CAM'}{\sin \angle ACB}} = \frac{\sin \angle BAM'}{\sin \angle ACD} \frac{\sin \angle ABD}{\sin \angle CAM'}$$
$$= \frac{\sin \angle CAD}{\sin \angle ACD} \frac{\sin \angle ABD}{\sin \angle BAD} = \frac{CD}{AD} \frac{AD}{BD} = 1.$$
Thus AM' is the median and AD is the symmedian.

Now by the lemma, in our original problem, AP is the symmedian. Now since $\angle PDA = \angle PEA = 90^\circ$ we have that $PDAE$ is a cyclic quad. So, $\angle EAK = \angle PAD = 90^\circ - \angle APD = 90^\circ - \angle AEK$. Thus $\angle AKE = 90^\circ \Rightarrow AK \perp DE$.

Again, $PB = PC$ in $\triangle PBC$ and M is the midpoint of BC. So, $PM \perp BC \Rightarrow \angle PMB = 90^\circ$. Also we have $\angle PDB = 90^\circ$. So $PDBM$ is a cyclic quad too.

Now,
$\angle ADL = \angle BDM = \angle BPM = 90^\circ - \angle PBM = 90^\circ - \angle PBC = 90^\circ - \angle BAC$
$= 90^\circ - \angle DAL \Rightarrow \angle DLA = 90^\circ$.
So $DL \perp EA$.

Thus M is the orthocenter of $\triangle ADE$.

SmartClown
#9 Jul 21, 2014, 9:46 pm

Since PB and PC are tangents $PB = PC$.Let M be the midpoint of BC.Since $\angle PMC = 90$ and $\angle PEC = 90$ quadriletal $PMCE$ is cyclic so $\angle PME = \angle PCE = \angle ABC$.Let ME intersect AB at K.Then $\angle KMB = 90 - \angle PME = 90 - \angle ABC$.Because of that it is $\angle BKM = 90$ so $EM \perp AB$.We similarly prove for DM so M is the orthocenter.

junioragd
#10 Jul 21, 2014, 9:57 pm • 1

Let M be the midpoint of BC.Now,since PB=PC,we have <PMB = <PMC =90,and now we have two cyclics,DPMB and and PECM,and the rest is angle chase

esque
#11 Aug 1, 2014, 3:08 am • 1

From angle chasing, we know $\angle PBC = \angle A$ so $\angle PBD = \angle C$ and $\angle DPB = 90^\circ - \angle C$. Let H be the foot of the perpendicular from D to AE, and let M be the point where DH intersects BC. Since $\angle MAH = \angle C$, we must have $\angle DMB = 90^\circ - \angle C$. We note $\angle DPB = \angle DMB$ so quadrilateral $DPMB$ is cyclic. Then from $\angle BAH = \angle A$, it follows $\angle BPM = \angle BDM = 90^\circ - \angle A$. Hence $\angle BMP = 90^\circ$, so M is the foot of the altitude from P to BC, which is the midpoint of BC. Similarly, the intersection of the altitude from E to AD and BC is the midpoint of BC. Thus the orthocenter of $\triangle ADE$ is M.

algemania
#12 Aug 1, 2014, 9:10 am

simple angle chasing. M is BC's midpoint, and angleBOD=angle CPD=90-C, so PO is perpendicular to AC

infiniteturtle
#13 Aug 3, 2014, 1:06 am

It's trivial; M the midpoint of BC and then $PMBD$ cyclic so $\angle ADM = \frac{\pi}{2} - \angle MDP = \frac{\pi}{2} - \angle MBP = \frac{\pi}{2} - \angle BAC$ so $DM \perp AC$.

thecmd999
#14 Sep 23, 2014, 11:59 am

Solution

utkarshgupta
#15 Dec 31, 2014, 5:42 pm

A not so nice solution to the nice problem

Call M the midpoint of BC

It is also well known that AP is the extended symmedian of ABC
$\implies \angle PAE = \angle MAD$ and since $PEAD$ are concyclic $\angle APE = \angle ADE$
But since $\angle PAE + \angle APE = 90$
$\implies AM \perp DE$
And now

> *infiniteturtle wrote:*
> It's trivial; M the midpoint of BC and then $PMBD$ cyclic so $\angle ADM = \frac{\pi}{2} - \angle MDP = \frac{\pi}{2} - \angle MBP = \frac{\pi}{2} - \angle BAC$ so $DM \perp AC$.

Thus we are done

euclid1998
#16 Jan 1, 2015, 12:52 am • 2

Let M be the midpoint and O be the center of the circle,
let EM extended cut AB at G
By easy angle chasing we see that $\angle ECP = B$, $\angle PBD = C$, $\angle MBP = \angle MCP = A$,
$\angle OBM = 90 - A$, $\angle OBA = 90 - C$
Quadrilateral $ECMP$ is cyclic so $\angle CME = \angle CPE = 90 - B = \angle BMG$,

Also $\angle MBA = \angle CBA = B$ so $\angle BGM = 90$ i.e $EM \perp AB$
Similarly $DM \perp AC$ so M is the orthocenter of ΔAED
Proved

aditya21

#17 Jan 2, 2015, 10:59 am • 1

my solution =
let
let M be midpoint of BC than $BM = MC$.
also let EM meet AB at R and similarly let DM meet AC at Q.
now as $\angle PMB = 90 = \angle CEP$ thus $MPEC$ is cyclic quad.
thus $\angle MPC = \angle MEC$
since $\angle MCP = \angle CAB$ thus we get that $\angle MPC = 90 - \angle CAB = \angle CME = \angle CER$
thus we easily get in triangle AER that $\angle ARE = 90$
analogously we prove that $\angle ADM = \angle ADQ = 90$
thus M being intersection of altituded DQ and ER is the orthocentre of triangle ADE
HENCE PROVED

jayme

#18 Apr 4, 2015, 7:41 pm

Dear Mathlinkers,
we can avoid all calculation by using the Reim's theorem in different ways...
Sincerely
Jean-Louis

jlammy

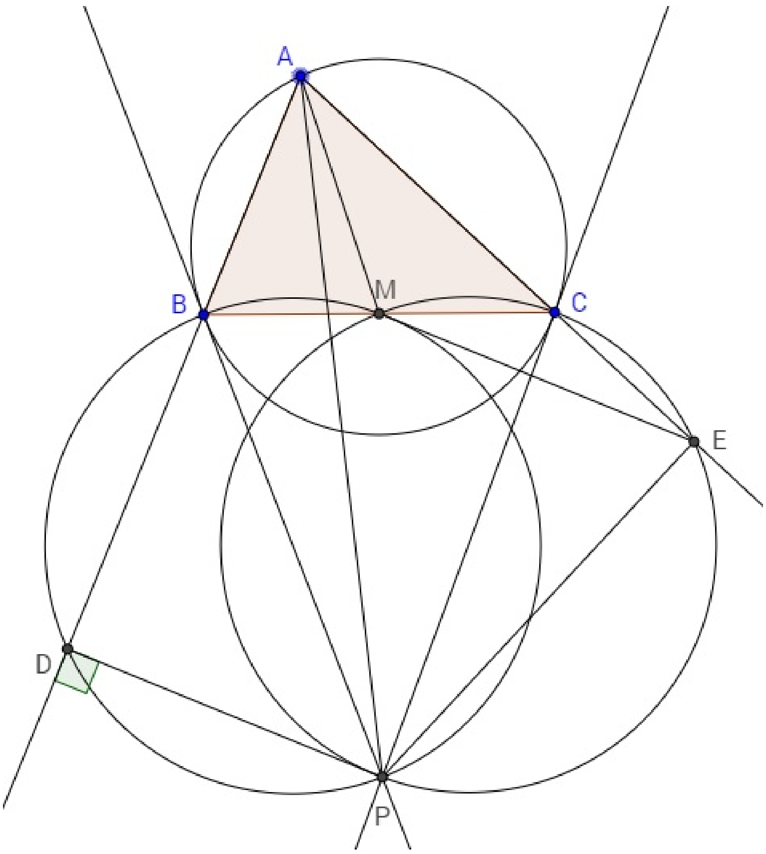
#19 Apr 4, 2015, 8:30 pm

Let M be the midpoint of $\overline{BC}$. Since $\overline{AP}$ is the A-diameter of (ADE), and it is isogonal to the median AM (wrt $\triangle ABC$), it follows AM is an altitude of $\triangle ADE$.

Incomplete finish

Angle chasing finishes the problem.

Attachments:



This post has been edited 1 time. Last edited by jlammy, Apr 4, 2015, 9:10 pm
Reason: Incomplete stuff hidden; diagram added.

KudouShinichi
#20 Apr 24, 2015, 3:05 pm

Drop perpendicular from D to AC and name the new point as K .

KD intersects BC at H

As BP is the tangent so $\angle PBH = \angle A$

so $\angle PBD = \angle C$, as $ADPE$ cyclic so $\angle DPE = 180 - \angle A$

and $DK \parallel PE$ so $\angle HDP = \angle A$ so $HBDP$ cyclic

Then by angle chasing we get $CHPE$ cyclic which implies $\angle DHE = 180 - \angle A$

so H is the orthocentre of $\triangle ADE$

As $PB = PC$ (equal common tangents) we get $\triangle PBH \cong \triangle PCH$

which implies H is the midpoint of BC

npip99
#21 Jul 25, 2015, 11:43 pm

Barycentric Solution

bobthesmartypants
#22 Mar 12, 2016, 5:21 am

My solution:

Let M be the midpoint of BC and Q, R be the feet of the altitudes from M to AB, AC respectively. We have

$$\angle QMR \overset{AQMR}{\underset{\text{cyclic}}{=}} 180 - \angle A = 180 - \angle BCP \overset{BC \perp MP}{=} 180 - (90 - \angle CMP) \overset{CEPM}{\underset{\text{cyclic}}{=}} 180 - (90 - \angle CEM) = 180 - \angle EMR \implies E, M, Q \text{ collinear}$$

Similarly, D, M, R collinear so M is the orthocenter of $\triangle ADE$, done.

quangMavis1999
#23 Mar 12, 2016, 10:45 am

Generalization : Let triangle ABC and circumcenter O. Let P is arbitrary point lie on $\odot(BOC)$. Points D and E are projections of P to AB, AC. Prove that the orthocentre of $\triangle ADE$ moves on fixed circle when P moves.

TelvCohl
#24 Mar 12, 2016, 2:03 pm

> *quangMavis1999 wrote:*
> **Generalization :** Let triangle ABC and circumcenter O. Let P is arbitrary point lie on $\odot(BOC)$. Points D and E are projections of P to AB, AC. Prove that the orthocentre of $\triangle ADE$ moves on fixed circle when P moves.

Let T be the orthocenter of $\triangle ADE$ and let $D^* \equiv TD \cap OP, E^* \equiv TE \cap OP$. From $\measuredangle BPD^* = \measuredangle BCO = \measuredangle BDD^* \Longrightarrow B, D, D^*, P$ are concyclic, so D^* is the projection of B on OP. Similarly, we can prove E^* is the projection of C on OP, so we conclude that T is the orthopole of OP WRT $\triangle ABC \Longrightarrow T$ lies on the 9-point circle of $\triangle ABC$.

Quick Reply

High School Olympiads

Prove concurrent

Reply

oscarloi

#1 May 8, 2013, 3:03 pm

$(O_1), (O_2), (O_3)$ are the A, B, C mixtilinear incircles of ΔABC. They touch (ABC) at D, E, F reps. FO_1 cuts DO_3 at Z. FO_2 cuts EO_3 at X. DO_2 cuts EO_1 at Y. Prove that YO_3, XO_1, ZO_2 are concurrent.

Luis González

#2 May 19, 2013, 7:43 am

Since $U \equiv EF \cap BC, V \equiv FD \cap CA$ and $W \equiv DE \cap AB$ are the exsimilicenters of $(O_2) \sim (O_3), (O_3) \sim (O_1)$ and $(O_1) \sim (O_2)$, respectively, then $U, V, W,$ are collinear on a homothety axis τ of $(O_1), (O_2), (O_3)$.

If N, L denote the midpoints of the arcs CA, AB of $\odot(ABC)$, then $N \in EV$ and $L \in FW$ (this was proved during the solution of the problem Mixtilinear Incircles Parallels). Thus, by Pascal theorem for $CLFEAB$, the intersections $R \equiv CL \cap EA, W \equiv LF \cap AB$ and $U \equiv FE \cap BC$ are collinear, i.e. $R \in \tau$. Analogously, $Q \equiv BN \cap AF$ is on τ. If $I \equiv BN \cap CL$ is the incenter of $\triangle ABC$, then it follows that $\triangle IO_2O_3$ and $\triangle AFE$ are perspective through τ, thus by Desargues theorem $X \equiv AI \cap EO_3 \cap FO_2$. Similarly, the lines YO_3 and ZO_2 pass through I.

Quick Reply

High School Olympiads

regular hexagon geometry

Reply

school5

#1 May 17, 2013, 9:23 pm

Given is regular hexagon $ABCDEF$ with center O. M and N are midpoints of CD and DE and L intersection of AM and BN. Prove that:

(1) ABL and $DMLN$ have equal area
(2) $ALO = OLN = 60$
(3) $OLD = 90$

Luis González

#2 May 18, 2013, 12:25 am

Obviously, $\triangle OMA \cong \triangle ONB$ are congruent by SSS criterion $\Longrightarrow \angle OML = \angle ONL \Longrightarrow OLMDN$ is cyclic and O, D are midpoints of the arcs MLN and MDN of its circumcircle $\Longrightarrow LO, LD$ bisect $\angle MLN \Longrightarrow \angle OLD = 90^\circ$. Since $\angle MLN = \angle MON = 60^\circ$, then $\angle ALO = \angle OLN = 60^\circ$.

If AM cuts the hexagon circumcircle (O) again at P, then clearly $\triangle BLP$ is L-isosceles with 120º-30º-30º, thus $LB = LP$. If K is the reflection of N on the bisector LD of $\angle MLN$, we have $\angle MKD = \angle LND = \angle DMK \Longrightarrow \triangle DMK$ is D-isosceles with D-altitude DP, i.e. P is midpoint of $\overline{MK} \Longrightarrow$

$$LB = LP = \tfrac{1}{2}(LM + LK) = \tfrac{1}{2}(LM + LN).$$

If X, Y, Z denote the projections of L on AB, CD, DE, then clearly $\triangle LBX \sim \triangle LMY \sim \triangle LNZ$. Therefore the latter expression gives rise to $LX = \tfrac{1}{2}(LY + LZ) \Longrightarrow [ABL] = \tfrac{1}{2}([CDL] + [EDL])$, but $[CDL] = 2[MDL]$ and $[EDL] = 2[NDL] \Longrightarrow [ABL] = [MDL] + [NDL] = [DMLN]$.

Quick Reply

High School Olympiads

Inscribed Quadrilateral and Colinearity

Reply

Source: Unknown

jowramos

#1 May 11, 2013, 9:35 am

Let ABC be a triangle, with a circle k that passes through B, C and intersects AB at M and AC at N. Let P, Q be points, respectively, on the lines MN and BC, such that the bisector of the angle BAC is also a bisector of the angle PAQ. If the midpoint of MC is R, the midpoint of NB is S and the midpoint of PQ is T, prove that R, S, T are colinear.

Luis González

#2 May 17, 2013, 9:23 pm

Assume that BCNM is convex and T is inside of it, remaning cases are proved similarly.

$$[BPC] = [BCNM] - [BPM] - [CPN] =$$

$$= [BCNM] - \frac{MP}{MN} \cdot [BMN] - \frac{NP}{MN} \cdot [CNM] \quad (1).$$

Similarly, we get $[MQN] = [BCNM] - \frac{BQ}{BC} \cdot [BCM] - \frac{CQ}{BC} \cdot [BCN] \quad (2).$

Since $\triangle ABC$ and $\triangle ANM$ are similar with corresponding cevians AQ, AP, then $\frac{NP}{MN} = \frac{BQ}{BC}$ and $\frac{MP}{MN} = \frac{CQ}{BC}$. Hence, adding (1) and (2) together gives

$$[MQN] + [BPC] = 2[BCNM] - [BCNM] \left(\frac{NP + MP}{MN} \right) = [BCNM].$$

$$\Longrightarrow [TMN] + [TBC] = \tfrac{1}{2}([MQN] + [BPC]) = \tfrac{1}{2}[BCNM] \Longrightarrow T \in RS.$$

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=532017.

vittasko

#3 May 18, 2013, 6:32 pm

EQUIVALENT PROBLEM. - A cyclic quadrilateral $ABCD$ is given and let be the points $E \equiv AB \cap CD$ and $F \equiv AD \cap BC$. Through the point E, we draw an arbitrary line which intersects the line $AD,\ BC$, at poins $A',\ B'$, respectively and suppose that B' is between $F,\ B$. The line through E and isogonal to EA', with respect to the angle $\angle AED$, intersects the lines $BC,\ AD$, at points $C',\ D'$, respectively. Prove that the Gauss-Newton line of the quadrilateral $A'B'C'D'$ is coincided with the one of $ABCD$.

• Let $L,\ N$ be, the points of intersection of $AD,\ BC$ respectively, form the angle bisector of $\angle AED$.

Because of the cyclic quadrilateral $ABCD$, we have the similarity of the triangles $\triangle\ EAD,\ \triangle\ ECB$ and then, $\frac{EB}{ED} = \frac{EC}{EA}$
$\Longrightarrow \frac{EB}{EC} = \frac{ED}{EA} \quad ,(1)$

But, $\frac{EB}{EC} = \frac{NB}{NC} \quad ,(2)$ and $\frac{ED}{EA} = \frac{LD}{LA} \quad ,(3)$ because of the line ENL bisects the angle $\angle AED$.

From $(1),\ (2),\ (3) \Longrightarrow \frac{NB}{NC} = \frac{LD}{LA} \quad ,(4)$

We consider the non-convex quadrilateral $ACBD$ and from (4), based on the **ERIQ** (= **E**qual **R**atios **I**n **Q**uadrilateral) **theorem** (*), we have that the points $P,\ Q,\ M,$ as the midpoints of $AC,\ BD,\ LN$ respectively, are collinear.

That is, the points $P,\ Q,$ lie on the constant line $MT,$ where T is the midpoint of $EF.$

● It is easy to show that the quadrilateral $A'B'C'D'$ is also cyclic, because of $\angle B'C'E = \angle BCE - \angle CEC' = \angle BAD - \langle AEA' = \angle B'A'D'$ and so, from the similar triangles $\triangle\ EA'D',\ \triangle\ EC'B'$ we have as before $\frac{NB'}{NC'} = \frac{LD'}{LA'}$ $,(5)$ because of the line ENL bisects the angle $\angle A'ED'$ too.

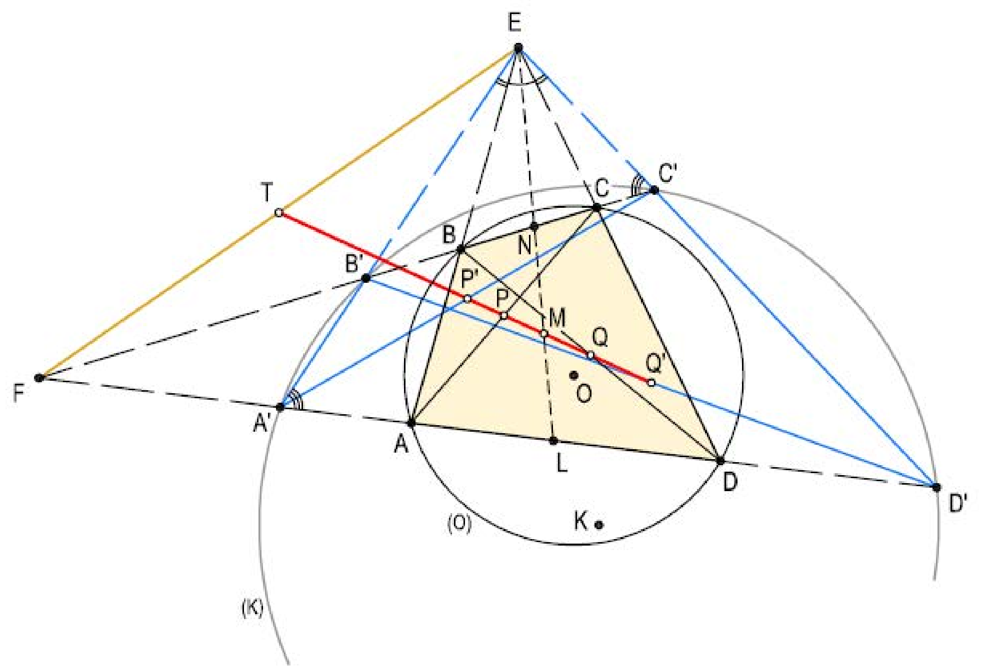
So, based again on the **ERIQ theorem**, we conclude that $P',\ Q',$ as the midppoints of $A'C',\ B'D'$ respectively, also lie on the line MT and the proof of the equivalent problem is completed.

Kostas Vittas.

(*) Two elementary proofs of this powerful theorem, have been posted Here.

Kostas Vittas.

Attachments:



High School Olympiads

isosceles trapezoid

Pirkuliyev Rovsen

#1 May 1, 2013, 3:31 pm

A circle is inscribed in an isosceles trapezoid $ABCD$.The diagonal AC intersects the circle at K and L, in the order A,K,L,C.

Find the value of $\sqrt[4]{\frac{AL\cdot KC}{AK\cdot LC}}$.

Azerbaijan Land of the Fire

Luis González

#2 May 2, 2013, 3:23 am • 1

This ratio is equal for any tangential quadrilateral whose intouch quadrilateral is harmonic, the isosceles trapezoids are just particular cases. If P,Q,R,S denote the tangency points of the incirle τ with $AB,BC,CD,DA,$ then there exists a projective transformation (central projection followed by a parallel projection) that carries $PQRS$ with its circumcircle τ into a square $P^*Q^*R^*S^*$ inscribed in another circle $\tau^* \Longrightarrow ABCD$ is also transformed into a square. Since the cross ratio (A,L,K,C) is preserved, then it suffices to find the ratio for a square $ABCD.$ If O denotes then the center of $ABCD,$ we have

$$AK=OA-OK=\tfrac{\sqrt{2}}{2}AB-\tfrac{1}{2}AB=\tfrac{\sqrt{2}-1}{2}AB$$

$$CK=AC-AK=\sqrt{2}AB-\tfrac{\sqrt{2}-1}{2}AB=\tfrac{\sqrt{2}+1}{2}AB$$

$$(A,L,K,C)=\frac{AL}{AK}\cdot\frac{CK}{CL}=\frac{CK^2}{AK^2}=\frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)^2}=(\sqrt{2}+1)^4.$$

sunken rock

#3 May 2, 2013, 3:55 am

That's 1999 Belarus NO, 4th round, pr. 10.4 (overthere, without radical)!

Best regards,
sunken rock

High School Olympiads

right triangle

Reply

Source: own

leader

#1 May 1, 2013, 6:27 am

Problem 1: In right $\triangle ABC(\angle BAC=90)$ E is the midpoint of AC. G is a variable point such that $\angle GBE=90$. If P,Q are feet of tangents from G to $k(C,CA)$ prove that there exist 2 fixed points X,Y (different from C) such that P,Q,X,Y are con-cyclic.
P2 describes my way of construction for P1
Problem 2: In right triangle $BAC(\angle BAC=90)$. G is variable such that $GE\perp BC(E$ is the midpoint of $BA)$. if P,Q are feet of tangents from G to $k(C,CA)$ prove that circle BPQ passes through a fixed point.

This post has been edited 3 times. Last edited by leader, May 1, 2013, 2:49 pm

Luis González

#2 May 1, 2013, 7:30 am • 1

Actually, G can run on any line ℓ through B not cutting the circle k (not necessarily the perpendicular to BE at B, as the problem states). We denote X the projection of A on BC and AX cuts ℓ at Y. We prove that P,Q,X,Y are concyclic.

If D is the projection of C on ℓ, then the polars PQ and AX of G and B WRT k meet at the pole $M\in DC$ of ℓ WRT k. Since $\angle CPG,\angle CQG$ and $\angle CDG$ are right, then C,P,Q,G,D are concyclic $\Longrightarrow MP\cdot MQ=MC\cdot MD$. But since $CXDY$ is cyclic due to the right angles at X,D, we have $MC\cdot MD=MX\cdot MY\Longrightarrow MP\cdot MQ=MX\cdot MY\Longrightarrow$ P,Q,X,Y are concyclic, as desired.

leader

#3 May 1, 2013, 2:41 pm

I guess this proves there are infinitely many pairs of points X,Y such that P,Q,X,Y are con-cyclic.
Anyway in my first post i will add the problem using the construction i had in mind.

Quick Reply

High School Olympiads

The locus of intersection points of two lines

Reply

Source: Czech-Polish-Slovak 2001 Q4

djb86
#1 Apr 30, 2013, 1:06 pm

Distinct points A and B are given on the plane. Consider all triangles ABC in this plane on whose sides BC, CA points D, E respectively can be taken so that
(i) $\frac{BD}{BC}=\frac{CE}{CA}=\frac{1}{3}$;
(ii) points A, B, D, E lie on a circle in this order.

Find the locus of the intersection points of lines AD and BE.

Luis González
#2 May 1, 2013, 2:20 am

$CE\cdot CA=\frac{1}{3}CA\cdot CA=CD\cdot CB=\frac{2}{3}CB\cdot CB\Longrightarrow CA^2=2\cdot CB^2\Longrightarrow \frac{CA}{CB}=\sqrt{2}\Longrightarrow$ ratio of distances from C to A, B is constant $\Longrightarrow C$ moves on the Apollonius circle $\mathcal{O}$ of $\overline{AB}$ referred to that ratio. Let $P\equiv AD\cap BE$ and $F\equiv CP\cap AB.$ Using Ceva's theorem for the concurrent cevians AD, BE, CF and Menelaus' theorem for $\triangle CBF$ cut by $APD,$ we get

$$\frac{AF}{FB}=\frac{DC}{BD}\cdot\frac{EA}{CE}=2\cdot 2=4\ ,\ \frac{PF}{CP}=\frac{AF}{AB}\cdot\frac{BD}{DC}=\frac{4}{5}\cdot\frac{1}{2}=\frac{2}{5}\Longrightarrow\frac{FP}{FC}=\frac{2}{7}.$$

Hence, F is fixed and the ratio $\frac{FP}{FC}$ is constant $\Longrightarrow$ locus of P is the homothetic circle of $\mathcal{O}$ under homothety with center F and ratio $\frac{2}{7}$.

mathuz
#3 May 12, 2013, 10:23 am

it's easy by Appalonious circle. Locus is circle.

Quick Reply

High School Olympiads

D,Q,F are collinear

tdl

#1 Apr 30, 2013, 8:44 am

Let a circle (O) with a diameter AB. $C \in (O)$ so that $AC > BC$. Two tangent lines at A, C intersect at D.
$BD \cap (O) = E$. CH is perpendicular with AB at H.
a) $DH \cap AE = I, CI \cap AD = K$. Prove that: KE is tangent line of (O).
b) $F \in (O) : EF||AD, S \in KB : ES||AB, OS \cap AE = Q$. Prove that: D, Q, F are collinear.

Luis González

#2 Apr 30, 2013, 10:42 pm

a) Let AE cut CH at M. AC is polar of D WRT $(O) \Longrightarrow A(B,E,C,D) = -1 \Longrightarrow A(H,M,C,D) = -1$, but since $AD \parallel CH$ (both perpendicular to AB), it follows that C is the midpoint of $\overline{MH}$, thus CI cuts AD at its midpoint K. Since $\angle AED$ is right, then $\triangle AKE$ is K-isosceles $\Longrightarrow KE$ is tangent to (O).

b) Let $P \equiv EF \cap AB$. Since $EP \parallel AD$, the B-median BK of $\triangle BDA$ bisects $\overline{EP}$. Together with $ES \parallel BP$, then $EBPS$ is a parallelogram $\Longrightarrow ES = PB$. Hence

$$\frac{AQ}{QE} = \frac{OA}{ES} = \frac{OB}{PB} = \frac{1}{2} \cdot \frac{AB}{PB} = \frac{1}{2} \cdot \frac{AD}{PE} = \frac{AD}{EF} \Longrightarrow D, Q, F \text{ are collinear.}$$

High School Olympiads

DX bisects BI

Reply

Source: own

leader

#1 Apr 30, 2013, 7:54 pm

Please give your opinion on the problem if you have one because i am new at problem making.

P, I are midpoint of arc BAC and the incenter of ABC. PI meets circle ABC again at D and X is on BC such that $IX||BP$. Prove that DX bisects BI.

Luis González

#2 Apr 30, 2013, 8:52 pm • 1

$\angle IXC = \angle PBC = \angle PDC \equiv \angle IDC \Longrightarrow CIXD$ is cyclic. Moreover, $\angle BXI = 180^\circ - \angle PBC = 180^\circ - (90^\circ - \frac{1}{2}\angle A) = 90^\circ + \frac{1}{2}\angle A = \angle BIC \Longrightarrow \angle BIX = \angle ICB \Longrightarrow IB$ is tangent to $\odot(CIX)$. $\angle CIX = 90^\circ + \frac{1}{2}\angle A - \frac{1}{2}\angle C \Longrightarrow$ $\angle BDX = 180^\circ - \angle A - (90^\circ - \frac{1}{2}\angle A + \frac{1}{2}\angle C) = \frac{1}{2}\angle B = \angle IBC \Longrightarrow IB$ is tangent to $\odot(BDX)$. Hence DX is radical axis of $\odot(CIXD)$ and $\odot(BDX)$, bisecting their common tangent $\overline{IB}$.

leader

#3 May 1, 2013, 4:50 am

Here's my original solution.
Extend AI to meet circle ABC at K again. then since $\angle XID = \angle BPD = \angle XCD$ we get that $CIXD$ is cyclic. but if M is the midpoint of BI since $\angle KIB = (\angle BAC + \angle ABC)/2 = \angle KBI$ and therefore $KI = KB$ then $\angle KMI = 90 = \angle PDK = \angle IDK$ so $IMDK$ is cyclic. now $\angle MDI = \angle MKI = \angle BKA/2 = \angle BCA/2 = \angle ICX = \angle IDX$ so D, X, M are collinear.

Quick Reply

High School Olympiads

Four beautiful properties

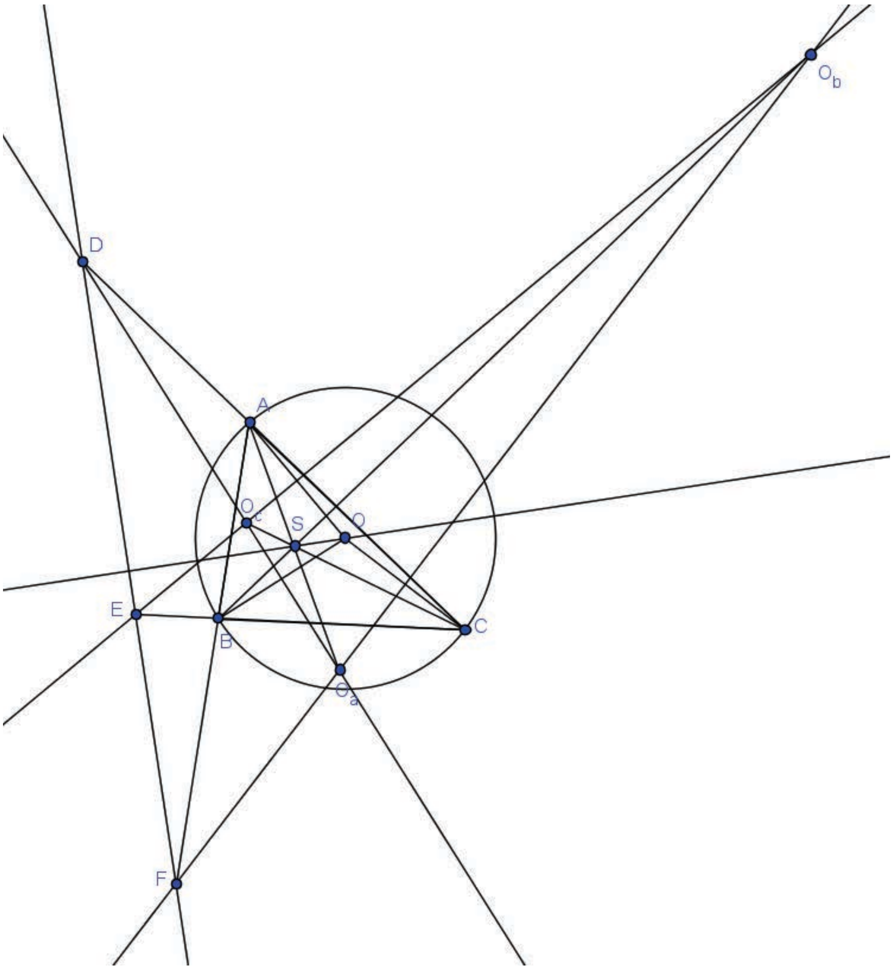
Reply

CTK9CQT

#1 Apr 30, 2013, 3:40 pm

Let ABC is an triangle and (O) be its circumcircle. The perpendicular bisector of OA cuts BC and the perpendicular of OB at E and O_c, respectively. We define (D, O_a) ; (F, O_b) similarly. Prove that:
a) D, E, F are collinear
b) AO_a, BO_b and CO_c and are concurrent at a point S
c) X and S are two isogonal conjugate points, where X is the center of the nine points circle of the triangle ABC
d) OS is perpendicular to DEF
(I did not draw the point X, for the pure figure)

Attachments:



Luis González

#2 Apr 30, 2013, 8:11 pm

Please use the search for old or well-known problems.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=1395
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=142469
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=373509
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=462456

Quick Reply

High School Olympiads

N,C,I are collinear

Reply

tdl

#1 Apr 30, 2013, 9:00 am

Let a circle (O) with a diameter AB. $C \in (O)$ so that $AC > BC$. Two tangent lines at A, C intersect at D.
$BD \cap (O) = E, M \in BC : OM||AD$.
a) $AE \cap DM = N, AC \cap OD = H$. Prove that: $HN||OC$.
b) $AC \cap DM = S, BS \cap (O) = I$. Prove that: N, C, I are collinear.

Luis González

#2 Apr 30, 2013, 11:05 am • 1

a) Trivially $MD \perp AD$. Since $\angle DEA$ is right, we have $ND^2 = NE \cdot NA \Longrightarrow N$ has equal power WRT (O) and the circle (D) with zero radius $\Longrightarrow N$ is on radical axis of $(O), (D) \Longrightarrow N$ is on perpendicular bisector of $\overline{HD}$, i.e. $\triangle NDH$ is N-isosceles $\Longrightarrow \angle NHD = \angle NDH = \angle DAC = \angle DOC \Longrightarrow HN \parallel OC$.

b) $J \equiv AC \cap OM$ is the pole of the line DMN WRT $(O) \Longrightarrow I(A, C, J, S) = -1$, but since $AECB$ is harmonic, we have $I(A, C, E, S) = -1 \Longrightarrow I, E, J$ are collinear. Now, from the complete cyclic quadrilateral $EAIC$, the lines AE, IC meet on the polar DM of J WRT (O), i.e. N, C, I are collinear.

Quick Reply

High School Olympiads

O is orthocenter of triangle A'B'C

Locked

tdl

#1 Apr 30, 2013, 7:28 am

Let a triangle ABC inscribed in circle (O) and $A' \in BC, B' \in CA, C' \in AB$ so that two triangle ABC and $A'B'C'$ are similar. Prove that: O is orthocenter of triangle $A'B'C'$.

Luis González

#2 Apr 30, 2013, 8:57 am

Posted several times before. It's Turkey TST 2007; day 2 problem 1.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=26494
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=143090
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=355657
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=364950

High School Olympiads

Place!

Reply

erfan2108

#1 Apr 29, 2013, 8:01 pm

Find the place of the all points P on the plane of a quadrilateral ABCD, such that S(APB)+S(CPD)=S(BPC)+S(APD). Where S(XYZ) denotes the area of triangle XYZ.

Luis González

#2 Apr 29, 2013, 11:38 pm

Locus of points P is a whole line ℓ if we consider oriented areas. Put $ABCD$ on a rectangular reference (x,y) and let $\delta_1,\delta_2,\delta_3,\delta_4$ denote the oriented distances from P to $AB,BC,CD,DA.$ Equation is obtained as $\delta_1\cdot|AB|+\delta_3\cdot|CD|=\delta_2\cdot|BC|+\delta_4\cdot|DA|.$ Since the formula of the oriented distance from $P(x,y)$ to a line is a linear function of $x,y,$ it follows that δ_i are linear fuctions of $x,y \Longrightarrow$ locus ℓ of $P(x,y)$ is a linear fuction of $x,y,$ i.e. it is either a line or the whole plane. Midpoints M,N of AC,BD obviously fulfill the property, thus ℓ is the Newton line of $ABCD$ passing through $M,N.$

See also http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=141467.

Quick Reply

High School Olympiads

Czech-Polish-Slovak 2003 Geometry

Reply

Source: Czech-Polish-Slovak 2003 Q2

djb86

#1 Apr 28, 2013, 7:41 pm

In an acute-angled triangle ABC the angle at B is greater than 45°. Points D, E, F are the feet of the altitudes from A, B, C respectively, and K is the point on segment AF such that $\angle DKF = \angle KEF$.
(a) Show that such a point K always exists.
(b) Prove that $KD^2 = FD^2 + AF \cdot BF$.

Luis González

#2 Apr 29, 2013, 11:35 am • 1

Since AB is the external bisector of $\angle DFE,$ it follows that the reflection P of D across AB is the second intersection of EF with the circle $\odot(AEDB)$ with diameter $\overline{AB}$. $\angle PKF = \angle DKF = \angle KEF \Longrightarrow PK$ is tangent to $\odot(EFK) \Longrightarrow$ $PK^2 = PF \cdot PE \Longrightarrow K$ is always the intersection of $\overline{AF}$ with the circle with center P and radius $\sqrt{PF \cdot PE}$.

$$KD^2 = KP^2 = PF \cdot PE = DF \cdot (DF + FE) = DF^2 + DF \cdot FE =$$

$$= DF^2 + PF \cdot FE = DF^2 + AF \cdot BF.$$

Quick Reply

High School Olympiads

AK, HS, (LBD) are concurrent

Reply

tdl

#1 Apr 28, 2013, 5:04 pm

Let an acute triangle ABC so that $AB < AC$ inscribed circle (O). Altitude BD, CE intersect at H. AI is a diameter of circle (O). $DE \cap (O) = S$ in small arc AB. $SI \cap BC = K$, $HS \cap BC = L$. Prove that:
a) AK is perpendicular with HS.
b) $AK, HS, (LBD)$ are concurrent.

Luis González

#2 Apr 28, 2013, 9:55 pm

Let F be the foot of the A-altitude and T the second intersection of ED with (O). $AO \perp DE$, is then perpendicular bisector of $\overline{ST}$, i.e. $AS = AT$. The inversion with center A and radius $AS = AT$ swaps (O) and $ED \Longrightarrow$ $AS^2 = AE \cdot AB = AH \cdot AF \Longrightarrow H$ is the inverse of $F \Longrightarrow$ perpendicular BC to AH through F is the polar of H WRT (A, AS). Since $\angle ISA$ is right, then $\overline{IKS}$ is tangent of (A, AS). Hence, polar of K WRT (A, AS) will pass through S and $H \Longrightarrow AK \perp HS$ at $U \Longrightarrow U$ is on the circumcircle of the cyclic $ADHE \Longrightarrow \angle BDU = \angle FAU = \angle BLU \Longrightarrow$ $U \in \odot(LBD)$, i.e. $AK, HS, \odot(LBD)$ concur.

Quick Reply

High School Olympiads

Concurrency of lines

Reply

61plus

#1 Apr 27, 2013, 3:51 pm

Let $ACBD$ be a quadrilateral. E and F are intersections of AC and BD and BC and AD. Choose a point O on any circle ω. Let OA, OB, OC, OD, OE, OF intersect ω again at A', B', C', D', E', F' .Show that $A'B', C'D', E'F'$ concurrent.

Luis González

#2 Apr 28, 2013, 10:18 am

Let OB, OD, OF cut AC at B_0, D_0, F_0, respectively. By Desargues theorem, the line AC cuts the opposite sidelines of the complete quadrilateral $OBFD$ at pairs of point in involution, i.e. $E \mapsto F_0, A \mapsto B_0, C \mapsto D_0$ is an involution $\Longrightarrow$ $OE \mapsto OF, OA \mapsto OB$ and $OC \mapsto OD$ is an involutive pencil. Let $P \equiv E'F' \cap C'D'$ and $A'P$ cuts ω again at B''. $C' \mapsto D', E' \mapsto F', A' \mapsto B''$ is a circular involution with center $P \Longrightarrow OE' \mapsto OF', OC' \mapsto OD', OA' \mapsto OB''$ is an involution, therefore lines OB and OB'' coincide $\Longrightarrow B' \equiv B'' \Longrightarrow A'B', C'D', E'F'$ concur.

Quick Reply

High School Olympiads

Locus of points for which products of areas are equal

Reply

Source: Czech-Polish-Slovak 2005 Q5

djb86

#1 Apr 28, 2013, 12:58 am

Given a convex quadrilateral $ABCD$, find the locus of the points P inside the quadrilateral such that

$$S_{PAB} \cdot S_{PCD} = S_{PBC} \cdot S_{PDA}$$

(where S_X denotes the area of triangle X).

Luis González

#2 Apr 28, 2013, 4:45 am • 1

Taking away the restrictions imposed in the problem, i.e. considering 4 points A,B,C,D in general position with no three of them collinear, then the locus will be the union of a conic through A,B,C,D and the lines AC,BD. Using barycentric coordinates WRT $\triangle ABC$, and denoting $(x:y:z), (p:q:r)$ the coordinates of P, D, the areas of $\triangle PAB, \triangle PBC, \triangle PCD, \triangle PDA$ are given by

$$\frac{[PAB]}{[ABC]} = \left|\frac{z}{x+y+z}\right| \ , \ \frac{[PBC]}{[ABC]} = \left|\frac{x}{x+y+z}\right|$$

$$\frac{[PCD]}{[ABC]} = \left|\frac{qx-py}{(x+y+z)(p+q+r)}\right| \ , \ \frac{[PDA]}{[ABC]} = \left|\frac{ry-qz}{(x+y+z)(p+q+r)}\right|$$

$$[PAB] \cdot [PCD] = [PBC] \cdot [PDA] \Longrightarrow z|qx-py| = x|ry-qz| \Longrightarrow$$

$$z^2(qx-py)^2 = x^2(ry-qz)^2 \Longrightarrow y(rx-pz)(pyz-2qzx+rxy) = 0.$$

Either $y=0 \Longrightarrow P \in AC, rx-pz=0 \Longrightarrow P \in BD$, or $pyz-2qzx+rxy=0 \Longrightarrow P$ is on a circumconic $\mathcal{C}$ of $\triangle ABC$ through D. $\mathcal{C}$ is the isogonal of the line D^*U WRT $\triangle ABC$, where D^* is the isogonal conjugate of D WRT $\triangle ABC$ and U is on AC, such that $U \equiv \left(-\frac{r}{c^2} : 0 : \frac{p}{a^2}\right)$.

mathuz

#3 Apr 30, 2013, 9:35 pm • 1

Let $\angle APB = x_1$, $\angle BPC = y_1$, $\angle CPD = x_2$, $\angle DPA = y_2$.
So, math=inline]$ APB $[/mathmath=inline]$ BPC $[/math=math=inline]$ CPD $[/mathmath=inline]$ DPA $[/math $\Rightarrow$
$\sin x_1 \sin x_2 = \sin y_1 \sin y_2 \Rightarrow \cos(x_1 - x_2) = \cos(y_1 - y_2)$.
Hence, locus of the point P is dioganals of the quadriteral $ABCD$.

Quick Reply