A geometry problem set

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Introduction

So this is a compilation of some nice geometry problems which may provide food for thought. The first 6 problems are from the source mentioned and the last 6 are due the author himself. So have fun solving these!!!

The Problems

Problem 1: Given an acute triangle ABC inscribed in (O), incircle (I). The tangent points of (I) on BC, CA, AB are respectively D, E, F. Let (O_a) be the A-excircle, and it is tangent to BC, CA, AB at D', E', F' respectively.

1/Prove that AD, BE, CF are concurrent; AD', BE', CE' are concurrent at I_0.

2/Prove that $D'C = DB$

3/ $ID \cap EF = D_1$. Prove that AD_1 passes through the midpoint M of BC.

4/ ID cuts (I) at $\{D, D_2\}$. Prove that AD_2 passes through D'.

5/ $BI \cap EF = B'$, $CI \cap EF = C'$. Prove that (B, I, F, C'), (C, I, E, B') and (B, C, B', C') are the sets of concyclic points.

6/ AI cuts (BIC) at $\{A, A_0\}$. I_0 is symmetric to I wrt BC. Let K' be the foot of the attitude from A of triangle ABC. Prove that $K'I_0$ passes through A_0.

7/ $E'D' \cap DF = K$. Prove that A, K, K' are collinear (Paul Yiu's theorem).

8/ $A*$ is symmetric to A wrt O. $A*I \cap EF = W$. Prove that DW is perpendicular to EF (mathandyou).

9/ $AI \cap BC = A_1$. Prove that $A_1D \cap A*I \in (O)$

10/ AI cuts (O) at $\{A, A_2\}$. Prove that A_2 is the circumcenter of triangle BIC.

11/ Let R be an arbitrary point lying on minor arc BC. R_1R_2 is the polar of R wrt (I). BC cuts RR_1 and RR_2 at R'_1 and R'_2 respectively.The A-mixtilinear incircle is tangent to (O) at Z.Prove that $Z \in (DA_1A_2) \cap (RR'_1R'_2)$. (Cosmin Pohoata)

12/ $IC \cap AK = C_0, IB \cap AK = B_0$. K_1 is the midpoint of AK'.Prove that K_1 lies on the radical axis of (C_0EC) and (B_0FB).

13/ Prove that (O) and (IAZ) are orthogonal.

14/ AZ cuts (DA_1A_2) at Z and Z'.Prove that $Z' \in (O_a)$. (USAMO 2015).

15/ Prove that OO_a is perpendicular to EF.

16/ (OI) cuts (IAB) at I, J.Prove that IJ is parallel to DE. (LeVietAn)

17/ (OI) cuts (OAB) at $\{O, J_1\}$.Prove that IJ_1 passes through the midpoint M_1 of DE. (LeVietAn)

18/ Let I_1 be the projection of I onto AD. $I_1M_1 \cap AC = N$.Prove that ND is parallel to EF.

19/ The line passing through A and parallel to BC cuts EF at T.Let T_1 be the midpoint of AT.Prove that T_1M is tangent to (I).

20/ Let H_b be the orthocenter of triangle IAC.Prove that H_bD is perpendicular to IM. (India MO 2014).

21/ Draw the diameters EE_2, FF_2 of (I). E_2F_2 cuts BC at P.Prove that $\widehat{MIP} = 90$.

22/ IB, IC cut E_2F_2 at E_3, F_3 respectively.Prove that the perpendicular bisector of E_3F_3 passes through the symmetric point of M wrt A_1. (VMO 2013).

23/ Prove that I is the incenter of triangle $D_2E_3F_3$. (buratinogiggle).

24/ DK' cuts (I) at $\{D, K_1\}$.Prove that K_1D is the angle bisector of triangle $\widehat{BK_1C}$.

25/ Prove that (BK_1C) is tangent to (I). (IMO SL 2004).

26/ Let V be the midpoint of $ID. A*V$ cuts (O) at $\{A*, V\}$.Prove that BC is tangent to (VAD). (Taiwan MO 2015).

27/ EF cuts (O) at E_4, F_4.Let A_e, A_f be respectively the projections of A onto IE_4, IF_4.Prove that DA_e, DA_f are isogonal conjugate wrt $\widehat{EDF}$.

28/ Prove that the center of (DE_4F_4) lies on the perpendicular bisector of IA. (rkm0959).

29/ If $AB + AC = 3BC$ then prove that $(IB'C')$ is tangent to (IBC). (Iran MO 2004).

30/ $(O_aBF) \cap (O_aCE) = O_a, W_a$.Prove that (BCW_a) is tangent to(I) (Telv Cohl).

31/ Similarly define W_b, W_c, then prove that W_aD, W_bE, W_cF, OI are concurrent. (High School for the Gifted's TST 2014).

32/ The angle bisector of $\widehat{AW_cB}$ cuts (AW_cB) at W_c', the angle bisector of $\widehat{AW_bC}$ cuts (AW_bC) at W_b'.Prove that W_b, W_c, W_b', W_c' are concyclic. (High School for the Gifted's TST 2014).

33/ Construct E_5, F_5 such that vector $EE_5 =$vector $FF_5 =$vector BC.Let I' be the intersection of BE, CF.Prove that $I'E_5 = I'F_5$.

34/ Construct B_1, C_1 on the rays BA, CA respectively such that $BB_1 = CC_1 = BC$.Let I_1, O_1 be respectively the incenter and circumcenter of triangle AB_1C_1.Prove that:
34.1/ IO is perpendicular to B_1C_1.
34.2/ IO' is perpendicular to BC.
34.3/ II_1 is parallel to OO_1.
34.4/ $IO \cap I_1O1 \in (O)$. (Telv Cohl).

35/ Let $B*, C*$ be respectively the midpoint of arc ABC, ACB.Prove that $B*, C*$ are respectively the centers of $(O_aBA), (O_aCA)$.

36/ Prove that $B*C*$ passes through I iff $AB + AC = 3BC$.(A.Polyansky).

37/ Prove that $D_2A = I_0D'$. (Sharygin Olympiad 2008).

Problem 2:

Given a cyclic quadrilateral $ABCD$, AC cuts BD at E, AB cuts CD at F .Let M, N be the midpoint of AD, BC respectively.Prove that EF is tangent to (MEN).

Problem 3:

Given triangle ABC,incircle (I) touches AB, AC at F, E respectively.Construct G, H satisying that $\overrightarrow{EG} = \overrightarrow{FH} = \overrightarrow{BC}$.Let S be the intersection of BE, CF.Prove that $SG = SH$.

Problem 4:

Given triangle ABC and an arbitrary point M inside such that MAB, MBC, MCA are not equilateral triangle.Let $O_a, G_a, O_b, G_b, O_c, G_c$ be the circumscribed circle's centers and centroids of MAB, MBC, MCA respectively. Prove that O_aG_a, O_bG_b, O_cG_c are concurrent or couple parallel iff AO_a, BO_b, CO_c are concurrent or couple parallel.

Problem 5:

Let ABC is a triangle inscribed (O) and (I) is the incircle which tangent to BC, CA, AB at D, E, F resp. Let $B_1 = BI \cap (O) \neq B$, $C_1 = CI \cap (O) \neq C$, $EF \cap (O) = X, Y$. Prove that the circumcenter of $\triangle DXY$ is the midpoint of C_1B_1.

Problem 6:

Given a circle (O) and AB is its chord. Let (S) be a circle which tangents to (O) at C and tangents to AB at D. Prove that CD is the angle bisector of the angle ACB.

Problem 7:

If D, E, F are the touch pts of the incircle with the sides BC, CA, AB resp, then prove that the circumcircles of AID, BIE and CIF are coaxial with OI as the radical axis where I is the incenter and O is the circumcenter of ABC.

Problem 8:

Let ABC be a triangle and A'B'C' be the medial triangle. Let the circle tangent to (AB'C') internally and to (BC'A') and (CB'A') externally touch (AB'C') at X. Define Y, Z similarly. Prove that A'X, B'Y and C'Z are concurrent.

Problem 9:

Let H be the orthocenter of the triangle ABC and M be the midpt of BC.
Let T be the foot of perp from A to HM.
Let E, F, X be the feet of the B and C altitudes and the intersection of the circumcircle of BHC with AM. Prove that HX, $EF$and AT concur on BC.

Problem 10:

Let ABC be a triangle with all angles$>$ 45°. D, E, F are the feet of the altitudes from A, B, C. G is the centroid. Intersection of DG and (ABC) is A'
. Define B' and C' analogously.$A'B'$ intersects with AC at L_{ac} and $A'B'$ with BC at L_{bc}.
Define the L's with other subscripts similarly.
Let (O_a) be the circle passing thru A and L_{ac} and tangent to $A'B'$. (O'_a) thru A and L_{ab} and tangent to $C'A'$. Define the circles with the other subscripts in the same manner.
Let the circumcentre of $AO_aO'_a$ be O''_a.
Define other centres similarly.
Extend $A'O''_a$ and the like to form a triangle $A_1B_1C_1$. Let the mixtilinear incircle touch pts with its circumcircle be $X, Y,$ and Z.Prove that the cevians A_1X etc concur at the point P such that the circumcentre of the cevian triangle of the isotomic conjugate of isogonal conjugate of P wrt $A_1B_1C_1$ is the circumcentre of ABC.

Problem 11:

Let ABC be a triangle with all angles$>$ 45°. D, E, F are the feet of the altitudes from A, B, C. G is the centroid. Intersection of DG and (ABC) is A'
. Define B' and C' analogously.$A'B'$ intersects with AC at L_{ac} and $A'B'$ with BC at L_{bc}.
Define the L's with other subscripts similarly.
Let (O_a) be the circle passing thru A and L_{ac} and tangent to $A'B'$. (O'_a) thru A and L_{ab} and tangent to $C'A'$. Define the circles with the other subscripts in the same manner.

Prove that O is the radical centre of $(O_aO_bO'_aO'_b)$ etc, $(O_aO'_aA')$ etc, $(O_aO'_aB')$ etc and $(O_aO'_aC')$ etc.

Problem 12:

Let the perimeter of a triangle ABC be 2 and let BC be the smallest side. Let P and Q be on AC and AB such that AP+PB=AQ+QC=1
A line parallel to the internal angle bisector of B thru P meets the perp bisector of BC at T.
BP intersects QC at W.
Prove that A,W,T are collinear iff AB=AC

References:

AoPS user huynguyen's blog

Further Solving:

See the AoPS forum for more problems.
Also see "150 nice geometry problems – Amir Hossein Parvardi", and I F Sharygin's "Problems in plane geometry".