High School Olympiads

If And Only If

Headhunter
#1 Dec 2, 2010, 1:51 am

Hello.

Let BE, CF be two altitudes of a triangle ABC, and H is its orthocenter.
Let l be the perpendicular to CA, (passing A).
Show that BC, EF, l are concurrent if and only if H is the midpoint of BE

Luis González
#2 Dec 2, 2010, 11:32 am

Let D be the foot of the A-altitude and $P \equiv EF \cap BC. P \in \ell \Longleftrightarrow AP \parallel BE \Longleftrightarrow \frac{PB}{PC} = \frac{AE}{AC}$. But since $(B, C, D, P) = -1$, it follows that $\frac{DB}{DC} = \frac{AE}{AC}$. Keeping in mind that $\triangle BHD \sim \triangle AHE \sim \triangle ACD$, we get

$$P \in \ell \Longleftrightarrow \frac{AE}{DB} = \frac{AH}{BH} = \frac{AC}{DC} = \frac{AH}{HE} \Longleftrightarrow BH = HE.$$

yunxiu
#3 Dec 3, 2010, 6:26 am

Let D be the foot of the A-altitude and and $P = EF \cap BC$□$Q = BE \cap AP$□Since $(B, E; H, Q) = -1$,So $BH = HE \Longleftrightarrow Q = \infty \Longleftrightarrow BE \| AP \Longleftrightarrow AP \perp AC$.

High School Olympiads

If And Only If*

Reply

Headhunter

#1 Dec 2, 2010, 2:24 am

Hello.

For a triangle ABC (incenter=I), let D, E, F be on AI, BI, CI
Three perpendicular bisectors of AD, BE, CF meet one another at K, L, M
Show that two circumcenters of $\triangle ABC, \triangle KLM$ coincide if and only if I is the orthocenter of $\triangle DEF$

Luis González

#2 Dec 2, 2010, 9:52 am

Let rays AI, BI, CI cut the circumcircle (O) of $\triangle ABC$ at $P, Q, R.$ Then it's well-known that P, Q, R are the circumcenters of $\triangle BIC, \triangle CIA, \triangle AIB.$ Assume that I is the orthocenter of $\triangle DEF,$ i.e. $EF \perp IP, FD \perp IQ$ and $DE \perp IR.$ Thus, EF is antiparallel to BC WRT $IB, IC \Longrightarrow B, C, E, F$ are concyclic. Therefore, perpendicular bisectors of BE, CF meet at the center of the circle passing through $B, C, E, F \Longrightarrow K \in OP.$ By similar reasoning, $L \in OQ$ and $R \in OM \Longrightarrow \triangle PQR$ and $\triangle KLM$ are homothetic, and $OP = OQ = OR$ implies that $OK = OL = OM,$ i.e. O is the circumcenter of $\triangle KLM.$ The converse is proved with the same arguments.

Quick Reply

High School Olympiads

Segments Relation

Reply

Headhunter

#1 Dec 2, 2010, 2:12 am • 1

Hello.

For a tangential quadrilateral $ABCD$, where $\angle BAD+\angle ADC>\pi$, let its incenter be I
A line passing I meet AB, CD at X, Y. Show that if $IX=IY$, then $AX\cdot DY=BX\cdot CY$

Luis González

#2 Dec 2, 2010, 7:11 am • 1

Let $U \equiv AB \cap CD.$ Because of $\overline{IX}=-\overline{IY},$ we deduce that $\triangle UXY$ is isosceles with apex $U.$ Since I becomes U-excenter of $\triangle UBC,$ it follows that $\angle CIY=\angle IBX,$ but $\angle UXI=\angle UYI$ implies that $\triangle BIX$ and $\triangle ICY$ are similar $\Longrightarrow \frac{BX}{IY}=\frac{IX}{CY} \Longrightarrow BX \cdot CY=\frac{1}{4}XY^2.$ By analogous reasoning, $AX \cdot DY=\frac{1}{4}XY^2$ and the result follows.

Quick Reply

High School Olympiads

Triangle ABC and F in it

sororak

#1 Dec 1, 2010, 5:16 pm

Let ABC be a triangle and points D,E are on the segments BC,AC, respectively. Let AD,BE meet at F. Prove that:

$$\frac{S_{ABC}}{S_{ABF}}=\frac{AC}{AE}+\frac{BC}{BD}-1.$$

Denote S_{XYZ} by the surface of the triangle XYZ.

Luis González

#2 Dec 2, 2010, 1:21 am

$$\frac{|\triangle ABC|}{|\triangle ABF|}=\frac{\mathrm{d}(C,AB)}{\mathrm{d}(F,AB)}=\frac{CP}{FP}=\frac{FP+CF}{FP}=1+\frac{CF}{FP}$$

On the other hand, we have: $\frac{CF}{FP}=\frac{CE}{EA}+\frac{CD}{DB}$ (Van Aubel's theorem), hence

$$\frac{|\triangle ABC|}{|\triangle ABF|}=1+\frac{CF}{FB}=1+\frac{AC-AE}{AE}+\frac{BC-BD}{BD}=\frac{AC}{AE}+\frac{BC}{BD}-1.$$

High School Olympiads

Concurrent

Reply

77ant

#1 Dec 1, 2010, 8:53 pm • 1

Dear everyone
For an acute $\triangle ABC$ with its orthocenter H, let $\overline{AD}$,$\overline{BE}$,$\overline{CF}$ be three altitudes.
K,L,M are on $\overline{EF}$,$\overline{FD}$,$\overline{DE}$. Prove the following proposition. I long for elementary ways, as possible.
If K,L,M are the feet of three altitudes of $\triangle DEF$, then $\overleftrightarrow{AK}$,$\overleftrightarrow{BL}$,$\overleftrightarrow{CM}$ are concurrent on the **Euler** line of $\triangle ABC$.

Luis González

#2 Dec 1, 2010, 9:39 pm

Lemma: Orthic triangle $\triangle H_aH_bH_c$ and excentral triangle $\triangle I_aI_bI_c$ of $\triangle ABC$ are perspective with perspector lying on the diacentral line IO of $\triangle ABC$.

Incircle (I) and A-excircle touch BC at X, Y and let Y' be the antipode of Y on (I_a). Since A is the exsimilicenter of $(I), (I_a)$, it follows that A, X, Y' are collinear. Thus, midpoints of AH_a, YY' and X are collinear, i.e. XI_a cuts AH_a at its midpoint. Since ray AI cuts circumcircle (O) at the midpoint M of II_a, then it follows that reflection I_1 of I across BC lies on $I_aH_a \Longrightarrow XM$ is the I-midline of $\triangle II_1I_a$. From the parallel radii $OM \parallel IX$, we deduce that $U \equiv IO \cap MX$ is the exsimilicenter of $(I), (O)$. Hence, $I_aI_1 \equiv I_aH_a$ pass through the reflection V of I about U. Likewise, lines I_bH_b and I_cH_c pass through $V \Longrightarrow \triangle H_aH_bH_c$ and $\triangle I_aI_bI_c$ are perspective with perspector on the diacentral line IO.

Back to the problem. Use the lemma for $\triangle DEF$ and its excentral triangle $\triangle ABC$. Then, orthic triangle $\triangle KLM$ of $\triangle DEF$ and $\triangle ABC$ are perspective with perspector on the diacentral line of $\triangle DEF$, i.e. the Euler line of $\triangle ABC$.

77ant

#3 Dec 1, 2010, 10:03 pm

Thank you so much. Is there something known about the point?
At the first look, it was so impressive.

Luis González

#4 Dec 1, 2010, 10:32 pm

77ant, it's easy to prove that the concurrency point has barycentric coordinates

$$(\tan \widehat{A} - \sin 2\widehat{A} : \tan \widehat{B} - \sin 2\widehat{B} : \tan \widehat{C} - \sin 2\widehat{C})$$

This corresponds to Kimberling center X_{24}, so for more properties you can check Encyclopedia of triangle centers and The Triangles Web (TTW).

Quick Reply

High School Olympiads

Explore Locus

Reply

Headhunter
#1 Dec 1, 2010, 10:44 am

Hello.

There is a hyperbola α, and draw an arbitrary line (but with constant direction) cutting α at two points P, Q
Find the locus of the midpoint M of $\overline{PQ}$. Just **without coordinates system**.

Luis González
#2 Dec 1, 2010, 12:10 pm

Let α be any conic section. T_{∞} denotes the infinity point of direction $PQ.$ Since cross ratio (P,Q,M,T_{∞}) is harmonic, then M moves on the polar of T_{∞} with respect to $\alpha \Longrightarrow$ Locus of M is a diameter of $\alpha.$ If α is a parabola then, locus of M is a ray parallel to its focal axis.

Quick Reply

High School Olympiads

Inequalituy with the incenter [Iran Second Round 1991]

Reply

Amir Hossein

#1 Dec 1, 2010, 2:02 am • 1

Triangle ABC is inscribed in circle C. The bisectors of the angles A, B and C meet the circle C again at the points A', B', C'. Let I be the incenter of ABC, prove that

$$\frac{IA'}{IA}+\frac{IB'}{IB}+\frac{IC'}{IC}\geq 3$$

$$,IA'+IB'+IC'\geq IA+IB+IC$$

Luis González

#2 Dec 1, 2010, 4:51 am • 1

Let I_a, I_b, I_c be the excenters of $\triangle ABC$ against A, B, C. Circumcircle (O,R) and incenter I of $\triangle ABC$ become 9-point circle and orthocenter of $\triangle I_aI_bI_c$, thus A', B', C' are the midpoints of II_a, II_b, II_c.

● From Euler's inequality $R\geq 2r$, we deduce that $16R^2-8Rr\geq 24Rr$. Now, substituting the well-known relation $(II_a)^2+(II_b)^2+(II_c)^2=16R^2-8Rr$ gives

$$(II_a)^2+(II_b)^2+(II_c)^2=4[(IA')^2+(IB')^2+(IC')^2]\geq 24Rr$$

Power of I WRT (O) is given by $IA\cdot IA'=IB\cdot IB'=IC\cdot IC'=2Rr$. Thus

$$IA'\cdot\frac{2Rr}{IA}+IB'\cdot\frac{2Rr}{IB}+IC'\cdot\frac{2Rr}{IC}\geq 6Rr\Longrightarrow\frac{IA'}{IA}+\frac{IB'}{IB}+\frac{IC'}{IC}\geq 3$$

● Using Erdos-Mordell inequality for the acute $\triangle I_aI_bI_c$ and its orthocenter I, we get

$$II_a+II_b+II_c=2(IA'+IB'+IC')\geq 2(IA+IB+IC)$$

Quick Reply

High School Olympiads

Construct a triangle assuming OH || AB

Reply

Amir Hossein

#1 Sep 14, 2010, 4:25 pm

Construct a triangle ABC given the side AB and the distance OH from the circumcenter O to the orthocenter H, assuming that OH and AB are parallel.

Luis González

#2 Nov 30, 2010, 12:47 am

Let M be the midpoint of AB and D the foot of the C-altitude. Since $MD=OH,$ then D is constructible. Assume that D lies on the ray MA and C lies on the upper half-plane determined by $AB.$ Ray AD cuts circumcircle (O) at the reflection F of H across $AB.$ Thus from power of D to $(O),$ it follows that $DA \cdot DB=DF \cdot DC=DH \cdot DC.$ But if G is the centroid of $\triangle ABC,$ we have $OH \parallel AB \Longleftrightarrow \frac{DH}{DC}=\frac{MG}{MC}=\frac{1}{3} \Longrightarrow DC^2=3DA \cdot DB \Longrightarrow$ length of the C-altitude is constructible. Then vertex C can be located on the perpendicular to AB through $D.$

Quick Reply

High School Olympiads

Parallel lines

Reply

jgnr

#1 Nov 28, 2010, 6:31 pm • 1

Point I is the incenter of triangle ABC where $AB > AC$. The midpoints of AB and AC are N and M respectively. Points D and E are on line AC and AB respectively such that $BD \parallel IM$ and $CE \parallel IN$. Point P is on BC such that $DE \parallel IP$. The orthogonal projection of P onto line AI is point Q. Prove that $ABQC$ is cyclic.

Luis González

#2 Nov 29, 2010, 12:43 am

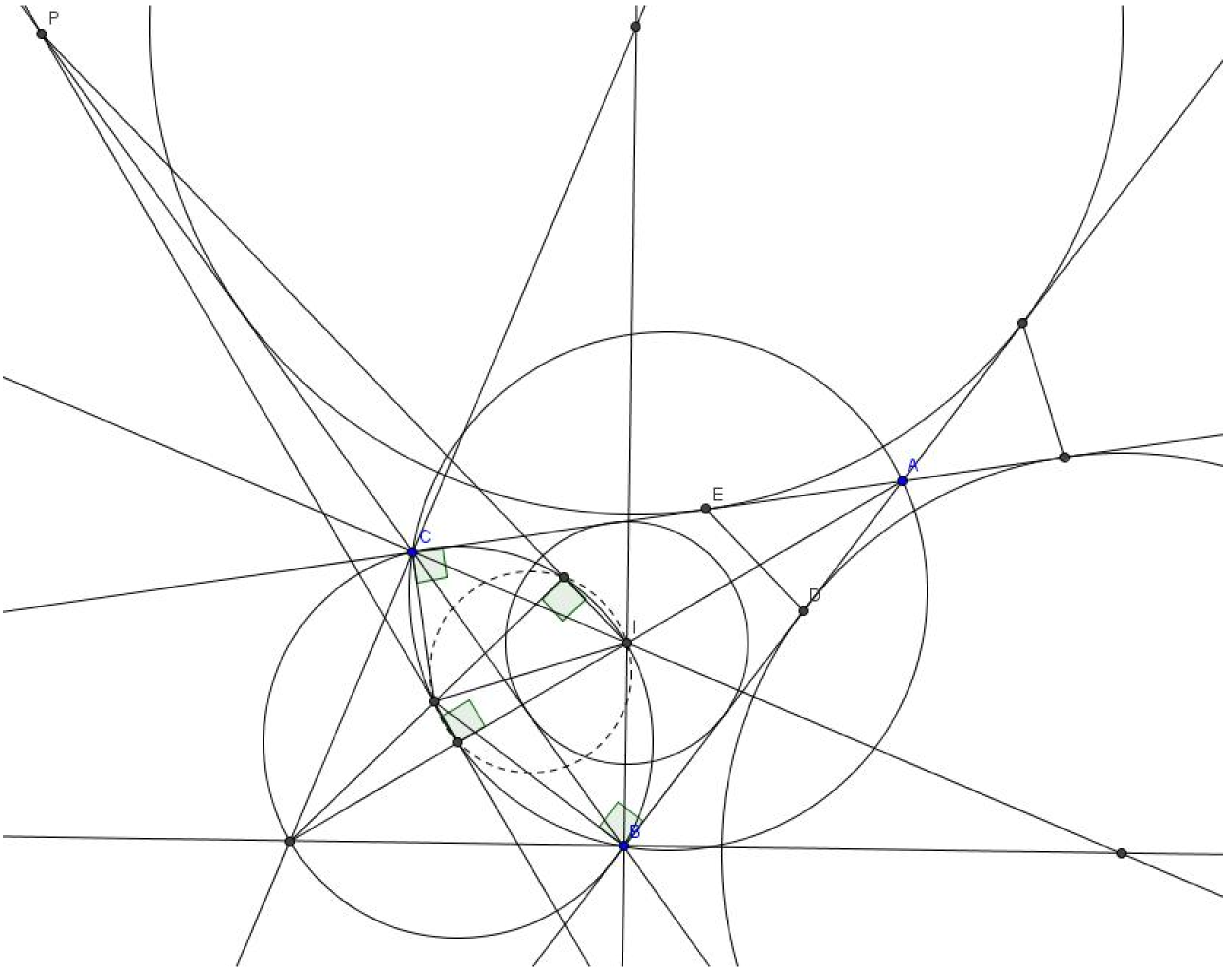
It is well known that BD, CE are the B- and C-Nagel cevians of $\triangle ABC$. For instance, let B-excircle (I_b) and incircle (I) touch AC at D', F. Then IF cuts BD' at the antipode F' of F WRT incircle (I) and since $\overline{MF} = -\overline{MD'} \Longrightarrow IM$ is the F-midline of $\triangle FD'F' \Longrightarrow IM \parallel BD'$, i.e. $D \equiv D'$. Now, the configuration is the same as question 2 of TST Peru 2007 discussed in the topic Excircle and generalized in the topic Parallel 1.

skytin

#3 Nov 29, 2010, 12:45 am

picture for solution

Attachments:

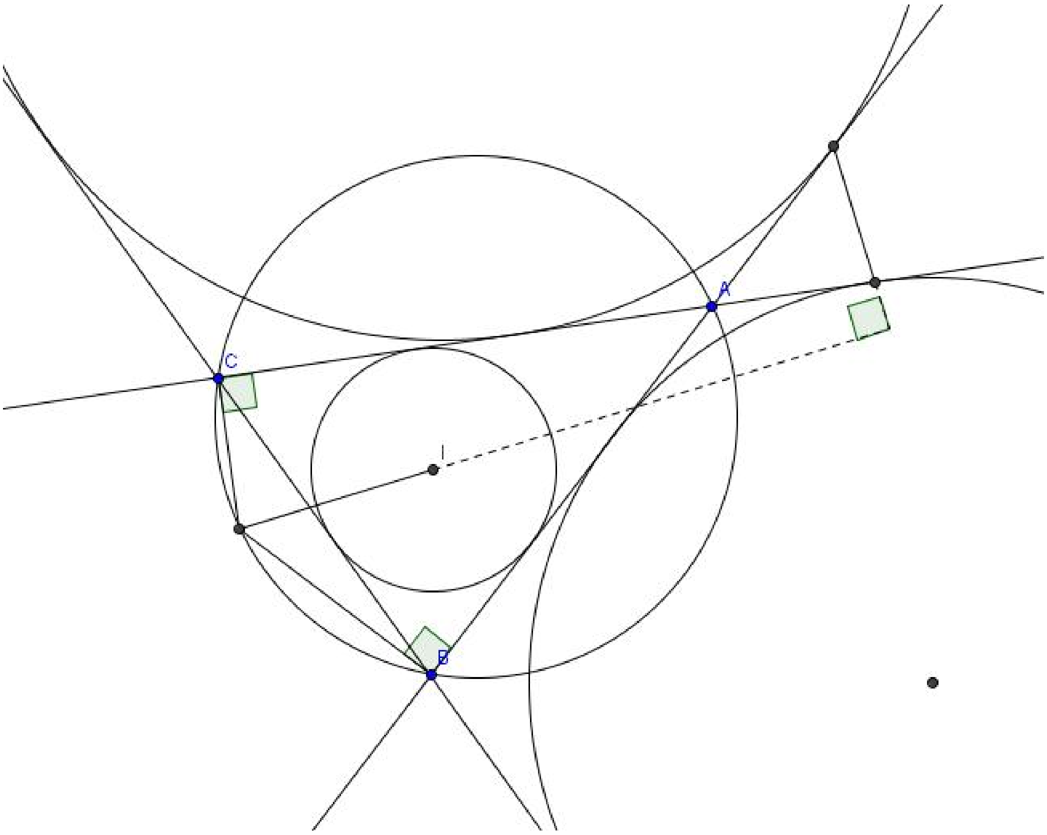


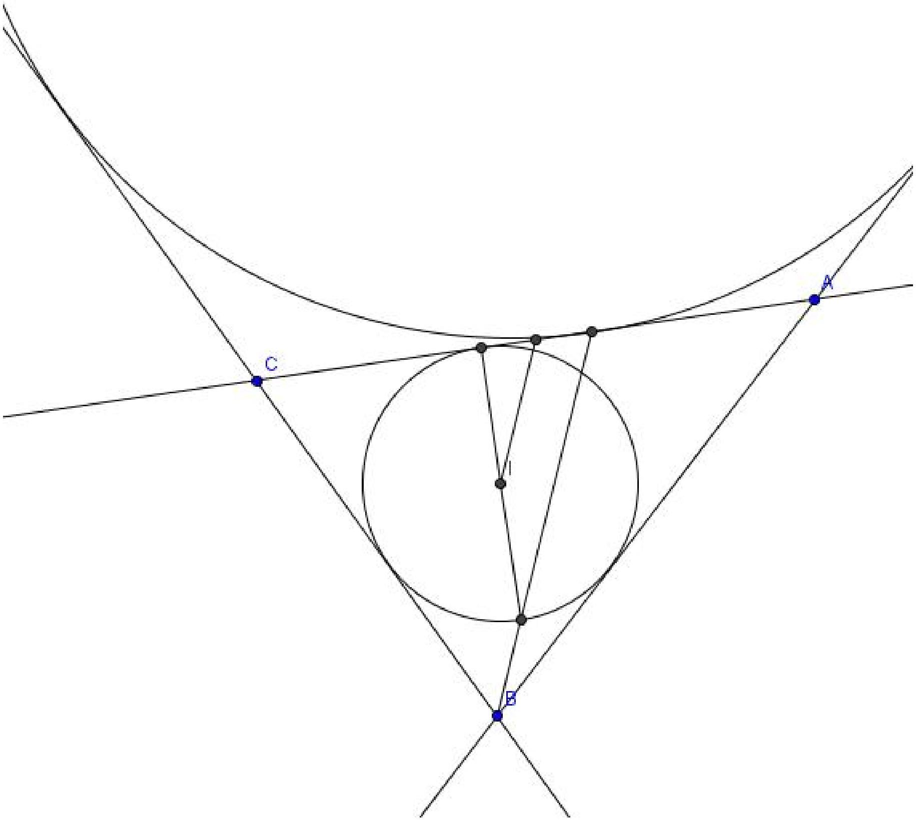
skytin

#4 Nov 29, 2010, 12:57 am

pictures for lemmas

Attachments:





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jgnr

#5 Dec 3, 2010, 8:45 pm

Thanks for the link, Luis.

I haven't managed to understand skytin's solution.

oneplusone

#6 Dec 4, 2010, 11:57 am

I think this is skytin's solution.

Lemma 1 just shows that D is the point of tangency of the excircle opposite B with the side AC, and similarly for E. The lemma is well known.

Lemma 2: The excircle opposite B touches the extension of BA at D', and similarly define E'. If AF is the diameter of the circumcircle of $\triangle ABC$, then $FI \perp D'E'$. Easily proven by showing $D'I^2 - IE'^2 = D'F^2 - FE'^2$. I think a better lemma would be if X is the excenter opposite A, then $XF \perp DE$. Also proven similarly.

Suppose Q is the intersection of AI with the circumcircle, and G is on XF such that $XF \perp IG$. Then it suffices to show IG, QF, CB are concurrent, since $\angle FQA = 90$. But since $BGICX, BFQCA, GIQF$ are all cyclic, by radical axis theorem IG, QF, CB are concurrent and we are done.

I think this is his proof, quite nice.

jgnr

#7 Dec 4, 2010, 1:18 pm

Thanks. That's really nice.

Quick Reply

High School Olympiads

China Mathematics Olympiads (National Round) 2008 Problem 1

Reply

Lei Lei

#1 Nov 28, 2010, 11:14 am

Suppose $\triangle ABC$ is scalene. O is the circumcenter and A' is a point on the extension of segment AO such that $\angle BA'A = \angle CA'A$. Let point A_1 and A_2 be foot of perpendicular from A' onto AB and AC. H_A is the foot of perpendicular from A onto BC. Denote R_A to be the radius of circumcircle of $\triangle H_AA_1A_2$. Similiarly we can define R_B and R_C. Show that:

$$\frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C}=\frac{2}{R}$$

where R is the radius of circumcircle of $\triangle ABC$.

Luis González

#2 Nov 28, 2010, 11:16 pm • 1

> *Lei Lei wrote:*
> Let point A_1 and A_2 be foot of perpendicular from A_1 onto AB and AC.

This is clearly a typo, thus, for convenience I'll denote the orthogonal projections of A' on AC, AB as Y, Z and X is the orthogonal projection of A' onto BC. Circumcenter O of $\triangle ABC$ is the intersection of the internal bisector AA' of $\angle BA'C$ and the perpendicular bisector of $BC \Longrightarrow O$ lies on the circumcircle of $\triangle A'BC$ and it's the midpoint of its arc BC. Hence, circumcircle (O) of $\triangle ABC$ cuts AA' at the incenter K and A-excenter A of $\triangle A'BC \Longrightarrow CA, BA$ are the external bisectors of $\angle BCA'$ and $\angle CBA' \Longrightarrow$ orthogonal projections of A' onto CA, BA lie then on the A'-midline of $\triangle A'BC$. Consequently, if ray AH_A cuts $\odot(AYZ)$ at D, then D is the reflection of H_A across YZ, i.e. H_A is the orthocenter of $\triangle AYZ$ $(*)$. On the other hand, from cyclic quadrilaterals $YXCA'$ and $ZBXA'$, we have

$$\angle A'XY = \angle A'CY = \angle ACB \ , \ \angle A'XZ = \angle A'BZ = \angle ABC$$

$$\Longrightarrow \angle ZXY = \angle ACB + \angle ABC = \angle YA'Z$$

Thus, $\odot(XYZ)$ is the reflection of $\odot(AYZ)$ across YZ, hence $(*)$ implies that $H_A \in \odot(XYZ)$ and $AA' = 2R_A$. Let AA', BB', CC' cut BC, CA, AC at P, Q, R. From $(A', P, K, A) = -1$, we get $OA^2 = R^2 = OP \cdot OA'$.

$$AA' = R + OA' \Longrightarrow 2R_A = R + OA' = R + \frac{R^2}{OP} \Longrightarrow 2R_A = R \cdot \frac{AP}{OP}$$

By similar reasoning, we get $: \ 2R_B = R \cdot \frac{BQ}{OQ} \ , \ 2R_C = R \cdot \frac{CR}{OR}$

$$\Longrightarrow \frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C}=\frac{2}{R}\left(\frac{OP}{AP}+\frac{OQ}{BQ}+\frac{OR}{CR}\right)$$

$$\frac{OP}{AP}+\frac{OQ}{BQ}+\frac{OR}{CR}=\frac{|\triangle BOC|}{|\triangle ABC|}+\frac{|\triangle COA|}{|\triangle ABC|}+\frac{|\triangle AOB|}{|\triangle ABC|}=\frac{|\triangle ABC|}{|\triangle ABC|}=1$$

$$\Longrightarrow \frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C}=\frac{2}{R}.$$

Quick Reply

High School Olympiads

Prove M P · OA = BC · OQ

Reply

Source: IberoAmerican 1989 Q4

WakeUp

#1 Nov 28, 2010, 12:39 am

The incircle of the triangle ABC is tangent to sides AC and BC at M and N, respectively. The bisectors of the angles at A and B intersect MN at points P and Q, respectively. Let O be the incentre of $\triangle ABC$. Prove that $MP\cdot OA=BC\cdot OQ$.

Luis González

#2 Nov 28, 2010, 2:39 am

For convenience, rename $I\equiv O$ and let D,E be the midpoints of BC,BA. It's well known that AI,MN,DE and the perpendicular dropped from B to AI concur at P. For a short proof you may see the topic *Another unlikely concurrence* here. Hence, it follows that $AQ\perp BI$ and $PE\parallel CA$ implies that $\triangle CMN\sim\triangle DNP$. Thus

$$\frac{MP}{MN}=\frac{CD}{CN}=\frac{BC}{2CN}\Longrightarrow\frac{MP}{BC}=\frac{MN}{2CN}$$

But if $T\equiv MN\cap CI$, we have $TN=\frac{1}{2}MN\Longrightarrow\frac{MP}{BC}=\frac{TN}{CN}$.

Since $\angle QAI=90^\circ-\frac{1}{2}(\angle A+\angle B)=\angle TCN$, we get $\triangle CNT\sim\triangle AIQ$

$$\Longrightarrow\frac{IQ}{IA}=\frac{TN}{CN}=\frac{MP}{BC}\Longrightarrow MP\cdot IA=BC\cdot IQ.$$

omar31415

#3 Feb 8, 2012, 11:31 pm

Correct me if I'm wrong, for convenience also, let I be the incenter. <AMI=90 so <IMC=90 also<INC=90 so MCNI is cyclic so <IMN=<INM=<C/2. Then in triangle APM we obtain that <APM=180-(<A/2+90+<C/2)=B/2, similarly in BQN we obtain that <BQN=<BQP=<A/2 . Then, quadrilateral ABPQ is cyclic, so BIP ~ AIQ so, IQ/AI=PI/BI, but we have to prove IQ/IA=MP/BC so it is enough to prove PI/BI=MP/BC, but this is true because triangles IMP ~ IBC (<IMP=<ICB=<C/2 and <IPM=<IBC=<B/2) so MP/BC=PI/BI so we're done

Virgil Nicula

#4 Feb 10, 2012, 1:21 am

PP. Let $\triangle ABC$ with incircle $w=C(I,r)$ and $\begin{array}{ccc}M\in AC\cap w & ; & N\in BC\cap w\\ P\in AI\cap MN & ; & Q\in BI\cap MN\end{array}$. Prove that $\left\{\begin{array}{c}MP\cdot IA=BC\cdot IQ\\ NQ\cdot IB=AC\cdot IP\end{array}\right\|$.

Proof. From an well-known property $\left\{\begin{array}{c}QA\perp QB\\ PA\perp PB\end{array}\right\|$ obtain that $AQPB$ is inscribed in the circle θ with diameter $[AB]$.

Thus,

$$\left\{\begin{array}{ccc}\triangle BIC\sim\triangle PIM & \Longrightarrow & \frac{BC}{PM}=\frac{IB}{IP}\\ \triangle AIC\sim\triangle QIN & \Longrightarrow & \frac{AC}{QN}=\frac{IA}{IQ}\\ p_\theta(I)=IA\cdot IP=IO\cdot IB & \Longrightarrow & \frac{IB}{IP}=\frac{IA}{IQ}\end{array}\right\|\Longrightarrow\left\{\begin{array}{c}\frac{BC}{PM}=\frac{IA}{IQ}\\ \frac{AC}{QN}=\frac{IB}{IP}\end{array}\right\|\Longrightarrow\left\{\begin{array}{c}MP\cdot IA=BC\cdot IQ\\ NQ\cdot IB=AC\cdot IP\end{array}\right\|.$$

See **PP5** from **here**.

Quick Reply

High School Olympiads

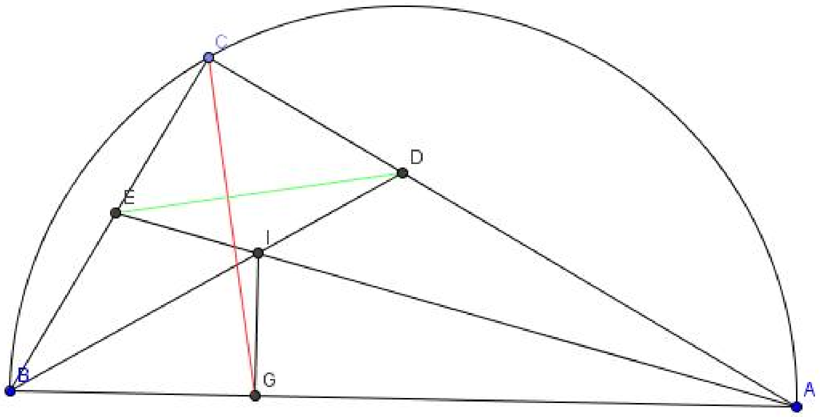
Looks easy

Reply

skytin
#1 Nov 26, 2010, 10:04 pm

Let angle C of trangle ABC is 90 . and bissectors of angles A and B intersects sides at points E and D , and I is incenter . If H is altitude from I on BA prove that ED is perpendicular to HC .

Attachments:



Luis González
#2 Nov 27, 2010, 12:36 am

Incircle (I) of $\triangle ABC$ touches AB, BC, CA at H, K, L and let $\triangle A_0B_0C_0$ be the antimedial triangle of $\triangle HKL$, such that A_0, B_0, C_0 correspond to K, L, H, respectively. Then, C_0 is the pole of ED WRT $(I) \Longrightarrow IC_0 \perp DE$. Hence, it's enough to show that $IC_0 \parallel HC$. I becomes 9-point center of $\triangle A_0B_0C_0$ and let U be the circumcenter of $\triangle A_0B_0C_0$. Because of $\angle A_0C_0B_0 = \angle KHL = 45^\circ$, it follows that circumcenter of the isosceles right $\triangle UA_0B_0$ is the midpoint H of A_0B_0. Thereby, isogonal line C_0I of C_0H WRT $\angle A_0C_0B_0$ is identical with the C_0-symmedian of $\triangle A_0B_0C_0$. But HC is the H-symmedian of its medial triangle $\triangle KLH \Longrightarrow IC_0 \parallel HC$.

oneplusone
#3 Nov 28, 2010, 10:27 am

This is my solution.

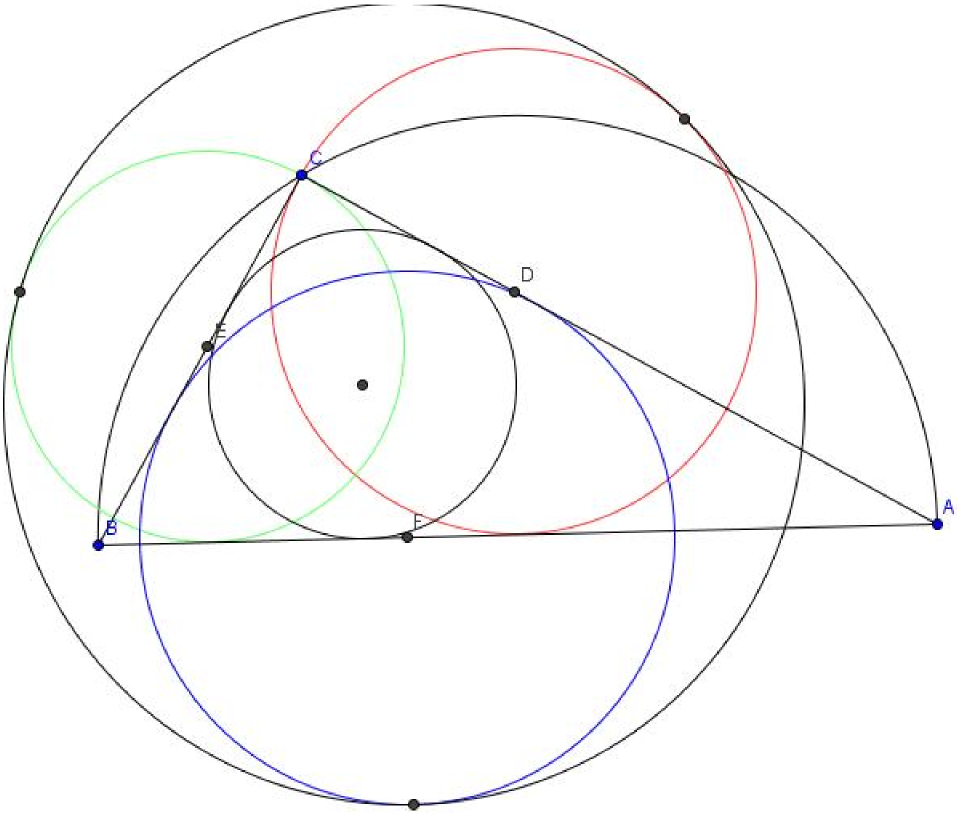
Let P, Q be feet of perpendiculars from E, D to AB. Then since $\angle ECD = 90$, we have $EP = EC$, $DC = DQ$ and $IP = IC = IQ$, thus $HP = HQ$. Now consider two circles C_1, C_2, C_1 has center E radius EC while C_2 has center D radius DC. Then HP, HQ are tangent to C_1, C_2 respectively, and both have equal length. Thus HC is the radical axis of C_1, C_2, so $DE \perp HC$.

skytin
#4 Nov 28, 2010, 6:19 pm

Good solution oneplusone , i have got same solution .
Another problem for you :
Let angle C of trangle ABC is 90 . and bissectors of angles A and B intersects sides at points E and D and bissector of angle BCA intersect side BA at point F . Let a is circle with center F and tangent to BC and let b is circle wich is tangent to C_1 , C_2

and a . Prove that R(b) = R(a)*(3/2)

Attachments:



![avatar]

dgreenb801

#5 Nov 29, 2010, 6:06 am

My solution:
Note that $\angle CED + \angle CDE = 90 = \angle BCH + \angle ACH$. We wish to show that $\angle BCH = \angle CDE$, in which case we would have $CH \perp DE$. But we can instead show that $\frac{\sin BCH}{\sin ACH} = \frac{\sin CDE}{\sin CED}$ since $f(x) = \frac{sinx}{sin(90-x)}$ is strictly increasing for $0 < x < 90$. But $\frac{\sin CDE}{\sin CED} = \frac{CE}{ED}$, so we want to show $\frac{\sin BCH}{\sin ACH} = \frac{CE}{ED}$.

Note that $\frac{[BCH]}{[ACH]} = \frac{BH}{HA}$.
Also, $\frac{[BCH]}{[ACH]} = \frac{\frac{1}{2}(BC)(CH)(\sin BCH)}{\frac{1}{2}(AC)(CH)(\sin ACH)}$.
Combining the two and simplfying, we get
$\frac{\sin BCH}{\sin ACH} = \frac{AC \cdot BH}{BC \cdot AH}$
But since I is the incenter, $BH = \frac{1}{2}(AB + BC - CA)$. Similarly $AH = \frac{1}{2}(AB + AC - BC)$. Thus
$\frac{\sin BCH}{\sin ACH} = \frac{CA(AB + BC - CA)}{CB(AB + AC - BC)}$

By the angle bisector theorem, $CE = \frac{(CB)(AC)}{AC + AB}$ and $CD = \frac{(AC)(CB)}{BC + AB}$, thus $\frac{CE}{ED} = \frac{BC + AB}{AC + AB}$

$$\frac{CE}{ED} = \frac{\sin BCH}{\sin ACH} \iff$$
$$\frac{BC + AB}{AC + AB} = \frac{CA(AB + BC - CA)}{CB(AB + AC - BC)} \iff$$
$$CB(AB^2 - BC^2 + AC(BC + AB)) = CA(AB^2 - AC^2 + BC(AC + AB)) \iff$$
So we have $CB(AC^2 + AC(BC + AB)) = CA(BC^2 + BC(AC + AB))$ $\iff$ (by the Pythagorean Theorem) $(AC)(BC)(AB + BC + CA) = (AC)(BC)(AB + BC + CA)$
Which is obviously true.

Quick Reply

High School Olympiads

locus question

Bledimat94

#1 Nov 26, 2010, 7:18 pm

We examine the circumscribed and inscribed circles of a triangle ABC.It is known that there can be drawn an infinite number of triangles for which this given circles are circumscribed and inscribed.
Find the locus of the intersection points of the medians of this triangles.

Luis González

#2 Nov 26, 2010, 7:58 pm

Let $(I,r),(O,R)$ be the fixed incircle and circumcircle of the poristic triangles $\triangle ABC. H,N,G$ denote the orthocenter, 9-point center and centroid of $\triangle ABC.$ Let O' be the reflection of O about $I.$ Then $HO'=2NI,$ but since (I) and 9-point circle (N) are internally tangent, it follows that $NI=\frac{1}{2}R-r \Longrightarrow HO'=R-2r.$ Thus, locus of H is a circle ω with center O' and radius $R-2r.$ Since $\overline{OG}:\overline{OH}=1:3,$ locus of G is the image ω' of ω under the homothety $(O,\frac{1}{3}).$

High School Olympiads

Area problem 1

Reply

hEatLove

#1 Nov 24, 2010, 8:51 pm

There are AE and BF diameter are perpendicular in R radiuced circle. There is C point in EF arc and $[CA] \cap [BF] = P$, $[CB] \cap [AE] = Q$
Prove that : $S_{APQB} = R^2$

Luis González

#2 Nov 26, 2010, 8:58 am

$\angle ACB = \angle QAB = \angle PAB = 45^\circ \Longrightarrow \triangle ACB \sim \triangle QAB \sim \triangle ABP$

$$\Longrightarrow \frac{AQ}{AB} = \frac{AB}{BP} \Longrightarrow \ AQ \cdot BP = AB^2 = 2R^2$$

$[APQB] = [APB] + [QPB] = \frac{1}{2}BP \cdot (AO + QO) = \frac{1}{2}AQ \cdot BP = R^2.$

Quick Reply

High School Olympiads

Triangle center - open problem 1

Reply

borislav_mirchev
#1 Nov 25, 2010, 7:22 pm

Incircle k of acute-angled triangle ABC is tangent to BC, CA, AB at the points L1, L2, L3 respectively.
The height through A intersect with k at the points A' and A" (A' is between A and A").
The height through B intersect with k at the points B' and B" (B' is between B and B").
The height through C intersect with k at the points C' and C" (C' is between C and C").
M1, M2, M3 are the middles of A'A", B'B", C'C" respectively.
Prove that L1M1, L2M2, L3M3 intersects at a common point - **C1 center**.

I discovered this fact as a statement and would like to know how to solve the problem and
if is it a well known fact. What is its level of difficulty?

This post has been edited 1 time. Last edited by borislav_mirchev, Nov 26, 2010, 1:07 pm

Luis González
#2 Nov 26, 2010, 1:15 am • 1

Ray AI cuts circumcircle (O) of $\triangle ABC$ at the midpoint of its arc BC. Rays ML_1, MO cut (O) at E and the antipode D of M. Inversion with pole M and radius $MB = MC = MI$ takes midpoint of BC into D, L_1 into E and I is double $\Longrightarrow$ $MI^2 = ME \cdot ML_1 \Longrightarrow \angle MIL_1 = \angle IEM$, but $\angle MIL_1 = \angle MAA'$. Therefore, if $R \equiv AA' \cap EL_1$, then it follows that A, E, R, I are concyclic, but $\angle AED = \angle AMD = \angle IEM$ implies that $\angle AEI = \angle DEM = 90^\circ \Longrightarrow$ $\angle ARI = 90^\circ \Longrightarrow R$ coincides with the orthogonal projection M_1 of I onto A-altitude. Hence, from the parallel radii $OM \parallel IL_1$, it follows that $\overline{ML_1M_1}$ cuts IO at the exsimilicenter X_{56} of $(O), (I)$. Likewise, lines L_2M_2 and L_3M_3 pass through X_{56}.

borislav_mirchev
#3 Nov 26, 2010, 1:34 am

I don't know inversion but tooking a look at other details of your proof I think it's great.
Almost all high-school geometry was used in clever manner.
What is X_{56}? I think it comes to say the statement of this problem is not new. Am I right?
Do you like the problem?

I'm interested to see more solutions of the problem if it is possible without using complex numbers, pole and polars, inversion, barycentric coordinates, vectors going beyound highschool material.

Luis González
#4 Nov 26, 2010, 2:31 am • 1

> *borislav_mirchev wrote:*
> What is X_{56} ? I think it comes to say the statement of this problem is not new.

A great number of triangle centers are labeled X(i) on ETC. For instance, i=56 corresponds to the exsimilicenter of the incircle and circumcircle of ABC.

Quick Reply

High School Olympiads

Paralell line in convex quadrilateral

euler1990

#1 Nov 25, 2010, 1:17 am

Let $ABCD$ be a convex quadrilateral and S the intersection of its diagonals. Let the line throught S parallel to AB intersect CD at P, AD at R and BC at Q. Prove that $|PS|^2=|PQ|\cdot|PR|$.

Luis González

#2 Nov 25, 2010, 5:16 am • 1

Denote $E\equiv BC\cap AD, F\equiv BA\cap CD$ and $U\equiv FS\cap AD$. Let T be the reflection of S about P and $E'\equiv FT\cap AD$. Since $FB\parallel ST$, it follows that line pencil $F(S,T,P,B)$ is harmonic $\Longrightarrow (U,E',D,A)=-1$, but $(U,E,D,A)=-1\Longrightarrow E\equiv E'\Longrightarrow E(C,D,S,F)=(R,Q,S,T)=-1\Longleftrightarrow PT^2=PS^2=PQ\cdot PR.$

High School Olympiads

anticenter, orthic axis

Reply

wild_sloth
#1 Nov 24, 2010, 3:29 pm

let ABCD cyclic, T is the anticenter
let foot of perpendicular from T to ABC, BCD, CDA, DAB 's orthic axis be E, F, G, H
proove E, F, G, H concylic

Luis González
#2 Nov 25, 2010, 12:28 am

For the sake of ease, let us consider that $ABCD$ is convex. Let M,N,L,K be the midpoints of $AB,BC,CD,DA.$ Because of $TM \perp CD$ and $TK \perp CB,$ it follows that $\angle MTK=\angle BAD \Longrightarrow T$ lies on the 9-point circle (N_a) of $\triangle BAD.$ By similar reasoning, T lies on 9-point circles $(N_b),(N_c),(N_d)$ of $\triangle ABC, \triangle BCD$ and $\triangle CDA,$ i.e. T coincides with the Poncelet point of $ABCD.$ It's known that orthic axes $\ell_a,\ell_b,\ell_c,\ell_d$ of $\triangle DAB, \triangle ABC, \triangle BCD$ and $\triangle CDA$ are the radical axes of their common circumcircle (O) with their 9-point circles. If $P \equiv \ell_d \cap \ell_a, Q \equiv \ell_a \cap \ell_b,$ $R \equiv \ell_b \cap \ell_c$ and $S \equiv \ell_c \cap \ell_d,$ then P,Q,R,S have equal powers WRT $(N_d)(N_a),(N_a)(N_b),(N_b)(N_c)$ and $(N_c)(N_d)$ $\Longrightarrow P,Q,R,S$ lie on TK,TM,TN,TL respectively. From the cyclic $THQE,TERF,TFSG,TGPH$ we get

$$\angle THE=\angle TQR \ , \ \angle THG=\angle TPS \ , \ \angle TFE=\angle TRQ \ , \ \angle TFG=\angle TSP$$

$$\Longrightarrow \angle GHE=\angle TQR+\angle TPS \ , \ \angle GFE=\angle TRQ+\angle TSP$$

But because of $\angle QTR=\angle MTN=180^\circ-\angle KTL=180^\circ-\angle PTS,$ it follows that $\angle GHE+\angle GFE=180^\circ \Longrightarrow$ E,F,G,H are concyclic.

Quick Reply

High School Olympiads

KA=KE

Reply

Mahan17

#1 Nov 23, 2010, 11:11 pm

In a triangel ABC, E is a point on the line BC such that AE is tangent to the circumcircle of ABC. D is a point on the side BC such that AD=AB. and M is the midpoint of AD, K is a point on MC, EK and AB are parallel lines, prove that KA=KE.

Luis González

#2 Nov 24, 2010, 6:39 am • 1

Since $\angle AEB = \angle CAD,$ it follows that CA is tangent to the circumcircle of $\triangle EAD$ through $A.$ Together with $\triangle EAB \sim \triangle ECA,$ we obtain the relation

$$\frac{CE}{CD} = \frac{AE^2}{AD^2} = \frac{AE^2}{AB^2} = \frac{EC^2}{AC^2}$$

Let $N \equiv CM \cap AE.$ By Menelaus' theorem for $\triangle AED$ cut by $NMC,$ we get

$$\frac{EN}{NA} = \frac{CE}{CD} \cdot \frac{MD}{AM} = \frac{CE}{CD} = \frac{EC^2}{AC^2}$$

Thus, $CN \equiv CK$ is the C-symmedian of $\triangle EAC.$ But since $\angle KEA = \angle ACB \Longrightarrow KE$ is tangent to $\odot(EAC)$ through $E \Longrightarrow KA = KE.$

Quick Reply

High School Olympiads

nice collinear

Reply

KDS

#1 Nov 23, 2010, 7:32 pm

Given $\triangle ABC$ with its circumcircle (O) and its orthocenter H
Let A_1 be an arbitary point on side BC.
Let A_2 be the reflection of A wrt O.
A_1A_2 cuts (O) at A_3.
Let A_4 be the reflection of A wrt the line OA_1.
Let A_5 be the point on (O) such that $A_4A_5 \perp BC$.
Prove that A_3, A_5, H are collinear.

Luis González

#2 Nov 24, 2010, 4:20 am • 1

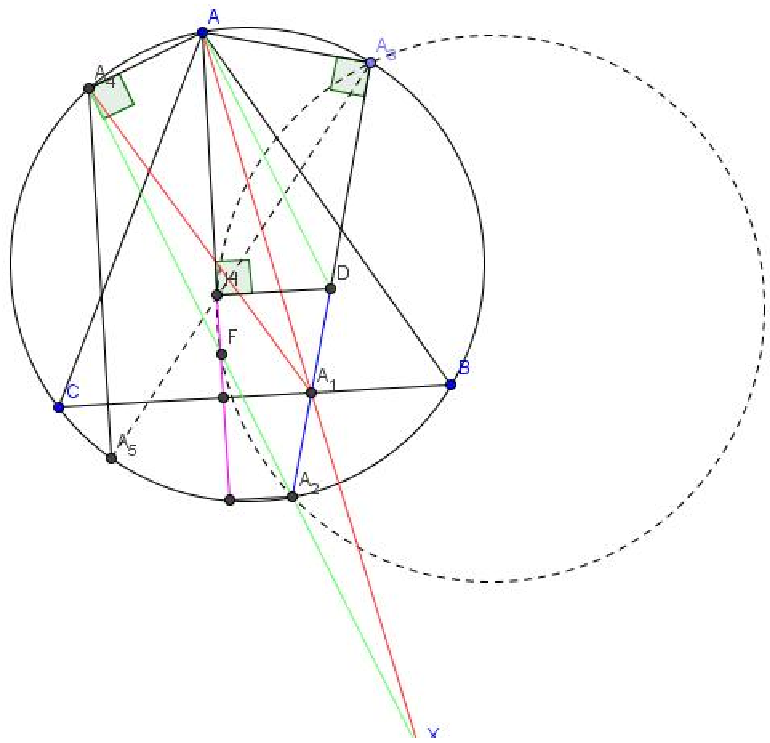
Ray AH cuts BC and circumcircle (O) at D, E, respectively. Since $\angle BAE = \angle CAA_2$, it follows that $EA_2 \parallel BC$. By Pascal theorem for nonconvex cyclic hexagon $EA_2A_3A_5A_4A$, the intersections $P \equiv EA_2 \cap A_4A_5, Q \equiv A_2A_3 \cap AA_4$ and $H' \equiv A_3A_5 \cap AE$ are collinear $(*)$. Denote $R \equiv BC \cap PQ$ and let M be the midpoint of AA_4. Then A_1M is the perpendicular bisector of $AA_4 \Longrightarrow MA_1 \parallel A_4A_2$. But due to $PA_2 \parallel RA_1$ and $(*)$, we conclude that $\triangle RMA_1$ and $\triangle PA_4A_2$ are homothetic through $Q \equiv RP \cap MA_4 \cap A_1A_2 \Longrightarrow MR \parallel AD \parallel A_4A_5 \Longrightarrow MR$ is the midline of the trapezoid AA_4PH', i.e. R is the midpoint of $PH' \Longrightarrow H'$ is the reflection of E about $BC \Longrightarrow H' \equiv H$.

skytin

#3 Nov 26, 2010, 12:23 am • 1

Let X is on A_1A and XA_1 = A_1A , easy to see that A_1A_4 = A_1A , so angle XA_4A = 90 = A_2A_4A , so A_2 is lie on XA_4 . Let XA_4 intersect HA at point F , let D is on A_2A_3 and A_2A_1 = A_1D , easy to see that angle DHA = 90 (parallel lines) , so DHAA_3 is cyclic A_2A_4 || DA , so A_2FHA_3 is cyclic and A_4A_5 || FH , so easy to see that H is on A_5A_3 . done

Attachments:



TelvCohl

#4 Oct 25, 2014, 9:57 am • 1

My solution:

Let $D = AD \cap BC, E = BH \cap CA, F = CH \cap AB$.
Invert with center H and factor $AH * HD = BH * HE = CH * HF$.
Denote A_3' as the image of A_3 under this inversion .

Since $\angle A_1DA = \angle A_1A_3A = 90$ and $AH * HD = A_3H * HA_3'$,
so A, A_3, A_3', D all lie on (AA_1) . ie. the pedal circle of A_1 ,
hence A_3' is the second intersection of the nine point circle of $\triangle ABC$ and (AA_1) (different from D) ,
so we get A_3 is the orthopole of OA_1 with respect to $\triangle ABC$.
ie. the Anti-steiner point of OA_1 with respect to the medial triangle of $\triangle ABC$

Since A_5 is the pole of the Simson line which is perpendicular to OA_1 ,
so H, A_3', A_5 are collinear (notice that H is the exsimilicenter of $(DEF) \sim (ABC)$).
ie. A_3, A_5, H are collinear

Q.E.D

Quick Reply

High School Olympiads

find the locus of X

Source: Trinh Bang Giang

saeedghodsi
#1 Mar 31, 2010, 2:18 pm

let X be an interior point of traingle ABC. lines XA, XB, XC meet BC, CA, AB respectively at M, N, P find the locus of X such that :

$$\sum_{cyc}[(S_{BXM})^2-(S_{CXM})^2]=0$$

Luis González
#2 Nov 22, 2010, 9:48 pm

Let $(x:y:z)$ be the barycentric coordinates of X WRT $\triangle ABC$. Then we have

$$\frac{|\triangle BXM|}{|\triangle ABC|}=\frac{xz}{(x+y+z)(y+z)} \ , \ \frac{|\triangle CXM|}{|\triangle ABC|}=\frac{xy}{(x+y+z)(y+z)}$$

$$\Longrightarrow |\triangle BXM|^2-|\triangle CXM|^2=|\triangle ABC|^2\cdot\frac{x^2(z-y)}{(z+y)(x+y+z)^2}$$

Similarly, we have the expressions

$$|\triangle CXN|^2-|\triangle AXN|^2=|\triangle ABC|^2\cdot\frac{y^2(x-z)}{(x+z)(x+y+z)^2}$$

$$|\triangle AXP|^2-|\triangle BXP|^2=|\triangle ABC|^2\cdot\frac{z^2(y-x)}{(y+x)(x+y+z)^2}$$

$$|\triangle BXM|^2-|\triangle CXM|^2+|\triangle CXN|^2-|\triangle AXN|^2+|\triangle AXP|^2-|\triangle BXP|^2=0$$

$$\Longleftrightarrow \frac{x^2(z-y)}{(z+y)(x+y+z)^2}+\frac{y^2(x-z)}{(x+z)(x+y+z)^2}+\frac{z^2(y-x)}{(y+x)(x+y+z)^2}=0$$

$$\Longleftrightarrow (x-y)(y-z)(z-x)(x+y+z)^2=0$$

If X does not lie on the line at infinity $x+y+z=0$, then locus of X is the union of the three medians of $\triangle ABC$. Namely, $y=z, z=x$ and $x=y$.

High School Olympiads

Name of concurent point

Reply

novae

#1 Nov 21, 2010, 4:44 pm

Given triangle ABC, orthocenter H, altitudes AA', BB', CC' $(A', B', C' \in BC, CA, AB \text{ respectively})$.
I think we have a well-known result that 3 Euler lines of $\Delta AB'C', \Delta BC'A', \Delta CA'B'$, 3 Euler circles of $\Delta AHO, \Delta BHO, \Delta CHO$ and Euler circle of ΔABC concurrent.
Who can prove and show me the name of that concurent point?

Luis González

#2 Nov 22, 2010, 1:29 am

9-point circle (N_a) of $\triangle AOH$ passes through 9-point center N of $\triangle ABC$ (midpoint of OH), the second intersection E of (N), AA' (midpoint of AH) and the reflection of N across the A-midline of $\triangle ABC$ (midpoint of AO). Let M_a, M_b, M_c denote the midpoints of BC, CA, A and D the orthogonal projection of A onto OH, which lies on the circle $\odot(AM_bM_c)$ with diameter AO. Since $\odot(AM_bM_c)$ is the reflection of (N) about the midline M_bM_c, it follows that reflection U of D across M_bM_c lies on (N). Thus, $UDNA_1$ is an isosceles trapezoid with $DU \parallel NA_1 \Longrightarrow U \in (N_a)$ and U is the anti-Steiner point of OH WRT $\triangle M_aM_bM_c$. Likewise, 9-point circles of $\triangle BOH$ and $\triangle COH$ cut (N) at U.

Since $B'C'$ is antiparallel to BC WRT AB, AC, then Euler lines of $\triangle ABC$ and $\triangle AB'C'$ are antiparallel WRT AO, AH. Since E is the circumcenter of $\triangle AB'C'$, it is enough to show that EU and OH are antiparallel WRT AO, AH. Let O_a be the circumcenter of $\triangle AOH$. If (N_a) cuts AH again at L, then LD is antiparallel to AO WRT $HA, HO \Longrightarrow HO_a \perp LD$, but since $\angle HAO_a = AHO_a$ and $\angle UEA = \angle DLH$ (due to the isosceles trapezoid EUDL), it follows that $EU \perp AO_a$ $\Longrightarrow EU$ is antiparallel to OH WRT AO, AH. Therefore, EU is the Euler line of $\triangle AB'C'$. Analogously, Euler lines of $\triangle BC'A'$ and $\triangle CA'B'$ pass through U and the proof is completed.

P.S. $U \equiv X_{125}$ is also the anticomplement of the Feuerbach point of $\triangle A'B'C'$ with respect to $\triangle A'B'C'$. For a proof, you may see the topic With the Feuerbach point. Thus, there are several ways to name/characterize the point U, e.g. Euler's reflection point of the medial triangle $\triangle M_aM_bM_c$, orthopole of the Euler line with respect to $\triangle ABC$, anticomplement of the Feuerbach point of $\triangle A'B'C'$ with respect to $\triangle A'B'C'$, center of the rectangular circum-hyperbola through A, B, C, O, H (Jerabek hyperbola), etc.

dan23

#3 Nov 28, 2013, 11:23 pm

Prove :
length of one of $X_{125}A'$,$X_{125}B'$,$X_{125}C'$ is equal to sum of others
i.e.
$X_{125}A' = X_{125}B' + X_{125}C'$

jayme

#4 Nov 29, 2013, 6:44 pm

Dear Mathlinkers,
for the first point of concur, the result comes from Thébault... I saw in the geometric litterature: "Thébault's point"
For the second point of concur, the result comes from Boubals... and we have "the first Boubals point"
An article is encoded on my site... next.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Two perspective triangles

Reply

Source: By Zhonghao Ye

Fang-jh

#1 Apr 2, 2009, 8:09 pm

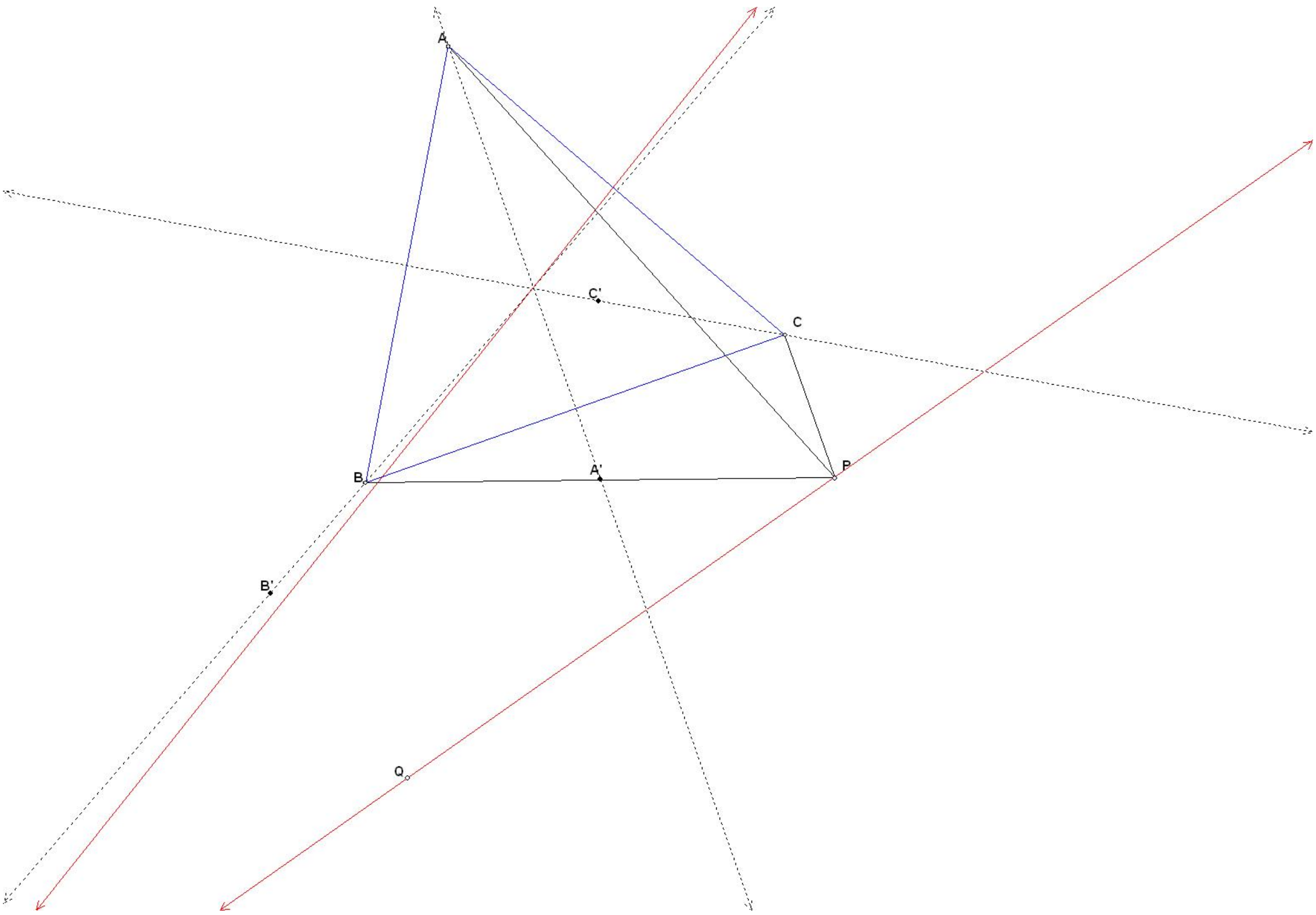
Let P be a point in the plane of triangle ABC. Let Q be the isogonal conjugate of P with respect to triangle ABC. Prove that line PQ is parallel to the Euler line of triangle ABC if and only if the triangle formed by the circumcenters of triangles PBC, PCA, PAB and triangle ABC are perspective.

Leonhard Euler

#2 Apr 11, 2009, 3:32 pm

I think this problem is wrong. See below

Attachments:

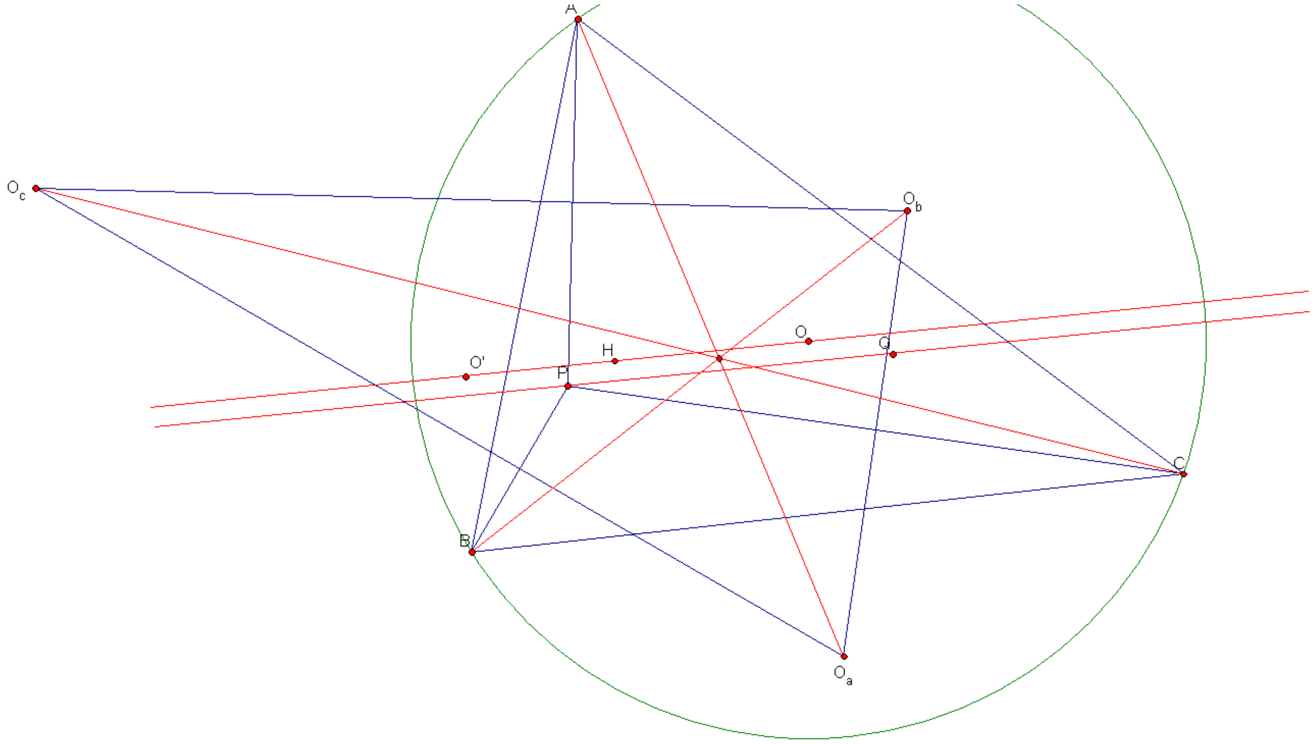


Fang-jh

#3 Apr 11, 2009, 4:49 pm

Dear Han-sol Shin
I think it is true! also see the following picture:

Attachments:



darij grinberg

#4 Apr 11, 2009, 5:32 pm

The problem should state that:

the triangle formed by the circumcenters of triangles PBC, PCA, PAB and triangle ABC are perspective if and only if the line PQ is parallel to the Euler line of triangle ABC or one of the points P and Q lies on the circumcircle of triangle ABC.

(The locus of all points P such that the line PQ is parallel to the Euler line of triangle ABC is called the *Neuberg cubic* of triangle ABC. The above is one of the properties of this cubic. I don't remember a synthetic proof, but many problems have been solved after I left geometry, so this doesn't mean no proof is known...)

darij

Luis González

#5 Nov 19, 2010, 11:53 pm

According to the topic Find locus, $\triangle ABC$ and the triangle formed by the circumcenters of $\triangle PBC, \triangle PCA$ and $\triangle PAB$ are perspective $\Longleftrightarrow$ Either P lies on the Neuberg cubic $\mathcal{N}$ of $\triangle ABC$ or it lies on the circumcircle $\mathcal{O}$. Thus, it remains to show that PQ is parallel to the Euler line of $\triangle ABC \Longleftrightarrow P \in \mathcal{N}$.

Equation of line τ passing through $P \equiv (u:v:w)$ and $Q \equiv \left(\frac{a^2}{u} : \frac{b^2}{v} : \frac{c^2}{w}\right)$ is:

$$\tau \equiv u(c^2v^2 - b^2w^2)x + v(a^2w^2 - c^2u^2)y + w(b^2u^2 - a^2v^2)z = 0$$

τ is parallel to the Euler line of $\triangle ABC \Longleftrightarrow \tau$ goes through its infinity point X_{30}

$$(S_AS_B + S_AS_C - 2S_BS_C : S_BS_C + S_BS_A - 2S_CS_A : S_CS_A + S_CS_B - 2S_AS_B)$$

Hence, setting $(u, v, w) \longrightarrow (x, y, z)$, locus $f(x, y, z) = 0$ of P is

$\sum_{\text{cyclic}} (S_AS_B + S_AS_C - 2S_AS_B)(c^2y^2 - b^2z^2)x = 0$, i.e. the Neuberg cubic $\mathcal{N}$.

lym

#6 Nov 20, 2010, 2:20 pm

About this problem there are many equivalent propositions□and it have been totally solved in useing geometric method by chinese expert zhonghao Ye□I hope him or his student can come here to explan this Problems.

You can reference here□**Neuberg**□□http://bbs.cnool.net/topic_show.jsp?id=6265276&thesisid=494&flag=topic1

High School Olympiads

Tangent Circles

Reply

chris!!!

#1 Nov 19, 2010, 5:06 pm

Let $ABCD$ be a quadrilateral inscribed in a circle (ω_1) and let M,N the midpoints of its diagonals AC,BD respectively.If $\{K\}\equiv AD\cap BC,\{L\}\equiv AB\cap CD$ prove that KL is tangent t the circumcircles of the triangles $\triangle KMN$ and $\triangle LMN.$

Luis González

#2 Nov 19, 2010, 8:32 pm

Let U be the midpoint of $KL.$ Then M,N,U are collinear on the Newton line of $ABCD.$ Since K,L are conjugate points with respect to $\omega_1,$ it follows that circle (U) with diameter KL is orthogonal to $(O)\equiv \omega_1$ and the circle γ with diameter $OP,$ where $P\equiv AC\cap BD$ is the pole of KL with respect to $\omega_1.$ Therefore, $UK^2=UL^2=UN\cdot UM \Longrightarrow KL$ is tangent to circles $\odot(KMN)$ and $\odot(LMN).$

Quick Reply

High School Olympiads

5 concyclic points A,B,C,D,E have AB||EC, AC||ED

Reply

Source: Baltic Way 2001

WakeUp
#1 Nov 18, 2010, 12:55 am

The points A, B, C, D, E lie on the circle c in this order and satisfy $AB \parallel EC$ and $AC \parallel ED$. The line tangent to the circle c at E meets the line AB at P. The lines BD and EC meet at Q. Prove that $|AC| = |PQ|$.

Luis González
#2 Nov 18, 2010, 10:06 am

By Pascal theorem for the degenerate cyclic hexagon $EEDBAC$, the intersections $P \equiv EE \cap AB, ED \cap AC$ and $Q \equiv DB \cap CE$ are collinear, but $ED \cap AC$ is the infinity point of the parallel lines $ED, AC \Longrightarrow ED \parallel AC \parallel PQ$. Since $PA \parallel QC$, it follows that $PACQ$ is a parallelogram $\Longrightarrow$ Segments AC and PQ are congruent and parallel.

erfan_Ashorion
#3 Oct 1, 2011, 3:55 am

we know that $AE = BC$ and $AE = DC \longrightarrow DC = BC$
so
$BDC = CBD \longrightarrow CQ = DC^2 / EC$
other wise from 2 triangle PAE and AEC we know that $PA = AE^2 / EC$
so $PA = QC$
so $PAQC$ is a parallelogram!
im so slothy excuse me for my bad write
if you not catch say to explain it!!

Quick Reply

High School Olympiads

Determine the angles of ABC

Reply

Source: Baltic Way 2001

WakeUp
#1 Nov 18, 2010, 1:23 am

In a triangle ABC, the bisector of $\angle BAC$ meets the side BC at the point D. Knowing that $|BD| \cdot |CD| = |AD|^2$ and $\angle ADB = 45^\circ$, determine the angles of triangle ABC.

Luis González
#2 Nov 18, 2010, 9:12 am

Let (O, R) be the circumcircle of $\triangle ABC$, M the midpoint of BC and P the foot of the A-altitude. $\angle ADB = \angle PAD = 45^\circ \Longrightarrow \angle PAO = 90^\circ \Longrightarrow AO \parallel BC$, i.e. $OM = AP$. On the other hand, we have:

$$AD^2 = AB \cdot AC - BD \cdot CD = AB \cdot AC - AD^2 \Longrightarrow 2 \cdot AD^2 = AB \cdot AC$$

$$\Longrightarrow AD^2 = 2 \cdot AP^2 = R \cdot AP \Longrightarrow 2 \cdot OM = R \Longrightarrow \angle OCM = 30^\circ$$

Therefore, $\angle A = 60^\circ$, $\angle B = 105^\circ$ and $\angle C = 15^\circ$.

Quick Reply

High School Olympiads

Locus of P with PE^2=PD*PF

Reply

Kunihiko_Chikaya
#1 Nov 16, 2010, 9:58 pm

Three points $A\ (0,\ 1),\ B\ (-1,\ 0),\ C\ (1,\ 0)$ are on the x-y plane. Denote $PD,\ PE,\ PF$ the perpendiculars drawn from the point P, which are not on three lines $AB,\ BC,\ CA$, to $AB,\ BC,\ CA$ respectively. Find the locus of P such that the lengths of line segments $PD,\ PE,\ PF$ form geometric progressions in this order, then sketch the locus with three lines $AB,\ BC,\ CA$.

castigioni
#2 Nov 17, 2010, 9:40 pm

The locus is an ellipse

Kunihiko_Chikaya
#3 Nov 18, 2010, 12:12 am

Could you explain the details?

Luis González
#4 Nov 18, 2010, 7:02 am

Let $\triangle ABC$ be any scalene triangle and $(x:y:z)$ stand for the barycentric coordinates of point P with respect to $\triangle ABC$. Then $PE^2=PD\cdot PF \Longleftrightarrow bcx^2-a^2yz=0$. This is a conic $\mathcal{K}$ passing through the incenter $I\equiv(a:b:c)$, A-excenter $I_a\equiv(-a:b:c)$ and tangent to AB,AC at B,C. For a construction of $\mathcal{K}$ you may see the last reply here.

Quick Reply

High School Olympiads

Circle through A of parallelogram ABCD

Source: Baltic Way 2001

WakeUp

#1 Nov 18, 2010, 1:07 am

Given a parallelogram $ABCD$. A circle passing through A meets the line segments AB, AC and AD at inner points M, K, N, respectively. Prove that

$$|AB| \cdot |AM| + |AD| \cdot |AN| = |AK| \cdot |AC|$$

Luis González

#2 Nov 18, 2010, 6:25 am

Let ω be the object circle and τ the perpendicular line to AC through C. τ is the inverse of ω under the inversion with pole A and power $\overline{AK} \cdot \overline{AC} \Longrightarrow M' \equiv AB \cap \tau$ and $N' \equiv AD \cap \tau$ are the inverses of M, N. Thus $\overline{AK} \cdot \overline{AC} = \overline{AM} \cdot \overline{AM'} = \overline{AN} \cdot \overline{AN'}$. But from $\triangle M'BC \sim \triangle M'AN' \sim \triangle CDN'$, we deduce that

$$\frac{\overline{AB}}{\overline{AM'}} + \frac{\overline{AD}}{\overline{AN'}} = 1 \Longrightarrow \overline{AB} \cdot \frac{\overline{AM}}{\overline{AK} \cdot \overline{AC}} + \overline{AD} \cdot \frac{\overline{AN}}{\overline{AK} \cdot \overline{AC}} = 1$$

$$\Longrightarrow \overline{AB} \cdot \overline{AM} + \overline{AD} \cdot \overline{AN} = \overline{AK} \cdot \overline{AC}.$$

hatchguy

#3 Nov 19, 2010, 6:38 am

By Ptolemy: $AM * NK + AN * KM = AK * NM$ (1)

Also $\angle KMN = \angle KAN = \angle ACB$ and $\angle MNK = \angle MAK$ so ACB is similar to NMK and therefore:

$$\frac{AC}{MN} = \frac{AB}{NK} = \frac{BC}{MK} = \frac{AD}{MK} = k$$

which implies, $NK = AB * \frac{1}{k}$, $KM = AD * \frac{1}{k}$ and $NM = AC * \frac{1}{k}$. Substituting this into (1) we get the desired result.

High School Olympiads

Beautiful problem

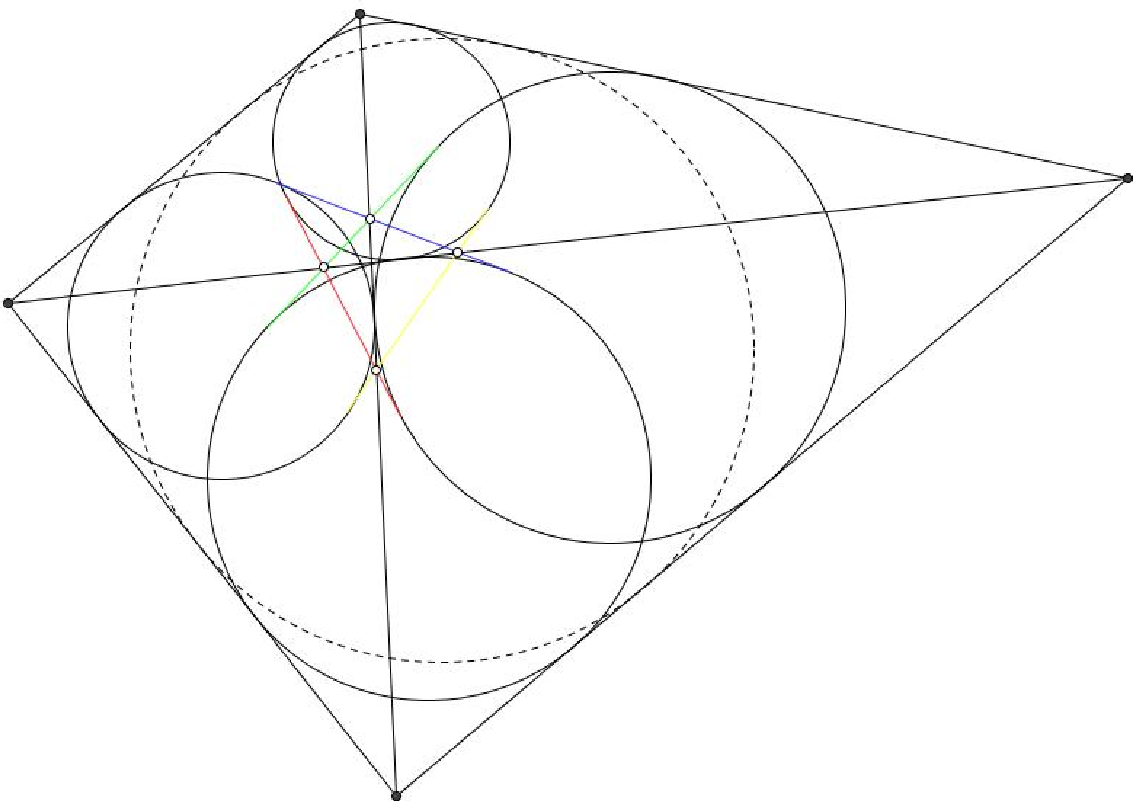
Reply

skytin

#1 Nov 16, 2010, 9:32 pm • 1

Let ABCD has incircle .
a , b , c , d are incircles of trangles ABC BCD CDA DAB , prove that other tangents to a b , b c , c d , d a meets on diagonals of ABCD

Attachments:



Luis González

#2 Nov 17, 2010, 2:46 am • 1

Let $(I_1), (I_2), (I_3)$ be the incircles of $\triangle ABC, \triangle BDC$ and $\triangle CDA$. $(I_1), (I_2)$ touch BC at M, N and $(I_2), (I_3)$ touch CD at P, Q. Common external tangents ℓ_1 and ℓ_2 of $(I_1), (I_2)$ and $(I_2), (I_3)$, different from BC, CD, intersect at E. ℓ_1 touches $(I_1), (I_2)$ at U, V and ℓ_2 touches $(I_2), (I_3)$ at X, Y. We have then

$$UV = MN = CM - CN = \frac{1}{2}(CA + CB - AB) - CN$$

$$XY = PQ = CQ - CP = \frac{1}{2}(CA + CD - AD) - CP$$

$CB - AB = CD - AD$ and $CN = CP$ implies that $(I_1), (I_3)$ are externally tangent and $UV = XY$. Since $EX = EV$, then $EU = EY \Longrightarrow E$ has equal power WRT $(I_1), (I_3) \Longrightarrow E$ lies on the radical axis AC of $(I_1), (I_3)$.

lym

#3 Nov 17, 2010, 3:39 am • 1

Really very nice problem□we can use radical axis to solve as luis said.

lym

#4 Nov 17, 2010, 5:12 am

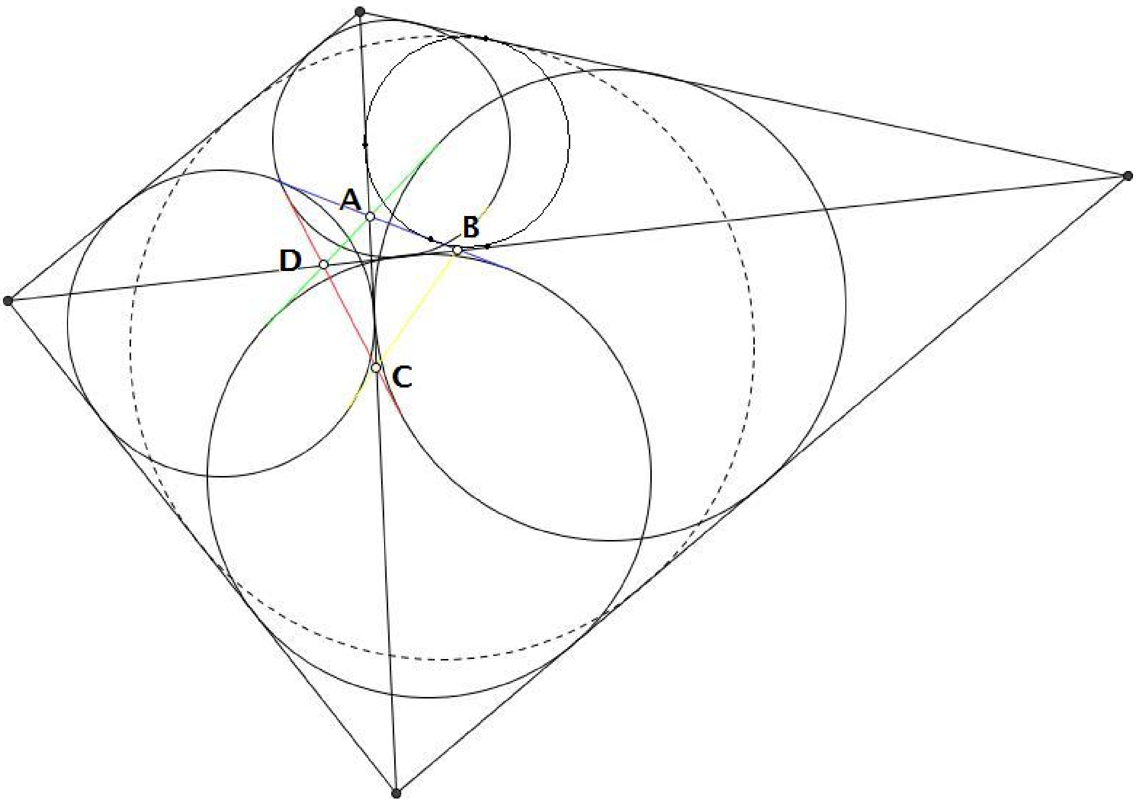
Continue to prove□
If one tangent parallel to the side of $ABCD$□then the other trangles also parallel to the sides of $ABCD$.
Click to reveal hidden text

skytin

#5 Nov 18, 2010, 12:09 am

Good work luisgeometria and lym!
(1) Prove that incircle of one of four little trangle is tangent to AB (see picture)
(2) Prove that ABCD is tangent to fixed circle

Attachments:



Luis González

#6 Nov 18, 2010, 1:26 am

> *skytin wrote:*
> (1) Prove that incircle of one of four little trangle is tangent to AB (see picture)
> (2) Prove that ABCD is tangent to fixed circle

For 1) see this theorem, 2) is a degenerate case.

lym

#7 Nov 18, 2010, 10:58 pm • 1

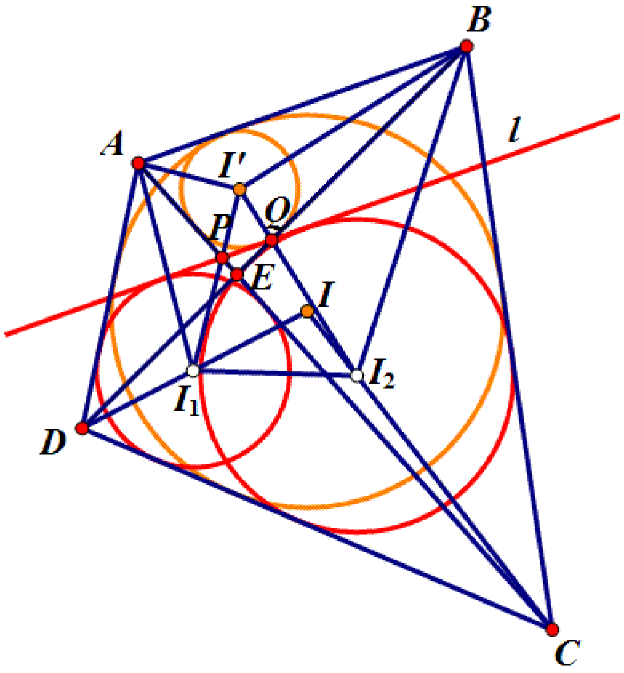
Becase of $\angle I_1II_2 + \angle AIB = \pi$ and $\angle IAI_1 = \angle I'AB$□$\angle IBI_2 = \angle I'BA$□So I' is the isogonal conjugate of I□

so $\angle I_1I'I_2 + \angle AI'B = \pi$□then I' is the excenter of $\triangle EPQ$□so $\odot I'$□$\odot I_1$□$\odot I_2$ have the common tangent.

As a result of (1)□(2) is obvious

.

Attachments:



Quick Reply

High School Olympiads

Prove that D≤OF

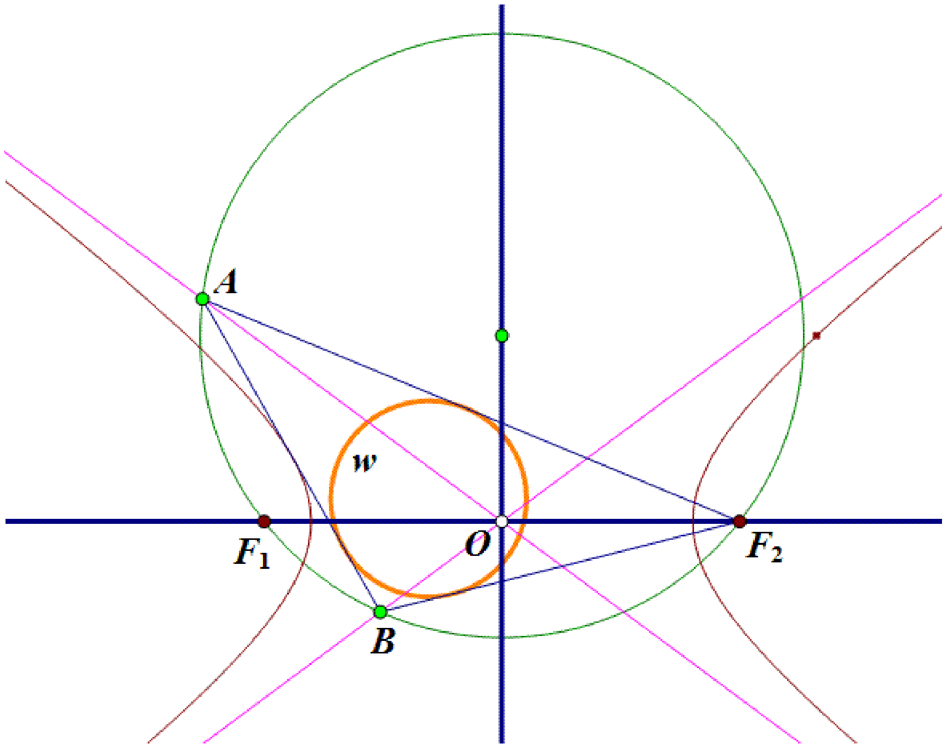
lym

#1 Nov 17, 2010, 1:52 am

Figure□Given an hyperbola $\mathcal{K}$□O□F respectively are its center and focus□Let A□B respectively are on the different asymptotes and AB tangent to $\mathcal{K}$. Denote $\mathcal{D}$ is the incircle's diameter of $\triangle FAB$□Prove that $\mathcal{D} \leq OF$.

Click to reveal hidden text

Attachments:



Luis González

#2 Nov 17, 2010, 5:14 am • 1

Rename the foci of the hyperbola $\mathcal{K}$ as E, F and u, v denote its asymptotes. Tangent line ℓ of $\mathcal{K}$ through arbitrary D on the E-branch cuts u, v at A, B. Asymptotes u, v are hyperbola tangents at their infinite points U_∞ and V_∞. By Brianchon's theorem BU_∞, AV_∞ and OD concur at a point T, but from $BT \parallel u$ and $AT \parallel v$, it follows that $ATBO$ is a parallelogram. Thus AB and OT bisect each other $\Longrightarrow D$ is the midpoint AB. Normal line τ of $\mathcal{K}$ at D becomes perpendicular bisector of AB and focal axis EF is the bisector of the angle $\angle u, v$, different from the conjugate axis. Hence $P \equiv EF \cap \tau$ lies on $\odot(OAB)$ and it's the midpoint of the arc AB. Since ℓ and τ bisect $\angle EDF$, then $(E, F, Q, P) = -1$, where $Q \equiv EF \cap \ell \Longrightarrow$ $QE \cdot QF = QO \cdot QP = QB \cdot QA \Longrightarrow E$ lies on the circumcircle ω of $\triangle FAB$. Further, AB is the polar of P WRT ω $\Longrightarrow FE$ is the F-symmedian of $\triangle FAB$. Therefore $FA \cdot FB = FD \cdot FE$ $(*)$. Let R, r, h, m denote the circumradius, inradius, F-altitude and median FD of $\triangle FAB$. Then $(*)$ is equivalent to

$OF = \frac{FE}{2} = R \cdot \frac{h}{m}$. But from Panaitopol's inequality, it follows that : $\frac{h}{m} \geq \frac{2r}{R}$

$$\Longrightarrow \frac{OF}{R} \geq \frac{2r}{R} \Longrightarrow OF \geq 2r.$$

lym

#3 Nov 17, 2010, 5:51 am

Thx Dear luis，you are very shrewd.
I don't know panaitopol's inequality ，I just use another geometric way to prove in the end ，it's equivalent with Panaitopol's inequality. Next I'll show it.

lym

#4 Nov 17, 2010, 5:59 pm

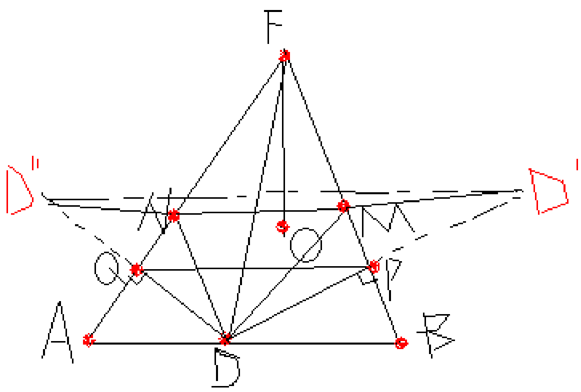
See the picture，I prove the last part：

D，M and N are the midpoint，D'，D'' are the symmetrical points of D aboat BF，AF respectively，then

$2OF \cdot FD = FA \cdot FB = \frac{2S}{sin\angle AFB}$，therefore $OF = \frac{S}{\frac{D'D''}{2}} \geq \frac{S}{\frac{DM+DN+MN}{2}} = 2r = \mathcal{D}$.

Actually，we also have proven **Panaitopol's inequality**.

Attachments:



Quick Reply

High School Olympiads

ABCD be a parallelogram

Reply

aktyw19

#1 Mar 6, 2010, 2:58 am

Let ABCD be a parallelogram, points $E \in (AB)$ and $F \in (AD)$ such that $EF \parallel BD$,

$G \in CE \cap BD, H \in CF \cap BD, P \in FG \cap BC, Q \in EH \cap CD$. Prove that $\frac{PQ}{EF} = 1 + \frac{BE}{BA}$.

Luis González

#2 Nov 16, 2010, 10:06 am

Clearly AC bisects the parallel segments $EF, PQ \Longrightarrow M \equiv PF \cap QE \cap AC$ and the similar $\triangle BEP$ and $\triangle DCF$ are homothetic $\Longrightarrow EP \parallel CF$. Let L be the reflection of F about M, thus $EL \parallel CA$. If $K \equiv EP \cap BD$, then $\triangle EKG$ and $\triangle CHG$ are homothetic and their cevians EL and CA are homologous $\Longrightarrow EL$ bisects GK. Hence, line pencil $E(G, K, L, F)$ is harmonic $\Longrightarrow (G, P, L, F) = -1$. Then

$$ML^2 = MF^2 = MG \cdot MP \Longrightarrow \frac{MP}{MF} = \frac{PQ}{EF} = \frac{MF}{MG} = \frac{CE}{GC}$$

$$\Longrightarrow \frac{PQ}{EF} = \frac{CE}{GC} = \frac{GC + GE}{GC} = 1 + \frac{GE}{GC} = 1 + \frac{BE}{DC} = 1 + \frac{BE}{BA}.$$

Quick Reply

High School Olympiads

Find locus

Reply

novae

#1 Oct 23, 2010, 4:57 pm

Let ABC be a triangle, M is an arbitrary point, D is the center of (MBC), E is the center of (MCA), F is the center of (MAB). Find locus of M such that AD, BE, CF concurrent.
I think about this problem for a long time but until now, I have no idea to solve this problem
Note that O - the center of (ABC), I - the incenter of ΔABC and H - the orthocenter of ΔABC are satisfy the feature above.
Click to reveal hidden text

This post has been edited 1 time. Last edited by novae, Oct 23, 2010, 9:50 pm

Luis González

#2 Oct 23, 2010, 9:43 pm

> *novae wrote:*
> D is the center of (MBC), E is the center of (MCA), F is the center of (MCA).

There's a little typo, since F should be the center of circle (MAB). Locus of M such that lines AD,BE,CF concur is the union of a circular self-isogonal cubic known as Neuberg cubic, the circumcircle of ABC and the line at infinity.

novae

#3 Oct 23, 2010, 9:52 pm

Yes, you are right what about the solution

Luis González

#4 Nov 16, 2010, 8:21 am

Sorry for my late reply. Let $(u : v : w)$ be the barycentric coordinates of M with respect to $\triangle ABC$. Barycentric equation of the circle (D) passing through B, C, M is given by

$$(D) \equiv a^2yz + b^2xz + c^2xy - x(x+y+z)\frac{(a^2vw + b^2wu + c^2uv)}{u(u+v+w)} = 0$$

Setting $\lambda = \dfrac{(a^2vw + b^2wu + c^2uv)}{(u+v+w)} = 0$ to simplify subsequent expressions, we get

$$D \equiv \left(-a^2\frac{\lambda}{u} + a^2S_A \ : \ S_C\frac{\lambda}{u} + b^2S_B \ : \ S_B\frac{\lambda}{u} + c^2S_C\right)$$

$$\Longrightarrow AD \equiv \left(S_B\frac{\lambda}{u} + c^2S_C\right)y - \left(S_C\frac{\lambda}{u} + b^2S_B\right)z = 0$$

By cyclic permutations of elements we get the equations of lines BE, CF as

$$BE \equiv \left(S_A\frac{\lambda}{v} + c^2S_C\right)x - \left(S_C\frac{\lambda}{v} + a^2S_A\right)z = 0$$

$$CF \equiv \left(S_A\frac{\lambda}{w} + b^2S_B\right)x - \left(S_B\frac{\lambda}{w} + a^2S_A\right)y = 0$$

Lines AD, BE, CF concur if and only if

$$\begin{bmatrix} 0 & S_B\frac{\lambda}{u}+c^2S_C & -S_C\frac{\lambda}{u}-b^2S_B \\ S_A\frac{\lambda}{v}+c^2S_C & 0 & -S_C\frac{\lambda}{v}-a^2S_A \\ S_A\frac{\lambda}{w}+b^2S_B & -S_B\frac{\lambda}{w}-a^2S_A & 0 \end{bmatrix} = 0 \implies$$

$$a^2S_C{S_A}^2\frac{\lambda^2}{uv}+c^2S_B{S_C}^2\frac{\lambda^2}{uw}+a^2c^2S_A{S_C}^2\frac{\lambda}{u}+b^2S_A{S_B}^2\frac{\lambda^2}{vw}+$$

$$a^2b^2S_B{S_A}^2\frac{\lambda}{v}+b^2c^2S_C{S_B}^2\frac{\lambda}{w}-c^2S_A{S_C}^2\frac{\lambda^2}{wv}-b^2S_C{S_B}^2\frac{\lambda^2}{uv}-$$

$$b^2c^2S_B{S_C}^2\frac{\lambda}{v}-a^2S_B{S_A}^2\frac{\lambda^2}{uw}-a^2c^2S_C{S_A}^2\frac{\lambda}{w}-a^2b^2S_A{S_B}^2\frac{\lambda}{u}=0$$

$$\implies \lambda\sum_{\text{cyclic}} S_A(b^2{S_B}^2-c^2{S_C}^2)\left(\frac{\lambda}{wv}-\frac{a^2}{u}\right)=0$$

Plugging the value of λ and $(u,v,w)\longrightarrow(x,y,z)$, subsequent simplifications yield

$$(a^2yz+b^2xz+c^2xy)\sum_{\text{cyclic}}(S_AS_B+S_AS_C-2S_BS_C)(c^2y^2-b^2z^2)x=0$$

$$\mathcal{N}\equiv\sum_{\text{cyclic}}(S_AS_B+S_AS_C-2S_BS_C)(c^2y^2-b^2z^2)x=0$$

$$\mathcal{O}\equiv a^2yz+b^2xz+c^2xy=0$$

Therefore, either M lies on the circumcircle $\mathcal{O}$ of $\triangle ABC$ or it lies on the circular self-isogonal cubic $\mathcal{N}$ with pivot $X_{30}=S_AS_B+S_AS_C-2S_BS_C$, i.e. the infinite point of the Euler line of $\triangle ABC$.

Quick Reply

High School Olympiads

Parallel 1

buratinogigle

#1 Oct 20, 2009, 10:47 pm

Let ABC be a triangle and point P

a) Through midpoint of BC, CA, AB draw lines parallel to PA, PB, PC prove that those lines are concurrent at point O.

b) Let $A'B'C'$ be cevian triangle of P. Through midpoint of PA draw line parallell to BC, it cuts $B'C'$ at Q. $OB \cap A'C' = \{B_1\}, OC \cap A'B' = \{C_1\}$ .Prove that $PQ \parallel B_1C_1$

Note that, it is generalization of the problem on the post Excircles

Luis González

#2 Nov 14, 2010, 9:08 am

For convenience rename $O \equiv U$. P_1 denotes the midpoint of PA and ℓ_a the parallel from P_1 to BC. Parallels from the midpoints of BC, CA, AB to PA, PB, PC concur at the complement U of P WRT $\triangle ABC$. Let $(u : v : w)$ be the barycentric coordinates of P WRT $\triangle ABC$. Then we have

$U : (v + w : w + u : u + v)$, $P_1 : (2u + v + w : v : w)$

$\Longrightarrow \ell_a : (v + w)x - (2u + v + w)y - (2u + v + w)z = 0$

Thus, ℓ_a cuts line $B'C' : -vwx + uwy + uvz = 0$ at Q with coordinates

$Q : (u(w - v)(2u + v + w) : v(wv - uv + uw + w^2) : w(-uv + uw - vw - v^2))$

$PQ : vw(v + w)^2x + uw(uv - uw - vw - w^2)y + uv(-uv + uw - vw - v^2)z = 0$

Coordinates of the infinite point R_∞ of line PQ are then

$R_\infty : (u(v - w)(v + w) : -v(uv - uw + vw + w^2) : w(uw - uv + vw + v^2))$

On the other hand, we have

$BU : (u + v)x - (v + w)z = 0$, $CU : (u + w)x - (v + w)y = 0$

$A'C' : vwx - uwy + uvz = 0$, $A'B' : vwx + uwy - uvz = 0$

Thus, coordinates of $B_1 \equiv BU \cap A'C'$ and $C_1 \equiv CU \cap A'B'$ are then

$B_1 : (uw(v + w) : v(u^2 + uv + vw + w^2) : uw(u + v))$

$C_1 : (uv(v + w) : uv(u + w) : w(u^2 + uw + vw + v^2))$

Hence, coordinates of the infinite point S_∞ of B_1C_1 are given by

$S_\infty : (u(v - w)(v + w) : -v(uv - uw + vw + w^2) : w(uw - uv + vw + v^2))$

Which are identical to the coordinates of the infinity point R_∞ of $PQ \Longrightarrow B_1C_1 \parallel PQ$.

jayme

#3 Nov 14, 2010, 10:46 am

#3 Nov 14, 2010, 10:46 am

Dear Mathlinkers,
for a) we can think to two parallelogic triangles.
Sincerely
Jean-Louis

jayme

#4 Nov 14, 2010, 7:16 pm

Dear Mathlinkers,
for part b) I have found some parallels, but I have not manly idea how to resolve this problem synthetically.
Any ideas?
Sincerely
Jean-Louis

pacoga

#5 Nov 15, 2010, 9:39 pm

> *Luis González wrote:*
> For convenience rename $O \equiv U \ldots$

In fact, we can take U here as any point on the line joining the midpoint of BC and the complement of P.

High School Olympiads

In a trapezoid MF=MA <==> BN || FD .

Reply

Virgil Nicula

#1 Oct 11, 2010, 2:26 pm

Let $ABCD$ be a trapezoid, where $AD\parallel BC$ and $BC<AD$. For a point $M\in (AB)$ denote $N\in (CD)$ for which $MN\parallel AD$, $I\in MC\cap NB$ and $F\in AB$ for which $FI\parallel AD$. Prove that $MF=MA\iff BN\parallel FD$.

Luis González

#2 Nov 11, 2010, 10:17 am

Let $G \equiv FI \cap DC.$ From $\triangle MIF \sim \triangle MCB$ and $\triangle NIG \sim \triangle NBC,$ we obtain $\frac{FI}{BC}=\frac{MI}{MC}$ and $\frac{GI}{BC}=\frac{NI}{NB}.$ But $\frac{MI}{MC}=\frac{NI}{NB},$ due to $\triangle MIN \sim \triangle CIB.$ Therefore, $FI=GI,$ i.e. I is the midpoint of $FG.$

Assume that M is the midpoint of $AF.$ Then $L \equiv FD \cap MN$ is the midpoint of $FD \Longrightarrow IL$ is the F-midline of $\triangle DFG$ $\Longrightarrow IL \parallel CN$ and since IF is parallel to $CB,$ it follows that $\triangle LIF$ and $\triangle NCB$ are homothetic through $M \Longrightarrow FLD$ is parallel to $BN.$ The converse is proved analogously.

Quick Reply

High School Olympiads

Collinear

Reply

77ant

#1 Nov 11, 2010, 1:49 am

Dear everyone
For a triangle ABC with its incircle (I), let (I) touch BC, CA, AB at D, E, F respectively.
M, N lie on $\overline{EC}, \overline{FA}$ respectively, where $EM = FN$. Circumcircles of $\triangle EMD, \triangle EFN$ meet each other at K.
Prove that E, K, B are collinear.

Luis González

#2 Nov 11, 2010, 4:42 am

Let L be the second intersection of $\odot(EMD)$ with BC. Since $\triangle CED$ is isosceles with apex C, it follows that quadrilateral $EMLD$ is an isosceles trapezoid with legs $EM = DL$. Therefore, $FN = EM = DL$ and because of $BD = BF$, then $BN = BL \Longrightarrow \overline{BN} \cdot \overline{BF} = \overline{BL} \cdot \overline{BD} \Longrightarrow B$ has equal power WRT $\odot(EMD)$ and $\odot(EFN) \Longrightarrow B$ lies on the radical axis KE of $\odot(EMD)$ and $\odot(EFN)$.

77ant

#3 Nov 11, 2010, 1:42 pm

Dear luisgeometra
Thanks for a nice proof, :)

Quick Reply

High School Olympiads

Concurrent

Reply

77ant

#1 Nov 11, 2010, 1:57 am

Dear everyone
Prove that if three triangles have a common center of perspective, their axes are concurrent.

Luis González

#2 Nov 11, 2010, 3:43 am

Assume that triangles $\triangle ABC, \triangle DEF$ and $\triangle GHI$ are perspective (in that order) through P, i.e. AEH, BDG and CFI concur at $P.$ Let AB cut ED, HG at M, N and AC cut EF, HI at K, L respectively. Then MK and NL are the perspective axes of $\triangle ABC, \triangle DEF$ and $\triangle ABC, \triangle GHI.$ Since $\triangle MEK$ and $\triangle NHL$ are perspective through $A,$ by Desargues theorem, it follows that intersections $Q \equiv EK \cap HL, R \equiv MK \cap NL$ and $S \equiv EM \cap HN$ are collinear, but QS is the perspective axis of $\triangle DEF$ and $\triangle GHI.$ Thus, pairwise perpectives axes of $\triangle ABC, \triangle DEF, \triangle GHI$ concur at $R.$

77ant

#3 Nov 11, 2010, 1:52 pm

Dear luisgeometra
Thanks a lot. Mine is not as concise as yours.

Quick Reply

High School Olympiads

may be well known....

Reply

earth
#1 Nov 8, 2010, 11:20 pm

In $\triangle ABC$,cevians BD and CE are drawn such that $\angle EBD=10, \angle DBC=70, \angle DCE=20, \angle ECB=60$. Find $\angle BDE$.

Luis González
#2 Nov 10, 2010, 7:33 am

$\triangle ABC$ is isosceles with $\angle ABC=\angle ACB=80^\circ$. Let R be a point on $\overline{AC}$ such that $\angle RBC=50^\circ$. From the "infamous" Langley's problem we know that $\angle REC=30^\circ$. Further, if $S \equiv BD \cap CE$, then $\angle SDR=30^\circ=\angle RES$ and $\angle SCR=\angle SBR=20^\circ \Longrightarrow$ quadrilaterals $EDRS$ and $CRSB$ are both cyclic. Thus

$\angle BDE=\angle ERS=\angle ERC-\angle SRC=130^\circ-(180^\circ-70^\circ)=20^\circ.$

jayme
#3 Nov 10, 2010, 5:41 pm

Dear Mathlinkers,
this adventitious angles problem can be solved in a remarkable approach as you can see on
http://perso.orange.fr/jl.ayme vol. 2 In memorian of Juan Carlos Salazar.
Sincerely
jean-Louis

Quick Reply

High School Olympiads

The circumcentre of triangle AED

Reply

Source: Italy TST 2002

WakeUp
#1 Nov 10, 2010, 1:58 am

A scalene triangle ABC is inscribed in a circle Γ. The bisector of angle A meets BC at E. Let M be the midpoint of the arc BAC. The line ME intersects Γ again at D. Show that the circumcentre of triangle AED coincides with the intersection point of the tangent to Γ at D and the line BC.

Luis González
#2 Nov 10, 2010, 6:34 am

$F \equiv AM \cap BC$ is the foot of the A-external bisector. Thus circle Γ_a with diameter EF is the A-Apollonius circle of $\triangle ABC$, which is orthogonal to its circumcircle Γ. Hence, it is enough to show that $D \in \Gamma_a$.

$$\angle AFE = |\angle B - \angle FAB| = |\angle B - 90^\circ + \tfrac{1}{2}\angle A| = \tfrac{1}{2}|\angle B - \angle C|$$

$$\angle ADE = \angle ACM = |\angle MCB - \angle C| = |90^\circ - \tfrac{1}{2}\angle A - \angle C| = \tfrac{1}{2}|\angle B - \angle C|$$

$$\Longrightarrow \angle AFE = \angle ADE \Longrightarrow D \in \Gamma_a.$$

jayme
#3 Nov 10, 2010, 6:05 pm

Dear Mathlinkers,
1. F is the point of intersection of AM and BC.
2. Remark that the tangent Tm to "Gamma" at M is parallel to EF.
3. According to a converse of Reim's theorem
applied to Gamma with the basic points A and D, the borning monians MAF and MDF, the parallels Tm
and FE, the points A, D, E and F are concyclic on the A-Apollonius circle of ABC.
4. This last circle being orthogonal to "Gamma", we are done.
Sincerely
Jean-Louis

dgreenb801
#4 Nov 11, 2010, 5:48 am

Note that
$$\angle EDK = \angle MDK = \frac{1}{2}\hat{M}AD = \frac{1}{2}(\hat{D}B + \hat{B}M) = \frac{1}{2}(\hat{D}B + \hat{M}C) = \angle KED.$$
Thus, K lies on the perpendicular bisector of ED. Also,
$$\angle DAE = \frac{1}{2}(2\angle DAE) = \frac{1}{2}(\angle DAE + \angle NAC + \angle DAE - \angle BAN) = \frac{1}{2}(\angle DAC - \angle BAD) = \frac{1}{2}[\frac{1}{2}\hat{C}D - \frac{1}{2}\hat{B}D] = \frac{1}{2}\angle DKE$$
Thus, K is the circumcenter of $\triangle AED$.

Quick Reply

High School Olympiads

F,C,H are collinear

Reply

WakeUp
#1 Nov 10, 2010, 12:26 am

The circle Γ and the line ℓ have no common points. Let AB be the diameter of Γ perpendicular to ℓ, with B closer to ℓ than A. An arbitrary point $C \neq A, B$ is chosen on Γ. The line AC intersects ℓ at D. The line DE is tangent to Γ at E, with B and E on the same side of AC. Let BE intersect ℓ at F, and let AF intersect Γ at $G \neq A$. Let H be the reflection of G in AB. Show that F, C, and H are collinear.

Luis González
#2 Nov 10, 2010, 5:27 am

Let $T \equiv \ell \cap AB$ and $G' \equiv FC \cap \Gamma$, different fom C. By Pascal theorem for the degenerate cyclic hexagon $EEBACG'$, the intersections $D \equiv EE \cap AC, F \equiv EB \cap CG'$ and $T' \equiv AB \cap G'E$ are collinear $\Longrightarrow T' \equiv AB \cap \ell$, hence $T \equiv T'$. Quadrilateral $FAET$ is cyclic on account of $\angle AEF = \angle ATF = 90^\circ$. Therefore $\angle AFT = \angle AEG' = \angle AGG' \Longrightarrow GG' \parallel FT \Longrightarrow G'$ is the reflection of G across AB, i.e. $H \equiv G'$.

thanhnam2902
#3 Nov 10, 2010, 3:11 pm

The same probolem at here:
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=358712

jayme
#4 Nov 10, 2010, 6:11 pm

Dear Mathlinkers,
according my sources, this problem comes from
Problème 5, German pre TST 2005
Sincerely
Lean-Louis

djmathman
#5 Dec 26, 2013, 11:22 pm

Let $X = AB \cap \ell$, and let O be the center of Γ. Since DE is tangent to Γ, we have $\angle DEB = \angle BAE$, and since $\angle DXO = \angle DEO$ we have $DXEO$ cyclic, which implies $\angle AOE = \angle EDF$. Therefore $\triangle AOE \sim \triangle EDF$, and since $OA = OB$ we have $ED = DF$. Next, by Power of a Point,

$$DC \cdot DA = DE^2 = DF^2 \implies \frac{DF}{DC} = \frac{DA}{DF}.$$

Therefore $\triangle DCF \sim \triangle DFA$. Finally, since $GH \perp AB$, we have $GH \parallel \ell$, so $\angle DCF = \angle DFA = \angle HGA = \angle HCA$, whence H, C, F are collinear.

Quick Reply

High School Olympiads

Tangential quadrilateral II

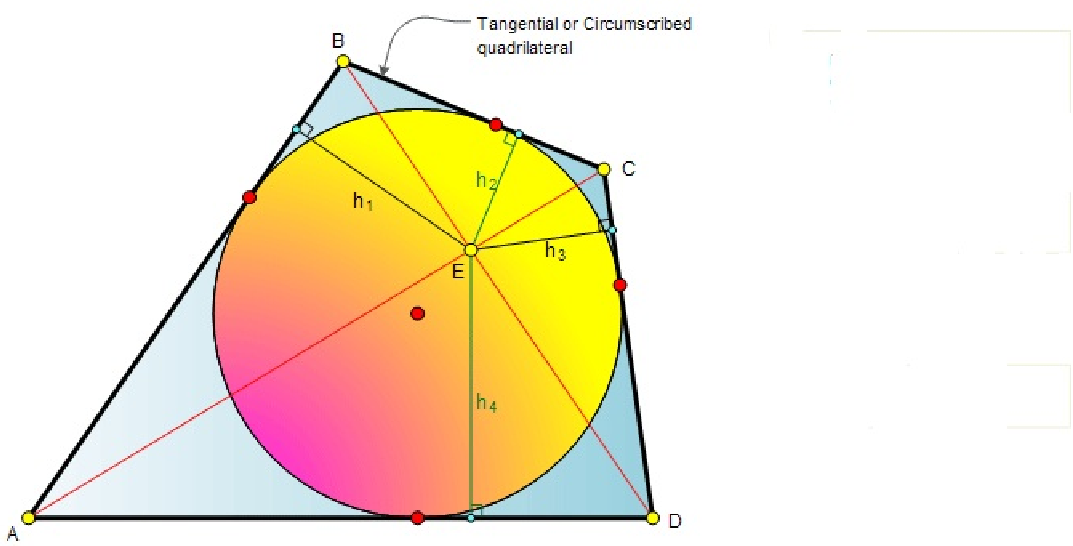
Speed2001

#1 Nov 9, 2010, 6:38 am

Solve the problem given in the attachement below

Attachments:

The figure shows a tangential or circumscribed quadrilateral ABCD. The diagonals AC and BD meet at E. If h_1, h_2, h_3, and h_4 are perpendicular to AB, BC, CD, and AD, respectively, prove that $\frac{1}{h_1}+\frac{1}{h_3}=\frac{1}{h_2}+\frac{1}{h_4}$.



Luis González

#2 Nov 9, 2010, 9:03 am • 1

Incircle (I,ϱ) of $ABCD$ touches AB,BC,CD,DA through $M,N,L,P.$ Denote $X\equiv AD\cap CB$ and $Y\equiv AB\cap CD$ and let M_0,N_0,L_0,P_0 be the orthogonal projections of M,N,L,P on line $XY.$ $E\equiv AC\cap BD$ is the pole of XY WRT $(I),$ thus $IE\perp XY$ through point $F.$ But since pencil $X(P,N,E,F)$ is harmonic, it follows that FE bisects $\angle PFN.$ As a result, PN_0 and NP_0 pass through the midpoint of $FE.$ Similarly, lines LM_0 and ML_0 pass through the midpoint of $EF \Longrightarrow$ trapezoids NPP_0N_0 and MLL_0M_0 share the same harmonic mean $EF.$ Thus

$$\frac{1}{MM_0}+\frac{1}{LL_0}=\frac{1}{NN_0}+\frac{1}{PP_0}=\frac{2}{EF} \ \ (\ast)$$

By Salmon's theorem for E,M,N,L,P and their polars WRT $(I),$ we get

$$\frac{\varrho}{MM_0}=\frac{EI}{h_1} \ , \ \frac{\varrho}{LL_0}=\frac{EI}{h_3} \ , \ \frac{\varrho}{NN_0}=\frac{EI}{h_2} \ , \ \frac{\varrho}{PP_0}=\frac{EI}{h_4}$$

$$\varrho\left(\frac{1}{MM_0}+\frac{1}{LL_0}\right)=EI\left(\frac{1}{h_1}+\frac{1}{h_3}\right) \ , \ \varrho\left(\frac{1}{NN_0}+\frac{1}{PP_0}\right)=EI\left(\frac{1}{h_2}+\frac{1}{h_4}\right)$$

Combining the latter expressions with $(\ast)$ yields : $\frac{1}{h_1}+\frac{1}{h_3}=\frac{1}{h_2}+\frac{1}{h_4}.$

This post has been edited 2 times. Last edited by Luis González, Nov 9, 2010, 9:29 am

Speed2001

#3 Nov 9, 2010, 9:11 am

Yes , Thx for you solution

jayme

#4 Nov 9, 2010, 9:52 am

Dear Mathlinkers,
nice problem...
But it would be nice if a figure from a site, the reference of the author will follow...
Sincerely
Jean-Louis

nsato

#5 Nov 18, 2010, 12:49 pm

Also, see Theorem 17 in http://www.cip.ifi.lmu.de/~grinberg/CircumRev.pdf.

Quick Reply

High School Olympiads

With a symmedian again (own)

Reply

jayme

#1 Nov 5, 2010, 8:43 pm • 1

Dear Mathlinkers,
ABC an acute triangle,
DEF the orthic triangle of ABC,
Sa the A-symmedian of ABC,
A* the foot of Sa,
A+ the midpoint of the segment AA*,
K the symmedian point of ABC
and U, V the points of intersection of FD and BA+, DE and CA+, resp..
Prove : UKV is parallel to EF.
Sincerely
Jean-Louis

Luis González

#2 Nov 6, 2010, 7:03 am • 1

The problem can be generalized, but I only have a proof with barycentric coordinates.

P is an arbitrary point in the plane of $\triangle ABC$ and $\triangle DEF$ is the cevian triangle of $P, D \in BC, E \in CA$ and $F \in AB. P_0$ is the isotomcomplement of P WRT $\triangle ABC$ and A_0 is its trace on the sideline $BC. Q$ is the midpoint of AA_0 and lines QB, QC cut FD, DE at $U, V,$ respectively. Then $UP_0V \parallel EF.$

Luis González

#3 Nov 8, 2010, 10:18 pm • 1

Let $(u : v : w)$ be the barycentric coordinates of P with respect to $\triangle ABC.$ Then

$$P : (u : v : w), \ P_0 : \left(\frac{1}{v} + \frac{1}{w} : \frac{1}{w} + \frac{1}{u} : \frac{1}{u} + \frac{1}{v}\right), \ A_0 : (0 : v(u+w) : w(u+v))$$

$$Q : (uv + uw + 2vw : v(u+w) : w(u+v))$$

$$FD \equiv wvx - uwy + uvz = 0 \ , \ DE \equiv wvx + uwy - uvz = 0$$

$$BQ \equiv w(u+v)x - (uv + uw + 2vw)z = 0$$

$$CQ \equiv v(u+w)x - (uv + uw + 2vw)y = 0$$

Coordinates of intersections $U \equiv BQ \cap FD$ and $V \equiv CQ \cap DE$ are then

$$U : (u(uv + uw + 2vw) : v(u+2v)(u+w) : uw(u+v))$$

$$V : (u(uv + uw + 2vw) : uv(u+w) : w(u+2w)(u+v))$$

$$\Longrightarrow UV \equiv wv(u+w)(u+v)x - uw^2(u+v)y - uv^2(u+w)z = 0$$

We verify that U, V, P_0 are collinear since P_0 satisfies the equation of $UV.$ On the other hand, the coordinates of the infinity point P_∞ of the line UV are $P_\infty : (uw - uv : uv + vw : -uw - wv),$ which are identical to the coordinates of the infinity point of $EF \Longrightarrow UP_0V \parallel EF.$

Quick Reply

High School Olympiads

About Mittenpunkt

Stephen
#1 Nov 7, 2010, 3:46 pm

Prove that a triangle's incenter, the symmedian point, and the mittenpunkt lies on one line.

jayme
#2 Nov 7, 2010, 5:31 pm

Dear Mathlinkers,
1. The basic result : orthocenter, Lemoine's point of ABC are collinear with the Lemoine's point of the orthic triangle of ABC.
2. Now transpose this result to the excentral triangle of ABC and we are done.

But we have to prove the basic result.
Sincerely
Jean-Louis

Luis González
#3 Nov 7, 2010, 11:01 pm

Let $O, H, I, K, M_t, N_a, G_e, R$ denote the circumcenter, orthocenter, incenter, symmedian point, Mittenpunkt, Nagel point, Gergonne point and Retrocenter of $\triangle ABC$, respectively. It's well-known that I, K, M_t become Nagel point, Retrocenter and Gergonne point of the medial triangle of $\triangle ABC$. Thus, it is enough to show that N_a, G_e, R are collinear.

H, N_a, G_e lie on a same circumhyperbola $\mathcal{F}$, namely, the Feuerbach hyperbola of $\triangle ABC$, since their isogonal conjugates: O and the two homothetic centers of $(I), (O)$ are collinear. Then isotomic conjugation in $\triangle ABC$ takes N_a, G_e into each other and H into $R \Longrightarrow N_a, G_e, R$ are collinear on the isotomic line of $\mathcal{F}$.

jayme
#4 Jan 20, 2011, 8:57 pm

Dear Mathlinkers,
just to remind a link
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=192071
A remark : a complete synthetic proof is possible.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Two circles

mathlink

#1 Nov 7, 2010, 10:42 am

Two circles (O_1, R_1) and (O_2, R_2) intersect at two points A and B. The line l passes through B, meet (O_1) and (O_2) at P, Q respectively.
1) Prove that: The perpendicular bisector of PQ goes through a fixed point when the line l changes
2) Find the location of the line l to maximum the area of triangle APQ

Luis González

#2 Nov 7, 2010, 11:36 am

1) Let M be the midpoint of $\overline{PQ}$. Then powers of M with respect to (O_1) and (O_2) are in the same ratio, namely -1, since $\overline{MP} \cdot \overline{MB} = -\overline{MQ} \cdot \overline{MB}$. Thus M moves on a the circle (U) coaxal with $(O_1), (O_2)$ whose center is the midpoint U of O_1O_2. Thereby, perpendicular bisector of $\overline{PQ}$ always passes through the reflection of B about U.

2) Since triangles $\triangle APQ$ are all similar, then $[\triangle APQ]$ will be maximum iff length PQ is maximum. Let D, E be the midpoints of BP, BQ and let the parallel from B to O_1O_2 cut $(O_1), (O_2)$ again at X, Y. From the right trapezoid O_1DEO_2, it follows that $O_1O_2 \geq DE \Longrightarrow XY \geq PQ \Longrightarrow [\triangle AXY] \geq [\triangle APQ]$. Hence, ℓ is parallel to O_1O_2.

High School Olympiads

A new result about Steiner chains?. (own)

Luis González

#1 Nov 7, 2010, 5:28 am

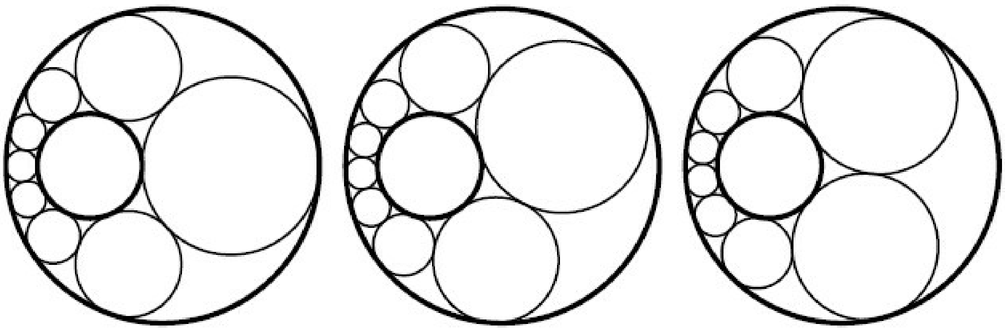
Nonintersecting circles (O) and (I) with radii R, r are given, such that (I) lies inside (O). If n $(n = 3, 4, 5, 6, 7...)$ tangent circles ω_i with radii ϱ_i can be inscribed around the region between $(O), (I)$ such that the final circle is tangent to the first, then $(O), (I)$ form a *Steiner chain* with length n and this will happen for any position of the first circle ω_1. (See example below)

Assume that $(O), (I)$ admit Steiner chains of length $n \geq 3$. Then prove: The sum of the curvatures of the n circles ω_i is constant, i.e. it is independent of the first circle chosen to form the chain. Further, show that

$$\frac{1}{2n}\sum_{i=1}^{n}\frac{1}{\varrho_i} = \frac{\mathcal{K}_O + R^2}{\mathcal{K}_O{}^2|\frac{R}{\mathcal{K}_O} - \frac{r}{\mathcal{K}_I}|} + \frac{1}{4}|\frac{R}{\mathcal{K}_O} - \frac{r}{\mathcal{K}_I}|\cot^2\frac{\pi}{n}$$

K is the external limiting point of $(O), (I)$ and $\mathcal{K}_O, \mathcal{K}_I$ denote its powers to $(O), (I)$.

Attachments:



Ivan_Borsenco

#2 Jul 3, 2014, 12:02 pm

Let $\mathcal{K}$ be the limiting center and let $I(K, k^2)$ be inversion that transforms the Steiner configuration into concentric circles with center O^*. Then

$$KO^* = k^2 \cdot \frac{KO}{\mathcal{K}_O} = k^2 \cdot \frac{KI}{\mathcal{K}_I}.$$

The radii of the concentric circles are R^* and r^* and are equal to

$$R^* = k^2 \cdot \frac{R}{\mathcal{K}_O}, \qquad r^* = k^2 \cdot \frac{r}{\mathcal{K}_I}.$$

Let $\omega_i^*(O_i^*, \rho_i^*)$ be inverted circles of ω_i in the Steiner chain.

They have the same radius $\rho_i^* = \frac{1}{2}|R^* - r^*| = \rho^*$.

If we apply the same inversion, circles ω_i^* are transformed back to ω_i.

Therefore

$$\rho_i = k^2 \cdot \frac{\rho_i^*}{\mathcal{K}_{O_i^*}} = k^2 \cdot \frac{\rho^*}{(KO_i^*)^2 - (\rho^*)^2},$$

yielding

$$\frac{1}{\rho_i} = \frac{(KO_i^*)^2}{k^2\rho^*} - \frac{\rho^*}{k^2},$$

Because O^* is the gravity center of O_I^*, we get

$$\sum_{i=1}^{n}(KO_i^*)^2 = \sum_{i=1}^{n}(O^*O_i^*)^2 + n(KO^*)^2 = n(\frac{1}{2}(R^*+r^*))^2 + n(KO^*)^2.$$

The sum of the curvatures is equal to

$$\sum_{i=1}^{n}\frac{1}{\rho_i} = \frac{1}{k^2\rho^*}\cdot(n(\frac{1}{2}(R^*+r^*))^2 + n(KO^*)^2) - n\frac{\rho^*}{k^2},$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\rho_i} = \frac{1}{k^2\cdot\frac{1}{2}|R^*-r^*|}\cdot(\frac{1}{4}(R^*+r^*)^2 + (KO^*)^2) - \frac{\frac{1}{2}|R^*-r^*|}{k^2},$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\rho_i} = \frac{R^*r^* + (KO^*)^2}{k^2\cdot\frac{1}{2}|R^*-r^*|},$$

$$\frac{1}{2n}\sum_{i=1}^{n}\frac{1}{\rho_i} = \frac{R^*r^* + (KO^*)^2}{k^4}\cdot\frac{1}{|\frac{R}{K_O} - \frac{r}{K_I}|} = \frac{Rr + KO\cdot KI}{\mathcal{K}_O\mathcal{K}_I}\cdot\frac{1}{|\frac{R}{\mathcal{K}_O} - \frac{r}{\mathcal{K}_I}|}.$$

We get that the sum of the curvatures is constant. My expression is a bit different from the proposed one.
I wonder if they are equivalent and I am interested to see the proposer's solution.

Also, from Soddy circles (case $n = 3$), we have

$$2\sum_{i=1}^{3}\frac{1}{\rho_i} = \frac{1}{R} + \frac{1}{r}$$

and it would be nice to find an expression for $\sum_{i=1}^{n}\frac{1}{\rho_i}$ only in terms of (R, r).
I also have an idea how to do that! But I will write it later))

Quick Reply

High School Olympiads

Intersecting lines

Reply

borislav_mirchev
#1 Nov 7, 2010, 4:01 am

It is given the triangle ABC. A1, B1, C1 are the middles of the sides BC, CA, AB respectively. The angle bisectors of the angles <A, <B, <C intersects incircle at the points A2, B2, C2 respectively. Prove that A1A2, B1B2, C1C2 intersects at a common point.

I would like to know how this problem can be solved and if it is a new statement.

Luis González
#2 Nov 7, 2010, 4:45 am

Let X,Y,Z be the tangency points of the incircle (I) with BC,CA,AB. Assuming that A2,B2,C2 lie on the small arcs YZ,ZX,XY (otherwise, there is no concurrency), then A1A2, B1B2 and C1C2 concur at Kimberling center X(178); the second mid-arc point of ABC. The proof with barycentric coordinates is quite straightforward, yet I like the synthetic proof presented by Jean- Louis on his web-site. See Volume 5 "The second mid-arc point".

borislav_mirchev
#3 Nov 7, 2010, 5:05 am

Thank you for the valuable comments. Today I reinvented 4 geometric problems but I think it is the hardest and more interesting one.

jayme
#4 Nov 8, 2010, 7:53 pm

Dear Mathlinkers,
a generalization of this hard problem in order to find a synthetic proof can be seen on
http://forumgeom.fau.edu/FG2006volume6/FG200629.pdf
Sincerely
Jean-Louis

borislav_mirchev
#5 Nov 9, 2010, 7:00 pm

Thank you for the excellent resource! I saw all the resources.
If it is possible can someone post synthetic proof using only highschool material
with as less as possible unknown to the pupils facts?

Quick Reply

High School Olympiads

TI-(x+y=45)

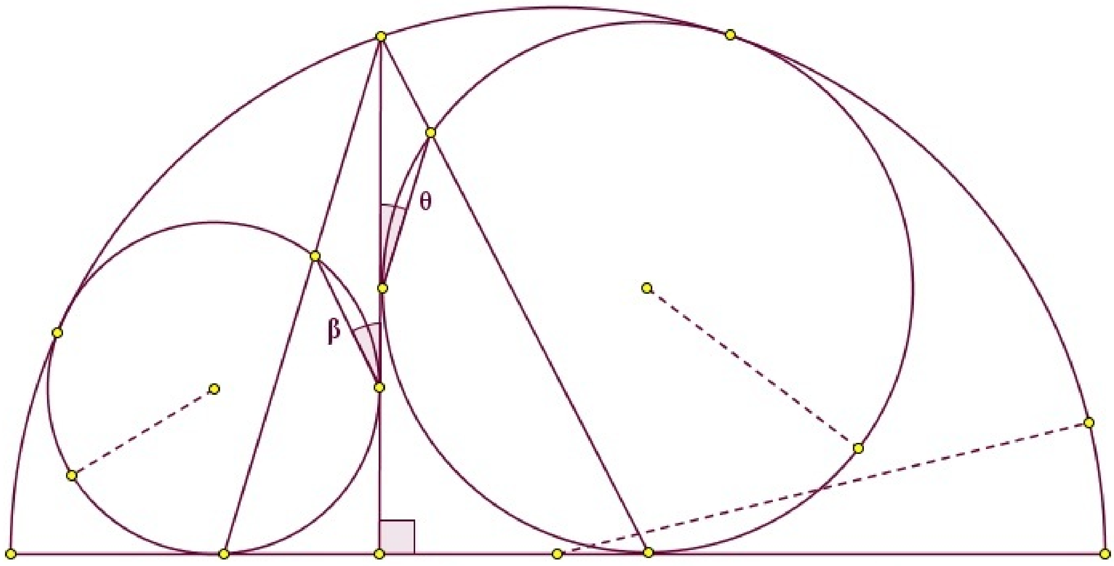
Reply

Speed2001

#1 Nov 5, 2010, 9:33 am

Pure Geometry :

Attachments:



In the figure shown , Prove that β + θ = 45°.

Luis González

#2 Nov 5, 2010, 11:14 am

Problem. $\triangle ABC$ is right with $\angle ACB=90^{\circ}$, D is the orthogonal projection of C on AB and let ω_1,ω_2 be the two Thebault circles of CD. ω_1 is tangent to $\overline{DB},\overline{DC}$ at X,P and ω_2 is tangent to $\overline{DA},\overline{DC}$ at Y,Q. Rays XC,YC cut ω_1 and ω_2 at $M,N,$ respectively. Then $\angle CPM+\angle CQN=45^{\circ}$.

Let C',ω'_1 be the reflections of C,ω_1 about AB. Then BD bisects $\angle CBC'$ and ω'_1 is tangent to $\odot(ABC)$. By Thebault theorem, we deduce that $X\equiv \omega_1\cap\omega'_1$ is the incenter of $\triangle CBC' \Longrightarrow CX$ bisects $\angle BCD$. Likewise, CY bisects $\angle ACD$. Consequently, we get

$$\angle CPM=\angle CXP=\angle CXD-45^{\circ}=(90^{\circ}-\tfrac{1}{2}\angle DCB)-45^{\circ}$$

$$\angle CQN=\angle CYQ=\angle CYD-45^{\circ}=(90^{\circ}-\tfrac{1}{2}\angle DCA)-45^{\circ}$$

$$\Longrightarrow \angle CPM+\angle CQN=90^{\circ}-\tfrac{1}{2}(\angle DCB+\angle DCA)=90^{\circ}-45^{\circ}=45^{\circ}.$$

phiReKaLk6781

#3 Nov 6, 2010, 4:56 am

Do not double post, especially, if the two forums you post the same problem in are a Middle School forum and an Olympiad forum.

Speed2001

#4 Nov 6, 2010, 5:04 am

Hi PhireKaLk6781 , I post in forum Olympiad and then there was error on the website.

Quick Reply

High School Olympiads

Difficult calculation problem

Reply

delegat

#1 Nov 5, 2010, 4:46 am

Given is triangle $\triangle ABC$ with circumcenter O and circumradius R. Prove that sum of squares of distances between circumcenter and tangent points of incircle (in total 3 summands) plus sqares of distances of circumcenter to excircles tangent points (in total 9 summands) plus square of distance between orthocenter and circumcenter is equal to $21R^2$.

Luis González

#2 Nov 5, 2010, 6:36 am

Let X,Y,Z and M,N,L be the tangency points of the incircle (I) and A-excircle (I_a) with BC,CA,AB, respectively. λ and λ_a stand for the sum of the square distances from O to the vertices of $\triangle XYZ$ and $\triangle MNL$. λ_b, λ_c are defined cyclically. From the powers of X,Y,Z and M,N,L WRT (O) we obtain

$$R^2-OX^2=(s-b)(s-c)$$

$$R^2-OY^2=(s-c)(s-a)$$

$$R^2-OZ^2=(s-a)(s-b)$$

$$\Longrightarrow \lambda=3R^2-(s-b)(s-c)-(s-c)(s-a)-(s-a)(s-b) \ \ (1)$$

$$ON^2-R^2=s(s-b) \ , \ OL^2-R^2=s(s-c) \ , \ R^2-OM^2=(s-b)(s-c)$$

$$\Longrightarrow \lambda_a=3R^2+a\cdot s-(s-b)(s-c)$$

By cyclic permutation of a,b,c we get λ_b and λ_c. Then

$$\lambda_a+\lambda_b+\lambda_c=9R^2+2s^2-(s-b)(s-c)-(s-c)(s-a)-(s-a)(s-b) \ \ (2)$$

Adding the expressions (1) and (2) gives

$$\lambda+\lambda_a+\lambda_b+\lambda_c=12R^2+2s^2-2\sum_{\text{cyclic}}(s-b)(s-c)$$

$$\lambda+\lambda_a+\lambda_b+\lambda_c=12R^2+(a^2+b^2+c^2)$$

Since $OH^2=9R^2-(a^2+b^2+c^2)$, we obtain $\lambda+\lambda_a+\lambda_b+\lambda_c+OH^2=21R^2$

Quick Reply

High School Olympiads

Prove that projections of vertex of pyramid are concyclic

Reply

Goutham
#1 Nov 3, 2010, 10:48 am

Given a pyramid whose base is an n-gon inscribable in a circle, let H be the projection of the top vertex of the pyramid to its base. Prove that the projections of H to the lateral edges of the pyramid lie on a circle.

Luis González
#2 Nov 4, 2010, 12:19 am

Let A be the apex of the pyramid. Denote the vertices of the cyclic n-gon $P_1P_2P_3...P_n$ and the projections of H onto $AP_1, AP_2, AP_3, ..., AP_n$ as $H_1, H_2, H_3, ..., H_n$.

$$\overline{AH}^2 = \overline{AP_1} \cdot \overline{AH_1} = \overline{AP_2} \cdot \overline{AH_2} = \overline{AP_3} \cdot \overline{AH_3} = \overline{AP_n} \cdot \overline{AH_n}$$

$\Longrightarrow$ Points $H_1, H_2, H_3, ..., H_n$ lie on the inverse image of the circumcircle $\mathcal{C}$ of the n-gon, under the inversion with center A and radius $\overline{AH}$. The spherical surface $\mathcal{E}$ passing through $A, \mathcal{C}$ is taken into a plane π' and the plane π containing the base of the pyramid is taken into a spherical surface $\mathcal{E}'$ passing through $A \Longrightarrow H_1, H_2, H_3, ..., H_n$ lie on the intersection (circumference) $\mathcal{C}' \equiv \mathcal{E}' \cap \pi'$. Moreover, $\mathcal{C}$ and $\mathcal{C}'$ lie on a same spherical surface.

Quick Reply

High School Olympiads

geometry

Reply

huyhoang

#1 Nov 1, 2010, 9:01 pm

From point A outside the circle (O), construct two tangent lines AB, AC to (O). Denote D be the intersection of AO and (O). Construct $BX \perp CD$ and Y be the midpoint of BX. Let Z be the intersection of DY and (O).
a. Prove that the circumcircle of triangle AZD tangents with BD
b. Prove that $AZ \perp CZ$

This post has been edited 1 time. Last edited by huyhoang, Nov 2, 2010, 7:05 pm

tkrass

#2 Nov 2, 2010, 12:53 am

What is Z?

basketball9

#3 Nov 2, 2010, 2:41 am

The intersectoin between DZ and O

huyhoang

#4 Nov 2, 2010, 7:06 pm

> *tkrass wrote:*
> What is Z?

Z is the intersection of DY and (O)

Luis González

#5 Nov 3, 2010, 11:13 pm

Assuming that D is the midpoint of the big arc BC, let $E \equiv BC \cap AD$ (midpoint of BC). Then EY is the B-midline of $\triangle BXC \Longrightarrow EY \parallel CD$. Thus, $\angle BYE = \angle BXC = 90^\circ$ and $\angle BZY = \angle BCD = \angle BEY \Longrightarrow Z, Y$ lie on the circle with diameter EB, i.e. $\angle BZE = 90^\circ$. Since $\angle BZC = \angle(CA, CB)$, it follows that $\angle BZC = 90^\circ + \angle EAC \Longrightarrow \angle EZC = 90^\circ + \angle EAC - 90^\circ = \angle EAC \Longrightarrow$ Quadrilateral $AZEC$ is cyclic. Then $\angle AZC = \angle AEC = 90^\circ$, i.e. $AZ \perp CZ$ and $\angle ZAE = \angle ZCB = \angle ZDB \Longrightarrow BD$ is tangent to $\odot(AZD)$ through D.

huyhoang

#6 Nov 5, 2010, 9:22 pm

thanks, but my solution is using inversion.
anyway, thank you

Quick Reply

High School Olympiads

Victory.US

#1 Nov 1, 2010, 4:22 pm

Let triangle ABC. $AA_1 \perp BC$. Similar $BB_1; CC_1$. Circle (C) go through B_1C_1 and tangent to circumcircle of ABC at A_2. Similar BB_2, CC_2. Prove that A_2A_1, B_2B_1, C_2C_1 are concurrent

Luis González

#2 Nov 3, 2010, 8:13 am

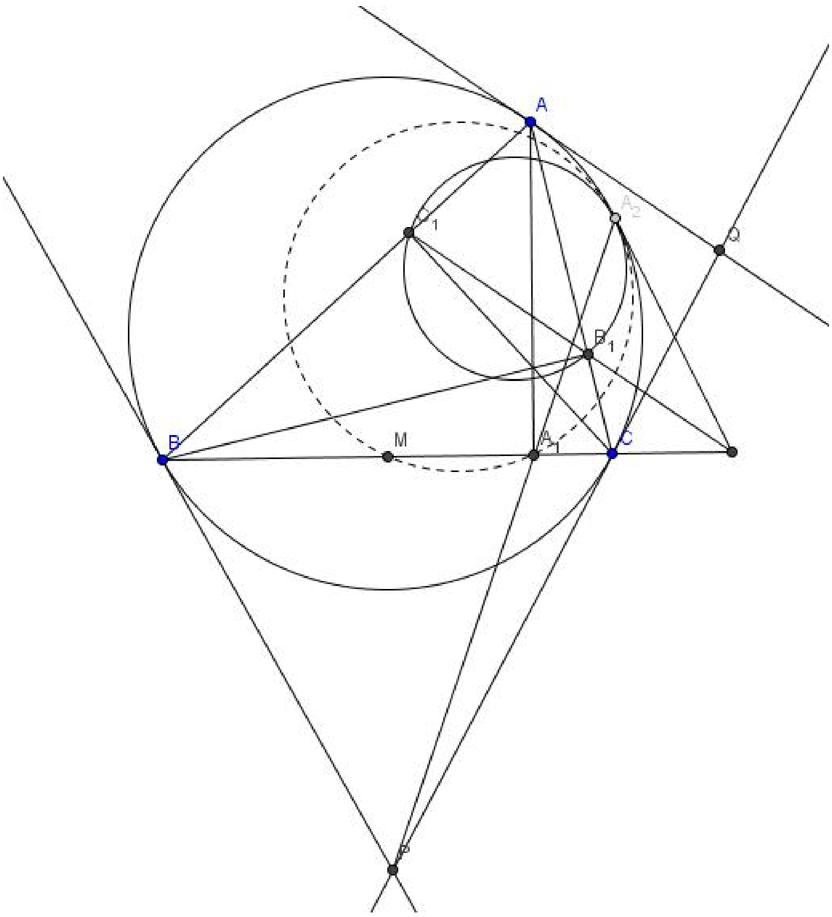
Let ω_a be a circle passing through B_1, C_1 and tangent to the circumcircle (O) at A_2. Tangent line τ_a of (O) through A_2 is the radical axis of $(O), \omega_a$, line B_1C_1 is the radical axis of ω_a and the circle (M_a) with diameter BC and sideline BC is the radical axis of $(O), (M_a) \Longrightarrow \tau_a, B_1C_1$ and BC concur at the radical center A_c of $(O), (M_a), \omega_a$. But since the cross ratio (B, C, A_1, A_0) is harmonic, it follows that A_1A_2 is the polar of A_c with respect to (O). Likewise, B_1B_2 and C_1C_2 are the polars of $B_0 \equiv C_1A_1 \cap CA$ and $C_0 \equiv A_1B_1 \cap AB$. Points A_0, B_0, C_0 are collinear on the trilinear polar of the orthocenter H, i.e. orthic axis of $\triangle ABC$, which is perpendicular to its Euler line e. Therefore, lines A_1A_2, B_1B_2, C_1C_2 and e concur at the pole of $A_0B_0C_0$ with respect to (O), i.e. Gob's point X_{25} of $\triangle ABC$.

skytin

#3 Nov 3, 2010, 8:01 pm

Let tangents to (ABC) thru A B C intersects at points P Q R , easy to see that (MA_1A_2) is tangent to (ABC) , so angles BA_2M = A_1A_2C so A_2A_1 is simmedian of trangle BA_2C , so A_2A_1 goes thru point P , like the same other lines and easy to see that trangles PQR and ABC are homotetiv , so PA_1 QB_1 and RC_1 intersects at one point . done

Attachments:



Quick Reply

High School Olympiads

Circumcentres make a parallelogram

Reply

WakeUp
#1 Nov 2, 2010, 1:57 am • 1

In a triangle ABC with $AB < AC < BC$, the perpendicular bisectors of AC and BC intersect BC and AC at K and L, respectively. Let O, O_1, and O_2 be the circumcentres of triangles ABC, CKL, and OAB, respectively. Prove that OCO_1O_2 is a parallelogram.

Luis González
#2 Nov 2, 2010, 10:59 pm

From the isosceles $\triangle ACK, \triangle BCL$ with apices $K, L,$ we have $\angle AKB = 2\angle ACB$ and $\angle ALB = 2\angle ACB \pmod{\pi}$ $\Longrightarrow A, B, O, K, L$ are concyclic. Now, since lines AB and KL are antiparallel with respect to $CA, CB,$ it follows that $CO_1 \perp AB$ and $CO \perp KL,$ i.e. $CO \parallel O_1O_2$ and $CO_1 \parallel OO_2 \Longrightarrow OCO_1O_2$ is a parallelogram.

FantasyLover
#3 Nov 3, 2010, 3:28 am

Solution

xeroxia
#4 Jan 19, 2013, 8:39 pm

Since $\angle ALB = 2 \cdot \angle ACB = \angle AKB = \angle AOB$, A, L, O, K, B are cyclic, and their circumcenter is O_2. And we have $\angle CLK = \angle ABC$.
O_1O_2 is perpendicular to LK at its midpoint. Similarly, OO_2 is perpendicular to AB at its midpoint.
$\angle CO_1O_2 = \angle CO_1K + \angle KO_1O_2 = 2\angle CLK + \angle KCL = 2\angle ABC + \angle ACB$
Similarly,
$\angle COO_2 = \angle COA + \angle AOO_2 = 2\angle CBA + \angle ACB$
So we have
$\angle COO_2 = \angle CO_1O_2 = 2\angle CBA + (180^\circ - \angle CBA - \angle CAB) = 180^\circ - (\angle CAB - \angle CBA)$
Also, $\angle O_1CO = \angle O_1CK - \angle OCK = 90^\circ - \angle CBA - (90^\circ - \angle CAB) = \angle CAB - \angle CBA$
Since $\angle CO_1O_2 = \angle COO_2 = 180^\circ - \angle O_1CO$, the quadrilateral O_1O_2OC is a parallelogram.

Quick Reply

High School Olympiads

Prove that SD = SM - Poland 2003 Third Round

Reply

Amir Hossein

#1 Nov 2, 2010, 1:30 am

In an acute-angled triangle ABC, CD is the altitude. A line through the midpoint M of side AB meets the rays CA and CB at K and L respectively such that $CK = CL$. Point S is the circumcenter of the triangle CKL. Prove that $SD = SM$.

Luis González

#2 Nov 2, 2010, 2:19 am • 1

Recently posted in the current geometry marathon with similar notations.

> barcelona wrote:
>
> It is given $\triangle ABC$ and M is the midpoint of AB. Let ℓ pass through M. $K \equiv \ell \cap AC$ and $L \equiv \ell \cap BC$, such that $CK = CL$. Let $CD \perp AB, D \in AB$ and O is the center of the circumcircle of $\triangle CKL$. Prove that $OM = OD$.

Since $MA = MB$ and $CK = CL$, by Menelaus' theorem for $\triangle ABC$ cut by the straight line KML, we get $AK = BL$. Thus, $E \equiv \odot(CKL) \cap \odot(ABC) \not\equiv C$ is the center of the rotation carrying AK into BL, and due to $EA = EB$, then E is the midpoint of the arc AB of $\odot(ABC)$. Obviously, E lies on the internal bisector of $\angle ACB \Longrightarrow O \in CE$. Since CE is the circumdiameter of $\triangle CKL$ perpendicular to KL, it follows that O is the midpoint of CE. Moreover, if A', B' denote the midpoints of CB, CA, then O is the intersection of CE with the perpendicular bisector of segment $A'B'$, since the positive homothety with center C and factor $\frac{1}{2}$ transforms A, B into A', B', E into O and the perpendicular bisector EM of AB into the perpendicular bisector of $A'B'$. Now, it remains to see that quadrilateral $A'B'DM$ is an isosceles trapezoid, in which O lies on its symmetry axis. Indeed, $\triangle MA'B'$ and $\triangle DB'A'$ are symmetric about the perpendicular bisector of $A'B'$, hence $OD = OM$.

jgnr

#3 Nov 3, 2010, 6:05 pm

Let the line through B parallel to CA meet line KL at P. So $\angle BPL = \angle LKC = \angle CLK = \angle BLP$, so $BL = BP$. We also have $\triangle AMK \cong \triangle BMP$, so $BP = AK$, therefore $AK = BL$. WLOG assume $a \geq b$, so $AK = BL = \frac{a-b}{2}$. Let X, Y, Z be the projections of S onto AB, BC, CA respectively. We get $CY = \frac{a+b}{4}$, $BY = \frac{3a-b}{4}$, $CZ = \frac{a+b}{4}$, $ZA = \frac{3b-a}{4}$. Note that $CY^2 + BX^2 + AZ^2 = BY^2 + AX^2 + CZ^2$, so $BX^2 - AX^2 = BY^2 - AZ^2 \Longrightarrow (BX - AX)(BX + AX) = \frac{9a^2 - 6ab + b^2}{16} - \frac{9b^2 - 6ab + a^2}{16} \Longrightarrow (2BX - c)c = \frac{a^2 - b^2}{2}$. Therefore $BX = \frac{2c^2 + a^2 - b^2}{4c}$. Also note that $BM + BD = \frac{c}{2} + \frac{a^2 + c^2 - b^2}{2c} = \frac{2c^2 + a^2 - b^2}{2c} = \frac{BX}{2}$. So X is the midpoint of MD and hence $SD = SM$, as desired.

jayme

#4 Nov 3, 2010, 8:01 pm

Dear Mathlinkers,

1. (0) the circumcircle of ABC and O its center
2. (1) the circumcircle of CKL and O1 its center
3. (2) the circumcircle of ADM and O2 its center
4. E the second point of intersection of (1) and (2)
5. P, N the second points of intersection of AO1 with (1), (2) resp.
6. K, L the points of intersection of MN with CA, CB resp.

7. P, M and E are collinear (by a converse of the Reim's theorem)
8. O1O2 // PM (by the midpoint theorem)
9. If we prove that (0) goes through P, then O is on PME and O1O2 perpendicular to DM
10. O1O2 being the mediator of DM, we are done.

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Prove that S{AQC}/S{CMT} = (sin B / cos C)^2

Source: Mediterranean MO 2004

Amir Hossein
#1 Oct 31, 2010, 7:25 pm

In a triangle ABC, the altitude from A meets the circumcircle again at T. Let O be the circumcenter. The lines OA and OT intersect the side BC at Q and M, respectively. Prove that

$$\frac{S_{AQC}}{S_{CMT}} = \left(\frac{\sin B}{\cos C}\right)^2.$$

Luis González
#2 Oct 31, 2010, 11:31 pm

Since $\angle BCT = \angle BAT = \angle CAQ$ and $\angle MTC = \angle OCT = \angle QCA$, it follows that $\triangle AQC \sim \triangle CMT$. Thus

$ \frac{|\triangle AQC|}{|\triangle CMT|}=\left (\frac{AC}{CT}^\right)^2= \left (\frac{ \sin \widehat{ATC}}{\sin \widehat{TAC}} \right)^2= \left (\frac{ \sin B}{\sin (90^{\circ}-C)} \right)^2=\left (\frac{\sin B}{ \cos C} \right)^2$

High School Olympiads

Three lines concur on the Brocard axis

Source: 0

Luis González

#1 Jul 15, 2009, 9:03 am

Let K be the symmedian point of $\triangle ABC$ and $\triangle K_aK_bK_c$ the cevian triangle of $K.$ The circle passing through A tangent to BC at K_a cuts the circumcircle (O) again at $A'.$ Define cyclically B',C' and the following points: $X \equiv CC' \cap BB',$ $Y \equiv AA' \cap CC'$ and $Z \equiv AA' \cap BB'.$ Prove that AX,BY,CZ and the Brocard axis OK concur.

pacoga

#2 Oct 31, 2010, 4:36 am

I don't know a synthetic solution of this problem, but it can be generalized substituting the symmedian point K by any other point $P=(u:v:w)$. We'll get that the lines AX, BY and CZ always concur in the barycentric square $Q=(u^2:v^2:w^2)$ of P.

The locus of points P such that O, P and Q are collinear is the cubic K168, through the symmedian point.

Luis González

#3 Oct 31, 2010, 10:39 am

Thanks Francisco, I was already aware of the general configuration of cevians AX,BY,CZ intersecting at the barycentric square of P $(\star).$ But I'm not able to prove synthetically that the barycentric square of the symmedian point (third power point), defined as the trilinear pole of the line passing through the exsimilicenters of the pairs of circles with diameters $(BK_a,CK_a),$ (CK_b,AK_b) and $(AK_c,BK_c),$ lies on the Brocard axis. Nevertheless, we can proof $(\star)$ prescinding from calculations with barycentrics.

Replace symmedian point K by an arbitrary point P and let $\triangle P_aP_bP_c$ be the cevian triangle of $P,$ the remaining notations are the same. Let lines AA',BB' and CC' cut BC,CA and AB at $M,N,L,$ respectively. AM is radical axis of (O) and the circle passing through A and tangent to BC at $P_a \Longrightarrow MA \cdot MA'=MB \cdot MC=MP_a^2.$ Hence, inversion with center M and radius MP_a transforms the circles with diameters BP_a and CP_a into each other $\Longrightarrow M$ is their exsimilicenter. Likewise, we have that N,L are the exsimilicenters of the circles with diameters CP_b,AP_b and $AP_c,BP_c \Longrightarrow MNL$ is the trilinear polar of the barycentric square Q of $P.$ But since the pencil $A(L,N,X,M)$ is harmonic, it follows that AX passes through $Q.$ Similarly, lines BY,CZ pass through $Q.$

Quick Reply

High School Olympiads

Ratio of incircle and excircle ? - JBMO Shortlist

Reply

WakeUp
#1 Oct 31, 2010, 2:19 am

A triangle ABC is inscribed in the circle $\mathcal{C}(O,R)$. Let $\alpha<1$ be the ratio of the radii of the circles tangent to $\mathcal{C}$, and both of the rays $(AB$ and $(AC$. The numbers $\beta<1$ and $\gamma<1$ are defined analogously. Prove that $\alpha+\beta+\gamma=1$.

Luis González
#2 Oct 31, 2010, 4:25 am

$(I,r),(I_a,r_a),(I_b,r_b),(I_c,r_c)$ denote the incircle and three excircles against $A,B,C.$ Let $(U_a,\rho_a),(U_b,\rho_b),(U_b,\rho_c)$ be the mixtilinear incircles against A,B,C and $(V_a,\varrho_a),(V_b,\varrho_b),(V_c,\varrho_c)$ their corresponding mixtilinear excircles. (U_a,ρ_a) and (V_a,ϱ_a) are tangent to rays AB,AC through M,N and X,Y respectively. It's well-known that lines MN and XY pass through the incenter I and A-excenter I_a, respectively. Thus, from the similar kites $AMU_aN \sim AXV_aY$, it follows that

$$\alpha=\frac{\rho_a}{\varrho_a}=\frac{AI}{AI_a}=\frac{r}{r_a}$$

Similarly, we have the ratios: $\beta=\frac{\rho_b}{\varrho_b}=\frac{r}{r_b}$, $\gamma=\frac{\rho_c}{\varrho_c}=\frac{r}{r_c}$

$$\Longrightarrow \alpha+\beta+\gamma=\frac{\rho_a}{\varrho_a}+\frac{\rho_b}{\varrho_b}+\frac{\rho_c}{\varrho_c}=r\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)=r\cdot\frac{1}{r}=1.$$

Quick Reply

High School Olympiads

Hard

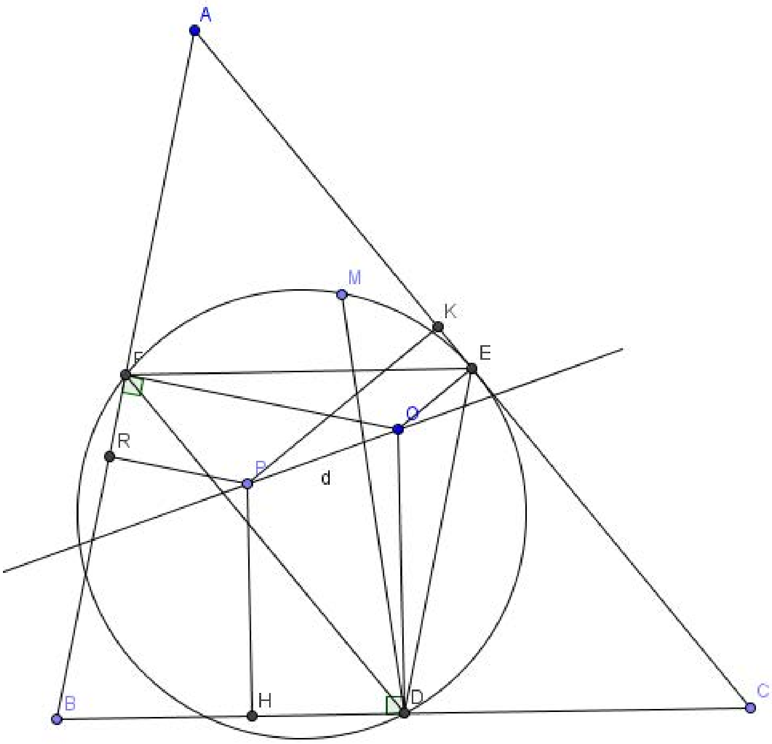
Reply

ndk09

#1 Oct 29, 2010, 4:56 pm

Let ABC triangle, D, E, F are the midpoints of BC, CA, AB respectively. $M \in (DEF)$, d is the Steiner line of M. P is the point on d, H, K, R are the foots of perpendicular from P to BC, CA, AB. Prove that M, H, K, R is cycilc.

Attachments:



Luis González

#2 Oct 30, 2010, 11:05 am

Steiner line d of M with respect to $\triangle DEF$ goes through the circumcenter O of $\triangle ABC$ (orthocenter of DEF), thus according to 2nd Fontené theorem, pedal circle $\odot(HKR)$ of P passes through the orthopole of d with respect to $\triangle ABC$. Thus, it's enough to show that M is the orthopole of d. Let A', B', C' be the reflections of M across lines EF, FD, DE. Since $\odot(DEF)$ and $\odot(AEF)$ are symmetric about EF, it follows that $A' \in \odot(AEF) \Longrightarrow \angle AA'O$ is right, i.e. $AA' \perp d$. Similarly, $BB' \perp d$ and $CC' \perp d \Longrightarrow M$ is the orthopole of d with respect to $\triangle ABC$.

Quick Reply

High School Olympiads

Prove that there exist four points A, B, C, D (22)

Reply

Amir Hossein

#1 Oct 29, 2010, 10:02 pm

A circle centered at a point F and a parabola with focus F have two common points. Prove that there exist four points A, B, C, D on the circle such that the lines AB, BC, CD and DA touch the parabola.

Luis González

#2 Oct 30, 2010, 7:31 am

Let circle (F) and parabola $\mathcal{F}$ with focus F meet at $X, Y.$ Let A, B, C, D be four points on (F) (not lying on its arc XY cut by the angle XFY) such that AB, BC, CD are tangent to $\mathcal{F}.$ Denote $P \equiv AB \cap CD.$ Reflections F_1, F_2, F_3 of focus F on tangents PB, PC, BC lie on the directrix τ of $\mathcal{F} \Longrightarrow F \in \odot(PBC),$ this is $\tau \equiv F_1F_2F_3$ is the Steiner line of F with respect to $\triangle PBC.$ Consequently, $\angle BPC = \angle BFC = 2\angle BAC \Longrightarrow \triangle APC$ is isosceles with apex $P.$ Thus, $ADBC$ is an isosceles trapezoid with $AC \parallel BD.$ By axial symmetry, $F \in \odot(PAD) \Longrightarrow \tau$ is also the Steiner line of F WRT $\triangle PAD,$ i.e. Reflection F_4 of F across DA lies on $\tau \Longrightarrow DA$ is tangent to $\mathcal{F}.$

Quick Reply

High School Olympiads

Prove that XV=YU (19)

Reply

Amir Hossein

#1 Oct 29, 2010, 9:50 pm

A quadrilateral $ABCD$ is inscribed into a circle with center O. Points P and Q are opposite to C and D respectively. Two tangents drawn to that circle at these points meet the line AB in points E and F. (A is between E and B, B is between A and F). The line EO meets AC and BC in points X and Y respectively, and the line FO meets AD and BD in points U and V respectively. Prove that $XV = YU$.

Luis González

#2 Oct 30, 2010, 1:35 am

Let B' be the antipode of B WRT the circle (O) and define $X' \equiv PB' \cap AC$. By Pascal theorem for degenerate cyclic hexagon $PPB'BAC$, it follows that intersections $E \equiv PP \cap AB, X' \equiv PB' \cap AC$ and $O \equiv BB' \cap PC$ are collinear $\Longrightarrow X$ and X' coincide. Therefore, $XPB' \parallel BCY \Longrightarrow O$ is the midpoint of XY. With the same argument we'll have that O is the midpoint of $UV \Longrightarrow XVYU$ is a parallelogram $\Longrightarrow XV = YU$.

This post has been edited 1 time. Last edited by Luis González, Oct 30, 2010, 3:29 am

mathVNpro

#3 Oct 30, 2010, 1:40 am • 1

> *Quote:*
>
> A quadrilateral $ABCD$ is inscribed into a circle with center O. Points P and Q are opposite to C and D respectively. Two tangents drawn to that circle at these points meet the line AB in points E and F. (A is between E and B, B is between A and F). The line EO meets AC and BC in points X and Y respectively, and the line FO meets AD and BD in points U and V respectively. Prove that $XV = YU$.

$\textbf{Lemma :}$ Let $\triangle ABC$ inscribed circumcenter (O) and P_a be the symmentric point of A with respect to O.The tangent at P_a intersects BC at K. Let KO intersects AB and AC respectively at M and N. Prove that O is the midpoint of MN.

Proof. Let the second tangent Kx from K to (O) intersects (O) at P'_a. Then $\angle xP'_aA = \angle P'_aP_aA = \angle P'_aKO$, which implies $P'A \parallel KO$. However, since P'_aBP_aC is a harmonic quadrilateral, thus $(AB, AC, AP_1, AP'_a) = -1$. Or in other words, $(AM, AN, AO, AP'_a) = -1$. As a result, O is the midpoint of MN.

Back to our problem, by the *lemma*, we have known that O respectively is the midpoint of XY and UV. Thus, $XVYU$ is a parallelogram. Therefore, $XV = YU$. Our proof is completed then. $\square$

Martin N.

#4 Oct 31, 2010, 8:08 pm

> *mathVNpro wrote:*
>
> $\textbf{Lemma:}$ Let $\triangle ABC$ inscribed circumcenter (O) and P_a be the symmentric point of A with respect to O.The tangent at P_a intersects BC at K. Let KO intersects AB and AC respectively at M and N. Prove that O is the midpoint of MN.

I proved exactly the same thing - but in a different way:
Let l be the line perpendicular to KO through K and denote the intersections of l with P_aB and P_aC by X and Y, respectively. Then, by the butterfly theorem, we know that $XK = KY$.
As $\sphericalangle YKN = \sphericalangle YCN = \frac{\pi}{2}$ and $\sphericalangle XKM = \sphericalangle XBM = \frac{\pi}{2}$, $YKCN$ and $XKMB$ are cyclic, implying $\sphericalangle P_aXK = \sphericalangle BXK = \sphericalangle KMA = \sphericalangle NMA$ and $\sphericalangle P_aYK = \sphericalangle CYK = \sphericalangle CNK = \sphericalangle ANM$, i.e. the triangles $\triangle P_aXY$ and $\triangle AMN$ are similar. As furthermore $\sphericalangle KP_aY = \sphericalangle P_aAC = \sphericalangle OAN$, we know that $\frac{MO}{ON} = \frac{XK}{KY} = 1$ or $MO = ON$ as required.

Quick Reply

High School Olympiads

1/BX = 1/AX + 1/CX

Reply

sororak

#1 Oct 28, 2010, 5:25 pm

Let ABC be a triangle and $AB < BC < AC$. E, D are on the sides AB, AC, respectively such that BD, CE are the angle bisectors of B, C, respectively. Let ED meet the arc AB of the circumcircle of ΔABC at X. Prove that

$$\frac{1}{BX} = \frac{1}{AX} + \frac{1}{CX}.$$

sororak

#2 Oct 29, 2010, 1:34 pm

Doesn't anybody have any solutions for this problem?

Luis González

#3 Oct 29, 2010, 10:22 pm

For a proof of the proposed problem see here. Now, let's prove a slight generalization.

Proposition. Point U lies inside $\triangle ABC$ and has trilinear coordinate $(p : q : r)$ WRT $\triangle ABC$. BU, CU cut AC, AB at D, E and ray DE cuts the circumcircle of $\triangle ABC$ at X. Then we have

$$\frac{1}{q \cdot BX} = \frac{1}{p \cdot AX} + \frac{1}{r \cdot CX}$$

Proof. Let P, Q, R be orthogonal projections of X on sidelines BC, CA, AB. From $\angle XBP = \angle XAQ$ and $\angle XCP = \angle XAR$, it follows that $\triangle XBP \sim \triangle XAQ$ and $\triangle XCP \sim \triangle XAR \Longrightarrow$

$$\frac{XB}{XA} = \frac{XP}{XQ} \quad (1) \quad , \quad \frac{XC}{XA} = \frac{XP}{XR} \quad (2)$$

Equation of line DE is $\frac{\alpha}{p} - \frac{\beta}{q} - \frac{\gamma}{r} = 0 \Longrightarrow \frac{XP}{p} = \frac{XQ}{q} - \frac{XR}{r} \quad (3)$

Substituting XQ, XR from (1) and (2) into (3) yields $\frac{1}{q \cdot BX} = \frac{1}{p \cdot AX} + \frac{1}{r \cdot CX}$.

Quick Reply

High School Olympiads

The line AK passes through circumcenter (6)

Amir Hossein

#1 Oct 29, 2010, 2:33 am

Points M and N lie on the side BC of the regular triangle ABC (M is between B and N), and $\angle MAN = 30^\circ$. The circumcircles of triangles AMC and ANB meet at a point K. Prove that the line AK passes through the circumcenter of triangle AMN.

Luis González

#2 Oct 29, 2010, 12:00 pm

$\angle BKC = \angle AKC + \angle AKB = \angle AMN + \angle ANB = 180^\circ - 30^\circ = 150^\circ$. Then it follows that A is the circumcenter of $\triangle BKC \Longrightarrow$ triangle $\triangle AKC$ is isosceles with apex A. But because of $\angle AKN = \angle ABN = \angle ACN = 60^\circ$, we deduce that AN is the perpendicular bisector of $\overline{KC}$, i.e. $AN \perp KC \Longrightarrow \angle KAN = 90^\circ - \angle AMN$. Therefore, AK passes through the circumcenter of $\triangle AMN$.

High School Olympiads

Prove that O_1M=O_2M (7)

Reply

Amir Hossein

#1 Oct 29, 2010, 2:59 am

The line passing through the vertex B of a triangle ABC and perpendicular to its median BM intersects the altitudes dropped from A and C (or their extensions) in points K and $N.$ Points O_1 and O_2 are the circumcenters of the triangles ABK and CBN respectively. Prove that $O_1M=O_2M.$

Luis González

#2 Oct 29, 2010, 7:35 am • 1

Let D,E be the feet of the altitudes issuing from $A,C,$ respectively. By Butterfly theorem for the chords AE,CD in the circle (M) with diameter $AC,$ the perpendicular to MB through B cuts lines AD,CE at K,N such that $\overline{BN}=-\overline{BK}.$ Since $\angle BAK=\angle BCN,$ it follows that $\odot(ABK) \equiv (O_1)$ and $\odot(CBN) \equiv (O_2)$ are symmetric about $BM.$ Let P be the 2nd intersection of (O_1) and $(O_2).$ Thus, $P \in BM$ and $\angle BPC=\angle ABM \Longrightarrow PC \parallel AB \Longrightarrow \overline{MO_1}=-\overline{MO_2}.$

dyn

#3 Dec 1, 2010, 2:37 pm

Let point D be such that $ABCD$ is a parallelogram. Since $\angle AKB=\angle DBC=\angle BDA$, it follows that $DABK$ is cyclic and O_1 is the midpoint of KD. Similarly, O_2 is the midpoint of DN. Up to now, M lies on the line segment O_1O_2. It suffices to prove that circles O_1 and O_2 are equal, which is obvious since they are the circumcircles of the triangles ABD and BCD, where $ABCD$ is a parallelogram.

Quick Reply

High School Olympiads

A property of symmedians.

Reply

Virgil Nicula

#1 Oct 27, 2010, 11:42 pm

Let $\triangle ABC$ with centroid G , circumcenter O and symmedian center S .

Prove that AS is a bisector of $\angle BSC \iff GO \perp GA$.

Luis González

#2 Oct 28, 2010, 11:27 pm

$AG \perp OG \Longleftrightarrow A, G$ and infinity point of the orthic axis of $\triangle ABC$ are collinear. We use barycentric coordinates with respect to $\triangle ABC$. Then $A(1:0:0), G(1:1:1)$ and $P_{\infty}(S_B - S_C : S_C - S_A : S_A - S_B)$ are collinear.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ S_B - S_C & S_C - S_A & S_A - S_B \end{pmatrix} = 0 \Longleftrightarrow 2S_A = S_B + S_C \ \ (1)$$

A-symmedian bisects $\angle BSC \Longleftrightarrow S(a^2 : b^2 : c^2)$ lies on the line connecting the circumcenter $O(a^2S_A : b^2S_B : c^2S_C)$ with the center of the A-Apollonius circle $O_a(0 : b^2 : -c^2)$

$$\begin{pmatrix} a^2 & b^2 & c^2 \\ a^2S_A & b^2S_B & c^2S_C \\ 0 & b^2 & -c^2 \end{pmatrix} = 0 \Longleftrightarrow 2S_A = S_B + S_C \ \ (2)$$

Since (1) and (2) are identical, it follows that AS bisects $\angle BSC \Longleftrightarrow AG \perp OG.$

Quick Reply

High School Olympiads

A hard problem aboat Equilateral Triangle (oWn)

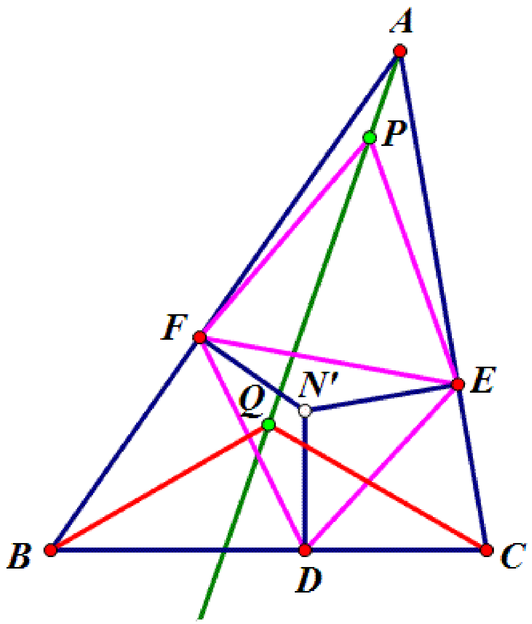
Reply

lym

#1 Oct 17, 2010, 2:18 pm

Figure□in a $\triangle ABC$□N' is the isogonal conjugate point of **9-point center**□
$\triangle DEF$ is the pedal triangle of N' with respect $\triangle ABC$□$\triangle PEF$ is equilateral triangle .
prove that□AP pass through an **napoleon point** .

Attachments:



vittasko

#2 Oct 21, 2010, 3:10 pm

What I have now, is an equivalent problem as Follows:

EQUIVALENT PROBLEM. - A triangle $\triangle ABC$ is given with circumcircle (O) and suppose that A is a mobile point on it. We draw the line through B and perpendicular to AN, where N is the Nine point center of $\triangle ABC$, which intersects AC, at point so be it D. Let E be, the point in to the same side of BC as A, such that the triangle $\triangle BDE$ to be equilateral. Prove that $BP = 2MP$, where M is the midpoint of the side-segment BC and $P \equiv OM \cap AE$, where O is the circumcenter of $\triangle ABC$.

Kostas Vittas.

PS. But, I have not in mind any solution of the above equivalent problem.

Attachments:
t=372318.pdf (6kb)

Luis González

#3 Oct 28, 2010, 6:28 am

Unfortunately, I don't have a synthetic proof to this nice problem but the following self-contained approach. Let $\triangle XYZ$ be the pedal triangle of 9-point center N and P' be the vertex of the equilateral triangle $\triangle P'YZ$ constructed outside $\triangle XYZ$. If AP passes through the 2nd Napoleon point X, then AP' must pass through its isogonal conjugate X. Let Y, Z be the centers passes through the 2nd Napoleon point X_{18}, then AP' must pass through its isogonal conjugate X_{62}. Let Y_1, Z_1 be the centers of the B- and C- inner Fermat triangles of $\triangle ABC$. Rays AN, AY_1, AZ_1 cut BC at U, B_0, C_0, respectively.

$$\frac{\sin \widehat{CAP'}}{\sin \widehat{BAP'}} = \frac{\sin \widehat{AYP'}}{\sin \widehat{AZP'}} = \frac{\sin \widehat{UAC_0}}{\sin \widehat{UAB_0}} = \frac{C_0U}{UB_0} \cdot \frac{BB_0}{C_0C} \cdot \frac{AC}{AB} = \frac{C_0U}{UB_0} \cdot \frac{B_0C}{BC_0} \cdot \frac{AB}{AC}$$

$$\Longrightarrow \frac{\sin \widehat{CAP'}}{\sin \widehat{BAP'}} \cdot \frac{AC}{AB} = \frac{|\triangle CAP'|}{|\triangle BAP'|} = \frac{C_0U}{UB_0} \cdot \frac{B_0C}{BC_0} \quad (*)$$

Now, let us use barycentric coordinates WRT $\triangle ABC$ to compute such ratios

$$N(S^2 + S_BS_C : S^2 + S_CS_A : S^2 + S_AS_B)$$

$$\Longrightarrow U\left(0 : \frac{S^2 + S_CS_A}{2S^2 + a^2S_A} : \frac{S^2 + S_AS_B}{2S^2 + a^2S_A}\right)$$

$$Y_1(S_C - \sqrt{3}S : -b^2 : S_A - \sqrt{3}S) \Longrightarrow B_0\left(0 : \frac{-b^2}{-S_C - \sqrt{3}S} : \frac{S_A - \sqrt{3}S}{-S_C - \sqrt{3}S}\right)$$

$$Z_1(S_B - \sqrt{3}S : S_A - \sqrt{3}S : -c^2) \Longrightarrow C_0\left(0 : \frac{S_A - \sqrt{3}S}{-S_B - \sqrt{3}S} : \frac{-c^2}{-S_B - \sqrt{3}S}\right)$$

From normalized coordinates of U, B_0, C_0 we obtain the ratios $\frac{C_0U}{UB_0}$, $\frac{B_0C}{BC_0}$

$$\frac{B_0C}{BC_0} = \frac{b^2}{c^2} \cdot \frac{S_B + \sqrt{3}S}{S_C + \sqrt{3}S} \ , \ \frac{C_0U}{UB_0} = \frac{S_B - \sqrt{3}S}{S_B + \sqrt{3}S} \cdot \frac{S_C + \sqrt{3}S}{S_C - \sqrt{3}S}$$

Combining these two latter expresions with $(*)$ gives the equation of the line AP' as

$$\frac{y}{z} = \frac{b^2}{c^2} \cdot \frac{S_B + \sqrt{3}S}{S_C + \sqrt{3}S} \cdot \frac{S_B - \sqrt{3}S}{S_B + \sqrt{3}S} \cdot \frac{S_C + \sqrt{3}S}{S_C - \sqrt{3}S} = \frac{b^2(S_B - \sqrt{3}S)}{c^2(S_C - \sqrt{3}S)}$$

Which is the A-cevian of $X_{62}\left(a^2(S_A - \sqrt{3}S) : b^2(S_B - \sqrt{3}S) : c^2(S_C - \sqrt{3}S)\right)$

skytin

#4 Aug 26, 2011, 6:35 pm

Solution:
Reflect Q wrt BC and get point P
(BPQ) intersect AB at points B' and B
(CPQ) intersects with CA at points C , C'
BB_1 and CC_1 are diameters of (BPQ) , (CPQ)
B_1B' intersect C_1C' at point H'
N is midpoint of BC
H is orthocenter of ABC
Easy to see that H' is on NH and NH' = H'H/2
O is circumcenter of ABC
reflect O wrt BC and get point O'
Well known that O'N = HA/2
NH'/H'H = O'N/HA and O'N || HA , so O' is on H'A
Let N'' is center of nine point circle ABC
N'' is midpoint of O'A
PC' intersect AB at point C''
(C''QC') intersect AB at points X and C'' , CA at points B'' , C'
Angle B''C''Q = QC'A = PC'C = B''QC'' = 60 , so B''C''Q is equilateral
Easy to see that QB''P ~ QC''B , so PB'' intersect B'B at point X and X is on (PBQ)
X = B'
B''C''B'QC' is cyclic
H'A goes thru circumcenter of AB'C' , so N''A is altitude of AB''C''
B''C'' || EF
A is homotety center of triangles B''C''Q and EFP , so Q is on PA . done

High School Olympiads

about bisectors of triangle

Reply

creatorvn

#1 Oct 26, 2010, 8:48 pm

Ex1:
i)Let be given triangle ABC. P is the midpoint of arc AC(not including B) of the circumcircle (O) . Prove that $PA=PC=PI=PI_2$(where I,I_2 are incenter and center of escribed circle of B respectively)
ii) Let be given triangle ABC. Q is the midpoint of arc AC(including B) of the circumcircle (O) . Prove that $QA=QC=QI_1=Q_3$(where I_1,I_3 are centers of escribed circle of A and C respectively)

Ex2:
Let be given tri ABC. O and I are circumcenter and incenter respectively. Letting AX,BY,CZ are external bisectors. We all know that X,Y,Z are on straight line called *l*.Prove that *l* is perpendicular to OI
Also express the distance from I to *l* in term of R,r (which are radii of circumcircle and incircle)

to be continued...

Luis González

#2 Oct 26, 2010, 10:12 pm

Incenter I and circumcircle (O) of $\triangle ABC$ become orthocenter and 9-point circle of its excentral triangle $\triangle I_1I_2I_3.$ Thus, midpoint P of arc AC is also the midpoint of $\overline{II_2} \Longrightarrow PA=PC=PI=PI_2.$ Circumcircle (O) cuts $\overline{I_1I_3}$ again at its midpoint, i.e. Q is the center of the circle with diameter $I_1I_3 \Longrightarrow QA=QC=QI_1=QI_3.$ On the other hand, let U be the circumcenter of $\triangle I_1I_2I_3$ (reflection of I about O). XYZ becomes orthic axis of $\triangle I_2I_2I_3,$ i.e. radical axis of $(O),(U) \Longrightarrow$ $XYZ \perp IOU.$ By Euler's theorem we have $d=OI=OU=\sqrt{R^2-2Rr}$ and let $V \equiv IO \cap XYZ.$ Since V has equal power with respect to $(O),(U),$ it follows that

$$UV^2-4R^2=OV^2-R^2 \Longrightarrow (d+OV)^2-3R^2=OV^2$$

$$\Longrightarrow d^2+OV^2+2d \cdot OV-3R^2=OV^2 \Longrightarrow OV=\frac{R(R+r)}{d}$$

$$\Longrightarrow IV=OV-d=\frac{R(R+r)}{\sqrt{R^2-2Rr}}-\sqrt{R^2-2Rr}=\frac{3Rr}{\sqrt{R^2-2Rr}}$$

Quick Reply

High School Olympiads

Nice geometric result (own)

truongtansang89
#1 Oct 26, 2010, 9:42 am

Problem: Let a quadrilateral with inscribed circle (I) and circumcircle (O). AB, BC, CD, DA tangent (I) at M, N, P, Q.
Prove that : OI passes through centroid of $MNPQ$.

Remark : If quadrilateral $ABCD$ becomes triangle ABC. Then, we have this well-known result :
Let ΔABC with inscribed circle (I) and circumcircle (0). AB, BC, CA tangent (I) at M, N, P. Prove that OI passes through centroid of ΔMNP

Luis González
#2 Oct 26, 2010, 11:22 am

I don't know whether you mean the centroid G of vertices M, N, P, Q (center of the Varignon parallelogram of MNPQ) or the centroid G_0 of the homogeneous area $MNPQ$. However, we'll show that both G and G_0 lie on IO.

Let A', B', C', D' be the midpoints of QM, MN, NP, PQ. Inversion WRT (I) takes A', B', C', D' into A, B, C, D. Since A, B, C, D are concyclic, then Varignon parallelogram $A'B'C'D'$ of $MNPQ$ is cyclic $\Longrightarrow A'B'C'D'$ is a rectangle, thus $MP \perp NQ$ at K. Moreover, it's well-known that points I, O, K are collinear, since by Newton's theorem $K \equiv AC \cap BD$. Then K is the common pole of the third diagonal of the complete quadrangle $ABCD$ with respect to (I) and (O). Let $U \equiv D'K \cap MN$. Then it follows that $\angle QKD' = \angle D'QK = \angle KMN \Longrightarrow \angle KMN = \angle UKN$, i.e. $D'K \perp MN$. Similarly, we have $B'K \perp PQ$, thus $D'K \parallel IB'$ and $B'K \parallel ID' \Longrightarrow B'KD'I$ is a parallelogram. Hence diacentral IKO bisects $B'D'$. In other words, center G of the rectangle $A'B'C'D'$ (centroid of vertices M,N,P,Q) coincides with the midpoint of IK.

On the other hand, let G_1, G_2, G_3, G_4 be the centroids of triangles $\triangle MNQ, \triangle NPM, \triangle PQN, \triangle QMP$. Then $G_0 \equiv G_1G_3 \cap G_2G_4$ is the centroid of the homogeneous area $MNPQ$. Moreover, $PQMN \sim G_1G_2G_3G_4$ are centrally similar with coefficient $-\frac{1}{3}$. But line $B'D'$ bisects the pairs of homologous sides G_1G_2, PQ and G_3G_4, MN. Similarly, line $A'C'$ bisects the pairs of homologous sides G_2G_3, QM and $G_1G_4, PN \Longrightarrow G \equiv A'C' \cap B'D'$ is similarity center of $PQMN$ and $G_1G_2G_3G_4$. Since their diagonal intersections K, G_0 are homologous, it follows that $G_0 \in KGOI$.

truongtansang89
#3 Oct 27, 2010, 9:32 am

Wow, that's so cool, **Luis** COOL!

My proof is different from yours.
Another proof ?

High School Olympiads

Triangle with an arbitrary point inside

Reply

Source: Ukrainian TST 2008 Problem 9

April

#1 Feb 12, 2009, 1:07 pm

Given $\triangle ABC$ with point D inside. Let $A_0 = AD \cap BC$, $B_0 = BD \cap AC$, $C_0 = CD \cap AB$ and A_1, B_1, C_1, A_2, B_2, C_2 are midpoints of BC, AC, AB, AD, BD, CD respectively. Two lines parallel to A_1A_2 and C_1C_2 and passes through point B_0 intersects B_1B_2 in points A_3 and C_3respectively. Prove that $\frac{A_3B_1}{A_3B_2} = \frac{C_3B_1}{C_3B_2}$.

Luis González

#2 Oct 25, 2010, 11:50 pm

Let lines A_1C_1 and AA_1 cut BB_0 at $E, V,$ respectively. Parallel from A_1 to C_1C_2 cuts BB_0 and C_1A_2 at U, Q. Since $C_1A_2 \parallel A_1C_2 \parallel BB_0$ and $C_1A_1 \parallel A_2C_2 \parallel AC$, it follows that $C_1A_1C_2A_2$ is a parallelogram $\Longrightarrow$ Segments A_1A_2 and C_1C_2 bisect each other. Therefore, C_1 is the midpoint of $QA_2 \Longrightarrow$ Median A_1C_1 of $\triangle QA_1A_2$ bisects the parallel segment $UV \parallel QA_2 \Longrightarrow E$ is the midpoint of UV. Consequently, pencil of lines $A_1(U,V,E,C_2)$ is harmonic. From $B_0C_3 \parallel A_1U$, $B_0A_3 \parallel A_1V, B_0B_2 \parallel A_1C_2$ and $B_0B_1 \parallel A_1E$, it follows that pencils $A_1(U,V,E,C_2)$ and $B_0(C_3,A_3,B_1,B_2)$ are similar. Therefore, cross ratio (C_3,A_3,B_1,B_2) is harmonic $\Longrightarrow \frac{B_1A_3}{B_1C_3} = -\frac{B_2A_3}{B_2C_3}$.

Issl

#3 Aug 9, 2011, 12:36 pm

Maybe there is a typo :

> Quote:
> Let lines A_1C_1 and AA_1 cut BB_0 at $E, V,$ respectively

it should be A_1A_2 cut BB_0 at V .

skytin

#4 Aug 21, 2011, 7:14 pm

let BB_0 intersect C_1C_2 , A_2A_1 , C_1A_1 , A_2C_2 at points X , Y , P , Q
Easy to see that PQ || C_1A_2 || A_1C_2 , so YD = DX , make homotety with center B_2 and k = B_2B_1/B_2D and let Y -> Y' , X -> X' , so Y'B_0 = B_0X' and B_2B_0/B_0Y' = B_2A_3/B_0A_3 = B_2B_0/B_0X' = B_2C_3/C_3B_1 . done

Quick Reply

High School Olympiads

M is variable on arc BC

Reply

sororak
#1 Oct 24, 2010, 12:59 am

Let ABC be a triangle and the changing point M is on arc BC from the circumcircle of ΔABC. K, L are 2 points on AB, AC (or the stretch of them), respectively, such that $MK \perp AB$ and $ML \perp AC$. Find the location of M where $MK + ML$ is maximum.

Luis González
#2 Oct 24, 2010, 10:01 am

Let I, O be the incenter and circumcenter of $\triangle ABC.$ Perpendicular line from M to AI cuts the rays AB, AC at $D, E \Longrightarrow$ Triangle $\triangle ADE$ is isosceles with apex $A.$ If F denotes the orthogonal projection of A onto $DE,$ it follows that $MK + ML = AF \cdot \sin A \cdot \sec \frac{A}{2}.$ Thereby, sum of distances $MK + ML$ is maximum $\Longleftrightarrow AF$ is maximum, in other words, when DE is the farthest line from A cutting the circumcircle. That is when DE is tangent to $(O) \Longrightarrow OM \parallel AI.$

Quick Reply

High School Olympiads

Inradii have the same total

chaotic_iak
#1 Oct 22, 2010, 12:48 pm

Let M be an arbitrary point inside the equilateral triangle ABC, and A', B', C' be the projections of M to the sides BC, CA, AB respectively. Let $r_1, r_2, r_3, r'_1, r'_2, r'_3$ be the length of the inradii of the triangles $MAC', MBA', MCB', MAB', MBC', MCA'$ respectively. Prove that $r_1 + r_2 + r_3 = r'_1 + r'_2 + r'_3$.

Luis González
#2 Oct 23, 2010, 12:27 am

Generalization. M is a point inside the scalene triangle $\triangle ABC$. A', B', C' denote the orthogonal projections of M on sidelines BC, CA, AB. Then the sum of the inradii of $\triangle MAC', \triangle MBA', \triangle MCB'$ equals the sum of the inradii of $\triangle MAB', \triangle MCA', \triangle MBC' \Longleftrightarrow M$ lies on the diacentral line IO of $\triangle ABC$.

Inradii condition is equivalent to $\overline{AC'} + \overline{BA'} + \overline{CB'} = \frac{1}{2}(a+b+c)$.

Let $(x:y:z)$ be the barycentric coordinates of M with respect to $\triangle ABC$. Therefore, coordinates of its projections A', B', C' on BC, CA, AB, in Conway's notation, are $A' : (0 : a^2y + xS_C : a^2z + xS_B)$, $B' : (b^2x + yS_C : 0 : b^2z + yS_A)$ and $C' : (c^2x + zS_B : c^2y + zS_A : 0)$. From these, we deduce

$$\overline{AC'} = \frac{c^2y + zS_A}{c(x+y+z)} \ , \ \overline{BA'} = \frac{a^2z + xS_B}{a(x+y+z)} \ , \ \overline{CB'} = \frac{b^2x + yS_C}{b(x+y+z)}$$

Thus, locus $f(x, y, z) = 0$ of M is given by the linear equation

$$\frac{(a+b+c)(x+y+z)}{2} = \frac{c^2y + zS_A}{c} + \frac{a^2z + xS_B}{a} + \frac{b^2x + yS_C}{b}$$

Now, it's easy to figure out without using the latter equation that f contains the circumcenter O and incenter I of $\triangle ABC$. Indeed, if M_a, M_b, M_c are the midpoints of BC, CA, AB and X, Y, Z are tangency points of the incircle (I) with BC, CA, AB, then

$$\overline{AM_c} + \overline{BM_a} + \overline{CM_b} = \frac{1}{2}(a+b+c) \Longrightarrow O \in f$$

$$\overline{AZ} + \overline{BX} + \overline{CY} = (s-a) + (s-b) + (s-c) = \frac{1}{2}(a+b+c) \Longrightarrow I \in f.$$

If $a = b = c$, then the locus of M is the interior of $\triangle ABC$.

High School Olympiads

Iwanttolive

#1 Oct 19, 2010, 11:51 pm

Given a convex hexagon $ABCDEF$. The point Y lies inside the hexagon.
Points K, L, M, N, P, Q are the midpoints of sides AB, BC, CD, DE, EF, FA . Prove that the sum of the squares of fields $QAKY, LCMY, NEPY$ does not depend on the choice point Y.

This post has been edited 2 times. Last edited by Luis González, Apr 14, 2015, 7:55 am

Luis González

#2 Oct 20, 2010, 5:09 am

Lemma. If P is an arbitrary point on the plane of $\triangle ABC$, whose centroid is G, then one of the triangles $\triangle PAG, \triangle PBG$, $\triangle PCG$ is equivalent to the sum of the other two.

WLOG assume that line PG separates segment BC from vertex A. Let M be the midpoint of BC and let X, Y, Z, U be the orthogonal projections of A, B, C, M onto PG. Then UM is the median of the right trapezoid $BYZC$ and $\triangle MUG \sim \triangle AZG$ are similar with similarity coefficient $\frac{GM}{AG} = \frac{1}{2}$. Therefore

$$BY + CZ = 2 \cdot MU = AX \Longrightarrow PG \cdot BY + PG \cdot CZ = PG \cdot AX$$

$$\Longrightarrow [\triangle PBG] + [\triangle PCG] = [\triangle PAG].$$

● By similar reasoning, it's easy to show that the distance from G to an abitrary line ℓ in the plane ABC equals the arithmetic mean of the directed distances from A, B, C to ℓ.

Back to the problem, since $[\triangle AQK] = \frac{1}{4}[\triangle AFB], [\triangle CLM] = \frac{1}{4}[\triangle CBD]$ and $[\triangle ENP] = \frac{1}{4}[\triangle EDF]$ are constant, then it's enough to show that the sum of areas $[\triangle YLM] + [\triangle YNP] + [\triangle YQK]$ is constant. Let G, G' be the centroids of $\triangle KMP$ and $\triangle LNQ$. Notation $\delta(P)$ stands for the distance from a point P to the line AF.

$$3 \cdot \delta(G) = \delta(K) + \delta(M) + \delta(P) = \frac{1}{2}[\delta(B) + \delta(C) + \delta(D) + \delta(E)]$$

$$\Longrightarrow 3 \cdot \delta(G) = \delta(L) + \delta(N) = 3 \cdot \delta(G').$$

Since, the same relation occurs with respect to the remaining sides of the hexagon, then we deduce that G and G' coincide. In other words, $\triangle KMP$ and $\triangle LNQ$ share the same centroid G. Now, WLOG assume that G lies inside $\triangle MYN$ and that line YG separates K, L, M from N, P, Q. Using previous lemma in $\triangle KMP$ and $\triangle LNQ$, we get

$$[\triangle YQG] + [\triangle YNG] = [\triangle YLG] \ \ (1) \ , \ [\triangle YMG] + [\triangle YKG] = [\triangle YPG] \ \ (2)$$

On the other hand, by adding areas we obtain

$$[\triangle YLM] = [\triangle YLG] + [\triangle LGM] - [\triangle YMG] \ \ (3)$$

$$[\triangle YNP] = [\triangle YPG] + [\triangle NGP] - [\triangle YNG] \ \ (4)$$

$$[\triangle YQK] = [\triangle QGK] - [\triangle YQG] - [\triangle YKG] \ \ (5)$$

Adding the expressions $(1), (2), (3), (4), (5)$ properly gives

$$[\triangle YLM] + [\triangle YNP] + [\triangle YQK] = [\triangle LGM] + [\triangle NGP] + [\triangle QGK] = \text{const}.$$

High School Olympiads

Pascal Line

Headhunter
#1 Oct 16, 2010, 3:04 am • 1

Hello.

A hexagon is inscribed in a circle.
Show that the continued products of the perpendiculars from any point on the Pascal line on the alternate sides are equal.

Luis González
#2 Oct 19, 2010, 1:21 pm • 1

Let $ABCDEF$ be the vertices of the subject hexagon inscribed in the circle (O,R). Denote the lines AB, BC, CD, DE, EF, FA as a,b,c,d,e,f. Notation $\delta(P,\ell)$ stands for the distance from point P to the line ℓ. Pairwise tangents of (O) through A,B,C,D,E,F meet at M,N,L,Q,R,S. This is, M,N,L,Q,R,S are the poles of a,b,c,d,e,f WRT (O). Let $U \equiv MQ \cap NR \cap LS$ be the Brianchon point of $MNLQRS$. Then we have

$$\frac{QU}{MU}=\frac{|\triangle NQR|}{|\triangle NMR|}=\frac{\delta(N,RQ)\cdot RQ}{\delta(R,MN)\cdot MN} \quad (*)$$

By Salmon's theorem for the pairs $(N,E),(R,B)$ and their polars BC, RQ and FE, MN respectively, we get

$$\frac{\delta(N,RQ)}{NO}=\frac{\delta(E,b)}{R} \ \ , \ \ \frac{\delta(R,MN)}{RO}=\frac{\delta(B,e)}{R}$$

Combining these two latter expressions with $(*)$ yields

$$\frac{QU}{MU}=\frac{NO}{RO}\cdot\frac{RQ}{MN}\cdot\frac{\delta(E,b)}{\delta(B,e)}=\frac{NO}{RO}\cdot\frac{RQ}{MN}\cdot\frac{BE\cdot CE\cdot 2R}{FB\cdot BE\cdot 2R}=\frac{NO}{RO}\cdot\frac{RQ}{MN}\cdot\frac{CE}{FB} \quad (1)$$

By analogous reasoning we obtain the expressions

$$\frac{SU}{LU}=\frac{QO}{MO}\cdot\frac{SM}{LQ}\cdot\frac{AE}{DB} \quad (2) \ , \ \frac{NU}{RU}=\frac{SO}{LO}\cdot\frac{NL}{SR}\cdot\frac{CA}{FD} \quad (3)$$

Multiplying the expressions (1), (2) and (3) gives

$$\frac{QU}{MU}\cdot\frac{SU}{LU}\cdot\frac{NU}{RU}=\frac{NO}{RO}\cdot\frac{QO}{MO}\cdot\frac{SO}{LO}\cdot\frac{CE}{FB}\cdot\frac{AE}{DB}\cdot\frac{CA}{FD}\cdot\frac{RQ}{MN}\cdot\frac{SM}{LQ}\cdot\frac{NL}{SR} \quad (4)$$

Inversion in (O) takes midpoints of AB, BC, CD, DE, EF, FA into M,N,L,Q,R,S.

$$MN=\frac{CA\cdot R^2}{2\cdot\delta(O,a)\cdot\delta(O,b)} \ \ , \ \ RQ=\frac{FD\cdot R^2}{2\cdot\delta(O,d)\cdot\delta(O,e)}$$

$$LQ=\frac{CE\cdot R^2}{2\cdot\delta(O,c)\cdot\delta(O,d)} \ \ , \ \ SM=\frac{FB\cdot R^2}{2\cdot\delta(O,a)\cdot\delta(O,f)}$$

$$SR=\frac{AE\cdot R^2}{2\cdot\delta(O,e)\cdot\delta(O,f)} \ \ , \ \ NL=\frac{DB\cdot R^2}{2\cdot\delta(O,b)\cdot\delta(O,c)}$$

$$\Longrightarrow \frac{MN}{RQ}\cdot\frac{LQ}{SM}\cdot\frac{SR}{NL}=\frac{CA}{FD}\cdot\frac{CE}{FB}\cdot\frac{AE}{DB}$$

Together with (4) we obtain : $\frac{QU}{MU}\cdot\frac{SU}{LU}\cdot\frac{NU}{RU}=\frac{NO}{RO}\cdot\frac{QO}{MO}\cdot\frac{SO}{LO}$ (5)

Let τ be the Pascal line defined by the intersections $BC\cap EF, CD\cap FA$ and $DE\cap AB.$ Hence, Brianchon point U of $MNLQRS$ is the pole of τ WRT $(O).$ Let P be an arbitrary point of τ and let ℓ be its polar WRT $(O),$ which goes through the pole U of $\tau.$ By Salmon's theorem for P and each $M,N,L,Q,R,S,$ we get the expressions

$$\frac{\delta(P,a)}{PO}=\frac{\delta(M,\ell)}{MO} \ , \ \frac{\delta(P,c)}{PO}=\frac{\delta(L,\ell)}{LO} \ , \ \frac{\delta(P,e)}{PO}=\frac{\delta(R,\ell)}{RO}$$

$$\frac{\delta(P,b)}{PO}=\frac{\delta(N,\ell)}{NO} \ , \ \frac{\delta(P,d)}{PO}=\frac{\delta(Q,\ell)}{QO} \ , \ \frac{\delta(P,f)}{PO}=\frac{\delta(S,\ell)}{SO}$$

$$\Longrightarrow \frac{\delta(P,a)\cdot\delta(P,c)\cdot\delta(P,e)}{\delta(P,b)\cdot\delta(P,d)\cdot\delta(P,f)}=\frac{NO}{RO}\cdot\frac{QO}{MO}\cdot\frac{SO}{LO}\cdot\frac{\delta(M,\ell)}{\delta(Q,\ell)}\cdot\frac{\delta(L,\ell)}{\delta(S,\ell)}\cdot\frac{\delta(R,\ell)}{\delta(N,\ell)}$$

But, on the other hand, we have that

$$\frac{\delta(M,\ell)}{\delta(Q,\ell)}=\frac{MU}{QU} \ , \ \frac{\delta(L,\ell)}{\delta(S,\ell)}=\frac{LU}{SU} \ , \ \frac{\delta(R,\ell)}{\delta(N,\ell)}=\frac{RU}{NU}$$

$$\Longrightarrow \frac{\delta(P,a)\cdot\delta(P,c)\cdot\delta(P,e)}{\delta(P,b)\cdot\delta(P,d)\cdot\delta(P,f)}=\frac{NO}{RO}\cdot\frac{QO}{MO}\cdot\frac{SO}{LO}\cdot\frac{MU}{QU}\cdot\frac{LU}{SU}\cdot\frac{RU}{NU} \quad (6)$$

$(5)\cap(6)\Longrightarrow \delta(P,a)\cdot\delta(P,c)\cdot\delta(P,e)=\delta(P,b)\cdot\delta(P,d)\cdot\delta(P,f).$

Headhunter

#3 Oct 23, 2010, 6:23 pm

danke für gute Lösung.

Quick Reply

High School Olympiads

PO=HQ in triangle ABC

Reply

sororak
#1 Oct 17, 2010, 8:21 am

Let ABC be a triangle and $AB < AC$. Let H, O be the orthocenter and circumcenter of ΔABC, respectively. Let the line HO meet AB, AC at P, Q, respectively. Prove that:

$$PO = HQ.$$

jayme
#2 Oct 17, 2010, 11:29 am

Dear sororak and Mathlinkers,
are you sure of your conjecture?
Sincerely
Jean-Louis

Luis González
#3 Oct 17, 2010, 1:35 pm

> sororak wrote:
> Let ABC be a triangle and $AB < AC$. Let H, O be the orthocenter and circumcenter of ΔABC, respectively. Let the line HO meet AB, AC at P, Q, respectively. Prove that:
>
> $$PO = HQ.$$

I think that, either $\angle BAC = 60^\circ$ or $\angle BAC = 120^\circ$ is necessary.

pacoga
#4 Oct 17, 2010, 3:18 pm

There are other triangles satisfying $OP = HQ$, those that
$a^6 - 3a^4b^2 + 3a^2b^4 - b^6 + a^4c^2 + 3a^2b^2c^2 - 4b^4c^2 - 5a^2c^4 + 2b^2c^4 + 3c^6 = 0$.
Of course, the rectangle triangles at B satisfy $AB < AC$ and $OP = HQ = 0$.

Quick Reply

High School Olympiads

The incircle and concurrent lines!

Reply

sororak

#1 Oct 15, 2010, 9:18 pm

Let ABC be a triangle and its interior incircle Γ meets segments AB,AC,BC at E,F,D, respectively. An arbitrary diameter of Γ meets the segments DE,DF at M,N, respectively. Prove that the segments CM,BN and the perpendicular line from D to MN are concurrent.

Luis González

#2 Oct 17, 2010, 1:19 pm

Let I be the center of $\Gamma, P \equiv MN \cap BC$ and $L \equiv BN \cap CM.$ Then it is enough to show that DL is the polar of P WRT $\Gamma.$ Let U,V be the orthogonal projections of B,C onto MN and let $R \equiv BU \cap DF, S \equiv CV \cap DE \Longrightarrow R,S$ are the poles of CM,BN WRT $\Gamma \Longrightarrow RS$ is the polar of L WRT Γ $(*).$ On the other hand, let X,Y be the orthogonal projections of B,C onto $DF,DE.$ Because of $\angle VIC=\angle BRX$ and $\angle UIB=\angle CSY,$ it follows that $\triangle RBX \sim \triangle CIV$ and $\triangle CSY \sim \triangle BIU.$ Consequently, we have:

$$\frac{BR}{IC}=\frac{BX}{CV} \ , \ \frac{CS}{IB}=\frac{CY}{BU} \Longrightarrow \frac{BR}{CS}=\frac{IC}{IB} \cdot \frac{BU}{CV} \cdot \frac{BX}{CY}=\frac{BU}{CV}$$

$\Longrightarrow P \in RS.$ Together with $(*),$ we deduce that DL is the polar of P with respect to $\Gamma,$ i.e. $DL \perp MN.$ In addition, locus of the intersection L of lines CM,BN as MN spins around I is the D-midine of $\triangle DEF.$

ndk09

#3 Oct 25, 2010, 6:01 pm

Let $MN \cap BC$ at S, $DH \perp MN$
It's easy to check that $IB \perp DE; IC \perp DF; ID \perp BC; IH \perp DH$ so we have
$I(SDBC)=D(SHMN) \Longrightarrow (SDBC)=(SHMN)$ so BM,CN and DH are concurrent

Quick Reply

High School Olympiads

Two perspective triangles

jayme
#1 Oct 16, 2010, 8:46 pm

Dear Luis and Mathlinkers,
Let ABC be a triangle, G the centroid of ABC, H the orthocenter of ABC, A'B'C' the orthic triangle of ABC, G' the centroid of A'B'C', K the symmedian point of ABC and K* the isotomic of K wrt ABC.
Prove that the triangle GHK and G'OK* are perspective.
Sincerely
Jean-Louis

pacoga
#2 Oct 16, 2010, 11:40 pm

Maybe I'm misunderstanding something, but I think that this result is true for isosceles triangles only.

Let us change H by an arbitrary point P:

Let ABC be a triangle, G, O the centroid and the circumcenter of ABC, $A'B'C'$ the pedal triangle of P with respect ABC, G' the centroid of $A'B'C'$, K the symmedian point of ABC and K^* the isotomic conjugate of K with respect ABC.

The locus of points P such that the triangles GPK and $G'OK^*$ are perspective is formed by two lines through O. One of these lines is the line OK. I am not been able to identify the other at the moment. At the moment I only have its barycentric equation:

$$a^6b^4x - a^2b^8x - 4a^4b^4c^2x + 2a^2b^6c^2x - a^6c^4x + 4a^4b^2c^4x - 4b^6c^4x - 2a^2b^2c^6x + 4b^4c^6x + a^2c^8x + a^8b^2y - a^4b^6y - 2a^6b^2c^2y + 4a^4b^4c^2y + 4a^6c^4y - 4a^2b^4c^4y + b^6c^4y - 4a^4c^6y + 2a^2b^2c^6y - b^2c^8y - 4a^6b^4z + 4a^4b^6z - a^8c^2z + 2a^6b^2c^2z - 2a^2b^6c^2z + b^8c^2z - 4a^4b^2c^4z + 4a^2b^4c^4z + a^4c^6z - b^4c^6z = 0.$$

Luis González
#3 Oct 17, 2010, 2:47 am

Jean Louis probably meant that $\triangle GHK$ and $\triangle G'K'O$ (in that order) are perspective through the infinite point of the Brocard axis OK. Let R be the Retrocenter of $\triangle ABC$, i.e. isotomic conjugate of H WRT $\triangle ABC$. Since A, B, C, O, H, R, K lie on a same hyperbola $\mathcal{J}$, namely the Jerabek hyperbola of $\triangle ABC$ (Isogonal conjugate of Euler line OH), then isotomic conjugate of $\mathcal{J}$ is the line $HR \Longrightarrow$ Isotomic conjugate K' of K lies on HR. Since R, H become the symmedian point and circumcenter of the anticomplementary triangle of ABC, then line HR is the anticomplement of Brocard axis $OK \Longrightarrow OK \parallel HR \equiv HK'$. But according to the topic 4 centroids on a parallel line to the Brocard axis, GG' is the image of OK under the homothety $(H, \frac{2}{3})$, thus $HK' \parallel GG' \parallel OK$ and the conclusion follows.

pacoga
#4 Oct 17, 2010, 3:41 am

Now I have caculated the new locus (points P such that GPK and $G'K^*O$ are perspective) and it is a hyperbola through O, H, K and K^*.

jayme
#5 Oct 17, 2010, 11:06 am

Dear Luis, Francisco and Mathlinkers,
yes, for the GHK and G'K'O... as you have seen, it is a consequence of your nice result about the 4 centroids.
I have proved synthetically that HK'//OK that you will see in a future article.
Sincerely

Jean-Louis

jayme
#6 Oct 17, 2010, 6:31 pm

Dear Mathlinkers,
for a prove of HK'//OK you can see
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=182853
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

A,B,C,A_1,B_1,C_1

Reply

sororak
#1 Oct 15, 2010, 9:28 pm

Let ABC be a triangle and A_1, B_1, C_1 be the midpoints of the arcs BC, AC, AB from the circumcircle of ΔABC. Prove that:

$$S_{A_1B_1C_1} = \frac{1}{2} S_{AC_1BA_1CB_1} \geq S_{ABC}$$

where $S_{A_1A_2\cdots A_n}$ is the surface of the $n-$gonal $A_1A_2\cdots A_n$.

Luis González
#2 Oct 16, 2010, 6:47 am

Let I, I_a, I_b, I_c be the incenter and three excenters of $\triangle ABC$ against A, B, C. $\triangle ABC$ and its circumcircle (O) become orthic triangle and 9-point circle of $\triangle I_aI_bI_c$, thus $\triangle A_1B_1C_1 \sim \triangle I_aI_bI_c$ are homothetic with factor $\frac{1}{2}$. Then, $[\triangle A_1B_1C_1] = \frac{1}{4}[\triangle I_aI_bI_c]$ and $\triangle CA_1B_1, \triangle AB_1C_1, \triangle BC_1A_1$ become the I-antimedial triangles of $\triangle II_aI_b, \triangle II_bI_c$ and $\triangle II_cI_a$. Consequently:

$[\triangle CA_1B_1] = \frac{1}{4}[\triangle II_aI_b]$, $[\triangle AB_1C_1] = \frac{1}{4}[\triangle II_bI_c]$, $[\triangle BC_1A_1] = \frac{1}{4}[\triangle II_cI_a]$

Summing up the areas gives $[AC_1BA_1CB_1A] = \frac{1}{2}[\triangle I_aI_bI_c] = 2[\triangle A_1B_1C_1]$.

Medial triangle $\triangle A_0B_0C_0$ of $\triangle I_aI_bI_c$ is its pedal triangle with maximum area, thus

$[\triangle A_0B_0C_0] = \frac{1}{4}[\triangle I_aI_bI_c] = [\triangle A_1B_1C_1] \geq [\triangle ABC]$.

Quick Reply

High School Olympiads

Yatpp(yet another triangle-point problem)- parallels by p

Reply

Source: Problem 2, Brazilian MO, 1991

Johann Peter Dirichlet

#1 Mar 19, 2006, 11:04 pm

P is a point inside the triangle ABC. The line through P parallel to AB meets AC A_C and BC at B_0. Similarly, the line through P parallel to CA meets AB at A_1 and BC at C_1, and the line through P parallel to BC meets AB at B_2 and AC at C_2. Find the point P such that $A_0B_0 = A_1B_1 = A_2C_2$.

zzz123

#2 Mar 29, 2010, 3:28 pm

Where is B_1 and A_2??
or you want to mean $A_0B_0 = A_1C_1 = B_2C_2$

Luis González

#3 Oct 16, 2010, 6:03 am

I also think that it should be $A_0B_0 = A_1C_1 = B_2C_2$. Let $(u : v : w)$ be the normalized barycentric coordinates of P WRT $\triangle ABC$. Then lengths of the paralell sections from P to BC, CA, AB are given by

$$B_2C_2 = a(1-u) \ \ , \ A_1C_1 = b(1-v) \ \ , \ A_0B_0 = c(1-w)$$

Solving $a(1-u) = b(1-v) = c(1-w)$, together with $u+v+w=1$ gives

$$P \equiv \left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a} : \frac{1}{a}+\frac{1}{c}-\frac{1}{b} : \frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right),$$

which is the anticomplement of the isotomic conjugate of the incenter of $\triangle ABC$.

underzero

#4 Feb 23, 2013, 2:22 am

Hint:
You can use of this lemma :
$(A_0B_0/AB) + (A_1C_1/AC) + (B_2C_2/BC) = 2$

Quick Reply

High School Olympiads

An inequality and an identity in a triangle ABC.

Reply

Virgil Nicula

#1 Oct 15, 2010, 12:06 am

In $\triangle ABC$ with the circumcircle w and the incenter I denote the projections X , Y , Z of I on the tangent lines to w in A , B , C respectively. Prove that $IA+IB+IC\ge$ $\boxed{IX+IY+IZ\ge 6r}$ and $\boxed{a\cdot IX+b\cdot IY+c\cdot IZ=4S}$, where $S=[ABC]$.

Luis González

#2 Oct 15, 2010, 12:07 pm

Let h_a,h_b,h_c be the lengths of the altitudes of $\triangle ABC$ issuing from $A,B,C.$ Using this result for $D\equiv B$ we get $r+IY=AB\cdot\sin A=h_b\Longrightarrow IY=h_b-r.$ Likewise, we have $IX=h_a-r$ and $IZ=h_c-r.$ Thus

$$a\cdot IX=a\cdot h_a-a\cdot r=2|\triangle ABC-\triangle IBC|$$

$$b\cdot IY=b\cdot h_b-b\cdot r=2|\triangle ABC-\triangle ICA|$$

$$c\cdot IZ=c\cdot h_c-c\cdot r=2|\triangle ABC-\triangle IAB|$$

$$\Longrightarrow a\cdot IX+b\cdot IY+c\cdot IZ=6|\triangle ABC|-2|\triangle ABC|=4|\triangle ABC|$$

On the other hand, $IX+IY+IZ=h_a+h_b+h_c-3r.$ But by AM-HM we get

$$h_a+h_b+h_c\ge 9\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)^{-1}=9\left(\frac{1}{r}\right)^{-1}=9r$$

$$\Longrightarrow IX+IY+IZ-3r\ge 9r\Longrightarrow\ IX+IY+IZ\ge 6r.$$

Quick Reply

High School Olympiads

little problem

Reply

ndk09
#1 Oct 15, 2010, 9:23 am

Let triangle ABC , M is the midpoint of BC, $ME, ME \perp AC, AB$ at E, F
O is the center of (ABC) , I is the mid point of EF, prove that $MI \parallel AO$.

Luis González
#2 Oct 15, 2010, 10:30 am

Let D be the intersection of the tangents of (O) through B, C and let U be the midpoint of DM. If C' denotes the projection of C onto AB, from $\angle BAC = \angle BCD$, it follows that $\frac{CA}{CD} = \frac{CC'}{DM} = \frac{MF}{MU} \Longrightarrow \triangle ACD \sim \triangle FMU$ (SAS). Therefore $\angle UFM = \angle CAD = \angle FEM \Longrightarrow UF$ is tangent to $\odot(MFE)$ through F. Likewise, UE is tangent to $\odot(MFE)$ through $E \Longrightarrow UM \equiv UO$ is the M-symmedian of $\triangle MFE$, thus $\angle EMI = \angle OMF$. Consequently, $\angle IMC = 90^\circ - |\angle C - \angle B|$, which is precisely the measure of the angle between lines AO and $BC \Longrightarrow AO \parallel ME$.

jayme
#3 Oct 15, 2010, 7:46 pm

Dear ndk09 and Mathlinkers,
I think you have rediscovered the nice lemma 1 of Luis
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=316488.
Sincerely
Jean-Louis

pacoga
#4 Oct 16, 2010, 4:26 pm

The result is also true for any point N on the $A-$median: Let N be on the $A-$ median of ABC, let E, F be the orthogonal projections of N on CA, AB, respectively, and let I be the midpoint of EF. Then the lines NI and AO are parallel.

Quick Reply

High School Olympiads

Extension of N.M.O. Iran, 1998 (own ?!).

Reply

Virgil Nicula

#1 Oct 14, 2010, 4:53 pm

Proposed problem. Let ABC be a triangle with the circumcircle w and the incenter I . Consider $D\in w$ for which the sideline BC separates A , D . Denote the projections E , F of I on the line BD , CD respectively. Prove that $IE+IF=AD\cdot\sin A$

Remark. For $A\in\{30^{\circ},150^{\circ}\}$ obtain the proposed prblem from **N.M.O. IRAN**, **1998**.

Luis González

#2 Oct 15, 2010, 7:25 am • 1

Let R be the radius of the circumcircle ω and let rays BI,CI cut ω at $M,N.$ By Ptolemy's theorem for $AMDN,$ we get $DN\cdot AM+DM\cdot AN=AD\cdot MN \ (*)$

Since AM and AN are circumradii of $\triangle AIC$ and $\triangle AIB,$ it follows that:

$$IC=2AM\cdot \sin \frac{A}{2} \ \ (1) \ , \ \ IB=2AN\cdot \sin \frac{A}{2} \ \ (2)$$

On the other hand, let I_a,I_b,I_c be the excenters of $\triangle ABC$ against vertices $A,B,C.$ Since $\triangle ABC,\omega$ are the orthic triangle and 9-point circle of $\triangle I_aI_bI_c,$ we deduce that:

$$MN=\frac{I_bI_c}{2}=\frac{1}{2}\sec \widehat{I_bI_aI_c}\cdot BC=\frac{1}{2}\sec \left(\frac{\pi-A}{2}\right)\cdot BC=\frac{1}{2}\csc \frac{A}{2}\cdot BC \ (3)$$

Combining the expressions $(1),(2),(3)$ with $(*)$ yields:

$$DN\cdot IC\cdot \csc \frac{A}{2}+DM\cdot IB\cdot \csc \frac{A}{2}=AD\cdot BC\cdot \csc \frac{A}{2}$$

$$\Longrightarrow \ IC\cdot DN+IB\cdot DM=AD\cdot BC \ \ (4)$$

$$IF=IC\cdot \sin \widehat{NCD}=IC\cdot \frac{DN}{2R} \Longrightarrow \ IC\cdot DN=2R\cdot IF$$

$$IE=IB\cdot \sin \widehat{MBD}=IB\cdot \frac{DM}{2R} \Longrightarrow \ IB\cdot DM=2R\cdot IE$$

Substituting these two latter expressions into (4) yields:

$$2R\cdot (IE+IF)=AD\cdot BC \Longrightarrow \ IE+IF=AD\cdot \frac{BC}{2R}=AD\cdot \sin A.$$

oneplusone

#3 Oct 15, 2010, 8:01 pm

You can actually use the same method as the Iran problem here http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=369927 without a lot of algebraic manipulation.

Virgil Nicula

#4 Oct 16, 2010, 1:21 am

> *oneplusone wrote:*
> You can actually use the same method as the Iran problem here http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=369927 without a lot of algebraic manipulation.

I know this nice proof. Try you and post here.

High School Olympiads

About Fermat point!

tdl

#1 Nov 12, 2009, 7:47 am

Contruct equilateral triangle BCD, CAE, ABF outside a triangle ABC. Call M, N, P are intersection of AD, BE, CF with circle (DEF).
Prove that: $AM + BN + CP = AD$.

shoki

#2 Nov 13, 2009, 1:28 am

use these :
if a point P is inside a circle and three distinct chords called AB, CD, EF passes through it such that $\angle APC = \angle CPE = 60^\circ$ then we have :
$PA + PD + PE = PB + PC + PF$
if a point P is on the circle (O) and ABC is an equilateral triangle with circumcenter (O) then one of the distances PB, PA, PC is equal to the sum of two others.

livetolove212

#3 Nov 13, 2009, 11:04 am

An interesting result:
Denote J, L, K the reflections of M, N, P wrt A, B, C, respectively. Let T, O be Fermat point and circumcenter of triangle ABC. Prove that the center of (JLK) lies on OT.

Luis González

#4 Oct 14, 2010, 6:02 am

$T \equiv AD \cap BE \cap CF$ is the 1st Fermat point of $\triangle ABC$, i.e. T is the common point of circles $\odot(DBC), \odot(ECA), \odot(FAB)$. It's clear that T is also the 1st Fermat point of $\triangle DEF$ and $\triangle MNP$. Let P_1, Q_1, R_1 be the circumcenters of $\triangle TNP, \triangle TPM, \triangle TMN$ and let P_2, Q_2, R_2 be the circumcenters of $\triangle TEF, \triangle TFD, \triangle TDE$. Equilateral $\triangle P_1Q_1R_1$ and $\triangle P_2Q_2R_2$ are the outer Napoleon triangles of $\triangle MNP$ and $\triangle DEF$, respectively.

Let U be the second intersection of circumcircles $(R_1), (Q_2)$ and $V \equiv DE \cap TU$. Since $\angle MDV = \angle MNE = \angle MUV$, it follows that $V \in \odot(UMD)$. Similarly, $MP \cap UT$ lies on $\odot(UMD) \Longrightarrow V \equiv MP \cap DE$ lies on UT. Analogously, intersections $[T, S] \equiv (R_2) \cap (Q_1), W \equiv MN \cap DF$ and T are collinear. But, by Pascal theorem for the cyclic hexagon $DFPMNE$, the points V, W, T are collinear $\Longrightarrow U, S, T$ are collinear. Thus, it follows that $R_1Q_2 \perp UTS \perp Q_1R_2$. Since $Q_1R_1 \perp AT \perp R_2Q_2$, then $R_1Q_1R_2Q_2$ is a parallelogram $\Longrightarrow R_1Q_1$ and R_2Q_2 are congruent and parallel $\Longrightarrow \triangle P_1Q_1R_1$ and $\triangle P_2Q_2R_2$ are congruent. Thus, by Viviani's theorem for T inside $\triangle P_1Q_1R_1$ and $\triangle P_2Q_2R_2$ we get

$$TM + TN + TP = \sqrt{3} \cdot R_1Q_1 = \sqrt{3} \cdot R_2Q_2 = TD + TE + TF$$

$$\Longrightarrow \ TA + TB + TC + AM + BN + CP = TD + TE + TF \ \ (\star)$$

By Ptolemy's theorem for cyclic quadrilaterals $TBDC, TCEA, TAFB$ we obtain:

$$TD = TB + TC \ , \ TE = TC + TA \ , \ TF = TA + TB$$

Combining these with $(\star)$, we get $AM + BN + CP = TA + TB + TC = AD$.

High School Olympiads

Regular heptagon

Reply

gammaduc
#1 Oct 12, 2010, 8:41 am

Let $ABCDEFG$ be a regular heptagon and let lengths $AB = a, AC = b, AD = c$. Then find $\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}$.

seasonal squirrel
#2 Oct 13, 2010, 4:24 am

Solution

Luis González
#3 Oct 13, 2010, 8:56 am

By Ptolemy's theorem for quadrilaterals $ABDC, ABDE, ABDF, ABCE$ we have

$$a^2 + a \cdot c = b^2 \Longrightarrow \frac{b^2}{a^2} = 1 + \frac{c}{a} \ \ (1)$$

$$b \cdot c + a^2 = c^2 \Longrightarrow \frac{a^2}{c^2} = 1 - \frac{b}{c} \ \ (2)$$

$$a \cdot b + b^2 = c^2 \Longrightarrow \frac{c^2}{b^2} = \frac{a}{b} + 1 \ \ (3)$$

$$a \cdot b + a \cdot c = b \cdot c \Longrightarrow a = \frac{b \cdot c}{b + c} \ \ (4)$$

Adding the expressions $(1), (2), (3)$ together and then combining with (4) yields

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 3 + \frac{c}{a} + \frac{a}{b} - \frac{b}{c} = 4 + \frac{c}{b} + \frac{c}{b + c} - \frac{b}{c}$$

But $(3) \cap (4)$ yields $: \frac{c}{b} - \frac{b}{c} = 1 - \frac{c}{b + c} \Longrightarrow \frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5.$

staymytime
#4 Nov 15, 2015, 10:01 am

the polynomial $8x^4 + 4x^3 - 8x^2 - 3x + 1$.How did you get the polynomial?

Quick Reply

High School Olympiads

With the A-circle of Mention (own)

Reply

jayme

#1 Oct 12, 2010, 12:50 pm

Dear Mathlinkers,
let ABC be a triangle, (0) the circumcircle of ABC, X the midpoint of the arch BC which doesn't contain A,
(1) the circle with center at X and passing through B (this is the A-circle of Mention),
(2) the circle with diameter AC,
E the second point of intersection of (1) and (2),
D the point of intersection of the A-bisector of ABC with BC.
Prove that BE goes through the midpoint of AD.
Sincerely
Jean-Louis

Luis González

#2 Oct 12, 2010, 9:46 pm

Let ray BE cut circumcircle (O) at F and denote $M \equiv XF \cap AC$. Since $\angle BEC = 90^\circ + \frac{1}{2}\angle BAC$, it follows that $\angle AEF = \angle BFX = \frac{1}{2}\angle BAC \Longrightarrow AE \parallel XF$. Then $XF \perp EC$ is the perpendicular bisector of segment $EC \Longrightarrow M$ is the midpoint of AC $(\ast)$. On the other hand, let L be the midpoint of AD. Then $ML \parallel BC$. Because of $\angle ALM = \angle ADC$, $\angle AFX = \angle ACX$ and $\angle ADC + \angle ACX = 180^\circ$, it follows that $AFML$ is cyclic $\Longrightarrow$ $\angle LFX = \angle XAC = \angle XAB$, hence B, L, F are collinear. Together with $(\ast)$ we deduce that BE passes through the midpoint L of AD.

Quick Reply

High School Olympiads

Super collinear!

Reply

oneplusone
#1 Oct 2, 2010, 7:49 am

Let $ABCD$ be a cyclic quadrilateral with circumcircle Γ. AB intersect DC at E. EF, EG are tangents to Γ with F, G on Γ. DA intersect CB at H. AC intersect BD at I. Circumcircles of $\triangle AID, \triangle BIC$ intersect again at J. The tangents to Γ at A, B intersect at K. Tangents to Γ at C, D intersect at L. Prove that

$$F, G, H, I, J, K, L$$

lie on a straight line.

Luis González
#2 Oct 11, 2010, 5:41 am

It's clear that F, G, H, I, K, L lie on the polar τ of E WRT Γ, thus it remains to show that τ is the radical axis of circumcircles of $\triangle IBC$ and $\triangle AID$. Denote the circumcircles of $\triangle IBC$ and $\triangle EBC$ as ω_1 and ω_2, respectively. Line $a \equiv AD$ is the image of ω_1 and ω_2 under the inversions with poles I, E and powers $p(I, \Gamma)$ and $p(E, \Gamma)$, respectively. By conformity, it follows that $\angle(a, \Gamma) = \angle(\omega_1, \Gamma) = \angle(\omega_2, \Gamma)$. But since B, C are double points under the inversion $\mathcal{I}$ WRT Γ, it follows that $\mathcal{I} : \omega_1 \mapsto \omega_2$. Let O be the center of Γ and $U \equiv EO \cap \tau$. Since $\mathcal{I} : U \mapsto E$, it implies that $U \in \odot(IBC)$. Analogous reasoning yields $U \in \odot(IAD)$, hence $U \equiv J \Longrightarrow J \in \tau$.

ACCCGS8
#3 Sep 6, 2012, 11:22 am

We need to show that IJ is the polar of E. By the Radical Axis Theorem, H, I, J are collinear and since HI is the polar of E, so is IJ and we are done.

Quick Reply

High School Olympiads

4 centroids on a parallel line to the Brocard axis

Reply

Source: 0

Luis González

#1 Dec 6, 2009, 1:49 pm • 1

In the scalene triangle $\triangle ABC,$ let $\triangle H_aH_bH_c$ be its orthic triangle and $\triangle G_aG_bG_c$ the pedal triangle of its centroid $G.$ $\triangle A_0B_0C_0$ is its anticomplementary triangle, whose vertices are the reflections of A,B,C about the midpoints of BC,CA,AB and A',B',C' are the othogonal projections of A_0,B_0,C_0 onto $BC,CA,AB.$ Show that centroids of $\triangle ABC, \triangle H_aH_bH_c, \triangle G_aG_bG_c, \triangle A'B'C'$ lie on a parallel line to the Brocard axis of $\triangle ABC.$

Luis González

#2 Dec 7, 2009, 12:37 am

The problem is a consequence of a more general configuration

Lemma. Let $\triangle ABC$ be a triangle and P a variable point on a conic $\mathcal{H}$ in its plane. Then locus of the centroid of the pedal triangle of P WRT $\triangle ABC$ is another conic. When P moves on a line (degenerate conic) the locus is another line and if this line coincides with the Euler line of $\triangle ABC,$ then the locus is parallel to its Brocard axis.

lym

#3 Dec 7, 2009, 8:42 am

that can be said□
P is an any point on eular line of □ABC□the pedal triangle of P is □DEF□then the locus of centroid of □DEF is a line perpendicular to the Brocard axis of □ABC
there is a little not complicated way to sovle it□use vector.
and what's your way?

Luis González

#4 Dec 7, 2009, 9:47 am

Lym, I get you, that was indeed the first result that I observed (see the lemma above). However, I think you're confusing the Brocard axis, which passes through the circumcenter and symmedian point, with the Brocard line, which passes through the two Brocard points of ABC. By the way, I do not have a synthetic proof for the lemma.

Luis González

#5 Dec 8, 2009, 7:36 am

I'll prove in the upper lemma the case when P moves on a line, which I think is the most fruitful result. The case when P moves on a fixed general conic $\mathcal{H}$ can be approached with the same method.

Lemma: Let P be a varible point on a fixed line ℓ on the plane of $\triangle ABC.$ Let $\triangle XYZ$ be the pedal triangle of P with respect to $\triangle ABC.$ Then the locus of the centroid G' of $\triangle XYZ$ is another line.

Define the orthogonal reference xy where the sidelines of $\triangle ABC$ are given by the equations $ax+by+c=0,$ $dx+ey+f=0$ and $gx+hy+k=0$ and w.l.o.g asumme that ℓ is identical to the y-axis $\Longrightarrow P(0,\varrho).$

Coordinates of the orthogonal projections X,Y,Z of P on BC,CA,AB are given by

$$X\left(\frac{-ac-ab\varrho}{a^2+b^2},\frac{a^2\varrho-bc}{a^2+b^2}\right)$$

$$Y\left(\frac{-df-de\varrho}{d^2+e^2},\frac{d^2\varrho-ef}{d^2+e^2}\right)$$

$$Z\left(\frac{-gk-gh\varrho}{g^2+h^2},\frac{g^2\varrho-hk}{g^2+h^2}\right)$$

Coordinates of the centroid $G'(\overline{x},\overline{y})$ of $\triangle XYZ$ are then

$$\overline{x}=-\frac{\varrho}{3}\left(\frac{ab}{a^2+b^2}+\frac{de}{d^2+e^2}+\frac{gh}{g^2+h^2}\right)-\frac{1}{3}\left(\frac{ac}{a^2+b^2}-\frac{df}{d^2+e^2}-\frac{gk}{g^2+h^2}\right)$$

$$\overline{y}=\frac{\varrho}{3}\left(\frac{a^2}{a^2+b^2}+\frac{d^2}{d^2+e^2}+\frac{g^2}{g^2+h^2}\right)-\frac{1}{3}\left(\frac{bc}{a^2+b^2}+\frac{ef}{d^2+e^2}+\frac{hk}{g^2+h^2}\right)$$

Locus of $G'(\overline{x},\overline{y})$ clearly represents a parametric equation of a line.

Using the lemma in the proposed problem, where G, G_1, G_2, G_3 are the centroids of the pedal triangles of the circumcenter O, the orthocenter H, the centroid G and the De Longchamps point X_{20}, all lying on the Euler line of $\triangle ABC$, we get that G, G_1, G_2, G_3 are collinear. Thus it only remains to show that this line is parallel to the Brocard axis of $\triangle ABC$.

Now, let us use barycentric coordinates WRT $\triangle ABC$. Brocard axis passing through the symmedian point $K\ (a^2:b^2:c^2)$ and the circumcenter $O\ (a^2S_A:b^2S_B:c^2S_C)$ has infinite point T_∞ with coordinates

$$(a^2(c^4+b^4-a^2c^2-a^2b^2):b^2(a^4+c^4-a^2b^2-b^2c^2):c^2(a^4+b^4-b^2c^2-a^2c^2))$$

On the other hand, coordinates of the orthogonal projections of $G(1:1:1)$ on BC, AC, AB are

$$G_a(0:a^2+S_C:a^2+S_B),\ G_b(b^2+S_C:0:b^2+S_A),\ G_c(c^2+S_B:c^2+S_A:0)$$

Therefore, the centroid G_2 of $\triangle G_aG_bG_c$ has coordinates

$$G_2\left(\frac{b^2+S_C}{b^2}+\frac{c^2+S_B}{c^2}:\frac{a^2+S_C}{a^2}+\frac{c^2+S_A}{c^2}:\frac{a^2+S_B}{a^2}+\frac{b^2+S_A}{b^2}\right)$$

These coordinates can be re-expressed as

$$(a^2(c^2S_C+b^2S_B+2b^2c^2):b^2(a^2S_A+c^2S_C+2a^2c^2):c^2(a^2S_A+b^2S_B+2a^2b^2))$$

The infinite point R_∞ of the line GG_2 is then

$$(a^2(c^4+b^4-a^2c^2-a^2b^2):b^2(a^4+c^4-a^2b^2-b^2c^2):c^2(a^4+b^4-b^2c^2-a^2c^2))$$

$\Longrightarrow T_\infty\equiv R_\infty$, hence the line passing trough G, G_1, G_2, G_3 is parallel to OK.

lym

#6 Dec 8, 2009, 9:02 am

Is OK the Brocard axis□cause I'm still think a pair Brocard points are the Brocard axis□OH□I got an elementary mistaken

“P is an any point on eular line of □ABC□the pedal triangle of P is □DEF□then the locus of centroid of □DEF is a line perpendicular to the Brocard axis of □ABC ”
Dear luis□aboat this□I have got a geometric method□next I will post here□ I'm sovling a three circle problem

The point **G** with respect □GaGbGc is calld **X1**□**G** is a new point aboat □**GaGbGc** □**X2** is the isogonal conjugate point of **X1** with respect to □**GaGbGc**
X1□**X2** found by chinese expert **Ye zhonghao**□let **G'** □**G2**□be the centroid of □**GaGbGc**□**F1**□**F2** are the fermat points of □**GaGbGc**□ **S1**□**S2** are the isodynamic points of □**GaGbGc**□Then □
X1X2//OK□OK belong □ABC□□**G'**□**G2**□ is the midpoint of **X1X2**□**F1**□**F2**□**X1**□**X2** are on a circle□**S1**□**S2**□**X1**□**X2** are also on a circle □and **lester circle**
X1□**G**□□**X2** are also the focal points of **steiner ellipse** aboat □**GaGbGc** □**F1F2X1X2** □**S1S2X1X2** both are Harmonic Quadrilateral
all of this is new Conclusion

tell you a good information□I have solved lester circle with good geometric method□
only “**F1**□**F2**□**X1**□**X2** □**S1**□**S2**□**X1**□**X2** are on a circle” is not solved with geometric method

livetolove212

#7 Dec 8, 2009, 12:02 pm

Denote L the symmedian point, A_1, B_1, C_1 the midpoints of BC, CA, AB; A_2, B_2, C_2 the midpoints of AH_a, BH_b, CH_c, respectively.

We have $\vec{GG_3} = \frac{1}{3}(\vec{G_aA_1} + \vec{G_bB_1} + \vec{G_cC_1}) = \frac{1}{9}(\vec{H_aA_1} + \vec{H_BB_1} + \vec{H_cC_1})$

$= \frac{2}{9}.(\vec{A_1A_2} + \vec{B_1B_2} + \vec{C_1C_2})$

$= \frac{2}{9}.\sum_{cyc}(\vec{B_1C_1}.(\frac{A_1B_2}{A_1C_1} - \frac{A_1C_2}{A_1B_1})$

$= \frac{-1}{9}.\sum_{cyc}.(\vec{BC}.(\frac{CH_b}{b} - \frac{BH_c}{c})) = \frac{-1}{9}.\sum_{cyc}.(\vec{BC}.(\frac{a\cos C}{b} - \frac{a\cos B}{c})(1)$

We know that $L(a^2, b^2, c^2)$; $O(\sin 2A, \sin 2B, \sin 2C)$ then $\vec{OL}$ has the direction:

$\vec{BC}.(b^2.\sin 2C - c^2.\sin 2B) + \vec{CA}.(c^2.\sin 2A - a^2.\sin 2C) + \vec{AB}.(a^2.\sin 2B - b^2.\sin 2A)(2)$

From (1) and (2) it's easy to show that $\vec{GG_3}//\vec{OL}$.

For an arbitrary point P which lies on Euler's line of trangle ABC, denote $A_pB_pC_p$ the pedal triangle of P, G_p the centroid of triangle $A_pB_pC_p$.Since $\frac{A_1G_a}{A_1A_p} = \frac{B_1G_b}{B_1B_p} = \frac{C_1G_c}{C_1C_p} = \frac{OG}{OP}$ We get

$\vec{GG_p} = \frac{1}{3}(\vec{A_1A_p} + \vec{B_1B_p} + \vec{C_1C_p}) = k(\vec{A_1G_a} + \vec{B_1G_b} + \vec{C_1G_c})$. Therefore G, G_3, G_p are collinear. So the locus of the centroid of the pedal triangle of a variable point on Euler line is a line through G and parallel to OL.

Remark: (1): From this problem we get an interesting lemma: Let X and Y be two isotomic conjugate points wrt ΔABC; $X_aX_bX_c$ the cevian triangle of X. Denote $\vec{u}$ so that $\vec{u} = \vec{AX_a} + \vec{BX_b} + \vec{CX_c}$Then $\vec{XY}//\vec{u}$.

(2): I see a beautiful contaction of this problem and a property of **Fontene**'s theorem:
Given triangle ABC with its circumcenter O, its centroid G. Let $M_aM_bM_c$, $G_aG_bG_c$ be the pedal triangles of G and O wrt ΔABC.$(G_aG_bG_c)$ intesects the Nine-point circle of triangle ABC at two points. Then:
a, One point of the intersections (we call X) is the Anti-Steiner point of OG wrt $\Delta M_aM_bM_c$.
b, Another point of the intersections is called Y. Then two Steiner lines l_1, l_2 of Y wrt $\Delta M_aM_bM_c$ and $\Delta G_aG_bG_c$ are parallel.
c, The locus of the centroid of the pedal triangle of a variable point P on OG wrt ΔABC is a line which bisects the distance of two lines l_1, l_2.

It's another way to show this problem 🙂

k.l.l4ever

#8 Dec 8, 2009, 4:55 pm

This problem could be solved easily by using the E.R.I.Q Theorem.(http://www.artofproblemsolving.com/Forum/viewtopic.php?t=97493)
I will return later with more detail.(sorry I'm busy now)

Luis González

#9 Oct 10, 2010, 9:58 am • 1 👍

After quite some time, I think I've found a nice synthetic solution to this problem and a couple of additional things like: G_1 is the anticomplement of G WRT $\triangle G_aG_bG_c$ and G is the midpoint of G_1G_3. But first of all let us introduce some general lemmata.

Lemma 1: In $\triangle ABC$ with circumcircle (O), M is the midpoint of BC and N, L denote the orthogonal projections of M on AB, AC. If E is the midpoint of NL, then $ME \parallel AO$.

Proof: Let D be the intersection of the tangents of (O) through B, C and let U be the midpoint of DM. If C' denotes the projection of C onto AB, from $\angle BAC = \angle BCD$, it follows that $\frac{CA}{CD} = \frac{CC'}{DM} = \frac{MN}{MU} \Longrightarrow \triangle ACD \sim \triangle NMU$ (s.a.s). Thereby $\angle UNM = \angle CAD = \angle NLM \Longrightarrow UN$ is tangent to $\odot(MNL)$ through N. Likewise, UL is tangent to $\odot(MNL)$ through $L \Longrightarrow UM \equiv UO$ is the M-symmedian of $\triangle MNL$, thus $\angle LME = \angle OMN$. Consequently, $\angle EMC = 90^\circ - |\angle C - \angle B|$, which is precisely the measure of the angle between AO and $BC \Longrightarrow AO \parallel ME$.

Lemma 2: (M, M'), (N, N') and (L, L') are three pairs of points on the sidelines BC, CA, AB of $\triangle ABC$, respectively. D, E, F denote the midpoints of MM', NN' and LL'. Then centroid G of $\triangle DEF$ is the midpoint of the segment connecting the centroids U, U' of $\triangle MNL$ and $\triangle M'N'L'$.

Proof: Let $\delta(P)$ denote the directed distance from point P to the sideline BC. Then

$$3\delta(U) = \delta(N) + \delta(L)\ ,\ \ 3\delta(U') = \delta(N') + \delta(L')$$

$$\Longrightarrow \delta(U) + \delta(U') = \tfrac{1}{3}[\delta(N) + \delta(N') + \delta(L) + \delta(L')] = \tfrac{2}{3}[\delta(E) + \delta(F)] = 2\delta(G)$$

By similar reasoning we deduce that G is the midpoint of $\overline{UU'}$.

Lemma 3: Mittenpunkt M, Spieker point S and Bevan point B_e of any scalene $\triangle ABC$ are collinear.

Proof: Let $(I_a), (I_b), (I_c)$ be the excircles of $\triangle ABC$ against vertices A, B, C. Incircle (I) and excircle (I_a) are tangent to BC through X, Y and let D, E, F be the midpoints of BC, CA, AB. A-Gergonne ray AX cuts (I_a) at the antipode Y' of Y WRT (I_a). Since D is also the midpoint of XY, it follows that DI_a is the Y-midline of $\triangle XYY' \Longrightarrow DI_a \parallel AX$, i.e. DI_a is the complement of AX. In other words, tangency points X_1, X_2, X_3 of the Spieker circle (S) with EF, FD, DE lie on I_aD, I_bE, I_cF. On the other hand, perpendicular lines dropped from I_a, I_b, I_c to BC, CA, AB concur at the Bevan point B_e, i.e. circumcenter of $\triangle I_aI_bI_c$. From the parallel radii $B_eI_a \parallel SX_1, B_eI_b \parallel SX_2$ and $B_eI_c \parallel SX_3$, it follows that Mittenpunkt $M \equiv I_aD \cap I_bE \cap I_cF$ is the insimilicenter of (S) and $\odot(I_aI_bI_c)$.

Back to the problem, let M_a, M_b, M_c be the midpoints of BC, CA, AB and D_a, D_b, D_c the midpoints of segments G_bG_c, G_cG_a, G_aG_b. Since $\triangle GG_bG_c$ and the pedal triangle of M_a WRT $\triangle ABC$ are homothetic, their G- and Ma- medians are parallel. From Lemma 1, we deduce then $GD_a \parallel AO\ \ (*)$ where O is the circumcenter of $\triangle ABC$.

On the other hand, let M_1 be the midpoint of H_bH_c. Hence M_aM_1 is the perpendicular bisector of $\overline{H_bH_c}$, thus

$$\frac{H_aG_1}{G_1M_1} = \frac{AG}{GM_a} = \frac{H_aG_a}{G_aM_a} = 2 \Longrightarrow G_aG_1 \parallel M_aM_1 \parallel AO$$

Together with $(*)$, we have $G_aG_1 \parallel D_aG$.

Similarly, $D_bG \parallel G_bG_1$ and $D_cG \parallel G_cG_1 \Longrightarrow G_1$ is the anticomplement of G with respect to $\triangle G_aG_bG_c$, i.e. G, G_1, G_2 are collinear, such that $\overline{G_2G} : \overline{G_2G_1} = -1 : 2$. Since A', B', C' are the reflections of H_a, H_b, H_c about M_a, M_b, M_c, from Lemma 2 we deduce that centroids G, G_1, G_3 are collinear such that $\overline{GG_1} = -\overline{GG_3}$.

Circumcenter O and symmedian point K of $\triangle ABC$ become Bevan point and Mittenpunkt of $\triangle H_aH_bH_c$. From Lemma 3, it follows that Spieker point S' of $\triangle H_aH_bH_c$ lies on OK. Further, if $\triangle ABC$ is acute, then H becomes the incenter of $\triangle H_aH_bH_c \Longrightarrow HG_1$ cuts Brocard axis OK at S'. Now, because of $\frac{HG}{HO} = \frac{HG_1}{HS'} = \frac{2}{3}$, we conclude that GG_1 is parallel to $OS' \equiv OK \Longrightarrow$ Centroids G, G_1, G_2, G_3 lie on the image ℓ of the Brocard axis OK under the homothety $(H, \frac{2}{3})$.

jayme

#10 Apr 9, 2012, 1:33 pm

Dear Mathlinkers,
a synthetic proof concerning this nice result can be seen in the article "Du théorème d'Ernesto Cesàro à une droite parallèle à l'axe de Brocard" that has been put on my website with some developpments.

http://perso.orange.fr/jl.ayme vol. 6, p. 13, 20, 22, 23.

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Prove the fraction on cotangents

Reply

Amir Hossein

#1 Oct 8, 2010, 1:14 am

Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD intersect in a point P. Prove that

$$\frac{AP}{PC}=\frac{\cot\angle BAC+\cot\angle DAC}{\cot\angle BCA+\cot\angle DCA}$$

Luis González

#2 Oct 8, 2010, 6:48 am

$$\cot\widehat{BAC}+\cot\widehat{DAC}=\frac{\sin(\widehat{BAC}+\widehat{DAC})}{\sin\widehat{BAC}\cdot\sin\widehat{DAC}}=\frac{\sin\widehat{DAB}}{\sin\widehat{BAC}\cdot\sin\widehat{DAC}}$$

$$\cot\widehat{BCA}+\cot\widehat{DCA}=\frac{\sin(\widehat{BCA}+\widehat{DCA})}{\sin\widehat{BCA}\cdot\sin\widehat{DCA}}=\frac{\sin\widehat{BCD}}{\sin\widehat{BCA}\cdot\sin\widehat{DCA}}$$

$$\frac{\cot\widehat{BAC}+\cot\widehat{DAC}}{\cot\widehat{BCA}+\cot\widehat{DCA}}=\frac{\sin\widehat{DAB}}{\sin\widehat{BCD}}\cdot\frac{\sin\widehat{BCA}}{\sin\widehat{BAC}}\cdot\frac{\sin\widehat{DCA}}{\sin\widehat{DAC}}$$

$$\frac{\cot\widehat{BAC}+\cot\widehat{DAC}}{\cot\widehat{BCA}+\cot\widehat{DCA}}=\frac{AB\cdot AD\cdot\sin\widehat{DAB}}{BC\cdot DC\cdot\sin\widehat{BCD}}=\frac{|\triangle DAB|}{|\triangle BCD|}=\frac{\overline{AP}}{\overline{PC}}.$$

Quick Reply

High School Olympiads

Triangle and intersecting lines

Reply

borislav_mirchev
#1 Oct 6, 2010, 11:14 pm

Let $\triangle ABC$ be a triangle. The incircle of triangle $\triangle ABC$ touches side BC at A'. Let segment AA' meet the incircle again at P. Segments BP, CP meet the incircle at M, N, respectively. Show that lines AA', BN, CM are concurrent.

Luis González
#2 Oct 7, 2010, 1:00 am • 1

The result is true for any P on segment $\overline{AA'}$. Let the incircle (I) touch CA, AB at B', C' and lines PB, PC cut either its arc $B'C'$ or $B'A'C'$ at M, N, respectively. Define the point $Q \equiv BC \cap B'C'$ and let QM intersect (I) again at N'. Since AA' is the polar of Q WRT (I), it follows that $P(M, N', A', Q) = -1$, but $P(B, C, A', Q) = -1$, which implies that N', P, C are collinear $\Longrightarrow N$ and N' coincide. Hence, lines BN and CM meet on AA'.

borislav_mirchev
#3 Oct 7, 2010, 1:23 am

Thank you very much. I rediscovered the problem at my own it is the reason I posted it in this section.
I like most the following solution:
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1989323#p1989323
Can you explain why the quadrilaterals mentioned are complete?

Luis González
#4 Oct 7, 2010, 1:50 am • 1

> borislav_mirchev wrote:
> Can you explain why the quadrilaterals mentioned are complete?

I think you meant harmonic quadrilaterals. A quadrilateral ABCD with circumcircle (O) is said to be harmonic iff tangents to (O) through A,C meet on the line BD. Hence, tangents to (O) through B,D meet on the line AC. As a result, AC is the common symmedian of triangles ABD and CBD issuing from A,C. Likewise, BD is the common symmedian of triangles DAC and BAC issuing from D,B. Thus, a quadrilateral is harmonic iff the products of its opposite sides are equal.

borislav_mirchev
#5 Oct 7, 2010, 2:41 am

Thank you for the explanation.

Quick Reply

High School Olympiads

bicentric quadrilateral inscribed in a circle

Reply

nguyenvuthanhha

#1 Oct 6, 2010, 9:02 am

Let $ABCD$ be a bicentric quadrilateral
inscribed in a circle with center I and circumcribed about a circle with
center O. A line through I, parallel to a side of $ABCD$, intersects its
two opposite sides at M and N. Prove that the length of MN does not
depend on the choice of side to which the line is parallel

Luis González

#2 Oct 6, 2010, 11:38 am

This is an old problem, but it is incorrectly stated. In general, it is true for the incenter of ABCD, but not for its circumcenter. Thus, for convencience let I be the incenter of ABCD and let us introduce a previous lemma, which is actually q2 of 9th IBMO Brazil.

Lemma. $ABCD$ is a cyclic quadrangle and there exists a circumference centered on AB that is tangent to the other three sides of the quadrangle. Then $AB = AD + BC.$

Let O be the center of the subject circumference centered on $AB.$ Rays OC and OD are bisectors of $\angle BCD = \theta$ and $\angle CDA = \varphi.$ Taking a point M on AB such that $AD = AM,$ then $\triangle ADM$ is isosceles, where

$$\angle DMA = \frac{\pi - \angle DAB}{2} = \frac{\pi}{2} - \frac{\pi - \theta}{2} = \frac{\theta}{2}$$

Since $\angle DCO = \frac{\theta}{2} \Longrightarrow DCOM$ is cyclic, consequently $\angle CMB = \angle ODC = \frac{\varphi}{2}$

On the other hand, in $\triangle CMB$ we have $\angle MCB = \pi - \frac{\varphi}{2} - (\pi - \varphi) = \frac{\varphi}{2} \Longrightarrow \angle CMB = \angle MCB = \frac{\varphi}{2}.$ Thus, triangle $\triangle CMB$ is isosceles with apex $B \Longrightarrow AB = AM + BM = AD + BC.$

Let the parallel from the incenter I to CD cut BC, AD at $M, N.$ Then $ABMN$ is a cyclic quadrangle with a circle (I) centered on MN and tangent to $MB, BA, AN.$ From the previous lemma, we get $MN = MB + NA,$ but $\triangle IMC$ and $\triangle IND$ are isosceles with legs $MI = MC$ and $NI = ND.$ Therefore

$$MN = \frac{MC + MB + ND + NA}{2} = \frac{DA + BC}{2} = \frac{AB + BC + CD + DA}{4}$$

skytin

#3 Oct 7, 2010, 7:29 pm

if K L are on AB P Q on DC and I is on KQ and PL and PL is parallel to AD and KQ is parallel to BC then easy to see that KLQP is cyclic and KLI ~ PIQ , if IH and IM are altitudes from I on AB and DC then easy to see that IH = IM so KIL = PIQ so LI = IQ and KI = IP easy to se that LI = LA KI = KB QI = QC PI = PD so LP + KQ = AB + CD = AD + BC = double of other segments

Quick Reply

High School Olympiads

PA+PC+PE=PB+PD

Reply

Kunihiko_Chikaya
#1 Sep 30, 2010, 11:37 am

Given a regular pentagon inscribed in a circle. Take a point P in the circumference.
Prove that $PA+PC+PE=PB+PD.$

Luis González
#2 Oct 5, 2010, 10:35 pm

The relation is only true for points P on the small arc $EA,$ otherwise, appropriate choice of signs have to be considered. Let e,d denote the side and diagonal of $ABCDE.$ By Ptolemy's theorem for $PACE, PBCD, ABDE$ we get

$$PA \cdot d+PE \cdot d=PC \cdot e \Longrightarrow \ PA+PE+PC=PC\left(1+\frac{e}{d}\right) \ \ (1)$$

$$PB \cdot e+PD \cdot e=PC \cdot d \Longrightarrow \ PB+PD=PC \cdot \frac{d}{e} \ \ (2)$$

$$d^2=d \cdot e+e^2 \Longrightarrow \ \frac{d}{e}=1+\frac{e}{d} \ \ (3)$$

Combining $(1),(2)$ and (3) yields $PA+PE+PC=PB+PD.$

Kunihiko_Chikaya
#3 Oct 6, 2010, 5:12 am

That's correct.

Quick Reply

High School Olympiads

Construct pentagon given pentagon by external angle bisector

Reply

Goutham

#1 Oct 4, 2010, 8:50 pm

$(NET5)$ The bisectors of the exterior angles of a pentagon $B_1B_2B_3B_4B_5$ form another pentagon $A_1A_2A_3A_4A_5$. Construct $B_1B_2B_3B_4B_5$ from the given pentagon $A_1A_2A_3A_4A_5$.

Luis González

#2 Oct 5, 2010, 7:39 am

Rename the unknown pentagon as $ABCDE$ and let a, b, c, d, e be the external bisectors of its angles at A, B, C, D, E, respectively. $\mathcal{A}(\ell)$ denotes the axial symmetry across ℓ. Oriented lines AB, EA are homologous under the composition $\mathcal{A} \equiv \mathcal{A}(b) \circ \mathcal{A}(c) \circ \mathcal{A}(d) \circ \mathcal{A}(e)$. But, $\mathcal{A}(b) \circ \mathcal{A}(c)$ is a rotation $\mathcal{R}_1$ with center $b \cap c$ and $\mathcal{A}(d) \circ \mathcal{A}(e)$ is another rotation $\mathcal{R}_2$ with center $d \cap e$. Therefore, AB, EA are homologous under certain rotation $\mathcal{R}_3 \equiv \mathcal{R}_1 \circ \mathcal{R}_2$. To define $\mathcal{R}_3$, pick two arbitrary points U, V in the plane and construct their images U', V' under $\mathcal{A}$. Then, perpendicular bisectors of UU' and VV' meet at the center O of $\mathcal{R}_3$ and $\angle UOU' = \angle VOV' = \omega$ is its rotational angle. Consequently, A is the orthogonal projection of O onto a and $\angle(a, AB) = \frac{1}{2}\omega$. Once the point A and the line passing through A, B are constructed, the construction of the remaining vertices B, C, D, E is straightforward.

Quick Reply

High School Olympiads

S_PQR/S_A'B'C' - Iran NMO 1999 (Second Round) Problem5

Reply

sororak
#1 Oct 4, 2010, 1:42 pm

Let ABC be a triangle and points P,Q,R be on the sides AB,BC,AC, respectively. Now, let A',B',C' be on the segments PR,QP,RQ in a way that $AB||A'B'$, $BC||B'C'$ and $AC||A'C'$. Prove that:

$$\frac{AB}{A'B'}=\frac{S_{PQR}}{S_{A'B'C'}}.$$

Where S_{XYZ} is the surface of the triangle XYZ.

Luis González
#2 Oct 4, 2010, 10:41 pm

$O \equiv AA' \cap BB' \cap CC'$ is the homothetic center of $\triangle ABC \sim \triangle A'B'C'$ and let X,Y,Z be the orthogonal projections of O onto $BC,CA,AB.$ Then

$$[PQR]=[OC'QB']+[OB'PA']+[OA'RC'] \ , \ (1)$$

$$[ABC]=[OBC]+[OCA]+[OAB]=\frac{OX \cdot BC+OY \cdot CA+OZ \cdot AB}{2} \ , \ (2)$$

On the other hand, we have the expressions:

$$[OC'QB']=\frac{OX \cdot B'C'}{2} \ , \ [OB'PA']=\frac{OZ \cdot A'B'}{2} \ , \ [OA'RC']=\frac{OY \cdot C'A'}{2}$$

Substituting these latter expressions into (1) gives

$$[PQR]=\frac{OX \cdot B'C'+OY \cdot C'A'+OZ \cdot A'B'}{2}$$

If $k>1$ denotes the similarity coefficient of $\triangle ABC \sim \triangle A'B'C'$, we have

$[PQR]=\dfrac{OX \cdot BC+OY \cdot CA+OZ \cdot AB}{2k}$. Together with (2) we obtain

$$[PQR]=\frac{[ABC]}{k}=k \cdot [A'B'C'] \Longrightarrow \frac{[PQR]}{[A'B'C']}=k=\frac{AB}{A'B'}$$

Quick Reply

High School Olympiads

Three concurrent circles (own)

Petry

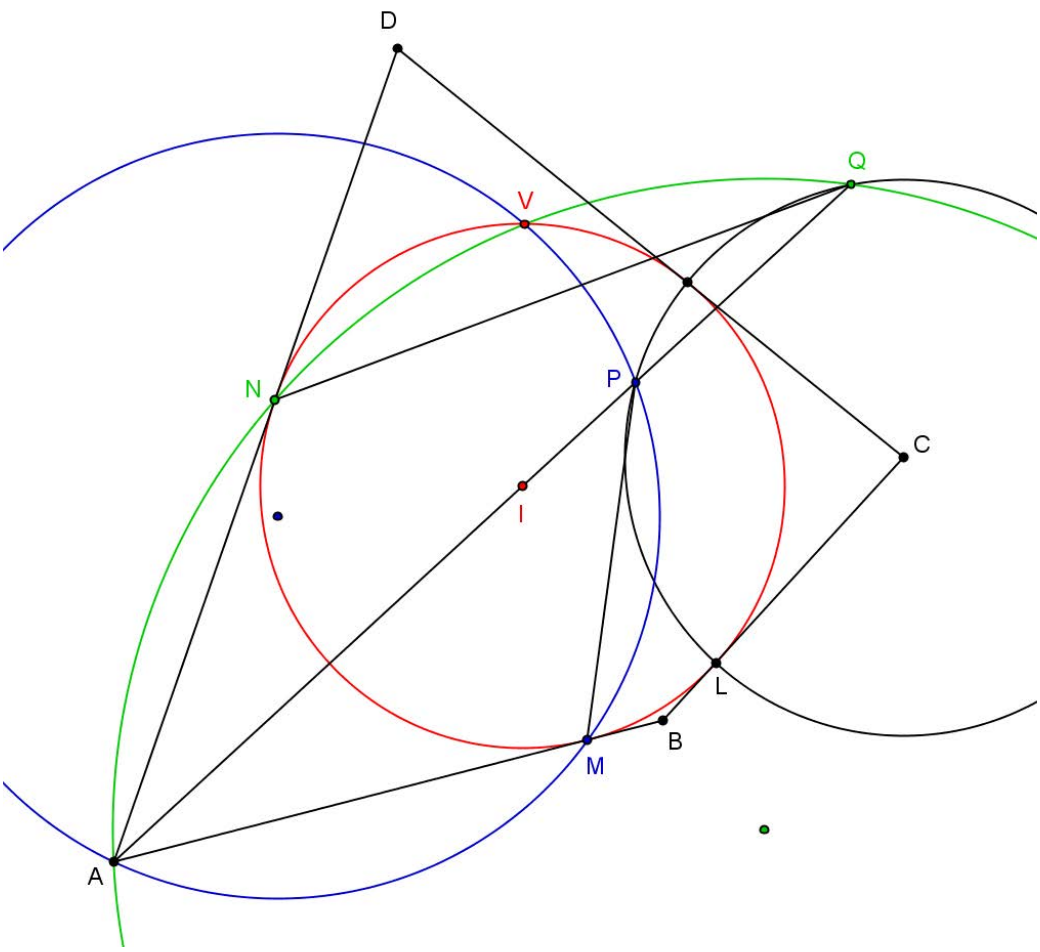
#1 Oct 3, 2010, 11:53 pm

Hello!

Let $ABCD$ be a quadrilateral circumscribed around a circle (I) and $\{M\} = AB \cap (I)$, $\{N\} = AD \cap (I)$, $\{L\} = BC \cap (I)$. If $\{P, Q\} = AI \cap (C; CL)$ then prove that the circle (I) and the circumcircles of ΔAMP and ΔANQ are concurrent.

Best regards,
Petrisor Neagoe :)

Attachments:



Luis González

#2 Oct 4, 2010, 7:32 am • 1

Since (C, CL) is orthogonal to the incircle (I), then (C, CL) is double under the inversion WRT $(I) \Longrightarrow$ Circles $\odot(AMP)$ and $\odot(ANQ)$ are taken into the circles $\odot(EMQ)$ and $\odot(ENP)$ respectively, where E is the midpoint of MN. Hence, it is enough to show that $\odot(EMQ), \odot(ENP)$ and (I) have a common point. Let U be the second intersection of $\odot(EMQ)$ with (I). Line AI cuts (I) at R, F (R is inside triangle AMN). Because of $\angle RUF = 90^\circ$ and $\angle MEQ = \angle MUQ = 90^\circ$, it follows that $\angle FUQ = \angle RUM$. But since R is the midpoint of the arc MN of (I), then $\angle NUM = 2\angle FUQ$ $(*)$.

On the other hand, from $IF^2 = IR^2 = IP \cdot IQ \Longrightarrow (P, Q, F, R) = -1$. Since $\angle RUF = 90^\circ$, we deduce that UF bisects $\angle PUQ \Longrightarrow \angle PUQ = 2\angle FUQ$. Together with $(*)$, it follows that $\angle NUM = \angle PUQ$ which implies that $\angle MUQ = \angle NUP = 90^\circ$, thus $U \in \odot(ENP)$. Consequently, $(I), \odot(EMQ)$ and $\odot(ENP)$ concur at U.

Petry

#3 Oct 4, 2010, 10:31 am • 1

My solution:

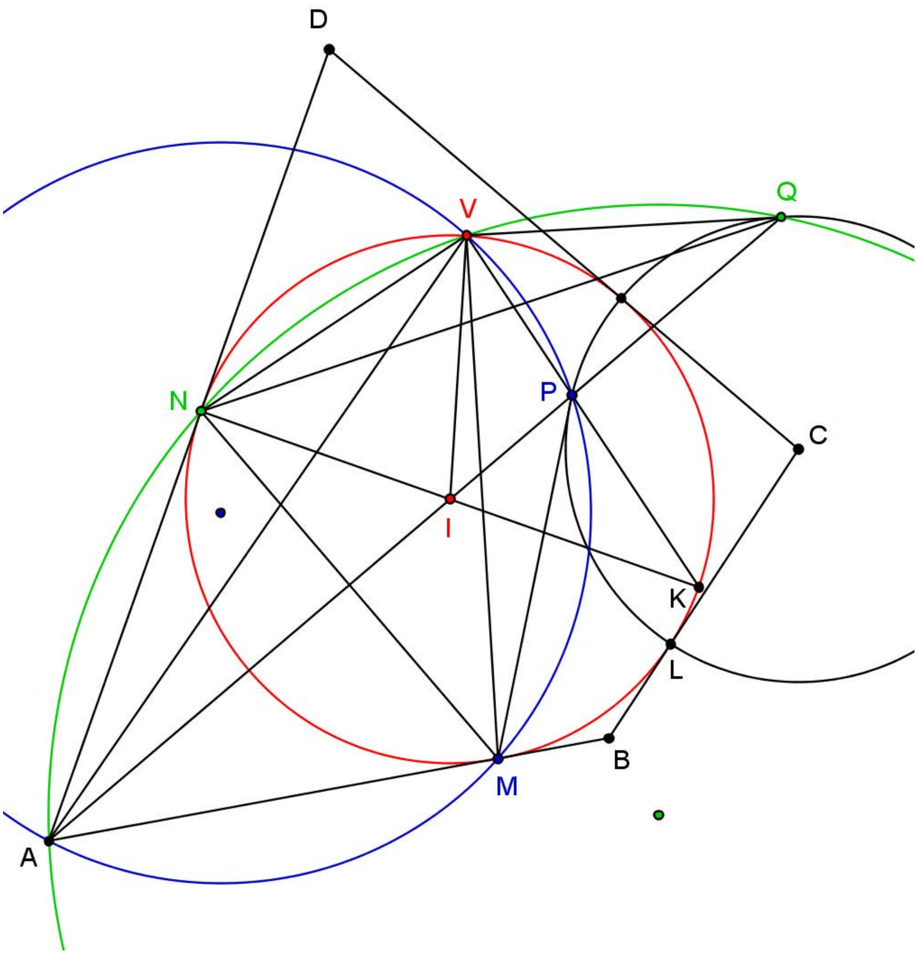
$\{K, N\} = IN \cap (I)$ and $\{V, K\} = KP \cap (I)$
$\angle MAP = \angle MAI = \angle MNI = \angle MNK = \angle MVK = \angle MVP \Rightarrow$
$\Rightarrow$ the quadrilateral $MAVP$ is cyclic.
So, the point V lies on the circumcircle of ΔAMP. (*)

$IL \perp BC \Rightarrow IL$ is tangent to the circle $(C; CL) \Rightarrow$
$\Rightarrow IP \cdot IQ = IL^2 = IV^2 \Rightarrow IV$ is tangent to the circumcircle of ΔPVQ at $V \Rightarrow$
$\Rightarrow \angle IVP = \angle IQV$
$\angle AQV = \angle IQV = \angle IVP = \angle IVK = \angle IKV = \angle DNV \Rightarrow$
$\Rightarrow$ the quadrilateral $NAQV$ is cyclic $\Rightarrow$
So, the point V lies on the circumcircle of ΔANQ. (**)

(*),(**)$\Rightarrow$ the circle (I) and the circumcircles of ΔAMP and ΔANQ are concurrent at V.

Best regards,
Petrisor Neagoe

Attachments:



Quick Reply

High School Olympiads

Two perspective triangles (own)

jayme
#1 Oct 3, 2010, 11:45 am

Dear Mathlinkers,
let ABC be a triangle, I the incenter, D, E, F the feet of the A, B, C-bisectors,
Ma the perpendicular bisector of AD,
Q, R the points of intersection of BI, CI with Ma,
and A* the point of intersection of EF and QR.
Cyclically, B*, C*.
Prove : AA*, BB* and CC* are concurrent or ABC and A*B*C* are perspective
Sincerely
Jean-Louis

Luis González
#2 Oct 4, 2010, 1:32 am

Let us use barycentric coordinates WRT $\triangle ABC$. The perpendicular bisector of segment AD passes through the center of the A-Apollonius circle $U(0:b^2:-c^2)$ and the infinity point $P_\infty(c-b:b:-c)$ of the external bisector of $\angle A$. Thus, equations of lines EF and RQ are given by

\[EF : \left(\begin{array}{ccc} a & b & 0\\ a & 0 & c\\ x & y & z\end{array}\right)=0 \, \ \[RQ:\left(\begin{array}{ccc} c-b & b & -c\\ 0 & b^2 & -c^2 \\ x & y & z\end{array}\right)=0\]

These meet at $A_0 \equiv (-a(b-c)(c-b) : b(b-c)(b-a) : c(c-a)(c-b))$

By cyclic permutation of a, b, c we get the coordinates of B_0, C_0, and therefore the lines AA_0, BB_0, CC_0 concur at the Feuerbach point of the antimedial triangle of $\triangle ABC$, namely

$$X_{100} \equiv \left(\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}\right)$$

Quick Reply

High School Olympiads

Three other concurrent lines (own)

Reply

jayme
#1 Oct 2, 2010, 5:48 pm

Dear Mathlinkers,
Let ABC be a triangle, I the incenter, D, E, F the feet of the A, B, C-bisectors, Ma the perpendicular bisector of AD,
Q, R the points of intersection of BI, CI with Ma,
B' the point of intersection of DQ and AC,
C' the point of intersection of DR and AB,
Prove : B'C', QR and EF are concurrent.
Sincerely
Jean-Louis

Luis González
#2 Oct 2, 2010, 11:16 pm • 2

Since quadrilaterals $ARDC$ and $AQDB$ are cyclic, it follows that $\angle RAD=\angle RCB$ and $\angle QAD=\angle QBC \Longrightarrow$ $\angle RIQ+\angle RAQ=\pi$, thus $I \in \odot(ARQ).$ Let B_1,C_1 be the second intersections of $\omega \equiv \odot(ARQ)$ with $AC,AB.$ Since $(O),\omega$ are internally tangent at $A,$ from $\angle ARC_1=\angle ACB, \angle ARC=\angle ADC, \angle DRC=\angle DAC,$ it follows that D,R,C_1 are collinear, i.e. $C' \equiv C_1.$ Likewise, we get $B' \equiv B_1.$ Consequently, intersections $E \equiv IQ \cap AB',$ $F \equiv IR \cap AC'$ and $B'C' \cap RQ$ lie on the polar of D WRT $\omega \Longrightarrow$ Lines $B'C',QR,EF$ concur.

Quick Reply

High School Olympiads

Three concurrent lines (own)

Reply

jayme

#1 Oct 2, 2010, 5:35 pm • 1

Dear Mathlinkers,
Let ABC be a triangle, I the incenter, D the foot of the A-bissector, Ma the perpendicular bisector of AD,
Q, R the points of intersection of BI, CI with Ma,
(1) the circle passing through A, Q, R
and Ta the tangent to (1) at A.
Prove : Ta, QR and BC are concurrent.
Sincerely
Jean-Louis

Luis González

#2 Oct 2, 2010, 9:24 pm

It's well-known that AI passes through the circuncenter of $\triangle BIC,$ namely, the midpoint of the arc BC of the circumcircle (O) of $\triangle ABC.$ Hence RQ is antiparallel to BC with respect to $IB,IC,$ i.e. B,C,Q,R are concyclic and $U \equiv QR \cap BC$ is the center of the A-Apollonius circle of $\triangle ABC \Longrightarrow \overline{UR} \cdot \overline{UQ}=\overline{UB} \cdot \overline{UC}.$ But since (U) is orthogonal to the circumcircle $(O),$ then $\overline{UB} \cdot \overline{UC}=\overline{UD}^2=\overline{UA}^2 \Longrightarrow$ tangents of $(O), \odot(AQR)$ at A coincide.

Quick Reply

High School Olympiads

Fixed Angle

Reply

Headhunter

#1 Sep 29, 2010, 6:32 am • 1

Hello.

For a parallelogram $ABCD$, a line through A meet BC,CD at X,Y.
Let K,L be the excenters of $\triangle ABX$, $\triangle AYD$. Show that $\angle KCL$ is constant.

Luis González

#2 Oct 2, 2010, 10:20 am • 1

Let us consider the configuration where X lies on $\overrightarrow{BC}$ and $Y \in \overline{CD}$, the remaining cases are treated analogously. Let I be incenter of $\triangle ABX.$ Since B, I, X, K are concyclic and $XI \parallel AL,$ it follows that $\angle AKB = \angle IXB = \angle DAL.$ But since $\angle ADL = \angle KBA,$ then $\triangle ADL \sim \triangle KBA.$ Hence $\frac{DL}{AB} = \frac{AD}{BK} \Longrightarrow \frac{DL}{BC} = \frac{BC}{BK}.$

Since $\angle LDC = \angle CBK = 90^\circ - \frac{1}{2}\angle ADC,$ we deduce that $\triangle DLC \sim \triangle BCK.$ Then $\angle BCK = \angle DLC$ implies

$$\angle KCL = 360^\circ - \angle BCK - \angle LCD - \angle DCB = \angle LDC + \angle ADC = \angle ADL$$

$$\Longrightarrow \angle KCL = 90^\circ + \frac{1}{2}\angle ADC = \text{const}.$$

Quick Reply

High School Olympiads

Geometry problem No:5 (Is this hard problem?)

Reply

krenkovr
#1 Sep 22, 2010, 4:16 am

[geogebra]01973927ce31f45d909ab1ed8cdff0430b6bc55d[/geogebra]

Let AD, BE, CF are altitude in triangle ABC. If R, R1, R2, R3 are radius of circle described in triangles ABC, AEF, BFD, CED and r are radius of circle inscribed in triangle DEF then proof that R1^2+R2^2+R3^2=Rx(R-r) (see picture).

krenkovr
#2 Sep 25, 2010, 5:22 pm

> krenkovr wrote:
> [geogebra]01973927ce31f45d909ab1ed8cdff0430b6bc55d[/geogebra]
>
> Let AD, BE, CF are altitude in triangle ABC. If R, R1, R2, R3 are radius of circle described in triangles ABC, AEF, BFD, CED and r are radius of circle inscribed in triangle DEF then proof that R1^2+R2^2+R3^2=Rx(R-r) (see picture).

I still wait solution.

krenkovr
#3 Oct 1, 2010, 2:33 am

Can somebody send me shortly solution. I have long solution.

Solving
#4 Oct 1, 2010, 3:34 am

Post ur solution, i think that yetti have the beautiful solution, as always... =)

Luis González
#5 Oct 1, 2010, 4:41 am

For convenience, denote the inradius of $\triangle DEF$ as r_0. Let H, N, O be the orthocenter, 9-point center and circumcenter of $\triangle ABC$. Assuming that $\triangle ABC$ is acute, H, N become incenter and circumcenter of $\triangle DEF$. Then, by Euler theorem

$$HN^2=\left(\frac{R}{2}\right)^2-2\left(\frac{R\cdot r_0}{2}\right)=\tfrac{1}{4}R^2-Rr_0$$

Using this result for $P\equiv H \Longrightarrow \tfrac{1}{4}(HA^2+HB^2+HC^2)=HN^2+\tfrac{3}{4}R^2$

$$\tfrac{1}{4}(HA^2+HB^2+HC^2)={R_1}^2+{R_2}^2+{R_3}^2=\tfrac{1}{4}R^2-Rr_0+\tfrac{3}{4}R^2=R(R-r_0)$$

krenkovr
#6 Oct 1, 2010, 9:07 pm

> Solving wrote:

Post ur solution, i think that yetti have the beautiful solution, as always... =)

This is my solution.

Let A_1, B_1, C_1 are point of altitude lie on page BC, AC, AB. For area of triangle ABC (S) we have $(a*AA_1)/2+(b*BB_1)/2+(c*CC_1)/2=3S$. Now we have $(a*AH+a*HA_1)/2+(b*BH+b*HB_1)/2+(c*CH+c*HC_1)/2=3S$, $(a*2R_1+a*HA_1)/2+(b*2R_2+b*HB_1)/2+(c*2R_3+c*HC_1)/2=3S$, $a*R_1+b*R_2+c*R_3+S=3S$ follow $(a*R_1+b*R_2+c*R_3)/2S=1$ ...(1). Let $[AC_1B_1]=S_1, [C_1BA_1]=S_2, [A_1B_1C]=S_3$. From similarity on triangle $AC_1B_1, A_1BC_1, A_1B_1C$ with triangle ABC get $S_1=(R_1^2/R^2)*S$ ...(2), $S_2=(R_2^2/R^2)*S$ ...(3), $S_3=(R_3^2/R^2)*S$ ...(4). Analogy have $B_1C_1=(a/R)*R_1$ ...(5), $C_1A_1=(b/R)*R_2$ ...(6), $A_1B_1=(c/R)*R_3$ ...(7) Use the equation (5),(6),(7) and (1) getting $[A_1B_1C_1]=[(B_1C_1+C_1A_1+A_1B_1)/2]*r=(S/R)*r$ ...(8). From equation $S_1+S_2+S_3+[A_1B_1C_]=S$ using equation (2),(3),(4) and (8) getting $R_1^2+R_2^2+R_3^2+R*r=R^2$ from where follow solution.

Please see site http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=368008 and send solution

Quick Reply

High School Olympiads

Prove that MB=MT

Reply

Goutham

#1 Sep 30, 2010, 12:42 am

$(FRA4)$ A right-angled triangle OAB has its right angle at the point B. An arbitrary circle with center on the line OB is tangent to the line OA. Let AT be the tangent to the circle different from OA (T is the point of tangency). Prove that the median from B of the triangle OAB intersects AT at a point M such that $MB = MT$.

Luis González

#2 Sep 30, 2010, 10:56 am

For convenience rename $O \equiv C$ and denote the subject circle as (U). Let P be the tangency point of (U) with AC and D the midpoint of AC. Since $\angle APU, \angle ATU$ and $\angle ABU$ are right, then A, B, T, U, P are concyclic on a circle ω. Let N be the second intersection of BD with ω. Since $\triangle BDA$ is isosceles with apex D, it follows that $BPNA$ is an isosceles trapezoid with legs $AN = BU \Longrightarrow BN = AP$, but $AP = AT$. Thus, $BTNA$ is an isosceles trapezoid with legs $BA = TN \Longrightarrow$ $\triangle MBT$ is isosceles with apex M.

kalantzis

#3 Oct 1, 2010, 9:16 pm

I have a simplier solution only with angle chase :)
Again, using luisgeometria's notation,we get that A, B, T, U, P are concyclic.
Using this (the next are angles):
$MBT = MBU + UBT = DBC + UAT = DCB + UAP = (90 - BAC) + CAU = 90 - BAU$
and $MTB = ATB = AUB = 90 - BAU$
QED

dgreenb801

#4 Oct 3, 2010, 1:57 am

I have another solution.

Like before, $ABTUP$ is cyclic, and the center of the circumscribed circle is the midpoint R of AU. Since $\triangle BDA$ is isoceles, R lies on the bisector of $\angle BDA$. Also, R lies on the bisector of $\angle PAT$. So R is the incenter of $\triangle MDA$, so RM bisects $\angle TMB$, and since R is the center of the circle, $MT = MB$.

Quick Reply

High School Olympiads

Identity between Brocard angle and median angle

Reply

Goutham

#1 Sep 30, 2010, 12:23 am

$(BUL4)$ Let M be the point inside the right-angled triangle $ABC(\angle C = 90^\circ)$ such that $\angle MAB = \angle MBC = \angle MCA = \phi$. Let Ψ be the acute angle between the medians of AC and BC. Prove that $\frac{\sin(\phi+\Psi)}{\sin(\phi-\Psi)} = 5.$

Luis González

#2 Sep 30, 2010, 4:56 am • 2

RHS should be -5 as LHS is defined. Rename $\angle MAB = \angle MBC = \angle MCA = \omega$ the Brocard angle of $\triangle ABC$ and let G be the centroid of $\triangle ABC$. Tangent of the Brocard angle can be found using the fact that a median and a symmedian issuing from two vertices of any triangle intersect on a Brocard ray issuing from the third vertex.

$$\tan\omega = \frac{4[\triangle ABC]}{AB^2+BC^2+CA^2} = \frac{4[\triangle ABC]}{AB^2+AB^2} = \frac{2[\triangle ABC]}{AB^2}$$

$$\cot\Psi = -\frac{GA^2+GB^2-AB^2}{4[\triangle GAB]} = -\frac{3(\frac{2}{3}AB^2-GC^2-AB^2)}{4[\triangle ABC]} = \frac{AB^2}{3[\triangle ABC]}$$

$$\frac{\sin(\omega+\Psi)}{\sin(\omega-\Psi)} = \frac{\sin\omega\cos\Psi+\cos\omega\sin\Psi}{\sin\omega\cos\Psi-\cos\omega\sin\Psi} = \frac{\tan\omega+\tan\Psi}{\tan\omega-\tan\Psi} = \frac{2+3}{2-3} = -5.$$

Quick Reply

High School Olympiads

Line passes through fixed point on normal

Reply

Goutham

#1 Sep 29, 2010, 11:58 pm

$(BEL4)$ Let O be a point on a nondegenerate conic. A right angle with vertex O intersects the conic at points A and B. Prove that the line AB passes through a fixed point located on the normal to the conic through the point O.

Luis González

#2 Sep 30, 2010, 1:01 am

Label the given conic $\mathcal{H}$. Let τ be the tangent line to $\mathcal{H}$ through O and let ω be an arbitrary fixed circle tangent to τ through O. Point O is center of a homology $\mathcal{U} : \omega \mapsto \mathcal{H}$. Let OA, OB cut ω again at A', B', i.e. the homologous points of A, B under homology $\mathcal{U}$. $\angle A'OB' \equiv \angle AOB = 90^\circ \Longrightarrow$ Lines $A'B'$ always pass through the fixed center P of ω, thus AB always passes through the image P' of P under $\mathcal{U}$. Consequently, chords AB pass through a fixed point on the normal line to $\mathcal{H}$ through O.

Quick Reply

High School Olympiads

Prove all conics are hyperbolas and find locus of centers.

Reply

Goutham

#1 Sep 29, 2010, 11:59 pm

$(BEL5)$ Let G be the centroid of the triangle OAB.
(a) Prove that all conics passing through the points O, A, B, G are hyperbolas.
(b) Find the locus of the centers of these hyperbolas.

Luis González

#2 Sep 30, 2010, 12:50 am

Rename $C \equiv O$. Isogonal conjugation with respect to $\triangle ABC$ takes its circumcircle ω into the line at infinity, takes centroid G into the symmedian point K of $\triangle ABC$ and circumconics $\mathcal{K} \equiv ABCG$ into a pencil ℓ of lines passing though K. Since K is inside ω, then each ℓ cut ω at two points $\Longrightarrow$ each $\mathcal{K}$ has two points at infinite, i.e $\mathcal{K}$ are all hyperbolas.

According to About Nine-point conic, locus of centers of $\mathcal{K}$ is the conic passing through midpoints of BC, CA, AB and midpoints of GA, GB, GB, i.e. the Steiner inellipse of ABC.

Quick Reply

High School Olympiads

Indonesia National Science Olympiad 2010 - Day 1 Problem 2

Reply

wangsacl

#1 Sep 28, 2010, 11:14 am

Given an acute triangle ABC with $AC > BC$ and the circumcenter of triangle ABC is O. The altitude of triangle ABC from C intersects AB and the circumcircle at D and E, respectively. A line which passed through O which is parallel to AB intersects AC at F. Show that the line CO, the line which passed through F and perpendicular to AC, and the line which passed through E and parallel with DO are concurrent.

Fajar Yuliawan, Bandung

Luis González

#2 Sep 28, 2010, 10:08 pm

Denote R the intersection of CO with the perpendicular line to AC through F. Then it is enough to show that $ER \parallel DO$. Let the ray CO cut AB and the circumcircle (O) at P, Q, respectively. Since $QA \perp CA$, it follows that $\triangle CFR$ and $\triangle CAQ$ are homothetic, and their cevians FO and AP are homologous $\Longrightarrow \frac{CP}{PQ} = \frac{CO}{OR}\ (\ast)$. Because of $\angle ACQ = \angle BCE$, it follows that $EQ \parallel AB$. Hence, by Thales theorem $\frac{CP}{PQ} = \frac{CD}{DE}$, together with $(\ast)$ we obtain $\frac{CO}{OR} = \frac{CD}{DE} \Longrightarrow ER \parallel DO$.

jayme

#3 Sep 29, 2010, 11:35 am

Dear Mathlinkers,
following the notation of Luis,
1. let P the point of intersection of the parallel to B passing through O with CED.
2. According to the Desargues' week theorem applied to the homothetic triangles FRB and AQD (center C), RP // QD.
3. According to the little Pappus theorem applied to the hexagone REQDOPR, RE//DO.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Indonesia National Science Olympiad 2010 - Day 2 Problem 8

Reply

wangsacl
#1 Sep 28, 2010, 11:17 am • 1

Given an acute triangle ABC with circumcenter O and orthocenter H. Let K be a point inside ABC which is not O nor H. Point L and M are located outside the triangle ABC such that $AKCL$ and $AKBM$ are parallelogram. At last, let BL and CM intersects at N, and let J be the midpoint of HK. Show that $KONJ$ is also a parallelogram.

Raja Oktovin, Pekanbaru

Luis González
#2 Sep 28, 2010, 12:26 pm • 1

Let E,F be the midpoints of sides $CA,AB.$ It is clear that $ML \parallel EF \parallel BC$ and $ML=BC=\frac{1}{2}EF.$ Segment $\overline{BC}$ is the image of segment $\overline{EF}$ under the composition $\mathcal{G}$ of the homotheties $\mathcal{K}(K,2) \circ \mathcal{N}(N,-1).$ Coefficent of $\mathcal{G}$ equals -2 and its center (centroid G) lies on the line $KN,$ such that $\overline{GN}:\overline{GK}=-1:2.$ Since $\overline{GO}:\overline{GH}=-1:2,$ it follows that $ON \parallel HK$ and $ON=\frac{1}{2}HK=JK \Longrightarrow KONJ$ is a parallelogram.

sunken rock
#3 Feb 24, 2014, 1:23 pm • 1

Let D be midpoint of BC; $AH \parallel OD, AH=2OD$. N is the midpoint of diagonals BL, CM, so $ND \parallel BM \parallel AK, AK=BM=2ND$, so $\triangle ODN \sim \triangle AHK$ and their similitude ratio is $\frac{1}{2}$, resulting $KH \parallel ON, 2KJ=KH=2ON$, done.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Concurrence in a triangle (own).

Reply

Virgil Nicula

#1 Sep 25, 2010, 3:55 pm

Let ABC be triangle for which $b\ne c$. Denote the midpoint D of $[BC]$ and construct $E\in (BD)$, $F\in (DC)$ so that $\frac {EB}{ED}=\frac {FD}{FC}=\frac {AB}{AC}$. For a mobile point $M\in AD$ define $U\in BM\cap AE$, $V\in CM\cap AF$. Prove that UV pass through a fixed point $S\in BC$ for which $\frac {SE}{SF}=\frac {AB}{AC}$ and $\frac {SB}{SC}=\left(\frac {AB}{AC}\right)^2$, i.e. the sideline BC , the exterior A-angle bisector of $\triangle\ EAF$ and the tangent line in A to the circumcircle of $\triangle\ ABC$ are concurrently.

Luis González

#2 Sep 28, 2010, 5:00 am

$$\frac{MV}{VC}=\frac{AM}{AC} \cdot \frac{\sin \widehat{DAF}}{\sin \widehat{CAF}}=\frac{AM}{AC} \cdot \frac{DF}{FC} \cdot \frac{AC}{AD}=\frac{AM}{AD} \cdot \frac{AB}{AC}$$

$$\frac{BU}{UM}=\frac{AB}{AM} \cdot \frac{\sin \widehat{BAE}}{\sin \widehat{DAE}}=\frac{AB}{AM} \cdot \frac{BE}{ED} \cdot \frac{AD}{AB}=\frac{AD}{AM} \cdot \frac{AB}{AC}$$

Let $S \equiv UV \cap BC.$ By Menelaus' theorem for $\triangle BMC$ cut by $\overline{SUV},$ we get

$$\frac{SB}{SC}=\frac{BU}{UM} \cdot \frac{MV}{VC}=\frac{AD}{AM} \cdot \frac{AB}{AC} \cdot \frac{AM}{AD} \cdot \frac{AB}{AC}=\left (\frac{AB}{AC} \right)^2$$

By Menelaus' theorem for $\triangle ADF$ and $\triangle ADE$ cut by $\overline{CVM}$ and $\overline{BUM},$ we obtain

$$\frac{AV}{VF}=\frac{CD}{CF} \cdot \frac{MA}{DM} \ , \ \frac{AU}{UE}=\frac{BD}{BE} \cdot \frac{MA}{DM}$$

Thus, by Meneleaus' theorem for $\triangle AEF$ cut by $\overline{SUV},$ we get

$$\frac{SE}{SF}=\frac{AV}{VF} \cdot \frac{UE}{AU}=\frac{CD}{CF} \cdot \frac{MA}{DM} \cdot \frac{BE}{BD} \cdot \frac{DM}{MA}=\frac{AB \cdot BD}{AB+AC} \cdot \frac{AB+AC}{AC \cdot DC}=\frac{AB}{AC}$$

Quick Reply

High School Olympiads

An interestig perpendicularity (own)

Reply

jayme
#1 Sep 27, 2010, 4:31 pm

Dear Mathlinkers,
Let ABC be a triangle, (1) the incircle, I the center of (1), D the contact poibnt of (1) with BC,
(1a) the circle passing trough B, C ant tangent to (1) at T,
U the second intersection point of AT with (1), and A' the midpoint of the arch BC which contains T.
Prove that A'I is perpendicular to DU.
Sincerely
Jean-Louis

vntbqpqh234
#2 Sep 27, 2010, 6:01 pm

I don't understand that(2) pass B,C then don't tangent with (1)

jayme
#3 Sep 27, 2010, 6:55 pm

Dear Mathlinkers,
Let ABC be a triangle, (1) the incircle, I the center of (1), D the contact point of (1) with BC,
(1a) the circle passing trough B, C and tangent to (1) at T :

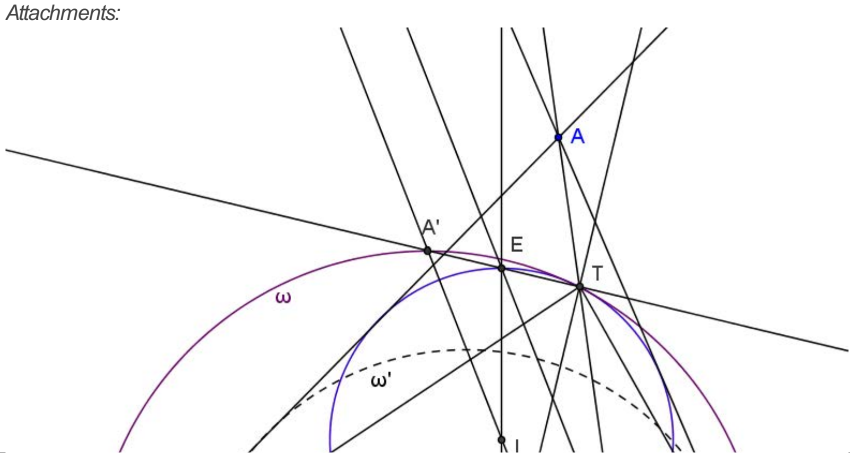
(1) and (1a) are internally tangent at a point noted T.

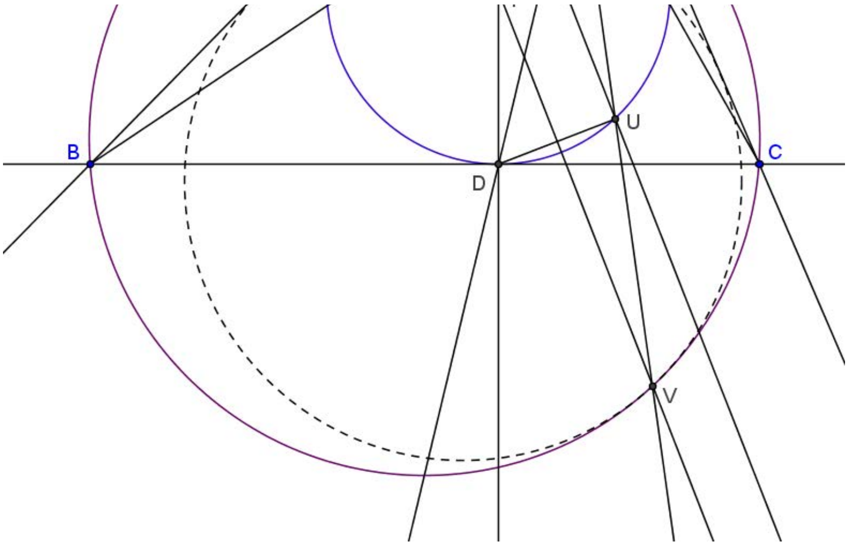
Sincerely
Jean-Louis

Luis González
#4 Sep 28, 2010, 1:54 am • 1

Rename ω the circle passing through B, C and tangent to incircle (I) through T. Let ω' be the circle tangent to lines AB, AC and internally tangent to ω through V. Hence, A, T, V are the exsimilicenters of $(I) \sim \omega'$, $(I) \sim \omega$ and $\omega \sim \omega'$, respectively. By Monge & d'alembert theorem, A, T, V are collinear. From the *Remarquable résultat de Vladimir Protassov*, it follows that VI bisects $\angle BVC \Longrightarrow V, I, A'$ are collinear. On the other hand, let E be the antipode of D WRT the incircle (I). Since TD, TA' are the internal and external bisectors of $\angle BTC$, together with $\angle ETD = 90^\circ$, it follows that $E \in TA'$. By Reim's theorem for $(I), \omega$ cut by the two secants TEA' and TUV, it follows that $EU \parallel A'V$, but $EU \perp DU \Longrightarrow A'IV \perp DU$.

Attachments:





Quick Reply

High School Olympiads

Problem 5, Iberoamerican Olympiad 2010

Reply

Concyclicboy
#1 Sep 25, 2010, 11:54 pm

Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD are perpendicular. Let O be the circumcenter of $ABCD$, K the intersection of the diagonals, $L \neq O$ the intersection of the circles circumscribed to OAC and OBD, and G the intersection of the diagonals of the quadrilateral whose vertices are the midpoints of the sides of $ABCD$. Prove that O, K, L and G are collinear

Luis González
#2 Sep 26, 2010, 1:25 am • 1

Let P, Q, R, S be the midpoints AB, BC, CD, DA and U the orthogonal projection of K on segment CD. Then we have $\angle UKD = \angle KCD = \angle KBP = \angle PKB \Longrightarrow P, K, U$ are collinear, i.e. $PK \perp CD$. Likewise, we'll have $RK \perp AB$. Since $OR \perp CD$ and $OP \perp AB$, it follows that $PORK$ is a parallelogram $\Longrightarrow PR$ passes through the midpoint of OK. Similarly, SQ passes through the midpoint of OK. Hence, G is the midpoint of segment OK. Now, BD is the radical axis of $(O), \odot(OBD)$ and AC is the radical axis of $(O), \odot(OAC) \Longrightarrow OGK$ is the radical axis of $\odot(OBD), \odot(OAC)$. Thus, 2nd intersection L of $\odot(OBD)$ and $\odot(OAC)$ lies on the line OGK.

Jorge Miranda
#3 Sep 26, 2010, 7:11 pm

Solution

Joao Pedro Santos
#4 Sep 27, 2010, 6:07 am

A nice generalization of the "hard" part of the problem:

> Quote:
> Let $ABCD$ be a quadrilateral whose diagonals AC and BD are perpendicular. Let O be the intersection of the perpendicular bisectors of AC and BD and let G be the intersection of the diagonals of the quadrilaterals whose vertices are the midpoints of the sides of $ABCD$. Prove that G is the midpoint of OK.

Solution

daniel73
#5 Sep 27, 2010, 7:52 pm

> Joao Pedro Santos wrote:
> Let $ABCD$ be a quadrilateral whose diagonals AC and BD are perpendicular. Let O be the intersection of the perpendicular bisectors of AC and BD and let G be the intersection of the diagonals of the quadrilaterals whose vertices are the midpoints of the sides of $ABCD$. Prove that G is the midpoint of OK.

I solved the "easy" part of the problem (ie, $K \in OL$) saying that OL is clearly the radical axis of the circumcirles of OAC and OBD, and K is on that radical axis since $KA \cdot KC = KB \cdot KD$ because $ABCD$ is cyclic. Note that we do not need the diagonals of $ABCD$ to be perpendicular for this part, but if K is on the radical axis, then necessarily $KA \cdot KC = KB \cdot KD$, ie $K \in OL$ iff $ABCD$ is cyclic, regardless of the angle between the diagonals.

I propose a different proof (no coordinates) for the "improved" version of the "hard" part of the problem (ie G is the midpoint of

I propose a different proof (no coordinates) for the "improved" version of the "hard" part of the problem (ie G is the midpoint of OK). Denote M the midpoint of OK, and T the midpoint of AB. Clearly $KT = TA = TB = \frac{AB}{2}$ is the circumradius of right-angled triangle AKB, and denote by r the circumradius of $ABCD$. Applying the median theorem twice,

$$MT^2 = \frac{OT^2 + KT^2}{2} - \frac{OK^2}{4} = \frac{OA^2 + OB^2}{4} - \frac{AB^2}{8} + \frac{AB^2}{8} - \frac{OK^2}{4} = \frac{r^2}{2} - \frac{OK^2}{4}.$$

By cyclic symmetry, the result is the same for the midpoints U, V, W of BC, CD, DA, or $MT = MU = MV = MW$. But $GT = GU = GV = GW$, since the midpoints of the sides of $ABCD$ form a parallelogram whose sides are parallel to diagonals AC, BD, which are in turn perpendicular, ie, $TUVW$ is a rectangle and its diagonals meet at point G which is at the same distance from all vertices.

JG

#6 Jan 22, 2011, 9:55 am

Since $ABCD$ is cyclic $AK \cdot KC = BK \cdot KD$ and since K is inside the circumcircles of AOC and BOD we get that the power of K with respect to them is the same, this implies the collinearity of O, K and L. Now it's really easy to prove that the perpendicularity of the diagonals of $ABCD$ implies that its anticenter is K. Now, the center, barycenter and anticenter of a cyclic quadrilateral are collinear. Done.

Quick Reply

High School Olympiads

Locus Of Intersection

Reply

Headhunter

#1 Sep 24, 2010, 10:03 am

Hello.

The line l is tangent to the circle S at the point A.
B and C are two points on l on opposite sides of A.
The other tangents from B, C to S intersect at a point P.
B, C move along l in such a way that $|AB| \cdot |AC|$ is constant.
Find the locus of P.

I solved it in an elementary way, but want to know how to solve it in inversion.

Luis González

#2 Sep 25, 2010, 5:51 am

Let us rename the point $A \equiv P$ and vice-versa, in order to use the common ABC-triangle notation. Thus $PB \cdot PC = (s-a)(s-b) = k^2$. Let r be the radius of S. Then

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \Longrightarrow r^2 = \frac{(s-a)k^2}{s} \Longrightarrow \frac{s}{a} = \frac{k^2}{k^2-r^2}$$

Let h_a be the length of the altitude issuing from vertex A. Then we have

$$a \cdot h_a = 2r \cdot s \Longrightarrow h_a = \frac{2r \cdot s}{a} = \frac{2r \cdot k^2}{k^2-r^2} = \text{const}.$$

Locus of A is a parallel line ℓ' to ℓ in the half-plane of S such that $\text{dist}(\ell, \ell') = \dfrac{2r \cdot k^2}{k^2-r^2}$

Quick Reply

High School Olympiads

bisector of XAY - Iran NMO 2004 (Second Round) - Problem5

Reply

sororak
#1 Sep 24, 2010, 11:22 pm

The interior bisector of $\angle A$ from $\triangle ABC$ intersects the side BC and the circumcircle of ΔABC at D, M, respectively. Let ω be a circle with center M and radius MB. A line passing through D, intersects ω at X, Y. Prove that AD bisects $\angle XAY$.

Luis González
#2 Sep 25, 2010, 12:58 am

It's well-known that the circle ω with center M and radius $MB = MC$ goes through the incenter I and A- excenter I_a of $\triangle ABC$. Since $(A, D, I, I_a) = -1$, it follows that external bisector ℓ_a of $\angle BAC$ is the polar of D with respect to ω. If XY cuts ℓ_a at Z, then cross ratio (Z, D, X, Y) is harmonic. Together with $DA \perp AZ$, we deduce that AD, ℓ_a are the internal and external bisector of $\angle XAY$.

jgnr
#3 Sep 26, 2010, 6:24 pm

Let AE and AF be tangents to ω (E and F are the tangency points). It is easy to see that D, E, F are collinear. From here, triangles AEF, ABC, and AXY share the same incenter, hence AD bisects $\angle XAY$.

math_genie
#4 Sep 28, 2010, 8:34 pm

Let G_1 be the circumcircle of $\triangle ABC$.

Let G_2 be the circumcircle of $\triangle XYM$.

Since $\angle XYM = \angle YXM$ as $MX = MY$.

Thus it suffices to prove that the 4 points A, Y, M, X are concyclic.
(as then $\angle XAD = \angle MAY$)

If A, Y, M, X are not concyclic, then suppose G_1 and G_2 intersect at Q.
(Q not equal to A)

Now, by radical axis theorem, MQ, BC, XY are concurrent which cannot be true as MA, BC, XY are concurrent.
Hence $Q = A$ and A, Y, M, X are concyclic and we're done.

jayme
#5 Sep 28, 2010, 10:22 pm

Dear Mathlinkers,
1. According to Monge's theorem, A, M, X and Y are concyclic
2. MX=MY and we are done...
Sincerely
Jean-Louis

huynguyen
#6 Aug 9, 2015, 9:08 am • 1

Sorry for reviving, but today I found another solution to this

Sorry for reviving, but today I found another solution to this.
Let R be the radius of (M).
Construct the tangent lines of (M) passing through A, called AE, AF (E, F lie on (O)).
Triangle MBD, MAB are congruent ($\widehat{BMD} = \widehat{AMB}, \widehat{MBD} = \widehat{MAC} = \widehat{MAB}$)
$\Rightarrow \dfrac{MD}{MB} = \dfrac{MB}{MA}$
$\Rightarrow MD.MA = MB^2 = R^2$
$\Rightarrow D$ is the image of A under the inversion (M, R^2).
$\Rightarrow D$ lies on EF and D is the midpoint of EF.
So now it remains to prove that AX, AY cut EF at two points that is symmetry to each other wrt D. (1)
Let AY cuts (M) at a second point G.
GD cuts (M) at a second point H.
Let AY, AX cuts EF at U, V respectively.By Butterfly's theorem, we conclude that U, V are symmetric wrt (D)
We have DU is the angle bisector of $\widehat{GDI}$ (well-known) and since DA is perpendicular to DU, we conclude that (A, U, G, Y)=-1.
$\Rightarrow D(A, U, G, Y) = -1$
$\Rightarrow D(A, V, X, H) = -1$
$\Rightarrow A, V, X, H$ are collinear.
So (1) has been proved.The proof is completed.
Q,E,D

leeky

#7 Aug 9, 2015, 12:55 pm

D lies on the radical axis of the 2 circles, $DA \times DM = DB \times DC = DX \times DY$ so A, Y, M, X concyclic, since $MX = MY$ we are done.

High School Olympiads

Show that sum of areas is equal

Reply

Amir Hossein

#1 Sep 14, 2010, 5:17 pm

Let $\Gamma_i, i = 0, 1, 2, \dots$, be a circle of radius r_i inscribed in an angle of measure 2α such that each Γ_i is externally tangent to Γ_{i+1} and $r_{i+1} < r_i$. Show that the sum of the areas of the circles Γ_i is equal to the area of a circle of radius $r = \frac{1}{2} r_0(\sqrt{\sin\alpha} + \sqrt{\csc\alpha})$.

Luis González

#2 Sep 24, 2010, 11:55 pm

Let a, b be the sides of the given angle with vertex P and denote by X_i the tangency point of $\Gamma_i(O_i, r_i)$ with a. From the similar triangles $\triangle POX_0 \sim \triangle PO_1X_1$ we get

$$\frac{O_1X_1}{O_0X_0} = \frac{r_1}{r_0} = \frac{PO_1}{PO_0} = \frac{PO_0 - O_0O_1}{PO_0} = \frac{PO_0 - r_0 - r_1}{PO_0}$$

Plugging $PO_0 = r_0 \cdot \csc\alpha$ yields $\frac{r_1}{r_0} = \frac{\csc\alpha - 1}{\csc\alpha + 1} = \frac{1 - \sin\alpha}{1 + \sin\alpha}$

Radii $r_0, r_1, r_2, r_3 \dots$ form a decreasing geometric progression with ratio $k = \frac{1 - \sin\alpha}{1 + \sin\alpha} \Longrightarrow$ Areas S_i of Γ_i form another decreasing geometric progession with ratio k^2.

$$\sum_{i=0}^{\infty} S_i = \frac{S_0}{1 - k^2} = \frac{\pi {r_0}^2}{1 - \left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^2} = \left(\frac{1 + \sin\alpha}{2\sqrt{\sin\alpha}}\right)^2 \cdot \pi {r_0}^2$$

Let ϱ be the radius of the circle equivalent to the chain of circles Γ_i. Then

$$\pi\varrho^2 = \left(\frac{1 + \sin\alpha}{2\sqrt{\sin\alpha}}\right)^2 \cdot \pi {r_0}^2 \Longrightarrow \varrho = \tfrac{1}{2} r_0(\sqrt{\sin\alpha} + \sqrt{\csc\alpha}).$$

Quick Reply

High School Olympiads

Geometry problem

Reply

LastPrime
#1 Sep 1, 2010, 8:39 pm

Circle O is the circumcircle of non-isosceles triangle ABC. The tangent lines to circle O at points B and C intersect at L_a, and the tangents at A and C intersect at L_b. The external angle bisectors of triangle ABC at B and C meet at I_a and the external bisectors at A and C intersect at I_b. Prove that lines L_aI_a, L_bI_b, and AB are concurrent.

Luis González
#2 Sep 24, 2010, 3:47 am

Let us use barycentric coordinates with respect to $\triangle ABC$. Then we have

$$L_a(-a^2:b^2:c^2)\ ,\ L_b(a^2:-b^2:c^2)\ ,\ I_a(-a:b:c)\ ,\ I_b(a:-b:c)$$

Therefore, equations of the straight lines L_aI_a and L_bI_b are given by

$$L_aI_a \equiv bc(b-c)x+ac(a-c)y+ab(b-a)z=0$$

$$L_bI_b \equiv bc(c-b)x+ac(c-a)y+ab(b-a)z=0$$

Lines L_aI_a, L_bI_b meet at the point $U\equiv (a(a-c):b(c-b):0) \Longrightarrow U\in AB.$

Goutham
#3 Sep 24, 2010, 7:53 am

> Luis González wrote:
> $L_a(-a^2:b^2:c^2)\ ,\ L_b(a^2:-b^2:c^2)$

Could you explain how you got this? I know that AL_a, BL_b are symmedians.

Luis González
#4 Sep 24, 2010, 9:26 pm

Points L_a, L_b, L_c are known as ex-symmedian points of $\triangle ABC$. They can be found as the intersections of the pairwise tangent lines ℓ_a, ℓ_b and ℓ_c of the circumcircle of $\triangle ABC$ through the vertices $A, B, C,$ respectively. Namely, $\ell_a \equiv c^2y+b^2z=0$ and cyclic permutations. Alternatively, if $K(a^2:b^2:c^2)$ is the symmedian point, we can use the fact that AB, AC, AK, ℓ_a is a harmonic bundle to find the equation of ℓ_a and similarly the equations of ℓ_b, ℓ_c.

Quick Reply

High School Olympiads

Find the locus of the point P

Amir Hossein

#1 Sep 22, 2010, 1:09 pm

Let a tetrahedron $ABCD$ be inscribed in a sphere S. Find the locus of points P inside the sphere S for which the equality

$$\frac{AP}{PA_1}+\frac{BP}{PB_1}+\frac{CP}{PC_1}+\frac{DP}{PD_1}=4$$

holds, where A_1, B_1, C_1, and D_1 are the intersection points of S with the lines AP, BP, CP, and DP, respectively.

Luis González

#2 Sep 24, 2010, 3:19 am

Let the circumsphere S have center O and radius R. Then we have

$$\frac{PA}{PA_1}+\frac{PB}{PB_1}+\frac{PC}{PC_1}+\frac{PD}{PD_1}=\frac{PA^2+PB^2+PC^2+PD^2}{p(P,S)}=4$$

$$\Longrightarrow PA^2+PB^2+PC^2+PD^2=4(R^2-PO^2) \ \ (1)$$

Let G be the centroid of $ABCD$ and a,b,c,d,e,f denote its edges. By Leibniz theorem for P,G and O,G we get

$$PA^2+PB^2+PC^2+PD^2=4PG^2+\tfrac{1}{4}(a^2+b^2+c^2+d^2+e^2+f^2) \ \ (2)$$

$$OA^2+OB^2+OC^2+OD^2=4OG^2+\tfrac{1}{4}(a^2+b^2+c^2+d^2+e^2+f^2) \ \ (3)$$

From (2) and (3) we get $PA^2+PB^2+PC^2+PD^2=4(PG^2+R^2-OG^2) \ \ (4)$

Combining (1) and (4) yields: $4(PG^2+R^2-OG^2)=4(R^2-PO^2)$

$$\Longrightarrow PG^2+PO^2=OG^2 \Longrightarrow \angle OPG=90^\circ$$

Therefore, locus of P is the spherical surface with diameter $\overline{OG}$.

High School Olympiads

XYZ is the polar of G WRT (O)

Reply

Luis González

#1 Sep 22, 2010, 2:40 am

G, O are the centroid and circumcenter of the scalene triangle $\triangle ABC$. D, E, F denote the midpoints of BC, CA, AB and U is the symmedian point of $\triangle DEF$. Lines AU, BU, CU cut BC, CA, AB at M, N, L and let M', N', L' be the reflections of M, N, L about D, E, F, respectively. Lines $N'L', L'M', M'N'$ cut BC, CA, AB at X, Y, Z. Show that X, Y, Z are collinear on the polar of G WRT circumcircle (O).

Quick Reply

High School Olympiads

Nine Point Center, Orthocenter

Headhunter

#1 Sep 14, 2010, 12:03 am • 1

Hello.

N=nine point center, R=radius of circumcircle, H=orthocenter
Show that $NA^2+NB^2+NC^2+NH^2=3R^2$
I have no solution.

Luis González

#2 Sep 18, 2010, 2:44 am • 4

The proposed relation can be generalized for any point P.

Proposition. $\triangle ABC$ is scalene with circumcircle (O,R), orthocenter H and 9-point center N. For any point P we have the following relation:

$$PA^2+PB^2+PC^2+PH^2=4PN^2+3R^2$$

Proof. Let G be the centroid of $\triangle ABC$. Since $NH=\frac{3}{4}HG$ and $NG=\frac{1}{4}HG$, by Stewart theorem for the cevian PN in $\triangle PGH$, we obtain the expression:

$4PN^2=3PG^2+PH^2-\frac{3}{4}HG^2 \ \ (1)$

By Leibniz theorem for P, G we obtain the relation

$PG^2=\frac{1}{3}(PA^2+PB^2+PC^2)-\frac{1}{9}(a^2+b^2+c^2)$

Substituting PG^2 from this latter expression into (1) yields

$4PN^2=PA^2+PB^2+PC^2-\frac{1}{3}(a^2+b^2+c^2)+PH^2-\frac{3}{4}HG^2 \ \ (2)$

By Leibniz theorem for O, G, keeping in mind that $HG=2GO$, we have

$GO^2=\frac{1}{3}(OA^2+OB^2+OC^2)-\frac{1}{9}(a^2+b^2+c^2)=R^2-\frac{1}{9}(a^2+b^2+c^2)$

$\Longrightarrow HG^2=4R^2-\frac{4}{9}(a^2+b^2+c^2)$

Substituting HG^2 from this latter expression into (2) yields

$4PN^2=PA^2+PB^2+PC^2+PH^2-3R^2.$

High School Olympiads

Euler line ratio problem

Reply

delegat

#1 Sep 15, 2010, 5:14 pm

Euler line of triangle $\triangle ABC$ intersects side AB at point D. Calculate $\frac{AD}{DB}$ as function of sidelengths. (it can be assumed that D lies between A and B)

castigioni

#2 Sep 17, 2010, 2:17 am

I found the relation :(a^2-b^2-c^2)(b^2-c^2)/((a^2-b^2+c^2)(a^2-c^2))

Luis González

#3 Sep 17, 2010, 2:59 am

Barycentric equation of Euler line e of $\triangle ABC$ in Conway's notation is given by:

$$S_A(S_B-S_C)x+S_B(S_C-S_A)y+S_C(S_A-S_B)z=0$$

Intersection $D \equiv AB \cap e$ has coordinates $D(S_B(S_C-S_A):S_A(S_C-S_B):0)$

$$\Longrightarrow \frac{\overline{DA}}{\overline{DB}}=-\frac{S_A(S_C-S_B)}{S_B(S_C-S_A)}=-\frac{(b^2+c^2-a^2)(b^2-c^2)}{(a^2+c^2-b^2)(a^2-c^2)}.$$

This post has been edited 1 time. Last edited by Luis González, Sep 18, 2010, 2:21 am

delegat

#4 Sep 17, 2010, 3:01 am

You mean that fraction is

$$\frac{\left(c^2+b^2-a^2\right)\left(c^2-b^2\right)}{\left(c^2+a^2-b^2\right)\left(c^2-a^2\right)}$$

Can you post your solution (or at least main ideas) ?

delegat

#5 Sep 17, 2010, 3:03 am

Oh I did not see your post luisg (you posted it while I was writing my previous post) Thanks

jayme

#6 Sep 17, 2010, 8:08 pm

Dear Luis and Mathlinkers,
an article concerning a triangle and a transversal with analog questions has been written on the Monthly (I think from some geometer of Princeton). But I have lost the reference...
Any idea?
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

International competition SRMC 2003 P-2

Reply

Ovchinnikov Denis
#1 Sep 8, 2010, 5:53 pm

Let $s=\frac{AB+BC+AC}{2}$ be half-perimeter of triangle ABC. Let L and Nbe a point's on ray's AB and CB, for which $AL=CN=s$. Let K is point, symmetric of point B by circumcenter of ABC. Prove, that perpendicular from K to NL passes through incenter of ABC.

Solution for problem here

Luis González
#2 Sep 9, 2010, 5:09 am

Let I_a, I_b, I_c be the excenters of $\triangle ABC$ against $A, B, C.$ Circumcircle (O) and incenter I of $\triangle ABC$ become 9-point circle and orthocenter of $\triangle I_aI_bI_c.$ If D, E denote the tangency points of the excircles $(I_a), (I_c)$ with $CB, BA,$ then $U \equiv DI_a \cap EI_c$ is the circumcenter of $\triangle I_aI_bI_c \Longrightarrow IOU$ is the Euler line of $\triangle I_aI_bI_c,$ thus $\overline{OU}=-\overline{OI} \Longrightarrow BUKI$ is a parallelogram with center $O,$ i.e. $KI \parallel BU \ (\ast).$ Since $\angle BDU=\angle BEU=90^\circ,$ it follows that circumcenter V of $\triangle BED$ is the midpoint of $\overline{UB}.$ But, it's clear that L, N are the tangency points of $(I_a), (I_c)$ with $AB, BC \Longrightarrow NL, ED$ are antiparallel WRT $BA, BC,$ since $NLDE$ is an isosceles trapezoid with legs $NL=ED.$ Thus, $BV \perp NL$ and together with $(\ast),$ it follows that $KI \perp NL.$

Quick Reply

High School Olympiads

A problem...in trouble

Reply

simon89889

#1 Sep 7, 2010, 10:12 pm

A convex quadrilateral $ABCD$ has $AD = CD$ and $\angle DAB = \angle ABC$.The line through D and the midpoint of BC intersects line AB in point E. Prove that $\angle BEC = \angle DAC$.

frenchy

#2 Sep 7, 2010, 11:39 pm

Let us note $\angle CAE = \phi$ $\angle DAC = \gamma$ $\angle ACB = \alpha$ and $\angle BCE = x$.It is easy to observe that $2 \cdot \phi + \gamma + \alpha = 180$ so our aim is to prove that $x = \phi$.We easily conclude that $\angle CBE = \phi + \alpha$ also $\angle DAE = \phi + \gamma$ and $\angle DCE = \gamma + \alpha + x$.It is well-known that if a point X is situated on the median line from B of $\triangle ABC$ then the next relation holds:$\frac{d(X,AB)}{d(X,BC)} = \frac{BC}{AB}$.Let us note the projection of D onto AE and EC as Q and P,we know that $AD = DC$ so we get that for point D the relation will be:$\frac{sin(\gamma+\alpha)}{sin(\gamma+\alpha+x)} = \frac{sin(\phi+\alpha)}{sinx}$.Taking into acount that $sin(\gamma+\phi) = sin(\phi+\alpha)$ we easily conclude that $\angle BCE = \phi$.
So we are done.

simon89889

#3 Sep 8, 2010, 7:18 pm

Excuse me

How could we get this equation: $\frac{sin(\gamma+\alpha)}{sin(\gamma+\alpha+x)} = \frac{sin(\phi+\alpha)}{sinx}$????

frenchy

#4 Sep 8, 2010, 8:05 pm

Sorry that i did not explain carefully,I was in rush.We heve the relation.$\frac{d(X,AB)}{d(X,BC)} = \frac{BC}{AB}$.Appyling the sine theorem in $\triangle DQA$ we get that $DQ = DA \cdot sin(\gamma+\phi)$ and applying the same theorem in $\triangle DCP$ we get that $DP = DC \cdot sin(\gamma+\alpha+x)$ because we know that $AD = DC$ we conclude that $\frac{d(X,AB)}{d(X,BC)} = \frac{sin(\gamma+\phi)}{sin(\gamma+\alpha+x)}$.Let us now pay attention to $\triangle BCE$,applying sine theorem we get $\frac{BC}{AB} = \frac{sin(\phi+\alpha)}{sinx}$ after combining both relations we get $\frac{sin(\gamma+\alpha)}{sin(\gamma+\alpha+x)} = \frac{sin(\phi+\alpha)}{sinx}$
So we are done.

simon89889

#5 Sep 8, 2010, 8:28 pm

It means D is on the median line from B to $\triangle ABC$??? Something weird,maybe I misunderstand...

Actually I try it with geometry tranformation at school all day = =.
But I don't know how to use the midpoint of BC....

frenchy

frenchy

#6 Sep 9, 2010, 2:28 am

Lemma:Let us consider $\triangle ABC$ let M be the middle point of AC if D is on BM then the next relation holds:$\dfrac{d(D,AB)}{d(D,BC)}=\dfrac{BC}{AB}$

Proof:

Let $\angle BAC=\phi$ and $\angle ACB=\gamma$.Draw a line through D which intersects AB and BC at X and Y respectively such that $XY \parallel CA$.We easily conclude that $\angle BXY=\phi$ and $\angle XYB=\gamma$ and D is the middle point of XY.Let us note the projections of D on XB and YB as P and Q respectively.Now we get that $PD=XD\cdot sin\phi$ and $DQ=DY\cdot sin\gamma$.So we get that $\dfrac{DP}{DQ}=\dfrac{sin\phi}{sin\gamma}$,from $\triangle ABC$ applying the sine theorem:$\dfrac{AB}{BC}=\dfrac{sin\gamma}{sin\phi}$ so we get that $\dfrac{d(D,AB)}{d(D,BC)}=\dfrac{BC}{AB}$

So we are done.

Luis González

#7 Sep 9, 2010, 3:45 am

Let T be the midpoint of BC and $P\equiv DA\cap CB.$ By Menelaus' theorem for $\triangle PTD$ cut by the transversal $\overline{ABE},$ keeping in mind that $\triangle PAB$ is isosceles due to $\angle DAB=\angle ABC,$ we get $\frac{ET}{ED}=\frac{TB}{AD}=\frac{TC}{DC}.$ Thus, by the converse of angle bisector theorem, CE is the external bisector of $\angle DCB \Longrightarrow \angle BCE=90^\circ-\frac{1}{2}\angle BCD.$ On the other hand, by angle chasing we get

$$\angle BEC=\angle DAB-(90^\circ-\tfrac{1}{2}\angle BCD)=\angle DAB-90^\circ+\tfrac{1}{2}\angle BCD$$

$$\angle DAC=90^\circ-\tfrac{1}{2}(360^\circ-2\angle DAB-\angle BCD)=\angle DAB-90^\circ+\tfrac{1}{2}\angle BCD$$

$$\Longrightarrow \angle DAC=\angle BEC.$$

Quick Reply

High School Olympiads

International competition SRMC 2004 P-3

Reply

Ovchinnikov Denis

#1 Sep 7, 2010, 11:56 pm • 1

In-circle of ABC with center I touch AB and AC at P and Q respectively. BI and CI intersect PQ at K and L respectively. Prove, that circumcircle of ILK touch incircle of ABC iff $|AB|+|AC|=3|BC|$.

Luis González

#2 Sep 8, 2010, 3:55 am • 1

Let (I) touch BC at $R.$ Since $\angle RIC+\angle LKI=\angle RIC+\angle ICR=90^{\circ},$ it follows that RI passes through the circumcenter of $\triangle LIK.$ Therefore, if $\odot(ILK)$ is tangent to (I) through $N,$ then N is the antipode of R WRT $(I).$

Let line CN cut (I) again at $T.$ Then we have $\angle PNT=\angle RNC+\angle RNP=\angle RNC+\angle NCR=90^{\circ}$. Hence, T is the antipode of P WRT the incircle $(I) \Longrightarrow N$ is the Nagel point of $\triangle ABC.$ Now, it is well-known that N lies on the small arc PQ of (I) if and only if $AB+AC=3BC.$

The inverse excercise is proved analogously. Condition $AB+AC=3BC$ implies that the antipode N of R WRT (I) coincides with the Nagel point of $\triangle ABC.$ Then $BI \cap NC$ coincides with K (orthogonal projection of C onto BI) $\Longrightarrow N$ is the antipode of I WRT the circle $\odot(ILK) \Longrightarrow \odot(ILK)$ is tangent to (I) through $N.$

mahanmath

#3 Sep 8, 2010, 4:45 am • 1

As I heard there are several results from condition $|AB|+|AC|=3|BC|$, Can someone help me with finding some pages about it ?

Luis González

#4 Sep 8, 2010, 5:26 am • 2

@mahanmath. These are the ones that I know, there must be additional results

1) $IG \perp BC,$ where I,G are the incenter and centroid of $\triangle ABC.$

2) Nagel point lies on the incircle $(I).$ Further, it's the antipode of $R \equiv (I) \cap BC.$

3) Mittenpunkt X_9 of $\triangle ABC$ lies on the A-sideline of its incentral triangle.

4) Circumcenter O is equidistant from I and the midpoint of $\overline{IA}.$

We have some metric and trigonometric relations such as:

$$AI^2=\frac{AB \cdot AC}{2}=4Rr \ , \ r=\frac{r_a}{2}=\frac{IB \cdot IC}{IA}$$

$$\cot \frac{A}{2}=\cot \frac{B}{2}+\cot \frac{C}{2} \ , \ \tan \frac{B}{2} \cdot \tan \frac{C}{2}=\frac{1}{2}$$

$$\sin \frac{A}{2}=\sin \frac{B}{2} \cdot \sin \frac{C}{2}=\frac{1}{3} \cos \left(\frac{B-C}{2}\right)=\sqrt{\frac{r}{4R}}$$

This post has been edited 1 time. Last edited by Luis González, Sep 8, 2010, 5:39 am

mahanmath

#5 Sep 8, 2010, 5:39 am • 1

Thanks a lot !

Quick Reply

High School Olympiads

Collinear Points

Reply

Headhunter
#1 Sep 7, 2010, 1:50 am

Hello.

$\triangle DEF$ is the tangential triangle of $\triangle ABC$.
On the sides of $\triangle DEF$, take two equal segments AG, BH (A-G-E,B-H-F)
Circle($\triangle ACG$) meet Circle($\triangle ABH$) at Q. Circle mean circumcircle.
Show that A,Q,D are collinear.

Luis González
#2 Sep 7, 2010, 9:53 am

Let M be the second intersection of $\odot(ACG)$ with line DE. Since $\triangle EAC$ is isosceles with apex E, it follows that $AGMC$ is an isosceles trapezoid with $GM \parallel AC \Longrightarrow AG = CM$. Since $DC = DB$, then we deduce that $DM = DH$. Therefore, $\overline{DC} \cdot \overline{DM} = \overline{DB} \cdot \overline{DH} \Longrightarrow D$ has equal power with respect to circles $\odot(ACG)$ and $\odot(ABH) \Longrightarrow D$ lies on the radical axis AQ of $\odot(ACG)$ and $\odot(ABH)$.

Quick Reply

High School Olympiads

Passing Fixed Points

Reply

Headhunter
#1 Sep 7, 2010, 1:35 am

Hello.

Two circles (A), (B) are outside each other.
There are infinitely many circles orthogonal to both of (A) and (B).
Show the infinitely many circles pass two fixed points.

basketball9
#2 Sep 7, 2010, 2:48 am

What do you mean by a circle is orthogonal to another circle?Thanks

Jason27603
#3 Sep 7, 2010, 2:52 am

If two circles A, B intersect at P, Q, they are orthogonal if and only if AP and AQ are tangent to circle B and BP and BQ are tangent to circle A.

basketball9
#4 Sep 7, 2010, 2:56 am

Oh ok thanks.

Luis González
#5 Sep 7, 2010, 6:56 am

Let τ be the radical axis of $(A),(B)$ and let (P,ϱ) represent the orthogonal pencil of $(A),(B)$. Circle (P) cuts (A) and (B) at (A_1,A_2) and (B_1,B_2), respectively. Since $PA_1=PB_1=\varrho$, it follows that P has equal power with respect to $(A),(B)$ $\Longrightarrow$ centers P of the orthogonal pencil (P) lie on τ. In other words, radical axis τ of $(A),(B)$ becomes the center line of the pencil (P) and the other way around.

Since the circle $\mathcal{M}$ centered on the intersection $U\equiv \ell \cap \tau$ and orthogonal to $(A),(B)$ belows to the pencil (P), then all circles (P) must pass through the intersections M,N of $\mathcal{M}$ with their radical axis ℓ. These points M,N are known as the limiting points of the pencil $(A),(B)$.

Quick Reply

High School Olympiads

An interesting own extension of problem from ... t=206229.

Virgil Nicula

#1 Sep 6, 2010, 11:48 pm

Let AS be the A-symmedian in the triangle ABC , where $S\in (BC)$. Denote the midpoints M , N of $[AC]$, $[AB]$ respectively and

$D\in BM\cap AS$, $E\in CN\cap AS$. Prove that $m\left(\widehat{ACD}\right)=m\left(\widehat{ABE}\right)=\phi$, where $\tan\phi=\frac{4S}{a^2+b^2+c^2}$ and $\phi\le 30^{\circ}$.

See here.

Luis González

#2 Sep 7, 2010, 1:09 am

It follows from the fact that, in any triangle a median and a symmedian issuing from two different vertices intersect on a Brocard ray issuing from the the third vertex. The proof is quite straightforward using barycentric coordinates. Therefore, in this case, ray CD passes through the first Brocard point Ω_1 and ray BE passes through the 2nd Brocard point $\Omega_2 \Longrightarrow \angle ACD=\angle ABE$ is the Brocard angle ω of $\triangle ABC.$ Hence

$$\cot \omega=\frac{a^2+b^2+c^2}{4[\triangle ABC]}=\cot A+\cot B+\cot C \Longrightarrow \ \omega \le 30^{\circ}.$$

Virgil Nicula

#3 Sep 7, 2010, 8:07 am

> Quote:
> **Generalization 1.** Let AS be the A-cevian in the triangle ABC , where $S\in (BC)$. Denote the midpoints M , N of the sides $[AC]$, $[AB]$ respectively and $D\in BM\cap AS$, $E\in CN\cap AS$. Prove that $m\left(\widehat{ACD}\right)=m\left(\widehat{ABE}\right)\iff$ the cevian AS is the A-symmedian and in this case $\tan\phi=\frac{4S}{a^2+b^2+c^2}$, where $\phi\le 30^{\circ}$.

Proof. Let $\begin{array}{c}G\in AB\cap CD\\ F\in AC\cap BE\end{array}$. Since D , E belong to the medians BM , CN respectively obtain that $SG\parallel AC$ and $SF\parallel AB$, i.e. $AGSF$ is a parallelogram.

$\boxed{\Longleftarrow}$ Suppose that S is the foot of the A-symmedian. Using the point (1) from the lower lemma obtain GF is antiparallel to $BC\implies GBFC$ is cyclically $\implies \widehat{GBF}\equiv\widehat{GCF}\implies\widehat{ABE}\equiv\widehat{ACD}=\phi$. Observe that $\frac{c^2}{b^2}=\frac{SB}{SC}=\frac{FA}{FC}=$ $\frac{BA}{BC}\cdot\frac{\sin\widehat{FBA}}{\sin\widehat{FBC}}=\frac{c\cdot\sin\phi}{a\cdot\sin(B-\phi)}\implies ac\cdot\sin(B-\phi)=b^2\cdot\sin\phi\implies ac\cdot\sin B-ac\cdot\cos B\tan\phi=$ $b^2\cdot\tan\phi\implies\tan\phi=\frac{ac\cdot\sin B}{ac\cdot\cos B+b^2}\implies\boxed{\tan\phi=\frac{4S}{a^2+b^2+c^2}}$. Using the remarkable inequality $a^2+b^2+c^2\ge 4S\sqrt 3$ obtain that and $\phi\le 30^{\circ}$.

$\boxed{\Longleftrightarrow}$ Denote $m(\widehat{ABE})=\phi$, $m(\widehat{ACD})=\psi$ and $\frac{SB}{SC}=m$. Observe that

$\left\|\ m=\frac{FA}{FC}=\frac{c}{a}\cdot\frac{\sin\phi}{\sin(B-\phi)}\implies\tan\phi=\frac{ma\sin B}{c+ma\cos B}\ \right\|$. Therefore, $\phi=\psi\iff\tan\phi=\tan\psi\iff$

$\| \; m = \frac{GB}{GA} = \frac{a}{b} \cdot \frac{\sin(C-\psi)}{\sin\psi} \implies \tan\psi = \frac{a\sin C}{mb + a\cos C} \; \|$

$b\sin B \cdot \underline{m}^2 + a\sin(B-C) \cdot \underline{m} - c\sin C = 0$. Observe that (-1) verifies the previous equation in m because $b\sin B + a\sin(B-C) - c\sin C = 0 \iff \sin^2 B - \sin^2 C = \sin A \sin(B-C)$, what is truly.

In conclusion, $m = \dfrac{c\sin C}{b\sin B}$, i.e. $m = \dfrac{c^2}{b^2} \iff$ the point S is the foot of the A-symmedian.

▶ **Lemma (well-known).** Consider in $\triangle ABC$ the points $M \in (BC)$, $N \in (CA)$ and $P \in (AB)$. Find the point M for which :

$(1).\ MN \parallel AB\ ,\ MP \parallel AC \implies NP$ is antiparallel to BC. **Answer** : M is the foot of A-symmedian.
$(2).\ MN$ is antiparallel to AB , MP is antiparallel to $AC \implies NP \parallel BC$. **Answer** : M is the foot of A-symmedian.
$(3).\ MN$ is antiparallel to AB , MP is antiparallel to $AC \implies NP$ is antiparallel to BC. **Answer** : M is the foot of A-altitude.

Note. Say that in $\triangle ABC$ the line MN is antiparallel to $AB \iff M \in CA$, $N \in CB$ and $AMNB$ is cyclically.

> Quote:
>
> **Generalization 2.** Consider the points $S \in (BC)$ and $M \in (CA)$, $N \in (AB)$ so that $\frac{MA}{MC} \cdot \frac{NA}{NC} = 1$. Denote $D \in BM \cap AS$, $E \in CN \cap AS$. Prove that $m\left(\widehat{ACD}\right) = m\left(\widehat{ABE}\right) \iff \frac{SB}{SC} = \left(\frac{AB}{AC}\right)^2 \cdot \frac{MA}{MC}$.

Quick Reply

High School Olympiads

XY passes through the Gergonne poin of triangle ABC.

Reply

Source: !!

ma 29

#1 Dec 13, 2009, 11:35 pm

Problem1 :Let ABC be a triangle and let D,E,F be the points of tangency of the incircle (I) of ABC with the sides BC,CA,AB , respectively.Let A',B',C' be three points on the rays ID,IE,IF respectively such that $IA'=IB'=IC'$.Denote by M,N,P the midpoints of EF,DF,DE.Prove that AA',BB',CC' concur at the one point X ; $A'M,B'N,C'P$ concur at the one point Y and the lineXY passes through the **Gergonne** poin of triangleABC.

Reference
A topic of Linh :http://www.mathlinks.ro/viewtopic.php?t=315748

Luis González

#2 Sep 5, 2010, 10:22 pm

Denote $\frac{ID}{IA'}=\frac{IE}{IB'}=\frac{IF}{IC'}=k.$ Then lines AA',BB',CC' concur at X due to Kariya's theorem. Nevertheless, we'll dispense with this theorem to show that XY passes through the Gergonne point G_e of $\triangle ABC.$ Since $\triangle DEF$ and $\triangle A'B'C'$ are centrally similar with similarity coefficient $k,$ then it follows that $\triangle MNP$ and $\triangle A'B'C'$ are centrally similar with similarity coefficient $-\frac{k}{2}\Longrightarrow Y\equiv MA'\cap NB'\cap PC'$ is the similarity center of $\triangle A'B'C'$ and $\triangle MNP.$ Let AA' cut the line connecting Y and the Gergonne point G_e at $X,$ we'll show that BB',CC' pass through the same point $X.$ We have then

$$\frac{\overline{XY}}{\overline{XG_e}}=-\frac{AM}{MA'}\cdot\frac{YA'}{AG_e}\cdot\frac{\sin\widehat{IAA'}}{\sin\widehat{DAA'}}=-\frac{IA'}{A'D}\cdot\frac{AD}{AI}\cdot\frac{AM}{MA'}\cdot\frac{YA'}{AG_e}$$

$$\Longrightarrow\frac{\overline{XY}}{\overline{XG_e}}=-\frac{2}{(k-1)(k+2)}\cdot\frac{AM}{AI}\cdot\frac{AD}{AG_e}$$

Then, substituting : $\frac{AM}{AI}=\frac{s(s-a)}{bc}\ ,\ \frac{AD}{AG_e}=\frac{(s-b)(s-c)+a(s-a)}{a(s-a)}$

$$\frac{\overline{XY}}{\overline{XG_e}}=\frac{2}{(1-k)(k+2)}\cdot\frac{s[(s-b)(s-c)+a(s-a)]}{abc}$$

$$\Longrightarrow\frac{\overline{XY}}{\overline{XG_e}}=\frac{1}{(1-k)(k+2)}\cdot\left(2+\frac{r}{2R}\right)$$

Since this latter expressions is obviously independent of the chosen vertex, it follows that BB',CC' intersect YG_e at the same point $X,$ and the proof is completed.

Luis González

#3 Sep 6, 2010, 11:54 pm

There's a remarkable corollary coming from this result. If A',B',C' coincide with the points at infinity of $ID,IE,IF,$ then X,Y coincide with the orthocenter H of $\triangle ABC$ and 9-point center U of the intouch triangle $\triangle DEF,$ moreover $k=0.$ Therefore

In any scalene triangle $\triangle ABC,$ its orthocenter $H,$ Gergonne point G_e and the nine point center U of its intouch triangle $\triangle DEF$ are collinear such that $\frac{\overline{HU}}{\overline{HG_e}}=1+\frac{r}{4R}.$

Quick Reply

High School Olympiads

IMO LongList 1987-B'C' intersects incircle in two points

Reply

Amir Hossein

#1 Sep 6, 2010, 1:45 pm

The bisectors of the angles B, C of a triangle ABC intersect the opposite sides in B', C' respectively. Prove that the straight line $B'C'$ intersects the inscribed circle in two different points.

Attachments:

Luis González

#2 Sep 6, 2010, 9:22 pm • 1

Let X, Y, Z be the tangency points of the incircle (I) with BC, CA, AB. Let the tangent line from C' to (I), different from AB, touch the incircle (I) at M. Since $MZ \perp CI \perp XY \Longrightarrow ZM \parallel XY$, thus MZ passes through the reflection U of X about the midpoint of YZ, i.e. Polar of C' WRT (I) goes through U. Similarly, polar of B' WRT (I) goes through $U \Longrightarrow U$ is the pole of $B'C'$ WRT (I). Since $\angle ZXY = \angle ZUY = 90^\circ - \frac{1}{2}\angle BAC$ is indeed acute, then U lies outside $(I) \Longrightarrow$ Polar $B'C'$ of U WRT (I) cuts (I) at two points.

Quick Reply

High School Olympiads

Kazakhstan NMO 2009 11 grade P 2

Reply

Ovchinnikov Denis

#1 Sep 5, 2010, 11:32 pm

In triangle ABC $AA_1; BB_1; CC_1$-altitudes. Let I_1 and I_2 be in-centers of triangles AC_1B_1 and CA_1B_1 respectively. Let in-circle of ABC touch AC in B_2.
Prove, that quadrilateral $I_1I_2B_1B_2$ inscribed in a circle.

Luis González

#2 Sep 6, 2010, 1:52 am

The key is to show that B_2 is equidistant from I_1, I_2. For instance, a step of this proof (see post #2) shows that B_2 lies on the perpendicular bisector of segment $I_1I_2 \Longrightarrow B_2I_1 = B_2I_2$. Since line $AC \equiv B_1B_2$ is the external bisector of $\angle I_1B_1I_2$, it follows that B_2 lies on $\odot(I_1I_2B_1)$ and it's the midpoint of the arc $I_1B_1I_2$.

Quick Reply

High School Olympiads

Whirling Right Angle

Reply

Headhunter
#1 Sep 5, 2010, 8:00 pm

Hello.

A right angle is rotated about a fixed vertex (inside a circle). The legs of the right angle meet the circle at two points. Draw two tangent lines to the circle at the two points. Find the locus of the intersection of the two tangent lines.

Luis González
#2 Sep 5, 2010, 8:34 pm

Label the fixed circle and the fixed point $(O,R), P.$ Points A,B lie on (O) such that $\angle APB=90^\circ$ and tangents to (O) through A,B meet at $Q.$ PA,PB cut (O) again at $C,D.$ By Brahmagupta's theorem for the cyclic quadrilateral $ABCD$ with perpendicular diagonals $AC,BD,$ it follows that midpoints of AB,BC,CD,DA lie on a fixed circle ω whose center is the midpoint of $PO.$ Let M denote the midpoint of $AB.$ Since $\overline{OM}\cdot\overline{OQ}=R^2 \Longrightarrow Q$ moves on the inverse circle of ω under the inversion WRT $(O).$

77ant
#3 Sep 6, 2010, 12:18 am

Dear Luis
Seeing your so beautiful proof, I can't believe my eyes.

Quick Reply

High School Olympiads

Kazakhstan NMO 2010 11 grade 5P

Reply

Ovchinnikov Denis

#1 Sep 5, 2010, 2:29 am

Let O be the circumcircle of acute triangle ABC, AD-altitude of ABC ($D \in BC$), $AD \cap CO = E$, M-midpoint of AE, F-feet of perpendicular from C to AO.
Proved that point of intersection OM and BC lies on circumcircle of triangle BOF

This post has been edited 4 times. Last edited by Ovchinnikov Denis, Sep 5, 2010, 3:31 pm

Luis González

#2 Sep 5, 2010, 6:18 am

U_{∞} is the infinity point of the line AD, L is the midpoint of BC and let $P \equiv OM \cap BC, Q \equiv AO \cap BC.$ Then $O(A, E, M, U_{\infty}) \equiv O(Q, C, P, L) = -1.$ Therefore, P, F and the orthogonal projection of Q onto OC are collinear $\Longrightarrow$ Lines PF, BC are antiparallel with respect to lines $OA, OC.$ Therefore:

$$\angle FPQ = \angle OQC - \angle OCB = (\angle B + 90^\circ - \angle C) - (90^\circ - \angle A) = 180^\circ - 2\angle C$$

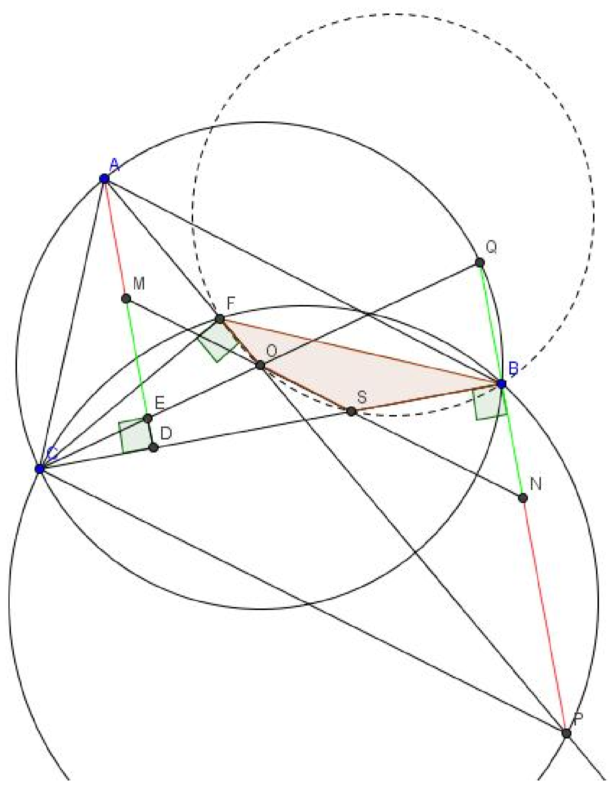
$$\angle FPQ = \angle BOF = 180^\circ - 2\angle C \implies P \in \odot(BOF)$$

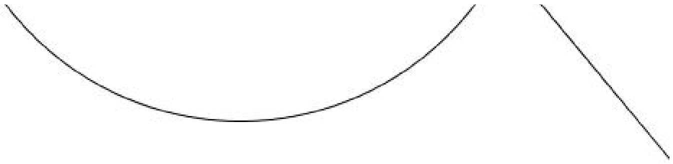
skytin

#3 Sep 5, 2010, 2:51 pm

Let line l is going from point B and l is perpendicular to CB, let AO intersect l at point P and line CO intersect l at point Q easy tose that CO = OQ and Q is on circle arround ABC. AE is parallel to QP so if N is midpoint of QP than easy to see that O is on line MN. Let line MN intersect CB at point S. QN = NP and QO =OC so easy to see that ON is parallel to CP , so angle OPC = SOP and CFBP is cyclic so angle FBC = FPC = SOP so FBSO is cyclic

Attachments:





littletush

#4 Oct 9, 2011, 10:48 am

let the expansion of CF intersect circle O at T.then $CF = FT$.
notiice that □BTC□□OEM,then what follows is trivial.

jayme

#5 Dec 14, 2012, 7:53 pm

Dear Littletush and Mathlinkers,
very nice idea... which can lead to more interesting results...
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Intouch triangle and trigonometric identity.

Reply

Goutham

#1 Sep 4, 2010, 11:36 pm

Given triangle ABC. The incircle Γ tangents BC, CA, AB in points P, Q, R respectively. O is the circumcentre of ABC and I is the incentre of PQR. PI intersects Γ at M. F is a point on segment QR and E is a point on the arc MR such that $\angle QPF = \angle RPE < \frac{1}{2}\angle QPR$. We have G to be the midpoint of IF. If MG and IE intersect at O, prove that $\cos A + \cos B + \cos C = \sqrt{2}$

Luis González

#2 Sep 5, 2010, 2:23 am

According to IMO 2010 problem 2, the intersection $MG \cap IE$ lies on the circumcircle of $\triangle PQR \Longrightarrow O \in \Gamma.$ Then, from Euler's theorem $r^2 = R^2 - 2Rr \Longrightarrow R = r(1+\sqrt{2}).$

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R} = 1 + \frac{1}{1+\sqrt{2}} = \sqrt{2}.$$

Quick Reply

High School Olympiads

circles and lines

Reply

duck1606

#1 Sep 4, 2010, 4:12 pm

Given a triangle ABC . A line d cut AB,BC,CA at R,P,Q. A point O (not in d). OA,OB,OC cut d at X,Y,Z. Prove that three circles (OPX),(OQY),(ORZ) have the common point (they all through a point M).

livetolove212

#3 Sep 4, 2010, 9:12 pm

Dear duck1606,
This is the generalization of this problem :
Given a triangle ABC. A line d cut AB, BC, CA at R, P, Q. Let O and I be arbitrary points in the plane (not in d). OA, OB, OC cut d at X, Y, Z. Prove that three circles $(IPX), (IQY), (IRZ)$ have the second common point.

Luis González

#4 Sep 5, 2010, 1:46 am

> livetolove212 wrote:
> Given a triangle ABC. A line d cut AB, BC, CA at R, P, Q. Let O and I be arbitrary points in the plane (not in d). OA, OB, OC cut d at X, Y, Z. Prove that three circles $(IPX), (IQY), (IRZ)$ have the second common point.

The reasoning is exactly the same, since we have $\frac{\overline{ZP} \cdot \overline{ZX}}{\overline{ZQ} \cdot \overline{ZY}} = \frac{\overline{RP} \cdot \overline{RX}}{\overline{RQ} \cdot \overline{RY}} \Longrightarrow$ For any point I not lying on d, circles $\odot(IPX), \odot(IQY)$ and $\odot(IRZ)$ are coaxal.

Quick Reply

High School Olympiads

R_a + R_b + R_c >= PA + PB + PC

Mateescu Constantin

#1 Aug 31, 2010, 4:51 am

Let P be an interior point of $\triangle\ ABC$. Denote R_a , R_b , R_c the circumradii of the triangles

PBC , PCA and PAB respectively. Prove that : $R_a+R_b+R_c\ge PA+PB+PC$.

Luis González

#2 Sep 4, 2010, 9:38 am • 2

Let M,N,L be the midpoints of $PA,PB,PC.$ Perpendicular lines to PA,PB,PC through M,N,L pairwise meet at the circumcenters X,Y,Z of $\triangle PBC, \triangle PCA$ and $\triangle PAB.$ By Erdős-Mordell inequality for $\triangle XYZ \cup P$ we get

$$PX+PY+PZ=R_a+R_b+R_c \ge 2(PM+PN+PL)=PA+PB+PC$$

High School Olympiads

Locus is equilateral hyperbola - OIMU 2005 Problem 4

Reply

Jorge Miranda
#1 Sep 4, 2010, 12:21 am

A variable tangent t to the circle C_1, of radius r_1, intersects the circle C_2, of radius r_2 in A and B. The tangents to C_2 through A and B intersect in P.
Find, as a function of r_1 and r_2, the distance between the centers of C_1 and C_2 such that the locus of P when t varies is contained in an equilateral hyperbola.

Note: A hyperbola is said to be *equilateral* if its asymptotes are perpendicular.

Luis González
#2 Sep 4, 2010, 2:27 am

Locus of midpoints M of the chords AB is the pedal curve of $\mathcal{C}_1$ with respect to O_2, i.e. a Pascal's limaçon $\mathcal{L}$ with cusp O_2 embracing $\mathcal{C}_1$. Since $\overline{O_2M} \cdot \overline{O_2P} = (r_2)^2$, it follows that locus of points P is the inverse of $\mathcal{L}$ under the inversion with respect to $\mathcal{C}_2 \Longrightarrow P$ moves on a conic $\mathcal{H}$ with focus O_2 and focal axis O_1O_2. $\mathcal{H}$ becomes an equilateral hyperbola, iff its eccentricity e equals $\sqrt{2}$. By taking O_2 as the pole of the polar coordinate system (r, θ) and O_2O_1 as the polar axis $\theta = 0^\circ$, we have $O_1 \equiv (d, 0)$. Thereby, equations of $\mathcal{L}$ and $\mathcal{H}$ are:

$$r = r_1 + d\cos\theta \ \wedge \ r = \frac{(r_2)^2}{r_1 + d\cos\theta} \Longrightarrow e = \sqrt{2} = \frac{d}{r_1}$$

Quick Reply

High School Olympiads

Sequence of circles and inequality - ILL 1992 THA1

Reply

Amir Hossein

#1 Sep 2, 2010, 11:42 pm

Let two circles A and B with unequal radii r and R, respectively, be tangent internally at the point A_0. If there exists a sequence of distinct circles (C_n) such that each circle is tangent to both A and B, and each circle C_{n+1} touches circle C_n at the point A_n, prove that

$$\sum_{n=1}^{\infty} |A_{n+1}A_n| < \frac{4\pi Rr}{R+r}.$$

Luis González

#2 Sep 3, 2010, 11:49 am

Let M, N be the antipodes of A_0 in $\mathcal{A}$ and $\mathcal{B}$, respectively. The inversion through pole A_0 with power $\overline{A_0M} \cdot \overline{A_0N} = 4R \cdot r$ transforms $\mathcal{A}$ and $\mathcal{B}$ into two lines a, b perpendicular to A_0N through N, M, respectively and the Pappus chain C_n into a chain of congruent circles U_n tangent to each other and tangent to the parallels a, b. Hence, contact points of U_n lie on the midline ℓ of $a, b \Longrightarrow$ Points A_n are concyclic on a circle $\mathcal{C}$ with radius ϱ, i.e. the inverse image of ℓ. If L is the midpoint of MN, we have

$$\overline{A_0M} \cdot \overline{A_0N} = 4R \cdot r = 2\varrho \cdot A_0L = 2\varrho \cdot (R+r) \Longrightarrow \varrho = \frac{2Rr}{R+r}$$

Since $A_0A_1A_2...A_n$ is inscribed in $\mathcal{C}$, then its perimeter $\sum_{n=1}^{\infty} |A_nA_{n+1}|$ is less than $\mathcal{C}$'s

$$\Longrightarrow 2\pi \cdot \varrho = \frac{4\pi Rr}{R+r} > \sum_{n=1}^{\infty} |A_nA_{n+1}|$$

Quick Reply

High School Olympiads

Geometry - sum of sides

Reply

Ramchandran

#1 Aug 31, 2010, 6:13 pm

ABC is an equilateral triangle with P in the interior. Another triangle XYZ with sides equal to PA,PB,PC has an interior point M such that angles XMY , YMZ ,ZMX are all 120 , Prove that - XM+YM+ZM = AB

Ramchandran

#2 Sep 2, 2010, 9:21 am

any ideas ?
thanks in advance.

Luis González

#3 Sep 2, 2010, 11:09 am

Let X,Y,Z be the points on BC,CA,AB such that $PX \parallel CA, PY \parallel AB$ and $PZ \parallel BC.$ Then quadrilaterals $PXCY, PYAZ$ and $PZBX$ are isosceles trapezoids with legs $XC = PY, YA = PZ$ and $ZB = PX.$ Therefore, $PA = YZ, PB = ZX$ and $PC = XY \Longrightarrow$ Side-segments of $\triangle XYZ$ equal $PA, PB, PC.$ Furthermore, we have $\angle YPZ = \angle ZPX = \angle XPY = 120^\circ \Longrightarrow P \equiv M.$ Let $U \equiv PZ \cap AC.$ Since $\triangle PYU$ is equilateral and $PUCX$ is a parallelogram, it follows that $PY = YU, PX = UC$ and $PZ = YA \Longrightarrow AB = AC = YA + YU + UC \Longrightarrow PX + PY + PZ = AB \Longrightarrow MX + MY + MZ = AB.$

sunken rock

#4 Feb 10, 2012, 2:33 pm

Imagine the problem this way:
Take 2 identical sheets of paper of equilateral triangle shape perfectly overlapping, their common vertices being A, B, C, pick them at a point P and rotate the one on top by 60° about P; Call D, E the new positions of A, B respectively, see that $DE \parallel BC, DE = BC$, while triangles $\triangle APD, \triangle BPE$ are equilateral.
Easily to see, $\triangle APE$ has its sides equal to AP, BP, CP, hence it is our $\triangle XYZ$.
Let $\{\ F\ \} \in AB \cap DE$; as constructed, $\widehat{BFE} = 60^\circ$, hence $BPFE$ is cyclic; applying Ptolemy one gets $EF + FP = BF$ and, easily, $\widehat{PFE} = 120^\circ$. Further, with $AF + EF + PF = AF + BF = AB$ and $\widehat{AFE} = 120^\circ$, $F \equiv M$ we are done.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Turkey TST 2010 Q5

Reply

crazyfehmy
#1 Sep 1, 2010, 5:33 pm

For an interior point D of a triangle $ABC,$ let Γ_D denote the circle passing through the points $A, \ E, \ D, \ F$ if these points are concyclic where $BD \cap AC = \{E\}$ and $CD \cap AB = \{F\}.$ Show that all circles Γ_D pass through a second common point different from A as D varies.

jayme
#2 Sep 1, 2010, 8:14 pm

Dear Mathlinkers,
the problem seems to me not clear. But, I can be wrong...
All the circles pass through A and D.???
Sincerely
Jean-Louis

crazyfehmy
#3 Sep 1, 2010, 8:52 pm

> *jayme wrote:*
> Dear Mathlinkers,
> the problem seems to me not clear. But, I can be wrong...
> All the circles pass through A and D.???
> Sincerely
> Jean-Louis

Dear Jean-Louis
The problem says that D is not a constant point, it varies, we have to find a constant point different from $A.$

shoki
#4 Sep 1, 2010, 9:21 pm

hint

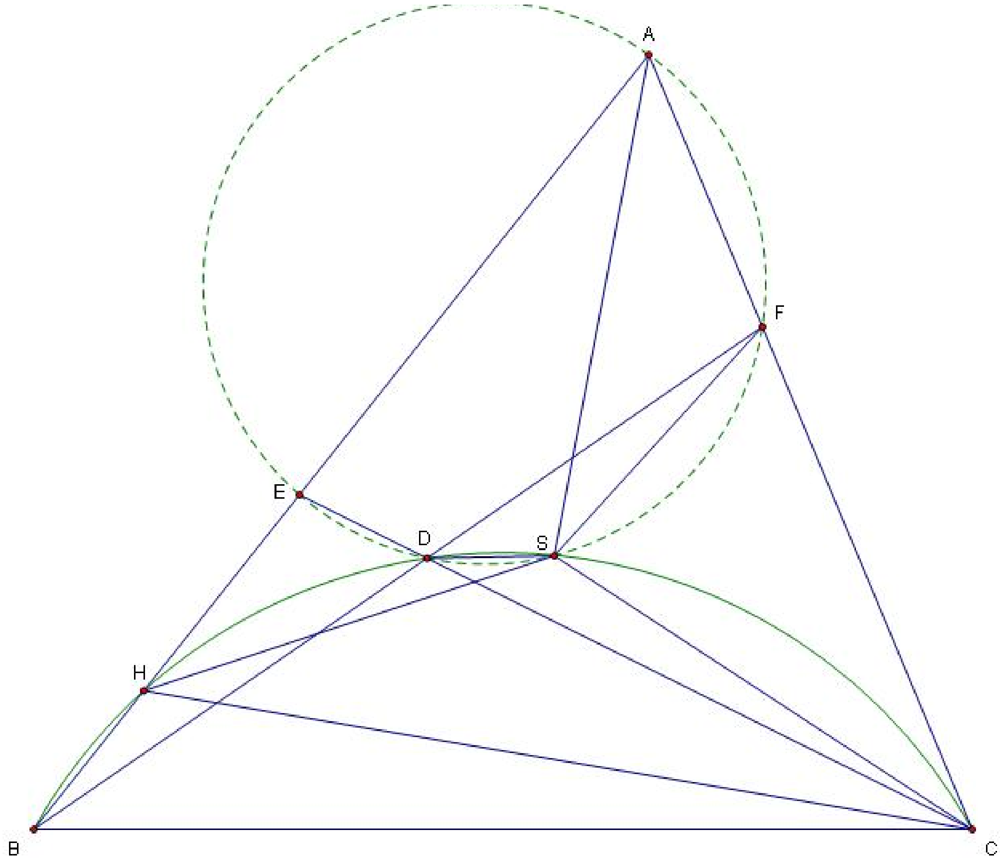
tuanh208
#5 Sep 1, 2010, 11:56 pm

I have a solution for this problem.

First we will construct S be constant point such that $S \in \Gamma_D$.
Let $H \in AB$ such that $CA = CH$ and let w be the circle pass through A, H and tangent to CA, CH.
$w \cap (B, H, C) = S(S \neq H)$.

Now we will prove that $ADSF$ is cyclic.
We have $\angle BHC = 180^0 - \angle AHC = 180^0 - \angle BAC = \angle BDC$ so $BHDC$ is cyclic but $BHSC$ is cyclic hence $BHDS$ is cyclic.
Thus $\angle SDF = 180^0 - \angle BDS = 180^0 - \angle BHS = \angle SHA = \angle SAF$ so $ADSF$ is cyclic and the problem is done 🙂

Attachments:



Luis González

#6 Sep 2, 2010, 12:11 am

$B \equiv AF \cap ED$ and $C \equiv AE \cap FD$ are conjugate points WRT Γ_D. Then it follows that circle (M) with diameter BC is orthogonal to Γ_D. Thus, inversion through pole A with power equal to the power of A WRT (M) takes (M) into itself and Γ_D into a straight line γ orthogonal to (M), due to the conformity $\Longrightarrow M \in \gamma$. Hence, Γ_D passes through the image of M under the referred inversion, i.e. the orthogonal projection of the orthocenter of $\triangle ABC$ on AM.

motal

#7 Sep 2, 2010, 7:42 am

> *Luis González wrote:*
> $B \equiv AF \cap ED$ and $C \equiv AE \cap FD$ are conjugate points WRT Γ_D

What do you mean by 'conjugate points with respect to a circle' and why does it imply orthogonality?

cnyd

#8 Oct 21, 2010, 6:20 pm

> *shoki wrote:*
> hint

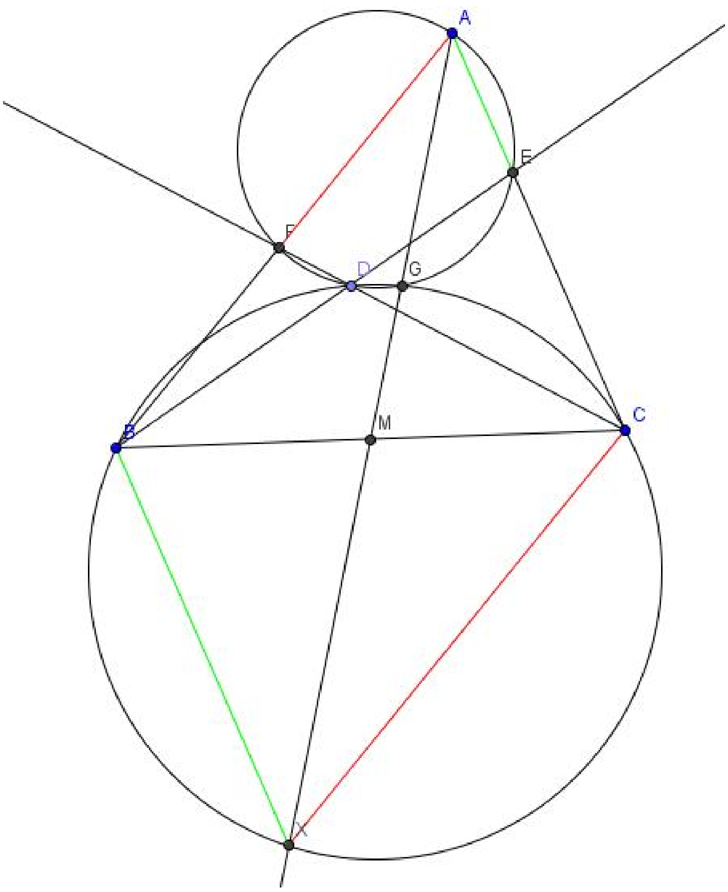
Shoki ,could you explain better ?

skytin

#9 Oct 21, 2010, 10:40 pm

Let circle (AFE) intersect (BDC) at D and G and line AG intersect (BDC) at X easy to see that XB || AE and XC || AF , let AX intersect BC at point M easy to see that M is midpoint of BC and (BDC) is constant , so G is fixed

Attachments:



shoki

#10 Nov 1, 2010, 7:18 pm

> Quote:
>
> Let H be the orthocenter.Let H_b, H_c be the feet of perpendiculars from B, C to AC, AB. the problem is equivalent to prove that $\frac{H_bE}{H_cF}$ is constant.

@cnyd, see here

War-Hammer

#11 May 6, 2013, 12:54 am

Hi ;

This Point Is Also On AM Where M Is The Midpoint Of BC Such That If We Connect It To I Then We Have IK Is Perpendicular To AM

Best Regard

robinpark

#12 May 6, 2013, 4:07 am • 1

We use barycentric coordinates. Let $BC = a, CA = b, AB = c$, and let $A = (1,0,0), B = (0,1,0), C = (0,0,1)$.

Let $D = (p:q:r)$. We claim that Γ_D always passes through $P(a^2 : b^2 + c^2 - a^2 : b^2 + c^2 - a^2)$, which is independent of D. Since E and F are the traces of D onto CA and AB, we have $E = (p:0:r)$ and $F = (p:q:0)$. We are given that A, E, D, F are concyclic; that is, the equations of the circumcircles of AED and AFD are equal.

$$\odot AED : a^2yz + b^2zx + c^2xy - (x+y+z)\left(\frac{a^2r(p+r) - b^2rp + c^2p(p+r)}{(p+q+r)(p+r)}y + \frac{b^2p}{p+r}z\right) = 0$$

$$\odot AFD : a^2yz + b^2zx + c^2xy - (x+y+z)\left(\frac{c^2p}{\ldots}y + \frac{a^2q(p+q) + b^2p(p+q) - c^2pq}{\ldots}z\right) = 0$$

$$\odot AFD : a^2yz + b^2zx + c^2xy - (x+y+z)(\frac{\quad}{p+q}y + \frac{\quad}{(p+q+r)(p+q)}z) = 0$$

Therefore $(p+q)(a^2r(p+r) - b^2rp + c^2p(p+r)) = (p+q+r)(p+r)c^2p$, or $c^2p(p+r) = (p+q)(a^2(p+r) - b^2p)$. Similarly, $b^2p(p+q+r)(p+q) = (p+r)(a^2q(p+q) + b^2p(p+q) - c^2pq)$, or $b^2p(p+q) = (p+r)(a^2(p+q) - c^2p)$. Multiplying the two yields

$$a^2(p+r)(p+q) = b^2p(p+q) + c^2p(p+r)$$

Solving for $c^2p(p+r)$ and plugging this into the equation of the circumcircle of AED yields

$$a^2yz + b^2zx + c^2xy - (x+y+z)(a^2y + \frac{b^2p}{p+r}(z-y))) = 0$$

Substituting P into this equation yields

$$a^2(b^2+c^2-a^2)(b^2+c^2-a^2) + b^2(b^2+c^2-a^2)(a^2) + c^2(a^2)(b^2+c^2-a^2) - (-a^2+2b^2+2c^2)(a^2(b^2+c^2-a^2)) = 0,$$

which is true. Therefore the circumcircle of $AEDF$ always passes through P, as claimed.

This was a lot more manageable than I'd thought. I didn't have to use Mathematica to expand the last expression, since the assumed condition $y = z$ allowed me to find the x and y-coordinates of P in the first place.

proglote

#13 May 7, 2013, 9:39 am

> *Luis González wrote:*
> $B \equiv AF \cap ED$ and $C \equiv AE \cap FD$ are conjugate points WRT Γ_D. Then it follows that circle (M) with diameter BC is orthogonal to Γ_D.

Alternatively, we can finish here by noting that Γ_D passes through the second limit point of the coaxial system, different from A. But of course a characterization of this point gives a nicer solution.

leader

#14 May 26, 2013, 7:04 pm

let $ABHC$ be a parallelogram and let $k = \odot HBC$ meet AH again at a fixed point L.
$\angle BDC = \angle EDF = 180 - \angle BAC = 180 - \angle BHC$ so $D \in k$. Now
$\angle DLA = 180 - \angle DLH = \angle HBD = \angle AEB$ so $L \in \Gamma_D$ Q.E.D

NewAlbionAcademy

#15 Sep 8, 2013, 12:56 am • 4

We do not use barycentric coordinates. Clearly the locus of D is tha arc of a circle ω with chord BC such that $\angle BDC = \pi - \angle A$. Fix any point D on this locus, and let Γ_D intersect ω again at X. I claim X is the desired common point. Take another point D' on the locus, and let the circumcircle of $\triangle AD'X$ intersect AB and AC at F' and E' respectively. If we show C, D', and F' are collinear then similarly B, D',, and E' are collinear, so Γ'_D goes through X.

We have $\angle F'D'X + \angle XD'C = \angle F'AX + \pi - \angle XDC = \pi$, so C, D', and F' are collinear as desired, and we are done. QED.
(of course, leader's solution is much more elegant and gives a nice characterization of X, but it is harder to find)

This was a lot more manageable than I'd thought. I didn't have to use Mathematica to expand the last expression, since I did not use barycentric coordinates in the first place.

Quick Reply

High School Olympiads

A'B', B'C', C'A intersects the incircle in two points

Reply

Amir Hossein

#1 Aug 29, 2010, 5:47 pm

Let AA', BB', CC' be the bisectors of the angles of a triangle ABC $(A' \in BC, B' \in CA, C' \in AB)$. Prove that each of the lines $A'B', B'C', C'A'$ intersects the incircle in two points.

Luis González

#2 Sep 1, 2010, 11:08 am

Let X, Y, Z be the tangency points of the incircle (I) with $BC, CA, AB.$ Let the tangent line from C' to (I), different from AB, touch the incircle (I) at $M.$ Since $MZ \perp CI \perp XY \Longrightarrow ZM \parallel XY$, thus MZ passes through the reflection U of X about the midpoint of YZ, i.e. polar of C' WRT (I) goes through $U.$ Similarly, polar of B' WRT (I) goes through $U \Longrightarrow U$ is the pole of $B'C'$ WRT $(I).$ Since $\angle ZXY = \angle ZUY = 90^\circ - \frac{1}{2}\angle BAC$ is indeed acute, then U lies outside $(I) \Longrightarrow$ polar $B'C'$ of U WRT (I) cuts (I) at two points. Likewise, lines $A'B'$, $A'C'$ always cut the incircle at two points.

Quick Reply

High School Olympiads

On A Line

Reply

Headhunter
#1 Sep 1, 2010, 7:34 am

Hello.

Let H_1 and H_2 be the feet of perpendiculars from the orthocenter H of the triangle ABC to the bisectors of external and internal angles at the vertex C.
Show that the midpoint of AB is on H_1H_2.

Luis González
#2 Sep 1, 2010, 8:48 am

O is the circumcenter of $\triangle ABC$ and M the midpoint of $AB.$ Since CH_1HH_2 is a rectangle, it follows that H_1H_2 passes through the midpoint D of $CH.$ Let P be the reflection of H about the external bisector of $\angle ACB.$ Then CP, CH are isogonal WRT $\angle C \Longrightarrow O \in CP.$ Thus, H_1DH_2 is the H-midline of $\triangle HPO \Longrightarrow H_1H_2$ passes through the midpoint N of OH (9 point center) of $\triangle ABC,$ but we know that D, N, M are indeed collinear, i.e. H_1, H_2, M, N are collinear.

Quick Reply

High School Olympiads

Parallelepiped

Reply

Headhunter
#1 Aug 30, 2010, 10:43 pm

Hello.

For a **hexahedron** with all of its faces '**quadrilaterals**'
Let the sum of the **squares** of its **diagonals** be equal to the sum of the **squares** of its **segments**.
Show that the hexahedron is a **parallelepiped**.

Luis González
#2 Sep 1, 2010, 3:41 am • 1

Lemma. The sum of the squares of the edges of a hexahedron is always greater than the sum of the squares of its diagonals, with equality if and only if the hexahedron is a parallelepiped.

Label the hexahedron $ABCDEFGH,$ where the faces $ABCD,EFGH,ABEH,BCFE,CDGF$ and $DAHG$ are either planar or skew quadrilateral. Denote p,q the length of the segments connecting the midpoints of the diagonals EG,ED and $AF,HC,$ and denote m,n the length of the segments connecting the midpoints of the diagonals of the faces $ABCD,EFGH,$ respectively. Then, in the quadrilaterals $BDGE$ and $ACFH,$ we have the following relations

$$BE^2+DG^2+EG^2+BD^2=ED^2+BG^2+4p^2$$

$$AH^2+CF^2+AC^2+HF^2=AF^2+HC^2+4q^2$$

Likewise, in the quadrilaterals $ABCD$ and $EFGH$ we have the relations

$$AB^2+BC^2+DC^2+AD^2=AC^2+BD^2+4m^2$$

$$HE^2+EF^2+FG^2+GH^2=HF^2+EG^2+4n^2$$

Susbstituting (AC^2+BD^2) and (HF^2+EG^2) from these two latter expressions into the sum of the two expressions above yields the equation:

$$\sum(\text{edges})^2=\sum(\text{diagonals})^2+4(p^2+q^2+m^2+n^2)$$

Since $p,q,m,n\geq 0,$ it follows that $\sum(\text{edges})^2\geq\sum(\text{diagonals})^2$

Therefore, equality occurs if and only if the quadrilaterals $ABCD,EFGH,ACFH$ and $BDGE$ are parallelograms. Which also means that planes $ABCD,EFGH$ are parallel $\Longrightarrow$ Hexahedron $ABCDEFGH$ is a parallelepiped.

Headhunter
#3 Sep 1, 2010, 7:05 am

@luis, Many thanks.

Quick Reply

High School Olympiads

crazyfehmy

#1 Aug 31, 2010, 6:53 pm

Let Γ be the circumcircle of a triangle ABC, and let D and E be two points different from the vertices on the sides AB and AC, respectively. Let A' be the second point where Γ intersects the bisector of the angle BAC, and let P and Q be the second points where Γ intersects the lines $A'D$ and $A'E$, respectively. Let R and S be the second points of intersection of the lines AA' and the circumcircles of the triangles APD and AQE, respectively.
Show that the lines $DS,\ ER$ and the tangent line to Γ through A are concurrent.

Luis González

#2 Sep 1, 2010, 1:00 am • 2

We assume that D, E lie on the side-segments AB, AC. Denote $\omega_1 \equiv \odot(APD)$ and $\omega_2 \equiv \odot(AQE)$. Because of $\angle SQE = \angle SAE = \angle SAB$, it follows that B, S, Q are collinear. Likewise, C, R, P are collinear. Thus, $SE \parallel RD \parallel BC$ $\Longrightarrow$ arcs ES and RD of ω_2 and ω_1 are homologous $\Longrightarrow ER, DS$ intersect at the exsimilicenter of ω_1, ω_2. On the other hand, since the tangent line τ to Γ through A, AA' and BC bound an isosceles triangle, then τ cuts ω_2 again at F, such that $AFES$ is an isosceles trapezoid, i.e. $EF \parallel RA \Longrightarrow F, A$ are homologous points under the positive homothety that takes ω_1 into ω_2. Hence, DS, ER, τ concur at the exsimilicenter of ω_1, ω_2.

yunustuncbilek

#3 Nov 8, 2011, 9:09 pm

> Quote:
> Arcs ES and RD of ω_2 and ω_1 are homologous $\Longrightarrow ER, DS$ intersect at the exsimilicenter of ω_1, ω_2.

Luis, what is homologous and why ER, DS intersect at the exsimilicenter?

Jeje

#4 Oct 9, 2014, 6:57 pm

What ia homogolous???

Is there another sinplere and more elemntary solution????

TelvCohl

#5 Oct 9, 2014, 8:11 pm • 3

My solution:

Let T be the intersection of DS and RE
Let X be the intersection of TA and (AQE)

Easy to see $DR \parallel SE \parallel BC$ (Reim theorem).

Since arc DR = arc SE ,
so DR and SE are corresponding chord in (APD) and (AQE) ,
hence we get T is the exsimilicenter of $(APD) \sim (AQE)$.
Since $\triangle ADR$ and $\triangle XSE$ are homothetic with center T ,
so $\angle TAB = \angle TXS = 180^\circ - \angle AES = 180^\circ - \angle ACB$.
ie. TA is tangent to (ABC)

Q.E.D

High School Olympiads

Find the angle between planes

Amir Hossein

#1 Aug 29, 2010, 6:35 pm

Let $ABCD$ be a tetrahedron and O its incenter, and let the line OD be perpendicular to AD. Find the angle between the planes DOB and $DOC.$

Luis González

#2 Aug 31, 2010, 12:55 am

Let α,β,γ denote the planes $DBC,DAC,DAB.$ For conveniece, let's cut the trihedron α,β,γ with the spherical surface $\mathcal{E}$ with center D and radius $DO.$ Rays DA,DB,DC cut $\mathcal{E}$ at A',B',C', thus $A'O,B'O,C'O$ become internal angle bisectors of the spherical triangle $A'B'C'.$ Great circle (D,DO) cuts $A'C',A'B'$ at M,N such that $\angle OMA'=\angle ONA'=90^\circ \Longrightarrow M,N$ coincide with the projections of O on $A'C',A'B'.$ If L is the projection of O on $B'C'$, then $\angle LOB'=\frac{1}{2}\angle NOL$ and $\angle LOC'=\frac{1}{2}MOL.$ Thus $\angle B'OC'=\frac{1}{2}(180^\circ)=90^\circ.$

High School Olympiads

Two lines intersect in a point on the nine-point circle

Reply

Amir Hossein

#1 Aug 29, 2010, 6:08 pm • 1

Two straight lines perpendicular to each other meet each side of a triangle in points symmetric with respect to the midpoint of that side. Prove that these two lines intersect in a point on the nine-point circle.

Luis González

#2 Aug 30, 2010, 8:58 am • 1

Label the given triangle $\triangle ABC$ and p,q the two perpendicular lines at $K.M,N,L$ denote the midpoints of $BC,CA,AB.$ Thus, lines p,q cut BC,CA at P,R and $Q,S,$ respectively, such that $\overline{MP}=-\overline{MQ}$ and $\overline{NR}=-\overline{NS}.$ If $\angle KPM=\theta$ and $\angle KSN=\lambda,$ we have then: $\angle PKM=\theta, \angle SKN=\lambda \Longrightarrow \angle MKN=90^\circ+\theta+\lambda \ (*).$ On the other hand, we have $\angle KRC=90^\circ+\lambda$ and $\angle KQC=90^\circ+\theta \Longrightarrow 90^\circ=\angle ACB+\theta+\lambda \Longrightarrow \angle MLN=90^\circ-(\theta+\lambda).$ Together with $(*),$ it follows that $\angle MKN=\angle MLN, \pmod \pi) \Longrightarrow K\in \odot(MNL).$

Quick Reply

High School Olympiads

The radius of the sphere S

Amir Hossein

#1 Aug 29, 2010, 2:33 pm

Determine the radius of a sphere S that passes through the centroids of each face of a given tetrahedron T inscribed in a unit sphere with center O. Also, determine the distance from O to the center of S as a function of the edges of T.

Luis González

#2 Aug 30, 2010, 1:52 am

For the sake of generality, let $\mathcal{O} \equiv (O, R)$ be the circumsphere of the tetrahedron $ABCD$ and $\mathcal{S} \equiv (S, R')$ be the circumsphere of the tetrahedron whose vertices are the centroids A', B', C', D' of its faces against A, B, C, D. Segments AA', BB', CC', CC', DD' concur at the centroid G of $ABCD$, such that G divides them in the same ratio $1:3 \Longrightarrow$ Tetrahedra $A'B'C'D'$ and $ABCD$ are homothetic through the homothety with center G and coefficient $-\frac{1}{3}$. Therefore, $R' = \frac{1}{3}R$ and $OS = \frac{4}{3}OG$. Let a, b, c, d, e, f be the edges of $ABCD$. By Leibniz theorem for the circumcenter O of $ABCD$, we have

$$OG^2 = \frac{1}{4}(OA^2 + OB^2 + OC^2 + OD^2) - \frac{1}{16}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2)$$

$$OG^2 = R^2 - \frac{1}{16}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2)$$

$$\Longrightarrow OS^2 = \frac{16}{9}R^2 - \frac{1}{9}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2)$$

High School Olympiads

Min (Sum Of Ratios)

Headhunter
#1 Aug 29, 2010, 1:51 am

Hello.

For a tetrahedron $ABCD$, let O be on the inside.
$AO\cap\triangle BCD=A_1$, $BO\cap\triangle CDA=B_1$
$CO\cap\triangle DAB=C_1$, $DO\cap\triangle ABC=D_1$

What is the Min of $\sum_{cyclic}\frac{AA_1}{A_1O}$?

Luis González
#2 Aug 29, 2010, 6:36 am

Let S,S_A,S_B,S_C,S_D be the volume of the tetrahedra ABCD,OBCD,OACD,OADB,OABC.

$$\frac{AA_1}{OA_1}+\frac{BB_1}{OB_1}+\frac{CC_1}{OC_1}+\frac{DD_1}{OD_1}=\frac{S}{S_A}+\frac{S}{S_B}+\frac{S}{S_C}+\frac{S}{S_D}$$

$$\frac{AA_1}{OA_1}+\frac{BB_1}{OB_1}+\frac{CC_1}{OC_1}+\frac{DD_1}{OD_1}=S\left(\frac{1}{S_A}+\frac{1}{S_B}+\frac{1}{S_C}+\frac{1}{S_D}\right)$$

By AM-HM on the positive numbers S_A,S_B,S_C,S_D, we get

$$\frac{4}{\frac{1}{S_A}+\frac{1}{S_B}+\frac{1}{S_C}+\frac{1}{S_D}}\le\frac{S_A+S_B+S_C+S_D}{4}=\frac{S}{4}$$

$$\Longrightarrow\frac{AA_1}{OA_1}+\frac{BB_1}{OB_1}+\frac{CC_1}{OC_1}+\frac{DD_1}{OD_1}\ge 16$$

Therefore, minimum value occurs when O coincides with the centroid of $ABCD$.

High School Olympiads

China TST 2010, Problem 1

orl

#1 Aug 29, 2010, 1:44 am

Given acute triangle ABC with $AB > AC$, let M be the midpoint of BC. P is a point in triangle AMC such that $\angle MAB = \angle PAC$. Let O, O_1, O_2 be the circumcenters of $\triangle ABC, \triangle ABP, \triangle ACP$ respectively. Prove that line AO passes through the midpoint of O_1O_2.

orl

#2 Aug 29, 2010, 2:53 am • 2

Approach by vladimir92:

Observe that AP is symmedian and let it intersect BC at E and the circumcircle of $\triangle ABC$ at G and denote $N \equiv (AO) \cap (O_1O_2)$. An easy angle chasing gives that: $\angle NAO_1 = \angle PCG$ and $\angle NAO_2 = \angle PCB$. Therefor, $\frac{NO_1}{NO_2} = \frac{AO_1}{AO_2} \cdot \frac{sin(\angle PCG)}{sin(\angle PBG)}$, since AO_1 is the circumraduis of $\odot(APC)$ we have $2AO_1 = \frac{PC}{sin(\angle PAC)}$ similary $2AO_2 = \frac{PB}{sin(\angle PAB)}$. Then $\frac{AO_1}{AO_2} = \frac{PC}{PB} \cdot \frac{GB}{GC}$ It follow that $\frac{NO_1}{NO_2} = \frac{EC}{EB} \cdot \left(\frac{BG}{CG}\right)^2 = \left(\frac{AC}{CG} \cdot \frac{AB}{BG}\right)^2$ because AE is symmedian. Another angle chasing gives that $\triangle ABG \sim \triangle AMC$ and $\triangle ACG \sim \triangle AMB$ from which follow immediatly that $\frac{NO_1}{NO_2} = 1$ meaning that AO passe trough the midpoint of O_1O_2.

Luis González

#3 Aug 29, 2010, 4:47 am • 3

Let the tangents of $\odot(ABC) \equiv (O)$ through B, C intersect at D. Then $P \in AD$. Let N, L be the orthogonal projections of M on AB, AC and let (O') be the circumcircle of $ANML$. C' denotes the orthogonal projection of C onto AB and U denotes the midpoint of DM.

From $\angle BCD = \angle BAC$, we have $\frac{AC}{DC} = \frac{CC'}{DM} = \frac{MN}{MU} \Longrightarrow \triangle ACD \sim \triangle NMU$ by SAS criterion. Thus, $\angle UNM = \angle DAC = \angle NLM \Longrightarrow UN$ is tangent to (O') through N. Likewise, UL is tangent to (O') through $L \Longrightarrow OM \equiv MU$ is the M-symmedian of $\triangle MNL$. If E is the midpoint of NL, then $\angle LME = \angle NMO$ yields $AO \parallel ME$. Now, since $O_1O_2 \perp AP \perp NL, OO_1 \parallel MN$ and $OO_2 \parallel ML$, it follows that $\triangle MNL$ and $\triangle OO_1O_2$ are homothetic with corresponding cevians ME, OA. Therefore, ray OA is the O-median of $\triangle OO_1O_2$.

This post has been edited 1 time. Last edited by Luis González, Aug 30, 2010, 11:14 am

77ant

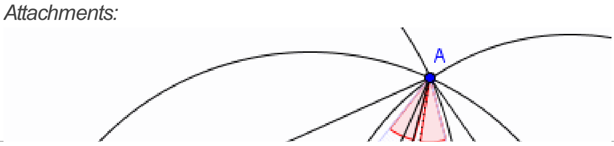
#4 Aug 29, 2010, 4:57 am • 1

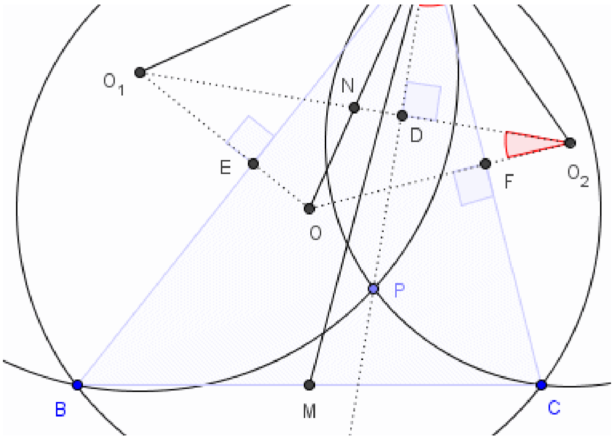
$$\frac{O_1N}{\sin\angle O_1ON} = \frac{ON}{\sin\angle NO_1O} \Rightarrow O_1N = \frac{ON\cdot\sin\angle C}{\sin\angle BAP} \quad \text{Similarly} \quad O_2N = \frac{ON\cdot\sin\angle B}{\sin\angle PAC} \quad (1)$$

$$\frac{BA}{\sin\angle AMB} = \frac{BM}{\sin\angle BAM} \text{ and } \frac{CA}{\sin\angle AMC} = \frac{CM}{\sin\angle CAM} \Rightarrow \frac{\sin\angle B}{\sin\angle C} = \frac{\sin\angle PAC}{\sin\angle BAP} \quad (2)$$

$\therefore O_1N = O_2N$

Attachments:





k.l.l4ever

#5 Sep 2, 2010, 12:02 am • 1

Let the perpendicular line from B cut $(O),(O_1)$ at second points E, I,respectively.Denote the points F,J in the same way.
Obviously O_1, O_2 are midpoints of AI, AJ.Hence it's suffice to prove that AE pass through the midpoint of IJ.
Also,AD is the A-symmedian of triangle ABC and $D \in (O)$.So A, D, B, C form a harmonic quadilateral.It's equivalent with $(ED, EA, EI, EF) = -1$.Because $IJ \perp AD \perp ED$ so $IJ \| ED$.From those we deduce that EA pass through the midpoint of IJ (Q.E.D)

Image not found

limes123

#6 Oct 24, 2010, 4:14 am

Let AM intersect circumcircle of triangle ABC in N. It's easy to prove, that $\triangle BNC \sim \triangle O_1OO_2$ and $\angle AOO_2 = \angle ANC$ which proves that AO is median in triangle OO_1O_2.

vladimir92

#7 Jul 10, 2011, 9:51 pm

This is another approach.
Let lines CA , BA and AP meet (O_1) , (O_2) and (O) at B_1 , C_1 and S respectively. Easy to see that $\triangle PC_1C \sim \triangle PBB_1$ and $\triangle PC_1B \sim \triangle SCB \sim \triangle PCB_1$, So,
$\frac{BB_1}{CC_1} = \frac{PB}{PC_1} = \frac{SB}{SC} = \frac{AB}{AC}$ because quadrilateral $CABS$ is harmonic, thus $(B_1C_1) \parallel (BC)$. Now let C_2 and B_2 be the orthogonale projections of O_1 and O_2 into CA and BA respectively, So $(B_2C_2) \parallel (M_bM_c)$ where M_b and M_c are midpoints of AC and AB respectively. Denote $X \equiv (O_1C_2) \cap (O_2B_2)$ and $Y \equiv (AO) \cap (O_2C_2$ So
$\angle C_2XA = \angle C_2B_2A = \angle AM_cM_b = \angle AOM_b = \angle C_2YA$, Then $X \equiv Y$, or again $A \in OX$ and since O_2OO_1X is a parallelogram, we deduce that AO bissect segement O_1O_2.

cwein3

#8 Nov 28, 2011, 4:37 am

Let AP intersect the circumcircle of ABC at D, let AM intersect the circumcircle of ABC at E. I will show that $\triangle O_1OO_2 \sim \triangle BEC$, from which we can use the fact that $\angle O_1OA = \angle ACB = \angle BEM$ to conclude that OA intersects O_1O_2 at its midpoint.

Note that we have a spiral similarity making $\triangle DOC \sim \triangle O_2PC$, so $\frac{DP}{OO_2} = \frac{DC}{OC}$. Similarly, we can get $\frac{DP}{OO_1} = \frac{DB}{OB}$.
Combining these two equations and using $OB = OC$, we obtain $\frac{DC}{DB} = \frac{OO_1}{OO_2}$. But we also have $DC = EB$ and $DB = EC$ since AE and AD are isogonal lines. In addition, $\angle BEC = 180 - \angle BAC = \angle O_1OO_2$, so by SAS similarity, $\triangle O_1OO_2 \sim \triangle BEC$, as desired.

simplependulum

#9 Nov 28, 2011, 5:07 pm

We just need to prove that the angle between the $A-$median and $A-$attitude of ΔAO_1O_2 is equal to the angle between AO and AP, or the angle between AM and $A-$attitude of ΔABC ...

and AP , of the angle between AM and $A-$attitude of ΔABC .

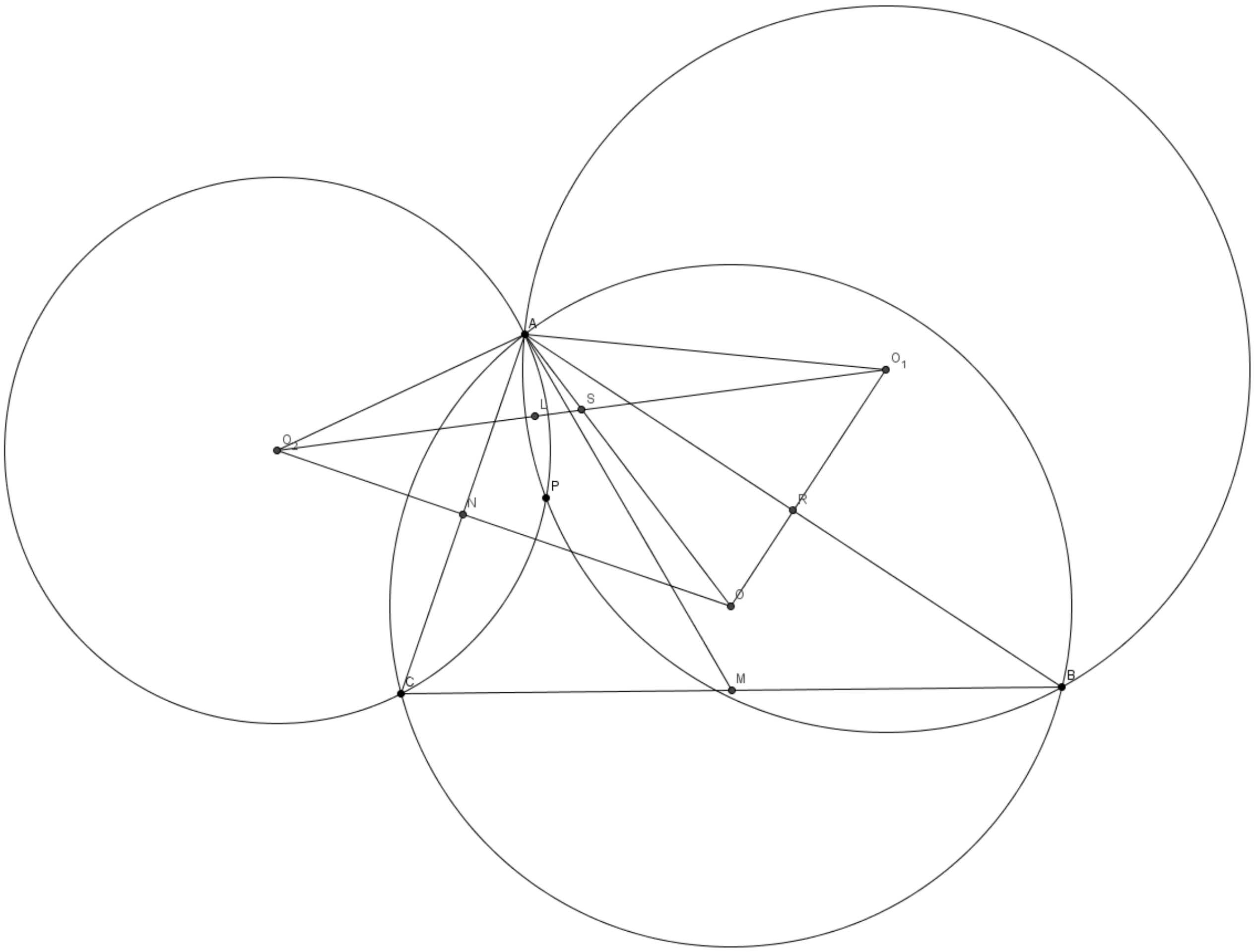
Consider the isogonal conjugate of P w.r.t. ΔABC , Q , which is on segment AM . We have $\angle QBC = \angle AO_1O_2$, $\angle QCB = \angle AO_2O_1$ so $\Delta AO_1O_2 \sim \Delta QBC$. Since $AO_1 > AO_2$, we can see that the $A-$median of ΔAO_1O_2 and AO are of the same side of AP . At the same time ,the angle between the $A-$median and $A-$attitude of ΔAO_1O_2 $=$ the angle between the $A-$median and $A-$attitude of ΔQBC $=$ the angle between AM and $A-$attitude of ΔABC , done .

andria

#10 Jun 15, 2015, 12:15 pm

My solution:
Let $AO \cap O_1O_2 = S, AP \cap O_1O_2 = L$ and R, N are midpoints of AB, AC Note that OO_1, OO_2 are perpendicular bisectors of $AB, AC \longrightarrow \angle AOO_1 = \angle C, \angle AOO_2 = \angle B$ so in $\triangle O_1OO_2$: $\frac{SO_2}{SO_1} = \frac{\sin B}{\sin C} \cdot \frac{\sin OO_2O_1}{\sin OO_1O_2}$(1) but observe that in cyclic quadrilaterals $ALNO_2, ALRO_1$:
$\angle O_1O_2O = \angle CAP, \angle O_2O_1O = \angle PAB \longrightarrow \frac{\sin OO_2O_1}{\sin OO_1O_2} = \frac{\sin PAC}{\sin PAB} = \frac{\sin MAB}{\sin MAC} = \frac{\sin C}{\sin B}$(2) combining (1),(2) we get the result.
DONE

Attachments:



tranquanghuy7198

#11 Jun 15, 2015, 3:31 pm

My solution:
O is the center of (ABC)
t is the line passing through A which is perpendicular to AO
x is the line passing through O which is perpendicular to $AP \Rightarrow x \parallel O_1O_2$
According to the subject: AP is the symmedian of $\triangle ABC$
$\Rightarrow A(BCPt) = -1 \Rightarrow O(O_1O_2xA) = -1$ (orthogonal harmonic pencil)
$\Rightarrow OA$ bisects O_1O_2 (because $Ox \parallel O_1O_2$)
Q.E.D

Dukejukem

Dukejukem

#12 Jun 16, 2015, 9:53 am

Let AO cut O_1O_2 at K, and let X, Y be the midpoints of $\overline{AB}, \overline{AC}$, respectively. Consider the inversion $\mathcal{T}: X \mapsto X'$, composed of an inversion with pole A and radius $r = \sqrt{bc}$ combined with a reflection in the A-angle bisector. It is easy to see that $B' \equiv C$ and $C' \equiv B$. Then since P lies on the A-symmedian in $\triangle ABC$, it follows that P' lies on the A-median in $\triangle AB'C'$. Furthermore note that X' is the reflection of A in B', and Y' is the reflection of A in C'. Then because $\angle AXO = \angle AYO = 90^\circ$, it follows under inversion that $\angle AO'X' = \angle AO'Y' = 90^\circ \implies O'$ is the projection of A onto $X'Y'$. Hence, O' is the reflection of A in $B'C'$. Similarly, we find that O_1' is the reflection of A in $B'P'$ and O_2' is the reflection of A in $C'P'$. Meanwhile, K' is the second intersection of AO' and $\odot(AO_1'O_2')$.

From the inversive distance formula, we have

$$K'O_1' = KO_1 \cdot \frac{r^2}{AK \cdot AO_1} \quad \text{and} \quad K'O_2' = KO_2 \cdot \frac{r^2}{AK \cdot AO_2} \implies \frac{K'O_1'}{K'O_2'} = \frac{KO_1}{KO_2} \cdot \frac{AO_2}{AO_1}.$$

Furthermore,

$$AO_1 \cdot AO_1' = AO_2 \cdot AO_2' = r^2 \implies \frac{AO_2}{AO_1} = \frac{AO_1'}{AO_2'}.$$

Therefore, in order to show that K is the midpoint of $\overline{O_1O_2}$, we need only show that $\frac{K'O_1'}{K'O_2'} = \frac{AO_1'}{AO_2'}$, i.e. show that quadrilateral $AO_1'K'O_2'$ is harmonic. To see this, let O_1^*, O_2^* be the projections of A onto $B'P', C'P'$, respectively, and let K^* be the midpoint of $\overline{AK'}$. By considering the homothety with center A and ratio $1/2$, it is sufficient to prove that quadrilateral $AO_1^*K^*O_2^*$ is harmonic. Because $AO_1^* \perp P'O_1^*$ and $AO_2^* \perp P'O_2^*$, it is clear that A, O_1^*, O_2^*, P' are inscribed in the circle ω of diameter $\overline{AP'}$. Hence, K^* also lies on $\omega \implies P'K^* \perp AK^* \implies P'K^* \parallel B'C'$. Then if P_∞ denotes a point at infinity on line $B'C'$ and M^* is the midpoint of $\overline{B'C'}$, the division $(B', C'; M^*, P_\infty)$ is harmonic. By taking perspective at P' onto ω, we obtain the desired result. $\square$

This post has been edited 1 time. Last edited by Dukejukem, Jun 16, 2015, 9:56 am

hayoola

#13 Sep 22, 2015, 11:14 am

use that in triangel $O2O1O$if AO is median

$\angle O_1O_2O = \angle CAP, \angle O_2O_1O = \angle PAB \longrightarrow \frac{\sin OO_2O_1}{\sin OO_1O_2} = \frac{\sin PAC}{\sin PAB} = \frac{\sin MAB}{\sin MAC} = \frac{\sin C}{\sin B}$

utkarshgupta

#15 Dec 12, 2015, 10:26 pm

Let M, N be the midpoints of AB, BC respectively and let l be a line perpendicular to the $A-$symmedian passing through A.
Let $OM \cap l = X$ and $ON \cap l = Y$
Then obviously $\triangle OXY$ is homothetic to $\triangle OO_1O_2$ with O the centre of homothety.

Thus we are left to prove that A is the midpoint of XY.
This is easy trigo.

aditya21

#16 Dec 16, 2015, 1:49 pm

my solution = we have $OO_1 \perp AB$ and $OO_2 \perp AC$ and $O_1O_2 \perp AP$
also angle chasing we get $\angle O_2O_1O = \angle BAP$ and $\angle O_1O_2O = \angle PAC$.
let $AO \cap O_1O_2 = D$.
so by sine law in triangle DO_1O and triangle DO_2O we get

$$\frac{O_1D}{O_2D} = \frac{sinC.sinBAP}{sinB.sinPAC} = 1$$

as by sine law in triangle ABP and ACP along with using $AM = BM$ we get $\frac{sinC}{sinB} = \frac{sinPAC}{sinBAP}$

so we are done. 😁

This post has been edited 1 time. Last edited by aditya21, Dec 16, 2015, 1:50 pm
Reason: e

Quick Reply

High School Olympiads

An interesting relation

Reply

jayme

#1 Aug 28, 2010, 5:33 pm

Dear Mathlinkers,
let ABC a triangle, O the center of the circumcircle, H the orthocenter of ABC, G the centroid,
Fe the Feuerbach point, the Na the Nagel point,
X the point of intersection of HFe with ONa,
Y the antipole of Fe wrt Euler's circle of ABC.
Prove : GX=2.GY.
Sincerely
Jean-Louis

Luis González

#2 Aug 28, 2010, 9:06 pm

Let $\triangle A_0B_0C_0$ be the anticomplementary triangle of $\triangle ABC.$ Vertices A_0, B_0, C_0 against $A, B, C,$ respectively. Then, $N_a, (O)$ become incenter and 9-point circle of $\triangle A_0B_0C_0.$ If I denotes the incenter of $\triangle ABC,$ then ON_a is the anticomplement of NIF_e WRT $\triangle ABC,$ i.e. $ON_a \parallel NF_e.$ But since H is the exsimilicenter of $(N) \sim (O),$ it follows that intersection X of HF_e and ON_a lies on the circumcircle $(O).$ Thus, X is the antipode of the Feuerbach point of $\triangle A_0B_0C_0$ WRT its 9-point circle $(O),$ i.e. the anticomplement of Y WRT $\triangle ABC.$ Therefore, $\overline{GX} = -2 \cdot \overline{GY}.$

Quick Reply

High School Olympiads

Similarity II.

Reply

Virgil Nicula

#1 Aug 25, 2010, 6:29 am

Let P be an interior point for $\triangle ABC$. Draw through P the lines DE , FG , HI which are anti-parallel to sidelines BC , CA , AB respectively so that $DE = FG = HI = x$, where $\left\{\begin{array}{c} \{I, F\} \subset (BC) \\ \{E, H\} \subset (CA) \\ \{G, D\} \subset (AB) \end{array}\right.$. Ascertain x in terms of a , b , c .

Answer

Luis González

#2 Aug 27, 2010, 10:58 pm

Assume that FG, HI are equal and antiparallel to $AC, AB,$ respectively. $P \equiv FG \cap IH.$ Then the quadrilateral $GHFI$ is an isosceles trapezoid with $GH \parallel IF \equiv BC$ and since $\angle PGH = \angle PHG = \angle BAC,$ it follows that AP is identical to the A-symmedian of $\triangle ABC.$ Thus, the point P of "equal antiparallels" is the symmedian point K of $\triangle ABC,$ i.e. $D, E, F, G, H,$ I lie on a circle with center $K.$ Let S be the foot of the A-symmedian and M be the midpoint of $BC.$ AK, AM are homologous medians of the similar triangles $\triangle ADE \sim \triangle ACB.$ Consequently, we have the following well-known expressions

$$\frac{DE}{BC} = \frac{AK}{AM} \ , \ \frac{AK}{AS} = \frac{b^2+c^2}{a^2+b^2+c^2} \ , \ \frac{AS}{AM} = \frac{2bc}{b^2+c^2}$$

$$\Longrightarrow DE = FG = HI = \frac{2abc}{b^2+c^2} \cdot \frac{b^2+c^2}{a^2+b^2+c^2} = \frac{2abc}{a^2+b^2+c^2}$$

Quick Reply

High School Olympiads

XA_1 , YB_1 and ZC_1 are concurrent

Reply

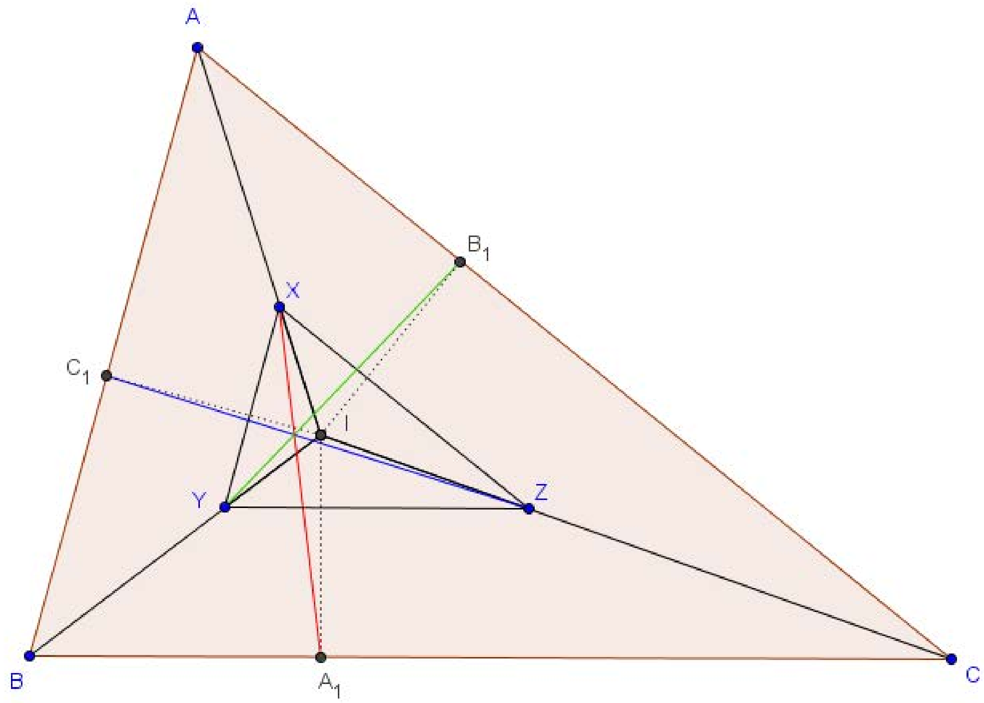
Mateescu Constantin

#1 Aug 26, 2010, 2:34 am

Let ABC be a triangle with incenter I for which denote its projections A_1 , B_1 , C_1 on the sidelines BC , CA and AB respectively .

Define the points $X\in (IA)$, $Y\in (IB)$, $Z\in (IC)$ so that $\frac{IX}{IA}=\frac{IY}{IB}=\frac{IZ}{IC}=\lambda>0$. Prove that the lines XA_1 , YB_1 , ZC_1 are concurrent .

Attachments:



Luis González

#2 Aug 26, 2010, 5:29 am

$\triangle ABC$ and $\triangle XYZ$ are homothetic through the homothety with center I (their common incenter) and coefficient k. Lines IA_1, IB_1, IC_1 cut YZ, ZX, XY at their tangency points D, E, F with the incircle of $\triangle XYZ$ and $\frac{ID}{IA_1}=\frac{IE}{IB_1}=\frac{IF}{IC_1}=k$. Hence, by Kariya's theorem for $\triangle XYZ$, the lines XA_1, YB_1, ZC_1 concur. This well-known theorem has been mentioned and posted several times in the forum. The proof with barycentric coordinates is quite straightforward, but you may see the nice Darij's approach in the topic Help [Kariya point].

jayme

#3 Aug 26, 2010, 7:18 pm

Dear Mathlinkers,
if the k of Luis is equal to 1/2, we have the Gray point which is a special case of the Kariya's points...
For a synthetical proof, you can see
http://perso.orange.fr/jl.ayme vol. 2, Le point de Gray p. 7.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Perpendicular bisector

Reply

Source: Romania TST 2009, Day 3, Problem 3

Maxim Bogdan

#1 Jul 27, 2009, 1:22 am

Let ABC be a non-isosceles triangle, in which $X,Y,$ and Z are the tangency points of the incircle of center I with sides BC,CA and AB respectively. Denoting by O the circumcircle of $\triangle ABC$, line OI meets BC at a point $D.$ The perpendicular dropped from X to YZ intersects AD at E. Prove that YZ is the perpendicular bisector of $[EX]$.

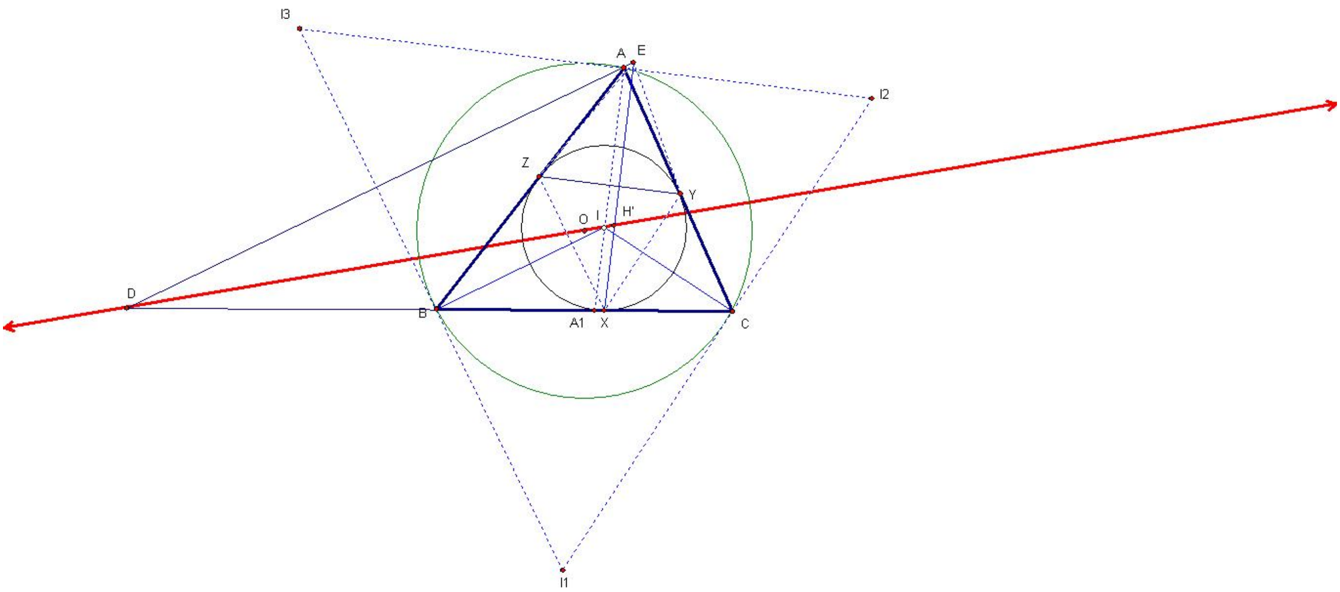
plane geometry

#2 Jul 27, 2009, 6:15 pm

1.Denote the reflection of X w.r.t YZ is E, we will prove the EA,OI,BC are concurrent.
2.Let I1I2I3 be the excentral triangle. The lines Y Z and I2I3 are parallel because both are perpendicular to AI. Similarly, ZX//I3I1 and XY//I1I2. Hence, the excentral triangle and the intouch triangle are homothetic and their Euler lines are parallel. Now, I and O are the orthocenter and nine-point center of the excentral triangle. On the other hand, I is the circumcenter of the intouch triangle. Therefore, the line OI is their common Euler line, contains the orthocenter H' of XY Z.
3.AI/IA1=BA/BA1 XH'=2r*cos(A/2) XE=2h EH'= XE- XH'
where h, r are the attitude from X to YZ and the incircle radii
we can calculate AI/IA1= EH'/ XH'= (sinB + sinC)/sinA

Therefore, we complete the proof.

Attachments:



April

#3 Jul 28, 2009, 5:07 am

> Maxim Bogdan wrote:
> Let ABC be a non-isosceles triangle, in which $X,Y,$ and Z are the tangency points of the incircle of center I with sides