Forum Geometricorum
Volume 1 (2001) 1–6.

FORUM GEOM

Friendship Among Triangle Centers

Floor van Lamoen

Abstract. If we erect on the sides of a scalene triangle three squares, then at the vertices of the triangle we find new triangles, the *flanks*. We study pairs of triangle centers X and Y such that the triangle of Xs in the three flanks is perspective with ABC at Y, and vice versa. These centers X and Y we call *friends*. Some examples of friendship among triangle centers are given.

1. Flanks

Given a triangle ABC with side lengths $BC = a$, $CA = b$, and $AB = c$. By erecting squares AC_aC_bB, BA_bA_cC, and CB_cB_aA externally on the sides, we form new triangles AB_aC_a, BC_bA_b, and CA_cB_c, which we call the *flanks* of ABC. See Figure 1.

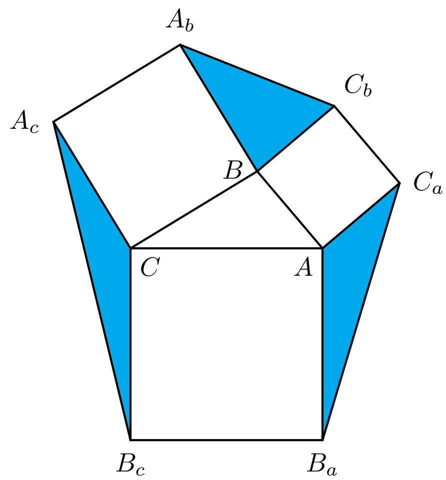


Figure 1

If we rotate the A-flank (triangle AB_aC_a) by $\frac{\pi}{2}$ about A, then the image of C_a is B, and that of B_a is on the line CA. Triangle ABC and the image of the A-flank form a larger triangle in which BA is a median. From this, ABC and the A-flank have equal areas. It is also clear that ABC is the A-flank triangle of the A-flank triangle. These observations suggest that there are a close relationship between ABC and its flanks.

2. Circumcenters of flanks

If P is a triangle center of ABC, we denote by P_A, P_B, and P_C the *same* center of the A-, B-, and C- flanks respectively.

Publication Date: January 24, 2001. Communicating Editor: Paul Yiu.

Let O be the circumcenter of triangle ABC. Consider the triangle $O_AO_BO_C$ formed by the circumcenters of the flanks. By the fact that the circumcenter is the intersection of the perpendicular bisectors of the sides, we see that $O_AO_BO_C$ is homothetic (parallel) to ABC, and that it bisects the squares on the sides of ABC. The distances between the corresponding sides of ABC and $O_AO_BO_C$ are therefore $\frac{a}{2}$, $\frac{b}{2}$ and $\frac{c}{2}$.

3. Friendship of circumcenter and symmedian point

Now, homothetic triangles are perspective at their center of similitude. The distances from the center of similitude of ABC and $O_AO_BO_C$ to the sides of ABC are proportional to the distances between the corresponding sides of the two triangles, and therefore to the sides of ABC. This perspector must be the *symmedian point* K. [1]

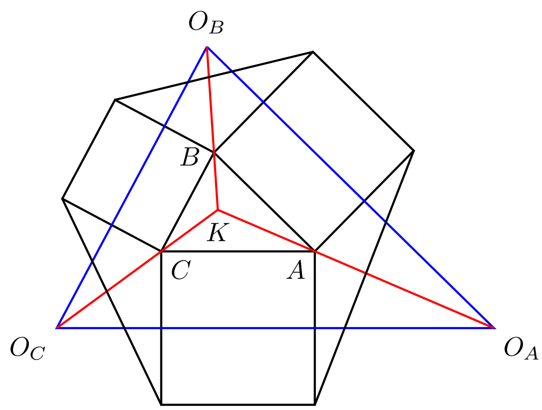


Figure 2

The triangle $O_AO_BO_C$ of *circumcenters* of the flanks is perspective with ABC at the *symmedian point* K of ABC. In particular, the A-Cevian of K in ABC (the line AK) is the same line as the A-Cevian of O_A in the A-flank. Since ABC is the A-flank of triangle AB_aC_a, the A-Cevian of K_A in the A-flank is the same line as the A-Cevian of O in ABC as well. Clearly, the same statement can be made for the B- and C-flanks. The triangle $K_AK_BK_C$ of *symmedian points* of the flanks is perspective with ABC at the *circumcenter* O.

For this relation we call the triangle centers O and K *friends*. See Figure 3. More generally, we say that P *befriends* Q if the triangle $P_AP_BP_C$ is perspective with ABC at Q. Such a friendship relation is always symmetric since, as we have remarked earlier, ABC is the A-, B-, C-flank respectively of its A-, B-, C-flanks.

---

[1] This is X_6 in [2, 3].

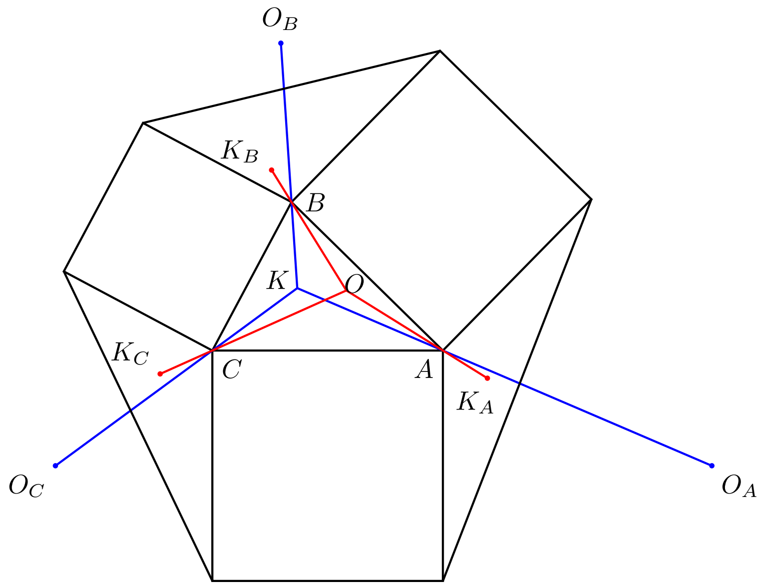


Figure 3

4. Isogonal conjugacy

It is easy to see that the bisector of an angle of ABC also bisects the corresponding angle of its flank. The incenter of a triangle, therefore, *befriends* itself.

Consider two friends P and Q. By reflection in the bisector of angle A, the line PAQ_A is mapped to the line joining the isogonal conjugates of P and Q_A.[2] We conclude:

Proposition. If two triangle centers are friends, then so are their isogonal conjugates.

Since the centroid G and the orthocenter H are respectively the isogonal conjugates of the symmedian point K and the circumcenter O, we conclude that G and H are friends.

5. The Vecten points

The centers of the three squares AC_aC_bB, BA_bA_cC and CB_cB_aA form a triangle perspective with ABC. The perspector is called the *Vecten point* of the triangle. [3] By the same token the centers of three squares constructed *inwardly* on the three sides also form a triangle perspective with ABC. The perspector is called the *second Vecten point*. [4] We show that each of the Vecten points befriends itself.

---

[2]For Q_A, this is the same line when isogonal conjugation is considered both in triangle ABC and in the A-flank.

[3]This is the point X_{485} of [3].

[4]This is the point X_{486} of [3], also called the *inner* Vecten point.

6. The Second Vecten points

O. Bottema [1] has noted that the position of the midpoint M of segment B_cC_b depends only on B, C, but not on A. More specifically, M is the apex of the isosceles right triangle on BC pointed towards A. [5]

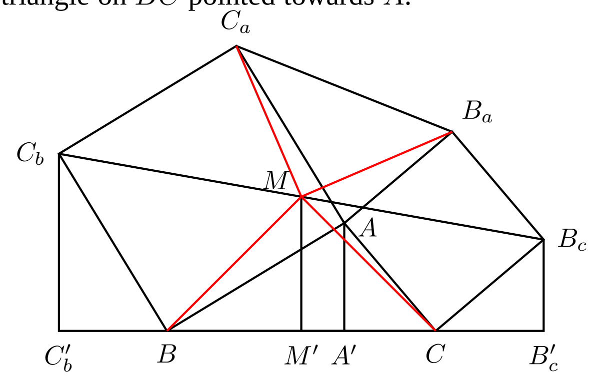


Figure 4

To see this, let A', M', B'_c and C'_b be the orthogonal projections of A, M, B_c and C_b respectively on the line BC. See Figure 4. Triangles $AA'C$ and CB'_cB_c are congruent by rotation through $\pm\frac{\pi}{2}$ about the center of the square CB_cB_aA. Triangles $AA'B$ and BC'_bC_b are congruent in a similar way. So we have $AA' = CB'_c = BC'_b$. It follows that M' is also the midpoint of BC. And we see that $C'_bC_b + B'_c + B_c = BA' + A'C = a$ so $MM' = \frac{a}{2}$. And M is as desired.

By symmetry M is also the apex of the isosceles right triangle on B_aC_a pointed towards A.

We recall that the triangle of apexes of similar isosceles triangles on the sides of ABC is perspective with ABC. The triangle of apexes is called a *Kiepert triangle*, and the *Kiepert perspector* $K(\phi)$ depends on the base angle ϕ (mod π) of the isosceles triangle. [6]

We conclude that AM is the A-Cevian of $K(-\frac{\pi}{4})$, also called the *second Vecten point* of both ABC and the A-flank. From similar observations on the B- and C-flanks, we conclude that the second Vecten point befriends itself.

7. Friendship of Kiepert perspectors

Given any real number t, Let X_t and Y_t be the points that divide CB_c and BC_b such that $CX_t : CB_c = BY_t : BC_b = t : 1$, and let M_t be their midpoint. Then BCM_t is an isosceles triangle, with base angle $\arctan t = \angle BAY_t$. See Figure 5.

Extend AX_t to X'_t on B_aB_c, and AY_t to Y'_t on C_aC_b and let M'_t be the midpoint of $X'_tY'_t$. Then $B_aC_aM'_t$ is an isosceles triangle, with base angle $\arctan\frac{1}{t} = \angle Y'_tAC_a = \frac{\pi}{2} - \angle BAY_t$. Also, by the similarity of triangles AX_tY_t and $AX'_tY'_t$

[5]Bottema introduced this result with the following story. Someone had found a treasure and hidden it in a complicated way to keep it secret. He found three marked trees, A, B and C, and thought of rotating BA through 90 degrees to BC_b, and CA through -90 degrees to CB_c. Then he chose the midpoint M of C_bB_c as the place to hide his treasure. But when he returned, he could not find tree A. He decided to guess its position and try. In a desperate mood he imagined numerous

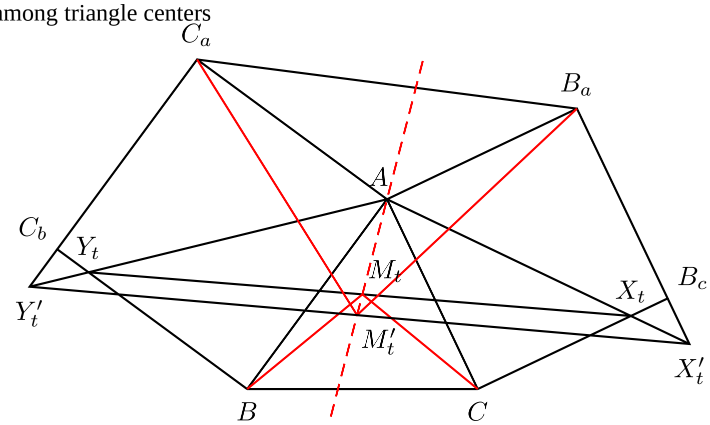


Figure 5

we see that A, M_t and M'_t are collinear. This shows that the Kiepert perspectors $K(\phi)$ and $K(\frac{\pi}{2} - \phi)$ are friends.

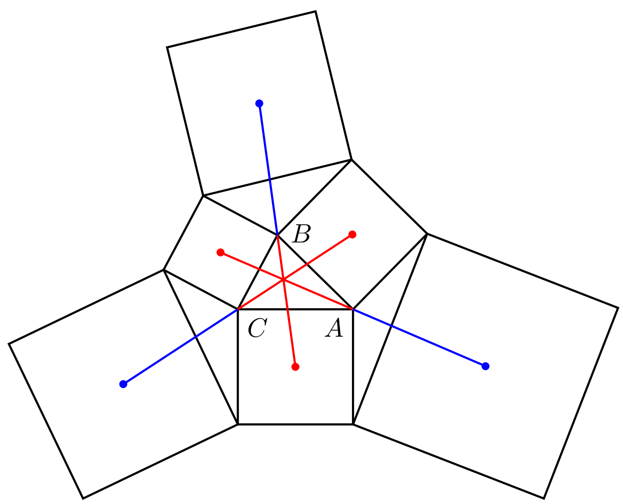


Figure 6

In particular, the first Vecten point $K(\frac{\pi}{4})$ also befriends itself. See Figure 6. The Fermat points $K(\pm\frac{\pi}{3})$ [7] are friends of the the Napoleon points $K(\frac{\pi}{6})$. [8]

Seen collectively, the *Kiepert hyperbola*, the locus of Kiepert perspectors, befriends itself; so does its isogonal transform, the Brocard axis OK.

---

diggings without result. But, much to his surprise, he was able to recover his treasure on the very first try!

[6]By convention, ϕ is positive or negative according as the isosceles triangles are pointing outwardly or inwardly.

[7]These are the points X_{13} and X_{14} in [2, 3], also called the isogenic centers.

[8]These points are labelled X_{17} and X_{18} in [2, 3]. It is well known that the Kiepert triangles are equilateral.

References

[1] O. Bottema, Verscheidenheid XXXVIII, in *Verscheidenheden*, p.51, Nederlandse Vereniging van Wiskundeleraren / Wolters Noordhoff, Groningen (1978).
[2] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.
[3] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://cedar.evansville.edu/~ck6/encyclopedia/.

Floor van Lamoen: Statenhof 3, 4463 TV Goes, The Netherlands
E-mail address: f.v.lamoen@wxs.nl

Forum Geometricorum
Volume 1 (2001) 7–8.

FORUM GEOM

Another Proof of the Erdős-Mordell Theorem

Hojoo Lee

Abstract. We give a proof of the famous Erdős-Mordell inequality using Ptolemy's theorem.

The following neat inequality is well-known:

Theorem. If from a point O inside a given triangle ABC perpendiculars OD, OE, OF are drawn to its sides, then $OA+OB+OC \geq 2(OD+OE+OF)$. Equality holds if and only if triangle ABC is equilateral.

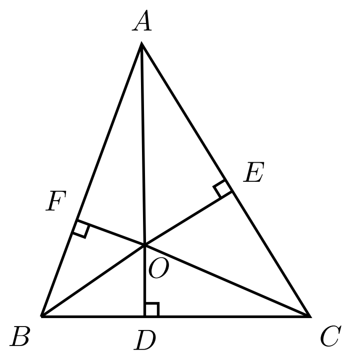


Figure 1

This was conjectured by Paul Erdős in 1935, and first proved by Louis Mordell in the same year. Several proofs of this inequality have been given, using Ptolemy's theorem by André Avez [5], angular computations with similar triangles by Leon Bankoff [2], area inequality by V. Komornik [6], or using trigonometry by Mordell and Barrow [1]. The purpose of this note is to give another elementary proof using Ptolemy's theorem.

Proof. Let HG denote the orthogonal projections of BC on the line FE. See Figure 2. Then, we have $BC \geq HG = HF + FE + EG$. It follows from $\angle BFH = \angle AFE = \angle AOE$ that the right triangles BFH and AOE are similar and $HF = \frac{OE}{OA}BF$. In a like manner we find that $EG = \frac{OF}{OA}CE$. Ptolemy's theorem applied to $AFOE$ gives

$$OA \cdot FE = AF \cdot OE + AE \cdot OF \quad \text{or} \quad FE = \frac{AF \cdot OE + AE \cdot OF}{OA}.$$

Combining these, we have

$$BC \geq \frac{OE}{OA}BF + \frac{AF \cdot OE + AE \cdot OF}{OA} + \frac{OF}{OA}CE,$$

Publication Date: January 29, 2001. Communicating Editor: Paul Yiu.

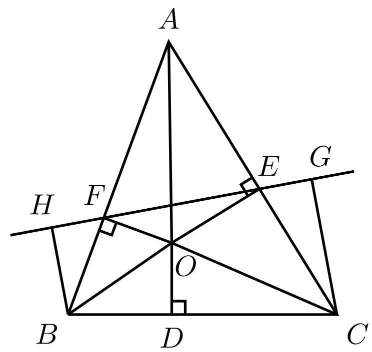


Figure 2

or

$$BC \cdot OA \geq OE \cdot BF + AF \cdot OE + AE \cdot OF + OF \cdot CE = OE \cdot AB + OF \cdot AC.$$

Dividing by BC, we have $OA \geq \frac{AB}{BC}OE + \frac{AC}{BC}OF$.

Applying the same reasoning to other projections, we have

$$OB \geq \frac{BC}{CA}OF + \frac{BA}{CA}OD \quad \text{and} \quad OC \geq \frac{CA}{AB}OD + \frac{CB}{AB}OE.$$

Adding these inequalities, we have

$$OA + OB + OC \geq \left(\frac{BA}{CA} + \frac{CA}{AB}\right)OD + \left(\frac{AB}{BC} + \frac{CB}{AB}\right)OE + \left(\frac{AC}{BC} + \frac{BC}{CA}\right)OF.$$

It follows from this and the inequality $\frac{x}{y} + \frac{y}{x} \geq 2$ (for positive real numbers x, y) that

$$OA + OB + OC \geq 2(OD + OE + OF).$$

It is easy to check that equality holds if and only if $AB = BC = CA$ and O is the circumcenter of ABC. □

References

[1] P. Erdős, L.J. Mordell, and D.F. Barrow, Problem 3740, *Amer. Math. Monthly*, 42 (1935) 396; solutions, *ibid.*, 44 (1937) 252 – 254.

[2] L. Bankoff, An elementary proof of the Erdős-Mordell theorem, *Amer. Math. Monthly*, 65 (1958) 521.

[3] A. Oppenheim, The Erdős inequality and other inequalities for a triangle, *Amer. Math. Monthly*, 68 (1961), 226 - 230.

[4] L. Carlitz, Some inequalities for a triangle, *Amer. Math. Monthly*, 71 (1964) 881 – 885.

[5] A. Avez, A short proof of a theorem of Erdős and Mordell, *Amer. Math. Monthly*, 100 (1993) 60 – 62.

[6] V. Komornik, A short proof of the Erdős-Mordell theorem, *Amer. Math. Monthly*, 104 (1997) 57 – 60.

Hojoo Lee: Department of Mathematics, Kwangwoon University, Wolgye-Dong, Nowon-Gu, Seoul 139-701, Korea

E-mail address: leehojoo@hotmail.com

Forum Geometricorum
Volume 1 (2001) 9–16.

FORUM GEOM

Perspective Poristic Triangles

Edward Brisse

Abstract. This paper answers a question of Yiu: given a triangle ABC, to construct and enumerate the triangles which share the same circumcircle and incircle and are perspective with ABC. We show that there are exactly three such triangles, each easily constructible using ruler and compass.

1. Introduction

Given a triangle ABC with its circumcircle $O(R)$ and incircle $I(r)$, the famous Poncelet - Steiner porism affirms that there is a continuous family of triangles with the same circumcircle and incircle [1, p.86]. Every such triangle can be constructed by choosing an arbitrary point A' on the circle (O), drawing the two tangents to (I), and extending them to intersect (O) again at B' and C'. Yiu [3] has raised the enumeration and construction problems of poristic triangles perspective with triangle ABC, namely, those poristic triangles $A'B'C'$ with the lines AA', BB', CC' intersecting at a common point. We give a complete solution to these problems in terms of the limit points of the coaxial system of circles generated by the circumcircle and the incircle.

Theorem 1. *The only poristic triangles perspective with ABC are:*

(1) *the reflection of ABC in the line OI, the perspector being the infinite point on a line perpendicular to OI,*
(2) *the circumcevian triangles of the two limit points of the coaxial system generated by the circumcircle and the incircle.*

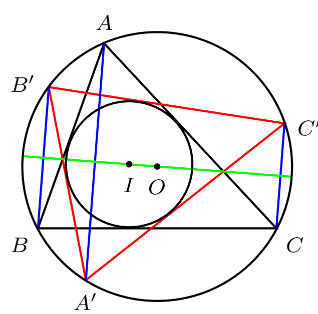


Figure 1

In (1), the lines AA', BB', CC' are all perpendicular to the line OI. See Figure 1. The perspector is the infinite point on a line perpendicular to OI. One such line

Publication Date: February 8, 2001. Communicating Editor: Paul Yiu.

is the trilinear polar of the incenter $I = (a : b : c)$, with equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$$

in homogeneous barycentric coordinates. The perspector is therefore the point $(a(b-c) : b(c-a) : c(a-b))$. We explain in §§2, 3 the construction of the two triangles in (2), which are symmetric with respect to the line OI. See Figure 2. In §4 we justify that these three are the only poristic triangles perspective with ABC.

2. Poristic triangles from an involution in the upper half-plane

An easy description of the poristic triangles in Theorem 1(2) is that these are the circumcevian triangles of the common poles of the circumcircle and the incircle. There are two such points; each of these has the same line as the polar with respect the circumcircle and the incircle. These common poles are symmetric with respect

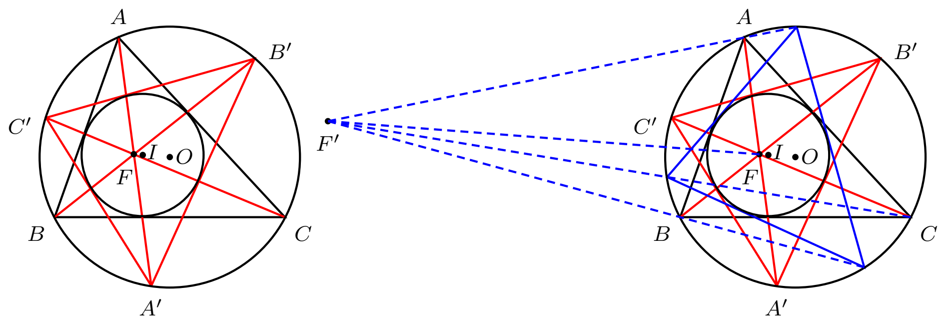


Figure 2

to the radical axis of the circles (O) and (I), and are indeed the *limit points* of the coaxial system of circles generated by (O) and (I).[1] This is best explained by the introduction of an involution of the upper half-plane. Let $a > 0$ be a fixed real number. Consider in the upper half-plane $\mathcal{R}^2_+ := \{(x, y) : y > 0\}$ a family of circles

$$\mathcal{C}_b : \qquad x^2 + y^2 - 2by + a^2 = 0, \qquad b \geq a.$$

Each circle $\mathcal{C}_b$ has center $(0, b)$ and radius $\sqrt{b^2 - a^2}$. See Figure 3. Every point in $\mathcal{R}^2_+$ lies on a unique circle $\mathcal{C}_b$ in this family. Specifically, if

$$b(x, y) = \frac{x^2 + y^2 + a^2}{2y},$$

the point (x, y) lies on the circle $\mathcal{C}_{b(x,y)}$. The circle $\mathcal{C}_a$ consists of the single point $F = (0, a)$. We call this the limit point of the family of circles. Every pair of circles in this family has the x-axis as radical axis. By reflecting the system of circles about the x-axis, we obtain a complete coaxial system of circles. The reflection of F, namely, the point $F' = (0, -a)$, is the other limit point of this system. Every circle through F and F' is orthogonal to every circle $\mathcal{C}_b$.

[1]The common polar of each one of these points with respect to the two circles passes through the other.

Consider a line through the limiting point F, with slope m, and therefore equation $y = mx + a$. This line intersects the circle $\mathcal{C}_b$ at points whose y-coordinates are the roots of the quadratic equation

$$(1 + m^2)y^2 - 2(a + bm^2)y + a^2(1 + m^2) = 0.$$

Note that the two roots multiply to a^2. Thus, if one of the intersections is (x, y), then the other intersection is $(-\frac{ax}{y}, \frac{a^2}{y})$. See Figure 4. This defines an involution on the upper half plane:

$$P^* = (-\frac{ax}{y}, \frac{a^2}{y}) \quad \text{for} \quad P = (x, y).$$

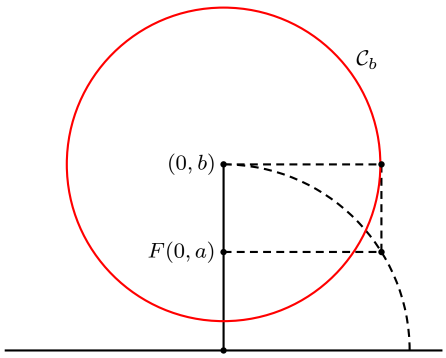


Figure 3

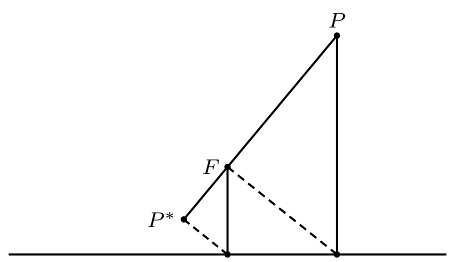


Figure 4

Proposition 2. *(1) $P^{**} = P$.*
(2) P and P^* belong to the same circle in the family $\mathcal{C}_b$. In other words, if P lies on the circle $\mathcal{C}_b$, then the line FP intersects the same circle again at P^*.
(3) The line PF' intersects the circle $\mathcal{C}_b$ at the reflection of P^* in the y-axis.

Proof. (1) is trivial. (2) follows from $b(P) = b(P^*)$. For (3), the intersection is the point $(\frac{ax}{y}, \frac{a^2}{y})$. □

Lemma 3. *Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be two points on the same circle $\mathcal{C}_b$. The segment AB is tangent to a circle $\mathcal{C}_{b'}$ at the point whose y-coordinate is $\sqrt{y_1 y_2}$.*

Proof. This is clear if $y_1 = y_2$. In the generic case, extend AB to intersect the x-axis at a point C. The segment AB is tangent to a circle $\mathcal{C}_{b'}$ at a point P such that $CP = CF$. It follows that $CP^2 = CF^2 = CA \cdot CB$. Since C is on the x-axis, this relation gives $y^2 = y_1 y_2$ for the y-coordinate of P. □

Theorem 4. *If a chord AB of $\mathcal{C}_b$ is tangent to $\mathcal{C}_{b'}$ at P, then the chord A^*B^* is tangent to the same circle $\mathcal{C}_{b'}$ at P^*.*

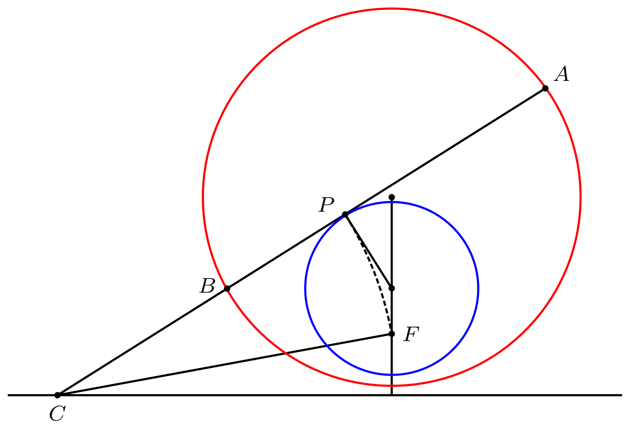


Figure 5

Proof. That P and P^* lie on the same circle is clear from Proposition 2(1). It remains to show that P^* is the correct point of tangency. This follows from noting that the y-coordinate of P^*, being $\frac{a^2}{\sqrt{y_1 y_2}}$, is the geometric mean of those of A^* and B^*. □

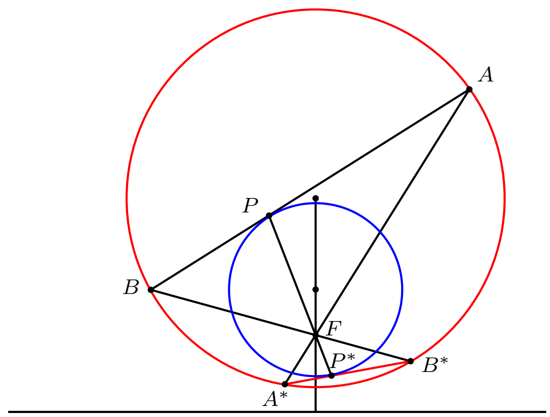


Figure 6

Consider the circumcircle and incircle of triangle ABC. These two circles generate a coaxial system with limit points F and F'.

Corollary 5. *The triangle $A^*B^*C^*$ has $I(r)$ as incircle, and is perspective with ABC at F.*

Corollary 6. *The reflection of the triangle $A^*B^*C^*$ in the line OI also has $I(r)$ as incircle, and is perspective with ABC at the point F'.*

Proof. This follows from Proposition 2 (3). □

It remains to construct the two limit points F and F', and the construction of the two triangles in Theorem 1(2) would be complete.

Proposition 7. *Let XY be the diameter of the circumcircle through the incenter I. If the tangents to the incircle from these two points are XP, XQ', YQ, and YP' such that P and Q are on the same side of OI, then PP' intersects OI at F (so does QQ'), and PQ intersects OI at F' (so does $P'Q'$).*

Proof. This follows from Theorem 4 by observing that Y = X*. □

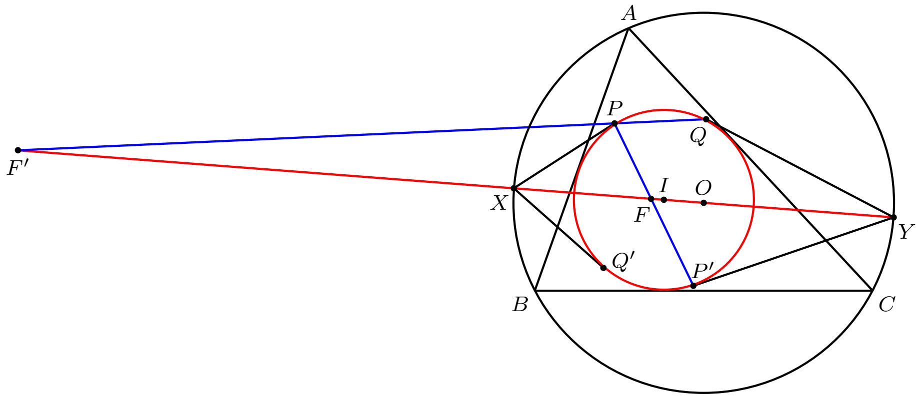


Figure 7

3. Enumeration of perspective poristic triangles

In this section, we show that the poristic triangles constructed in the preceding sections are the only ones perspective with ABC. To do this, we adopt a slightly different viewpoint, by searching for circumcevian triangles which share the same incircle with ABC. We work with homogeneous barycentric coordinates. Recall that if a, b, c are the lengths of the sides BC, CA, AB respectively, then the circumcircle has equation

$$a^2yz + b^2zx + c^2xy = 0,$$

and the incircle has equation

$$\begin{aligned}(s-a)^2x^2 + (s-b)^2y^2 + (s-c)^2z^2 - 2(s-b)(s-c)yz\\ -2(s-c)(s-a)zx - 2(s-a)(s-b)xy = 0,\end{aligned}$$

where $s = \frac{1}{2}(a+b+c)$.

We begin with a lemma.

Lemma 8. *The tangents from a point $(u : v : w)$ on the circumcircle (O) to the incircle (I) intersect the circumcircle again at two points on the line*

$$\frac{(s-a)u}{a^2}x + \frac{(s-b)v}{b^2}y + \frac{(s-c)w}{c^2}z = 0.$$

Remark: This line is tangent to the incircle at the point

$$\left(\frac{a^4}{(s-a)u^2}\ :\ \frac{b^4}{(s-b)v^2}\ :\ \frac{c^4}{(s-c)w^2}\right).$$

Given a point $P = (u : v : w)$ in homogeneous barycentric coordinates, the circumcevian triangle $A'B'C'$ is formed by the *second* intersections of the lines AP, BP, CP with the circumcircle. These have coordinates

$$A' = (\frac{-a^2vw}{b^2w+c^2v} : v : w), \quad B' = (u : \frac{-b^2wu}{c^2u+a^2w} : w), \quad C' = (u : v : \frac{-c^2uv}{a^2v+b^2u}).$$

Applying Lemma 8 to the point A', we obtain the equation of the line $B'C'$ as

$$\frac{-(s-a)vw}{b^2w+c^2v}x + \frac{(s-b)v}{b^2}y + \frac{(s-c)w}{c^2}z = 0.$$

Since this line contains the points B' and C', we have

$$-\frac{(s-a)uvw}{b^2w+c^2v} - \frac{(s-b)uvw}{c^2u+a^2w} + \frac{(s-c)w^2}{c^2} = 0, \tag{1}$$

$$-\frac{(s-a)uvw}{b^2w+c^2v} + \frac{(s-b)v^2}{b^2} - \frac{(s-c)uvw}{b^2u+a^2v} = 0. \tag{2}$$

The difference of these two equations gives

$$\frac{a^2vw+b^2wu+c^2uv}{b^2c^2(b^2u+a^2v)(c^2u+a^2w)} \cdot f = 0, \tag{3}$$

where

$$f = -b^2c^2(s-b)uv + b^2c^2(s-c)wu - c^2a^2(s-b)v^2 + a^2b^2(s-c)w^2.$$

If $a^2vw + b^2wu + c^2uv = 0$, the point $(u : v : w)$ is on the circumcircle, and both equations (1) and (2) reduce to

$$\frac{s-a}{a^2}u^2 + \frac{s-b}{b^2}v^2 + \frac{s-c}{c^2}w^2 = 0,$$

clearly admitting no real solutions. On the other hand, setting the quadratic factor f in (3) to 0, we obtain

$$u = \frac{-a^2}{b^2c^2} \cdot \frac{c^2(s-b)v^2 - b^2(s-c)w^2}{(s-b)v-(s-c)w}.$$

Substitution into equation (1) gives

$$\frac{vw(c(a-b)v-b(c-a)w)}{b^2c^2(c^2v^2-b^2w^2)(v(s-b)-w(s-c))} \cdot g = 0, \tag{4}$$

where

$$\begin{aligned} g \ &= \ c^3(s-b)(a^2+b^2-c(a+b))v^2 + b^3(s-c)(c^2+a^2-b(c+a))w^2 \\ &\quad +2bc(s-b)(s-c)(b^2+c^2-a(b+c))vw. \end{aligned}$$

There are two possibilities.

(i) If $c(a-b)v - b(c-a)w = 0$, we obtain $v : w = b(c-a) : c(a-b)$, and consequently, $u : v : w = a(b-c) : b(c-a) : c(a-b)$. This is clearly an infinite point, the one on the line $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$, the trilinear polar of the incenter. This line is perpendicular to the line OI. This therefore leads to the triangle in Theorem 1(1).

(ii) Setting the quadratic factor g in (4) to 0 necessarily leads to the two triangles constructed in §2. The corresponding perspectors are the two limit points of the coaxial system generated by the circumcircle and the incircle.

4. Coordinates

The line OI has equation

$$\frac{(b-c)(s-a)}{a}x + \frac{(c-a)(s-b)}{b}y + \frac{(a-b)(s-c)}{c}z = 0.$$

The radical axis of the two circles is the line

$$(s-a)^2x + (s-b)^2y + (s-c)^2z = 0.$$

These two lines intersect at the point

$$\left(\frac{a(a^2(b+c) - 2a(b^2-bc+c^2) + (b-c)^2(b+c))}{b+c-a} : \cdots : \cdots\right),$$

where the second and third coordinates are obtained from the first by cyclic permutations of a, b, c. This point is not found in [2].

The coordinates of the common poles F and F' are

$$(a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)) + t(a : b : c)$$

where

$$t = \frac{1}{2}\left(-2abc + \sum_{\text{cyclic}}(a^3 - bc(b+c))\right) \pm 2\triangle\sqrt{2ab+2bc+2ca-a^2-b^2-c^2}, \tag{5}$$

and $\triangle$ = area of triangle ABC. This means that the points F and F' divide harmonically the segment joining the incenter $I(a : b : c)$ to the point whose homogeneous barycentric coordinates are

$$(a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)) + \frac{1}{2}\left(-2abc + \sum_{\text{cyclic}}(a^3 - bc(b+c))\right)(a : b : c).$$

This latter point is the triangle center

$$X_{57} = \left(\frac{a}{b+c-a} : \frac{b}{c+a-b} : \frac{c}{a+b-c}\right)$$

in [2], which divides the segment OI in the ratio $OX_{57} : OI = 2R+r : 2R-r$. The common poles F and F', it follows from (5) above, divide the segment IX_{57} harmonically in the ratio $2R-r : \pm\sqrt{(4R+r)r}$.

References

[1] N. Altshiller-Court, *College Geometry*, 2nd edition, 1952, Barnes and Noble, New York.
[2] C. Kimberling, *Encyclopedia of Triangle Centers*,
http://cedar.evansville.edu/ ck6/encyclopedia/ .
[3] P. Yiu, Hyacinthos messages 999 and 1004, June, 2000,
http://groups.yahoo.com/group/Hyacinthos.

Edward Brisse: 5628 Saint-André, Montréal, Québec, H2S 2K1, Canada
E-mail address: ebcfgl@videotron.ca

Forum Geometricorum
Volume 1 (2001) 17–24.

FORUM GEOM

Heron Triangles:
A Gergonne-Cevian-and-Median Perspective

K.R.S. Sastry

Abstract. We give effective constructions of Heron triangles by considering the intersection of a median and a cevian through the Gergonne point.

1. Introduction

Heron gave the triangle area formula in terms of the sides a, b, c:

$$(*) \qquad \triangle = \sqrt{s(s-a)(s-b)(s-c)}, \qquad s = \frac{1}{2}(a+b+c).$$

He is further credited with the discovery of the integer sided and integer area triangle (13,14,15;84). Notice that this is a non-Pythagorean triangle, *i.e.*, it does not contain a right angle. We might as well say that with this discovery he challenged us to determine triangles having integer sides and area, *Heron triangles*. Dickson [3] sketches the early attempts to meet this challenge. The references [1, 4, 5, 6, 7, 8, 10, 11] describe recent attempts in that direction. The present discussion uses the intersection point of a Gergonne cevian (the line segment between a vertex and the point of contact of the incircle with the opposite side) and a median to generate Heron triangles. Why do we need yet another description? The answer is simple: Each new description provides new ways to solve, and hence to acquire new insights into, earlier Heron problems. More importantly, they pose new Heron challenges. We shall illustrate this. Dickson uses the name Heron triangle to describe one having rational sides and area. However, these rationals can always be rendered integers. Therefore for us a Heron triangle is one with integer sides and area except under special circumstances.

We use the standard notation: a, b, c for the sides BC, CA, AB of triangle ABC. We use the word side also in the sense of the length of a side. Furthermore, we assume $a \geq c$. No generality is lost in doing so because we may relabel the vertices if necessary.

Publication Date: February 12, 2001. Communicating Editor: Paul Yiu.

2. A preliminary result

We first solve this problem: Suppose three cevians of a triangle concur at a point. How does one determine the ratio in which the concurrence point sections one of them? The answer is given by

Theorem 1 (van Aubel [2, p.163])**.** *Let the cevians AD, BE, CF of triangle ABC concur at the point S. Then*

$$\frac{AS}{SD} = \frac{AE}{EC} + \frac{AF}{FB}.$$

Proof. Let $[T]$ denote the area of triangle T. We use the known result: if two triangles have a common altitude, then their areas are proportional to the corresponding bases. Hence, from Figure 1,

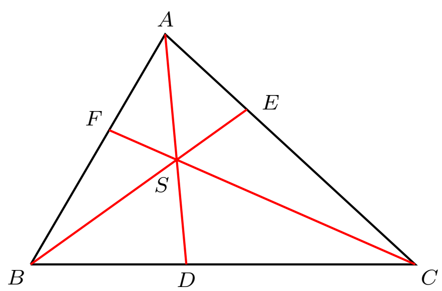


Figure 1

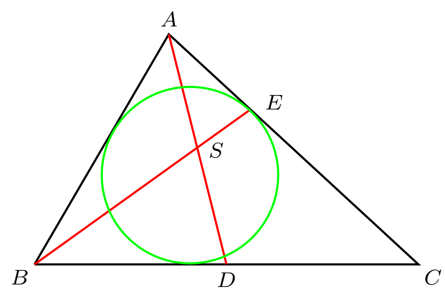


Figure 2

$$\frac{AS}{SD} = \frac{[ABS]}{[SBD]} = \frac{[ASC]}{[SDC]} = \frac{[ABS]+[ASC]}{[SBD]+[SDC]} = \frac{[ABS]}{[SBC]} + \frac{[ASC]}{[SBC]}. \tag{1}$$

But

$$\frac{AE}{EC} = \frac{[ABE]}{[EBC]} = \frac{[ASE]}{[ESC]} = \frac{[ABE]-[ASE]}{[EBC]-[ESC]} = \frac{[ABS]}{[SBC]}, \tag{2}$$

and likewise,

$$\frac{AF}{FB} = \frac{[ASC]}{[SBC]}. \tag{3}$$

Now, (1), (2), (3) complete the proof. □

In the above proof we used a property of equal ratios, namely, if $\frac{p}{q} = \frac{r}{s} = k$, then $k = \frac{p \pm q}{r \pm s}$. From Theorem 1 we deduce the following corollary that is important for our discussion.

Corollary 2. *In Figure 2, let AD denote the median, and BE the Gergonne cevian. Then* $\frac{AS}{SD} = \frac{2(s-a)}{s-c}$.

Proof. The present hypothesis implies $BD = DC$, and E is the point where the incircle is tangent with AC. It is well - known that $AE = s - a$, $EC = s - c$. Now, Ceva's theorem, $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$, yields $\frac{AF}{FB} = \frac{s-a}{s-c}$. Then Theorem 1 upholds the claim of Corollary 2. $\square$

In the case of a Heron triangle, a, b, c and s are natural numbers. Therefore, $\frac{AS}{SD} = \frac{2(s-a)}{s-c} = \lambda$ is a rational ratio. Of course this will be true more generally even if $\triangle$ is not an integer; but that is beside the main point. Also, $a \geq c$ implies that $0 < \lambda \leq 2$. Next we show how each rational number λ generates an infinite family, a $\lambda-$family of Heron triangles.

3. Description of $\lambda-$family of Heron triangles

Theorem 3 gives expressions for the sides of the Heron triangle in terms of λ. At present we do not transform these rational sides integral. However, when we specify a rational number for λ then we do express a, b, c integral such that $\gcd(a, b, c) = 1$. An exception to this common practice may be made in the solution of a Heron problem that requires $\gcd(a, b, c) > 1$, in (*D1*) later, for example.

Theorem 3. *Let λ be a rational number such that $0 < \lambda \leq 2$. The $\lambda-$family of Heron triangles is described by*

$$(a, b, c) = (2(m^2 + \lambda^2 n^2),\ (2 + \lambda)(m^2 - 2\lambda n^2),\ \lambda(m^2 + 4n^2)),$$

m, n being relatively prime natural numbers such that $m > \sqrt{2\lambda} \cdot n$.

Proof. From the definition we have

$$\frac{2(s-a)}{s-c} = \lambda \quad \text{or} \quad b = \frac{2+\lambda}{2-\lambda}(a-c).$$

If $\lambda \neq 2$, we assume $a - c = (2 - \lambda)p$. This gives $b = (2 + \lambda)p$. If $\lambda = 2$, then we define $b = 4p$. The rest of the description is common to either case. Next we calculate

$$\begin{aligned} a &= (2-\lambda)p + c, \quad s = c + 2p, \quad \text{and from } (*), \\ \triangle^2 &= 2\lambda p^2 (c + 2p)(c - \lambda p). \end{aligned} \tag{4}$$

To render (a, b, c) Heron we must have $(c + 2p)(c - \lambda p) = 2\lambda q^2$. There is no need to distinguish two cases: 2λ itself a rational square or not. This fact becomes clearer later when we deduce Corollary 5. With the help of a rational number $\frac{m}{n}$ we may write down

$$c + 2p = \frac{m}{n} q, \quad \text{and} \quad c - \lambda p = \frac{n}{m}(2\lambda q).$$

We solve the above simultaneous equations for p and c:

$$p = \frac{m^2 - 2\lambda n^2}{(2+\lambda)mn} \cdot q, \qquad c = \frac{\lambda(m^2 + 4n^2)}{(2+\lambda)mn} \cdot q.$$

This yields

$$\frac{p}{m^2 - 2\lambda n^2} = \frac{q}{(2+\lambda)mn} = \frac{c}{\lambda(m^2+4n^2)}.$$

Since p, q, c, λ, m, n are positive we must have $m > \sqrt{2\lambda} \cdot n$. We may ignore the constant of proportionality so that

$$p = m^2 - 2\lambda n^2, \qquad q = (2+\lambda)mn \qquad c = \lambda(m^2+4n^2).$$

These values lead to the expressions for the sides a, b, c in the statement of Theorem 3. Also, $\triangle = 2\lambda(2+\lambda)mn(m^2 - 2\lambda n^2)$, see (4), indicates that the area is rational. □

Here is a numerical illustration. Let $\lambda = 1$, $m = 4$, $n = 1$. Then Theorem 3 yields $(a, b, c) = (34, 42, 20)$. Here $\gcd(a, b, c) = 2$. In the study of Heron triangles often gcd$(a, b, c) > 1$. In such a case we divide the side length values by the gcd to list primitive values. Hence, $(a, b, c) = (17, 21, 10)$.

Now, suppose $\lambda = \frac{3}{2}$, $m = 5$, $n = 2$. Presently, Theorem 3 gives $(a, b, c) = (68, \frac{91}{2}, \frac{123}{2})$. As it is, the sides b and c are not integral. In this situation we render the sides integral (and divide by the gcd if it is greater than 1) so that $(a, b, c) = (136, 91, 123)$.

We should remember that Theorem 3 yields the same Heron triangle more than once if we ignore the order in which the sides appear. This depends on the number of ways in which the sides a, b, c may be permuted preserving the constraint $a \geq c$. For instance, the $(17, 21, 10)$ triangle above for $\lambda = 1$, $m = 4$, $n = 1$ may also be obtained when $\lambda = \frac{3}{7}$, $m = 12$, $n = 7$, or when $\lambda = \frac{6}{7}$, $m = 12$, $n = 7$. The verification is left to the reader. It is time to deduce a number of important corollaries from Theorem 3.

Corollary 4. *Theorem 3 yields the Pythagorean triangles* $(a, b, c) = (u^2+v^2, u^2-v^2, 2uv)$ *for* $\lambda = \dfrac{2v}{u}$, $m = 2$, $n = 1$.

Incidentally, we observe that the famous generators u, v of the Pythagorean triples/triangles readily tell us the ratio in which the Gergonne cevian BE intersects the median AD. Similar observation may be made throughout in an appropriate context.

Corollary 5. *Theorem 3 yields the isosceles Heron triangles* $(a, b, c) = (m^2 + n^2, 2(m^2 - n^2), m^2 + n^2)$ *for* $\lambda = 2$.

Actually, $\lambda = 2$ yields $(a, b, c) = (m^2+4n^2, 2(m^2-4n^2), m^2+4n^2)$. However, the transformation $m \mapsto 2m$, $n \mapsto n$ results in the more familiar form displayed in Corollary 5.

Corollary 6. *Theorem 3 describes the complete set of Heron triangles.*

This is because the Gergonne cevian BE must intersect the median AD at a unique point. Therefore for all Heron triangles $0 < \lambda \leq 2$. Suppose first we fix λ at such a rational number. Then Theorem 3 gives the entire $\lambda-$family of Heron triangles each member of which has BE intersecting AD in the same ratio, that is

λ. Next we vary λ over rational numbers $0 < \lambda \le 2$. By successive applications of the preceding remark the claim of Corollary 6 follows.

Corollary 7. *[Hoppe's Problem] Theorem 3 yields Heron triangles* $(a, b, c) = (m^2 + 9n^2, 2(m^2 + 3n^2), 3(m^2 + n^2))$ *having the sides in arithmetic progression for* $\lambda = \dfrac{m^2}{6n^2}$.

Here too a remark similar to the one following Corollary 5 applies. Corollaries 4 through 7 give us the key to the solution, often may be partial solutions of many Heron problems: Just consider appropriate $\lambda-$family of Heron triangles. We will continue to amplify on this theme in the sections to follow. To richly illustrate this we prepare a table of $\lambda-$families of Heron triangles. In Table 1, π denotes the perimeter of the triangle.

Table 1. $\lambda-$families of Heron triangles

λ	a	b	c	π	$\triangle$
1	$2(m^2+n^2)$	$3(m^2-2n^2)$	m^2+4n^2	$6m^2$	$6mn(m^2-2n^2)$
$1/2$	$4m^2+n^2$	$5(m^2-n^2)$	m^2+4n^2	$10m^2$	$10mn(m^2-n^2)$
$1/3$	$2(9m^2+n^2)$	$7(3m^2-2n^2)$	$3(m^2+4n^2)$	$42m^2$	$42mn(3m^2-2n^2)$
$2/3$	$9m^2+4n^2$	$4(3m^2-4n^2)$	$3(m^2+4n^2)$	$24m^2$	$24mn(3m^2-4n^2)$
$1/4$	$16m^2+n^2$	$9(2m^2-n^2)$	$2(m^2+4n^2)$	$36m^2$	$36mn(2m^2-n^2)$
$3/4$	$16m^2+9n^2$	$11(2m^2-3n^2)$	$6(m^2+4n^2)$	$44m^2$	$132mn(2m^2-3n^2)$
$1/5$	$2(25m^2+n^2)$	$11(5m^2-2n^2)$	$5(m^2+4n^2)$	$110m^2$	$110mn(5m^2-2n^2)$
$2/5$	$25m^2+4n^2$	$6(5m^2-4n^2)$	$5(m^2+4n^2)$	$60m^2$	$60mn(5m^2-4n^2)$
$3/5$	$2(25m^2+9n^2)$	$13(5m^2-6n^2)$	$15(m^2+4n^2)$	$130m^2$	$390mn(5m^2-6n^2)$
$4/5$	$25m^2+16n^2$	$7(5m^2-8n^2)$	$10(m^2+4n^2)$	$70m^2$	$140mn(5m^2-8n^2)$
$3/2$	$4m^2+9n^2$	$7(m^2-3n^2)$	$3(m^2+4n^2)$	$14m^2$	$42mn(m^2-3n^2)$
$4/3$	$9m^2+16n^2$	$5(3m^2-8n^2)$	$6(m^2+4n^2)$	$30m^2$	$60mn(3m^2-8n^2)$
$5/3$	$2(9m^2+25n^2)$	$11(3m^2-10n^2)$	$15(m^2+4n^2)$	$66m^2$	$330mn(3m^2-10n^2)$
$5/4$	$16m^2+25n^2$	$13(2m^2-5n^2)$	$10(m^2+4n^2)$	$52m^2$	$260mn(2m^2-5n^2)$
$7/4$	$16m^2+49n^2$	$15(2m^2-7n^2)$	$14(m^2+4n^2)$	$60m^2$	$420mn(2m^2-7n^2)$

4. Heron problems and solutions

In what follows we omit the word "determine" from each problem statement. "Heron triangles" will be contracted to HT, and we do *not* provide solutions in detail.

A. Involving sides. A1. HT in which two sides differ by a desired integer. In fact finding one such triangle is equivalent to finding an infinity! This is because they depend on the solution of the so-called Fermat-Pell equation $x^2 - dy^2 = e$, where e is an integer and d not an integer square. It is well-known that Fermat-Pell equations have an infinity of solutions (x, y) (i) when $e = 1$ and (ii) when $e \neq 1$ if there is one. The solution techniques are available in an introductory number theory text, or see [3].

HT in which the three sides are consecutive integers are completely given by Corollary 7. For example, $m = 3$, $n = 1$ gives the (3,4,5); $m = 2$, $n = 1$, the (13,14,15), and so on. Here two sides differ by 1 and incidentally, two sides by 2. However, there are other HT in which two sides differ by 1 (or 2). For another partial solution, consider $\lambda = 1$ family from Table 1. Here $a - c = 1 \Longleftrightarrow m^2 - 2n^2 = 1$. $m = 3$, $n = 2$ gives the $(26, 3, 25)$. $m = 17$, $n = 12$, the (866,3,865) triangle and so on. We observe that 3 is the common side of an infinity of HT. Actually, it is known that *every* integer greater than 2 is a common side of an infinity of HT [1, 3].

To determine a HT in which two sides differ by 3, take $\lambda = \frac{1}{2}$ family and set $b - a = 3$. This leads to the equation $m^2 - 6n^2 = 3$. The solution $(m, n) = (3, 1)$ gives $(a, b, c) = (37, 40, 13)$; $(m, n) = (27, 11)$ gives $(3037, 3040, 1213)$ and so on. This technique can be extended.

A2. A pair of HT having a common side. Consider the pairs $\lambda = 1$, $\lambda = \frac{1}{2}$; $\lambda = \frac{1}{3}$, $\lambda = \frac{2}{3}$; or some two distinct $\lambda-$families that give identical expressions for a particular side. For instance, $m = 3$, $n = 1$ in $\lambda = \frac{1}{3}$ and $\lambda = \frac{2}{3}$ families yields a pair $(164, 175, 39)$ and $(85, 92, 39)$. It is now easy to obtain as many pairs as one desires. This is a quicker solution than the one suggested by *A1*.

A3. A pair of HT in which a pair of corresponding sides are in the ratio $1 : 2$, $1 : 3$, $2 : 3$ etc. The solution lies in the column for side c.

A4. A HT in which two sides sum to a square. We consider $\lambda = \frac{1}{2}$ family where $a + c = 5(m^2 + n^2)$ is made square by $m = 11$, $n = 2$; $(488, 585, 137)$. It is now a simple matter to generate any number of them.

B. Involving perimeter. The perimeter column shows that it is a function of the single parameter m. This enables us to pose, and solve almost effortlessly, many perimeter related problems. To solve such problems by traditional methods would often at best be extremely difficult. Here we present a sample.

B1. A HT in which the perimeter is a square. A glance at Table 1 reveals that $\pi = 36m^2$ for $\lambda = \frac{1}{4}$ family. An infinity of primitive HT of this type is available.

B2. A pair of HT having equal perimeter. An infinity of solution is provided by the $\lambda = \frac{2}{5}$ and $\lambda = \frac{7}{4}$ families. All that is needed is to substitute identical values for m and suitable values to n to ensure the outcome of primitive HT.

B3. A finite number of HT all with equal perimeter. The solution is unbelievably simple! Take *any* λ family and put sufficiently large constant value for m and then vary the values of n only.

A pair of HT in which one perimeter is twice, thrice, ... another, or three or more HT whose perimeters are in arithmetic progression, or a set of four HT such that the sum or the product of two perimeters equals respectively the sum or the product of the other two perimeters are simple games to play. More extensive tables of $\lambda-$family HT coupled with a greater degree of observation ensures that ingenius problem posing solving activity runs wild.

C. Involving area. The $\lambda = \frac{1}{2}$ family has $\triangle = 10mn(m^2 - n^2)$. Now, $mn(m^2 - n^2)$ gives the area of the Pythagorean triangle $(m^2 - n^2, 2mn, m^2 + n^2)$. Because of this an obvious problem has posed and solved itself:

C1. Given a Pythagorean triangle there exists a non-Pythagorean Heron triangle such that the latter area is ten times the former.

It may happen that sometimes one of them may be primitive and the other not, or both may not be primitive. Also, for $m = 2$, $n = 1$, both are Pythagorean. However, there is the $(6, 25, 29)$ Heron triangle with $\triangle = 60$. This close relationship should enable us to put known vast literature on Pythagorean problems to good use, see the following problem for example.

C2. Two Heron triangles having equal area; two HT having areas in the ratio $r : s$.

In [3], pp. 172 – 175, this problem has been solved for right triangles. The primitive solutions are not guaranteed.

D. Miscellaneous problems. In this section we consider problems involving both perimeter and area.

D1. HT in which perimeter equals area. This is such a popular problem that it continues to resurface. It is known that there are just five such HT. The reader is invited to determine them. Hint: They are in $\lambda = \frac{1}{4}, \frac{1}{3}, \frac{2}{5}$, 1 and $\frac{4}{3}$ families. Possibly elsewhere too, see the remark preceding Corollary 4.

D2. HT in which π and $\triangle$ are squares. In $\lambda = \frac{1}{4}$ family we put $m = 169$, $n = 1$.

D3. Pairs of HT with equal perimeter and equal area in each pair. An infinity of such pairs may be obtained from $\lambda = \frac{1}{2}$ family. We put $m = u^2 + uv + v^2$, $n_1 = u^2 - v^2$ and $m = u^2 + uv + v^2$, $n_2 = 2uv + v^2$. For instance, $u = 3$, $v = 1$ *i.e.*, $m = 13$, $n_1 = 8$, $n_2 = 7$ produces a desired pair $(148, 105, 85)$ and $(145, 120, 73)$. They have $\pi_1 = \pi_2 = 338$ and $\triangle_1 = \triangle_2 = 4368$.

If we accept pairs of HT that may not be primitive then we may consider $\lambda = \frac{2}{3}$ family. Here, $m = p^2 + 3q^2$, $n_1 = p^2 - 3q^2$ and $m = p^2 + 3q^2$, $n_2 = \frac{1}{2}(-p^2 + 6pq + 3q^2)$.

E. Open problems. We may look upon the problem (D3) as follows: $\dfrac{\pi_2}{\pi_1} = \dfrac{\triangle_2}{\triangle_1} = 1$. This immediately leads to the following

Open problem 1. Suppose two HTs have perimeters π_1, π_2 and areas $\triangle_1$, $\triangle_2$ such that $\dfrac{\pi_2}{\pi_1} = \dfrac{\triangle_2}{\triangle_1} = \dfrac{p}{q}$, a rational number. Prove or disprove the existence of an infinity of HT such that for each pair $\dfrac{\pi_2}{\pi_1} = \dfrac{\triangle_2}{\triangle_1} = \dfrac{p}{q}$ holds.

For instance, $\lambda_1 = \frac{1}{5}$, (odd) $m_1 > 4k$, $n_1 = 4k$ and $\lambda_2 = \frac{4}{5}$, $m_2 > 4k$ (again odd), $m_2 = m_1$, $n_2 = 2k$ yield $\dfrac{\pi_2}{\pi_1} = \dfrac{\triangle_2}{\triangle_1} = \dfrac{11}{7}$ for $k = 1, 2, 3, \ldots$.

With some effort it is possible to find an infinity of pairs of HT such that for each pair, $\dfrac{\triangle_2}{\triangle_1} = e \cdot \dfrac{\pi_2}{\pi_1}$ for certain natural numbers e. This leads to

Open problem 2. Let e be a given natural number. Prove or disprove the existence of an infinity of pairs of HT such that for each pair $\dfrac{\triangle_2}{\triangle_1} = e \cdot \dfrac{\pi_2}{\pi_1}$ holds.

5. Conclusion

The present description of Heron triangles did provide simple solutions to certain Heron problems. Additionally it suggested new ones that arose naturally in our discussion. The reader is encouraged to try other $\lambda-$families for different solutions from the presented ones. There is much scope for problem posing and solving activity. Non-standard problems such as: find three Heron triangles whose perimeters (areas) are themselves the sides of a Heron triangle or a Pythagorean triangle. Equally important is to pose unsolved problems. A helpful step in this direction would be to consider Heron analogues of the large variety of existing Pythagorean problems.

References

[1] J.R. Carlson, Determination of Heronian triangles, *Fibonnaci Quarterly*, 8 (1970) 499 – 506, 551.
[2] N. Altshiller-Court, *College Geometry*, 2nd edition, 1952, Barnes and Noble, New York.
[3] L.E. Dickson, *History of the Theory of Numbers*, vol. II, Chelsea, New York, New York, 1971; pp.171 – 201.
[4] K.R.S. Sastry, Heron problems, *Math. Comput. Ed.*, 29 (1995) 192 – 202.
[5] K.R.S. Sastry, Heron triangles: a new perspective, *Aust. Math. Soc. Gazette*, 26 (1999) 160 – 168.
[6] K.R.S. Sastry, Heron triangles: an incenter perspective, *Math. Mag.*, 73 (2000) 388 – 392.
[7] K.R.S. Sastry, A Heron difference, *Crux Math. with Math. Mayhem*, 27 (2001) 22 – 26.
[8] K.R.S. Sastry, Heron angles, to appear in *Math. Comput. Ed.*.
[9] D. Singmaster, Some corrections to Carlson's "Determination of Heronian triangles", *Fibonnaci Quarterly*, 11 (1973) 157 – 158.
[10] P. Yiu, Isosceles triangles equal in perimeter and area, *Missouri J. Math. Sci.*, 10 (1998) 106 – 111.
[11] P. Yiu, Construction of indecomposable Heronian triangles, *Rocky Mountain Journal of Mathematics*, 28 (1998) 1189 – 1201.

K.R.S. Sastry: Jeevan Sandhya, DoddaKalsandra Post, Raghuvana Halli, Bangalore, 560 062, India.

Forum Geometricorum
Volume 1 (2001) 25–32.

FORUM GEOM

Equilateral Triangles Intercepted by Oriented Parallelians

Sabrina Bier

Abstract. Given a point P in the plane of triangle ABC, we consider rays through P parallel to the side lines. The intercepts on the sidelines form an equilateral triangle precisely when P is a Brocardian point of one of the Fermat points. There are exactly four such equilateral triangles.

1. Introduction

The construction of an interesting geometric figure is best carried out after an analysis. For example, given a triangle ABC, how does one construct a point P through which the parallels to the three sides make equal intercepts? A very simple analysis of this question can be found in [6, 7]. It is shown that there is only one such point P,[1] which has homogeneous barycentric coordinates

$$\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}:\frac{1}{c}+\frac{1}{a}-\frac{1}{b}:\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\sim(ca+ab-bc:ab+bc-ca:bc+ca-ab).$$

This leads to a very easy construction of the point[2] and its three equal parallel intercepts. See Figure 1. An interesting variation is to consider equal "semi-parallel

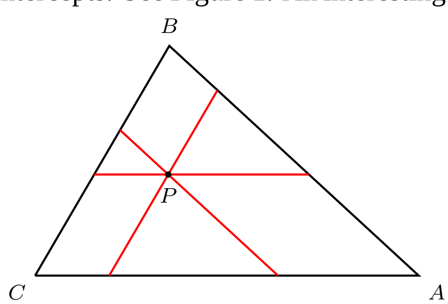


Figure 1

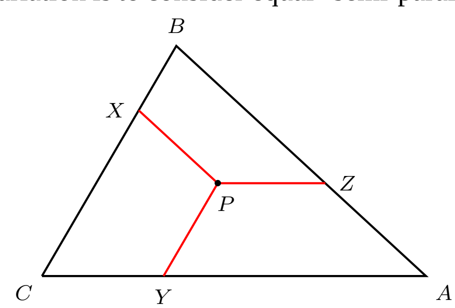


Figure 2

intercepts". Suppose through a point P in the plane of triangle ABC, parallels to the sides AB, BC, CA intersect BC, CA, AB are X, Y, Z respectively. How should one choose P so that the three "semi-parallel intercepts" PX, PY, PZ have equal lengths? (Figure 2). A simple calculation shows that the only point satisfying this requirement, which we denote by $L_{\rightarrow}$, has coordinates $(\frac{1}{c}:\frac{1}{a}:\frac{1}{b})$.

Publication Date: February 19, 2001. Communicating Editor: Paul Yiu.

The results in this paper were obtained in the fall semester, 2000, in a Directed Independent Study under Professor Paul Yiu. This paper was prepared with the assistance of Professor Yiu, who also contributed §5 and the Appendix.

[1]In [3], this is the equal-parallelian point X_{192}. In [7], this is called the equal-intercept point.

[2]If G is the centroid and I' the isotomic conjugate of the incenter of triangle ABC, then $I'P = 3\cdot I'G$.

If we reverse the orientations of the parallel rays, we obtain another point $I_{\leftarrow}$ with coordinates $(\frac{1}{b} : \frac{1}{c} : \frac{1}{a})$. See Figure 5. These two points are called the Jerabek points; they can be found in [2, p.1213]. For a construction, see §4.

2. Triangles intercepted by forward parallelians

Given a triangle ABC, we mean by a *parallelian* a directed ray parallel to one of the sides, *forward* if it is along the direction of AB, BC, or CA, and *backward* if it is along BA, CB, or AC. In this paper we study the question: how should one choose the point P so that so that the *triangle XYZ intercepted by forward parallelians through P* is equilateral? See Figure 2. [3] We solve this problem by performing an analysis using homogeneous barycentric coordinates. If $P = (u : v : w)$, then X, Y, and Z have coordinates

$$X = (0 : u + v : w), \quad Y = (u : 0 : v + w), \quad Z = (w + u : v : 0).$$

The lengths of AY and AZ are respectively $\frac{(v+w)b}{u+v+w}$ and $\frac{vc}{u+v+w}$. By the law of cosines, the square length of YZ is

$$\frac{1}{(u+v+w)^2}((v+w)^2b^2 + v^2c^2 - (v+w)v(b^2+c^2-a^2)).$$

Similarly, the square lengths of ZX and XY are respectively

$$\frac{1}{(u+v+w)^2}((w+u)^2c^2 + w^2a^2 - (w+u)w(c^2+a^2-b^2))$$

and

$$\frac{1}{(u+v+w)^2}((u+v)^2a^2 + u^2b^2 - (u+v)u(a^2+b^2-c^2)).$$

The triangle XYZ is equilateral if and only if

$$\begin{aligned} & (v+w)^2b^2 + v^2c^2 - (v+w)v(b^2+c^2-a^2) \\ = \; & (w+u)^2c^2 + w^2a^2 - (w+u)w(c^2+a^2-b^2) \\ = \; & (u+v)^2a^2 + u^2b^2 - (u+v)u(a^2+b^2-c^2). \end{aligned} \tag{1}$$

By taking differences of these expressions, we rewrite (1) as a system of two homogeneous quadratic equations in three unknowns:

$$\mathcal{C}_1 : \qquad a^2v^2 - b^2w^2 - ((b^2+c^2-a^2)w - (c^2+a^2-b^2)v)u = 0,$$

and

$$\mathcal{C}_2 : \qquad b^2w^2 - c^2u^2 - ((c^2+a^2-b^2)u - (a^2+b^2-c^2)w)v = 0.$$

[3]Clearly, a solution to this problem can be easily adapted to the case of "backward triangles", as we shall do at the end §4.

3. Intersections of two conics

3.1. *Representation by symmetric matrices.* We regard each of the two equations $\mathcal{C}_1$ and $\mathcal{C}_2$ as defining a conic in the plane of triangle ABC. The question is therefore finding the intersections of two conics. This is done by choosing a suitable combination of the two conics which degenerates into a pair of straight lines. To do this, we represent the two conics by symmetric 3×3 matrices

$$M_1 = \begin{pmatrix} 0 & c^2+a^2-b^2 & -(b^2+c^2-a^2) \\ c^2+a^2-b^2 & 2a^2 & 0 \\ -(b^2+c^2-a^2) & 0 & -2b^2 \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} -2c^2 & -(c^2+a^2-b^2) & 0 \\ -(c^2+a^2-b^2) & 0 & a^2+b^2-c^2 \\ 0 & a^2+b^2-c^2 & 2b^2 \end{pmatrix},$$

and choose a combination $M_1 - tM_2$ whose determinant is zero.

3.2. *Reduction to the intersection with a pair of lines.* Consider, therefore, the matrix

$$M_1 - tM_2 = \begin{pmatrix} 2tc^2 & (1+t)(c^2+a^2-b^2) & -(b^2+c^2-a^2) \\ (1+t)(c^2+a^2-b^2) & 2a^2 & -t(a^2+b^2-c^2) \\ -(b^2+c^2-a^2) & -t(a^2+b^2-c^2) & -2(1+t)b^2 \end{pmatrix}. \tag{2}$$

Direct calculation shows that the matrix $M_1 - tM_2$ in (2) has determinant

$$-32\triangle^2((b^2-c^2)t^3 - (c^2+a^2-2b^2)t^2 - (c^2+a^2-2b^2)t - (a^2-b^2)),$$

where $\triangle$ denotes the area of triangle ABC. The polynomial factor further splits into

$$((b^2-c^2)t - (a^2-b^2))(t^2+t+1).$$

We obtain $M_1 - tM_2$ of determinant zero by choosing $t = \dfrac{a^2-b^2}{b^2-c^2}$. This matrix represents a quadratic form which splits into two linear forms. In fact, the combination $(b^2-c^2)\mathcal{C}_1 - (a^2-b^2)\mathcal{C}_2$ leads to

$$((a^2-b^2)u + (b^2-c^2)v + (c^2-a^2)w)(c^2u + a^2v + b^2w) = 0.$$

From this we see that the intersections of the two conics $\mathcal{C}_1$ and $\mathcal{C}_2$ are the same as those of any one of them with the pairs of lines

$$\ell_1 : \quad (a^2-b^2)u + (b^2-c^2)v + (c^2-a^2)w = 0,$$

and

$$\ell_2 : \quad c^2u + a^2v + b^2w = 0.$$

3.3. *Intersections of $\mathcal{C}_1$ with ℓ_1 and ℓ_2.* There is an easy parametrization of points on the line ℓ_1. Since it clearly contains the points $(1:1:1)$ (the centroid) and $(c^2:a^2:b^2)$, every point on ℓ_1 is of the form $(c^2+t:a^2+t:b^2+t)$ for some real number t. Direct substitution shows that this point lies on the conic $\mathcal{C}_1$ if and only if

$$3t^2+3(a^2+b^2+c^2)t+(a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2)=0.$$

In other words,

$$\begin{aligned} t &= \frac{-(a^2+b^2+c^2)}{2}\pm\frac{1}{2\sqrt{3}}\sqrt{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4} \\ &= \frac{-(a^2+b^2+c^2)}{2}\pm\frac{2\triangle}{\sqrt{3}}. \end{aligned}$$

From these, we conclude that the conic $\mathcal{C}_1$ and the line ℓ_1 intersect at the points

$$P^{\pm}=\left(\frac{a^2+b^2-c^2}{2}\pm\frac{2\triangle}{\sqrt{3}}:\frac{b^2+c^2-a^2}{2}\pm\frac{2\triangle}{\sqrt{3}}:\frac{c^2+a^2-b^2}{2}\pm\frac{2\triangle}{\sqrt{3}}\right). \quad (3)$$

The line ℓ_2, on the other hand, does not intersect the conic $\mathcal{C}_1$ at real points. [4] It follows that the conics $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect only at the two real points $P^{\pm}$ given in (3) above. [5]

4. Construction of the points $P^{\pm}$

The coordinates of $P^{\pm}$ in (3) can be rewritten as

$$\begin{aligned} P^{\pm} &= (ab\cos C\pm\frac{1}{\sqrt{3}}ab\sin C:bc\cos A\pm\frac{1}{\sqrt{3}}bc\sin A:ca\cos B\pm\frac{1}{\sqrt{3}}ca\sin B) \\ &= (\frac{2ab}{\sqrt{3}}\sin(C\pm\frac{\pi}{3}):\frac{2bc}{\sqrt{3}}\sin(A\pm\frac{\pi}{3}):\frac{2ca}{\sqrt{3}}\sin(B\pm\frac{\pi}{3})) \\ &\sim (\frac{1}{c}\cdot\sin(C\pm\frac{\pi}{3}):\frac{1}{a}\cdot\sin(A\pm\frac{\pi}{3}):\frac{1}{b}\cdot\sin(B\pm\frac{\pi}{3})). \end{aligned}$$

A simple interpretation of these expressions, via the notion of Brocardian points [5], leads to an easy construction of the points $P^{\pm}$.

Definition. The Brocardian points of a point $Q=(x:y:z)$ are the two points

$$Q_{\rightarrow}=(\frac{1}{z}:\frac{1}{x}:\frac{1}{y}) \quad \text{and} \quad Q_{\leftarrow}=(\frac{1}{y}:\frac{1}{z}:\frac{1}{x}).$$

We distinguish between these two by calling $Q_{\rightarrow}$ the *forward* Brocardian point and $Q_{\leftarrow}$ the *backward* one, and justify these definitions by giving a simple construction.

Proposition 1. *Given a point Q, construct through the traces A_Q, B_Q, C_Q forward parallelians to AB, BC, CA, intersecting CA, AB, BC at Y, Z and X respectively. The lines AX, BY, CZ intersect at $Q_{\rightarrow}$. On the other hand, if the*

[4] Substitution of $u=\frac{-(a^2v+b^2w)}{c^2}$ into $(\mathcal{C}_1)$ gives $a^2v^2+(a^2+b^2-c^2)vw+b^2w^2=0$, which has no real roots since $(a^2+b^2-c^2)^2-4a^2b^2=a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2=-16\triangle^2<0$.

[5] See Figure 9 in the Appendix for an illustration of the conics and their intersections.

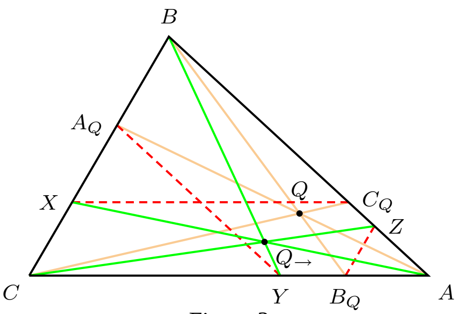


Figure 3a

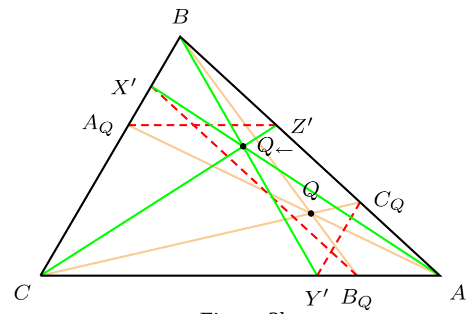


Figure 3b

backward parallelians through A_Q, B_Q, C_Q to CA, AB, BC, intersect AB, BC, CA at Z', X', Y' respectively, then, the lines AX', BY', CZ' intersect at $Q_{\leftarrow}$.

Proof. Suppose $Q = (x : y : z)$ in homogeneous barycentric coordinates. In Figure 3a, $BX : XC = BC_Q : C_QA = x : y$ since $C_Q = (x : y : 0)$. It follows that $X = (0 : y : x) \sim (0 : \frac{1}{x} : \frac{1}{y})$. Similarly, $Y = (\frac{1}{z} : 0 : \frac{1}{x})$ and $Z = (\frac{1}{z} : \frac{1}{y} : 0)$. From these, the lines AX, BY, and CZ intersect at the point $(\frac{1}{z} : \frac{1}{x} : \frac{1}{y})$, which we denote by $Q_{\rightarrow}$. The proof for $Q_{\leftarrow}$ is similar; see Figure 3b. □

Examples. If $Q = K = (a^2 : b^2 : c^2)$, the symmedian point, the Brocardian points $K_{\rightarrow}$ and $K_{\leftarrow}$ are the Brocard points [6] satisfying

$$\angle K_{\rightarrow}BA = \angle K_{\rightarrow}CB = \angle K_{\rightarrow}AC = \omega = \angle K_{\leftarrow}CA = \angle K_{\leftarrow}AB = \angle K_{\leftarrow}BC,$$

where ω is the Brocard angle given by $\cot\omega = \cot A + \cot B + \cot C$. These points lie on the circle with OK as diameter, O being the circumcenter of triangle ABC. See Figure 4.

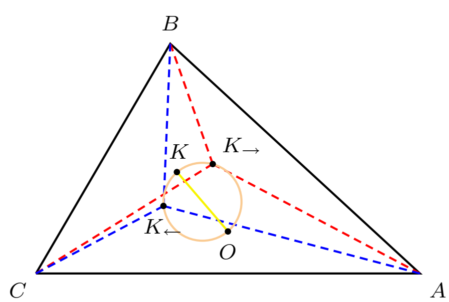


Figure 4

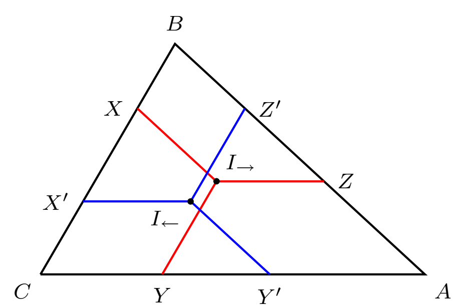


Figure 5

On the other hand, the Brocardian points of the incenter $I = (a : b : c)$ are the Jerabek points $I_{\rightarrow}$ and $I_{\leftarrow}$ mentioned in §1. See Figure 5.

[6] These points are traditionally labelled Ω (for $K_{\rightarrow}$) and Ω' (for $K_{\leftarrow}$) respectively. See [1, pp.274–280.]

Proposition 2. *The points $P^{\pm}$ are the* forward *Brocardian points of the Fermat points* [7]

$$F^{\pm} = \left(\frac{a}{\sin(A \pm \frac{\pi}{3})} : \frac{b}{\sin(B \pm \frac{\pi}{3})} : \frac{c}{\sin(C \pm \frac{\pi}{3})}\right).$$

By reversing the orientation of the parallelians, we obtain two more equilateral triangles, corresponding to the *backward* Brocardian points of the same two Fermat points $F^{\pm}$.

Theorem 3. *There are exactly four equilateral triangles intercepted by oriented parallelians, corresponding to the four points $F^{\pm}_{\rightarrow}$ and $F^{\pm}_{\leftarrow}$.*

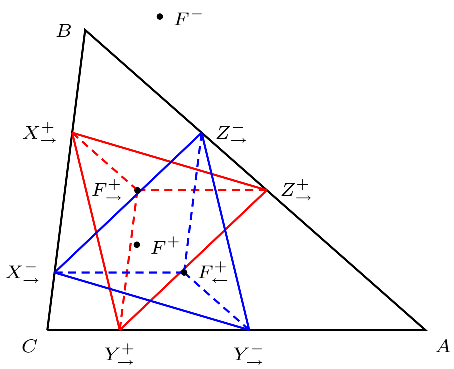


Figure 6a

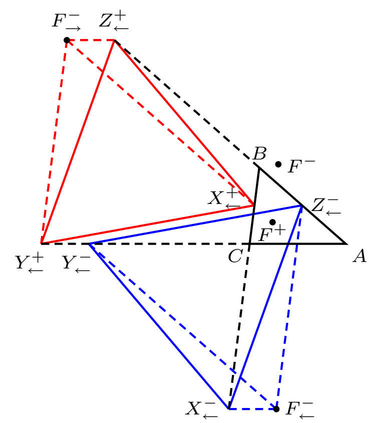


Figure 6b

5. Some further results

The two equilateral triangles $X^+_\rightarrow Y^+_\rightarrow Z^+_\rightarrow$ and $X^+_\leftarrow Y^+_\leftarrow Z^+_\leftarrow$ corresponding to the Fermat point F^+ are congruent; so are $X^-_\rightarrow Y^-_\rightarrow Z^-_\rightarrow$ and $X^-_\leftarrow Y^-_\leftarrow Z^-_\leftarrow$. In fact, they are homothetic at the common midpoint of the segments $X^+_\rightarrow Y^+_\leftarrow$, $Y^+_\rightarrow Z^+_\leftarrow$, and $Z^+_\rightarrow X^+_\leftarrow$, and their sides are parallel to the corresponding cevians of the Fermat point. This is indeed a special case of the following proposition.

Proposition 4. *For every point Q not on the side lines of triangle ABC, the triangle intercepted by the forward parallelians through $Q_\rightarrow$ and that by the backward parallelians through $Q_\leftarrow$ are homothetic at $(u(v+w) : v(w+u) : w(u+v))$, with ratio $1 : -1$. Their corresponding sides are parallel to the cevians AQ, BQ, and CQ respectively.*

[7]The Fermat point F^+ (respectively F^-) of triangle ABC is the intersection of the lines AX, BY, CZ, where XBC, YCA and ZAB are equilateral triangles constructed externally (respectively internally) on the sides BC, CA, AB of the triangle. This is the point X_{13} (respectively X_{14}) in [3].

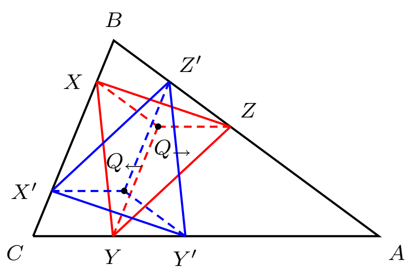


Figure 7

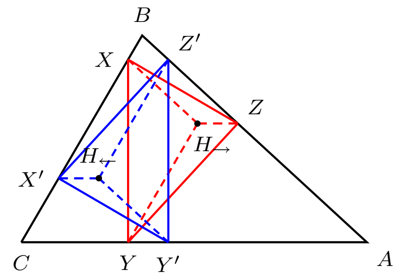


Figure 8

These two triangles are the only inscribed triangles whose sides are parallel to the respective cevians of Q. See Figure 7. They are the Bottema triangles in [4]. Applying this to the orthocenter H, we obtain the two congruent inscribed triangles whose sides are perpendicular to the sides of ABC (Figure 8).

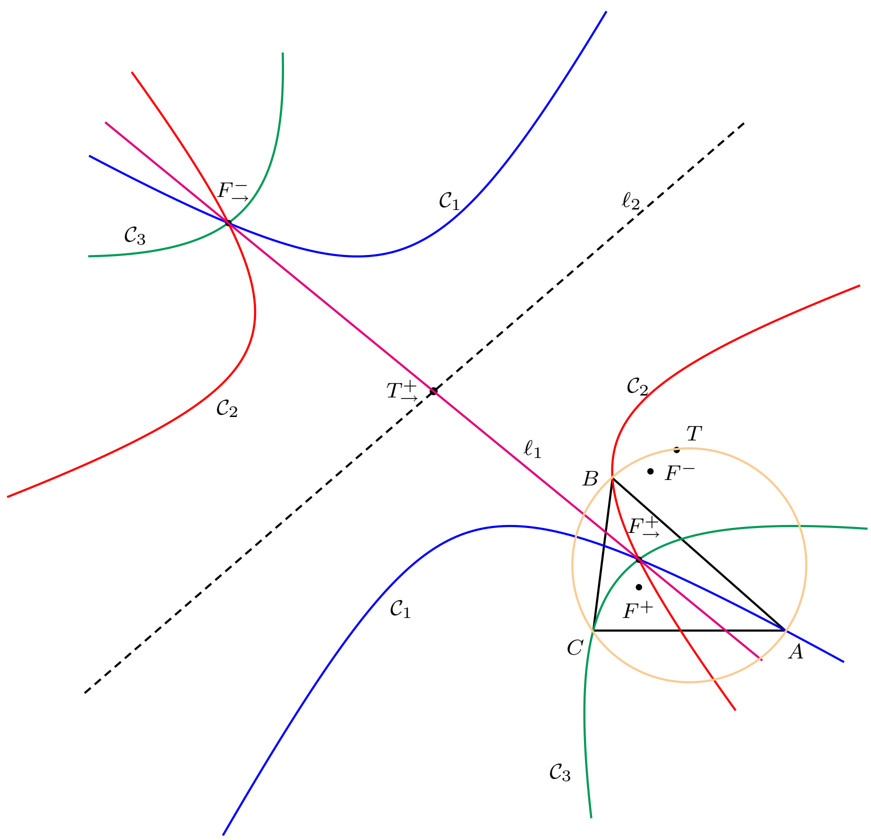


Figure 9

Appendix

Figure 9 illustrates the intersections of the two conics $\mathcal{C}_1$ and $\mathcal{C}_2$ in §2, along with a third conic $\mathcal{C}_3$ which results from the difference of the first two expressions in (1), namely,

$$\mathcal{C}_3 : \qquad c^2u^2 - a^2v^2 - ((a^2 + b^2 - c^2)v - (b^2 + c^2 - a^2)u)w = 0.$$

These three conics are all hyperbolas, and have a common center $T^+_\rightarrow$, which is the forward Brocardian point of the Tarry point T, and is the midpoint between the common points $F^+_\rightarrow$ and $F^-_\rightarrow$. In other words, $F^+_\rightarrow F^-_\rightarrow$ is a common diameter of the three hyperbolas. We remark that the Tarry point T is the point X_{98} of [3], and is the fourth intersection of the Kiepert hyperbola and the circumcircle of triangle ABC. The fact that ℓ_1 and ℓ_2 intersect at $T_\rightarrow$ follows from the observation that these lines are respectively the loci of the forward Brocardians of points on the Kiepert hyperbola $\frac{b^2-c^2}{u} + \frac{c^2-a^2}{v} + \frac{a^2-b^2}{w} = 0$ and the circumcircle $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$ respectively. The tangents to the hyperbolas $\mathcal{C}_1$ at A, $\mathcal{C}_2$ at B, and $\mathcal{C}_3$ at C intersect at the point $H_\rightarrow$, the forward Brocardian of the orthocenter.

References

[1] N. Altshiller-Court, *College Geometry*, 2nd edition, 1952, Barnes and Noble, New York.
[2] F.G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, Paris, 1991.
[3] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000, http://cedar.evansville.edu/~ck6/encyclopedia/.
[4] F.M. van Lamoen, Bicentric triangles, *Nieuw Archief voor Wiskunde*, 17 (1999) 363–372.
[5] E. Vigarie, Géométrie du triangle: étude bibliographique et terminologique, *Journal de Math. Spéc.*, (1887) 154–157.
[6] P. Yiu, *Euclidean Geometry*, Florida Atlantic University Lecture Notes, 1998.
[7] P. Yiu, The uses of homogeneous barycentric coordinates in plane euclidean geometry, *Int. J. Math. Educ. Sci. Technol.*, 31 (2000) 569 – 578.

Sabrina Bier: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida, 33431-0991, USA

E-mail address: true_pisces2000@hotmail.com

Forum Geometricorum
Volume 1 (2001) 33–42.

FORUM GEOM

The Isogonal Tripolar Conic

Cyril F. Parry

Abstract. In trilinear coordinates with respect to a given triangle ABC, we define the isogonal tripolar of a point $P(p, q, r)$ to be the line p: $p\alpha + q\beta + r\gamma = 0$. We construct a unique conic Φ, called the isogonal tripolar conic, with respect to which p is the polar of P for all P. Although the conic is imaginary, it has a real center and real axes coinciding with the center and axes of the real orthic inconic. Since ABC is self-conjugate with respect to Φ, the imaginary conic is harmonically related to every circumconic and inconic of ABC. In particular, Φ is the reciprocal conic of the circumcircle and Steiner's inscribed ellipse. We also construct an analogous isotomic tripolar conic Ψ by working with barycentric coordinates.

1. Trilinear coordinates

For any point P in the plane ABC, we can locate the right projections of P on the sides of triangle ABC at P_1, P_2, P_3 and measure the distances PP_1, PP_2 and PP_3. If the distances are directed, i.e., measured positively in the direction of each vertex to the opposite side, we can identify the distances $\underline{\alpha} = \overrightarrow{PP_1}$, $\underline{\beta} = \overrightarrow{PP_2}$, $\underline{\gamma} = \overrightarrow{PP_3}$ (Figure 1) such that

$$a\underline{\alpha} + b\underline{\beta} + c\underline{\gamma} = 2\triangle$$

where a, b, c, $\triangle$ are the side lengths and area of triangle ABC. This areal equation for all positions of P means that the ratio of the distances is sufficient to define the *trilinear coordinates* of $P(\alpha, \beta, \gamma)$ where

$$\alpha : \beta : \gamma = \underline{\alpha} : \underline{\beta} : \underline{\gamma}.$$

For example, if we consider the coordinates of the vertex A, the incenter I, and the first excenter I_1, we have absolute $\underline{\alpha\beta\gamma}-$coordinates : $A(h_1, 0, 0)$, $I(r, r, r)$, $I_1(-r_1, r_1, r_1)$, where h_1, r, r_1 are respectively the altitude from A, the inradius and the first exradius of triangle ABC. It follows that the trilinear $\alpha\beta\gamma-$coordinates in their simplest form are $A(1, 0, 0)$, $I(1, 1, 1)$, $I_1(-1, 1, 1)$. Let R be the circumradius, and h_1, h_2, h_3 the altitudes, so that $ah_1 = bh_2 = ch_3 = 2\triangle$. The absolute coordinates of the circumcenter O, the orthocenter H, and the median point [1] G are $O(R\cos A, R\cos B, R\cos C)$, $H(2R\cos B\cos C, 2R\cos C\cos A,$

Publication Date: February 26, 2001. Communicating Editor: Clark Kimberling.

[1]The median point is also known as the centroid.

$2R\cos A\cos B$), and $G(\frac{h_1}{3}, \frac{h_2}{3}, \frac{h_3}{3})$, giving trilinear coordinates: $O(\cos A, \cos B, \cos C)$, $H(\sec A, \sec B, \sec C)$, and $G(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$.

2. Isogonal conjugate

For any position of P we can define its isogonal conjugate Q such that the directed angles $(AC, AQ) = (AP, AB) = \theta_1$, $(BA, BP) = (BQ, BC) = \theta_2$, $(CB, CP) = (CQ, CA) = \theta_3$ as shown in Figure 1. If the absolute coordinates of Q are $\underline{\alpha}' = \overrightarrow{QQ_1}$, $\underline{\beta}' = \overrightarrow{QQ_2}$, $\underline{\gamma}' = \overrightarrow{QQ_3}$, then

$$\frac{PP_2}{PP_3} = \frac{AP\sin(A-\theta_1)}{AP\sin\theta_1} \quad \text{and} \quad \frac{QQ_2}{QQ_3} = \frac{AQ\sin\theta_1}{AQ\sin(A-\theta_1)}$$

so that that $PP_2 \cdot QQ_2 = PP_3 \cdot QQ_3$, implying $\underline{\beta\beta'} = \underline{\gamma\gamma'}$. Similarly, $\underline{\alpha\alpha'} = \underline{\beta\beta'}$ and $\underline{\gamma\gamma'} = \underline{\alpha\alpha'}$, so that $\underline{\alpha\alpha'} = \underline{\beta\beta'} = \underline{\gamma\gamma'}$. Consequently, $\alpha\alpha' = \beta\beta' = \gamma\gamma'$.

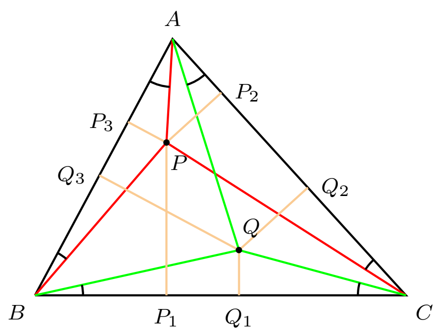


Figure 1

Hence, Q is the triangular inverse of P; i.e., if P has coordinates (α, β, γ), then its isogonal conjugate Q has coordinates $(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma})$. It will be convenient to use the notation $\hat{P}$ for the isogonal conjugate of P. We can immediately note that $O(\cos A, \cos B, \cos C)$ and $H(\sec A, \sec B, \sec C)$ are isogonal conjugates. On the other hand, the symmedian point K, being the isogonal conjugate of $G(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$, has coordinates $K(a, b, c)$, i.e., the distances from K to the sides of triangle ABC are proportional to the side lengths of ABC.

3. Tripolar

We can now define the *line coordinates* (l, m, n) of a given line ℓ in the plane ABC, such that any point P with coordinates (α, β, γ) lying on ℓ must satisfy the linear equation $l\alpha + m\beta + n\gamma = 0$. In particular, the side lines BC, CA, AB have line coordinates (1,0,0), (0,1,0), (0,0,1), with equations $\alpha = 0$, $\beta = 0$, $\gamma = 0$ respectively.

A specific line that may be defined is the harmonic or trilinear polar of Q with respect to ABC, which will be called the *tripolar* of Q.

In Figure 2, $L'M'N'$ is the tripolar of Q, where LMN is the diagonal triangle of the quadrangle $ABCQ$; and $L'M'N'$ is the axis of perspective of the triangles ABC and LMN. Any line through Q meeting two sides of ABC at U, V and meeting $L'M'N'$ at W creates an harmonic range $(UV;QW)$. To find the line coordinates of $L'M'N'$ when Q has coordinates (p',q',r'), we note $L = AQ \cap BC$ has coordinates $(0,q',r')$, since $\frac{LL_2}{LL_3} = \frac{QQ_2}{QQ_3}$. Similarly for $M(p',0,r')$ and $N(p',q',0)$. Hence the equation of the line MN is

$$\frac{\alpha}{p'} = \frac{\beta}{q'} + \frac{\gamma}{r'} \tag{1}$$

since the equation is satisfied when the coordinates of M or N are substituted for α, β, γ in (1). So the coordinates of $L' = MN \cap BC$ are $L'(0,q',-r')$. Similarly for $M'(p',0,-r')$ and $N'(p',-q',0)$, leading to the equation of the line $L'M'N'$:

$$\frac{\alpha}{p'} + \frac{\beta}{q'} + \frac{\gamma}{r'} = 0. \tag{2}$$

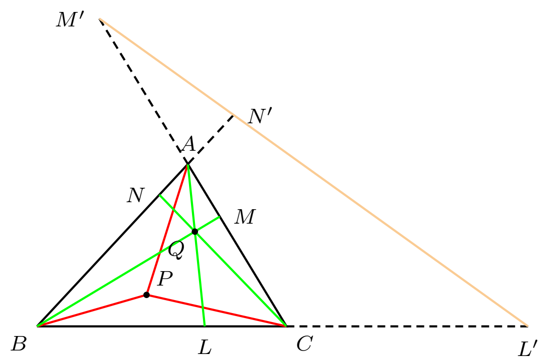


Figure 2

Now from the previous analysis, if $P(p,q,r)$ and $Q(p',q',r')$ are isogonal conjugates then $pp' = qq' = rr'$ so that from (2) the equation of the line $L'M'N'$ is $p\alpha + q\beta + r\gamma = 0$. In other words, the line coordinates of the tripolar of Q are the trilinear coordinates of P. We can then define the *isogonal tripolar* of $P(p,q,r)$ as the line $L'M'N'$ with equation $p\alpha + q\beta + r\gamma = 0$.

For example, for the vertices $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$, the isogonal tripolars are the corresponding sides BC ($\alpha = 0$), CA ($\beta = 0$), AB ($\gamma = 0$). For the notable points $O(\cos A, \cos B, \cos C)$, $I(1,1,1)$, $G(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$, and $K(a,b,c)$, the corresponding isogonal tripolars are

$$\begin{aligned}
\mathsf{o}: &\quad \alpha\cos A + \beta\cos B + \gamma\cos C = 0,\\
\mathsf{i}: &\quad \alpha + \beta + \gamma = 0,\\
\mathsf{g}: &\quad \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 0,\\
\mathsf{k}: &\quad a\alpha + b\beta + c\gamma = 0.
\end{aligned}$$

Here, o, i, g, k are respectively the orthic axis, the anti-orthic axis, Lemoine's line, and the line at infinity, i.e., the tripolars of H, I, K, and G. Clark Kimberling has assembled a catalogue of notable points and notable lines with their coordinates in a contemporary publication [3].

4. The isogonal tripolar conic Φ

Now consider a point $P_2(p_2, q_2, r_2)$ on the isogonal tripolar of $P_1(p_1, q_1, r_1)$, i.e., the line

$$\mathsf{p}_1: \qquad p_1\alpha + q_1\beta + r_1\gamma = 0.$$

Obviously P_1 lies on the isogonal tripolar of P_2 since the equality $p_1p_2 + q_1q_2 + r_1r_2 = 0$ is the condition for both incidences. Furthermore, the line P_1P_2 has equation

$$(q_1r_2 - q_2r_1)\alpha + (r_1p_2 - r_2p_1)\beta + (p_1q_2 - p_2q_1)\gamma = 0,$$

while the point $\mathsf{p}_1 \cap \mathsf{p}_2$ has coordinates $(q_1r_2 - q_2r_1, r_1p_2 - r_2p_1, p_1q_2 - p_2q_1)$. It follows that $\mathsf{t} = P_1P_2$ is the isogonal tripolar of $T = \mathsf{p}_1 \cap \mathsf{p}_2$. These isogonal tripolars immediately suggest the classical polar reciprocal relationships of a geometrical conic. In fact, the triangle P_1P_2T has the analogous properties of a self-conjugate triangle with respect to a conic, since each side of triangle P_1P_2T is the isogonal tripolar of the opposite vertex. This means that a significant conic could be drawn self-polar to triangle P_1P_2T. But an infinite number of conics can be drawn self-polar to a given triangle; and a further point with its polar are required to identify a unique conic [5]. We can select an arbitrary point P_3 with its isogonal tripolar p_3 for this purpose. Now the equation to the general conic in trilinear coordinates is [4]

$$\mathcal{S}: \qquad l\alpha^2 + m\beta^2 + n\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0$$

and the polar of $P_1(p_1, q_1, r_1)$ with respect to $\mathcal{S}$ is

$$\mathsf{s}_1: \quad (lp_1 + hq_1 + gr_1)\alpha + (hp_1 + mq_1 + fr_1)\beta + (gp_1 + fq_1 + nr_1)\gamma = 0.$$

By definition we propose that for $i = 1, 2, 3$, the lines p_i and s_i coincide, so that the line coordinates of p_i and s_i must be proportional; i.e.,

$$\frac{lp_i + hq_i + gr_i}{p_i} = \frac{hp_i + mq_i + fr_i}{q_i} = \frac{gp_i + fq_i + nr_i}{r_i}.$$

Solving these three sets of simultaneous equations, after some manipulation we find that $l = m = n$ and $f = g = h = 0$, so that the equation of the required conic is $\alpha^2 + \beta^2 + \gamma^2 = 0$. This we designate the *isogonal tripolar conic* Φ.

From the analysis Φ is the unique conic which reciprocates the points P_1, P_2, P_3 to the lines p_1, p_2, p_3. But any set of points P_i, P_j, P_k with the corresponding isogonal tripolars p_i, p_j, p_k could have been chosen, leading to the same equation for the reciprocal conic. We conclude that *the isogonal tripolar of any point P in the plane ABC is the polar of P with respect to* Φ. Any triangle $P_iP_jT_k$ with $T_k = \mathsf{p}_i \cap \mathsf{p}_j$ is self-conjugate with respect to Φ. In particular, the basic triangle ABC is self-conjugate with respect to Φ, since each side is the isogonal tripolar of its opposite vertex.

From the form of the equation $\alpha^2 + \beta^2 + \gamma^2 = 0$, the isogonal tripolar conic Φ is obviously an imaginary conic. So the conic exists on the complex projective plane. However, it will be shown that the imaginary conic has a real center and real axes; and that Φ is the reciprocal conic of a pair of notable real conics.

5. The center of Φ

To find the center of Φ, we recall that the polar of the center of a conic with respect to that conic is the line at infinty ℓ_∞ which we have already identified as $\mathsf{k}:\ a\alpha+b\beta+c\gamma=0$, the isogonal tripolar of the symmedian point $K(a,b,c)$. So the center of Φ and the center of its director circle are situated at K. From Gaskin's Theorem, the director circle of a conic is orthogonal to the circumcircle of every self-conjugate triangle. Choosing the basic triangle ABC as the self-conjugate triangle with circumcenter O and circumradius R, we have $\rho^2+R^2=OK^2$, where ρ is the director radius of Φ. But it is known [2] that $R^2-OK^2=3\mu^2$, where $\mu=\dfrac{abc}{a^2+b^2+c^2}$ is the radius of the cosine circle of ABC. From this,

$$\rho=i\sqrt{3}\mu=i\sqrt{3}\cdot\frac{abc}{a^2+b^2+c^2}.$$

6. Some lemmas

To locate the axes of Φ, some preliminary results are required which can be found in the literature [1] or obtained by analysis.

Lemma 1. *If a diameter of the circumcircle of ABC meets the circumcircle at X, Y, then the isogonal conjugates of X and Y (designated $\hat{X}$, $\hat{Y}$) lie on the line at infinity; and for arbitrary P, the line $P\hat{X}$ and $P\hat{Y}$ are perpendicular.*

Here is a special case.

Lemma 2. *If the chosen diameter is the Euler line OGH, then $\hat{X}\hat{Y}$ lie on the asymptotes of Jerabek's hyperbola $\mathcal{J}$, which is the locus of the isogonal conjugate of a variable point on the Euler line OGH (Figure 3).*

Lemma 3. *If the axes of a conic $\mathcal{S}$ with center Q meets ℓ_∞ at E, F, then the polars of E, F with respect to $\mathcal{S}$ are the perpendicular lines QF, QE; and E, F are the only points on ℓ_∞ with this property.*

Lemma 4. *If UGV is a chord of the circumcircle Γ through G meeting Γ at U, V, then the tripolar of U is the line $K\hat{V}$ passing through the symmedian point K and the isogonal conjugate of V.*

7. The axes of Φ

To proceed with the location of the axes of Φ, we start with the conditions of Lemma 2 where X, Y are the common points of OGH and Γ.

From Lemma 4, since XGY are collinear, the tripolars of X, Y are respectively $K\hat{Y}$, $K\hat{X}$, which are perpendicular from Lemma 1. Now from earlier definitions, the tripolars of X, Y are the isogonal tripolars of $\hat{X}$, $\hat{Y}$, so that the isogonal tripolars of $\hat{X}$, $\hat{Y}$ are the perpendiculars $K\hat{Y}$, $K\hat{X}$ through the center of Φ. Since $\hat{X}\hat{Y}$ lie on ℓ_∞, $K\hat{X}$, $K\hat{Y}$ must be the axes of Φ from Lemma 3. And these axes are parallel to the asymptotes of $\mathcal{J}$ from Lemma 2.

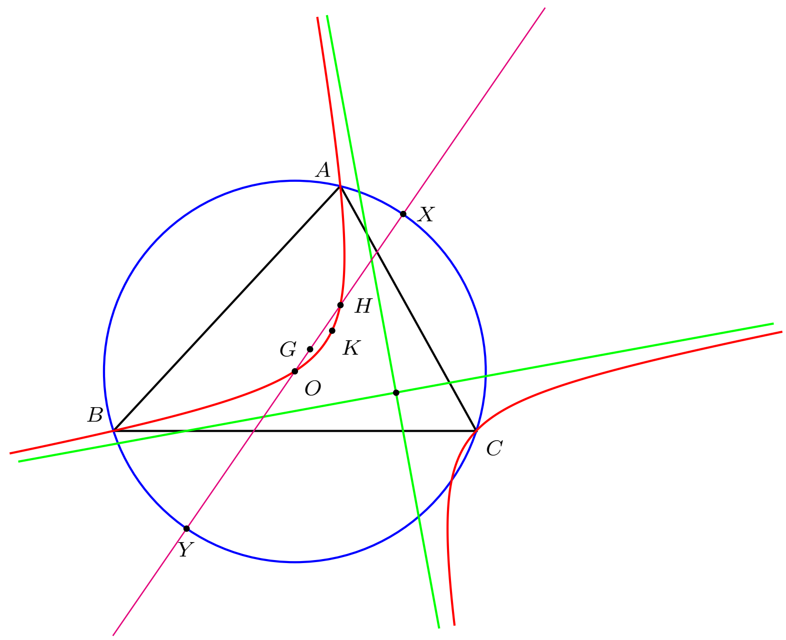


Figure 3. The Jerabek hyperbola

Now it is known [1] that the asymptotes of $\mathcal{J}$ are parallel to the axes of the orthic inconic (Figure 4). The orthic triangle has its vertices at H_1, H_2, H_3 the feet of the altitudes AH, BH, CH. The orthic inconic has its center at K and touches the sides of triangle ABC at the vertices of the orthic triangle. So the axes of the imaginary conic Φ coincide with the axes of the real orthic inconic.

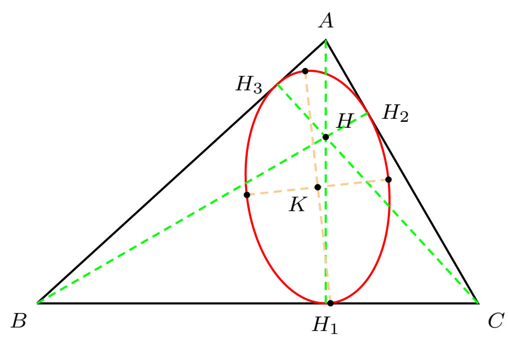


Figure 4. The orthic inconic

8. Φ as a reciprocal conic of two real conics

Although the conic Φ is imaginary, every real point P has a polar p with respect to Φ. In particular if P lies on the circumcircle Γ, its polar p touches Steiner's inscribed ellipse σ with center G. This tangency arises from the known theorem

[1] that the tripolar of any point on ℓ_∞ touches σ. From Lemma 1 this tripolar is the isogonal tripolar of the corresponding point of Γ. Now the basic triangle ABC (which is self-conjugate with respect to Φ) is inscribed in Γ and tangent to σ, which touches the sides of ABC at their midpoints (Figure 5). In the language of classical geometrical conics, the isogonal tripolar conic Φ is harmonically inscribed to Γ and harmoncially circumscribed to σ. From the tangency described above, Φ is the reciprocal conic to $\Gamma \rightleftharpoons \sigma$. Furthermore, since ABC is self-conjugate with respect to Φ, an infinite number of triangles $P_iP_jP_k$ can be drawn with its vertices inscribed in Γ, its sides touching σ, and self-conjugate with respect to Φ. Since Φ is the reciprocal conic of $\Gamma \rightleftharpoons \sigma$, for any point on σ, its polar with respect to Φ (i.e., its isogonal tripolar) touches Γ. In particular, if the tangent p_i touches σ at $T_i(u_i, v_i, w_i)$ for $i = 1, 2, 3$, then t_i, the isogonal tripolar of T_i, touches Γ at P_i (Figure 5).

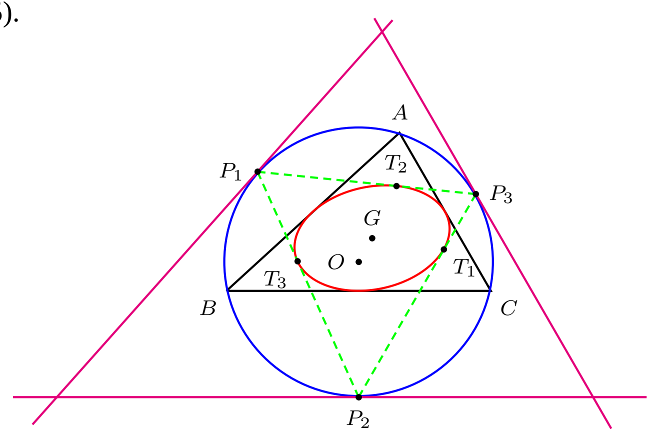


Figure 5

Now, the equation to the circumcircle Γ is $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$. The equation of the tangent to Γ at $P_i(p_i, q_i, r_i)$ is

$$(cq_i + br_i)\alpha + (ar_i + cp_i)\beta + (bp_i + aq_i)\gamma = 0.$$

If this tangent coincides with t_i, the isogonal tripolar of T_i, then the coordinates of T_i are

$$u_i = cq_i + br_i, \qquad v_i = ar_i + cp_i, \qquad w_i = bp_i + aq_i. \tag{3}$$

So, if t_i is the tangent at $P_i(p_i, q_i, r_i)$ to Γ, and simultaneously the isogonal tripolar of T_i, then the coordinates of T_i are as shown in (3). But this relationship can be generalized for any P_i in the plane of ABC, since the equation to the polar of P_i with respect to Γ is identical to the equation to the tangent at P_i (in the particular case that P_i lies on Γ). In other words, the isogonal tripolar of $T_i(u_i, v_i, w_i)$ with the coordinates shown at (3) is the polar of $P_i(p_i, q_i, r_i)$ with respect to Γ, for any P_i, T_i in the plane of ABC.

9. The isotomic tripolar conic Ψ

To find an alternative description of the transformation $P \mapsto T$, we define the *isotomic conjugate* and the *isotomic tripolar*.

In the foregoing discussion we have used trilinear coordinates (α, β, γ) to define the point P and its isogonal tripolar p. However, we could just as well use *barycentric* (areal) coordinates (x, y, z) to define P. With $\underline{x}$ = area (PBC), $\underline{y}$ = area (PCA), $\underline{z}$ = area (PAB), and $\underline{x} + \underline{y} + \underline{z}$ = area (ABC), comparing with trilinear coordinates of P we have

$$a\underline{\alpha} = 2\underline{x}, \qquad b\underline{\beta} = 2\underline{y}, \qquad c\underline{\gamma} = 2\underline{z}.$$

Using directed areas, i.e., positive area (PBC) when the perpendicular distance PP_1 is positive, the ratio of the areas is sufficient to define the (x, y, z) coordinates of P, with $x : \underline{x} = y : \underline{y} = z : \underline{z}$. The absolute coordinates $(\underline{x}, \underline{y}, \underline{z})$ can then be found from the areal coordinates (x, y, z) using the areal identity $\underline{x} + \underline{y} + \underline{z} = \triangle$. For example, the barycentric coordinates of A, I, I_1, O, H, G, K are $A(1, 0, 0)$, $I(a, b, c)$, $I_1(-a, b, c)$, $O(a\cos A, b\cos B, c\cos C)$, $H(a\sec A, b\sec B, c\sec C)$, $G(1, 1, 1)$, $K(a^2, b^2, c^2)$ respectively.

In this barycentric system we can identity the coordinates (x', y', z') of the isotomic conjugate $\overline{P}$ of P as shown in Figure 6, where $\overrightarrow{BL}{=}\overrightarrow{L'C}$, $\overrightarrow{CM}{=}\overrightarrow{M'A}$, $\overrightarrow{AN}{=}\overrightarrow{N'B}$. We find by the same procedure that $xx' = yy' = zz'$ for P, $\overline{P}$, so that the areal coordinates of $\overline{P}$ are $(\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$, explaining the alternative description that $\overline{P}$ is the triangular reciprocal of P.

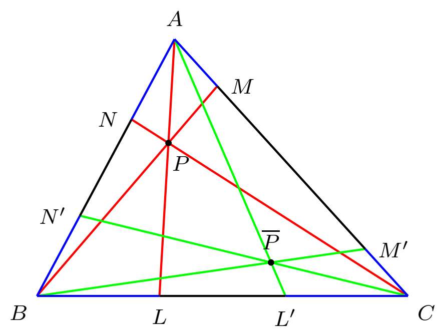


Figure 6

Following the same argument as heretofore, we can define the *isotomic tripolar* of $P(p, q, r)$ as the tripolar of $\overline{P}$ with barycentric equation $px + qy + rz = 0$, and then identify the imaginary *isotomic tripolar conic* Ψ with equation $x^2 + y^2 + z^2 = 0$. The center of Ψ is the median point $G(1, 1, 1)$ since the isotomic tripolar of G is the ℓ_∞ with barycentric equation $x + y + z = 0$. By analogous procedure we can find the axes of Ψ which coincide with the real axes of Steiner's inscribed ellipse σ.

Again, we find that the basic triangle ABC is self conjugate with respect to Ψ, and from Gaskin's Theorem, the radius of the imaginary director circle d is given by $d^2 + R^2 = OG^2$. From this, $d^2 = OG^2 - R^2 = -\frac{1}{9}(a^2 + b^2 + c^2)$, giving

$$d = \frac{i}{3}\sqrt{a^2 + b^2 + c^2}.$$

In the analogous case to Figure 5, we find that in Figure 7, if P is a variable point on Steiner's circum-ellipse θ (with center G), then the isotomic tripolar of P is tangent to σ, and Ψ is the reciprocal conic of $\theta \rightleftharpoons \sigma$. Generalizing this relationship as before, we find that the polar of P (pqr) with respect to θ is the isotomic tripolar of T with barycentric coordinates $(q{+}r, r{+}p, p{+}q)$. Furthermore, we can describe the transformation $P \mapsto T$ in vector geometry as $\overrightarrow{PG} = 2\ \overrightarrow{GT}$, or more succinctly that T is the complement of P [2]. The inverse transformation $T \mapsto P$ is given by $\overrightarrow{TG} = \frac{1}{2}\ \overrightarrow{GP}$, where P is the anticomplement of T. So the transformation of point T to the isotomic tripolar t can be described as

$$\begin{aligned} \mathsf{t} &= \text{isotomic tripole of } T \\ &= \text{polar of } T \text{ with respect to } \Psi \\ &= \text{polar of } P \text{ with respect to } \theta, \end{aligned}$$

where $\overrightarrow{PG} = 2\ \overrightarrow{GT}$. In other words, the transformation of a point $P(p, q, r)$ to its isotomic tripolar $px + qy + rz = 0$ is a dilatation $(G, -2)$ followed by polar reciprocation in θ, Steiner's circum-ellipse.

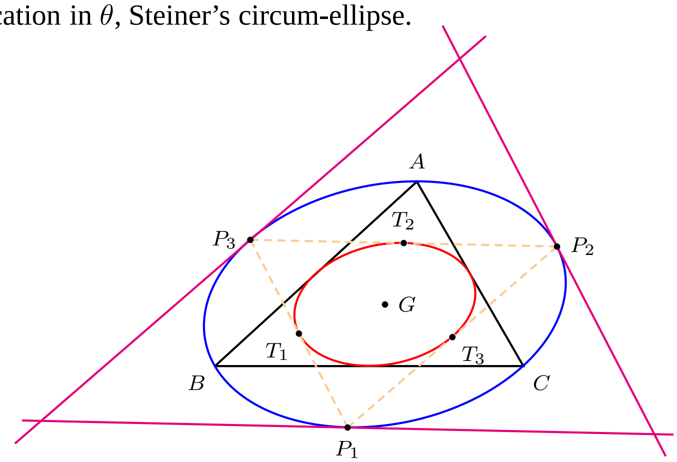


Figure 7

To find the corresponding transformation of a point to its isogonal tripolar, we recall that the polar of $P(p, q, r)$ with respect to Γ is the isogonal tripolar of T, where T has trilinear coordinates $(cq + br, ar + cp, bp + aq)$ from (3). Now, $\overline{\hat{P}}$, the isotomic conjugate of the isogonal conjugate of P, has coordinates $(\frac{p}{a^2}, \frac{q}{b^2}, \frac{r}{c^2})$ [3]. Putting $R = \overline{\hat{P}}$, the complement of R has coordinates $(cq + br, ar + cp, bp + aq)$, which are the coordinates of T. So the transformation of point T to its isogonal tripolar t can be described as

$$\begin{aligned} \mathsf{t} &= \text{isogonal tripolar of } T \\ &= \text{polar of } T \text{ with respect to } \Phi \\ &= \text{polar of } P \text{ with respect to } \Gamma, \end{aligned}$$

where $\overrightarrow{RG} = 2\ \overrightarrow{GT}$, and $P = \hat{\overline{R}}$, the isogonal conjugate of the isotomic conjugate of R. In other words, the transformation of a point P with trilinear coordinates

(p, q, r) to its isogonal tripolar $(p\alpha + q\beta + r\gamma = 0)$ is a dilatation $(G, -2)$, followed by isotomic transformation, then isogonal transformation, and finally polar reciprocation in the circumcircle Γ.

We conclude with the remark that the two well known systems of homogeneous coordinates, viz. trilinear (α, β, γ) and barycentric (x, y, z), generate two analogous imaginary conics Φ and Ψ, whose real centers and real axes coincide with the corresponding elements of notable real inconics of the triangle. In each case, the imaginary conic reciprocates an arbitrary point P to the corresponding line p, whose line coordinates are identical to the point coordinates of P. And in each case, reciprocation in the imaginary conic is the equivalent of well known transformations of the real plane.

References

[1] J. Casey, *A Sequel to Euclid*, 6th edition, Hodges & Friggis, Dublin, 1892.
[2] N. Altshiller-Court, *College Geometry*, 2nd edition, Barnes & Noble, New York, 1952.
[3] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1 – 295.
[4] E.A. Maxwell, *General Homogeneous Coordinates*, Cambridge University Press, Cambridge, 1957.
[5] J.W. Russell, *Pure Geometry*, Oxford University Press, Oxford, 1893.

Cyril F. Parry: 73 Scott Drive, Exmouth, Devon, England EX8 3LF
E-mail address: simplysorted@cableinet.co.uk

Forum Geometricorum
Volume 1 (2001) 43–50.

FORUM GEOM
ISSN 1534-1178

The Malfatti Problem

Oene Bottema

Abstract. A solution is given of Steiner's variation of the classical Malfatti problem in which the triangle is replaced by three circles mutually tangent to each other externally. The two circles tangent to the three given ones, presently known as Soddy's circles, are encountered as well.

In this well known problem, construction is sought for three circles C_1', C_2' and C_3', tangent to each other pairwise, and of which C_1' is tangent to the sides A_1A_2 and A_1A_3 of a given triangle $A_1A_2A_3$, while C_2' is tangent to A_2A_3 and A_2A_1 and C_3' to A_3A_1 and A_3A_2. The problem was posed by Malfatti in 1803 and solved by him with the help of an algebraic analysis. Very well known is the extraordinarily elegant geometric solution that Steiner announced without proof in 1826. This solution, together with the proof Hart gave in 1857, one can find in various textbooks.[1] Steiner has also considered extensions of the problem and given solutions. The first is the one where the lines A_2A_3, A_3A_1 and A_1A_2 are replaced by circles. Further generalizations concern the figures of three circles on a sphere, and of three conic sections on a quadric surface. In the nineteenth century many mathematicians have worked on this problem. Among these were Cayley (1852) [2], Schellbach (who in 1853 published a very nice goniometric solution), and Clebsch (who in 1857 extended Schellbach's solution to three conic sections on a quadric surface, and for that he made use of elliptic functions). If one allows in Malfatti's original problem also escribed and internally tangent circles, then there are a total of 32 (real) solutions. One can find all these solutions mentioned by Pampuch (1904).[3] The generalizations mentioned above even have, as appears from investigation by Clebsch, 64 solutions.

Publication Date: March 6, 2001. Communicating Editor: Paul Yiu.

Translation by Floor van Lamoen from the Dutch original of O. Bottema, Het vraagstuk van Malfatti, *Euclides*, 25 (1949-50) 144–149. Permission by Kees Hoogland, Chief Editor of *Euclides*, of translation into English is gratefully acknowledged.

The present article is one, *Verscheidenheid* XXVI , in a series by Oene Bottema (1901-1992) in the periodical *Euclides* of the Dutch Association of Mathematics Teachers. A collection of articles from this series was published in 1978 in form of a book [1]. The original article does not contain any footnote nor bibliography. All annotations, unless otherwise specified, are by the translator. Some illustrative diagrams are added in the Appendix.

[1] See, for examples, [3, 5, 7, 8, 9].

[2] Cayley's paper [4] was published in 1854.

[3] Pampuch [11, 12].

The literature about the problem is so vast and widespread that it is hardly possible to consult completely. As far as we have been able to check, the following special case of the generalization by Steiner has not drawn attention. It is attractive by the simplicity of the results and by the possibility of a certain stereometric interpretation.

The problem of Malfatti-Steiner is as follows: Given are three circles C_1, C_2 and C_3. Three circles C_1', C_2' and C_3' are sought such that C_1' is tangent to C_2, C_3, C_2' and C_3', the circle C_2' to C_3, C_1, C_3' and C_1', and, C_3' to C_1, C_2, C_1' and C_2'. Now we examine the special case, where the *three given circles* C_1, C_2, C_3 *are pairwise tangent as well*.

This problem certainly can be solved following Steiner's general method. We choose another route, in which the simplicity of the problem appears immediately. If one applies an *inversion* with center the point of tangency of C_2 and C_3, then these two circles are transformed into two parallel lines ℓ_2 and ℓ_3, and C_1 into a circle K tangent to both (Figure 1). In this figure the construction of the required circles K_i is very simple. If the distance between ℓ_2 and ℓ_3 is $4r$, then the radii of K_2 and K_3 are equal to r, that of K_1 equal to $2r$, while the distance of the centers of K and K_1 is equal to $4r\sqrt{2}$. Clearly, the problem always has *two* (real) solutions.[4]

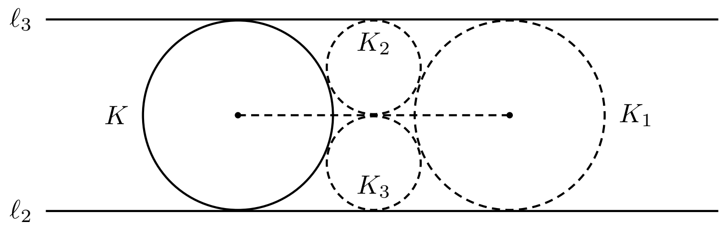


Figure 1

Our goal is the computation of the radii R_1', R_2' and R_3' of C_1', C_2' and C_3' if the radii R_1, R_2 and R_3 of the given circles C_1, C_2 and C_3 (which fix the figure of these circles) are given. For this purpose we let the objects in Figure 1 undergo an arbitrary inversion. Let O be the center of inversion and we choose a rectangular grid with O as its origin and such that ℓ_2 and ℓ_3 are parallel to the $x-$axis. For the power of inversion we can without any objection choose the unit. The inversion is then given by

$$x' = \frac{x}{x^2+y^2}, \qquad y' = \frac{y}{x^2+y^2}.$$

From this it is clear that the circle with center (x_0, y_0) and radius ρ is transformed into a circle of radius

$$\left|\frac{\rho}{x_0^2+y_0^2-\rho^2}\right|.$$

[4]See Figure 2 in the Appendix, which we add in the present translation.

If the coordinates of the center of K are (a,b), then those of K_1 are $(a+4r\sqrt{2},\ b)$. From this it follows that

$$R_1 = \left|\frac{2r}{a^2+b^2-4r^2}\right|, \qquad R_1' = \left|\frac{2r}{(a+4r\sqrt{2})^2+b^2-4r^2}\right|.$$

The lines ℓ_2 and ℓ_3 are inverted into circles of radii

$$R_2 = \frac{1}{2|b-2r|}, \qquad R_3 = \frac{1}{2|b+2r|}.$$

Now we first assume that O is chosen between ℓ_2 and ℓ_3, and outside K. The circles C_1, C_2 and C_3 then are pairwise tangent *externally*. One has $b-2r<0$, $b+2r>0$, and $a^2+b^2>4r^2$, so that

$$R_2 = \frac{1}{2(2r-b)}, \qquad R_3 = \frac{1}{2(2r+b)}, \qquad R_1 = \frac{2r}{a^2+b^2-4r^2}.$$

Consequently,

$$a = \pm\frac{1}{2}\sqrt{\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}}, \quad b = \frac{1}{4}\left(\frac{1}{R_3}-\frac{1}{R_2}\right), \quad r = \frac{1}{8}\left(\frac{1}{R_3}+\frac{1}{R_2}\right),$$

so that one of the solutions has

$$\frac{1}{R_1'} = \frac{1}{R_1}+\frac{2}{R_2}+\frac{2}{R_3}+2\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)},$$

and in the same way

$$\begin{aligned}\frac{1}{R_2'} &= \frac{2}{R_1}+\frac{1}{R_2}+\frac{2}{R_3}+2\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)},\\ \frac{1}{R_3'} &= \frac{2}{R_1}+\frac{2}{R_2}+\frac{1}{R_3}+2\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)},\end{aligned} \tag{1}$$

while the second solution is found by replacing the square roots on the right hand sides by their opposites and then taking absolute values. The first solution consists of three circles which are pairwise tangent externally. For the second there are different possibilities. It may consist of three circles tangent to each other externally, or of three circles, two tangent externally, and with a third circle tangent internally to each of them.[5] One can check the correctness of this remark by choosing O outside each of the circles K_1, K_2 and K_3 respectively, or inside these. According as one chooses O on the circumference of one of the circles, or at the point of tangency of two of these circles, respectively one , or two, straight lines[6] appear in the solution.

Finally, if one takes O outside the strip bordered by ℓ_2 and ℓ_3, or inside K, then the resulting circles have two internal and one external tangencies. If the circle C_1 is tangent *internally* to C_2 and C_3, then one should replace in solution (1) R_1 by $-R_1$, and the same for the second solution. In both solutions the circles are tangent

[5]See Figures 2 and 3 in the Appendix.

[6]See Figures 4, 5, and 6 in the Appendix.

to each other externally.[7] Incidentally, one can take (1) and the corresponding expression, where the sign of the square root is taken oppositely, as the general solution for each case, if one agrees to accept also negative values for a radius and to understand that two externally tangent circles have radii of equal signs and internally tangent circles of opposite signs.

There are two circles that are tangent to the three given circles. [8] This also follows immediately from Figure 1. In this figure the radii of these circles are both $2r$, the coordinates of their centers $(a \pm 4r,\ b)$. After inversion one finds for the radii of these 'inscribed' circles of the figure C_1, C_2, C_3:

$$\frac{1}{\rho_{1,2}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \pm 2\sqrt{\frac{1}{R_2R_3} + \frac{1}{R_3R_1} + \frac{1}{R_1R_2}}, \tag{2}$$

expressions showing great analogy to (1). One finds these already in Steiner [9] (*Werke* I, pp. 61 – 63, with a clarifying remark by Weierstrass, p.524). [10] While ρ_1 is always positive, $\frac{1}{\rho_2}$ can be greater than, equal to, or smaller than zero. One of the circles is tangent to all the given circles externally, the other is tangent to them all externally, or all internally, (or in the transitional case a straight line). One can read these properties easily from Figure 1 as well. Steiner proves (2) by a straightforward calculation with the help of a formula for the altitude of a triangle.

From (1) and (2) one can derive a large number of relations among the radii R_i of the given circles, the radii R_i' of the Malfatti circles, and the radii ρ_i of the tangent circles. We only mention

$$\frac{1}{R_1} + \frac{1}{R_1'} = \frac{1}{R_2} + \frac{1}{R_2'} = \frac{1}{R_3} + \frac{1}{R_3'}.$$

About the formulas (1) we want to make some more remarks. After finding for the figure $\mathcal{S}$ of given circles C_1, C_2, C_3 one of the two sets $\mathcal{S}'$ of Malfatti circles C_1', C_2', C_3', clearly one may repeat the same construction to $\mathcal{S}'$. One of the two sets of Malfatti circles that belong to $\mathcal{S}'$ clearly is $\mathcal{S}$. Continuing this way in two directions *a chain of triads of circles* arises, with the property that each of two consecutive triples is a Malfatti figure of the other.

By *iteration* of formula (1) one can express the radii of the circles in the n^{th} triple in terms of the radii of the circles one begins with. If one applies (1) to $\frac{1}{R_i'}$, and chooses the negative square root, then one gets back $\frac{1}{R_i}$. For the new set we find

$$\frac{1}{R_1''} = \frac{17}{R_1} + \frac{16}{R_2} + \frac{16}{R_3} + 20\sqrt{2\left(\frac{1}{R_2R_3} + \frac{1}{R_3R_1} + \frac{1}{R_1R_2}\right)}$$

---

[7] See Figure 7 in the Appendix.

[8] See Figure 8 in the Appendix.

[9] Steiner [15].

[10] This formula has become famous in modern times since the appearance of Soddy [5]. See [6]. According to Boyer and Merzbach [2], however, an equivalent formula was already known to René Descartes, long before Soddy and Steiner.

and cyclic permutations. For the next sets,

$$\begin{aligned}
\frac{1}{R_1^{(3)}} &= \frac{161}{R_1}+\frac{162}{R_2}+\frac{162}{R_3}+198\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)}\\
\frac{1}{R_1^{(4)}} &= \frac{1601}{R_1}+\frac{1600}{R_2}+\frac{1600}{R_3}+1960\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)}
\end{aligned}$$

If one takes

$$\begin{aligned}
\frac{1}{R_1^{(2p)}} &= \frac{a_{2p}+1}{R_1}+\frac{a_{2p}}{R_2}+\frac{a_{2p}}{R_3}+b_{2p}\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)}\\
\frac{1}{R_1^{(2p+1)}} &= \frac{a_{2p+1}+1}{R_1}+\frac{a_{2p+1}+2}{R_2}+\frac{a_{2p+1}+2}{R_3}\\
&\quad +b_{2p+1}\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)},
\end{aligned}$$

then one finds the recurrences [11]

$$\begin{aligned}
a_{2p+1} &= 10a_{2p}-a_{2p-1},\\
a_{2p} &= 10a_{2p-1}-a_{2p-2}+16,\\
b_k &= 10b_{k-1}-b_{k-2},
\end{aligned}$$

from which one can compute the radii of the circles in the triples.

The figure of three pairwise tangent circles C_1, C_2, C_3 forms with a set of Malfatti circles C_1', C_2', C_3' a configuration of six circles, of which each is tangent to four others. If one maps the circles of the plane to points in a three dimensional projective space, where the point-circles correspond with the points of a quadric surface Ω, then the configuration matches with an octahedron, of which the edges are tangent to Ω. The construction that was under discussion is thus the same as the following problem: *around a quadric surface Ω (for instance a sphere) construct an octahedron, of which the edges are tangent to Ω, and the vertices of one face are given.* This problem therefore has two solutions. And with the above chain corresponds a chain of triangles, all circumscribing Ω, and having the property that two consecutive triangles are opposite faces of a circumscribing octahedron.

From the formulas derived above for the radii it follows that these are decreasing if one goes in one direction along the chain, and increasing in the other direction, a fact that is apparent from the figure. Continuing in one direction, the triple of circles will eventually converge to a single point. With the question of how this point is positioned with respect to the given circles, we wish to end this modest contribution to the knowledge of the curious problem of Malfatti.

[11]These are the sequences A001078 and A053410 in N.J.A. Sloane's *Encyclopedia of Integer Sequences* [13].

Appendix

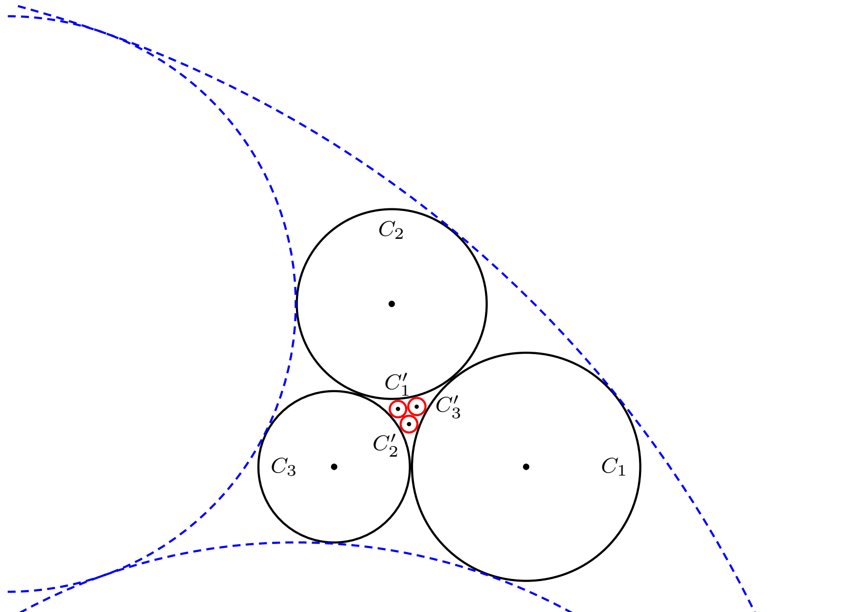


Figure 2

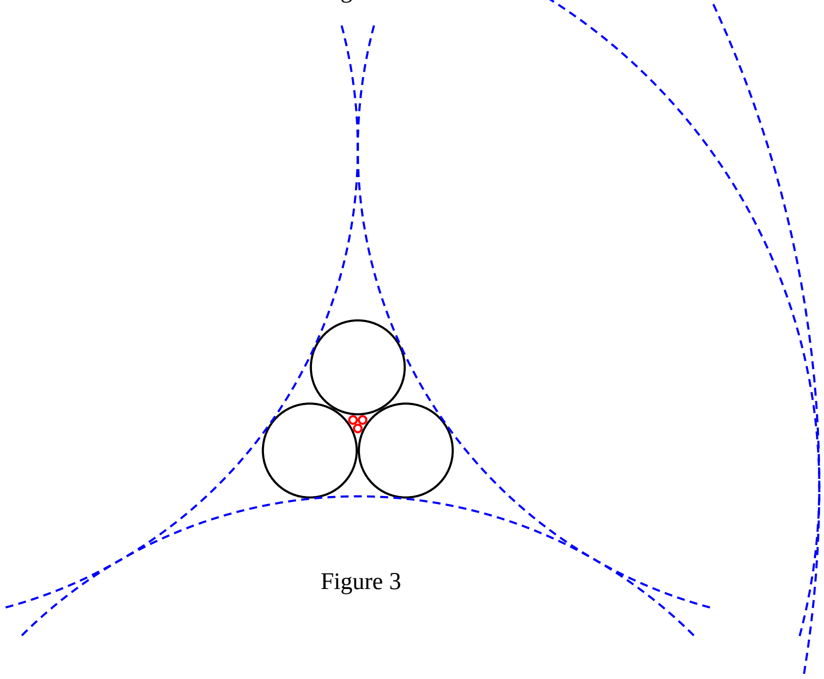


Figure 3

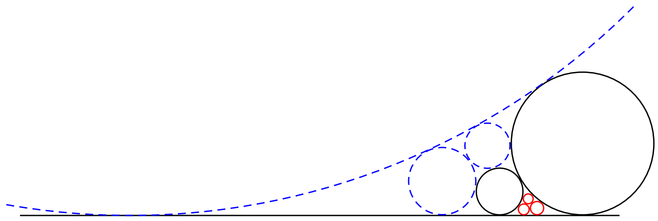


Figure 4

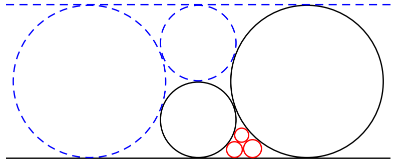


Figure 5

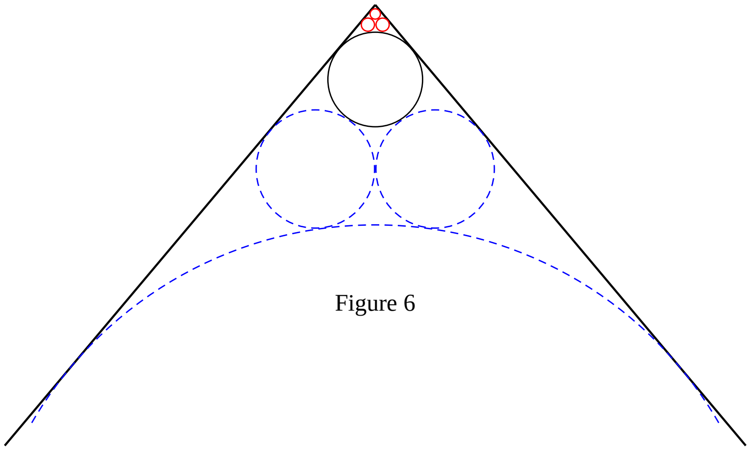


Figure 6

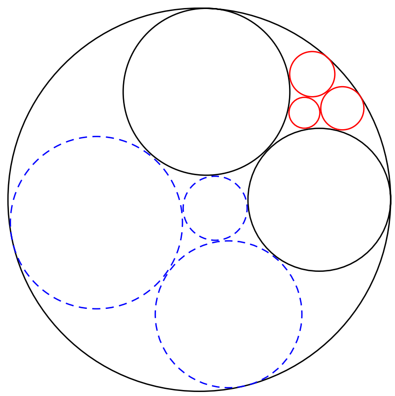


Figure 7

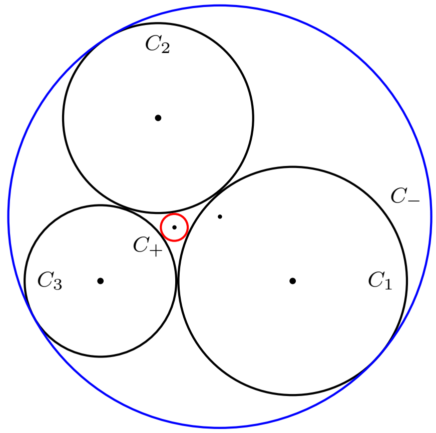


Figure 8

References

[1] O. Bottema, *Verscheidenheden*, Nederlandse Vereniging van Wiskundeleraren / Wolters Noordhoff, Groningen, 1978.
[2] C.B. Boyer and U.C. Merzbach, *A History of Mathematics, 2nd ed.*, Wiley, New York, 1991.
[3] J. Casey, *A sequel to the First Six Books of the Elements of Euclid, Containing an Easy Introduction to Modern Geometry with Numerous Examples, 5th ed.*, 1888, Hodges, Figgis & Co., Dublin.
[4] A. Cayley, Analytical researches connected with Steiner's extension of Malfatti's problem, *Phil. Trans.* (1854) 253 - 278.
[5] J.L. Coolidge, *Treatise on the Circle and the Sphere*, 1916, Chelsea reprint, New York.
[6] H.S.M. Coxeter, *Introduction to Geometry*, 1961; reprinted as Wiley classics, 1996.
[7] F.G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, Paris, 1991.
[8] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, Charles Babbage Research Centre, Winnipeg, 1989.
[9] Hart, Geometrical investigation of Steiner's solution of Malfatti's problem, *Quart. J. Math.*, 1 (1856) 219.
[10] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000, http://cedar.evansville.edu/~ck6/encyclopedia/.
[11] A. Pampuch, Die 32 Lösungen des Malfattischen Problems, *Arch. der Math. u. Phys.*, (3) 8 (1904) 36-49.
[12] A. Pampuch, *Das Malfatti - Steinersche Problem*, Pr. Bischöfl. Gymn. St. Stephan, Straßburg. 53 S. 10 Taf. 4°.
[13] N.J.A. Sloane (ed.), *On-Line Encyclopedia of Integer Sequences*, http://www.research.att.com/~njas/sequences/.
[14] F. Soddy, The Kiss Precise, *Nature*, 137 (1936) 1021.
[15] J. Steiner, *Gesammelte Werke*, 2 volumes, edited by K. Weierstrass, 1881; Chelsea reprint.

Translated by FLOOR VAN LAMOEN

Floor van Lamoen, Statenhof 3, 4463 TV Goes, The Netherlands
E-mail address: f.v.lamoen@wxs.nl

Forum Geometricorum
Volume 1 (2001) 50a

FORUM GEOM
ISSN 1534-1178

The Malfatti Problem

Oene Bottema

Supplement

Julio Gonzalez Cabillon has kindly supplied the following details for references mentioned in the text.

Additional References

[16] A. Clebsch, Anwendung der elliptischen Funktionen auf ein Problem der Geometrie des Raumes, *Crelle Journal*, 53 (1857) 292–308.
[17] K.H. Schellbach, Eine Lösung der Malfattischen Aufgabe, *Crelle Journal*, 45 (1853) 91–92.
[18] K.H. Schellbach, Eine Erweiterung der Malfattischen Aufgabe, *Crelle Journal*, 45 (1853) 186–187.

Forum Geometricorum
Volume 1 (2001) 51–58.

FORUM GEOM
ISSN 1534-1178

A Morley Configuration

Jean-Pierre Ehrmann and Bernard Gibert

Abstract. Given a triangle, the isogonal conjugates of the infinite points of the side lines of the Morley (equilateral) triangle is an equilateral triangle PQR inscribed in the circumcircle. Their isotomic conjugates form another equilateral triangle $P'Q'R'$ inscribed in the Steiner circum-ellipse, homothetic to PQR at the Steiner point. We show that under the one-to-one correspondence $P \mapsto P'$ between the circumcircle and the Steiner circum-ellipse established by isogonal and then isotomic conjugations, this is the only case when both PQR and $P'Q'R'$ are equilateral.

1. Introduction

Consider the Morley triangle $M_aM_bM_c$ of a triangle ABC, the equilateral triangle whose vertices are the intersections of pairs of angle trisectors adjacent to a side. Under *isogonal* conjugation, the infinite points of the Morley lines M_bM_c, M_cM_a, M_aM_b correspond to three points G_a, G_b, G_c on the circumcircle. These three points form the vertices of an equilateral triangle. This phenomenon is true for any three lines making 60° angles with one another.[1]

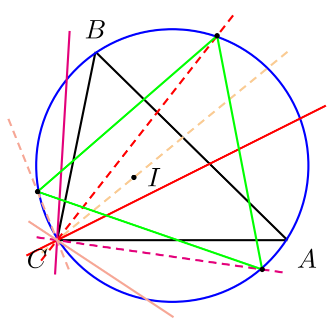


Figure 1

Under *isotomic* conjugation, on the other hand, the infinite points of the same three Morley lines correspond to three points T_a, T_b, T_c on the Steiner circum-ellipse. It is interesting to note that these three points also form the vertices of an equilateral triangle. Consider the mapping that sends a point P to P', the isotomic conjugate of the isogonal conjugate of P. This maps the circumcircle onto the Steiner circum-ellipse. The main result of this paper is that $G_aG_bG_c$ is the only equilateral triangle PQR for which $P'Q'R'$ is also equilateral.

Publication Date: March 22, 2001. Communicating Editor: Paul Yiu.

[1]In Figure 1, the isogonal conjugates of the infinite points of the three lines through A are the intersections of the circumcircle with their reflections in the bisector of angle A.

Main Theorem. Let PQR be an equilateral triangle inscribed in the circumcircle. The triangle $P'Q'R'$ is equilateral if and only if P, Q, R are the isogonal conjugates of the infinite points of the Morley lines.

Before proving this theorem, we make some observations and interesting applications.

2. Homothety of $G_aG_bG_c$ and $T_aT_bT_c$

The two equilateral triangles $G_aG_bG_c$ and $T_aT_bT_c$ are homothetic at the Steiner point S, with ratio of homothety $1 : 4\sin^2\Omega$, where Ω is the Brocard angle of triangle ABC. The circumcircle of the equilateral triangle $T_aT_bT_c$ has center at the third Brocard point [2], the isotomic conjugate of the symmedian point, and is tangent to the circumcircle of ABC at the Steiner point S. In other words, the circle centered at the third Brocard point and passing through the Steiner point intersects the Steiner circum-ellipse at three other points which are the vertices of an equilateral triangle homothetic to the Morley triangle. This circle has radius $4R\sin^2\Omega$ and is smaller than the circumcircle, except when triangle ABC is equilateral.

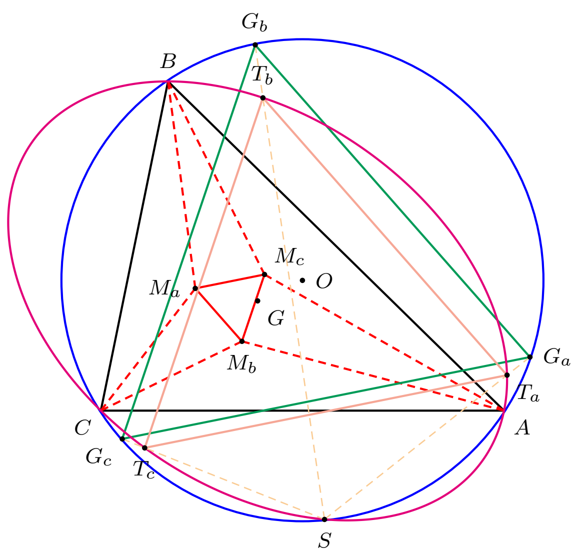


Figure 2

The triangle $G_aG_bG_c$ is the circum-tangential triangle in [3]. It is homothetic to the Morley triangle. From this it follows that the points G_a, G_b, G_c are the points of tangency with the circumcircle of the deltoid which is the envelope of the axes of inscribed parabolas.[3]

[2] This point is denoted by X_{76} in [3].

[3] The axis of an inscribed parabola with focus F is the perpendicular from F to its Simson line, or equivalently, the homothetic image of the Simson line of the antipode of F on the circumcircle, with homothetic center G and ratio -2. In [5], van Lamoen has shown that the points of contact of Simson lines tangent to the nine-point circle also form an equilateral triangle homothetic to the Morley triangle.

3. Equilateral triangles inscribed in an ellipse

Let $\mathcal{E}$ be an ellipse centered at O, and U a point on $\mathcal{E}$. With homothetic center O, ratio $-\frac{1}{2}$, maps U to u. Construct the parallel through u to its polar with respect to $\mathcal{E}$, to intersect the ellipse at V and W. The circumcircle of UVW intersects $\mathcal{E}$ at the Steiner point S of triangle UVW. Let M be the third Brocard point of UVW. The circle, center M, passing through S, intersects $\mathcal{E}$ at three other points which form the vertices of an equilateral triangle. See Figure 3.

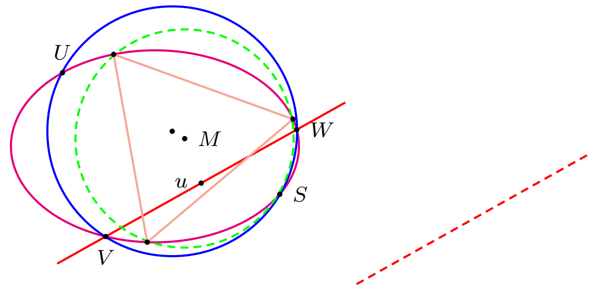


Figure 3

From this it follows that the locus of the centers of equilateral triangles inscribed in the Steiner circum-ellipse of ABC is the ellipse

$$\sum_{\text{cyclic}} a^2(a^2+b^2+c^2)x^2 + (a^2(b^2+c^2) - (2b^4+b^2c^2+2c^4))yz = 0$$

with the same center and axes.

4. Some preliminary results

Proposition 1. *If a circle through the focus of a parabola has its center on the directrix, there exists an equilateral triangle inscribed in the circle, whose side lines are tangent to the parabola.*

Proof. Denote by p the distance from the focus F of the parabola to its directrix. In polar coordinates with the pole at F, let the center of the circle be the point $(\frac{p}{\cos\alpha}, \alpha)$. The radius of the circle is $R = \frac{p}{\cos\alpha}$. See Figure 4. If this center is at a distance d to the line tangent to the parabola at the point $(\frac{p}{1+\cos\theta}, \theta)$, then

$$\frac{d}{R} = \left|\frac{\cos(\theta-\alpha)}{2\cos\frac{\theta}{2}}\right|.$$

Thus, for $\theta = \frac{2}{3}\alpha$, $\frac{2}{3}(\alpha+\pi)$ and $\frac{2}{3}(\alpha-\pi)$, we have $d = \frac{R}{2}$, and the lines tangent to the parabola at these three points form the required equilateral triangle. $\square$

Proposition 2. *If P lies on the circumcircle, then the line PP' passes through the Steiner point S.* [4]

[4] More generally, if $u+v+w=0$, the line joining $(\frac{p}{u} : \frac{q}{v} : \frac{r}{w})$ to $(\frac{l}{u} : \frac{m}{v} : \frac{n}{w})$ passes through the point $(\frac{1}{qn-rm} : \frac{1}{rl-pn} : \frac{1}{pm-ql})$ which is the fourth intersection of the two circumconics $\frac{p}{u}+\frac{q}{v}+\frac{r}{w}=0$ and $\frac{l}{u}+\frac{m}{v}+\frac{n}{w}=0$.

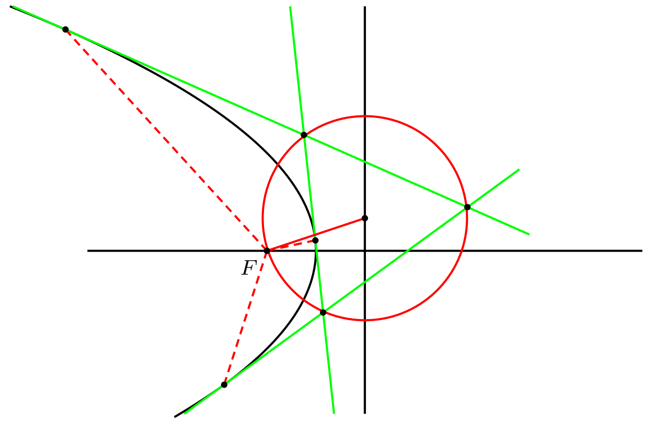


Figure 4

It follows that a triangle PQR inscribed in the circumcircle is always perspective with $P'Q'R'$ (inscribed in the Steiner circum-ellipse) at the Steiner point. The perspectrix is a line parallel to the tangent to the circumcircle at the focus of the Kiepert parabola. [5]

We shall make use of the Kiepert parabola

$$\mathcal{P}: \qquad \sum (b^2-c^2)^2x^2 - 2(c^2-a^2)(a^2-b^2)yz = 0.$$

This is the inscribed parabola with perspector the Steiner point S, focus $S' = (\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2})$, [6] and the Euler line as directrix. For more on inscribed parabolas and inscribed conics in general, see [1].

Proposition 3. *Let PQ be a chord of the circumcircle. The following statements are equivalent:* [7]

(a) PQ and $P'Q'$ are parallel.

(b) The line PQ is tangent to the Kiepert parabola $\mathcal{P}$.

(c) The Simson lines $s(P)$ and $s(Q)$ intersect on the Euler line.

Proof. If the line PQ is $ux+vy+wz=0$, then $P'Q'$ is $a^2ux+b^2vy+c^2wz=0$. These two lines are parallel if and only if

$$\frac{b^2-c^2}{u}+\frac{c^2-a^2}{v}+\frac{a^2-b^2}{w}=0, \tag{1}$$

which means that PQ is tangent to the the Kiepert parabola.

The common point of the Simson lines $s(P)$ and $s(Q)$ is $(x:y:z)$, where

$$\begin{aligned} x \;=\; & (2b^2(c^2+a^2-b^2)v+2c^2(a^2+b^2-c^2)w-(c^2+a^2-b^2)(a^2+b^2-c^2)u) \\ & \cdot((a^2+b^2-c^2)v+(c^2+a^2-b^2)w-2a^2u), \end{aligned}$$

---

[5] This line is also parallel to the trilinear polars of the two isodynamic points.

[6] This is the point X_{110} in [3].

[7] These statements are also equivalent to (d): The orthopole of the line PQ lies on the Euler line.

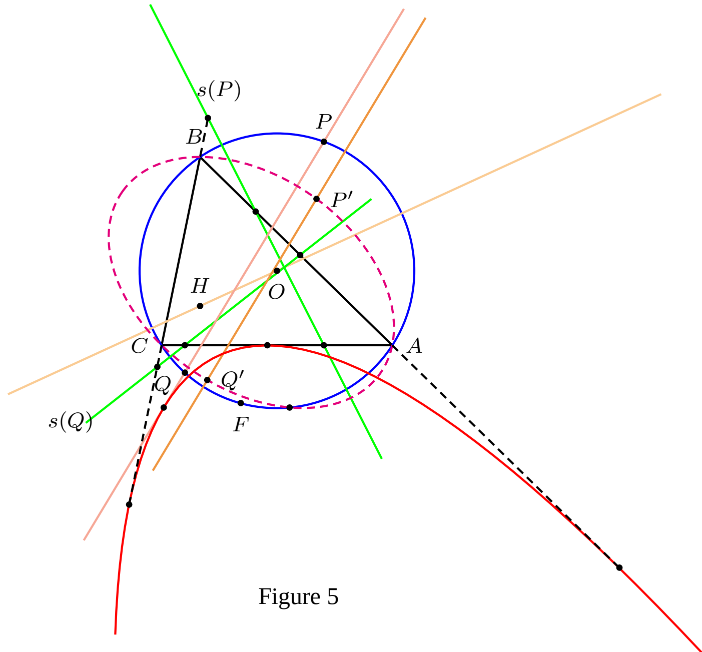


Figure 5

and y and z can be obtained by cyclically permuting a, b, c, and u, v, w. This point lies on the Euler line if and only if (1) is satisfied. □

In the following proposition, (ℓ_1, ℓ_2) denotes the directed angle between two lines ℓ_1 and ℓ_2. This is the angle through which the line ℓ_1 must be rotated in the positive direction in order to become parallel to, or to coincide with, the line ℓ_2. See [2, §§16–19.].

Proposition 4. *Let P, Q, R be points on the circumcircle. The following statements are equivalent.*

(a) The Simson lines $s(P)$, $s(Q)$, $s(R)$ are concurrent.
(b) $(AB, PQ) + (BC, QR) + (CA, RP) = 0 \pmod{\pi}$.
(c) $s(P)$ and QR are perpendicular; so are $s(Q)$ and RP; $s(R)$ and PQ.

Proof. See [4, §§2.16–20]. □

Proposition 5. *A line ℓ is parallel to a side of the Morley triangle if and only if*

$$(AB, \ell) + (BC, \ell) + (CA, \ell) = 0 \pmod{\pi}.$$

Proof. Consider the Morley triangle $M_a M_b M_c$. The line BM_c and CM_b intersecting at P, the triangle PM_bM_c is isoceles and $(M_cM_b, M_cP) = \frac{1}{3}(B + C)$. Thus, $(BC, M_bM_c) = \frac{1}{3}(B - C)$. Similarly, $(CA, M_bM_c) = \frac{1}{3}(C - A) + \frac{\pi}{3}$, and $(AB, M_bM_c) = \frac{1}{3}(A - B) - \frac{\pi}{3}$. Thus

$$(AB, M_bM_c) + (BC, M_bM_c) + (CA, M_bM_c) = 0 \pmod{\pi}.$$

There are only three directions of line ℓ for which $(AB, \ell)+(BC, \ell)+(CA, \ell) = 0$. These can only be the directions of the Morley lines. □

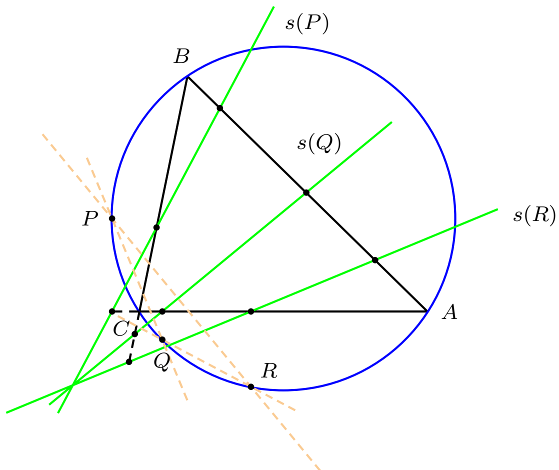


Figure 6

5. Proof of Main Theorem

Let $\mathcal{P}$ be the Kiepert parabola of triangle ABC. By Proposition 1, there is an equilateral triangle PQR inscribed in the circumcircle whose sides are tangent to $\mathcal{P}$. By Propositions 2 and 3, the triangle $P'Q'R'$ is equilateral and homothetic to PQR at the Steiner point S. By Proposition 3 again, the Simson lines $s(P)$, $s(Q)$, $s(R)$ concur. It follows from Proposition 4 that $(AB, PQ) + (BC, QR) + (CA, RP) = 0 \pmod{\pi}$. Since the lines PQ, QR, and RP make 60° angles with each other, we have

$$(AB, PQ) + (BC, PQ) + (CA, PQ) = 0 \pmod{\pi},$$

and PQ is parallel to a side of the Morley triangle by Proposition 5. Clearly, this is the same for QR and RP. By Proposition 4, the vertices P, Q, R are the isogonal conjugates of the infinite points of the Morley sides.

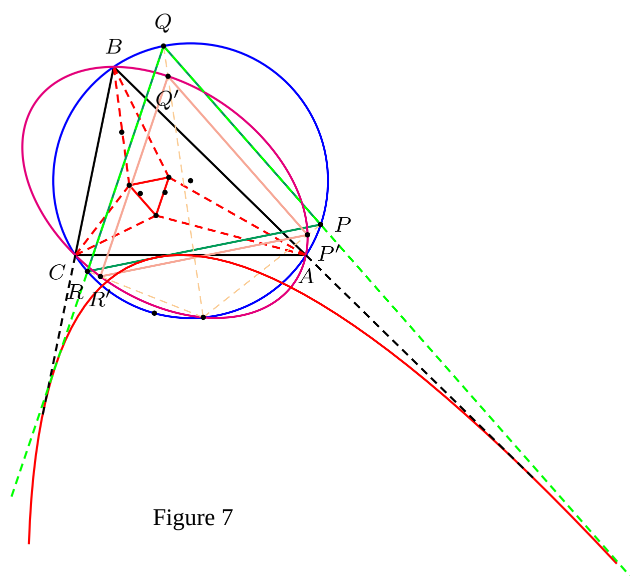


Figure 7

Uniqueness: For $M(x:y:z)$, let

$$f(M) = \frac{x+y+z}{\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}}.$$

The determinant of the affine mapping $P \mapsto P'$, $Q \mapsto Q'$, $R \mapsto R'$ is

$$\frac{f(P)f(Q)f(R)}{a^2b^2c^2}.$$

This determinant is positive for P, Q, R on the circumcircle, which does not intersect the Lemoine axis $\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}=0$. Thus, if both triangles are equilateral, the similitude $P \mapsto P'$, $Q \mapsto Q'$, $R \mapsto R'$ is a *direct* one. Hence,

$$(SP', SQ') = (SP, SQ) = (RP, RQ) = (R'P', R'Q'),$$

and the circle $P'Q'R'$ passes through S. Now, through any point on an ellipse, there is a unique circle intersecting the ellipse again at the vertices of an equilateral triangle. This establishes the uniqueness, and completes the proof of the theorem.

6. Concluding remarks

We conclude with a remark and a generalization.

(1) The reflection of $G_aG_bG_c$ in the circumcenter is another equilateral triangle PQR (inscribed in the circumcircle) whose sides are parallel to the Morley lines.[8] This, however, does not lead to an equilateral triangle inscribed in the Steiner circum-ellipse.

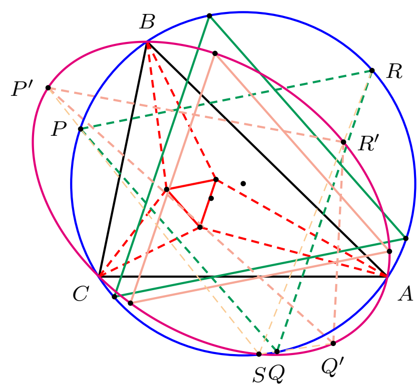


Figure 8

(2) Consider the circum-hyperbola $\mathcal{C}$ through the centroid G and the symmedian point K.[9] For any point P on $\mathcal{C}$, let $\mathcal{C}_P$ be the circumconic with perspector P, intersecting the circumcircle again at a point S_P. [10] For every point M on the

[8]This is called the circumnormal triangle in [3].

[9]The center of this hyperbola is the point $(a^4(b^2-c^2)^2 : b^4(c^2-a^2)^2 : c^4(a^2-b^2)^2)$.

[10]The perspector of a circumconic is the perspector of the triangle bounded by the tangents to the conic at the vertices of ABC. If $P=(u:v:w)$, the circumconic $\mathcal{C}_P$ has center $(u(v+w-u) : v(w+u-v) : w(u+v-w))$, and S_P is the point $(\frac{1}{b^2w-c^2v} : \frac{1}{c^2u-a^2w} : \frac{1}{a^2v-b^2u})$. See Footnote 4.

circumcircle, denote by M' the second common point of $\mathcal{C}_U$ and the line MS_P. Then, if G_a, G_b, G_c are the isogonal conjugates of the infinite points of the Morley lines, $G'_aG'_bG'_c$ is homothetic to $G_aG_bG_c$ at S_U. The reason is simple: Proposition 3 remains true. For $U = G$, this gives the equilateral triangle $T_aT_bT_c$ inscribed in the case of the Steiner circum-ellipse. Here is an example. For $U = (a(b+c) : b(c+a) : c(a+b))$, [11] we have the circumellipse with center the Spieker center $(b+c : c+a : a+b)$. The triangles $G_aG_bG_c$ and $G'_aG'_bG'_c$ are homothetic at $X_{100} = (\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b})$, and the circumcircle of $G'_aG'_bG'_c$ is the incircle of the anticomplementary triangle, center the Nagel point, and ratio of homothety $R : 2r$.

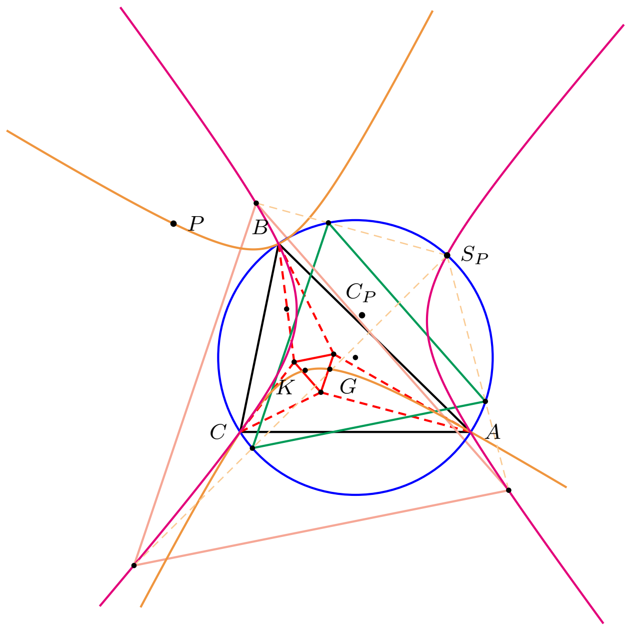


Figure 9

References

[1] J.P. Ehrmann, Steiner's note on the complete quadrilateral, *Forum Geom*, to appear.
[2] R.A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.
[3] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–295.
[4] T. Lalesco, *La Géométrie du Triangle*, Bucharest, 1937; Gabay reprint, 1987.
[5] F.M. van Lamoen, Morley related triangles on the nine-point circle, *Amer. Math. Monthly*, 107 (2000) 941–945.

Jean-Pierre Ehrmann: 6 rue des Cailloux, 92110 - Clichy, France
E-mail address: Jean-Pierre.EHRMANN@wanadoo.fr

Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France
E-mail address: b.gibert@free.fr

[11] This is the point X_{37} in [3].

Forum Geometricorum
Volume 1 (2001) 59–68.

FORUM GEOM
ISSN 1534-1178

Concurrency of Four Euler Lines

Antreas P. Hatzipolakis, Floor van Lamoen, Barry Wolk, and Paul Yiu

Abstract. Using tripolar coordinates, we prove that if P is a point in the plane of triangle ABC such that the Euler lines of triangles PBC, APC and ABP are concurrent, then their intersection lies on the Euler line of triangle ABC. The same is true for the Brocard axes and the lines joining the circumcenters to the respective incenters. We also prove that the locus of P for which the four Euler lines concur is the same as that for which the four Brocard axes concur. These results are extended to a family $\mathcal{L}_n$ of lines through the circumcenter. The locus of P for which the four $\mathcal{L}_n$ lines of ABC, PBC, APC and ABP concur is always a curve through 15 finite real points, which we identify.

1. Four line concurrency

Consider a triangle ABC with incenter I. It is well known [13] that the Euler lines of the triangles IBC, AIC and ABI concur at a point on the Euler line of ABC, the Schiffler point with homogeneous barycentric coordinates[1]

$$\left(\frac{a(s-a)}{b+c} : \frac{b(s-b)}{c+a} : \frac{c(s-c)}{a+b}\right).$$

There are other notable points which we can substitute for the incenter, so that a similar statement can be proven relatively easily. Specifically, we have the following interesting theorem.

Theorem 1. *Let P be a point in the plane of triangle ABC such that the Euler lines of the component triangles PBC, APC and ABP are concurrent. Then the point of concurrency also lies on the Euler line of triangle ABC.*

When one tries to prove this theorem with homogeneous coordinates, calculations turn out to be rather tedious, as one of us has noted [14]. We present an easy analytic proof, making use of tripolar coordinates. The same method applies if we replace the Euler lines by the Brocard axes or the OI-lines joining the circumcenters to the corresponding incenters.

Publication Date: April 9, 2001. Communicating Editor: Jean-Pierre Ehrmann.

[1]This appears as X_{21} in Kimberling's list [7]. In the expressions of the coordinates, s stands for the semiperimeter of the triangle.

2. Tripolar coordinates

Given triangle ABC with $BC = a$, $CA = b$, and $AB = c$, consider a point P whose distances from the vertices are $PA = \lambda$, $PB = \mu$ and $PC = \nu$. The precise relationship among λ, μ, and ν dates back to Euler [4]:

$$\begin{aligned}&(\mu^2+\nu^2-a^2)^2\lambda^2+(\nu^2+\lambda^2-b^2)^2\mu^2+(\lambda^2+\mu^2-c^2)^2\nu^2\\ &-(\mu^2+\nu^2-a^2)(\nu^2+\lambda^2-b^2)(\lambda^2+\mu^2-c^2)-4\lambda^2\mu^2\nu^2=0.\end{aligned}$$

See also [1, 2]. Geometers in the 19th century referred to the triple (λ, μ, ν) as the *tripolar* coordinates of P. A comprehensive introduction can be found in [12]. [2] This series begins with the following easy theorem.

Proposition 2. *An equation of the form $\ell\lambda^2 + m\mu^2 + n\nu^2 + q = 0$ represents a circle or a line according as $\ell + m + n$ is nonzero or otherwise.*

The center of the circle has homogeneous barycentric coordinates $(\ell : m : n)$. If $\ell + m + n = 0$, the line is orthogonal to the direction $(\ell : m : n)$. Among the applications one finds the equation of the Euler line in tripolar coordinates [op. cit. §26]. [3]

Proposition 3. *The tripolar equation of the Euler line is*

$$(b^2-c^2)\lambda^2+(c^2-a^2)\mu^2+(a^2-b^2)\nu^2=0. \tag{1}$$

We defer the proof of this proposition to §5 below. Meanwhile, note how this applies to give a simple proof of Theorem 1.

3. Proof of Theorem 1

Let P be a point with tripolar coordinates (λ, μ, ν) such that the Euler lines of triangles PBC, APC and ABP intersect at a point Q with tripolar coordinates (λ', μ', ν'). We denote the distance PQ by ρ.

Applying Proposition 3 to the triangles PBC, APC and ABP, we have

$$\begin{aligned}(\nu^2-\mu^2)\rho^2+(\mu^2-a^2)\mu'^2+(a^2-\nu^2)\nu'^2&=0,\\ (b^2-\lambda^2)\lambda'^2+(\lambda^2-\nu^2)\rho^2+(\nu^2-b^2)\nu'^2&=0,\\ (\lambda^2-c^2)\lambda'^2+(c^2-\mu^2)\mu'^2+(\mu^2-\lambda^2)\rho^2&=0.\end{aligned}$$

Adding up these equations, we obtain (1) with λ', μ', ν' in lieu of λ, μ, ν. This shows that Q lies on the Euler line of ABC.

---

[2] [5] and [8] are good references on tripolar coordinates.

[3] The tripolar equations of the lines in §§5 – 7 below can be written down from the barycentric equations of these lines. The calculations in these sections, however, do not make use of these barycentric equations.

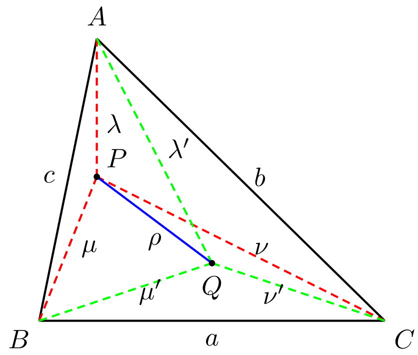


Figure 1

4. Tripolar equations of lines through the circumcenter

O. Bottema [2, pp.37–38] has given a simple derivation of the equation of the Euler line in tripolar coordinates. He began with the observation that since the point-circles

$$\lambda^2 = 0, \qquad \mu^2 = 0, \qquad \nu^2 = 0,$$

are all orthogonal to the circumcircle,[4] for arbitrary t_1, t_2, t_3, the equation

$$t_1\lambda^2 + t_2\mu^2 + t_3\nu^2 = 0 \tag{2}$$

represents a circle orthogonal to the circumcircle. By Proposition 2, this represents a line through the circumcenter if and only if $t_1 + t_2 + t_3 = 0$.

5. Tripolar equation of the Euler line

Consider the centroid G of triangle ABC. By the Apollonius theorem, and the fact that G divides each median in the ratio $2 : 1$, it is easy to see that the tripolar coordinates of G satisfy

$$\lambda^2 : \mu^2 : \nu^2 = 2b^2 + 2c^2 - a^2 : 2c^2 + 2a^2 - b^2 : 2a^2 + 2b^2 - c^2.$$

It follows that the Euler line OG is defined by (2) with t_1, t_2, t_3 satisfying

$$\begin{array}{rcrcrcl} t_1 & + & t_2 & + & t_3 & = & 0, \\ (2b^2 + 2c^2 - a^2)t_1 & + & (2c^2 + 2a^2 - b^2)t_2 & + & (2a^2 + 2b^2 - c^2)t_3 & = & 0, \end{array}$$

or $$t_1 : t_2 : t_3 = b^2 - c^2 : c^2 - a^2 : a^2 - b^2.$$

This completes the proof of Proposition 3.

[4] These point-circles are evidently the vertices of triangle ABC.

6. Tripolar equation of the OI-line

For the incenter I, we have

$$\lambda^2 : \mu^2 : \nu^2 = \csc^2 \frac{A}{2} : \csc^2 \frac{B}{2} : \csc^2 \frac{C}{2} = \frac{s-a}{a} : \frac{s-b}{b} : \frac{s-c}{c},$$

where $s = \frac{a+b+c}{2}$. The tripolar equation of the OI-line is given by (2) with t_1, t_2, t_3 satisfying

$$t_1 + t_2 + t_3 = 0, \qquad \frac{s-a}{a}t_1 + \frac{s-b}{b}t_2 + \frac{s-c}{c}t_3 = 0.$$

From these, $t_1 : t_2 : t_3 = \frac{1}{b} - \frac{1}{c} : \frac{1}{c} - \frac{1}{a} : \frac{1}{a} - \frac{1}{b}$, and the tripolar equation of the OI-line is

$$\left(\frac{1}{b} - \frac{1}{c}\right)\lambda^2 + \left(\frac{1}{c} - \frac{1}{a}\right)\mu^2 + \left(\frac{1}{a} - \frac{1}{b}\right)\nu^2 = 0.$$

The same reasoning in §3 yields Theorem 1 with the Euler lines replaced by the OI-lines.

7. Tripolar equation of the Brocard axis

The Brocard axis is the line joining the circumcenter to the symmedian point. Since this line contains the two isodynamic points, whose tripolar coordinates, by definition, satisfy

$$\lambda : \mu : \nu = \frac{1}{a} : \frac{1}{b} : \frac{1}{c},$$

it is easy to see that the tripolar equation of the Brocard axis is[5]

$$\left(\frac{1}{b^2} - \frac{1}{c^2}\right)\lambda^2 + \left(\frac{1}{c^2} - \frac{1}{a^2}\right)\mu^2 + \left(\frac{1}{a^2} - \frac{1}{b^2}\right)\nu^2 = 0.$$

The same reasoning in §3 yields Theorem 1 with the Euler lines replaced by the Brocard axes.

8. The lines $\mathcal{L}_n$

The resemblance of the tripolar equations in §§5 – 7 suggests consideration of the family of lines through the circumcenter:

$$\mathcal{L}_n : \qquad (b^n - c^n)\lambda^2 + (c^n - a^n)\mu^2 + (a^n - b^n)\nu^2 = 0,$$

for nonzero integers n. The Euler line, the Brocard axis, and the OI-line are respectively $\mathcal{L}_n$ for $n = 2, -2$, and -1. In homogeneous barycentric coordinates,

[5]The same equation can be derived directly from the tripolar distances of the symmedian point: $AK^2 = \frac{b^2c^2(2b^2+2c^2-a^2)}{(a^2+b^2+c^2)^2}$ etc. This can be found, for example, in [11, p.118].

the equation of $\mathcal{L}_n$ is[6]

$$\sum_{\text{cyclic}}(a^n(b^2-c^2)-(b^{n+2}-c^{n+2}))x=0.$$

The line $\mathcal{L}_1$ contains the points [7]

$$(2a+b+c:a+2b+c:a+b+2c)$$

and

$$(a(b+c)-(b-c)^2:b(c+a)-(c-a)^2:c(a+b)-(a-b)^2).$$

Theorem 1 obviously applies when the Euler lines are replaced by $\mathcal{L}_n$ lines for a fixed nonzero integer n.

9. Intersection of the $\mathcal{L}_n$ lines

It is known that the locus of P for which the Euler lines ($\mathcal{L}_2$) of triangles PBC, APC and ABP are concurrent is the union of the circumcircle and the Neuberg cubic.[8] See [10, p.200]. Fred Lang [9] has computed the locus for the Brocard axes ($\mathcal{L}_{-2}$) case, and found exactly the same result. The coincidence of these two loci is a special case of the following theorem.

Theorem 4. *Let n be a nonzero integer. The $\mathcal{L}_n$ lines of triangles PBC, APC and ABP concur (at a point on $\mathcal{L}_n$) if and only if the $\mathcal{L}_{-n}$ lines of the same triangles concur (at a point on $\mathcal{L}_{-n}$).*

Proof. Consider the component triangles PBC, APC and ABP of a point P. If P has tripolar coordinates (L,M,N), then the $\mathcal{L}_n$ lines of these triangles have tripolar equations

$$\mathcal{L}_n(PBC):\qquad (N^n-M^n)\rho^2+(M^n-a^n)\mu^2+(a^n-N^n)\nu^2=0,$$

$$\mathcal{L}_n(APC):\qquad (b^n-L^n)\lambda^2+(L^n-N^n)\rho^2+(N^n-b^n)\nu^2=0,$$

$$\mathcal{L}_n(ABP):\qquad (L^n-c^n)\lambda^2+(c^n-M^n)\mu^2+(M^n-L^n)\rho^2=0,$$

where ρ is the distance between P and a variable point (λ,μ,ν).[9] These equations can be rewritten as

---

[6]This can be obtained from the tripolar equation by putting

$$\lambda^2=\frac{1}{(x+y+z)^2}(c^2y^2+(b^2+c^2-a^2)yz+b^2z^2)$$

and analogous expressions for μ^2 and ν^2 obtained by cyclic permutations of a, b, c and x, y, z.

[7]These are respectively the midpoint between the incenters of ABC and its medial triangle, and the symmedian point of the excentral triangle of the medial triangle.

[8]The Neuberg cubic is defined as the locus of points P such that the line joining P to its isogonal conjugate is parallel to the Euler line.

[9]See Figure 1, with λ, μ, ν replaced by L, M, N, and λ', μ', ν' by λ, μ, ν respectively.

$$\begin{array}{rcrcrcl} & - & (M^n - a^n)(\rho^2 - \mu^2) & + & (N^n - a^n)(\rho^2 - \nu^2) & = & 0, \\ (L^n - b^n)(\rho^2 - \lambda^2) & & & - & (N^n - b^n)(\rho^2 - \nu^2) & = & 0, \\ -(L^n - c^n)(\rho^2 - \lambda^2) & + & (M^n - c^n)(\rho^2 - \mu^2) & & & = & 0. \end{array} \tag{3}$$

One trivial solution to these equations is $\rho = \lambda = \mu = \nu$, which occurs only when the variable point is the circumcenter O, with P on the circumcircle. In this case the $\mathcal{L}_n$ lines all concur at the point O, for all n. Otherwise, we have a solution to (3) with at least one of the values $\rho^2 - \lambda^2$, $\rho^2 - \mu^2$, and $\rho^2 - \nu^2$ being non-zero. And the condition for a solution of this kind is

$$(L^n - b^n)(M^n - c^n)(N^n - a^n) = (L^n - c^n)(M^n - a^n)(N^n - b^n). \tag{4}$$

This condition is clearly necessary. Conversely, take P satisfying (4). This says that (3), as linear homogeneous equations in $\rho^2 - \lambda^2$, $\rho^2 - \mu^2$, and $\rho^2 - \nu^2$, have a nontrivial solution (u, v, w), which is determined up to a scalar multiple. Then the equations of the $\mathcal{L}_n$ lines of triangles ABP and PBC can be rewritten as $(\frac{1}{u} - \frac{1}{v})XP^2 - \frac{1}{u}XA^2 + \frac{1}{v}XB^2 = 0$ and $(\frac{1}{v} - \frac{1}{w})XP^2 - \frac{1}{v}XB^2 + \frac{1}{w}XC^2 = 0$. If X is a point common to these two lines, then it satisfies

$$\frac{XP^2 - XA^2}{u} = \frac{XP^2 - XB^2}{v} = \frac{XP^2 - XC^2}{w}$$

and also lies on the $\mathcal{L}_n$ line of triangle APC.

Note that (4) is clearly equivalent to

$$\left(\frac{1}{L^n} - \frac{1}{b^n}\right)\left(\frac{1}{M^n} - \frac{1}{c^n}\right)\left(\frac{1}{N^n} - \frac{1}{a^n}\right) = \left(\frac{1}{L^n} - \frac{1}{c^n}\right)\left(\frac{1}{M^n} - \frac{1}{a^n}\right)\left(\frac{1}{N^n} - \frac{1}{b^n}\right),$$

which, by exactly the same reasoning, is the concurrency condition for the $\mathcal{L}_{-n}$ lines of the same triangles. □

Corollary 5. *The locus of P for which the Brocard axes of triangles PBC, APC and ABP are concurrent (at a point on the Brocard axis of triangle ABC) is the union of the circumcircle and the Neuberg cubic.*

Let $\mathcal{C}_n$ be the curve with tripolar equation

$$(\lambda^n - b^n)(\mu^n - c^n)(\nu^n - a^n) = (\lambda^n - c^n)(\mu^n - a^n)(\nu^n - b^n),$$

so that together with the circumcircle, it constitutes the locus of points P for which the four $\mathcal{L}_n$ lines of triangles PBC, APC, ABP and ABC concur.[10] The symmetry of equation (4) leads to the following interesting fact.

Corollary 6. *If P lies on the $\mathcal{C}_n$ curve of triangle ABC, then A (respectively B, C) lies on the $\mathcal{C}_n$ curve of triangle PBC (respectively APC, ABP).*

Remark. The equation of $\mathcal{C}_n$ can also be written in one of the following forms:

$$\sum_{\text{cyclic}} (b^n - c^n)(a^n\lambda^n + \mu^n\nu^n) = 0$$

[10] By Theorem 4, it is enough to consider n positive.

or

$$\begin{vmatrix} \lambda^n + a^n & \mu^n + b^n & \nu^n + c^n \\ a^n\lambda^n & b^n\mu^n & c^n\nu^n \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

10. Points common to $\mathcal{C}_n$ curves

Proposition 7. *A complete list of finite real points common to all $\mathcal{C}_n$ curves is as follows:*

(1) the vertices A, B, C and their reflections on the respective opposite side,
(2) the apexes of the six equilateral triangles erected on the sides of ABC,
(3) the circumcenter, and
(4) the two isodynamic points.

Proof. It is easy to see that each of these points lies on $\mathcal{C}_n$ for every positive integer n. For the isodynamic points, recall that $\lambda : \mu : \nu = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$. We show that $\mathcal{C}_1$ and $\mathcal{C}_2$ meet precisely in these 15 points. From their equations

$$(\lambda - b)(\mu - c)(\nu - a) = (\lambda - c)(\mu - a)(\nu - b) \tag{5}$$

and

$$(\lambda^2 - b^2)(\mu^2 - c^2)(\nu^2 - a^2) = (\lambda^2 - c^2)(\mu^2 - a^2)(\nu^2 - b^2). \tag{6}$$

If both sides of (5) are zero, it is easy to list the various cases. For example, solutions like $\lambda = b, \mu = a$ lead to a vertex and its reflection through the opposite side (in this case C and its reflection in AB); solutions like $\lambda = b$, $\nu = b$ lead to the apexes of equilateral triangles erected on the sides of ABC (in this case on AC). Otherwise we can factor and divide, getting

$$(\lambda + b)(\mu + c)(\nu + a) = (\lambda + c)(\mu + a)(\nu + b).$$

Together with (5), this is easy to solve. The only solutions in this case are $\lambda = \mu = \nu$ and $\lambda : \mu : \nu = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$, giving respectively $P = O$ and the isodynamic points. □

Remarks. (1) If P is any of the points listed above, then this result says that the triangles ABC, PBC, APC, and ABP have concurrent $\mathcal{L}_n$ lines, for all non-zero integers n. There is no degeneracy in the case where P is an isodynamic point, and we then get an infinite sequence of four-fold concurrences.

(2) The curve $\mathcal{C}_4$ has degree 7, and contains the two circular points at infinity, each of multiplicity 3. These, together with the 15 finite real points above, account for all 21 intersections of $\mathcal{C}_2$ and $\mathcal{C}_4$.

11. Intersections of Euler lines and of Brocard axes

For $n = \pm 2$, the curve $\mathcal{C}_n$ is the Neuberg cubic

$$\sum_{\text{cyclic}} ((b^2 - c^2)^2 + a^2(b^2 + c^2) - 2a^4)x(c^2y^2 - b^2z^2) = 0$$

in homogeneous barycentric coordinates. Apart from the points listed in Proposition 7, this cubic contains the following notable points: the orthocenter, incenter and excenters, the Fermat points, and the Parry reflection point.[11] A summary of interesting properties of the Neuberg cubic can be found in [3]. Below we list the corresponding points of concurrency, giving their coordinates. For points like the Fermat points and Napoleon points resulting from erecting equilateral triangles on the sides, we label the points by $\epsilon = +1$ or -1 according as the equilateral triangles are constructed exterior to ABC or otherwise. Also, Δ stands for the area of triangle ABC. For functions like F_a, F_b, F_c indexed by a, b, c, we obtain F_b and F_c from F_a by cyclic permutations of a, b, c.

P	Intersection of Euler lines	Intersection of Brocard axes
Circumcenter	Circumcenter	Circumcenter
Reflection of vertex on opposite side	Intercept of Euler line on the side line	Intercept of Brocard axis on the side line
Orthocenter	Nine-point center	Orthocenter of orthic triangle
Incenter	Schiffler point	Isogonal conjugate of Spieker center
Excenters $I_a = (-a : b : c)$ $I_b = (a : -b : c)$ $I_c = (a : b : -c)$	 $(\frac{as}{b+c} : \frac{b(s-c)}{c-a} : \frac{c(s-b)}{-a+b})$ $(\frac{a(s-c)}{-b+c} : \frac{bs}{c+a} : \frac{c(s-a)}{a-b})$ $(\frac{a(s-b)}{b-c} : \frac{b(s-a)}{-c+a} : \frac{cs}{a+b})$	 $(\frac{a^2}{b+c} : \frac{b^2}{c-a} : \frac{c^2}{-a+b})$ $(\frac{a^2}{-b+c} : \frac{b^2}{c+a} : \frac{c^2}{a-b})$ $(\frac{a^2}{b-c} : \frac{b^2}{-c+a} : \frac{c^2}{a+b})$
ϵ-Fermat point	centroid	Isogonal conjugate of $(-\epsilon)$-Napoleon point
ϵ-isodynamic point		Isogonal conjugate of ϵ-Napoleon point

Apexes of ϵ-equilateral triangles erected on the sides of ABC. Let P be the apex of an equilateral triangle erected the side BC. This has coordinates

$$\left(-2a^2 : a^2 + b^2 - c^2 + \epsilon \cdot \frac{4}{\sqrt{3}}\Delta : c^2 + a^2 - b^2 + \epsilon \cdot \frac{4}{\sqrt{3}}\Delta \right).$$

The intersection of the Euler lines has coordinates

$$\begin{aligned}\Big(-a^2(a^2-b^2)(a^2-c^2) \quad &: \quad (a^2-b^2)(a^2b^2 + \epsilon \cdot \frac{4}{\sqrt{3}}\Delta(a^2+b^2-c^2)) \\ &: \quad (a^2-c^2)(a^2c^2 + \epsilon \cdot \frac{4}{\sqrt{3}}\Delta(c^2+a^2-b^2)) \Big),\end{aligned}$$

and the Brocard axis intersection is the point

$$\begin{aligned} & \Big(a^2(a^2-b^2)(a^2-c^2)(-\epsilon(b^2+c^2-a^2) + 4\sqrt{3}\Delta) \\ : \quad & b^2(a^2-b^2)(-\epsilon(a^4+2b^4+3c^4-5b^2c^2-4c^2a^2-3a^2b^2) + 4\sqrt{3}\Delta(c^2+a^2)) \\ : \quad & c^2(a^2-c^2)(-\epsilon(a^4+3b^4+2c^4-5b^2c^2-3c^2a^2-4a^2b^2) + 4\sqrt{3}\Delta(a^2+b^2)) \Big).\end{aligned}$$

---

[11] Bernard Gibert has found that the Fermat points of the anticomplementary triangle of ABC also lie on the Neuberg cubic. These are the points X_{616} and X_{617} in [7]. Their isogonal conjugates (in triangle ABC) clearly lie on the Neuberg cubic too. Ed.

Isodynamic points. For the ϵ-isodynamic point, the Euler line intersections are

$$\begin{array}{l}(a^2(\sqrt{3}b^2c^2 + \epsilon\cdot 4\Delta(b^2+c^2-a^2))\\ : b^2(\sqrt{3}c^2a^2 + \epsilon\cdot 4\Delta(c^2+a^2-b^2))\\ : c^2(\sqrt{3}a^2b^2 + \epsilon\cdot 4\Delta(a^2+b^2-c^2))).\end{array}$$

These points divide the segment GO harmonically in the ratio $8\sin A\sin B\sin C : 3\sqrt{3}$.[12] The Brocard axis intersections for the Fermat points and the isodynamic points are illustrated in Figure 2.

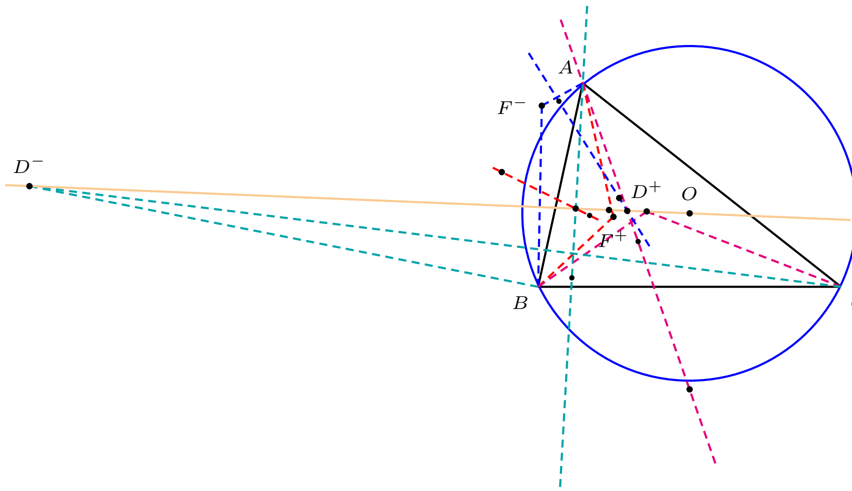


Figure 2

The Parry reflection point. This is the reflection of the circumcenter in the focus of the Kiepert parabola. [13] Its coordinates, and those of the Euler line and Brocard axis intersections, can be described with the aids of three functions.

(1) Parry reflection point: $(a^2P_a : b^2P_b : c^2P_c)$,

(2) Euler line intersection: $(a^2P_af_a : b^2P_bf_b : c^2P_cf_c)$,

(3) Brocard axis intersection: $(a^2f_ag_a : b^2f_bg_b : c^2f_cg_c)$, where

$$\begin{array}{rcl}P_a &=& a^8 - 4a^6(b^2+c^2) + a^4(6b^4+b^2c^2+6c^4)\\ && -a^2(b^2+c^2)(4b^4-5b^2c^2+4c^4) + (b^2-c^2)^2(b^4+4b^2c^2+c^4),\\ f_a &=& a^6 - 3a^4(b^2+c^2) + a^2(3b^4-b^2c^2+3c^4) - (b^2-c^2)^2(b^2+c^2),\\ g_a &=& 5a^8 - 14a^6(b^2+c^2) + a^4(12b^4+17b^2c^2+12c^4)\\ && -a^2(b^2+c^2)(2b^2+c^2)(b^2+2c^2) - (b^2-c^2)^4.\end{array}$$

---

[12]These coordinates, and those of the Brocard axis intersections, can be calculated by using the fact that triangle PBC has $(-\epsilon)$-isodynamic point at the vertex A and circumcenter at the point

$$(a^2((b^2+c^2-a^2)-\epsilon\cdot 4\sqrt{3}\Delta) : b^2((c^2+a^2-b^2)+\epsilon\cdot 4\sqrt{3}\Delta) : c^2((a^2+b^2-c^2)+\epsilon\cdot 4\sqrt{3}\Delta)).$$

[13]The Parry reflection point is the point X_{399} in [6]. The focus of the Kiepert parabola is the point on the circumcircle with coordinates $(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2})$.

This completes the identification of the Euler line and Brocard axis intersections for points on the Neuberg cubic. The identification of the locus for the $\mathcal{L}_{\pm 1}$ problems is significantly harder. Indeed, we do not know of any interesting points on this locus, except those listed in Proposition 7.

References

[1] O. Bottema, On the distances of a point to the vertices of a triangle, *Crux Math.*, 10 (1984) 242 – 246.

[2] O. Bottema, *Hoofdstukken uit de Elementaire Meetkunde*, 2nd ed. 1987, Epsilon Uitgaven, Utrecht.

[3] Z. Čerin, Locus properties of the Neuberg cubic, *Journal of Geometry*, 63 (1998), 39–56.

[4] L. Euler, De symptomatibus quatuor punctorum in eodem plano sitorum, *Acta Acad. sci. Petropolitanae*, 6 I (1782:I), 1786, 3 – 18; opera omnia, ser 1, vol 26, pp. 258 – 269.

[5] W. Gallatly, *The Modern Geometry of the Triangle*, 2nd ed. 1913, Francis Hodgson, London.

[6] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 140 (1998) 1–295.

[7] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000, http://cedar.evansville.edu/˜ck6/encyclopedia/.

[8] T. Lalesco, *La Géométrie du Triangle*, 2nd ed., 1952; Gabay reprint, 1987, Paris.

[9] F. Lang, Hyacinthos message 1599, October, 2000, http://groups.yahoo.com/group/Hyacinthos.

[10] F. Morley and F.V. Morley, *Inversive Geometry*, Oxford, 1931.

[11] I. Panakis, *Trigonometry*, volume 2 (in Greek), Athens, 1973.

[12] A. Poulain, Des coordonnées tripolaires, *Journal de Mathématiques Spéciales*, ser 3, 3 (1889) 3 – 10, 51 – 55, 130 – 134, 155 – 159, 171 – 172.

[13] K. Schiffler, G.R. Veldkamp, and W.A.van der Spek, Problem 1018 and solution, *Crux Math.*, 11 (1985) 51; 12 (1986) 150 – 152.

[14] B. Wolk, Posting to Math Forum, Geomety-puzzles groups, April 15, 1999, http://mathforum.com/epigone/geometry-puzzles/skahvelnerd.

Antreas P. Hatzipolakis: 81 Patmou Street, Athens 11144, Greece
E-mail address: xpolakis@otenet.gr

Floor van Lamoen: Statenhof 3, 4463 TV Goes, The Netherlands
E-mail address: f.v.lamoen@wxs.nl

Barry Wolk: Department of Mathematics, University of Manitoba, Wennipeg, Manitoba, R3T 2N2, Canada
E-mail address: wolkb@cc.umanitoba.ca

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida, 33431-0991, USA
E-mail address: yiu@fau.edu

Forum Geometricorum
Volume 1 (2001) 69–74.

FORUM GEOM
ISSN 1534-1178

Some Remarkable Concurrences

Bruce Shawyer

Abstract. In May 1999, Steve Sigur, a high school teacher in Atlanta, Georgia, posted on The Math Forum, a notice stating that one of his students (Josh Klehr) noticed that

> *Given a triangle with mid-point of each side. Through each mid-point, draw a line whose slope is the reciprocal to the slope of the side through that mid-point. These lines concur.*

Sigur then stated that "we have proved this". Here, we extend this result to the case where the slope of the line through the mid-point is a constant times the reciprocal of the slope of the side.

In May 1999, Steve Sigur, a high school teacher in Atlanta, Georgia, posted on The Math Forum, a notice stating that one of his students (Josh Klehr) noticed that

> *Given a triangle with mid-point of each side. Through each mid-point, draw a line whose slope is the reciprocal to the slope of the side through that mid-point. These lines concur.*

Sigur then stated that "we have proved this".

There was a further statement that another student (Adam Bliss) followed up with a result on the concurrency of reflected line, with the point of concurrency lying on the nine-point circle. This was subsequently proved by Louis Talman [2]. See also the variations, using the feet of the altitudes in place of the mid-points and different reflections in the recent paper by Floor van Lamoen [1].

Here, we are interested in a generalization of Klehr's result.

At the mid-point of each side of a triangle, we construct the line such that the product of the slope of this line and the slope of the side of the triangle is a fixed constant. To make this clear, the newly created lines have slopes of the fixed constant times the reciprocal of the slopes of the sides of the triangle with respect to a given line (parallel to the x–axis used in the Cartesian system). We show that the three lines obtained are always concurrent.

Further, the locus of the points of concurrency is a rectangular hyperbola. This hyperbola intersects the side of the triangles at the mid-points of the sides, and each side at another point. These three other points, when considered with the vertices of the triangle opposite to the point, form a Ceva configuration. Remarkably, the point of concurrency of these Cevians lies on the circumcircle of the original triangle.

Publication Date: May 3, 2001. Communicating Editor: Floor van Lamoen.

Since we are dealing with products of slopes, we have restricted ourselves to a Cartesian proof.

Suppose that we have a triangle with vertices $(0,0)$, $(2a,2b)$ and $(2c,2d)$.

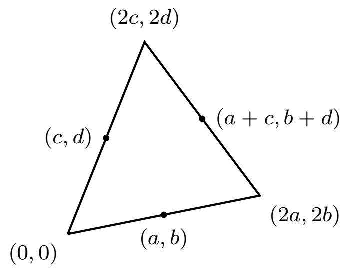


Figure 1

In order to ensure that the triangle is not degenerate, we assume that $ad-bc \neq 0$. For ease of proof, we also take $0 \neq a \neq c \neq 0$ and $0 \neq b \neq d \neq 0$ to avoid division by zero. However, by continuity, the results obtained here can readily be extended to include these cases.

At the mid-point of each side, we find the equations of the new lines:

Mid-point	Slope	Equation
(a,b)	$\frac{\lambda a}{b}$	$y = \frac{\lambda a}{b}x + \frac{b^2-\lambda a^2}{b}$
(c,d)	$\frac{\lambda c}{d}$	$y = \frac{\lambda c}{d}x + \frac{d^2-\lambda c^2}{d}$
$(a+c,b+d)$	$\frac{\lambda(c-a)}{d-b}$	$y = \frac{\lambda(c-a)}{d-b}x + \frac{(a^2-c^2)\lambda+(d^2-b^2)}{d-b}$

With the aid of a computer algebra program, we find that the first two lines meet at

$$\left(\frac{\lambda(a^2d-bc^2)+bd(d-b)}{\lambda(ad-bc)}, \frac{\lambda ac(a-c)+(ad^2-b^2c)}{(ad-bc)}\right),$$

which it is easy to verify lies on the third line.

By eliminating λ from the equations

$$x \;=\; \frac{\lambda(a^2d-bc^2)+bd(d-b)}{\lambda(ad-bc)}, \qquad y \;=\; \frac{\lambda ac(a-c)+(ad^2-b^2c)}{(ad-bc)},$$

we find that the locus of the points of concurrency is

$$y \;=\; \frac{abcd(a-c)(d-b)}{ad-bc}\cdot\frac{1}{(ad-bc)x-(a^2d-bc^2)}+\frac{ad^2-b^2c}{ad-bc}.$$

This is a rectangular hyperbola, with asymptotes

$$x \;=\; \frac{a^2d-bc^2}{ad-bc}, \qquad y \;=\; \frac{ad^2-b^2c}{ad-bc}.$$

Now, this hyperbola meets the sides of the given triangle as follows:

from	to	mid-point	new-point
$(0,0)$	$(2a,2b)$	(a,b)	$\left(\dfrac{ad+bc}{b}, \dfrac{ad+bc}{a}\right)$
$(0,0)$	$(2c,2d)$	(c,d)	$\left(\dfrac{ad+bc}{d}, \dfrac{ad+bc}{c}\right)$
$(2a,2b)$	$(2c,2d)$	$(a+c,b+d)$	$\left(\dfrac{ad-bc}{d-b}, \dfrac{ad-bc}{a-c}\right)$

The three lines joining the three points (new-point, in each case) to the vertices opposite are concurrent! (Again, easily shown by computer algebra.) The point of concurrency is

$$\left(2(a-c)\left(\frac{ad+bc}{ad-bc}\right),\ 2(d-b)\left(\frac{ad+bc}{ad-bc}\right)\right).$$

It is easy to check that this point is not on the hyperbola. However, it is also easy to check that this point lies on the circumcircle of the original triangle. (Compare this result with the now known result that the point on the hyperbola corresponding to $\lambda = 1$ lies on the nine-point circle. See [2].)

In Figures 2, 3, 4 below, we illustrate the original triangle ABC, the rectangular hyperbola $YWLPXQVZ$ (where $\lambda < 0$) and $MSOUN$ (where $\lambda > 0$), the asymptotes (dotted lines), the circumcircle and the nine-point circle, and the first remarkable point K.

Figure 2 shows various lines through the mid-points of the sides being concurrent on the hyperbola, and also the concurrency of AX, BY, CZ at K.

Figure 3 shows the lines concurrent through the second remarkable point J, where we join points with parameters λ and $-\lambda$. This point J is indeed the center of the rectangular hyperbola.

Figure 4 shows the parallel lines (or lines concurrent at infinity), where we join points with parameters λ and $-\frac{1}{\lambda}$.

Now, this is a purely Cartesian demonstration. As a result, there are several questions that I have not (yet) answered:

(1) Does the Cevian intersection point have any particular significance?
(2) What significant differences (if any) would occur if the triangle were to be rotated about the origin?
(3) Are there variations of these results along the lines of Floor van Lamoen's paper [1]?

Acknowledgement. The author expresses his sincere thanks to the Communicating Editor for valuable comments that improved this presentation.

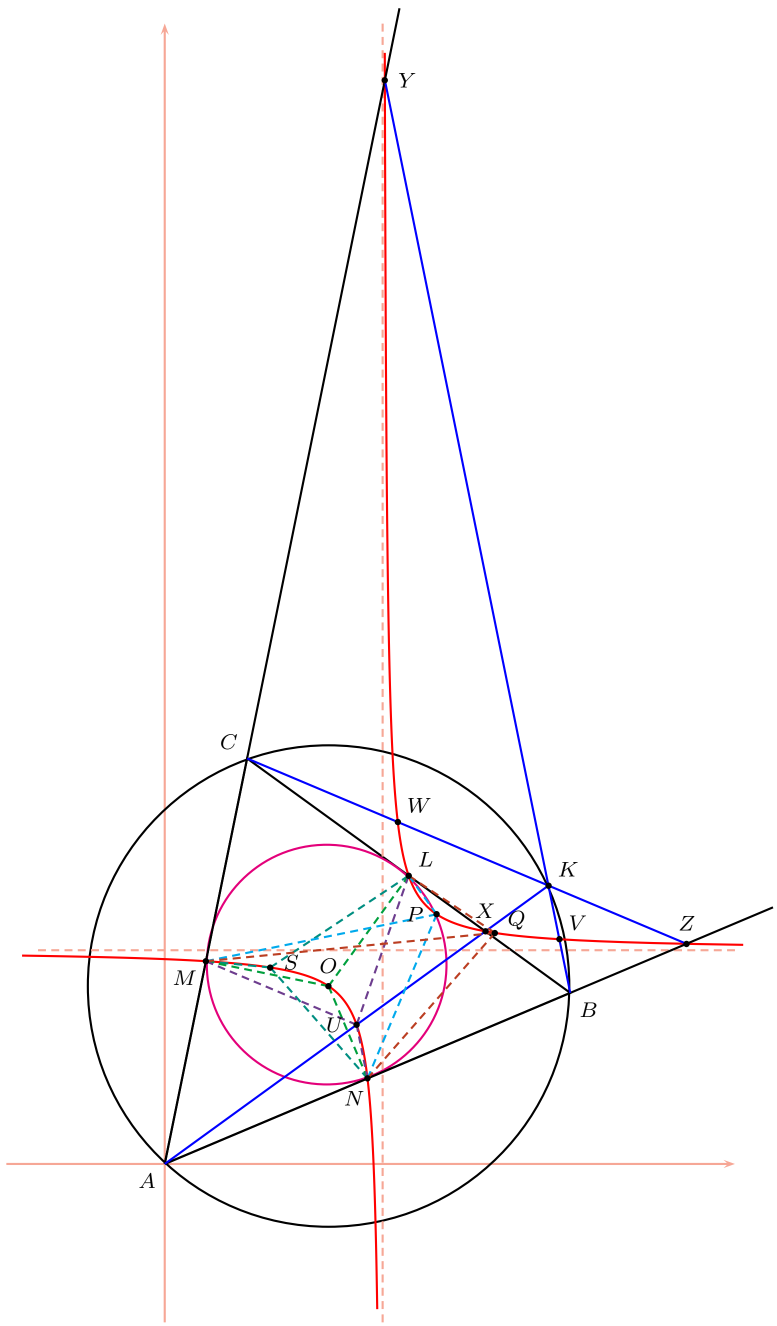


Figure 2

λ	1	$\frac{ac}{bd}$	$\frac{bd}{ac}$	$\frac{b(b-d)}{a(a-c)}$	$\frac{d(b-d)}{c(a-c)}$	$\frac{(b-d)^2}{(a-c)^2}$	$\frac{d^2}{c^2}$	$\frac{b^2}{a^2}$
Point from λ	P	Q	L	M	N	X	Y	Z
Point from $-\lambda$	O	S	U	V	W			

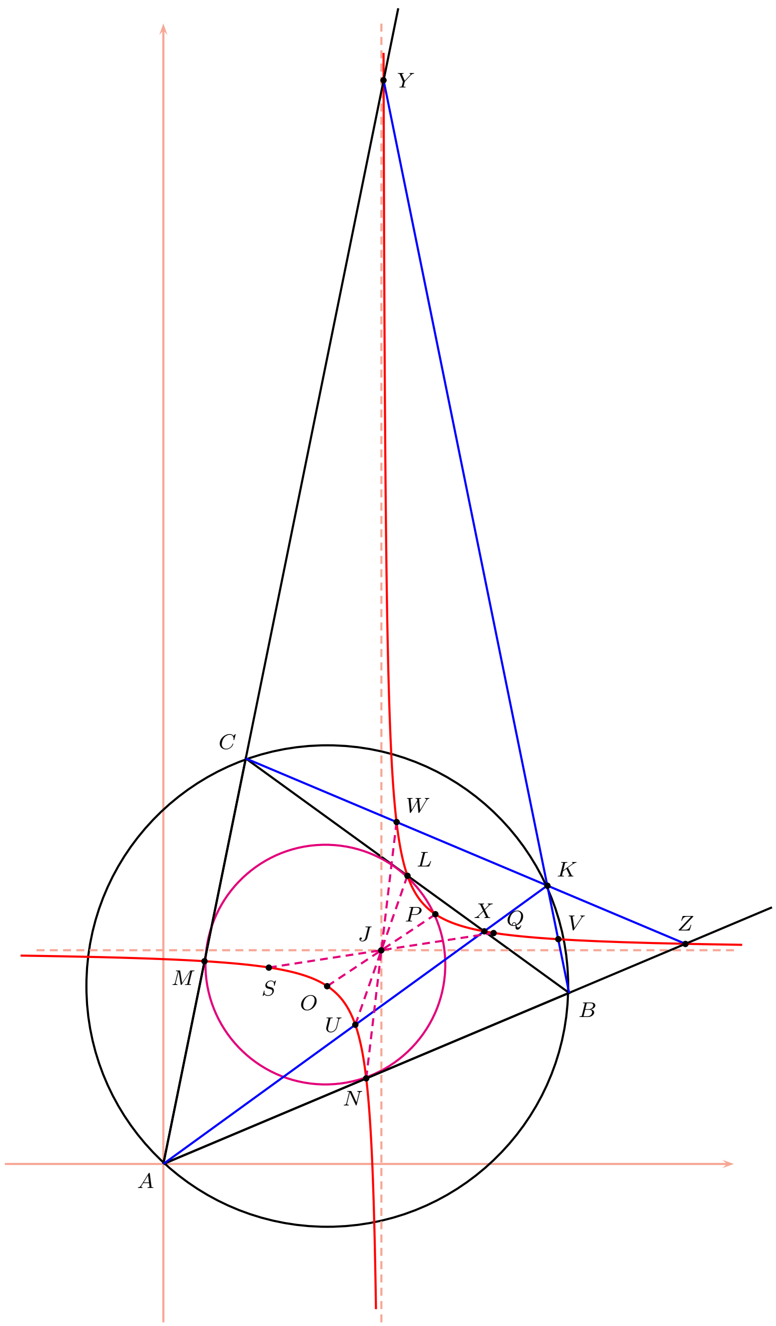


Figure 3

λ	1	$\frac{ac}{bd}$	$\frac{bd}{ac}$	$\frac{b(b-d)}{a(a-c)}$	$\frac{d(b-d)}{c(a-c)}$	$\frac{(b-d)^2}{(a-c)^2}$	$\frac{d^2}{c^2}$	$\frac{b^2}{a^2}$
Point from λ	P	Q	L	M	N	X	Y	Z
Point from $-\lambda$	O	S	U	V	W			

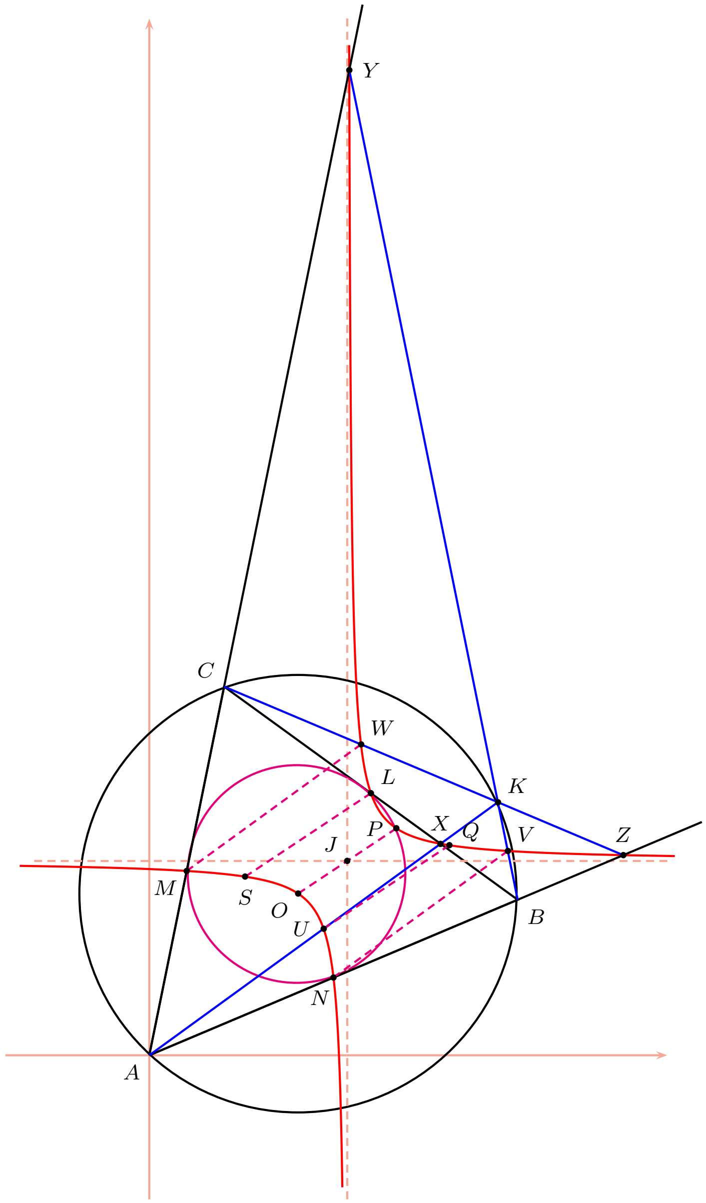


Figure 4

References

[1] F.M. van Lamoen, Morley related triangles on the nine-point circle, *Amer. Math. Monthly*, 107 (2000) 941–945.

[2] L.A. Talman, A remarkable concurrence, http://clem.mscd.edu/∼talmanl, 1999.

Bruce Shawyer: Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NF, Canada A1C 5S7

E-mail address: bshawyer@math.mun.ca

Forum Geometricorum
Volume 1 (2001) 75–79.

FORUM GEOM
ISSN 1534-1178

The Gergonne problem

Nikolaos Dergiades

Abstract. An effective method for the proof of geometric inequalities is the use of the dot product of vectors. In this paper we use this method to solve some famous problems, namely Heron's problem, Fermat's problem and the extension of the previous problem in space, the so called Gergonne's problem. The solution of this last is erroneously stated, but not proved, in F.G.-M.

1. Introduction

In this paper whenever we write AB we mean the length of the vector $\mathbf{AB}$, i.e. $AB = |\mathbf{AB}|$. The method of using the dot product of vectors to prove geometric inequalities consists of using the following well known properties:

(1) $\mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}||\mathbf{b}|$.

(2) $\mathbf{a} \cdot \mathbf{i} \leq \mathbf{a} \cdot \mathbf{j}$ if $\mathbf{i}$ and $\mathbf{j}$ are unit vectors and $\angle(\mathbf{a}, \mathbf{i}) \geq \angle(\mathbf{a}, \mathbf{j})$.

(3) If $\mathbf{i} = \frac{\mathbf{AB}}{|\mathbf{AB}|}$ is the unit vector along $\mathbf{AB}$, then the length of the segment AB is given by

$$AB = \mathbf{i} \cdot \mathbf{AB}.$$

2. The Heron problem and the Fermat point

2.1. *Heron's problem.* A point O on a line XY gives the smallest sum of distances from the points A, B (on the same side of XY) if $\angle XOA = \angle BOY$.

Proof. If M is an arbitrary point on XY (see Figure 1) and $\mathbf{i}, \mathbf{j}$ are the unit vectors of $\mathbf{OA}, \mathbf{OB}$ respectively, then the vector $\mathbf{i} + \mathbf{j}$ is perpendicular to XY since it bisects the angle between $\mathbf{i}$ and $\mathbf{j}$. Hence $(\mathbf{i} + \mathbf{j}) \cdot \mathbf{OM} = 0$ and

$$\begin{aligned} OA + OB &= \mathbf{i} \cdot \mathbf{OA} + \mathbf{j} \cdot \mathbf{OB} \\ &= \mathbf{i} \cdot (\mathbf{OM} + \mathbf{MA}) + \mathbf{j} \cdot (\mathbf{OM} + \mathbf{MB}) \\ &= (\mathbf{i} + \mathbf{j}) \cdot \mathbf{OM} + \mathbf{i} \cdot \mathbf{MA} + \mathbf{j} \cdot \mathbf{MB} \\ &= \mathbf{i} \cdot \mathbf{MA} + \mathbf{j} \cdot \mathbf{MB} \\ &\leq |\mathbf{i}||\mathbf{MA}| + |\mathbf{j}||\mathbf{MB}| \\ &= MA + MB. \end{aligned}$$

□

Publication Date: May 10, 2001. Communicating Editor: Michael Lambrou.

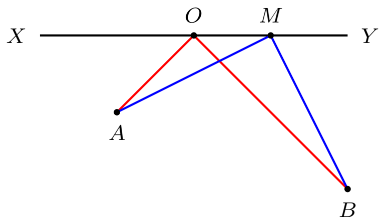


Figure 1

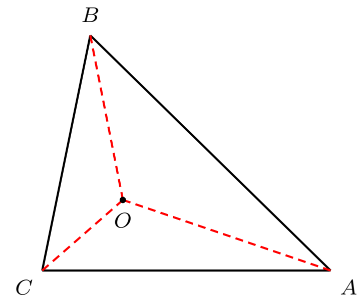


Figure 2

2.2. *The Fermat point.* If none of the angles of a triangle ABC exceeds $120°$, the point O inside a triangle ABC such that $\angle BOC = \angle COA = \angle AOB = 120°$ gives the smallest sum of distances from the vertices of ABC. See Figure 2.

Proof. If M is an arbitrary point and **i**, **j**, **k** are the unit vectors of $\mathbf{OA}$, $\mathbf{OB}$, $\mathbf{OC}$, then $\mathbf{i} + \mathbf{j} + \mathbf{k} = \mathbf{0}$ since this vector does not changes by a $120°$ rotation. Hence,

$$\begin{aligned} OA + OB + OC &= \mathbf{i} \cdot \mathbf{OA} + \mathbf{j} \cdot \mathbf{OB} + \mathbf{k} \cdot \mathbf{OC} \\ &= \mathbf{i} \cdot (\mathbf{OM} + \mathbf{MA}) + \mathbf{j} \cdot (\mathbf{OM} + \mathbf{MB}) + \mathbf{k} \cdot (\mathbf{OM} + \mathbf{MC}) \\ &= (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot \mathbf{OM} + \mathbf{i} \cdot \mathbf{MA} + \mathbf{j} \cdot \mathbf{MB} + \mathbf{k} \cdot \mathbf{MC} \\ &= \mathbf{i} \cdot \mathbf{MA} + \mathbf{j} \cdot \mathbf{MB} + \mathbf{k} \cdot \mathbf{MC} \\ &\leq |\mathbf{i}||\mathbf{MA}| + |\mathbf{j}||\mathbf{MB}| + |\mathbf{k}||\mathbf{MC}| \\ &= MA + MB + MC. \end{aligned}$$

□

3. The Gergonne problem

Given a plane π and a triangle ABC not lying in the plane, the Gergonne problem [3] asks for a point O on a plane π such that the sum $OA + OB + OC$ is minimum. This is an extention of Fermat's problem to 3 dimensions. According to [2, pp. 927–928], [1] this problem had hitherto been unsolved (for at least 90 years). Unfortunately, as we show in §4.1 below, the solution given there, for the special case when the planes π and ABC are parallel, is erroneous. We present a solution here in terms of the centroidal line of a trihedron. We recall the definition which is based on the following fact. See, for example, [1, p.43].

Proposition and Definition. *The three planes determined by the edges of a trihedral angle and the internal bisectors of the respective opposite faces intersect in a line. This line is called the centroidal line of the trihedron.*

Theorem 1. *If O is a point on the plane π such that the centroidal line of the trihedron $O.ABC$ is perpendicular to π, then $OA + OB + OC \leq MA + MB + MC$ for every point M on π.*

[1] Problem 742-III, especially 1901 c3.

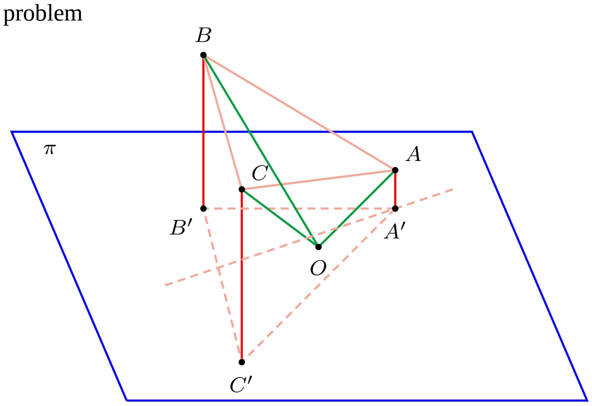


Figure 3

Proof. Let M be an arbitrary point on π, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the unit vectors along $\mathbf{OA}$, $\mathbf{OB}$, $\mathbf{OC}$ respectively. The vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is parallel to the centroidal line of the trihedron $O.ABC$. Since this line is perpendicular to π by hypothesis we have

$$(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\mathbf{OM}=0. \tag{1}$$

Hence,

$$\begin{aligned} OA+OB+OC &= \mathbf{i}\cdot\mathbf{OA}+\mathbf{j}\cdot\mathbf{OB}+\mathbf{k}\cdot\mathbf{OC}\\ &= \mathbf{i}\cdot(\mathbf{OM}+\mathbf{MA})+\mathbf{j}\cdot(\mathbf{OM}+\mathbf{MB})+\mathbf{k}\cdot(\mathbf{OM}+\mathbf{MC})\\ &= (\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\mathbf{OM}+\mathbf{i}\cdot\mathbf{MA}+\mathbf{j}\cdot\mathbf{MB}+\mathbf{k}\cdot\mathbf{MC}\\ &= \mathbf{i}\cdot\mathbf{MA}+\mathbf{j}\cdot\mathbf{MB}+\mathbf{k}\cdot\mathbf{MC}\\ &\leq |\mathbf{i}||\mathbf{MA}|+|\mathbf{j}||\mathbf{MB}|+|\mathbf{k}||\mathbf{MC}|\\ &= MA+MB+MC. \end{aligned}$$

□

4. Examples

We set up a rectangular coordinate system such that A, B, C, are the points $(a,0,p)$, $(0,b,q)$ and $(0,c,r)$. Let A', B', C' be the orthogonal projections of A, B, C on the plane π. Write the coordinates of O as $(x,y,0)$. The x- and y-axes are the altitude from A' and the line $B'C'$ of triangle $A'B'C'$ in the plane π. Since

$$\begin{aligned} \mathbf{i} &= \frac{-1}{\sqrt{(x-a)^2+y^2+p^2}}(x-a,y,-p),\\ \mathbf{j} &= \frac{-1}{\sqrt{x^2+(y-b)^2+q^2}}(x,y-b,-q),\\ \mathbf{k} &= \frac{-1}{\sqrt{x^2+(y-c)^2+r^2}}(x,y-c,-r), \end{aligned}$$

it is sufficient to put in (1) for $\mathbf{OM}$ the vectors $(1,0,0)$ and $(0,1,0)$. From these, we have

$$\begin{aligned} \frac{x-a}{\sqrt{(x-a)^2+y^2+p^2}} + \frac{x}{\sqrt{x^2+(y-b)^2+q^2}} + \frac{x}{\sqrt{x^2+(y-c)^2+r^2}} &= 0, \\ \frac{y}{\sqrt{(x-a)^2+y^2+p^2}} + \frac{y-b}{\sqrt{x^2+(y-b)^2+q^2}} + \frac{y-c}{\sqrt{x^2+(y-c)^2+r^2}} &= 0. \end{aligned} \tag{2}$$

The solution of this system cannot in general be expressed in terms of radicals, as it leads to equations of high degree. It is therefore in general not possible to construct the point O using straight edge and compass. We present several examples in which O is constructible. In each of these examples, the underlying geometry dictates that $y = 0$, and the corresponding equation can be easily written down.

4.1. *π parallel to ABC.* It is very easy to mistake for O the Fermat point of triangle $A'B'C'$, as in [2, loc. cit.]. If we take $p = q = r = 3$, $a = 14$, $b = 2$, and $c = -2$, the system (2) gives $y = 0$ and

$$\frac{x-14}{\sqrt{(x-14)^2+9}} + \frac{2x}{\sqrt{x^2+13}} = 0, \qquad x > 0.$$

This leads to the quartic equation

$$3x^4 - 84x^3 + 611x^2 + 364x - 2548 = 0.$$

This quartic polynomial factors as $(x-2)(3x^3 - 78x^2 + 455x + 1274)$, and the only positive root of which is $x = 2$. [2] Hence $\angle B'OC' = 90°$, $\angle A'OB' = 135°$, and $\angle A'OC' = 135°$, showing that O is not the Fermat point of triangle $A'B'C'$. [3]

4.2. *ABC isosceles with A on π and BC parallel to π.* In this case, $p = 0$, $q = r$, $c = -b$, and we may assume $a > 0$. The system (2) reduces to $y = 0$ and

$$\frac{x-a}{|x-a|} + \frac{2x}{\sqrt{x^2+b^2+q^2}} = 0.$$

Since $0 < x < a$, we get

$$(x, y) = \left(\sqrt{\frac{b^2+q^2}{3}},\ 0\right)$$

with $b^2 + q^2 < 3a^2$. Geometrically, since $OB = OC$, the vectors $\mathbf{i}$, $\mathbf{j} - \mathbf{k}$ are parallel to π. We have

$$\mathbf{i} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0, \qquad (\mathbf{i} + \mathbf{j} + \mathbf{k})(\mathbf{j} - \mathbf{k}) = 0.$$

Equivalently,

$$\mathbf{i} \cdot \mathbf{j} + \mathbf{i} \cdot \mathbf{k} = -1, \qquad \mathbf{i} \cdot \mathbf{j} - \mathbf{i} \cdot \mathbf{k} = 0.$$

Thus, $\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = -\frac{1}{2}$ or $\angle AOB = \angle AOC = 120°$, a fact that is a generalization of the Fermat point to 3 dimensions.

[2]The cubic factor has one negative root ≈ -2.03472, and two non-real roots. If, on the other hand, we take $p = q = r = 2$, the resulting equation becomes $3x^4 - 84x^3 + 596x^2 + 224x - 1568 = 0$, which is irreducible over rational numbers. It roots are not constructible using ruler and compass. The positive real root is $x \approx 1.60536$. There is a negative root ≈ -1.61542 and two non-real roots.

[3]The solution given in [2] assumes erroneously OA, OB, OC equally inclined to the planes π and of triangle ABC.

If $b^2 + q^2 \geq 3a^2$, the centroidal line cannot be perpendicular to π, and Theorem 1 does not help. In this case we take as point O to be the intersection of x-axis and the plane MBC. It is obvious that

$$MA + MB + MC \geq OA + OB + OC = |x - a| + 2\sqrt{x^2 + b^2 + q^2}.$$

We write $f(x) = |x - a| + 2\sqrt{x^2 + b^2 + q^2}$.

If $0 < a < x$, then $f'(x) = 1 + \frac{2x}{\sqrt{x^2+b^2+q^2}} > 0$ and f is an increasing function. For $x \leq 0$, $f'(x) = -1 + \frac{2x}{\sqrt{x^2+b^2+q^2}} < 0$ and f is a decreasing function.

If $0 < x \leq a \leq \sqrt{\frac{b^2+q^2}{3}}$, then $4x^2 \leq x^2 + b^2 + q^2$ so that $f'(x) = -1 + \frac{2x}{\sqrt{x^2+b^2+q^2}} \leq 0$ and f is a decreasing function. Hence we have minimum when $x = a$ and $O \equiv A$.

4.3. *B, C on π.* If the points B, C lie on π, then the vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is perpendicular to the vectors $\mathbf{j}$ and $\mathbf{k}$. From these, we obtain the interesting equality $\angle AOB = \angle AOC$. Note that they are not neccesarily equal to $120°$, as in Fermat's case. Here is an example. If $a = 10$, $b = 8$, $c = -8$, $p = 3$, $q = r = 0$ the system (2) gives $y = 0$ and

$$\frac{x - 10}{\sqrt{(x - 10)^2 + 9}} + \frac{2x}{\sqrt{x^2 + 64}} = 0, \qquad 0 < x < 10,$$

which leads to the equation

$$3x^4 - 60x^3 + 272x^2 + 1280x - 6400 = 0.$$

This quartic polynomial factors as $(x - 4)(3x^3 - 48x^2 + 80x + 1600)$. It follows that the only positive root is $x = 4$. [4] Hence we have

$$\angle AOB = \angle AOC = \arccos(-\frac{2}{5}) \qquad \text{and} \qquad \angle BOC = \arccos(-\frac{3}{5}).$$

References

[1] N. Altshiller-Court, *Modern Pure Solid Geometry*, 2nd ed., Chelsea reprint, 1964.
[2] F.G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, 1991, Paris.
[3] J.-D. Gergonne, *Annales mathématiques de Gergonne*, 12 (1821-1822) 380.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: dernik@ccf.auth.gr

[4] The cubic factor has one negative root ≈ -4.49225, and two non-real roots.

Forum Geometricorum
Volume 1 (2001) 81–90.

FORUM GEOM
ISSN 1534-1178

Pedal Triangles and Their Shadows

Antreas P. Hatzipolakis and Paul Yiu

Abstract. The pedal triangle of a point P with respect to a given triangle ABC casts equal shadows on the side lines of ABC if and only if P is the internal center of similitude of the circumcircle and the incircle of triangle ABC or the external center of the circumcircle with one of the excircles. We determine the common length of the equal shadows. More generally, we construct the point the shadows of whose pedal triangle are proportional to given p, q, r. Many interesting special cases are considered.

1. Shadows of pedal triangle

Let P be a point in the plane of triangle ABC, and $A'B'C'$ its pedal triangle, *i.e.*, A', B', C' are the pedals (orthogonal projections) of A, B, C on the side lines BC, CA, AB respectively. If B_a and C_a are the pedals of B' and C' on BC, we call the segment B_aC_a the *shadow* of $B'C'$ on BC. The shadows of $C'A'$ and $A'B'$ are segments C_bA_b and A_cB_c analogously defined on the lines CA and AB. See Figure 1.

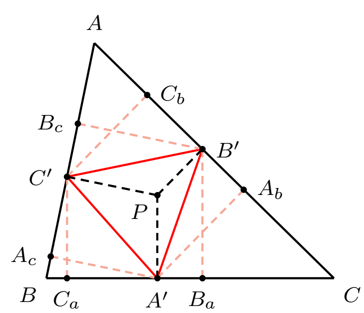


Figure 1

In terms of the *actual* normal coordinates x, y, z of P with respect to ABC,[1] the length of the shadow C_aB_a can be easily determined:

$$C_aB_a = CaA' + A'B_a = z\sin B + y\sin C. \tag{1}$$

In Figure 1, we have shown P as interior point of triangle ABC. For generic positions of P, we regard C_aB_a as a directed segment so that its length given by (1) is signed. Similarly, the shadows of $C'A'$ and $A'B'$ on the respective side lines have signed lengths $x\sin C + z\sin A$ and $y\sin A + x\sin B$.

Publication Date: May 25, 2001. Communicating Editor: Jean-Pierre Ehrmann.

[1]Traditionally, normal coordinates are called trilinear coordinates. Here, we follow the usage of the old French term *coordonnées normales* in F.G.-M. [1], which is more suggestive. The actual normal (trilinear) coordinates of a point are the *signed* distances from the point to the three side lines.

Theorem 1. *The three shadows of the pedal triangle of P on the side lines are equal if and only if P is the internal center of similitude of the circumcircle and the incircle of triangle ABC, or the external center of similitude of the circumcircle and one of the excircles.*

Proof. These three shadows are equal if and only if

$$\epsilon_1(y\sin C + z\sin B) = \epsilon_2(z\sin A + x\sin C) = \epsilon_3(x\sin B + y\sin A)$$

for an appropriate choice of signs $\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1$ subject to the convention

($\star$) *at most one of* ϵ_1, ϵ_2, ϵ_3 *is negative.*

It follows that

$$\begin{array}{rcrcrcl} x\epsilon_2\sin C & - & y\epsilon_1\sin C & + & z(\epsilon_2\sin A - \epsilon_1\sin B) & = & 0,\\ x(\epsilon_3\sin B - \epsilon_2\sin C) & + & y\epsilon_3\sin A & - & z\epsilon_2\sin A & = & 0. \end{array}$$

Replacing, by the law of sines, $\sin A$, $\sin B$, $\sin C$ by the side lengths a, b, c respectively, we have

$$\begin{aligned} x:y:z &= \begin{vmatrix} -\epsilon_1 c & \epsilon_2 a - \epsilon_1 b \\ \epsilon_3 a & -\epsilon_2 a \end{vmatrix} : - \begin{vmatrix} \epsilon_2 c & \epsilon_2 a - \epsilon_1 b \\ \epsilon_3 b - \epsilon_2 c & -\epsilon_2 a \end{vmatrix} : \begin{vmatrix} \epsilon_2 c & -\epsilon_1 c \\ \epsilon_3 b - \epsilon_2 c & \epsilon_3 a \end{vmatrix} \\ &= a(\epsilon_3\epsilon_1 b + \epsilon_1\epsilon_2 c - \epsilon_2\epsilon_3 a) : b(\epsilon_1\epsilon_2 c + \epsilon_2\epsilon_3 a - \epsilon_3\epsilon_1 b) : c(\epsilon_2\epsilon_3 a + \epsilon_3\epsilon_1 b - \epsilon_1\epsilon_2 c) \\ &= a(\epsilon_2 b + \epsilon_3 c - \epsilon_1 a) : b(\epsilon_3 c + \epsilon_1 a - \epsilon_2 b) : c(\epsilon_1 a + \epsilon_2 b - \epsilon_3 c). \end{aligned} \tag{2}$$

If $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, this is the point X_{55} in [4], the internal center of similitude of the circumcircle and the incircle. We denote this point by T. See Figure 2A. We show that if one of ϵ_1, ϵ_2, ϵ_3 is negative, then P is the external center of similitude of the circumcircle and one of the excircles.

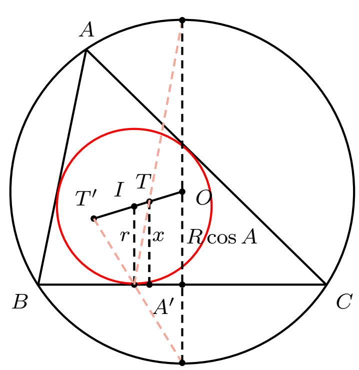


Figure 2A

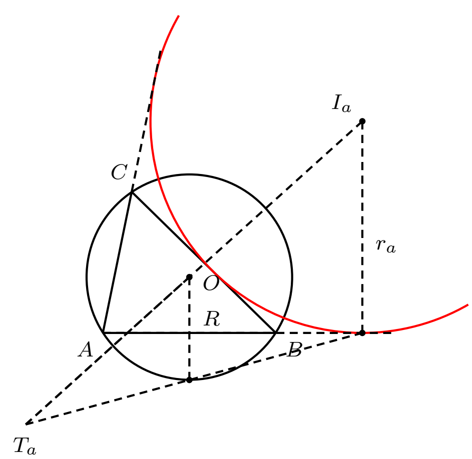


Figure 2B

Let R denote the circumradius, s the semiperimeter, and r_a the radius of the A-excircle. The actual normal coordinates of the circumcenter are $R\cos A$, $R\cos B$,

$R\cos C$, while those of the excenter I_a are $-r_a, r_a, r_a$. See Figure 2B. The external center of similitude of the two circles is the point T_a dividing I_aO in the ratio $I_aT_a : T_aO = r_a : -R$. As such, it is the point $\frac{1}{r_a-R}(r_a \cdot O - R \cdot I_a)$, and has normal coordinates

$$\begin{aligned}
& -(1+\cos A) : 1-\cos B : 1-\cos C \\
= & -\cos^2\frac{A}{2} : \sin^2\frac{B}{2} : \sin^2\frac{C}{2} \\
= & -a(a+b+c) : b(a+b-c) : c(c+a-b).
\end{aligned}$$

This coincides with the point given by (2) for $\epsilon_1 = -1$, $\epsilon_2 = \epsilon_3 = 1$. The cases for other choices of signs are similar, leading to the external centers of similitude with the other two excircles. □

Remark. With these coordinates, we easily determine the common length of the equal shadows in each case. For the point T, this common length is

$$\begin{aligned}
y\sin C + z\sin B &= \frac{Rr}{R+r}((1+\cos B)\sin C + (1+\cos C)\sin B) \\
&= \frac{Rr}{R+r}(\sin A + \sin B + \sin C) \\
&= \frac{1}{R+r}\cdot\frac{1}{2}(a+b+c)r \\
&= \frac{\triangle}{R+r},
\end{aligned}$$

where $\triangle$ denotes the area of triangle ABC. For T_a, the common length of the equal shadows is $\left|\frac{\triangle}{r_a-R}\right|$; similarly for the other two external centers of similitudes.

2. Pedal triangles with shadows in given proportions

If the signed lengths of the shadows of the sides of the pedal triangle of P (with normal coordinates $(x:y:z)$) are proportional to three given quantities p, q, r, then

$$\frac{cy+bz}{p} = \frac{az+cx}{q} = \frac{bx+ay}{r}.$$

From these, we easily obtain the normal of coordinates of P:

$$(a(-ap+bq+cr) : b(ap-bq+cr) : c(ap+bq-cr)). \tag{3}$$

This follows from a more general result, which we record for later use.

Lemma 2. *The solution of*

$$f_1x+g_1y+h_1z = f_2x+g_2y+h_2z = f_3x+g_3y+h_3z$$

is

$$x:y:z = \begin{vmatrix} 1 & g_1 & h_1 \\ 1 & g_2 & h_2 \\ 1 & g_3 & h_3 \end{vmatrix} : \begin{vmatrix} f_1 & 1 & h_1 \\ f_2 & 1 & h_2 \\ f_3 & 1 & h_3 \end{vmatrix} : \begin{vmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \\ f_3 & g_3 & 1 \end{vmatrix}.$$

Proof. since there are two linear equations in three indeterminates, solution is unique up to a proportionality constant. To verify that this is the correct solution, note that for $i = 1, 2, 3$, substitution into the i-th linear form gives

$$-\begin{vmatrix} 0 & f_i & g_i & h_i \\ 1 & f_1 & g_1 & h_1 \\ 1 & f_2 & g_2 & h_2 \\ 1 & f_3 & g_3 & h_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & f_1 & g_1 & h_1 \\ 1 & f_2 & g_2 & h_2 \\ 1 & f_3 & g_3 & h_3 \end{vmatrix} = \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix}$$

up to a constant. □

Proposition 3. *The point the shadows of whose pedal triangle are in the ratio $p : q : r$ is the perspector of the cevian triangle of the point with normal coordinates $(\frac{1}{p} : \frac{1}{q} : \frac{1}{r})$ and the tangential triangle of ABC.*

Proof. If Q is the point with normal coordinates $(\frac{1}{p} : \frac{1}{q} : \frac{1}{r})$, then P, with coordinates given by (3), is the Q-Ceva conjugate of the symmedian point $K = (a : b : c)$. See [3, p.57]. □

If we assume p, q, r positive, there are four points satisfying

$$\frac{cy + bz}{\epsilon_1 p} = \frac{az + cx}{\epsilon_2 q} = \frac{bx + ay}{\epsilon_3 r},$$

for signs ϵ_1, ϵ_2, ϵ_3 satisfying $(\star)$. Along with P given by (3), there are

$$\begin{aligned} P_a &= (-a(ap + bq + cr) : b(ap + bq - cr) : c(ap - bq + cr)), \\ P_b &= (a(-ap + bq + cr) : -b(ap + bq + cr) : c(-ap + bq + cr)), \\ P_c &= (a(ap - bq + cr) : b(-ap + bq + cr) : -c(ap + bq + cr)). \end{aligned}$$

While it is clear that $P_aP_bP_c$ is perspective with ABC at

$$\left(\frac{a}{-ap + bq + cr} : \frac{b}{ap - bq + cr} : \frac{c}{ap + bq - cr}\right),$$

the following observation is more interesting and useful in the construction of these points from P.

Proposition 4. *$P_aP_bP_c$ is the anticevian triangle of P with respect to the tangential triangle of ABC.*

Proof. The vertices of the tangential triangle are

$$A' = (-a : b : c), \qquad B' = (a : -b : c), \qquad C' = (a : b : -c).$$

From

$$\begin{aligned} & (a(-ap + bq + cr),\ b(ap - bq + cr),\ c(ap + bq - cr)) \\ = \ & ap(-a, b, c) + (a(bq + cr),\ -b(bq - cr),\ c(bq - cr)), \end{aligned}$$

and

$$\begin{aligned} & (-a(ap + bq + cr),\ b(ap + bq - cr),\ c(ap - bq + cr)) \\ = \ & ap(-a, b, c) - (a(bq + cr),\ -b(bq - cr),\ c(bq - cr)), \end{aligned}$$

we conclude that P and P_a divide A' and $A'' = (a(bq + r) : -b(bq - cr) : c(bq - cr))$ harmonically. But since

$$(a(bq + cr),\ -b(bq - cr),\ c(bq - cr)) = bq(a, -b, c) + cr(a, b, -c),$$

the point A'' is on the line $B'C'$. The cases for P_b and P_c are similar, showing that triangle $P_aP_bP_c$ is the anticevian triangle of P in the tangential triangle. $\square$

3. Examples

3.1. *Shadows proportional to side lengths.* If $p : q : r = a : b : c$, then P is the circumcenter O. The pedal triangle of O being the medial triangle, the lengths of the shadows are halves of the side lengths. Since the circumcenter is the incenter or one of the excenters of the tangential triangle (according as the triangle is acute- or obtuse-angled), the four points in question are the circumcenter and the excenters of the tangential triangle. [2]

3.2. *Shadows proportional to altitudes.* If $p : q : r = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$, then P is the symmedian point $K = (a : b : c)$. [3] Since K is the Gergonne point of the tangential triangle, the other three points, with normal coordinates $(3a : -b : -c)$, $(-a : 3b : -c)$, and $(-a : -b : 3c)$, are the Gergonne points of the excircles of the tangential triangle. These are also the cases when the shadows are inversely proportional to the distances from P to the side lines, or, equivalently, when the triangles PB_aC_a, PC_aB_a and PA_cB_c have equal areas. [4]

3.3. *Shadows inversely proportional to exradii.* If $p : q : r = \frac{1}{r_a} : \frac{1}{r_b} : \frac{1}{r_c} = b + c - a : c + a - b : a + b - c$, then P is the point with normal coordinates $(\frac{a}{b+c-a} : \frac{b}{c+a-b} : \frac{c}{a+b-c}) = (ar_a : br_b : cr_c)$. This is the *external* center of similitude of the circumcircle and the incircle, which we denote by T'. See Figure 2A. This point appears as X_{56} in [4]. The other three points are the *internal* centers of similitude of the circumcircle and the three excircles.

3.4. *Shadows proportional to exradii.* If $p : q : r = r_a : r_b : r_c = \tan\frac{A}{2} : \tan\frac{B}{2} : \tan\frac{C}{2}$, then P has normal coordinates

$$\begin{aligned} & a(b\tan\frac{B}{2} + c\tan\frac{C}{2} - a\tan\frac{A}{2}) : b(c\tan\frac{C}{2} + a\tan\frac{A}{2} - b\tan\frac{B}{2}) : c(a\tan\frac{A}{2} + b\tan\frac{B}{2} - c\tan\frac{C}{2}) \\ \sim & 2a(\sin^2\frac{B}{2} + \sin^2\frac{C}{2} - \sin^2\frac{A}{2}) : 2b(\sin^2\frac{C}{2} + \sin^2\frac{A}{2} - \sin^2\frac{B}{2}) : 2c(\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2}) \\ \sim & a(1 + \cos A - \cos B - \cos C) : b(1 + \cos B - \cos C - \cos A) : c(1 + \cos C - \cos A - \cos B). \end{aligned} \tag{4}$$

---

[2] If ABC is right-angled, the tangential triangle degenerates into a pair of parallel lines, and there is only one finite excenter.

[3] More generally, if $p : q : r = a^n : b^n : c^n$, then the normal coordinates of P are

$$(a(b^{n+1} + c^{n+1} - a^{n+1}) : b(c^{n+1} + a^{n+1} - b^{n+1}) : c(a^{n+1} + b^{n+1} - c^{n+1})).$$

[4] For signs $\epsilon_1, \epsilon_2, \epsilon_3$ satisfying $(\star)$, the equations $\epsilon_1 x(cy + bz) = \epsilon_2 y(az + cx) = \epsilon_3 z(bx + ay)$ can be solved for $yz : zx : xy$ by an application of Lemma 2. From this it follows that $x : y : z = (\epsilon_2 + \epsilon_3 - \epsilon_1)a : (\epsilon_3 + \epsilon_1 - \epsilon_2)b : (\epsilon_1 + \epsilon_2 - \epsilon_3)c$.

This is the point X_{198} of [4]. It can be constructed, according to Proposition 3, from the point with normal coordinates $(\frac{1}{r_a}:\frac{1}{r_b}:\frac{1}{r_c}) = (s-a:s-b:s-c)$, the Mittenpunkt. [5]

4. A synthesis

The five triangle centers we obtained with special properties of the shadows of their pedal triangles, namely, O, K, T, T', and the point P in §3.4, can be organized together in a very simple way. We take a closer look at the coordinates of P given in (4) above. Since

$$1-\cos A+\cos B+\cos C = 2-4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 2-\frac{r_a}{R},$$

the normal coordinates of P can be rewritten as

$$(a(2R-r_a):b(2R-r_b):c(2R-r_c)).$$

These coordinates indicate that P lies on the line joining the symmedian point $K(a:b:c)$ to the point $(ar_a:br_b:cr_c)$, the point T' in §3.3, with division ratio

$$\begin{aligned} T'P:PK &= 2R(a^2+b^2+c^2):-(a^2r_a+b^2r_b+c^2r_c) \\ &= R(a^2+b^2+c^2):-2(R-r)s^2. \end{aligned} \tag{5}$$

To justify this last expression, we compute in two ways the distance from T' to the line BC, and obtain

$$\frac{2\triangle}{a^2r_a+b^2r_b+c^2r_c}\cdot ar_a = \frac{Rr}{R-r}(1-\cos A).$$

From this,

$$\begin{aligned} a^2r_a+b^2r_b+c^2r_c &= \frac{2\triangle(R-r)}{Rr}\cdot\frac{ar_a}{1-\cos A} \\ &= \frac{2\triangle(R-r)}{Rr}\cdot\frac{4R\sin\frac{A}{2}\cos\frac{A}{2}\cdot s\tan\frac{A}{2}}{2\sin^2\frac{A}{2}} \\ &= 4(R-r)s^2. \end{aligned}$$

This justifies (5) above.

Consider the intersection X of the line TP with OK. See Figure 3. Applying Menelaus' theorem to triangle OKT' with transversal TXP, we have

$$\frac{OX}{XK} = -\frac{OT}{TT'}\cdot\frac{T'P}{PK} = \frac{R-r}{2r}\cdot\frac{R(a^2+b^2+c^2)}{2(R-r)s^2} = \frac{R(a^2+b^2+c^2)}{4\triangle s}.$$

This expression has an interesting interpretation. The point X being on the line OK, it is the isogonal conjugate of a point on the Kiepert hyperbola. Every point on this hyperbola is the perspector of the apexes of similar isosceles triangles constructed on the sides of ABC. If this angle is taken to be $\arctan\frac{s}{R}$, and the

[5] This appears as X_9 in [4], and can be constructed as the perspector of the excentral triangle and the medial triangle, *i.e.*, the intersection of the three lines each joining an excenter to the midpoint of the corresponding side of triangle ABC.

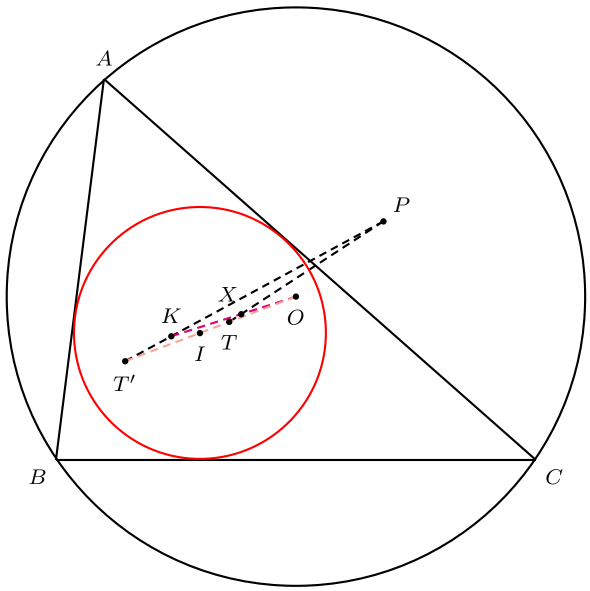


Figure 3

isosceles triangles constructed externally on the sides of triangle ABC, then the isogonal conjugate of the perspector is precisely the point X.

This therefore furnishes a construction for the point P.[6]

5. Two more examples

5.1. *Shadows of pedal triangle proportional to distances from circumcenter to side lines.* The point P is the perspector of the tangential triangle and the cevian triangle of $(\frac{1}{\cos A} : \frac{1}{\cos B} : \frac{1}{\cos C})$, which is the orthic triangle of ABC. The two triangles are indeed homothetic at the Gob perspector on the Euler line. See [2, pp.259–260]. It has normal coordinates $(a \tan A : b \tan B : c \tan C)$, and appears as X_{25} in [4].

5.2. *Shadows of pedal triangles proportional to distances from orthocenter to side lines.* In this case, P is the perspector of the tangential triangle and the cevian triangle of the circumcenter. This is the point with normal coordinates

$$(a(-\tan A+\tan B+\tan C) : b(\tan A-\tan B+\tan C) : c(\tan A+\tan B-\tan C)),$$

and is the centroid of the tangential triangle. It appears as X_{154} in [4]. The other three points with the same property are the vertices of the anticomplementary triangle of the tangential triangle.

[6]The same P can also be constructed as the intersection of KT' and the line joining the incenter to Y on OK, which is the isogonal conjugate of the perspector (on the Kiepert hyperbola) of apexes of similar isosceles triangles with base angles $\arctan \frac{s}{2R}$ constructed externally on the sides of ABC.

6. The midpoints of shadows as pedals

The midpoints of the shadows of the pedal triangle of $P = (x : y : z)$ are the pedals of the point

$$P' = (x + y\cos C + z\cos B : \; y + z\cos A + x\cos C : \; z + x\cos B + y\cos A) \quad (6)$$

in normal coordinates. This is equivalent to the concurrency of the perpendiculars from the midpoints of the sides of the pedal triangle of P to the corresponding sides of ABC.[7] See Figure 4.

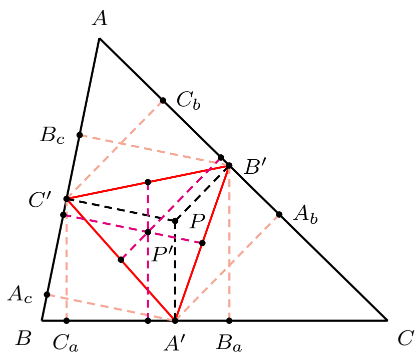


Figure 4

If P is the symmedian point, with normal coordinates $(\sin A : \sin B : \sin C)$, it is easy to see that P' is the same symmedian point.

Proposition 5. *There are exactly four points for each of which the midpoints of the sides of the pedal triangle are equidistant from the corresponding sides of ABC.*

Proof. The midpoints of the sides of the pedal triangle have *signed* distances

$$x + \frac{1}{2}(y\cos C + z\cos B), \quad y + \frac{1}{2}(z\cos A + x\cos C), \quad z + \frac{1}{2}(x\cos B + y\cos A)$$

from the respective sides of ABC. The segments joining the midpoints of the sides and their shadows are equal in length if and only if

$$\epsilon_1(2x + y\cos C + z\cos B) = \epsilon_2(2y + z\cos A + x\cos C) = \epsilon_3(2z + x\cos B + y\cos A)$$

for ϵ_1, ϵ_2, ϵ_3 satisfying $(\star)$. From these, we obtain the four points.

For $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, this gives the point

$$\begin{aligned} M = ((2 - \cos A)(2 + \cos A - \cos B - \cos C) \\ :(2 - \cos B)(2 + \cos B - \cos C + \cos A) \\ :(2 - \cos C)(2 + \cos C - \cos A + \cos B)) \end{aligned}$$

in normal coordinates, which can be constructed as the incenter-Ceva conjugate of

$$Q = (2 - \cos A : 2 - \cos B : 2 - \cos C),$$

[7] If x, y, z are the actual normal coordinates of P, then those of P' are halves of those given in (6) above, and P' is $\frac{x}{2}$, $\frac{y}{2}$, and $\frac{z}{2}$ below the midpoints of the respective sides of the pedal triangle.

See [3, p.57]. This point Q divides the segments OI externally in the ratio $OQ : QI = 2R : -r$. See Figures 5A and 5B.

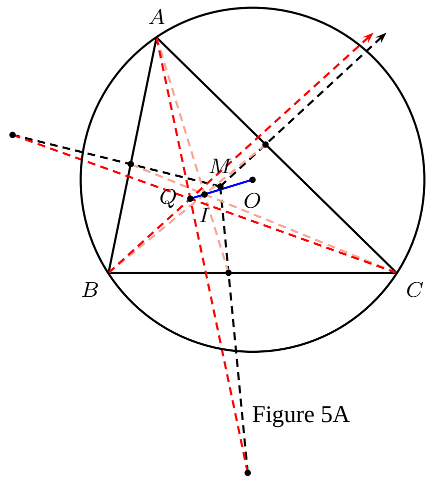


Figure 5A

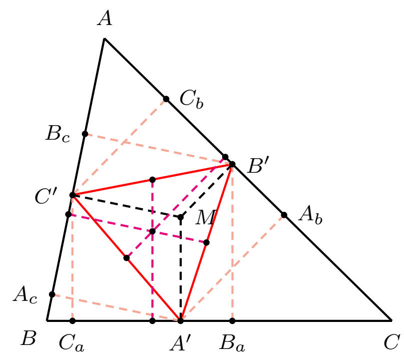


Figure 5B

There are three other points obtained by choosing one negative sign among ϵ_1, ϵ_2, ϵ_3. These are

$$\begin{aligned} M_a = (&-(2-\cos A)(2+\cos A+\cos B+\cos C) \\ &:(2+\cos B)(2-\cos A-\cos B+\cos C) \\ &:(2+\cos C)(2-\cos A+\cos B-\cos C)), \end{aligned}$$

and M_b, M_c whose coordinates can be written down by appropriately changing signs. It is clear that $M_aM_bM_c$ and triangle ABC are perspective at

$$M' = \left(\frac{2+\cos A}{2+\cos A-\cos B-\cos C} : \frac{2+\cos B}{2-\cos A+\cos B-\cos C} : \frac{2+\cos C}{2-\cos A-\cos B+\cos C}\right).$$

□

The triangle centers Q, M, and M' in the present section apparently are not in [4].

Appendix: Pedal triangles of a given shape

The side lengths of the pedal triangle of P are given by $AP \cdot \sin A$, $BP \cdot \sin B$, and $CP \cdot \sin C$. [2, p.136]. This is similar to one with side lengths $p : q : r$ if and only if the *tripolar* coordinates of P are

$$AP : BP : CP = \frac{p}{a} : \frac{q}{b} : \frac{r}{c}.$$

In general, there are two such points, which are common to the three generalized Apollonian circles associated with the point $(\frac{1}{p} : \frac{1}{q} : \frac{1}{r})$ in *normal* coordinates. See, for example, [5]. In the case of equilateral triangles, these are the isodynamic points.

Acknowledgement. The authors express their sincere thanks to the Communicating Editor for valuable comments that improved this presentation.

References

[1] F.G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, 1991, Paris.
[2] R.A. Johnson, *Advanced Euclidean Geometry*, Dover reprint 1960.
[3] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–295.
[4] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000, http://cedar.evansville.edu/˜ck6/encyclopedia/.
[5] P. Yiu, Generalized Apollonian circles, *Forum Geom.*, to appear.

Antreas P. Hatzipolakis: 81 Patmou Street, Athens 11144, Greece
E-mail address: xpolakis@otenet.gr

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida, 33431-0991, USA
E-mail address: yiu@fau.edu

Forum Geometricorum
Volume 1 (2001) 91–97.

FORUM GEOM
ISSN 1534-1178

Some Properties of the Lemoine Point

Alexei Myakishev

Abstract. The Lemoine point, K, of $\triangle ABC$ has special properties involving centroids of pedal triangles. These properties motivate a definition of Lemoine field, F, and a coordinatization of the plane of $\triangle ABC$ using perpendicular axes that pass through K. These principal axes are symmetrically related to two other lines: one passing through the isodynamic centers, and the other, the isogonic centers.

1. Introduction

Let $A'B'C'$ be the pedal triangle of an arbitrary point Z in the plane of a triangle ABC, and consider the vector field $\mathbf{F}$ defined by $\mathbf{F}(Z) = \mathbf{ZA'} + \mathbf{ZB'} + \mathbf{ZC'}$. It is well known that $\mathbf{F}(Z)$ is the zero vector if and only if Z is the Lemoine point, K, also called the symmedian point. We call $\mathbf{F}$ the *Lemoine field* of $\triangle ABC$ and K the *balance point* of $\mathbf{F}$.

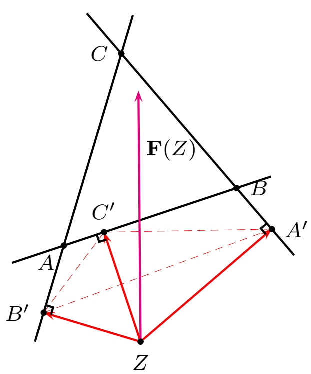


Figure 1

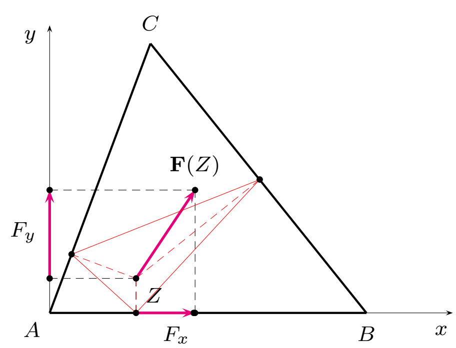


Figure 2

The Lemoine field may be regarded as a physical force field. Any point Z in this field then has a natural motion along a certain curve, or trajectory. See Figure 1. We shall determine parametric equations for these trajectories and find, as a result, special properties of the lines that bisect the angles between the line of the isogonic centers and the line of the isodynamic centers of $\triangle ABC$.

Publication Date: June 21, 2001. Communicating Editor: Clark Kimberling.
Я благодарю дорогую Лену за оказанную мне моральную поддержку. Также я очень признателен профессору Кимберлингу за разрешение многочисленных проблем, касающихся английского языка. The author dedicates his work to *Helen* and records his appreciation to the Communicating Editor for assistance in translation.

2. The Lemoine equation

In the standard cartesian coordinate system, place $\triangle ABC$ so that $A = (0,0)$, $B = (c,0)$, $C = (m,n)$, and write $Z = (x,y)$. For any line $Px + Qy + R = 0$, the vector H from Z to the projection of Z on the line has components

$$h_x = \frac{-P}{P^2+Q^2}(Px+Qy+R), \qquad h_y = \frac{-Q}{P^2+Q^2}(Px+Qy+R).$$

From these, one find the components of the three vectors whose sum defines $\mathbf{F}(Z)$:

vector	$x-\text{component}$	$y-\text{component}$
$\mathbf{ZA}'$	$\frac{-n(nx+y(c-m)-cn)}{n^2+(c-m)^2}$	$\frac{(m-c)(nx+y(c-m)-cn)}{n^2+(c-m)^2}$
$\mathbf{ZB}'$	$\frac{-n(nx-my)}{m^2+n^2}$	$\frac{m(nx-my)}{m^2+n^2}$
$\mathbf{ZC}'$	0	$-y$

The components of the Lemoine field $\mathbf{F}(Z) = \mathbf{ZA}' + \mathbf{ZB}' + \mathbf{ZC}'$ are given by

$$F_x = -(\alpha x + \beta y) + d_x, \qquad F_y = -(\beta x + \gamma y) + d_y,$$

where

$$\begin{aligned} &\alpha = \tfrac{n^2}{m^2+n^2} + \tfrac{n^2}{n^2+(c-m)^2}, && \beta = \tfrac{-mn}{m^2+n^2} + \tfrac{n(c-m)}{n^2+(c-m)^2}, \\ &\gamma = 1 + \tfrac{m^2}{m^2+n^2} + \tfrac{(c-m)^2}{n^2+(c-m)^2}; && \\ &d_x = \tfrac{cn^2}{n^2+(c-m)^2}, && d_y = \tfrac{cn(c-m)}{n^2+(c-m)^2}. \end{aligned}$$

See Figure 2. Assuming a unit mass at each point Z, Newton's Second Law now gives a system of differential equations:

$$x'' = -(\alpha x + \beta y) + d_x, \qquad y'' = -(\beta x + \gamma y) + d_y,$$

where the derivatives are with respect to time, t. We now translate the origin from $(0,0)$ to the balance point (d_x, d_y), which is the Lemoine point K, thereby obtaining the system

$$x'' = -(\alpha x + \beta y), \qquad y'' = -(\beta x + \gamma y),$$

which has the matrix form

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = -M \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1}$$

where $M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$. We shall refer to (1) as the *Lemoine equation.*

3. Eigenvalues of the matrix M

In order to solve equation (1), we first find eigenvalues λ_1 and λ_2 of M. These are the solutions of the equation $|M - \lambda I| = 0$, *i.e.*, $(\alpha - \lambda)(\gamma - \lambda) - \beta^2 = 0$, or

$$\lambda^2 - (\alpha + \gamma)\lambda + (\alpha\gamma - \beta^2) = 0.$$

Thus

$$\lambda_1 + \lambda_2 = \alpha + \gamma = 1 + \frac{m^2+n^2}{m^2+n^2} + \frac{n^2+(c-m)^2}{n^2+(c-m)^2} = 3.$$

Writing a, b, c for the sidelengths $|BC|$, $|CA|$, $|AB|$ respectively, we find the determinant

$$|M| = \alpha\gamma - \beta^2 = \frac{n^2}{a^2b^2}(a^2+b^2+c^2) > 0.$$

The discriminant of the characteristic equation $\lambda^2 - (\alpha+\gamma)\lambda + (\alpha\gamma - \beta^2) = 0$ is given by

$$D = (\alpha+\gamma)^2 - 4(\alpha\gamma - \beta^2) = (\alpha-\gamma)^2 + 4\beta^2 \geq 0. \qquad (2)$$

Case 1: equal eigenvalues $\lambda_1 = \lambda_2 = \frac{3}{2}$. In this case, $D = 0$ and (2) yields $\beta = 0$ and $\alpha = \gamma$. To reduce notation, write $p = c - m$. Then since $\beta = 0$, we have $\frac{m}{m^2+n^2} = \frac{p}{p^2+n^2}$, so that

$$(m-p)(mp-n^2) = 0. \qquad (3)$$

Also, since $\alpha = \gamma$, we find after mild simplifications

$$n^4 - (m^2+p^2)n^2 - 3m^2p^2 = 0. \qquad (4)$$

Equation (3) imples that $m = p$ or $mp = n^2$. If $m = p$, then equation (4) has solutions $n = \sqrt{3}m = \sqrt{3}p$. Consequently, $C = \left(\frac{1}{2}c, \frac{\sqrt{3}}{2}c\right)$, so that $\triangle ABC$ is equilateral. However, if $mp = n^2$, then equation (4) leads to $(m+p)^2 = 0$, so that $c = 0$, a contradiction. Therefore from equation (3) we obtain this conclusion: *if the eigenvalues are equal, then $\triangle ABC$ is equilateral.*

Case 2: distinct eigenvalues $\lambda_{1,2} = \frac{3\pm\sqrt{D}}{2}$. Here $D > 0$, and $\lambda_{1,2} > 0$ according to (2). We choose to consider the implications when

$$\beta = 0, \qquad \alpha \neq \gamma. \qquad (5)$$

We omit an easy proof that these conditions correspond to $\triangle ABC$ being a right triangle or an isosceles triangle. In the former case, write $c^2 = a^2 + b^2$. Then the characteristic equation yields eigenvalues α and γ, and

$$\alpha = \frac{n^2}{b^2} + \frac{n^2}{a^2} = \frac{n^2(a^2+b^2)}{a^2b^2} = \frac{n^2c^2}{a^2b^2} = 1,$$

since $ab = nc =$ twice the area of the right triangle. Since $\alpha + \gamma = 3$, $\gamma = 2$.

4. General solution of the Lemoine equation

According to a well known theorem of linear algebra, rotation of the coordinate system about K gives the system $x'' = -\lambda_1 x$, $y'' = -\lambda_2 y$. Let us call the axes of this coordinate system the *principal axes* of the Lemoine field.

Note that if $\triangle ABC$ is a right triangle or an isosceles triangle (cf. conditions (5)), then the angle of rotation is zero, and K is on an altitude of the triangle. In this case, one of the principal axes is that altitude, and the other is parallel to the corresponding side. Also if $\triangle ABC$ is a right triangle, then K is the midpoint of that altitude.

In the general case, the solution of the Lemoine equation is given by

$$x = c_1 \cos \omega_1 t + c_2 \sin \omega_2 t, \qquad y = c_3 \cos \omega_1 t + c_4 \sin \omega_2 t, \tag{6}$$

where $\omega_1 = \sqrt{\lambda_1}$, $\omega_2 = \sqrt{\lambda_2}$. Initial conditions $x(0) = x_0$, $y(0) = y_0$, $x'(0) = 0$, $y'(0) = 0$ reduce (6) to

$$x = x_0 \cos \omega_1 t, \qquad y = y_0 \cos \omega_2 t, \tag{7}$$

with $\omega_1 > 0$, $\omega_2 > 0$, $\omega_1^2 + \omega_2^2 = 3$. Equations (7) show that each trajectory is bounded. If $\lambda_1 = \lambda_2$, then the trajectory is a line segment; otherwise, (7) represents a Lissajous curve or an almost-everywhere rectangle-filling curve, according as $\frac{\omega_1}{\omega_2}$ is rational or not.

5. Lemoine sequences and centroidal orbits

Returning to the Lemoine field, **F**, suppose Z_0 is an arbitrary point, and G_{Z_0} is the centroid of the pedal triangle of Z_0. Let Z_0' be the point to which **F** translates Z_0. It is well known that G_{Z_0} lies on the line Z_0Z_0' at a distance $\frac{1}{3}$ of that from Z_0 to Z_0'. With this in mind, define inductively the *Lemoine sequence* of Z_0 as the sequence $(Z_0, Z_1, Z_2, \ldots)$, where Z_n, for $n \geq 1$, is the centroid of the pedal triangle of Z_{n-1}. Writing the centroid of the pedal triangle of Z_0 as $Z_1 = (x_1, y_1)$, we obtain $3(x_1 - x_0) = -\lambda_1 x_0$ and

$$x_1 = \frac{1}{3}(3 - \lambda_1)x_0 = \frac{1}{3}\lambda_2 x_0; \qquad y_1 = \frac{1}{3}\lambda_1 y_0.$$

Accordingly, the Lemoine sequence is given with respect to the principal axes by

$$Z_n = \left(x_0 \left(\frac{\lambda_2}{3} \right)^n, \; y_0 \left(\frac{\lambda_1}{3} \right)^n \right). \tag{8}$$

Since $\frac{1}{3}\lambda_1$ and $\frac{1}{3}\lambda_2$ are between 0 and 1, the points Z_n approach $(0, 0)$ as $n \to \infty$. That is, the Lemoine sequence of every point converges to the Lemoine point.

Representation (8) shows that Z_n lies on the curve $(x, y) = (x_0 u^t, y_0 v^t)$, where $u = \frac{1}{3}\lambda_2$ and $v = \frac{1}{3}\lambda_1$. We call this curve the *centroidal orbit* of Z_0. See Figure 3. Reversing the directions of axes if necessary, we may assume that $x_0 > 0$ and $y_0 > 0$, so that elimination of t gives

$$\frac{y}{y_0} = \left(\frac{x}{x_0} \right)^k, \qquad k = \frac{\ln v}{\ln u}. \tag{9}$$

Equation (9) expresses the centroidal orbit of $Z_0 = (x_0, y_0)$. Note that if $\omega_1 = \omega_2$, then $v = u$, and the orbit is a line. Now let X_Z and Y_Z be the points in which line ZG_Z meets the principal axes. By (8),

$$\frac{|ZG_Z|}{|G_Z X_Z|} = \frac{\lambda_2}{\lambda_1}, \qquad \frac{|ZG_Z|}{|G_Z Y_Z|} = \frac{\lambda_1}{\lambda_2}. \tag{10}$$

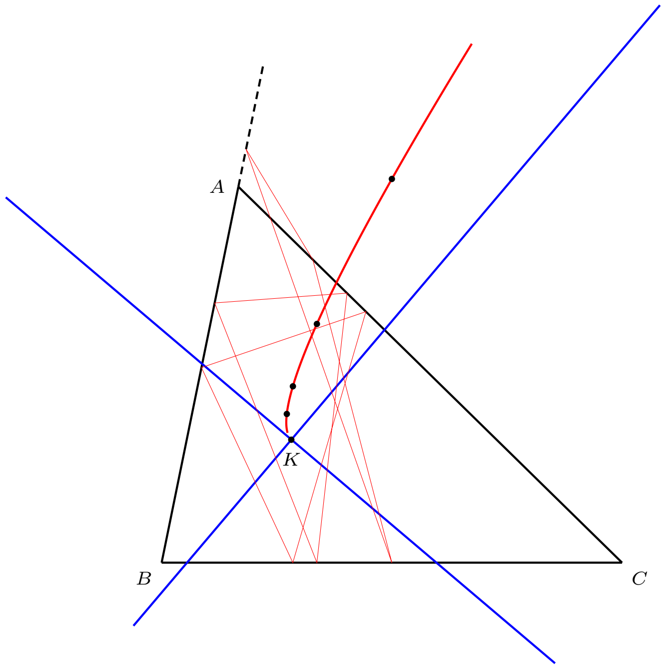


Figure 3

These equations imply that if $\triangle ABC$ is equilateral with center O, then the centroid G_Z is the midpoint of segment OG_Z.

As another consequence of (10), suppose $\triangle ABC$ is a right triangle; let H be the line parallel to the hypotenuse and passing through the midpoint of the altitude H' to the hypotenuse. Let X and Y be the points in which line ZG_Z meets H and H', respectively. Then $|ZG_Z| : |XG_Z| = |YG_Z| : |ZG_Z| = 2 : 1$.

6. The principal axes of the Lemoine field

Physically, the principal axes may be described as the locus of points in the plane of $\triangle ABC$ along which the "direction" of the Lemoine sequence remains constant. That is, if Z_0 lies on one of the principal axes, then all the points $Z_1, Z_2, \ldots$ lie on that axis also.

In this section, we turn to the geometry of the principal axes. Relative to the coordinate system adopted in §5, the principal axes have equations $x = 0$ and $y = 0$. Equation (8) therefore shows that if Z_0 lies on one of these two perpendicular lines, then Z_n lies on that line also, for all $n \geq 1$.

Let A_1, A_2 denote the isodynamic points, and F_1, F_2 the isogonic centers, of $\triangle ABC$. Call lines A_1A_2 and F_1F_2 the *isodynamic axis* and the *isogonic axis* respectively. [1]

Lemma 1. *Suppose Z and Z' are a pair of isogonal conjugate points. Let O and O' be the circumcircles of the pedal triangles of Z and Z'. Then $O = O'$, and the center of O is the midpoint between Z and Z'.*

[1]The points F_1, F_2, A_1, A_2 are indexed as X_{13}, X_{14}, X_{15}, X_{16} and discussed in [2].

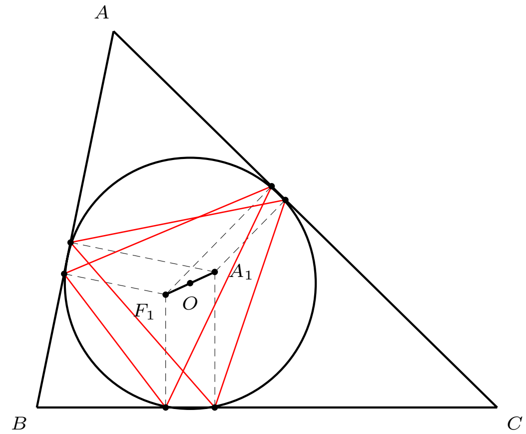


Figure 4

A proof is given in Johnson [1, pp.155–156]. See Figure 4.

Now suppose that $Z = A_1$. Then $Z' = F_1$, and, according to Lemma 1, the pedal triangles of Z and Z' have the same circumcircle, whose center O is the midpoint between A_1 and F_1. Since the pedal triangle of A_1 is equilateral, the point O is the centroid of the pedal triangle of A_1.

Next, suppose L is a line not identical to either of the principal axes. Let L' be the reflection of L about one of the principal axes. Then L' is also the reflection of L about the other principal axis. We call L and L' a *symmetric pair of lines*.

Lemma 2. *Suppose that G_P is the centroid of the pedal triangle of a point P, and that Q is the reflection of P in G_P. Then there exists a symmetric pair of lines, one passing through P and the other passing through Q.*

Proof. With respect to the principal axes, write $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$. Then $G_P = (\frac{1}{3}\lambda_2 x_P,\ \frac{1}{3}\lambda_1 y_P)$, and $\frac{2}{3}\lambda_2 x_P = x_P + x_Q$, so that

$$x_Q = \left(\frac{2}{3}\lambda_2 - 1\right)x_P = \frac{1}{3}(2\lambda_2 - (\lambda_1 + \lambda_2))x_P = \frac{1}{3}(\lambda_2 - \lambda_1)x_P.$$

Likewise, $y_Q = \frac{1}{3}y_P(\lambda_1 - \lambda_2)$. It follows that $\frac{x_P}{y_P} = -\frac{x_Q}{y_Q}$. This equation shows that the line $y = \frac{y_P}{x_P} \cdot x$ passing through P and the line $y = \frac{y_Q}{x_Q} \cdot x$ passing through Q are symmetric about the principal axes $y = 0$ and $x = 0$. See Figure 5. □

Theorem. *The principal axes of the Lemoine field are the bisectors of the angles formed at the intersection of the isodynamic and isogonic axes in the Lemoine point.*

Proof. In Lemma 2, take $P = A_1$ and $Q = F_1$. The symmetric pair of lines are then the isodynamic and isogonic axes. Their symmetry about the principal axes is equivalent to the statement that these axes are the asserted bisectors. □

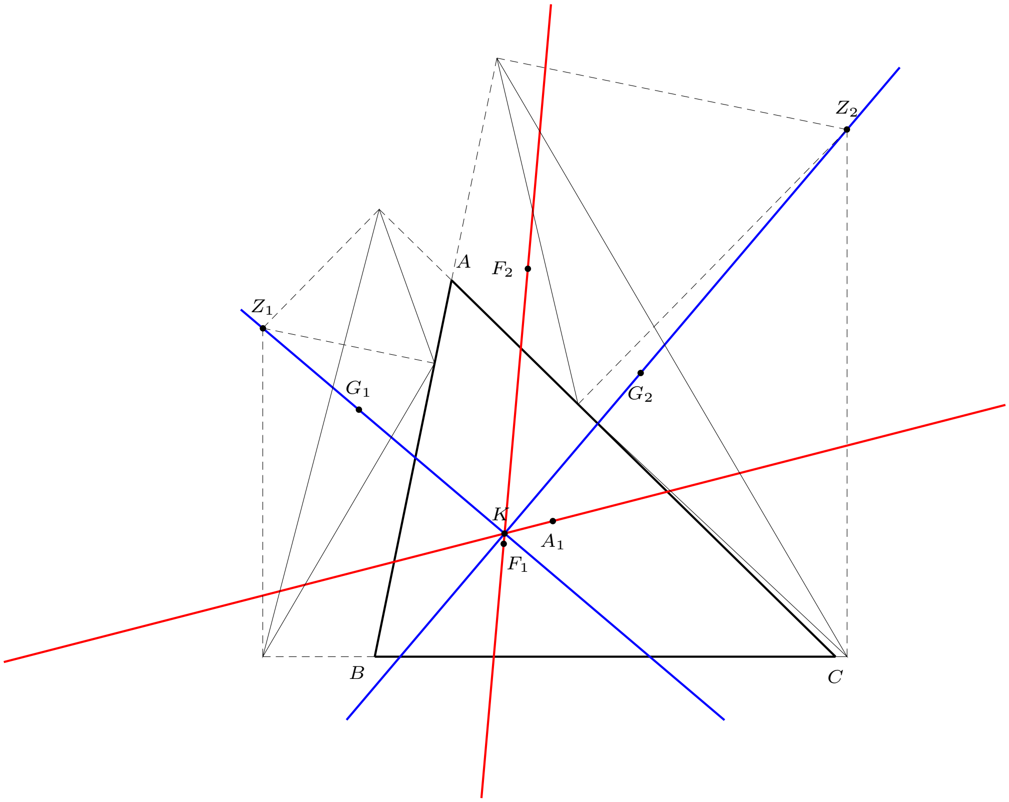


Figure 5

References

[1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, 1960.
[2] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000, http://cedar.evansville.edu/˜ck6/encyclopedia/.

Alexei Myakishev: Smolnaia 61-2, 138, Moscow, Russia, 125445.
E-mail address: alex_geom@mtu-net.ru

Forum Geometricorum
Volume 1 (2001) 99–105.

FORUM GEOM
ISSN 1534-1178

Multiplying and Dividing Curves by Points

Clark Kimberling

Abstract. Pointwise products and quotients, defined in terms of barycentric and trilinear coordinates, are extended to products $P \cdot \Gamma$ and quotients Γ/P, where P is a point and Γ is a curve. In trilinears, for example, if Γ_0 denotes the circumcircle, then $P \cdot \Gamma_0$ is a parabola if and only if P lies on the Steiner inscribed ellipse. Barycentric division by the triangle center X_{110} carries Γ_0 onto the Kiepert hyperbola Γ'; if P is on Γ_0, then the point $P' = P/X_{110}$ is the point, other than the Tarry point, X_{98}, in which the line PX_{98} meets Γ', and if Ω_1 and Ω_2 denote the Brocard points, then $|P'\Omega_1|/|P'\Omega_2| = |P\Omega_1|/|P\Omega_2|$; that is, P' and P lie on the same Apollonian circle with respect to Ω_1 and Ω_2.

1. Introduction

Paul Yiu [7] gives a magnificent construction for a product $P \cdot Q$ of points in the plane of triangle ABC. If

$$P = \alpha_1 : \beta_1 : \gamma_1 \text{ and } Q = \alpha_2 : \beta_2 : \gamma_2 \tag{1}$$

are representations in homogeneous barycentric coordinates, then the Yiu product is given by

$$P \cdot Q = \alpha_1\alpha_2 : \beta_1\beta_2 : \gamma_1\gamma_2 \tag{2}$$

whenever $\{\alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2\} \neq \{0\}$.

Cyril Parry [3] constructs an analogous product using trilinear coordinates. In view of the applicability of both the Yiu and Parry products, the notation in equations (1) and (2) will represent general homogeneous coordinates, as in [6, Chapter 1], unless otherwise noted. We also define the quotient

$$P/Q := \alpha_1\beta_2\gamma_2 : \beta_1\gamma_2\alpha_2 : \gamma_1\alpha_2\beta_2$$

whenever $Q \notin \{A, B, C\}$. Specialization of coordinates will be communicated by phrases such as those indicated here:

$$\left\{ \begin{array}{c} \text{barycentric} \\ \text{trilinear} \end{array} \right\} \left\{ \begin{array}{c} \text{multiplication} \\ \text{product} \\ \text{division} \\ \text{quotient} \end{array} \right\}.$$

Publication Date: July 12, 2001. Communicating Editor: Floor van Lamoen.

If S is a set of points, then $P \cdot S := \{P \cdot Q : \; Q \in S\}$. In particular, if S is a curve Γ, then $P \cdot \Gamma$ and Γ/P are curves, except for degenerate cases, such as when $P \in \{A, B, C\}$.

In all that follows, suppose $P = p : q : r$ is a point not on a sideline of triangle ABC, so that $pqr \neq 0$, and consequently, $U/P = \frac{u}{p} : \frac{v}{q} : \frac{w}{r}$ for all $U = u : v : w$.

Example 1. If Γ is a line $\ell\alpha + m\beta + n\gamma = 0$, then $P \cdot \Gamma$ is the line $(\ell/p)\alpha + (m/q)\beta + (n/r)\gamma = 0$ and Γ/P is the line $p\ell\alpha + qm\beta + rn\gamma = 0$. Given the line QR of points Q and R, it is easy to check that $P \cdot QR$ is the line of $P \cdot Q$ and $P \cdot R$. In particular, $P \cdot \triangle ABC = \triangle ABC$, and if T is a cevian triangle, then $P \cdot T$ is a cevian triangle.

2. Conics and Cubics

Each conic Γ in the plane of triangle ABC is given by an equation of the form

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0. \tag{3}$$

That $P \cdot \Gamma$ is the conic

$$(u/p^2)\alpha^2 + (v/q^2)\beta^2 + (w/r^2)\gamma^2 + 2(f/qr)\beta\gamma + 2(g/rp)\gamma\alpha + 2(h/pq)\alpha\beta = 0 \tag{4}$$

is clear, since $\alpha : \beta : \gamma$ satifies (3) if and only if $p\alpha : q\beta : r\gamma$ satisfies (4). In the case of a circumconic Γ given in general form by

$$\frac{f}{\alpha} + \frac{g}{\beta} + \frac{h}{\gamma} = 0, \tag{5}$$

the product $P \cdot \Gamma$ is the circumconic

$$\frac{pf}{\alpha} + \frac{qg}{\beta} + \frac{rh}{\gamma} = 0.$$

Thus, if X is the point such that $X \cdot \Gamma$ is a given circumconic $\frac{u}{\alpha} + \frac{v}{\beta} + \frac{w}{\gamma} = 0$, then $X = \frac{u}{f} : \frac{v}{g} : \frac{w}{h}$.

Example 2. In trilinears, the circumconic Γ in (5) is the isogonal transform of the line L given by $f\alpha + g\beta + h\gamma = 0$. The isogonal transform of $P \cdot L$ is Γ/P.

Example 3. Let $U = u : v : w$. The conic $W(U)$ given in [1, p. 238] by

$$u^2\alpha^2 + v^2\beta^2 + w^2\gamma^2 - 2vw\beta\gamma - 2wu\gamma\alpha - 2uv\alpha\beta = 0$$

is inscribed in triangle ABC. The conic $P \cdot W(U)$ given by

$$(u/p)^2\alpha^2 + (v/q)^2\beta^2 + (w/r)^2\gamma^2 - 2(vw/qr)\beta\gamma - 2(wu/rp)\gamma\alpha - 2(uv/pq)\alpha\beta = 0$$

is the inscribed conic $W(U/P)$. In trilinears, we start with Γ = incircle, given by

$$u = u(a, b, c) = a(b + c - a), v = u(b, c, a), w = u(c, a, b),$$

and find [1]

[1]The conics in Example 3 are discussed in [1, p.238] as examples of a type denoted by $W(X_i)$, including incircle = $W(X_{55})$, Steiner inscribed ellipse = $W(X_6)$, Kiepert parabola = $W(X_{512})$, and Yff parabola = $W(X_{647})$. A list of X_i including trilinears, barycentrics, and remarks is given in [2].

Conic	Trilinear product	Barycentric product
Steiner inscribed ellipse	$X_9 \cdot \Gamma$	$X_8 \cdot \Gamma$
Kiepert parabola	$X_{643} \cdot \Gamma$	$X_{645} \cdot \Gamma$
Yff parabola	$X_{644} \cdot \Gamma$	$X_{646} \cdot \Gamma$

Example 4. Here we combine notions from Examples 1-3. The circumcircle, Γ_0, may be regarded as a special circumconic, and every circumconic has the form $P \cdot \Gamma_0$. We ask for the locus of a point P for which the circumconic $P \cdot \Gamma_0$ is a parabola. As such a conic is the isogonal transform of a line tangent to Γ_0, we begin with this statement of the problem: find $P = p : q : r$ (trilinears) for which the line L given by $\frac{p}{\alpha} + \frac{q}{\beta} + \frac{r}{\gamma} = 0$ meets Γ_0, given by $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 0$ in exactly one point. Eliminating γ leads to

$$\frac{\alpha}{\beta} = \frac{cr - ap - bq \pm \sqrt{(ap + bq - cr)^2 - 4abpq}}{2bp}.$$

We write the discriminant as

$$\Phi(p, q, r) = a^2p^2 + b^2q^2 + c^2r^2 - 2bcqr - 2carp - 2abpq.$$

In view of Example 3 and [5, p.81], we conclude that if $W(X_6)$ denotes the Steiner inscribed ellipse, with trilinear equation $\Phi(\alpha, \beta, \gamma) = 0$, then

$$P \cdot \Gamma_0 \text{ is a } \left\{ \begin{array}{c} \text{hyperbola} \\ \text{parabola} \\ \text{ellipse} \end{array} \right\} \text{ according as } P \text{ lies } \left\{ \begin{array}{rl} \text{inside} & W(X_6) \\ \text{on} & W(X_6) \\ \text{outside} & W(X_6) \end{array} \right\}. \tag{6}$$

Returning to the case that L is tangent to Γ_0, it is easy to check that the point of tangency is $(X_1/P)©X_6$. (See Example 7 for Ceva conjugacy, denoted by ©.)

If the method used to obtain statement (6) is applied to barycentric multiplication, then a similar conclusion is reached, in which the role of $W(X_6)$ is replaced by the inscribed conic whose barycentric equation is

$$\alpha^2 + \beta^2 + \gamma^2 - 2\beta\gamma - 2\gamma\alpha - 2\alpha\beta = 0,$$

that is, the ellipse $W(X_2)$.

Example 5. Suppose points P and Q are given in trilinears: $P = p : q : r$, and $U = u : v : w$. We shall find the locus of a point $X = \alpha : \beta : \gamma$ such that $P \cdot X$ lies on the line UX. This on-lying is equivalent to the determinant equation

$$\begin{vmatrix} u & v & w \\ \alpha & \beta & \gamma \\ p\alpha & q\beta & r\gamma \end{vmatrix} = 0,$$

expressible as a circumconic:

$$\frac{u(q-r)}{\alpha} + \frac{v(r-p)}{\beta} + \frac{w(p-q)}{\gamma} = 0. \tag{7}$$

One may start with the line X_1P, form its isogonal transform Γ, and then recognize (7) as $U \cdot \Gamma$. For example, in trilinears, equation (7) represents the hyperbolas of

Kiepert, Jerabek, and Feuerbach according as $(P, U) = (X_{31}, X_{75})$, (X_6, X_{48}), and (X_1, X_3); or, in barycentrics, according as $(P, U) = (X_6, X_{76})$, (X_1, X_3), and (X_2, X_{63}).

Example 6. Again in trilinears, let Γ be the self-isogonal cubic $Z(U)$ given in [1, p. 240] by

$$u\alpha(\beta^2 - \gamma^2) + v\beta(\gamma^2 - \alpha^2) + w\gamma(\alpha^2 - \beta^2) = 0.$$

This is the locus of points X such that X, X_1/X, and U are collinear; the point U is called the *pivot* of $Z(U)$. The quotient Γ/P is the cubic

$$up\alpha(q^2\beta^2 - r^2\gamma^2) + vq\beta(r^2\gamma^2 - p^2\alpha^2) + wr\gamma(p^2\alpha^2 - q^2\beta^2) = 0.$$

Although Γ/P is not generally self-isogonal, it is self-conjugate under the P^2-isoconjugacy defined (e.g., [4]) by $X \to X_1/(X \cdot P^2)$.

Example 7. Let $X©P$ denote the X-Ceva conjugate of P, defined in [1, p.57] for $X = x : y : z$ and $P = p : q : r$ by

$$X©P = p(-\frac{p}{x} + \frac{q}{y} + \frac{r}{z}) : q(-\frac{q}{y} + \frac{r}{z} + \frac{p}{x}) : r(-\frac{r}{z} + \frac{p}{x} + \frac{q}{y}).$$

Assume that $X \neq P$. It is easy to check that the locus of a point X for which $X©P$ lies on the line XP is given by

$$\frac{\alpha}{p}(\frac{\beta^2}{q^2} - \frac{\gamma^2}{r^2}) + \frac{\beta}{q}(\frac{\gamma^2}{r^2} - \frac{\alpha^2}{p^2}) + \frac{\gamma}{r}(\frac{\alpha^2}{p^2} - \frac{\beta^2}{q^2}) = 0. \tag{8}$$

In trilinears, equation (8) represents the product $P \cdot \Gamma$ where Γ is the cubic $Z(X_1)$. The locus of X for which $P©X$ lies on XP is also the cubic (8).

3. Brocard Points and Apollonian Circles

Here we discuss some special properties of the triangle centers X_{98} (the Tarry point) and X_{110} (the focus of the Kiepert parabola). X_{98} is the point, other than A, B, C, that lies on both the circumcircle and the Kiepert hyperbola.

Let ω be the Brocard angle, given by

$$\cot\omega = \cot A + \cot B + \cot C.$$

In trilinears,

$$\begin{aligned} X_{98} &= \sec(A+\omega) : \sec(B+\omega) : \sec(C+\omega), \\ X_{110} &= \frac{a}{b^2 - c^2} : \frac{b}{c^2 - a^2} : \frac{c}{a^2 - b^2}. \end{aligned}$$

Theorem. *Barycentric division by X_{110} carries the circumcircle Γ_0 onto the Kiepert hyperbola Γ'. For every point P on Γ_0, the line joining P to the Tarry point X_{98} (viz., the tangent at X_{98} if $P = X_{98}$) intersects Γ' again at $P' = P/X_{110}$. Furthermore, P/X_{110} lies on the Apollonian circle of P with respect to the two Brocard points Ω_1 and Ω_2; that is*

$$\frac{|P'\Omega_1|}{|P'\Omega_2|} = \frac{|P\Omega_1|}{|P\Omega_2|}. \tag{9}$$

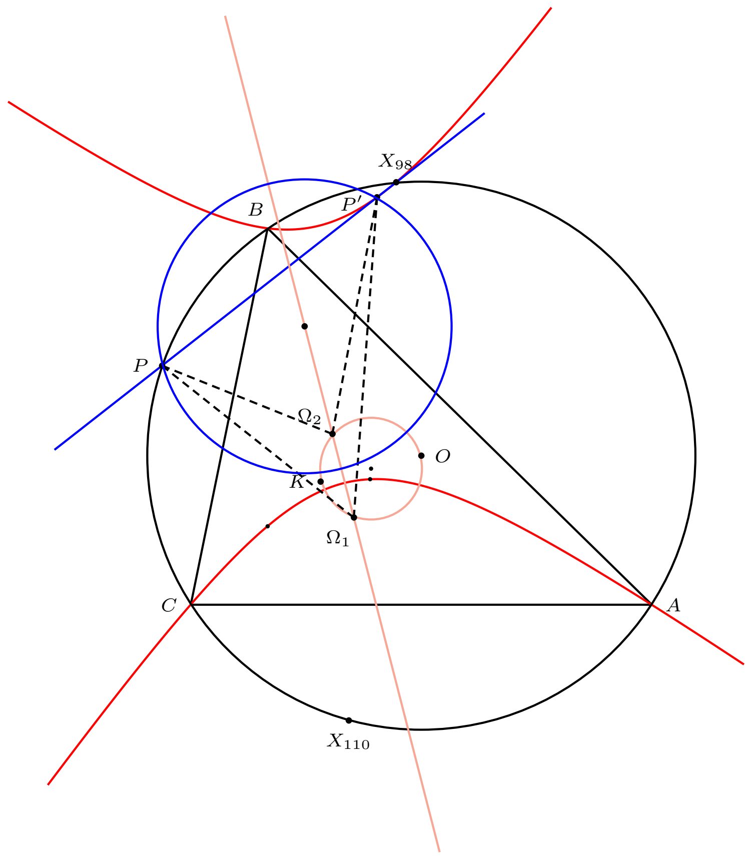


Figure 1

Proof. In barycentrics, Γ_0 and Γ' are given by

$$\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} = 0 \text{ and } \frac{b^2-c^2}{\alpha} + \frac{c^2-a^2}{\beta} + \frac{a^2-b^2}{\gamma} = 0,$$

and, also in barycentrics,

$$X_{110} = \frac{a}{b^2-c^2} : \frac{b}{c^2-a^2} : \frac{c}{a^2-b^2}$$

so that $\Gamma' = \Gamma_0/X_{110}$.

For the remainder of the proof, we use trilinears. A parametric representation for Γ_0 is given by

$$P = P(t) = a(1-t) : bt : ct(t-1), \tag{10}$$

for $-\infty < t < \infty$, and the barycentric product P/X_{110} is given in trilinears by

$$(1-t)\frac{b^2-c^2}{a} : t\frac{c^2-a^2}{b} : t(t-1)\frac{a^2-b^2}{c}.$$

That this point lies on line PX_{98} is equivalent to the following easily verified identity:

$$\begin{vmatrix} (1-t)\frac{b^2-c^2}{a} & t\frac{c^2-a^2}{b} & t(t-1)\frac{a^2-b^2}{c} \\ a(1-t) & bt & ct(t-1) \\ \sec(A+\omega) & \sec(B+\omega) & \sec(C+\omega) \end{vmatrix} = 0.$$

We turn now to a formula [1, p.31] for the distance between two points expressed in normalized[2] trilinears (α, β, γ) and $(\alpha', \beta', \gamma')$:

$$\frac{1}{2\sigma}\sqrt{abc[a\cos A(\alpha-\alpha')^2 + b\cos B(\beta-\beta')^2 + c\cos C(\gamma-\gamma')^2]}, \qquad (11)$$

where σ denotes the area of triangle ABC. Let

$$\begin{aligned} D &= c^2t^2 - (c^2+a^2-b^2)t + a^2, \\ S &= a^2b^2 + b^2c^2 + c^2a^2. \end{aligned}$$

Normalized trilinears for (10) and the two Brocard points follow:

$$P = ((1-t)ha, thb, t(t-1)hc),$$

where $h = \frac{2\sigma}{D}$, and

$$\Omega_1 = \left(\frac{h_1 c}{b}, \frac{h_1 a}{c}, \frac{h_1 b}{a}\right), \qquad \Omega_2 = \left(\frac{h_1 b}{c}, \frac{h_1 c}{a}, \frac{h_1 a}{b}\right),$$

where and $h_1 = \frac{2abc\sigma}{S}$.

Abbreviate $a\cos A$, $b\cos B$, $c\cos C$, and $1-t$ as a', b', c', and t' respectively, and write

$$E = a'\left(\frac{t'ha - h_1c}{b}\right)^2 + b'\left(\frac{thb - h_1a}{c}\right)^2 + c'\left(\frac{tt'hc - h_1b}{a}\right)^2, \qquad (12)$$

$$F = a'\left(\frac{t'ha - h_1b}{c}\right)^2 + b'\left(\frac{thb - h_1c}{a}\right)^2 + c'\left(\frac{tt'hc - h_1a}{b}\right)^2. \qquad (13)$$

Equation (11) then gives

$$\frac{|P\Omega_1|^2}{|P\Omega_2|^2} = \frac{E}{F}. \qquad (14)$$

In (12) and (13), replace $\cos A$ by $(b^2+c^2-a^2)/2bc$, and similarly for $\cos B$ and $\cos C$, obtaining from (14) the following:

$$\frac{|P\Omega_1|^2}{|P\Omega_2|^2} = \frac{t^2a^2 - t(a^2+b^2-c^2) + b^2}{t^2b^2 - t(b^2+c^2-a^2) + c^2}.$$

[2]Sometimes trilinear coordinates are called normal coordinates. We prefer "trilinears", so that we can say "normalized trilinears," not "normalized normals." One might say that the latter double usage of "normal" can be avoided by saying "actual normal distances", but this would be unsuitable for normalization of points at infinity. Another reason for retaining "trilinear" and "quadriplanar"—not replacing both with "normal"— is that these two terms distinguish between lines and planes as the objects with respect to which normal distances are defined. In discussing points relative to a tetrahedron, for example, one could have both trilinears and quadriplanars in the same sentence.

Note that if the numerator in the last fraction is written as $f(t,a,b,c)$, then the denominator is $t^2 f(\frac{1}{t},c,b,a)$. Similarly,

$$\frac{|P'\Omega_1|^2}{|P'\Omega_2|^2} = \frac{g(t,a,b,c)}{t^4 g(\frac{1}{t},c,b,a)},$$

where

$$g(t,a,b,c) = t^4 e_4 + t^3 e_3 + t^2 e_2 + te_1 + e_0,$$

and

$$\begin{aligned}
e_4 &= a^4b^2(a^2-b^2)^2,\\
e_3 &= a^2(a^2-b^2)(b^6+c^6+2a^2b^2c^2-2a^4b^2-2a^2c^4-2b^2c^4+a^4c^2+a^2b^4),\\
e_2 &= b^2c^2(b^2-c^2)^3+a^2c^2(c^2-a^2)^3+a^2b^2(a^6+2b^6-3a^2b^4)\\
&\quad + a^2b^2c^2(b^4+c^4-2a^4-4b^2c^2+2a^2c^2+2a^2b^2),\\
e_1 &= b^2(c^2-b^2)(a^6+c^6-3b^2c^4+2b^4c^2-2a^4b^2-2a^4c^2+2a^2b^2c^2+a^2b^4),\\
e_0 &= b^4c^2(b^2-c^2)^2.
\end{aligned}$$

One may now verify directly, using a computer algebra system, or manually with plenty of paper, that

$$t^2 f(t,a,b,c)g(\frac{1}{t},c,b,a) = f(\frac{1}{t},c,b,a)g(t,a,b,c),$$

which is equivalent to the required equation (9). □

References

[1] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://cedar.evansville.edu/~ck6/encyclopedia/.

[3] C. Kimberling and C. Parry, Products, square roots, and layers in triangle geometry, *Mathematics and Informatics Quarterly*, 10 (2000) 9-22.

[4] C. Kimberling, Conjugacies in the plane of a triangle, *Aequationes Mathematicae*, 61 (2001) forthcoming.

[5] S.L. Loney, *The Elements of Coordinate Geometry, Part II: Trilinear Coordinates, Etc.*, Macmillan, London, 1957.

[6] E.A. Maxwell, *The Methods of Plane Projective Geometry Based on the Use of General Homogeneous Coordinates*, Cambridge University Press, 1957.

[7] P. Yiu, The uses of homogeneous barycentric coordinates in plane euclidean geometry, *Int. J. Math. Educ. Sci. Technol.*, 31 (2000) 569 – 578.

Clark Kimberling: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, USA

E-mail address: ck6@evansville.edu

Forum Geometricorum
Volume 1 (2001) 107–114.

FORUM GEOM
ISSN 1534-1178

The Simson cubic

Jean-Pierre Ehrmann and Bernard Gibert

Abstract. The Simson cubic is the locus of the trilinear poles of the Simson lines. There exists a conic such that a point M lies on the Simson cubic if and only if the line joining M to its isotomic conjugate is tangent to this conic. We also characterize cubics which admit pivotal conics for a given isoconjugation.

1. Introduction

Antreas P. Hatzipolakis [2] has raised the question of the locus of points for which the triangle bounded by the pedal cevians is perspective. More precisely, given triangle ABC, let $A_{[P]}B_{[P]}C_{[P]}$ be the pedal triangle of a point P, and consider the intersection points

$$Q_a := BB_{[P]} \cap CC_{[P]}, \qquad Q_b := CC_{[P]} \cap AA_{[P]}, \qquad Q_c := AA_{[P]} \cap BB_{[P]}.$$

We seek the locus of P for which the triangle $Q_aQ_bQ_c$ is perspective with ABC. See Figure 1. This is the union of

(1a) the Darboux cubic consisting of points whose pedal triangles are cevian,[1]
(1b) the circumcircle together with the line at infinity.

The loci of the perspector in these cases are respectively

(2a) the Lucas cubic consisting of points whose cevian triangles are pedal,[2]
(2b) a cubic related to the Simson lines.

We give an illustration of the Darboux and Lucas cubics in the Appendix. Our main interest is in the singular case (2b) related to the Simson lines of points on the circumcircle. The curve in (2b) above is indeed the locus of the tripoles[3] of the Simson lines. Let P be a point on the circumcircle, and $\mathsf{t}(P) = (u : v : w)$ the tripole of its Simson line $\mathsf{s}(P)$. This means that the perpendicular to the sidelines at the points

$$U = (0 : v : -w), \quad V = (-u : 0 : w), \quad W = (u : -v : 0) \tag{1}$$

Publication Date: August 24, 2001. Communicating Editor: Paul Yiu.

[1] This is the isogonal cubic with pivot the de Longchamps point, the reflection of the orthocenter in the circumcenter. A point P lies on this cubic if and only if its the line joining P to its isogonal conjugate contains the de Longchamps point.

[2] This is the isotomic cubic with pivot $\mathsf{i}(H)$, the isotomic conjugate of the orthocenter. A point P lies on this cubic if and only if its the line joining P to its isotomic conjugate contains the point $\mathsf{i}(H)$.

[3] We use the term tripole as a short form of trilinear pole.

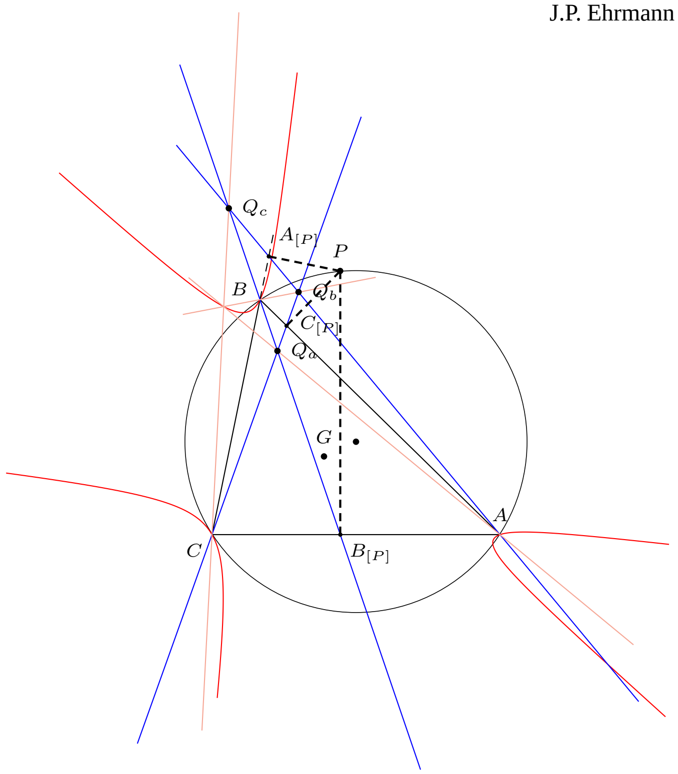


Figure 1

are concurrent (at P on the circumcircle). In the notations of John H. Conway,[4] the equations of these perpendiculars are

$$\begin{array}{ccccccc} (S_Bv+S_Cw)x & + & a^2vy & + & a^2wz & = & 0, \\ b^2ux & + & (S_Cw+S_Au)y & + & b^2wz & = & 0, \\ c^2ux & + & c^2vy & + & (S_Au+S_Bv)z & = & 0. \end{array}$$

Elimination of x, y, z leads to the cubic

$$\mathcal{E}:\ S_Au(v^2+w^2)+S_Bv(w^2+u^2)+S_Cw(u^2+v^2)-(a^2+b^2+c^2)uvw=0.$$

This is clearly a self-isotomic cubic, *i.e.*, a point P lies on the cubic if and only if its isotomic conjugate does. We shall call $\mathcal{E}$ the *Simson cubic* of triangle ABC.

2. A parametrization of the Simson cubic

It is easy to find a rational parametrization of the Simson cubic. Let P be a point on the circumcircle. Regarded as the isogonal conjugate of the infinite point of a

[4]For a triangle ABC with side lengths a, b, c, denote

$$S_A=\frac{1}{2}(b^2+c^2-a^2),\quad S_B=\frac{1}{2}(c^2+a^2-b^2),\quad S_C=\frac{1}{2}(a^2+b^2-c^2).$$

These satisfy a number of basic relations. We shall, however, only need the obvious relations

$$S_B+S_C=a^2,\qquad S_C+S_A=b^2,\quad S_A+S_B=c^2.$$

line $px + qy + rz = 0$, the point P has homogeneous barycentric coordinates

$$\left(\frac{a^2}{q-r} : \frac{b^2}{r-p} : \frac{c^2}{p-q}\right).$$

The pedals of P on the side lines are the points U, V, W in (1) with

$$\begin{aligned} u &= \frac{1}{q-r}(-a^2p + S_Cq + S_Br), \\ v &= \frac{1}{r-p}(S_Cp - b^2q + S_Ar), \\ w &= \frac{1}{p-q}(S_Bp + S_Aq - c^2r). \end{aligned} \tag{2}$$

This means that the tripole of the Simson line $\mathsf{s}(P)$ of P is the point $\mathsf{t}(P) = (u : v : w)$. The system (2) therefore gives a rational parametrization of the Simson cubic. It also shows that $\mathcal{E}$ has a singularity, which is easily seen to be an isolated singularity at the centroid. [5]

3. Pivotal conic of the Simson cubic

We have already noted that $\mathcal{E}$ is a self-isotomic cubic. In fact, the isotomic conjugate of $\mathsf{t}(P)$ is the point $\mathsf{t}(P')$, where P' is the antipode of P (with respect to the circumcircle). [6]

It is well known that the Simson lines of antipodal points intersect (orthogonally) on the nine-point circle. As this intersection moves on the nine-point circle, the line joining the tripoles $\mathsf{t}(P)$, $\mathsf{t}(P')$ of the orthogonal Simson lines $\mathsf{s}(P)$, $\mathsf{s}(P')$ envelopes the conic $\mathcal{C}$ dual to the nine-point circle. This conic has equation[7]

$$\sum_{\text{cyclic}} (b^2 - c^2)^2 x^2 - 2(b^2c^2 + c^2a^2 + a^2b^2 - a^4)yz = 0,$$

and is the inscribed ellipse in the anticomplementary triangle with center the symmedian point of triangle ABC, $K = (a^2 : b^2 : c^2)$. The Simson cubic $\mathcal{E}$ can therefore be regarded as an isotomic cubic with the ellipse $\mathcal{C}$ as pivot. See Figure 2.

Proposition 1. *The pivotal conic $\mathcal{C}$ is tritangent to the Simson cubic $\mathcal{E}$ at the tripoles of the Simson lines of the isogonal conjugates of the infinite points of the Morley sides.*

---

[5] If P is an infinite point, its pedals are the infinite points of the side lines. The triangle $Q_aQ_bQ_c$ in question is the anticomplementary triangle, and has perspector at the centroid G.

[6] The antipode of P has coordinates

$$\left(\frac{a^2}{-a^2p + S_Cq + S_Br} : \frac{b^2}{S_Cp - b^2q + S_Ar} : \frac{c^2}{S_Bp + S_Aq - c^2r}\right).$$

[7] The equation of the nine-point circle is $\sum_{\text{cyclic}} S_Ax^2 - a^2yz = 0$. We represent this by a symmetric matrix A. The dual conic is then represented by the adjoint matrix of A.

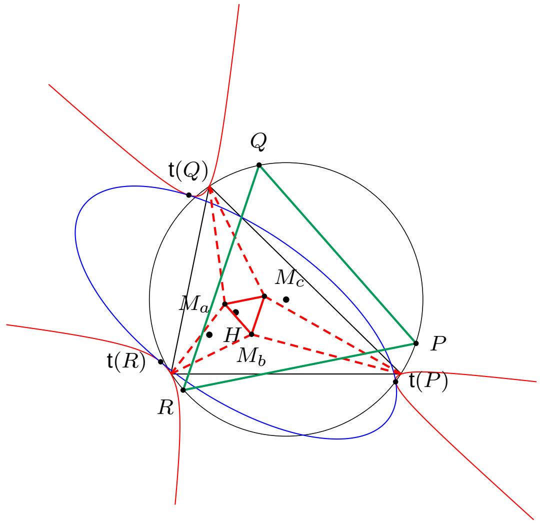


Figure 2

Proof. Since $\mathcal{C}$ is the dual of the nine-point circle, the following statements are equivalent:

(1) $\mathsf{t}(P)$ lies on $\mathcal{C} \cap \mathcal{E}$.

(2) $\mathsf{s}(P')$ is tangent to the nine-point circle.

(3) $\mathsf{s}(P)$ passes through the nine-point center.

Thus, $\mathcal{C}$ and $\mathcal{E}$ are tangent at the three points $\mathsf{t}(P)$ for which the Simson lines $S(P)$ pass through the nine-point center. If P, Q, R are the isogonal conjugates of the infinite points of the side lines of the Morley triangle, then PQR is an equilateral triangle and the Simson lines $\mathsf{s}(P)$, $\mathsf{s}(Q)$, $\mathsf{s}(R)$ are perpendicular to QR, RP, PQ respectively. See [1]. Let H be the orthocenter of triangle ABC, and consider the midpoints P_1, Q_1, R_1 of HP, HQ, HR. Since $\mathsf{s}(P)$, $\mathsf{s}(Q)$, $\mathsf{s}(R)$ pass through P_1, Q_1, R_1 respectively, these Simson lines are the altitudes of the triangle $P_1Q_1R_1$. As this triangle is equilateral and inscribed in the nine-point circle, the Simson lines $\mathsf{s}(P)$, $\mathsf{s}(Q)$, $\mathsf{s}(R)$ pass through the nine-point center. $\square$

Remarks. (1) The triangle PQR is called the circum-tangential triangle of ABC in [3].

(2) The ellipse $\mathcal{C}$ intersects the Steiner circum - ellipse at the four points

$$\left(\frac{1}{b-c} : \frac{1}{c-a} : \frac{1}{a-b}\right), \qquad \left(\frac{1}{b-c} : \frac{1}{c+a} : \frac{-1}{a+b}\right),$$
$$\left(\frac{-1}{b+c} : \frac{1}{c-a} : \frac{1}{a+b}\right), \qquad \left(\frac{1}{b+c} : \frac{-1}{c+a} : \frac{1}{a-b}\right).$$

These points are the perspectors of the four inscribed parabolas tangent respectively to the tripolars of the incenter and of the excenters. In Figure 3, we illustrate the parabolas for the incenter and the B-excenter. The foci are the isogonal conjugates of the infinite points of the lines $\pm ax \pm by \pm cz = 0$, and the directrices are the corresponding lines of reflections of the foci in the side lines of triangle ABC.

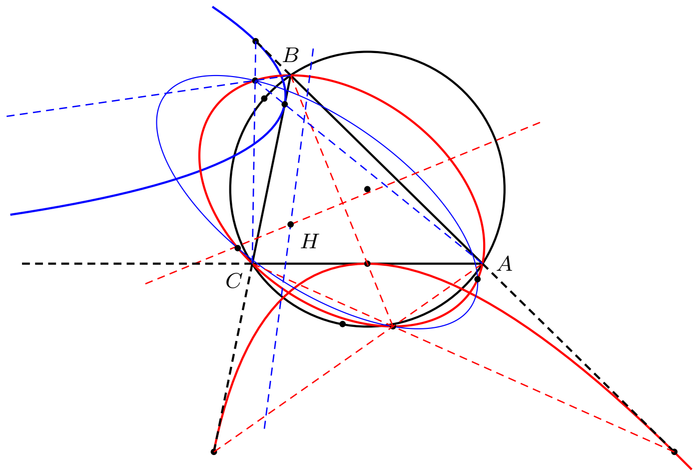


Figure 3

4. Intersection of $\mathcal{E}$ with a line tangent to $\mathcal{C}$

Consider again the Simson line of P and P' intersecting orthogonally at a point N on the nine-point circle. There is a third point Q on the circumcircle whose Simson line $\mathsf{s}(Q)$ passes through N.

- Q is the intersection of the line HN with the circumcircle, H being the orthocenter.
- The line $\mathsf{t}(P)\mathsf{t}(P')$ intersects again the cubic at $\mathsf{t}(Q')$.
- The tangent lines at $\mathsf{t}(P)$ and $\mathsf{t}(P')$ to the cubic intersect at $\mathsf{t}(Q)$ on the cubic.

If the line $\mathsf{t}(P)\mathsf{t}(P')$ touches $\mathcal{C}$ at S, then

(i) S and $\mathsf{t}(Q')$ are harmonic conjugates with respect to $\mathsf{t}(P)$ and $\mathsf{t}(P')$;

(ii) the isotomic conjugate of S is the tripole of the line tangent at N to the nine-point circle.

See Figure 4.

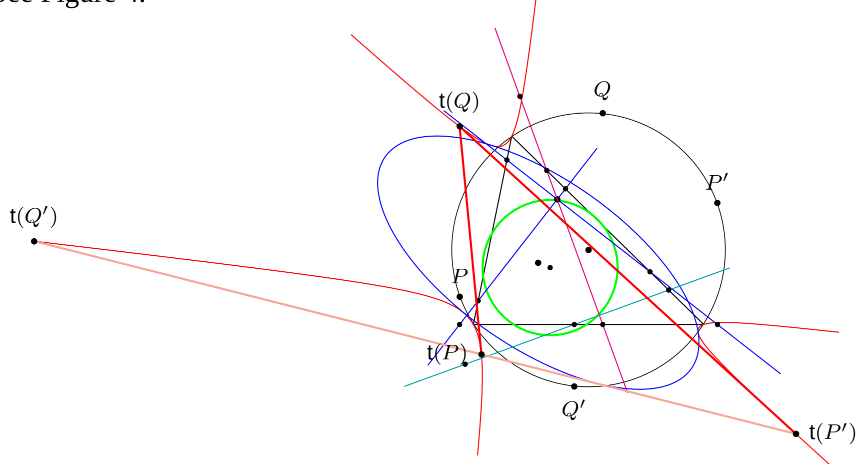


Figure 4

5. Circumcubic invariant under a quadratic transformation

Let $\mathcal{E}$ be a circumcubic invariant under a quadratic transformation τ defined by

$$\tau(x:y:z) = \left(\frac{f^2}{x} : \frac{g^2}{y} : \frac{h^2}{z}\right).$$

The fixed points of τ are the points $(\pm f : \pm g : \pm h)$, which form a harmonic quadruple.

Consider a circumcubic $\mathcal{E}$ invariant under τ. Denote by U, V, W the "third" intersections of $\mathcal{E}$ with the side lines. Then, either U, V, W lie on same line or UVW is perspective with ABC.

The latter case is easier to describe. If UVW is perspective with ABC at P, then $\mathcal{E}$ is the $\tau-$cubic with pivot P, *i.e.*, a point Q lies on $\mathcal{E}$ if and only if the line joining Q and $\tau(Q)$ passes through P.

On the other hand, if U, V, W are collinear, their coordinates can be written as in (1) for appropriate choice of u, v, w, so that the line containing them is the tripolar of the point $(u:v:w)$. In this case, then the equation of $\mathcal{E}$ is

$$\sum_{\text{cyclic}} f^2yz(wy+vz) + txyz = 0$$

for some t.

(a) If $\mathcal{E}$ contains exactly one of the fixed points $F = (f:g:h)$, then

$$t = -2(ghu + hfv + fgw).$$

In this case, $\mathcal{E}$ has a singularity at F. If $M = (x:y:z)$ in barycentric coordinates with respect to ABC, then with respect to the precevian triangle of F (the three other invariant points), the tangential coordinates of the line joining M to $\tau(M)$ are

$$(p:q:r) = \left(\frac{gz-hy}{(g+h-f)(gz+hy)} : \frac{hx-fz}{(h+f-g)(hx+fz)} : \frac{fy-gx}{(f+g-h)(fy+gx)}\right).$$

As the equation of $\mathcal{E}$ can be rewritten as

$$\frac{p_0}{p} + \frac{q_0}{q} + \frac{r_0}{r} = 0,$$

where

$$p_0 = \frac{f(hv+gw)}{g+h-f}, \quad q_0 = \frac{g(fw+hu)}{h+f-g}, \quad r_0 = \frac{h(gu+fv)}{f+g-h},$$

it follows that the line $M\tau(M)$ envelopes a conic inscribed in the precevian triangle of F.

Conversely, if $\mathcal{C}$ is a conic inscribed in the precevian triangle $A^FB^FC^F$, the locus of M such as the line $M\tau(M)$ touches $\mathcal{C}$ is a $\tau-$cubic with a singularity at F. The tangent lines to $\mathcal{E}$ at F are the tangent lines to $\mathcal{C}$ passing through P.

Note that if F lies on $\mathcal{C}$, and T the tangent to $\mathcal{C}$ at P, then $\mathcal{E}$ degenerates into the union of T and T^*.

(b) If $\mathcal{E}$ passes through two fixed points F and, A^F, then it degenerates into the union of FA^F and a conic.

(c) If the cubic $\mathcal{E}$ contains none of the fixed points, each of the six lines joining two of these fixed points contains, apart from a vertex of triangle ABC, a pair of points of $\mathcal{E}$ conjugate under τ. In this case, the lines $M\tau(M)$ cannot envelope a conic, because this conic should be tangent to the six lines, which is clearly impossible.

We close with a summary of the results above.

Proposition 2. *Let $\mathcal{E}$ be a circumcubic and τ a quadratic transformation of the form*

$$\tau(x:y:z) = (f^2yz : g^2zx : h^2xy).$$

The following statements are equivalent.

(1) $\mathcal{E}$ is $\tau-$invariant with pivot a conic.

(2) $\mathcal{E}$ passes through one and only one fixed points of τ, has a singularity at this point, and the third intersections of $\mathcal{E}$ with the side lines lie on a line ℓ.

In this case, if $\mathcal{E}$ contains the fixed point $F = (f : g : h)$, and if ℓ is the tripolar of the point $(u : v : w)$, then the equation of $\mathcal{E}$ is

$$-2(ghu + hfv + fgw)xyz + \sum_{\text{cyclic}} ux(h^2y^2 + g^2z^2) = 0. \tag{3}$$

The pivotal conic is inscribed in the precevian triangle of F and has equation[8]

$$\sum_{\text{cyclic}} (gw - hv)^2x^2 - 2(ghu^2 + 3fu(hv + gw) + f^2vw)yz = 0.$$

Appendix

Proposition 3. *Let ℓ be the tripolar of the point $(u : v : w)$, intersecting the sidelines of triangle ABC at U, V, W with coordinates given by (1), and $F = (f : g : h)$ a point not on ℓ nor the side lines of the reference triangle. The locus of M for which the three intersections $AM \cap FU$, $BM \cap FV$ and $CM \cap FW$ are collinear is the cubic $\mathcal{E}$ defined by (3) above.*

Proof. These intersections are the points

$$\begin{aligned}
AM \cap FU &= (f(wy + vz) : (hv + gw)y : (hv + gw)z),\\
BM \cap FV &= ((fw + hu)x : g(uz + wx) : (fw + hu)z),\\
CM \cap FW &= ((gu + fv)x : (gu + fv)y : h(vx + uy)).
\end{aligned}$$

The corresponding determinant is $(fvw + gwu + huv)R$ where R is the expression on the left hand side of (3). □

The Simson cubic is the particular case $F = G$, the centroid, and ℓ the line

$$\frac{x}{S_A} + \frac{y}{S_B} + \frac{z}{S_C} = 0,$$

which is the tripolar of the isotomic conjugate of the orthocenter H.

---

[8]The center of this conic is the point $(f(v+w-u)+u(g+h-f) : g(w+u-v)+v(h+f-g) : h(u+v-w)+w(f+g-h))$.

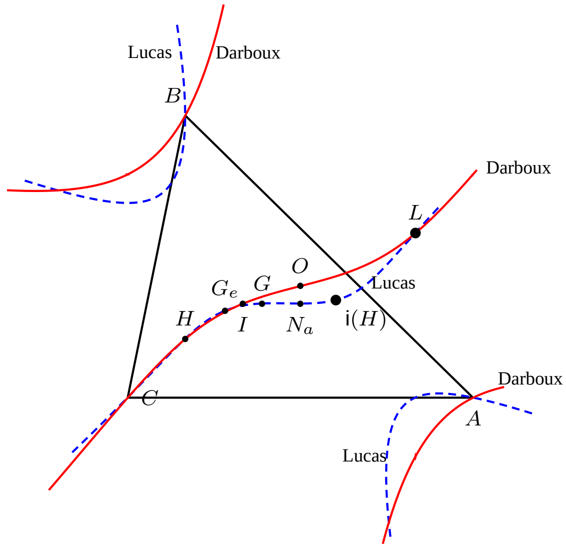


Figure 5. The Darboux and Lucas cubics

References

[1] J.P. Ehrmann and B. Gibert, A Morley configuration, *Forum Geom.*, 1 (2001) 51–58.
[2] A.P. Hatzipolakis, Hyacinthos message 1686, October 29, 2000.
[3] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

Jean-Pierre Ehrmann: 6, rue des Cailloux, 92110 - Clichy, France
E-mail address: `Jean-Pierre.EHRMANN@wanadoo.fr`

Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France
E-mail address: `b.gibert@free.fr`

Forum Geometricorum
Volume 1 (2001) 115–120.

FORUM GEOM
ISSN 1534-1178

Geometric Construction of Reciprocal Conjugations

Keith Dean and Floor van Lamoen

Abstract. Two conjugation mappings are well known in the geometry of the triangle: the isogonal and isotomic conjugations. These two are members of the family of reciprocal conjugations. In this paper we provide an easy and general construction for reciprocal conjugates of a point, given a pair of conjugate points. A connection is made to desmic configurations.

1. Introduction

Let ABC be a triangle. To represent a point in the plane of ABC we make use of homogeneous coordinates. Two such coordinate systems are well known, barycentric and normal (trilinear) coordinates. See [1] for an introduction on normal coordinates,[1] and [4] for barycentric coordinates. In the present paper we work with homogeneous barycentric coordinates exclusively.

Consider a point $X = (x : y : z)$. The isogonal conjugate X^* of X is represented by $(a^2yz : b^2xz : c^2yz)$, which we loosely write as $(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z})$ for X outside the sidelines of ABC (so that $xyz \neq 0$). In the same way the isotomic conjugate $X^\bullet$ of X is represented by $(\frac{1}{x} : \frac{1}{y} : \frac{1}{z})$. For both X^* and $X^\bullet$ the coordinates are the products of the reciprocals of those of X and the constant 'coordinates' from a certain homogeneous triple, $(a^2 : b^2 : c^2)$ and $(1 : 1 : 1)$ respectively. [2]

With this observation it is reasonable to generalize the two famous conjugations to *reciprocal conjugations*, where the homogeneous triple takes a more general form $(\ell : m : n)$ with $\ell mn \neq 0$. By the $(\ell : m : n)$-*reciprocal conjugation* or simply $(\ell : m : n)$-*conjugation*, we mean the mapping

$$\tau : (x : y : z) \mapsto \left(\frac{\ell}{x} : \frac{m}{y} : \frac{n}{z}\right).$$

It is clear that for any point X outside the side lines of ABC, $\tau(\tau(X)) = X$. A reciprocal conjugation is uniquely determined by any one pair of conjugates: if $\tau(x_0 : y_0 : z_0) = (x_1 : y_1 : z_1)$, then $\ell : m : n = x_0x_1 : y_0y_1 : z_0z_1$. It is convenient to regard $(\ell : m : n)$ as the coordinates of a point P_0, which we call the *pole* of the conjugation τ. The poles of the isogonal and isotomic conjugations,

Publication Date: August 28, 2001. Communicating Editor: Jean-Pierre Ehrmann.

[1]What we call normal coordinates are traditionally called *trilinear* coordinates; they are the ratio of signed distances of the point to the side lines of the reference triangle.

[2]Analogous results hold when normal coordinates are used instead of barycentrics.

for example, are the symmedian point and the centroid respectively. In this paper we address the questions of (i) construction of τ given a pair of points P and Q conjugate under τ, (ii) construction of $\tau(P)$ given the pole R_0.

2. The parallelogram construction

2.1. *Isogonal conjugation.* There is a construction of the isogonal conjugate of a point that gives us a good opportunity for generalization to all reciprocal conjugates.

Proposition 1. *Let P be a point and let $A'B'C'$ be its pedal triangle. Let A'' be the point such that $B'PC'A''$ is a parallelogram. In the same way construct B'' and C''. Then the perspector of triangles ABC and $A''B''C''$ is the isogonal conjugate P^* of P.*

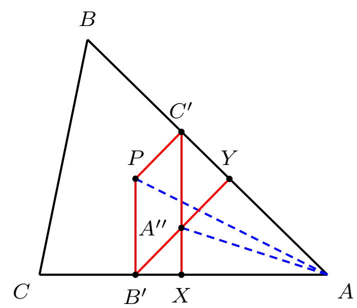


Figure 1

Proof. This is equivalent to the construction of the isogonal conjugate of P as the point of concurrency of the perpendiculars through the vertices of ABC to the corresponding sides of the pedal triangle. Here we justify it directly by noting that AP and AA'' are isogonal lines. Let $C'A''$ and $B'A''$ intersect AC and AB at X and Y respectively. Clearly,

$$B'P : YA'' = A''C' : YA'' = B'A : YA.$$

From this we conclude that triangles APB' and $AA''Y$ are similar, so that AP and AA'' are indeed isogonal lines. □

2.2. *Construction of $(\ell : m : n)$-conjugates.* Observe that the above construction depends on the 'altitudes' forming the pedal triangle $A'B'C'$. When these altitudes are replaced by segments parallel to the cevians of a point $H = (f : g : h)$, we obtain a generalization to the reciprocal conjugation

$$\tau : (x : y : z) \mapsto \left(\frac{f(g+h)}{x} : \frac{g(f+h)}{y} : \frac{h(f+g)}{z}\right).$$

In this way we get the complete set of reciprocal conjugations . In particular, given $(\ell : m : n)$, by choosing H to be the point with (homogeneous barycentric) coordinates

$$\left(\frac{1}{m+n-\ell} : \frac{1}{n+\ell-m} : \frac{1}{\ell+m-n}\right),$$

this construction gives the $(\ell : m : n)$-conjugate of points.[3]

3. The perspective triangle construction

The parallelogram construction depended on a triple of directions of cevians. These three directions can be seen as a degenerate triangle on the line at infinity, perspective to ABC. We show that this triangle can be replaced by any triangle $A_1B_1C_1$ perspective to ABC, thus making the notion of reciprocal conjugation *projective*.

Proposition 2. *A triangle $A_1B_1C_1$ perspective with ABC induces a reciprocal conjugation : for every point M not on the side lines of ABC and $A_1B_1C_1$, construct*

$$A' = A_1M \cap BC, \qquad B' = B_1M \cap CA, \qquad C' = C_1M \cap AB;$$
$$A'' = B_1C' \cap C_1B', \qquad B'' = C_1A' \cap A_1C', \qquad C'' = A_1B' \cap B_1A'.$$

Triangle $A''B''C''$ is perspective with ABC at a point N, and the correspondence $M \mapsto N$ is a reciprocal conjugation.

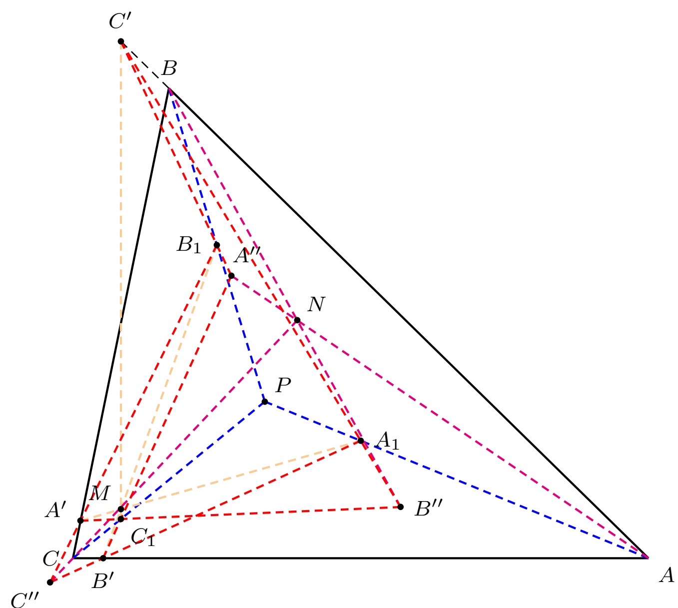


Figure 2

Proof. Since $A_1B_1C_1$ is perspective with ABC, we may write the coordinates of its vertices in the form

$$A_1 = (U : v : w), \qquad B_1 = (u : V : w), \qquad C_1 = (u : v : W). \qquad (1)$$

The perspector is $P = (u : v : w)$. Let $M = (f : g : h)$ be a point outside the sidelines of ABC and $A_1B_1C_1$. Explicitly,

$$A' = (0 : gU - fv : hU - fw) \quad \text{and} \quad B' = (fV - gu : 0 : hV - gw).$$

[3]Let X be the point with coordinates $(\ell : m : n)$. The point H can be taken as the isotomic conjugate of the point Y which divides the segment XG in the ratio $XG : GY = 1 : 2$.

The lines A_1B' and B_1A' are given by [4]

$$\begin{aligned}(gw-hV)vx+(hUV-gwU-fwV+guw)y+(fV-gu)vz &= 0,\\ (gwU-fvw-hUV+fwV)x+(hU-fw)uy+(fv-gU)uz &= 0.\end{aligned}$$

These lines intersect in the point C'' with coordinates

$$(guU:fvV:fwV+gwU-hUV)\sim\left(\frac{uU}{f}:\ \frac{vV}{g}:\ \frac{fwV+gwU-hUV}{fg}\right).$$

With similar results for A'' and B'' we have the perspectivity of $A''B''C''$ and ABC at the point

$$N=\left(\frac{uU}{f}:\frac{vV}{g}:\frac{wW}{h}\right).$$

The points M and N clearly correspond to one another under the reciprocal $(uU:vV:wW)$-conjugation. □

Theorem 3. *Let P, Q, R be collinear points. Denote by X, Y, Z the traces of R on the side lines BC, CA, AB of triangle ABC, and construct triangle $A_1B_1C_1$ with vertices*

$$A_1=PA\cap QX,\quad B_1=PB\cap QY,\quad C_1=PC\cap QZ. \tag{2}$$

Triangle $A_1B_1C_1$ is perspective with ABC, and induces the reciprocal conjugation under which P and Q correspond.

Proof. If $P=(u:v:w)$, $Q=(U:V:W)$, and $R=(1-t)P+tQ$ for some $t\neq 0,1$, then

$$A_1=\left(\frac{-tU}{1-t}:v:w\right),\qquad B_1=\left(u:\frac{-tV}{1-t}:w\right),\qquad C_1=\left(u:v:\frac{-tW}{1-t}\right).$$

The result now follows from Proposition 2 and its proof. □

Proposition 2 and Theorem 3 together furnish a construction of $\tau(M)$ for an arbitrary point M (outside the side lines of ABC) under the conjugation τ defined by two distinct points P and Q. In particular, the pole P_0 can be constructed by applying to the triangle $A_1B_1C_1$ in (2) and M the centroid of ABC in the construction of Proposition 2.

Corollary 4. *Let P_0 be a point different from the centroid G of triangle ABC, regarded as the pole of a reciprocal conjugation τ. To construct $\tau(M)$, apply the construction in Theorem 3 to $(P,Q)=(G,P_0)$. The choice of R can be arbitrary, say, the midpoint of GP_0.*

Remark. This construction does not apply to isotomic conjugation, for which the pole is the centroid.

[4]Here we have made use of the fact that the line B_1C_1 is given by the equation

$$w(v-V)x+w(u-U)y+(UV-uv)z=0,$$

so that we indeed can divide by $fw(v-V)+gw(u-U)+h(UV-uv)$.

4. Desmic configuration

We take a closer look of the construction in Proposition 2. Given triangle $A_1B_1C_1$ with perspector P, it is known that the triangle $A_2B_2C_2$ with vertices

$$A_2 = BC_1 \cap CB_1, \qquad B_2 = AC_1 \cap CA_1, \qquad C_2 = AB_1 \cap BA_1,$$

is perspective to both ABC and $A_1B_1C_1$, say at points Q and R respectively, and that the perspectors P, Q, R are collinear. See, for example, [2]. Indeed, if the vertices of $A_1B_1C_1$ have coordinates given by (1), those of $A_2B_2C_2$ have coordinates

$$A_2 = (u : V : W), \quad B_2 = (U : v : W), \qquad C_2 = (U : V : w).$$

From these, it is clear that

$$Q = (U : V : W) \qquad \text{and} \qquad R = (u + U : v + V : w + W).$$

Triangle $A_2B_2C_2$ is called the *desmic mate* of triangle $A_1B_1C_1$. The three triangles, their perspectors, and the connecting lines form a *desmic configuration, i.e.*, each line contains 3 points and each point is contained in 4 lines. This configuration also contains the three desmic quadrangles: $ABCR$, $A_1B_1C_1Q$ and $A_2B_2C_2P$.

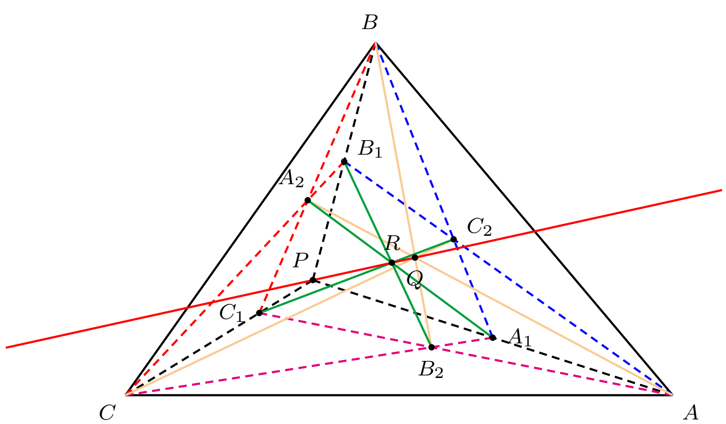


Figure 3

The construction in the preceding section shows that given collinear points P, Q, and R, there is a desmic configuration as above in which the quadrangles $A_1B_1C_1Q$ and $A_2B_2C_2P$ are perspective at R. The reciprocal conjugations induced by $A_1B_1C_1$ and $A_2B_2C_2$ are the same, and is independent of the choice of R.

Barry Wolk [3] has observed that these twelve points all lie on the *iso*-$(uU : vV : wW)$ cubic with pivot R: [5]

$$x(u+U)(wWy^2 - vVz^2) + y(v+V)(uUz^2 - wWx^2) + z(w+W)(vVx^2 - uUy^2) = 0.$$

[5] An iso-$(\ell : m : n)$ cubic with pivot R is the locus of points X for which X and its iso-$(\ell : m : n)$-conjugate are collinear with R.

Since reciprocal conjugates link the vertices of ABC to their opposite sides, clearly the traces of R are also on the cubic. By the symmetry of the desmic configurations, the traces of Q in $A_1B_1C_1$ and P in $A_2B_2C_2$ are also on the desmic cubic.

References

[1] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1–295.
[2] F.M. van Lamoen, Bicentric triangles, *Nieuw Archief voor Wiskunde*, 17 (1999) 363–372.
[3] B. Wolk, Hyacinthos message 462, http://groups.yahoo.com/group/Hyacinthos/message/462
[4] P. Yiu, The uses of homogeneous barycentric coordinates in plane euclidean geometry, *International Journal of Mathematical Education in Science and Technology*, 31 (2000) 569–578.

Keith Dean: KCP Technologies, 1150 65th Street, Emeryville, California 94662-9652, USA
E-mail address: kdean@keypress.com

Floor van Lamoen: Statenhof 3, 4463 TV Goes, The Netherlands
E-mail address: f.v.lamoen@wxs.nl

Forum Geometricorum
Volume 1 (2001) 121–124.

FORUM GEOM
ISSN 1534-1178

A Note on the Feuerbach Point

Lev Emelyanov and Tatiana Emelyanova

Abstract. The circle through the feet of the internal bisectors of a triangle passes through the Feuerbach point, the point of tangency of the incircle and the nine-point circle.

The famous Feuerbach theorem states that the nine-point circle of a triangle is tangent internally to the incircle and externally to each of the excircles. Given triangle ABC, the Feuerbach point F is the point of tangency with the incircle. There exists a family of cevian circumcircles passing through the Feuerbach point. Most remarkable are the cevian circumcircles of the incenter and the Nagel point.[1] In this note we give a geometric proof in the incenter case.

Theorem. *The circle passing through the feet of the internal bisectors of a triangle contains the Feuerbach point of the triangle.*

The proof of the theorem is based on two facts: the triangle whose vertices are the feet of the internal bisectors and the Feuerbach triangle are (a) similar and (b) perspective.

Lemma 1. *In Figure 1, circle $O(R)$ is tangent externally to each of circles $O_1(r_1)$ and $O_2(r_2)$, at A and B respectively. If A_1B_1 is a segment of an external common tangent to the circles (O_1) and (O_2), then*

$$AB = \frac{R}{\sqrt{(R+r_1)(R+r_2)}} \cdot A_1B_1. \tag{1}$$

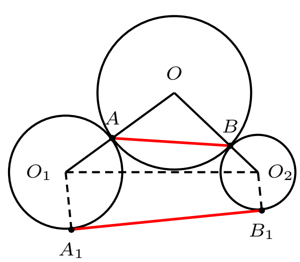


Figure 1

Proof. In the isosceles triangle AOB, $\cos AOB = \frac{2R^2-AB^2}{2R^2} = 1-\frac{AB^2}{2R^2}$. Applying the law of cosines to triangle O_1OO_2, we have

Publication Date: September 4, 2001. Communicating Editor: Paul Yiu.

[1]The cevian feet of the Nagel point are the points of tangency of the excircles with the corresponding sides.

$$\begin{aligned} O_1O_2^2 &= (R+r_1)^2 + (R+r_2)^2 - 2(R+r_1)(R+r_2)\left(1 - \frac{AB^2}{2R^2}\right) \\ &= (r_1-r_2)^2 + (R+r_1)(R+r_2)\left(\frac{AB}{R}\right)^2. \end{aligned}$$

From trapezoid $A_1O_1O_2B_1$, $O_1O_2^2 = (r_1-r_2)^2 + A_1B_1^2$. Comparison now gives A_1B_1 as in (1). □

Consider triangle ABC with side lengths $BC = a$, $CA = b$, $AB = c$, and circumcircle $O(R)$. Let $I_3(r_3)$ be the excircle on the side AB.

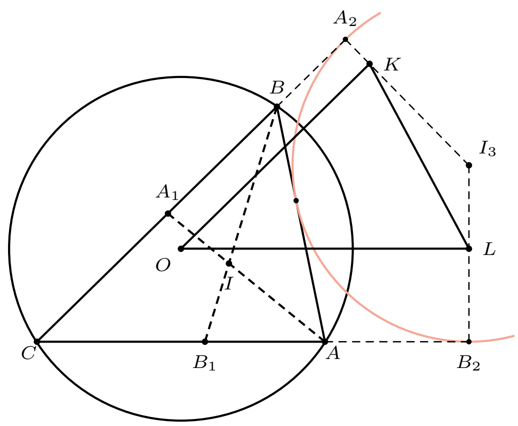


Figure 2

Lemma 2. *If A_1 and B_1 are the feet of the internal bisectors of angles A and B, then*

$$A_1B_1 = \frac{abc\sqrt{R(R+2r_3)}}{(c+a)(b+c)R}. \tag{2}$$

Proof. In Figure 2, let K and L be points on I_3A_2 and I_3B_2 such that $OK//CB$, and $OL//CA$. Since $CA_2 = CB_2 = \frac{a+b+c}{2}$,

$$OL = \frac{a+b+c}{2} - \frac{b}{2} = \frac{c+a}{2}, \qquad OK = \frac{a+b+c}{2} - \frac{a}{2} = \frac{b+c}{2}.$$

Also,

$$CB_1 = \frac{ba}{c+a}, \qquad CA_1 = \frac{ab}{b+c},$$

and

$$\frac{CB_1}{CA_1} = \frac{b+c}{c+a} = \frac{OK}{OL}.$$

Thus, triangle CA_1B_1 is similar to triangle OLK, and

$$\frac{A_1B_1}{LK} = \frac{CB_1}{OK} = \frac{2ab}{(c+a)(b+c)}. \tag{3}$$

Since OI_3 is a diameter of the circle through O, L, K, by the law of sines,

$$LK = OI_3 \cdot \sin LOK = OI_3 \cdot \sin C = OI_3 \cdot \frac{c}{2R}. \tag{4}$$

Combining (3), (4) and Euler's formula $OI_3^2 = R(R+2r_3)$, we obtain (2). □

Now, we prove the main theorem.

(a) Consider the nine-point circle $N(\frac{R}{2})$ tangent to the A- and B-excircles. See Figure 3. The length of the external common tangent of these two excircles is

$$XY = AY + BX - AB = \frac{a+b+c}{2} + \frac{a+b+c}{2} - c = a+b.$$

By Lemma 1,

$$F_1F_2 = \frac{(a+b)\cdot\frac{R}{2}}{\sqrt{(\frac{R}{2}+r_1)(\frac{R}{2}+r_2)}} = \frac{(a+b)R}{\sqrt{(R+2r_1)(R+2r_2)}}.$$

Comparison with (2) gives

$$\frac{A_1B_1}{F_1F_2} = \frac{abc\sqrt{R(R+2r_1)(R+2r_2)(R+2r_3)}}{(a+b)(b+c)(c+a)R^2}.$$

The symmetry of this ratio in a, b, c and the exradii shows that

$$\frac{A_1B_1}{F_1F_2} = \frac{B_1C_1}{F_2F_3} = \frac{C_1A_1}{F_3F_1}.$$

It follows that the triangles $A_1B_1C_1$ and $F_1F_2F_3$ are similar.

(b) We prove that the points F, B_1 and F_2 are collinear. By the Feuerbach theorem, F is the homothetic center of the incircle and the nine-point circle, and F_2 is the internal homothetic center of the nine-point circle and the B- excircle. Note that B_1 is the internal homothetic center of the incircle and the B-excircle. These three homothetic centers divide the side lines of triangle I_2NI in the ratios

$$\frac{NF}{FI} = -\frac{R}{2r}, \qquad \frac{IB_1}{B_1I_2} = \frac{r}{r_2}, \qquad \frac{I_2F_2}{F_2N} = \frac{2r_2}{R}.$$

Since

$$\frac{NF}{FI} \cdot \frac{IB_1}{B_1I_2} \cdot \frac{I_2F_2}{F_2N} = -1,$$

by the Menelaus theorem, F, B_1, and F_2 are collinear. Similarly F, C_1, F_3 are collinear, as are F, A_1, F_1. This shows that triangles $A_1B_1C_1$ and $F_1F_2F_3$ are perspective at F.

From (a) and (b) it follows that

$$\angle C_1FA_1 + \angle C_1B_1A_1 = \angle F_3FF_1 + \angle F_3F_2F_1 = 180^\circ,$$

i.e., the circle $A_1B_1C_1$ contains the Feuerbach point F.

This completes the proof of the theorem.

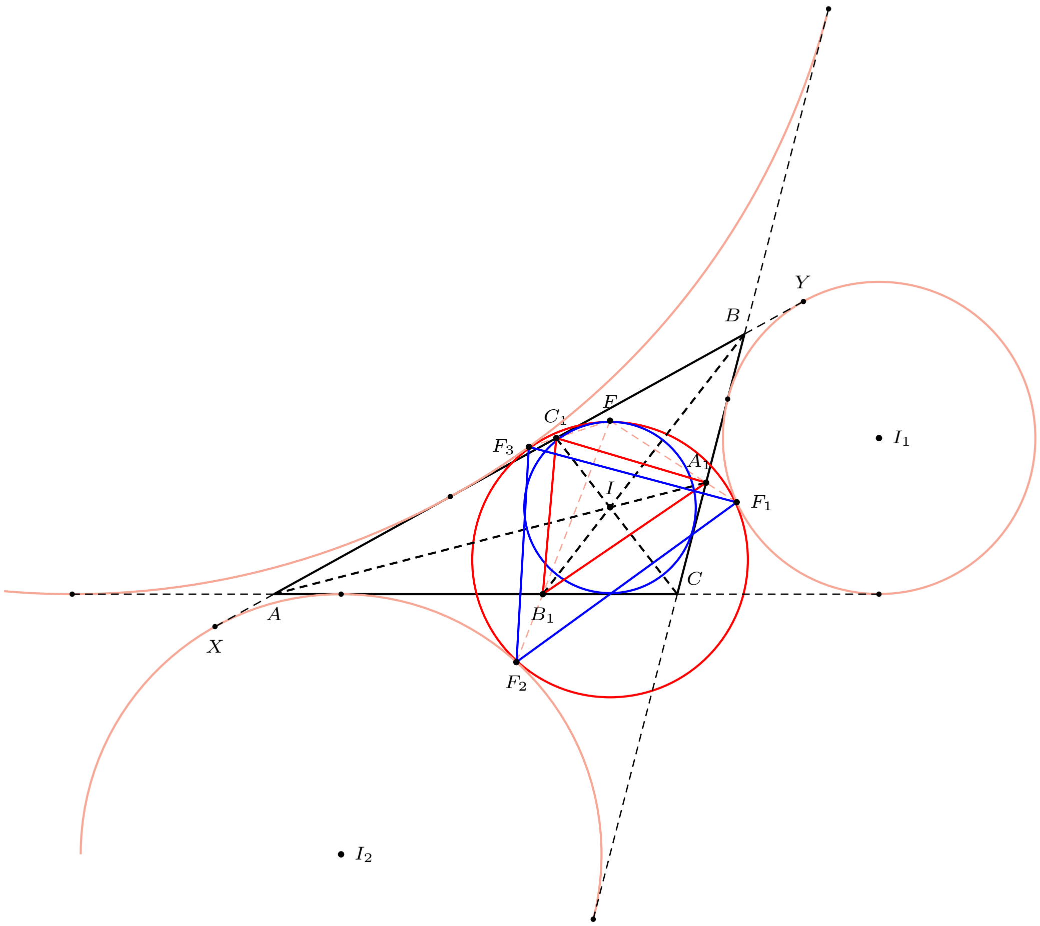


Figure 3

Lev Emelyanov: 18-31 Proyezjaia Street, Kaluga, Russia 248009
E-mail address: `emelyanov@kaluga.ru`

Tatiana Emelyanova: 18-31 Proyezjaia Street, Kaluga, Russia 248009
E-mail address: `emelyanov@kaluga.ru`

Forum Geometricorum
Volume 1 (2001) 125–132.

FORUM GEOM
ISSN 1534-1178

The Kiepert Pencil of Kiepert Hyperbolas

Floor van Lamoen and Paul Yiu

Abstract. We study Kiepert triangles $\mathcal{K}(\phi)$ and their iterations $\mathcal{K}(\phi, \psi)$, the Kiepert triangles $\mathcal{K}(\psi)$ relative to Kiepert triangles $\mathcal{K}(\phi)$. For arbitrary ϕ and ψ, we show that $\mathcal{K}(\phi, \psi) = \mathcal{K}(\psi, \phi)$. This iterated Kiepert triangle is perspective to each of ABC, $\mathcal{K}(\phi)$, and $\mathcal{K}(\psi)$. The Kiepert hyperbolas of $\mathcal{K}(\phi)$ form a pencil of conics (rectangular hyperbolas) through the centroid, and the two infinite points of the Kiepert hyperbola of the reference triangle. The centers of the hyperbolas in this Kiepert pencils are on the line joining the Fermat points of the medial triangle of ABC. Finally we give a construction of the degenerate Kiepert triangles. The vertices of these triangles fall on the parallels through the centroid to the asymptotes of the Kiepert hyperbola.

1. Preliminaries

Given triangle ABC with side lengths a, b, c, we adopt the notation of John H. Conway. Let S denote *twice* the area of the triangle, and for every θ, write $S_\theta = S \cdot \cot\theta$. In particular, from the law of cosines,

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2}.$$

The sum $S_A + S_B + S_C = \frac{1}{2}(a^2 + b^2 + c^2) = S_\omega$ for the Brocard angle ω of the triangle. See, for example, [2, p.266] or [3, p.47]. For convenience, a product $S_\phi \cdot S_\psi \cdots$ is simply written as $S_{\phi\psi\cdots}$. We shall make use of the following fundamental formulae.

Lemma 1 (Conway)**.** *The following relations hold:*
(a) $a^2 = S_B + S_C$, $b^2 = S_C + S_A$, *and* $c^2 = S_A + S_B$*;*
(b) $S_A + S_B + S_C = S_\omega$*;*
(c) $S_{AB} + S_{BC} + S_{CA} = S^2$*;*
(d) $S_{ABC} = S^2 \cdot S_\omega - a^2b^2c^2$.

Proposition 2 (Distance formula)**.** *The square distance between two points with* absolute *barycentric coordinates* $P = x_1A + y_1B + z_1C$ *and* $Q = x_2A + y_2B + z_2C$ *is given by*

$$|PQ|^2 = S_A(x_1 - x_2)^2 + S_B(y_1 - y_2)^2 + S_C(z_1 - z_2)^2. \tag{1}$$

Publication Date: September 11, 2001. Communicating Editor: Jean-Pierre Ehrmann.

Proposition 3 (Conway). *Let P be a point such that the directed angles PBC and PCB are respectively ϕ and ψ. The homogeneous barycentric coordinates of P are*

$$(-a^2 \ : \ S_C + S_\psi \ : \ S_B + S_\phi).$$

Since the cotangent function has period π, we may always choose ϕ and ψ in the range $-\frac{\pi}{2} < \phi, \ \psi \leq \frac{\pi}{2}$. See Figure 1.

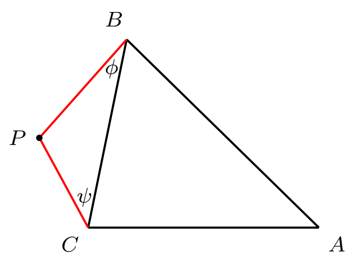


Figure 1

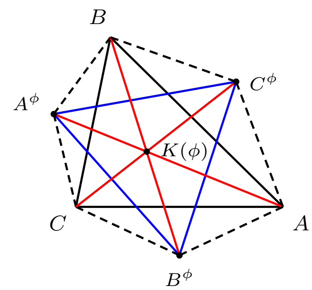


Figure 2

2. The Kiepert triangle $\mathcal{K}(\phi)$

Given an angle ϕ, let A^ϕ, B^ϕ, C^ϕ be the apexes of isosceles triangles on the sides of ABC with base angle ϕ. These are the points

$$\begin{array}{rcl} A^\phi &=& (-a^2 : S_C + S_\phi : S_B + S_\phi), \\ B^\phi &=& (S_C + S_\phi : -b^2 : S_A + S_\phi), \\ C^\phi &=& (S_B + S_\phi : S_A + S_\phi : -c^2). \end{array} \tag{2}$$

They form the *Kiepert triangle* $\mathcal{K}(\phi)$, which is perspective to ABC at the *Kiepert perspector*

$$K(\phi) = \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C + S_\phi} \right). \tag{3}$$

See Figure 2. If $\phi = \frac{\pi}{2}$, this perspector is the orthocenter H. The Kiepert triangle $\mathcal{K}(\frac{\pi}{2})$ is one of three degenerate Kiepert triangles. Its vertices are the infinite points in the directions of the altitudes. The other two are identified in §2.3 below.

The Kiepert triangle $\mathcal{K}(\phi)$ has the same centroid $G = (1 : 1 : 1)$ as the reference triangle ABC. This is clear from the coordinates given in (2) above.

2.1. *Side lengths.* We denote by a_ϕ, b_ϕ, and c_ϕ the lengths of the sides $B^\phi C^\phi$, $C^\phi A^\phi$, and $A^\phi B^\phi$ of the Kiepert triangle $\mathcal{K}(\phi)$. If $\phi \neq \frac{\pi}{2}$, these side lengths are given by

$$\begin{array}{rcl} 4S_\phi^2 \cdot a_\phi^2 &=& a^2 S_\phi^2 + S^2(4S_\phi + S_\omega + 3S_A), \\ 4S_\phi^2 \cdot b_\phi^2 &=& b^2 S_\phi^2 + S^2(4S_\phi + S_\omega + 3S_B), \\ 4S_\phi^2 \cdot c_\phi^2 &=& c^2 S_\phi^2 + S^2(4S_\phi + S_\omega + 3S_C). \end{array} \tag{4}$$

Here is a simple relation among these side lengths.

Proposition 4. *If $\phi \neq \frac{\pi}{2}$,*

$$b_\phi^2 - c_\phi^2 = \frac{1 - 3\tan^2\phi}{4} \cdot (b^2 - c^2);$$

similarly for $c_\phi^2 - a_\phi^2$ and $a_\phi^2 - b_\phi^2$.

If $\phi = \pm\frac{\pi}{6}$, we have $b_\phi^2 = c_\phi^2 = a_\phi^2$, and the triangle is equilateral. This is Napoleon's theorem.

2.2. *Area.* Denote by S' twice the area of $\mathcal{K}(\phi)$. If $\phi \neq \frac{\pi}{2}$,

$$S' = \frac{S}{(2S_\phi)^3}\begin{vmatrix} -a^2 & S_C + S_\phi & S_B + S_\phi \\ S_C + S_\phi & -b^2 & S_A + S_\phi \\ S_B + S_\phi & S_A + S_\phi & -c^2 \end{vmatrix} = \frac{S}{4S_\phi^2}(S_\phi^2 + 2S_\omega S_\phi + 3S^2). \tag{5}$$

2.3. *Degenerate Kiepert triangles.* The Kiepert triangle $\mathcal{K}(\phi)$ degenerates into a line when $\phi = \frac{\pi}{2}$ as we have seen above, or $S' = 0$. From (5), this latter is the case if and only if $\phi = \omega_\pm$ for

$$\cot\omega_\pm = -\cot\omega \pm \sqrt{\cot^2\omega - 3}. \tag{6}$$

See §5.1 and Figures 8A,B for the construction of the two finite degenerate Kiepert triangles.

2.4. *The Kiepert hyperbola.* It is well known that the locus of the Kiepert perspectors is the Kiepert hyperbola

$$\mathcal{K}: \qquad (b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy = 0.$$

See, for example, [1]. In this paper, we are dealing with the Kiepert hyperbolas of various triangles. This particular one (of the reference triangle) will be referred to as the *standard* Kiepert hyperbola. It is the rectangular hyperbola with asymptotes the Simson lines of the intersections of the circumcircle with the Brocard axis OK (joining the circumcenter and the symmedian point). Its center is the point $((b^2 - c^2)^2 : (c^2 - a^2)^2 : (a^2 - b^2)^2)$ on the nine-point circle. The asymptotes, regarded as infinite points, are the points $K(\phi)$ for which

$$\frac{1}{S_A + S_\phi} + \frac{1}{S_B + S_\phi} + \frac{1}{S_C + S_\phi} = 0.$$

These are $I_\pm = K(\frac{\pi}{2} - \omega_\pm)$ for $\omega_\pm$ given by (6) above.

3. Iterated Kiepert triangles

Denote by A', B', C' the magnitudes of the angles A^ϕ, B^ϕ, C^ϕ of the Kiepert triangle $\mathcal{K}(\phi)$. From the expressions of the side lengths in (4), we have

$$S'_{A'} = \frac{1}{4S_\phi^2}(S_A S_\phi^2 + 2S^2 S_\phi + S^2(2S_\omega - 3S_A)) \tag{7}$$

together with two analogous expressions for $S'_{B'}$ and $S'_{C'}$.

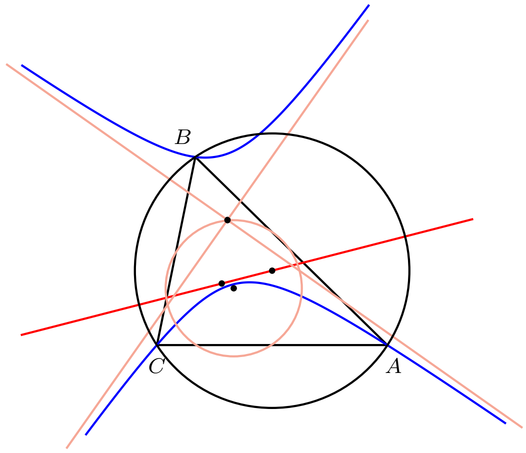


Figure 3

Consider the Kiepert triangle $\mathcal{K}(\psi)$ of $\mathcal{K}(\phi)$. The coordinates of the apex $A^{\phi,\psi}$ with respect to $\mathcal{K}(\phi)$ are $(-a_\phi^2 : S'_{C'} + S'_\psi : S'_{B'} + S'_\psi)$. Making use of (5) and (7), we find the coordinates of the vertices of $\mathcal{K}(\phi,\psi)$ with reference to ABC, as follows.

$$A^{\phi,\psi} = (-(2S^2 + a^2(S_\phi + S_\psi) + 2S_{\phi\psi}) : S^2 - S_{\phi\psi} + S_C(S_\phi + S_\psi) : S^2 - S_{\phi\psi} + S_B(S_\phi + S_\psi)),$$
$$B^{\phi,\psi} = (S^2 - S_{\phi\psi} + S_C(S_\phi + S_\psi) : -(2S^2 + b^2(S_\phi + S_\psi) + 2S_{\phi\psi}) : S^2 - S_{\phi\psi} + S_A(S_\phi + S_\psi)),$$
$$C^{\phi,\psi} = (S^2 - S_{\phi\psi} + S_B(S_\phi + S_\psi) : S^2 - S_{\phi\psi} + S_A(S_\phi + S_\psi) : -(2S^2 + c^2(S_\phi + S_\psi) + 2S_{\phi\psi})).$$

From these expressions we deduce a number of interesting properties of the iterated Kiepert triangles.

1. The symmetry in ϕ and ψ of these coordinates shows that the triangles $\mathcal{K}(\phi,\psi)$ and $\mathcal{K}(\psi,\phi)$ coincide.

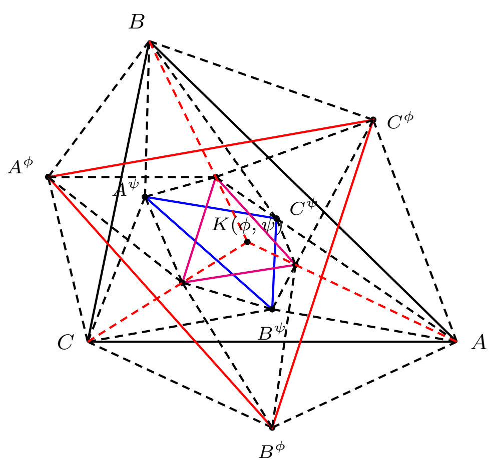


Figure 4

2. It is clear that the iterated Kiepert triangle $\mathcal{K}(\phi,\psi)$ is perspective with each of $\mathcal{K}(\phi)$ and $\mathcal{K}(\psi)$, though the coordinates of the perspectors $K_\phi(\psi)$ and $K_\psi(\phi)$ are very tedious. It is interesting, however, to note that $\mathcal{K}(\phi,\psi)$ is also perspective with ABC. See Figure 4. The perspector has relatively simple coordinates:

$$K(\phi,\psi)=\left(\frac{1}{S^2+S_A(S_\phi+S_\psi)-S_{\phi\psi}}:\frac{1}{S^2+S_B(S_\phi+S_\psi)-S_{\phi\psi}}:\frac{1}{S^2+S_C(S_\phi+S_\psi)-S_{\phi\psi}}\right).$$

3. This perspector indeed lies on the Kiepert hyperbola of ABC; it is the Kiepert perspector $K(\theta)$, where

$$\cot\theta=\frac{1-\cot\phi\cot\psi}{\cot\phi+\cot\psi}=-\cot(\phi+\psi).$$

In other words,

$$K(\phi,\psi)=K(-(\phi+\psi)). \tag{8}$$

From this we conclude that the Kiepert hyperbola of $\mathcal{K}(\phi)$ has the same infinite points of the standard Kiepert hyperbola, *i.e.*, their asymptotes are parallel.

4. The triangle $\mathcal{K}(\phi,-\phi)$ is homothetic to ABC at G, with ratio of homothety $\frac{1}{4}(1-3\tan^2\phi)$. Its vertices are

$$\begin{aligned}
A^{\phi,-\phi} &= (-2(S^2-S_\phi^2):S^2+S_\phi^2:S^2+S_\phi^2),\\
B^{\phi,-\phi} &= (S^2+S_\phi^2:-2(S^2-S_\phi^2):S^2+S_\phi^2),\\
C^{\phi,-\phi} &= (S^2+S_\phi^2:S^2+S_\phi^2:-2(S^2-S_\phi^2)).
\end{aligned}$$

See also [4].

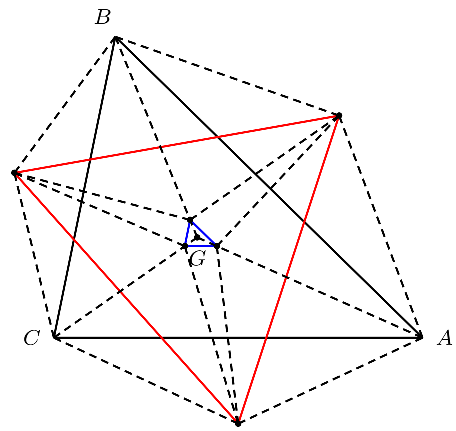


Figure 5

4. The Kiepert hyperbola of $\mathcal{K}(\phi)$

Since the Kiepert triangle $\mathcal{K}(\phi)$ has centroid G, its Kiepert hyperbola $\mathcal{K}_\phi$ contains G. We show that it also contains the circumcenter O.

Proposition 5. *If* $\phi\neq\frac{\pi}{2},\ \pm\frac{\pi}{6}$, $O=K_\phi(-(\frac{\pi}{2}-\phi))$.

Proof. Let $\psi = -(\frac{\pi}{2} - \phi)$, so that $S_\psi = -\frac{S^2}{S_\phi}$. Note that

$$A^{\phi,\psi} = \left(-a^2 : \frac{2S^2 S_\phi}{S^2 + S_\phi^2} + S_C : \frac{2S^2 S_\phi}{S^2 + S_\phi^2} + S_B\right),$$

while

$$A^\phi = (-a^2 : S_C + S_\phi : S_B + S_\phi).$$

These two points are distinct unless $\phi = \frac{\pi}{2}, \pm\frac{\pi}{6}$. Subtracting these two coordinates we see that the line $\ell_a := A^\phi A^{\phi,\psi}$ passes through $(0 : 1 : 1)$, the midpoint of BC. This means, by the construction of A^ϕ, that ℓ_a is indeed the perpendicular bisector of BC, and thus passes through O. By symmetry this proves the proposition. □

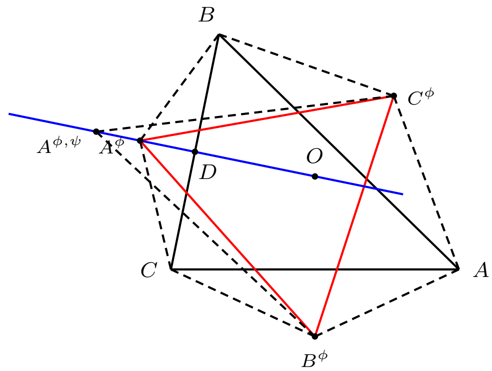


Figure 6

The Kiepert hyperbolas of the Kiepert triangles therefore form the pencil of conics through the centroid G, the circumcenter O, and the two infinite points of the standard Kiepert hyperbola. The Kiepert hyperbola $\mathcal{K}_\phi$ is the one in the pencil that contains the Kiepert perspector $K(\phi)$, since $K(\phi) = K_\phi(-2\phi)$ according to (8). Now, the line containing $K(\phi)$ and the centroid has equation

$$(b^2 - c^2)(S_A + S_\phi)x + (c^2 - a^2)(S_B + S_\phi)y + (a^2 - b^2)(S_C + S_\phi)z = 0.$$

It follows that the equation of $\mathcal{K}_\phi$ is of the form

$$\sum_{\text{cyclic}}(b^2 - c^2)yz + \lambda(x + y + z)(\sum_{\text{cyclic}}(b^2 - c^2)(S_A + S_\phi)x) = 0,$$

where λ is determined by requiring that the conic passes through the circumcenter $O = (a^2 S_A : b^2 S_B : c^2 S_C)$. This gives $\lambda = \frac{1}{2S_\phi}$, and the equation of the conic can be rewritten as

$$2S_\phi(\sum_{\text{cyclic}}(b^2 - c^2)yz) + (x + y + z)(\sum_{\text{cyclic}}(b^2 - c^2)(S_A + S_\phi)x) = 0.$$

Several of the hyperbolas in the pencil are illustrated in Figure 7.

The locus of the centers of the conics in a pencil is in general a conic. In the case of the Kiepert pencil, however, this locus is a line. This is clear from Proposition 4 that the center of $\mathcal{K}_\phi$ has coordinates

$$((b^2 - c^2)^2 : (c^2 - a^2)^2 : (a^2 - b^2)^2)$$

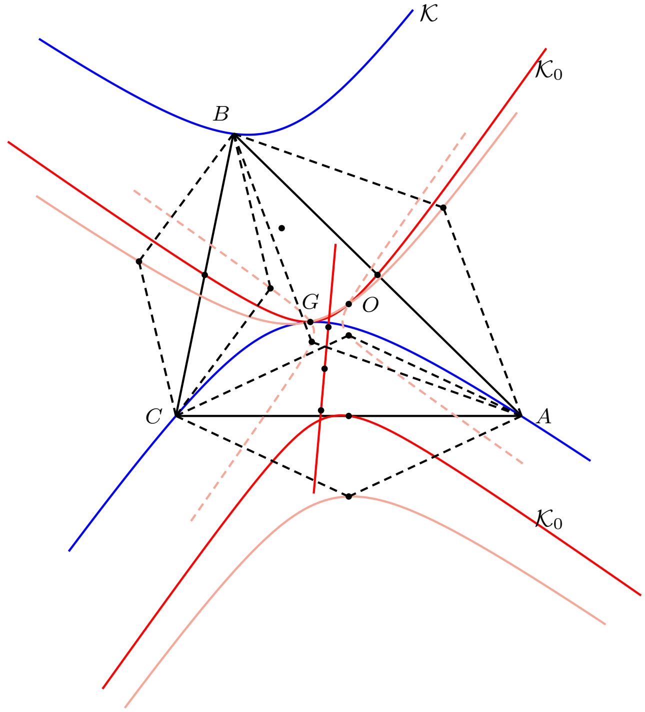


Figure 7

relative to $A^\phi B^\phi C^\phi$, and from (2) that the coordinates of A^ϕ, B^ϕ, C^ϕ are linear functions of S_ϕ. This is the line joining the Fermat points of the medial triangle.

5. Concluding remarks

5.1. *Degenerate Kiepert conics.* There are three degenerate Kiepert triangles corresponding to the three degenerate members of the Kiepert pencil, which are the three pairs of lines connecting the four points G, O, $I_\pm = K(\frac{\pi}{2} - \omega_\pm)$ defining the pencil. The Kiepert triangles $\mathcal{K}(\omega_\pm)$ degenerate into the straight lines $GI_\mp$. The vertices are found by intersecting the line with the perpendicular bisectors of the sides of ABC. The centers of these degenerate Kiepert conics are also on the circle with OG as diameter.

5.2. *The Kiepert hyperbolas of the Napoleon triangles.* The Napoleon triangles $\mathcal{K}(\pm\frac{\pi}{6})$ being equilateral do not posses Kiepert hyperbolas, the centroid being the only finite Kiepert perspector. The rectangular hyperbolas $\mathcal{K}_{\pm\pi/6}$ in the pencil are the circumconics through this common perspector G and O. The centers of these rectangular hyperbolas are the Fermat points of the medial triangle.

5.3. *Kiepert coordinates.* Every point outside the standard Kiepert hyperbola $\mathcal{C}$, and other than the circumcenter O, lies on a unique member of the Kiepert pencil, *i.e.*, it can be *uniquely* written as $K_\phi(\psi)$. As an example, the symmedian point

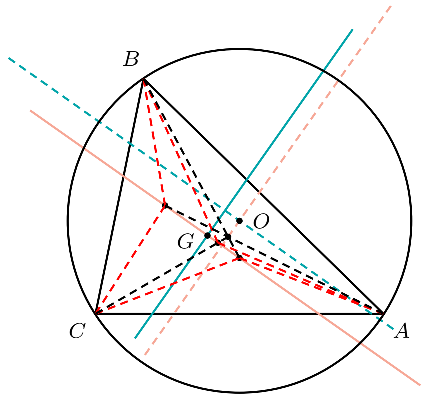


Figure 8A

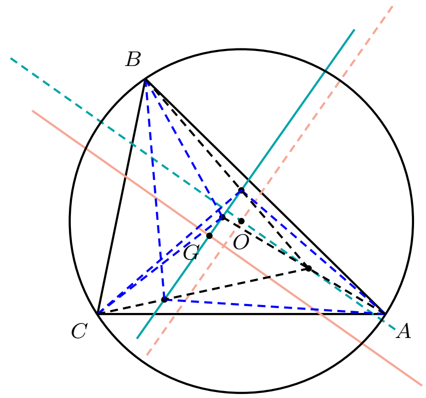


Figure 8B

$K = K_\phi(\psi)$ for $\phi = \omega$ (the Brocard angle) and $\psi = \text{arccot}(\frac{1}{3}\cot\omega)$. We leave the details to the readers.

References

[1] R.H. Eddy and R. Fritsch, The conics of Ludwig Kiepert: a comprehensive lesson in the geometry of the triangle, *Mathematics Magazine*, 67 (1994) 188–205.

[2] R.A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, 1960.

[3] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1 – 295.

[4] F.M. van Lamoen, Circumrhombi to a triangle, to appear in *Forum Geom.*

Floor van Lamoen: Statenhof 3, 4463 TV Goes, The Netherlands
E-mail address: `f.v.lamoen@wxs.nl`

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida, 33431-0991, USA
E-mail address: `yiu@fau.edu`

Forum Geometricorum
Volume 1 (2001) 132a

FORUM GEOM
ISSN 1534-1178

September 11, 2001

We express our deep sympathy for the families and friends of the victims of terrorism on this day, and extend our sincere thanks to all fellow human beings who participate in the rescue and relief work.

The Editors,
Forum Geometricorum

Forum Geometricorum
Volume 1 (2001) 133–136.

FORUM GEOM
ISSN 1534-1178

Simple Constructions of the Incircle of an Arbelos

Peter Y. Woo

Abstract. We give several simple constructions of the incircle of an arbelos, also known as a shoemaker's knife.

Archimedes, in his *Book of Lemmas*, studied the arbelos bounded by three semicircles with diameters AB, AC, and CB, all on the same side of the diameters.[1] See Figure 1. Among other things, he determined the radius of the incircle of the arbelos. In Figure 2, GH is the diameter of the incircle parallel to the base AB, and G', H' are the (orthogonal) projections of G, H on AB. Archimedes showed that $GHH'G'$ is a square, and that AG', $G'H'$, $H'B$ are in geometric progression. See [1, pp. 307–308].

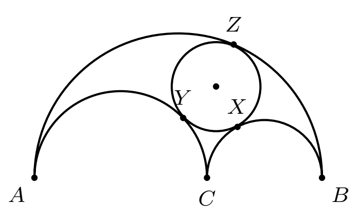


Figure 1

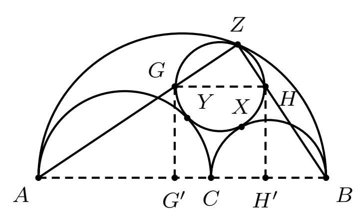
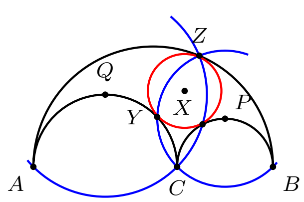
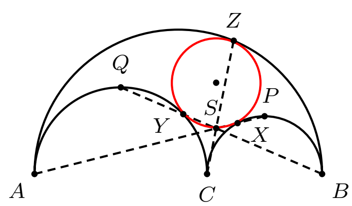


Figure 2

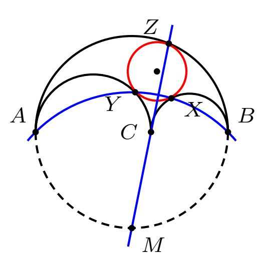
In this note we give several simple constructions of the incircle of the arbelos. The elegant Construction 1 below was given by Leon Bankoff [2]. The points of tangency are constructed by drawing circles with centers at the midpoints of two of the semicircles of the arbelos. In validating Bankoff's construction, we obtain Constructions 2 and 3, which are easier in the sense that one is a ruler-only construction, and the other makes use only of the midpoint of one semicircle.



Construction 1



Construction 2



Construction 3

Publication Date: September 18, 2001. Communicating Editor: Paul Yiu.

[1]The arbelos is also known as the shoemaker's knife. See [3].

Theorem 1 (Bankoff [2])**.** *Let P and Q be the midpoints of the semicircles (BC) and (AC) respectively. If the incircle of the arbelos is tangent to the semicircles (BC), (AC), and (AB) at X, Y, Z respectively, then*

(i) A, C, X, Z lie on a circle, center Q;

(ii) B, C, Y, Z lie on a circle, center P.

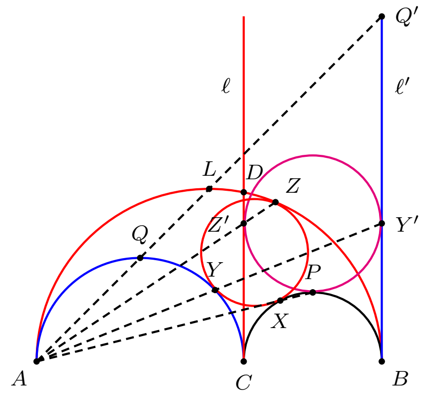


Figure 3

Proof. Let D be the intersection of the semicircle (AB) with the line perpendicular to AB at C. See Figure 3. Note that $AB \cdot AC = AD^2$ by Euclid's proof of the Pythagorean theorem. [2] Consider the inversion with respect to the circle $A(D)$. This interchanges the points B and C, and leaves the line AB invariant. The inversive images of the semicircles (AB) and (AC) are the lines ℓ and ℓ perpendicular to AB at C and B respectively. The semicircle (BC), being orthogonal to the invariant line AB, is also invariant under the inversion. The incircle XYZ of the arbelos is inverted into a circle tangent to the semicircle (BC), and the lines ℓ, ℓ, at P, Y', Z' respectively. Since the semicircle (BC) is invariant, the points A, X, and P are collinear. The points Y' and Z' are such that BPZ' and CPY' are lines making $45°$ angles with the line AB. Now, the line BPZ' also passes through the midpoint L of the semicircle (AB). The inversive image of this line is a circle passing through A, C, X, Z. Since inversion is conformal, this circle also makes a $45°$ angle with the line AB. Its center is therefore the midpoint Q of the semicircle (AC). This proves that the points X and Z lies on the circle $Q(A)$.

The same reasoning applied to the inversion in the circle $B(D)$ shows that Y and Z lie on the circle $P(B)$. □

Theorem 1 justifies Construction 1. The above proof actually gives another construction of the incircle XYZ of the arbelos. It is, first of all, easy to construct the circle $PY'Z'$. The points X, Y, Z are then the intersections of the lines AP, AY', and AZ' with the semicircles (BC), (CA), and (AB) respectively. The following two interesting corollaries justify Constructions 2 and 3.

[2]*Euclid's Elements*, Book I, Proposition 47.

Corollary 2. *The lines AX, BY, and CZ intersect at a point S on the incircle XYZ of the arbelos.*

Proof. We have already proved that A, X, P are collinear, as are B, Y, Q. In Figure 4, let S be the intersection of the line AP with the circle XYZ. The inversive image S' (in the circle $A(D)$) is the intersection of the same line with the circle $PY'Z'$. Note that

$$\angle AS'Z' = \angle PS'Z' = \angle PY'Z' = 45^\circ = \angle ABZ'$$

so that A, B, S', Z' are concyclic. Considering the inversive image of this circle, we conclude that the line CZ contains S. In other words, the lines AP and CZ intersect at the point S on the circle XYZ. Likewise, BQ and CZ intersect at the same point. $\square$

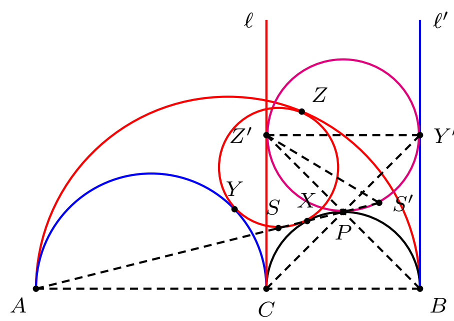


Figure 4

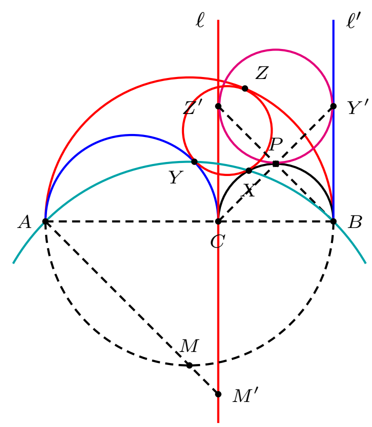


Figure 5

Corollary 3. *Let M be the midpoint of the semicircle (AB) on the opposite side of the arbelos.*

(i) The points A, B, X, Y lie on a circle, center M.

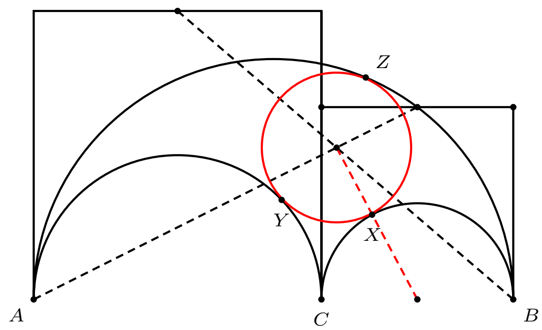
(ii) The line CZ passes through M.

Proof. Consider Figure 5 which is a modification of Figure 3. Since C, P, Y' are on a line making a 45° angle with AB, its inversive image (in the circle $A(D)$) is a circle through A, B, X, Y, also making a 45° angle with AB. The center of this circle is necessarily the midpoint M of the semicircle AB on the opposite side of the arbelos.

Join A, M to intersect the line ℓ at M'. Since $\angle BAM' = 45^\circ = \angle BZ'M'$, the four points A, Z', B, M' are concyclic. Considering the inversive image of the circle, we conclude that the line CZ passes through M. $\square$

The center of the incircle can now be constructed as the intersection of the lines joining X, Y, Z to the centers of the corresponding semicircles of the arbelos.

However, a closer look into Figure 4 reveals a simpler way of locating the center of the incircle XYZ. The circles XYZ and $PY'Z'$, being inversive images, have the center of inversion A as a center of similitude. This means that the center of the incircle XYZ lies on the line joining A to the midpoint of $Y'Z'$, which is the opposite side of the square erected on BC, on the same side of the arbelos. The same is true for the square erected on AC. This leads to the following Construction 4 of the incircle of the arbelos:



Construction 4

References

[1] T.L. Heath, *The Works of Archimedes with the Method of Archimedes*, 1912, Dover reprint; also in *Great Books of the Western World*, 11, Encyclopædia Britannica Inc., Chicago, 1952.

[2] L. Bankoff, A mere coincide, *Mathematics Newsletter*, Los Angeles City College, November 1954; reprinted in *College Math. Journal* 23 (1992) 106.

[3] C.W. Dodge, T. Schoch, P.Y. Woo, and P. Yiu, Those ubiquitous Archimedean circles, *Mathematics Magazine* 72 (1999) 202–213.

Peter Y. Woo: Department of Mathematics, Biola University, 13800 Biola Avenue, La Mirada, California 90639, USA

E-mail address: `woobiola@yahoo.com`

Forum Geometricorum
Volume 1 (2001) 137–140.

FORUM GEOM
ISSN 1534-1178

Euler's Formula and Poncelet's Porism

Lev Emelyanov and Tatiana Emelyanova

1. Introduction

It is well known [2, p. 187] that two intersecting circles $O(R)$ and $O_1(R_1)$ are the circumcircle and an excircle respectively of a triangle if and only if the Euler formula

$$d^2 = R^2 + 2RR_1, \tag{1}$$

where $d = |OO_1|$, holds. We present a possibly new proof and an application to the Poncelet porism.

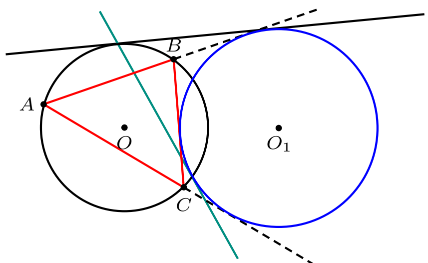


Figure 1

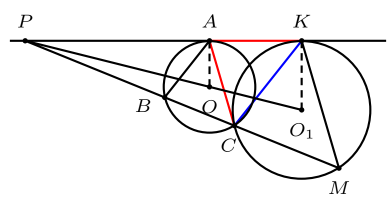


Figure 2

Theorem 1. *Intersecting circles (O) and (O_1) are the circumcircle and an excircle of a triangle if and only if the tangent to (O_1) at an intersection of the circles meets (O) again at the touch point of a common tangent.*

Proof. (Sufficiency) Let $O(R)$ and $O_1(R_1)$ be intersecting circles. (These circles are not assumed to be related to a triangle as in Figure 1.) Of the two lines tangent to both circles, let AK be one of them, as in Figure 2. Let $P = AK \cap OO_1$. Of the two points of intersection of (O) and (O_1), let C be the one not on the same side of line OO_1 as point A. Line AC is tangent to circle $O_1(R_1)$ if and only if $|AC| = |AK|$. Let B and M be the points other than C where line PC meets circles $O(R)$ and $O_1(R_1)$, respectively. Triangles ABC and KCM are homothetic with ratio $\frac{R}{R_1}$, so that $\dfrac{|AB|}{|CK|} = \dfrac{R}{R_1}$. Also, triangles ABC and CAK are similar, since $\angle ABC = \angle CAK$ and $\angle BAC = \angle ACK$. Therefore, $\dfrac{|AB|}{|AC|} = \dfrac{|AC|}{|CK|}$, so that $\dfrac{|CK|}{|AC|} \cdot \dfrac{R}{R_1} = \dfrac{|AC|}{|CK|}$, and

$$|CK| = |AC|\sqrt{\frac{R_1}{R}}. \tag{2}$$

Publication Date: October 8, 2001. Communicating Editor: Clark Kimberling.

Also,

$$\begin{aligned} |AK| &= |AC|\cos(\angle CAK) + |CK|\cos(\angle CKA) \\ &= |AC|\sqrt{1-\frac{|AC|^2}{4R^2}} + |CK|\sqrt{1-\frac{|CK|^2}{4R_1^2}}. \end{aligned} \tag{3}$$

If $|AC| = |AK|$, then equations (2) and (3) imply

$$|AK| = |AK|\sqrt{1-\frac{|AK|^2}{4R^2}} + |AK|\sqrt{\frac{R_1}{R}-\frac{|AK|^2}{4R^2}},$$

which simplifies to $|AK|^2 = 4RR_1 - R_1^2$. Since $|AK|^2 = d^2 - (R-R_1)^2$, where $d = |OO_1|$, we have the Euler formula given in (1). $\square$

We shall prove the converse below from Poncelet's porism.

2. Poncelet porism

Suppose triangle ABC has circumcircle $O(R)$ and incircle $I(r)$. The Poncelet porism is the problem of finding all triangles having the same circumcircle and incircle, and the well known solution is an infinite family of triangles. Unless triangle ABC is equilateral, these triangles vary in shape, but even so, they may be regarded as "rotating" about a fixed incircle and within a fixed circumcircle.

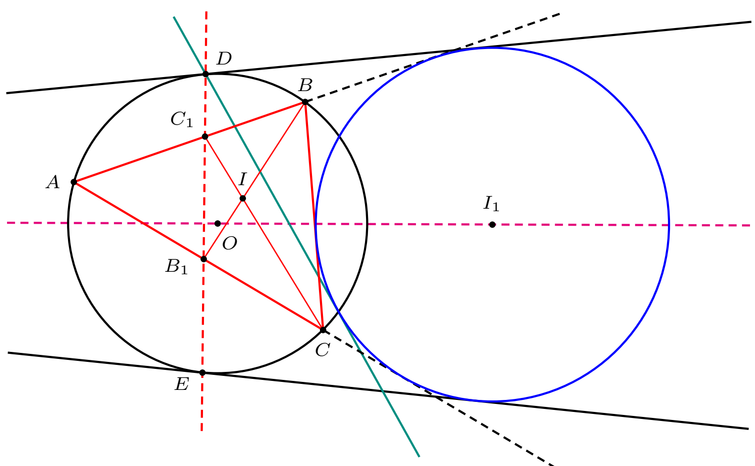


Figure 3

Continuing with the proof of the necessity part of Theorem 1, let $I_1(r_1)$ be the excircle corresponding to vertex A. Since Euler's formula holds for this configuration, the conditions for the Poncelet porism (e.g. [2, pp. 187-188]) hold. In the family of rotating triangles ABC there is one whose vertices A and B coincide in a point, D, and the limiting line AB is, in this case, tangent to the excircle. Moreover, lines CA and BC coincide as the line tangent to the excircle at a point of intersection of the circles, as in Figure 3. This completes the proof of Theorem 1.

Certain points of triangle ABC, other than the centers of the two fixed circles, stay fixed during rotation ([1, p.16-19]). We can also find a fixed line in the Poncelet porism.

Theorem 2. *For each of the rotating triangles ABC with fixed circumcircle and excircle corresponding to vertex A, the feet of bisectors BB_1 and CC_1 traverse line DE, where E is the touch point of the second common tangent.*

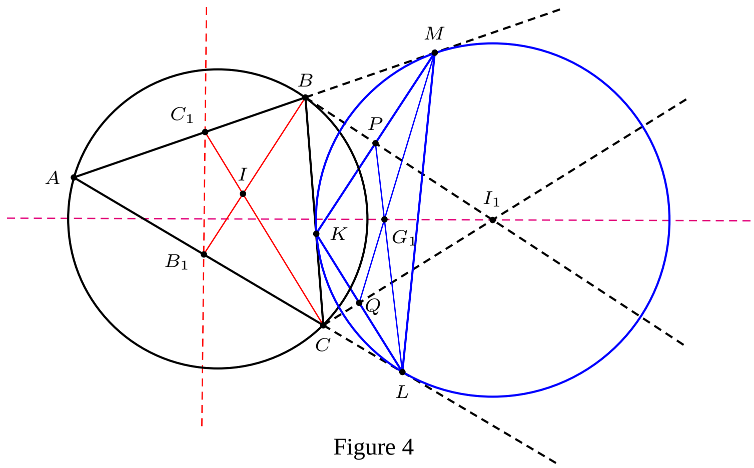


Figure 4

3. Proof of Theorem 2

We begin with the pole-polar correspondence between points and lines for the excircle with center I_1, as in Figure 4.

The polars of A, B, C are LM, MK, KL, respectively, where ΔKLM is the A-extouch triangle. As BB_1 is the internal bisector of angle B and BI_1 is the external bisector, we have $BB_1 \perp BI_1$, and the pole of BB_1 lies on the polar of B, namely MK. Therefore the pole of BB_1 is the midpoint P of segment MK. Similarly, the pole of the bisector CC_1 is the midpoint Q of segment KL. The polar of B_1 is the line passing through the poles of BB_1 and LB_1, i.e. line PL. Likewise, MQ is the polar of C_1, and the pole of B_1C_1 is centroid of triangle KLM, which we denote as G_1.

We shall prove that G_1 is fixed by proving that the orthocenter H_1 of triangle KLM is fixed. (Gallatly [1] proves that the orthocenter of the intouch triangle stays fixed in the Poncelet porism with fixed circumcircle and incircle; we offer a different proof, which applies also to the circumcircle and an excircle.)

Lemma 3. *The orthocenter H_1 of triangle KLM stays fixed as triangle ABC rotates.*

Proof. Let KLM be the extouch triangle of triangle ABC, let RST be the orthic triangle of triangle KLM, and let H_1 and E_1 be the orthocenter and nine-point center, respectively, of triangle KLM, as in Figure 5.

(1) The circumcircle of triangle RST is the nine-point circle of triangle KLM, so that its radius is equal to $\frac{1}{2}R_1$, and its center E_1 is on the Euler line I_1H_1 of triangle KLM.

(2) It is known that altitudes of an obtuse triangle are bisectors (one internal and two external) of its orthic triangle, so that H_1 is the R-excenter of triangle RST.

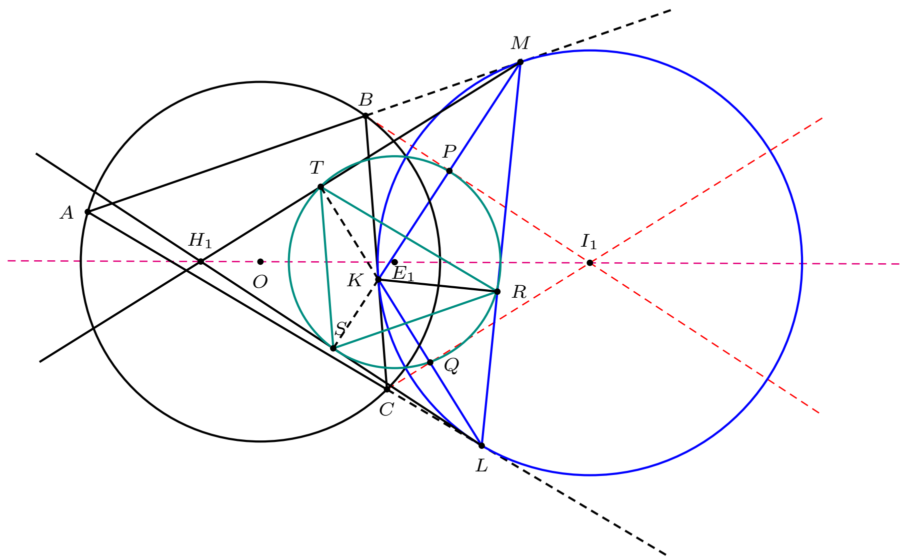


Figure 5

(3) Triangle RST and triangle ABC are homothetic. To see, for example, that $AB||RS$, we have $\angle KRL = \angle KSL = 90^\circ$, so that L, R, S, K are concyclic. Thus, $\angle KLR = \angle KSR = \angle RSM$. On the other hand, $\angle KLR = \angle KLM = \angle KMB$ and $\angle RSM = \angle SMB$. Consequently, $AB||RS$.

(4) The ratio k of homothety of triangle ABC and triangle RST is equal to the ratio of their circumradii, i.e. $k = \frac{2R}{R_1}$. Under this homothety, $O \to E_1$ (the circumcenters) and $I_1 \to H_1$ (the excenter). It follows that $OI_1||E_1H_1$. Since E_1, I_1, H_1 are collinear, O, I_1, H_1 are collinear. Thus OI_1 is the fixed Euler line of every triangle KLM.

The place of H stays fixed on OI because $EH = \frac{OI}{k}$ remains constant. Therefore the centroid of, triangle KLM also stays fixed. $\square$

To complete the proof of Theorem 2, note that by Lemma 3, G_1 is fixed on line OI_1. Therefore, line B_1C_1, as the polar of G_1, is fixed. Moreover, $B_1C_1 \perp OI_1$. Considering the degenerate case of the Poncelet porism, we conclude that B_1C_1 coincides with DE, as in Figure 3.

References

[1] W. Gallatly, *The Modern Geometry of the Triangle*, 2nd edition, Francis Hodgson, London, 1913.
[2] R.A. Johnson, *Modern Geometry*, Houghton Mifflin, Boston, 1929.

Lev Emelyanov: 18-31 Proyezjaia Street, Kaluga, Russia 248009
E-mail address: `emelyanov@kaluga.ru`

Tatiana Emelyanova: 18-31 Proyezjaia Street, Kaluga, Russia 248009
E-mail address: `emelyanov@kaluga.ru`

Forum Geometricorum
Volume 1 (2001) 141–150.

FORUM GEOM
ISSN 1534-1178

Conics Associated with a Cevian Nest

Clark Kimberling

Abstract. Various mappings in the plane of $\triangle ABC$ are defined in the context of a cevian nest consisting of $\triangle ABC$, a cevian triangle, and an anticevian triangle. These mappings arise as Ceva conjugates, cross conjugates, and cevapoints. Images of lines under these mappings and others, involving trilinear and conic-based poles and polars, include certain conics that are the focus of this article.

1. Introduction

Suppose L is a line in the plane of $\triangle ABC$, but not a sideline BC, CA, AB, and suppose a variable point Q traverses L. The isogonal conjugate of Q traces a conic called the isogonal transform of L, which, as is well known, passes through the vertices A, B, C. In this paper, we shall see that for various other transformations, the transform of L is a conic. These include Ceva and cross conjugacies, cevapoints, and pole-to-pole mappings[1]. Let

$$P = p_1 : p_2 : p_3 \tag{1}$$

be a point[2] not on a sideline of $\triangle ABC$. Suppose

$$U = u_1 : u_2 : u_3 \quad \text{and} \quad V = v_1 : v_2 : v_3 \tag{2}$$

are distinct points on L. Then L is given parametrically by

$$Q_t = u_1 + v_1 t : u_2 + v_2 t : u_3 + v_3 t, \; -\infty < t \leq \infty, \tag{3}$$

where $Q_\infty := V$. The curves in question can now be represented by the form $P*Q_t$ (or $P_t * Q$), where $*$ represents any of the various mappings to be considered. For any such curve, a parametric representation is given by the form

$$x_1(t) : x_2(t) : x_3(t),$$

Publication Date: October 18, 2001. Communicating Editor: Peter Yff.

[1]The cevian triangle of a point P not on a sideline of ABC is the triangle $A'B'C'$, where $A' = PA \cap BC$, $B' = PB \cap CA$, $CB' = PC \cap AB$. The name *cevian* (pronounced cheh´vian) honors Giovanni Ceva (pronounced Chay´va). We use a lower case c in adjectives such as *anticevian* (cf. *nonabelian*) and a capital when the name stands alone, as in *Ceva conjugate.* The name *anticevian* derives from a special case called the *anticomplementary triangle,* so named because its vertices are the anticomplements of A, B, C.

[2]Throughout, coordinates for points are homogeneous trilinear coordinates.

where the coordinates are polynomials in t having no common nonconstant polynomial factor. The degree of the curve is the maximum of the degrees of the polynomials. When this degree is 2, the curve is a conic, and the following theorem (e.g. [5, pp. 60–65]) applies.

Theorem 1. *Suppose a point $X = x_1 : x_2 : x_3$ is given parametrically by*

$$x_1 = d_1t^2 + e_1t + f_1 \tag{4}$$
$$x_2 = d_2t^2 + e_2t + f_2 \tag{5}$$
$$x_3 = d_3t^2 + e_3t + f_3, \tag{6}$$

where the matrix

$$M = \begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix}$$

is nonsingular with adjoint (cofactor) matrix

$$M^{\#} = \begin{pmatrix} D_1 & D_2 & D_3 \\ E_1 & E_2 & E_3 \\ F_1 & F_2 & F_3 \end{pmatrix}.$$

Then X lies on the conic:

$$(E_1\alpha + E_2\beta + E_3\gamma)^2 = (D_1\alpha + D_2\beta + D_3\gamma)(F_1\alpha + F_2\beta + F_3\gamma). \tag{7}$$

Proof. Since M is nonsingular, its determinant δ is nonzero, and $M^{-1} = \frac{1}{\delta}M^{\#}$. Let

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix},$$

so that $X = MT$ and $M^{-1}X = T$. This second equation is equivalent to the system

$$\begin{array}{ccccccc} D_1x_1 & + & D_2x_2 & + & D_3x_3 & = & \delta t^2 \\ E_1x_1 & + & E_2x_2 & + & E_3x_3 & = & \delta t \\ F_1x_1 & + & F_2x_2 & + & F_3x_3 & = & \delta. \end{array}$$

The equal quotients $\delta t^2/\delta t$ and $\delta t/\delta$ yield

$$\frac{D_1x_1 + D_2x_2 + D_3x_3}{E_1x_1 + E_2x_2 + E_3x_3} = \frac{E_1x_1 + E_2x_2 + E_3x_3}{F_1x_1 + F_2x_2 + F_3x_3}.$$

□

For a first example, suppose $Q = q_1 : q_2 : q_3$ is a point not on a sideline of $\triangle ABC$, and let L be the line $q_1\alpha + q_2\beta + q_3\gamma = 0$. The P-isoconjugate of Q, is (e.g., [4, Glossary]) the point

$$P * Q = \frac{1}{p_1q_1} : \frac{1}{p_2q_2} : \frac{1}{p_3q_3}.$$

The method of proof of Theorem 1 shows that the P-isoconjugate of L (i.e., the set of points $P * R$ for R on L) is the circumconic

$$\frac{q_1}{p_1\alpha} + \frac{q_2}{p_2\beta} + \frac{q_3}{p_3\gamma} = 0.$$

We shall see that the same method applies to many other configurations.

2. Cevian nests and two conjugacies

A fruitful configuration in the plane of $\triangle ABC$ is the cevian nest, consisting of three triangles T_1, T_2, T_3 such that T_2 is a cevian triangle of T_1, and T_3 is a cevian triangle of T_2. In this article, $T_2 = \triangle ABC$, so that T_1 is the anticevian triangle of some point P, and T_3 is the cevian triangle of some point Q. It is well known (e.g. [1, p.165]) that if any two pairs of such triangles are perspective pairs, then the third pair are perspective also[3]. Accordingly, for a cevian nest, given two of the perspectors, the third may be regarded as the value of a binary operation applied to the given perspectors. There are three such pairs, hence three binary operations. As has been noted elsewhere ([2, p. 203] and [3, Glossary]), two of them are involutory: Ceva conjugates and cross conjugates.

2.1. *Ceva conjugate.* The *P-Ceva conjugate of Q*, denoted by $P©Q$, is the perspector of the cevian triangle of P and the anticevian triangle of Q; for $P = p_1 : p_2 : p_3$ and $Q = q_1 : q_2 : q_3$, we have

$$P©Q = q_1\left(-\frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{q_3}{p_3}\right) : q_2\left(\frac{q_1}{p_1} - \frac{q_2}{p_2} + \frac{q_3}{p_3}\right) : q_3\left(\frac{q_1}{p_1} + \frac{q_2}{p_2} - \frac{q_3}{p_3}\right).$$

Theorem 2. *Suppose P, U, V, Q_t are points as in (1)-(3); that is, Q_t traverses line UV. The locus of $P©Q_t$ is the conic*

$$\frac{\alpha^2}{p_1q_1} + \frac{\beta^2}{p_2q_2} + \frac{\gamma^2}{p_3q_3} - \left(\frac{1}{p_2q_3} + \frac{1}{p_3q_2}\right)\beta\gamma - \left(\frac{1}{p_3q_1} + \frac{1}{p_1q_3}\right)\gamma\alpha - \left(\frac{1}{p_1q_2} + \frac{1}{p_2q_1}\right)\alpha\beta = 0, \quad (8)$$

where $Q := q_1 : q_2 : q_3$, the trilinear pole of the line UV, is given by

$$q_1 : q_2 : q_3 = \frac{1}{u_2v_3 - u_3v_2} : \frac{1}{u_3v_1 - u_1v_3} : \frac{1}{u_1v_2 - u_2v_1}.$$

This conic[4] *passes through the vertices of the cevian triangles of P and Q.*

Proof. First, it is easy to verify that equation (8) holds for $\alpha : \beta : \gamma$ equal to any of these six vertices:

$$0 : p_2 : p_3, \quad p_1 : 0 : p_3, \quad p_1 : p_2 : 0, \quad 0 : q_2 : q_3, \quad q_1 : 0 : q_3, \quad q_1 : q_2 : 0 \qquad (9)$$

[3]Peter Yff has observed that in [1], Court apparently overlooked the fact that $\triangle ABC$ and any inscribed triangle are triply perspective, with perspectors A, B, C. For these cases, Court's result is not always true. It seems that he intended his inscribed triangles to be cevian triangles.

[4]The general equation (8) for the circumconic of two cevian triangles is one of many interesting equations in Peter Yff's notebooks.

A conic is determined by any five of its points, so it suffices to prove that the six vertices are of the form $P©Q_t$. Putting $x_1 = 0$ in (4) gives roots

$$t_a = \frac{-e_1 \pm \sqrt{e_1^2 - 4d_1 f_1}}{2d_1}, \tag{10}$$

where

$$\begin{aligned} d_1 &= v_1(-\frac{v_1}{p_1} + \frac{v_2}{p_2} + \frac{v_3}{p_3}), &(11)\\ e_1 &= -\frac{2u_1v_1}{p_1} + \frac{u_1v_2 + u_2v_1}{p_2} + \frac{u_1v_3 + u_3v_1}{p_3}, &(12)\\ f_1 &= u_1(-\frac{u_1}{p_1} + \frac{u_2}{p_2} + \frac{u_3}{p_3}). &(13)\end{aligned}$$

The discriminant in (10) is a square, and t_a simplifies:

$$t_a = \frac{-e_1p_2p_3q_2q_3 \pm (p_3q_2 - p_2q_3)}{2d_1p_2p_3q_2q_3}.$$

If the numerator is $-e_1p_2p_3q_2q_3 + (p_3q_2 - p_2q_3)$, then (5) and (6), and substitutions for $d_2, e_2, f_2, d_3, e_3, f_3$ obtained cyclically from (11)-(13), give $x_2/x_3 = p_2/p_3$, so that $P©Q_{t_a} = 0 : p_2 : p_3$. On the other hand, if the numerator is $-e_1p_2p_3q_2q_3 - (p_3q_2 - p_2q_3)$, then $x_2/x_3 = q_2/q_3$ and $P©Q_{t_a} = 0 : q_2 : q_3$. Likewise, the roots t_b and t_c of (5) and (6) yield a proof that the other four vertices in (9) are of the form $P©Q_t$. □

Corollary 2.1. *Suppose $P = p_1 : p_2 : p_3$ is a point and L given by $\ell_1\alpha + \ell_2\beta + \ell_3\gamma = 0$ is a line. Suppose the point Q_t traverses L. The locus of $P©Q_t$ is the conic*

$$\frac{\ell_1\alpha^2}{p_1} + \frac{\ell_2\beta^2}{p_2} + \frac{\ell_3\gamma^2}{p_3} - (\frac{\ell_3}{p_2} + \frac{\ell_2}{p_3})\beta\gamma - (\frac{\ell_1}{p_3} + \frac{\ell_3}{p_1})\gamma\alpha - (\frac{\ell_2}{p_1} + \frac{\ell_1}{p_2})\alpha\beta = 0. \tag{14}$$

Proof. Let U, V be distinct points on L, and apply Theorem 2. □

Corollary 2.2. *The conic (14) is inscribed to $\triangle ABC$ if and only if the line $L = UV$ is the trilinear pole of P.*

Proof. In this case, $\ell_1 : \ell_2 : \ell_3 = 1/p_1 : 1/p_2 : 1/p_3$, so that $P = Q$. The cevian triangles indicated by (9) are now identical, and the six pass-through points are three tangency points. □

One way to regard Corollary 2.2 is to start with an inscribed conic Γ. It follows from the general equation for such a conic (e.g., [2, p.238]) that the three touch points are of the form $0 : p_2 : p_3, p_1 : 0 : p_3, p_1 : p_2 : 0$, for some $P = p_1 : p_2 : p_3$. Then Γ is the locus of $P©Q_t$ as Q_t traverses L.

Example 1. Let $P =$ centroid and $Q =$ orthocenter. Then line UV is given by

$$(\cos A)\alpha + (\cos B)\beta + (\cos C)\gamma = 0,$$

and the conic (8) is the nine-point circle. The same is true for $P =$ orthocenter and $Q =$ centroid.

Example 2. Let P = orthocenter and $Q = X_{648}$, the trilinear pole of the Euler line, so that UV is the Euler line. The conic (8) passes through the vertices of the orthic triangle, and X_4, X_{113}, X_{155}, X_{193}, which are the P-Ceva conjugates of X_4, X_{30}, X_3, X_2, respectively.[5]

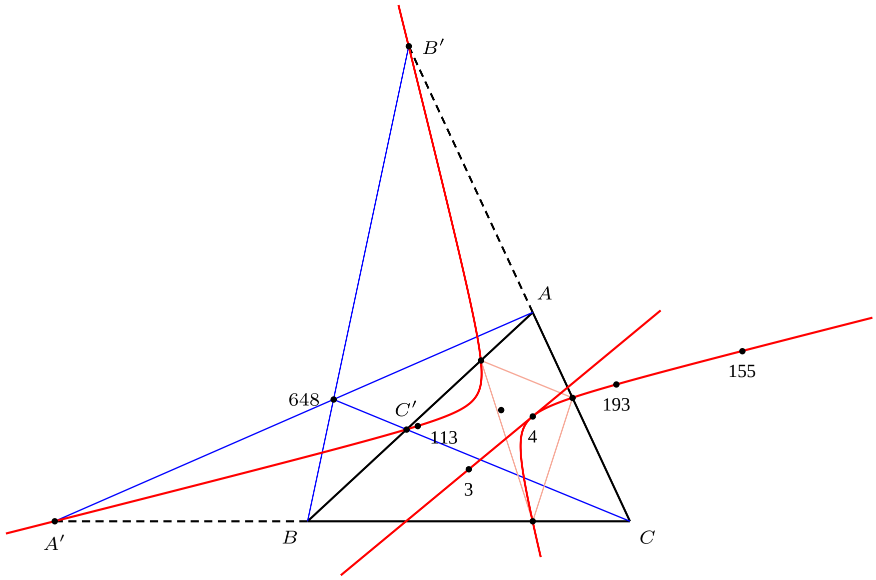


Figure 1

2.2. *Cross conjugate.* Along with Ceva conjugates, cevian nests proffer cross conjugates. Suppose $P = p_1 : p_2 : p_3$ and $Q = q_1 : q_2 : q_3$ are distinct points, neither lying on a sideline of $\triangle ABC$. Let $A'B'C'$ be the cevian triangle of Q. Let

$$A'' = PA' \cap B'C', \quad B'' = PB' \cap C'A', \quad C'' = PC' \cap A'B',$$

so that $A''B''C''$ is the cevian triangle (in $\triangle A'B'C'$) of P. The *cross conjugate* $P \otimes Q$ is the perspector of $\triangle ABC$ and $\triangle A''B''C''$. It has coordinates

$$\frac{q_1}{-\frac{p_1}{q_1}+\frac{p_2}{q_2}+\frac{p_3}{q_3}} : \frac{q_2}{-\frac{p_2}{q_2}+\frac{p_3}{q_3}+\frac{p_1}{q_1}} : \frac{q_3}{-\frac{p_3}{q_3}+\frac{p_1}{q_1}+\frac{p_2}{q_2}}.$$

It is easy to verify directly that $\otimes$ is a conjugacy; i.e., $P \otimes (P \otimes Q) = Q$, or to reach the same conclusion using the identity

$$X \otimes P = (X^{-1} © P^{-1})^{-1},$$

where $(\ )^{-1}$ signifies isogonal conjugation.

The locus of $P \otimes Q_t$ is generally a curve of degree 5. However, on switching the roles of P and Q, we obtain a conic, as in Theorem 3. Specifically, let $Q = q_1 : q_2 : q_3$ remain fixed while

$$P_t = u_1 + v_1 t : u_2 + v_2 t : u_3 + v_3 t, \ -\infty < t \leq \infty,$$

ranges through the line UV.

[5]Indexing of triangle centers is as in [3].

Theorem 3. *The locus of the $P_t \otimes Q$ is the circumconic*

$$\left(\frac{p_3}{q_2}+\frac{p_2}{q_3}\right)\beta\gamma+\left(\frac{p_1}{q_3}+\frac{p_3}{q_1}\right)\gamma\alpha+\left(\frac{p_2}{q_1}+\frac{p_1}{q_2}\right)\alpha\beta=0, \tag{15}$$

where line UV is represented as

$$p_1\alpha+p_2\beta+p_3\gamma=(u_2v_3-u_3v_2)\alpha+(u_3v_1-u_1v_3)\beta+(u_1v_2-u_2v_1)\gamma=0.$$

Proof. Following the proof of Theorem 1, let

$$u_1'=-\frac{u_1}{q_1}+\frac{u_2}{q_2}+\frac{u_3}{q_3},\ v_1'=-\frac{v_1}{q_1}+\frac{v_2}{q_2}+\frac{v_3}{q_3},$$

and similarly for u_2', u_3', v_2', v_3'. Then

$$d_1=q_1v_2'v_3',\quad e_1=q_1(u_2'v_3'+u_3'v_2'),\quad f_1=q_1u_2'u_3',$$

and similarly for $d_i,\ e_i,\ f_i,\ i=2,3$. The nine terms $d_i,\ e_i,\ f_i$, yield the nine cofactors D_i, E_i, F_i, which then yield 0 for the coefficients of $\alpha^2, \beta^2, \gamma^2$ in (7) and the other three coefficients as asserted in (15). □

Example 3. Regarding the conic (15), suppose $P=p_1:p_2:p_3$ is an arbitrary triangle center and Γ is an arbitrary circumconic $\ell/\alpha+m/\beta+n/\gamma=0$. Let

$$\begin{aligned}Q &= q_1:q_2:q_3\\ &= \frac{1}{p_1(-p_1\ell+p_2m+p_3n)}:\frac{1}{p_2(-p_2m+p_3n+p_1\ell)}:\frac{1}{p_3(-p_3n+p_1\ell+p_2m)}.\end{aligned}$$

For P_t ranging through the line L given by $p_1\alpha+p_2\beta+p_3\gamma=0$, the locus of $P_t\otimes Q$ is then Γ, since

$$\frac{p_3}{q_2}+\frac{p_2}{q_3}:\frac{p_1}{q_3}+\frac{p_3}{q_1}:\frac{p_2}{q_1}+\frac{p_1}{q_2}=\ell:m:n.$$

In other words, given P and L, there exists Q such that $P_t\otimes Q$ ranges through any prescribed circumconic. In fact, Q is the isogonal conjugate of $P©L'$, where L' denotes the pole of line L. Specific cases are summarized in the following table.

P	Q	ℓ	pass-through points, X_i, for $i=$
X_1	X_1	1	$88, 100, 162, 190$ (Steiner ellipse)
X_1	X_2	$b+c$	$80, 100, 291$ (ellipse)
X_1	X_6	$a(b+c)$	$101, 190, 292$ (ellipse)
X_1	X_{57}	a	$74, 98, 99, ..., 111, 112,$ ...(circumcircle)
X_1	X_{63}	$\sin 2A$	$109, 162, 163, 293$ (ellipse)
X_1	X_{100}	$b-c$	$1, 2, 28, 57, 81, 88, 89, 105,$ ...(hyperbola)
X_1	X_{101}	$a(b-c)(b+c-a)$	$6, 9, 19, 55, 57, 284, 333,$ (hyperbola)
X_1	X_{190}	$a(b-c)$	$1, 6, 34, 56, 58, 86, 87, 106,$ ...(hyperbola)

3. Poles and polars

In this section, we shall see that, in addition to mappings discussed in §2, certain mappings defined in terms of poles and polars are nicely represented in terms of Ceva conjugates and cross conjugates

We begin with definitions. Suppose $A'B'C'$ is the cevian triangle of a point P not on a sideline of $\triangle ABC$. By Desargues's Theorem, the points $BC \cap B'C'$, $CA \cap C'A'$, $AB \cap A'B'$ are collinear. Their line is the *trilinear polar of* P. Starting with a line L, the steps reverse, yielding the *trilinear pole of* L. If L is given by $x\alpha + y\beta + z\gamma = 0$ then the trilinear pole of L is simply $1/x : 1/y : 1/z$.

Suppose Γ is a conic and X is a point. For each U on Γ, let V be the point other than U in which the line UX meets Γ, and let X' be the harmonic conjugate of X with respect to U and V. As U traverses Γ, the point X' traverses a line, the polar of X with respect to Γ, or Γ-*based polar of* X. Here, too, as with the trilinear case, for given line L, the steps reverse to define the Γ-*based pole of* L.

In §2, two mappings were defined in the context of a cevian nest. We return now to the cevian nest to define a third mapping. Suppose $P = p : q : r$ and $X = x : y : z$ are distinct points, neither lying on a sideline of $\triangle ABC$. Let $A''B''C''$ be the anticevian triangle of X. Let

$$A' = PA'' \cap BC, \quad B' = PB'' \cap CA, \quad C' = PC'' \cap AB.$$

The *cevapoint* of P and X is the perspector, R, of triangles ABC and $A'B'C'$. Trilinears are given by

$$R = \frac{1}{qz + ry} : \frac{1}{rx + pz} : \frac{1}{py + qx}. \qquad (16)$$

It is easy to verify that $P = R©X$.

The general conic Γ is given by the equation

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2p\beta\gamma + 2q\gamma\alpha + 2r\alpha\beta = 0,$$

and the Γ-based polar of $X = x : y : z$ is given (e.g., [5]) by

$$(ux + ry + qz)\alpha + (vy + pz + rx)\beta + (wz + qx + py)\gamma = 0. \qquad (17)$$

Example 4. Let Γ denote the circumconic $p/\alpha + q/\beta + r/\gamma = 0$, that is, the circumconic having as pivot the point $P = p : q : r$. The Γ-based polar of X is the trilinear polar of the cevapoint of P and X, given by

$$(qz + ry)\alpha + (rx + pz)\beta + (py + qx)\gamma = 0.$$

In view of (16), (trilinear polar of X) = (Γ-based polar of $X©P$).

Example 5. Let Γ denote conic determined as in Theorem 2 by points P and Q. The conic is inscribed in $\triangle ABC$ if and only if $P = Q$, and in this case, the Γ-based polar of X is given by

$$\frac{1}{p}\left(-\frac{x}{p} + \frac{y}{q} + \frac{z}{r}\right)\alpha + \frac{1}{q}\left(\frac{x}{p} - \frac{y}{q} + \frac{z}{r}\right)\beta + \frac{1}{r}\left(\frac{x}{p} + \frac{y}{q} - \frac{z}{r}\right)\gamma = 0.$$

In other words, $(\Gamma\text{-based polar of } X) = (\text{trilinear polar of } X \otimes P)$. In particular, choosing $P = X_7$, we obtain the incircle-based polar of X:

$$f(A,B,C)\alpha + f(B,C,A)\beta + f(C,A,B)\gamma = 0,$$

where

$$f(A,B,C) = \frac{\sec^2 \frac{A}{2}}{-x\cos^2 \frac{A}{2} + y\cos^2 \frac{B}{2} + z\cos^2 \frac{C}{2}}.$$

Suppose now that Γ is a conic and L a line. As a point

$$X = p_1 + q_1 t : p_2 + q_2 t : p_3 + q_3 t \tag{18}$$

traverses L, a mapping is defined by the trilinear pole of the Γ-based polar of X. This pole has trilinears found directly from (17):

$$\frac{1}{g_1(t)} : \frac{1}{g_2(t)} : \frac{1}{g_3(t)},$$

where $g_1(t) = u(p_1+q_1t) + r(p_2+q_2t) + q(p_3+q_3t)$, and similarly for $g_2(t)$ and $g_3(t)$. The same pole is given by

$$g_2(t)g_3(t) : g_3(t)g_1(t) : g_1(t)g_2(t), \tag{19}$$

and Theorem 1 applies to form (19). With certain exceptions, the resulting conic (7) is a circumconic; specifically, if $uq_1 + rq_2 + qq_3 \neq 0$, then $g_1(t)$ has a root for which (19) is vertex A, and similarly for vertices B and C.

Example 6. For $P = u : v : w$, let $\Gamma(P)$ be the circumconic $u\beta\gamma + v\gamma\alpha + w\alpha\beta = 0$. Assume that at least one point of $\Gamma(P)$ lies inside $\triangle ABC$; in other words, assume that $\Gamma(P)$ is not an ellipse. Let $\widehat{\Gamma}(P)$ be the conic[6]

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0. \tag{20}$$

For each $\alpha : \beta : \gamma$ on the line $u\alpha + v\beta + w\gamma = 0$ and inside or on a side of $\triangle ABC$, let $P = p : q : r$, with $p \geq 0$, $q \geq 0$, $r \geq 0$, satisfy

$$\alpha = p^2, \quad \beta = q^2, \quad \gamma = r^2,$$

and define

$$\sqrt{P} := \sqrt{p} : \sqrt{q} : \sqrt{r} \tag{21}$$

and

$$P_A := -\sqrt{p} : \sqrt{q} : \sqrt{r}, \quad P_B := \sqrt{p} : -\sqrt{q} : \sqrt{r}, \quad P_C := \sqrt{p} : \sqrt{q} : -\sqrt{r} \tag{22}$$

Each point in (21) and (22) satisfies (20), and conversely, each point satisfying (20) is of one of the forms in (21) and (22). Therefore, the conic (20) consists

[6]Let $\Phi = vwa^2 + wub^2 + uvc^2$. Conic (20) is an ellipse, hyperbola, or parabola according as $\Phi > 0$, $\Phi < 0$, or $\Phi = 0$. Yff [6, pp.131-132], discusses a class of conics of the form (20) in connection with self-isogonal cubics and orthocentric systems.

of all points as in (21) and (22). Constructions[7] for $\sqrt{P}$ are known, and points P_A, P_B, P_C are constructible as harmonic conjugates involving $\sqrt{P}$ and vertices A, B, C; e.g., P_A is the harmonic conjugate of P with respect to A and the point $BC \cap AP$. Now suppose that L is a line, given by $\ell\alpha + m\beta + n\gamma = 0$. For X as in (18) traversing L, we have $g_1(t) = u(p_1 + q_1 t)$, leading to nine amenable coefficients in (4)-(6) and on to amenable cofactors, as indicated by

$$D_1 = up_1^2 r_1, \qquad E_1 = -up_1 q_1 r_1, \qquad F_1 = uq_1^2 r_1,$$

where $r_1 = p_2 q_3 - p_3 q_2$. The nine cofactors and (7) yield this conclusion: the Γ-based pole of X traverses the circumconic

$$\frac{\ell}{u\alpha} + \frac{m}{v\beta} + \frac{n}{w\gamma} = 0. \tag{23}$$

For example, taking line $u\alpha + v\beta + w\gamma = 0$ to be the trilinear polar of X_{100} and L that of X_{101}, the conic (23) is the Steiner circumellipse. In this case, the conic (20) is the hyperbola passing through X_i for $i = 1, 43, 165, 170, 365,$ and 846. Another notable choice of (20) is given by $P = X_{798}$, which has first trilinear $(\cos^2 B - \cos^2 C)\sin^2 A$. Points on this hyperbola include X_i for $i = 1, 2, 20, 63, 147, 194, 478, 488, 616, 617, 627,$ and 628.

Of course, for each $X = x : y : z$ on a conic $\widehat{\Gamma}(P)$, the points

$$-x : y : z, \qquad x : -y : z, \qquad x : y : -z$$

are also on $\widehat{\Gamma}(P)$, and if X also lies inside $\triangle ABC$, then X_1/X^2 lies on $\Gamma(P)$.

Example 7. Let Γ be the circumcircle, given by $a/\alpha + b/\beta + c/\gamma = 0$, and let L be the Brocard axis, which is the line passing through the points $X_6 = a : b : c$ and $X_3 = \cos A : \cos B : \cos C$. Using notation in Theorem 1, we find

$$d_1 = bc, \qquad e_1 = 2a(b^2 + c^2), \qquad f_1 = 4a^2 bc$$

and

$$D_1 = 8ab^2c^2(c^2 - b^2), \qquad E_1 = 4a^2bc(b^2 - c^2), \qquad F_1 = 2a^3(c^2 - b^2),$$

leading to this conclusion: the circumcircle-based pole of X traversing the Brocard axis traverses the circumhyperbola

$$\frac{a(b^2 - c^2)}{\alpha} + \frac{b(c^2 - a^2)}{\beta} + \frac{c(a^2 - b^2)}{\gamma} = 0,$$

namely, the isogonal transform of the trilinear polar of the Steiner point.

---

[7]The trilinear square root is constructed in [4]. An especially attractive construction of barycentric square root in [7] yields a second construction of trilinear square root. We describe the latter here. Suppose $P = p : q : r$ in trilinears; then in barycentric, $P = ap : bq : cr$, so that the barycentric square root of P is $\sqrt{ap} : \sqrt{bq} : \sqrt{cr}$. Barycentric multiplication (as in [7]) by $\sqrt{a} : \sqrt{b} : \sqrt{c}$ gives $a\sqrt{p} : b\sqrt{q} : c\sqrt{r}$, these being barycentrics for the trilinear square root of P, which in trilinears is $\sqrt{p} : \sqrt{q} : \sqrt{r}$.

References

[1] N.A. Court, *College Geometry*, Barnes & Noble, New York, 1969.
[2] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, v. 129, i-xxv, 1-295. Utilitas Mathematica, University of Manitoba, Winnipeg, 1998.
[3] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://cedar.evansville.edu/~ck6/encyclopedia/.
[4] C. Kimberling and C. Parry, Products, square roots, and layers in triangle geometry, *Mathematics and Informatics Quarterly*, 10 (2000) 9-22.
[5] E.A. Maxwell, *General Homogeneous Coordinates*, Cambridge University Press, Cambridge, 1957.
[6] P. Yff, Two families of cubics associated with a triangle, in *In Eves' Circles*, J.M. Anthony, editor, Notes Number 34, Mathematical Association of America, 1994, 127-137.
[7] P. Yiu, The uses of homogeneous barycentric coordinates in plane euclidean geometry, *Int. J. Math. Educ. Sci. Technol.*, 31 (2000) 569-578.

Clark Kimberling: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, USA

E-mail address: ck6@evansville.edu

Forum Geometricorum
Volume 1 (2001) 151–160.

FORUM GEOM
ISSN 1534-1178

$P\ell$-Perpendicularity

Floor van Lamoen

Abstract. It is well known that perpendicularity yields an involution on the line at infinity $\mathcal{L}^\infty$ mapping perpendicular directions to each other. Many notions of triangle geometry depend on this involution. Since in projective geometry the perpendicular involution is not different from other involutions, theorems using standard perpendicularity in fact are valid more generally.

In this paper we will classify alternative perpendicularities by replacing the orthocenter H by a point P and $\mathcal{L}^\infty$ by a line ℓ. We show what coordinates undergo with these changes and give some applications.

1. Introduction

In the Euclidean plane we consider a reference triangle ABC. We shall perform calculations using homogeneous barycentric coordinates. In these calculations $(f : g : h)$ denotes the barycentrics of a point, while $[l : m : n]$ denotes the line with equation $lx + my + nz = 0$. The line at infinity $\mathcal{L}^\infty$, for example, has coordinates $[1 : 1 : 1]$.

Perpendicularity yields an involution on the line at infinity, mapping perpendicular directions to each other. We call this involution *the standard perpendicularity*, and generalize it by replacing the orthocenter H by another point P with coordinates $(f : g : h)$, stipulating that the cevians of P be "perpendicular" to the corresponding sidelines of ABC. To ensure that P is outside the sidelines of ABC, we assume $fgh \neq 0$.

Further we let the role of $\mathcal{L}^\infty$ be taken over by another line $\ell = [l : m : n]$ not containing P. To ensure that ℓ does not pass through any of the vertices of ABC, we assume $lmn \neq 0$ as well. We denote by $[L]^\ell$ the intersection of a line L with ℓ. When we replace H by P and $\mathcal{L}^\infty$ by ℓ, we speak of *$P\ell$-perpendicularity*.

Many notions of triangle geometry, like rectangular hyperbolas, circles, and isogonal conjugacy, depend on the standard perpendicularity. Replacing the standard perpendicularity by $P\ell$-perpendicularity has its effects on these notions. Also, with the replacement of the line of infinity $\mathcal{L}^\infty$, we have to replace affine notions like midpoint and the center of a conic by their projective generalizations. So it may seem that there is a lot of triangle geometry to be redone, having to prove many generalizations. Nevertheless, there are at least two advantages in making

Publication Date: November 8, 2001. Communicating Editor: Bernard Gibert.

The author wants to express his sincere thanks to the communicating editor, without whom this paper would not have been the same.

calculations in generalized perpendicularities. (1) Calculations using coordinates in P-perpendicularity are in general easier and more transparent than when we use specific expressions for the orthocenter H. (2) We give ourselves the opportunity to work with different perpendicularities simultaneously. Here, we may find new interesting views to the triangle in the Euclidean context.

2. $P\ell$-Perpendicularity

In the following we assume some basic results on involutions. These can be found in standard textbooks on projective geometry, such as [2, 3, 8].

2.1. *$P\ell$-rectangular conics.* We generalize the fact that all hyperbolas from the pencil through A, B, C, H are rectangular hyperbolas. Let $\mathcal{P}$ be the pencil of circumconics through P. The elements of $\mathcal{P}$ we call $P\ell$-rectangular conics. According to Desargues' extended Involution Theorem (see, for example, [2, §16.5.4], [8, p.153], or [3, §6.72]) each member of $\mathcal{P}$ must intersect a line ℓ in two points, which are images under an involution $\tau_{P\ell}$. This involution we call the $P\ell$-perpendicularity.

Since an involution is determined by two pairs of images, $\tau_{P\ell}$ can be defined by the degenerate members of the pencil, the pairs of lines (BC, PA), (AC, PB), and (AB, PC). Two of these pairs are sufficient.

If two lines L and M intersect ℓ in a pair of images of $\tau_{P\ell}$, then we say that they are $P\ell$-perpendicular, and write $L \perp_{P\ell} M$. Note that for any ℓ, this perpendicularity replaces the altitudes of a triangle by the cevians of P as lines $P\ell$-perpendicular to the corresponding sides.

The involution $\tau_{P\ell}$ has two fixed points J_1 and J_2, real if $\tau_{P\ell}$ is hyperbolic, and complex if $\tau_{P\ell}$ is elliptic.

Again, by Desargues' Involution Theorem, every nondegenerate triangle $P_1P_2P_3$ has the property that the lines through the vertices $P\ell$-perpendicular to the opposite sides are concurrent at a point. We call this point of concurrence the $P\ell$-orthocenter of the triangle.

Remark. In order to be able to make use of the notions of parallelism and midpoints, and to perform calculations with simpler coordinates, it may be convenient to only replace H by P, but not $\mathcal{L}^\infty$ by another line. In this case we speak of P-perpendicularity. Each $P\ell$-perpendicularity corresponds to the Q-perpendicularity for an appropriate Q by the mappings $(x : y : z) \leftrightarrow (lx : my : nz)$.[1]

2.2. *Representation of $\tau_{P\ell}$ in coordinates.*

Theorem 1. *For $P = (f : g : h)$ and $\ell = [l : m : n]$, the $P\ell$-perpendicularity is given by*

$$\tau_{P\ell} : (f_L : g_L : h_L) \mapsto \left(\frac{f(gh_L - hg_L)}{l} : \frac{g(hf_L - fh_L)}{m} : \frac{h(fg_L - gf_L)}{n} \right). \tag{1}$$

[1]These mappings can be constructed by the $(l : m : n)$-reciprocal conjugacy followed by isotomic conjugacy and conversely, as explained in [4].

Proof. Let L be a line passing through C with $[L]^\ell = (f_L : g_L : h_L)$, and let $B_L = L \cap AB = (f_L : g_L : 0)$. We will consider triangle AB_LC. We have noted above that the $P\ell$-altitudes of triangle AB_LC are concurrent. Two of them are very easy to identify. The $P\ell$-altitude from C simply is $CP = [-g : f : 0]$. On the other hand, since $[BP]^\ell = (fm : -lf - hn : hm)$, the $P\ell$-altitude from B_L is $[-hmg_L : hmf_L : fmg_L + flf_L + hnf_L]$. These two $P\ell$-altitudes intersect in the point: [2]

$$X = (f(fmg_L + flf_L + hnf_L) : gn(hf_L - fh_L) : hm(fg_L - gf_Lg)).$$

Finally, we find that the third $P\ell$-altitude meets ℓ in

$$[AX]^\ell = \left(\frac{f(gh_L - hg_L)}{l} : \frac{g(hf_L - fh_L)}{m} : \frac{h(fg_L - gf_L)}{n}\right),$$

which indeed satisfies (1). $\square$

3. $P\ell$-circles

Generalizing the fact that in the standard perpendicularity, all circles pass through the two circular points at infinity, we define a $P\ell$-circle to be any conic through the fixed points J_1 and J_2 of the involution $\tau_{P\ell}$. This viewpoint leads to another way of determining the involution, based on the following well known fact, which can be found, for example, in [2, §5.3]:

> Let a conic $\mathcal{C}$ intersect a line L in two points I and J. The involution τ on L with fixed points I and J can be found as follows: Let X be a point on L, then $\tau(X)$ is the point of intersection of L and the polar of X with respect to $\mathcal{C}$.

It is clear that applying this to a $P\ell$-circle with line ℓ we get the involution $\tau_{P\ell}$. In particular this shows us that in any $P\ell$-circle $\mathcal{C}$ a radius and the tangent to $\mathcal{C}$ through its endpoint are P-perpendicular. Knowing this, and restricting ourselves to P-circles, i.e. $\ell = \mathcal{L}^\infty$, we can conclude that all P-circles are homothetic in the sense that parallel radii of two P-circles have parallel tangents, or equivalently, that two parallel radii of two P-circles have a ratio that is independent of its direction.[3]

We now identify the most important $P\ell$-circle.

Theorem 2. *The conic $\mathcal{O}_{P\ell}$:*

$$f(gm + hn)yz + g(fl + hn)xz + h(fl + gm)xy = 0 \tag{2}$$

is the $P\ell$-circumcircle.

Proof. Clearly A, B and C are on the conic given by the equation. Let $J = (f_1 : g_1 : h_1)$, then with (1) the condition that J is a fixed point of $\tau_{P\ell}$ gives

$$\left(\frac{fgh_1 - fg_1h}{l} : \frac{f_1gh - fgh_1}{m} : \frac{fg_1h - f_1gh}{n}\right) = (f_1 : g_1 : h_1)$$

---

[2]In computing the coordinates of X, we have have used of the fact that $lf_L + mg_L + nh_L = 0$.

[3]Note here that the ratio might involve a real radius and a complex radius. This happens for instance when we have in the real plane two hyperbolas sharing asymptotes, but on alternative sides of these asymptotes.

which, under the condition $f_1 l + g_1 m + h_1 n = 0$, is equivalent to (2). This shows that the fixed points J_1 and J_2 of τ_P lie on $\mathcal{C}_P$ and proves the theorem. $\square$

As the 'center' of $\mathcal{O}_{P\ell}$ we use the pole of ℓ with respect to $\mathcal{O}_{P\ell}$. This is the point

$$O_{P\ell} = \left(\frac{mg+nh}{l} : \frac{lf+nh}{m} : \frac{lf+mg}{n}\right).$$

3.1. *$P\ell$-Nine Point Circle.* The 'centers' of $P\ell$-rectangular conics, *i.e.*, elements of the pencil $\mathcal{P}$ of conics through A, B, C, P, form a conic through the traces of P, [4] the 'midpoints' [5] of the triangle sides, and also the 'midpoints' of AP, BP and CP. This conic $\mathcal{N}_{P\ell}$ is an analogue of the nine-point conics, its center is the 'midpoint' of P and $O_{P\ell}$.

The conic through A, B, C, P, and J_1 (or J_2) clearly must be tangent to ℓ, so that J_1 (J_2) is the 'center' of this conic. So both J_1 and J_2 lie on $\mathcal{N}_{P\ell}$, which makes it a $P\ell$-circle.

4. $P\ell$-conjugacy

In standard perpendicularity we have the isogonal conjugacy τ_H as the natural (reciprocal) conjugacy. It can be defined by combining involutions on the pencils of lines through the vertices of ABC. The involution that goes with the pencil through A is defined by two pairs of lines. The first pair is AB and AC, the second pair is formed by the lines through A perpendicular to AB and to AC. Of course this involution maps to each other lines through A making opposite angles to AB and AC respectively. Similarly we have involutions on the pencil through B and C. The isogonal conjugacy is found by taking the images of the cevians of a point P under the three involutions. These images concur in the isogonal conjugate of P.

This isogonal conjugacy finds its P-perpendicular cognate in the following reciprocal conjugacy:

$$\tau_{P\ell c} : (x : y : z) \mapsto \left(\frac{f(mg+nh)}{lx} : \frac{g(lf+nh)}{my} : \frac{h(lf+mg)}{nz}\right), \qquad (3)$$

which we will call the $P\ell$-conjugacy. This naming is not unique, since for each line ℓ' there is a point Q so that the $P\ell$- and $Q\ell'$-conjugacies are equal. In particular, if $\ell = \mathcal{L}^\infty$, this reciprocal conjugacy is

$$(x : y : z) \mapsto \left(\frac{f(g+h)}{x} : \frac{g(h+f)}{y} : \frac{h(f+g)}{z}\right).$$

[4] These are the 'centers' of the degenerate elements of $\mathcal{P}$.

[5] The 'midpoint' of XY is the harmonic conjugate of $[XY]^\ell$ with respect to X and Y. The 'midpoints' of the triangle sides are also the traces of the trilinear pole of ℓ.

Clearly the $P\ell$-conjugacy maps P to $O_{P\ell}$. This provides us with a construction of the conjugacy. See [4]. [6] From (2) it is also clear that this conjugacy transforms $\mathcal{C}_{P\ell}$ into ℓ and back.

Now we note that any reciprocal conjugacy maps any line to a circumconic of ABC, and conversely. In particular, any line through $O_{P\ell}$ is mapped to a conic from the pencil $\mathcal{P}$, a $P\ell$-rectangular conic. This shows that $\tau_{P\ell c}$ maps the $P\ell$-perpendicularity to the involution on $\mathcal{O}_{P\ell}$ mapping each point X to the second point of intersection of $O_{P\ell}X$ with $\mathcal{C}_{P\ell}$.

The four points

$$\left(\pm\sqrt{\frac{f(mg+nh)}{l}} : \pm\sqrt{\frac{g(lf+nh)}{m}} : \pm\sqrt{\frac{h(lf+mg)}{n}}\right)$$

are the fixed points of the $P\ell$-conjugacy. They are the centers of the $P\ell$-circles tritangent to the sidelines of ABC.

5. Applications of P-perpendicularity

As mentioned before, it is convenient not to change the line at infinity $\mathcal{L}^\infty$ into ℓ and speak only of P-perpendicularity. This notion is certainly less general. Nevertheless, it works with simpler coordinates and it allows one to make use of parallelism and ratios in the usual way. For instance, the Euler line is generalized quite easily, because the coordinates of O_P are $(g+h : f+h : f+g)$, so that it is easy to see that $PG : GO_P = 2:1$.

We give a couple of examples illustrating the convenience of the notion of P-perpendicularity in computations and understanding.

5.1. *Construction of ellipses.* Note that the equation (1) does not change when we exchange $(f:g:h)$ and $(x:y:z)$. So we have:

Proposition 3. *P lies on $\mathcal{O}_{Q\ell}$ if and only if Q lies on $\mathcal{O}_{P\ell}$.*

When we restrict ourselves to P-perpendicularity, Proposition 3 is helpful in finding the axes of a circumellipse of a triangle. Let's say that the ellipse is $\mathcal{O}_P$. [7] If we find the fourth intersection X of a circumellipse and the circumcircle, then the X-circumcircle $\mathcal{O}_X$ passes through H as well as P, and thus it is a rectangular hyperbola as well as a P-rectangular conic. This means that the asymptotes of $\mathcal{O}_X$ must correspond to the directions of the axes of the ellipse. These yield indeed the only diameters of the ellipse to which the tangents at the endpoints are (standard) perpendicular. Note also that this shows that all conics through A, B, C, X, apart from the circumcircle have parallel axes. Figure 1 illustrates the case when $P = G$, the centroid, and X = Steiner point. Here, $\mathcal{O}_X =$ is the Kiepert hyperbola.

The knowledge of P-perpendicularity can be helpful when we try to draw conics in dynamic geometry software. This can be done without using foci.

---

[6] In [4] we can find more ways to construct the $P\ell$-conjugacy, for instance, by using the degenerate triangle where AP, BP and CP meet ℓ.

[7] When we know the center O_P of $\mathcal{O}_P$, we can find P by the ratio $O_PG : GP = 1:2$.

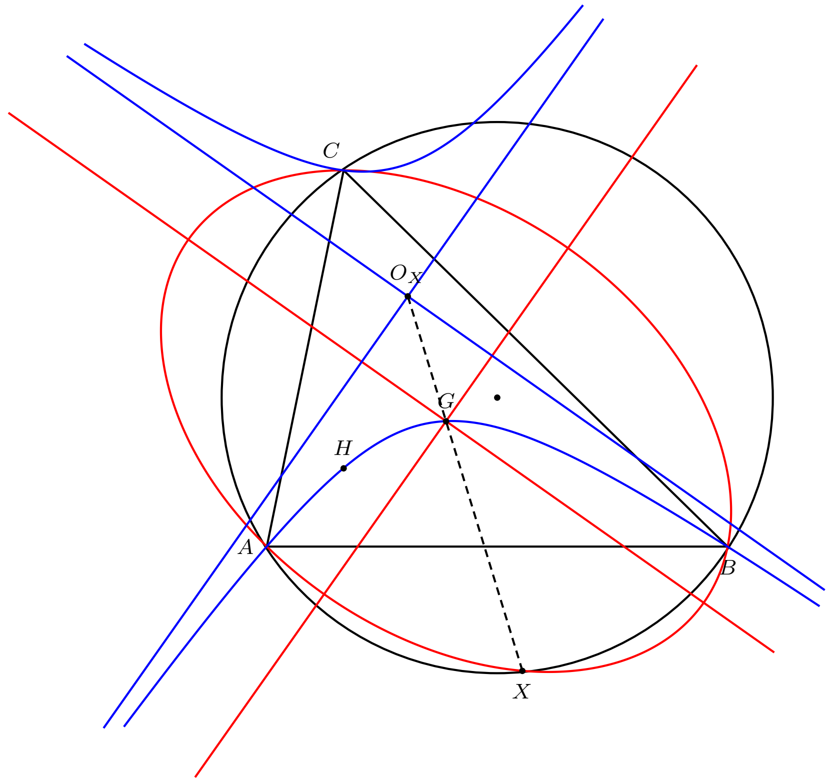


Figure 1

If we have the center O_P of a conic through three given points, say ABC, we easily find P as well. Also by reflecting one of the vertices, say A, through O_P we have the endpoints of a diameter, say AA_r. Then if we let a line m go through A, and a line n which is P-perpendicular to m through A_r. Their point of intersection lies on the P-circle through ABC. See Figure 2.

5.2. *Simson-Wallace lines.* Given a generic finite point $X = (x : y : z)$, let $A' \in BC$ be the point such that $A'X \parallel AP$, and let B' and C' be defined likewise, then we call $A'B'C'$ the *triangle of P-traces* of X. This triangle is represented by the following matrix:

$$M = \begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} 0 & gx + (g+h)y & hx + (g+h)z \\ fy + (f+h)x & 0 & hy + (f+h)z \\ fz + (f+g)x & gz + (f+g)y & 0 \end{pmatrix} \quad (4)$$

We are interested in the conic that plays a role similar to the circumcircle in the occurrence of Simson-Wallace lines.[8] To do so, we find that $A'B'C'$ is degenerate

[8] In [5] Miguel de Guzmán generalizes the Simson-Wallace line more drastically. He allows three arbitrary directions of projection, with the only restriction that these directions are not all equal, each not parallel to the side to which it projects.

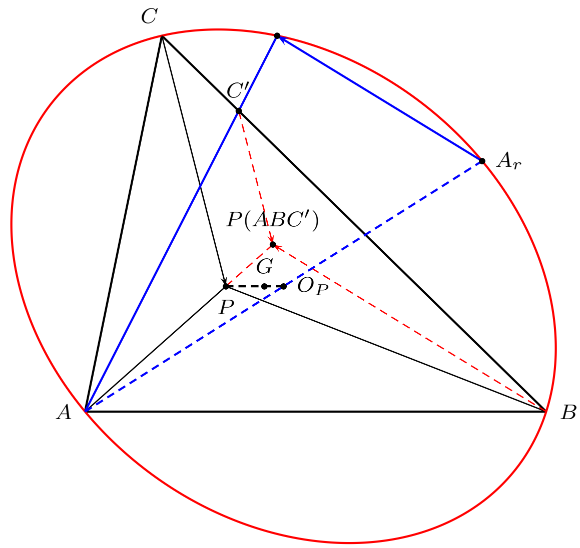


Figure 2

iff determinant $|M| = 0$, which can be rewritten as

$$(f + g + h)(x + y + z)(\tilde{f}yz + \tilde{g}xz + \tilde{h}xy) = 0, \qquad (5)$$

where

$$\tilde{f} = f(g + h), \qquad \tilde{g} = g(f + h), \qquad \tilde{h} = h(f + g).$$

Using that X and P are finite points, (5) can be rewritten into (2), so that the locus is the P-circle $\mathcal{C}_P$. See Figure 3.

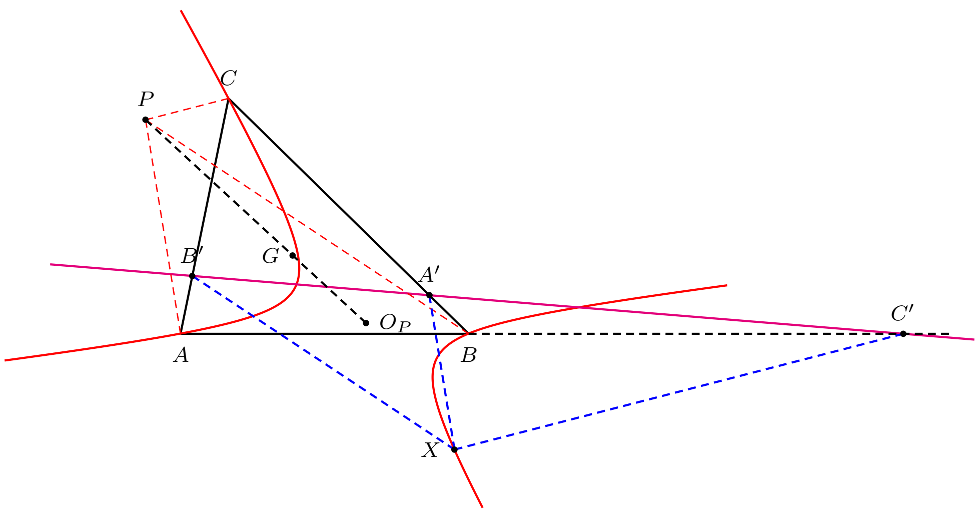


Figure 3

We further remark that since the rows of matrix M in (4) add up to $(f+g+h)X+(x+y+z)P$, the P-Simson-Wallace line $A'B'C'$ bisects the segment XP when $X \in \mathcal{C}_P$. Thus, the point of intersection of $A'B'C'$ and XP lies on $\mathcal{N}_P$.

5.3. *The Isogonal Theorem.* The following theorem generalizes the Isogonal Theorem. [9] We shall make use of the involutions τ_{PA}, τ_{PB} and τ_{PC} that the P-conjugacy causes on the pencil of lines through A, B and C respectively.

Theorem 4. *For $I \in \{A, B, C\}$, consider lines l_I and l'_I unequal to sidelines of ABC that are images under τ_{PI}. Let $A_1 = l_B \cap l'_C$, $B_1 = l_C \cap l'_A$ and $C_1 = l_A \cap l'_B$. We call $A_1B_1C_1$ a P-*conjugate triangle*. Then triangles ABC and $A_1B_1C_1$ are perspective.*

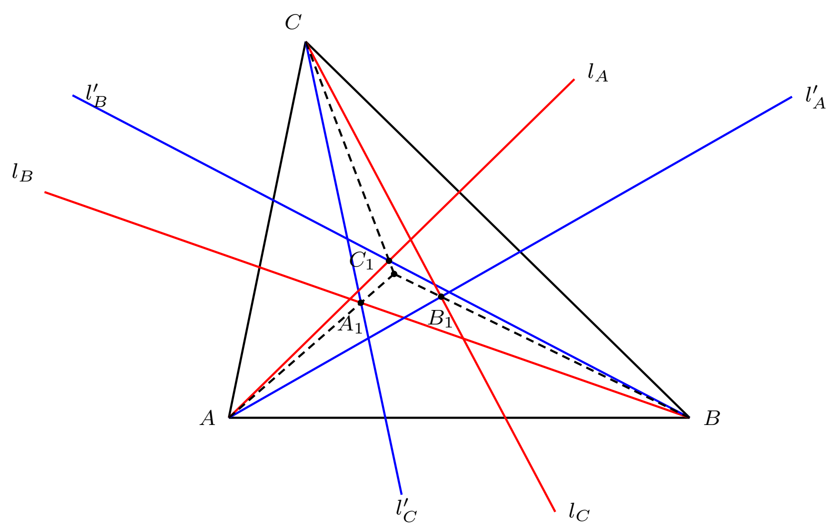


Figure 4

Proof. For $I \in \{A, B, C\}$, let $P_I = (x_I : y_I : z_I) \in l_I$ be a point different from I. We find, for instance, $l_A = [0 : z_A : -y_A]$ and $l'_C = [\tilde{g}/y_C : -\tilde{f}/x_C : 0]$. Consequently $B_1 = (\tilde{f}y_A/x_C : \tilde{g}y_A/y_C : \tilde{g}z_A/y_C)$. In the same way we find coordinates for A_1 and C_1 so that the P-conjugate triangle $A_1B_1C_1$ is given by

$$\begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix} = \begin{pmatrix} \tilde{f}x_C/x_B & \tilde{f}y_C/x_B & \tilde{h}x_C/z_B \\ \tilde{f}y_A/x_C & \tilde{g}y_A/y_C & \tilde{g}z_A/y_C \\ \tilde{h}x_B/z_A & \tilde{g}z_B/y_A & \tilde{h}z_B/z_A \end{pmatrix}.$$

With these coordinates it is not difficult to verify that $A_1B_1C_1$ is perspective to ABC. This we leave to the reader. □

[9] This theorem states that a triangle $A_1B_1C_1$ with $\angle BAC_1 = \angle CAB_1$, $\angle CBA_1 = \angle ABC_1$ and $\angle ACB_1 = \angle BCA_1$ is perspective to ABC. See [1, p.55], also [6, 9], and [7, Theorem 6D].

Interchanging the lines l_I and l'_I in Theorem 4 above, we see that the P-conjugates of $A_1B_1C_1$ form a triangle $A_2B_2C_2$ perspective to ABC as well. This is its desmic mate. [10] Now, each triangle perspective to ABC is mapped to its desmic mate by a reciprocal conjugacy. From this and Theorem 4 we see that the conditions 'perspective to ABC' and 'desmic mate is also an image under a reciprocal conjugacy' are equivalent.

5.3.1. Each P-conjugate triangle can be written in coordinates as

$$\begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix} = M_1 = \begin{pmatrix} \tilde{f} & w & v \\ w & \tilde{g} & u \\ v & u & \tilde{h} \end{pmatrix}.$$

Let a second P-conjugate triangle be given by

$$\begin{pmatrix} A_2 \\ B_2 \\ C_2 \end{pmatrix} = M_2 = \begin{pmatrix} \tilde{f} & W & V \\ W & \tilde{g} & U \\ V & U & \tilde{h} \end{pmatrix}.$$

Considering linear combinations $tM_1 + uM_2$ it is clear that the following proposition holds.

Proposition 5. *Let $A_1B_1C_1$ and $A_2B_2C_2$ be two distinct P-conjugate triangles. Define $A' = A_1A_2 \cap BC$ and B', C' analogously. Then $A'B'C'$ is a cevian triangle. In fact, if $A''B''C''$ is such that the cross ratios $(A_1A_2A'A'')$, $(B_1B_2B'B'')$ and $(C_1C_2C'C'')$ are equal, then $A''B''C''$ is perspective to ABC as well.*

The following corollary uses that the points where the cevians of P meet $\mathcal{L}^\infty$ is a P-conjugate triangle.

Corollary 6. *Let $A_1B_1C_1$ be a P-conjugate triangle. Let A' be the P-perpendicular projections of A_1 on BC, B_1 on CA, and C_1 on AB respectively. Let $A''B''C''$ be such that $A'A_1 : A''A_1 = B'B_1 : B''B_1 = C'C_1 : C''C_1 = t$, then $A''B''C''$ is perspective to ABC. As t varies, the perspector traverses the P-rectangular circumconic through the perspector of $A_1B_1C_1$.*

5.4. *The Darboux cubic.* We conclude with an observation on the analogues of the Darboux cubic. It is well known that the locus of points X whose pedal triangles are perspective to ABC is a cubic curve, the Darboux cubic. We generalize this to triangles of P-traces.

First, let us consider the lines connecting the vertices of ABC and the triangle of P-traces of X given in (4). Let μ_{ij} denote the entry in row i and column j of (4), then we find as matrix of coefficients of these lines

$$N = \begin{pmatrix} 0 & -\mu_{13} & \mu_{12} \\ \mu_{23} & 0 & -\mu_{21} \\ -\mu_{32} & \mu_{31} & 0 \end{pmatrix}. \tag{6}$$

[10] See for instance [4].

These lines concur iff $\det N = 0$. This leads to the cubic equation

$$(-f+g+h)x(\tilde{h}y^2-\tilde{g}z^2)+(f-g+h)y(\tilde{f}z^2-\tilde{h}x^2)+(f+g-h)z(\tilde{g}x^2-\tilde{f}y^2)=0. \tag{7}$$

We will refer to this cubic as the *P-Darboux cubic*. The cubic consists of the points Q such that Q and its P-conjugate are collinear with the point $(-f+g+h : f-g+h : f+g-h)$, which is the reflection of P in O_P.

It is seen easily from (4) and (6) that if we interchange $(f : g : h)$ and $(x : y : z)$, then (7) remains unchanged. From this we can conclude:

Proposition 7. *For two points P and Q be two points not on the sidelines of triangle ABC, P lies on the Q-Darboux cubic if and only if Q lies on the P-Darboux cubic.*

This example, and others in §5.1, demonstrate the fruitfulness of considering different perpendicularities simultaneously.

References

[1] P. Baptist, *Die Entwicklung der Neueren Dreiecksgeometrie*, Mannheim: B.I. Wissenschaftsverlag, 1992.

[2] M. Berger, *Geometry* II, 2nd edition (translated from French), Springer-Verlag, Berlin (1996).

[3] H.S.M. Coxeter, *The Real Projective Plane*, 1949; German translation by R. Oldenbourg, München (1955).

[4] K. Dean and F.M. van Lamoen, Construction of Reciprocal Conjugates, *Forum Geom.*, 1 (2001) 115 – 120.

[5] M. de Guzmán, An extension of the Wallace-Simson theorem: Projecting in arbitrary directions, *Amer. Math. Monthly*, 106 (1999) 574 – 580.

[6] R.A. Johnson, *Advanced Euclidean Geometry*, Dover Reprint, New York (1960).

[7] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1 – 295.

[8] M. Kindt, *Lessen in Projectieve Meetkunde*, (Dutch) Epsilon Uitgaven, Utrecht (1996).

[9] M. de Villiers, A generalization of the Fermat-Torricelli point, *Math. Gazette*, 79 (1995) 374 – 378.

Floor van Lamoen: Statenhof 3, 4463 TV Goes, The Netherlands

E-mail address: `f.v.lamoen@wxs.nl`

Forum Geometricorum
Volume 1 (2001) 161–171.

FORUM GEOM
ISSN 1534-1178

Cubics Associated with Triangles of Equal Areas

Clark Kimberling

Abstract. The locus of a point X for which the cevian triangle of X and that of its isogonal conjugate have equal areas is a cubic that passes through the 1st and 2nd Brocard points. Generalizing from isogonal conjugate to P-isoconjugate yields a cubic $Z(U,P)$ passing through U; if X is on $Z(U,P)$ then the P-isoconjugate of X is on $Z(U,P)$ and this point is collinear with X and U. A generalized equal areas cubic $\Gamma(P)$ is presented as a special case of $Z(U,P)$. If $\sigma =$ area$(\triangle ABC)$, then the locus of X whose cevian triangle has prescribed oriented area $K\sigma$ is a cubic $\Lambda(P)$, and P is determined if K has a certain form. Various points are proved to lie on $\Lambda(P)$.

1. Introduction

For any point $X = \alpha : \beta : \gamma$ (homogeneous trilinear coordinates) not a vertex of $\triangle ABC$, let

$$T = \begin{pmatrix} 0 & \beta & \gamma \\ \alpha & 0 & \gamma \\ \alpha & \beta & 0 \end{pmatrix} \qquad \text{and} \qquad T' = \begin{pmatrix} 0 & \gamma & \beta \\ \gamma & 0 & \alpha \\ \beta & \alpha & 0 \end{pmatrix},$$

so that T is the cevian triangle of X, and T' is the cevian triangle of the isogonal conjugate of X. Let σ be the area of $\triangle ABC$, and assume that X does not lie on a sideline $\triangle ABC$. Then oriented areas are given (e.g. [3, p.35]) in terms of the sidelengths a, b, c by

$$\text{area}(T) = \frac{abc}{8\sigma^2}\begin{vmatrix} 0 & k_1\beta & k_1\gamma \\ k_2\alpha & 0 & k_2\gamma \\ k_3\alpha & k_3\beta & 0 \end{vmatrix}, \qquad \text{area}(T') = \frac{abc}{8\sigma^2}\begin{vmatrix} 0 & l_1\gamma & l_1\beta \\ l_2\gamma & 0 & l_2\alpha \\ l_3\beta & l_3\alpha & 0 \end{vmatrix},$$

where k_i and l_i are normalizers. [1] Thus,

$$\text{area}(T) = \frac{k_1k_2k_3\alpha\beta\gamma abc}{4\sigma^2} \qquad \text{and} \qquad \text{area}(T') = \frac{l_1l_2l_3\alpha\beta\gamma abc}{8\sigma^2},$$

Publication Date: December 7, 2001. Communicating Editor: Floor van Lamoen.
The author thanks Edward Brisse, Bernard Gibert, and Floor van Lamoen for insightful communications.

[1]If $P = \alpha : \beta : \gamma$ is not on the line $\mathcal{L}^\infty$ at infinity, then the normalizer h makes $h\alpha, h\beta, h\gamma$ the directed distances from P to sidelines BC, CA, AB, respectively, and $h = 2\sigma/(a\alpha + b\beta + c\gamma)$. If P is on $\mathcal{L}^\infty$ and $\alpha\beta\gamma \neq 0$, then the normalizer is $h := 1/\alpha + 1/\beta + 1/\gamma$; if P is on $\mathcal{L}^\infty$ and $\alpha\beta\gamma = 0$, then $h := 1$.

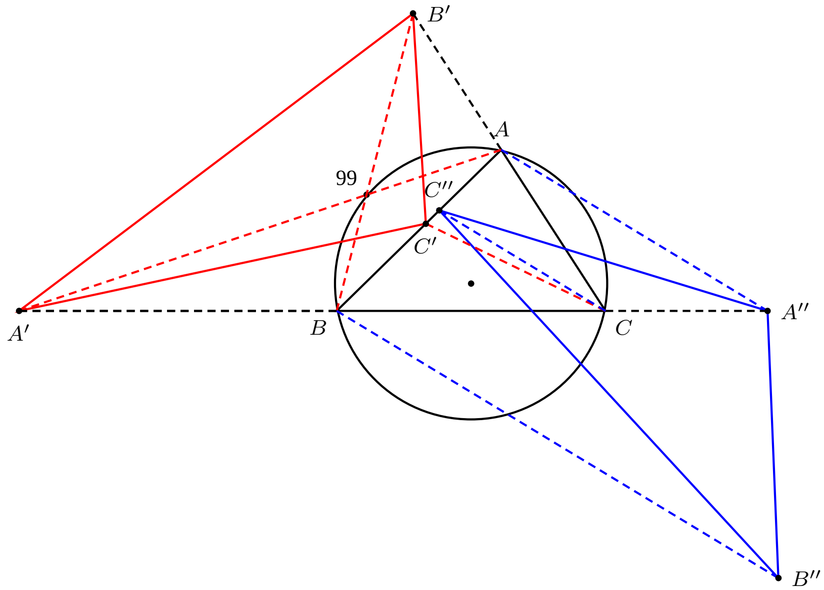


Figure 1. Triangles $A'B'C'$ and $A''B''C''$ have equal areas

so that area(T) = area(T') if and only if $k_1k_2k_3 = l_1l_2l_3$. Substituting yields

$$\frac{1}{b\beta+c\gamma}\cdot\frac{1}{c\gamma+a\alpha}\cdot\frac{1}{a\alpha+b\beta}=\frac{1}{b\gamma+c\beta}\cdot\frac{1}{c\alpha+a\gamma}\cdot\frac{1}{a\beta+b\alpha},$$

which simplifies to

$$a(b^2-c^2)\alpha(\beta^2-\gamma^2)+b(c^2-a^2)\beta(\gamma^2-\alpha^2)+c(a^2-b^2)\gamma(\alpha^2-\beta^2)=0. \quad (1)$$

In the parlance of [4, p.240], equation (1) represents the self-isogonal cubic $Z(X_{512})$, and, in the terminology of [1, 2], the auto-isogonal cubic having pivot X_{512}. [2] It is easy to verify that the following 24 points lie on this cubic:[3]

vertices A, B, C,
incenter X_1 and excenters,
Steiner point X_{99} and its isogonal conjugate X_{512} (see Figure 1),
vertices of the cevian triangle of X_{512},
1st and 2nd Brocard points Ω_1 and Ω_2,
X_{512}©X_1 and X_{512}©X_{99}, where © denotes Ceva conjugate,
$(X_{512}$©$X_1)^{-1}$ and $(X_{512}$©$X_{99})^{-1}$, where $()^{-1}$ denotes isogonal conjugate,
vertices of triangle T_1 below,
vertices of triangle T_2 below.

[2] X_i is the ith triangle center as indexed in [5].

[3] This “equal-areas cubic” was the subject of a presentation by the author at the CRCC geometry meeting hosted by Douglas Hofstadter at Indiana University, March 23-25, 1999.

The vertices of the bicentric[4] triangle T_1 are

$$-ab : a^2 : bc, \qquad ca : -bc : b^2, \qquad c^2 : ab : -ca, \tag{2}$$

and those of T_2 are

$$-ac : bc : a^2, \qquad b^2 : -ba : ca, \qquad ab : c^2 : -cb. \tag{3}$$

Regarding (2), $-ab : a^2 : bc$ is the point other than A and Ω_1 in which line $A\Omega_1$ meets $Z(X_{512})$. Similarly, lines $A\Omega_1$ and $C\Omega_1$ meet $Z(X_{512})$ in the other two points in (2). Likewise, the points in (3) lie on lines $A\Omega_2$, $B\Omega_2, C\Omega_2$. The points in (3) are isogonal conjugates of those in (2).

Vertex $A' := -ab : a^2 : bc$ is the intersection of the C-side of the anticomplementary triangle and the B-exsymmedian, these being the lines $a\alpha + b\beta = 0$ and $c\alpha + a\gamma = 0$. The other five vertices are similarly constructed.

Other descriptions of $Z(X_{512})$ are easy to check: (i) the locus of a point Q collinear with its isogonal conjugate and X_{512}, and (ii) the locus of Q for which the line joining Q and its isogonal conjugate is parallel to the line $\Omega_1\Omega_2$.

2. Isoconjugates and reciprocal conjugates

In the literature, isoconjugates are defined in terms of trilinears and reciprocal conjugates are defined in terms of barycentrics. We shall, in this section, use the notations $(x : y : z)_t$ and $(x : y : z)_b$ to indicate trilinears and barycentrics, respectively. [5]

Definition 1. [6] Suppose $P = (p : q : r)_t$ and $X = (x : y : z)_t$ are points, neither on a sideline of $\triangle ABC$. The P-isoconjugate of X is the point

$$(P \cdot X)_t^{-1} = (qryz : rpzx : pqxy)_t.$$

On the left side, the subscript t signifies trilinear multiplication and division.

Definition 2. [3] Suppose $P = (p : q : r)_b$ and $X = (x : y : z)_b$ are points not on a sideline of $\triangle ABC$. The P-reciprocal conjugate of X is the point

$$(P/X)_b = (pyz : qzx : rxy)_b.$$

In keeping with the meanings of "iso-" and "reciprocal",

$$\begin{aligned} X\text{-isoconjugate of } P &= P\text{-isoconjugate of } X, \\ X\text{-reciprocal conjugate of } P &= \frac{G}{P\text{-reciprocal conjugate of } X}, \end{aligned}$$

where G, the centroid, is the identity corresponding to barycentric division.

---

[4]Definitions of bicentric triangle, bicentric pair of points, and triangle center are given in [5, Glossary]. If $f(a,b,c) : g(a,b,c) : h(a,b,c)$ is the A-vertex of a bicentric triangle, then the B-vertex is $h(b,c,a) : f(b,c,a) : g(b,c,a)$ and the C-vertex is $g(c,a,b) : h(c,a,b) : f(c,a,b)$.

[5]A point X with trilinears $\alpha : \beta : \gamma$ has barycentrics $a\alpha : b\beta : c\gamma$. For points not on $\mathcal{L}^\infty$, trilinears are proportional to the directed distances between X and the sidelines BC, CA, AB, respectively, whereas barycentrics are proportional to the oriented areas of triangles XBC, XCA, XAB, respectively.

3. The cubic $Z(U, P)$

In this section, all coordinates are trilinears; for example, $(\alpha : \beta : \gamma)_t$ appears as $\alpha : \beta : \gamma$. Suppose $U = u : v : w$ and $P = p : q : r$ are points, neither on a sideline of $\triangle ABC$. We generalize the cubic $Z(U)$ defined in [4, p.240] to a cubic $Z(U, P)$, defined as the locus of a point $X = \alpha : \beta : \gamma$ for which the points U, X, and the P-isoconjugate of X are collinear. This requirement is equivalent to

$$\begin{vmatrix} u & v & w \\ \alpha & \beta & \gamma \\ qr\beta\gamma & rp\gamma\alpha & pq\alpha\beta \end{vmatrix} = 0, \tag{4}$$

hence to

$$up\alpha(q\beta^2 - r\gamma^2) + vq\beta(r\gamma^2 - p\alpha^2) + wr\gamma(q\alpha^2 - r\beta^2) = 0.$$

Equation (4) implies these properties:

(i) $Z(U, P)$ is self P-isoconjugate;
(ii) $U \in Z(U, P)$;
(iii) if $X \in Z(U, P)$, then X, U, and $(P \cdot X)_t^{-1}$ are collinear.

The following ten points lie on $Z(U, P)$:

the vertices A, B, C;
the vertices of the cevian triangle of U, namely,

$$0 : v : w, \quad u : 0 : w, \qquad u : v : 0; \tag{5}$$

and the points invariant under P-isoconjugation:

$$\frac{1}{\sqrt{p}} : \frac{1}{\sqrt{q}} : \frac{1}{\sqrt{r}}, \tag{6}$$

$$\frac{-1}{\sqrt{p}} : \frac{1}{\sqrt{q}} : \frac{1}{\sqrt{r}}, \quad \frac{1}{\sqrt{p}} : \frac{-1}{\sqrt{q}} : \frac{1}{\sqrt{r}}, \quad \frac{1}{\sqrt{p}} : \frac{1}{\sqrt{q}} : \frac{-1}{\sqrt{r}}. \tag{7}$$

As an illustration of (i), the cubics $Z(U, X_1)$ and $Z(U, X_{31})$ are self-isogonal conjugate and self-isotomic conjugate, respectively. Named cubics of the type $Z(U, X_1)$ include the Thomson ($U = X_2$), Darboux ($U = X_{20}$), Neuberg ($U = X_{30}$), Ortho ($U = X_4$), and Feuerbach ($U = X_5$). The Lucas cubic is $Z(X_{69}, X_{31})$, and the Spieker, $Z(X_8, X_{58})$. Table 1 offers a few less familiar cubics.

It is easy to check that the points

$$U \copyright X_1 = -u + v + w : u - v + w : u + v - w,$$
$$U \copyright U^{-1} = \frac{1}{u}\left(-\frac{1}{u^2} + \frac{1}{v^2} + \frac{1}{w^2}\right) : \frac{1}{v}\left(\frac{1}{u^2} - \frac{1}{v^2} + \frac{1}{w^2}\right) : \frac{1}{w}\left(\frac{1}{u^2} + \frac{1}{v^2} - \frac{1}{w^2}\right)$$

lie on $Z(U)$. Since their isogonal conjugates also lie on $Z(U)$, we have four more points on $Z(U, P)$ in the special case that $P = X_1$.

U	P	Centers on cubic $Z(U,P)$
X_{385}	X_1	$X_1, X_2, X_6, X_{32}, X_{76}, X_{98}, X_{385}, X_{511}, X_{694}$
X_{395}	X_1	$X_1, X_2, X_6, X_{14}, X_{16}, X_{18}, X_{62}, X_{395}$
X_{396}	X_1	$X_1, X_2, X_6, X_{13}, X_{15}, X_{17}, X_{61}, X_{396}$
X_{476}	X_1	$X_1, X_{30}, X_{74}, X_{110}, X_{476}, X_{523}, X_{526}$
X_{171}	X_2	$X_2, X_{31}, X_{42}, X_{43}, X_{55}, X_{57}, X_{81}, X_{171}, X_{365}, X_{846}, X_{893}$
X_{894}	X_6	$X_6, X_7, X_9, X_{37}, X_{75}, X_{86}, X_{87}, X_{192}, X_{256}, X_{366}, X_{894}, X_{1045}$
X_{309}	X_{31}	$X_2, X_{40}, X_{77}, X_{189}, X_{280}, X_{309}, X_{318}, X_{329}, X_{347}, X_{962}$
X_{226}	X_{55}	$X_2, X_{57}, X_{81}, X_{174}, X_{226}, X_{554}, X_{559}, X_{1029}, X_{1081}, X_{1082}$
X_{291}	X_{239}	$X_1, X_6, X_{42}, X_{57}, X_{239}, X_{291}, X_{292}, X_{672}, X_{894}$
X_{292}	X_{238}	$X_1, X_2, X_{37}, X_{87}, X_{171}, X_{238}, X_{241}, X_{291}, X_{292}$

Table 1

4. Trilinear generalization: $\Gamma(P)$

Next we seek the locus of a point $X = \alpha : \beta : \gamma$ (trilinears) for which the cevian triangle T and the cevian triangle

$$\widehat{T} = \begin{pmatrix} 0 & r\gamma & q\beta \\ r\gamma & 0 & p\alpha \\ q\beta & p\alpha & 0 \end{pmatrix}$$

of the P-isoconjugate of X have equal areas. For this, the method leading to (1) yields a cubic denoted by $\Gamma(P)$:

$$ap(rb^2-qc^2)\alpha(q\beta^2-r\gamma^2)+bq(pc^2-ra^2)\beta(r\gamma^2-p\alpha^2)+cr(qa^2-pb^2)\gamma(p\alpha^2-q\beta^2)=0, \tag{8}$$

except for $P = X_{31} = a^2 : b^2 : c^2$; that is, except when P-isoconjugation is isotomic conjugation, for which the two triangles have equal areas for all X. The cubic (8) is $Z(U,P)$ for

$$U = U(P) = a(rb^2 - qc^2) : b(pc^2 - ra^2) : c(qa^2 - pb^2),$$

a point on $\mathcal{L}^\infty$. As in Section 3, the vertices A, B, C and the points (5)-(7) lie on $\Gamma(P)$.

Let U^* denote the P-isoconjugate of U. This is the trilinear pole of the line XX_2, where $X = \frac{a}{p} : \frac{b}{q} : \frac{c}{r}$, the P-isoconjugate of X_2. Van Lamoen has noted that since U lies on the trilinear polar, L, of the P-isoconjugate of the centroid (i.e., L has equation $\frac{p\alpha}{a} + \frac{q\beta}{b} + \frac{r\gamma}{c} = 0$), and U also lies on $\mathcal{L}^\infty$, we have U^* lying on the Steiner circumellipse and on the conic

$$\frac{pa}{\alpha} + \frac{qb}{\beta} + \frac{rc}{\gamma} = 0, \tag{9}$$

this being the P-isoconjugate of $\mathcal{L}^\infty$.

Theorem 1. *Suppose P_1 and P_2 are distinct points, collinear with but not equal to X_{31}. Then $U(P_2) = U(P_1)$.*

Proof. Write $P_1 = p_1 : q_1 : r_1$ and $P_2 = p_2 : q_2 : r_2$. Then for some $s = s(a,b,c) \neq 0$,

$$a^2 = sp_1 + p_2, \qquad b^2 = sq_1 + q_2, \qquad c^2 = sr_1 + r_2,$$

so that for $f(a,b,c) := a[(c^2 - sr_1)b^2 - (b^2 - sq_1)c^2]$, we have

$$\begin{aligned} U(P_2) &= f(a,b,c) : f(b,c,a) : f(c,a,b) \\ &= a(sc^2q_1 - sb^2r_1) : b(sa^2r_1 - sc^2p_1) : c(sb^2p_1 - sa^2q_1) \\ &= U(P_1). \end{aligned}$$

□

Example 1. For each point P on the line $X_1X_{31,}$ the pivot $U(P)$ is the isogonal conjugate (X_{512}) of the Steiner point (X_{99}). Such points P include the Schiffler point (X_{21}), the isogonal conjugate (X_{58}) of the Spieker center, and the isogonal conjugate (X_{63}) of the Clawson point.

The cubic $\Gamma(P)$ meets $\mathcal{L}^\infty$ in three points. Aside from U, the other two are where $\mathcal{L}^\infty$ meets the conic (9). If (9) is an ellipse, then the two points are nonreal. In case P is the incenter, so that the cubic is the equal areas cubic, the two points are given in [6, p.116] by the ratios [6]

$$e^{\pm iB} : e^{\mp iA} : -1.$$

Theorem 2. *The generalized Brocard points defined by*

$$\frac{qc}{b} : \frac{ra}{c} : \frac{pb}{a} \qquad \text{and} \qquad \frac{rb}{c} : \frac{pc}{a} : \frac{qa}{b} \tag{10}$$

lie on $\Gamma(P)$.

Proof. Writing ordered triples for the two points, we have

$$\begin{aligned} & (a(rb^2 - qc^2),\ b(pc^2 - ra^2),\ c(qa^2 - pb^2)) \\ = \ & abc(\frac{qc}{b}, \frac{ra}{c}, \frac{pb}{a}) + abc(\frac{rb}{c}, \frac{pc}{a}, \frac{qa}{b}), \end{aligned}$$

showing U as a linear combination of the points in (10). Since those two are isogonal conjugates collinear with U, they lie on $\Gamma(P)$. □

If P is a triangle center, then the generalized Brocard points (10) comprise a bicentric pair of points. In §8, we offer geometric constructions for such points.

[6] The pair is also given by $-1 : e^{\pm iC} : e^{\mp iB}$ and by $e^{\mp iC} : -1 : e^{\pm iA}$. Multiplying the three together and then by -1 gives cubes in "central form" with first coordinates

$$\cos(B - C) \pm i \sin(B - C).$$

The other coordinates are now given from the first by cyclic permutations.

5. Barycentric generalization: $\hat{\Gamma}(P)$

Here, we seek the locus of a point $X = \alpha : \beta : \gamma$ (barycentrics) for which the cevian triangle of the P-reciprocal conjugate of X and that of X have equal areas. The method presented in §1 yields a cubic that we denote by $\hat{\Gamma}(P)$:

$$p(q-r)\alpha(r\beta^2-q\gamma^2)+q(r-p)\beta(p\gamma^2-r\alpha^2)+r(p-q)\gamma(q\alpha^2-p\beta^2)=0, \quad (11)$$

In particular, the equal areas cubic (1) is given by (11) using

$$(p:q:r)_b = (a^2 : b^2 : c^2)_b.$$

In contrast to (11), if equation (1) is written as $s(a, b, c, \alpha, \beta, \gamma) = 0$, then

$$s(\alpha, \beta, \gamma, a, b, c) = s(a, b, c, \alpha, \beta, \gamma),$$

a symmetry stemming from the use of trilinear coordinates and isogonal conjugation.

6. A sextic

For comparison with the cubic $\Gamma(P)$ of §4, it is natural to ask about the locus of a point X for which the anticevian triangle of X and that of its isogonal conjugate have equal areas. The result is easily found to be the self-isogonal sextic

$$\begin{aligned} & \alpha\beta\gamma(-a\alpha+b\beta+c\gamma)(a\alpha-b\beta+c\gamma)(a\alpha+b\beta-c\gamma) \\ = \quad & (-a\beta\gamma+b\gamma\alpha+c\alpha\beta)(a\beta\gamma-b\gamma\alpha+c\alpha\beta)(a\beta\gamma+b\gamma\alpha-c\alpha\beta), \end{aligned}$$

on which lie A, B, C, the incenter, excenters, and the two Brocard points. Remarkably, the vertices A, B, C are triple points of this sextic.

7. Prescribed area cubic: $\Lambda(P)$

Suppose $P = p : q : r$ (trilinears) is a point, and let $K\sigma$ be the oriented area of the cevian triangle of P. The method used in §1 shows that if $X = \alpha : \beta : \gamma$, then the cevian triangle of X has area $K\sigma$ if

$$k_1k_2k_3abc\alpha\beta\gamma = 8K\sigma^3, \quad (12)$$

where $k_1 = \frac{2\sigma}{b\beta+c\gamma}$ and k_2 and k_3 are obtained cyclically. Substituting into (12) and simplifying gives

$$K = 2 \cdot \frac{pa}{bq+cr} \cdot \frac{qb}{cr+ap} \cdot \frac{rc}{ap+bq}. \quad (13)$$

The locus of X for which (13) holds is therefore given by the equation

$$(bq+cr)(cr+ap)(ap+bq)\alpha\beta\gamma - pqr(b\beta+c\gamma)(c\gamma+a\alpha)(a\alpha+b\beta) = 0. \quad (14)$$

We call this curve the *prescribed area cubic for* P (or for K) and denote it by $\Lambda(P)$. One salient feature of $\Lambda(P)$, easily checked by substituting

$$\frac{1}{a^2\alpha}, \ \frac{1}{b^2\beta}, \ \frac{1}{c^2\gamma}$$

for α, β, γ, respectively, into the left side of (14), is that $\Lambda(P)$ is self-isotomic. That is, if X lies on $\Lambda(P)$ but not on a sideline of $\triangle ABC$, then so does its isotomic conjugate, which we denote by $\widetilde{X}$. (Of course, we already know that $\Lambda(P)$ is self-isotomic, by the note just after (8)).

If $(bq-cr)(cr-ap)(ap-bq) \neq 0$, then the line $P\widetilde{P}$ meets $\Lambda(P)$ in three points, namely P, $\widetilde{P}$, and the point

$$P' := \frac{a^2p^2 - bcqr}{a^2p(bq-cr)} \;:\; \frac{b^2q^2 - carp}{b^2q(cr-ap)} \;:\; \frac{c^2r^2 - abpq}{c^2r(ap-bq)}.$$

If P is a triangle center on $\Lambda(P)$, then $\widetilde{P}$, P', and $\widetilde{P'}$ are triangle centers on $\Lambda(P)$. Since $\widetilde{P'}$ is not collinear with the others, three triangle centers on $\Lambda(P)$ can be found as points of intersection of $\Lambda(P)$ with the lines joining $\widetilde{P'}$ to P, $\widetilde{P}$, and P'. Then more central lines are defined, bearing triangle centers that lie on $\Lambda(P)$, and so on. Some duplication of centers thus defined inductively can be expected, but one wonders if, for many choices of P, this scheme accounts for infinitely many centers lying on $\Lambda(P)$.

It is easy to check that $\Lambda(P)$ meets the line at infinity in the following points:

$$A' := 0 : c : -b, \qquad B' := -c : 0 : a, \qquad C' := b : -a : 0.$$

Three more points are found by intersecting lines PA', PB', PC' with $\Lambda(P)$:

$$A'' := bcp : c^2r : b^2q, \quad B'' := c^2r : caq : a^2p, \quad C'' := b^2q : a^2p : abr.$$

A construction for A'' is given by the equation $A'' = PA' \cap \widetilde{P}A$.

Line AP meets $\Lambda(P)$ in the collinear points A, P, and, as is easily checked, the point

$$\frac{bcqr}{pa^2} : q : r.$$

Writing this and its cyclical cousins integrally, we have these points on $\Lambda(P)$:

$$bcqr : a^2pq : a^2rp, \qquad b^2pq : carp : b^2qr, \qquad c^2rp : c^2qr : abpq.$$

We have seen for given P how to form K. It is of interest to reverse these. Suppose a prescribed area is specified as $K\sigma$, where K has the form

$$k(a,b,c)k(b,c,a)k(c,a,b)$$

in which $k(a,b,c)$ is homogeneous of degree zero in a,b,c.[7] We abbreviate the factors as k_a, k_b, k_c and seek a point $P = p : q : r$ satisfying

$$K = k_ak_bk_c = \frac{2abcpqr}{(bq+cr)(cr+ap)(ap+bq)}.$$

Solving the system obtained cyclically from

$$k_a = \frac{\sqrt[3]{2}ap}{bq+cr} \tag{15}$$

yields

$$p : q : r = \frac{k_a}{a(\sqrt[3]{2}+k_a)} : \frac{k_b}{b(\sqrt[3]{2}+k_b)} : \frac{k_c}{c(\sqrt[3]{2}+k_c)}$$

[7]That is, $k(ta,tb,tc) = k(a,b,c)$, where t is an indeterminate.

except for $k_a = -\sqrt[3]{2}$, which results from (15) with $P = X_{512}$. The following table offers a variety of examples:

P	$k_a/\sqrt[3]{2}$
X_1	$\frac{a}{b+c}$
X_2	1
X_3	$\frac{\sin 2A}{\sin 2B+\sin 2C}$
X_4	$\frac{\tan A}{\tan B+\tan C}$
X_{10}	$\frac{b+c}{a}$
X_{57}	$-\frac{a}{b+c}$
X_{870}	$\frac{bc}{b^2+c^2}$
X_{873}	$\frac{2bc}{b^2+c^2}$

Table 2

Next, suppose $U = u : v : w$ is a point, not on a sideline of $\triangle ABC$, and let

$$P = \frac{vc}{b} : \frac{wa}{c} : \frac{ub}{a}.$$

Write out K as in (13), and use not (15), but instead, put

$$k_a = \frac{\sqrt[3]{2}a^2u}{b^2w + c^2v},$$

corresponding to the point $U \cdot X_6 = ua : vb : wc$, in the sense that the cevian triangle of $U \cdot X_6$ and that of P have equal areas. Likewise, the cevian triangle of the point

$$P' = \frac{wb}{c} : \frac{uc}{a} : \frac{va}{b},$$

has the same area, $K\sigma$. The points P and P' are essentially those of Theorem 2.

Three special cases among the cubics $\Lambda(P)$ deserve further comment. First, for $K = 2$, corresponding to $P = X_{512}$, equation (14) takes the form

$$(a\alpha + b\beta + c\gamma)(bc\beta\gamma + ca\gamma\alpha + ab\alpha\beta) = 0. \qquad (16)$$

Since $\mathcal{L}^\infty$ is given by the equation $a\alpha + b\beta + c\gamma = 0$ and the Steiner circumellipse is given by

$$bc\beta\gamma + ca\gamma\alpha + ab\alpha\beta = 0,$$

the points satisfying (16) occupy the line and the ellipse together. J.H. Weaver [8] discusses the cubic.

Second, when $K = \frac{1}{4}$, the cubic $\Lambda(P)$ is merely a single point, the centroid. Finally, we note that $\Lambda(X_6)$ passes through these points:

$$a : b : c, \quad a : c : b, \quad b : c : a, \quad b : a : c, \quad c : a : b, \quad c : b : a. \qquad (17)$$

8. Constructions

In the preceding sections, certain algebraically defined points, as in (17), have appeared. In this section, we offer Euclidean constructions for such points.For given $U = u : v : w$ and $X = x : y : z$ and let us begin with the trilinear product, quotient, and square root, denoted respectively by $U \cdot X$, U/X, and $\sqrt{X}$.

Constructions for closely related barycentric product, quotient, and square root are given in [9], and these constructions are easily adapted to give the trilinear results.

We turn now to a construction from X of the point $x : z : y$. In preparation, decree as *positive* the side of line AB that contains C, and also the side of line CA that contains B. The opposite sides will be called *negative*.

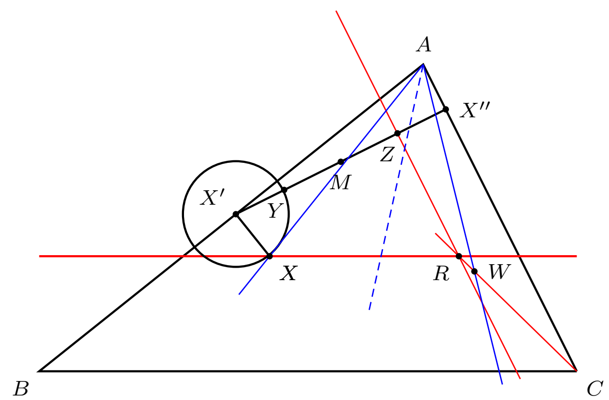


Figure 2. Construction of $W = x : z : y$ from $X = x : y : z$

Let X' be the foot of the perpendicular from X on line AB, and let X'' be the foot of the perpendicular from X' on line CA. Let M be the midpoint of segment $X'X''$, and let $\mathcal{O}$ be the circle centered at X' and passing through X. Line $X'X''$ meets circle $\mathcal{O}$ in two points; let Y be the one closer to M, as in Figure 2, and let Z' be the reflection of Y in M. If X is on the positive side of AB and Z' is on the positive side of CA, or if X is on the negative side of AB and Z' is on the negative side of CA, then let $Z = Z'$; otherwise let Z be the reflection of Z' in line CA.

Now line L through Z parallel to line CA has directed distance kz from line CA, where kx is the directed distance from line BC of the line L' through X parallel to BC. Let $R = L \cap L'$. Line CR has equation $z\alpha = x\beta$. Let L'' be the reflection of line AX about the internal angle bisector of $\angle CAB$. This line has equation $y\beta = z\gamma$. Geometrically and algebraically, it is clear that $x : z : y = CR \cap L''$, labeled W in Figure 2.

One may similarly construct the point $y : z : x$ as the intersection of lines $x\beta = z\gamma$ and $z\alpha = y\beta$. Then any one of the six points

$$x : y : z, \quad x : z : y, \quad y : z : x, \quad y : x : z, \quad z : x : y, \quad z : y : x,$$

can serve as a starting point for constructing the other five. (A previous appearance of these six points is [4, p.243], where an equation for the Yff conic, passing through the six points, is given.)

The methods of this section apply, in particular, to the constructing of the generalized Brocard points (10); e.g., for given $P = p : q : r$, construct $P' := q : r : p$, and then construct $P' \cdot \Omega_1$.

References

[1] H.M. Cundy and C.F. Parry, Some cubic curves associated with a triangle, *Journal of Geometry*, 53 (1995) 41–66.

[2] H.M. Cundy and C.F. Parry, Geometrical properties of some Euler and circular cubics. Part 1, *Journal of Geometry*, 66 (1999) 72–103.

[3] K. Dean and F.M. van Lamoen, Geometric construction of reciprocal conjugations, *Forum Geom.*, 1 (2001) 115-120.

[4] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1-285.

[5] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://cedar.evansville.edu/˜ck6/encyclopedia/.

[6] C. Kimberling, Conics associated with a cevian nest, *Forum Geom.*, 1 (2001) 141–150.

[7] C.A. Scott, *Projective Methods in Plane Analytic Geometry*, third edition, Chelsea, New York, 1961.

[8] J.H. Weaver, On a cubic curve associated with a triangle, *Mathematics Magazine*, 11 (1937) 293–296.

[9] P. Yiu, The uses of homogeneous barycentric coordinates in plane euclidean geometry, *Int. J. Math. Educ. Sci. Technol.*, 31 (2000) 569–578.

Clark Kimberling: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, USA

E-mail address: ck6@evansville.edu

Forum Geometricorum
Volume 1 (2001) 173–175.

FORUM GEOM
ISSN 1534-1178

A Feuerbach Type Theorem on Six Circles

Lev Emelyanov

According to the famous Feuerbach theorem there exists a circle which is tangent internally to the incircle and externally to each of the excircles of a triangle. This is the nine-point circle of the triangle. We obtain a similar result by replacing the excircles with circles each tangent internally to the circumcircle and to the sides at the traces of a point. We make use of Casey's theorem. See, for example, [1, 2].

Theorem (Casey)**.** *Given four circles $\mathcal{C}_i$, $i = 1, 2, 3, 4$, let t_{ij} be the length of a common tangent between $\mathcal{C}_i$ and $\mathcal{C}_j$. The four circles are tangent to a fifth circle (or line) if and only if for appropriate choice of signs,*

$$t_{12}t_{34} \pm t_{13}t_{42} \pm t_{14}t_{23} = 0.$$

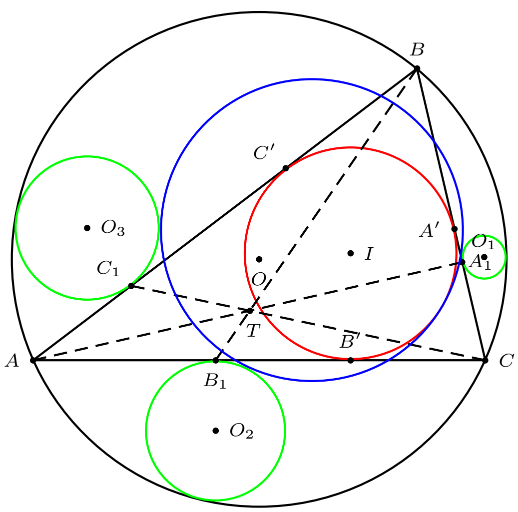


Figure 1

In this note we establish the following theorem. Let ABC be a triangle of side lengths $BC = a$, $CA = b$, and $AB = c$.

Theorem. *Let points A_1, B_1 and C_1 be on the sides BC, CA and AB respectively of triangle ABC. Construct three circles (O_1), (O_2) and (O_3) outside the triangle which is tangent to the sides of ABC at A_1, B_1 and C_1 respectively and also tangent to the circumcircle of ABC. The circle tangent externally to these three circles is also tangent to the incircle of triangle ABC if and only if the lines AA_1, BB_1 and CC_1 are concurrent.*

Publication Date: December 13, 2001. Communicating Editor: Paul Yiu.

Proof. Let in our case $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ be the circles (O_1), (O_2), (O_3) and the incircle respectively. With reference to Figure 1, we show that

$$t_{12}t_{34} - t_{13}t_{42} - t_{14}t_{23} = 0, \tag{1}$$

where t_{12}, t_{13} and t_{23} are the lengths of the common extangents, t_{34}, t_{24} and t_{14} are the lengths of the common intangents.

Let (A) be the degenerate circle $A(0)$ (zero radius) and $t_i(A)$ be the length of the tangent from A to $\mathcal{C}_i$. Similar notations apply to vertices B and C. Applying Casey's theorem to circles (A), (B), (O_1) and (C), which are all tangent to the circumcircle, we have

$$t_1(A) \cdot a = c \cdot CA_1 + b \cdot A_1B.$$

From this we obtain $t_1(A)$, and similarly $t_2(B)$ and $t_3(C)$:

$$\begin{aligned} t_1(A) &= \frac{c \cdot CA_1 + b \cdot A_1B}{a}, && (2)\\ t_2(B) &= \frac{a \cdot AB_1 + c \cdot B_1C}{b}, && (3)\\ t_3(C) &= \frac{b \cdot BC_1 + a \cdot C_1A}{c}. && (4) \end{aligned}$$

Applying Casey's theorem to circles (B), (C), (O_2) and (O_3), we have

$$t_2(B)t_3(C) = a \cdot t_{23} + CB_1 \cdot C_1B.$$

Using (3) and (4), we obtain t_{23}, and similarly, t_{13} and t_{12}:

$$\begin{aligned} t_{23} &= \frac{a \cdot C_1A \cdot AB_1 + b \cdot AB_1 \cdot BC_1 + c \cdot AC_1 \cdot CB_1}{bc}, && (5)\\ t_{13} &= \frac{b \cdot A_1B \cdot BC_1 + c \cdot BC_1 \cdot CA_1 + a \cdot BA_1 \cdot AC_1}{ca}, && (6)\\ t_{12} &= \frac{c \cdot B_1C \cdot CA_1 + a \cdot CA_1 \cdot AB_1 + b \cdot CB_1 \cdot BA_1}{ab}. && (7) \end{aligned}$$

In the layout of Figure 1, with A', B', C' the touch points of the incircle with the sides, the lengths of the common tangents of the circles (O_1), (O_2), (O_3) with the incircle are

$$\begin{aligned} t_{14} &= A_1A' = -CA_1 + CA' = -CA_1 + \frac{a+b-c}{2}, && (8)\\ t_{24} &= B_1B' = -AB_1 + AB' = -AB_1 + \frac{b+c-a}{2}, && (9)\\ t_{34} &= C_1C' = BC_1 - BC' = BC_1 - \frac{c+a-b}{2}. && (10) \end{aligned}$$

Substituting (5)-(10) into (1) and simplifying, we obtain

$$t_{12}t_{34} - t_{13}t_{24} - t_{14}t_{23} = \frac{F(a,b,c)}{abc} \cdot (AB_1 \cdot BC_1 \cdot CA_1 - A_1B \cdot B_1C \cdot C_1A),$$

where

$$F(a,b,c) = 2bc + 2ca + 2ab - a^2 - b^2 - c^2.$$

Since $F(a,b,c)$ can be rewritten as

$$(c+a-b)(a+b-c)+(a+b-c)(b+c-a)+(b+c-a)(c+a-b),$$

it is clearly nonzero. It follows that $t_{12}t_{34}-t_{13}t_{24}-t_{14}t_{23}=0$ if and only if

$$AB_1 \cdot BC_1 \cdot CA_1 - A_1B \cdot B_1C \cdot C_1A = 0. \tag{11}$$

By the Ceva theorem, (11) is the condition for the concurrency of AA_1, BB_1 and CC_1. It is clear that for different positions of the touch points of circles (O_1), (O_2) and (O_3) relative to those of the incircle, the proofs are analogous. □

References

[1] J. L. Coolidge, *A Treatise on Circles and Spheres*, 1917, Chelsea reprint.

[2] I. M. Yaglom, *Geometric Transformations*, 3 volumes, Mathematical Association of America, 1968.

Lev Emelyanov: 18-31 Proyezjaia Street, Kaluga, Russia 248009

E-mail address: emelyanov@kaluga.ru

Forum Geometricorum
Volume 1 (2001) 176.

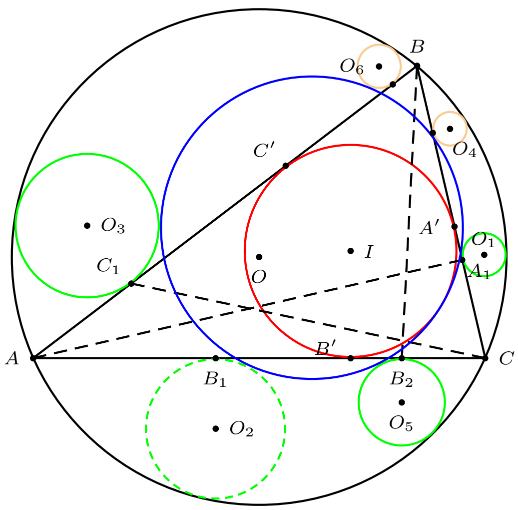
FORUM GEOM
ISSN 1534-1178

Correction to

A Feuerbach Type Theorem on Six Circles

Lev Emelyanov

Floor van Lamoen has kindly pointed out that the necessity part of the main theorem of [1] does not hold. In the layout of Figure 1 there, it is possible to have a circle (O_5) outside the triangle, tangent to both the circumcircle and the "new" circle, but to AC at a point B_2 between B' and C. The points of tangency of the circles (O_1), (O_5) and (O_3) with the sides of triangles do not satisfy Ceva's theorem. Likewise, it is also possible to place circles (O_4) and (O_6) on the sides BC and AB so that the points of tangency do not satisfy Ceva's theorem.



We hereby modify the statement of the theorem as follows.

Theorem. *Let A_1, B_1, C_1 be the traces of an interior point T on the side lines of triangle ABC. Construct three circles (O_1), (O_2) and (O_3) outside the triangle which are tangent to the sides at A_1, B_1, C_1 respectively and also tangent to the circumcircle of ABC. The circle tangent externally to these three circles is also tangent to the incircle of triangle ABC.*

References

[1] L. Emelyanov, A Feuerbach type theorem on six circles, *Forum Geom.*, 1 (2001) 173 – 175.

Lev Emelyanov: 18-31 Proyezjaia Street, Kaluga, Russia 248009
E-mail address: emelyanov@kaluga.ru

Publication Date: December 31, 2001. Communicating Editor: Paul Yiu.

FORUM GEOMETRICORUM

A Journal on Classical Euclidean Geometry and Related Areas

published by

Department of Mathematical Sciences
Florida Atlantic University

Volume 2
2002

http://forumgeom.fau.edu

ISSN 1534-1178

Editorial Board

Advisors:

John H. Conway	Princeton, New Jersey, USA
Julio Gonzalez Cabillon	Montevideo, Uruguay
Richard Guy	Calgary, Alberta, Canada
George Kapetis	Thessaloniki, Greece
Clark Kimberling	Evansville, Indiana, USA
Kee Yuen Lam	Vancouver, British Columbia, Canada
Tsit Yuen Lam	Berkeley, California, USA
Fred Richman	Boca Raton, Florida, USA

Editor-in-chief:

Paul Yiu	Boca Raton, Florida, USA

Editors:

Clayton Dodge	Orono, Maine, USA
Roland Eddy	St. John’s, Newfoundland, Canada
Jean-Pierre Ehrmann	Paris, France
Lawrence Evans	La Grange, Illinois, USA
Chris Fisher	Regina, Saskatchewan, Canada
Rudolf Fritsch	Munich, Germany
Bernard Gibert	St Etiene, France
Antreas P. Hatzipolakis	Athens, Greece
Michael Lambrou	Crete, Greece
Floor van Lamoen	Goes, Netherlands
Fred Pui Fai Leung	Singapore, Singapore
Daniel B. Shapiro	Columbus, Ohio, USA
Steve Sigur	Atlanta, Georgia, USA
Man Keung Siu	Hong Kong, China
Peter Woo	La Mirada, California, USA

Technical Editors:

Yuandan Lin	Boca Raton, Florida, USA
Aaron Meyerowitz	Boca Raton, Florida, USA
Xiao-Dong Zhang	Boca Raton, Florida, USA

Consultants:

Frederick Hoffman	Boca Raton, Floirda, USA
Stephen Locke	Boca Raton, Florida, USA
Heinrich Niederhausen	Boca Raton, Florida, USA

Table of Contents

Forum Geometricorum
Volume 2 (2002) 1–4.

FORUM GEOM
ISSN 1534-1178

A Pair of Kiepert Hyperbolas

Jean-Pierre Ehrmann

Abstract. The solution of a locus problem of Hatzipolakis can be expressed in terms of a simple relationship concerning points on a pair of Kiepert hyperbolas associated with a triangle. We study a generalization.

Let P be a finite point in the plane of triangle ABC. Denote by a, b, c the lengths of the sides BC, CA, AB respectively, and by A_H, B_H, C_H the feet of the altitudes. We consider rays through P in the directions of the altitudes AA_H, BB_H, CC_H, and, for a nonzero constant k, choose points A', B', C' on these rays such that

$$PA' = ka, \qquad PB' = kb, \qquad PC' = kc. \tag{1}$$

Antreas P. Hatzipolakis [1] has asked, for $k = 1$, for the locus of P for which triangle $A'B'C'$ is perspective with ABC.

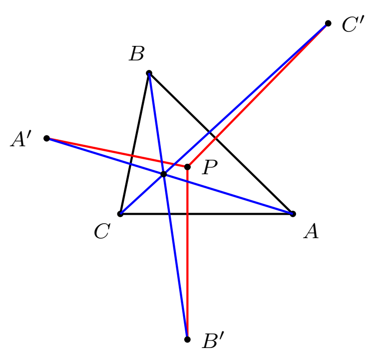


Figure 1

We tackle the general case by making use of homogeneous barycentric coordinates with respect to ABC. Thus, write $P = (u : v : w)$. In the notations introduced by John H. Conway, [1]

$$\begin{aligned}
A' &= (uS - k(u+v+w)a^2 : vS + k(u+v+w)S_C : wS + k(u+v+w)S_B),\\
B' &= (uS + k(u+v+w)S_C : vS - k(u+v+w)b^2 : wS + k(u+v+w)S_A),\\
C' &= (uS + k(u+v+w)S_B : vS + k(u+v+w)S_A : wS - k(u+v+w)c^2).
\end{aligned}$$

Publication Date: January 18, 2002. Communicating Editor: Paul Yiu.
The author expresses his sincere thanks to Floor van Lamoen and Paul Yiu for their help and their valuable comments.

[1] Let ABC be a triangle of side lengths a, b, c, and area $\frac{1}{2}S$. For each ϕ, $S_\phi := S \cdot \cot\phi$. Thus, $S_A = \frac{1}{2}(b^2+c^2-a^2)$, $S_B = \frac{1}{2}(c^2+a^2-b^2)$, and $S_C = \frac{1}{2}(a^2+b^2-c^2)$. These satisfy $S_AS_B + S_BS_C + S_CS_A = S^2$ and other simple relations. For a brief summary, see [3, §1].

The equations of the lines AA', BB', CC' are

$$
\begin{aligned}
(wS+k(u+v+w)S_B)y-(vS+k(u+v+w)S_C)z &=0, && (2)\\
-(wS+k(u+v+w)S_A)x \qquad +(uS+k(u+v+w)S_C)z &=0, && (3)\\
(vS+k(u+v+w)S_A)x-(uS+k(u+v+w)S_B)y &=0. && (4)
\end{aligned}
$$

These three lines are concurrent if and only if

$$
\begin{vmatrix}
0 & wS+k(u+v+w)S_B & -(vS+k(u+v+w)S_C)\\
-(wS+k(u+v+w)S_A) & 0 & uS+k(u+v+w)S_C\\
vS+k(u+v+w)S_A & -(uS+k(u+v+w)S_B) & 0
\end{vmatrix} = 0.
$$

This condition can be rewritten as

$$kS(u+v+w)(S\cdot K(u,v,w)-k(u+v+w)L(u,v,w))=0,$$

where

$$
\begin{aligned}
K(u,v,w) &=(b^2-c^2)vw+(c^2-a^2)wu+(a^2-b^2)uv, && (5)\\
L(u,v,w) &=(b^2-c^2)S_Au+(c^2-a^2)S_Bv+(a^2-b^2)S_Cw. && (6)
\end{aligned}
$$

Note that $K(u,v,w)=0$ and $L(u,v,w)=0$ are respectively the equations of the Kiepert hyperbola and the Euler line of triangle ABC. Since P is a finite point and k is nonzero, we conclude, by writing $k=\tan\phi$, that the locus of P for which $A'B'C'$ is perspective with ABC is the rectangular hyperbola

$$S_\phi K(u,v,w)-(u+v+w)L(u,v,w)=0 \qquad (7)$$

in the pencil generated by the Kiepert hyperbola and the Euler line.

Floor van Lamoen [2] has pointed out that this hyperbola (7) is the Kiepert hyperbola of a Kiepert triangle of the dilated (anticomplementary) triangle of ABC. Specifically, let $\mathcal{K}(\phi)$ be the Kiepert triangle whose vertices are the apexes of similar isosceles triangles of base angles ϕ constructed on the sides of ABC. It is shown in [3] that the Kiepert hyperbola of $\mathcal{K}(\phi)$ has equation

$$2S_\phi(\sum_{\text{cyclic}}(b^2-c^2)yz)+(x+y+z)(\sum_{\text{cyclic}}(b^2-c^2)(S_A+S_\phi)x)=0.$$

If we replace x, y, z respectively by $v+w$, $w+u$, $u+v$, this equation becomes (7) above. This means that the hyperbola (7) is the Kiepert hyperbola of the Kiepert triangle $\mathcal{K}(\phi)$ of the dilated triangle of ABC.[2]

The orthocenter H and the centroid G are always on the locus. Trivially, if $P=H$, the perspector is the same point H. For $P=G$, the perspector is the point [3]

$$\left(\frac{1}{3kS_A+S}:\frac{1}{3kS_B+S}:\frac{1}{3kS_C+S}\right),$$

[2]The Kiepert triangle $\mathcal{K}(\phi)$ of the dilated triangle of ABC is also the dilated triangle of the Kiepert triangle $\mathcal{K}(\phi)$ of triangle ABC.

[3]In the notations of [3], this is the Kiepert perspector $K(\arctan 3k)$.

the second common point of Kiepert hyperbola and the tangent at P to the locus of P, the Kiepert hyperbola of the dilated triangle of $\mathcal{K}(\phi)$.

Now we identify the perspector when P is different from G. Addition of the equations (2,3,4) of the lines AA', BB', CC' gives

$$(v-w)x+(w-u)y+(u-v)z=0.$$

This is the equation of the line joining P to the centroid G, showing that the perspector lies on the line GP.

We can be more precise. Reorganize the equations (2,3,4) as

$$\begin{aligned} k(S_By-S_Cz)u+&(k(S_By-S_Cz)-Sz)v+(k(S_By-S_Cz)+Sy)w=0, && (8)\\ (k(S_Cz-S_Ax)+Sz)u+&\qquad k(S_Cz-S_Ax)v+(k(S_Cz-S_Ax)-Sx)w=0, && (9)\\ (k(S_Ax-S_By)-Sy)u+&(k(S_Ax-S_By)+Sx)v+\qquad k(S_Ax-S_By)w=0. && (10)\end{aligned}$$

Note that the combination x·(8) + y·(9) + z·(10) gives

$$k(u+v+w)(x(S_By-S_Cz)+y(S_Cz-S_Ax)+z(S_Ax-S_By))=0.$$

Since k and $u+v+w$ are nonzero, we have

$$(S_C-S_B)yz+(S_A-S_C)zx+(S_B-S_A)xy=0,$$

or equivalently, $(b^2-c^2)yz+(c^2-a^2)zx+(a^2-b^2)xy=0$. It follows that the perspector is also on the Kiepert hyperbola.

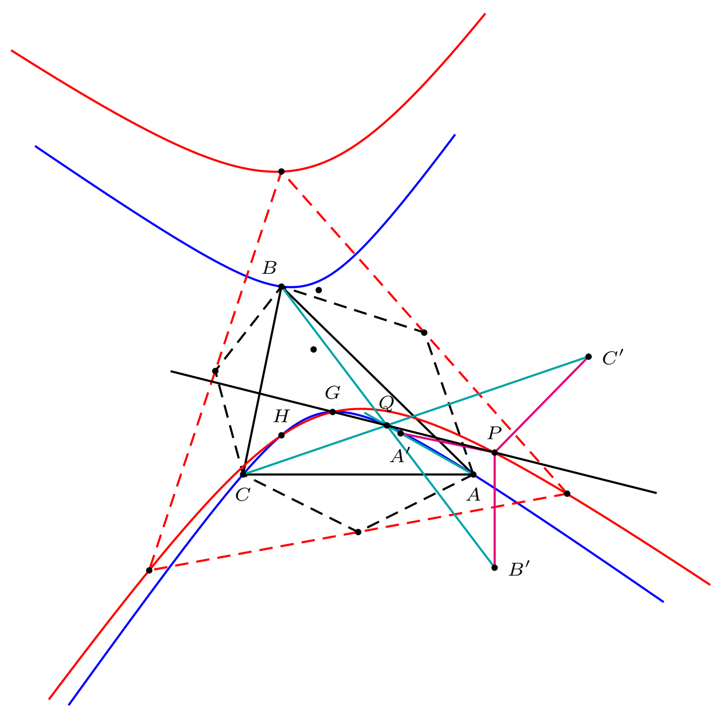


Figure 2

We summarize these results in the following theorem.

Theorem. *Let $k = \tan\phi$ be nonzero, and points A', B', C' be given by (1) along the rays through P parallel to the altitudes. The lines AA', BB', CC' are concurrent if and only if P lies on the Kiepert hyperbola of the Kiepert triangle $\mathcal{K}(\phi)$ of the dilated triangle. The intersection of these lines is the second intersection of the line GP and the Kiepert hyperbola of triangle ABC.*

If we change, for example, the orientation of PA', the locus of P is the rectangular hyperbola with center at the apex of the isosceles triangle on BC of base angle ϕ, [4] asymptotes parallel to the A-bisectors, and passing through the orthocenter H (and also the A-vertex $A^G = (-1 : 1 : 1)$ of the dilated triangle). For $P = A^G$, the perspector is the point $\left(\frac{1}{kS_A + S} : \frac{1}{kS_B - S} : \frac{1}{kS_C - S}\right)$, and for $P \neq A^G$, the second common point of the line PA^G and the rectangular circum-hyperbola with center the midpoint of BC.

We conclude by noting that for a positive k, the locus of P for which we can choose points A', B', C' on the perpendiculars through P to BC, CA, AB such that the lines AA', BB', CC' concur and the distances from P to A', B', C' are respectively k times the lengths of the corresponding side is the union of 8 rectangular hyperbolas.

References

[1] A. P. Hatzipolakis, Hyacinthos message 2510, March 1, 2001.
[2] F. M. van Lamoen, Hyacinthos message 2541, March 6, 2001.
[3] F. M. van Lamoen and P. Yiu, The Kiepert pencil of Kiepert hyperbolas, *Forum Geom.*, 1 (2001) 125–132.

Jean-Pierre Ehrmann: 6, rue des Cailloux, 92110 - Clichy, France
E-mail address: Jean-Pierre.EHRMANN@wanadoo.fr

[4]This point has coordinates $(-a^2 : S_C + S_\phi : S_B + S_\phi)$.

Forum Geometricorum
Volume 2 (2002) 5–13.

FORUM GEOM
ISSN 1534-1178

Some Concurrencies from Tucker Hexagons

Floor van Lamoen

Abstract. We present some concurrencies in the figure of Tucker hexagons together with the centers of their Tucker circles. To find the concurrencies we make use of extensions of the sides of the Tucker hexagons, isosceles triangles erected on segments, and special points defined in some triangles.

1. The Tucker hexagon $\mathcal{T}_\phi$ and the Tucker circle $\mathcal{C}_\phi$

Consider a scalene (nondegenerate) reference triangle ABC in the Euclidean plane, with sides $a = BC$, $b = CA$ and $c = AB$. Let B_a be a point on the sideline CA. Let C_a be the point where the line through B_a antiparallel to BC meets AB. Then let A_c be the point where the line through C_a parallel to CA meets BC. Continue successively the construction of parallels and antiparallels to complete a hexagon $B_aC_aA_cB_cC_bA_b$ of which B_aC_a, A_cB_c and C_bA_b are antiparallel to sides BC, CA and AB respectively, while B_cC_b, A_cC_a and A_bB_a are parallel to these respective sides.

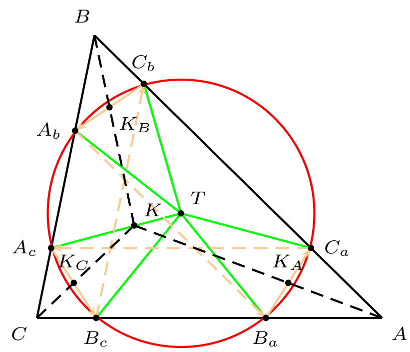


Figure 1

This is the well known way to construct a *Tucker hexagon*. Each Tucker hexagon is circumscribed by a circle, the *Tucker circle*. The three antiparallel sides are congruent; their midpoints K_A, K_B and K_C lie on the symmedians of ABC in such a way that $AK_A : AK = BK_B : BK = CK_C : CK$, where K denotes the symmedian point. See [1, 2, 3].

1.1. *Identification by central angles.* We label by $\mathcal{T}_\phi$ the specific Tucker hexagon in which the congruent central angles on the chords B_aC_a, C_bA_b and A_cB_c have measure 2ϕ. The circumcircle of the Tucker hexagon is denoted by $\mathcal{C}_\phi$, and its radius by r_ϕ. In this paper, the points B_a, C_a, A_b, C_b, A_c and B_c are the vertices of $\mathcal{T}_\phi$, and T denotes the center of the Tucker circle $\mathcal{C}_\phi$.

Publication Date: January 25, 2002. Communicating Editor: Paul Yiu.
The author thanks the Communicating Editor for his assistance with §7.

Let M_a, M_b and M_c be the midpoints of A_bA_c, B_aB_c and C_aC_b respectively. Since

$$\angle M_bTM_c = B + C, \quad \angle M_cTM_a = C + A, \quad \angle M_aTM_b = A + B,$$

the top angles of the isosceles triangles TA_bA_c, TB_cB_a and TC_aC_b have measures $2(A-\phi)$, $2(B-\phi)$, and $2(C-\phi)$ respectively. [1]

From these top angles, we see that the distances from T to the sidelines of triangle ABC are $r_\phi \cos(A-\phi)$, $r_\phi \cos(B-\phi)$ and $r_\phi \cos(C-\phi)$ respectively, so that in homogeneous barycentric coordinates,

$$T = (a\cos(A-\phi) : b\cos(B-\phi) : c\cos(C-\phi)).$$

For convenience we write $\overline{\phi} := \frac{\pi}{2} - \phi$. In the notations introduced by John H. Conway, [2]

$$T = (a^2(S_A + S_{\overline{\phi}}) : b^2(S_B + S_{\overline{\phi}}) : c^2(S_C + S_{\overline{\phi}})). \tag{1}$$

This shows that T is the isogonal conjugate of the Kiepert perspector $K(\overline{\phi})$.[3] We shall, therefore, write $K^*(\overline{\phi})$ for T. It is clear that $K^*(\overline{\phi})$ lies on the Brocard axis, the line through the circumcenter O and symmedian point K.

Some of the most important $K^*(\overline{\phi})$ are listed in the following table, together with the corresponding number in Kimberling's notation of [4, 5]. We write ω for the Brocard angle.

ϕ	$K^*(\overline{\phi})$	Kimberling's Notation
0	Circumcenter	X_3
ω	Brocard midpoint	X_{39}
$\pm\frac{\pi}{4}$	Kenmotu points	X_{371}, X_{372}
$\pm\frac{\pi}{3}$	Isodynamic centers	X_{15}, X_{16}
$\frac{\pi}{2}$	Symmedian point	X_6

1.2. *Coordinates.* Let K' and C_b' be the feet of the perpendiculars from $K^*(\overline{\phi})$ and C_b to BC. By considering the measures of sides and angles in $C_bC_b'K'K^*(\overline{\phi})$ we find that the (directed) distances α from C_b to BC as

$$\begin{aligned} \alpha &= r_\phi(\cos(A-\phi) - \cos(A+\phi)) \\ &= 2r_\phi \sin A \sin\phi. \end{aligned} \tag{2}$$

In a similar fashion we find the (directed) distance β from C_b to AC as

$$\begin{aligned} \beta &= r_\phi(\cos(B-\phi) + \cos(A-C+\phi)) \\ &= 2r_\phi \sin C \sin(A+\phi). \end{aligned} \tag{3}$$

[1] Here, a negative measure implies a negative orientation for the isosceles triangle.

[2] For an explanation of the notation and a brief summary, see [7, §1].

[3] This is the perspector of the triangle formed by the apexes of isosceles triangles on the sides of ABC with base angles $\overline{\phi}$. See, for instance, [7].

Combining (2) and (3) we obtain the barycentric coordinates of C_b:

$$\begin{aligned} C_b =&(a^2 \sin\phi : bc(\sin(A+\phi) : 0) \\ =&(a^2 : S_A + S_\phi : 0). \end{aligned}$$

In this way we find the coordinates for the vertices of the Tucker hexagon as

$$\begin{array}{ll} B_a = (S_C + S_\phi : 0 : c^2), & C_a = (S_B + S_\phi : b^2 : 0), \\ A_c = (0 : b^2 : S_B + S_\phi), & B_c = (a^2 : 0 : S_A + S_\phi), \\ C_b = (a^2 : S_A + S_\phi : 0), & A_b = (0 : S_C + S_\phi : c^2). \end{array} \tag{4}$$

Remark. The radius of the Tucker circle is $r_\phi = \frac{R \sin\omega}{\sin(\phi+\omega)}$.

2. Triangles of parallels and antiparallels

With the help of (4) we find that the three antiparallels from the Tucker hexagons bound a triangle $A_1B_1C_1$ with coordinates:

$$\begin{aligned} A_1 =& \left(\frac{a^2(S_A - S_\phi)}{S_A + S_\phi} : b^2 : c^2 \right), \\ B_1 =& \left(a^2 : \frac{b^2(S_B - S_\phi)}{S_B + S_\phi} : c^2 \right), \\ C_1 =& \left(a^2 : b^2 : \frac{c^2(S_C - S_\phi)}{S_C + S_\phi} \right). \end{aligned} \tag{5}$$

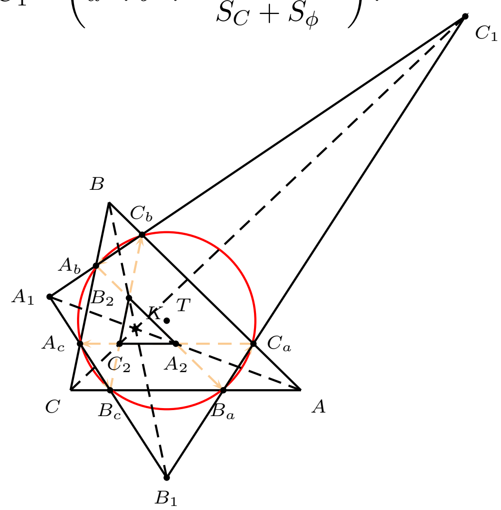


Figure 2

In the same way the parallels bound a triangle $A_2B_2C_2$ with coordinates:

$$\begin{aligned} A_2 =&(-(S_A - S_\phi) : b^2 : c^2), \\ B_2 =&(a^2 : -(S_B - S_\phi) : c^2), \\ C_2 =&(a^2 : b^2 : -(S_C - S_\phi)). \end{aligned} \tag{6}$$

It is clear that the three triangles are perspective at the symmedian point K. See Figure 2. Since ABC and $A_2B_2C_2$ are homothetic, we have a very easy construction of Tucker hexagons without invoking antiparallels: construct a triangle homothetic to ABC through K, and extend the sides of this triangles to meet the sides of ABC in six points. These six points form a Tucker hexagon.

3. Congruent rhombi

Fix ϕ. Recall that K_A, K_B and K_C are the midpoints of the antiparallels B_aC_a, A_bC_b and A_cB_c respectively. With the help of (4) we find

$$\begin{aligned} K_A =&(a^2 + 2S_\phi : b^2 : c^2), \\ K_B =&(a^2 : b^2 + 2S_\phi : c^2), \\ K_C =&(a^2 : b^2 : c^2 + 2S_\phi). \end{aligned} \tag{7}$$

Reflect the point $K^*(\overline{\phi})$ through K_A, K_B and K_C to A_ϕ, B_ϕ and C_ϕ respectively. These three points are the opposite vertices of three congruent rhombi from the point $T = K^*(\overline{\phi})$. Inspired by the figure of the *Kenmotu point* X_{371} in [4, p.268], which goes back to a collection of *Sangaku problems* from 1840, the author studied these rhombi in [6] without mentioning their connection to Tucker hexagons.

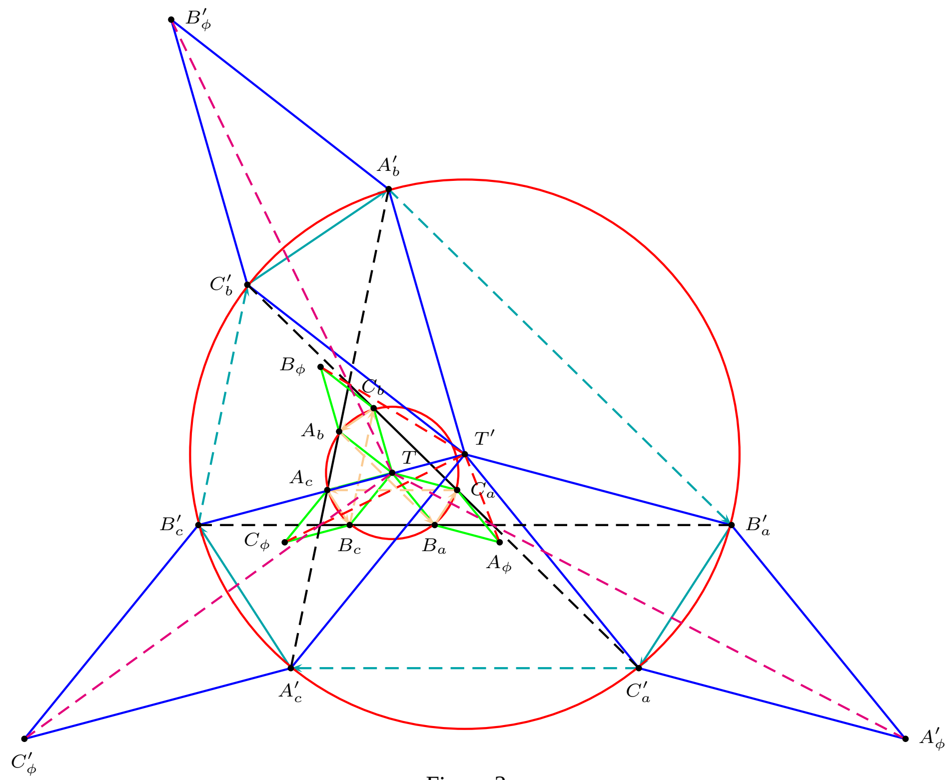


Figure 3

With the help of the coordinates for $K^*(\overline{\phi})$ and K_A found in (1) and (7) we find after some calculations,

$$\begin{aligned}A_\phi =&(a^2(S_A - S_{\overline{\phi}}) - 4S^2 : b^2(S_B - S_{\overline{\phi}}) : c^2(S_C - S_{\overline{\phi}})),\\ B_\phi =&(a^2(S_A - S_{\overline{\phi}}) : b^2(S_B - S_{\overline{\phi}}) - 4S^2 : c^2(S_C - S_{\overline{\phi}})),\\ C_\phi =&(a^2(S_A - S_{\overline{\phi}}) : b^2(S_B - S_{\overline{\phi}}) : c^2(S_C - S_{\overline{\phi}}) - 4S^2).\end{aligned} \tag{8}$$

From these, it is clear that ABC and $A_\phi B_\phi C_\phi$ are perspective at $K^*(-\overline{\phi})$.

The perspectivity gives spectacular figures, because the rhombi formed from $\mathcal{T}_\phi$ and $\mathcal{T}_{-\phi}$ are parallel. See Figure 3. In addition, it is interesting to note that $K^*(\overline{\phi})$ and $K^*(-\overline{\phi})$ are *harmonic conjugates* with respect to the circumcenter O and the symmedian point K.

4. Isosceles triangles on the sides of $A_bA_cB_cB_aC_aC_b$

Consider the hexagon $A_bA_cB_cB_aC_aC_b$. Define the points A_3, B_3, C_3, A_4, B_4 and C_4 as the apexes of isosceles triangles $A_cA_bA_3$, $B_aB_cB_3$, $C_bC_aC_3$, $B_aC_aA_4$, $C_bA_bB_4$ and $A_cB_cC_4$ of base angle ψ, where all six triangles have positive orientation when $\psi > 0$ and negative orientation when $\psi < 0$. See Figure 4.

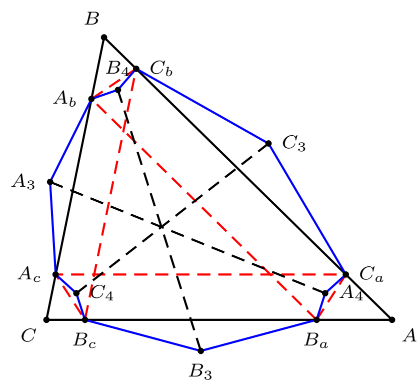


Figure 4

Proposition 1. *The lines A_3A_4, B_3B_4 and C_3C_4 are concurrent.*

Proof. Let $B_aC_a = C_bA_b = A_cB_c = 2t$, where t is given positive sign when C_aB_a and BC have equal directions, and positive sign when these directions are opposite. Note that $K_AK_BK_C$ is homothetic to ABC and that $K^*(\overline{\phi})$ is the circumcenter of $K_AK_BK_C$. Denote the circumradius of $K_AK_BK_C$ by ρ. Then we find the following:

- the signed distance from K_AK_c to AC is $t\sin B = t\frac{|K_AK_C|}{2\rho}$;
- the signed distance from AC to B_3 is $\frac{1}{2}\tan\psi|K_AK_C| - t\tan\psi\cos B$;
- the signed distance from A_4C_4 to K_AK_C is $t\tan\psi\cos B$.

Adding these signed distances we find that the signed distance from A_4C_4 to B_3 is equal to $(\frac{t}{2\rho} + \frac{\tan\psi}{2})|K_AK_C|$. By symmetry we see the signed distances from the sides B_4C_4 and A_4B_4 to A_3 and C_3 respectively are $|K_BK_C|$ and $|K_AK_B|$ multiplied by the same factor. Since triangles $K_AK_BK_C$ and $A_4B_4C_4$ are similar, the three distances are proportional to the sidelengths of $A_4B_4C_4$. Thus, $A_3B_3C_3$ is a Kiepert triangle of $A_4B_4C_4$. From this, we conclude that A_3A_4, B_3B_4 and C_3C_4 are concurrent. $\square$

5. Points defined in *pap* triangles

Let ϕ vary and consider the triangle $A_2C_aB_a$ formed by the lines B_aA_b, B_aC_a and C_aA_c. We call this the A-*pap* triangle, because it consists of a **p**arallel, an **a**ntiparallel and again a **p**arallel. Let the parallels B_aA_b and C_aA_c intersect in A_2. Then, A_2 is the reflection of A through K_A. It clearly lies on the $A-$symmedian. See also §2. The A-*pap* triangle $A_2C_aB_a$ is oppositely similar to ABC. Its vertices are

$$\begin{aligned} A_2 =&(-(S_A - S_\phi) : b^2 : c^2), \\ C_a =&(S_B + S_\phi : b^2 : 0), \\ B_a =&(S_C + S_\phi : 0 : c^2). \end{aligned} \tag{9}$$

Now let $P = (u : v : w)$ be some point given in homogeneous barycentric coordinates with respect to ABC. For $X \in \{A, B, C\}$, the locus of the counterpart of P in the X-*pap* triangles for varying ϕ is a line through X. This can be seen from the fact that the quadrangles $AC_aA_2B_a$ in all Tucker hexagons are similar. Because the sums of coordinates of these points given in (9) are equal, we find that the A-counterpart of P, namely, P evaluated in $A_2C_aB_a$, say P_{A-pap}, has coordinates

$$\begin{aligned} P_{A-pap} \sim& u \cdot A_2 + v \cdot C_a + w \cdot B_a \\ \sim& u(-(S_A - S_\phi) : b^2 : c^2) + v(S_B + S_\phi : b^2 : 0) + w(S_C + S_\phi : 0 : c^2) \\ \sim& (-S_Au + S_Bv + S_Cw + (u + v + w)S_\phi : b^2(u + v) : c^2(u + w)). \end{aligned}$$

From this, it is clear that P_{A-pap} lies on the line $A\widetilde{P}$ where

$$\widetilde{P} = \left(\frac{a^2}{v+w} : \frac{b^2}{w+u} : \frac{c^2}{u+v}\right).$$

Likewise, we consider the counterparts of P in the B-*pap* and C-*pap* triangles $C_bB_2A_b$ and $B_cA_cC_2$. By symmetry, the loci of P_{B-pap} and P_{C-pap} are the B- and C-cevians of $\widetilde{P}$.

Proposition 2. *For every ϕ, the counterparts of P in the three pap-triangles of the Tucker hexagon $\mathcal{T}_\phi$ form a triangle perspective with ABC at the point $\widetilde{P}$.*

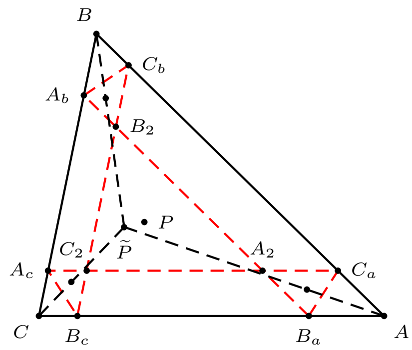


Figure 5

6. Circumcenters of *apa* triangles

As with the *pap*-triangles in the preceding section, we name the triangle $A_1B_cC_b$ formed by the **a**ntiparallel B_cC_b, the **p**arallel A_bC_b, and the **a**ntiparallel A_cB_c the A-*apa* triangle. The other two *apa*-triangles are $A_cB_1C_a$ and $A_bB_aC_1$. Unlike the *pap*-triangles, these are in general not similar to ABC. They are nevertheless isosceles triangles. We have the following interesting results on the circumcenters.

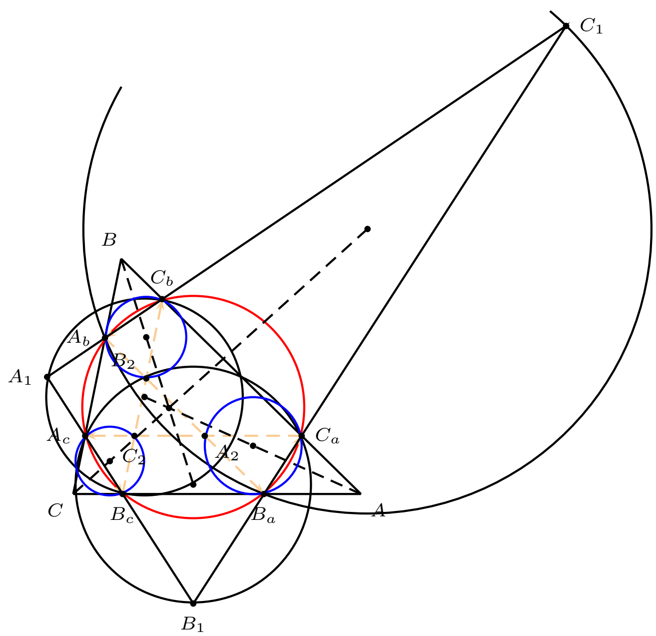


Figure 6

We note that the quadrangles $BA_cO_{B-apa}C_a$ for all possible ϕ are homothetic through B. Therefore, the locus of O_{B-apa} is a line through B. To identify this line, it is sufficient to find O_{B-apa} for one ϕ. Thus, for one special Tucker hexagon, we take the one with $C_a = A$ and $A_c = C$. Then the B-*apa* triangle is the isosceles triangle erected on side b and having a base angle of B, and its circumcenter

O_{B-apa} is the apex of the isosceles triangle erected on the same side with base angle $2B - \frac{\pi}{2}$. Using the identity [4]

$$S^2 = S_{AB} + S_{AC} + S_{BC},$$

we find that

$$\begin{aligned} O_{B-apa} =& (S_C + S_{2B-\frac{\pi}{2}} : -b^2 : S_A + S_{2B-\frac{\pi}{2}}) \\ =& (a^2(a^2 S_A + b^2 S_B) : b^2(S_{BB} - SS) : c^2(b^2 S_B + c^2 S_C)), \end{aligned}$$

after some calculations. From this, we see that the O_{B-apa} lies on the line BN^*, where

$$N^* = \left(\frac{a^2}{b^2 S_B + c^2 S_C} : \frac{b^2}{a^2 S_A + c^2 S_C} : \frac{c^2}{c^2 S_C + b^2 S_B}\right)$$

is the isogonal conjugate of the nine point center N. Therefore, the locus of O_{B-apa} for all Tucker hexagons is the B-cevian of N^*. By symmetry, we see that the loci of O_{A-apa} and O_{C-apa} are the A- and C-cevians of N^* respectively. This, incidentally, is the same as the perspector of the circumcenters of the *pap*-triangles in the previous section.

Proposition 3. *For $X \in \{A, B, C\}$, the line joining the circumcenters of the X-*pap-*triangle and the X-*apa-*triangle passes through X. These three lines intersect at the isogonal conjugate of the nine point center of triangle ABC.*

7. More circumcenters of isosceles triangles

From the center $T = K^*(\overline{\phi})$ of the Tucker circle and the vertices of the Tucker hexagon $\mathcal{T}_\phi$, we obtain six isosceles triangles. Without giving details, we present some interesting results concerning the circumcenters of these isosceles triangles.

(1) The circumcenters of the isosceles triangles TB_aC_a, TC_bA_b and TA_cB_c form a triangle perspective with ABC at

$$K^*(\overline{2\phi}) = (a^2(S_A + S \cdot \tan 2\phi) : b^2(S_B + S \cdot \tan 2\phi) : c^2(S_C + S \cdot \tan 2\phi)).$$

See Figure 7, where the Tucker hexagon $\mathcal{T}_{2\phi}$ and Tucker circle $\mathcal{C}_{2\phi}$ are also indicated.

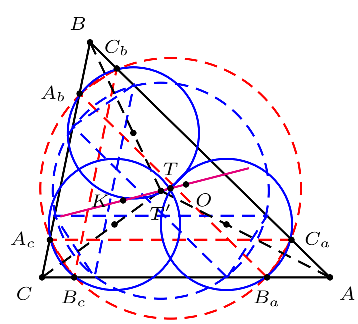


Figure 7

[4]Here, S_{XY} stands for the product $S_X S_Y$.

(2) The circumcenters of the isosceles triangles TA_bA_c, TB_cB_a and TC_aC_b form a triangle perspective with ABC at

$$\left(\frac{a^2}{S^2(3S^2 - S_{BC}) + 2a^2S^2 \cdot S_\phi + (S^2 + S_{BC})S_{\phi\phi}} : \cdots : \cdots\right).$$

See Figure 8.

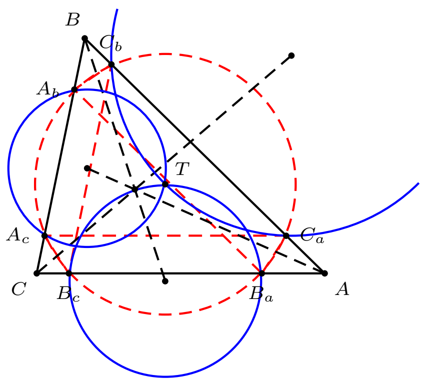


Figure 8

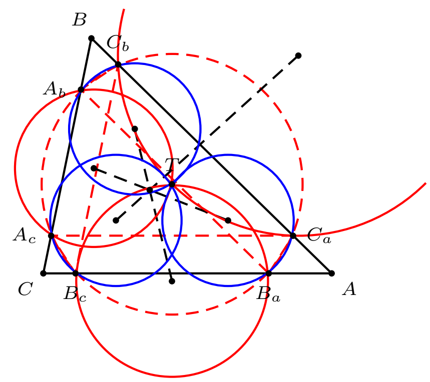


Figure 9

(3) The three lines joining the circumcenters of TB_aC_a, TA_bA_c; ... are concurrent at the point

$$(a^2(S^2(3S^2 - S_{\omega A}) + 2S^2(S_\omega + S_A)S_\phi + (2S^2 - S_{BC} + S_{AA})S_{\phi\phi}) : \cdots : \cdots).$$

See Figure 9.

References

[1] N. A. Court, *College Geometry, An Introduction to the Modern Geometry of the Triangle and the Circle*, Barnes and Noble, New York (1952).

[2] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, Mathematical Association of America. Washington D.C. (1995).

[3] R. A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, New York (1965).

[4] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1 – 295.

[5] C. Kimberling, *Encyclopedia of Triangle Centers*, http://cedar.evansville.edu/˜ck6/encyclopedia/, (2000).

[6] F. M. v. Lamoen, Triangle Centers Associated with Rhombi, *Elem. Math.*, 55 (2000) 102 – 109.

[7] F. M. v. Lamoen and P. Yiu, The Kiepert pencil of Kiepert hyperbolas, *Forum Geom.*, 1 (2001) 125 – 132.

Floor van Lamoen: Statenhof 3, 4463 TV Goes, The Netherlands

E-mail address: `f.v.lamoen@wxs.nl`

Forum Geometricorum
Volume 2 (2002) 15–19.

FORUM GEOM
ISSN 1534-1178

Congruent Inscribed Rectangles

Jean-Pierre Ehrmann

Abstract. We solve the construction problem of an interior point P in a given triangle ABC with congruent rectangles inscribed in the subtriangles PBC, PCA and PAB.

1. Congruent inscribed rectangles

Given a triangle with sidelengths a, b, c, let $L_m = \min(a, b, c)$; $L \in (0, L_m)$ and $\mu > 0$. Let P be a point inside ABC with distances d_a, d_b, d_c to the sidelines of ABC. Suppose that a rectangle with lengths of sides L and μL is inscribed in the triangle PBC, with two vertices with distance L on the segment BC, the other vertices on the segments PB and PC. Then, $\dfrac{L}{d_a - \mu L} = \dfrac{a}{d_a}$, or $d_a = \dfrac{\mu a L}{a - L}$.

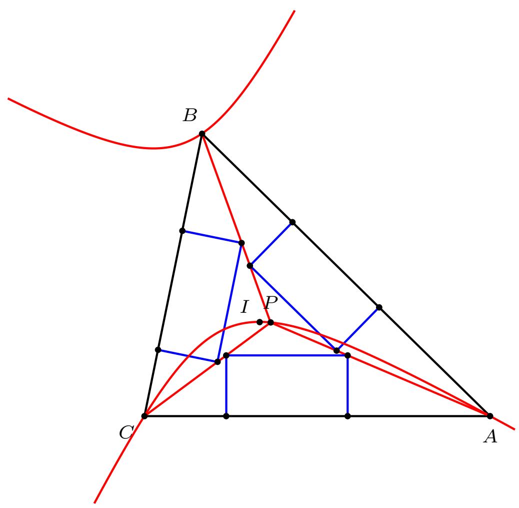


Figure 1

If we can inscribe congruent rectangles with side lengths L and μL in the three triangles PBC, PCA, PAB, we have necessarily

$$f_\mu(L) := \frac{a^2}{a-L} + \frac{b^2}{b-L} + \frac{c^2}{c-L} - \frac{2\Delta}{\mu L} = 0, \tag{1}$$

where Δ is the area of triangle ABC. This is because $ad_a + bd_b + cd_c = 2\Delta$.

Publication Date: February 7, 2002. Communicating Editor: Bernard Gibert.

The function $f_\mu(L)$ increases from $-\infty$ to $+\infty$ when L moves on $(0, L_m)$. The equation $f_\mu(L) = 0$ has a unique root L_μ in $(0, L_m)$ and the point

$$P_\mu = \left(\frac{a^2}{a - L_\mu} : \frac{b^2}{b - L_\mu} : \frac{c^2}{c - L_\mu}\right)$$

in homogeneous barycentric coordinates is the only point P inside ABC for which we can inscribe congruent rectangles with side lengths L_μ and μL_μ in the three triangles PBC, PCA, PAB. If $\mathcal{H}_0$ is the circumhyperbola through I (incenter) and K (symmedian point), the locus of P_μ when μ moves on $(0, +\infty)$ is the open arc Ω of $\mathcal{H}_0$ from I to the vertex of ABC opposite to the shortest side. See Figure 1. For $\mu = 1$, the smallest root L_1 of $f_1(L) = 0$ leads to the point P_1 with congruent inscribed squares.

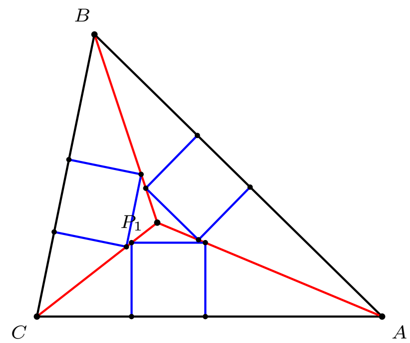


Figure 2

2. Construction of congruent inscribed rectangles

Consider $P \in \Omega$, Q and E the reflections of P and C with respect to the line IB. The parallel to AB through Q intersects BP at F. The lines EF and AP intersect at X. Then the parallel to AB through X is a sideline of the rectangle inscribed in PAB. The reflections of this line with respect to AI and BI will each give a sideline of the two other rectangles. [1]

Proof. We have $\dfrac{\overline{BE}}{\overline{BA}} = \dfrac{a}{c}$, $\dfrac{\overline{BP}}{\overline{BF}} = \dfrac{d_c}{d_a} = \dfrac{c}{a}\dfrac{a - L_\mu}{c - L_\mu}$. Applying the Menelaus theorem to triangle PAB and transversal EFX, we have

$$\frac{\overline{XA}}{\overline{XP}} = \frac{\overline{FB}}{\overline{FP}}\frac{\overline{EA}}{\overline{EB}} = \frac{L_\mu - c}{L_\mu}.$$

More over, the sidelines of the rectangles parallel to BC, CA, AB form a triangle homothetic at I with ABC. $\square$

[1]This construction was given by Bernard Gibert.

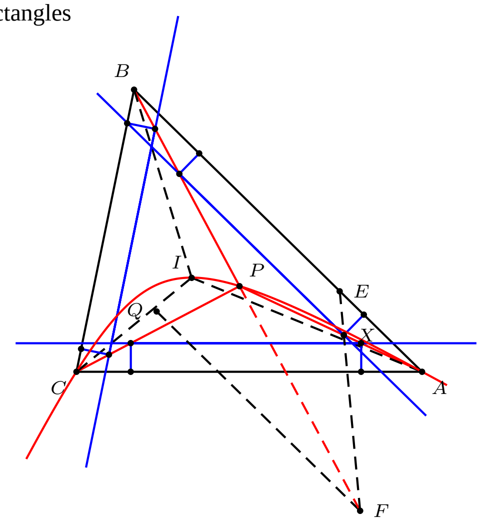


Figure 3

3. Construction of P_μ

The point P_μ is in general not constructible with ruler and compass. We give here a construction as the intersection of the arc Ω with a circle.

Consider the points

$$X_{100} = \left(\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}\right)$$

and

$$X_{106} = \left(\frac{a^2}{b+c-2a} : \frac{b^2}{c+a-2b} : \frac{c^2}{a+b-2c}\right)$$

on the circumcircle. [2] Note that the line $X_{100}X_{106}$ passes through the incenter I. The line joining X_{106} to the symmedian point K intersects the circumcircle again at

$$X_{101} = \left(\frac{a^2}{b-c} : \frac{b^2}{c-a} : \frac{c^2}{a-b}\right).$$

Construction. Draw outwardly a line ℓ parallel to AC at a distance μb from AC, intersecting the line CK at S. The parallel at S to the line CX_{101} intersects the line $KX_{101}X_{106}$ at Y_μ. Then P_μ is the intersection of the arc Ω with the circle through X_{100}, X_{106}, and Y_μ. See Figure 4.

Proof. From

$$L = \frac{2a\Delta x}{2\Delta x + \mu a^2(x+y+z)} = \frac{2b\Delta y}{2\Delta x + \mu b^2(x+y+z)} = \frac{2c\Delta z}{2\Delta z + \mu a^2(x+y+z)},$$

---

[2] We follow the notations of [1]. Here, X_{100} is the isogonal conjugate of the infinite point of the trilinear polar of the incenter, and X_{106} is the isogonal conjugate of the infinite point of the line GI joining the centroid and the incenter.

we note that P_μ lies on the three hyperbolas $\mathcal{H}_a$, $\mathcal{H}_b$ and $\mathcal{H}_c$ with equations

$$\mu bc(x+y+z)(cy-bz)+2\Delta(b-c)yz=0, \qquad (\mathcal{H}_a)$$
$$\mu ca(x+y+z)(az-cx)+2\Delta(c-a)zx=0, \qquad (\mathcal{H}_b)$$
$$\mu ab(x+y+z)(bx-ay)+2\Delta(a-b)xy=0. \qquad (\mathcal{H}_c)$$

Computing $a^2(a-b)(c-a)(\mathcal{H}_a)+b^2(b-c)(a-b)(\mathcal{H}_b)+c^2(c-a)(b-c)(\mathcal{H}_c)$, we see that P_μ lies on the circle Γ_μ:

$$\mu abc(x+y+z)\Lambda+2\Delta(a-b)(b-c)(c-a)(a^2yz+b^2zx+c^2xy)=0,$$

where

$$\Lambda=bc(b-c)(b+c-2a)x+ca(c-a)(c+a-2b)y+ab(a+b-2c)(a-b)z.$$

As $\Lambda=0$ is the line $X_{100}X_{106}$, the circle Γ_μ passes through X_{100} and X_{106}.

Now, as ℓ is the line $2\Delta y+\mu b^2(x+y+z)=0$, we have

$$S=\left(a^2:b^2:-\left(a^2+b^2+\frac{2\Delta}{\mu}\right)\right).$$

The parallel through S to CX_{101} is the line

$$\mu(b+a-2c)(x+y+z)+2\Delta\left(\frac{(b-c)x}{a^2}+\frac{(a-c)y}{b^2}\right)=0,$$

and KX_{101} is the line

$$b^2c^2(b-c)(b+c-2a)x+c^2a^2(c-a)(c+a-2b)y+a^2b^2(a-b)(a+b-2c)z=0.$$

We can check that these two lines intersect at the point

$$\begin{aligned}Y_\mu=&(a^2(2\Delta(c-a)(a-b)+\mu(-a^2(b^2+c^2)+2abc(b+c)+(b^4-2b^3c-2bc^3+c^4))\\&:b^2(2\Delta(a-b)(b-c)+\mu(-b^2(c^2+a^2)+2abc(c+a)+(c^4-2c^4a-2ca^3+a^4))\\&:c^2(2\Delta(b-c)(c-a)+\mu(-c^2(a^2+b^2)+2abc(a+b)+(a^4-2a^3b-2ab^3+b^4)))\end{aligned}$$

on the circle Γ_μ. □

Remark. The circle through X_{100}, X_{106} and P_μ is the only constructible circle through P_μ, and there is no constructible line through P_μ.

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://cedar.evansville.edu/~ck6/encyclopedia/.

Jean-Pierre Ehrmann: 6, rue des Cailloux, 92110 - Clichy, France
E-mail address: Jean-Pierre.EHRMANN@wanadoo.fr

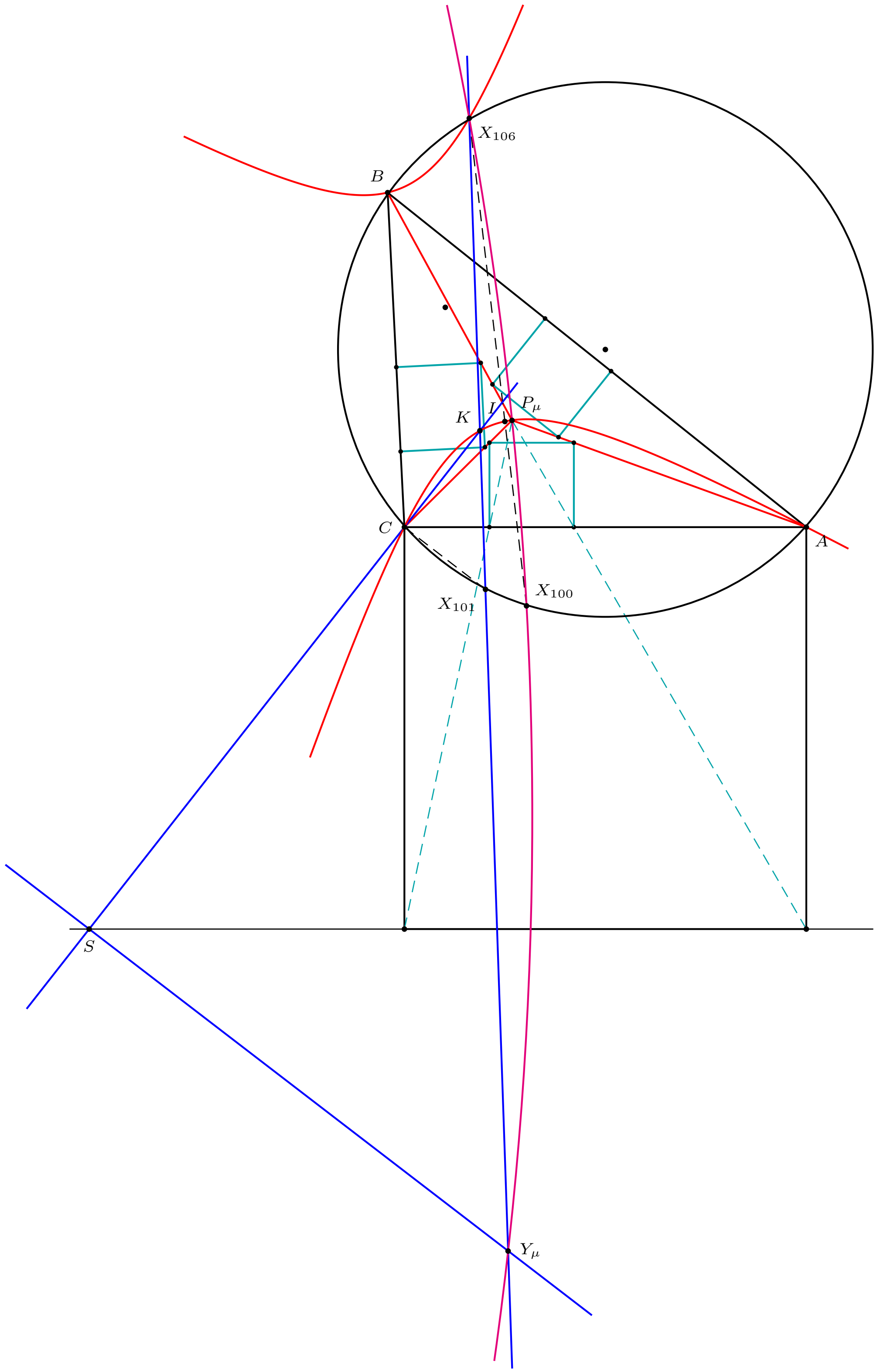


Figure 4

Forum Geometricorum
Volume 2 (2002) 21–32.

FORUM GEOM
ISSN 1534-1178

Collineations, Conjugacies, and Cubics

Clark Kimberling

Abstract. If F is an involution and φ a suitable collineation, then $\varphi \circ F \circ \varphi^{-1}$ is an involution; this form includes well-known conjugacies and new conjugacies, including *aleph, beth, complementary,* and *anticomplementary*. If $Z(U)$ is the self-isogonal cubic with pivot U, then φ carries $Z(U)$ to a pivotal cubic. Particular attention is given to the Darboux and Lucas cubics, D and L, and conjugacy-preserving mappings between D and L are formulated.

1. Introduction

The defining property of the kind of mapping called *collineation* is that it carries lines to lines. Matrix algebra lends itself nicely to collineations as in [1, Chapter XI] and [5]. In order to investigate collineation-induced conjugacies, especially with regard to triangle centers, suppose that an arbitrary point P in the plane of $\triangle ABC$ has homogeneous trilinear coordinates $p : q : r$ relative to $\triangle ABC$, and write

$$A = 1 : 0 : 0, \quad B = 0 : 1 : 0, \quad C = 0 : 0 : 1,$$

so that

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Suppose now that suitably chosen points $P_i = p_i : q_i : r_i$ and $P'_i = p'_i : q'_i : r'_i$ for $i = 1, 2, 3, 4$ are given and that we wish to represent the unique collineation φ that maps each P_i to P'_i. (Precise criteria for "suitably chosen" will be determined soon.) Let

$$\mathbb{P} = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}, \qquad \mathbb{P}' = \begin{pmatrix} p'_1 & q'_1 & r'_1 \\ p'_2 & q'_2 & r'_2 \\ p'_3 & q'_3 & r'_3 \end{pmatrix}.$$

We seek a matrix $\mathbb{M}$ such that $\varphi(X) = X\mathbb{M}$ for every point $X = x : y : z$, where X is represented as a 1×3 matrix:

$$X = \begin{pmatrix} x & y & z \end{pmatrix}$$

In particular, we wish to have

$$\mathbb{PM} = \mathbb{DP}' \quad \text{and} \quad P_4\mathbb{M} = \begin{pmatrix} gp'_4 & gq'_4 & gr'_4 \end{pmatrix},$$

Publication Date: March 11, 2002. Communicating Editor: Paul Yiu.

where

$$\mathbb{D} = \begin{pmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{pmatrix}$$

for some multipliers d, e, f, g. By homogeneity, we can, and do, put $g = 1$. Then substituting $\mathbb{P}^{-1}\mathbb{D}\mathbb{P}'$ for $\mathbb{M}$ gives $P_4\mathbb{P}^{-1}\mathbb{D} = \mathbb{P}'_4(\mathbb{P}')^{-1}$. Writing out both sides leads to

$$\begin{aligned}
d &= \frac{(q'_2r'_3 - q'_3r'_2)p'_4 + (r'_2p'_3 - r'_3p'_2)q'_4 + (p'_2q'_3 - p'_3q'_2)r'_4}{(q_2r_3 - q_3r_2)p_4 + (r_2p_3 - r_3p_2)q_4 + (p_2q_3 - p_3q_2)r_4}, \\
e &= \frac{(q'_3r'_1 - q'_1r'_3)p'_4 + (r'_3p'_1 - r'_1p'_3)q'_4 + (p'_3q'_1 - p'_1q'_3)r'_4}{(q_3r_1 - q_1r_3)p_4 + (r_3p_1 - r_1p_3)q_4 + (p_3q_1 - p_1q_3)r_4}, \\
f &= \frac{(q'_1r'_2 - q'_2r'_1)p'_4 + (r'_1p'_2 - r'_2p'_1)q'_4 + (p'_1q'_2 - p'_2q'_1)r'_4}{(q_1r_2 - q_2r_1)p_4 + (r_1p_2 - r_2p_1)q_4 + (p_1q_2 - p_2q_1)r_4}.
\end{aligned}$$

The point $D := d : e : f$ is clearly expressible as quotients of determinants:

$$D = \frac{\begin{vmatrix} p'_4 & q'_4 & r'_4 \\ p'_2 & q'_2 & r'_2 \\ p'_3 & q'_3 & r'_3 \end{vmatrix}}{\begin{vmatrix} p_4 & q_4 & r_4 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}} : \frac{\begin{vmatrix} p'_4 & q'_4 & r'_4 \\ p'_3 & q'_3 & r'_3 \\ p'_1 & q'_1 & r'_1 \end{vmatrix}}{\begin{vmatrix} p_4 & q_4 & r_4 \\ p_3 & q_3 & r_3 \\ p_1 & q_1 & r_1 \end{vmatrix}} : \frac{\begin{vmatrix} p'_4 & q'_4 & r'_4 \\ p'_1 & q'_1 & r'_1 \\ p'_2 & q'_2 & r'_2 \end{vmatrix}}{\begin{vmatrix} p_4 & q_4 & r_4 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}.$$

With $\mathbb{D}$ determined[1], we write

$$\mathbb{M} = \mathbb{P}^{-1}\mathbb{D}\mathbb{P}'$$

and are now in a position to state the conditions to be assumed about the eight initial points:

(i) $\mathbb{P}$ and $\mathbb{P}'$ are nonsingular;

(ii) the denominators in the expressions for d, e, f are nonzero;

(iii) $def \neq 0$.

Conditions (i) and (ii) imply that the collineation φ is given by $\varphi(X) = X\mathbb{M}$, and (iii) ensures that $\varphi^{-1}(X) = X\mathbb{M}^{-1}$. A collineation φ satisfying (i)-(iii) will be called *regular*. If φ is regular then clearly φ^{-1} is regular.

If the eight initial points are centers (*i.e.*, triangle centers) for which no three P_i are collinear and no three P'_i are collinear, then for every center X, the image $\varphi(X)$ is a center. If P_1, P_2, P_3 are respectively the A-, B-, C- vertices of a central triangle [3, pp. 53-57] and P_4 is a center, and if the same is true for P'_i for $i = 1$, 2, 3, 4, then in this case, too, φ carries centers to centers.

[1]A geometric realization of D follows. Let $\hat{P}$ denote the circle

$$(p_1\alpha + p_2\beta + p_3\gamma)(a\alpha + b\beta + c\gamma) + p_4(a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0,$$

and let $\hat{Q}$, $\hat{R}$, $\hat{P}'$, $\hat{Q}'$, $\hat{R}'$ be the circles likewise formed from the points P_i and P'_i. Following [3, p.225], let Λ and Λ' be the radical centers of circles $\hat{P}, \hat{Q}, \hat{R}$ and $\hat{P}', \hat{Q}', \hat{R}'$, respectively. Then D is the trilinear quotient Λ/Λ'.

The representation $\varphi(X) = X\mathbb{M}$ shows that for $X = x : y : z$, the image $\varphi(X)$ has the form

$$f_1x + g_1y + h_1z : f_2x + g_2y + h_2z : f_3x + g_3y + h_3z.$$

Consequently, if Λ is a curve homogeneous of degree $n \geq 1$ in α, β, γ, then $\varphi(\Lambda)$ is also a curve homogeneous of degree n in α, β, γ. In particular, φ carries a circumconic onto a conic that circumscribes the triangle having vertices $\varphi(A)$, $\varphi(B)$, $\varphi(C)$, and likewise for higher order curves. We shall, in §5, concentrate on cubic curves.

Example 1. Suppose

$$P = p : q : r, \qquad U = u : v : w, \qquad U' = u' : v' : w'$$

are points, none lying on a sideline of $\triangle ABC$, and U' is not on a sideline of the cevian triangle of P (whose vertices are the rows of matrix $\mathbb{P}'$ shown below). Then the collineation φ that carries ABC to $\mathbb{P}'$ and U to U' is regular. We have

$$\mathbb{P}' = \begin{pmatrix} 0 & q & r \\ p & 0 & r \\ p & q & 0 \end{pmatrix}, \quad \text{and} \quad (\mathbb{P}')^{-1} = \frac{1}{|\mathbb{P}'|}\begin{pmatrix} -p & q & r \\ p & -q & r \\ p & q & -r \end{pmatrix},$$

leading to

$$\varphi(X) = X\mathbb{M} = p(ey + fz) : q(fz + dx) : r(dx + ey), \tag{1}$$

where

$$d : e : f = \frac{1}{u}\left(-\frac{u'}{p} + \frac{v'}{q} + \frac{w'}{r}\right) : \frac{1}{v}\left(\frac{u'}{p} - \frac{v'}{q} + \frac{w'}{r}\right) : \frac{1}{w}\left(\frac{u'}{p} + \frac{v'}{q} - \frac{w'}{r}\right). \tag{2}$$

Example 2. Continuing from Example 1, φ^{-1} is the collineation given by

$$\varphi^{-1}(X) = X\mathbb{M}^{-1} = \frac{1}{d}\left(-\frac{x}{p} + \frac{y}{q} + \frac{z}{r}\right) : \frac{1}{e}\left(\frac{x}{p} - \frac{y}{q} + \frac{z}{r}\right) : \frac{1}{f}\left(\frac{x}{p} + \frac{y}{q} - \frac{z}{r}\right). \tag{3}$$

2. Conjugacies induced by collineations

Suppose F is a mapping on the plane of $\triangle ABC$ and φ is a regular collineation, and consider the following diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \varphi(X) \arrow[d] \\ F(X) \arrow[r] & \varphi(F(X)) \end{tikzcd}$$

On writing $\varphi(X)$ as P, we have $m(P) = \varphi(F(\varphi^{-1}(P)))$. If $F(F(X)) = X$, then $m(m(P)) = P$; in other words, if F is an involution, then m is an involution. We turn now to Examples 3-10, in which F is a well-known involution and φ is the collineation in Example 1 or a special case thereof. In Examples 11 and 12, φ is complementation and anticomplementation, respectively.

Example 3. For any point $X = x : y : z$ not on a sideline of $\triangle ABC$, the isogonal conjugate of X is given by

$$F(X) = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

Suppose P, U, φ are as in Example 1. The involution m given by $m(X) = \varphi(F(\varphi^{-1}(X)))$ will be formulated: equation (3) implies

$$F(\varphi^{-1}(X)) = \frac{d}{-\frac{x}{p} + \frac{y}{q} + \frac{z}{r}} : \frac{e}{\frac{x}{p} - \frac{y}{q} + \frac{z}{r}} : \frac{f}{\frac{x}{p} + \frac{y}{q} - \frac{z}{r}},$$

and substituting these coordinates into (1) leads to

$$m(X) = m_1 : m_2 : m_3, \tag{4}$$

where

$$m_1 = m_1(p, q, r, x, y, z) = p \left(\frac{e^2}{\frac{x}{p} - \frac{y}{q} + \frac{z}{r}} + \frac{f^2}{\frac{x}{p} + \frac{y}{q} - \frac{z}{r}} \right) \tag{5}$$

and m_2 and m_3 are determined cyclically from m_1; for example, $m_2(p, q, r, x, y, z) = m_1(q, r, p, y, z, x)$.

In particular, if $U = 1 : 1 : 1$ and $U' = p : q : r$, then from equation (2), we have $d : e : f = 1 : 1 : 1$, and (5) simplifies to

$$m(X) = x \left(-\frac{x}{p} + \frac{y}{q} + \frac{z}{r} \right) : y \left(\frac{x}{p} - \frac{y}{q} + \frac{z}{r} \right) : z \left(\frac{x}{p} + \frac{y}{q} - \frac{z}{r} \right).$$

This is the P-Ceva conjugate of X, constructed [3, p. 57] as the perspector of the cevian triangle of P and the anticevian triangle of X.

Example 4. Continuing with isogonal conjugacy for F and with φ as in Example 3 (with $U = 1 : 1 : 1$ and $U' = p : q : r$), here we use φ^{-1} in place of φ, so that $m(X) = \varphi^{-1}(F(\varphi(X))$. The result is (4), with

$$m_1 = -q^2r^2x^2 + r^2p^2y^2 + p^2q^2z^2 + (-q^2r^2 + r^2p^2 + p^2q^2)(yz + zx + xy).$$

In this case, $m(X)$ is the *P-aleph conjugate of* X.

Let

$$n(X) = \frac{1}{y+z} : \frac{1}{z+x} : \frac{1}{x+y}.$$

Then $X = n(X)$-aleph conjugate of X. Another easily checked property is that a necessary and sufficient condition that

$$X = X\text{-aleph conjugate of the incenter}$$

is that $X =$ incenter or else X lies on the conic $\beta\gamma + \gamma\alpha + \alpha\beta = 0$.

In [4], various triples $(m(X), P, X)$ are listed. A selection of these permuted to $(X, P, m(X))$ appears in Table 1. The notation X_i refers to the indexing of triangle centers in [4]. For example,

$$X_{57} = \frac{1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c} = \tan\frac{A}{2} : \tan\frac{B}{2} : \tan\frac{C}{2},$$

abbreviated in Table 1 and later tables as "57, $\tan\frac{A}{2}$". In Table 1 and the sequel, the area σ of $\triangle ABC$ is given by

$$16\sigma^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c).$$

Table 1. Selected aleph conjugates

center, X	P	P-aleph conj. of X
$57, \tan\frac{A}{2}$	$7, \sec^2\frac{A}{2}$	$57, \tan\frac{A}{2}$
$63, \cot A$	$2, \frac{1}{a}$	$1, 1$
$57, \tan\frac{A}{2}$	$174, \sec\frac{A}{2}$	$1, 1$
$2, \frac{1}{a}$	$86, \frac{bc}{b+c}$	$2, \frac{1}{a}$
$3, \cos A$	$21, \frac{1}{\cos B+\cos C}$	$3, \cos A$
$43, ab+ac-bc$	$1, 1$	$9, b+c-a$
$610, \sigma^2 - a^2\cot A$	$2, \frac{1}{a}$	$19, \tan A$
$165, \tan\frac{B}{2}+\tan\frac{C}{2}-\tan\frac{A}{2}$	$100, \frac{1}{b-c}$	$101, \frac{a}{b-c}$

Example 5. Here, F is reflection about the circumcenter:

$$F(x:y:z) = 2R\cos A - hx : 2R\cos B - hy : 2R\cos C - hz,$$

where $R =$ circumradius, and h normalizes[2] X. Keeping φ as in Example 4, we find

$$m_1(x,y,z) = 2abc(\cos B + \cos C)\left(\frac{x(b+c-a)}{p} + \frac{y(c+a-b)}{q} + \frac{z(a+b-c)}{r}\right) - 16\sigma^2 x,$$

which, via (4), defines the *P-beth conjugate of* X.

Table 2. Selected beth conjugates

center, X	P	P-beth conj. of X
$110, \frac{a}{b^2-c^2}$	$643, \frac{b+c-a}{b^2-c^2}$	$643, \frac{b+c-a}{b^2-c^2}$
$6, a$	$101, \frac{a}{b-c}$	$6, a$
$4, \sec A$	$8, \csc^2\frac{A}{2}$	$40, \cos B+\cos C-\cos A-1$
$190, \frac{bc}{b-c}$	$9, b+c-a$	$292, a/(a^2-bc)$
$11, 1-\cos(B-C)$	$11, 1-\cos(B-C)$	$244, (1-\cos(B-C))\sin^2\frac{A}{2}$
$1, 1$	$99, \frac{bc}{b^2-c^2}$	$85, \frac{b^2c^2}{b+c-a}$
$10, \frac{b+c}{a}$	$100, \frac{1}{b-c}$	$73, \cos A(\cos B+\cos C)$
$3, \cos A$	$21, \frac{1}{\cos B+\cos C}$	$56, 1-\cos A$

Among readily verifiable properties of beth-conjugates are these:

(i) $\varphi(X_3)$ is collinear with every pair $\{X, m(X)\}$.

(ii) Since each line $\mathcal{L}$ through X_3 has two points fixed under reflection in X_3, the line $\varphi(\mathcal{L})$ has two points that are fixed by m, namely $\varphi(X_3)$ and $\varphi(\mathcal{L}\cap\mathcal{L}^\infty)$.

[2]If $X \notin \mathcal{L}^\infty$, then $h = 2\sigma/(ax+by+cz)$; if $X \in \mathcal{L}^\infty$ and $xyz \neq 0$, then $h = 1/x + 1/y + 1/z$; otherwise, $h = 1$. For $X \notin \mathcal{L}^\infty$, the nonhomogeneous representation for X as the ordered triple (hx, hy, hz) gives the actual directed distances hx, hy, hz from X to sidelines BC, CA, AB, respectively.

(iii) When $P = X_{21}$, φ carries the Euler line $L(3,4,20,30)$ to $L(1,3,56,36)$, on which the m-fixed points are X_1 and X_{36}, and φ carries the line $L(1,3,40,517)$ to $L(21,1,58,1078)$, on which the m-fixed points are X_1 and X_{1078}.

(iv) If X lies on the circumcircle, then the X_{21}-beth conjugate, X', of X lies on the circumcircle. Such pairs (X, X') include (X_i, X_j) for these (i,j): $(99,741)$, $(100,106)$, $(101,105)$, $(102,108)$, $(103,934)$, $(104,109)$, $(110,759)$.

(v) $P = P$-beth conjugate of X if and only if $X = P \cdot X_{56}$ (trilinear product).

Example 6. Continuing Example 5 with φ^{-1} in place of φ leads to the P-*gimel conjugate of* X, defined via (4) by

$$m_1(x,y,z) = 2abc\left(-\frac{\cos A}{p} + \frac{\cos B}{q} + \frac{\cos C}{r}\right)S - 8\sigma^2 x,$$

where $S = x(bq+cr) + y(cr+ap) + z(ap+bq)$.

It is easy to check that if $P \in \mathcal{L}^\infty$, then $m(X_1) = X_1$.

Table 3. Selected gimel conjugates

center, X	P	P-gimel conjugate of X
$1, 1$	$3, \cos A$	$1, 1$
$3, \cos A$	$283, \frac{\cos A}{\cos B + \cos C}$	$3, \cos A$
$30, \cos A - 2\cos B \cos C$	$8, \csc^2 \frac{A}{2}$	$30, \cos A - 2\cos B \cos C$
$4, \sec A$	$21, \frac{1}{\cos B + \cos C}$	$4, \sec A$
$219, \cos A \cot \frac{A}{2}$	$63, \cot A$	$6, a$

Example 7. For distinct points $X' = x' : y' : z'$ and $X = x : y : z$, neither lying on a sideline of $\triangle ABC$, the X'-Hirst inverse of X is defined [4, Glossary] by

$$y'z'x^2 - x'^2 yz : z'x'y^2 - y'^2 zx : x'y'z^2 - z'^2 xy.$$

We choose $X' = U = U' = 1:1:1$. Keeping φ as in Example 4, for $X \neq P$ we obtain m as in expression (4), with

$$m_1(x,y,z) = p\left(\frac{y}{q} - \frac{z}{r}\right)^2 + x\left(\frac{2x}{p} - \frac{y}{q} - \frac{z}{r}\right).$$

In this example, $m(X)$ defines the P-*daleth conjugate of* X. The symbol ω in Table 5 represents the Brocard angle of $\triangle ABC$.

Table 4. Selected daleth conjugates

center, X	P	P-daleth conjugate of X
$518, b^2 + c^2 - a(b+c)$	$1, 1$	$37, b+c$
$1, 1$	$1, 1$	$44, b+c-2a$
$511, \cos(A+\omega)$	$3, \cos A$	$216, \sin 2A \cos(B-C)$
$125, \cos A \sin^2(B-C)$	$4, \sec A$	$125, \cos A \sin^2(B-C)$
$511, \cos(A+\omega)$	$6, a$	$39, a(b^2+c^2)$
$672, a(b^2+c^2-a(b+c))$	$6, a$	$42, a(b+c)$
$396, \cos(B-C) + 2\cos(A - \frac{\pi}{3})$	$13, \csc(A + \frac{\pi}{3})$	$30, \cos A - 2\cos B \cos C$
$395, \cos(B-C) + 2\cos(A + \frac{\pi}{3})$	$14, \csc(A - \frac{\pi}{3})$	$30, \cos A - 2\cos B \cos C$

Among properties of daleth conjugacy that can be straightforwardly demonstrated is that for given P, a point X satisfies the equation

$$P = P\text{-daleth conjugate of } X$$

if and only if X lies on the trilinear polar of P.

Example 8. Continuing Example 7, we use φ^{-1} in place of φ and define the resulting image $m(X)$ as the *P-he conjugate of X*.[3] We have m as in (4) with

$$\begin{aligned} m_1(x,y,z) &= -p(y+z)^2 + q(z+x)^2 + r(x+y)^2 \\ &+ \frac{qr}{p}(x+y)(x+z) - \frac{rp}{q}(y+z)(y+x) - \frac{pq}{r}(z+x)(z+y). \end{aligned}$$

Table 5. Selected he conjugates

center, X	P	P-he conjugate of X
$239, bc(a^2-bc)$	$2, \frac{1}{a}$	$9, b+c-a$
$36, 1-2\cos A$	$6, a$	$43, \csc B + \csc C - \csc A$
$514, \frac{b-c}{a}$	$7, \sec^2\frac{A}{2}$	$57, \tan\frac{A}{2}$
$661, \cot B - \cot C$	$21, \frac{1}{\cos B+\cos C}$	$3, \cos A$
$101, \frac{a}{b-c}$	$100, \frac{1}{b-c}$	$101, \frac{a}{b-c}$

Example 9. The X_1-Ceva conjugate of X not lying on a sideline of is $\triangle ABC$ is the point

$$-x(-x+y+z) : y(x-y+z) : z(x+y-z).$$

Taking this for F and keeping φ as in Example 4 leads to

$$m_1(x,y,z) = p(x^2q^2r^2 + 2p^2(ry-qz)^2 - pqr^2xy - pq^2rxz),$$

which via m as in (4) defines the *P-waw conjugate of X*.

Table 6. Selected waw conjugates

center, X	P	P-waw conjugate of X
$37, b+c$	$1, 1$	$354, (b-c)^2 - ab - ac$
$5, \cos(B-C)$	$2, \frac{1}{a}$	$141, bc(b^2+c^2)$
$10, \frac{b+c}{a}$	$2, \frac{1}{a}$	$142, b+c-\frac{(b-c)^2}{a}$
$53, \tan A\cos(B-C)$	$4, \sec A$	$427, (b^2+c^2)\sec A$
$51, a^2\cos(B-C)$	$6, a$	$39, a(b^2+c^2)$

Example 10. Continuing Example 9 with φ^{-1} in place of φ gives

$$m_1(x,y,z) = p(y+z)^2 - ry^2 - qz^2 + (p-r)xy + (p-q)xz,$$

which via m as in (4) defines the *P-zayin conjugate of X*. When $P =$ incenter, this conjugacy is isogonal conjugacy. Other cases are given in Table 7.

[3]The fifth letter of the Hebrew alphabet is *he*, homophonous with *hay*.

Table 7. Selected zayin conjugates

center, X	P	P-zayin conjugate of X
$9, b+c-a$	$2, \frac{1}{a}$	$9, b+c-a$
$101, \frac{a}{b-c}$	$2, \frac{1}{a}$	$661, \cot B - \cot C$
$108, \frac{\sin A}{\sec B - \sec C}$	$3, \cos A$	$656, \tan B - \tan C$
$109, \frac{\sin A}{\cos B - \cos C}$	$4, \sec A$	$656, \tan B - \tan C$
$43, ab+ac-bc$	$6, a$	$43, ab+ac-bc$
$57, \tan \frac{A}{2}$	$7, \sec^2 \frac{A}{2}$	$57, \tan \frac{A}{2}$
$40, \cos B + \cos C - \cos A - 1$	$8, \csc^2 \frac{A}{2}$	$40, \cos B + \cos C - \cos A - 1$

Example 11. The complement of a point X not on $\mathcal{L}^\infty$ is the point X' satisfying the vector equation

$$\overrightarrow{X'X_2} = \frac{1}{2}\overrightarrow{X_2X}.$$

If $X = x : y : z$, then

$$X' = \frac{by+cz}{a} : \frac{cz+ax}{b} : \frac{ax+by}{c}. \tag{6}$$

If $X \in \mathcal{L}^\infty$, then (6) defines the complement of X. The mapping $\varphi(X) = X'$ is a collineation. Let $P = p : q : r$ be a point not on a sideline of $\triangle ABC$, and let

$$F(X) = \frac{1}{px} : \frac{1}{qy} : \frac{1}{rz},$$

the P-isoconjugate of X. Then m as in (4) is given by

$$m_1(x,y,z) = \frac{1}{a}\left(\frac{b^2}{q(ax-by+cz)} + \frac{c^2}{r(ax+by-cz)}\right)$$

and defines the *P-complementary conjugate of X*. The X_1-complementary conjugate of X_2, for example, is the symmedian point of the medial triangle, X_{141}, and X_{10} is its own X_1-complementary conjugate. Moreover, X_1-complementary conjugacy carries $\mathcal{L}^\infty$ onto the nine-point circle. Further examples follow:

Table 8. Selected complementary conjugates

center X	P	P-complementary conjugate of X
$10, \frac{b+c}{a}$	$2, \frac{1}{a}$	$141, bc(b^2+c^2)$
$10, \frac{b+c}{a}$	$3, \cos A$	$3, \cos A$
$10, \frac{b+c}{a}$	$4, \sec A$	$5, \cos(B-C)$
$10, \frac{b+c}{a}$	$6, a$	$2, \frac{1}{a}$
$141, bc(b^2+c^2)$	$7, \sec^2 \frac{A}{2}$	$142, b+c-\frac{(b-c)^2}{a}$
$9, b+c-a$	$9, b+c-a$	$141, bc(b^2+c^2)$
$2, \frac{1}{a}$	$19, \tan A$	$5, \cos(B-C)$
$125, \cos A \sin^2(B-C)$	$10, \frac{b+c}{a}$	$513, b-c$

Example 12. The anticomplement of a point X is the point X'' given by

$$X'' = \frac{-ax+by+cz}{a} : \frac{ax-by+cz}{b} : \frac{ax+by-cz}{c}.$$

Keeping F and φ as in Example 11, we have $\varphi^{-1}(X) = X''$ and define m by $m = \varphi^{-1} \circ F \circ \varphi$, Thus, $m(X)$ is determined as in (4) from

$$m_1(x,y,z) = \frac{1}{a}\left(\frac{b^2}{q(ax+cz)} + \frac{c^2}{r(ax+by)} - \frac{a^2}{p(by+cz)}\right).$$

Here, $m(X)$ defines the *P-anticomplementary conjugate of* X. For example, the centroid is the X_1-anticomplementary conjugate of X_{69} (the symmedian point of the anticomplementary triangle), and the Nagel point, X_8, is its own self X_1-anticomplementary conjugate. Moreover, X_1-anticomplementary conjugacy carries the nine-point circle onto $\mathcal{L}^\infty$. Further examples follow:

Table 9. Selected anticomplementary conjugates

center, X	P	P-anticomplementary conj. of X
$3, \cos A$	$1, 1$	$4, \sec A$
$5, \cos(B-C)$	$1, 1$	$20, \cos A - \cos B \cos C$
$10, \frac{b+c}{a}$	$2, \frac{1}{a}$	$69, bc(b^2+c^2-a^2)$
$10, \frac{b+c}{a}$	$3, \cos A$	$20, \cos A - \cos B \cos C$
$10, \frac{b+c}{a}$	$4, \sec A$	$4, \sec A$
$10, \frac{b+c}{a}$	$6, a$	$2, \frac{1}{a}$
$5, \cos(B-C)$	$19, \tan A$	$2, \frac{1}{a}$
$125, \cos A \sin^2(B-C)$	$10, \frac{b+c}{a}$	$513, b-c$

3. The Darboux cubic, D

This section formulates a mapping Ψ on the plane of $\triangle ABC$; this mapping preserves two pivotal properties of the Darboux cubic D. In Section 4, $\Psi(D)$ is recognized as the Lucas cubic. In Section 5, collineations will be applied to D, carrying it to cubics having two pivotal configurations with properties analogous to those of D.

The Darboux cubic is the locus of a point X such that the pedal triangle of X is a cevian triangle. The pedal triangle of X has for its A-vertex the point in which the line through X perpendicular to line BC meets line BC, and likewise for the B- and C- vertices. We denote these three vertices by X_A, X_B, X_C, respectively. To say that their triangle is a cevian triangle means that the lines AX_A, BX_B, CX_C concur. Let $\Psi(P)$ denote the point of concurrence. In order to obtain a formula for Ψ, we begin with the pedal triangle of P:

$$\begin{pmatrix} X_A \\ X_B \\ X_C \end{pmatrix} = \begin{pmatrix} 0 & \beta+\alpha c_1 & \gamma+\alpha b_1 \\ \alpha+\beta c_1 & 0 & \gamma+\beta a_1 \\ \alpha+\gamma b_1 & \beta+\gamma a_1 & 0 \end{pmatrix},$$

where $a_1 = \cos A$, $b_1 = \cos B$, $c_1 = \cos C$. Then

$$\begin{aligned}
BX_B \cap CX_C =&(\alpha + \beta c_1)(\alpha + \gamma b_1) : (\beta + \gamma a_1)(\alpha + \beta c_1) : (\gamma + \beta a_1)(\alpha + \gamma b_1),\\
CX_C \cap AX_A =&(\alpha + \gamma b_1)(\beta + \alpha c_1) : (\beta + \gamma a_1)(\beta + \alpha c_1) : (\gamma + \alpha b_1)(\beta + \gamma a_1),\\
AX_A \cap BX_B =&(\alpha + \beta c_1)(\gamma + \alpha b_1) : (\beta + \alpha c_1)(\gamma + \beta a_1) : (\gamma + \alpha b_1)(\gamma + \beta a_1).
\end{aligned}$$

Each of these three points is $\Psi(X)$. Multiplying and taking the cube root gives the following result:

$$\Psi(X) = \psi(\alpha, \beta, \gamma, a, b, c) : \psi(\beta, \gamma, \alpha, b, c, a) : \psi(\gamma, \alpha, \beta, c, a, b),$$

where

$$\psi(\alpha, \beta, \gamma, a, b, c) = [(\alpha + \beta c_1)^2(\alpha + \gamma b_1)^2(\beta + \alpha c_1)(\gamma + \alpha b_1)]^{1/3}.$$

The Darboux cubic is one of a family of cubics $Z(U)$ given by the form (e.g., [3, p.240])

$$u\alpha(\beta^2 - \gamma^2) + v\beta(\gamma^2 - \alpha^2) + w\gamma(\alpha^2 - \beta^2) = 0, \tag{7}$$

where the point $U = \ u : v : w$ is called the pivot of $Z(U)$, in accord with the collinearity of U, X, and the isogonal conjugate, X^{-1}, of X, for every point $X = \alpha : \beta : \gamma$ on $Z(U)$. The Darboux cubic is $Z(X_{20})$; that is,

$$(a_1 - b_1c_1)\alpha(\beta^2 - \gamma^2) + (b_1 - c_1a_1)\beta(\gamma^2 - \alpha^2) + (c_1 - a_1b_1)\gamma(\alpha^2 - \beta^2) = 0.$$

This curve has a secondary pivot, the circumcenter, X_3, in the sense that if X lies on D, then so does the reflection of X in X_3. Since X_3 itself lies on D, we have here a second system of collinear triples on D.

The two types of pivoting lead to chains of centers on D:

$$X_1 \xrightarrow{\text{refl}} X_{40} \xrightarrow{\text{isog}} X_{84} \xrightarrow{\text{refl}} \cdots \tag{8}$$

$$X_3 \xrightarrow{\text{isog}} X_4 \xrightarrow{\text{refl}} X_{20} \xrightarrow{\text{isog}} X_{64} \xrightarrow{\text{refl}} \cdots . \tag{9}$$

Each of the centers in (8) and (9) has a trilinear representation in polynomials with all coefficients integers. One wonders if all such centers on D can be generated by a finite collection of chains using reflection and isogonal conjugation as in (8) and (9).

4. The Lucas cubic, L

Transposing the roles of pedal and cevian triangles in the description of D leads to the Lucas cubic, L, i.e., the locus of a point $X = \alpha : \beta : \gamma$ whose cevian triangle is a pedal triangle. Mimicking the steps in Section 3 leads to

$$\Psi^{-1}(X) = \lambda(\alpha, \beta, \gamma, a, b, c) : \lambda(\beta, \gamma, \alpha, b, c, a) : \lambda(\gamma, \alpha, \beta, c, a, b),$$

where $\lambda(\alpha, \beta, \gamma, a, b, c) =$

$$\{[\alpha^2 - (\alpha a_1 - \gamma c_1)(\alpha a_1 - \beta b_1)]([(\alpha\beta + \gamma(\alpha a_1 - \beta b_1)][(\alpha\gamma + \beta(\alpha a_1 - \gamma c_1)]\}^{1/3}.$$

It is well known [1, p.155] that "the feet of the perpendiculars from two isogonally conjugate points lie on a circle; that is, isogonal conjugates have a common pedal circle ...” Consequently, L is self-cyclocevian conjugate [3, p. 226]. Since L is also self-isotomic conjugate, certain centers on L are generated in chains:

$$X_7 \xrightarrow{\text{isot}} X_8 \xrightarrow{\text{cycl}} X_{189} \xrightarrow{\text{isot}} X_{329} \xrightarrow{\text{cycl}} \cdots \tag{10}$$

$$X_2 \xrightarrow{\text{cycl}} X_4 \xrightarrow{\text{isot}} X_{69} \xrightarrow{\text{cycl}} X_{253} \xrightarrow{\text{isot}} X_{20} \xrightarrow{\text{cycl}} \cdots . \tag{11}$$

The mapping Ψ, of course, carries D to L, isogonal conjugate pairs on D to cyclocevian conjugate pairs on L, reflection-in-circumcenter pairs on D to isotomic conjugate pairs on L, and chains (8) and (9) to chains (10) and (11).

5. Cubics of the form $\varphi(Z(U))$

Every line passing through the pivot of the Darboux cubic D meets D in a pair of isogonal conjugates, and every line through the secondary pivot X_3 of D meets D in a reflection-pair. We wish to obtain generalizations of these pivotal properties by applying collineations to D. As a heuristic venture, we apply to D trilinear division by a point $P = p : q : r$ for which $pqr \neq 0$: the set D/P of points X/P as X traverses D is easily seen to be the cubic

$$\begin{aligned}&(a_1 - b_1c_1)px(q^2y^2 - r^2z^2) + (b_1 - c_1a_1)qy(r^2z^2 - p^2x^2)\\ +&(c_1 - a_1b_1)rz(p^2x^2 - q^2y^2) = 0.\end{aligned}$$

This is the self-P-isoconjugate cubic with pivot X_{20}/P and secondary pivot X_3/P. The cubic D/P, for some choices of P, passes through many “known points,” of course, and this is true if for D we substitute any cubic that passes through many “known points”.

The above preliminary venture suggests applying a variety of collineations to various cubics $Z(U)$. To this end, we shall call a regular collineation φ a *tricentral collineation* if there exists a mapping m_1 such that

$$\varphi(\alpha : \beta : \gamma) = m_1(\alpha : \beta : \gamma) : m_1(\beta : \gamma : \alpha) : m_1(\gamma : \alpha : \beta) \tag{12}$$

for all $\alpha : \beta : \gamma$. In this case, φ^{-1} has the form given by

$$n_1(\alpha : \beta : \gamma) : n_1(\beta : \gamma : \alpha) : n_1(\gamma : \alpha : \beta),$$

hence is tricentral.

The tricentral collineation (12) carries $Z(U)$ in (7) to the cubic $\varphi(Z(U))$ having equation

$$u\hat{\alpha}(\hat{\beta}^2 - \hat{\gamma}^2) + v\hat{\beta}(\hat{\gamma}^2 - \hat{\alpha}^2) + w\hat{\gamma}(\hat{\alpha}^2 - \hat{\beta}^2) = 0, \tag{13}$$

where

$$\hat{\alpha} : \hat{\beta} : \hat{\gamma} = n_1(\alpha : \beta : \gamma) : n_1(\beta : \gamma : \alpha) : n_1(\gamma : \alpha : \beta).$$

Example 13. Let

$$\varphi(\alpha : \beta : \gamma) = p(\beta + \gamma) : q(\gamma + \alpha) : r(\alpha + \beta),$$

so that

$$\varphi^{-1}(\alpha : \beta : \gamma) = -\frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} : \frac{\alpha}{p} - \frac{\beta}{q} + \frac{\gamma}{r} : \frac{\alpha}{p} + \frac{\beta}{q} - \frac{\gamma}{r}.$$

In accord with (13), the cubic $\varphi(Z(U))$ has equation

$$\frac{u\alpha}{p}\left(-\frac{\alpha}{p}+\frac{\beta}{q}+\frac{\gamma}{r}\right)\left(\frac{\beta}{q}-\frac{\gamma}{r}\right)+\frac{v\beta}{q}\left(\frac{\alpha}{p}-\frac{\beta}{q}+\frac{\gamma}{r}\right)\left(\frac{\gamma}{r}-\frac{\alpha}{p}\right)$$
$$+\quad \frac{w\gamma}{r}\left(\frac{\alpha}{p}+\frac{\beta}{q}-\frac{\gamma}{r}\right)\left(\frac{\alpha}{p}-\frac{\beta}{q}\right)=0.$$

Isogonic conjugate pairs on $Z(U)$ are carried as in Example 3 to P-Ceva conjugate pairs on $\varphi(Z(U))$. Indeed, each collinear triple X, U, X^{-1} is carried to a collinear triple, so that $\varphi(U)$ is a pivot for $\varphi(Z(U))$.

If $U = X_{20}$, so that $Z(U)$ is the Darboux cubic, then collinear triples X, X_3, $\tilde{X}$, where $\tilde{X}$ denotes the reflection of X in X_3, are carried to collinear triples $\varphi(X)$, $\varphi(X_3)$, $\varphi(\tilde{X})$, where $\varphi(\tilde{X})$ is the P-beth conjugate of X, as in Example 5.

Example 14. Continuing Example 13 with φ^{-1} in place of φ, the cubic $\varphi^{-1}(Z(U))$ is given by

$$s(u,v,w,p,q,r,\alpha,\beta,\gamma)+s(v,w,u,q,r,p,\beta,\gamma,\alpha)+s(w,u,v,r,p,q,\gamma,\alpha,\beta)=0,$$

where

$$s(u,v,w,p,q,r,\alpha,\beta,\gamma)=up(\beta+\gamma)(q^2(\gamma+\alpha)^2-r^2(\alpha+\beta)^2).$$

Collinear triples X, U, X^{-1} on $Z(U)$ yield collinear triples on $\varphi^{-1}(Z(U))$, so that $\varphi^{-1}(U)$ is a pivot for $\varphi^{-1}(Z(U))$. The point $\varphi^{-1}(X^{-1})$ is the P-aleph conjugate of X, as in Example 4.

On the Darboux cubic, collinear triples $X, X_3, \tilde{X}$, yield collinear triples $\varphi^{-1}(X)$, $\varphi^{-1}(X_3)$, $\varphi^{-1}(\tilde{X})$, this last point being the P-gimel conjugate of X, as in Example 6.

References

[1] M. Bôcher, *Introduction to Higher Algebra*, Macmillan, New York, 1931.
[2] R. Johnson, *Advanced Euclidean Geometry*, Dover, New York, 1960.
[3] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1-285.
[4] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://cedar.evansville.edu/˜ck6/encyclopedia/.
[5] R. Winger, *An Introduction to Projective Geometry*, Dover, New York, 1962.

Clark Kimberling: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, USA
E-mail address: ck6@evansville.edu

Forum Geometricorum
Volume 2 (2002) 33–37.

FORUM GEOM
ISSN 1534-1178

Equilateral Chordal Triangles

Floor van Lamoen

Abstract. When a circle intersects each of the sidelines of a triangle in two points, we can pair the intersections in such a way that three chords not along the sidelines bound a triangle, which we call a *chordal triangle*. In this paper we show that equilateral chordal triangles are homothetic to Morley's triangle, and identify all cases.

1. Chordal triangles

Let $T = ABC$ be the triangle of reference, and let a circle γ intersect side a in points B_a and C_a, side b in A_b and C_b and side c in A_c and B_c. The chords $a' = C_bB_c$, $b' = A_cC_a$ and $c' = A_bB_a$ enclose a triangle T', which we call a *chordal triangle*. See Figure 1. We begin with some preliminary results. In writing these the expression (ℓ_1, ℓ_2) denotes the directed angle from ℓ_1 to ℓ_2.

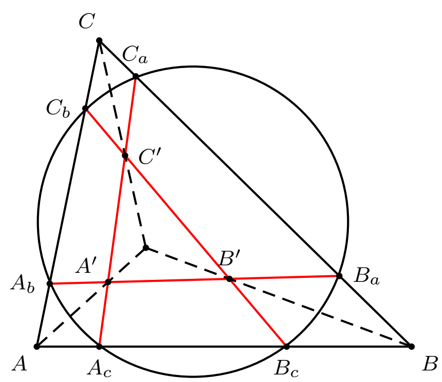


Figure 1

Proposition 1. *The sides of the chordal triangle T' satisfy*

$$(a', a) + (b', b) + (c', c) = 0 \pmod{\pi}.$$

Proof. Note that $(a', c) = (B_cC_b, B_cA)$ and

$$(c', b) = -(A_bA, A_bB_a) = (B_cB_a, B_cC_b) \pmod{\pi}$$

while also

$$(b', a) = -(C_aC, C_aA_c) = (B_cA_c, B_cB_a) \pmod{\pi}.$$

We conclude that $(a', c) + (c', b) + (b', a) = 0 \pmod{\pi}$, and the proposition follows from the fact that the internal directed angles of a triangle have sum π. $\square$

Publication Date: April 4, 2002. Communicating Editor: Jean-Pierre Ehrmann.

Proposition 2. *The triangle T' is perspective to ABC.*

Proof. From Pascal's hexagon theorem applied to $C_aB_aA_bC_bB_cA_c$ we see that the points of intersection $C_aB_a \cap C_bB_c$, $B_cA_c \cap B_aA_b$ and $A_bC_b \cap A_cC_a$ are collinear. Therefore, triangles ABC and $A'B'C'$ are line perspective, and by Desargues' two-triangle theorem, they are point perspective as well. □

The triangle T'' enclosed by the lines $a'' = (a \cap b') \cup (a' \cap b)$ and similarly defined b'' and c'' is also a chordal triangle, which we will call the *alternative chordal triangle* of T'. [1]

Proposition 3. *The corresponding sides of T' and T'' are antiparallel with respect to triangle T.*

Proof. From the fact that $B_cA_cA_bC_b$ is a cyclic quadrilateral, immediately we see $\angle AB_cC_b = \angle AA_bA_c$, so that a' and a'' are antiparallel. By symmetry this proves the proposition. □

We now see that there is a family of chordal triangles homothetic to T'. From a starting point on one of the sides of ABC we can construct segments to the next sides alternately parallel to corresponding sides of T' and T''.[2] Extending the segments parallel to T' we get a chordal triangle homothetic to T'. [3]

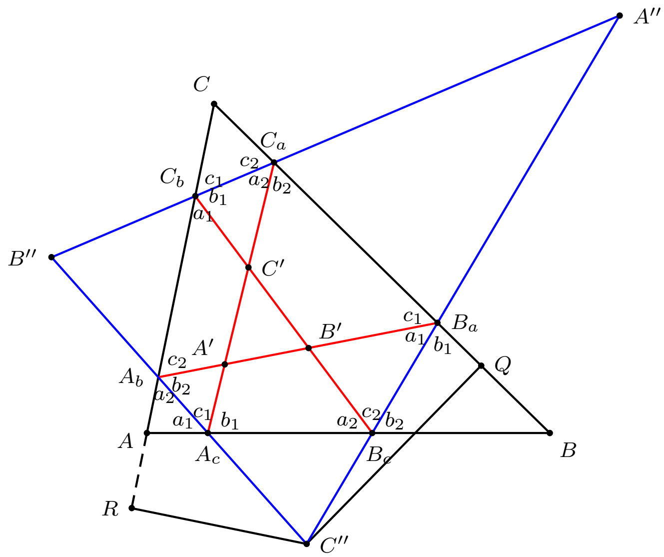


Figure 2

[1]This is the triangle enclosed by the lines A_bA_c, B_cB_a and C_aC_b in Figure 2. The definition of T'' from T and T' is exactly dual to the definition of 'desmic mate' (see [1, §4]). This yields also that T, T' and T'' are perspective through one perspector, which will be shown differently later this section, in order to keep this paper self contained.

[2]This is very similar to the well known construction of the Tucker hexagon.

[3]In fact it is easy to see that starting with any pair of triangles T' and T'' satisfying Propositions 1 and 3, we get a family of chordal triangles with this construction.

With the knowledge of Propositions 1-3 we can indicate angles as in Figure 2. In this figure we have also drawn altitudes $C''Q$ and $C''R$ to BC and AC respectively.

Note that

$$\begin{aligned} C''Q &= \sin(b_1)\sin(b_2)\csc(C'')\cdot A_bB_a, \\ C''R &= \sin(a_1)\sin(a_2)\csc(C'')\cdot A_bB_a. \end{aligned}$$

This shows that the (homogeneous) normal coordinates[4] for C'' are of the form

$$(\csc(a_1)\csc(a_2) : \csc(b_1)\csc(b_2) : \cdots).$$

From this we see that T and T'' have perspector

$$(\csc(a_1)\csc(a_2) : \csc(b_1)\csc(b_2) : \csc(c_1)\csc(c_2)).$$

Clearly this perspector is independent from choice of T' or T'', and depends only on the angles a_1, a_2, b_1, b_2, c_1 and c_2.

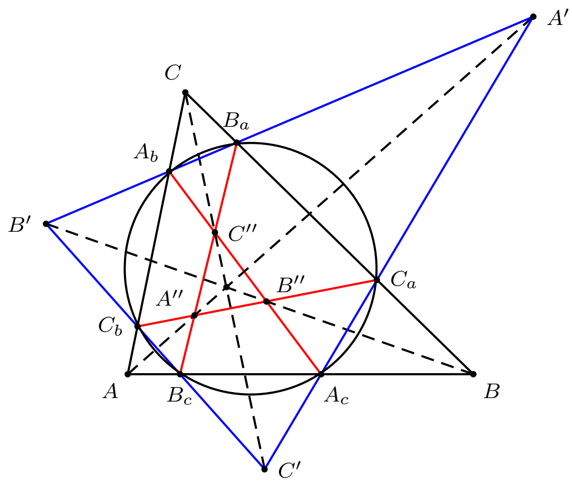


Figure 3

Proposition 4. *All chordal triangles homothetic to a chordal triangle T', as well as all chordal triangles homothetic to the alternative chordal triangle of T', are perspective to T through one perspector.*

2. Equilateral chordal triangles

Jean-Pierre Ehrmann and Bernard Gibert have given a magnificently elegant characterization of lines parallel to sides of Morley's trisector triangle.

Proposition 5. [2, Proposition 5] *A line ℓ is parallel to a side of Morley's trisector triangle if and only if*

$$(\ell, a) + (\ell, b) + (\ell, c) = 0 \pmod{\pi}.$$

[4]These are traditionally called (homogeneous) trilinear coordinates.

An interesting consequence of Proposition 5 in combination with Proposition 1 is that Morley triangles of chordal triangles are homothetic to the Morley triangle of ABC. Furthermore, equilateral chordal triangles themselves are homothetic to Morley's triangle. This means that they are not in general constructible by ruler and compass.

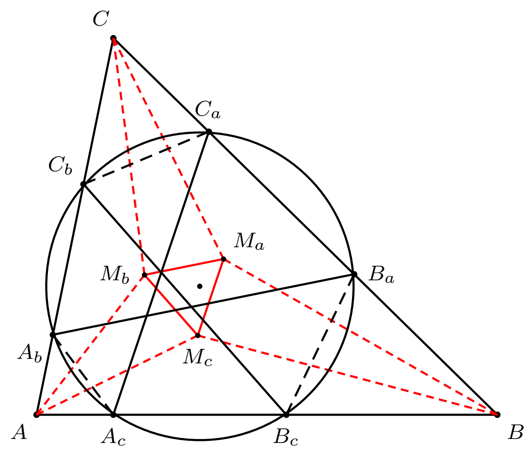


Figure 4

With this knowledge we can identify all equilateral chordal triangles. First we can specify the angles (a', c), (c', b), and (b', a). There are six possibilities. Now we can fix one point, say B_a, and use the specified angles and Proposition 3 to find the other vertices of hexagon $C_aB_aA_bC_bB_cA_c$. The rest is easy.

We can now identify all equilateral chordal triangles by (homogeneous) normal coordinates. As an example we will study the family

$$(a', c) = \frac{2}{3}B + \frac{1}{3}C, \quad (c', b) = \frac{2}{3}A + \frac{1}{3}B, \quad (b', a) = \frac{2}{3}C + \frac{1}{3}A.$$

From the derivation of Proposition 4 we see that the perspector of this family has normal coordinates

$$\left(\csc\frac{2B+C}{3}\csc\frac{B+2C}{3} : \csc\frac{2A+C}{3}\csc\frac{A+2C}{3} : \csc\frac{2A+B}{3}\csc\frac{A+2B}{3}\right).$$

Writing (TU) for the directed arc from T to U, and defining

$$\begin{array}{lll} t_a = (C_bB_c), & t_b = (A_cC_a), & t_c = (B_aA_b), \\ u_a = (C_aB_a), & u_b = (A_bC_b), & u_c = (B_cA_c), \end{array}$$

we find the following system of equations

$$\begin{array}{ll} (C_bA_c) = t_a + u_c = \frac{4}{3}B + \frac{2}{3}C, & (A_bB_c) = u_b + t_a = \frac{4}{3}C + \frac{2}{3}B, \\ (B_aC_b) = t_c + u_b = \frac{4}{3}A + \frac{2}{3}B, & (C_aA_b) = u_a + t_c = \frac{4}{3}B + \frac{2}{3}A, \\ (A_cB_a) = t_b + u_a = \frac{4}{3}C + \frac{2}{3}A, & (B_cC_a) = u_c + t_b = \frac{4}{3}A + \frac{2}{3}C. \end{array}$$

This system can be solved with one parameter τ to be

$$t_a = \tfrac{4(B+C)}{3} - 2\tau \quad t_b = \tfrac{4(C+A)}{3} - 2\tau \quad t_c = \tfrac{4(A+B)}{3} - 2\tau$$

$$u_a = -\tfrac{2A}{3} + 2\tau \qquad u_b = -\tfrac{2B}{3} + 2\tau \qquad u_c = -\tfrac{2C}{3} + 2\tau$$

The coordinates of the centers of these circles are now given by[5]

$$\left(\pm\cos\left(\frac{A}{3}+\tau\right) : \pm\cos\left(\frac{B}{3}+\tau\right) : \pm\cos\left(\frac{C}{3}+\tau\right)\right).$$

Assuming all cosines positive, these centers describe a line, which passes (take $\tau = 0$) through the perspector of the adjoint Morley triangle and ABC, in [3,4] numbered as X_{358}. By taking $\tau = \frac{\pi}{2}$ we see the line also passes through the point

$$\left(\sin\frac{A}{3} : \sin\frac{B}{3} : \sin\frac{C}{3}\right).$$

Hence, the equation of this line through the centers of the circles is

$$\sum_{\text{cyclic}} \left(\sin\frac{B}{3}\cos\frac{C}{3} - \cos\frac{B}{3}\sin\frac{C}{3}\right) x = 0,$$

or simply

$$\sum_{\text{cyclic}} \sin\frac{B-C}{3} x = 0.$$

One can find the other families of equilateral chordal triangles by adding and/or subtracting appropriate multiples of $\frac{\pi}{3}$ to the inclinations of the sides of T' with respect to T. The details are left to the reader.

References

[1] K. R. Dean and F. M. van Lamoen, Geometric construction of reciprocal conjugations, *Forum Geom.*, 1 (2001) 115–120.

[2] J.-P. Ehrmann and B. Gibert, A Morley configuration, *Forum Geom.*, 1 (2001) 51–58.

[3] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

[4] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://cedar.evansville.edu/˜ck6/encyclopedia/.

Floor van Lamoen: Statenhof 3, 4463 TV Goes, The Netherlands.

E-mail address: f.v.lamoen@wxs.nl

[5]We have to be careful with this type of conclusion. We cannot blindly give signs to the coordinates. In particular, we cannot blindly follow the signs of the cosines below - if we would add 360 degrees to u_a, this would yield a change of sign for the first coordinate for the same figure. To establish signs, one can shuffle the hexagon $C_aB_aA_bC_bB_cA_c$ in such a way that the central angles on the segments on the sides are all positive and the sum of central angles is exactly 2π. From this we can draw conclusions on the location of the center with respect to the sides.

Forum Geometricorum
Volume 2 (2002) 39–46.

FORUM GEOM
ISSN 1534-1178

The Napoleon Configuration

Gilles Boutte

Abstract. It is an elementary fact in triangle geometry that the two Napoleon triangles are equilateral and have the same centroid as the reference triangle. We recall some basic properties of the Fermat and Napoleon configurations, and use them to study equilateral triangles bounded by cevians. There are two families of such triangles, the triangles in the two families being oppositely oriented. The locus of the circumcenters of the triangles in each family is one of the two Napoleon circles, and the circumcircles of each family envelope a conchoid of a circle.

1. The Fermat-Napoleon configuration

Consider a reference triangle ABC, with side lengths a, b, c. Let F_a^+ be the point such that CBF_a^+ is equilateral with the same orientation as ABC; similarly for F_b^+ and F_c^+. See Figure 1.

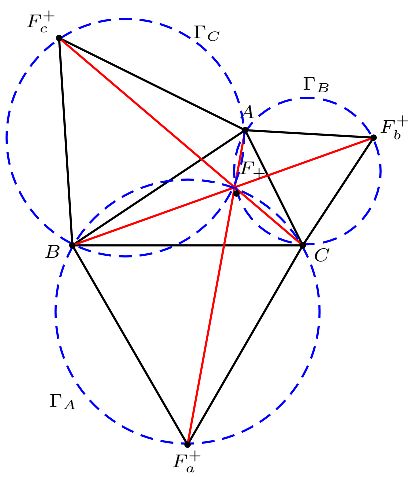


Figure 1. The Fermat configuration

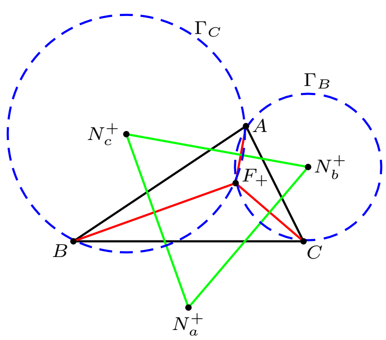


Figure 2. The Napoleon configuration

The triangle $F_a^+F_b^+F_c^+$ is called the *first Fermat triangle*. It is an elementary fact that triangles $F_a^+F_b^+F_c^+$ and ABC are perspective at the *first Fermat point* F_+. We define similarly the *second Fermat triangle* $F_a^-F_b^-F_c^-$ in which CBF_a^-, ACF_b^- and BAF_c^- are equilateral triangles with opposite orientation of ABC. This is perspective with ABC at the *second Fermat point* F_-. [1] Denote by Γ_A the circumcircle of CBF_a^+, and N_a^+ its center; similarly for Γ_B, Γ_C, N_b^+, N_c^+. The

Publication Date: April 30, 2002. Communicating Editor: Paul Yiu.

[1]In [1], these are called the isogonic centers, and are referenced as X_{13} and X_{14}.

triangle $N_a^+N_b^+N_c^+$ is called the *first Napoleon triangle*, and is perspective with ABC at the *first Napoleon point* N_+. Similarly, we define the *second Napoleon triangle* $N_a^-N_b^-N_c^-$ perspective with ABC at the *second Napoleon point* N_-.[2] See Figure 2. Note that N_a^- is the antipode of F_a^+ on Γ_A.

We summarize some of the important properties of the Fermat and Napoleon points.

Theorem 1. *Let ABC be a triangle with side lengths a, b, c and area $\triangle$.*

(1) *The first Fermat point F_+ is the common point to Γ_A, Γ_B and Γ_C.*
(2) *The segments AF_a^+, BF_b^+, CF_c^+ have the same length ℓ given by*

$$\ell^2 = \frac{1}{2}(a^2 + b^2 + c^2 + 4\sqrt{3}\triangle).$$

(3) *The first Napoleon triangle $N_a^+N_b^+N_c^+$ is equilateral with the same orientation as ABC. Its circumradius is $\frac{\ell}{3}$.*
(4) *The Fermat and Napoleon triangles have the same centroid G as ABC.*
(5) *The first Fermat point lies on the circumcircle of the second Napoleon triangle. We shall call this circle the* second Napoleon circle.
(6) *The lines $N_b^+N_c^+$ and AF_+ are respectively the line of centers and the common chord of Γ_B and Γ_C. They are perpendicular.*

Remarks. (i) Similar statements hold for the second Fermat and Napoleon points F_- and N_-, with appropriate changes of signs.

(ii) (4) is an easy corollary of the following important result: Given a triangle $A'B'C'$ with ABC', BCA', CAB' positively similar. Thus ABC and $A'B'C'$ have the same centroid. See, for example, [3, p.462].

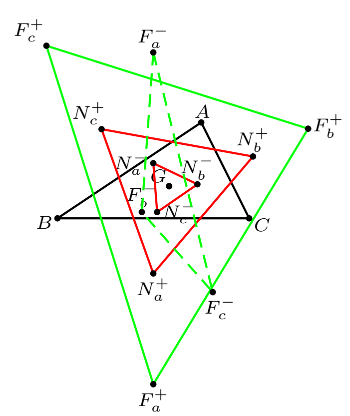


Figure 3. The Fermat and Napoleon triangles

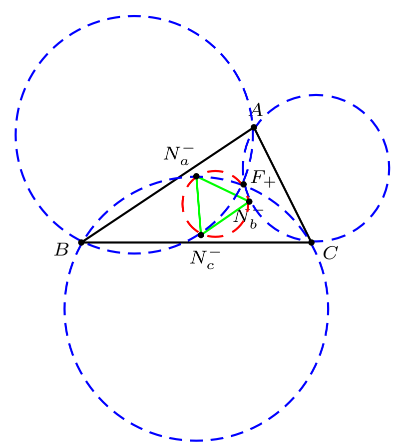


Figure 4. The Fermat point on the second Napoleon circle

[2] In [1], the Napoleon points as X_{17} and X_{18}.

2. Equilateral triangles bounded by cevians

Let $A_1B_1C_1$ be an equilateral triangle, with the same orientation as ABC and whose sides are cevian lines in ABC, i.e. A lies on B_1C_1, B lies on C_1A_1, C lies on A_1B_1. See Figure 5. Thus, CB is seen from A_1 at an angle $\frac{\pi}{3}$, *i.e.*, $\angle CA_1B = \frac{\pi}{3}$, and A_1 lies on Γ_A. Similarly B_1 lies on Γ_B and C_1 lies on Γ_C. Conversely, let A_1 be any point on Γ_A. The line A_1B intersects Γ_C at B and C_1, the line A_1C intersects Γ_B at C and B_1.

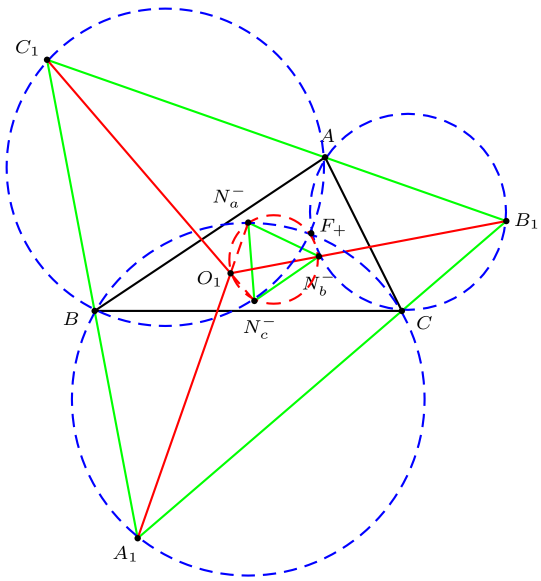


Figure 5. Equilateral triangle bounded by cevians

Three of the angles of the quadrilateral $A_1B_1AC_1$ are $\frac{\pi}{3}$; so A lies on B_1C_1 and $A_1B_1C_1$ is equilateral with the same orientation as ABC. We obtain an equilateral triangle bounded by cevians. There is an infinity of such triangles.

Let O_1 be the center of $A_1B_1C_1$. BO_1 is seen from A_1 at an angle $\frac{\pi}{6} \pmod{\pi}$; similarly for BN_a^-. The line A_1O_1 passes through N_a^-. Similarly the lines B_1O_1 and C_1O_1 pass through N_b^- and N_c^- respectively. It follows that $N_b^-N_c^-$ and B_1C_1 are seen from O_1 at the same angle $\frac{2\pi}{3} = -\frac{\pi}{3} \pmod{\pi}$, and the point O_1 lies on the circumcircle of $N_a^-N_b^-N_c^-$. Thus we have:

Theorem 2. *The locus of the center of equilateral triangles bounded by cevians, and with the same orientation as ABC, is the second Napoleon circle.*

Similarly, the locus of the center of equilateral triangles bounded by cevians, and with the opposite orientation of ABC, is the first Napoleon circle.

3. Pedal curves and conchoids

We recall the definitions of pedal curves and conchoids from [2].

Definitions. Given a curve $\mathcal{C}$ and a point O,

(1) the *pedal curve* of $\mathcal{C}$ with respect to O is the locus of the orthogonal projections of O on the tangent lines of $\mathcal{C}$;
(2) for a positive number k, the *conchoid* of $\mathcal{C}$ with respect to O and with *offset* k is the locus of the points P for which there exists M on $\mathcal{C}$ with O, M, P collinear and $MP = k$.

For the constructions of normal lines, we have

Theorem 3. *Let $\mathcal{P}_O$ be the pedal curve of $\mathcal{C}$ with respect to O. For any point M on $\mathcal{C}$, if P is the projection of O on the tangent to $\mathcal{C}$ at M, and Q is such that $OPMQ$ is a rectangle, then the line PQ is normal to $\mathcal{P}_O$ at P.*

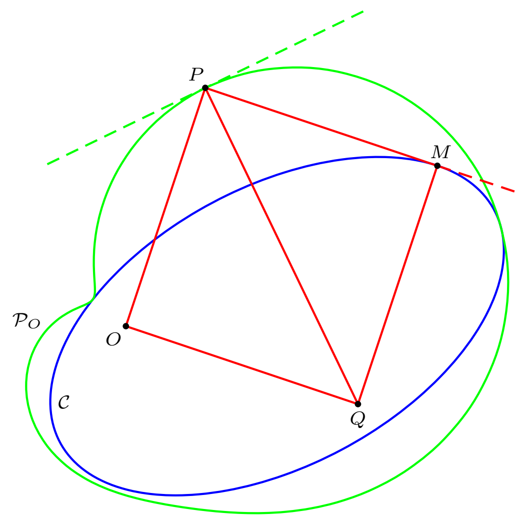


Figure 6. The normal to a pedal curve

Theorem 4. *Let $\mathcal{C}_{O,k}$ be the conchoid of $\mathcal{C}$ with respect to O and offset k. For any point P on $\mathcal{C}_{O,k}$, if M is the intersection of the line OP with $\mathcal{C}$, then the mormal lines to $\mathcal{C}_{O,k}$ at P and to $\mathcal{C}$ at M intersect on the perpendicular to OP at O.*

4. Envelope of the circumcircles

Consider one of the equilateral triangles with the same orientation of ABC. Let $\mathcal{C}_1$ be the circumcircle of $A_1B_1C_1$, R_1 its radius. Its center O_1 lies on the Napoleon circle and the vertex A_1 lies on the circle Γ_A. The latter two circles pass through F_+ and N_a^-. The angles $\angle N_a^- A_1 F_+$ and $\angle N_a^- O_1 F_+$ have constant magnitudes. The shape of triangle $A_1O_1F_+$ remains unchanged when O_1 traverses the second Napoleon circle $\mathcal{N}$. The ratio $\dfrac{O_1A_1}{O_1F_+} = \dfrac{R_1}{O_1F_+}$ remains constant, say, λ.

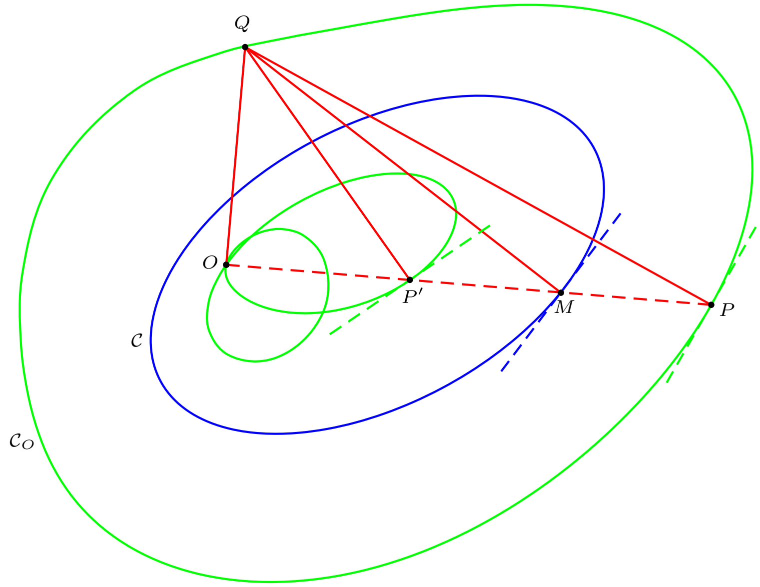


Figure 7. The normal to a conchoid

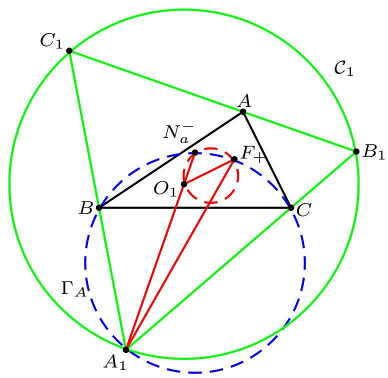


Figure 8. Equilateral triangle bounded by cevians and its circumcircle

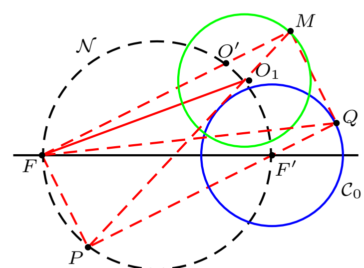


Figure 9. The pedal of $\mathcal{C}_0$ with respect to F

For convenience we denote by $\mathcal{N}$ the Napoleon circle which is the locus of O_1, F the Fermat point lying on this circle, F' the antipode of F on $\mathcal{N}$, and $\mathcal{C}_0$ the particular position of $\mathcal{C}_1$ when O_1 and F' coincide. See Figure 7. Let P be any point on $\mathcal{N}$, the line PF' intersects $\mathcal{C}_0$ at Q and Q' (F' between P and Q), we construct the point M such that $FPQM$ is a rectangle. The locus $\mathcal{P}_F$ of M is the pedal curve of $\mathcal{C}_0$ with respect to F and, by Theorem 3, the line MP is the normal to $\mathcal{P}_F$ at M. The line MP intersects $\mathcal{N}$ at P and O_1 and the circle through M with center O_1 is tangent to $\mathcal{P}_F$ at M.

The triangles FMO_1 and FQF' are similar since $\angle FMO_1 = \angle FQF'$ and $\angle FO_1M = \angle FF'Q$.[3] It follows that $\dfrac{O_1M}{O_1F} = \dfrac{F'Q}{F'F} = \lambda$, and $O_1M = R_1$. The circle through M with center O_1 is one in the family of circle for which we search the envelope.

Furthermore, the line FM intersects $\mathcal{N}$ at F and O', and $O'MQF'$ is a rectangle. Thus, $O'M = F'Q$, the radius of $\mathcal{C}_0$. It follows that M lies on the external branch of the conchoid of $\mathcal{N}$ with respect to F and the length $R =$ radius of $\mathcal{C}_0$.

By the same reasoning for the point Q', we obtain M' on $\mathcal{P}_F$, but on the internal branch of the conchoid. Each circle $\mathcal{C}_1$ touches both branches of the conchoid.

Theorem 5. *Let $\mathcal{F}$ be the family of circumcircles of equilateral triangles bounded by cevians whose locus of centers is the Napoleon circle $\mathcal{N}$ passing through the Fermat point F. The envelope of this family $\mathcal{F}$ is the pedal with respect to F of the circle $\mathcal{C}_0$ of $\mathcal{F}$ whose center is the antipode of F on $\mathcal{N}$, i.e. the conchoid of $\mathcal{N}$ with respect to F and offset the radius of $\mathcal{C}_0$. Each circle of $\mathcal{F}$ is bitangent to the envelope.*

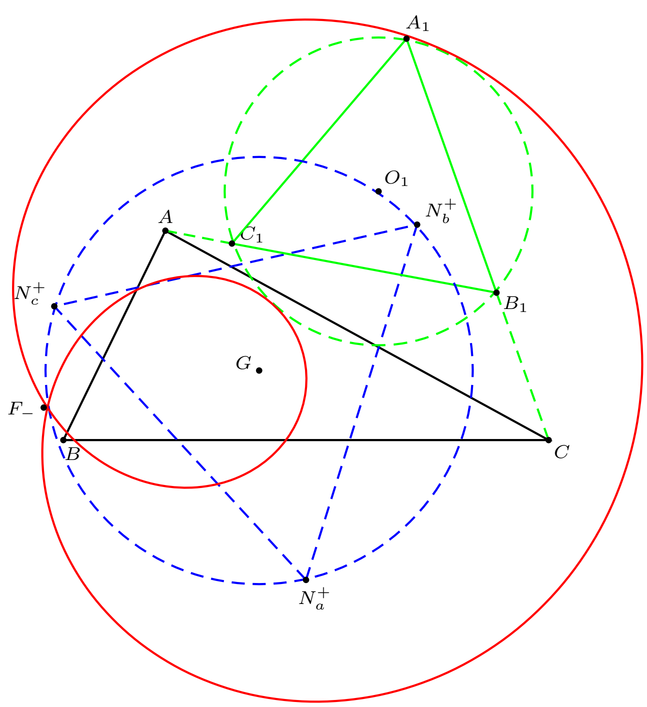


Figure 10. The envelope of the circumcirles ($\lambda < 1$)

[3] FP is seen at the same angle from O_1 and from F'.

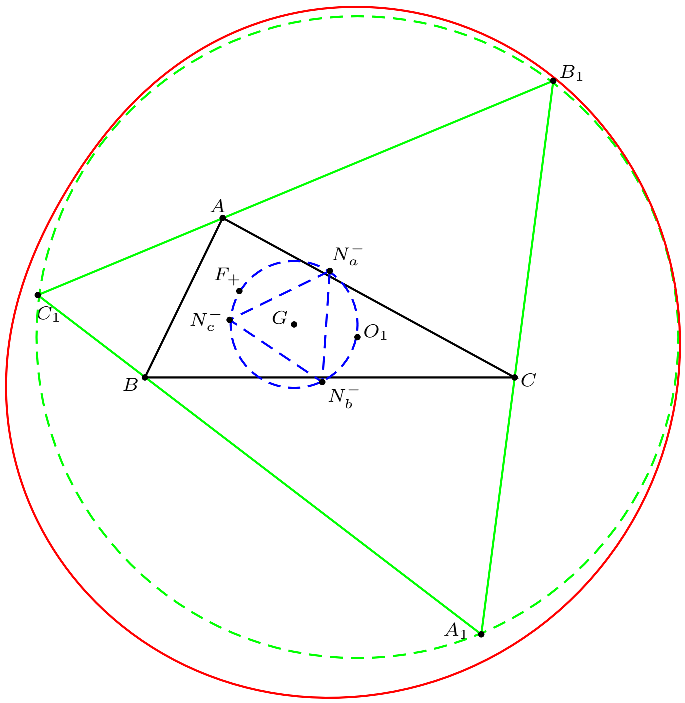


Figure 11. The envelope of the circumcirles ($\lambda > 1$)

Let i be the inversion with respect to a circle $\mathcal{C}$ whose center is F and such that $\mathcal{C}_0$ is invariant under it. The curve $i(\mathcal{P}_F)$ is the image of $\mathcal{C}_0$ by the reciprocal polar transformation with respect to $\mathcal{C}$, i.e., a conic with one focus at F. This conic is :

(1) a hyperbola for $\lambda < 1$ (F is exterior at $\mathcal{C}_0$) ;
(2) a parabola for $\lambda = 1$ (F lies on $\mathcal{C}_0$) ;
(3) an ellipse for $\lambda > 1$ (F is interior at $\mathcal{C}_0$).

So the envelope $\mathcal{P}$ of the circumcircles $\mathcal{C}_1$ is the inverse of this conic with respect to one of its foci, i.e., a conchoid of circle which is :

(1) a limaçon of Pascal for $\lambda < 1$: the hyperbola $i(\mathcal{P})$ as two asymptotes, so F is a node on $\mathcal{P}$;
(2) a cardioid for $\lambda = 1$: the parabola $i(\mathcal{P})$ is tangent to the line at infinity, so F is a cusp on $\mathcal{P}$;
(3) a curve without singularity for $\lambda > 1$: all points of the ellipse $i(\mathcal{P})$ are at finite distance.

We illustrate (1) and (3) in Figures 12 and 13. [4] It should be of great interest to see if always $\lambda > 1$ for F_+ (and < 1 for F_-). We think that the answer is affirmative, and that $\lambda = 1$ is possible if and only if A, B, C are collinear.

[4] Images of inversion of the limaçon of Pascal and the cardioid can also be found in the websites http://www-history.mcs.st-andrews.ac.uk/history/Curves and http://xahlee.org/SpecialPlaneCurves.

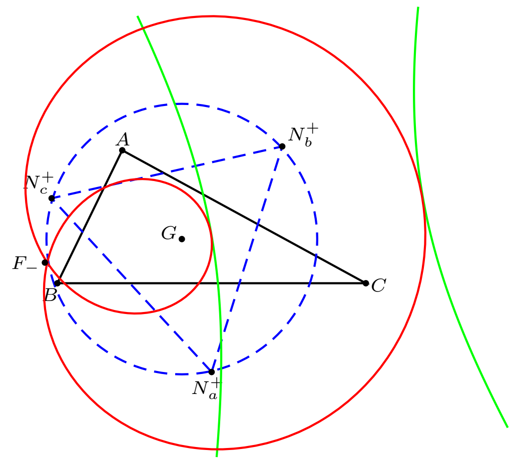


Figure 12. The inverse of the envelope ($\lambda < 1$)

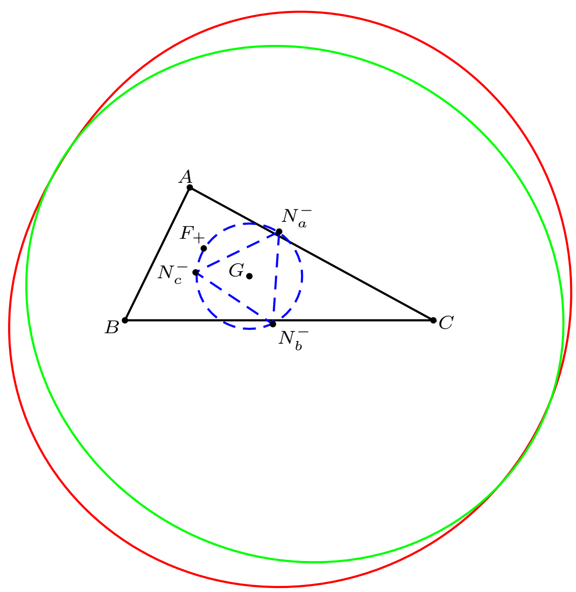


Figure 13. The inverse of the envelope ($\lambda > 1$)

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, http://www2.evansville.edu/ck6/encyclopedia/ (2000).
[2] J. D. Lawrence, *A Catalog of Special Plane Curves*, Dover, 1972.
[3] E. Rouché et Ch. de Comberousse, *Traité de Géométrie*, Gauthier-Villars, Paris, 6e edition, 1891.

Gilles Boutte: Le Sequoia 118, rue Crozet-Boussingault, 42100 Saint-Etienne, France
E-mail address: `g.boutte@free.fr`

Forum Geometricorum
Volume 2 (2002) 47–63.

FORUM GEOM
ISSN 1534-1178

The Lemoine Cubic and Its Generalizations

Bernard Gibert

Abstract. For a given triangle, the Lemoine cubic is the locus of points whose cevian lines intersect the perpendicular bisectors of the corresponding sides of the triangle in three collinear points. We give some interesting geometric properties of the Lemoine cubic, and study a number of cubics related to it.

1. The Lemoine cubic and its constructions

In 1891, Lemoine published a note [5] in which he very briefly studied a cubic curve defined as follows. Let M be a point in the plane of triangle ABC. Denote by M_a the intersection of the line MA with the perpendicular bisector of BC and define M_b and M_c similarly. The locus of M such that the three points M_a, M_b, M_c are collinear on a line $\mathcal{L}_M$ is the cubic in question. We shall denote this cubic by $\mathcal{K}(O)$, and follow Neuberg [8] in referring to it as the Lemoine cubic. Lemoine claimed that the circumcenter O of the reference triangle was a triple point of $\mathcal{K}(O)$. As pointed out in [7], this statement is false. The present paper considerably develops and generalizes Lemoine's note.

We use homogeneous barycentric coordinates, and adopt the notations of [4] for triangle centers. Since the second and third coordinates can be obtained from the first by cyclic permutations of a, b, c, we shall simply give the first coordinates. For convenience, we shall also write

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2}.$$

Thus, for example, the circumcenter is $X_3 = [a^2 S_A]$.

Figure 1 shows the Lemoine cubic $\mathcal{K}(O)$ passing through A, B, C, the orthocenter H, the midpoints A', B', C' of the sides of triangle ABC, the circumcenter O, and several other triangle centers such as $X_{32} = [a^4]$, $X_{56} = \left[\frac{a^2}{b+c-a}\right]$ and its extraversions. [1] Contrary to Lemoine's claim, the circumcenter is a node. When M traverses the cubic, the line $\mathcal{L}_M$ envelopes the Kiepert parabola with focus

Publication Date: May 10, 2002. Communicating Editor: Paul Yiu.

The author sincerely thanks Edward Brisse, Jean-Pierre Ehrmann and Paul Yiu for their friendly and efficient help. Without them, this paper would never have been the same.

[1] The three extraversions of a point are each formed by changing in its homogeneous barycentric coordinates the signs of one of a, b, c. Thus, $X_{56a} = \left(\frac{a^2}{b+c+a} : \frac{b^2}{c-a-b} : \frac{c^2}{-a+b-c}\right)$, and similarly for X_{56b} and X_{56c}.

$F = X_{110} = \left[\frac{a^2}{b^2-c^2}\right]$ and directrix the Euler line. The equation of the Lemoine cubic is

$$\sum_{\text{cyclic}} a^4 S_A yz(y-z) + (a^2-b^2)(b^2-c^2)(c^2-a^2)xyz = 0.$$

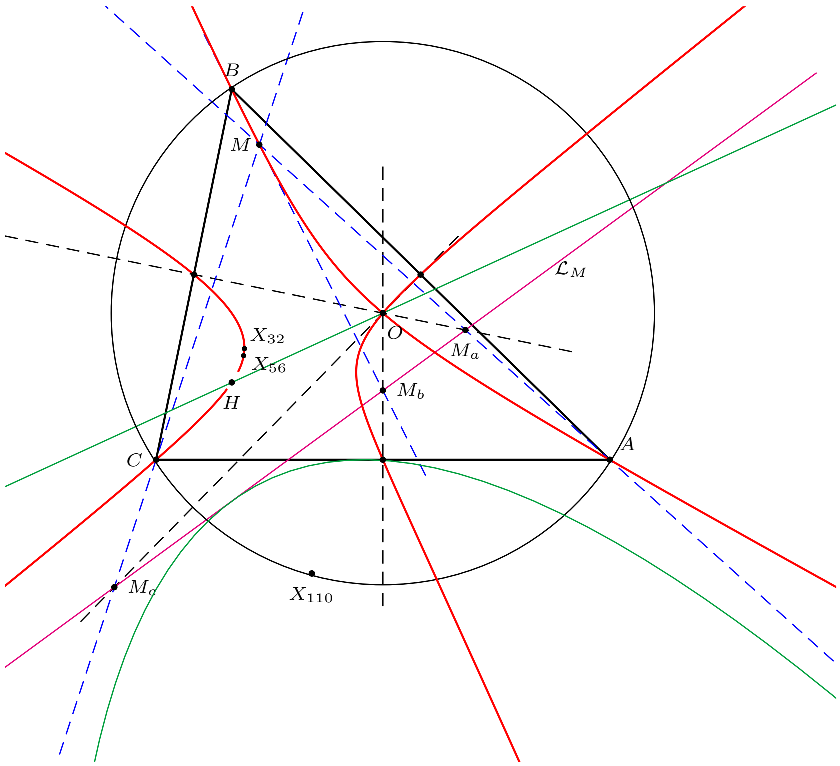


Figure 1. The Lemoine cubic with the Kiepert parabola

We give two equivalent constructions of the Lemoine cubic.

Construction 1. For any point Q on the line GK, the trilinear polar q of Q meets the perpendicular bisectors OA', OB', OC' at Q_a, Q_b, Q_c respectively. [2] The lines AQ_a, BQ_b, CQ_c concur at M on the cubic $\mathcal{K}(O)$.

For $Q = (a^2+t : b^2+t : c^2+t)$, this point of concurrency is

$$M = \left(\frac{a^2+t}{b^2c^2+(b^2+c^2-a^2)t} : \frac{b^2+t}{c^2a^2+(c^2+a^2-b^2)t} : \frac{c^2+t}{a^2b^2+(a^2+b^2-c^2)t}\right).$$

[2]The tripolar q envelopes the Kiepert parabola.

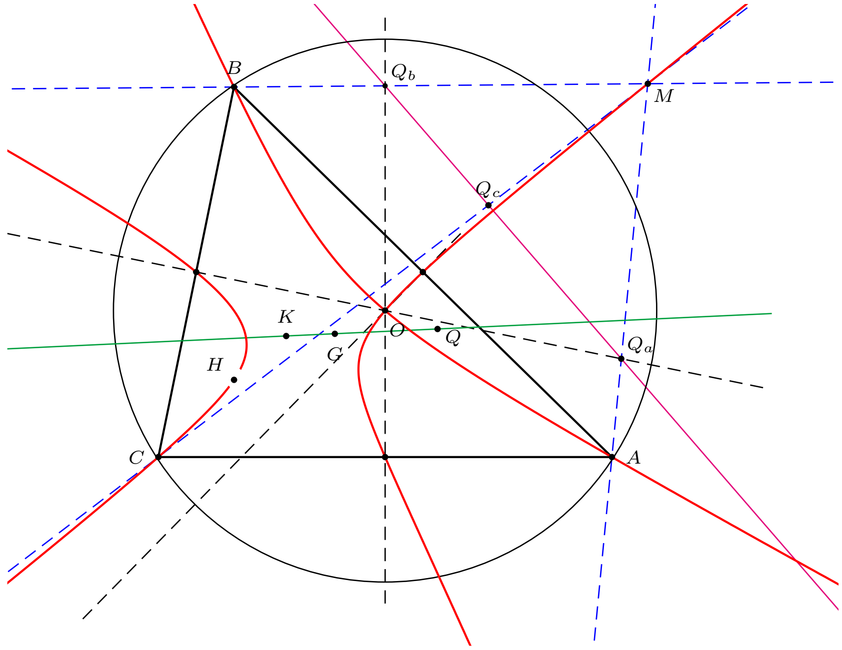


Figure 2. The Lemoine cubic as a locus of perspectors (Construction 1)

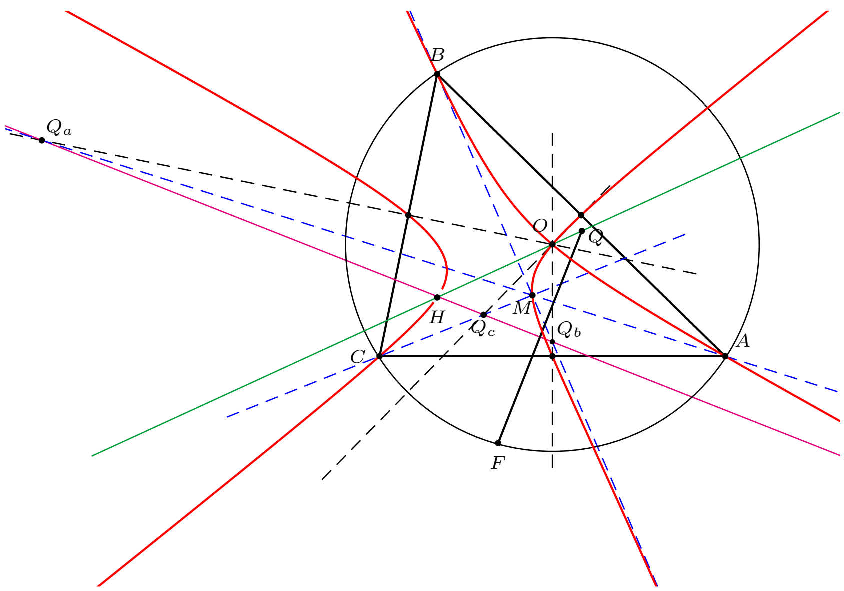


Figure 3. The Lemoine cubic as a locus of perspectors (Construction 2)

This gives a parametrization of the Lemoine cubic. This construction also yields the following points on $\mathcal{K}(O)$, all with very simple coordinates, and are not in [4].

i	$Q = X_i$	$M = M_i$
69	S_A	$\frac{S_A}{b^4+c^4-a^4}$
86	$\frac{1}{b+c}$	$\frac{1}{a(b+c)-(b^2+bc+c^2)}$
141	b^2+c^2	$\frac{b^2+c^2}{b^4+b^2c^2+c^4-a^4}$
193	$b^2+c^2-3a^2$	$S_A(b^2+c^2-3a^2)$

Construction 2. For any point Q on the Euler line, the perpendicular bisector of FQ intersects the perpendicular bisectors OA', OB', OC' at Q_a, Q_b, Q_c respectively. The lines AQ_a, BQ_b, CQ_c concur at M on the cubic $\mathcal{K}(O)$.

See Figure 3 and Remark following Construction 4 on the construction of tangents to $\mathcal{K}(O)$.

2. Geometric properties of the Lemoine cubic

Proposition 1. *The Lemoine cubic has the following geometric properties.*

(1) *The two tangents at O are parallel to the asymptotes of the Jerabek hyperbola.*

(2) *The tangent at H passes through the center $X_{125} = [(b^2-c^2)^2 S_A]$ of the Jerabek hyperbola.* [3]

(3) *The tangents at A, B, C concur at $X_{184} = [a^4 S_A]$, the inverse of X_{125} in the Brocard circle.*

(4) *The asymptotes are parallel to those of the orthocubic,* i.e., *the pivotal isogonal cubic with pivot H.*

(5) *The "third" intersections H_A, H_B, H_C of $\mathcal{K}(O)$ and the altitudes lie on the circle with diameter OH.* [4] *The triangles $A'B'C'$ and $H_AH_BH_C$ are perspective at a point*

$$Z_1 = \left[a^4 S_A(a^4+b^4+c^4-2a^2(b^2+c^2))\right]$$

on the cubic. [5]

(6) *The "third" intersections A'', B'', C'' of $\mathcal{K}(O)$ and the sidelines of the medial triangle form a triangle perspective with $H_AH_BH_C$ at a point*

$$Z_2 = \left[\frac{a^4 S_A^2}{3a^4-2a^2(b^2+c^2)-(b^2-c^2)^2}\right]$$

on the cubic. [6]

(7) *$\mathcal{K}(O)$ intersects the circumcircle of ABC at the vertices of the circumnormal triangle of ABC.* [7]

---

[3] This is also tangent to the Jerabek hyperbola at H.

[4] In other words, these are the projections of O on the altitudes. The coordinates of H_A are

$$\left(\frac{2a^4 S_A}{a^2(b^2+c^2)-(b^2-c^2)^2} : S_C : S_B\right).$$

[5] Z_1 is the isogonal conjugate of X_{847}. It lies on a large number of lines, 13 using only triangle centers from [4], for example, X_2X_{54}, X_3X_{49}, X_4X_{110}, X_5X_{578}, $X_{24}X_{52}$ and others.

[6] This point Z_2 is not in the current edition of [4]. It lies on the lines X_3X_{64}, X_4X_{122} and $X_{95}X_{253}$.

[7] These are the points U, V, W on the circumcircle for which the lines UU^*, VV^*, WW^* (joining each point to its own isogonal conjugate) all pass through O. As such, they are, together with the vertices, the intersections of the circumcircle and the McCay cubic, the isogonal cubic with pivot the circumcenter O. See [3, p.166, §6.29].

We illustrate (1), (2), (3) in Figure 4, (4) in Figure 5, (5), (6) in Figure 6, and (7) in Figure 7 below.

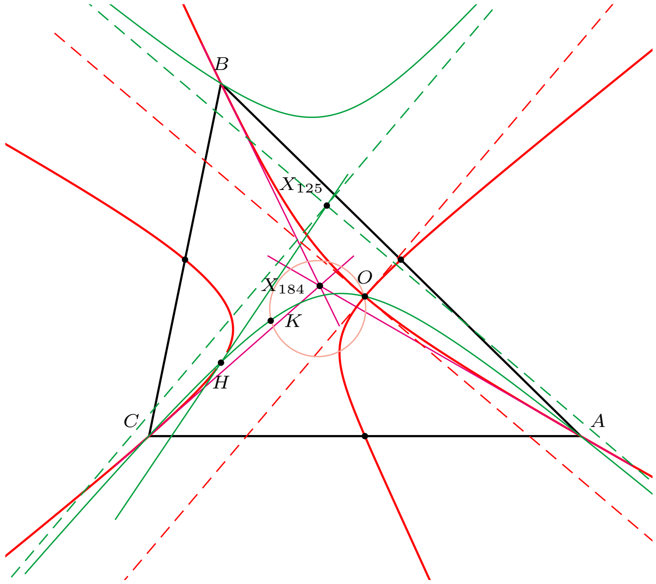


Figure 4. The tangents to the Lemoine cubic at O and the Jerabek hyperbola

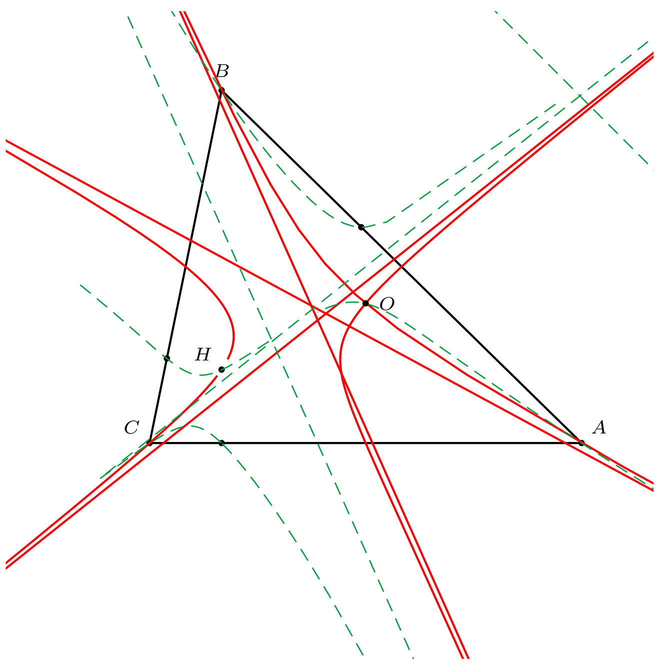


Figure 5. The Lemoine cubic and the orthocubic have parallel asymptotes

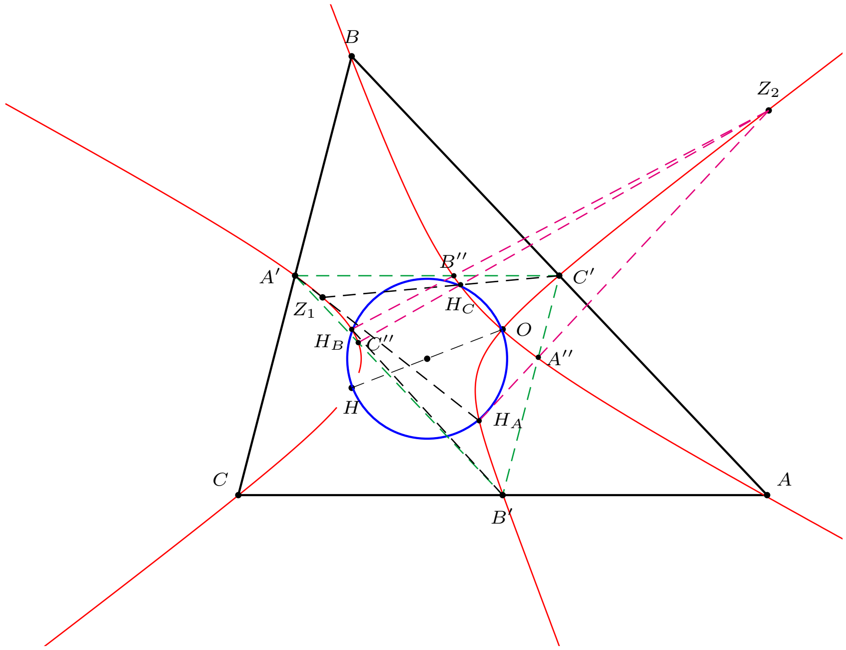


Figure 6. The perspectors Z_1 and Z_2

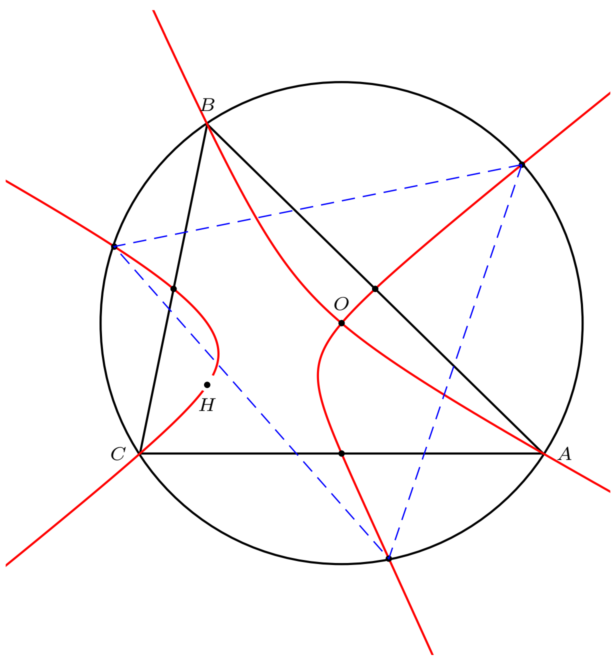


Figure 7. The Lemoine cubic with the circumnormal triangle

3. The generalized Lemoine cubic

Let P be a point distinct from H, not lying on any of the sidelines of triangle ABC. Consider its pedal triangle $P_aP_bP_c$. For every point M in the plane, let $M_a = PP_a \cap AM$. Define M_b and M_c similarly. The locus of M such that the three points M_a, M_b, M_c are collinear on a line $\mathcal{L}_M$ is a cubic $\mathcal{K}(P)$ called the generalized Lemoine cubic associated with P. This cubic passes through A, B, C, H, P_a, P_b, P_c, and P which is a node. Moreover, the line $\mathcal{L}_M$ envelopes the inscribed parabola with directrix the line HP and focus F the antipode (on the circumcircle) of the isogonal conjugate of the infinite point of the line HP.[8] The perspector S is the second intersection of the Steiner circum-ellipse with the line through F and the Steiner point $X_{99} = \left[\frac{1}{b^2-c^2}\right]$.

With $P = (p : q : r)$, the equation of $\mathcal{K}(P)$ is

$$\sum_{\text{cyclic}} x\left(r(c^2p + S_Br)y^2 - q(b^2p + S_Cq)z^2\right) + \left(\sum_{\text{cyclic}} a^2p(q-r)\right) xyz = 0.$$

The two constructions in §1 can easily be adapted to this more general situation.

Construction 3. For any point Q on the trilinear polar of S, the trilinear polar q of Q meets the lines PP_a, PP_b, PP_c at Q_a, Q_b, Q_c respectively. The lines AQ_a, BQ_b, CQ_c concur at M on the cubic $\mathcal{K}(P)$.

Construction 4. For any point Q on the line HP , the perpendicular bisector of FQ intersects the lines PP_a, PP_b, PP_c at Q_a, Q_b, Q_c respectively. The lines AQ_a, BQ_b, CQ_c concur at M on the cubic $\mathcal{K}(P)$.

Remark. The tangent at M to $\mathcal{K}(P)$ can be constructed as follows: the perpendicular at Q to the line HP intersects the perpendicular bisector of FQ at N, which is the point of tangency of the line through Q_a, Q_b, Q_c with the parabola. The tangent at M to $\mathcal{K}(P)$ is the tangent at M to the circum-conic through M and N. Given a point M on the cubic, first construct $M_a = AM \cap PP_a$ and $M_b = BM \cap PP_b$, then Q the reflection of F in the line M_aM_b, and finally apply the construction above.

Jean-Pierre Ehrmann has noticed that $\mathcal{K}(P)$ can be seen as the locus of point M such that the circum-conic passing through M and the infinite point of the line PM is a rectangular hyperbola. This property gives another very simple construction of $\mathcal{K}(P)$ or the construction of the "second" intersection of $\mathcal{K}(P)$ and any line through P.

Construction 5. A line ℓ_P through P intersects BC at P_1. The parallel to ℓ_P at A intersects HC at P_2. AB and P_1P_2 intersect at P_3. Finally, HP_3 intersects ℓ_P at M on the cubic $\mathcal{K}(P)$.

Most of the properties of the Lemoine cubic $\mathcal{K}(O)$ also hold for $\mathcal{K}(P)$ in general.

---

[8]Construction of F : draw the perpendicular at A to the line HP and reflect it about a bisector passing through A. This line meets the circumcircle at A and F.

Proposition 2. *Let $\mathcal{K}(P)$ be the generalized Lemoine cubic.*

(1) *The two tangents at P are parallel to the asymptotes of the rectangular circum-hyperbola passing through P.*
(2) *The tangent at H to $\mathcal{K}(P)$ is the tangent at H to the rectangular circum-hyperbola which is the isogonal image of the line OF. The asymptotes of this hyperbola are perpendicular and parallel to the line HP.*
(3) *The tangents at A, B, C concur if and only if P lies on the Darboux cubic.*[9]
(4) *The asymptotes are parallel to those of the pivotal isogonal cubic with pivot the anticomplement of P.*
(5) *The "third" intersections H_A, H_B, H_C of $\mathcal{K}(P)$ with the altitudes are on the circle with diameter HP. The triangles $P_aP_bP_c$ and $H_AH_BH_C$ are perspective at a point on $\mathcal{K}(P)$.*[10]
(6) *The "third" intersections A'', B'', C'' of $\mathcal{K}(P)$ and the sidelines of $P_aP_bP_c$ form a triangle perspective with $H_AH_BH_C$ at a point on the cubic.*

Remarks. (1) The tangent of $\mathcal{K}(P)$ at H passes through the center of the rectangular hyperbola through P if and only if P lies on the isogonal non-pivotal cubic $\mathcal{K}_H$

$$\sum_{\text{cyclic}} x\,(c^2y^2 + b^2z^2) - \Phi\, x\, y\, z = 0$$

where

$$\Phi = \frac{\sum_{\text{cyclic}} \left(2b^2c^2(a^4 + b^2c^2) - a^6(2b^2 + 2c^2 - a^2)\right)}{4S_AS_BS_C}.$$

We shall study this cubic in §6.3 below.

(2) The polar conic of P can be seen as a degenerate rectangular hyperbola. If $P \neq X_5$, the polar conic of a point is a rectangular hyperbola if and only if it lies on the line PX_5. From this, there is only one point (apart from P) on the curve whose polar conic is a rectangular hyperbola. Very obviously, the polar conic of H is a rectangular hyperbola if and only if P lies on the Euler line. If $P = X_5$, all the points in the plane have a polar conic which is a rectangular hyperbola. This very special situation is detailed in §4.2.

4. Special Lemoine cubics

4.1. *$\mathcal{K}(P)$ with concuring asymptotes.* The three asymptotes of $\mathcal{K}(P)$ are concurrent if and only if P lies on the cubic $\mathcal{K}_{conc}$

$$\sum_{\text{cyclic}} \left(S_B\left(c^2(a^2+b^2) - (a^2-b^2)^2\right) y - S_C\left(b^2(a^2+c^2) - (a^2-c^2)^2\right) z\right) x^2$$
$$- 2(a^2-b^2)(b^2-c^2)(c^2-a^2)xyz = 0.$$

[9] The Darboux cubic is the isogonal cubic with pivot the de Longchamps point X_{20}.
[10] The coordinates of this point are $(p^2(-S_Ap + S_Bq + S_Cr) + a^2pqr : \cdots : \cdots)$.

The three asymptotes of $\mathcal{K}(P)$ are all real if and only if P lies inside the Steiner deltoid $\mathcal{H}_3$. [11] For example, the point $X_{76} = \left[\frac{1}{a^2}\right]$ lies on the cubic $\mathcal{K}_{conc}$ and inside the Steiner deltoid. The cubic $\mathcal{K}(X_{76})$ has three real asymptotes concurring at a point on X_5X_{76}. See Figure 8. On the other hand, the de Longchamps point X_{20} also lies on $\mathcal{K}_{conc}$, but it is not always inside $\mathcal{H}_3$. See Figure 10. The three asymptotes of $\mathcal{K}(X_{20})$, however, intersect at the real point X_{376}, the reflection of G in O.

We shall study the cubic $\mathcal{K}_{conc}$ in more detail in §6.1 below.

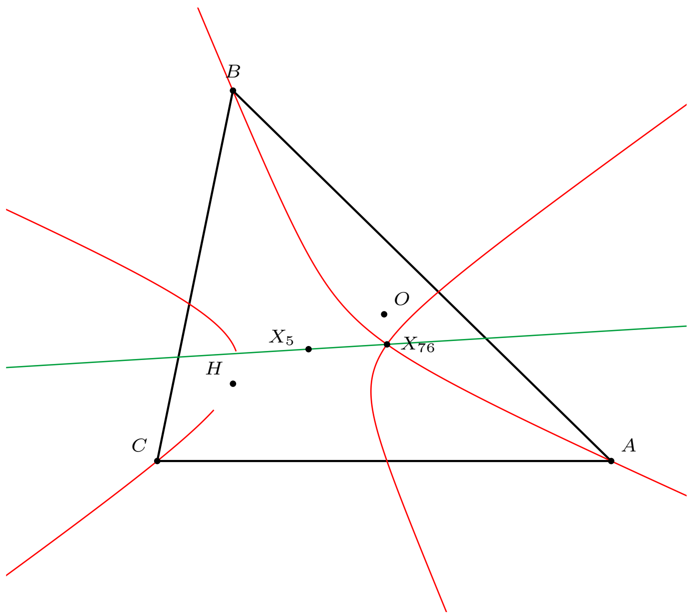


Figure 8. $\mathcal{K}(X_{76})$ with three concurring asymptotes

4.2. $\mathcal{K}(P)$ *with asymptotes making* 60° *angles with one another.* $\mathcal{K}(P)$ has three real asymptotes making 60° angles with one another if and only if P is the nine-point center X_5. See Figure 9. The asymptotes of $\mathcal{K}(X_5)$ are parallel again to those of the McCay cubic and their point of concurrence is [12]

$$Z_3 = \left[a^2((b^2-c^2)^2 - a^2(b^2+c^2))(a^4 - 2a^2(b^2+c^2) + b^4 - 5b^2c^2 + c^4)\right].$$

[11] Cf. Cundy and Parry [1] have shown that for a pivotal isogonal cubic with pivot P, the three asymptotes are all real if and only if P lies inside a certain "critical deltoid" which is the anticomplement of $\mathcal{H}_3$, or equivalently, the envelope of axes of inscribed parabolas.

[12] Z_3 is not in the current edition of [4]. It is the common point of several lines, e.g. X_5X_{51}, $X_{373}X_{549}$ and $X_{511}X_{547}$.

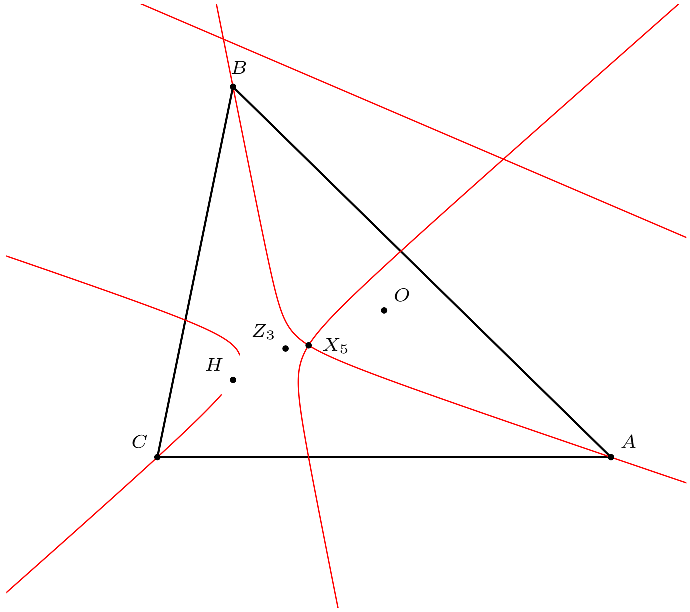


Figure 9. $\mathcal{K}(X_5)$ with three concurring asymptotes making 60° angles

4.3. *Generalized Lemoine isocubics.* $\mathcal{K}(P)$ is an isocubic if and only if the points P_a, P_b, P_c are collinear. It follows that P must lie on the circumcircle. The line through P_a, P_b, P_c is the Simson line of P and its trilinear pole R is the root of the cubic. When P traverses the circumcircle, R traverses the Simson cubic. See [2]. The cubic $\mathcal{K}(P)$ is a conico-pivotal isocubic: for any point M on the curve, its isoconjugate M^* (under the isoconjugation with fixed point P) lies on the curve and the line MM^* envelopes a conic. The points M and M^* are obtained from two points Q and Q' (see Construction 4) on the line HP which are inverse with respect to the circle centered at P going through F, focus of the parabola in §2. (see remark in §5 for more details)

5. The construction of nodal cubics

In §3, we have seen how to construct $\mathcal{K}(P)$ which is a special case of nodal cubic. More generally, we give a very simple construction valid for any nodal circum-cubic with a node at P, intersecting the sidelines again at any three points P_a, P_b, P_c. Let R_a be the trilinear pole of the line passing through the points $AB \cap PP_b$ and $AC \cap PP_c$. Similarly define R_b and R_c. These three points are collinear on a line $\mathcal{L}$ which is the trilinear polar of a point S. For any point Q on the line $\mathcal{L}$, the trilinear polar q of Q meets PP_a, PP_b, PP_c at Q_a, Q_b, Q_c respectively. The lines AQ_a, BQ_b, CQ_c concur at M on the sought cubic and, as usual, q envelopes the inscribed conic γ with perspector S.

Remarks. (1) The tangents at P to the cubic are those drawn from P to γ. These tangents are

(i) real and distinct when P is outside γ and is a "proper" node,
(ii) imaginary when P is inside γ and is an isolated point, or
(iii) identical when P lies on γ and is a cusp, the cuspidal tangent being the tangent at P to γ.

It can be seen that this situation occurs if and only if P lies on the cubic tangent at P_a, P_b, P_c to the sidelines of ABC and passing through the points $BC \cap P_bP_c$, $CA \cap P_cP_a$, $AB \cap P_aP_b$. In other words and generally speaking, there is no cuspidal circum-cubic with a cusp at P passing through P_a, P_b, P_c.

(2) When P_a, P_b, P_c are collinear on a line ℓ, the cubic becomes a conico-pivotal isocubic invariant under isoconjugation with fixed point P: for any point M on the curve, its isoconjugate M^* lies on the curve and the line MM^* envelopes the conic Γ inscribed in the anticevian triangle of P and in the triangle formed by the lines AP_a, BP_b, CP_c. The tangents at P to the cubic are tangent to both conics γ and Γ.

6. Some cubics related to $\mathcal{K}(P)$

6.1. *The cubic $\mathcal{K}_{conc}$.* The circumcubic $\mathcal{K}_{conc}$ considered in §4.1 above contains a large number of interesting points: the orthocenter H, the nine-point center X_5, the de Longchamps point X_{20}, X_{76}, the point

$$Z_4 = [a^2 S_A^2 (a^2(b^2+c^2) - (b^2-c^2)^2)]$$

which is the anticomplement of X_{389}, the center of the Taylor circle. [13] The cubic $\mathcal{K}_{conc}$ also contains the traces of X_{69} on the sidelines of ABC, the three cusps of the Steiner deltoid, and its contacts with the altitudes of triangle ABC.[14] Z is also the common point of the three lines each joining the trace of X_{69} on a sideline of ABC and the contact of the Steiner deltoid with the corresponding altitude. See Figure 10.

Proposition 3. *The cubic $\mathcal{K}_{conc}$ has the following properties.*

(1) *The tangents at A, B, C concur at X_{53}, the Lemoine point of the orthic triangle.*
(2) *The tangent at H is the line HK.*
(3) *The tangent at X_5 is the Euler line of the orthic triangle, the tangential being the point Z_4.* [15]
(4) *The asymptotes of $\mathcal{K}_{conc}$ are parallel to those of the McCay cubic and concur at a point* [16]

$$Z_5 = \left[a^2(a^2(b^2+c^2) - (b^2-c^2)^2)(2S_A^2 + b^2c^2)\right].$$

[13] The point Z_4 is therefore the center of the Taylor circle of the antimedial triangle. It lies on the line X_4X_{69}.

[14] The contact with the altitude AH is the reflection of its trace on BC about the midpoint of AH.

[15] This line also contains X_{51}, X_{52} and other points.

[16] Z_5 is not in the current edition of [4]. It is the common point of quite a number of lines, e.g. X_3X_{64}, X_5X_{51}, $X_{113}X_{127}$, $X_{128}X_{130}$, and $X_{140}X_{185}$. The three asymptotes of the McCay cubic are concurrent at the centroid G.

(5) *$\mathcal{K}_{conc}$ intersects the circumcircle at A, B, C and three other points which are the antipodes of the points whose Simson lines pass through X_{389}.*

We illustrate (1), (2), (3) in Figure 11, (4) in Figure 12, and (5) in Figure 13.

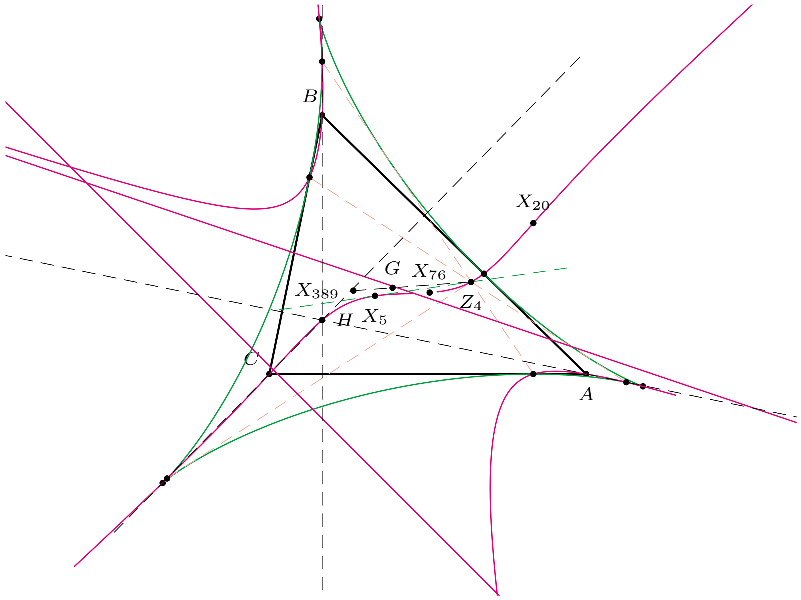


Figure 10. $\mathcal{K}_{conc}$ with the Steiner deltoid

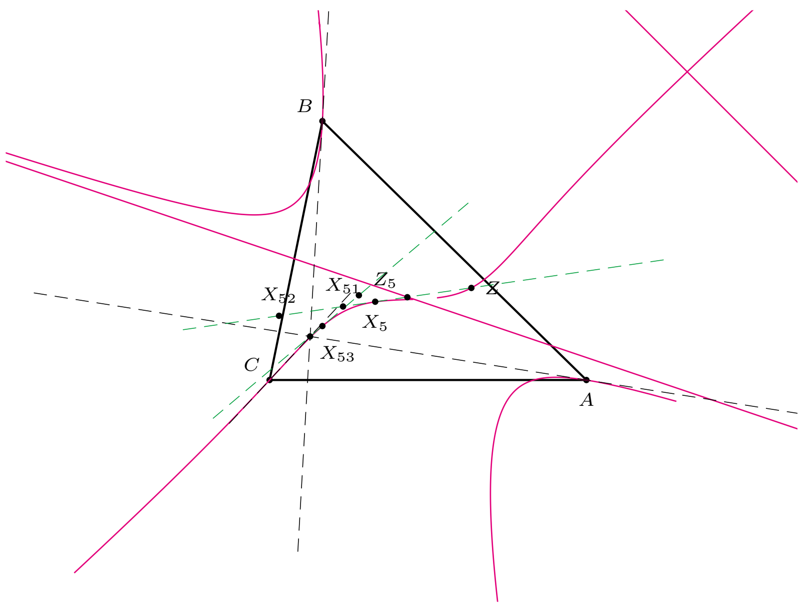


Figure 11. Tangents of $\mathcal{K}_{conc}$

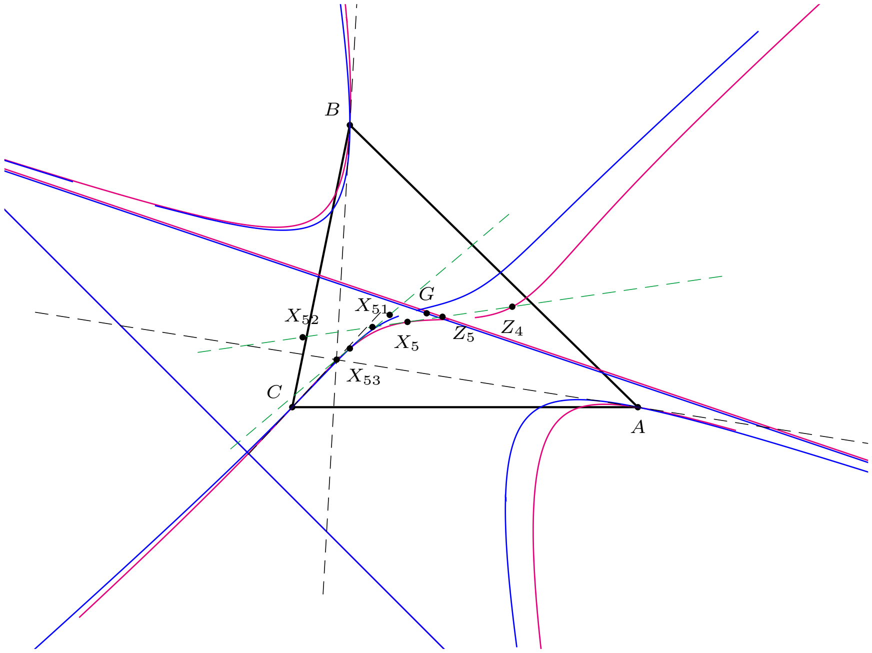


Figure 12. $\mathcal{K}_{conc}$ with the McCay cubic

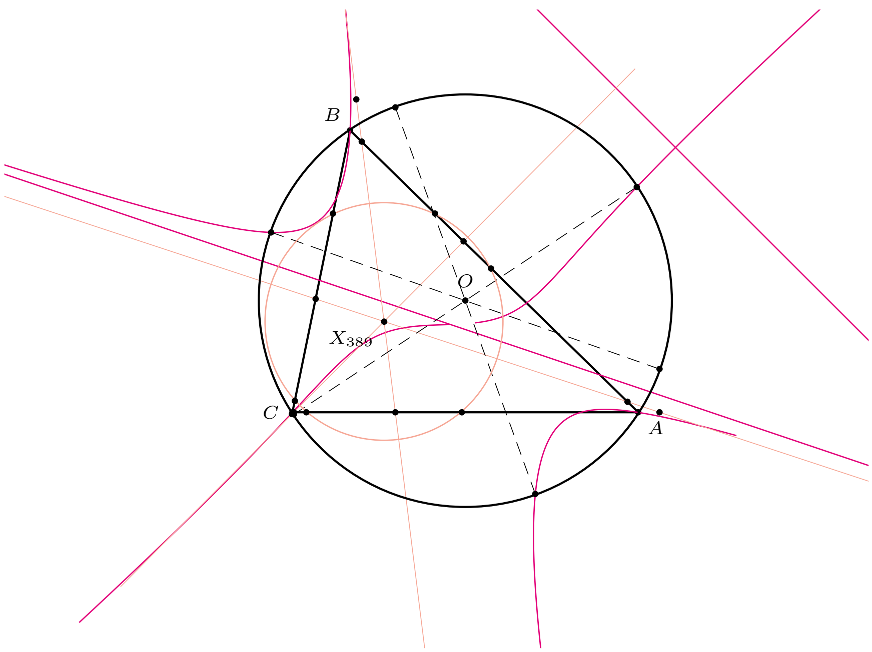


Figure 13. $\mathcal{K}_{conc}$ with the circumcircle and the Taylor circle

6.2. *The isogonal image of* $\mathcal{K}(O)$. Under isogonal conjugation, $\mathcal{K}(O)$ transforms into another nodal circum-cubic

$$\sum_{\text{cyclic}} b^2c^2\,x\,(S_By^2 - S_Cz^2) + (a^2-b^2)(b^2-c^2)(c^2-a^2)xyz = 0.$$

The node is the orthocenter H. The cubic also passes through O, X_8 (Nagel point) and its extraversions, X_{76}, $X_{847} = Z_1^*$, and the traces of $X_{264} = \left[\frac{1}{a^2 S_A}\right]$. The tangents at H are parallel to the asymptotes of the Stammler rectangular hyperbola[17]. The three asymptotes are concurrent at the midpoint of GH,[18] and are parallel to those of the McCay cubic.

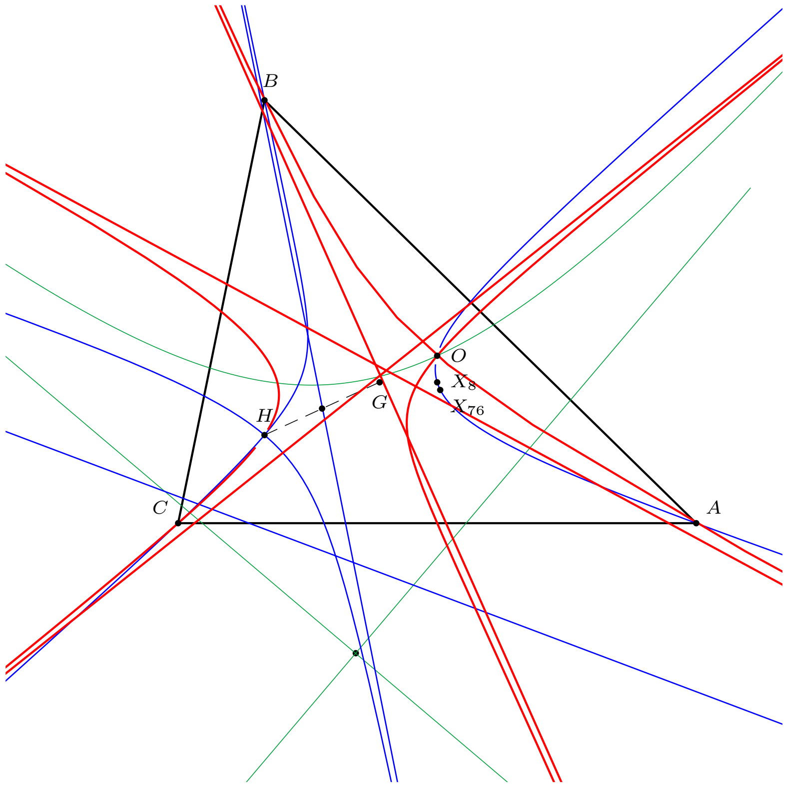


Figure 14. The Lemoine cubic and its isogonal

This cubic was already known by J. R. Musselman [6] although its description is totally different. We find it again in [9] in a different context. Let P be a point on the plane of triangle ABC, and P_1, P_2, P_3 the orthogonal projections of P on the perpendicular bisectors of BC, CA, AB respectively. The locus of P such that the triangle $P_1P_2P_3$ is in perspective with ABC is the Stammler hyperbola and the locus of the perspector is the cubic which is the isogonal transform of $\mathcal{K}(O)$. See Figure 15.

[17]The Stammler hyperbola is the rectangular hyperbola through the circumcenter, incenter, and the three excenters. Its asymptotes are parallel to the lines through X_{110} and the two intersections of the Euler line and the circumcircle

[18]This is $X_{381} = [a^2(a^2 + b^2 + c^2) - 2(b^2 - c^2)^2]$.

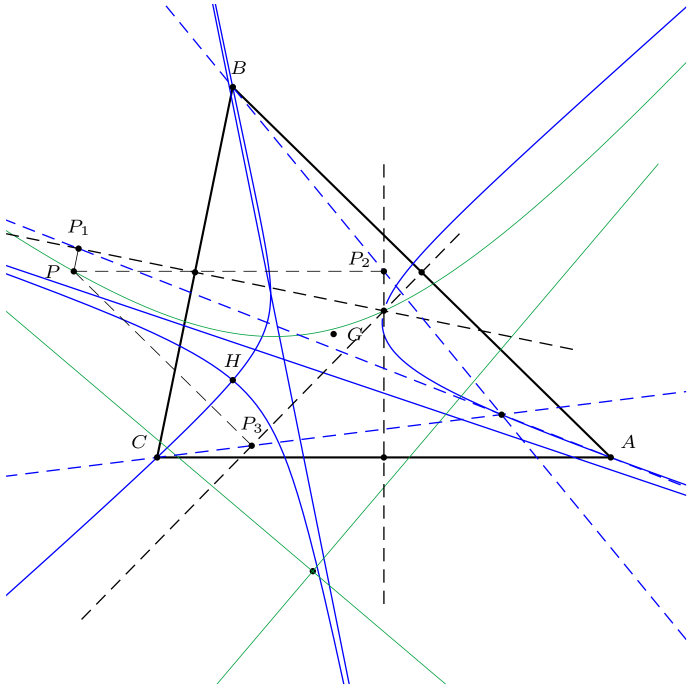


Figure 15. The isogonal of $\mathcal{K}(O)$ with the Stammler hyperbola

6.3. *The cubic* $\mathcal{K}_H$. Recall from Remark (1) following Proposition 2 that the tangent at H to $\mathcal{K}(P)$ passes through the center of the rectangular circum-hyperbola passing through P if and only if P lies on the cubic $\mathcal{K}_H$. This is a non-pivotal isogonal circum-cubic with root at G. See Figure 14.

Proposition 4. *The cubic* $\mathcal{K}_H$ *has the following geometric properties.*

(1) $\mathcal{K}_H$ *passes through* A, B, C, O, H, *the three points* H_A, H_B, H_C *and their isogonal conjugates* H_A^*, H_B^*, H_C^*. [19]
(2) *The three real asymptotes are parallel to the sidelines of* ABC.
(3) *The tangents of* $\mathcal{K}_H$ *at* A, B, C *are the sidelines of the tangential triangle. Hence,* $\mathcal{K}_H$ *is tritangent to the circumcircle at the vertices* A, B, C.
(4) *The tangent at* A *(respectively* B, C*) and the asymptote parallel to* BC *(respectively* CA, AB*) intersect at a point* $\widetilde{A}$ *(respectively* $\widetilde{B}$, $\widetilde{C}$*) on* $\mathcal{K}_H$.
(5) *The three points* $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ *are collinear on the perpendicular* $\mathcal{L}$ *to the line* OK *at the inverse of* X_{389} *in the circumcircle.*[20]

[19] The points H_A, H_B, H_C are on the circle, diameter OH. See Proposition 1(5). Their isogonal conjugates are on the lines OA. OB, OC respectively.

[20] In other words, the line $\mathcal{L}$ is the inversive image of the circle with diameter OX_{389}. Hence, $\widetilde{A}$ is the common point of $\mathcal{L}$ and the tangent at A to the circumcircle, and the parallel through $\widetilde{A}$ to BC is an asymptote of $\mathcal{K}_H$.

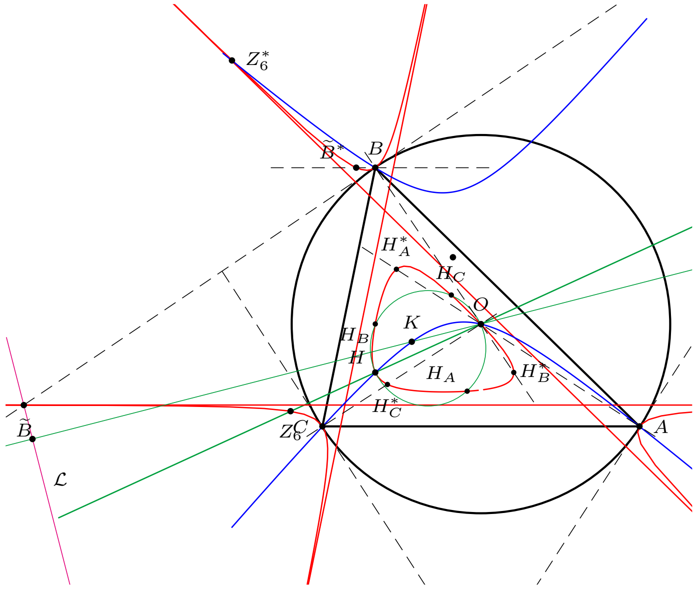


Figure 16. The cubic $\mathcal{K}_H$ with the Jerabek hyperbola

(6) *The isogonal conjugate of* $\widetilde{A}$ *is the "third" intersection of* $\mathcal{K}_H$ *with the parallel to* BC *through* A*; similarly for the isogonal conjugates of* $\widetilde{B}$ *and* $\widetilde{C}$.

(7) *The third intersection with the Euler line, apart from* O *and* H*, is the point* [21]

$$Z_6 = \left[\frac{(b^2-c^2)^2 + a^2(b^2+c^2-2a^2)}{(b^2-c^2)\,S_A}\right].$$

(8) *The isogonal conjugate of* Z_6 *is the sixth intersection of* $\mathcal{K}_H$ *with the Jerabek hyperbola.*

We conclude with another interesting property of the cubic $\mathcal{K}_H$. Recall that the polar circle of triangle ABC is the unique circle with respect to which triangle ABC is self-polar. This is in the coaxal system generated by the circumcircle and the nine-point circle. It has center H, radius ρ given by

$$\rho^2 = 4R^2 - \frac{1}{2}(a^2+b^2+c^2),$$

and is real only when triangle ABC is obtuse angled. Let $\mathcal{C}$ be the concentric circle with radius $\frac{\rho}{\sqrt{2}}$.

Proposition 5. *$\mathcal{K}_H$ is the locus of point M whose pedal circle is orthogonal to circle $\mathcal{C}$.*

---

[21] This is not in [4]. It is the homothetic of X_{402} (Gossard perspector) in the homothety with center G, ratio 4 or, equivalently, the anticomplement of the anticomplement of X_{402}.

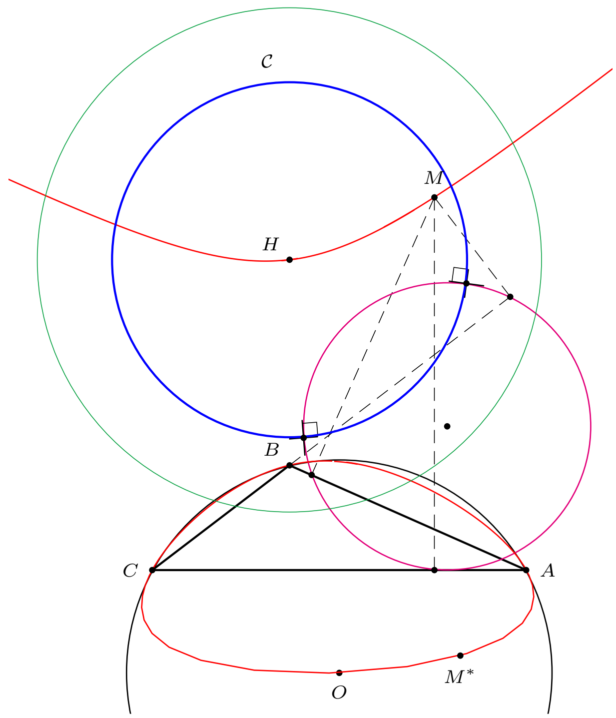


Figure 17. The cubic $\mathcal{K}_H$ for an obtuse angled triangle

In fact, more generally, every non-pivotal isogonal cubic can be seen, in a unique way, as the locus of point M such that the pedal circle of M is orthogonal to a fixed circle, real or imaginary, proper or degenerate.

References

[1] H. M. Cundy and C. F. Parry, Some cubic curves associated with a triangle, *Journal of geometry*, 53 (1995) 41–66.

[2] J.P. Ehrmann and B. Gibert, The Simson cubic, *Forum Geom.*, 1 (2001) 107 – 114.

[3] C. Kimberling, Triangle Centers and Central Triangles,*Congressus Numerantium*, 129 (1998) 1–295.

[4] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://www2.evansville.edu/ck6/encyclopedia/.

[5] E. Lemoine, *A. F.* (Association Française pour l'Avancement des Sciences) (1891) 149, and (1892) 124.

[6] J. R. Musselman, Some loci connected with a triangle, *Amer. Math. Monthly*, 47 (1940) pp. 354–361.

[7] J. Neuberg et A. Mineur, Sur la cubique de Lemoine, *Mathesis* 39 (1925) 64–65 .

[8] J. Neuberg, Sur les cubiques de Darboux, de Lemoine et de Thomson, *Annales Société Sc. Bruxelles* 44 (1925) 1–10.

[9] P. Yiu, Hyacinthos, message 1299, August 28, 2000.

Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France
E-mail address: b.gibert@free.fr

Forum Geometricorum
Volume 2 (2002) 65–66.

FORUM GEOM
ISSN 1534-1178

A Simple Construction of the Golden Section

Kurt Hofstetter

Abstract. We construct the golden section by drawing 5 circular arcs.

We denote by $P(Q)$ the circle with P as center and PQ as radius. Figure 1 shows two circles $A(B)$ and $B(A)$ intersecting at C and D. The line AB intersects the circles again at E and F. The circles $A(F)$ and $B(E)$ intersect at two points X and Y. It is clear that C, D, X, Y are on a line. It is much more interesting to note that D divides the segment CX in the golden ratio, *i.e.*,

$$\frac{CD}{CX} = \frac{\sqrt{5}-1}{2}.$$

This is easy to verify. If we assume AB of length 2, then $CD = 2\sqrt{3}$ and $CX = \sqrt{15}+\sqrt{3}$. From these,

$$\frac{CD}{CX} = \frac{2\sqrt{3}}{\sqrt{15}+\sqrt{3}} = \frac{2}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{2}.$$

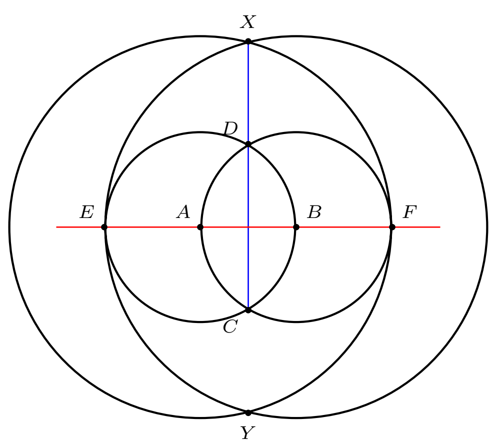


Figure 1

This shows that to construct three collinear points in golden section, we need four circles and one line. It is possible, however, to replace the line AB by a circle, say $C(D)$. See Figure 2. Thus, *the golden section can be constructed with compass only, in 5 steps*.

Publication Date: May 31, 2002. Communicating Editor: Paul Yiu.

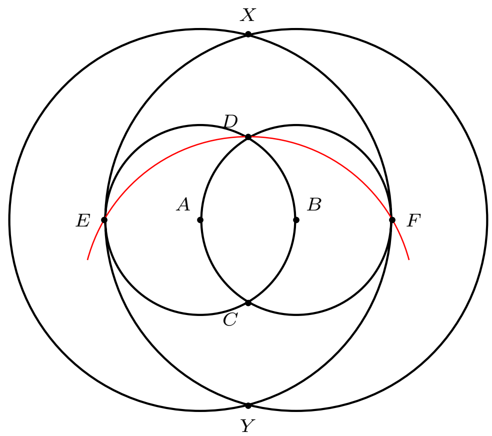


Figure 2

It is interesting to compare this with Figure 3 which also displays the golden section. See [1, p.105, note on 3.5(b)] and [2].[1] Here, ABC is an equilateral triangle. The line joining the midpoints D, E of two sides intersects the circumcircle at F. Then E divides DF in the golden section, *i.e.*,

$$\frac{DE}{DF} = \frac{\sqrt{5}-1}{2}.$$

However, it is unlikely that this diagram can be constructed in fewer than 5 steps, using ruler and compass, or compass alone.

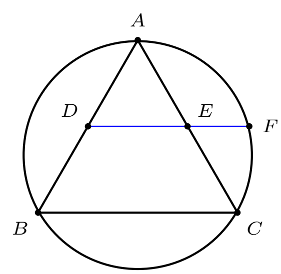


Figure 3

References

[1] D. H. Fowler, *The Mathematics of Plato's Academy*, Oxford University Press, 1988.

[2] G. Odom and J. van de Craats, Elementary Problem 3007, *American Math. Monthly*, 90 (1983) 482; solution, 93 (1986) 572.

Kurt Hofstetter: Object Hofstetter, Media Art Studio, Langegasse 42/8c, A-1080 Vienna, Austria

E-mail address: `pendel@sunpendulum.at`

[1] I am indebted to a referee for these references.

Forum Geometricorum
Volume 2 (2002) 67–70.

FORUM GEOM
ISSN 1534-1178

A Rapid Construction of Some Triangle Centers

Lawrence S. Evans

Abstract. We give a compass and ruler construction of fifteen centers associated with a triangle by drawing 6 circles and 23 lines.

Given triangle T with vertices A, B, and C, draw a red circle centered at A passing through B, another centered at B going through C, and a third centered at C going through A. Now, draw a blue circle centered at A passing through C, one centered at C going through B, and one centered at B going through A. There will be 12 intersections of red circles with blue ones. Three of them are A, B, and C. Three are apices of equilateral triangles erected on the sides of T and pointing outward. Denote such an apex by A_+, B_+, C_+. Three are the apices of equilateral triangles erected on the sides pointing inward. Denote them by A_-, B_-, C_-. The last three are the reflections of the vertices of T in the opposite sides, which we shall call A^*, B^*, C^*.

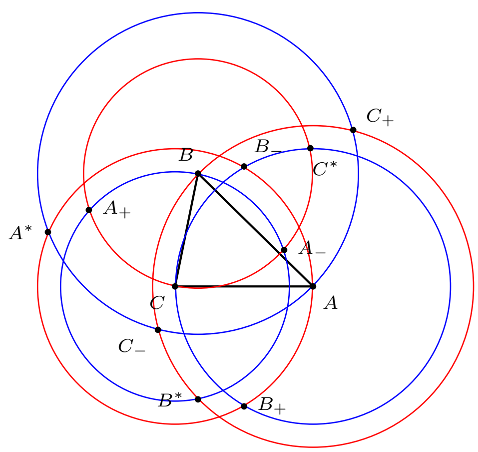


Figure 1. Construction of $A_\pm$, $B_\pm$, $C_\pm$, A^*, B^*, C^*

The four triangles $T = ABC$, $T_+ = A_+B_+C_+$, $T_- = A_-B_-C_-$, and $T^* = A^*B^*C^*$ are pairwise in perspective. The 6 centers of perspectivity are

(1) $[T, T_+] = F_+$, the inner Fermat point,
(2) $[T, T_-] = F_-$, the outer Fermat point,
(3) $[T, T^*] = H$, the orthocenter,

Publication Date: June 18, 2002. Communicating Editor: Paul Yiu.
Expanded from a presentation at the MAA Meeting, Urbana, Illinois, March 22, 2001.

(4) $[T_+, T_-] = O$, the circumcenter,
(5) $[T_+, T^*] = J_-$, the outer isodynamic point,
(6) $[T_-, T^*] = J_+$, the inner isodynamic point.

Only two lines, AA_+ and BB_+, are needed to determine F_+ by intersection. Likewise, 10 more are necessary to determine the other 5 centers F_-, H, O, J_- and J_+. We have drawn twelve lines so far.[1] See Figure 2, where the green lines only serve to indicate perspectivity; they are not necessary for the constructions of the triangle centers.

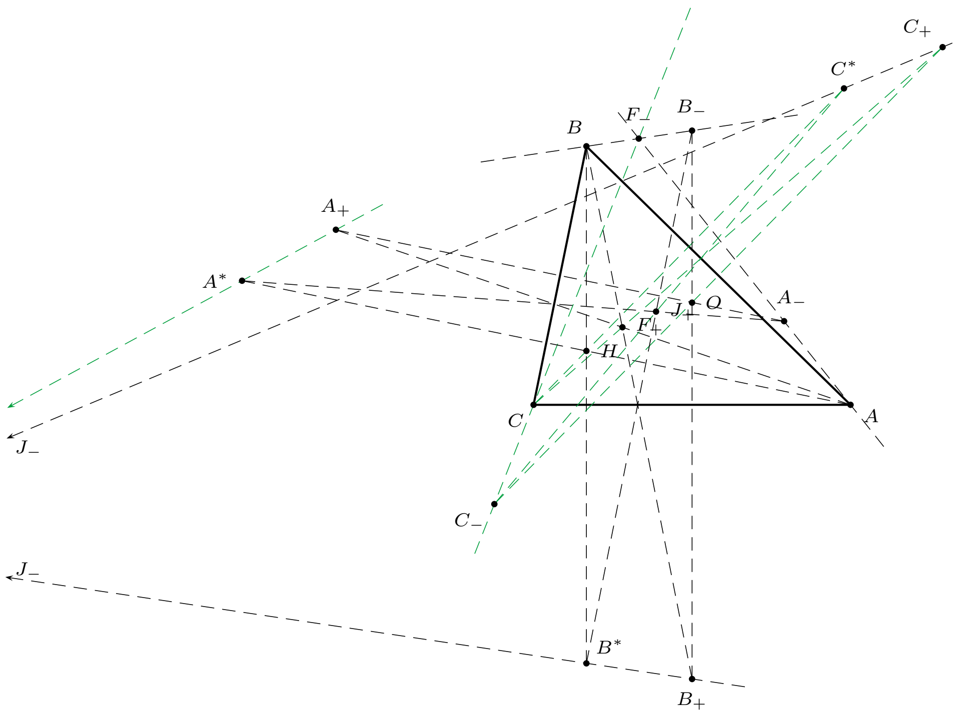


Figure 2. Construction of $F_\pm$, H, O, $J_\pm$

Define three more lines: the Euler line OH, the Fermat line F_+F_-, and the Apollonius line J_+J_-. The Apollonius line J_+J_- is also known as the Brocard axis. It contains the circumcenter O and the (Lemoine) symmedian point K. Then,

(7) $K = J_+J_- \cap F_+F_-$;
(8) $D = OH \cap F_+F_-$ is the center of orthocentroidal circle, the midpoint of between the centroid and the orthocenter.

We construct six more lines to locate four more centers:

(9) the outer Napoleon point is $N_+ = HJ_+ \cap OF_+$,

[1]The 18 points A, $A_\pm$, A^*, B, $B_\pm$, B^*, C, $C_\pm$, C^*, H, O, $F\pm$, $J_\pm$ all lie on a third degree curve called the Neuberg cubic.

(10) the inner Napoleon point is $N_- = HJ_- \cap OF_-$;
(11) the centroid $G = OH \cap J_+F_-$ (or $OH \cap J_-F_+$);
(12) the nine-point center $N_p = OH \cap N_-F_+$ (or $OH \cap N_+F-$).

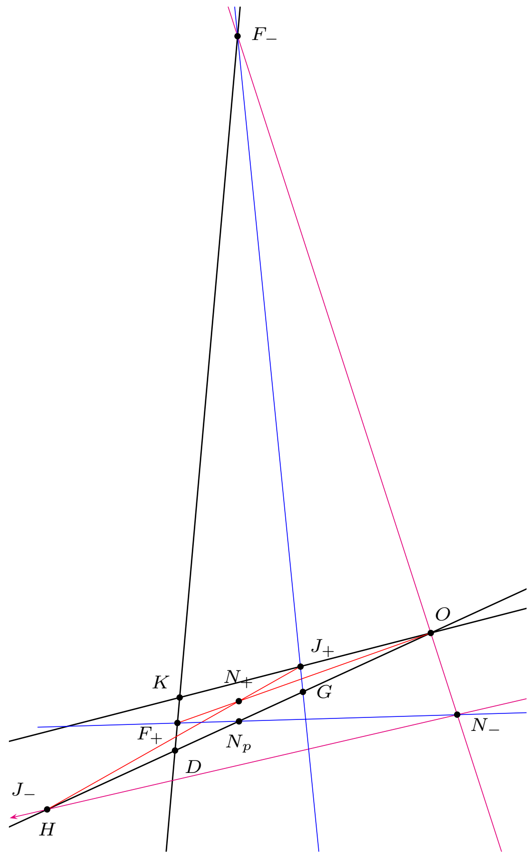


Figure 3. Construction of K, D, $N_\pm$, G, N_p

The line N_-F_+ (used in (12) above to locate N_p) intersects $OK = J_+J_-$ at the isogonal conjugate of N_-. Likewise, the lines N_+F_- and OK intersect at the isogonal conjugate of N_+. We also note that the line J_+N_- intersects the Euler line OH at the nine-point center N_p' of the medial triangle. Thus,

(13) $N_+^* = N_+F_- \cap OK$,
(14) $N_-^* = N_-F_+ \cap OK$, and
(15) $N_p' = J_+N_- \cap OH$ (or $J_-N_+ \cap OH$).

See Figure 4, in which we note that the points G, N_+ and N_-^* are collinear, so are G, N_- and N_+^*.

We have therefore constructed 15 centers with 6 circles and 23 lines: 12 to determine O, H, $F_\pm$, $J_\pm$ as the 6 centers of perspectivity of T, $T_\pm$ and T^*; then 9 to determine K, D, $N_\pm$, G, N_p, N_-^*, and finally 2 more to give N_+^* and N_p'.

Remark. The triangle centers in this note appear in [1, 2] as X_n for n given below.

center	O	H	F_+	F_-	J_+	J_-	K	D	N_+	N_-	G	N_p	N_+^*	N_-^*	N_p'
n	3	4	13	14	15	16	6	381	17	18	2	5	61	62	140

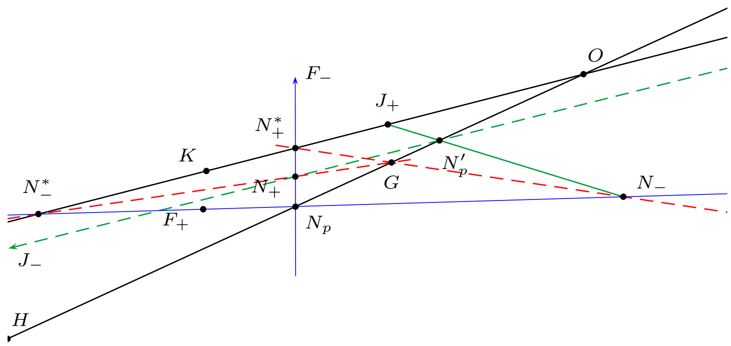


Figure 4. Construction of N_+^*, N_-^*, and N_p'

This construction uses Kimberling's list [1] of collinearities among centers. It can be implemented on a dynamic software like the Geometer's Sketchpad. After hiding the circles and lines, one is left with T and the centers, which can be observed to move in concert as one drags a vertex of T on the computer screen. Some important centers we do not get here are the incenter, the Gergonne and the Nagel points.

References

[1] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1 – 295.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://www2.evansville.edu/ck6/encyclopedia/.

Lawrence S. Evans: 910 W. 57th Street, La Grange, Illinois 60525, USA
E-mail address: 75342.3052@compuserve.com

Forum Geometricorum
Volume 2 (2002) 71–87.

FORUM GEOM
ISSN 1534-1178

A Generalization of the Tucker Circles

Peter Yff

Abstract. Let hexagon $PQRSTU$ be inscribed in triangle $A_1A_2A_3$ (ordered counterclockwise) such that P and S are on line A_3A_1, Q and T are on line A_1A_2, and R and U are on line A_2A_3. If PQ, RS, and TU are respectively parallel to A_2A_3, A_1A_2, and A_3A_1, while QR, ST, and UP are antiparallel to A_3A_1, A_2A_3, and A_1A_2 respectively, the vertices of the hexagon are on one circle. Now, let hexagon $P'Q'R'S'T'U'$ be described as above, with each of its sides parallel to the corresponding side of $PQRSTU$. Again the six vertices are concyclic, and the process may be repeated indefinitely to form an infinite family of circles (Tucker [3]). This family is a coaxaloid system, and its locus of centers is the Brocard axis of the triangle, passing through the circumcenter and the symmedian point. J. A. Third ([2]) extended this idea by relaxing the conditions for the directions of the sides of the hexagon, thus finding infinitely many coaxaloid systems of circles. The present paper defines a further extension by allowing the directions of the sides to be as arbitrary as possible, resulting in families of homothetic conics with properties analogous to those of the Tucker circles.

1. Circles of Tucker and Third

The system of Tucker circles is a special case of the systems of Third circles. In a Third system the directions of PQ, QR, and RS may be taken arbitrarily, while ST is made antiparallel to PQ (with respect to angle $A_2A_1A_3$). Similarly, TU and UP are made antiparallel to QR and RS respectively. The hexagon may then be inscribed in a circle, and a different starting point P' with the same directions produces another circle. It should be noted that the six vertices need not be confined to the sides of the triangle; each point may lie anywhere on its respective sideline. Thus an infinite family of circles may be obtained, and Third shows that this is a coaxaloid system. That is, it may be derived from a coaxal system of circles by multiplying every radius by a constant. (See Figures 1a and 1b). In particular, the Tucker system is obtained from the coaxal system of circles through the Brocard points Ω and Ω' by multiplying the radius of each circle by $\dfrac{R}{O\Omega}$, R being the circumradius of the triangle and O its circumcenter ([1, p.276]). In general, the line of centers of a Third system is the perpendicular bisector of the segment joining the pair of isogonal conjugate points which are the common points of the corresponding coaxal system. Furthermore, although the coaxal system has no envelope, it

Publication Date: July 5, 2002. Communicating Editor: Clark Kimberling.

will be seen later that the envelope of the coaxaloid system is a conic tangent to the sidelines of the triangle, whose foci are the points common to the coaxal circles.

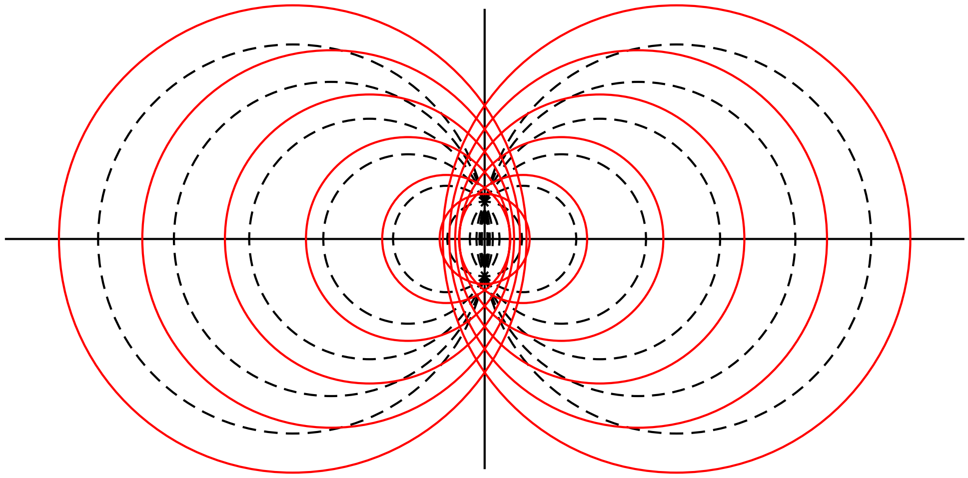


Figure 1a: Coaxaloid system with elliptic envelope, and its corresponding coaxal system

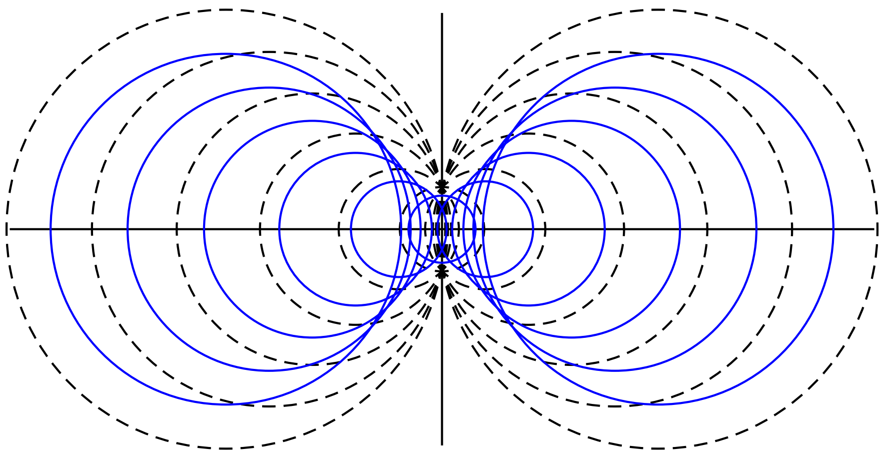


Figure 1b: Coaxaloid system with hyperbolic envelope, and its corresponding coaxal system

2. Two-circuit closed paths in a triangle

2.1. Consider a polygonal path from P on A_3A_1 to Q on A_1A_2 to R on A_2A_3 to S on A_3A_1 to T on A_1A_2 to U on A_2A_3, and back to P. Again the six points may be selected anywhere on their respective sidelines. The vertices of the triangle are numbered counterclockwise, and the lengths of the corresponding sides are denoted by a_1, a_2, a_3. Distances measured along the perimeter of the triangle in the counterclockwise sense are regarded as positive. The length of PA_1 is designated by λ, which is negative in case A_1 is between A_3 and P. Thus, $A_3P = a_2 - \lambda$, and the barycentric coordinates of P are $(a_2 - \lambda : 0 : \lambda)$. Also, six "directions" w_i are defined:

$$w_1 = \frac{PA_1}{A_1Q}, \qquad w_2 = \frac{QA_2}{A_2R}, \qquad w_3 = \frac{RA_3}{A_3S},$$
$$w_4 = \frac{SA_1}{A_1T}, \qquad w_5 = \frac{TA_2}{A_2U}, \qquad w_6 = \frac{UA_3}{A_3P}.$$

Any direction may be positive or negative depending on the signs of the directed segments. Then, $A_1Q = \dfrac{\lambda}{w_1}$, $QA_2 = \dfrac{a_3w_1 - \lambda}{w_1}$, $A_2R = \dfrac{a_3w_1 - \lambda}{w_1w_2}$, and so on.

2.2. A familiar example is that in which PQ and ST are parallel to A_2A_3, QR and TU are parallel to A_3A_1, and RS and UP are parallel to A_1A_2 (Figure 2). Then

$$w_1 = w_4 = \frac{a_2}{a_3}, \qquad w_2 = w_5 = \frac{a_3}{a_1}, \qquad w_3 = w_6 = \frac{a_1}{a_2}.$$

It is easily seen by elementary geometry that this path closes after two circuits around the sidelines of the triangle.

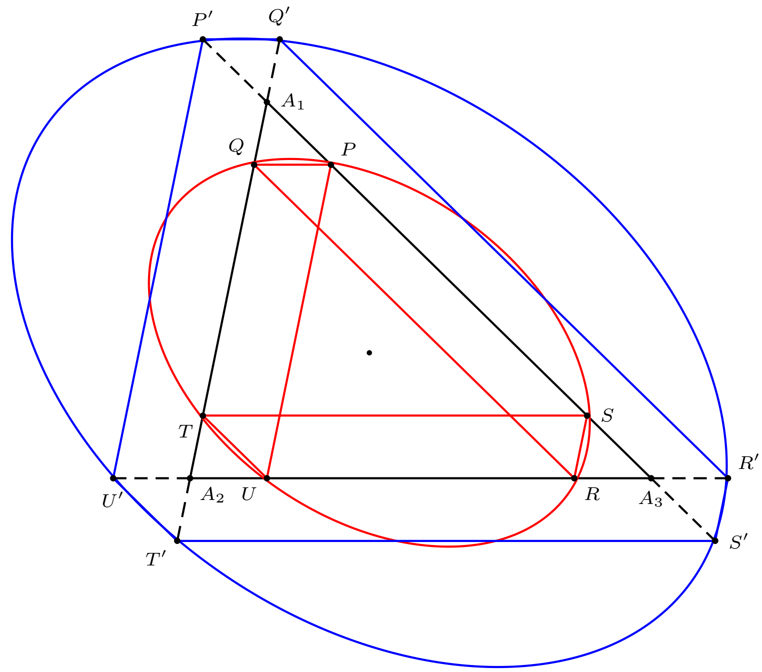


Figure 2. Hexagonal paths formed by parallels

2.3. Closure is less obvious, but still not difficult to prove, when "parallel" in the first example is replaced by "antiparallel" (Figure 3). Here,

$$w_1 = w_4 = \frac{a_3}{a_2}, \qquad w_2 = w_5 = \frac{a_1}{a_3}, \qquad w_3 = w_6 = \frac{a_2}{a_1}.$$

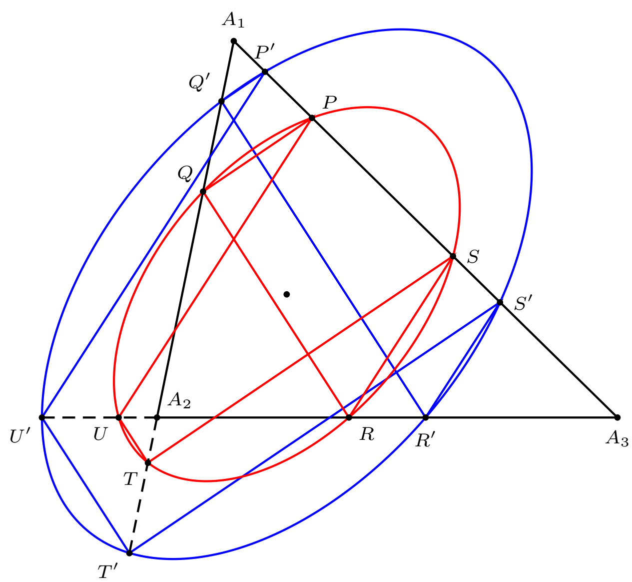


Figure 3. Hexagonal paths formed by antiparallels

2.4. Another positive result is obtained by using isoscelizers ([1, p.93]). That is, $PA_1 = A_1Q$, $QA_2 = A_2R$, $RA_3 = A_3S, \ldots, UA_3 = A_3P$. Therefore,

$$w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = 1.$$

2.5. These examples suggest that, if $w_1 = w_4$, $w_2 = w_5$, $w_3 = w_6$, the condition $w_1w_2w_3 = 1$ is sufficient to close the path after two circuits. Indeed, by computing lengths of segments around the triangle, one obtains

$$A_3P = \frac{UA_3}{w_3} = \frac{a_1w_1^2w_2^2w_3 - a_3w_1^2w_2w_3 + a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - \lambda}{w_1^2w_2^2w_3^2}.$$

But also $A_3P = a_2 - \lambda$, and equating the two expressions yields

$$(1 - w_1w_2w_3)(a_1w_1w_2 - a_2w_1w_2w_3 - a_3w_1 + \lambda(1 + w_1w_2w_3)) = 0. \qquad (1)$$

In order that (1) may be satisfied for all values of λ, the solution is $w_1w_2w_3 = 1$.

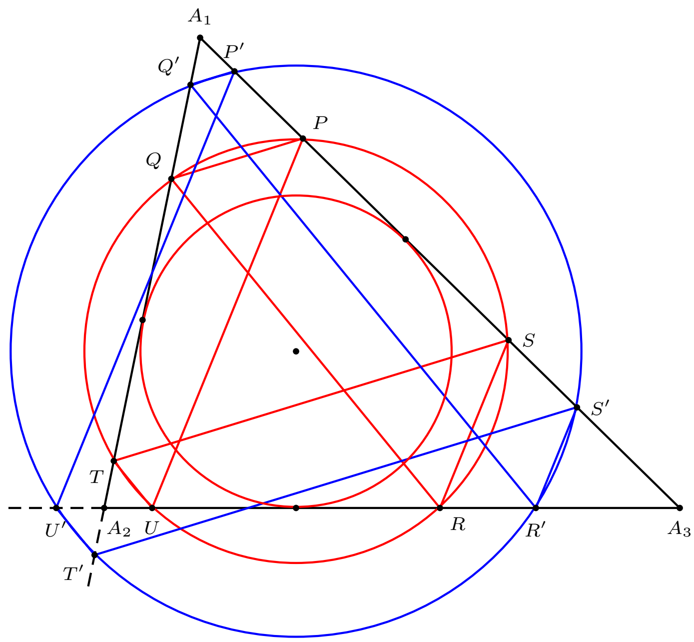


Figure 4. Hexagonal paths formed by isoscelizers

2.6. As a slight digression, the other factor in (1) gives the special solution

$$\lambda = \frac{w_1(a_2w_2w_3 - a_1w_2 + a_3)}{1 + w_1w_2w_3},$$

and calculation shows that this value of λ causes the path to close after only one circuit, that is $S = P$. For example, if antiparallels are used, and if P is the foot of the altitude from A_2, the one-circuit closed path is the orthic triangle of $A_1A_2A_3$.

Furthermore, if also $w_1w_2w_3 = 1$, the special value of λ becomes

$$\frac{a_2 - a_1w_1w_2 + a_3w_1}{2},$$

and the cevians A_1R, A_2P, and A_3Q are concurrent at the point (in barycentric coordinates, as throughout this paper)

$$\left(\frac{1}{-a_1w_1w_2 + a_2 + a_3w_1} : \frac{1}{a_1w_1w_2 - a_2 + a_3w_1} : \frac{1}{a_1w_1w_2 + a_2 - a_3w_1}\right). \tag{2}$$

It follows that there exists a conic tangent to the sidelines of the triangle at P, Q, R. The coordinates of the center of the conic are $(a_1w_1w_2 : a_2 : a_3w_1)$.

2.7. Returning to the conditions $w_1w_2w_3 = 1$, $w_1 = w_4$, $w_2 = w_5$, $w_3 = w_6$, the coordinates of the six points may be found:

$$
\begin{aligned}
P =&(a_2-\lambda:0:\lambda),\\
Q =&(a_3w_1-\lambda:\lambda:0),\\
R =&(0:a_1w_1w_2-a_3w_1+\lambda:a_3w_1-\lambda),\\
S =&(a_1w_1w_2-a_3w_1+\lambda:0:a_2-a_1w_1w_2+a_3w_1-\lambda),\\
T =&(a_1w_1w_2-a_2+\lambda:a_2-a_1w_1w_2+a_3w_1-\lambda:0),\\
U =&(0:a_2-\lambda:a_1w_1w_2-a_2+\lambda).
\end{aligned}
$$

These points are on one conic, given by the equation

$$
\begin{aligned}
&\lambda(a_2-a_1w_1w_2+a_2w_1-\lambda)x_1^2\\
&+(a_3w_1-\lambda)(a_1w_1w_2-a_2+\lambda)x_2^2\\
&+(a_2-\lambda)(a_1w_1w_2-a_3w_1+\lambda)x_3^2\\
&-(a_1^2w_1^2w_2^2+2a_2a_3w_1-a_3a_1w_1^2w_2-a_1a_2w_1w_2\\
&\qquad+2(a_1w_1w_2-a_2-a_3w_1)\lambda+2\lambda^2)x_2x_3\\
&-(a_2^2+a_2a_3w_1-a_1a_2w_1w_2+2(a_1w_1w_2-a_2-a_3w_1)\lambda+2\lambda^2)x_3x_1\\
&-(a_3^2w_1^2+a_2a_3w_1-a_3a_1w_1^2w_2+2(a_1w_1w_2-a_2-a_3w_1)\lambda+2\lambda^2)x_1x_2\\
=&0. \qquad (3)
\end{aligned}
$$

This equation may also be written in the form

$$
\begin{aligned}
&\lambda(a_2-a_1w_1w_2+a_3w_1-\lambda)(x_1+x_2+x_3)^2\\
&+a_3w_1(a_1w_1w_2-a_2)x_2^2+a_2w_1(a_1w_2-a_3)x_3^2\\
&-(a_1^2w_1^2w_2^2+2a_2a_3w_1-a_3a_1w_1^2w_2-a_1a_2w_1w_2)x_2x_3\\
&-a_2(a_2-a_1w_1w_2+a_3w_1)x_3x_1\\
&-a_3w_1(a_2-a_1w_1w_2+a_3w_1)x_1x_2\\
=&0. \qquad (4)
\end{aligned}
$$

As λ varies, (3) or (4) represents an infinite family of conics. However, λ appears only when multiplied by $(x_1+x_2+x_3)^2$, so it has no effect at infinity, where $x_1+x_2+x_3=0$. Hence all conics in the system are concurrent at infinity. If they have two real points there, they are hyperbolas with respectively parallel asymptotes. This is not sufficient to make them all homothetic to each other, but it will be shown later that this is indeed the case. If the two points at infinity coincide, all of the conics are tangent to the line at infinity at that point. Therefore they are parabolas with parallel axes, forming a homethetic set. Finally, if the points at infinity are imaginary, the conics are ellipses and their asymptotes are imaginary. As in the hyperbolic case, any two conics have respectively parallel asymptotes and are homothetic to each other.

2.8. The center of (3) may be calculated by the method of [1, p.234], bearing in mind the fact that the author uses trilinear coordinates instead of barycentric. But it is easily shown that the addition of any multiple of $(x_1+x_2+x_3)^2$ to the equation of a conic has no effect on its center. Therefore (4) shows that the expression containing λ may be ignored, leaving all conics with the same center. Moreover, this center has already been found, because the conic tangent to the sidelines at $P=S$, $Q=T$, $R=U$ is a special member of (3), obtained when λ has the value $\frac{1}{2}(a_2-a_1w_1w_2+a_3w_1)$. Thus, the common center of all the members of (3) is $(a_1w_1w_2 : a_2 : a_3w_1)$; and if they are homothetic, any one of them may be obtained from another by a dilatation about this point. (See Figures 2, 3, 4).

Since the locus of centers is not a line, this system differs from those of Tucker and Third and may be regarded as degenerate in the context of the general thery. One case worthy of mention is that in which the sides of the hexagon are isocelizers, so that

$$w_1=w_2=w_3=w_4=w_5=w_6=1.$$

Exceptionally this is a Third system, because every isocelizer is both parallel and antiparallel to itself. Therefore, the conics are concentric circles, the smallest real one being the incircle of the triangle (Figure 4).

Since the "center" of a parabola is at infinity, (3) consists of parabolas only when $a_1w_1w_2+a_2+a_3w_1=0$. This can happen if some of the directions are negative, which was seen earlier as a possibility.

2.9. Some perspectivities will now be mentioned. If

$$PU\cap QR=B_1,\quad ST\cap PU=B_2,\quad QR\cap ST=B_3,$$

the three lines A_iB_i are concurrent at (2) for every value of λ. Likewise, if

$$RS\cap TU=C_1,\quad PQ\cap RS=C_2,\quad TU\cap PQ=C_3,$$

the lines A_iC_i also concur at (2). Thus for each i the points B_i andC_i move on a fixed line through A_i.

2.10. Before consideration of the general case it may be noted that whenever the directions w_1, w_2, w_3 lead to a conic circumscribing hexagon $PQRSTU$ (that is, $w_1w_2w_3=1$), any permutation of them will do the same. Any permutation of w_1^{-1}, w_2^{-1}, w_3^{-1} will also work. Other such triples may be invented, such as $\frac{w_2}{w_3}$, $\frac{w_3}{w_1}$, $\frac{w_1}{w_2}$.

3. The general case

3.1. Using all six directions w_i, one may derive the following expressions for the lengths of segments:

$$
\begin{aligned}
A_1Q =& w_1^{-1}\lambda,\\
QA_2 =& w_1^{-1}(a_3w_1-\lambda),\\
A_2R =& w_1^{-1}w_2^{-1}(a_3w_1-\lambda),\\
RA_3 =& w_1^{-1}w_2^{-1}(a_1w_1w_2-a_3w_1+\lambda),\\
A_3S =& w_1^{-1}w_2^{-1}w_3^{-1}(a_1w_1w_2-a_3w_1+\lambda),\\
SA_1 =& w_1^{-1}w_2^{-1}w_3^{-1}(a_2w_1w_2w_3-a_1w_1w_2+a_3w_1-\lambda),\\
A_1T =& w_1^{-1}w_2^{-1}w_3^{-1}w_4^{-1}(a_2w_1w_2w_3-a_1w_1w_2+a_3w_1-\lambda),\\
TA_2 =& w_1^{-1}w_2^{-1}w_3^{-1}w_4^{-1}(a_3w_1w_2w_3w_4-a_2w_1w_2w_3+a_1w_1w_2-a_3w_1+\lambda).
\end{aligned}
$$

Then working clockwise from P to U to T,

$$
\begin{aligned}
A_3P =& a_2-\lambda,\\
UA_3 =& w_6(a_2-\lambda),\\
A_2U =& a_1-a_2w_6+w_6\lambda,\\
TA_2 =& w_5(a_1-a_2w_6+w_6\lambda).
\end{aligned}
$$

Equating the two expressions for TA_2 shows that, if the equality is to be independent of λ, the product $w_1w_2w_3w_4w_5w_6$ must equal 1. From this it follows that

$$w_5=\frac{a_1w_1w_2+a_2(1-w_1w_2w_3)-a_3w_1(1-w_2w_3w_4)}{a_1w_1w_2w_3w_4}. \tag{5}$$

Hence w_5 and w_6 may be expressed in terms of the other directions. Given P and the first four directions, points Q, R, S, T are determined, and the five points determine a conic. Independence of λ, used above, ensures that U is also on this conic.

Now the coordinates of the six points may be calculated:

$$
\begin{aligned}
P =& (a_2-\lambda:0:\lambda),\\
Q =& (a_3w_1-\lambda:\lambda:0),\\
R =& (0:a_1w_1w_2-a_3w_1+\lambda:a_3w_1-\lambda),\\
S =& (a_1w_1w_2-a_3w_1+\lambda:0:a_2w_1w_2w_3-a_1w_1w_2+a_3w_1-\lambda),\\
T =& (a_3w_1w_2w_3w_4-a_2w_1w_2w_3+a_1w_1w_2-a_3w_1+\lambda\\
&\quad :a_2w_1w_2w_3-a_1w_1w_2+a_3w_1-\lambda:0),\\
U =& (0:a_2-\lambda:a_3w_1w_2w_3w_4-a_2w_1w_2w_3+a_1w_1w_2-a_3w_1+\lambda).
\end{aligned}
$$

3.2. These points are on the conic whose equation may be written in the form

$$
\begin{aligned}
&\lambda(a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - \lambda)(x_1 + x_2 + x_3)^2\\
&+a_3w_1(a_3w_1w_2w_3w_4 - a_2w_1w_2w_3 + a_1w_1w_2 - a_3w_1 + (1 - w_2w_3w_4)\lambda)x_2^2\\
&+a_2(a_1w_1w_2 - a_3w_1 + (1 - w_1w_2w_3)\lambda)x_3^2\\
&-(a_1^2w_1^2w_2^2 + a_3^2w_1^2(1 - w_2w_3w_4) + a_2a_3w_1(1 + w_1w_2w_3)\\
&\qquad - a_3a_1w_1^2w_2(2 - w_2w_3w_4) - a_1a_2w_1^2w_2^2w_3 - (a_2(1 - w_1w_2w_3)\\
&\qquad + a_3w_1(1 - w_2w_3w_4))\lambda)x_2x_3\\
&-a_2(a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - (1 - w_1w_2w_3)\lambda)x_3x_1\\
&-a_3w_1(a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - (1 - w_2w_3w_4)\lambda)x_1x_2\\
&=0. \qquad (6)
\end{aligned}
$$

The part of (6) not containing the factor $(x_1 + x_2 + x_3)^2$, being linear in λ, represents a pencil of conics. Each of these conics is transformed by a dilatation about its center, induced by the expression containing $(x_1 + x_2 + x_3)^2$. Thus (6) suggests a system of conics analogous to a coaxaloid system of circles. In order to establish the analogy with the Tucker circles, it will be necessary to find a dilatation which transforms every conic by the same ratio of magnification and also transforms (6) into a pencil of conics.

First, if (6) be solved simultaneously with $x_1 + x_2 + x_3 = 0$, it will be found that all terms containing λ vanish. As in the special case, all conics are concurrent at infinity, and it will be shown that all of them are homothetic to each other.

3.3. It is also expected that the centers of the conics will be on one line. When the coordinates $(y_1 : y_2 : y_3)$ of the center are calculated, the results are too long to be displayed here. Suffice it to say that each coordinate is linear in λ, showing that the locus of $(y_1 : y_2 : y_3)$ is a line. If this line is represented by $c_1x_1+c_2x_2+c_3x_3 = 0$, the coefficients, after a large common factor of degree 4 has been removed, may be written as

$$
\begin{aligned}
c_1 &=a_2a_3w_1w_3(w_1 - w_4)(a_2w_2w_3 - a_1w_2 + a_3),\\
c_2 &=a_3w_1(a_3w_3w_4 - a_2w_3 + a_1)(a_1w_1w_2(1 - y) + a_2(1 - x) - a_3w_1(1 - y)),\\
c_3 &=a_2(a_1w_1w_2 - a_3w_1 + a_2)(-a_1(1 - x) + a_2w_3(1 - x) - a_3w_1w_3(1 - y)).
\end{aligned}
$$

For brevity the products $w_1w_2w_3$ and $w_2w_3w_4$ have been represented by the letters x and y respectively.

3.4. As has been seen, addition of any multiple of $(x_1+x_2+x_3)^2$ to the equation of a conic apparently induces a dilatation of the conic about its center. What must now be done, in order to establish an analogy with the system of Tucker circles, is to select a number σ such that the addition of $\sigma(x_1+x_2+x_3)^2$ to (6) dilates every conic by the same ratio ρ and transforms the system of conics into a pencil with two common points besides the two at infinity.

Using a formula for the distance between two points (*e.g.*, [1, p.31]), it may be shown that a dilatation with center $(y_1 : y_2 : y_3)$ sending $(x_1 : x_2 : x_3)$ to $(\overline{x_1} : \overline{x_2} : \overline{x_3})$ with ratio ρ is expressed by $\overline{x_i} \sim y_i + kx_i$, $(i = 1, 2, 3)$, where

$$k = \frac{\pm\rho(y_1 + y_2 + y_3)}{(1 \mp \rho)(x_1 + x_2 + x_3)}$$

or

$$\rho = \frac{\pm(x_1 + x_2 + x_3)}{(y_1 + kx_1) + (y_2 + kx_2) + (y_3 + kx_3)}.$$

In particular, if the conic $\sum a_{ij}x_ix_j = 0$ is dilated about its center $(y_1 : y_2 : y_3)$ with ratio ρ, so that the new equation is

$$\sum a_{ij}x_ix_j + \sigma(x_1 + x_2 + x_3)^2 = 0,$$

then

$$\rho^2 = 1 + \frac{\sigma(y_1 + y_2 + y_3)^2}{\sum a_{ij}y_iy_j}.$$

Here the ambiguous sign is avoided by choosing the a_{ij} so that the denominator of the fraction is positive.

Since it is required that ρ be the same for all conics in (6), it must be free of the parameter λ. For the center $(y_1 : y_2 : y_3)$ of (6), whose coordinates are linear in λ, it may be calculated that $y_1 + y_2 + y_3$ is independent of λ. As for $\sum a_{ij}y_iy_j$, let it first be noted that

$$\begin{aligned}\sum a_{ij}y_jx_i =&(a_{11}y_1 + a_{12}y_2 + a_{13}y_3)x_1\\ &+(a_{12}y_1 + a_{22}y_2 + a_{23}y_3)x_2\\ &+(a_{13}y_1 + a_{23}y_2 + a_{33}y_3)x_3.\end{aligned}$$

(By convention, $a_{ij} = a_{ji}$). Also, $\sum a_{ij}y_jx_i = 0$ is the equation of the polar line of the center with respect to the conic, but this is the line at infinity $x_1 + x_2 + x_3 = 0$. Therefore the coefficients of x_1, x_2, x_3 in the above equation are all equal, and it follows that

$$\sum a_{ij}y_iy_j = (a_{11}y_1 + a_{12}y_2 + a_{13}y_3)(y_1 + y_2 + y_3),$$

and

$$\rho^2 = 1 + \frac{\sigma(y_1 + y_2 + y_3)}{a_{11}y_1 + a_{12}y_2 + a_{13}y_3}.$$

Since the a_{ij} are quadratic in λ, and the y_i are linear, the denominator of the fraction is at most cubic in λ. Calculation shows that

$$a_{11}y_1 + a_{12}y_2 + a_{13}y_3 = M(A\lambda^2 + B\lambda + C),$$

in which

$$\begin{aligned}M =&a_1w_1(a_2w_2w_3 - a_1w_2 + a_3)\cdot\\&(-a_1^2w_1w_2 + a_2^2w_3 + a_3^2w_1w_3w_4 - a_2a_3w_3(w_1 + w_4)\\&\quad + a_3a_1w_1(1 - y) - a_1a_2(1 - x)),\\A =&a_1w_1w_2 + a_2 + a_3w_1w_2w_3w_4,\\B =&w_1(a_1^2w_1w_2^2 - a_2^2w_2w_3 - a_3^2w_1w_2w_3w_4 - 2a_2a_3\\&\quad - a_3a_1w_1w_2(1 - y) + a_1a_2w_2(1 - x)),\\C =&a_2a_3w_1^2(a_2w_2w_3 - a_1w_2 + a_3).\end{aligned}$$

3.5. If the system (6) is to become a pencil of conics, the equation

$$\sum a_{ij}x_ix_j + \sigma(x_1 + x_2 + x_3)^2 = 0$$

must be linear in λ. Since λ^2 appears in (6) as $-\lambda^2(x_1+x_x+x_3)^2$, this will vanish only if the coefficient of λ^2 in σ is 1. Therefore, to eliminate λ from the fractional part of ρ^2, it follows that

$$\sigma = \lambda^2 + \frac{B}{A}\lambda + \frac{C}{A}.$$

With this value of σ, if $\sigma(x_1 + x_2 + x_3)^2$ be added to (6), the equation becomes

$$\begin{aligned}&\lambda(-a_2a_3w_1(1 - xy)(x_1 + x_2 + x_3)^2 + (a_1w_1w_2 + a_2 + a_3w_1y)\cdot\\&\quad(a_3w_1(1 - y)x_2^2 + a_2(1 - x)x_3^2 + (a_2(1 - x) + a_3w_1(1 - y))x_2x_3\\&\quad + a_2(1 - x)x_3x_1 + a_3w_1(1 - y)x_1x_2))\\&+a_2a_3w_1^2(a_2w_2w_3 - a_1w_2 + a_3)(x_1 + x_2 + x_3)^2\\&+(a_1w_1w_2 + a_2 + a_3w_1y)(a_3w_1(a_1w_1w_2 - a_2x - a_3w_1(1 - y))x_2^2\\&+a_2w_1(a_1w_2 - a_3)x_3^2\\&-(a_1^2w_1^2w_2^2 + a_3^2w_1^2(1 - y) + a_2a_3w_1(1 + x)\\&\quad - a_3a_1w_1^2w_2(2 - y) - a_1a_2w_1w_2x)x_2x_3\\&+a_2(a_1w_1w_2 - a_2x - a_3w_1)x_3x_1\\&+a_3w_1(a_1w_1w_2 - a_2x - a_3w_1)x_1x_2)\\&=0. \end{aligned} \tag{7}$$

Since (7) is linear in λ, it represents a pencil of conics. These conics should have four points in common, of which two are known to be at infinity. In order to facilitate finding the other two points, it is noted that a pencil contains three degenerate conics, each one consisting of a line through two of the common points, and the line of the other two points. In this pencil the line at infinity and the line of the other two common points comprise one such degenerate conic. Its equation may be given by setting equal to zero the product of $x_1 + x_2 + x_3$ and a second linear factor. Since it is known that the coefficient of λ vanishes at infinity, the conic represented by $\lambda = \infty$ in (7) must be the required one. The coefficient of λ does indeed factor as follows:

$$\begin{aligned}&(x_1 + x_2 + x_3)(-a_2 a_3 w_1 (1 - xy) x_1 \\ &\quad + (a_3 w_1 (1 - y)(a_1 w_1 w_2 + a_2 + a_3 w_1 y) - a_2 a_3 w_1 (1 - xy)) x_2 \\ &\quad + (a_2 (1 - x)(a_1 w_1 w_2 + a_2 + a_3 w_1 y) - a_2 a_3 w_1 (1 - xy)) x_3).\end{aligned}$$

Therefore the second linear factor equated to zero must represent the line through the other two fixed points of (7).

3.6. These points may be found as the intersection of this line and any other conic in the system, for example, the conic given by $\lambda = 0$. To solve simultaneously the equations of the line and the conic, x_1 is eliminated, reducing the calculation to

$$a_3^2 w_1 w_2 w_3 w_4 x_2^2 - a_2 a_3 (1 + w_1 w_2^2 w_3^2 w_4) x_2 x_3 + a_2^2 w_2 w_3 x_3^2 = 0$$

or

$$(a_3 x_2 - a_2 w_2 w_3 x_3)(a_3 w_1 w_2 w_3 w_4 x_2 - a_2 x_3) = 0.$$

Therefore,

$$\frac{x_2}{x_3} = \frac{a_2 w_2 w_3}{a_3} \quad \text{or} \quad \frac{a_2}{a_3 w_1 w_2 w_3 w_4}.$$

The first solution gives the point

$$\Lambda = (a_1 w_2 w_3 w_4 w_5 : a_2 w_2 w_3 : a_3)$$

and the second solution gives

$$\Lambda' = (a_1 w_1 w_2 : a_2 : a_3 w_1 w_2 w_3 w_4).$$

Thus the dilatation of every conic of (6) about its center with ratio ρ transforms (6) into pencil (7) with common points Λ and Λ'.

3.7. Returning to the question of whether the conics of (6) are all homothetic to each other, this was settled in the case of parabolas. As for hyperbolas, it was found that they all have respectively parallel asymptotes, but a hyperbola could be enclosed in the acute sectors formed by the asymptotes, or in the obtuse sectors. However, when (6) is transformed to (7), there are at least two hyperbolas in the pencil that are homothetic. Since the equation of any hyperbola in the pencil may be expressed as a linear combination of the equations of these two homothetic ones, it follows that all hyperbolas in the pencil, and therefore in system (6), are homothetic to each other. A similar argument shows that, if (6) consists of ellipses, they must all be homothetic. Figure 5 shows a system (6) of ellipses, with one hexagon left in place. In Figure 6 the same system has been transformed into a pencil with two common points. Figure 7 shows two hyperbolas of a system (6), together with their hexagons. The related pencil is not shown.

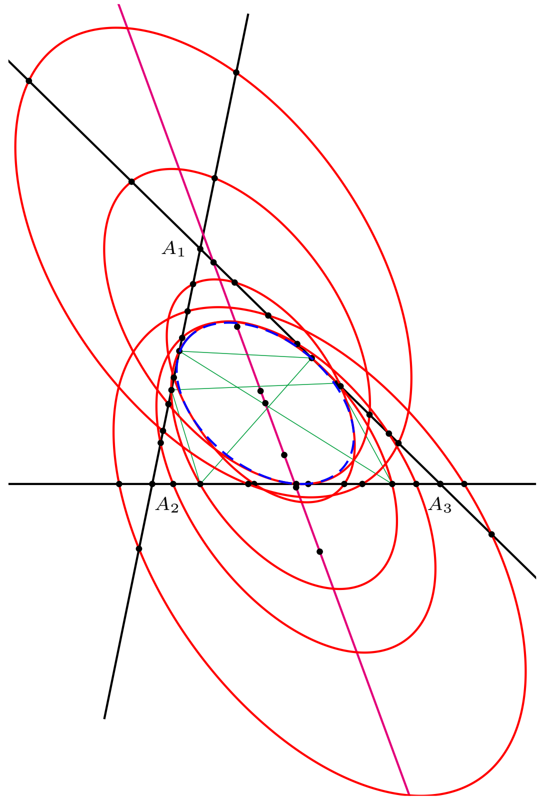


Figure 5

3.8. In the barycentric coordinate system, the midpoint $(v_1 : v_2 : v_3)$ of $(x_1 : x_2 : x_3)$ and $(y_1 : y_2 : y_3)$ is given by

$$v_i \sim \frac{x_i}{x_1 + x_2 + x_3} + \frac{y_i}{y_1 + y_2 + y_3}, \quad i = 1, 2, 3.$$

Thus it may be shown that the coordinates of the midpoint of $\Lambda\Lambda'$ are

$$(2a_1w_1w_2 + a_2(1-x) - a_3w_1(1-y) : a_2(1+x) : a_3w_1(1+y)).$$

This point is on the line of centers of (6), expressed earlier as

$$c_1x_1 + c_2x_2 + c_3x_3 = 0,$$

so the segment $\Lambda\Lambda'$ is bisected by the line of centers. However, it is not the perpendicular bisector unless (6) consists of circles. This case has already been disposed of, because if a circle cuts the sidelines of $A_1A_2A_3$, PQ and ST must be antiparallel to each other, as must QR and TU, and RS and UP. This would mean that (6) is a Third system.

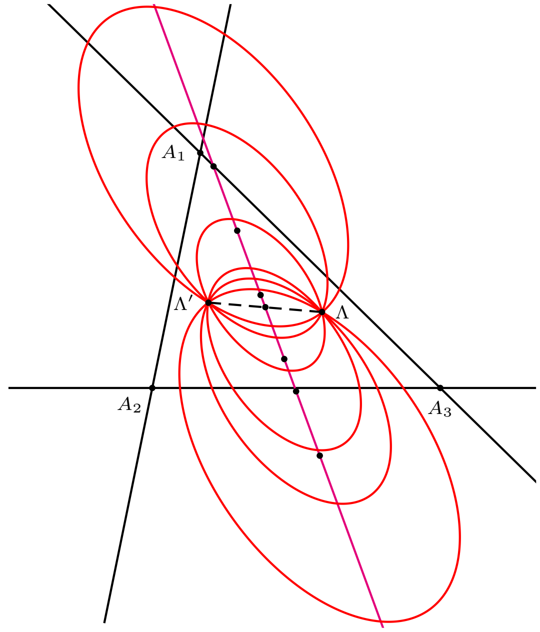


Figure 6

In system (6) the lines PQ, RS, TU are concurrent for a unique value of λ, which has been calculated but will not be written here. The point of concurrence is

$$\left(\frac{1}{-a_1w_1w_2 + a_2x + a_3w_1} : \frac{1}{a_1w_1w_2 - a_2x + a_3w_1} : \frac{1}{a_1w_1w_2 + a_2x - a_3w_1}\right),$$

which is a generalization of (2). The same point is obtained when QR, ST. UP are concurrent. It will also be written as $\left(\frac{1}{F_1} : \frac{1}{F_2} : \frac{1}{F_3}\right)$.

3.9. System (6) has an envelope which may be found by writing (6) as a quadratic equation in λ. Setting its discriminant equal to zero gives an equation of the envelope. The discriminant contains the factor $(x_1 + x_2 + x_3)^2$, which may be deleted, leaving

$$\sum F_i^2 x_i^2 - 2F_jF_kx_jx_k = 0.$$

This is an equation of the conic which touches the sidelines of $A_1A_2A_3$ at $L_1 = (0 : F_3 : F_2)$, $L_2 = (F_3 : 0 : F_1)$, and $L_3 = (F_2 : F_1 : 0)$. The cevians A_iL_i are concurrent at $\left(\frac{1}{F_1} : \frac{1}{F_2} : \frac{1}{F_3}\right)$.

The center of the envelope is the midpoint of $\Lambda\Lambda'$, but Λ and Λ' are not foci unless they are isogonal conjugates. This happens when $(w_1w_2)(w_2w_3w_4w_5) = (1)(w_2w_3) = (w_1w_2w_3w_4)(1)$, for which the solution is

$$w_1w_4 = w_2w_5 = w_3w_6 = 1.$$

Since this defines a Third system, it follows that Λ and Λ' are isogonal conjugates (and foci of the envelope of (6)) if and only if the conics are circles.

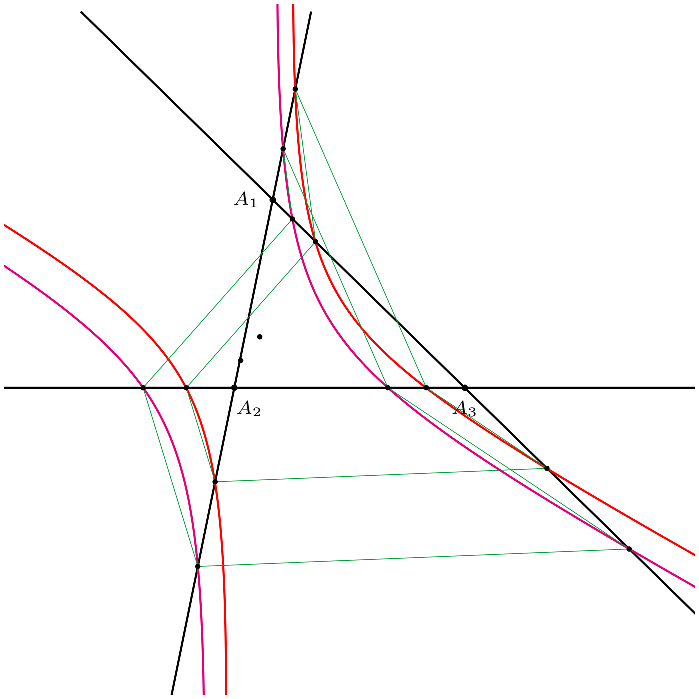


Figure 7

4. The parabolic case

There remains the question of whether (6) can be a system of parabolas. This is because the dilatations used above were made from the centers of the conics, whereas the centers of parabolas may be regarded as being at infinity. If the theory still holds true, the dilatations would have to be translations. That such cases actually exist may be demonstrated by the following example.

Let the triangle have sides $a_1 = 4$, $a_2 = 2$, $a_3 = 3$, and let

$$w_1 = \frac{2}{3}, \quad w_2 = \frac{3}{4}, \quad w_3 = \frac{1}{2}, \quad w_4 = \frac{2}{3}, \quad w_5 = 3, \quad w_6 = 2.$$

Substitution of these values in (6) gives the equation (after multiplication by 2)

$$\begin{aligned} &\lambda(1-2\lambda)(x_1+x_2+x_3)^2 + 3\lambda x_2^2 + 3\lambda x_3^2 \\ &+2(3\lambda-4)x_2x_3 + (3\lambda-2)x_3x_1 + (3\lambda-2)x_1x_2 = 0. \end{aligned} \tag{8}$$

To verify that this is a system of parabolas, solve (8) simultaneously with $x_1 + x_2 + x_3 = 0$, and elimination of x_1 gives the double solution $x_2 = x_3$. This shows that for every λ the conic is tangent to the line at infinity at the point $(-2 : 1 : 1)$. Hence, every nondegenerate conic in the system is a parabola, and all are homothetic to each other. (See Figure 8).

The formulae for σ gives the value $\left(\lambda - \frac{2}{3}\right)^2$, but (8) was obtained after multiplication by 2. Therefore $2\left(\lambda - \frac{2}{3}\right)^2 (x_1 + x_2 + x_3)^2$ is added to (8), yielding the

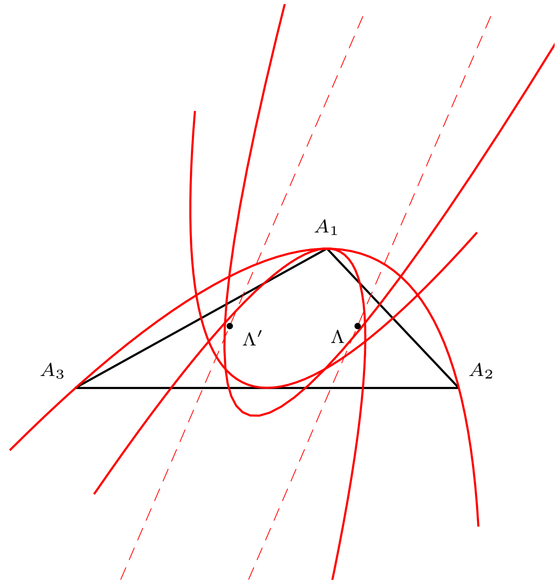


Figure 8. A system of parabolas

equation

$$(8-15\lambda)x_1^2+4(2+3\lambda)(x_2^2+x_3^2)-8(7-3\lambda)x_2x_3-(2+3\lambda)(x_3x_1+x_1x_2)=0,$$

which is linear in λ and represents a pencil of parabolas. The parabola $\lambda=\infty$ is found by using only terms containing λ, which gives the equation

$$-15x_1^2+12x_2^2+12x_3^2+24x_2x_3-3x_3x_1-3x_1x_2=0$$

or

$$3(x_1+x_2+x_3)(-5x_1+4x_2+4x_3)=0.$$

Thus it is the degenerate conic consisting of the line at infinity and the line $-5x_1+4x_2+4x_3=0$. Calculation shows that this line intersects every parabola of the pencil at $\Lambda(4:1:4)$ and $\Lambda'(4:4:1)$. (See Figure 9). The parallel dashed lines in both figures form the degenerate parabola $\lambda=\frac{2}{3}$, which is invariant under the translation which transformed the system into a pencil.

Finally, since all of the "centers" coincide, this is another exception to the rule that the line of centers of (6) bisects the segment $\Lambda\Lambda'$.

References

[1] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1–295.
[2] J. A. Third, Systems of circles analogous to Tucker circles, *Proc. Edinburgh Math. Soc.*, 17 (1898) 70–99.
[3] R. Tucker, On a group of circles, *Quart. J. Math.* 20 (1885) 57.
[4] P. Yff, Unpublished notes, 1976.

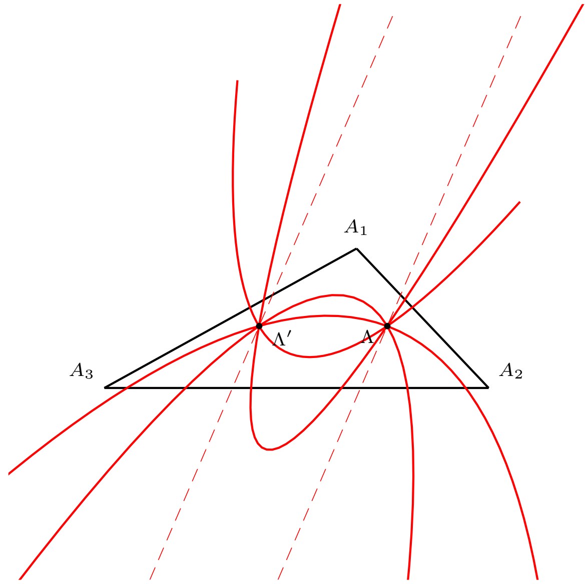


Figure 9. A pencil of parabolas

Peter Yff: 10840 Cook Ave., Oak Lawn, Illinois, 60453, USA
E-mail address: pjyff@aol.com

Forum Geometricorum
Volume 2 (2002) 89–92.

FORUM GEOM
ISSN 1534-1178

A Conic Through Six Triangle Centers

Lawrence S. Evans

Abstract. We show that there is a conic through the two Fermat points, the two Napoleon points, and the two isodynamic points of a triangle.

1. Introduction

It is always interesting when several significant triangle points lie on some sort of familiar curve. One recently found example is June Lester's circle, which passes through the circumcenter, nine-point center, and inner and outer Fermat (isogonic) points. See [8], also [6]. The purpose of this note is to demonstrate that there is a conic, apparently not previously known, which passes through six classical triangle centers.

Clark Kimberling's book [6] lists 400 centers and innumerable collineations among them as well as many conic sections and cubic curves passing through them. The list of centers has been vastly expanded and is now accessible on the internet [7]. Kimberling's definition of triangle center involves trilinear coordinates, and a full explanation would take us far afield. It is discussed both in his book and journal publications, which are readily available [4, 5, 6, 7]. Definitions of the Fermat (isogonic) points, isodynamic points, and Napoleon points, while generally known, are also found in the same references. For an easy construction of centers used in this note, we refer the reader to Evans [3]. Here we shall only require knowledge of certain collinearities involving these points. When points $X, Y, Z, \dots$ are collinear we write $\mathcal{L}(X, Y, Z, \dots)$ to indicate this and to denote their common line.

2. A conic through six centers

Theorem 1. *The inner and outer Fermat, isodynamic, and Napoleon points lie on a conic section.*

Proof. Let O denote the circumcenter of a triangle, H its orthocenter, and G its centroid. Denote the inner Fermat point by F_+, the inner isodynamic point by J_+, and the inner Napoleon point by N_+. Similarly denote the outer Fermat, isodynamic, and Napoleon points by F_-, J_-, and N_-.

Consider the hexagon whose vertices are F_+, N_+, J_+, F_-, N_-, and J_-. Kimberling lists many collineations of triangle centers which are readily verified when

Publication Date: July 22, 2002. Communicating Editor: Paul Yiu.
Presented at AMS Meeting, Urbana, Illinois, March 20, 1999.

the centers are given in homogeneous trilinear coordinates. Within the list are these collinearities involving the sides of the hexagon and classical centers on the Euler line: $\mathcal{L}(H, N_+, J_+)$, $\mathcal{L}(H, N_-, J_-)$, $\mathcal{L}(O, F_-, N_-)$, $\mathcal{L}(O, F_+, N_+)$, $\mathcal{L}(G, J_+, F_-)$, and $\mathcal{L}(G, J_-, F_+)$. These six lines pass through opposite sides of the hexagon and concur in pairs at H, O, and G. But we know that H, O, and G are collinear, lying on the Euler line. So, by the converse of Pascal's theorem there is a conic section through the six vertices of the hexagon. □

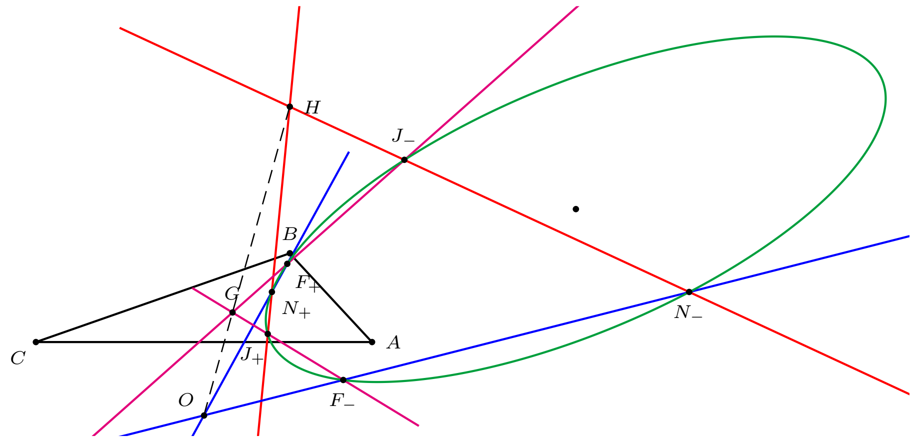


Figure 1. The conic through $F_\pm$, $N_\pm$ and $J_\pm$

Remark. In modern texts one sometimes sees Pascal's theorem stated as an "if and only if" theorem, omitting proper attribution for its converse, first proved independently by Braikenridge and by MacLaurin (See [2]). In the proof above, the Euler line plays the role of the Pascal line for the hexagon.

In Figure 1 the conic is shown as an ellipse, but it can also take the shape of a parabola or hyperbola. Since its announcement, several geometers have contributed knowledge about it. Peter Yff has calculated the equation of this conic [9], Paul Yiu has found criteria for it to be an ellipse, parabola, or a hyperbola [10],[1] and John H. Conway has generalized the conic [1].

3. Another conic

From Kimberling's list of collinearities, there is at least one more set of six points to which similar reasoning applies. We assume the reader is familiar with the concept of isogonal conjugate, fully explained in [6, 7].

Theorem 2. *The inner and outer Fermat (isogonic) and Napoleon points along with the isogonal conjugates of the Napoleon points all lie on a conic consisting of two lines intersecting at the center of the nine-point circle.*

[1]This conic is an ellipse, a parabola, or a hyperbola according as the Brocard angle is less than, equal to, or greater than $\arctan \frac{1}{3}$.

Proof. Denote the isogonal conjugates of the inner and outer Napoleon points by N_+^* and N_-^* respectively. Consider the hexagon with vertices F_+, F_-, N_+, N_-, N_+^*, and N_-^*. Kimberling lists these collinearities: $\mathcal{L}(G, N_+, N_-^*)$, $\mathcal{L}(G, N_-, N_+^*)$, $\mathcal{L}(O, F_+, N_+)$, $\mathcal{L}(O, F_-, N_-)$, $\mathcal{L}(H, F_+, N_+^*)$, $\mathcal{L}(H, F_-, N_-^*)$, so the converse of Pascal's theorem applies with the role of the Pascal line played by the Euler line, $\mathcal{L}(O, G, H)$. The conic is degenerate, consisting of two lines $L(F_-, N_+, N_+^*, N_p)$ and $\mathcal{L}(F_+, N_-, N_-^*, N_p)$, meeting at the nine-point center N_p . □

Second proof. The two collinearities $\mathcal{L}(F_-, N_+, N_+^*, N_p)$ and $\mathcal{L}(F_+, N_-, N_-^*, N_p)$ are in Kimberling's list, which *a fortiori* says that the six points in question lie on the degenerate conic consisting of the two lines. See Figure 2.

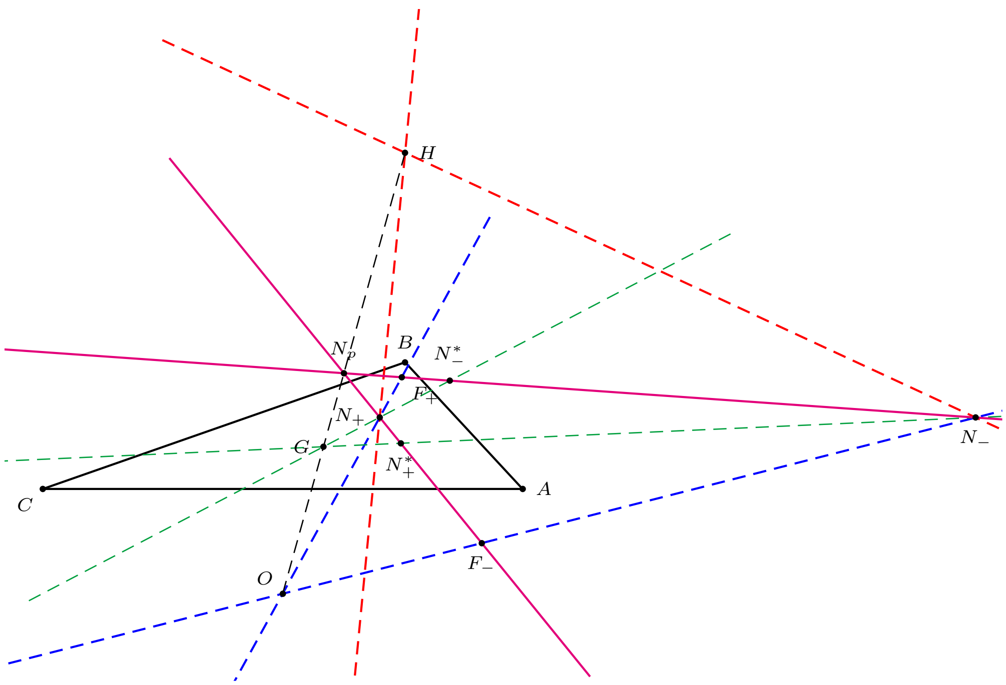


Figure 2. The degenerate conic through $F_\pm$, $N_\pm$ and $N_\pm^*$

References

[1] J. H. Conway, Hyacinthos, message 459, March 3, 2000.
[2] H. S. M. Coxeter and S. L. Greitzer, *Geometery Revisted*, Math. Assoc. America, 1967.
[3] L. S. Evans, A rapid construction of some triangle centers, *Forum Geom.*, 2 (2002) 67–70.
[4] C. Kimberling, Central points and central lines in the plane of a triangle, *Math. Magazine*, 67 (1994) 163–187.
[5] C. Kimberling, Major centers of triangles, *Amer. Math. Monthly*, 104 (1997) 431–488.
[6] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

[7] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000
http://www2.evansville.edu/ck6/encyclopedia/.

[8] J. A. Lester, Triangles, I, *Aequationes Math.*, 52 (1996) 30–54; II, *ibid.* 214–245; III, *ibid.* 53 (1997) 4–35.

[9] P. Yff, Personal correspondence, 1998.

[10] P. Yiu, Personal correspondence, 2001.

Lawrence S. Evans: 910 W. 57th Street, La Grange, Illinois 60525, USA
E-mail address: 75342.3052@compuserve.com

Forum Geometricorum
Volume 2 (2002) 93–104.

FORUM GEOM
ISSN 1534-1178

Paper-folding and Euler's Theorem Revisited

Benedetto Scimemi

Abstract. Given three points O, G, I, we give a simple construction by paper-folding for a triangle having these points as circumcenter, centroid, and incenter. If two further points H and N are defined by $\mathbf{OH} = 3\mathbf{OG} = 2\mathbf{ON}$, we prove that this procedure is successful if and only if I lies inside the circle on GH as diameter and differs from N. This locus for I is also independently derived from a famous paper of Euler, by complementing his calculations and properly discussing the reality of the roots of an algebraic equation of degree 3.

1. Introduction

The so-called *Modern Geometry of the Triangle* can be said to have been founded by Leonhard Euler in 1765, when his article [2] entitled *Easy Solution to some Very Difficult Geometrical Problems* was published in St. Petersburg. In this famous paper the distances between the main notable points of the triangle (centroid G, circumcenter O, orthocenter H, incenter I) are calculated in terms of the side lengths, so that several relationships regarding their mutual positions can be established. Among Euler's results, two have become very popular and officially bear his name: the vector equation $\mathbf{OH} = 3\mathbf{OG}$, implying the collinearity of G, O, H on the Euler line, and the scalar equation $OI^2 = R(R - 2r)$ involving the radii of the circumcircle and the incircle. Less attention has been given to the last part of the paper, though it deals with the problem Euler seems most proud to have solved in a very convenient [1] way, namely, the "determination of the triangle" from its points O, G, H, I. If one wants to avoid the "tedious calculations" which had previously prevented many geometricians from success, says Euler in his introduction, "everything comes down to choosing proper quantities". This understatement hides Euler's masterly use of symmetric polynomials, for which he adopts a cleverly chosen basis and performs complicated algebraic manipulations.

A modern reader, while admiring Euler's far-sightedness and skills, may dare add a few critical comments:

(1) Euler's §31 is inspired by the correct intuition that, given O, G, H, the location of I cannot be free. In fact he establishes the proper algebraic conditions but does not tell what they geometrically imply, namely that I must always lie inside the circle on GH as diameter. Also, a trivial mistake

Publication Date: August 19, 2002. Communicating Editor: Clark Kimberling.

[1]Latin: commodissime.

leads Euler to a false conclusion; his late editor's formal correction[2] does not lead any further.

(2) As for the determination of the triangle, Euler reduces the problem of finding the side lengths to solving an algebraic equation of degree 3. However, no attention is given to the crucial requirements that the three roots - in order to be side lengths - be real positive and the triangle inequalities hold. On the other hand, Euler's equation clearly suggests to a modern reader that the problem cannot be solved by ruler and compass.

(3) In Euler's words (§20) the main problem is described as follows: *Given the positions of the four points ..., to construct the triangle*. But finding the side lengths does not imply determining the location of the triangle, given that of its notable points. The word *construct* also seems improperly used, as this term's traditional meaning does not include solving an algebraic equation. It should rather refer, if not to ruler and compass, to some alternative geometrical techniques.

The problem of the locus of the incenter (and the excenters) has been independently settled by Andrew P. Guinand in 1982, who proved in his nice paper [5] that I must lie inside the critical circle on GH as diameter[3] (Theorem 1) and, conversely, any point inside this circle - with a single exception - is eligible for I (Theorem 4). In his introduction, Guinand does mention Euler's paper, but he must have overlooked its final section, as he claims that in all previous researches "the triangle was regarded as given and the properties of the centers were investigated" while in his approach "the process is reversed".

In this paper we give an alternative treatment of Euler's problem, which is independent both of Euler's and Guinand's arguments. Euler's crucial equation, as we said, involves the side lengths, while Guinand discusses the cosines of the angles. We deal, instead, with the coefficients for equations of the sides. But an independent interest in our approach may be found in the role played by the Euler point of the triangle, a less familiar notable point.[4] Its properties are particularly suitable for reflections and suggest a most natural paper-folding reconstruction procedure. Thus, while the first part (locus) of the following theorem is well-known, the construction mentioned in the last statement is new:

Theorem 1. *Let O, G, I be three distinct points. Define two more points H, N on the line OG by letting $\mathbf{OH} = 3\mathbf{OG} = 2\mathbf{ON}$. Then there exists a nondegenerate, nonequilateral triangle $\mathcal{T}$ with centroid G, circumcenter O, orthocenter H, and incenter I, if and only if I lies inside the circle on GH as diameter and differs from N. In this case the triangle $\mathcal{T}$ is unique and can be reconstructed by paper-folding, starting with the points O, G, I.*

---

[2] A. Speiser in [2, p.155, footnote].

[3] This is also known as the orthocentroidal circle. See [7]. This term is also used by Varilly in [9]. The author thanks the referee for pointing out this paper also treats this subject.

[4] This point is the focus of the Kiepert parabola, indexed as X_{110} in [7], where the notable points of a triangle are called triangle centers.

We shall find the sides of the triangle sides as proper creases, *i.e.*, reflecting lines, which simultaneously superimpose two given points onto two given lines. This can be seen as constructing the common tangents to two parabolas, whose foci and directrices are given. Indeed, the extra power of paper-folding, with respect to ruler-and-compass, consists in the feasibility of constructing such lines. See [4, 8].

The reconstruction of a triangle from three of its points (e.g. one vertex, the foot of an altitude and the centroid G) is the subject of an article of William Wernick [10], who in 1982 listed 139 triplets, among which 41 corresponded to problems still unsolved. Our procedure solves items 73, 80, and 121 of the list, which are obviously equivalent. [5] It would not be difficult to make slight changes in our arguments in order to deal with one of the excenters in the role of the incenter I.

As far as we know, paper-folding, which has been successfully applied to trisecting an angle and constructing regular polygons, has never yet produced any significant contribution to the geometry of the triangle.

This paper is structured as follows: in §2 we reformulate the well-known properties of the Simson line of a triangle in terms of side reflections and apply them to paper-folding. In §3 we introduce the Euler point E and study its properties. The relative positions of E, O, G are described by analytic geometry. This enables us to establish the locus of E and a necessary and sufficient condition for the existence of the triangle. [6] An immediate paper-folding construction of the triangle from E, O, G is then illustrated. In §4 we use complex variables to relate points E and I. In §5 a detailed ruler-and-compass construction of E from I, O, G is described.[7] The expected incenter locus is proved in §6 by reducing the problem to the former results on E, so that the proof of Theorem 1 is complete. In §7 we take up Euler's standpoint and interpret his formulas to find once more the critical circle locus as a necessary condition. Finally, we discuss the discriminant of Euler's equation and complete his arguments by supplying the missing algebraic calculations which imply sufficiency. Thus a third, independent, proof of the first part of Theorem 1 is achieved.

2. Simson lines and reflections

In this section we shall reformulate well-known results on the Simson line in terms of reflections, so that applications to paper-folding constructions will be natural. The following formulation was suggested by a paper of Longuet-Higgins [6].

Theorem 2. *Let H be the orthocenter, $\mathcal{C}$ the circumcircle of a triangle $\mathcal{T} = A_1A_2A_3$.*

(i) *For any point P, let P_i denote the reflection of P across the side A_jA_h of $\mathcal{T}$. (Here, i, j, h is a permutation of 1, 2, 3). Then the points P_i are collinear on a line $r = r(P)$ if and only if P lies on $\mathcal{C}$. In this case H lies on r.*

---

[5] Given I and two of O, G, H.

[6] Here too, as in the other approaches, the discussion amounts to evaluating the sign of a discriminant.

[7] A ruler and compass construction always entails a paper-folding construction. See [4, 8].

(ii) *For any line r, let r_i denote the reflection of r across the side A_jA_h. Then the lines r_i are concurrent at a point $P = P(r)$ if and only if H lies on r. In this case P lies on $\mathcal{C}$. When P describes an arc of angle α on $\mathcal{C}$, $r(P)$ rotates in the opposite direction around H by an angle $-\frac{\alpha}{2}$.*

All these statements are easy consequences of well-known properties of the Simson line, which is obviously parallel to $r(P)$. See, for example, [1, Theorems 2.5.1, 2.7.1,2]. This theorem defines a bijective mapping $P \mapsto r(P)$. Thus, given any line e through H, there exists a unique point E on $\mathcal{C}$ such that $r(E) = e$.

We now recall the basic assumption of paper-folding constructions, namely the possibility of determining a line, *i.e.*, folding a crease, which simultaneously reflects two given points A, B onto points which lie on two given lines a, b. It is proved in [4, 8] that this problem has either one or three solutions. We shall discuss later how these two cases can be distinguished, depending on the relative positions of the given points and lines. For the time being, we are interested in the case that three such lines (creases) are found. The following result is a direct consequence of Theorem 2.

Corollary 3. *Given two points A, B and two (nonparallel) lines a, b, assume that there exist three different lines r such that A (respectively B) is reflected across r onto a point A' (respectively B') lying on a (respectively b). These lines are the sides of a triangle $\mathcal{T}$ such that*

(i) *a and b intersect at the orthocenter H of $\mathcal{T}$;*
(ii) *A and B lie on the circumcircle of $\mathcal{T}$;*
(iii) *the directed angle $\angle AOB$ is twice the directed angle from b to a. Here, O denotes the circumcenter of $\mathcal{T}$.*

3. The Euler point

We shall now consider a notable point whose behaviour under reflections makes it especially suitable for paper-folding applications. The Euler point E is the unique point which is reflected across the three sides of the triangle onto the Euler line OG. Equivalently, the three reflections of the Euler line across the sides are concurrent at E. [8]

We first prove that for any nonequilateral, nondegenerate triangle with prescribed O and G (hence also H), the Euler point E lies outside a region whose boundary is a cardioid, a closed algebraic curve of degree 4, which is symmetric with respect to the Euler line and has the centroid G as a double-point (a cusp; see Figure 3). If we choose cartesian coordinates such that such that $G = (0, 0)$ and $O = (-1, 0)$ (so that $H = (2, 0)$), then this curve is represented by

$$(x^2 + y^2 + 2x)^2 - 4(x^2 + y^2) = 0 \quad \text{or} \quad \rho = 2(1 - \cos\theta). \tag{1}$$

Since this cardioid is uniquely determined by the choice of the two (different) points G, O, we shall call it the GO-cardioid. As said above, we want to prove that the locus of Euler point E for a triangle is the exterior of the GO-cardioid.

[8] This point can also be described as the Feuerbach point of the tangential triangle.

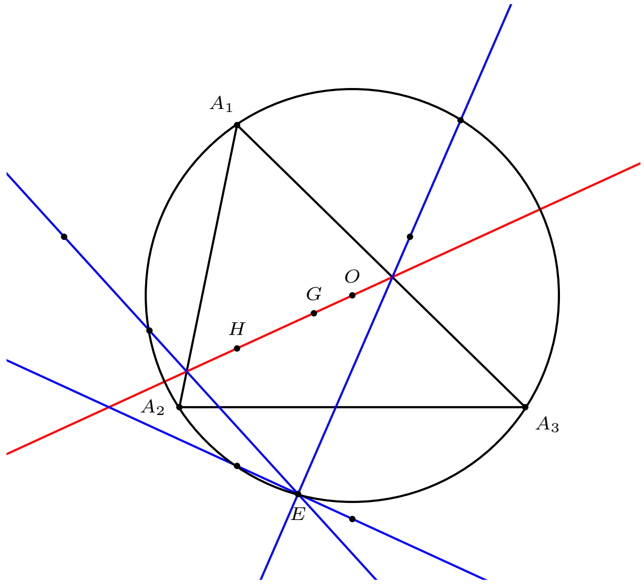


Figure 1. The Euler point of a triangle

Theorem 4. *Let G, O, E be three distinct points. Then there exists a triangle $\mathcal{T}$ whose centroid, circumcenter and Euler point are G, O, E, respectively, if and only if E lies outside the GO-cardioid. In this case the triangle $\mathcal{T}$ is unique and can be constructed by paper-folding, from the points G, O, E .*

Proof. Let us first look at isosceles (nonequilateral) triangles, which can be treated within ruler-and-compass geometry. [9] Here, by symmetry, the Euler point E lies on the Euler line; indeed, by definition, it must be one of the vertices, say $A_3 = E = (e, 0)$. Then being external to the GO-cardioid is equivalent to lying outside the segment GH_O, where $H_O = (-4, 0)$ is the symmetric of H with respect to O. Now the side A_1A_2 must reflect the orthocenter H into the point $E_O = (-2-e, 0)$, symmetric of E with respect to O, and therefore its equation is $x = -\frac{e}{2}$. This line has two intersections with the circumcircle $(x+1)^2 + y^2 = (e+1)^2$ if and only if $e(e+4) > 0$, which is precisely the condition for E to be outside GH_O. Conversely, given any two distinct points O, G, define H and H_O by $\mathbf{GH_O} = -2\mathbf{GH} = 4\mathbf{GO}$. Then for any choice of E on line OG, outside the segment GH_O, we can construct an isosceles triangle having E, O, G, H as its notable points as follows: first construct the (circum)-circle centered at O, through E, and let E_O be diametrically opposite to E. Then, under our assumptions on E, the perpendicular bisector of HE_O intersects the latter circle at two points, say A_1, A_2, and the isosceles triangle $\mathcal{T} = A_1A_2E$ fulfills our requirements.

We now deal with the nonisosceles case. Let $E = (u, v)$, $v \neq 0$ be the Euler point of a triangle $\mathcal{T}$. By definition, E is reflected across the three sides of the triangle into points E' which lie on the line $y = 0$. Now the line which reflects

[9]The case of the isosceles triangle is also studied separately by Euler in [2, §§25–29].

$E(u,v)$ onto a point $E'(t,0)$ has equation $2(u-t)x+2vy-(u^2+v^2-t^2)=0$. If the same line must also reflect (according to Theorem 2) point $E_O=(-u-2,-v)$ onto the line $x=2$ which is orthogonal to the Euler line through $H(2,0)$, then a direct calculation yields the following condition:

$$t^3-3(u^2+v^2)t+2u(u^2+v^2)-4v^2=0. \tag{2}$$

Hence we find three different reflecting lines if and only if this polynomial in t has three different real roots. The discriminant is

$$\Delta(u,v)=108v^2((u^2+v^2+2u)^2-4(u^2+v^2)).$$

Since $v\neq 0$, the inequality $\Delta(u,v)>0$ holds only if and only if E lies outside the cardioid, as we wanted.

The preceding argument can be also used for sufficiency: the assumed locus of E guarantees that (2) has three real roots. Therefore, three different lines exist which simultaneously reflect E onto line $a=OH$ and E_O onto the line b through H, perpendicular to OH. According to Corollary 3, these three lines are the sides of a triangle $\mathcal{T}$ which fulfills our requirements. In fact, H is the intersection of lines a and b and therefore H is the orthocenter of $\mathcal{T}$; a and b are perpendicular, hence E and E_O must be diametrically opposite points on the circumcircle of $\mathcal{T}$, so that their midpoint O is the circumcenter of $\mathcal{T}$. The three sides reflect E onto the x-axis, that is the Euler line of $\mathcal{T}$. Hence, by definition, E is the Euler point of $\mathcal{T}$. Since a polynomial of degree 3 cannot have more than 3 roots, the triangle is uniquely determined. □

Let us summarize the procedure for the reconstruction of the sides from the points O, G, E:

(1) Construct points H and E_O such that $\mathbf{GH}=-2\mathbf{GO}$ and $\mathbf{OE_O}=-\mathbf{OE}$.
(2) Construct line a through O, H and line b through H, perpendicular to a.
(3) Construct three lines that simultaneously reflect E on to a and E_O on to b.

4. Coordinates

The preceding results regarding the Euler point E are essential in dealing with the incenter I. In fact we shall construct E from G, O and I, so that Theorem 1 will be reduced to Theorem 3. To this end, we introduce the Gauss plane and produce complex variable equations relating I and E.[10] The cartesian coordinates will be different from the one we used in §3, but this seems unavoidable if we want to simplify calculations. A point $Z=(x,y)$ will be represented by the complex number $z=x+iy$. We write $Z=z$ and sometimes indicate operations as if they were acting directly on points rather than on their coordinates. We also write $z^*=x-iy$ and $|z|^2=x^2+y^2$.

Let $A_i=a_i$ be the vertices of a nondegenerate, nonequilateral triangle $\mathcal{T}$. Without loss of generality, we can assume for the circumcenter that $O=0$ and $|a_i|=1$, so that $a_i^{-1}=a_i^*$. Now the orthocenter H and the Euler point E have the following simple expressions in terms of elementary symmetric polynomials σ_1, σ_2, σ_3.

[10] A good reference for the use of complex variables in Euclidean geometry is [3].

$$\begin{aligned} H =& a_1 + a_2 + a_3 = \sigma_1, \\ E =& \frac{a_1 a_2 + a_2 a_3 + a_3 a_1}{a_1 + a_2 + a_3} = \frac{\sigma_2}{\sigma_1}. \end{aligned}$$

The first formula is trivial, as $G = \frac{1}{3}\sigma$ and $H = 3G$. As for E, the equation for a side, say $A_1 A_2$, is $z + a_1 a_2 z^* = a_1 + a_2$, and the reflection across this line takes a point $T = t$ on to $T' = a_1 + a_2 - a_1 a_2 t^*$. An easy calculation shows that E' lies on the Euler line $z\sigma_1^* - z^*\sigma_1 = 0$. This holds for all sides, and this property characterizes E by Theorem 2. Notice that $\sigma_1 \neq 0$, as we have assumed $G \neq O$.[11]

We now introduce $\sigma_3 = a_1 a_2 a_3$, $k = |OH|$ and calculate

$$\sigma_1^* = \sigma_2 \sigma_3^{-1}, \qquad |\sigma_3|^2 = 1, \quad |\sigma_1|^2 = |\sigma_2|^2 = \sigma_1 \sigma_2 \sigma_3^{-1} = k^2.$$

Hence,

$$\sigma_3 = \frac{\sigma_1 \sigma_2}{k^2} = \left(\frac{\sigma_1}{k}\right)^2 \cdot \frac{\sigma_2}{\sigma_1} = \frac{H}{|H|} \cdot E.$$

In order to deal with the incenter I, let $B_i = b_i$ denote the (second) intersection of the circumcircle with the internal angle bisector of A_i. Then $b_i^{-1} = b_i^*$ and $b_i^2 = a_j a_k$, and $b_1 b_2 b_3 = -a_1 a_2 a_3$. Since I is the orthocenter of triangle $B_1 B_2 B_3$, we have, as above, $I = b_1 + b_2 + b_3$. Likewise, we define

$$\tau_1 = b_1 + b_2 + b_3, \quad \tau_2 = b_1 b_2 + b_2 b_3 + b_3 b_1, \quad \tau_3 = b_1 b_2 b_3, \quad f = |OI|,$$

and calculate

$$\tau_1^* = \tau_2 \tau_3^{-1}, \qquad |\tau_3|^2 = 1, \quad |\tau_1|^2 = |\tau_2|^2 = \tau_1 \tau_2 \tau_3^{-1} = f^2.$$

From the definition of b_i, we derive

$$\begin{aligned} \tau_2^2 =& \sigma_3(\sigma_1 - 2\tau_1), \\ \tau_3 =& -\sigma_3 = \left(\frac{\tau_3}{\tau_2}\right)^2 \cdot \frac{\tau_2^2}{\tau_3} = -\left(\frac{\tau_1}{f^2}\right)^2 (\sigma_1 - 2\tau_1). \end{aligned}$$

Equivalently,

$$\begin{aligned} \sigma_3 = -\tau_3 = \left(\frac{\tau_1}{f}\right)^2 \cdot \frac{\sigma_1 - 2\tau_1}{f^2}, \\ \left(\frac{H}{|H|}\right)^2 \cdot E = \left(\frac{I}{|I|}\right)^2 \cdot \frac{H - 2I}{|I|^2}, \\ \left(\frac{G}{|G|}\right)^2 \cdot E = \left(\frac{I}{|I|}\right)^2 \cdot \frac{3G - 2I}{|I|^2}, \end{aligned} \tag{3}$$

where the Euler equation $H = 3G$ has been used.

[11] The triangle is equilateral if $G = O$.

5. Construction of the Euler point

The last formulas suggest easy constructions of E from O, G (or H) and I. Since $H - 2I = 3G - 2I = G - 2(I - G)$, our attention moves from $\mathcal{T}$ to its antimedial triangle $\mathcal{T}^*$ (the midpoints of whose sides are the vertices of $\mathcal{T}$) and the homothetic mapping: $Z \mapsto G - 2(Z - G)$. Thus $I^* = 3G - 2I$ is the incenter of $\mathcal{T}^*$. Note that multiplying by a *unit* complex number $\cos\theta + i\sin\theta$ is equivalent to rotating around O by an angle θ. Since $G/|G|$ and $I/|I|$ are unit complex numbers, multiplication by $\dfrac{(I/|I|)^2}{(G/|G|)^2}$ represents a rotation which is the product of two reflections, first across the line OG, then across OI. Since $|I|^2 = f^2$, dividing f^2 by I^* is equivalent to inverting I^* in the circle with center O and radius OI. Altogether, we conclude that E can be constructed from O, G and I by the following procedure. See Figure 2.

(1) Construct lines OG and OI; construct I^* by the equation $\mathbf{GI}^* = -2\mathbf{GI}$.
(2) Construct the circle Ω centered at O through I. By inverting I^* with respect to this circle, construct F^*. Note that this inversion is possible if and only if $I^* \neq O$, or, equivalently, $I \neq N$. [12]
(3) Construct E: first reflect F^* in line OG, and then its image in line OI.

Note that all these steps can be performed by ruler and compass.

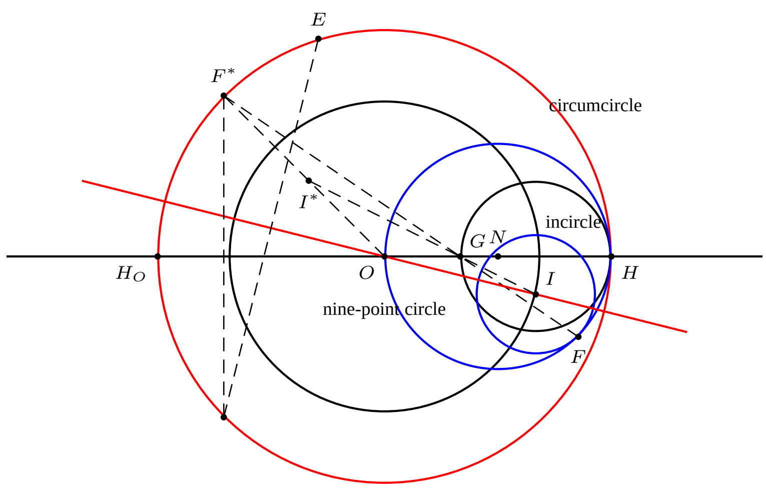


Figure 2. Construction of the Euler point from O, G, I

[12]It will appear that F^* is the Feuerbach point of $\mathcal{T}^*$. Thus, at this stage we have both the circumcircle (center O, through F^*) and the incircle of $\mathcal{T}$ (center I, through F, as defined by $\mathbf{GF}^* = -2\mathbf{GF}$).

6. The locus of incenter

As we know from §3, one can now apply paper-folding to O, G, E and produce the sides of $\mathcal{T}$. But in order to prove Theorem 1 we must show that the critical circle locus for I is equivalent to the existence of three different good creases. To this end we check that I lies inside the orthocentroidal GH-circle if and only if E lies outside the GO-cardioid. If we show that the two borders correspond under the transformation $I \mapsto E$ described by (3) for given O, G, H, then, by continuity, the two ranges, the interior of the circle and the exterior of the cardioid, will also correspond.

We first notice that the right side of (3) can be simplified when I lies on the GH-circle, as $|IO| = 2|IN| = |I^*O|$ implies that the inversion (step 2) does not affect I^*. In order to compare the transformation $I \mapsto E$ with our previous results, we must change scale and return to the cartesian coordinates used in §2, where $G = (0, 0)$, $H = (2, 0)$. If we set $I = (r, s)$, then $I^* = (-2r, -2s)$. The first reflection (across the Euler line) maps I^* on to $(-2r, 2s)$; the second reflection takes place across line OI: $s(x + 1) - (r + 1)y = 0$ and yields $E(u, v)$, where

$$(u, v) = \left(\frac{-2(r^3 - 3rs^2 + r + 2r^2 - s^2)}{(r + 1)^2 + s^2}, \frac{-2s(3r^2 - s^2 + 3r)}{(r + 1)^2 + s^2}\right).$$

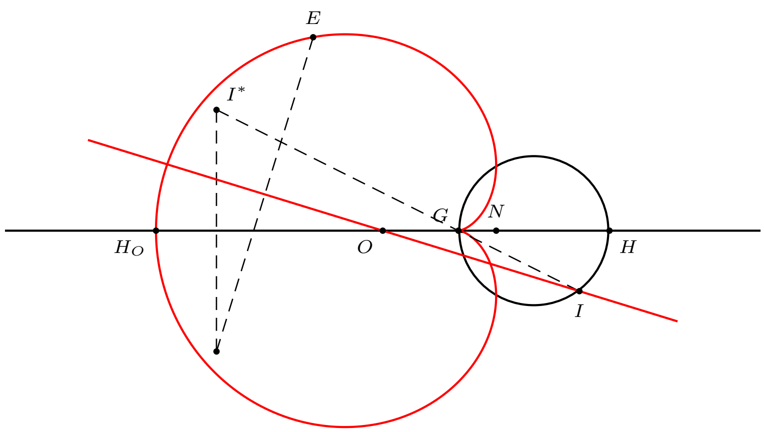


Figure 3. Construction of the GO-cardioid from the GH-circle

Notice that $I \neq O$ implies $(r + 1)^2 + s^2 \neq 0$. Then, by direct calculations, we have

$$((u^2 + v^2 + 2u)^2 - 4(u^2 + v^2))((r + 1)^2 + s^2) = 16(r^2 + s^2 - 2r)(r^2 + s^2 + r)^2$$

and conclude that $I(r, s)$ lies on the GH-circle $x^2 + y^2 - 2x = 0$ whenever E lies on the GO-cardioid (1), as we wanted. Thus the proof of Theorem 1 is complete.

7. Euler's theorem revisited

We shall now give a different proof of the first part of Theorem 1 by exploiting Euler's original ideas and complementing his calculations.

Necessity. In [2] Euler begins (§§1-20) with a nonequilateral, nondegenerate triangle and calculates the "notable" lengths

$$|HI| = e, \quad |OI| = f, \quad |OG| = g, \quad |GI| = h, \quad |HO| = k$$

as functions of the side lengths a_1, a_2, a_3. From those expressions he derives a number of algebraic equalities and inequalities, whose geometrical interpretations he only partially studies. [13] In particular, in §31, by observing that some of his quantities can only assume positive values, Euler explicitly states that the two inequalities

$$k^2 < 2e^2 + 2f^2, \tag{4}$$

$$k^2 > 2e^2 + f^2 \tag{5}$$

must hold. However, rather than studying their individual geometrical meaning, he tries to combine them and wrongly concludes, owing to a trivial mistake, that the inequalities $19f^2 > 8e^2$ and $13f^2 < 19e^2$ are also necessary conditions. Speiser's correction of Euler's mistake [2, p.155, footnote] does not produce any interesting result. On the other hand, if one uses the main result $\mathbf{OH} = 3\mathbf{OG}$, defines the nine-point center N (by letting $\mathbf{OH} = 2\mathbf{ON}$) and applies elementary geometry (Carnot's and Apollonius's theorems), it is very easy to check that the two original inequalities (4) and (5) are respectively equivalent to

(4′) I is different from N, and
(5′) I lies inside the GH-circle.

These are precisely the conditions of Theorem 1. It is noteworthy that Euler, unlike Guinand, could not use Feuerbach's theorem.

Sufficiency. In §21 Euler begins with three positive numbers f, g, h and derives a real polynomial of degree 3, whose roots a_1, a_2, a_3 - in case they are sides of a triangle - produce indeed f, g, h for the notable distances. It remains to prove that, under the assumptions of Theorem 1, these roots are real positive and satisfy the triangle inequalities. In order to complete Euler's work, we need a couple of lemmas involving symmetric polynomials.

Lemma 5. *(a) Three real numbers a_1, a_2, a_3 are positive if and only if $\sigma_1 = a_1 + a_2 + a_3$, $\sigma_2 = a_1a_2 + a_2a_3 + a_3a_1$ and $\sigma_3 = a_1a_2a_3$ are all positive.*

---

[13] The famous result on the collinearity of O, G, H and the equation $\mathbf{OH} = 3\mathbf{OG}$ are explicitly described in [2]. The other famous formula $OI^2 = R(R - 2r)$ is not explicitly given, but can be immediately derived, by applying the well known formulas for the triangle area $\frac{1}{2}r(a_1 + a_2 + a_3) = \frac{a_1a_2a_3}{4R}$.

(b) Three positive real numbers a_1, a_2, a_3 *satisfy the triangle inequalities* $a_1+a_2 \geq a_3$, $a_2+a_3 \geq a_1$ *and* $a_3+a_1 \geq a_2$ *if and only if*[14]

$$\tau(a_1,a_2,a_3) = (a_1+a_2+a_3)(-a_1+a_2+a_3)(a_1-a_2+a_3)(a_1+a_2-a_3) \geq 0.$$

Now suppose we are given three different points I, O, N and define two more points G, H by $3\mathbf{OG} = 2\mathbf{ON} = \mathbf{OH}$. Assume that I is inside the GH-circle. If we let

$$m = |ON|, \quad n = |IN|, \quad f = |OI|,$$

then we have $m > 0$, $n > 0$, $f > 0$, $n+f-m \geq 0$, and also, according to Lemma 5(b), $\tau(f,m,n) \geq 0$. Moreover, the assumed locus of I within the critical circle implies, by Apollonius, $f - 2n > 0$ so that $f^2 - 4n^2 = b^2$ for some real $b > 0$. We now introduce the same quantities p, q, r of Euler,[15] but rewrite their defining relations in terms of the new variables m, n, f as follows:

$$\begin{aligned} n^2 r =& f^4, \\ 4n^2 q =& b^2 f^2, \\ 9h^2 =& (f-2n)^2 + 2((n+f)^2 - m^2), \\ 4n^2 p =& 27b^4 + 128n^2 b^2 + 144h^2 n^2. \end{aligned}$$

Notice that, under our assumptions, all these functions assume positive values, so that we can define three more positive quantities[16]

$$\sigma_1 = \sqrt{p}, \quad \sigma_2 = \frac{p}{4} + 2q + \frac{q^2}{r}, \quad \sigma_3 = q\sqrt{p}.$$

Now let a_1, a_2, a_3 be the (complex) roots of the polynomial $x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3$. The crucial point regards the discriminant

$$\begin{aligned} \Delta(a_1,a_2,a_3) =& (a_1-a_2)^2(a_2-a_3)^2(a_3-a_1)^2 \\ =& \sigma_1^2\sigma_2^2 + 18\sigma_1\sigma_2\sigma_3 - 4\sigma_1^3\sigma_3 - 4\sigma_2^3 - 27\sigma_3^2. \end{aligned}$$

By a tedious but straightforward calculation, involving a polynomial of degree 8 in m, n, f, one finds

$$n^2\Delta(a_1,a_2,a_3) = b^4\tau(f,m,n).$$

Since, by assumption, $n \neq 0$, this implies $\Delta(a_1,a_2,a_3) \geq 0$, so that a_1, a_2, a_3 are real. By Lemma 5(a), since σ_1, σ_2, $\sigma_3 > 0$, we also have a_1, a_2, $a_3 > 0$. A final calculation yields $\tau(a_1,a_2,a_3) = \dfrac{4pq^2}{r} > 0$, ensuring, by Lemma 5(b) again, that the triangle inequalities hold. Therefore, under our assumptions, there exists a triangle with a_1, a_2, a_3 as sides lengths, which is clearly nondegenerate and nonequilateral, and whose notable distances are f, m, n. Thus the alternative proof the first part of Theorem 1 is complete. Of course, the last statement on

---

[14]The expression $\tau(a_1,a_2,a_3)$ appears under the square root in Heron's formula for the area of a triangle.

[15]These quantities read P, Q, R in [2, p.149].

[16]These quantities read p, q, r in [2, p.144].

construction is missing: the actual location of the triangle, in terms of the location of its notable points, cannot be studied by this approach.

References

[1] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Math. Assoc. America, 1967.

[2] L. Euler, Solutio facilis problematum quorundam geometricorum difficillimorum, *Novi Comm. Acad. Scie. Petropolitanae* 11 (1765); reprinted in *Opera omnia*, serie prima, vol.26 (ed. by A. Speiser), (n.325) 139–157.

[3] L. Hahn, *Complex Numbers & Geometry*, Math. Assoc. America, 1994.

[4] H. Huzita and B. Scimemi. The algebra of paper-folding, *Proceedings of 1st Int. Meeting on Origami Science and Techn.*, Ferrara, 1989, 215–222.

[5] A. P. Guinand, Euler lines, tritangent centers, and their triangles, *Amer. Math. Monthly*, 91 (1984) 290–300.

[6] M. S. Longuet-Higgins, Reflections on a triangle, *Math. Gazette*, 57 (1973) 293–296.

[7] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1–295.

[8] G. Martin, *Geometric Constructions*, Springer, 1998; Chapter 10.

[9] A. Varilly, Location of incenters and Fermat points in variable triangles, *Math. Magazine*, 74 (2001) 123–129.

[10] W. Wernick, Triangle constructions with three located points, *Math. Magazine*, 55 (1982) 227–230.

Benedetto Scimemi: Department of Mathematics, University of Padova, 35100 Padova - via Belzoni, 7, Italy

E-mail address: scimemi@math.unipd.it

Forum Geometricorum
Volume 2 (2002) 105–113.

FORUM GEOM
ISSN 1534-1178

Loci Related to Variable Flanks

Zvonko Čerin

Abstract. Let BR_1R_2C, CR_3R_4A, AR_5R_6B be rectangles built on the sides of a triangle ABC such that the oriented distances $|BR_1|$, $|CR_3|$, $|AR_5|$ are $\lambda\,|BC|$, $\lambda\,|CA|$, $\lambda\,|AB|$ for some real number λ. We explore relationships among the central points of triangles ABC, AR_4R_5, BR_6R_1, and CR_2R_3. Our results extend recent results by Hoehn, van Lamoen, C. Pranesachar and Venkatachala who considered the case when $\lambda = 1$ (with squares erected on sides).

1. Introduction

In recent papers (see [2], [5], and [6]), L. Hoehn, F. van Lamoen, and C. R. Pranesachar and B. J. Venkatachala have considered the classical geometric configuration with squares BS_1S_2C, CS_3S_4A, and AS_5S_6B erected on the sides of a triangle ABC and studied relationships among the central points (see [3]) of the base triangle $\tau = ABC$ and of three interesting triangles $\tau_A = AS_4S_5$, $\tau_B = BS_6S_1$, $\tau_C = CS_2S_3$ (called *flanks* in [5] and *extriangles* in [2]). In order to describe their main results, recall that triangles ABC and XYZ are *homologic* provided that the lines AX, BY, and CZ are concurrent. The point P in which they concur is their *homology center* and the line ℓ containing the intersections of the pairs of lines $(BC,\, YZ)$, $(CA,\, ZX)$, and $(AB,\, XY)$ is their *homology axis*. In this situation we use the notation $ABC \underset{\ell}{\overset{P}{\bowtie}} XYZ$, where ℓ or both ℓ and P may be omitted. Let $X_i = \underline{X}_i(\tau)$, $X_i^j = \underline{X}_i(\tau_j)$ (for $j = A,\, B,\, C$), and $\sigma_i = X_i^A X_i^B X_i^C$, where $\underline{X}_i$ (for $i = 1, \ldots$) is any of the triangle central point functions from Kimberling's lists [3] or [4].

Instead of homologic, homology center, and homology axis many authors use the terms *perspective*, *perspector*, and *perspectrix*. Also, it is customary to use letters I, G, O, H, F, K, and L instead of X_1, X_2, X_3, X_4, X_5, X_6, and X_{20} to denote the incenter, the centroid, the circumcenter, the orthocenter, the center of the nine-point circle, the symmedian (or Grebe-Lemoine) point, and the de Longchamps point (the reflection of H about O), respectively.

In [2] Hoehn proved $\tau \bowtie \sigma_3$ and $\tau \overset{X_j}{\bowtie} \sigma_i$ for $(i,\, j) = (1,\, 1),\, (2,\, 4),\, (4,\, 2)$. In [6] C. R. Pranesachar and B. J. Venkatachala added some new results because they showed that $\tau \overset{X_j}{\bowtie} \sigma_i$ for $(i,\, j) = (1,\, 1),\, (2,\, 4),\, (4,\, 2),\, (3,\, 6),\, (6,\, 3)$. Moreover,

Publication Date: September 5, 2002. Communicating Editor: Floor van Lamoen.

they observed that if $\tau \overset{X}{\bowtie} X_A X_B X_C$, and Y, Y_A, Y_B, and Y_C are the isogonal conjugates of points X, X_A, X_B, and X_C with respect to triangles τ, τ_A, τ_B, and τ_C respectively, then $\tau \overset{Y}{\bowtie} Y_A Y_B Y_C$. Finally, they also answered negatively the question by Prakash Mulabagal of Pune if $\tau \bowtie XYZ$, where X, Y, and Z are the points of contact of the incircles of triangles τ_A, τ_B, and τ_C with the sides opposite to A, B, and C, respectively.

In [5] van Lamoen said that X_i *befriends* X_j when $\tau \overset{X_j}{\bowtie} \sigma_i$ and showed first that $\tau \overset{X_j}{\bowtie} \sigma_i$ implies $\tau \overset{X_n}{\bowtie} \sigma_m$ where X_m and X_n are the isogonal conjugates of X_i and X_j. Also, he proved that $\tau \overset{X_j}{\bowtie} \sigma_i$ is equivalent to $\tau \overset{X_i}{\bowtie} \sigma_j$, and that $\tau \overset{X_j}{\bowtie} \sigma_i$ for $(i,\, j) = (1,\, 1),\ (2,\, 4),\ (3,\, 6),\ (4, 2),\ (6,\, 3)$. Then he noted that $\tau \overset{K(\frac{\pi}{2}-\phi)}{\bowtie} K(\phi)$, where $K(\phi)$ denotes the homology center of τ and the Kiepert triangle formed by apexes of similar isosceles triangles with the base angle ϕ erected on the sides of ABC. This result implies that $\tau \overset{X_i}{\bowtie} \sigma_i$ for $i = 485,\, 486$ (Vecten points – for $\phi = \pm\frac{\pi}{4}$), and $\tau \overset{X_j}{\bowtie} \sigma_i$ for $(i,\, j) = (13,\, 17),\ (14,\, 18)$ (isogonic or Fermat points X_{13} and X_{14} – for $\phi = \pm\frac{\pi}{3}$, and Napoleon points X_{17} and X_{18} – for $\phi = \pm\frac{\pi}{6}$). Finally, van Lamoen observed that the Kiepert hyperbola (the locus of $K(\phi)$) befriends itself; so does its isogonal transform, the Brocard axis OK.

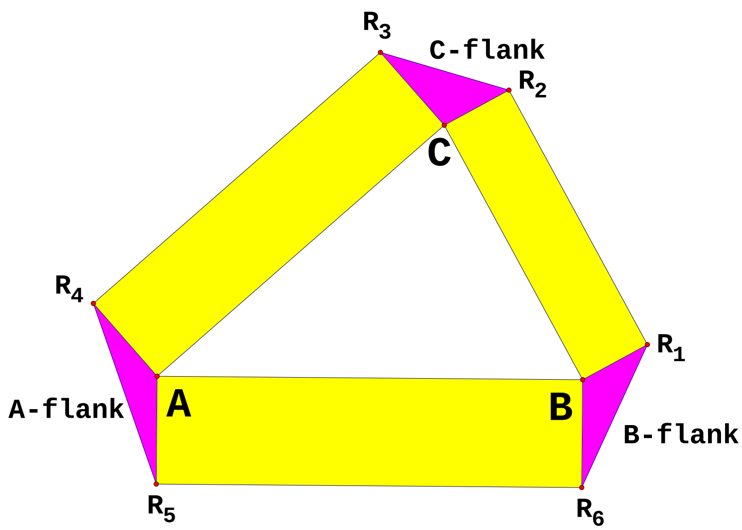


Figure 1. Triangle ABC with three rectangles and three flanks.

The purpose of this paper is to extend and improve the above results by replacing squares with rectangles whose ratio of nonparallel sides is constant. More precisely, let BR_1R_2C, CR_3R_4A, AR_5R_6B be rectangles built on the sides of a triangle ABC such that the oriented distances $|BR_1|$, $|CR_3|$, $|AR_5|$ are $\lambda\,|BC|$,

$\lambda\,|CA|$, $\lambda\,|AB|$ for some real number λ. Let $\tau_A^\lambda = AR_4R_5$, $\tau_B^\lambda = BR_6R_1$, and $\tau_C^\lambda = CR_2R_3$ and let $X_i^j(\lambda)$ and σ_i^λ, for $j = A, B, C$, have obvious meaning. The most important central points have their traditional notations so that we shall often use these because they might be easier to follow. For example, $H^A(\lambda)$ is the orthocenter of the flank τ_A^λ and σ_G^λ is the triangle $G^A(\lambda)G^B(\lambda)G^C(\lambda)$ of the centroids of flanks.

Since triangles AS_4S_5 and AR_4R_5 are homothetic and the vertex A is the center of this homothety (and similarly for pairs BS_6S_1, BR_6R_1 and CS_2S_3, CR_2R_3), we conclude that $\{A,\, X_i^A,\, X_i^A(\lambda)\}$, $\{B,\, X_i^B,\, X_i^B(\lambda)\}$, and $\{C,\, X_i^C,\, X_i^C(\lambda)\}$ are sets of collinear points so that all statements from [2], [6], and [5] concerning triangles σ_i are also true for triangles σ_i^λ.

However, since in our approach instead of a single square on each side we have a family of rectangles it is possible to get additional information. This is well illustrated in our first theorem.

Theorem 1. *The homology axis of* ABC *and* $G^A(\lambda)G^B(\lambda)G^C(\lambda)$ *envelopes the Kiepert parabola of* ABC.

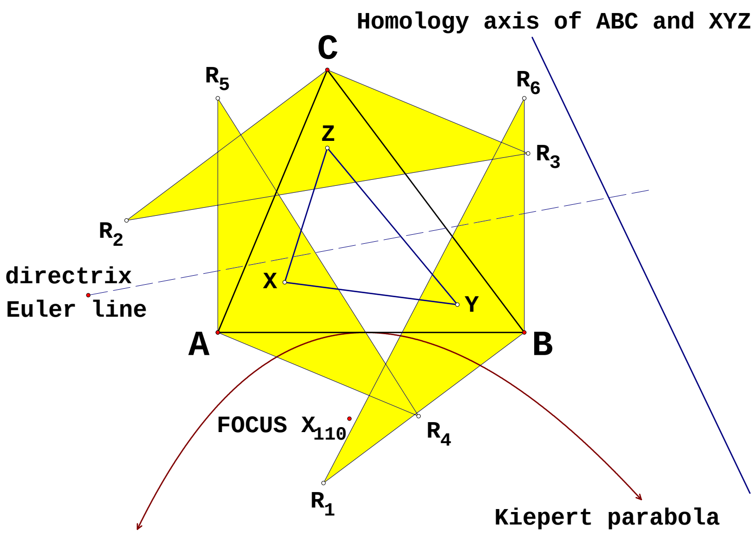


Figure 2. The homology axis of ABC and XYZ envelopes the Kiepert parabola of ABC.

Proof. In our proofs we shall use trilinear coordinates. The advantage of their use is that a high degree of symmetry is present so that it usually suffices to describe part of the information and the rest is self evident. For example, when we write $X_1(1)$ or $I(1)$ or simply say I is 1 this indicates that the incenter has trilinear coordinates $1 : 1 : 1$. We give only the first coordinate while the other two are cyclic permutations of the first. Similarly, $X_2(\frac{1}{a})$, or $G(\frac{1}{a})$, says that the centroid has has trilinears $\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$, where a, b, c are the lengths of the sides of ABC. The expressions in terms of sides a, b, c can be shortened using the following notation.

$$d_a = b-c, \quad d_b = c-a, \quad d_c = a-b, \quad z_a = b+c, \quad z_b = c+a, \quad z_c = a+b,$$
$$t = a+b+c, \quad t_a = b+c-a, \quad t_b = c+a-b, \quad t_c = a+b-c,$$
$$m = abc, \quad m_a = bc, \quad m_b = ca, \quad m_c = ab, \quad T = \sqrt{t t_a t_b t_c},$$

For an integer n, let $t_n = a^n + b^n + c^n$ and $d_{na} = b^n - c^n$, and similarly for other cases. Instead of t_2, t_{2a}, t_{2b}, and t_{2c} we write k, k_a, k_b, and k_c.

In order to achieve even greater economy in our presentation, we shall describe coordinates or equations of only one object from triples of related objects and use cyclic permutations φ and ψ below to obtain the rest. For example, the first vertex A_a of the anticomplementary triangle $A_aB_aC_a$ of ABC has trilinears $-\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$. Then the trilinears of B_a and C_a need not be described because they are easily figured out and memorized by relations $B_a = \varphi(A_a)$ and $C_a = \psi(A_a)$. One must remember always that transformations φ and ψ are not only permutations of letters but also of positions, *i.e.*,

$$\varphi : \; a,\, b,\, c,\, 1,\, 2,\, 3 \mapsto b,\, c,\, a,\, 2,\, 3,\, 1$$

and

$$\psi : \; a,\, b,\, c,\, 1,\, 2,\, 3 \mapsto c,\, a,\, b,\, 3,\, 1,\, 2.$$

Therefore, the trilinears of B_a and C_a are $\frac{1}{a} : -\frac{1}{b} : \frac{1}{c}$ and $\frac{1}{a} : \frac{1}{b} : -\frac{1}{c}$.

The trilinears of the points R_1 and R_2 are equal to $-2\lambda m : c(T+\lambda k_c) : \lambda b k_b$ and $-2\lambda m : \lambda c k_c : b(T+\lambda k_b)$ (while $R_3 = \varphi(R_1)$, $R_4 = \varphi(R_2)$, $R_5 = \psi(R_1)$, and $R_6 = \psi(R_2)$). It follows that the centroid $X_2^A(\lambda)$ or $G^A(\lambda)$ of the triangle AR_4R_5 is $\frac{3T+2a^2\lambda}{-a} : \frac{k_c\lambda}{b} : \frac{k_b\lambda}{c}$.

Hence, the line $G^B(\lambda)G^C(\lambda)$ has equation

$$a(T\,\lambda^2 + 6\,z_{2a}\,\lambda + 9\,T)\,x + b\,\lambda(T\,\lambda + 3\,k_c)\,y + c\,\lambda(T\,\lambda + 3\,k_b)\,z = 0.$$

It intersects the line BC whose equation is $x = 0$ in the point $0 : \frac{T\,\lambda + 3\,k_b}{b} : \frac{T\,\lambda+3\,k_c}{-c}$. Joining this point with its related points on lines CA and/or AB we get the homology axis of triangles ABC and $G^A(\lambda)G^B(\lambda)G^C(\lambda)$ whose equation is

$$\sum a(T^2\,\lambda^2 + 6\,a^2\,T\,\lambda + 9\,k_b\,k_c)\,x = 0.$$

When we differentiate this equation with respect to λ and solve for λ we get $\lambda = \frac{-3\,(\sum\, a^3\,x)}{T\,(\sum\, a\,x)}$. Substituting this value back into the above equation of the axis we obtain the equation

$$\sum(a^2\,d_{2a}^2\,x^2 - 2\,m_a\,d_{2b}\,d_{2c}\,y\,z) = 0$$

of their envelope. It is well-known (see [1]) that this is in fact the equation of the Kiepert parabola of ABC. □

Recall that triangles ABC and XYZ are *orthologic* provided the perpendiculars from the vertices of ABC to the sides YZ, ZX, and XY of XYZ are concurrent. The point of concurrence of these perpendiculars is denoted by $[ABC,\, XYZ]$. It is well-known that the relation of orthology for triangles is reflexive and symmetric.

Hence, the perpendiculars from the vertices of XYZ to the sides BC, CA, and AB of ABC are concurrent at a point $[XYZ,\, ABC]$.

Since G (the centroid) befriends H (the orthocenter) it is clear that triangles τ and σ_G^λ are orthologic and $[\sigma_G^\lambda,\, \tau] = H$. Our next result shows that point $[\tau,\, \sigma_G^\lambda]$ traces the Kiepert hyperbola of τ.

Theorem 2. *The locus of the orthology center $[\tau,\, \sigma_G^\lambda]$ of τ and σ_G^λ is the Kiepert hyperbola of ABC.*

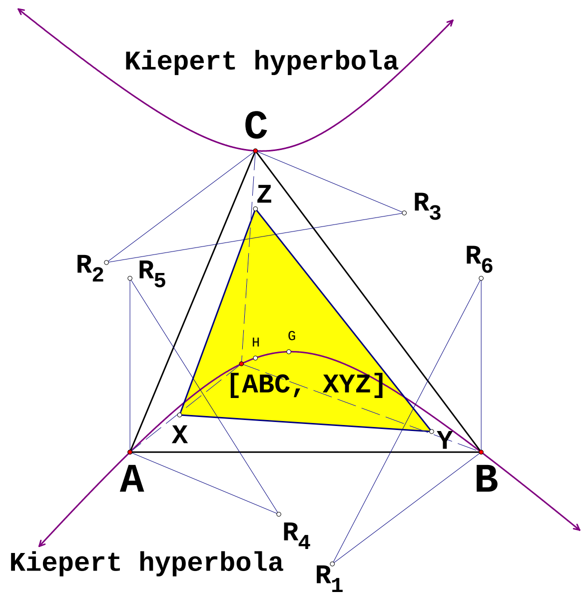


Figure 3. The orthology center $[ABC,\, XYZ]$ of triangles $\tau = ABC$ and $\sigma_G^\lambda = XYZ$ traces the Kiepert hyperbola of ABC.

Proof. The perpendicular from A onto the line $G^B(\lambda)G^C(\lambda)$ has equation

$$b(T\,\lambda + 3\,k_b)y - c(T\,\lambda + 3\,k_c)z = 0.$$

It follows that $[\tau,\, \sigma_G^\lambda]$ is $\frac{1}{a(T\lambda+3k_a)}$. This point traces the conic with equation $\sum m_a\, d_{2a}\, y\, z = 0$. The verification that this is the Kiepert hyperbola is easy because we must only check that it goes through A, B, C, $H(\frac{1}{a\,k_a})$, and $G(\frac{1}{a})$. $\square$

Theorem 3. *For every $\lambda \in \mathbb{R}$, the triangles τ and σ_O^λ are homothetic, with center of homothety at the symmedian point K. Hence, they are homologic with homology center K and their homology axis is the line at infinity.*

Proof. The point $\frac{T+z_{2a}\,\lambda}{-abc} : \frac{\lambda}{c} : \frac{\lambda}{b}$ is the circumcenter $O^A(\lambda)$ of the flank AR_4R_5. Since the determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ \frac{T+z_{2a}\,\lambda}{-abc} & \frac{\lambda}{c} & \frac{\lambda}{b} \end{vmatrix}$$

is obviously zero, we conclude that the points A, K, and $O^A(\lambda)$ are collinear. In a similar way it follows that $\{B,\, K,\, O^B(\lambda)\}$ and $\{C,\, K,\, O^C(\lambda)\}$ are triples of collinear points. Hence, $\tau \overset{K}{\bowtie} \sigma_O^\lambda$. For $\lambda = -\frac{T}{k}$, the points $O^A(\lambda)$, $O^B(\lambda)$, and $O^C(\lambda)$ coincide with the symmedian point K. For $\lambda \neq -\frac{T}{k}$, the line $O^B(\lambda)O^C(\lambda)$ has equation $(a^2\,\lambda + T)x + \lambda\, a\, b\, y + \lambda\, c\, a\, z = 0$ and is therefore parallel to the sideline BC. Hence, the triangles τ and σ_A^λ are homothetic and the center of this homothety is the symmedian point K of τ. □

Theorem 4. *For every $\lambda \in \mathbb{R}$, the triangles τ and σ_O^λ are orthologic. The orthology center $[\,\tau,\, \sigma_O^\lambda\,]$ is the orthocenter H while the orthology center $[\,\sigma_O^\lambda,\, \tau\,]$ traces the line HK joining the orthocenter with the symmedian point.*

Proof. Since the triangles τ and σ_O^λ are homothetic and their center of similitude is the symmedian point K, it follows that τ and σ_O^λ are orthologic and that $[\,\tau,\, \sigma_O^\lambda\,] = H$. On the other hand, the perpendicular $p(O^A(\lambda),\, BC)$ from $O^A(\lambda)$ onto BC has equation

$$\lambda\, a\, d_{2a}\, k_a\, x + b\,(\lambda\, d_{2a}\, k_a - T\, k_b)\, y + c\,(\lambda\, d_{2a}\, k_a + T\, k_c)\, z = 0.$$

It follows that $[\,\sigma_O^\lambda,\, \tau\,]$ (= the intersection of $p(O^A(\lambda),\, BC)$ and $p(O^B(\lambda),\, CA)$) is the point $\frac{T\, k_b\, k_c + (2\, a^6 - z_{2a}\, a^4 - z_{2a}\, d_{2a}^2)\, \lambda}{a}$. This point traces the line with equation $\sum\, a\, d_{2a}\, k_a^2\, x = 0$. One can easily check that the points H and K lie on it. □

Theorem 5. *The homology axis of τ and σ_H^λ envelopes the parabola with directrix the line HK and focus the central point X_{112}.*

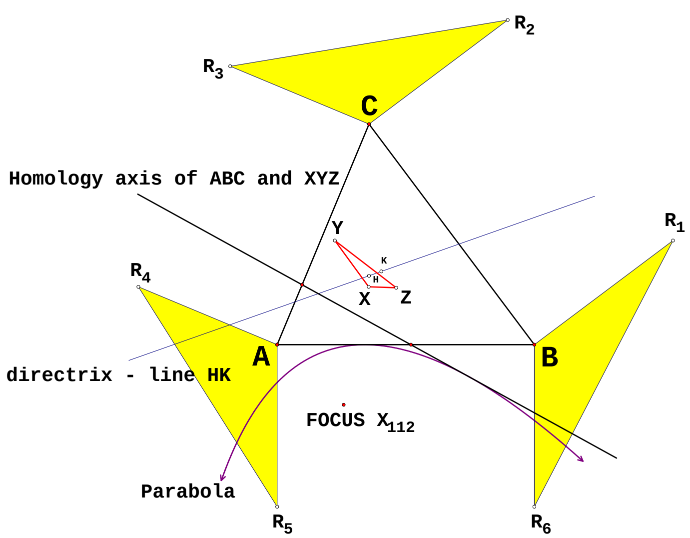


Figure 4. The homology axis of triangles $\sigma_H^\lambda = XYZ$ and $\tau = ABC$ envelopes the parabola with directrix HK and focus X_{112}.

Proof. The orthocenter $H^A(\lambda)$ of the flank AR_4R_5 is $\frac{T-2\lambda k_a}{a\,k_a} : \frac{\lambda}{b} : \frac{\lambda}{c}$. The line $H^B(\lambda)H^C(\lambda)$ has equation

$$a\,(3\,k_b\,k_c\,\lambda^2 - 4\,a^2\,T\,\lambda + T^2)\,x + b\,\lambda\,k_b\,(3\,k_c\,\lambda - T)\,y + c\,\lambda\,k_c\,(3\,k_b\,\lambda - T)\,z = 0.$$

It intersects the sideline BC in the point $0 : \frac{k_c\,(T-3\,k_b\,\lambda)}{b} : \frac{k_b\,(3\,k_c\,\lambda - T)}{c}$. We infer that the homology axis of the triangles τ and σ_H^λ has equation

$$\sum a\,k_a\,(9\,k_b\,k_c\,\lambda^2 - 6\,a^2\,T\,\lambda + T^2)\,x = 0.$$

It envelopes the conic with equation

$$\sum (a^2\,d_{2a}^2\,k_a^2\,x^2 - 2\,m_a\,d_{2b}\,d_{2c}\,k_b\,k_c\,y\,z) = 0.$$

It is easy to check that the above is an equation of a parabola because it intersects the line at infinity $\sum a\,x = 0$ only at the point $\frac{d_{2a}\,k_a}{a}$. On the other hand, we obtain the same equation when we look for the locus of all points P which are at the same distance from the central point $X_{112}(\frac{a}{d_{2a}\,k_a})$ and from the line HK. Hence, the above parabola has the point X_{112} for focus and the line HK for directrix. □

Theorem 6. *For every real number λ the triangles τ and σ_H^λ are orthologic. The locus of the orthology center $[\tau,\,\sigma_H^\lambda]$ is the Kiepert hyperbola of ABC. The locus of the orthology center $[\sigma_H^\lambda,\,\tau]$ is the line HK.*

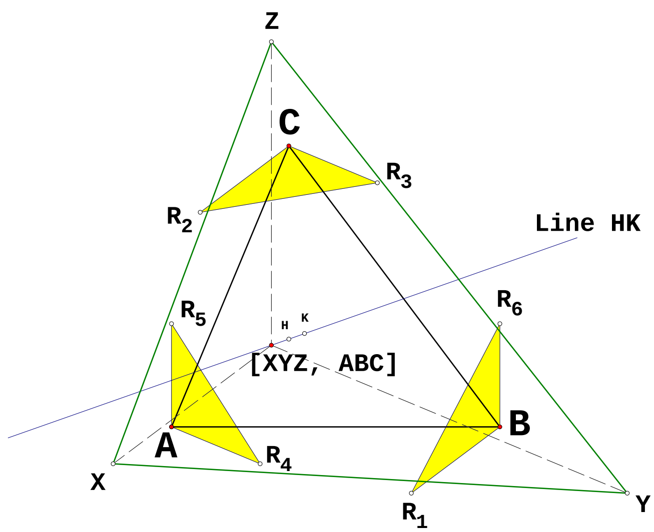


Figure 5. The orthology centers $[\sigma_4^\lambda,\,\tau]$ are on the line HK.

Proof. The perpendicular $p(A,\,H^B(\lambda)H^C(\lambda))$ from A onto the line $H^B(\lambda)H^C(\lambda)$ has equation $M_-(b,\,c)\,y - M_+(c,\,b)\,z = 0$, where

$$M_\pm(b,\,c) = b\,[(3\,a^4 \pm 2\,d_{2a}\,a^2 \pm d_{2a}(b^2 + 3\,c^2))\,\lambda - k_b\,T\,].$$

The lines $p(A,\, H^B(\lambda)H^C(\lambda))$, $p(B,\, H^C(\lambda)H^A(\lambda))$, and $p(C,\, H^A(\lambda)H^B(\lambda))$ concur at the point $\frac{1}{a\,[(a^4+2\,z_{2a}\,a^2-2\,m_{2a}-3\,z_{4a})\,\lambda+k_a\,T\,]}$. Just as in the proof of Theorem 2 we can show that this point traces the Kiepert hyperbola of ABC.

The perpendicular $p(H^A(\lambda),\, BC)$ from $H^A(\lambda)$ onto BC has equation

$$2\,\lambda\,a\,d_{2a}\,k_a\,x+b\,(2\,d_{2a}\,k_a\,\lambda+k_b\,T)\,y+c\,(2\,d_{2a}\,k_a\,\lambda-k_c\,T)\,z=0.$$

The lines $p(H^A(\lambda),\, BC)$, $p(H^B(\lambda),\, CA)$, and $p(H^C(\lambda),\, AB)$ concur at the point $\frac{2\,(2\,a^6-z_{2a}\,a^4-z_{2a}\,d_{2a}^2)\,\lambda-k_b\,k_c\,T}{a}$. We infer that the orthology center $[\sigma_H^\lambda,\, \tau]$ traces the line HK because we get its equation by eliminating the parameter λ from the equations $x=x_0$, $y=y_0$, and $z=z_0$, where x_0, y_0, and z_0 are the trilinears of $[\sigma_H^\lambda,\, \tau]$. □

Theorem 7. *For every $\lambda\in\mathbb{R}\setminus\{0\}$, the triangles ABC and $F^A(\lambda)F^B(\lambda)F^C(\lambda)$ are homologic if and only if the triangle ABC is isosceles.*

Proof. The center $F^A(\lambda)$ of the nine-point circle of the flank AR_4R_5 is

$$\frac{(k_a-a^2)\,\lambda-2\,T}{a}\ :\ \frac{\lambda\,d_{2b}}{-b}\ :\ \frac{\lambda\,d_{2c}}{c}.$$

The line $AF^A(\lambda)$ has equation $b\,d_{2c}\,y+c\,d_{2b}\,z=0$. Hence, the condition for these three lines to concur (expressed in terms of the side lengths) is $2\,m\,d_{2a}\,d_{2b}\,d_{2c}=0$, which immediately implies our claim. □

When triangle ABC is scalene and isosceles, one can show easily that the homology center of ABC and $F^A(\lambda)F^B(\lambda)F^C(\lambda)$ is the midpoint of the base while the homology axis envelopes again the Kiepert parabola of ABC (which agrees with the line parallel to the base through the opposite vertex).

The following two theorems have the same proofs as Theorem 6 and Theorem 1, respectively.

Theorem 8. *For every real number λ the triangles ABC and $F^A(\lambda)F^B(\lambda)F^C(\lambda)$ are orthologic. The orthology centers $[\,\sigma_F^\lambda,\, \tau\,]$ and $[\,\tau,\, \sigma_F^\lambda\,]$ trace the line HK and the Kiepert hyperbola, respectively.*

Theorem 9. *The homology axis of the triangles ABC and $K^A(\lambda)K^B(\lambda)K^C(\lambda)$ envelopes the Kiepert parabola of ABC.*

Theorem 10. *For every $\lambda\in\mathbb{R}\setminus\{0\}$, the triangles ABC and $K^A(\lambda)K^B(\lambda)K^C(\lambda)$ are orthologic if and only if the triangle ABC is isosceles.*

Proof. The symmedian point $K^A(\lambda)$ of the flank AR_4R_5 is

$$\frac{(d_{2a}^2-a^2\,z_{2a})\lambda-T\,(3\,k_a+2\,a^2)}{a}\ :\ \lambda\,b\,k_b\ :\ \lambda\,c\,k_c.$$

It follows that the perpendicular $p(K^A(\lambda),\, BC)$ from $K^A(\lambda)$ to BC has equation

$$\lambda\,a\,d_{2a}\,T\,x+b\,(\lambda\,d_{2a}\,T-k_b\,(3\,k_a+2\,a^2))\,y+c\,(\lambda\,d_{2a}\,T+k_c\,(3\,k_a+2\,a^2))\,z=0.$$

The triangles ABC and $K^A(\lambda)K^B(\lambda)K^C(\lambda)$ are orthologic if and only if the coefficient determinant of the equations of the lines $p(K^A(\lambda),\, BC)$, $p(K^B(\lambda),\, CA)$, and $p(K^C(\lambda),\, AB)$ is zero. But, this determinant is equal to $-16\lambda m d_{2a} d_{2b} d_{2c} T^6$, which immediately implies that our claim is true. □

When the triangle ABC is scalene and isosceles one can show easily that the orthology centers of ABC and $K^A(\lambda)K^B(\lambda)K^C(\lambda)$ both trace the perpendicular bisector of the base.

The proofs of the following two theorems are left to the reader because they are analogous to proofs of Theorem 1 and Theorem 6, respectively. However, the expressions that appear in them are considerably more complicated.

Theorem 11. *The homology axis of τ and σ_x^λ envelopes the Kiepert parabola of ABC for $x = 15,\ 16,\ 61,\ 62$.*

Theorem 12. *For every real number λ the triangles τ and σ_L^λ are orthologic. The loci of the orthology centers $[\,\tau,\, \sigma_L^\lambda\,]$ and $[\,\sigma_L^\lambda,\, \tau\,]$ are the Kiepert hyperbola and the line HK, respectively.*

References

[1] R. H. Eddy and R. Fritsch, The conics of Ludwig Kiepert: A comprehensive lesson in the geometry of the triangle, *Math. Magazine*, 67 (1994), 188–205.

[2] L. Hoehn, Extriangles and excevians, *Math. Magazine*, 74 (2001), 384–388.

[3] C. Kimberling, Central points and central lines in the plane of a triangle, *Math. Magazine*, 67 (1994) 163–187.

[4] C. Kimberling, *Encyclopedia of Triangle Centers*, http://www2.evansville.edu/ck6/encyclopedia/.

[5] F. M. van Lamoen, Friendship among triangle centers, *Forum Geom.*, 1 (2001) 1 – 6.

[6] C. R. Pranesachar and B. J. Venkatachala, On a Curious Duality in Triangles, *Samasyā*, 7 (2001), number 2, 13–19.

Zvonko Čerin: Kopernikova 7, 10010 Zagreb, Croatia

E-mail address: cerin@math.hr

Forum Geometricorum
Volume 2 (2002) 115–128.

FORUM GEOM
ISSN 1534-1178

Napoleon-like Configurations and Sequences of Triangles

Barukh Ziv

Abstract. We consider the sequences of triangles where each triangle is formed out of the apices of three similar triangles built on the sides of its predecessor. We show under what conditions such sequences converge in shape, or are periodic.

1. Introduction

The well-known geometrical configuration consisting of a given triangle and three equilateral triangles built on its sides, all outwardly or inwardly, has many interesting properties. The most famous is the theorem attributed to Napoleon that the centers of the three equilateral triangles built on the sides are vertices of another equilateral triangle [3, pp. 60–65]. Numerous works have been devoted to this configuration, including various generalizations [6, 7, 8] and converse problems [10].

Some authors [5, 9, 1] considered the iterated configurations where construction of various geometrical objects (e.g. midpoints) on the sides of polygons is repeated an arbitrary number of times. Douglass [5] called such constructions *linear polygon transformations* and showed their relation with circulant matrices. In this paper, we study the sequence of triangles obtained by a modification of such a configuration. Each triangle in the sequence is called a *base* triangle, and is obtained from its predecessor by two successive transformations: (1) the classical construction on the sides of the base triangle triangles similar to a given (*transformation*) triangle and properly oriented, (2) a normalization which is a direct similarity transformation on the apices of these new triangles so that one of the images lies on a fixed circle. The three points thus obtained become the vertices of the new base triangle. The normalization step is the feature that distinguishes the present paper from earlier works, and it gives rise to interesting results. The main result of this study is that under some general conditions the sequence of base triangles converges to an equilateral triangle (in a sense defined at the end of §2). When the limit does not exist, we study the conditions for periodicity. We study two types of sequences of triangles: in the first, the orientation of the transformation triangle is given a priori; in the second, it depends on the orientation of the base triangle.

Publication Date: October 4, 2002. Communicating Editor: Floor van Lamoen.
The author is grateful to Floor van Lamoen and the referee for their valuable suggestions in improving the completeness and clarity of the paper.

The rest of the paper is organized as follows. In §2, we explain the notations and definitions used in the paper. In §3, we give a formal definition of the transformation that generates the sequence. In §4, we study the first type of sequences mentioned above. In §5, we consider the exceptional case when the transformation triangle degenerates into three collinear points. In §6, we consider the second type of sequences mentioned above. In §7, we study a generalization for arbitrary polygons.

2. Terminology and definitions

We adopt the common notations of complex arithmetic. For a complex number z, Re(z) denotes its real part, Im(z) its imaginary part, $|z|$ its modulus, arg(z) its argument (chosen in the interval $(-\pi, \pi]$), and $\overline{z}$ its conjugate. The primitive complex n-th root of unity $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, is denoted by ζ_n. Specifically, we write $\omega = \zeta_3$ and $\eta = \zeta_6$. The important relation between the two is $\omega^2 + \eta = 0$.

A triangle is oriented if an ordering of its vertices is specified. It is positively (negatively) oriented if the vertices are ordered counterclockwise (clockwise). Two oriented triangles are directly (oppositely) similar if they have the same (opposite) orientation and pairs of corresponding vertices may be brought into coincidence by similarity transformations.

Throughout the paper, we coordinatized points in a plane by complex numbers, using the same letter for a point and its complex number coordinate. An oriented triangle is represented by an ordered triple of complex numbers. To obtain the orientation and similarity conditions, we define the following function $z : \mathbb{C}^3 \to \mathbb{C}$ on the set of all vectors $V = (A, B, C)$ by

$$z[V] = z(A, B, C) = \frac{C - A}{B - A}. \tag{1}$$

Triangle ABC is positively or negatively oriented according as arg($z(A, B, C)$) is positive or negative. Furthermore, every complex number z defines a class of directly similar oriented triangles ABC such that $z(A, B, C) = z$. In particular, if ABC is a positively (respectively negatively) oriented equilateral triangle, then $z(A, B, C) = \eta$ (respectively $\overline{\eta}$).

Finally, we define the convergence of triangles. An infinite sequence of triangles $(A_nB_nC_n)$ converges to a triangle ABC if the sequence of complex numbers $z(A_n, B_n, C_n)$ converges to $z(A, B, C)$.

3. The transformation f

We describe the transformations that generate the sequence of triangles we study in the paper. We start with a base triangle $A_0B_0C_0$ and a transformation triangle XYZ. Let G be the centroid of $A_0B_0C_0$, and Γ the circle centered at G and passing through the vertex farthest from G. (Figure 1a). For every $n > 0$, triangle $A_nB_nC_n$ is obtained from its predecessor $A_{n-1}B_{n-1}C_{n-1}$ by $f = f_2 \circ f_1$, where

(i) f_1 maps $A_{n-1}B_{n-1}C_{n-1}$ to $A'_nB'_nC'_n$, by building on the sides of triangle $A_{n-1}B_{n-1}C_{n-1}$, three triangles $B_{n-1}C_{n-1}A'_n$, $C_{n-1}A_{n-1}B'_n$, $A_{n-1}B_{n-1}C'_n$ similar to XYZ and all with the same orientation, [1] (Figure 1b);

(ii) f_2 transforms by similarity with center G the three points A'_n, B'_n, C'_n so that the image of the farthest point lies on the circle Γ, (Figure 1c).

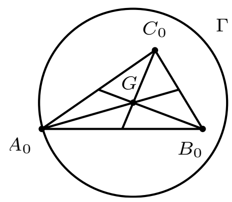


Figure 1a

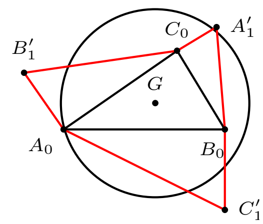


Figure 1b

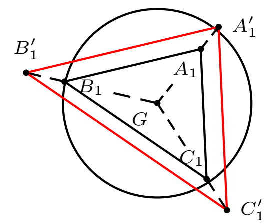


Figure 1c

The three points so obtained are the vertices of the next base triangle $A_nB_nC_n$. We call this the *normalization* of triangle $A'_nB'_nC'_n$. In what follows, it is convenient to coordinatize the vertices of triangle $A_0B_0C_0$ so that its centroid G is at the origin, and Γ is the unit circle. In this setting, normalization is simply division by

$$r_n = \max(|A'_n|, |B'_n|, |C'_n|).$$

It is easy to see that f may lead to a degenerate triangle. Figure 2 depicts an example of a triple of collinear points generated by f_1. Nevertheless, f is well defined, except only when $A_{n-1}B_{n-1}C_{n-1}$ degenerates into the point G. But it is readily verified that this happens only if triangle $A_{n-1}B_{n-1}C_{n-1}$ is equilateral, in which case we stipulate that $A_nB_nC_n$ coincides with $A_{n-1}B_{n-1}C_{n-1}$.

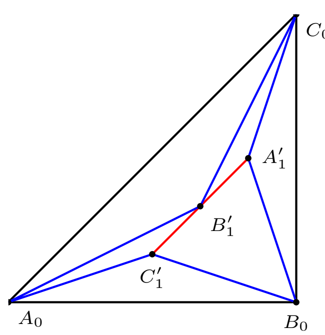


Figure 2

The normalization is a crucial part of this transformation. While preserving direct similarity of the triangles $A'_nB'_nC'_n$ and $A_nB_nC_n$, it prevents the latter from

[1] We deliberately do not specify the orientation of those triangles with respect to the transformation triangle, since they are specific for the different types of sequences we discuss later in this paper.

converging to a single point or diverging to infinity (since every triangle after normalization lies inside a fixed circle, and at least one of its vertices lies on the circle), and the convergence of triangles receives a definite geometrical meaning. Also, since f_1 and f_2 leave G fixed, we have a rather expected important property that G is a fixed point of the transformation.

4. The first sequence

We first keep the orientation of the transformation triangle fixed and independent from the base triangle.

Theorem 1. *Let $A_0B_0C_0$ be an arbitrary base triangle, and XYZ a non-degenerate transformation triangle. The sequence $(A_nB_nC_n)$ of base triangles generated by the transformation f in §3 (with $B_{n-1}C_{n-1}A'_n$, $C_{n-1}A_{n-1}B'_n$, $A_{n-1}B_{n-1}C'_n$ directly similar to XYZ) converges to the equilateral triangle with orientation opposite to XYZ, except when $A_0B_0C_0$, and the whole sequence, is equilateral with the same orientation as XYZ.*

Proof. Without loss of generality let XYZ be positively oriented. We treat the special cases first. The exceptional case stated in the theorem is verified straightforwardly; also it is obvious that we may exclude the cases where $A_nB_nC_n$ is positively oriented equilateral. Hence in what follows it is assumed that $z(A_0, B_0, C_0) \neq \eta$, and $r_n \neq 0$ for every n.

Let $z(X, Y, Z) = t$. Since for every n, triangle $B_{n-1}C_{n-1}A'_n$ is directly similar to XYZ, by (1)

$$A'_n = (1-t)B_{n-1} + tC_{n-1},$$

and similarly for B'_n and C'_n. After normalization,

$$V_n = \frac{1}{r_n} T V_{n-1}, \tag{2}$$

where $V_n = \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix}$, T is the circulant matrix $\begin{pmatrix} 0 & 1-t & t \\ t & 0 & 1-t \\ 1-t & t & 0 \end{pmatrix}$, and $r_n = \max(|A'_n|, |B'_n|, |C'_n|)$. By induction,

$$V_n = \frac{1}{r_1 \cdots r_n} T^n V_0.$$

We use the standard diagonalization procedure to compute the powers of T. Since T is circulant, its eigenvectors are the columns of the Fourier matrix ([4, pp.72–73])

$$F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 \\ \omega^0 & \omega^2 & \omega^4 \end{pmatrix},$$

and the corresponding eigenvalues are λ_0, λ_1, λ_2 are [2]

$$\lambda_j = (1-t)\omega^j + t\omega^{2j}, \tag{3}$$

[2]Interestingly enough, ordered triples $(\omega, \omega^2, \lambda_1)$ and $(\omega^2, \omega, \lambda_2)$ form triangles directly similar to XYZ.

for $j = 0, 1, 2$. With these, we have

$$T = F_3 U F_3^{-1},$$

where U is the diagonal matrix $\begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$.

Let $S = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix}$ be a vector of points in the complex plane that is transformed into V_0 by the Fourier matrix, *i.e.*,

$$V_0 = F_3 S. \tag{4}$$

Since $A_0 + B_0 + C_0 = 3G = 0$, we get $s_0 = 0$, and

$$V_0 = s_1 F_{3,1} + s_2 F_{3,2}, \tag{5}$$

where $F_{3,j}$ is the j-th column of F_3. After n iterations,

$$V_n \sim T^n V_0 = F_3 U^n F_3^{-1}(s_1 F_{3,1} + s_2 F_{3,2}) = s_1 \lambda_1^n F_{3,1} + s_2 \lambda_2^n F_{3,2}. \tag{6}$$

According to (3) and the assumption that XYZ is negatively oriented,

$$|\lambda_2|^2 - |\lambda_1|^2 = \lambda_2 \overline{\lambda_2} - \lambda_1 \overline{\lambda_1} = 2\sqrt{3}\mathrm{Im}(t) < 0,$$

so that $\dfrac{|\lambda_2|}{|\lambda_1|} < 1$, and $\dfrac{|\lambda_2^n|}{|\lambda_1^n|} \to 0$ when $n \to \infty$. Also, expressing $z(A_0, B_0, C_0)$ in terms of s_1, s_2, we get

$$z(A_0, B_0, C_0) = \frac{s_1 \eta + s_2}{s_1 + s_2 \eta}, \tag{7}$$

so that $z(A_0, B_0, C_0) \neq \overline{\eta}$ implies $s_1 \neq 0$. Therefore,

$$\lim_{n\to\infty} z(A_n, B_n, C_n) = \lim_{n\to\infty} z[V_n] = z[F_{3,1}] = \eta.$$

□

Are there cases when the sequence converges after a finite number of iterations? Because the columns of the Fourier matrix F_3 are linearly independent, this may happen if and only if the second term in (6) equals 0. There are two cases:

(i) $s_2 = 0$: this, according to (7), corresponds to a base triangle $A_0 B_0 C_0$ which is equilateral and positively oriented;

(ii) $\lambda_2 = 0$: this, according to (3), corresponds to a transformation triangle XYZ which is isosceles with base angle $\frac{\pi}{6}$. In this case, one easily recognizes the triangle of the Napoleon theorem.

We give a geometric interpretation of the values s_1, s_2. Changing for a while the coordinates of the complex plane so that A_0 is at the origin, we get from (4):

$$|s_1| = |B_0 - C_0 \eta|, \qquad |s_2| = |B_0 - C_0 \overline{\eta}|,$$

and we have the following construction: On the side $A_0 C_0$ of the triangle $A_0 B_0 C_0$ build two oppositely oriented equilateral triangles (Figure 3), then $|s_1| = B_0 B'$,

$|s_2| = B_0B''$. After some computations, we obtain the following symmetric formula for the ratio $\frac{s_1}{s_2}$ in terms of the angles α, β, γ of triangle $A_0B_0C_0$:

$$\left|\frac{s_1}{s_2}\right|^2 = \frac{\sin\alpha\sin(\alpha+\frac{\pi}{3}) + \sin\beta\sin(\beta+\frac{\pi}{3}) + \sin\gamma\sin(\gamma+\frac{\pi}{3})}{\sin\alpha\sin(\alpha-\frac{\pi}{3}) + \sin\beta\sin(\beta-\frac{\pi}{3}) + \sin\gamma\sin(\gamma-\frac{\pi}{3})}.$$

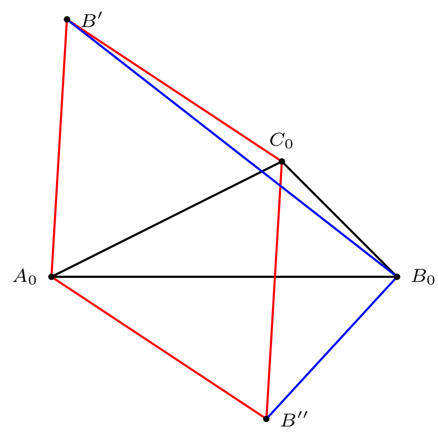


Figure 3

5. An exceptional case

In this section we consider the case t is a real number. Geometrically, it means that the transformation triangle XYZ degenerates into a triple of collinear points, so that A'_n, B'_n, C'_n divide the corresponding sides of triangle $A_{n-1}B_{n-1}C_{n-1}$ in the ratio $1-t:t$. (Figure 4 depicts an example for $t=\frac{1}{3}$). Can the sequence of triangles still converge in this case? To settle this question, notice that when t is real, λ_1 and λ_2 are complex conjugates, and rewrite (6) as follows:

$$V_n \sim \lambda_1^n \left(s_1F_{3,1} + \frac{\lambda_2^n}{\lambda_1^n} s_2F_{3,2} \right), \tag{8}$$

and because $\dfrac{\lambda_2}{\lambda_1}$ defines a *rotation*, it is clear that it does not have a limit unless $\dfrac{\lambda_2}{\lambda_1} = 1$, in which case the sequence consists of directly similar triangles. Now, $\lambda_1 = \lambda_2$ implies $t = \frac{1}{2}$, so we have the well-known result that the triangle is similar to its medial triangle [3, p. 19].

Next, we find the conditions under which the sequence has a finite period m. Geometrically, it means that m is the least number such that triangles $A_nB_nC_n$ and $A_{n+m}B_{n+m}C_{n+m}$ are directly similar for every $n \geq 0$. The formula (8) shows that it happens when $\dfrac{\lambda_2}{\lambda_1} = \zeta_m^k$, and k, m are co-prime. Plugging this into

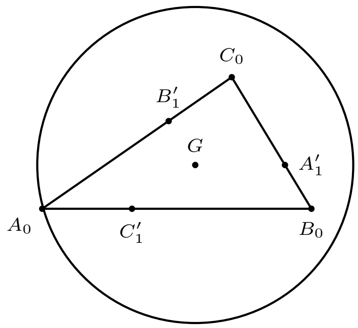


Figure 4a

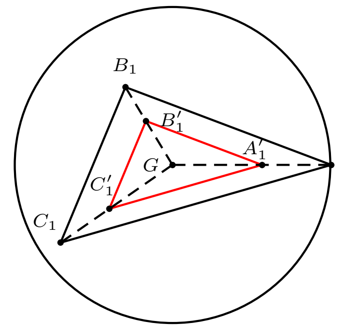


Figure 4b

(3) and solving for t, we conclude that the sequence of triangles with period m exists for t of the form

$$t(m) = \frac{1}{2} + \frac{1}{2\sqrt{3}} \tan \frac{k\pi}{m}. \tag{9}$$

Several observations may be made from this formula. First, the periodic sequence with finite t exists for every $m \neq 2$. (The case $m = 2$ corresponds to transformation triangle with two coinciding vertices X, Y). The number of different sequences of a given period m is $\phi(m)$, Euler's totient function [2, pp.154–156]. Finally, the case $m = 1$ yields $t = \frac{1}{2}$, which is the case of medial triangles.

Also, several conclusions may be drawn about the position of corresponding triangles in a periodic sequence. Comparing (8) with (5), we see that triangle $A_mB_mC_m$ is obtained from triangle $A_0B_0C_0$ by a rotation about their common centroid G through angle $m \cdot \arg(\lambda_1)$. Because 2arg(λ_1) = 0 $\mathrm{mod} 2\pi$, it follows that $A_mB_mC_m$ coincides with $A_0B_0C_0$, or is a half-turn. In both cases, the triangle $A_{2m}B_{2m}C_{2m}$ will always coincide with $A_0B_0C_0$. We summarize these results in the following theorem.

Theorem 2. *Let a triangle $A_0B_0C_0$ and a real number t be given. The sequence $(A_nB_nC_n)$ of triangles constructed by first dividing the sides of each triangle in the ratio $1 - t : t$ and then normalizing consists of similar triangles with period m if and only if t satisfies (9) for some k relatively prime to m. In this case, triangles $A_nB_nC_n$ and $A_{n+2m}B_{n+2m}C_{n+2m}$ coincide for every $n \geq 0$. In all other cases the sequence never converges, unless $A_0B_0C_0$, and hence every $A_nB_nC_n$, is equilateral.*

6. The second sequence

In this section we study another type of sequence, where the orientation of transformation triangles depends on the base triangle. More precisely, we consider two cases: when triangles built on the sides of the base triangle are oppositely or equally oriented to it. The main results of this section will be derived using the following important lemma.

Lemma 3. *Let ABC be any triangle, and $V = (A, B, C)$ the corresponding vector of points in the complex plane with the centroid of ABC at the origin. If S is defined as in (4), then ABC is positively (negatively) oriented when $|s_1| > |s_2|$ ($|s_1| < |s_2|$), and A, B, C are collinear if $|s_1| = |s_2|$.*

Proof. According to (7), Im$(z(A, B, C)) \sim s_1\overline{s_1} - s_2\overline{s_2} = |s_1|^2 - |s_2|^2$. □

Before proceeding, we extend notations. As the orientation of the transformation triangle may change throughout the sequence, $z(X, Y, Z)$ equals t or $\overline{t}$, depending on the orientation of the base triangle. So, for the transformation matrix we shall use the notation $T(t)$ or $T(\overline{t})$ accordingly. Note that if the eigenvalues of $T(t)$ are λ_0, λ_1, λ_2, then the eigenvalues of $T(\overline{t})$ are $\overline{\lambda_0}$, $\overline{\lambda_1}$, $\overline{\lambda_2}$. The first result concerning the case of the oppositely oriented triangles is as follows.

Theorem 4. *Let $A_0B_0C_0$ be the base triangle, and XYZ the transformation triangle. If the sequence of triangles $A_nB_nC_n$ is generated as described in §3 with every triangle $B_{n-1}C_{n-1}A'_n$ etc. oppositely oriented to $A_{n-1}B_{n-1}C_{n-1}$, then the sequence converges to the equilateral triangle that has the same orientation as $A_0B_0C_0$.*

Proof. Without loss of generality, we may assume $A_0B_0C_0$ positively oriented. It is sufficient to show that triangle $A_nB_nC_n$ is positively oriented for every n. Then, every triangle $B_{n-1}C_{n-1}A'_n$ etc. is negatively oriented, and the result follows immediately from Theorem 1.

We shall show this by induction. Assume triangles $A_0B_0C_0, \ldots, A_{n-1}B_{n-1}C_{n-1}$ are positively oriented, then they all are the base for the *negatively* oriented directly similar triangles to build their successors, so arg$(t) < 0$, and $|\lambda_1^n| > |\lambda_2^n|$. Also, $|s_1| > |s_2|$, and according to (6) and the above lemma, $A_nB_nC_n$ is positively oriented. □

We proceed with the case when triangles are constructed with the same orientation of the base triangle. In this case, the behavior of the sequence turns out to be much more complicated. Like in the first case, assume $A_0B_0C_0$ positively oriented. If $s_2 = 0$, which corresponds to the equilateral triangle, then all triangles $A_nB_nC_n$ are positively oriented and, of course, equilateral. Otherwise, because arg$(t) > 0$, and therefore $|\lambda_1| < |\lambda_2|$, it follows that $|s_1\lambda_1^n| - |s_2\lambda_2^n|$ eventually becomes negative, and the sequence of triangles changes the orientation. Specifically, it happens exactly after ℓ steps, where

$$\ell = \left\lceil \frac{\ln \frac{s_2}{s_1}}{\ln \frac{\lambda_1}{\lambda_2}} \right\rceil . \tag{10}$$

What happens next? We know that $A_\ell B_\ell C_\ell$ is the first negatively oriented triangle in the sequence, therefore triangles $B_\ell C_\ell A'_{\ell+1}$ etc. built on its sides are also negatively oriented. Thus, $z(B_\ell, C_\ell, A'_{\ell+1}) = \overline{t}$. Therefore, according to (3) and

(6),

$$V_{\ell+1} \sim T(t)^{\ell} T(\bar{t}) V_0 = s_1 \lambda_1^{\ell} \overline{\lambda_2} F_{3,1} + s_2 \lambda_2^{\ell} \overline{\lambda_1} F_{3,2}.$$

Since

$$|s_1 \lambda_1^{\ell} \overline{\lambda_2}| = |s_1 \lambda_1^{\ell-1}||\lambda_1 \lambda_2| > |s_2 \lambda_2^{\ell-1}||\lambda_1 \lambda_2| = |s_2 \lambda_2^{\ell} \overline{\lambda_1}|,$$

triangle $A_{\ell+1}B_{\ell+1}C_{\ell+1}$ is positively oriented. Analogously, we get that for $n \geq 0$, every triangle $A_{\ell+2n}B_{\ell+2n}C_{\ell+2n}$ is negatively oriented, while its successor $A_{\ell+2n+1}B_{\ell+2n+1}C_{\ell+2n+1}$ is positively oriented.

Consider now the sequence $(A_{\ell+2n}B_{\ell+2n}C_{\ell+2n})$ consisting of negatively oriented triangles. Clearly, the transformation matrix for this sequence is the product of $T(t)$ and $T(\bar{t})$, which is a circulant matrix

$$\begin{pmatrix} t+\bar{t}-2t\bar{t} & t\bar{t} & 1-t-\bar{t}+t\bar{t} \\ 1-t-\bar{t}+t\bar{t} & t+\bar{t}-2t\bar{t} & t\bar{t} \\ t\bar{t} & 1-t-\bar{t}+t\bar{t} & t+\bar{t}-2t\bar{t} \end{pmatrix}$$

with eigenvalues

$$\lambda_j' = t+\bar{t}-2t\bar{t}+t\bar{t}\omega^j + (1-t-\bar{t}+t\bar{t})\omega^{2j}, \qquad j=0,1,2. \tag{11}$$

Since this matrix is real, the sequence $(A_{\ell+2n}B_{\ell+2n}C_{\ell+2n})$ of triangles does not converge. It follows at once that the sequence $(A_{\ell+2n+1}B_{\ell+2n+1}C_{\ell+2n+1})$ of successors does not converge either.

Finally, we consider the conditions when these two sequences are periodic. Clearly, only even periods $2m$ may exist. In this case, λ_1' and λ_2' must satisfy $\dfrac{\lambda_1'}{\lambda_2'} = \zeta_m^k$ for k relatively prime to m. Since λ_1', λ_2' are complex conjugates, this is equivalent to arg$(\lambda_1') = \frac{k\pi}{m}$. Applying (11), we arrive at the following condition:

$$\tan\frac{k\pi}{m} = \frac{1}{\sqrt{3}} \cdot \frac{\operatorname{Re}(t)-\frac{1}{2}}{\operatorname{Re}(t)-|t|^2-\frac{1}{6}}.$$

Several interesting properties about periodic sequences may be derived from this formula. First, for a given pair of numbers k, m, the locus of t is a *circle* centered at the point O on a real axis, and radius R defined as follows:

$$O(m) = \frac{1}{2} - \frac{1}{2\sqrt{3}}\cot\frac{k\pi}{m}, \qquad R(m) = \frac{1}{2\sqrt{3}}\csc\frac{k\pi}{m}. \tag{12}$$

Furthermore, all the circles have the two points $\frac{1}{3}(1+\overline{\eta})$ and $\frac{1}{3}(1+\eta)$ in common. This is clear if we note that they correspond to the cases $\lambda_1' = 0$ and $\lambda_2' = 0$ respectively, *i.e.*, when the triangle becomes equilateral after the first iteration (see the discussion following Theorem 1 in §3).

Summarizing, we have the following theorem.

Theorem 5. *Let $A_0B_0C_0$ be the base triangle, and XYZ the transformation triangle. The sequence $(A_nB_nC_n)$ of triangles constructed by the transformation f ($B_{n-1}C_{n-1}A_n'$, $C_{n-1}A_{n-1}B_n'$, $A_{n-1}B_{n-1}C_n'$ with the same orientation of $A_{n-1}B_{n-1}C_{n-1}$) converges only if $A_0B_0C_0$ is equilateral (and so is the whole sequence). Otherwise the orientation of $A_0B_0C_0$ is preserved for first $\ell-1$ iterations, where ℓ is determined by (10); afterwards, it is reversed in each iteration.*

The sequence consists of similar triangles with an even period $2m$ *if and only if* $t = z(X, Y, Z)$ *lies on a circle* $O(R)$ *defined by (12) for some* k *relatively prime to* m*. In this case, triangles* $A_n B_n C_n$ *and* $A_{n+4m} B_{n+4m} C_{n+4m}$ *coincide for every* $n \geq \ell$*.*

We conclude with a demonstration of the last theorem's results. Setting $m = 1$ in (12), both O and R tend to infinity, and the circle degenerates into line $\mathrm{Re}(t) = \frac{1}{2}$, that corresponds to any isosceles triangle. Figures 5a through 5d illustrate this case when XYZ is the right isosceles triangle, and $A_0 B_0 C_0$ is also isosceles positively oriented with base angle $\frac{3\pi}{8}$. According to (10), $\ell = 2$. Indeed, $A_2 B_2 C_2$ is the first negatively oriented triangle in the sequence, $A_3 B_3 C_3$ is again positively oriented and similar to $A_1 B_1 C_1$. The next similar triangle $A_5 B_5 C_5$ will coincide with $A_1 B_1 C_1$.

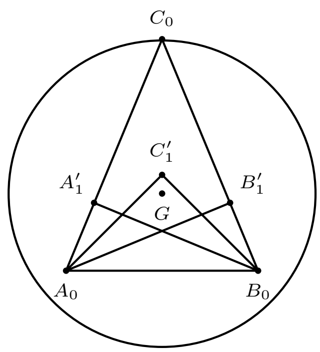


Figure 5a

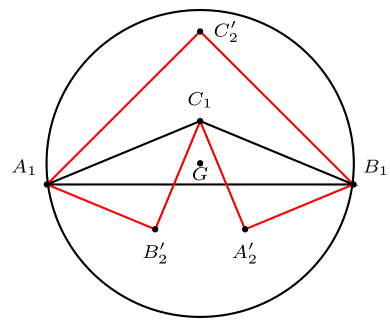


Figure 5b

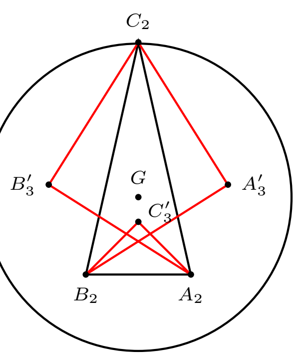


Figure 5c

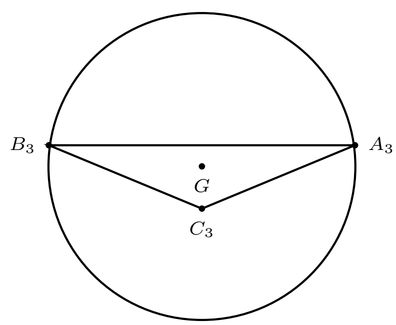


Figure 5d

7. Generalization to polygons

In this section, we generalize the results in §4 by replacing the sequences of triangles by sequences of polygons. The transformation performed at every iteration remains much the same as in §3, with triangles built on every side of the base polygon directly similar to a given transformation triangle. We seek the conditions under which the resulting sequence of polygons converges in shape.

Let the unit circle be divided into k equal parts by the points P_0, P_1, $\ldots$, P_{k-1}. We call the polygon regular k-gon of q-type if it is similar to the polygon $P_0P_q \cdots P_{(k-1)q}$, where the indices are taken modulo k [5, p. 558]. The regular 1-type and $(k-1)$-type polygons are simply the convex regular polygons in an ordinary sense. Other regular k-gons may be further classified into

(i) star-shaped if q, k are co-prime, (for example, a pentagram is a 2-type regular pentagon, Figure 6a), and

(ii) multiply traversed polygons with fewer vertices if q, k have a common divisor, (for example, a regular hexagon of 2-type is an equilateral triangle traversed twice, Figure 6b).

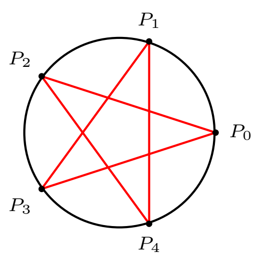


Figure 6a

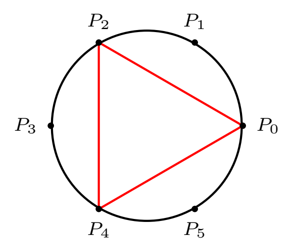


Figure 6b

In general, regular k-gons of q-type and $(k-q)$-type are equally shaped and oppositely oriented. It is also evident that $(-q)$-type and $(k-q)$-type k-gons are identical. We shall show that under certain conditions the sequence of polygons converges to regular polygons so defined.

Let $\Pi_0 = P_{0,0}P_{1,0} \cdots P_{k-1,0}$ be an arbitrary k-gon, and XYZ the non-degenerate transformation triangle, and let the sequence of k-gons $\Pi_n = P_{0,n}P_{1,n} \cdots P_{k-1,n}$ be generated as in §3, with triangles $P_{0,n-1}P_{1,n-1}P'_{0,n}, \ldots P_{k-1,n-1}P_{0,n-1}P'_{k-1,n}$ built on the sides of Π_{n-1} directly similar to XYZ and then normalized. The transformation matrix T_k for such a sequence is a circulant $k \times k$ matrix with the first row

$$\begin{pmatrix} 1-t & t & 0 & \cdots & 0 \end{pmatrix},$$

whose eigenvectors are columns of Fourier matrix

$$F_k = \frac{1}{\sqrt{k}}(\zeta_k^{ij}), \quad i,j = 0,\ldots,k-1,$$

and the eigenvalues:

$$\lambda_i = (1-t) + t\zeta_k^i, \quad i = 0,\ldots,k-1. \tag{13}$$

Put Π_0 into the complex plane so that its centroid $G = \frac{1}{k}\sum_{i=0}^{k-1} P_{i,0}$ is at the origin, and let V_n be a vector of points corresponding to Π_n. If $S = (s_0, \dots, s_{k-1})$ is a vector of points that is transformed into V_0 by Fourier matrix, *i.e.*, $S = \overline{F_k} V_0$, then similar to (6), we get:

$$V_n \sim \sum_{i=0}^{k-1} s_i \lambda_i^n F_{k,i}. \tag{14}$$

Noticing that the column vectors $F_{k,q}$ correspond to regular k-gons of q-type, we have the following theorem:

Theorem 6. *The sequence of k-gons Π_n converges to a regular k-gon of q-type, if and only if $|\lambda_q| > |\lambda_i|$ for every $i \neq q$ such that $s_i \neq 0$.*

As in the case of triangles, we proceed to the cases when the sequence converges after a finite number of iterations. As follows immediately from (14), we may distinguish between two possibilities:

(i) $s_q \neq 0$ and $s_i = 0$ for every $i \neq q$. This corresponds to Π_0 - and the whole sequence - being regular of q-type.

(ii) There are two integers q, r such that $\lambda_r = 0$, $s_q, s_r \neq 0$, and $s_i = 0$ for every $i \neq q, r$. In this case, Π_0 turns into regular k-gon of q-type after the first iteration. An example will be in order here. Let $k = 4$, $q = 1$, $\lambda_2 = 0$ and $S = (0, 1, 1, 0)$. Then, $t = \frac{1}{2}$ and Π_0 is a concave kite-shaped quadrilateral; the midpoints of its sides form a square, as depicted in Figure 7.

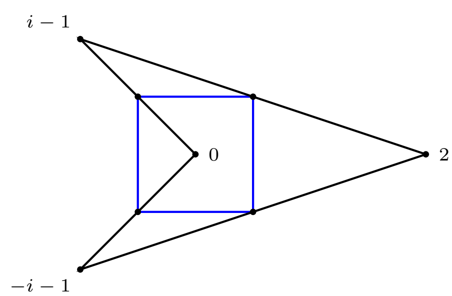


Figure 7

The last theorem shows that the convergence of the sequence of polygons depends on the shapes of both the transformation triangle and the original polygon Π_0. Let us now consider for what transformation triangles the sequence converges for any Π_0? Obviously, this will be the case if no two eigenvalues (13) have equal moduli. That is, for every pair of distinct integers q, r,

$$|(1-t) + t\zeta_k^q| \neq |(1-t) + t\zeta_k^r|.$$

Dividing both sides by $1 - t$, we conclude that $\frac{t}{1-t}\zeta_k^q$ and $\frac{t}{1-t}\zeta_k^r$ should not be complex conjugates, that is:

$$\arg\left(\frac{t}{1-t}\right) \neq -\frac{q+r}{k}\pi, \qquad 0 \leq q, r \leq k. \tag{15}$$

Solving for t and designating ℓ for $(q+r) \bmod k$, we get:

$$\frac{\operatorname{Im}(t)}{\operatorname{Re}(t)-|t|^2} \neq \frac{\ell}{k}\pi, \quad 0 \le \ell < k.$$

This last inequality is given a geometric interpretation in the following final theorem.

Theorem 7. *The sequence of k-gons converges to a regular k-gon for every Π_0 if and only if $t = z(X, Y, Z)$ does not lie on any circle $O(R)$ defined as follows:*

$$O = \left(\frac{1}{2},\ \frac{1}{2}\cot\frac{\ell}{k}\pi\right), \quad R = \frac{1}{2}\csc\frac{\ell}{k}\pi, \qquad 0 \le \ell < k.$$

We conclude with a curious application of the last result. Let $k = 5$, and XYZ be a negatively oriented equilateral triangle, *i.e.*, $t = \overline{\eta}$. It follows from (15) that the sequence of pentagons converges for any given Π_0. Let Π_0 be similar to

$$(1+\epsilon,\ \zeta_5,\ \zeta_5^2,\ \zeta_5^3,\ \zeta_5^4).$$

Taking $\epsilon \neq 0$ sufficiently small, Π_0 may be made as close to the regular convex pentagon as we please. The striking fact is that $q = 2$! Figures 8 depict this transforming of an “almost regular” convex pentagon into an “almost regular” pentagram in just 99 iterations.

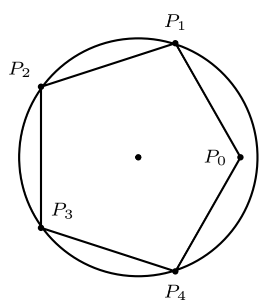


Figure 8a: $n = 0$

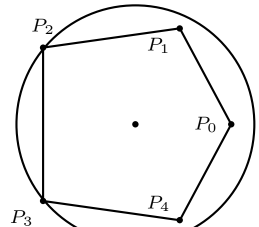


Figure 8b: $n = 20$

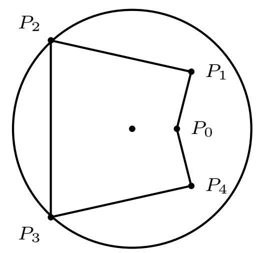


Figure 8c: $n = 40$

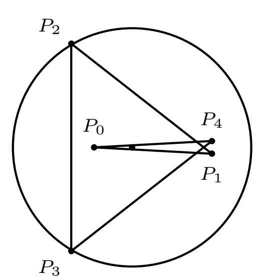


Figure 8d: $n = 60$

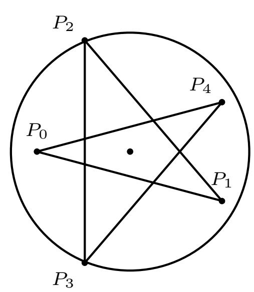


Figure 8e: $n = 80$

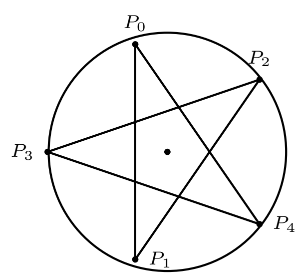


Figure 8f: $n = 99$

References

[1] E. R. Berlekamp, E. N. Gilbert, and F. W. Sinden, A polygon problem, *Amer. Math. Monthly*, 72 (1965) 233–241.
[2] J. H. Conway and R. K. Guy, *The Book of Numbers*, Springer-Verlag, 1996.
[3] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, New Math. Library, vol. 19, Random House and L. W. Singer, NY, 1967; reprinted by Math. Assoc. America.
[4] P. J. Davis, *Circulant matrices*, John Wiley & Sons, NY, 1979.
[5] J. Douglass, On linear polygon transformation, *Bull. Amer. Math. Soc.*, 46 (1940), 551–560.
[6] L. Gerber, Napoleon's theorem and the parallelogram inequality for affine-regular polygons, *Amer. Math. Monthly*, 87 (1980) 644–648.
[7] J. G. Mauldon, Similar triangles, *Math. Magazine*, 39 (1966) 165–174.
[8] J. F. Rigby, Napoleon revisited. *Journal of Geometry*, 33 (1988) 129–146.
[9] I. J. Schoenberg, The finite Fourier series and elementary geometry, *Amer. Math. Monthly*, 57 (1950) 390–404.
[10] J. E. Wetzel, Converses of Napoleon's theorem, *Amer. Math. Monthly*, 99 (1992) 339–351.

Barukh Ziv: Havatzelet 4-5, Nesher 36730, Israel
E-mail address: `zbaruh@yahoo.co.uk`

Forum Geometricorum
Volume 2 (2002) 129–130.

FORUM GEOM
ISSN 1534-1178

An Elementary Proof of the Isoperimetric Inequality

Nikolaos Dergiades

Abstract. We give an elementary proof of the isoperimetric inequality for polygons, simplifying the proof given by T. Bonnesen.

We present an elementary proof of the known inequality $\mathcal{L}^2 \geq 4\pi\mathcal{A}$, where $\mathcal{L}$ and $\mathcal{A}$ are the perimeter and the area of a polygon. It simplifies the proof given by T. Bonnesen [1, 2].

Theorem. *In every polygon with perimeter $\mathcal{L}$ and area $\mathcal{A}$ we have $\mathcal{L}^2 \geq 4\pi\mathcal{A}$.*

Proof. It is sufficient to prove the inequality for a convex polygon $ABM \cdots Z$. From the vertex A of the polygon we can draw the segment AQ dividing the polygon in two polygons such that

(1) $AB + BM + \cdots + PQ = \frac{\mathcal{L}}{2}$, and
(2) the area $\mathcal{A}_1$ of the polygon $ABM \cdots PQA$ satisfies $\mathcal{A}_1 \geq \frac{\mathcal{A}}{2}$.

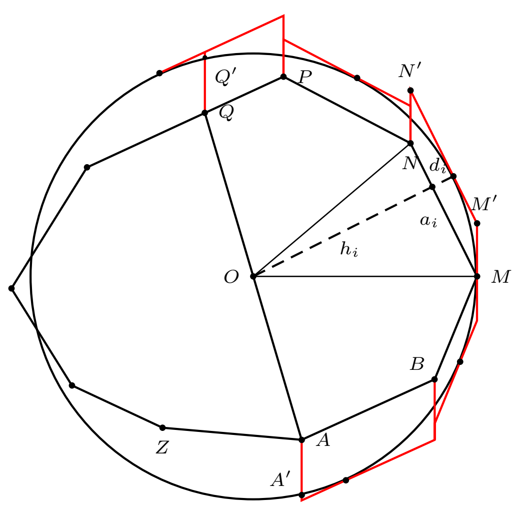


Figure 1

Let O be the mid-point of AQ, and let M be the vertex of $ABM \cdots PQA$ farthest from O, with $OM = \mathcal{R}$. Draw the circle $(O, \mathcal{R})$, and from the points A and Q draw perpendiculars to OM to meet the circle at A', Q' respectively. Because of symmetry, the part of the circle $AA'MQ'QA$ has area $\mathcal{S}$ equal to half of the area of the circle, *i.e.*, $\mathcal{S} = \frac{1}{2}\pi\mathcal{R}^2$. Outside the polygon $ABM \cdots PQ$ construct parallelograms touching the circle, with bases the sides such as $MN = a_i$ and

Publication Date: October 28, 2002. Communicating Editor: Michael Lambrou.

other sides parallel to AA'. If h_i is the altitude of triangle OMN and d_i is the height of the parallelogram $MM'N'N$, then $h_i + d_i = \mathcal{R}$. Note that $\mathcal{A}_1$ is the sum of the areas of triangles $OAB, \ldots, OMN, \ldots, OPQ$, *i.e.*,

$$\mathcal{A}_1 = \frac{1}{2} \sum_i a_i h_i.$$

If we denote by $\mathcal{A}_2$ the sum of the areas of the parallelograms, we have

$$\mathcal{A}_2 = \sum_i a_i d_i = \sum_i a_i (\mathcal{R} - h_i) = \mathcal{R} \cdot \frac{\mathcal{L}}{2} - 2\mathcal{A}_1.$$

Since $\mathcal{A}_1 + \mathcal{A}_2 \geq \mathcal{S}$, we have $\mathcal{R} \cdot \frac{\mathcal{L}}{2} - \mathcal{A}_1 \geq \frac{1}{2}\pi\mathcal{R}^2$, and so $\pi\mathcal{R}^2 - \mathcal{L}\mathcal{R} + 2\mathcal{A}_1 \leq 0$. Rewriting this as

$$\pi \left(\mathcal{R} - \frac{\mathcal{L}}{2\pi} \right)^2 - \left(\frac{\mathcal{L}^2}{4\pi} - 2\mathcal{A}_1 \right) \leq 0,$$

we conclude that $\mathcal{L}^2 \geq 4\pi \cdot 2\mathcal{A}_1 \geq 4\pi\mathcal{A}$. $\square$

The above inequality, by means of limits can be extended to a closed curve. Since for the circle the inequality becomes equality, we conclude that of all closed curves with constant perimeter $\mathcal{L}$, the curve that contains the maximum area is the circle.

References

[1] T. Bonnesen, *Les Problèmes des Isopérimètres et des Isépiphanes*, Paris, Gauthier-Villars 1929; pp. 59-61.

[2] T. Bonnesen and W. Fenchel, *Theorie der Convexen Körper*, Chelsea Publishing, New York, 1948; S.111-112.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr

Forum Geometricorum
Volume 2 (2002) 131–134.

FORUM GEOM
ISSN 1534-1178

The Perimeter of a Cevian Triangle

Nikolaos Dergiades

Abstract. We show that the cevian triangles of certain triangle centers have perimeters not exceeding the semiperimeter of the reference triangle. These include the incenter, the centroid, the Gergonne point, and the orthocenter when the given triangle is acute angled.

1. Perimeter of an inscribed triangle

We begin by establishing an inequality for the perimeter of a triangle inscribed in a given triangle ABC.

Proposition 1. *Consider a triangle ABC with $a \le b \le c$. Denote by X, Y, Z the midpoints of the sides BC, CA, and AB respectively. Let D, E, F be points on the sides BC, CA, AB satisfying the following two conditions:*

(1.1) *D is between X and C, E is between Y and C, and F is between Z and B.*
(1.2) *$\angle CDE \le \angle BDF$, $\angle CED \le \angle AEF$, and $\angle BFD \le \angle AFE$.*

Then the perimeter of triangle DEF does not exceed the semiperimeter of triangle ABC.

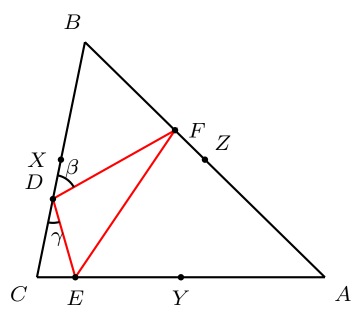


Figure 1

Proof. Denote by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the unit vectors along $\mathbf{EF}$, $\mathbf{FD}$, $\mathbf{DE}$. See Figure 1. Since $\angle BFD \le \angle AFE$, we have $\mathbf{i}\cdot\mathbf{ZF} \le \mathbf{j}\cdot\mathbf{ZF}$. Similarly, since $\angle CDE \le \angle BDF$ and $\angle CED \le \angle AEF$, we have $\mathbf{j}\cdot\mathbf{XD} \le \mathbf{k}\cdot\mathbf{XD}$ and $\mathbf{i}\cdot\mathbf{EY} \le \mathbf{k}\cdot\mathbf{EY}$. Now, we have

$$\begin{aligned} EF+FD+DE &= \mathbf{i}\cdot\mathbf{EF}+\mathbf{j}\cdot\mathbf{FD}+\mathbf{k}\cdot\mathbf{DE} \\ &= \mathbf{i}\cdot(\mathbf{EY}+\mathbf{YZ}+\mathbf{ZF})+\mathbf{j}\cdot(\mathbf{FZ}+\mathbf{ZX}+\mathbf{XD})+\mathbf{k}\cdot\mathbf{DE} \\ &\le (\mathbf{k}\cdot\mathbf{EY}+\mathbf{i}\cdot\mathbf{YZ}+\mathbf{j}\cdot\mathbf{ZF}) \\ &\quad +(\mathbf{j}\cdot\mathbf{FZ}+\mathbf{j}\cdot\mathbf{ZX}+\mathbf{k}\cdot\mathbf{XD})+\mathbf{k}\cdot\mathbf{DE} \end{aligned}$$

Publication Date: November 1, 2002. Communicating Editor: Paul Yiu.

$$\begin{aligned}
&= \mathbf{i}\cdot\mathbf{YZ}+\mathbf{j}\cdot\mathbf{ZX}+\mathbf{k}\cdot\mathbf{XY}\\
&\le |\mathbf{i}||\mathbf{YZ}|+|\mathbf{j}||\mathbf{ZX}|+|\mathbf{k}||\mathbf{XY}| \qquad (1)\\
&= YZ+ZX+XY\\
&= \frac{1}{2}(AB+BC+CA).
\end{aligned}$$

Equality holds in (1) only when the triangles DEF and XYZ have parallel sides, *i.e.*, when the points D, E, F coincide with the midpoints X, Y, Z respectively, as is easily seen. □

2. Cevian triangles

Proposition 2. *Suppose the side lengths of triangle ABC satisfy $a \le b \le c$. Let P be an interior point with (positive) homogeneous barycentric coordinates $(x:y:z)$ satisfying*

(2.1) $x \le y \le z$,
(2.2) $x\cot A \ge y\cot B \ge z\cot C$.

Then the perimeter of the cevian triangle of P does not exceed the perimeter of the medial triangle of ABC, i.e.*, the cevian triangle of the centroid.*

Proof. In Figure 1, $BD = \frac{az}{y+z}$, $DC = \frac{ay}{y+z}$, and $BF = \frac{cx}{x+y}$. Since $y \le z$, it is clear that $BD \ge DC$. Similarly, $AE \ge EC$, and $AF \ge FB$. Condition (1.1) is satisfied. Applying the law of sines to triangle BDF, we have $\frac{\sin(B+\beta)}{\sin\beta} = \frac{BD}{BF}$. It follows that

$$\frac{\sin(B+\beta)}{\sin B\sin\beta} = \frac{\sin(B+C)}{\sin B\sin C}\cdot\frac{z(x+y)}{x(y+z)}.$$

From this, $\cot\beta+\cot B = (\cot B+\cot C)\cdot\frac{z(x+y)}{x(y+z)}$. Similarly, $\cot\gamma+\cot C = (\cot B+\cot C)\cdot\frac{y(z+x)}{x(y+z)}$. Consequently,

$$\cot\gamma-\cot\beta = \frac{2(y\cot B - z\cot C)}{y+z},$$

so that $\beta \ge \gamma$ provided $y\cot B \ge z\cot C$. The other two inequalities in (1.2) can be similarly established. The result now follows from Proposition 1. □

This applies, for example, to the following triangle centers. For the case of the orthocenter, we require the triangle to be acute-angled. [1] It is easy to see that the barycentrics of each of these points satisfy condition (2.1).

P	$(x:y:z)$	$x\cot A \ge y\cot B \ge z\cot C$
Incenter	$(a:b:c)$	$\cos A \ge \cos B \ge \cos C$
Centroid	$(1:1:1)$	$\cot A \ge \cot B \ge \cot C$
Orthocenter	$(\tan A:\tan B:\tan C)$	$1 \ge 1 \ge 1$
Gergonne point	$\left(\tan\frac{A}{2}:\tan\frac{B}{2}:\tan\frac{C}{2}\right)$	$\frac{1}{2}(1-\tan^2\frac{A}{2}) \ge \frac{1}{2}(1-\tan^2\frac{B}{2})$ $\ge \frac{1}{2}(1-\tan^2\frac{C}{2})$

[1] For the homogeneous barycentric coordinates of triangle centers, see [1].

The perimeter of the cevian triangle of each of these points does not exceed the semiperimeter of ABC.[2] The case of the incenter can be found in [2].

3. Another example

The triangle center with homogeneous barycentric coordinates $(\sin\frac{A}{2} : \sin\frac{B}{2} : \sin\frac{C}{2})$ provides another example of a point P the perimeter of whose cevian triangle not exceeding the semiperimeter of ABC. It clearly satisfies (2.1). Since $\sin\frac{A}{2}\cot A = \cos\frac{A}{2} - \frac{1}{2\cos\frac{A}{2}}$, it also satisfies condition (2.2). In [1], this point appears as X_{174} and is called the Yff center of congruence. Here is another description of this triangle center [3]:

The tangents to the incircle at the intersections with the angle bisectors farther from the vertices intersect the corresponding sides at the traces of the point with homogeneous barycentric coordinates $(\sin\frac{A}{2} : \sin\frac{B}{2} : \sin\frac{C}{2})$.

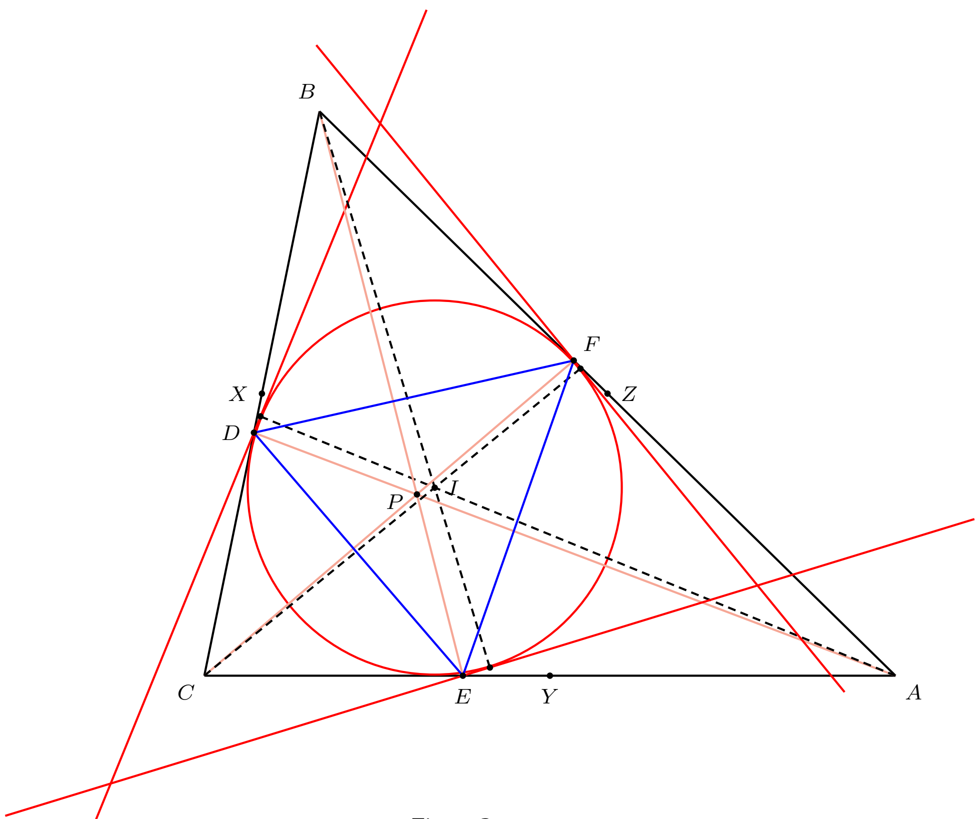


Figure 2

[2]The Nagel point, with homogeneous barycentric coordinates $(\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2})$, also satisfies (2.2). However, it does not satisfy (2.1) so that the conclusion of Proposition 2 does not apply. The same is true for the circumcenter.

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000
http://www2.evansville.edu/ck6/encyclopedia/.
[2] T. Seimiya and M. Bataille, Problem 2502 and solution, *Crux Math.*, 26 (2000) 45; 27 (2001) 53–54.
[3] P. Yiu, Hyacinthos message 2114, http://groups.yahoo.com/group/Hyacinthos/message/2114, December 18, 2000.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr

Forum Geometricorum
Volume 2 (2002) 135–146.

FORUM GEOM
ISSN 1534-1178

Geometry and Group Structures of Some Cubics

Fred Lang

Abstract. We review the group structure of a cubic in the projective complex plane and give group theoretic formulations of some geometric properties of a cubic. Then, we apply them to pivotal isocubics, in particular to the cubics of Thomson, Darboux and Lucas. We use the group structure to identify different transformations of cubics. We also characterize equivalence of cubics in terms of the Salmon cross ratio.

1. The group structure of a cubic

Let Γ be a nonsingular cubic curve in the complex projective plane, *i.e.*, Γ has no cusp and no node. It is well known that Γ has a group structure, which does not depend on the choice of a neutral element O on the cubic. In other words, the group structures on Γ for various choices of the neutral elements are isomorphic.

If P and Q are points of a cubic Γ, we denote by $P \cdot Q$ the third intersection of the line PQ with Γ. In particular, $P \cdot P := P_t$ is the *tangential* of P, the second intersection of Γ with the tangent at P.

Proposition 1. *The operation $\cdot$ is commutative but not associative. For P, Q, R on Γ,*

(1) $(P \cdot Q) \cdot P = Q$,
(2) $P \cdot Q = R \cdot Q \Longleftrightarrow P = R$,
(3) $P \cdot Q = R \Longleftrightarrow P = R \cdot Q$.

Convention: When we write $P \cdot Q \cdot R$, we mean $(P \cdot Q) \cdot R$.

We choose a point O on Γ as the neutral point,[1] and define a group structure $+$ on Γ by

$$P + Q = (P \cdot Q) \cdot O.$$

We call the tangential of O, the point $N = O_t = O \cdot O$, the *constant point* of Γ. Note that $-N = N_t$, since $N + N_t = N \cdot N_t \cdot O = N \cdot O = O$.

We begin with a fundamental result whose proof can be found in [4, p.392].

Theorem 2. *$3k$ points P_i, $1 \leq i \leq 3k$, of a cubic Γ are on a curve of order k if and only if $\sum P_i = kN$.*

For $k = 1, 2, 3$, we have the following corollary.

Corollary 3. *Let $P, Q, R, S, T, U, V, W, X$ be points of Γ.*

(1) *P, Q, R are collinear if and only if $P + Q + R = N$.*
(2) *P, Q, R, S, T, U are on a conic if and only if $P+Q+R+S+T+U = 2N$.*

Publication Date: November 8, 2002. Communicating Editor: Paul Yiu.

[1] O is not necessarily an inflexion point (a flex).

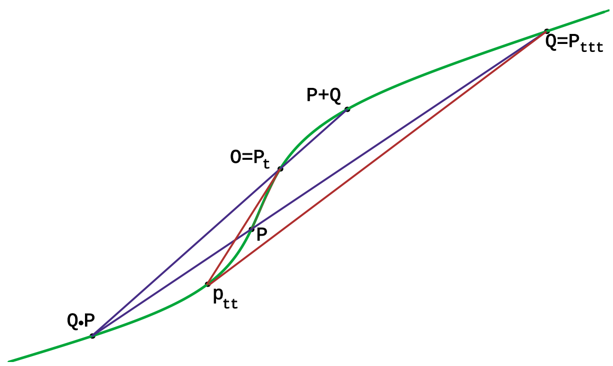


Figure 1. The first three tangentials of P and $P+Q$

(3) *P, Q, R, S, T, U, V, W, X are on a cubic if and only if* $P+Q+R+S+T+U+V+W+X=3N$.

Remark. The case $k=2$ is equivalent to the following property.

Geometric formulation	Group theoretic formulation
Let P, Q, R, S, T, U be six points of a cubic Γ, and let $X=P\cdot Q$, $Y=R\cdot S$, $Z=T\cdot U$, then P, Q, R, S, T, U are on a conic if and only if X, Y, Z are collinear.	Let P, Q, R, S, T, U be six points of a cubic Γ, and let $P+Q+X=N$, $R+S+Y=N$, $T+U+Z=N$, then $P+Q+R+S+T+U=2N$ if and only if $X+Y+Z=N$.

A geometrical proof is given [8, p.135]; an algebraic proof is a straightforward calculation.

We can do normal algebraic calculations in the group, but have to be careful to torsion points: for example $2P=O$ does not imply $P=O$. The group of Γ has non zero torsion points, i.e, points with the property $kP=O$, for $P\neq O$. Indeed the equation $kX=Q$ has k^2 (complex) solutions for the point X. See [10, 17].

The tangential P_t of P is $N-2P$, since P, P, and P_t are collinear. The *second tangential* P_{tt} of P is $N-2(N-2P)=-N+4P$. The *third tangential* is $N-2(-N+4P)=3N-8P$.

2. A sample of theorems on cubics

We give a sample of theorems on cubics, in both geometric and group-theoretic formulations. Most of the theorems can be found in [8, p.135]. In the following table, all points are on a cubic Γ. A point $P\in\Gamma$ is a *sextatic* point if there is a conic through P with contact of order 6 with Γ at P.

	Geometric formulation	Group theoretic formulation
1	P and Q have the same tangential.	$2P = 2Q$ or $2(P - Q) = O$
2	There are four tangents from P.	$2X + P = N$ has four solutions
3	P is a flex	$3P = N$
4	Γ has nine flexes	$3P = N$ has nine solutions
5	If P and Q are flexes, then $R = P \cdot Q$ is another flex. If $P \neq Q$, then $R \neq P,\ Q$.	$3P = N$, $3Q = N$, and $P + Q + R = N$ $\Longrightarrow 3R = N$.
6	Let P_1, P_2, P_3 and P_4 be fixed on Γ. If a variable conic intersects Γ at $P_1 \ldots, P_6$, then the line P_5P_6 passes through a fixed point Q on Γ, which we call the *coresidual* of P_1, P_2, P_3, P_4.	$P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 2N$ and $P_5 + P_6 + Q = N$ $\Longrightarrow Q = -N + P_1 + P_2 + P_3 + P_4$, which is fixed.
7	If a conic intersects Γ at $P_1, \ldots P_6$, then the tangentials $Q_1, \ldots, Q_6$ are on another conic	$\sum P_i = 2N$, $2P_i + Q_i = N$ for $i = 1, \ldots, 6$ $\Longrightarrow \sum Q_i = 2N$.
8	Let Ω be a conic *tritangent* to Γ at P, Q, R, and let Ψ be another conic which intersects Γ at P, Q, R, P', Q', R', then there exists a conic Λ tangent to Γ at P', Q', R'.	$2P + 2Q + 2R = 2N$ and $P + Q + R + P' + Q' + R' = 2N$ $\Longrightarrow 2P' + 2Q' + 2R' = 2N$.
9	A conic Ω is tritangent to Γ at P, Q, R if and only if the tangentials P', Q', R' of P, Q, R are collinear.	For $2P + P' = N$, $2Q + Q' = N$, and $2R + R' = N$, $2P + 2Q + 2R = 2N$ $\Longleftrightarrow P' + Q' + R' = N$.
10	If Q, R, S are given points, there exist 9 points X such that a conic *osculates* at X and passes through Q, R, S	The equation $3X + Q + R + S = 2N$ has nine solutions.
11	P is sextatic if and only if the tangent at P contains a flex Q different from P.	For $2P + Q = N$, $6P = 2N \Longleftrightarrow 3Q = N$.
12	P is sextatic if and only if P is the tangential of a flex Q.	$6P = 2N \Longleftrightarrow$ $2Q + P = N$ and $3Q = N$.
13	There are 27 sextatic points on a cubic.	$6P = 2N$ has 36 solutions, nine are the flexes, the others 27 are the sextatic points.
14	If P and Q are sextatic, then $R = P \cdot Q$ is sextatic.	$6P = 2N$, $6Q = 2N$ and $P + Q + R = N$ $\Longrightarrow 6R = 2N$.

Remarks. The coresidual in (6) is called the *gegenüberliegende Punkt* in [8, p.140].

3. The group structure of a pivotal isocubic

Let $P \mapsto P^*$ be a given isoconjugation in the plane of the triangle ABC (with trilinear coordinates). See, for example, [5]. For example, $P(x : y : z) \mapsto P^*(\frac{1}{x} : \frac{1}{y} : \frac{1}{z})$ is the isogonal transformation and $P(x : y : z) \mapsto (\frac{1}{a^2x} : \frac{1}{b^2y} : \frac{1}{c^2z})$ is the isotomic transformation. We shall also consider the notion of cevian quotient. For any two points P and Q, the cevian triangle of P and the precevian triangle of

Q are always perspective. We call their perspector the *cevian quotient* P/Q. See [11].

Let Γ be a pivotal isocubic with pivot F. See, for example, [6, 7, 14]. Take the pivot F for the neutral element O of the group. The constant point is $N = F_t$.

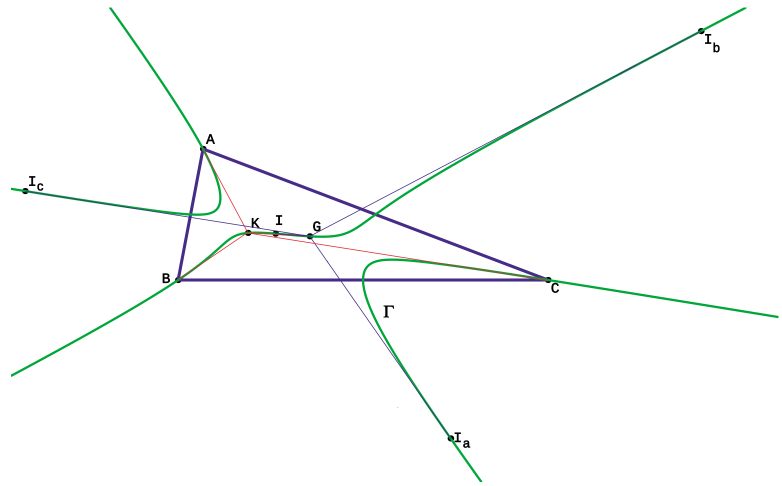


Figure 2. Two tangential quadruples on the Thomson cubic

Definition. Four points of Γ form a *tangential quadruple* if they have the same tangential point.

Theorem 4. *Consider the group structure on a pivotal isocubic with the pivot F as neutral element. The constant point is $N = F_t$.*

(1) $P \cdot P^* = F$, $P \cdot F = P^*$, $P^* \cdot F = P$.
(2) $F_t = F^*$.
(3) $P + P^* = F_t$.
(4) $P + Q = (P \cdot Q)^*$ *or* $P \cdot Q = (P + Q)^*$.
(5) P, Q, R *are collinear if and only if* $P + Q + R = F_t$.
(6) $-P = P \cdot F_t$.
(7) $-P = F/P$.
(8) *If* (P, Q, R, S) *is a tangential quadruple then* (P^*, Q^*, R^*, S^*) *is also a tangential quadruple.*
(9) *Every tangential quadruple is of the form* $(P, P + A, P + B, P + C)$.
(10) A, B, C *are points of order 2,* i.e., $2A = 2B = 2C = F$.

Proof. (1) F is the pivot, so P, P^* and F are collinear.
(2) Put $P = F$ in (1).
(3) $P + P^* = (P \cdot P^*) \cdot F = F \cdot F = F_t$.
(4) $P + Q = (P \cdot Q) \cdot F = (P \cdot Q)^*$. (use (1))
(5) This is Corollary 3.
(6) $P + (P \cdot F_t) = (P \cdot (P \cdot F_t)) \cdot F = F_t \cdot F = F$.

(7) If the pivot F has trilinear coordinates $(u : v : w)$ and $P(x : y : z)$, then the Cevian quotient F/P is the point

$$(x(-vwx + uwy + uvz) :\ y(vwx - uwy + uvz) :\ z(vwx + uwy - uvz)).$$

We can verify that it is on Γ and is collinear with P and F_t.

(8) We have to prove that, if P and Q have a common tangential T, then P^* and Q^* have a common tangential U. Let U be the tangential of P^*, then (5) and (2) give

$$U + 2P^* = F_t = F^*.$$

Since F, P, P^* are collinear, and so are F, Q, Q^*, we have

$$P + P^* = F^* \quad \text{and} \quad Q + Q^* = F^*.$$

Since T is the common tangential of P and Q,

$$2P + T = F^* \quad \text{and} \quad 2Q + T = F^*.$$

From these,

$$\begin{aligned} U + 2Q^* =&(F^* - 2P^*) + 2Q^* \\ =&F^* - 2(F^* - P) + 2(F^* - Q) \\ =&F^* + 2P - 2Q \\ =&F^* + F^* - T - F^* + T \\ =&F^*, \end{aligned}$$

and U is the tangential of Q^* too.

(9) We have to prove that, if P is on the cubic, P and $P + A$ have the same tangential. Let Q and Q_a be the tangential of P and $P + A$ respectively. By property (3), $P + P + Q = F^*$ and $(P + A) + (P + A) + Q_a = F^*$. Hence $Q = Q_a \Longleftrightarrow 2A = 0 \Longleftrightarrow A = -A$. By properties (6) and (2), $-A = A \cdot F^*$, hence we have to prove that $A = A \cdot F^*$, i.e. the tangential of A is F^*. The equation of the tangent to the cubic at A is $r^2vy = q^2wz$, and $F^*(p^2vw : q^2uw : r^2uv)$ is clearly on this line. But F^* is on Γ. Hence it is the tangential point of A.

(10) $2A = A + A = A \cdot A = A_t^* = F^{**} = F$. □

A consequence of (10) is that the cubic is not connected. See, for example, [10, p.20].

4. The Thomson, Darboux and Lucas cubics

These well-known pivotal cubics have for pivots G (centroid), L (de Longchamps point) and K_+ (isotomic of the orthocenter H). Thomson and Darboux are isogonal cubics and Lucas is an isotomic one. We study the subgroups generated by the points G, I, A, B, C for Thomson, L, I', A, B, C for Darboux and K_+, N_o, A, B, C for Lucas. For a generic triangle, [2] these groups are isomorphic to $\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

[2]This may be false for some particular triangles. For example, if ABC has a right angle at A, then $H = A$ and for Thomson, $H = 4I$.

Notation. For each point P, we denote by P_a, P_b, P_c the points $P + A$, $P + B$, $P + C$ respectively. We use the notations of [12] for triangle centers, but adopt the following for the common ones.

G	centroid
K	symmedian (Lemoine) point
H	orthocenter
O	circumcenter
I	incenter; I_a, I_b, I_c are the excenters
L	de Longchamps point
M	Mittenpunkt
G_o	Gergonne point
N_o	Nagel point

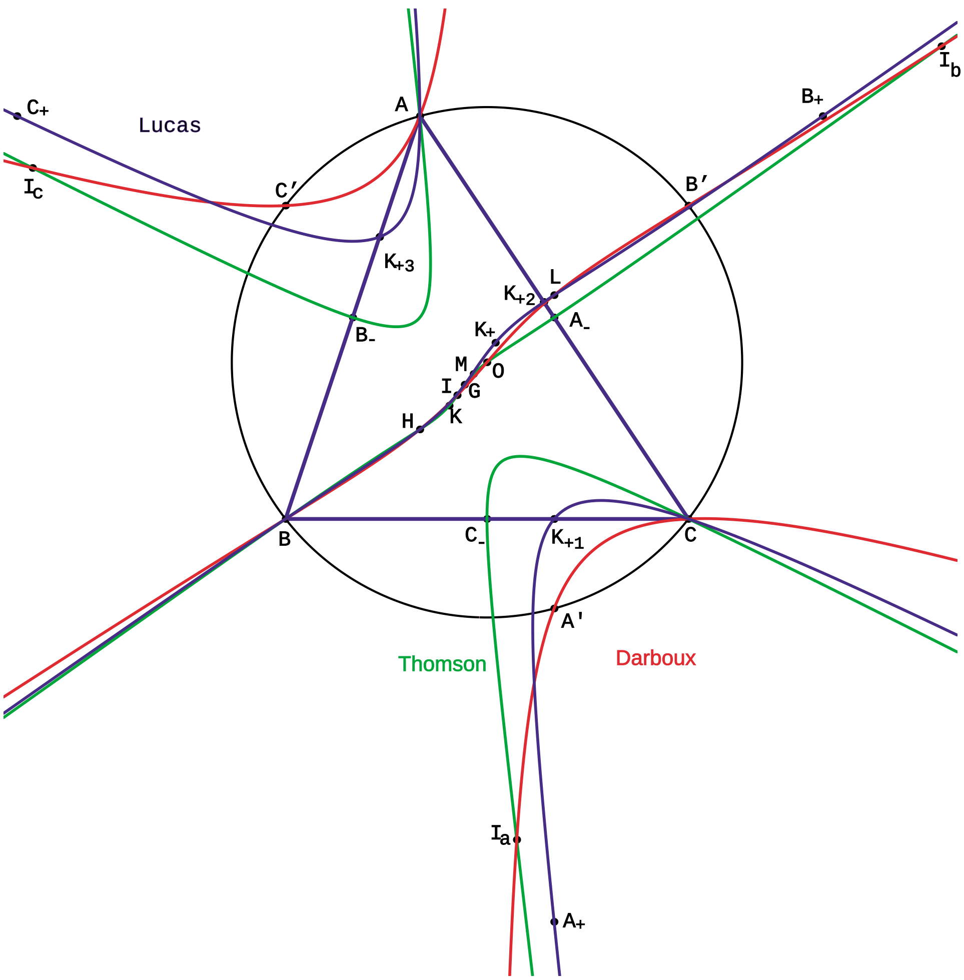


Figure 3. The Thomson, Darboux and Lucas cubics

In the following table, the lines represent the $\mathbb{Z}$-part, and the columns the $\mathbb{Z}_2 \times \mathbb{Z}_2$-part. The last column give the tangential point of the tangential quadruple of the corresponding line. The line number 0 is the subgroup generated by the pivot and A, B, C. It is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

4.1. *The Thomson cubic.*

	P	P_a	P_b	P_c	tangential
-6	H_t	H_{ta}	H_{tb}	H_{tc}	
-5	$-X_{282}$	$-X_{282a}$	$-X_{282b}$	$-X_{282c}$	
-4	O_t^*	O_{ta}^*	O_{tb}^*	O_{tc}^*	
-3	X_{223}	X_{223a}	X_{223b}	X_{223c}	
-2	O	A_-	B_-	C_-	O_t
-1	M	M_a	M_b	M_c	H
0	G	A	B	C	K
1	I	I_a	I_b	I_c	G
2	K	A_-	B_-	C_-	O
3	M^*	M_a^*	M_b^*	M_c^*	O_t^*
4	H	H_a	H_b	H_c	H_t
5	X_{282}	X_{282a}	X_{282b}	X_{282c}	
6	O_t	O_{ta}	O_{tb}	O_{tc}	O_{tt}

(1) Neutral = pivot = G = centroid.
(2) Constant $= G_t = G^* = K$.
(3) Three points are collinear if and only if their sum is 2.
(4) Examples of calculation:
  (a) $I + K = (I \cdot K) \cdot G = M \cdot G = M^*$.
  (b) $A + A = (A \cdot A) \cdot G = K \cdot G = G$.
  (c) To find the intersection X of the line OM with the cubic, we have to solve the equation $x + (-2) + (-1) = 2$. Hence, $x = 5$ and $X = X_{282}$.
(5) A_-, B_-, C_- are the midpoints of the sides of ABC, diagonal triangle of $GABC$.
(6) A^-, B^-, C^- are the midpoints of the altitudes of ABC, diagonal triangle of $KA_-B_-C_-$.
(7) O_t is the isoconjugate of the circumcenter O relative to the pencil of conics through the points K, A_-, B_-, C_-.
(8) O_{tt} is the isoconjugate of O_t relative to the pencil of conics through the points O, A^-, B^-, C^-.
(9) $O_{ta}O_{tb}O_{tc}$ is the diagonal triangle of $OA^-B^-C^-$.
(10) $H_a = A^{-*} = OA \cap A^-G = B^-C \cap C^-B$.
(11) $X_{223} = -(M^*)$ is the third intersection of the line IH and Γ. (Proof: $I + H + X_{223} = 1 + (-3) + 4 = 2 =$ constant).
(12) If a point X has the line number x, then the points X^*, X_t and G/X have line numbers $2 - x$, $2 - 2x$ and $-x$.

4.2. *The Darboux cubic.*

	P	P_a	P_b	P_c	tangential
-6	L_t^*				
-5	$-I^{'*}$				
-4	$-H$				
-3	$I^{'*'*'}$	$I_a^{'*'*'}$	$I_b^{'*'*'}$	$I_c^{'*'*'}$	
-2	$L^{*'}$	L'_a	L'_b	L'_c	
-1	$I^{'*'}$	$I_a^{'*'}$	$I_b^{'*'}$	$I_c^{'*'}$	
0	L	A	B	C	L^*
1	I'	I'_a	I'_b	I'_c	H
2	O	$H_{1\infty}$	$H_{2\infty}$	$H_{3\infty}$	O
3	I	I_a	I_b	I_c	L
4	H	A'	B'	C'	$L_{*'}$
5	$I^{'*}$	$I_a^{'*}$	$I_b^{'*}$	$I_c^{'*}$	
6	L^*	L_1	L_2	L_3	L_t^*
7	$I^{'*'*}$	$I_a^{'*'*}$	$I_b^{'*'*}$	$I_c^{'*'*}$	
8	$L^{*'*}$				

(1) Neutral = pivot = L = de Longchamps point = symmetric of H relative to O; constant point = L^*.
(2) Three points are collinear if and only if their sum is 6.
(3) L_1, L_2, L_3 = Cevian points of L.
(4) $H_{1\infty}$ = infinite point in the direction of the altitude AH.
(5) P' is the symmetric of P relative to O. (Symmetry relative to line 2)
(6) $P^{'*}$ gives the translation of $+2$ and $P^{*'}$ of -2. Three points are collinear if and only if their sum is 6.
(7) If a point X has the line number x, then the points X^*, X_t, X' and G/X have line numbers $6-x$, $6-2x$, $4-x$ and $-x$.

4.3. *The Lucas cubic.*

	P	P_a	P_b	P_c	tangential
-4	K_{+t}				
-3	$-G_o$	$-G_a$	$-G_b$	$-G_c$	
-2	L	L_a	L_b	L_c	X_{1032}
-1	X_{329}	X_{329a}	X_{329b}	X_{329c}	L^r
0	K_+	A	B	C	H
1	N_o	N_a	N_b	N_c	G
2	G	A_+	B_+	C_+	K_+
3	G_o	G_a	G_b	G_c	L
4	H	K_{+1}	K_{+2}	K_{+3}	K_{+t}
5	X_{189}	X_{189a}	X_{189b}	X_{189c}	
6	L^r	L_a^r	L_b^r	L_c^r	
7	X_{1034}	X_{1034a}	X_{1034b}	X_{1034c}	
8	X_{1032}				

(1) Neutral = K_+ = Lemoine point of the precevian triangle $A_+B_+C_+$ of ABC = isotomic of H; Constant point = H. Three points are collinear if and only if their sum is 4.
(2) P^r = isotomic of P (symmetry relative to line 2).
(3) K_{+1}, K_{+2}, K_{+3} = cevian points of K_+ = intersections of Lucas cubic with the sides of ABC.
(4) X_{329} = intersection of the lines N_oH and G_oG with the cubic.
(5) If a point X has line number x, then the points X^*, X_t and G/X have line numbers $4-x$, $4-2x$ and $-x$.
(6) $X_{329}^r = X_{189}$.

5. Transformations of pivotal isocubics

We present here some general results without proofs. See [16, 15, 9, 4].

5.1. *Salmon cross ratio.* The Salmon cross ratio of a cubic is the cross ratio of the four tangents issued from a point P of Γ. It is defined up to permutations of the tangents. We shall therefore take it to be a set of the form

$$\left\{\lambda,\ \lambda-1,\ \frac{1}{\lambda},\ \frac{1}{\lambda-1},\ \frac{\lambda}{\lambda-1},\ \frac{\lambda-1}{\lambda}\right\},$$

since if λ is a Salmon cross ratio, then we obtain the remaining five values of permutation of the tangents.

A cubic Γ is *harmonic* if $\lambda = -1$; it is *equiharmonic* if λ satisfies $\lambda^2 - \lambda + 1 = 0$.

The Salmon cross ratio is independent of the choice of P.

5.2. *Birational equivalence.* A transformation [9] of Γ is *birational* if the transformation and its inverse are given by rational functions of the coordinates. [3] Two cubics Γ_1 and Γ_2 are equivalent if there is a birational transformation $\Gamma_1 \to \Gamma_2$.

Theorem 5. *A birational transformation of a cubic Γ onto itself induces a transformation of its group of the form $x \mapsto ux + k$, where*
(1) $u^2 = 1$ *for a general cubic,*
(2) $u^4 = 1$ *for a harmonic cubic, and*
(3) $u^6 = 1$ *for an equiharmonic cubic.*

Theorem 6. *Two equivalent cubics have isomorphic groups.*

Examples:
1) The groups of the cubics of Darboux, Thomson and Lucas are isomorphic.
2) The transformation that associate to a point its tangential is given by $X \mapsto N - 2X$ and is not birational.

Theorem 7. *Two cubics Γ_1 and Γ_2 are equivalent if and only if their Salmon cross ratios are equal.*

[3] Cautions: Two different transformations of the projective plane may induce the same transformation on curves. see [15].

If the isoconjugation has fixed point $(p : q : r)$, it is easy to prove the following result:

Theorem 8. *A pivotal isocubic of pivot* $(u : v : w)$ *has Salmon cross ratio*

$$\frac{q^2(r^2u^2 - p^2w^2)}{r^2(q^2u^2 - p^2v^2)}.$$

For example, the cubics of Darboux, Thomson, Lucas all have Salmon cross ratio

$$\frac{b^2(a^2 - c^2)}{c^2(a^2 - b^2)}.$$

Are Thomson, Darboux and Lucas the only equivalent pivotal cubics? No! Here is a counter-example. Take the isoconjugation with fixed point X_{63}. The pivotal isocubic of pivot X_{69} (the same as Lucas) is equivalent to Thomson.

6. Examples of birational transformations of cubics

We give now a list of birational transformations, with the corresponding effects on the lines of the group table. Recall that N is the tangential of the pivot, *i.e.*, the constant point.

6.1. *Projection:* $\Gamma \to \Gamma$. Let $P \in \Gamma$. A projection of Γ on itself from P gives a transformation $X \mapsto X'$ so that P, X, X' are collinear:

$$x \mapsto n - p - x.$$

6.2. *Cevian quotient:* $\Gamma \to \Gamma$. Let F be the pivot of Γ, then the involution $X \mapsto F/X$ gives the transformation: $x \mapsto -x$.

6.3. *Isoconjugation:* $\Gamma \to \Gamma$. Since F, X, X^* are collinear, the isoconjugation is a projection from the pivot $F : x \mapsto n - x$.

6.4. *Pinkernell's quadratic transformations.* We recall the definition of the $d-$pedal cubics Γ_d and of the $d-$cevian cubics Δ_d. If P has *absolute* trilinear coordinates (x, y, z), then define P_A, P_B, P_C on the perpendiculars from P to the sides such that $PP_A = dx$, etc. The locus of P for which $P_AP_BP_C$ is perspective to ABC is a cubic Γ_d, and the locus of the perspector is another cubic Δ_d. Hence we have a birational transformation $f_d : \Gamma_d \to \Delta_d$.

The $d-$pedal is different from the $(-d)$-pedal, but the d-cevian is the same as the $(-d)$-cevian.

For example: $\Gamma_1 =$ Darboux , $\Gamma_{-1} =$ Thomson, and $\Delta_1 =$ Lucas.

Let L_d be the pivot of Γ_d and X on Γ_d. Since L_d, X and $f_d(X)$ are collinear we can identify f_d as a projection of Γ_d to Δ_d from the pivot L_d.

These transformations are birational. Hence the groups of the cubics Γ_d, Γ_{-d} and Δ_d are isomorphic.

For $d = 1$, X and $f_d(X)$ are on the same line in the group table: $x \mapsto x$.

6.5. *The quadratic transformations* $h_d : \Gamma_d \longrightarrow \Gamma_{-d}$. Let g_d be the inverse of f_d. Define $h_d = g_d \circ f_d$.

$$x \mapsto x.$$

For $d = 1$, we have a map from Darboux to Thomson. In this case, a simple construction of h_1 is given by: Let P be a point on Darboux and P_i the perpendicular projections of P on the sides of ABC, let A^-, B^-, C^- be the midpoint of the altitudes of ABC, then $Q = h_1(P)$ is the intersection of the lines P_1A^-, P_2B^- and P_3C^-.

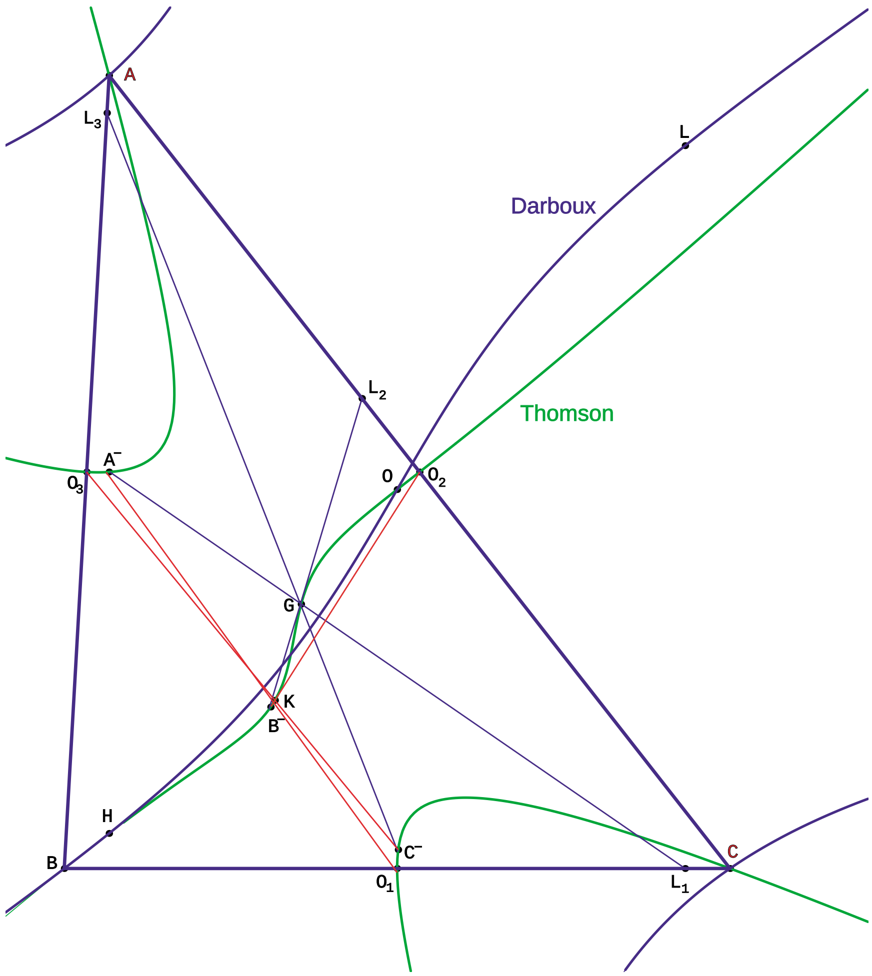


Figure 4. $G = h_1(L)$ and $K = h_1(O)$, h_1 : Darboux $\longrightarrow$ Thomson

6.6. *Cevian, precevian, pedal and prepedal quadratic transformations.* 1. The Lucas cubic is the set of points P such that the cevian triangle of P is the pedal triangle of Q. The locus of Q is the Darboux cubic and the transformation is $g_1 : x \mapsto x$.

2. The Lucas cubic is the set of points P such that the cevian triangle of P is the prepedal triangle of Q. The locus of Q is the Darboux cubic and the transformation is the isogonal of $g_1 : x \mapsto 6 - x$.

3. The Thomson cubic is the set of points P such that the precevian triangle of P is the pedal triangle of Q. The locus of Q is the Darboux cubic and the transformation is the inverse of $h_1 : x \mapsto x$.

4. The Thomson cubic is the set of points P such that the precevian triangle of P is the prepedal triangle of Q. The locus of Q is the Darboux cubic and the transformation is the symmetric of the inverse of $h_1 : x \mapsto 4 - x$.

This last transformation commutes with isogonality:

Proof: $x \mapsto 4-x \mapsto 6-(4-x) = 2+x$ and $x \mapsto 2-x \mapsto 4-(2-x) = 2+x$.

6.7. *Symmetry of center O of the Darboux cubic and induced transformations on Thomson and Lucas.* The symmetry is a linear transformation of the Darboux cubic: $x \mapsto 4 - x$. It induces via f_d and f_{-d} a quadratic involution of the Thomson cubic: $x \mapsto 4 - x$. And, via f_d and g_d, a quadratic involution of the Lucas cubic: $x \mapsto 4 - x$.

6.8. *Cyclocevian transformation.* The cyclocevian transformation [12] is an involution of the Lucas cubic. It is the symmetry relative to the line 3 of the group table: $x \mapsto 6 - x$.

References

[1] P. E. Appell and E. J.-P. Goursat, *Théorie des fonctions algébriques et de leurs intégrales*, Paris 1895, pages 295 et 474.
[2] R. Bix, *Conics and Cubics*, Springer 1998.
[3] H. M. Cundy and C. F. Parry, Geometrical Properties of some Euler and circular cubics, part 1, *Journal of Geometry*, 66 (1999) 72–103.
[4] A. Clebsch: *Leçons sur la géométrie*, tome II, Paris 1880.
[5] K. R. Dean and F. M. van Lamoen, Geometric Construction of reciprocal conjugations, *Forum Geom.*, 1 (2001) 115 – 120.
[6] R. Deaux, Cubiques anallagmatiques, *Mathesis*, 62 (1953) 193–204
[7] L. Droussent, Cubiques anallagmatiques, *Mathesis*, 62 (1953) 204–215.
[8] H. Durège: *Die Ebenen Curven Dritter Ordung*, Leipzig, 1871.
[9] L. Godeaux, *Les transformations birationnelles du plan*, Paris, 1953.
[10] D. Husemoller: *Elliptic Curves*, Springer, 1987.
[11] J. H. Conway, Hyacinthos, message 1018.
[12] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://www2.evansville.edu/ck6/encyclopedia/.
[13] G. Pinkernell, Cubics curves in the triangle plane, *Journal of Geometry*, 55 (1996) 141–161.
[14] P. Rubio, Anallagmatic cubics through quadratic involutive transformations I, *Journal of Geometry*, 48 (1993) 184.
[15] G. Salmon, *Higher Plane Curves*, Dublin, 1873; Chelsea reprint.
[16] P. Du Val, *Elliptic Functions and Elliptic Curves*, Cambridge University Press. 1973.
[17] P. L. Walker, *Elliptic Functions*, John Wiley and sons, 1996; p.190.

Fred Lang: Ecole d'ingénieurs du Canton de Vaud, Route de Cheseaux 1, CH-1400 Yverdon-les-Bains, Switzerland

E-mail address: Fred.Lang@eivd.ch

Forum Geometricorum
Volume 2 (2002) 147–149.

FORUM GEOM
ISSN 1534-1178

On Some Remarkable Concurrences

Charles Thas

Abstract. In [2], Bruce Shawyer proved the following result : "At the midpoint of each side of a triangle, we construct the line such that the product of the slope of this line and the slope of the side of the triangle is a fixed constant. We show that the three lines obtained are always concurrent. Further, the locus of the points of concurrency is a rectangular hyperbola. This hyperbola intersects the sides of the triangle at the midpoints of the sides, and each side at another point. These three other points, when considered with the vertices of the triangle opposite to the point, form a Ceva configuration. Remarkably, the point of concurrency of these Cevians lies on the circumcircle of the original triangle". Here, we extend these results in the projective plane and give a short synthethic proof.

We work in the complex or the real complexified projective plane $\mathcal{P}$. The conic through five points A, B, C, D, E is denoted by $\mathcal{C}(A, B, C, D, E)$ and $(XYZW)$ is the notation for the cross-ratio of four collinear points X, Y, Z, W.

Theorem 1. *Consider a triangle ABC and a line l, not through A, B or C, in $\mathcal{P}$. Put $AB \cap l = C'', BC \cap l = A''$, $CA \cap l = B''$ and construct the points A', B' and C' for which $(BCA'A'') = (CAB'B'') = (ABC'C'') = -1$. Then, take two different points I and I' on l (both different from A'', B'', C'') and consider the points A''', B''' and C''' such that $(II'A''A''') = (II'B''B''') = (II'C''C''') = -1$. Then the lines $A'A''', B'B'''$ and $C'C'''$ are concurrent at a point L.*

Proof. The line $A'A'''$ is clearly the polar line of A'' with respect to the conic $\mathcal{C}(A, B, C, I, I')$ and likewise for the line $B'B'''$ and B'', and for the line $C'C'''$ and C''. Thus, $A'A''', B'B'''$ and $C'C'''$ concur at the polar point L of l with respect to $\mathcal{C}(A, B, C, I, I')$. □

Theorem 2. *If I, I' are variable conjugate points in an involution Ω on the line l with double (or fixed) points D and D', then the locus of the point L is the conic $\mathcal{L} = \mathcal{C}(A', B', C', D, D')$. Moreover, putting $\mathcal{L} \cap AB = \{C', Z\}$, $\mathcal{L} \cap BC = \{A', X\}$ and $\mathcal{L} \cap CA = \{B', Y\}$, the triangles ABC and XYZ form a Ceva configuration. The point K of concurrency of the Cevians AX, BY, CZ is the fourth basis point (besides A, B, C) of the pencil of conics $\mathcal{C}(A, B, C, I, I')$.*

Publication Date: November 18, 2002. Communicating Editor: Roland Eddy.

Proof. Since the conics $\mathcal{C}(A, B, C, I, I')$ intersect the line l in the variable conjugate points I, I' of an involution on l, these conics must belong to a pencil with basis points A, B, C and a fourth point K : this follows from the Theorem of Desargues-Sturm (see [1], page 63). So, the locus $\mathcal{L}$ is the locus of the polar point L of the line l with respect to the conics of this pencil. Now, it is not difficult to prove (or even well known) that such locus is the conic through the points $A', B',' C', D, D'$ and through the points K', K'', K''' which are determined by $(AKK'K_1) = (BKK''K_2) = (CKK'''K_3) = -1$, where $K_1 = l \cap KA, K_2 = l \cap KB$ and $K_3 = l \cap KC$, and finally, through the singular points $X = KA \cap BC$, $Y = KB \cap CA, Z = KC \cap AB$ of the degenerate conics of the pencil. This completes the proof. □

Next, let us consider a special case of the foregoing theorems in the Euclidean plane Π. Take a triangle ABC in Π and let $l = l_\infty$ be the line at infinity, while the points D and D' of theorem 2 are the points at infinity of the X-axis and the Y-axis of the rectangular coordinate system in Π, respectively.
Homogeneous coordinates in Π are (x, y, z) and $z = 0$ is the line l_∞; the points D and D' have coordinates $(1, 0, 0)$ and $(0, 1, 0)$, respectively. A line with slope a has an equation $y = ax + bz$ and point at infinity $(1, a, 0)$. Now, if (in Theorem 1) the product of the slopes of the lines BC and $A'A'''$, CA and $B'B'''$, AB and $C'C'''$ is a fixed constant $\lambda(\neq 0)$, then the points at infinity of these lines (i.e. A'' and A''', B'' and B''', C'' and C''') have coordinates of the form $(1, t, 0)$ and $(1, t', 0)$, with $tt' = \lambda$. This means that A'' and A''', B'' and B''', C'' and C''' are conjugate points in the involution on l_∞ with double points $I(1, -\sqrt{\lambda}, 0)$ and $I'(1, \sqrt{\lambda}, 0)$ and thus $(II'A''A''') = (II'B''B''') = (II'C''C''') = -1$. If we let λ be variable, the points I and I' are variable conjugate points in the involution on l_∞ with double points D and D', the latter occuring for $t = 0$ and $t' = \infty$ respectively.

Now all the results of [2], given in the abstract, easily follow from Theorems 1 and 2. For instance, the locus $\mathcal{L}$ is the rectangular hyperbola $\mathcal{C}(A', B', C', D, D')$ (also) through the points K', K'', K''', X, Y, Z. Remark that the basis point K belongs to any conic $\mathcal{C}(A, B, C, I, I')$ and for $\lambda = -1$, we get that $I(1, i, 0)$ and $I'(1, -i, 0)$ are the cyclic points, so that $\mathcal{C}(A, B, C, I, I')$ becomes the circumcircle of ΔABC. For $\lambda = -1$, we have $A'A''' \perp BC, B'B''' \perp CA$ and $C'C''' \perp AB$, and $A'A''', B'B''', C'C'''$ concur at the center O of the circumcircle of ABC.

Remark also that O is the orthocenter of $\Delta A'B'C'$ and that any conic (like $\mathcal{L}$) through the vertices of a triangle and through its orthocenter is always a rectangular hyperbola.

At the end of his paper, B. Shawyer asks the following question : Does the Cevian intersection point K have any particular significance? It follows from the foregoing that K is a point of the parabolas through A, B, C and with centers $D(1, 0, 0)$ and $D'(0, 1, 0)$, the points at infinity of the X-axis and the Y-axis. And from this it follows that the circumcircle of any triangle ABC is the locus of the fourth common point of the two parabolas through A, B, C with variable orthogonal axes.

Next, we look for an (other) extension of the results of B. Shawyer : At the midpoint of each side of a triangle, construct the line such that the slope of this line and the slope of the side of the triangle satisfy the equation $ctt' - a(t+t') - b = 0$, with a, b and c constant and $a^2 + bc \neq 0$. Then these three lines are concurrent. This follows from Theorem 1, since the given equation determines a general non-singular involution. Shawyer's results correspond with $a = 0$ (and $\frac{b}{c} = \lambda$ and λ variable). Now, consider the special case where $c = 0$ and put $-\frac{b}{a} = \lambda$; the sum of the slopes is a constant λ or $t + t' = \lambda$. On the line l_∞ at infinity we get the corresponding points $(1, t, 0)$ and $(1, t', 0)$ and the fixed points of the involution on l_∞ determined by $t + t' = \lambda$ are $I(0, 1, 0)$ (or the point at infinity of the Y-axis) and $I'(1, \frac{\lambda}{2}, 0)$. In this case, the locus $\mathcal{L}$ of the point of concurrency L is the locus of the polar point L of the line l_∞ with respect to the conics of the pencil with basis points A, B, C and $I(0, 1, 0)$. A straightforward calculation shows that this locus $\mathcal{L}$ is the parabola through the midpoints A', B', C' of BC, CA, AB, respectively, and with center I. The second intersection points of this parabola $\mathcal{L}$ with the sides of the triangle are $X = IA \cap BC, Y = IB \cap CA$ and $Z = IC \cap AB$. Remark that IA, IB, IC are the lines parallel with the Y-axis through A, B, C, respectively.

References

[1] P. Samuel, *Projective Geometry*, Undergraduate Texts in Mathematics, Springer Verlag 1988.
[2] B. Shawyer, Some remarkable concurrences, *Forum Geom.*, 1 (2001) 69–74.

Charles Thas: Department of Pure Mathematics and Computer Algebra, Krijgslaan 281-S22, B-9000 Gent, Belgium

E-mail address: `charles.thas@rug.ac.be`

Forum Geometricorum
Volume 2 (2002) 151–161.

FORUM GEOM
ISSN 1534-1178

The Stammler Circles

Jean-Pierre Ehrmann and Floor van Lamoen

Abstract. We investigate circles intercepting chords of specified lengths on the sidelines of a triangle, a theme initiated by L. Stammler [6, 7]. We generalize his results, and concentrate specifically on the Stammler circles, for which the intercepts have lengths equal to the sidelengths of the given triangle.

1. Introduction

Ludwig Stammler [6, 7] has investigated, for a triangle with sidelengths a, b, c, circles that intercept chords of lengths μa, μb, μc ($\mu > 0$) on the sidelines BC, CA and AB respectively. He called these circles *proportionally cutting circles*,[1] and proved that their centers lie on the rectangular hyperbola through the circumcenter, the incenter, and the excenters. He also showed that, depending on μ, there are 2, 3 or 4 circles cutting chords of such lengths.

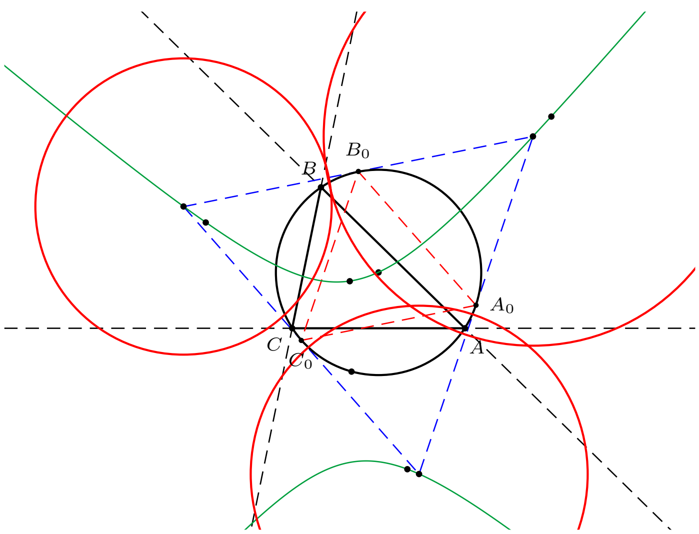


Figure 1. The three Stammler circles with the circumtangential triangle

As a special case Stammler investigated, for $\mu = 1$, the three proportionally cutting circles apart from the circumcircle. We call these the *Stammler circles*. Stammler proved that the centers of these circles form an equilateral triangle, circumscribed to the circumcircle and homothetic to Morley's (equilateral) trisector

Publication Date: November 22, 2002. Communicating Editor: Bernard Gibert.

[1]Proportionalschnittkreise in [6].

triangle. In fact this triangle is tangent to the circumcircle at the vertices of the circumtangential triangle. [2] See Figure 1.

In this paper we investigate the circles that cut chords of specified lengths on the sidelines of ABC, and obtain generalizations of results in [6, 7], together with some further results on the Stammler circles.

2. The cutting circles

We define a (u, v, w)-cutting circle as one that cuts chords of lengths u, v, w on the sidelines BC, CA, AB of ABC respectively. This is to be distinguished from a $(u : v : w)$-cutting circle, which cuts out chords of lengths in the proportion $u : v : w$.

2.1. Consider a $(\mu u, \mu v, \mu w)$-cutting circle with center P, whose (signed) distances to the sidelines of ABC are respectively X, Y, Z. [3] It is clear that

$$Y^2 - Z^2 = \left(\frac{\mu}{2}\right)^2 (w^2 - v^2). \tag{1}$$

If $v \neq w$, this equation describes a rectangular hyperbola with center A and asymptotes the bisectors of angle A. In the same way, P also lies on the conics (generally rectangular hyperbolas)

$$Z^2 - X^2 = \left(\frac{\mu}{2}\right)^2 \left(u^2 - w^2\right) \tag{2}$$

and

$$X^2 - Y^2 = \left(\frac{\mu}{2}\right)^2 \left(v^2 - u^2\right). \tag{3}$$

These three hyperbolas generate a pencil which contains the conic with barycentric equation

$$\frac{(v^2 - w^2)x^2}{a^2} + \frac{(w^2 - u^2)y^2}{b^2} + \frac{(u^2 - v^2)z^2}{c^2} = 0. \tag{4}$$

This is a rectangular hyperbola through the incenter, excenters and the points $(\pm au : \pm bv : \pm cw)$.

Theorem 1. *The centers of the* $(u : v : w)$*-cutting circles lie on the rectangular hyperbola through the incenter and the excenters and the points with homogeneous barycentric coordinates* $(\pm au : \pm bv : \pm cw)$.

Remarks. 1. When $u = v = w$, the centers of $(u : v : w)$-cutting circles are the incenter and excenters themselves.

2. Triangle ABC is self polar with respect to the hyperbola (4).

[2] The vertices of the circumtangential triangle are the triple of points X on the circumcircle for which the line through X and its isogonal conjugate is tangent to the circumcircle. These are the isogonal conjugates of the infinite points of the sidelines of the Morley trisector triangle. See [4] for more on the circumtangential triangle.

[3] We say that the point P has *absolute* normal coordinates (X, Y, Z) with respect to triangle ABC.

2.2. Since (1) and (2) represent two rectangular hyperbolas with distinct asymptote directions, these hyperbolas intersect in four points, of which at least two are real points. Such are the centers of $(\mu u, \mu v, \mu w)$-cutting circles. The limiting case $\mu = 0$ always yields four real intersections, the incenter and excenters. As μ increases, there is some $\mu = \mu_0$ for which the hyperbolas (1) and (2) are tangent, yielding a double point. For $\mu > \mu_0$, the hyperbolas (1, 2, 3) have only two real common points. When there are four real intersections, these form an orthocentric system. From (1), (2) and (3) we conclude that A, B, C must be on the nine point circle of this orthocentric system.

Theorem 2. *Given positive real numbers u, v, w, there are four (u, v, w)-cutting circles, at least two of which are real. When there are four distinct real circles, their centers form an orthocentric system, of which the circumcircle is the nine point circle. When two of these centers coincide, they form a right triangle with its right angle vertex on the circumcircle.*

2.3. Let (O_1) and (O_2) be two (u, v, w)-cutting circles with centers O_1 and O_2. Consider the midpoint M of O_1O_2. The orthogonal projection of M on BC clearly is the midpoint of the orthogonal projections of O_1 and O_2 on the same line. Hence, it has equal powers with respect to the circles (O_1) and (O_2), and lies on the radical axis of these circles. In the same way the orthogonal projections of M on AC and AB lie on this radical axis as well. It follows that M is on the circumcircle of ABC, its Simson-Wallace line being the radical axis of (O_1) and (O_2). See Figure 2.

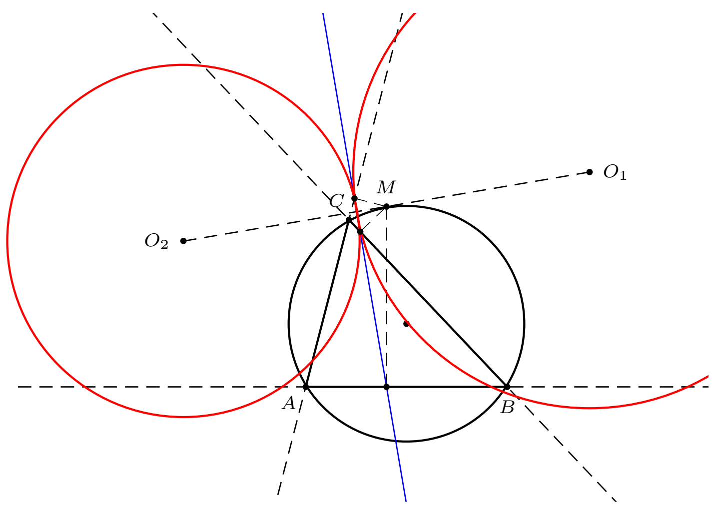


Figure 2. The radical axis of (O_1) and (O_2) is the Simson-Wallace line of M

2.4. Let Q be the reflection of the De Longchamps point L in M.[4] It lies on the circumcircle of the dilated (anticomplementary) triangle. The Simson-Wallace line

[4]The de Longchamps point L is the reflection of the orthocenter H in the circumcenter O. It is also the orthocenter of the dilated (anticomplementary) triangle.

of Q in the dilated triangle passes through M and is perpendicular to the Simson-Wallace line of M in ABC. It is therefore the line O_1O_2, which is also the same as MM^*, where M^* denotes the isogonal conjugate of M (in triangle ABC).

Theorem 3. *The lines connecting centers of (u, v, w)-cutting circles are Simson-Wallace lines of the dilated triangle. The radical axes of (u, v, w)-cutting circles are Simson-Wallace lines of ABC. When there are four real (u, v, w)-cutting circles, their radical axes form the sides of an orthocentric system perpendicular to the orthocentric system formed by the centers of the circles, and half of its size.*

2.5. For the special case of the centers O_1, O_2 and O_3 of the Stammler circles, we immediately see that they must lie on the circle $(O, 2R)$, where R is the circumradius. Since the medial triangle of $O_1O_2O_3$ must be circumscribed by the circumcircle, we see in fact that $O_1O_2O_3$ must be an equilateral triangle circumscribing the circumcircle. The sides of $O_1O_2O_3$ are thus Simson-Wallace lines of the dilated triangle, tangent to the nine point circle of the dilated triangle. See Figure 3.

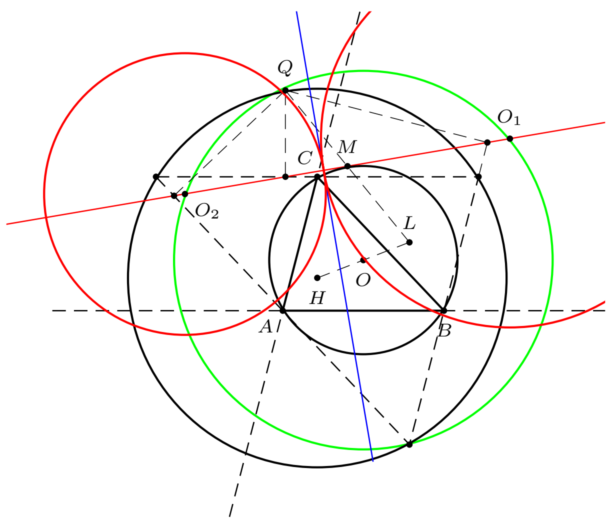


Figure 3. The line O_1O_2 is the dilated Simson-Wallace line of Q

Corollary 4. *The centers of the Stammler circles form an equilateral triangle circumscribing the circumcircle of ABC, and tangent to the circumcircle at the vertices $A_0B_0C_0$ of the circumtangential triangle. The radical axes of the Stammler circles among themselves are the Simson-Wallace lines of A_0, B_0, C_0.* [5] *The radical axes of the Stammler circles with the circumcircle are the sidelines of triangle $A_0B_0C_0$ translated by* $\mathbf{ON}$*, where N is the nine-point center of triangle ABC.*

[5]These are the three Simson-Wallace lines passing through N, *i.e.*, the cevian lines of N in the triangle which is the translation of $A_0B_0C_0$ by $\mathbf{ON}$. They are also the tangents to the Steiner deltoid at the cusps.

Remark. Since the nine-point circle of an equilateral triangle is also its incircle, we see that the centers of the Stammler circles are the only possible equilateral triangle of centers of (u, v, w)-cutting circles.

3. Constructions

3.1. Given a (u, v, w)-cutting circle with center P, let P' be the reflection of P in the circumcenter O. The centers of the other (u, v, w)-cutting circles can be found by intersecting the hyperbola (4) with the circle $P'(2R)$. One of the common points is the reflection of P in the center of the hyperbola.[6] The others are the required centers. This gives a *conic* construction. In general, the points of intersection are not constructible by ruler and compass. See Figure 4.

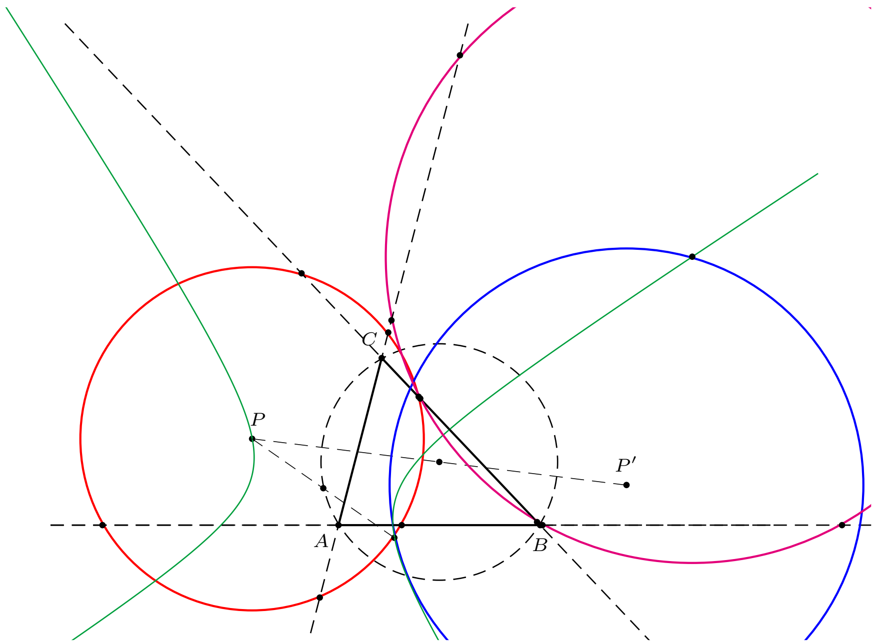


Figure 4. Construction of (u, v, w)-cutting circles

3.2. The same method applies when we are only given the magnitudes u, v, w. The centers of (u, v, w)-cutting circles can be constructed as the common points of the hyperbolas (1), (2), (3) with $\mu = 1$. If we consider two points T_A, T_B lying respectively on the lines CB, CA and such as $CT_A = u$, $CT_B = v$, the hyperbola (3) passes through the intersection M_0 of the perpendicular bisectors of CT_A and CT_B. Its asymptotes being the bisectors of angle C, a variable line through M_0 intersects these asymptotes at D, D'. The reflection of M_0 with respect to the midpoint of DD' lies on the hyperbola.

[6]The center of the hyperbola (4) is the point $\left(\frac{a^2}{v^2-w^2} : \frac{b^2}{w^2-u^2} : \frac{c^2}{u^2-v^2}\right)$ on the circumcircle.

3.3. When two distinct centers P and P' are given, then it is easy to construct the remaining two centers. Intersect the circumcircle and the circle with diameter PP', let the points of intersection be U and U'. Then the points $Q = PU \cap P'U'$ and $Q' = PU' \cap P'U$ are the points desired.

When one center P on the circumcircle is given, then P must in fact be a double point, and thus the right angle vertex of a right triangle containing the three $(u, v, w)-$intercepting circles. As the circumcircle of ABC is the nine point circle of the right triangle, the two remaining vertices must lie on the circle through P with P_r as center, where P_r is the reflection of P through O. By the last sentence before Theorem 3, we also know that the two remaining centers must lie on the line $P_r P_r^*$. Intersection of circle and line give the desired points.

3.4. Let three positive numbers u, v and w be given, and let P be a point on the hyperbola of centers of $(u : v : w)$-cutting circles. We can construct the circle with center P intercepting on the sidelines of ABC chords of lengths μu, μv and μw respectively for some μ.

We start from the point Q with barycentrics $(au : bv : cw)$. Let X, Y and Z be the distances from P to BC, AC and AB respectively. Since P satisfies (4) we have

$$(v^2 - w^2)X^2 + (w^2 - u^2)Y^2 + (u^2 - v^2)Z^2 = 0, \tag{5}$$

which is the equation in normal coordinates of the rectangular hyperbola through Q, the incenter and the excenters.

Now, the parallel through Q to AC (respectively AB) intersects AB (respectively AC) in Q_1 (respectively Q_2). The line perpendicular to Q_1Q_2 through P intersects AQ at U. The power p_a of P with respect to the circle with diameter AU is equal to $\dfrac{w^2Y^2 - v^2Z^2}{w^2 - v^2}$. Similarly we find powers p_b and p_c.

As P lies on the hyperbola given by (5), we have $p_a = p_b = p_c$. Define ρ by $\rho^2 = p_a$. Now, the circle (P, ρ) intercepts chords of with lengths L_a, L_b, L_c respectively on the sidelines of ABC, where

$$\left(\frac{L_a}{L_b}\right)^2 = \frac{\rho^2 - X^2}{\rho^2 - Y^2} = \frac{p_c - X^2}{p_c - Y^2} = \left(\frac{u}{v}\right)^2$$

and similarly

$$\left(\frac{L_b}{L_c}\right)^2 = \left(\frac{v}{w}\right)^2.$$

Hence this circle (P, ρ), if it exists and intersects the side lines, is the required circle. To construct this circle, note that if U' is the midpoint of AU, the circle goes through the common points of the circles with diameters AU and PU'.

4. The Stammler circles

For some particular results on the Stammler circles we use complex number coordinates. Each point is identified with a complex number $\rho \cdot e^{i\theta}$ called its *affix*. Here, (ρ, θ) are the polar coordinates with the circumcenter O as pole, scaled in such a way that points on the circumcircle are represented by unit complex numbers. Specifically, the vertices of the circumtangential triangle are represented by the cube roots of unity, namely,

$$A_0 = 1, \quad B_0 = \omega, \quad C_0 = \omega^2 = \overline{\omega},$$

where $\omega^3 = 1$. In this way, the vertices A, B, C have as affixes unit complex numbers $A = e^{i\theta}$, $B = e^{i\varphi}$, $C = e^{i\psi}$ satisfying $\theta + \varphi + \psi \equiv 0 \pmod{2\pi}$. In fact, we may take

$$\theta = \frac{2}{3}(\beta - \gamma), \quad \varphi = \frac{2}{3}(\beta + 2\gamma), \quad \psi = -\frac{2}{3}(2\beta + \gamma), \tag{6}$$

where α, β, γ are respectively the measures of angles A, B, C. In this setup the centers of the Stammler circles are the points

$$\Omega_A = -2, \qquad \Omega_B = -2\omega, \qquad \Omega_C = -2\overline{\omega}.$$

4.1. The intersections of the A-Stammler circle with the sidelines of ABC are

$$\begin{array}{ll} A_1 = B + \overline{A} - 1, & A_2 = C + \overline{A} - 1, \\ B_1 = C + \overline{B} - 1, & B_2 = A + \overline{B} - 1, \\ C_1 = A + \overline{C} - 1, & C_2 = B + \overline{C} - 1. \end{array}$$

The reflections of A, B, C in the line B_0C_0 are respectively

$$A' = -1 - \overline{A}, \qquad B' = -1 - \overline{B}, \qquad C' = -1 - \overline{C}.$$

The reflections of A', B', C' respectively in BC, CA, AB are

$$A'' = (1 + B)(1 + C), \qquad B'' = (1 + C)(1 + A), \qquad C'' = (1 + A)(1 + B).$$

Now,

$$\begin{aligned} B'' - A'' &= B_2 - A_1 = \frac{2}{\sqrt{3}}\left(\sin\theta - \sin\varphi\right)\left(C_0 - B_0\right), \\ C'' - B'' &= C_2 - B_1 = \frac{2}{\sqrt{3}}(\sin\varphi - \sin\psi)(C_0 - B_0), \\ A'' - C'' &= A_2 - C_1 = \frac{2}{\sqrt{3}}(\sin\psi - \sin\theta)(C_0 - B_0). \end{aligned}$$

Moreover, as the orthocenter $H = A + B + C$, the points $\Omega_B + H$ and $\Omega_C + H$ are collinear with $A''B''C''$.

4.2. Let R_A be the radius of the A-Stammler circle. It is easy to check that the twelve segments $A'B$, $A'C$, $A''B$, $A''C$, $B'C$, $B'A$, $B''C$, $B''A$, $C'A$, $C'B$, $C''A$, $C''B$ all have length equal to $R_A = \Omega_A A_1$. See Figure 6. Making use of the affixes, we easily obtain

$$R_A^2 = 3 + 2\left(\cos\theta + \cos\varphi + \cos\psi\right). \tag{7}$$

Theorem 5. *From the points of intersection of each of the Stammler circles with the sidelines of ABC three chords can be formed, with the condition that each chord is parallel to the side of Morley's triangle corresponding to the Stammler circle. The smaller two of these chords together are as long as the greater one.*

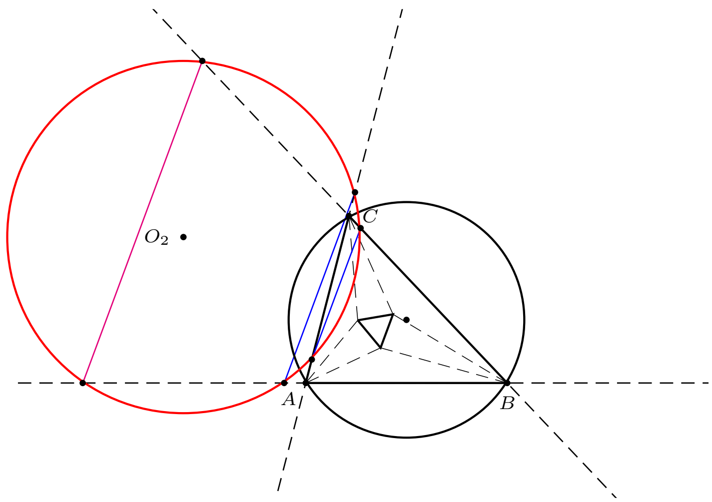


Figure 5. Three parallel chords on the $B-$Stammler circle

Remark. This is indeed true for any conic intercepting chords of lengths a, b, c on the sidelines.

4.3. We investigate the triangles $P_AP_BP_C$ with P_AB, P_AC, P_BA, P_BC, P_CA, P_CB all of length $\rho = \sqrt{\nu}$, which are perspective to ABC through P. Let P have homogeneous barycentric coordinates $(p : q : r)$. The line AP and the perpendicular bisector of BC meet in the point

$$P_A = \left(-(q-r)a^2 : q(b^2-c^2) : r(b^2-c^2)\right).$$

With the distance formula, [7] we have

$$|P_AB|^2 = a^2\frac{(a^2(c^2q^2+b^2r^2)+((b^2-c^2)^2-a^2(b^2+c^2))qr}{((a^2-b^2+c^2)q-(a^2+b^2-c^2)r)^2}$$

Similarly we find expressions for the squared distances $|P_BC|^2$ and $|P_CA|^2$.

Now let $|P_AB|^2 = |P_BC|^2 = |P_CA|^2 = \nu$. From these three equations we can eliminate q and r. When we simplify the equation assuming that ABC is nonisosceles and nondegenerate, this results in

$$p\nu(-16\triangle^2\nu+a^2b^2c^2)(-16\triangle^2\nu^3+a^2b^2c^2(9\nu^2-3(a^2+b^2+c^2)\nu+a^2b^2+b^2c^2+a^2c^2)) = 0. \tag{8}$$

Here, $\triangle$ is the area of triangle ABC. One real solution is clearly $\rho = \frac{a^2b^2c^2}{16\triangle^2} = R^2$. The other nonzero solutions are the roots of the cubic equation

$$\nu^3 - R^2(9\nu^2-3(a^2+b^2+c^2)\nu+a^2b^2+b^2c^2+a^2c^2) = 0. \tag{9}$$

[7]See for instance [5, Proposition 2].

As $A'B'C'$ is a particular solution of the problem, the roots of this cubic equation are the squares of the radii of the Stammler circles. A simple check of cases shows that the mentioned solutions are indeed the only ones.

Theorem 6. *Reflect the vertices of ABC through one of the sides of the circumtangential triangle to A', B' and C'. Then $A'B'C'$ lie on the perpendicular bisectors. In particular, together with O as a triple point and the reflections of O through the sides of ABC these are the only triangles perspective to ABC with $A'B = A'C = B'A = B'C = C'A = C'B$, for nonisosceles (and nondegenerate) ABC.*

Remark. Theorem 6 answers a question posed by A. P. Hatzipolakis [3].

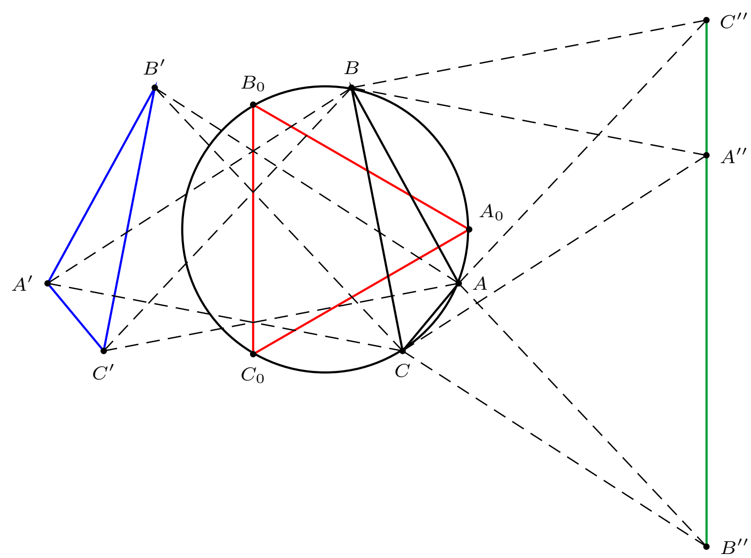


Figure 6. A perspective triangle $A'B'C'$ and the corresponding degenerate $A''B''C''$

4.4. Suppose that three points U, V, W lie on a same line ℓ and that $UB = UC = VC = VA = WA = WB = r \neq R$.

Let z_a the signed distance from A to ℓ. We have $\tan(\ell, BC) = 2 \cdot \dfrac{z_b - z_c}{\overline{VW}}$ and $z_a^2 = r^2 - \frac{1}{4}\overline{VW}^2$. It follows that

$$(\ell, BC) + (\ell, CA) + (\ell, AB) = 0,$$

and ℓ is parallel to a sideline of the Morley triangle of ABC. See [2, Proposition 5]. Now, U, V, W are the intersections of ℓ with the perpendicular bisectors of ABC and, for a fixed direction of ℓ, there is only one position of ℓ for which $VA = WA \neq R$. Hence the degenerate triangles $A''B''C''$, together with O as a triple point, are the only solutions in the collinear cases with $A''B = A''C = B''A = B''C = C''A = C''B$.

Theorem 7. *Reflect $A'B'C'$ through the sides of ABC respectively to A'', B'', C''. Then $A''B''C''$ are contained in the same line ℓ_i parallel to the side L_i of the circumtangential triangle. Together with O as a triple point these are the only degenerate triangles $A''B''C''$ satisfying the condition $A''B = A''C = B''A = B''C = C''A = C''B$. The lines ℓ_A, ℓ_B, ℓ_C bound the triangle which is the translation of $\Omega_A\Omega_B\Omega_C$ through the vector $\mathbf{OH}$.*

The three segments from $A''B''C''$ are congruent to the chords of Theorem 5. See Figure 6.

4.5. With θ, φ, ψ given by (6), we obtain from (7), after some simplifications,

$$\left(\frac{R_A}{R}\right)^2 = 1 + 8\cos\frac{\beta-\gamma}{3}\cos\frac{\beta+2\gamma}{3}\cos\frac{2\beta+\gamma}{3}.$$

Since $\left(\frac{OH}{R}\right)^2 = 1 - 8\sin\alpha\sin\beta\sin\gamma$, (see, for instance, [1, Chapter XI]), this shows that the radius R_A can be constructed, allowing angle trisection. R_A is the distance from O to the orthocenter of the triangle $AB'C'$, where B' is the image of B after rotation through $\frac{2(\beta-\gamma)}{3}$ about O, and C' is the image of A after rotation through $\frac{2(\gamma-\beta)}{3}$ about O.

The barycentric coordinates of Ω_A are

$$\left(a\left(\cos\alpha - 2\cos\frac{\beta-\gamma}{3}\right) : b\left(\cos\beta + 2\cos\frac{\beta+2\gamma}{3}\right) : c\left(\cos\gamma + 2\cos\frac{2\beta+\gamma}{3}\right)\right).$$

We find the distances

$$\begin{array}{ll}
B_1C_2 = 2a\cos\frac{\beta-\gamma}{3}, & BA_1 = CA_2 = 2R\sin\frac{|\beta-\gamma|}{3}, \\
C_1A_2 = 2b\cos\frac{\beta+2\gamma}{3}, & CB_1 = AB_2 = 2R\sin\frac{\beta+2\gamma}{3}, \\
A_1B_2 = 2c\cos\frac{2\beta+\gamma}{3}, & AC_1 = BC_2 = 2R\sin\frac{2\beta+\gamma}{3}.
\end{array}$$

Finally we mention the following relations of the Stammler radii. These follow easily from the fact that they are the roots of the cubic equation (9).

$$\begin{aligned}
&R_A^2 + R_B^2 + R_C^2 = 9R^2; \\
&\frac{1}{R_A^2} + \frac{1}{R_B^2} + \frac{1}{R_C^2} = \frac{3(a^2+b^2+c^2)}{a^2b^2+a^2c^2+b^2c^2}; \\
&R_AR_BR_C = R\sqrt{a^2b^2+b^2c^2+c^2a^2}.
\end{aligned}$$

References

[1] O. Bottema, *Hoofdstukken uit de Elementaire Meetkunde*, 2nd ed. 1987, Epsilon Uitgaven, Utrecht.

[2] J.-P. Ehrmann and B. Gibert, A Morley configuration, *Forum Geom.*, 1 (2001) 51-58.

[3] A.P. Hatzipolakis, Hyacinthos message 4714, January 30, 2002.

[4] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1–295.

[5] F.M. van Lamoen and P. Yiu, The Kiepert Pencil of Kiepert Hyperbolas, *Forum Geom.*, 1 (2001) 125-132.

[6] L. Stammler, Dreiecks-Proportionalschnittkreise, ihre Mittenhyperbel und ein Pendant zum Satz von Morley, *Elem. Math.*, 47 (1992) 158-168.

[7] L. Stammler, Cutting Circles and the Morley Theorem, *Beitr. Alg. Geom.*, 38 (1997) 91-93. http://www.emis.de/journals/BAG/vol.38/no.1/7.html

Jean-Pierre Ehrmann: 6 rue des Cailloux, 92110 - Clichy, France
E-mail address: `Jean-Pierre EHRMANN@wanadoo.fr`

Floor van Lamoen: 4463 Statenhof 3, TV Goes, The Netherlands
E-mail address: `f.v.lamoen@wxs.nl`

Forum Geometricorum
Volume 2 (2002) 163–166.

FORUM GEOM
ISSN 1534-1178

Some Similarities Associated with Pedals

Jean-Pierre Ehrmann and Floor van Lamoen

Abstract. The pedals of a point divide the sides of a triangle into six segments. We build on these segments six squares and obtain some interesting similarities.

Given a triangle ABC, the pedals of a point P are its orthogonal projections A', B', C' on the sidelines BC, CA, AB of the triangle. We build on the segments AC', $C'B$, BA', $A'C$, CB' and $B'A$ squares with orientation opposite to that of ABC.

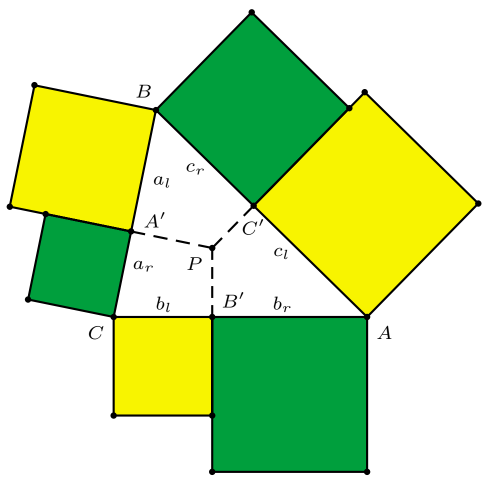


Figure 1

About this figure, O. Bottema [1, §77] showed that the sum of the areas of the squares on BA', CB' and AC' is equal to the sum of the areas of the squares on $A'C$, $B'A$ and $C'B$, namely,

$$a_l^2 + b_l^2 + c_l^2 = a_r^2 + b_r^2 + c_r^2.$$

See also [2, p.112]. Bottema showed conversely that when this equation holds, $A'B'C'$ is indeed a pedal triangle. While this can be easily established by applying the Pythagorean Theorem to the right triangles $AB'P$, $AC'P$, $BA'P$, $BC'P$, $CA'P$ and $CB'P$, we find a few more interesting properties of the figure. We adopt the following notations.

Publication Date: December 2, 2002. Communicating Editor: Paul Yiu.

O	circumcenter	
K	symmedian point	
$\triangle$	area of triangle ABC	
ω	Brocard angle	$\cot\omega = \frac{a^2+b^2+c^2}{4\triangle}$
Ω_1	Brocard point	$\angle BA\Omega_1 = \angle CB\Omega_1 = \angle AC\Omega_1 = \omega$
Ω_2	Brocard point	$\angle AB\Omega_2 = \angle BC\Omega_2 = \angle CA\Omega_2 = \omega$
$\mathsf{h}(P, r)$	homothety with center P and ratio r	
$\rho(P, \theta)$	rotation about P through an angle θ	

Let $A_1B_1C_1$ be the triangle bounded by the lines containing the sides of the squares opposite to BA', CB', AC' respectively. Similarly, let $A_2B_2C_2$ be the one bounded by the lines containing the sides of the squares opposite to $A'C$, $B'A$ and $C'B$.

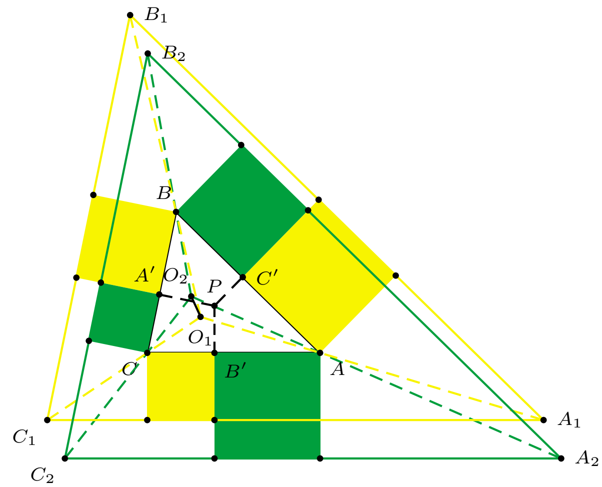


Figure 2

Theorem. *Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are each homothetic to ABC. Let O_1, and O_2 be the respective centers of homothety.*

(1) *The ratio of homothety in each case is $1 + \cot\omega$. Therefore, $A_1B_1C_1$ and $A_2B_2C_2$ are homothetic and congruent.*
(2) *The mapping $P \mapsto O_1$ is the direct similarity which is the rotation $\rho(\Omega_1, \frac{\pi}{2})$ followed by the homothety $\mathsf{h}(\Omega_1, \tan\omega)$. Likewise, The mapping $P \mapsto O_2$ is the direct similarity which is the rotation $\rho(\Omega_2, -\frac{\pi}{2})$ followed by the homothety $\mathsf{h}(\Omega_2, \tan\omega)$.*
(3) *The midpoint of the segment O_1O_2 is the symmedian point K.*
(4) *The vector of translation $A_1B_1C_1 \mapsto A_2B_2C_2$ is the image of $2\mathbf{OP}$ under the rotation $\rho(O, \frac{\pi}{2})$.*

Proof. We label the directed distances $a_l = BA'$, $a_r = A'C$, $b_l = CB'$, $b_r = B'A$, $c_l = AC'$ and $c_r = C'B$ as in Figure 1. Because ABC and $A_1B_1C_1$ are homothetic through O_1, the distances f, g, h of O_1 to the respective sides of ABC are in the same ratio as the distances between the corresponding sides of ABC and $A_1B_1C_1$. We have $f : g : h = a_l : b_l : c_l$. See Figure 2. Furthermore, the sum of the areas of triangles O_1BC, AO_1C and ABO_1 is equal to the area $\triangle$ of ABC, so that $af + bg + ch = 2\triangle$. But we also have

$$\begin{aligned} a_l^2 + b_l^2 + c_l^2 &= a_r^2 + b_r^2 + c_r^2 \\ &= (a - a_l)^2 + (b - b_l)^2 + (c - c_l)^2, \end{aligned}$$

from which we find

$$aa_l + bb_l + cc_l = \frac{a^2 + b^2 + c^2}{2} = 2\triangle \cot \omega.$$

This shows that $\frac{a_l}{f} = \frac{b_l}{g} = \frac{c_l}{h} = \cot \omega$, and thus that the ratio of homothety of $A_1B_1C_1$ to ABC is $1 + \cot \omega$. By symmetry, we find the same ratio of homothety of $A_2B_2C_2$ to ABC. This proves (1).

Now suppose that $P = O_1$. Then $\tan \angle CBO_1 = \frac{f}{a_l} = \tan \omega$. By symmetry this shows that P must be the Brocard point Ω_1.

To investigate the mapping $P \mapsto O_1$, we imagine that P moves through a line perpendicular to BC. For all points P on this line a_l is the same, so that for all images O_1 the distance f is the same. Therefore, O_1 traverses a line parallel to BC. Now imagine that P travels a distance d in the direction $A'P$. Then $AC' = c_l$ decreases with $d/\sin B$. The distance h of O_1 to AB thus decreases with $\frac{d \tan \omega}{\sin B}$, and O_1 must have travelled in the direction $\mathbf{CB}$ through $d \tan \omega$. Of course we can find similar results by letting P move through a line perpendicular to AC or AB.

Now any point P can be reached from Ω_1 by first going through a certain distance perpendicular to BC and then through another distance perpendicular to AC. Since Ω_1 is a fixed point of $P \mapsto O_1$, we can combine the results of the previous paragraph to conclude that $P \mapsto O_1$ is the rotation $\rho(\Omega_1, \frac{\pi}{2})$ followed by the homothety $\mathsf{h}(\Omega_1, \tan \omega)$.

In a similar fashion we see that $P \mapsto O_2$ is the rotation $\rho(\Omega_2, -\frac{\pi}{2})$ followed by the homothety $\mathsf{h}(\Omega_2, \tan \omega)$. This proves (2).

Now note that the pedal triangle of O is the medial triangle, so that the images of O under both mappings are identical. This image must be the point for which the distances to the sides are proportional to the corresponding sides, well known to be the symmedian point K. Now the segment OP is mapped to KO_1 and KO_2 respectively under the above mappings, while the image segments are congruent and make an angle of π. This proves (3).

More precisely the ratio of lengths $|KO_1| : |OP| = \tan \omega : 1$, so that $|O_1O_2| : |OP| = 2 \tan \omega : 1$. By (1), we also know that $|O_1O_2| : |A_1A_2| = \tan \omega : 1$. Together with the observation that O_1O_2 and A_1A_2 are oppositely parallel, this proves (4). □

We remark that (1) can be generalized to *inscribed* triangles $A'B'C'$. Since $BA' + A'C = BC$ it is clear that the line midway between B_1C_1 and B_2C_2 is at distance $\frac{a}{2}$ from BC, it is the line passing through the apex of the isosceles right triangle erected outwardly on BC. We conclude that the midpoints of A_1A_2, B_1B_2

and C_1C_2 form a triangle independent from $A'B'C'$, homothetic to ABC through K with ratio $1 + \cot\omega$. But then since $A_1B_1C_1$ and $A_2B_2C_2$ are homothetic to each other, as well as to ABC, it follows that the sum of their homothety ratios is $2(1 + \cot\omega)$.

References

[1] O. Bottema, *De Elementaire Meetkunde van het Platte Vlak*, 1938, P. Noordhoff, Groningen-Batavia.
[2] R. Deaux, *Compléments de Géométrie Plane*, A. de Boeck, Brussels, 1945.

Jean-Pierre Ehrmann: 6 rue des Cailloux, 92110 - Clichy, France
E-mail address: Jean-Pierre.EHRMANN@wanadoo.fr

Floor van Lamoen: Statenhof 3, 4463 TV Goes, The Netherlands
E-mail address: f.v.lamoen@wxs.nl

Forum Geometricorum
Volume 2 (2002) 167–173.

FORUM GEOM
ISSN 1534-1178

Brahmagupta Quadrilaterals

K. R. S. Sastry

Abstract. The Indian mathematician Brahmagupta made valuable contributions to mathematics and astronomy. He used Pythagorean triangles to construct general Heron triangles and cyclic quadrilaterals having integer sides, diagonals, and area, *i.e.*, Brahmagupta quadrilaterals. In this paper we describe a new numerical construction to generate an infinite family of Brahmagupta quadrilaterals from a Heron triangle.

1. Introduction

A triangle with integer sides and area is called a Heron triangle. If some of these elements are rationals that are not integers then we call it a rational Heron triangle. More generally, a polygon with integer sides, diagonals and area is called a Heron polygon. A rational Heron polygon is analogous to a rational Heron triangle. Brahmagupta's work on Heron triangles and cyclic quadrilaterals intrigued later mathematicians. This resulted in Kummer's complex construction to generate Heron quadrilaterals outlined in [2]. By a Brahmagupta quadrilateral we mean a cyclic Heron quadrilateral. In this paper we give a construction of Brahmagupta quadrilaterals from rational Heron triangles.

We begin with some well known results from circle geometry and trigonometry for later use.

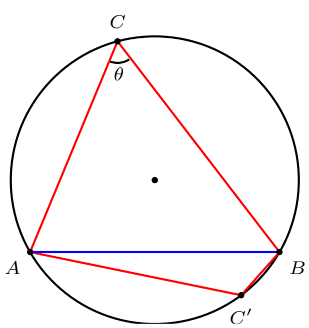


Figure 1

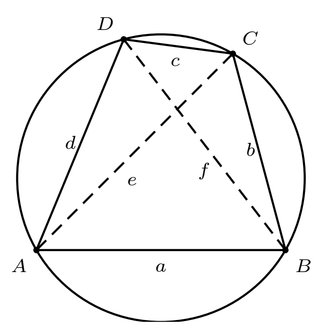


Figure 2

Figure 1 shows a chord AB of a circle of radius R. Let C and C' be points of the circle on opposite sides of AB. Then,

$$\angle ACB + \angle AC'B = \pi;$$
$$AB = 2R\sin\theta. \tag{1}$$

Publication Date: December 9, 2002. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for the help rendered in the preparation of this paper.

Throughout our discussion on Brahmagupta quadrilaterals the following notation remains standard. $ABCD$ is a cyclic quadrilateral with vertices located on a circle in an order. $AB = a$, $BC = b$, $CD = c$, $DA = d$ represent the sides or their lengths. Likewise, $AC = e$, $BD = f$ represent the diagonals. The symbol $\triangle$ represents the area of $ABCD$. Brahmagupta's famous results are

$$e = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}, \tag{2}$$

$$f = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}, \tag{3}$$

$$\triangle = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \tag{4}$$

where $s = \frac{1}{2}(a+b+c+d)$.

We observe that $d = 0$ reduces to Heron's famous formula for the area of triangle in terms of a, b, c. In fact the reader may derive Brahmagupta's expressions in (2), (3), (4) independently and see that they give two characterizations of a cyclic quadrilateral. We also observe that Ptolemy's theorem, viz., *the product of the diagonals of a cyclic quadrilateral equals the sum of the products of the two pairs of opposite sides*, follows from these expressions. In the next section, we give a construction of Brahmagupta quadrilaterals in terms of Heron angles. A Heron angle is one with rational sine and cosine. See [4]. Since

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2},$$

for $t = \tan\frac{\theta}{2}$, the angle θ is Heron if and only $\tan\frac{\theta}{2}$ is rational. Clearly, sums and differences of Heron angles are Heron angles. If we write, for triangle ABC, $t_1 = \tan\frac{A}{2}$, $t_2 = \tan\frac{B}{2}$, and $t_3 = \tan\frac{C}{2}$, then

$$a : b : c = t_1(t_2+t_3) : t_2(t_3+t_1) : t_3(t_1+t_2).$$

It follows that a triangle is rational if and only if its angles are Heron.

2. Construction of Brahmagupta quadrilaterals

Since the opposite angles of a cyclic quadrilateral are supplementary, we can always label the vertices of one such quadrilateral $ABCD$ so that the angles $A,\ B \leq \frac{\pi}{2}$ and $C, D \geq \frac{\pi}{2}$. The cyclic quadrilateral $ABCD$ is a rectangle if and only if $A = B = \frac{\pi}{2}$; it is a trapezoid if and only if $A = B$. Let $\angle CAD = \angle CBD = \theta$. The cyclic quadrilateral $ABCD$ is rational if and only if the angles A, B and θ are Heron angles.

If $ABCD$ is a Brahmagupta quadrilateral whose sides AD and BC are not parallel, let E denote their intersection. [1] In Figure 3, let $EC = \alpha$ and $ED = \beta$. The triangles EAB and ECD are similar so that $\dfrac{AB}{CD} = \dfrac{EB}{ED} = \dfrac{EA}{EC} = \lambda$, say.

[1]Under the assumption that $A,\ B \leq \frac{\pi}{2}$, these lines are parallel only if the quadrilateral is a rectangle.

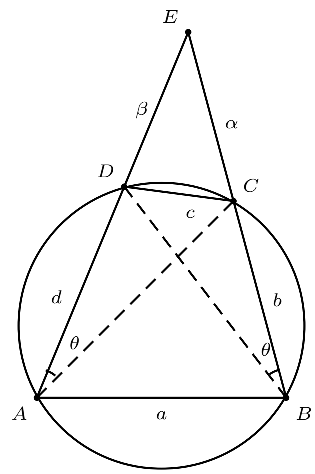


Figure 3

That is,

$$\frac{a}{c} = \frac{\alpha + b}{\beta} = \frac{\beta + d}{\alpha} = \lambda,$$

or

$$a = \lambda c, \quad b = \lambda\beta - \alpha, \quad d = \lambda\alpha - \beta, \qquad \lambda > \max\left(\frac{\alpha}{\beta}, \frac{\beta}{\alpha}\right). \tag{5}$$

Furthermore, from the law of sines, we have

$$e = 2R\sin B = 2R\sin D = \frac{R}{\rho}\cdot\alpha, \quad f = 2R\sin A = 2R\sin C = \frac{R}{\rho}\cdot\beta. \tag{6}$$

where ρ is the circumradius of triangle ECD. Ptolemy's theorem gives $ac + bd = ef$, and

$$\frac{R^2}{\rho^2}\cdot\alpha\beta = c^2\lambda + (\beta\lambda - \alpha)(\alpha\lambda - \beta)$$

This equation can be rewritten as

$$\begin{aligned}\left(\frac{R}{\rho}\right)^2 =& \lambda^2 - \frac{\alpha^2 + \beta^2 - c^2}{\alpha\beta}\lambda + 1\\ =& \lambda^2 - 2\lambda\cos E + 1\\ =& (\lambda - \cos E)^2 + \sin^2 E,\end{aligned}$$

or

$$\left(\frac{R}{\rho} - \lambda + \cos E\right)\left(\frac{R}{\rho} + \lambda - \cos E\right) = \sin^2 E.$$

Note that $\sin E$ and $\cos E$ are rational since E is a Heron angle. In order to obtain rational values for R and λ we put

$$\frac{R}{\rho} - \lambda - \cos E = t \sin E,$$
$$\frac{R}{\rho} + \lambda + \cos E = \frac{\sin E}{t},$$

for a rational number t. From these, we have

$$R = \frac{\rho}{2} \sin E \left(t + \frac{1}{t}\right) = \frac{c}{4}\left(t + \frac{1}{t}\right),$$
$$\lambda = \frac{1}{2} \sin E \left(\frac{1}{t} - t\right) - \cos E.$$

From the expression for R, it is clear that $t = \tan\frac{\theta}{2}$. If we set

$$t_1 = \tan\frac{D}{2} \quad \text{and} \quad t_2 = \tan\frac{C}{2}$$

for the Heron angles C and D, then

$$\cos E = \frac{(t_1+t_2)^2 - (1-t_1t_2)^2}{(1+t_1^2)(1+t_2^2)}$$

and

$$\sin E = \frac{2(t_1+t_2)(1-t_1t_2)}{(1+t_1^2)(1+t_2^2)}.$$

By choosing $c = t(1+t_1^2)(1+t_2^2)$, we obtain from (6)

$$\alpha = \frac{tt_1(1+t_1^2)(1+t_2^2)^2}{(t_1+t_2)(1-t_1t_2)}, \quad \beta = \frac{tt_2(1+t_1^2)^2(1+t_2^2)}{(t_1+t_2)(1-t_1t_2)},$$

and from (5) the following simple rational parametrization of the sides and diagonals of the cyclic quadrilateral:

$$a = (t(t_1+t_2) + (1-t_1t_2))(t_1+t_2 - t(1-t_1t_2)),$$
$$b = (1+t_1^2)(t_2 - t)(1+tt_2),$$
$$c = t(1+t_1^2)(1+t_2^2),$$
$$d = (1+t_2^2)(t_1 - t)(1+tt_1),$$
$$e = t_1(1+t^2)(1+t_2^2),$$
$$f = t_2(1+t^2)(1+t_1^2).$$

This has area

$$\triangle = t_1t_2(2t(1-t_1t_2) - (t_1+t_2)(1-t^2))(2(t_1+t_2)t + (1-t_1t_2)(1-t^2)),$$

and is inscribed in a circle of diameter

$$2R = \frac{(1+t_1^2)(1+t_2^2)(1+t^2)}{2}.$$

Replacing $t_1 = \frac{n}{m}$, $t_2 = \frac{q}{p}$, and $t = \frac{v}{u}$ for integers m, n, p, q, u, v in these expressions, and clearing denominators in the sides and diagonals, we obtain Brahmagupta quadrilaterals. Every Brahmagupta quadrilateral arises in this way.

3. Examples

Example 1. By choosing $t_1 = t_2 = \frac{n}{m}$ and putting $t = \frac{v}{u}$, we obtain a generic Brahmagupta trapezoid:

$$\begin{aligned} a =&(m^2u - n^2u + 2mnv)(2mnu - m^2v + n^2v), \\ b = d =&(m^2 + n^2)(nu - mv)(mu + nv), \\ c =&(m^2 + n^2)^2uv, \\ e = f =&mn(m^2 + n^2)(u^2 + v^2), \end{aligned}$$

This has area

$$\triangle = 2m^2n^2(nu-mv)(mu+nv)((m+n)u-(m-n)v)((m+n)v-(m-n)u),$$

and is inscribed in a circle of diameter

$$2R = \frac{(m^2 + n^2)^2(u^2 + v^2)}{2}.$$

The following Brahmagupta trapezoids are obtained from simple values of t_1 and t, and clearing common divisors.

t_1	t	a	$b = d$	c	$e = f$	$\triangle$	$2R$
$1/2$	$1/7$	25	15	7	20	192	25
$1/2$	$2/9$	21	10	9	17	120	41
$1/3$	$3/14$	52	15	28	41	360	197
$1/3$	$3/19$	51	20	19	37	420	181
$2/3$	$1/8$	14	13	4	15	108	$65/4$
$2/3$	$3/11$	21	13	11	20	192	61
$2/3$	$9/20$	40	13	30	37	420	$1203/4$
$3/4$	$2/11$	25	25	11	30	432	61
$3/4$	$1/18$	17	25	3	26	240	$325/12$
$3/5$	$2/9$	28	17	12	25	300	$164/3$

Example 2. Let ECD be the rational Heron triangle with $c : \alpha : \beta = 14 : 15 : 13$. Here, $t_1 = \frac{2}{3}$, $t_2 = \frac{1}{2}$ (and $t_3 = \frac{4}{7}$). By putting $t = \frac{v}{u}$ and clearing denominators, we obtain Brahmagupta quadrilaterals with sides

$$a = (7u-4v)(4u+7v),\ b = 13(u-2v)(2u+v),\ c = 65uv,\ d = 5(2u-3v)(3u+2v),$$

diagonals

$$e = 30(u^2 + v^2), \quad f = 26(u^2 + v^2),$$

and area

$$\triangle = 24(2u^2 + 7uv - 2v^2)(7u^2 - 8uv - 7v^2).$$

If we put $u = 3$, $v = 1$, we generate the particular one:

$$(a, b, c, d, e, f; \triangle) = (323, 91, 195, 165, 300, 260; 28416).$$

On the other hand, with $u = 11$, $v = 3$, we obtain a quadrilateral whose sides and diagonals are multiples of 65. Reduction by this factor leads to

$$(a, b, c, d, e, f; \triangle) = (65, 39, 33, 25, 52, 60; 1344).$$

This is inscribed in a circle of diameter 65. This latter Brahmagupta quadrilateral also appears in Example 4 below.

Example 3. If we take ECD to be a right triangle with sides $CD : EC : ED = m^2 + n^2 : 2mn : m^2 - n^2$, we obtain

$$\begin{aligned}
a =&(m^2 + n^2)(u^2 - v^2),\\
b =&((m - n)u - (m + n)v)((m + n)u + (m - n)v),\\
c =&2(m^2 + n^2)uv,\\
d =&2(nu - mv)(mu + nv),\\
e =&2mn(u^2 + v^2),\\
f =&(m^2 - n^2)(u^2 + v^2);\\
\triangle =&mn(m^2 - n^2)(u^2 + 2uv - v^2)(u^2 - 2uv - v^2).
\end{aligned}$$

Here, $\frac{u}{v} > \frac{m}{n}, \frac{m+n}{m-n}$. We give two very small Brahmagupta quadrilaterals from this construction.

n/m	v/u	a	b	c	d	e	f	$\triangle$	$2R$
$1/2$	$1/4$	75	13	40	36	68	51	966	85
$1/2$	$1/5$	60	16	25	33	52	39	714	65

Example 4. If the angle θ is chosen such that $A + B - \theta = \frac{\pi}{2}$, then the side BC is a diameter of the circumcircle of $ABCD$. In this case,

$$t = \tan\frac{\theta}{2} = \frac{1 - t_3}{1 + t_3} = \frac{t_1 + t_2 - 1 + t_1t_2}{t_1 + t_2 + 1 - t_1t_2}.$$

Putting $t_1 = \frac{n}{m}$, $t_2 = \frac{q}{p}$, and $t = \frac{(m+n)q-(m-n)p}{(m+n)p-(m-n)q}$, we obtain the following Brahmagupta quadrilaterals.

$$\begin{aligned}
a =&(m^2 + n^2)(p^2 + q^2),\\
b =&(m^2 - n^2)(p^2 + q^2),\\
c =&((m + n)p - (m - n)q)((m + n)q - (m - n)p),\\
d =&(m^2 + n^2)(p^2 - q^2),\\
e =&2mn(p^2 + q^2),\\
f =&2pq(m^2 + n^2).
\end{aligned}$$

Here are some examples with relatively small sides.

t_1	t_2	t	a	b	c	d	e	f	$\triangle$
$2/3$	$1/2$	$3/11$	65	25	33	39	60	52	1344
$3/4$	$1/2$	$1/3$	25	7	15	15	24	20	192
$3/4$	$1/3$	$2/11$	125	35	44	100	120	75	4212
$6/7$	$1/3$	$1/4$	85	13	40	68	84	51	1890
$7/9$	$1/3$	$1/5$	65	16	25	52	63	39	1134
$8/9$	$1/2$	$3/7$	145	17	105	87	144	116	5760
$7/11$	$1/2$	$1/4$	85	36	40	51	77	68	2310
$8/11$	$1/3$	$1/6$	185	57	60	148	176	111	9240
$11/13$	$1/2$	$2/5$	145	24	100	87	143	116	6006

References

[1] J. R. Carlson, Determination of Heronian triangles, *Fibonnaci Quarterly*, 8 (1970) 499 – 506, 551.

[2] L. E. Dickson, *History of the Theory of Numbers*, vol. II, Chelsea, New York, New York, 1971; pp.171 – 201.

[3] C. Pritchard, Brahmagupta, *Math. Spectrum*, 28 (1995–96) 49–51.

[4] K. R. S. Sastry, Heron angles, *Math. Comput. Ed.*, 35 (2001) 51 – 60.

[5] K. R. S. Sastry, Heron triangles: a Gergonne cevian and median perspective, *Forum Geom.*, 1 (2001) 25 – 32.

[6] K. R. S. Sastry, Polygonal area in the manner of Brahmagupta, *Math. Comput. Ed.*, 35 (2001) 147–151.

[7] D. Singmaster, Some corrections to Carlson's "Determination of Heronian triangles", *Fibonnaci Quarterly*, 11 (1973) 157 – 158.

K. R. S. Sastry: Jeevan Sandhya, DoddaKalsandra Post, Raghuvana Halli, Bangalore, 560 062, India.

Forum Geometricorum
Volume 2 (2002) 175–182.

FORUM GEOM
ISSN 1534-1178

The Apollonius Circle as a Tucker Circle

Darij Grinberg and Paul Yiu

Abstract. We give a simple construction of the circular hull of the excircles of a triangle as a Tucker circle.

1. Introduction

The Apollonius circle of a triangle is the circular hull of the excircles, the circle internally tangent to each of the excircles. This circle can be constructed by making use of the famous Feuerbach theorem that the nine-point circle is tangent *externally* to each of the excircles, and that the radical center of the excircles is the Spieker point X_{10}, the incenter of the medial triangle. If we perform an inversion with respect to the radical circle of the excircles, which is the circle orthogonal to each of them, the excircles remain invariant, while the nine-point circle is inverted into the Apollonius circle. The points of tangency of the Apollonius circle, being the inversive images of the points of tangency of the nine-point circle, can be constructed by joining to these latter points to Spieker point to intersect the respective excircles again. [1] See Figure 1. In this paper, we give another simple construction of the Apollonius circle by identifying it as a Tucker circle.

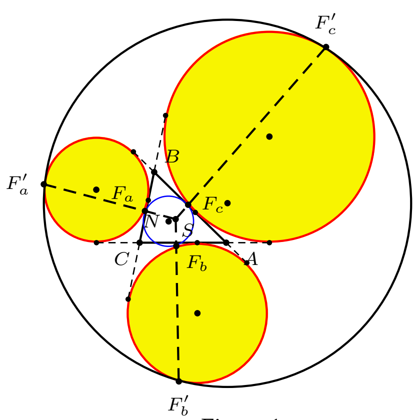


Figure 1

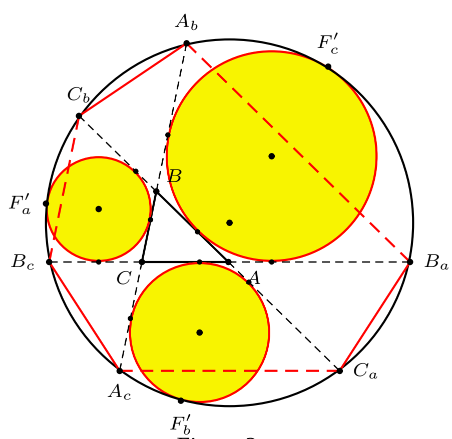


Figure 2

Theorem 1. *Let B_a and C_a be points respectively on the extensions of CA and BA beyond A such that B_aC_a is antiparallel to BC and has length s, the semiperimeter of triangle ABC. Likewise, let C_b, A_b be on the extensions of AB and CB beyond*

Publication Date: December 16, 2002. Communicating Editor: Jean-Pierre Ehrmann.

[1]The tangency of this circle with each of the excircles is internal because the Spieker point, the center of inversion, is contained in nine-point circle.

B, with C_bA_b antiparallel to CA and of length s, A_c, B_c on the extensions of BC and AC beyond C, with A_cB_c is antiparallel to AB and of length s. Then the six points A_b, B_a, C_a, A_c, B_c, C_b are concyclic, and the circle containing them is the Apollonius circle of triangle ABC.

The vertices of the Tucker hexagon can be constructed as follows. Let X_b and X_c be the points of tangency of BC with excircles (I_b) and (I_c) respectively. Since BX_b and CX_c each has length s, the parallel of AB through X_b intersects AC at C', and that of AC through X_c intersects AB at B' such that the segment $B'C'$ is parallel to BC and has length s. The reflections of B' and C' in the line I_bI_c are the points B_a and C_a such that triangle AB_aC_a is similar to ABC, with $B_aC_a = s$. See Figure 3. The other vertices can be similarly constructed. In fact, the Tucker circle can be constructed by locating A_c as the intersection of BC and the parallel through C_a to AC.

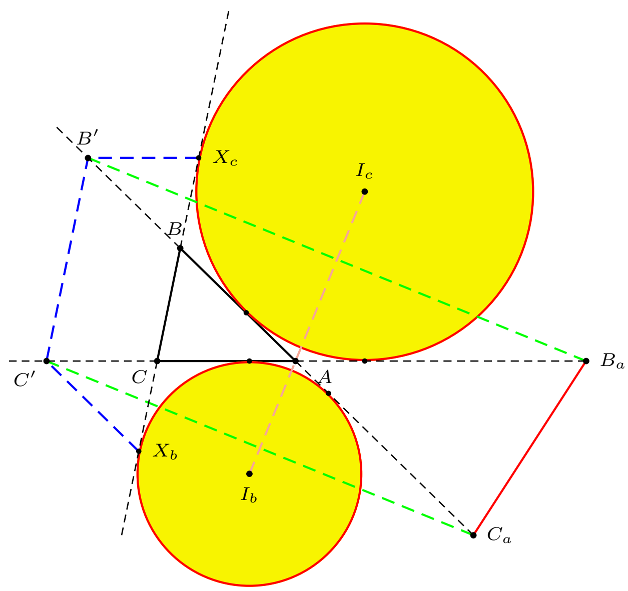


Figure 3

2. Some basic results

We shall denote the side lengths of triangle ABC by a, b, c.

R	circumradius
r	inradius
s	semiperimeter
$\triangle$	area
ω	Brocard angle

The Brocard angle is given by

$$\cot\omega = \frac{a^2+b^2+c^2}{4\triangle}.$$

Lemma 2. (1) $abc = 4Rrs$;
(2) $ab + bc + ca = r^2 + s^2 + 4Rr$;
(3) $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$;
(4) $(a + b)(b + c)(c + a) = 2s(r^2 + s^2 + 2Rr)$.

Proof. (1) follows from the formulae $\triangle = rs$ and $R = \frac{abc}{4\triangle}$.

(2) follows from the Heron formula $\triangle^2 = s(s - a)(s - b)(s - c)$ and

$$s^3 - (s - a)(s - b)(s - c) = (ab + bc + ca)s + abc.$$

(3) follows from (2) and $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$.

(4) follows from $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc$. $\square$

Unless explicitly stated, all coordinates we use in this paper are *homogeneous barycentric coordinates*. Here are the coordinates of some basic triangle centers.

circumcenter	O	$(a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2))$
incenter	I	$(a : b : c)$
Spieker point	S	$(b + c : c + a : a + b)$
symmedian point	K	$(a^2 : b^2 : c^2)$

Note that the sum of the coordinates of O is $16\triangle^2 = 16r^2s^2$. [2] We shall also make use of the following basic result on circles, whose proof we omit.

Proposition 3. *Let p_1, p_2, p_3 be the powers of A, B, C with respect to a circle $\mathcal{C}$. The power of a point with homogeneous barycentric coordinates $(x : y : z)$ with respect to the same circle is*

$$\frac{(x + y + z)(p_1x + p_2y + p_3z) - (a^2yz + b^2zx + c^2xy)}{(x + y + z)^2}.$$

Hence, the equation of the circle is

$$a^2yz + b^2zx + c^2xy = (x + y + z)(p_1x + p_2y + p_3z).$$

3. The Spieker radical circle

The fact that the radical center of the excircles is the Spieker point S is well known. See, for example, [3]. We verify this fact by computing the power of S with respect to the excircles. This computation also gives the radius of the radical circle.

Theorem 4. *The radical circle of the excircles has center at the Spieker point $S = (b + c : c + a : a + b)$, and radius $\frac{1}{2}\sqrt{r^2 + s^2}$.*

[2]This is equivalent to the following version of Heron's formula:

$$16\triangle^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4.$$

Proof. We compute the power of $(b+c:c+a:a+b)$ with respect to the A-excircle. The powers of A, B, C with respect to the A-excircle are clearly

$$p_1 = s^2, \qquad p_2 = (s-c)^2, \qquad p_3 = (s-b)^2.$$

With $x = b+c$, $y = c+a$, $z = a+b$, we have $x+y+z = 4s$ and

$$\begin{aligned}
&(x+y+z)(p_1x+p_2y+p_3z) - (a^2yz+b^2zx+c^2xy)\\
=&4s(s^2(b+c)+(s-c)^2(c+a)+(s-b)^2(a+b))\\
&-(a^2(c+a)(a+b)+b^2(a+b)(b+c)+c^2(b+c)(c+a))\\
=&2s(2abc+(a+b+c)(a^2+b^2+c^2)) - 2s(a^3+b^3+c^3+abc)\\
=&2s(abc+a^2(b+c)+b^2(c+a)+c^2(a+b))\\
=&4s^2(r^2+s^2),
\end{aligned}$$

and the power of the Spieker point with respect to the A-excircle is $\frac{1}{4}(r^2+s^2)$. This being symmetric in a, b, c, it is also the power of the same point with respect to the other two excircles. The Spieker point is therefore the radical center of the excircles, and the radius of the radical circle is $\frac{1}{2}\sqrt{r^2+s^2}$. □

We call this circle the Spieker radical circle, and remark that the Spieker point is the inferior of the incenter, namely, the image of the incenter under the homothety $\mathsf{h}(G, -\frac{1}{2})$ at the centroid G.

4. The Apollonius circle

To find the Apollonius circle it is more convenient to consider its superior, *i.e.*, its homothetic image $\mathsf{h}(G,-2)$ in the centroid G with ratio -2. This homothety transforms the nine-point circle and the Spieker radical circle into the circumcircle $O(R)$ and the circle $I(\sqrt{r^2+s^2})$ respectively.

Let d be the distance between O and I. By Euler's theorem, $d^2 = R^2 - 2Rr$. On the line OI we treat I as the origin, and O with coordinate R. The circumcircle intersects the line IO at the points $d \pm R$. The inversive images of these points have coordinates $\frac{r^2+s^2}{d\pm R}$. The inversive image is therefore a circle with radius

$$\frac{1}{2}\left|\frac{r^2+s^2}{d-R} - \frac{r^2+s^2}{d+R}\right| = \left|\frac{R(r^2+s^2)}{d^2-R^2}\right| = \frac{r^2+s^2}{2r}.$$

The center is the point Q' with coordinate

$$\frac{1}{2}\left(\frac{r^2+s^2}{d-R} + \frac{r^2+s^2}{d+R}\right) = \frac{d(r^2+s^2)}{d^2-R^2} = -\frac{r^2+s^2}{2Rr}\cdot d.$$

In other words,

$$IQ' : IO = -(r^2+s^2) : 2Rr.$$

Explicitly,

$$Q' = I - \frac{r^2+s^2}{2Rr}(O-I) = \frac{(r^2+s^2+2Rr)I - (r^2+s^2)O}{2Rr}.$$

From this calculation we make the following conclusions.

(1) The radius of the Apollonius circle is $\rho = \frac{r^2+s^2}{4r}$.
(2) The Apollonius center, being the homothetic image of Q' under $\mathsf{h}(G, -\frac{1}{2})$, is the point [3]

$$Q = \frac{1}{2}(3G - Q') = \frac{6Rr \cdot G + (r^2+s^2)O - (r^2+s^2+2Rr)I}{4Rr}.$$

Various authors have noted that Q lies on the Brocard axis OK, where the centers of Tucker circles lie. See, for example, [1, 9, 2, 7]. In [1], Aeppli states that if d_A, d_B, d_C are the distances of the vertices A, B, C to the line joining the center of the Apollonius circle with the circumcenter of ABC, then

$$d_A : d_B : d_C = \frac{b^2-c^2}{a^2} : \frac{c^2-a^2}{b^2} : \frac{a^2-b^2}{c^2}.$$

It follows that the barycentric equation of the line is

$$\frac{b^2-c^2}{a^2}x + \frac{c^2-a^2}{b^2}y + \frac{a^2-b^2}{c^2}z = 0.$$

This is the well known barycentric equation of the Brocard axis. Thus, the Apollonius center lies on the Brocard axis. Here, we write Q explicitly in terms of O and K.

Proposition 5. $Q = \frac{1}{4Rr}\left((s^2-r^2)O - \frac{1}{2}(a^2+b^2+c^2)K\right)$.

Proof.

$$\begin{aligned}
Q =& \frac{1}{4Rr}\left((r^2+s^2)O + 6Rr\cdot G - (r^2+s^2+2Rr)I\right)\\
=& \frac{1}{4Rr}\left((s^2-r^2)O + 2r^2\cdot O + 6Rr\cdot G - (r^2+s^2+2Rr)I\right)\\
=& \frac{1}{16Rrs^2}\left(4s^2(s^2-r^2)O + 8r^2s^2\cdot O + 24Rrs^2\cdot G - 4s^2(r^2+s^2+2Rr)I\right).
\end{aligned}$$

Consider the sum of the last three terms. By Lemma 2, we have

$$\begin{aligned}
& 8r^2s^2\cdot O + 24Rrs^2\cdot G - 4s^2(r^2+s^2+2Rr)I\\
=& 8r^2s^2\cdot O + abc\cdot 2s\cdot 3G - 2s(a+b)(b+c)(c+a)I\\
=& \frac{1}{2}(a^2(b^2+c^2-a^2), b^2(c^2+a^2-b^2), c^2(a^2+b^2-c^2))\\
& + (a+b+c)abc(1,1,1) - (a+b)(b+c)(c+a)(a,b,c).
\end{aligned}$$

[3]This point is X_{970} of [7].

Consider the first component.

$$\begin{aligned}
&\frac{1}{2}\left(a^2(b^2+c^2-a^2)+2abc(a+b+c)-2(a+b)(b+c)(c+a)a\right)\\
=&\frac{1}{2}\left(a^2(b^2+2bc+c^2-a^2)+2abc(a+b+c)-2a((a+b)(b+c)(c+a)+abc)\right)\\
=&\frac{1}{2}\left(a^2(a+b+c)(b+c-a)+2abc(a+b+c)-2a(a+b+c)(ab+bc+ca)\right)\\
=&s(a^2(b+c-a)+2abc-2a(ab+bc+ca))\\
=&s(a^2(b+c-a)-2a(ab+ca))\\
=&a^2s(b+c-a-2(b+c))\\
=&-a^2\cdot 2s^2.
\end{aligned}$$

Similarly, the other two components are $-b^2\cdot 2s^2$ and $-c^2\cdot 2s^2$. It follows that

$$\begin{aligned}
Q=&\frac{1}{16Rrs^2}\left(4s^2(s^2-r^2)O-2s^2(a^2,b^2,c^2)\right)\\
=&\frac{1}{4Rr}\left((s^2-r^2)O-\frac{1}{2}(a^2+b^2+c^2)K\right).
\end{aligned}\tag{1}$$

□

5. The Apollonius circle as a Tucker circle

It is well known that the centers of Tucker circles also lie on the Brocard axis. According to [8], a Tucker hexagon/circle has three principal parameters:

- the chordal angle $\phi\in\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right]$,
- the radius of the Tucker circle
$$r_\phi=\left|\frac{R}{\cos\phi+\cot\omega\sin\phi}\right|,$$
- the length of the equal antiparallels
$$d_\phi=2r_\phi\cdot\sin\phi.$$

This length d_ϕ is negative for $\phi<0$. In this way, for a given d_ϕ, there is one and only one Tucker hexagon with d_ϕ as the length of the antiparallel segments. In other words, a Tucker circle can be uniquely identified by d_ϕ. The center of the Tucker circle is the isogonal conjugate of the Kiepert perspector $K\left(\frac{\pi}{2}-\phi\right)$. Explicitly, this is the point

$$\frac{4\triangle\cot\phi\cdot O+(a^2+b^2+c^2)K}{4\triangle\cot\phi+(a^2+b^2+c^2)}.$$

Comparison with (1) shows that $4\triangle\cot\phi=-2(s^2-r^2)$. Equivalently,

$$\tan\phi=-\frac{2rs}{s^2-r^2}.$$

This means that $\phi = -2\arctan\frac{r}{s}$. Clearly, since $s > r$,

$$\cos\phi = \frac{s^2 - r^2}{r^2 + s^2}, \qquad \sin\phi = -\frac{2rs}{r^2 + s^2}.$$

Now, the radius of the Tucker circle with chordal angle $\phi = -2\arctan\frac{r}{s}$ is given by

$$r_\phi = \left|\frac{R}{\cos\phi + \cot\omega\sin\phi}\right| = \frac{r^2 + s^2}{4r}.$$

This is exactly the radius of the Apollonius circle. We therefore conclude that the Apollonius circle is the Tucker circle with chordal angle $-2\arctan\frac{r}{s}$. The common length of the antiparallels is

$$d_\phi = 2r_\phi \cdot \sin\phi = 2 \cdot \frac{r^2 + s^2}{4r} \cdot \frac{-2rs}{r^2 + s^2} = -s.$$

This proves Theorem 1 and justifies the construction in Figure 3.

6. Concluding remarks

We record the coordinates of the vertices of the Tucker hexagon.[4]

$$\begin{aligned} B_c &= (-as : 0 : as + bc), & C_b &= (-as : as + bc : 0),\\ A_b &= (0 : cs + ab : -cs), & B_a &= (cs + ab : 0 : -cs),\\ C_a &= (bs + ca : -bs : 0), & A_c &= (0 : -bs : bs + ca). \end{aligned}$$

From these, the power of A with respect to the Apollonian circle is

$$-\frac{cs}{a}\left(b + \frac{as}{c}\right) = \frac{-s(bc + as)}{a}.$$

Similarly, by computing the powers of B and C, we obtain the equation of the Apollonius circle as

$$a^2yz + b^2zx + c^2xy + s(x + y + z)\sum_{\text{cyclic}}\frac{bc + as}{a}x = 0.$$

Finally, with reference to Figure 1, Iwata and Fukagawa [5] have shown that triangles $F_a'F_b'F_c'$ and ABC are perspective at a point P on the line IQ with $IP : PQ = -r : \rho$.[5] They also remarked without proof that according to a Japanese wooden tablet dating from 1797,

$$\rho = \frac{1}{4}\left(\frac{s^4}{r_ar_br_c} + \frac{r_ar_br_c}{s^2}\right),$$

which is equivalent to $\rho = \frac{r^2+s^2}{4r}$ established above.

---

[4]These coordinates are also given by Jean-Pierre Ehrmann [2].

[5]This perspector is the Apollonius point $X_{181} = \left(\frac{a^2(b+c)^2}{s-a} : \frac{b^2(c+a)^2}{s-b} : \frac{c^2(a+b)^2}{s-c}\right)$ in [7]. In fact, the coordinates of F_a' are $(-a^2(a(b+c)+(b^2+c^2))^2 : 4b^2(c+a)^2s(s-c) : 4c^2(a+b)^2s(s-b))$; similarly for F_b' and F_c'.

References

[1] A. Aeppli: Das Taktionsproblem von Apollonius, angewandt auf die vier Berührungskreise eines Dreiecks, *Elemente der Mathematik*, 13 (1958) 25–30.
[2] J.-P. Ehrmann, Hyacinthos message 4620, January 1, 2002.
[3] E. Rouché et Ch. de Comberousse, *Traité de Géométrie*, Gauthier-Villars, Paris, 1935.
[4] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.
[5] C. Kimberling, S. Iwata, H. Fukagawa and T. Seimiya, Problem 1091 and solution, *Crux Math.*, 11 (1985) 324; 13 (1987) 128–129, 217–218.
[6] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.
[7] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://www2.evansville.edu/ck6/encyclopedia/.
[8] F. M. van Lamoen, Some concurrencies from Tucker hexagons, *Forum Geom.*, 2 (2002) 5–13.
[9] R. Stärk, Ein Satz über Eckentfernungen beim Dreieck, *Elemente der Mathematik*, 45 (1990) 155–165.

Darij Grinberg: Geroldsäckerweg 7, D-76139 Karlsruhe, Germany
E-mail address: darij_grinberg@web.de

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida, 33431-0991, USA
E-mail address: yiu@fau.edu

Forum Geometricorum
Volume 2 (2002) 183–185.

FORUM GEOM
ISSN 1534-1178

An Application of Thébault's Theorem

Wilfred Reyes

Abstract. We prove the "Japanese theorem" as a very simple corollary of Thébault's theorem.

Theorem 1 below is due to the French geometer Victor Thébault [8]. See Figure 1. It had been a long standing problem, but a number of proofs have appeared since the early 1980's. See, for example, [7, 6, 1], and also [5] for a list of proofs in Dutch published in the 1970's. A very natural and understandable proof based on Ptolemy's theorem can be found in [3].

Theorem 1 (Thébault)**.** *Let E be a point on the side of triangle ABC such that $\angle AEB = \theta$. Let $O_1(r_1)$ be a circle tangent to the circumcircle and to the segments EA, EB. Let $O_2(r_2)$ be also tangent to the circumcircle and to EA, EC. If $I(\rho)$ is the incircle of ABC, then*

(1.1) *I lies on the segment O_1O_2 and* $\dfrac{O_1I}{IO_2} = \tan^2 \frac{\theta}{2}$,

(1.2) $\rho = r_1 \cos^2 \frac{\theta}{2} + r_2 \sin^2 \frac{\theta}{2}$.

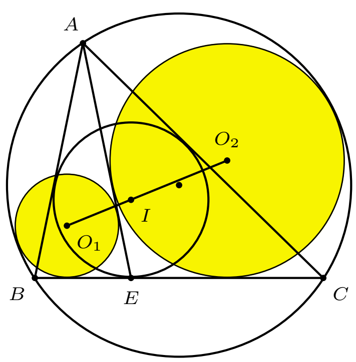


Figure 1

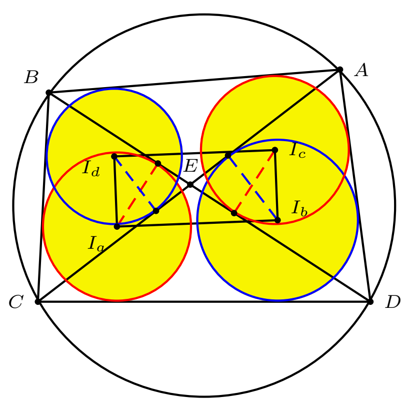


Figure 2

Theorem 2 below is called the "Japanese Theorem" in [4, p.193]. See Figure 2. A very long proof can be found in [2, pp.125–128]. In this note we deduce the Japanese Theorem as a very simple corollary of Thébault's Theorem.

Publication Date: December 30, 2002. Communicating Editor: Paul Yiu.

Theorem 2. *Let $ABCD$ be a convex quadrilateral inscribed in a circle. Denote by $I_a(\rho_a)$, $I_b(\rho_b)$, $I_c(\rho_c)$, $I_d(\rho_d)$ the incircles of the triangles BCD, CDA, DAB, and ABC.*

(2.1) *The incenters form a rectangle.*
(2.2) $\rho_a + \rho_c = \rho_b + \rho_d$.

Proof. In $ABCD$ we have the following circles: $O_{cd}(r_{cd})$, $O_{da}(r_{da})$, $O_{ab}(r_{ab})$, and $O_{bc}(r_{bc})$ inscribed respectively in angles AEB, BEC, CED, and DEA, each tangent internally to the circumcircle. Let $\angle AEB = \angle CED = \theta$ and $\angle BEC = \angle DEA = \pi - \theta$.

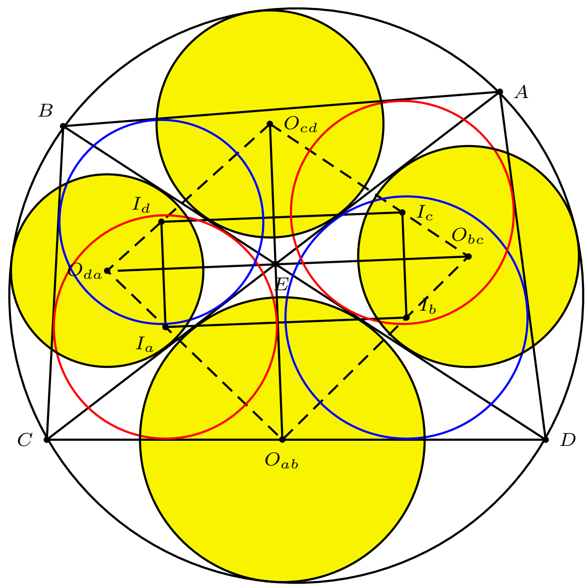


Figure 3

Now, by Theorem 1, the centers I_a, I_b, I_c, I_d lie on the lines $O_{da}O_{ab}$, $O_{ab}O_{bc}$, $O_{bc}O_{cd}$, $O_{cd}O_{da}$ respectively. Furthermore,

$$\frac{O_{da}I_a}{I_aO_{ab}} = \frac{O_{bc}I_c}{I_cO_{cd}} = \tan^2\left(\frac{\pi-\theta}{2}\right) = \cot^2\frac{\theta}{2},$$
$$\frac{O_{ab}I_b}{I_bO_{bc}} = \frac{O_{cd}I_d}{I_dO_{da}} = \tan^2\frac{\theta}{2}.$$

From these, we have

$$\frac{O_{da}I_a}{I_aO_{ab}} = \frac{O_{bc}I_b}{I_bO_{ab}}, \qquad \frac{O_{ab}I_b}{I_bO_{bc}} = \frac{O_{cd}I_c}{I_cO_{bc}},$$
$$\frac{O_{bc}I_c}{I_cO_{cd}} = \frac{O_{da}I_d}{I_dO_{cd}}, \qquad \frac{O_{cd}I_d}{I_dO_{da}} = \frac{O_{ab}I_a}{I_aO_{da}}.$$

These proportions imply the following parallelism:

$$I_aI_b//O_{da}O_{bc}, \quad I_bI_c//O_{ab}O_{cd}, \quad I_cI_d//O_{bc}O_{da}, \quad I_dI_a//O_{cd}O_{ab}.$$

As the segments $O_{cd}O_{ab}$ and $O_{da}O_{bc}$ are perpendicular because they are along the bisectors of the angles at E, $I_aI_bI_cI_d$ is an inscribed rectangle in $O_{ab}O_{bc}O_{cd}O_{da}$, and this proves (2.1).

Also, the following relation results from (1.2):

$$\rho_a + \rho_c = (r_{ab} + r_{cd})\cos^2\frac{\theta}{2} + (r_{da} + r_{bc})\sin^2\frac{\theta}{2}.$$

This same expression is readily seen to be equal to $\rho_b + \rho_d$ as well. This proves (2.2). □

References

[1] S. Dutta, Thébault's problem via euclidean geometry, *Samayā*, 7 (2001) number 2, 2–7.

[2] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry*, Charles Babbage Research Centre, Manitoba, Canada, 1989.

[3] S. Gueron, Two applications of the generalized Ptolemy theorem, *Amer. Math. Monthly*, 109 (2002) 362–370.

[4] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.

[5] F. M. van Lamoen, Hyacinthos message 5753, July 2, 2002.

[6] R. Shail, *A proof of Thébault's theorem*, *Amer. Math. Monthly*, 108 (2001) 319–325.

[7] K. B. Taylor, Solution of Problem 3887, *Amer. Math. Monthly*, 90 (1983) 486–187.

[8] V. Thébault, Problem 3887, *Amer. Math. Monthly*, 45 (1948) 482–483.

Wilfred Reyes: Departamento de Ciencias Básicas, Universidad del Bío-Bío, Chillán, Chile

E-mail address: wreyes@ubiobio.cl

Forum Geometricorum
Volume 2 (2002) 187.

FORUM GEOM
ISSN 1534-1178

Author Index

FORUM GEOMETRICORUM

A Journal on Classical Euclidean Geometry and Related Areas

published by

Department of Mathematical Sciences
Florida Atlantic University

Volume 3
2003

http://forumgeom.fau.edu

ISSN 1534-1178

Editorial Board

Advisors:

John H. Conway	Princeton, New Jersey, USA
Julio Gonzalez Cabillon	Montevideo, Uruguay
Richard Guy	Calgary, Alberta, Canada
George Kapetis	Thessaloniki, Greece
Clark Kimberling	Evansville, Indiana, USA
Kee Yuen Lam	Vancouver, British Columbia, Canada
Tsit Yuen Lam	Berkeley, California, USA
Fred Richman	Boca Raton, Florida, USA

Editor-in-chief:

Paul Yiu	Boca Raton, Florida, USA

Editors:

Clayton Dodge	Orono, Maine, USA
Roland Eddy	St. John's, Newfoundland, Canada
Jean-Pierre Ehrmann	Paris, France
Lawrence Evans	La Grange, Illinois, USA
Chris Fisher	Regina, Saskatchewan, Canada
Rudolf Fritsch	Munich, Germany
Bernard Gibert	St Etiene, France
Antreas P. Hatzipolakis	Athens, Greece
Michael Lambrou	Crete, Greece
Floor van Lamoen	Goes, Netherlands
Fred Pui Fai Leung	Singapore, Singapore
Daniel B. Shapiro	Columbus, Ohio, USA
Steve Sigur	Atlanta, Georgia, USA
Man Keung Siu	Hong Kong, China
Peter Woo	La Mirada, California, USA

Technical Editors:

Yuandan Lin	Boca Raton, Florida, USA
Aaron Meyerowitz	Boca Raton, Florida, USA
Xiao-Dong Zhang	Boca Raton, Florida, USA

Consultants:

Frederick Hoffman	Boca Raton, Floirda, USA
Stephen Locke	Boca Raton, Florida, USA
Heinrich Niederhausen	Boca Raton, Florida, USA

Table of Contents

Forum Geometricorum
Volume 3 (2003) 1–27.

FORUM GEOM
ISSN 1534-1178

Orthocorrespondence and Orthopivotal Cubics

Bernard Gibert

Abstract. We define and study a transformation in the triangle plane called the orthocorrespondence. This transformation leads to the consideration of a family of circular circumcubics containing the Neuberg cubic and several hitherto unknown ones.

1. The orthocorrespondence

Let P be a point in the plane of triangle ABC with barycentric coordinates $(u : v : w)$. The perpendicular lines at P to AP, BP, CP intersect BC, CA, AB respectively at P_a, P_b, P_c, which we call the *orthotraces* of P. These orthotraces lie on a line $\mathcal{L}_P$, which we call the *orthotransversal* of P.[1] We denote the trilinear pole of $\mathcal{L}_P$ by $P^{\perp}$, and call it the *orthocorrespondent* of P.

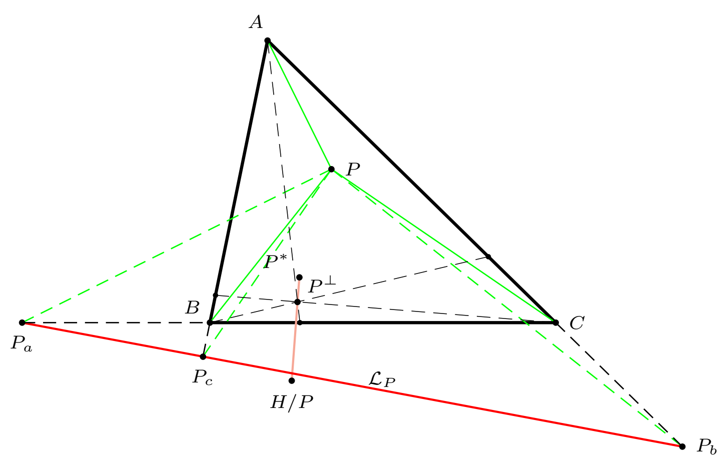


Figure 1. The orthotransversal and orthocorrespondent

In barycentric coordinates, [2]

$$P^{\perp} = (u(-uS_A + vS_B + wS_C) + a^2vw : \cdots : \cdots), \tag{1}$$

Publication Date: January 21, 2003. Communicating Editor: Paul Yiu.
We sincerely thank Edward Brisse, Jean-Pierre Ehrmann, and Paul Yiu for their friendly and valuable helps.

[1]The homography on the pencil of lines through P which swaps a line and its perpendicular at P is an involution. According to a Desargues theorem, the points are collinear.

[2]All coordinates in this paper are homogeneous barycentric coordinates. Often for triangle centers, we list only the first coordinate. The remaining two can be easily obtained by cyclically permuting a, b, c, and corresponding quantities. Thus, for example, in (1), the second and third coordinates are $v(-vS_B + wS_C + uS_A) + b^2wu$ and $w(-wS_C + uS_A + vS_B) + c^2uv$ respectively.

where, a, b, c are respectively the lengths of the sides BC, CA, AB of triangle ABC, and, in J.H. Conway's notations,

$$S_A = \frac{1}{2}(b^2 + c^2 - a^2),\ S_B = \frac{1}{2}(c^2 + a^2 - b^2),\ S_C = \frac{1}{2}(a^2 + b^2 - c^2). \quad (2)$$

The mapping $\Phi : P \mapsto P^\perp$ is called the *orthocorrespondence* (with respect to triangle ABC).

Here are some examples. We adopt the notations of [5] for triangle centers, except for a few commonest ones. Triangle centers without an explicit identification as X_n are not in the current edition of [5].

(1) $I^\perp = X_{57}$, the isogonal conjugate of the Mittenpunkt X_9.
(2) $G^\perp = (b^2 + c^2 - 5a^2 : \cdots : \cdots)$ is the reflection of G about K, and the orthotransversal is perpendicular to GK.
(3) $H^\perp = G$.
(4) $O^\perp = (\cos 2A : \cos 2B : \cos 2C)$ on the line GK.
(5) More generally, the orthocorrespondent of the Euler line is the line GK. The orthotransversal envelopes the Kiepert parabola.
(6) $K^\perp = \left(a^2(b^4 + c^4 - a^4 - 4b^2c^2) : \cdots : \cdots\right)$ on the Euler line.
(7) $X_{15}^\perp = X_{62}$ and $X_{16}^\perp = X_{61}$.
(8) $X_{112}^\perp = X_{115}^\perp = X_{110}$.

See §2.3 for points on the circumcircle and the nine-point circle with orthocorrespondents having simple barycentric coordinates.

Remarks. (1) While the geometric definition above of $P^\perp$ is not valid when P is a vertex of triangle ABC, by (1) we extend the orthocorrespondence Φ to cover these points. Thus, $A^\perp = A$, $B^\perp = B$, and $C^\perp = C$.

(2) The orthocorrespondent of P is not defined if and only if the three coordinates of $P^\perp$ given in (1) are simultaneously zero. This is the case when P belongs to the three circles with diameters BC, CA, AB.[3] There are only two such points, namely, the circular points at infinity.

(3) We denote by P^* the isogonal conjugate of P and by H/P the cevian quotient of H and P.[4] It is known that

$$H/P = (u(-uS_A + vS_B + wS_C) : \cdots : \cdots).$$

This shows that $P^\perp$ lies on the line through P^* and H/P. In fact,

$$(H/P)P^\perp : (H/P)P^* = a^2vw + b^2wu + c^2uv : S_Au^2 + S_Bv^2 + S_Cw^2.$$

In [6], Jim Parish claimed that this line also contains the isogonal conjugate of P with respect to its anticevian triangle. We add that this point is in fact the harmonic conjugate of $P^\perp$ with respect to P^* and H/P. Note also that the line through P and H/P is perpendicular to the orthotransversal $\mathcal{L}_P$.

(4) The orthocorrespondent of any (real) point on the line at infinity $\mathcal{L}^\infty$ is G.

[3] See Proposition 2 below.

[4] H/P is the perspector of the cevian triangle of H (orthic triangle) and the anticevian triangle of P.

(5) A straightforward computation shows that the orthocorrespondence Φ has exactly five fixed points. These are the vertices A, B, C, and the two Fermat points X_{13}, X_{14}. Jim Parish [7] and Aad Goddijn [2] have given nice synthetic proofs of this in answering a question of Floor van Lamoen [3]. In other words, X_{13} and X_{14} are the only points whose orthotransversal and trilinear polar coincide.

Theorem 1. *The orthocorrespondent $P^\perp$ is a point at infinity if and only if P lies on the Monge (orthoptic) circle of the inscribed Steiner ellipse.*

Proof. From (1), $P^\perp$ is a point at infinity if and only if

$$\sum_{\text{cyclic}} S_A x^2 - 2a^2yz = 0. \tag{3}$$

This is a circle in the pencil generated by the circumcircle and the nine-point circle, and is readily identified as the Monge circle of the inscribed Steiner ellipse.[5] □

It is obvious that $P^\perp$ is at infinity if and only if $\mathcal{L}_P$ is tangent to the inscribed Steiner ellipse. [6]

Proposition 2. *The orthocorrespondent $P^\perp$ lies on the sideline BC if and only if P lies on the circle Γ_{BC} with diameter BC. The perpendicular at P to AP intersects BC at the harmonic conjugate of $P^\perp$ with respect to B and C.*

Proof. $P^\perp$ lies on BC if and only if its first barycentric coordinate is 0, *i.e.*, if and only if $u(-uS_A + vS_B + wS_C) + a^2vw = 0$ which shows that P must lie on Γ_{BC}. □

2. Orthoassociates and the critical conic

2.1. *Orthoassociates and antiorthocorrespondents.*

Theorem 3. *Let Q be a finite point. There are exactly two points P_1 and P_2 (not necessarily real nor distinct) such that $Q = P_1^\perp = P_2^\perp$.*

Proof. Let Q be a finite point. The trilinear polar ℓ_Q of Q intersects the sidelines of triangle ABC at Q_a, Q_b, Q_c. The circles Γ_a, Γ_b, Γ_c with diameters AQ_a, BQ_b, CQ_c are in the same pencil of circles since their centers O_a, O_b, O_c are collinear (on the Newton line of the quadrilateral formed by the sidelines of ABC and ℓ_Q), and since they are all orthogonal to the polar circle. Thus, they have two points P_1 and P_2 in common. These points, if real, satisfy $P_1^\perp = Q = P_2^\perp$. [7] □

We call P_1 and P_2 the *antiorthocorrespondents* of Q and write $Q^\top = \{P_1, P_2\}$. We also say that P_1 and P_2 are *orthoassociates*, since they share the same orthocorrespondent and the same orthotransversal. Note that P_1 and P_2 are homologous

---

[5]The Monge (orthoptic) circle of a conic is the locus of points whose two tangents to the conic are perpendicular to each other. It has the same center of the conic. For the inscribed Steiner ellipse, the radius of the Monge circle is $\frac{\sqrt{2}}{6}\sqrt{a^2+b^2+c^2}$.

[6]The trilinear polar of a point at infinity is tangent to the in-Steiner ellipse since it is the in-conic with perspector G.

[7]P_1 and P_2 are not always real when ABC is obtuse angled, see §2.2 below.

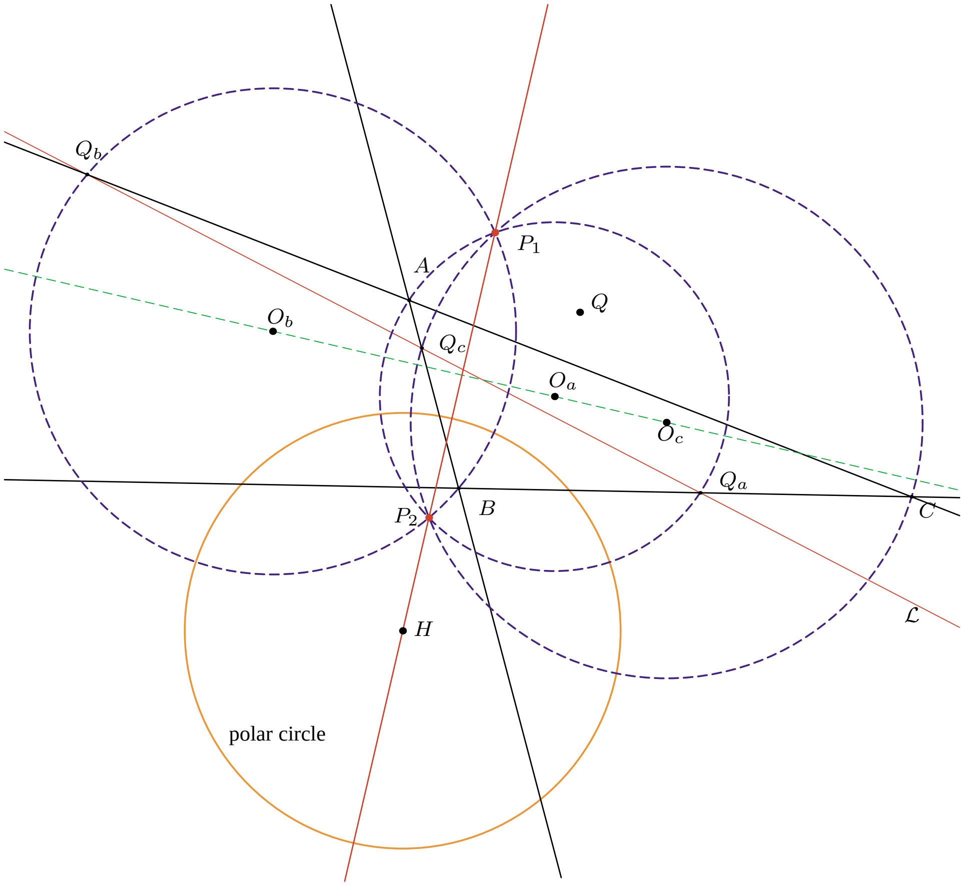


Figure 2. Antiorthocorrespondents

under the inversion ι_H with pole H which swaps the circumcircle and the nine-point circle.

Proposition 4. *The orthoassociate* $\overline{P}$ *of* $P(u:v:w)$ *has coordinates*

$$\left(\frac{S_Bv^2+S_Cw^2-S_Au(v+w)}{S_A}:\frac{S_Cw^2+S_Au^2-S_Bv(w+u)}{S_B}:\frac{S_Au^2+S_Bv^2-S_Cw(u+v)}{S_C}\right). \tag{4}$$

Let S denote *twice* of the area of triangle ABC. In terms of S_A, S_B, S_C in (2), we have

$$S^2=S_AS_B+S_BS_C+S_CS_A.$$

Proposition 5. *Let*

$$K(u,v,w)=S^2(u+v+w)^2-4(a^2S_Avw+b^2S_Bwu+c^2S_Cuv).$$

The antiorthocorrespondents of $Q = (u : v : w)$ are the points with barycentric coordinates

$$((u-w)(u+v-w)S_B+(u-v)(u-v+w)S_C \pm \frac{\sqrt{K(u,v,w)}}{S}((u-w)S_B+(u-v)S_C) : \cdots : \cdots). \quad (5)$$

These are real points if and only if $K(u, v, w) \geq 0$.

2.2. *The critical conic $\mathcal{C}$.* Consider the *critical conic $\mathcal{C}$* with equation

$$S^2(x+y+z)^2 - 4\sum_{\text{cyclic}} a^2 S_A yz = 0, \quad (6)$$

which is degenerate, real, imaginary according as triangle ABC is right-, obtuse-, or acute-angled. It has center the Lemoine point K, and the same infinite points as the circumconic

$$a^2S_Ayz + b^2S_Bzx + c^2S_Cxy = 0,$$

which is the isogonal conjugate of the orthic axis $S_Ax + S_By + S_Cz = 0$, and has the same center K. This critical conic is a hyperbola when it is real. Clearly, if Q lies on the critical conic, its two real antiorthocorrespondents coincide.

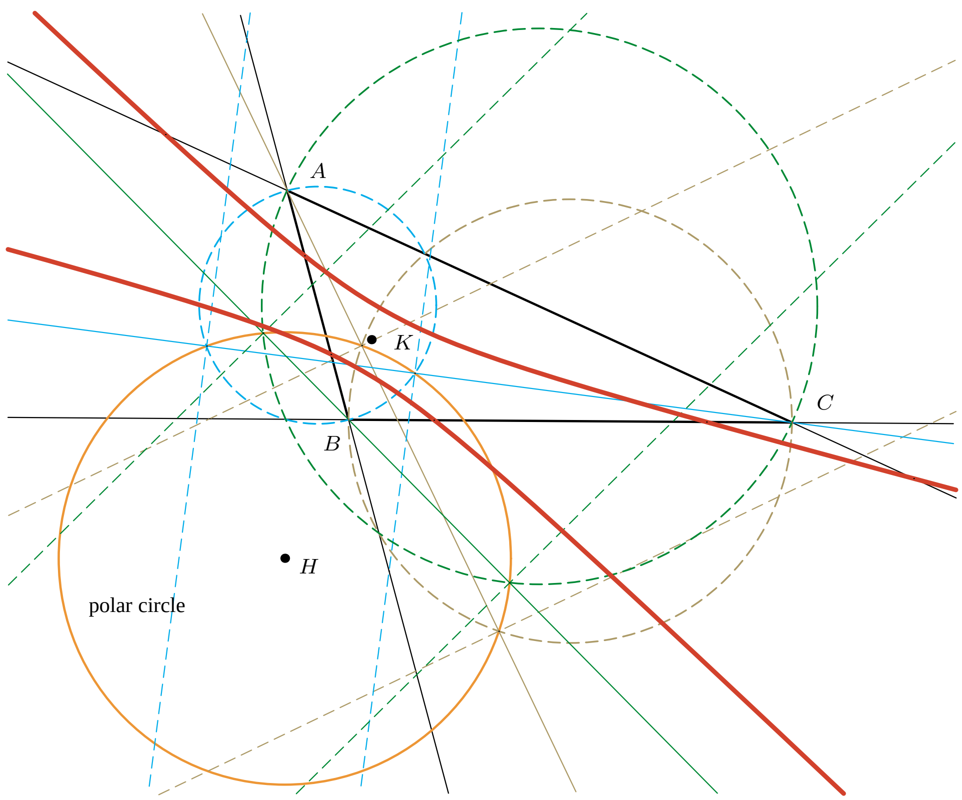


Figure 3. The critical conic

Proposition 6. *The antiorthocorrespondents of Q are real if and only if one of the following conditions holds.*
(1) Triangle ABC is acute-angled.
(2) Triangle ABC is obtuse-angled and Q lies in the component of the critical hyperbola not containing the center K.

Proposition 7. *The critical conic is the orthocorrespondent of the polar circle. When it is real, it intersects each sideline of ABC at two points symmetric about the corresponding midpoint. These points are the orthocorrespondents of the intersections of the polar circle and the circles Γ_{BC}, Γ_{CA}, Γ_{AB} with diameters BC, CA, AB.*

2.3. *Orthocorrespondent of the circumcircle.* Let P be a point on the circumcircle. Its orthotransversal passes through O, and $P^{\perp}$ lies on the circumconic centered at K. [8] The orthoassociate $\overline{P}$ lies on the nine-point circle. The table below shows several examples of such points. [9]

P	P^*	$\overline{P}$	$P^{\perp}$
X_{74}	X_{30}	X_{133}	$a^2S_A/((b^2-c^2)^2+a^2(2S_A-a^2))$
X_{98}	X_{511}	X_{132}	X_{287}
X_{99}	X_{512}	$(b^2-c^2)^2(S_A-a^2)/S_A$	$S_A/(b^2-c^2)$
X_{100}	X_{513}		$aS_A/(b-c)$
X_{101}	X_{514}		$a^2S_A/(b-c)$
X_{105}	X_{518}		$aS_A/(b^2+c^2-ab-ac)$
X_{106}	X_{519}		$a^2S_A/(b+c-2a)$
X_{107}	X_{520}	X_{125}	$X_{648}=X^*_{647}$
X_{108}	X_{521}	X_{11}	$X_{651}=X^*_{650}$
X_{109}	X_{522}		$a^2S_A/((b-c)(b+c-a))$
X_{110}	X_{523}	X_{136}	$a^2S_A/(b^2-c^2)$
X_{111}	X_{524}		$a^2S_A/(b^2+c^2-2a^2)=X^*_{468}$
X_{112}	X_{525}	X_{115}	$X_{110}=X^*_{523}$
X_{675}	X_{674}		$S_A/(b^3+c^3-a(b^2+c^2))$
X_{689}	X_{688}		$S_A/(a^2(b^4-c^4))$
X_{691}	X_{690}		$a^2S_A/((b^2-c^2)(b^2+c^2-2a^2))$
P_1	P_1^*	X_{114}	X^*_{230}

Remark. The coordinates of P_1 can be obtained from those of X_{230} by making use of the fact that X^*_{230} is the barycentric product of P_1 and X_{69}. Thus,

$$P_1=\left(\frac{a^2}{S_A((b^2-c^2)^2-a^2(b^2+c^2-2a^2))}:\cdots:\cdots\right).$$

[8]If $P=(u:v:w)$ lies on the circumcircle, then $P^{\perp}=(uS_A:vS_B:wS_C)$ is the barycentric product of P and X_{69}. See [9]. The orthotransversal is the line $\frac{x}{uS_A}+\frac{y}{vS_B}+\frac{z}{wS_C}=0$ which contains O.

[9]The isogonal conjugates are trivially infinite points.

2.4. *The orthocorrespondent of a line.* The orthocorrespondent of a sideline, say BC, is the circumconic through G and its projection on the corresponding altitude. The orthoassociate is the circle with the segment AH as diameter.

Consider a line ℓ intersecting BC, CA, AB at X, Y, Z respectively. The orthocorrespondent $\ell^\perp$ of ℓ is a conic containing the centroid G (the orthocorrespondent of the infinite point of ℓ) and the points $X^\perp$, $Y^\perp$, $Z^\perp$. [10] A fifth point can be constructed as $P^\perp$, where P is the pedal of G on ℓ. [11] These five points entirely determine the conic. According to Proposition 2, $\ell^\perp$ meets BC at the orthocorrespondents of the points where ℓ intersects the circle Γ_{BC}. [12] It is also the orthocorrespondent of the circle through H which is the orthoassociate of ℓ.

If the line ℓ contains H, the conic $\ell^\perp$ degenerates into a double line containing G. If ℓ also contains $P = (u : v : w)$ other than H, then this line has equation

$$(S_Bv - S_Cw)x + (S_Cw - S_Au)y + (S_Au - S_Bv)z = 0.$$

This double line passes through the second intersection of ℓ with the Kiepert hyperbola. [13] It also contains the point $(uS_A : vS_B : wS_C)$. The two lines intersect at the point

$$\left(\frac{S_B - S_C}{S_Bv - S_Cw} : \frac{S_C - S_A}{S_Cw - S_Au} : \frac{S_A - S_B}{S_Au - S_Bv}\right).$$

The orthotransversals of points on ℓ envelope the inscribed parabola with directrix ℓ and focus the antipode (on the circumcircle) of the isogonal conjugate of the infinite point of ℓ.

2.5. *The antiorthocorrespondent of a line.* Let ℓ be the line with equation $lx + my + nz = 0$.

When ABC is acute angled, the antiorthocorrespondent $\ell^\top$ of ℓ is the circle centered at $\Omega_\ell = (m+n : n+l : l+m)$[14] and orthogonal to the polar circle. It has square radius

$$\frac{S_A(m+n)^2 + S_B(n+l)^2 + S_C(l+m)^2}{4(l+m+n)^2}$$

and equation

$$(x+y+z)\left(\sum_{\text{cyclic}} S_A lx\right) - (l+m+n)\left(\sum_{\text{cyclic}} a^2yz\right) = 0.$$

When ABC is obtuse angled, $\ell^\top$ is only a part of this circle according to its position with respect to the critical hyperbola $\mathcal{C}$. This circle clearly degenerates

---

[10] These points can be easily constructed. For example, $X^\perp$ is the trilinear pole of the perpendicular at X to BC.

[11] $P^\perp$ is the antipode of G on the conic.

[12] These points can be real or imaginary, distinct or equal.

[13] In particular, the orthocorrespondent of the tangent at H to the Kiepert hyperbola, *i.e.*, the line HK, is the Euler line.

[14] Ω_ℓ is the complement of the isotomic conjugate of the trilinear pole of ℓ.

into the union of $\mathcal{L}^\infty$and a line through H when G lies on ℓ. This line is the directrix of the inscribed conic which is now a parabola.

Conversely, any circle centered at Ω (proper or degenerate) orthogonal to the polar circle is the orthoptic circle of the inscribed conic whose perspector P is the isotomic conjugate of the anticomplement of the center of the circle. The orthocorrespondent of this circle is the trilinear polar ℓ_P of P. The table below shows a selection of usual lines and inscribed conics.[15]

P	Ω	ℓ	inscribed conic
X_1	X_{37}	antiorthic axis	ellipse, center I
X_2	X_2	$\mathcal{L}^\infty$	Steiner in-ellipse
X_4	X_6	orthic axis	ellipse, center K
X_6	X_{39}	Lemoine axis	Brocard ellipse
X_7	X_1	Gergonne axis	incircle
X_8	X_9		Mandart ellipse
X_{13}	X_{396}		Simmons conic
X_{76}	X_{141}	de Longchamps axis	
X_{110}	X_{647}	Brocard axis	
X_{598}	X_{597}		Lemoine ellipse

2.6. *Orthocorrespondent and antiorthocorrespondent of a circle.* In general, the orthocorrespondent of a circle is a conic. More precisely, two orthoassociate circles share the same orthocorrespondent conic, or the part of it outside the critical conic $\mathcal{C}$ when ABC is obtuse-angled. For example, the circumcircle and the nine-point circle have the same orthocorrespondent which is the circumconic centered at K. The orthocorrespondent of each circle (and its orthoassociate) of the pencil generated by circumcircle and the nine-point circle is another conic also centered at K and homothetic of the previous one. The axis of these conics are the parallels at K to the asymptotes of the Kiepert hyperbola. The critical conic is one of them since the polar circle belongs to the pencil.

This conic degenerates into a double line (or part of it) if and only if the circle is orthogonal to the polar circle. If the radical axis of the circumcircle and this circle is $lx+my+nz=0$, this double line has equation $\frac{l}{S_A}x+\frac{m}{S_B}y+\frac{n}{S_C}z=0$. This is the trilinear polar of the barycentric product X_{69} and the trilinear pole of the radical axis.

The antiorthocorrespondent of a circle is in general a bicircular quartic.

[15]The conics in this table are entirely defined either by their center or their perspector in the table. See [1]. In fact, there are two Simmons conics (and not ellipses as Brocard and Lemoyne wrote) with perspectors (and foci) X_{13} and X_{14}.

3. Orthopivotal cubics

For a given a point P with barycentric coordinates $(u : v : w)$, the locus of point M such that P, M, $M^\perp$ are collinear is the cubic curve $\mathcal{O}(P)$:

$$\sum_{\text{cyclic}} x\left((c^2u - 2S_Bw)y^2 - (b^2u - 2S_Cv)z^2\right) = 0. \tag{7}$$

Equivalently, $\mathcal{O}(P)$ is the locus of the intersections of a line through P with the circle which is its antiorthocorrespondent. See §2.5. We shall say that $\mathcal{O}(P)$ is an *orthopivotal* cubic, and call P its *orthopivot*.

Equation (7) can be rewritten as

$$\sum_{\text{cyclic}} u\left(x(c^2y^2 - b^2z^2) + 2yz(S_By - S_Cz)\right) = 0. \tag{8}$$

Accordingly, we consider the cubic curves

$$\begin{array}{ll} \Sigma_a : & x(c^2y^2 - b^2z^2) + 2yz(S_By - S_Cz) = 0, \\ \Sigma_b : & y(a^2z^2 - c^2x^2) + 2zx(S_Cz - S_Ax) = 0, \\ \Sigma_c : & z(b^2x^2 - a^2y^2) + 2xy(S_Ax - S_By) = 0, \end{array} \tag{9}$$

and very loosely write (8) in the form

$$u\Sigma_a + v\Sigma_b + w\Sigma_c = 0. \tag{10}$$

We study the cubics Σ_a, Σ_b, Σ_c in §6.5 below, where we shall see that they are strophoids. We list some basic properties of the $\mathcal{O}(P)$.

Proposition 8. (1) *The orthopivotal cubic $\mathcal{O}(P)$ is a circular circumcubic*[16] *passing through the Fermat points, P, the infinite point of the line GP, and*

$$P' = \left(\frac{b^2 - c^2}{v - w} : \frac{c^2 - a^2}{w - u} : \frac{a^2 - b^2}{u - v}\right), \tag{11}$$

which is the second intersection of the line GP and the Kiepert hyperbola.[17]

(2) *The "third" intersection of $\mathcal{O}(P)$ and the Fermat line $X_{13}X_{14}$ is on the line PX_{110}.*

(3) *The tangent to $\mathcal{O}(P)$ at P is the line $PP^\perp$.*

(4) *$\mathcal{O}(P)$ intersects the sidelines BC, CA, AB at U, V, W respectively given by*

$$\begin{aligned} U &= (0 : 2S_Cu - a^2v : 2S_Bu - a^2w), \\ V &= (2S_Cv - b^2u : 0 : 2S_Av - b^2w), \\ W &= (2S_Bw - c^2u : 2S_Aw - c^2v : 0). \end{aligned}$$

(5) *$\mathcal{O}(P)$ also contains the (not always real) antiorthocorrespondents P_1 and P_2 of P.*

[16] This means that the cubic passes through the two circular points at infinity common to all circles, and the three vertices of the reference triangle.

[17] This is therefore the sixth intersection of $\mathcal{O}(P)$ with the Kiepert hyperbola.

Here is a simple construction of the intersection U in (4) above. If the parallel at G to BC intersects the altitude AH at H_a, then U is the intersection of PH_a and BC. [18]

4. Construction of $\mathcal{O}(P)$ and other points

Let the trilinear polar of P intersect the sidelines BC, CA, AB at X, Y, Z respectively. Denote by Γ_a, Γ_b, Γ_c the circles with diameters AX, BY, CZ and centers O_a, O_b, O_c. They are in the same pencil $\mathbb{F}$ whose radical axis is the perpendicular at H to the line $\mathcal{L}$ passing through O_a, O_b, O_c, and the points P_1 and P_2 seen above. [19]

For an arbitrary point M on $\mathcal{L}$, let Γ be the circle of $\mathbb{F}$ passing through M. The line $PM^\perp$ intersects Γ at two points N_1 and N_2 on $\mathcal{O}(P)$. From these we note the following.

(1) $\mathcal{O}(P)$ contains the second intersections A_2, B_2, C_2 of the lines AP, BP, CP with the circles Γ_a, Γ_b, Γ_c.
(2) The point P' in (11) lies on the radical axis of $\mathbb{F}$.
(3) The circle of $\mathbb{F}$ passing through P meets the line $PP^\perp$ at $\widetilde{P}$, tangential of P.
(4) The perpendicular bisector of N_1N_2 envelopes the parabola with focus F_P (see §5 below) and directrix the line GP. This parabola is tangent to $\mathcal{L}$ and to the two axes of the inscribed Steiner ellipse.

This yields another construction of $\mathcal{O}(P)$: a tangent to the parabola meets $\mathcal{L}$ at ω. The perpendicular at P to this tangent intersects the circle of $\mathbb{F}$ centered at ω at two points on $\mathcal{O}(P)$.

5. Singular focus and an involutive transformation

The singular focus of a circular cubic is the intersection of the two tangents to the curve at the circular points at infinity. When this singular focus lies on the curve, the cubic is said to be a focal cubic. The singular focus of $\mathcal{O}(P)$ is the point

$$F_P = \left(a^2(v^2 + w^2 - u^2 - vw) + b^2u(u + v - 2w) + c^2u(u + w - 2v) \;:\cdots:\cdots\right).$$

If we denote by F_1 and F_2 the foci of the inscribed Steiner ellipse, then F_P is the inverse of the reflection of P in the line F_1F_2 with respect to the circle with diameter F_1F_2.

Consider the mapping $\Psi : P \mapsto F_P$ in the affine plane (without the centroid G) which transforms a point P into the singular focus F_P of $\mathcal{O}(P)$. This is clearly an involution: F_P is the singular focus of $\mathcal{O}(P)$ if and only if P is the singular focus of $\mathcal{O}(F_P)$. It has exactly two fixed points, *i.e.*, F_1 and F_2. [20]

---

[18] H_a is the "third" intersection of AH with the Napoleon cubic, the isogonal cubic with pivot X_5.

[19] This line $\mathcal{L}$ is the trilinear polar of the isotomic conjugate of the anticomplement of P.

[20] The two cubics $\mathcal{O}(F_1)$ and $\mathcal{O}(F_2)$ are central focals with centers at F_1 and F_2 respectively, with inflexional tangents through K, sharing the same real asymptote F_1F_2.

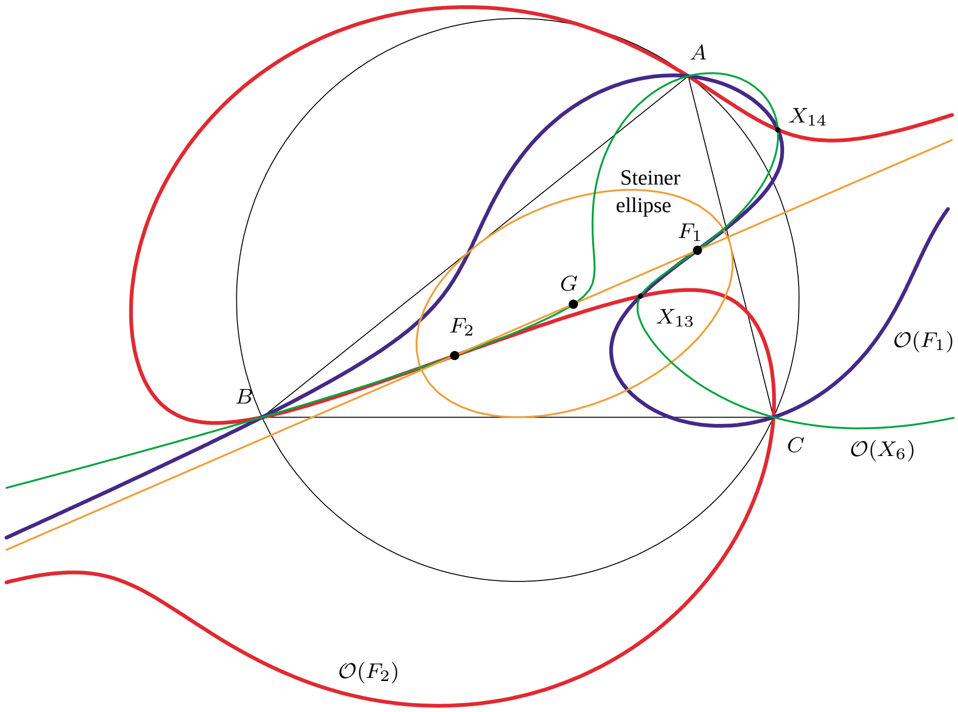


Figure 4. $\mathcal{O}(F_1)$ and $\mathcal{O}(F_2)$

The table below shows a selection of homologous points under Ψ, most of which we shall meet in the sequel. When P is at infinity, $F_P = G$, *i.e.*, all $\mathcal{O}(P)$ with orthopivot at infinity have G as singular focus.

P	X_1	X_3	X_4	X_6	X_{13}	X_{15}	X_{23}	X_{69}
F_P	X_{1054}	X_{110}	X_{125}	X_{111}	X_{14}	X_{16}	X_{182}	X_{216}

P	X_{100}	X_{184}	X_{187}	X_{352}	X_{616}	X_{617}	X_{621}	X_{622}
F_P	X_{1083}	X_{186}	X_{353}	X_{574}	X_{619}	X_{618}	X_{624}	X_{623}

The involutive transformation Ψ swaps

(1) the Euler line and the line through GX_{110}, [21]
(2) more generally, any line GP and its reflection in F_1F_2,
(3) the Brocard axis OK and the Parry circle.
(4) more generally, any line OP (which is not the Euler line) and the circle through G, X_{110}, and F_P,
(5) the circumcircle and the Brocard circle,
(6) more generally, any circle not through G and another circle not through G.

[21]The nine-point center is swapped into the anticomplement of X_{110}.

The involutive transformation Ψ leaves the second Brocard cubic $\mathcal{B}_2$ [22]

$$\sum_{\text{cyclic}} (b^2 - c^2)x(c^2y^2 + b^2z^2) = 0$$

globally invariant. See §6.4 below. More generally, Ψ leaves invariant the pencil of circular circumcubics through the vertices of the second Brocard triangle (they all pass through G). [23] There is another cubic from this pencil which is also globally invariant, namely,

$$(a^2b^2c^2 - 8S_AS_BS_C)xyz + \sum_{\text{cyclic}} (b^2 + c^2 - 2a^2)x(c^2S_Cy^2 + b^2S_Bz^2) = 0.$$

We call this cubic $\mathcal{B}_6$. It passes through X_3, X_{110}, and X_{525}.

If $\mathcal{O}(P)$ is nondegenerate, then its real asymptote is the homothetic image of the line GP under the homothety $\mathsf{h}(F_P, 2)$.

6. Special orthopivotal cubics

6.1. *Degenerate orthopivotal cubics.* There are only two situations where we find a degenerate $\mathcal{O}(P)$. A cubic can only degenerate into the union of a line and a conic. If the line is $\mathcal{L}^\infty$, we find only one such cubic. It is $\mathcal{O}(G)$, the union of $\mathcal{L}^\infty$ and the Kiepert hyperbola. If the line is not $\mathcal{L}^\infty$, there are ten different possibilities depending of the number of vertices of triangle ABC lying on the conic above which now must be a circle.

(1) $\mathcal{O}(X_{110})$ is the union of the circumcircle and the Fermat line. [24]
(2) $\mathcal{O}(P)$ is the union of one sideline of triangle ABC and the circle through the remaining vertex and the two Fermat points when P is the "third" intersection of an altitude of ABC with the Napoleon cubic. [25]
(3) $\mathcal{O}(P)$ is the union of a circle through two vertices of ABC and one Fermat point and a line through the remaining vertex and Fermat point when P is a vertex of one of the two Napoleon triangles. See [4, §6.31].

6.2. *Isocubics $\mathcal{O}(P)$.* We denote by $p\mathcal{K}$ a *pivotal* isocubic and by $n\mathcal{K}$ a *non-pivotal* isocubic. Consider an orthopivotal circumcubic $\mathcal{O}(P)$ intersecting the sidelines of triangle ABC at U, V, W respectively. The cubic $\mathcal{O}(P)$ is an isocubic in the two following cases.

---

[22] The second Brocard cubic $\mathcal{B}_2$ is the locus of foci of inscribed conics centered on the line GK. It is also the locus of M for which the line $MM^\perp$ contains the Lemoine point K.

[23] The inversive image of a circular cubic with respect to one of its points is another circular cubic through the same point. Here, Ψ swaps ABC and the second Brocard triangle $A_2B_2C_2$. Hence, each circular cubic through A, B, C, A_2, B_2, C_2 and G has an inversive image through the same points.

[24] X_{110} is the focus of the Kiepert parabola.

[25] The Napoleon cubic is the isogonal cubic with pivot X_5. These third intersections are the intersections of the altitudes with the parallel through G to the corresponding sidelines.

6.2.1. *Pivotal $\mathcal{O}(P)$.*

Proposition 9. *An orthopivotal cubic $\mathcal{O}(P)$ is a pivotal circumcubic $p\mathcal{K}$ if and only if the triangles ABC and UVW are perspective,* i.e., *if and only if P lies on the Napoleon cubic (isogonal $p\mathcal{K}$ with pivot X_5). In this case,*

(1) *the* pivot Q *of $\mathcal{O}(P)$ lies on the cubic $\mathcal{K}_n$:* [26] *it is the perspector of ABC and the (-2)-pedal triangle of P,* [27] *and lies on the line PX_5;*
(2) *the* pole Ω *of the isoconjugation lies on the cubic*

$$\mathcal{C}_o: \qquad \sum_{\text{cyclic}} (4S_A^2 - b^2c^2)x^2(b^2z - c^2y) = 0.$$

The Ω-isoconjugate Q^* of Q lies on the Neuberg cubic and is the inverse in the circumcircle of the isogonal conjugate of Q. The Ω-isoconjugate P^* of P lies on $\mathcal{K}_n$ and is the third intersection with the line QX_5.

Here are several examples of such cubics.

(1) $\mathcal{O}(O) = \mathcal{O}(X_3)$ is the Neuberg cubic.
(2) $\mathcal{O}(X_5)$ is $\mathcal{K}_n$.
(3) $\mathcal{O}(I) = \mathcal{O}(X_1)$ has pivot $X_{80} = ((2S_C - ab)(2S_B - ac) : \cdots : \cdots)$, pole $(a(2S_C - ab)(2S_B - ac) : \cdots : \cdots)$, and singular focus

$$(a(2S_A + ab + ac - 3bc) : \cdots : \cdots).$$

(4) $\mathcal{O}(H) = \mathcal{O}(X_4)$ has pivot H, pole M_o the intersection of HK and the orthic axis, with coordinates

$$\left(\frac{a^2(b^2+c^2-2a^2)+(b^2-c^2)^2}{S_A} : \cdots : \cdots\right),$$

and singular focus X_{125}, center of the Jerabek hyperbola.

$\mathcal{O}(H)$ is a very remarkable cubic since every point on it has orthocorrespondent on the Kiepert hyperbola. It is invariant under the inversion with respect to the conjugated polar circle and is also invariant under the isogonal transformation with respect to the orthic triangle. It is an isogonal $p\mathcal{K}$ with pivot X_{30} with respect to this triangle.

6.2.2. *Non-pivotal $\mathcal{O}(P)$.*

Proposition 10. *An orthopivotal cubic $\mathcal{O}(P)$ is a non-pivotal circumcubic $n\mathcal{K}$ if and only if its "third" intersections with the sidelines* [28] *are collinear,* i.e., *if and only if P lies on the isogonal $n\mathcal{K}$ with root X_{30}:* [29]

$$\sum_{\text{cyclic}} \left((b^2-c^2)^2 + a^2(b^2+c^2-2a^2)\right) x(c^2y^2+b^2z^2) + 2(8S_AS_BS_C - a^2b^2c^2)xyz = 0.$$

We give two examples of such cubics.

---

[26] $\mathcal{K}_n$ is the 2-cevian cubic associated with the Neuberg and the Napoleon cubics. See [8].

[27] For any non-zero real number t, the t-pedal triangle of P is the image of its pedal triangle under the homothety $\mathsf{h}(P, t)$.

[28] These are the points U, V, W in Proposition 8(4).

[29] This passes through G, K, X_{110}, and X_{523}.

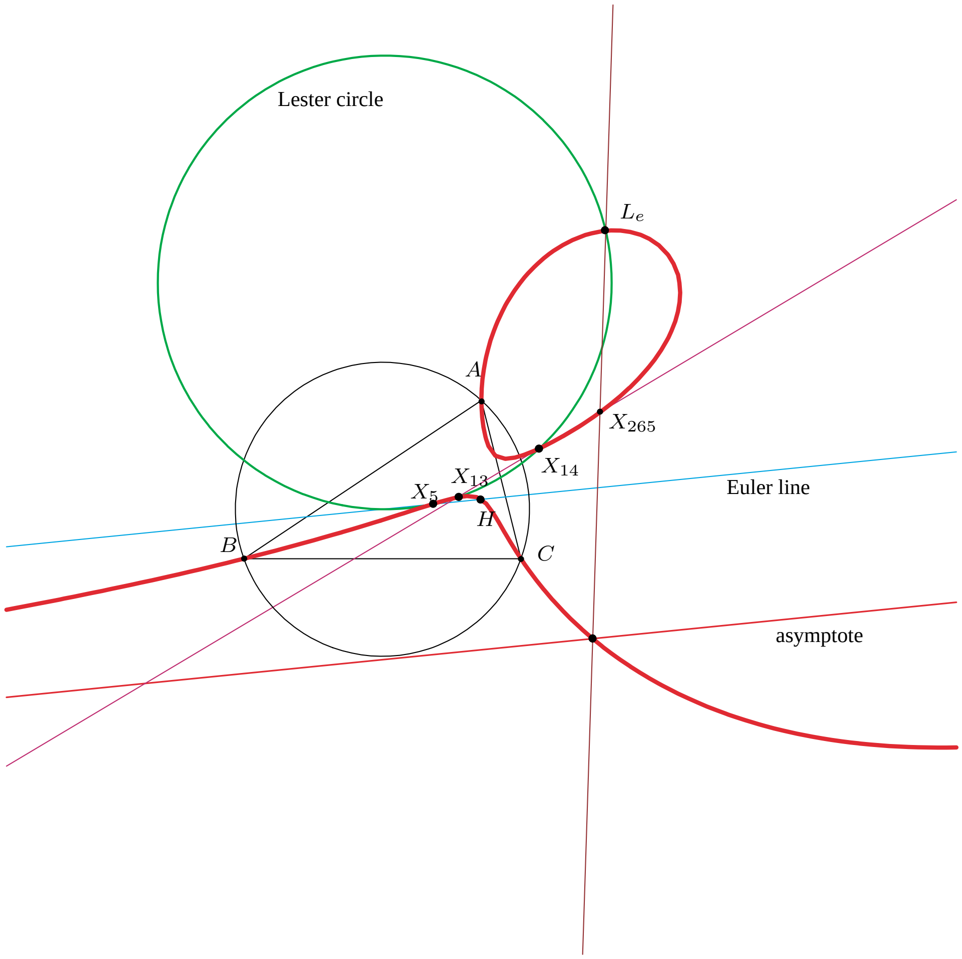


Figure 5. $\mathcal{K}_n$

(1) $\mathcal{O}(K) = \mathcal{O}(X_6)$ is the second Brocard cubic $\mathcal{B}_2$.
(2) $\mathcal{O}(X_{523})$ is a $n\mathcal{K}$ with pole and root both at the isogonal conjugate of X_{323}, and singular focus G: [30]

$$\sum_{\text{cyclic}} (4S_A^2 - b^2c^2)x^2(y+z) = 0$$

6.3. *Isogonal* $\mathcal{O}(P)$. There are only two)$\mathcal{O}(P)$ which are isogonal cubics, one pivotal and one non-pivotal:

(i) $\mathcal{O}(X_3)$ is the Neuberg cubic (pivotal),
(ii) $\mathcal{O}(X_6)$ is $\mathcal{B}_2$ (nonpivotal).

[30] $\mathcal{O}(X_{523})$ meets the circumcircle at the Tixier point X_{476}.

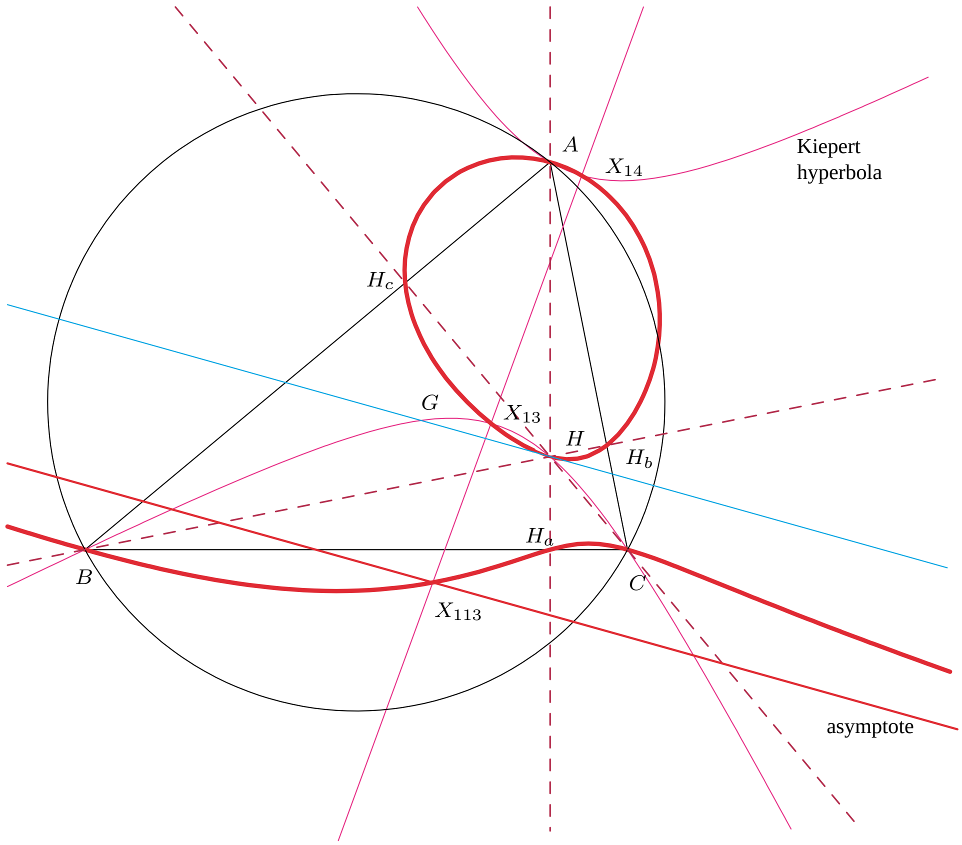


Figure 6. $\mathcal{O}(X_4)$

6.4. *Orthopivotal focals.* Recall that a focal is a circular cubic containing its own singular focus. [31]

Proposition 11. *An orthopivotal cubic $\mathcal{O}(P)$ is a focal if and only if P lies on $\mathcal{B}_2$.*

This is the case of $\mathcal{B}_2$ itself, which is an isogonal focal cubic passing through the following points: A, B, C, G, K, X_{13}, X_{14}, X_{15}, X_{16}, X_{111} (the singular focus), X_{368}, X_{524}, the vertices of the second Brocard triangle and their isogonal conjugates. All those points are orthopivots of orthopivotal focals. When the orthopivot is a fixed point of the orthocorrespondence, we shall see in §6.5 below that $\mathcal{O}(P)$ is a strophoid.

We have seen in §5 that F_1 and F_2 are invariant under Ψ. These two points lie on $\mathcal{B}_2$ (and also on the Thomson cubic). The singular focus of an orthopivotal focal $\mathcal{O}(P)$ always lies on $\mathcal{B}_2$; it is the "third" point of $\mathcal{B}_2$ and the line KP.

[31]Typically, a focal is the locus of foci of conics inscribed in a quadrilateral. The only focals having double points (nodes) are the strophoids.

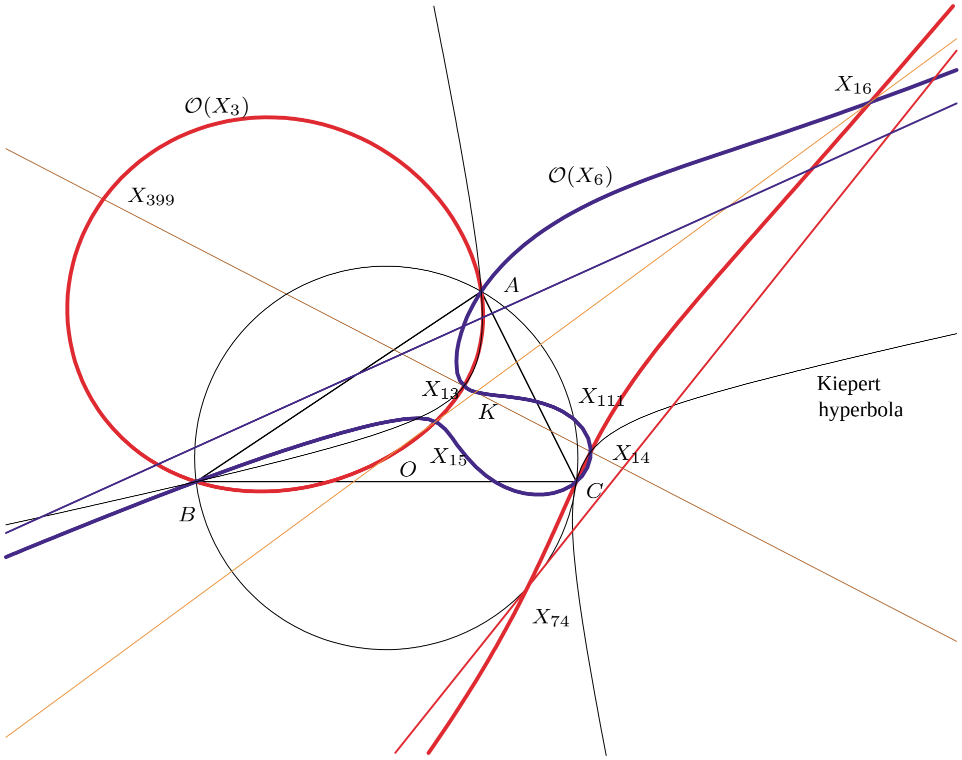


Figure 7. $\mathcal{O}(X_3)$ and $\mathcal{O}(X_6)$

One remarkable cubic is $\mathcal{O}(X_{524})$: it is another central cubic with center and singular focus at G and the line GK as real asymptote. This cubic passes through X_{67} and obviously the symmetrics of A, B, C, X_{13}, X_{14}, X_{67} about G. Its equation is

$$\sum_{\text{cyclic}} x\left(\left(b^2+c^4-a^4-c^2(a^2+2b^2-2c^2)\right)y^2-\left(b^4+c^4-a^4-b^2(a^2-2b^2+2c^2)\right)z^2\right)=0.$$

Another interesting cubic is $\mathcal{O}(X_{111})$ with K as singular focus. Its equation is

$$\sum_{\text{cyclic}} (b^2+c^2-2a^2)x^2\left(c^2(a^4-b^2c^2+3b^4-c^4-2a^2b^2)y-b^2(a^4-b^2c^2+3c^4-b^4-2a^2c^2)z\right)=0.$$

The sixth intersection with the Kiepert hyperbola is X_{671}, a point on the Steiner circumellipse and on the line through X_{99} and X_{111}.

6.5. *Orthopivotal strophoids.* It is easy to see that $\mathcal{O}(P)$ is a strophoid if and only if P is one of the five real fixed points of the orthocorrespondence, namely, A, B, C, X_{13}, X_{14}, the fixed point being the double point of the curve. This means that the mesh of orthopivotal cubics contains five strophoids denoted by $\mathcal{O}(A)$, $\mathcal{O}(B)$, $\mathcal{O}(C)$, $\mathcal{O}(X_{13})$, $\mathcal{O}(X_{14})$.

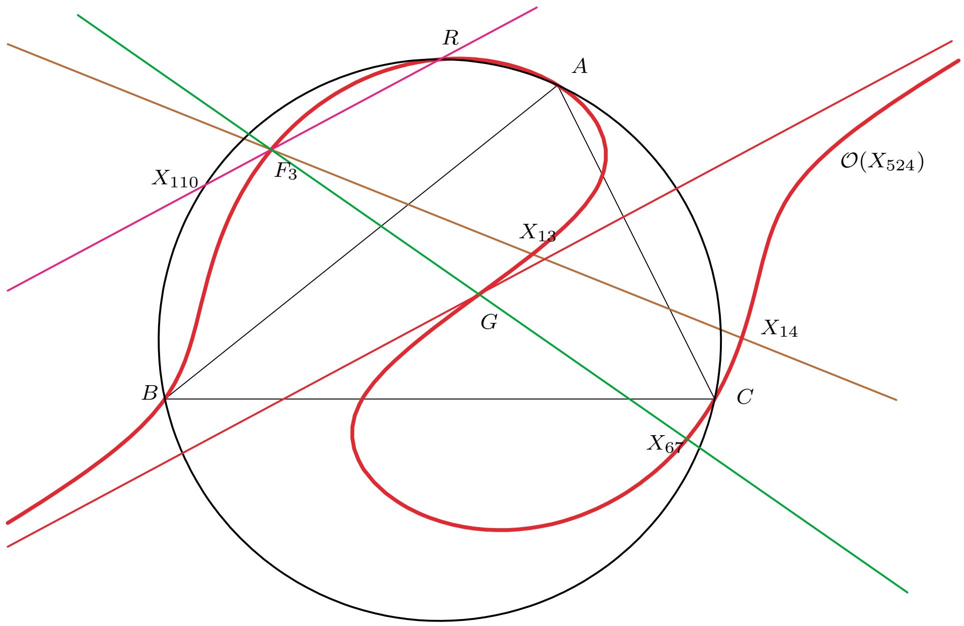


Figure 8. $\mathcal{O}(X_{524})$

6.5.1. *The strophoids* $\mathcal{O}(A)$, $\mathcal{O}(B)$, $\mathcal{O}(C)$. These are the cubics Σ_a, Σ_b, Σ_c with equations given in (9). It is enough to consider $\mathcal{O}(A) = \Sigma_a$. The bisectors of angle A are the tangents at the double point A. The singular focus is the corresponding vertex of the second Brocard triangle, namely, the point $A_2 = (2S_A : b^2 : c^2)$. [32] The real asymptote is parallel to the median AG, being the homothetic image of AG under $\mathsf{h}(A_2, 2)$.

Here are some interesting properties of $\mathcal{O}(A) = \Sigma_a$.

(1) Σ_a is the isogonal conjugate of the Apollonian A-circle

$$\mathcal{C}_A : \qquad a^2(b^2z^2 - c^2y^2) + 2x(b^2S_Bz - c^2S_Cy) = 0, \tag{12}$$

which passes through A and the two isodynamic points X_{15} and X_{16}.

(2) The isogonal conjugate of A_2 is the point $A_4 = (a^2 : 2S_A : 2S_A)$ on the Apollonian circle $\mathcal{C}_A$, which is the projection of H on AG. The isogonal conjugate of the antipode of A_4 on $\mathcal{C}_A$ is the intersection of Σ_a with its real asymptote. [33]

(3) $\mathcal{O}(A) = \Sigma_a$ is the pedal curve with respect to A of the parabola with focus at the second intersection of $\mathcal{C}_A$ and the circumcircle and with directrix the median AG.

---

[32]This is the projection of O on the symmedian AK, the tangent at A_2 being the reflection about OA_2 of the parallel at A_2 to AG.

[33]This isogonal conjugate is on the perpendicular at A to AK, and on the tangent at A_2 to Σ_a.

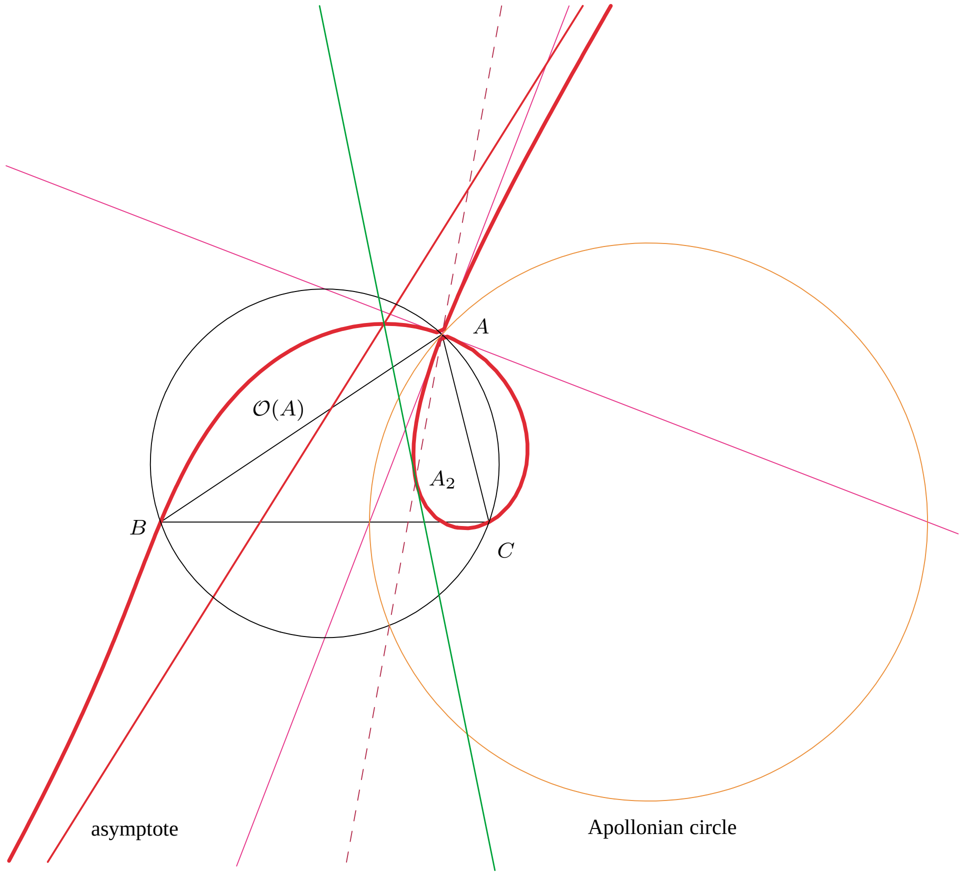


Figure 9. The strophoid $\mathcal{O}(A)$

6.5.2. *The strophoids* $\mathcal{O}(X_{13})$ *and* $\mathcal{O}(X_{14})$. The strophoid $\mathcal{O}(X_{13})$ has singular focus X_{14}, real asymptote the parallel at X_{99} to the line GX_{13}, [34] The circle centered at X_{14} passing through X_{13} intersects the parallel at X_{14} to GX_{13} at D_1 and D_2 which lie on the nodal tangents. The perpendicular at X_{14} to the Fermat line meets the bisectors of the nodal tangents at E_1 and E_2 which are the points where the tangents are parallel to the asymptote and therefore the centers of anallagmaty of the curve. [35]

$\mathcal{O}(X_{13})$ is the pedal curve with respect to X_{13} of the parabola with directrix the line GX_{13} and focus X'_{13}, the symmetric of X_{13} about X_{14}.

[34]The "third intersection" of this asymptote with the cubic lies on the perpendicular at X_{13} to the Fermat line. The intersection of the perpendicular at X_{13} to GX_{13} and the parallel at X_{14} to GX_{13} is another point on the curve.

[35]This means that E_1 and E_2 are the centers of two circles through X_{13} and the two inversions with respect to those circles leave $\mathcal{O}(X_{13})$ unchanged.

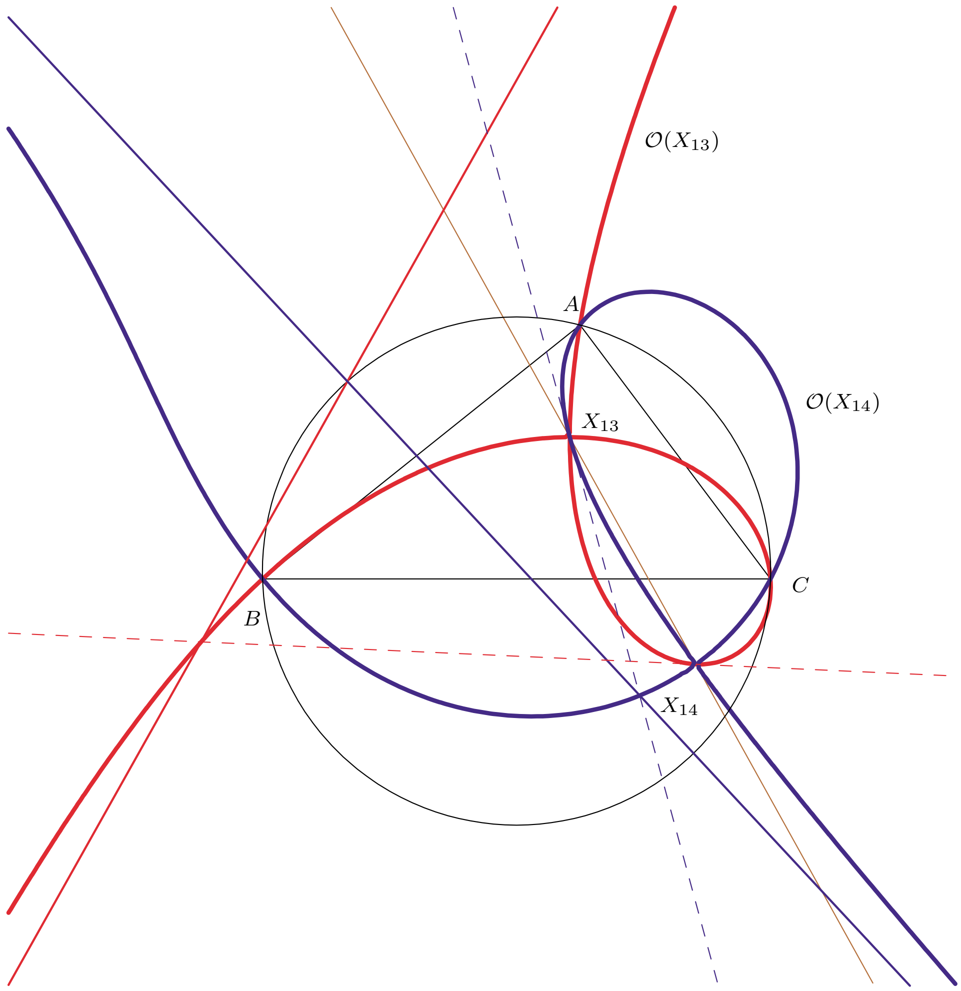


Figure 10. $\mathcal{O}(X_{13})$ and $\mathcal{O}(X_{14})$

The construction of $\mathcal{O}(X_{13})$ is easy to realize. Draw the parallel ℓ at X_{14} to GX_{13} and take a variable point M on it. The perpendicular at M to MX'_{13} and the parallel at X_{13} to MX'_{13} intersect at a point on the strophoid.

We can easily adapt all these to $\mathcal{O}(X_{14})$.

6.6. *Other remarkable* $\mathcal{O}(P)$. The following table gives a list triangle centers P with $\mathcal{O}(P)$ passing through the Fermat points X_{13}, X_{14}, and at least four more triangle centers of [5]. Some of them are already known and some others will be detailed in the next section. The very frequent appearance of X_{15}, X_{16} is explained in §7.3 below.

P	centers	P	centers
X_1	$X_{10,80,484,519,759}$	X_{182}	$X_{15,16,98,542}$
X_3	Neuberg cubic	X_{187}	$X_{15,16,598,843}$
X_5	$X_{4,30,79,80,265,621,622}$	X_{354}	$X_{1,105,484,518}$
X_6	$X_{2,15,16,111,368,524}$	X_{386}	$X_{10,15,16,519}$
X_{32}	$X_{15,16,83,729,754}$	X_{511}	$X_{15,16,262,842}$
X_{39}	$X_{15,16,76,538,755}$	X_{569}	$X_{15,16,96,539}$
X_{51}	$X_{61,62,250,262,511}$	X_{574}	$X_{15,16,543,671}$
X_{54}	$X_{3,96,265,539}$	X_{579}	$X_{15,16,226,527}$
X_{57}	$X_{1,226,484,527}$	X_{627}	$X_{17,532,617,618,622}$
X_{58}	$X_{15,16,106,540}$	X_{628}	$X_{18,533,616,619,621}$
X_{61}	$X_{15,16,18,533,618}$	X_{633}	$X_{18,533,617,623}$
X_{62}	$X_{15,16,17,532,619}$	X_{634}	$X_{17,532,616,624}$

7. Pencils of $\mathcal{O}(P)$

7.1. *Generalities.* The orthopivotal cubics with orthopivots on a given line ℓ form a pencil $\mathbb{F}_\ell$ generated by any two of them. Apart from the vertices, the Fermat points, and two circular points at infinity, all the cubics in the pencil pass through two fixed points depending on the line ℓ. Consequently, all the orthopivotal cubics passing through a given point Q have their orthopivots on the tangent at Q to $\mathcal{O}(Q)$, namely, the line $QQ^\perp$. They all pass through another point Q' on this line which is its second intersection with the circle which is its antiorthocorrespondent. For example, $\mathcal{O}(Q)$ passes through G, O, or H if and only if Q lies on GK, OX_{54}, or the Euler line respectively.

7.2. *Pencils with orthopivot on a line passing through G.* If ℓ contains the centroid G, every orthopivotal cubic in the pencil $\mathbb{F}_\ell$ passes through its infinite point and second intersection with the Kiepert hyperbola. As P traverses ℓ, the singular focus of $\mathcal{O}(P)$ traverses its reflection about F_1F_2 (see §5).

The most remarkable pencil is the one with ℓ the Euler line. In this case, the two fixed points are the infinite point X_{30} and the orthocenter H. In other words, all the cubics in this pencil have their asymptote parallel to the Euler line. In this pencil, we find the Neuberg cubic and $\mathcal{K}_n$. The singular focus traverses the line GX_{98}, X_{98} being the Tarry point.

Another worth noticing pencil is obtained when ℓ is the line GX_{98}. In this case, the two fixed points are the infinite point X_{542} and X_{98}. The singular focus traverses the Euler line. This pencil contains the two degenerate cubics $\mathcal{O}(G)$ and $\mathcal{O}(X_{110})$ seen in §6.1.

When ℓ is the line GK, the two fixed points are the infinite point X_{524} and the centroid G. The singular focus lies on the line GX_{99}, X_{99} being the Steiner point. This pencil contains $\mathcal{B}_2$ and the central cubic seen in §6.4.

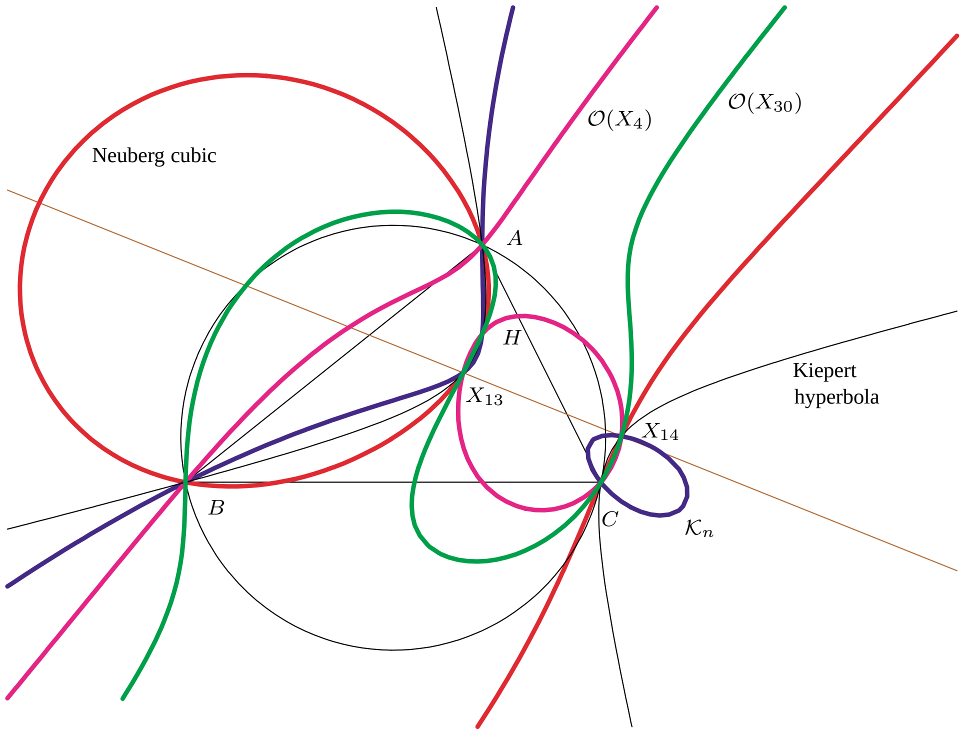


Figure 11. The Euler pencil

7.3. *Pencils with orthopivots on a line not passing through G.* If ℓ is a line not through G, the orthopivotal cubics in the pencil $\mathbb{F}_\ell$ pass through the two (not necessarily real nor distinct) intersections of ℓ with the circle which is its antiorthocorrespondent of. See §2.5 and §3. The singular focus lies on a circle through G, and the real asymptote envelopes a deltoid tangent to the line F_1F_2 and tritangent to the reflection of this circle about G.

According to §6.2.1, §6.2.2, §6.4, this pencil contains at least one, at most three $p\mathcal{K}$, $n\mathcal{K}$, focal(s) depending of the number of intersections of ℓ with the cubics met in those paragraphs respectively.

Consider, for example, the Brocard axis OK. We have seen in §6.3 that there are two and only two isogonal $\mathcal{O}(P)$, the Neuberg cubic and the second Brocard cubic $\mathcal{B}_2$ obtained when the orthopivots are O and K respectively. The two fixed points of the pencil are the isodynamic points.[36]

The singular focus lies on the Parry circle (see §5) and the asymptote envelopes a deltoid tritangent to the reflection of the Parry circle about G.

The pencil $\mathbb{F}_{OK}$ is invariant under isogonal conjugation, the isogonal conjugate of $\mathcal{O}(P)$ being $\mathcal{O}(Q)$, where Q is the harmonic conjugate of P with respect to

[36]The antiorthocorrespondent of the Brocard axis is a circle centered at X_{647}, the isogonal conjugate of the trilinear pole of the Euler line.

O and K. It is obvious that the Neuberg cubic and $\mathcal{B}_2$ are the only cubic which are "self-isogonal" and all the others correspond two by two. Since OK intersects the Napoleon cubic at O, X_{61} and X_{62}, there are only three $p\mathcal{K}$ in this pencil, the Neuberg cubic and $\mathcal{O}(X_{61})$, $\mathcal{O}(X_{62})$. [37]

$\mathcal{O}(X_{61})$ passes though X_{18}, X_{533}, X_{618}, and the isogonal conjugates of X_{532} and X_{619}.

$\mathcal{O}(X_{62})$ passes though X_{17}, X_{532}, X_{619}, and the isogonal conjugates of X_{533} and X_{618}. There are only three focals in the pencil $\mathbb{F}_{OK}$, namely, $\mathcal{B}_2$ and $\mathcal{O}(X_{15})$, $\mathcal{O}(X_{16})$ (with singular foci X_{16}, X_{15} respectively).

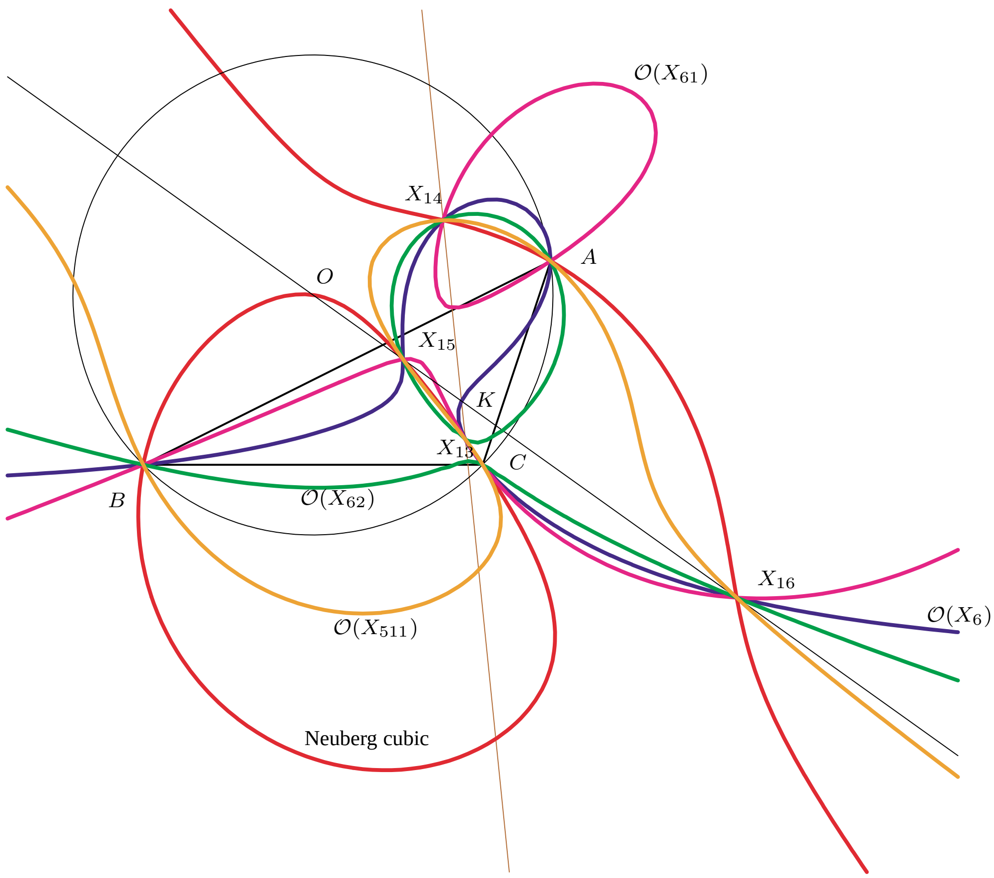


Figure 12. The Brocard pencil

An interesting situation is found when $P = X_{182}$, the midpoint of OK. Its harmonic conjugate with respect to OK is the infinite point $Q = X_{511}$. $\mathcal{O}(X_{511})$ passes through X_{262} which is its intersection with its real asymptote parallel at G

[37] $\mathcal{O}(X_{61})$ and $\mathcal{O}(X_{62})$ are isogonal conjugates of each other. Their pivots are X_{14} and X_{13} respectively and their poles are quite complicated and unknown in [5].

to OK. Its singular focus is G. The third intersection with the Fermat line is U_1 on $X_{23}X_{110}$ and the last intersection with the circumcircle is $X_{842} = X^*_{542}$.[38]

$\mathcal{O}(X_{182})$ is the isogonal conjugate of $\mathcal{O}(X_{511})$ and passes through X_{98}, X_{182}. Its singular focus is X_{23}, inverse of G in the circumcircle. Its real asymptote is parallel to the Fermat line at X_{323} and the intersection is the isogonal conjugate of U_1.

The following table gives several pairs of harmonic conjugates P and Q on OK. Each column gives two cubics $\mathcal{O}(P)$ and $\mathcal{O}(Q)$, each one being the isogonal conjugate of the other.

P	X_{32}	X_{50}	X_{52}	X_{58}	X_{187}	X_{216}	X_{284}	X_{371}	X_{389}	X_{500}
Q	X_{39}	X_{566}	X_{569}	X_{386}	X_{574}	X_{577}	X_{579}	X_{372}	X_{578}	X_{582}

8. A quintic and a quartic

We present a pair of interesting higher degree curves associated with the orthocorrespondence.

Theorem 12. *The locus of point P whose orthotransversal $\mathcal{L}_P$ and trilinear polar ℓ_P are parallel is the circular quintic*

$$\mathcal{Q}_1 : \qquad \sum_{\text{cyclic}} a^2y^2z^2(S_By - S_Cz) = 0.$$

Equivalently, $\mathcal{Q}_1$ is the locus of point P for which

(1) *the lines PP^* and ℓ_P (or $\mathcal{L}_P$) are perpendicular,*
(2) *P lies on the Euler line of the pedal triangle of P^*,*
(3) *P, P^*, H/P (and $P^\perp$) are collinear,*
(4) *P lies on $\mathcal{O}(P^*)$.*

Note that $\mathcal{L}_P$ and ℓ_P coincide when P is one of the Fermat points.[39]

Theorem 13. *The isogonal transform of the quintic $\mathcal{Q}_1$ is the circular quartic*

$$\mathcal{Q}_2 : \qquad \sum_{\text{cyclic}} a^4S_Ayz(c^2y^2 - b^2z^2) = 0,$$

which is also the locus of point P such that

(1) *the lines PP^* and ℓ_{P^*} (or $\mathcal{L}_{P^*}$) are perpendicular,*
(2) *P lies on the Euler line of its pedal triangle,*
(3) *P, P^*, H/P^* are collinear,*
(4) *P^* lies on $\mathcal{O}(P)$.*

These two curves $\mathcal{Q}_1$ and $\mathcal{Q}_2$ contain a large number of interesting points, which we enumerate below.

Proposition 14. *The quintic $\mathcal{Q}_1$ contains the 58 following points:*

[38] This is on $X_{23}X_{110}$ too. It is the reflection of the Tarry point X_{98} about the Euler line and the reflection of X_{74} about the Brocard line.

[39] See §1, Remark (5).

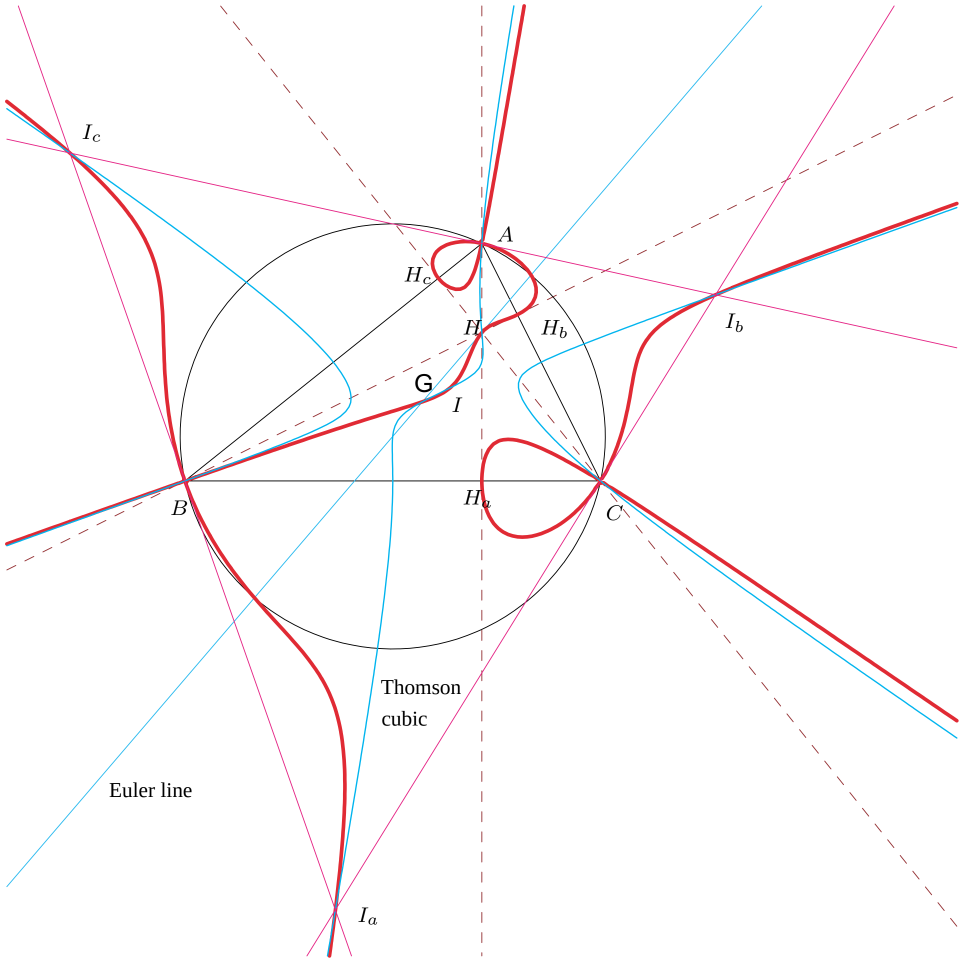


Figure 13. The quintic $\mathcal{Q}_1$

(1) *the vertices* A, B, C, *which are singular points with the bisectors as tangents,*
(2) *the circular points at infinity and the singular focus* G,[40]
(3) *the three infinite points of the Thomson cubic,*[41]
(4) *the in/excenters* I, I_a, I_b, I_c, *with tangents passing through* O, *and the isogonal conjugates of the intersections of these tangents with the trilinear polars of the corrresponding in/excenters,*
(5) H, *with tangent the Euler line,*

[40]The tangent at G passes through the isotomic conjugate of $G^{\perp}$, the point with coordinates $\left(\frac{1}{b^2+c^2-5a^2} : \dots : \dots\right)$.
[41]In other words, $\mathcal{Q}_1$ has three real asymptotes parallel to those of the Thomson cubic.

(6) *the six points where a circle with diameter a side of ABC intersects the corresponding median,* [42]
(7) *the feet of the altitudes, the tangents being the altitudes,*
(8) *the Fermat points X_{13} and X_{14},*
(9) *the points X_{1113} and X_{1114} where the Euler line meets the circumcircle,*
(10) *the perspectors of the 27 Morley triangles and ABC.* [43]

Proposition 15. *The quartic $\mathcal{Q}_2$ contains the 61 following points:*

(1) *the vertices A, B, C,* [44]
(2) *the circular points at infinity,* [45]
(3) *the three points where the Thomson cubic meets the circumcircle again,*
(4) *the in/excenters I, I_a, I_b, I_c, with tangents all passing through O, and the intersections of these tangents OI_x with the trilinear polars of the corresponding in/excenters,*
(5) *O and K,* [46]
(6) *the six points where a symmedian intersects a circle centered at the corresponding vertex of the tangential triangle passing through the remaining two vertices of ABC,* [47]
(7) *the six feet of bisectors,*
(8) *the isodynamic points X_{15} and X_{16}, with tangents passing through X_{23},*
(9) *the two infinite points of the Jerabek hyperbola,* [48]
(10) *the isogonal conjugates of the perspectors of the 27 Morley triangles and ABC.* [49]

We give a proof of (10). Let k_1, k_2, $k_3 = 0, \pm 1$, and consider

$$\varphi_1 = \frac{A + 2k_1\pi}{3}, \quad \varphi_2 = \frac{B + 2k_2\pi}{3}, \quad \varphi_3 = \frac{C + 2k_3\pi}{3}.$$

Denote by M one of the 27 points with barycentric coordinates

$$(a\cos\varphi_1 : b\cos\varphi_2 : c\cos\varphi_3).$$

---

[42] The two points on the median AG have coordinates

$$(2a : -a \pm \sqrt{2b^2 + 2c^2 - a^2} : -a \pm \sqrt{2b^2 + 2c^2 - a^2}).$$

[43] The existence of the these points was brought to my attention by Edward Brisse. In particular, X_{357}, the perspector of ABC and first Morley triangle.

[44] These are inflection points, with tangents passing through O.

[45] The singular focus is the inverse X_{23} of G in the circumcircle. This point is not on the curve $\mathcal{Q}_2$.

[46] Both tangents at O and K pass through the point $Z = (a^2S_A(b^2 + c^2 - 2a^2) : \cdots : \cdots)$, the intersection of the trilinear polar of O with the orthotransversal of X_{110}. The tangent at O is also tangent to the Jerabek hyperbola and the orthocubic.

[47] The two points on the symmedian AK have coordinates $(-a^2 \pm a\sqrt{2b^2 + 2c^2 - a^2} : 2b^2 : 2c^2)$.

[48] The two real asymptotes of $\mathcal{Q}_2$ are parallel to those of the Jerabek hyperbola and meet at Z in footnote 46 above.

[49] In particular, the Morley-Yff center X_{358}.

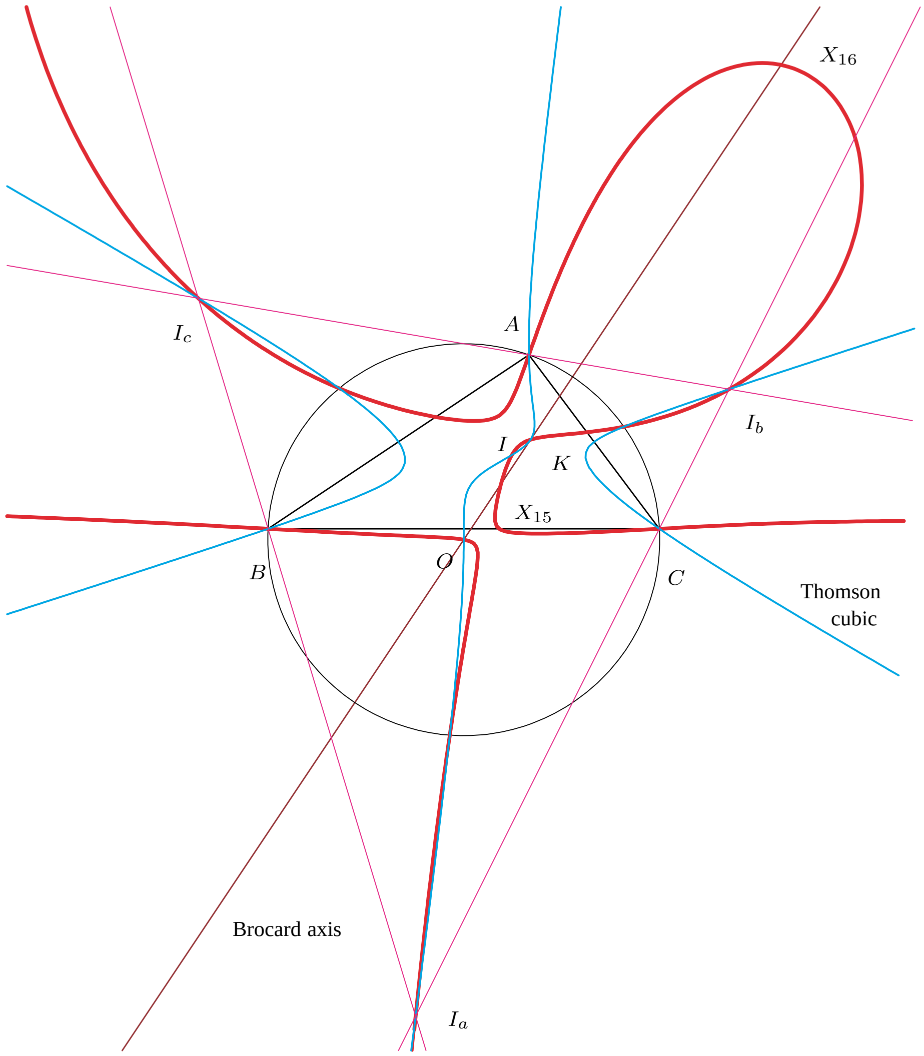


Figure 14. The quartic $\mathcal{Q}_2$

The isogonal conjugate of M is the perspector of ABC and one of the 27 Morley triangles. [50] We show that M lies on the quartic $\mathcal{Q}_2$. [51] Since $\cos A = \cos 3\varphi_1 = 4\cos^3\varphi_1 - 3\cos\varphi_1$, we have $\cos^3\varphi_1 = \frac{1}{4}(\cos A + 3\cos\varphi_1)$ and similar identities for $\cos^3\varphi_2$ and $\cos^3\varphi_3$. From this and the equation of $\mathcal{Q}_2$, we obtain

$$\sum_{\text{cyclic}} a^4 S_A b\cos\varphi_2 \; c\cos\varphi_3 \; (c^2b^2\cos^2\varphi_2 - b^2c^2\cos^2\varphi_3)$$

[50]For example, with $k_1 = k_2 = k_3 = 0$, $M^* = X_{357}$ and $M = X_{358}$.
[51]Consequently, M^* lies on the quintic $\mathcal{Q}_1$. See Proposition 14(10).

$$\begin{aligned}
&= \sum_{\text{cyclic}} a^4b^3c^3S_A(\cos\varphi_3\ \cos^3\varphi_2 - \cos\varphi_2\ \cos^3\varphi_3) \\
&= \sum_{\text{cyclic}} \frac{1}{4}\, a^4b^3c^3S_A(\cos\varphi_3\cos B - \cos\varphi_2\cos C) \\
&= \sum_{\text{cyclic}} \frac{1}{4}\, a^4b^3c^3S_A\left(\frac{S_B}{ac}\cos\varphi_3 - \frac{S_C}{ab}\cos\varphi_2\right) \\
&= \frac{1}{4}\, a^3b^3c^3S_AS_BS_C \sum_{\text{cyclic}} \left(\frac{\cos\varphi_3}{c\,S_C} - \frac{\cos\varphi_2}{b\,S_B}\right) \\
&= 0.
\end{aligned}$$

This completes the proof of (10).

Remark. $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are *strong* curves in the sense that they are invariant under extraversions: any point lying on one of them has its three extraversions also on the curve. [52]

References

[1] H. Brocard and T. Lemoyne, *Courbes Géométriques Remarquables*, Librairie Albert Blanchard, Paris, third edition, 1967.

[2] A. Goddijn, Hyacinthos message 6226, December 29, 2002.

[3] F. M. van Lamoen, Hyacinthos message 6158, December 13, 2002.

[4] C. Kimberling, Triangle Centers and Central Triangles,*Congressus Numerantium*, 129 (1998) 1–295.

[5] C. Kimberling, *Encyclopedia of Triangle Centers*, August 22, 2002 edition, available at http://www2.evansville.edu/ck6/encyclopedia/; January 14, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[6] J. Parish, Hyacinthos message 1434, September 15, 2000.

[7] J. Parish, Hyacinthos messages 6161, 6162, December 13, 2002.

[8] G. M. Pinkernell, Cubic curves in the triangle plane, *Journal of Geometry*, 55 (1996) 142–161.

[9] P. Yiu, The uses of homogeneous barycentric coordinates in plane euclidean geometry, *Int. J. Math. Educ. Sci. Technol.*, 31 (2000) 569 – 578.

Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France

E-mail address: b.gibert@free.fr

[52] The extraversions of a point are obtained by replacing one of a, b, c by its opposite. For example, the extraversions of the incenter I are the three excenters and I is said to be a *weak* point. On the contrary, K is said to be a ”strong” point.

Forum Geometricorum
Volume 3 (2003) 29–34.

FORUM GEOM
ISSN 1534-1178

On the Procircumcenter and Related Points

Alexei Myakishev

Abstract. Given a triangle ABC, we solve the construction problem of a point P, together with points B_c, C_b on BC, C_a, A_c on CA, and A_b, B_a on AB such that PB_aC_a, A_bPC_b, and A_cB_cP are congruent triangles similar to ABC. There are altogether seven such triads. If these three congruent triangles are all oppositely similar to ABC, then P must be the procircumcenter, with trilinear coordinates $(a^2 \cos A : b^2 \cos B : c^2 \cos C)$. If at least one of the triangles in the triad is directly similar to ABC, then P is either a vertex or the midpoint of a side of the tangential triangle. We also determine the ratio of similarity in each case.

1. Introduction

Given a triangle ABC, we consider the construction of a point P, together with points B_c, C_b on BC, C_a, A_c on CA, and A_b, B_a on AB such that PB_aC_a, A_bPC_b, and A_cB_cP are congruent triangles similar to ABC. We first consider in §§2,3 the case when these triangles are all *oppositely* similar to ABC. See Figure 1. In §4, the possibilities when at least one of these congruent triangles is directly similar to ABC are considered. See, for example, Figure 2.

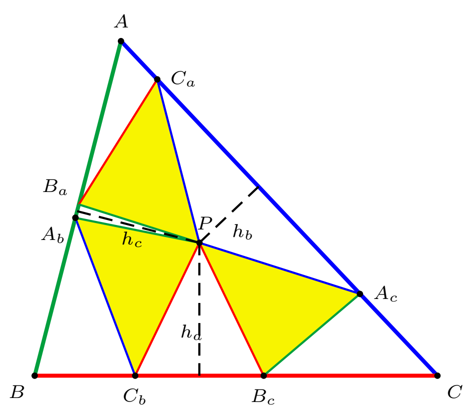


Figure 1

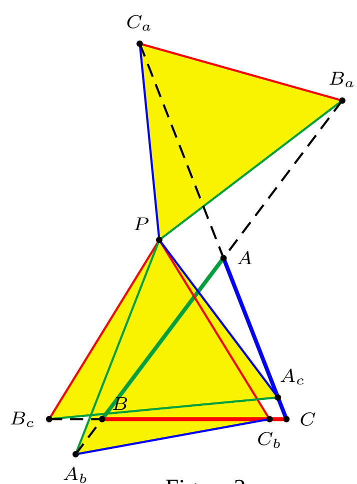


Figure 2

Publication Date: January 27, 2003. Communicating Editor: Jean-Pierre Ehrmann.

The author thanks Jean-Pierre Ehrmann and Paul Yiu for their helps in the preparation of this paper.

2. The case of opposite similarity: construction of P

With reference to Figure 1, we try to find the trilinear coordinates of P. As usual, we denote the lengths of the sides opposite to angles A, B, C by a, b, c. Denote the *oriented* angles C_bPB_c by φ_a, A_cPC_a by φ_b, and B_aPA_b by φ_c. [1] Since $PC_b = PB_c$, $\angle PB_cC_b = \frac{1}{2}(\pi - \varphi_a)$. Since also $\angle PB_cA_c = B$, we have $\angle A_cB_cC = \frac{1}{2}(\pi + \varphi_a) - B$. For the same reason, $\angle B_cA_cC = \frac{1}{2}(\pi + \varphi_b) - A$. Considering the sum of the angles in triangle A_cB_cC, we have $\frac{1}{2}(\varphi_a + \varphi_b) = \pi - 2C$. Since $\varphi_a + \varphi_b + \varphi_c = \pi$, we have $\varphi_c = 4C - \pi$. Similarly, $\varphi_a = 4A - \pi$ and $\varphi_b = 4B - \pi$.

Let k be the ratio of similarity of the triangles PB_aC_a, A_bPC_b, and A_cB_cP with ABC, *i.e.*, $B_aC_a = PC_b = B_cP = k \cdot BC = ka$. The perpendicular distance from P to the line BC is

$$h_a = ka \cos \frac{\varphi_a}{2} = ka \cos \left(2A - \frac{\pi}{2}\right) = ka \sin 2A.$$

Similarly, the perpendicular distances from P to CA and AB are $h_b = kb \sin 2B$ and $h_c = kc \sin 2C$. It follows that P has trilinear coordinates,

$$(a \sin 2A : b \sin 2B : c \sin 2C) \sim (a^2 \cos A : b^2 \cos B : c^2 \cos C). \tag{1}$$

Note that we have found not only the trilinears of P, but also the angles of isosceles triangles PC_bB_c, PA_cC_a, PB_aA_b. It is therefore easy to construct the triangles by ruler and compass from P. Now, we easily identify P as the isogonal conjugate of the isotomic conjugate of the circumcenter O, which has trilinear coordinates $(\cos A : \cos B : \cos C)$. We denote this point by $\overline{O}$ and follow John H. Conway in calling it the *procircumcenter* of triangle ABC. We summarize the results in the following proposition.

Proposition 1. *Given a triangle ABC* not *satisfying (2), the point P for which there are congruent triangles PB_aC_a, A_bPC_b, and A_cB_cP oppositely similar to ABC (with B_c, C_b on BC, C_a, A_c on CA, and A_b, B_a on AB) is the procircumcenter $\overline{O}$. This is a finite point unless the given triangle satisfies*

$$a^4(b^2 + c^2 - a^2) + b^4(c^2 + a^2 - b^2) + c^4(a^2 + b^2 - c^2) = 0. \tag{2}$$

The procircumcenter $\overline{O}$ appears as X_{184} in [3], and is identified as the inverse of the Jerabek center X_{125} in the Brocard circle. A simple construction of $\overline{O}$ is made possible by the following property discovered by Fred Lang.

Proposition 2 (Lang [4])**.** *Let the perpendicular bisectors of BC, CA, AB intersect the other pairs of sides at B_1, C_1, C_2, A_2, A_3, B_3 respectively. The perpendicular bisectors of B_1C_1, C_2A_2 and A_3B_3 bound a triangle homothetic to ABC at the procircumcenter $\overline{O}$.*

---

[1] We regard the orientation of triangle ABC as positive. The oriented angles are defined modulo 2π.

3. The case of opposite similarity: ratio of similarity

We proceed to determine the ratio of similarity k. We shall make use of the following lemmas.

Lemma 3. *Let $\triangle$ denote the area of triangle ABC, and R its circumradius.*
(1) $\triangle = 2R^2 \sin A \sin B \sin C$;
(2) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$;
(3) $\sin 4A + \sin 4B + \sin 4C = -4 \sin 2A \sin 2B \sin 2C$;
(4) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$.

Proof. (1) By the law of sines,

$$\triangle = \frac{1}{2} bc \sin A = \frac{1}{2}(2R \sin B)(2R \sin C) \sin A = 2R^2 \sin A \sin B \sin C.$$

For (2),

$$\begin{aligned}
&\sin 2A + \sin 2B + \sin 2C \\
=&2 \sin A \cos A + 2 \sin(B+C) \cos(B-C) \\
=&2 \sin A(-\cos(B+C) + \cos(B-C)) \\
=&4 \sin A \sin B \sin C.
\end{aligned}$$

The proof of (3) is similar. For (4),

$$\begin{aligned}
&\sin^2 A + \sin^2 B + \sin^2 C \\
=&\sin^2 A + 1 - \frac{1}{2}(\cos 2B + \cos 2C) \\
=&\sin^2 A + 1 - \cos(B+C)\cos(B-C) \\
=&2 - \cos^2 A + \cos A \cos(B-C) \\
=&2 + \cos A(\cos(B+C) + \cos(B-C)) \\
=&2 + 2\cos A \cos B \cos C.
\end{aligned}$$

□

Lemma 4. $a^2 + b^2 + c^2 = 9R^2 - OH^2$*, where R is the circumradius, and O, H are respectively the circumcenter and orthocenter of triangle ABC.*

This was originally due to Euler. An equivalent statement

$$a^2 + b^2 + c^2 = 9(R^2 - OG^2),$$

where G is the centroid of triangle ABC, can be found in [2, p.175].

Proposition 5 (Dergiades [1])**.** *The ratio of similarity of $\overline{O}B_aC_a$, $A_b\overline{O}C_b$, and $A_cB_c\overline{O}$ with ABC is*

$$k = \left| \frac{R^2}{3R^2 - OH^2} \right|.$$

Proof. Since $2\triangle = a\cdot h_a + b\cdot h_b + c\cdot h_c$, and $h_a = ka\sin 2A$, $h_b = kb\sin 2B$, and $h_c = kc\sin 2C$, the ratio of similarity is the absolute value of

$$
\begin{aligned}
& \frac{2\triangle}{a^2\sin 2A + b^2\sin 2B + c^2\sin 2C} \\
= & \frac{4R^2\sin A\sin B\sin C}{4R^2(\sin^2 A\sin 2A + \sin^2 B\sin 2B + \sin^2 C\sin 2C)} && \text{[Lemma 3(1)]} \\
= & \frac{2\sin A\sin B\sin C}{(1-\cos 2A)\sin 2A + (1-\cos 2B)\sin 2B + (1-\cos 2C)\sin 2C} \\
= & \frac{4\sin A\sin B\sin C}{2(\sin 2A + \sin 2B + \sin 2C) - (\sin 4A + \sin 4B + \sin 4C)} \\
= & \frac{4\sin A\sin B\sin C}{8\sin A\sin B\sin C + 4\sin 2A\sin 2B\sin 2C} && \text{[Lemma 3(2,3)]} \\
= & \frac{1}{2 + 8\cos A\cos B\cos C} \\
= & \frac{1}{4(\sin^2 A + \sin^2 B + \sin^2 C) - 6} && \text{[Lemma 3(4)]} \\
= & \frac{R^2}{a^2+b^2+c^2-6R^2} \\
= & \frac{R^2}{3R^2 - OH^2}
\end{aligned}
$$

by Lemma 4. □

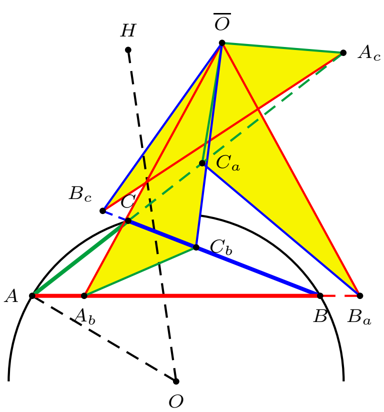


Figure 3: $OH = 2R$

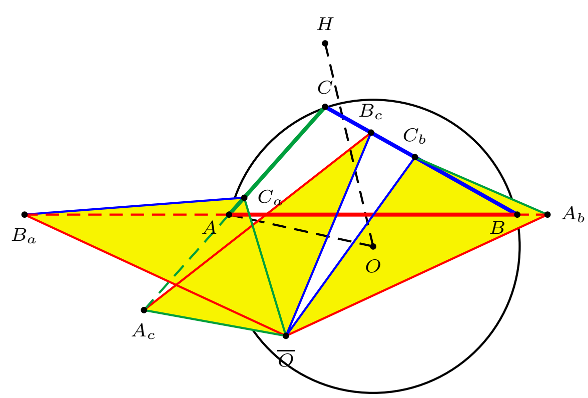


Figure 4: $OH = \sqrt{2}R$

From Proposition 5, we also infer that $\overline{O}$ is an infinite point if and only if $OH = \sqrt{3}R$. More interesting is that for triangles satisfying $OH = 2R$ or $\sqrt{2}R$, the congruent triangles in the triad are also congruent to the reference triangle ABC. See Figures 3 and 4. These are triangles satisfying

$$a^4(b^2+c^2-a^2) + b^4(c^2+a^2-b^2) + c^4(a^2+b^2-c^2) = \pm a^2b^2c^2.$$

4. Cases allowing direct similarity with ABC

As Jean-Pierre Ehrmann has pointed out, by considering all possible orientations of the triangles PB_aC_a, A_bPC_b, A_cB_cP, there are other points, apart from the procircumcenter $\overline{O}$, that yield triads of congruent triangles similar to ABC.

4.1. *Exactly one of the triangles oppositely similar to* ABC. Suppose, for example, that among the three congruent triangles, only PB_aC_a be oppositely similar to ABC, the other two, A_bPC_b and A_cB_cP being directly similar. We denote by P_a^+ the point P satisfying these conditions. Modifying the calculations in §2, we have

$$\varphi_a = \pi + 2A, \quad \varphi_b = \pi - 2A, \quad \varphi_c = \pi - 2A.$$

From these, we obtain the trilinears of P_a^+ as

$$(-a \sin A : b \sin A : c \sin A) = (-a : b : c).$$

It follows that P_a^+ is the A-vertex of the tangential triangle of ABC. See Figure 5.

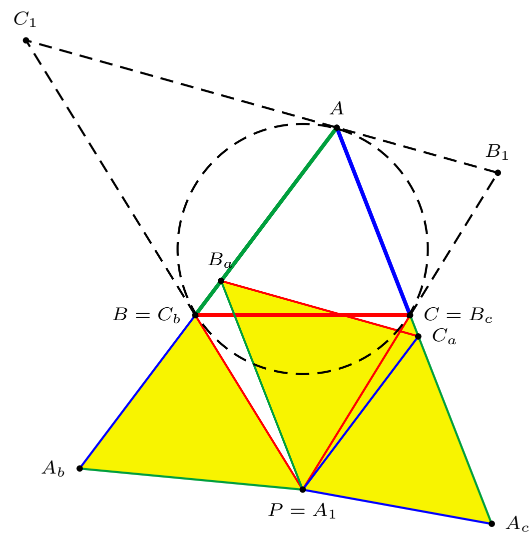


Figure 5

The ratio of similarity, by a calculation similar to that performed in §3, is $k = \left|\frac{1}{2\cos A}\right|$. This is equal to 1 only when $A = \frac{\pi}{3}$ or $\frac{2\pi}{3}$. In these cases, the three triangles are congruent to ABC.

Clearly, there are two other triads of congruent triangles corresponding to the other two vertices of the tangential triangle.

4.2. *Exactly one of triangles directly similar to* ABC. Suppose, for example, that among the three congruent triangles, only PB_aC_a be directly similar to ABC, the other two, A_bPC_b and A_cB_cP being oppositely similar. We denote by P_a^- the point P satisfying these conditions. See Figure 6. In this case, we have

$$\varphi_a = 2A - \pi, \quad \varphi_b = \pi + 2B - 2C, \quad \varphi_c = \pi + 2C - 2B.$$

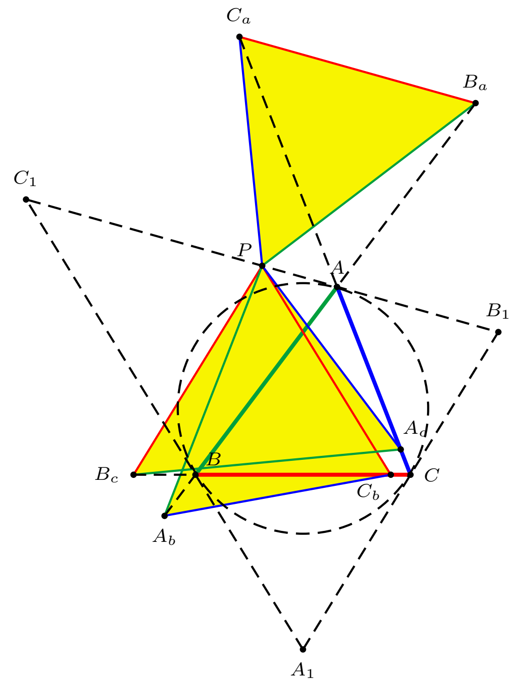


Figure 6

From these, we obtain the trilinears of P_a^- as

$$(-a\sin A : b\sin(B-C) : c\sin(C-B)) = (-a^3 : b(b^2-c^2) : c(c^2-b^2)).$$

It is easy to check that this is the midpoint of the side B_1C_1 of the tangential triangle of ABC. In this case, the ratio of similarity is $k = \left|\frac{1}{4\cos B\cos C}\right|$.

Clearly, there are two other triads of congruent triangles corresponding to the midpoints of the remaining two sides of the tangential triangle.

We conclude with the remark that it is not possible for all three of the congruent triangles to be directly similar to ABC, since this would require $\varphi_a = \varphi_b = \varphi_c = \pi$.

References

[1] N. Dergiades, Hyacinthos message 5437, May 10, 2002.

[2] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.

[3] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

[4] F. Lang, Hyacinthos message 1190, August 13, 2000.

Alexei Myakishev: Smolnaia 61-2, 138, Moscow, Russia, 125445
E-mail address: alex.geom@mtu-net.ru

Forum Geometricorum
Volume 3 (2003) 35–47.

FORUM GEOM
ISSN 1534-1178

Bicentric Pairs of Points and Related Triangle Centers

Clark Kimberling

Abstract. Bicentric pairs of points in the plane of triangle ABC occur in connection with three configurations: (1) cevian traces of a triangle center; (2) points of intersection of a central line and central circumconic; and (3) vertex-products of bicentric triangles. These bicentric pairs are formulated using trilinear coordinates. Various binary operations, when applied to bicentric pairs, yield triangle centers.

1. Introduction

Much of modern triangle geometry is carried out in in one or the other of two homogeneous coordinate systems: barycentric and trilinear. Definitions of triangle center, central line, and bicentric pair, given in [2] in terms of trilinears, carry over readily to barycentric definitions and representations. In this paper, we choose to work in trilinears, except as otherwise noted.

Definitions of *triangle center* (or simply *center*) and *bicentric pair* will now be briefly summarized. A triangle center is a point (as defined in [2] as a function of variables a, b, c that are sidelengths of a triangle) of the form

$$f(a, b, c) : f(b, c, a) : f(c, a, b),$$

where f is homogeneous in a, b, c, and

$$|f(a, c, b)| = |f(a, b, c)|. \tag{1}$$

If a point satisfies the other defining conditions but (1) fails, then the points

$$\begin{aligned} F_{ab} &:= f(a, b, c) : f(b, c, a) : f(c, a, b), \\ F_{ac} &:= f(a, c, b) : f(b, a, c) : f(c, b, a) \end{aligned} \tag{2}$$

are a *bicentric pair*. An example is the pair of Brocard points,

$$c/b : a/c : b/a \quad \text{and} \quad b/c : b/a : c/b.$$

Seven binary operations that carry bicentric pairs to centers are discussed in §§2, 3, along with three bicentric pairs associated with a center. In §4, bicentric pairs associated with cevian traces on the sidelines BC, CA, AB will be examined. §§6–10 examine points of intersection of a central line and central circumconic; these points are sometimes centers and sometimes bicentric pairs. §11 considers

Publication Date: February, 2003. Communicating Editor: Jean-Pierre Ehrmann.

bicentric pairs associated with bicentric triangles. §5 supports §6, and §12 revisits two operations discussed in §3.

2. Products: trilinear and barycentric

Suppose $U = u : v : w$ and $X = x : y : z$ are points expressed in general homogeneous coordinates. Their product is defined by

$$U \cdot X = ux : vy : wz.$$

Thus, when coordinates are specified as trilinear or barycentric, we have here two distinct product operations, corresponding to constructions of barycentric product [8] and trilinear product [6]. Because we have chosen trilinears as the primary means of representation in this paper, it is desirable to write, for future reference, a formula for barycentric product in terms of trilinear coordinates. To that end, suppose $u : v : w$ and $x : y : z$ are trilinear representations, so that in barycentrics,

$$U = au : bv : cw \quad \text{and} \quad X = ax : by : cz.$$

Then the barycentric product is $a^2ux : b^2vy : c^2wz$, and we conclude as follows: the trilinear representation for the barycentric product of $U = u : v : w$ and $X = x : y : z$, these being trilinear representations, is given by

$$U \cdot_\flat X = aux : bvy : cwz.$$

3. Other centralizing operations

Given a bicentric pair, aside from their trilinear and barycentric products, various other binary operations applied to the pair yield a center. Consider the bicentric pair (2). In [2, p. 48], the points

$$F_{ab} \oplus F_{ac} := f_{ab} + f_{ac} : f_{bc} + f_{ba} : f_{ca} + f_{cb} \tag{3}$$

and

$$F_{ab} \ominus F_{ac} := f_{ab} - f_{ac} : f_{bc} - f_{ba} : f_{ca} - f_{cb} \tag{4}$$

are observed to be triangle centers. See §8 for a geometric discussion.

Next, suppose that the points F_{ab} and F_{ac} do not lie on the line at infinity, $\mathcal{L}^\infty$, and consider normalized trilinears, represented thus:

$$F'_{ab} = (k_{ab}f_{ab}, k_{ab}f_{bc}, k_{ab}f_{ca}), \quad F'_{ac} = (k_{ac}f_{ac}, k_{ac}f_{ba}, k_{ac}f_{cb}), \tag{5}$$

where

$$k_{ab} := \frac{2\sigma}{af_{ab} + bf_{bc} + cf_{ca}},\ k_{ac} := \frac{2\sigma}{af_{ac} + bf_{ba} + cf_{cb}},\ \sigma := \text{area}(\triangle ABC).$$

These representations give

$$F'_{ab} \oplus F'_{ac} = k_{ab}f_{ab} + k_{ac}f_{ac} : k_{ab}f_{bc} + k_{ac}f_{ba} : k_{ab}f_{ca} + k_{ac}f_{cb}, \tag{6}$$

which for many choices of $f(a, b, c)$ differs from (3). In any case, (6) gives the the midpoint of the bicentric pair (2), and the harmonic conjugate of this midpoint with respect to F_{ab} and F_{ac} is the point in which the line $F_{ab}F_{ac}$ meets $\mathcal{L}^\infty$.

We turn now to another centralizing operation on the pair (2). Their line is given by the equation

$$\begin{vmatrix} \alpha & \beta & \gamma \\ f_{ab} & f_{bc} & f_{ca} \\ f_{ac} & f_{ba} & f_{cb} \end{vmatrix} = 0$$

and is a central line. Its trilinear pole, P, and the isogonal conjugate of P, given by

$$f_{bc}f_{cb} - f_{ca}f_{ba} : f_{ca}f_{ac} - f_{ab}f_{cb} : f_{ab}f_{ba} - f_{bc}f_{ac},$$

are triangle centers.

If

$$X := x : y : z = f(a,b,c) : f(b,c,a) : f(c,a,b)$$

is a triangle center other than X_1, then the points

$$Y := y : z : x \quad \text{and} \quad Z := z : x : y$$

are clearly bicentric. The operations discussed in §§2,3, applied to $\{Y, Z\}$, yield the following centers:

- trilinear product $= X_1/X$ (the indexing of centers as X_i follows [3]);
- barycentric product $= X_6/X$ (here, "/" signifies trilinear division);
- $Y \oplus Z = y + z : z + x : x + y$;
- $Y \ominus Z = y - z : z - x : x - y$;
- midpoint $= m(a,b,c) : m(b,c,a) : m(c,a,b)$, where

$$m(a,b,c) = cy^2 + bz^2 + 2ayz + x(by + cz);$$

- $YZ \cap \mathcal{L}^\infty = n(a,b,c) : n(b,c,a) : n(c,a,b)$, where

$$n(a,b,c) = cy^2 - bz^2 + x(by - cz);$$

- (isogonal conjugate of trilinear pole of YZ)

$$\begin{aligned} &= x^2 - yz : y^2 - zx : z^2 - xy \\ &= (X_1\text{-Hirst inverse of } X). \end{aligned}$$

The points Z/Y and Y/Z are bicentric and readily yield the centers with first coordinates $x(y^2 + z^2)$, $x(y^2 - z^2)$, and $x^3 - y^2z^2/x$. One more way to make bicentric pairs from triangle centers will be mentioned: if $U = r : s : t$ and $X := x : y : z$ are centers, then ([2, p.49])

$$U \otimes X := sz : tx : ry, \qquad X \otimes U := ty : rz : sx$$

are a bicentric pair. For example, $(U \otimes X) \ominus (X \otimes U)$ has for trilinears the coefficients for line UX.

4. Bicentric pairs associated with cevian traces

Suppose P is a point in the plane of $\triangle ABC$ but not on one of the sidelines BC, CA, AB and not on $\mathcal{L}^\infty$. Let A', B', C' be the points in which the lines AP, BP, CP meet the sidelines BC, CA, AB, respectively. The points A', B', C' are the *cevian traces* of P. Letting $|XY|$ denote the directed length of a segment from a point X to a point Y, we recall a fundamental theorem of triangle geometry (often called Ceva's Theorem, but Hogendijk [1] concludes that it was stated and proved by an ancient king) as follows:

$$|BA'| \cdot |CB'| \cdot |AC'| = |A'C| \cdot |B'A| \cdot |C'B|.$$

(The theorem will not be invoked in the sequel.) We shall soon see that if P is a center, then the points

$$P_{BC} := |BA'| : |CB'| : |AC'| \quad \text{and} \quad P_{CB} := |A'C| : |B'A| : |C'B|$$

comprise a bicentric pair, except for $P = centroid$, in which case both points are the incenter. Let σ denote the area of $\triangle ABC$, and write $P = x : y : z$. Then the actual trilinear distances are given by

$$B = \left(0, \frac{2\sigma}{b}, 0\right) \quad \text{and} \quad A' = \left(0, \frac{2\sigma y}{by+cz}, \frac{2\sigma z}{by+cz}\right).$$

Substituting these into a distance formula (e.g. [2, p. 31]) and simplifying give

$$|BA'| = \frac{z}{b(by+cz)};$$

$$P_{BC} = \frac{z}{b(by+cz)} : \frac{x}{c(cz+ax)} : \frac{y}{a(ax+by)}; \tag{7}$$

$$P_{CB} = \frac{y}{c(by+cz)} : \frac{z}{a(cz+ax)} : \frac{x}{b(ax+by)}. \tag{8}$$

So represented, it is clear that P_{BC} and P_{CB} comprise a bicentric pair if P is a center other that the centroid. Next, let

$$P'_{BC} = \frac{|BA'|}{|CA'|} : \frac{|CB'|}{|AB'|} : \frac{|AC'|}{|BC'|} \quad \text{and} \quad P'_{CB} = \frac{|CA'|}{|BA'|} : \frac{|AB'|}{|CB'|} : \frac{|BC'|}{|AC'|}.$$

Equation (7) implies

$$P'_{BC} = \frac{cz}{by} : \frac{ax}{cz} : \frac{by}{ax} \quad \text{and} \quad P'_{CB} = \frac{by}{cz} : \frac{cz}{ax} : \frac{ax}{by}. \tag{9}$$

Thus, using "/" for trilinear quotient, or for barycentric quotient in case the coordinates in (7) and (8) are barycentrics, we have $P'_{BC} = P_{BC}/P_{CB}$ and $P'_{CB} = P_{CB}/P_{BC}$. The pair of isogonal conjugates in (9) generalize the previously mentioned Brocard points, represented by (9) when $P = X_1$.

As has been noted elsewhere, the trilinear (and hence barycentric) product of a bicentric pair is a triangle center. For both kinds of product, the representation is given by

$$P_{BC} \cdot P_{CB} = \frac{a}{x(by+cz)^2} : \frac{b}{y(cz+ax)^2} : \frac{c}{z(ax+by)^2}.$$

P	X_2	X_1	X_{75}	X_4	X_{69}	X_7	X_8
$P_{BC} \cdot P_{CB}$	X_{31}	X_{593}	X_{593}	X_{92}	X_{92}	X_{57}	X_{57}
$P_{BC} \cdot_\flat P_{CB}$	X_{32}	X_{849}	X_{849}	X_4	X_4	X_{56}	X_{56}

Table 1. Examples of trilinear and barycentric products

The line of a bicentric pair is clearly a central line. In particular, the line $P'_{BC}P'_{CB}$ is given by the equation

$$\left(\frac{a^2x^2}{bcyz} - \frac{bcyz}{a^2x^2}\right)\alpha + \left(\frac{b^2y^2}{cazx} - \frac{cazx}{b^2y^2}\right)\beta + \left(\frac{c^2z^2}{abxy} - \frac{abxy}{c^2z^2}\right)\gamma = 0.$$

This is the trilinear polar of the isogonal conjugate of the E-Hirst inverse of F, where $E = ax : by : cz$, and F is the isogonal conjugate of E. In other words, the point whose trilinears are the coefficients for the line $P'_{BC}P'_{CB}$ is the E-Hirst inverse of F.

The line $P_{BC}P_{CB}$ is given by $x'\alpha + y'\beta + z'\gamma = 0$, where

$$x' = bc(by + cz)(a^2x^2 - bcyz),$$

so that $P_{BC}P_{CB}$ is seen to be a certain product of centers if P is a center.

Regarding a euclidean construction for P_{BC}, it is easy to transfer distances for this purpose. Informally, we may describe P_{BC} and P'_{BC} as points constructed "by rotating through 90° the corresponding relative distances of the cevian traces from the vertices A, B, C".

5. The square of a line

Although this section does not involve bicentric pairs directly, the main result will make an appearance in §7, and it may also be of interest *per se*.

Suppose that $U_1 = u_1 : v_1 : w_1$ and $U_2 = u_2 : v_2 : w_2$ are distinct points on an arbitrary line L, represented in general homogeneous coordinates relative to $\triangle ABC$. For each point

$$X := u_1 + u_2t : v_1 + v_2t : w_1 + w_2t,$$

let

$$X^2 := (u_1 + u_2t)^2 : (v_1 + v_2t)^2 : (w_1 + w_2t)^2.$$

The locus of X^2 as t traverses the real number line is a conic section. Following the method in [4], we find an equation for this locus:

$$l^4\alpha^2 + m^4\beta^2 + n^4\gamma^2 - 2m^2n^2\beta\gamma - 2n^2l^2\gamma\alpha - 2l^2m^2\alpha\beta = 0, \qquad (10)$$

where l, m, n are coefficients for the line U_1U_2; that is,

$$l : m : n = v_1w_2 - w_1v_2 : w_1u_2 - u_1w_2 : u_1v_2 - v_1u_2.$$

Equation (10) represents an inscribed ellipse, which we denote by L^2. If the coordinates are trilinears, then the center of the ellipse is the point

$$bn^2 + cm^2 : cl^2 + an^2 : am^2 + bl^2.$$

6. (Line L)$\cap$(Circumconic Γ), two methods

Returning to general homogeneous coordinates, suppose that line L, given by $l\alpha + m\beta + n\gamma = 0$, meets circumconic Γ, given by $u/\alpha + v/\beta + w/\gamma = 0$. Let R and S denote the points of intersection, where $R = S$ if L is tangent to Γ. Substituting $-(m\beta + n\gamma)/l$ for α yields

$$mw\beta^2 + (mv + nw - lu)\beta\gamma + nv\gamma^2 = 0, \tag{11}$$

with discriminant

$$D := l^2u^2 + m^2v^2 + n^2w^2 - 2mnvw - 2nlwu - 2lmuv, \tag{12}$$

so that solutions of (11) are given by

$$\frac{\beta}{\gamma} = \frac{lu - mv - nw \pm \sqrt{D}}{2mw}. \tag{13}$$

Putting β and γ equal to the numerator and denominator, respectively, of the right-hand side (13), putting $\alpha = -(m\beta + n\gamma)/l$, and simplifying give for R and S the representation

$$x_1 : y_1 : z_1 = m(mv - lu - nw \mp \sqrt{D}) : l(lu - mv - nw \pm \sqrt{D}) : 2lmw. \tag{14}$$

Cyclically, we obtain two more representations for R and S:

$$x_2 : y_2 : z_2 = 2mnu : n(nw - mv - lu \mp \sqrt{D}) : m(mv - nw - lu \pm \sqrt{D}) \tag{15}$$

and

$$x_3 : y_3 : z_3 = n(nw - lu - mv \pm \sqrt{D}) : 2nlv : l(lu - nw - mv \mp \sqrt{D}). \tag{16}$$

Multiplying the equal points in (14)-(16) gives R^3 and S^3 as

$$x_1x_2x_3 : y_1y_2y_3 : z_1z_2z_3$$

in cyclic form. The first coordinates in this form are

$$2m^2n^2u(mv - lu - nw \mp \sqrt{D})(nw - lu - mv \pm \sqrt{D}),$$

and these yield

$$\begin{aligned} (\text{1st coordinate of } R^3) &= m^2n^2u[l^2u^2 - (mv - nw - \sqrt{D})^2] && (17)\\ (\text{1st coordinate of } S^3) &= m^2n^2u[l^2u^2 - (mv - nw + \sqrt{D})^2]. && (18) \end{aligned}$$

The 2nd and 3rd coordinates are determined cyclically.

In general, products (as in §2) of points on Γ intercepted by a line are notable: multiplying the first coordinates shown in (17) and (18) gives

$$(\text{1st coordinate of } R^3 \cdot S^3) = l^2m^5n^5u^4vw,$$

so that

$$R \cdot S = mnu : nlv : lmw.$$

Thus, on writing $L = l : m : n$ and $U = u : v : w$, we have $R \cdot S = U/L$.

The above method for finding coordinates of R and S in symmetric form could be called the multiplicative method. There is also an additive method.[1] Adding the coordinates of the points in (14) gives

$$m(mv - lu - nw) : l(lu - mv - nw) : 2lmw.$$

Do the same using (15) and (16), then add coordinates of all three resulting points, obtaining the point $U = u_1 : u_2 : u_3$, where

$$\begin{aligned} u_1 &= (lm + ln - 2mn)u + (m - n)(nw - mv) \\ u_2 &= (mn + ml - 2nl)v + (n - l)(lu - nw) \\ u_3 &= (nl + nm - 2lm)w + (l - m)(mv - lu). \end{aligned}$$

Obviously, the point

$$V = v_1 : v_2 : v_3 = m - n : n - l : l - m$$

also lies on L, so that L is given parametrically by

$$u_1 + tv_1 : u_2 + tv_2 : u_3 + tv_3. \tag{19}$$

Substituting into the equation for Γ gives

$$u(u_2 + tv_2)(u_3 + tv_3) + v(u_3 + tv_3)(u_1 + tv_1) + w(u_1 + tv_1)(u_2 + tv_2) = 0.$$

The expression of the left side factors as

$$(t^2 - D)F = 0, \tag{20}$$

where

$$F = u(n - l)(l - m) + v(l - m)(m - n) + w(m - n)(n - l).$$

Equation (20) indicates two cases:

Case 1: $F = 0$. Here, V lies on both L and Γ, and it is then easy to check that the point

$$W = mnu(n - l)(l - m) : nlv(l - m)(m - n) : lmw(m - n)(n - l)$$

also lies on both.

Case 2: $F \neq 0$. By (20), the points of intersection are

$$u_1 \pm v_1\sqrt{D} : u_2 \pm v_2\sqrt{D} : u_3 \pm v_3\sqrt{D}. \tag{21}$$

As an example to illustrate Case 1, take $u(a, b, c) = (b - c)^2$ and $l(a, b, c) = a$. Then $D = (b - c)^2(c - a)^2(a - b)^2$, and the points of intersection are $b - c : c - a : a - b$ and $(b - c)/a : (c - a)/b : (a - b)/c$.

[1] I thank the Jean-Pierre Ehrmann for describing this method and its application.

7. $L \cap \Gamma$ **when** $D = 0$

The points R and S are identical if and only if $D = 0$. In this case, if in equation (12) we regard either $l : m : n$ or $u : v : w$ as a variable $\alpha : \beta : \gamma$, then the resulting equation is that of a conic inscribed in $\triangle ABC$. In view of equation (10), we may also describe this locus in terms of squares of lines; to wit, if $u : v : w$ is the variable $\alpha : \beta : \gamma$, then the locus is the set of squares of points on the four lines indicated by the equations

$$\sqrt{|l|}\alpha \pm \sqrt{|m|}\beta \pm \sqrt{|n|}\gamma = 0.$$

Taking the coordinates to be trilinears, examples of centers $X_i = l : m : n$ and $X_j = u : v : w$ for which $D = 0$ are given in Table 2. It suffices to show results for $i \leq j$, since L and U are interchangeable in (12).

i	j
1	$244, 678$
2	$1015, 1017$
3	125
6	115
11	$55, 56, 181, 202, 203, 215$
31	$244, 1099, 1109, 1111$
44	44

Table 2. Examples for $D = 0$

8. $L \cap \Gamma$ **when** $D \neq 0$ **and** $l : m : n = u : v : w$

Returning to general homogeneous coordinates, suppose now that $l : m : n$ and $u : v : w$ are triangle centers for which $D \neq 0$. Then, sometimes, R and S are centers, and sometimes, a bicentric pair. We begin with the case $l : m : n = u : v : w$, for which (12) gives

$$D := (u + v + w)(u - v + w)(u + v - w)(u - v - w).$$

This factorization shows that if $u + v + w = 0$, then $D = 0$. We shall prove that converse also holds. Suppose $D = 0$ but $u + v + w \neq 0$. Then one of the other three factors must be 0, and by symmetry, they must all be 0, so that $u = v + w$, so that

$$\begin{aligned} u(a, b, c) &= v(a, b, c) + w(a, b, c) \\ u(a, b, c) &= u(b, c, a) + u(c, a, b) \\ u(b, c, a) &= u(c, a, b) + u(a, b, c). \end{aligned}$$

Applying the third equation to the second gives $u(a, b, c) = u(c, a, b) + u(a, b, c) + u(c, a, b)$, so that $u(a, b, c) = 0$, contrary to the hypothesis that U is a triangle center.

Writing the roots of (11) as r_2/r_3 and s_2/s_3, we find

$$\frac{r_2 s_2}{r_3 s_3} = \frac{(u^2 - v^2 - w^2 + \sqrt{D})(u^2 - v^2 - w^2 - \sqrt{D})}{4v^2 w^2} = 1,$$

which proves that R and S are a conjugate pair (isogonal conjugates in case the coordinates are trilinears). Of particular interest are cases for which these points are polynomial centers, as listed in Table 3, where, for convenience, we put

$$E := (b^2 - c^2)(c^2 - a^2)(a^2 - b^2).$$

u	$\sqrt{D}$	r_1	s_1
$a(b^2 - c^2)$	E	a	bc
$a(b^2 - c^2)(b^2 + c^2 - a^2)$	$16\sigma^2 E$	$\sec A$	$\cos A$
$a(b - c)(b + c - a)$	$4abc(b - c)(c - a)(a - b)$	$\cot(A/2)$	$\tan(A/2)$
$a^2(b^2 - c^2)(b^2 + c^2 - a^2)$	$4a^2b^2c^2E$	$\tan A$	$\cot A$
$bc(a^4 - b^2c^2)$	$(a^4 - b^2c^2)(b^4 - c^2a^2)(c^4 - a^2b^2)$	b/c	c/b

Table 3. Points $R = r_1 : r_2 : r_3$ and $S = s_1 : s_2 : s_3$ of intersection

In Table 3, the penultimate row indicates that for $u : v : w = X_{647}$, the Euler line meets the circumconic $u/\alpha + v/\beta + w/\gamma = 0$ in the points X_4 and X_3. The final row shows that R and S can be a bicentric pair.

9. $L \cap \Gamma$: Starting with Intersection Points

It is easy to check that a point R lies on Γ if and only if there exists a point $x : y : z$ for which

$$R = \frac{u}{by - cz} : \frac{v}{cz - ax} : \frac{w}{ax - by}.$$

From this representation, it follows that every line that meets Γ in distinct points

$$\frac{u}{by_i - cz_i} : \frac{v}{cz_i - ax_i} : \frac{w}{ax_i - by_i},\ i = 1, 2,$$

has the form

$$\frac{(by_1 - cz_1)(by_2 - cz_2)\alpha}{u} + \frac{(cz_1 - ax_1)(cz_2 - ax_2)\beta}{v} + \frac{(ax_1 - by_1)(ax_2 - by_2)\gamma}{w} = 0. \tag{22}$$

and conversely. In this case,

$$D = u^2v^2w^2 \begin{vmatrix} bc & ca & ab \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}^2,$$

indicating that $D = 0$ if and only if the points $x_i : y_i : z_i$ are collinear with the $bc : ca : ab$, which, in case the coordinates are trilinears, is the centroid of $\triangle ABC$.

Example 1. Let

$$x_1 : y_1 : z_1 = c/b : a/c : b/a \quad \text{and} \quad x_2 : y_2 : z_2 = b/c : c/a : a/b.$$

These are the 1st and 2nd Brocard points in case the coordinates are trilinears, but in any case, (22) represents the central line

$$\frac{\alpha}{ua^2(b^2 - c^2)} + \frac{\beta}{vb^2(c^2 - a^2)} + \frac{\gamma}{wc^2(a^2 - b^2)} = 0,$$

meeting Γ in the bicentric pair

$$\frac{u}{b^2(a^2-c^2)}:\frac{v}{c^2(b^2-a^2)}:\frac{w}{a^2(c^2-b^2)},\ \frac{u}{c^2(a^2-b^2)}:\frac{v}{a^2(b^2-c^2)}:\frac{w}{b^2(c^2-a^2)}.$$

Example 2. Let $X = x : y : z$ be a triangle center other than X_1, so that $y : z : x$ and $z : x : y$ are a bicentric pair. The points

$$\frac{u}{bz-cx}:\frac{v}{cx-ay}:\frac{w}{ay-bz},\quad \text{and} \quad \frac{u}{cy-bx}:\frac{v}{az-cy}:\frac{w}{bx-az}$$

are the bicentric pair in which the central line

$$vw(bx-cy)(cx-bz)\alpha+wu(cy-az)(ay-cx)\beta+uv(az-bx)(bz-ay)\gamma=0$$

meets Γ.

10. $L \cap \Gamma$: Euler Line and Circumcircle

Example 3. Using trilinears, the circumcircle is given by $u(a, b, c) = a$ and the Euler line by

$$l(a,b,c)=a(b^2-c^2)(b^2+c^2-a^2).$$

The discriminant $D = 4a^2b^2c^2d^2$, where

$$d=\sqrt{a^6+b^6+c^6+3a^2b^2c^2-b^2c^2(b^2+c^2)-c^2a^2(c^2+a^2)-a^2b^2(a^2+b^2)}.$$

Substitutions into (17) and (18) and simplification give the points of intersection, centers R and S, represented by 1st coordinates

$$\left\{\frac{[ca(a^2-c^2)\pm bd][ba(a^2-b^2)\pm cd]}{(b^2-c^2)^2(b^2+c^2-a^2)^2}\right\}^{1/3}.$$

11. Vertex-products of bicentric triangles

Suppose that $f(a, b, c) : g(b, c, a) : h(c, a, b)$ is a point, as defined in [2] We abbreviate this point as $f_{ab} : g_{bc} : h_{ca}$ and recall from [5, 7] that bicentric triangles are defined by the forms

$$\begin{pmatrix} f_{ab} & g_{bc} & h_{ca} \\ h_{ab} & f_{bc} & g_{ca} \\ g_{ab} & h_{bc} & f_{ca} \end{pmatrix} \text{ and } \begin{pmatrix} f_{ac} & h_{ba} & g_{cb} \\ g_{ac} & f_{ba} & h_{cb} \\ g_{ac} & h_{ba} & f_{cb} \end{pmatrix}.$$

The vertices of the first of these two triangles are the rows of the first matrix, etc. We assume that $f_{ab}g_{ab}h_{ab} \neq 0$. Then the product of the three vertices, namely

$$f_{ab}g_{ab}h_{ab} : f_{bc}g_{bc}h_{bc} : f_{ca}g_{ca}h_{ca} \tag{23}$$

and the product of the vertices of the second triangle, namely

$$f_{ac}g_{ac}h_{ac} : f_{ba}g_{ba}h_{ba} : f_{cb}g_{cb}h_{cb} \tag{24}$$

clearly comprise a bicentric pair if they are distinct, and a triangle center otherwise.

Examples of bicentric pairs thus obtained will now be presented. An inductive method [6] of generating the non-circle-dependent objects of triangle geometry enumerates such objects in sets formally of size six. When the actual size is six, which means that no two of the six objects are identical, the objects form a pair of bicentric triangles. The least such pair for which $f_{ab}g_{ab}h_{ab} \neq 0$ are given by Objects 31-36:

$$\begin{pmatrix} b & c\cos B & -b\cos B \\ -c\cos C & c & a\cos C \\ b\cos A & -a\cos A & a \end{pmatrix} \text{ and } \begin{pmatrix} c & -c\cos C & b\cos C \\ c\cos A & a & -a\cos A \\ -b\cos B & a\cos B & b \end{pmatrix}.$$

In this example, the bicentric pair of points (23) and (24) are

$$\frac{b}{a\cos B} : \frac{c}{b\cos C} : \frac{a}{c\cos A} \quad \text{and} \quad \frac{c}{a\cos C} : \frac{a}{b\cos A} : \frac{b}{c\cos B},$$

and the product of these is the center $\cos A\csc^3 A : \cos B\csc^3 B : \cos C\csc^3 C$.

This example and others obtained successively from Generation 2 of the aforementioned enumeration are presented in Table 4. Column 1 tells the Object numbers in [5]; column 2, the A-vertex of the least Object; column 3, the first coordinate of point (23) after canceling a symmetric function of (a, b, c); and column 4, the first coordinate of the product of points (23) and (24) after canceling a symmetric function of (a, b, c). In Table 4, $\cos A, \cos B, \cos C$ are abbreviated as a_1, b_1, c_1, respectively.

Objects	$f_{ab} : g_{ab} : h_{ab}$	$[f_{ab}g_{ab}h_{ab}]$	$[f_{ab}f_{ac}g_{ab}g_{ac}h_{ab}h_{ac}]$
31-36	$b : cb_1 : -bb_1$	b/ab_1	a_1/a^3
37-42	$bc_1 : -ca_1 : ba_1$	bc_1/aa_1	$(aa_1)^{-3}$
43-48	$bb_1 : c : -b$	bb_1/a	$(a_1a^3)^{-1}$
49-54	$ab : -c^2 : bc$	b/c	1
58-63	$c + ba_1 : cc_1 : -bc_1$	$(ba_1 + c)/ac_1$	$a_1(ba_1 + c)(ca_1 + b)a^{-2}$
71-76	$-b_1^2 : c_1 : b_1$	b_1^2/a_1	a_1^{-4}
86-91	$c_1 - a_1b_1 : c_1^2 : b_1c_1$	$b_1(c_1 - a_1b_1)$	$[a_1(a_1 - b_1c_1)]^{-1}$
92-97	$a_1b_1 : 1 : -a_1$	b_1/c_1	1
98-103	$1 : -c_1 : c_1a_1$	b_1/c_1	1
104-109	$aa_1 : -c : ca_1$	a/cc_1	a^3a_1
110-115	$a : b : -ba_1$	ab_1/b	a^3/a_1
116-121	$c_1 - a_1b_1 : 1 : -a_1$	$b_1(c_1 - a_1b_1)$	$[a_1(a_1 - b_1c_1)]^{-1}$
122-127	$1 + a_1^2 : c_1 : -c_1a_1$	$b_1(1 + a_1^2)/c_1$	$(1 + a_1^2)^2$
128-133	$2a_1 : -b_1 : a_1b_1$	a_1	a_1^2

Table 4. Bicentric triangles, bicentric points, and central vertex-products

Table 4 includes examples of interest: (i) bicentric triangles for which (23) and (24) are identical and therefore represent a center; (ii) distinct pairs of bicentric triangles that yield the identical bicentric pairs of points; and (iii) cases in which the pair (23) and (24) are isogonal conjugates. Note that Objects 49-54 yield for (23) and (24) the 2nd Brocard point, $\Omega_2 = b/c : c/a : a/b$ and the 1st Brocard point, $\Omega_1 = c/b : a/c : b/a$.

12. Geometric discussion: $\oplus$ and $\ominus$

Equations (3) and (4) define operations $\oplus$ and $\ominus$ on pairs of bicentric points. Here, we shall consider the geometric meaning of these operations. First, note that one of the points in (2) lies on $\mathcal{L}^\infty$ if and only if the other lies on $\mathcal{L}^\infty$, since the transformation $(a, b, c) \to (a, c, b)$ carries each of the equations

$$af_{ab} + bf_{bc} + cf_{ca} = 0, \quad af_{ac} + bf_{ba} + cf_{cb} = 0$$

to the other. Accordingly, the discussion breaks into two cases.

Case 1: F_{ab} not on $\mathcal{L}^\infty$. Let k_{ab} and k_{ac} be the normalization factors given in §3. Then the actual directed trilinear distances of F_{ab} and F_{ac} (to the sidelines BC, CA, AB) are given by (5). The point F that separates the segment $F_{ab}F_{ac}$ into segments satisfying

$$\frac{|F_{ab}F|}{|FF_{ac}|} = \frac{k_{ab}}{k_{ac}},$$

where | | denotes directed length, is then

$$\frac{k_{ac}}{k_{ab} + k_{ac}} F'_{ab} + \frac{k_{ab}}{k_{ab} + k_{ac}} F'_{ac} = \frac{k_{ac}k_{ab}}{k_{ab} + k_{ac}} F_{ab} + \frac{k_{ab}k_{ac}}{k_{ab} + k_{ac}} F_{ac},$$

which, by homogeneity, equals $F_{ab} \oplus F_{ac}$. Similarly, the point "constructed" as

$$\frac{k_{ac}}{k_{ab} + k_{ac}} F'_{ab} - \frac{k_{ab}}{k_{ab} + k_{ac}} F'_{ac}$$

equals $F_{ab} \ominus F_{ac}$. These representations show that $F_{ab} \oplus F_{ac}$ and $F_{ab} \ominus F_{ac}$ are a harmonic conjugate pair with respect to F_{ab} and F_{ac}.

Case 2: F_{ab} on $\mathcal{L}^\infty$. In this case, the isogonal conjugates F_{ab}^{-1} and F_{ac}^{-1} lie on the circumcircle, so that Case 1 applies:

$$F_{ab}^{-1} \oplus F_{ac}^{-1} = \frac{f_{ab} + f_{ac}}{f_{ab}f_{ac}} : \frac{f_{bc} + f_{ba}}{f_{bc}f_{ba}} : \frac{f_{ca} + f_{cb}}{f_{ca}f_{cb}}.$$

Trilinear multiplication [6] by the center $F_{ab} \cdot F_{ac}$ gives

$$F_{ab} \oplus F_{ac} = (F_{ab}^{-1} \oplus F_{ac}^{-1}) \cdot F_{ab} \cdot F_{ac}.$$

In like manner, $F_{ab} \ominus F_{ac}$ is "constructed".

It is easy to prove that a pair P_{ab} and P_{ac} of bicentric points on $\mathcal{L}^\infty$ are necessarily given by

$$P_{ab} = bf_{ca} - cf_{bc} : cf_{ab} - af_{ca} : af_{bc} - bf_{ab}$$

for some bicentric pair as in (2). Consequently,

$$\begin{aligned} P_{ab} \oplus P_{ac} &= g(a, b, c) : g(b, c, a) : g(c, a, b), \\ P_{ab} \ominus P_{ac} &= h(a, b, c) : h(b, c, a) : h(c, a, b), \end{aligned}$$

where

$$\begin{aligned} g(a, b, c) &= b(f_{ca} + f_{cb}) - c(f_{bc} + f_{ba}), \\ h(a, b, c) &= b(f_{ca} - f_{cb}) + c(f_{ba} - f_{bc}). \end{aligned}$$

Example 4. We start with $f_{ab} = c/b$, so that F_{ab} and F_{ac} are the Brocard points, and P_{ab} and P_{ac} are given by 1st coordinates $a - c^2/a$ and $a - b^2/a$, respectively, yielding 1st coordinates $(2a^2 - b^2 - c^2)/a$ and $(b^2 - c^2)/a$ for $P_{ab} \oplus P_{ac}$ and $P_{ab} \ominus P_{ac}$. These points are the isogonal conjugates of X_{111} (the Parry point) and X_{110} (focus of the Kiepert parabola), respectively.

References

[1] J. P. Hogendijk, Al-Mu'taman ibn Hūd, 11th century king of Saragossa and brilliant mathematician, *Historia Mathematica*, 22 (1995) 1–18.
[2] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1–295.
[3] C. Kimberling, *Encyclopedia of Triangle Centers*, August 22, 2002 edition, available at http://www2.evansville.edu/ck6/encyclopedia.
[4] C. Kimberling, Conics associated with cevian nests, *Forum Geom.*, 1 (2001) 141–150.
[5] C. Kimberling, Enumerative triangle geometry, part 1: the primary system, S, *Rocky Mountain Journal of Mathematics*, 32 (2002) 201–225.
[6] C. Kimberling and C. Parry, Products, square roots, and layers in triangle geometry, *Mathematics and Informatics Quarterly*, 10 (2000) 9–22.
[7] F. M. van Lamoen, Bicentric triangles, *Nieuw Archief voor Wiskunde*, (4) 17 (1999) 363–372.
[8] P. Yiu, The uses of homogeneous barycentric coordinates in plane euclidean geometry, *Int. J. Math. Educ. Sci. Technol.*, 31 (2000) 569 – 578.

Clark Kimberling: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, USA
E-mail address: ck6@evansville.edu

Forum Geometricorum
Volume 3 (2003) 49–56.

FORUM GEOM
ISSN 1534-1178

Some Configurations of Triangle Centers

Lawrence S. Evans

Abstract. Many collections of triangle centers and symmetrically defined triangle points are vertices of configurations. This illustrates a high level of organization among the points and their collinearities. Some of the configurations illustrated are inscriptable in Neuberg's cubic curve and others arise from Monge's theorem.

1. Introduction

By a configuration $\mathcal{K}$ we shall mean a collection of p points and g lines with r points on each line and q lines meeting at each point. This implies the relationship $pq = gr$. We then say that $\mathcal{K}$ is a (p_q, g_r) configuration. The simplest configuration is a point with a line through it. Another example is the triangle configuration, $(3_2, 3_2)$ with $p = g = 3$ and $q = r = 2$. When $p = g$, $\mathcal{K}$ is called *self-dual*, and then we must also have $q = r$. The symbol for the configuration is now simplified to read (p_q). The smallest (n_3) self-dual configurations exist combinatorially, when the "lines" are considered as suitable triples of points (vertices), but they cannot be realized with lines in the Euclidean plane. Usually when configurations are presented graphically, the lines appear as segments to make the figure compact and easy to interpret. Only one (7_3) configuration exists, the Fano plane of projective geometry, and only one (8_3) configuration exists, the Möbius-Kantor configuration. Neither of these can be realized with straight line segments. For larger n, the symbol may not determine a configuration uniquely. The smallest (n_3) configurations consisting of line segments in the Euclidean plane are (9_3), and there are three of them, one of which is the familiar Pappus configuration [4, pp.94–170]. The number of distinct (n_3) configurations grows rapidly with n. For example, there are 228 different (12_3) configurations [11, p.40]. In the discussion here, we shall only be concerned with configurations lying in a plane.

While configurations have long been studied as combinatorial objects, it does not appear that in any examples the vertices have been identified with triangle-derived points. In recent years there has been a resurgence of interest in triangle geometry along with the recognition of many new special points defined in different very ways. Since each point is defined from original principles, it is somewhat surprising that so many of them are collinear in small sets. An even higher level of relationship among special points is seen when they can be incorporated into

Publication Date: February 24, 2003. Communicating Editor: Steve Sigur.

certain configurations of moderate size. Then the collinearities and their incidences are summarized in a tidy, symmetrical, and graphic way. Here we exhibit several configurations whose vertices are naturally defined by triangles and whose lines are collinearities among them. It happens that the general theory for the first three examples was worked out long ago, but then the configurations were not identified as consisting of familiar triangle points and their collinearities.

2. Some configurations inscriptable in a cubic

First let us set the notation for several triangles. Given a triangle $\mathbf{T}$ with vertices A, B, and C, let A^* be the reflection of vertex A in side BC, A_+ the apex of an equilateral triangle erected outward on BC, and A_- the apex of an equilateral triangle erected inward on BC. Similarly define the corresponding points for B and C. Denote the triangle with vertices A^*, B^*, C^* as $\mathbf{T}^*$ and similarly define the triangles $\mathbf{T}_+$ and $\mathbf{T}_-$. Using trilinear coordinates it is straightforward to verify that the four triangles above are pairwise in perspective to one another. The points of perspective are as follows.

	$\mathbf{T}$	$\mathbf{T}^*$	$\mathbf{T}_+$	$\mathbf{T}_-$
$\mathbf{T}$		H	F_+	F_-
$\mathbf{T}^*$	H		J_-	J_+
$\mathbf{T}_+$	F_+	J_-		O
$\mathbf{T}_-$	F_-	J_+	O	

Here, O and H are respectively the circumcenter and orthocenter, $F_\pm$ the isogonic (Fermat) points, and $J_\pm$ the isodynamic points. They are triangle centers as defined by Kimberling [5, 6, 7, 8], who gives their trilinear coordinates and discusses their geometric significance. See also the in §5. For a simple simultaneous construction of all these points, see Evans [2].

To assemble the configurations, we first need to identify certain sets of collinear points. Now it is advantageous to introduce a notation for collinearity. Write $\mathcal{L}(X, Y, Z, \dots)$ to denote the line containing X, Y, Z, ... The key to identifying configurations among all the previously mentioned points depends on the observation that A^* , B_+, and C_- are always collinear, so we may write $\mathcal{L}(A^*, B_+, C_-)$. One can easily verify this using trilinear coordinates. This is also true for any permutation of A, B, and C, so we have

(I): the 6 lines $\mathcal{L}(A^*, B_+, C_-)$, $\mathcal{L}(A^*, B_-, C_+)$, $\mathcal{L}(B^*, C_+, A_-)$, $\mathcal{L}(B^*, C_-, A_+)$, $\mathcal{L}(C^*, A_+, B_-)$, $\mathcal{L}(C^*, A_-, B_+)$.

They all occur in Figures 1, 2, and 3. In fact the nine points A_+, A_-, A^*, ... themselves form the vertices of a $(9_2, 6_3)$ configuration.

It is easy to see other collinearities, namely 3 from each pair of triangles in perspective. For example, triangles $\mathbf{T}_+$ and $\mathbf{T}_-$ are in perspective from O, so we have

(II): the 3 lines $\mathcal{L}(A_+, O, A_-)$, $\mathcal{L}(B_+, O, B_-)$ and $\mathcal{L}(C_+, O, C_-)$.

See Figure 2.

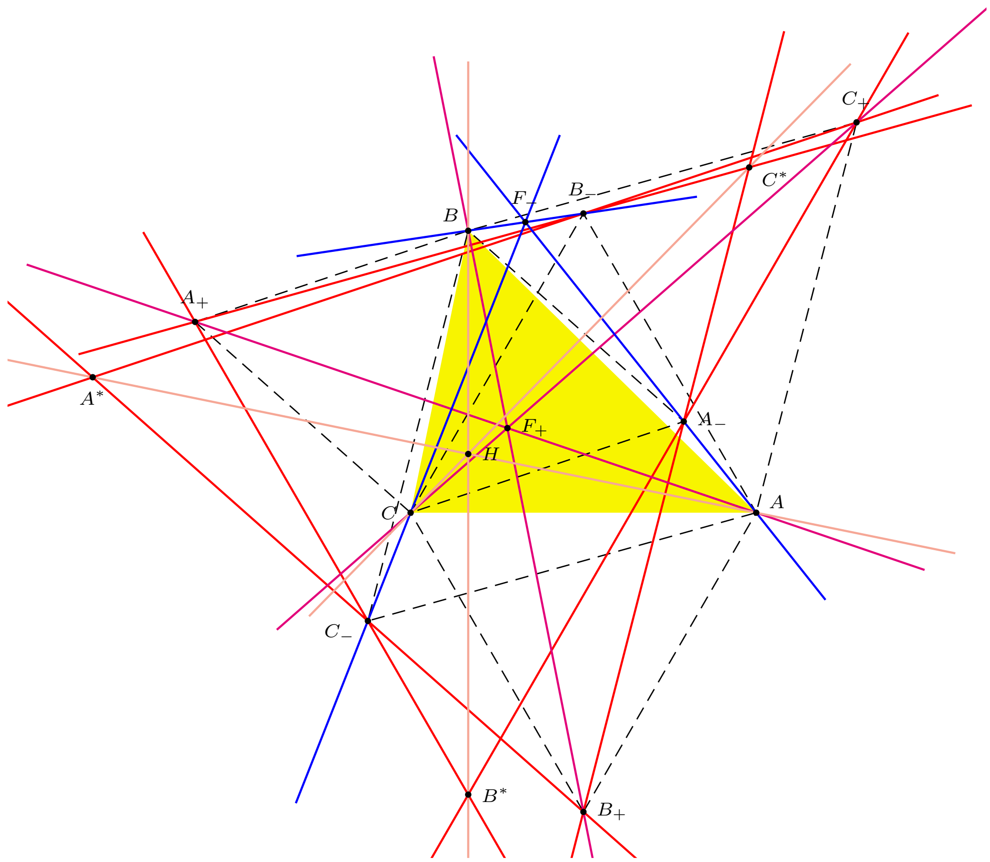


Figure 1. The Cremona-Richmond configuration

2.1. *The Cremona-Richmond configuration* (15_3). Consider the following sets of collinearities of three points:

- **(III):** the 3 lines $\mathcal{L}(A, F_+, A_+)$, $\mathcal{L}(B, F_+, B_+)$ and $\mathcal{L}(C, F_+, C_+)$;
- **(IV):** the 3 lines $\mathcal{L}(A, F_-, A_-)$, $\mathcal{L}(B, F_-, B_-)$ and $\mathcal{L}(C, F_-, C_-)$;
- **(V):** the 3 lines $\mathcal{L}(A, H, A^*)$, $\mathcal{L}(B, H, B^*)$ and $\mathcal{L}(C, H, C^*)$.

The 15 points (A, B, C, A^*, B^*, C^*, $A_\pm$, $B_\pm$, $C_\pm$, H, $F_\pm$) and 15 lines in **(I), (III), (IV)**, and **(V)** form a figure which is called the Cremona-Richmond configuration [7]. See Figure 1. It has 3 lines meeting at each point with 3 points on each line, so it is self-dual with symbol (15_3). Inspection reveals that this configuration itself contains no triangles.

The reader may have noticed that the fifteen points in the configuration all lie on Neuberg's cubic curve, which is known to contain many triangle centers [7]. Recently a few papers, such as Pinkernell's [10] discussing Neuberg's cubic have

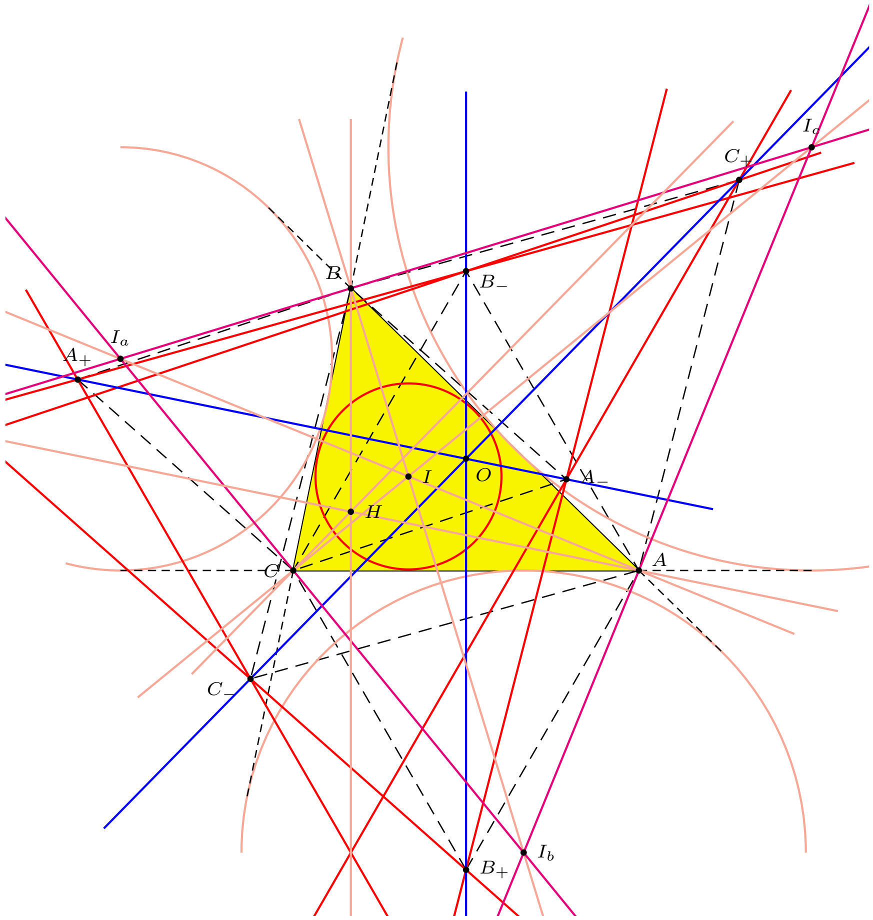


Figure 2

appeared, so we shall not elaborate on the curve itself. It has been known for a long time that many configurations are inscriptable in cubic curves, possibly first noticed by Schoenflies circa 1888 according to Feld [3]. However, it does not appear to be well-known that Neuberg's cubic in particular supports such configurations of familiar points. We shall exhibit two more configurations inscriptable in Neuberg's cubic.

2.2. *A* (18_3) *associated with the excentral triangle.* For another configuration, this one of the type (18_3), we employ the excentral triangle, that is, the triangle whose vertices are the excenters of $\mathbf{T}$. Denote the excenter opposite vertex A by I_a, etc., and denote the extriangle as $\mathbf{T}_\mathrm{x}$. Triangles $\mathbf{T}$ and $\mathbf{T}_\mathrm{x}$ are in perspective from the incenter, I. This introduces two more sets of collinearities involving the excenters:

(VI): the 3 lines $\mathcal{L}(A, I, I_a)$, $\mathcal{L}(B, I, I_b)$ and $\mathcal{L}(C, I, I_c)$;
(VII): the 3 lines $\mathcal{L}(I_b, A, I_c)$, $\mathcal{L}(I_c, B, I_a)$ and $\mathcal{L}(I_a, C, I_b)$.

The 18 lines of **(I), (II), (V), (VI), (VII)** and the 18 points A, B, C, I_a, I_b, I_c, A^*, B^*, C^*, $A_\pm$, $B_\pm$, $C_\pm$, O, H, and I form an (18_3) configuration. See Figure 2. There are enough points to suggest the outline of Neuberg's cubic, which is bipartite. The 10 points in the lower right portion of the figure lie on the ovoid portion of the curve. The 8 other points lie on the serpentine portion, which has an asymptote parallel to Euler's line (dashed). For other shapes of the basic triangle T, these points will not necessarily lie on the same components of the curve.

2.3. *A configuration* $(12_4, 16_3)$. Now we define two more sets of collinearities involving the isodynamic points:

(VIII): the 3 lines $\mathcal{L}(A^*, J_-, A_+)$, $\mathcal{L}(B^*, J_-, B_+)$ and $\mathcal{L}(C^*, J_-, C_+)$;
(IX): the 3 lines $\mathcal{L}(A^*, J_+, A_-)$, $\mathcal{L}(B^*, J_+, B_-)$ and $\mathcal{L}(C^*, J_+, C_-)$.

Among the centers of perspective we have defined so far, there is an additional collinearity, $\mathcal{L}(J_+, O, J_-)$, which is the Brocard axis. See Figure 3.

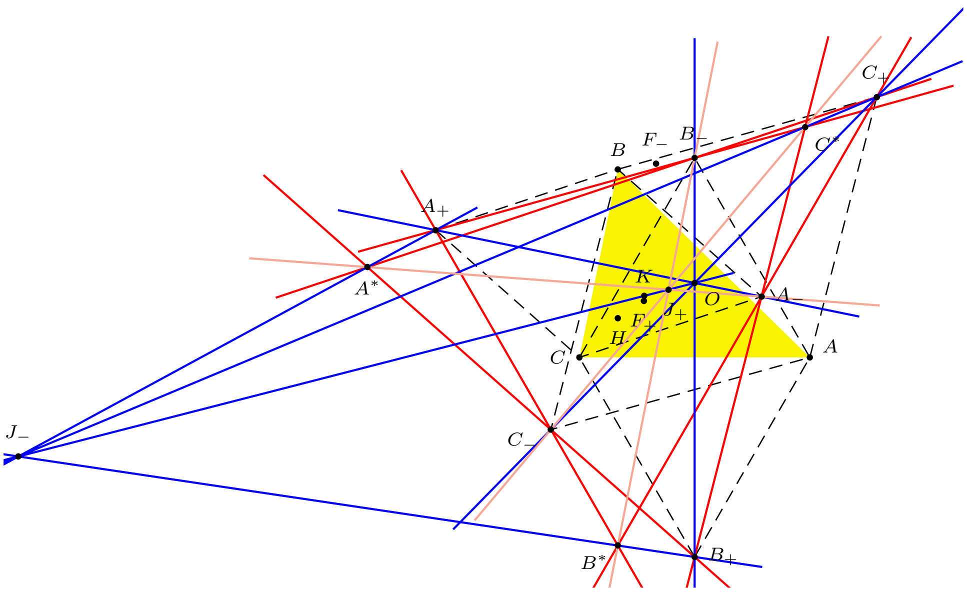


Figure 3

Using Weierstrass elliptic functions, Feld proved that within any bipartite cubic, a real configuration can be inscribed which has 12 points and 16 lines, with 4 lines meeting at each point and 3 points on each line [11], so that is, its symbol is $(12_4, 16_3)$. Now the Neuberg cubic of a non-equilateral triangle is bipartite, consisting of an ovoid portion and a serpentine portion whose asymptote is parallel to the Euler line of the triangle. Here one such inscriptable configuration consists of the following sets of lines: **(I), (II), (VIII), (IX)**, and the line, $\mathcal{L}(J_+, O, J_-)$. See Figure 3. The three triangles $\mathbf{T}_+$, $\mathbf{T}_-$, and $\mathbf{T}^*$ are pair-wise in perspective with collinear perspectors J_+, J_-, and O. The vertices of the basic triangle $\mathbf{T}$ are not in this configuration.

3. A Desargues configuration with triangle centers as vertices

There are so many collinearities involving triangle centers that we can also exhibit a Desargues (10_3) configuration with vertices consisting entirely of basic centers. Let K denote the symmedian (Lemoine's) point, N_p the center of the nine-point circle, G the centroid, N_+ the first Napoleon point, and N_- the second Napoleon point. Then the ten points F_+, F_-, J_+, J_-, N_+, N_-, K, G, H and N_p form the vertices of such a configuration. This is seen on noting that the triangles $F_-J_+N_+$ and $F_+J_-N_-$ are in perspective from K with the line of perspective $\mathcal{L}(G, N_p, H)$, which is Euler's line. See Figure 4. In a Desargues configuration any vertex may be chosen as the center of perspective of two suitable triangles. For simplicity we have chosen $\mathcal{K}$ in this example. Unlike the previous examples, Desargues configurations are not inscriptable in cubic curves [9].

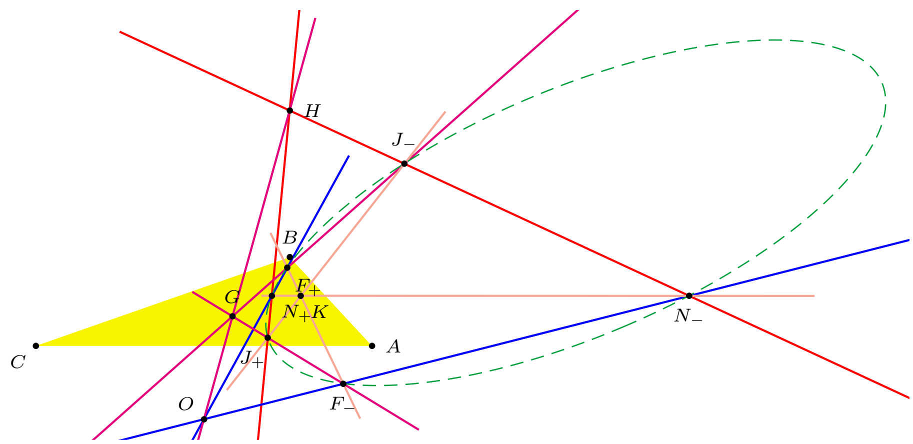


Figure 4

4. Configurations from Monge's theorem

Another way triangle centers form vertices of configurations arises from Monge's theorem [4, 11]. This theorem states that if we have three circles, then the 3 external centers of simitude (ecs) are collinear and that each external center of simitude is collinear with two of the internal centers of simitude (ics). These 4 collinearities form a $(4_3, 6_2)$ configuration, *i.e.*, a complete quadrilateral with the centers of similitude as vertices. This is best illustrated by an example. Suppose we have the circumcircle, the nine-point circle, and the incircle of a triangle. The ics of the circumcircle and the nine-point circle is the centroid, G, and their ecs is the orthocenter, H. The ics of the nine-point circle and the incircle is X_{12} in Kimberling's list and the ecs is Feuerbach's point, X_{11}. The ics of the circumcircle and the incircle is X_{55}, and the ecs is X_{56}. The lines of the configuration are then $\mathcal{L}(H, X_{56}, X_{11})$, $\mathcal{L}(G, X_{55}, X_{11})$, $\mathcal{L}(G, X_{56}, X_{12})$, and $\mathcal{L}(H, X_{55}, X_{12})$. This construction, of course, applies to any group of three circles related to the triangle. In the example given, the circles can be nested, so it may not be easy to see the centers of similitude. In such a case, the radii of the circles can be reduced in the same proportion to make the circles small enough that they do not overlap. The ecs's and ics's remain the same. The ecs of two such circles is the point where the two common external tangents meet, and the ics is the point where the two common internal tangents meet. When two of the circles have the same radii, their ics is the midpoint of the line joining their centers and their ecs is the point at infinity in the direction of the line joining their centers.

One may ask what happens when a fourth circle whose center is not collinear with any other two is also considered. Monge's theorem applies to each group of three circles. First it happens that the four lines containing only ecs's themselves form a $(6_2, 4_3)$ configuration. Second, when the twelve lines containing an ecs and two ics's are annexed, the result is a $(12_4, 16_3)$ configuration. This is a projection onto the plane of Reye's three-dimensional configuration, which arises from a three-dimensional analog of Monge's theorem for four spheres [4]. This is illustrated in Figure 5 with the vertices labelled with the points of Figure 3, which shows that these two $(12_4, 16_3)$ configurations are actually the same even though the representation in Figure 5 may not be inscriptable in a bipartite cubic. Evidently larger configurations arise by the same process when yet more circles are considered.

5. Final remarks

We have see that certain collections of collinear triangle points can be knitted together into highly symmetrical structures called configurations. Furthermore some relatively large configuratons such as the (18_3) shown above are inscriptable in low degree algebraic curves, in this case a cubic.

General information about configurations can be found in Hilbert and Cohn-Vossen [4]. Also we recommend Coxeter [1], which contains an extensive bibliography of related material pre-dating 1950.

The centers here appear in Kimberling [5, 6, 7, 8] as X_n for n below.

center	I	G	O	H	N_p	K	F_+	F_-	J_+	J_-	N_+	N_-
n	1	2	3	4	5	6	13	14	15	16	17	18

While not known by eponyms, X_{12}, X_{55}, and X_{56} are also geometrically significant in elementary ways [7, 8].

References

[1] H. S. M. Coxeter, Self-dual configurations and regular graphs, *Bull. Amer. Math. Soc.*, 56 (1950) 413–455; reprinted in *The Beauty of Geometry: Twelve Essays*, Dover, Mineola, New York, 1999, which is a reprint of *Twelve Geometric Essays*, Southern Illinois University Press, Carbondale, 1968.

[2] L. S. Evans, A rapid construction of some triangle centers, *Forum Geom.*, 2 (2002) 67–70.

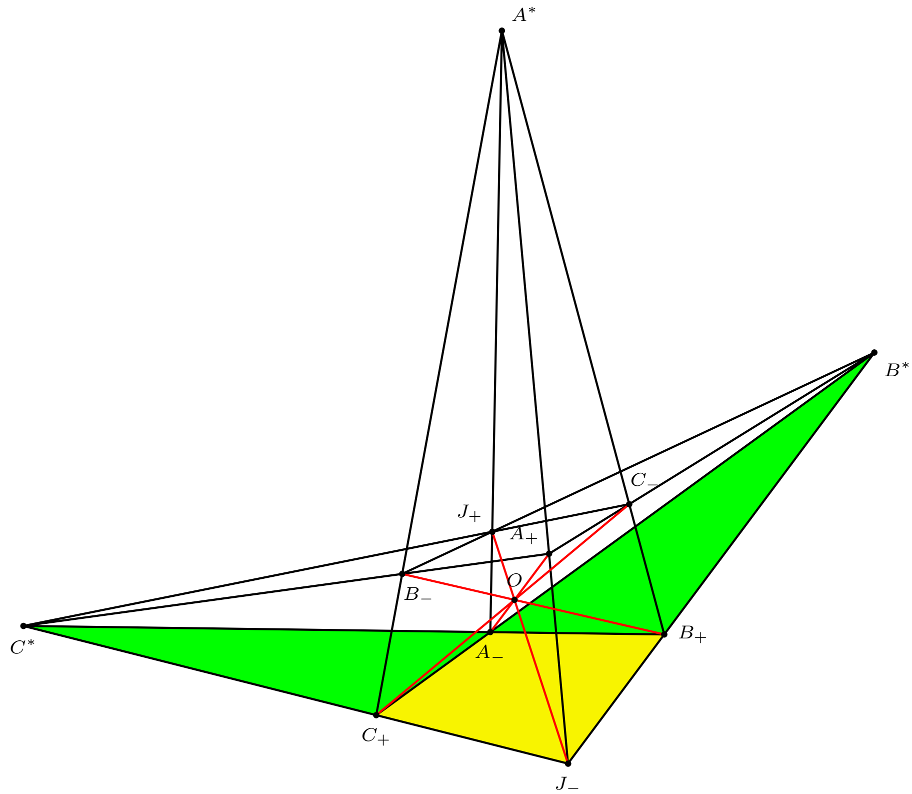


Figure 5

[3] J. M. Feld, Configurations inscriptable in a plane cubic curve, *Amer. Math. Monthly*, 43 (1936) 549–555.

[4] D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination*, 2nd. ed., Chelsea (1990), New York.

[5] C. Kimberling, Central points and central lines in the plane of a triangle, *Math. Magazine*, 67 (1994) 163–187.

[6] C. Kimberling, Major centers of triangles, *Amer. Math. Monthly*, 104 (1997) 431–488.

[7] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

[8] C. Kimberling, *Encyclopedia of Triangle Centers*, August 22, 2002 edition, available at http://www2.evansville.edu/ck6/encyclopedia/; February 17, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[9] N. S. Mendelsohn, R. Padmanabhan and B. Wolk, Placement of the Desargues configuration on a cubic curve, *Geom. Dedicata*, 40 (1991) 165–170.

[10] G. Pinkernell, Cubic curves in the triangle plane, *Journal of Geometry*, 55 (1996) 141–161.

[11] D. Wells, *The Penguin Dictionary of Curious and Interesting Geometry*, (1991), Penguin, Middlesex.

Lawrence S. Evans: 910 W. 57th Street, La Grange, Illinois 60525, USA
E-mail address: `75342.3052@compuserve.com`

Forum Geometricorum
Volume 3 (2003) 57–63.

FORUM GEOM
ISSN 1534-1178

On the Circumcenters of Cevasix Configurations

Alexei Myakishev and Peter Y. Woo

Abstract. We strengthen Floor van Lamoen's theorem that the 6 circumcenters of the cevasix configuration of the centroid of a triangle are concyclic by giving a proof which at the same time shows that the converse is also true with a minor qualification, *i.e.*, the circumcenters of the cevasix configuration of a point P are concyclic if and only if P is the centroid or the orthocenter of the triangle.

1. Introduction

Let P be a point in the plane of triangle ABC, with traces A', B', C' on the sidelines BC, CA, AB respectively. We assume that P does not lie on any of the sidelines. Triangle ABC is then divided by its cevians AA', BB', CC' into six triangles, giving rise to what Clark Kimberling [2, pp.257–260] called the *cevasix configuration* of P. See Figure 1. Floor van Lamoen has discovered that when P is the centroid of triangle ABC, the 6 circumcenters of the cevasix configuration are concyclic. See Figure 2. This was posed as a problem in the *American Mathematical Monthly* [3]. Solutions can be found in [3, 4]. In this note we study the converse.

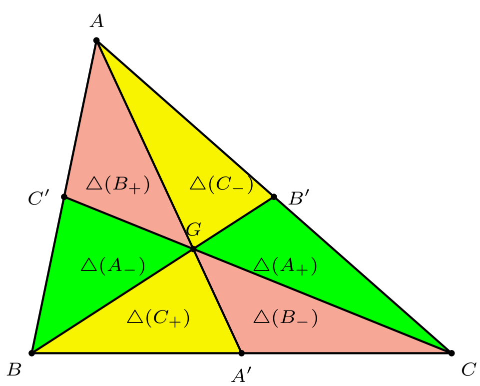


Figure 1

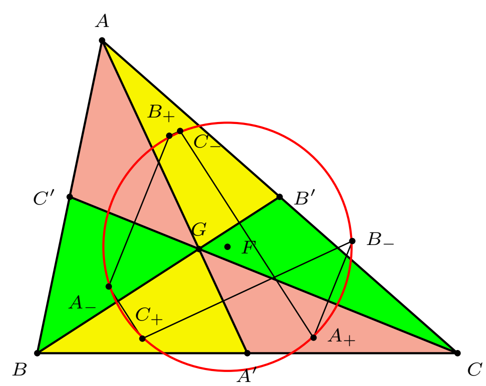


Figure 2

Theorem 1. *The circumcenters of the cevasix configuration of P are concyclic if and only if P is the centroid or the orthocenter of triangle ABC.*

Publication Date: March 3, 2003. Communicating Editor: Paul Yiu.

2. Preliminary results

We adopt the following notations.

Triangle	PCB'	$PC'B$	PAC'	$PA'C$	PBA'	$PB'A$
Notation	$\triangle(A_+)$	$\triangle(A_-)$	$\triangle(B_+)$	$\triangle(B_-)$	$\triangle(C_+)$	$\triangle(C_-)$
Circumcenter	A_+	A_-	B_+	B_-	C_+	C_-

It is easy to see that two of these triangle may possibly share a common circumcenter only when they share a common vertex of triangle ABC.

Lemma 2. *The circumcenters of triangles APB' and APC' coincide if and only if P lies on the reflection of the circumcircle in the line BC.*

Proof. Triangles APB' and APC' have the same circumcenter if and only if the four points A, B', P, C' are concyclic. In terms of directed angles, $\angle BPC = \angle B'PC' = \angle B'AC' = \angle CAB = -\angle BAC$. See, for example, [1, §§16–20]. It follows that the reflection of A in the line BC lies on the circumcircle of triangle PBC, and P lies on the reflection of the circumcircle in BC. The converse is clear. $\square$

Thus, if $B_+ = C_-$ and $C_+ = A_-$, then necessarily P is the orthocenter H, and also $A_+ = B_-$. In this case, there are only three distinct circumcenters. They clearly lie on the nine-point circle of triangle ABC. We shall therefore assume $P \neq H$, so that there are at least five distinct points in the set $\{A_\pm, B_\pm, C_\pm\}$.

The next proposition appears in [2, p.259].

Proposition 3. *The 6 circumcenters of the cevasix configuration of P lie on a conic.*

Proof. We need only consider the case when these 6 circumcenters are all distinct. The circumcenters B_+ and C_- lie on the perpendicular bisector of the segment AP; similarly, B_- and C_+ lie on the perpendicular bisector of PA'. These two perpendicular bisectors are clearly parallel. This means that B_+C_- and B_-C_+ are parallel. Similarly, $C_+A_-//C_-A_+$ and $A_+B_-//A_-B_+$. The hexagon $A_+C_-B_+A_-C_+B_-$ has three pairs of parallel opposite sides. By the converse of Pascal's theorem, there is a conic passing through the six vertices of the hexagon. $\square$

Proposition 4. *The vertices of a hexagon $A_+C_-B_+A_-C_+B_-$ with parallel opposite sides $B_+C_-//C_+B_-$, $C_+A_-//A_+C_-$, $A_+B_-//B_+A_-$ lie on a circle if and only if the main diagonals A_+A_-, B_+B_- and C_+C_- have equal lengths.*

Proof. If the vertices are concyclic, then $A_+C_-A_-C_+$ is an isosceles trapezoid, and $A_+A_- = C_+C_-$. Similarly, $C_+B_-C_-B_+$ is also an isosceles trapezoid, and $C_+C_- = B_+B_-$.

Conversely, consider the triangle XYZ bounded by the three diagonals A_+A_-, B_+B_- and C_+C_-. If these diagonals are equal in length, then the trapezoids $A_+C_-A_-C_+$, $C_+B_-C_-B_+$ and $B_+A_-B_-A_+$ are isosceles. From these we immediately conclude that the common perpendicular bisector of A_+C_- and A_-C_+

is the bisector of angle XYZ. Similarly, the common perpendicular bisector of B_+C_- and B_-C_+ is the bisector of angle X, and that of A_+B_- and A_-B_+ the bisector of angle Z. These three perpendicular bisectors clearly intersect at a point, the incenter of triangle XYZ, which is equidistant from the six vertices of the hexagon. □

Proposition 5. *The vector sum* $\mathbf{AA'} + \mathbf{BB'} + \mathbf{CC'} = \mathbf{0}$ *if and only if* P *is the centroid.*

Proof. Suppose with reference to triangle ABC, the point P has *absolute* barycentric coordinates $uA + vB + wC$, where $u + v + w = 1$. Then,

$$A' = \frac{1}{v+w}(vB + wC), \quad B' = \frac{1}{w+u}(wC + uA), \quad C' = \frac{1}{u+v}(uA + vB).$$

From these,

$$\begin{aligned} &\mathbf{AA'} + \mathbf{BB'} + \mathbf{CC'} \\ =&(A' + B' + C') - (A + B + C) \\ =&\frac{u^2 - vw}{(w+u)(u+v)} \cdot A + \frac{v^2 - wu}{(u+v)(v+w)} \cdot B + \frac{w^2 - uv}{(v+w)(w+u)} \cdot C. \end{aligned}$$

This is zero if and only if

$$u^2 - vw = v^2 - wu = w^2 - uv = 0,$$

and $u = v = w = \frac{1}{3}$ since they are all real, and $u + v + w = 1$. □

We denote by π_a, π_b, π_c the orthogonal projections on the lines AA', BB', CC' respectively.

Proposition 6.

$$\begin{aligned} \pi_b(\mathbf{A_+A_-}) &= -\tfrac{1}{2}\mathbf{BB'}, & \pi_c(\mathbf{A_+A_-}) &= \tfrac{1}{2}\mathbf{CC'}, \\ \pi_c(\mathbf{B_+B_-}) &= -\tfrac{1}{2}\mathbf{CC'}, & \pi_a(\mathbf{B_+B_-}) &= \tfrac{1}{2}\mathbf{AA'}, \\ \pi_a(\mathbf{C_+C_-}) &= -\tfrac{1}{2}\mathbf{AA'}, & \pi_b(\mathbf{C_+C_-}) &= \tfrac{1}{2}\mathbf{BB'}. \end{aligned} \tag{1}$$

Proof. The orthogonal projections of A_+ and A_- on the cevian BB' are respectively the midpoints of the segments PB' and BP. Therefore,

$$\pi_b(\mathbf{A_+A_-}) = \frac{B+P}{2} - \frac{P+B'}{2} = -\frac{B'-B}{2} = -\frac{1}{2}\mathbf{BB'}.$$

The others follow similarly. □

3. Proof of Theorem 1

Sufficiency part. Let P be the centroid G of triangle ABC. By Proposition 4, it is enough to prove that the diagonals A_+A_-, B_+B_- and C_+C_- have equal lengths. By Proposition 5, we can construct a triangle $A^*B^*C^*$ whose sides as vectors $\mathbf{B^*C^*}$, $\mathbf{C^*A^*}$ and $\mathbf{A^*B^*}$ are equal to the medians $\mathbf{AA'}$, $\mathbf{BB'}$, $\mathbf{CC'}$ respectively.

Consider the vector $\mathbf{A}^*\mathbf{Q}$ equal to $\mathbf{A}_+\mathbf{A}_-$. By Proposition 6, the orthogonal projections of $\mathbf{A}_+\mathbf{A}_-$ on the two sides C^*A^* and A^*B^* are the midpoints of the sides. This means that Q is the circumcenter of triangle $A^*B^*C^*$, and the length of $\mathbf{A}_+\mathbf{A}_-$ is equal to the circumradius of triangle $A^*B^*C^*$. The same is true for the lengths of $\mathbf{B}_+\mathbf{B}_-$ and $\mathbf{C}_+\mathbf{C}_-$. The case $P = H$ is trivial.

Necessity part. Suppose the 6 circumcenters $A_\pm$, $B_\pm$, $C_\pm$ lie on a circle. By Proposition 3, the diagonals A_+A_-, B_+B_-, and C_+C_- have equal lengths. We show that $\mathbf{AA}' + \mathbf{BB}' + \mathbf{CC}' = 0$, so that P is the centroid of triangle ABC by Proposition 5. In terms of scalar products, we rewrite equation (1) as

$$\begin{aligned} \mathbf{A}_+\mathbf{A}_- \cdot \mathbf{BB}' &= -\tfrac{1}{2}\mathbf{BB}' \cdot \mathbf{BB}', & \mathbf{A}_+\mathbf{A}_- \cdot \mathbf{CC}' &= \tfrac{1}{2}\mathbf{CC}' \cdot \mathbf{CC}', \\ \mathbf{B}_+\mathbf{B}_- \cdot \mathbf{CC}' &= -\tfrac{1}{2}\mathbf{CC}' \cdot \mathbf{CC}', & \mathbf{B}_+\mathbf{B}_- \cdot \mathbf{AA}' &= \tfrac{1}{2}\mathbf{AA}' \cdot \mathbf{AA}', \qquad (2) \\ \mathbf{C}_+\mathbf{C}_- \cdot \mathbf{AA}' &= -\tfrac{1}{2}\mathbf{AA}' \cdot \mathbf{AA}', & \mathbf{C}_+\mathbf{C}_- \cdot \mathbf{BB}' &= \tfrac{1}{2}\mathbf{BB}' \cdot \mathbf{BB}'. \end{aligned}$$

From these, $(\mathbf{B}_+\mathbf{B}_- + \mathbf{C}_+\mathbf{C}_-) \cdot \mathbf{AA}' = 0$, and $\mathbf{AA}'$ is orthogonal to $\mathbf{B}_+\mathbf{B}_- + \mathbf{C}_+\mathbf{C}_-$. Since $\mathbf{B}_+\mathbf{B}_-$, and $\mathbf{C}_+\mathbf{C}_-$ have equal lengths, $\mathbf{B}_+\mathbf{B}_- + \mathbf{C}_+\mathbf{C}_-$ and $\mathbf{B}_+\mathbf{B}_- - \mathbf{C}_+\mathbf{C}_-$ are orthogonal. We may therefore write $\mathbf{B}_+\mathbf{B}_- - \mathbf{C}_+\mathbf{C}_- = k\mathbf{AA}'$ for a scalar k. From (2) above,

$$\begin{aligned} k\mathbf{AA}' \cdot \mathbf{AA}' =& (\mathbf{B}_+\mathbf{B}_- - \mathbf{C}_+\mathbf{C}_-) \cdot \mathbf{AA}' \\ =& \frac{1}{2}\mathbf{AA}' \cdot \mathbf{AA}' + \frac{1}{2}\mathbf{AA}' \cdot \mathbf{AA}' \\ =& \mathbf{AA}' \cdot \mathbf{AA}'. \end{aligned}$$

From this, $k = 1$ and we have

$$\mathbf{AA}' = \mathbf{B}_+\mathbf{B}_- - \mathbf{C}_+\mathbf{C}_-.$$

The same reasoning shows that

$$\begin{aligned} \mathbf{BB}' =& \mathbf{C}_+\mathbf{C}_- - \mathbf{A}_+\mathbf{A}_-, \\ \mathbf{CC}' =& \mathbf{A}_+\mathbf{A}_- - \mathbf{B}_+\mathbf{B}_-. \end{aligned}$$

Combining the three equations, we have

$$\mathbf{AA}' + \mathbf{BB}' + \mathbf{CC}' = \mathbf{0}.$$

It follows from Proposition 5 that P must be the centroid of triangle ABC.

4. An alternative proof of Theorem 1

We present another proof of Theorem 1 by considering an auxiliary hexagon. Let $\mathcal{L}_a$ and $\mathcal{L}'_a$ be the lines perpendicular to AA' at A and A' respectively; similarly, $\mathcal{L}_b$, $\mathcal{L}'_b$, and $\mathcal{L}_c$ and $\mathcal{L}'_c$. Consider the points

$$\begin{aligned} X_+ &= \mathcal{L}_c \cap \mathcal{L}'_b, & X_- &= \mathcal{L}_b \cap \mathcal{L}'_c, \\ Y_+ &= \mathcal{L}_a \cap \mathcal{L}'_c, & Y_- &= \mathcal{L}_c \cap \mathcal{L}'_a, \\ Z_+ &= \mathcal{L}_b \cap \mathcal{L}'_a, & Z_- &= \mathcal{L}_a \cap \mathcal{L}'_b. \end{aligned}$$

Note that the circumcenters $A_\pm$, $B_\pm$, $C_\pm$ are respectively the midpoints of $PX_\pm$, $PY_\pm$, $PZ_\pm$. Hence, the six circumcenters are concyclic if and only if $X_\pm$, $Y_\pm$, $Z_\pm$ are concyclic.

In Figure 3, let $\angle CPA' = \angle APC' = \alpha$. Since angles $PA'Y_-$ and PCY_- are both right angles, the four points P, A', C, Y_- are concyclic and $\angle Z_+Y_-X_+ = \angle A'Y_-X_+ = \angle A'PC = \alpha$. Similarly, $\angle CPB' = \angle BPC' = \angle Y_-X_+Z_-$, and we denote the common measure by β.

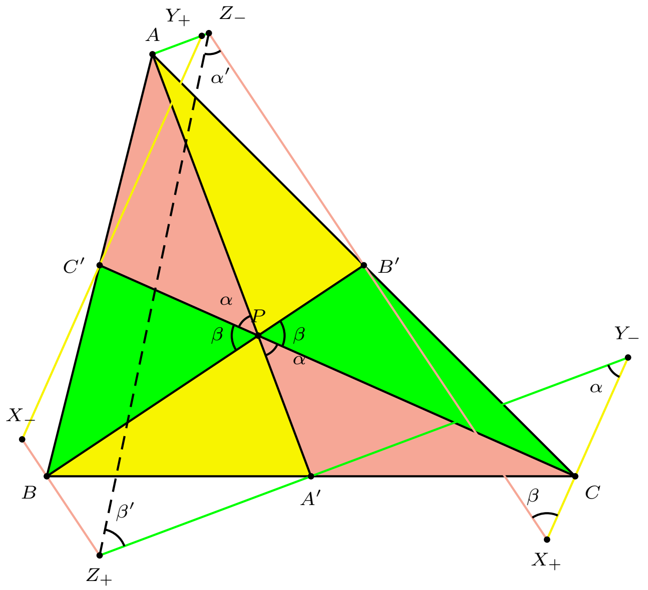


Figure 3

Lemma 7. *If the four points X_+, Y_-, Z_+, Z_- are concyclic, then P lies on the median through C.*

Proof. Let $x = \frac{AP}{AA'}$ and $y = \frac{BP}{BB'}$. If the four points X_+, Y_-, Z_+, Z_- are concyclic, then $\angle Z_+Z_-X_+ = \alpha$ and $\angle Y_-Z_+Z_- = \beta$. Now,

$$\frac{|BB'|}{|AA'|} = \frac{|Z_+Z_-| \cdot \sin \alpha'}{|Z_+Z_-| \cdot \sin \beta'} = \frac{\sin \alpha}{\sin \beta} = \frac{\frac{|AC'|}{|AP|}}{\frac{|BC'|}{|BP|}}.$$

It follows that

$$\frac{|BP|}{|BB'| \cdot |BC'|} = \frac{|AP|}{|AA'| \cdot |AC'|},$$

and, as a ratio of signed lengths,

$$\frac{BC'}{AC'} = -\frac{y}{x}. \tag{3}$$

Now applying Menelaus' theorem to triangle APC' with transversal $A'CB$, and triangle BGA' with transversal $B'CA$, we have

$$\frac{AA'}{A'P} \cdot \frac{PC}{CC'} \cdot \frac{C'B}{BA} = -1 = \frac{BB'}{B'P} \cdot \frac{PC}{CC'} \cdot \frac{C'A}{AB}.$$

From this, $\frac{AA'}{A'P} \cdot BC' = \frac{BB'}{B'P} \cdot AC'$, or

$$\frac{BC'}{1-x} = -\frac{AC'}{1-y}. \tag{4}$$

Comparing (3) and (4), we have $\frac{1-x}{1-y} = \frac{y}{x}$, $(x-y)(x+y-1) = 0$. Either $x = y$ or $x + y = 1$. It is easy to eliminate the possibility $x + y = 1$. If P has homogeneous barycentric coordinates $(u : v : w)$ with reference to triangle ABC, then $x = \frac{v+w}{u+v+w}$ and $y = \frac{w+u}{u+v+w}$. Thus, $x + y = 1$ requires $w = 0$ and P lies on the sideline AB, contrary to the assumption. It follows that $x = y$, and from (3), C' is the midpoint of AB, and P lies on the median through C. □

The necessity part of Theorem 1 is now an immediate corollary of Lemma 7.

5. Concluding remark

We conclude with a remark on triangles for which two of the circumcenters of the cevasix configuration of the centroid coincide. Clearly, $B_+ = C_-$ if and only if A, B', G, C' are concyclic. Equivalently, the image of G under the homothety $\mathsf{h}(A, 2)$ lies on the circumcircle of triangle ABC. This point has homogeneous barycentric coordinates $(-1 : 2 : 2)$. Since the circumcircle has equation

$$a^2yz + b^2zx + c^2xy = 0,$$

we have $2a^2 = b^2 + c^2$. There are many interesting properties of such triangles. We simply mention that it is similar to its own triangle of medians. Specifically,

$$m_a = \frac{\sqrt{3}}{2}a, \quad m_b = \frac{\sqrt{3}}{2}c, \quad m_c = \frac{\sqrt{3}}{2}b.$$

Editor's endnote. John H. Conway [5] has located the center of the Van Lamoen circle (of the circumcenters of the cevasix configuration of the centroid) as

$$F = \mathsf{m}N + \frac{\cot^2\omega}{12} \cdot (G - K),$$

where $\mathsf{m}N$ is the medial Ninecenter, [1] G the centroid, K the symmedian point, and ω the Brocard angle of triangle ABC. In particular, the parallel through F to the symmedian line GK hits the Euler line in $\mathsf{m}N$. See Figure 4. The point F has homogeneous barycentric coordinates

$$(10a^4 - 13a^2(b^2 + c^2) + (4b^4 - 10b^2c^2 + 4c^4) : \cdots : \cdots).$$

This appears as X_{1153} of [6].

[1] This is the point which divides OH in the ratio $1 : 3$.

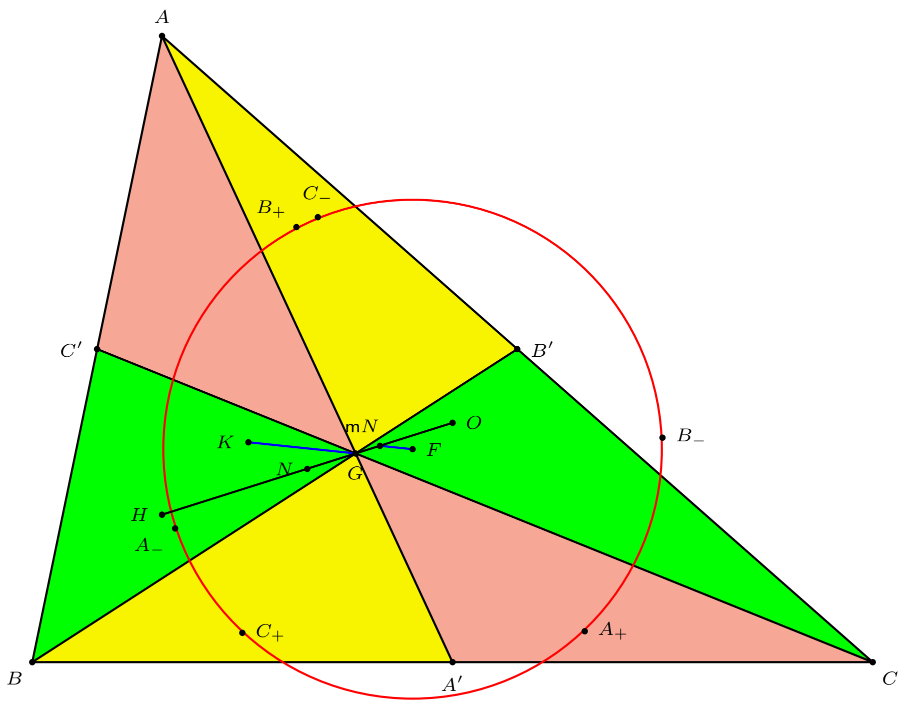


Figure 4

References

[1] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.
[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.
[3] F. M. van Lamoen, Problem 10830, *Amer. Math. Monthly*, 2000 (107) 863; solution by the Monthly editors, 2002 (109) 396–397.
[4] K. Y. Li, Concyclic problems, *Mathematical Excalibur*, 6 (2001) Number 1, 1–2; available at http://www.math.ust.hk/excalibur.
[5] J. H. Conway, Hyacinthos message 5555, May 24, 2002.
[6] C. Kimberling, *Encyclopedia of Triangle Centers*, August 22, 2002 edition, available at http://www2.evansville.edu/ck6/encyclopedia/; February 17, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Alexei Myakishev: Smolnaia 61-2, 138, Moscow, Russia, 125445
E-mail address: `alex_geom@mtu-net.ru`

Peter Y. Woo: Department of Mathematics, Biola University, 13800 Biola Avenue, La Mirada, California 90639, USA
E-mail address: `woobiola@yahoo.com`

Forum Geometricorum
Volume 3 (2003) 65–71.

FORUM GEOM
ISSN 1534-1178

Napoleon Triangles and Kiepert Perspectors

Two examples of the use of complex number coordinates

Floor van Lamoen

Abstract. In this paper we prove generalizations of the well known Napoleon Theorem and Kiepert Perspectors. We use complex numbers as coordinates to prove the generalizations, because this makes representation of isosceles triangles built on given segments very easy.

1. Introduction

In [1, XXVII] O. Bottema describes the famous (first) Fermat-Torricelli point of a triangle ABC. This point is found by attaching outwardly equilateral triangles to the sides of ABC. The new vertices form a triangle $A'B'C'$ that is perspective to ABC, that is, AA', BB' and CC' have a common point of concurrency, the perspector of ABC and $A'B'C'$. A lot can be said about this point, but for this paper we only need to know that the lines AA', BB' and CC' make angles of 60 degrees (see Figure 1), and that this is also the case when the equilateral triangles are pointed inwardly, which gives the second Fermat-Torricelli point.

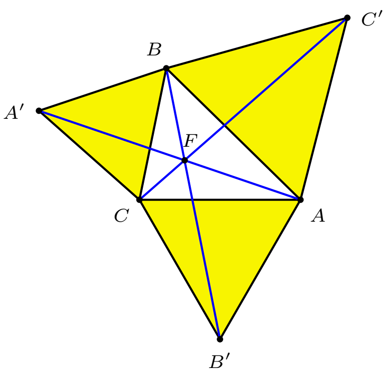


Figure 1

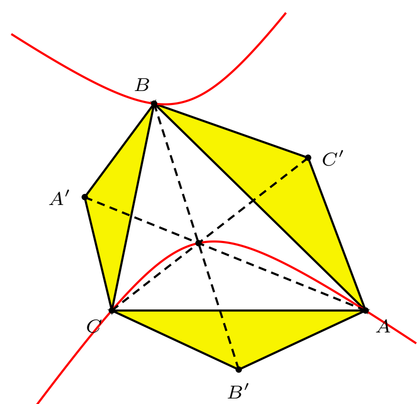


Figure 2

It is well known that to yield a perspector, the triangles attached to the sides of ABC do not need to be equilateral. For example they may be isosceles triangles with base angle ϕ, like Bottema tells us in [1, XI]. It was Ludwig Kiepert who studied these triangles - the perspectors with varying ϕ lie on a rectangular hyperbola

Publication Date: March 10, 2003. Guest Editor: Dick Klingens.

The Dutch version of this paper, *Napoleons driehoeken en Kieperts perspectors*, appeared in *Euclides*, 77 (2002) nr 4, 182–187. This issue of *Euclides* is a tribute to O. Bottema (1901-1992). Permission from the editors of *Euclides* to publish the present English version is gratefully acknowledged.

named after Kiepert. See [4] for some further study on this hyperbola, and some references. See Figure 2. However, it is already sufficient for the lines AA', BB', CC' to concur when the attached triangles have oriented angles satisfying

$$\angle BAC' = \angle CAB', \qquad \angle ABC' = \angle CBA', \qquad \angle ACB = \angle BCA'.$$

When the attached triangles are equilateral, there is another nice geometric property: *the centroids of the triangles $A'BC$, $AB'C$ and ABC' form a triangle that is equilateral itself*, a fact that is known as Napoleon's Theorem. The triangles are referred to as the *first* and *second Napoleon triangles* (for the cases of outwardly and inwardly pointed attached triangles). See Figures 3a and 3b. The perspectors of these two triangles with ABC are called *first* and *second Napoleon points*. General informations on Napoleon triangles and Kiepert perspectors can be found in [2, 3, 5, 6].

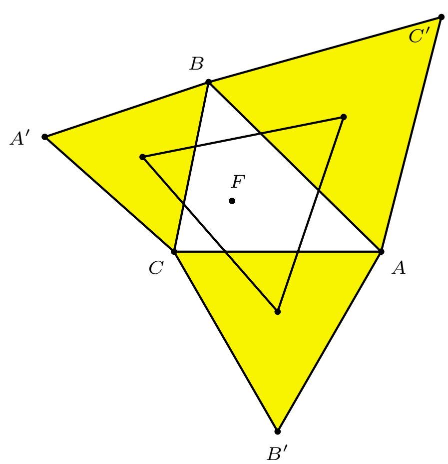


Figure 3a

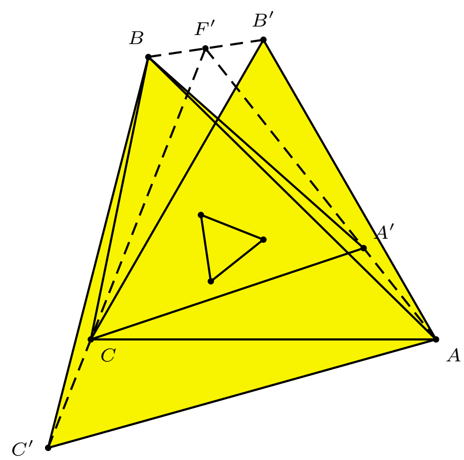


Figure 3b

2. The equation of a line in the complex plane

Complex coordinates are not that much different from the rectangular (x, y) - the two directions of the axes are now hidden in one complex number, that we call the *affix* of a point. Of course such an affix just exists of a real (x) and imaginary (y) part - the complex number $z = p + qi$ in fact resembles the point (p, q).

If $z = p + qi$, then the number $\overline{z} = p - qi$ is called complex conjugate of z. The combination of z and $\overline{z}$ is used to make formulas, since we do not have x and y anymore! A parametric formula for the line through the points a_1 and a_2 is $z = a_1 + t(a_2 - a_1)$, where t runs through the *real* numbers. The complex conjugate of this formula is $\overline{z} = \overline{a_1} + t(\overline{a_2} - \overline{a_1})$. Elimination of t from these two formulas gives the formula for the line through the points with affixes a_1 and a_2:

$$z(\overline{a_1 - a_2}) - \overline{z}(a_1 - a_2) + (a_1\overline{a_2} - \overline{a_1}a_2) = 0.$$

3. Isosceles triangle on a segment

Let the points A and B have affixes a and b. We shall find the affix of the point C for which ABC is an isosceles triangle with base angle ϕ and apex C. The midpoint of AB has affix $\frac{1}{2}(a+b)$. The distance from this midpoint to C is equal to $\frac{1}{2}|AB|\tan\phi$. With this we find the affix for C as

$$c = \frac{a+b}{2} + i\tan\phi \cdot \frac{b-a}{2} = \frac{1-i\tan\phi}{2}a + \frac{1+i\tan\phi}{2}b = \overline{\chi}a + \chi b$$

where $\chi = \frac{1}{2} + \frac{i}{2}\tan\phi$, so that $\chi + \overline{\chi} = 1$.

The special case that ABC is equilateral, yields for χ the sixth root of unity $\zeta = \frac{1}{2} + \frac{i}{2}\sqrt{3} = e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$. This number ζ is a sixth root of unity, because it satisfies

$$\zeta^6 = e^{2i\pi} = \cos 2\pi + i\sin 2\pi = 1.$$

It also satisfies the identities $\zeta^3 = -1$ and $\zeta \cdot \overline{\zeta} = \zeta + \overline{\zeta} = 1$. Depending on orientation one can find two vertices C that together with AB form an equilateral triangle, for which we have respectively $c = \zeta a + \overline{\zeta} b$ (negative orientation) and $c = \overline{\zeta} a + \zeta b$ (positive orientation). From this one easily derives

Proposition 1. *The complex numbers a, b and c are affixes of an equilateral triangle if and only if*

$$a + \zeta^2 b + \zeta^4 c = 0$$

for positive orientation or

$$a + \zeta^4 b + \zeta^2 c = 0$$

for negative orientation.

4. Napoleon triangles

We shall generalize Napoleon's Theorem, by extending the idea of the use of centroids. Napoleon triangles were indeed built in a triangle ABC by attaching to the sides of a triangle equilateral triangles, and taking the centroids of these. We now start with two triangles $A_kB_kC_k$ for $k = 1,\ 2$, and attach equilateral triangles to the connecting segments between the A's, the B's and the C's. This seems to be entirely different, but Napoleon's Theorem will be a special case by starting with triangles BCA and CAB.

So we start with two triangles $A_kB_kC_k$ for $k = 1,\ 2$ with affixes a_k, b_k and c_k for the vertices. The centroids Z_k have affixes $z_k = \frac{1}{3}(a_k + b_k + c_k)$. Now we attach positively orientated equilateral triangles to segments A_1A_2, B_1B_2 and C_1C_2 having A_{3+}, B_{3+}, C_{3+} as third vertices. In the same way we find A_{3-}, B_{3-}, C_{3-} from equilateral triangles with negative orientation. We find as affixes

$$a_{3+} = \zeta a_2 + \overline{\zeta} a_1$$

and

$$a_{3-} = \zeta a_1 + \overline{\zeta} a_2,$$

and similar expressions for b_{3+}, b_{3-}, c_{3+} and c_{3-}. The centroids Z_{3+} and Z_{3-} now have affixes $z_{3+} = \zeta z_2 + \overline{\zeta} z_1$ and $z_{3-} = \zeta z_1 + \overline{\zeta} z_2$ respectively, from which we

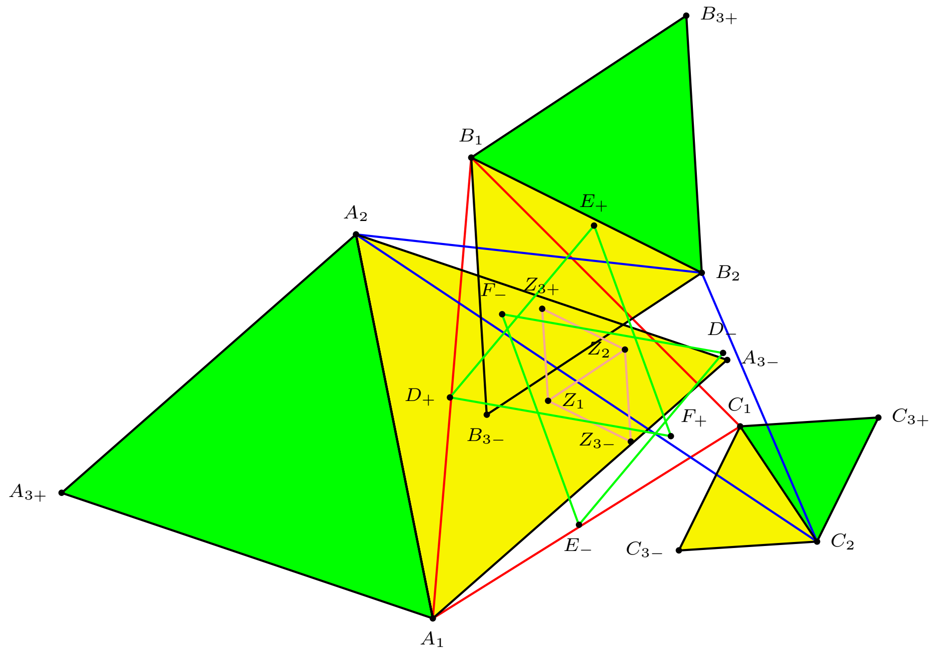


Figure 4

see that $Z_1Z_2Z_{3+}$ and $Z_1Z_2Z_{3-}$ are equilateral triangles of positive and negative orientation respectively.

We now work with the following centroids:
D_+, E_+ and F_+ of triangles $B_1C_2A_{3+}$, $C_1A_2B_{3+}$ and $A_1B_2C_{3+}$ respectively;
D_-, E_- and F_- of triangles $C_1B_2A_{3-}$, $A_1C_2B_{3-}$ and $B_1A_2C_{3-}$ respectively.

For these we claim

Theorem 2. *Given triangles $A_kB_kC_k$ and points Z_k for $k = 1,\ 2,\ 3+,\ 3-$ and $D_\pm E_\pm F_\pm$ as described above, triangles $D_+E_+F_+$ and $D_-E_-F_-$ are equilateral triangles of negative orientation, congruent and parallel, and their centroids coincide with the centroids of $Z_1Z_2Z_{3+}$ and $Z_1Z_2Z_{3-}$ respectively. (See Figure 4).*

Proof. To prove this we find the following affixes

$$d_+ = \tfrac{1}{3}(b_1 + c_2 + \zeta a_2 + \overline{\zeta} a_1), \qquad d_- = \tfrac{1}{3}(b_2 + c_1 + \zeta a_1 + \overline{\zeta} a_2),$$

$$e_+ = \tfrac{1}{3}(c_1 + a_2 + \zeta b_2 + \overline{\zeta} b_1), \qquad e_- = \tfrac{1}{3}(c_2 + a_1 + \zeta b_1 + \overline{\zeta} b_2),$$

$$f_+ = \tfrac{1}{3}(a_1 + b_2 + \zeta c_2 + \overline{\zeta} c_1), \qquad f_- = \tfrac{1}{3}(a_2 + b_1 + \zeta c_1 + \overline{\zeta} c_2).$$

Using Proposition 1 it is easy to show that $D_+E_+F_+$ and $D_-E_-F_-$ are equilateral triangles of negative orientation. For instance, the expression $d_+ + \zeta^4 e_+ + \zeta^2 f_+$ has as 'coefficient' of b_1 the number $\frac{1}{3}(1 + \zeta^4\overline{\zeta}) = 0$. We also find that

$$d_+ - e_+ = e_- - d_- = \overline{\zeta}(a_1 - a_2) + \zeta(b_1 - b_2) + (c_2 - c_1),$$

from which we see that D_+E_+ and D_-E_- are equal in length and directed oppositely. Finally it is easy to check that $\frac{1}{3}(d_+ + e_+ + f_+) = \frac{1}{3}(z_1 + z_2 + z_{3+})$ and $\frac{1}{3}(d_- + e_- + f_-) = \frac{1}{3}(z_1 + z_2 + z_{3-})$, and the theorem is proved. $\square$

We can make a variation of Theorem 2 if in the creation of $D_\pm E_\pm F_\pm$ we interchange the roles of $A_{3+}B_{3+}C_{3+}$ and $A_{3-}B_{3-}C_{3-}$. The roles of Z_{3+} and Z_{3-} change as well, and the equilateral triangles found have positive orientation.

We note that if the centroids Z_1 and Z_2 coincide, then they coincide with Z_{3+} and Z_{3-}, so that $D_+E_-F_+D_-E_+F_-$ is a regular hexagon, of which the center coincides with Z_1 and Z_2.

Napoleon's Theorem is a special case. If we take $A_1B_1C_1 = BCA$ and $A_2B_2C_2 = CAB$, then $D_+E_+F_+$ is the second Napoleon Triangle, and indeed appears equilateral. We get as a bonus that $D_+E_-F_+D_-E_+F_-$ is a regular hexagon. Now D_- is the centroid of AAA_{3-}, that is, D_- is the point on AA_{3-} such that $AD_- : D_-A_{3-} = 1 : 2$. In similar ways we find E_- and F_-. Triangles ABC and $A_{3-}B_{3-}C_{3-}$ have the first point of Fermat-Torricelli F_1 as perspector, and the lines AA_{3-}, BB_{3-} and CC_{3-} make angles of 60 degrees. From this it is easy to see (congruent inscribed angles) that F_1 must be on the circumcircle of $D_-E_-F_-$ and thus also on the circumcircle of $D_+E_+F_+$. See Figure 5. In the same way, now using the variation of Theorem 2, we see that the second Fermat-Torricelli point lies on the circumcircle of the first Napoleon Triangle.

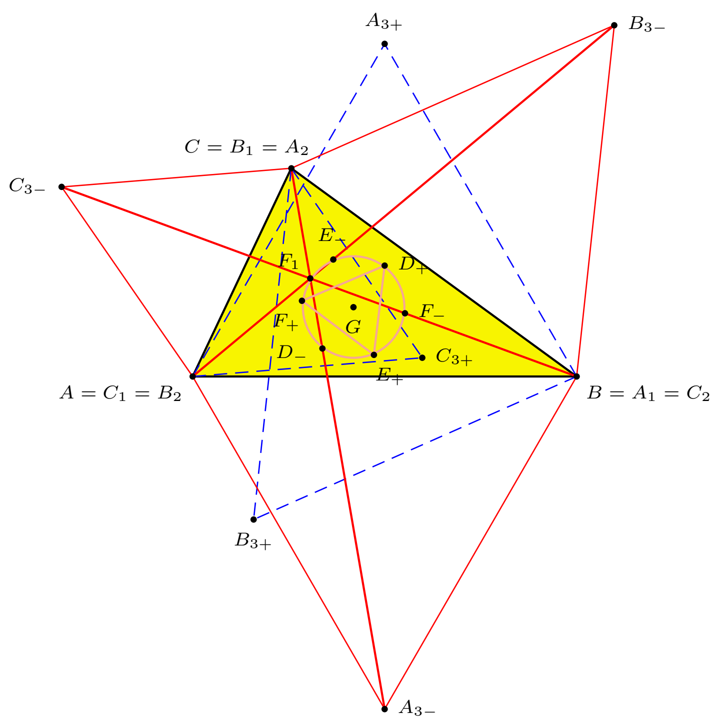


Figure 5

5. Kiepert perspectors

To generalize the Kiepert perspectors we start with two triangles as well. We label these ABC and $A'B'C'$ to distinguish from Theorem 2. These two triangles we take to be directly congruent (hence A corresponds to A', etc.) and of the same orientation. This means that the two triangles can be mapped to each other by a combination of a rotation and a translation (in fact one of both is sufficient). We now attach isosceles triangles to segments connecting ABC and $A'B'C'$. While we usually find Kiepert perspectors on a line, for example, from A to the apex of an isosceles triangle built on BC, now we start from the apex of an isosceles triangle on AA' and go to the apex of an isosceles triangle on on BC'. This gives the following theorem:

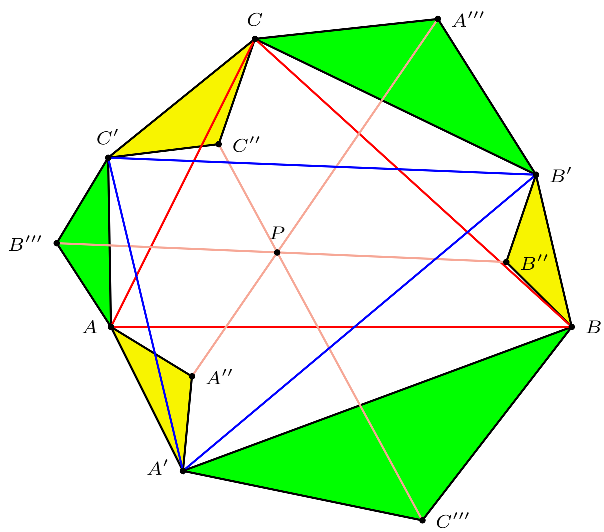


Figure 6

Theorem 3. *Given two directly congruent triangles ABC and $A'B'C'$ with the same orientation, attach to the segments AA', BB', CC', CB', AC' and BA' similar isosceles triangles with the same orientation and apexes A'', B'', C'', A''', B''' and C'''. The lines $A''A'''$, $B''B'''$ and $C''C'''$ are concurrent, so triangles $A''B''C''$ and $A'''B'''C'''$ are perspective. (See Figure 6).*

Proof. For the vertices A, B and C we take the affixes a, b and c. Because triangles ABC and $A'B'C'$ are directly congruent and of equal orientation, we can get $A'B'C'$ by applying on ABC a rotation about the origin, followed by a translation. This rotation about the origin can be represented by multiplication by a number τ on the unit circle, so that $\tau\overline{\tau} = 1$. The translation is represented by addition with a number σ. So the affixes of A', B' and C' are the numbers $\tau a + \sigma$, $\tau b + \sigma$ and $\tau c + \sigma$.

We take for the base angles of the isosceles triangle ϕ again, and we let $\chi = \frac{1}{2} + \frac{i}{2}\tan\phi$, so that the affix for A'' is $(\overline{\chi}+\chi\tau)a+\chi\sigma$. For A''' we find $\overline{\chi}c+\chi\tau b+\chi\sigma$. The equation of the line $A''A'''$ we can find after some calculations as

$$\begin{aligned}
&(\chi\overline{a}+\overline{\chi\tau a}-\chi\overline{c}-\overline{\chi\tau b})z-(\overline{\chi}a+\chi\tau a-\overline{\chi}c-\chi\tau b)\overline{z}\\
&+(\overline{\chi}+\chi\tau)a(\chi\overline{c}+\overline{\chi\tau b})+\chi\sigma(\chi\overline{c}+\overline{\chi\tau b})+\overline{\chi\sigma}(\overline{\chi}+\chi\tau)a\\
&-(\chi+\overline{\chi\tau})\overline{a}(\overline{\chi}c+\chi\tau b)-\overline{\chi\sigma}(\overline{\chi}c+\chi\tau b)-\chi\sigma(\chi+\overline{\chi\tau})\overline{a}\\
&=0.
\end{aligned}$$

In a similar fashion we find for $B''B'''$,

$$\begin{aligned}
&(\chi\overline{b}+\overline{\chi\tau b}-\chi\overline{a}-\overline{\chi\tau c})z-(\overline{\chi}b+\chi\tau b-\overline{\chi}a-\chi\tau c)\overline{z}\\
&+(\overline{\chi}+\chi\tau)b(\chi\overline{a}+\overline{\chi\tau c})+\chi\sigma(\chi\overline{a}+\overline{\chi\tau c})+\overline{\chi\sigma}(\overline{\chi}+\chi\tau)b\\
&-(\chi+\overline{\chi\tau})\overline{b}(\overline{\chi}a+\chi\tau c)-\overline{\chi\sigma}(\overline{\chi}a+\chi\tau c)-\chi\sigma(\chi+\overline{\chi\tau})\overline{b}\\
&=0,
\end{aligned}$$

and for $C''C'''$,

$$\begin{aligned}
&(\chi\overline{c}+\overline{\chi\tau c}-\chi\overline{b}-\overline{\chi\tau a})z-(\overline{\chi}c+\chi\tau c-\overline{\chi}b-\chi\tau a)\overline{z}\\
&+(\overline{\chi}+\chi\tau)c(\chi\overline{b}+\overline{\chi\tau a})+\chi\sigma(\chi\overline{b}+\overline{\chi\tau a})+\overline{\chi\sigma}(\overline{\chi}+\chi\tau)c\\
&-(\chi+\overline{\chi\tau})\overline{c}(\overline{\chi}b+\chi\tau a)-\overline{\chi\sigma}(\overline{\chi}b+\chi\tau a)-\chi\sigma(\chi+\overline{\chi\tau})\overline{c}\\
&=0.
\end{aligned}$$

We must do some more effort to see what happens if we add the three equations. Our effort is rewarded by noticing that the sum gives $0 = 0$. The three equations are dependent, so the lines are concurrent. This proves the theorem. □

We end with a question on the locus of the perspector for varying ϕ. It would have been nice if the perspector would, like in Kiepert's hyperbola, lie on an equilateral hyperbola. This, however, does not seem to be generally the case.

References

[1] O. Bottema, *Hoofdstukken uit de elementaire meetkunde*, 2e druk, Epsilon Utrecht, 1987.

[2] H. S. M. Coxeter and S. L. Greitzer, *Geometery Revisted*, Math. Assoc. America, 1967.

[3] D. Klingens, Homepage, http://www.pandd.demon.nl/meetkunde.htm

[4] F. M. van Lamoen and P. Yiu, The Kiepert pencil of Kiepert hyperbolas, *Forum Geom.*, 1 (2001) 125–132.

[5] E. Weisstein, *MathWorld*, available at http://mathworld.wolfram.com/.

[6] D. Wells, *The Penguin Dictionary of Curious and Interesting Geometry*, Penguin, London, 1991.

Floor van Lamoen: Statenhof 3, 4463 TV Goes, The Netherlands

E-mail address: `f.v.lamoen@wxs.nl`

Forum Geometricorum
Volume 3 (2003) 73–81.

FORUM GEOM
ISSN 1534-1178

On the Fermat Lines

Paul Yiu

Abstract. We study the triangle formed by three points each on a Fermat line of a given triangle, and at equal distances from the vertices. For two specific values of the common distance, the triangle degenerates into a line. The two resulting lines are the axes of the Steiner ellipse of the triangle.

1. The Fermat lines

This paper is on a variation of the theme of Bottema [2]. Bottema studied the triangles formed by three points each on an *altitude* of a given triangle, at equal distances from the respective vertices. See Figure 1. He obtained many interesting properties of this configuration. For example, these three points are collinear when the common distance is $R \pm d$, where R is the circumradius and d the distance between the circumcenter and the incenter of the reference triangle. The two lines containing the two sets of collinear points are perpendicular to each other at the incenter, and are parallel to the asymptotes of the Feuerbach hyperbola, the rectangular hyperbola through the vertices, the orthocenter, and the incenter. See Figure 2.

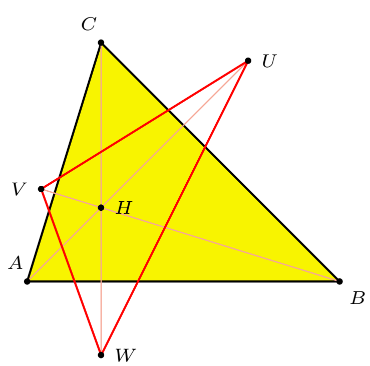


Figure 1

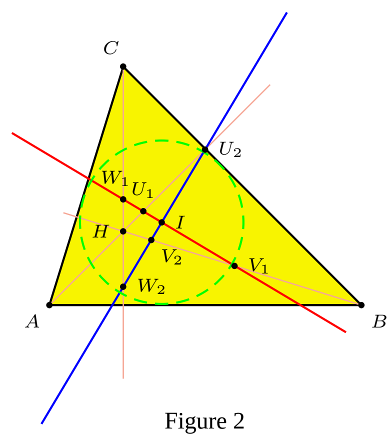


Figure 2

In this paper we consider the *Fermat lines*, which are the lines joining a vertex of the given triangle ABC to the apex of an equilateral triangle constructed on its opposite side. We label these triangles BCA_ϵ, CAB_ϵ, and ABC_ϵ, with $\epsilon = +1$

Publication Date: March 10, 2003. Guest Editor: Dick Klingens.

This paper is an extended revision of its Dutch version, *Over de lijnen van Fermat*, Euclides, 77 (2002) nr. 4, 188–193. This issue of *Euclides* is a tribute to O. Bottema (1900 – 1992). The author thanks Floor van Lamoen for translation into Dutch, and the editors of *Euclides* for permission to publish the present English version.

for those erected externally, and $\epsilon = -1$ otherwise. There are 6 of such lines, AA_+, BB_+, CC_+, AA_-, BB_-, and CC_-. See Figure 3. The reason for choosing these lines is that, for $\epsilon = \pm 1$, the three segments AA_ϵ, BB_ϵ, and CC_ϵ have equal lengths τ_ϵ given by

$$\tau_\epsilon^2 = \frac{1}{2}(a^2 + b^2 + c^2) + \epsilon \cdot 2\sqrt{3}\triangle,$$

where a, b, c are the side lengths, and $\triangle$ the area of triangle ABC. See, for example, [1, XXVII.3].

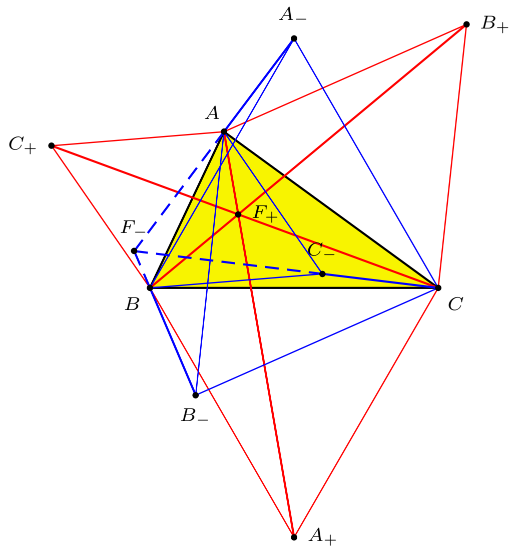


Figure 3

It is well known that the three Fermat lines AA_ϵ, BB_ϵ, and CC_ϵ intersect each other at the ϵ-Fermat point F_ϵ at 60° angles. The centers of the equilateral triangles BCA_ϵ, CAB_ϵ, and ABC_ϵ form the ϵ-Napoleon equilateral triangle. The circumcircle of the ϵ-Napoleon triangle has radius $\frac{\tau_\epsilon}{3}$ and passes through the $(-\epsilon)$-Fermat point. See, for example, [5].

2. The triangles $\mathcal{T}_\epsilon(t)$

We shall label points on the Fermat lines by their distances from the corresponding vertices of ABC, positive in the direction from the vertex to the Fermat point, negative otherwise. Thus, $A_+(t)$ is the unique point X on the positive Fermat line AF_+ such that $AX = t$. In particular,

$$A_\epsilon(\tau_\epsilon) = A_\epsilon, \qquad B_\epsilon(\tau_\epsilon) = B_\epsilon, \qquad C_\epsilon(\tau_\epsilon) = C_\epsilon.$$

We are mainly interested in the triangles $\mathcal{T}_\epsilon(t)$ whose vertices are $A_\epsilon(t)$, $B_\epsilon(t)$, $C_\epsilon(t)$, for various values of t. Here are some simple observations.

(1) The centroid of AA_+A_- is G. This is because the segments A_+A_- and BC have the same midpoint.

(2) The centers of the equilateral triangles BCA_+ and BCA_- trisect the segment A_+A_-. Therefore, the segment joining $A_\epsilon(\frac{\tau_\epsilon}{3})$ to the center of $BCA_{-\epsilon}$ is parallel to the Fermat line $AA_{-\epsilon}$ and has midpoint G.

(3) This means that $A_\epsilon(\frac{\tau_\epsilon}{3})$ is the reflection of the A-vertex of the $(-\epsilon)$-Napoleon triangle in the centroid G. See Figure 4, in which we label $A_+(\frac{\tau_+}{3})$ by X and $A_-(\frac{\tau_-}{3})$ by X' respectively.

This is the same for the other two points $B_\epsilon(\frac{\tau_\epsilon}{3})$ and $C_\epsilon(\frac{\tau_\epsilon}{3})$.

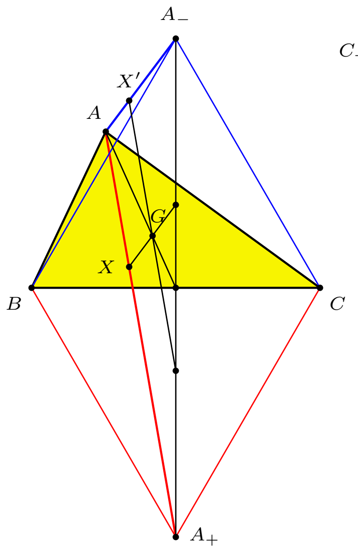


Figure 4

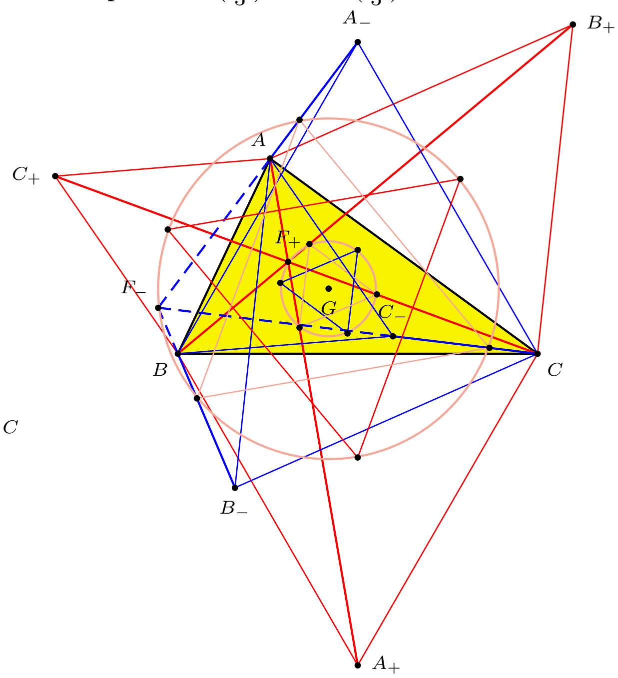


Figure 5

(4) It follows that the triangle $\mathcal{T}_\epsilon(\frac{\tau_\epsilon}{3})$ is the reflection of the $(-\epsilon)$-Napoleon triangle in G, and is therefore equilateral.

(5) The circle through the vertices of $\mathcal{T}_\epsilon(\frac{\tau_\epsilon}{3})$ and the $(-\epsilon)$-Napoleon triangle has radius $\frac{\tau_{-\epsilon}}{3}$ and also passes through the Fermat point F_ϵ. See Figure 5.

Since $GA_\epsilon(\frac{\tau_\epsilon}{3}) = \frac{\tau_{-\epsilon}}{3}$, (see Figure 4), the circle, center X, radius $\frac{\tau_{-\epsilon}}{3}$, passes through G. See Figure 6A. Likewise, the circle, center X', radius $\frac{\tau_\epsilon}{3}$ also passes through G. See Figure 6B. In these figures, we label

$$Y = A_+\left(\frac{\tau_+ - \tau_-}{3}\right), \qquad Z = A_+\left(\frac{\tau_+ + \tau_-}{3}\right),$$
$$Y' = A_-\left(\frac{\tau_- - \tau_+}{3}\right), \qquad Z' = A_-\left(\frac{\tau_+ + \tau_-}{3}\right).$$

It follows that GY and GZ are perpendicular to each other; so are GY' and GZ'.

(6) For $\epsilon = \pm 1$, the lines joining the centroid G to $A_\epsilon(\frac{\tau_\epsilon+\tau_{-\epsilon}}{3})$ and $A_\epsilon(\frac{\tau_\epsilon-\tau_{-\epsilon}}{3})$ are perpendicular to each other. Similarly, the lines joining G to $B_\epsilon(\frac{\tau_\epsilon+\tau_{-\epsilon}}{3})$ and $B_\epsilon(\frac{\tau_\epsilon-\tau_{-\epsilon}}{3})$ are perpendicular to each other; so are the lines joining G to $C_\epsilon(\frac{\tau_\epsilon+\tau_{-\epsilon}}{3})$ and $C_\epsilon(\frac{\tau_\epsilon-\tau_{-\epsilon}}{3})$.

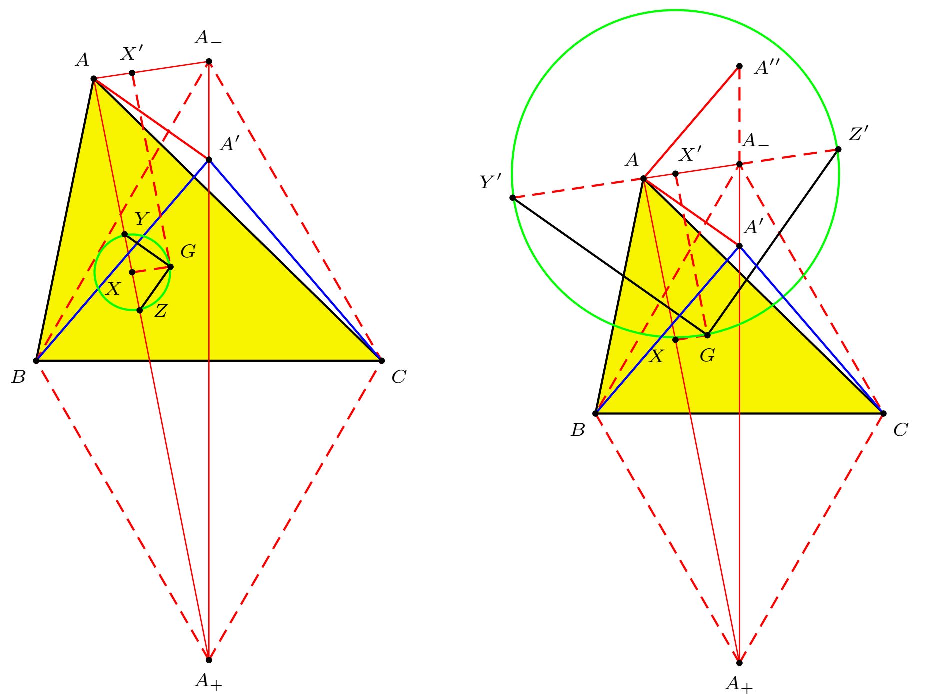


Figure 6A Figure 6B

In Figure 6A, since $\angle XGY = \angle XYG$ and $AXGX'$ is a parallelogram, the line GY is the bisector of angle XGX', and is parallel to the bisector of angle A_+AA_-. If the internal bisector of angle A_+AA_- intersects A_+A_- at A', then it is easy to see that A' is the apex of the isosceles triangle constructed *inwardly* on BC with base angle φ satisfying

$$\cot\varphi = \frac{\tau_+ + \tau_-}{\sqrt{3}(\tau_+ - \tau_-)}. \qquad (\dagger)$$

Similarly, in Figure 6B, the line GZ' is parallel to the external bisector of the same angle. We summarize these as follows.

(7) The lines joining $A_+(\frac{\tau_+-\tau_-}{3})$ to $A_-(\frac{\tau_--\tau_+}{3})$ and $A_+(\frac{\tau_++\tau_-}{3})$ to $A_-(\frac{\tau_++\tau_-}{3})$ are perpendicular at G, and are respectively parallel to the internal and external bisectors of angle A_+AA_-. Similarly, the two lines joining $B_+(\frac{\tau_+-\tau_-}{3})$ to $B_-(\frac{\tau_--\tau_+}{3})$ and $B_+(\frac{\tau_++\tau_-}{3})$ to $B_-(\frac{\tau_++\tau_-}{3})$ are perpendicular at G, being parallel to the internal and external bisectors of angle B_+BB_-; so are the lines joining

$C_+(\frac{\tau_+-\tau_-}{3})$ to $C_-(\frac{\tau_--\tau_+}{3})$, and $C_+(\frac{\tau_++\tau_-}{3})$ to $C_-(\frac{\tau_++\tau_-}{3})$, being parallel to the internal and external bisectors of angle C_+CC_-.

3. Collinearity

What is interesting is that these 3 pairs of perpendicular lines in (7) above form the same right angles at the centroid G. Specifically, the six points

$$A_+(\frac{\tau_++\tau_-}{3}), B_+(\frac{\tau_++\tau_-}{3}), C_+(\frac{\tau_++\tau_-}{3}), A_-(\frac{\tau_++\tau_-}{3}), B_-(\frac{\tau_++\tau_-}{3}), C_-(\frac{\tau_++\tau_-}{3})$$

are collinear with the centroid G on a line $\mathcal{L}_+$; so are the 6 points

$$A_+(\frac{\tau_+-\tau_-}{3}), B_+(\frac{\tau_+-\tau_-}{3}), C_+(\frac{\tau_+-\tau_-}{3}), A_-(\frac{\tau_--\tau_+}{3}), B_-(\frac{\tau_--\tau_+}{3}), C_-(\frac{\tau_--\tau_+}{3})$$

on a line $\mathcal{L}_-$ through G. See Figure 7. To justify this, we consider the triangle $\mathcal{T}_\epsilon(t) := A_\epsilon(t)B_\epsilon(t)C_\epsilon(t)$ for varying t.

(8) For $\epsilon = \pm 1$, the triangle $\mathcal{T}_\epsilon(t)$ degenerates into a line containing the centroid G if and only if $t = \frac{\tau_\epsilon+\delta\tau_{-\epsilon}}{3}$, $\delta = \pm 1$.

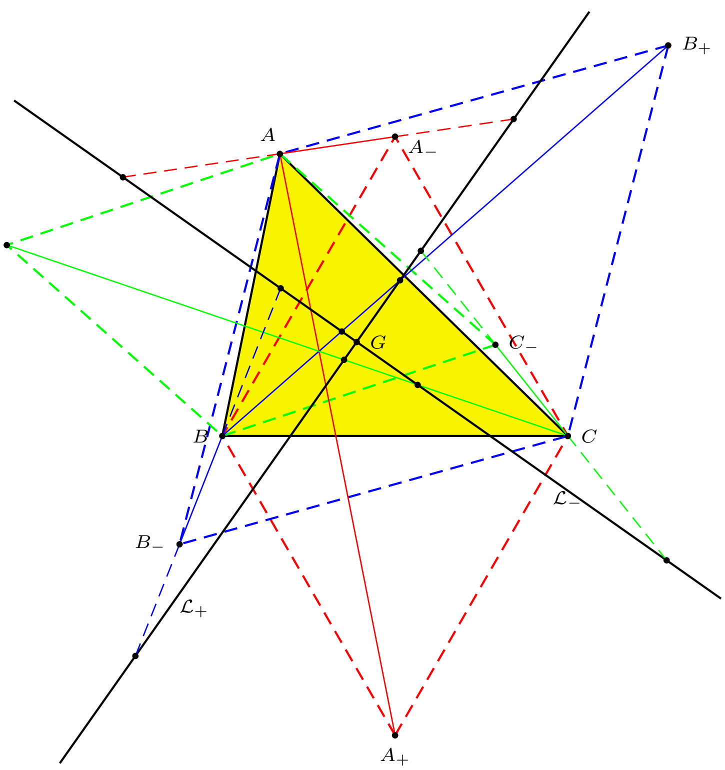


Figure 7

4. Barycentric coordinates

To prove (8) and to obtain further interesting geometric results, we make use of coordinates. Bottema has advocated the use of homogeneous barycentric coordinates. See [3, 6]. Let P be a point in the plane of triangle ABC. With reference to ABC, the homogeneous barycentric coordinates of P are the ratios of signed areas

$$(\triangle PBC : \triangle PCA : \triangle PAB).$$

The coordinates of the vertex A_+ of the equilateral triangle BCA_+, for example, are $(-\frac{\sqrt{3}}{4}a^2 : \frac{1}{2}ab\sin(C+60^\circ) : \frac{1}{2}ca\sin(B+60^\circ))$, which can be rewritten as

$$A_+ = (-2\sqrt{3}a^2 : \sqrt{3}(a^2+b^2-c^2)+4\triangle : \sqrt{3}(c^2+a^2-b^2)+4\triangle).$$

More generally, for $\epsilon = \pm 1$, the vertices of the equilateral triangles erected on the sides of triangle ABC are the points

$$\begin{aligned}
A_\epsilon =&(-2\sqrt{3}a^2 : \sqrt{3}(a^2+b^2-c^2)+4\epsilon\triangle : \sqrt{3}(c^2+a^2-b^2)+4\epsilon\triangle),\\
B_\epsilon =&(\sqrt{3}(a^2+b^2-c^2)+4\epsilon\triangle : -2\sqrt{3}b^2 : \sqrt{3}(b^2+c^2-a^2)+4\epsilon\triangle),\\
C_\epsilon =&(\sqrt{3}(c^2+a^2-b^2)+4\epsilon\triangle : \sqrt{3}(b^2+c^2-a^2)+4\epsilon\triangle : -2\sqrt{3}c^2).
\end{aligned}$$

Note that in each case, the coordinate sum is $8\epsilon\triangle$. From this we easily compute the coordinates of the centroid by simply adding the corresponding coordinates of the three vertices.

(9A) For $\epsilon = \pm 1$, triangles $A_\epsilon B_\epsilon C_\epsilon$ and ABC have the same centroid.

Sometimes it is convenient to work with *absolute* barycentric coordinates. For a finite point $P = (u : v : w)$, we obtain the absolute barycentric coordinates by normalizing its homogeneous barycentric coordinates, namely, by dividing by the coordinate sum. Thus,

$$P = \frac{1}{u+v+w}(uA+vB+wC),$$

provided $u+v+w$ is nonzero.

The absolute barycentric coordinates of the point $A_\epsilon(t)$ can be easily written down. For each value of t,

$$A_\epsilon(t) = \frac{1}{\tau_\epsilon}((\tau_\epsilon - t)A + t\cdot A_\epsilon),$$

and similarly for $B_\epsilon(t)$ and $C_\epsilon(t)$.

This, together with (9A), leads easily to the more general result.

(9B) For arbitrary t, the triangles $\mathcal{T}_\epsilon(t)$ and ABC have the same centroid.

5. Area of $\mathcal{T}_\epsilon(t)$

Let $X = (x_1 : x_2 : x_3)$, $Y = (y_1 : y_2 : y_3)$ and $Z = (z_1 : z_2 : z_3)$ be finite points with homogeneous coordinates with respect to triangle ABC. The *signed* area of the oriented triangle XYZ is

$$\frac{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}}{(x_1+x_2+x_3)(y_1+y_2+y_3)(z_1+z_2+z_3)} \cdot \triangle.$$

A proof of this elegant formula can be found in [1, VII] or [3]. A direct application of this formula yields the area of triangle $\mathcal{T}_\epsilon(t)$.

(10) The area of triangle $\mathcal{T}_\epsilon(t)$ is

$$\frac{3\sqrt{3}\epsilon}{4}\left(t - \frac{\tau_\epsilon + \tau_{-\epsilon}}{3}\right)\left(t - \frac{\tau_\epsilon - \tau_{-\epsilon}}{3}\right)\triangle.$$

Statement (8) follows immediately from this formula and (9B).

(11) $\mathcal{T}_\epsilon(t)$ has the same area as ABC if and only if $t = 0$ or $\frac{2\tau_\epsilon}{3}$. In fact, the two triangles $\mathcal{T}_+(\frac{2\tau_+}{3})$ and $\mathcal{T}_-(\frac{2\tau_-}{3})$ are symmetric with respect to the centroid. See Figures 8A and 8B.

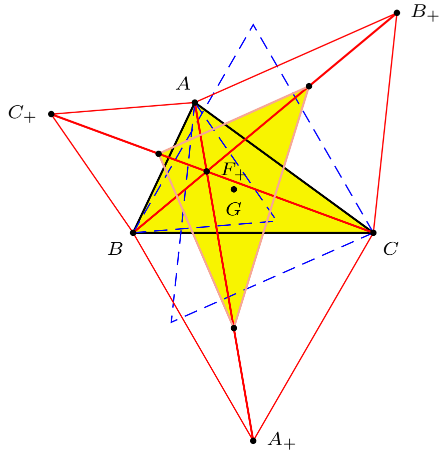


Figure 8A

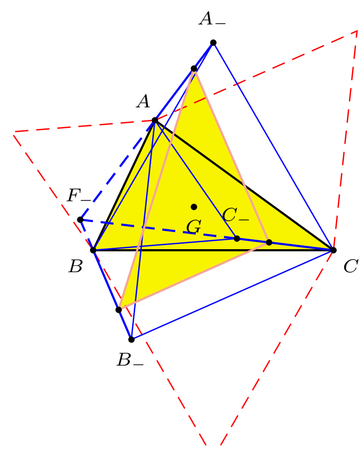


Figure 8B

6. Kiepert hyperbola and Steiner ellipse

The existence of the line $\mathcal{L}_-$ (see §3) shows that the internal bisectors of the angles A_+AA_-, B_+BB_-, and C_+CC_- are parallel. These bisectors contain the the apexes A', B', C' of isosceles triangles constructed inwardly on the sides with the same base angle given by (†). It is well known that $A'B'C'$ and ABC are perspective at a point on the Kiepert hyperbola, the rectangular circum-hyperbola through the orthocenter and the centroid. This perspector is necessarily an infinite point (of an asymptote of the hyperbola). In other words, the line $\mathcal{L}_-$ is parallel to an asymptote of this rectangular hyperbola.

(12) The lines $\mathcal{L}_\pm$ are the parallels through G to the asymptotes of the Kiepert hyperbola.

(13) It is also known that the asymptotes of the Kiepert hyperbola are parallel to the axes of the Steiner in-ellipse, (see [4]), the ellipse that touches the sides of triangle ABC at their midpoints, with center at the centroid G. See Figure 9.

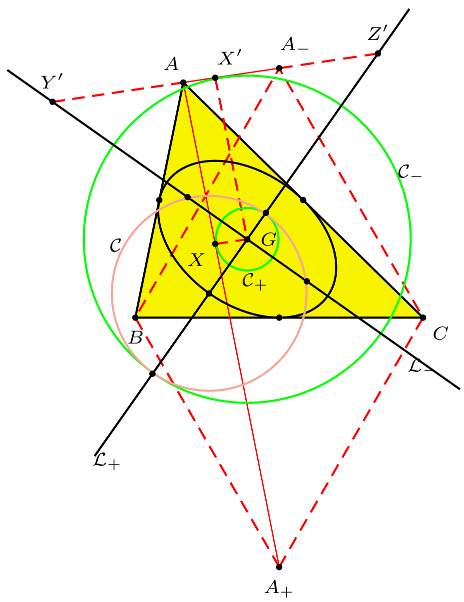


Figure 9

(14) The Steiner in-ellipse has major and minor axes of lengths $\frac{\tau_+ \pm \tau_-}{3}$. From this, we have the following construction of its foci. See Figure 9.

- Construct the concentric circles $\mathcal{C}_\pm$ at G through $A_\epsilon(\frac{\tau_\epsilon}{3})$.
- Construct a circle $\mathcal{C}$ with center on $\mathcal{L}_+$ tangent to the circles $\mathcal{C}_+$ internally and $\mathcal{C}_-$ externally. There are two such circles; any one of them will do.
- The intersections of the circle $\mathcal{C}$ with the line $\mathcal{L}_-$ are the foci of Steiner in-ellipse.

We conclude by recording the homogeneous barycentric coordinates of the two foci of the Steiner in-ellipse. Let

$$Q = a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2.$$

The line $\mathcal{L}_-$ containing the two foci has infinite point

$$\begin{aligned} I_-^\infty = ((b-c)(a(a+b+c) - (b^2+bc+c^2) - \sqrt{Q}), \\ (c-a)(b(a+b+c) - (c^2+ca+a^2) - \sqrt{Q}), \\ (a-b)(c(a+b+c) - (a^2+ab+b^2) - \sqrt{Q})). \end{aligned}$$

As a vector, this has square length $2\sqrt{Q}(f+g\sqrt{Q})$, where

$$f = \sum_{\text{cyclic}} a^6 - bc(b^4+c^4) + a^2bc(ab+ac-bc),$$

$$g = \sum_{\text{cyclic}} a^4 - bc(b^2+c^2-a^2).$$

Since the square distance from the centroid to each of the foci is $\frac{1}{9}\sqrt{Q}$, these two foci are the points

$$G \pm \frac{1}{3\sqrt{2(f+g\sqrt{Q})}} I_-^\infty.$$

References

[1] O. Bottema, *Hoofdstukken uit de Elementaire Meetkunde*, 2nd ed. 1987, Epsilon Uitgaven, Utrecht.

[2] O. Bottema, Verscheidenheden LV: Zo maar wat in een driehoek, *Euclides* 39 (1963/64) 129–137; reprinted in *Verscheidenheden*, 93–101, Groningen, 1978.

[3] O. Bottema, On the area of a triangle in barycentric coordinates, *Crux Mathematicorum*, 8 (1982) 228–231.

[4] J. H. Conway, Hyacinthos message 1237, August 18, 2000.

[5] F. M. van Lamoen, Napoleon triangles and Kiepert perspectors, *Forum Geom.*, 3 (2003) 65–71; Dutch version, *Euclides*, 77 (2002) 182–187.

[6] P. Yiu, The use of homogeneous barycentric coordinates in plane euclidean geometry, *J. Math. Edu. Sci. Technol.*, 31 (2000) 569–578.

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida, 33431-0991, USA

E-mail address: yiu@fau.edu

Forum Geometricorum
Volume 3 (2003) 83–93.

FORUM GEOM
ISSN 1534-1178

Triangle Centers Associated with the Malfatti Circles

Milorad R. Stevanović

Abstract. Various formulae for the radii of the Malfatti circles of a triangle are presented. We also express the radii of the excircles in terms of the radii of the Malfatti circles, and give the coordinates of some interesting triangle centers associated with the Malfatti circles.

1. The radii of the Malfatti circles

The Malfatti circles of a triangle are the three circles inside the triangle, mutually tangent to each other, and each tangent to two sides of the triangle. See Figure 1. Given a triangle ABC, let a, b, c denote the lengths of the sides BC, CA, AB, s the semiperimeter, I the incenter, and r its inradius. The radii of the Malfatti circles of triangle ABC are given by

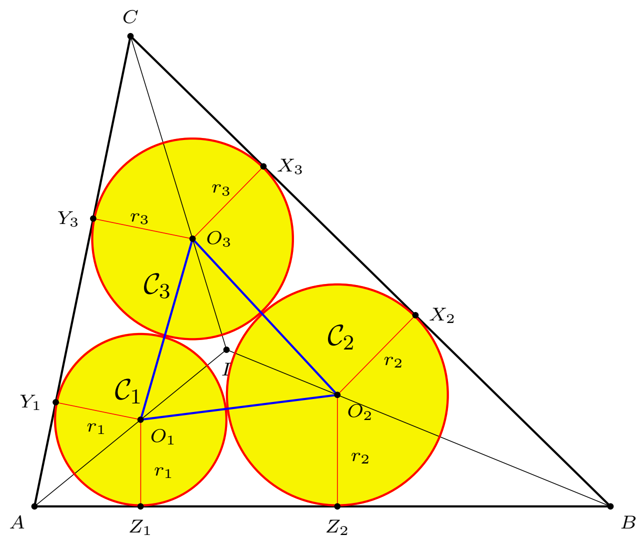


Figure 1

$$
\begin{aligned}
r_1 &= \frac{r}{2(s-a)}\left(s - r - (IB + IC - IA)\right), \\
r_2 &= \frac{r}{2(s-b)}\left(s - r - (IC + IA - IB)\right), \qquad (1)\\
r_3 &= \frac{r}{2(s-c)}\left(s - r - (IA + IB - IC)\right).
\end{aligned}
$$

Publication Date: March 24, 2003. Communicating Editor: Paul Yiu.
The author is grateful to the referee and the editor for their valuable comments and helps.

According to F.G.-M. [1, p.729], these results were given by Malfatti himself, and were published in [7] after his death. See also [6]. Another set of formulae give the same radii in terms of a, b, c and r:

$$\begin{aligned} r_1 =& \frac{(IB + r - (s-b))(IC + r - (s-c))}{2(IA + r - (s-a))}, \\ r_2 =& \frac{(IC + r - (s-c))(IA + r - (s-a))}{2(IB + r - (s-b))}, \\ r_3 =& \frac{(IA + r - (s-a))(IB + r - (s-b))}{2(IC + r - (s-c))}. \end{aligned} \tag{2}$$

These easily follow from (1) and the following formulae that express the radii r_1, r_2, r_3 in terms of r and trigonometric functions:

$$\begin{aligned} r_1 =& \frac{\left(1 + \tan\frac{B}{4}\right)\left(1 + \tan\frac{C}{4}\right)}{1 + \tan\frac{A}{4}} \cdot \frac{r}{2}, \\ r_2 =& \frac{\left(1 + \tan\frac{C}{4}\right)\left(1 + \tan\frac{A}{4}\right)}{1 + \tan\frac{B}{4}} \cdot \frac{r}{2}, \\ r_3 =& \frac{\left(1 + \tan\frac{A}{4}\right)\left(1 + \tan\frac{B}{4}\right)}{1 + \tan\frac{C}{4}} \cdot \frac{r}{2}. \end{aligned} \tag{3}$$

These can be found in [10]. They can be used to obtain the following formula which is given in [2, pp.103–106]. See also [12].

$$\frac{2}{r} = \frac{1}{\sqrt{r_1 r_2}} + \frac{1}{\sqrt{r_2 r_3}} + \frac{1}{\sqrt{r_1 r_2}} - \sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}}. \tag{4}$$

2. Exradii in terms of Malfatti radii

Antreas P. Hatzipolakis [3] asked for the exradii r_a, r_b, r_c of triangle ABC in terms of the Malfatti radii r_1, r_2, r_3 and the inradius r.

Proposition 1.

$$\begin{aligned} r_a - r_1 =& \frac{\frac{2}{r} - \frac{1}{\sqrt{r_2 r_3}}}{\left(\frac{2}{r} - \frac{1}{\sqrt{r_3 r_1}}\right)\left(\frac{2}{r} - \frac{1}{\sqrt{r_1 r_2}}\right)}, \\ r_b - r_2 =& \frac{\frac{2}{r} - \frac{1}{\sqrt{r_3 r_1}}}{\left(\frac{2}{r} - \frac{1}{\sqrt{r_1 r_2}}\right)\left(\frac{2}{r} - \frac{1}{\sqrt{r_2 r_3}}\right)}, \\ r_c - r_3 =& \frac{\frac{2}{r} - \frac{1}{\sqrt{r_1 r_2}}}{\left(\frac{2}{r} - \frac{1}{\sqrt{r_2 r_3}}\right)\left(\frac{2}{r} - \frac{1}{\sqrt{r_3 r_1}}\right)}. \end{aligned} \tag{5}$$

Proof. For convenience we write

$$t_1 := \tan\frac{A}{4}, \qquad t_2 := \tan\frac{B}{4}, \qquad t_3 := \tan\frac{C}{4}.$$

Note that from $\tan\left(\frac{A}{4}+\frac{B}{4}+\frac{C}{4}\right)=1$, we have

$$1-t_1-t_2-t_3-t_1t_2-t_2t_3-t_3t_1+t_1t_2t_3=0. \tag{6}$$

From (3) we obtain

$$\begin{aligned}\frac{2}{r}-\frac{1}{\sqrt{r_2r_3}} &= \frac{t_1}{1+t_1}\cdot\frac{2}{r},\\ \frac{2}{r}-\frac{1}{\sqrt{r_3r_1}} &= \frac{t_2}{1+t_2}\cdot\frac{2}{r},\\ \frac{2}{r}-\frac{1}{\sqrt{r_1r_2}} &= \frac{t_3}{1+t_3}\cdot\frac{2}{r}.\end{aligned} \tag{7}$$

For the exradius r_a, we have

$$r_a=\frac{s}{s-a}\cdot r=\cot\frac{B}{2}\cot\frac{C}{2}\cdot r=\frac{(1-t_2^2)(1-t_3^2)}{4t_2t_3}\cdot r.$$

It follows that

$$\begin{aligned}r_a-r_1 &= (1+t_2)(1+t_3)\cdot\frac{r}{2}\left(\frac{(1-t_2)(1-t_3)}{2t_2t_3}-\frac{1}{1+t_1}\right)\\ &= (1+t_2)(1+t_3)\cdot\frac{r}{2}\cdot\frac{(1+t_1)(1-t_2)(1-t_3)-2t_2t_3}{2t_2t_3(1+t_1)}\\ &= (1+t_2)(1+t_3)\cdot\frac{r}{2}\cdot\frac{2t_1}{2t_2t_3(1+t_1)} \qquad \text{(from (6))}\\ &= \frac{t_1}{1+t_1}\cdot\frac{1+t_2}{t_2}\cdot\frac{1+t_3}{t_3}\cdot\frac{r}{2}.\end{aligned}$$

Now the result follows from (7). □

Note that with the help of (4), the exradii r_a, r_b, r_c can be explicitly written in terms of the Malfatti radii r_1, r_2, r_3. We present another formula useful in the next sections in the organization of coordinates of triangle centers.

Proposition 2.

$$\frac{1}{r_1}-\frac{1}{r_a}=\frac{a}{rs}\cdot\frac{(1+\cos\frac{B}{2})(1+\cos\frac{C}{2})}{1+\cos\frac{A}{2}}.$$

3. Triangle centers associated with the Malfatti circles

Let A' be the point of tangency of the Malfatti circles $\mathcal{C}_2$ and $\mathcal{C}_3$. Similarly define B' and C'. It is known ([4, p.97]) that triangle $A'B'C'$ is perspective with ABC at the *first Ajima-Malfatti point* X_{179}. See Figure 3. We work out the details here and construct a few more triangle centers associated with the Malfatti circles. In particular, we find two new triangle centers P_+ and P_- which divide the incenter I and the first Ajima-Malfatti point harmonically.

3.1. *The centers of the Malfatti circles.* We begin with the coordinates of the centers of the Malfatti circles.

Since O_1 divides the segment AI_a in the ratio $AO_1 : O_1I_a = r_1 : r_a - r_1$, we have $\frac{O_1}{r_1} = \left(\frac{1}{r_1} - \frac{1}{r_a}\right) A + \frac{1}{r_a} \cdot I_A$. With $r_a = \frac{rs}{s-a}$ we rewrite the absolute barycentric coordinates of O_1, along with those of O_2 and O_3, as follows.

$$
\begin{aligned}
\frac{O_1}{r_1} &= \left(\frac{1}{r_1} - \frac{1}{r_a}\right) A + \frac{s-a}{rs} \cdot I_a, \\
\frac{O_2}{r_2} &= \left(\frac{1}{r_2} - \frac{1}{r_b}\right) B + \frac{s-b}{rs} \cdot I_b, \\
\frac{O_3}{r_3} &= \left(\frac{1}{r_3} - \frac{1}{r_c}\right) C + \frac{s-c}{rs} \cdot I_c.
\end{aligned}
\tag{8}
$$

From these expressions we have, in homogeneous barycentric coordinates,

$$
\begin{aligned}
O_1 &= \left(2rs\left(\frac{1}{r_1} - \frac{1}{r_a}\right) - a : b : c\right), \\
O_2 &= \left(a : 2rs\left(\frac{1}{r_2} - \frac{1}{r_b}\right) - b : c\right), \\
O_3 &= \left(a : b : 2rs\left(\frac{1}{r_3} - \frac{1}{r_c}\right) - c\right).
\end{aligned}
$$

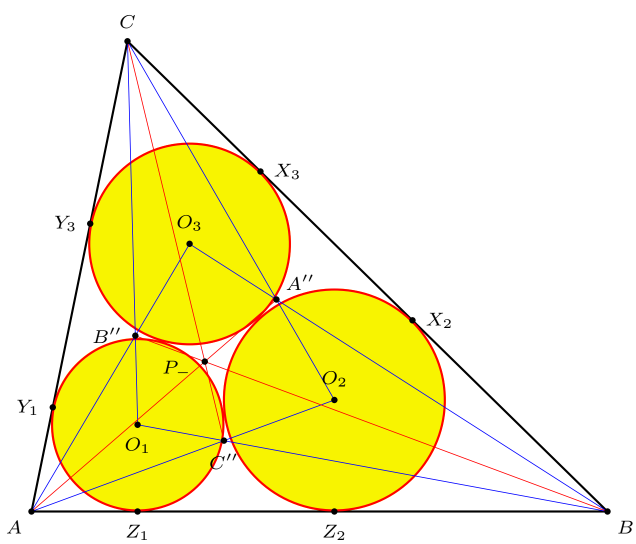


Figure 2

3.2. *The triangle center P_-.* It is clear that $O_1O_2O_3$ is perspective with ABC at the incenter $(a:b:c)$. However, it also follows that if we consider

$$A'' = BO_3 \cap CO_2, \quad B'' = CO_1 \cap AO_3, \quad C'' = AO_2 \cap BO_1,$$

then triangle $A''B''C''$ is perspective with ABC at

$$\begin{aligned}
P_- &= \left(2rs\left(\frac{1}{r_1} - \frac{1}{r_a}\right) - a : 2rs\left(\frac{1}{r_2} - \frac{1}{r_b}\right) - b : 2rs\left(\frac{1}{r_3} - \frac{1}{r_c}\right) - c\right) \\
&= \left(\frac{1}{r_1} - \frac{1}{r_a} - \frac{a}{2rs} : \frac{1}{r_2} - \frac{1}{r_b} - \frac{b}{2rs} : \frac{1}{r_3} - \frac{1}{r_c} - \frac{c}{2rs}\right) \\
&= \left(a\left(\frac{(1+\cos\frac{B}{2})(1+\cos\frac{C}{2})}{1+\cos\frac{A}{2}} - \frac{1}{2}\right) : \cdots : \cdots\right)
\end{aligned} \tag{9}$$

by Proposition 2. See Figure 2.

Remark. The point P_- appears in [5] as the *first Malfatti-Rabinowitz point* X_{1142}.

3.3. *The first Ajima-Malfatti point.* For the points of tangency of the Malfatti circles, note that A' divides O_2O_3 in the ratio $O_2A' : A'O_3 = r_2 : r_3$. We have, in absolute barycentric coordinates,

$$\left(\frac{1}{r_2} + \frac{1}{r_3}\right)A' = \frac{O_2}{r_2} + \frac{O_3}{r_3} = \frac{a}{rs} \cdot A + \left(\frac{1}{r_2} - \frac{1}{r_b}\right)B + \left(\frac{1}{r_3} - \frac{1}{r_c}\right)C;$$

similarly for B' and C'. In homogeneous coordinates,

$$\begin{aligned}
A' &= \left(\frac{a}{rs} : \frac{1}{r_2} - \frac{1}{r_b} : \frac{1}{r_3} - \frac{1}{r_c}\right), \\
B' &= \left(\frac{1}{r_1} - \frac{1}{r_a} : \frac{b}{rs} : \frac{1}{r_3} - \frac{1}{r_c}\right), \\
C' &= \left(\frac{1}{r_1} - \frac{1}{r_a} : \frac{1}{r_2} - \frac{1}{r_b} : \frac{c}{rs}\right).
\end{aligned} \tag{10}$$

From these, it is clear that $A'B'C'$ is perspective with ABC at

$$\begin{aligned}
P &= \left(\frac{1}{r_1} - \frac{1}{r_a} : \frac{1}{r_2} - \frac{1}{r_b} : \frac{1}{r_3} - \frac{1}{r_c}\right) \\
&= \left(\frac{a(1+\cos\frac{B}{2})(1+\cos\frac{C}{2})}{1+\cos\frac{A}{2}} : \frac{b(1+\cos\frac{C}{2})(1+\cos\frac{A}{2})}{1+\cos\frac{B}{2}} \right. \\
&\qquad \left. : \frac{c(1+\cos\frac{A}{2})(1+\cos\frac{B}{2})}{1+\cos\frac{C}{2}}\right) \\
&= \left(\frac{a}{\left(1+\cos\frac{A}{2}\right)^2} : \frac{b}{\left(1+\cos\frac{B}{2}\right)^2} : \frac{c}{\left(1+\cos\frac{C}{2}\right)^2}\right)
\end{aligned} \tag{11}$$

by Proposition 2. The point P appears as X_{179} in [4, p.97], with trilinear coordinates

$$\left(\sec^4 \frac{A}{4} : \sec^4 \frac{B}{4} : \sec^4 \frac{C}{4}\right)$$

computed by Peter Yff, and is named the *first Ajima-Malfatti point*. See Figure 3.

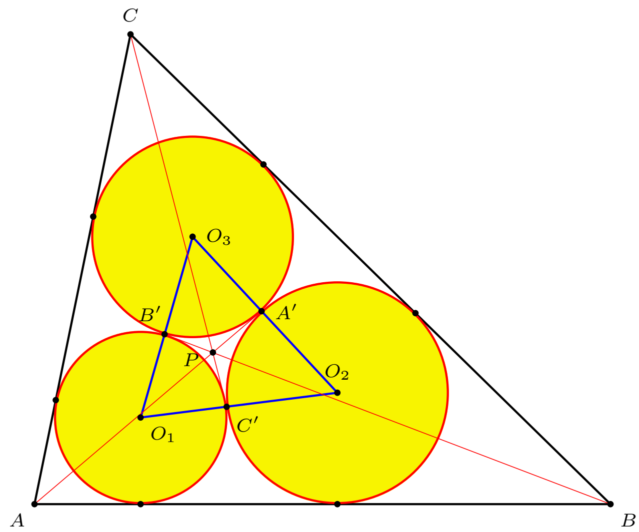


Figure 3

3.4. *The triangle center* P_+. Note that the circle through A', B', C' is orthogonal to the Malfatti circles. It is the radical circle of the Malfatti circles, and is the incircle of $O_1O_2O_3$. The lines O_1A', O_2B', O_3C' are concurrent at the Gergonne point of triangle $O_1O_2O_3$. See Figure 4. As such, this is the point P_+ given by

$$\begin{aligned}
&\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)P_+ = \frac{O_1}{r_1}+\frac{O_2}{r_2}+\frac{O_3}{r_3}\\
=&\left(\frac{1}{r_1}-\frac{1}{r_a}\right)A+\frac{I_a}{r_a}+\left(\frac{1}{r_2}-\frac{1}{r_b}\right)B+\frac{I_b}{r_b}+\left(\frac{1}{r_3}-\frac{1}{r_c}\right)C+\frac{I_c}{r_c}\\
=&\left(\frac{1}{r_1}-\frac{1}{r_a}\right)A+\left(\frac{1}{r_2}-\frac{1}{r_b}\right)B+\left(\frac{1}{r_3}-\frac{1}{r_c}\right)C+\frac{I}{r}\\
=&\left(\frac{1}{r_1}-\frac{1}{r_a}\right)A+\left(\frac{1}{r_2}-\frac{1}{r_b}\right)B+\left(\frac{1}{r_3}-\frac{1}{r_c}\right)C+\frac{1}{2rs}(aA+bB+cC)\\
=&\left(\frac{1}{r_1}-\frac{1}{r_a}+\frac{a}{2rs}\right)A+\left(\frac{1}{r_2}-\frac{1}{r_b}+\frac{b}{2rs}\right)B+\left(\frac{1}{r_3}-\frac{1}{r_c}+\frac{c}{2rs}\right)C.
\end{aligned}$$

It follows that in homogeneous coordinates,

$$\begin{aligned} P_+ &= \left(\frac{1}{r_1} - \frac{1}{r_a} + \frac{a}{2rs} : \frac{1}{r_2} - \frac{1}{r_b} + \frac{b}{2rs} : \frac{1}{r_3} - \frac{1}{r_c} + \frac{c}{2rs}\right) \\ &= \left(a\left(\frac{(1+\cos\frac{B}{2})(1+\cos\frac{C}{2})}{1+\cos\frac{A}{2}} + \frac{1}{2}\right) : \cdots : \cdots\right) \end{aligned} \tag{12}$$

by Proposition 2.

Proposition 3. *The points P_+ and P_- divide the segment IP harmonically.*

Proof. This follows from their coordinates given in (12), (9), and (11). □

From the coordinates of P, P_+ and P_-, it is easy to see that P_+ and P_- divide the segment IP harmonically.

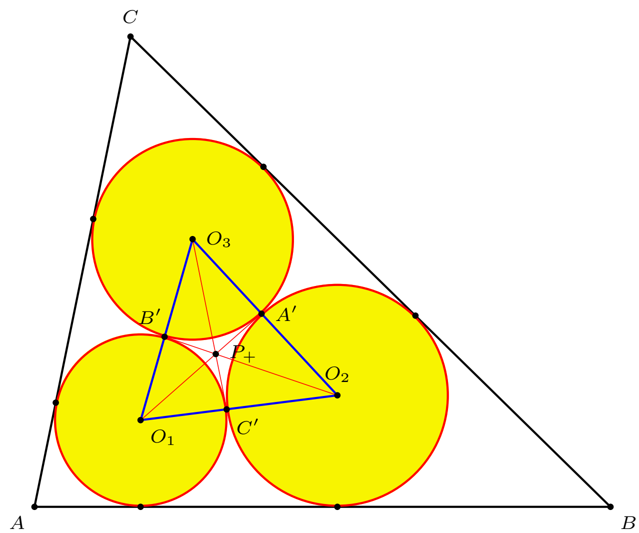


Figure 4

3.5. *The triangle center Q.* Let the Malfatti circle $\mathcal{C}_1$ touch the sides CA and AB at Y_1 and Z_1 respectively. Likewise, let $\mathcal{C}_2$ touch AB and BC at Z_2 and X_2, $\mathcal{C}_3$ touch BC and CA at X_3 and Y_3 respectively. Denote by X, Y, Z the midpoints of the segments X_2X_3, Y_3Y_1, Z_1Z_2 respectively. Stanley Rabinowitz [9] asked if the lines AX, BY, CZ are concurrent. We answer this in the affirmative.

Proposition 4. *The lines AX, BY, CZ are concurrent at a point Q with homogeneous barycentric coordinates*

$$\left(\tan\frac{A}{4} : \tan\frac{B}{4} : \tan\frac{C}{4}\right). \tag{13}$$

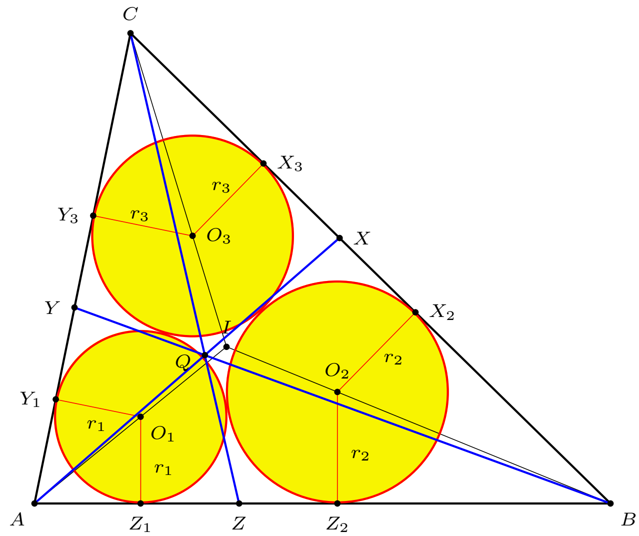


Figure 5

Proof. In Figure 5, we have

$$\begin{aligned} BX =& \frac{1}{2}(a + BX_2 - X_3C) \\ =& \frac{1}{2}\left(a + \frac{r_2}{r}(s-b) - \frac{r_3}{r}(s-c)\right) \\ =& \frac{1}{2}(a + IB - IC) \qquad \text{(from (1))} \\ =& \frac{1}{2}\left(2R\sin A + \frac{r}{\sin\frac{B}{2}} - \frac{r}{\sin\frac{C}{2}}\right) \\ =& 4R\sin\frac{A}{2}\cos\frac{B}{4}\sin\frac{C}{4}\cos\frac{B+C}{4} \end{aligned}$$

by making use of the formula

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Similarly,

$$XC = \frac{1}{2}(a - BX_2 + X_3C) = 4R\sin\frac{A}{2}\sin\frac{B}{4}\cos\frac{C}{4}\cos\frac{B+C}{4}.$$

It follows that

$$\frac{BX}{XC} = \frac{\cos\frac{B}{4}\sin\frac{C}{4}}{\sin\frac{B}{4}\cos\frac{C}{4}} = \frac{\tan\frac{C}{4}}{\tan\frac{B}{4}}.$$

Likewise,

$$\frac{CY}{YA} = \frac{\tan\frac{A}{4}}{\tan\frac{C}{4}} \quad \text{and} \quad \frac{AZ}{ZB} = \frac{\tan\frac{B}{4}}{\tan\frac{A}{4}},$$

and it follows from Ceva's theorem that AX, BY, CZ are concurrent since

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

In fact, we can easily identify the homogeneous barycentric coordinates of the intersection Q as given in (13) above since those of X, Y, Z are

$$X = \left(0 : \tan\frac{B}{4} : \tan\frac{C}{4}\right),$$
$$Y = \left(\tan\frac{A}{4} : 0 : \tan\frac{C}{4}\right),$$
$$Z = \left(\tan\frac{A}{4} : \tan\frac{B}{4} : 0\right).$$

□

Remark. The coordinates of Q can also be written as

$$\left(\frac{\sin\frac{A}{2}}{1+\cos\frac{A}{2}} : \frac{\sin\frac{B}{2}}{1+\cos\frac{B}{2}} : \frac{\sin\frac{C}{2}}{1+\cos\frac{C}{2}}\right)$$

or

$$\left(\frac{a}{(1+\cos\frac{A}{2})\cos\frac{A}{2}} : \frac{b}{(1+\cos\frac{B}{2})\cos\frac{B}{2}} : \frac{c}{(1+\cos\frac{C}{2})\cos\frac{C}{2}}\right).$$

3.6. *The radical center of the Malfatti circles.* Note that the common tangent of $\mathcal{C}_2$ and $\mathcal{C}_3$ at A' passes through X. This means that $A'X$ is perpendicular to O_2O_3 at A'. This line therefore passes through the incenter I' of $O_1O_2O_3$. Now, the homogeneous coordinates of A' and X can be rewritten as

$$A' = \left(\frac{a}{(1+\cos\frac{A}{2})(1+\cos\frac{B}{2})(1+\cos\frac{C}{2})} : \frac{b}{(1+\cos\frac{B}{2})^2} : \frac{c}{(1+\cos\frac{C}{2})^2}\right),$$
$$X = \left(0 : \frac{b}{(1+\cos\frac{B}{2})\cos\frac{B}{2}} : \frac{c}{(1+\cos\frac{C}{2})\cos\frac{C}{2}}\right).$$

It is easy to verify that these two points lie on the line

$$\frac{(1+\cos\frac{A}{2})(\cos\frac{B}{2}-\cos\frac{C}{2})}{a}x - \frac{(1+\cos\frac{B}{2})\cos\frac{B}{2}}{b}y + \frac{(1+\cos\frac{C}{2})\cos\frac{C}{2}}{c}z = 0,$$

which also contains the point

$$\left(\frac{a}{1+\cos\frac{A}{2}} : \frac{b}{1+\cos\frac{B}{2}} : \frac{c}{1+\cos\frac{C}{2}}\right).$$

Similar calculations show that the latter point also lies on the lines $B'Y$ and $C'Z$. It is therefore the incenter I' of triangle $O_1O_2O_3$. See Figure 6. This point appears in [5] as X_{483} , the radical center of the Malfatti circles.

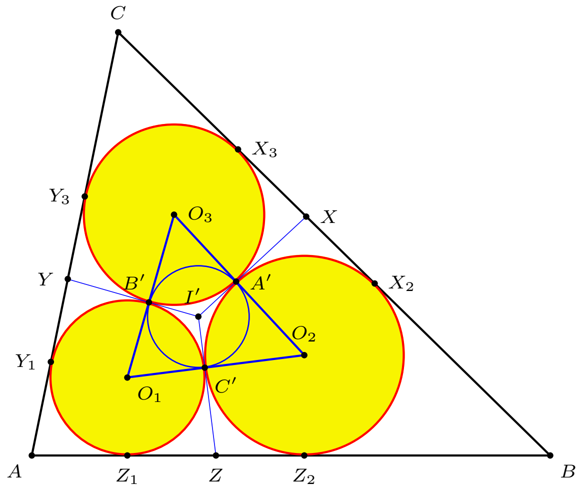


Figure 6

Remarks. (1) The line joining Q and I' has equation

$$\frac{(1+\cos\frac{A}{2})(\cos\frac{B}{2}-\cos\frac{C}{2})}{\sin\frac{A}{2}}x+\frac{(1+\cos\frac{B}{2})(\cos\frac{C}{2}-\cos\frac{A}{2})}{\sin\frac{B}{2}}y$$
$$+\frac{(1+\cos\frac{C}{2})(\cos\frac{A}{2}-\cos\frac{B}{2})}{\sin\frac{C}{2}}z=0.$$

This line clearly contains the point $\left(\sin\frac{A}{2}:\sin\frac{B}{2}:\sin\frac{C}{2}\right)$, which is the point X_{174}, the Yff center of congruence in [4, pp.94–95].

(2) According to [4], the triangle $A'B'C'$ in §3.3 is also perspective with the excentral triangle. This is because cevian triangles and anticevian triangles are always perspective. The perspector

$$\left(\frac{a\left((2+\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2})^2+\cos\frac{A}{2}(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}-(2+\cos\frac{A}{2})^2)\right)}{1+\cos\frac{A}{2}}:\cdots:\cdots\right)$$

is named the *second Ajima-Malfatti* point X_{180}. For the same reason, the triangle XYZ in §3.5 is also perspective with the excentral triangle. The perspector is the point

$$\left(a\left(-\cos\frac{A}{2}\left(1+\cos\frac{A}{2}\right)+\cos\frac{B}{2}\left(1+\cos\frac{B}{2}\right)+\cos\frac{C}{2}\left(1+\cos\frac{C}{2}\right)\right):\cdots:\cdots\right).$$

This point and the triangle center P_+ apparently do not appear in the current edition of [5].

Editor's endnote. The triangle center Q in §3.5 appears in [5] as the *second Malfatti-Rabinowitz point* X_{1143}. Its coordinates given by the present editor [13] were not correct owing to a mistake in a sign in the calculations. In the notations of [13], if

α, β, γ are such that

$$\sin^2\alpha = \frac{a}{s}, \quad \sin^2\beta = \frac{b}{s}, \quad \sin^2\gamma = \frac{c}{s},$$

and $\lambda = \frac{1}{2}(\alpha+\beta+\gamma)$, then the homogeneous barycentric coordinates of Q are

$$(\cot(\lambda-\alpha) : \cot(\lambda-\beta) : \cot(\lambda-\gamma)).$$

These are equivalent to those given in (13) in simpler form.

References

[1] F. G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, Paris, 1991.
[2] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, Charles Babbage Research Centre, Winnipeg, 1989.,p.p.103-106.
[3] A. P. Hatzipolakis, Hyacinthos message 2375, January 8, 2001.
[4] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.
[5] C. Kimberling, *Encyclopedia of Triangle Centers*, August 22, 2002 edition, available at http://www2.evansville.edu/ck6/encyclopedia/; February 26, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[6] U. J. Knisely, Solution to problem Problem 186. *The Mathematical Visitor*, 1 (1881) 189; reprinted in [8, pp.148–149].
[7] Malfatti, *Annales mathématiques de Gergonne*, 1 (1811) 347.
[8] S. Rabinowitz, *Problems and solutions from the Mathematical Visitor, 1877–1896*, MathPro Press, Westford, MA, 1996.
[9] S. Rabinowitz, Hyacinthos message 4610, December 30, 2001.
[10] E. B. Seitz, Solution to problem 186. *The Mathematical Visitor*, 1 (1881) 190; reprinted in [8, p.150].
[11] M. R. Stevanović, Hyacinthos message 4613, December 31, 2001.
[12] G. Tsintsifas, Solution to problem 618, *Crux Math.*, 8 (1982) 82–85.
[13] P. Yiu, Hyacinthos message 4615, December 30, 2001.

Milorad R. Stevanović: Technical Faculty, Svetog Save 65, 32000 Čačak, Serbia, Yugoslavia
E-mail address: milmath@ptt.yu

Forum Geometricorum
Volume 3 (2003) 95–100.

FORUM GEOM
ISSN 1534-1178

The Lucas Circles and the Descartes Formula

Wilfred Reyes

Abstract. We determine the radii of the three circles each tangent to the circumcircle of a given triangle at a vertex, and mutually tangent to each other externally. The calculations are then reversed to give the radii of the two Soddy circles associated with three circles tangent to each other externally.

1. The Lucas circles

Consider a triangle ABC with circumcircle $\mathcal{C}$. We set up a coordinate system with the circumcenter O at the origin and A, B, C represented by complex numbers of moduli R, the circumradius. If the lengths of the sides BC, CA, AB are a, b, c respectively, then

$$||A-B|| = c \quad \text{and} \quad \langle A, B\rangle = R^2 - \frac{c^2}{2}. \tag{1}$$

Analogous relations hold for the pairs B, C and C, A. Let $0 \leq \alpha < R$, and consider the circle $\mathcal{C}_A(\alpha)$ with center $\frac{R-\alpha}{R}\cdot A$ and radius α. This is internally tangent to the circumcircle at A, and is the image of $\mathcal{C}$ under the homothety $\mathsf{h}(A, \frac{\alpha}{R})$. See Figure 1. For real numbers β, γ satisfying $0 \leq \beta, \gamma < R$, we consider the circles $\mathcal{C}_B(\beta)$ and $\mathcal{C}_C(\gamma)$ analogously defined. Now, the circles $\mathcal{C}_A(\alpha)$ and $\mathcal{C}_B(\beta)$ are tangent externally if and only if

$$\left|\left|\frac{R-\alpha}{R}A - \frac{R-\beta}{R}B\right|\right| = \alpha+\beta.$$

This is equivalent, by a simple application of (1), to

$$c^2 = \frac{4\alpha\beta}{(R-\alpha)(R-\beta)}.$$

Therefore, the three circles $\mathcal{C}_A(\alpha)$, $\mathcal{C}_B(\beta)$ and $\mathcal{C}_C(\gamma)$ are tangent externally to each other if and only if

$$a^2 = \frac{4R^2\beta\gamma}{(R-\beta)(R-\gamma)}, \quad b^2 = \frac{4R^2\gamma\alpha}{(R-\gamma)(R-\alpha)}, \quad c^2 = \frac{4R^2\alpha\beta}{(R-\alpha)(R-\beta)}. \tag{2}$$

Publication Date: March 31, 2003. Communicating Editor: Paul Yiu.
The author thanks Professor Paul Yiu for his helps in the preparation of this paper.

These equations can be solved for the radii α, β, and γ in terms of a, b, c, and R. In fact, multiplying the equations in (2), we obtain

$$abc = \frac{8R^3\alpha\beta\gamma}{(R-\alpha)(R-\beta)(R-\gamma)}.$$

Consequently,

$$\frac{\alpha}{R-\alpha} = \frac{bc}{2Ra}, \quad \frac{\beta}{R-\beta} = \frac{ca}{2Rb}, \quad \frac{\gamma}{R-\gamma} = \frac{ab}{2Rc}.$$

From these, we obtain

$$\alpha = \frac{bc}{2Ra+bc}\cdot R, \quad \beta = \frac{ca}{2Rb+ca}\cdot R, \quad \gamma = \frac{ab}{2Rc+ab}\cdot R. \tag{3}$$

Denote by $\triangle$ the area of triangle ABC, and h_a, h_b, h_c its three altitudes. We have $2\triangle = a\cdot h_a = b\cdot h_b = c\cdot h_c$. Since $abc = 4R\triangle$, the expression for α in (3) can be rewritten as

$$\frac{\alpha}{R} = \frac{abc}{2Ra^2+abc} = \frac{4R\triangle}{2Ra^2+4R\triangle} = \frac{2\triangle}{a^2+2\triangle} = \frac{a\cdot h_a}{a^2+a\cdot h_a} = \frac{h_a}{a+h_a}.$$

Therefore, the homothety $\mathsf{h}(A, \frac{\alpha}{R})$ is the one that contracts the square on the side BC (externally) into the inscribed square on this side. See Figure 1. The same is true for the other two circles. The three circles $\mathcal{C}_A(\alpha)$, $\mathcal{C}_B(\beta)$, $\mathcal{C}_C(\gamma)$ are therefore the Lucas circles considered in [3]. See Figure 2.

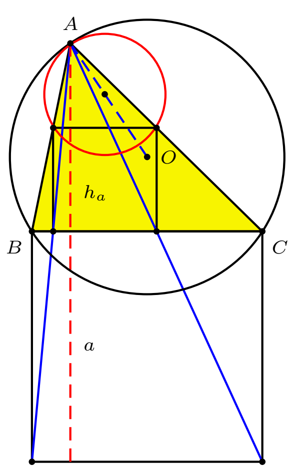


Figure 1

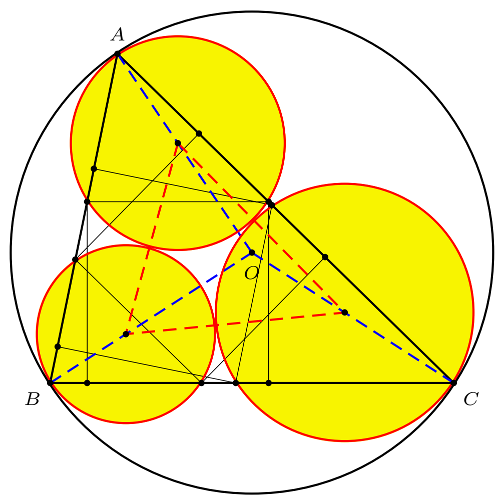


Figure 2

2. Another triad of circles

A simple modification of the above calculations shows that for positive numbers α', β', γ', the images of the circumcircle $\mathcal{C}$ under the homotheties $\mathsf{h}(A, -\frac{\alpha'}{R})$, $\mathsf{h}(B, -\frac{\beta'}{R})$ and $\mathsf{h}(C, -\frac{\gamma'}{R})$ (each tangent to $\mathcal{C}$ at a vertex) are tangent to each other if and only if

$$\alpha' = \frac{bc}{2Ra - bc} \cdot R, \quad \beta' = \frac{ca}{2Rb - ca} \cdot R, \quad \gamma' = \frac{ab}{2Rc - ab} \cdot R. \tag{4}$$

The tangencies are all external provided $2Ra - bc$, $2Rb - ca$ and $2Rc - ab$ are all positive. These quantities are essentially the excesses of the sides over the corresponding altitudes:

$$2Ra - bc = \frac{bc}{a}(a - h_a), \quad 2Rb - ca = \frac{ca}{b}(b - h_b), \quad 2Rc - ab = \frac{ab}{c}(c - h_c).$$

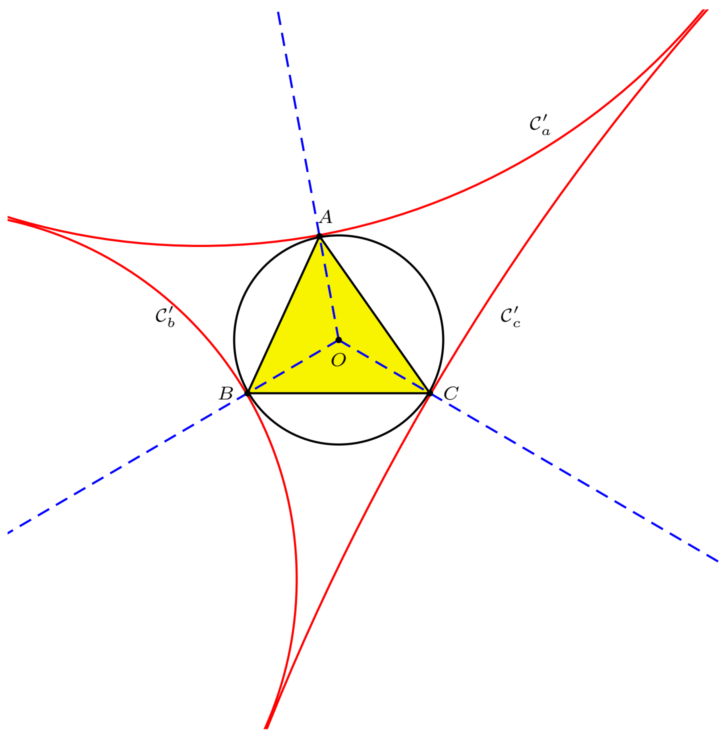


Figure 3

It may occur that one of them is negative. In that case, the tangencies with the corresponding circle are all internal. See Figure 4.

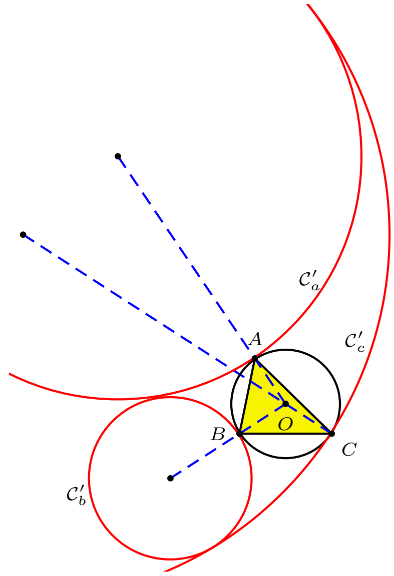


Figure 4

3. Inscribed squares

Consider the triad of circles in §2. The homothety $\mathsf{h}(A, -\frac{\alpha'}{R})$ transforms the square erected on BC on the same side of A into an inscribed square since $\frac{-\alpha'}{R} = \frac{-h_a}{a-h_a}$. See Figure 5.

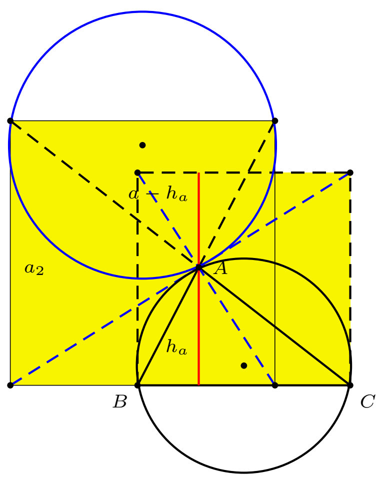


Figure 5

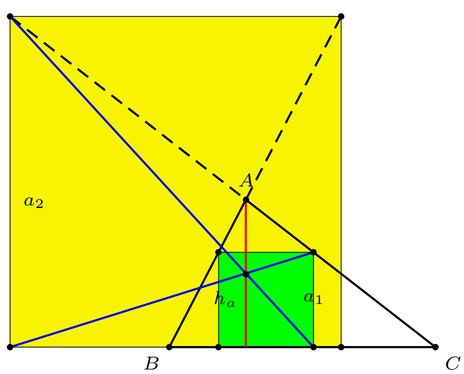


Figure 6

Denote by a_1 and a_2 the lengths of sides of the two inscribed squares on BC, under the homotheties $\mathsf{h}(A, \frac{\alpha}{R})$ and $\mathsf{h}(A, -\frac{\alpha'}{R})$ respectively, *i.e.*, $a_1 = \frac{\alpha}{R} \cdot a$ and

$a_2 = \frac{\alpha'}{R} \cdot a$. Making use of (3) and (4), we have

$$\frac{1}{a_1} + \frac{1}{a_2} = \left(\frac{1}{\alpha} + \frac{1}{\alpha'}\right)\frac{R}{a} = \frac{4a}{bc} \cdot \frac{R}{a} = \frac{a}{\triangle} = \frac{2}{h_a}.$$

This means that the altitude h_a is the harmonic mean of the lengths of the sides of the two inscribed squares on the side BC. See Figure 6.

4. The Descartes formula

We reverse the calculations in §§1,2 to give a proof of the Descartes formula. See, [2, pp.90–92]. Given three circles of radii α, β, γ tangent to each other externally, we determine the radii of the two Soddy circles tangent to each of them. See, for example, [1, pp.13–16]. We first seek the radius R of the circle tangent *internally* to each of them, the *outer* Soddy circle. Regard, in equation (3), R, a, b, c as unknowns, and write $\triangle$ for the area of the unknown triangle ABC whose vertices are the points of tangency. Thus, by Heron's formula,

$$16\triangle^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4. \tag{5}$$

In terms of $\triangle$, (3) can be rewritten as

$$\alpha = \frac{2\triangle}{a^2 + 2\triangle} \cdot R, \qquad \beta = \frac{2\triangle}{b^2 + 2\triangle} \cdot R, \qquad \gamma = \frac{2\triangle}{c^2 + 2\triangle} \cdot R,$$

or

$$a^2 = \frac{2(R-\alpha)\triangle}{\alpha}, \qquad b^2 = \frac{2(R-\beta)\triangle}{\beta}, \qquad c^2 = \frac{2(R-\gamma)\triangle}{\gamma}. \tag{6}$$

Substituting these into (5) and simplifying, we obtain

$$\begin{aligned}&\alpha^2\beta^2\gamma^2 + 2\alpha\beta\gamma(\beta\gamma + \gamma\alpha + \alpha\beta)R\\ &+(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 - 2\alpha^2\beta\gamma - 2\alpha\beta^2\gamma - 2\alpha\beta\gamma^2)R^2 = 0.\end{aligned}$$

Dividing throughout by $\alpha^2\beta^2\gamma^2 \cdot R^2$, we have

$$\frac{1}{R^2} + 2\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)\frac{1}{R} + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{2}{\alpha\beta} - \frac{2}{\beta\gamma} - \frac{2}{\gamma\alpha}\right) = 0.$$

Since $R > \alpha, \beta, \gamma$, we have

$$\frac{1}{R} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} - 2\sqrt{\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta}}.$$

This is positive if and only if

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{2}{\alpha\beta} - \frac{2}{\beta\gamma} - \frac{2}{\gamma\alpha} > 0. \tag{7}$$

This is the condition necessary and sufficient for the existence of a circle tangent *internally* to each of the given circles.

By reversing the calculations in §2, the radius of the circle tangent to the three given circles externally, the *inner* Soddy circle, is given by

$$\frac{1}{R'} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + 2\sqrt{\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta}}.$$

If condition (7) is not satisfied, both Soddy circles are tangent to each of the given circles externally.

References

[1] H. S. M. Coxeter, *Introduction to Geometry*, 1961; reprinted as Wiley classics, 1996.
[2] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, Charles Babbage Research Centre, Winnipeg, 1989.
[3] A. P. Hatzipolakis and P. Yiu, The Lucas circles, *Amer. Math. Monthly*, 108 (2001) 444–446.

Wilfred Reyes: Departamento de Ciencias Básicas, Universidad del Bío-Bío, Chillán, Chile
E-mail address: wreyes@ubiobio.cl

Forum Geometricorum
Volume 3 (2003) 101–104.

FORUM GEOM
ISSN 1534-1178

Similar Pedal and Cevian Triangles

Jean-Pierre Ehrmann

Abstract. The only point with similar pedal and cevian triangles, other than the orthocenter, is the isogonal conjugate of the Parry reflection point.

1. Introduction

We begin with notation. Let ABC be a triangle with sidelengths a, b, c, orthocenter H, and circumcenter O. Let K_A, K_B, K_C denote the vertices of the tangential triangle, O_A, O_B, O_C the reflections of O in A, B, C, and A_S, B_S, C_S the reflections of the vertices of A in BC, of B in CA, and of C in AB. Let

$M^* =$ isogonal conjugate of a point M;
$\overline{M} =$ inverse of M in the circumcircle;
$\measuredangle LL' =$ the measure, modulo π, of the directed angle of the lines L, L';
$S_A = bc\cos A = \frac{1}{2}(b^2+c^2-a^2)$, with S_B and S_C defined cyclically;
$x : y : z =$ barycentric coordinates relative to triangle ABC;
$\Gamma_A =$ circle with diameter K_AO_A, with circles Γ_B and Γ_C defined cyclically.

The circle Γ_A passes through the points B_S, C_S and is the locus of M such that $\measuredangle B_SMC_S = -2\measuredangle BAC$. An equation for Γ_A, in barycentrics, is

$$2S_A\left(a^2yz+b^2zx+c^2xy\right)+\left(b^2c^2x+2c^2S_Cy+2b^2S_Bz\right)(x+y+z)=0.$$

Consider a triangle $A'B'C'$, where A', B', C' lie respectively on the sidelines BC, CA, AB. The three circles $AB'C'$, $BC'A'$, $CA'B'$ meet in a point S called the Miquel point of $A'B'C'$. See [2, pp.131–135]. The point S (or $\overline{S}$) is the only point whose pedal triangle is directly (or indirectly) similar to $A'B'C'$.

The circles Γ_A, Γ_B, Γ_C have a common point T: the Parry reflection point, X_{399} in [3]; the three radical axes TA_S, TB_S, TC_S are the reflections with respect to a sideline of ABC of the parallel to the Euler line going through the opposite vertex. See [3, 4], and Figure 1. T lies on the circle $(O, 2R)$, on the Neuberg cubic, and is the antipode of O on the Stammler hyperbola. See [1].

2. Similar triangles

Let $A'B'C'$ be the cevian triangle of a point $P = p : q : r$.

Lemma 1. *The pedal and cevian triangles of P are directly (or indirectly) similar if and only if P (or $\overline{P}$) lies on the three circles $AB'C'$, $BC'A'$, $CA'B'$.*

Proof. This is an immediate consequence of the properties of the Miquel point above. □

Lemma 2. *A, B', C', P are concyclic if and only if P lies on the circle BCH.*

Publication Date: April 7, 2003. Communicating Editor: Clark Kimberling.

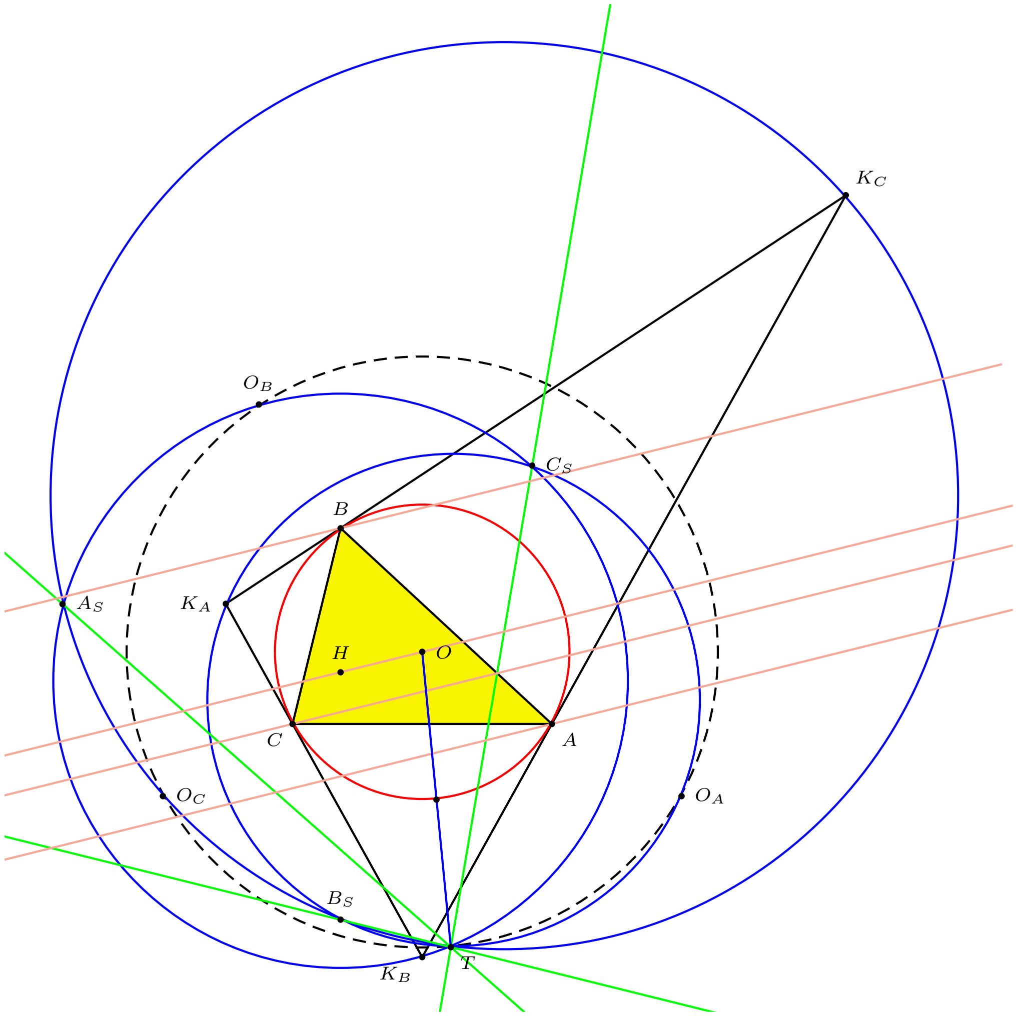


Figure 1

Proof. A, B', C' and P are concyclic $\Leftrightarrow \measuredangle B'PC' = \measuredangle B'AC' \Leftrightarrow \measuredangle BPC = \measuredangle BHC \Leftrightarrow P$ lies on the circle BCH. $\square$

Proposition 3. *The pedal and cevian triangles of P are directly similar only in the trivial case of $P = H$.*

Proof. By Lemma 1, the pedal and cevian triangles of P are directly similar if and only if P lies on the three circles $AB'C'$, $BC'A'$, $CA'B'$. By Lemma 2, P lies on the three circles BCH, CAH, ABH. Hence, $P = H$. $\square$

Lemma 4. A, B', C', $\overline{P}$ *are concyclic if and only if* P^* *lies on the circle* Γ_A.

Proof. If $P = p : q : r$, the circle Φ_A passing through A, B', C' is given by

$$a^2yz + b^2zx + c^2xy - p(x+y+z)\left(\frac{c^2}{p+q}y + \frac{b^2}{p+r}z\right) = 0,$$

and its inverse in the circumcircle is the circle $\overline{\Phi}_A$ given by

$$\begin{aligned} &(a^2(p^2-qr) + (b^2-c^2)p(q-r))(a^2yz + b^2zx + c^2xy) \\ - \quad & pa^2(x+y+z)(c^2(p+r)y + b^2(p+q)z) = 0. \end{aligned}$$

Since Φ_A contains $\overline{P}$, its inverse $\overline{\Phi}_A$ contains P. Changing (p,q,r) to (x,y,z) gives the locus of P satisfying $\overline{P} \in \Phi_A$. Then changing (x,y,z) to $\left(\frac{a^2}{x}, \frac{b^2}{y}, \frac{c^2}{z}\right)$ gives the locus $\widehat{\overline{\Phi}}_A$ of the point P^* such that $\overline{P} \in \Phi_A$. By examination, $\widehat{\overline{\Phi}}_A = \Gamma_A$. $\square$

Proposition 5. *The pedal and cevian triangles of P are indirectly similar if and only if P is the isogonal conjugate of the Parry reflection point.*

Proof. By Lemma 1, the pedal and cevian triangles of P are indirectly similar if and only if $\overline{P}$ lies on the three circles $AB'C'$, $BC'A'$, $CA'B'$. By Lemma 4, P^* lies on each of the circles Γ_A, Γ_B, Γ_C. Hence, $P^* = T$, and $P = T^*$. $\square$

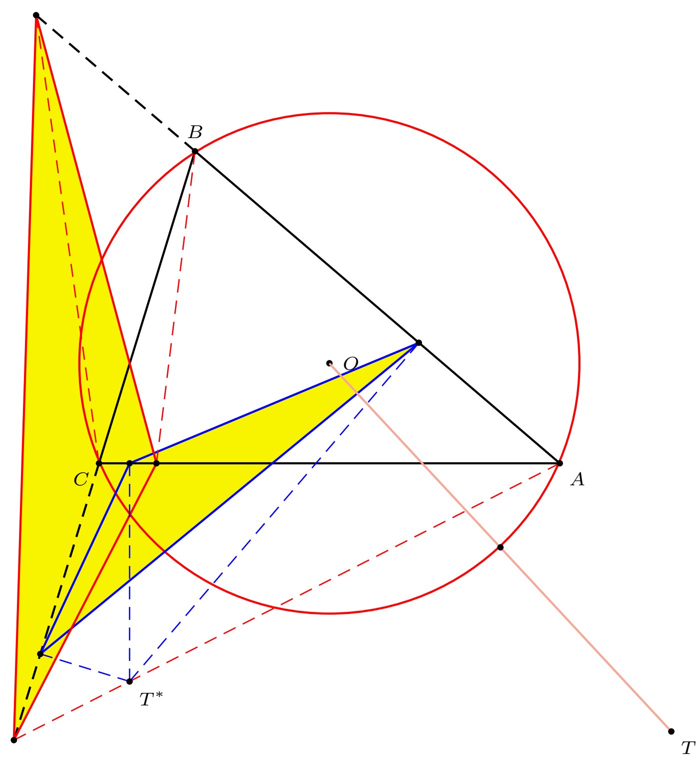


Figure 2

Remarks. (1) The isogonal conjugate of X_{399} is X_{1138} in [3]: this point lies on the Neuberg cubic.

(2) We can deduce Lemma 4 from the relation $\measuredangle B'\overline{P}C' - \measuredangle B_s P^* C_s = \measuredangle BAC$, which is true for every point P in the plane of ABC except the vertices A, B, C.

(3) As two indirectly similar triangles are orthologic and as the pedal and cevian triangles of P are orthologic if and only if P^* lies on the Stammler hyperbola, a point with indirectly similar cevian and pedal triangles must be the isogonal conjugate of a point of the Stammler hyperbola.

References

[1] J.-P. Ehrmann and F. M. van Lamoen, The Stammler circles, *Forum Geom.*, 2 (2002) 151 – 161.

[2] R. A. Johnson, *Modern Geometry*, 1929; reprinted as *Advanced Euclidean Geometry*, Dover Publications, 1960.

[3] C. Kimberling, *Encyclopedia of Triangle Centers*, August 22, 2002 edition, available at http://www2.evansville.edu/ck6/encyclopedia/; March 30, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[4] C.F. Parry, Problem 10637, *Amer. Math. Monthly*, 105 (1998) 68.

Jean-Pierre Ehrmann: 6, rue des Cailloux, 92110 - Clichy, France
E-mail address: Jean-Pierre.EHRMANN@wanadoo.fr

Forum Geometricorum
Volume 3 (2003) 105–111.

FORUM GEOM
ISSN 1534-1178

On the Kosnita Point and the Reflection Triangle

Darij Grinberg

Abstract. The Kosnita point of a triangle is the isogonal conjugate of the nine-point center. We prove a few results relating the reflections of the vertices of a triangle in their opposite sides to triangle centers associated with the Kosnita point.

1. Introduction

By the Kosnita point of a triangle we mean the isogonal conjugate of its nine-point center. The name Kosnita point originated from J. Rigby [5].

Theorem 1 (Kosnita)**.** *Let ABC be a triangle with the circumcenter O, and X, Y, Z be the circumcenters of triangles BOC, COA, AOB. The lines AX, BY, CZ concur at the isogonal conjugate of the nine-point center.*

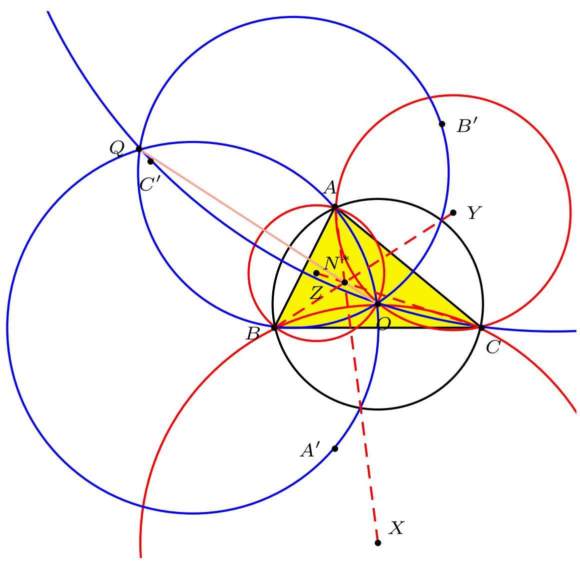


Figure 1

We denote the nine-point center by N and the Kosnita point by N^*. Note that N^* is an infinite point if and only if the nine-point center is on the circumcircle. We study this special case in §5 below. The points N and N^* appear in [3] as X_5 and X_{54} respectively. An old theorem of J. R. Musselman [4] relates the Kosnita

Publication Date: April 18, 2003. Communicating Editor: Paul Yiu.
The author thanks the communicating editor for simplifications, corrections and numerous helpful comments, particularly in §§4-5.

point to the reflections A', B', C' of A, B, C in their opposite sides BC, CA, AB respectively.

Theorem 2 (Musselman)**.** *The circles AOA', BOB', COC' pass through the inversive image of the Kosnita point in the circumcircle of triangle ABC.*

This common point of the three circles is the triangle center X_{1157} in [3], which we denote by Q in Figure 1. The following theorem gives another triad of circles containing this point. It was obtained by Paul Yiu [7] by computations with barycentric coordinates. We give a synthetic proof in §2.

Theorem 3 (Yiu)**.** *The circles $AB'C'$, $BC'A'$, $CA'B'$ pass through the inversive image of the Kosnita point in the circumcircle.*

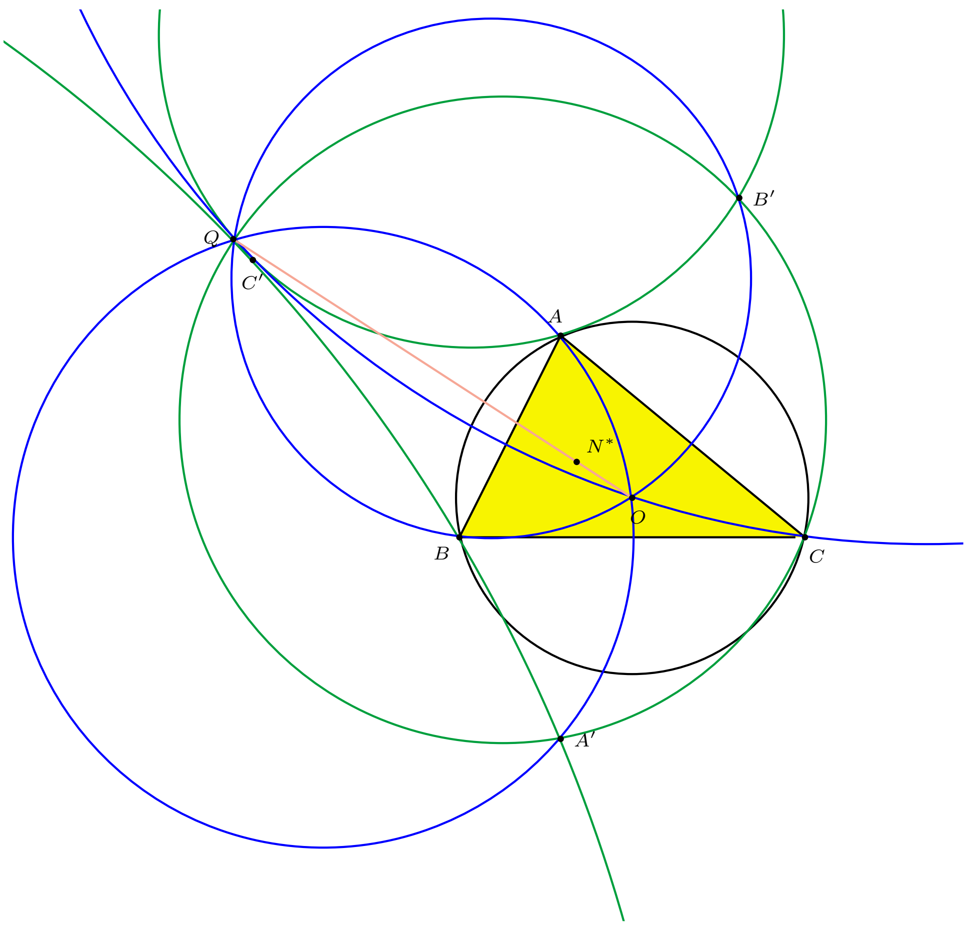


Figure 2

On the other hand, it is clear that the circles $A'BC$, $B'CA$, and $C'AB$ pass through the orthocenter of triangle ABC. It is natural to inquire about the circumcenter of the *reflection triangle* $A'B'C'$. A very simple answer is provided by the following characterization of $A'B'C'$ by G. Boutte [1].

Theorem 4 (Boutte)**.** *Let G be the centroid of ABC. The reflection triangle $A'B'C'$ is the image of the pedal triangle of the nine-point center N under the homothety* $\mathsf{h}(G, 4)$.

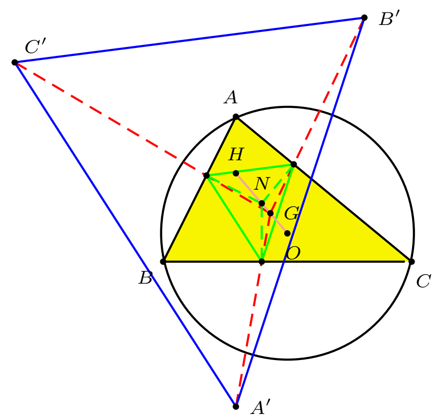


Figure 3

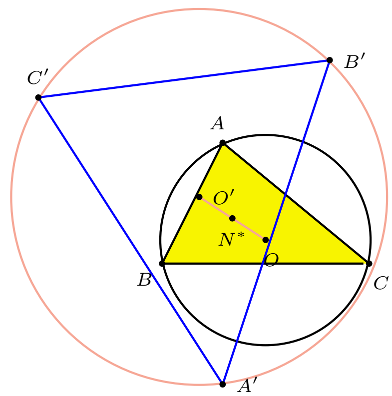


Figure 4

Corollary 5. *The circumcenter of the reflection triangle $A'B'C'$ is the reflection of the circumcenter in the Kosnita point.*

2. Proof of Theorem 3

Denote by Q the inverse of the Kosnita point N^* in the circumcircle. By Theorem 2, Q lies on the circles BOB' and COC'. So $\angle B'QO = \angle B'BO$ and $\angle C'QO = \angle C'CO$. Since $\angle B'QC' = \angle B'QO + \angle C'QO$, we get

$$\begin{aligned}\angle B'QC' =& \angle B'BO + \angle C'CO\\ =& \left(\angle CBB' - \angle CBO\right) + \left(\angle BCC' - \angle BCO\right)\\ =& \angle CBB' + \angle BCC' - (\angle CBO + \angle BCO)\\ =& \angle CBB' + \angle BCC' - (\pi - \angle BOC)\\ =& \angle CBB' + \angle BCC' - \pi + \angle BOC.\end{aligned}$$

But we have $\angle CBB' = \frac{\pi}{2} - C$ and $\angle BCC' = \frac{\pi}{2} - B$. Moreover, from the central angle theorem we get $\angle BOC = 2A$. Thus,

$$\begin{aligned}\angle B'QC' =& \left(\frac{\pi}{2} - C\right) + \left(\frac{\pi}{2} - B\right) - \pi + 2A\\ =& \pi - B - C - \pi + 2A = 2A - B - C\\ =& 3A - (A + B + C) = 3A - \pi,\end{aligned}$$

and consequently

$$\pi - \angle B'QC' = \pi - (3A - \pi) = 2\pi - 3A.$$

But on the other hand, $\angle BAC' = \angle BAC = A$ and $\angle CAB' = A$, so

$$\angle B'AC' = 2\pi - \left(\angle BAC' + \angle BAC + \angle CAB'\right) = 2\pi - (A + A + A) = 2\pi - 3A.$$

Consequently, $\angle B'AC' = \pi - \angle B'QC'$. Thus, Q lies on the circle $AB'C'$. Similar reasoning shows that Q also lies on the circles $BC'A'$ and $CA'B'$.

This completes the proof of Theorem 3.

Remark. In general, if a triangle ABC and three points A', B', C' are given, and the circles $A'BC$, $B'CA$, and $C'AB$ have a common point, then the circles $AB'C'$, $BC'A'$, and $CA'B'$ also have a common point. This can be proved with some elementary angle calculations. In our case, the common point of the circles $A'BC$, $B'CA$, and $C'AB$ is the orthocenter of ABC, and the common point of the circles $AB'C'$, $BC'A'$, and $CA'B'$ is Q.

3. Proof of Theorem 4

Let A_1, B_1, C_1 be the midpoints of BC, CA, AB, and A_2, B_2, C_2 the midpoints of B_1C_1, C_1A_1, A_1B_1. It is clear that $A_2B_2C_2$ is the image of ABC under the homothety $\mathsf{h}(G, \frac{1}{4})$. Denote by X the image of A' under this homothety. We show that this is the pedal of the nine-point center N on BC.

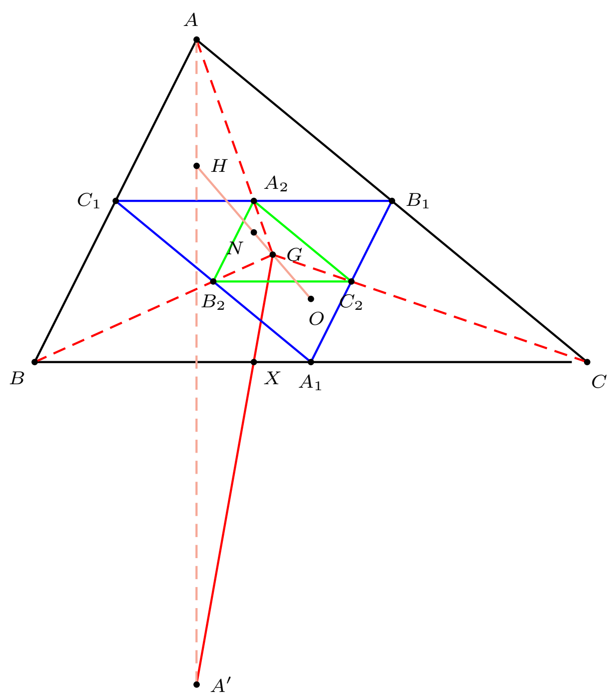


Figure 5

First, note that X, being the reflection of A_2 in B_2C_2, lies on BC. This is because A_2X is perpendicular to B_2C_2 and therefore to BC. The distance from

X to A_2 is twice of that from A_2 to B_2C_2. This is equal to the distance between the parallel lines B_2C_2 and BC.

The segment A_2X is clearly the perpendicular bisector of B_1C_1. It passes through the circumcenter of triangle $A_1B_1C_1$, which is the nine-point N of triangle ABC. It follows that X is the pedal of N on BC. For the same reasons, the images of B', C' under the same homothety $\mathsf{h}(G, \frac{1}{4})$ are the pedals of N on CA and AB respectively.

This completes the proof of Theorem 4.

4. Proof of Corollary 5

It is well known that the circumcenter of the pedal triangle of a point P is the midpoint of the segment PP^*, P^* being the isogonal conjugate of P. See, for example, [2, pp.155–156]. Applying this to the nine-point center N, we obtain the circumcenter of the reflection triangle $A'B'C'$ as the image of the midpoint of NN^* under the homothety $\mathsf{h}(G, 4)$. This is the point

$$\begin{aligned} G + 4\left(\frac{N+N^*}{2} - G\right) &= 2(N+N^*) - 3G \\ &= 2N^* + 2N - 3G \\ &= 2N^* + (O+H) - (2\cdot O + H) \\ &= 2N^* - O, \end{aligned}$$

the reflection of O in the Kosnita point N^*. Here, H is orthocenter, and we have made use of the well known facts that N is the midpoint of OH and G divides OH in the ratio $HG : GO = 2 : 1$.

This completes the proof of Corollary 5.

This point is the point X_{195} of [3]. Barry Wolk [6] has verified this theorem by computer calculations with barycentric coordinates.

5. Triangles with nine-point center on the circumcircle

Given a circle $O(R)$ and a point N on its circumference, let H be the reflection of O in N. For an arbitrary point P on the minor arc of the circle $N(\frac{R}{2})$ *inside* $O(R)$, let (i) A be the intersection of the *segment* HP with $O(R)$, (ii) the perpendicular to HP at P intersect $O(R)$ at B and C. Then triangle ABC has nine-point center N on its circumcircle $O(R)$. See Figure 6. This can be shown as follows. It is clear that $O(R)$ is the circumcircle of triangle ABC. Let M be the midpoint of BC so that OM is orthogonal to BC and parallel to PH. Thus, $OMPH$ is a (self-intersecting) trapezoid, and the line joining the midpoints of PM and OH is parallel to PH. Since the midpoint of OH is N and PH is orthogonal to BC, we conclude that N lies on the perpendicular bisector of PM. Consequently, $NM = NP = \frac{R}{2}$, and M lies on the circle $N(\frac{R}{2})$. This circle is the nine-point circle of triangle ABC, since it passes through the pedal P of A on BC and through the midpoint M of BC and has radius $\frac{R}{2}$.

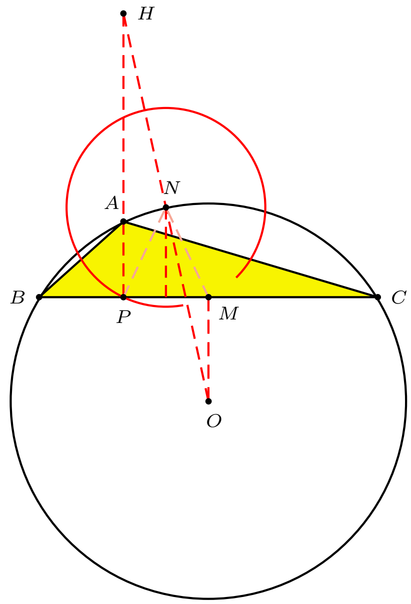


Figure 6

Remark. As P traverses the minor arc which the intersection of $N(\frac{R}{2})$ with the interior of $O(R)$, the line $\mathcal{L}$ passes through a fixed point, which is the reflection of O in H.

Theorem 6. *Suppose the nine-point center N of triangle ABC lies on the circumcircle.*

(1) *The reflection triangle $A'B'C'$ degenerates into a line $\mathcal{L}$.*
(2) *If X, Y, Z are the centers of the circles BOC, COA, AOB, the lines AX, BY, CZ are all perpendicular to $\mathcal{L}$.*
(3) *The circles AOA', BOB', COC' are mutually tangent at O. The line joining their centers is the parallel to $\mathcal{L}$ through O.*
(4) *The circles $AB'C'$, $BC'A'$, $CA'B'$ pass through O.*

References

[1] G. Boutte, Hyacinthos message 3997, September 28, 2001.
[2] R. A. Johnson, *Modern Geometry*, 1929; reprinted as *Advanced Euclidean Geometry*, Dover Publications, 1960.
[3] C. Kimberling, *Encyclopedia of Triangle Centers*, April 16, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[4] J. R. Musselman and R. Goormaghtigh, Advanced Problem 3928, *Amer. Math. Monthly*, 46 (1939) 601; solution, 48 (1941) 281–283.
[5] J. Rigby, Brief notes on some forgotten geometrical theorems, *Mathematics & Informatics Quarterly*, 7 (1997) 156–158.
[6] B. Wolk, Hyacinthos message 6432, January 26, 2003.
[7] P. Yiu, Hyacinthos message 4533, December 12, 2001.

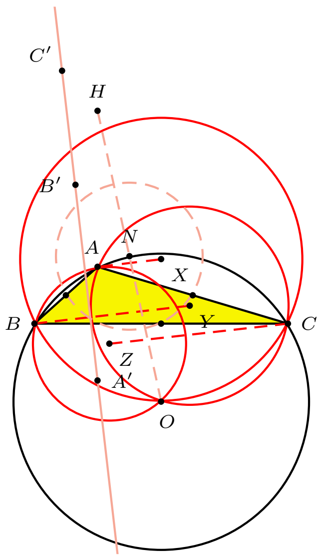


Figure 7

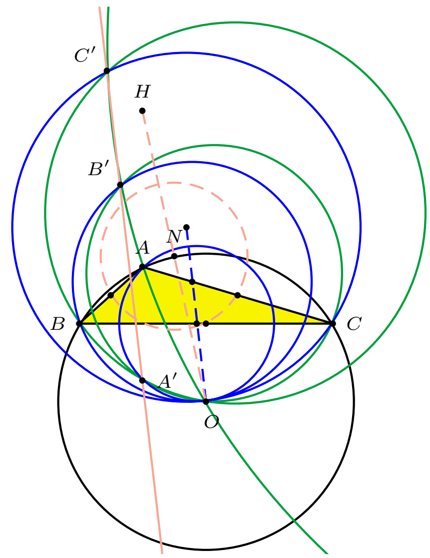


Figure 8

Added in proof: Bernard Gibert has kindly communincated the following results. Let A_1 be the intersection of the lines OA' and $B'C'$, and similarly define B_1 and C_1. Denote, as in §1, by Q be the inverse of the Kosnita point in the circumcircle.

Theorem 7 (Gibert)**.** *The lines AA_1, BB_1, CC_1 concur at the isogonal conjugate of Q.*

This is the point X_{1263} in [3]. The points A, B, C, A', B', C', O, Q, A_1, B_1, C_1 all lie on the Neuberg cubic of triangle ABC, which is the isogonal cubic with pivot the infinite point of the Euler line. This cubic is also the locus of all points whose reflections in the sides of triangle ABC form a triangle perspective to ABC. The point Q is the unique point whose triangle of reflections has perspector on the circumcircle. This perspector, called the Gibert point X_{1141} in [3], lies on the line joining the nine-point center to the Kosnita point.

Darij Grinberg: Geroldsäckerweg 7, D-76139 Karlsruhe, Germany
E-mail address: `darij_grinberg@web.de`

Forum Geometricorum
Volume 3 (2003) 113–116.

FORUM GEOM
ISSN 1534-1178

A Note on the Schiffler Point

Lev Emelyanov and Tatiana Emelyanova

Abstract. We prove two interesting properties of the Schiffler point.

1. Main results

The Schiffler point is the intersection of four Euler lines. Let I be the incenter of triangle ABC. The Schiffler point S is the point common to the Euler lines of triangles IBC, ICA, IAB, and ABC. See [1, p.70]. Not much is known about S. In this note, we prove two interesting properties of this point.

Theorem 1. *Let A and I_1 be the circumcenter and A-excenter of triangle ABC, and A_1 the intersection of OI_1 and BC. Similarly define B_1 and C_1. The lines AA_1, BB_1 and CC_1 concur at the Schiffler point S.*

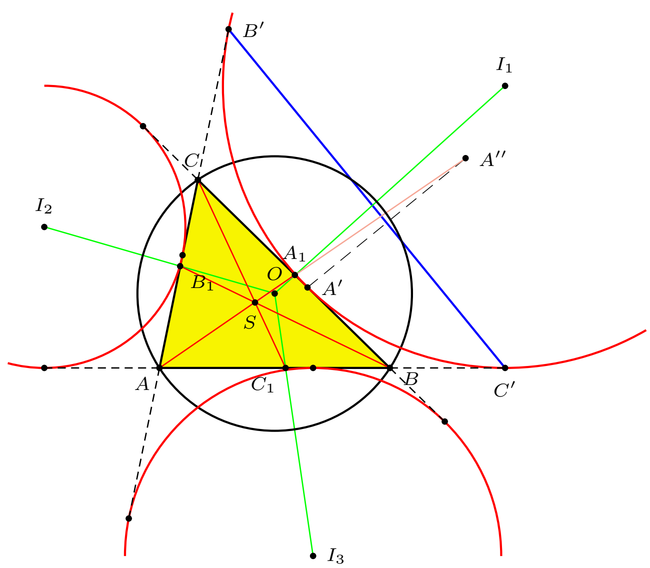


Figure 1

Theorem 2. *Let A', B', C' be the touch points of the A-excircle and BC, CA, AB respectively, and A'' the reflection of A' in $B'C'$. Similarly define B'' and C''. The lines AA'', BB'' and CC'' concur at the Schiffler point S.*

Publication Date: May 16, 2003. Communicating Editor: Paul Yiu.

We make use of trilinear coordinates with respect to triangle ABC. According to [1, p.70], the Schiffler point has coordinates

$$\left(\frac{1}{\cos B + \cos C} : \frac{1}{\cos C + \cos A} : \frac{1}{\cos A + \cos B}\right).$$

2. Proof of Theorem 1

We show that AA_1 passes through the Schiffler point S. Because

$$O = (\cos A : \cos B : \cos C) \quad \text{and} \quad I_1 = (-1 : 1 : 1),$$

the line OI_1 is given by

$$(\cos B - \cos C)\alpha - (\cos C + \cos A)\beta + (\cos A + \cos B)\gamma = 0.$$

The line BC is given by $\alpha = 0$. Hence the intersection of OI_1 and BC is

$$A_1 = (0 : \cos A + \cos B : \cos A + \cos C).$$

The collinearity of A_1, S and A follows from

$$\begin{aligned}&\begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos A + \cos B & \cos A + \cos C \\ \frac{1}{\cos B+\cos C} & \frac{1}{\cos C+\cos A} & \frac{1}{\cos A+\cos B} \end{vmatrix} \\ =&\begin{vmatrix} \cos A + \cos B & \cos A + \cos C \\ \frac{1}{\cos C+\cos A} & \frac{1}{\cos A+\cos B} \end{vmatrix} \\ =&0.\end{aligned}$$

This completes the proof of Theorem 1.

Remark. It is clear from the proof above that more generally, if P is a point with trilinear coordinates $(p : q : r)$, and A_1, B_1, C_1 the intersections of PI_a with BC, PI_2 with CA, PI_3 with AB, then the lines AA_1, BB_1, CC_1 intersect at a point with trilinear coordinates $\left(\frac{1}{q+r} : \frac{1}{r+p} : \frac{1}{p+q}\right)$. If P is the symmedian point, for example, this intersection is the point $X_{81} = \left(\frac{1}{b+c} : \frac{1}{c+a} : \frac{1}{a+b}\right)$.

3. Proof of Theorem 2

We deduce Theorem 2 as a consequence of the following two lemmas.

Lemma 3. *The line OI_1 is the Euler line of triangle $A'B'C'$.*

Proof. Triangle ABC is the tangential triangle of $A'B'C'$. It is known that the circumcenter of the tangential triangle lies on the Euler line. See, for example, [1, p.71]. It follows that OI_1 is the Euler line of triangle $A'B'C'$. □

Lemma 4. *Let A^* be the reflection of vertex A of triangle ABC with respect to BC, $A_1B_1C_1$ be the tangential triangle of ABC. Then the Euler line of ABC and line A_1A^* intersect line B_1C_1 in the same point.*

Proof. As is well known, the vertices of the tangential triangle are given by

$$A_1 = (-a : b : c), \quad B_1 = (a : -b : c), \quad C_1 = (a : b : -c).$$

The line B_1C_1 is given by $c\beta + b\gamma = 0$. According to [1, p.42], the Euler line of triangle ABC is given by

$$a(b^2-c^2)(b^2+c^2-a^2)\alpha+b(c^2-a^2)(c^2+a^2-b^2)\beta+c(a^2-b^2)(a^2+b^2-c^2)\gamma = 0.$$

Now, it is not difficult to see that

$$\begin{aligned} A^* =&(-1 : 2\cos C : 2\cos B) \\ =&(-abc : c(a^2+b^2-c^2) : b(c^2+a^2-b^2)). \end{aligned}$$

The equation of the line A^*A_1 is then

$$\begin{vmatrix} -abc & 2c(a^2+b^2-c^2) & 2b(c^2+a^2-b^2) \\ -a & b & c \\ \alpha & \beta & \gamma \end{vmatrix} = 0.$$

After simplification, this is

$$-(b^2-c^2)(b^2+c^2-a^2)\alpha + ab(a^2-b^2)\beta - ac(a^2-c^2)\gamma = 0.$$

Now, the lines B_1C_1, A^*A_1, and the Euler line are concurrent if the determinant

$$\begin{vmatrix} 0 & c & b \\ -(b^2-c^2)(b^2+c^2-a^2) & ab(a^2-b^2) & -ac(a^2-c^2) \\ a(b^2-c^2)(b^2+c^2-a^2) & b(c^2-a^2)(c^2+a^2-b^2) & c(a^2-b^2)(a^2+b^2-c^2) \end{vmatrix}$$

is zero. Factoring out $(b^2-c^2)(b^2+c^2-a^2)$, we have

$$\begin{aligned} &\begin{vmatrix} 0 & c & b \\ -1 & ab(a^2-b^2) & -ac(a^2-c^2) \\ a & b(c^2-a^2)(c^2+a^2-b^2) & c(a^2-b^2)(a^2+b^2-c^2) \end{vmatrix} \\ =&-c\begin{vmatrix} -1 & -ac(a^2-c^2) \\ a & c(a^2-b^2)(a^2+b^2-c^2) \end{vmatrix} + b\begin{vmatrix} -1 & ab(a^2-b^2) \\ a & b(c^2-a^2)(c^2+a^2-b^2) \end{vmatrix} \\ =&c^2((a^2-b^2)(a^2+b^2-c^2) - a^2(a^2-c^2)) \\ &-b^2((c^2-a^2)(c^2+a^2-b^2) + a^2(a^2-b^2)) \\ =&c^2 \cdot b^2(c^2-b^2) - b^2 \cdot c^2(c^2-b^2) \\ =&0. \end{aligned}$$

This confirms that the three lines are concurrent. □

To prove Theorem 2, it is enough to show that the line AA'' in Figure 1 contains S. Now, triangle $A'B'C'$ has tangential triangle ABC and Euler line OI_1 by Lemma 3. By Lemma 4, the lines OI_1, AA'' and BC are concurrent. This means that the line AA'' contains A_1. By Theorem 1, this line contains S.

Reference

[1] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

Lev Emelyanov: 18-31 Proyezjaia Street, Kaluga, Russia 248009
E-mail address: emelyanov@kaluga.ru

Tatiana Emelyanova: 18-31 Proyezjaia Street, Kaluga, Russia 248009
E-mail address: emelyanov@kaluga.ru

Forum Geometricorum
Volume 3 (2003) 117–124.

FORUM GEOM
ISSN 1534-1178

Harcourt's Theorem

Nikolaos Dergiades and Juan Carlos Salazar

Abstract. We give a proof of Harcourt's theorem that if the signed distances from the vertices of a triangle of sides a, b, c to a tangent of the incircle are a_1, b_1, c_1, then $aa_1 + bb_1 + cc_1$ is twice of the area of the triangle. We also show that there is a point on the circumconic with center I whose distances to the sidelines of ABC are precisely a_1, b_1, c_1. An application is given to the extangents triangle formed by the external common tangents of the excircles.

1. Harcourt's Theorem

The following interesting theorem appears in F. G.-M.[1, p.750] as Harcourt's theorem.

Theorem 1 (Harcourt)**.** *If the distances from the vertices A, B, C to a tangent to the incircle of triangle ABC are a_1, b_1, c_1 respectively, then the algebraic sum $aa_1 + bb_1 + cc_1$ is twice of the area of triangle ABC.*

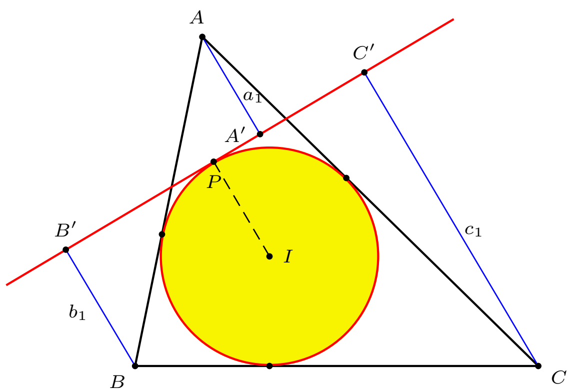


Figure 1

The distances are signed. Distances to a line from points on opposite sides are opposite in sign, while those from points on the same side have the same sign. For the tangent lines to the incircle, we stipulate that the distance from the incenter is positive. For example, in Figure 1, when the tangent line ℓ separates the vertex A from B and C, a_1 is negative while b_1 and c_1 are positive. With this sign convention, Harcourt's theorem states that

$$aa_1 + bb_1 + cc_1 = 2\triangle, \tag{1}$$

Publication Date: June 2, 2003. Communicating Editor: Paul Yiu.

The authors thank the editor for his valuable comments, helps and strategic improvements. JCS also thanks Francisco Bellot Rosado and Darij Grinberg for their helpful remarks in an early stage of the preparation of this paper.

where $\triangle$ is the area of triangle ABC.

We give a simple proof of Harcourt's theorem by making use of homogeneous barycentric coordinates with reference to triangle ABC. First, we establish a fundamental formula.

Proposition 2. *Let ℓ be a line passing through a point P with homogeneous barycentric coordinates $(x : y : z)$. If the signed distances from the vertices A, B, C to a line ℓ are d_1, d_2, d_3 respectively, then*

$$d_1 x + d_2 y + d_3 z = 0. \tag{2}$$

Proof. It is enough to consider the case when ℓ separates A from B and C. We take d_1 as negative, and d_2, d_3 positive. See Figure 2. If A' is the trace of P on the side line BC, it is well known that

$$\frac{AP}{PA'} = \frac{x}{y+z}.$$

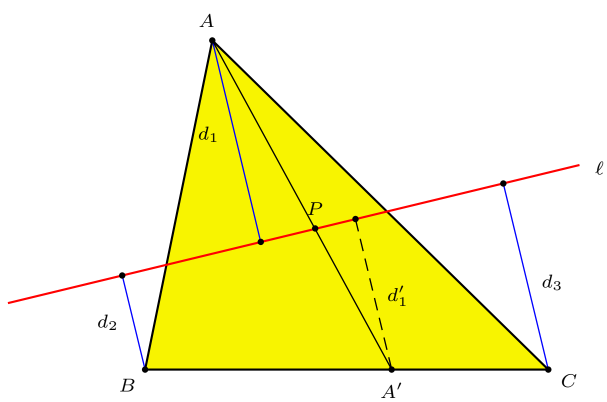


Figure 2

Since $\frac{BA'}{A'C} = \frac{z}{y}$, the distance from A' to ℓ is

$$d'_1 = \frac{yd_2 + zd_3}{y+z}.$$

Since $\frac{-d_1}{d'_1} = \frac{AP}{PA'} = \frac{y+z}{x}$, the equation (2) follows. $\square$

Proof of Harcourt's theorem. We apply Proposition 2 to the line ℓ through the incenter $I = (a : b : c)$ parallel to the tangent. The signed distances from A, B, C to ℓ are $d_1 = a_1 - r$, $d_2 = a_2 - r$, and $d_3 = a_3 - r$. From these,

$$\begin{aligned} aa_1 + bb_1 + cc_1 =& a(d_1 + r) + b(d_2 + r) + c(d_3 + r) \\ =&(ad_1 + bd_2 + cd_3) + (a + b + c)r \\ =&2\triangle, \end{aligned}$$

since $ad_1 + bd_2 + cd_3 = 0$ by Proposition 2.

2. Harcourt's theorem for the excircles

Harcourt's theorem for the incircle and its proof above can be easily adapted to the excircles.

Theorem 3. *If the distances from the vertices A, B, C to a tangent to the A-excircle of triangle ABC are a_1, b_1, c_1 respectively, then $-aa_1 + bb_1 + cc_1 = 2\triangle$. Analogous statements hold for the B- and C-excircles.*

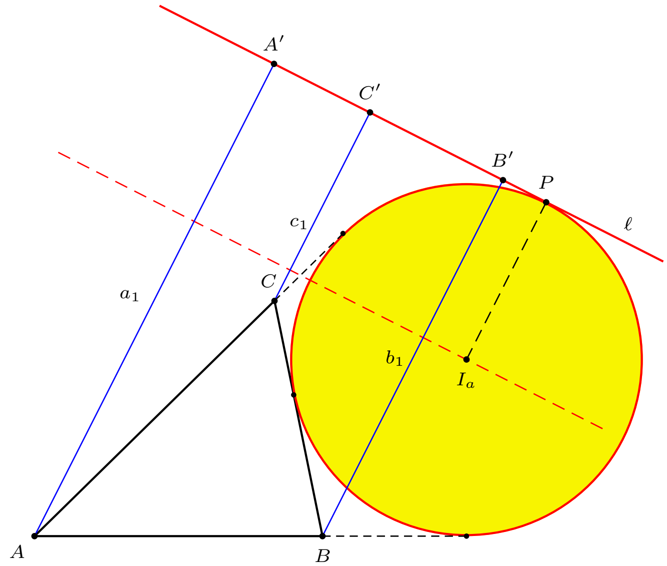


Figure 3

Proof. Apply Proposition 2 to the line ℓ through the excenter $I_a = (-a : b : c)$ parallel to the tangent. If the distances from A, B, C to ℓ are d_1, d_2, d_3 respectively, then

$$-ad_1 + bd_2 + cd_3 = 0.$$

Since $a_1 = d_1 + r_1$, $b_1 = d_2 + r_1$, $c_1 = d_3 + r_1$, where r_1 is the radius of the excircle, it easily follows that

$$\begin{aligned}-aa_1 + bb_1 + cc_1 =& - a(d_1 + r_1) + b(d_2 + r_1) + c(d_3 + r_1)\\ =&(-ad_1 + bd_2 + cd_3) + r_1(-a + b + c)\\ =&r_1(-a + b + c)\\ =&2\triangle.\end{aligned}$$

□

Consider the external common tangents of the excircles of triangle ABC. Let ℓ_a be the external common tangent of the B- and C-excircles. Denote by d_{a1}, d_{a2}, d_{a3} the distances from the A, B, C to this line. Clearly, $d_{a1} = h_a$, the altitude on BC. Similarly define ℓ_b, ℓ_c and the associated distances.

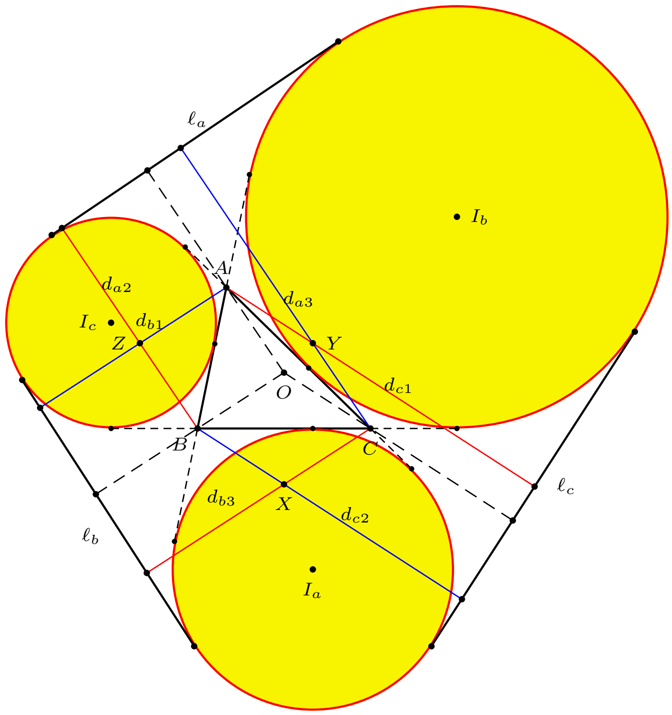


Figure 4

Theorem 4. $d_{a2}d_{b3}d_{c1} = d_{a3}d_{b1}d_{c2}$.

Proof. Applying Theorem 3 to the tangent ℓ_a of the B-excircle (respectively the C-excircle), we have

$$ad_{a1} - bd_{a2} + cd_{a3} = 2\triangle,$$
$$ad_{a1} + bd_{a2} - cd_{a3} = 2\triangle.$$

From these it is clear that $bd_{a2} = cd_{a3}$, and

$$\frac{d_{a2}}{d_{a3}} = \frac{c}{b}.$$

Similarly,

$$\frac{d_{b3}}{d_{b1}} = \frac{a}{c} \quad \text{and} \quad \frac{d_{c1}}{d_{c2}} = \frac{b}{a}.$$

Combining these three equations we have $d_{a2}d_{b3}d_{c1} = d_{a3}d_{b1}d_{c2}$. □

It is clear that the perpendiculars from A to ℓ_a, being the reflection of the A-altitude, passes through the circumcenter; similarly for the perpendiculars from B to ℓ_b and from C to ℓ_c.

Let X be the intersection of the perpendiculars from B to ℓ_c and from C to ℓ_b. Note that OB and CX are parallel, so are OC and BX. Since $OB = OC$, it follows that $OBXC$ is a rhombus, and $BX = CX = R$, the circumradius of triangle ABC. It also follows that X is the reflection of O in the side BC. Similarly, if Y is the intersection of the perpendiculars from C to ℓ_a and from A to ℓ_c, and Z that of the perpendiculars from A to ℓ_b and from B to ℓ_a, then XYZ is the triangle of reflections of the circumcenter O. As such, it is oppositely congruent to ABC, and the center of homothety is the nine-point center of triangle ABC.

3. The circum-ellipse with center I

Consider a tangent $\mathcal{L}$ to the incircle at a point P. If the signed distances from the vertices A, B, C to $\mathcal{L}$ are a_1, b_1, c_1, then by Harcourt's theorem, there is a point $P^{\#}$ whose signed distances to the sides BC, CA, AB are precisely a_1, b_1, c_1. What is the locus of the point $P^{\#}$ as P traverses the incircle? By Proposition 2, the barycentric equation of $\mathcal{L}$ is

$$a_1x + b_1y + c_1z = 0.$$

This means that the point with homogeneous barycentric coordinates $(a_1 : b_1 : c_1)$ is a point on the dual conic of the incircle, which is the circumconic with equation

$$(s-a)yz + (s-b)zx + (s-c)xy = 0. \tag{3}$$

The point $P^{\#}$ in question has barycentric coordinates $(aa_1 : bb_1 : cc_1)$. Since (a_1, b_1, c_1) satisfies (3), if we put $(x, y, z) = (aa_1, bb_1, cc_1)$, then

$$a(s-a)yz + b(s-b)zx + c(s-c)xy = 0.$$

Thus, the locus of $P^{\#}$ is the circumconic with perspector $(a(s-a) : b(s-b) : c(s-c))$. [1] It is an ellipse, and its center is, surprisingly, the incenter I. [2] We denote this circum-ellipse by $\mathcal{C}_I$. See Figure 5.

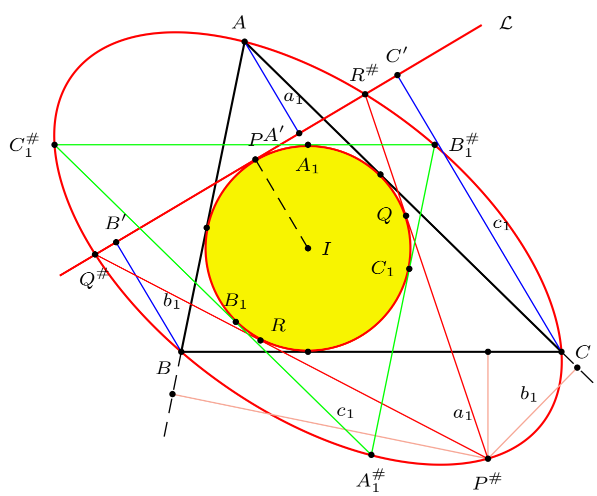


Figure 5

[1]This is the Mittenpunkt, the point X_9 in [4]. It can be constructed as the intersection of the lines joining the excenters to the midpoints of the corresponding sides of triangle ABC.

[2]In general, the center of the circumconic $pyz + qzx + rxy = 0$ is the point with homogeneous barycentric coordinates $(p(q + r - p) : q(r + p - q) : r(p + q - r))$.

Let A_1, B_1, C_1 be the antipodes of the points of tangency of the incircle with the sidelines. It is quite easy to see that $A_1^\#$, $B_1^\#$, $C_1^\#$ are the antipodes of A, B, C in the circum-ellipse $\mathcal{C}_I$. Note that $A_1^\# B_1^\# C_1^\#$ and ABC are oppositely congruent at I. It follows from Steiner's porism that if we denote the intersections of $\mathcal{L}$ and this ellipse by $Q^\#$ and $R^\#$, then the lines $P^\# Q^\#$ and $P^\# R^\#$ are tangent to the incircle at Q and R. This leads to the following construction of $P^\#$.

Construction. If the tangent to the incircle at P intersects the ellipse $\mathcal{C}_I$ at two points, the second tangents from these points to the incircle intersect at $P^\#$ on $\mathcal{C}_I$.

If the point of tangency P has coordinates $\left(\frac{u^2}{s-a} : \frac{v^2}{s-b} : \frac{w^2}{s-c}\right)$, with $u+v+w = 0$, then $P^\#$ is the point $\left(\frac{a(s-a)}{u} : \frac{b(s-b)}{v} : \frac{c(s-c)}{w}\right)$. In particular, if $\mathcal{L}$ is the common tangent of the incircle and the nine-point circle at the Feuerbach point, which has coordinates $((s-a)(b-c)^2 : (s-b)(c-a)^2 : (s-c)(a-b)^2)$, then $P^\#$ is the point $\left(\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}\right)$. This is X_{100} of [3, 4]. It is a point on the circumcircle, lying on the half line joining the Feuerbach point to the centroid of triangle ABC. See [3, Figure 3.12, p.82].

4. The extangents triangle

Consider the external common tangent ℓ_a of the excircles (I_b) and (I_c). Let d_{a1}, d_{a2}, d_{a3} be the distances from A, B, C to this line. We have shown that $\frac{d_{a2}}{d_{a3}} = \frac{c}{b}$. On the other hand, it is clear that $\frac{d_{a1}}{d_{a2}} = \frac{b}{b+c}$. See Figure 6. It follows that

$$d_{a1} : d_{a2} : d_{a3} = bc : c(b+c) : b(b+c).$$

By Proposition 2, the barycentric equation of ℓ_a is

$$bcx + c(b+c)y + b(b+c)z = 0.$$

Similarly, the equations of ℓ_b and ℓ_c are

$$\begin{aligned} c(c+a)x + cay + a(c+a)z &= 0, \\ b(a+b)x + a(a+b)y + abz &= 0. \end{aligned}$$

These three external common tangents bound a triangle called the *extangents triangle* in [3]. The vertices are the points [3]

$$\begin{aligned} A' &= (-a^2 s : b(c+a)(s-c) : c(a+b)(s-b)), \\ B' &= (a(b+c)(s-c) : -b^2 s : c(a+b)(s-a)), \\ C' &= (a(b+c)(s-b) : b(c+a)(s-a) : -c^2 s). \end{aligned}$$

Let I_a' be the incenter of the reflection of triangle ABC in A. It is clear that the distances from A and I_a' to ℓ_a are respectively h_a and r. Since A is the midpoint of II_a', the distance from I to ℓ_a is $2h_a - r$.

[3] The *trilinear* coordinates of these vertices given in [3, p.162, §6.17] are not correct. The diagonal entries of the matrices should read $1 + \cos A$ etc. and $\frac{-a(a+b+c)}{(a-b+c)(a+b-c)}$ etc. respectively.

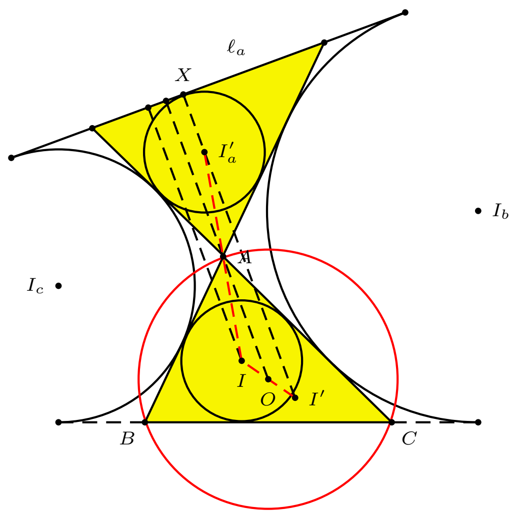


Figure 6

Now consider the reflection of I in O. We denote this point by I'. [4] Since the distances from I and O to ℓ_a are respectively $2h_a - r$ and $R + h_a$, it follows that the distance from I' to ℓ_a is $2(R + h_a) - (2h_a - r) = 2R + r$. For the same reason, the distances from I' to ℓ_b and ℓ_c are also $2R + r$. From this we deduce the following interesting facts about the extangents triangle.

Theorem 5. *The extangent triangle bounded by ℓ_a, ℓ_b, ℓ_c*

(1) *has incenter I' and inradius $2R + r$;*
(2) *is perspective with the excentral triangle at I';*
(3) *is homothetic to the tangential triangle at the internal center of similitude of the circumcircle and the incircle of triangle ABC, the ratio of the homothety being $\frac{2R+r}{R}$.*

Proof. It is enough to locate the homothetic center in (3). This is the point which divides $I'O$ in the ratio $2R + r : -R$, *i.e.*,

$$\frac{(2R + r)O - R(2O - I)}{R + r} = \frac{r \cdot O + R \cdot I}{R + r},$$

the internal center of similitude of the circumcircle and incircle of triangle ABC.[5] $\square$

Remarks. (1) The statement that the extangents triangle has inradius $2R + r$ can also be found in [2, Problem 2.5.4].

(2) Since the excentral triangle has circumcenter I' and circumradius $2R$, it follows that the excenters and the incenters of the reflections of triangle ABC in A, B, C are concyclic. It is well known that since ABC is the orthic triangle of the

[4] This point appears as X_{40} in [4].
[5] This point appears as X_{55} in [4].

excentral triangle, the circumcircle of ABC is the nine-point circle of the excentral triangle.

(3) If the incircle of the extangents triangle touches its sides at X, Y, Z respectively, [6] then triangle XYZ is homothetic to ABC, again at the internal center of similitude of the circumcircle and the incircle.

(4) More generally, the reflections of the traces of a point P in the respective sides of the excentral triangle are points on the sidelines of the extangents triangle. They form a triangle perspective with ABC at the isogonal conjugate of P. For example, the reflections of the points of tangency of the excircles (traces of the Nagel point $(s-a : s-b : s-c)$) form a triangle with perspector $\left(\frac{a^2}{s-a} : \frac{b^2}{s-b} : \frac{c^2}{s-c}\right)$, the external center of similitude of the circumcircle and the incircle.[7]

References

[1] F. G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, Paris, 1991.

[2] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, Charles Babbage Research Centre, Winnipeg, 1989.

[3] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

[4] C. Kimberling, *Encyclopedia of Triangle Centers*, May 23, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr

Juan Carlos Salazar: Calle Maturín N°C 19, Urb., Mendoza, Puerto Ordaz 8015, Estado Bolívar, Venezuela
E-mail address: caisersal@yahoo.com

[6] These are the reflections of the traces of the Gergonne point in the respective sides of the excentral triangle.

[7] This point appears as X_{56} in [4].

Forum Geometricorum
Volume 3 (2003) 125–134.

FORUM GEOM
ISSN 1534-1178

Isotomic Inscribed Triangles and Their Residuals

Mario Dalcín

Abstract. We prove some interesting results on inscribed triangles which are isotomic. For examples, we show that the triangles formed by the centroids (respectively orthocenters) of their residuals have equal areas, and those formed by the circumcenters are congruent.

1. Isotomic inscribed triangles

The starting point of this investigation was the interesting observation that if we consider the points of tangency of the sides of a triangle with its incircle and excircles, we have two triangles of equal areas.

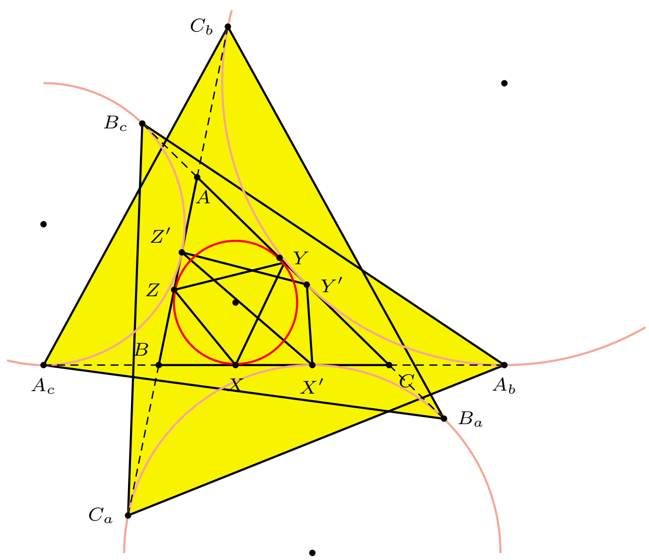


Figure 1

In Figure 1, X, Y, Z are the points of tangency of the incircle with the sides BC, CA, AB of triangle ABC, and X', Y', Z' those with the corresponding excircles. In [2], XYZ and $X'Y'Z'$ are called the intouch and extouch triangles of ABC respectively. That these two triangles have equal areas is best explained by the fact that each pair of points X, X'; Y, Y'; Z, Z' are isotomic on their respective sides, *i.e.*,

$$BX = X'C, \quad CY = Y'A, \quad AZ = Z'B. \tag{1}$$

Publication Date: June 16, 2003. Communicating Editor: Paul Yiu.

We shall say that XYZ and $X'Y'Z'$ are isotomic inscribed triangles. The following basic proposition follows from simple calculations with barycentric coordinates.

Proposition 1. *Isotomic inscribed triangles have equal areas.*

Proof. Let X, Y, Z be points on the sidelines BC, CA, AB dividing the sides in the ratios

$$BX : XC = x : 1-x, \quad CY : YA = y : 1-y, \quad AZ : ZB = z : 1-z.$$

In terms of barycentric coordinates with respect to ABC, we have

$$X = (1-x)B + xC, \quad Y = (1-y)C + yA, \quad Z = (1-z)A + zB. \qquad (2)$$

The area of triangle XYZ, in terms of the area $\triangle$ of ABC, is

$$\begin{aligned} \triangle XYZ = & \begin{vmatrix} 0 & 1-x & x \\ y & 0 & 1-y \\ 1-z & z & 0 \end{vmatrix} \triangle \\ = & (1-(x+y+z)+(xy+yz+zx))\triangle \\ = & (xyz+(1-x)(1-y)(1-z))\triangle. \end{aligned} \qquad (3)$$

See, for example, [4, Proposition 1]. If X', Y', Z' are points satisfying (1), then

$$BX' : X'C = 1-x : x, \quad CY' : Y'A = 1-y : y, \quad AZ' : Z'B = 1-z : z, \qquad (4)$$

and

$$X' = xB + (1-x)C, \quad Y' = yC + (1-y)A, \quad Z' = zA + (1-z)B. \qquad (5)$$

The area of triangle $X'Y'Z'$ can be obtained from (3) by replacing x, y, z by $1-x$, $1-y$, $1-z$ respectively. It is clear that this results in the same expression. This completes the proof of the proposition. □

Proposition 2. *The centroids of isotomic inscribed triangles are symmetric with respect to the centroid of the reference triangle.*

Proof. The expressions in (2) allow one to determine the centroid of triangle XYZ easily. This is the point

$$G_{XYZ} = \frac{1}{3}(X+Y+Z) = \frac{(1+y-z)A+(1+z-x)B+(1+x-y)C}{3}. \qquad (6)$$

On the other hand, with the coordinates given in (5), the centroid of triangle $X'Y'Z'$ is

$$G_{X'Y'Z'} = \frac{1}{3}(X'+Y'+Z') = \frac{(1-y+z)A+(1-z+x)B+(1-x+y)C}{3}. \qquad (7)$$

It follows easily that

$$\frac{1}{2}(G_{XYZ}+G_{X'Y'Z'}) = \frac{1}{3}(A+B+C) = G,$$

the centroid of triangle ABC. □

Corollary 3. *The intouch and extouch triangles have equal areas, and the midpoint of their centroids is the centroid of triangle ABC.*

Proof. These follow from the fact that the intouch triangle XYZ and the extouch triangle $X'Y'Z'$ are isotomic, as is clear from the following data, where a, b, c denote the lengths of the sides BC, CA, AB of triangle ABC, and $s = \frac{1}{2}(a+b+c)$.

$$\begin{aligned} BX &= X'C = s-b, & BX' &= XC = s-c,\\ CY &= Y'A = s-c, & CY' &= YA = s-a,\\ AZ &= Z'B = s-a, & AZ' &= ZB = s-b. \end{aligned}$$

□

In fact, we may take

$$x = \frac{s-b}{a}, \quad y = \frac{s-c}{b}, \quad z = \frac{s-a}{c},$$

and use (3) to obtain

$$\triangle XYZ = \triangle X'Y'Z' = \frac{2(s-a)(s-b)(s-c)}{abc}\triangle.$$

Let R and r denote respectively the circumradius and inradius of triangle ABC. Since $\triangle = rs$ and

$$R = \frac{abc}{4\triangle}, \qquad r^2 = \frac{(s-a)(s-b)(s-c)}{s},$$

we have

$$\triangle XYZ = \triangle X'Y'Z' = \frac{r}{2R}\cdot\triangle.$$

If we denote by A_b and A_c the points of tangency of the line BC with the B- and C-excircles, it is easy to see that A_b and A_c are isotomic points on BC. In fact,

$$BA_b = A_cC = s, \quad BA_c = A_bC = -(s-a).$$

Similarly, the other points of tangency B_c, B_a, C_a, C_b form pairs of isotomic points on the lines CA and AB respectively. See Figure 1.

Corollary 4. *The triangles $A_bB_cC_a$ and $A_cB_aC_b$ have equal areas. The centroids of these triangles are symmetric with respect to the centroid G of triangle ABC.*

These follow because $A_bB_cC_a$ and $A_cB_bC_a$ are isotomic inscribed triangles. Indeed,

$$\begin{aligned} BA_b : A_bC =& s : -(s-a) = 1 + \frac{s-a}{a} : -\frac{s-a}{a} = CA_c : A_cB,\\ CB_c : B_cA =& s : -(s-b) = 1 + \frac{s-b}{b} : -\frac{s-b}{b} = AB_a : B_aC,\\ AC_a : C_aB =& s : -(s-c) = 1 + \frac{s-c}{c} : -\frac{s-c}{c} = BC_b : C_bA. \end{aligned}$$

Furthermore, the centroids of the four triangles XYZ, $X'Y'Z'$, $A_bB_cC_a$ and $A_cB_aC_b$ form a parallelogram. See Figure 2.

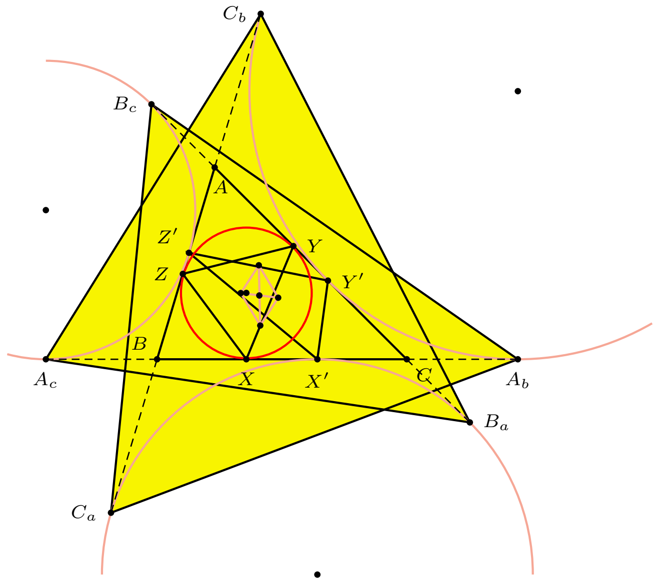


Figure 2

2. Triangles of residual centroids

For an inscribed triangle XYZ, we call the triangles AYZ, BZX, CXY its residuals. From (2, 5), we easily determine the centroids of these triangles.

$$G_{AYZ} = \frac{1}{3}((2+y-z)A + zB + (1-y)C),$$
$$G_{BZX} = \frac{1}{3}((1-z)A + (2+z-x)B + xC),$$
$$G_{CXY} = \frac{1}{3}(yA + (1-x)B + (2+x-y)C).$$

We call these the residual centroids of the inscribed triangle XYZ.

The following two propositions are very easily to established, by making the interchanges $(x, y, z) \leftrightarrow (1-x, 1-y, 1-z)$.

Proposition 5. *The triangles of residual centroids of isotomic inscribed triangles have equal areas.*

Proof. From the coordinates given above, we obtain the area of the triangle of residual centroids as

$$\frac{1}{27}\begin{vmatrix} 2+y-z & z & 1-y \\ 1-z & 2+z-x & x \\ y & 1-x & 2+x-y \end{vmatrix}\triangle$$
$$= \frac{1}{9}(3-x-y-z+xy+yz+zx)\triangle$$
$$= \frac{1}{9}(2+xyz+(1-x)(1-y)(1-z))\triangle$$

By effecting the interchanges $(x, y, z) \leftrightarrow (1-x, 1-y, 1-z)$, we obtain the area of the triangle of residual centroids of the isotomic inscribed triangle $X'Y'Z'$. This clearly remains unchanged. □

Proposition 6. *Let XYZ and $X'Y'Z'$ be isotomic inscribed triangles of ABC. The centroids of the following five triangles are collinear:*

- *G of triangle ABC,*
- *G_{XYZ} and $G_{X'Y'Z'}$ of the inscribed triangles,*
- *$\widetilde{G}$ and $\widetilde{G'}$ of the triangles of their residual centroids.*

Furthermore,

$$G_{XYZ}\widetilde{G} : \widetilde{G}G : G\widetilde{G'} : \widetilde{G'}G_{X'Y'Z'} = 1 : 2 : 2 : 1.$$

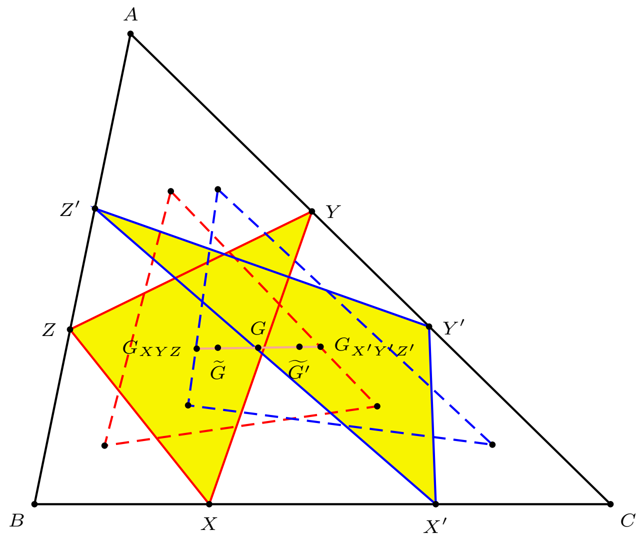


Figure 3

Proof. The centroid $\widetilde{G}$ is the point

$$\widetilde{G} = \frac{1}{9}((3+2y-2z)A + (3+2z-2x)B + (3+2x-2y)C).$$

We obtain the centroid $\widetilde{G'}$ by interchanging $(x, y, z) \leftrightarrow (1-x, 1-y, 1-z)$. From these coordinates and those given in (6,7), the collinearity is clear, and it is easy to figure out the ratios of division. □

3. Triangles of residual orthocenters

Proposition 7. *The triangles of residual orthocenters of isotomic inscribed triangles have equal areas.*

See Figure 4. This is an immediate corollary of the following proposition (see Figure 5), which in turn is a special case of a more general situation considered in Proposition 8 below.

Proposition 8. *An inscribed triangle and its triangle of residual orthocenters have equal areas.*

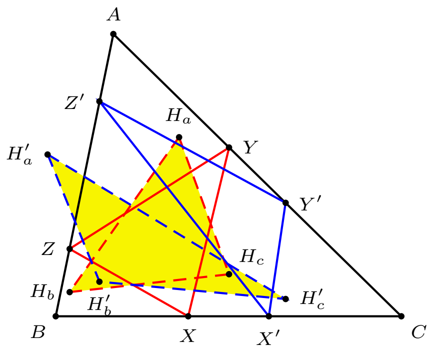


Figure 4

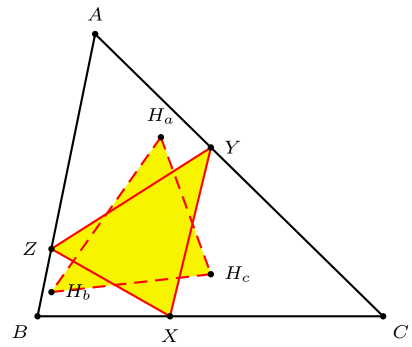


Figure 5

Proposition 9. *Given a triangle ABC, if pairs of parallel lines $\mathcal{L}_{1B}$, $\mathcal{L}_{1C}$ through B, C, $\mathcal{L}_{2C}$, $\mathcal{L}_{2A}$ through C, A, and $\mathcal{L}_{3A}$, $\mathcal{L}_{3B}$ through A, B are constructed, and if*

$$P_a = \mathcal{L}_{2C} \cap \mathcal{L}_{3B}, \qquad P_b = \mathcal{L}_{3A} \cap \mathcal{L}_{1C}, \qquad P_c = \mathcal{L}_{1B} \cap \mathcal{L}_{2A},$$

then the triangle $P_aP_bP_c$ has the same area as triangle ABC.

Proof. We write $Y = \mathcal{L}_{2C} \cap \mathcal{L}_{3A}$ and $Z = \mathcal{L}_{2A} \cap \mathcal{L}_{3B}$. Consider the parallelogram AZP_aY in Figure 6. If the points B and C divide the segments ZP_a and YP_a in the ratios

$$ZB : BP_a = v : 1 - v, \qquad YC : CP_a = w : 1 - w,$$

then it is easy to see that

$$\text{Area}(ABC) = \frac{1 + vw}{2} \cdot \text{Area}(AZP_aY). \tag{8}$$

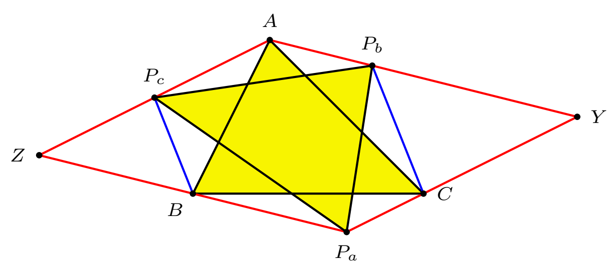


Figure 6

Now, P_b and P_c are points on AY and AZ such that BP_c and CP_b are parallel. If

$$YP_b : P_bA = v' : 1 - v', \qquad ZP_c : P_cA = w' : 1 - w',$$

then from the similarlity of triangles BZP_c and P_bYC, we have

$$ZB : ZP_c = YP_b : YC.$$

This means that $v : w' = v' : w$ and $v'w' = vw$. Now, in the same parallelogram AZP_aY, we have

$$\text{Area}(P_aP_bP_c) = \frac{1+v'w'}{2} \cdot \text{Area}(AZP_aY).$$

From this we conclude that $P_aP_bP_c$ and ABC have equal areas.

□

4. Triangles of residual circumcenters

Consider the circumcircles of the residuals of an inscribed triangle XYZ. By Miquel's theorem, the circles AYZ, BZX, and CXY have a common point. Furthermore, the centers O_a, O_b, O_c of these circles form a triangle similar to ABC. See, for example, [1, p.134]. We prove the following interesting theorem.

Theorem 10. *The triangles of residual circumcenters of the isotomic inscribed triangles are congruent.*

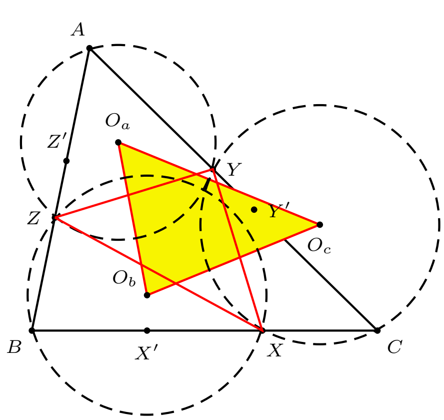


Figure 7A

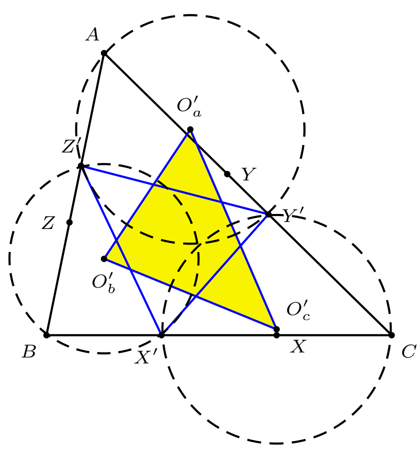


Figure 7B

We prove this theorem by calculations.

Lemma 11. *Let X, Y, Z be points on BC, CA, AB such that*

$$BX : XC = w : v, \quad CY : YA = u_c : w, \quad AZ : ZB = v : u_b.$$

The distance between the circumcenters O_b and O_c is the hypotenuse of a right triangle with one side $\frac{a}{2}$ and another side

$$\frac{(v-w)(u_b+v)(u_c+w)a^2 + (v+w)(w-u_c)(u_b+v)b^2 + (v+w)(w+u_c)(u_b-v)c^2}{8\triangle(u_b+v)(v+w)(w+u_c)} \cdot a. \tag{9}$$

Proof. The distance between O_b and O_c along the side BC is clearly $\frac{a}{2}$. We calculate their distance along the altitude on BC. The circumradius of BZX is clearly $R_b = \frac{ZX}{2\sin B}$. The distance of O_b above BC is

$$\begin{aligned}
R_b \cos BZX =& \frac{ZX\cos BZX}{2\sin B} = \frac{2BZ\cdot ZX\cos BZX}{4BZ\sin B} = \frac{BZ^2+ZX^2-BX^2}{4BZ\sin B}\\
=& \frac{BZ^2+BZ^2+BX^2-2BZ\cdot BX\cos B-BX^2}{4BZ\sin B}\\
=& \frac{BZ-BX\cos B}{2\sin B} = \frac{c(BZ-BX\cos B)}{4\triangle}\cdot a\\
=& \frac{c\left(\frac{u_b}{u_b+v}c-\frac{w}{v+w}a\cos B\right)}{4\triangle}\cdot a\\
=& \frac{u_b(v+w)2c^2-w(u_b+v)(c^2+a^2-b^2)}{8\triangle(u_b+v)(v+w)}\cdot a\\
=& \frac{-(u_b+v)w(a^2-b^2)+(2u_bv+u_bw-vw)c^2}{8\triangle(u_b+v)(v+w)}\cdot a
\end{aligned}$$

By making the interchanges $b \leftrightarrow c$, $v \leftrightarrow w$, and $u_b \leftrightarrow u_c$, we obtain the distance of O_c above the same line as

$$\frac{-(u_c+w)v(a^2-c^2)+(2u_cw+u_cv-vw)b^2}{8\triangle(u_c+w)(v+w)}\cdot a.$$

The difference between these two is the expression given in (9) above. $\square$

Consider now the isotomic inscribed triangle $X'Y'Z'$. We have

$$\begin{aligned}
BX' : X'C =& v : w,\\
CY' : Y'A =& w : u_c = \frac{vw}{u_c} : v,\\
AZ' : Z'B =& u_b : v = w : \frac{vw}{u_b}.
\end{aligned}$$

Let O_b' and O_c' be the circumcenters of $BZ'X'$ and $CX'Y'$. By making the following interchanges

$$v \leftrightarrow w, \qquad u_b \leftrightarrow \frac{vw}{u_b}, \qquad u_c \leftrightarrow \frac{vw}{u_c}$$

in (9), we obtain the distance between O_b' and O_c' along the altitude on BC as

$$\begin{aligned}
&\frac{(w-v)(\frac{vw}{u_b}+w)(\frac{vw}{u_c}+v)a^2+(v+w)(v-\frac{vw}{u_c})(\frac{vw}{u_b}+w)b^2+(v+w)(v+\frac{vw}{u_c})(\frac{vw}{u_b}-w)c^2}{8\triangle(\frac{vw}{u_b}+w)(v+w)(v+\frac{vw}{u_c})}\cdot a\\
=&\frac{(w-v)(v+u_b)(w+u_c)a^2+(v+w)(u_c-w)(v+u_b)b^2+(v+w)(w+u_c)(v-u_b)c^2}{8\triangle(v+u_b)(v+w)(u_c+w)}\cdot a.
\end{aligned}$$

Except for a reversal in sign, this is the same as (9).

From this we easily conclude that the segments O_bO_c and $O'_bO'_c$ are congruent. The same reasoning also yields the congruences of O_cO_a, $O'_cO'_a$, and of O_aO_b, $O'_aO'_b$. It follows that the triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are congruent. This completes the proof of Theorem 9.

5. Isotomic conjugates

Let XYZ be the cevian triangle of a point P, *i.e.*, X, Y, Z are respectively the intersections of the line pairs AP, BC; BP, CA; CP, AB. By the residual centroids ((respectively orthocenters, circumcenters) of P, we mean those of its cevian triangle. If we construct points X', Y', Z' satisfying (1), then the lines AX', BY', CZ' intersect at a point P' called the isotomic conjugate of P. If the point P has homogeneous barycentric coordinates $(x : y : z)$, then P' has homogeneous barycentric coordinates $\left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$. All results in the preceding sections apply to the case when XYZ and $X'Y'Z'$ are the cevian triangles of two isotomic conjugates. In particular, in the case of residual circumcenters in §4 above, if XYZ is the cevian triangle of P with homogeneous barycentric coordinates $(u : v : w)$, then

$$BX : XC = w : v, \qquad CY : YA = u : w, \qquad AZ : ZB = v : u.$$

By putting $u_b = u_c = u$ in (9) we obtain a necessary and sufficient condition for the line O_bO_c to be parallel to BC, namely,

$$(v-w)(u+v)(u+w)a^2+(v+w)(w-u)(u+v)b^2+(v+w)(w+u)(u-v)c^2 = 0.$$

This can be reorganized into the form

$$(b^2+c^2-a^2)u(v^2-w^2)+(c^2+a^2-b^2)v(w^2-u^2)+(a^2+b^2-c^2)w(u^2-v^2) = 0.$$

This is the equation of the Lucas cubic, consisting of points P for which the line joining P to its isotomic conjugate P' passes through the orthocenter H. The symmetry of this equation leads to the following interesting theorem.

Theorem 12. *The triangle of residual circumcenters of P is homothetic to ABC if and only if P lies on the Lucas cubic.*

It is well known that the Lucas cubic is the locus of point P whose cevian triangle is also the pedal triangle of a point Q. In this case, the circumcircles of AYZ, BZX and CXY intersect at Q, and the circumcenters O_a, O_b, O_c are the midpoints of the segments AQ, BQ, CQ. The triangle $O_aO_bO_c$ is homothetic to ABC at Q.

For example, if P is the Gergonne point, then $O_aO_bO_c$ is homothetic to ABC at the incenter I. The isotomic conjugate of P is the Nagel point, and $O'_aO'_bO'_c$ is homothetic to ABC at the reflection of I in the circumcenter O.

References

[1] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.

[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

[3] C. Kimberling, *Encyclopedia of Triangle Centers*, May 23 edition, available at http://www2.evansville.edu/ck6/encyclopedia/.

[4] P. Yiu, The uses of homogeneous barycentric coordinates in plane euclidean geometry, *Int. J. Math. Educ. Sci. Technol.*, 31 (2000) 569 – 578.

Mario Dalcín: Caribes 2364, C.P.11.600, Montevideo, Uruguay
E-mail address: filomate@adinet.com.uy

Forum Geometricorum
Volume 3 (2003) 135–144.

FORUM GEOM
ISSN 1534-1178

The M-Configuration of a Triangle

Alexei Myakishev

Abstract. We give an easy construction of points A_a, B_a, C_a on the sides of a triangle ABC such that the figure M path $BC_aA_aB_aC$ consists of 4 segments of equal lengths. We study the configuration consisting of the three figures M of a triangle, and define an interesting mapping of triangle centers associated with such an M-configuration.

1. Introduction

Given a triangle ABC, we consider points A_a on the line BC, B_a on the half line CA, and C_a on the half line BA such that $BC_a = C_aA_a = A_aB_a = B_aC$. We shall refer to $BC_aA_aB_aC$ as M$_a$, because it looks like the letter M when triangle ABC is acute-angled. See Figures 1a. Figure 1b illustrates the case when the triangle is obtuse-angled. Similarly, we also have M$_b$ and M$_c$. The three figures M$_a$, M$_b$, M$_c$ constitute the M-configuration of triangle ABC. See Figure 2.

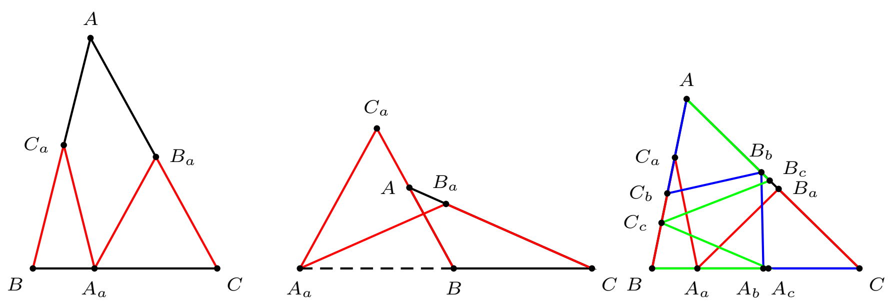


Figure 1a Figure 1b Figure 2

Proposition 1. *The lines AA_a, BB_a, CC_a concur at the point with homogeneous barycentric coordinates*

$$\left(\frac{1}{\cos A} : \frac{1}{\cos B} : \frac{1}{\cos C}\right).$$

Proof. Let l_a be the length of $BC_a = C_aA_a = A_aB_a = B_aC$. It is clear that the directed length $BA_a = 2l_a \cos B$ and $A_aC = 2l_a \cos C$, and $BA_a : A_aC = \cos B : \cos C$. For the same reason, $CB_b : B_bA = \cos C : \cos A$ and $AC_c : C_cB = \cos A : \cos B$. It follows by Ceva's theorem that the lines AA_a, BB_a, CC_a concur at the point with homogeneous barycentric coordinates given above.[1] □

Publication Date: June 30, 2003. Communicating Editor: Paul Yiu.
The author is grateful to the editor for his help in the preparation of this paper.

[1]This point appears in [3] as X_{92}.

Remark. Since $2l_a \cos B + 2l_a \cos C = a = 2R \sin A$, where R is the circumradius of triangle ABC,

$$l_a = \frac{a}{2(\cos B + \cos C)} = \frac{R \sin A}{\cos B + \cos C} = \frac{R \cos \frac{A}{2}}{\cos \frac{B-C}{2}}. \tag{1}$$

For later use, we record the absolute barycentric coordinates of A_a, B_a, C_a in terms of l_a:

$$\begin{aligned} A_a =& \frac{2l_a}{a}(\cos C \cdot B + \cos B \cdot C), \\ B_a =& \frac{1}{b}(l_a \cdot A + (b - l_a)C), \\ C_a =& \frac{1}{c}(l_a \cdot A + (c - l_a)B). \end{aligned} \tag{2}$$

2. Construction of $\mathbf{M}_a$

Proposition 2. *Let A' be the intersection of the bisector of angle A with the circumcircle of triangle ABC.*

(a) *A_a is the intersection of BC with the parallel to AA' through the orthocenter H.*

(b) *B_a (respectively C_a) is the intersection of CA (respectively BA) with the parallel to CA' (respectively BA') through the circumcenter O.*

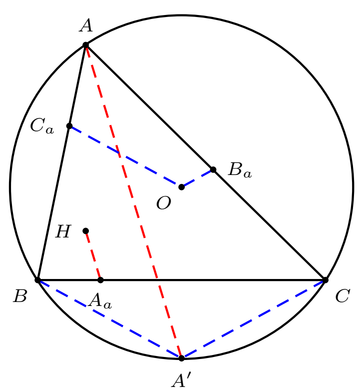


Figure 3a

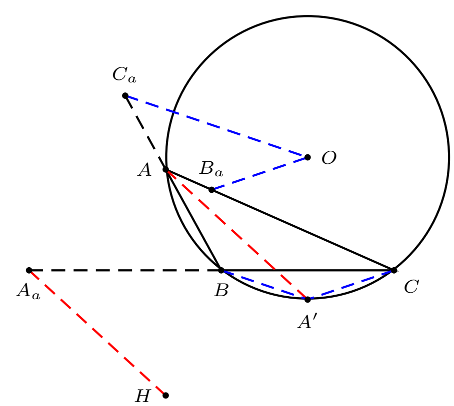


Figure 3b

Proof. (a) The line joining $A_a = (0 : \cos C : \cos B)$ to $H = \left(\frac{a}{\cos A} : \frac{b}{\cos B} : \frac{c}{\cos C}\right)$ has equation

$$\begin{vmatrix} 0 & \cos C & \cos B \\ \frac{a}{\cos A} & \frac{b}{\cos B} & \frac{c}{\cos C} \\ x & y & z \end{vmatrix} = 0.$$

This simplifies to

$$-(b-c)x\cos A + a(y\cos B - z\cos C) = 0.$$

It has infinite point

$$\begin{aligned}&(-a(\cos B+\cos C) : a\cos C-(b-c)\cos A : (b-c)\cos A + a\cos B)\\ =&(-a(\cos B+\cos C) : b(1-\cos A) : c(1-\cos A)).\end{aligned}$$

It is clear that this is the same as the infinite point $(-(b+c) : b : c)$, which is on the line joining A to the incenter.

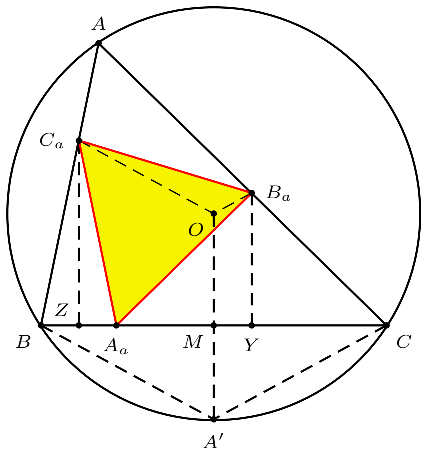


Figure 4

(b) Let M be the midpoint of BC, and Y, Z the pedals of B_a, C_a on BC. See Figure 4. We have

$$\begin{aligned}OM =&\frac{a}{2}\cot A = l_a(\cos B+\cos C)\cot A,\\ C_aZ =&l_a\sin B,\\ MZ =&\frac{a}{2} - l_a\cos B = l_a(\cos B+\cos C) - l_a\cos B = l_a\cos C.\end{aligned}$$

From this the acute angle between the line C_aO and BC has tangent ratio

$$\begin{aligned}\frac{C_aZ-OM}{MZ} =&\frac{\sin B-(\cos B+\cos C)\cot A}{\cos C}\\ =&\frac{\sin B\sin A-(\cos B+\cos C)\cos A}{\cos C\sin A}\\ =&\frac{-\cos(A+B)-\cos C\cos A}{\cos C\sin A} = \frac{\cos C(1-\cos A)}{\cos C\sin A}\\ =&\frac{1-\cos A}{\sin A} = \tan\frac{A}{2}.\end{aligned}$$

It follows that C_aO makes an angle $\frac{A}{2}$ with the line BC, and is parallel to BA'. The same reasoning shows that B_aO is parallel to CA'. $\square$

3. Circumcenters in the M-configuration

Note that $\angle B_aA_aC_a = \angle A$. It is clear that the circumcircles of $B_aA_aC_a$ and B_aAC_a are congruent. The circumradius is

$$R_a = \frac{l_a}{2\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)} = \frac{l_a}{2\cos\frac{A}{2}} = \frac{R}{2\cos\frac{B-C}{2}} \tag{3}$$

from (1).

Proposition 3. *The circumcircle of triangle AB_aC_a contains* (i) *the circumcenter O of triangle ABC,* (ii) *the orthocenter H_a of triangle $A_aB_aC_a$, and* (iii) *the midpoint of the arc BAC.*

Proof. (i) is an immediate corollary of Proposition 2(b) above.

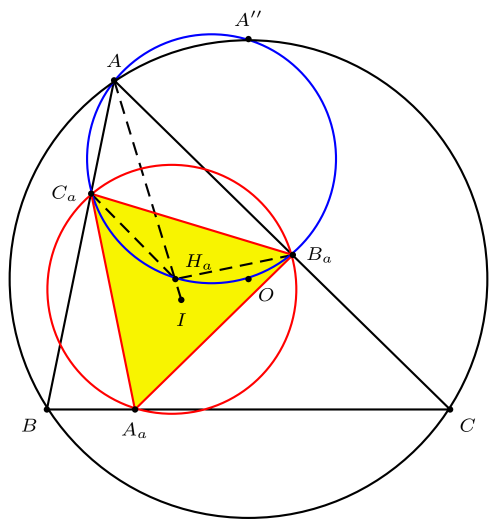


Figure 5

(ii) Let H_a be the orthocenter of triangle $A_aB_aC_a$. It is clear that

$$\angle B_aH_aC_a = \pi - \angle B_aA_aC_a = \pi - \angle BAC = \pi - \angle C_aAB_a.$$

It follows that H_a lies on the circumcircle of AB_aC_a. See Figure 5. Since the triangle $A_aB_aC_a$ is isosceles, $B_aH_a = C_aH_a$, and the point H_a lies on the bisector of angle A.

(iii) Let A'' be the midpoint of the arc BAC. By a simple calculation, $\angle AA''O = \frac{\pi}{2} - \frac{1}{2}|B-C|$. Also, $\angle AC_aO = \frac{\pi}{2} + \frac{1}{2}|B-C|$. [2] This shows that A'' also lies on the circle AB_aOC_a. □

The points B_a and C_a are therefore the intersections of the circle OAA'' with the sidelines AC and AB. This furnishes another simple construction of the figure M$_a$.

---

[2]This is $C + \frac{A}{2}$ if $C \geq B$ and $B + \frac{A}{2}$ otherwise.

Remarks. (1) If we take into consideration also the other figures M_b and M_c, we have three triangles AB_aC_a, BC_bA_b, CA_cB_c with their circumcircles intersecting at O.

(2) We also have three triangles A_aB_aCa, $A_bB_bC_b$, $A_cB_cC_c$ with their orthocenters forming a triangle perspective with ABC at the incenter I.

Proposition 4. *The circumcenter O_a of triangle $A_aB_aC_a$ is equidistant from O and H.*

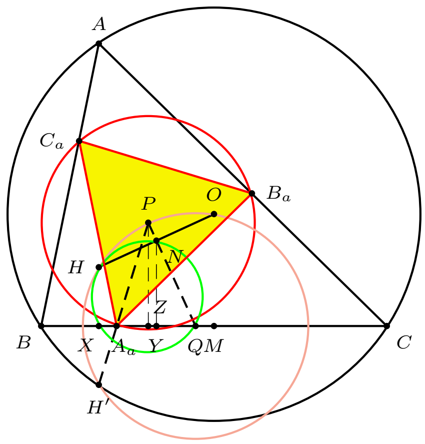


Figure 6

Proof. Construct the circle through O and H with center Q on the line BC. We prove that the midpoint P of the arc OH on the opposite side of Q is the circumcenter O_a of triangle $A_aB_aC_a$. See Figure 6. It will follow that O_a is equidistant from O and H. Let N be the midpoint of OH. Suppose the line PQ makes an angle φ with BC. Let X, Y, and M be the pedals of H, N, O on the line BC.

Since H, X, Q, N are concyclic, and the diameter of the circle containing them is $QH = \frac{NX}{\sin\varphi} = \frac{R}{2\sin\varphi}$. This is the radius of the circle OPH.

By symmetry, the circle OPH contains the reflection H' of H in the line BC.

$$\angle HH'P = \frac{1}{2}\angle HQP = \frac{1}{2}\angle HQN = \frac{1}{2}\angle HXN = \frac{1}{2}|B-C|.$$

Therefore, the angle between $H'P$ and BC is $\frac{\pi}{2} - \frac{1}{2}|B-C|$. It is obvious that the angle between A_aO_a and BC is the same. But from Proposition 2(a), the angle between HA_a and BC is the same too, so is the angle between the reflection $H'A_a$ and BC. From these we conclude that H', A_a, O_a and P are collinear. Now, let Z be the pedal of P on BC.

$$A_aP = \frac{PZ}{\cos\frac{1}{2}(B-C)} = \frac{QP\sin\varphi}{\cos\frac{1}{2}(B-C)} = \frac{R}{2\cos\frac{1}{2}(B-C)} = R_a.$$

Therefore, P is the circumcenter O_a of triangle $A_aB_aC_a$. □

Applying this to the other two figures M_b and M_c, we obtain the following remarkable theorem about the M-configuration of triangle ABC.

Theorem 5. *The circumcenters of triangles $A_aB_aC_a$, $A_bB_bC_b$, and $A_cB_cC_c$ are collinear. The line containing them is the perpendicular bisector of the segment OH.*

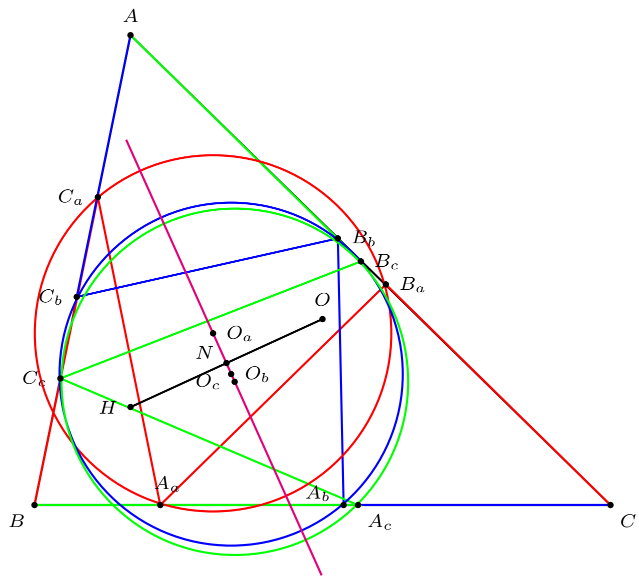


Figure 7

One can check without much effort that in homogeneous barycentric coordinates, the equation of this line is

$$\frac{\sin 3A}{\sin A}x + \frac{\sin 3B}{\sin B}y + \frac{\sin 3C}{\sin C}z = 0.$$

4. A central mapping

Let P be a triangle center in the sense of Kimberling [2, 3], given in homogeneous barycentric coordinates $(f(a,b,c) : f(b,c,a) : f(c,a,b))$ where $f = f_P$ satisfies $f(a,b,c) = f(a,c,b)$. If the reference triangle ABC is isosceles, say, with $AB = AC$, then P lies on the perpendicular bisector of BC and has coordinates of the form $(g_P : 1 : 1)$. The coordinate g depends only on the shape of the isosceles triangle, and we express it as a function of the *base* angle. We shall call $g = g_P$ the *isoscelized form* of the triangle center function f_P. Let P^* denote the isogonal conjugate of P.

Lemma 6. $g_{P^*}(B) = \frac{4\cos^2 B}{g_P(B)}$.

Proof. If $P = (g_P(B) : 1 : 1)$ for an isosceles triangle ABC with $B = C$, then

$$P^* = \left(\frac{\sin^2 A}{g_P(B)} : \sin^2 B : \sin^2 B\right) = \left(\frac{4\cos^2 B}{g_P(B)} : 1 : 1\right)$$

since $\sin^2 A = \sin^2(\pi - 2B) = \sin^2 2B = 4\sin^2 B\cos^2 B$. $\square$

Here are some examples.

Center	f_P	g_P
centroid	1	1
incenter	a	$2\cos B$
circumcenter	$a^2(b^2+c^2-a^2)$	$-2\cos 2B$
orthocenter	$\frac{1}{b^2+c^2-a^2}$	$\frac{-2\cos^2 B}{\cos 2B}$
symmedian point	a^2	$4\cos^2 B$
Gergonne point	$\frac{1}{s-a}$	$\frac{\cos B}{1-\cos B}$
Nagel point	$s-a$	$\frac{1-\cos B}{\cos B}$
Mittenpunkt	$a(s-a)$	$2(1-\cos B)$
Spieker point	$b+c$	$\frac{2}{1+2\cos B}$
X_{55}	$a^2(s-a)$	$4\cos B(1-\cos B)$
X_{56}	$\frac{a^2}{s-a}$	$\frac{4\cos^3 B}{1-\cos B}$
X_{57}	$\frac{a}{s-a}$	$\frac{2\cos^2 B}{1-\cos B}$

Consider a triangle center given by a triangle center function with isoscelized form $g = g_P$. The triangle center of the isosceles triangle C_aBA_a is the point $P_{a,b}$ with coordinates $(g(B) : 1 : 1)$ relative to C_aBA_a. Making use of the absolute barycentric coordinates of A_a, B_a, C_a given in (2), it is easy to see that this is the point

$$P_{a,b} = \left(\frac{g(B)l_a}{c} : \frac{g(B)(c-l_a)}{c} + 1 + \frac{2l_a}{a}\cos C : \frac{2l_a}{a}\cos B\right).$$

The same triangle center of the isosceles triangle B_aA_aC is the point

$$P_{a,c} = \left(\frac{g(C)l_a}{b} : \frac{2l_a}{a}\cos C : \frac{g(C)(b-l_a)}{b} + \frac{2l_a}{a}\cos B + 1\right).$$

It is clear that the lines $BP_{a,b}$ and $CP_{a,c}$ intersect at the point

$$\begin{aligned}
P_a &= \left(\frac{g(B)g(C)l_a^2}{bc} : \frac{2g(B)l_a^2\cos C}{ca} : \frac{2g(C)l_a^2\cos B}{ab}\right)\\
&= (ag(B)g(C) : 2bg(B)\cos C : 2cg(C)\cos B)\\
&= \left(\frac{ag(B)g(C)}{2\cos B\cos C} : \frac{bg(B)}{\cos B} : \frac{cg(C)}{\cos C}\right).
\end{aligned}$$

Figure 8 illustrates the case of the Gergonne point.

In the M-configuration, we may also consider the same triangle center (given in isoscelized form g_P of the triangle center function) in the isosceles triangles . These are the point $P_{b,c}$, $P_{b,a}$, $P_{c,a}$, $P_{c,b}$. The pairs of lines $CP_{b,c}$, $AP_{b,a}$ intersecting at P_b and $AP_{c,a}$, $BP_{c,b}$ intersecting at P_c. The coordinates of P_b and P_c can be

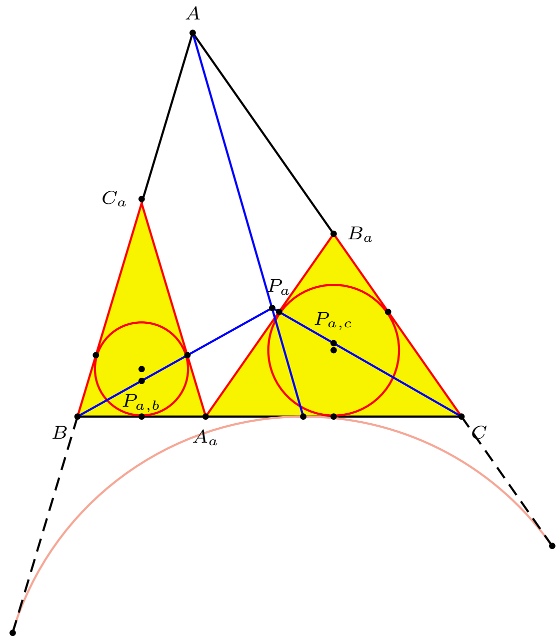


Figure 8

written down easily from those of P_a. From these coordinates, we easily conclude that that $P_aP_bP_c$ is perspective with triangle ABC at the point

$$\begin{aligned}\Phi(P) &= \left(\frac{ag_P(A)}{\cos A} : \frac{bg_P(B)}{\cos B} : \frac{cg_P(C)}{\cos C}\right)\\ &= \left(g_P(A)\tan A : g_P(B)\tan B : g_P(C)\tan C\right).\end{aligned}$$

Proposition 7. $\Phi(P^*) = \Phi(P)^*$.

Proof. We make use of Lemma 6.

$$\begin{aligned}\Phi(P^*) =& (g_{P^*}(A)\tan A : g_{P^*}(B)\tan B : g_{P^*}(C)\tan C)\\ =& \left(\frac{4\cos^2 A}{g_P(A)}\tan A : \frac{4\cos^2 B}{g_P(B)}\tan B : \frac{4\cos^2 C}{g_P(C)}\tan C\right)\\ =& \left(\frac{\sin^2 A}{g_P(A)\tan A} : \frac{\sin^2 B}{g_P(B)\tan B} : \frac{\sin^2 C}{g_P(C)\tan C}\right)\\ =& \Phi(P)^*.\end{aligned}$$

□

We conclude with some examples.

P	$\Phi(P)$	P^*	$\Phi(P^*) = \Phi(P)^*$
incenter	incenter		
centroid	orthocenter	symmedian point	circumcenter
circumcenter	X_{24}	orthocenter	X_{68}
Gergonne point	Nagel point	X_{55}	X_{56}
Nagel point	X_{1118}	X_{56}	$X_{1259} = X^*_{1118}$
Mittenpunkt	X_{34}	X_{57}	$X_{78} = X^*_{34}$

For the Spieker point, we have

$$\begin{aligned}\Phi(X_{10}) &= \left(\frac{\tan A}{1+2\cos A} : \frac{\tan B}{1+2\cos B} : \frac{\tan C}{1+2\cos C}\right)\\ &= \left(\frac{1}{a(b^2+c^2-a^2)(b^2+c^2-a^2+bc)} : \cdots : \cdots\right).\end{aligned}$$

This triangle center does not appear in the current edition of [3].

Remark. For $P = X_8$, the Nagel point, the point P_a has an alternative description. Antreas P. Hatzipolakis [1] considered the incircle of triangle ABC touching the sides CA and AB at Y and Z respectively, and constructed perpendiculars from Y, Z to BC intersecting the incircle again at Y' and Z'. See Figure 9. It happens that B, Z', $P_{a,b}$ are collinear; so are C, Y', $P_{a,c}$. Therefore, BZ' and CY' intersect at P_a. The coordinates of Y' and Z' are

$$\begin{aligned}Y' &= (a^2(b+c-a)(c+a-b) : (a^2+b^2-c^2)^2 : (b+c)^2(a+b-c)(c+a-b)),\\ Z' &= (a^2(b+c-a)(a+b-c) : (b+c)^2(c+a-b)(a+b-c) : (a^2-b^2+c^2)^2).\end{aligned}$$

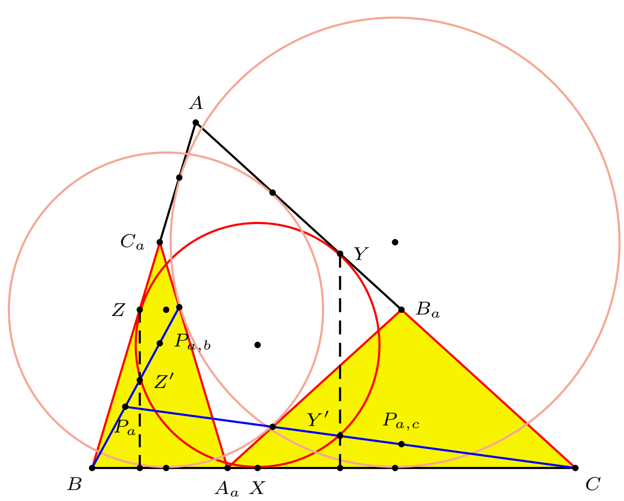


Figure 9

The lines BZ' and CY' intersect at

$$P_a = \left(a^2(b+c-a) : \frac{(a^2+b^2-c^2)^2}{c+a-b} : \frac{(a^2-b^2+c^2)^2}{a+b-c}\right)$$
$$= \left(\frac{a^2(b+c-a)}{(a^2-b^2+c^2)^2(a^2+b^2-c^2)^2} : \frac{1}{(c+a-b)(a^2-b^2+c^2)^2} : \frac{1}{(a+b-c)(a^2+b^2-c^2)^2}\right).$$

It was in this context that Hatzipolakis constructed the triangle center

$$X_{1118} = \left(\frac{1}{(b+c-a)(b^2+c^2-a^2)^2} : \cdots : \cdots\right).$$

References

[1] A. P. Hatzipolakis, Hyacinthos message 5321, April 30, 2002.
[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.
[3] C. Kimberling, *Encyclopedia of Triangle Centers*, May 23, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Alexei Myakishev: Smolnaia 61-2, 138, Moscow, Russia, 125445
E-mail address: alex_geom@mtu-net.ru

Forum Geometricorum
Volume 3 (2003) 145–159.

FORUM GEOM
ISSN 1534-1178

Rectangles Attached to Sides of a Triangle

Nikolaos Dergiades and Floor van Lamoen

Abstract. We study the figure of a triangle with a rectangle attached to each side. In line with recent publications on special cases we find concurrencies and study homothetic triangles. Special attention is given to the cases in which the attached rectangles are similar, have equal areas and have equal perimeters, respectively.

1. Introduction

In recent publications [3, 4, 10, 11, 12] the configurations have been studied in which rectangles or squares are attached to the sides of a triangle. In these publications the rectangles are all similar. In this paper we study the more general case in which the attached rectangles are not necessarily similar. We consider a triangle ABC with attached rectangles BCA_cA_b, CAB_aB_c and ABC_bC_a. Let u be the length of CA_c, positive if A_c and A are on opposite sides of BC, otherwise negative. Similarly let v and w be the lengths of AB_a and BC_b. We describe the shapes of these rectangles by the ratios

$$U = \frac{a}{u}, \qquad V = \frac{b}{v}, \qquad W = \frac{c}{w}. \tag{1}$$

The vertices of these rectangles are [1]

$$\begin{aligned} A_b &= (-a^2 : S_C + SU : S_B), & A_c &= (-a^2 : S_C : S_B + SU), \\ B_a &= (S_C + SV : -b^2 : S_A), & B_c &= (S_C : -b^2 : S_A + SV), \\ C_a &= (S_B + SW : S_A : -c^2), & C_b &= (S_B : S_A + SW : -c^2). \end{aligned}$$

Consider the flank triangles AB_aC_a, A_bBC_b and A_cB_cC. With the same reasoning as in [10], or by a simple application of Ceva's theorem, we can see that the triangle $H_aH_bH_c$ of orthocenters of the flank triangles is perspective to ABC with perspector

$$P_1 = \left(\frac{a}{u} : \frac{b}{v} : \frac{c}{w}\right) = (U : V : W). \tag{2}$$

Publication Date: August 25, 2003. Communicating Editor: Paul Yiu.

[1] All coordinates in this note are homogeneous barycentric coordinates. We adopt J. H. Conway's notation by letting $S = 2\Delta$ denote twice the area of ABC, while $S_A = \frac{-a^2+b^2+c^2}{2} = S\cot A$, $S_B = S\cot B$, $S_C = S\cot C$, and generally $S_{XY} = S_XS_Y$.

See Figure 1. On the other hand, the triangle $O_aO_bO_c$ of circumcenters of the flank triangles is clearly homothetic to ABC, the homothetic center being the point

$$P_2 = (au : bv : cw) = \left(\frac{a^2}{U} : \frac{b^2}{V} : \frac{c^2}{W}\right). \tag{3}$$

Clearly, P_1 and P_2 are isogonal conjugates.

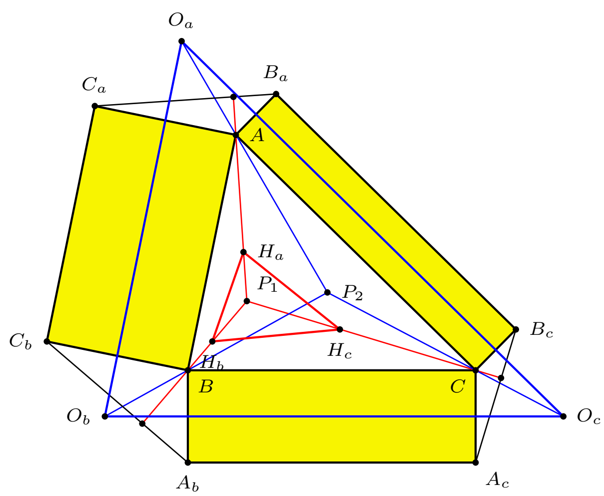


Figure 1

Now the perpendicular bisectors of B_aC_a, A_bC_b and A_cB_c pass through O_a, O_b and O_c respectively and are parallel to AP_1, BP_1 and CP_1 respectively. This shows that these perpendicular bisectors concur in a point P_3 on P_1P_2 satisfying

$$P_2P_1 : P_1P_3 = 2S : au + bv + cw,$$

where S is twice the area of ABC. See Figure 2. More explicitly,

$$\begin{aligned} P_3 =&(-a^2VW(V+W) + U^2(b^2W + c^2V) + 2SU^2VW \\ &: -b^2WU(W+U) + V^2(c^2U + a^2W) + 2SUV^2W) \\ &: -c^2UV(U+V) + W^2(a^2V + b^2U) + 2SUVW^2) \end{aligned} \tag{4}$$

This concurrency generalizes a similar result by Hoehn in [4], and was mentioned by L. Lagrangia [9]. It was also a question in the Bundeswettbewerb Mathematik Deutschland (German National Mathematics Competition) 1996, Second Round.

From the perspectivity of ABC and the orthocenters of the flank triangles, we see that ABC and the triangle $A'B'C'$ enclosed by the lines B_aC_a, A_bC_b and A_cB_c are orthologic. This means that the lines from the vertices of $A'B'C'$ to the corresponding sides of ABC are concurrent as well. The point of concurrency is the reflection of P_1 in O, *i.e.*,

$$P_4 = (-S_{BC}U + a^2 S_A(V+W) : \cdots : \cdots). \quad (5)$$

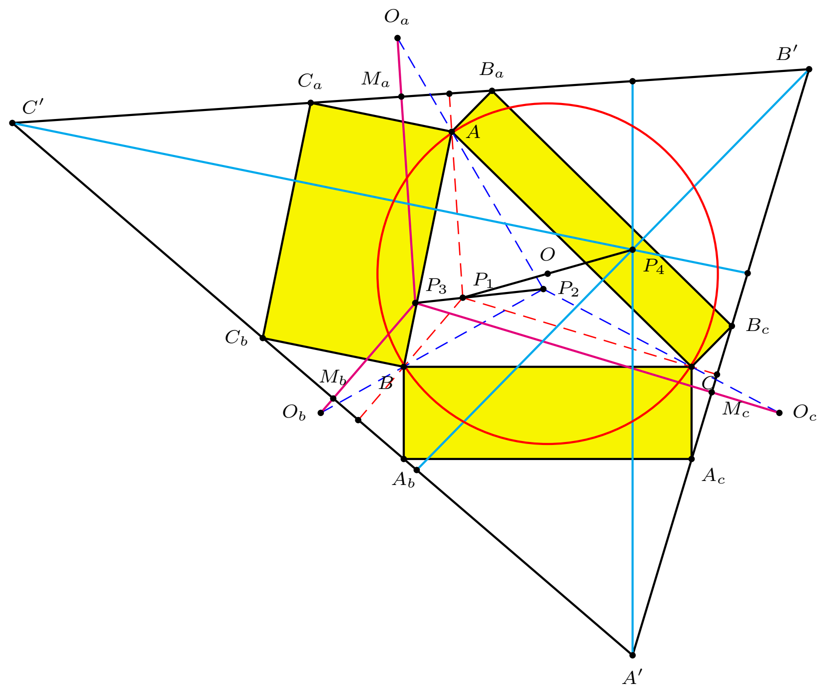


Figure 2

Remark. We record the coordinates of A'. Those of B' and C' can be written down accordingly.

$$\begin{aligned} A' =&(-(a^2S(U+V+W)+(a^2V+S_CU)(a^2W+S_BU)) \\ &:S_CS(U+V+W)+(b^2U+S_CV)(a^2W+S_BU) \\ &:S_BS(U+V+W)+(a^2V+S_CU)(c^2U+S_BW)). \end{aligned}$$

2. Special cases

We are mainly interested in three special cases.

2.1. *The similarity case.* This is the case when the rectangles are similar, *i.e.*, $U = V = W = t$ for some t. In this case, $P_1 = G$, the centroid, and $P_2 = K$, the symmedian point. As t varies,

$$P_3 = (b^2+c^2-2a^2+2St : c^2+a^2-2b^2+2St : a^2+b^2-2c^2+2St)$$

traverses the line GK. The point P_4, being the reflection of G in O, is X_{376} in [7]. The triangle $M_aM_bM_c$ is clearly perspective with ABC at the orthocenter H. More interestingly, it is also perspective with the medial triangle at

$$((S_A+St)(a^2+2St) : (S_B+St)(b^2+2St) : (S_C+St)(c^2+2St)),$$

which is the complement of the Kiepert perspector

$$\left(\frac{1}{S_A+St}:\frac{1}{S_B+S_t}:\frac{1}{S_C+St}\right).$$

It follows that as t varies, this perspector traverses the Kiepert hyperbola of the medial triangle. See [8].

The case $t=1$ is the *Pythagorean* case, when the rectangles are squares erected externally. The perspector of $M_aM_bM_c$ and the medial triangle is the point

$$O_1=(2a^4-3a^2(b^2+c^2)+(b^2-c^2)^2-2(b^2+c^2)S:\cdots:\cdots),$$

which is the center of the circle through the centers of the squares. See Figure 3. This point appears as X_{641} in [7].

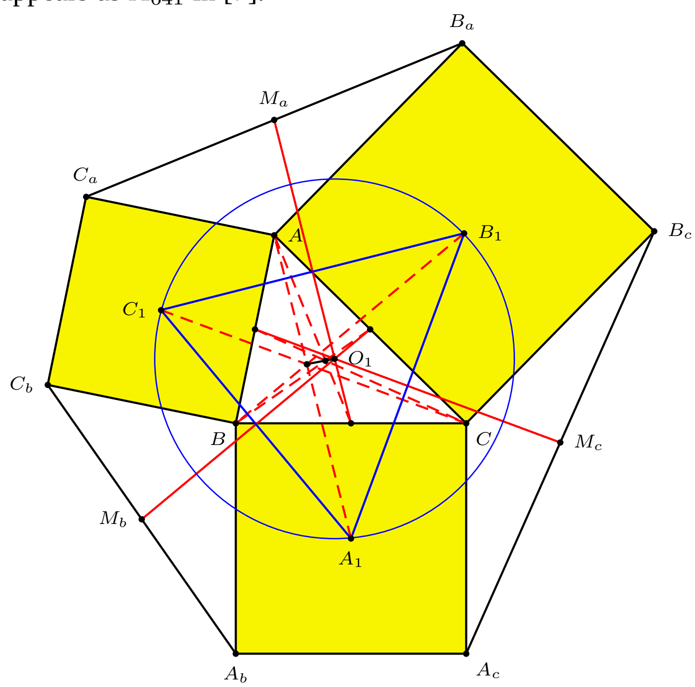


Figure 3

2.2. *The equiareal case.* When the rectangles have equal areas $\frac{T}{2}$, *i.e.*, $(U,V,W)=\left(\frac{2a^2}{T},\frac{2b^2}{T},\frac{2c^2}{T}\right)$, it is easy to see that $P_1=K$, $P_2=G$, and

$$\begin{aligned}P_4=&(a^2(-S_{BC}+S_A(b^2+c^2)):\cdots:\cdots)\\=&(a^2(a^4+2a^2(b^2+c^2)-(3b^4+2b^2c^2+3c^4)):\cdots:\cdots)\end{aligned}$$

is the reflection of K in O.[2] The *special equiareal case* is when $T=S$, the rectangles having the same area as triangle ABC. See Figure 4. In this case,

$$P_3=(6a^2-b^2-c^2:6b^2-c^2-a^2:6c^2-a^2-b^2).$$

[2]This point is not in the current edition of [7].

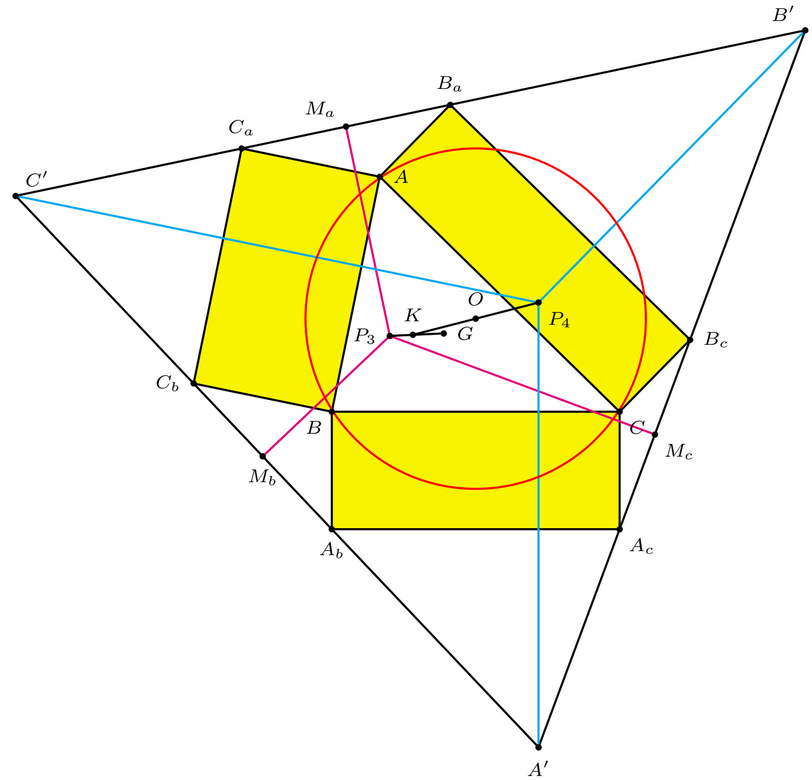


Figure 4

2.3. *The isoperimetric case.* This is the case when the rectangles have equal perimeters $2p$, *i.e.*, $(u, v, w) = (p - a, p - b, p - c)$. The *special isoperimetric* case is when $p = s$, the semiperimeter, the rectangles having the same perimeter as triangle ABC. In this case, $P_1 = X_{57}$, $P_2 = X_9$, the Mittenpunkt, and

$$\begin{aligned}P_3 =&(a(bc(2a^2 - a(b+c) - (b-c)^2) + 4(s-b)(s-c)S) : \cdots : \cdots),\\ P_4 =&(a(a^6 - 2a^5(b+c) - a^4(b^2 - 10bc + c^2) + 4a^3(b+c)(b^2 - bc + c^2)\\ &- a^2(b^4 + 8b^3c - 2b^2c^2 + 8c^3b + c^4) - 2a(b+c)(b-c)^2(b^2+c^2)\\ &+ (b+c)^2(b-c)^4) : \cdots : \cdots).\end{aligned}$$

These points can be described in terms of division ratios as follows.[3]

$$\begin{aligned}P_3X_{57} : X_{57}X_9 =& 4R + r : 2s,\\ P_4I : IX_{57} =& 4R : r.\end{aligned}$$

3. A pair of homothetic triangles

Let A_1, B_1 and C_1 be the centers of the rectangles BCA_cA_b, CAB_aB_c and ABC_bC_a respectively, and $A_2B_2C_2$ the triangle bounded by the lines B_cC_b, C_aA_c and A_bB_a. Since, for instance, segments B_1C_1 and B_cC_b are homothetic through

[3]These points are not in the current edition of [7].

A, the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are homothetic. See Figure 5. Their homothetic center is the point

$$P_5 = \left(-a^2S_A(V+W) + U(S_B+SW)(S_C+SV) : \cdots : \cdots\right).$$

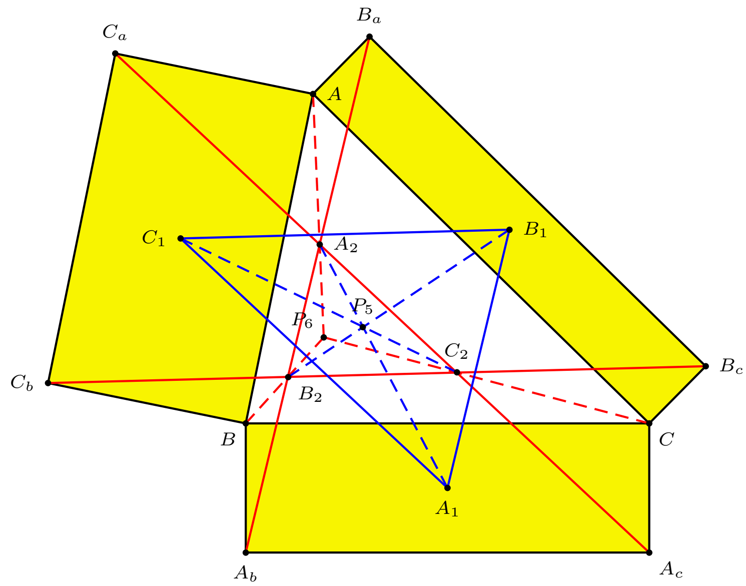


Figure 5

For the Pythagorean case with squares attached to triangles, *i.e.*, $U = V = W = 1$, Toshio Seimiya and Peter Woo [12] have proved the beautiful result that the areas Δ_1 and Δ_2 of $A_1B_1C_1$ and $A_2B_2C_2$ have geometric mean Δ. See Figure 5. We prove a more general result by computation using two fundamental area formulae.

Proposition 1. *For $i = 1, 2, 3$, let P_i be finite points with homogeneous barycentric coordinates $(x_i : y_i : z_i)$ with respect to triangle ABC. The oriented area of the triangle $P_1P_2P_3$ is*

$$\frac{\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}}{(x_1+y_1+z_1)(x_2+y_2+z_2)(x_3+y_3+z_3)} \cdot \Delta.$$

A proof of this proposition can be found in [1, 2].

Proposition 2. *For $i = 1, 2, 3$, let ℓ_i be a finite line with equation $p_ix + q_iy + r_iz = 0$. The oriented area of the triangle bounded by the three lines ℓ_1, ℓ_2, ℓ_3 is*

$$\frac{\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}^2}{D_1 \cdot D_2 \cdot D_3} \cdot \Delta,$$

where

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} p_1 & q_1 & r_1 \\ 1 & 1 & 1 \\ p_3 & q_3 & r_3 \end{vmatrix}, \quad D_3 = \begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ 1 & 1 & 1 \end{vmatrix}.$$

A proof of this proposition can be found in [5].

Theorem 3. $\frac{\Delta_1\Delta_2}{\Delta^2} = \frac{(U+V+W-UVW)^2}{4(UVW)^2}$.

Proof. The coordinates of A_1, B_1, C_1 are

$$\begin{aligned} A_1 =& (-a^2 : S_C + SU : S_B + SU), \\ B_1 =& (S_C + SV : -b^2 : S_A + SV), \\ C_1 =& (S_B + SW : S_A + SW : -c^2). \end{aligned}$$

By Proposition 1, the area of triangle $A_1B_1C_1$ is

$$\Delta_1 = \frac{S(U+V+W+UVW) + (a^2VW + b^2WU + c^2UV)}{4SUVW} \cdot \Delta. \tag{6}$$

The lines B_cC_b, C_aA_c, A_bB_a have equations

$$\begin{aligned} (S(1-VW) - S_A(V+W))x + (S+S_BV)y + (S+S_CW)z =& 0, \\ (S+S_AU)x + (S(1-WU) - S_B(W+U))y + (S+S_CW)z =& 0, \\ (S+S_AU)x + (S+S_BV)y + (S(1-UV) - S_C(U+V))z =& 0. \end{aligned}$$

By Proposition 2, the area of the triangle bounded by these lines is

$$\Delta_2 = \frac{S(U+V+W-UVW)^2}{UVW(S(U+V+W+UVW) + (a^2VW + b^2WU + c^2UV))} \cdot \Delta. \tag{7}$$

From (6, 7), the result follows. $\square$

Remarks. (1) The ratio of homothety is

$$\frac{-S(U+V+W-UVW)}{2(S(U+V+W+UVW) + (a^2VW + b^2WU + c^2UV))}.$$

(2) We record the coordinates of A_2 below. Those of B_2 and C_2 can be written down accordingly.

$$\begin{aligned} A_2 =& (-a^2((S+S_AU)(V+W) + SU(1-VW)) + (S_B+SW)(S_C+SV)U^2 \\ & : (S+S_AU)(SUV + S_C(U+V+W)) \\ & : (S+S_AU)(SUW + S_B(U+V+W))). \end{aligned}$$

From the coordinates of $A_2B_2C_2$ we see that this triangle is perspective to ABC at the point

$$P_6 = \left(\frac{1}{S_A(U+V+W) + SVW} : \cdots : \cdots\right).$$

4. Examples

4.1. *The similarity case.* If the rectangles are similar, $U = V = W = t$, then

$$P_6 = \left(\frac{1}{3S_A + St} : \frac{1}{3S_B + St} : \frac{1}{3S_C + St}\right)$$

traverses the Kiepert hyperbola. In the Pythagorean case, the homothetic center P_5 is the point

$$((S_B - S)(S_C - S) - 4S_{BC} : (S_C - S)(S_A - S) - 4S_{CA} : (S_A - S)(S_B - S) - 4S_{AB}).$$

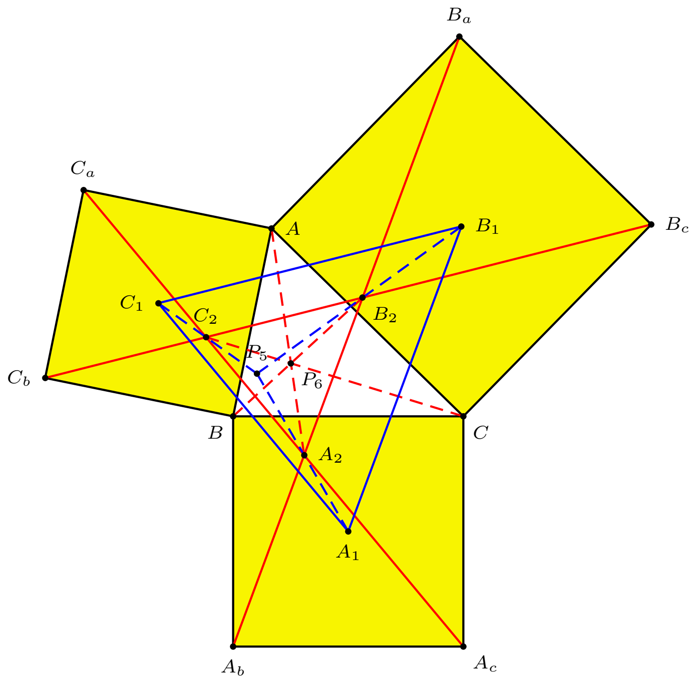


Figure 6

4.2. *The equiareal case.* For $(U, V, W) = (\frac{2a^2}{T}, \frac{2b^2}{T}, \frac{2c^2}{T})$, we have

$$P_6 = \left(\frac{1}{T(a^2 + b^2 + c^2)S_A + 2Sb^2c^2} : \cdots : \cdots\right).$$

This traverses the Jerabek hyperbola as T varies. When the rectangles have the same area as the triangle, the homothetic center P_5 is the point

$$(a^2((a^2 + 3b^2 + 3c^2)^2 - 4(4b^4 - b^2c^2 + 4c^4)) : \cdots : \cdots).$$

5. More homothetic triangles

Let $\mathcal{C}_A$, $\mathcal{C}_B$ and $\mathcal{C}_C$ be the circumcricles of rectangles BCA_cA_b, CAB_aB_c and ABC_bC_a respectively. See Figure 7. Since the circle $\mathcal{C}_A$ passes through B and C, its equation is of the form

$$a^2yz + b^2zx + c^2xy - px(x + y + z) = 0.$$

Since the same circle passes through A_b, we have $p = \frac{S_A U + S}{U} = S_A + \frac{S}{U}$. By the same method we derive the equations of the three circles:

$$a^2yz + b^2zx + c^2xy = (S_A + \frac{S}{U})x(x+y+z),$$
$$a^2yz + b^2zx + c^2xy = (S_B + \frac{S}{V})y(x+y+z),$$
$$a^2yz + b^2zx + c^2xy = (S_C + \frac{S}{W})z(x+y+z).$$

From these, the radical center of the three circles is the point

$$J = \left(\frac{1}{S_A + \frac{S}{U}} : \frac{1}{S_B + \frac{S}{V}} : \frac{1}{S_C + \frac{S}{W}}\right) = \left(\frac{U}{S_AU + S} : \frac{V}{S_BV + S} : \frac{W}{S_CW + S}\right).$$

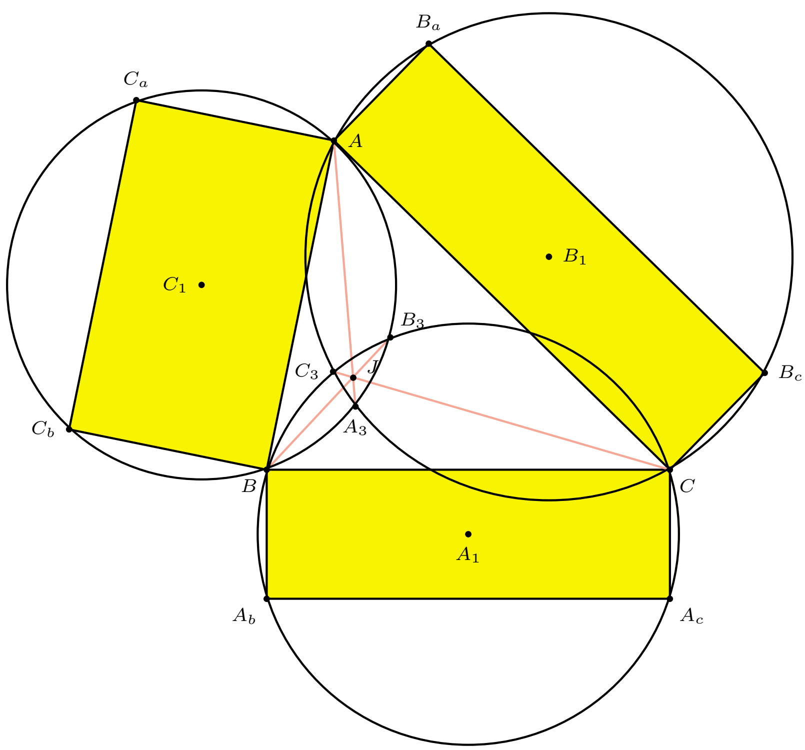


Figure 7

Note that the isogonal conjugate of J is the point

$$J^* = \left(a^2S_A + S \cdot \frac{a^2}{U} : b^2S_B + S \cdot \frac{b^2}{V} : c^2S_C + S \cdot \frac{c^2}{W}\right).$$

It lies on the line joining O to P_2. In fact,

$$P_2J^* : J^*O = 2S : au + bv + cw = P_2P_1 : P_1P_3.$$

The circles $\mathcal{C}_B$ and $\mathcal{C}_C$ meet at A and a second point A_3, which is the reflection of A in B_1C_1. See Figure 8. In homogeneous barycentric coordinates,

$$A_3 = \left(\frac{V+W}{S_A(V+W)-S(1-VW)} : \frac{V}{S_BV+S} : \frac{W}{S_CW+S}\right).$$

Similarly we have points B_3 and C_3. Clearly, the radical center J is the perspector of ABC and $A_3B_3C_3$.

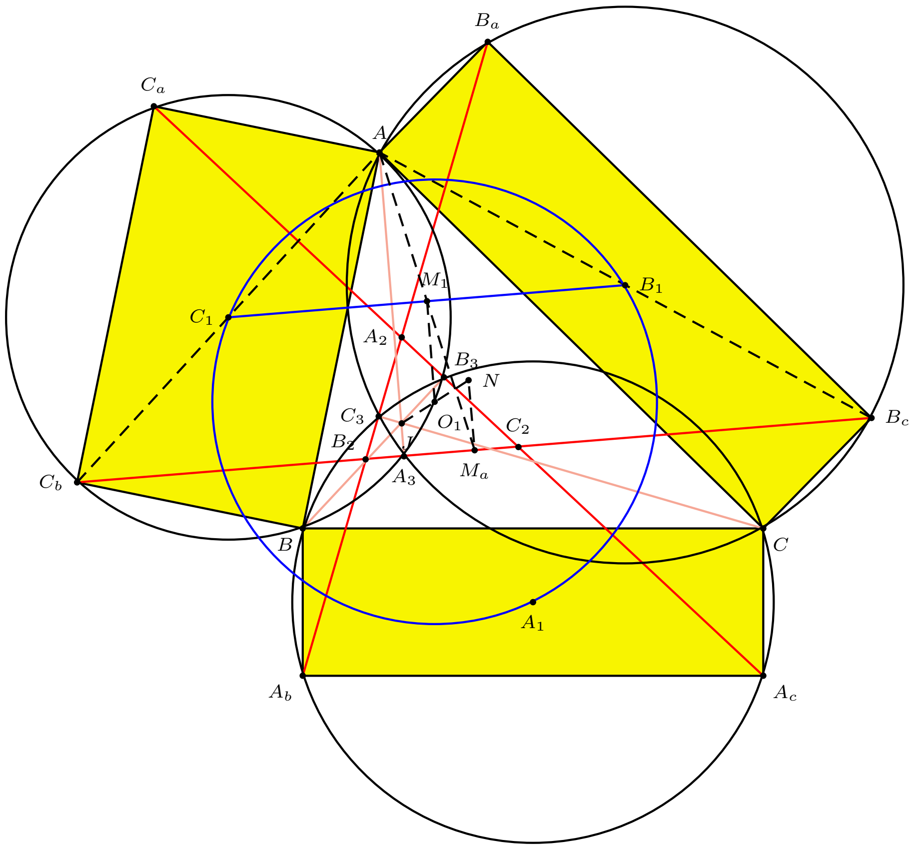


Figure 8

Proposition 4. *The triangles ABC and $A_2B_2C_2$ are orthologic. The perpendiculars from the vertices of one triangle to the corresponding lines of the other triangle concur at the point J.*

Proof. As C_1B_1 bisects AA_3, we see A_3 lies on B_cC_b and $AJ \perp B_cC_b$. Similarly, we have $BJ \perp C_aA_c$ and $CJ \perp A_bB_a$. The perpendiculars from A, B, C to the corresponding sides of $A_2B_2C_2$ concur at J.

On the other hand, the points B, C_3, B_3, C are concyclic and B_3C_3 is antiparallel to BC with respect to triangle JBC. The quadrilateral $JB_3A_2C_3$ is cyclic, with JA_2 as a diameter. It is known that every perpendicular to JA_2 is antiparallel to

B_3C_3 with respect to triangle JB_3C_3. Hence, $A_2J \perp BC$. Similarly, $B_2J \perp CA$ and $C_2J \perp AB$. □

It is clear that the perpendiculars from A_3, B_3, C_3 to the corresponding sides of triangle $A_2B_2C_2$ intersect at J. Hence, the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are orthologic.

Proposition 5. *The perpendiculars from A_2, B_2, C_2 to the corresponding sides of $A_3B_3C_3$ meet at the reflection of J in the circumcenter O_3 of triangle $A_3B_3C_3$.*

Proof. Since triangle $A_3B_3C_3$ is the pedal triangle of J in $A_2B_2C_2$, and A_2J passes through the circumcenter of triangle $A_2B_3C_3$, the perpendicular from A_2 to B_3C_3 passes through the orthocenter of $A_2B_3C_3$ and is isogonal to A_2J in triangle $A_2B_2C_2$. This line therefore passes through the isogonal conjugate of J in $A_2B_2C_2$. We denote this point by $J^!$. Similarly, the perpendiculars from B_2, C_2 to the sides C_3A_3 and A_3B_3 pass through $J^!$. The circumcircle of $A_3B_3C_3$ is the pedal circle of J. Hence, its circumcenter O_3 is the midpoint of $JJ^!$. It follows that $J^!$ is the reflection of J in O_3. □

Remark. The point J and the circumcenters O and O_3 of triangles ABC and $A_3B_3C_3$ are collinear. This is because $|JA \cdot JA_3| = |JB \cdot JB_3| = |JC \cdot JC_3|$, say, $= d^2$, and an inversion in the circle (J, d) transforms ABC into $A_3B_3C_3$ or its reflection in J.

Theorem 6. *The perpendicular bisectors of B_cC_b, C_aA_c, A_bB_a are concurrent at a point which is the reflection of J in the circumcenter O_1 of triangle $A_1B_1C_1$.*

Proof. Let M_1 and M_a be the midpoints of B_1C_1 and B_cC_b respectively. Note that M_1 is also the midpoint of AM_a. Also, let O_1 be the circumcenter of $A_1B_1C_1$, and the perpendicular bisector of B_cC_b meet JO_1 at N. See Figure 8. Consider the trapezium AM_aNJ. Since O_1M_1 is parallel to AJ, we conclude that O_1 is the midpoint of JN. Similarly the perpendicular bisectors of C_aA_c, A_bB_a pass through N, which is the reflection of J in O_1. □

We record the coordinates of O_1:

$$\begin{aligned}
&((c^2U^2V - a^2VW(V+W) + b^2WU(W+U) \\
&\quad + UVW((S_A+3S_B)UV + (S_A+3S_C)UW))S \\
&\quad + c^2S_BU^2V^2 + b^2S_CU^2W^2 - a^4V^2W^2 \\
&\quad + (S^2+S_{BC})U^2V^2W^2 + 4S^2U^2VW) \\
&: \cdots : \cdots)
\end{aligned}$$

In the Pythagorean case, the coordinates of O_1 are given in §2.1.

6. More triangles related to the attached rectangles

Write $U = \tan\alpha$, $V = \tan\beta$, and $W = \tan\gamma$ for angles α, β, γ in the range $(-\frac{\pi}{2}, \frac{\pi}{2})$. The point A_4 for which the swing angles CBA_4 and BCA_4 are β and γ

respectively has coordinates

$$(-a^2 : S_C + S \cdot \cot\gamma : S_B + S \cdot \cot\beta) = \left(-a^2 : S_C + \frac{S}{W} : S_B + \frac{S}{V}\right).$$

It is clear that this point lies on the line AJ. See Figure 9. If B_4 and C_4 are analogously defined, the triangles $A_4B_4C_4$ and ABC are perspective at J.

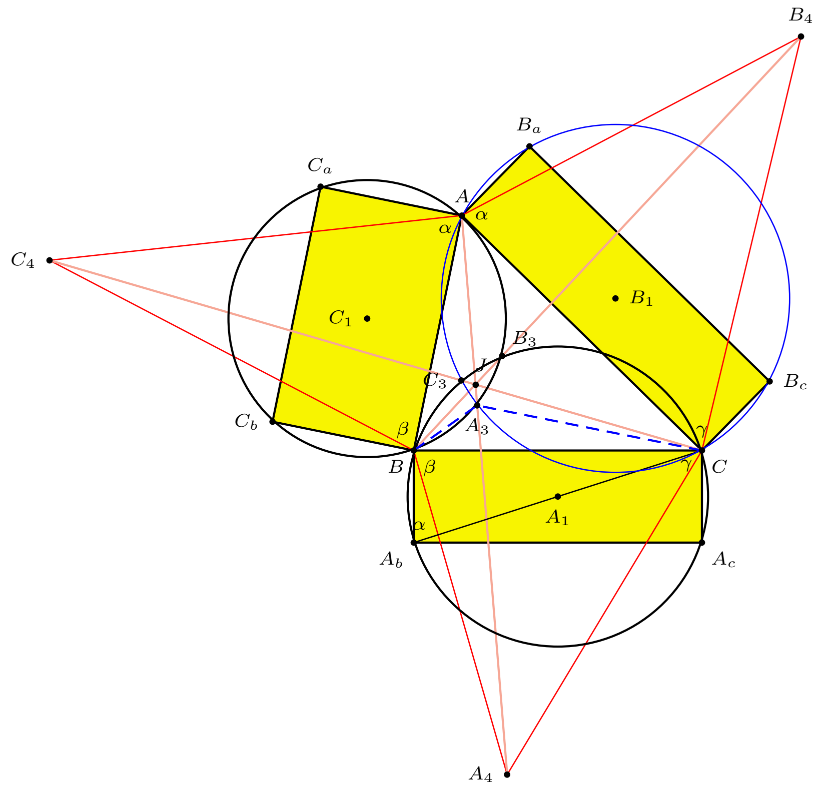


Figure 9

Note that A_3, B, A_4, C are concyclic since $\angle A_4BC = \beta = \angle AB_cV = \angle A_4A_3C$.

Let $d_1 = B_cC_b$, $d_2 = C_aA_c$, $d_3 = A_bB_a$, $d'_1 = AA_4$, $d'_2 = BB_4$, $d'_3 = CC_4$.

Proposition 7. *The ratios* $\frac{d_i}{d'_i}$, $i = 1, 2, 3$, *are independent of triangle* ABC. *More precisely,*

$$\frac{d_1}{d'_1} = \frac{1}{V} + \frac{1}{W}, \qquad \frac{d_2}{d'_2} = \frac{1}{W} + \frac{1}{U}, \qquad \frac{d_3}{d'_3} = \frac{1}{U} + \frac{1}{V}.$$

Proof. Since $AA_4 \perp C_bB_c$, the circumcircle of the cyclic quadrilateral A_3BA_4C meets C_bB_c besides A_3 at the antipode A_5 of A_4. See Figure 10. Let f, g, h denote, for vectors, the compositions of a rotation by $\frac{\pi}{2}$, and homotheties of ratios

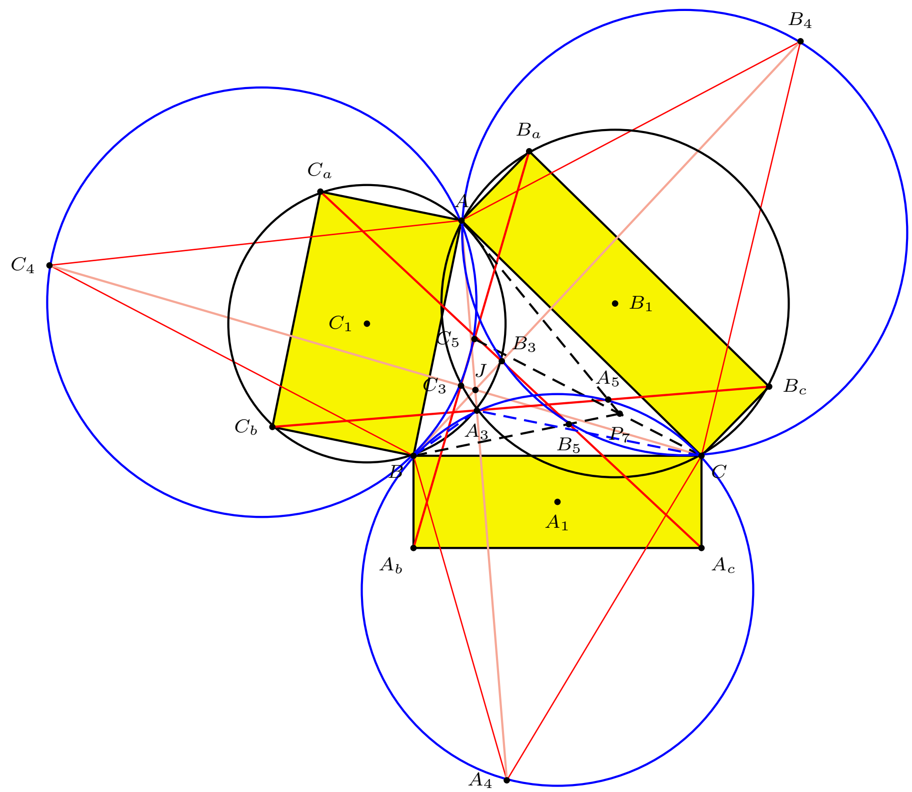


Figure 10

$\frac{1}{U}$, $\frac{1}{V}$, and $\frac{1}{W}$ respectively. Then

$$g(\overrightarrow{AA_4}) = g(\overrightarrow{AC}) + g(\overrightarrow{CA_4}) = \overrightarrow{CB_c} + \overrightarrow{A_5C} = \overrightarrow{A_5B_c},$$

and $\frac{A_5B_c}{AA_4} = \frac{1}{V}$. Similarly, $h(\overrightarrow{AA_4}) = \overrightarrow{C_bA_5}$, and $\frac{C_bA_5}{AA_4} = \frac{1}{W}$. It follows that $\frac{d_1}{d_1'} = \frac{1}{V} + \frac{1}{W}$. $\square$

The coordinates of A_5 can be seen immediately: Since A_4A_5 is a diameter of the circle (A_4BC), we see that $\angle BCA_5 = -\frac{\pi}{2} + \angle BCA_4$, and

$$A_5 = (-a^2 : S_C - SW : S_B - SV).$$

Similarly, we have the coordinates of B_5 and C_5. From these, it is clear that $A_5B_5C_5$ and ABC are perspective at

$$P_7 = \left(\frac{1}{S_A - SU} : \frac{1}{S_B - SV} : \frac{1}{S_C - SW}\right) = \left(\frac{1}{\cot A - U} : \frac{1}{\cot B - V} : \frac{1}{\cot C - W}\right).$$

For example, in the similarity case it is obvious from the above proof that the points A_5, B_5, C_5 are the midpoints of B_cC_b, C_aA_c, A_bB_a. Clearly in the Pythagorean case, the points A_4, B_4, C_4 coincide with A_1, B_1, C_1 respectively.

In this case, J is the Vecten point and from the above proof we have $d_1 = 2d_1'$, $d_2 = 2d_2'$, $d_3 = 2d_3'$ and $P_7 = X_{486}$.

7. Another interesting special case

If $\alpha + \beta + \gamma = \pi$, then $U + V + W = UVW$. From Theorem 3 we conclude that $\triangle_2 = 0$, and the points A_2, B_2, C_2, A_3, B_3, C_3 coincide with J, which now is the common point of the circumcircles of the three rectangles. Also, the points A_4, B_4, C_4 lie on the circles $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$ respectively.

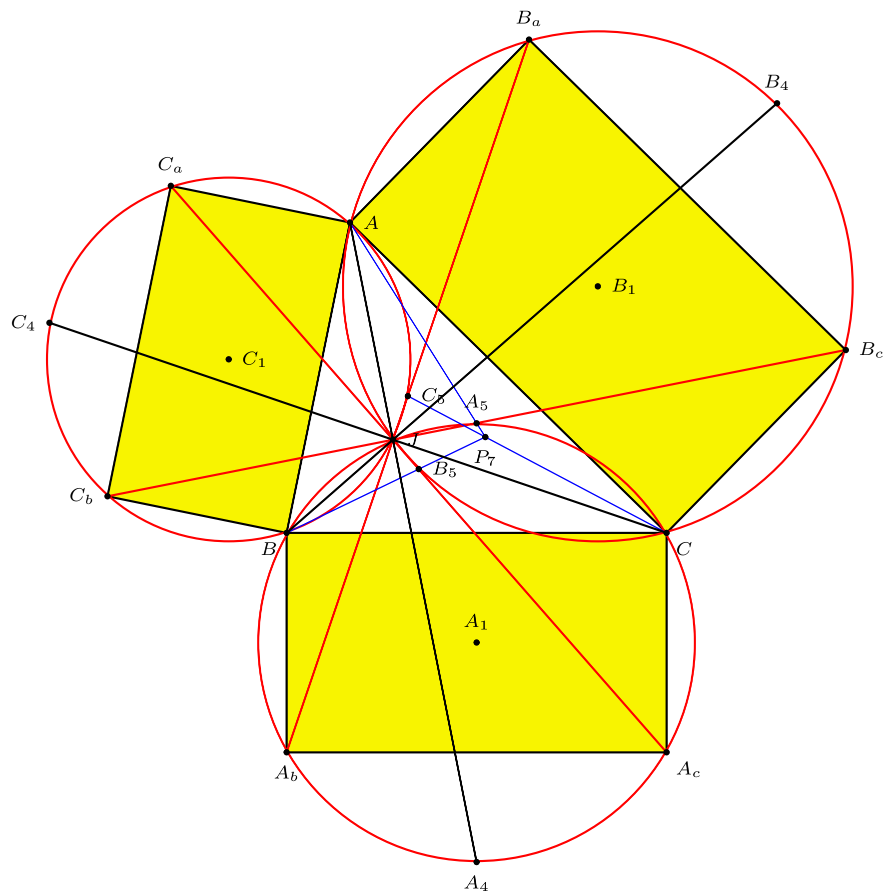


Figure 11

In Figure 11 we illustrate the case $\alpha = \beta = \gamma = \frac{\pi}{3}$. In this case, J is the Fermat point. The triangles BCA_4, CAB_4, ABC_4 are the Fermat equilateral triangles, and the angles of the lines AA_4, BB_4, CC_4, B_cC_b, C_aA_c, A_bB_a around J are $\frac{\pi}{6}$. The points A_5, B_5, C_5 are the mid points of B_cC_b, C_aA_c, A_bB_a. Also, $d_1' = d_2' = d_3'$, and $d_1 = d_2 = d_3 = \frac{2\sqrt{3}}{3}d_1'$. In this case, P_7 is the second Napoleon point, the point X_{18} in [7].

References

[1] O. Bottema, *Hoofdstukken uit de elementaire meetkunde*, 2nd ed. 1987, Epsilon Uitgaven, Utrecht.
[2] O. Bottema, On the area of a triangle in barycentric coordinates, *Crux Math.*, 8 (1982) 228–231.
[3] Z. Čerin, Loci related to variable flanks, *Forum Geom.*, 2 (2002) 105–113.
[4] L. Hoehn, Extriangles and excevians, *Math. Magazine*, 67 (2001) 188–205.
[5] G. A. Kapetis, *Geometry of the Triangle*, vol. A (in Greek), Zitis, Thessaloniki, 1996.
[6] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1–285.
[7] C. Kimberling, *Encyclopedia of Triangle*, *Centers*, July 1, 2003 edition, available at http://faculty.evansville.edu/ck6/encyclopedia, (2000-2003).
[8] S. Kotani, H. Fukagawa, and P. Penning, Problem 1759, *Crux Math.*, 16 (1990) 240; solution, 17 (1991) 307–309.
[9] L. Lagrangia, Hyacinthos 6948, April 13, 2003.
[10] F. M. van Lamoen, Friendship among triangle centers, *Forum Geom.*, 1 (2001) 1–6.
[11] C. R. Panesachar and B. J. Venkatachala, On a curious duality in triangles, *Samasyā*, 7 (2001), number 2, 13–19.
[12] T. Seimiya and P. Woo, Problem 2635, *Crux Math.*, 27 (2001) 215; solution, 28 (2002) 264–266.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr

Floor van Lamoen: St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands
E-mail address: f.v.lamoen@wxs.nl

Forum Geometricorum
Volume 3 (2003) 161–167.

FORUM GEOM
ISSN 1534-1178

A Generalization of the Lemoine Point

Charles Thas

Abstract. It is known that the Lemoine point K of a triangle in the Euclidean plane is the point of the plane where the sum of the squares of the distances d_1, d_2, and d_3 to the sides of the triangle takes its minimal value. There are several ways to generalize the Lemoine point. First, we can consider $n \geq 3$ lines u_1, $\dots$, u_n instead of three in the Euclidean plane and search for the point which minimalizes the expression $d_1^2 + \cdots + d_n^2$, where d_i is the distance to the line u_i, $i = 1, \dots, n$. Second, we can work in the Euclidean m-space R^m and consider n hyperplanes in R^m with $n \geq m+1$. In this paper a combination of these two generalizations is presented.

1. Introduction

Let us start with a triangle $A_1A_2A_3$ in the Euclidean plane R^2 and suppose that its sides $a_1 = A_2A_3$, $a_2 = A_3A_1$, and $a_3 = A_1A_2$ have length l_1, l_2, and l_3, respectively. The easiest way to deal with the Lemoine point K of the triangle is to work with trilinear coordinates with regard to $A_1A_2A_3$ (also called normal coordinates). See [1, 5, 6]. These are homogeneous projective coordinates (x_1, x_2, x_3) such that A_1, A_2, A_3, and the incenter I of the triangle, have coordinates $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$, respectively. If (a_i^1, a_i^2) are the non-homogeneous coordinates (x, y) of the point A_i with respect to an orthonormal coordinate system in R^2, $i = 1, 2, 3$, then the relationship between homogeneous cartesian coordinates (x, y, z) and trilinear coordinates (x_1, x_2, x_3) is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l_1a_1^1 & l_2a_2^1 & l_3a_3^1 \\ l_1a_1^2 & l_2a_2^2 & l_3a_3^2 \\ l_1 & l_2 & l_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

This follows from the fact that the position vector of the incenter I of $A_1A_2A_3$ is given by

$$\vec{r} = \frac{l_1\vec{r}_1 + l_2\vec{r}_2 + l_3\vec{r}_3}{l_1 + l_2 + l_3},$$

with $\vec{r}_i$ the position vector of A_i. Remark also that $z = 0$ corresponds with $l_1x_1 + l_2x_2 + l_3x_3 = 0$, which is the equation in trilinear coordinates of the line at infinity

Publication Date: August 29, 2003. Communicating Editor: J. Chris Fisher.

of R^2. If (x_1, x_2, x_3) are normal coordinates of any point P of R^2 with regard to $A_1A_2A_3$, then the so-called absolute normal coordinates of P are

$$(d_1, d_2, d_3) = \left(\frac{2Fx_1}{l_1x_1 + l_2x_2 + l_3x_3}, \frac{2Fx_2}{l_1x_1 + l_2x_2 + l_3x_3}, \frac{2Fx_3}{l_1x_1 + l_2x_2 + l_3x_3} \right),$$

where F is the area of $A_1A_2A_3$. It is well known that d_i is the relative distance from P to the side a_i of the triangle (d_i is positive or negative, according as P lies at the same side or opposite side as A_i, with regard to a_i).

Next, consider the locus of the points of R^2 for which $d_1^2 + d_2^2 + d_3^2 = k$, with k a given value. In trilinear coordinates this locus is given by

$$F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - k(l_1x_1 + l_2x_2 + l_3x_3)^2 = 0. \tag{1}$$

For variable k, we get a pencil of homothetic ellipses (they all have the same points at infinity, the same asymptotes, the same center and the same axes), and the center of these ellipses is the Lemoine point K of the triangle $A_1A_2A_3$. A straightforward calculation gives that (l_1, l_2, l_3) are trilinear coordinates of K and the minimal value of $d_1^2 + d_2^2 + d_3^2$ reached at K is $\dfrac{4F^2}{l_1^2 + l_2^2 + l_3^2}$.

Remark also that K is the singular point of the degenerate ellipse of the pencil (1) corresponding with $k = \frac{1}{l_1^2+l_2^2+l_3^2}$ (set $\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial x_3} = 0$).

More properties and constructions of the Lemoine point K can be found in [1]. And in [3] and [7] constructions for the axes of the ellipses (1) are given, while [7] contains a lot of generalizations.

Next, the foregoing can immediately be generalized to higher dimensions as follows. Consider in the Euclidean m-space R^m $(m \geq 2)$, $m+1$ hyperplanes not through a point and no two parallel; this determines an m-simplex with vertices $A_1, \ldots, A_{m+1}$. Let us denote the $(m-1)$-dimensional volume of the "face" a_i with vertices $A_1, \ldots, \hat{A}_i, \ldots, A_{m+1}$ by F_i, $i = 1, \ldots, m+1$. Then the position vector of the incenter I of $A_1A_2 \ldots A_{m+1}$ (= center of the hypersphere of R^m inscribed in $A_1 \ldots A_{m+1}$) is given by

$$\vec{r} = \frac{F_1\vec{r}_1 + F_2\vec{r}_2 + \cdots + F_{m+1}\vec{r}_{m+1}}{F_1 + F_2 + \cdots + F_{m+1}},$$

where $\vec{r}_i$ is the position vector of A_i, and normal coordinates $(x_1, \ldots, x_{m+1})$ with respect to $A_1 \ldots, A_{m+1}$ are homogeneous projective coordinates such that $A_1, \ldots, A_{m+1}$, and I, have coordinates $(1, 0, \ldots, 0)$, …, $(0, \ldots, 0, 1)$, and $(1, 1, \ldots, 1)$, respectively. If $(a_i^1, a_i^2, \ldots, a_i^m)$ are cartesian coordinates (with respect to an orthonormal coordinate system) of A_i, $i = 1, \ldots, m+1$, the coordinate transformation between homogeneous cartesian coordinates $(z_1, \ldots, z_{m+1})$ and normal coordinates $(x_1, \ldots, x_{m+1})$ with respect to $A_1 \ldots A_{m+1}$ is given by

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \\ z_{m+1} \end{pmatrix} = \begin{pmatrix} F_1 a_1^1 & F_2 a_2^1 & \dots & F_{m+1} a_{m+1}^1 \\ F_1 a_1^2 & F_2 a_2^2 & \dots & F_{m+1} a_{m+1}^2 \\ \vdots & \vdots & & \vdots \\ F_1 a_1^m & F_2 a_2^m & \dots & F_{m+1} a_{m+1}^m \\ F_1 & F_2 & \dots & F_{m+1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ x_{m+1} \end{pmatrix}.$$

In normal coordinates the hyperplane at infinity of R^m has the equation $F_1 x_1 + \cdots + F_{m+1} x_{m+1} = 0$. Absolute normal coordinates of a point P of R^m with respect to $A_1, A_2, \dots, A_{m+1}$ are $d_i = \dfrac{mFx_i}{F_1 x_1 + \cdots + F_{m+1} x_{m+1}}$, $i = 1, \dots, m+1$, where F is the m-dimensional volume of $A_1 A_2 \dots A_{m+1}$ and d_i is the relative distance from P to the face a_i (d_i is positive or negative, according as P lies at the same side or at the opposite face as A_i, with regard to a_i). Remark that $F_1 d_1 + \cdots + F_{m+1} d_{m+1} = mF$.

The locus of the points of R^m for which $d_1^2 + \cdots + d_{m+1}^2 = k$ now determines a pencil of hyperquadrics (hyperellipsoids) with equation

$$x_1^2 + x_2^2 + \cdots + x_{m+1}^2 - k(F_1 x_1 + \cdots + F_{m+1} x_{m+1})^2 = 0 \tag{2}$$

and all these (homothethic) hyperellipsoids have the same axes, the same points at infinity and the same center K, which we call the Lemoine point of $A_1 \dots A_{m+1}$ and which obviously has normal coordinates $(F_1, F_2, \dots, F_{m+1})$. The minimal value of $d_1^2 + \cdots + d_{m+1}^2$, reached at K is given by $\dfrac{m^2 F^2}{F_1^2 + \cdots + F_{m+1}^2}$. Remark that K is the singular point of the singular hyperquadric (hypercone) corresponding in the pencil (2) with the value $k = \frac{1}{F_1^2 + \cdots + F_{m+1}^2}$.

Remark. Some characterizations and constructions of the Lemoine point K of a triangle in the plane R^2 are no longer valid in higher dimensions. For instance, K is the perspective center of the triangle $A_1 A_2 A_3$ and the triangle $A_1' A_2' A_3'$ whose sides are the tangents of the circumscribed circle of $A_1 A_2 A_3$ at A_1, A_2, and A_3 (in trilinear coordinates the circumcircle has equation $l_1 x_2 x_3 + l_2 x_3 x_1 + l_3 x_1 x_2 = 0$). This construction is, in general, not correct in R^3 : a tetrahedron $A_1 A_2 A_3 A_4$ and its so called tangential tetrahedron, which is the tetrahedron $A_1' A_2' A_3' A_4'$ consisting of the tangent planes of the circumscribed sphere of $A_1 A_2 A_3 A_4$ at A_1, A_2, A_3, and A_4, are, in general, not perspective. If they are perspective, the tetrahedron is a special one, an *isodynamic* tetrahedron in which the three products of the three pairs of opposite edges are equal. The lines joining the vertices of an isodynamic tetrahedron to the Lemoine points of the respective opposite faces have a point in common and this common point is the perspective center of the isodynamic tetrahedron and its tangential tetrahedron (see [2]). It is not difficult to prove that this point of an isodynamic tetrahedron coincides with the Lemoine point K of the tetrahedron obtained with our definition of “Lemoine point”.

2. The main theorem

First we give some notations. Consider n hyperplanes, denoted by $u_1, \ldots, u_n$ in the Euclidean space R^m ($m \geq 2, n \geq m+1$), in general position (this means : no two are parallel and no $m+1$ are concurrent). The “figure” consisting of these n hyperplanes is called an n-hyperface (examples: for $m = 2, n = 3$ it determines a triangle in R^2, for $m = 2$, $n = 4$ it is an quadrilateral in R^2, and for $m = 3$, $n = 4$ it is a tetrahedron in R^3). The Lemoine point K of this n-hyperface is, by definition, the point of R^m for which the sum of the squares of the distances to the n hyperplanes $u_1, \ldots, u_n$ is minimal. The uniqueness of K follows from the proof of the next theorem.

Next, K^i is the Lemoine point of the $(n-1)$-hyperface $u_1 u_2 \ldots \hat{u}_i \ldots u_n$, $i = 1, \ldots, n$. And $K^{rs} = K^{sr}$ is the Lemoine point of the $(n-2)$-hyperface $u_1 u_2 \ldots \hat{u}_r \ldots \hat{u}_s \ldots u_n$, with $r, s = 1, \ldots, n$, $r \neq s$ (only defined if $n > m+1$).

Now, for an $(m+1)$-hyperface or m-simplex in R^m (a triangle in R^2, a tetrahedron in $R^3, \ldots$) we know the position (the normal coordinates) of the Lemoine point (see §1). The following theorem gives us a construction for the Lemoine point K of a general n-hyperface in R^m ($m \geq 2$ and $n > m+1$):

Theorem 1. *Working with an n-hyperface in R^m, we have, with the notations given above that $K^i K \cap u_j = K^j K^{ji} \cap u_j$, $i, j = 1, \ldots, n$ and $n > m+1$.*

Proof. In this proof, we work with cartesian coordinates $(x_1, \ldots, x_m)$ or homogeneous $(x_1, \ldots, x_{m+1})$ with respect to an orthonormal coordinate system in R^m. Suppose that the hyperplane u_r has equation $a_r^1 x_1 + a_r^2 x_2 + \cdots + a_r^m x_m + a_r^{m+1} = 0$, with $(a_r^1)^2 + (a_r^2)^2 + \cdots + (a_r^m)^2 = 1, r = 1, \ldots, n$. Then the Lemoine point K of the n-hyperface $u_1 u_2 \ldots u_n$ is the center of the hyperquadrics of the pencil with equation

$$\mathcal{F}(x_1, \ldots, x_m) = \sum_{r=1}^{n} (a_r^1 x_1 + a_r^2 x_2 + \cdots + a_r^m x_m + a_r^{m+1})^2 - k = 0, \qquad (3)$$

where k is a parameter. Indeed, since the coordinates of K minimize the expression $\sum_{r=1}^{n} \left(a_r^1 x_r + \cdots + a_r^{m+1}\right)^2$, they are a (the) solution of $\frac{\partial \mathcal{F}}{\partial x_1} = \frac{\partial \mathcal{F}}{\partial x_2} = \cdots = \frac{\partial \mathcal{F}}{\partial x_m} = 0$. In homogeneous coordinates, (3) becomes

$$\mathcal{F}(x_1, \ldots, x_{m+1}) = \sum_{r=1}^{n} (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})^2 - k x_{m+1}^2 = 0. \qquad (4)$$

Next, the Lemoine point K^i of $u_1 u_2 \ldots \hat{u}_i \ldots u_n$ is the center of the hyperquadrics of the pencil given by (we use the same notation k for the parameter)

$$\mathcal{F}^i(x_1, \ldots, x_{m+1}) = \sum_{\substack{r=1 \\ r \neq i}}^{n} (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})^2 - k x_{m+1}^2 = 0. \qquad (5)$$

The diameter of the hyperquadrics (5), conjugate with respect to the direction of the ith hyperplane u_i has the equations (consider the polar hyperplanes of the

$m-1$ points at infinity with coordinates $(a_i^2, -a_i^1, 0, \ldots, 0)$, $(a_i^3, 0, -a_i^1, 0, \ldots, 0)$, $(a_i^4, 0, 0, -a_i^1, 0, \ldots, 0)$, ..., $(a_i^m, 0, \ldots, 0, -a_i^1, 0)$ of the hyperplane u_i):

$$\begin{cases} \sum_{r=1}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^2 - a_r^2 a_i^1) = 0, \\ \sum_{r=1}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^3 - a_r^3 a_i^1) = 0, \\ \quad \vdots \\ \sum_{r=1}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^m - a_r^m a_i^1) = 0. \end{cases} \tag{6}$$

But the first side of each of these equations becomes zero for $r = i$, and thus (6) gives us also the conjugate diameter with respect to the hyperplane u_i of the hyperquadrics of the pencil (5). It follows that (6) determines the line KK^i.

Next, the Lemoine point K^j is the center of the hyperquadrics of the pencil

$$\mathcal{F}^j(x_1, \ldots, x_{m+1}) = \sum_{\substack{r=1 \\ r \neq j}}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})^2 - k x_{m+1}^2 = 0, \tag{7}$$

and K^{ji} is the center of the hyperquadrics:

$$\mathcal{F}^{ji}(x_1, \ldots, x_{m+1}) = \sum_{\substack{r=1 \\ r \neq j,i}}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})^2 - k x_{m+1}^2 = 0. \tag{8}$$

The diameter of the hyperquadrics (7), conjugate with respect to the direction of u_i is given by

$$\begin{cases} \sum_{\substack{r=1 \\ r \neq j}}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^2 - a_r^2 a_i^1) = 0 \\ \quad \vdots \\ \sum_{\substack{r=1 \\ r \neq j}}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^m - a_r^m a_i^1) = 0. \end{cases} \tag{9}$$

And this gives us also the diameter of the hyperquadrics (8) conjugate with regard to the direction of u_i; in other words, (9) determines the line $K^j K^{ji}$.

Finally, the coordinates of the point $K^i K \cap u_j$ are the solutions of the linear system

$$\begin{cases} (6) \\ a_j^1 x_1 + \cdots + a_j^{m+1} x_{m+1} = 0, \end{cases}$$

while the point $K^j K^{ji} \cap u_j$ is given by

$$\begin{cases} (9) \\ a_j^1 x_1 + \cdots + a_j^{m+1} x_{m+1} = 0. \end{cases}$$

It is obvious that this gives the same point and the proof is complete. □

3. Applications

3.1. Let us first consider the easiest example for trying out our construction: the case where $m = 2$ and $n = 4$, or four lines u_1, u_2, u_3, u_4 in general position (they form a quadrilateral) in R^2. Using orthonormal coordinates (x, y, z) in R^2, the homogeneous equation of u_r is $a_r x + b_r y + c_r z = 0$ with $a_r^2 + b_r^2 = 1$,

$r = 1, 2, 3, 4$. Where lies the Lemoine point K of the quadrilateral $u_1u_2u_3u_4$? For instance K^1 is the Lemoine point of the triangle with sides (lines) u_2, u_3, u_4; K^2 of the triangle with sides u_1, u_3, u_4, and so on ... . We may assume that we can construct the Lemoine point of a triangle. But which point is, for instance, the point K^{12} : it is the Lemoine point of the 2-side u_3u_4, *i.e.*, it is the point $u_3 \cap u_4$.

Let us denote the six vertices of the quadrilateral as follows : $u_1 \cap u_2 = C, u_2 \cap u_3 = A, u_3 \cap u_4 = F, u_1 \cap u_4 = D, u_2 \cap u_4 = E$, and $u_1 \cap u_3 = B$, then $K^{12} = K^{21} = F$, $K^{23} = D, K^{34} = C, K^{14} = A, K^{24} = B$, and $K^{13} = E$. Now, from $K^iK \cap u_j = K^jK^{ji} \cap u_j$, we find, for instance for $i = 1$ and $j = 2$:

$$K^1K \cap u_2 = K^2K^{21} \cap u_2 = K^2F \cap u_2$$

and for $i = 2$ and $j = 1$: $K^2K \cap u_1 = K^1K^{12} \cap u_1 = K^1F \cap u_1$, with K^1 (K^2, resp.) the Lemoine point of the triangle AFE (of the triangle BFD, resp.). This allows us to construct the point K.
In particular, we can construct the diameters KK^1, KK^2, KK^3, and KK^4 of the ellipses of the pencil $\sum_{r=1}^{4}(a_rx + b_ry + c_rz)^2 = kz^2$, which are conjugate to the directions of the lines u_1, u_2, u_3, and u_4, respectively. In other words, we have four pairs of conjugate diameters of these ellipses : (KK^i, KI^i_∞), where I^i_∞ is the point at infinity of the line $u_i, i = 1, \ldots, 4$. From this, we can construct the axes of the conics of this bundle (in fact, two pairs of conjugate diameters are sufficient): consider any circle $\mathcal{C}$ through K and project the involution of conjugate diameters onto $\mathcal{C}$; if S is the center of this involution on $\mathcal{C}$ and if the diameter of $\mathcal{C}$ through S intersects $\mathcal{C}$ at the points S_1 and S_2, then KS_1 and KS_2 are the axes.

In the case of a triangle in R^2, constructions of the common axes of the ellipses determined by $d_1^2 + d_2^2 + d_3^2 = k$ with center the Lemoine point of the triangle, are given in [3] and [7]. In [3], J. Bilo proved that the axes are the perpendicular lines through K on the Simson lines of the common points of the Euler line and the circumscribed circle of the triangle. And in [7], we proved that these axes are the orthogonal lines through K which cut the sides of the triangle in pairs of points whose midpoints are three collinear points. Moreover [7] contains a lot of generalizations for pencils whose conics have any point P of the plane as common center and whose common axes are constructed in the same way.

3.2. In the case $m = 2$ and $n \geq 4$, we can construct the n diameters $KK^1, \ldots, KK^n$ of the ellipses $\sum_{r=1}^{n}(a_rx + b_ry + c_rz)^2 = kz^2$ which are conjugate to the directions of the n lines $u_1, \ldots, u_n$.

3.3. The easiest example in space is the case where $m = 3$ and $n = 5$, or five planes in R^3. Assume that the planes have equations $a_rx + b_ry + c_rz + d_ru = 0$, with $a_r^2 + b_r^2 + c_r^2 = 1, r = 1, 2, \ldots, 5$. We look for the Lemoine point K of the "5-plane" $u_1u_2u_3u_4u_5$ in R^3 and assume that we know the position of the Lemoine point of any tetrahedron in R^3 (we know its normal coordinates). The points $K^1, \ldots, K^5$ are the Lemoine points of the tetrahedra $u_2u_3u_4u_5, \ldots, u_1u_2u_3u_4$, respectively. And, for instance K^{12} is the Lemoine point of the "3-plane" $u_3u_4u_5$, *i.e.*, it is the common point of these three planes u_3, u_4, and u_5. Now, for instance from

$$K^1K \cap u_2 = K^2K^{21} \cap u_2 \quad \text{and} \quad K^2K \cap u_1 = K^1K^{12} \cap u_1,$$

we can construct the lines K^1K and K^2K, and thus the point K. In fact, we can construct the diameters $KK^1, \dots, KK^5$ conjugate to the plane directions of $u_1, \dots, u_5$, respectively, of the quadrics with center K of the pencil given by $d_1^2 + \cdots + d_5^2 = k$ or

$$\sum_{r=1}^{5}(a_r x + b_r y + c_r z + d_r u)^2 = ku^2.$$

Finally, the construction of the point K in the general case $n > m+1$, $m \geq 2$ is obvious.

References

[1] N. Altshiller-Court, *College geometry, An introduction to the modern geometry of the triangle and the circle*, Barnes and Noble, New York, 1952.
[2] N. Altshiller-Court, *Modern pure solid geometry*, Chelsea Publ., New York 1964.
[3] J. Bilo, Over een bundel homothetische ellipsen om het punt van Lemoine, *Nieuw Tijdschrift voor Wiskunde*, (1987), 74.
[4] O. Bottema, Om het punt van Lemoine, *Euclides*, (1972-73) 48.
[5] W. Gallatly, *The modern geometry of the triangle*, Francis Hodgson, London, 1935.
[6] A. C. Jones, *An introduction to algebraical geometry*, Oxford University Press, 1937
[7] C. Thas, On ellipses with center the Lemoine point and generalizations, *Nieuw Archief voor Wiskunde*, ser. 4, 11 (1993) 1–7.

Charles Thas: Department of Pure Mathematics and Computer Algebra, Krijgslaan 281-S22, B-9000 Gent, Belgium
E-mail address: charles.thas@UGent.be

Forum Geometricorum
Volume 3 (2003) 169–180.

FORUM GEOM
ISSN 1534-1178

The Parasix Configuration and Orthocorrespondence

Bernard Gibert and Floor van Lamoen

Abstract. We introduce the parasix configuration, which consists of two congruent triangles. The conditions of these triangles to be orthologic with ABC or a circumcevian triangle, to form a cyclic hexagon, to be equilateral or to be degenerate reveal a relation with orthocorrespondence, as defined in [1].

1. The parasix configuration

Consider a triangle ABC of reference with finite points P and Q not on its sidelines. Clark Kimberling [2, §§9.7,8] has drawn attention to configurations defined by six triangles. As an example of such configurations we may create six triangles using the lines ℓ_a, ℓ_b and ℓ_c through Q parallel to sides a, b and c respectively. The triples of lines (ℓ_a, b, c), (a, ℓ_b, c) and (a, b, ℓ_c) bound three triangles which we refer to as the *great paratriple*. Figure 1a shows the A-triangle of the great paratriple. On the other hand, the triples (a, ℓ_b, ℓ_c), (ℓ_a, b, ℓ_c) and (ℓ_a, ℓ_b, c) bound three triangles which we refer to as the *small paratriple*. See Figure 1b.

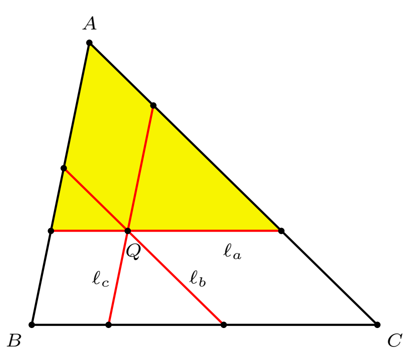


Figure 1a

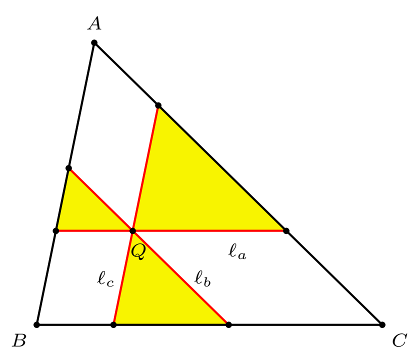


Figure 1b

Clearly these six triangles are all homothetic to ABC, and it is very easy to find the homothetic images of P in these triangles, A_g in the A-triangle bounded by (ℓ_a, b, c) in the great paratriple, and A_s in the A-triangle bounded by (a, ℓ_b, ℓ_c) in the small paratriple; similarly for B_g, C_g, B_s, C_s. These six points form the *parasix configuration of P with respect to Q*, or shortly $\mathsf{Parasix}(P, Q)$. See Figure 2. If in homogeneous barycentric coordinates with reference to ABC, $P = (u : v : w)$ and $Q = (f : g : h)$, then these are the points

Publication Date: September 5, 2003. Communicating Editor: Paul Yiu.

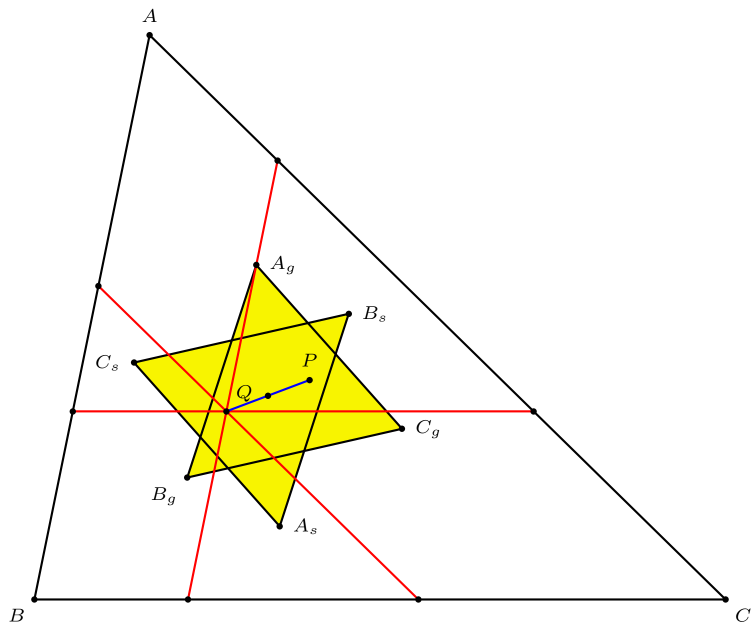


Figure 2. Parasix(P, Q)

$$
\begin{aligned}
A_g =&(u(f+g+h)+f(v+w) : v(g+h) : w(g+h)),\\
B_g =&(u(f+h) : g(u+w)+v(f+g+h) : w(f+h)),\\
C_g =&(u(f+g) : v(f+g) : h(u+v)+w(f+g+h));\\
A_s =&(uf : g(u+w)+v(f+g) : h(u+v)+w(f+h)),\\
B_s =&(u(f+g)+f(v+w) : vg : h(u+v)+w(g+h)),\\
C_s =&(u(f+h)+f(v+w) : g(u+w)+v(g+h) : wh).
\end{aligned} \tag{1}
$$

Proposition 1. (1) *Triangles $A_gB_gC_g$ and $A_sB_sC_s$ are symmetric about the midpoint of segment PQ.*
(2) *The six points of a parasix configuration lie on a central conic.*
(3) *The centroids of triangles $A_gB_gC_g$ and $A_sB_sC_s$ trisect the segment PQ.*

Proof. It is clear from the coordinates given above that the segments A_gA_s, B_gB_s, C_gC_s, PQ have a common midpoint

$$(f(u+v+w)+u(f+g+h) : \cdots : \cdots).$$

The six points therefore lie on a conic with this common midpoint as center. For (3), it is enough to note that the centroids G_g and G_s of $A_gB_gC_g$ and $A_sB_sC_s$ are the points

$$
\begin{aligned}
G_g =&(2u(f+g+h)+f(u+v+w) : \cdots : \cdots),\\
G_s =&(u(f+g+h)+2f(u+v+w) : \cdots : \cdots).
\end{aligned}
$$

It follows that vectors $\overrightarrow{PG_g} = \frac{1}{3}\,\overrightarrow{PQ}$ and $\overrightarrow{PG_s} = \frac{2}{3}\,\overrightarrow{PQ}$. $\square$

While $\mathsf{Parasix}(P,Q)$ consists of the two triangles $A_gB_gC_g$ and $A_sB_sC_s$, we write $\widetilde{A}_g\widetilde{B}_g\widetilde{C}_g$ and $\widetilde{A}_s\widetilde{B}_s\widetilde{C}_s$ for the two corresponding triangles of $\mathsf{Parasix}(Q,P)$. From (1) we easily derive their coordinates by interchanging the roles of f, g, h, and u, v, w. Note that $\widetilde{G}_s = G_g$ and $\widetilde{G}_g = G_s$.

Let P_A and Q_A be the the points where AP and AQ meet BC respectively, and let $AP : PP_A = t_P : 1 - t_P$ while $AQ : QQ_A = t_Q : 1 - t_Q$. Then it is easy to see that

$$AA_g : A_gP_A = A\widetilde{A}_g : \widetilde{A}_gQ_A = t_Pt_Q : 1 - t_Pt_Q$$

so that the line $A_g\widetilde{A}_g$ is parallel to BC. By Proposition 1, $A_s\widetilde{A_s}$ is also parallel to BC.

Proposition 2. (a) *The lines $A_g\widetilde{A}_g$, $B_g\widetilde{B}_g$ and $C_g\widetilde{C}_g$ bound a triangle homothetic to ABC. The center of homothety is the point*

$$(f(u+v+w)+u(g+h) : g(u+v+w)+v(h+f) : h(u+v+w)+w(f+g))\,.$$

The ratio of homothety is

$$-\frac{fu+gv+hw}{(f+g+h)(u+v+w)}.$$

(b) *The lines $A_s\widetilde{A}_s$, $B_s\widetilde{B}_s$ and $C_s\widetilde{C}_s$ bound a triangle homothetic to ABC with center of homothety $(uf : vg : wh)$* [1] *The ratio of homothety is*

$$1-\frac{fu+gv+hw}{(f+g+h)(u+v+w)}.$$

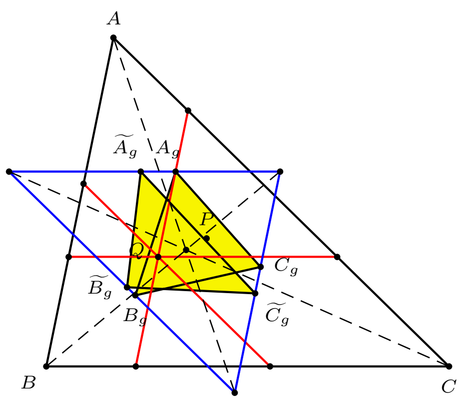


Figure 3a

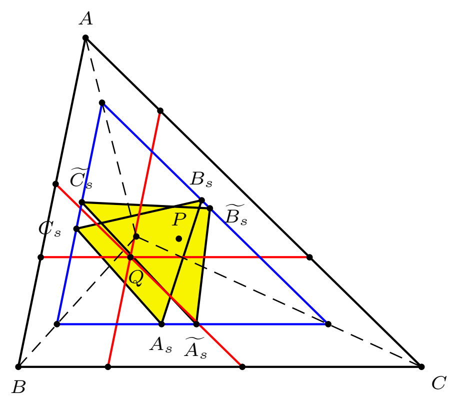


Figure 3b

[1] This point is called the barycentric product of P and Q. Another construction was given by P. Yiu in [4]. These homothetic centers are collinear with the midpoint of PQ.

2. Parasix loci

We present a few line and conic loci associated with parasix configurations. For $P = (u : v : w)$, we denote by

(i) $\mathcal{L}_P$ the trilinear polar of P, which has equation

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0;$$

(ii) $\mathcal{C}_P$ the circumconic with perspector P, which has equation

$$\frac{u}{x} + \frac{v}{y} + \frac{w}{z} = 0.$$

2.1. *Area of parasix triangles.* The parasix triangles $A_gB_gC_g$ and $A_sB_sC_s$ have a common area

$$\frac{ghu + hfv + fgw}{(f+g+h)^2(u+v+w)}. \tag{2}$$

Proposition 3. (a) *For a given Q, the locus of P for which the triangles $A_gB_gC_g$ and $A_sB_sC_s$ have a fixed (signed) area is a line parallel to $\mathcal{L}_P$.*

(b) *For a given P, the locus of Q for which the triangles $A_gB_gC_g$ and $A_sB_sC_s$ have a fixed (signed) area is a conic homothetic to $\mathcal{C}_P$ at its center.*

In particular, the parasix triangles degenerate into two parallel lines if and only if

$$\frac{u}{f} + \frac{v}{g} + \frac{w}{h} = 0. \tag{$*$}$$

This condition can be construed in two ways: $P \in \mathcal{L}_Q$, or equivalently, $P \in \mathcal{C}_P$. See §6.

2.2. *Perspectivity with the pedal triangle.*

Proposition 4. (a) *Given P, the locus of Q so that $A_sB_sC_s$ is perspective to the pedal triangle of Q is the line*[2]

$$\sum_{\text{cyclic}} S_A(S_Bv - S_Cw)(-uS_A + vS_B + wS_C)x = 0.$$

This line passes through the orthocenter H and the point

$$\left(\frac{1}{S_A(-uS_A + vS_B + wS_C)} : \cdots : \cdots\right),$$

which can be constructed as the perspector of ABC and the cevian triangle of P in the orthic triangle.

[2]Here we adopt J.H. Conway's notation by writing S for *twice* of the area of triangle ABC and

$$S_A = S{\cdot}\cot A = \frac{b^2 + c^2 - c^2}{2},\ S_B = S{\cdot}\cot B = \frac{c^2 + a^2 - b^2}{2},\ S_C = S{\cdot}\cot C = \frac{a^2 + b^2 - c^2}{2}.$$

These satisfy $S_{AB} + S_{BC} + S_{CA} = S^2$. The expressions S_{AB}, S_{BC}, S_{CA} stand for S_AS_B, S_BS_C, S_CS_A respectively.

2.3. *Parallelogy.* A triangle is said to be parallelogic to a second triangle if the lines through the vertices of the triangle parallel to the corresponding opposite sides of the second triangle are concurrent.

Proposition 5. (a) *Given $P = (u : v : w)$, the locus of Q for which ABC is parallelogic to $A_gB_gC_g$ (respectively $A_sB_sC_s$) is the line $(v+w)x+(w+u)y+(u+v)z = 0$, which can be constructed as the trilinear polar of the isotomic conjugate of the complement of P.*

(b) *Given $Q = (f : g : h)$, the locus of P for which ABC is parallelogic to $A_gB_gC_g$ (respectively $A_sB_sC_s$) is the line $(g+h)x+(h+f)y+(f+g)z = 0$, which can be constructed as the trilinear polar of the isotomic conjugate of the complement of Q.*

2.4. *Perspectivity with ABC.* Clearly $A_gB_gC_g$ is perspective to ABC at P. The perspectrix is the line $gh(g+h)x+fh(f+h)y+fg(f+g)z = 0$, parallel to the trilinear polar of Q. Given P, the locus of Q such that $A_sB_sC_s$ is perspective to ABC is the cubic

$$(v+w)x(wy^2-vz^2)+(u+w)y(uz^2-wx^2)+(u+v)z(vx^2-uy^2) = 0,$$

which is the isocubic with pivot $(v+w : w+u : u+v)$ and pole P. For $P = K$, the symmedian point, this is the isogonal cubic with pivot $X_{141} = (b^2+c^2 : c^2+a^2 : a^2+b^2)$.

3. Orthology

Some interesting loci associated with the orthology of triangles attracted our attention because of their connection with the orthocorrespondence defined in [1]. We recall that two triangles are orthologic if the perpendiculars from the vertices of one triangle to the opposite sides of the corresponding vertices of the other triangle are concurrent.

First, consider the locus of Q, given P, such that the triangles $A_gB_gC_g$ and $A_sB_sC_s$ are orthologic to ABC. We can find this locus by simple calculation since this is also the locus such that $A_gB_gC_g$ is perspective to the triangle of the infinite points of the altitudes, with coordinates

$$H_A^\infty = (-a^2, S_C, S_B), \quad H_B^\infty = (S_C, -b^2, S_A), \quad H_C^\infty = (S_B, S_A, -c^2).$$

The lines $A_gH_A^\infty$, $V_gH_B^\infty$ and $C_gH_C^\infty$ concur if and only if Q lies on the line

$$(S_Bv-S_Cw)x+(S_Cw-S_Au)y+(S_Au-S_Bv)z = 0, \tag{3}$$

which is the line through the centroid G and the orthocorrespondent of P, namely, the point [3]

$$P^\perp = (u(-S_Au+S_Bv+S_Cw)+a^2vw : \cdots : \cdots).$$

The line (3) is the orthocorrespondent of the line HP. See [1, §2.4].

[3]The lines perpendicular at P to AP, BP, CP intersect the respective sidelines at three collinear points. The orthocorrespondent of P is the trilinear pole $P^\perp$ of the line containing these three intersections.

For the second locus problem, we let Q be given, and ask for the locus of P such that the triangles $A_gB_gC_g$ and $A_sB_sC_s$ are orthologic to ABC. The computations are similar, and again we find a line as the locus:

$$S_A(g-h)x + S_B(h-f)y + S_C(f-g)z = 0.$$

This is the line through H, and the two anti-orthocorrespondents of Q. See [1, Figure 2]. It is the anti-orthocorrespondent of the line GQ.

Given P, for both $A_gB_gC_g$ and $\widetilde{A}_g\widetilde{B}_g\widetilde{C}_g$ to be orthologic to ABC, the point Q has to be the intersection of the line $GP^{\perp}$ ((3) above) and

$$S_A(v-w)x + S_B(w-u)y + S_C(u-v)z = 0,$$

the anti-orthocorrespondent of GP. This is the point

$$\tau(P) = \big(S_A(c^2-b^2)u^2 + (S_{AC}-S_{BB})uv - (S_{AB}-S_{CC})uw + a^2(c^2-b^2)vw \\ : \cdots : \cdots\big).$$

The point $\tau(P)$ is not well defined if all three coordinates of $\tau(P)$ are equal to zero, which is the case exactly when P is either K, the orthocenter H, or the centroid G. The pre-images of these points are lines: GH (the Euler line), GK, and HK for K, G and H respectively. Outside these lines the mapping $P \longmapsto \tau(P)$ is an involution. Note that P and $\tau(P)$ are collinear with the symmedian point K.

The fixed points of τ are the points of the Kiepert hyperbola

$$(b^2-c^2)yz + (c^2-a^2)xz + (a^2-b^2)xy = 0.$$

More precisely, the line joining $\tau(P)$ to H meets GP on the Kiepert hyperbola. Therefore we may characterize $\tau(P)$ as the intersection of the line PK with the polar of P in the Kiepert hyperbola. [4]

In the table below we give the first coordinates of some well known triangle centers and their images under τ. The indexing of triangle centers follows [3].

P	first coordinate	$\tau(P)$	first coordinate
X_1	a	X_9	$a(s-a)$
X_7	$(s-b)(s-c)$	X_{948}	$(s-b)(s-c)F$
X_8	$s-a$		$a^2+(b+c)^2$
X_{19}			aG
X_{34}			$a(s-b)(s-c)(a^2+(b+c)^2)$
X_{37}		X_{72}	$a(b+c)S_A$
X_{42}	$a^2(b+c)$	X_{71}	$a^2(b+c)S_A$
X_{57}	$a/(s-a)$	X_{223}	$a(s-b)(s-c)F$
X_{58}		X_{572}	a^2G

[4] This is also called the *Hirst inverse* of P with respect to K. See the glossary of [3].

Here,

$$F = a^3 + a^2(b+c) - a(b+c)^2 - (b+c)(b-c)^2,$$
$$G = a^3 + a^2(b+c) + a(b+c)^2 + (b+c)(b-c)^2,$$

We may also wonder, given P outside the circumcircle, for which Q are the $\mathsf{Parasix}(P,Q)$ triangles $A_gB_gC_g$ and $A_sB_sC_s$ orthologic to the circumcevian triangle of P. The A-vertex of the circumcevian triangle of P has coordinates

$$\left(-a^2yz : (b^2z+c^2y)y : (b^2z+c^2y)z\right).$$

Hence we find that the lines from the vertices of the circumcevian triangle of P perpendicular to the corresponding sides of $A_gB_gC_g$ concur if and only if

$$(uyz + vxz + wxy)L = 0, \tag{4}$$

where

$$L = \sum_{\text{cyclic}} (c^2v^2 + 2S_Avw + b^2w^2)((c^2S_Cv - b^2S_Bw)u^2 + a^2((c^2v^2 - b^2w^2)u + (S_Bv - S_Cw)vw))x.$$

The first factor in (4) represents the circumconic with perspector P, and when Q is on this conic, $\mathsf{Parasix}(P,Q)$ is degenerate, see §6 below. The second factor L yields the locus we are looking for, a line passing through $P^\perp$.[5]

A point X lies on the line $L = 0$ if and only if P lies on a bicircular circumquintic through the in- and excenters[6]. For the special case $X = G$ this quintic decomposes into $\mathcal{L}_\infty$ (with multiplicity 2) and the McCay cubic.[7] In other words, for any P on the McCay cubic, the circumcevian triangle of P is orthologic to the $\mathsf{Parasix}(P,Q)$ triangles if and only if Q lies on the line $GP^\perp$.

4. Concyclic $\mathsf{Parasix}(P,Q)$-hexagons

We may ask, given P, for which Q the parasix configuration yields a cyclic hexagon. This is equivalent to the circumcenter of $A_gB_gC_g$ being equal to the midpoint of segment PQ. Now the midpoint of PQ lies on the perpendicular bisector of B_gC_g if and only if Q lies on the line

$$-(w(S_Au + S_Bv - S_Cw) + c^2uv)y + (v(S_Au - S_Bv + S_Cw)v + b^2wu)z = 0,$$

which is indeed the cevian line $AP^\perp$. Remarkably, we find the same cevian line as locus for Q satisfying the condition that $B_gC_g \perp AP$.

Proposition 6. *The following statements are equivalent.*

(1) $\mathsf{Parasix}(P,Q)$ *yields a cyclic hexagon.*

---

[5]The line $L = 0$ is not defined when P is an in/excenter. This means that, for any Q, triangles $A_gB_gC_g$ and $A_sB_sC_s$ in $\mathsf{Parasix}(P,Q)$ are orthologic to the circumcevian triangle of P. This is not surprising since P is the orthocenter of its own circumcevian triangle. For $P = X_3$, $L = 0$ is the line GK, while for $P = X_{13}, X_{14}$, it is the parallel at P to the Euler line.

[6]This quintic has equation $Q_Ax + Q_By + Q_Cz = 0$ where Q_A represents the union of the circle center A, radius 0 and the Van Rees focal which is the isogonal pivotal cubic with pivot the infinite point of AH and singular focus A.

[7]The McCay cubic is the isogonal cubic with pivot O given by the equation $\sum_{\text{cyclic}} a^2S_Ax(c^2y^2 - b^2z^2) = 0$.

(2) *$A_gB_gC_g$ and $A_sB_sC_s$ are homothetic to the antipedal triangle of P.*
(3) *Q is the orthocorrespondent of P.*

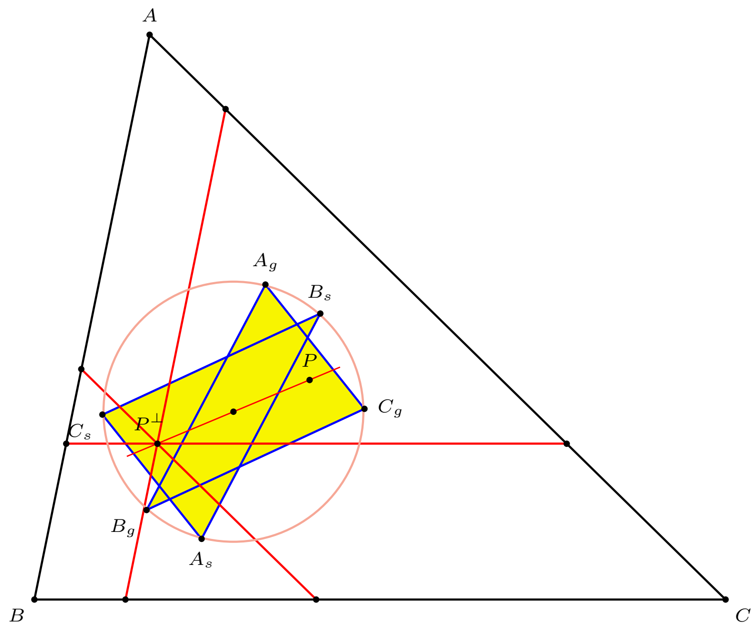


Figure 4

The center of the circle containing the 6 points is the midpoint of PQ.

The homothetic centers and the circumcenter of the cyclic hexagon are collinear.

A nice example is the circle around Parasix(H, G). It is homothetic to the circumcircle and nine point circle through H with factors $\frac{1}{3}$ and $\frac{2}{3}$ respectively. The center of the circle divides OH in the ratio $2 : 1$.[8] The antipedal triangle of H is clearly the anticomplementary triangle of ABC. The two homothetic centers divide the same segment in the ratios $5 : 2$ and $3 : 2$ respectively.[9] See Figure 5.

As noted in [1], $P = P^\perp$ only for the Fermat-Torricelli points X_{13} and X_{14}. The vertices of parasix(X_{13}, X_{13}) and Parasix(X_{14}, X_{14}) form regular hexagons. See Figure 6.

5. Equilateral triangles

The last example raises the question of finding, for given P, the points Q for which the triangles $A_gB_gC_g$ and $A_sB_sC_s$ are equilateral. We find that the A-median of $A_gB_gC_g$ is also an altitude in this triangle if and only if Q lies on the

---

[8]This is also the midpoint of GH, the center of the orthocentroidal circle, the point X_{381} in [3].

[9]These have homogeneous barycentric coordinates $(3a^4+2a^2(b^2+c^2)-5(b^2-c^2)^2 : \cdots : \cdots)$ and $(a^4 - 2a^2(b^2+c^2) + 3(b^2-c^2)^2 : \cdots : \cdots)$ respectively. They are not in the current edition of [3].

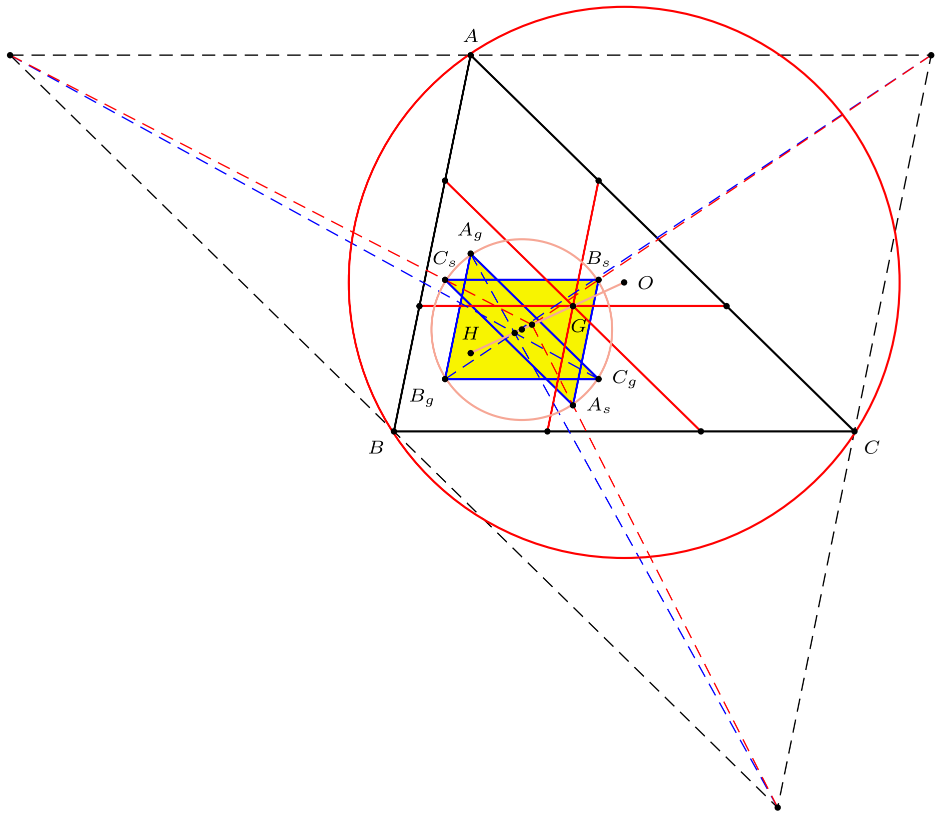


Figure 5. Parasix(H, G)

conic

$$-2((S_A u + S_B v - S_C w)w + c^2 uv)xy + 2((S_A u - S_B v + S_C w)v + b^2 uw)xz$$
$$-(c^2u^2 + a^2w^2 + 2S_B uw)y^2 + (b^2u^2 + a^2v^2 + 2S_C uv)z^2 = 0.$$

We find an analogous conic for the B-median of $A_gB_gC_g$ to be an altitude. The two conics intersect in four points: two imaginary points and the points

$$Q_{1,2} = \left((-S_A u + S_B v + S_C w)u + a^2 vw \pm \frac{1}{3}\sqrt{3}Su(u+v+w) : \cdots : \cdots\right).$$

Proposition 7. *Given P, there are two (real) points Q for which triangles $A_gB_gC_g$ and $A_sB_sC_s$ are equilateral. These two points divide $PP^{\perp}$ harmonically.*

The points $Q_{1,2}$ from Proposition 7 can be constructed in the following way, using the fact that P, G_s, G_g and $P^{\perp}$ are collinear.

Start with a point G' on $PP^{\perp}$. We shall construct an equilateral triangle $A'B'C'$ with vertices on AP, BP and CP respectively and centroid at G'. This triangle must be homothetic to one of the equilateral triangles $A_gB_gC_g$ of Proposition 7 through P.

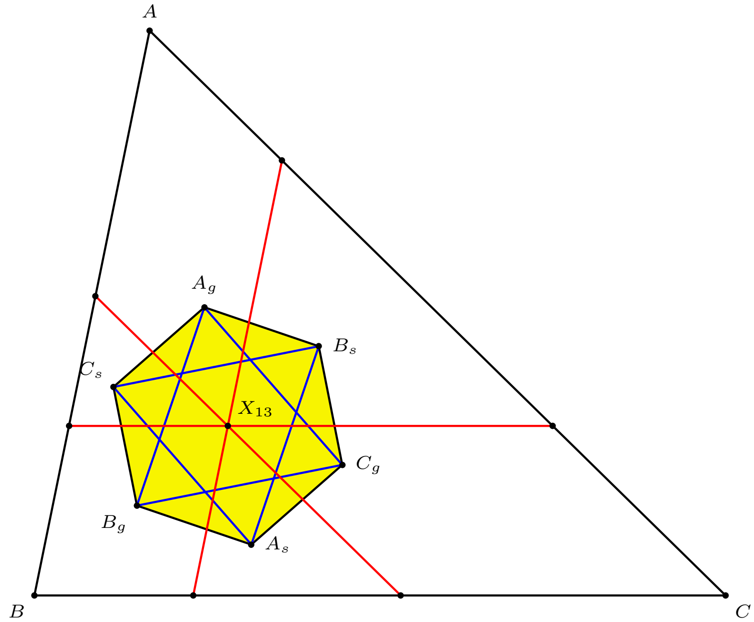


Figure 6. The parasix configuration $\mathsf{Parasix}(X_{13}, X_{13})$

Consider the rotation ρ about G' through $\pm\frac{2\pi}{3}$. The image of AP intersects BP in a point B'. Now let C' be the image of B' and A' the image of C'. Then $A'B'C'$ is equilateral, A' lies on AP, G' is the centroid and C' must lie on CP.

The homothety with center A that maps P to A' also maps BC to a line ℓ_a. Similarly we find ℓ_b and ℓ_c. These lines enclose a triangle $A''B''C''$ homothetic to ABC. We of course want to find the case for which $A''B''C''$ degenerates into one point, which is the Q we are looking for. Since all possible equilateral $A'B'C'$ of the same orientation are homothetic through P, the trianges $A''B''C''$ are all homothetic to ABC through the same point. So the homothety center of $A''B''C''$ and ABC is the point Q we are looking for.

6. Degenerate parasix triangles

We begin with a simple interesting fact.

Proposition 8. *Every line through P intersects the circumconic $\mathcal{C}_P$ at two real points.*

Proof. For the special case of the symmedian point K this is clear, since K is the interior of the circumcircle. Now, there is a homography φ fixing A, B, C and transforming $P = (u : v : w)$ into $K = (a^2 : b^2 : c^2)$. It is given by

$$\varphi(x : y : z) = \left(\frac{a^2}{u}x : \frac{b^2}{v}y : \frac{c^2}{w}z\right),$$

and is a projective transformation mapping $\mathcal{C}_P$ into the circumcircle and any line through P into a line through K. If ℓ is a line through P, then $\varphi(\ell)$ is a line through K, intersecting the circumcircle at two real points q_1 and q_2. The circumcircle and the circumconic $\mathcal{C}_P$ have a fourth real point Z in common, which is the trilinear pole of the line PK. For any point M on $\mathcal{C}_P$, the points Z, M, $\varphi(M)$ are collinear. The second intersections of the lines Zq_1 and Zq_2 are common points of ℓ and the circumconic $\mathcal{C}_P$. $\square$

In §2, we have seen that the parasix triangles are degenerate if and only if $P \in \mathcal{L}_Q$ or equivalently, $Q \in \mathcal{C}_P$. This means that for each line ℓ_P through P intersecting the circumconic $\mathcal{C}_P$ at Q_1 and Q_2, the triangles of $\mathsf{Parasix}(P, Q_i)$, $i = 1, 2$, are degenerate.

Theorem 9. *For $i = 1, 2$, the two lines containing the degenerate triangles of the parasix configuration $\mathsf{Parasix}(P, Q_i)$ are parallel to a tangent from P to the inscribed conic $\mathcal{C}_\ell$ with perspector the trilinear pole of ℓ_P. The two tangents for $i = 1$, 2 are perpendicular if and only if the line ℓ_P contains the orthocorrespondent $P^\perp$.*

For example, for $P = K$, the symmedian point, the circumconic $\mathcal{C}_P$ is the circumcircle. The orthocorrespondent is the point

$$K^\perp = (a^2(a^4 - b^4 + 4b^2c^2 - c^4) : \cdots : \cdots)$$

on the Euler line. The line ℓ joining K to this point has equation

$$\frac{(b^2-c^2)(b^2+c^2-2a^2)}{a^2}x + \frac{(c^2-a^2)(c^2+a^2-2b^2)}{b^2}y + \frac{(a^2-b^2)(a^2+b^2-2c^2)}{c^2}z = 0.$$

The inscribed conic $\mathcal{C}_\ell$ has center

$$(a^2(b^2-c^2)(a^4 - b^4 + b^2c^2 - c^4) : \cdots : \cdots).$$

The tangents from K to the conic $\mathcal{C}_\ell$ are the Brocard axis OK and its perpendicular at K.[10] The points of tangency are

$$\left(\frac{a^2(2a^2-b^2-c^2)}{b^2-c^2} : \frac{b^2(2b^2-c^2-a^2)}{c^2-a^2} : \frac{c^2(2c^2-a^2-b^2)}{a^2-b^2}\right)$$

on the Brocard axis and

$$\left(\frac{a^2(b^2-c^2)}{2a^2-b^2-c^2} : \frac{b^2(c^2-a^2)}{2b^2-c^2-a^2} : \frac{c^2(a^2-b^2)}{2c^2-a^2-b^2}\right)$$

on the perpendicular tangent. See Figure 7. The line ℓ intersects the circumcircle at the point

$$X_{110} = \left(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2}\right)$$

and the Parry point

$$X_{111} = \left(\frac{a^2}{b^2+c^2-2a^2} : \frac{b^2}{c^2+a^2-2b^2} : \frac{c^2}{a^2+b^2-2c^2}\right).$$

The lines containing the degenerate triangles of $\mathsf{Parasix}(K, X_{110})$ are parallel to the Brocard axis, while those for $\mathsf{Parasix}(K, X_{111})$ are parallel to the tangent from K which is perpendicular to the Brocard axis.

[10]The infinite points of these lines are respectively X_{511} and X_{512}.

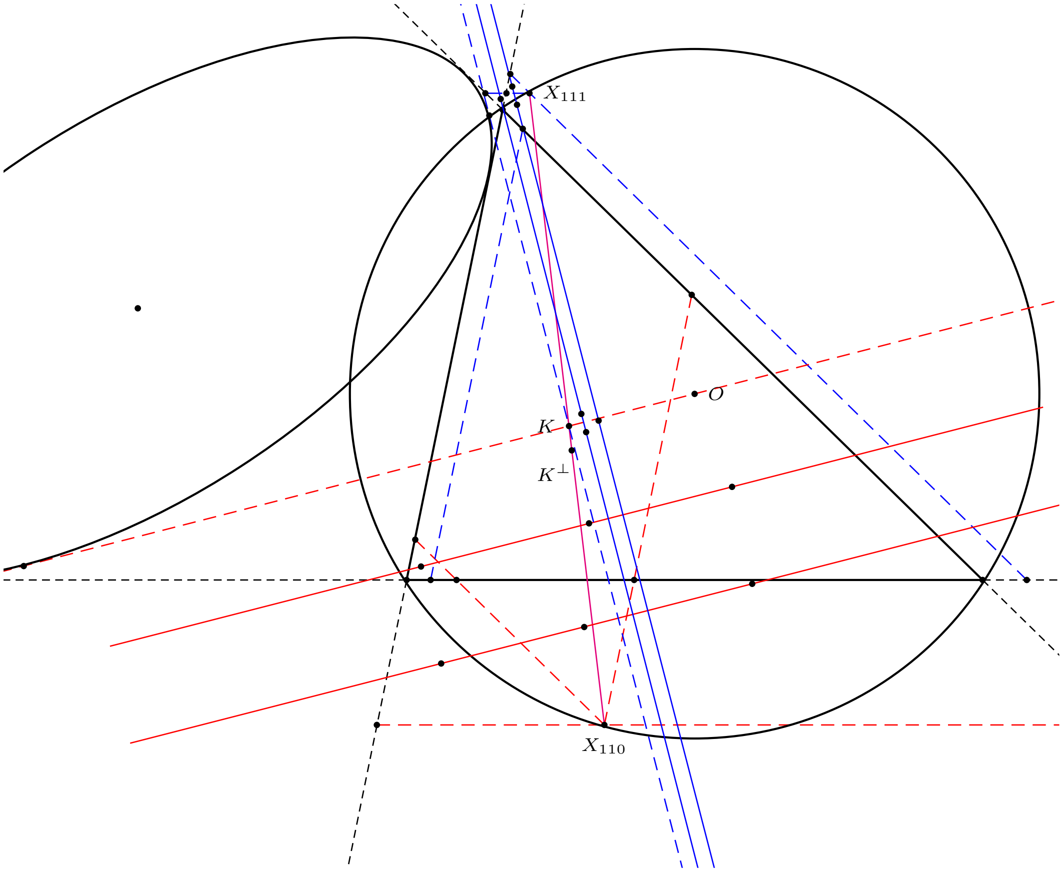


Figure 7. Degenerate Parasix(K, X_{110}) and Parasix(K, X_{111})

References

[1] B. Gibert, Orthocorrespondence and Orthopivotal Cubics, *Forum Geom.*, 3 (2003) 1–27.

[2] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[3] C. Kimberling, *Encyclopedia of Triangle Centers*, July 1, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[4] P. Yiu, The uses of homogeneous barycentric coordinates in plane euclidean geometry, *Int. J. Math. Educ. Sci. Technol.*, 31 (2000) 569 – 578.

Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France
E-mail address: bg42@wanadoo.fr

Floor van Lamoen: St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands
E-mail address: f.v.lamoen@wxs.nl

Forum Geometricorum
Volume 3 (2003) 181–186.

FORUM GEOM
ISSN 1534-1178

A Tetrahedral Arrangement of Triangle Centers

Lawrence S. Evans

Abstract. We present a graphic scheme for indexing 25 collinearities of 17 triangle centers three at a time. The centers are used to label vertices and edges of nested polyhedra. Two new triangle centers are introduced to make this possible.

1. Introduction

Collinearities of triangle centers which are defined in apparently different ways has been of interest to geometers since it was first noticed that the orthocenter, centroid, and circumcenter are collinear, lying on Euler's line. Kimberling [3] lists a great many collinearites, including many more points on Euler's line. The object of this note is to present a three-dimensional graphical summary of 25 three-center collinearities involving 17 centers, in which the centers are represented as vertices and edge midpoints of nested polyhedra: a tetrahedron circumscribing an octahedron which then circumscribes a cubo-octahedron. Such a symmetric collection of collinearities may be a useful mnemonic. Probably the reason why this has not been recognized before is that two of the vertices of the tetrahedron represent previously undescribed centers. First we describe two new centers, which Kimberling lists as X_{1276} and X_{1277} in his *Encyclopedia of Triangle Centers* [3]. Then we describe the tetrahedron and work inward to the cubo-octahedron.

2. Perspectors and the excentral triangle

The excentral triangle, $\mathbf{T}_\text{x}$, of a triangle $\mathbf{T}$ is the triangle whose vertices are the excenters of $\mathbf{T}$. Let $\mathbf{T}_+$ be the triangle whose vertices are the apices of equilateral triangles erected outward on the sides of $\mathbf{T}$. Similarly let $\mathbf{T}_-$ be the triangle whose vertices are the apices of equilateral triangles erected inward on the sides of $\mathbf{T}$. It happens that $\mathbf{T}_\text{x}$ is in perspective from $\mathbf{T}_+$ from a point V_+, a previously undescribed triangle center now listed as X_{1276} in [3], and that $\mathbf{T}_\text{x}$ is also in perspective from $\mathbf{T}_-$ from another new center V_- listed as X_{1277} in [3]. See Figure 1.

For $\varepsilon = \pm 1$, the homogeneous trilinear coordinates of V_ε are

$$1 - v_a + v_b + v_c : 1 + v_a - v_b + v_c : 1 + v_a + v_b - v_c,$$

where $v_a = -\frac{2}{\sqrt{3}} \sin(A + \varepsilon \cdot 60^\circ)$ etc.

It is well known that $\mathbf{T}_\text{x}$ and $\mathbf{T}$ are in perspective from the incenter I. Define $\mathbf{T}^*$ as the triangle whose vertices are the reflections of the vertices of $\mathbf{T}$ in the opposite sides. Then $\mathbf{T}_\text{x}$ and $\mathbf{T}^*$ are in perspective from a point W listed as X_{484} in [3]. See Figure 2. The five triangles $\mathbf{T}$, $\mathbf{T}_\text{x}$, $\mathbf{T}_+$, $\mathbf{T}_-$, and $\mathbf{T}^*$ are pairwise in perspective, giving 10 perspectors. Denote the perspector of two triangles by enclosing the two triangles in brackets, so, for example $[\mathbf{T}_\text{x}, \mathbf{T}] = I$.

Publication Date: September 19, 2003. Communicating Editor: Paul Yiu.

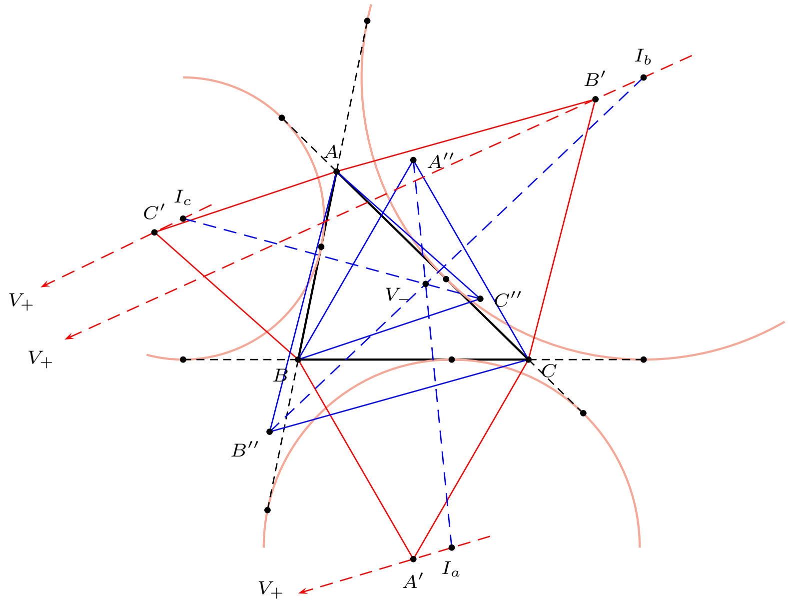


Figure 1

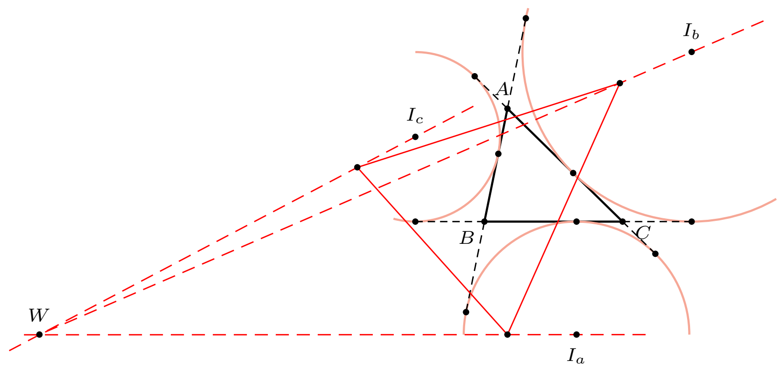


Figure 2

Here is a list of the 10 perspectors with their names and ETC numbers:

$[\mathbf{T}, \mathbf{T}_+]$	F_+	First Fermat point	X_{13}
$[\mathbf{T}, \mathbf{T}_-]$	F_-	Second Fermat point	X_{14}
$[\mathbf{T}, \mathbf{T}^*]$	H	Orthocenter	X_4
$[\mathbf{T}, \mathbf{T}_\mathrm{x}]$	I	Incenter	X_1
$[\mathbf{T}_+, \mathbf{T}_-]$	O	Circumcenter	X_3
$[\mathbf{T}_+, \mathbf{T}^*]$	J_-	Second isodynamic point	X_{16}
$[\mathbf{T}_-, \mathbf{T}^*]$	J_+	First isodynamic point	X_{15}
$[\mathbf{T}_\mathrm{x}, \mathbf{T}^*]$	W	First Evans perspector	X_{484}
$[\mathbf{T}_\mathrm{x}, \mathbf{T}_+]$	V_+	Second Evans perspector	X_{1276}
$[\mathbf{T}_\mathrm{x}, \mathbf{T}_-]$	V_-	Third Evans perspector	X_{1277}

3. Collinearities among the ten perspectors

As in [2], we shall write $\mathcal{L}(X, Y, Z, \dots)$ to denote the line containing X, Y, Z, ... The following collinearities may be easily verified:

$$\begin{array}{lll} \mathcal{L}(I, O, W), & \mathcal{L}(I, J_-, V_-), & \mathcal{L}(I, J_+, V_+), \\ \mathcal{L}(V_+, H, V_-), & \mathcal{L}(W, F_+, V_-), & \mathcal{L}(W, F-, V_+). \end{array}$$

What is remarkable is that all five triangles are involved in each collinearity, with $\mathbf{T}_\mathrm{x}$ used twice. For example, rewrite $\mathcal{L}(I, O, W)$ as

$$\mathcal{L}([\mathbf{T}, \mathbf{T}_\mathrm{x}], [\mathbf{T}_+, \mathbf{T}_-], [\mathbf{T}_\mathrm{x}, \mathbf{T}^*])$$

to see this. The six collinearites have been stated so that the first and third perspectors involve $\mathbf{T}_\mathrm{x}$, with the perspector of the remaining two triangles listed second. This lends itself to a graphical representation as a tetrahedron with vertices labelled with I, V_+, V_-, and W, and the edges labelled with the perspectors collinear with the vertices. See Figure 3. When these centers are actually constructed, they may not be in the order listed in these collinearities. For example, O is not necessarily between I and W. There is another collinearity which we do not use, however, namely, $\mathcal{L}(O, J_+, J_-)$, which is the Brocard axis. Triangle $\mathbf{T}_\mathrm{x}$ is not involved in any of the perspectors in this collinearity.

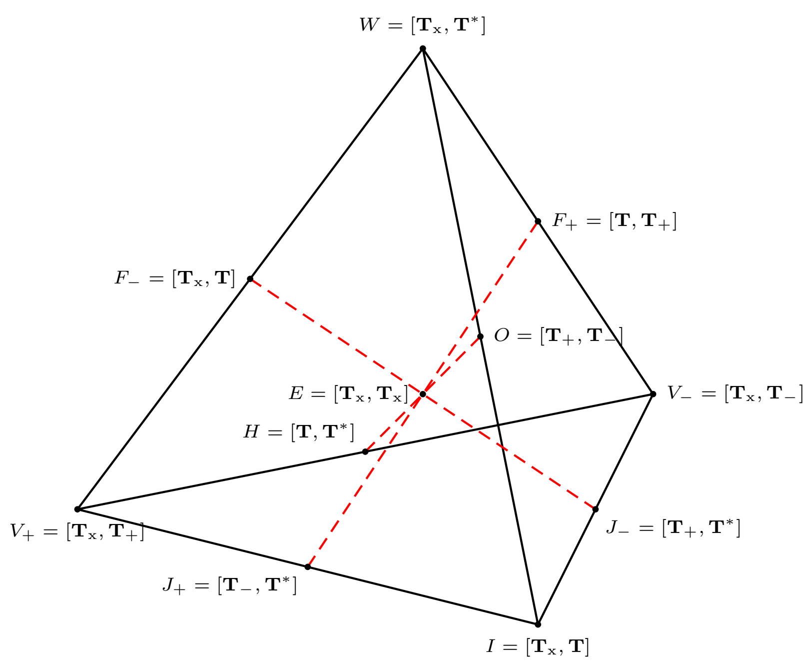


Figure 3

If we label each edge of the tetrahedron at its midpoint by the middle center listed in each of the collinearities above, then opposite edge midpoints are pairs of isogonal conjugates: H and O, J_+ and F_+, and J_- and F_-. Also the lines $\mathcal{L}(O, H)$, $\mathcal{L}(F_+, J_+)$, and $\mathcal{L}(F_-, J_-)$ are parallel to the Euler line, and may be interpreted as intersecting at the Euler infinity point E, listed as X_{30} in [3]. This adds three more collinearities to the tetrahedral scheme:

$$\mathcal{L}(O,E,H),\ \mathcal{L}(F_+,E,J_+),\ \mathcal{L}(F_-,E,J_-).$$

The five triangles $\mathbf{T}$, $\mathbf{T}_+$, $\mathbf{T}_-$, $\mathbf{T}^*$, and $\mathbf{T}_{\mathrm{x}}$ are all inscribed in Neuberg's cubic curve. Now consider a triangle $\mathbf{T}'_{\mathrm{x}}$ in perspective with $\mathbf{T}_{\mathrm{x}}$ and inscribed in the cubic with vertices very close to those of $\mathbf{T}_{\mathrm{x}}$ (the excenters of $\mathbf{T}$). The lines of perspective of $\mathbf{T}'_{\mathrm{x}}$ and $\mathbf{T}_{\mathrm{x}}$ approach the tangents to Neuberg's cubic at the vertices of $\mathbf{T}_{\mathrm{x}}$ as $\mathbf{T}'_{\mathrm{x}}$ approaches $\mathbf{T}_{\mathrm{x}}$. These tangents are known to be parallel to the Euler line and may be thought of as converging at the Euler point at infinity, $E = X_{30}$. So we can write $E = [\mathbf{T}_{\mathrm{x}}, \mathbf{T}_{\mathrm{x}}]$, interpreting this to mean that $\mathbf{T}_{\mathrm{x}}$ is in perspective from itself from E. I propose the term "ipseperspector" for such a point, from the Latin "ipse" for self. Note that the notion of ipseperspector is dependent on the curve circumscribing the triangle $\mathbf{T}$. A well-known example of an ipseperspector for a triangle curcumscribed in Neuberg's cubic is X_{74}, this being the point where the tangents to the curve at the vertices of $\mathbf{T}$ intersect.

4. Further nested polyhedra

We shall encounter other named centers, which are listed here for reference:

G	Centroid	X_2
K	Symmedian (Lemoine) point	X_6
N_+	First Napoleon point	X_{17}
N_-	Second Napoleon point	X_{18}
N_+^*	Isogonal conjugate of N_+	X_{61}
N_-^*	Isogonal conjugate of N_-	X_{62}

The six midpoints of the edges of the tetrahedron may be considered as the vertices of an inscribed octahedron. This leads to indexing more collinearities in the following way: label the midpoint of each edge of the octahedron by the point where the lines indexed by opposite edges meet. For example, opposite edges of the octahedron $\mathcal{L}(F_+,J_-)$ and $\mathcal{L}(F_-,J_+)$ meet at the centroid G. We can then write two 3-point collinearities as $\mathcal{L}(F_+,G,J_-)$ and $\mathcal{L}(F_-,G,J_+)$. Now the edges adjacent to both of these edges index the lines $\mathcal{L}(F_+,F_-)$ and $\mathcal{L}(J_+,J_-)$, which meet at the symmedian point K. This gives two more 3-point collinearities, $\mathcal{L}(F_+,K,F_-)$ and $\mathcal{L}(J_+,K,J_-)$. Note that G and K are isogonal conjugates. This pattern persists with the other pairs of opposite edges of the octahedron.

The intersections of other lines represented as opposite edges intersect at the Napoleon points and their isogonal conjugates. When we consider the four vertices O, F_-, H, and J_- of the octahedron, four more 3-point collinearities are indexed in the same manner: $\mathcal{L}(O,N_-^*,J_-)$, $\mathcal{L}(H,N_-^*,F_-)$, $\mathcal{L}(O,N_-,F_-)$, and $\mathcal{L}(H,N_-,J_-)$. Similarly, from vertices O, F_+ , H, and J_+, four more 3-point collinearites arise in the same indexing process: $\mathcal{L}(O,N_+^*,J_+)$, $\mathcal{L}(H,N_+^*,F_+)$, $\mathcal{L}(O,N_+,F_+)$, and $\mathcal{L}(H,N_+,J_+)$. So each of the twelve edges of the octahedron indexes a different 3-point collinearity.

Let us carry this indexing scheme further. Now consider the midpoints of the edges of the octahedron to be the vertices of a polyhedron inscribed in the octahedron. This third nested polyhedron is a cubo-octahedron: it has eight triangular faces, each of which is coplanar with a face of the octahedron, and six square faces. Yet again more 3-point collinearities are indexed, but this time by the triangular faces of the cubo-octahedron. It happens that the three vertices of each triangular face of the cubo-octahedron, which inherit their labels as edges of the octahedron, are collinear in the plane of the basic triangle $\mathbf{T}$. Opposite edges of the octahedron have the same point labelling their midpoints, so opposite triangular faces of the cubo-octahedron are labelled by the same three centers. This means that there are four instead of eight collinearities indexed by the triangular faces: $\mathcal{L}(G, N_+, N_-^*)$,$\mathcal{L}(G, N_-, N_+^*)$, $\mathcal{L}(K, N_+, N_-)$, and $\mathcal{L}(K, N_-^*, N_+^*)$. See Figure 4.

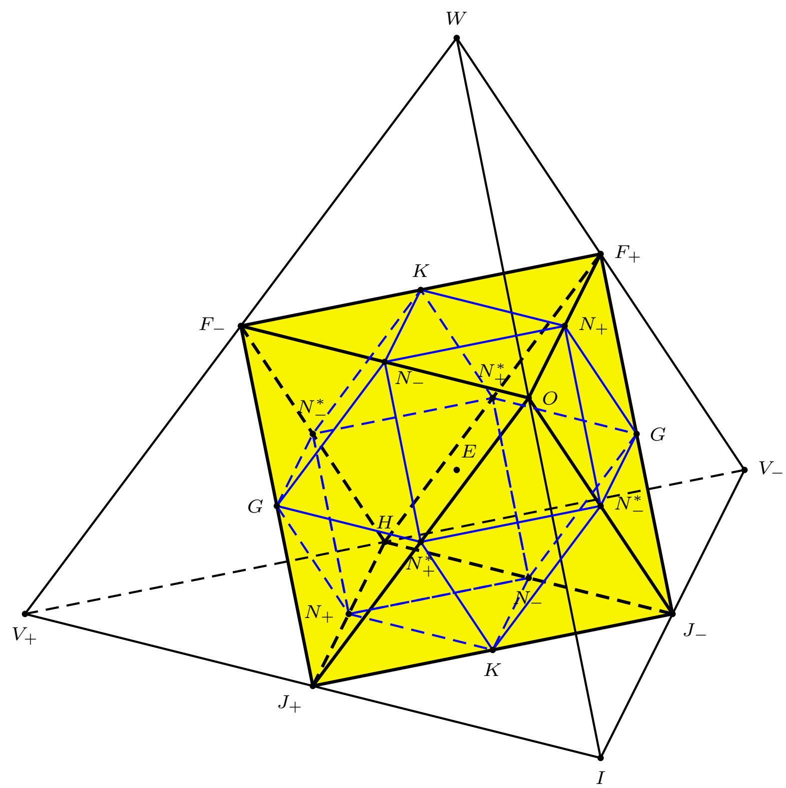


Figure 4

So we have 6 collinearites indexed by edges of the tetrahedron, 3 more by its diagonals, 12 by the inscribed octahedron, and 4 more by the further inscribed cubo-octahedron, for a total of 25.

5. Concluding remarks

In a sense, the location of each center entering into this graphical scheme places it in equal importance to the other centers in similar locations. So the four centers I, U, V, and W, which arose as perspectors with the excentral triangle are on one level. On the next level we may place the six centers O, H, J_+, J_-, F_+, and F_- which index the edges of the tetrahedron and the vertices of the inscribed octahedron. It is interesting that these six centers are the first to appear in the construction given by the author [1], and that the subsequent centers indexed by the midpoints of the edges of the octahedron arise as intersections of lines they determine. The Euler infinity point, E, is the only point at the third level of construction. Centers I, V_+, V_-, W, O, H, F_+ , J_+, F_-, J_-, and E all lie on Neuberg's cubic curve. The Euler line appears as the collinearity $\mathcal{L}(O, E, H)$, with no indication that G lies on the line. The Brocard axis appears four times as $\mathcal{L}(J_+, K, J_-)$, $\mathcal{L}(K, N^*_-, N^*_+)$, $\mathcal{L}(O, N^*_+, J_+)$, and $\mathcal{L}(O, N^*_-, J-)$, but the better-known collinearity $\mathcal{L}(O, J_+, J_-)$ does not.

References

[1] L. S. Evans, A rapid construction of some triangle centers, *Forum Geom.*, 2 (2002) 67–70.
[2] L. S. Evans, Some configurations of triangle centers, *Forum Geom.*, 3 (2003) 49–56.
[3] C. Kimberling, *Encyclopedia of Triangle Centers*, August 16, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[4] D. Wells, *The Penguin Dictionary of Curious and Interesting Geometry*, Penguin, London, 1991.

Lawrence S. Evans: 910 W. 57th Street, La Grange, Illinois 60525, USA
E-mail address: 75342.3052@compuserve.com

Forum Geometricorum
Qolume 3 (2003) 187–195.

FORUM GEOM
ISSN 1534-1178

The Apollonius Circle and Related Triangle Centers

Milorad R. Stevanović

Abstract. We give a simple construction of the Apollonius circle without directly invoking the excircles. This follows from a computation of the coordinates of the centers of similitude of the Apollonius circle with some basic circles associated with a triangle. We also find a circle orthogonal to the five circles, circumcircle, nine-point circle, excentral circle, radical circle of the excircles, and the Apollonius circle.

1. The Apollonius circle of a triangle

The Apollonius circle of a triangle is the circle tangent internally to each of the three excircles. Yiu [5] has given a construction of the Apollonius circle as the inversive image of the nine-point circle in the radical circle of the excircles, and the coordinates of its center Q. It is known that this radical circle has center the Spieker center S and radius $\rho = \frac{1}{2}\sqrt{r^2 + s^2}$. See, for example, [6, Theorem 4]. Ehrmann [1] found that this center can be constructed as the intersection of the Brocard axis and the line joining S to the nine-point center N. See Figure 1. A proof of this fact was given in [2], where Grinberg and Yiu showed that the Apollonius circle is a

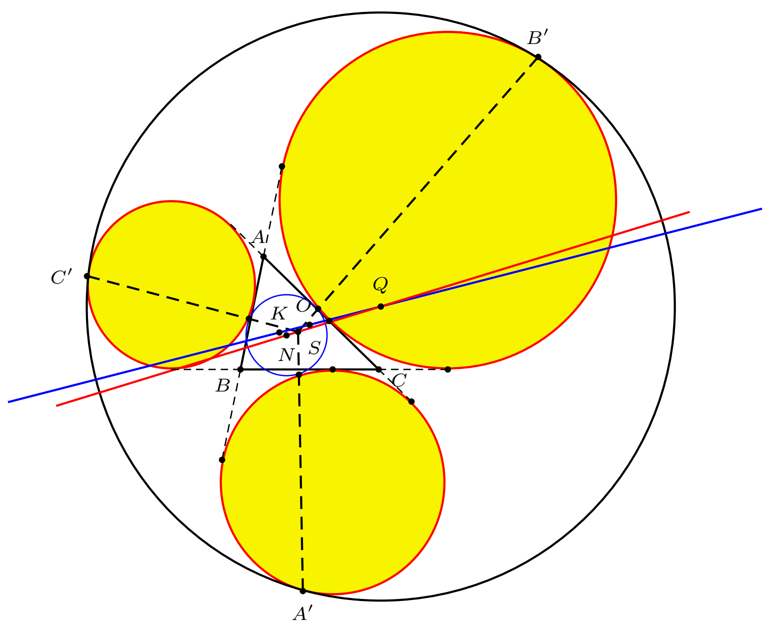


Figure 1

Publication Date: October 15, 2003. Communicating Editor: Paul Yiu.
The author thanks the editor for his helps in the preparation of this paper.

Tucker circle. In this note we first verify these results by expressing the coordinates of Q in terms of R, r, and s, (the circumradius, inradius, and semiperimeter) of the triangle. By computing some homothetic centers of circles associated with the Apollonius circle, we find a simple construction of the Apollonius circle without directly invoking the excircles. See Figure 4.

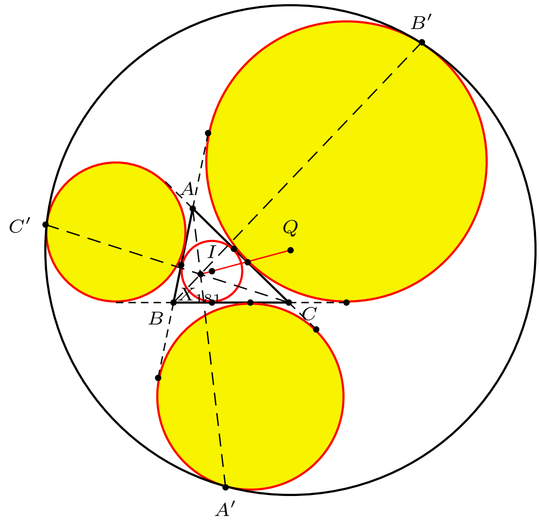


Figure 2

For triangle centers we shall adopt the notation of Kimberling's *Encyclopedia of Triangle Centers* [3], except for the most basic ones:

G	centroid	O	circumcenter
I	incenter	H	orthocenter
N	nine-point center	K	symmedian point
S	Spieker center	I'	reflection of I in O

We shall work with barycentric coordinates, absolute and homogeneous. It is known that if the Apollonius circle touches the three excircles respectively at A', B', C', then the lines AA', BB', CC' concur in the point [1]

$$X_{181} = \left(\frac{a^2(b+c)^2}{s-a} : \frac{b^2(c+a)^2}{s-b} : \frac{c^2(a+b)^2}{s-c}\right).$$

We shall make use of the following simple lemma.

Lemma 1. *Under inversion with respect to a circle, center P, radius ρ, the image of the circle center P', radius ρ', is the circle, radius $\left|\frac{\rho^2}{d^2-\rho'^2}\cdot\rho'\right|$ and center Q which divides the segment PP' in the ratio*

$$PQ : QP' = \rho^2 : d^2 - \rho^2 - \rho'^2,$$

[1]The trilinear coordinates of X_{181} were given by Peter Yff in 1992.

where d is the distance between P and P'. Thus,

$$Q = \frac{(d^2 - \rho^2 - \rho'^2)P + \rho^2 \cdot P'}{d^2 - \rho'^2}.$$

Theorem 2. *The Apollonius circle has center*

$$Q = \frac{1}{4Rr}\left((r^2 + 4Rr + s^2)O + 2Rr \cdot H - (r^2 + 2Rr + s^2)I\right)$$

and radius $\frac{r^2+s^2}{4r}$.

Proof. It is well known that the distance between O and I is given by

$$OI^2 = R^2 - 2Rr.$$

Since S and N divide the segments IG and OG in the ratio $3 : -1$,

$$SN^2 = \frac{R^2 - 2Rr}{4}.$$

Applying Lemma 1 with

$$\begin{array}{ll} P = S = \frac{1}{2}(3G - I) = \frac{1}{2}(2O + H - I), & P' = N = \frac{1}{2}(O + H), \\ \rho^2 = \frac{1}{4}(r^2 + s^2), & \rho'^2 = \frac{1}{4}R^2, \\ d^2 = SN^2 = \frac{1}{4}(R^2 - 2Rr), & \end{array}$$

we have

$$Q = \frac{1}{4Rr}\left((r^2 + 4Rr + s^2)O + 2Rr \cdot H - (r^2 + 2Rr + s^2)I\right).$$

The radius of the Apollonius circle is $\frac{r^2+s^2}{4r}$. □

The point Q appears in Kimberling's *Encyclopedia of Triangle Centers* [3] as

$$\begin{aligned} X_{970} =& (a^2(a^3(b+c)^2 + a^2(b+c)(b^2+c^2) - a(b^4 + 2b^3c + 2bc^3 + c^4) \\ & - (b+c)(b^4+c^4)) : \cdots : \cdots). \end{aligned}$$

We verify that it also lies on the Brocard axis.

Proposition 3.

$$\overrightarrow{OQ} = -\frac{s^2 - r^2 - 4Rr}{4Rr} \cdot \overrightarrow{OK}.$$

Proof. The oriented areas of the triangles KHI, OKI, and OHK are as follows.

$$\begin{aligned} \triangle(KHI) =& \frac{(a-b)(b-c)(c-a)f}{16(a^2+b^2+c^2) \cdot \triangle}, \\ \triangle(OKI) =& \frac{abc(a-b)(b-c)(c-a)}{8(a^2+b^2+c^2) \cdot \triangle}, \\ \triangle(OHK) =& \frac{-(a-b)(b-c)(c-a)(a+b)(b+c)(c+a)}{8(a^2+b^2+c^2) \cdot \triangle}, \end{aligned}$$

where $\triangle$ is the area of triangle ABC and

$$\begin{aligned} f =& a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) + 2abc \\ =& 8rs(2R+r). \end{aligned}$$

Since $abc = 4Rrs$ and $(a+b)(b+c)(c+a) = 2s(r^2+2Rr+s^2)$, it follows that, with respect to OHI, the symmedian point K has homogeneous barycentric coordinates

$$\begin{aligned} & f : 2abc : -2(a+b)(b+c)(c+a) \\ =& 8rs(2R+r) : 8Rrs : -4s(r^2+2Rr+s^2) \\ =& 2r(2R+r) : 2Rr : -(r^2+2Rr+s^2). \end{aligned}$$

Therefore,

$$K = \frac{1}{4Rr+r^2-s^2}\left(2r(2R+r)O + 2Rr\cdot H - (r^2+2Rr+s^2)I\right),$$

and

$$\begin{aligned} \overrightarrow{OK} =& \frac{1}{4Rr+r^2-s^2}\left((r^2+s^2)O + 2Rr\cdot H - (r^2+2Rr+s^2)I\right) \\ =& -\frac{4Rr}{s^2-r^2-4Rr}\cdot\overrightarrow{OQ}. \end{aligned}$$

$\square$

2. Centers of similitude

We compute the coordinates of the centers of similitude of the Apollonius circle with several basic circles. Figure 3 below shows the Apollonius circle with the circumcircle, incircle, nine-point circle, excentral circle, and the radical circle (of the excircles). Recall that the excentral circle is the circle through the excenters of the triangle. It has center I' and radius $2R$.

Lemma 4. *Two circles with centers P, P', and radii ρ, ρ' respectively have internal center of similitude* $\dfrac{\rho'\cdot P+\rho\cdot P'}{\rho'+\rho}$ *and external center of similitude* $\dfrac{\rho'\cdot P-\rho\cdot P'}{\rho'-\rho}$.

Proposition 5. *The homogeneous barycentric coordinates (with respect to triangle ABC) of the centers of similitude of the Apollonius circle with the various circles are as follows.*

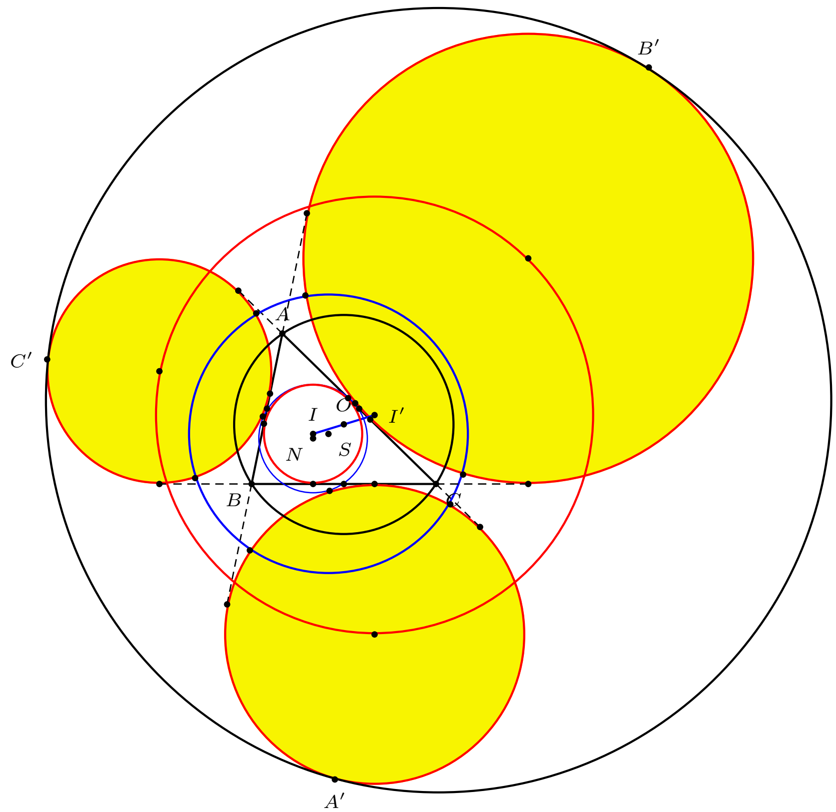


Figure 3

circumcircle	
internal X_{573}	$a^2(a^2(b+c) - abc - (b^3+c^3)) : \cdots : \cdots$
external X_{386}	$a^2(a(b+c) + b^2 + bc + c^2) : \cdots : \cdots$
incircle	
internal X_{1682}	$a^2(s-a)(a(b+c)+b^2+c^2)^2 : \cdots : \cdots$
external X_{181}	$\frac{a^2(b+c)^2}{s-a} : \cdots : \cdots$
nine − point circle	
internal S	$b+c : c+a : a+b$
external X_{2051}	$\frac{1}{a^3-a(b^2-bc+c^2)-bc(b+c)} : \cdots : \cdots$
excentral circle	
internal X_{1695}	$a \cdot F : \cdots : \cdots$
external X_{43}	$a(a(b+c) - bc) : \cdots : \cdots$

where

$$\begin{aligned} F =& a^5(b+c) + a^4(4b^2+7bc+4c^2) + 2a^3(b+c)(b^2+c^2) \\ & - 2a^2(2b^4+3b^3c+3bc^3+2c^4) - a(b+c)(3b^4+2b^2c^2+3c^4) - bc(b^2-c^2)^2. \end{aligned}$$

Proof. The homogenous barycentric coordinates (with respect to triangle OHI) of the centers of similitude of the Apollonius circle with the various circles are as follows.

circumcircle	
internal X_{573}	$2(r^2+2Rr+s^2):2Rr:-(r^2+2Rr+s^2)$
external X_{386}	$4Rr:2Rr:-(r^2+2Rr+s^2)$
incircle	
internal X_{1682}	$-r(r^2+4Rr+s^2):-2Rr^2:r^3+Rr^2-(R-r)s^2$
external X_{181}	$-r(r^2+4Rr+s^2):-2Rr^2:r^3+3Rr^2+(R+r)s^2$
nine $-$ point circle	
internal S	$2:1:-1$
external X_{2051}	$-4Rr:r^2-2Rr+s^2:r^2+2Rr+s^2$
excentral circle	
internal X_{1695}	$4(r^2+2Rr+s^2):4Rr:-(3r^2+4Rr+3s^2)$
external X_{43}	$8Rr:4Rr:-(r^2+4Rr+s^2)$

Using the relations

$$r^2=\frac{(s-a)(s-b)(s-c)}{s}\quad\text{and}\quad R=\frac{abc}{4rs},$$

and the following coordinates of O, H, I (with equal coordinate sums),

$$\begin{aligned}O =&(a^2(b^2+c^2-a^2),b^2(c^2+a^2-b^2),c^2(a^2+b^2-c^2)),\\ H =&((c^2+a^2-b^2)(a^2+b^2-c^2),(a^2+b^2-c^2)(b^2+c^2-a^2),\\ &(b^2+c^2-a^2)(c^2+a^2-b^2)),\\ I =&(b+c-a)(c+a-b)(a+b-c)(a,b,c),\end{aligned}$$

these can be converted into those given in the proposition. □

Remarks. 1. $X_{386}=OK\cap IG$.
2. $X_{573}=OK\cap HI'=OK\cap X_{55}X_{181}$.
3. $X_{43}=IG\cap X_{57}X_{181}$.

From the observation that the Apollonius circle and the nine-point circle have S as internal center of similitude, we have an easy construction of the Apollonius circle without directly invoking the excircles.

Construct the center Q of Apollonius circle as the intersection of OK and NS. Let D be the midpoint of BC. Join ND and construct the parallel to ND through Q (the center of the Apollonius circle) to intersect DS at A', a point on the Apollonius circle, which can now be easily constructed. See Figure 4.

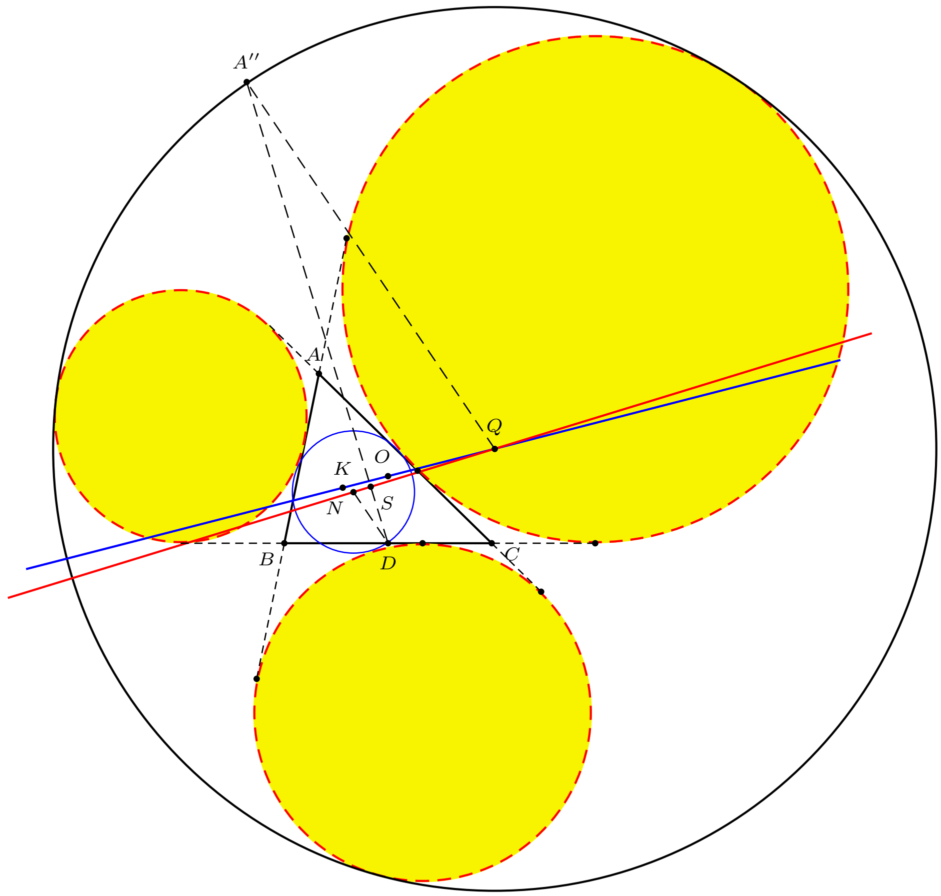


Figure 4

Proposition 6. *The center Q of the Apollonius circle lies on the each of the lines $X_{21}X_{51}$, $X_{40}X_{43}$ and $X_{411}X_{185}$. More precisely,*

$$X_{51}X_{21} : X_{21}Q = 2r : 3R,$$
$$X_{43}X_{40} : X_{43}Q = 8Rr : r^2 + s^2,$$
$$X_{185}X_{411} : X_{411}Q = 2r : R.$$

Remark. The Schiffler point X_{21} is the intersection of the Euler lines of the four triangles ABC, IBC, ICA and IAB. It divides OH in the ratio

$$OX_{21} : X_{21}H = R : 2(R + r).$$

The harmonic conjugate of X_{21} in OH is the triangle center

$$\begin{aligned} X_{411} =&(a(a^6 - a^5(b+c) - a^4(2b^2 + bc + 2c^2) + 2a^3(b+c)(b^2 - bc + c^2) \\ &+ a^2(b^2+c^2)^2 - a(b-c)^2(b+c)(b^2+c^2) + bc(b-c)^2(b+c)^2) \\ &: \cdots : \cdots). \end{aligned}$$

3. A circle orthogonal to 5 given ones

We write the equations of the circles encountered above in the form

$$a^2yz + b^2zx + c^2xy + (x + y + z)L_i = 0,$$

where L_i, $1 \leq i \leq 5$, are linear forms given below.

i	circle	L_i
1	circumcircle	0
2	nine − point circle	$-\frac{1}{4}((b^2 + c^2 - a^2)x + (c^2 + a^2 - b^2)y + (a^2 + b^2 - c^2)z)$
3	excentral circle	$bcx + cay + abz$
4	radical circle	$(s - b)(s - c)x + (s - c)(s - a)y + (s - a)(s - b)z$
5	Apollonius	$s\left((s + \frac{bc}{a})x + (s + \frac{ca}{b})y + (s + \frac{ab}{c})z\right)$

Remark. The equations of the Apollonius circle was computed in [2]. The equations of the other circles can be found, for example, in [6].

Proposition 7. *The four lines $L_i = 0$, $i = 2, 3, 4, 5$, are concurrent at the point*

$$X_{650} = (a(b - c)(s - a) : b(c - a)(s - b) : c(a - b)(s - c)).$$

It follows that this point is the radical center of the five circles above. From this we obtain a circle orthogonal to the five circles.

Theorem 8. *The circle*

$$a^2yz + b^2zx + c^2xy + (x + y + z)L = 0,$$

where

$$L = \frac{bc(b^2 + c^2 - a^2)}{2(c - a)(a - b)}x + \frac{ca(c^2 + a^2 - b^2)}{2(a - b)(b - c)}y + \frac{ab(a^2 + b^2 - c^2)}{2(b - c)(c - a)}z,$$

is orthogonal to the circumcircle, excentral circle, Apollonius circle, nine-point circle, and the radical circle of the excircles. It has center X_{650} and radius the square root of

$$\frac{abc \cdot G}{4(a - b)^2(b - c)^2(c - a)^2},$$

where

$$\begin{aligned} G =& abc(a^2 + b^2 + c^2) - a^4(b + c - a) - b^4(c + a - b) - c^4(a + b - c) \\ =& 16r^2s(r^2 + 5Rr + 4R^2 - s^2). \end{aligned}$$

This is an interesting result because among these five circles, only three are coaxal, namely, the Apollonius circle, the radical circle, and the nine-point circle.

Remark. X_{650} is also the perspector of the triangle formed by the intersections of the corresponding sides of the orthic and intouch triangles. It is the intersection of the trilinear polars of the Gergonne and Nagel points.

4. More centers of similitudes with the Apollonius circle

We record the coordinates of the centers of similitude of the Apollonius circle with the Spieker radical circle. These are

$$(a^2(-a^3(b+c)^2 - a^2(b+c)(b^2+c^2) + a(b^4+2b^3c+2bc^3+c^4) + (b+c)(b^4+c^4))$$
$$\pm\, abc(b+c)\sqrt{(b+c-a)(c+a-b)(a+b-c)(a^2(b+c)+b^2(c+a)+c^2(a+b)+abc)}$$
$$: \cdots : \cdots)$$

It turns out that the centers of similitude with the Spieker circle (the incircle of the medial triangle) and the Moses circle (the one tangent internally to the nine-point circle at the center of the Kiepert hyperbola) also have rational coordinates in a, b, c:

Spieker circle	
internal	$a(b+c-a)(a^2(b+c)^2 + a(b+c)(b^2+c^2) + 2b^2c^2)$
external	$a(a^4(b+c)^2 + a^3(b+c)(b^2+c^2) - a^2(b^4-4b^2c^2+c^4))$ $-a(b+c)(b^4-2b^3c-2b^2c^2-2bc^3+c^4) + 2b^2c^2(b+c)^2)$
Moses circle	
internal	$a^2(b+c)^2(a^3 - a(2b^2-bc+2c^2) - (b^3+c^3))$
external	$a^2(a^3(b+c)^2 + 2a^2(b+c)(b^2+c^2) - abc(b-c)^2$ $-(b-c)^2(b+c)(b^2+bc+c^2))$

References

[1] J-P. Ehrmann, Hyacinthos message 4620, January 2, 2002

[2] D. Grinberg and P. Yiu, The Apollonius circle as a Tucker circle, *Forum Geom.*, 2 (2002) 175–182.

[3] C. Kimberling, *Encyclopedia of Triangle Centers*, September 29, 2003 edition, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[4] C. Kimberling, S. Iwata, and H. Fukagawa, Problem 1091 and solution, *Crux Math.*, 13 (1987) 128-129, 217-218.

[5] P. Yiu, Hyacinthos message 4619, January 1, 2002.

[6] P. Yiu, *Introduction to the Geometry of the triangle*, Florida Atlantic University Lecture Notes, 2001, available at http://www.math.fau.edu/yiu/geometry.html.

Milorad R. Stevanović: Technical Faculty, Svetog Save 65, 32000 Čačak, Serbia and Montenegro
E-mail address: milmath@ptt.yu

Forum Geometricorum
Volume 3 (2003) 197–203.

FORUM GEOM
ISSN 1534-1178

Two Triangle Centers Associated with the Excircles

Milorad R. Stevanović

Abstract. The triangle formed by the second intersections of the bisectors of a triangle and the respective excircles is perspective to each of the medial and intouch triangles. We identify the perspectors. In the former case, the perspector is closely related to the Yff center of congruence.

1. Introduction

In this note we construct two triangle centers associated with the excircles. Given a triangle ABC, let A' be the "second" intersection of the bisector of angle A with the A-excircle, which is outside the segment AI_a, I_a being the A-excenter. Similarly, define B' and C'.

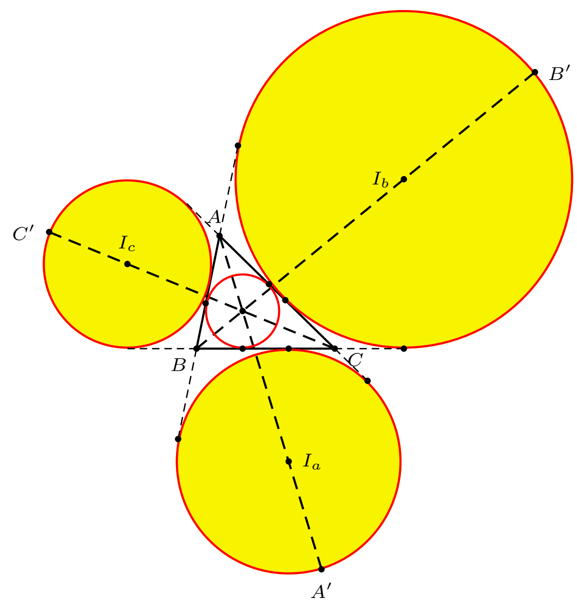


Figure 1

Theorem 1. *Triangle $A'B'C'$ is perspective with the medial triangle at the Yff center of congruence of the latter triangle, namely, the point P with homogeneous barycentric coordinates*

$$\left(\sin\frac{B}{2}+\sin\frac{C}{2}:\sin\frac{C}{2}+\sin\frac{A}{2}:\sin\frac{A}{2}+\sin\frac{B}{2}\right)$$

with respect to ABC.

Publication Date: October 22, 2003. Communicating Editor: Paul Yiu.

Theorem 2. *Triangle $A'B'C'$ is perspective with the intouch triangle at the point Q with homogeneous barycentric coordinates*

$$\left(\tan\frac{A}{2}\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right):\tan\frac{B}{2}\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right):\tan\frac{C}{2}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\right).$$

Remark. These triangle centers now appear as X_{2090} and X_{2091} in [2].

2. Notations and preliminaries

We shall make use of the following notations. In a triangle ABC of sidelengths a, b, c, circumradius R, inradius r, and semiperimeter s, let

$$s_a = \sin\tfrac{A}{2},\quad s_b = \sin\tfrac{B}{2},\quad s_c = \sin\tfrac{C}{2};$$

$$c_a = \cos\tfrac{A}{2},\quad c_b = \cos\tfrac{B}{2},\quad c_c = \cos\tfrac{C}{2}.$$

The following formulae can be found, for example, in [1].

$$\begin{array}{lll} r = 4Rs_as_bs_c, & s = 4Rc_ac_bc_c; & \\ s-a = 4Rc_as_bs_c, & s-b = 4Rs_ac_bs_c, & s-c = 4Rs_as_bc_c. \end{array}$$

2.1. *The medial triangle.* The medial triangle $A_1B_1C_1$ has vertices the midpoints of the sides BC, CA, AB of triangle ABC. From

$$\mathbf{A_1} = \frac{\mathbf{B}+\mathbf{C}}{2},\qquad \mathbf{B}_1 = \frac{\mathbf{C}+\mathbf{A}}{2},\qquad \mathbf{C}_1 = \frac{\mathbf{A}+\mathbf{B}}{2},$$

we have

$$\mathbf{A} = \mathbf{B}_1+\mathbf{C}_1-\mathbf{A}_1,\qquad \mathbf{B} = \mathbf{C}_1+\mathbf{A}_1-\mathbf{B}_1,\qquad \mathbf{C} = \mathbf{A}_1+\mathbf{B}_1-\mathbf{C}_1. \quad (1)$$

Lemma 3. *The barycentric coordinates of the excenters with respect to the medial triangle are*

$$\begin{aligned} \mathbf{I}_a &= \frac{s\cdot\mathbf{A}_1-(s-c)\mathbf{B}_1-(s-b)\mathbf{C}_1}{s-a}, \\ \mathbf{I}_b &= \frac{-(s-c)\mathbf{A}_1+s\cdot\mathbf{B}_1-(s-a)\mathbf{C}_1}{s-b}, \\ \mathbf{I}_c &= \frac{-(s-b)\mathbf{A}_1-(s-a)\mathbf{B}_1+s\cdot\mathbf{C}_1}{s-c}. \end{aligned}$$

Proof. It is enough to compute the coordinates of the excenter I_a:

$$\begin{aligned} \mathbf{I}_a &= \frac{-a\cdot\mathbf{A}+b\cdot\mathbf{B}+c\cdot\mathbf{C}}{b+c-a} \\ &= \frac{-a(\mathbf{B}_1+\mathbf{C}_1-\mathbf{A}_1)+b(\mathbf{C}_1+\mathbf{A}_1-\mathbf{B}_1)+c(\mathbf{A}_1+\mathbf{B}_1-\mathbf{C}_1)}{b+c-a} \\ &= \frac{(a+b+c)\mathbf{A}_1-(a+b-c)\mathbf{B}_1-(c+a-b)\mathbf{C}_1}{b+c-a} \\ &= \frac{s\cdot\mathbf{A}_1-(s-c)\mathbf{B}_1-(s-b)\mathbf{C}_1}{s-a}. \end{aligned}$$

□

2.2. *The intouch triangle.* The vertices of the intouch triangle are the points of tangency of the incircle with the sides. These are

$$\mathbf{X}=\frac{(s-c)\mathbf{B}+(s-b)\mathbf{C}}{a},\quad \mathbf{Y}=\frac{(s-c)\mathbf{A}+(s-a)\mathbf{C}}{b},\quad \mathbf{Z}=\frac{(s-b)\mathbf{A}+(s-a)\mathbf{B}}{c}.$$

Equivalently,

$$\begin{aligned}\mathbf{A}=&\frac{-a(s-a)\mathbf{X}+b(s-b)\mathbf{Y}+c(s-c)\mathbf{Z}}{2(s-b)(s-c)},\\ \mathbf{B}=&\frac{a(s-a)\mathbf{X}-b(s-b)\mathbf{Y}+c(s-c)\mathbf{Z}}{2(s-c)(s-a)},\\ \mathbf{C}=&\frac{a(s-a)\mathbf{X}+b(s-b)\mathbf{Y}-c(s-c)\mathbf{Z}}{2(s-a)(s-b)}.\end{aligned}\tag{2}$$

Lemma 4. *The barycentric coordinates of the excenters with respect to the intouch triangle are*

$$\begin{aligned}\mathbf{I}_a=&\frac{a(bc-(s-a)^2)\mathbf{X}-b(s-b)^2\mathbf{Y}-c(s-c)^2\mathbf{Z}}{2(s-a)(s-b)(s-c)},\\ \mathbf{I}_b=&\frac{-a(s-a)^2\mathbf{X}+b(ca-(s-b)^2)\mathbf{Y}-c(s-c)^2\mathbf{Z}}{2(s-a)(s-b)(s-c)},\\ \mathbf{I}_c=&\frac{-a(s-a)^2\mathbf{X}-b(s-b)^2\mathbf{Y}+c(ab-(s-c)^2)\mathbf{Z}}{2(s-a)(s-b)(s-c)}.\end{aligned}$$

3. Proof of Theorem 1

We compute the barycentric coordinates of A' with respect to the medial triangle. Note that A' divides AI_a externally in the ratio $AA':A'I_a=1+s_a:-s_a$. It follows that

$$\begin{aligned}\mathbf{A}'=&(1+s_a)\mathbf{I}_a-s_a\cdot\mathbf{A}\\ =&\frac{1+s_a}{s-a}(s\cdot\mathbf{A}_1-(s-c)\mathbf{B}_1-(s-b)\mathbf{C}_1)-s_a(\mathbf{B}_1+\mathbf{C}_1-\mathbf{A}_1).\end{aligned}$$

From this, the homogeneous barycentric coordinates of A' with respect to $A_1B_1C_1$ are

$$\begin{aligned}&(1+s_a)s+s_a(s-a):-(1+s_a)(s-c)-s_a(s-a)\\ &\qquad :-(1+s_a)(s-b)-s_a(s-a)\\ =&s+s_a(b+c):-((s-c)+s_ab):-((s-b)+s_ac)\\ =&4Rc_ac_bc_c+4Rs_a(s_bc_b+s_cc_c):-4R(s_as_bc_c+s_as_bc_b):-4R(s_ac_bs_c+s_as_cc_c)\\ =&-\frac{c_ac_bc_c+s_a(s_bc_b+s_cc_c)}{s_a(c_b+c_c)}:s_b:s_c.\end{aligned}$$

Similarly,

$$B' = \left(s_a : -\frac{c_a c_b c_c + s_b(s_c c_c + s_a c_a)}{s_b(c_c + c_a)} : s_c \right),$$

$$C' = \left(s_a : s_b : -\frac{c_a c_b c_c + s_c(s_a c_a + s_b c_b)}{s_c(c_a + c_b)} \right).$$

From these, it is clear that $A'B'C'$ and the medial triangle are perspective at the point with coordinates $(s_a : s_b : s_c)$ relative to $A_1B_1C_1$. This is clearly the Yff center of congruence of the medial triangle. See Figure 2. Its coordinates with respect to ABC are

$$(s_b + s_c : s_c + s_a : s_a + s_b).$$

This completes the proof of Theorem 1.

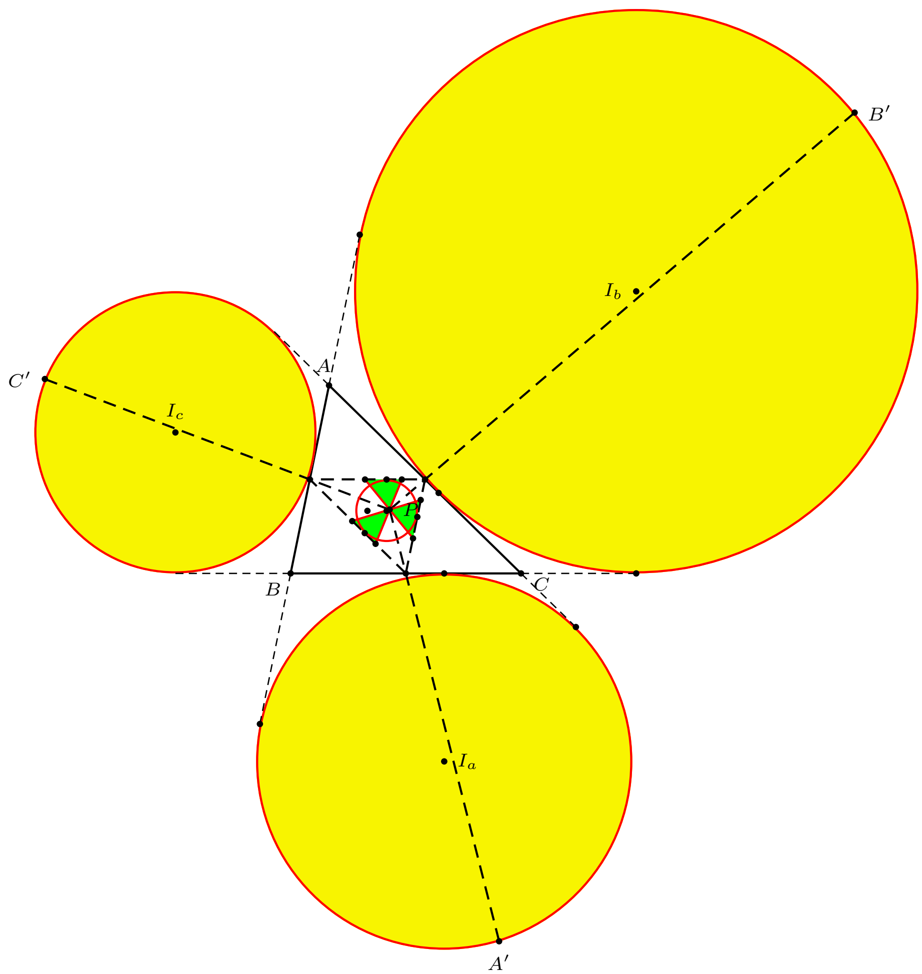


Figure 2

Remark. In triangle ABC, let A'', B'', C'' be the feet of the bisectors of angles BIC, CIA, AIB respectively on sides BC, CA, AB. Triangles $A''B''C''$ and ABC are perspective at the Yff center of congruence X_{174}, *i.e.*, if the perpendiculars from X_{174} to the bisectors of the angles of ABC intersect the sides of triangle ABC at X_b, X_c, Y_a, Y_c, Z_a, Z_b (see Figure 3), then the triangles $X_{174}X_bX_c$, $Y_aX_{174}Y_c$ and $Z_aZ_bX_{174}$ are congruent. See [3].

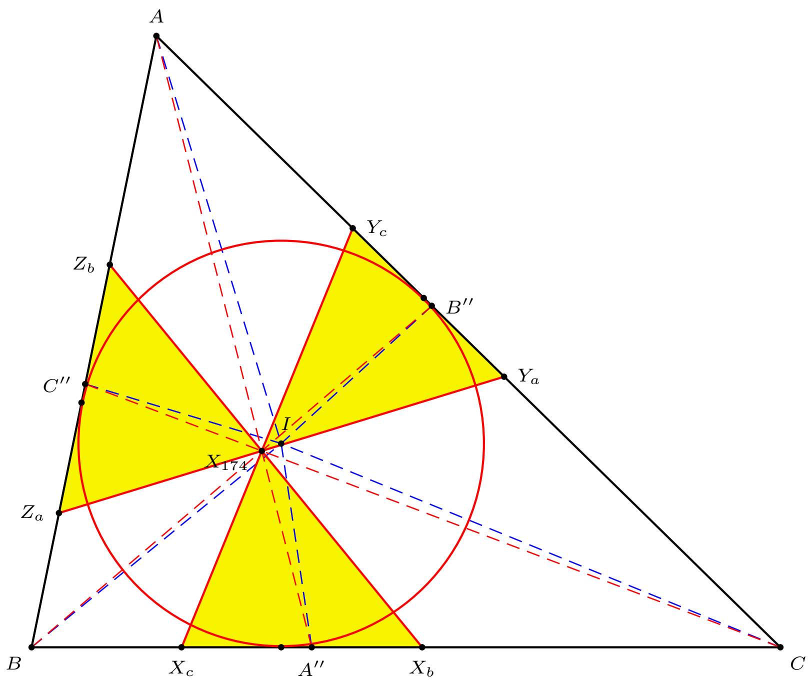


Figure 3

4. Proof of Theorem 2

Consider the coordinates of $\mathbf{A}' = (1+s_a)\mathbf{I}_a - s_a \cdot \mathbf{A}$ with respect to the intouch triangle XYZ. By Lemma 3, the Y-coordinate is

$$\begin{aligned}&\frac{-(1+s_a)b(s-b)^2 - s_a b(s-a)(s-b)}{2(s-a)(s-b)(s-c)}\\ =&\frac{-b(s-b)((1+s_a)(s-b)+s_a(s-a))}{2(s-a)(s-b)(s-c)}\\ =&\frac{-b(s-b)(s-b+s_a\cdot c)}{2(s-a)(s-b)(s-c)}\\ =&\frac{-(c_b+c_c)}{2c_ac_bc_c}\cdot\frac{c_b^2}{s_b}.\end{aligned}$$

Similarly for the Z-coordinate is $\frac{-(c_b+c_c)}{2c_ac_bc_c}\cdot\frac{c_c^2}{s_c}$. Therefore, $A'B'C'$ is perspective with XYZ at

$$Q=\left(\frac{c_a^2}{s_a}:\frac{c_b^2}{s_b}:\frac{c_c^2}{s_c}\right).$$

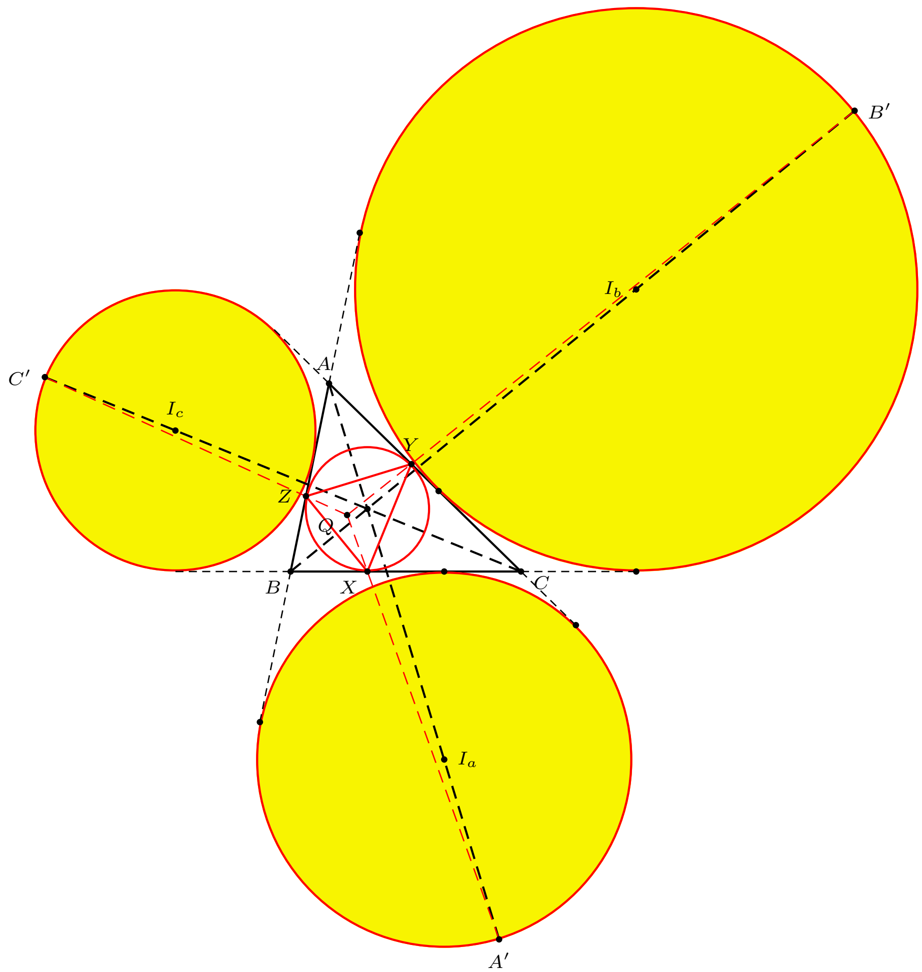


Figure 4

Note that the angles of the intouch triangles are $X=\frac{B+C}{2}$, $Y=\frac{C+A}{2}$, and $Z=\frac{A+B}{2}$. This means

$$s_a=\cos\frac{B+C}{2}=\cos X,\qquad c_a=\sin\frac{B+C}{2}=\sin X,$$

etc. It follows that Q has homogeneous barycentric coordinates

$$\left(\frac{\sin^2 X}{\cos X}:\frac{\sin^2 Y}{\cos Y}:\frac{\sin^2 Z}{\cos Z}\right)$$

and is the Clawson point of the intouch triangle XYZ. With respect to triangle ABC, this perspector Q has coordinates given by

$$\begin{aligned}
&\left(\frac{a(s-a)}{s_a}+\frac{b(s-b)}{s_b}+\frac{c(s-c)}{s_c}\right)\mathbf{Q}\\
=&\frac{a(s-a)\mathbf{X}}{s_a}+\frac{b(s-b)\mathbf{Y}}{s_b}+\frac{c(s-c)\mathbf{Z}}{s_c}\\
=&\frac{(s-b)(s-c)(s_b+s_c)}{s_bs_c}\mathbf{A}+\frac{(s-c)(s-a)(s_c+s_a)}{s_cs_a}\mathbf{B}+\frac{(s-a)(s-b)(s_a+s_b)}{s_as_b}\mathbf{C}\\
=&(4R)^2s_a^2c_bc_c(s_b+s_c)\mathbf{A}+(4R)^2s_b^2c_cc_a(s_c+s_a)\mathbf{B}+(4R)^2s_c^2c_ac_b(s_a+s_b)\mathbf{C}\\
=&(4R)^2c_ac_bc_c\left(\frac{s_a^2(s_b+s_c)}{c_a}\cdot\mathbf{A}+\frac{s_b^2(s_c+s_a)}{c_b}\cdot\mathbf{B}+\frac{s_c^2(s_a+s_b)}{c_c}\cdot\mathbf{C}\right).
\end{aligned}$$

Therefore, the homogeneous barycentric coordinates of Q with respect to ABC are

$$\begin{aligned}
&\left(\frac{s_a^2(s_b+s_c)}{c_a}:\frac{s_b^2(s_c+s_a)}{c_b}:\frac{s_c^2(s_a+s_b)}{c_c}\right)\\
=&\left(\tan\frac{A}{2}\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right):\tan\frac{B}{2}\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right):\tan\frac{C}{2}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\right).
\end{aligned}$$

This completes the proof of Theorem 2.

Inasmuch as Q is the Clawson point of the intouch triangle, it is interesting to point out that the congruent isoscelizers point X_{173}, a point closely related to the Yff center of congruence X_{174} and with coordinates

$$(a(-c_a+c_b+c_c):b(c_a-c_b+c_c):c(c_a+c_b-c_c)),$$

is the Clawson point of the excentral triangle $I_aI_bI_c$ (which is homothetic to the intouch triangle at X_{57}). This fact was stated in an earlier edition of [2], and can be easily proved by the method of this paper.

References

[1] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.
[2] C. Kimberling, *Encyclopedia of Triangle Centers*, October 6, 2003 edition, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[3] M. Stevanović, Hyacinthos, message 6837, March 30, 2003.

Milorad R. Stevanović: Technical Faculty, Svetog Save 65, 32000 Čačak, Serbia and Montenegro
E-mail address: milmath@ptt.yu

Forum Geometricorum
Volume 3 (2003) 205–206.

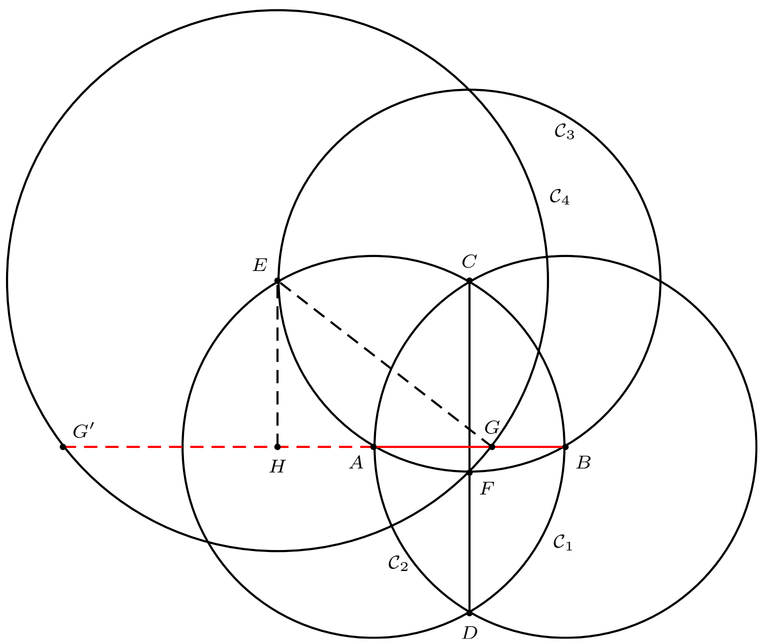
FORUM GEOM
ISSN 1534-1178

A 5-step Division of a Segment in the Golden Section

Kurt Hofstetter

Abstract. Using ruler and compass only in five steps, we divide a given segment in the golden section.

Inasmuch as we have given in [1] a construction of the golden section by drawing 5 circular arcs, we present here a very simple division of a given segment in the golden section, in 5 euclidean steps, using ruler and compass only. For two points P and Q, we denote by $P(Q)$ the circle with P as center and PQ as radius.



Construction. Given a segment AB, construct

(1) $\mathcal{C}_1 = A(B)$,
(2) $\mathcal{C}_2 = B(A)$, intersecting $\mathcal{C}_1$ at C and D,
(3) $\mathcal{C}_3 = C(A)$, intersecting $\mathcal{C}_1$ again at E,
(4) the segment CD to intersect $\mathcal{C}_3$ at F,
(5) $\mathcal{C}_4 = E(F)$ to intersect AB at G.

The point G divides the segment AB in the golden section.

Publication Date: November 26, 2003. Communicating Editor: Paul Yiu.

Proof. Suppose AB has unit length. Then $CD = \sqrt{3}$ and $EG = EF = \sqrt{2}$. Let H be the orthogonal projection of E on the line AB. Since $HA = \frac{1}{2}$, and $HG^2 = EG^2 - EH^2 = 2 - \frac{3}{4} = \frac{5}{4}$, we have $AG = HG - HA = \frac{1}{2}(\sqrt{5} - 1)$. This shows that G divides AB in the golden section. □

Remark. The other intersection G' of $\mathcal{C}_4$ and the line AB is such that $G'A : AB = \frac{1}{2}(\sqrt{5} + 1) : 1$.

References

[1] K. Hofstetter, A simple construction of the golden section, *Forum Geom.*, 2 (2002) 65–66.

Kurt Hofstetter: Object Hofstetter, Media Art Studio, Langegasse 42/8c, A-1080 Vienna, Austria
E-mail address: pendel@sunpendulum.at

Forum Geometricorum
Volume 3 (2003) 207–214.

FORUM GEOM
ISSN 1534-1178

Circumcenters of Residual Triangles

Eckart Schmidt

Abstract. This paper is an extension of Mario Dalcín's work on isotomic inscribed triangles and their residuals [1]. Considering the circumcircles of residual triangles with respect to isotomic inscribed triangles there are two congruent triangles of circumcenters. We show that there is a rotation mapping these triangles to each other. The center and angle of rotation depend on the Miquel points. Furthermore we give an interesting generalization of Dalcin's definitive example.

1. Introduction

If X, Y, Z are points on the sides of a triangle ABC, there are three residual triangles AZY, BXZ, CYX. The circumcenters of these triangles form a triangle $O_aO_bO_c$ similar to the reference triangle ABC [2]. The circumcircles have a common point M by Miquel's theorem. The lines MX, MY, MZ and the corresponding side lines have the same angle of intersection $\mu = (AY, YM) = (BZ, ZM) = (CX, XM)$. The angles are directed angles measured between 0 and π.

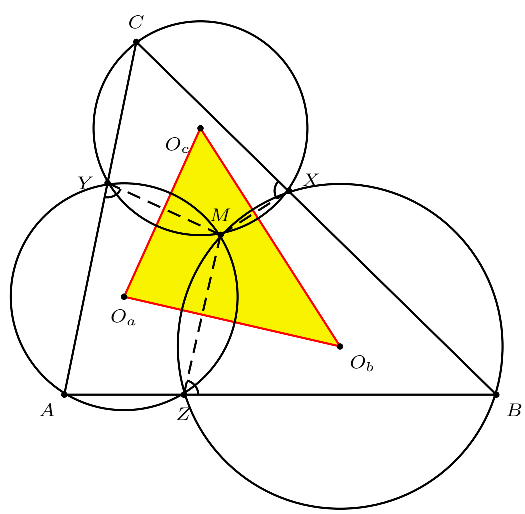


Figure 1

Dalcín considers isotomic inscribed triangles XYZ and $X'Y'Z'$. Here, X', Y', Z' are the reflections of X, Y, Z in the midpoints of the respective sides. The triangle XYZ may or may not be cevian. If it is the cevian triangle of a point P, then $X'Y'Z'$ is the cevian triangle of the isotomic conjugate of P. The

Publication Date: December 8, 2003. Communicating Editor: Paul Yiu.
The author thanks the editor for his helps in the preparation of this paper.

corresponding Miquel point M' of X', Y', Z' has Miquel angle $\mu' = \pi - \mu$. The circumcircles of the residual triangles $AZ'Y'$, $BX'Z'$, $CY'X'$ give further points of intersection. The intersections A' of the circles AZY and $AZ'Y'$, B' of BXZ and $BX'Z'$, and C' of CYX and $CY'X'$ form a triangle $A'B'C'$ perspective to the reference triangle ABC with the center of perspectivity Q. See Figure 2. It can be shown that the points M, M', A', B', C', Q and the circumcenter O of the reference triangle lie on a circle with the diameter OQ.

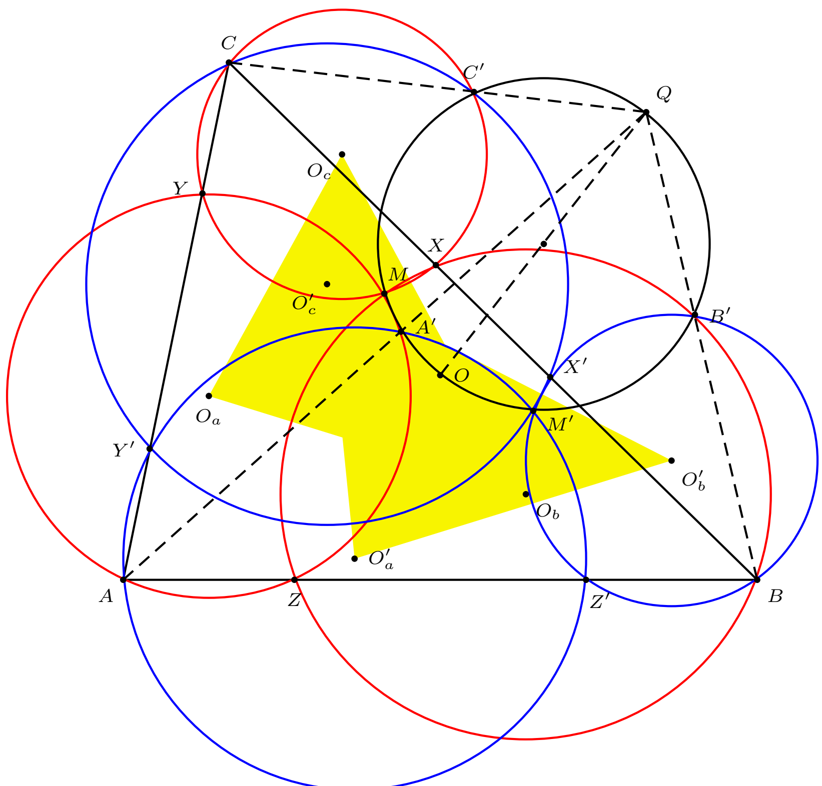


Figure 2

These results can be proved by analytical calculations. We make use of homogeneous barycentric coordinates. Let X, Y, Z divide the sides BC, CA, AB respectively in the ratios

$$BX : XC = x : 1, \qquad CY : YA = y : 1, \qquad AZ : ZB = z : 1.$$

These points have coordinates

$$\begin{aligned} X &= (0 : 1 : x), \quad & Y &= (y : 0 : 1), \quad & Z &= (1 : z : 0); \\ X' &= (0 : x : 1), \quad & Y' &= (1 : 0 : y), \quad & Z' &= (z : 1 : 0). \end{aligned}$$

The circumcenter, the Miquel points, and the center of perspectivity are the points

$$\begin{aligned} O =&(a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)),\\ M =&(a^2x(1+y)(1+z) - b^2xy(1+x)(1+z) - c^2(1+x)(1+y) : \cdots : \cdots),\\ M' =&(a^2x(1+y)(1+z) - b^2(1+x)(1+z) - c^2xz(1+x)(1+y) : \cdots : \cdots),\\ Q =&\left(\frac{(1-x)a^2}{1+x} : \frac{(1-y)b^2}{1+y} : \frac{(1-z)c^2}{1+z}\right). \end{aligned}$$

The Miquel angle μ is given by

$$\cot\mu = \frac{1-yz}{(1+y)(1+z)}\cot A + \frac{1-zx}{(1+z)(1+x)}\cot B + \frac{1-xy}{(1+x)(1+y)}\cot C.$$

For example, let X, Y, Z divide the sides in the same ratio k, *i.e.*, $x=y=z=k$, then we have

$$\begin{aligned} M =&(a^2(-c^2+a^2k-b^2k^2) : b^2(-a^2+b^2k-c^2k^2) : c^2(-b^2+c^2k-a^2k^2)),\\ M' =&(a^2(-b^2+a^2k-c^2k^2) : b^2(-c^2+b^2k-a^2k^2) : c^2(-a^2+c^2k-b^2k^2)),\\ Q =&(a^2 : b^2 : c^2) = X_6(\text{ Lemoine point});\\ \cot\mu =&\frac{1-k}{1+k}\cot\omega, \end{aligned}$$

where ω is the Brocard angle.

2. Two triangles of circumcenters

Considering the circumcenters of the residual triangles for XYZ and $X'Y'Z'$, Dalcín ([1, Theorem 10]) has shown that the triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are congruent. We show that there is a rotation mapping $O_aO_bO_c$ to $O'_aO'_bO'_c$. This rotation also maps the Miquel point M to the circumcenter O, and O to the other Miquel point M'. See Figure 3. The center of rotation is therefore the midpoint of OQ. This center of rotation is situated with respect to $O_aO_bO_c$ and $O'_aO'_bO'_c$ as the center of perspectivity with respect to the reference triangle ABC. The angle φ of rotation is given by

$$\varphi = \pi - 2\mu.$$

The similarity ratio of triangles $O_aO_bO_c$ and ABC is

$$\frac{1}{2\cos\frac{\varphi}{2}} = \frac{1}{2\sin\mu},$$

similarly for triangle $O'_aO'_bO'_c$.

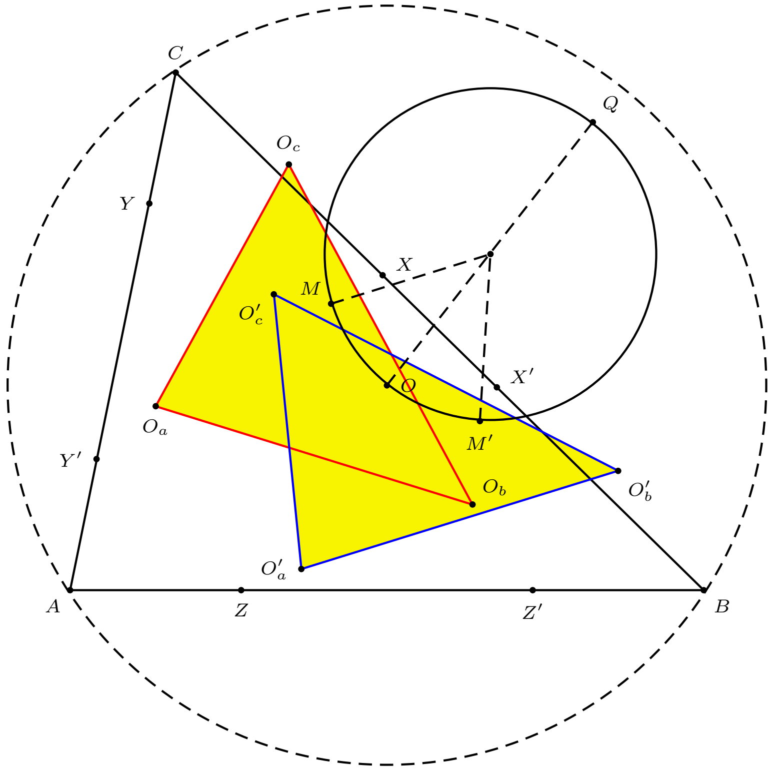


Figure 3

3. Dalcín's example

If we choose X, Y, Z as the points of tangency of the incircle with the sides, XYZ is the cevian triangle of the Gergonne point G_e and $X'Y'Z'$ is the cevian triangle of the Nagel point N_a. The Miquel point M is the incenter I and the Miquel point M' is the reflection of I in O, *i.e.*,

$$X_{40} = (a(a^3 - b^3 - c^3 + (a - b)(a - c)(b + c)) : \cdots : \cdots).$$

In this case, $O_aO_bO_c$ is homothetic to ABC at M, with factor $\frac{1}{2}$. This is also the case when XYZ is the cevian triangle of the Nagel point, with $M = X_{40}$.

Therefore, the circle described in §2, degenerates into a line. The center of perspectivity $Q(a(b - c) : b(c - a) : c(a - b))$ is a point of infinity. The triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are homothetic to the triangle ABC at the Miquel points M and M' with factor $\frac{1}{2}$. There is a parallel translation mapping $O_aO_bO_c$ to $O'_aO'_bO'_c$.

The fact that ABC is homothetic to $OaObOc$ with the factor $\frac{1}{2}$ does not only hold for the Gergonne and Nagel points. Here are further examples.

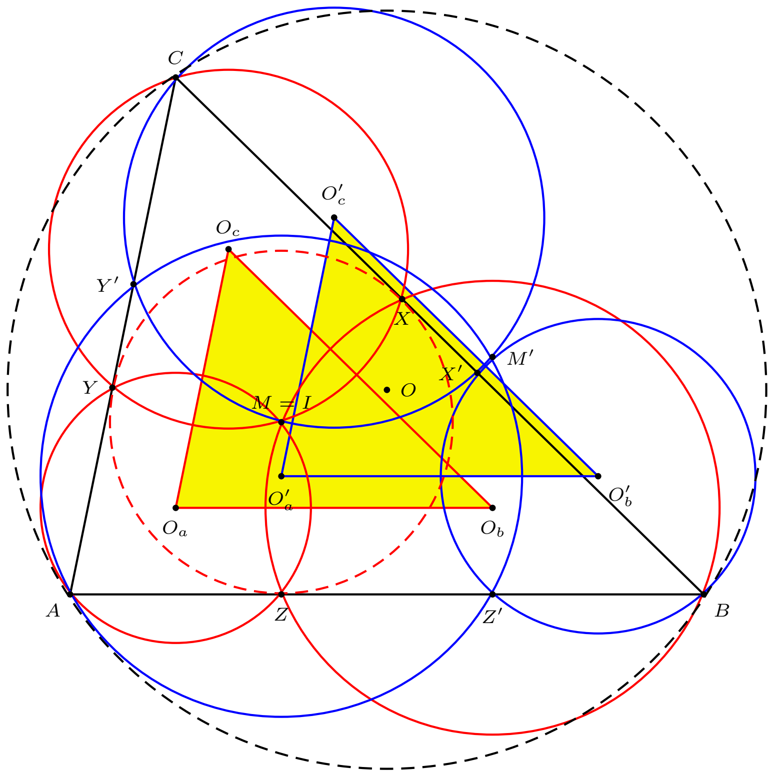


Figure 4

P	Homothetic center and Miquel point M
centroid G	circumcenter O
orthocenter H	H
X_{69}	X_{20}
X_{189}	X_{84}
X_{253}	X_{64}
X_{329}	X_{1490}

These points $P(u : v : w)$, whose cevian triangle is also the pedal triangle of the point M, lie on the Lucas cubic [1]

$$(b^2+c^2-a^2)u(v^2-w^2)+(c^2+a^2-b^2)v(w^2-u^2)+(a^2+b^2-c^2)w(u^2-v^2) = 0.$$

The points M lie on the Darboux cubic. [2] Isotomic points P and $P^\wedge$ on the Lucas cubic have corresponding points M and M' on the Darboux cubic symmetric with respect to the circumcenter. Isogonal points M and M^* on the Darboux cubic have

[1]The Lucas cubic is invariant under the isotomic conjugation and the isotomic conjugate X_{69} of the orthocenter is the pivot point.

[2]The Darboux cubic is invariant under the isogonal conjugation and the pivot point is the DeLongchamps point X_{20}, the reflection of the orthocenter in the circumcenter. It is symmetric with respect to the circumcenter.

corresponding points P and P' on the Lucas cubic with $P' = P^{\wedge*\wedge}$. Here, $()^*$ is the isogonal conjugation with respect to the anticomplementary triangle of ABC. The line PM and MM^* all correspond with the DeLongchamps point X_{20} and so the points P, $P^{\wedge*\wedge}$, M, M^* and X_{20} are collinear. For example, for $P = N_a$, the five points N_a, X_{189}, X_{40}, X_{84}, X_{20} are collinear.

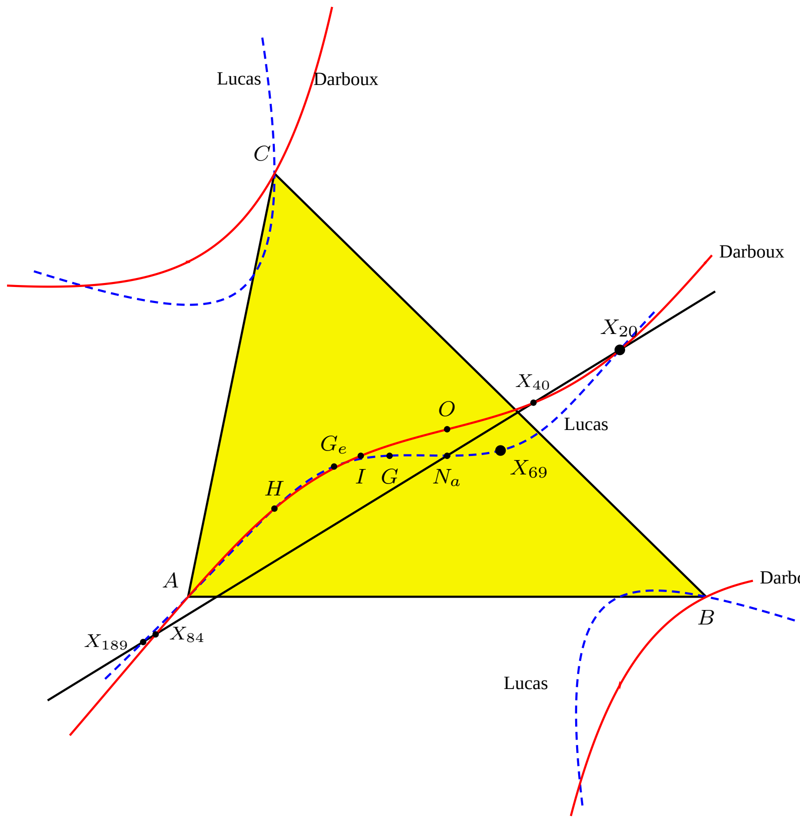


Figure 5. The Darboux and Lucas cubics

4. Further results

Dalcín's example can be extended. The cevian triangle of the Gergonne point G_e is the triangle of tangency of the incircle, the cevian triangle of the Nagel point N_a is the triangle of the inner points of tangency of the excircles. Consider the points of tangency of the excircles with the sidelines:

$A-$excircle	$B_a = (-a+b-c : 0 : a+b+c)$	with CA
	$C_a = (-a-b+c : a+b+c : 0)$	with AB
$B-$excircle	$A_b = (0 : a-b-c : a+b+c)$	with BC
	$C_b = (a+b+c : -a-b+c : 0)$	with AB
$C-$excircle	$A_c = (0 : a+b+c : a-b-c)$	with BC
	$B_c = (a+b+c : 0 : -a+b-c)$	with CA

The point pairs (A_b, A_c), (B_c, B_a) and (C_a, C_b) are symmetric with respect to the corresponding midpoints of the sides. If $XYZ = A_bB_cC_a$, then $X'Y'Z' = A_cB_aC_b$. See Figure 6.

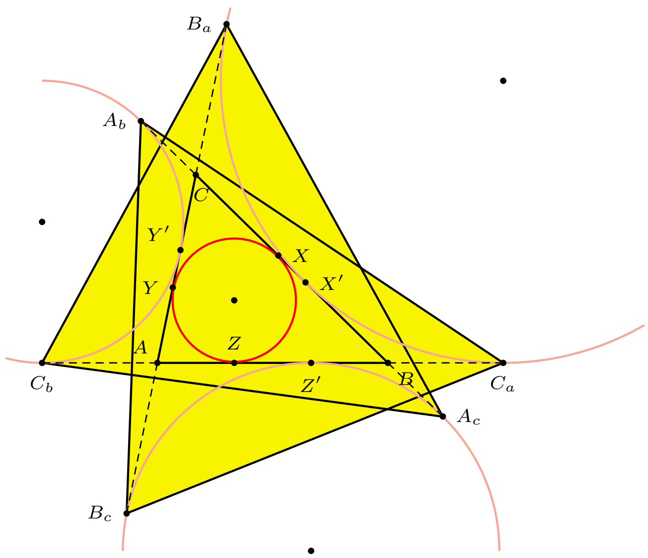


Figure 6

Consider the residual triangles of $A_bB_cC_a$ and those of $A_cB_aC_b$, with the circumcenters. The two congruent triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ have a common area

$$\frac{\triangle}{4} + \frac{(ab+bc+ca)^2}{16\triangle}.$$

The center of perspectivity is

$$Q = (a(b+c) : b(c+a) : c(a+b)) = X_{37}.$$

The center of rotation which maps $O_aO_bO_c$ to $O'_aO'_bO'_c$ is the midpoint of OQ. The point X_{37} of a triangle is the complement of the isotomic conjugate of the incenter. The center of rotation is the common point X_{37} of $O_aO_bO_c$ and $O'_aO'_bO'_c$. The angle of rotation is given by

$$\tan\frac{\varphi}{2} = \frac{ab+bc+ca}{2\triangle} = \frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}.$$

References

[1] M. Dalcín, Isotomic inscribed triangles and their residuals, *Forum Geom.*, 3 (2003) 125–134.
[2] E. Donath: *Die merkwürdigen Punkte und Linien des ebenen Dreiecks*, VEB Deutscher Verlag der Wissenschaften, Berlin 1976.
[3] G. M. Pinkernell, Cubic curves in the triangle plane, *Journal of Geometry*, 55 (1996), 144–161.
[4] C. Kimberling, *Encyclopedia of Triangle Centers*, August 22, 2002 edition, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Eckart Schmidt: Hasenberg 27 - D 24223 Raisdorf, Germany
E-mail address: eckart_schmidt@t-online.de

Forum Geometricorum
Volume 3 (2003) 215–223.

FORUM GEOM
ISSN 1534-1178

Circumrhombi

Floor van Lamoen

Abstract. We consider rhombi circumscribing a given triangle ABC in the sense that one vertex of the rhombus coincides with a vertex of ABC, while the sidelines of the rhombus opposite to this vertex pass through the two remaining vertices of ABC respectively. We construct some new triangle centers associated with these rhombi.

1. Introduction

In this paper we further study the rhombi circumscribing a given reference triangle ABC that the author defined in [4]. These rhombi circumscribe ABC in the sense that each of them shares one vertex with ABC, with its two opposite sides passing through the two remaining vertices of ABC. These rhombi will depend on a fixed angle ϕ and its complement $\overline{\phi} = \frac{\pi}{2} - \phi$. More precisely, for a given ϕ, the rhombus $\mathcal{R}_A(\phi) = AA_cA_aA_b$ will be such that $\angle A_bAA_c = 2\phi$, $B \in A_cA_a$ and $C \in A_bA_a$. Similarly there are rhombi $BB_aB_bB_c$ and $CC_bC_cC_a$.

In [4] it was shown that the vertices of the rhombi opposite to ABC form a triangle $A_aB_bC_c$ perspective to ABC, and that their perspector lies on the Kiepert hyperbola. We give another proof of this result (Theorem 3).

We denote by $\mathcal{K}(\phi) = A^\phi B^\phi C^\phi$ the Kiepert triangle formed by isosceles triangles built on the sides of ABC with base angles ϕ. When the isosceles triangles are constructed outwardly, $\phi > 0$. Otherwise, $\phi < 0$. These vertices have homogeneous barycentric coordinates [1]

$$\begin{aligned}
A^\phi =&(-(S_B+S_C) : S_C+S_\phi : S_B+S_\phi),\\
B^\phi =&(S_C+S_\phi : -(S_C+S_A) : S_A+S_\phi),\\
C^\phi =&(S_B+S_\phi : S_A+S_\phi : -(S_A+S_B)).
\end{aligned}$$

From these it is clear that $\mathcal{K}(\phi)$ is perspective with ABC at the point

$$K(\phi) = \left(\frac{1}{S_A+S_\phi} : \frac{1}{S_B+S_\phi} : \frac{1}{S_C+S_\phi}\right).$$

Publication Date: December 15, 2003. Communicating Editor: Paul Yiu.

[1]For the notations, see [5].

2. Circumrhombi to a triangle

Theorem 1. *Consider* $\triangle ABC$ *and* $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\}$. *There are unique rhombi* $\mathcal{R}_A(\phi) = AA_cA_aA_b$, $\mathcal{R}_B(\phi) = BB_aB_bB_c$ *and* $\mathcal{R}_C(\phi) = CC_bC_cC_a$ *with*

$$\angle A_bAA_c = \angle B_cBB_a = \angle C_aCC_b = 2\phi,$$

and $B \in A_cA_a$ *and* $C \in A_bA_a$. *Similarly there are rhombi* $C \in B_aB_b$, $A \in B_cB_b$, $A \in C_bC_c$, $B \in C_aC_c$.

Proof. It is enough to show the construction of $\mathcal{R}_A = \mathcal{R}_A(\phi)$.

Let B_r be the image of B after a rotation through $-2\overline{\phi}$ about A, and C_r the image of C after a rotation through $2\overline{\phi}$ about A. Then let $A_a = B_rC \cap C_rB$. Points $A_c \in C_rA_a$ and $A_b \in B_rA_a$ can be constructed in such a way that $AA_cA_aA_b$ is a parallelogram. Observe that $\triangle AC_rB \equiv \triangle ACB_r$, so that the perpendicular distances from A to lines B_rA_a and C_rA_a are equal. And $AA_cA_aA_b$ must be a rhombus. See Figure 1.

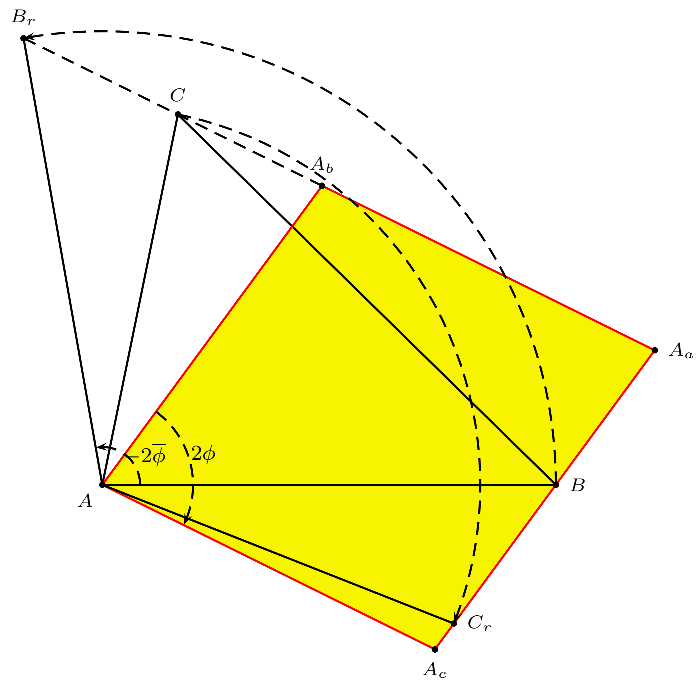


Figure 1

Note that line $B_rC = A_aA_b$ is the image of line $C_rB = A_aA_c$ after rotation through $2\overline{\phi}$ about A, so that the directed angle $\angle A_cA_aA_b = 2\phi$. It follows that $AA_cA_aA_b$ is the rhombus desired in the theorem.

It is easy to see that this is the unique rhombus fulfilling these requirements. When we rotate the complete figure of $\triangle ABC$ and rhombus $AA_cA_aA_b$ through $-2\overline{\phi}$ about A, and let B_r be the image of B again, we see immediately that $B_r \in A_aC$. In the same way we see that the image of C after rotation through $2\overline{\phi}$ about A must be on the line A_aB. $\square$

Consider $\mathcal{R}_A$ and $\mathcal{R}_B$. We note that $\angle AA_aB \equiv \phi \mod \pi$ and also $\angle AB_bB \equiv \phi \mod \pi$. This means that ABA_aB_b is cyclic. The center P of its circle should be the apex of the isosceles triangle built on AB such that $\angle APB = 2\phi$,[2] so that $P = C^{\overline{\phi}}$. This shows that $C^{\overline{\phi}}$ lies on the perpendicular bisectors of AA_a and BB_b, hence $A_bA_c \cap B_aB_c = C^{\overline{\phi}}$. See Figure 2.

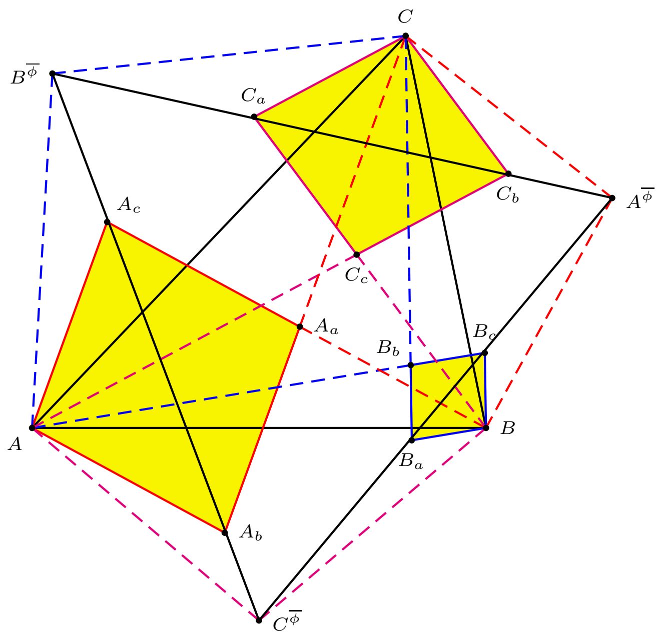


Figure 2

Theorem 2. *The diagonals A_bA_c, B_aB_c and C_aC_b of the circumrhombi $\mathcal{R}_A(\phi)$, $\mathcal{R}_B(\phi)$, $\mathcal{R}_C(\phi)$ bound the Kiepert triangle $\mathcal{K}(\overline{\phi})$.*

3. Radical center of a triad of circles

It is now interesting to further study the circles $A^{\overline{\phi}}(B)$, $B^{\overline{\phi}}(C)$ and $C^{\overline{\phi}}(A)$ with centers at the apices of $\mathcal{K}(\overline{\phi})$, passing through the vertices of ABC. Since the circle $A^{\overline{\phi}}(B)$ passes through B and C, it is represented by an equation of the form

$$a^2yz + b^2zx + c^2xy - kx(x + y + z) = 0.$$

Since it also passes through $A^{-\phi/2} = (-(S_B + S_C) : S_C - S_{\phi/2} : S_B - S_{\phi/2})$, we find

$$k = \frac{S_\phi^2 + 2S_AS_{\phi/2} - S^2}{2S_{\phi/2}} = S_A + S_\phi.$$

[2]Hence, when ϕ is negative, the apex is on the same side of AB as the vertex C.

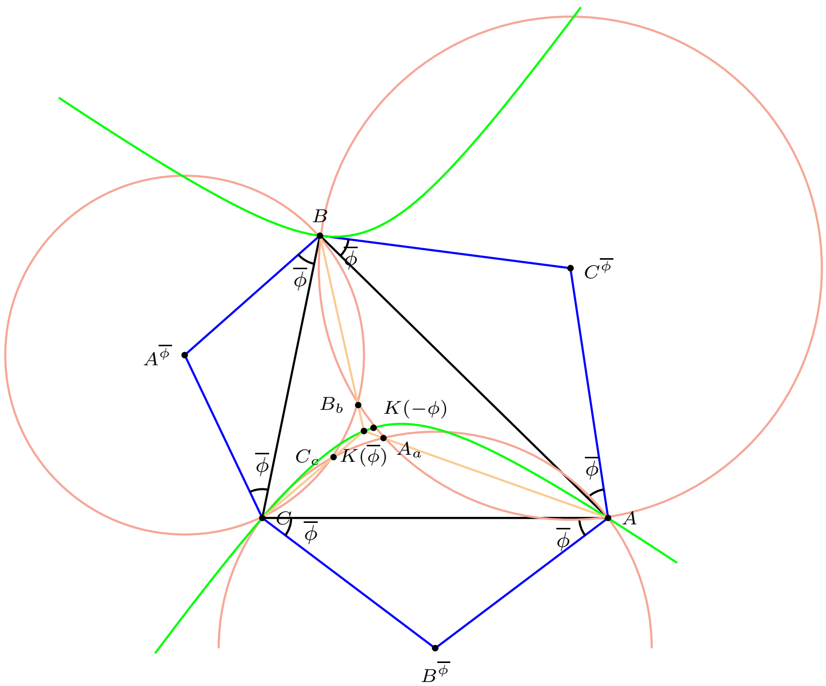


Figure 3

The equations of the three circles are thus

$$\begin{aligned}
a^2yz + b^2zx + c^2xy - (S_A + S_\phi)x(x + y + z) &= 0,\\
a^2yz + b^2zx + c^2xy - (S_B + S_\phi)y(x + y + z) &= 0,\\
a^2yz + b^2zx + c^2xy - (S_C + S_\phi)z(x + y + z) &= 0.
\end{aligned}$$

From this, it is clear that the radical center of the three circles is the point

$$K(\phi) = \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C + S_\phi}\right).$$

The intersections of the circles apart from A, B and C are the points

$$\begin{aligned}
A_a &= \left(\frac{1}{S_A - S_{2\phi}} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C + S_\phi}\right),\\
B_b &= \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B - S_{2\phi}} : \frac{1}{S_C + S_\phi}\right),\\
C_c &= \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C - S_{2\phi}}\right).
\end{aligned} \tag{1}$$

Theorem 3. *The triangle $A_aB_bC_c$ is perspective to ABC and the perspector is $K(\phi)$.*

Remark. For $\phi = \pm\frac{\pi}{3}$, triangle $A_aB_bC_c$ degenerates into the Fermat point $K\left(\pm\frac{\pi}{3}\right)$.

The coordinates of the circumcenter of $A_aB_bC_c$ are too complicated to record here, even in the case of circumsquares. However, we prove the following interesting collinearity.

Theorem 4. *The circumcenters of triangles ABC and $A_aB_bC_c$ are collinear with $K(\phi)$.*

Proof. Since $P = K(\phi)$ is the radical center of $A^\phi(B)$, $B^\phi(C)$ and $C^\phi(A)$ we see that

$$\overline{PA} \cdot \overline{PA_a} = \overline{PB} \cdot \overline{PB_b} = \overline{PC} \cdot \overline{PC_c},$$

which product we will denote by Γ. When $\Gamma > 0$ then the inversion with center P and radius $\sqrt{\Gamma}$ maps A to A_a, B to B_b and C to C_c. Consequently the circumcircles of ABC and $A_aB_bC_c$ are inverses of each other, and the centers of these circles are collinear with the center of inversion.

When $\Gamma < 0$ then the inversion with center P and radius $\sqrt{-\Gamma}$ maps A, B and C to the reflections of A_a, B_b and C_c through P. And the collinearity follows in the same way as above.

When $\Gamma = 0$ the theorem is trivial. $\square$

4. Coordinates of the vertices of the circumrhombi

Along with the coordinates given in (1), we record those of the remaining vertices of the circumrhombi.

$$\begin{aligned}
A_b =&((b^2 + S\csc 2\phi)(S_B + S_\phi) : (S_A - S_{2\phi})(b^2 + S\csc 2\phi) : -(S_A - S_{2\phi})^2),\\
A_c =&\left((c^2 + S\csc 2\phi)(S_C + S_\phi) : -(S_A - S_{2\phi})^2 : (S_A - S_{2\phi})(c^2 + S\csc 2\phi)\right);\\
B_c =&(-(S_B - S_{2\phi})^2 : (c^2 + S\csc 2\phi)(S_C + S_\phi) : (S_B - S_{2\phi})(c^2 + S\csc 2\phi)),\\
B_a =&\left((S_B - S_{2\phi})(a^2 + S\csc 2\phi) : (a^2 + S\csc 2\phi)(S_A + S_\phi) : -(S_B - S_{2\phi})^2\right);\\
C_a =&((S_C - S_{2\phi})(a^2 + S\csc 2\phi) : -(S_C - S_{2\phi})^2 : (a^2 + S\csc 2\phi)(S_A + S_\phi)),\\
C_b =&\left(-(S_C - S_{2\phi})^2 : (S_C - S_{2\phi})(b^2 + S\csc 2\phi) : (b^2 + S\csc 2\phi)(S_B + S_\phi)\right).
\end{aligned} \tag{2}$$

5. The triangle $A'B'C'$

Let $A' = CC_a \cap BB_a$, $B' = CC_b \cap AA_b$ and $C' = AA_c \cap BB_c$. The coordinates of A', using (2), are

$$\begin{aligned}
A' &= \left(a^2 + S\csc 2\phi : -(S_C - S_{2\phi}) : -(S_B - S_{2\phi})\right)\\
&= \left(\frac{a^2 + S\csc 2\phi}{(S_B - S_{2\phi})(S_C - S_{2\phi})} : \frac{-1}{S_B - S_{2\phi}} : \frac{-1}{S_C - S_{2\phi}}\right);
\end{aligned}$$

Similarly for B' and C'. It is clear that $A'B'C'$ is perspective to ABC at $K(-2\phi)$. Note that in absolute barycentric coordinates,

$$\begin{aligned}&S(\csc 2\phi + 2\cot 2\phi)A'\\ =&\left(a^2 + S\csc 2\phi, -(S_C - S_{2\phi}), -(S_B - S_{2\phi})\right)\\ =&(S_B + S_C, -(S_C + S_{\overline{\phi}}), -(S_B + S_{\overline{\phi}})) + S(\csc 2\phi, \cot 2\phi + \tan\phi, \cot 2\phi + \tan\phi)\\ =&(S_B + S_C, -(S_C + S_{\overline{\phi}}), -(S_B + S_{\overline{\phi}})) + S\csc 2\phi(1,1,1)\\ =&S(-2\tan\phi A^{\overline{\phi}} + 3\csc 2\phi G).\end{aligned}$$

Now, $\frac{-2\tan\phi}{-2\tan\phi+3\csc 2\phi} = \frac{4}{1-3\cot^2\phi}$. It follows that

$$A' = \mathsf{h}\left(G, \frac{4}{1-3\cot^2\phi}\right)(A^{\overline{\phi}}).$$

Similarly for B' and C'.

Proposition 5. *Triangles $A'B'C'$ and $\mathcal{K}(\overline{\phi})$ are homothetic at G.*

Corollary 6. *ABC is the Kiepert triangle $\mathcal{K}(-\phi)$ with respect to $A'B'C'$.*

See [5, Proposition 4].

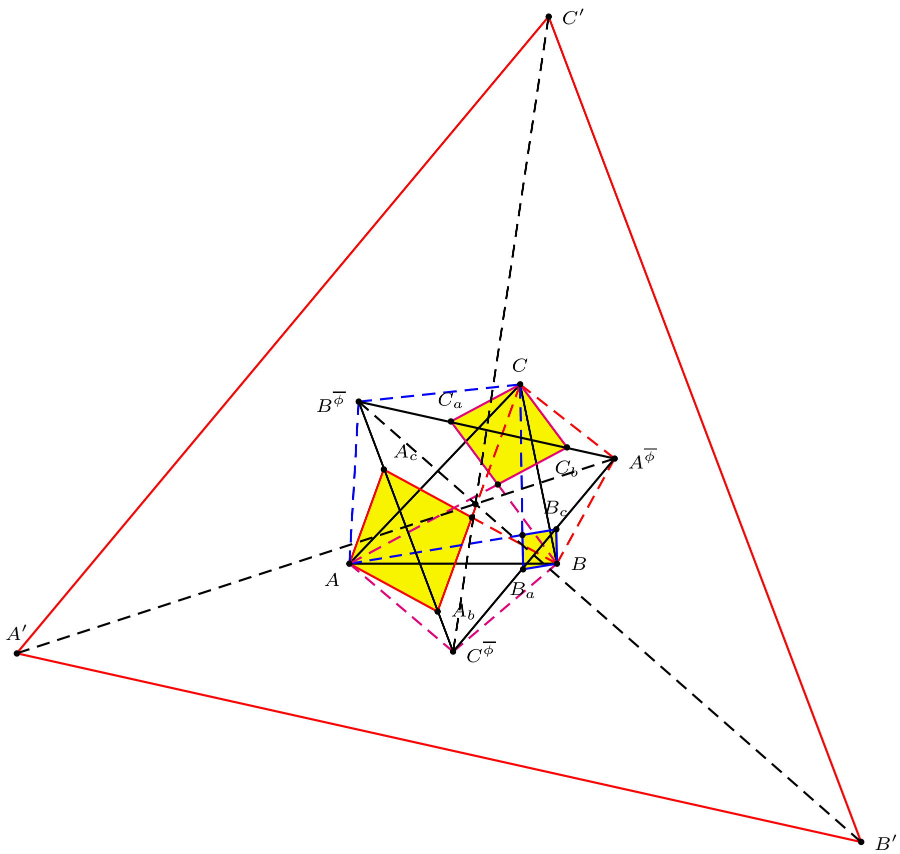


Figure 4

6. The desmic mates

Let XYZ be a triangle perspective with ABC at $P = (u : v : w)$. Its vertices have coordinates

$$X = (x : v : w), \qquad Y = (u : y : w), \qquad Z = (u : v : z),$$

for some x, y, z. The desmic mate of XYZ is the triangle with vertices $X' = BZ \cap CY$, $Y' = CX \cap AZ$, $Z' = AY \cap BX$. These have coordinates

$$X' = (u : y : z), \qquad Y' = (x : v : z), \qquad Z' = (x : y : w).$$

Lemma 7. *The triangle $X'Y'Z'$ is perspective to ABC at $(x : y : z)$ and to XYZ at $(u + x : v + y : w + z)$.*

See, for example, [1, §4].

The desmic mate of $A_aB_bC_a$ has vertices

$$\begin{aligned} A'_a &= \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B - S_{2\phi}} : \frac{1}{S_C - S_{2\phi}}\right), \\ B'_b &= \left(\frac{1}{S_A - S_{2\phi}} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C - S_{2\phi}}\right), \\ C'_c &= \left(\frac{1}{S_A - S_{2\phi}} : \frac{1}{S_B - S_{2\phi}} : \frac{1}{S_C + S_\phi}\right). \end{aligned} \tag{3}$$

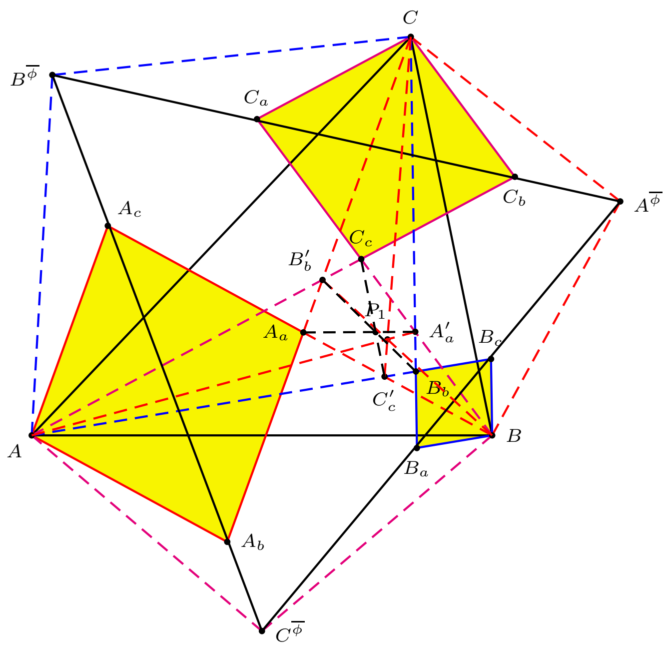


Figure 5

Proposition 8. *Triangle $A'_aB'_bC'_c$ is perspective to ABC at $K(-2\phi)$. It is also perspective to $A_aB_bC_c$ at*

$$P_1(\phi) = \left(\frac{2S_A + S\csc 2\phi}{(S_A+S_\phi)(S_A+S_{2\phi})} : \frac{2S_B + S\csc 2\phi}{(S_B+S_\phi)(S_B+S_{2\phi})} : \frac{2S_C + S\csc 2\phi}{(S_C+S_\phi)(S_C+S_{2\phi})}\right).$$

See Figure 5.

The desmic mate of $A'B'C'$ has vertices

$$\begin{aligned} A'' =&(-(S_B-S_{2\phi})(S_C-S_{2\phi}) : (S_B-S_{2\phi})(b^2+S\csc 2\phi)\\ &: (S_C-S_{2\phi})(c^2+S\csc 2\phi));\\ B'' =&((S_A-S_{2\phi})(a^2+S\csc 2\phi) : -(S_C-S_{2\phi})(S_A-S_{2\phi})\\ &: (S_C-S_{2\phi})(c^2+S\csc 2\phi)),\\ C'' =&((S_A-S_{2\phi})(a^2+S\csc 2\phi) : (S_B-S_{2\phi})(b^2+S\csc 2\phi)\\ &: -(S_A-S_{2\phi})(S_B-S_{2\phi})). \end{aligned} \tag{4}$$

Proposition 9. *Triangle $A''B''C''$ is perspective to*

(1) ABC *at*

$$P_2(\phi) = \left((S_A-S_{2\phi})(a^2+S\csc 2\phi) : \cdots : \cdots\right),$$

(2) $A'B'C'$ *at*

$$P_3(\phi) = ((a^2S_A - S_{BC}) - S\csc 2\phi(S_A - S_\phi\cos 2\phi) : \cdots : \cdots),$$

(3) *the dilated triangle*[3] *at*

$$P_4(\phi) = (S_B + S_C - S_A - S_{2\phi} : \cdots : \cdots).$$

Proof. (1) is clear from the coordinates given in (4). Since

$$\begin{aligned} &(a^2+S\csc 2\phi)(S_A - S\cot 2\phi) - (S_B - S\cot 2\phi)(S_C - S\cot 2\phi)\\ =&(a^2S_A - S_{BC}) + S^2\csc 2\phi\cot A - S\cot 2\phi(a^2 + S\csc 2\phi - (S_B+S_C) + S\cot 2\phi)\\ =&(a^2S_A - S_{BC}) + S^2\csc 2\phi\cot A - S^2\cot 2\phi\cot\phi\\ =&(a^2S_A - S_{BC}) - S_AS\csc 2\phi + S_{2\phi}S_\phi\\ =&(a^2S_A - S_{BC}) - S\csc 2\phi(S_A - S_\phi\cos 2\phi), \end{aligned}$$

it follows from Lemma 7 that $A''B''C''$ is perspective to $A'B'C'$ at

$$\begin{aligned} &\left(\frac{a^2+S\csc 2\phi}{(S_B-S_{2\phi})(S_C-S_{2\phi})} - \frac{1}{S_A-S_{2\phi}} : \cdots : \cdots\right)\\ =&((a^2S_A - S_{BC}) - S\csc 2\phi(S_A - S_\phi\cos 2\phi) : \cdots : \cdots). \end{aligned}$$

[3] This is also called the anticomplementary triangle, it is formed by the lines through the vertices of ABC, parallel to the corresponding opposite sides.

This proves (2). For (3), we rewrite the coordinates for A'' as

$$\begin{aligned}A'' =&(S_B - S_{2\phi})(S_C - S_{2\phi}) \cdot (-1, 1, 1)\\ &+ (S \csc(2\phi) + S_{2\phi} + S_A) \cdot (0, S_B - S_{2\phi}, S_C - S_{2\phi})\\ =&(S_B - S_{2\phi})(S_C - S_{2\phi}) \cdot (-1, 1, 1)\\ &+ (S_A + S_\phi) \cdot (0, S_B - S_{2\phi}, S_C - S_{2\phi})\end{aligned}$$

From this we see that A'' is on the line connecting the A-vertices of the dilated triangle and the cevian triangle of the isotomic conjugate of $K(-2\phi)$, namely, the point

$$K^\bullet(-2\phi) = (S_A - S_{2\phi} : S_B - S_{2\phi} : S_C - S_{2\phi}).$$

This shows that $A''B''C''$ is perspective to both triangles, and that the perspector is the *cevian quotient* $K^\bullet(-2\phi)/G$, [4] where G denotes the centroid. It is easy to see that this is the superior of $K^\bullet(-2\phi)$. Equivalently, it is $K^\bullet(-2\phi)$ of the dilated triangle, with coordinates

$$(S_B + S_C - S_A - S_{2\phi} : \cdots : \cdots).$$

$\square$

We conclude with a table showing the triangle centers associated with the circumsquares, when $\phi = \pm\frac{\pi}{4}$.

k	$P_k(\frac{\pi}{4})$	$P_k(-\frac{\pi}{4})$
1	$K(\frac{\pi}{4})$	$K(-\frac{\pi}{4})$
2	circumcenter	circumcenter
3	de Longchamps point	de Longchamps point
4	X_{193}	X_{193}

References

[1] K. Dean and F. M. van Lamoen, Geometric construction of reciprocal conjugations, *Forum Geom.*, 1 (2001) 115–120.

[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–295.

[3] C. Kimberling, *Encyclopedia of Triangle Centers*, November 4, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[4] F. M. van Lamoen, Triangle centers associated with rhombi, *Elem. Math.*, 55 (2000) 102–109.

[5] F. M. van Lamoen and P. Yiu, The Kiepert Pencil of Kiepert Hyperbolas, *Forum Geom.*, 1 (2001) 125–132.

Floor van Lamoen: St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands

E-mail address: `f.v.lamoen@wxs.nl`

[4]The cevian quotient X/Y is the perspector of the cevian triangle of X and the precevian triangle of Y. This is the X-Ceva conjugate of Y in the terminology of [2].

Forum Geometricorum
Volume 3 (2003) 225–229.

FORUM GEOM
ISSN 1534-1178

Sawayama and Thébault's theorem

Jean-Louis Ayme

Abstract. We present a purely synthetic proof of Thébault's theorem, known earlier to Y. Sawayama.

1. Introduction

In 1938 in a "Problems and Solutions" section of the Monthly [24], the famous French problemist Victor Thébault (1882-1960) proposed a problem about three circles with collinear centers (see Figure 1) to which he added a correct ratio and a relation which finally turned out to be wrong. The date of the first three metric

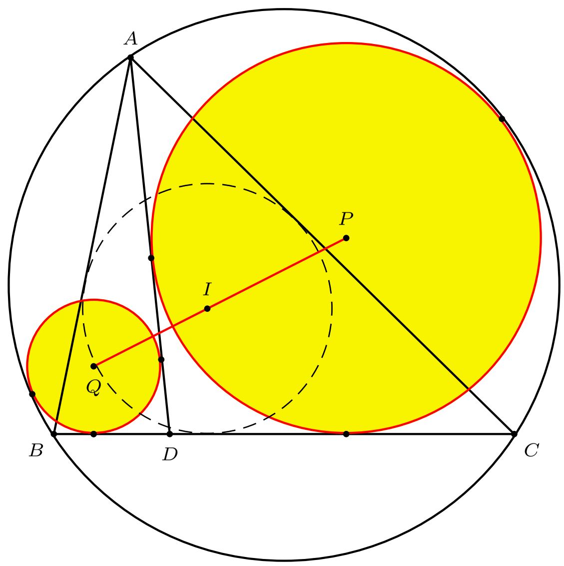


Figure 1

solutions [22] which appeared discreetly in 1973 in the Netherlands was more widely known in 1989 when the Canadian revue *Crux Mathematicorum* [27] published the simplified solution by Veldkamp who was one of the two first authors to prove the theorem in the Netherlands [26, 5, 6]. It was necessary to wait until the end of this same year when the Swiss R. Stark, a teacher of the Kantonsschule of Schaffhausen, published in the Helvetic revue *Elemente der Mathematik* [21] the first synthetic solution of a "more general problem" in which the one of Thébault's appeared as a particular case. This generalization, which gives a special importance to a rectangle known by J. Neuberg [15], citing [4], has been pointed out in 1983 by the editorial comment of the *Monthly* in an outline publication about the supposed

Publication Date: December 22, 2003. Communicating Editor: Floor van Lamoen.

first metric solution of the English K. B. Taylor [23] which amounted to 24 pages. In 1986, a much shorter proof [25], due to Gerhard Turnwald, appeared. In 2001, R. Shail considered in his analytic approach, a “more complete” problem [19] in which the one of Stark appeared as a particular case. This last generalization was studied again by S. Gueron [11] in a metric and less complete way. In 2003, the *Monthly* published the angular solution by B. J. English, received in 1975 and “lost in the mists of time” [7].

Thanks to *JSTOR*, the present author has discovered in an anciant edition of the *Monthly* [18] that the problem of Shail was proposed in 1905 by an instructor Y. Sawayama of the central military School of Tokyo, and geometrically resolved by himself, mixing the synthetic and metric approach. On this basis, we elaborate a new, purely synthetic proof of Sawayama-Thébault theorem which includes several theorems that can all be synthetically proved. The initial step of our approach refers to the beginning of the Sawayama’s proof and the end refers to Stark’s proof. Furthermore, our point of view leads easily to the Sawayama-Shail result.

2. A lemma

Lemma 1. *Through the vertex A of a triangle ABC, a straight line AD is drawn, cutting the side BC at D. Let P be the center of the circle $\mathcal{C}_1$ which touches DC, DA at E, F and the circumcircle $\mathcal{C}_2$ of ABC at K. Then the chord of contact EF passes through the incenter I of triangle ABC.*

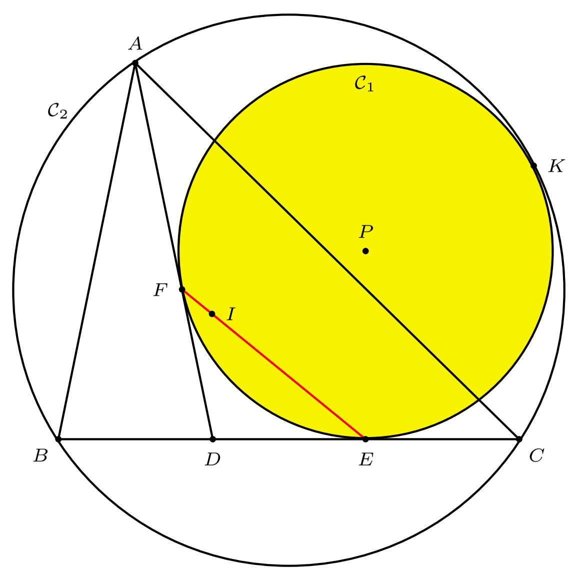


Figure 2

Proof. Let M, N be the points of intersection of KE, KF with $\mathcal{C}_2$, and J the point of intersection of AM and EF (see Figure 3). KE is the internal bisector of $\angle BKC$ [8, Théorème 119]. The point M being the midpoint of the arc BC which does not contain K, AM is the A-internal bisector of ABC and passes through I.

The circles $\mathcal{C}_1$ and $\mathcal{C}_2$ being tangent at K, EF and MN are parallel.

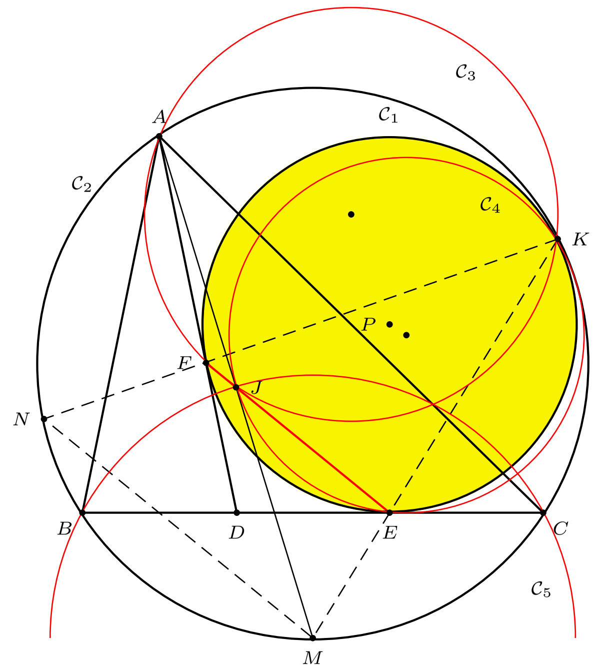


Figure 3

The circle $\mathcal{C}_2$, the basic points A and K, the lines MAJ and NKF, the parallels MN and JF, lead to a converse of Reim's theorem ([8, Théorème 124]). Therefore, the points A, K, F and J are concyclic. This can also be seen directly from the fact that angles FJA and FKA are congruent.

Miquel's pivot theorem [14, 9] applied to the triangle AFJ by considering F on AF, E on FJ, and J on AJ, shows that the circle $\mathcal{C}_4$ passing through E, J and K is tangent to AJ at J. The circle $\mathcal{C}_5$ with center M, passing through B, also passes through I ([2, Livre II, p.46, théorème XXI] and [12, p.185]). This circle being orthogonal to circle $\mathcal{C}_1$ [13, 20] is also orthogonal to circle $\mathcal{C}_4$ ([10, 1]) as KEM is the radical axis of circles $\mathcal{C}_1$ and $\mathcal{C}_4$. [1] Therefore, $MB = MJ$, and $J = I$. Conclusion: the chord of contact EF passes through the incenter I. □

Remark. When D is at B, this is the theorem of Nixon [16].

3. Sawayama-Thébault theorem

Theorem 2. *Through the vertex A of a triangle ABC, a straight line AD is drawn, cutting the side BC at D. I is the center of the incircle of triangle ABC. Let P be the center of the circle which touches DC, DA at E, F, and the circumcircle of ABC, and let Q be the center of a further circle which touches DB, DA in G, H and the circumcircle of ABC. Then P, I and Q are collinear.*

[1] From $\angle BKE = \angle MAC = \angle MBE$, we see that he circumcircle of BKE is tangent to BM at B. So circle $\mathcal{C}_5$ is orthogonal to this circumcircle and consequently also to $\mathcal{C}_1$ as M lies on their radical axis.

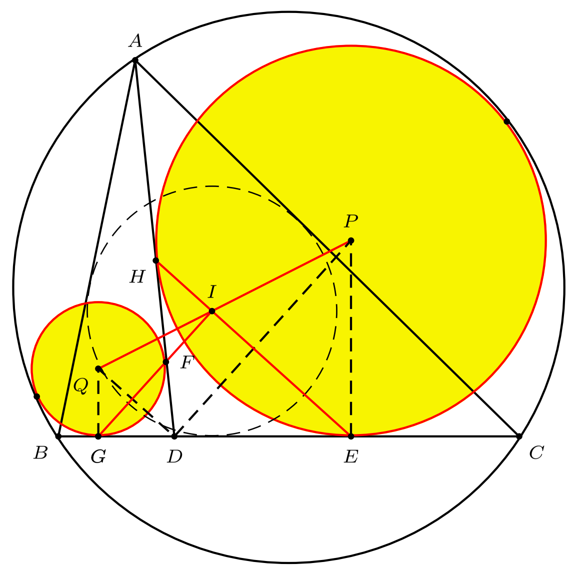


Figure 4

Proof. According to the hypothesis, $QG \perp BC$, $BC \perp PE$; so $QG // PE$. By Lemma 1, GH and EF pass through I. Triangles DHG and QGH being isosceles in D and Q respectively, DQ is

(1) the perpendicular bisector of GH,

(2) the D-internal angle bisector of triangle DHG.

Mutatis mutandis, DP is

(1) the perpendicular bisector of EF,

(2) the D-internal angle bisector of triangle DEF.

As the bisectors of two adjacent and supplementary angles are perpendicular, we have $DQ \perp DP$. Therefore, $GH//DP$ and $DQ//EF$. Conclusion: using the converse of Pappus's theorem ([17, Proposition 139] and [3, p.67]), applied to the hexagon $PEIGQDP$, the points P, I and Q are collinear. □

References

[1] N. Altshiller-Court, *College Geometry*, Barnes & Noble, 205.

[2] E. Catalan, Théorèmes et problèmes de Géométrie élémentaires, 1879.

[3] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Math. Assoc. America, 1967.

[4] *Archiv der Mathematik und Physik* (1842) 328.

[5] B. C. Dijkstra-Kluyver, Twee oude vraagstukken in één klap opgelost, *Nieuw Tijdschrift voor Wiskunde*, 61 (1973-74) 134–135.

[6] B. C. Dijkstra-Kluyver and H. Streefkerk, Nogmaals het vraagstuk van Thébault, *Nieuw Tijdschrift voor Wiskunde*, 61 (1973-74) 172–173.

[7] B. J. English, Solution of Problem 3887, *Amer. Math. Monthly*, 110 (2003) 156–158.

[8] F. G.-M., *Exercices de Géométrie*, sixième édition, 1920, J. Gabay reprint.

[9] H. G. Forder, *Geometry*, Hutchinson, 1960.

[10] L. Gaultier (de Tours), Les contacts des cercles, *Journal de l'Ecole Polytechnique*, Cahier 16 (1813) 124–214.

[11] S. Gueron, Two Applications of the Generalized Ptolemy Theorem, *Amer. Math. Monthly*, 109 (2002) 362–370.

[12] R. A. Johnson, *Advanced Euclidean Geometry*, Dover, 1965.
[13] *Leybourn's Mathematical Repository* (Nouvelle série) 6 tome I, 209.
[14] A. Miquel, Théorèmes de Géométrie, *Journal de mathématiques pures et appliquées de Liouville*, 3 (1838) 485-487.
[15] J. Neuberg, *Nouvelle correspondance mathématique*, 1 (1874) 96.
[16] R. C. J. Nixon, Question 10693, *Reprints of Educational Times*, London (1863-1918) 55 (1891) 107.
[17] Pappus, *La collection mathématique*, 2 volumes, French translation by Paul Ver Eecker, Paris, Desclée de Brouver, 1933.
[18] Y. Sawayama, A new geometrical proposition, *Amer. Math. Monthly*, 12 (1905) 222–224.
[19] R. Shail., A proof of Thébault's Theorem, *Amer. Math. Monthly*, 108 (2001) 319–325.
[20] S. Shirali, On the generalized Ptolemy theorem, *Crux Math.*, 22 (1996) 48–53.
[21] R. Stark, Eine weitere Lösung der Thébault'schen Aufgabe, *Elem. Math.*, 44 (1989) 130–133.
[22] H. Streefkerk, Waarom eenvoudig als het ook ingewikkeld kan?, *Nieuw Tijdschrift voor Wiskunde*, 60 (1972-73) 240–253.
[23] K. B. Taylor, Solution of Problem 3887, *Amer. Math. Monthly*, 90 (1983) 482–487.
[24] V. Thébault, Problem 3887, Three circles with collinear centers, *Amer. Math. Monthly*, 45 (1938) 482–483.
[25] G. Turnwald, Über eine Vermutung von Thébault, *Elem. Math.*, 41 (1986) 11–13.
[26] G. R. Veldkamp, Een vraagstuk van Thébault uit 1938, *Nieuw Tijdschrift voor Wiskunde*, 61 (1973-74) 86–89.
[27] G. R. Veldkamp, Solution to Problem 1260, *Crux Math.*, 15 (1989) 51–53.

Jean-Louis Ayme: 37 rue Ste-Marie, 97400 St.-Denis, La Réunion, France
E-mail address: `jeanlouisayme@yahoo.fr`

Forum Geometricorum
Volume 3 (2003) 231–249.

FORUM GEOM
ISSN 1534-1178

Antiorthocorrespondents of Circumconics

Bernard Gibert

Abstract. This note is a complement to our previous paper [3]. We study how circumconics are transformed under antiorthocorrespondence. This leads us to consider a pencil of pivotal circular cubics which contains in particular the Neuberg cubic of the orthic triangle.

1. Introduction

This paper is a complement to our previous paper [3] on the orthocorrespondence. Recall that in the plane of a given triangle ABC, the orthocorrespondent of a point M is the point $M^{\perp}$ whose trilinear polar intersects the sidelines of ABC at the orthotraces of M. If $M = (p : q : r)$ in homogeneous barycentric coordinates, then [1]

$$M^{\perp} = (p(-pS_A + qS_B + rS_C) + a^2qr : \cdots : \cdots). \tag{1}$$

The antiorthocorrespondents of M consists of the two points M_1 and M_2, not necessarily real, for which $M_1^{\perp} = M = M_2^{\perp}$. We write $M^{\top} = \{M_1, M_2\}$, and say that M_1 and M_2 are orthoassociates. We shall make use of the following basic results.

Lemma 1. *Let $M = (p : q : r)$ and $M^{\top} = \{M_1, M_2\}$.*

(1) *The line M_1M_2 [2] has equation*

$$S_A(q - r)x + S_B(r - p)y + S_C(p - q)z = 0.$$

It always passes through the orthocenter H, and intersects the line GM at the point

$$\big((b^2 - c^2)/(q - r) : \cdots : \cdots\big)$$

on the Kiepert hyperbola.

(2) *The perpendicular bisector ℓ_M of the segment M_1M_2 is the trilinear polar of the isotomic conjugate of the anticomplement of M,* i.e.,

$$(q + r - p)x + (r + p - q)y + (p + q - r)z = 0.$$

Publication Date: December 29, 2003. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his helps in the preparation of the present paper.

[1] Throughout this paper, we use the same notations in [3]. All coordinates are barycentric coordinates with respect to the reference triangle ABC.

[2] M_1M_2 is the Steiner line of the isogonal conjugate of the infinite point of the trilinear polar of the isotomic conjugate of M.

We study how circumconics are transformed under antiorthocorrespondence. Let $P = (u : v : w)$ be a point not lying on the sidelines of ABC. Denote by Γ_P the circumconic with perspector P, namely,

$$\frac{u}{x} + \frac{v}{y} + \frac{w}{z} = 0.$$

This has center [3]

$$G/P = (u(-u+v+w) : v(-v+w+u) : w(u+v-w)),$$

and is the locus of trilinear poles of lines passing through P.

A point $(x : y : z)$ is the orthocorrespondent of a point on Γ_P if and only if

$$\sum_{\text{cyclic}} \frac{u}{x(-xS_A + yS_B + zS_C) + a^2yz} = 0. \tag{2}$$

The antiorthocorrespondent of Γ_P is therefore in general a quartic $\mathcal{Q}_P$. It is easy to check that $\mathcal{Q}_P$ passes through the vertices of the orthic triangle and the pedal triangle of P. It is obviously invariant under orthoassociation, *i.e.*, inversion with respect to the polar circle. See [3, §2]. It is therefore a special case of anallagmatic fourth degree curve.

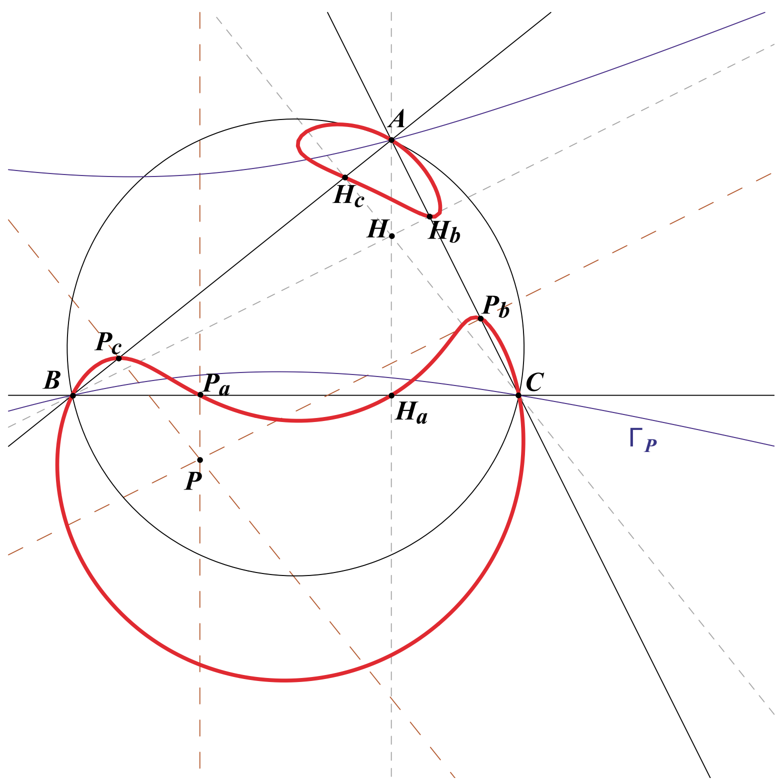


Figure 1. The bicircular circum-quartic $\mathcal{Q}_P$

[3]This is the perspector of the medial triangle and anticevian triangle of P.

The equation of $\mathcal{Q}_P$ can be rewritten as

$$(u+v+w)\mathcal{C}^2 - \left(\sum_{\text{cyclic}} (v+w)S_A\, x\right)\mathcal{L}\mathcal{C} - \left(\sum_{\text{cyclic}} uS_BS_C\, yz\right)\mathcal{L}^2 = 0, \quad (3)$$

with

$$\mathcal{C} = a^2yz + b^2zx + c^2xy, \qquad \mathcal{L} = x+y+z.$$

From this it is clear that $\mathcal{Q}_P$ is a bicircular quartic if and only if $u+v+w \neq 0$; equivalently, Γ_P does not contain the centroid G. We shall study this case in §3 below, and the case $G \in \Gamma_P$ in §4.

2. The conic γ_P

A generic point on the conic Γ_P is

$$M = M(t) = \left(\frac{u}{(v-w)(u+t)} : \frac{v}{(w-u)(v+t)} : \frac{w}{(u-v)(w+t)}\right).$$

As M varies on the circumconic Γ_P, the perpendicular bisector ℓ_M of M_1M_2 envelopes the conic γ_P:

$$\sum((u+v+w)^2 - 4vw)x^2 - 2(u+v+w)(v+w-u)yz = 0.$$

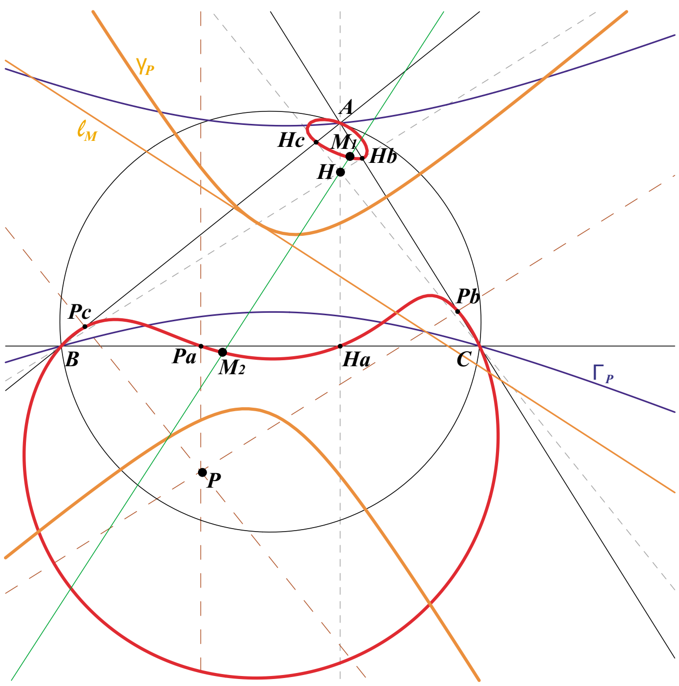


Figure 2. The conic γ_P

The point of tangency of γ_P and the perpendicular bisector of M_1M_2 is

$$T_M = (v(u-v)^2(w+t)^2 + w(u-w)^2(v+t)^2 : \cdots : \cdots).$$

The conic γ_P is called the déférente of Γ_P in [1]. It has center $\omega_P = (2u+v+w : \cdots : \cdots)$, and is homothetic to the circumconic with perspector $((v+w)^2 : (w+u)^2 : (u+v)^2)$. [4] It is therefore a circle when P is the Nagel point or one of its extraversions. This circle is the Spieker circle. We shall see in §3.5 below that $\mathcal{Q}_P$ is an oval of Descartes.

It is clear that γ_P is a parabola if and only if ω_P and therefore P are at infinity. In this case, Γ_P contains the centroid G. See §4 below.

3. Antiorthocorrespondent of a circumconic not containing the centroid

Throughout this section we assume P a finite point so that the circumconic Γ_P does not contain the centroid G.

Proposition 2. *Let ℓ be a line through G intersecting Γ_P at two points M and N. The antiorthocorrespondents of M and N are four collinear points on $\mathcal{Q}_P$.*

Proof. Let M_1, M_2 be the antiorthocorrespondents of M, and N_1, N_2 those of N. By Lemma 1, each of the lines M_1M_2 and N_1N_2 intersects ℓ at the same point on the Kiepert hyperbola. Since they both contain H, M_1M_2 and N_1N_2 are the same line. □

Corollary 3. *Let the medians of ABC meet Γ_P again at A_g, B_g, C_g. The antiorthocorrespondents of these points are the third and fourth intersections of $\mathcal{Q}_P$ with the altitudes of ABC.* [5]

Proof. The antiorthocorrespondents of A are A and H_a. □

In this case, the third and fourth points on AH are symmetric about the second tangent to γ_P which is parallel to BC. The first tangent is the perpendicular bisector of AH_a with contact $(v+w : v : w)$, the contact with this second tangent is $(u(v+w) : uw + (v+w)^2 : uv + (v+w)^2)$ while $A_g = (-u : v+w : v+w)$.

For distinct points P_1 and P_2, the circumconics Γ_{P_1} and Γ_{P_2} have a "fourth" common point T, which is the trilinear pole of the line P_1P_2. Let $T^\top = \{T_1, T_2\}$. The conics Γ_{P_1} and Γ_{P_2} generate a pencil $\mathcal{F}$ consisting of Γ_P for P on the line P_1P_2. The antiorthocorrespondent of every conic $\Gamma_P \in \mathcal{F}$ contains the following 16 points:
(i) the vertices of ABC and the orthic triangle $H_aH_bH_c$,
(ii) the circular points at infinity with multiplicity 4, [6]
(iii) the antiorthocorrespondents $T^\top = \{T_1, T_2\}$.

Proposition 4. *Apart from the circular points at infinity and the vertices of ABC and the orthic triangle, the common points of the quartics $\mathcal{Q}_{P_1}$ and $\mathcal{Q}_{P_2}$ are the antiorthocorrespondents of the trilinear pole of the line P_1P_2.*

---

[4] It is inscribed in the medial triangle; its anticomplement is the circumconic with center the complement of P, with perspector the isotomic conjugate of P.

[5] They are not always real when ABC is obtuse angle.

[6] Think of $\mathcal{Q}_{P_1}$ as the union of two circles and $\mathcal{Q}_{P_2}$ likewise. These have at most 8 real finite points and the remaining 8 are the circular points at infinity, each counted with multiplicity 4.

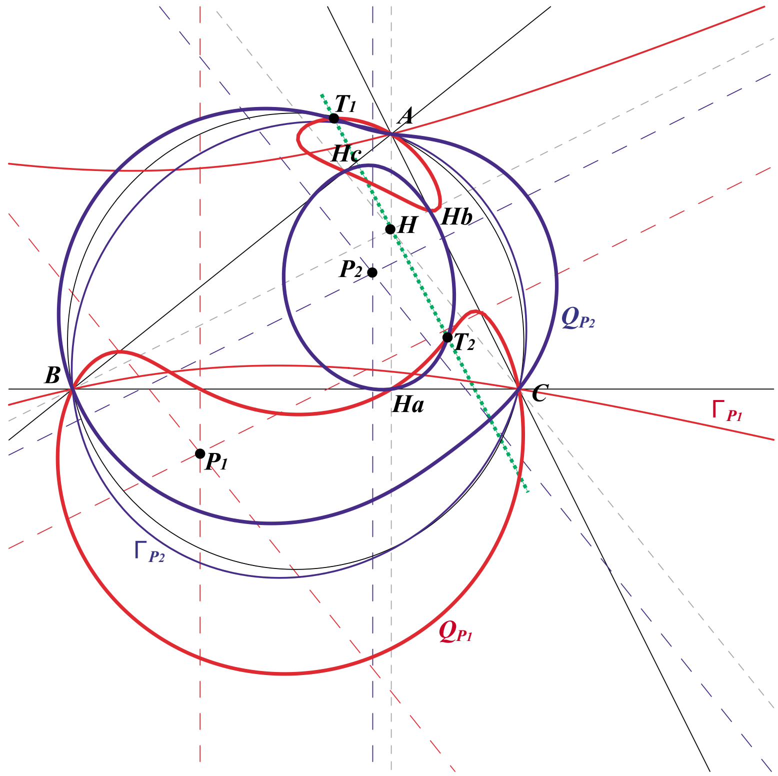


Figure 3. The bicircular quartics $\mathcal{Q}_{P_1}$ and $\mathcal{Q}_{P_2}$

Remarks. 1. T_1 and T_2 lie on the line through H which is the orthocorrespondent of the line GT. See [3, §2.4]. This line T_1T_2 is the directrix of the inscribed (in ABC) parabola tangent to the line P_1P_2.

2. The pencil $\mathcal{F}$ contains three degenerate conics $BC \cup AT$, $CA \cup BT$, and $AB \cup CT$. The antiorthocorrespondent of $BC \cup AT$, for example, degenerates into the circle with diameter BC and another circle through A, H_a, T_1 and T_2 (see [3, Proposition 2]).

3.1. *The points* S_1 *and* S_2. Since $\mathcal{Q}_P$ and the circumcircle have already seven common points, the vertices A, B, C, and the circular points at infinity, each of multiplicity 2, they must have an eighth common point, namely,

$$S_1 = \left(\frac{a^2}{\frac{v}{b^2 S_B} - \frac{w}{c^2 S_C}} : \cdots : \cdots \right), \tag{4}$$

which is the isogonal conjugate of the infinite point of the line

$$\frac{u}{a^2 S_A} x + \frac{v}{b^2 S_B} y + \frac{w}{c^2 S_C} z = 0.$$

Similarly, $\mathcal{Q}_P$ and the nine-point circle also have a real eighth common point

$$S_2 = ((S_B(u-v+w) - S_C(u+v-w))(c^2S_Cv - b^2S_Bw) : \cdots : \cdots), \quad (5)$$

which is the inferior of

$$\left(\frac{a^2}{S_B(u-v+w) - S_C(u+v-w)} : \cdots : \cdots\right)$$

on the circumcircle.

We know that the orthocorrespondent of the circumcircle is the circum-ellipse Γ_O, with center K, the Lemoine point, ([3, §2.6]). If $P \neq O$, this ellipse meets Γ_P at A, B, C and a fourth point

$$S = S(P) = \left(\frac{1}{c^2S_Cv - b^2S_Bw} : \cdots : \cdots\right), \quad (6)$$

which is the trilinear pole of the line OP. The point S lies on the circumcircle if and only if P is on the Brocard axis OK.

Proposition 5. $S^\top = \{S_1, S_2\}$.

Corollary 6. *$S(P) = S(P')$ if and only if P, P' and O are collinear.*

Remark. When $P = O$ (circumcenter), Γ_P is the circum-ellipse with center K. In this case $\mathcal{Q}_P$ decomposes into the union of the circumcircle and the nine point circle.

3.2. *Bitangents.*

Proposition 7. *The points of tangency of the two bitangents to $\mathcal{Q}_P$ passing through H are the antiorthocorrespondents of the points where the polar line of G in Γ_P meets Γ_P.*

Proof. Consider a line ℓ_H through H which is supposed to be tangent to $\mathcal{Q}_P$ at two (orthoassociate) points M and N. The orthocorrespondents of M and N must lie on Γ_P and on the orthocorrespondent of ℓ_H which is a line through G. Since M and N are double points, the line through G must be tangent to Γ_P and MN is the polar of G in Γ_P. □

Remark. M and N are not necessarily real. If $M^\top = \{M_1, M_2\}$ and $N^\top = \{N_1, N_2\}$, the perpendicular bisectors of M_1M_2 and N_1N_2 are the asymptotes of γ_P.[7] The four points M_1, M_2, N_1, N_2 are concyclic and the circle passing through them is centered at ω_P.

Denote by H_1, H_2, H_3 the vertices of the triangle which is self polar in both the polar circle and γ_P. The orthocenter of this triangle is obviously H. For $i = 1,\ 2,\ 3$, let $\mathcal{C}_i$ be the circle centered at H_i orthogonal to the polar circle and Γ_i the circle centered at ω_P orthogonal to $\mathcal{C}_i$. The circle Γ_i intersects $\mathcal{Q}_P$ at the circular points at infinity (with multiplicity 2) and four other points two by two homologous in the inversion with respect to $\mathcal{C}_i$ which are the points of tangency of the (not

[7]The union of the line at infinity and a bitangent is a degenerate circle which is bitangent to $\mathcal{Q}_P$. Its center must be an infinite point of γ_P.

always real) bitangents drawn from H_i to $\mathcal{Q}_P$. The orthocorrespondent of Γ_i is a conic (see [3, §2.6]) intersecting Γ_P at four points whose antiorthocorrespondents are eight points, two by two orthoassociate. Four of them lie on Γ_i and are the required points of tangency. The remaining four are their orthoassociates and they lie on the circle which is the orthoassociate of Γ_i. Figure 4 below shows an example of $\mathcal{Q}_P$ with three pairs of real bitangents.

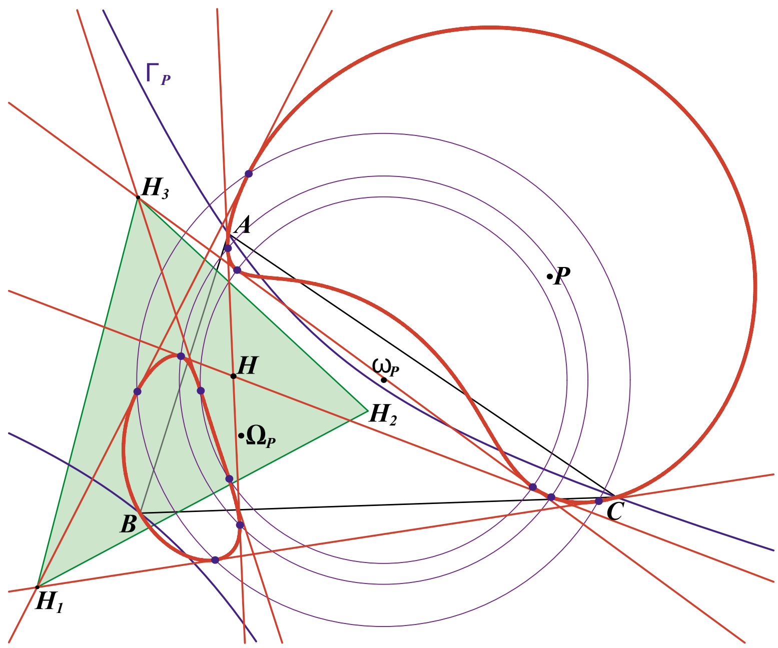


Figure 4. Bitangents to $\mathcal{Q}_P$

Proposition 8. *$\mathcal{Q}_P$ is tangent at H_a, H_b, H_c to BC, CA, AB if and only if $P = H$.*

3.3. *$\mathcal{Q}_P$ as an envelope of circles.*

Theorem 9. *The circle $\mathcal{C}_M$ centered at T_M passing through M_1 and M_2 is bitangent to $\mathcal{Q}_P$ at those points and orthogonal to the polar circle.*

This is a consequence of the following result from [1, tome 3, p.170]. A bicircular quartic is a special case of "plane cyclic curve". Such a curve always can be considered in four different ways as the envelope of circles centered on a conic (déférente) cutting orthogonally a fixed circle. Here the fixed circle is the polar circle with center H, and since M_1 and M_2 are anallagmatic (inverse in the polar circle) and collinear with H, there is a circle passing through M_1, M_2, centered on the déférente, which must be bitangent to the quartic.

Corollary 10. *$\mathcal{Q}_P$ is the envelope of circles $\mathcal{C}_M$, $M \in \Gamma_P$, centered on γ_P and orthogonal to the polar circle.*

Construction. It is easy to draw γ_P since we know its center ω_P. For m on γ_P, draw the tangent t_m at m to γ_P. The perpendicular at m to Hm meets the perpendicular bisector of AH_a at a point which is the center of a circle through A (and H_a). This circle intersects Hm at two points which lie on the circle centered at m and orthogonal to the polar circle. This circle intersects the perpendicular at H to t_m at two points of $\mathcal{Q}_P$.

Corollary 11. *The tangents at M_1 and M_2 to $\mathcal{Q}_P$ are the tangents to the circle $\mathcal{C}_M$ at these points.*

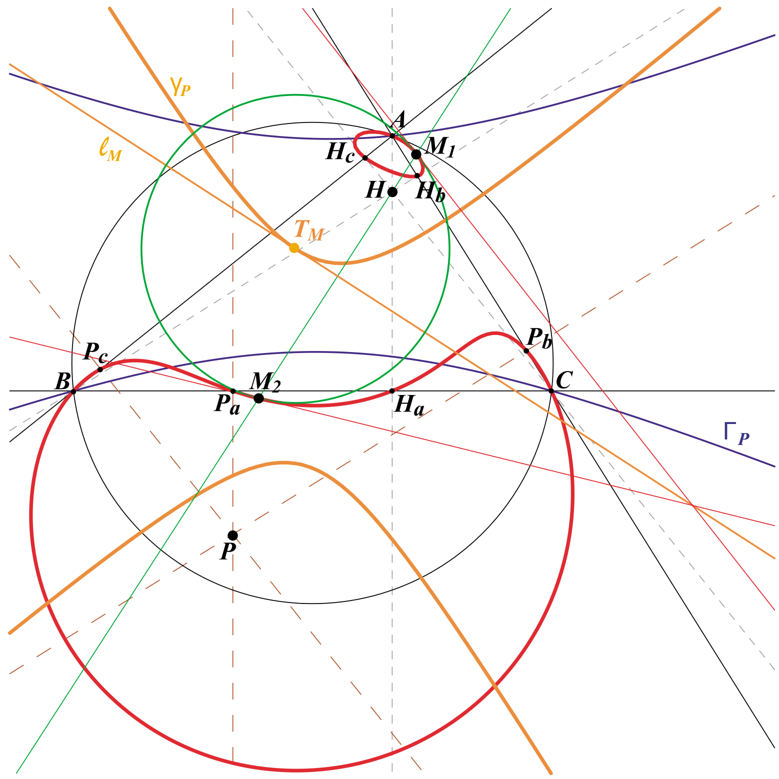


Figure 5. $\mathcal{Q}_P$ as an envelope of circles

3.4. *Inversions leaving $\mathcal{Q}_P$ invariant.*

Theorem 12. *$\mathcal{Q}_P$ is invariant under three other inversions whose poles are the vertices of the triangle which is self-polar in both the polar circle and γ_P.*

Proof. This is a consequence of [1, tome 3, p.172].

Construction: Consider the transformation ϕ which maps any point M of the plane to the intersection M' of the polars of M in both the polar circle and γ_P. Let Σ_a, Σ_b, Σ_c be the conics which are the images of the altitudes AH, BH, CH under ϕ. The conic Σ_a is entirely defined by the following five points:

(1) the point at infinity of BC.
(2) the point at infinity of the polar of H in γ_P.
(3) the foot on BC of the polar of A in γ_P.
(4) the intersection of the polar of H_a in γ_P with the parallel at A to BC.
(5) the pole of AH in γ_P.

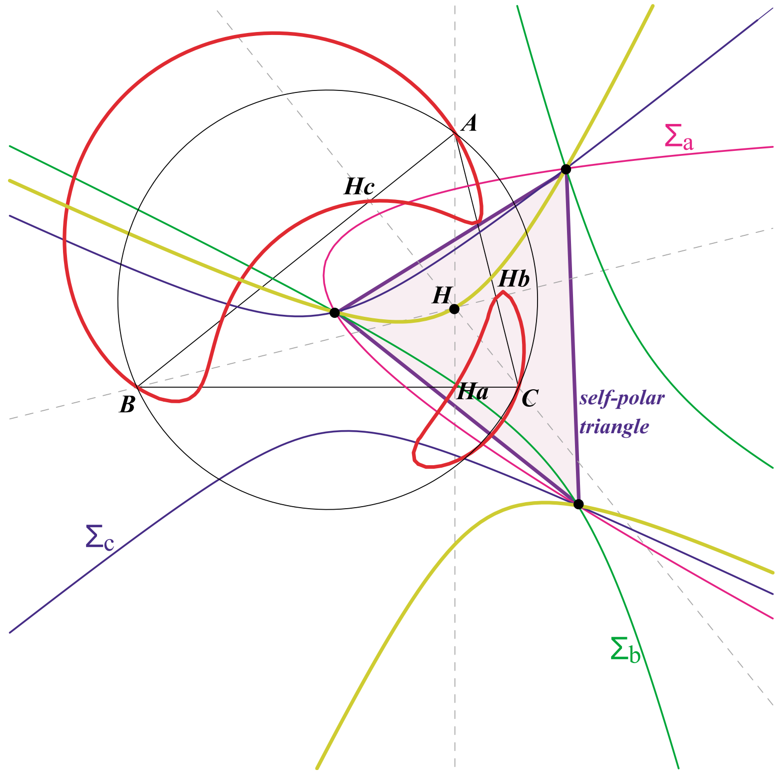


Figure 6. The conics Σ_a, Σ_b, Σ_c

Similarly, we define the conics Σ_b and Σ_c. These conics are in the same pencil and meet at four points: one of them is the point at infinity of the polar of H in γ_P and the three others are the required poles. The circles of inversion are centered at those points and are orthogonal to the polar circle. Their radical axes with the polar circle are the sidelines of the self-polar triangle. □

Another construction is possible : the transformation of the sidelines of triangle ABC under ϕ gives three other conics σ_a, σ_b, σ_c but not defining a pencil since the three lines are not now concurrent. σ_a passes through A, the two points where the trilinear polar of P^+ (anticomplement of P) meets AB and AC, the pole of the line BC in γ_P, the intersection of the parallel at A to BC with the polar of H_a in γ_P. See Figure 7.

Remark. The Jacobian of σ_a, σ_b, σ_c is a degenerate cubic consisting of the union of the sidelines of the self-polar triangle.

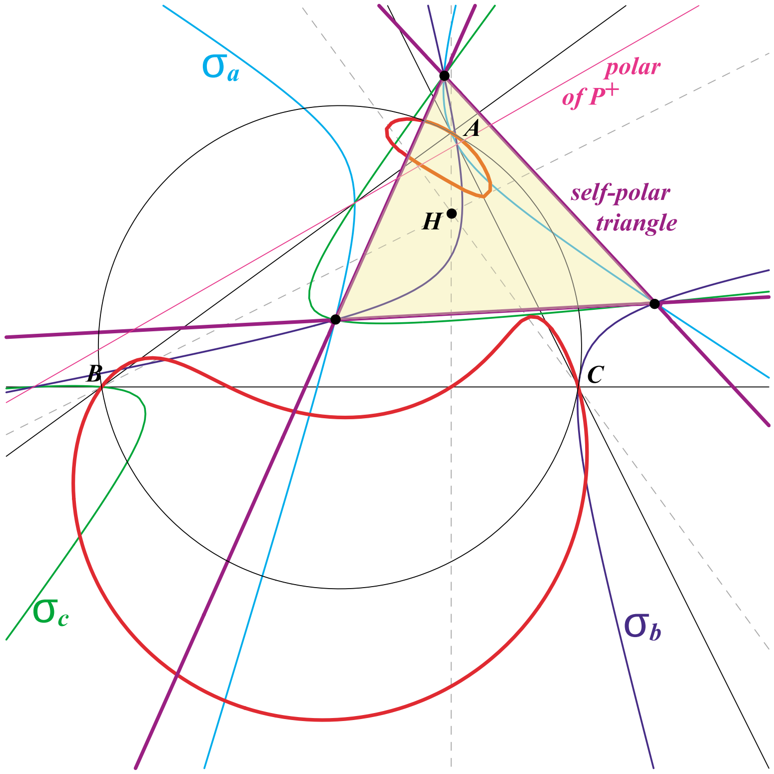


Figure 7. The conics σ_a, σ_b, σ_c

3.5. *Examples.* We provide some examples related to common centers of ABC.

P	S	S_1	S_2	Γ_P	Remark
H	X_{648}	X_{107}	X_{125}		see Figure 8
K	X_{110}	X_{112}	X_{115}	circumcircle	
G	X_{648}	X_{107}	S_{125}	Steiner circum − ellipse	
X_{647}				Jerabek hyperbola	

Remarks. 1. For $P = H$, $\mathcal{Q}_P$ is tangent at H_a, H_b, H_c to the sidelines of ABC. See Figure 8.

2. $P = X_{647}$, the isogonal conjugate of the tripole of the Euler line: Γ_P is the Jerabek hyperbola.

3. $\mathcal{Q}_P$ has two axes of symmetry if and only if P is the point such that $\overrightarrow{OP} = 3\,\overrightarrow{OH}$ (this is a consequence of [1, tome 3, p.172, §15]. Those axes are the parallels at H to the asymptotes of the Kiepert hyperbola. See Figure 9.

4. When $P = X_8$ (Nagel point), γ_P is the incircle of the medial triangle (its center is X_{10} = Spieker center) and Γ_P the circum-conic centered at $\Omega_P = ((b + c - a)(b + c - 3a) : \cdots : \cdots)$. Since the déférente is a circle, $\mathcal{Q}_P$ is now an oval

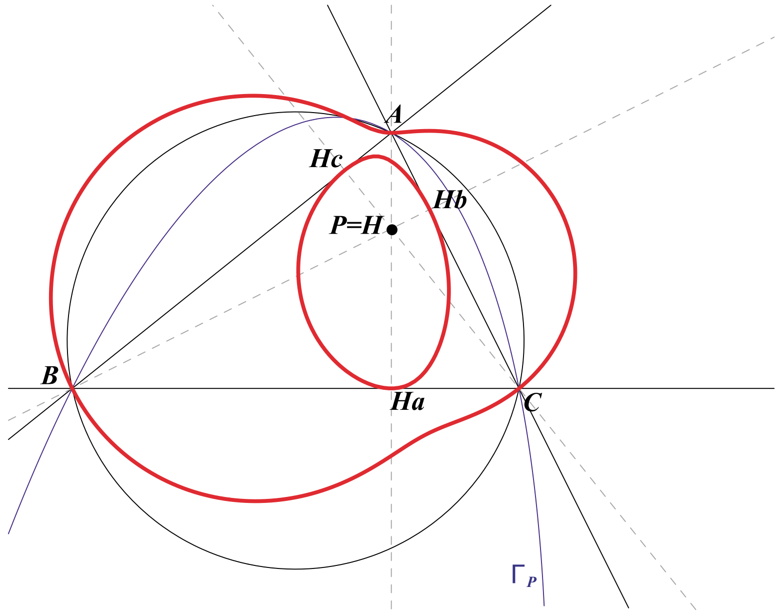


Figure 8. The quartic $\mathcal{Q}_H$

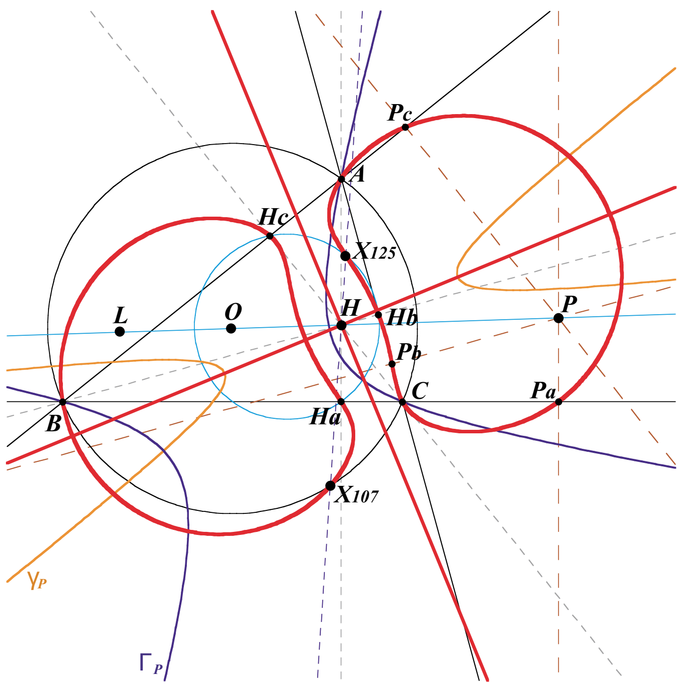


Figure 9. $\mathcal{Q}_P$ with two axes of symmetry

of Descartes (see [1, tome 1, p.8]) with axis the line HX_{10}. We obtain three more ovals of Descartes if X_8 is replaced by one of its extraversions. See Figure 10.

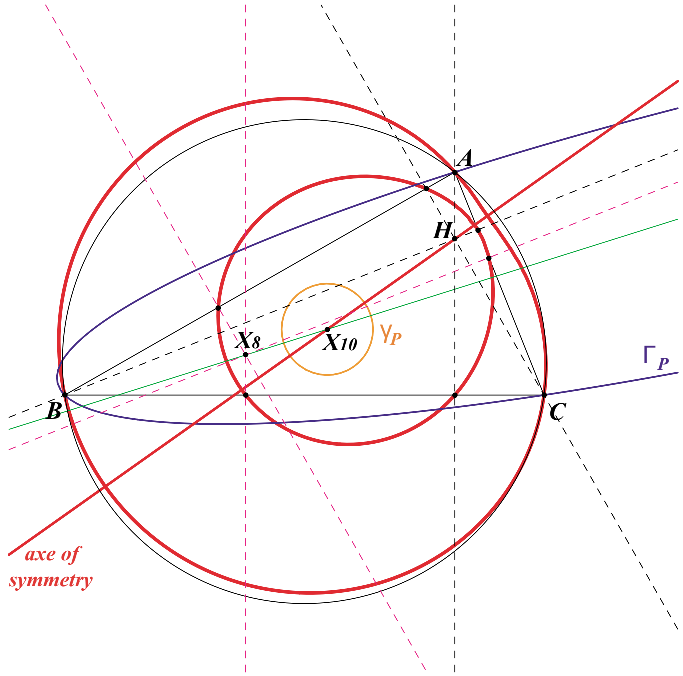


Figure 10. $\mathcal{Q}_P$ as an oval of Descartes

4. Antiorthocorrespondent of a circum-conic passing through G

We consider the case when the circumconic Γ_P contains the centroid G; equivalently, $P = (u : v : w)$ is an infinite point. In this case, Γ_P has center $(u^2 : v^2 : w^2)$ on the inscribed Steiner ellipse. The trilinear polar of points $Q \neq G$ on Γ_P are all parallel, and have infinite point P. It is clear from (3) that the curve $\mathcal{Q}_P$ decomposes into the union of the line at infinity $\mathcal{L}^\infty : x + y + z = 0$ and the cubic $\mathcal{K}_P$

$$\sum x(S_B(S_Au - S_Bv)y^2 - S_C(S_Cw - S_Au)z^2) = 0. \tag{7}$$

This is the pivotal isocubic $\mathrm{p}\mathcal{K}(\Omega_P, H)$, with pivot H and pole

$$\Omega_P = \left(\frac{S_Bv - S_Cw}{S_A} : \frac{S_Cw - S_Au}{S_B} : \frac{S_Au - S_Bv}{S_C}\right).$$

Since the orthocorrespondent of the line at infinity is the centroid G, we shall simply say that the antiorthocorrespondent of Γ_P is the cubic $\mathcal{K}_P$. The orthocenter H is the only finite point whose orthocorrespondent is G. We know that $\mathcal{Q}_P$ has already the circular points (counted twice) on $\mathcal{L}^\infty$. This means that the cubic $\mathcal{K}_P$ is also a circular cubic. In fact, equation (7) can be rewritten as

$$\begin{aligned}(uS_Ax + vS_By + wS_Cz)(a^2yz + b^2zx + c^2xy)\\ +(x + y + z)(uS_{BC}yz + vS_{CA}zx + wS_{AB}xy) = 0.\end{aligned} \tag{8}$$

As P traverses $\mathcal{L}^\infty$, these cubics $\mathcal{K}_P$ form a pencil of circular pivotal isocubics since they all contain A, B, C, H, H_a, H_b, H_c and the circular points at infinity. The poles of these isocubics all lie on the orthic axis.

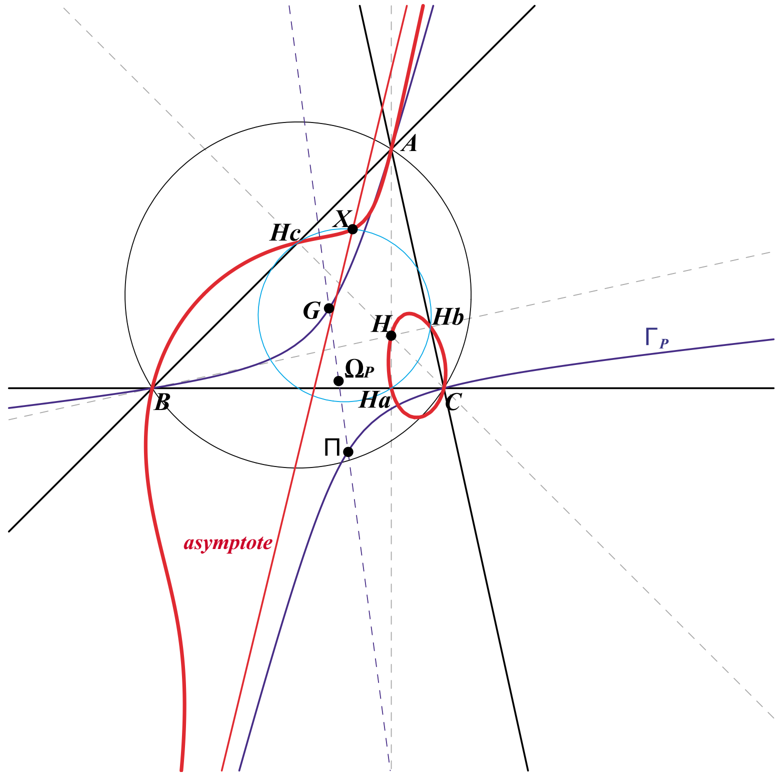


Figure 11. The circular pivotal cubic $\mathcal{K}_P$

4.1. *Properties of* $\mathcal{K}_P$.

(1) $\mathcal{K}_P$ is invariant under orthoassociation: the line through H and M on $\mathcal{K}_P$ meets $\mathcal{K}_P$ again at M' simultaneously the Ω_P-isoconjugate and orthoassociate of M. $\mathcal{K}_P$ is also invariant under the three inversions with poles A, B, C which swap H and H_a, H_b, H_c respectively. [8] See Figure 11.

(2) The real asymptote of $\mathcal{K}_P$ is the line ℓ_P

$$\frac{u}{S_Bv - S_Cw}x + \frac{v}{S_Cw - S_Au}y + \frac{w}{S_Au - S_Bv}z = 0. \tag{9}$$

It has infinite point

$$P' = (S_Bv - S_Cw : S_Cw - S_Au : S_Au - S_Bv),$$

[8] H, H_a, H_b, H_c are often called the centers of anallagmaty of the circular cubic.

and is parallel to the tangents at A, B, C, and H.[9] It is indeed the Simson line of the isogonal conjugate of P. It is therefore tangent to the Steiner deltoid.

(3) The tangents to $\mathcal{K}_P$ at H_a, H_b, H_c are the reflections of those at A, B, C, about the perpendicular bisectors of AH_a, BH_b, CH_c respectively.[10] They concur on the cubic at the point

$$X = \left(\frac{S_B v - S_C w}{u} \left(\frac{b^2 S_B}{v} - \frac{c^2 S_C}{w} \right) : \cdots : \cdots \right),$$

which is also the intersection of ℓ_P and the nine point circle. This is the inferior of the isogonal conjugate of P'. It is also the image of P^*, the isogonal conjugate of P, under the homothety $\mathsf{h}(H, \frac{1}{2})$.

(4) The antipode F of X on the nine point circle is the singular focus of $\mathcal{K}_P$:

$$F = (u(b^2 v - c^2 w) : v(c^2 w - a^2 u) : w(a^2 u - b^2 v)).$$

(5) The orthoassociate Y of X is the "last" intersection of $\mathcal{K}_P$ with the circumcircle, apart from the vertices and the circular points at infinity.

(6) The second intersection of the line XY with the circumcircle is $Z = P^*$.

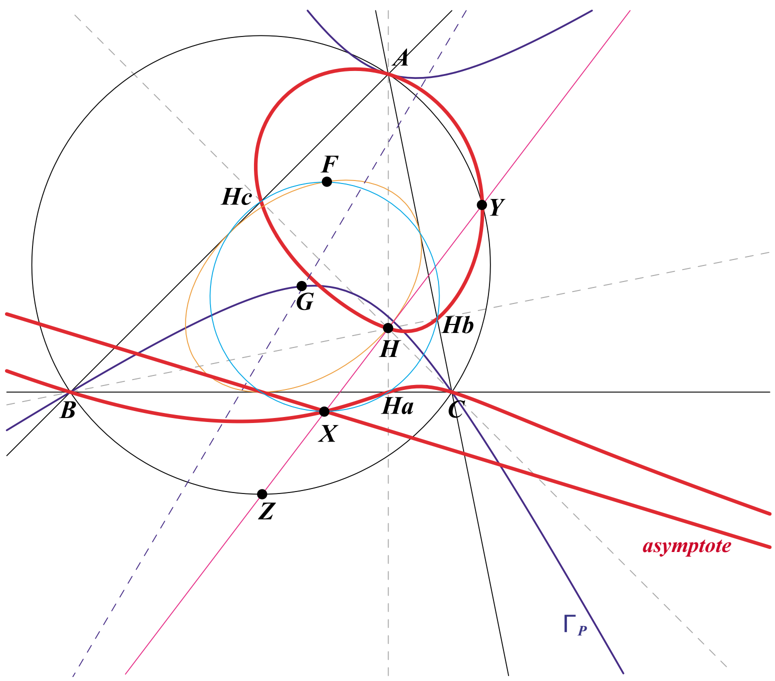


Figure 12. The points X, Y, Z and $\mathcal{K}_P$ for $P = X_{512}$

[9]The latter is the line $uS_A x + vS_B y + wS_C z = 0$.

[10]These are the lines $S^2 ux - (S_B v - S_C w)(S_B y - S_C z) = 0$ etc.

(7) $\mathcal{K}_P$ intersects the sidelines of the orthic triangle at three points lying on the cevian lines of Y in ABC.
(8) $\mathcal{K}_P$ is the envelope of circles centered on the parabola $\mathcal{P}_P$ (focus F, directrix the parallel at O to the Simson line of Z) and orthogonal to the polar circle. See Figure 13.

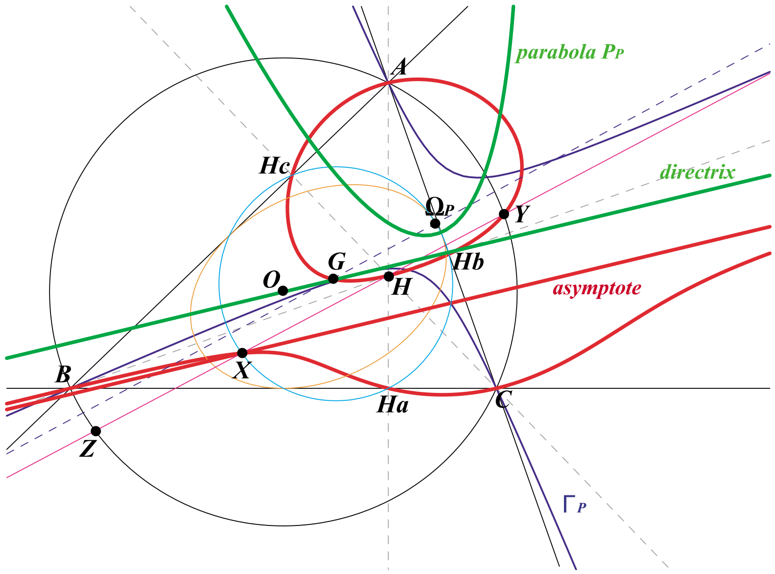


Figure 13. $\mathcal{K}_P$ and the parabola $\mathcal{P}_P$

(9) Γ_P meets the circumcircle again at

$$S=\left(\frac{1}{b^2v-c^2w}:\frac{1}{c^2w-a^2u}:\frac{1}{a^2u-b^2v}\right)$$

and the Steiner circum-ellipse again at

$$R=\left(\frac{1}{v-w}:\frac{1}{w-u}:\frac{1}{u-v}\right).$$

The antiorthocorrespondents of these two points S are four points on $\mathcal{K}_P$. They lie on a same circle orthogonal to the polar circle. See [3, §2.5] and Figure 14.

4.2. *$\mathcal{K}_P$ passing through a given point.* Since all the cubics form a pencil, there is a unique $\mathcal{K}_P$ passing through a given point Q which is not a base-point of the pencil. The circumconic Γ_P clearly contains G and $Q^\perp$, the orthocorrespondent of Q. It follows that P is the infinite point of the tripolar of $Q^\perp$.

Here is another construction of P. The circumconic through G and $Q^\perp$ intersects the Steiner circum-ellipse at a fourth point R. The midpoint M of GR is the center of Γ_P. The anticevian triangle of M is perspective to the medial triangle at P. The lines through their corresponding vertices are parallel to the tangents to

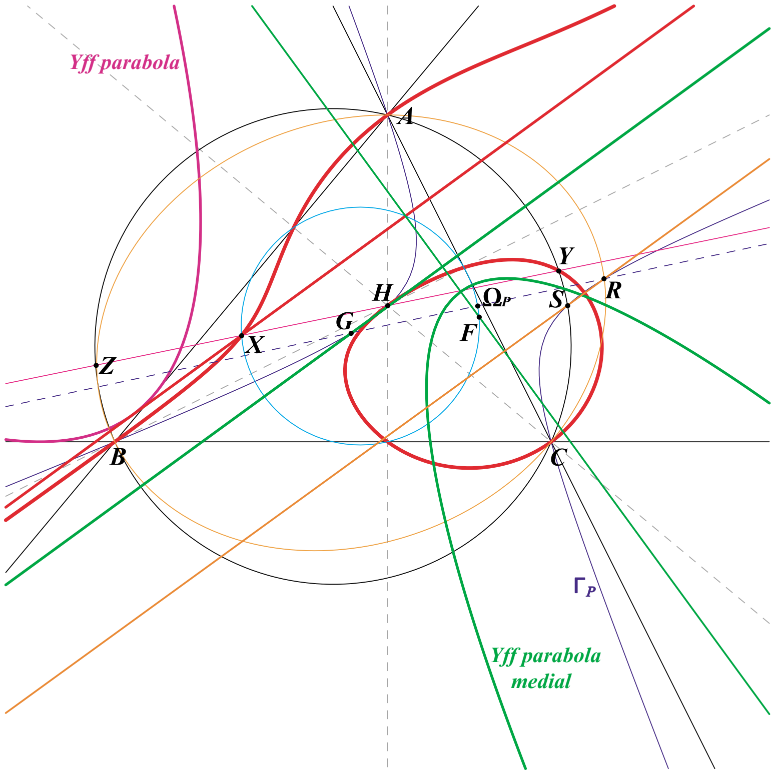


Figure 14. The points R, S and $\mathcal{K}_P$ for $P = X_{514}$

$\mathcal{K}_P$ at A, B, C. The point at infinity of these parallel lines is the point P for which $\mathcal{K}_P$ contains Q.

In particular, if Q is a point on the circumcircle, P is simply the isogonal conjugate of the second intersection of the line HQ with the circumcircle.

4.3. *Some examples and special cases.*

(1) The most remarkable circum-conic through G is probably the Kiepert rectangular hyperbola with perspector $P = X_{523}$, point at infinity of the orthic axis. Its antiorthocorrespondent is $\mathrm{p}\mathcal{K}(X_{1990}, H)$, identified as the orthopivotal cubic $\mathcal{O}(H)$ in [3, §6.2.1]. See Figure 15.

(2) With $P =$ isogonal conjugate of X_{930} [11], $\mathcal{K}_P$ is the Neuberg cubic of the orthic triangle. We have $F = X_{137}$, $X = X_{128}$, $Y =$ isogonal conjugate of X_{539}, $Z = X_{930}$. The cubic contains X_5, X_{15}, X_{16}, X_{52}, X_{186}, X_{1154} (at infinity). See Figure 16.

(3) $\mathcal{K}_P$ degenerates when P is the point at infinity of one altitude. For example, with the altitude AH, $\mathcal{K}_P$ is the union of the sideline BC and the circle through A, H, H_b, H_c.

[11] $P = (a^2(b^2 - c^2)(4S_A^2 - 3b^2c^2) : \cdots : \cdots)$. The point X_{930} is the anticomplement of X_{137} which is X_{110} of the orthic triangle.

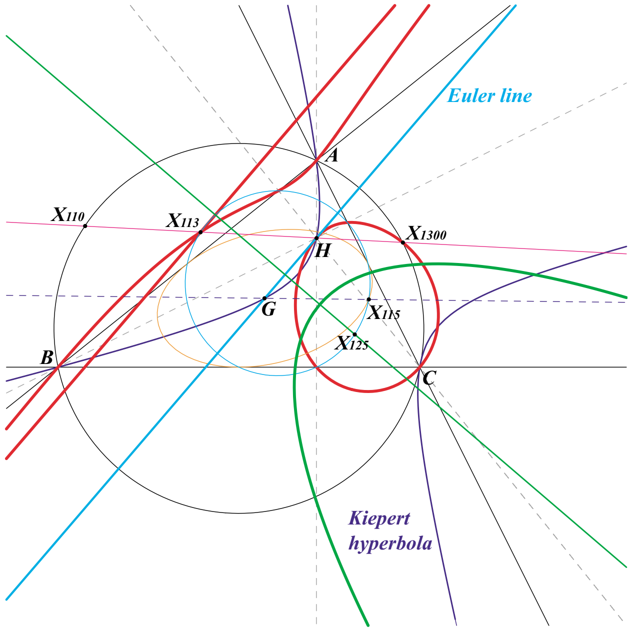


Figure 15. $\mathcal{O}(H)$ or $\mathcal{K}_P$ for $P = X_{523}$

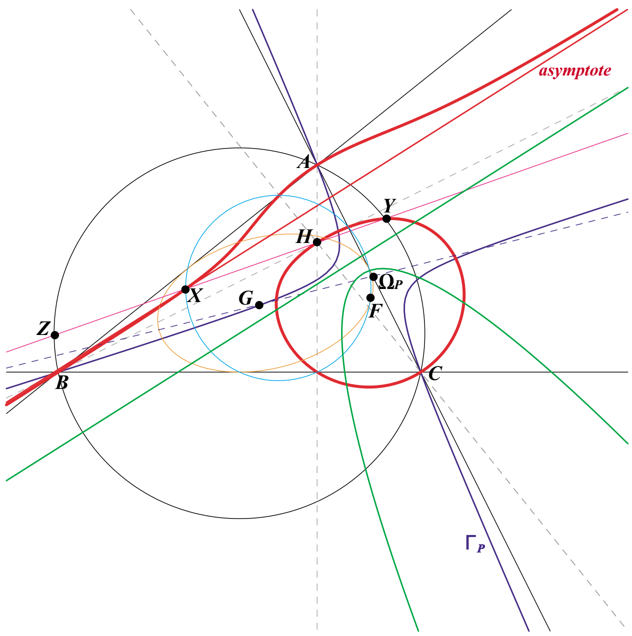


Figure 16. $\mathcal{K}_P$ as the Neuberg cubic of the orthic triangle

(4) $\mathcal{K}_P$ is a focal cubic if and only if P is the point at infinity of one tangent to the circumcircle at A, B, C. For example, with A, $\mathcal{K}_P$ is the focal cubic

denoted $\mathcal{K}_a$ with singular focus H_a and pole the intersection of the orthic axis with the symmedian AK. The tangents at A, B, C, H are parallel to the line OA. Γ_P is the isogonal conjugate of the line passing through K and the midpoint of BC. $\mathcal{P}_P$ is the parabola with focus H_a and directrix the line OA.

$\mathcal{K}_a$ is the locus of point M from which the segments BH_b, CH_c are seen under equal or supplementary angles. It is also the locus of contacts of tangents drawn from H_a to the circles centered on H_bH_c and orthogonal to the circle with diameter H_bH_c. See Figure 17.

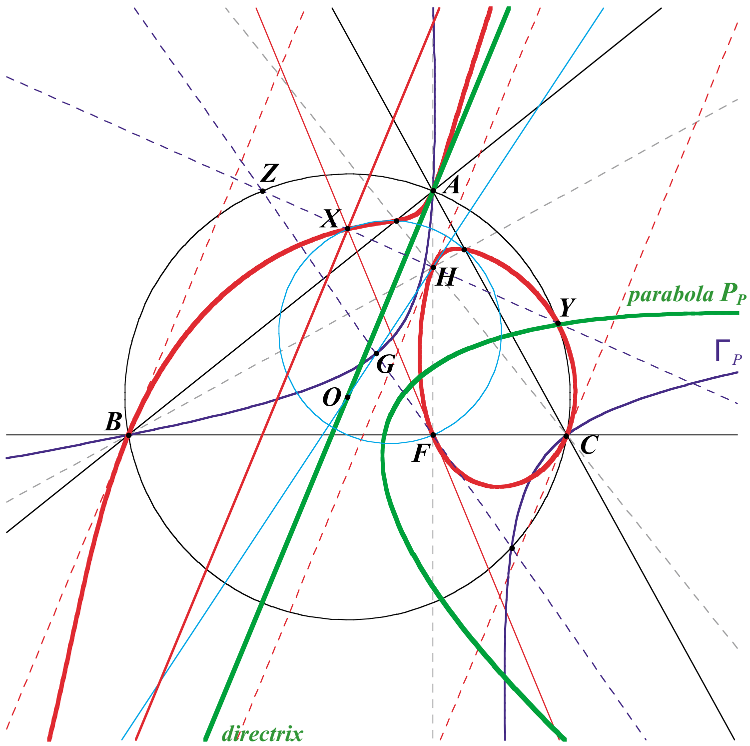


Figure 17. The focal cubic $\mathcal{K}_a$

4.4. *Conclusion.* We conclude with the following table showing the repartition of the points we met in the study above in some particular situations. Recall that P', X, Y always lie on $\mathcal{K}_P$, Y, Z, S on the circumcircle, X, F on the nine point circle, R on the Steiner circum-ellipse. When the point is not mentioned in [6], its first barycentric coordinate is given, as far as it is not too complicated. M^* denotes the isogonal conjugate of M, and $M^\#$ denotes the isotomic conjugate of M.

P	P'	X	Y	Z	F	S	R	Remark
X_{30}	X_{523}	X_{125}	X_{107}	X_{74}	X_{113}	X_{1302}	X_{648}	
X_{523}	X_{30}	X_{113}	X_{1300}	X_{110}	X_{125}	X_{98}	X_{671}	(1)
X_{514}	X_{516}	X_{118}	X_{917}	X_{101}	X_{116}	X_{675}	X_{903}	(2)
X_{511}	X_{512}	X_{115}	X_{112}	X_{98}	X_{114}	X_{110}	M_1	
X_{512}	X_{511}	X_{114}	M_2	X_{99}	X_{115}	X_{111}	$X^{\#}_{538}$	(3)
X_{513}	X_{517}	X_{119}	X_{915}	X_{100}	X_{11}	X_{105}	$X^{\#}_{536}$	(4)
X_{524}	X_{1499}	M_3	M_4	X_{111}	X_{126}	X_{99}	X_{99}	
X_{520}	X^*_{1294}	X_{133}	X_{74}	X_{107}	X_{122}	X_{1297}		
X_{525}	X_{1503}	X_{132}	X_{98}	X_{112}	X_{127}	$X^{\#}_{858}$	$X^{\#}_{30}$	
X^*_{930}	X_{1154}	X_{128}	X^*_{539}	X_{930}	X_{137}			
X_{515}	X_{522}	X_{124}	M_5	X_{102}	X_{117}			
X_{516}	X_{514}	X_{116}	M_6	X_{103}	X_{118}		M_7	

Remarks. (1) $\Omega_P = X_{115}$. Γ_P is the Kiepert hyperbola. $\mathcal{P}_P$ is the Kiepert parabola of the medial triangle with directrix the Euler line. See Figure 15.

(2) $\Omega_P = X_{1086}$. $\mathcal{P}_P$ is the Yff parabola of the medial triangle. See Figure 14.

(3) $\Omega_P = X_{1084}$. The directrix of $\mathcal{P}_P$ is the Brocard line.

(4) $\Omega_P = X_{1015}$. The directrix of $\mathcal{P}_P$ is the line OI.

The points $M_1, \dots, M_7$ are defined by their first barycentric coordinates as follows.

M_1	$1/[(b^2-c^2)(a^2S_A+b^2c^2)]$
M_2	$a^2/[S_A((b^2-c^2)^2-a^2(b^2+c^2-2a^2))]$
M_3	$(b^2-c^2)^2(b^2+c^2-5a^2)(b^4+c^4-a^4-4b^2c^2)$
M_4	$1/[S_A(b^2-c^2)(b^4+c^4-a^4-4b^2c^2)]$
M_5	$S_A(b-c)(b^3+c^3-a^2b-a^2c+abc)$
M_6	$1/[S_A(b-c)(b^2+c^2-ab-ac+bc)]$
M_7	$1/[(b-c)(3b^2+3c^2-a^2-2ab-2ac+2bc)]$

References

[1] H. Brocard and T. Lemoyne , *Courbes Géométriques Remarquables*, Librairie Albert Blanchard, Paris, third edition, 1967.

[2] B. Gibert, *Cubics in the Triangle Plane*, available at http://perso.wanadoo.fr/bernard.gibert/index.html.

[3] B. Gibert, *Orthocorrespondence and Orthopivotal Cubics*, Forum Geometricorum, vol.3, pp.1-27, 2003.

[4] F. Gomes Teixeira, *Traité des Courbes Spéciales Remarquables* , Reprint Editions Jacques Gabay, Paris, 1995.

[5] C. Kimberling, *Triangle Centers and Central Triangles*, Congressus Numerantium, 129 (1998) 1–295.

[6] C. Kimberling, *Encyclopedia of Triangle Centers*, November 4, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France
E-mail address: bg42@wanadoo.fr

Forum Geometricorum
Volume 3 (2003) 251.

FORUM GEOM
ISSN 1534-1178

Author Index

FORUM GEOMETRICORUM

A Journal on Classical Euclidean Geometry and Related Areas

published by

Department of Mathematical Sciences
Florida Atlantic University

FORUM GEOM

Volume 4
2004

http://forumgeom.fau.edu

ISSN 1534-1178

Editorial Board

Advisors:

John H. Conway	Princeton, New Jersey, USA
Julio Gonzalez Cabillon	Montevideo, Uruguay
Richard Guy	Calgary, Alberta, Canada
Clark Kimberling	Evansville, Indiana, USA
Kee Yuen Lam	Vancouver, British Columbia, Canada
Tsit Yuen Lam	Berkeley, California, USA
Fred Richman	Boca Raton, Florida, USA

Editor-in-chief:

Paul Yiu	Boca Raton, Florida, USA

Editors:

Clayton Dodge	Orono, Maine, USA
Roland Eddy	St. John’s, Newfoundland, Canada
Jean-Pierre Ehrmann	Paris, France
Lawrence Evans	La Grange, Illinois, USA
Chris Fisher	Regina, Saskatchewan, Canada
Rudolf Fritsch	Munich, Germany
Bernard Gibert	St Etiene, France
Antreas P. Hatzipolakis	Athens, Greece
Michael Lambrou	Crete, Greece
Floor van Lamoen	Goes, Netherlands
Fred Pui Fai Leung	Singapore, Singapore
Daniel B. Shapiro	Columbus, Ohio, USA
Steve Sigur	Atlanta, Georgia, USA
Man Keung Siu	Hong Kong, China
Peter Woo	La Mirada, California, USA

Technical Editors:

Yuandan Lin	Boca Raton, Florida, USA
Aaron Meyerowitz	Boca Raton, Florida, USA
Xiao-Dong Zhang	Boca Raton, Florida, USA

Consultants:

Frederick Hoffman	Boca Raton, Floirda, USA
Stephen Locke	Boca Raton, Florida, USA
Heinrich Niederhausen	Boca Raton, Florida, USA

Table of Contents

Forum Geometricorum
Volume 4 (2004) 1–20.

FORUM GEOM
ISSN 1534-1178

Antiparallels and Concurrent Euler Lines

Nikolaos Dergiades and Paul Yiu

Abstract. We study the condition for concurrency of the Euler lines of the three triangles each bounded by two sides of a reference triangle and an antiparallel to the third side. For example, if the antiparallels are concurrent at P and the three Euler lines are concurrent at Q, then the loci of P and Q are respectively the tangent to the Jerabek hyperbola at the Lemoine point, and the line parallel to the Brocard axis through the inverse of the deLongchamps point in the circumcircle. We also obtain an interesting cubic as the locus of the point P for which the three Euler lines are concurrent when the antiparallels are constructed through the vertices of the cevian triangle of P.

1. Thébault's theorem on Euler lines

We begin with the following theorem of Victor Thébault [8] on the concurrency of three Euler lines.

Theorem 1 (Thébault)**.** *Let $A'B'C'$ be the orthic triangle of ABC. The Euler lines of the triangles $AB'C'$, $BC'A'$, $CA'B'$ are concurrent at the Jerabek center.*[1]

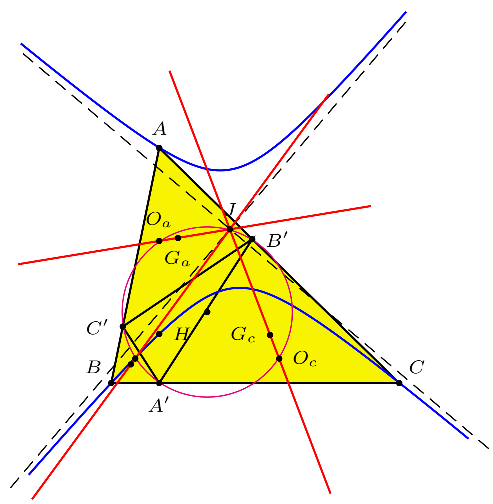


Figure 1. Thébault's theorem on the concurrency of Euler lines

We shall make use of homogeneous barycentric coordinates. With reference to triangle ABC, the vertices of the orthic triangle are the points

$$A' = (0 : S_C : S_B), \qquad B' = (S_C : 0 : S_A), \qquad C' = (S_B : S_A : 0).$$

Publication Date: February 3, 2004. Communicating Editor: Antreas P. Hatzipolakis.

[1]Thébault [8] gave an equivalent characterization of this common point. See also [7].

These are the traces of the orthocenter $H = (S_{BC} : S_{CA} : S_{AB})$.

The centroid of $AB'C'$ is the point

$$(S_{AA} + 2S_{AB} + 2S_{AC} + 3S_{BC} : S_A(S_C + S_A) : S_A(S_A + S_B)).$$

The circumcenter of $A'BC$, being the midpoint of AH, has coordinates

$$(S_{CA} + S_{AB} + 2S_{BC} : S_{AC} : S_{AB}).$$

It is straightforward to verify that these two points lie on the line

$$S_{AA}(S_B-S_C)(x+y+z) = (S_A+S_B)(S_{AB}+S_{BC}-2S_{CA})y-(S_C+S_A)(S_{BC}+S_{CA}-2S_{AB})z, \tag{1}$$

which is therefore the Euler line of triangle $AB'C'$. Furthermore, the line (1) also contains the point

$$J = (S_A(S_B - S_C)^2 : S_B(S_C - S_A)^2 : S_C(S_A - S_B)^2),$$

which is the center of the Jerabek hyperbola.[2] Similar reasoning gives the equations of the Euler lines of triangles $BC'A'$ and $A'B'C$, and shows that these contain the same point J. This completes the proof of Thébault's theorem.

2. Triangles intercepted by antiparallels

Since the sides of the orthic triangles are antiparallel to the respective sides of triangle ABC, we consider the more general situation when the residuals of the orthic triangle are replaced by triangles intercepted by lines ℓ_1, ℓ_2, ℓ_3 antiparallel to the sidelines of the reference triangle, with the following intercepts on the sidelines

	BC	CA	AB
ℓ_1		B_a	C_a
ℓ_2	A_b		C_b
ℓ_3	A_c	B_c	

These lines are parallel to the sidelines of the orthic triangle $A'B'C'$. We shall assume that they are the images of the lines $B'C'$, $C'A'$, $A'B'$ under the homotheties $\mathsf{h}(A, 1-t_1)$, $\mathsf{h}(B, 1-t_2)$, and $\mathsf{h}(C, 1-t_3)$ respectively. The points B_a, C_a etc. have homogeneous barycentric coordinates

$$\begin{array}{ll} B_a = (t_1S_A + S_C : 0 : (1-t_1)S_A), & C_a = (t_1S_A + S_B : (1-t_1)S_A : 0), \\ C_b = ((1-t_2)S_B : t_2S_B + S_A : 0), & A_b = (0 : t_2S_B + S_C : (1-t_2)S_B), \\ A_c = (0 : (1-t_3)S_C : t_3S_C + S_B), & B_c = ((1-t_3)S_C : 0 : t_3S_C + S_A). \end{array}$$

[2]The point J appears as X_{125} in [4].

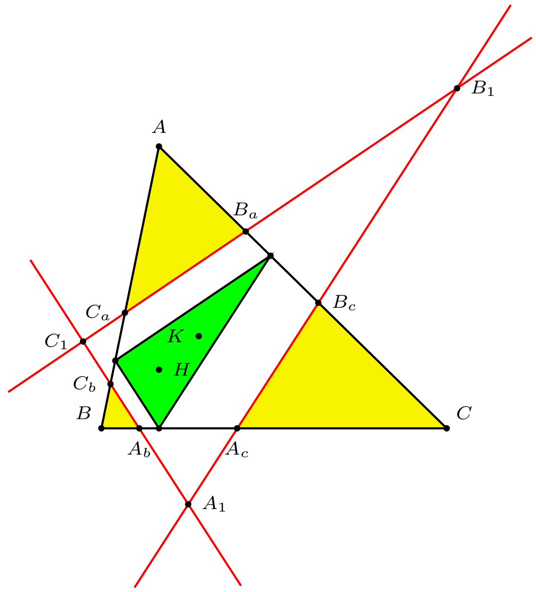


Figure 2. Triangles intercepted by antiparallels

2.1. *The Euler lines* $\mathcal{L}_i$, $i = 1, 2, 3$. Denote by $\mathbf{T}_1$ the triangle AB_aC_a intercepted by ℓ_1; similarly $\mathbf{T}_2$ and $\mathbf{T}_3$. These are oppositely similar to ABC. We shall study the condition of the concurrency of their Euler lines.

Proposition 2. *With reference to triangle ABC, the barycentric equations of the Euler lines of $\mathbf{T}_i$, $i = 1, 2, 3$, are*

$$\begin{aligned}(1-t_1)S_{AA}(S_B-S_C)(x+y+z) &= c^2(S_{AB}+S_{BC}-2S_{CA})y - b^2(S_{BC}+S_{CA}-2S_{AB})z,\\(1-t_2)S_{BB}(S_C-S_A)(x+y+z) &= a^2(S_{BC}+S_{CA}-2S_{AB})z - c^2(S_{CA}+S_{AB}-2S_{BC})x,\\(1-t_3)S_{CC}(S_A-S_B)(x+y+z) &= b^2(S_{CA}+S_{AB}-2S_{BC})x - a^2(S_{AB}+S_{BC}-2S_{CA})y.\end{aligned}$$

Proof. It is enough to establish the equation of the Euler line $\mathcal{L}_1$ of $\mathbf{T}_1$. This is the image of the Euler line $\mathcal{L}_1'$ of triangle $AB'C'$ under the homothety $\mathsf{h}(A, 1-t_1)$. A point $(x : y : z)$ on $\mathcal{L}_1$ corresponds to the point $((1-t_1)x - t_1(y+z) : y : z)$ on $\mathcal{L}_1'$. The equation of $\mathcal{L}_1$ can now by obtained from (1). □

From the equations of these Euler lines, we easily obtain the condition for their concurrency.

Theorem 3. *The three Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent if and only if*

$$t_1a^2(S_B-S_C)S_{AA} + t_2b^2(S_C-S_A)S_{BB} + t_3c^2(S_A-S_B)S_{CC} = 0. \qquad (2)$$

Proof. From the equations of $\mathcal{L}_i$, $i = 1, 2, 3$, given in Proposition 2, it is clear that the condition for concurrency is

$$(1-t_1)a^2(S_B-S_C)S_{AA} + (1-t_2)b^2(S_C-S_A)S_{BB} + (1-t_3)c^2(S_A-S_B)S_{CC} = 0.$$

This simplifies into (2) above. □

2.2. *Antiparallels with given common point of $\mathcal{L}_i$, $i = 1, 2, 3$* . We shall assume triangle ABC scalene, *i.e.*, its angles are unequal and none of them is a right angle. For such triangles, the Euler lines of the residuals of the orthic triangle and the corresponding altitudes intersect at finite points.

Theorem 4. *Given a point Q in the plane of a scalene triangle ABC, there is a unqiue triple of antiparallels ℓ_i, $i = 1, 2, 3$, for which the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent at Q.*

Proof. Construct the parallel through Q to the Euler line of $AB'C'$ to intersect the line AH at O_a. The circle through A with center O_a intersects AC and AB at B_a and C_a respectively. The line B_aC_a is parallel to $B'C'$. It follows that its Euler line is parallel to that of $AB'C'$. This is the line O_aQ. Similar constructions give the other two antiparallels with corresponding Euler lines passing through Q. □

We make a useful observation here. From the equations of the Euler lines given in Proposition 2 above, the intersection of any two of them have coordinates expressible in linear functions of t_1, t_2, t_3. It follows that if t_1, t_2, t_3 are linear functions of a parameter t, and the three Euler lines are concurrent, then as t varies, the common point traverses a straight line. In particular, $t_1 = t_2 = t_3 = t$, the Euler lines are concurrent by Theorem 3. The locus of the intersection of the Euler lines is a straight line. Since this intersection is the Jerabek center when $t = 0$ (Thébault's theorem), and the orthocenter when $t = -1$,[3] this is the line

$$\mathcal{L}_{\mathrm{c}}: \qquad \sum_{\text{cyclic}} S_{AA}(S_B - S_C)(S_{CA} + S_{AB} - 2S_{BC})x = 0.$$

We give a summary of some of the interesting loci of common points of Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, when the lines ℓ_i, $i = 1, 2, 3$, are subjected to some further conditions. In what follows, $\mathbf{T}$ denotes the triangle bounded by the lines ℓ_i, $i = 1, 2, 3$.

Line	Construction	Condition	Reference
$\mathcal{L}_{\mathrm{c}}$	HJ	$\mathbf{T}$ homothetic to orthic triangle at X_{25}	
$\mathcal{L}_{\mathrm{q}}$	Remark below	ℓ_i, $i = 1, 2, 3$, concurrent	§3.2
$\mathcal{L}_{\mathrm{t}}$	KX_{74}	ℓ_i are the antiparallels of a Tucker hexagon	§6
$\mathcal{L}_{\mathrm{f}}$	X_5X_{184}	$\mathcal{L}_i$ intersect on Euler line of $\mathbf{T}$	§7.2
$\mathcal{L}_{\mathrm{r}}$	GX_{110}	$\mathbf{T}$ and ABC perspective	§8.3

Remark. $\mathcal{L}_{\mathrm{q}}$ can be constructed as the line parallel to the Brocard axis through the intersection of the inverse of the deLongchamps point in the circumcircle.

[3]For $t = 1$, this intersection is the point X_{74} on the circumcircle, the isogonal conjugate of the infinite point of the Euler line.

3. Concurrent antiparallels

In this section we consider the case when the antiparallels ℓ_1, ℓ_2, ℓ_3 all pass through a point $P = (u : v : w)$. In this case,

$$\begin{aligned}
B_a &= ((S_C + S_A)u - (S_B - S_C)v : 0 : (S_A + S_B)v + (S_C + S_A)w),\\
C_a &= ((S_A + S_B)u + (S_B - S_C)w : (S_A + S_B)v + (S_C + S_A)w : 0),\\
C_b &= ((S_B + S_C)w + (S_A + S_B)u : (S_A + S_B)v - (S_C - S_A)w : 0),\\
A_b &= (0 : (S_B + S_C)v + (S_C - S_A)u : (S_B + S_C)w + (S_A + S_B)u),\\
A_c &= (0 : (S_C + S_A)u + (S_B + S_C)v : (S_B + S_C)w - (S_A - S_B)u),\\
B_c &= ((S_C + S_A)u + (S_B + S_C)v : 0 : (S_C + S_A)w + (S_A - S_B)v).
\end{aligned}$$

For example, when $P = K$, these are the vertices of the second cosine circle.

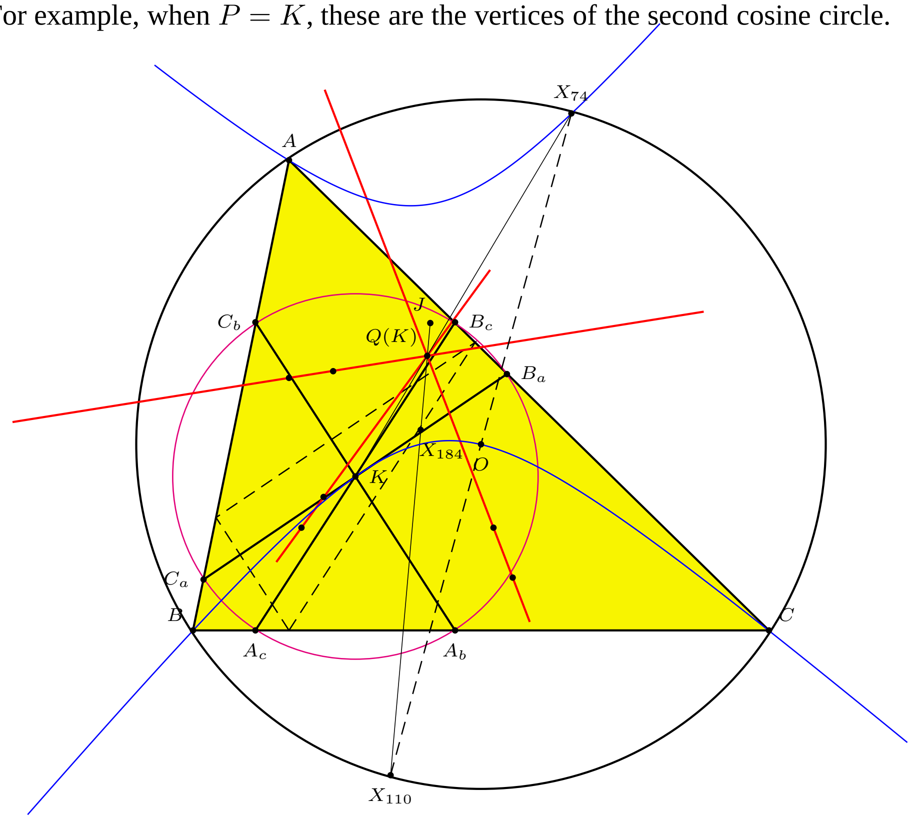


Figure 3. $Q(K)$ and the second Lemoine circle

Proposition 5. *The Euler lines of triangles $\mathbf{T}_i$, $i = 1, 2, 3$, are concurrent if and only if P lies on the line*

$$\mathcal{L}_{\rm p}: \qquad \frac{S_A(S_B - S_C)}{a^2}x + \frac{S_B(S_C - S_A)}{b^2}y + \frac{S_C(S_A - S_B)}{c^2}z = 0.$$

When P traverses $\mathcal{L}_{\rm p}$, the intersection Q of the Euler lines traverses the line

$$\mathcal{L}_{\rm q}: \qquad \sum_{\text{cyclic}} \frac{(b^2 - c^2)(a^2(S_{AA} + S_{BC}) - 4S_{ABC})}{a^2}x = 0.$$

For a point P on the line $\mathcal{L}_{\rm p}$, we denote by $Q(P)$ the corresponding point on $\mathcal{L}_{\rm q}$.

Proposition 6. *For points P_1, P_2, P_3 on $\mathcal{L}_{\rm p}$, $Q(P_1)$, $Q(P_2)$, $Q(P_3)$ are points on $\mathcal{L}_{\rm q}$ satisfying*

$$Q(P_1)Q(P_2) : Q(P_2)Q(P_3) = P_1P_2 : P_2P_3.$$

3.1. *The line $\mathcal{L}_{\rm p}$.* The line $\mathcal{L}_{\rm p}$ contains K and is the tangent to the Jerabek hyperbola at K. See Figure 4. It also contains, among others, the following points.

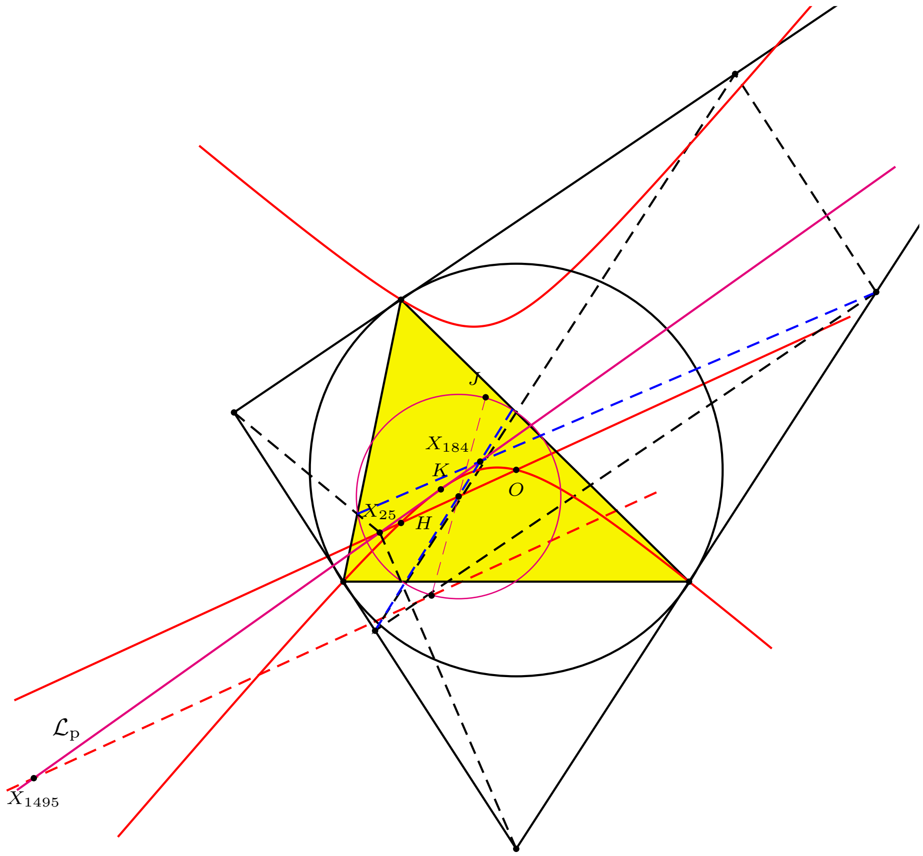


Figure 4. The line $\mathcal{L}_{\rm p}$

(1) $X_{25} = \left(\frac{a^2}{S_A} : \frac{b^2}{S_B} : \frac{c^2}{S_C}\right)$ which is on the Euler line of ABC, and is the homothetic center of the orthic and the tangential triangles,[4]

(2) $X_{184} = (a^4S_A : b^4S_B : c^4S_C)$ which is the homothetic center of the orthic triangle and the medial tangential triangle,[5]

[4] See also §4.1.

[5] For other interesting properties of X_{184}, see [6], where it is named the procircumcenter of triangle ABC.

(3) $X_{1495} = (a^2(S_{CA} + S_{AB} - 2S_{BC}) : \cdots : \cdots)$ which lies on the parallel to the Euler line through the antipode of the Jerabek center on the nine-point circle. [6]

3.2. *The line $\mathcal{L}_q$* . The line $\mathcal{L}_q$ is parallel to the Brocard axis. See Figure 5. It contains the following points.

(1) $Q(K) = (a^2 S_A(b^2c^2(S_{BB} - S_{BC} + S_{CC}) - 2a^2 S_{ABC}) : \cdots : \cdots)$. It can be constructed as the intersection of the lines joining K to X_{74}, and J to X_{110}. See Figure 3 and §6 below. The line $\mathcal{L}_q$ can therefore be constructed as the parallel through this point to the Brocard axis.

(2) $Q(X_{1495}) = (a^2 S_A(a^2 S^2 - 6S_{ABC}) : \cdots : \cdots)$, which is on the line joining O to X_{184} (on $\mathcal{L}_p$).

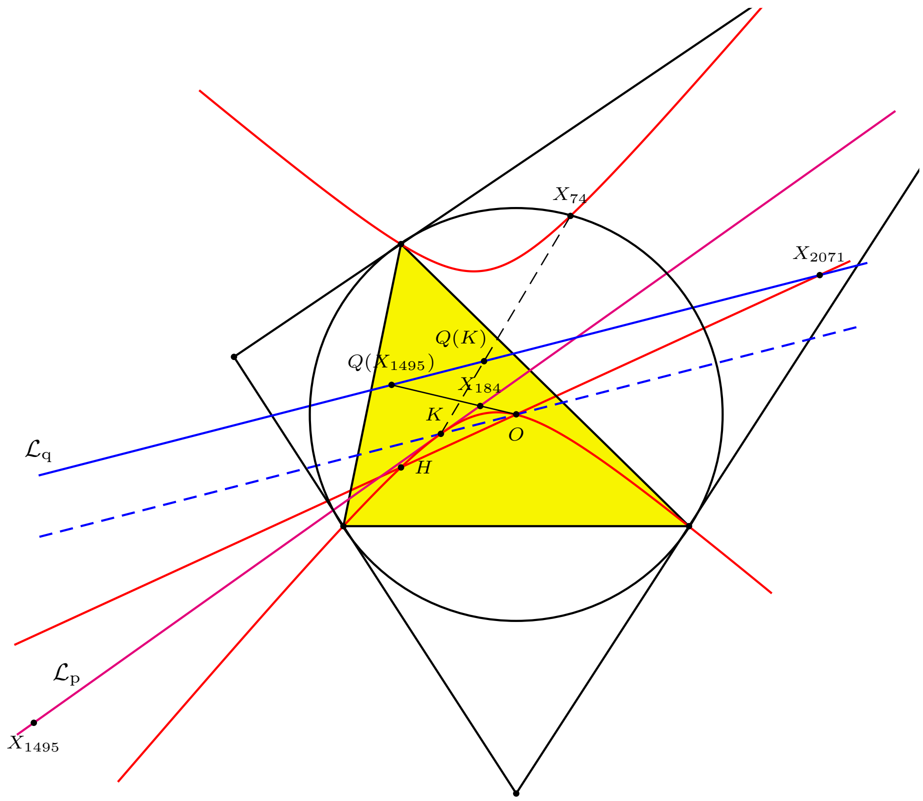


Figure 5. The line $\mathcal{L}_q$

The line $\mathcal{L}_q$ intersects the Euler line of ABC at the point

$$X_{2071} = (a^2(a^2 S_{AAA} + S_{AA}(S_{BB} - 3S_{BC} + S_{CC}) - S_{BBCC}) : \cdots : \cdots),$$

[6] This is the point X_{113}.

which is the inverse of the de Longchamps point in the circumcircle. This corresponds to the antiparallels through

$$P_{2071} = (a^4((a^2S_{AAA} + S_{AA}(S_{BB} - 3S_{BC} + S_{CC}) - S_{BBCC}) : \cdots : \cdots)$$

on the line $\mathcal{L}_{\mathrm{p}}$. This point can be constructed by a simple application of Theorem 4 or Proposition 6. (See also Remark 2 following Theorem 12).

3.3. *The intersection of $\mathcal{L}_{\mathrm{p}}$ and $\mathcal{L}_{\mathrm{q}}$.* The lines $\mathcal{L}_{\mathrm{p}}$ and $\mathcal{L}_{\mathrm{q}}$ intersect at the point

$$M = (a^2S_A(S_{AB} + S_{AC} + S_{BB} - 4S_{BC} + S_{CC}) : \cdots : \cdots).$$

(1) $Q(M)$ is the point on $\mathcal{L}_{\mathrm{q}}$ with coordinates

$$(a^2S_A(S_{AA}(S_{BB}+S_{CC})+a^2S_A(S_{BB}-3S_{BC}+S_{CC})+S_{BC}(S_B-S_C)^2) : \cdots : \cdots).$$

(2) The point P on $\mathcal{L}_{\mathrm{p}}$ for which $Q(P) = M$ has coordinates

$$(a^2(a^2(2S_{AA} - S_{BC}) + 2S_A(S_{BB} - 3S_{BC} + S_{CC})) : \cdots : \cdots).$$

4. The triangle $\mathbf{T}$ bounded by the antiparallels

We assume the line ℓ_i, $i = 1, 2, 3$, nonconcurrent so that they bound a nondegenerate triangle $\mathbf{T} = A_1B_1C_1$. Since these lines have equations

$$\begin{aligned}
-t_1S_A(x+y+z) &= -S_Ax + S_By + S_Cz,\\
-t_2S_B(x+y+z) &= S_Ax - S_By + S_Cz,\\
-t_3S_C(x+y+z) &= S_Ax + S_By - S_Cz,
\end{aligned}$$

the vertices of $\mathbf{T}$ are the points

$$\begin{aligned}
A_1 =&(-a^2(t_2S_B + t_3S_C) : 2S_{CA} + t_2b^2S_B + t_3S_C(S_C - S_A)\\
&: 2S_{AB} + t_2S_B(S_B - S_A) + t_3c^2S_C),\\
B_1 =&(2S_{BC} + t_3S_C(S_C - S_B) + t_1a^2S_A : -b^2(t_3S_C + t_1S_A)\\
&: 2S_{AB} + t_3c^2S_C + t_1S_A(S_A - S_B))\\
C_1 =&(2S_{BC} + t_1a^2S_A + t_2S_B(S_B - S_C) : 2S_{CA} + t_1S_A(S_A - S_C) + t_2b^2S_B\\
&: -c^2(t_1S_A + t_2S_B)).
\end{aligned}$$

4.1. *Homothety with the orthic triangle .* The triangle $\mathbf{T} = A_1B_1C_1$ is homothetic to the orthic triangle $A'B'C'$. The center of homothety is the point

$$P(\mathbf{T}) = \left(\frac{t_2S_B + t_3S_C}{S_A} : \frac{t_3S_C + t_1S_A}{S_B} : \frac{t_1S_A + t_2S_B}{S_C}\right), \tag{3}$$

and the ratio of homothety is

$$1 + \frac{t_1a^2S_{AA} + t_2b^2S_{BB} + t_3c^2S_{CC}}{2S_{ABC}}.$$

Proposition 7. *If the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent, the homothetic center $P(\mathbf{T})$ of $\mathbf{T}$ and the orthic triangle lies on the line $\mathcal{L}_{\mathrm{p}}$.*

Proof. If we write $P(\mathbf{T}) = (x : y : z)$. From (3), we obtain

$$t_1 = \frac{-xS_A + yS_B + zS_C}{2S_A}, \quad t_2 = \frac{-yS_B + zS_C + xS_A}{2S_B}, \quad t_3 = \frac{-zS_C + xS_A + yS_B}{2S_C}.$$

Substitution in (2) yields the equation of the line $\mathcal{L}_{\rm p}$. □

For example, if $t_1 = t_2 = t_3 = t$, $P(\mathbf{T}) = X_{25} = \left(\frac{a^2}{S_A} : \frac{b^2}{S_B} : \frac{c^2}{S_C}\right)$.[7] If the ratio of homothety is 0, triangle $\mathbf{T}$ degenerates into the point X_{25} on $\mathcal{L}_{\rm p}$. The intersection of $\mathcal{L}_{\rm c}$ and $\mathcal{L}_{\rm q}$ is the point

$$\begin{aligned} Q(X_{25}) =& (a^2S_A(b^4S_B^4 + c^4S_C^4 + a^2S_{AAA}(S_B - S_C)^2 \\ & - S_{ABC}(4a^2S_{BC} + 3S_A(S_B - S_C)^2)) : \cdots : \cdots). \end{aligned}$$

Remark. The line $\mathcal{L}_{\rm p}$ is also the locus of the centroid of $\mathbf{T}$ for which the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, concur.

4.2. *Common point of $\mathcal{L}_i$, $i = 1, 2, 3$, on the Brocard axis.* We consider the case when the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, intersect on the Brocard axis. A typical point on the Brocard axis, dividing the segment OK in the ratio $t : 1-t$, has coordinates

$$(a^2(S_A(S_A + S_B + S_C) + (S_{BC} - S_{AA})t) : \cdots : \cdots).$$

This point lies on the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, if and only if we choose

$$\begin{aligned} t_1 =& \frac{-(S_A + S_B + S_C)(S^2 - S_{AA}) + b^2c^2(S_B + S_C - 2S_A)t}{2S_{AA}(S_A + S_B + S_C)}, \\ t_2 =& \frac{-(S_A + S_B + S_C)(S^2 - S_{BB}) + c^2a^2(S_C + S_A - 2S_B)t}{2S_{BB}(S_A + S_B + S_C)}, \\ t_3 =& \frac{-(S_A + S_B + S_C)(S^2 - S_{CC}) + a^2b^2(S_A + S_B - 2S_C)t}{2S_{CC}(S_A + S_B + S_C)}. \end{aligned}$$

The corresponding triangle $\mathbf{T}$ is homothetic to the orthic triangle at the point

$$(a^2(-(S_A + S_B + S_C) \cdot a^2S_A + t(-(2S_A + S_B + S_C)S_{BC} + b^2S_{CA} + c^2S_{AB}) : \cdots : \cdots),$$

which divides the segment $X_{184}K$ in the ratio $2t : 1-2t$. The ratio of homothety is $-\frac{a^2b^2c^2}{4S_{ABC}}$. These triangles are all directly congruent to the medial tangential triangle of ABC. We summarize this in the following proposition.

Proposition 8. *Corresponding to the family of triangles directly congruent to the medial tangential triangle, homothetic to orthic triangle at points on the line $\mathcal{L}_{\rm p}$, the common points of the Euler lines of $\mathcal{L}_i$, $i = 1, 2, 3$, all lie on the Brocard axis.*

[7] See also §3.1(1). The tangential triangle is $\mathbf{T}$ with $t = 1$.

5. Perspectivity of $\mathbf{T}$ with ABC

Proposition 9. *The triangles* $\mathbf{T}$ *and* ABC *are perspective if and only if*

$$\sum_{\text{cyclic}}(S_B-S_C)(t_1S_{AA}-t_2t_3S_{BC})=0. \tag{4}$$

Proof. From the coordinates of the vertices of $\mathbf{T}$, it is straightforward to check that $\mathbf{T}$ and ABC are perspective if and only if

$$t_1a^2S_{AA}+t_2b^2S_{BB}+t_3c^2S_{CC}+2S_{ABC}=0$$

or (4) holds. Since the area of triangle $\mathbf{T}$ is

$$\frac{(t_1a^2S_{AA}+t_2b^2S_{BB}+t_3c^2S_{CC}+2S_{ABC})^2}{a^2b^2c^2S_{ABC}}$$

times that of triangle ABC, we assume $t_1a^2S_{AA}+t_2b^2S_{BB}+t_3c^2S_{CC}+2S_{ABC}\neq 0$ and (4) is the necessary and sufficient condition for perspectivity. □

Theorem 10. *If the triangle* $\mathbf{T}$ *is nondegenerate and is perspective to* ABC*, then the perspector lies on the Jerabek hyperbola of* ABC.

Proof. If triangles $A_1B_1C_1$ and ABC are perspective at $P=(x:y:z)$, then

$$A_1=(u+x:y:z),\quad B_1=(x:v+y:z),\quad C_1=(x:y:w+z)$$

for some u, v, w. Since the line B_1C_1 is parallel to $B'C'$, which has infinite point $(S_B-S_C:-(S_C+S_A):S_A+S_B)$, we have

$$\begin{vmatrix} S_B-S_C & -(S_C+S_A) & S_A+S_B \\ x & y+v & z \\ x & y & z+w \end{vmatrix}=0,$$

and similarly for the other two lines. These can be rearranged as

$$\begin{aligned}
\frac{(S_C+S_A)x-(S_B-S_C)y}{v}-\frac{(S_B-S_C)z+(S_A+S_B)x}{w} &= S_B-S_C,\\
\frac{(S_A+S_B)y-(S_C-S_A)z}{w}-\frac{(S_C-S_A)x+(S_B+S_C)y}{u} &= S_C-S_A,\\
\frac{(S_B+S_C)z-(S_A-S_B)x}{u}-\frac{(S_A-S_B)y+(S_C+S_A)z}{v} &= S_A-S_B.
\end{aligned}$$

Multiplying these equations respectively by

$$S_A(S_B+S_C)yz,\qquad S_B(S_C+S_A)zx,\qquad S_C(S_A+S_B)xy$$

and adding up, we obtain

$$\left(1+\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right)\sum_{\text{cyclic}}S_A(S_{BB}-S_{CC})yz=0.$$

Since the area of triangle $\mathbf{T}$ is

$$uvw\left(1+\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right)$$

times that of triangle ABC, we must have $1 + \frac{x}{u} + \frac{y}{v} + \frac{z}{w} \neq 0$. It follows that

$$\sum_{\text{cyclic}} S_A(S_{BB} - S_{CC})yz = 0.$$

This means that P lies on the Jerabek hyperbola. $\square$

We shall identify the locus of the common points of Euler lines in §8.3 below. In the meantime, we give a construction for the point Q from the perspector on the Jerabek hypebola.

Construction. Given a point P on the Jerabek hyperbola, construct parallels to $A'B'$ and $A'C'$ through an arbitrary point A_1' on the line AP. Let M_1 be the intersection of the Euler lines of the triangles formed by these antiparallels and the sidelines of ABC. With another point A_1'' obtain a point M_2 by the same construction. Similarly, working with two points B_1' and B_1'' on BP, we construct another line M_3M_4. The intersection of M_1M_2 and M_3M_4 is the common point Q of the Euler lines corresponding the antiparallels that bound a triangle perspective to ABC at P.

6. The Tucker hexagons and the line $\mathcal{L}_t$

It is well known that if the antiparallels, together with the sidelines of triangle ABC, bound a Tucker hexagon, the vertices lie on a circle whose center is on the Brocard axis. If this center divides the segment OK in the ratio $t : 1-t$, the antiparallels pass through the points dividing the symmedians in the same ratio. The vertices of the Tucker hexagon are

$$\begin{array}{ll} B_a = (S_C + (1-t)c^2 : 0 : tc^2), & C_a = (S_B + (1-t)b^2 : tb^2 : 0), \\ C_b = (ta^2 : S_A + (1-t)a^2 : 0), & A_b = (0 : S_C + (1-t)c^2 : tc^2), \\ A_c = (0 : tb^2 : S_B + (1-t)b^2), & B_c = (ta^2 : 0 : S_A + (1-t)a^2). \end{array}$$

In this case,

$$1-t_1 = \frac{t \cdot b^2c^2}{S_A(S_A+S_B+S_C)}, \quad 1-t_2 = \frac{t \cdot c^2a^2}{S_B(S_A+S_B+S_C)}, \quad 1-t_3 = \frac{t \cdot a^2b^2}{S_C(S_A+S_B+S_C)}.$$

It is clear that the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent. As t varies, this common point traverses a straight line $\mathcal{L}_t$. We show that this is the line joining K to $Q(K)$.

(1) For $t = 1$, this Tucker circle is the second Lemoine circle with center K, the triangle $\mathbf{T}$ degenerates into the point K. The common point of the Euler lines is therefore the point $Q(K)$. See §3.2 and Figure 3.

(2) For $t = \frac{3}{2}$, the vertices of the Tucker hexagon are

$$\begin{array}{ll} B_a = (a^2 + b^2 - 2c^2 : 0 : 3c^2), & C_a = (c^2 + a^2 - 2b^2 : 3b^2 : 0), \\ C_b = (3a^2 : b^2 + c^2 - 2a^2 : 0), & A_b = (0 : a^2 + b^2 - 2c^2 : 3c^2), \\ A_c = (0 : 3b^2 : c^2 + a^2 - 2b^2), & B_c = (3a^2 : 0 : b^2 + c^2 - 2a^2). \end{array}$$

The triangles $\mathbf{T}_i$, $i = 1, 2, 3$, have a common centroid K, which is therefore the common point of their Euler lines. The corresponding Tucker center is the point X_{576} (which divides OK in the ratio $3 : -1$).

From these, we obtain the equation of the line

$$\mathcal{L}_{\rm t} : \qquad \sum_{\rm cyclic} b^2c^2 S_A(S_B - S_C)(S_{CA} + S_{AB} - 2S_{BC})x = 0.$$

Remarks. (1) The triangle $\mathbf{T}$ is perspective to ABC at K. See, for example, [5].

(2) The line $\mathcal{L}_{\rm t}$ also contains X_{74} which we may regard as corresponding to $t = 0$.

For more about Tucker hexagons, see §8.2.

7. Concurrency of four or more Euler lines

7.1. *Common point of $\mathcal{L}_i$, $i = 1, 2, 3$, on the Euler line of ABC.* We consider the case when the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, intersect on the Euler line of ABC. A typical point on the Euler line axis divides the segment OH in the ratio $t : 1 - t$, has coordinates

$$(a^2S_A - (S_{CA} + S_{AB} - 2S_{BC})t) : \cdots : \cdots).$$

This lies on the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, if and only if we choose

$$\begin{aligned}
t_1 =& \frac{-(S^2 - S_{AA}) + (S^2 - 3S_{AA})t}{2S_{AA}},\\
t_2 =& \frac{-(S^2 - S_{BB}) + (S^2 - 3S_{BB})t}{2S_{BB}},\\
t_3 =& \frac{-(S^2 - S_{CC}) + (S^2 - 3S_{CC})t}{2S_{CC}}.
\end{aligned}$$

Independently of t, the corresponding triangle $\mathbf{T}$ is always homothetic to the medial tangential triangle at the point P_{2071} on the line $\mathcal{L}_{\rm p}$ for which $Q(P_{2071}) = X_{2071}$, the intersection of $\mathcal{L}_{\rm q}$ with the Euler line. See the end of §3.2 above. The ratio of homothety is $1 + t - \frac{8S_{ABC}}{a^2b^2c^2}t$. We summarize this in the following proposition.

Proposition 11. *Let P_{2071} be the point on $\mathcal{L}_{\rm p}$ such that $Q(P_{2071}) = X_{2071}$. The Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, corresponding to the sidelines of triangles homothetic at P_{2071} to the medial tangential triangle intersect on the Euler line of ABC.*

7.2. *The line $\mathcal{L}_{\rm f}$.* The Euler line of triangle $\mathbf{T}$ is the line

$$\begin{aligned}
&(x + y + z) \sum_{\rm cyclic} t_1 a^2 S_{AA}(S_B - S_C)(S^2 + S_{BC})(S^2 - S_{AA})\\
=&2S_{ABC} \sum_{\rm cyclic} (S^2 + S_{CA})(S^2 + S_{AB})x. \qquad (5)
\end{aligned}$$

Theorem 12. *The Euler lines of the four triangles* $\mathbf{T}$ *and* $\mathbf{T}_i$, $i = 1, 2, 3$, *are concurrent if and only if*

$$t_1 = -\frac{16S^2 \cdot S_{ABC} + t(a^2b^4c^4 - 4S_{ABC}(3S^2 - S_{AA}))}{4S_{AA}(a^2b^2c^2 + 4S_{ABC})},$$
$$t_2 = -\frac{16S^2 \cdot S_{ABC} + t(a^4b^2c^4 - 4S_{ABC}(3S^2 - S_{BB}))}{4S_{BB}(a^2b^2c^2 + 4S_{ABC})},$$
$$t_3 = -\frac{16S^2 \cdot S_{ABC} + t(a^4b^4c^2 - 4S_{ABC}(3S^2 - S_{CC}))}{4S_{CC}(a^2b^2c^2 + 4S_{ABC})},$$

with $t \neq \frac{-24a^2b^2c^2S_{ABC}}{(a^2b^2c^2-8S_{ABC})(3(S_A+S_B+S_C)S^2+S_{ABC})}$. *The locus of the common point of the four Euler lines is the line* $\mathcal{L}_\mathrm{f}$ *joining the nine-point center of* ABC *to* X_{184}, *with the intersection with* $\mathcal{L}_\mathrm{q}$ *deleted.*

Proof. The equation of the Euler line $\mathcal{L}_i$, $i = 1, 2, 3$, can be rewritten as

$$\begin{aligned} &t_1S_A(S_B - S_C)(x + y + z) + S_{AA}(S_B - S_C)x \\ &+(S_{AB}(S_B - S_C) - (S_{AA} - S_{BB})S_C)y + (S_{AC}(S_B - S_C) + (S_{AA} - S_{CC})S_B)z = 0, \quad (6) \\ &t_2S_A(S_B - S_C)(x + y + z) + S_{BB}(S_C - S_A)y \\ &+(S_{BA}(S_C - S_A) + (S_{BB} - S_{AA})S_C)x + (S_{BC}(S_C - S_A) - (S_{BB} - S_{CC})S_A)z = 0, \quad (7) \\ &t_3S_C(S_A - S_B)(x + y + z) + S_{CC}(S_A - S_B)z \\ &+(S_{CA}(S_A - S_B) - (S_{CC} - S_{AA})S_B)x + (S_{CB}(S_A - S_B) + (S_{CC} - S_{BB})S_A)y = 0. \quad (8) \end{aligned}$$

Multiplying (4), (5), (6) respectively by

$$a^2S_A(S^2+S_{BC})(S^2-S_{AA}), \quad b^2S_B(S^2+S_{CA})(S^2-S_{BB}), \quad c^2S_C(S^2+S_{AB})(S^2-S_{CC}),$$

and adding, we obtain by Theorem 10 the equation of the line

$$\mathcal{L}_\mathrm{f} : \qquad \sum_{\text{cyclic}} (S_B - S_C)(S^2(2S_{AA} - S_{BC}) + S_{ABC} \cdot S_A)x = 0$$

which contains the common point of the Euler lines of $\mathbf{T}_i$, $i = 1, 2, 3$, if it also lies on the Euler line $\mathcal{L}$ of $\mathbf{T}$. The line $\mathcal{L}_\mathrm{f}$ contains the nine-point center X_5 and $X_{184} = (a^4S_A : b^4S_B : c^4S_C)$. Let Q_t be the point which divides the segment $X_{184}X_5$ in the ratio $t : 1 - t$. It has coordinates

$$\begin{aligned} &((1 - t)4S^2 \cdot a^4S_A + t(a^2b^2c^2 + 4S_{ABC})(S_{CA} + S_{AB} + 2S_{BC}) \\ &: (1 - t)4S^2 \cdot b^4S_B + t(a^2b^2c^2 + 4S_{ABC})(2S_{CA} + S_{AB} + S_{BC}) \\ &: (1 - t)4S^2 \cdot c^4S_C + t(a^2b^2c^2 + 4S_{ABC})(S_{CA} + 2S_{AB} + S_{BC})). \end{aligned}$$

The point Q_t lies on the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, respectively if we choose t_1, t_2, t_3 given above.

If Q lies on $\mathcal{L}_{\rm q}$, then $Q_t = Q(P)$ for some point P on $\mathcal{L}_{\rm p}$.[8] In this case, the triangle $\mathbf{T}$ degenerates into the point $P \neq Q$ and its Euler line is not defined. It should be excluded from $\mathcal{L}_{\rm f}$. The corresponding value of t is as given in the statement above. □

Here are some interesting points on $\mathcal{L}_{\rm f}$.

(1) For $t = 0$, $\mathbf{T}$ is perspective with ABC at X_{74}, and the common point of the four Euler lines is X_{184}. The antiparallels are drawn through the intercepts of the trilinear polars of $X_{186} = \left(\frac{a^2}{S_A(S^2-3S_{AA})} : \cdots : \cdots\right)$, the inversive image of the orthocenter in the circumcircle.

(2) For $t = 1$, this common point is the nine-point of triangle ABC. The triangle $\mathbf{T}$ is homothetic to the orthic triangle at X_{51} and to the medial tangential triangle at the point P_{2071} in §3.2.

(3) $t = -\frac{a^2b^2c^2}{4S_{ABC}}$ gives X_{156}, the nine-point center of the tangential triangle.

In these two cases, we have the concurrency of five Euler lines.

(4) The line $\mathcal{L}_{\rm f}$ intersects the Brocard axis at X_{569}. This corresponds to $t = \frac{2a^2b^2c^2}{3a^2b^2c^2+4S_{ABC}}$.

Proposition 13. *The triangle $\mathbf{T}$ is perspective with ABC and its Euler line contains the common point of the Euler lines of $\mathbf{T}_i$, $i = 1, 2, 3$ precisely in the following three cases.*

(1) $t = 0$, *with perspector X_{74} and common point of Euler line X_{184}.*

(2) $t = \frac{-12a^2b^2c^2S_{ABC}}{a^4b^4c^4-12a^2b^2c^2S_{ABC}-16(S_{ABC})^2}$, *with perspector K.*

Remarks. (1) In the first case,

$$t_1 = \frac{k}{S_{AA}}, \qquad t_2 = \frac{k}{S_{BB}}, \qquad t_3 = \frac{k}{S_{CC}}$$

for $k = -\frac{4S^2 \cdot S_{ABC}}{a^2b^2c^2+4S_{ABC}}$. The antiparallels pass through the intercepts of the trilinear polar of X_{186}, the inversive image of H in the circumcircle.

(2) In the second case, the antiparallels bound a Tucker hexagon. The center of the Tucker circle divides OK in the ratio $t : 1-t$, where

$$t = \frac{S^2(S_A+S_B+S_C)(a^2b^2c^2-16S_{ABC})}{a^4b^4c^4-12a^2b^2c^2S_{ABC}-16(S_{ABC})^2}.$$

It follows that the common point of the Euler lines is the intersection of the lines $\mathcal{L}_{\rm f} = X_5X_{184}$ and $\mathcal{L}_{\rm t}$.

8. Common points of $\mathcal{L}_i$, $i = 1, 2, 3$, when $\mathbf{T}$ is perspective

If the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent, then, according to (2) we may put

$$t_1 = \frac{k(\lambda+S_A)}{a^2S_{AA}}, \qquad t_2 = \frac{k(\lambda+S_B)}{b^2S_{BB}}, \qquad t_3 = \frac{k(\lambda+S_C)}{c^2S_{CC}}$$

[8] This point is the intersection of $\mathcal{L}_{\rm p}$ with the line joining the Jerabek center J to X_{323}, the reflection in X_{110} of the inversive image of the centroid in the circumcircle.

for some λ and k. If, also, the $\mathbf{T}$ is perspective, (4) gives

$$k(k\lambda + S_{ABC})(\lambda + S_A + S_B + S_C)(k(3\lambda + S_A + S_B + S_C) + 2S_{ABC}) = 0.$$

If $k = 0$, $\mathbf{T}$ is the orthic triangle. We consider the remaining three cases below.

8.1. *The case* $k(S_A + S_B + S_C + 3\lambda) + 2S_{ABC} = 0$. In this case,

$$\begin{aligned} t_1 &= -\frac{2S_{ABC} + k(S_B + S_C - 2S_A)}{3a^2S_{AA}}, \\ t_2 &= -\frac{2S_{ABC} + k(S_C + S_A - 2S_B)}{3b^2S_{BB}}, \\ t_3 &= -\frac{2S_{ABC} + k(S_A + S_B - 2S_C)}{3c^2S_{CC}}. \end{aligned}$$

The antiparallels are concurrent.

8.2. *The case* $k\lambda + S_{ABC} = 0$. In this case,

$$t_1 = \frac{k - S_{BC}}{a^2S_A}, \qquad t_2 = \frac{k - S_{CA}}{b^2S_B}, \qquad t_3 = \frac{k - S_{AB}}{c^2S_C}.$$

In this case, the perspector is the Lemoine point K. The antiparallels bound a Tucker hexagon. The locus of the common point of Euler lines is the line $\mathcal{L}_{\mathrm{t}}$. Here are some more interesting points on this line.

(1) For $k = 0$, we have

$$t_1 = -\frac{S_{BC}}{S_A(S_B + S_C)}, \qquad t_2 = -\frac{S_{CA}}{S_B(S_C + S_A)}, \qquad t_3 = -\frac{S_{AB}}{S_C(S_A + S_B)}.$$

This gives the Tucker hexagon with vertices

$$\begin{aligned} B_a &= (S_{CC} : 0 : S^2), \quad & C_a &= (S_{BB} : S^2 : 0), \\ C_b &= (S^2 : S_{AA} : 0), \quad & A_b &= (0 : S_{CC} : S^2), \\ A_c &= (0 : S^2 : S_{BB}), \quad & B_c &= (S^2 : 0 : S_{AA}). \end{aligned}$$

These are the pedals of A', B', C' on the sidelines. The Tucker circle is the Taylor circle. The triangle $\mathbf{T}$ is the medial triangle of the orthic triangle. The corresponding Euler lines intersect at X_{974}, which is the intersection of $\mathcal{L}_{\mathrm{t}} = KX_{74}$ with X_5X_{125}. See [2].

(2) For $k = \frac{S_{ABC}}{S_A+S_B+S_C}$, we have

$$t_1 = -\frac{S_{BC}}{S_A(S_A + S_B + S_C)}, \qquad t_2 = -\frac{S_{CA}}{S_B(S_A + S_B + S_C)}, \qquad t_3 = -\frac{S_{AB}}{S_C(S_A + S_B + S_C)}.$$

The Tucker circle is the second Lemoine circle, considered in §6.

(3) The line $\mathcal{L}_{\mathrm{t}}$ intersects the Euler line at

$$X_{378} = \left(\frac{a^2(S^2 + 3S_{AA})}{S_A} : \cdots : \cdots\right).$$

The corresponding Tucker circle has center

$$(S^2(S_B+S_C)(S_C-S_A)(S_A-S_B)+3(S_A+S_B)(S_B+S_C)(S_C+S_A)S_{BC}:\cdots:\cdots)$$

which is the intersection of the Brocard axis and the line joining the orthocenter to X_{110}.

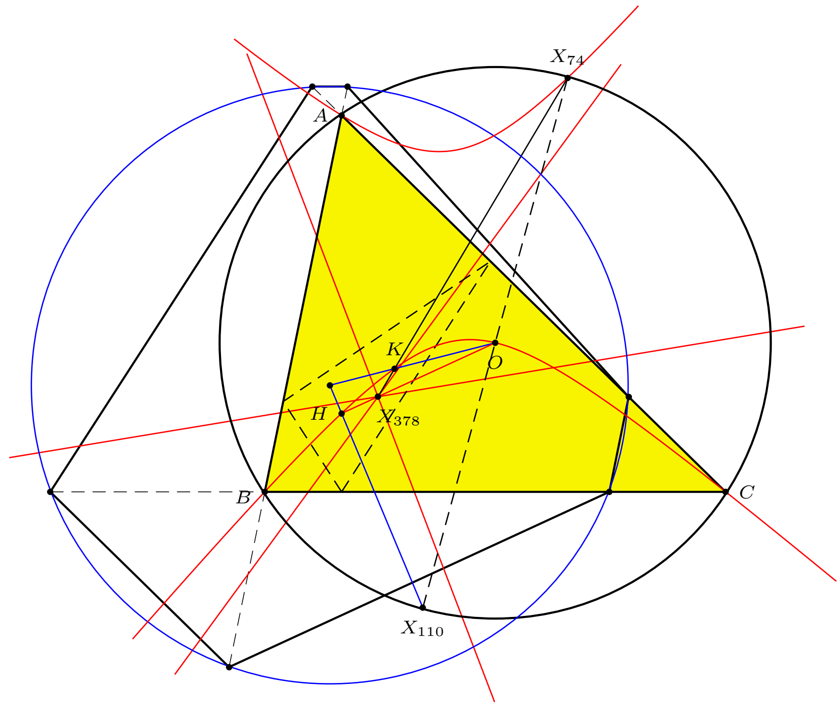


Figure 6. Intersection of 4 Euler lines at X_{378}

8.3. *The case* $\lambda = -(S_A+S_B+S_C)$. In this case, we have

$$t_1 = -\frac{k}{S_{AA}}, \qquad t_2 = -\frac{k}{S_{BB}}, \qquad t_3 = -\frac{k}{S_{CC}}.$$

In this case, the perspector is the point

$$\left(\frac{1}{2S_{ABC}\cdot S_A - k(b^2c^2-2S_{BC})}:\cdots:\cdots\right)$$

on the Jerabek hyperbola. If the point on the Jerabek hyperbola is the isogonal conjugate of the point which divides OH in the ratio $t:1-t$, then

$$k = \frac{4tS^2\cdot S_{ABC}}{a^2b^2c^2(1+t)+4t\cdot S_{ABC}}.$$

The locus of the intersection of the Euler lines $\mathcal{L}_i$, $i=1,2,3$, is clearly a line. Since this intersection is the Jerabek center for $k=0$ (Thébault's theorem) and the centroid for $k = \frac{S^2}{3}$, this is the line

$$\mathcal{L}_{\mathrm{r}}: \quad \sum_{\text{cyclic}} (S_B - S_C)(S_{BC} - S_{AA})x = 0.$$

This line also contains, among other points, X_{110} and X_{184}. We summarize the general situation in the following theorem.

Theorem 14. *Let P be a point on the Euler line other than the centroid G. The antiparallels through the intercepts of the trilinear polar of P bound a triangle perspective with ABC (at a point on the Jerabek hyperbola). The Euler lines of the triangles $\mathbf{T}_i$, $i = 1, 2, 3$, are concurrent (at a point Q on the line L_{r} joining the centroid G to X_{110}).*

Here are some interesting examples with P easily constructed on the Euler line.

P	Perspector	Q
H	H	X_{125}
O	$X_{64} = X_{20}^*$	X_{110}
X_{30}	X_{2071}^*	G
X_{186}	X_{74}	X_{184}
X_{403}	$X_{265} = X186^*$	X_{1899}
X_{23}	$X_{1177} = X_{858}^*$	X_{182}
X_{858}		X_{1352}
X_{1316}		X_{98}

Remarks. (1) X_{186} is the inversive image of H in the circumcircle.
(2) X_{403} is the midpoint between H and X_{186}.
(3) X_{23} is the inversive image of G in the circumcircle.
(4) X_{858} is the inferior of X_{23}.
(5) X_{182} is the midpoint of OK, the center of the Brocard circle.
(6) X_{1352} is the reflection of K in the nine-point center.
(7) X_{1316} is the intersection of the Euler line and the Brocard circle apart from O.

9. Two loci: a line and a cubic

We conclude this paper with a brief discussion on two locus problems.

9.1. *Antiparallels through the vertices of a pedal triangle.* Suppose the antiparallels ℓ_i, $i = 1, 2, 3$, are constructed through the vertices of the pedal triangle of a finite point P. Then the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent if and only if P lies on the line

$$\sum_{\text{cyclic}} S_A(S_B - S_C)(S_{AA} - S_{BC})x = 0.$$

This is the line containing H and the Tarry point X_{98}. For $P = H$, the common point of the Euler line is

$$X_{185} = (a^2 S_A(S_A(S_{BB} + S_{CC}) + a^2 S_{BC}) : \cdots : \cdots).$$

9.2. *Antiparallels through the vertices of a cevian triangle.* If, instead, the antiparallels ℓ_i, $i = 1, 2, 3$, are constructed through the vertices of the cevian triangle of P, then the locus of P for which the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent is the cubic

$$\mathcal{K}: \quad \frac{S_A + S_B + S_C}{S_{ABC}} xyz + \sum_{\text{cyclic}} \frac{x}{S_A(S_B - S_C)} \left(\frac{S_A + S_B}{S_C} y^2 - \frac{S_C + S_A}{S_B} z^2 \right) = 0.$$

This can also be written in the form

$$\begin{aligned} &\Big(\sum_{\text{cyclic}} (S_B + S_C) yz\Big)\Big(\sum_{\text{cyclic}} S_A(S_B - S_C)(S_B + S_C - S_A)x\Big) \\ =&\Big(\sum_{\text{cyclic}} S_A(S_B - S_C)x\Big)\Big(\sum_{\text{cyclic}} S_A(S_B + S_C)yz\Big). \end{aligned}$$

From this, we obtain the following points on $\mathcal{K}$:

- the orthocenter H (as the intersection of the Euler line and the line $\sum_{\text{cyclic}} S_A(S_B - S_C)(S_B + S_C - S_A)x) = 0$),
- the Euler reflection point X_{110} (as the "fourth" intersection of the circumcircle and the circumconic $\sum_{\text{cyclic}} S_A(S_B + S_C)yz = 0$ with center K),
- the intersections of the Euler line with the circumcircle, the points X_{1113} and X_{1114}.

Corresponding to $P = X_{110}$, the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, intersect at the circumcenter O. On the other hand, X_{1113} and X_{1114} are the points

$$(a^2 S_A + \lambda(S_{CA} + S_{AB} - 2S_{BC}) : \cdots : \cdots)$$

for $\lambda = -\frac{abc}{\sqrt{a^2b^2c^2 - 8S_{ABC}}}$ and $\lambda = \frac{abc}{\sqrt{a^2b^2c^2 - 8S_{ABC}}}$ respectively. The antiparallels through the traces of each of these points correspond to

$$t_1 = t_2 = t_3 = \frac{\lambda - 1}{\lambda + 1}.$$

This means that the corresponding intersections of Euler lines lie on the line $\mathcal{L}_c = HJ$ in §2.2.

9.3. *The cubic $\mathcal{K}$.* The infinite points of the cubic $\mathcal{K}$ can be found by rewriting the equation of $\mathcal{K}$ in the form

$$\begin{aligned} &\Big(\sum_{\text{cyclic}} S_A(S_B - S_C)(S_B + S_C)yz\Big)\Big(\sum_{\text{cyclic}} (S_B + S_C)x\Big) \\ =&(x + y + z)\Big(\sum_{\text{cyclic}} (S_B + S_C)(S_B - S_C)(S_A(S_A + S_B + S_C) - S_{BC})yz\Big) \end{aligned}$$

They are the infinite points of the Jerabek hyperbola and the line $(S_B + S_C)x + (S_C + S_A)y + (S_A + S_B)z = 0$. The latter is $X_{523} = (S_B - S_C : S_C - S_A : S_A - S_B)$. The asymptotes of $\mathcal{K}$ are

- the parallels to the asymptotes of Jerabek hyperbola through the antipode the Jerabek center on the nine-point circle, *i.e.*,

$$X_{113} = ((S_{CA}+S_{AB}-2S_{BC})(b^2S_{BB}+c^2S_{CC}-a^2S_{AA}-2S_{ABC}) : \cdots : \cdots),$$

- the perpendicular to the Euler line (of ABC) at the circumcenter O, intersecting $\mathcal{K}$ again at

$$Z = \left(\frac{S_{CA}+S_{AB}-2S_{BC}}{b^2S_{BB}+c^2S_{CC}-a^2S_{AA}-2S_{ABC}} : \cdots : \cdots\right),$$

  which also lies on the line joining H to X_{110}. See Figure 7. [9]

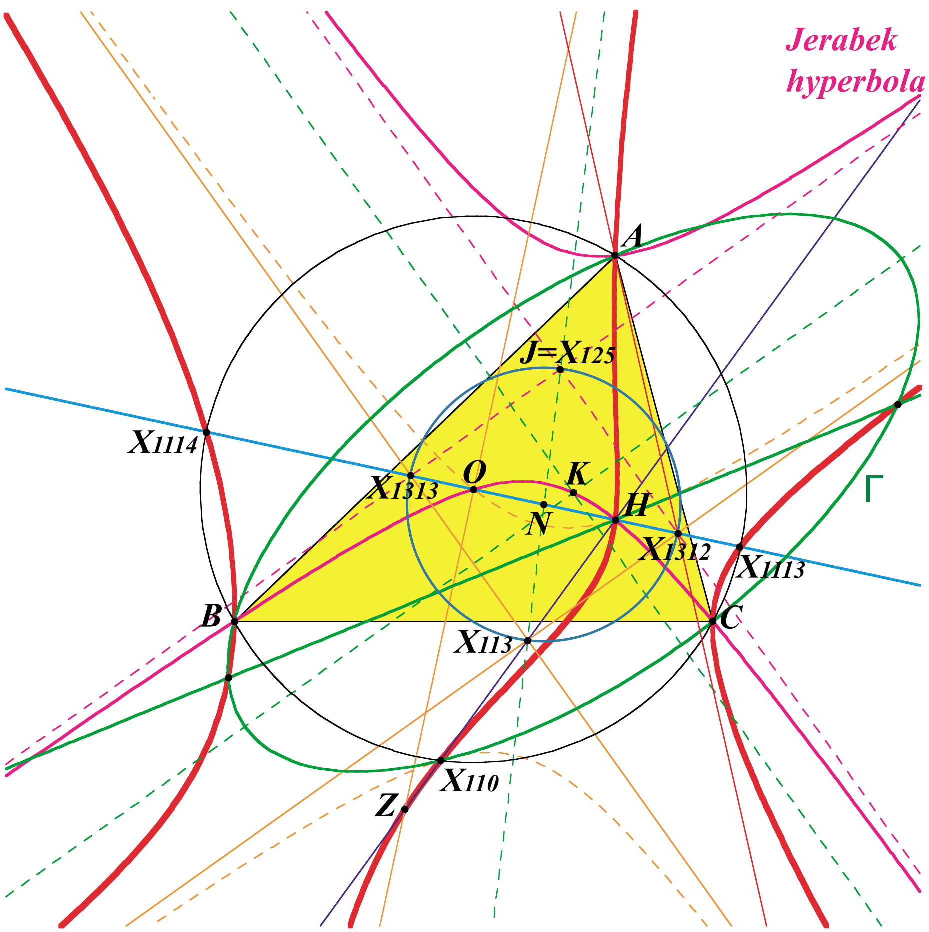


Figure 7. The cubic $\mathcal{K}$

[9] We thank Bernard Gibert for providing the sketch of $\mathcal{K}$ in Figure 7.

Remark. The asymptotes of $\mathcal{K}$ and the Jerabek hyperbola bound a rectangle inscribed in the nine-point circle. Two of the vertices and $J = X_{125}$ and its antipode X_{113}. The other two are the points X_{1312} and X_{1313} on the Euler line.

References

[1] N. Dergiades, Hyacinthos 7777, September 4, 2003.
[2] J.-P. Ehrmann, Hyacinthos 3693, September 1, 2001.
[3] D. Grinberg, Hyacinthos 7781, September 4, 2003.
[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[5] F. M. van Lamoen, Concurrencies from Tucker hexagons, *Forum Geom.*, 2 (2002) 5–13.
[6] A. Myakishev, On the procircumcenter and related points, *Forum Geom.*, 3 (2003) 29–34.
[7] V. Thébault, Concerning the Euler line of a triangle, *Amer. Math. Monthly*, 54 (1947) 447–453.
[8] V. Thébault, O. J. Rammler, and R. Goormaghtigh, Problem 4328, *Amer. Math. Monthly*, 56 (1949) 39; solution, (1951) 45.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: `ndergiades@yahoo.gr`

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida, 33431-0991, USA
E-mail address: `yiu@fau.edu`

Forum Geometricorum
Volume 4 (2004) 21–22.

FORUM GEOM
ISSN 1534-1178

Another 5-step Division of a Segment in the Golden Section

Kurt Hofstetter

Abstract. We give one more 5-step division of a segment into golden section, using ruler and compass only.

Inasmuch as we have given in [1, 2] 5-step constructions of the golden section we present here another very simple method using ruler and compass only. It is fascinating to discover how simple the golden section appears. For two points P and Q, we denote by $P(Q)$ the circle with P as center and PQ as radius.

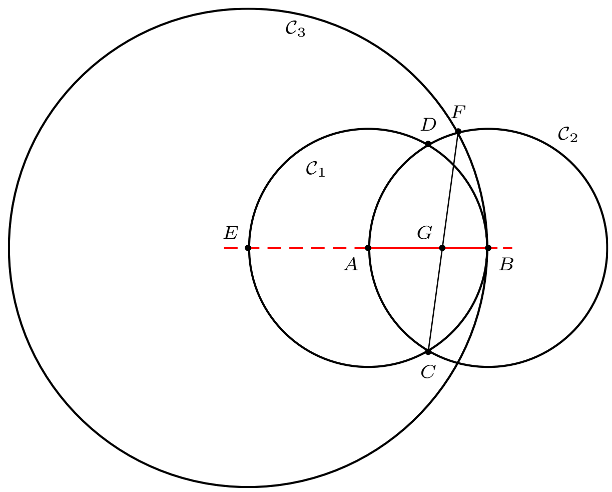


Figure 1

Construction. Given a segment AB, construct

(1) $\mathcal{C}_1 = A(B)$,
(2) $\mathcal{C}_2 = B(A)$, intersecting $\mathcal{C}_1$ at C and D,
(3) the line AB to intersect $\mathcal{C}_1$ at E (apart from B),
(4) $\mathcal{C}_3 = E(B)$ to intersect $\mathcal{C}_2$ at F (so that C and F are on opposite sides of AB),
(5) the segment CF to intersect AB at G.

The point G divides the segment AB in the golden section.

Publication Date: February 10, 2004. Communicating Editor: Paul Yiu.

Proof. Suppose AB has unit length. It is enough to show that $AG = \frac{1}{2}(\sqrt{5}-1)$.

Extend BA to intersect $\mathcal{C}_3$ at H. Let CD intersect AB at I, and let J be the orthogonal projection of F on AB. In the right triangle HFB, $BH = 4$, $BF = 1$. Since $BF^2 = BJ \times BH$, $BJ = \frac{1}{4}$. Therefore, $IJ = \frac{1}{4}$. It also follows that $JF = \frac{1}{4}\sqrt{15}$.

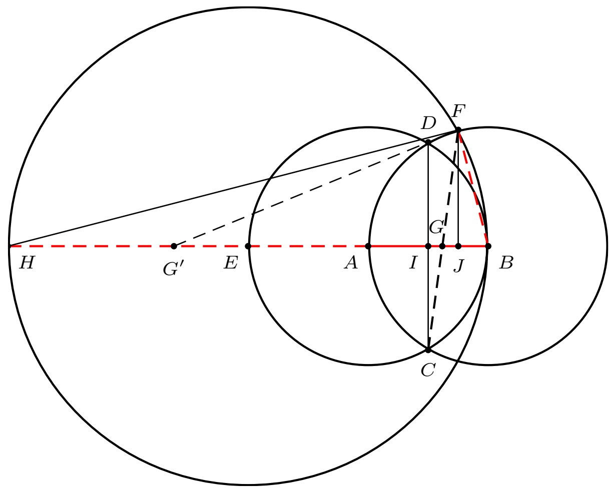


Figure 2

Now, $\frac{IG}{GJ} = \frac{IC}{JF} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{4}\sqrt{15}} = \frac{2}{\sqrt{5}}$. It follows that $IG = \frac{2}{\sqrt{5}+2} \cdot IJ = \frac{\sqrt{5}-2}{2}$, and $AG = \frac{1}{2} + IG = \frac{\sqrt{5}-1}{2}$. This shows that G divides AB in the golden section. $\square$

Remark. If FD is extended to intersect AH at G', then G' is such that $G'A : AB = \frac{1}{2}(\sqrt{5}+1) : 1$.

After the publication of [2], Dick Klingens and Marcello Tarquini have kindly written to point out that the same construction had appeared in [3, p.51] and [4, S.37] almost one century ago.

References

[1] K. Hofstetter, A simple construction of the golden section, *Forum Geom.*, 2 (2002) 65–66.
[2] K. Hofstetter, A 5-step division of a segment in the golden section, *Forum Geom.*, 3 (2003) 205–206.
[3] E. Lemoine, *Géométrographie ou Art des Constructions Géométriques*, C. Naud, Paris, 1902.
[4] J. Reusch, *Planimetricsche Konstruktionen in Geometrographischer Ausführung*, Teubner, Leipzig, 1904.

Kurt Hofstetter: Object Hofstetter, Media Art Studio, Langegasse 42/8c, A-1080 Vienna, Austria
E-mail address: pendel@sunpendulum.at

Forum Geometricorum
Volume 4 (2004) 23–26.

FORUM GEOM
ISSN 1534-1178

Extreme Areas of Triangles in Poncelet's Closure Theorem

Mirko Radić

Abstract. Among the triangles with the same incircle and circumcircle, we determine the ones with maximum and miniumum areas. These are also the ones with maximum and minimum perimeters and sums of altitudes.

Given two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ of radii r and R whose centers are at a distance d apart satisfying Euler's relation

$$R^2 - d^2 = 2Rr, \tag{1}$$

by Poncelet's closure theorem, for every point A_1 on the circle $\mathcal{C}_2$, there is a triangle $A_1A_2A_3$ with incircle $\mathcal{C}_1$ and circumcircle $\mathcal{C}_2$. In this article we determine those triangles with extreme areas, perimeters, and sum of altitudes.

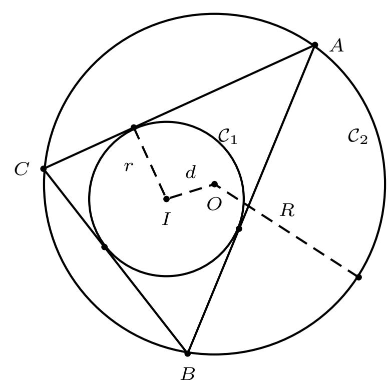


Figure 1a

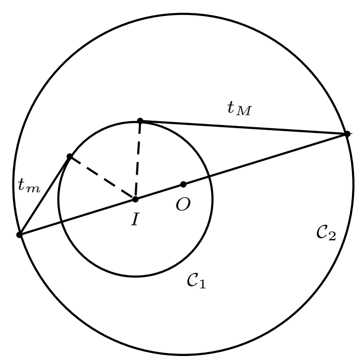


Figure 1b

Denote by t_m and t_M respectively the lengths of the shortest and longest tangents that can be drawn from C_2 to C_1. These are given by

$$t_m = \sqrt{(R-d)^2 - r^2}, \qquad t_M = \sqrt{(R+d)^2 - r^2}. \tag{2}$$

We shall use the following result given in [2, Theorem 2.2]. Let t_1 be any given length satisfying

$$t_m \le t_1 \le t_M, \tag{3}$$

Publication Date: February 23, 2004. Communicating Editor: Paul Yiu.

and let t_2 and t_3 be given by

$$t_2 = \frac{2Rrt_1 + \sqrt{D}}{r^2 + t_1^2}, \qquad t_3 = \frac{2Rrt_1 - \sqrt{D}}{r^2 + t_1^2}, \tag{4}$$

where

$$D = 4R^2r^2t_1^2 - r^2(r^2 + t_1^2)(4Rr + r^2 + t_1^2).$$

Then there is a triangle $A_1A_2A_3$ with incircle $\mathcal{C}_1$ and circumcircle $\mathcal{C}_2$ with side lengths

$$a_i = |A_iA_{i+1}| = t_i + t_{i+1}, \qquad i = 1, 2, 3. \tag{5}$$

Here, the indices are taken modulo 3. It is easy to check that

$$\begin{aligned} (t_1 + t_2 + t_3)r^2 =& t_1t_2t_3, \\ t_1t_2 + t_2t_3 + t_3t_1 =& 4Rr + r^2, \end{aligned}$$

and that these are necessary and sufficient for C_1 and C_2 to be the incircle and circumcircle of triangle $A_1A_2A_3$.

Denote by $J(t_1)$ the area of triangle $A_1A_2A_3$. Thus,

$$J(t_1) = r(t_1 + t_2 + t_3). \tag{6}$$

Note that $D = 0$ when $t_1 = t_m$ or $t_1 = t_M$. In these cases,

$$t_2 = t_3 = \begin{cases} \frac{2Rrt_m}{r^2+t_m^2}, & \text{if } t_1 = t_m, \\ \\ \frac{2Rrt_M}{r^2+t_M^2}, & \text{if } t_1 = t_M. \end{cases}$$

For convenience, we shall write

$$\widehat{t_m} = \frac{2Rrt_m}{r^2 + t_m^2} \quad \text{and} \quad \widehat{t_M} = \frac{2Rrt_M}{r^2 + t_M^2}. \tag{7}$$

Theorem 1. *$J(t_1)$ is maximum when $t_1 = t_M$ and minimum when $t_1 = t_m$. In other words, $J(t_m) \le J(t_1) \le J(t_M)$ for $t_m \le t_1 \le t_M$.*

Proof. It follows from (6) and (4) that

$$J(t_1) = r\left(t_1 + \frac{4Rrt_1}{r^2 + t_1^2}\right).$$

From $\frac{d}{dt_1}J(t_1) = 0$, we obtain the equation

$$t_1^4 - 2(2Rr - r^2)t_1^2 + 4Rr^3 + r^4 = 0,$$

and

$$t_1^2 = 2Rr - r^2 \pm 2r\sqrt{R^2 - 2Rr} = 2Rr - r^2 \pm 2rd.$$

Since $4R^2r^2 = (R^2 - d^2)^2$, we have

$$
\begin{aligned}
&2Rr - r^2 + 2rd - \widehat{t_m}^2 \\
=&2Rr - r^2 + 2rd - \frac{(R+d)^2((R-d)^2 - r^2)}{(R-d)^2} \\
=&\frac{(R-d)^2(2Rr - r^2 + 2rd) - (R+d)^2((R-d)^2 - r^2)}{(R-d)^2} \\
=&\frac{((R+d)^2 - (R-d)^2)r^2 + 2r(R+d)(R-d)^2 - (R^2-d^2)^2}{(R-d)^2} \\
=&\frac{4Rdr^2 + 2r(R-d)(2Rr) - (2Rr)^2}{(R-d)^2} \\
=&0.
\end{aligned}
$$

Similarly, $2Rr - r^2 - 2rd - \widehat{t_M}^2 = 0$. It follows that $\frac{d}{dt_1}J(t_1) = 0$ for $t_1 = \widehat{t_m}, \widehat{t_M}$. The maximum of J occurs at $t_1 = t_M$ and $\widehat{t_M}$ while the minimum occurs at $t_1 = t_m$ and $\widehat{t_m}$.

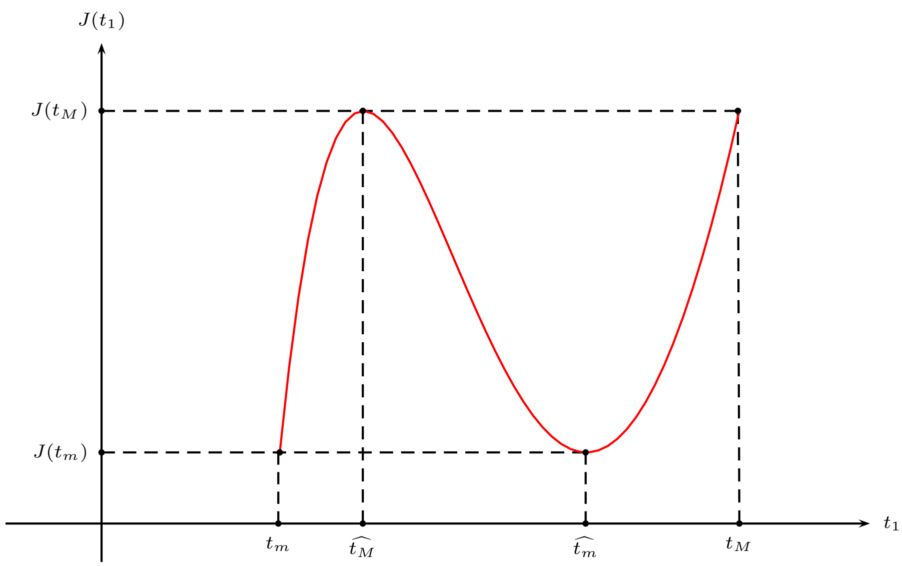


Figure 2

The triangle determined by $\widehat{t_m}$ (respectively $\widehat{t_M}$) is exactly the one determined by t_m (respectively t_M).

$\square$

We conclude with an interesting corollary. Let h_1, h_2, h_3 be the altitudes of the triangle $A_1A_2A_3$. Since

$$2R(h_1 + h_2 + h_3) = a_1a_2 + a_2a_3 + a_3a_1 = (t_1 + t_2 + t_3)^2 + 4Rr + r^2,$$

the following are equivalent:

- the triangle has maximum (respectively minimum) area,
- the triangle has maximum (respectively minimum) perimeter,
- the triangle has maximum (respectively minimum) sum of altitudes.

It follows that these are precisely the two triangles determined by t_M and t_m.

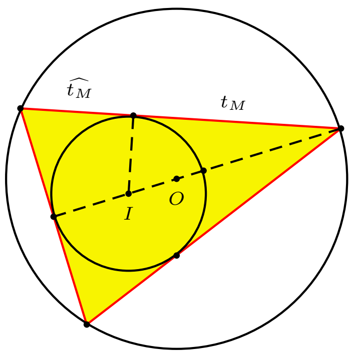


Figure 3a

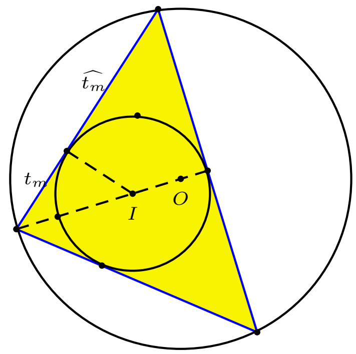


Figure 3b

References

[1] H. Dörrie, *100 Great Problems of Elementary Mathematics*, Dover, 1965.

[2] M. Radić, Some relations concerning triangles and bicentric quadrilaterals in connection with Poncelet's closure theorem, *Math. Maced.* 1 (2003) 35–58.

Mirko Radić: Department of Mathematics, Faculty of Philosophy, University of Rijeka, 51000 Rijeka, Omladinska 14, Croatia

E-mail address: mradic@pefri.hr

Forum Geometricorum
Volume 4 (2004) 27–34.

FORUM GEOM
ISSN 1534-1178

The Archimedean Circles of Schoch and Woo

Hiroshi Okumura and Masayuki Watanabe

Abstract. We generalize the Archimedean circles in an arbelos (shoemaker's knife) given by Thomas Schoch and Peter Woo.

1. Introduction

Let three semicircles α, β and γ form an arbelos, where α and β touch externally at the origin O. More specifically, α and β have radii $a, b > 0$ and centers $(a, 0)$ and $(-b, 0)$ respectively, and are erected in the upper half plane $y \geq 0$. The y-axis divides the arbelos into two curvilinear triangles. By a famous theorem of Archimedes, the inscribed circles of these two curvilinear triangles are congruent and have radii $r = \frac{ab}{a+b}$. See Figure 1. These are called the twin circles of Archimedes. Following [2], we call circles congruent to these twin circles Archimedean circles.

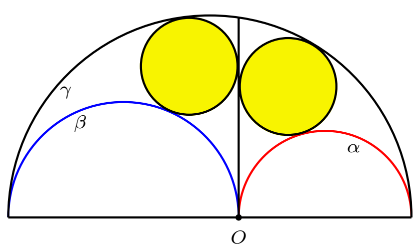


Figure 1

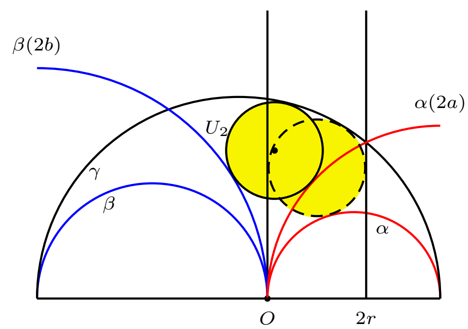


Figure 2

For a real number n, denote by $\alpha(n)$ the semicircle in the upper half-plane with center $(n, 0)$, touching α at O. Similarly, let $\beta(n)$ be the semicircle with center $(-n, 0)$, touching β at O. In particular, $\alpha(a) = \alpha$ and $\beta(b) = \beta$. T. Schoch has found that

(1) the distance from the intersection of $\alpha(2a)$ and γ to the y-axis is $2r$, and
(2) the circle U_2 touching γ internally and each of $\alpha(2a)$, $\beta(2b)$ externally is Archimedean. See Figure 2.

P. Woo considered the Schoch line $\mathcal{L}_s$ through the center of U_2 parallel to the y-axis, and showed that for every nonnegative real number n, the circle U_n with center on $\mathcal{L}_s$ touching $\alpha(na)$ and $\beta(nb)$ externally is also Archimedean. See Figure 3. In this paper we give a generalization of Schoch's circle U_2 and Woo's circles U_n.

Publication Date: March 3, 2004. Communicating Editor: Paul Yiu.

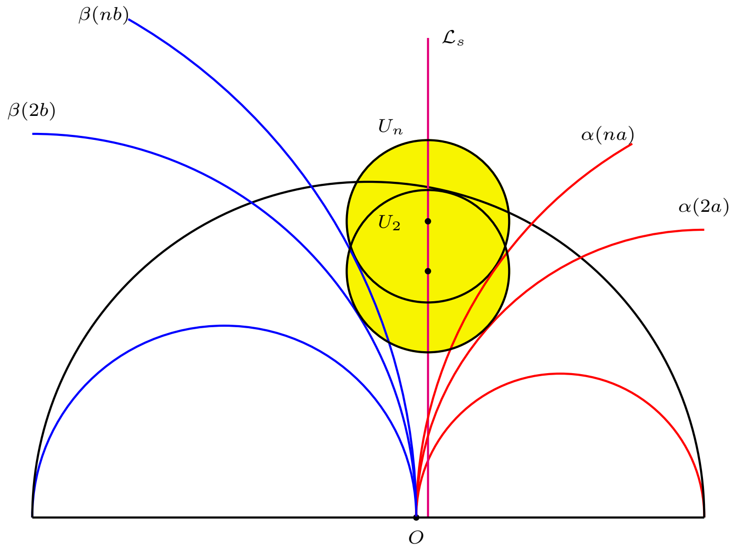


Figure 3

2. A generalization of Schoch's circle U_2

Let a' and b' be real numbers. Consider the semicircles $\alpha(a')$ and $\beta(b')$. Note that $\alpha(a')$ touches α internally or externally according as $a' > 0$ or $a' < 0$; similarly for $\beta(b')$ and β. We assume that the image of $\alpha(a')$ lies on the right side of the image of $\beta(b')$ when these semicircles are inverted in a circle with center O. Denote by $\mathcal{C}(a', b')$ the circle touching γ internally and each of $\alpha(a')$ and $\beta(b')$ at a point different from O.

Theorem 1. *The circle $\mathcal{C}(a', b')$ has radius* $\frac{ab(a'+b')}{aa'+bb'+a'b'}$.

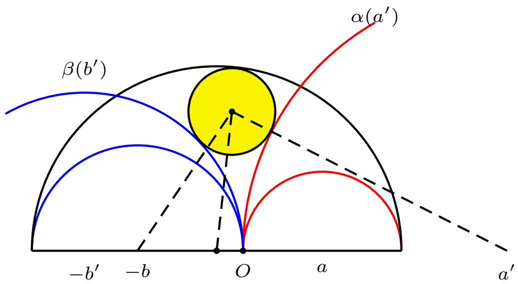


Figure 4a

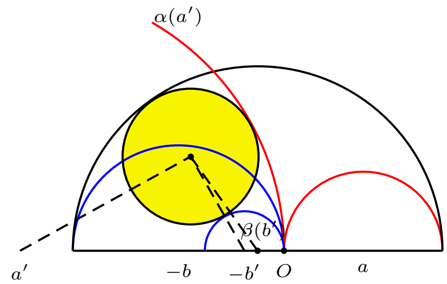


Figure 4b

Proof. Let x be the radius of the circle touching γ internally and also touching $\alpha(a')$ and $\beta(b')$ each at a point different from O. There are two cases in which this circle touches both $\alpha(a')$ and $\beta(b')$ externally (see Figure 4a) or one internally and the other externaly (see Figure 4b). In any case, we have

$$\frac{(a-b+b')^2+(a+b-x)^2-(b'+x)^2}{2(a-b+b')(a+b-x)} = -\frac{(a'-(a-b))^2+(a+b-x)^2-(a'+x)^2}{2(a'-(a-b))(a+b-x)},$$

by the law of cosines. Solving the equation, we obtain the radius given above. □

Note that the radius $r = \frac{ab}{a+b}$ of the Archimedean circles can be obtained by letting $a' = a$ and $b' \to \infty$, or $a' \to \infty$ and $b' = b$.

Let $P(a')$ be the external center of similitude of the circles γ and $\alpha(a')$ if $a' > 0$, and the internal one if $a' < 0$, regarding the two as complete circles. Define $P(b')$ similarly.

Theorem 2. *The two centers of similitude $P(a')$ and $P(b')$ coincide if and only if*

$$\frac{a}{a'} + \frac{b}{b'} = 1. \tag{1}$$

Proof. If the two centers of similitude coincide at the point $(t, 0)$, then by similarity,

$$a' : t - a' = a + b : t - (a - b) = b' : t + b'.$$

Eliminating t, we obtain (1). The converse is obvious by the uniqueness of the figure.

□

From Theorems 1 and 2, we obtain the following result.

Theorem 3. *The circle $\mathcal{C}(a', b')$ is an Archimedean circle if and only if $P(a')$ and $P(b')$ coincide.*

When both a' and b' are positive, the two centers of similitude $P(a')$ and $P(b')$ coincide if and only if the three semicircles $\alpha(a')$, $\beta(b')$ and γ share a common external tangent. Hence, in this case, the circle $\mathcal{C}(a', b')$ is Archimedean if and only if $\alpha(a')$, $\beta(b')$ and γ have a common external tangent. Since $\alpha(2a)$ and $\beta(2b)$ satisfy the condition of the theorem, their external common tangent also touches γ. See Figure 5. In fact, it touches γ at its intersection with the y-axis, which is the midpoint of the tangent. The original twin circles of Archimedes are obtained in the limiting case when the external common tangent touches γ at one of the intersections with the x-axis, in which case, one of $\alpha(a')$ and $\beta(b')$ degenerates into the y-axis, and the remaining one coincides with the corresponding α or β of the arbelos.

Corollary 4. *Let m and n be nonzero real numbers. The circle $\mathcal{C}(ma, nb)$ is Archimedean if and only if*

$$\frac{1}{m} + \frac{1}{n} = 1.$$

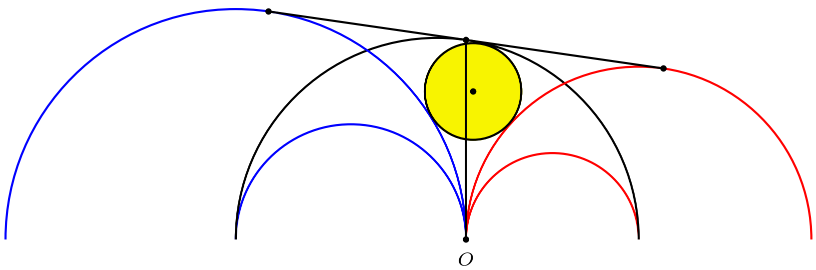


Figure 5

3. Another characterizaton of Woo's circles

The center of the Woo circle U_n is the point

$$\left(\frac{b-a}{b+a}r,\ 2r\sqrt{n+\frac{r}{a+b}}\right). \tag{2}$$

Denote by $\mathcal{L}$ the half line $x = 2r,\ y \geq 0$. This intersects the circle $\alpha(na)$ at the point

$$\left(2r,\ 2\sqrt{r(na-r)}\right). \tag{3}$$

In what follows we consider β as the complete circle with center $(-b, 0)$ passing through O.

Theorem 5. *If T is a point on the line $\mathcal{L}$, then the circle touching the tangents of β through T with center on the Schoch line $\mathcal{L}_s$ is an Archimedean circle.*

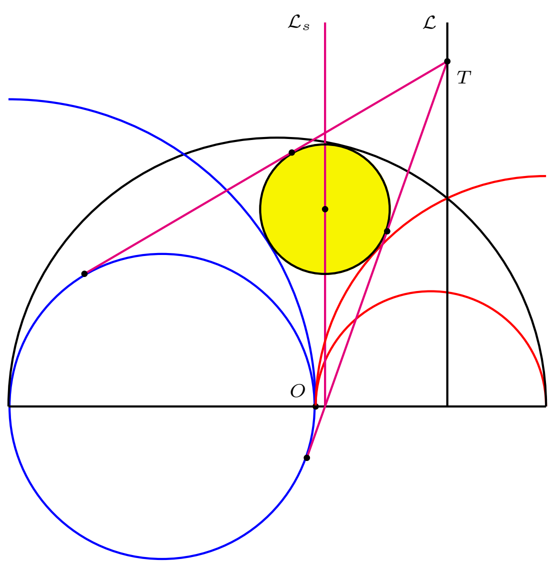


Figure 6

Proof. Let x be the radius of this circle. By similarity (see Figure 6),

$$b+2r : b = 2r - \frac{b-a}{b+a}r : x.$$

From this, $x = r$. □

The set of Woo circles is a proper subset of the set of circles determined in Theorem 5 above. The external center of similitude of U_n and β has y-coordinate

$$2a\sqrt{n+\frac{r}{a+b}}.$$

When U_n is the circle touching the tangents of β through a point T on $\mathcal{L}$, we shall say that it is determined by T. The y-coordinate of the intersection of α and $\mathcal{L}$ is $2a\sqrt{\frac{r}{a+b}}$. Therefore we obtain the following theorem (see Figure 7).

Theorem 6. *U_0 is determined by the intersection of α and the line $\mathcal{L} : x = 2r$.*

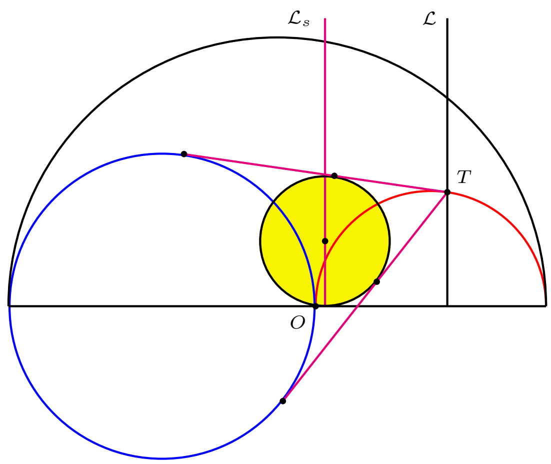


Figure 7

As stated in [2] as the property of the circle labeled as W_{11}, the external tangent of α and β also touches U_0 and the point of tangency at α coincides with the intersection of α and $\mathcal{L}$. Woo's circles are characterized as the circles determined by the points on $\mathcal{L}$ with y-coordinates greater than or equal to $2a\sqrt{\frac{r}{a+b}}$.

4. Woo's circles U_n with $n < 0$

Woo considered the circles U_n for nonnegative numbers n, with U_0 passing through O. We can, however, construct more Archimedean circles passing through points on the y-axis below O using points on $\mathcal{L}$ lying below the intersection with α. The expression (2) suggests the existence of U_n for

$$-\frac{r}{a+b} \le n < 0. \tag{4}$$

In this section we show that it is possible to define such circles using $\alpha(na)$ and $\beta(nb)$ with negative n satisfying (4).

Theorem 7. *For n satisfying (4), the circle with center on the Schoch line touching $\alpha(na)$ and $\beta(nb)$ internally is an Archimedean circle.*

Proof. Let x be the radius of the circle with center given by (2) and touching $\alpha(na)$ and $\beta(nb)$ internally, where n satisfies (4). Since the centers of $\alpha(na)$ and $\beta(nb)$ are $(na, 0)$ and $(-nb, 0)$ respectively, we have

$$\left(\frac{b-a}{b+a}r - na\right)^2 + 4r^2\left(n + \frac{r}{a+b}\right) = (x+na)^2,$$

and

$$\left(\frac{b-a}{b+a}r + nb\right)^2 + 4r^2\left(n + \frac{r}{a+b}\right) = (x+nb)^2.$$

Since both equations give the same solution $x = r$, the proof is complete. □

5. A generalization of U_0

We conclude this paper by adding an infinite set of Archimedean circles passing through O. Let x be the distance from O to the external tangents of α and β. By similarity,

$$b - a : b + a = x - a : a.$$

This implies $x = 2r$. Hence, the circle with center O and radius $2r$ touches the tangents and the lines $x = \pm 2r$. We denote this circle by $\mathcal{E}$. Since U_0 touches the external tangents and passes through O, the circles U_0, $\mathcal{E}$ and the tangent touch at the same point. We easily see from (2) that the distance between the center of U_n and O is $\sqrt{4n+1}r$. Therefore, U_2 also touches $\mathcal{E}$ externally, and the smallest circle touching U_2 and passing through O, which is the Archimedean circle W_{27} in [2] found by Schoch, and U_2 touches $\mathcal{E}$ at the same point. All the Archimedean circles pass through O also touch $\mathcal{E}$. In particular, Bankoff's third circle [1] touches $\mathcal{E}$ at a point on the y-axis.

Theorem 8. *Let $\mathcal{C}_1$ be a circle with center O, passing through a point P on the x-axis, and $\mathcal{C}_2$ a circle with center on the x-axis passing through O. If $\mathcal{C}_2$ and the vertical line through P intersect, then the tangents of $\mathcal{C}_2$ at the intersection also touches $\mathcal{C}_1$.*

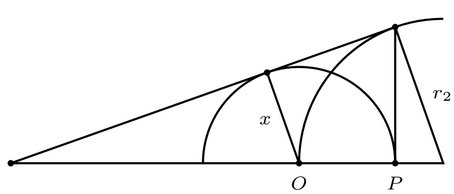


Figure 8a

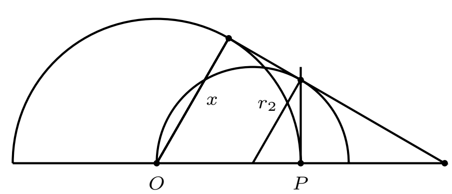


Figure 8b

Proof. Let d be the distance between O and the intersection of the tangent of $\mathcal{C}_2$ and the x-axis, and let x be the distance between the tangent and O. We may assume $r_1 \neq r_2$ for the radii r_1 and r_2 of the circles $\mathcal{C}_1$ and $\mathcal{C}_2$. If $r_1 < r_2$, then

$$r_2 - r_1 : r_2 = r_2 + d = x : d.$$

See Figure 8a. If $r_1 > r_2$, then

$$r_1 - r_2 : r_2 = r_2 : d - r_2 = x : d.$$

See Figure 8b. In each case, $x = r_1$. □

Let t_n be the tangent of $\alpha(na)$ at its intersection with the line $\mathcal{L}$. This is well defined if $n \geq \frac{b}{a+b}$. By Theorem 8, t_n also touches $\mathcal{E}$. This implies that the smallest circle touching t_n and passing through O is an Archimedean circle, which we denote by $\mathcal{A}(n)$. Similarlary, another Archimedean circle $\mathcal{A}'(n)$ can be constructed, as the smallest circle through O touching the tangent t'_n of $\beta(nb)$ at its intersection with the line $\mathcal{L}' : x = -2r$. See Figure 9 for $\mathcal{A}(2)$ and $\mathcal{A}'(2)$. Bankoff's circle is $\mathcal{A}\left(\frac{2r}{a}\right) = \mathcal{A}'\left(\frac{2r}{b}\right)$, since it touches $\mathcal{E}$ at $(0, 2r)$. On the other hand, $U_0 = \mathcal{A}(1) = \mathcal{A}'(1)$ by Theorem 6.

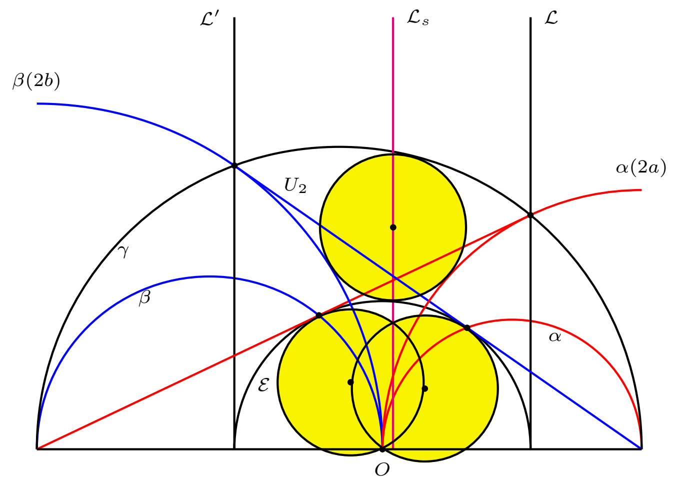


Figure 9

Theorem 9. *Let m and n be positive numbers. The Archimedean circles $\mathcal{A}(m)$ and $\mathcal{A}'(n)$ coincide if and only if m and n satisfy*

$$\frac{1}{ma} + \frac{1}{nb} = \frac{1}{r} = \frac{1}{a} + \frac{1}{b}. \tag{5}$$

Proof. By (3) the equations of the tangents t_m and t'_n are

$$-(ma + (m-2)b)x + 2\sqrt{b(ma + (m-1)b)}y = 2mab,$$
$$(nb + (n-2)a)x + 2\sqrt{a(nb + (n-1)a)}y = 2nab.$$

These two tangents coincide if and only if (5) holds. □

The line t_2 has equation

$$-ax + \sqrt{b(2a+b)}y = 2ab. \tag{6}$$

It clearly passes through $(-2b, 0)$, the point of tangency of γ and β (see Figure 9). Note that the point

$$\left(-\frac{2r}{a+b}a,\ \frac{2r}{a+b}\sqrt{b(2a+b)}\right)$$

lies on $\mathcal{E}$ and the tangent of $\mathcal{E}$ is also expressed by (6). Hence, t_2 touches $\mathcal{E}$ at this point. The point also lies on β. This means that $\mathcal{A}(2)$ touches t_2 at the intersection of β and t_2. Similarly, $\mathcal{A}'(2)$ touches t_2' at the intersection of α and t_2'. The Archimedean circles $\mathcal{A}(2)$ and $\mathcal{A}'(2)$ intersect at the point

$$\left(\frac{b-a}{b+a}r,\ \frac{r}{a+b}(\sqrt{a(a+2b)} + \sqrt{b(2a+b)})\right)$$

on the Schoch line.

References

[1] L. Bankoff, Are the twin circles of Archimedes really twin?, *Math. Mag.*, 47 (1974) 134–137.
[2] C. W. Dodge, T. Schoch, P. Y. Woo and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.*, 72 (1999) 202–213.

Hiroshi Okumura: Department of Information Engineering, Maebashi Institute of Technology, 460-1 Kamisadori Maebashi Gunma 371-0816, Japan
E-mail address: okumura@maebashi-it.ac.jp

Masayuki Watanabe: Department of Information Engineering, Maebashi Institute of Technology, 460-1 Kamisadori Maebashi Gunma 371-0816, Japan
E-mail address: watanabe@maebashi-it.ac.jp

Forum Geometricorum
Volume 4 (2004) 35–52.

FORUM GEOM
ISSN 1534-1178

Steiner's Theorems on the Complete Quadrilateral

Jean-Pierre Ehrmann

Abstract. We give a translation of Jacob Steiner's 1828 note on the complete quadrilateral, with complete proofs and annotations in barycentric coordinates.

1. Steiner's note on the complete quadrilateral

In 1828, Jakob Steiner published in Gergonne's *Annales* a very short note [9] listing ten interesting and important theorems on the complete quadrilateral. The purpose of this paper is to provide a translation of the note, to prove these theorems, along with annotations in barycentric coordinates. We begin with a translation of Steiner's note.

Suppose four lines intersect two by two at six points.

(1) These four lines, taken three by three, form four triangles whose circumcircles pass through the same point F.
(2) The centers of the four circles (and the point F) lie on the same circle.
(3) The perpendicular feet from F to the four lines lie on the same line $\mathcal{R}$, and F is the only point with this property.
(4) The orthocenters of the four triangles lie on the same line $\mathcal{R}'$.
(5) The lines $\mathcal{R}$ and $\mathcal{R}'$ are parallel, and the line $\mathcal{R}$ passes through the midpoint of the segment joining F to its perpendicular foot on $\mathcal{R}'$.
(6) The midpoints of the diagonals of the complete quadrilateral formed by the four given lines lie on the same line $\mathcal{R}''$ (Newton).
(7) The line $\mathcal{R}''$ is a common perpendicular to the lines $\mathcal{R}$ and $\mathcal{R}'$.
(8) Each of the four triangles in (1) has an incircle and three excircles. The centers of these 16 circles lie, four by four, on eight new circles.
(9) These eight new circles form two sets of four, each circle of one set being orthogonal to each circle of the other set. The centers of the circles of each set lie on a same line. These two lines are perpendicular.
(10) Finally, these last two lines intersect at the point F mentioned above.

The configuration formed by four lines is called a complete quadrilateral. Figure 1 illustrates the first 7 theorems on the complete quadrilateral bounded by the four lines UVW, UBC, AVC, and ABW. The diagonals of the quadrilateral are the

Publication Date: March 17, 2004. Communicating Editor: Paul Yiu.

segments AU, BV, CW. The four triangles ABC, AVW, BWU, and CUV are called the associated triangles of the complete quadrilateral. We denote by

- H, H_a, H_b, H_c their orthocenters,
- Γ, Γ_a, Γ_b, Γ_c their circumcircles, and
- O, O_a, O_b, O_c the corresponding circumcenters.

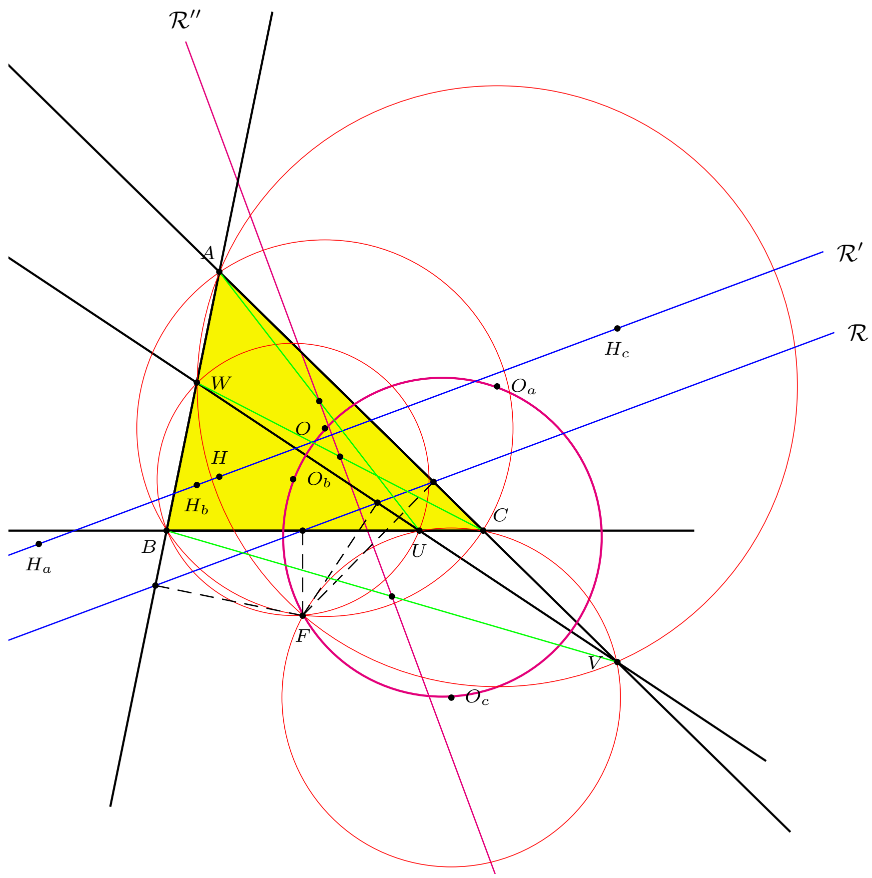


Figure 1.

2. Geometric preliminaries

2.1. *Directed angles.* We shall make use of the notion of *directed* angles. Given two lines ℓ and ℓ', the directed angle (ℓ, ℓ') is the angle through which ℓ must be rotated in the positive direction in order to become parallel to, or to coincide with, the line ℓ'. See [3, §§16–19]. It is defined modulo π.

Lemma 1. (1) $(\ell, \ell'') = (\ell, \ell') + (\ell', \ell'')$.

(2) *Four noncollinear points* P, Q, R, S *are concyclic if and only if* $(PR, PS) = (QR, QS)$.

2.2. *Simson-Wallace lines.* The pedals [1] of a point M on the lines BC, CA, AB are collinear if and only if M lies on the circumcircle Γ of ABC. In this case, the Simson-Wallace line passes through the midpoint of the segment joining M to the orthocenter H of triangle ABC. The point M is the isogonal conjugate (with respect to triangle ABC) of the infinite point of the direction orthogonal to its own Simson-Wallace line.

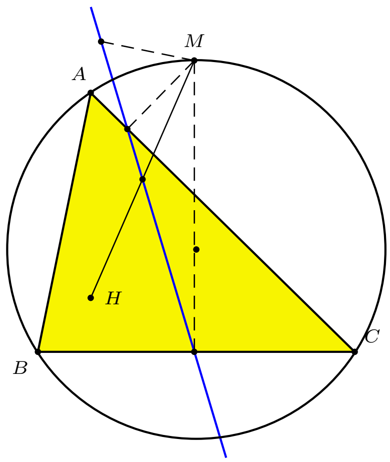


Figure 2

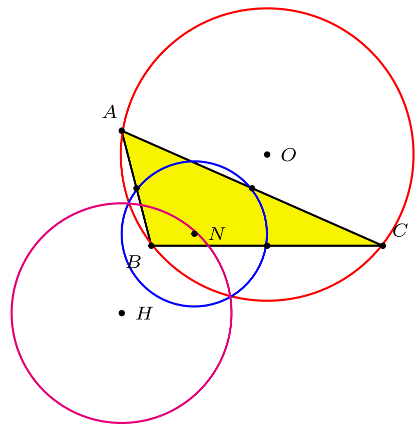


Figure 3

2.3. *The polar circle of a triangle.* There exists one and only one circle with respect to which a given triangle ABC is self polar. The center of this circle is the orthocenter of ABC and the square of its radius is

$$-4R^2 \cos A \cos B \cos C.$$

This *polar* circle is real if and only if ABC is obtuse-angled. It is orthogonal to any circle with diameter a segment joining a vertex of ABC to a point of the opposite sideline. The inversion with respect the polar circle maps a vertex of ABC to its pedal on the opposite side. Consequently, this inversion swaps the circumcircle and the nine-point circle.

2.4. *Center of a direct similitude.* Suppose that a direct similitude with center Ω maps M to M' and N to N', and that the lines MM' and NN' intersect at S. If Ω does not lie on the line MN, then M, N, Ω, S are concyclic; so are M', N', Ω, S. Moreover, if $MN \perp M'N'$, the circles $MN\Omega S$ and $M'N'\Omega S$ are orthogonal.

[1]In this paper we use the word pedal in the sense of orthogonal projection.

3. Steiner's Theorems 1–7

3.1. *Steiner's Theorem 1 and the Miquel point.* Let F be the second common point (apart from A) of the circles Γ and Γ_a. Since

$$(FB, FW) = (FB, FA) + (FA, FW) = (CB, CA) + (VA, VW) = (UB, UW),$$

we have $F \in \Gamma_b$ by Lemma 1(2). Similarly $F \in \Gamma_c$. This proves (1).

We call F the *Miquel point* of the complete quadrilateral.

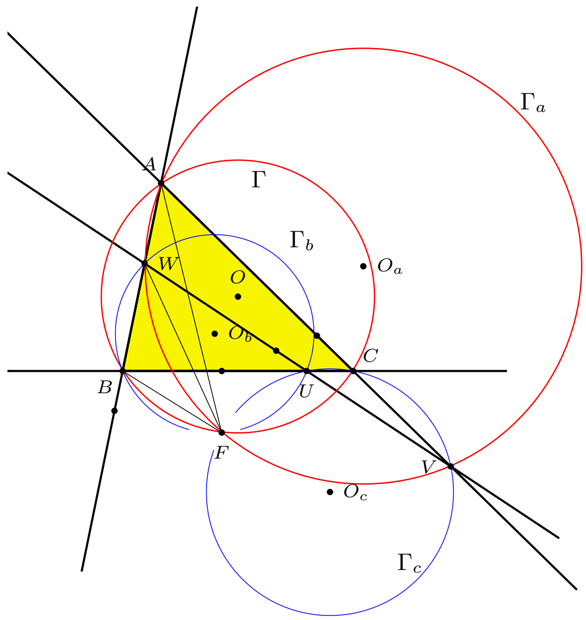


Figure 4.

3.2. *Steiner's Theorem 3 and the pedal line.* The point F has the same Simson-Wallace line with respect to the four triangles of the complete quadrilateral. See Figure 5. Conversely, if the pedals of a point M on the four sidelines of the complete quadrilateral lie on a same line, M must lie on each of the four circumcircles. Hence, $M = F$. This proves (3).

We call the line $\mathcal{R}$ the *pedal line* of the quadrilateral.

3.3. *Steiner's Theorems 4, 5 and the orthocentric line.* As the midpoints of the segments joining F to the four orthocenters lie on $\mathcal{R}$, the four orthocenters lie on a line $\mathcal{R}'$, which is the image of $\mathcal{R}$ under the homothety $\mathsf{h}(F, 2)$. This proves (4) and (5). See Figure 5.

We call the line $\mathcal{R}'$ the *orthocentric line* of the quadrilateral.

Remarks. (1) As U, V, W are the reflections of F with respect to the sidelines of the triangle $O_aO_bO_c$, the orthocenter of this triangle lies on $\mathcal{L}$.

(2) We have $(BC, FU) = (CA, FV) = (AB, FW)$ because, for instance, $(BC, FU) = (UB, UF) = (WB, WF) = (AB, FW)$.

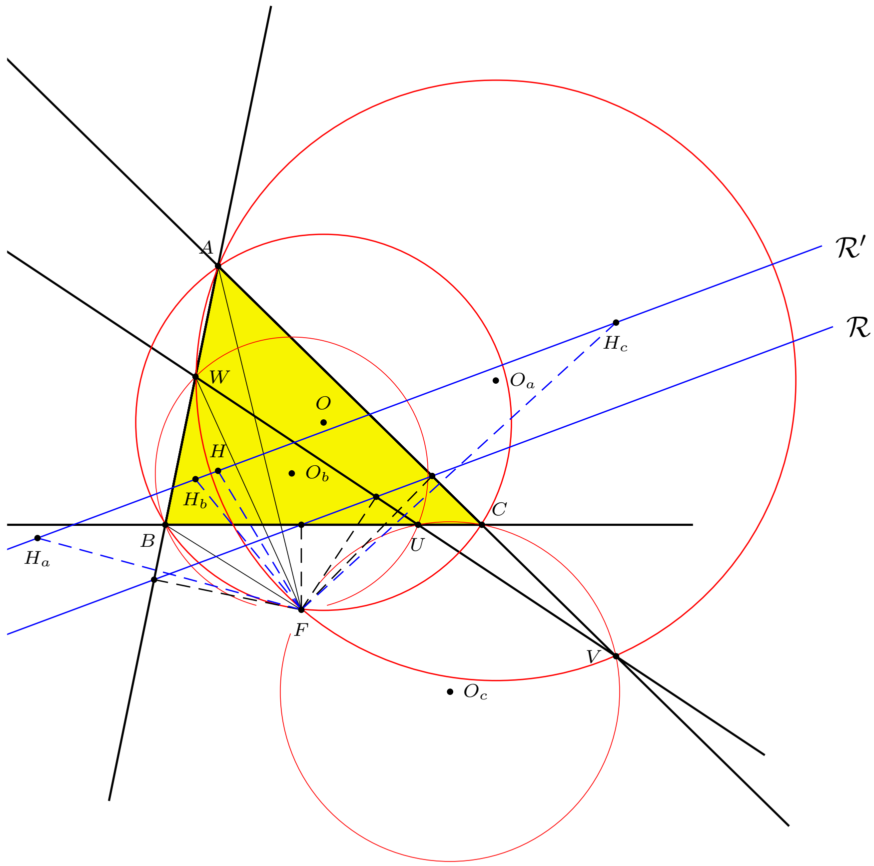


Figure 5.

(3) Let P_a, P_b, P_c be the projections of F upon the lines BC, CA, AB. As P_a, P_b, C, F are concyclic, it follows that F is the center of the direct similitude mapping P_a to U and P_b to V. Moreover, by (2) above, this similitude maps P_c to W.

3.4. *Steiner's Theorem 2 and the Miquel circle.* By Remark (3) above, if F_a, F_b, F_c are the reflections of F with respect to the lines BC, CA, AB, a direct similitude σ with center F maps F_a to U, F_b to V, F_c to W. As A is the circumcenter of FF_bF_c, it follows that $\sigma(A) = O_a$; similarly, $\sigma(B) = O_b$ and $\sigma(C) = O_c$. As A, B, C, F are concyclic, so are O_a, O_b, O_c, F. Hence F and the circumcenters of three associated triangles are concyclic. It follows that O, O_a, O_b, O_c, F lie on the same circle, say, Γ_m. This prove (2).

We call Γ_m the *Miquel circle* of the complete quadrilateral. See Figure 6.

3.5. *The Miquel perspector.* Now, by §2.4, the second common point of Γ and Γ_m lies on the three lines AO_a, BO_b, CO_c. Hence,

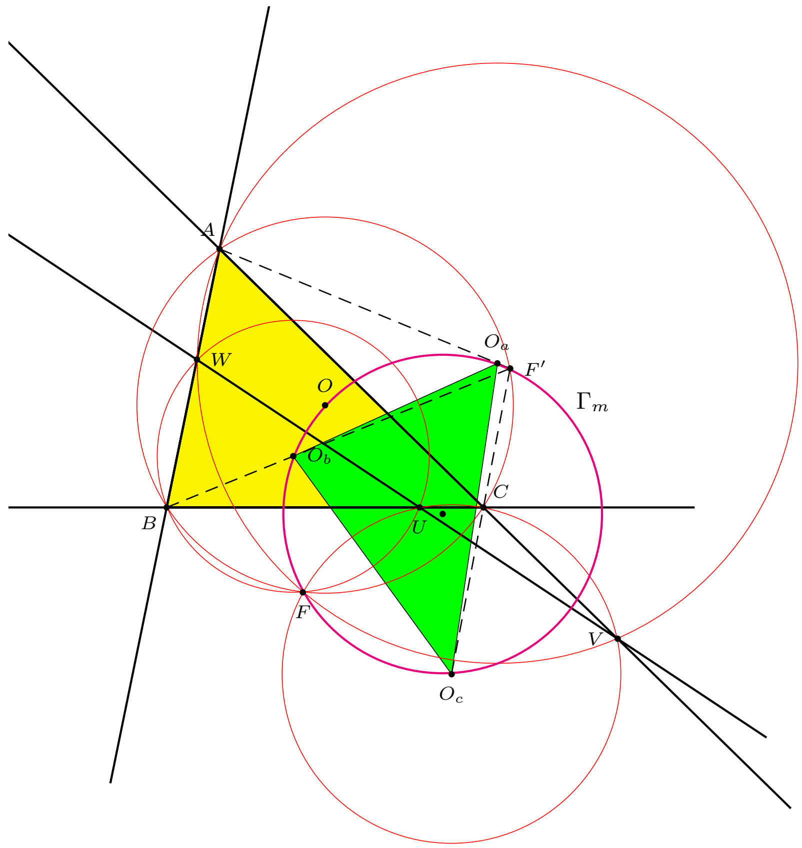


Figure 6.

Proposition 2. *The triangle $O_aO_bO_c$ is directly similar and perspective with ABC. The center of similitude is the Miquel point F and the perspector is the second common point F' of the Miquel circle and the circumcircle Γ of triangle ABC.*

We call F' the *Miquel perspector* of the triangle ABC.

3.6. *Steiner's Theorems 6, 7 and the Newton line.* We call *diagonal triangle* the triangle $A'B'C'$ with sidelines AU, BV, CW.

Lemma 3. *The polar circles of the triangles ABC, AVW, BWU, CUV and the circumcircle of the diagonal triangle are coaxal. The three circles with diameter AU, BV, CW are coaxal. The corresponding pencils of circles are orthogonal.*

Proof. By §2.3, each of the four polar circles is orthogonal to the three circles with diameter AU, BV, CW. More over, as each of the quadruples (A, U, B', C'), (B, V, C', A') and (C, W, A', B') is harmonic, the circle $A'B'C'$ is orthogonal to the three circles with diameter AU, BV and CW. $\square$

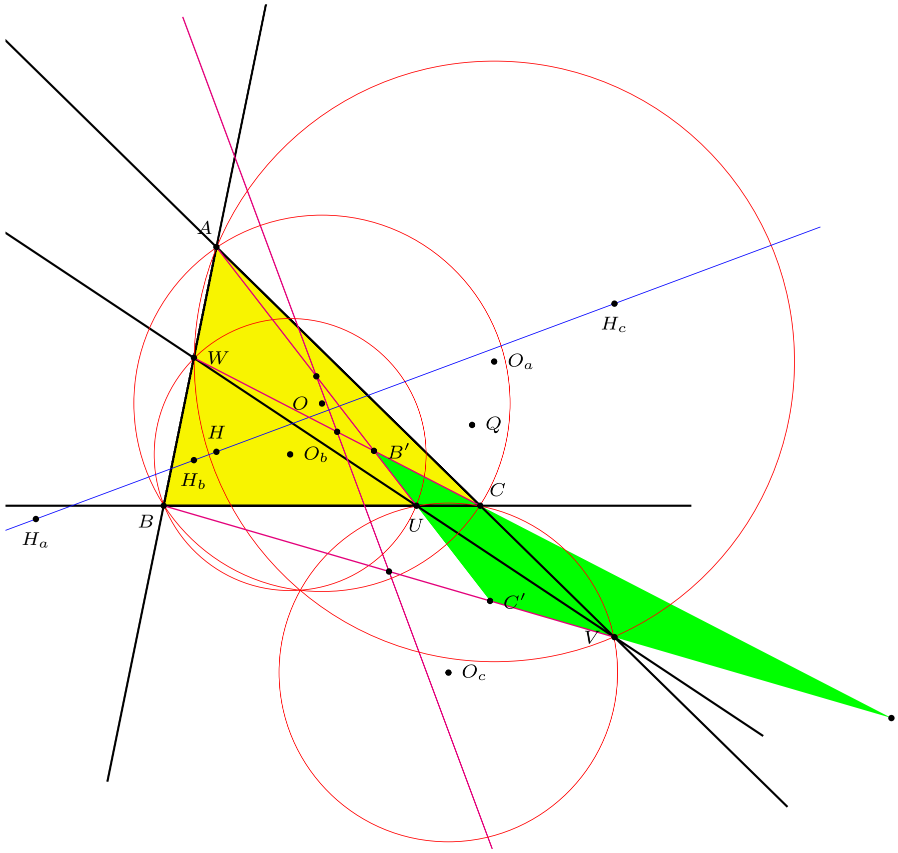


Figure 7.

As the line of centers of the first pencil of circles is the orthocentric line $\mathcal{R}'$, it follows that the midpoints of AU, BV and CW lie on a same line $\mathcal{R}''$ perpendicular to $\mathcal{R}'$. This proves (6) and (7).

4. Some further results

4.1. *The circumcenter of the diagonal triangle.*

Proposition 4. *The circumcenter of the diagonal triangle lies on the orthocentric line.*

This follows from Lemma 3 and §2.3.

We call the line $\mathcal{R}''$ the *Newton line* of the quadrilateral. As the Simson-Wallace line $\mathcal{R}$ of F is perpendicular to $\mathcal{R}''$, we have

Proposition 5. *The Miquel point is the isogonal conjugate of the infinite point of the Newton line with respect to each of the four triangles* ABC, AVW, BWU, CUV.

4.2. *The orthopoles.* Recall that the three lines perpendicular to the sidelines of a triangle and going through the projection of the opposite vertex on a given line go through a same point : the *orthopole* of the line with respect to the triangle.

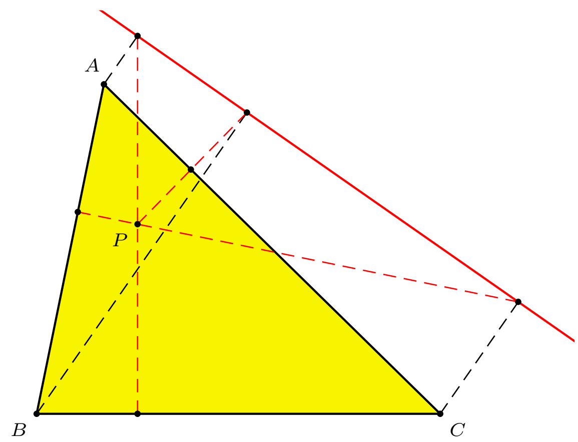


Figure 8

Proposition 6 (Goormaghtigh)**.** *The orthopole of a sideline of the complete quadrilateral with respect to the triangle bounded by the three other sidelines lies on the orthocentric line.*

Proof. See [1, pp.241–242]. □

5. Some barycentric coordinates and equations

5.1. *Notations.* Given a complete quadrilateral, we consider the triangle bounded by three of the four given lines as a reference triangle ABC, and construe the fourth line as the trilinear polar with respect to ABC of a point Q with homogeneous barycentric coordinates $(u : v : w)$, *i.e.*, the line

$$\mathcal{L}: \qquad \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.$$

The intercepts of $\mathcal{L}$ with the sidelines of triangle ABC are the points

$$U = (0 : v : -w), \qquad V = (-u : 0 : w), \qquad W = (u : -v : 0).$$

The lines AU, BV, CW bound the diagonal triangle with vertices

$$A' = (-u : v : w), \qquad B' = (u : -v : w), \qquad C' = (u : v : -w).$$

Triangles ABC and $A'B'C'$ are perspective at Q.

We adopt the following notations. If a, b, c stand for the lengths of the sides BC, CA, AB, then

$$S_A = \frac{1}{2}(b^2 + c^2 - a^2), \quad S_B = \frac{1}{2}(c^2 + a^2 - b^2), \quad S_C = \frac{1}{2}(a^2 + b^2 - c^2).$$

We shall also denote by S *twice* of the signed area of triangle ABC, so that

$$S_A = S \cdot \cot A, \quad S_B = S \cdot \cot B, \quad S_C = S \cdot \cot C,$$

and

$$S_{BC} + S_{CA} + S_{AB} = S^2.$$

Lemma 7. (1) *The infinite point of the line $\mathcal{L}$ is the point*

$$(u(v-w) : v(w-u) : w(u-v)).$$

(2) *Lines perpendicular to $\mathcal{L}$ have infinite point $(\lambda_a : \lambda_b : \lambda_c)$, where*

$$\begin{aligned} \lambda_a &= S_B v(w-u) - S_C w(u-v), \\ \lambda_b &= S_C w(u-v) - S_A u(v-w), \\ \lambda_c &= S_A u(v-w) - S_B v(w-u). \end{aligned}$$

Proof. (1) is trivial. (2) follows from (1) and the fact that two lines with infinite points $(p : q : r)$ and $(p' : q' : r')$ are perpendicular if and only if

$$S_A pp' + S_B qq' + S_C rr' = 0.$$

Consequently, given a line with infinite point $(p : q : r)$, lines perpendicular to it all have the infinite point $(S_B q - S_C r : S_C r - S_A p : S_A p - S_B q)$. $\square$

5.2. *Coordinates and equations.* We give the barycentric coordinates of points and equations of lines and circles in Steiner's theorems.

(1) The Miquel point:

$$F = \left(\frac{a^2}{v-w} : \frac{b^2}{w-u} : \frac{c^2}{u-v}\right).$$

(2) The pedal line:

$$\mathcal{R} : \quad \frac{v-w}{S_C v + S_B w - a^2 u}x + \frac{w-u}{S_A w + S_C u - b^2 v}y + \frac{u-v}{S_B u + S_A v - c^2 w}z = 0.$$

(3) The orthocentric line:

$$\mathcal{R}' : \quad (v-w)S_A x + (w-u)S_B y + (u-v)S_C z = 0.$$

(4) The Newton line:

$$\mathcal{R}'' : \quad (v+w-u)x + (w+u-v)y + (u+v-w)z = 0.$$

(5) The equation of the Miquel circle:

$$a^2yz + b^2zx + c^2xy + \frac{2R^2(x+y+z)}{(v-w)(w-u)(u-v)}\left(\frac{v-w}{a^2}\lambda_a x + \frac{w-u}{b^2}\lambda_b y + \frac{u-v}{c^2}\lambda_c z\right) = 0.$$

(6) The Miquel perspector, being the isogonal conjugate of the infinite point of the direction orthogonal to $\mathcal{L}$, is

$$F' = \left(\frac{a^2}{\lambda_a} : \frac{b^2}{\lambda_b} : \frac{c^2}{\lambda_c}\right).$$

The Simson-Wallace line of F' is parallel to ℓ.

(7) The orthopole of $\mathcal{L}$ with respect to ABC is the point

$$(\lambda_a(-S_B S_C vw + b^2 S_B wu + c^2 S_C uv) : \cdots : \cdots).$$

5.3. *Some metric formulas* . Here, we adopt more symmetric notations. Let ℓ_i, $i = 1, 2, , 3, 4$, be four given lines.

- For distinct i and j, $A_{i,j} = \ell_i \cap \ell_j$,
- $\mathcal{T}_i$ the triangle bounded by the three lines other than ℓ_i, O_i its circumcenter, R_i its circumradius.
- $F_i = O_j A_{k,l} \cap O_k A_{l,j} \cap O_l A_{j,k}$ its Miquel perspector, *i.e.*, the second intersection (apart from F) of its circumcircle with the Miquel circle; R_m is the radius of the Miquel circle.

Let d be the distance from F to the pedal line $\mathcal{R}$ and $\theta_i = (\mathcal{R}, \ell_i)$. Up to a direct congruence, the complete quadrilateral is characterized by d, θ_1, θ_2, θ_3, and θ_4.

(1) The distance from F to ℓ_i is $\dfrac{d}{|\cos\theta_i|}$.
(2) $|FA_{i,j}| = \dfrac{d}{|\cos\theta_i \cos\theta_j|}$.
(3) $|A_{k,i}A_{k,j}| = d\left|\dfrac{\sin(\theta_j - \theta_i)}{\cos\theta_i \cos\theta_j \cos\theta_k}\right|$.
(4) The directed angle $(FA_{k,i}, FA_{k,j}) = (\ell_i, \ell_j) = \theta_j - \theta_i \mod \pi$.
(5) $R_m = \dfrac{d}{4\,|\cos\theta_1 \cos\theta_2 \cos\theta_3 \cos\theta_4|} = \dfrac{R_i}{2\,|\cos\theta_i|}$ for $i = 1, 2, 3, 4$.
(6) $|FA_{1,2}| \cdot |FA_{3,4}| = |FA_{1,3}| \cdot |FA_{2,4}| = |FA_{1,4}| \cdot |FA_{2.3}| = 4dR_m$.
(7) $|FF_i| = 2R_i |\sin\theta_i|$.
(8) The *oriented* angle between the vectors $\mathbf{O_i F}$ and $\mathbf{O_i F_i} = -2\theta_i \mod 2\pi$.
(9) The distance from F to $\mathcal{R}''$ is

$$\frac{d}{2}\left|\tan\theta_1 + \tan\theta_2 + \tan\theta_3 + \tan\theta_4\right|.$$

6. Steiner's Theorems 8 – 10

At each vertex M of the complete quadrilateral, we associate the pair of angle bisectors m and m'. These lines are perpendicular to each other at M. We denote the intersection of two bisectors m and n by $\mathsf{m} \cap \mathsf{n}$.

- $\mathbf{T}(\mathsf{m}, \mathsf{n}, \mathsf{p})$ denotes the triangle bounded by a bisector at M, one at N, and one at P.
- $\Gamma(\mathsf{m}, \mathsf{n}, \mathsf{p})$ denotes the circumcircle of $\mathbf{T}(\mathsf{m}, \mathsf{n}, \mathsf{p})$.

Consider three bisectors $\mathsf{a}, \mathsf{b}, \mathsf{c}$ intersecting at a point J, the incenter or one of the excenters of ABC. Suppose two bisectors v and w intersect on a. Then so do v' and w'. Now, the line joining $\mathsf{b} \cap \mathsf{w}$ and $\mathsf{c} \cap \mathsf{v}$ is a U-bisector. If we denote this line by u, then u' the line joining $\mathsf{b} \cap \mathsf{w}'$ and $\mathsf{c} \cap \mathsf{v}'$.

The triangles $\mathbf{T}(\mathsf{a}', \mathsf{b}', \mathsf{c}')$, $\mathbf{T}(\mathsf{u}, \mathsf{v}, \mathsf{w})$, and $\mathbf{T}(\mathsf{u}', \mathsf{v}', \mathsf{w}')$ are perspective at J. Hence, by Desargues' theorem, the points $\mathsf{a}' \cap \mathsf{u}$, $\mathsf{b}' \cap \mathsf{v}$, and $\mathsf{c}' \cap \mathsf{w}$ are collinear; so are $\mathsf{a}' \cap \mathsf{u}'$, $\mathsf{b}' \cap \mathsf{v}'$, and $\mathsf{c}' \cap \mathsf{w}'$. Moreover, as the corresponding sidelines of triangles $\mathbf{T}(\mathsf{u}, \mathsf{v}, \mathsf{w})$, and $\mathbf{T}(\mathsf{u}', \mathsf{v}', \mathsf{w}')$ are perpendicular, it follows from §2.4 that their circumcircles $\Gamma(\mathsf{u},\mathsf{v},\mathsf{w})$, and $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}')$ are orthogonal and pass through J. See Figure 9. [2]

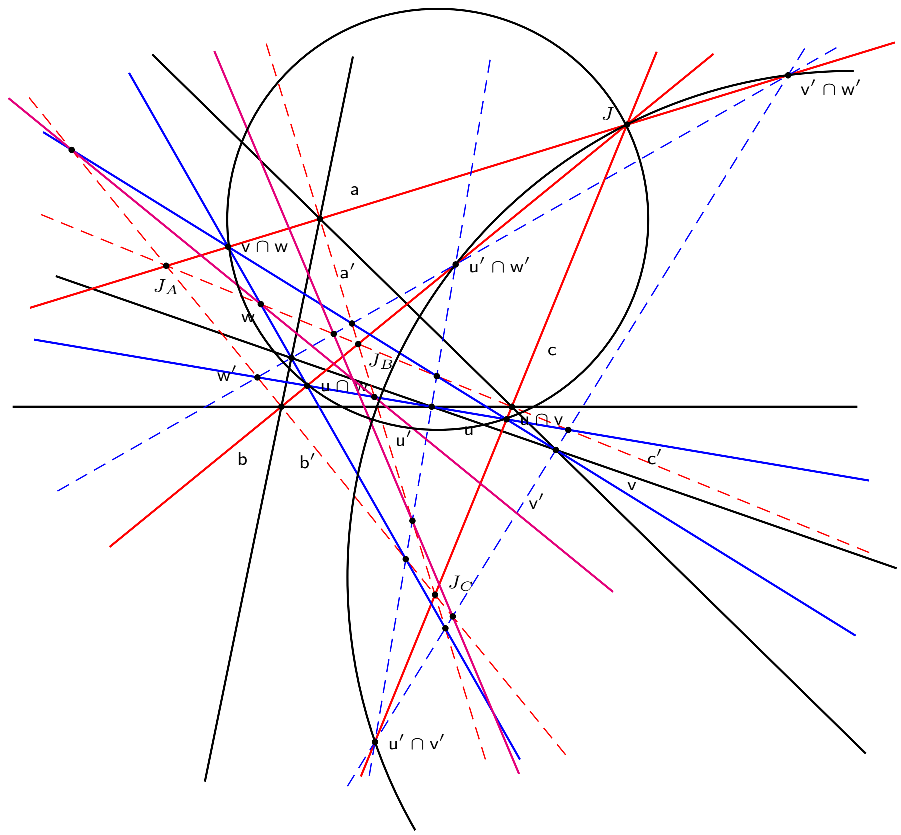


Figure 9

As a intersects the circle $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}')$ at J and $\mathsf{v}'\cap\mathsf{w}'$ and u' intersects the circle $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}')$ at $\mathsf{u}'\cap\mathsf{v}'$ and $\mathsf{u}'\cap\mathsf{w}'$, it follows that the polar line of $\mathsf{a}\cap\mathsf{u}'$ with respect to $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}')$ passes through $\mathsf{b}\cap\mathsf{v}'$ and $\mathsf{c}\cap\mathsf{w}'$. Hence $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}')$ is the polar circle of the triangle with vertices $\mathsf{a}\cap\mathsf{u}'$, $\mathsf{b}\cap\mathsf{v}'$, $\mathsf{c}\cap\mathsf{w}'$. Similarly, $\Gamma(\mathsf{u},\mathsf{v},\mathsf{w})$ is the polar circle of the triangle with vertices $\mathsf{a}\cap\mathsf{u}$, $\mathsf{b}\cap\mathsf{v}$, $\mathsf{c}\cap\mathsf{w}$.

By the same reasoning, we obtain the following.

(a) As the triangles $\mathbf{T}(\mathsf{a}',\mathsf{b},\mathsf{c})$, $\mathbf{T}(\mathsf{u},\mathsf{v}',\mathsf{w}')$, and $\mathbf{T}(\mathsf{u}',\mathsf{v},\mathsf{w})$ are perspective at $J_A=\mathsf{a}\cap\mathsf{b}'\cap\mathsf{c}'$, it follows that

- the circles $\Gamma(\mathsf{u},\mathsf{v}',\mathsf{w}')$ and $\Gamma(\mathsf{u}',\mathsf{v},\mathsf{w})$ are orthogonal and pass through J_A,
- the points $\mathsf{a}'\cap\mathsf{u}$, $\mathsf{b}\cap\mathsf{v}'$, and $\mathsf{c}\cap\mathsf{w}'$ are collinear; so are $\mathsf{a}'\cap\mathsf{u}'$, $\mathsf{b}\cap\mathsf{v}$, and $\mathsf{c}\cap\mathsf{w}$,

[2] In Figures 9 and 10, at each of the points A, B, C, U, V, W are two bisectors, one shown in solid line and the other in dotted line. The bisectors in solid lines are labeled a, b, c, u, v, w, and those in dotted line labeled a', b', c', u', v', w'. Other points are identified as intersections of two of these bisectors. Thus, for example, $J=\mathsf{a}\cap\mathsf{b}$, and $J_A=\mathsf{b}'\cap\mathsf{c}'$.

- the circle $\Gamma(\mathsf{u},\mathsf{v}',\mathsf{w}')$ is the polar circle of the triangle with vertices $\mathsf{a}\cap\mathsf{u}$, $\mathsf{b}'\cap\mathsf{v}'$, $\mathsf{c}'\cap\mathsf{w}'$, and $\Gamma(\mathsf{u}',\mathsf{v},\mathsf{w})$ is the polar circle of the triangle with vertices $\mathsf{a}\cap\mathsf{u}'$, $\mathsf{b}'\cap\mathsf{v}$, $\mathsf{c}'\cap\mathsf{w}$.

(b) As the triangles $\mathbf{T}(\mathsf{a},\mathsf{b}',\mathsf{c})$, $\mathbf{T}(\mathsf{u}',\mathsf{v},\mathsf{w}')$, and $\mathbf{T}(\mathsf{u},\mathsf{v}',\mathsf{w})$ are perspective at $J_B=\mathsf{a}'\cap\mathsf{b}\cap c'$, it follows that

- the circles $\Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}')$ and $\Gamma(\mathsf{u},\mathsf{v}',\mathsf{w})$ are orthogonal and pass through J_B,
- the points $\mathsf{a}\cap\mathsf{u}'$, $\mathsf{b}'\cap\mathsf{v}$, and $\mathsf{c}\cap\mathsf{w}'$ are collinear; so are $\mathsf{a}\cap\mathsf{u}$, $\mathsf{b}'\cap\mathsf{v}'$, and $\mathsf{c}\cap\mathsf{w}$,
- the circle $\Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}')$ is the polar circle of the triangle with vertices $\mathsf{a}'\cap\mathsf{u}'$, $\mathsf{b}\cap\mathsf{v}$, $\mathsf{c}'\cap\mathsf{w}'$, and $\Gamma(\mathsf{u},\mathsf{v}',\mathsf{w})$ is the polar circle of the triangle with vertices $\mathsf{a}'\cap\mathsf{u}$, $\mathsf{b}\cap\mathsf{v}'$, $\mathsf{c}'\cap\mathsf{w}$.

(c) As the triangles $\mathbf{T}(\mathsf{a},\mathsf{b},\mathsf{c}')$, $\mathbf{T}(\mathsf{u}',\mathsf{v}',\mathsf{w})$, and $\mathbf{T}(\mathsf{u},\mathsf{v},\mathsf{w}')$ are perspective at $J_C=\mathsf{a}'\cap\mathsf{b}'\cap c$, it follows that

- the circles $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w})$ and $\Gamma(\mathsf{u},\mathsf{v},\mathsf{w}')$ are orthogonal and pass through J_C,
- the points $\mathsf{a}\cap\mathsf{u}'$, $\mathsf{b}\cap\mathsf{v}'$, and $\mathsf{c}'\cap\mathsf{w}$ are collinear; so are $\mathsf{a}\cap\mathsf{u}$, $\mathsf{b}\cap\mathsf{v}$, and $\mathsf{c}'\cap\mathsf{w}'$,
- the circle $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w})$ is the polar circle of the triangle with vertices $\mathsf{a}'\cap\mathsf{u}'$, $\mathsf{b}'\cap\mathsf{v}'$, $\mathsf{c}\cap\mathsf{w}$, and $\Gamma(\mathsf{u},\mathsf{v},\mathsf{w}')$ is the polar circle of the triangle with vertices $\mathsf{a}'\cap\mathsf{u}$, $\mathsf{b}'\cap\mathsf{v}$, $\mathsf{c}\cap\mathsf{w}'$.

Therefore, we obtain two new complete quadrilaterals:

(1) $\mathcal{Q}_1$ with sidelines those containing the triples of points

$$(\mathsf{a}'\cap\mathsf{u},\ \mathsf{b}'\cap\mathsf{v},\ \mathsf{c}'\cap\mathsf{w}),\ (\mathsf{a}'\cap\mathsf{u},\ \mathsf{b}\cap\mathsf{v}',\ \mathsf{c}\cap\mathsf{w}'),\ (\mathsf{a}\cap\mathsf{u}',\ \mathsf{b}'\cap\mathsf{v},\ \mathsf{c}\cap\mathsf{w}'),\ (\mathsf{a}\cap\mathsf{u}',\ \mathsf{b}\cap\mathsf{v}',\ \mathsf{c}'\cap\mathsf{w}),$$

(2) $\mathcal{Q}_2$ with sidelines those containing the triples of points

$$(\mathsf{a}'\cap\mathsf{u}',\ \mathsf{b}'\cap\mathsf{v}',\ \mathsf{c}'\cap\mathsf{w}'),\ (\mathsf{a}'\cap\mathsf{u}',\ \mathsf{b}\cap\mathsf{v},\ \mathsf{c}\cap\mathsf{w}),\ (\mathsf{a}\cap\mathsf{u},\ \mathsf{b}'\cap\mathsf{v}',\ \mathsf{c}\cap\mathsf{w}),\ (\mathsf{a}\cap\mathsf{u},\ \mathsf{b}\cap\mathsf{v},\ \mathsf{c}'\cap\mathsf{w}').$$

The polar circles of the triangles associated with $\mathcal{Q}_1$ are

$$\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}),\ \Gamma(\mathsf{u},\mathsf{v}',\mathsf{w}),\ \Gamma(\mathsf{u},\mathsf{v},\mathsf{w}').$$

These circles pass through J, J_A, J_B, J_C respectively.

The polar circles of the triangles associated with $\mathcal{Q}_2$ are

$$\Gamma(\mathsf{u},\mathsf{v},\mathsf{w}),\ \Gamma(\mathsf{u},\mathsf{v}',\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}).$$

These circles pass through J, J_A, J_B, J_C respectively. Moreover, by §2.4, the circles in the first group are orthogonal to those in the second group. For example, as u and u' are perpendicular to each other, the circles $\Gamma(\mathsf{u},\mathsf{v},\mathsf{w})$ and $\Gamma(\mathsf{u}',\mathsf{v},\mathsf{w})$ are orthogonal. Now it follows from Lemma 3 applied to $\mathcal{Q}_1$ and $\mathcal{Q}_2$ that

Proposition 8 (Mention [4]). *(1) The following seven circles are members of a pencil Φ:*

$$\Gamma(\mathsf{u},\mathsf{v},\mathsf{w}),\ \Gamma(\mathsf{u},\mathsf{v}',\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}),$$

and those with diameters

$$(\mathsf{a}\cap\mathsf{u}')(\mathsf{a}'\cap\mathsf{u}),\ (\mathsf{b}\cap\mathsf{v}')(\mathsf{b}'\cap\mathsf{v}),\ (\mathsf{c}\cap\mathsf{w}')(\mathsf{c}'\cap\mathsf{w}).$$

(2) The following seven circles are members of a pencil Φ'*:*

$$\Gamma(\mathsf{u}', \mathsf{v}', \mathsf{w}'),\ \Gamma(\mathsf{u}', \mathsf{v}, \mathsf{w}),\ \Gamma(\mathsf{u}, \mathsf{v}', \mathsf{w}),\ \Gamma(\mathsf{u}, \mathsf{v}, \mathsf{w}'),$$

and those with diameters

$$(\mathsf{a} \cap \mathsf{u})(\mathsf{a}' \cap \mathsf{u}'),\ (\mathsf{b} \cap \mathsf{v})(\mathsf{b}' \cap \mathsf{v}'),\ (\mathsf{c} \cap \mathsf{w})(\mathsf{c}' \cap \mathsf{w}').$$

(3) The circles in the two pencils Φ *and* Φ' *are orthogonal.*

This clearly gives Steiner's Theorems 8 and 9.

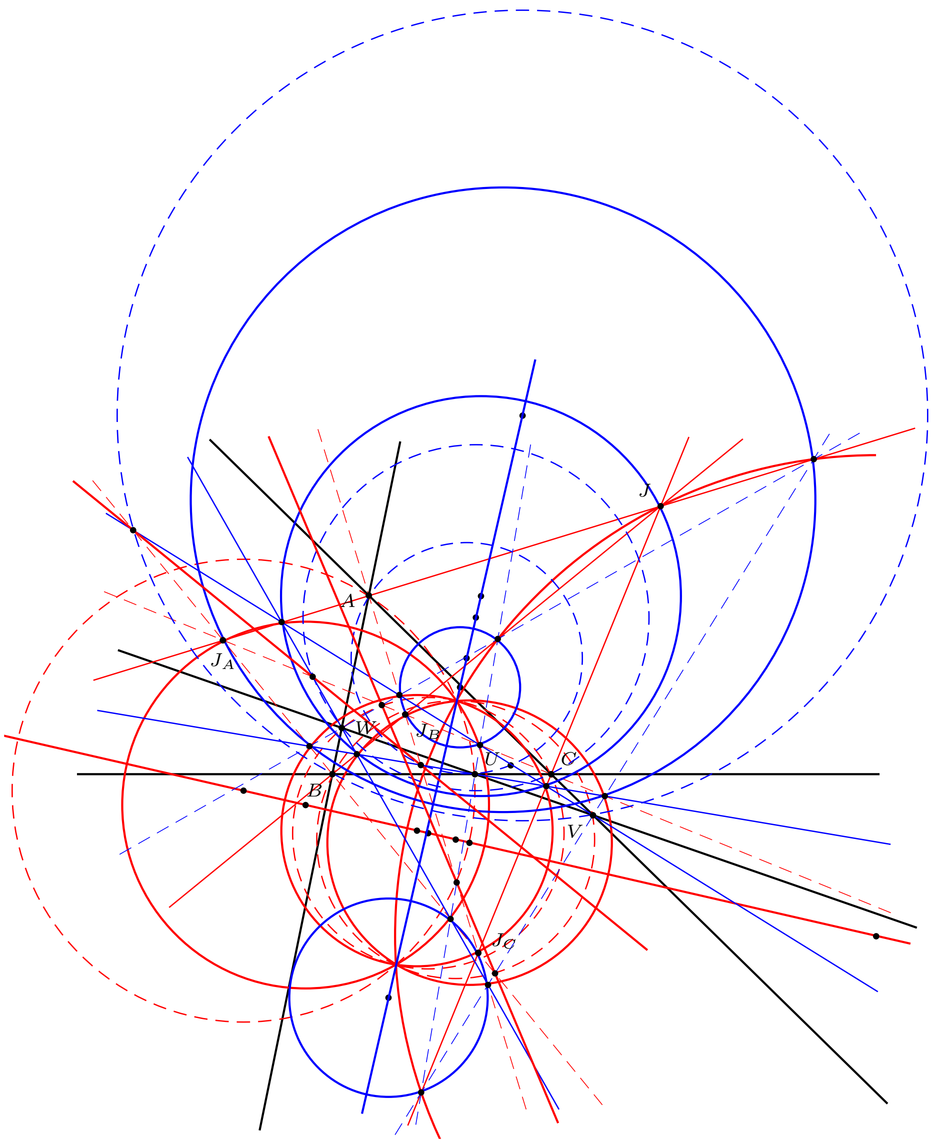


Figure 10

Let P be the midpoint of the segment joining $\mathsf{a} \cap \mathsf{u}'$ and $\mathsf{a}' \cap \mathsf{u}$, and P' the midpoint of the segment joining $\mathsf{a} \cap \mathsf{u}$ and $\mathsf{a}' \cap \mathsf{u}'$. The nine-point circle of the orthocentric system

$$\mathsf{a} \cap \mathsf{u}, \quad \mathsf{a}' \cap \mathsf{u}', \quad \mathsf{a} \cap \mathsf{u}', \quad \mathsf{a}' \cap \mathsf{u}$$

is the circle with diameter PP'. This circle passes through A and U. See Figure 11. Furthermore, P and P' are the midpoints of the two arcs AU of this circle. As P is the center of the circle passing through A, U, $\mathsf{a} \cap \mathsf{u}'$ and $\mathsf{a}' \cap \mathsf{u}$, we have

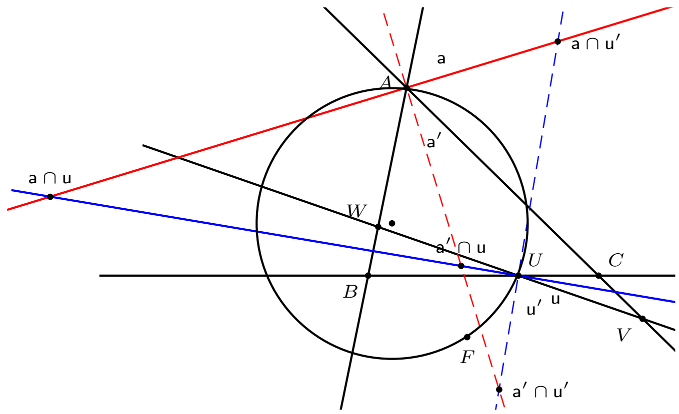


Figure 11.

$$\begin{aligned}
(PA,\ PU) =& 2((\mathsf{a} \cap \mathsf{u}')A,\ (\mathsf{a} \cap \mathsf{u}')U) \\
=& 2((\mathsf{a} \cap \mathsf{u}')A,\ AB) + 2(AB,\ UV) + 2(UV,\ (\mathsf{a} \cap \mathsf{u}')U) \\
=& (AC,\ AB) + 2(AB,\ UV) + (UV, BC) \\
=& (CA,\ CB) + (AB,\ UV) \\
=& (CA,\ CB) + (WB,\ WU) \\
=& (FA,\ FB) + (FB,\ FU) \\
=& (FA,\ FU).
\end{aligned}$$

Hence, F lies on the circle with diameter PP', and the lines FP, FP' bisect the angles between the lines FA and FU.

As the central lines of the pencils Φ and Φ' are perpendicular and pass respectively through P and P', their common point lies on the circle FAU. Similarly, this common point must lie on the circles FBV and FCW. Hence, this common point is F. This proves Steiner's Theorem 10 and the following more general result.

Proposition 9 (Clawson)**.** *The central lines of the pencils Φ and Φ' are the common bisectors of the three pairs of lines $(FA,\ FU)$, $(FB,\ FV)$, and $(FC,\ FW)$.*

Note that, as $(FA,\ FB) = (FV,\ FU) \ = (CA,\ CB)$, it is clear that the three pairs of lines $(FA,\ FU)$, $(FB,\ FV)$, $(FC,\ FW)$ have a common pair of bisectors $(\mathsf{f}, \mathsf{f}')$. These bisectors are called the *incentric lines* of the complete quadrilateral. With the notations of §5.3, we have

$$2(\mathcal{R},\ \mathsf{f}) = 2(\mathcal{R},\ \mathsf{f}') = \theta_1 + \theta_2 + \theta_3 + \theta_4 \bmod \pi.$$

7. Inscribed conics

7.1. *Centers and foci of inscribed conics.* We give some classical properties of the conics tangent to the four sidelines of the complete quadrilateral.

Proposition 10. *The locus of the centers of the conics inscribed in the complete quadrilateral is the Newton line $\mathcal{R}''$.*

Proposition 11. *The locus of the foci of these conics is a circular focal cubic* (van Rees focal).

This cubic γ passes through A, B, C, U, V, W, F, the circular points at infinity I_∞, J_∞ and the feet of the altitudes of the diagonal triangle.

The real asymptote is the image of the Newton line under the homothety $\mathsf{h}(F, 2)$, and the imaginary asymptotes are the lines FI_∞ and FJ_∞. In other words, F is the singular focus of γ. As F lies on the γ, γ is said to be *focal*. The cubic γ is self isogonal with respect to each of the four triangles ABC, AVW, BWU, CUV. It has barycentric equation

$$\begin{aligned} &ux\left(c^2y^2+b^2z^2\right)+vy\left(a^2z^2+c^2x^2\right)+wz\left(b^2x^2+a^2y^2\right)\\ &+2\left(S_Au+S_Bv+S_Cw\right)xyz=0. \end{aligned}$$

If we denote by $\overline{PQRS}$ the van Rees focal of P, Q, R, S, *i.e.*, the locus of M such as $(MP, MQ) = (MR, MS)$, then

$$\gamma = \overline{ABVU} = \overline{BCWV} = \overline{CAUW} = \overline{AVBU} = \overline{BWCV} = \overline{CUAW}.$$

Here is a construction of the cubic γ.

Construction. Consider a variable circle through the pair of isogonal conjugate points on the Newton line. [3] Draw the lines through F tangent to the circle. The locus of the points of tangency is the cubic γ. See Figure 12

7.2. *Orthoptic circles.* Recall that the Monge (or orthoptic) circle of a conic is the locus of M from which the tangents to the conic are perpendicular.

Proposition 12 (Oppermann)**.** *The circles of the pencil generated by the three circles with diameters AU, BV, CW are the Monge circle's of the conics inscribed in the complete quadrilateral.*

Proof. See [5, pp.60–61]. □

[3] These points are not necessarily real.

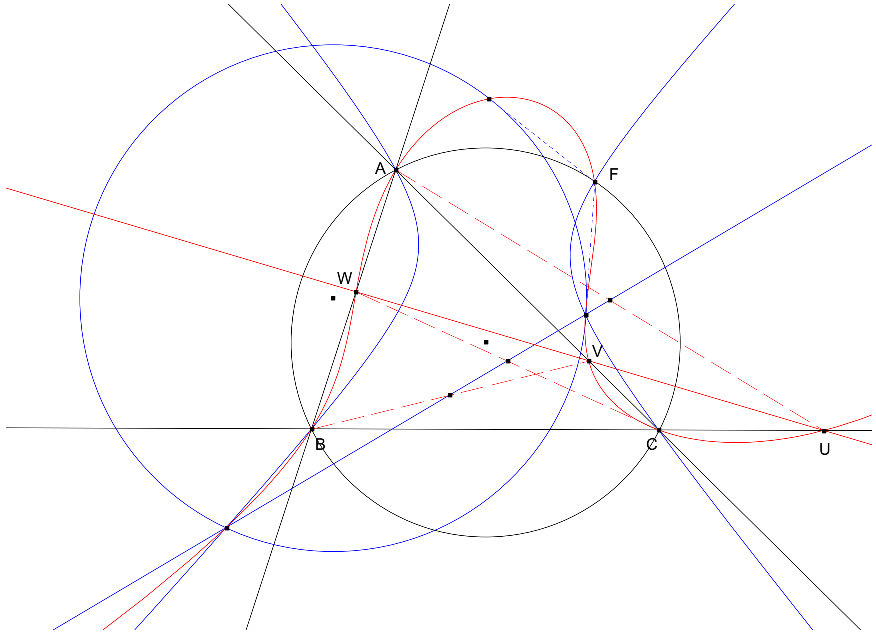


Figure 12.

7.3. *Coordinates and equations.* Recall that the perspector (or Brianchon point) of a conic inscribed in the triangle ABC is the perspector of ABC and the contact triangle. Suppose the perspector is the point $(p : q : r)$.

(1) The center of the conic is the point

$$(p(q+r) : q(r+p) : r(p+q)).$$

(2) The equation of the conic is

$$\frac{x^2}{p^2}+\frac{y^2}{q^2}+\frac{z^2}{r^2}-2\frac{xy}{pq}-2\frac{yz}{qr}-2\frac{zx}{rp}=0.$$

(3) The line $\frac{x}{u}+\frac{y}{v}+\frac{z}{w}=0$ is tangent to the conic if and only if $\frac{u}{p}+\frac{v}{q}+\frac{w}{r}=0$.

(4) The equation of the Monge circle of the conic is

$$\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)(a^2yz+b^2zx+c^2xy)=(x+y+z)\left(\frac{S_A}{p}x+\frac{S_B}{q}y+\frac{S_C}{r}z\right).$$

The locus of the perspectors of the conics inscribed in the complete quadrilateral is the circumconic

$$\frac{u}{x}+\frac{v}{y}+\frac{w}{z}=0,$$

i.e., the circumconic with perspector Q.

7.4. *Inscribed parabola.*

Proposition 13. *The only parabola inscribed in the quadrilateral is the parabola with focus F and directrix the orthocentric line $\mathcal{R}'$.*

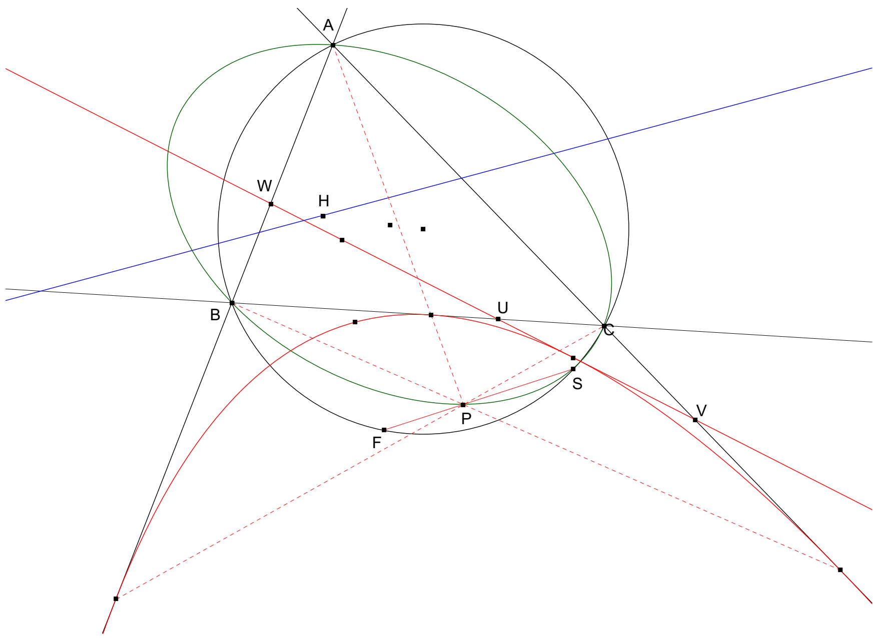


Figure 13

The perspector of the parabola has barycentric coordinates

$$\left(\frac{1}{v-w} : \frac{1}{w-u} : \frac{1}{u-v}\right).$$

This point is the isotomic conjugate of the infinite point of the Newton line. It is also the second common point (apart from the Steiner point S of triangle ABC) of the line SF and the Steiner circum-ellipse.

If a line ℓ' tangent to the parabola intersects the lines BC, CA, AB respectively at U', V', W', we have

$$(FU,\ FU') = (FV,\ FV') = (FW,\ FW') = (\ell,\ \ell').$$

If four points P, Q, R, S lie respectively on the sidelines BC, CA, AB, ℓ and verify

$$(FP,\ BC) = (FQ,\ CA) = (FR,\ AB) = (FS,\ \ell),$$

then these four points lie on the same line tangent to the parabola. This is a generalization of the pedal line.

References

[1] J. W. Clawson, The complete quadrilateral, *Annals of Mathematics*, ser. 2, 20 (1919) 232–261.
[2] F. G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, Paris, 1991.
[3] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.
[4] Mention, *Nouvelles Annales de Mathematiques*, (1862) 76.
[5] A. Oppermann, *Premiers elements d'une theorie du quadrilatere complet*, Gauthier-Villars, Paris, 1919.
[6] L. Ripert, *Compte rendu de l'Association pour l'avancement des Sciences*, 30 (1901) part 2, 91.
[7] L. Sancery, *Nouvelles Annales de Mathematiques*, (1875) 145.
[8] P. Serret, *Nouvelles Annales de Mathematiques* (1848) p. 214.
[9] J. Steiner, *Annales de Gergonne*, XVIII (1827) 302; reprinted in *Gesammelte Werke*, 2 volumes, edited by K. Weierstrass, 1881; Chelsea reprint.
[10] P. Terrier, *Nouvelles Annales de Mathematiques* (1875) 514.
[11] Van Rees, *Correspondance mathematique et physique*, V (1829) 361–378.

Jean-Pierre Ehrmann: 6, rue des Cailloux, 92110 - Clichy, France
E-mail address: `Jean-Pierre.EHRMANN@wanadoo.fr`

Forum Geometricorum
Volume 4 (2004) 53–59.

FORUM GEOM
ISSN 1534-1178

Orthopoles and the Pappus Theorem

Atul Dixit and Darij Grinberg

Abstract. If the vertices of a triangle are projected onto a given line, the perpendiculars from the projections to the corresponding sidelines of the triangle intersect at one point, the orthopole of the line with respect to the triangle. We prove several theorems on orthopoles using the Pappus theorem, a fundamental result of projective geometry.

1. Introduction

Theorems on orthopoles are often proved with the help of coordinates or complex numbers. In this note we prove some theorems on orthopoles by using a well-known result from projective geometry, the Pappus theorem. Notably, we need not even use it in the general case. What we need is a simple affine theorem which is a special case of the Pappus theorem. We denote the intersection of two lines g and g$'$ by g $\cap$ g$'$. Here is the Pappus theorem in the general case.

Theorem 1. *Given two lines in a plane, let A, B, C be three points on one line and A', B', C' three points on the other line. The three points*

$$BC' \cap CB', \qquad CA' \cap AC', \qquad AB' \cap BA'$$

are collinear.

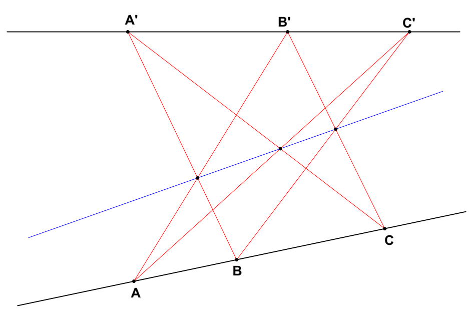


Figure 1

Theorem 1 remains valid if some of the points A, B, C, A', B', C' are projected to infinity, even if one of the two lines is the line at infinity. In this paper, the only case we need is the special case if the points A', B', C' are points at infinity. For the sake of completeness, we give a proof of the Pappus theorem for this case.

Publication Date: March 30, 2004. Communicating Editor: Paul Yiu.

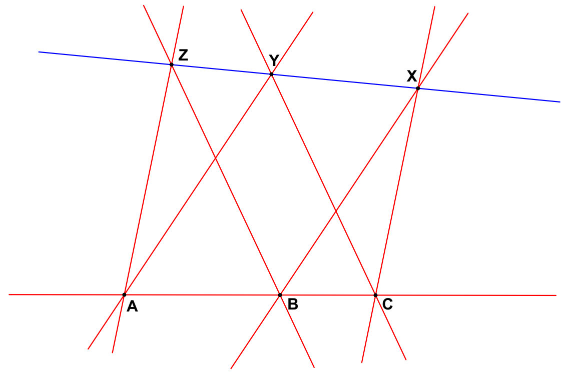


Figure 2

Let $X = BC' \cap CB'$, $Y = CA' \cap AC'$, $Z = AB' \cap BA'$. The points A', B', C' being infinite points, we have $CY \parallel BZ$, $AZ \parallel CX$, and $BX \parallel AY$. See Figure 2. We assume the lines ZX and ABC intersect at a point P, and leave the easy case $ZX \parallel ABC$ to the reader. In Figure 3, let $Y' = ZX \cap AY$. We show that $Y' = Y$. Since $AY \parallel BX$, we have $\frac{PA}{PB} = \frac{PY'}{PX}$ in signed lengths. Since $AZ \parallel CX$, we have $\frac{PC}{PA} = \frac{PX}{PZ}$. From these, $\frac{PC}{PB} = \frac{PY'}{PZ}$, and $CY' \parallel BZ$. Since $CY \parallel BZ$, the point Y' lies on the line CY. Thus, $Y' = Y$, and the points X, Y, Z are collinear.

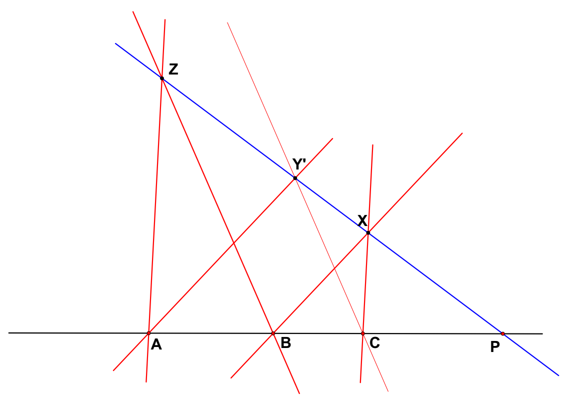


Figure 3

2. The orthocenters of a fourline

We denote by $\Delta\mathsf{abc}$ the triangle bounded by three lines a, b, c. A complete quadrilateral, or, simply, a fourline is a set of four lines in a plane. The fourline consisting of lines a, b, c, d, is denoted by $\square\mathsf{abcd}$. If g is a line, then all lines perpendicular to g have an infinite point in common. This infinite point will be called $\overline{\mathsf{g}}$. With this notation, $P\overline{\mathsf{g}}$ is the perpendicular from P to g. Now, we establish the well-known Steiner's theorem.

Theorem 2 (Steiner)**.** *If* a, b, c, d *are any four lines, the orthocenters of* $\Delta\mathsf{bcd}$, $\Delta\mathsf{acd}$, $\Delta\mathsf{abd}$, $\Delta\mathsf{abc}$ *are collinear.*

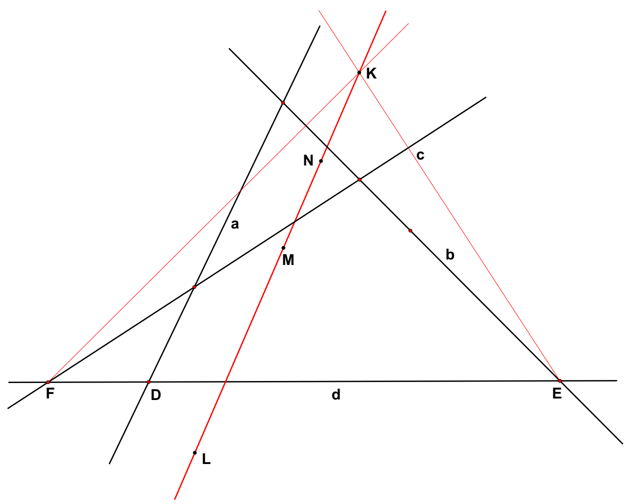


Figure 4

Proof. Let D, E, F be the intersections of d with a, b, c, and K, L, M, N the orthocenters of $\Delta\mathsf{bcd}$, $\Delta\mathsf{acd}$, $\Delta\mathsf{abd}$, and $\Delta\mathsf{abc}$. Note that $K = E\overline{\mathsf{c}} \cap F\overline{\mathsf{b}}$, being the intersection of the perpendiculars from E to c and from F to b. Similarly, $L = F\overline{\mathsf{a}} \cap D\overline{\mathsf{c}}$ and $M = D\overline{\mathsf{b}} \cap E\overline{\mathsf{a}}$. The points D, E, F being collinear and the points $\overline{\mathsf{a}}$, $\overline{\mathsf{b}}$, $\overline{\mathsf{c}}$ being infinite, we conclude from the Pappus theorem that K, L, M are collinear. Similarly, L, M, N are collinear. The four orthocenters lie on the same line. $\square$

The line $KLMN$ is called the Steiner line of the fourline $\square ABCD$. Theorem 2 is usually associated with Miquel points [6, §9] and proved using radical axes. A consequence of such proofs is the fact that the Steiner line of the fourline $\square\mathsf{abcd}$ is the radical axis of the circles with diameters AD, BE, CF, where $A = \mathsf{b} \cap \mathsf{c}$, $B = \mathsf{c} \cap \mathsf{a}$, $C = \mathsf{a} \cap \mathsf{b}$, $D = \mathsf{d} \cap \mathsf{a}$, $E = \mathsf{d} \cap \mathsf{b}$, $F = \mathsf{d} \cap \mathsf{c}$. Also, the Steiner line is the directrix of the parabola touching the four lines a, b, c, d. The Steiner line is also called four-orthocenter line in [6, §11] or the orthocentric line in [5], where it is studied using barycentric coordinates.

3. The orthopole and the fourline

We prove the theorem that gives rise to the notion of orthopole.

Theorem 3. *Let ΔABC be a triangle and d a line. If A', B', C' are the pedals of A, B, C on d, then the perpendiculars from A', B', C' to the lines BC, CA, AB intersect at one point.*

This point is the orthopole of the line d with respect to ΔABC.

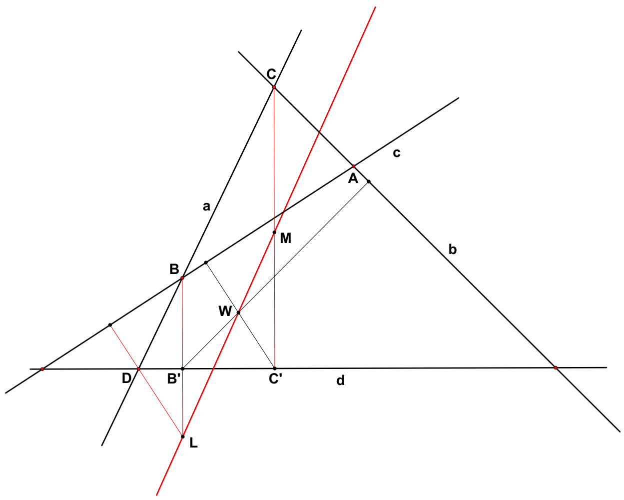


Figure 5

Proof. Denote by a, b, c the lines BC, CA, AB. By Theorem 2, the orthocenters K, L, M, N of triangles $\Delta\mathsf{bcd}$, $\Delta\mathsf{acd}$, $\Delta\mathsf{abd}$, $\Delta\mathsf{abc}$ lie on a line. Let $D = \mathsf{d} \cap \mathsf{a}$, and $W = B'\overline{\mathsf{b}} \cap C'\overline{\mathsf{c}}$. The orthocenter L of $\Delta\mathsf{acd}$ is the intersection of the perpendiculars from D to c and from B to d. Since the perpendicular from B to d is also the perpendicular from B' to d, $L = D\overline{\mathsf{c}} \cap B'\overline{\mathsf{d}}$. Analogously, $M = D\overline{\mathsf{b}} \cap C'\overline{\mathsf{d}}$. By the Pappus theorem, the points W, M, L are collinear. Hence, W lies on the line $KLMN$. Since $W = B'\overline{\mathsf{b}} \cap C'\overline{\mathsf{c}}$, the intersection W of the lines $KLMN$ and $B'\overline{\mathsf{b}}$ lies on $C'\overline{\mathsf{c}}$. Similarly, this intersection W lies on $A'\overline{\mathsf{a}}$. Hence, the point W is the common point of the four lines $A'\overline{\mathsf{a}}$, $B'\overline{\mathsf{b}}$, $C'\overline{\mathsf{c}}$, and $KLMN$. Since $A'\overline{\mathsf{a}}$, $B'\overline{\mathsf{b}}$, $C'\overline{\mathsf{c}}$ are the perpendiculars from A', B', C' to a, b, c respectively, the perpendiculars from A', B', C' to BC, CA, AB and the line $KLMN$ intersect at one point. This already shows more than the statement of the theorem. In fact, we conclude that the orthopole of d with respect to triangle ΔABC lies on the Steiner line of the complete quadrilateral $\square\mathsf{abcd}$. $\square$

The usual proof of Theorem 3 involves similar triangles ([1], [10, Chapter 11]) and does not directly lead to the fourline. Theorem 4 originates from R. Goormaghtigh, published as a problem [7]. It was also mentioned in [5, Proposition 6], with reference to [2]. The following corollary is immediate.

Corollary 4. *Given a fourline* $\square$abcd*, the orthopoles of* a, b, c, d *with respect to* $\triangle$bcd, $\triangle$acd, $\triangle$abd, $\triangle$abc *lie on the Steiner line of the fourline.*

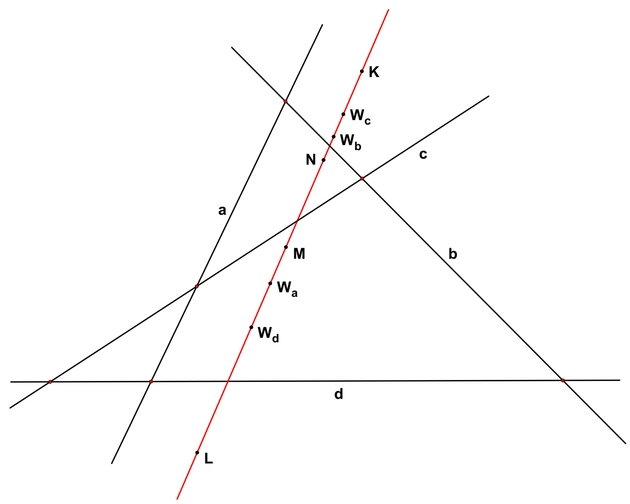


Figure 6

4. Two theorems on the collinarity of quadruples of orthopoles

Theorem 5. *If A, B, C, D are four points and* e *is a line, then the orthopoles of* e *with respect to triangles ΔBCD, ΔCDA, ΔDAB, ΔABC are collinear.*

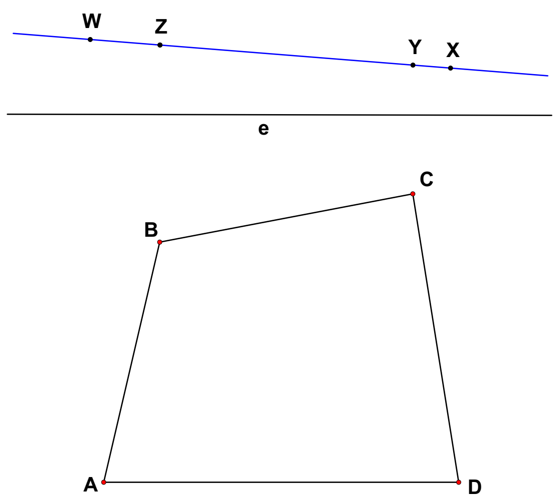


Figure 7

Proof. Denote these orthopoles by X, Y, Z, W respectively. If A', B', C', D' are the pedals of A, B, C, D on e, then $X = B'\overline{CD} \cap C'\overline{BD}$. Similarly, $Y = C'\overline{AD} \cap A'\overline{CD}$, $Z = A'\overline{BD} \cap B'\overline{AD}$. Now, A', B', C' lie on one line, and $\overline{AD}$, $\overline{BD}$, $\overline{CD}$ lie on the line at infinity. By Pappus' theorem, the points X, Y, Z are collinear. Likewise, Y, Z, W are collinear. We conclude that all four points X, Y, Z, W are collinear. $\square$

Theorem 5 was also proved using coordinates by N. Dergiades in [3] and by R. Goormaghtigh in [8, p.178]. A special case of Theorem 5 was shown in [11] using the Desargues theorem.[1] Another theorem surprisingly similar to Theorem 5 was shown in [9] using complex numbers.

Theorem 6. *Given five lines* a, b, c, d, e, *the orthopoles of* e *with respect to* $\Delta\mathsf{bcd}$, $\Delta\mathsf{acd}$, $\Delta\mathsf{abd}$, $\Delta\mathsf{abc}$ *are collinear.*

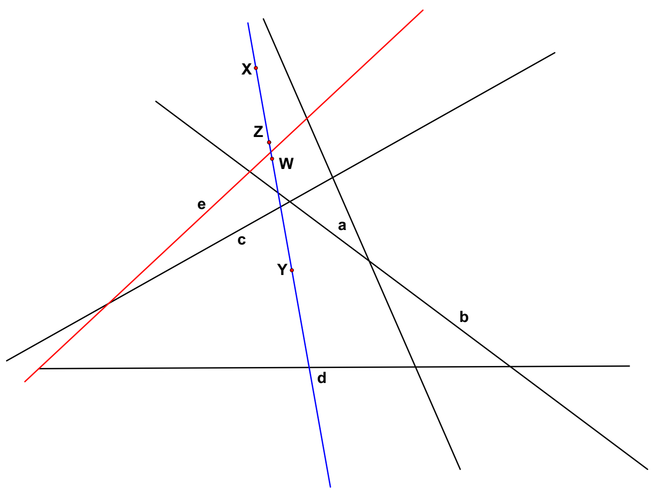


Figure 8

Proof. Denote these orthopoles by X, Y, Z, W respectively. Let the line d intersect a, b, c at D, E, F, and let D', E', F' be the pedals of D, E, F on e.

Since $E = \mathsf{b} \cap \mathsf{d}$ and $F = \mathsf{c} \cap \mathsf{d}$ are two vertices of triangle $\Delta\mathsf{bcd}$, and E' and F' are the pedals of these vertices on e, the orthopole $X = E'\overline{\mathsf{c}} \cap F'\overline{\mathsf{b}}$. Similarly, $Y = F'\overline{\mathsf{a}} \cap D'\overline{\mathsf{c}}$, and $Z = D'\overline{\mathsf{b}} \cap E'\overline{a}$. Since D', E', F' lie on one line, and $\overline{\mathsf{a}}$, $\overline{\mathsf{b}}$, $\overline{\mathsf{c}}$ lie on the line at infinity, the Pappus theorem yields the collinearity of the points X, Y, Z. Analogously, the points Y, Z, W are collinear. The four points X, Y, Z, W are on the same line. $\square$

[1] In [11], Witczyński proves Theorem 5 for the case when A, B, C, D lie on one circle and the line e crosses this circle. Instead of orthopoles, he equivalently considers Simson lines. The Simson lines of two points on the circumcircle of a triangle intersect at the orthopole of the line joining the two points.

References

[1] A. Bogomolny, Orthopole, http://cut-the-knot.com/Curriculum/Geometry/Orthopole.shtml.
[2] J. W. Clawson, The complete quadrilateral, *Annals of Mathematics*, Ser. 2, 20 (1919) 232–261.
[3] N. Dergiades, Hyacinthos message 3352, August 5, 2001.
[4] A. Dixit, Hyacinthos message 3340, August 4, 2001.
[5] J.-P. Ehrmann, Steiner's theorems on the complete quadrilateral, *Forum Geom.*, 4 (2004) 35–52.
[6] W. Gallatly, *The Modern Geometry of the Triangle*, 2nd ed. 1913, Francis Hodgson, London.
[7] R. Goormaghtigh, Question 2388, *Nouvelles Annales de Mathématiques*, Série 4, 19 (1919) 39.
[8] R. Goormaghtigh, A study of a quadrilateral inscribed in a circle, *Amer. Math. Monthly*, 49 (1942) 174–181.
[9] C. Hsu, On a certain collinearity of orthopoles, and of isopoles, *Soochow J. Math.*, 10 (1984) 27–31.
[10] R. Honsberger, *Episodes of 19th and 20th Century Euclidean Geometry*, Math. Assoc. America, 1995.
[11] K. Witczyński, On collinear Griffiths points, *Journal of Geometry*, 74 (2002) 157–159.

Atul Abhay Dixit: 32, Snehbandhan Society, Kelkar Road, Ramnagar, Dombivli (East) 421201, Mumbai, Maharashtra, India
E-mail address: `atul_dixie@hotmail.com`

Darij Grinberg: Geroldsäckerweg 7, D-76139 Karlsruhe, Germany
E-mail address: `darij_grinberg@web.de`

Forum Geometricorum
Volume 4 (2004) 61–65.

FORUM GEOM
ISSN 1534-1178

On the Areas of the Intouch and Extouch Triangles

Juan Carlos Salazar

Abstract. We prove an interesting relation among the areas of the triangles whose vertices are the points of tangency of the sidelines with the incircle and excircles.

1. The intouch and extouch triangles

Consider a triangle ABC with incircle touching the sides BC, CA, AB at A_0, B_0, C_0 respectively. The triangle $A_0B_0C_0$ is called the intouch triangle of ABC. Likewise, the triangle formed by the points of tangency of an excircle with the sidelines is called an extouch triangle. There are three of them, the A-, B-, C-extouch triangles, [1] as indicated in Figure 1. For $i = 0, 1, 2, 3$, let T_i denote the area of triangle $A_iB_iC_i$. In this note we present two proofs of a simple interesting relation among the areas of these triangles.

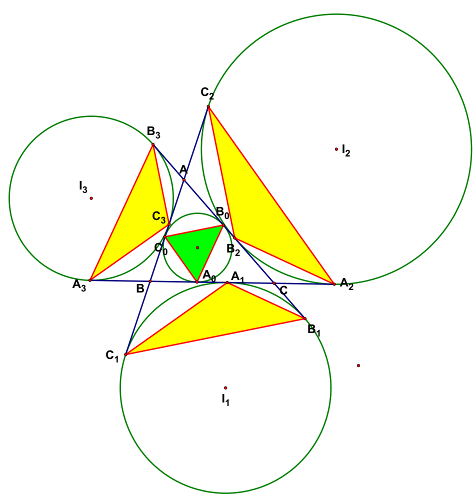


Figure 1

Theorem 1. $\frac{1}{T_0} = \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3}$.

Publication Date: April 14, 2004. Communicating Editor: Paul Yiu.

[1]These qualified extouch triangles are not the same as the extouch triangle in [2, §6.9], which means triangle $A_1B_2C_3$ in Figure 1. For a result on this unqualified extouch triangle, see §3.

Proof. Let I be the incenter and r the inradius of triangle ABC. Consider the excircle on the side BC, with center I_1, tangent to the lines BC, CA, AB at A_1, B_1, C_1 respectively. See Figure 2. It is easy to see that triangles $I_1A_1C_1$ and BA_0C_0 are similar isosceles triangles; so are triangles $I_1A_1B_1$ and CA_0B_0. From these, it easily follows that the angles $B_0A_0C_0$ and $B_1I_1C_1$ are supplementary. It follows that

$$\frac{T_0}{T_1} = \frac{A_0B_0 \cdot A_0C_0}{A_1B_1 \cdot A_1C_1} = \frac{IC}{I_1C} \cdot \frac{IB}{I_1B} = \frac{IB \cdot IC}{I_1B \cdot I_1C}.$$

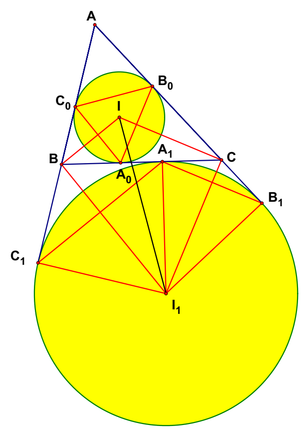


Figure 2

Now, in the cyclic quadrilateral IBI_1C with diameter II_1,

$$IB \cdot IC = IB \cdot II_1 \sin II_1C = II_1 \cdot IA_0 = r \cdot II_1.$$

Similarly, $I_1B \cdot I_1C = II_1 \cdot r_1$, where r_1 is the radius of the A-excircle. It follows that

$$\frac{T_0}{T_1} = \frac{r}{r_1}. \tag{1}$$

Likewise, $\frac{T_0}{T_2} = \frac{r}{r_2}$ and $\frac{T_0}{T_3} = \frac{r}{r_3}$, where r_2 and r_3 are respectively the radii of the B- and C-excircles. From these,

$$\frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} = \left(\frac{r}{r_1} + \frac{r}{r_2} + \frac{r}{r_3}\right)\frac{1}{T_0} = \frac{1}{T_0},$$

since $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$. $\square$

Corollary 2. *Let $ABCD$ be a quadrilateral with an incircle $I(r)$ tangent to the sides at W, X, Y, Z. If the excircles $I_W(r_W)$, $I_X(r_X)$, $I_Y(r_Y)$, $I_Z(r_Z)$ have areas T_W, T_X, T_Y, T_Z respectively, then*

$$\frac{T_W}{r_W} + \frac{T_Y}{r_Y} = \frac{T_X}{r_X} + \frac{T_Z}{r_Z} = \frac{T}{r},$$

where T is the area of the intouch quadrilateral $WXYZ$. See Figure 3.

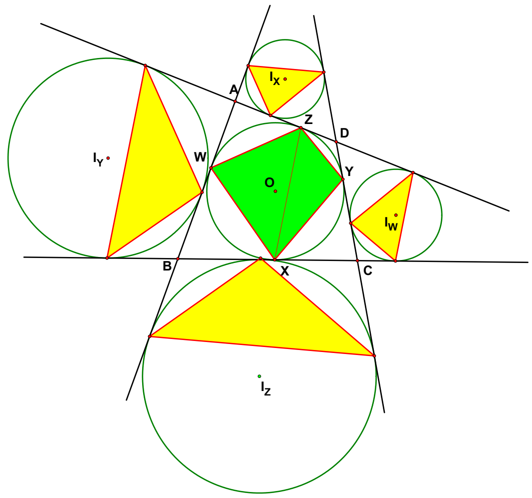


Figure 3

Proof. By (1) above, we have $\frac{T_W}{r_W} = \frac{\text{Area } XYZ}{r}$ and $\frac{T_Y}{r_Y} = \frac{\text{Area } ZWX}{r}$ so that

$$\frac{T_W}{r_W} + \frac{T_Y}{r_Y} = \frac{\text{Area } XYZ + \text{Area } ZWX}{r} = \frac{T}{r}.$$

Similarly, $\frac{T_X}{r_X} + \frac{T_Z}{r_Z} = \frac{T}{r}$. □

2. An alternative proof using barycentric coordinates

The area of a triangle can be calculated easily from its barycentric coordinates. Denote by Δ the area of the reference triangle ABC. The area of a triangle with vertices $A' = (x_1 : y_1 : z_1)$, $B' = (x_2 : y_2 : z_2)$, $C' = (x_3 : y_3 : z_3)$ is given by

$$\frac{\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Delta}{(x_1+y_1+z_1)(x_2+y_2+z_2)(x_3+y_3+z_3)}. \tag{2}$$

Note that this area is signed. It is positive or negative according as triangle $A'B'C'$ has the same or opposite orientation as the reference triangle. See, for example, [3]. In particular, the area of the cevian triangle of a point with coordinates $(x : y : z)$ is

$$\frac{\begin{vmatrix} 0 & y & z \\ x & 0 & z \\ x & y & 0 \end{vmatrix} \Delta}{(y+z)(z+x)(z+y)} = \frac{2xyz\Delta}{(y+z)(z+x)(z+y)}. \tag{3}$$

Let s denote the semiperimeter of triangle ABC, *i.e.*, $s = \frac{1}{2}(a+b+c)$.

The barycentric coordinates of the vertices of the intouch triangle are

$$A_0 = (0 : s-c : s-b), \quad B_0 = (s-c : 0 : s-a), \quad C_0 = (s-b : s-a : 0). \tag{4}$$

The area of the intouch triangle is

$$\begin{aligned} T_0 =& \frac{1}{abc}\begin{vmatrix} 0 & s-c & s-b \\ s-c & 0 & s-a \\ s-b & s-a & 0 \end{vmatrix}\Delta \\ =& \frac{2(s-a)(s-b)(s-c)}{abc}\Delta. \end{aligned}$$

For the A-extouch triangle $A_1B_1C_1$,

$$A_1 = (0 : s-b : s-c), \quad B_1 = (-(s-b) : 0 : s), \quad C_1 = (-(s-c) : s : 0), \tag{5}$$

the area is

$$\frac{1}{abc}\begin{vmatrix} 0 & s-b & s-c \\ -(s-b) & 0 & s \\ -(s-c) & s & 0 \end{vmatrix}\Delta = \frac{-2s(s-b)(s-c)}{abc}\Delta.$$

Similarly, the areas of the B- and C-extouch triangles are $\frac{-2s(s-c)(s-a)}{abc}\Delta$ and $\frac{-2s(s-a)(s-b)}{abc}\Delta$ respectively. Note that these are all negative. Disregarding signs, we have

$$\begin{aligned} \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} =& \frac{abc}{2s(s-a)(s-b)(s-c)}\left((s-a)+(s-b)+(s-c)\right) \cdot \frac{1}{\Delta} \\ =& \frac{abc}{2(s-a)(s-b)(s-c)} \cdot \frac{1}{\Delta} \\ =& \frac{1}{T_0}. \end{aligned}$$

3. A generalization

Using the area formula (3) it is easy to see that the (unqualifed) extouch triangle $A_1B_2C_3$ has the same area T_0 as the intouch triangle. This is noted, for example, in [1]. The use of coordinates in §2 also leads to a more general result. Replace the incircle by the inscribed conic with center $P = (p : q : r)$, and the excircles by those with centers

$$P_1 = (-p : q : r), \quad P_2 = (p : -q : r), \quad P_3 = (p : q : -r),$$

respectively. These are the vertices of the anticevian triangle of P, and the four inscribed conics are homothetic. See Figure 4. The coordinates of their points of tangency with the sidelines can be obtained from (4) and (5) by replacing a, b, c by p, q, r respectively. It follows that the areas of intouch and extouch triangles for these conics bear the same relation given in Theorem 1.

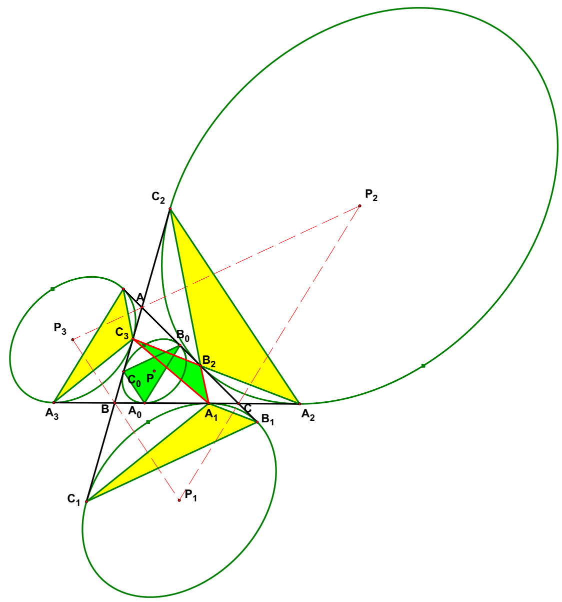


Figure 4

References

[1] M. Dalcín, Isotomic inscribed triangles and their residuals, *Forum Geom.*, 3 (2003) 125–134.

[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[3] P. Yiu, The uses of homogeneous barycentric coordinates in plane euclidean geometry, *Int. J. Math. Educ. Sci. Technol.*, 31 (2000) 569–578.

Juan Carlos Salazar: Calle Maturín N$^{\mathrm{o}}$C 19, Urb., Mendoza, Puerto Ordaz 8015, Estado Bolívar, Venezuela

E-mail address: `caisersal@yahoo.com`

Forum Geometricorum
Volume 4 (2004) 67–68.

FORUM GEOM
ISSN 1534-1178

Signed Distances and the Erdős-Mordell Inequality

Nikolaos Dergiades

Abstract. Using signed distances from the sides of a triangle we prove an inequality from which we get the Erdős-Mordell inequality as a simple consequence.

Let P be an arbitrary point in the plane of triangle ABC. Denote by x_1, x_2, x_3 the distances of P from the vertices A, B, C, and d_1, d_2, d_3 the *signed* distances of P from the sidelines BC, CA, AB respectively. Let a, b, c be the lengths of these sides. We establish an inequality from which the famous Erdős-Mordell inequality easily follows.

Theorem.

$$x_1 + x_2 + x_3 \geq \left(\frac{b}{c} + \frac{c}{b}\right) d_1 + \left(\frac{c}{a} + \frac{a}{c}\right) d_2 + \left(\frac{a}{b} + \frac{b}{a}\right) d_3; \qquad (1)$$

equality holds if and only if P is the circumcenter of ABC.

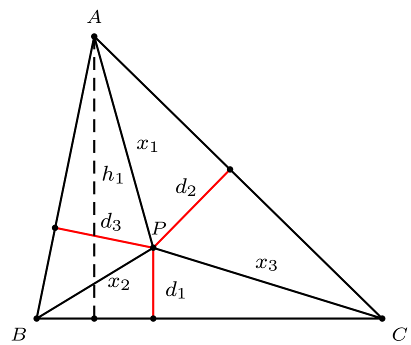


Figure 1

Proof. Let h_1 be the length of the altitude from A to BC, and Δ the area of ABC. Clearly,

$$2\Delta = ah_1 = ad_1 + bd_2 + cd_3.$$

Note that $x_1 + d_1 \geq h_1$. This is true even if $d_1 < 0$, *i.e.*, when P is not an interior point of the triangle. Also, equality holds if and only if P lies on the line containing the A-altitude. We have $ax_1 + ad_1 \geq ah_1 = ad_1 + bd_2 + cd_3$, or

$$ax_1 \geq bd_2 + cd_3. \qquad (2)$$

If we apply inequality (2) to triangle $AB'C'$ symmetric to ABC with respect to the A-bisector of ABC we get

$$ax_1 \geq cd_2 + bd_3$$

Publication Date: April 28, 2004. Communicating Editor: Paul Yiu.

or

$$x_1 \geq \frac{c}{a}d_2 + \frac{b}{a}d_3. \tag{3}$$

Equality holds only when P lies on the A-altitude of $AB'C'$, *i.e.*, the line passing through A and the circumcenter of ABC.

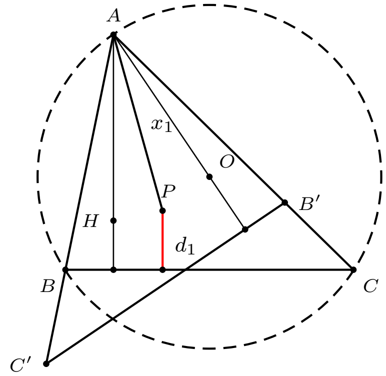


Figure 2

Similarly we get

$$x_2 \geq \frac{a}{b}d_3 + \frac{c}{b}d_1, \tag{4}$$

$$x_3 \geq \frac{b}{c}d_1 + \frac{a}{c}d_2, \tag{5}$$

and by addition of (3), (4), (5), we get the inequality (1). Equality holds only when P is the circumcenter of ABC. □

If P is an internal point of ABC, $d_1,\ d_2,\ d_3 > 0$. Since $\frac{b}{c} + \frac{c}{b} \geq 2$, $\frac{c}{a} + \frac{a}{c} \geq 2$, $\frac{a}{b} + \frac{b}{a} \geq 2$, we have

$$x_1 + x_2 + x_3 \geq 2(d_1 + d_2 + d_3).$$

This is the famous Erdős-Mordell inequality. The equality holds only when $a = b = c$, *i.e.*, ABC is equilateral, and P is the circumcenter of ABC.

There are numerous proofs of the Erdős-Mordell inequality. See, for example, [3] and the bibliography therin. In Mordell's original proof [2], the inequality (1) was established assuming $d_1, d_2, d_3 > 0$. See also [1, §12.13]. Our proof of (1) is more transparent and covers all positions of P.

References

[1] O. Bottema et al, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.
[2] P. Erdős and L. J. Mordell, Problem 3740, *Amer. Math. Monthly*, 42 (1935) 396; solutions, *ibid.*, 44 (1937) 252.
[3] H.J Lee, Another Proof of the Erdős-Mordell Theorem, *Forum Geom.*, 1 (2001) 7–8.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: `ndergiades@yahoo.gr`

Forum Geometricorum
Volume 4 (2004) 69–71.

FORUM GEOM
ISSN 1534-1178

A Simple Construction of the Congruent Isoscelizers Point

Eric Danneels

Abstract. We give a very simple construction of the congruent isoscelizers point as an application of the cevian nest theorem.

1. Construction of the congruent isoscelizers point

Given a triangle, an isoscelizer is a segment intercepted in the interior of the triangle by a line perpendicular to an angle bisector. There is a unique point through which the three isoscelizers have equal lengths. This is the congruent isoscelizers points X_{173} of [4]. In this note we present a very simple construction of this triangle center.

Theorem 1. *Let $A'B'C'$ be the intouch triangle of ABC, and $A''B''C''$ the intouch triangle of $A'B'C'$. The triangles $A''B''C''$ and ABC are perspective at the congruent isoscelizers point of ABC.*

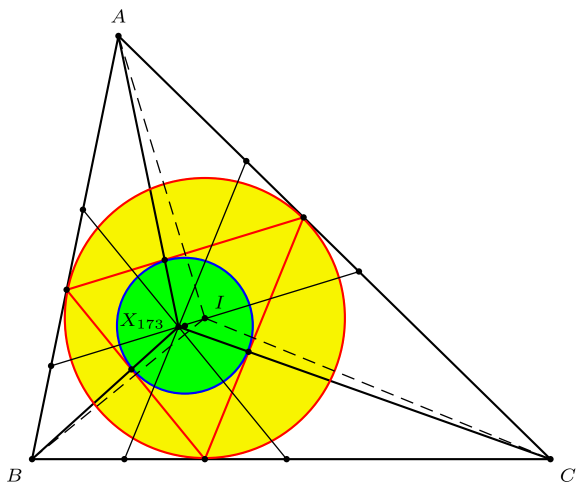


Figure 1

The proof is a simple application of the following cevian nest theorem.[1]

Theorem 2. *Let $A'B'C'$ be the cevian triangle of P in triangle ABC with homogeneous barycentric coordinates $(u : v : w)$ with respect to ABC, and $A''B''C''$*

Publication Date: May 12, 2004. Communicating Editor: Paul Yiu.

[1]Theorem 2 appears in [1, p.165, Supplementary Exercise 7] as follows: The triangle $(Q) = DEF$ is inscribed in the triangle $(P) = ABC$, and the triangle $(K) = KLM$ is inscribed in (Q). Show that if any two of these triangles are perspective to the third, they are perspective to each other.

the cevian triangle of Q in triangle $A'B'C'$, with homogeneous barycentric coordinates $(x : y : z)$ with respect to triangle $A'B'C'$. Triangle $A''B''C''$ is the cevian triangle of

$$Q(P) = \left(\frac{u(v+w)}{x} : \frac{v(w+u)}{y} : \frac{w(u+v)}{z}\right) \tag{1}$$

with respect to triangle ABC.

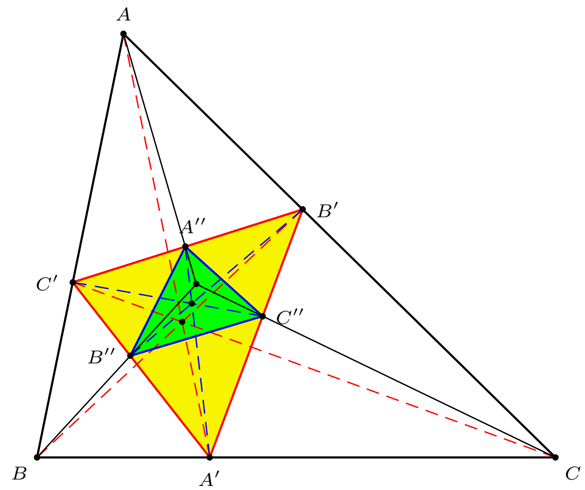


Figure 2

The concurrency of the lines AA'', BB'', CC'' follows from the fact every cevian triangle and every anticevian triangle with respect to $A'B'C'$ are perspective. See, for example, [3, §2.12]. The cevian and anticevian triangles in question are $A''B''C''$ and ABC respectively.

Proof. We compute the absolute barycentric coordinates explicitly.

$$\begin{aligned} A'' =& \frac{yB' + zC'}{y+z} = \frac{y \cdot \frac{wC+uA}{w+u} + z \cdot \frac{uA+vB}{u+v}}{y+z} \\ =& \frac{(y(u+v)+z(w+u))uA + zv(w+u)B + yw(u+v)C}{(y+z)(w+u)(u+v)}. \end{aligned}$$

It is clear that the line AA'' intersects BCat the point with homogeneous barycentric coordinates

$$(0 : zv(w+u) : yw(u+v)) = \left(0 : \frac{v(w+u)}{y} : \frac{w(u+v)}{z}\right).$$

Similarly, the intersections of BB'' with CA, CC'' with AB are the points

$$\left(\frac{u(v+w)}{x} : 0 : \frac{w(u+v)}{z}\right) \quad \text{and} \quad \left(\frac{u(v+w)}{x} : \frac{v(w+u)}{y} : 0\right)$$

respectively. It is clear that the lines AA'', BB'', CC'' intersect at the point given by (1) above. $\square$

2. Proof of Theorem 1

Let P be the Gergonne point, and $A'B'C'$ the intouch triangle. The sidelengths are in the proportions of

$$B'C' : C'A' : A'B' = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}.$$

If Q is the Gergonne point of $A'B'C'$, then we have

$$Q(P) = \left(a\left(-\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right) : \cdots : \cdots\right).$$

This is the point X_{173}, the congruent isoscelizers point.

3. Another example

Let P be the incenter of triangle ABC, with cevian triangle $A'B'C'$, and Q the centroid of $A'B'C'$. Then

$$Q(P) = (a(b+c) : b(c+a) : c(a+b)).$$

This is the triangle center X_{37} of [4].

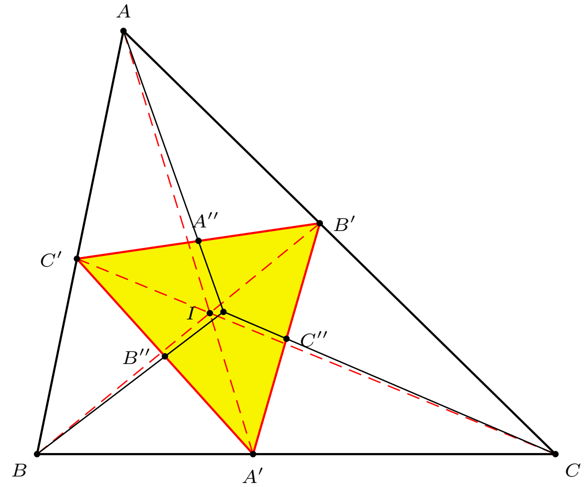


Figure 3

References

[1] N. Altshiller-Court, *College Geometry*, 2nd edition, 1952, Barnes and Noble, New York.
[2] E. Danneels, Hyacinthos message 7892, September 13, 2003.
[3] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.
[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Eric Danneels: Hubert d'Ydewallestraat 26, 8730 Beernem, Belgium
E-mail address: eric.danneels@pandora.be

Forum Geometricorum
Volume 4 (2004) 73–80.

FORUM GEOM
ISSN 1534-1178

Triangles with Special Isotomic Conjugate Pairs

K. R. S. Sastry

Abstract. We study the condition for the line joining a pair of isotomic conjugates to be parallel to a side of a given triangle. We also characterize triangles in which the line joining a specified pair of isotomic conjugates is parallel to a side.

1. Introduction

Two points in the plane of a given triangle ABC are called isotomic conjugates if the cevians through them divide the opposite sides in ratios that are reciprocals to each other. See [3], also [1]. We study the condition for the line joining a pair of isotomic conjugates to be parallel to a side of a given triangle. We also characterize triangles in which the line joining a specified pair of isotomic conjugates is parallel to a side.

2. Some background material

The standard notation is used throughout: a, b, c for the sides or the lengths of BC, CA, AB respectively of triangle ABC. The median and the altitude through A (and their lengths) are denoted by m_a and h_a respectively. We denote the centroid, the incenter, and the circumcenter by G, I, and O respectively.

2.1. *The orthic triangle.* The triangle formed by the feet of the altitudes is called its orthic triangle. It is the cevian triangle of the orthocenter H. Its sides are easily calculated to be the absolute values of $a\cos A$, $b\cos B$, $c\cos C$.

2.2. *The Gergonne and symmedian points.* The Gergonne point Γ is the concurrence point of the cevians that connect the vertices of triangle ABC to the points of contact of the opposite sides with the incircle.

The symmedian point K is the Gergonne point of the tangential triangle which is bounded by the tangents to the circumcircle at A, B, C.

Publication Date: May 24, 2004. Communicating Editor: Paul Yiu.
The author thanks the referee and Paul Yiu for their kind suggestions to improve the presentation of this paper.

2.3. *The Brocard points.* The Crelle-Brocard points Ω_+ and Ω_- are the interior points such that

$$\angle\Omega_+AB = \angle\Omega_+BC = \angle\Omega_+CA = \omega,$$
$$\angle\Omega_-AC = \angle\Omega_-BA = \angle\Omega_-CB = \omega,$$

where ω is the Crelle-Brocard angle.

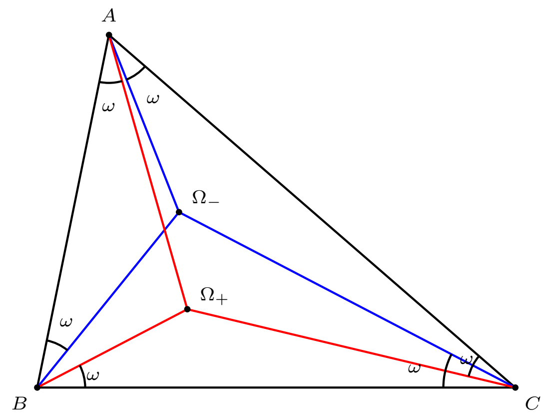


Figure 1

It is known that

$$\cot\omega = \cot A + \cot B + \cot C.$$

See, for example, [3, 5]. According to [4],

$$A + \omega = \frac{\pi}{2} \text{ if and only if } \tan^2 A = \tan B \tan C. \tag{1}$$

2.4. *Self-altitude triangles.* The sides a, b, c of a triangle are in geometric progression if and only if they are proportional to h_a, h_b, h_c in some order. Such a triangle is called a self-altitude triangle in [6]. It has a number of interesting properties. Suppose $a^2 = bc$. Then

(1) Ω_+ and Ω_- are the perpendicular feet of the symmedian point K on the perpendicular bisectors of AC and AB.
(2) The line $\Omega_+\Omega_-$ coincides with the bisector AI.
(3) $B\Omega_+$ and $C\Omega_-$ are tangent to the Brocard circle which has diameter OK.
(4) The median BG and the symmedian CK intersect on AI; so do CG and BK.

See Figure 2.

2.5. *A generalization of a property of equilateral triangles.* An equilateral triangle ABC has this easily provable property: if P is any point on the minor arc BC of the circumcircle of ABC, then $AP = BP+PC$. Surprisingly, however, if triangle ABC is non-isosceles, then there exists a unique point P on the arc BC (not containing the vertex A) such that $AP = BP + PC$ if and only if $a = \frac{mb^2+nc^2}{mb+nc}$.

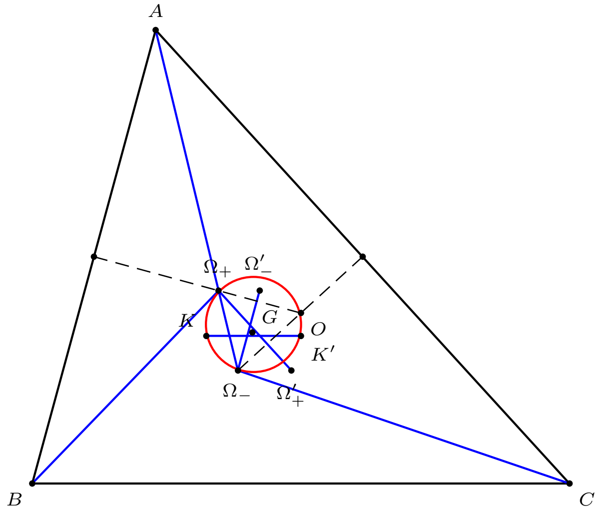


Figure 2

See [8]. Here, $\frac{m}{n}$ is the ratio in which AP divides the side BC. In particular, the extension AP of the median m_a has the preceding property if and only if

$$a = \frac{b^2 + c^2}{b + c}. \tag{2}$$

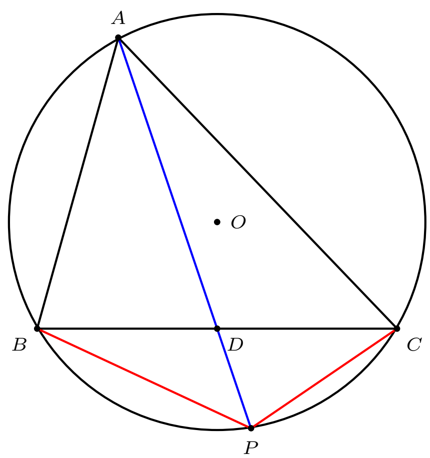


Figure 3.

3. Homogeneous barycentric coordinates

With reference to triangle ABC, every point in the plane is specified by a set of homogeneous barycentric coordinates. See, for example, [9]. If P is a point (not on any of the side lines of triangle ABC) with coordinates $(x : y : z)$, its isotomic conjugate P' has coordinates $\left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$. Here are the coordinates of some of classical triangle centers.

Point	Coordinates
centroid G	$(1 : 1 : 1)$
incenter I	$(a : b : c)$
circumcenter O	$(a \cos A : b \cos B : c \cos C)$
orthocenter H	$(\tan A : \tan B : \tan C)$
symmedian point K	$(a^2 : b^2 : c^2)$
Gergonne point Γ	$\left(\frac{1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c}\right)$
Brocard point Ω_+	$\left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}\right)$
Brocard point Ω_-	$\left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right)$

The isotomic conjugate of the Gergonne point is the Nagel point N, which is the concurrence points of the cevians joining the vertices to the point of tangency of its opposite side with the excircle on that side. It has coordinates $(b+c-a : c+a-b : a+b-c)$.

The homogeneous barycentric coordinate of a point can be normalized to give its *absolute* homogeneous barycentric coordinate, provided the sum of the coordinates is nonzero. If $P = (x : y : z)$, we say that in absolute barycentric coordinates,

$$P = \frac{xA + yB + zC}{x + y + z},$$

provided $x + y + z \neq 0$. Points $(x : y : z)$ with $x + y + z = 0$ are called infinite points. The isotomic conjugate of $P = (x : y : z)$ is an infinite point if and only if $xy + yz + zx = 0$. This is the Steiner circum-ellipse which has center at the centroid G of triangle ABC. Another fruitful way is to view an infinite point as the difference $Q - P$ of the absolute barycentric coordinates of two points P and Q. As such, it represents the vector $\overrightarrow{PQ}$.

4. The basic results

The segment joining P to its isotomic conjugate is represented by the infinite point

$$\begin{aligned} PP' =& \frac{yzA + zxB + xyC}{xy + yz + zx} - \frac{xA + yB + zC}{x + y + z} \\ =& \frac{(y+z)(yz - x^2)A + (z+x)(zx - y^2)B + (x+y)(xy - z^2)C}{(x+y+z)(xy+yz+zx)}. \end{aligned} \quad (3)$$

This is parallel to the line BC if it is a multiple of the infinte point of BC, namely, $-B + C$. This is the case if and only if

$$(y + z)(x^2 - yz) = 0. \quad (4)$$

The equation $y + z = 0$ represents the line through A parallel to BC. It is clear that this line is invariant under isotomic conjugation. Every finite point on this line has coordinates $(x : 1 : -1)$ for a nonzero x. Its isotomic conjugate is the point $(\frac{1}{x} : 1 : -1)$ on the same line. On the other hand, the equation $x^2 - yz = 0$ represent an ellipse homothetic to the Steiner circum-ellipse. It passes through $B = (0 : 1 : 0)$, $C = (0 : 0 : 1)$, $G = (1 : 1 : 1)$, and $(-1 : 1 : 1)$. It is tangent to AB and AC at B and C respectively. It is obtained by translating the Steiner circum-ellipse along the vector $\overrightarrow{AG}$. We summarize this in the following theorem.

Theorem 1. *Let P be a finite point. The line joining P to its isotomic conjugate if parallel to BC if and only if P lies on the line through A parallel to BC or the ellipse through the centroid tangent to AB and AC at B and C respectively. In the latter case, the isotomic conjugate P' is the second intersection of the ellipse with the line through P parallel to BC.*

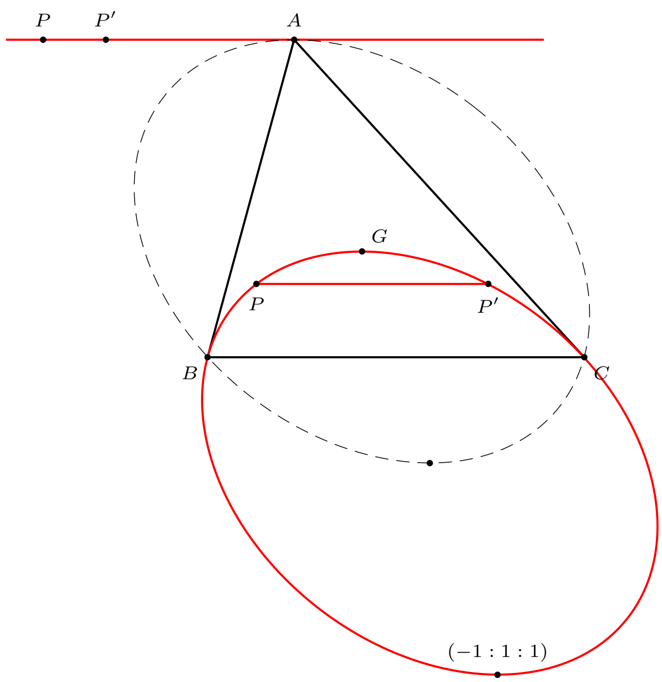


Figure 4

Now we consider the possibility for PP' not only to be parallel to BC, but also equal to one half of its length. This means that the vector PP' is $\pm\frac{1}{2}(C - B)$. If P is a finite point on the parallel to BC through A, we write $P = (x : 1 : -1)$, $x \neq 0$. From (3), we have $PP' = \frac{(1-x^2)(-B+C)}{x} = \frac{1}{2}(-B + C)$ if and only if $x = \frac{-1\pm\sqrt{17}}{4}$. These give the first two pairs of isotomic conjugates listed in Theorem 2 below.

By Theorem 1, P may also lie on the ellipse $x^2 - yz = 0$. It is convenient to use a parametrization

$$x = \mu, \quad y = \mu^2, \quad z = 1. \tag{5}$$

Setting the coefficient of C in (3) to $\frac{1}{2}$, simplifying, we obtain

$$\frac{\mu^2 - \mu - 3}{2(\mu^2 + \mu + 1)} = 0.$$

The only possibilities are $\mu = \frac{1}{2}\left(1 \pm \sqrt{13}\right)$. These give the last two pairs in Theorem 2 below.

Theorem 2. *There are four pairs of isotomic conjugates P, P' for which the segment PP' is parallel to BC and has half of its length.*

i	P_i	P'_i
1	$(\sqrt{17}-1:4:-4)$	$(\sqrt{17}+1:4:-4)$
2	$(\sqrt{17}+1:-4:4)$	$(\sqrt{17}-1:-4:4)$
3	$(\sqrt{13}+1:\ \sqrt{13}+7:\ 2)$	$(\sqrt{13}+1:\ 2:\ \sqrt{13}+7)$
4	$(-(\sqrt{13}-1):\ 7-\sqrt{13}:\ 2)$	$(-(\sqrt{13}-1):\ 2:\ 7-\sqrt{13})$

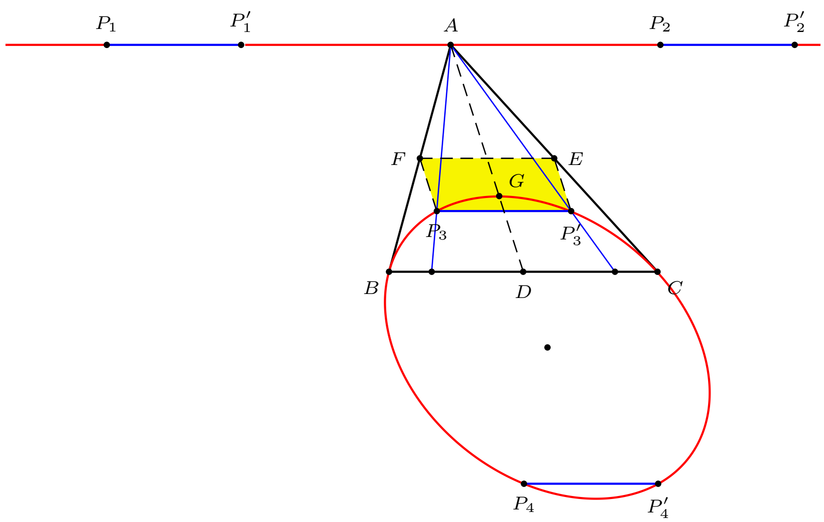


Figure 5

Among these four pairs, only the pair $(P_3,\ P'_3)$ are interior points. The segments FP_3 and EP'_3 are parallel to the median AD, and $P_3P'_3EF$ is a parallelogram with $FP_3 = EP'_3 = \frac{(5-\sqrt{13})m_a}{6}$.

5. Triangles with specific PP' parallel to BC

We examine the condition under which the line joining a pair of isotomic conjugates is parallel to C. We shall exclude the trivial case of equilateral triangles.

5.1. *The incenter.* Since the incenter has coordinates $(a : b : c)$, if II' is parallel to BC, we must have, according to (5), $a^2 - bc = 0$. Therefore, the triangle is self-altitude. See §2.4. It is, however, not possible to have II' equal to half of the side BC, since the coordinates of P_3 in Theorem 2 do not satisfy the triangle inequality.

5.2. *The symmedian and Brocard points.* Likewise, for the symmedian point K, the line KK' is parallel to BC if and only if $a^4 = b^2c^2$, or $a^2 = bc$. In other words, the triangle is self-altitude again. In fact, the following statements are equivalent.

(1) $a^2 = bc$.
(2) K is on the ellipse $x^2 - yz = 0$; KK' is parallel to BC.
(3) Ω_+ is on the ellipse $z^2 - xy = 0$; $\Omega_+\Omega'_+$ is parallel to CA.
(4) Ω_- is on the ellipse $y^2 - zx = 0$; $\Omega_-\Omega'_-$ is parallel to BA.

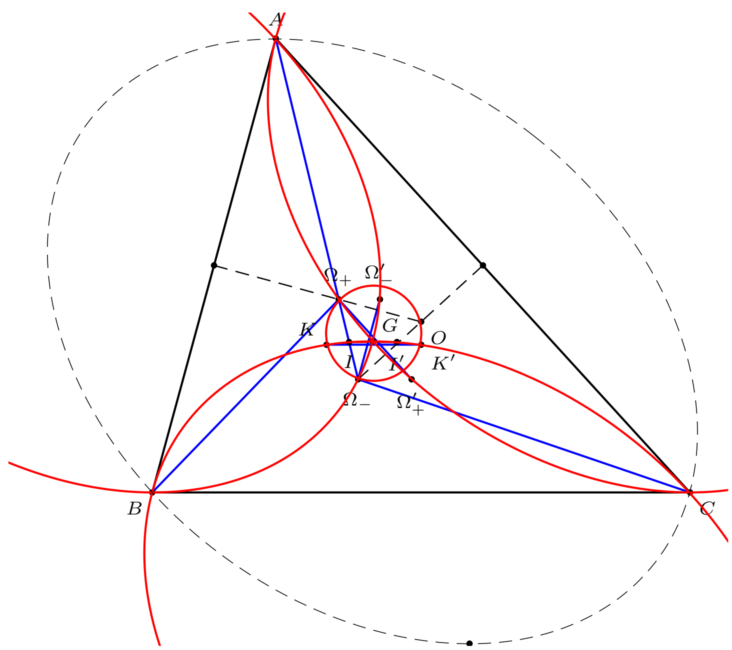


Figure 6

The self-altitude triangle with sides

$$a : b : c = \sqrt{2(1+\sqrt{13})} : 1+\sqrt{13} : 2$$

has $KK' = \frac{1}{2}BC$.

5.3. *The circumcenter.* Unlike the incenter, the circumcenter may be outside the triangle. If O lies on the line $y + z = 0$, then $b\cos B + c\cos C = 0$. From this we deduce $\cos(B - C) = 0$, and $|B - C| = \pm\frac{\pi}{2}$. (This also follows from [2] by noting that the nine-point center lies on BC).

The homogeneous barycentric coordinates of the circumcenter are proportional to the sides of the orthic triangle (the pedal triangle of the orthocenter). To construct such a triangle, we take a self-altitude triangle $A'B'C'$ with incenter I_0, and construct the perpendiculars to $I'A'$, $I'B'$, $I'C'$ at A', B', C' respectively. These bound a triangle ABC whose orthocenter is I_0. Its circumcenter O is such that OO' is parallel to BC.

5.4. *The orthocenter.* The orthocenter has barycentric coordinates $(\tan A : \tan B : \tan C)$. If the triangle is acute, the condition $\tan^2 A = \tan B \tan C$ is equivalent to $A + \omega = \frac{\pi}{2}$ according to (1).

5.5. *The Gergonne and Nagel points.* The line joining the Gergonne and Nagel points is parallel to BC if and only if $(b+c-a)^2 = (c+a-b)(a+b-c)$. This is equivalent to (2). Hence, we have a characterization of such a triangle: the extension of the median m_a intersects the minor arc BC at a point P such that $AP = BP + CP$.

Since the Gergonne and Nagel points are interior points, there is a triangle (up to similarity) with ΓN parallel to BC and half in length. From

$$b+c-a : c+a-b : a+b-c = \sqrt{13}+1 : \ 2 : \ \sqrt{13}+7,$$

we obtain

$$a : b : c = \sqrt{13}+9 : 2\sqrt{13}+8 : \sqrt{13}+3 = 3\sqrt{13}-7 : \sqrt{13}+1 : 2.$$

References

[1] M. Dalcín, Isotomic inscribed triangles and their residuals, *Forum Geom.*, 3 (2003) 125–134.

[2] A. P. Hatzipolakis, P. Yiu, N. Dergiades and D. Loeffler, Problem 2525 (April), *Crux Math.*, 26 (2000) 177; solution, 27 (2001) 270–271.

[3] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.

[4] M. S. Klamkin, K. R. S. Sastry and D. Loffler, Problem 2848, *Crux Math.*, 29 (2003) 242; solution, 30 (2004) 255–256.

[5] D. S. Mitrinović et al, *Recent Advances in Geometric Inequalities*, Kluwer, Dordrecht, 1989.

[6] K. R. S. Sastry, Self-altitude triangles, *Math. Spectrum*, 22 (1989-90) 88–90.

[7] K. R. S. Sastry, Heron triangles: A Gergonne-Cevian-and-median perspective, *Forum Geom.*, 1 (2001) 17–24.

[8] K. R. S. Sastry, Analogies are interesting! *Elem. Math.*, 59 (2004) 29–36.

[9] P. Yiu, The uses of homogeneous barycentric coordinates in plane euclidean geometry, *Int. J. Math. Educ. Sci. Technol.*, 31 (2000) 569–578.

K. R. S. Sastry: Jeevan Sandhya, DoddaKalsandra Post, Raghuvana Halli, Bangalore, 560 062, India.

Forum Geometricorum
Volume 4 (2004) 81–84.

FORUM GEOM
ISSN 1534-1178

On the Intercepts of the OI-Line

Lev Emelyanov

Abstract. We prove a new property of the intercepts of the line joining the circumcenter and the incenter on the sidelines of a triangle.

Given a triangle ABC with circumcenter O and incenter I, consider the intouch triangle XYZ. Let X' be the reflection of X in YZ, and similarly define Y' and Z'.

Theorem 1. *The intersections of AX' with BC, BY' with CA, and CZ' with AB are all on the line OI.*

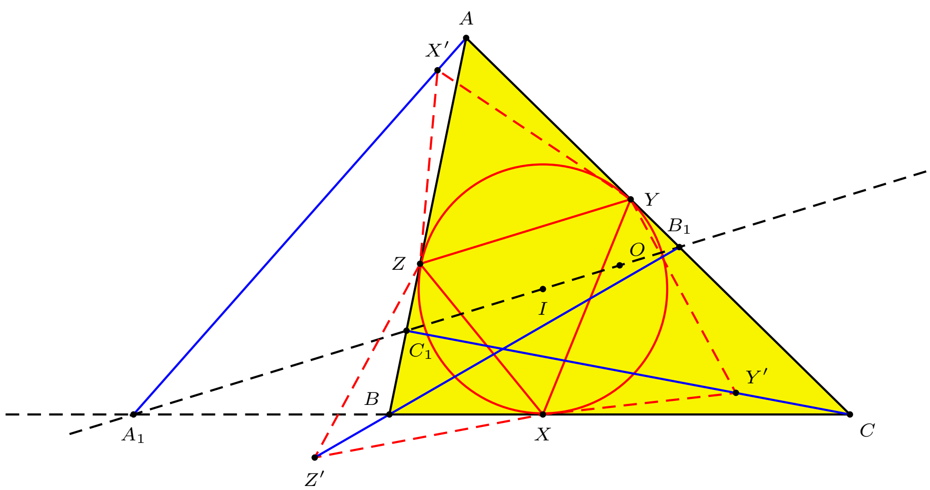


Figure 1.

Lemma 2. *The orthocenter H' of the intouch triangle lies on the line OI.*

Proof. Let $I_1I_2I_3$ be the excentral triangle. The lines YZ and I_2I_3 are parallel because both are perpendicular to AI. Similarly, $ZX//I_3I_1$ and $XY//I_1I_2$. See Figure 2. Hence, the excentral triangle and the intouch triangle are homothetic and their Euler lines are parallel. Now, I and O are the orthocenter and nine-point center of the excentral triangle. On the other hand, I is the circumcenter of the intouch triangle. Therefore, the line OI is their common Euler line, contains the orthocenter H' of XYZ. $\square$

Publication Date: June 8, 2004. Communicating Editor: Paul Yiu.

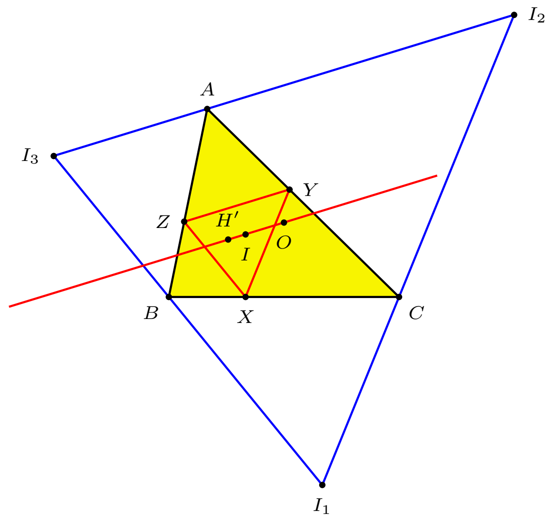


Figure 2.

Proof of Theorem 1. To prove that the intersection point A_1 of OI and AX' lies BC it is sufficient to show that $\frac{X'H'}{H'X} = \frac{AI}{IA_2}$, where A_2 is the foot of the bisector AI. See Figure 3.

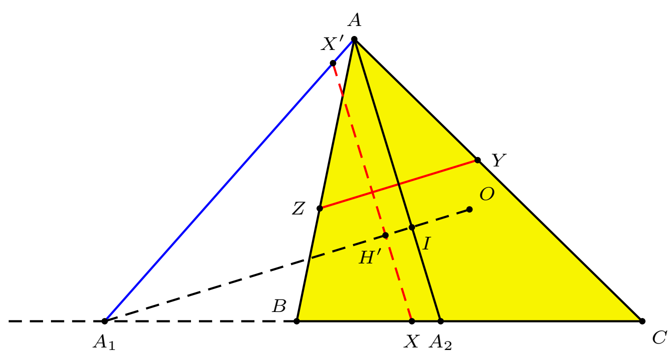


Figure 3.

It is known that

$$\frac{AI}{IA_2} = \frac{CA+AB}{BC} = \frac{\sin B + \sin C}{\sin A}.$$

For any acute triangle, $AH = 2R\cos A$. The angles of the intouch triangle are

$$X = \frac{B+C}{2}, \quad Y = \frac{C+A}{2}, \quad Z = \frac{A+B}{2}.$$

It is clear that triangle XYZ is always acute, and

$$XH' = 2r\cos X = 2r\cos\frac{B+C}{2} = 2r\sin\frac{A}{2},$$

where r is the inradius of triangle ABC.

$$\frac{X'H'}{H'X} = \frac{X'X - H'X}{H'X} = \frac{X'X \cdot YZ}{H'X \cdot YZ} - 1$$
$$= \frac{2 \cdot \text{area of } XYZ}{H'X \cdot YZ} - 1$$
$$= \frac{2r^2(\sin 2X + \sin 2Y + \sin 2Z)}{2r \sin X \cdot 2r \cos X} - 1$$
$$= \frac{\sin 2Y + \sin 2Z}{\sin 2X} = \frac{\sin B + \sin C}{\sin A}.$$

This completes the proof of Theorem 1.

Similar results hold for the extouch triangle. In part it is in [1]. The following corollaries are clear.

Corollary 3. *The line joining A_1 to the projection of X on YZ passes through the midpoint of the bisector of angle A.*

Proof. In Figure 3, $X'X$ is parallel to the bisector of angle A and its midpoint is the projection of X on YZ. □

Corollary 4. *The OI-line is parallel to BC if and only if the projection of X on YZ lies on the line joining the midpoints of AB and AC.*

Corollary 5. *Let XYZ be the tangential triangle of ABC, X' the reflection of X in BC. If A_1 is the intersection of the Euler line and XX', then AA_1 is tangent to the circumcircle.*

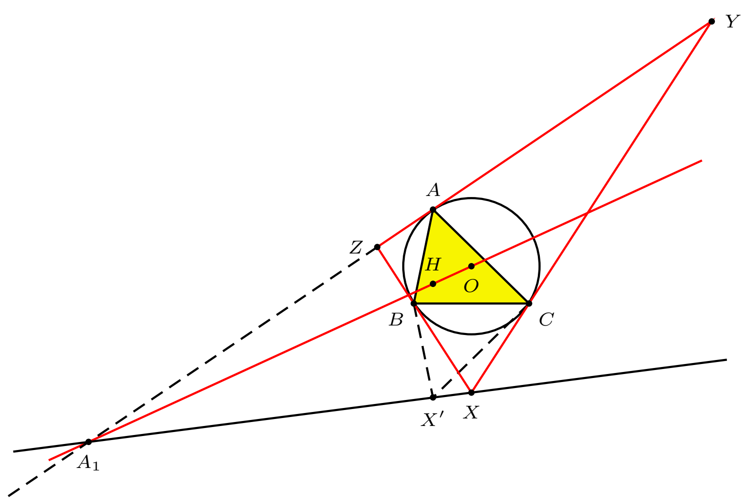


Figure 4.

References

[1] L. Emelyanov and T. Emelyanova, A note on the Schiffler point, *Forum Geom.*, 3 (2003) 113–116.

Lev Emelyanov: 18-31 Proyezjaia Street, Kaluga, Russia 248009
E-mail address: `emelyanov@kaluga.ru`

Forum Geometricorum
Volume 4 (2004) 85–95.

FORUM GEOM
ISSN 1534-1178

On the Schiffler center

Charles Thas

Abstract. Suppose that ABC is a triangle in the Euclidean plane and I its incenter. Then the Euler lines of ABC, IBC, ICA, and IAB concur at a point S, the Schiffler center of ABC. In the main theorem of this paper we give a projective generalization of this result and in the final part, we construct Schiffler-like points and a lot of other related centers. Other results in connection with the Schiffler center can be found in the articles [1] and [3].

1. Introduction

We recall some formulas and tools of projective geometry, which will be used in §2. Although we focus our attention on the real projective plane, it will be convenient to work in the complex projective plane $\mathcal{P}$.

1.1. Suppose that (x_1, x_2) are projective coordinates on a complex projective line and that two pairs of points are given as follows: P_1 and P_2 by the quadratic equation

$$ax_1^2 + 2bx_1x_2 + cx_2^2 = 0 \tag{1}$$

and Q_1 and Q_2 by

$$a'x_1^2 + 2b'x_1x_2 + c'x_2^2 = 0. \tag{2}$$

Then the cross-ratio $(P_1P_2Q_1Q_2)$ equals -1 iff

$$ac' - 2bb' + a'c = 0. \tag{3}$$

Proof. Put $t = \frac{x_1}{x_2}$ and assume that t_1, t_2 (t'_1, t'_2 respectively) are the solutions of (1) ((2) respectively), divided by x_2^2. Then $(t_1\, t_2\, t'_1\, t'_2) = -1$ is equivalent to $2(t_1t_2 + t'_1t'_2) = (t_1 + t_2)(t'_1 + t'_2)$ or $2(\frac{c}{a} + \frac{c'}{a'}) = (-\frac{2b}{a})(-\frac{2b'}{a'})$, which gives (3). □

1.2.1. Consider a triangle ABC in the complex projective plane $\mathcal{P}$ and assume that ℓ is a line in $\mathcal{P}$, not through A, B, or C. Put $AB \cap \ell = M'_C, BC \cap \ell = M'_A$, and $CA \cap \ell = M'_B$ and determine the points M_C, M_A, and M_B by $(ABM'_CM_C) = (BCM'_AM_A) = (CAM'_BM_B) = -1$, then AM_A, BM_B, and CM_C concur at a point Z, the so-called trilinear pole of ℓ with regard to ABC.

Publication Date: June 28, 2004. Communicating Editor: J. Chris Fisher.

Proof. If $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$, and ℓ is the unit line $x_1 + x_2 + x_3 = 0$, then $M'_C = (1,-1,0)$, $M'_A = (0,1,-1)$, $M'_B = (1,0,-1)$, and $M_C = (1,1,0)$, $M_A = (0,1,1), M_B = (1,0,1)$, and Z is the unit point $(1,1,1)$. $\square$

1.2.2. The trilinear pole Z_C of the unit-line ℓ with regard to ABQ, where $A = (1,0,0), B = (0,1,0)$, and $Q = (\mathcal{A}, \mathcal{B}, \mathcal{C})$, has coordinates $(2\mathcal{A} + \mathcal{B} + \mathcal{C}, \mathcal{A} + 2\mathcal{B} + \mathcal{C}, \mathcal{C})$.

Proof. The point Z_C is the intersection of the line QM_C and BM_{QA}, with $M_C = (1,1,0)$, and M_{QA} the point of QA, such that $(Q\ A\ M_{QA}\ M'_{QA}) = -1$, with $M'_{QA} = QA \cap \ell$. We find for M_{QA} the coordinates $(2\mathcal{A} + \mathcal{B} + \mathcal{C}, \mathcal{B}, \mathcal{C})$, and a straightforward calculation completes the proof. $\square$

1.3. Consider a non-degenerate conic $\mathcal{C}$ in the complex projective plane $\mathcal{P}$, and two points A, Q, not on $\mathcal{C}$, whose polar lines with respect to $\mathcal{C}$, intersect $\mathcal{C}$ at T_1, T_2, and I_1, I_2 respectively. Then Q lies on one of the lines ℓ_1, ℓ_2 through A which are determined by $(AT_1\ AT_2\ \ell_1\ \ell_2) = (AI_1\ AI_2\ \ell_1\ \ell_2) = -1$.

Proof. This follows immediately from the fact that the pole of the line AQ with respect to $\mathcal{C}$ is the point $T_1T_2 \cap I_1I_2$. $\square$

1.4. For any triangle ABC of $\mathcal{P}$ and line ℓ not through a vertex, the Desargues-Sturm involution theorem ([7, p.341], [8, p.63]) provides a one-to-one correspondence between the involutions on ℓ and the points P in $\mathcal{P}$ that lie neither on ℓ nor on a side of the triangle. Specifically, the conics of the pencil $\mathcal{B}(A, B, C, P)$ intersect ℓ in pairs of points that are interchanged by an involution with fixed points I and J. Conversely, P is the fourth intersection point of the conics through A, B, and C that are tangent to ℓ at I and J. The point P can easily be constructed from A, B, C, I, and J as the point of intersection of the lines AA', and BB', where A' is the harmonic conjugate of $BC \cap \ell$ with respect to I and J, and B' is the harmonic conjugate of $AC \cap \ell$ with respect to I and J.

1.5. Denote the pencil of conics through the four points A_1, A_2, A_3, and A_4 by $\mathcal{B}(A_1, A_2, A_3, A_4)$, and assume that ℓ is a line not through A_i, $i = 1, \ldots, 4$. Put $M'_{12} = A_1A_2 \cap \ell$, and let M_{12} be the harmonic conjugate of M'_{12} with respect to A_1 and A_2, and define the points M_{23}, M_{34}, M_{13}, M_{14}, and M_{24} likewise. Let X, Y, and Z be the points $A_1A_2 \cap A_3A_4$, $A_2A_3 \cap A_1A_4$, and $A_1A_3 \cap A_2A_4$ respectively. Finally, let I and J be the tangent points with ℓ of the two conics of the pencil which are tangent at ℓ. Then the eleven points M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34}, X, Y, Z, I, and J belong to a conic ([8, p.109]).

Proof. We prove that this conic is the locus $\mathcal{C}$ of the poles of the line ℓ with regard to the conics of the pencil $\mathcal{B}(A_1, A_2, A_3, A_4)$. But first, let us prove that this locus is indeed a conic: if we represent the pencil by $F_1 + tF_2 = 0$, where $F_1 = 0$ and $F_2 = 0$ are two conics of the pencil, the equation of the locus is obtained by eliminating t from two linear equations which represent the polar lines of two points of ℓ, which gives a quadratic equation. Then, call A'_3 the point which is the harmonic conjugate of A_3 with respect to $M_{12}A_3 \cap \ell$ and M_{12}, and consider the conic of the pencil through A'_3: the pole of ℓ with respect to this conic clearly is M_{12}, which means that M_{12}, and thus also M_{ij}, is a point of the locus. Next, X, Y and Z are points of the locus, since they are singular points of the three degenerate conics of the pencil. And finally, I and J belong to the locus, because they are the poles of ℓ with regard to the two conics of the pencil which are tangent to ℓ. □

1.6. Consider again a triangle ABC in $\mathcal{P}$, and a point P not on a side of ABC. The *Ceva triangle* of P is the triangle with vertices $AP \cap BC$, $BP \cap CA$, and $CP \cap AB$. Example: with the notation of §1.2.1 the Ceva triangle of Z is $M_A M_B M_C$.

Next, assume that I and J are any two (different) points, not on a side of ABC, on a line ℓ, not through a vertex, and that P is the point which corresponds (according to 1.4) to the involution on ℓ with fixed points I and J. Let $H'_A H'_B H'_C$ be the Ceva triangle of P, let A' (B', and C' respectively) be the harmonic conjugate of $PA \cap \ell$ ($PB \cap \ell$, and $PC \cap \ell$ respectively) with respect to A and P (B and P, and C and P, respectively), and let $M_A M_B M_C$ be the Ceva triangle of the trilinear pole Z of ℓ with regard to ABC. Then there is a conic through I, J, and the triples $H'_A H'_B H'_C$, $A'B'C'$, and $M_A M_B M_C$. This conic is known as the *eleven-point conic* of ABC with regard to I and J ([7, pp.342–343]).

Proof. Apply 1.5 to the pencil $\mathcal{B}(A, B, C, P)$. □

2. The main theorem

Theorem. *Let ABC be a triangle in the complex projective plane $\mathcal{P}$, ℓ be a line not through a vertex, and I and J be any two (different) points of ℓ not on a side of the triangle. Choose $\mathcal{C}$ to be one of the four conics through I and J that are tangent to the sides of triangle ABC, and define Q to be the pole of ℓ with respect to $\mathcal{C}$. If Z, Z_A, Z_B, and Z_C are the trilinear poles of ℓ with respect to the triangles ABC, QBC, QCA, and QAB respectively, while P, P_A, P_B, and P_C respectively, are the points determined by these triangles and the involution on ℓ whose fixed points are I and J (see 1.4), then the lines PZ, $P_A Z_A$, $P_B Z_B$, and $P_C Z_C$ concur at a point S_P.*

Proof. We choose our projective coordinate system in $\mathcal{P}$ as follows : $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$, and ℓ is the unit line with equation $x_1 + x_2 + x_3 = 0$. The point P has coordinates (α, β, γ).

Two degenerate conics of the pencil $\mathcal{B}(A, B, C, P)$ are (CP, AB) and (BP, CA), which intersect ℓ at the points $(-\alpha, -\beta, \alpha+\beta)$, $(1, -1, 0)$ and $(-\alpha, \alpha+\gamma, -\gamma)$, $(1, 0, -1))$ respectively. Joining these points to A, we find the lines $(\alpha+\beta)x_2 + \beta x_3 = 0$, $x_3 = 0$ and $\gamma x_2 + (\alpha+\gamma)x_3 = 0$, $x_2 = 0$, or as quadratic equations $(\alpha+\beta)x_2x_3 + \beta x_3^2 = 0$ and $\gamma x_2^2 + (\alpha+\gamma)x_2x_3 = 0$ respectively. Therefore, the lines AI and AJ are given by $kx_2^2 + 2lx_2x_3 + mx_3^2 = 0$ whereby k, l, and m are solution of (see 1.1):

$$\begin{cases} \beta k - (\alpha+\beta)l = 0 \\ -(\alpha+\gamma)l + \gamma m = 0, \end{cases}$$

and thus $(k, l, m) = (\gamma(\alpha+\beta), \beta\gamma, \beta(\alpha+\gamma))$. Next, the lines through A which form together with AI, AJ and with AB, AC an harmonic quadruple, are determined by $px_2^2 + 2qx_2x_3 + rx_3^2 = 0$ with p, q, r solutions of (see again 1.1)

$$\begin{cases} \beta(\alpha+\gamma)p - 2\beta\gamma q + \gamma(\alpha+\beta)r = 0 \\ q = 0, \end{cases}$$

and thus these lines are given by $\gamma(\alpha+\beta)x_2^2 - \beta(\alpha+\gamma)x_3^2 = 0$. In the same way, we find the quadratic equation of the two lines through B (C, respectively) which form together with BI, BJ and with BC, BA (with CI, CJ and with CA, CB respectively) an harmonic quadruple : $\alpha(\beta+\gamma)x_3^2 - \gamma(\beta+\alpha)x_1^2 = 0$ ($\beta(\gamma+\alpha)x_1^2 - \alpha(\gamma+\beta)x_2^2 = 0$ respectively). The intersection points of these three pairs of lines through A, B, and C are the poles Q_1, Q_2, Q_3, Q_4 of ℓ with respect to the four conics through I and J that are tangent to the sides of triangle ABC (see 1.3) and their coordinates are $Q_1(\mathcal{A}, \mathcal{B}, \mathcal{C})$, $Q_2(-\mathcal{A}, \mathcal{B}, \mathcal{C})$, $Q_3(\mathcal{A}, -\mathcal{B}, \mathcal{C})$, and $Q_4(\mathcal{A}, \mathcal{B}, -\mathcal{C})$, where

$$\mathcal{A} = \sqrt{\alpha(\beta+\gamma)}, \quad \mathcal{B} = \sqrt{\beta(\gamma+\alpha)}, \quad \mathcal{C} = \sqrt{\gamma(\alpha+\beta)}.$$

For now, let us choose for Q the point $Q_1(\mathcal{A}, \mathcal{B}, \mathcal{C})$.

The coordinates of the points Z, Z_A, Z_B, and Z_C are $(1,1,1)$, $(\mathcal{A}, \mathcal{A}+2\mathcal{B}+\mathcal{C}, \mathcal{A}+\mathcal{B}+2\mathcal{C})$, $(2\mathcal{A}+\mathcal{B}+\mathcal{C}, \mathcal{B}, \mathcal{A}+\mathcal{B}+2\mathcal{C})$, and $(2\mathcal{A}+\mathcal{B}+\mathcal{C}, \mathcal{A}+2\mathcal{B}+\mathcal{C}, \mathcal{C})$ (see 1.2.2).

Now, in connection with the point P_C, remark that $(AP_C \cap \ell\ (QB \cap \ell)\ I\ J) = -1$. But $(Q_2Q_4 \cap \ell\ (Q_1Q_3 \cap \ell)\ I\ J) = -1$ and $Q_2Q_4 = Q_2B$, $Q_1Q_3 = Q_1B$, so that $AP_C \cap \ell = Q_2B \cap \ell$, and since Q_2B has equation $\mathcal{C}x_1 + \mathcal{A}x_3 = 0$, the point $AP_C \cap \ell$ has coordinates $(\mathcal{A}, \mathcal{C}-\mathcal{A}, -\mathcal{C})$ and the line AP_C has equation $\mathcal{C}x_2 + (\mathcal{C}-\mathcal{A})x_3 = 0$. In the same way, we find the equation of the line BP_C: $\mathcal{C}x_1 + (\mathcal{C}-\mathcal{B})x_3 = 0$, and the common point of these two lines is the point P_C with coordinates $(\mathcal{B}-\mathcal{C}, \mathcal{A}-\mathcal{C}, \mathcal{C})$.

Finally, the line P_CZ_C has equation :

$$\mathcal{C}(\mathcal{B}+\mathcal{C})x_1 - \mathcal{C}(A+\mathcal{C})x_2 + (\mathcal{A}^2 - \mathcal{B}^2)x_3 = 0,$$

and cyclic permutation gives us the equations of P_AZ_A and P_BZ_B.

Now, P_AZ_A, P_BZ_B, and P_CZ_C are concurrent if the determinant

$$\begin{vmatrix} \mathcal{B}^2 - \mathcal{C}^2 & \mathcal{A}(\mathcal{C}+\mathcal{A}) & -\mathcal{A}(\mathcal{B}+\mathcal{A}) \\ -\mathcal{B}(\mathcal{C}+\mathcal{B}) & \mathcal{C}^2 - \mathcal{A}^2 & \mathcal{B}(\mathcal{A}+\mathcal{B}) \\ \mathcal{C}(\mathcal{B}+\mathcal{C}) & -\mathcal{C}(\mathcal{A}+\mathcal{C}) & \mathcal{A}^2 - \mathcal{B}^2 \end{vmatrix}$$

is zero, which is obviously the case, since the sum of the rows gives us three times zero. Then, the line PZ has equation $(\beta-\gamma)x_1 + (\gamma-\alpha)x_2 + (\alpha-\beta)x_3 = 0$. But $\mathcal{A}^2 = \alpha(\beta+\gamma)$, $\mathcal{B}^2 = \beta(\gamma+\alpha)$, and $\mathcal{C}^2 = \gamma(\alpha+\beta)$, so that $(\mathcal{B}^2 - \mathcal{C}^2)(-\mathcal{A}^2 + \mathcal{B}^2 + \mathcal{C}^2) = 2\alpha\beta\gamma(\beta-\gamma)$, and PZ has also the following equation

$$\begin{aligned} &(\mathcal{B}^2 - \mathcal{C}^2)(-\mathcal{A}^2 + \mathcal{B}^2 + \mathcal{C}^2)x_1 + (\mathcal{C}^2 - \mathcal{A}^2)(\mathcal{A}^2 - \mathcal{B}^2 + \mathcal{C}^2)x_2 \\ &+ (\mathcal{A}^2 - \mathcal{B}^2)(\mathcal{A}^2 + \mathcal{B}^2 - \mathcal{C}^2)x_3 = 0. \end{aligned}$$

For PZ, P_AZ_A, and P_BZ_B to be concurrent, the following determinant must vanish :

$$\begin{vmatrix} (\mathcal{B}^2-\mathcal{C}^2)(-\mathcal{A}^2+\mathcal{B}^2+\mathcal{C}^2) & (\mathcal{C}^2-\mathcal{A}^2)(\mathcal{A}^2-\mathcal{B}^2+\mathcal{C}^2) & (\mathcal{A}^2-\mathcal{B}^2)(\mathcal{A}^2+\mathcal{B}^2-\mathcal{C}^2) \\ \mathcal{B}^2-\mathcal{C}^2 & \mathcal{A}(\mathcal{C}+\mathcal{A}) & -\mathcal{A}(\mathcal{B}+\mathcal{A}) \\ -\mathcal{B}(\mathcal{C}+\mathcal{B}) & \mathcal{C}^2-\mathcal{A}^2 & \mathcal{B}(\mathcal{A}+\mathcal{B}) \end{vmatrix}$$
$$\begin{aligned}
=&(\mathcal{B}+\mathcal{C})(\mathcal{C}+\mathcal{A})(\mathcal{A}+\mathcal{B})(\mathcal{A}(\mathcal{B}-\mathcal{C})(-\mathcal{A}^2+\mathcal{B}^2+\mathcal{C}^2)(-\mathcal{A}+\mathcal{B}+\mathcal{C})\\
&+\mathcal{B}(\mathcal{C}-\mathcal{A})(\mathcal{A}^2-\mathcal{B}^2+\mathcal{C}^2)(\mathcal{A}-\mathcal{B}+\mathcal{C})+\mathcal{C}(\mathcal{A}-\mathcal{B})(\mathcal{A}^2+\mathcal{B}^2-\mathcal{C}^2)(\mathcal{A}+\mathcal{B}-\mathcal{C}))\\
=&0.
\end{aligned}$$

We may conclude that PZ, P_AZ_A, P_BZ_B, and P_CZ_C are concurrent. This completes the proof. □

Remarks. (1) If Q is chosen as the point Q_2 (Q_3, or Q_4, respectively), then $\mathcal{A}$ ($\mathcal{B}$, or $\mathcal{C}$ respectively) must be replaced by $-\mathcal{A}$ ($-\mathcal{B}$, or $-\mathcal{C}$ respectively) in the foregoing proof.

(2) The coordinates of the common point S_P of the lines PZ, P_AZ_A, P_BZ_B, and P_CZ_C are $(\mathcal{A}\frac{-\mathcal{A}+\mathcal{B}+\mathcal{C}}{\mathcal{B}+\mathcal{C}}, \mathcal{B}\frac{\mathcal{A}-\mathcal{B}+\mathcal{C}}{\mathcal{C}+\mathcal{A}}, \mathcal{C}\frac{\mathcal{A}+\mathcal{B}-\mathcal{C}}{\mathcal{A}+\mathcal{B}})$.

(3) Of course, when we work in the real (complexified) projective plane $\mathcal{P}$ with a real triangle ABC, a real line ℓ and a real point P, the points Q and S_P, are not always real. That depends on the values of α, β, and γ and thus on the position of the point P in the plane. For instance, in example 5.5 of §5, the points Q and S_P will be imaginary.

(4) The conic through A, B, C, and through the points I, J on ℓ has equation

$$\alpha(\beta+\gamma)x_2x_3+\beta(\gamma+\alpha)x_3x_1+\gamma(\alpha+\beta)x_1x_2=0$$

or

$$\mathcal{A}^2x_2x_3+\mathcal{B}^2x_3x_1+\mathcal{C}^2x_1x_2=0.$$

Indeed, eliminating x_1 from this equation and from $x_1+x_2+x_3=0$, gives us $\gamma(\alpha+\beta)x_2^2+2\gamma\beta x_2x_3+\beta(\gamma+\alpha)x_3^2=0$, which determines the lines AI and AJ (see the proof of the theorem).

The pole of the line ℓ with respect to this conic is the point $Y(\beta+\gamma, \gamma+\alpha, \alpha+\beta)$, which clearly is a point of the line PZ. We denote this conic by (Y).

(5) The locus of the poles of the line ℓ with respect to the conics of the pencil $\mathcal{B}(A, B, C, P)$ is the conic with equation

$$\beta\gamma x_1^2+\gamma\alpha x_2^2+\alpha\beta x_3^2-\alpha(\gamma+\beta)x_2x_3-\beta(\alpha+\gamma)x_3x_1-\gamma(\beta+\alpha)x_1x_2=0.$$

It is the eleven-point conic of triangle ABC with regard to I and J (see 1.6): it is the conic through the points $M_A(0,1,1)$, $M_B(1,0,1)$, $M_C(1,1,0)$, $AP\cap BC=H'_A(0,\beta,\gamma)$, $BP\cap CA=H'_B(\alpha,0,\gamma)$, $CP\cap AB=H'_C(\alpha,\beta,0)$, $A'(2\alpha+\beta+\gamma,\beta,\gamma)$, $B'(\alpha,\alpha+2\beta+\gamma,\gamma)$, $C'(\alpha,\beta,\alpha+\beta+2\gamma)$, I, and J. The pole of the line ℓ with regard to this conic is the point $Y'(2\alpha+\beta+\gamma,\ \alpha+2\beta+\gamma,\ \alpha+\beta+2\gamma)$, which is also a point of the line PZ. We denote this conic by (Y').

Here is an alternative formulation of the main theorem.

Theorem. *Let ABC be a triangle in the complex projective plane $\mathcal{P}$, ℓ be a line not through a vertex, and I and J be any two (different) points of ℓ not on a side* *of the triangle. Denote by Q the pole of ℓ with respect to one of the four conics through I and J that are tangent to the sides of the triangle. If Y, Y_A, Y_B, and Y_C are the poles of ℓ with respect to the conics determined by I, J, and the triples ABC, QBC, QCA, and QAB respectively, while Y', Y'_A, Y'_B, and Y'_C are the respective poles with respect to their eleven-point conics with regard to I and J, then YY', $Y_AY'_A$, $Y_BY'_B$, and $Y_CY'_C$ concur at a point S.*

3. The Euclidean case

In this section we give applications of the main theorem in the Euclidean plane Π. Throughout the following sections, we only consider a general real triangle ABC in Π, i.e., the side-lengths a, b, and c are distinct and the triangle has no right angle.

Corollary 1. *Let ABC be a triangle in Π and assume that ℓ is the line at infinity of Π. Suppose that P coincides with the orthocenter H of ABC; then the conics of the pencil $\mathcal{B}(A, B, C, H)$ are rectangular hyperbolas and the involution on ℓ, determined by H (see 1.4), becomes the absolute (or orthogonal) involution with fixed points the cyclic points (or circle points) J and J' of Π. The four conics through J, J' and tangent to the sidelines of ABC are now the incircle and the excircles of ABC, and the points $Q = Q_1$, Q_2, Q_3, Q_4 become the incenter I, and the excenters I_A (the line II_A contains A), I_B, and I_C, respectively.*

Next, the points Z, Z_A, Z_B, and Z_C, are the centroids of ABC, IBC, ICA, and of IAB respectively. Finally, P_A, P_B, P_C are the orthocenters H_A, H_B, H_C of IBC, ICA, and IAB respectively. Then the lines HZ, H_AZ_A, H_BZ_B, and H_CZ_C concur at a point S_H.

Remark that HZ, H_AZ_A, H_BZ_B, and H_CZ_C are the Euler lines of the triangles ABC, IBC, ICA, and IAB, respectively. The point of concurrence of these Euler lines is known as the Schiffler point S ([9]), but we prefer in this paper the notation S_H, since it results from setting $P = H$.

In connection with Remarks 4 and 5 of the foregoing section, and again working with ℓ as the line at infinity and J, J' the cyclic points, the conic (Y) becomes the circumcircle (O) of ABC, (Y') becomes its nine-point circle (O'), and OO' is the Euler line.

In connection with Remark 5, we recall that the locus of the centers of the rectangular hyperbolas through A, B, C (and H) is the nine-point circle (O') of ABC and that, for each point U of the circumcircle (O), the midpoint of HU is a point of (O') (and O' is the midpoint of HO on the Euler line).

The main theorem allows us to generalize the foregoing corollary as follows:

Corollary 2. *Let ABC be a triangle and let ℓ be the line at infinity in Π. Choose a general point P (*i.e., *not on a sideline of ABC, not on ℓ and different from the centroid of ABC) and call J, J' the tangent points on ℓ of the two conics of the pencil $\mathcal{B}(A, B, C, P)$ which are tangent to ℓ (these are the centers of the parabolas through A, B, C and P). Denote by Q the center of one of the four conics through J and J', which are tangent at the sidelines of ABC. Next, Z is the centroid* *of ABC and Z_A, Z_B, Z_C are the centroids of the triangles QBC, QCA, QAB respectively. Finally, P_A (P_B, and P_C respectively) is the fourth common point of the two parabolas through Q, B, C (through Q, C, A, and through Q, A, B respectively) and tangent to ℓ at J and J'. Then the lines PZ, P_AZ_A, P_BZ_B, and P_CZ_C concur at a point S_P.*

4. The use of trilinear coordinates

From now on, we work with trilinear coordinates (x_1, x_2, x_3) with respect to the real triangle ABC in the Euclidean plane Π ([2, 5]): A, B, C, and the incenter I of ABC, have coordinates $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$ respectively. The line at infinity ℓ has equation $ax_1 + bx_2 + cx_3 = 0$, where a, b, c are the side-lengths of ABC. The orthocenter H, the centroid Z, the circumcenter O, and the center of the nine-point circle O', have trilinear coordinates $\left(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}\right)$, $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, $(\cos A, \cos B, \cos C)$, and $(bc(a^2b^2 + a^2c^2 - (b^2 - c^2)^2), ca(b^2c^2 + b^2a^2 - (c^2 - a^2)^2), ab(c^2a^2 + c^2b^2 - (a^2 - b^2)^2))$ respectively. The equations of the circumcircle (O) and the nine-point circle (O') are $ax_2x_3 + bx_3x_1 + cx_1x_2 = 0$ and $x_1^2 \sin 2A + x_2^2 \sin 2B + x_3^2 \sin 2C - 2x_2x_3 \sin A - 2x_3x_1 \sin B - 2x_1x_2 \sin C = 0$.

The Schiffler point $S = S_H$ (the common point of the Euler lines of ABC, IBC, ICA, and IAB) has trilinear coordinates $\left(\frac{-a+b+c}{b+c}, \frac{a-b+c}{c+a}, \frac{a+b-c}{a+b}\right)$.

If T is a point of Π, not on a sideline of ABC, reflect the line AT about the line AI, and reflect BT and CT about the corresponding bisectors BI and CI. The three reflections concur in the isogonal conjugate T^{-1} of T, and T^{-1} has trilinear coordinates (t_2t_3, t_3t_1, t_1t_2) or $\left(\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}\right)$ if T has trilinear coordinates (t_1, t_2, t_3). Examples: the circumcenter O is the isogonal conjugate of the orthocenter H, and the centroid Z is the isogonal conjugate of the Lemoine point (or symmedian point) $K(a, b, c)$.

Let us now interpret the main theorem (or Corollary 2) in the Euclidean case using trilinear coordinates, with $\ell : ax_1 + bx_2 + cx_3 = 0$ as line at infinity and with $P(\alpha, \beta, \gamma)$ a general point of Π. In fact, the only thing that we have to do, is to replace in the proof of the main theorem the equation $x_1 + x_2 + x_3 = 0$ of ℓ, by $ax_1 + bx_2 + cx_3 = 0$, and a straightforward calculation gives us the following trilinear coordinates for the point Q: $(\sqrt{bc\alpha(b\beta + c\gamma)}, \sqrt{ca\beta(c\gamma + a\alpha)}, \sqrt{ab\gamma(a\alpha + b\beta)}) = (\mathcal{A}, \mathcal{B}, \mathcal{C})$. Next, the points Z, Z_A, Z_B, and Z_C are the centroids of ABC, QBC, QCA and QAB with trilinear coordinates $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, $(bc\mathcal{A}, c(a\mathcal{A}+2b\mathcal{B}+c\mathcal{C}), b(a\mathcal{A}+b\mathcal{B}+2c\mathcal{C}))$, $(c(2a\mathcal{A}+b\mathcal{B}+c\mathcal{C}), ca\mathcal{B}, a(a\mathcal{A}+b\mathcal{B}+2c\mathcal{C}))$, $(b(2a\mathcal{A}+b\mathcal{B}+c\mathcal{C}), a(a\mathcal{A}+2b\mathcal{B} + c\mathcal{C}), ba\mathcal{C})$, respectively. Now, for the points P_A, P_B, P_C, again after a straightforward calculation, we find the coordinates: $P_A(bc\mathcal{A}, c(c\mathcal{C} - a\mathcal{A}), b(b\mathcal{B} - a\mathcal{A}))$, $P_B(c(c\mathcal{C} - b\mathcal{B}), ca\mathcal{B}, a(a\mathcal{A} - b\mathcal{B}))$ and $P_C(b(b\mathcal{B} - c\mathcal{C}), a(a\mathcal{A} - c\mathcal{C}), ab\mathcal{C})$.

And finally, we find the trilinear coordinates of the point S_P, corresponding to Q:

$$\left(\frac{\mathcal{A}(-a\mathcal{A} + b\mathcal{B} + c\mathcal{C})}{b\mathcal{B} + c\mathcal{C}}, \frac{\mathcal{B}(a\mathcal{A} - b\mathcal{B} + c\mathcal{C})}{c\mathcal{C} + a\mathcal{A}}, \frac{\mathcal{C}(a\mathcal{A} + b\mathcal{B} - c\mathcal{C})}{a\mathcal{A} + b\mathcal{B}}\right).$$

Remark that we find for the case $P(\alpha,\beta,\gamma) = H(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C})$:

$$\begin{aligned}\mathcal{A} &= \sqrt{bc\alpha(b\beta+c\gamma)} = \sqrt{\tfrac{bc}{\cos A}(\tfrac{b}{\cos B}+\tfrac{c}{\cos C})} = \sqrt{\tfrac{bc(b\cos C+c\cos B)}{\cos A\cos B\cos C}} \\ &= \sqrt{\tfrac{abc}{\cos A\cos B\cos C}} = \mathcal{B} = \mathcal{C}\end{aligned}$$

and $Q(\mathcal{A},\mathcal{B},\mathcal{C}) = I(1,1,1)$, while since $\mathcal{A}=\mathcal{B}=\mathcal{C}$, we get for S_H the coordinates $\left(\frac{-a+b+c}{b+c}, \frac{a-b+c}{c+a}, \frac{a+b-c}{a+b}\right)$, which gives us the Schiffler point S.

Let us also calculate the trilinear coordinates of the points Y and Y', defined above as the centers of the conic (Y) through A, B, C, J and J', and of the conic (Y') through the midpoints of the sides of ABC and through J, J' (or the eleven-point conic of ABC with regard to J and J'; remark that J and J' are the cyclic points only when $P = H$):

(Y) has equation $\alpha(b\beta+c\gamma)x_2x_3 + \beta(c\gamma+a\alpha)x_3x_1 + \gamma(a\alpha+b\beta)x_1x_2 = 0$ and center $Y(bc(b\beta+c\gamma), ca(c\gamma+a\alpha), ab(a\alpha+b\beta))$,

(Y') has equation $a\beta\gamma x_1^2 + b\gamma\alpha x_2^2 + c\alpha\beta x_3^2 - \alpha(\gamma c+b\beta)x_2x_3 - \beta(a\alpha+c\gamma)x_3x_1 - \gamma(b\beta+a\alpha)x_1x_2 = 0$ and center $Y'(bc(2a\alpha+b\beta+c\gamma), ca(a\alpha+2b\beta+c\gamma), ab(\alpha+b\beta+2c\gamma))$.

Remark that $Q = \sqrt{P*Y}$, with the notation $\sqrt{(x_1,x_2,x_3)*(y_1,y_2,y_3)} = (\sqrt{x_1y_1}, \sqrt{x_2y_2}, \sqrt{x_3y_3})$.
Recall that the coordinate transformation between trilinear coordinates (x_1,x_2,x_3) with regard to $\triangle ABC$ and trilinear coordinates (x_1',x_2',x_3') with regard to the medial triangle $M_AM_BM_C$, is given by ([5, p.207]):

$$\begin{pmatrix} ax_1 \\ bx_2 \\ cx_3 \end{pmatrix} = \begin{pmatrix} 0 & b & c \\ a & 0 & c \\ a & b & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}.$$

Now, this gives for (x_1,x_2,x_3) the coordinates of the point Y, if (x_1',x_2',x_3') are the coordinates (α,β,γ) of P and it gives for (x_1,x_2,x_3) the coordinates of Y' if (x_1',x_2',x_3') are the coordinates of Y. Moreover, $\triangle ABC$ and its medial triangle are homothetic. As a corollary, we have that if P (Y, respectively) is *triangle center* $X(k)$ for $\triangle ABC$ (for the definition of triangle center, see [5, p.46]), then Y (Y' respectively) is center $X(k)$ for $\triangle M_AM_BM_C$.

5. Applications

In this section we choose $P(\alpha,\beta,\gamma)$ as a triangle center of the triangle ABC and calculate the coordinates of the corresponding points Y, Y', Q and S_P (sometimes Y' and S_P are not given).

Remark that P must be different from the centroid Z of ABC. The triangle centers are taken from Kimberling's list : $X(1)$, $X(2)$, $\ldots$, $X(2445)$ (list until 29 March 2004, see [6]). When we found the points Y, Y', Q or S_P in this list, we give the number $X(\cdots)$ and if possible, the name of the center. But, without doubt, we overlooked some centers and more points Y, Y', Q, S_P than indicated will occur in Kimberling's list. Several times, only the first trilinear coordinate is given: the second and the third are obtained by cyclic permutations.

5.1. The first example is of course:
$P(\alpha,\beta,\gamma) = H(\frac{1}{\cos A},\frac{1}{\cos B},\frac{1}{\cos C}) = X(4)$ (orthocenter),
$Y = O(\cos A, \cos B, \cos C) = X(3)$ (circumcenter),
$Y' = O'(bc(a^2b^2 + a^2c^2 - (b^2 - c^2)^2), \cdots, \cdots) = X(5)$ (nine-point center),
$Q = I(1,1,1) = X(1)$ (incenter), and
$S_H = S\left(\frac{-a+b+c}{b+c}, \cdots, \cdots\right) = X(21)$ (Schiffler point).

5.2. $P(\alpha,\beta,\gamma) = I(1,1,1) = X(1)$,
$Y = (\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}) = X(10)$ (Spieker point = incenter of the medial triangle $M_AM_BM_C$),
$Y' = (\frac{2a+b+c}{a}, \cdots, \cdots) = X(1125)$ (Spieker point of the medial triangle),
$Q = (\sqrt{bc(b+c)}, \cdots, \cdots)$, and
$S_I = (\sqrt{bc(b+c)}\frac{-a\sqrt{bc(b+c)}+b\sqrt{ca(c+a)}+c\sqrt{ab(a+b)}}{b\sqrt{ca(c+a)}+c\sqrt{ab(a+b)}}, \cdots, \cdots)$.

5.3. $P(\alpha,\beta,\gamma) = K(a,b,c) = X(6)$ (Lemoine point),
$Y = (\frac{b^2+c^2}{a}, \cdots, \cdots) = X(141)$ = Lemoine point of medial triangle,
$Y' = (\frac{2a^2+b^2+c^2}{a}, \cdots, \cdots)$,
$Q = (\sqrt{b^2+c^2}, \cdots, \cdots)$, and
$S_K = (\sqrt{b^2+c^2}\frac{-a\sqrt{b^2+c^2}+b\sqrt{c^2+a^2}+c\sqrt{a^2+b^2}}{b\sqrt{c^2+a^2}+c\sqrt{a^2+b^2}}, \cdots, \cdots)$.

5.4. $P(\alpha,\beta,\gamma) = (\frac{1}{a(-a+b+c)}, \cdots, \cdots) = X(7)$ (Gergonne point),
$Y = (-a+b+c, a-b+c, a+b-c) = X(9)$ (Mittenpunkt = Lemoine point of the excentral triangle $I_AI_BI_C$ = Gergonne point of medial triangle),
$Y' = (bc(a(b+c)-(b-c)^2), \cdots, \cdots) = X(142)$ (Mittenpunkt of medial triangle),
$Q = (\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}) = X(366)$, and
$S_{X(7)} = (\frac{1}{\sqrt{a}}\frac{-\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{b}+\sqrt{c}}, \cdots, \cdots)$.

5.5. $P(\alpha,\beta,\gamma) = (\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}) = X(100)$,
$Y = (bc(b-c)^2(-a+b+c), \cdots, \cdots) = X(11)$ (Feuerbach point = $X(100)$ of medial triangle),
$Y' = (bc((a-b)^2(a+b-c) + (c-a)^2(a-b+c)), \cdots, \cdots)$ (Feuerbach point of medial triangle), and $Q = (\sqrt{bc(b-c)(-a+b+c)}, \cdots, \cdots)$.

In the foregoing examples, the coordinates of the point S_P are mostly rather complicated. Another method is to start with the coordinates of the point Q: if (k,l,m) are the trilinear coordinates of Q, then a short calculation shows that it corresponds with the point $P(\frac{1}{a(-a^2k^2+b^2l^2+c^2m^2)}, \cdots, \cdots)$ and S_P becomes the point $(k\frac{-ak+bl+cm}{bl+cm}, \cdots, \cdots)$. Finally, the coordinates of Y and Y' are $(ak^2(-a^2k^2 + b^2l^2 + c^2m^2), \cdots, \cdots)$, and $(bc(a^2k^2(b^2l^2+c^2m^2) - (b^2l^2 - c^2m^2)^2), \cdots, \cdots)$, respectively. Here are some examples.

5.6. $Q(k,l,m) = K(a,b,c) = X(6)$(Lemoine point),
$P = (\frac{1}{a(-a^4+b^4+c^4)}, \cdots, \cdots) = X(66) = X(22)^{-1}$ ($X(22)$ is the Exeter point),
$Y = (a^3(-a^4+b^4+c^4), \cdots, \cdots) = X(206)$ ($X(66)$ of medial triangle),
$Y' = (bc(a^4(b^4+c^4) - (b^4-c^4)^2), \cdots, \cdots)$ ($X(206)$ of medial triangle), and
$S_{X(66)} = (\frac{a(-a^2+b^2+c^2)}{b^2+c^2}, \cdots, \cdots) = (\frac{\cos A}{b^2+c^2}, \cdots, \cdots) = X(1176)$.

5.7. $Q(k,l,m) = H(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}) = X(4)$ (orthocenter),
$P = (\frac{1}{a(-\frac{a^2}{\cos^2 A}+\frac{b^2}{\cos^2 B}+\frac{c^2}{\cos^2 C})}, \cdots, \cdots)$,
$Y = (\frac{a}{\cos^2 A}(-\frac{a^2}{\cos^2 A} + \frac{b^2}{\cos^2 B} + \frac{c^2}{\cos^2 C}), \cdots, \cdots)$, and
$S_P = (\frac{\cos A - \cos B \cos C}{\cos^2 A}, \cdots, \cdots)$.

5.8. $Q(k,l,m) = (\frac{b+c}{a}, \cdots, \cdots) = X(10)$ (Spieker point),
$P = (\frac{1}{a(-(b+c)^2+(c+a)^2+(a+b)^2)}, \cdots, \cdots) = X(596)$,
$Y = \left(\frac{(b+c)^2}{a}(-(b+c)^2 + (c+a)^2 + (a+b)^2), \cdots, \cdots\right)$ ($X(596)$ of medial triangle), and
$S_P = (\frac{b+c}{2a+b+c}, \frac{c+a}{a+2b+c}, \frac{a+b}{a+b+2c})$.

We also can start with the coordinates of the point $Y(p,q,r)$, then
$P = (\frac{-ap+bq+cr}{a}, \cdots, \cdots)$,
$Y'(bc(bq+cr), \cdots, \cdots)$, and
$Q = \sqrt{P * Y} = (\sqrt{\frac{p(-ap+bq+cr)}{a}}, \cdots, \cdots)$. Here are some examples.

5.9. $Y(p,q,r) = I(1,1,1) = X(1)$, $P = (\frac{-a+b+c}{a}, \frac{a-b+c}{b}, \frac{a+b-c}{c}) = X(8)$ (Nagel point),
$Y' = \left(\frac{b+c}{a}, \cdots, \cdots\right) = X(10)$ (Spieker point = incenter of medial triangle),
$Q = \left(\sqrt{\frac{-a+b+c}{a}}, \cdots, \cdots\right) = X(188)$, and
$S_P = (\mathcal{A}\frac{-a\mathcal{A}+b\mathcal{B}+c\mathcal{C}}{b\mathcal{B}+c\mathcal{C}}, \cdots, \cdots)$ with $Q(\mathcal{A}, \mathcal{B}, \mathcal{C})$.

5.10. $Y = K(a,b,c) = X(6)$ (Lemoine point),
$P = (\frac{-a^2+b^2+c^2}{a}, \cdots, \cdots) = (\frac{cosA}{a^2}, \cdots, \cdots) = X(69)$,
$Y' = (\frac{b^2+c^2}{a}, \cdots, \cdots) = X(141)$ (Lemoine point of medial triangle), and
$Q = (\sqrt{-a^2+b^2+c^2}, \cdots, \cdots)$.

5.11. $Y = (\frac{2a+b+c}{a}, \cdots, \cdots) = X(1125)$ (Spieker point of medial triangle),
$P = (\frac{b+c}{a}, \cdots, \cdots) = X(10)$ (Spieker point),
$Y' = (\frac{2a+3b+3c}{a}, \cdots, \cdots)$ ($X(1125)$ of medial triangle), and
$Q = (bc\sqrt{(b+c)(2a+b+c)}, \cdots, \cdots)$.

References

[1] L. Emelyanov and T. Emelyenova, A note on the Schiffler point, *Forum Geom.*, 3 (2003) 113–116.

[2] W. Gallatly, *The Modern Geometry of the Triangle*, 2nd ed. 1913, Francis Hodgson, London.

[3] A. P. Hatzipolakis, F. M. van Lamoen, B. Wolk, and P. Yiu, Concurrency of four Euler lines, *Forum Geom.*, 1 (2001) 59–68.

[4] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, Math. Assoc. of America, 1995.

[5] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

[6] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[7] D. Pedoe, *A Course of Geometry for Colleges and Universities*, Cambridge Univ. Press, 1970.

[8] P. Samuel, *Projective Geometry*, Springer Verlag, 1988.

[9] K. Schiffler, G. R. Veldkamp, and W. A. van der Spek, Problem 1018, *Crux Math.*, 11 (1985) 51; solution 12 (1986) 150–152.

Charles Thas: Department of Pure Mathematics and Computer Algebra, Krijgslaan 281-S22, B-9000 Gent, Belgium

E-mail address: charles.thas@UGent.be

Forum Geometricorum
Volume 4 (2004) 97–109.

FORUM GEOM
ISSN 1534-1178

The Vertex-Midpoint-Centroid Triangles

Zvonko Čerin

Abstract. This paper explores six triangles that have a vertex, a midpoint of a side, and the centroid of the base triangle ABC as vertices. They have many interesting properties and here we study how they monitor the shape of ABC. Our results show that certain geometric properties of these six triangles are equivalent to ABC being either equilateral or isosceles.

Let A', B', C' be midpoints of the sides BC, CA, AB of the triangle ABC and let G be its centroid (*i.e.*, the intersection of medians AA', BB', CC'). Let G_a^-, G_a^+, G_b^-, G_b^+, G_c^-, G_c^+ be triangles BGA', CGA', CGB', AGB', AGC', BGC' (see Figure 1).

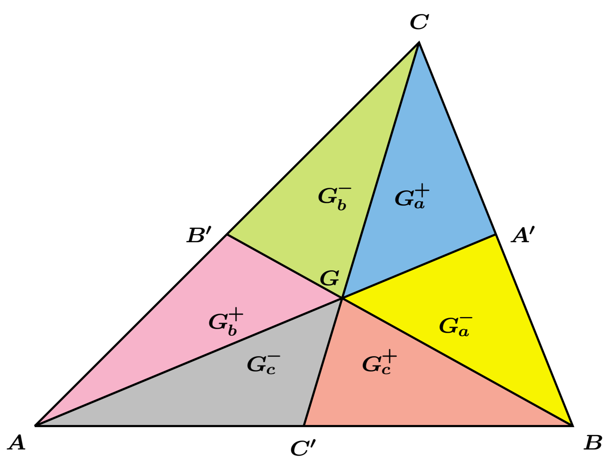


Figure 1. Six vertex–midpoint–centroid triangles of ABC.

This set of six triangles associated to the triangle ABC is a special case of the cevasix configuration (see [5] and [7]) when the chosen point is the centroid G. It has the following peculiar property (see [1]).

Theorem 1. *The triangle ABC is equilateral if and only if any three of the triangles from the set $\sigma_G = \{G_a^-, G_a^+, G_b^-, G_b^+, G_c^-, G_c^+\}$ have the same either perimeter or inradius.*

In this paper we wish to show several similar results. The idea is to replace perimeter and inradius with other geometric notions (like k-perimeter and Brocard angle) and to use various central points (like the circumcenter and the orthocenter – see [4]) of these six triangles.

Publication Date: July 14, 2004. Communicating Editor: Paul Yiu.

Let a, b, c be lengths of sides of the base triangle ABC. For a real number k, the sum $p_k = p_k(ABC) = a^k + b^k + c^k$ is called the k-*perimeter* of ABC. Of course, the 1-perimeter $p_1(ABC)$ is just the perimeter $p(ABC)$. The above theorem suggests the following problem.

Problem. *Find the set Ω of all real numbers k such that the following is true: The triangle ABC is equilateral if and only if any three of the triangles from σ_G have the same k-perimeter.*

Our first goal is to show that the set Ω contains some values of k besides the value $k = 1$. We start with $k = 2$ and $k = 4$.

Theorem 2. *The triangle ABC is equilateral if and only if any three of the triangles in σ_G have the same either 2-perimeter or 4-perimeter.*

Proof for $k = 2$. We shall position the triangle ABC in the following fashion with respect to the rectangular coordinate system in order to simplify our calculations. The vertex A is the origin with coordinates $(0, 0)$, the vertex B is on the x-axis and has coordinates $(r(f + g), 0)$, and the vertex C has coordinates $\left(\frac{rg(f^2-1)}{fg-1}, \frac{2rfg}{fg-1}\right)$. The three parameters r, f, and g are the inradius and the cotangents of half of angles at vertices A and B. Without loss of generality, we can assume that both f and g are larger than 1 (*i.e.*, that angles A and B are acute).

Nice features of this placement are that many important points of the triangle have rational functions in f, g, and r as coordinates and that we can easily switch from f, g, and r to side lengths a, b, and c and back with substitutions

$$a = \frac{rf\left(g^2+1\right)}{fg-1}, \quad b = \frac{rg\left(f^2+1\right)}{fg-1}, \quad c = r\,(f+g)\,,$$
$$f = \frac{(b+c)^2-a^2}{4\Delta}, \quad g = \frac{(a+c)^2-b^2}{4\Delta}, \quad r = \frac{2\Delta}{a+b+c},$$

where the area Δ is $\frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a-b+c)(a+b-c)}$.

There are 20 ways in which we can choose 3 triangles from the set σ_G. The following three cases are important because all other cases are similar to one of these.

Case 1: (G_a^-, G_a^+, G_b^-). When we compute the 2-perimeters $p_2(G_a^-)$, $p_2(G_a^+)$, and $p_2(G_b^-)$ and convert to lengths of sides we get

$$p_2(G_a^-) - p_2(G_a^+) = \frac{(c-b)(c+b)}{3},$$
$$p_2(G_a^-) - p_2(G_b^-) = \frac{a^2}{6} - \frac{b^2}{2} + \frac{c^2}{3}.$$

Both of these differences are by assumption zero. From the first we get $b = c$ and when we substitute this into the second the conclusion is $\frac{(a-c)(a+c)}{6} = 0$. Hence, $b = c = a$ so that ABC is equilateral.

Case 2: (G_a^-, G_a^+, G_b^+). Now we have

$$p_2(G_a^-) - p_2(G_a^+) = \frac{(c-b)(c+b)}{3},$$
$$p_2(G_a^-) - p_2(G_b^+) = \frac{(a-b)(a+b)}{2},$$

which makes the conclusion easy.

Case 3: (G_a^-, G_b^-, G_c^-). This time we have

$$p_2(G_a^-) - p_2(G_b^-) = \frac{a^2}{6} - \frac{b^2}{2} + \frac{c^2}{3},$$
$$p_2(G_a^-) - p_2(G_c^-) = \frac{a^2}{2} - \frac{b^2}{3} - \frac{c^2}{6}.$$

The only solution of this linear system in a^2 and b^2 is $a^2 = c^2$ and $b^2 = c^2$. Thus the triangle ABC is equilateral because the lengths of sides are positive. □

Recall that the Brocard angle ω of the triangle ABC satisfies the relation

$$\cot\omega = \frac{p_2(ABC)}{4\Delta}.$$

Since all triangles in σ_G have the same area, from Theorem 2 we get the following corollary.

Corollary 3. *The triangle ABC is equilateral if and only if any three of the triangles in σ_G have the same Brocard angle.*

On the other hand, when we put $k = -2$ then for $a = \sqrt{-5 + 3\sqrt{3}}$ and $b = c = 1$ we find that the triangles G_a^-, G_a^+, and G_b^- have the same (-2)-perimeter while ABC is not equilateral. In other words the value -2 is not in Ω.

The following result answers the final question in [1]. It shows that some pairs of triangles from the set σ_G could be used to detect if ABC is isosceles. Let τ denote the set whose elements are pairs (G_a^-, G_a^+) (G_a^-, G_b^+), (G_a^-, G_c^+), (G_a^+, G_b^-), (G_a^+, G_c^-), (G_b^-, G_b^+), (G_b^-, G_c^+), (G_b^+, G_c^-), (G_c^-, G_c^+).

Theorem 4. *The triangle ABC is isosceles if and only if triangles from some element of τ have the same perimeter.*

Proof. This time there are only two representative cases.

Case 1: (G_a^-, G_a^+). By assumption,

$$p(G_a^-) - p(G_a^+) = \frac{\sqrt{2a^2 - b^2 + 2c^2}}{3} - \frac{\sqrt{2a^2 + 2b^2 - c^2}}{3} = 0.$$

When we move the second term to the right then take the square of both sides and move everything back to the left we obtain $\frac{(c-b)(c+b)}{3} = 0$. Hence, $b = c$ and ABC is isosceles.

Case 2: (G_a^-, G_b^+). This time our assumption is

$$p(G_a^-) - p(G_b^+) = \frac{a-b}{2} + \frac{\sqrt{2a^2 - b^2 + 2c^2}}{6} - \frac{\sqrt{2c^2 + 2b^2 - a^2}}{6} = 0.$$

When we move the third term to the right then take the square of both sides and move the right hand side back to the left and bring the only term with the square root to the right we obtain

$$\frac{2a^2-3ab+b^2}{6} = \frac{(b-a)\sqrt{2a^2-b^2+2c^2}}{6}.$$

In order to eliminate the square root, we take the square of both sides and move the right hand side to the left to get $\frac{(a-b)^2(a-b-c)(a-b+c)}{18} = 0$. Hence, $a = b$ and the triangle ABC is again isosceles. □

Remark. The above theorem is true also when the perimeter is replaced with the 2-perimeter and the 4-perimeter. It is not true for $k = -2$ but it holds for any $k \neq 0$ when only pairs (G_a^-, G_a^+), (G_b^-, G_b^+), (G_c^-, G_c^+) are considered.

We continue with results that use various central points (see [4], [5, 6]) (like the centroid, the circumcenter, the orthocenter, the center of the nine-point circle, the symmedian or the Grebe-Lemoine point, and the Longchamps point) of the triangles from the set σ_G and try to detect when ABC is either equilateral or isosceles.

Recall that triangles ABC and XYZ are *homologic* provided lines AX, BY, and CZ are concurrent. The point in which they concur is their homology *center* and the line containing intersections of pairs of lines (BC, YZ), (CA, ZX), and (AB, XY) is their homology *axis*. Instead of homologic, homology center, and homology axis many authors use the terms *perspective*, *perspector*, and *perspectrix*.

The triangles ABC and XYZ are *orthologic* when the perpendiculars at vertices of ABC onto the corresponding sides of XYZ are concurrent. The point of concurrence is $[ABC, XYZ]$. It is well-known that the relation of orthology for triangles is reflexive and symmetric. Hence, the perpendiculars at vertices of XYZ onto corresponding sides of ABC are concurrent at a point $[XYZ, ABC]$.

By replacing in the above definition perpendiculars with parallels we get the analogous notion of *paralogic* triangles and two centers of paralogy $\langle ABC, XYZ\rangle$ and $\langle XYZ, ABC\rangle$.

The triangle ABC is paralogic to its first Brocard triangle $A_bB_bC_b$ which has the orthogonal projections of the symmedian point K onto the perpendicular bisectors of sides as vertices (see [2] and [3]).

Theorem 5. *The centroids $G_{G_a^-}$, $G_{G_a^+}$, $G_{G_b^-}$, $G_{G_b^+}$, $G_{G_c^-}$, $G_{G_c^+}$ of the triangles from σ_G lie on the image of the Steiner ellipse of ABC under the homothety $\mathsf{h}(G, \frac{\sqrt{7}}{6})$. This ellipse is a circle if and only if ABC is equilateral. The triangles $G_{G_a^-}G_{G_b^-}G_{G_c^-}$ and $G_{G_a^+}G_{G_b^+}G_{G_c^+}$ are both homologic and paralogic to triangles $A_bB_bC_b$, $B_bC_bA_b$ and $C_bA_bB_b$ and they share with ABC the centroid and the Brocard angle and both have $\frac{7}{36}$ of the area of ABC. They are directly similar to each other or to ABC if and only if ABC is an equilateral triangle. They are orthologic to either $A_bB_bC_b$, $B_bC_bA_b$ or $C_bA_bB_b$ if and only if ABC is an equilateral triangle.*

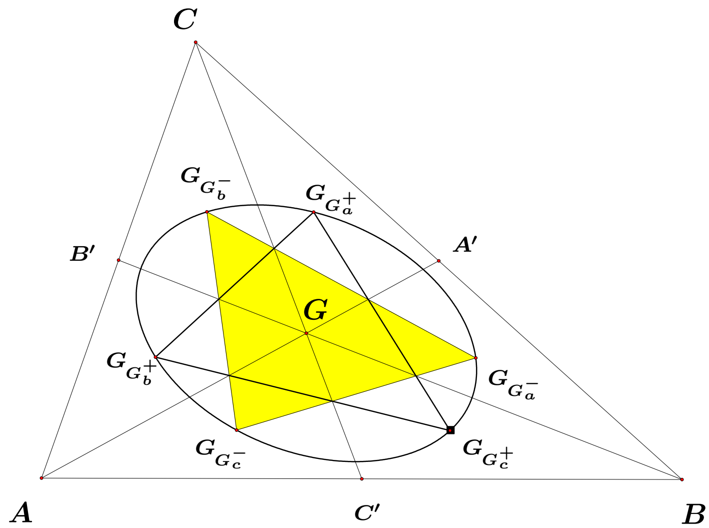


Figure 2. The ellipse containing vertices of $G_{G_a^-}G_{G_b^-}G_{G_c^-}$ and $G_{G_a^+}G_{G_b^+}G_{G_c^+}$.

Proof. We look for the conic through five of the centroids and check that the the sixth centroid lies on it. The trilinear coordinates of $G_{G_a^-}$ are $\frac{2}{a} : \frac{11}{b} : \frac{5}{c}$ while those of other centroids are similar. It follows that they all lie on the ellipse with the equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0,$$

where

$$\begin{aligned} &a_{11} = 432\Delta^2, \quad a_{12} = 108\Delta(a-b)(a+b), \\ &a_{22} = 27(a^4 + b^4 + 3c^4 - 2a^2b^2), \\ &a_{13} = -216\Delta^2 c, \quad a_{23} = -54\Delta c(a^2 - b^2 + c^2), \quad a_{33} = 116\Delta^2 c^2. \end{aligned}$$

Since $D_0 = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = \frac{3c^4}{16\Delta^2} > 0$, and $\frac{A_0}{I_0} = \frac{-7c^4}{72(a^2+b^2+c^2)} < 0$ with $I_0 = a_{11} + a_{22}$, and $A_0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$ it follows that this is an ellipse whose center is G. It will be a circle provided either $I_0^2 = 4D_0$ or $a_{11} = a_{22}$ and $a_{12} = 0$. This happens if and only if ABC is equilateral.

The precise identification of this ellipse is now easy. We take a point (p, q) which is on the Steiner ellipse of ABC (with the equation $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 0$ in trilinear coordinates) and denote its image under $\mathsf{h}(G, \frac{\sqrt{7}}{6})$ by (x, y). By eliminating p and q we check that this image satisfies the above equation (of the common Steiner ellipse of $G_{G_a^-}G_{G_b^-}G_{G_c^-}$ and $G_{G_a^+}G_{G_b^+}G_{G_c^+}$).

Since the trilinear coordinates of A_b are $abc : c^3 : b^3$, the line $A_bG_{G_a^-}$ has the equation

$$a(11b^2 - 5c^2)x + b(5a^2 - 2b^2)y + c(11a^2 - 2c^2)z = 0.$$

The lines $B_bG_{G_b^-}$ and $C_bG_{G_c^-}$ have similar equations. The determinant of the coefficients of these three lines is equal to zero so that we conclude that the triangles $G_{G_a^-}G_{G_b^-}G_{G_c^-}$ and $A_bB_bC_b$ are homologic. The other claims about homologies and paralogies are proved in a similar way. We note that $\langle G_{G_a^-}G_{G_b^-}G_{G_c^-}, A_bB_bC_b\rangle$ is on the (above) Steiner ellipse of $G_{G_a^-}G_{G_b^-}G_{G_c^-}$ while $\langle A_bB_bC_b, G_{G_a^-}G_{G_b^-}G_{G_c^-}\rangle$ is on the Steiner ellipse of $A_bB_bC_b$. The other centers behave accordingly.

When we substitute the coordinates of the six centroids into the conditions

$$x_1(v_2 - v_3) + x_2(v_3 - v_1) + x_3(v_1 - v_2) - u_1(y_2 - y_3) - u_2(y_3 - y_1) - u_3(y_1 - y_2) = 0,$$
$$x_1(u_2 - u_3) + x_2(u_3 - u_1) + x_3(u_1 - u_2) - y_1(v_2 - v_3) - y_2(v_3 - v_1) - y_3(v_1 - v_2) = 0,$$

for triangles with vertices at the points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (u_1, v_1), (u_2, v_2), (u_3, v_3) to be directly similar and convert to the side lengths, we get

$$\frac{4\Delta(a-b)(a+b+c)}{9c^2} = 0 \quad \text{and} \quad \frac{h(1,1,2,1,1,2)}{9c^2} = 0,$$

where

$$h(u, v, w, x, y, z) = ub^2c^2 + vc^2a^2 + wa^2b^2 - xa^4 - yb^4 - zc^4.$$

The first relation implies $a = b$, which gives $h(1, 1, 2, 1, 1, 2) = 2c^2(c - b)(c + b)$. Therefore, $b = c$ so that ABC is an equilateral triangle.

Substituting the coordinates of $G_{G_a^-}$, $G_{G_b^-}$, $G_{G_c^-}$, A_b, B_b, C_b into the left hand side of the condition

$$x_1(u_2-u_3)+x_2(u_3-u_1)+x_3(u_1-u_2)+y_1(v_2-v_3)+y_2(v_3-v_1)+y_3(v_1-v_2) = 0,$$

for triangles with vertices at the points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (u_1, v_1), (u_2, v_2), (u_3, v_3) to be orthologic, we obtain

$$\frac{-h(1,1,1,1,1,1)}{3p_2(ABC)} = \frac{(b^2-c^2)^2 + (c^2-a^2)^2 + (a^2-b^2)^2}{6p_2(ABC)}$$

so that the triangles $G_{G_a^-}G_{G_b^-}G_{G_c^-}$ and $A_bB_bC_b$ are orthologic if and only if ABC is equilateral.

The remaining statements are proved similarly or by substitution of coordinates into well-known formulas for the area, the centroid, and the Brocard angle. □

Let m_a, m_b, m_c be lengths of medians of the triangle ABC. The following result is for the most part already proved in [7]. The center of the circle is given in [6] as $X(1153)$.

Theorem 6. *The circumcenters $O_{G_a^-}$, $O_{G_a^+}$, $O_{G_b^-}$, $O_{G_b^+}$, $O_{G_c^-}$, $O_{G_c^+}$ of the triangles from σ_G lie on the circle whose center O_G is a central point with the first*

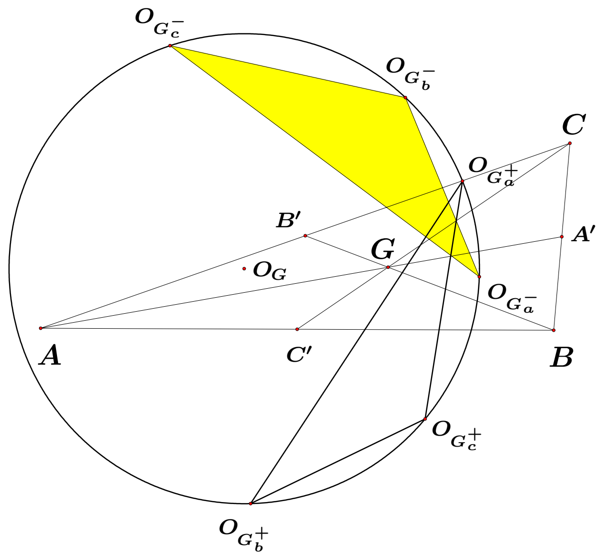


Figure 3. The vertices of $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_a^+}O_{G_b^+}O_{G_c^+}$ are on a circle.

trilinear coordinate

$$\frac{10a^4 - 13a^2(b^2+c^2) + 4b^4 + 4c^4 - 10b^2c^2}{a}$$

and whose radius is

$$\frac{m_a m_b m_c \sqrt{2(a^4+b^4+c^4) - 5(b^2c^2+c^2a^2+a^2b^2)}}{72\Delta}.$$

Also, $|O_G G| = \frac{m_a m_b m_c \sqrt{(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2}}{72\sqrt{2}\Delta}$.

Proof. The proof is conceptually simple but technically involved so that we shall only outline how it could be done on a computer. In order to find points $O_{G_a^-}$, $O_{G_a^+}$, $O_{G_b^-}$, $O_{G_b^+}$, $O_{G_c^-}$, $O_{G_c^+}$ we use the circumcenter function and evaluate it in vertices of the triangles from σ_G. Applying it again in points $O_{G_a^-}$, $O_{G_a^+}$, $O_{G_b^-}$ we obtain the point O_G. The remaining points $O_{G_b^+}$, $O_{G_c^-}$, $O_{G_c^+}$ are at the same distance from it as the vertex $O_{G_a^-}$ is. The remaining tasks are standard (they involve only the distance function and the conversion to the side lengths). □

The last sentence in Theorem 6 implies the following corollary.

Corollary 7. *The triangle ABC is equilateral if and only if the circumcenters of any three of the triangles in σ_G have the same distance from the centroid G.*

Let P, Q and R denote vertices of similar isosceles triangles BCP, CAQ and ABR.

Theorem 8. *(1) The triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_b^+}O_{G_c^+}O_{G_a^+}$ are congruent. They are orthologic to BCA and CAB, respectively.*

(2) The triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ *and* $O_{G_a^+}O_{G_b^+}O_{G_c^+}$ *are orthologic to* QRP *and* RPQ *if and only if* ABC *is an equilateral triangle.*

(3) The triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ *and* $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ *are orthologic if and only if the lengths of sides of* ABC *satisfy* $h(7,7,7,4,4,4)=0$*.*

(4) The line joining the centroids of triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ *and* $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ *will go through the centroid, the circumcenter, the orthocenter, the center of the nine-point circle, the Longchamps point, or the Bevan point of* ABC *(*i.e., $X(2)$*,* $X(3)$*,* $X(4)$*,* $X(5)$*,* $X(20)$*, or* $X(40)$ *in* [6]*) if and only if it is an equilateral triangle.*

(5) The line joining the symmedian points of $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ *and* $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ *goes through the centroid of* ABC*. It will go through the centroid of its orthic triangle (*i.e., $X(51)$ *in* [6]*) if and only if* ABC *is an equilateral triangle.*

(6) The centroids of triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ *and* $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ *have the same distance from* $X(2)$*,* $X(3)$*,* $X(4)$*,* $X(5)$*,* $X(6)$*,* $X(20)$*,* $X(39)$*,* $X(40)$*, or* $X(98)$ *if and only if* ABC *is an isosceles triangle.*

Proof. (1) The points $O_{G_a^-}$ and $O_{G_a^+}$ have trilinear coordinates

$$a(5c^2-a^2-b^2) : \frac{2h(3,3,5,2,2,1)}{b} : \frac{h(6,1,3,1,2,4)}{c},$$

$$a(5b^2-a^2-c^2) : \frac{h(6,3,1,1,4,2)}{b} : \frac{2h(3,5,3,2,1,2)}{c},$$

while the trilinears of the points $O_{G_b^-}$, $O_{G_c^-}$, $O_{G_b^+}$, $O_{G_c^+}$ are their cyclic permutations. We can show easily that $|O_{G_b^-}O_{G_c^-}|^2 - |O_{G_c^+}O_{G_a^+}|^2 = 0$, $|O_{G_c^-}O_{G_a^-}|^2 - |O_{G_a^+}O_{G_b^+}|^2 = 0$, and $|O_{G_a^-}O_{G_b^-}|^2 - |O_{G_b^+}O_{G_c^+}|^2 = 0$, so that $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_b^+}O_{G_c^+}O_{G_a^+}$ are indeed congruent.

Substituting the coordinates of $O_{G_a^-}$, $O_{G_b^-}$, $O_{G_c^-}$, B, C, A into the left hand side of the above condition for triangles to be orthologic we conclude that it holds. The same is true for the triangles $O_{G_a^+}O_{G_b^+}O_{G_c^+}$ and CAB.

(2) The point P has the trilinear coordinates

$$2ka : \frac{k(a^2+b^2-c^2)+2\Delta}{b} : \frac{k(a^2-b^2+c^2)+2\Delta}{c}$$

for some real number $k \neq 0$. The coordinates of Q and R are analogous. It follows that the triangles $O_{G_c^-}O_{G_a^-}O_{G_b^-}$ and QRP are orthologic provided

$$\frac{h(1,1,1,1,1,1)k}{8\Delta} = 0,$$

i.e., if and only if ABC is equilateral.

(3) The triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ are orthologic provided $\frac{p_2(ABC)h(7,7,7,4,4,4)}{384\Delta^2} = 0$. The triangle with lengths of sides $4, 4, 3\sqrt{2}+\sqrt{10}$ satisfies this condition.

(4) for $X(40)$. The first trilinear coordinates of the centroids of the triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ are

$$\frac{3a^4-(2b^2+7c^2)a^2+b^4-3b^2c^2+2c^4}{a}$$

and

$$\frac{3a^4-(7b^2+2c^2)a^2+2b^4-3b^2c^2+c^4}{a}.$$

The line joining these centroids will go through $X(40)$ with the first trilinear coordinate $a^3+(b+c)a^2-(b+c)^2a-(b+c)(b-c)^2$ provided

$$\frac{(a^2+b^2+c^2-bc-ca-ab)(3bc+3ca+3ab+a^2+b^2+c^2)}{96\Delta}=0.$$

Since $a^2+b^2+c^2-bc-ca-ab=\frac{1}{2}\left((b-c)^2+(c-a)^2+(a-b)^2\right)$ it follows that this will happen if and only if ABC is equilateral.

(5) The first trilinear coordinates of the symmedian points of $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ are

$$\frac{2a^6-(b^2+3c^2)a^4+(3b^4-12b^2c^2-7c^4)a^2+2c^2(b^2-c^2)(b^2-2c^2)}{a}$$

and

$$\frac{2a^6-(3b^2+c^2)a^4-(7b^4+12b^2c^2-3c^4)a^2+2b^2(b^2-c^2)(2b^2-c^2)}{a}.$$

The line joining these symmedian points will go through $X(51)$ with the first trilinear coordinate $a\left((b^2+c^2)a^2-(b^2-c^2)^2\right)$ provided

$$\frac{2\Delta h(1,1,1,0,0,0)h(1,1,1,1,1,1)}{9a^2b^2c^2(a^2+b^2+c^2)}=0.$$

Since $h(1,1,1,1,1,1)=\frac{1}{2}\left((b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2\right)$ we see that this will happen if and only if ABC is equilateral. The trilinear coordinates $\frac{1}{a}:\frac{1}{b}:\frac{1}{c}$ of the centroid G satisfy the equation of this line.

(6) for $X(40)$. Using the information from the proof of (4), we see that the difference of squares of distances from $X(40)$ to the centroids of the triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ is $\frac{(b-c)(c-a)(a-b)M}{192\Delta^2}$, where

$$M=2(a^3+b^3+c^3)+5(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b)+18abc$$

is clearly positive. Hence, these distances are equal if and only if ABC is isosceles. $\square$

With points $O_{G_a^-}$, $O_{G_a^+}$, $O_{G_b^-}$, $O_{G_b^+}$, $O_{G_c^-}$, $O_{G_c^+}$ we can also detect if ABC is isosceles as follows.

Theorem 9. *(1) The relation $b=c$ holds in ABC if and only if $O_{G_a^-}$ is on BG and/or $O_{G_a^+}$ is on CG.*

(2) The relation $c=a$ holds in ABC if and only if $O_{G_b^-}$ is on CG and/or $O_{G_b^+}$ is on AG.

(3) The relation $a = b$ holds in ABC if and only if $O_{G_c^-}$ is on AG and/or $O_{G_a^+}$ is on BG.

Proof. (1) for $O_{G_a^-}$. Since the trilinear coordinates of $O_{G_a^-}$, G and B are

$$a(5c^2 - a^2 - b^2) : \frac{2h(3,3,5,2,2,1)}{b} : \frac{h(6,1,3,1,2,4)}{c},$$

$\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$ and $(0 : 1 : 0)$, it follows that these points are collinear if and only if $\frac{m_b^2(b-c)(b+c)}{72\Delta} = 0$. □

For the following result I am grateful to an anonymous referee. It refers to the point T on the Euler line which divides the segment joining the circumcenter with the centroid in ratio k for some real number $k \neq -1$. Notice that for $k = 0, -\frac{3}{4}, -\frac{3}{2}, -3$ the point T will be the circumcenter, the Longchamps point, the orthocenter, and the center of the nine-point circle, respectively.

Theorem 10. *The triangles $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are directly similar to each other or to ABC if and only if ABC is equilateral.*

Proof. For $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$.

The point $T_{G_a^-}$ has $\frac{p_1}{a} : \frac{p_2}{b} : \frac{p_3}{c}$ as trilinear coordinates, where

$$\begin{aligned}
p_1 =& 3a^2(a^2 + b^2 - 5c^2) - 32\Delta^2 k,\\
p_2 =& 12a^4 - 6(5b^2 + 3c^2)a^2 + 6(b^2 - c^2)(2b^2 - c^2) - 176\Delta^2 k,\\
p_3 =& 12a^4 - 6(3b^2 + 5c^2)a^2 + 6(b^2 - c^2)(b^2 - 2c^2) - 176\Delta^2 k.
\end{aligned}$$

Applying the method of the proof of Theorem 4 we see that $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are directly similar if and only if

$$\frac{(a^2 - b^2)M}{288\Delta c^2(k+1)^2} = 0 \quad \text{and} \quad \frac{h(1,1,2,1,1,2)M}{1152S^2c^2(k+1)^2} = 0,$$

where $M = 128\Delta^2k^2 + 240\Delta^2k + h(15,15,15,6,6,6)$. The discriminant

$$-48\Delta^2 h(10,10,10,-11,-11,-11)$$

of the trinomial M is negative so that M is always positive. Hence, from the first condition it follows that $a = b$. Then the factor $h(1,1,2,1,1,2)$ in the second condition is $2c^2(c-b)(c+b)$ so that $b = c$ and ABC is equilateral. The converse is easy because for $a = b = c$ the left hand sides of both conditions are equal to zero.

For $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and ABC. The two conditions are

$$\begin{aligned}
&32\Delta^2(a^2 - b^2)k - a^6 + (4b^2 + 3c^2)a^4\\
&- (5b^4 + 2b^2c^2 + c^4)a^2 - 3b^4c^2 + 2b^2c^4 + 2b^6 + c^6 = 0
\end{aligned}$$

and

$$h(2,2,4,2,2,4)k + h(1,2,3,1,2,3) = 0.$$

When $a \neq b$, we can solve the first equation for k and substitute it into the second to obtain $\frac{c^4(a^2+b^2+c^2)h(1,1,1,1,1,1)}{8\Delta^2(a^2-b^2)} = 0$. This implies that $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and ABC are directly similar if and only if ABC is equilateral because the first condition is $c^2\,(b-c)\,(b+c)\left(c^2+2b^2\right) = 0$ for $a = b$. □

Theorem 11. *(1) $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are orthologic to ABC if and only if $k = -\frac{3}{2}$.*

(2) $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are orthologic to $A_bB_bC_b$ if and only if either ABC is equilateral or $k = -\frac{3}{4}$.

(3) $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are paralogic to either $A_bB_bC_b$, $B_bC_bA_b$ or $C_bA_bB_b$ if and only if ABC is equilateral.

(4) $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ is orthologic to $B_bC_bA_b$ if and only if either ABC is equilateral or $k = -\frac{3}{2}$ and to $C_bA_bB_b$ if and only if ABC is equilateral.

(5) $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ is orthologic to $B_bC_bA_b$ if and only if ABC is equilateral and to $C_bA_bB_b$ if and only if either ABC is equilateral or $k = -\frac{3}{2}$.

Proof. All parts have similar proofs. For example, in the first, we find that the triangles $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and ABC are orthologic if and only if $-\frac{(a^2+b^2+c^2)(2k+3)}{12(k+1)} = 0$. □

The orthocenters $H_{G_a^-}$, $H_{G_a^+}$, $H_{G_b^-}$, $H_{G_b^+}$, $H_{G_c^-}$, $H_{G_c^+}$ of the triangles from σ_G also monitor the shape of the triangle ABC.

Theorem 12. *The triangles $H_{G_a^-}H_{G_b^-}H_{G_c^-}$ and $H_{G_a^+}H_{G_b^+}H_{G_c^+}$ are orthologic if and only if ABC is an equilateral triangle.*

Proof. Substituting the coordinates of $H_{G_a^-}$, $H_{G_b^-}$, $H_{G_c^-}$, $H_{G_a^+}$, $H_{G_b^+}$, $H_{G_c^+}$ into the condition for triangles to be orthologic (see the proof of Theorem 6), we obtain

$$\frac{(a^2+b^2+c^2)[(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2]}{192\Delta^2} = 0.$$

Hence, $a = b = c$ and the triangle ABC is equilateral. □

Remark. Note that the triangles $H_{G_a^-}H_{G_b^-}H_{G_c^-}$ and $H_{G_a^+}H_{G_b^+}H_{G_c^+}$ have the same Brocard angle and both have the area equal to one fourth of the area of ABC.

The centers $F_{G_a^-}$, $F_{G_a^+}$, $F_{G_b^-}$, $F_{G_b^+}$, $F_{G_c^-}$, $F_{G_c^+}$ of the nine point circles of the triangles from σ_G allow the following analogous result.

Theorem 13. *The triangles $F_{G_a^-}F_{G_b^-}F_{G_c^-}$ and $F_{G_a^+}F_{G_b^+}F_{G_c^+}$ have the same Brocard angle and area. The triangle ABC is equilateral if and only if this area is $\frac{3}{16}$ of the area of ABC.*

Proof. Recall the formula $\frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)|$ for the area of the triangle with vertices (x_1,y_1), (x_2,y_2), (x_3,y_3). Since

$$\frac{3}{16}|ABC| - |F_{G_a^-}F_{G_b^-}F_{G_c^-}| = \frac{(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2}{1536\Delta},$$

the second claim is true. The proof of the first are also substitutions of coordinates into well-known formulas. □

The symmedian points $K_{G_a^-}$, $K_{G_a^+}$, $K_{G_b^-}$, $K_{G_b^+}$, $K_{G_c^-}$, $K_{G_c^+}$ of the triangles from σ_G play the similar role.

Theorem 14. *The triangles $K_{G_a^-}K_{G_b^-}K_{G_c^-}$ and $K_{G_a^+}K_{G_b^+}K_{G_c^+}$ have the area equal to $\frac{7}{64}$ of the area of ABC if and only if ABC is an equilateral triangle.*

Proof. The difference $|K_{G_a^-}K_{G_b^-}K_{G_c^-}| - \frac{7}{64}|ABC|$ is equal to

$$\frac{3\Delta T}{64(5b^2+8c^2-a^2)(5c^2+8a^2-b^2)(5a^2+8b^2-c^2)},$$

where

$$T = 40(a^6+b^6+c^6)+231(b^4c^2+c^4a^2+a^4b^2)-147(b^2c^4+c^2a^4+a^2b^4)-372a^2b^2c^2.$$

We shall argue that T is equal to zero if and only if $a=b=c$. We can assume that $a \le b \le c$, $a=\sqrt{d}$, $b=\sqrt{(1+h)d}$, $c=\sqrt{(1+h+k)d}$ for some positive real numbers d, h and k. In new variables $\frac{T}{d^3}$ is

$$164h^3+(204+57k)h^2+3k(68-9k)h+4k^2(51+10k).$$

The quadratic part has the discriminant $-3k^2(41616+30056k+2797k^2)$. Thus T is always positive except when $h=k=0$ which proves our claim. □

Theorem 15. *The triangles $K_{G_a^-}K_{G_b^-}K_{G_c^-}$ and $K_{G_a^+}K_{G_b^+}K_{G_c^+}$ have the same area if and only if the triangle ABC is isosceles.*

Proof. The difference $|K_{G_a^-}K_{G_b^-}K_{G_c^-}| - |K_{G_a^+}K_{G_b^+}K_{G_c^+}|$ is equal to

$$\frac{81\Delta(b-c)(b+c)(c-a)(c+a)(a-b)(a+b)T}{2t(-1,8,5)t(-1,5,8)t(8,-1,5)t(5,-1,8)t(8,5,-1)t(5,8,-1)},$$

where $t(u,v,w)=ua^2+vb^2+wc^2$ and

$$T = 10(a^6+b^6+c^6)-105(b^4c^2+c^4a^2+a^4b^2+b^2c^4+c^2a^4+a^2b^4)-156a^2b^2c^2.$$

We shall now argue that T is always negative. Without loss of generality we can assume that $a \le b \le c$ and that

$$a=\sqrt{d}, \quad b=\sqrt{(1+h)d}, \quad c=\sqrt{(1+h+k)d},$$

for some positive real numbers d, h and k. Since $a+b>c$ it follows that

$$k < 1+2\sqrt{h+1} \le h+3$$

because $\sqrt{h+1}=\sqrt{1\cdot(h+1)} \le \frac{1+(h+1)}{2}$. In new variables,

$$-\frac{T}{d^3} = 190h^3+(285k+936)h^2+(1512+936k+75k^2)h-10k^3+180k^2+756k+756.$$

For $k \le h$ it is obvious that the above polynomial is positive since $190h^3-10k^3 > 0$. On the other hand, when $k \in (h,h+3)$, then k can be represented as $(1-w)h+w(h+3)$ for some $w \in (0,1)$. The above polynomial for this k is

$$540h^3+(2052+1215w)h^2+(3888w+405w^2+2268)h-270w^3+1620w^2+2268w+756.$$

But, the free coefficient of this polynomial for w between 0 and 1 is positive. Thus T is always negative which proves our claim. $\square$

The Longchamps points (*i.e.*, the reflections of the orthocenters in the circumcenters) $L_{G_a^-}$, $L_{G_a^+}$, $L_{G_b^-}$, $L_{G_b^+}$, $L_{G_c^-}$, $L_{G_c^+}$ of the triangles from σ_G offer the following result.

Theorem 16. *The triangles $L_{G_a^-}L_{G_b^-}L_{G_c^-}$ and $L_{G_a^+}L_{G_b^+}L_{G_c^+}$ have the same areas and Brocard angles. This area is equal to $\frac{3}{4}$ of the area of ABC and/or this Brocard angle is equal to the Brocard angle of ABC if and only if ABC is an equilateral triangle.*

Proof. The common area is $\frac{h(10,10,10,1,1,1)}{112\Delta}$ while the tangent of the common Brocard angle is $\frac{h(10,10,10,1,1,1)}{4\Delta p_2(ABC)h(2,2,2,-7,-7,-7)}$. It follows that the difference

$$\frac{3}{4}|ABC| - |L_{G_a^-}L_{G_b^-}L_{G_c^-}| = \frac{h(1,1,1,1,1,1)}{24\Delta}$$

while the difference of tangents of the Brocard angles of the triangles $L_{G_a^-}L_{G_b^-}L_{G_c^-}$ and ABC is $\frac{32\Delta h(1,1,1,1,1,1)}{p_2(ABC)h(2,2,2,-7,-7,-7)}$. From here the conclusions are easy because $h(1,1,1,1,1,1) = \frac{1}{2}\left((b^2-c^2)^2 + (c^2-a^2)^2 + (a^2-b^2)^2\right)$. $\square$

References

[1] Ž. Hanjš and V. P. Volenec, A property of triangles, *Mathematics and Informatics Quarterly*, 12 (2002) 48–49.

[2] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, The Mathematical Association of America, New Mathematical Library no. 37 Washington, 1995.

[3] R. A. Johnson, *Advanced Euclidean Geometry*, Dover Publications (New York), 1960.

[4] C. Kimberling, Central points and central lines in the plane of a triangle, *Math. Mag.*, 67 (1994) 163–187.

[5] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[6] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[7] A. M. Myakishev and P. Y. Woo, On the circumcenters of cevasix configurations, *Forum Geom.*, 3 (2003) 57–63.

Zvonko Čerin: Kopernikova 7, 10010 Zagreb, Croatia

E-mail address: cerin@math.hr

Forum Geometricorum
Volume 4 (2004) 111–115.

FORUM GEOM
ISSN 1534-1178

Minimal Chords in Angular Regions

Nicolae Anghel

Abstract. We use synthetic geometry to show that in an angular region minimal chords having a prescribed direction form a ray which is constructible with ruler and compass.

Let P be a fixed point inside a circle of center O. It is well-known that among the chords containing P one of minimal length is perpendicular to the diameter through P, if $P \neq O$, or is any diameter, if $P = O$. Consequently, such a chord is always constructible with ruler and compass.

When it comes to geometrically constructing minimal chords through given points in convex regions the circle is in some sense a singular case. Indeed, as shown in [1] this task is impossible even in the case of the conics. However, in general *it is* possible to construct all the points inside a convex region which support minimal chords parallel to a given direction. We proved this in [1, 2] by analytical means, with special emphasis on the conics.

The purpose of this note is to prove the same thing for angular regions, via essentially a purely geometrical argument.

To this end let $\angle AOB$ be an angle of vertex O and sides $\overrightarrow{OA}$, $\overrightarrow{OB}$, such that O, A, and B are not colinear, and let P be a point inside the angle. By definition, a *chord* in this angle is a straight segment $\overline{MN}$ such that $M \in \overrightarrow{OA}$ and $N \in \overrightarrow{OB}$. A continuity argument makes clear that among the chords containing P there is at least one of minimal length, that is a minimal chord through P in the given angle.

Problem. *Given a direction in the plane of $\angle AOB$, construct with ruler and compass the geometric locus of all the points inside the angle which support minimal chords parallel to that direction.*

In order to solve this problem we need the following

Lemma. *Inside $\angle AOB$ consider the chord $\overline{MN}$, $M \in \overrightarrow{OA}$, $N \in \overrightarrow{OB}$, such that $\angle OMN$ and $\angle ONM$ are acute angles. If P is the foot of the perpendicular on $\overline{MN}$ through the point Q diametrically opposite O on the circle circumscribed about $\triangle OMN$, then $\overline{MN}$ is the unique minimal chord through P inside $\angle AOB$. P is seen to be the unique point inside $\overline{MN}$ such that $\overline{ML} \cong \overline{NP}$, where L is*

Publication Date: July 21, 2004. Communicating Editor: Michael Lambrou.
I would like to thank the editor for a number of very insightful comments which led to the improvement of the paper.

the foot of the perpendicular from O *on* $\overline{MN}$*. Moreover, any point on the ray* $\overrightarrow{OP}$ *supports an unique minimal chord, parallel to* $\overline{MN}$*.*

Proof. Clearly, Q is an interior point to $\angle AOB$, situated on the other side of the line $\overleftrightarrow{MN}$ with respect to O, and $\overline{MQ} \perp \overrightarrow{OA}$ and $\overline{NQ} \perp \overrightarrow{OB}$. Since $\angle OMN$ and $\angle ONM$ are acute angles, and $\angle QMN$ and $\angle QNM$ are acute angles too, as complements of acute angles , the points P and L described in the statement of the Lemma are interior points to the segment $\overline{MN}$. (See Figure 1).

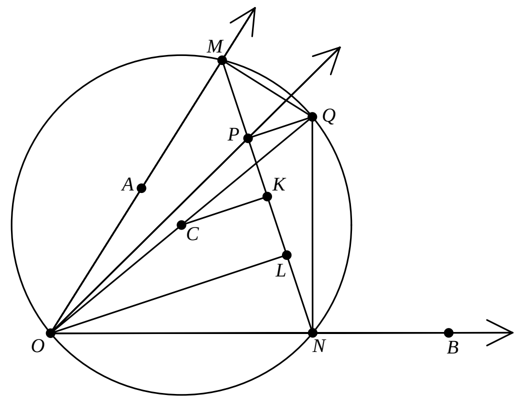


Figure 1

Let us prove first that $\overline{MN}$ is a minimal chord through P in $\angle AOB$. Let $\overline{M'N'}$, $M' \in \overrightarrow{OA}$, $N' \in \overrightarrow{OB}$, $P \in \overline{M'N'}$, be another chord through P (See Figure 2).

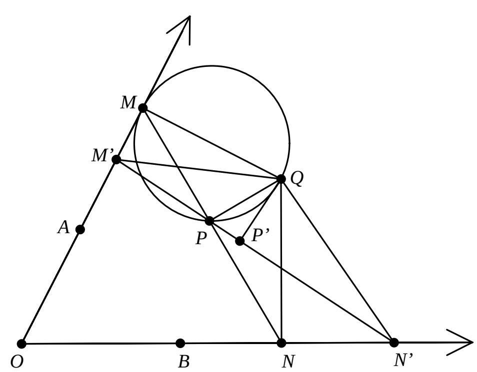


Figure 2

Notice now that the following angle inequalities hold:

$$\angle QM'P < \angle QMP, \;\; \angle QN'P < \angle QNP \tag{1}$$

Indeed, since the circle circumscribed about $\triangle MPQ$ is tangent to the ray $\overrightarrow{OA}$ at M, the point M' is located outside this circle. Now $\angle QMP$ and $\angle QM'P$ are precisely the angles the segment $\overline{PQ}$ is seen from M, respectively M'. Since M belongs to the circle circumscribed about $\triangle MPQ$ and M' is outside this circle, the inequality $\angle QM'P < \angle QMP$ becomes obvious. The other inequality (1) can be proved in a similar fashion.

The inequalities (1) prove that $\angle QM'N'$ and $\angle QN'M'$ are acute angles too, thus the foot P' of the perpendicular from Q on the line $\overleftrightarrow{M'N'}$ belongs to the interior of the segment $\overline{M'N'}$.

Notice now that

$$MQ < M'Q, \;\; NQ < N'Q, \;\; P'Q < PQ.$$

The above inequalities are obvious since in a right triangle a leg is shorter than the hypothenuse. Consequently, the Pythagorean Theorem yields

$$MP = \sqrt{MQ^2 - PQ^2} < \sqrt{M'Q^2 - P'Q^2} = M'P',$$

and similarly, $NP < N'P'$. In conclusion,

$$MN = MP + NP < M'P' + N'P' = M'N',$$

and so $\overline{MN}$ is indeed the unique minimal chord through P in $\angle AOB$.

The perpendicular line on $\overline{MN}$ through the center C of the circle circumscribed about the quadrilateral $OMQN$ intersects $\overline{MN}$ at its midpoint K (See Figure 1). Clearly, $\overline{KP} \cong \overline{KL}$, and so $\overline{ML} \cong \overline{NP}$ as stated.

Finally, the fact that any point on the ray $\overrightarrow{OP}$ supports an unique minimal chord parallel to $\overline{MN}$ is an immediate consequence of standard properties of similar triangles in the context of what was proved above. $\square$

To $\angle AOB$ we associate now another angle, $\angle A'OB'$, according to the following recipe:

$a)$ If $\angle AOB$ is *acute* then $\angle A'OB'$ is obtained by rotating $\angle AOB$ counter-clockwise 90° around O.

$b)$ If $\angle AOB$ is *not acute* (so it is either right or obtuse) then $\angle A'OB'$ is the supplementary angle to $\angle AOB$ along the line $\overleftrightarrow{OB}$ (See Figure 3).

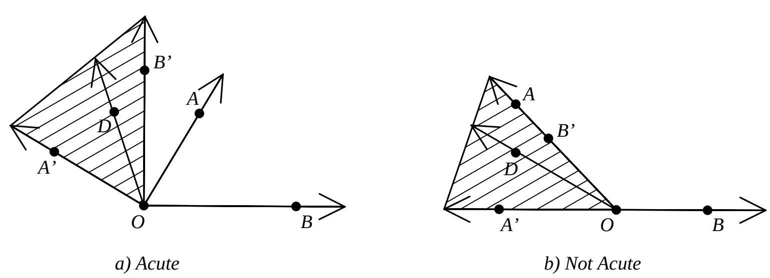


Figure 3

Definition. A ray $\overrightarrow{OD}$ is called an *admissible direction* for $\angle AOB$ if D is a point interior to $\angle A'OB'$.

It is easy to see that $\overrightarrow{OD}$ is an admissible direction for $\angle AOB$ if and only if any parallel line to $\overrightarrow{OD}$ through a point interior to $\angle AOB$ determines a chord $\overline{MN}$ such that $\angle OMN$ and $\angle ONM$ are acute angles.

Theorem. *Any point P inside $\angle AOB$ supports an unique minimal chord, parallel to an admissible direction. The geometric locus of all the points inside $\angle AOB$ which support minimal chords parallel to a given admissible direction can be constructed with ruler and compass as follows:*

$i)$ *Construct first the line $\overleftrightarrow{OL}$ perpendicular to the admissible direction, the point L being interior to $\angle AOB$.*

$ii)$ *Construct next the perpendicular through L to the line $\overleftrightarrow{OL}$, which intersects $\overrightarrow{OA}$ at M and $\overrightarrow{OB}$ at N.*

$iii)$ *Inside the segment $\overline{MN}$ construct the point P such that $\overline{NP} \cong \overline{ML}$.*

$iv)$ *Finally, construct the ray $\overrightarrow{OP}$, which is the desired geometric locus.*

Using the Lemma, an alternative construction can be provided by using the circle circumscribed about $\triangle OMN$, where the point M is chosen arbitrarily on $\overrightarrow{OA}$ and $N \in \overrightarrow{OB}$ is such that $\overline{MN}$ is parallel to the given admissible direction.

Proof. Let P be a fixed point inside $\angle AOB$. The proof splits naturally into two cases, according to $\angle AOB$ being acute or not.

$a)$ $\angle AOB$ *is acute*. Let $\overline{M_1N_1}$ be the perpendicular segment through P to $\overrightarrow{OA}$, $M_1 \in \overrightarrow{OA}$, $N_1 \in \overrightarrow{OB}$ and let $\overline{M_2N_2}$ be the perpendicular segment through P to $\overrightarrow{OB}$, $M_2 \in \overrightarrow{OA}$, $N_2 \in \overrightarrow{OB}$. Define now a function $f : \overline{M_1M_2} \longrightarrow \mathbf{R}$, by

$$f(M) = ML - NP, \;\; M \in \overline{M_1M_2}, \tag{2}$$

where N is the intersection point of the line $\overleftrightarrow{MP}$ with $\overrightarrow{OB}$, and L is the foot of the perpendicular from O to the segment $\overline{MN}$ (See Figure 4).

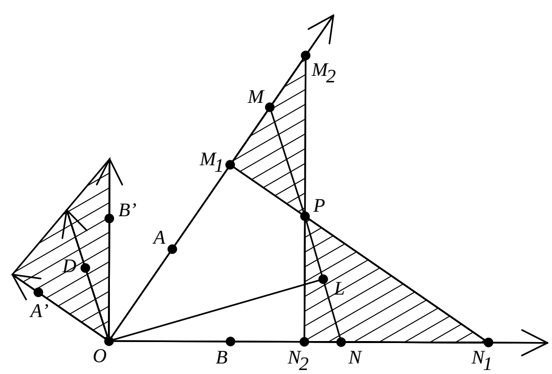


Figure 4

Clearly, this is a continuous function and $f(M_1) = -N_1P < 0$ and $f(M_2) = M_2P > 0$. By the intermediate value property there is some point $M \in \overline{M_1M_2}$ such that $f(M) = 0$, or equivalently $\overline{NP} \cong \overline{ML}$. According to the above Lemma, for this point M the chord $\overline{MN}$ is the unique minimal chord through P. It is also obvious that $\overline{MN}$ is parallel to an admissible direction.

$b)$ $\angle AOB$ *is not acute*. The proof in this case is a variant of that given at $a)$. Let M_0 be the point where the parallel line through P to $\overrightarrow{OB}$ intersects the ray $\overrightarrow{OA}$. Without loss of generality we can assume that M_0 is located between O and A. Defining now the function $f : \overrightarrow{M_0A} \longrightarrow \mathbf{R}$ by the same formula (2), we see that for points M close to M_0, $f(M)$ takes negative values and for points M far away on $\overrightarrow{M_0A}$, $f(M)$ takes positive values. One more time, the intermediate value property and the above Lemma guarantee the existence of an unique minimal chord through P, which is also parallel to an admissible direction.

Given now an admissible direction, the previous Lemma justifies the construction of the desired geometric locus as indicated in the statement of the theorem if we can prove that this locus does not contain points outside the ray $\overrightarrow{OP}$ described at $iv)$. Indeed this is the case since if there were other points then the equation $\overline{NP} \cong \overline{ML}$ would not hold. However, we have just proved that this equation is necessary for minimal chords. $\square$

References

[1] N. Anghel, On the constructibility with ruler and compass of a minimum chord in a parabola, *Libertas Math.*, XVII (1997) 9–12.

[2] N. Anghel, Geometric loci associated to certain minimal chords in convex regions, *J. Geom.*, 66 (1999) 1–16.

Nicolae Anghel: Department of Mathematics, University of North Texas, Denton, TX 76203
E-mail address: anghel@unt.edu

Forum Geometricorum
Volume 4 (2004) 117–124.

FORUM GEOM
ISSN 1534-1178

Three Pairs of Congruent Circles in a Circle

Li C. Tien

Abstract. Consider a closed chain of three pairs of congruent circles of radii a, b, c. The circle tangent internally to each of the 6 circles has radius $R = a + b + c$ if and only if there is a pair of congruent circles whose centers are on a diameter of the enclosing circle. Non-neighboring circles in the chain may overlap. Conditions for nonoverlapping are established. There can be a "central circle" tangent to four of the circles in the chain.

1. Introduction

Consider a closed chain of three pairs of congruent circles of radii a, b, c, as shown in Figure 1. Each of the circles is tangent internally to the enclosing circle (O) of radius R and tangent externally to its two neighboring circles.

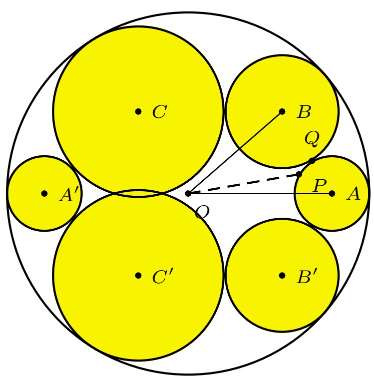


Figure 1A: $(abcacb)$

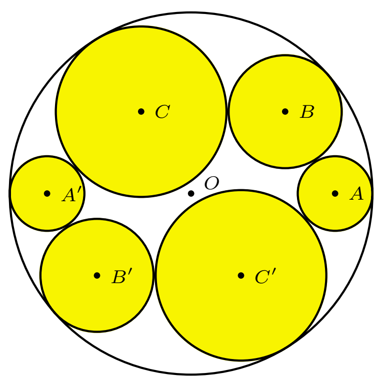


Figure 1B: $(abcabc)$

The essentially distinct arrangements, depending on the number of pairs of congruent neighboring circles, are

(A): $(aabcbc)$	(B): $(aacbbc)$
(C): $(aabbcc)$	(D): $(aaaabb)$
(E): $(abcabc)$, $(abcacb)$	(F): $(aabaab)$, $(aaabab)$
(G): $(aaaaaa)$	

Figures 1A and 1B illustrate the pattern (E). Patterns (D) and (F) have $c = a$. In pattern (G), $b = c = a$.

Publication Date: July 30, 2004. Communicating Editor: Paul Yiu.

According to [1, 3], in 1877 Sakuma proved $R = a + b + c$ for patterns (E). Hiroshi Okumura [1] published a much simpler proof. Unaware of this, Tien [4] rediscovered the theorem in 1995 and published a similar, simple proof. It is easy to see by symmetry that in each of the patterns (E), (F), (G), there is a pair of congruent circles with centers on a diameter of the enclosing circle. Let us call such a pair a *diametral* pair. Here is a stronger theorem:

Theorem 1. *In a closed chain of three pairs of congruent circles of radii a, b, c tangent internally to a circle of radius R, $R = a + b + c$ if and only if the closed chain contains a diametral pair of circles.*

In Figure 1, two non-neighboring circles intersect. The proof for $R = a + b + c$ does not forbid such an intersection. Sections 4 and 5 are about avoiding intersecting circles and about adding a "central" circle.

2. Preliminaries

In Figure 1, the enclosing circle (O) of radius R centers at O and the circles (A), (B), (C) of radii a, b, c, center at A, B, C, respectively. The circles (A'), (B'), (C') are also of radii a, b, c respectively.

Suppose two circles (A) and (B) of radii a and b are tangent externally each other, and each tangent internally to a circle $O(R)$. We denote the magnitude of angle AOB by θ_{ab}. See Figure 2A. This clearly depends on R. If $a < \frac{R}{2}$, then we can also speak of θ_{aa}. Note that the center O is outside each circle of radius a.

Lemma 2. (a) *If* $a < \frac{R}{2}$, $\sin\frac{\theta_{aa}}{2} = \frac{a}{R-a}$. (See Figure 2A).
(b) $\cos\theta_{bc} = \frac{(R-b)^2+(R-c)^2-(b+c)^2}{2(R-b)(R-c)}$. (See Figure 2B).

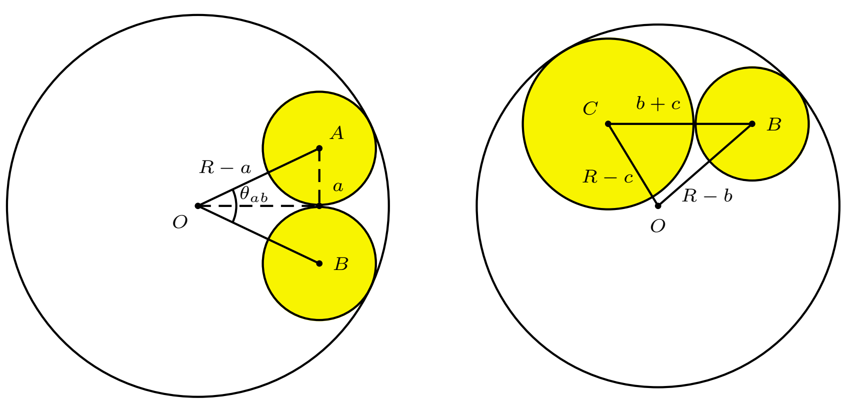


Figure 2A Figure 2B

Proof. These are clear from Figures 2A and 2B. □

Lemma 3. *If a and b are unequal and each $< \frac{R}{2}$, then $\theta_{aa} + \theta_{bb} > 2\theta_{ab}$.*

Proof. In Figure 1A, consider angle AOP, where P is a point on the circle (A). The angle AOP is maximum when line OP is tangent to the circle (A). This maximum is $\frac{\theta_{aa}}{2} \geq \angle AOQ$, where Q is the point of tangency of (A) and (B). Similarly, $\frac{\theta_{bb}}{2} \geq \angle BOQ$, and the result follows. □

Corollary 4. *If a, b, c are not the same, then* $\theta_{aa} + \theta_{bb} + \theta_{cc} > \theta_{ab} + \theta_{bc} + \theta_{ca}$.

Proof. Write

$$\theta_{aa} + \theta_{bb} + \theta_{cc} = \frac{\theta_{aa} + \theta_{bb}}{2} + \frac{\theta_{bb} + \theta_{cc}}{2} + \frac{\theta_{cc} + \theta_{aa}}{2}$$

and apply Lemma 3. □

3. Proof of Theorem 1

Sakuma, Okumura [1] and Tien [4] have proved the sufficiency part of the theorem. We need only the necessity part. This means showing that for distinct a, b, c in patterns (A) through (D) which do not have a diametral pair of circles, the assumption of $R = a + b + c$ causes contradictions. In patterns (E) with a pair of diametral circles and $R = a + b + c$, the sum of the angles around the center O of the enclosing circle is $2(\theta_{ab} + \theta_{bc} + \theta_{ca}) = 2\pi$, that is,

$$\theta_{ab} + \theta_{bc} + \theta_{ca} = \pi.$$

Pattern (A): $(aabcbc)$. The sum of the angles around O is

$$\begin{aligned}\theta_{aa} + \theta_{ab} + \theta_{bc} + \theta_{cb} + \theta_{bc} + \theta_{ca} =& \theta_{ab} + \theta_{bc} + \theta_{ca} + (\theta_{aa} + 2\theta_{bc}) \\ =& \pi + (\theta_{aa} + 2\theta_{bc}).\end{aligned}$$

This is 2π if and only if $(\theta_{aa} + 2\theta_{bc}) = \pi$, or $\frac{\pi}{2} - \frac{\theta_{aa}}{2} = \theta_{bc}$. The cosines of these angles, Lemma 2 and the assumption $R = a + b + c$ lead to

$$\frac{a}{b+c} = \frac{a^2 + ab + ac - bc}{(a+b)(a+c)},$$

which gives

$$(a-b)(a-c)(a+b+c) = 0,$$

an impossibility, if a, b, c are distinct.

Pattern (B): $(aacbbc)$. If $a > \frac{R}{2}$ or $b > \frac{R}{2}$, then the neighboring tangent circles of radii a or b, respectively, cannot fit inside the enclosing circle of radius $R = a + b + c$. For this equation to hold, it must be that $a \leq \frac{R}{2}$ and $b \leq \frac{R}{2}$. Then, O is outside $A(a)$ and $B(b)$. The sum of the angles around O exceeds 2π, by Lemma 3:

$$\begin{aligned}&\theta_{aa} + \theta_{ac} + \theta_{cb} + \theta_{bb} + \theta_{bc} + \theta_{ca} \\ =&(\theta_{aa} + \theta_{bb}) + 2(\theta_{bc} + \theta_{ca}) \\ >&2(\theta_{ab} + \theta_{bc} + \theta_{ca}) \\ =&2\pi.\end{aligned}$$

Patterns (C) and (D): $(aabbcc)$ *and* $(aaaabb)$. For $R = a + b + c$ to hold, O must be outside $A(a)$, $B(b)$, $C(c)$. Again, the sum of the angles around O exceeds 2π. For pattern (C),

$$\begin{aligned}&\theta_{aa} + \theta_{ab} + \theta_{bb} + \theta_{bc} + \theta_{cc} + \theta_{ca}\\ =&(\theta_{aa} + \theta_{bb} + \theta_{cc}) + (\theta_{ab} + \theta_{bc} + \theta_{ca})\\ >&(\theta_{ab} + \theta_{bc} + \theta_{ca}) + (\theta_{ab} + \theta_{bc} + \theta_{ca})\\ =&2\pi.\end{aligned}$$

Here, the inequality follows from Corollary 4 for a, b, c, not all the same.

For pattern (D) with $c = a$, the inequality remains true. This completes the proof of Theorem 1.

Remark. A narrower version of Theorem 1 treats a, b, c as variables, instead of any particular lengths. The proof for this version is simple. We see that when no pair of the enclosed circles is diametral, at least one pair has its two circles next to each other. Let these two be point circles and let the other four circles be of the same radius. Then the six circles become three equal tangent circles tangentially enclosed in a circle. In this special case $R = a + b + c = 0 + a + a$ is false. Then, a, b, c cannot be variables.

4. Nonoverlapping arrangements

Patterns (A) through (G) are adaptable to hands-on activities of trying to fit chains of three pairs of congruent circles into an enclosing circle of a fixed radius R. Most of the essential patterns have inessential variations. Assuming $a \le b \le c$, patterns (E) have four variations:

$$\begin{aligned}\mathrm{E}_1 :&\quad (abcabc)\\ \mathrm{E}_2 :&\quad (cabcba)\\ \mathrm{E}_3 :&\quad (abcacb)\\ \mathrm{E}_4 :&\quad (bcabac)\end{aligned}$$

For hands-on activities, it is desirable to find the conditions for the enclosed circles in patterns (E) not to overlap. We find the bounds of the ratio $\frac{a}{R}$ in these patterns.

4.1. *Patterns* E_1 *and* E_2. The largest circles (C) and $C')$ are diametral. For a nonoverlapping arrangement, Clearly, $a \le \frac{1}{3}R$ and $c \le \frac{1}{2}R$.

In Figure 3, a circle of radius b' is tangent externally to the two diametral circles of radii c, and internally to the enclosing circle of radius R. From

$$(b' + c)^2 = (R - b')^2 + (R - c)^2,$$

we have $b' = \frac{R(R-c)}{R+c}$. It follows that in a nonoverlapping patterns E_1 and E_2, with $\frac{1}{3}R \le c \le \frac{1}{2}R$, we have

$$b + c \le b' + c = \frac{R^2 + c^2}{R + c} \le \frac{5}{6}R.$$

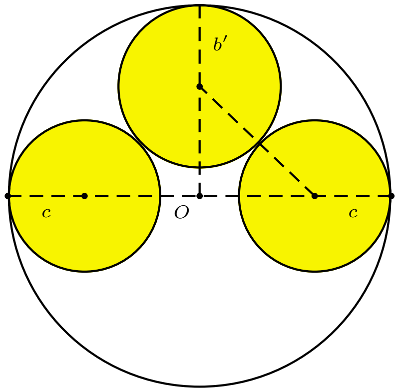


Figure 3

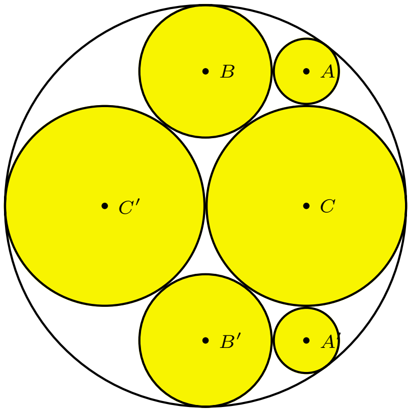


Figure 4

From this, $a \geq \frac{1}{6}R$. Figure 4 shows a nonoverlapping arrangement with $a = \frac{1}{6}R$, $b = \frac{1}{3}R$, $c = \frac{1}{2}R$. It is clear that for every a satisfying $\frac{1}{6}R \leq a \leq \frac{1}{3}R$, there are nonoverlapping patterns E_1 and E_2 (with $a \leq b \leq c$).

4.2. *Patterns* E_3 *and* E_4. In these cases the largest circles (C) and (C') are not diametral.

Lemma 5. *If three circles of radii x, z, z are tangent externally to each other, and are each tangent internally to a circle of radius R, then*

$$z = \frac{4Rx(R-x)}{(R+x)^2}.$$

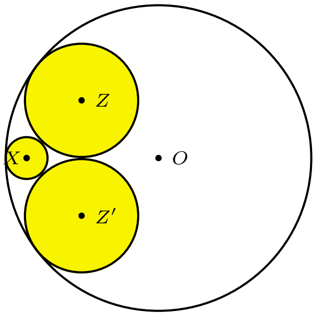


Figure 5

Proof. By the Descartes circle theorem [2], we have

$$2\left(\frac{1}{R^2} + \frac{1}{x^2} + \frac{2}{z^2}\right) = \left(-\frac{1}{R} + \frac{1}{x} + \frac{2}{z}\right)^2,$$

from which the result follows. □

Theorem 6. *For a given R, a nonoverlapping arrangement of pattern* $E_3(abcacb)$ *or* $E_4(bcabac)$ *with $a \leq b \leq c$ and $a+b+c=R$ exists if $\gamma R \leq a \leq \frac{1}{3}R$, where*

$$\gamma = \frac{1+\sqrt[3]{19+12\sqrt{87}}+\sqrt[3]{19-12\sqrt{87}}}{6} \approx 0.25805587\cdots .$$

Proof. For $b=a$ and the largest $c=R-2a$ for a nonoverlapping arrangement $E_3(abcacb)$, Lemma 5 gives

$$\frac{4Ra(R-a)}{(R+a)^2}-(R-2a)=\frac{f(\frac{a}{R})R^3}{(R+a)^2}=0,$$

where $f(x)=2x^3-x^2+4x-1$. It has a unique real root γ given above.

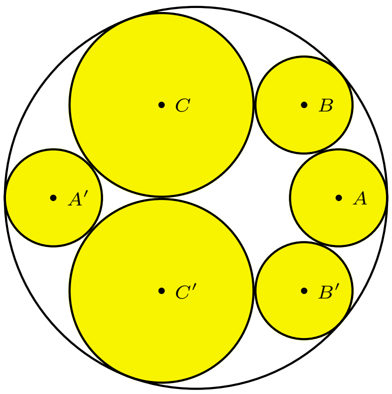


Figure 6

Figure 6 shows a nonoverlapping arrangement E_3 with $a=b=\gamma R$, and $c=(1-2\gamma)R$. For $\gamma R \leq a \leq \frac{1}{3}R$, from the figure we see that (C) and (C') and the other circles cannot overlap in arrangements of patterns $E_3(abcacb)$ and $E_4(bcabac)$. □

Corollary 7. *The sufficient condition $\gamma R \leq a \leq \frac{1}{3}R$ also applies to patterns E_1 and E_2.*

Outside the range $\gamma R \leq a \leq \frac{R}{3}$, patterns $E_3(abcacb)$ and $E_4(bcabac)$ still can have nonoverlapping circles. Both of the patterns involve Figure 5 and $z=\frac{4Rx(R-x)}{(R+x)^2}$, with $z=c$, $x=a$ or b, and $a \leq b \leq c$.

The equation gives the smallest $x=a=(3-2\sqrt{2})R \approx 0.1715\cdots R$ corresponding to the largest $b=c=(\sqrt{2}-1)R \approx 0.4142\cdots R$ and the largest $x=b=\frac{R}{3}$ corresponding to the largest $c=\frac{R}{2}$. Thus, the nonoverlapping conditions are $(3-2\sqrt{2})R \leq x \leq \frac{R}{3}$ and $c \leq \frac{4Rx(R-x)}{R+x)^2}$.

For $x \geq \frac{R}{3}$, circles (Z) and (Z') overlap with (X'), which is diametral with (X). Now Figure 3 and the associated $b'=\frac{R(R-c)}{R+c}$ are relevant. With b' replaced by c and c by b, the equation becomes $c=\frac{R(R-b)}{R+b}$. By this equation, when b varies from $\frac{R}{3}$ to $(\sqrt{2}-1)R$, $c \geq b$ varies from $\frac{R}{2}$ to $(\sqrt{2}-1)R$. Thus, the nonoverlapping conditions are $\frac{R}{3} \leq b \leq (\sqrt{2}-1)R$ and $c \leq \frac{R(R-b)}{R+b}$. The case of $b > (\sqrt{2}-1)R$ makes $b > c$ and the largest pair of circles diametral, already covered in §4.1.

5. The central circle and avoiding intersecting circles

Obviously, pattern (G) $(aaaaaa)$ admits a "central" circle tangent to all 6 circles of radii a. In patterns (F) $(aabaab)$, (aaabab), we can add a central circle tangent to the four circles of radius a. Figure 7 shows the less obvious central circle for $(abcacb)$ of pattern (E).

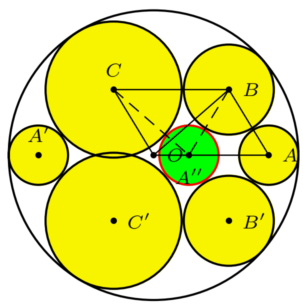


Figure 7

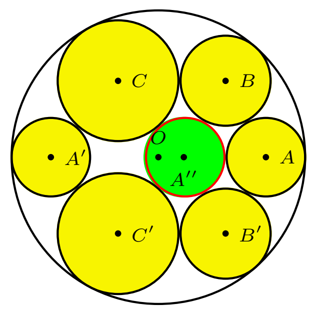


Figure 8

Theorem 8. *Consider a closed chain of pattern $(abcacb)$. There is a "central" circle of radius a tangent to the four circles of radii b and c. This circle does not overlap with the circle $A(a)$ if*

$$a \leq \frac{b(b+c)}{2c},$$

where $b \leq c$.

Proof. In Figure 7, the pattern of the chain tells that $R = a + b + c$. The central circle centered at A'' has radius a is tangent to $B(b)$, $B'(b)$, $C(c)$, $C'(c)$ because triangles $A''BC$ and OBC are mirror images of each other. When $b < c$, $A''(a)$ is closer to $A(a)$ than $A'(a)$. If $A''(a)$ and $A(a)$ are tangent to each other, then $AB^2 - a^2 = OB^2 - (OA - a)^2$. Now, $AB = a + b$ and $OB = a + c$, $OA = b + c$. This simplifies into $a = \frac{b(b+c)}{2c}$. If $a < \frac{b(b+c)}{2c}$, the circles $A(a)$ and $A''(a)$ are separate. $\square$

Figure 8 shows an arrangement $(abcacb)$ with a central circle touching 5 inner circles except (A').

References

[1] H. Okumura, Circle patterns arising from a seven-circle problem, *Crux Math.*, 21 (1995) 213–217.

[2] D. Pedoe, On a theorem of geometry, *Amer. Math. Monthly*, 74 (1967) 627–640.

[3] J. F. Rigby, Circle problems arising from Wasan, *Symmetry: Culture and Science*, 8 (1997) 68–73.
[4] L. C. Tien, Constant-sum figures, *Math. Intelligencer*, 23 (2001) no. 2, 15–16.

Li C. Tien: 4412 Huron Drive, Midland, Michigan, 48642-3503, USA
E-mail address: lichtien@aol.com

Forum Geometricorum
Volume 4 (2004) 125–134.

FORUM GEOM
ISSN 1534-1178

The Intouch Triangle and the OI-line

Eric Danneels

Abstract. We prove some interesting results relating the intouch triangle and the OI line of a triangle. We also give some interesting properties of the triangle center X_{57}, the homothetic center of the intouch and excentral triangles.

1. Introduction

L. Emelyanov [4] has recently given an interesting relation between the OI-line and the triangle of reflections of the intouch triangle. Here, O and I are respectively the circumcenter and incenter of the triangle. Given triangle ABC with intouch triangle XYZ, let X_2, Y_2, Z_2 be the reflections of X, Y, Z in their respective opposite sides YZ, ZX, XY. Then the lines AX_2, BY_2, CZ_2 intersect BC, CA, AB at the intercepts of the OI-line.

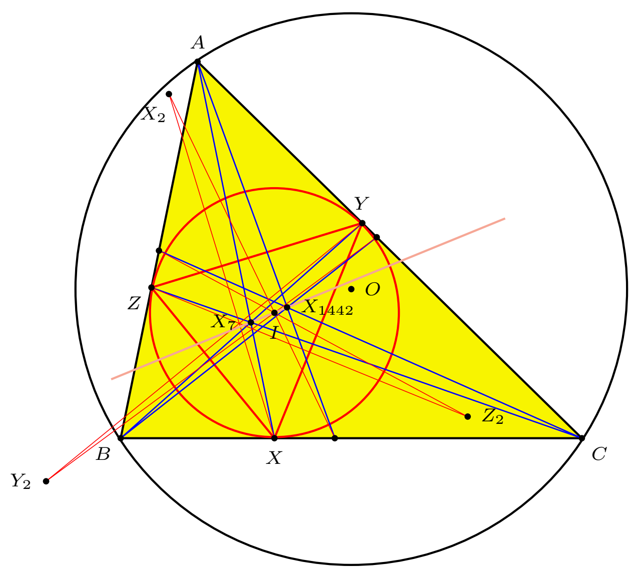


Figure 1.

Emelyanov [3] also noted that the intercepts of the points $IX_2 \cap BC$, $IY_2 \cap CA$, $IZ_2 \cap AB$ form a triangle perspective with ABC. See Figure 1. According to [7], this perspector is the point

$$X_{1442} = \left(\frac{a(b^2 + bc + c^2 - a^2)}{s-a} : \frac{b(c^2 + ca + a^2 - b^2)}{s-b} : \frac{c(a^2 + ab + b^2 - c^2)}{s-c} \right)$$

Publication Date: August 10, 2004. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this paper.

on the Soddy line joining the incenter and the Gergonne point.

In this paper we generalize these results. We work with barycentric coordinates with reference to triangle ABC.

2. The triangle center X_{57}

Let a, b, c be the lengths of the sides BC, CA, AB of triangle ABC, and $s = \frac{1}{2}(a+b+c)$ the semiperimeter. The intouch triangle XYZ and the excentral triangle (with the excenters as vertices) are clearly homothetic, since their corresponding sides are perpendicular to the same angle bisector of triangle ABC. These triangles are respectively the cevian triangle of the Gergonne point $\left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c}\right)$ and the anticevian triangle of the incenter $(a : b : c)$, their homothetic center has coordinates

$$\begin{aligned}&(a(-a(s-a)+b(s-b)+c(s-c)) : \cdots : \cdots)\\=&(2a(s-b)(s-c) : \cdots : \cdots)\\=&\left(\frac{a}{s-a} : \cdots : \cdots\right).\end{aligned}$$

This is the triangle center X_{57} in [6], defined as the isogonal conjugate of the Mittenpunkt $X_9 = (a(s-a) : b(s-b) : c(s-c))$. This is a point on the OI-line since the two triangles in question have circumcenters I and X_{40} (the reflection of I in O), [1]

We give some interesting properties of the triangle X_{57}.

Since ABC is the orthic triangle of the excentral triangle, it is homothetic to the orthic triangle $X_1Y_1Z_1$ of XYZ with the same homothetic center X_{57}. See Figure 2.

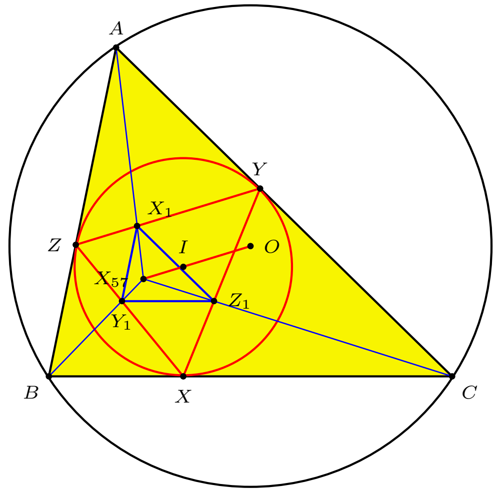


Figure 2.

[1]The circumcircle of ABC is the nine-point circle of the excentral triangle.

Let DEF be the circumcevian triangle of the incenter I, and D', E', F' the antipodes of D, E, F in the circumcircle. In other words, D and D' are the midpoints of the two arcs BC, D' on the arc containing the vertex A; similarly for the other two pairs. Clearly,

$$D = \left(\frac{a^2}{-(b+c)} : \frac{b^2}{b} : \frac{c^2}{c}\right) = (-a^2 : b(b+c) : c(b+c)).$$

Similarly,

$$E = (a(c+a) : -b^2 : c(c+a)) \quad \text{and} \quad F = (a(a+b) : b(a+b) : -c^2).$$

To compute the coordinates of D', E', F', we make use of the following formula.

Lemma 1. *Let $P = (a^2vw : b^2wu : c^2uv)$ be a point on the circumcircle (so that $u+v+w=0$). For a point $Q = (x : y : z)$ different from P and not lying on the circumcircle, the line PQ intersects the circumcircle again at the point $(a^2vw+tx : b^2wu+ty : c^2uv+tz)$, where*

$$t = \frac{b^2c^2u^2x + c^2a^2v^2y + a^2b^2w^2z}{a^2yz + b^2zx + c^2xy}. \tag{1}$$

Proof. Entering the coordinates

$$(\mathbb{X}, \mathbb{Y}, \mathbb{Z}) = (a^2vw + tx : b^2wu + ty : c^2uv + tz)$$

into the equation of the circumcircle

$$a^2\mathbb{Y}\mathbb{Z} + b^2\mathbb{Z}\mathbb{X} + c^2\mathbb{X}\mathbb{Y} = 0,$$

we obtain

$$\begin{aligned}
&(a^2yz + b^2zx + c^2xy)t^2\\
&+(b^2c^2u(v+w)x + c^2a^2v(w+u)y + a^2b^2w(u+v)z)t\\
&+a^2b^2c^2uvw(u+v+w) = 0.
\end{aligned}$$

Since $u+v+w=0$, this gives $t=0$ or the value given in (1) above. $\square$

Let $M = (0:1:1)$ be the midpoint of BC. Applying Lemma 1 to D and M, we obtain

$$D' = (-a^2 : b(b-c) : c(c-b)).$$

Similarly,

$$E' = (a(a-c) : -b^2 : c(c-a) \quad \text{and} \quad F' = (a(a-b) : b(b-a) : -c^2).$$

Applying Lemma 1 to D' and $X = (0 : a+b-c : c+a-b)$, (likewise to E' and Y, and to F' and Z), we obtain the points

$$\begin{aligned}
X' &= \left(\frac{-a^2}{a(b+c)-(b-c)^2} : \frac{b}{c+a-b} : \frac{c}{a+b-c}\right),\\
Y' &= \left(\frac{a}{b+c-a} : \frac{-b^2}{b(c+a)-(c-a)^2} : \frac{c}{a+b-c}\right),\\
Z' &= \left(\frac{a}{b+c-a} : \frac{b}{c+a-b} : \frac{-c^2}{c(a+b)-(a-b)^2}\right).
\end{aligned}$$

These are clearly the vertices of the circumcevian triangle of X_{57}. We summarize this in the following proposition.

Proposition 2. *If X' (respectively Y', Z') are the second intersections of $D'X$ (respectively $E'Y$, $F'Z$) and the circumcircle, then $X'Y'Z'$ is the circumcevian triangle of X_{57}.*

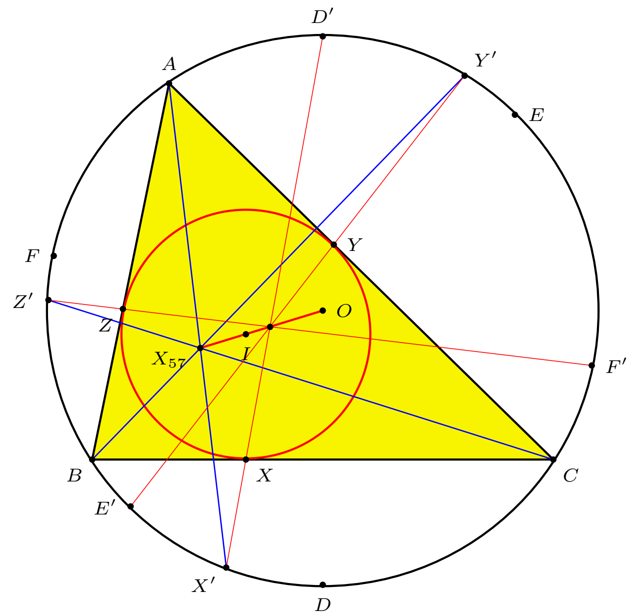


Figure 3.

Remark. The lines $D'X$, $E'Y$, $F'Z$ intersect at X_{55}, the internal center of similitude of the circumcircle and the incircle.

Proposition 3. *Let X'', Y'', Z'' be the second intersections of the circumcircle with the lines DX, EY, FZ respectively. The lines AX'', BY'', CZ'' bound the anticevian triangle of X_{57}.*

Proof. By Lemma 1, these are the points

$$X'' = \left(\frac{a^2}{s-a} : \frac{b(b-c)}{s-b} : \frac{c(c-b)}{s-c}\right),$$
$$Y'' = \left(\frac{a(a-c)}{s-a} : \frac{b^2}{s-b} : \frac{c(c-a)}{s-c}\right),$$
$$Z'' = \left(\frac{a(a-b)}{s-a} : \frac{b(b-a)}{s-b} : \frac{c^2}{s-c}\right).$$

The lines AX'', BY'', CZ'' have equations

$$\begin{array}{ccccccc} & & \frac{s-b}{b}y & + & \frac{s-c}{c}z & = & 0, \\ \frac{s-a}{a}x & & & + & \frac{s-c}{c}z & = & 0, \\ \frac{s-a}{a}x & + & \frac{s-b}{b}y & & & = & 0. \end{array}$$

They clearly bound the anticevian triangle of X_{57}. See Figure 4. $\square$

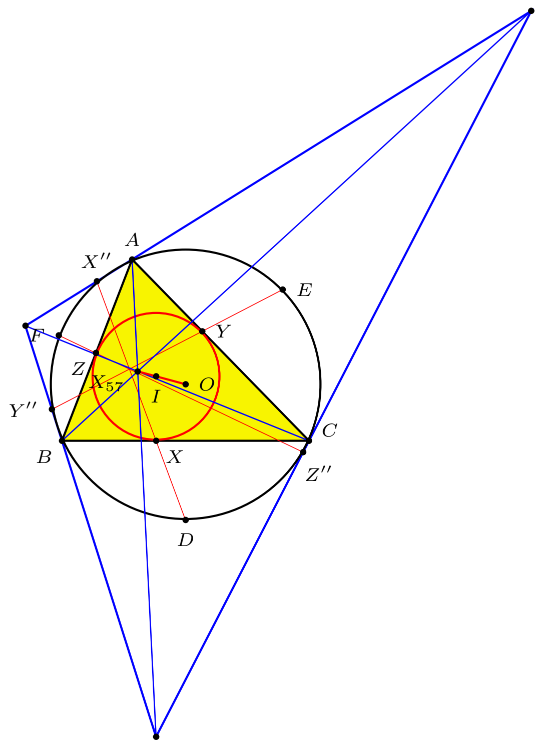


Figure 4.

Remark. The lines DX, EY, FZ intersect at X_{56}, the external center of similitude of the circumcircle and incircle.

Proposition 4. *X_{57} is the perspector of the triangle bounded by the polars of A, B, C with respect to the circle through the excenters.*

Proof. As is easily verified, the equation of the circumcircle of the excentral triangle is

$$a^2yz + b^2zx + c^2xy + (x + y + z)(bcx + cay + abz) = 0.$$

The polars are the lines

$$\begin{array}{ccccccc} \frac{x}{s} & + & \frac{y}{b} & + & \frac{z}{c} & = & 0, \\ \frac{x}{a} & + & \frac{y}{s} & + & \frac{z}{c} & = & 0, \\ \frac{x}{a} & + & \frac{y}{b} & + & \frac{z}{s} & = & 0. \end{array}$$

They bound a triangle with vertices

$$\left(-\frac{a(s^2-bc)}{s(s-b)(s-c)} : \frac{b}{s-b} : \frac{c}{s-c}\right),$$
$$\left(\frac{a}{s-a} : -\frac{b(s^2-ca)}{s(s-c)(s-a)} : \frac{c}{s-c}\right),$$
$$\left(\frac{a}{s-a} : \frac{s}{s-b} : -\frac{c(s^2-ab)}{s(s-a)(s-b)}\right).$$

This clearly has perspector X_{57}. □

Proposition 5. *X_{57} is the perspector of the reflections of the Gergonne point in the intouch triangle.*

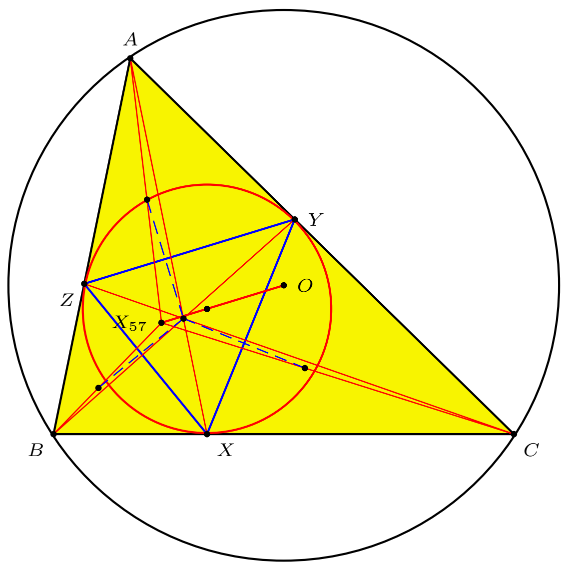


Figure 5.

More generally, the reflection triangle of $P = (u : v : w)$ in the cevian triangle of P is perspective with ABC at

$$\left(u\left(-\frac{a^2}{u^2}+\frac{b^2}{v^2}+\frac{c^2}{w^2}+\frac{b^2+c^2-a^2}{vw}\right) : \cdots : \cdots\right).$$

See [2]. For example, if P is the incenter, this perspector is the point

$$X_{35} = (a^2(b^2+bc+c^2-a^2) : b^2(c^2+ca+a^2-b^2) : c^2(a^2+ab+b^2-c^2))$$

which divides the segment OI in the ratio $OX_{35} : X_{35}I = R : 2r$.

Finally, we also mention from [5] that X_{57} is the orthocorrespondent of the incenter. This means that the trilinear polar of X_{57}, namely, the line

$$\frac{s-a}{a}x + \frac{s-b}{b}y + \frac{s-c}{c}z = 0$$

intersects the sidelines BC, CA, AB at X, Y, Z respectively such that $IX \perp IA$, $IY \perp IB$, and $IZ \perp IC$.

3. A locus of perspectors

As an extension of the result of [4], we consider, for a real number t, the triangle $X_tY_tZ_t$ with X_t, Y_t, Z_t dividing the segments XX_1, YY_1, ZZ_1 in the ratio

$$XX_t : X_tX_1 = YY_t : Y_tY_1 = ZZ_t : Z_tZ_1 = t : 1 - t.$$

Proposition 6. *The triangle $X_tY_tZ_t$ is perspective with ABC. The locus of the perspector is the Soddy line joining the incenter to the Gergonne point.*

Proof. We compute the coordinates of X_t, Y_t, Z_t. It is well known that $BX = s - b$, $XC = s - c$, *etc.*, so that, in absolute barycentric coordinates,

$$X = \frac{(s-c)B + (s-b)C}{a}, \qquad Y = \frac{(s-a)C + (s-c)A}{b}, \qquad Z = \frac{(s-b)A + (s-a)B}{c}.$$

Since the intouch triangle XYZ has (acute) angles $\frac{B+C}{2}$, $\frac{C+A}{2}$, and $\frac{A+B}{2}$ at X, Y, Z respectively, the pedal X_1 of X on YZ divides the segment in the ratio

$$YX_1 : X_1Z = \cot\frac{C+A}{2} : \cot\frac{A+B}{2} = \tan\frac{B}{2} : \tan\frac{C}{2} = s - c : s - b.$$

Similarly, Y_1 and Z_1 divide ZX and XY in the ratios

$$ZY_1 : Y_1X = s - a : s - c, \qquad XZ_1 : Z_1Y = s - b : s - a.$$

In absolute barycentric coordinates,

$$\begin{aligned} X_1 =& \frac{(s-b)Y + (s-c)Z}{a} \\ =& \frac{(b+c)(s-b)(s-c)A + b(s-c)(s-a)B + c(s-a)(s-b)C}{abc}. \end{aligned}$$

It follows that

$$\begin{aligned} X_t =& (1-t)X + tX_1 \\ =& \frac{t(b+c)(s-b)(s-c)A + b(s-c)(c - t(s-b))B + c(s-b)(b - t(s-c))C}{abc}. \end{aligned}$$

In homogeneous barycentric coordinates, this is

$$X_t = (t(b+c)(s-b)(s-c) : b(s-c)(c - t(s-b)) : c(s-b)(b - t(s-c))).$$

The line IX_t has equation

$$bc(b-c)(s-a)x + c(s-b)(ab - 2s(s-c)t)y - b(s-c)(ca - 2s(s-b)t)z = 0.$$

The line IX_t intersects BC at the point

$$\begin{aligned} X_t' =& (0 : b(s-c)(ca - 2s(s-b)t) : c(s-b)(ab - 2s(s-c)t)) \\ =& \left(0 : \frac{b(ca - 2s(s-b)t)}{s-b} : \frac{c(ab - 2s(s-c)t)}{s-c}\right). \end{aligned}$$

Similarly, the lines IY_t and IZ_t intersect CA and AB respectively at

$$Y'_t = \left(\frac{a(bc-2s(s-a)t)}{s-a} : 0 : \frac{c(ab-2s(s-c)t)}{s-c}\right),$$
$$Z'_t = \left(\frac{a(bc-2s(s-a)t)}{s-a} : \frac{b(ca-2s(s-b)t)}{s-b} : 0\right).$$

The triangle $X'_tY'_tZ'_t$ is perspective with ABC at the point

$$\left(\frac{a(bc-2s(s-a)t)}{s-a} : \frac{b(ca-2s(s-b)t)}{s-b} : \frac{c(ab-2s(s-c)t)}{s-c}\right).$$

As t varies, this perspector traverses a straight line. Since the perspector is the Gergonne point for $t=0$ and the incenter for $t=\infty$, this line is the Soddy line joining these two points. □

The Soddy line has equation

$$(b-c)(s-a)^2x + (c-a)(s-b)^2y + (a-b)(s-c)^2z = 0.$$

Here are some triangle centers on the Soddy line, with the corresponding values of t. The symbol r_a stands for the radius of the A-excircle.

t	perspector	first barycentric coordinate
1	X_{77}	$\frac{a(b^2+c^2-a^2)}{s-a}$
2	X_{1442}	$\frac{a(b^2+bc+c^2-a^2)}{s-a}$
$\frac{1}{2}$	X_{269}	$\frac{a}{(s-a)^2}$
$\frac{R}{s}$	X_{481}	$2r_a - a$
$\frac{-R}{s}$	X_{482}	$2r_a + a$
$\frac{2R}{s}$	X_{175}	$r_a - a$
$\frac{-2R}{s}$	X_{176}	$r_a + a$
$\frac{3R}{2s}$	X_{1372}	$4r_a - 3a$
$\frac{-3R}{2s}$	X_{1371}	$4r_a + 3a$
$\frac{R}{2s}$	X_{1374}	$4r_a - a$
$\frac{-R}{2s}$	X_{1373}	$4r_a + a$

The infinite point of the Soddy point is the point

$$X_{516} = (2a^3-(b+c)(a^2+(b-c)^2) : 2b^3-(c+a)(b^2+(c-a)^2) : 2c^3-(a+b)(c^2+(a-b)^2)).$$

It corresponds to $t = \frac{R(4R+r)}{s^2}$. The deLongchamps point X_{20} also lies on the Soddy line. It corresponds to $t = \frac{2R(2R+r)}{s^2}$.

4. Emelyanov's first problem

From the coordinates of X_t, we easily find the intersections

$$A_t = AX_t \cap BC, \quad B_t = BX_t \cap CA, \quad C_t = CX_t \cap AB.$$

These are

$$\begin{aligned} A_t =&(0 : b(s-c)(c-(s-b)t) : c(s-b)(b-(s-c)t),\\ B_t =&(a(s-c)(c-(s-a)t) : 0 : c(s-a)(a-(s-c)t),\\ C_t =&(a(s-b)(b-(s-a)t) : b(s-a)(a-(s-b)t) : 0). \end{aligned} \tag{2}$$

They are collinear if and only if

$$\begin{aligned} &(a-(s-b)t)(b-(s-c)t)(c-(s-a)t)\\ +&(a-(s-c)t)(b-(s-a)t)(c-(s-b)t) = 0. \end{aligned} \tag{3}$$

Since this is a cubic equation in t, there are three values of t for which A_t, B_t, C_t are collinear. One of these is $t = 2$ according to [4]. The other two roots are given by

$$abc - abct + 2(s-a)(s-b)(s-c)t^2 = 0. \tag{4}$$

Since $abc = 4Rrs$ and $(s-a)(s-b)(s-c) = r^2 s$, where R and r are respectively the circumradius and inradius, this becomes

$$2R - 2Rt + rt^2 = 0. \tag{5}$$

From this,

$$t = \frac{R \pm \sqrt{R^2 - 2Rr}}{r} = \frac{R \pm d}{r},$$

where d is the distance between O and I.

We identify the lines corresponding to these two values of t.

Proposition 7. *Corresponding to the two roots of (4), the lines containing A_t, B_t, C_t are the tangents to the incircle perpendicular to the OI-line.*

Lemma 8. *Consider a triangle ABC with intouch triangle XYZ, and a line $\mathcal{L}$ intersecting the sides BC, CA, AB at A', B', C' respectively. The line $\mathcal{L}$ is tangent to the incircle if and only if one of the following conditions holds.*

(1) *The intersection $BB' \cap CC'$ lies on the line YZ.*
(2) *The intersection $CC' \cap AA'$ lies on the line ZX.*
(3) *The intersection $AA' \cap BB'$ lies on the line XY.*

Proof. Let $A'B'$ be a tangent to the incircle. By Brianchon's theorem applied to the circumscribed hexagon $AYB'A'XB$ it immediately follows that AA', YX and $B'B$ are concurrent.

Now suppose AA', YX and $B'B$ are concurrent. Consider the tangent through A' (different from BC) to the incircle. Let B'' be the intersection of this tangent with AC. It follows from the preceeding that AA', YX and $B''B$ are concurrent. Therefore B'' must coincide with B'. This means that $A'B'$ is a tangent to the incircle. □

5. Proof of Proposition 7

The lines BB_t and CC_t intersect at the point

$$\begin{aligned} A'' = \bigg(& \frac{a}{s-a}(b-(s-a)t)(c-(s-a)t) \\ & : \frac{b}{s-b}(c-(s-a)t)(a-(s-b)t) \\ & : \frac{c}{s-c}(a-(s-c)t)(b-(s-a)t)\bigg). \end{aligned}$$

This point lies on the line $YZ: \ -(s-a)x+(s-b)y+(s-c)z=0$ if and only if

$$\begin{aligned} & -a(b-(s-a)t)(c-(s-a)t) \\ & +b(c-(s-a)t)(a-(s-b)t) \\ & +c(a-(s-c)t)(b-(s-a)t)=0. \end{aligned}$$

This reduces to equation (4) above. By Lemma 8, these two lines are tangent to the incircle. We claim that these are the tangents perpendicular to the line OI. From the coordinates given in (2), the equation of the line B_tC_t is

$$\begin{aligned} & -\frac{(s-a)(a-(s-b)t)(a-(s-c)t)}{a}x \\ & +\frac{(s-b)(a-(s-c)t)(b-(s-a)t)}{b}y \\ & +\frac{(s-c)(a-(s-b)t)(c-(s-a)t)}{c}z=0. \end{aligned}$$

According to [6], lines perpendicular to OI have infinite point

$$X_{513}=(a(b-c):b(c-a):c(a-b)).$$

The line B_tC_t contains the infinite point X_{513} if and only if the same equation (4) holds. This shows that the two lines in question are indeed the tangents to the incircle perpendicular to the OI-line.

References

[1] E. Danneels, Hyacinthos message 8670, November 18, 2003.
[2] J.-P. Ehrmann, Hyacinthos message 7999, September 24, 2003.
[3] L. Emelyanov, Hyacinthos message 8645, November 16, 2003.
[4] L. Emelyanov, On the intercepts of the OI-line, *Forum Geom.*, 4 (2004) 81–84.
[5] B. Gibert, Orthocorrespondence and orthopivotal cubics, *Forum Geom.*, 3 (2003) 1–27.
[6] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[7] P. Yiu, Hyacinthos message 8647, November 16, 2003.
[8] P. Yiu, Hyacinthos message 8675, November 18, 2003.

Eric Danneels: Hubert d'Ydewallestraat 26, 8730 Beernem, Belgium
E-mail address: eric.danneels@pandora.be

Forum Geometricorum
Volume 4 (2004) 135–141.

FORUM GEOM
ISSN 1534-1178

A Theorem on Orthology Centers

Eric Danneels and Nikolaos Dergiades

Abstract. We prove that if two triangles are orthologic, their orthology centers have the same barycentric coordinates with respect to the two triangles. For a point P with cevian triangle $A'B'C'$, we also study the orthology centers of the triangle of circumcenters of $PB'C'$, $PC'A'$, and $PA'B'$.

1. The barycentric coordinates of orthology centers

Let $A'B'C'$ be the cevian triangle of P with respect to a given triangle ABC. Denote by O_a, O_b, O_c the circumcenters of triangles $PB'C'$, $PC'A'$, $PA'B'$ respectively. Since O_bO_c, O_cO_a, and O_aO_b are perpendicular to AP, BP, CP, the triangles $O_aO_bO_c$ and ABC are orthologic at P. It follows that the perpendiculars from O_a, O_b, O_c to the sidelines BC, CA, AB are concurrent at a point Q. See Figure 1. We noted that the barycentric coordinates of Q with respect to triangle $O_aO_bO_c$ are the same as those of P with respect to triangle ABC. Alexey A. Zaslasky [7] pointed out that our original proof [3] generalizes to an arbitrary pair of orthologic triangles.

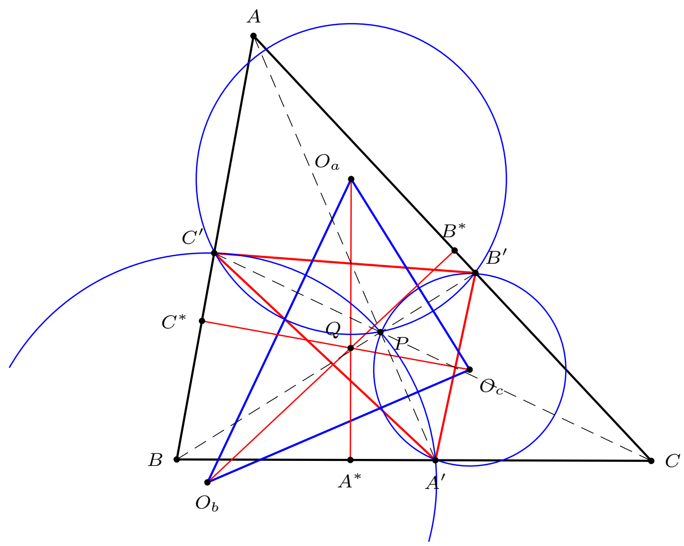


Figure 1

Theorem 1. *If triangles ABC and $A'B'C'$ are orthologic with centers P, P' then the barycentric coordinates of P with respect to ABC are equal to the barycentric coordinates of P' with respect to $A'B'C'$.*

Publication Date: September 15, 2004. Communicating Editor: Paul Yiu.

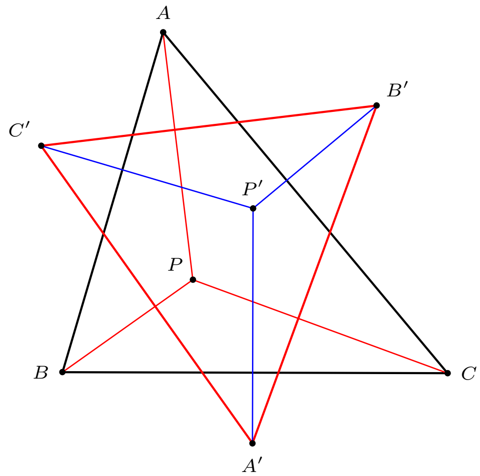


Figure 2

Proof. Since $A'P'$, $B'P'$, $C'P'$ are perpendicular to BC, CA, AB respectively, we have

$$\sin B'P'C' = \sin A, \quad \sin P'B'C' = \sin PAC, \quad \sin P'C'B' = \sin PAB.$$

Applying the law of sines to various triangles, we have

$$\begin{aligned}\frac{b}{P'B'} : \frac{c}{P'C'} =& \frac{1}{c \sin P'C'B'} : \frac{1}{b \sin P'B'C'}\\ =& \frac{1}{c \sin PAB} : \frac{1}{b \sin PAC}\\ =& \frac{1}{AP \cdot c \sin PAB} : \frac{1}{AP \cdot b \sin PAC}\\ =& \frac{1}{\text{area}(PAB)} : \frac{1}{\text{area}(PAC)}\\ =& \text{area}(PCA) : \text{area}(PAB).\end{aligned}$$

Similarly, $\frac{a}{P'A'} : \frac{b}{P'B'} = \text{area}(PBC) : \text{area}(PCA)$. It follows that the barycentric coordinates of P' with respect to triangle $A'B'C'$ are

$$\begin{aligned}&\text{area}(P'B'C') : \text{area}(P'C'A') : \text{area}(P'A'B')\\ =&(P'B')(P'C') \sin A : (P'C')(P'A') \sin B : (P'A')(P'B') \sin C\\ =&\frac{a}{P'A'} : \frac{b}{P'B'} : \frac{c}{P'C'}\\ =&\text{area}(PBC) : \text{area}(PCA) : \text{area}(PAB),\end{aligned}$$

the same as the barycentric coordinates of P with respect to triangle ABC. $\square$

This property means that if P is the centroid of ABC then P' is also the centroid of $A'B'C'$.

2. The orthology center of $O_aO_bO_c$

We compute explicitly the coordinates (with respect to triangle ABC) of the orthology center Q of the triangle of circumcenters $O_aO_bO_c$. See Figure 3. Let $P = (x : y : z)$ and $Q = (u : v : w)$ in homogeneous barycentric coordinates. then $BC' = \frac{cx}{x+y}$, $CB' = \frac{bx}{x+z}$. In the notations of John H. Conway, the pedal A^* of O_a on BC has homogeneous barycentric coordinates $(0 : uS_C + a^2v : uS_B + a^2w)$. See, for example, [6, pp.32, 49].

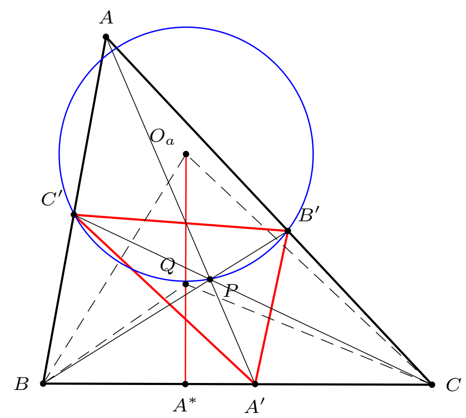


Figure 3

Note that $BA^* = \frac{uS_B+a^2w}{(u+v+w)a}$ and $A^*C = \frac{uS_C+a^2v}{(u+v+w)a}$. Also, by Stewart's theorem,

$$BB'^2 = \frac{c^2x^2 + a^2z^2 + (c^2 + a^2 - b^2)xz}{(x+z)^2},$$

$$CC'^2 = \frac{b^2x^2 + a^2y^2 + (a^2 + b^2 - c^2)xy}{(x+y)^2}.$$

Hence, if ρ is the circumradius of $PB'C'$, then

$$\begin{aligned}
&a(BA^* - A^*C)\\
=&(BA^* + A^*C)(BA^* - A^*C)\\
=&(BA^*)^2 - (A^*C)^2\\
=&(O_aB)^2 - (O_aA^*)^2 - (O_aC)^2 + (O_aA^*)^2\\
=&(O_aB)^2 - \rho^2 - (O_aC)^2 + \rho^2\\
=&BP \cdot BB' - CP \cdot CC'\\
=&\frac{c^2x^2 + a^2z^2 + (c^2 + a^2 - b^2)xz}{(x+z)(x+y+z)} - \frac{b^2x^2 + a^2y^2 + (a^2 + b^2 - c^2)xy}{(x+y)(x+y+z)}\\
=&-\frac{a^2(y-z)(x+y)(x+z) + b^2x(x+y)(x+2z) - c^2x(x+z)(x+2y)}{(x+y)(x+z)(x+y+z)}
\end{aligned}$$

since the powers of B and C with respect to the circle of $PB'C'$ are $BB' \cdot BP = (O_aB)^2 - \rho^2$ and $CC' \cdot CP = (O_aC)^2 - \rho^2$ respectively. In other words,

$$\begin{aligned}&\frac{(c^2-b^2)u - a^2(v-w)}{u+v+w}\\ &= -\frac{a^2(y-z)(x+y)(x+z) + b^2x(x+y)(x+2z) - c^2x(x+z)(x+2y)}{(x+y)(x+z)(x+y+z)},\end{aligned}$$

or

$$\begin{aligned}&(a^2(y-z)(x+y)(x+z) - b^2(x+y)(xy+yz+z^2) + c^2(x+z)(y^2+xz+yz))u\\ &-(a^2(x+y)(x+z)(x+2z) - b^2x(x+y)(x+2z) + c^2x(x+z)(x+2y))v\\ &+(a^2(x+y)(x+z)(x+2y) + b^2x(x+y)(x+2z) - c^2x(x+z)(x+2y))w = 0.\end{aligned}$$

By replacing x, y, z by y, z, x and u, v, w by v,w, u, we obtain another linear relation in u, v, w. From these we have $u : v : w$ given by

$$\begin{aligned}u =&(x^2-z^2)y^2S_{BB} + (x^2-y^2)z^2S_{CC} - x(2x+y)(x+z)(y+z)S_{AB}\\ &- x(2x+z)(x+y)(y+z)S_{CA} - 2(x+y)(x+z)(xy+yz+zx)S_{BC}.\end{aligned}$$

and v obtained from u by replacing x, y, z, S_A, S_B, S_C by v, w, u, S_B, S_C, S_A respectively, and w from v by the same replacements.

3. Examples

3.1. *The centroid.* For $P = G$,

$$\begin{aligned}O_a =&(5S_A(S_B+S_C) + 2(S_{BB} + 5S_{BC} + S_{CC})\\ &: 3S_{AB} + 4S_{AC} + S_{BC} - 2S_{CC}\\ &: 3S_{AC} + 4S_{AB} + S_{BC} - 2S_{BB}).\end{aligned}$$

Similarly, we write down the coordinates of O_b and O_c. The perpendiculars from O_a to BC, from O_b to CA, and from O_c to AB have equations

$$\begin{array}{rcrcrcl}(S_B-S_C)x & - & (3S_B+S_C)y & + & (S_B+3S_C)z & = & 0,\\ (S_C+3S_A)x & + & (S_C-S_A)y & - & (3S_C+S_A)z & = & 0,\\ -(3S_A+S_B)x & + & (S_A+3S_B)y & + & (S_A-S_B)z & = & 0.\end{array}$$

These three lines intersect at the nine-point center

$$X_5 = (S_{CA} + S_{AB} + 2S_{BC} : S_{AB} + S_{BC} + 2S_{CA} : S_{BC} + S_{CA} + 2S_{AB}),$$

which is the orthology center of $O_aO_bO_c$.

3.2. *The orthocenter.* If P is the orthocenter, the circumcenters O_a, O_b, O_c are simply the midpoints of the segments AP, BP, CP respectively. In this case, $Q = H$.

3.3. *The Steiner point.* If P is the Steiner point $\left(\frac{1}{S_B-S_C} : \frac{1}{S_C-S_A} : \frac{1}{S_A-S_B}\right)$, the perpendiculars from the circumcenters to the sidelines are

$$\begin{array}{ccccccc} (S_B-S_C)x & - & S_Cy & + & S_Bz & = & 0, \\ S_Cx & + & (S_C-S_A)y & - & S_Az & = & 0, \\ -S_Bx & + & S_Ay & + & (S_A-S_B)z & = & 0. \end{array}$$

These lines intersect at the deLongchamps point

$$X_{20} = (S_{CA}+S_{AB}-S_{BC} : S_{AB}+S_{BC}-S_{CA} : S_{BC}+S_{CA}-S_{AB}).$$

3.4. X_{671}. The point $P = X_{671} = \left(\frac{1}{S_B+S_C-2S_A} : \frac{1}{S_C+S_A-2S_B} : \frac{1}{S_A+S_B-2S_C}\right)$ is the antipode of the Steiner point on the Steiner circum-ellipse. It is also on the Kiepert hyperbola, with Kiepert parameter $-\operatorname{arccot}(\frac{1}{3}\cot\omega)$, where ω is the Brocard angle. In this case, the circumcenters are on the altitudes. This means that $Q = H$.

3.5. *An antipodal pair on the circumcircle.* The point X_{925} is the second intersection of the circumcircle with the line joining the deLongchamps point X_{20} to X_{74}, the isogonal conjugate of the Euler infinity point. It has coordinates

$$\left(\frac{1}{(S_B-S_C)(S^2-S_{AA})} : \frac{1}{(S_C-S_A)(S^2-S_{BB})} : \frac{1}{(S_A-S_B)(S^2-S_{CC})}\right).$$

For $P = X_{925}$, the orthology Q of $O_aO_bO_c$ is the point X_{68}, [1] which lies on the same line joining X_{20} to X_{74}.

The antipode of X_{925} is the point

$$X_{1300} = \left(\frac{1}{S_A((S_{AA}-S_{BC})(S_B+S_C)-S_A(S_B-S_C)^2)} : \cdots : \cdots\right).$$

It is the second intersection of the circumcircle with the line joining the orthocenter to the Euler reflection point [2] $X_{110} = \left(\frac{S_B+S_C}{S_B-S_C} : \frac{S_C+S_A}{S_C-S_A} : \frac{S_A+S_B}{S_A-S_B}\right)$. For $P = X_{1300}$, the orthology center Q of $O_aO_bO_c$ has first barycentric coordinate

$$\frac{S_{AA}(S_{BB}+S_{CC})(S_A(S_B+S_C)-(S_{BB}+S_{CC}))+S_{BC}(S_B-S_C)^2(S_{AA}-2S_A(S_B+S_C)-S_{BC}))}{S_A((S_B+S_C)(S_{AA}-S_{BC})-S_A(S_B-S_C)^2)}.$$

In this case, $O_aO_bO_c$ is also perspective to ABC at

$$X_{254} = \left(\frac{1}{S_A((S_{AA}-S_{BC})(S_B+S_C)-S_A(S_{BB}+S_{CC}))} : \cdots : \cdots\right).$$

By a theorem of Mitrea and Mitrea [5], this perspector lies on the line PQ.

---

[1] X_{68} is the perspector of the reflections of the orthic triangle in the nine-point center.

[2] The Euler reflection point is the intersection of the reflections of the Euler lines in the sidelines of triangle ABC.

3.6. More generally, for a generic point P on the circumcircle with coordinates $\left(\frac{S_B+S_C}{(S_A+t)(S_B-S_C)} : \cdots : \cdots\right)$, the center of orthology of $O_aO_bO_c$ is the point

$$\left(\frac{(S_B+S_C)(F(S_A,S_B,S_C)+G(S_A,S_B,S_C)t)}{S_A+t} : \cdots : \cdots\right),$$

where

$$\begin{aligned}F(S_A,S_B,S_C) =& S_{AA}(S_{BB}+S_{CC})(S_A+S_B+S_C)+S_{AABC}(S_B+S_C)\\&-S_{BB}S_{CC}(2S_A+S_B+S_C),\\G(S_A,S_B,S_C) =& 2(S_{AA}(S_{BB}+S_{BC}+S_{CC})-S_{BB}S_{CC}).\end{aligned}$$

Proposition 2. *If P lies on the circumcircle, the line joining P to Q always passes through the deLongchamps point X_{20}.*

Proof. The equation of the line PQ is

$$\begin{aligned}\sum_{\text{cyclic}}&(S_B-S_C)(S_A+t)(S_A^3(S_B-S_C)^2\\&+(S_B+S_C+2t)(S_{AA}(S_{BB}-S_{BC}+S_{CC})-S_{BB}S_{CC})x=0.\end{aligned}$$

□

3.7. *Some further examples.* We conclude with a few more examples of P with relative simple coordinates for Q, the orthology center of $O_aO_bO_c$.

P	first barycentric coordinate of Q
X_7	$4a^3+a^2(b+c)-2a(b-c)^2-3(b+c)(b-c)^2$
X_8	$4a^4-5a^3(b+c)-a^2(b^2-10bc+c^2)+5a(b-c)^2(b+c)-3(b^2-c^2)^2$
X_{69}	$3a^6-4a^4(b^2+c^2)+a^2(3b^4+2b^2c^2+3c^4)-2(b^2-c^2)^2(b^2+c^2)$
X_{80}	$\frac{4a^3-3a^2(b+c)-2a(2b^2-5bc+2c^2)+3(b-c)^2(b+c)}{(b^2+c^2-a^2-bc)}$

In each of the cases $P=X_7$ and X_{80}, the triangle $O_aO_bO_c$ is also perspective to ABC at the incenter.

References

[1] E. Danneels, Hyacinthos message 10068, July 12, 2004.
[2] N. Dergiades, Hyacinthos messages 10073, 10079, 10083, July 12, 13, 2004.
[3] N. Dergiades, Hyacinthos messages 10079, July 13, 2004.
[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[5] D. Mitrea and M. Mitrea, A generalization of a theorem of Euler, *Amer. Math. Monthly*, 101 (1994) 55–58.
[6] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University lecture notes, 2001.
[7] A. A. Zaslavsky, Hyacinthos message 10082, July 13, 2004.

Eric Danneels: Hubert d'Ydewallestraat 26, 8730 Beernem, Belgium
E-mail address: `eric.danneels@pandora.be`

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: `ndergiades@yahoo.gr`

Forum Geometricorum
Volume 4 (2004) 143–151.

FORUM GEOM
ISSN 1534-1178

A Grand Tour of Pedals of Conics

Roger C. Alperin

Abstract. We describe the pedal curves of conics and some of their relations to origami folding axioms. There are nine basic types of pedals depending on the location of the pedal point with respect to the conic. We illustrate the different pedals in our tour.

1. Introduction

The main 'axiom' of mathematical origami allows one to create a fold-line by sliding or folding point F onto a line L so that another point S is also folded onto yet another line M. One can regard this complicated axiom as making possible the folding of the common tangents to the parabola κ with focus F and directrix L and the parabola with focus S and directrix M. Since two parabolas have at most four common tangents in the projective plane and one of them is the line at infinity there are at most three folds in the Euclidean plane which will accomplish this origami operation. In the field theory associated to origami this operation yields construction methods for solving cubic equations, [1]. Hull has shown how to do the 'impossible' trisection of an angle using this folding, by a method due to Abe, [2]. In fact the trisection of Abe is quite similar to a classical method using Maclaurin's trisectrix, [3]. The trisectrix is one of the pedals along the tour.

One can simplify this origami folding operation into smaller steps: first fold S to the point P by reflection across the tangent of the parabola κ. The locus of points P for all the tangents of κ is a curve; finally, this locus is intersected with the line M. This 'origami locus' of points P is a cubic curve since intersecting with M will generally give three possible solutions. Since reflection of S across a line is just the double of the perpendicular projection S' of S onto L, this 'origami' locus is the scale by a factor of 2 of the locus of S', also known as the pedal curve of the parabola, [3]. As a generalization we shall investigate the pedal curves of an arbitrary conic; this pedal curve is generally a quartic curve.

Pedal of a conic. The points S' of the pedal curve lie on the lines through S at the places where the tangents to the curve are perpendicular to these lines. Suppose that S is at the origin. The line through the origin perpendicular to $\alpha x + \beta y + \gamma = 0$ is the line $\beta x - \alpha y = 0$; these meet when $x = -\frac{\alpha\gamma}{\alpha^2+\beta^2}$, $y = -\frac{\beta\gamma}{\alpha^2+\beta^2}$. This suggests

Publication Date: October 1, 2004. Communicating Editor: Paul Yiu.

using the inversion transform (at the origin), the map given by $x \to \frac{x}{x^2+y^2}$, $y \to \frac{y}{x^2+y^2}$.

A conic has the homogenous quadratic equation $F(x,y,z) = 0$ which can also be given by the matrix equation $F(x,y,z) = (x,y,z)A(x,y,z)^t = 0$ for a 3 by 3 symmetric matrix A. It is well-known that the dual curve of tangent lines to a conic is also a conic having homogeneous equation $F'(x,y,z) = 0$ obtained from the adjoint matrix A' of A. Thus the pedal curve has the (inhomogeneous) equation obtained by applying the inversion transform to $F'(x,y,-z) = 0$, evaluated at $z = 1$, [4].

The polar line of a point T is the line through the points U and V on the conic where the tangents from T meet the conic. It is important to realize the polar line of a point with respect to the conic κ having equation $F = 0$ can be expressed in terms of the matrix A. In terms of equations, if T has (projective) coordinates (u,v,w) then the dual line has the equation $(x,y,z)A(u,v,w)^t = 0$. For example, when S is placed at the origin the dual line is $(x,y,z)A(0,0,1)^t = 0$.

2. Equation of a pedal of a conic

Let S be at the origin. Suppose the (non-degenerate) conic equation is $F(x,y,z) = ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$. Applying the inversion to the adjoint equation gives after a bit of algebra the relatively simple equation

$$G = \Delta(x^2+y^2)^2 + (x^2+y^2)((4cd-2be)x + (4ae-2bd)y) + G_2 = 0$$

where $\Delta = 4ac - b^2$ is the discriminant of the conic; $\Delta = 0$ iff the conic is a parabola. In the case of a parabola, the pedal curve has a cubic equation. The origin is a singular point having as singular tangent lines the linear factors of the degree two term $G_2 = (4cf - e^2)x^2 + (2ed - 4bf)xy + (4af - d^2)y^2 = 0$.

3. Variety of pedals

Fix a (non-empty) real conic κ in the plane and a point S. There are two points U and V on the conic with tangents τ_U and τ_V meeting at S; the corresponding pedal point for each of these tangents is S. Thus S is a double point. The type of singularity or double point at S is either a node, cusp or acnode depending on whether or not the two tangents are real and distinct, real and equal or complex conjugates.

The perpendicular lines at S to τ_U and τ_V are the singular tangents. To see this notice that the dual line to $S = (0,0)$ is $(x,y,z)A(0,0,1)^t = 0$ or equivalently $dx + ey + 2f = 0$. This line meets the conic at the points U, V which are on the tangents from S. Determining the perpendiculars through the origin S to these tangents, and multiplying the two linear factors yields after a tedious calculation precisely the second degree terms G_2 of G.

The variety of pedals depending on the type of conic and the type of singularity, are displayed in Figures 1-9, along with their associated conics, the singular point S, the singular tangents, dual line and its intersections with the conic (whenever possible).

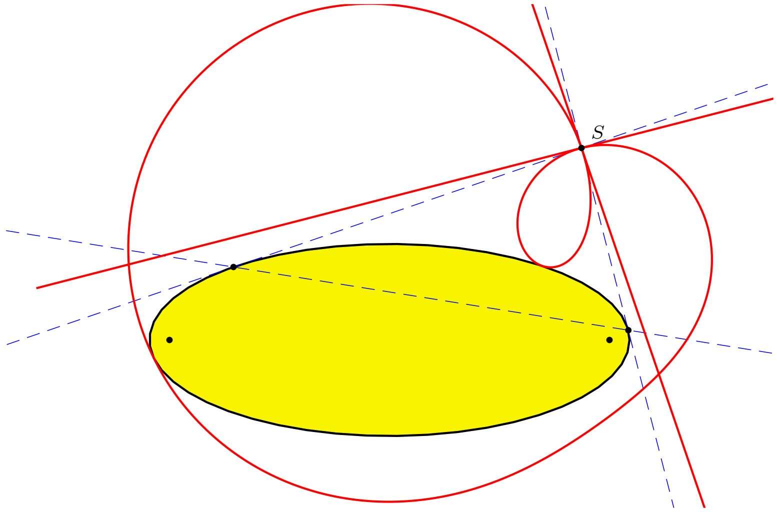


Figure 1. Elliptic node

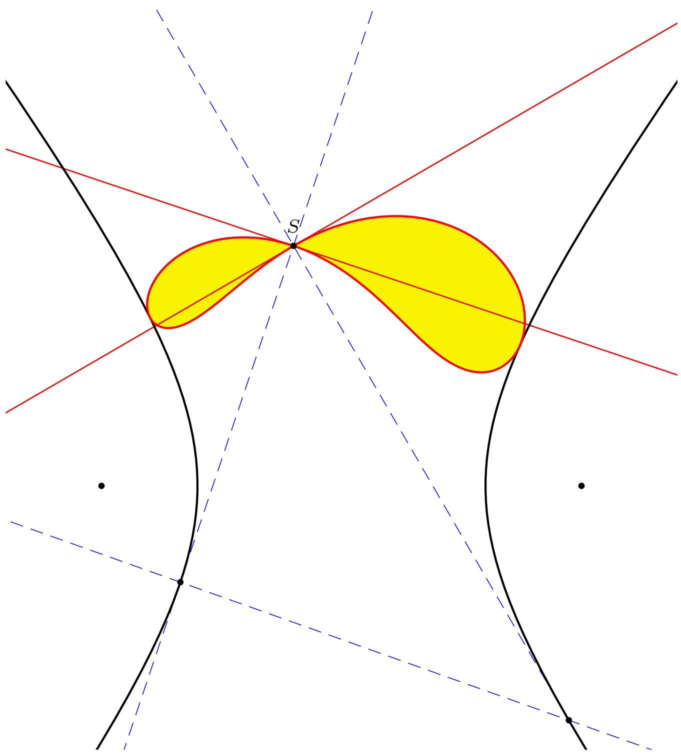


Figure 2. Hyperbolic node

Proposition 1. *The pedal of the real conic κ has a node, cusp or acnode depending on whether S is outside, on, or inside κ.*

Proof. By the calculation of the second degree terms of G, the singular tangents at the point S of the pedal are the perpendiculars to the two tangents from S to the conic κ. Thus the type of node depends on the position of S with respect to the conic since that determines how G_2 factors over the reals. $\square$

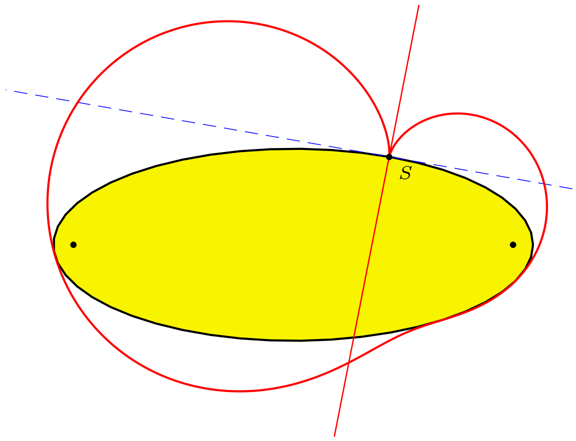


Figure 3. Elliptic cusp

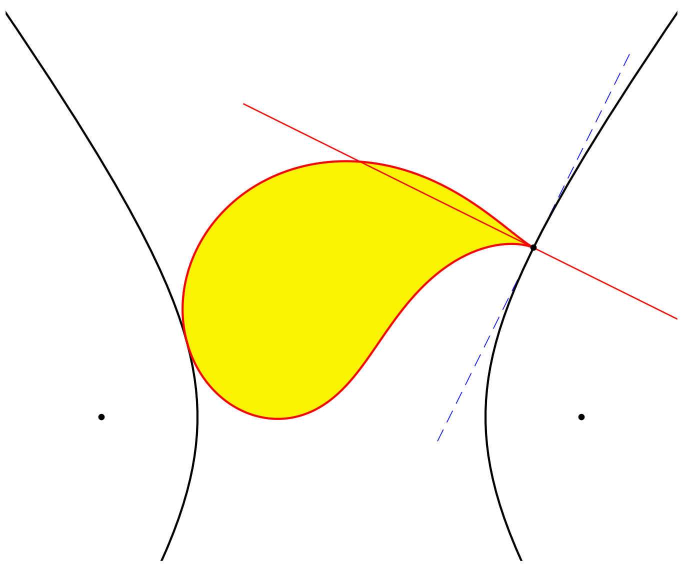


Figure 4. Hyperbolic cusp

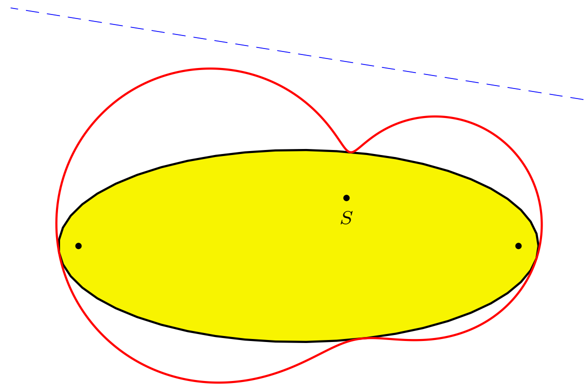


Figure 5. Elliptic acnode

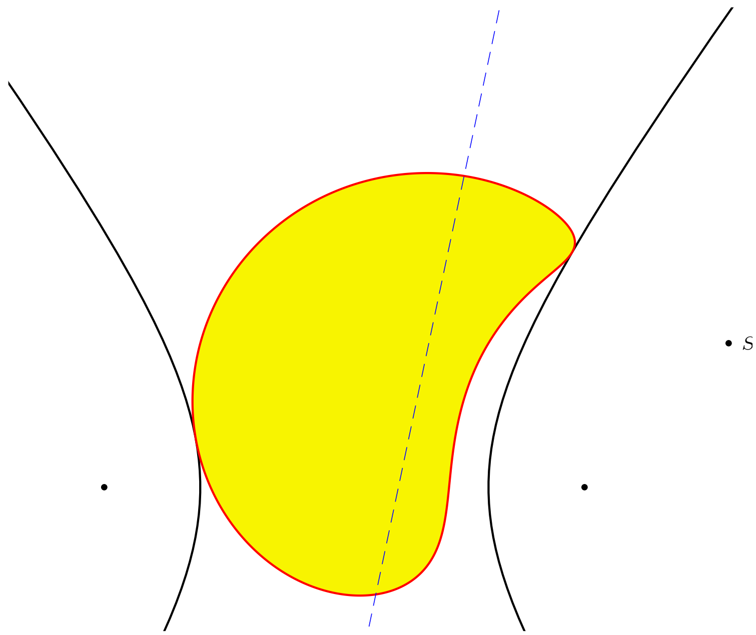


Figure 6. Hyperbolic acnode

4. Bicircular quartics

A quartic curve having circular double points is called bicircular.

Proposition 2. *A real quartic curve has the equation* $G = A(x^2 + y^2)^2 + (x^2 + y^2)(Bx + Cy) + Dx^2 + Exy + Fy^2 = 0$ *for* $A \neq 0$ *iff it is bicircular with double point at the origin. Thus the pedal of an ellipse or hyperbola is a bicircular quartic with a double point at* S.

Proof. A quartic has a double point at the origin iff there are no terms of degree less than 2 in the (inhomogeneous) equation $G = 0$. There are double points at the circular points iff $G(x, y, z)$ vanishes to second order when evaluated at the circular points; hence iff the gradient of G is zero at the circular points. Since $\frac{\partial G}{\partial z} = 2zG_2 + G_3$; this vanishes at the circular points iff G_3 is divisible by $x^2 + y^2$. Also G vanishes at the circular points iff G_4 is divisible by $x^2 + y^2$. Thus the homogeneous equation for the quartic is $G = (x^2 + y^2)(ux^2 + vxy + wy^2) + z(x^2 + y^2)(Bx + Cy) + z^2G_2 = 0$. Finally $\frac{\partial G}{\partial x}$ or equivalently $\frac{\partial G}{\partial y}$ will also vanish at the circular points iff $ux^2 + vxy + wy^2$ is divisible by $x^2 + y^2$. Hence a bircular quartic with a double point at the origin has the equation as specified in the proposition and conversely.

The conclusion for the pedal follows immediately from the equation given in Section 2. □

We now show that any real bicircular quartic having a third double point can be realized as the pedal of a conic.

Proposition 3. *A bicircular quartic is the pedal of an ellipse or hyperbola.*

Proof. Using the equation for the pedal of a conic as in Section 2 we consider the system of equations $A = 4ac-b^2$, $B = 4cd-2be$, $C = 4ae-2bdy$, $D = 4cf-e^2$, $E = 2ed - 4bf$, $F = 4af - d^2$. One can easily see that this is equivalent to a (symmetric) matrix equation $Y = X'$ where X' is the adjoint of X; we want to solve for X given Y. In our case here, Y involves the variables $A, B, \ldots$ and X involves $a, b, \ldots$ Certainly $det(Y) = det(X)^2$. Then we can solve using adjoints, $X = Y'$ iff the quadratic form $Q = Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2$ has positive determinant. However changing G to $-G$ changes the sign of this determinant so we can represent all these quartics by pedals. □

The type of singularity of a bicircular quartic with double point at S is determined from Proposition 1 and the previous Proposition. The type of singularity of the circular double points is determined by the low order terms of G when expanded at the circular points; since the circular point is complex it is nodal in general; a circular point is cuspidal when $BC = 8AE$ and $C^2 - B^2 = 16A(D - F)$ and then in fact both circular points are cusps.

5. Pedal of parabolas

In the case that the conic is a parabola ($\Delta = 0$) the pedal equation simplifies to a cubic equation. This pedal cubic is singular and circular.

Proposition 4. *A singular circular cubic with singularity at the origin has an equation $G = (x^2 + y^2)(Bx + Cy) + Dx^2 + Exy + Fy^2 = 0$ and conversely. This is the pedal of a parabola.*

Proof. The cubic is singular at the origin iff there are no terms of degree less than two; the curve is circular iff the cubic terms vanish at the circular points iff $x^2 + y^2$ is a factor of the cubic terms.

The pedal of a parabola having $\Delta = 4ac - b^2 = 0$, means the cubic equation is $G = (x^2 + y^2)((4cd - 2be)x + (4ae - 2bd)y) + (4cf - e^2)x^2 + (2ed - 4bf)xy +$

$(4af - d^2)y^2 = 0$. Solving the system of equations as in Proposition 3 we have a simpler system since $A = 0$ but similar methods give the desired result. $\square$

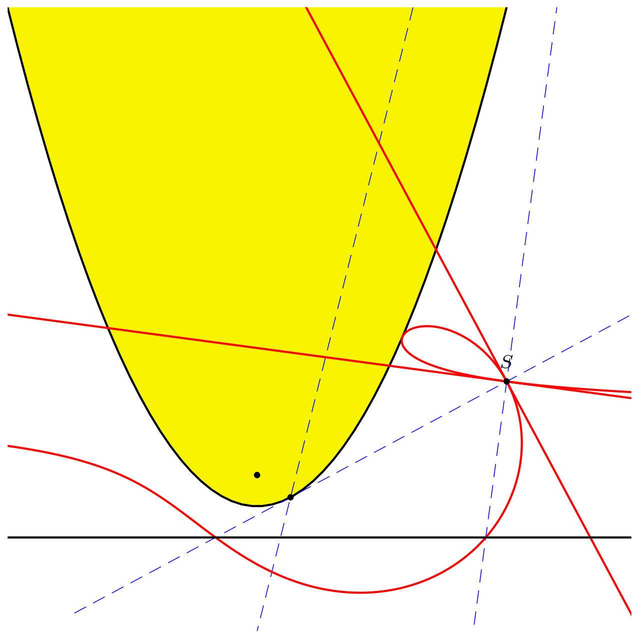


Figure 7. Parabolic node

6. Tangency of pedal and conic at their intersections

The pedal of a conic κ meets that conic at the places T iff the normal line to κ at that point passes through S. Thus the intersection occurs iff the line ST is a normal to the curve.

It follows from the fact that the conic and its pedal have a resultant which is a square (a horrendous calculation) that the pedal is tangent at all of its intersections with the conic. From Bezout's theorem, the conic and pedal have eight intersections (counted with multiplicity) and since each is a tangency there are at most four actual incidences just as expected from the figures.

Alternatively we can use elementary properties of a arbitrary curve $C(t)$ with unit speed parameterizations having tangent τ and normal η to see that when S is at the origin, the pedal $P(t)$ has a parametrization $P(t) = C(t) \cdot \eta(t)\eta(t)$ and tangent $P'(t) = -k(t)(C(t) \cdot \tau(t)\eta(t) + C(t) \cdot \eta(t)\tau(t))$ where $k(t)$ is the curvature. Thus the tangent to P is parallel to τ iff $C(t) \cdot \tau(t) = 0$ iff $C(t)$ is parallel to the normal $\eta(t)$ iff the normal passes through S.

7. Linear families of pedals

Because of the importance of a parabola in the origami axioms, we illustrate in Figure 10 a family of origami curves. Recall that the origami curve is the pedal of

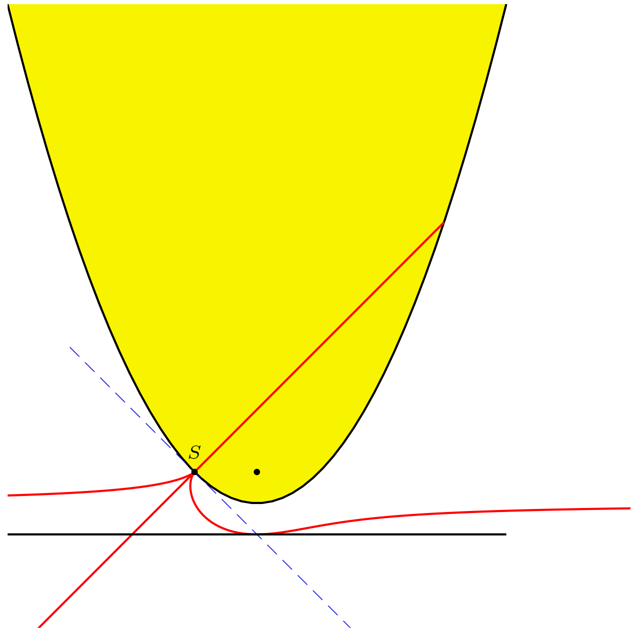


Figure 8. Parabolic cusp

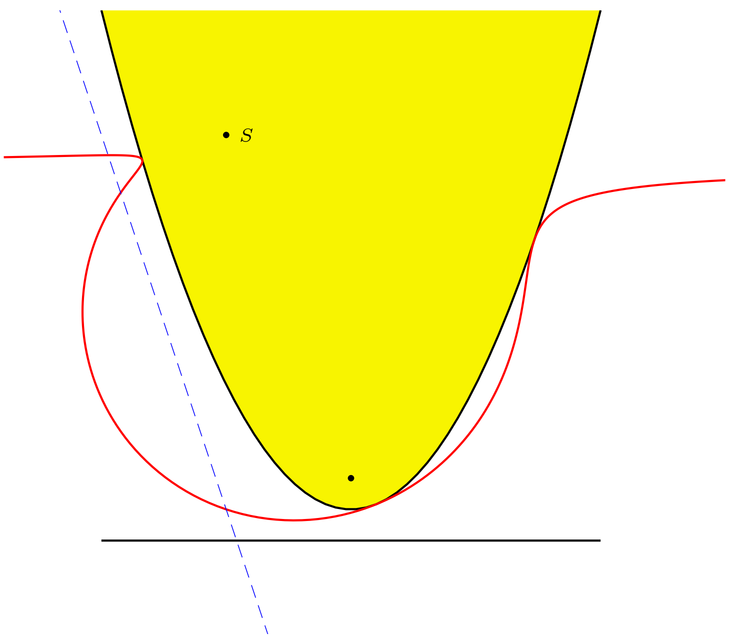


Figure 9. Parabolic acnode

a parabola scaled by 2 from the singular point S. The origami curves determined by a fixed parabola and S varying on a line parallel to the directrix are all tangent to a fixed circle of radius equal to the distance from S to the directrix. In case S varies on the directrix, then all the curves pass through the focus F.

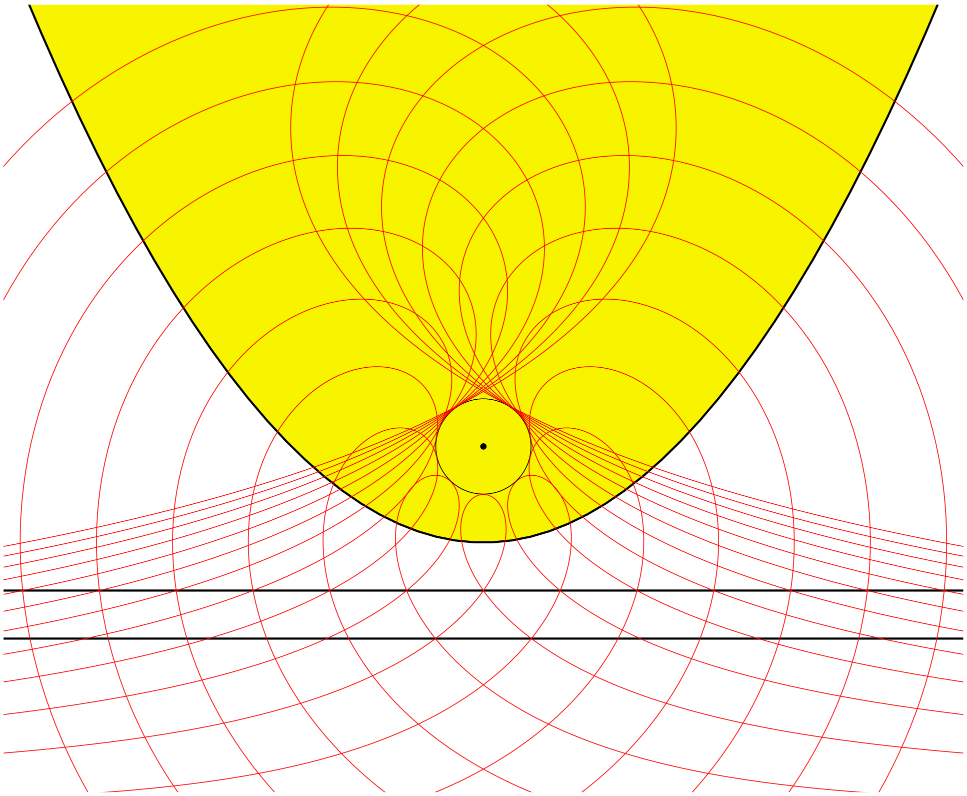


Figure 10. One parameter family of origami curves

References

[1] R. C. Alperin, A mathematical theory of origami constructions and numbers, *New York J. Math.*, 6, 119-133, 2000. *http://nyjm.albany.edu*

[2] T. Hull, A note on 'impossible' paper folding, *Amer. Math. Monthly*, 103 (199) 240–241.

[3] E. H. Lockwood, *A Book of Curves*, Cambridge University, 1963.

[4] P. Samuel, *Projective Geometry*, Springer-Verlag, 1988

Roger C. Alperin: Department of Mathematics, San Jose State University, San Jose, California 95192, USA

E-mail address: `alperin@math.sjsu.edu, alperin@jps.net`

Forum Geometricorum
Volume 4 (2004) 153–156.

FORUM GEOM
ISSN 1534-1178

Garfunkel's Inequality

Nguyen Minh Ha and Nikolaos Dergiades

Abstract. Let I be the incenter of triangle ABC and U, V, W the intersections of the segments IA, IB, IC with the incircle. If the centroid G is inside the incircle, and D, E, F the intersections of the segments GA, GB, GC with the incircle. Jack Garfunkel [1] asked for a proof that the perimeter of UVW is not greater than that of DEF. This problem is hitherto unsolved. We give a proof in this note.

Consider a triangle ABC with centroid G lying inside its incircle (I). Let the segments AG, BG, CG, AI, BI, CI intersect the incircle at D, E, F, U, V, W respectively. Garfunkel posed the inequality $\partial(UVW) \le \partial(DEF)$ as Problem 648(b) of *Crux Mathematicorum* [1, 2].[1] Here, $\partial(\cdot)$ denotes the perimeter of a triangle. The problem is hitherto unresolved. In this note we give a proof of this inequality. We adopt standard notations: a, b, c, are the sidelengths of triangle ABC, s the semiperimeter and r the inradius.

Lemma 1. *If the centroid G of the triangle ABC is inside the incircle (I), then*

$$a^2 < 4bc, \quad b^2 < 4ca, \quad c^2 < 4ab.$$

Proof. Because G is inside (I), we have $\overrightarrow{IG}^2 \le r^2$, $(\overrightarrow{AG} - \overrightarrow{AI})^2 \le r^2$, $\overrightarrow{AG}^2 + \overrightarrow{AI}^2 - 2\overrightarrow{AG}\cdot\overrightarrow{AI} \le r^2$. This inequality is equivalent to the following

$$\overrightarrow{AG}^2 + (\overrightarrow{AI}^2 - r^2) - \frac{2}{3}(\overrightarrow{AB} + \overrightarrow{AC})\cdot\overrightarrow{AI} \le 0$$

$$\frac{2(b^2+c^2) - a^2}{9} + (s-a)^2 - \frac{2(b+c)(s-a)}{3} \le 0$$

$$8(b^2+c^2) - 4a^2 + 9(b+c-a)^2 - 12(b+c)(b+c-a) \le 0$$

$$3(b+c-a)^2 + 2(b-c)^2 \le 2(4bc - a^2)$$

which implies $a^2 < 4bc$ and similarly $b^2 < 4ac$, $c^2 < 4ab$. □

Let the external bisectors of triangle UVW bound the triangle PQR, and intersect the incircle of ABC at U', V', W' respectively.

Publication Date: October 15, 2004. Communicating Editor: Paul Yiu.
The first author thanks Pham Van Thuan of Hanoi University for help in translation.

[1] Problem 648(a) asked for a proof of $\partial(XYZ) \le \partial(UVW)$, XYZ being the intouch triangle. See Figure 1. A proof by Garfunkel was given in [1].

Lemma 2. *If the centroid G of ABC is inside the incircle, then the points D, E, F are on the minor arcs UU', VV', WW' respectively.*

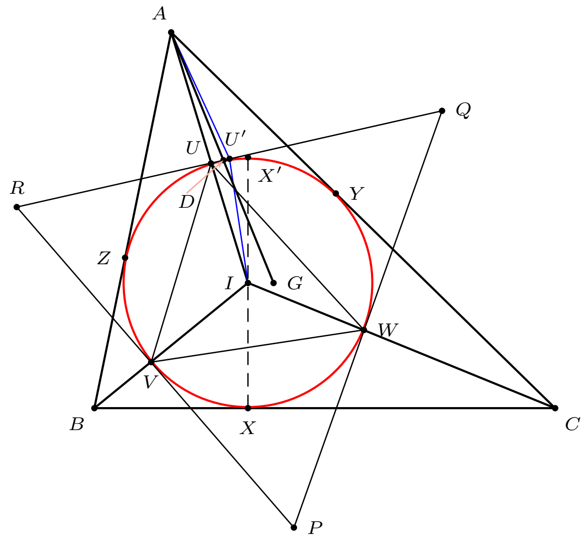


Figure 1

Proof. If $b = c$ then obviously U, D and U' are the same point.

Assume without loss of generality $b > c$. We set for brevity $\varphi = \frac{A}{2}$, $\theta = \frac{B-C}{4}$. Note that U' is the midpoint of the arc VUW. We have

$$\angle UIU' = \frac{1}{2}(\angle UIW - \angle UIV) = \frac{1}{2}\left(90^\circ + \frac{B}{2} - 90^\circ - \frac{C}{2}\right) = \theta.$$

Let X' be the antipode of the touch point X of the incircle with BC. Since $\angle UIV = \angle X'IW$, the point U' is the mid point of the arc UX'. We have

$$\begin{aligned}
\overrightarrow{AU'} =& \overrightarrow{AI} + \overrightarrow{IU'} = \overrightarrow{AI} + \frac{1}{2\cos\theta}\left(\overrightarrow{IU} + \overrightarrow{IX'}\right)\\
=& \overrightarrow{AI} + \frac{1}{2\cos\theta}\left(\sin\varphi\overrightarrow{IA} - \overrightarrow{IA} - \overrightarrow{AX}\right)\\
=& \left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\overrightarrow{AI} - \frac{1}{2\cos\theta}\overrightarrow{AX}\\
=& \left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\left(\frac{b}{2s}\overrightarrow{AB} + \frac{c}{2s}\overrightarrow{AC}\right)\\
&- \frac{1}{2\cos\theta}\left(\frac{s-c}{a}\overrightarrow{AB} + \frac{s-b}{a}\overrightarrow{AC}\right)\\
=& \left(\left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\frac{b}{2s} - \frac{1}{2\cos\theta}\cdot\frac{s-c}{a}\right)\overrightarrow{AB}\\
&+ \left(\left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\frac{c}{2s} - \frac{1}{2\cos\theta}\cdot\frac{s-b}{a}\right)\overrightarrow{AC}.
\end{aligned}$$

Since $b > c$, the centroid G lies inside the angle $\angle IAC$. To prove that D lies on the minor arc UU' it is sufficient to prove that the coefficient of $\overrightarrow{AC}$ is greater than that of $\overrightarrow{AB}$ in the above expression of $\overrightarrow{AU'}$. We need, therefore, to prove the inequality

$$\left(1-\frac{\sin\varphi-1}{2\cos\theta}\right)\frac{c}{2s}-\frac{1}{2\cos\theta}\cdot\frac{s-b}{a}>\left(1-\frac{\sin\varphi-1}{2\cos\theta}\right)\frac{b}{2s}-\frac{1}{2\cos\theta}\cdot\frac{s-c}{a}.$$

Factoring and grouping common terms, the inequality is equivalent to

$$\frac{1}{2\cos\theta}\cdot\frac{b-c}{a}-\left(1-\frac{\sin\varphi-1}{2\cos\theta}\right)\frac{b-c}{2s}>0$$
$$\frac{b-c}{4s\cos\theta}\left(\frac{b+c}{a}-2\cos\theta+\sin\varphi\right)>0$$
$$(b+c+a\sin\varphi)^2>4a^2\cos^2\theta. \tag{1}$$

Using the well-known identity $\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$, and $a\cos 2\theta=(b+c)\sin\varphi$ by the law of sines, inequality (1) can be written in the form

$$(b+c+a\sin\varphi)^2>2a^2+2a(b+c)\sin\varphi$$
$$(b+c)^2-a^2>a^2-a^2\sin^2\varphi$$
$$2bc+2bc\cos A>a^2\cos^2\varphi$$
$$4bc\cos^2(A/2)>a^2\cos^2\varphi$$
$$4bc>a^2.$$

This inequality holds by Lemma 1 since G is inside the incircle. This shows that D is on the minor arc UU'. The same reasoning also shows that E and F are on the minor arcs VV', WW' respectively. $\square$

Theorem (Garfunkel's inequality)**.** *If the centroid G lies inside the incircle, then $\partial(UVW)\le\partial(DEF)$.*

Proof. By Lemma 2, the points D, E, F lie on the minor arcs UU', VV', WW' respectively. Let X'' be the intersection point of DE and QR, Y'' be the intersection point of EF and RP, and Z'' be the intersection point of FD and PQ. Note that X'', Y'', Z'' belong to the segments DE, EF, FD respectively. See Figure 2. It follows that

$$\begin{aligned}\partial(DEF)&=DE+EF+FD\\&=DX''+X''E+EY''+Y''F+FZ''+Z''D\\&=(EX''+EY'')+(FY''+FZ'')+(DZ''+DX'')\\&\ge X''Y''+Y''Z''+Z''X''\\&=\partial(X''Y''Z'').\end{aligned}$$

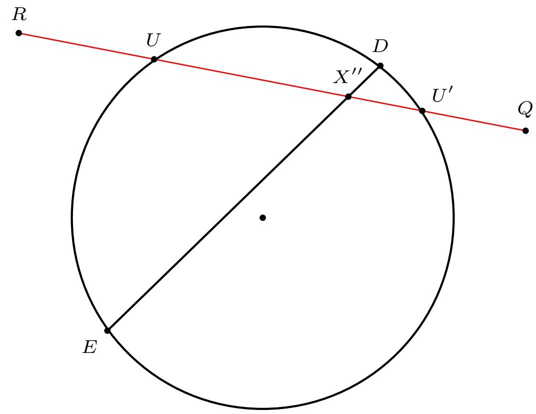


Figure 2

Therefore, $\partial(DEF) \geq \partial(X''Y''Z'')$. On the other hand, triangle PQR is acute and triangle UVW is its orthic triangle. See Figure 1. By Fagnano's theorem, we have $\partial(X''Y''Z'') \geq \partial(UVW)$. It follows that $\partial(DEF) \geq \partial(UVW)$. The equality holds if and only if triangle ABC is equilateral. $\square$

References

[1] J. Garfunkel, Problem 648, *Crux Math.*, 7 (1981) 178; solution, 8 (1982) 180–182.

[2] S. Rabinowitz, *Index to Mathematical Problems 1980-1984*, MathPro Press, Westford, Massachusetts USA 1992, p. 469.

Nguyen Minh Ha: Faculty of Mathematics, Hanoi University of Education, Xuan Thuy, Hanoi, Vietnam

E-mail address: `minhha27255@yahoo.com`

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece

E-mail address: `ndergiades@yahoo.gr`

Forum Geometricorum
Volume 4 (2004) 157–176.

FORUM GEOM
ISSN 1534-1178

On Some Actions of D_3 on a Triangle

Paris Pamfilos

Abstract. The action of the dihedral group D_3 on the equilateral triangle is generalized to various actions on general triangles.

1. Introduction

The equilateral triangle admits in a natural way the action of the dihedral group D_3. The elements f of the group act as reflexions (order 2: $f^2 = 1$) or as rotations (order 3: $f^3 = 1$). If we relax the property of f from being isometry, we can define similar actions on an arbitrary triangle. In fact there are infinitely many actions of D_3 on an arbitrary triangle, described by the following setting.

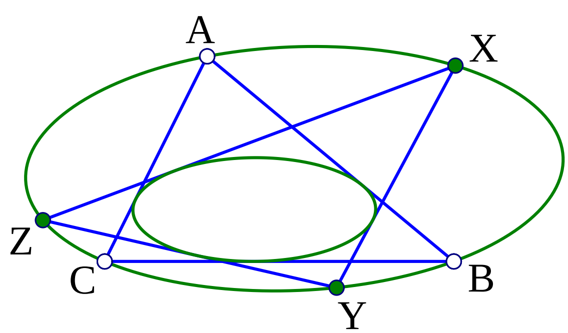


Figure 1. Projectivity preserving a conic

It is well known that given six points A, A', B, B', C, C' on a conic c, there is a unique projectivity preserving c and mapping A to A', B to B' and C to C'. Taking A', B', C' to be permutations of the set A, B, C we see that there is a group G of projectivities that permute the vertices of the triangle $t = (ABC)$ and preserve the conic c. It is not difficult to see that G is naturally isomorphic to the group of symmetries of the equilateral triangle. Thus from the algebraic point of view, the group action contains no significant information. But from the geometric point of view the situation is quite interesting. For example, fixing such a group, we can consider generalized rotations i.e. $f \in G$ of order three $f^3 = 1$, which applied to a point $X \in c$ generate an *orbital triangle* $X, Y = f(X), Z = f(f(X))$. All these orbital triangles envelope a second conic which is also invariant under the group G. For definitions, general facts on triangles, transformations and especially projectivities I refer to [1]. For special conics, circumscribed on a triangle, this setting unifies several dispersed properties and presents them under a new light.

Publication Date: October 28, 2004. Communicating Editor: Jean-Pierre Ehrmann.

I shall illustrate this aspect by applying the above method to two special cases. Then I shall discuss an exceptional, similar setting, which results by replacing the circumconic with the circumcircle of the triangle and the projectivities by Moebius transformations. The first case will be that of the exterior Steiner ellipse of the triangle.

2. Steiner dihedral group of a triangle

We start with a triangle $t = (ABC)$ and its exterior Steiner ellipse. Then we consider the projectivities that preserve this conic and permute the vertices of the triangle. First I shall state the facts. The group, which I call the *Steiner dihedral group* of the triangle, comprises two kinds of maps: involutions, that resemble to reflections, and cyclic permutations of the vertices that resemble to rotations.

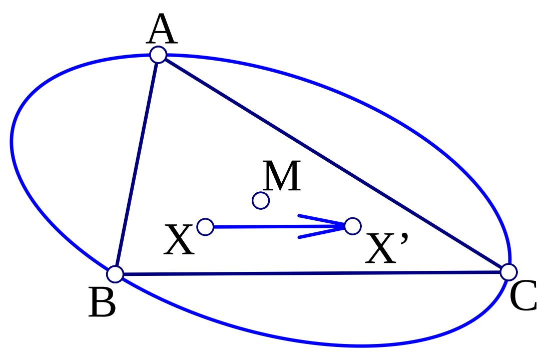


Figure 2. Isotomic conjugation

The involutions are related to the sides of the triangle and coincide with the isotomic conjugations with respect to the corresponding medians: Side a of the triangle defines an involution on the conic: $I_a(X) = Y$, where XY is parallel to the side a and bisected by the median to a. I_a has the median to a as its line of fixed points, which coincides with the conjugate diameter of a relative to the conic. The corresponding isolated fixed point (Fregier point of the involution) is the point at infinity of line a. Analogous definitions and properties have the involutions I_b, I_c.

More important seems to be the projectivity $f = I_b \circ I_a$, of order three $f^3 = 1$, that preserves the conic and cycles the vertices of the triangle. I call it the *isotomic rotation*.

As is the case with every projectivity f, preserving a conic, for all points X on c, the lines $[X, f(X)]$ envelope another conic, which in this case is the inner Steiner ellipse. By the same argument all *orbital* triangles i.e. triangles of the form $t' = (X, f(X), f(f(X)))$, are circumscribed on the inner Steiner ellipse. More precisely the following statements are valid and easy to prove:

(1) The centroid G of the triangle is the fixed point of f.

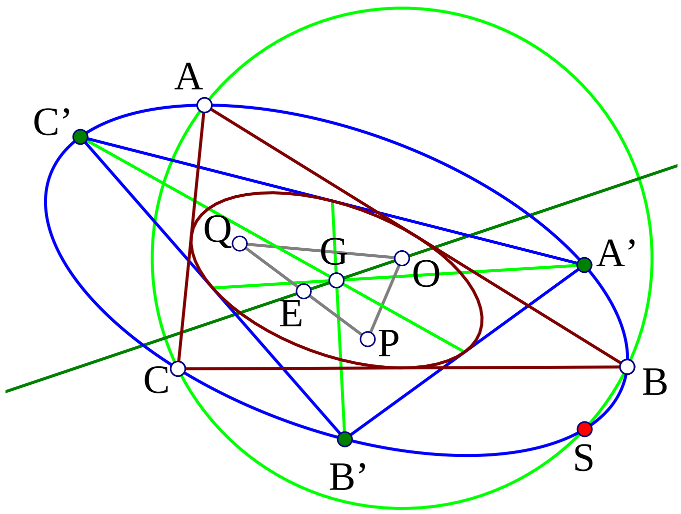


Figure 3. Isotomic rotation

(2) Every point X of the plane defines an *orbital* triangle

$$s = (X, f(X), f(f(X))),$$

which has G for its centroid.

(3) The orbital triangles s, as above, which have X on the external Steiner ellipse, are all circumscribed to the inner Steiner ellipse. They are precisely the only triangles that have these two ellipses as their external/internal Steiner ellipses.

(4) The inner and outer Steiner ellipses generate a family of homothetic conics, with homothety center the centroid G of the triangle. For every point X of the plane the orbital triangle s generated by X has the corresponding conics-family-member c, passing through X, as its outer Steiner ellipse. Besides, for all points X on c, the corresponding orbital triangles circumscribe another conics-family-member c', which is the inner Steiner ellipse of all these triangles.

(5) For a fixed orbital triangle $t = (ABC)$, the orbit of its circumcenter O, defines a triangle $u = (OPQ)$, whose median through O is the Euler line of the initial triangle t. The middle E of PQ is the center of the Euler circle of t.

(6) The trilinear coordinates of points $P = f(O)$ and $Q = f(P)$ are respectively:

$$P = \left(\frac{\sin 2C}{\sin A}, \frac{\sin 2A}{\sin B}, \frac{\sin 2B}{\sin C}\right),$$

$$Q = \left(\frac{\sin 2B}{\sin A}, \frac{\sin 2C}{\sin B}, \frac{\sin 2A}{\sin C}\right).$$

Deferring for a later moment the proofs, I shall pass now to the analogous group, of projectivities, which results by replacing the external ellipse with the circumcircle of the triangle. For a reason that will be made evident shortly I call the corresponding group the *Lemoine* dihedral group of the triangle.

3. Lemoine dihedral group of a triangle

We start with a triangle $t = (ABC)$ and its circumcircle c. Then we consider the projectivities that preserve c and permute the vertices of the triangle. There are again two kinds of such maps. Involutions, and maps of order three.

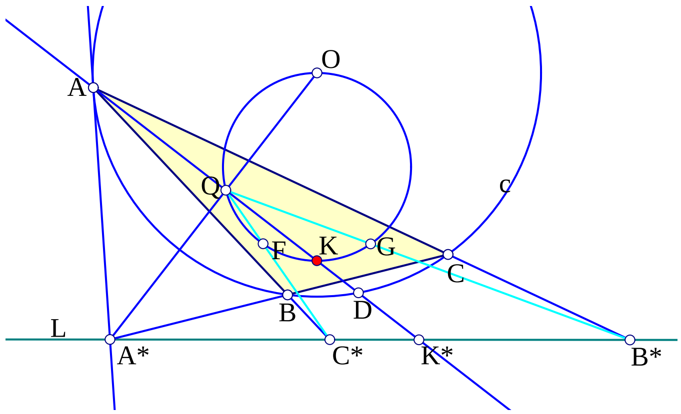


Figure 4. Lemoine reflexion

Side a of the triangle defines a projective involution $I_a(X) = X'$, by the properties $I_a(A) = A$ and $I_a(B) = C, I_a(C) = B$. Its line of fixed points, is the symmedian line AD. The corresponding isolated fixed point (Fregier point) is the pole A^* of the symmedian with respect to the circumcircle, which lies on the Lemoine axis L of the triangle. In the figure above, K is the symmedian point and Q is the projection of the circumcenter on the symmedian AD (is a vertex of the second Brocard triangle of t). From the invariance of cross-ratio and the fact that I_a maps L to itself, follows that $(C^*B^*K^*A^*) = 1$, hence the symmedian bisects the angle B^*QC^*. Joining Q with B^*, C^* we find the intersections F, G of these lines with the Brocard circle (with diameter OK). Below (in §6) we show that F, G coincide with the Brocard points of the triangle.

I_a could be called the *Lemoine reflexion* (on the symmedian through A). Analogous is the definition and the properties of the involutions I_b and I_c, corresponding to the other sides of the triangle.

More important seems to be the projectivity $f = I_b \circ I_a$, of order three $f^3 = 1$, which preserves the circumcircle and cycles the vertices of the triangle. I call it the *Lemoine rotation*.

As before, for all points X on c, the lines $[X, f(X)]$ envelope another conic, which in this case is the Brocard ellipse c' of the triangle t. By the same argument all *orbital* triangles i.e. triangles of the form $t' = (X, Y = f(X), Z = f(f(X)))$,

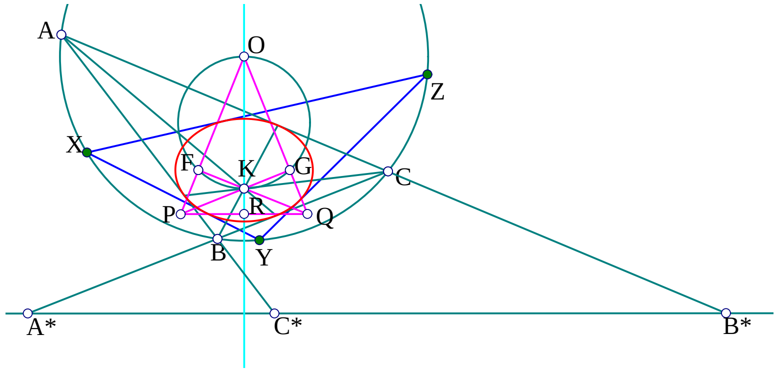


Figure 5. Lemoine rotation

are circumscribed on the Brocard ellipse. More precisely the following statements are valid and easy to prove:

(1) f leaves invariant each member of the family of conics generated by the circumcircle and the Brocard ellipse of t. In particular the Lemoine axis of t remains invariant under f, and permutes points A^*, B^*, C^*.

(2) The symmedian (or Lemoine) point K of the triangle is the fixed point of f.

(3) Every point X of the circle c defines an *orbital* triangle

$$s = (X, f(X), f(f(X))),$$

which has K as symmedian point.

(4) The orbital triangles s, as above, which have X on c, are all circumscribed to the Brocard ellipse c'. They are precisely the only triangles that have c and c' as circumcircle and Brocard ellipse, respectively.

(5) For a fixed orbital triangle $t = (ABC)$, the orbit of its circumcenter O, defines a triangle $u = (OPQ)$, whose median through O is the Brocard axis of the initial triangle t.

(6) The triangle u is isosceles and symmetric on the Brocard axis. The feet G, F of the altitudes of u from P and Q, respectively, coincide with the Brocard points of t.

(7) The triangles u, $u' = (PRF)$ and $u'' = (QRG)$ are similar. The similarity ratio of the two last to the first is equal to the sine of the Brocard angle.

Deferring once again the proofs at the end (§6), I shall pass to a third group, using now inversions instead of projectivities. For a reason that will be made evident shortly I call the corresponding group the *Brocard* dihedral group of the triangle.

4. Brocard dihedral group of a triangle

Once again we start with a triangle $t = (ABC)$ and its circumcircle c. Then we consider the Moebius transformations that permute the vertices of t. It is true that through such maps the sides are not mapped to sides. We do not have proper maps of the triangle's set of points onto itself, but we have a group that permutes its vertices, is isomorphic to D_3 and, as we will see, has intimate relations with the previous one and the geometry of the triangle.

Everything is based on the well known fact that a Moebius transformation is uniquely defined by prescribing three points and their images. Thus, fixing a vertex, A say, of the triangle and permuting the other two, we get a Moebius involution, I_a say. Analogously are defined the other two involutions I_b and I_c. I call them the *Brocard reflexions* of the triangle. Two of them generate the whole group. By the well known property of Moebius transformations, we know that all of them preserve the circumcircle c.

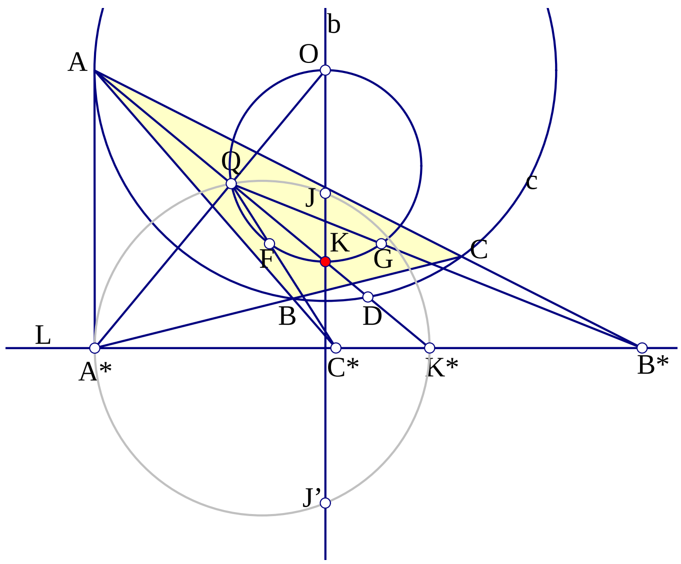


Figure 6. Brocard reflexion

I cite some properties of I_a that are easy to prove:

(1) On the points of the circumcircle the Brocard reflexion I_a coincides with the corresponding Lemoine reflexion.
(2) I_a leaves invariant each member of the bundle of circles through its fixed points A and D (D being the intersection of the symmedian from A with the circumcircle).
(3) I_a leaves invariant each member of the bundle of circles that is orthogonal to the previous one (i.e. the circles which are orthogonal to the symmedian AD and the circumcircle).
(4) In particular I_a leaves invariant the symmedian from A and maps the symmedian point K to the intersection K^* of the Lemoine axis with that symmedian.
(5) I_a permutes the circles of the bundle generated by the circumcircle and the Lemoine axis of t. The same happens with the orthogonal bundle to the previous one, which is the bundle generated by the Apollonian circles of t.

(6) I_a interchanges the circumcenter O with the pole A^* of the symmedian at A. It maps also the Brocard axis b onto the circle through the isodynamic points and A^*.
(7) All the circles through O, Q are mapped by I_a to lines through A^*. In particular the Brocard circle is mapped to the Lemoine axis.
(8) The line AB is mapped by I_a to the circle through A, C, tangent to this line at A.
(9) I_a maps the Brocard points F, G to the intersection points B^*, C^* of the sides AC and AB with the Lemoine axis respectively.

We pass now to the Moebius transformation that recycles the vertices of the triangle $t = (ABC)$. It is the product of two Brocard reflexions $f = I_b \circ I_a$. It is of order three: $f^3 = 1$ and I call it the *Brocard rotation*. The geometric properties of this transformation are related to the so called *characteristic parallelogram* of it. This is generally defined, for every Moebius transformation (may be degenarated), as the parallelogram whose vertices are the two fixed points and the poles of f and of its inverse f^{-1}. A short discussion of this parallelogram will be found in §8. Here are the main properties of our Brocard Rotation.

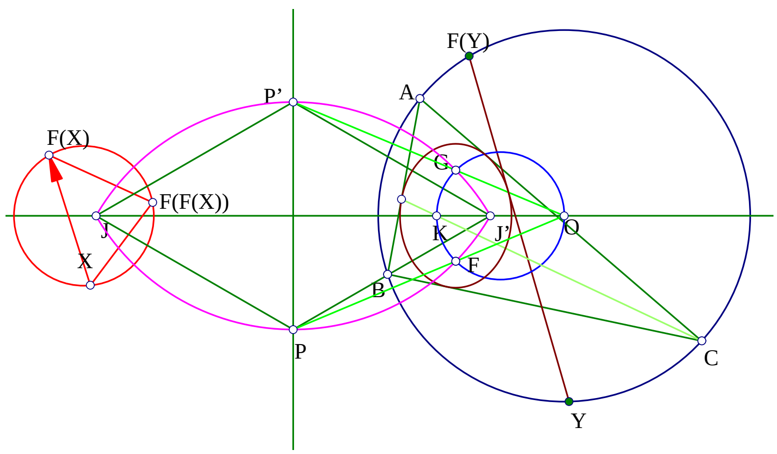


Figure 7. Brocard rotation

(10) On the points of the circumcircle c of t the Brocard Rotation coincides with the corresponding Lemoine rotation.
(11) The characteristic parallelogram of f is a rhombus with two angles of measure $\pi/3$. The vertices at these angles are the fixed points of f. They also coincide with the isodynamic points of the triangle. The other vertices of the parallelogram (angles $2\pi/3$) coincide with the inverses of the Brocard points with respect to the circumcircle.
(12) f leaves invariant every circle of the bundle of circles, generated by the circumcircle of t and its Brocard circle (circle throuch circumcenter and Brocard points).

(13) All circles of the bundle, which is orthogonal to the previous, pass through the isodynamic points J, J' of t. Each circle c of this bundle is mapped to a circle c' of the same bundle, which makes an angle of $\pi/6$ with c. In particular the Apollonian circles of the triangle are cyclically permuted by f.

(14) Every point X of the plane defines an *orbital* triangle

$$s = (X, f(X), f(f(X))),$$

which shares with t the same isodynamic points J, J', hence Brocard and Lemoine axes. Conversely, every triangle whose isodynamic points are J and J' is an orbital triangle of f.

(15) The Brocard points of all the above orbital triangles s fill the two $\pi/3$-angled arcs JPJ' and $JP'J'$ on the two circles with centers at the poles P, P' of f, joining the isodynamic points J and J'.

(16) The orbital triangles s, as above, which have X on the circumcircle of t, are all circumscribed to the Brocard ellipse c' of t. They are precisely the only triangles that have c and c' as their circumcircle and Brocard ellipse, respectively.

(17) The other two points of the orbital triangle of the circumcenter O, are the two Brocard points of t.

(18) The second Brocard triangle $A_2B_2C_2$ is an orbital triangle of f.

5. Proofs on Steiner

A convenient method to define the two Steiner ellipses of a triangle, is to use a projectivity F, that maps the vertices of an equilateral triangle $t' = (A'B'C')$ onto the vertices of an arbitrary triangle $t = (ABC)$ and the center of t' onto the centroid of t. As is well known, prescribing four points and their images, uniquely determines a projectivity of the plane. Thus the previous conditions uniquely determine F (up to permutation of vertices). Let a', b' be the circumcircle and incircle, correspondingly of t'. Their images $a = F(a')$ and $b = F(b')$ are correspondingly the exterior and interior Steiner ellipses of t.

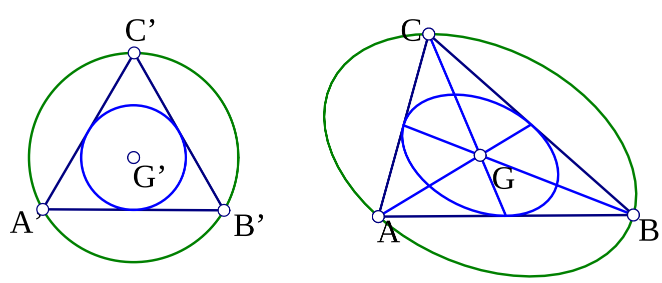


Figure 8. Creating the two Steiner ellipses of a triangle

From the general properties of projectivities result the main properties of Steiner's ellipses of the triangle t:

(1) From the invariance of cross-ratio, and the fact that F preserves the middles of the sides, follows that F preserves also the line at infinity. Thus, the images of circles are ellipses.
(2) The same reason implies, that the tangent to the outer ellipse at the vertex is parallel to the opposite side of the triangle.
(3) The same reason implies, that the centers of the two ellipses coincide with G and the ellipses are homothetic with ratio 2, with respect to that point.
(4) The invariance of cross-ratio implies also, that the *Steiner involution*, defined as the projectivity that fixes A and permutes B, C, coincides (on points of the conic) with the conjugation $X \mapsto Y$, where XY is parallel to a. It leaves the line at infinity fixed and coincides with the isotomic conjugation with respect to the median from A. The median being a conjugate direction to a with respect to the conic.
(5) The Fregier point of the involution I_a is the point at infinity of line $a = BC$ and the line of fixed points of I_a is the median from A.

The *isotomic rotation* is the projectivity $f = I_b \circ I_a$. One sees immediately that it has order three: $f^3 = 1$, that preserves the conic and cycles the vertices of the triangle. Besides it fixes the centroid G and cycles the middles of the sides. All the statements of §2, about orbital triangles, follow immediately from the previous facts and the property of f, to be conjugate, via F, to a rotation by $2\pi/3$ about G. For the statement on the particular orbital triangle of the circumcenter O of t, it suffices to do an easy calculation with trilinears. Actually the Euler line passes also through the symmetric O' of O with respect to G, which is one of the intersection points of the two conics of the figure below.

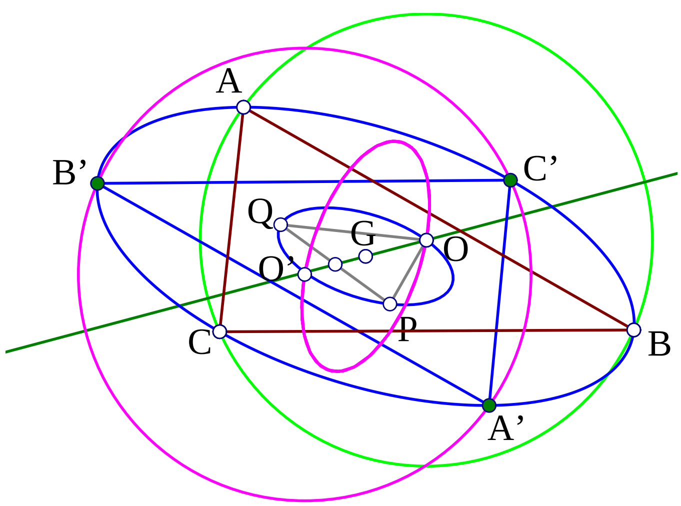


Figure 9. Circumcenters of orbital triangles

One of the conics is the member of the conics-family passing through O. The other ellipse has the same axes with the previous one and is the locus of the circumcenters of orbital triangles $u = (X, f(X), f^2(X))$, for X on the outer Steiner ellipse. O' is the circumcenter of the triangle $t' = (A'B'C')$ which is symmetric to t with respect to G.

6. Proofs on Lemoine

A convenient method to define the Brocard ellipse of a triangle, is to use a projectivity F, that maps the vertices of an equilateral triangle $t' = (A'B'C')$ onto the vertices of an arbitrary triangle $t = (ABC)$ and the center of t' onto the symmedian point of t. These conditions uniquely determine F (up to permutation of vertices).

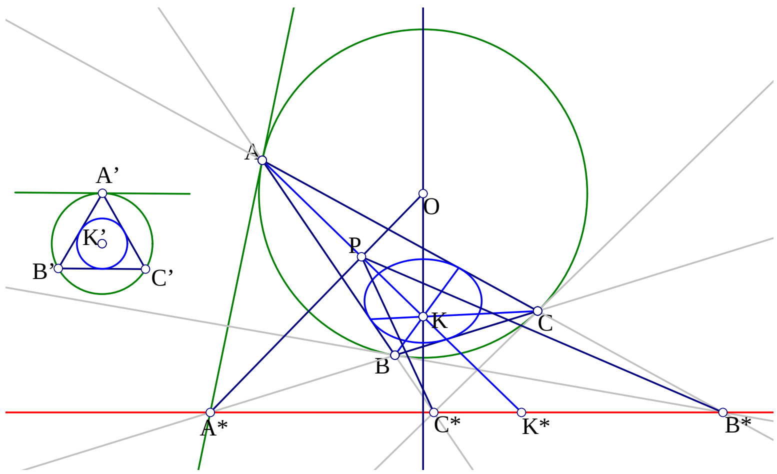


Figure 10. Creating the Brocard ellipse of a triangle

F maps the incircle of t' to the Brocard ellipse of t and the circumcircle of t' to the circumcircle of t. To see the later, notice that F preserves the cross ratio of a bundle of four lines through a point. Now the tangent of t' at A', the two sides $A'B'$, $A'C'$ and the median of t' from A' form a harmonic bundle. The same is true for the tangent of t at A the two sides AB, AC and the symmedian from A. Thus F maps the tangent of t' at A' to the tangent of t at A, and analogous properties hold for the other vertices. This forces the circumcircle of t to coincide with the image, under F, of the circumcircle of t'. The other statement, on the Brocard ellipse, follows from the fact, that this ellipse is characterized as the unique conic tangent to the sides of the triangle at the traces of the symmedians from the opposite vertices. The main properties of the *Lemoine reflexion* I_a result from the fact that it is conjugate, via F, to the reflexion of t' with respect to its median from A'. Thus the line of fixed points of I_a coincides with the symmedian from A. The intersection point A^* of the line BC with the tangent at A is the image, via F, of the point at infinity of the line $B'C'$. Analogous properties hold for the points

B^* and C^*. Since these points are known to be on the Lemoine axis, this implies that the line at infinity is mapped, via F, to the Lemoine axis of the triangle. All the lines through A^* remain invariant under I_a, hence this point coincides with the Fregier point of the involution.

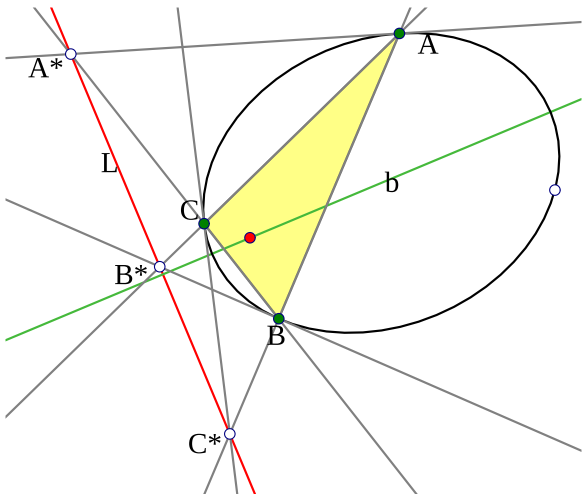


Figure 11. Orbital triangles

The *Lemoine rotation* is the projectivity $f = I_b \circ I_a$, of order three $f^3 = 1$, that preserves the circumcircle and cycles the vertices of the triangle. Besides it fixes the symmedian point K of the triangle and cycles the symmedians. f is conjugate, via F, to a rotation by $2\pi/3$ about K'. f leaves invariant the family of conics generated by the circumcircle and the Brocard ellipse. This family is the image, under F, of the bundle of concentric circles about K'. In particular the line at infinity is mapped onto the Lemoine axis of t, which is also invariant under f. The conics of the family, left invariant by f, are all symmetric with respect to the Brocard diameter b. Besides all orbital triangles $s = (A = X, B = f(X), C = f(f(X)))$ of f have the property shown in the above figure.

In this figure the point A^* is the intersection point of BC and the tangent at A of the conic-family member passing through A. Analogously are defined , B^* and C^*. The three points lie on the Lemoine axis L of t and are cyclically permuted by f. The proof is a repetition of the argument on harmonic bundles at the beginning of the paragraph. This has though a nice consequence. First, if A is on the Brocard diameter b of t, which is the symmetry axis of all the conics of the invariant family, then the coresponding orbital triangle s is symmetric. Besides the lines AB and AC pass through two fixed points C^* and B^* of L respectively. In fact, in that case, the tangent at A meets L at its point at infinity. Consequently the corresponding BC is parallel to L and s is isosceles. In addition, since f cycles the corresponding points A^*, B^*, C^*, the two last points are the image of the point at infinity of L, under f and its image respectively. Thus they are independent of the position of A on b.

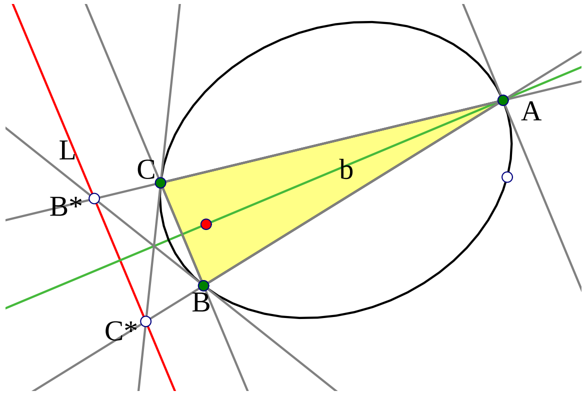


Figure 12. The orbit of the point at infinity of L

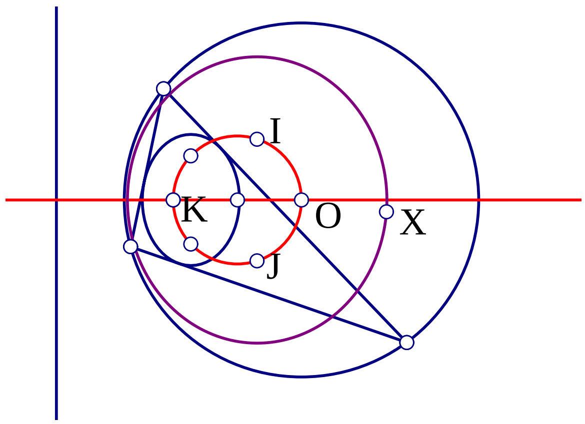


Figure 13. Focal points of the conics

Below B^*, C^* will be identified with the inverses of the Brocard points of t with respect to the circumcircle. Notice that the Brocard points of t are the focal points of the Brocard ellipse and they lie on the Brocard circle with diameter OK. It is well known, that in general the focal points of a family of conics lie on certain cubics. For a reference, see our paper with Apostolos Thoma [2], where we investigated such cubics from a geometric point of view. In the present case the family consists of conics that are symmetric with respect to the Brocard axis and the cubic must be reducible and equal to the product of a circle and a line. In fact a calculation shows that the cubic is the union of the Brocard cirlce and the Brocard axis. All points X inside the circumcircle of t define family members whose focal points are on the Brocard circle. All points X outside the circumcircle of t define family members whose focal points are on the Brocard axis. For X varying on b there are two positions, where the legs of the orbital isosceli contain the foci of the corresponding conic-member through X. One of these points is the center O of the circumcircle. Notice that the family of conics is generated also from the

Lemoine axis (squared) and the circumcircle. This representation makes simpler the computations of a proof of the last statements of §3, on the orbital triangle of the circumcenter. Another geometric proof of this fact may be derived from the arguments of the two next paragraphs.

7. Proofs on Brocard

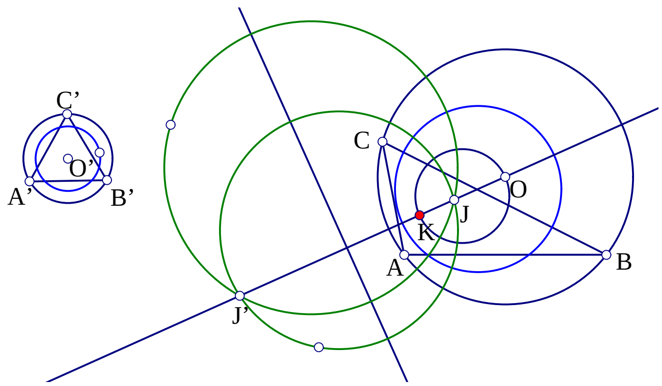


Figure 14. The isodynamic bundles of the triangle

In contrast to projectivities that need four, Moebius transformations are determined completely by three pairs of points. Imitating the procedures of the previous paragraphs, we define the Moebius transformation F that sends the vertices of an equilateral triangle $t' = (A'B'C')$ to the vertices of an arbitrary triangle $t = (ABC)$. Since Moebius transformations, preserve the set of circles and lines, the circumcircle of t' is mapped on the circumcircle of t. Moreover the bundle of concentric circles to the circumcircle of t' maps to the bundle Σ of circles generated by the circumcircle of t and its Lemoine axis. Below I call Σ the *Brocard bundle* of t. This is a hyperbolic bundle with focal (or limiting) points coinciding with the isodynamic points J, J' of t. Since F is conformal it maps the lines from O' to the circle bundle that is orthogonal to the previous one. All circles of this bundle pass through the isodynamic points. All these facts result immediately from the fact that the altitudes of t' map onto the corresponding Apollonian circles of t. This in turn follows from the invariance of the complex cross ratio, by considering the cross ratio of the vertices $(ABCD) = (A'B'C'D') = 1$. D on the circumcircle is uniquely determined by this condition and coincides with the trace of the symmedian from A. The conformality of Moebius transforms implies also that the Apollonian circles meet at J at angles equal to $\pi/3$. Below I call the bundle Σ' of circles through J, J' the *Apollonian bundle* of t. Now to the proofs of the statements in §4.

The first statement (1) is a general fact on Moebius transformations preserving a circle c. Given three pairs of points on c, there is a unique Moebius f and a unique projectivity f' preserving c and corresponding the points of the pairs. f and f' coincide on points $X \in c$. In fact, taking cross ratios $(ABCX)$ in complex or by projecting the points on a line, from a fixed point, $Z \in c$ say, gives the same result. The same is true for the images $(A'B'C'X')$ under both transformations, thus the images of X under f and f' coincide.

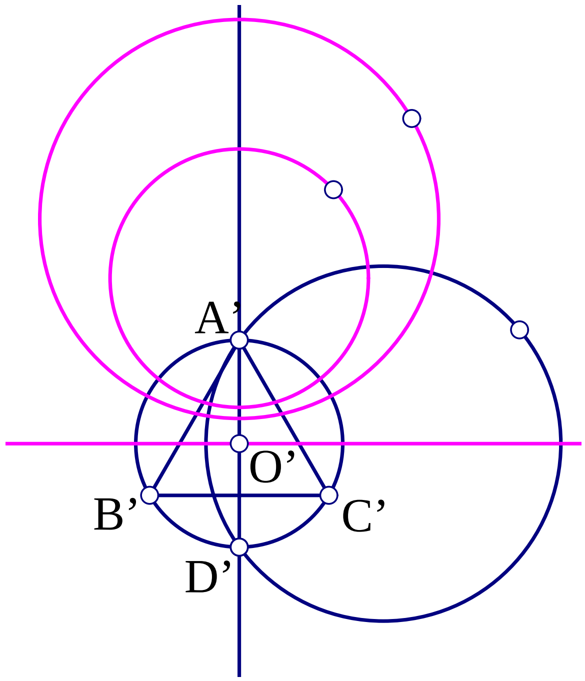


Figure 15. The I'_a invariant bundles

The next two statements (2,3) follow immediately from the fact that I_a is conjugate, via F, to the Moebius transformation I'_a fixing A', D' and mapping B' to C'. A short calculation shows that I'_a preserves the circles passing through A', D' and also preserves the circles of the orthogonal bundle to the previous one. These two I'_a-invariant bundles, map under F to the corresponding I_a-invariant bundles of the statements. The previous argument shows also that the bundle of concentric circles at O' is permuted by I'_a, consequently the same is true for the bundle of lines through O'. But these two bundles map under F to the main bundles of our configuration, the Brocard Σ and the Apollonian Σ' correspondingly. This proves also statement (4).

Next statement (5) follows from the invariance of cross ratio, along the I_a-invariant symmedian from A, and the fact that the Lemoine axis is the polar of the symmedian point with respect to the circumcircle. A consequence of this, taking into account that I_a permutes the Brocard bundle, is that the Brocard circle of t maps via I_a to the Lemoine axis.

From the previous considerations, on the Brocard and Apollonian bundles, follows that I_a does the following: (a) It interchanges O, P, (b) sends Q (the projection of the circumcenter on the symmedian) at the point at infinity, (c) maps the circles with center at Q to circles with the same property, (d) maps the lines e

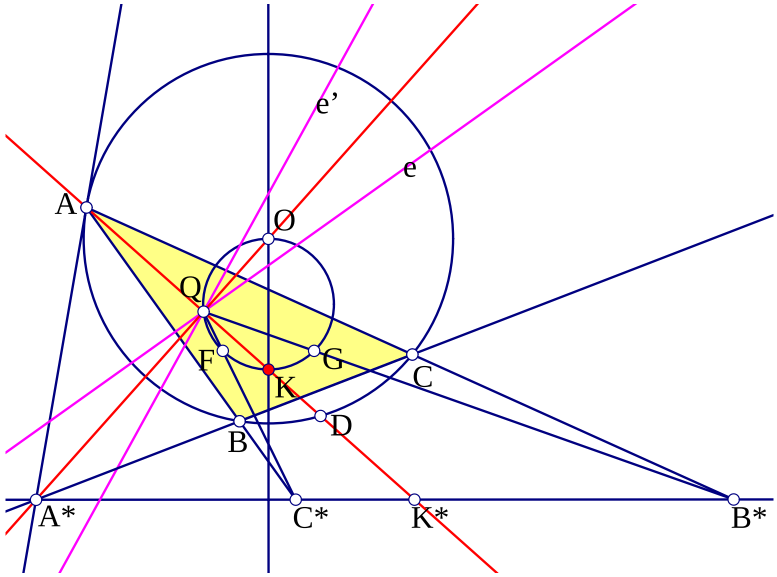


Figure 16. I_a on Brocard points

through Q to their symetrics e' with respect to PQ (or the symmedian at A). As a consequence I_a maps the line QB^* to the line QC^* and points G, F onto C^*, B^* correspondingly. Consider now the image of line AB via I_a. By the properties just described, points A, B, C^* are mapped onto A, C, G correspondingly. Also the point at infinity is mapped onto Q, thus the line maps to a circle r passing through the points (A, Q, C, G). It is trivial to show that the circle through the points (A, Q, C) is tangent to line AB at A. This identifies G with one of the two Brocard points of t. Statements (6-10) follow immediately from the previous remarks. Before to proceed to the proofs of the remaining statements of §4, let us review some facts about the characteristic parallelograms of Moebius transformations.

8. Characteristic parallelogram

For proofs of properties of Moebius transformations and their characteristic parallelogram I refer to Schwerdtfenger [3]. The characteristic parallelogram of a Moebius transformation f has one pair of opposite vertices coinciding with the fixed points of f, the other pair of vertices coinciding with the poles of f and f^{-1} respectively. The parallelogram can be degenerated or have infinite sides. It characterizes completely f, when we know which vertices are the fixed points and which are the poles. In the image below F, F' are the fixed points of f, P is its pole and P' is the pole of f^{-1}. Triangles zFP, $Fz'P'$ and $zz'F'$ are similar in that orientation. This defines the recipe by which we construct geometrically $z' = f(z)$.

Moebius transformations f permute the bundle Σ of circles which pass through their fixed points F, F'. Each circle a of Σ is mapped to a circle a' of the same

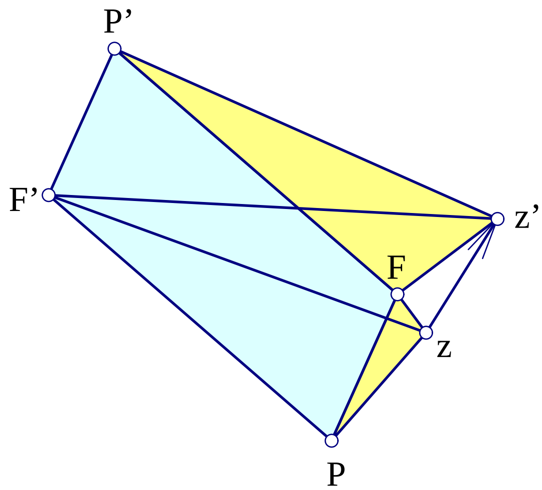


Figure 17. Building the image $z' = f(z)$

bundle, such that the angle at F is the same with the angle of the characteristic parallelogram at the pole P. In some sense the circles of Σ are *rotated* about the fixed points of f. The picture is complemented by the bundle Σ', which is orthogonal to the previous one. This is also permuted by f.

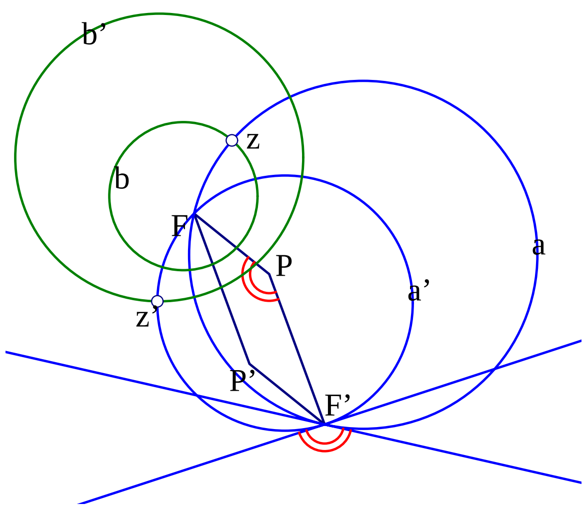


Figure 18. Characteristic bundles of a Moebius transformation

The *elliptic* Moebius transformations are characterized by their property to leave invariant a circle. The circle then belongs to the bundle Σ', whose all members remain also invariant by f. In fact, in that case f is conjugate to a rotation, and by this conjugation the two bundles correspond to the set of concentric circles about the rotation-center (Σ') and the set of lines through the rotation-center (Σ). In addition the parallelogram is then a rhombus.

Now to the proofs of the properties of Brocard rotations f of §4, preserving the notations introduced there. Since these transformations preserve the circumcircle of the triangle t, they are elliptic. Since they are conjugate, via the map F, to Rotations by $2\pi/3$, their characteristic parallelogram is a rhombus with an angle (at the pole) equal to $2\pi/3$. From the properties of F we know that the fixed points of f coincide with the isodynamic points of the triangle and the Apollonian circles are members of the bundle Σ, permuted by f. The Lemoine axis, being axis of symmetry of the isodynamic points, contains the other vertices of the rhombus. The other bundle Σ', of circles left invariant by f, coincides with the bundle generated by the circumcircle and the Lemoine axis. Later bundle contains the Brocard circle. The statement on orbital triangles follows from the corresponding property of Lemoine rotations, since the two maps coincide on the circumcircle.

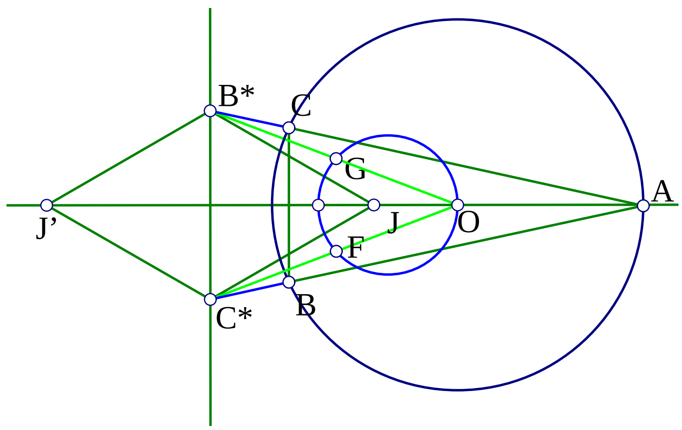


Figure 19. Projections of Brocard points on Lemoine axis

The fact that the circumcenter O, together with the two Brocard points F, G build an orbital triangle of f, follows now easily from the fact that $f = I_b \circ I_a$. In fact, from our discussion, on Brocard reflexions, we know that I_a maps the circumcenter onto A^*, the intersection of side $a = BC$ with the Lemoine axis. Then I_b, as shown there, maps A^* to one Brocard point. A similar argument proves that applying again f we get the other Brocard point. Analogously one proves that the second Brocard triangle is also an orbital triangle of f. All the statements (10-19) follow from the previous remarks.

Especially the statement about the fact that P, P' are the projections, from the circumcenter O, of the Brocard points, on the Lemoine axis, follows also easily from our arguments. In fact, the equibrocardian isosceles triangle $t = (ABC)$ of the previous picture, is also an orbital triangle of the corresponding Lemoine rotation. From there we know that its legs pass through the fixed points B^*, C^*. These points are identified as the images of the point at infinity of the Lemoine axis under the Lemoine Rotation. But this rotation coincides also with the Brocard rotation on that axis. This identifies P, P' with the other vertices of the characteristic parallelogram.

9. Remarks

(1) For every point P of the triangle's plane (e.g. some triangle center), one can define a projectivity F analogous to the one used in the two examples and establishing the conjugacy of the group G with the dihedral D_3. The projectivity F is required to map the vertices of the equilateral triangle to the vertices of the arbitrary triangle t. In addition, it is required to map the center P' of the equilateral to the selected point P. These conditions completely determine F and there are several phenomena, generalizing the previous examples. The bundle of circles centered at P' maps to a family Σ of conics. One of these conics, $c \in \Sigma$, circumcscribes t, one other being inscribed and touching the triangle's side at the feet of the cevians from P. One can define analogously the action of D_3, preserving c and permutting the vertices of the triangle. The properties of this action, reflect naturally properties of the point P with respect to triangle t. The action leaves invariant the whole family Σ.

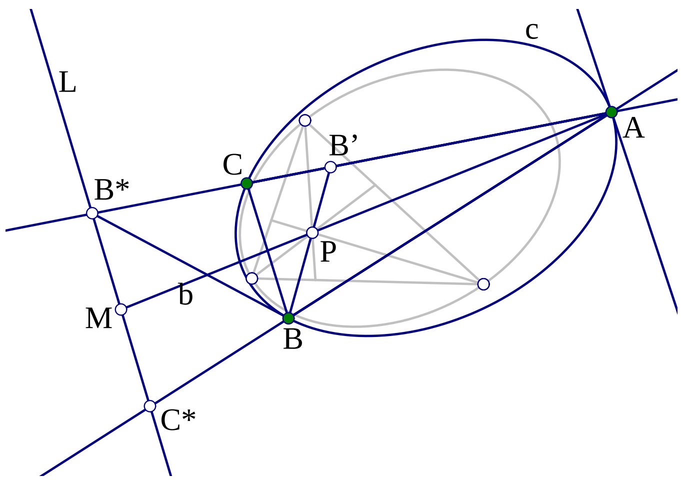


Figure 20. The limit points of the conics-family

Also, using essentially the same arguments as in the examples, one can show, that the line at infinity maps via F to the *trilinear polar* of P. The trilinear polar being then a singular member (double line) L of Σ. Besides all orbital triangles $t = (ABC)$ which have a side, BC say, parallel to this line, have the other two sides passing through two fixed points C^*, B^* of L, whereas the tangent to the member-conic c circumscribing the triangle at the other point A of the triangle is also parallel to L. The line $b = PA$, passes through the middle M of B^*C^* and is the conjugate direction to L, with respect to every conic of the family. In this case also the corresponding projective rotation f recycles points B^*, C^* and the point

at infinity of line L.
(2) The data L, P and the location of points B^*, C^* on L uniquely determine the invariant family of conics Σ and the related orbital triangles. In fact, once B^*, C^* are known, the line MP, where M is the middle of B^*, C^*, is conjugate to the direction of L, with respect to all the conics of Σ. A point A on this line can be determined, so that a special orbital triangle ABC can be constructed from the previous data. In fact, point B' on AB^* satisfies the condition that the four points $(ACB'B^*) = 1$, form a harmonic ratio. A triangle ABC is immediately constructed, so that BB^* and BB' are its bisectors and BC is parallel to L. Consequently the projectivity F can be defined, and from this the whole family is also constructed.

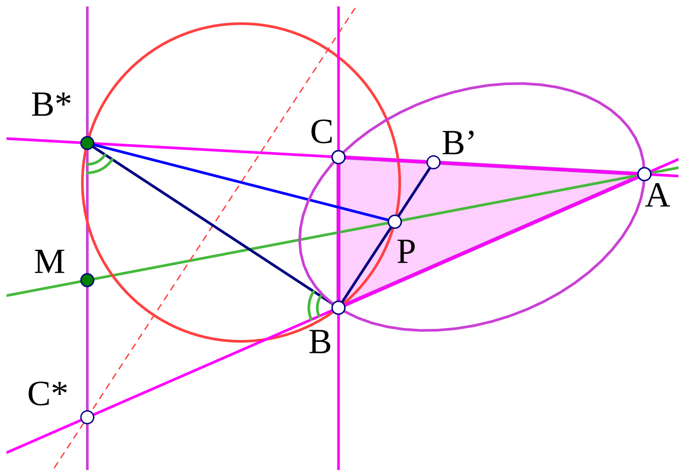


Figure 21. Special orbital triangle determined from B^*, C^*, P

(3) The previous considerations give a nice description of the set of triangles having a given line L and a given point $P \notin L$ as their trilinear polar with respect to P. They are orbital triangles of actions of the previous kind and they fall into families. Each family is characterized by the location of its limit points B^*, C^* on L.
(4) An easy calculation shows that the focal points of the members of Σ describe a singular cubic, self-intersecting at P. Besides the asymptotic line of this cubic coincides with b. When P is the Symmedian-point, the corresponding cubic coincides with the reducible one, consting of the Brocard circle and the Brocard line.
(5) Inscribed conics and corresponding actions of D_3, permutting their contact points with the sides of the triangle, could be also considered. They offer though nothing new, since they are equivalent to actions of the previous kind.
(6) In all the above groups of projectivities, the rotations are identical to the projectivities fixing the point P and cycling the vertices. One could start from such a projectivity and show the existence and invariance of the resepective family of conics. I prefer however the variant with the circumconics which introduces them

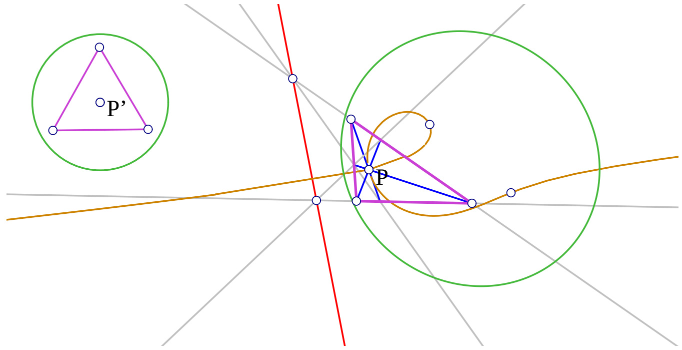


Figure 22. The focal cubic of the invariant family

into the play right from the beginning.
(7) The Brocard action is a singularium. It does not fit completely into the framework of circumconics and projectivities. As we have seen however, it has a close relationship to the Lemoine dihedral group. On Brocard Geometry there is an alternative exposition by John Conway [4], described in a letter to Hyacinthos .
(8) Finally a comment on the many figures used. They are produced with *EucliDraw*. This is a program, developed at the University of Crete, that does quickly the job of drawing interesting figures. It has many tools that do complicated jobs, reflecting the fact that it uses a conceptual granularity a bit wider than the very basic axioms. I am quite involved in its development and hope that other geometers will find it interesting, since it does quickly its job (sometimes even correctly), and new tools are continuously added. The program can be downloaded and tested from *www.euclidraw.com*.

References

[1] M. Berger, *Geometry* I, II, Springer 1991.
[2] P. Pamfilos and A. Thoma, Apollonian Cubics, *Math. Mag.*, 72 (5) (1999) 356–366.
[3] H. Schwerdtfenger, *Geometry of Complex Numbers*, Dover 1979.
[4] J. H. Conway, Letter to Hyacinthos.

Paris Pamfilos: Department of Mathematics, University of Crete, Crete, Greece
E-mail address: `pamfilos@math.uoc.gr`

Forum Geometricorum
Volume 4 (2004) 177–198.

FORUM GEOM
ISSN 1534-1178

Generalized Mandart Conics

Bernard Gibert

Abstract. We consider interesting conics associated with the configuration of three points on the perpendiculars from a point P to the sidelines of a given triangle ABC, all equidistant from P. This generalizes the work of H. Mandart in 1894.

1. Mandart triangles

Let ABC be a given triangle and $A'B'C'$ its medial triangle. Denote by Δ, R, r the area, the circumradius, the inradius of ABC. For any $t \in \mathbb{R} \cup \{\infty\}$, consider the points P_a, P_b, P_c on the perpendicular bisectors of BC, CA, AB such that the signed distances verify $A'P_a = B'P_b = C'P_c = t$ with the following convention: for $t > 0$, P_a lies in the half-plane bounded by BC which does not contain A. We call $\mathbf{T}_t = P_aP_bP_c$ the t-*Mandart triangle* with respect to ABC. H. Mandart has studied in detail these triangles and associated conics ([5, 6]). We begin a modernized review with supplementary results, and identify the triangle centers in the notations of [4]. In the second part of this paper, we generalize the Mandart triangles and conics.

The vertices of the Mandart triangle $\mathbf{T}_t$, in homogeneous barycentric coordinates, are

$$\begin{aligned} P_a &= -\,ta^2 : a\Delta + tS_C : a\Delta + tS_B, \\ P_b &= b\Delta + tS_C : -tb^2 : b\Delta + tS_A, \\ P_c &= c\Delta + tS_B : c\Delta + tS_A : -tc^2, \end{aligned}$$

where

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2}.$$

Proposition 1 ([6, §2]). *The points P_a, P_b, P_c are collinear if and only if* $t^2 + Rt + \frac{1}{2}Rr = 0$, i.e.,

$$t = \frac{R \pm \sqrt{R^2 - 2Rr}}{2} = \frac{R \pm OI}{2}.$$

The two lines containing those collinear points are the parallels at X_{10} (Spieker center) to the asymptotes of the Feuerbach hyperbola.

Publication Date: November 12, 2004. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this paper.

In other words, there are exactly two sets of collinear points on the perpendicular bisectors of ABC situated at the same (signed) distance from the sidelines of ABC. See Figure 1.

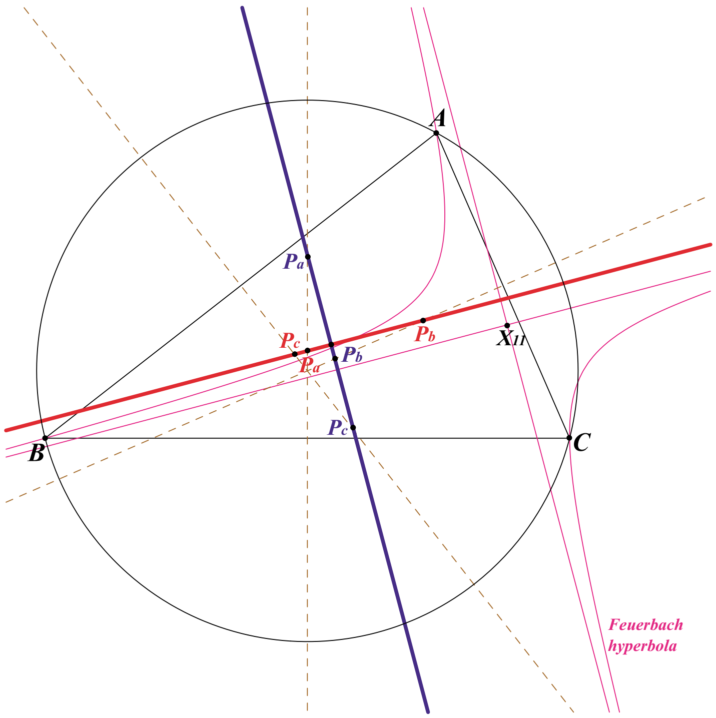


Figure 1. Collinear P_a, P_b, P_c

Proposition 2. *The triangles ABC and $P_aP_bP_c$ are perspective if and only if*
(1) $t = 0$: $P_aP_bP_c$ is the medial triangle, or
(2) $t = -r$: P_a, P_b, P_c are the projections of the incenter $I = X_1$ on the perpendicular bisectors.

In the latter case, P_a, P_b, P_c obviously lie on the circle with diameter IO. The two triangles are indirectly similar and their perspector is X_8 (Nagel point).

Remark. For any t, the triangle $Q_aQ_bQ_c$ bounded by the parallels at P_a, P_b, P_c to the sidelines BC, CA, AB is homothetic at I (incenter) to ABC.

Proposition 3. *The Mandart triangle $\mathbf{T}_t$ and the medial triangle $A'B'C'$ have the same area if and only if either :*
(1) $t = 0$: $\mathbf{T}_t$ is the medial triangle,
(2) $t = -R$,
(3) t is solution of: $t^2 + Rt + Rr = 0$.

This equation has two distinct (real) solutions when $R > 4r$, hence there are three Mandart triangles, distinct of $A'B'C'$, having the same area as $A'B'C'$. See Figure 2. In the very particular situation $R = 4r$, the equation gives the unique

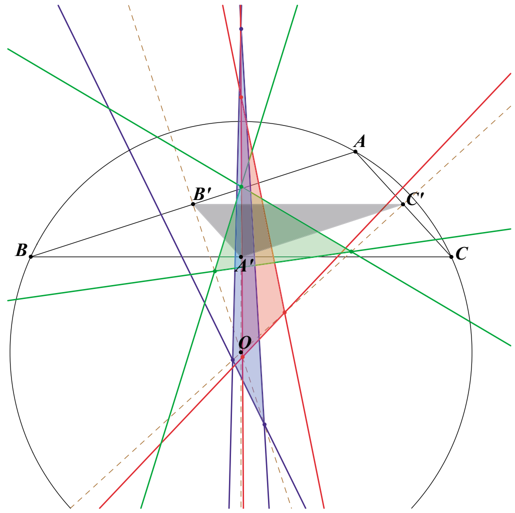


Figure 2. Three equal area triangles when $R > 4r$

solution $t = -2r = -\frac{R}{2}$ and we find only two such triangles. See Figure 3.

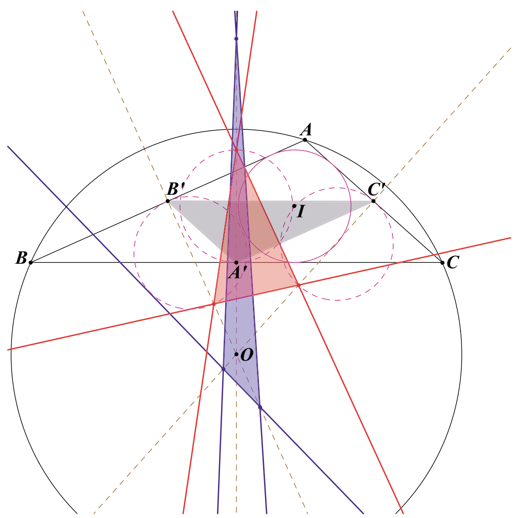


Figure 3. Only two equal area triangles when $R = 4r$

Proposition 4 ([5, §1]). *As t varies, the line P_bP_c envelopes a parabola $\mathcal{P}_a$.*

The parabola $\mathcal{P}_a$ is tangent to the perpendicular bisectors of AB and AC, to the line $B'C'$ and to the two lines met in proposition 1 above. Its focus F_a is the projection of O on the bisector AI. Its directrix ℓ_a is the bisector $A'X_{10}$ of the medial triangle. See Figure 4.

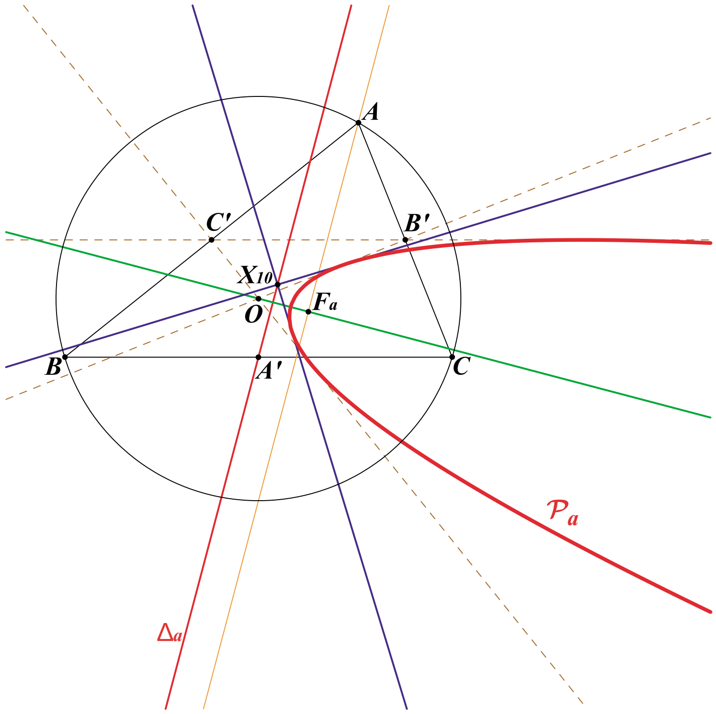


Figure 4. The parabola $\mathcal{P}_a$

Similarly, the lines P_cP_a and P_aP_b envelope parabolas $\mathcal{P}_b$ and $\mathcal{P}_c$ respectively. From this, we note the following.
(i) The foci of $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$ lie on the circle with diameter OI.
(ii) The directrices concur at X_{10}.
(iii) The axes concur at O.
(iv) The contacts of the lines P_bP_c, P_cP_a, P_aP_b with $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$ respectively are collinear. See Figure 5.

These three parabolas are generally not in the same pencil of conics since their jacobian is the union of the perpendicular at O to the line IX_{10} and the circle centered at X_{10} having the same radius as the Fuhrmann circle: the polar lines of any point on this circle in the parabolas concur on the line and conversely.

2. Mandart conics

Proposition 5 ([6, §7]). *The Mandart triangle* $\mathbf{T}_t$ *and the medial triangle are perspective at* O. *As* t *varies, the perspectrix envelopes the parabola* $\mathcal{P}_M$ *with focus* X_{124} *and directrix* X_3X_{10}.

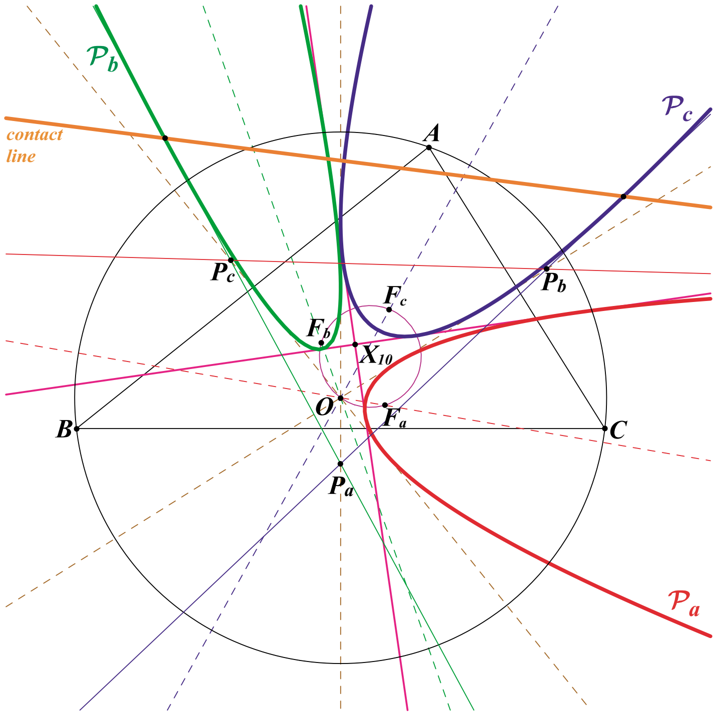


Figure 5. The three parabolas $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$

We call $\mathcal{P}_M$ the *Mandart parabola*. It has equation

$$\sum_{\text{cyclic}} \frac{x^2}{(b-c)(b+c-a)} = 0.$$

Triangle ABC is clearly self-polar with respect to $\mathcal{P}_M$. The directrix is the line X_3X_{10} and the focus is X_{124}. $\mathcal{P}_M$ is inscribed in the medial triangle with perspector

$$X_{1146} = ((b-c)^2(b+c-a)^2 : \cdots : \cdots),$$

the center of the circum-hyperbola passing through G and X_8 with respect to this triangle. The contacts of $\mathcal{P}_M$ with the sidelines of the medial triangle lie on the perpendiculars dropped from A, B, C to the directrix X_3X_{10}. $\mathcal{P}_M$ is the complement of the inscribed parabola with focus X_{109} and directrix the line IH. See Figure 6.

Proposition 6 ([5, 2, p.551])**.** *The Mandart triangle* $\mathbf{T}_t$ *and* ABC *are orthologic. The perpendiculars from* A, B, C *to the corresponding sidelines of* $P_aP_bP_c$ *are concurrent at*

$$Q_t = \left(\frac{a}{aS_A + 4\Delta t} : \cdots : \cdots\right).$$

As t *varies, the locus of* Q_t *is the Feuerbach hyperbola.*

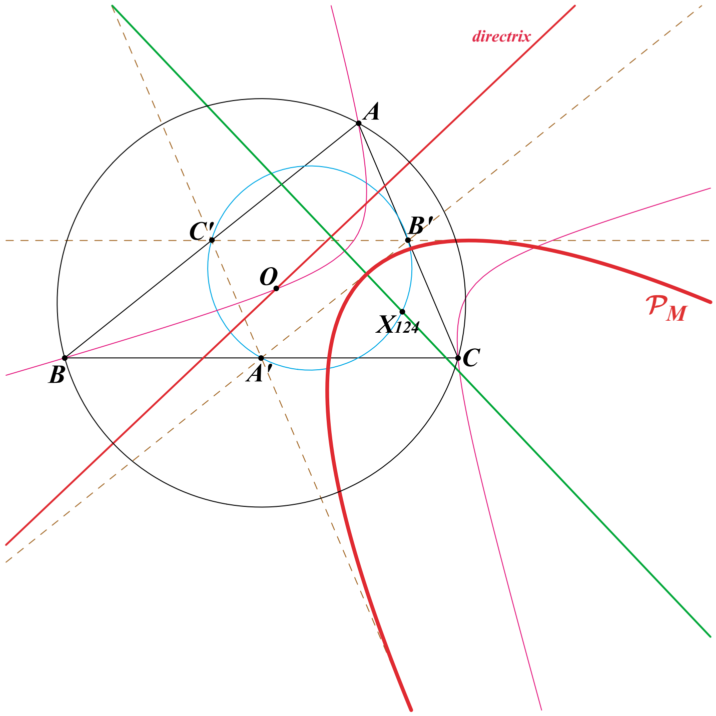


Figure 6. The Mandart parabola

Remark. The triangles $A'B'C'$ and $\mathbf{T}_t$ are also orthologic at Q'_t, the complement of Q_t.

Denote by $A_1B_1C_1$ the extouch triangle (see [3, p.158, §6.9]), *i.e.*, the cevian triangle of X_8 (Nagel point) or equivalently the pedal triangle of X_{40} (reflection of I in O). The circumcircle $\mathcal{C}_M$ of $A_1B_1C_1$ is called *Mandart circle*. $\mathcal{C}_M$ is therefore the pedal circle of X_{40} and X_{84} (isogonal conjugate of X_{40}), the cevian circumcircle of X_{189} (cyclocevian conjugate of X_8). $\mathcal{C}_M$ contains the Feuerbach point X_{11}. Its center is X_{1158}, intersection of the lines X_1X_{104} and X_8X_{40}. The second intersection with the incircle is X_{1364} and the second intersection with the nine-point circle is the complement of X_{934}. See Figure 7. The *Mandart ellipse* $\mathcal{E}_M$ (see [6, §§3,4]) is the inscribed ellipse with center X_9 (Mittenpunkt) and perspector X_8. It contains A_1, B_1, C_1, X_{11} and its axes are parallel to the asymptotes of the Feuerbach hyperbola. See Figure 7.

The equation of $\mathcal{E}_M$ is:

$$\sum_{\text{cyclic}} (c+a-b)^2(a+b-c)^2x^2 - 2(b+c-a)^2(c+a-b)(a+b-c)yz = 0$$

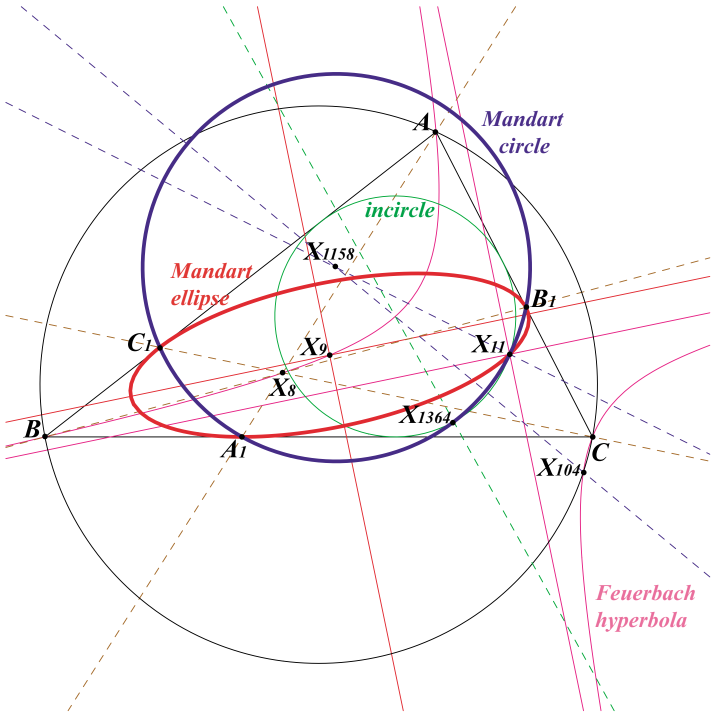


Figure 7. The Mandart circle and the Mandart ellipse

From this, we see that $\mathcal{C}_M$ is the Joachimsthal circle of X_{40} with respect to $\mathcal{E}_M$: the four normals drawn from X_{40} to $\mathcal{E}_M$ pass through A_1, B_1, C_1 and

$$F' = ((b+c-a)((b-c)^2 + a(b+c-2a))^2 : \cdots : \cdots),$$

the reflection X_{11} in X_9. [1]

The radical axis of $\mathcal{C}_M$ and the nine-point circle is the tangent at X_{11} to $\mathcal{E}_M$ and also the polar line of G in $\mathcal{P}_M$. The projection of X_9 on this tangent is the point X_{1364} we met above. Hence, $\mathcal{C}_M$, the nine-point circle and the circle with diameter X_9X_{11} belong to the same pencil of (coaxal) circles ([6, §§8,9]).

The radical axis of $\mathcal{C}_M$ and the incircle is the polar line of X_{10} in $\mathcal{P}_M$.

Proposition 7. [6, §§1,2] *The Mandart triangle* $\mathbf{T}_t$ *and the extouch triangle are orthologic. The perpendiculars drawn from* A_1, B_1, C_1 *to the corresponding sidelines of* $\mathbf{T}_t = P_aP_bP_c$ *are concurrent at* S*. As* t *varies, the locus of* S *is the rectangular hyperbola* $\mathcal{H}_M$ *passing through the traces of* X_8 *and* $X_{190} = \left(\frac{1}{b-c} : \frac{1}{c-a} : \frac{1}{a-b}\right)$

We call $\mathcal{H}_M$ the *Mandart hyperbola*. It has equation

$$\sum_{\text{cyclic}} (b-c)\left[(c+a-b)(a+b-c)x^2 + (b+c-a)^2 yz\right] = 0$$

[1] This point is not in the current edition of [4].

and contains the triangle centers X_8, X_9, X_{40}, X_{72}, X_{144}, X_{1145}, F', and F'' antipode of X_{11} on $\mathcal{C}_M$. Its asymptotes are parallel to those of the Feuerbach hyperbola. $\mathcal{H}_M$ is the Apollonian hyperbola of X_{40} with respect to $\mathcal{E}_M$. See Figure 8.

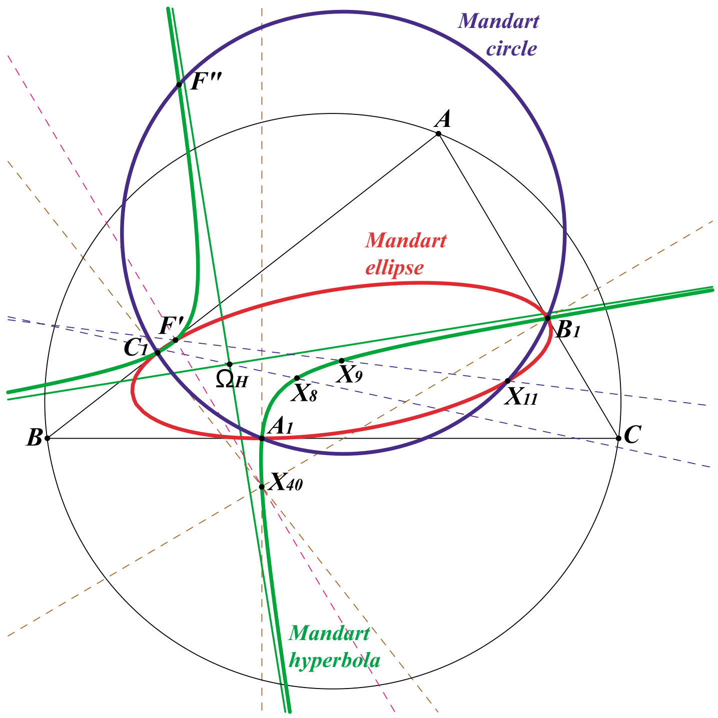


Figure 8. The Mandart hyperbola

3. Locus of some triangle centers in the Mandart triangles

We now examine the locus of some triangle centers of $\mathbf{T}_t = P_aP_bP_c$ when t varies. We shall consider the centroid, circumcenter, orthocenter, and Lemoine point.

Proposition 8. *The locus of the centroid of $\mathbf{T}_t$ is the parallel at G to the line OI.*

Proposition 9. *The locus of the circumcenter of $\mathbf{T}_t$ is the rectangular hyperbola passing through X_1, X_5, X_{10}, X_{21} (Schiffler point) and X_{1385}.* [2]

The equation of the hyperbola is

$$\sum_{\text{cyclic}} (b-c)\left[bc(b+c)x^2 + a(b^2+c^2-a^2+3bc)yz\right] = 0.$$

[2] X_{1385} is the midpoint of OI.

It has center X_{1125} (midpoint of IX_{10}) and asymptotes parallel to those of the Feuerbach hyperbola.

The locus of the orthocenter of $\mathbf{T}_t$ is a nodal cubic with node X_{10} passing through O, X_{1385}, meeting the line at infinity at X_{517} and the infinite points of the Feuerbach hyperbola. The line through the orthocenters of the t-Mandart triangle and the $(-t)$-Mandart triangle passes through a fixed point.

The locus of the Lemoine point of $\mathbf{T}_t$ is another nodal cubic with node X_{10}.

4. Generalized Mandart conics

Most of the results above can be generalized when X_8 is replaced by any point M on the Lucas cubic, the isotomic cubic with pivot X_{69}. The cevian triangle of such a point M is the pedal triangle of a point N on the Darboux cubic, the isogonal cubic with pivot the de Longchamps point X_{20}. [3]

For example, with $M = X_8$, we find $N = X_{40}$ and $M' = X_1 = I$.

Denote by $M_aM_bM_c$ the cevian triangle of M (on the Lucas cubic) and the pedal triangle of N (on the Darboux cubic). N^* is the isogonal conjugate of N also on the Darboux cubic. We now consider

– γ_M, inscribed conic in ABC with perspector M and center ω_M, which is the complement of the isotomic conjugate of M. It lies on the Thomson cubic and on the line KM' ($K = X_6$ is the Lemoine point),

– Γ_M, circumcircle of $M_aM_bM_c$ with center Ω_M, midpoint of NN^*. Γ_M is obviously the pedal circle of N and N^* and also the cevian circle of M°, cyclocevian conjugate of M (see [3, p.226, §8.12]). M° is a point on the Lucas cubic since this cubic is invariant under cyclocevian conjugation.

Since γ_M and Γ_M have already three points in common, they must have a fourth (always real) common point Z. Finally, denote by Z' the reflection of Z in ω_M. See Figure 9.

Table 1 gives examples for several known centers M on the Lucas cubic.[4] Those marked with $*$ are indicated in Table 2; those marked with $?$ are too complicated to give here.

<u>Table 1</u>

M	X_8	X_2	X_4	X_7	X_{20}	X_{69}	X_{189}	X_{253}	X_{329}	X_{1032}	X_{1034}
N	X_{40}	X_3	X_4	X_1	X_{1498}	X_{20}	X_{84}	X_{64}	X_{1490}	$*$	$*$
M'	X_1	X_2	X_3	X_9	X_4	X_6	X_{223}	X_{1249}	X_{57}	$*$	$*$
N^*	X_{84}	X_4	X_3	X_1	$*$	X_{64}	X_{40}	X_{20}	$*$	X_{1498}	X_{1490}
M°	X_{189}	X_4	X_2	X_7	X_{1032}	X_{253}	X_8	X_{69}	X_{1034}	X_{20}	X_{329}
ω_M	X_9	X_2	X_6	X_1	X_{1249}	X_3	X_{57}	X_4	X_{223}	X_{1073}	X_{282}
Ω_M	X_{1158}	X_5	X_5	X_1	$?$	$?$	X_{1158}	$?$	$?$	$?$	$?$
Z	X_{11}	X_{115}	X_{125}	X_{11}	X_{122}	X_{125}	$*$	X_{122}	$*$	$?$	$*$
Z'	$*$	$*$	$*$	X_{1317}	$*$	$*$	$*$	$*$	$*$	$?$	$*$

[3]It is also known that the complement of M is a point M' on the the Thomson cubic, the isogonal cubic with pivot $G = X_2$, the centroid.

[4]Two isotomic conjugates on the Lucas cubic are associated to the same point Z on the nine-point circle.

Table 2

Triangle center	First barycentric coordinate
$Z'(X_8)$	$(b+c-a)(2a^2-a(b+c)-(b-c)^2)^2$
$Z'(X_2)$	$(2a^2-b^2-c^2)^2$
$Z'(X_4)$	$\frac{(2a^2-b^2-c^2)^2}{S_A}$
$Z'(X_{20})$	$((3a^4-2a^2(b^2+c^2)-(b^2-c^2)^2)$ $\cdot(2a^8-a^6(b^2+c^2)-5a^4(b^2-c^2)^2+5a^2(b^2-c^2)^2(b^2+c^2)$ $-(b^2-c^2)^2(b^4+6b^2c^2+c^4))^2$
$Z'(X_{69})$	$S_A(2a^4-a^2(b^2+c^2)-(b^2-c^2)^2)^2$
$Z(X_{189})$	$(b-c)^2(b+c-a)^2(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2$
$Z'(X_{189})$	$\frac{(2a^2-a(b+c)-(b-c)^2)^2}{a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2}$
$Z'(X_{253})$	$\frac{(2a^4-a^2(b^2+c^2)-(b^2-c^2)^2)^2}{3a^4-2a^2(b^2+c^2)-(b^2-c^2)^2}$
$Z(X_{329})$	$(b-c)^2(b+c-a)^2(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2$
$Z'(X_{329})$	$(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)$ $\cdot(2a^5-a^4(b+c)-4a^3(b-c)^2+2a^2(b-c)^2(b+c)$ $+2a(b-c)^2(b^2+c^2)-(b-c)^2(b+c)^3)^2$
$N^*(X_{20})$	$1/(a^8-4a^6(b^2+c^2)+2a^4(3b^4-2b^2c^2+3c^4)$ $-4a^2(b^2-c^2)^2(b^2+c^2)+(b^2-c^2)^2(b^4+6b^2c^2+c^4))$
$N^*(X_{329})$	$a/(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b+c)(b-c)^2(b^2+c^2)+(b-c)^2(b+c)^4)$
$N(X_{1032})$	$1/(a^8-4a^6(b^2+c^2)+2a^4(3b^4-2b^2c^2+3c^4)$ $-4a^2(b^2-c^2)^2(b^2+c^2)+(b^2-c^2)^2(b^4+6b^2c^2+c^4))$
$M'(X_{1032})$	$(a^2(a^8-4a^6(b^2+c^2)+2a^4(3b^4-2b^2c^2+3c^4)$ $-4a^2(b^2-c^2)^2(b^2+c^2)+(b^2-c^2)^2(b^4+6b^2c^2+c^4))/$
	$(3a^4-2a^2(b^2+c^2)-(b^2-c^2)^2)$
$N(X_{1034})$	$a/(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$
	$-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4)$
$M'(X_{1034})$	$a(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4)/$ $(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)$
$Z(X_{1034})$	$(b-c)^2(b+c-a)(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)^2$ $\cdot(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4))$
$Z'(X_{1034})$	$(b+c-a)(2a^5-a^4(b+c)-4a^3(b-c)^2+2a^2(b-c)^2(b+c)$ $+2a(b-c)^2(b^2+c^2)-(b^2-c^2)^3)^2)/(a^6-2a^5(b+c)-a^4(b+c)^2$ $+4a^3(b+c)(b^2-bc+c^2)-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)$ $+(b-c)^2(b+c)^4)$
$M'(X_{1034})$	$a(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4)/$ $(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)$
$Z(X_{1034})$	$(b-c)^2(b+c-a)(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)^2$ $\cdot(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4))$
$Z'(X_{1034})$	$(b+c-a)(2a^5-a^4(b+c)-4a^3(b-c)^2+2a^2(b-c)^2(b+c)$ $+2a(b-c)^2(b^2+c^2)-(b^2-c^2)^3)^2)/(a^6-2a^5(b+c)-a^4(b+c)^2$ $+4a^3(b+c)(b^2-bc+c^2)-a^2(b^2-c^2)^2$ $-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4)$

Proposition 10. *Z is a point on the nine-point circle and Z' is the foot of the fourth normal drawn from N to γ_M.*

Proof. The lines NM_a, NM_b, NM_c are indeed already three such normals hence Γ_M is the Joachimsthal circle of N with respect to γ_M. This yields that Γ_M must pass through the reflection in ω_M of the foot of the fourth normal. See Figure 9. □

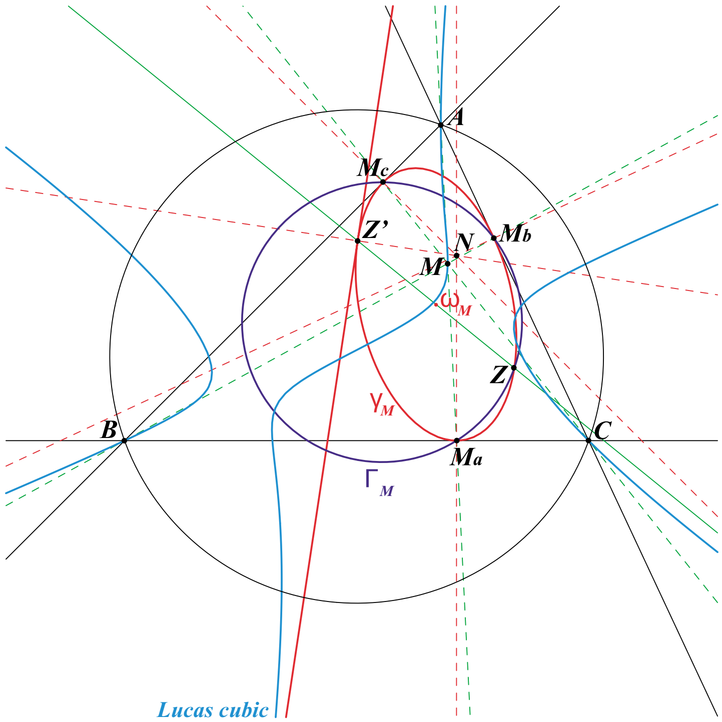


Figure 9. The generalized Mandart circle and conic

Remark. Z also lies on the cevian circumcircle of $M^{\#}$ isotomic conjugate of M and on the inscribed conic with perspector $M^{\#}$ and center M'.

Proposition 11. *The points M_a, M_b, M_c, M, N, ω_M and Z' lie on a same rectangular hyperbola whose asymptotes are parallel to the axes of γ_M.*

Proof. This hyperbola is the Apollonian hyperbola of N with respect to γ_M. □

Proposition 12. *The rectangular hyperbola passing through A, B, C, H and M is centered at Z. It also contains M', N^*, ω_M and $M^{\#}$. Its asymptotes are also parallel to the axes of γ_M.*

Remark. This hyperbola is the isogonal transform of the line ON and the isotomic transform of the line $X_{69}M$.

5. Generalized Mandart triangles

We now replace the circumcenter O by any finite point $P = (u : v : w)$ not lying on one sideline of ABC and we still call $A'B'C'$ its pedal triangle. For $t \in \mathbb{R} \cup \{\infty\}$, consider P_a, P_b, P_c defined as follows: draw three parallels to BC, CA, AB at the (signed) distance t with the conventions at the beginning of the paper. P_a, P_b, P_c are the projections of P on these parallels. See Figure 10.

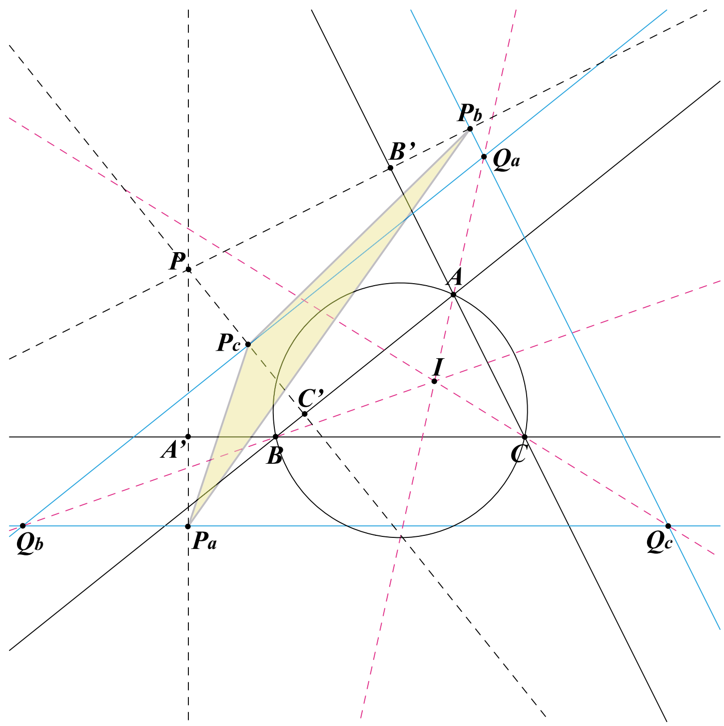


Figure 10. Generalized Mandart triangle

In homogeneous barycentric coordinates, these are the points

$$\begin{aligned}
P_a =& -a^3t : 2\Delta \cdot \frac{S_Cu + a^2v}{u+v+w} + taS_C : 2\Delta \cdot \frac{S_Bu + a^2w}{u+v+w} + taS_B,\\
P_b =& 2\Delta \cdot \frac{S_Cv + b^2u}{u+v+w} + tbS_C : -b^3t : 2\Delta \cdot \frac{S_Av + b^2w}{u+v+w} + tbS_A,\\
P_c =& 2\Delta \cdot \frac{S_Bw + c^2u}{u+v+w} + tcS_B : 2\Delta \cdot \frac{S_Aw + c^2v}{u+v+w} + tcS_A : -c^3t.
\end{aligned}$$

The triangle $\mathbf{T}_t(P) = P_aP_bP_c$ is called $t-$Mandart triangle of P.

Proposition 13. *For any P distinct from the incenter I, there are always two sets of collinear points P_a, P_b, P_c. The two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ containing the points are* *parallel to the asymptotes of the hyperbola which is the isogonal conjugate of the parallel to IP at X_{40}*[5]*. They meet at the point :*

$$(a((b+c)bcu + cS_C v + bS_B w) : \cdots : \cdots).$$

They are perpendicular if and only if P lies on OI.

Proof. P_a, P_b, P_c are collinear if and only if t is solution of the equation :

$$abc(a+b+c)t^2 + 2\Delta\,\Phi_1(u,v,w)\,t + 4\Delta^2\,\Phi_2(u,v,w) = 0 \tag{1}$$

where

$$\Phi_1(u,v,w) = \sum_{\text{cyclic}} bc(b+c)u \quad \text{and} \quad \Phi_2(u,v,w) = \sum_{\text{cyclic}} a^2vw.$$

We notice that $\Phi_1(u,v,w) = 0$ if and only if P lies on the polar line of I in the circumcircle and $\Phi_2(u,v,w) = 0$ if and only if P lies on the circumcircle.

The discriminant of (1) is non-negative for all P and null if and only if $P = I$. In this latter case, the points P_a, P_b, P_c are "collinear" if and only if they all coincide with I.

Considering now $P \neq I$, (1) always has two (real) solutions. □

Figure 11 shows the case $P = H$ with two (non-perpendicular) lines secant at X_{65} orthocenter of the intouch triangle.

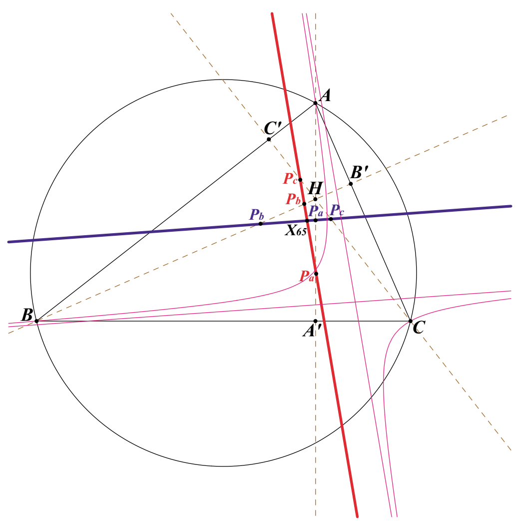


Figure 11. Collinear P_a, P_b, P_c with $P = H$

Figure 12 shows the case $P = X_{40}$ with two perpendicular lines secant at X_8 and parallel to the asymptotes of the Feuerbach hyperbola.

When P is a point on the circumcircle, equation (1) has a solution $t = 0$ and one of the two lines, say $\mathcal{L}_1$, is the Simson line of P: the triangle $A'B'C'$ degenerates

[5] X_{40} is the reflection of I in O.

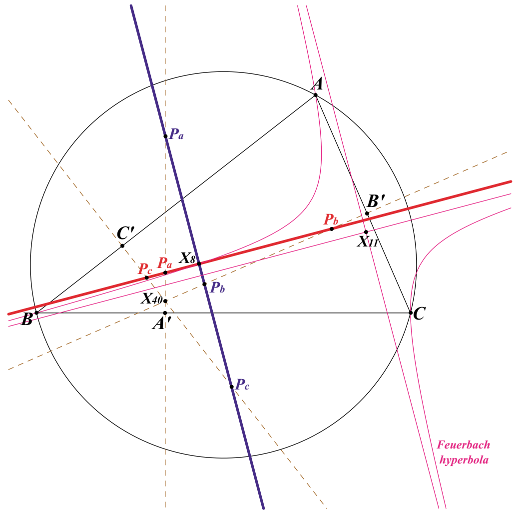


Figure 12. Collinear P_a, P_b, P_c with $P = X_{40}$

into this Simson line. $\mathcal{L}_1$ and $\mathcal{L}_2$ meet on the ellipse centered at X_{10} passing through X_{11}, the midpoints of ABC and the feet of the cevians of X_8. This ellipse is the complement of the circum-ellipse centered at I and has equation :

$$\sum_{\text{cyclic}} (a+b-c)(a-b+c)x^2 - 2a(b+c-a)yz = 0.$$

Figure 13 shows the case $P = X_{104}$ with two lines secant at X_{11}, one of them being the Simson line of X_{104}.

Following equation (1) again, we observe that, when P lies on the polar line of I in the circumcircle, we find to opposite values for t: the two corresponding points P_a are symmetric with respect to the sideline BC, P_b and P_c similarly. The most interesting situation is obtained with $P = X_{36}$ (inversive image of I in the circumcircle) since we find two perpendicular lines $\mathcal{L}_1$ and $\mathcal{L}_2$, parallel to the asymptotes of the Feuerbach hyperbola, intersecting at the midpoint of $X_{36}X_{80}$[6]. See Figure 14.

Construction of $\mathcal{L}_1$ and $\mathcal{L}_2$: the line IP[7] meets the circumcircle at S_1 and S_2. The parallels at P to OS_1 and OS_2 meet OI at T_1 and T_2. The homotheties with center I which map O to T_1 and T_2 also map the triangle ABC to the triangles $A_1B_1C_1$ and $A_2B_2C_2$. The perpendiculars PA', PB', PC' at P to the sidelines of ABC meet the corresponding sidelines of $A_1B_1C_1$ and $A_2B_2C_2$ at the requested points.

[6] X_{80} is the isogonal conjugate of X_{36}.
[7] We suppose $I \neq P$.

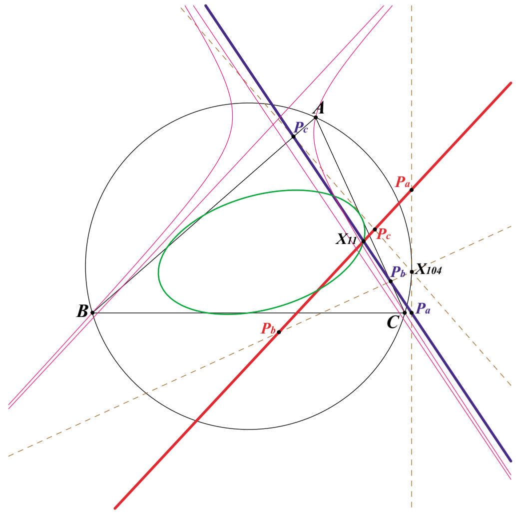


Figure 13. Collinear P_a, P_b, P_c with $P = X_{104}$

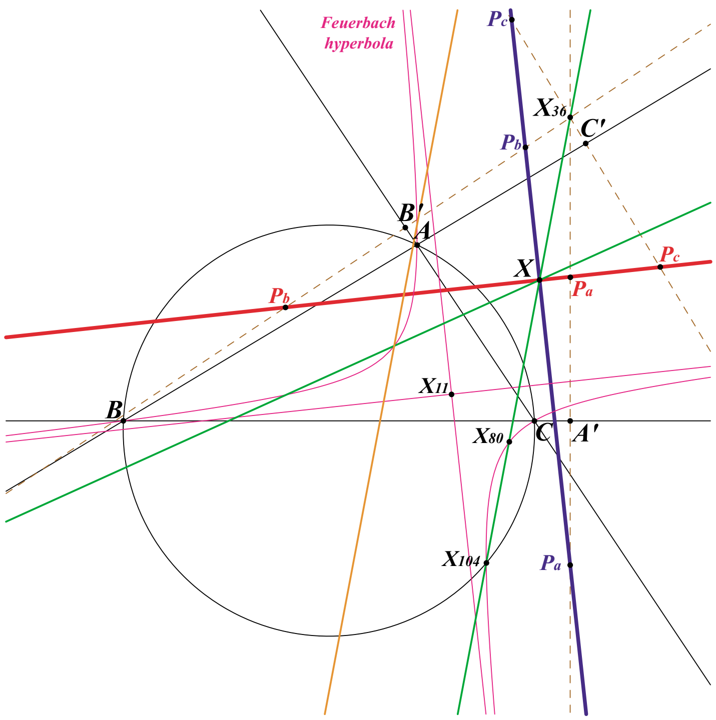


Figure 14. Collinear P_a, P_b, P_c with $P = X_{36}$

Proposition 14. *The triangles ABC and $P_aP_bP_c$ are perspective if and only if k is solution of :*

$$\Psi_2(u,v,w)\,t^2 + \Psi_1(u,v,w)\,t + \Psi_0(u,v,w) = 0 \tag{2}$$

where :

$$\Psi_2(u,v,w) = -\frac{1}{2}abc(a+b+c)(u+v+w)^2 \sum_{\text{cyclic}} (b-c)(b+c-a)S_A u,$$

$$\Psi_1(u,v,w) = \frac{1}{2}(a+b+c)(u+v+w)\Delta \sum_{\text{cyclic}} \left(-2bc(b-c)(b+c-a)S_A u^2 + a^2(b-c)(a+b+c)(b+c-a)^2 vw\right),$$

$$\Psi_0(u,v,w) = \Delta^2 \sum_{\text{cyclic}} (3a^4 - 2a^2(b^2+c^2) - (b^2-c^2)^2)u(c^2v^2 - b^2w^2).$$

Remarks. (1) $\Psi_2(u,v,w) = 0$ if and only if P lies on the line IH.

(2) $\Psi_1(u,v,w) = 0$ if and only if P lies on the hyperbola passing through I, H, X_{500}, X_{573}, X_{1742} [8] and having the same asymptotic directions as the isogonal transform of the line $X_{40}X_{758}$, *i.e.*, the reflection in O of the line X_1X_{21}.

(3) $\Psi_0(u,v,w) = 0$ if and only if P lies on the Darboux cubic. See Figure 15.

The equation (2) is clearly realized for all t if and only if $P = I$ or $P = H$: all $t-$Mandart triangles of I and H are perspective to ABC. Furthermore, if $P = H$ the perspector is always H, and if $P = I$ the perspector lies on the Feuerbach hyperbola. In the sequel, we exclude those two points and see that there are at most two real numbers t_1 and t_2 for which t_1- and t_2-Mandart triangles of P are perspective to ABC. Let us denote by R_1 and R_2 the (not always real) corresponding perspectors.

We explain the construction of these two perspectors with the help of several lemmas.

Lemma 15. *For a given P and a corresponding Mandart triangle $\mathbf{T}_t(P) = P_aP_bP_c$, the locus of $R_a = BP_b \cap CP_c$, when t varies, is a conic γ_a.*

Proof. The correspondence on the pencils of lines with poles B and C mapping the lines BP_b and CP_c is clearly an involution. Hence, the common point of the two lines must lie on a conic. □

This conic γ_a obviously contains B, C, H, $S_a = BB' \cap CC'$ and two other points B_1 on AB, C_1 on AC defined as follows. Reflect $AB \cap PB'$ in the bisector AI to get a point B_2 on AC. The parallel to AB at B_2 meets PC' at B_3. B_1 is the intersection of AB and CB_3. The point C_1 on AC is constructed similarly. See Figure 16.

Lemma 16. *The three conics γ_a, γ_b, γ_c have three points in common: H and the (not always real) sought perspectors R_1 and R_2. Their jacobian must degenerate*

[8] $X_{500} = X_1X_{30} \cap X_3X_6$, $X_{573} = X_4X_9 \cap X_3X_6$ and $X_{1742} = X_1X_7 \cap X_3X_{238}$.

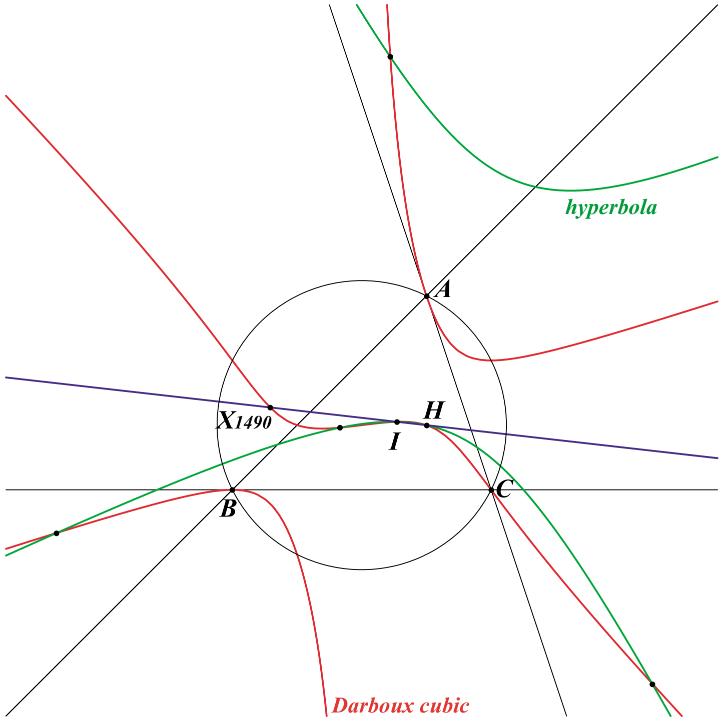


Figure 15. Proposition 14

into three lines, one always real $\mathcal{L}_P$ containing R_1 and R_2, two other passing through H.

Lemma 17. *$\mathcal{L}_P$ contains the Nagel point X_8. In other words, X_8, R_1 and R_2 are always collinear.*

With $P = (u : v : w)$, $\mathcal{L}_P$ has equation :

$$\sum_{\text{cyclic}} \frac{a(cv - bw)}{b + c - a}\, x = 0$$

$\mathcal{L}_P$ is the trilinear polar of the isotomic conjugate of point T, where T is the barycentric product of X_{57} and the isotomic conjugate of the trilinear pole of the line PI. The construction of R_1 and R_2 is now possible in the most general case with one of the conics and $\mathcal{L}_P$. Nevertheless, in three specific situations already mentioned, the construction simplifies as we see in the three following corollaries.

Corollary 18. *When P lies on IH, there is only one* (*always real*) *Mandart triangle $\mathbf{T}_t(P)$ perspective to ABC. The perspector R is the intersection of the lines HX_8 and PX_{78}.*

Proof. This is obvious since equation (2) is at most of the first degree when P lies on IH. □

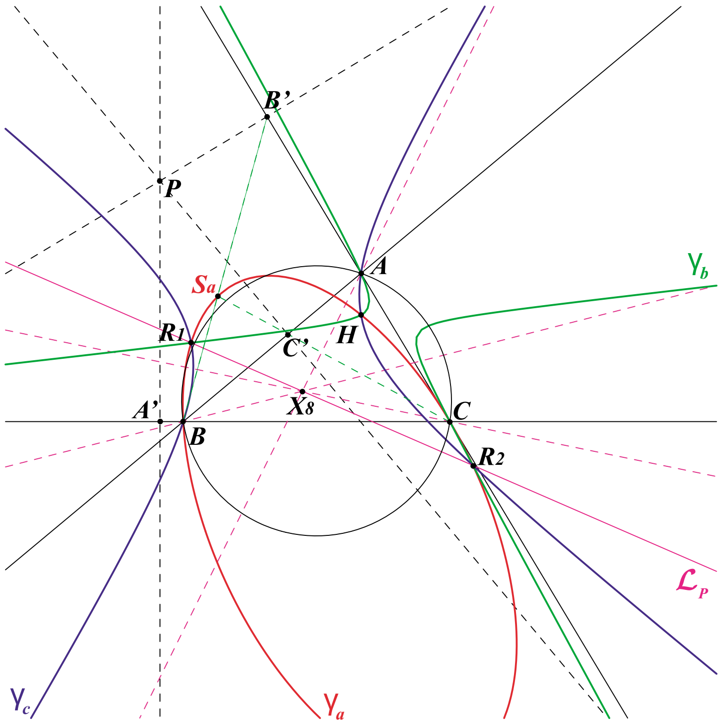


Figure 16. The three conics γ_a, γ_b, γ_c and the perspectors R_1, R_2

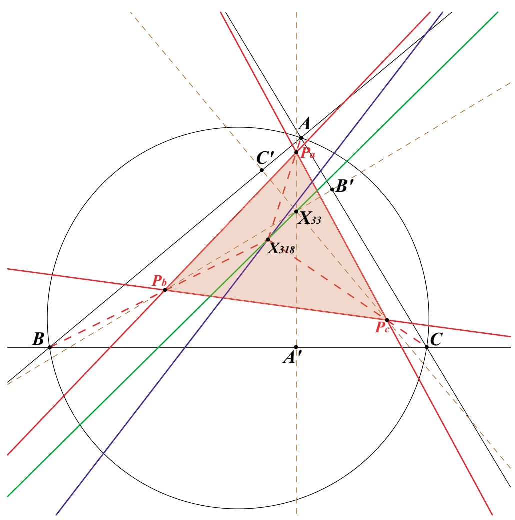


Figure 17. Only one triangle $P_aP_bP_c$ perspective to ABC when P lies on IH

In Figure 17, we have taken $P = X_{33}$ and $R = X_{318}$.

Remark. The line IH meets the Darboux cubic again at X_{1490}. The corresponding Mandart triangle $\mathbf{T}_t(P)$ is the pedal triangle of X_{1490} which is also the cevian triangle of X_{329}.

Corollary 19. *When P (different from I and H) lies on the conic seen above, there are two (not always real) Mandart triangles $\mathbf{T}_t(P)$ perspective to ABC obtained for two opposite values t_1 and t_2. The vertices of the triangles are therefore two by two symmetric in the sidelines of ABC.*

In the figure 18, we have taken $P = X_{500}$ (orthocenter of the incentral triangle).

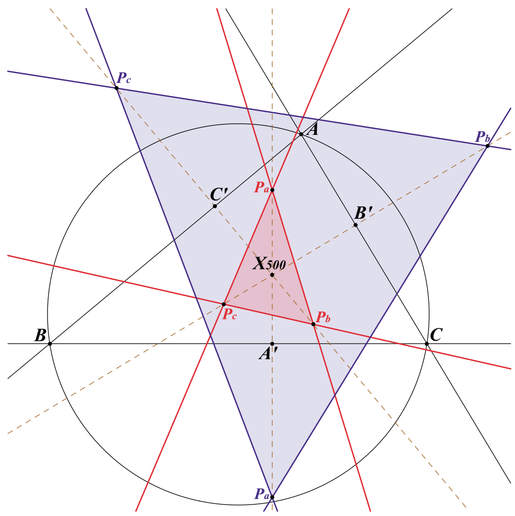


Figure 18. Two triangles $P_aP_bP_c$ perspective with ABC having vertices symmetric in the sidelines of ABC

Corollary 20. *When P (different from I, H, X_{1490}) lies on the Darboux cubic, there are two (always real) Mandart triangles $\mathbf{T}_t(P)$ perspective to ABC, one of them being the pedal triangle of P with a perspector on the Lucas cubic.*

Since one perspector, say R_1, is known, the construction of the other is simple: it is the "second" intersection of the line X_8R_1 with the conic $BCHS_aR_1$.

Table 3 gives P (on the Darboux cubic), the corresponding perspectors R_1 (on the Lucas cubic) and R_2.

Table 3

P	X_1	X_3	X_4	X_{20}	X_{40}	X_{64}	X_{84}	X_{1498}
R_1	X_7	X_2	X_4	X_{69}	X_8	X_{253}	X_{189}	X_{20}
R_2		X_8	X_4	X_{388}	X_{10}	$*$	X_{515}	$*$

Table 4

Triangle center	First barycentric coordinate
$R_2(X_{64})$	$\frac{a^8-4a^6(b+c)^2+2a^4(b+c)^2(3b^2-4bc+3c^2)-4a^2(b^2-c^2)^2(b^2+c^2)+(b-c)^2(b+c)^6}{b+c-a}$
$R_2(X_{1498})$	$\frac{a^4-2a^2(b+c)^2+(b^2-c^2)^2}{a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2}$

In Figure 19, we have taken $P = X_{40}$ (reflection of I in O).

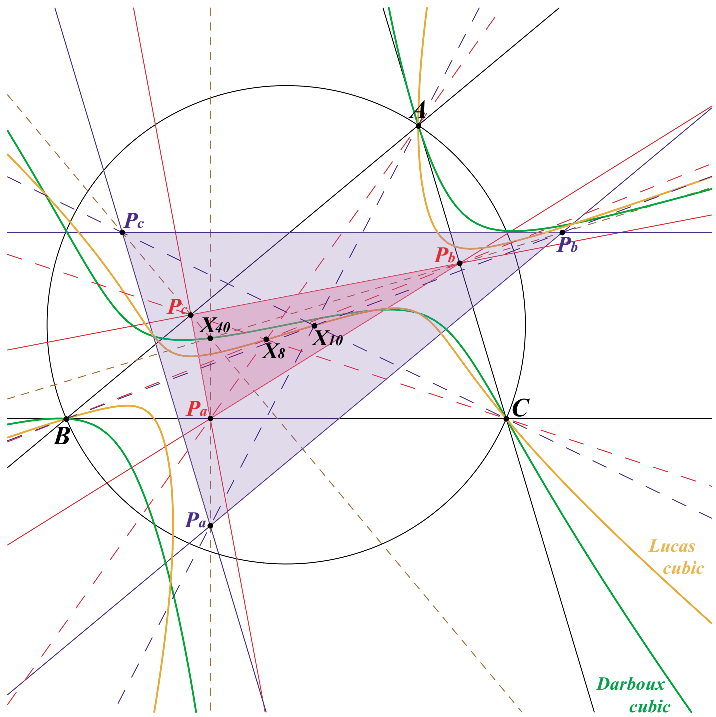


Figure 19. Two triangles $P_aP_bP_c$ perspective with ABC when $P = X_{40}$

Proposition 21. *The triangles $A'B'C'$ and $P_aP_bP_c$ have the same area if and only if*
(1) $t = 0$, or
(2) $t = -\frac{bc(b+c)u+ca(c+a)v+ab(a+b)w}{2R(a+b+c)(u+v+w)}$, [9]
(3) t is a solution of a quadratic equation [10] *whose discriminant has the same sign of*

$$f(u,v,w) = \sum_{\text{cyclic}} b^2c^2(b+c)^2u^2 + 2a^2bc(bc-3a(a+b+c))vw.$$

[9]This can be interpreted as $t = -\frac{d(P)}{d(O)} \cdot R$, where $d(X)$ denotes the distance from X to the polar line of I in the circumcircle.

[10]$abc(a+b+c)(u+v+w)^2t^2 + 2\Delta(u+v+w)\left(\sum_{\text{cyclic}} bc(b+c)u\right)t + 8\Delta^2(a^2vw + b^2wu + c^2uv) = 0.$

The equation $f(x, y, z) = 0$ represents an ellipse $\mathcal{E}$ centered at X_{35} [11] whose axes are parallel and perpendicular to the line OI. See Figure 20.

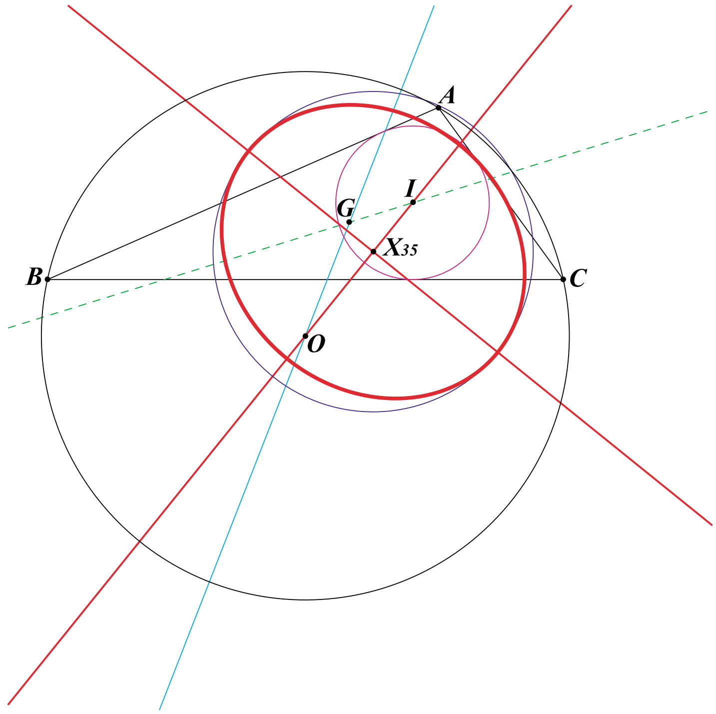


Figure 20. The "critical" ellipse $\mathcal{E}$

According to the position of P with respect to this ellipse, it is possible to have other triangles solution of the problem. More precisely, if P is
– inside $\mathcal{E}$, there is no other triangle,
– outside $\mathcal{E}$, there are two other (distinct) triangles,
– on $\mathcal{E}$, there is only one other triangle.

Proposition 22. *As t varies, each line P_bP_c, P_cP_a, P_aP_b still envelopes a parabola.*

Denote these parabolas by $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$ respectively. $\mathcal{P}_a$ has focus the projection F_a of P on AI and directrix ℓ_a parallel to AI at E_a such that $\overrightarrow{PE_a} = \cos A\ \overrightarrow{PF_a}$. Note that the direction of the directrix (and the axis) is independent of P. $\mathcal{P}_a$ is still tangent to the lines PB', PC', $B'C'$.

In this more general case, the directrices ℓ_a, ℓ_b, ℓ_c are not necessarily concurrent. This happens if and only if P lies on the line OI and, then, their common point lies on IG.

Proposition 23. *The Mandart triangle $\mathbf{T}_t(P)$ and the pedal triangle of P are perspective at P. As t varies, the envelope of their perspectrix is a parabola.*

[11] Let I'_a be the inverse-in-circumcircle of the excenter I_a, and define I'_b and I'_c similarly. The triangles ABC and $I'_aI'_bI'_c$ are perspective at X_{35} which is a point on the line OI.

The directrix of this parabola is parallel to the line IP^*. It is still inscribed in the pedal triangle $A'B'C'$ of P and is tangent to the two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ met in proposition 13.

Remark. Unlike the case $P = X_8$, ABC is not necessary self polar with respect to this Mandart parabola.

Proposition 24. *The Mandart triangle $\mathbf{T}_t(P)$ and ABC are orthologic. The perpendiculars from A, B, C to the corresponding sidelines of $P_aP_bP_c$ are concurrent at $Q = \left(\frac{a^2}{at+2\Delta u} : \cdots : \cdots\right)$. As t varies, the locus of Q is generally the circumconic which is the isogonal transform of the line IP.*

This conic has equation

$$\sum_{\text{cyclic}} a^2(cv - bw)yz = 0.$$

It is tangent at I to IP, and is a rectangular hyperbola if and only if P lies on the line OI ($P \neq I$). When $P = I$, the triangles are homothetic at I and the perpendiculars concur at I.

References

[1] Brocard H. and Lemoyne T. *Courbes Géométriques Remarquables*. Librairie Albert Blanchard, Paris, third edition, 1967.

[2] F.G.-M. *Exercices de géométrie*. 1920, Gabay reprint, 1991.

[3] Kimberling C. *Triangle Centers and Central Triangles*. Congressus Numerantium 129, Winnipeg, Canada, 1998.

[4] Kimberling C. *Encyclopedia of Triangle Centers*. http://www2.evansville.edu/ck6/encyclopedia.

[5] Mandart H. *Sur l'hyperbole de Feuerbach*, Mathesis, 1893, pp.81-89.

[6] Mandart H. *Sur une ellipse associée au triangle*, Mathesis, 1894, pp. 241-245.

Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France
E-mail address: `bgibert@wanadoo.fr`

Forum Geometricorum
Volume 4 (2004) 199–201.

FORUM GEOM
ISSN 1534-1178

Another Proof of Fagnano's Inequality

Nguyen Minh Ha

Abstract. We prove Fagnano's inequality using the scalar product of vectors.

In 1775, I. F. Fagnano, an Italian mathematician, proposed the following extremum problem.

Problem (Fagnano)**.** *In a given acute-angled triangle ABC, inscribe a triangle XYZ whose perimeter is as small as possible.*

Fagnano himself gave a solution to this problem using calculus. The second proof given in [1] repeatedly using reflections and the mirror property of the orthic triangle was due to L. Fejér. While H. A. Schwarz gave another proof in which reflection was also used, we give another proof by using the scalar product of two vectors.

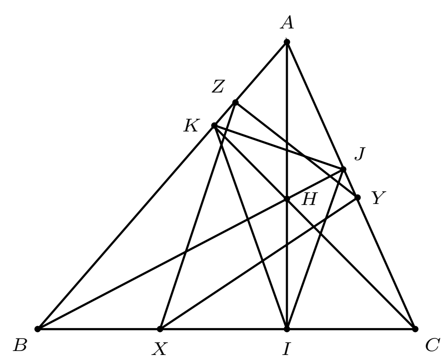


Figure 1

Let AI, BJ and CK be the altitudes of triangle ABC and H its orthocenter. Suppose that X, Y, Z are arbitrary points on the lines BC, CA and AB respectively. See Figure 1. We have

Publication Date: November 19, 2004. Communicating Editor: J. Chris Fisher.

$$
\begin{aligned}
&YZ+ZX+XY\\
=&\frac{YZ\cdot JK}{JK}+\frac{ZX\cdot KI}{KI}+\frac{XY\cdot IJ}{IJ}\\
\geqslant&\frac{\overrightarrow{YZ}\cdot\overrightarrow{JK}}{JK}+\frac{\overrightarrow{ZX}\cdot\overrightarrow{KI}}{KI}+\frac{\overrightarrow{XY}\cdot\overrightarrow{IJ}}{IJ}\\
=&\frac{(\overrightarrow{YJ}+\overrightarrow{JK}+\overrightarrow{KZ})\cdot\overrightarrow{JK}}{JK}+\frac{(\overrightarrow{ZK}+\overrightarrow{KI}+\overrightarrow{IX})\cdot\overrightarrow{KI}}{KI}+\frac{(\overrightarrow{XI}+\overrightarrow{IJ}+\overrightarrow{JY})\cdot\overrightarrow{IJ}}{IJ}\\
=&JK+KI+IJ+\overrightarrow{XI}\cdot\Big(\frac{\overrightarrow{IJ}}{IJ}+\frac{\overrightarrow{IK}}{IK}\Big)+\overrightarrow{YJ}\cdot\Big(\frac{\overrightarrow{JK}}{JK}+\frac{\overrightarrow{JI}}{JI}\Big)+\overrightarrow{ZK}\cdot\Big(\frac{\overrightarrow{KI}}{KI}+\frac{\overrightarrow{KJ}}{KJ}\Big).
\end{aligned}
$$

Since triangle ABC is acute-angled, its altitudes bisect the internal angles of its orthic triangle IJK. It follows that the vectors

$$\frac{\overrightarrow{IJ}}{IJ}+\frac{\overrightarrow{IK}}{IK}, \quad \frac{\overrightarrow{JK}}{JK}+\frac{\overrightarrow{JI}}{JI}, \quad \frac{\overrightarrow{KI}}{KI}+\frac{\overrightarrow{KJ}}{KJ}$$

are respectively perpendicular to the vectors $\overrightarrow{XI}$, $\overrightarrow{YJ}$, $\overrightarrow{ZK}$. It follows that

$$YZ+ZX+XY \geqslant JK+KI+IJ. \tag{1}$$

If the equality in (1) occurs, then the vectors $\overrightarrow{YZ}, \overrightarrow{ZX}, \overrightarrow{XY}$ point in the same directions of the vectors $\overrightarrow{JK}, \overrightarrow{KI}, \overrightarrow{IJ}$ respectively. Hence there exist positive numbers α, β and γ such that

$$\overrightarrow{YZ}=\alpha\overrightarrow{JK}, \quad \overrightarrow{ZX}=\beta\overrightarrow{KI}, \quad \overrightarrow{XY}=\gamma\overrightarrow{IJ}.$$

Now we have $\alpha\overrightarrow{JK}+\beta\overrightarrow{KI}+\gamma\overrightarrow{IJ}=\overrightarrow{0}$. It follows from this and the equality $\overrightarrow{JK}+\overrightarrow{KI}+\overrightarrow{IJ}=\overrightarrow{0}$ that $\alpha=\beta=\gamma$. Consequently,

$$\overrightarrow{YZ}=\alpha\overrightarrow{JK}, \ \overrightarrow{ZX}=\alpha\overrightarrow{KI}, \ \overrightarrow{XY}=\alpha\overrightarrow{IJ},$$

which implies that

$$YZ=\alpha JK, \ \ ZX=\alpha KI, \ \ XY=\alpha IJ,$$

and

$$YZ+ZX+XY=\alpha(JK+KI+IJ).$$

Note that the equality in (1) occurs, we have $\alpha=\beta=\gamma=1$. Then $\overrightarrow{YZ}=\overrightarrow{JK}$, $\overrightarrow{ZX}=\overrightarrow{KI}$, $\overrightarrow{XY}=\overrightarrow{IJ}$, which means that X, Y, Z respectively coincides with I, J, K.

Conversely, if X, Y, Z coincide with I, J, K respectively, then equality sign occurs in (1).

In conclusion, the triangle XYZ has the smallest possible perimeter when X, Y, Z coincide with I, J, K respectively.

Reference

[1] A. Bogomolny, http://www.cut-the-knot.org/Curriculum/Geometry/Fagnano.shtml

Nguyen Minh Ha: Faculty of Mathematics, Hanoi University of Education, Xuan Thuy, Hanoi, Vietnam

E-mail address: minhha27255@yahoo.com

Forum Geometricorum
Volume 4 (2004) 203–206.

FORUM GEOM
ISSN 1534-1178

Further Inequalities of Erdős-Mordell Type

Walther Janous

To the memory of Murray S. Klamkin

Abstract. We extend the recent generalization of the famous Erdős-Mordell inequality by Dar and Gueron in the *American Mathematical Monthly*.

1. Introduction

In the recent note [1] the following generalization of the famous Erdős - Mordell inequality has been established. (For a proof of the original inequality see for instance [2]). For a triangle $A_1A_2A_3$, we denote by a_i the length of the side opposite to A_i, $i = 1, 2, 3$. Let P be an interior point. Denote the distances of P from the vertices A_i by R_i and from the sides opposite A_i by r_i. For positive real numbers λ_1, λ_2, λ_3,

$$\lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3 \geq 2\sqrt{\lambda_1\lambda_2\lambda_3}\left(\frac{r_1}{\sqrt{\lambda_1}} + \frac{r_2}{\sqrt{\lambda_2}} + \frac{r_3}{\sqrt{\lambda_3}}\right), \tag{1}$$

This inequality appears in [3, p.318, Theorem 15] without proof and with an incorrect characterization for equality. In [3, Chapter XI] and [4, Chapter 12], there are quoted very many extensions and variations of the original Erdős - Mordell inequality. It is the goal of this note to prove a further generalization containing the results of [1] and to apply it to specific points in a triangle, resulting in new inequalities for several elements of triangles.

2. The inequalities

Let λ_1, λ_2, λ_3 and t denote positive real numbers, with $0 < t \leq 1$.

Theorem 1.

$$\lambda_1 R_1^t + \lambda_2 R_2^t + \lambda_3 R_3^t \geq 2^t\sqrt{\lambda_1\lambda_2\lambda_3}\left(\frac{r_1^t}{\sqrt{\lambda_1}} + \frac{r_2^t}{\sqrt{\lambda_2}} + \frac{r_3^t}{\sqrt{\lambda_3}}\right). \tag{2}$$

Equality holds if and only if $\lambda_1 : \lambda_2 : \lambda_3 = a_1^{2t} : a_2^{2t} : a_3^{2t}$ *and* P *is the circumcenter of triangle* $A_1A_2A_3$.

Publication Date: November 29, 2004. Communicating Editor: Paul Yiu.

Proof. As for instance in [1] we have

$$R_1 \geq \frac{a_3}{a_1} r_2 + \frac{a_2}{a_1} r_3, \qquad R_2 \geq \frac{a_1}{a_2} r_3 + \frac{a_3}{a_2} r_1, \qquad R_3 \geq \frac{a_2}{a_3} r_1 + \frac{a_1}{a_3} r_2.$$

Using the power means inequality we obtain (for $0 < t < 1$)

$$R_1^t \geq 2^t \left(\frac{\frac{a_3}{a_1} r_2 + \frac{a_2}{a_1} r_3}{2} \right)^t \geq 2^t \cdot \frac{\left(\frac{a_3}{a_1}\right)^t r_2^t + \left(\frac{a_2}{a_1}\right)^t r_3^t}{2}$$

and two similar inequalities. Applying several times the elementary estimation $x + \frac{1}{x} \geq 2$ for $x > 0$ we obtain

$$\begin{aligned} & \lambda_1 R_1^t + \lambda_2 R_2^t + \lambda_3 R_3^t \\ \geq & 2^t \left(\frac{\left(\frac{a_3}{a_2}\right)^t \lambda_2 + \left(\frac{a_2}{a_3}\right)^t \lambda_3}{2} r_1^t + \frac{\left(\frac{a_1}{a_3}\right)^t \lambda_3 + \left(\frac{a_3}{a_1}\right)^t \lambda_1}{2} r_2^t + \frac{\left(\frac{a_2}{a_1}\right)^t \lambda_1 + \left(\frac{a_1}{a_2}\right)^t \lambda_2}{2} r_3^t \right) \\ \geq & 2^t \left(\sqrt{\lambda_2 \lambda_3} r_1^t + \sqrt{\lambda_3 \lambda_1} r_2^t + \sqrt{\lambda_1 \lambda_2} r_3^t \right) \end{aligned}$$

as claimed. The conditions of equality are derived as in [1].

□

In view of the obvious inequality $(x+y)^t > x^t + y^t$ for $x, y > 0$, we have the following theorem.

Theorem 2. *For* $t > 1$,

$$\lambda_1 R_1^t + \lambda_2 R_2^t + \lambda_3 R_3^t \geq 2\sqrt{\lambda_1 \lambda_2 \lambda_3} \left(\frac{r_1^t}{\sqrt{\lambda_1}} + \frac{r_2^t}{\sqrt{\lambda_2}} + \frac{r_3^t}{\sqrt{\lambda_3}} \right). \tag{3}$$

As a consequence of Theorem 1 we get

Theorem 3.

$$\textstyle\sum_{i=1}^3 \frac{\lambda_i}{r_i^t} \geq 2^t \sqrt{\lambda_1 \lambda_2 \lambda_3} \sum_{i=1}^3 \frac{1}{\sqrt{\lambda_i} R_i^t}, \tag{4}$$

$$\textstyle\frac{\lambda_i}{R_i^t} \geq \frac{2^t \sqrt{\lambda_1 \lambda_2 \lambda_3}}{(R_1 R_2 R_3)^t \sum_{i=1}^3} \sum_{i=1}^3 \frac{(R_i r_i)^t}{\sqrt{\lambda_i}}, \tag{5}$$

$$\textstyle\sum_{i=1}^3 \lambda_i (R_i r_i)^t \geq 2^t \sqrt{\lambda_1 \lambda_2 \lambda_3} (r_1 r_2 r_3)^t \sum_{i=1}^3 \frac{1}{\sqrt{\lambda_i} r_i^t}, \tag{6}$$

$$\textstyle\sum_{i=1}^3 \lambda_i r_i^t \geq 2^t \sqrt{\lambda_1 \lambda_2 \lambda_3} (r_1 r_2 r_3)^t \sum_{i=1}^3 \frac{1}{\sqrt{\lambda_i} (R_i r_i)^t}, \tag{7}$$

$$\textstyle\sum_{i=1}^3 \frac{\lambda_i}{(R_i r_i)^t} \geq \frac{2^t \sqrt{\lambda_1 \lambda_2 \lambda_3}}{(R_1 R_2 R_3)^t} \sum_{i=1}^3 \frac{R_i^t}{\sqrt{\lambda_i}}. \tag{8}$$

The proofs of these inequalities follow from Theorem 1 upon application of transformations such as

(i) inversion with respect to the circle $\mathcal{C}(P, \sqrt{R_1 R_2 R_3})$ resulting in $R_i \mapsto \frac{R_1 R_2 R_3}{R_i}$ and $r_i \mapsto R_i r_i$ for $i = 1, 2, 3$,

(ii) reciprocation of $A_1 A_2 A_3$ yielding $R_i \mapsto \frac{r_1 r_2 r_3}{r_i}$ and $r_i \mapsto \frac{r_1 r_2 r_3}{R_i}$ for $i = 1, 2, 3$, and

(iii) isogonal conjugation.
For the details consult [3, pp. 293 - 295].

Remarks. (1) From (5) and (6) the following inequality is easily derived.

$$(R_1R_2R_3)^t \sum_{i=1}^{3} \frac{\lambda_i}{R_i^t} \geq 4^t \sqrt[4]{\lambda_1\lambda_2\lambda_3}(r_1r_2r_3)^t \sum_{i=1}^{3} \frac{\sqrt[4]{\lambda_i}}{r_i^t}. \tag{9}$$

whereas (7) and (8) lead to the "converse" of (9), *i.e.*,

$$\frac{1}{(r_1r_2r_3)^t} \sum_{i=1}^{3} \lambda_i r_i^t \geq \frac{4^t \sqrt[4]{\lambda_1\lambda_2\lambda_3}}{(R_1R_2R_3)^t} \sum_{i=1}^{3} \sqrt[4]{\lambda_i} R_i^t. \tag{10}$$

(2) We leave it as an exercise to the reader to derive an analogue of Theorem 2. It should be noted that the above inequalities include very many results of [3, 4] as special cases.

3. Applications to special triangle points

In this section we show that the theorems above, when specialized to suitably chosen interior points P, imply an abundance of new interesting triangle inequalities.

3.1. Let P be the incenter I of $A_1A_2A_3$. Then $r_1 = r_2 = r_3 = r$, the inradius of $A_1A_2A_3$, and $R_i = A_iI = r \csc \frac{A_i}{2}$, $i = 1, 2, 3$. Thus, from (8), we obtain, upon recalling that

$$\sin \frac{A_1}{2} \sin \frac{A_2}{2} \sin \frac{A_3}{2} = \frac{r}{4R},$$

the following inequality for $0 < t \leq 1$:

$$\sum_{i=1}^{3} \lambda_i \sin^t \frac{A_i}{2} \geq \sqrt{\lambda_1\lambda_2\lambda_3} \left(\frac{r}{2R}\right)^t \sum_{i=1}^{3} \frac{1}{\sqrt{\lambda_i}} \csc^t \frac{A_i}{2}. \tag{11}$$

3.2. Let P be the centroid G of $A_1A_2A_3$. Then $R_i = A_iG = \frac{2}{3}m_i$, and $r_i = \frac{h_i}{3}$, where, for $i = 1, 2, 3$, m_i and h_i denote respectively the median and altitude emanating from vertex A_i. Therefore, as an example, (4) becomes, for $0 < t \leq 1$,

$$\sum_{i=1}^{3} \frac{\lambda_i}{h_i^t} \geq \sqrt{\lambda_1\lambda_2\lambda_3} \sum_{i=1}^{3} \frac{1}{\sqrt{\lambda_i} m_i^t}. \tag{12}$$

If we put $\lambda_i = h_i^t$, $i = 1, 2, 3$, then

$$\left(\frac{\sqrt{h_2h_3}}{m_1}\right)^t + \left(\frac{\sqrt{h_3h_1}}{m_2}\right)^t + \left(\frac{\sqrt{h_1h_2}}{m_3}\right)^t \leq 3. \tag{13}$$

This inequality should be compared with the following one by Klamkin and Meir in [3, p. 215]:

$$\frac{\overline{h_1}}{m_1} + \frac{\overline{h_2}}{m_2} + \frac{\overline{h_3}}{m_3} \leq 3,$$

where $(\overline{h_1},\ \overline{h_2},\ \overline{h_3})$ is any permutation of (h_1, h_2, h_3).

Via the median - duality transforming an arbitrary triangle $A_1A_2A_3$ into one formed by its medians ([3, pp.109 - 111]), inequality (13) becomes

$$\left(\frac{h_1}{\sqrt{m_2m_3}}\right)^t+\left(\frac{h_2}{\sqrt{m_3m_1}}\right)^t+\left(\frac{h_3}{\sqrt{m_1m_2}}\right)^t\le 3. \tag{14}$$

Finally, in (12), we put $\lambda_i=\frac{1}{a_i^t}$ for $i=1,2,3$. A short calculation gives

$$3\left(\frac{R}{F}\right)^{\frac{t}{2}}\ge\sum_{i=1}^{3}\left(\frac{\sqrt{a_i}}{m_i}\right)^t. \tag{15}$$

Here, we make use of the identity $a_1a_2a_3=4RF$, where F denotes the area of $A_1A_2A_3$.

The median - dual of this inequality in turn reads

$$\sum_{i=1}^{3}\left(\frac{\sqrt{m_i}}{a_i}\right)^t\le 3\left(\frac{\sqrt{m_1m_2m_3}}{2F}\right)^t. \tag{16}$$

Of course, if in (12) had we put $\lambda_i=\frac{\mu_i}{a_i^t}$ with $\mu_i>0$, $i=1,2,3$, we would obtain an even more general but less elegant inequality.

Remarks. (1) Clearly, many further inequalities could be deduced by the methods of this section. We leave this as an exercise to the reader.

(2) As the right hand side of inequality (1) indeed reads $2(\sqrt{\lambda_2\lambda_3}r_1+\sqrt{\lambda_3\lambda_1}r_2+\sqrt{\lambda_1\lambda_2}r_3)$, it is enough to assume λ_1, λ_2, λ_2 nonnegative throughout this note.

References

[1] S. Dar and S. Gueron, A weighted Erdős-Mordell inequality, *Amer. Math. Monthly*, 108 (2001) 165–167.

[2] H. Lee, Another proof of the Erdős-Mordell inequality, *Forum Geom.*, 1 (2001) 7–8.

[3] D. S. Mitrinović, J. E. Pečarić and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Acad. Publ., Dordrecht 1989.

[4] O. Bottema, R.Ž. Djordjević, R. R. Janić, D. S. Mitrinovic and P. M. Vasic, *Geometric Inequalities*, Wolters-Noordhoff Publ., Groningen 1968.

Walther Janous: Ursulinengymnasium, Fürstenweg 86, A-6020 Innsbruck, Austria
E-mail address: walther.janous@tirol.com

Forum Geometricorum
Volume 4 (2004) 207–214.

FORUM GEOM
ISSN 1534-1178

Inscribed Squares

Floor van Lamoen

Abstract. We give simple constructions of various squares inscribed in a triangle, and some relations among these squares.

1. Inscribed squares

Given a triangle ABC, an inscribed square is one whose vertices are on the sidelines of ABC. Two of the vertices of an inscribed square must fall on a sideline. There are two kinds of inscribed squares.

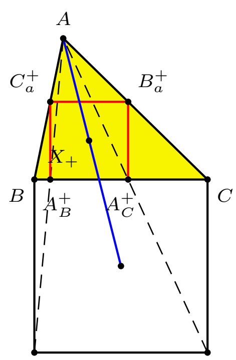


Figure 1A

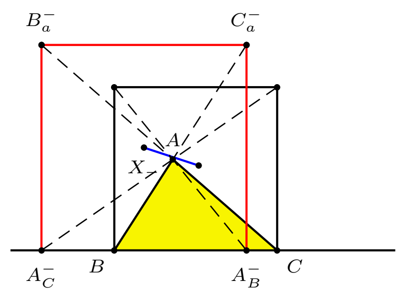


Figure 1B

1.1. *Inscribed squares of type I.* The Inscribed squares with two adjacent vertices on a sideline of ABC can be constructed easily from a homothety of a square erected on the side of ABC. Consider the two squares erected on the side BC. Their centers are the points with homogeneous barycentric coordinates $(-a^2 : S_C + \varepsilon S : S_B + \varepsilon S)$ for $\varepsilon = \pm 1$. Here, we use standard notations in triangle geometry. See, for example, [4, §1]. By applying the homothety $\mathsf{h}(A, \frac{\varepsilon S}{a^2+\varepsilon S})$, we obtain an inscribed square $\mathsf{Sq}^{\varepsilon}(A) = A_B^{\varepsilon} A_C^{\varepsilon} B_a^{\varepsilon} C_a^{\varepsilon}$ with center

$$\begin{aligned} X_{\varepsilon} =& \mathsf{h}(A, \frac{\varepsilon S}{a^2+\varepsilon S})(-a^2 : S_C + \varepsilon S : S_B + \varepsilon S) \\ =& (a^2 : S_C + \varepsilon S : S_B + \varepsilon S), \end{aligned}$$

and two vertices (A_B^{ε} and A_C^{ε}) on the sideline BC. See Figure 1. Similarly there are the inscribed squares $\mathsf{Sq}^{\varepsilon}(B)$ and $\mathsf{Sq}^{\varepsilon}(C)$.

We give the coordinates of the centers and vertices of these squares in Table 1 below.

Publication Date: December 6, 2004. Communicating Editor: Paul Yiu.

Table 1. Centers and vertices of inscribed squares of type I

$\mathsf{Sq}^\varepsilon(A)$	$\mathsf{Sq}^\varepsilon(B)$	$\mathsf{Sq}^\varepsilon(C)$
$X_\varepsilon = (a^2 : S_C + \varepsilon S : S_B + \varepsilon S)$	$Y_\varepsilon = (S_C + \varepsilon S : b^2 : S_A + \varepsilon S)$	$Z_\varepsilon = (S_B + \varepsilon S : S_A + \varepsilon S : c^2)$
$A_B^\varepsilon = (0 : S_C + \varepsilon S : S_B)$ $A_C^\varepsilon = (0 : S_C : S_B + \varepsilon S)$	$A_b^\varepsilon = (0 : b^2 : \varepsilon S)$	$A_c^\varepsilon = (0 : \varepsilon S : c^2)$
$B_a^\varepsilon = (a^2 : 0 : \varepsilon S)$,	$B_C^\varepsilon = (S_C : 0 : S_A + \varepsilon S)$ $B_A^\varepsilon = (S_C + \varepsilon S : 0 : S_A)$	$B_c^\varepsilon = (\varepsilon S : 0 : c^2)$
$C_a^\varepsilon = (a^2 : \varepsilon S : 0)$	$C_b^\varepsilon = (\varepsilon S : b^2 : 0)$	$C_A^\varepsilon = (S_B + \varepsilon S : S_A : 0)$ $C_B^\varepsilon = (S_B : S_A + \varepsilon S : 0)$

Proposition 1. *The triangle $X_\varepsilon Y_\varepsilon Z_\varepsilon$ and ABC perspective at the Vecten point*

$$V_\varepsilon = \left(\frac{1}{S_A + \varepsilon S} : \frac{1}{S_B + \varepsilon S} : \frac{1}{S_C + \varepsilon S}\right).$$

For V_+ and V_- are respectively X_{485} and X_{486} of [3].

1.2. *Inscribed squares of type II.* Another type of inscribed squares has two opposite vertices on a sideline of ABC. There are three such squares $\mathsf{Sq}^{\rm d}(A)$, $\mathsf{Sq}^{\rm d}(B)$, $\mathsf{Sq}^{\rm d}(C)$. The square $\mathsf{Sq}^{\rm d}(A)$ has two opposite vertices on the sideline BC. Its center X can be found as follows. The perpendicular at X to BC intersects CA and AB at B_a and C_a such that $B_aX + C_aX = 0$. If $X = (0 : v : w)$, it is easy to see that

$$\begin{aligned} B_aX =& CX \cdot \tan C = \frac{av}{S_C(v+w)}, \\ C_aX =& BX \cdot \tan B = \frac{aw}{S_B(v+w)}. \end{aligned}$$

It follows that $B_aX + C_aX = 0$ if and only if $v : w = -S_C : S_B$, and the center of $\mathsf{Sq}^{\rm d}(A)$ is the point $X = (0 : -S_C : S_B)$ on the line BC. The vertices can be easily determined, as given in Table 2 below.

Table 2. Centers and vertices of inscribed squares of type II

$\mathsf{Sq}^{\rm d}(A)$	$\mathsf{Sq}^{\rm d}(B)$	$\mathsf{Sq}^{\rm d}(C)$
$X = (0 : -S_C : S_B)$	$Y = (S_C : 0 : -S_A)$	$Z = (-S_B : S_A : 0)$
$A_+ = (0 : -S_C - S : S_B + S)$ $A_- = (0 : -S_C + S : S_B - S)$	$A_b = (0 : -b^2 : 2S_A)$	$A_c = (0 : 2S_A : -c^2)$
$B_a = (-a^2 : 0 : 2S_B)$	$B_+ = (S_C + S : 0 : -S_A - S)$ $B_- = (S_C - S : 0 : -S_A + S)$	$B_c = (2S_B : 0 : -c^2)$
$C_a = (-a^2 : 2S_C : 0)$	$C_b = (2S_C : -b^2 : 0)$	$C_+ = (-S_B - S : S_A + S : 0)$ $C_- = (-S_B + S : S_A - S : 0)$

2. Some collinearity relations

Proposition 2. (a) *The centers X, Y, Z are the intercepts of the orthic axis with the sidelines of triangle ABC.*
(b) *For $\varepsilon = \pm 1$, the points A_ε, B_ε and C_ε are collinear. The line containing them is parallel to the orthic axis.*

Proof. The line containing the points A_ε, B_ε and C_ε has equation

$$(S_A + \varepsilon S)x + (S_B + \varepsilon S)y + (S_C + \varepsilon S)z = 0.$$

See Figure 2. □

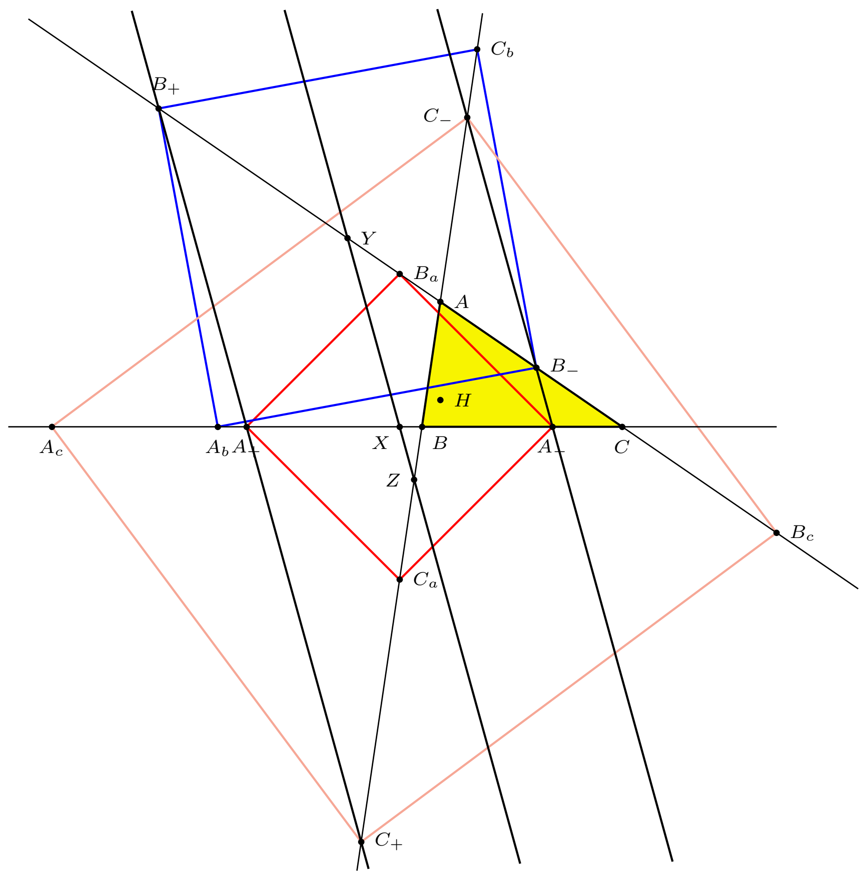


Figure 2

Proposition 3. (a) *The centers* X, Y_ε, Z_ε *of the squares* $\mathsf{Sq}^{\mathrm{d}}(A)$, $\mathsf{Sq}^{\varepsilon}(B)$, $\mathsf{Sq}^{\varepsilon}(C)$ *are collinear.*
(b) *The line* $B_C^\varepsilon C_B^\varepsilon$ *passes through the center* X *of* $\mathsf{Sq}^{\mathrm{d}}(A)$.
(c) *The line* $B_A^\varepsilon C_A^\varepsilon$ *passes through the point* A_ε.

Proof. (a) The line joining Y_ε and Z_ε has equation

$$-\varepsilon S x + S_B y + S_C z = 0$$

as is easily verified. This line clearly contains $X = (0 : -S_C : S_B)$.

(b) The line $B_C^\varepsilon C_B^\varepsilon$ has equation

$$-(S_A + \varepsilon S)x + S_B y + S_C z = 0.$$

It clearly passes through X.

(c) The line $B_A^\varepsilon C_A^\varepsilon$ has equation

$$-S_A x + (S_B + \varepsilon S)y + (S_C + \varepsilon S)z = 0.$$

It contains the point $A_\varepsilon = (0 : -S_C - \varepsilon S : S_B + \varepsilon S)$. See Figure 3 for $\varepsilon = 1$. $\square$

Remark. For $\varepsilon = \pm 1$, the lines in (b) and (c) above are parallel.

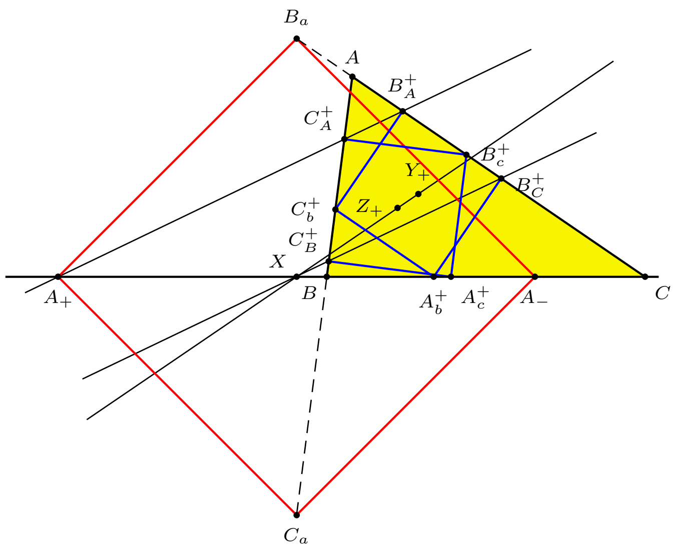


Figure 3

Let $T_A := B_A^+ C_A^+ \cap B_A^- C_A^- = (S_B - S_C : S_A : -S_A)$. The lines AT_A and BC are parallel. The three points T_A, T_B, T_C are collinear. The line connecting them has equation

$$S_A(S_B + S_C - S_A)x + S_B(S_C + S_A - S_B)y + S_C(S_A + S_B - S_C)z = 0.$$

Each of the squares of type II has a diagonal perpendicular to a sideline of triangle ABC. These diagonals clearly bound a triangle perspective to ABC with perspectrix the orthic axis. By [1] we know that the perspector lies on the circumcircle. Specifically, it is X_{74}, the Miquel perspector of the orthic axis.

The lines $B_A^\varepsilon C_A^\varepsilon$, $C_B^\varepsilon A_B^\varepsilon$, $A_C^\varepsilon B_C^\varepsilon$ bound a triangle perspective with ABC at the Kiepert perspector

$$K(\varepsilon \cdot \arctan 2) = \left(\frac{1}{2S_A + \varepsilon S} : \frac{1}{2S_B + \varepsilon S} : \frac{1}{2S_C + \varepsilon S} \right).$$

For $\varepsilon = +1$ and -1 respectively, these are X_{1131} and X_{1132} of [3]. The same perspector is found for the triangle bounded by the lines $B_C^\varepsilon C_B^\varepsilon$, $A_C^\varepsilon C_A^\varepsilon$, $A_B^\varepsilon B_A^\varepsilon$.

3. Inscribed squares and Miquel's theorem

We first recall Miquel's theorem.

Theorem 4 (Miquel)**.** *Let $A_1B_1C_1$ be a triangle inscribed in triangle ABC. There is a* pivot point *P such that $A_1B_1C_1$ is the image of the pedal triangle of P after a rotation about P followed by a homothety with center P. All inscribed triangles directly similar to $A_1B_1C_1$ have the same pivot point.*

A corollary of this theorem is for instance given in [2, Problem 8(ii), p.245].

Corollary 5. *Let X be a point defined with respect to the pedal triangle $A_PB_PC_P$ triangle of P. The images of X after the pivoting as in Miquel's theorem lie on a line.*

Proof. Let $A_2B_2C_2$ be the image of $A_PB_PC_P$ after pivoting, and let Y be the image of X. Clearly triangles PA_PA_2, PB_PB_2, PC_PC_2, and PXY are similar right triangles. This shows that Y lies on the line through X perpendicular to XP. □

Miquel's pivot theorem and Corollary 5 together give an easy explanation of Proposition 3(c). See Figure 4.

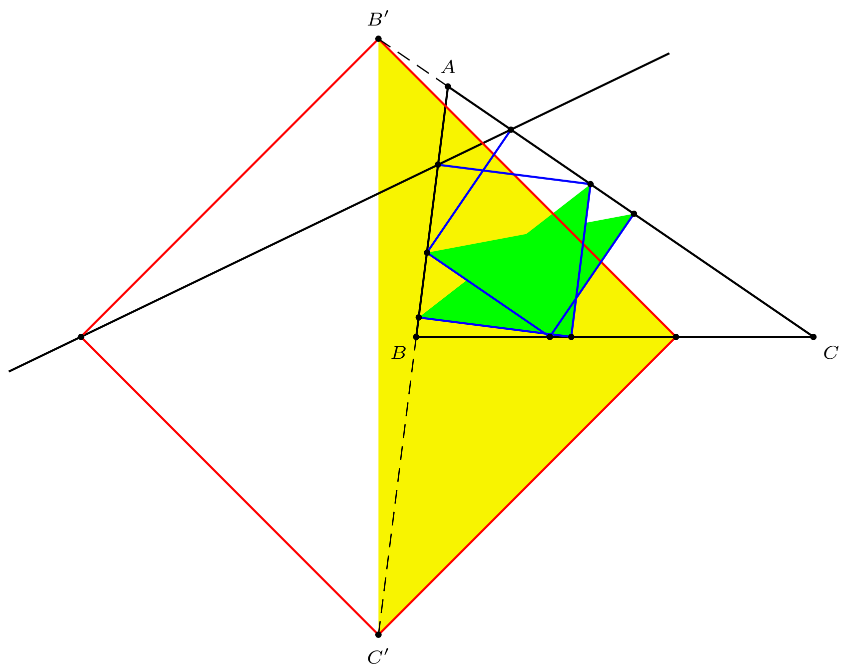


Figure 4

We have already seen that the centers of the inscribed squares of type II lie on the orthic axis. By Proposition 3(a), these centers are the intersections of the corresponding sides of the triangles $X_+Y_+Z_+$ and $X_-Y_-Z_-$ of the inscribed squares of type I. This means that the triangles $X_+Y_+Z_+$ and $X_-Y_-Z_-$ are perspective. The perspector is symmedian point $K = (a^2 : b^2 : c^2)$.

4. Squares with vertices on four given lines

Let us consider a fourth line in the plane of ABC. With the help of the inscribed squares of type I, we can construct two sets of three squares inscribing a fourline $\{\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}\}$, depending on the line containing the vertex opposite to that on d. Let ABC be the triangle bounded by the lines a, b, c. For $\varepsilon = \pm 1$, there is a square $\mathsf{Sq}^\varepsilon(\mathsf{a}) := A^\varepsilon_\mathsf{a} B^\varepsilon_\mathsf{a} D^\varepsilon_\mathsf{a} C^\varepsilon_\mathsf{a}$ with a pair of opposite vertices on a and d. The vertex on d is simply $D^\varepsilon_\mathsf{a} = B^\varepsilon_A C^\varepsilon_A \cap \mathsf{d}$. See the solution of Problem 55(a) of [5, p.146]. The other vertices of the square are determined by the same division ratio (of $B^\varepsilon_A C^\varepsilon_A$ by D^ε_a):

$$B^\varepsilon_A C^\varepsilon_A : C^\varepsilon_A D^\varepsilon_\mathsf{a} = A^\varepsilon_b A^\varepsilon_c : A^\varepsilon_c A^\varepsilon_\mathsf{a} = B^\varepsilon_C B^\varepsilon_c : B^\varepsilon_c B^\varepsilon_\mathsf{a} = C^\varepsilon_b C^\varepsilon_B : C^\varepsilon_B C^\varepsilon_\mathsf{a}.$$

See Figure 5 for $\varepsilon = +1$. In fact, if $D^\varepsilon_\mathsf{a} = (S_C + \varepsilon S, 0, S_A) + t(S_B + \varepsilon S, S_A, 0)$, then

$$\begin{aligned}
A^\varepsilon_\mathsf{a} =& (0, b^2, \varepsilon S) + t(0, \varepsilon S, c^2),\\
B^\varepsilon_\mathsf{a} =& (S_C, 0, S_A + \varepsilon S) + t(\varepsilon S, 0, c^2),\\
C^\varepsilon_\mathsf{a} =& (\varepsilon S, b^2, 0) + t(S_B, S_A + \varepsilon S, 0),
\end{aligned}$$

and the center of the square is the point

$$X^\varepsilon_\mathsf{a} = (S_C + \varepsilon S, b^2, S_A + \varepsilon S) + t(S_B + \varepsilon S, S_A + \varepsilon S, c^2).$$

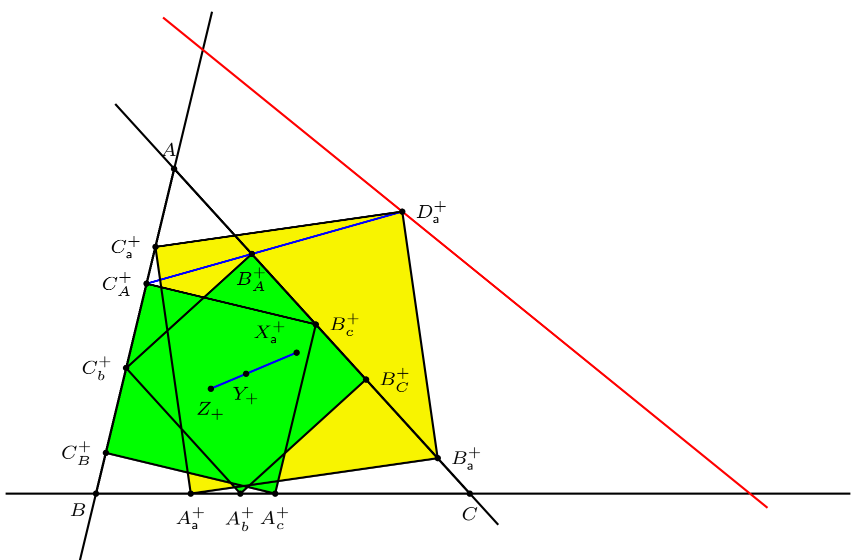


Figure 5

It is now clear that the position of A^ε_a relative to A^ε_b and A^ε_c fixes D^ε_a as well, even if we do not have a given line d. Similarly we D^ε_b and D^ε_c are fixed by B^ε_b and C^ε_c respectively. We may thus take A^ε_a, B^ε_b and C^ε_c to be the traces of a point $P = (u : v : w)$ and see if the corresponding D^ε_a, D^ε_b and D^ε_c are collinear. A simple calculation gives

$$D_{\mathsf{a}}^{\varepsilon} =((S_B - \varepsilon S)v + (S_C - \varepsilon S)w : \varepsilon Sv - b^2 w : \varepsilon Sw - c^2 v),$$
$$D_{\mathsf{b}}^{\varepsilon} =(\varepsilon Su - a^2 w : (S_C - \varepsilon S)w + (S_A - \varepsilon S)u : \varepsilon Sw - c^2 u),$$
$$D_{\mathsf{c}}^{\varepsilon} =(\varepsilon Su - a^2 v : \varepsilon Sv - b^2 u : (S_A - \varepsilon S)u + (S_B - \varepsilon S)v).$$

Also, the centers of the squares $\mathsf{Sq}^{\mathrm{d}}(A)$, $\mathsf{Sq}^{\mathrm{d}}(B)$, $\mathsf{Sq}^{\mathrm{d}}(C)$ are the points

$$X_{\mathsf{a}}^{\varepsilon} =(-(S_B - \varepsilon S)v - (S_C - \varepsilon S)w : (S_A - \varepsilon S)v + b^2 w : c^2 v + (S_A - \varepsilon S)w),$$
$$Y_{\mathsf{b}}^{\varepsilon} =(a^2 w + (S_B - \varepsilon S)u : -(S_C - \varepsilon S)w - (S_A - \varepsilon S)u : (S_B - \varepsilon S)w + c^2 u),$$
$$Z_{\mathsf{c}}^{\varepsilon} =((S_C - \varepsilon S)u + a^2 v : b^2 u + (S_C - \varepsilon S)v : -(S_A - \varepsilon S)u - (S_B - \varepsilon S)v).$$

Proposition 6. *Let $A_{\mathsf{a}}^{\varepsilon}$, $B_{\mathsf{b}}^{\varepsilon}$ and $C_{\mathsf{c}}^{\varepsilon}$ be the traces of a point $P = (u : v : w)$.* (a) *The three points $D_{\mathsf{a}}^{\varepsilon}$, $D_{\mathsf{b}}^{\varepsilon}$ and $D_{\mathsf{c}}^{\varepsilon}$ are collinear if and only if P lies on the circumcubic*

$$4a^2b^2c^2uvw + S^2 \sum_{\text{cyclic}} u((2S_A + S_B)v^2 + (2S_A + S_B)w^2)$$
$$=\varepsilon S \left(2S^2 uvw + \sum_{\text{cyclic}} u((2c^2a^2 - S_{AB})v^2 + (2a^2b^2 - S_{CA})w^2) \right).$$

(b) *The centers of the squares $\mathsf{Sq}^{\mathrm{d}}(A)$, $\mathsf{Sq}^{\mathrm{d}}(B)$, $\mathsf{Sq}^{\mathrm{d}}(C)$ are collinear if and only if*

$$2a^2b^2c^2uvw + S^2 \sum_{\text{cyclic}} u(c^2v^2 + b^2w^2)$$
$$=\varepsilon S \left(2S^2 uvw + \sum_{\text{cyclic}} a^2 u(c^2v^2 + b^2w^2) \right).$$

Remarks. (1) The locus of P for which $D_{\mathsf{a}}^{\varepsilon}D_{\mathsf{b}}^{\varepsilon}D_{\mathsf{c}}^{\varepsilon}$ and ABC are perspective is the isogonal cubic with pivot $(a^2 + \varepsilon S : b^2 + \varepsilon S : c^2 + \varepsilon S)$.

(2) The locus of P for which $X_{\mathsf{a}}^{\varepsilon}Y_{\mathsf{b}}^{\varepsilon}Z_{\mathsf{c}}^{\varepsilon}$ and ABC are perspective is the isogonal cubic with pivot H. Here are some examples of the perspectors for P on the cubic.

Table 3. Perspectors of $X_{\mathsf{a}}^{\varepsilon}Y_{\mathsf{b}}^{\varepsilon}Z_{\mathsf{c}}^{\varepsilon}$ for $\varepsilon = \pm 1$

P	$\varepsilon = +1$	$\varepsilon = -1$
I	I	I
O	$X_{372} = (a^2(S_A - S) : \cdots : \cdots)$	$X_{371} = (a^2(S_A + S) : \cdots : \cdots)$
H	$X_{486} = \left(\frac{1}{S_A - S} : \cdots : \cdots\right)$	$X_{485} = \left(\frac{1}{S_A + S} : \cdots : \cdots\right)$
X_{485}	G	$(a^2 + S : \cdots : \cdots)$
X_{486}	$(a^2 - S : \cdots : \cdots)$	G
X_{487}	$\left(\frac{1}{b^2c^2 + S_{BC} - (S_A + S_B + S_C)S} : \cdots : \cdots\right)$	$\left(\frac{S_A - a^2}{S_A - S} : \cdots : \cdots\right)$
X_{488}	$\left(\frac{S_A - a^2}{S_A + S} : \cdots : \cdots\right)$	$\left(\frac{1}{b^2c^2 + S_{BC} + (S_A + S_B + S_C)S} : \cdots : \cdots\right)$

(3) In comparison with Proposition 6 (a), if instead of traces, we take A_a^ε, B_b^ε and C_c^ε to be the *pedals* of a point P on the sidelines of ABC, then the locus of P for which D_a^ε, D_b^ε and D_c^ε are collinear turns out to be a conic, though with equation too complicated to record here.

References

[1] J. T. Groenman, Problem 578, *Nieuw Archief voor Wiskunde*, ser. 3, 28 (1980) 202; solution, 29 (1981) 115–118.

[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[3] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[4] F. M. van Lamoen and P. Yiu, The Kiepert pencil of Kiepert hyperbolas, *Forum Geom.*, 1 (2001) 125–132.

[5] I. M. Yaglom, *Geometric Transformations* II, Random House, New York, 1968.

Floor van Lamoen: St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands
E-mail address: fvanlamoen@planet.nl

Forum Geometricorum
Volume 4 (2004) 215–218.

FORUM GEOM
ISSN 1534-1178

On the Existence of Triangles with Given Lengths of One Side and Two Adjacent Angle Bisectors

Victor Oxman

Abstract. We give a necessary and sufficient condition for the existence of a triangle with given lengths of one side and the two adjacent angle bisectors.

1. Introduction

It is known that given three lengths ℓ_1, ℓ_2, ℓ_3, there is always a triangle whose three internal angle bisectors have lengths ℓ_1, ℓ_2, ℓ_3. See [1]. In this note we consider the question of existence and uniqueness of a triangle with given lengths of one side and the bisectors of the two angles adjacent to it. Recall that in a triangle ABC with sidelengths a, b, c, the bisector of angle A (with opposite side a) has length

$$\ell = \frac{2bc}{b+c}\cos\frac{A}{2} = \sqrt{bc\left(1-\frac{a^2}{(b+c)^2}\right)}. \tag{1}$$

We shall prove the following theorem.

Theorem 1. *Given a, ℓ_1, $\ell_2 > 0$, there is a unique triangle ABC with $BC = a$, and the lengths of the bisectors of angles B, C equal to ℓ_1 and ℓ_2 if and only if*

$$\sqrt{\ell_1^2+\ell_2^2} < 2a < \ell_1+\ell_2+\sqrt{\ell_1^2-\ell_1\ell_2+\ell_2^2}.$$

2. Uniqueness

First we prove that if such a triangle exists, then it is unique.

Denote the sidelengths of the triangle by a, x, y. If the angle bisectors on the sides x and y have lengths ℓ_1 and ℓ_2 respectively, then from (1) above,

$$y =(a+x)\sqrt{1-\frac{t_2}{x}}, \tag{2}$$

$$x =(a+y)\sqrt{1-\frac{t_1}{y}}, \tag{3}$$

where $t_1 = \frac{\ell_1^2}{a}$, $t_2 = \frac{\ell_2^2}{a}$, $(t_1 < y, t_2 < x)$.

Publication Date: December 13, 2004. Communicating Editor: Paul Yiu.

Let $t > 0$. We consider the function $y : (t, \infty) \to (0, \infty)$ defined by

$$y(x) = (a + x)\sqrt{1 - \frac{t}{x}}.$$

Obviously, y is a continuous function on the interval (t, ∞). It is increasing and has an oblique asymptote $y = x + a - \frac{t}{2}$. It is easy to check that $y'' < 0$ in (t, ∞), so that y is a *convex* funcion and its graph is below its oblique asymptote. See Figure 1.

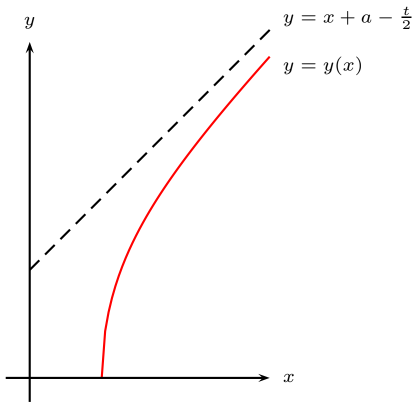


Figure 1

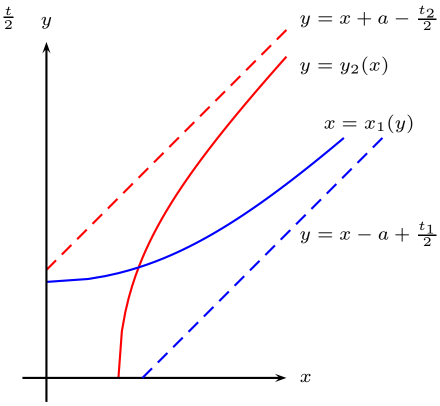


Figure 2

Now consider the system of equations

$$y = (a + x)\sqrt{1 - \frac{t}{x}}, \tag{4}$$

$$x = (a + y)\sqrt{1 - \frac{t}{y}}. \tag{5}$$

It is obvious that if a pair (x, y) satisfies (4), the pair (y, x) satisfies (5), and conversely. These equations therefore define inverse functions, and (5) defines a *concave* function $(0, \infty) \to (t, \infty)$ with an oblique asymptote $y = x - a + \frac{t}{2}$.

Applying to functions $y = y_2(x)$ and $x = x_1(y)$ defined by (2) and (3) respectively, we conclude that the system of equations (2), (3) cannot have more than one solution. See Figure 2.

Proposition 2. *If the side and the bisectors of the adjacent angles of triangle are respectively equal to the side and the bisectors of the adjacent angles of another triangle, then the triangles are congruent.*

Corollary 3 (Steiner-Lehmus theorem)**.** *If a triangle has two equal bisectors, then it is an isosceles triangle.*

Indeed, if the bisectors of the angles A and C of triangle ABC are equal, then triangle ABC is congruent to CBA, and so $AB = CB$.

3. Existence

Now we consider the question of existence of a triangle with given a, ℓ_1 and ℓ_2.

First of all note that in order for the system of equations (2), (3) to have a solution, it is necessary that $x + a - \frac{t_2}{2} > x - a + \frac{t_1}{2}$. Geometrically, this means that the asymptote of (2) is above that of (3). Thus, $2a > \frac{t_1+t_2}{2} = \frac{\ell_1^2+\ell_2^2}{2a}$, and

$$2a > \sqrt{\ell_1^2 + \ell_2^2}. \tag{6}$$

For the three lengths a, x, y to satisfy the triangle inequality, note that from (2) and (3), we have $y < a + x$ and $x < a + y$. If $x > a$ or $y > a$, then clearly $x + y > a$. We shall therefore restrict to $x < a$ and $y < a$.

Let BC be a given segment of length a. Consider a point Y in the plane such that the bisector of angle B of triangle YBC has a given length ℓ_1. It is easy to see from (1) that the length of BY is given by

$$y = \frac{a\ell_1}{2a\cos\frac{\theta}{2} - \ell_1} \quad \text{if } \angle CBY = \theta. \tag{7}$$

Let $\alpha = 2\arccos\frac{\ell_1}{2a}$. (7) defines a monotonic increasing function $y = y(\theta)$: $(0,\alpha) \to \left(\frac{a\ell_1}{2a-\ell_1}, \infty\right)$. It is easy to check that for $\theta \in (0,\alpha)$,

$$y > \frac{a\ell_1}{2a - \ell_1} > y\cos\theta.$$

The locus of Y is a continuous curve ξ_1 beginning at (but not including) a point M on the interval BC with $BM = \frac{a\ell_1}{2a-\ell_1}$. It has an oblique asymptote which forms an angle α with the line BC. See Figure 3. Since we are interested only in the case $y < a$, we may assume $a > \ell_1$. The angle α exceeds $\frac{2\pi}{3}$.

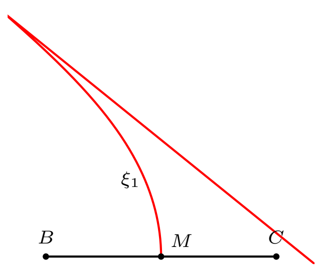


Figure 3

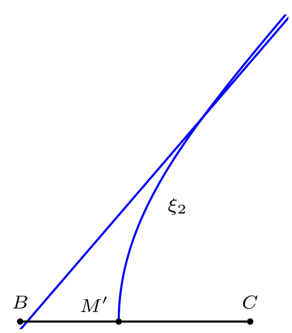


Figure 4

Consider now the locus of point Z such that the bisector of angle C of triangle ZBC has length $\ell_2 < a$. The same reasoning shows that this is a curve ξ_2 beginning at (but not including) a point M' on BC such that $M'C = \frac{a\ell_2}{2a-\ell_2}$, which has an oblique asymptote making an angle $2\arccos\frac{\ell_2}{2a}$ with CB. Again, this angle exceeds $\frac{2\pi}{3}$. See Figure 4.

The two curves ξ_1 and ξ_2 intersect if and only if $BM > BM'$, *i.e.*, $BM + M'C > a$. This gives

$$\frac{\ell_1}{2a-\ell_1} + \frac{\ell_2}{2a-\ell_2} > 1.$$

Simplifying, we have $4a^2 - 4a(\ell_1+\ell_2) + 3\ell_1\ell_2 < 0$, or

$$\ell_1 + \ell_2 - \sqrt{\ell_1^2 - \ell_1\ell_2 + \ell_2^2} < 2a < \ell_1 + \ell_2 + \sqrt{\ell_1^2 - \ell_1\ell_2 + \ell_2^2}.$$

Since $a > \ell_1, \ell_2$, the first inequality always holds. Comparison with (6) now completes the proof of Theorem 1.

In particular, for the existence of an isosceles triangle with base a and bisectors of the equal angles of length ℓ, it is necessary and sufficient that $\frac{\sqrt{2}}{2} < \frac{a}{\ell} < \frac{3}{2}$.

Reference

[1] P. Mironescu and L. Panaitopol, The existence of a triangle with prescribed angle bisector lengths, *Amer. Math. Monthly*, 101 (1994) 58–60.

Victor Oxman (Western Galilee College): Derech HaYam 191a, Haifa, 34890, Israel
E-mail address: voxman@research.haifa.ac.il

Forum Geometricorum
Volume 4 (2004) 219–224.

FORUM GEOM
ISSN 1534-1178

A Purely Synthetic Proof of the Droz-Farny Line Theorem

Jean-Louis Ayme

Abstract. We present a purely synthetic proof of the theorem on the Droz-Farny line, and a brief biographical note on Arnold Droz-Farny.

1. The Droz-Farny line theorem

In 1899, Arnold Droz-Farny published without proof the following remarkable theorem.

Theorem 1 (Droz-Farny [2]). *If two perpendicular straight lines are drawn through the orthocenter of a triangle, they intercept a segment on each of the sidelines. The midpoints of these three segments are collinear.*

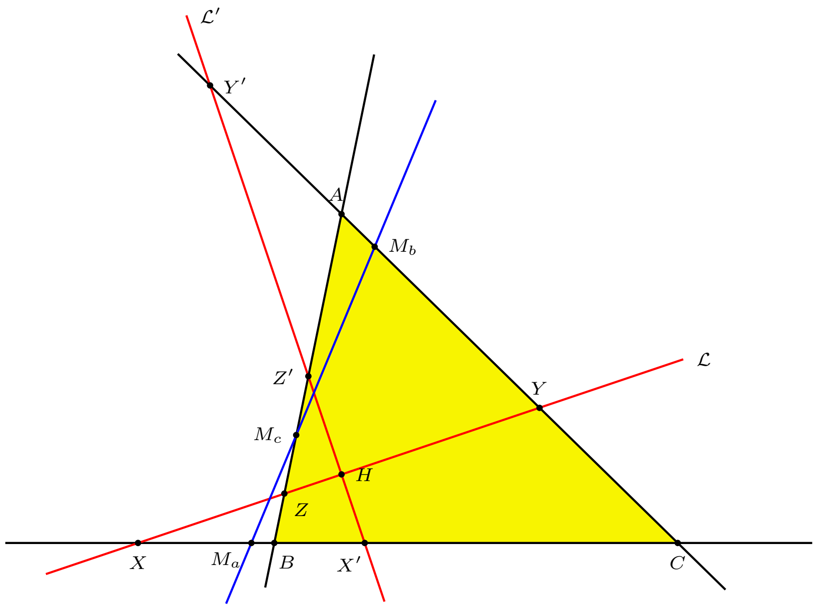


Figure 1.

Figure 1 illustrates the Droz-Farny line theorem. The perpendicular lines $\mathcal{L}$ and $\mathcal{L}'$ through the orthocenter H of triangle ABC intersect the sidelines BC at X, X', CA at Y, Y', and AB at Z, Z' respectively. The midpoints M_a, M_b, M_c of the segments XX', YY', ZZ' are collinear.

It is not known if Droz-Farny himself has given a proof. The Droz-Farny line theorem was presented again without any proof in 1995 by Ross Honsberger [9,

Publication Date: December 16, 2004. Communicating Editor: Floor van Lamoen.

p.72]. It also appeared in 1986 as Problem II 206 of [16, pp.111,311-313] without references but with an analytic proof. This "remarkable theorem", as it was named by Honsberger, has been the subject of many recent messages in the Hyacinthos group. If Nick Reingold [15] proposes a projective proof of it, he does not yet show that the considered circles intersect on the circumcircle. Darij Grinberg taking up an elegant idea of Floor van Lamoen presents a first trigonometric proof of this "rather difficult theorem" [5, 12, 3] which is based on the pivot theorem and applied on degenerated triangles. Grinberg also offers a second trigonometric proof, which starts from a generalization of the Droz-Farny's theorem simplifying by the way the one of Nicolaos Dergiades and gives a demonstration based on the law of sines [6]. Milorad Stevanović [17] presents a vector proof. Recently, Grinberg [8] picks up an idea in a newsgroup on the internet and proposes a proof using inversion and a second proof using angle chasing. In this note, we present a purely synthetic proof.

2. Three basic theorems

Theorem 2 (Carnot[1, p.101]). *The segment of an altitude from the orthocenter to the side equals its extension from the side to the circumcircle.*

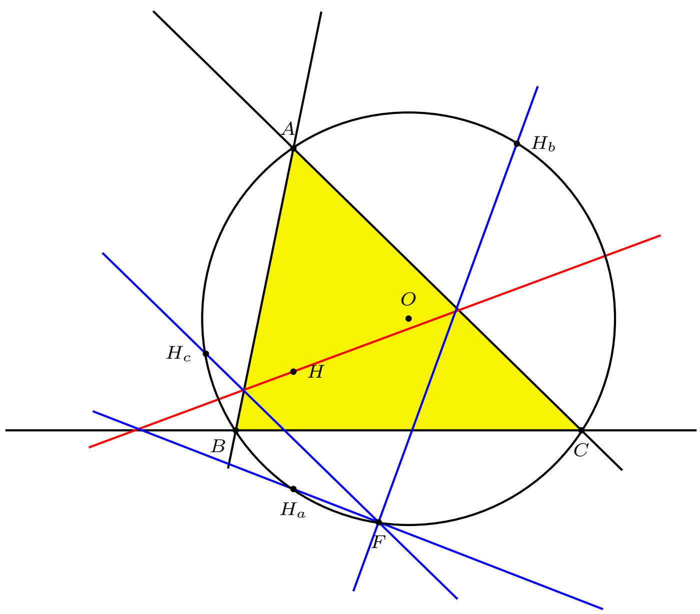


Figure 2.

Theorem 3. *Let $\mathcal{L}$ be a line through the orthocenter of a triangle ABC. The reflections of $\mathcal{L}$ in the sidelines of ABC are concurrent at a point on the circumcircle.*

See [11, p.99] or [10, §333].

Theorem 4 (Miquel's pivot theorem [13])**.** *If a point is marked on each side of a triangle, and through each vertex of the triangle and the marked points on the adjacent sides a circle is drawn, these three circles meet at a point.*

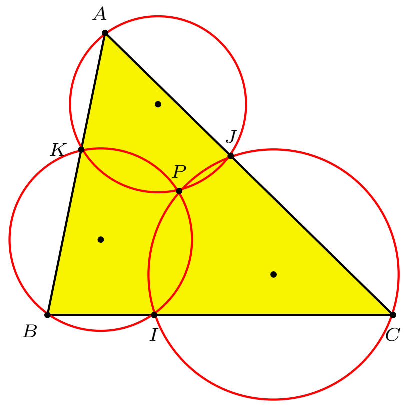


Figure 3.

See also [10, §184, p.131]. This result stays true in the case of tangency of lines or of two circles. Very few geometers contemporary to Miquel had realised that this result was going to become the spring of a large number of theorem.

3. A synthetic proof of Theorem 1

The right triangle case of the Droz-Farny theorem being trivial, we assume triangle ABC not containing a right angle. Let $\mathcal{C}$ be the circumcircle of ABC.

Let $\mathcal{C}_a$ (respectively $\mathcal{C}_b$, $\mathcal{C}_c$) be the circumcircle of triangle HXX' (respectively HYY', HZZ'), and H_a (respectively H_b, H_c) be the symmetric point of H in the line BC (respectively CA, AB). The circles $\mathcal{C}_a$, $\mathcal{C}_b$ and $\mathcal{C}_c$ have centers M_a, M_b and M_c respectively.

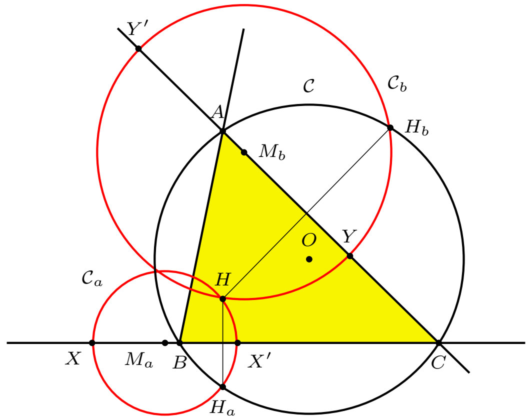


Figure 4.

According to Theorem 2, H_a is on the circle $\mathcal{C}$. XX' being a diameter of the circle $\mathcal{C}_a$, H_a is on the circle. Consequently, H_a is an intersection of $\mathcal{C}$ and $\mathcal{C}_a$, and the perpendicular to BC through H. In the same way, H_b is an intersection of $\mathcal{C}$ and $\mathcal{C}_b$, and the perpendicular to CA through H. See Figure 4.

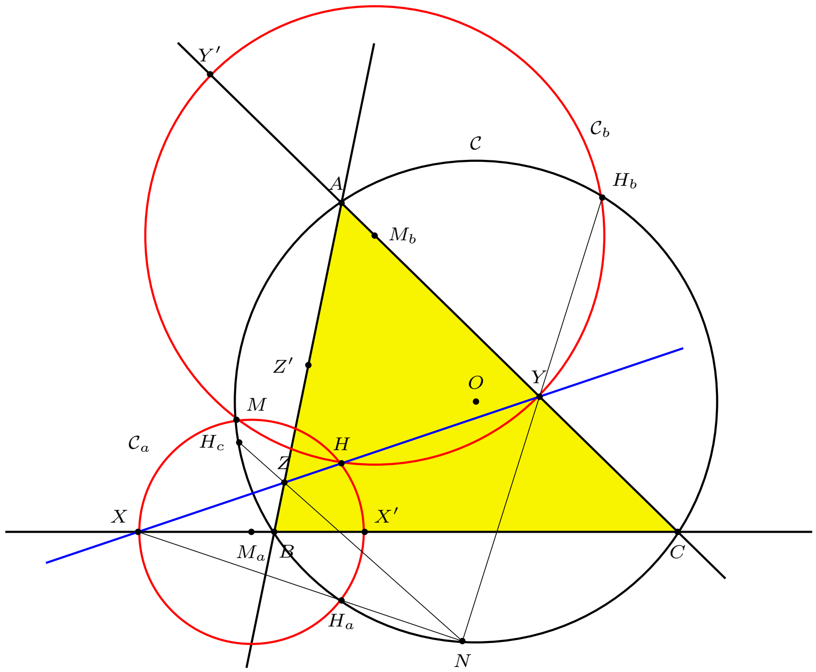


Figure 5.

Consider the point H_c, the symmetric of H in the line AB. According to Theorem 2, H_a is on the circle $\mathcal{C}$. Applying Theorem 3 to the line XYZ through H, we conclude that the lines H_aX, H_bY and H_cZ intersect at a point N on the circle $\mathcal{C}$. See Figure 5.

Applying Theorem 4 to the triangle XNY with the points H_a, H_b and H (on the lines XN, NY and YX respectively), we conclude that the circles $\mathcal{C}$, $\mathcal{C}_a$, and $\mathcal{C}_b$ pass through a common point M.

Mutatis mutandis, we show that the circles $\mathcal{C}$, $\mathcal{C}_b$, and $\mathcal{C}_c$ also pass through the same point M.

The circle $\mathcal{C}_a$, $\mathcal{C}_b$, and $\mathcal{C}_c$, all passing through H and M, are coaxial. Their centers are collinear. This completes the proof of Theorem 1.

4. A biographical note on Arnold Droz-Farny

Arnold Droz, son of Edouard and Louise Droz, was born in La Chaux-de-Fonds (Switzerland) on February 12, 1856. After his studies in the canton of Neufchatel, he went to Munich (Germany) where he attended lectures given by Felix Klein, but he finally preferred geometry. In 1880, he started teaching physics and mathematics in the school of Porrentruy (near Basel) where he stayed until 1908. He is known for having written four books between 1897 and 1909, two of them about geometry. He also published in the *Journal de Mathématiques Élementaires et*

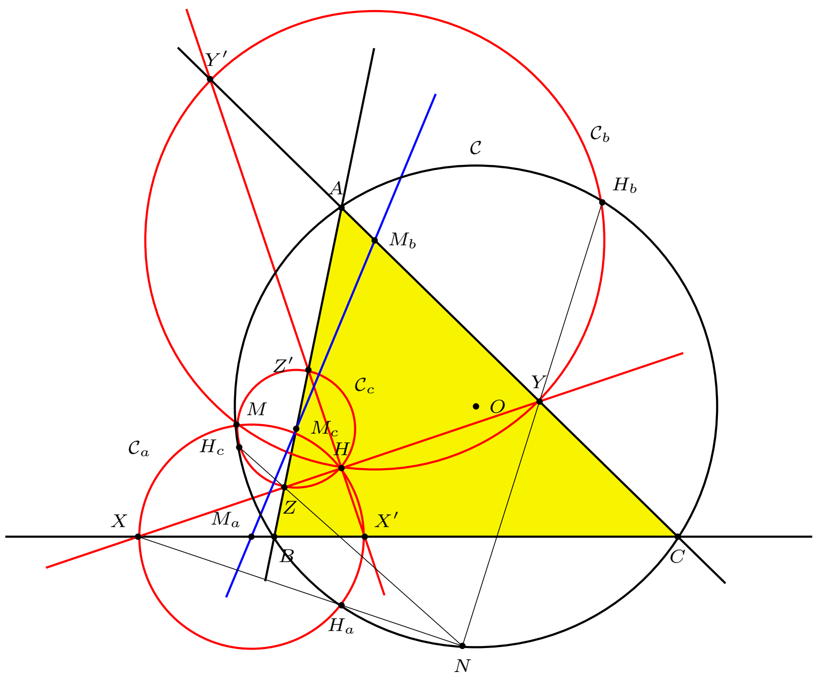


Figure 6.

Spéciales (1894, 1895), and in *L'intermédiaire des Mathématiciens* and in the *Educational Times* (1899) as well as in *Mathesis* (1901). As he was very sociable, he liked to be in contact with other geometers likes the Italian Virginio Retali and the Spanish Juan Jacobo Duran Loriga. In his free time, he liked to climb little mountains and to watch horse races. He was married to Lina Farny who was born also in La Chaux-de-Fonds. He died in Porrentruy on January 14, 1912 after having suffered from a long illness. See [4, 14].

References

[1] L. N. M. Carnot, *De la corrélation des figures géométriques*, 1801.
[2] A. Droz-Farny, Question 14111, *Ed. Times* 71 (1899) 89-90.
[3] J.-P. Ehrmann, Hyacinthos messages 6150, 6157, December 12, 2002.
[4] J. Gonzalez Cabillon, message to Historia Matematica, August 18, 2000, available at http://mathforum.org/epigone/historia_matematica/dityjerd.
[5] D. Grinberg, Hyacinthos messages 6128, 6141, 6245, December 10-11, 2002.
[6] D. Grinberg, From the complete quadrilateral to the Droz-Farny theorem, available from http://de.geocities.com/darij_grinberg.
[7] D. Grinberg, Hyacinthos message 7384, July 23, 2003.
[8] D. Grinberg, Hyacinthos message 9845, June 2, 2004.
[9] R. Honsberger, *Episodes of 19th and 20th Century Euclidean Geometry*, Math. Assoc. America, 1995.
[10] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.
[11] T. Lalesco, *La Géométrie du Triangle*, 1916; Jacques Gabay reprint, Paris, 1987.
[12] F. M. van Lamoen, Hyacinthos messages 6140, 6144, December 11, 2002.

[13] A. Miquel, Mémoire de Géométrie, *Journal de mathématiques pures et appliquées de Liouville* 1 (1838) 485-487.
[14] E. Ortiz, message to Historia Matematica, August 21, 2000, available at http://mathforum.org/epigone/historia_matematica/dityjerd.
[15] N. Reingold, Hyacinthos message 7383, July 22, 2003.
[16] I. Sharygin, *Problemas de Geometria*, (Spanish translation), Mir Edition, 1986.
[17] M. Stevanović, Hyacinthos message 9130, January 25, 2004.

Jean-Louis Ayme: 37 rue Ste-Marie, 97400 St.-Denis, La Réunion, France
E-mail address: jeanlouisayme@yahoo.fr

Forum Geometricorum
Volume 4 (2004) 225–227.

FORUM GEOM
ISSN 1534-1178

A Projective Generalization of the Droz-Farny Line Theorem

Jean-Pierre Ehrmann and Floor van Lamoen

Dedicated to the fifth anniversary of
the Hyacinthos group on triangle geometry

Abstract. We give a projective generalization of the Droz-Farny line theorem.

Ayme [1] has given a simple, purely synthetic proof of the following theorem by Droz-Farny.

Theorem 1 (Droz-Farny [1])**.** *If two perpendicular straight lines are drawn through the orthocenter of a triangle. they intercept a segment on each of the sidelines. The midpoints of these three segments are collinear.*

In this note we give and prove a projective generalization. We begin with a simple observation. Given triangle ABC and a point S, the perpendiculars to AS, BS, CS at A, B, C respectively concur if and only if S lies on the circumcircle of ABC. In this case, their common point is the antipode of S on the circumcircle.

Now, consider 5 points A, B, C, I, I' lying on a conic $\mathcal{E}$ and a point S not lying on the line II'. Using a projective transformation mapping the circular points at infinity to I and I', we obtain the following.

Proposition 2. *The polar lines of S with respect to the pairs of lines (AI, AI'), (BI, BI'), (CI, CI') concur if and only if S lies on $\mathcal{E}$. In this case, their common point lies on $\mathcal{E}$ and on the line joining S to the pole of II' with respect to $\mathcal{E}$.*

The dual form of this proposition is the following.

Theorem 3. *Let ℓ and ℓ' be two lines intersecting at P, tangent to the same inscribed conic $\mathcal{E}$, and d be a line not passing through P. Let X, Y, Z (respectively X', Y', Z'; X_{d}, Y_{d}, Z_{d}) be the intersections of ℓ (respectively ℓ', d) with the sidelines BC, CA, AB. If X'_{d} is the harmonic conjugate of X_{d} with respect to (X, X'), and similarly for Y'_{d} and Z'_{d}, then X'_{d}, Y'_{d}, Z'_{d} lie on a same line d' if and only if d touches $\mathcal{E}$. In this case, d' touches $\mathcal{E}$ and the intersection of d and d' lies on the polar of P with respect to $\mathcal{E}$.*

Publication Date: December 22, 2004. Communicating Editor: Bernard Gibert.
The authors thank Paul Yiu for his help in the preparation of this paper.

An equivalent condition is that A, B, C and the vertices of the triangle with sidelines ℓ, ℓ', d lie on a same conic.

More generally, consider points X'_{d}, Y'_{d} and Z'_{d} such that the cross ratios

$$(X, X', X_{\mathsf{d}}, X'_{\mathsf{d}}) = (Y, Y', Y_{\mathsf{d}}, Y'_{\mathsf{d}}) = (Z, Z', Z_{\mathsf{d}}, Z'_{\mathsf{d}}).$$

These points X'_{d}, Y'_{d}, Z'_{d} lie on a line d' if and only if d is tangent to $\mathcal{E}$. This follows easily from the dual of Steiner's theorem and its converse: two points P, Q lie on a conic through four given points A, B, C, D if and only if the cross ratios

$$(PA, PB, PC, PD) = (QA, QB, QC, QD).$$

If in Theorem 3 we take for d the line at infinity, we obtain the following.

Corollary 4. *The midpoints of XX', YY', ZZ' lie on a same line d' if and only if ℓ and ℓ' touch the same inscribed parabola. In this case, if ℓ and ℓ touch the parabola at M and M', d' is the tangent to the parabola parallel to MM'.*

An equivalent condition is that the circumhyperbola through the infinite points of ℓ and ℓ' passes through P.

We shall say that (ℓ, ℓ') is a pair of DF-lines if it satisfies the conditions of Corollary 4 above.

Now, if ℓ and ℓ' are perpendicular, we get immediately:

(a) if $P = H$, then (ℓ, ℓ') is a pair of DF-lines because H lies on any rectangular circumhyperbola, or, equivalently, on the directrix of any inscribed parabola. This is the Droz-Farny line theorem (Theorem 1 above).

(b) if $P \neq H$, then (ℓ, ℓ') is a pair of DF-lines if and only if they are the tangents from P to the inscribed parabola with directrix HP, or, equivalently, they are the parallels at P to the asymptotes of the rectangular circumhyperbola through P.

Remarks. (1) The focus of the inscribed parabola touching ℓ is the Miquel point F of the complete quadrilateral formed by AB, BC, CA, ℓ, and the directrix is the Steiner line of F. See [3].

(2) If the circle through F and with center P intersects the directrix at M, M', the tangents from P to the parabola are the perpendicular bisectors of FM and FM'.

(3) The tripoles of tangents to an inscribed parabola are collinear in a line through G.

(4) Let A_ℓ, B_ℓ, C_ℓ be the intercepts of ℓ on the sides of ABC. Let A_{r}, B_{r}, C_{r} be the reflections of these intercepts through the midpoints of the corresponding sides. Then A_{r}, B_{r}, and C_{r} are collinear on the "isotomic conjugate" of ℓ. Clearly, the isotomic conjugates of lines from a pencil are tangents to an inscribed conic and vice versa. In the case of inscribed parabolas, as above, the isotomic conjugates of the tangents are a pencil of parallel lines. It is trivial that lines dividing in equal ratios the intercepted segments by two parallel lines are again parallel. So, by isotomic conjugation of lines this holds for tangents to a parabola as well.

These remarks lead to a number of simple constructions of pairs of DF-lines satisfying a given condition.

References

[1] J.-L. Ayme, A synthetic proof of the Droz-Farny line theorem, *Forum Geom.*, 4 (2004) 219–224.
[2] H. S. M. Coxeter, *The Real Projective Plane*, 3rd edition, Springer-Verlag, 1992.
[3] J.-P. Ehrmann, Steiner's theorems on the complete quadrilateral, *Forum Geom.*, 4 (2004) 35–56.

Jean-Pierre Ehrmann: 6, rue des Cailloux, 92110 - Clichy, France
E-mail address: `Jean-Pierre.EHRMANN@wanadoo.fr`

Floor van Lamoen, St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands
E-mail address: `fvlamoen@stwillibrord.nl`

Forum Geometricorum
Volume 4 (2004) 229–251.

FORUM GEOM
ISSN 1534-1178

The Twin Circles of Archimedes in a Skewed Arbelos

Hiroshi Okumura and Masayuki Watanabe

Abstract. Any area surrounded by three mutually touching circles is called a skewed arbelos. The twin circles of Archimedes in the ordinary arbelos can be generalized to the skewed arbelos. The existence of several pairs of twin circles, under certain conditions, is demonstrated.

1. Introduction

Let O be an arbitrary point on the segment AB in the plane and α, β and γ the semicircles on the same side of the diameters AO, BO and AB, respectively. The area surrounded by the three semicircles is called an arbelos or a shoemaker's knife (see Figure 1). The common internal tangent of α and β divides the arbelos into two curvilinear triangles and the incircles of these triangles are congruent. They are called the twin circles of Archimedes or Archimedean twin circles. The authors of [3] pose the following question: Is it possible to find any interesting properties of a "skewed arbelos", in which the centers of the three circles α, β and γ are not collinear (see Figure 2), without resorting to trigonometry? In this article, we show several interesting properties of the skewed arbelos, one of them being the existence, in certain situations, of up to four pairs of twin circles. This property is a generalization of the existence of the twin circles of Archimedes in the ordinary arbelos.

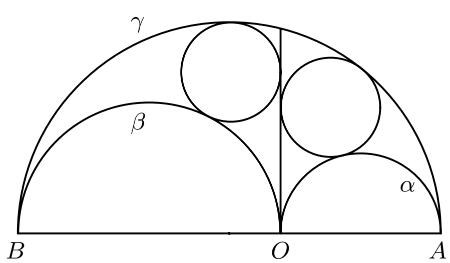


Figure 1.

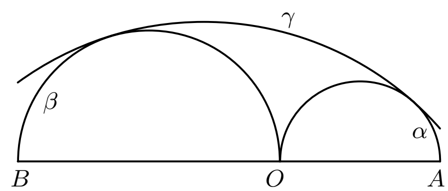


Figure 2.

Publication Date: December 29, 2004. Communicating Editor: Paul Yiu.
The authors express their sincere thanks to the referee for valuable useful comments that improved this paper.

2. The skewed arbelos

Throughout this paper, α and β are circles with centers $(a, 0)$ and $(0, -b)$ for positive real numbers a and b, touching externally at the origin O, and γ is another circle touching α and β at points different from O. We do not exclude the case, when γ touches α and β externally or when γ is one of the common external tangents of α and β. There are always two different areas surrounded by α, β and γ (if γ touches α and β externally, we still consider the exterior infinite area to be surrounded by these three circles). We select one of these areas in the following way (see Figure 3): If γ touches α and β externally from above, we choose the finite area, if γ touches α and β internally, we choose the upper area, and if γ touches α and β externally from below, we choose the infinite area. We call this area the *skewed arbelos* formed by the circles α, β and γ.

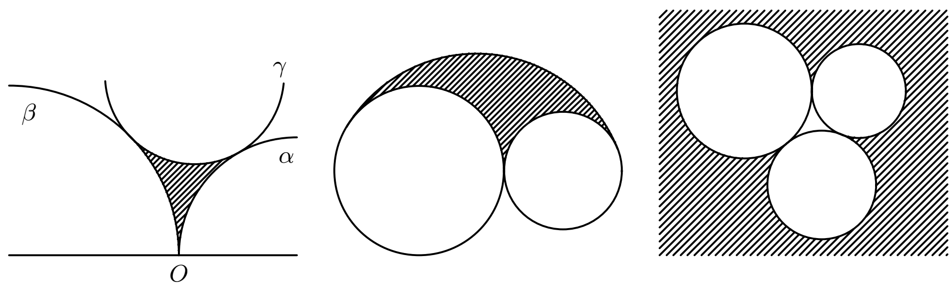


Figure 3.

Now we define four sets of tangent circles (or four chains of circles). If we include the lines parallel to the y-axis (circles of infinite radius) among the circles touching the y-axis, there are always two different circles touching γ, α and the y-axis, which do not pass through the tangency point of α and γ. We label the one inside of the skewed arbelos as α_0^+ and the other one as α_0^-. The circles β_0^+ and β_0^- touching γ, β and the y-axis are defined similarly (see Figure 4). There are also two circles touching α, α_0^+ and the y-axis, one intersecting γ and the other not. We label the former as α_{-1}^+ and the latter as α_1^+. The circles α_2^+, α_3^+, $\cdots$ can be defined inductively in the following way: Assuming the circles α_{i-1}^+ and α_i^+ are defined, α_{i+1}^+ is the circles touching α, α_i^+ and the y-axis and different from α_{i-1}^+. The circles α_{-2}^+, α_{-3}^+, $\cdots$ are defined similarly. Now the entire chain of circles

$$\{\cdots, \alpha_{-2}^+, \alpha_{-1}^+, \alpha_0^+, \alpha_1^+, \alpha_2^+, \cdots\}$$

is defined. The other three chains of circles

$$\{\cdots, \alpha_{-2}^-, \alpha_{-1}^-, \alpha_0^-, \alpha_1^-, \alpha_2^-, \cdots\},$$
$$\{\cdots, \beta_{-2}^+, \beta_{-1}^+, \beta_0^+, \beta_1^+, \beta_2^+, \cdots\},$$
$$\{\cdots, \beta_{-2}^-, \beta_{-1}^-, \beta_0^-, \beta_1^-, \beta_2^-, \cdots\},$$

where α_{-1}^-, β_{-1}^+ and β_{-1}^- intersect γ, are defined similarly. If α_i^+, α_i^-, β_i^+ and β_i^- are proper circles, there radii are denoted by a_i^+, a_i^-, b_i^+ and b_i^-, respectively. If, for example, α_i^+ is a line parallel to the y-axis, we consider the reciprocal value of its radius to be zero, even though we cannot define the radius a_i^+ itself.

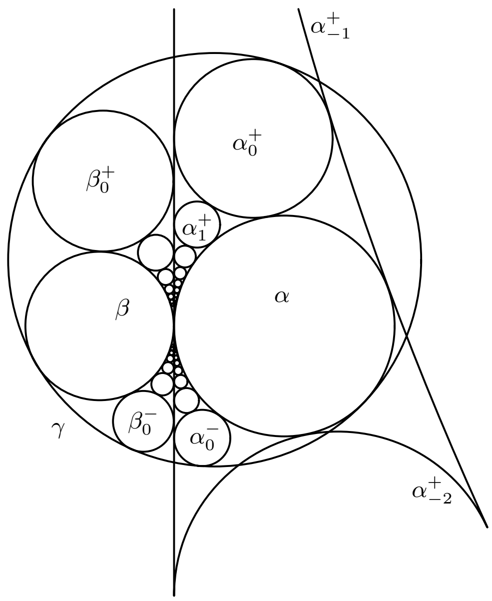


Figure 4.

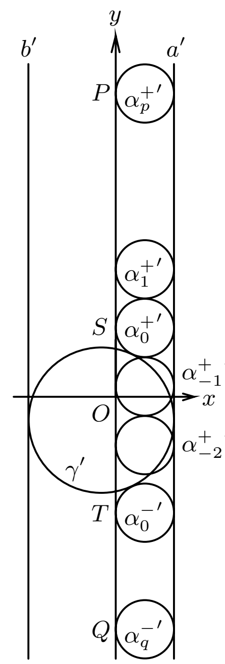


Figure 5.

If α_k^+ is a proper circle and the centers of α_k^+ and α_i^+ lie on the same side of the x-axis for all proper circles α_i^+ ($i > k$), we define $\sigma(\alpha_k^+) = 1$, otherwise we define $\sigma(\alpha_k^+) = -1$. If α_k^+ is a line parallel to the y-axis, we define $\sigma(\alpha_k^+) = 1$. The numbers $\sigma(\alpha_k^-)$, $\sigma(\beta_k^+)$, $\sigma(\beta_k^-)$ are defined similarly. If γ touches α and β internally, $\sigma(\alpha_0^+) = \sigma(\alpha_0^-) = 1$ and consequently, $\sigma(\alpha_i^+) = \sigma(\alpha_i^-) = 1$ for all non-negative integers i. Let s_i and t_j be the y-coordinates of the tangency points of the circles α_i^+ and α_j^- with the y-axis. If α_i^+ (or α_j^-) is a line, we consider $s_i = 0$ (or $t_j = 0$). We define $\sigma(\alpha_i^+, \alpha_j^-) = 1$, when $s_i t_j > 0$ and $s_i \leq t_j$, or when $s_i t_j \leq 0$ and $s_i \geq t_j$, otherwise $\sigma(\alpha_i^+, \alpha_j^-) = -1$. The number $\sigma(\beta_i^+, \beta_j^-)$ is defined similarly. If the centers of the three circles α, β and γ are collinear, we get an ordinary arbelos. In this case, the radii of the twin circles, which we denote as r_{A}, are equal to $ab/(a+b)$.

Theorem 1. *For any integers p and q,*

$$\sigma(\alpha_p^+, \alpha_q^-)\left(\frac{\sigma(\alpha_p^+)}{\sqrt{a_p^+}} + \frac{\sigma(\alpha_q^-)}{\sqrt{a_q^-}}\right) = \left|\frac{2}{\sqrt{r_{\mathrm{A}}}} + \frac{p+q}{\sqrt{a}}\right|$$

and for given circles α and β, the value on the right side does not depend on the circle γ.

Proof. Let p and q be arbitrary integers. We invert the figure in the circle with center O and radius $k = 2\sqrt{ab}$, and label the images of all circles with a prime (see Figure 5). The circles ${\alpha_0^+}'$ and ${\beta_0^+}'$ always lie above the circles ${\alpha_0^-}'$ and ${\beta_0^-}'$ respectively. $\sigma(\alpha_p^+) = 1$ (resp. $\sigma(\alpha_q^-) = 1$) is equivalent to the fact that the center of ${\alpha_p^+}'$ (resp. ${\alpha_q^-}'$) lies in the region $y \geq 0$ (resp. $y \leq 0$) and $\sigma(\alpha_p^+, \alpha_q^-) = 1$ is equivalent to the fact that the y-coordinate of the center of ${\alpha_p^+}'$ is greater than or equal to the y-coordinate of the center of ${\alpha_q^-}'$. Since α' is a line parallel to the y-axis, the circles ${\alpha_p^+}'$ and ${\alpha_q^-}'$ are congruent, and we denote their common radius as a'. Similarly, we denote the common radius of the circles ${\beta_p^+}'$ and ${\beta_q^-}'$ as b'. Let us assume that ${\alpha_0^+}'$, ${\alpha_0^-}'$, ${\alpha_p^+}'$ and ${\alpha_q^-}'$ touch the y-axis at the points S, T, P and Q. If α_p^+ is a proper circle, the inversion center O is also the center of homothety of the circles α_p^+ and ${\alpha_p^+}'$ with homothety coefficient equal to the square of the radius of the inversion circle (i.e., to the power of inversion) divided by the power $O({\alpha_p^+}')$ of the point O to the inverted circle ${\alpha_p^+}'$: $k^2/O({\alpha_p^+}')$. Hence, the radius of α_p^+ can be expressed as $a_p^+ = k^2 a'/O({\alpha_p^+}')$ [5, p. 50]. The reciprocal value of this radius is then $1/a_p^+ = |OP|^2/(4aba')$. The last equation holds even if α_p^+ is a line parallel to the y-axis. Similarly, the reciprocal value of the radius of the circle α_q^- is equal to $1/a_q^- = |OQ|^2/(4aba')$. The segment length of the common external tangent of the externally touching circles γ', ${\alpha_0^+}'$, or γ', ${\alpha_0^-}'$ between the tangency points is equal to $|ST|/2 = 2\sqrt{(a'+b')a'}$. Consequently,

$$\sigma(\alpha_p^+, \alpha_q^-)\left(\frac{\sigma(\alpha_p^+)}{\sqrt{a_p^+}} + \frac{\sigma(\alpha_q^-)}{\sqrt{a_q^-}}\right) = \sigma(\alpha_p^+, \alpha_q^-)\left(\frac{\sigma(\alpha_p^+)|OP| + \sigma(\alpha_q^-)|OQ|}{2\sqrt{ab}\sqrt{a'}}\right)$$

$$= \frac{|PQ|}{2\sqrt{ab}\sqrt{a'}} = \frac{||ST| + 2pa' + 2qa'|}{2\sqrt{ab}\sqrt{a'}} = \frac{\left|4\sqrt{(a'+b')a'} + 2(p+q)a'\right|}{2\sqrt{ab}\sqrt{a'}}.$$

Since $4aa' = 4bb' = 4ab$ by the definition of inversion, we get $a' = b$ and $b' = a$, and we finally obtain

$$\sigma(\alpha_p^+, \alpha_q^-)\left(\frac{\sigma(\alpha_p^+)}{\sqrt{a_p^+}} + \frac{\sigma(\alpha_q^-)}{\sqrt{a_q^-}}\right) = \left|2\sqrt{\frac{1}{a} + \frac{1}{b}} + \frac{p+q}{\sqrt{a}}\right|.$$

The proof of the theorem is now complete. □

We can get a similar expression for the radii of the circles β_r^+ and β_s^- for any integers s and r. According to the proof of Theorem 1, the circles α_p^+ and α_q^- coincide if and only if $P = Q$ and this is also equivalent to

$$\sqrt{1+\frac{a}{b}} = -\frac{p+q}{2}.$$

Hence, we obtain the following corollary:

Corollary 2. *The two chains* $\{\cdots, \alpha_{-2}^+, \alpha_{-1}^+, \alpha_0^+, \alpha_1^+, \alpha_2^+, \cdots\}$ *and* $\{\cdots, \alpha_{-2}^-, \alpha_{-1}^-, \alpha_0^-, \alpha_1^-, \alpha_2^-, \cdots\}$ *coincide if and only if there is an integer* n *such that*

$$\frac{a}{b} = \frac{n^2}{4} - 1.$$

In this event, $\alpha_p^+ = \alpha_{-|n|-p}^-$ *for any integer* p. *For given circles* α *and* β*, this property does not depend on the circle* γ *.*

From the inverted skewed arbelos (see Figure 5), it is easy to see that the circles α_p^+, α_p^-, β_q^+ and β_q^- have two common tangent circles for any integers p and q. The line passing through the center $O_{\gamma'}$ of the circle γ' and perpendicular to the y-axis is also perpendicular to the lines α' and β' and to the circle γ'. Let δ be the circle, which is inverted into this line. Since inversion preserves angles between circles or lines, the circle δ is centered on the y-axis and perpendicular to the circles α, β and γ. Consequently, the inversion in δ with positive power leaves the y-axis and these circles in place and exchanges α_p^+, α_p^- and β_q^+ and β_q^-, respectively. Since the inversion center is also the center of homothety of a circle and its image (external, if the inversion center is outside of the circle, and internal in the opposite case), the external center of similitude of the circles α_p^+ and α_p^- is the same point on the y-axis (the center of the circle δ) for any integer p. This point is also the external center of similitude of β_q^+ and β_q^- for any integer q.

Since $\sigma(\alpha_p^+, \alpha_{-p}^-) = \sigma(\beta_q^+, \beta_{-q}^-) = 1$ for any integers p and q, we get the following corollary:

Corollary 3. *For any integers* p *and* q*,*

$$\frac{\sigma(\alpha_p^+)}{\sqrt{a_p^+}} + \frac{\sigma(\alpha_{-p}^-)}{\sqrt{a_{-p}^-}} = \frac{\sigma(\beta_q^+)}{\sqrt{b_q^+}} + \frac{\sigma(\beta_{-q}^-)}{\sqrt{b_{-q}^-}} = \frac{2}{\sqrt{r_{\mathrm{A}}}}$$

and for given circles α *and* β*, the constant value on the right side does not depend on the circle* γ*.*

Corollary 4. *If* γ *touches* α *and* β *internally,*

$$\frac{1}{\sqrt{a_0^+}} + \frac{1}{\sqrt{a_0^-}} = \frac{1}{\sqrt{b_0^+}} + \frac{1}{\sqrt{b_0^-}} = \frac{2}{\sqrt{r_{\mathrm{A}}}}$$

and for given circles α *and* β*, the constant value on the right side does not depend on the circle* γ*.*

From the last corollary, it is obvious that Theorem 1 is a generalization of the existence of the twin circles of Archimedes in the ordinary arbelos.

3. The n-th twin circles of Archimedes (symmetrical case)

In this section, we demonstrate that in certain situations, a skewed arbelos also has a twin circle property, which is a generalization of the twin circles of Archimedes in an ordinary arbelos. We use the same notations as in the previous section. If one circle of the set $\{\alpha_n^+, \alpha_{-n}^-, \alpha_{-n}^+, \alpha_n^-\}$ is congruent to one circle from the set $\{\beta_n^+, \beta_{-n}^-, \beta_{-n}^+, \beta_n^-\}$ for some integer n, the congruent pair is called *a pair of the n-th twin circles of Archimedes*. The twin circles of Archimedes in the ordinary arbelos are represented by one pair of the 0-th twin circles.

If the circles α, β and γ form an ordinary arbelos, the intersection of γ with the y-axis in the region $y > 0$ has the coordinates $(0, 2\sqrt{ab})$. For a real number z, the point $(0, 2\sqrt{ab}/z)$ is denoted by V_z and we consider V_0 to be the point at infinity on the y-axis. We show that $V_{n\pm1}$ are closely related to some pairs of the n-th twin circles of Archimedes. There are also other points on the y-axis, related to pairs of the n-th twin circles of Archimedes. For a real number z, consider the following points with the y-coordinates

$$\begin{aligned}
W_z^{++}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{z\left(\sqrt{a}+\sqrt{b}\right)+2\sqrt{a+b}},\\
W_z^{--}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{z\left(\sqrt{a}+\sqrt{b}\right)-2\sqrt{a+b}},\\
W_z^{+-}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{z\left(\sqrt{a}-\sqrt{b}\right)+2\sqrt{a+b}},\\
W_z^{-+}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{z\left(\sqrt{a}-\sqrt{b}\right)-2\sqrt{a+b}}.
\end{aligned}$$

Reflecting the points V_z, W_z^{++} and W_z^{+-} in the x-axis, we get the points V_{-z}, W_{-z}^{--} and W_{-z}^{-+}. Since $\sqrt{2} \le 2\sqrt{a+b}/\left(\sqrt{a}+\sqrt{b}\right) < 2$, W_n^{++} and W_n^{--} cannot be the point at infinity on the y-axis for any integer n, but it can happen that each of W_n^{+-} and W_n^{-+} is identical with the point at infinity for some a, b and integer n. If the circle γ passes, for example, through both V_{n+1} and V_{n-1}, we say that γ passes through $V_{n\pm1}$.

Theorem 5. *Let n be an integer and $a \neq b$.*
(i) $1/a_n^+ = 1/b_n^+$ *if and only if the circle γ passes through $V_{n\pm1}$ or $W_{n\pm1}^{++}$. If γ passes through $V_{n\pm1}$,*

$$\frac{1}{a_n^+} = \frac{1}{b_n^+} = \left(n\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2 \tag{1}$$

and if γ passes through $W_{n\pm1}^{++}$,

$$\frac{1}{a_n^+} = \frac{1}{b_n^+} = \left(\left(n\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)\right)^2. \tag{2}$$

(ii) $1/a^-_{-n} = 1/b^-_{-n}$ *if and only if the circle* γ *passes through* $V_{n\pm1}$ *or* $W^{--}_{n\pm1}$*. If* γ *passes through* $V_{n\pm1}$*,*

$$\frac{1}{a^-_{-n}} = \frac{1}{b^-_{-n}} = \left(-n\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2$$

and if γ *passes through* $W^{--}_{n\pm1}$*,*

$$\frac{1}{a^-_{-n}} = \frac{1}{b^-_{-n}} = \left(\left(-n\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)\right)^2.$$

(iii) $1/a^+_{-n} = 1/b^-_{n}$ *if and only if the circle* γ *passes through* $V_{n\pm1}$ *or* $W^{+-}_{n\pm1}$*. If* γ *passes through* $V_{n\pm1}$*,*

$$\frac{1}{a^+_{-n}} = \frac{1}{b^-_{n}} = \left(-n\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2$$

and if γ *passes through* $W^{+-}_{n\pm1}$*,*

$$\frac{1}{a^+_{-n}} = \frac{1}{b^-_{n}} = \left(\left(-n\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)\right)^2.$$

(iv) $1/a^-_{n} = 1/b^+_{-n}$ *if and only if the circle* γ *passes through* $V_{n\pm1}$ *or* $W^{-+}_{n\pm1}$*. If* γ *passes through* $V_{n\pm1}$*,*

$$\frac{1}{a^-_{n}} = \frac{1}{b^+_{-n}} = \left(n\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2$$

and if γ *passes through* $W^{-+}_{n\pm1}$*,*

$$\frac{1}{a^-_{n}} = \frac{1}{b^+_{-n}} = \left(\left(n\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)\right)^2.$$

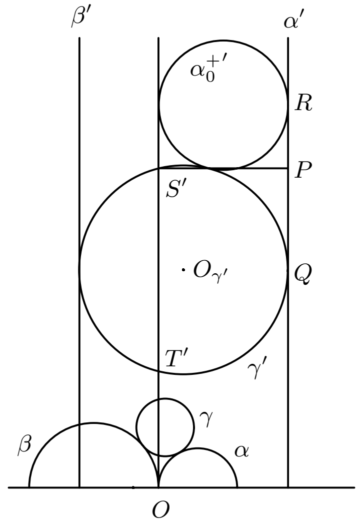


Figure 6.

Proof. Let S and T be the intersections of γ and the y-axis, where S lies on the arc or the line forming the boundary of the skewed arbelos. We denote the y-coordinates of S and T by s and t. If the circle γ touches α and β internally, $t < 0 < s$, otherwise $s < t$. We invert the figure in the circle centered at O and with radius $2\sqrt{ab}$ as in the proof of Theorem 1 (see Figure 6), label the images of all circles and points with a prime and denote the radii of $\alpha_n^{+\prime}$ and $\beta_n^{+\prime}$ by a' and b'. Then we obtain $a' = b$ and $b' = a$. Let the line parallel to the x-axis and passing through S' intersect the line α' at the point P. Let γ' and $\alpha_0^{+\prime}$ touch α' at the points Q and R, respectively, and let $O_{\gamma'}$ be the center of the circle γ'. From the right triangle formed by the lines $O_{\gamma'}S'$, $S'P$ and the line through $O_{\gamma'}$ parallel to the y-axis, we get $|PQ| = 2\sqrt{a'b'}$. The segment length of the common external tangent of the touching circles γ', $\alpha_0^{+\prime}$ between the tangency points is equal to $|QR| = 2\sqrt{(a'+b')a'}$. Hence, the reciprocal radius of α_n^+ is equal to

$$\begin{aligned}\frac{1}{a_n^+} &= \frac{O(\alpha_n^{+\prime})}{4aba'} = \frac{(s' - |PQ| + |QR| + 2na')^2}{4aba'}\\ &= \frac{(s' - 2\sqrt{a'b'} + 2\sqrt{(a'+b')a'} + 2na')^2}{4aba'}\\ &= \frac{(s' - 2\sqrt{ab} + 2\sqrt{(a+b)b} + 2nb)^2}{4ab^2},\end{aligned}$$

where s' is the y-coordinate of the point S' and $O(\alpha_n^{+\prime})$ is the power of the point O to the inverted circle $\alpha_n^{+\prime}$. Therefore, $1/a_n^+ = 1/b_n^+$ is equivalent to

$$\frac{(s' - 2\sqrt{ab} + 2\sqrt{(a+b)b} + 2nb)^2}{4ab^2} = \frac{(s' - 2\sqrt{ab} + 2\sqrt{(a+b)a} + 2na)^2}{4a^2b}.$$

This quadratic equation for s' has two roots:

$$s' = 2(n+1)\sqrt{ab}. \tag{3}$$

and

$$s' = -2(n-1)\sqrt{ab} - \frac{4\sqrt{ab(a+b)}}{\sqrt{a}+\sqrt{b}}. \tag{4}$$

Since $ss' = 4ab$, these are equivalent to

$$s = \frac{2\sqrt{ab}}{n+1}$$

and

$$s = \frac{-2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{(n-1)\left(\sqrt{a}+\sqrt{b}\right)+2\sqrt{a+b}}.$$

Hence, $1/a_n^+ = 1/b_n^+$ is equivalent to $S = V_{n+1}$ or $S = W_{n-1}^{++}$. If $S = V_{n+1}$, then

$$t' = s' - 2|PQ| = 2(n-1)\sqrt{ab},$$

where t' is the y-coordinate of the point T'. Hence,

$$t = \frac{4ab}{t'} = \frac{2\sqrt{ab}}{n-1},$$

and we obtain $T = V_{n-1}$. Similarly, $S = W_{n-1}^{++}$ implies $T = W_{n+1}^{++}$. Assume now that the circle γ passes through $V_{n\pm1}$. If $S = V_{n-1}$ and $T = V_{n+1}$, we would have

$$s' - t' = \frac{4ab}{s} - \frac{4ab}{t} = -4\sqrt{ab} < 0,$$

which contradicts to the fact $s' > t'$. Therefore, $S = V_{n+1}$ and s' is given by equation (3). Consequently, we arrive to equation (1):

$$\frac{1}{a_n^+} = \frac{\left(s' - 2\sqrt{ab} + 2\sqrt{(a+b)b} + 2nb\right)^2}{4ab^2} = \left(n\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)+\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2.$$

If γ passes through $W_{n\pm1}^{++}$, $S = W_{n-1}^{++}$. For if $S = W_{n+1}^{++}$, we would again have

$$s' - t' = \frac{4ab}{s} - \frac{4ab}{t} = -4\sqrt{ab} < 0,$$

which is a contradiction. Using equation (4), we arrive to equation (2):

$$\begin{aligned}\frac{1}{a_n^+} &= \frac{\left(-2n\sqrt{ab} + 2\sqrt{(a+b)b} + 2nb - \frac{4\sqrt{ab(a+b)}}{\sqrt{a}+\sqrt{b}}\right)^2}{4ab^2} \\ &= \left(\left(n\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)+\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)^2.\end{aligned}$$

Cases (ii), (iii) and (iv) can be proved similarly as case (i). The reciprocal radii $1/a^-_{-n}$, $1/a^+_{-n}$ and $1/a^-_n$ are equal to

$$\frac{1}{a^-_{-n}} = \frac{(s' - |PQ| - |QR| + 2na')^2}{4aba'} = \frac{(s' - 2\sqrt{ab} - 2\sqrt{(a+b)b} + 2nb)^2}{4ab^2},$$

$$\frac{1}{a^+_{-n}} = \frac{(s' - |PQ| + |QR| - 2na')^2}{4aba'} = \frac{(s' - 2\sqrt{ab} + 2\sqrt{(a+b)b} - 2nb)^2}{4ab^2},$$

$$\frac{1}{a^-_{n}} = \frac{(s' - |PQ| - |QR| - 2na')^2}{4aba'} = \frac{(s' - 2\sqrt{ab} - 2\sqrt{(a+b)b} - 2nb)^2}{4ab^2}.$$

One root of the quadratic equations corresponding to cases (ii), (iii) and (iv) is always given by equation (3) and the other roots are

$$s' = -2(n-1)\sqrt{ab} + \frac{4\sqrt{ab(a+b)}}{\sqrt{a}+\sqrt{b}}, \tag{5}$$

$$s' = -2(n-1)\sqrt{ab} - \frac{4\sqrt{ab(a+b)}}{\sqrt{a}-\sqrt{b}}, \tag{6}$$

$$s' = -2(n-1)\sqrt{ab} + \frac{4\sqrt{ab(a+b)}}{\sqrt{a}-\sqrt{b}}. \tag{7}$$

□

If the circle γ passes through the point $V_{n\pm1}$, we label the arbelos as $(V_{n\pm1})$. The arbeloi $(W^{++}_{n\pm1})$, $(W^{--}_{n\pm1})$, $(W^{+-}_{n\pm1})$ and $(W^{-+}_{n\pm1})$ are defined similarly. Reflecting the arbeloi $(V_{n\pm1})$, $(W^{++}_{n\pm1})$, $(W^{+-}_{n\pm1})$ in the x-axis yields the arbeloi $(V_{-n\pm1})$, $(W^{--}_{-n\pm1})$, $(W^{-+}_{-n\pm1})$, respectively. Equation (3) is obtained, when the signs of the expressions $s' - 2\sqrt{ab} + 2\sqrt{(a+b)b} + 2nb$ and $s' - 2\sqrt{ab} + 2\sqrt{(a+b)a} + 2na$ are the same. This implies that in $(V_{n\pm1})$, the centers of the circles α^+_n and β^+_n lie on the same side of the x-axis. On the other hand, equation (4) is obtained, when the signs of these expressions are different from each other. Consequently, in $(W^{++}_{n\pm1})$, the centers of α^+_n and β^+_n lie on the opposite sides of the x-axis. Similarly, we can find, on which sides of the x-axis lie the centers of the n-th twin circles of Archimedes in the remaining arbeloi. These results are arranged in Table 1.

	$(V_{n\pm1})$	$(W^{++}_{n\pm1})$	$(W^{--}_{n\pm1})$	$(W^{+-}_{n\pm1})$	$(W^{-+}_{n\pm1})$
same side	α^+_n, β^+_n $\alpha^-_{-n}, \beta^-_{-n}$			α^+_{-n}, β^-_n	α^-_n, β^+_{-n}
opposite side	α^+_{-n}, β^-_n α^-_n, β^+_{-n}	α^+_n, β^+_n	$\alpha^-_{-n}, \beta^-_{-n}$		

Table 1.

According to Theorem 5, there are four different pairs of the n-th twin circles of Archimedes in $(V_{n\pm1})$, for any non-zero integer n (see Figure 9). In this case, γ touches α and β externally from below for $n \le -1$, internally for $n = 0$, externally from above for $n \ge 1$. The twin circles of Archimedes in the ordinary arbelos $(V_{0\pm1})$ and their radii are obtained for $n = 0$. Figures 7 and 8 show the other pairs of the 0-th twin circles of Archimedes in the arbeloi $(W_{0\pm1}^{++})$ and $(W_{0\pm1}^{+-})$. The 0-th twin circles of Archimedes in $(W_{0\pm1}^{--})$ and $(W_{0\pm1}^{-+})$ are obtained by reflecting these figures in the x-axis and exchanging all plus and minus signs in the notation.

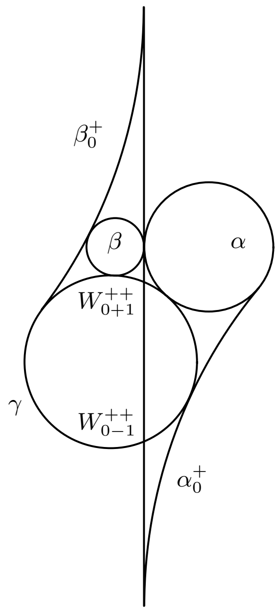


Figure 7. $a_0^+ = b_0^+$ for $(W_{0\pm1}^{++})$

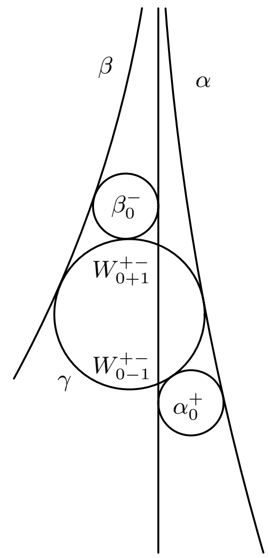


Figure 8. $a_0^+ = b_0^-$ for $(W_{0\pm1}^{+-})$

If γ is the common external tangent of α and β touching these circles from above, it passes through $V_{1\pm1}$, because this tangent bisects the segment OV_1 [2]. Hence, we get the following corollary (see Figure 9):

Corollary 6. *If γ is the common external tangent of α and β, touching these circles from above, then* (i) $a_1^+ = b_1^+$, (ii) $a_{-1}^- = b_{-1}^-$, (iii) $a_{-1}^+ = b_1^-$, (iv) $a_1^- = b_{-1}^+$, *and*

$$\text{(v)}\ \frac{1}{\sqrt{a_1^+}} = \frac{1}{\sqrt{a_1^-}} + \frac{1}{\sqrt{a_{-1}^+}} + \frac{1}{\sqrt{a_{-1}^-}} = \frac{1}{\sqrt{b_1^+}} = \frac{1}{\sqrt{b_1^-}} + \frac{1}{\sqrt{b_{-1}^+}} + \frac{1}{\sqrt{b_{-1}^-}}.$$

Proof. Since $1/\sqrt{a}$, $1/\sqrt{b}$, $1/\sqrt{r_A}$ satisfy the triangle inequality, relation (v) immediately follows from Theorem 5. □

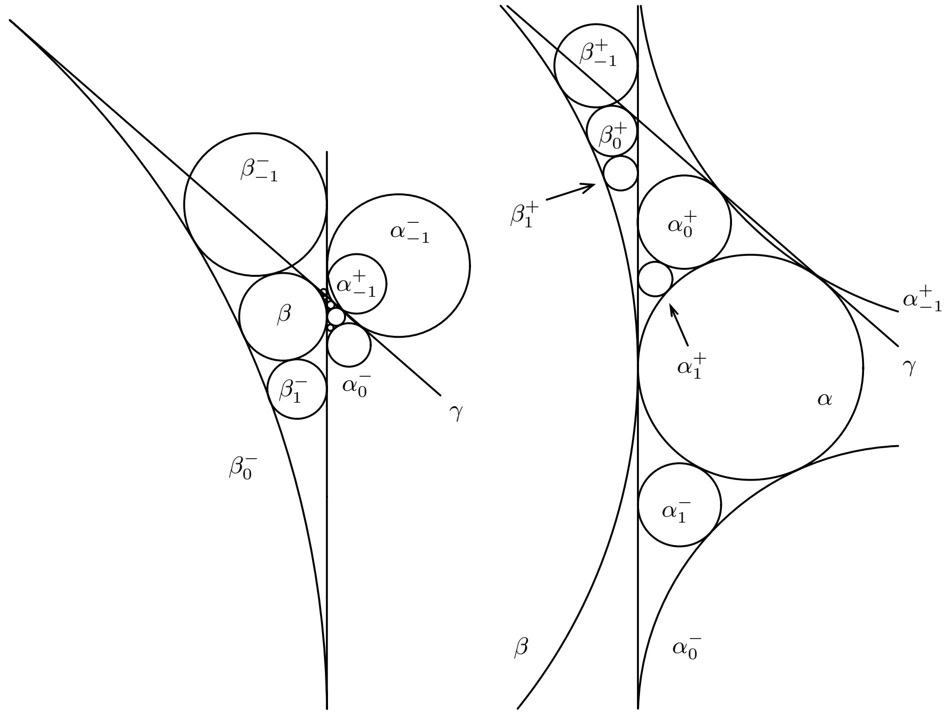


Figure 9. $a_{-1}^+ = b_1^-$, $a_{-1}^- = b_{-1}^-$ for $(V_{1\pm1})$ Magnified, $a_1^+ = b_1^+$, $a_1^- = b_{-1}^+$

Theorem 7. *Any circle touching α and β at points different from O passes through $V_{z\pm1}$ for some real number z. The proper circle touching α and β at points different from O and passing through $V_{z\pm1}$ for a real number $z \neq \pm1$ can be given by the equation*

$$\left(x - \frac{b-a}{z^2-1}\right)^2 + \left(y - \frac{2z\sqrt{ab}}{z^2-1}\right)^2 = \left(\frac{a+b}{z^2-1}\right)^2 \tag{8}$$

and conversely. The common external tangents of α and β can be expressed by the equations

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0, \tag{9}$$

which are obtained from equation (8) by approaching z to ±1.

Proof. We again invert the circles α, β and γ in the circle centered at O and with radius $2\sqrt{ab}$ as in the proofs of Theorems 1 and 5 and use the same notation. The circle γ is then carried into the circle γ' with radius $c' = a + b$, because $a' = b$ and $b' = a$. The intersection of the skewed arbelos boundary and the y-axis can be expressed as V_{z+1} for some real number z. Let t be the y-coordinate of the other intersection of γ and the y-axis. These intersections are carried into the intersections of γ' and the y-axis with the y-coordinates $s' = 4ab/s = 2(z+1)\sqrt{ab}$ and $t' = s' - 4\sqrt{ab} = 2(z-1)\sqrt{ab}$ (see the proof of Theorem 5), leading to $t = 4ab/t' = 2\sqrt{ab}/(z-1)$. Hence, the other intersection of γ and the y-axis is identical with the point V_{z-1}. Assume that γ is a proper circle passing through $V_{z\pm1}$ for a real number $z \neq \pm1$ and let (x_0, y_0) be the coordinates of the center of γ. Obviously, $y_0' = (s' + t')/2 = 2z\sqrt{ab}$ and $x_0' = (2a' - 2b')/2 = b - a$, where (x_0', y_0') are the coordinates of the center of γ'. The inversion center at the coordinate origin O is also the center of homothety of the circles γ and γ', with homothety coefficient equal to $h = 4ab/O(\gamma')$. Since $O(\gamma') = s't' = 4(z^2-1)ab$, this homothety coefficient is equal to $h = 1/(z-1)^2$. Hence, $x_0 = x_0'h = (b-a)/(z^2-1)$, $y_0 = y_0'h = 2z\sqrt{ab}/(z^2-1)$ and the radius of the circle γ is $c = c'h = (a+b)/|z^2-1|$, which leads to equation (8). The converse follows from the fact that (8) determines a circle touching α and β at points different from O and passing through V_{z+1} at the skewed arbelos boundary and this circle is then expressed by (8) again as we have already demonstrated. If $z \to \pm1$ and we neglect the terms quadratic in z^2-1 in (8), the remaining factors z^2-1 cancel out and we arrive to equation (9). $\square$

4. Relationship of two skewed arbeloi

In this section, we analyze further properties of the skewed arbeloi $(V_{n\pm1})$, $(W_{n\pm1}^{++})$, $(W_{n\pm1}^{--})$, $(W_{n\pm1}^{+-})$ and $(W_{n\pm1}^{-+})$ for an arbitrary integer n and also consider properties of the circle orthogonal to α and β. We assume that the circles α and β are fixed. For these arbeloi, the circles formerly denoted by α_m^+ for an integer m are now labeled explicitly as $\alpha_{n,m}^+$ and their radii as $a_{n,m}^+$. Similarly, we relabel the circles formerly denoted by α_m^-, β_m^+ and β_m^- and their radii. The circle passing through $V_{z\pm1}$ and touching α and β at points different from O is denoted by γ_z for a real number z. If γ_z is a proper circle, it is expressed by (8), and the circle γ_n forms $(V_{n\pm1})$ with α and β. Reflecting the arbeloi $(V_{n\pm1})$, $(W_{n\pm1}^{++})$ and $(W_{n\pm1}^{+-})$ in the x-axis yields the arbeloi $(V_{-n\pm1})$, $(W_{-n\pm1}^{--})$, $(W_{-n\pm1}^{-+})$, respectively. Therefore $1/a_{n,m}^{\pm} = 1/a_{-n,m}^{\mp}$ and $1/b_{n,m}^{\pm} = 1/b_{-n,m}^{\mp}$ in the arbelos pairs $(V_{n\pm1})$ and $(V_{-n\pm1})$; $(W_{n\pm1}^{++})$ and $(W_{-n\pm1}^{--})$; $(W_{n\pm1}^{+-})$ and $(W_{-n\pm1}^{-+})$, but this is trivial.

Since the y-coordinates of the points $V_{n\pm1}$, $W_{n\pm1}^{++}$ and $W_{n\pm1}^{--}$ are symmetrical in a and b, the radii $b_{n,m}^{\pm}$ can be obtained from $a_{n,m}^{\pm}$ by replacing a with b and b with a in the arbeloi $(V_{n\pm1})$, $(W_{n\pm1}^{++})$ and $(W_{n\pm1}^{--})$. On the other hand, the y-coordinates of the points $W_{n\pm1}^{+-}$ and $W_{n\pm1}^{-+}$ are not symmetrical in a and b. Hence, we cannot draw the same conclusion for the arbeloi $(W_{n\pm1}^{+-})$ and $(W_{n\pm1}^{-+})$. Using the same notations as in the proof of Theorem 5, from equation (3) for the arbelos $(V_{n\pm1})$, we get

$$\frac{1}{a_{n,m}^{\pm}} = \frac{\left(s' - 2\sqrt{ab} \pm 2\sqrt{(a+b)b} \pm 2mb\right)^2}{4ab^2} = \left(\frac{n}{\sqrt{b}} \pm \frac{m}{\sqrt{a}} \pm \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2.$$

Using equation (4) for the arbelos $(W_{n\pm1}^{++})$,

$$\frac{1}{a_{n,m}^{+}} = \left(\frac{n}{\sqrt{b}} - \frac{m}{\sqrt{a}} + \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2,$$

$$\frac{1}{a_{n,m}^{-}} = \left(\frac{n}{\sqrt{b}} + \frac{m}{\sqrt{a}} + \frac{3\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2.$$

Using equation (5) for the arbelos $(W_{n\pm1}^{--})$,

$$\frac{1}{a_{n,m}^{+}} = \left(\frac{n}{\sqrt{b}} - \frac{m}{\sqrt{a}} - \frac{3\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2,$$

$$\frac{1}{a_{n,m}^{-}} = \left(\frac{n}{\sqrt{b}} + \frac{m}{\sqrt{a}} - \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2.$$

Using equation (6) for the arbelos $(W_{n\pm1}^{+-})$,

$$\frac{1}{a_{n,m}^{+}} = \left(\frac{n}{\sqrt{b}} - \frac{m}{\sqrt{a}} + \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2,$$

$$\frac{1}{a_{n,m}^{-}} = \left(\frac{n}{\sqrt{b}} + \frac{m}{\sqrt{a}} + \frac{3\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2,$$

$$\frac{1}{b_{n,m}^{+}} = \left(\frac{n}{\sqrt{a}} - \frac{m}{\sqrt{b}} - \frac{3\sqrt{b}-\sqrt{a}}{\sqrt{b}-\sqrt{a}}\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2,$$

$$\frac{1}{b_{n,m}^{-}} = \left(\frac{n}{\sqrt{a}} + \frac{m}{\sqrt{b}} - \frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2.$$

Using equation (7) for the arbelos $(W_{n\pm1}^{-+})$,

$$\frac{1}{a_{n,m}^{+}} = \left(\frac{n}{\sqrt{b}} - \frac{m}{\sqrt{a}} - \frac{3\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2,$$

$$\frac{1}{a_{n,m}^{-}} = \left(\frac{n}{\sqrt{b}} + \frac{m}{\sqrt{a}} - \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2,$$

$$\frac{1}{b_{n,m}^{+}} = \left(\frac{n}{\sqrt{a}} - \frac{m}{\sqrt{b}} + \frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2,$$

$$\frac{1}{b_{n,m}^{-}} = \left(\frac{n}{\sqrt{a}} + \frac{m}{\sqrt{b}} + \frac{3\sqrt{b}-\sqrt{a}}{\sqrt{b}-\sqrt{a}}\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2.$$

By comparing the above equations, we obtain the following theorem (see Figure 10):

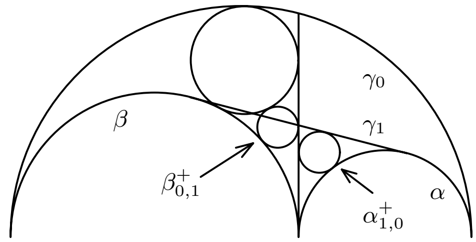


Figure 10. $a^+_{1,0} = b^+_{0,1}$ for $(V_{1\pm1})$ and $(V_{0\pm1})$

Theorem 8. *Let n and m be integers.*
(i) *For $(V_{n\pm1})$ and $(V_{m\pm1})$, we have $1/a^+_{n,m} = 1/b^+_{m,n}$, $1/b^+_{n,m} = 1/a^+_{m,n}$, $1/a^-_{n,-m} = 1/b^-_{m,-n}$, and $1/b^-_{n,-m} = 1/a^-_{m,-n}$.*
(ii) *For $(W^{++}_{n\pm1})$ and $(W^{++}_{m\pm1})$, we have $1/a^+_{n,m} = 1/b^+_{m,n}$ and $1/b^+_{n,m} = 1/a^+_{m,n}$.*
(iii) *For $(W^{--}_{n\pm1})$ and $(W^{--}_{m\pm1})$, we have $1/a^-_{n,-m} = 1/b^-_{m,-n}$ and $1/b^-_{n,-m} = 1/a^-_{m,-n}$.*
(iv) *For $(W^{+-}_{n\pm1})$ and $(W^{+-}_{m\pm1})$, we have $1/a^+_{n,-m} = 1/b^-_{m,n}$.*
(v) *For $(W^{-+}_{n\pm1})$ and $(W^{-+}_{m\pm1})$, we have $1/a^-_{n,m} = 1/b^+_{m,-n}$.*
(vi) *For $(W^{--}_{n\pm1})$ and $(W^{++}_{m\pm1})$, we have $1/a^-_{n,m} = 1/b^+_{m,-n}$ and $1/b^-_{n,m} = 1/a^+_{m,-n}$.*
(vii) *For $(W^{+-}_{n\pm1})$ and $(W^{-+}_{m\pm1})$, we have $1/a^+_{n,m} = 1/b^+_{m,n}$ and $1/b^-_{n,-m} = 1/a^-_{m,-n}$.*

For different real numbers z and w, $\zeta^\alpha_{z,w}$ is the circle touching α, γ_z and γ_w and passing through neither the tangency point of α and γ_z nor the tangency point of α and γ_w and different from β. Similarly the circle $\zeta^\beta_{z,w}$ is defined. In the figure formed by $(V_{0\pm1})$ and $(V_{1\pm1})$, two other congruent pairs of inscribed circles can be found (see Figure 11).

Theorem 9. *The circle inscribed in the curvilinear triangle formed by γ_0, the y-axis, and one of the twin circles of Archimedes touching β is congruent to $\zeta^\alpha_{0,1}$.*

To prove this theorem, we use the following result of the old Japanese geometry [7] (see Figure 12):

Lemma 10. *Assume that the circle C with radius r is divided by a chord t into two arcs and let h be the distance from the midpoint of one of the arcs to t. If two externally touching circles C_1 and C_2 with radii r_1 and r_2 also touch the chord t and the other arc of the circle C internally, then h, r, r_1 and r_2 are related as*

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{r_1 r_2 h}}.$$

Proof. The centers of C_1 and C_2 can be on the opposite sides of the normal dropped on t from the center of C or on the same side of this normal. From the right triangles formed by the centers of C and C_i $(i = 1, 2)$, the line parallel to t through the center of C, and the normal dropped on t from the center of C_i, we have

$$|\sqrt{(r-r_1)^2 - (h+r_1-r)^2} \pm \sqrt{(r-r_2)^2 - (h+r_2-r)^2}| = 2\sqrt{r_1 r_2},$$

where we used the fact that the segment length of the common external tangent of C_1 and C_2 between the tangency points is equal to $2\sqrt{r_1 r_2}$. The formula of the lemma follows from this equation. □

Now we can prove Theorem 9. The distance between the common external tangent of α and β and the midpoint of the minor arc of the circle γ_0 formed by this tangent is $2r_{\mathrm{A}}$ [2]. According to Lemma 10, the radii of the two inscribed circles are the root of the same quadratic equation

$$\frac{1}{r}+\frac{1}{a}+\frac{a+b}{ab}=2\sqrt{\frac{(a+b)^2}{a^2br}}.$$

From Figure 11, it is obvious that one root of this quadratic equation is equal to b. The other root is then $a^2b/(a+2b)^2$. □

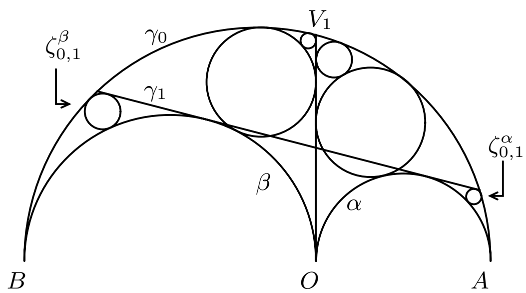


Figure 11. Two small congruent pairs

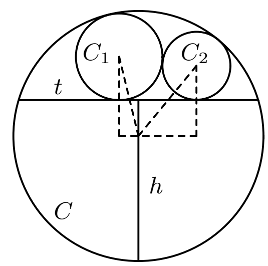


Figure 12.

Now we consider circles orthogonal to α and β. Let $t=(a+b)/\sqrt{ab}$ and let ϵ_z be the circle with a diameter OV_z for a real number z, where we consider ϵ_0 is identical with the x-axis. The mapping $\gamma_z \to \epsilon_z$ gives a one to one correspondence between the circles touching α and β at points different from O and the circles orthogonal to α and β. The circle ϵ_1 intersects α and γ_1 perpendicularly at their tangency point and the line segment AV_1 also passes through this point [2].

Theorem 11. *Let z and w be real numbers.*
(i) *The circle ϵ_z intersects α and γ_z perpendicularly at their tangency point and the line segment AV_z also passes through this point.*
(ii) *Let $w \neq 0$. The circle ϵ_z is orthogonal to any circle touching γ_{z-w} and γ_{z+w}. In particular ϵ_z intersects α and $\zeta^{\alpha}_{z-w,z+w}$ perpendicularly at their tangency point. If the two circles γ_{z-w} and γ_{z+w} intersect, ϵ_z also passes through their intersection.*
(iii) *The two circles γ_z and γ_w touch if and only if $z-w=\pm t$. The circle ϵ_z touches $\gamma_{z-t/2}$ and $\gamma_{z+t/2}$ at their tangency point.*
(iv) *The reciprocal radius of ϵ_{zt} is $|z|/r_{\mathrm{A}}$.*

Proof. We once again invert the circles in the circle centered at O and with radius $2\sqrt{ab}$ as in the proofs of Theorems 1, 5 and 7 and use the same notation.

The circle γ_z is then carried into the circle $\gamma_z{}'$ touching α' at a point with the y-coordinate $2z\sqrt{ab}$ as shown in the proof of Theorem 7 and ϵ_z is carried into the line $\epsilon_z{}'$: $y = 2z\sqrt{ab}$. This implies that ϵ_z intersects α and γ_z at their tangency point perpendicularly. The last part of (i) follows from the fact that the three points A', the tangency point of α' and $\gamma_z{}'$ and $V_z{}'$ lie on a circle passing through O in this order. (ii) follows from the fact that the two circles $\gamma_{z-w}{}'$ and $\gamma_{z+w}{}'$ are symmetrical in the line $\epsilon_z{}'$. The two circles $\gamma_z{}'$ and $\gamma_w{}'$ touch if and only if $2z\sqrt{ab} - 2w\sqrt{ab} = \pm 2(a+b)$ and this is equivalent to $z - w = \pm t$. This gives the first half part of (iii). The remaining part of (iii) and (iv) are now obvious. $\square$

The circle $\zeta^{\alpha}_{z-w,z+w}$ touches α at a fixed point for any non-zero real number w, which is the intersection of α and ϵ_z by (ii) of the theorem. For any chain of circles touching α and β, the reciprocals of the radii of their associated circles orthogonal to α and β and the circles in this chain form a geometric progression by the first half part of (iii) and (iv) of the theorem, where we assume that the radius of the associated circle touching the x-axis from below has minus sign. In particular, starting with the ordinary arbelos, we get the chain of circles

$$\{\cdots, \gamma_{-2t}, \gamma_{-t}, \gamma_0, \gamma_t, \gamma_{2t}, \cdots\}$$

and the reciprocal radius of the circle ϵ_{nt} associated with γ_{nt} in this chain is n/r_{A}. In the case $n = 1$, we get the well-known fact that the circle orthogonal to α, β and the inscribed circle of the ordinary arbelos is congruent to the twin circles of Archimedes in the ordinary arbelos [1]. Now let us consider some other special cases of Theorem 11. In Figure 11, the circle with center V_1 passing through O, i.e., $\epsilon_{1/2}$, intersects α and $\zeta^{\alpha}_{0,1}$ (also β and $\zeta^{\beta}_{0,1}$) perpendicularly at their tangency point and also intersects γ_0 and γ_1 at their intersections. These results are obtained by letting $z = w = 1/2$ in (ii). The circle $\epsilon_{(n+1/2)t}$ with radius $r_{\mathrm{A}}/(n+\frac{1}{2})$ touches γ_{nt} and $\gamma_{(n+1)t}$ at their tangency point by (iii) and (iv). In particular the circle $\epsilon_{t/2}$, which is double the size of the twin circles of Archimedes in the ordinary arbelos, intersects α and $\zeta^{\alpha}_{0,t}$ (also β and $\zeta^{\beta}_{0,t}$) perpendicularly at their tangency point and also touches γ_0 and γ_t at their tangency point (see Figure 13).

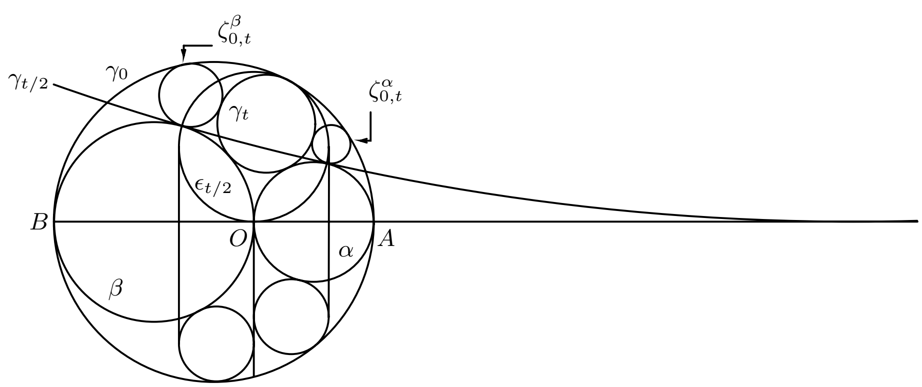


Figure 13.

There is a tangent between $\epsilon_{t/2}$ and each of the twin circles of Archimedes in the ordinary arbelos which is parallel to the y-axis. In order to avoid the overlapping circles, reflected twin circles of Archimedes in the x-axis are drawn in Figure 13. From (8) we can see that the circle $\gamma_{t/2}$ (also $\gamma_{-t/2}$) touches the x-axis.

5. The n-th twin circles of Archimedes (asymmetrical case)

To investigate further possibilities of the existence of pairs of the n-th twin circles of Archimedes, we define several other points on the y-axis, which are also related to some of those pairs. Consider the following points on the y-axis with given y-coordinates:

$$\begin{aligned}
X_{n,+}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{n\left(\sqrt{a}+\sqrt{b}\right)+\left(\sqrt{a}-\sqrt{b}\right)}, \\
X_{n,-}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{n\left(\sqrt{a}+\sqrt{b}\right)-\left(\sqrt{a}-\sqrt{b}\right)}; \\
Y_{n,+}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{n\left(\sqrt{a}-\sqrt{b}\right)+\left(\sqrt{a}+\sqrt{b}\right)}, \\
Y_{n,-}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{n\left(\sqrt{a}-\sqrt{b}\right)-\left(\sqrt{a}+\sqrt{b}\right)}.
\end{aligned}$$

Also,

$$\begin{aligned}
Z^{++}_{n,+}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{n\left(\sqrt{a}-\sqrt{b}\right)+\left(\sqrt{a}+\sqrt{b}\right)-2\sqrt{a+b}}, \\
Z^{++}_{n,-}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{n\left(\sqrt{a}-\sqrt{b}\right)-\left(\sqrt{a}+\sqrt{b}\right)-2\sqrt{a+b}}, \\
Z^{--}_{n,+}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{n\left(\sqrt{a}-\sqrt{b}\right)+\left(\sqrt{a}+\sqrt{b}\right)+2\sqrt{a+b}}, \\
Z^{--}_{n,-}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{n\left(\sqrt{a}-\sqrt{b}\right)-\left(\sqrt{a}+\sqrt{b}\right)+2\sqrt{a+b}}, \\
Z^{+-}_{n,+}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{n\left(\sqrt{a}+\sqrt{b}\right)+\left(\sqrt{a}-\sqrt{b}\right)-2\sqrt{a+b}}, \\
Z^{+-}_{n,-}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{n\left(\sqrt{a}+\sqrt{b}\right)-\left(\sqrt{a}-\sqrt{b}\right)-2\sqrt{a+b}}, \\
Z^{-+}_{n,+}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{n\left(\sqrt{a}+\sqrt{b}\right)+\left(\sqrt{a}-\sqrt{b}\right)+2\sqrt{a+b}}, \\
Z^{-+}_{n,-}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{n\left(\sqrt{a}+\sqrt{b}\right)-\left(\sqrt{a}-\sqrt{b}\right)+2\sqrt{a+b}}.
\end{aligned}$$

Reflecting the points $X_{n,+}$, $X_{n,-}$, $Y_{n,+}$, $Y_{n,-}$, $Z^{++}_{n,+}$, $Z^{++}_{n,-}$, $Z^{+-}_{n,+}$ and $Z^{+-}_{n,-}$ in the x-axis, we get the points $X_{-n,-}$, $X_{-n,+}$, $Y_{-n,-}$, $Y_{-n,+}$, $Z^{--}_{-n,-}$, $Z^{--}_{-n,+}$, $Z^{-+}_{-n,-}$ and $Z^{-+}_{-n,+}$, respectively. Since $-1 < (\sqrt{a}-\sqrt{b})/(\sqrt{a}+\sqrt{b}) < 1$, $X_{n,+}$ and $X_{n,-}$ cannot be the point at infinity on the y-axis for any integer n, if $a \neq b$. However, any of the other points can be identical with the point at infinity for some a and b and integer n. The proof of the next theorem is similar to the proof of Theorem 5.

Theorem 12. *Let n be an arbitrary integer and $a \neq b$.*

(i) $1/a_n^+ = 1/b_{-n}^+$ *if and only if the circle* γ *passes through* $X_{n,\pm}$ *or* $Z_{n,\pm}^{++}$. *If* γ *passes through* $X_{n,\pm}$,

$$\frac{1}{a_n^+} = \frac{1}{b_{-n}^+} = \left(n\frac{\sqrt{a+b}}{\sqrt{a}-\sqrt{b}} - 1\right)^2 \frac{1}{r_{\mathrm{A}}}$$

and if γ *passes through* $Z_{n,\pm}^{++}$,

$$\frac{1}{a_n^+} = \frac{1}{b_{-n}^+} = \left(\left(n\frac{\sqrt{a+b}}{\sqrt{a}-\sqrt{b}} - 1\right)\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)\right)^2 \frac{1}{r_{\mathrm{A}}}.$$

(ii) $1/a_{-n}^- = 1/b_n^-$ *if and only if the circle* γ *passes through* $X_{n,\pm}$ *or* $Z_{n,\pm}^{--}$. *If* γ *passes through* $X_{n,\pm}$,

$$\frac{1}{a_{-n}^-} = \frac{1}{b_n^-} = \left(n\frac{\sqrt{a+b}}{\sqrt{a}-\sqrt{b}} + 1\right)^2 \frac{1}{r_{\mathrm{A}}}$$

and if γ *passes through* $Z_{n,\pm}^{--}$,

$$\frac{1}{a_{-n}^-} = \frac{1}{b_n^-} = \left(\left(n\frac{\sqrt{a+b}}{\sqrt{a}-\sqrt{b}} + 1\right)\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)\right)^2 \frac{1}{r_{\mathrm{A}}}.$$

(iii) $1/a_{-n}^+ = 1/b_{-n}^-$ *if and only if the circle* γ *passes through* $Y_{n,\pm}$ *or* $Z_{n,\pm}^{+-}$. *If* γ *passes through* $Y_{n,\pm}$,

$$\frac{1}{a_{-n}^+} = \frac{1}{b_{-n}^-} = \left(n\frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} - 1\right)^2 \frac{1}{r_{\mathrm{A}}}$$

and if γ *passes through* $Z_{n,\pm}^{+-}$,

$$\frac{1}{a_{-n}^+} = \frac{1}{b_{-n}^-} = \left(\left(n\frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} - 1\right)\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)\right)^2 \frac{1}{r_{\mathrm{A}}}.$$

(iv) $1/a_n^- = 1/b_n^+$ *if and only if the circle* γ *passes through* $Y_{n,\pm}$ *or* $Z_{n,\pm}^{-+}$. *If* γ *passes through* $Y_{n,\pm}$,

$$\frac{1}{a_n^-} = \frac{1}{b_n^+} = \left(n\frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} + 1\right)^2 \frac{1}{r_{\mathrm{A}}}$$

and if γ *passes through* $Z_{n,\pm}^{-+}$,

$$\frac{1}{a_n^-} = \frac{1}{b_n^+} = \left(\left(n\frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} + 1\right)\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)\right)^2 \frac{1}{r_{\mathrm{A}}}.$$

Each of the propositions (i), (ii), (iii) and (iv) in Theorems 5 and 11 asserts the existence of two different pairs of the n-th twin circles of Archimedes in two different arbeloi, but the ratio of their radii is independent of n and the circle γ and always equal to $\left((\sqrt{a}+\sqrt{b})/(\sqrt{a}-\sqrt{b})\right)^{\pm 2}$.

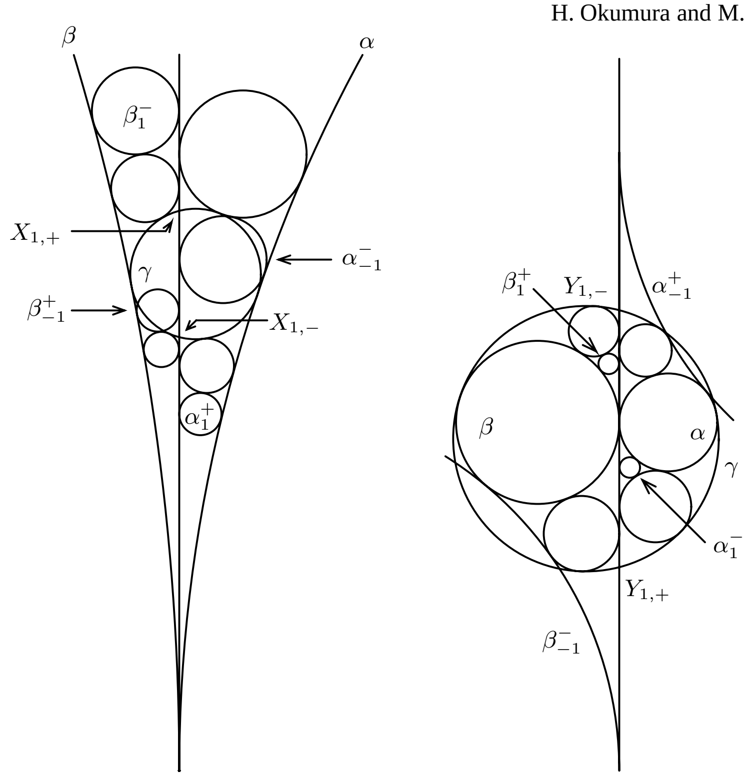


Figure 14. $a_1^+ = b_{-1}^+$, $a_{-1}^- = b_1^-$ for $(X_{1,\pm})$ $a_{-1}^+ = b_{-1}^-$, $a_1^- = b_1^+$ for $(Y_{1,\pm})$

If the circle γ passes through the points $X_{n,\pm}$, we label the arbelos as $(X_{n,\pm})$. The arbeloi $(Y_{n,\pm})$, $(Z_{n,\pm}^{++})$, $(Z_{n,\pm}^{--})$, $(Z_{n,\pm}^{+-})$ and $(Z_{n,\pm}^{-+})$ are defined similarly. Reflecting $(X_{n,\pm})$, $(Y_{n,\pm})$, $(Z_{n,\pm}^{++})$ and $(Z_{n,\pm}^{+-})$ in the x-axis, we get $(X_{-n,\pm})$, $(Y_{-n,\pm})$, $(Z_{-n,\pm}^{--})$ and $(Z_{-n,\pm}^{-+})$, respectively. Table 2 shows, on which sides of the x-axis lie the centers of the n-th twin circles of Archimedes in these arbeloi. According to Theorem 12, there are two pairs of the n-th twin circles of Archimedes in the arbeloi $(X_{n,\pm})$ and $(Y_{n,\pm})$ (see Figure 14).

	$(X_{n,\pm})$	$(Y_{n,\pm})$	$(Z_{n,\pm}^{++})$	$(Z_{n,\pm}^{--})$	$(Z_{n,\pm}^{+-})$	$(Z_{n,\pm}^{-+})$
same side	α_n^+, β_{-n}^+ α_{-n}^-, β_n^-				$\alpha_{-n}^+, \beta_{-n}^-$	α_n^-, β_n^+
opposite side		$\alpha_{-n}^+, \beta_{-n}^-$ α_n^-, β_n^+	α_n^+, β_{-n}^+	α_{-n}^-, β_n^-		

Table 2.

6. Another twin circle property

We demonstrate the existence of another pair of twin circles in the case, when the circle γ and the line joining the centers of α and β intersect. This pair of twin circles is a generalization of the circles W_6 and W_7 in [4]. A related result can be seen in [6]. We start by proving the following lemma:

Lemma 13. *Let A_0B_0 be the diameter of the circle γ parallel to the x-axis and intersecting the y-axis at the point O'. Let $a_0 = |A_0O'|$ and $b_0 = |B_0O'|$, where A_0 and B_0 lie on the same sides of the y-axis as the circles α and β, respectively. If γ touches α and β internally, $a/b = a_0/b_0$ and if γ touches α and β externally, $a/b = b_0/a_0$.*

Proof. Assume that γ touches α and β internally and $a < b$ (see Figure 15). Let O_α, O_β and O_γ be the centers of α, β and γ and F the foot of the normal dropped from O_γ to the x-axis. By Pythagorean theorem we get

$$|O_\gamma O_\alpha|^2 - |O_\alpha F|^2 = |O_\gamma O_\beta|^2 - |O_\beta F|^2.$$

Substituting $|O_\gamma O_\alpha| = (a_0 + b_0)/2 - a$, $|O_\gamma O_\beta| = (a_0 + b_0)/2 - b$, $|O_\alpha F| = a + |O_\gamma O'|$, $|O_\beta F| = b - |O_\gamma O'|$ and $|O_\gamma O'| = (a_0 + b_0)/2 - a_0$, we obtain $a/b = a_0/b_0$. The case, when γ touches α and β externally, can be proved in a similar way. $\square$

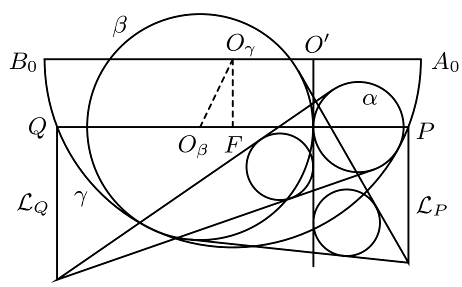


Figure 15.

Theorem 14. *Let AO and BO be the diameters of the circles α and β on the x-axis. Let P and Q be the intersections of the circle γ with the x-axis, choosing P and Q so that A, P, Q, B follow in this order on the x-axis, if we regard it as a circle of infinite radius closed through the point at infinity. Let $\mathcal{L}_P$ and $\mathcal{L}_Q$ be the lines through P and Q perpendicular to the x-axis. The circle touching the y-axis from the side opposite to β and the tangents to β from an arbitrary point on $\mathcal{L}_P$ is congruent to the circle touching the y-axis from the side opposite to α and the tangents to α from an arbitrary point on $\mathcal{L}_Q$.*

Proof. We use the same notation as in Lemma 13 and its proof. Assume that γ touches α and β internally and $a < b$. According to Lemma 13, there is a real number k, such that $a = a_0k$ and $b = b_0k$. Hence,

$$|O_\gamma F|^2 =((a_0 + b_0)/2 - b_0k)^2 - (b_0k - d)^2,$$
$$|QF|^2 =|O_\gamma Q|^2 - |O_\gamma F|^2 = 2a_0b_0k + d^2,$$

where $d = |OF| = (b_0 - a_0)/2$. Let r_b be the radius of the circle touching the y-axis from the side opposite to α and the common external tangents of α from an arbitrary point on $\mathcal{L}_Q$. Similarly, let r_a be the radius of the circle touching the y-axis from the side opposite to β and the common external tangent of β from an arbitrary point on $\mathcal{L}_P$. From the similarity of the circle with radius r_b and the circle α, we have

$$\frac{\sqrt{d^2 + 2a_0b_0k} + d - r_b}{r_b} = \frac{\sqrt{d^2 + 2a_0b_0k} + d + a_0k}{a_0k},$$

$$\frac{1}{r_b} = \frac{1}{a_0k} + \frac{\sqrt{d^2 + 2a_0b_0k} - d}{a_0b_0k}.$$

Similarly we obtain

$$\frac{1}{r_a} = \frac{1}{b_0k} + \frac{\sqrt{d^2 + 2a_0b_0k} + d}{a_0b_0k}.$$

But we can easily show that $1/r_a - 1/r_b = 0$ or $r_a = r_b$. The case, when γ touches α and β externally, can be proved in a similar way. □

Theorem 14 holds even in the case, when γ is one of the common external tangents of the circles α and β, if we consider γ to intersect the x-axis at the point at infinity. In this case, if $a < b$, these twin circles are congruent to α. If γ touches α and β internally, the minimum radii of these twin circles are equal to r_{A}, which is the case of the ordinary arbelos. If γ touches α and β externally, the radii of the twin circles are maximum in the case, when γ touches the x-axis. Let r be the maximum radius of the twin circles, c the radius of γ and d the distance of the tangency point of γ with the x-axis from the origin O and assume $a < b$. In this case

$$c^2 = (c + a)^2 - (d - a)^2 = (c + b)^2 - (d + b)^2.$$

Eliminating c and solving this equation for d, we get $d = 4ab/(b - a)$. From the similarity of the circle α and the corresponding twin circle, $(d-a)/a = (d+r)/r$, which implies $r = 2r_{\mathrm{A}}$. Consequently, we obtain that if $a < b$, $r_{\mathrm{A}} < a < 2r_{\mathrm{A}}$, and the the common radii of the twin circles take the minimum value r_{A} for the ordinary arbelos, a when γ is one of the common external tangents of α and β, and the maximum value $2r_{\mathrm{A}}$ when γ touches the x-axis. Since the circle γ touching the x-axis is identical with $\gamma_{\pm t/2}$ as mentioned at the end of §4, there is one more circle congruent to the twin circles in the last case, which is the circle $\epsilon_{\pm t/2}$ associated to $\gamma_{\pm t/2}$ by (iv) of Theorem 11 (see Figure 13).

7. Conclusion

We have demonstrated several interesting properties of the skewed arbelos, which could not have been found by consider the ordinary one. Since we confined our discussion largely to a generalization of the twin circles of Archimedes, it appears to be worth the effort to investigate other topics related to the skewed arbelos. We conclude our paper by proposing a problem. Let α, β and γ be three circles forming a skewed arbelos, i.e., γ is given by equations (8) or (9), and let δ be a circle touching α and β at their tangency point O and intersecting γ. The circle δ divides the skewed arbelos into two curvilinear triangles. Find (or construct) the circle δ, such that the incircles of the two curvilinear triangles are congruent (see Figure 16).

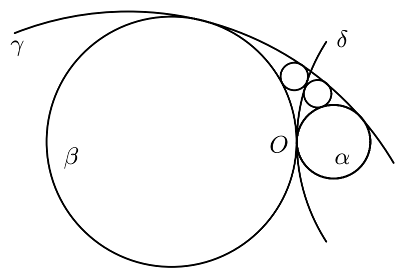


Figure 16.

References

[1] L. Bankoff, Are the twin circles of Archimedes really twins?, *Math. Mag.*, 47 (1974) 214–218.
[2] L. Bankoff, The marvelous arbelos, in *The lighter side of mathematics* 247-253, ed. R. K. Guy and R. E. Woodrow, Mathematical Association of America, 1994.
[3] L. Brother, F. S. C. Raphael, The shoemaker's knife, *Mathematics Teacher*, (1973) 319–323.
[4] C. W. Dodge, T. Schoch, P. Y. Woo and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.*, 72 (1999) 202–213.
[5] R. A. Johnson, *Advanced Euclidean geometry*, Dover, NY, 1960.
[6] H. Okumura, Two similar triangles, *Math. Gazette*, 79 (1995) 569–571.
[7] G. Yamamoto, *Sampō Jojutsu*, 1841.

Hiroshi Okumura: Department of Information Engineering, Maebashi Institute of Technology, 460-1 Kamisadori Maebashi Gunma 371-0816, Japan
E-mail address: okumura@maebashi-it.ac.jp

Masayuki Watanabe: Department of Information Engineering, Maebashi Institute of Technology, 460-1 Kamisadori Maebashi Gunma 371-0816, Japan
E-mail address: watanabe@maebashi-it.ac.jp

Forum Geometricorum
Volume 4 (2004) 253–260.

FORUM GEOM
ISSN 1534-1178

A Generalization of the Kiepert Hyperbola

Darij Grinberg and Alexei Myakishev

Abstract. Consider an arbitrary point P in the plane of triangle ABC with cevian triangle $A_1B_1C_1$. Erecting similar isosceles triangles on the segments BA_1, CA_1, CB_1, AB_1, AC_1, BC_1, we get six apices. If the apices of the two isosceles triangles with bases BA_1 and CA_1 are connected by a line, and the two similar lines for B_1 and C_1 are drawn, then these three lines form a new triangle, which is perspective to triangle ABC. For fixed P and varying base angle of the isosceles triangles, the perspector draws a hyperbola. Some properties of this hyperbola are studied in the paper.

1. Introduction

We consider the following configuration. Let P be a point in the plane of a triangle ABC, and AA_1, BB_1 and CC_1 be the three cevians of P. For an arbitrary nonzero angle φ satisfying $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, we erect two isosceles triangles BA_bA_1 and CA_cA_1 with the bases BA_1 and A_1C and base angle φ, both externally to triangle ABC if $\varphi > 0$, and internally otherwise. The same construction also gives the points B_c, B_a, C_a, C_b, with isosceles triangles all with base angle φ. This configuration depends on triangle ABC, the point P and $\varphi \in \left(-\frac{\pi}{2},\ \frac{\pi}{2}\right) \setminus \{0\}$.

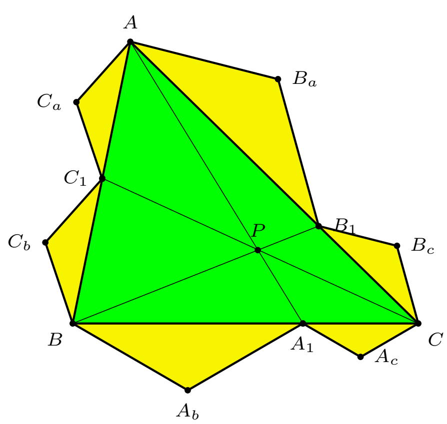


Figure 1.

We study an interesting locus problem associated with this configuration.

Publication Date: December 29, 2004. Communicating Editor: Paul Yiu.

The authors thank Professor Paul Yiu for his assistance and for the contribution of results which form a great part of §5.

2. The coordinates of the vertices

In this paper we work with homogeneous barycentric coordinates, and make use of John H. Conway's notations. See [1] for some basic properties of the Conway symbols. We begin by calculating the barycentric coordinates of the apices of our isosceles triangles. Let $(u : v : w)$ be the homogeneous barycentric coordinates of the point P.

Proposition 1. *The apices of the isosceles triangles on BA_1 and A_1C are the points*

$$A_b = (-a^2w : 2S_\varphi v + (S_C + S_\varphi)w : (S_B + S_\varphi)w), \tag{1}$$
$$A_c = (-a^2v : (S_C + S_\varphi)v : 2S_\varphi w + (S_B + S_\varphi)v). \tag{2}$$

Proof. Let A_φ be the apex of the isosceles triangle with base BC and base angle φ. It is well known that the point A_φ has the coordinates $(-a^2 : S_C + S_\varphi : S_B + S_\varphi)$. The line A_bA_1 is parallel to the line $A_\varphi C$; hence, using directed segments, we have $\dfrac{BA_b}{A_bA_\varphi} = \dfrac{BA_1}{A_1C} = \dfrac{w}{v}$, so that (identifying every point with the vector to the point from an arbitrarily chosen origin),

$$A_b = \frac{vB + wA_\varphi}{v+w} = \left(-\frac{a^2w}{2S_\varphi} : \frac{(S_C + S_\varphi)w}{2S_\varphi} + v : \frac{(S_B + S_\varphi)w}{2S_\varphi}\right).$$

Here, we have used the fact that the sum of the coordinates of the point A_φ is $-a^2 + (S_B + S_C) + 2S_\varphi = 2S_\varphi$.

This yields the coordinates of A_b given in (1) above. Similarly, A_c is as given in (2). The four remaining apices can be computed readily. □

Let $\mathcal{L}_a$ be the line joining the apices A_b and A_c. It is routine to compute the barycentric equation of the line $\mathcal{L}_a$.

Proposition 2. *The equation of the line $\mathcal{L}_a$ is*

$$(S_Bv^2 + S_Cw^2 + S_\varphi(v+w)^2)x + a^2w^2y + a^2v^2z = 0. \tag{3}$$

Proof. For $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}$, the equation of the line joining A_b and A_c is

$$\begin{vmatrix} x & y & z \\ -a^2w & 2S_\varphi v + (S_C + S_\varphi)w & (S_B + S_\varphi)w \\ -a^2v & (S_C + S_\varphi)v & 2S_\varphi w + (S_B + S_\varphi)v \end{vmatrix} = 0.$$

This simplifies into (3) above. □

Similarly, we define $\mathcal{L}_b$ and $\mathcal{L}_c$. Their equations can be easily written down:

$$b^2w^2x + (S_Cw^2 + S_Au^2 + S_\varphi(w+u)^2)y + b^2u^2z = 0, \tag{4}$$
$$c^2v^2x + c^2u^2y + (S_Au^2 + S_Bv^2 + S_\varphi(u+v)^2)z = 0. \tag{5}$$

3. The triangle formed by the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$

Consider the triangle bounded by the lines $\mathcal{L}_a$, $\mathcal{L}_b$ and $\mathcal{L}_c$. This has vertices

$$A_2 = \mathcal{L}_b \cap \mathcal{L}_c, \qquad B_2 = \mathcal{L}_c \cap \mathcal{L}_a, \qquad C_2 = \mathcal{L}_a \cap \mathcal{L}_b.$$

Theorem 3. *The triangle bounded by the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ is perspective with ABC. Their axis of perspectivity is the trilinear polar of the barycentric square of the point P.*

Proof. Let $A_0 = BC \cap \mathcal{L}_a$, $B_0 = CA \cap \mathcal{L}_b$, and $C_0 = AB \cap \mathcal{L}_c$. In homogeneous barycentric coordinates, these are the points

$$A_0 = (0 : -v^2 : w^2), \quad B_0 = (u^2 : 0 : -w^2), \quad C_0 = (-u^2 : v^2 : 0)$$

respectively, and are all on the line

$$\frac{x}{u^2} + \frac{y}{v^2} + \frac{z}{w^2} = 0. \tag{6}$$

It follows from the Desargues theorem that ABC and the triangle bounded by the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ are perspective. Note that the axis of perspectivity (6) is the trilinear polar of the point $(u^2 : v^2 : w^2)$, the barycentric square of P.[1] It is independent of φ. □

The perspector of the triangles, however, varies with φ. We work out its coordinates explicitly. The vertices of the triangle in question are

$$A_2 = \mathcal{L}_b \cap \mathcal{L}_c, \qquad B_2 = \mathcal{L}_c \cap \mathcal{L}_a, \qquad C_2 = \mathcal{L}_a \cap \mathcal{L}_b.$$

From (4) and (5), the line joining A_2 to A has equation

$$\begin{aligned} &c^2(v^2(S_C w^2 + S_A u^2 + 2S_\varphi (w+u)^2) - b^2 w^2 u^2) y \\ &- b^2(w^2(S_A u^2 + S_B v^2 + 2S_\varphi (u+v)^2) - c^2 u^2 v^2) z = 0. \end{aligned}$$

Similarly, by writing down the equations of the lines BB_2 and CC_2, we easily find the perspector of the triangles ABC and $A_2B_2C_2$.

Theorem 4. *For any point P and any angle φ, the perspector of the triangles ABC and $A_2B_2C_2$ is the point*

$$K_P(\varphi) = \left(\frac{a^2}{-S_B v^2(w^2-u^2) + S_C w^2(u^2-v^2) + u^2(v+w)^2 S_\varphi} : \cdots : \cdots \right) \tag{7}$$

Strictly speaking, the perspector $K_P(\varphi)$ is not defined in the cases $\varphi = 0$ and $\varphi = \frac{\pi}{2}$. However, in these two cases we can define the perspectors as the limits of the perspector when the angle approaches 0 and $\frac{\pi}{2}$, respectively. The coordinates of these limiting perspectors can be obtained from (7) by substituting $\varphi = 0$ and $\frac{\pi}{2}$ respectively.

---

[1] If we take the harmonic conjugates A_0', B_0' and C_0' of the points A_0, B_0, C_0 with respect to the sides BC, CA, AB respectively, then the lines AA_0', BB_0' and CC_0' concur at the trilinear pole of the line $A_0B_0C_0$, which is the barycentric square of P. This gives an interesting construction of the barycentric square of a point. For another construction, see [2].

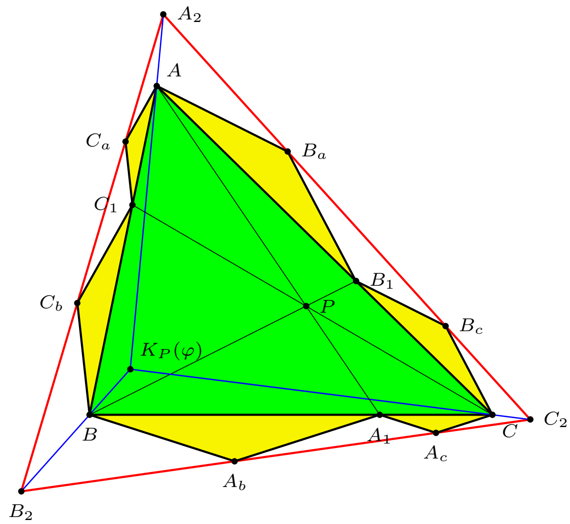


Figure 2.

$$K_P(0) = \left(\frac{a^2}{u^2(v+w)^2} : \frac{b^2}{v^2(w+u)^2} : \frac{c^2}{w^2(u+v)^2}\right),$$

$$K_P\left(\frac{\pi}{2}\right) = \left(\frac{a^2}{S_B\left(\frac{1}{w^2}-\frac{1}{u^2}\right) - S_C\left(\frac{1}{u^2}-\frac{1}{v^2}\right)} : \cdots : \cdots\right).$$

4. The locus of the perspector

From the coordinates of the perspector $K_P(\varphi)$ given in (7), it is clear that the point lies on the isogonal conjugate of the line joining the points

$$\begin{aligned} P_1 =& (u^2(v+w)^2 : v^2(w+u)^2 : w^2(u+v)^2), \\ P_2 =& (-S_B v^2(w^2-u^2) + S_C w^2(u^2-v^2) : \cdots : \cdots) \\ =& \left(S_B\left(\frac{1}{w^2}-\frac{1}{u^2}\right) - S_C\left(\frac{1}{u^2}-\frac{1}{v^2}\right) : \cdots : \cdots\right). \end{aligned}$$

Obviously P_1 is an interior point of triangle ABC. It is more interesting to note that P_2 is an infinite point, evidently of the line

$$S_A\left(\frac{1}{v^2}-\frac{1}{w^2}\right)x + S_B\left(\frac{1}{w^2}-\frac{1}{u^2}\right)y + S_C\left(\frac{1}{u^2}-\frac{1}{v^2}\right)z = 0. \qquad (8)$$

Note that $\left(\frac{1}{v^2}-\frac{1}{w^2} : \frac{1}{w^2}-\frac{1}{u^2} : \frac{1}{u^2}-\frac{1}{v^2}\right)$ is also an infinite point, of the line (6). From (8), these two lines are orthogonal. See [3, p.52].

Theorem 5. *Let $P = (u : v : w)$. The locus $\mathcal{K}_P$ is the isogonal conjugate of the line through the point $(u^2(v+w)^2 : v^2(w+u)^2 : w^2(u+v)^2)$ perpendicular to the trilinear polar of $(u^2 : v^2 : w^2)$.*

If P is not the centroid [2] and if this line does not pass through any of the vertices of ABC or its antimedial triangle, then the locus $\mathcal{K}_P$ is a circum-hyperbola, [3] which is rectangular if and only if P lies on the quintic

$$\begin{aligned}&a^2v^2w^2(v-w)+b^2w^2u^2(w-u)+c^2u^2v^2(u-v)\\=&uvw(u+v+w)((b^2-c^2)u+(c^2-a^2)v+(a^2-b^2)w).\end{aligned} \tag{9}$$

We shall study the degenerate case in §6 below.

5. Special cases

5.1. *The orthocenter.* If $P=H$, the orthocenter, $P_1=(a^4:b^4:c^4)$ and the trilinear polar of $\left(\frac{1}{S_{AA}}:\frac{1}{S_{BB}}:\frac{1}{S_{CC}}\right)$ is the line $S_{AA}x+S_{BB}y+S_{CC}z=0$. The perpendicular from P_1 to this line is the line

$$\frac{b^2-c^2}{a^2}x+\frac{c^2-a^2}{b^2}y+\frac{a^2-b^2}{c^2}z=0,$$

which is clearly the Brocard axis OK. The locus $\mathcal{K}_H$ is therefore the Kiepert hyperbola $\mathcal{K}$. A typical point on $\mathcal{K}$ is the Kiepert perspector

$$K(\theta)=\left(\frac{1}{S_A+S_\theta}:\frac{1}{S_B+S_\theta}:\frac{1}{S_C+S_\theta}\right)$$

which is the perspector of the triangle of apices of isosceles triangles of base angles θ erected on the sides of triangle ABC.

Theorem 6. *$K_H(\varphi)=K(\theta)$ if and only if*

$$\cot\varphi(\cot\omega+\cot\theta)+\cot\theta\cot\omega+1=0, \tag{10}$$

where ω is the Brocard angle of triangle ABC.

Proof. From (7),

$$K_H(\varphi)=\left(\frac{1}{S_{BC}-S_{AA}+a^2S_\varphi}:\cdots:\cdots\right).$$

This is the same as $K(\theta)$ if and only if

$$\begin{aligned}&((S_{CA}-S_{BB}+b^2S_\varphi)(S_{AB}-S_{CC}+c^2S_\varphi),\cdots,\cdots)\\=&k((S_B+S_\theta)(S_C+S_\theta),\cdots,\cdots)\end{aligned}$$

for some k. These conditions are satisfied if and only if

$$k=(S_A+S_B+S_C+S_\varphi)^2,$$

and

$$S_\theta S_\varphi+(S_A+S_B+S_C)(S_\theta+S_\varphi)+S^2=0.$$

This latter condition translates into (10) above. □

---

[2] $K_G(\varphi)=K$, the symmedian point, for every φ.

[3] In fact, being the isogonal conjugate of the line P_1P_2, this is a circumconic. Since the line P_1P_2 intersects the circumcircle of triangle ABC (as P_1 is an interior point), it is a circum-hyperbola. The isogonal conjugate of the point P_2 is the fourth point of intersection of the circumscribed hyperbola with the circumcircle of triangle ABC.

Note that the relation (10) is symmetric in φ, ω, and θ. From this we obtain the following interesting corollary.

Corollary 7. *$K_H(\varphi) = K(\theta)$ if and only if $K_H(\theta) = K(\varphi)$.*

Here are some examples of corresponding φ and θ.

φ	$\frac{\pi}{4}$	$-\frac{\pi}{4}$	ω	$-\omega$
θ	$-\frac{\pi}{4}$	$\frac{\pi}{4}$	$-\arctan(\sin 2\omega)$	0

5.2. *The incenter.* If $P = I$, the incenter, we have

$$P_1 = (a^2(b+c)^2 : b^2(c+a)^2 : c^2(a+b)^2) = X_{1500}.$$

The point P_2 is the infinite point of the perpendicular to the Lemoine axis, namely,

$$X_{511} = (a^2(a^2(b^2+c^2) - (b^4+c^4)) : \cdots : \cdots),$$

the same as the case $P = H$. The hyperbola $\mathcal{K}_I$ is the circum-hyperbola through X_{98} and

$$X_{1509} = \left(\frac{1}{(b+c)^2} : \frac{1}{(c+a)^2} : \frac{1}{(a+b)^2}\right).$$

The center of the hyperbola is the point

$$((b-c)^2 f(a,b,c)g(a,b,c) : (c-a)^2 f(b,c,a)g(b,c,a) : (a-b)^2 f(c,a,b)g(c,a,b)),$$

where

$$\begin{aligned} f(a,b,c) =& a^5 - a^3(b^2+bc+c^2) - a^2(b+c)(b^2+c^2) - abc(b+c)^2 - b^2c^2(b+c),\\ g(a,b,c) =& a^5 - a^3(2b^2+bc+2c^2) - a^2(b+c)(2b^2+bc+2c^2)\\ &- a(b^4 - b^3c - 2b^2c^2 - bc^3 + c^4) - bc(b+c)^3. \end{aligned}$$

5.3. *The Gergonne point.* If P is the Gergonne point, P_1 is the symmedian point K and the infinite point of the perpendicular to the trilinear polar of

$$X_{279} = \left(\frac{1}{(b+c-a)^2} : \frac{1}{(c+a-b)^2} : \frac{1}{(a+b-c)^2}\right)$$

is

$$X_{517} = (a(a^2(b+c) - 2abc - (b+c)(b-c)^2) : \cdots : \cdots).$$

The hyperbola passes through the centroid and X_{104} and has center

$$(a^2(b-c)^2(a^3 - a^2(b+c) - a(b-c)^2 + (b+c)(b^2+c^2))^2 : \cdots : \cdots).$$

5.4. $P = X_{671}$. The point $X_{671} = \left(\frac{1}{2a^2-b^2-c^2} : \frac{1}{2b^2-c^2-a^2} : \frac{1}{2c^2-a^2-b^2}\right)$ lies on the quintic (9). It is the reflection of the centroid in the Kiepert center X_{115}. If $P = X_{671}$, the locus $\mathcal{K}_P$ is the rectangular hyperbola whose center is the point

$$((b^2-c^2)^2(2a^2-b^2-c^2)(a^4-b^4+b^2c^2-c^4) : \cdots : \cdots)$$

on the nine-point circle.

5.5. *P on a sideline.* If P is a point on a sideline of triangle ABC, say, BC, then $P_1 = (0 : 1 : 1)$ is the midpoint of BC, and $P_2 = (-a^2 : S_C : S_B)$ is the infinite point of the A-altitude. It follows that P_1P_2 is the perpendicular bisector of BC. Its isogonal conjugate is the circum-hyperbola whose center is the midpoint of BC. It also passes through the antipode of A in the circumcircle.

6. The degenerate case

The locus $\mathcal{K}_P$ is a circum-hyperbola if and only if the line P_1P_2 does not contain a vertex of the triangle. The equation of the line P_1P_2 is of the form

$$U(P)x + V(P)y + W(P)z = 0,$$

where

$$\begin{aligned} U(P) = {} & v^2(w+u)^2(w^2(u^2S_A + v^2S_B) - c^2u^2v^2) \\ & - w^2(u+v)^2(v^2(u^2S_A + w^2S_C) - b^2u^2w^2), \end{aligned}$$

and $V(P)$ and $W(P)$ are obtained from $U(P)$ by cyclic permutations of (u, v, w), (a, b, c) and (S_A, S_B, S_C). The locus of the perspectors $\mathcal{K}_P$ is degenerate (*i.e.*, it is not a hyperbola) if and only if at least one of the three coefficients $U(P)$, $V(P)$ and $W(P)$ in the equation of the line P_1P_2 is zero, *i.e.*, if the point P lies on at least one of the three curves of 8 degree defined by the equations $U(P) = 0$, $V(P) = 0$ and $W(P) = 0$. Each of these three curves contains the vertices of the triangle ABC, its centroid G, and also the vertices of the antimedial triangle $G_aG_bG_c$. Moreover, for any two of these three curves, the only real common points are these 7 points just listed. We conclude with the following observations.

- If P is one of the vertices A, B and C, then the locus $\mathcal{K}_P$ is not defined. It is possibly an isolated singularity of one or more of the curves $U(P) = 0$, $V(P) = 0$, and $W(P) = 0$.
- If P is a vertex of the antimedial triangle, then the locus $\mathcal{K}_P$ is the corresponding sideline of the triangle ABC. For example, $\mathcal{K}_{G_a} = BC$.
- If $P = G$, the centroid of triangle ABC, then $\mathcal{K}_P$ consists of one single point, the symmedian point K of triangle ABC.
- In all other degenerate cases, the hyperbola degenerates into a pair of lines, one of them being a sideline of the triangle, while the other one passes through the opposite vertex (but does not coincide with a sideline).

  If we put

  $$(u, v, w) = \left(\frac{1}{y+z-x}, \frac{1}{z+x-y}, \frac{1}{x+y-z}\right),$$

  the equation $U(P) = 0$ defines the quartic curve

  $$x = \frac{yz(S_By^2 - S_Cz^2)}{S_A(y-z)(y^2+z^2) + S_By^3 - S_Cz^3}$$

  with respect to the antimedial triangle of ABC. Figure 3 shows an example of these curves in which the vertex B is an isolated singularity of the curves $U(P) = 0$ and $W(P) = 0$.

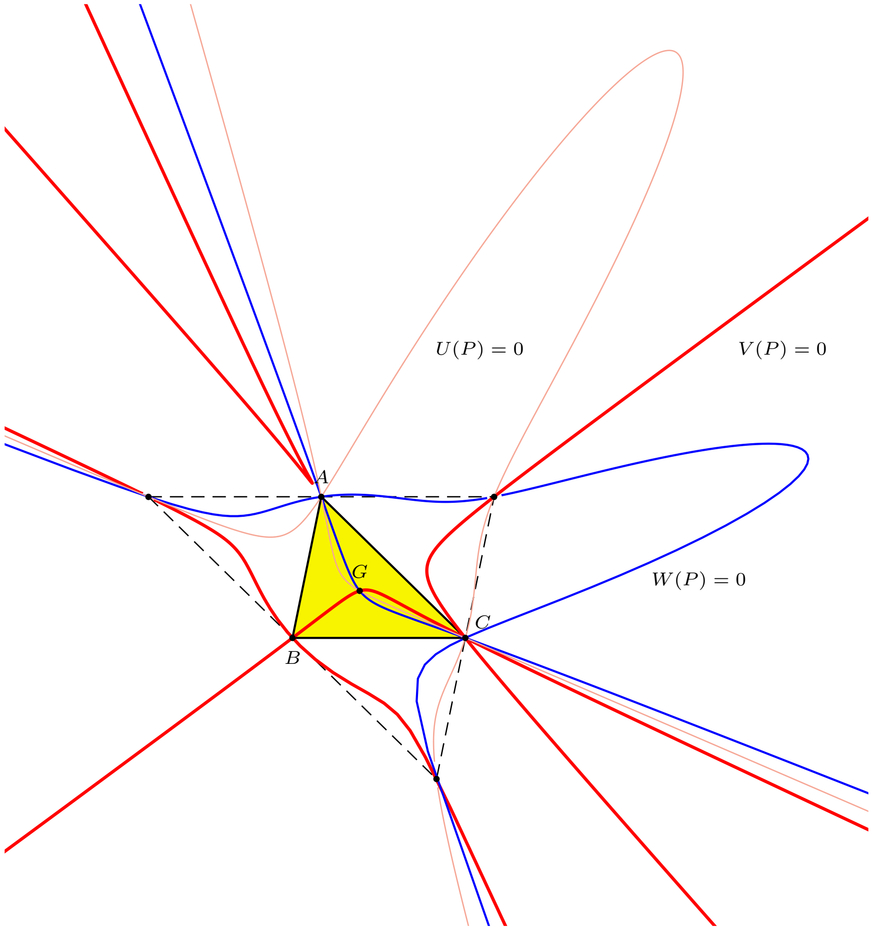


Figure 3.

References

[1] F. M. van Lamoen and P. Yiu, The Kiepert pencil of Kiepert hyperbolas, *Forum Geom.*, 1 (2001) 125–132.

[2] P. Yiu, The uses of homogeneous barycentric coordinates in plane euclidean geometry, *Int. J. Math. Educ. Sci. Technol.*, 31 (2000) 569–578.

[3] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University lecture notes, 2001.

Darij Grinberg: Geroldsäckerweg 7, D-76139 Karlsruhe, Germany
E-mail address: `darij_grinberg@web.de`

Alexei Myakishev: Smolnaia 61-2, 138, Moscow, Russia, 125445
E-mail address: `alex_geom@mtu-net.ru`

Forum Geometricorum
Volume 4 (2004) 261–262.

FORUM GEOM
ISSN 1534-1178

Author Index

FORUM GEOMETRICORUM

A Journal on Classical Euclidean Geometry and Related Areas

published by

Department of Mathematical Sciences
Florida Atlantic University

FORUM GEOM

Volume 5
2005

http://forumgeom.fau.edu

ISSN 1534-1178

Editorial Board

Advisors:

John H. Conway	Princeton, New Jersey, USA
Julio Gonzalez Cabillon	Montevideo, Uruguay
Richard Guy	Calgary, Alberta, Canada
Clark Kimberling	Evansville, Indiana, USA
Kee Yuen Lam	Vancouver, British Columbia, Canada
Tsit Yuen Lam	Berkeley, California, USA
Fred Richman	Boca Raton, Florida, USA

Editor-in-chief:

Paul Yiu	Boca Raton, Florida, USA

Editors:

Clayton Dodge	Orono, Maine, USA
Roland Eddy	St. John's, Newfoundland, Canada
Jean-Pierre Ehrmann	Paris, France
Chris Fisher	Regina, Saskatchewan, Canada
Rudolf Fritsch	Munich, Germany
Bernard Gibert	St Etiene, France
Antreas P. Hatzipolakis	Athens, Greece
Michael Lambrou	Crete, Greece
Floor van Lamoen	Goes, Netherlands
Fred Pui Fai Leung	Singapore, Singapore
Daniel B. Shapiro	Columbus, Ohio, USA
Steve Sigur	Atlanta, Georgia, USA
Man Keung Siu	Hong Kong, China
Peter Woo	La Mirada, California, USA

Technical Editors:

Yuandan Lin	Boca Raton, Florida, USA
Aaron Meyerowitz	Boca Raton, Florida, USA
Xiao-Dong Zhang	Boca Raton, Florida, USA

Consultants:

Frederick Hoffman	Boca Raton, Floirda, USA
Stephen Locke	Boca Raton, Florida, USA
Heinrich Niederhausen	Boca Raton, Florida, USA

Table of Contents

Forum Geometricorum
Volume 5 (2005) 1–15.

FORUM GEOM
ISSN 1534-1178

Where are the Conjugates?

Steve Sigur

Abstract. The positions and properties of a point in relation to its isogonal and isotomic conjugates are discussed. Several families of self-conjugate conics are given. Finally, the topological implications of conjugacy are stated along with their implications for pivotal cubics.

1. Introduction

The edges of a triangle divide the Euclidean plane into seven regions. For the projective plane, these seven regions reduce to four, which we call the central region, the a region, the b region, and the c region (Figure 1). All four of these regions, each distinguished by a different color in the figure, meet at each vertex. Equivalent structures occur in each, making the projective plane a natural background for fundamental triangle symmetries. In the sense that the projective plane can be considered a sphere with opposite points identified, the projective plane divided into four regions by the edges of a triangle can be thought of as an octahedron projected onto this sphere, a remark that will be helpful later.

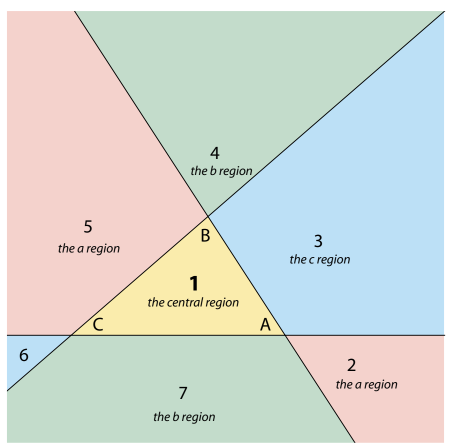


Figure 1. The plane of the triangle, Euclidean and projective views

A point $\mathbf{P}$ in any of the four regions has an harmonic associate in each of the others. Cevian lines through $\mathbf{P}$ and/or its harmonic associates traverse two of the these regions, there being two such possibilities at each vertex, giving 6 Cevian (including exCevian) lines. These lines connect the harmonic associates with the vertices in a natural way.

Publication Date: January 11, 2005. Communicating Editor: Paul Yiu.

Given two points in the plane there are two central points (a non-projective concept), the midpoint and a point at infinity. Given two lines there are two central lines, the angle bisectors. Where there is a sense of center, there is a sense of deviation from that center. For each point not at a vertex of the triangle there is a conjugate point defined using each of these senses of center. The isogonal conjugate is the one defined using angles and the isotomic conjugate is defined using distances. This paper is about the relation of a point to its conjugates.

We shall use the generic term *conjugate* when either type is implied. Other types of conjugacy are possible [2], and our remarks will hold for them as well.

Notation. Points and lines will be identified in bold type. John Conway's notation for points is used. The four incenters (the incenter and the three excenters) are $\mathbf{I}_o$, $\mathbf{I}_a$, $\mathbf{I}_b$, $\mathbf{I}_c$. The four centroids (the centroid and its harmonic associates) are $\mathbf{G}$, $\mathbf{A^G}$, $\mathbf{B^G}$, $\mathbf{C^G}$. We shall speak of equivalent structures around the four incenters or the four centroids. An angle bisector is identifed by the two incenters on it and a median by the two centroids on it as in "ob", or "ac". $\mathbf{A_P}$ is the Cevian trace of line $\mathbf{AP}$ and $\mathbf{A^P}$ is a vertex of the pre-Cevian triangle of $\mathbf{P}$. We shall often refer to this point as an "ex-"version of $\mathbf{P}$ or as an harmonic associate of $\mathbf{P}$. Coordinates are barycentric. $\mathrm{t}\mathbf{P}$ is the isotomic conjugate of $\mathbf{P}$, $\mathrm{g}\mathbf{P}$ the isogonal conjugate.

The isogonal of a line through a vertex is its reflection across either bisector through that vertex. The isogonal lines of the three Cevian lines of a point $\mathbf{P}$ concur in its conjugate $\mathrm{g}\mathbf{P}$. In the central region of a triangle, the relation of a point to its conjugate is simple. This region of the triangle is divided into 6 smaller regions by the three internal bisectors. If $\mathbf{P}$ is on a bisector, so is $\mathrm{g}\mathbf{P}$, with the incenter between them, making the bisectors fixed lines under isogonal conjugation. If $\mathbf{P}$ is not on a bisector, then $\mathrm{g}\mathbf{P}$ is in the one region of the six that is on the opposite side of each of the three bisectors. This allows us to color the central region with three colors so that a point and its conjugate are in regions of the same color (Figure 2). The isotomic conjugate behaves analogously with the medians serving as fixed or self-conjugate lines.

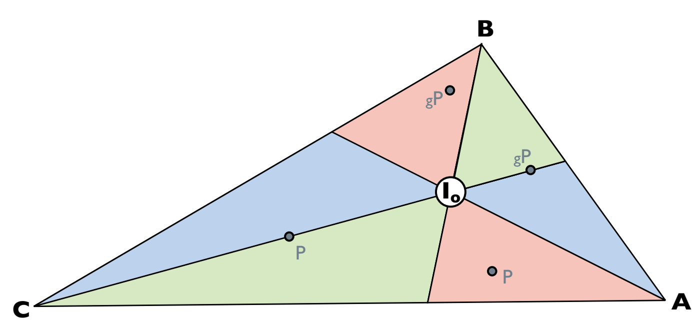


Figure 2. Angle bisctors divide the central region of the triangle into co-isotomic regions. The isogonal conjugate of a point on a bisector is also on that bisector. The conjugate of a point in one of the colored regions is in the other region of the same color.

2. Relation of conjugates to self-conjugate lines

The central region is all well and good, but the other three regions are locally identical in behavior and are to be considered structurally equivalent. Figure 3 shows the triangle with the incentral quadrangle. Each vertex of $\mathbf{ABC}$ hosts two bisectors, traditionally called internal and external. It is important to realize that an isogonal line through any vertex can be created by reflection in either bisector. This means that the three particular bisectors through any of the four incenters (one from each vertex) can be used to define the isogonal conjugate. Hence the behavior of conjugates around $\mathbf{I}_b$, say, is locally identical to that around $\mathbf{I}_o$, as shown in Figure 4.

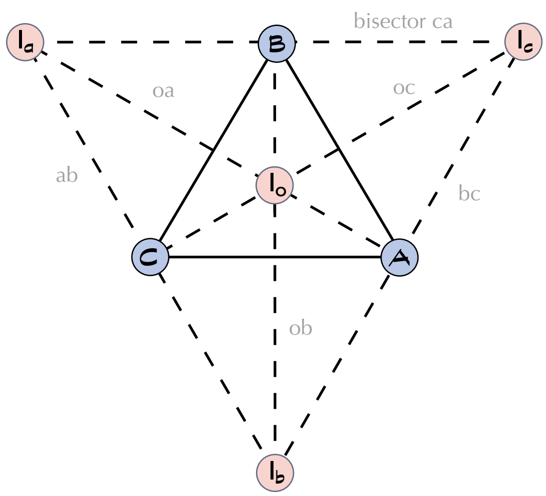


Figure 3. The triangle and its incentral quadrangle

If $\mathbf{P}$ is in the central region, the conjugate $\mathrm{g}\mathbf{P}$ is also; both are on the same side (the interior side) of each of the three external bisectors. So in the central region a point and its conjugate are on opposite sides of three bisectors (the internal ones) and on the same side of three others (the external ones). This is also true in the neighborhood of $\mathbf{I}_b$, although the particular bisectors have changed. No matter where in the plane, a point not on a bisector is on the opposite of three bisectors from its conjugate and on the same side for the other three bisectors. To some extent this statement is justified by the local equivalence of conjugate behavior mentioned above, but this assertion will be fully justified later in §10 on topological properties.

3. Formal properties of the conjugacy operation

Each type of conjugate has special fixed points and lines in the plane. As these properties are generally known, they will be stated without proof. Figures 5 and 8 show the mentioned structures.

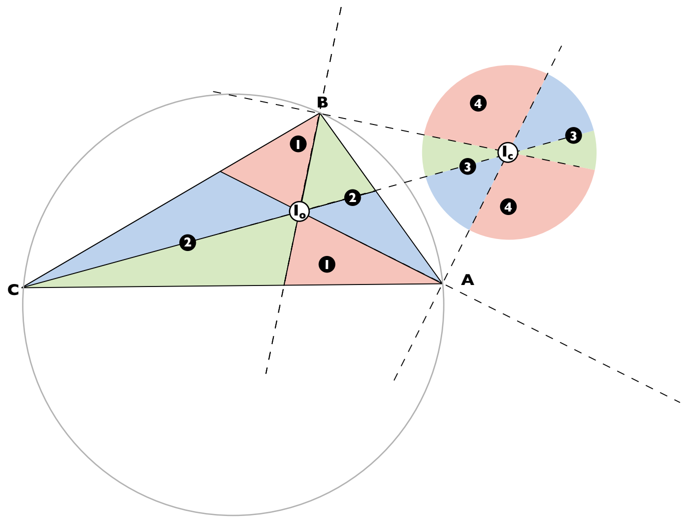


Figure 4. This picture shows the local equivalence of the region around $\mathbf{I}_o$ to that around $\mathbf{I}_c$. This equivalence appears to end at the circumcircle. Numbered points are co-conjugal, each being the conjugate of the other. For each region a pair of points both on and off a bisector is given.

conjugacy	fixed points	fixed lines	special curves	singularities
isotomic	centroid, its harmonic conjugates	medians and ex-medians	line at infinity, Steiner ellipse	vertices
isogonal	incenter, its harmonic conjugates	internal and external bisectors	line at infinity, circumcircle	vertices

For each type of conjugacy there are 4 points in the plane, harmonically related, that are fixed points under conjugacy. For isogonal conjugacy these are the 4 in/excenters. For isotomic conjugacy these are the centroid and its harmonic associates.In each case the six lines that connect the 4 fixed points are the fixed lines.

Special curves: Each point on the Steiner ellipse has the property that its isotomic Cevians are parallel, placing the isotomic conjugate at infinity. Similarly for any point on the circumcircle, its isogonal Cevians are parallel, again placing the isogonal conjugate at infinity. These special curves are very significant in the Euclidean plane, but not at all significant in the projective plane.

The conjugate of a point on an edge of $\mathbf{ABC}$ is at the corresponding vertex, an ∞ to 1 correspondence. This implies that the conjugate at a vertex is not defined, making the vertices the three points in the plane where this is true. This leads to a complicated partition of the Euclidean plane, as the behavior the conjugate of a point inside the Steiner ellipse or the circumcircle is different from that outside. We

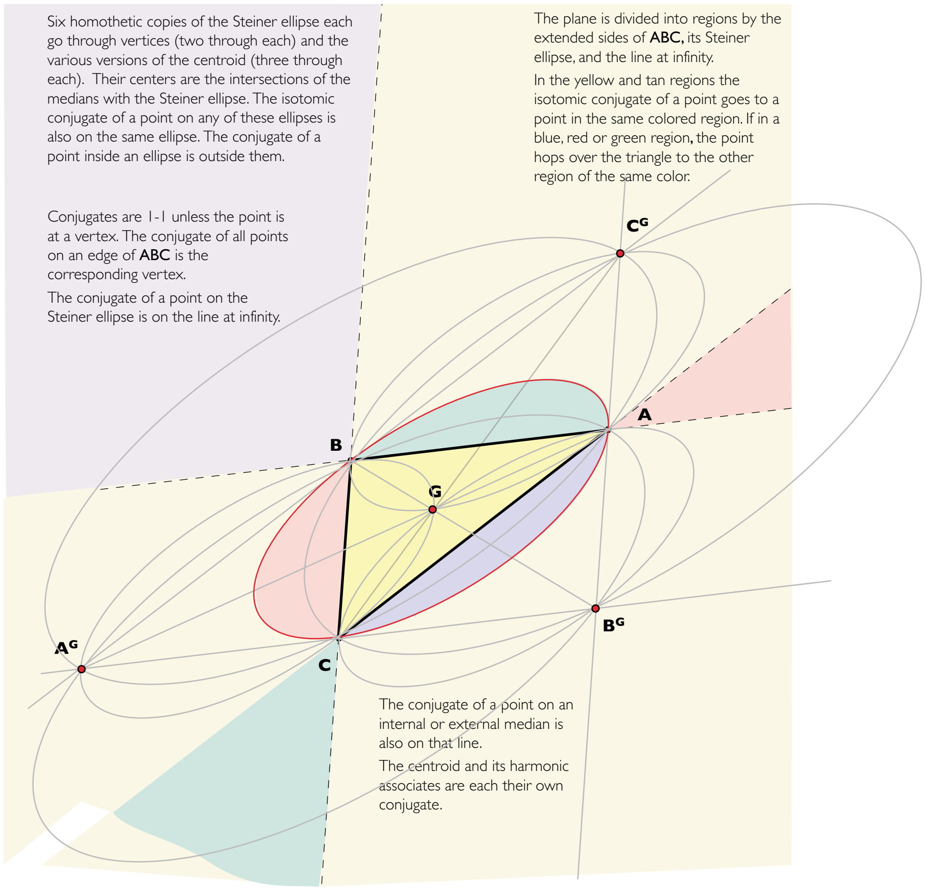


Figure 5. Isotomic conjugates

thus have the pictures of the regions of the plane in terms of conjugates as shown in Figures 5 and 8.

The colors in these two pictures show regions of the plane which are shared by the conjugates. The boundaries of these regions are the sides of the triangle, the circumconic and the line at infinity. The conjugate of a point in a region of a certain color is a region of the same color. For the red, green, and blue regions the conjugate is always in the other region of the same color.

These properties are helpful in locating a point in relation to the position of its conjugate, but there is more to this story.

4. Conjugate curves

4.1. *Lines.* The conjugate of a curve is found by taking the conjugate of each point on the curve. In general the conjugate of a straight line is a circumconic, but there are some exceptions.

Theorem 1. *If a line goes through a vertex of the reference triangle* $\mathbf{ABC}$*, the conjugate of this line is a line through the same vertex.*

Proof. Choose vertex $\mathbf{B}$. A line through this vertex has the form $nz - \ell x = 0$. The isotomic conjugate is $\frac{n}{z} - \frac{\ell}{x} = 0$, which is the same as $nx - \ell z = 0$, a line through the same vertex. The isogonal conjugate works analogously. □

This result is structurally useful. If a point approaches a vertex on a straight line (or a smooth curve, which must approximate one) its conjugate crosses an edge by the conjugate line ([3]).

4.2. *Self conjugate conics (isotomic case).* The isotomic conjugate of the general conic is a quartic curve, but again there are some interesting exceptions.

Theorem 2. *Conics through* $\mathbf{AGCB^G}$ *and* $\mathbf{ACC^GA^G}$ *are self-isotomic.*

Proof. The general conic is $\ell x^2 + my^2 + nz^2 + Lyz + Mzx + Nxy = 0$. Choosing the case $\mathbf{AGCB^G}$, since $\mathbf{A}$ and $\mathbf{C}$ are on the conic, we have that $\ell = n = 0$. From $\mathbf{G}$ and $\mathbf{B^G}$ we get the two equations $m \pm L + M \pm N = 0$, from which we get $M = -m$ and $N = -L$ giving $y^2 - zx + \lambda y(z - x) = 0$ as the family of conics through these two points. Replacing each coordinate with its reciprocal and assuming that $xyz \neq 0$, we see that this equation is self-isotomic.

For the case $\mathbf{CAA^GC^G}$ the equation is $y^2 + zx + \lambda y(z + x) = 0$, also self-isotomic.

□

Each family has one special conic homothetic to the Steiner ellipse and of special interest: $y^2 - zx = 0$, which goes through $\mathbf{AGCB^G}$, and $y^2 + zx + 2y(z+x) = 0$, which goes through $\mathbf{ACC^GA^G}$. Conics homothetic to the Steiner ellipse can be written as $yz + zx + xy + (Lx + My + Nz)(x + y + z) = 0$. Choosing $L = N = 0$ and $M = \pm 1$ gives the two conics of interest. The first of these has striking properties.

Theorem 3. *The ellipse* $y^2 - zx = 0$
(1) *goes through* $\mathbf{C}$, $\mathbf{A}$, $\mathbf{G}$, $\mathbf{A^G}$,
(2) *is tangent to edges* a *and* c,
(3) *contains the isotomic conjugate* $\mathrm{t}\mathbf{P}$ *of every point* $\mathbf{P}$ *on it,* (*and if one of* $\mathbf{P}$ *and* $\mathrm{t}\mathbf{P}$ *is inside, then the other is outside the ellipse; the line connecting a point on the ellipse with its conjugate is parallel to the* b *edge* [3]),
(4) *contains the* $\mathbf{B}$*-harmonic associate of every point on it,*
(5) *has center* $(2 : -1 : 2)$ *which is the intersection of the Steiner ellipse with the b-median,*
(6) *is the translation of the Steiner ellipse by the vector from* $\mathbf{B}$ *to* $\mathbf{G}$,
(7) *contains* $\mathbf{P}^n = (x^n : y^n : z^n)$ *for integer values of* n *if* $\mathbf{P} = (x : y : z)$, $(xyz \neq 0)$*, is on the curve,*
(8) *is the inverse in the Steiner ellipse of the b-edge of* **ABC** .

These last two properties are included for their interest, but have little to do with the topic at hand (other than that $n = -1$ is the isotomic conjugate). A second paper will be devoted to these properties of this curve.

Proof. (1) can be verified by substituting coordinates as done above.

(2) is true by the general principle that if an equation has the form (line 2)^2 = (line 1)$\cdot$(line 3), then the curve has a double intersection at the intersection of line 1 and line 2 and at the intersection of line 3 and line 2 and is tangent to lines 1 and 2 at those points.

For (3) we take the isotomic conjugage of a point on the curve to obtain $\frac{1}{y^2} - \frac{1}{zx} = 0$, which, since this curve only exists where the product zx is positive, is the same as $zx - y^2 = 0$, so that $\mathrm{t}\mathbf{P}$ is on the curve if $\mathbf{P}$ is, which also implies that the point and the conjugate are on different sides of the ellipse. $(yz : zx : xy)$ is the conjugate. If on the ellipse $zx = y^2$ we have $(yz : y^2 : xy) \sim (z : y : x)$. The vector from this point to $(x : y : z)$ is proportional to $(-1 : 0 : 1)$, which is in the direction of the b-edge.

(4) can be verified by noting that if (x, y, z) is on the ellipse, so is its harmonic associate $(x, -y, z)$.

(5) The center is found as the polar of the line at infinity.

(6) is verified by computing the translation $T : \mathbf{B} \to \mathbf{G}$, and computing $S(T^{-1}\mathbf{P})$, where $S(\mathbf{P})$ is the Steiner ellipse in terms of a point $\mathbf{P}$ on the curve.

(7) is verified since $(y^n)^2 - z^n x^n$ has $y^2 - zx$ as a factor, so that $\mathbf{P}^n$ is on the curve if $\mathbf{P}$ is.

(8) $(\cdots : y : \cdots) \to (\cdots : y^2 - zx : \cdots)$ is the Steiner inversion and takes $y = 0$ into $y^2 - zx = 0$. $\square$

5. The isotomic ellipses

Consider the three curves

$$x^2 - yz = 0,$$
$$y^2 - zx = 0,$$
$$z^2 - xy = 0,$$

which are translations of the Steiner ellipse, each through two vertices, and tangent to the edges of $\mathbf{ABC}$. Exactly as the three medians are self-isotomic and separate the central region of the triangle, so too do these ellipses. If a point is inside one, its conjugate is outside. The line from a point on one of these curves to its conjugate its parallel to a side of the triangle, or perhaps stated more correctly, to the an ex-median.

Consider the three curves

$$x^2 + yz + 2x(y + z) = 0,$$
$$y^2 + zx + 2y(z + x) = 0,$$
$$z^2 + xy + 2z(x + y) = 0$$

each homothetic to the Steiner ellipse. Each goes through two ex-centroids and two vertices and is centered at the other vertex. These are the exterior versions of the above three, rather as the ex-medians are external versions of the medians. They are self-isotomic and the line from a point to its conjugate is parallel to a median (proved below). These ellipses go through the ex-centroids and serve to define regions about them just as the others do for the central regions. They can also be seen in Figure 5. These six isotomic ellipses are all centered on the Steiner circumellipse of $\mathbf{ABC}$. Their tangents at the vertices are either parallel to the medians or the exmedians. For any point in the plane where the conjugate is defined, the point and its conjugate are on the same side (inside or outside) for three ellipses and on opposite sides for the other three (just as for the medians).

6. $\mathbf{P} - \mathrm{t}\mathbf{P}$ lines

For points on the interior versions (those that pass through $\mathbf{G}$) of these conics, the lines from a point to its conjugate are parallel to the ex-medians (and hence to the sides of $\mathbf{ABC}$). For points on the exterior ellipses, the line joining a point to its conjugate is parallel to a median of $\mathbf{ABC}$. This is illustrated in Figure 6.

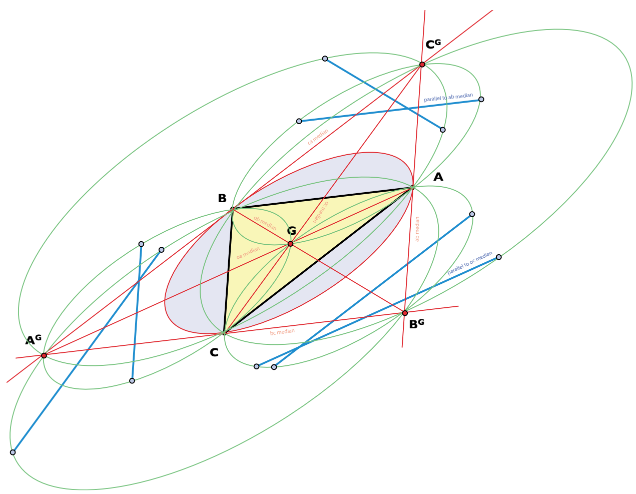


Figure 6. Points paired with their conjugates are connected by blue lines, each of which is parallel to a median or an ex-median of $\mathbf{ABC}$. The direction of the lines for the two ellipses through $\mathbf{A}$ and $\mathbf{B}$ are noted.

For the interior ellipses, this property has been proved. For the exterior ones the math is a bit harder. Note that a point and its conjugate can be written as $(x : y : z)$ and $(yz : zx : xy)$. The equation of the ellipse can be written as $zx = y^2 + 2y(z + x)$, so that the conjugate becomes

$$(yz : y^2 + 2y(z + x) : xy) \sim (z : y + 2(z + x) : x).$$

The vector between these two (normalized) points is

$$(x + y + z : -2(x + y + z) : x + y + z) \sim (1 : -2 : 1)$$

which is the direction of a median.

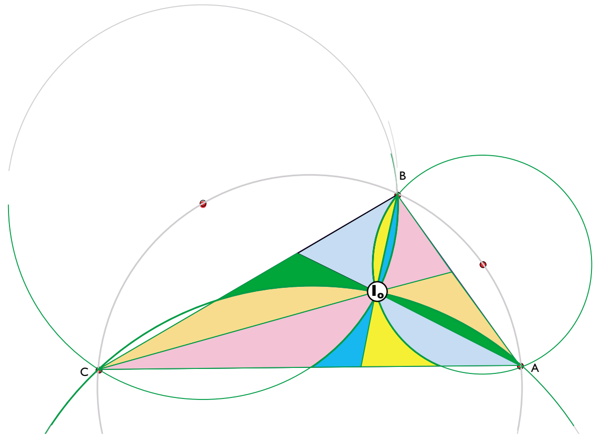


Figure 7. The central region divided by three bisectors and three self-isogonal circles.

7. The self-isogonal circles

Just as the ellipse homothetic to the Steiner ellipse through $\mathbf{CGAG^B}$ is isotomically self-conjugate, the circle through the corresponding set of points $\mathbf{CI_oAI}_b$ is isogonally self-conjugate, a very pretty result. Just as the there are six versions of the isotomic ellipses, each with a center on the Steiner ellipse, there are 6 isogonal circles, each centered on the circumcircle, also a pretty result (Figure 8).

We note that $\mathbf{I}_o\mathbf{CI}_b\mathbf{A}$ is cyclic because the bisector $\mathbf{AI}_b$ is perpendicular to the bisector $\mathbf{I}_o\mathbf{A}$. The angles at $\mathbf{A}$ and $\mathbf{C}$ are right angles so that opposite angles of the quadrilateral are supplementary. Hence there is a circle through $\mathbf{CI_oAI}_b$. It is in fact the diametral circle on $\mathbf{I}_o\mathbf{I}_b$.

The equation of a general circle is

$$a^2yz + b^2zx + c^2xy + (\ell x + my + nz)(x + y + z) = 0.$$

Demanding that it go through the above 4 points, we get

$$cay^2 - b^2zx - (a - c)(ayz - cxy) = 0$$

with center $(a(a + c) : -b^2 : c(a + c))$, the midpoint of $\mathbf{I}_o\mathbf{I}_a$. There are six such circles, each through 2 vertices and two incenters. Each pair of incenters determines one of these circles hence there are 6 of them. Just as each bisector goes through 2 incenters, so does each of these circles. Just as the bisectors separate a point from its conjugate, so do these circles, giving an even more detailed view of conjugacy in the neighborhood of an incenter (see Figure 7).

If a point on one of these six circles is connected to its conjugate, the line is parallel to one of the six bisectors, the circles through $\mathbf{I}_o$ pairing with exterior bisectors. The tangent lines at the vertices are also parallel to a bisector. These statements are proved just as for the isotomic ellipses.

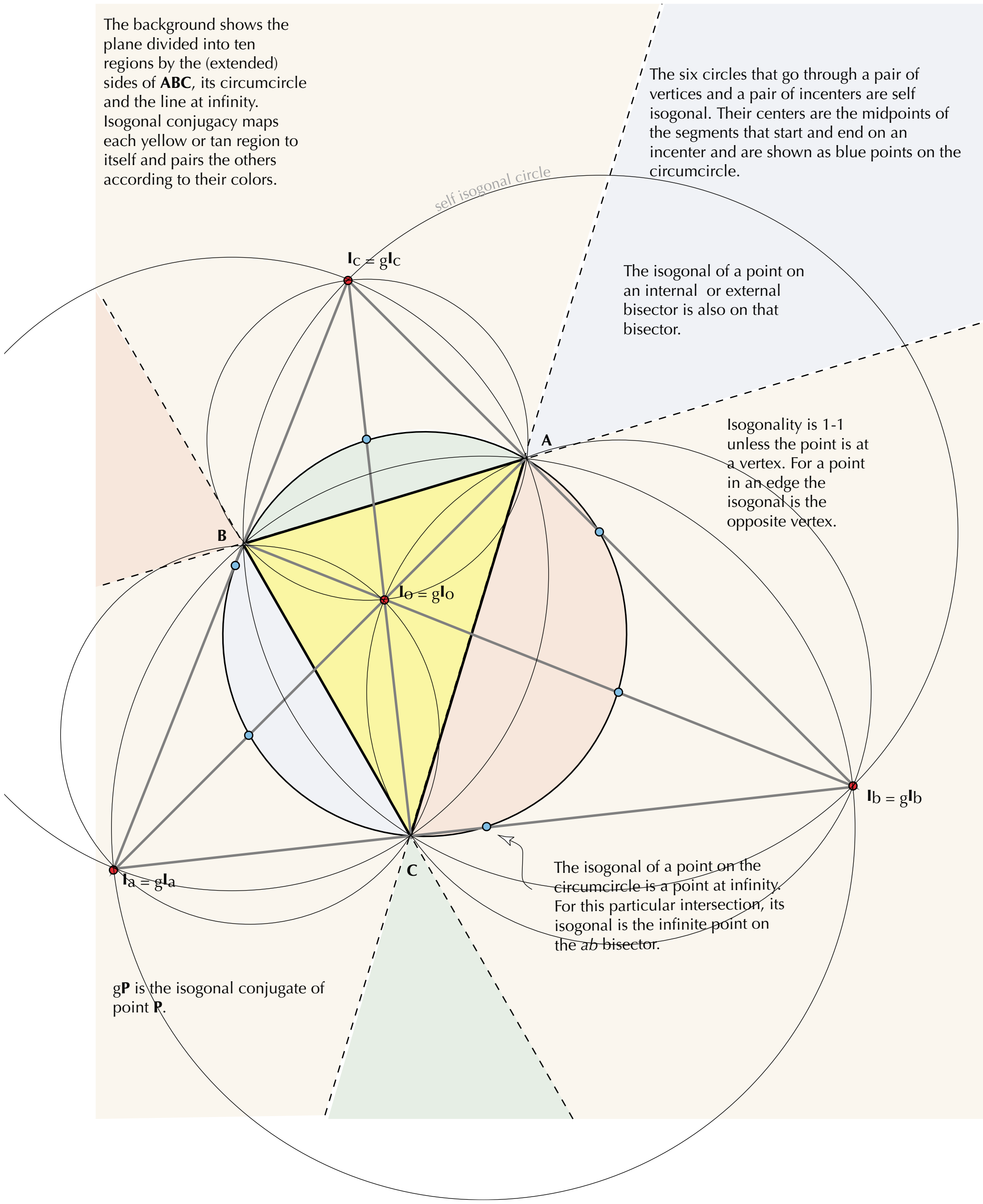


Figure 8. Isogonal conjuates

8. Self-isogonal conics

Demanding a conic go through $\mathbf{CI}_o\mathbf{AI}_b$, we get $cay^2 - b^2zx + \lambda y(az - cx) = 0$, which can be verified to be self- isogonal. Those through $\mathbf{CAI}_a\mathbf{I}_c$ have equation $cay^2 + b^2zx + \lambda y(az + cx) = 0$, and are similarly isogonal.

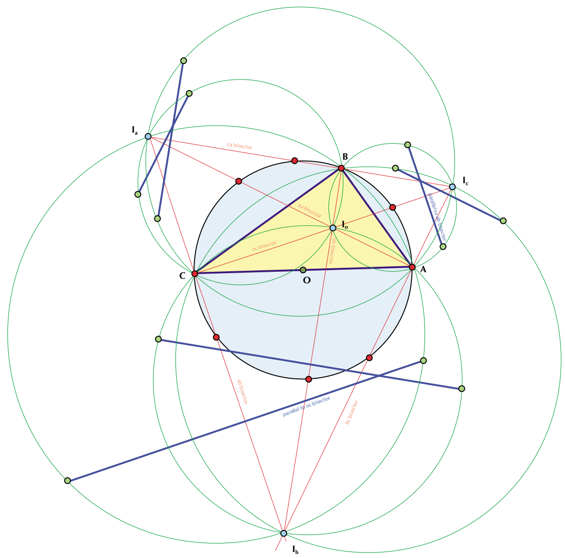


Figure 9. $\mathbf{P} - \mathrm{g}\mathbf{P}$ lines. On each isogonal circle the line from a point to its conjugate is parallel to one of the angle bisectors. If the circle goes through $\mathbf{I}_o$ the line is parallel to the corresponding external bisector. The red points on the circumcircle are the centers of the isogonal circles. For the two circles through $\mathbf{A}$ and $\mathbf{B}$, the directions of the $\mathbf{P} - \mathrm{g}\mathbf{P}$ line is noted.

9. The central region - an enhanced view

These self-conjugate circles thus help us place the isogonal conjugate of $\mathbf{P}$ just as do the median lines. If a point is on one of these circles, then so is its conjugate. If inside, the conjugate is outside and vice versa. This division of the plane into regions is very effective at giving the general location of the conjugate of a point (Figure 7). Of course this behavior around $\mathbf{I}_o$ is mimicked by that around the other incenters.

10. Topological considerations

There is a complication to the above analysis which leads to a very pretty picture of conjugacy in the projective plane. Conjugacy is 1-1 both ways except at the vertices where it blows up. This is in fact a topological blowup. To see this, let $\mathbf{P}$ move out of the central region across the b-edge, say. Near both $\mathbf{I}_o$ and $\mathbf{I}_b$, the behavior of a point to its conjugate is simple and known. In the central region, $\mathbf{P}$ and its conjugate $\mathbf{Q}$ were on opposite sides of the b-bisector; once $\mathbf{P}$ passed through the b-edge, $\mathbf{Q}$ passed through the $\mathbf{B}$-vertex, after which it is on the same side of the b-bisector as $\mathbf{P}$. We say that the plane of the triangle, underwent a Möbius-like twist at the $\mathbf{B}$-vertex. Continuing $\mathbf{P}$'s journey out of the central region through the b-edge towards $\mathbf{I}_b$, we encounter the second problem. As $\mathbf{P}$ nears the circumcircle, $\mathbf{Q}$ goes to infinity. As $\mathbf{P}$ crosses the circumcircle, $\mathbf{Q}$ crosses the line at infinity as well as the bisector, giving another twist to the plane as it passes. As

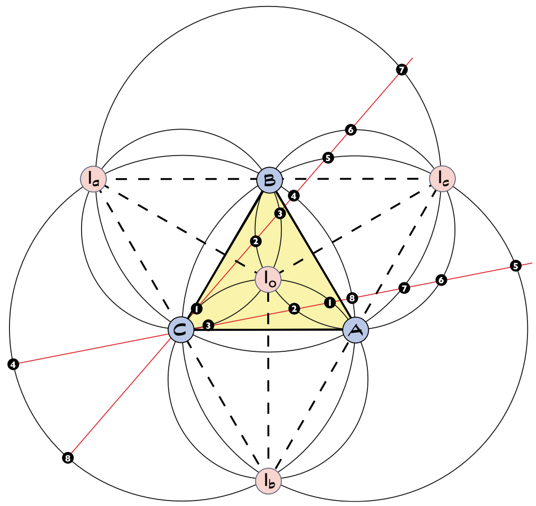


Figure 10. Here points numbered 18 are arranged on a line through **C**. The conjugates, numbered equally, are on the isogonal line through **C**, but are spaced wildly. The isogonal circles show and explain the unusual distribution of the conjugates.

P moves near $\mathbf{I}_b$, the center of the b-excircle, **Q** moves towards it, now again on the opposite side of the bisector. (This emphasis on topological properties is a result of a conversation about conjugacy with John Conway, one of the most interesting conversations about triangle geometry that I have ever had).

The isotomic conjugate behaves analogously at the vertices and at infinity with the Steiner ellipse taking the place of the circumcircle and the six medians replacing the six bisectors.

There is a way to tame the conjugacy operation at the three points in the plane which are not 1-1, and to throw light on the behavior of conjugates at the same time.

As a point approaches a vertex along a line, its conjugate goes to the point on the edge intersected by the isogonal line. Hence although the conjugate at a vertex is undefined, each direction into the vertex corresponds to a point on an edge.We represent this by letting the point "blowup", becoming a small disc. Each point on the edge of the disc represents a direction with respect to the center. Its antipodal point is on the same line so the disc has opposite points identified. This topological blowup replaces the vertex with a Möbius-like surface (a cross-cap), explaining the shift of the conjugate from the opposite side of a bisector to the same side.

Figure 11 shows the plane of the triangle from this point of view for the isogonal case. It is a very different view indeed. The important lines are the six bisectors and the important points are the three vertices and the four incenters. The edges of the triangle are only shown for orientation and the circumcircle is not relevant to the picture. The colors show co-isogonal regions - if a point is in a region of a certain color, so is its conjugate. The twists of the plane occur at the vertices

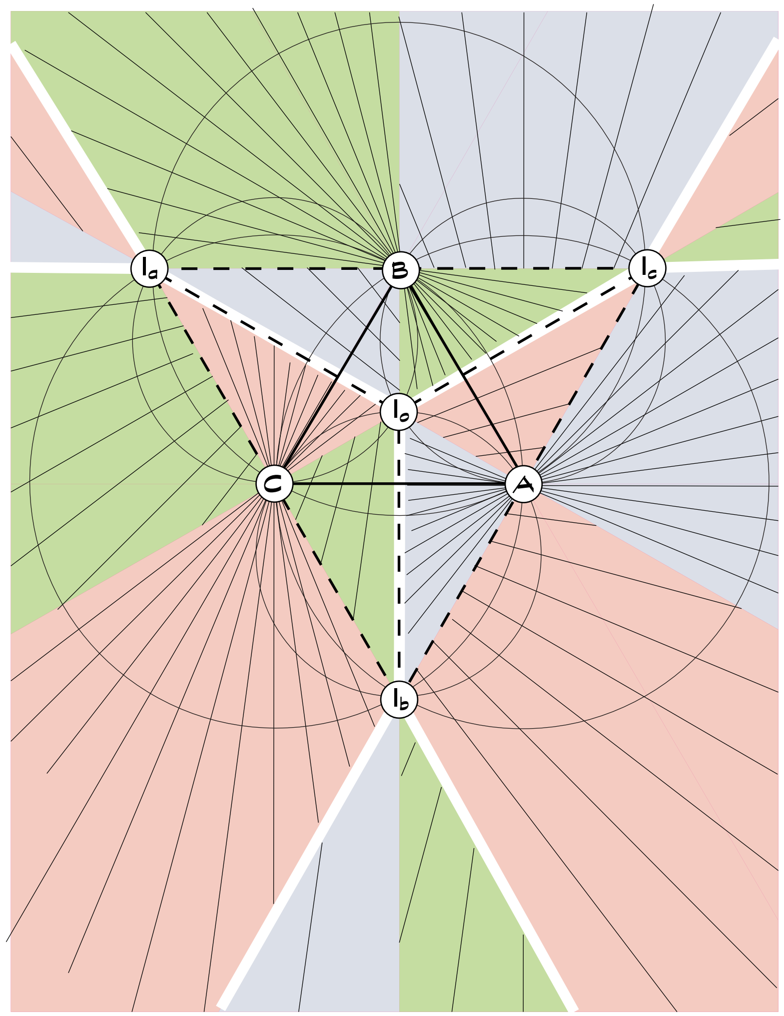


Figure 11. (Drawn with John Conway). Topological view of the location of conjugates The colors show co-isogonal regions. The lines issuing from the vertices show isogonal lines. The isogonal circles are shown. The white lines are the boundries of the three faces of a projective cube.

as shown by the colored regions converging on the vertices. In fact this figure forms a projective cube where the incenters are the four vertices that remain after anitpodes are identified. The view shown is directly toward the “vertex” $\mathbf{I}_o$ with the lines $\mathbf{I}_o\mathbf{I}_a$, $\mathbf{I}_o\mathbf{I}_b$, $\mathbf{I}_o\mathbf{I}_c$ being the three edges from that vertex. $\mathbf{I}_o\mathbf{I}_b\mathbf{I}_c\mathbf{I}_a$ form a face. The white lines are the edges of the cube. In the middle of each face is a cross-cap structure at a vertex. The final picture is of a projective cube with each face containing a crosscap singularity. The triangle $\mathbf{ABC}$ and its sides can be considered the projective octahedron inscribed to the cube with the four regions identified in the introductory paragraph being the four faces.

This leads to a nice view of pivotal cubics which are defined in terms of conjugates. The cubics go through all 7 relevant points.

11. Cubics

We can learn a bit about the shape of pivotal cubics from this topological picture of the conjugates. Pivotal cubics include both a point and its conjugate, so that each branch of the cubic must stay in co-isogonal regions, which are of a definite color on our topological picture.

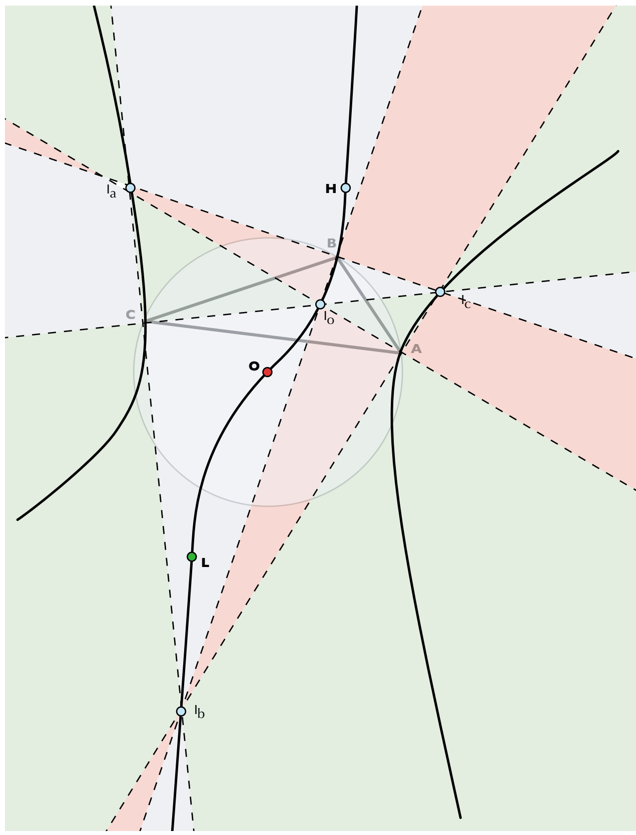


Figure 12. The Darboux cubic is a pivotal isogonal cubic, meaning that the isogonal conjugate of each point is on the cubic and colinear with the pivot point, which in this case is the deLongchamps point. The colored regions show the pattern of the conjugates. If a point is in a region of a certain color, so is its conjugate. This picture shows that the branches of the cubic turn to stay in regions of a particular color.

The Darboux cubic (Figure 12) has two branches, one through a single vertex, $\mathbf{I}_o$ and, in the illustration, $\mathbf{I}_b$. The other goes through $\mathbf{I}_c$, $\mathbf{I}_a$ and two vertices, wrapping around through the line at infinity. The Neuberg cubic (Figure 13) does the same. Its “circular component” being more visible since it does not pass through the line at infinity. We can understand the various “wiggles” of these cubics as necessary to stay in a self-conjugal region. Also we can see that a conjugate of a point on one branch cannot be on the other branch.

Geometry is fun.

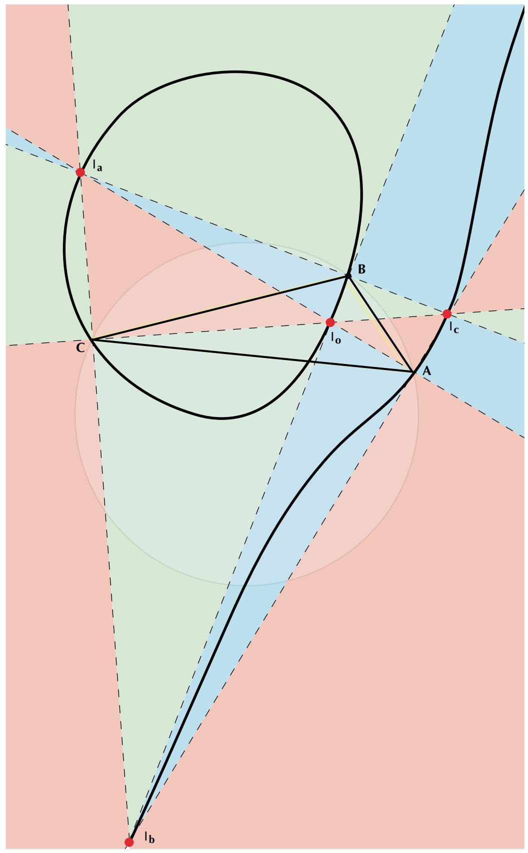


Figure 13. The Neuberg cubic is a pivotal isogonal cubic, meaning that the isogonal conjugate of each point is on the cubic and colinear with the pivot point, which in this case is the Euler infinity point. The colored regions show the pattern of the conjugates. If a point is in a region of a certain color, so its conjugate. This picture shows that the branches of the cubic turn to stay in regions of a particular color.

References

[1] J. H. Conway and S. Sigur, *The Triangle Book*, forthcoming.

[2] K. R. Dean and F. M. van Lamoen, Geometric construction of reciprocal conjugations, *Forum Geom.*, 1 (2001) 115–120.

[3] K. R. S. Sastry, Triangles with special isotomic conjugate pairs, *Forum Geom.*, 4 (2004) 73–80.

Steve Sigur: 1107 Oakdale Road, Atlanta, Georgia 30307, USA
E-mail address: `s.sigur@comcast.net`

Forum Geometricorum
Volume 5 (2005) 17–20.

FORUM GEOM
ISSN 1534-1178

A Synthetic Proof of Goormaghtigh's Generalization of Musselman's Theorem

Khoa Lu Nguyen

Abstract. We give a synthetic proof of a generalization by R. Goormaghtigh of a theorem of J. H. Musselman.

Consider a triangle ABC with circumcenter O and orthocenter H. Denote by A^*, B^*, C^* respectively the reflections of A, B, C in the side BC, CA, AB. The following interesting theorem was due to J. R. Musselman.

Theorem 1 (Musselman [2])**.** *The circles AOA^*, BOB^*, COC^* meet in a point which is the inverse in the circumcircle of the isogonal conjugate point of the nine point center.*

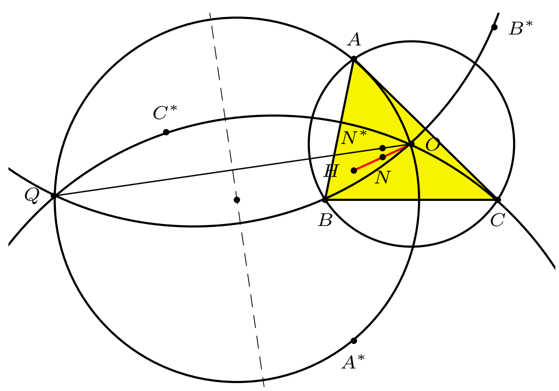


Figure 1

R. Goormaghtigh, in his solution using complex coordinates, gave the following generalization.

Theorem 2 (Goormaghtigh [2])**.** *Let A_1, B_1, C_1 be points on OA, OB, OC such that*

$$\frac{OA_1}{OA} = \frac{OB_1}{OB} = \frac{OC_1}{OC} = t.$$

(1) *The intersections of the perpendiculars to OA at A_1, OB at B_1, and OC at C_1 with the respective sidelines BC, CA, AB are collinear on a line ℓ.*

(2) *If M is the orthogonal projection of O on ℓ, M' the point on OM such that $OM' : OM = 1 : t$, then the inversive image of M' in the circumcircle of ABC*

Publication Date: January 24, 2005. Communicating Editor: Paul Yiu.

The author thanks the communicating editor for his help and also appreciates the great support of his teacher Mr. Timothy Do.

is the isogonal conjugate of the point P on the Euler line dividing OH in the ratio $OP : PH = 1 : 2t$. See Figure 1.

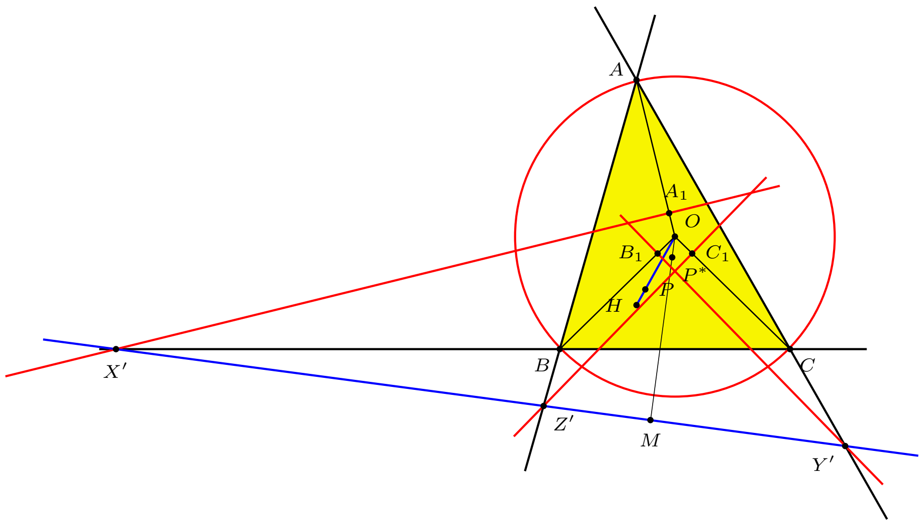


Figure 2

Musselman's Theorem is the case when $t = \frac{1}{2}$. Since the centers of the circles OAA^*, OBB^*, OCC^* are collinear, the three circles have a second common point which is the reflection of O in the line of centers. This is the inversive image of the isogonal conjugate of the nine-point center, the midpoint of OH.

By Desargues' theorem [1, pp.230–231], statement (1) above is equivalent to the perspectivity of ABC and the triangle bounded by the three perpendiculars in question. We prove this as an immediate corollary of Theorem 3 below. In fact, Goormaghtigh [2] remarked that (1) was well known, and was given in J. Neuberg's *Mémoir sur le Tétraèdre*, 1884, where it was also shown that the envelope of ℓ is the inscribed parabola with the Euler line as directrix (Kiepert parabola). He has, however, inadvertently omitted "the isogonal conjugate of " in statement (2).

Theorem 3. *Let $A'B'C'$ be the tangential triangle of ABC. Consider points X, Y, Z dividing OA', OB', OC' respectively in the ratio*

$$\frac{OX}{OA'} = \frac{OY}{OB'} = \frac{OZ}{OC'} = t. \qquad (\dagger)$$

The lines AX, BY, CZ are concurrent at the isogonal conjugate of the point P on the Euler line dividing OH in the ratio $OP : PH = 1 : 2t$.

Proof. Let the isogonal line of AX (with respect to angle A) intersect OA' at X'. The triangles OAX and $OX'A$ are similar. It follows that $OX \cdot OX' = OA^2$, and X, X' are inverse in the circumcircle. Note also that A' and M are inverse in the

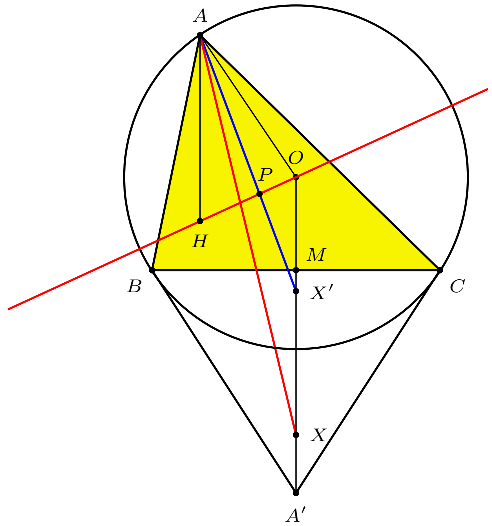


Figure 3

same circumcircle, and $OM \cdot OA' = OA^2$. If the isogonal line of AX intersects the Euler line OH at P, then

$$\frac{OP}{PH} = \frac{OX'}{AH} = \frac{OX'}{2 \cdot OM} = \frac{1}{2} \cdot \frac{OA'}{OX} = \frac{1}{2t}.$$

The same reasoning shows that the isogonal lines of BY and CZ intersect the Euler line at the same point P. From this, we conclude that the lines AX, BY, CZ intersect at the isogonal conjugate of P. □

For $t = \frac{1}{2}$, X, Y, Z are the circumcenters of the triangles OBC, OCA, OAB respectively. The lines AX, BY, CZ intersect at the isogonal conjugate of the midpoint of OH, which is clearly the nine-point center. This is Kosnita's Theorem (see [3]).

Proof of Theorem 2. Since the triangle XYZ bounded by the perpendiculars at A_1, B_1, C_1 is homothetic to the tangential triangle at O, with factor t. Its vertices X, Y, Z are on the lines OA', OB', OC' respectively and satisfy (†). By Theorem 3, the lines AX, BY, CZ intersect at the isogonal conjugate of P dividing OH in the ratio $OP : HP = 1 : 2t$. Statement (1) follows from Desargues' theorem. Denote by X' the intersection of BC and YZ, Y' that of CA and ZX, and Z' that of AB and XY. The points X', Y', Z' lie on a line ℓ.

Consider the inversion Ψ with center O and constant $t \cdot R^2$, where R is the circumradius of triangle ABC. The image of M under Ψ is the same as the inverse of M' (defined in statement (2)) in the circumcircle. The inversion Ψ clearly maps A, B, C into A_1, B_1, C_1 respectively. Let A_2, B_2, C_2 be the midpoints of BC, CA, AB respectively. Since the angles BB_1X and BA_2X are both right angles, the points B, B_1, A_2, X are concyclic, and

$$OA_2 \cdot OX = OB \cdot OB_1 = t \cdot R^2.$$

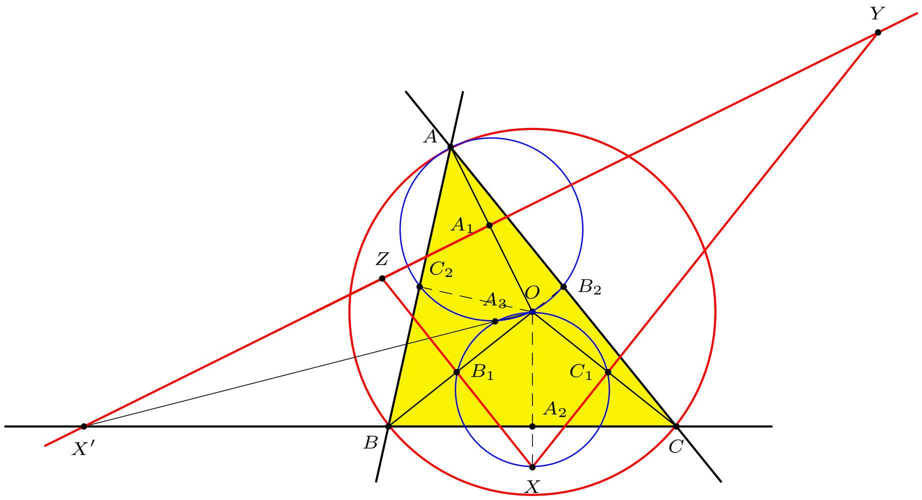


Figure 4

Similarly, $OB_2 \cdot OB_2' = OC_2 \cdot OC_2' = t \cdot R^2$. It follows that the inversion Ψ maps X, Y, Z into A_2, B_2, C_2 respectively.

Therefore, the image of X' under Ψ is the second common point A_3 of the circles OB_1C_1 and OB_2C_2. Likewise, the images of Y' and Z' are respectively the second common points B_3 of the circles OC_1A_1 and OC_2A_2, and C_3 of OA_1B_1 and OA_2B_2. Since X', Y', Z' are collinear on ℓ, the points O, A_3, B_3, C_3 are concyclic on a circle $\mathcal{C}$.

Under Ψ, the image of the line AX is the circle OA_1A_2, which has diameter OX' and contains M, the projection of O on ℓ. Likewise, the images of BY and CZ are the circles with diameters OY' and OZ' respectively, and they both contain the same point M. It follows that the common point of the lines AX, BY, CZ is the image of M under Ψ, which is the intersection of the line OM and $\mathcal{C}$. This is the antipode of O on $\mathcal{C}$.

References

[1] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.

[2] J. R. Musselman and R. Goormaghtigh, Advanced Problem 3928, *Amer. Math. Monthly*, 46 (1939) 601; solution, 48 (1941) 281 – 283.

[3] D. Grinberg, On the Kosnita point and the reflection triangle, *Forum Geom.*, 3 (2003) 105–111.

Khoa Lu Nguyen: 306 Arrowdale Dr, Houston, Texas, 77037-3005, USA
E-mail address: `treegoner@yahoo.com`

Forum Geometricorum
Volume 5 (2005) 21–22.

FORUM GEOM
ISSN 1534-1178

On the Existence of Triangles with Given Lengths of One Side, the Opposite and One Adjacent Angle Bisectors

Victor Oxman

Abstract. We give a necessary and sufficient condition for the existence of a triangle with given lengths of one sides, its opposite angle bisector, and one adjacent angle bisector.

In [1] the problem of existence of a triangle with given lengths of one side and two adjacent angle bisectors was solved. In this note we consider the same problem with one of the adjacent angle bisector replaced by the opposite angle bisector. We prove the following theorem.

Theorem 1. *Given a, ℓ_a, $\ell_b > 0$, there is a unique triangle ABC with $BC = a$ and lengths of bisectors of angles A and B equal to ℓ_a and ℓ_b respectively if and only if $\ell_b \leq a$ or*

$$a < \ell_b < 2a \quad \text{and} \quad \ell_a > \frac{4a\ell_b(\ell_b - a)}{(2a - \ell_b)(3\ell_b - 2a)}.$$

Proof. In a triangle ABC with $BC = a$ and given ℓ_a, ℓ_b, let $y = CA$ and $z = AB$. We have $\ell_b = \frac{2az}{a+z} \cos \frac{B}{2}$ and

$$z = \frac{a\ell_b}{2a \cos \frac{B}{2} - \ell_b}. \tag{1}$$

It follows that $\cos \frac{B}{2} > \frac{\ell_b}{2a}$, $\ell_b < 2a$, and

$$B < 2 \arccos \frac{\ell_b}{2a}. \tag{2}$$

Also,

$$y^2 = a^2 + z(z - 2a \cos B), \tag{3}$$

$$\ell_a^2 = yz \left(1 - \frac{a^2}{(y + z)^2}\right). \tag{4}$$

Publication Date: February 10, 2005. Communicating Editor: Paul Yiu.

Case 1: $\ell_b \leq a$. Clearly, (1) defines z as an increasing function of B on the open interval $\left(0, 2\arccos\frac{\ell_b}{2a}\right)$. As B increases from 0 to $2\arccos\frac{\ell_b}{2a}$, z increases from $\frac{a\ell_b}{2a-\ell_b}$ to ∞. At the same time, from (3), y increases from $a - \frac{a\cdot\ell_b}{2a-\ell_b} = \frac{2a(a-\ell_b)}{2a-\ell_b}$ to ∞. Correspondingly, the right hand side of (4) can be any positive number. From the intermediate value theorem, there exists a unique B for which (4) is satisfied. This proves the existence and uniqueness of the triangle.

Case 2: $a < \ell_b < 2a$. In this case, (1) defines the same increasing function z as before, but y increases from $\frac{a\cdot\ell_b}{2a-\ell_b} - a = \frac{2a(\ell_b-a)}{2a-\ell_b}$ to ∞. Correspondingly, the right hand side of (4) increases from

$$\frac{a\ell_b}{2a-\ell_b}\cdot\frac{2a(\ell_b-a)}{2a-\ell_b}\left(1-\frac{a^2}{\left(\frac{a\ell_b}{2a-\ell_b}+\frac{2a(\ell_b-a)}{2a-\ell_b}\right)^2}\right) = \frac{16a^2\ell_b^2(\ell_b-a)^2}{(2a-\ell_b)^2(3\ell_b-2a)^2}$$

to ∞. This means $\ell_a > \frac{4a\ell_b(\ell_b-a)}{(2a-\ell_b)(3\ell_b-2a)}$. Therefore, there is a unique value B for which (4) is satisfied. This proves the existence and uniqueness of the triangle. $\square$

Corollary 2. *For the existence of an isosceles triangle with equal sides a with opposite angle bisectors ℓ_a, it is necessary and sufficient that $\ell_a < \frac{4}{3}a$.*

Reference

[1] V. Oxman, On the existence of triangles with given lengths of one side and two adjacent angle bisectors, *Forum Geom.*, 4 (2004) 215–218.

Victor Oxman (Western Galilee College): Derech HaYam 191a, Haifa 34890, Israel
E-mail address: voxman@research.haifa.ac.il

Forum Geometricorum
Volume 5 (2005) 23–31.

FORUM GEOM
ISSN 1534-1178

On the Maximal Inflation of Two Squares

Thierry Gensane and Philippe Ryckelynck

Abstract. We consider two non-overlapping congruent squares q_1, q_2 and the homothetic congruent squares q_1^k, q_2^k obtained from two similitudes centered at the centers of the squares. We study the supremum of the ratios of these similitudes for which q_1^k, q_2^k are non-overlapping. This yields a function $\psi = \psi(q_1, q_2)$ for which the squares q_1^ψ, q_2^ψ are non-overlapping although their boundaries intersect. When the squares q_1 and q_2 are not parallel, we give a 8-step construction using straight edge and compass of the intersection $q_1^\psi \cap q_2^\psi$ and we obtain two formulas for ψ. We also give an angular characterization of a vertex which belongs to $q_1^\psi \cap q_2^\psi$.

1. Introduction and notation

We study here the problem of maximizing the *inflation* of two non-overlapping congruent squares $q_1 = q_{a_1,b_1,\theta_1,c}$ and $q_2 = q_{a_2,b_2,\theta_2,c}$. The square q_i has the four vertices

$$S_j(q_i) = (a_i, b_i) + c \cdot (\cos(\theta_i + j\tfrac{\pi}{2}), \sin(\theta_i + j\tfrac{\pi}{2})).$$

Let $q^k_{a,b,\theta,c} = q_{a,b,\theta,k}$ be the homothetic of ratio k/c of the square $q_{a,b,\theta,c}$. Our problem amounts to determining the supremum $\psi = \psi(q_1, q_2)$ of the numbers $k > 0$ for which q_1^k and q_2^k are disjoint.

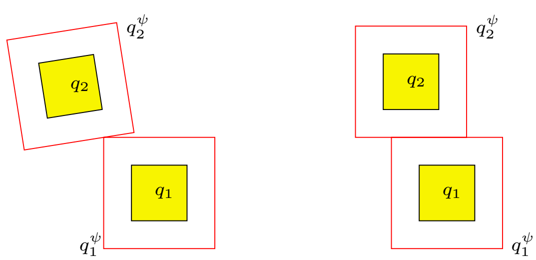


Figure 1

In [3, §4], $\psi = \psi(q_1, q_2)$ is called the *maximum inflation* of a configuration of two squares. It plays a central part in computation of dense packings of squares in a larger square. We refer to the paper of P. Erdős and R. Graham [1] who initiated the problem of maximizing the area sum of packings of an arbitrary square by unit

Publication Date: February 24, 2005. Communicating Editor: Paul Yiu.
We thank the referee for his valuable and helpful suggestions.

squares, see also the survey of E. Friedman [2]. We note that ψ is independent of c and that

$$k \leq \psi \quad \Leftrightarrow \quad \mathrm{int}(q_1^k) \cap \mathrm{int}(q_2^k) = \emptyset, \tag{1}$$

$$k \geq \psi \quad \Leftrightarrow \quad \partial q_1^k \cap \partial q_2^k \neq \emptyset, \tag{2}$$

where as usual, we denote by $\mathrm{int}(q)$ and ∂q the interior and the boundary of a square q. An explicit formula for $\psi = \psi(q_1, q_2)$ is given in [3, Prop.2] as follows. Let us define

$$\psi_0(a, b, \theta) = \min_{i=1,\ldots,4} \left\{ \frac{|a| + |b|}{\left|1 - \sqrt{2}\mathrm{sgn}(ab) \sin(\theta + \frac{\pi}{4} + i\frac{\pi}{2})\right|} \right\},$$

and

$$\rho(q_1, q_2) = \psi_0 \left(t \cos\theta_1 + t' \sin\theta_1, -t \sin\theta_1 + t' \cos\theta_1, \theta_2 - \theta_1 \right),$$

with $(t, t') = (a_2 - a_1, b_2 - b_1)$. The maximal inflation of two squares q_1 and q_2 is the maximum of $\rho(q_1, q_2)$ and $\rho(q_2, q_1)$. The minimum value, say $k = \rho(q_1, q_2) < \psi$, corresponds to the belongness of a vertex E of q_2^k to a straight line AB when $q_1^k = ABCD$, but without having E between A and B. This expression of ψ gives an efficient tool for doing calculations of maximal inflation of configurations of $n \geq 2$ squares.

In this paper, the two congruent squares q_1, q_2 are such that $q_1 \cap q_2 = \emptyset$ and their centers are denoted by $C_i = C(q_i)$. We say as in [3, §4], that q_2 *strikes* q_1 if the set $q_1^\psi \cap q_2^\psi$ contains a vertex of q_2^ψ. In §§3–5, we suppose that the squares q_1, q_2 are not parallel so that $q_1^\psi \cap q_2^\psi = \{P\}$, where the vertex P of q_1 or q_2 is the *percussion point*. However, at the end of each of these sections, we discuss the parallel case in a final remark. We find in §4 a 8-step construction using straight edge and compass of P. Since P is a vertex of q_1^ψ or q_2^ψ, the construction gives immediately the other vertices of q_1^ψ, q_2^ψ. At the same time, we choose a frame in which we obtain two simpler formulas for ψ. We give in §5 an angular characterization which allows to identify which square q_1 or q_2 strikes the other.

2. Quadrants defined by squares

If $q = q_{a,b,\theta,c}$ is a square, we define the two *axes* $A_1(q)$ and $A_2(q)$ of q as the straight lines through $(a, b) \in \mathbb{R}^2$ which are parallel to the sides of q. We define the four counterclockwise consecutive *rays* $D_i(q)$ as the half-lines with origin (a, b) and which contain the vertices of q; we set $D_0(q) = D_4(q)$. A couple of consecutive rays $D_i(q)$ and $D_{i+1}(q)$ defines the i^{th} quadrant $Q_i(q)$ in $\mathbb{R}^2$ associated to the square q.

If a point M, distinct from the center of q, belongs to $\mathrm{int}(Q_i(q))$, then we note $S(q, M) = Q_i(q)$. If the point M lies on the boundaries of two consecutive quadrants $Q_{i-1}(q)$ and $Q_i(q)$, then we choose indifferently $S(q, M)$ as one of the two quadrants $Q_{i-1}(q)$ or $Q_i(q)$. Note that $M \in \mathrm{int}(S(q, N))$ iff $N \in \mathrm{int}(S(q, M))$.

Lemma 1. *If the intersection set $q_1^\psi \cap q_2^\psi$ contains a vertex P of q_2^ψ, then $P \in S(q_2, C_1)$.*

Proof. Let D be the straight line containing a diagonal of q_2 and which does not contain P. Then the disc with center P and radius ψ contains C_1 and C_2 since $d(C_1,P) \le d(C_2,P) = \psi$. Hence there is only one half-plane $\mathcal{H}$, bounded by D, which contains this disc. Now, $\mathcal{H}$ is the union $\mathcal{S}_1 \cup \mathcal{S}_2$ of two quadrants associated to q_2 and the ray $D_i(q_2)$ through P is $\mathcal{S}_1 \cap \mathcal{S}_2$. If $C_1 \notin D_i(q_2)$, one of $\mathcal{S}_1$ and $\mathcal{S}_2$ is $S(q_2,C_1)$; but $P \in D_i(q_2) = \mathcal{S}_1 \cap \mathcal{S}_2$ gives $P \in S(q_2,C_1)$. If $C_1 \in D_i(q_2)$, then $P \in D_i(q_2) \subset S(q_2,C_1)$. $\square$

Lemma 2. *We have*

$$q_1^{\psi} \cap q_2^{\psi} \subset S(q_1,C_2) \cap S(q_2,C_1). \tag{3}$$

The intersection of the two quadrants is depicted in Figure 2.

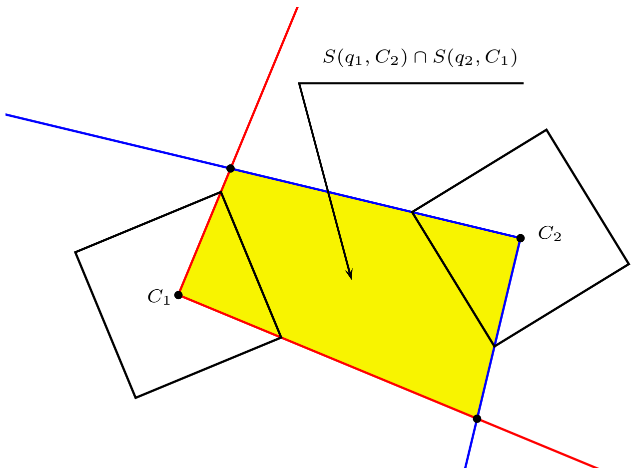


Figure 2

Proof. The proof is divided in three exclusive and exhaustive situations.
(i) First, we suppose that the intersection set $q_1^{\psi} \cap q_2^{\psi} = \{P\}$ where P is a common vertex of q_1^{ψ} and q_2^{ψ}. We readily obtain $P \in S(q_2,C_1)$ and $P \in S(q_1,C_2)$ from Lemma 1.
(ii) Second, we suppose that $q_1^{\psi} \cap q_2^{\psi}$ contains a vertex $P = (x_P, y_P)$ of q_2^{ψ} and that P is not a vertex of q_1^{ψ}. We denote by $ABCD$ the square q_1^{ψ} with $P \in]A,B[$ and let C_1A, C_1B be respectively the x-axis and the y-axis. For the interiors of the two squares to be disjoint, C_2 must be in $\{(x,y) \,:\, x \ge x_p \text{ and } y \ge y_p\}$ since the straight line $x+y=\psi$ separates the two squares. Hence the percussion point P and the center $C_2 = (a,b)$ of q_2 lie in the same quadrant $S(q_1,C_2)$. Due to Lemma 1, P is also in $S(q_2,C_1)$.
(iii) Third, when $q_1^{\psi} \cap q_2^{\psi}$ is a common edge of the two squares q_i^{ψ}, then $S(q_1,C_2) \cap S(q_2,C_1)$ is a square of size ψ and having vertices C_1, P_1, C_2, P_2. Since $q_1^{\psi} \cap q_2^{\psi}$ is a diagonal of this square, the inclusion (3) is obvious. $\square$

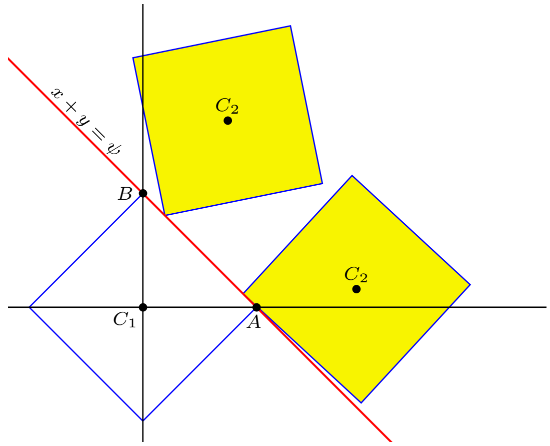


Figure 3

Remark. When the segment $[C_1, C_2]$ contains a vertex of q_1^ψ or q_2^ψ, say A, the statement in (3) can be strengthened: $q_1^\psi \cap q_2^\psi = \{A\}$ is the percussion point.

3. Location of the percussion point

We consider the integer $i_1 \in \{0, 1\}$ such that the axis $A_{i_1}(q_1)$ bounds an half-plane containing $S(q_1, C_2)$. Similarly, we consider the axis $A_{i_2}(q_2)$ which bounds an half-plane containing $S(q_2, C_1)$. Since $A_{i_1}(q_1)$, $A_{i_2}(q_2)$ are not parallel, we can set $A_{i_1}(q_1) \cap A_{i_2}(q_2) = \{W\}$. We use in §4 the point V which is the intersection of the axis $A_{j_2}(q_2)$ and WC_1 and where $j_2 \in \{0, 1\}$ is the integer different from i_2.

The two straight lines $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ define one dihedral angle which contains both C_1 and C_2, that we denote as $\angle C_1WC_2$. Let $\gamma = \gamma(q_1, q_2) = 2\omega = \widehat{C_1WC_2} \in [0, \pi]$ be the measure of this dihedral angle. We define now $B(q_1, q_2)$ as the half-line which bisects $\angle C_1WC_2$. We also note $\ell_1 = ||\overrightarrow{WC_1}||$ and $\ell_2 = ||\overrightarrow{WC_2}||$.

Lemma 3. *We have* $\gamma = \gamma(q_1, q_2) \in \left]0, \frac{\pi}{2}\right[$.

Proof. If $\gamma = 0$, the two axes $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ are equal to some straight line D. The centers C_1 and C_2 lie on D. But by construction $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ have to be perpendicular to the line D, contradiction.

If $\gamma = \frac{\pi}{2}$, the two axes $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ are perpendicular but this is excluded because the squares are not parallel.

We now suppose that $\frac{\pi}{2} < \gamma < \frac{3\pi}{4}$. The quadrant $S(q_2, C_1)$ intersects the axis $A_{i_1}(q_1)$ at a point M which belongs to the segment $[W, C_1]$ for C_1 lies in $S(q_2, C_1)$. Since the angle $\widehat{WMC_2} = \frac{3}{4}\pi - \gamma$ is strictly less than $\frac{\pi}{4}$, the quadrant $S(q_1, C_2)$ does not contain C_2, contradiction. See Figure 4.

The last case $\frac{3\pi}{4} \leq \gamma \leq \pi$ implies that $S(q_2, C_1)$ does not intersect the boundary of $\angle C_1WC_2$. This is in contradiction with $C_1 \in A_{i_1}(q_1) \cap S(q_2, C_1)$. $\square$

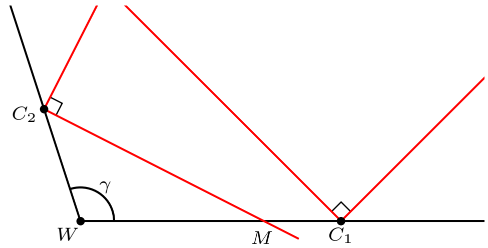


Figure 4

Lemma 4. *We have* $q_1^{\psi} \cap q_2^{\psi} \subset B(q_1, q_2)$.

Proof. Let $0 < k \leq \psi$. The homothetic square q_1^k (resp. q_2^k) has two vertices in $S(q_1, C_2)$ (resp. $S(q_2, C_1)$). The straight line passing through those vertices of q_1 (resp. q_2) is parallel at distance $k/\sqrt{2}$ to the axis $A_{i_1}(q_1)$ (resp. $A_{i_2}(q_2)$). The intersection of those two parallels belongs to $B(q_1, q_2)$ and, according to Lemma 2, allows to localize the point of percussion which is equal to $q_1^k \cap q_2^k$ when $k = \psi$. Thus $P \in B(q_1, q_2)$. $\square$

Remark. When q_1 and q_2 are parallel, Lemma 4 remains true provided $B(q_1, q_2)$ is replaced with the straight line containing the points equidistant from the two parallel axes $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$.

4. Construction of the percussion point

Two rays $D_i(q_1)$ and $D_{i+1}(q_1)$ intersect $B(q_1, q_2)$ at I_1, I_3. We use the natural order on $B(q_1, q_2)$ and we can suppose that $W < I_1 < I_3$. Similarly, we define $W < I_2 < I_4$ relatively to q_2.

Lemma 5. *We have*
(a) $\ell_1 = \ell_2 \Leftrightarrow I_1 = I_2 < I_3 = I_4$.
(b) $\ell_1 < \ell_2 \Leftrightarrow I_1 < I_2 < I_3 < I_4$.
(c) $\ell_2 < \ell_1 \Leftrightarrow I_2 < I_1 < I_4 < I_3$.

Proof. If $\ell_1 = \ell_2$ then $I_1 = I_2 < I_3 = I_4$. Shifting C_1 along WC_1 towards W causes $C_1 I_1$ and $C_1 I_3$ to slide in a parallel fashion, so that $I_1 < I_2$ and $I_3 < I_4$. Since $C_1 \in S(q_2, C_1)$, the point C_1 cannot pass the intersection C_ℓ of $C_2 I_2$ and WC_1. But when $C_1 = C_\ell$, we have $\widehat{WC_1 I_2} = \widehat{WC_\ell C_2} = 3\pi/4 - \gamma$. By Lemma 3, we deduce that $\pi/4 < \widehat{WC_1 I_2} < 3\pi/4$ and accordingly $I_2 < I_3$. The remaining implications are straightforward. $\square$

Theorem 6. (i) *Among the four points* $I_1, \ldots, I_4$*, the second one is the percussion point:* $P = q_1^{\psi} \cap q_2^{\psi} = \max\{I_1, I_2\}$*. We have*

$$\psi = \max\{\ell_1, \ell_2\} \frac{\sqrt{2}}{1 + \cot \omega}. \tag{4}$$

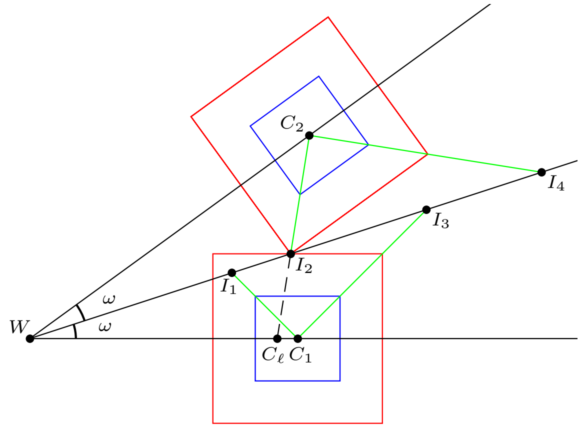


Figure 5

(ii) *If, say* $\ell_2 \geq \ell_1$, *then* q_2 *strikes* q_1 *at the point* P *which is the incenter of the triangle* C_2WV.

Proof. (i). We suppose first that $\ell_2 > \ell_1$. By Lemma 5 we have

$$d_1 = ||\overrightarrow{C_1I_1}|| < d_2 = ||\overrightarrow{C_2I_2}|| < d_3 = ||\overrightarrow{C_1I_3}|| < d_4 = ||\overrightarrow{C_2I_4}||.$$

We know from Lemma 4 that P is one of the four points $I_1, \ldots, I_4$ and thus the percussion occurs at $P = I_i$ if and only if $\psi = d_i$. It is impossible that $P = I_1$ because in that case $\psi = d_1 < d_2$ and then $P \in q_2^{d_1} \cap B(q_1, q_2) = \emptyset$. Hence $\psi > d_1$. If $\psi \geq d_3$ and since $I_2 \in]I_1, I_3[$ by Lemma 5, the point $I_2 \in q_2^{\psi}$ belongs also to the interior of q_1^{ψ} and then the two interiors are not disjoint. We get $\psi = d_2$ and $P = I_2 > I_1$. Easy calculations in the frame centered at $W = (0,0)$ and with x-axis WC_1, give $I_2 = \ell_2(1/(1+\tan\omega), \tan\omega/(1+\tan\omega))$ and (4).

The symmetric case $\ell_1 > \ell_2$ gives q_1 strikes q_2 at $P = I_1 > I_2$ and (4) again. Finally, if $\ell_1 = \ell_2$ the point $P = I_1 = I_2$ is effectively the percussion point.

(ii) If $\ell_2 \geq \ell_1$, by Lemma 4, the point $P = I_2$ belongs to the bisector ray $B(q_1, q_2)$ of the geometric angle $\angle C_1WC_2 = \angle VWC_2$. Now, since P is a vertex of q_2, we have $\widehat{VC_2P} = \widehat{PC_2W} = \pi/4$, so that P belongs to the bisector ray of the geometric angle $\angle VC_2W$. We conclude that P is the incenter of the triangle VC_2W. □

Corollary 7. *We have*

$$\begin{aligned}
\ell_1 < \ell_2 &\quad\Leftrightarrow\quad q_2 \textit{ strikes } q_1 \textit{ and } q_1 \textit{ does not strike } q_2,\\
\ell_2 < \ell_1 &\quad\Leftrightarrow\quad q_1 \textit{ strikes } q_2 \textit{ and } q_2 \textit{ does not strike } q_1,\\
\ell_1 = \ell_2 &\quad\Leftrightarrow\quad q_2 \textit{ strikes } q_1 \textit{ and } q_1 \textit{ strikes } q_2.
\end{aligned}$$

Proof. The three implications from left to right are direct consequences of Theorem 6 and its proof. Since the three cases are exclusive and exhaustive, the three converse implications readily follow. □

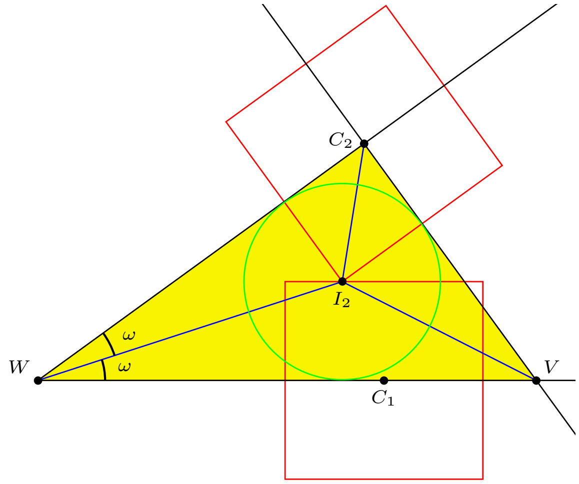


Figure 6

We now synthesize the whole preceding results. For two points M and N, we denote by $\Gamma(M,N)$ the circle with M as center and MN as radius.

Construction of P. Given the eight vertices of two congruent, non parallel and non-overlapping squares q_1 and q_2, construct

(1-2) the two centers C_1, C_2, intersection of the straight lines passing through opposite vertices of q_i, $i = 1, 2$,

(3-4) the axes $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ (this requires the determination of the quadrants $S(q_1, C_2)$ and $S(q_2, C_1)$ as much as two intermediate points),

(5) the point W, intersection of $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$,

(6) the point C_r, intersection of $\Gamma(W, C_2)$ and the half-line WC_1,

(7) the bisector $B(q_1, q_2)$ through W and $\Gamma(C_2, W) \cap \Gamma(C_r, W)$ (the four points $I_1, \cdots, I_4$ appear at this stage),

(8) the percussion point P, the second among the four points $I_1, \cdots, I_4$ on the oriented half-line $B(q_1, q_2)$.

Remarks. (1) We know that the area of the triangle VWC_2 is equal to $p \cdot r$ where p is the half-perimeter of the triangle and $r = \psi/\sqrt{2}$ the radius of the incircle. Now, we also have the formula

$$\psi = \frac{\sqrt{2}\mathrm{Area}(VWC_2)}{p} = \frac{\sqrt{2}VC_2 \cdot WC_2}{VC_2 + WC_2 + VW} = \ell_2 \frac{\sqrt{2}\sin\gamma}{\sin\gamma + \cos\gamma + 1}.$$

The last value is equal to (4) when $\ell_2 \geq \ell_1$.

(2) Let us suppose that the segment $[C_1, C_2]$ contains a vertex $S_i(q_2)$. This amounts to saying that $C_1 = C_\ell$, so that $S(q_2, C_1)$ has been chosen as one of two quadrants $Q_{i-1}(q_2)$, $Q_i(q_2)$. But these choices lead to consider the two dihedral angles $\angle C_1WC_2$ and $\angle C_1VC_2$. Due to the second part of Theorem 6, P and the formula for ψ are not altered by this choice.

(3) When q_1 and q_2 are parallel, the construction of the four points $I_1, \cdots, I_4$ makes sense using again the straight line $B(q_1, q_2)$ equidistant from the two axes

$A_{i_1}(q_1)$ and $A_{i_2}(q_2)$. We choose an order on $B(q_1, q_2)$ and next we label those four points in such a way that $[I_2, I_3] \subset [I_1, I_4]$ and we have $q_1^\psi \cap q_2^\psi = [I_2, I_3]$. In consequence, the steps (5-8) in the above Construction are replaced with the construction of the midpoint $(C_1+C_2)/2$ (three steps), of the straight line $B(q_1, q_2)$ (three steps) and lastly of the two points I_2, I_3.

5. An angular characterization of the percussion point

We define $\alpha(q_1, q_2)$ as the minimum of $\{S_i(q_1)\widehat{C(q_1)}C(q_2), 0 \le i \le 3\}$. This set contains two acute and two obtuse angles. We have $0 \le \alpha(q_1, q_2) \le \frac{\pi}{4}$ since $\alpha(q_1, q_2) \le \frac{\pi}{2} - \alpha(q_1, q_2)$.

Theorem 8. *The square q_2 strikes q_1 if and only if $\alpha(q_2, q_1) \le \alpha(q_1, q_2)$. The percussion point is the vertex of q_1 or q_2 which realizes the minimum of the eight angles appearing in $\alpha(q_1, q_2)$ and $\alpha(q_2, q_1)$.*

Proof. Suppose that q_2 strikes $q_1 = ABCD$ at P in the interior of side AB, see Figure 7. Let AB be the x-axis and P the origin. Then for the interiors of q_1 and q_2 to be disjoint, the center C_2 of q_2 must be in $\{(x, y) \,:\, y \ge |x|\}$. Also, C_2 lies on the arc $x^2 + y^2 = \psi^2$. Let C_0, C_ℓ, C_r be the three points on this arc which intersect the lines C_1P, $y = -x$ and $y = x$ respectively.

Letting C_2 moving along the arc from C_0 to C_r, the angle $\widehat{PC_2C_1}$ increases from $\widehat{PC_0C_1} = 0$ to $\widehat{PC_rC_1} = \widehat{BC_1C_r}$ and the angle $\widehat{BC_1C_2}$ decreases from $\widehat{BC_1C_0}$ to $\widehat{BC_1C_r}$. Hence throughout the move we have $\widehat{PC_2C_1} \le \widehat{BC_1C_r} \le \widehat{BC_1C_2}$. But we have obviously $\widehat{PC_2C_1} < \widehat{AC_1C_2}$ and thus $\widehat{PC_2C_1} \le \alpha(q_1, q_2)$. The same proof holds when C_2 moves on the arc C_0C_ℓ.

Since $\widehat{PC_2C_1} \le \pi/4$, we get $\alpha(q_2, q_1) = \widehat{PC_2C_1}$ and next $\alpha(q_2, q_1) \le \alpha(q_1, q_2)$. The angle $\widehat{PC_2C_1}$ realizes effectively the minimum of the eight angles. The converse implication holds because $\alpha(q_2, q_1) = \alpha(q_1, q_2)$ is equivalent to the fact that q_1 and q_2 strike each other at a common vertex. $\square$

Remark. In case q_1 and q_2 are parallel, q_1 strikes q_2 at P_1 and q_2 strikes q_1 at P_2. We have $\alpha(q_1, q_2) = \widehat{C_2C_1P_1} = \widehat{C_1C_2P_2} = \alpha(q_2, q_1)$. Hence the results in Theorem 8 remain true.

References

[1] P. Erdős and R. L. Graham, On packing squares with equal squares, *J. Combin. Theory Ser. A*, 19 (1975), 119–123.

[2] E. Friedman, Packing unit squares in squares: A survey and new results, *Electron. J. Combin.*, 7 (2000), #DS7.

[3] Th. Gensane and Ph. Ryckelynck, Improved dense packings of congruent squares in a square, *Discrete Comput. Geom.*, OF1-OF13 (2004) DOI: 10.1007/s00454-004-1129-z, and in press.

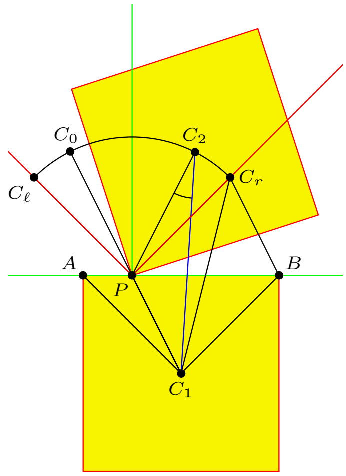


Figure 7

Thierry Gensane: Laboratoire de Mathématiques Pures et Appliquées J. Liouville, 50 rue F. Buisson, BP699, 62228 Calais cedex, France
E-mail address: `gensane@lmpa.univ-littoral.fr`

Philippe Ryckelynck: Laboratoire de Mathématiques Pures et Appliquées J. Liouville, 50 rue F. Buisson, BP699, 62228 Calais cedex, France
E-mail address: `ryckelyn@lmpa.univ-littoral.fr`

Forum Geometricorum
Volume 5 (2005) 33–36.

FORUM GEOM
ISSN 1534-1178

Triangle Centers with Linear Intercepts and Linear Subangles

Sadi Abu-Saymeh and Mowaffaq Hajja

Abstract. Let ABC be a triangle with side-lengths a, b, and c, and with angles A, B, and C. Let AA', BB', and CC' be the cevians through a point V, let x, y, and z be the lengths of the segments BA', CB', and AC', and let ξ, η, and ζ be the measures of the angles $\angle BAA'$, $\angle CBB'$, and $\angle ACC'$. The centers V for which x, y, and z are linear forms in a, b, and c are characterized. So are the centers for which ξ, η, and ζ are linear forms in A, B, and C.

Let ABC be a non-degenerate triangle with side-lengths a, b, and c, and let V be a point in its plane. Let AA', BB', and CC' be the cevians of ABC through V and let the intercepts x, y, and z be defined to be the directed lengths of the segments BA', CB', and AC', where x is positive or negative according as A' and C lie on the same side or on opposite sides of B, and similarly for y and z; see Figure 1. To avoid infinite intercepts, we assume that V does not lie on any of the three exceptional lines passing through the vertices of ABC and parallel to the opposite sides.

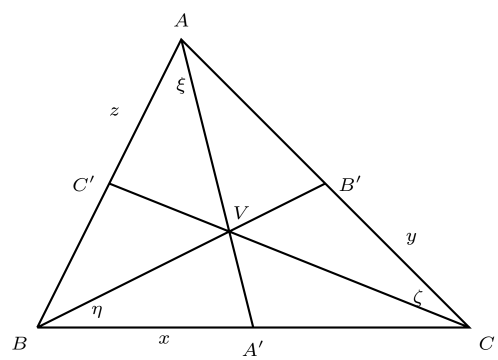


Figure 1

If V is the centroid of ABC, then the intercepts (x, y, z) are clearly given by $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$. It is also easy to see that the triples (x, y, z) determined by the Gergonne and Nagel points are

$$\left(\frac{a-b+c}{2}, \frac{a+b-c}{2}, \frac{-a+b+c}{2}\right), \ \left(\frac{a+b-c}{2}, \frac{-a+b+c}{2}, \frac{a-b+c}{2}\right),$$

Publication Date: March 10, 2005. Communicating Editor: Paul Yiu.
This work is supported by a research grant from Yarmouk University.

respectively. We now show that these are the only three centers whose corresponding intercepts (x, y, z) are linear forms in a, b, and c. Here, and in the spirit of [4] and [5], a center is a function that assigns to a triangle, in a family $\mathbf{U}$ of triangles, a point in its plane in a manner that is symmetric and that respects isometries and dilations. It is assumed that $\mathbf{U}$ has a non-empty interior, where $\mathbf{U}$ is thought of as a subset of $\mathbf{R}^3$ by identifying a triangle ABC with the point (a, b, c).

Theorem 1. *The triangle centers for which the intercepts x, y, z are linear forms in a, b, c are the centroid, the Gergonne and the Nagel points.*

Proof. Note first that if (x, y, z) are the intercepts corresponding to a center V, and if

$$x = \alpha a + \beta b + \gamma c,$$

then it follows from reflecting ABC about the perpendicular bisector of the segment BC that

$$a - x = \alpha a + \beta c + \gamma b.$$

Therefore $\alpha = \frac{1}{2}$ and $\beta + \gamma = 0$. Applying the permutation $(A\ B\ C) = (a\ b\ c) = (x\ y\ z)$, we see that

$$x = \alpha a + \beta b + \gamma c,\ y = \alpha b + \beta c + \gamma a,\ z = \alpha c + \beta a + \gamma b.$$

Substituting in the cevian condition $xyz = (a - x)(b - y)(c - z)$, we obtain the equation

$$\begin{aligned} & \left(\frac{a}{2} + \beta(b-c)\right)\left(\frac{b}{2} + \beta(c-a)\right)\left(\frac{c}{2} + \beta(a-b)\right) \\ = \quad & \left(\frac{a}{2} - \beta(b-c)\right)\left(\frac{b}{2} - \beta(c-a)\right)\left(\frac{c}{2} - \beta(a-b)\right) \end{aligned}$$

which simplifies into

$$\beta\left(\beta + \frac{1}{2}\right)\left(\beta - \frac{1}{2}\right)(a-b)(b-c)(c-a) = 0.$$

This implies the three possibilities $\beta = 0, -\frac{1}{2}$, or $\frac{1}{2}$ that correspond to the centroid, the Gergonne point and the Nagel point, respectively. □

In the same vein, the cevians through V define the subangles ξ, η, and ζ of the angles A, B, and C of ABC as shown in Figure 1. These are given by

$$\xi = \angle BAV, \qquad \eta = \angle CBV, \qquad \zeta = \angle ACV.$$

Here we temporaily take V to be inside ABC for simplicity, and treat the general case in Note 1 below. It is clear that the subangles (ξ, η, ζ) corresponding to the incenter of ABC are given by $(\frac{A}{2}, \frac{B}{2}, \frac{C}{2})$. Also, if ABC is acute-angled, then the orthocenter and circumcenter lie inside ABC and the triples (ξ, η, ζ) of subangles that they determine are given by

$$\left(\frac{A-B+C}{2}, \frac{A+B-C}{2}, \frac{-A+B+C}{2}\right), \left(\frac{A+B-C}{2}, \frac{-A+B+C}{2}, \frac{A-B+C}{2}\right), \tag{1}$$

or equivalently by

$$\left(\frac{\pi}{2}-B, \frac{\pi}{2}-C, \frac{\pi}{2}-A\right),\ \left(\frac{\pi}{2}-C, \frac{\pi}{2}-A, \frac{\pi}{2}-B\right), \tag{2}$$

respectively. Here again, we prove that these are the only centers whose corresponding subangles (ξ, η, ζ) are linear forms in A, B, and C. As before, we first show that the subangles (ξ, η, ζ) determined by such a center are of the form

$$\xi = \alpha A + \beta B + \gamma C,\ \eta = \alpha B + \beta C + \gamma A,\ \zeta = \alpha C + \beta A + \gamma B,$$

where $\alpha = \frac{1}{2}$ and $\beta + \gamma = 0$. Substituting in the trigonometric cevian condition

$$\sin \xi\ \sin\ \eta\ \sin\ \zeta\ =\ \sin\,(a-\xi)\ \sin\,(b-\eta)\ \sin\,(c-\zeta), \tag{3}$$

we obtain the equation

$$\begin{aligned}&\sin\left(\frac{A}{2}+\beta(B-C)\right)\ \sin\left(\frac{B}{2}+\beta(C-A)\right)\ \sin\left(\frac{C}{2}+\beta(A-B)\right)\\ =\ &\sin\left(\frac{A}{2}-\beta(B-C)\right)\ \sin\left(\frac{B}{2}-\beta(C-A)\right)\ \sin\left(\frac{C}{2}-\beta(A-B)\right).\end{aligned} \tag{4}$$

Using the facts that

$$\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\,,\ \beta(B-C)+\beta(C-A)+\beta(A-B)=0,$$

and the facts [3, Formulas 677, 678, page 166] that if $u+v+w=0$, then

$$\begin{aligned}4\ \cos\ u\ \cos\ v\ \sin\ w\ &=\ -\sin\ 2u\ -\ \sin\ 2v\ +\ \sin\ 2w,\\ 4\ \sin\ u\ \sin\ v\ \sin\ w\ &=\ -\sin\ 2u\ -\ \sin\ 2v\ -\ \sin\ 2w,\end{aligned}$$

and that if $u+v+w=\pi/2$, then

$$\begin{aligned}4\ \cos\ u\ \cos\ v\ \cos\ w\ &=\ \sin\ 2u\ +\ \sin\ 2v\ +\ \sin\ 2w,\\ 4\ \sin\ u\ \sin\ v\ \cos\ w\ &=\ \sin\ 2u\ +\ \sin\ 2v\ -\ \sin\ 2w,\end{aligned}$$

(4) simplifies into

$$\sin A\ \sin(2\beta(B-C))+\sin\ B\ \sin(2\beta(C-A))+\sin\ C\ \sin(2\beta(A-B))=0. \tag{5}$$

It is easy to check that for $\beta = -\frac{1}{2}$, 0, and $\frac{1}{2}$, this equation is satisfied for all triangles. Conversely, since (5) holds on a set **U** having a non-empty interior, it holds for all triangles, and in particular it holds for the triangle $(A, B, C) = (\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$. This implies that

$$\sin\frac{\beta\pi}{3}\ \left(\cos\frac{\beta\pi}{3}-\frac{\sqrt{3}}{2}\right)\ =\ 0.$$

Since $-\frac{3}{2} \le \beta \le \frac{3}{2}$ for this particular triangle, it follows that β must be $-\frac{1}{2}$, 0, or $\frac{1}{2}$. Thus the only solutions of (5) are $\beta = -\frac{1}{2}$, 0, and $\frac{1}{2}$. These correspond to the orthocenter, incenter and circumcenter, respectively. We summarize the result in the following theorem.

Theorem 2. *The triangle centers for which the subangles ξ, η, ζ are linear forms in A, B, C are the orthocenter, incenter, and circumcenter.*

Remarks. (1) Although the subangles ξ, η, and ζ of a given point V were defined for points that lie inside ABC only, it is possible to extend this definition to include exterior points also, without violating the trigonometric version (3) of Ceva's concurrence condition or the formulas (1) and (2) for the subangles corresponding to the orthocenter and the circumcenter. To do so, we let $\mathbf{H}_1$ and $\mathbf{H}_2$ be the open half planes determined by the line that is perpendicular at A to the internal angle-bisector of A, where we take $\mathbf{H}_1$ to be the half-plane containing B and C. For $V \in \mathbf{H}_1$, we define the subangle ξ to be the signed angle $\angle BAV$, where $\angle BAV$ is taken to be positive or negative according as the rotation within $\mathbf{H}_1$ that takes AB to AV has the same or opposite handedness as the one that takes AB to AC. For $V \in \mathbf{H}_2$, we stipulate that V and its reflection about A have the same subangle ξ. We define η and ζ similarly. Points on the three exceptional lines that are perpendicular at the vertices of ABC to the respective internal angle-bisectors are excluded.

(2) In terms of the intercepts and subangles, the first (respectively, the second) Brocard point of a triangle is the point whose subangles ξ, η, and ζ satisfy $\xi = \eta = \zeta$ (respectively, $A - \xi = B - \eta = C - \zeta$.) Similarly, the first and the second Brocard-like Yff points are the points whose intercepts x, y, and z satisfy $x = y = z$ and $a - x = b - y = c - z$, respectively. Other Brocard-like points corresponding to features other than intercepts and subangles are being explored by the authors.

(3) The requirement that the intercepts x, y, and z be linear in a, b, and c is quite restrictive, since the cevian condition has to be observed. It is thus tempting to weaken this requirement, which can be written in matrix form as $[x\ y\ z] = [a\ b\ c]L$, where L is a 3×3 matrix, to take the form $[x\ y\ z]M = [a\ b\ c]L$, where M is not necessarily invertible. The family of centers defined by this weaker requirement, together of course with the cevian condition, is studied in detail in [2]. So is the family obtained by considering subangles instead of intercepts.

References

[1] S. Abu-Saymeh and M. Hajja, In search of more triangle centers, preprint.
[2] S. Abu-Saymeh and M. Hajja, Linearly defined triangle centers, preprint.
[3] G. S. Carr, *Formulas and Theorems in Pure Mathematics*, 2nd edition, New York, Chelsea, 1970.
[4] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.
[5] C. Kimberling, Triangle centers as functions, *Rocky Mountain J. Math*. 23 (1993) 1269–1286.

Sadi Abu-Saymeh: Department of Mathematics, Yarmouk University, Irbid, Jordan
E-mail address: sade@yu.edu.jo

Mowaffaq Hajja: Department of Mathematics, Yarmouk University, Irbid, Jordan
E-mail address: mhajja@yu.edu.jo

Forum Geometricorum
Volume 5 (2005) 37–45.

FORUM GEOM
ISSN 1534-1178

The Arbelos in n-Aliquot Parts

Hiroshi Okumura and Masayuki Watanabe

Abstract. We generalize the classical arbelos to the case divided into many chambers by semicircles and construct embedded patterns of such arbelos.

1. Introduction and preliminaries

Let $\{\alpha,\ \beta,\ \gamma\}$ be an arbelos, that is, $\alpha,\ \beta,\ \gamma$ are semicircles whose centers are collinear and erected on the same side of this line, $\alpha,\ \beta$ are tangent externally, and γ touches α and β internally. In this paper we generalize results on the Archimedean circles of the arbelos. We take the line passing through the centers of $\alpha,\ \beta,\ \gamma$ as the x-axis and the line passing through the tangent point O of α and β and perpendicular to the x-axis as the y-axis. Let $\alpha_0 = \alpha,\ \alpha_1, \dots, \alpha_n = \beta$ be $n+1$ distinct semicircles touching α and β at O, where $\alpha_1, \dots, \alpha_{n-1}$ are erected on the same side as α and β, and intersect with γ. One of them may be the line perpendicular to the x-axis (i.e. y-axis). If the n inscribed circles in the curvilinear triangles bounded by $\alpha_{i-1},\ \alpha_i,\ \gamma$ are congruent we call this configuration of semicircles $\{\alpha_0 = \alpha,\ \alpha_1, \dots, \alpha_n = \beta,\ \gamma\}$ an arbelos in n-aliquot parts, and the inscribed circles the Archimedean circles in n-aliquot parts. In this paper we calculate the radii of the Archimedean circles in n-aliquot parts and construct embedded patterns of arbelos in aliquot parts.

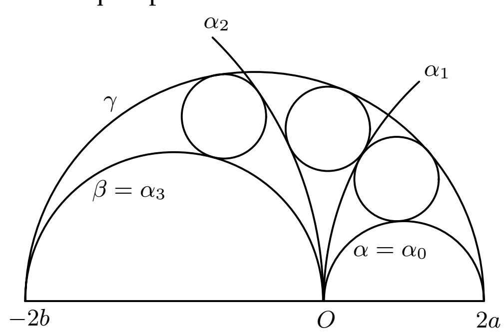


Figure 1. The case $n = 3$

For the arbelos $\{\alpha,\ \beta,\ \gamma\}$ we denote by $\Phi(\alpha, \beta, \gamma)$ the family of semicircles through O, having the common point with γ in the region $y \geq 0$ and with centers on the x-axis, together with the line perpendicular to the x-axis at O. Renaming if necessary we assume α in the region $x \geq 0$. Let $a,\ b$ be the radii of $\alpha,\ \beta$. The semicircle γ meets the x-axis at $-2b$ and $2a$.

For a semicircle $\alpha_i \in \Phi(\alpha, \beta, \gamma)$, let a_i be the x-coordinate of its center. Define $\mu(\alpha_i)$ as follows.

Publication Date: March 22, 2005. Communicating Editor: Paul Yiu.

If $a \neq b$,

$$\mu(\alpha_i) = \begin{cases} \dfrac{a_i - a + b}{a_i}, & \text{if } \alpha_i \text{ is a semi-circle,} \\ 1, & \text{if } \alpha_i \text{ is the line.} \end{cases}$$

If $a = b$,

$$\mu(\alpha_i) = \begin{cases} \dfrac{1}{a_i}, & \text{if } \alpha_i \text{ is a semi-circle,} \\ 0, & \text{if } \alpha_i \text{ is the line.} \end{cases}$$

In both cases $\mu(\alpha_i)$ depends only on α_i and the center of γ, but not on the radius of γ. For $\alpha_i,\ \alpha_j \in \Phi(\alpha, \beta, \gamma)$, the equality $\mu(\alpha_i) = \mu(\alpha_j)$ holds if and only if $\alpha_i = \alpha_j$. For any $\alpha_i \in \Phi(\alpha, \beta, \gamma)$,

$$\begin{aligned} \frac{b}{a} &= \mu(\alpha) \geq \mu(\alpha_i) \geq \mu(\beta) = \frac{a}{b} \text{ if } a < b, \\ \frac{1}{a} &= \mu(\alpha) \geq \mu(\alpha_i) \geq \mu(\beta) = -\frac{1}{a} \text{ if } a = b, \\ \frac{b}{a} &= \mu(\alpha) \leq \mu(\alpha_i) \leq \mu(\beta) = \frac{a}{b} \text{ if } a > b. \end{aligned}$$

For $\alpha_i,\ \alpha_j \in \Phi(\alpha, \beta, \gamma)$, define the order

$$\alpha_i < \alpha_j \text{ if and only if } \begin{cases} \mu(\alpha_i) > \mu(\alpha_j) & \text{if } a \leq b, \\ \mu(\alpha_i) < \mu(\alpha_j) & \text{otherwise}. \end{cases}$$

This means that α_i is nearer to α than α_j is. Throughout this paper we shall adopt these notations and assumptions.

2. An arbelos in aliquot parts

Lemma 1. *If α_i and α_j are semicircles in $\Phi(\alpha, \beta, \gamma)$ with $\alpha_i < \alpha_j$, the radius of the inscribed circle in the curvilinear triangle bounded by α_i, α_j and γ is*

$$\frac{ab(a_j - a_i)}{a_i a_j - a a_i + b a_j}\ .$$

Proof. Let $\mathcal{C}$ be the inscribed circle with radius r. First we invert $\{\alpha_i,\ \alpha_j,\ \gamma,\ \mathcal{C}\}$ in the circle with center O and radius k. Then α_i and α_j are inverted to the lines $\overline{\alpha_i}$ and $\overline{\alpha_j}$ perpendicular to the x-axis , γ is inverted to the semicircle $\overline{\gamma}$ erected on the x-axis and $\mathcal{C}$ is inverted to the circle $\overline{\mathcal{C}}$ tangent to $\overline{\gamma}$ externally. We write the x-coordinates of the intersections of $\overline{\alpha_i}$, $\overline{\alpha_j}$ and $\overline{\gamma}$ with the x-axis as $s,\ t$ and $p,\ q$ with $q < p$. Then $t < s$ since $a_i < a_j$.

By the definition of inversion we have

$$s = \frac{k^2}{2a_i}\ ,\ t = \frac{k^2}{2a_j}\ ,\ p = \frac{k^2}{2a}\ ,\ q = -\frac{k^2}{2b}\ . \tag{1}$$

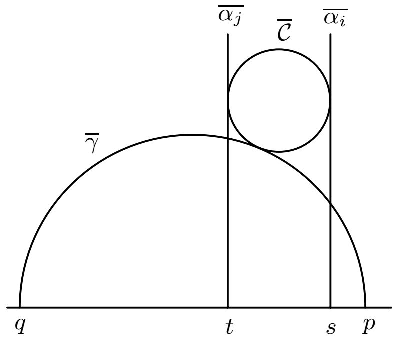


Figure 2

Since the x-coordinates of the center and the radius of $\overline{\mathcal{C}}$ are $\dfrac{s+t}{2}$ and $\dfrac{s-t}{2}$, and those of $\overline{\gamma}$ are $\dfrac{p+q}{2}$ and $\dfrac{p-q}{2}$, we have

$$\left(\frac{s+t}{2}-\frac{p+q}{2}\right)^2+d^2=\left(\frac{s-t}{2}+\frac{p-q}{2}\right)^2,$$

where d is the y-coordinate of the center of $\overline{\mathcal{C}}$. From this,

$$st-sp-tq+pq+d^2=0\ . \tag{2}$$

Since O is outside $\overline{\mathcal{C}}$, we have

$$r=\frac{k^2}{\left|\left(\frac{s+t}{2}\right)^2+d^2-\left(\frac{s-t}{2}\right)^2\right|}\cdot\frac{s-t}{2}=\frac{k^2}{\left(\frac{s+t}{2}\right)^2+d^2-\left(\frac{s-t}{2}\right)^2}\cdot\frac{s-t}{2}.$$

By using (1) and (2) we get the conclusion. □

Lemma 2. *If α_i (resp. α_j) is the line, then the radius of the inscribed circle is*

$$\frac{-ab}{a_j-a}(resp.\ \frac{ab}{a_i+b}).$$

Proof. Even in this case (2) in the proof of Lemma 1 holds with $s=0$ (resp. $t=0$), and we get the conclusion. □

Theorem 3. *Assume $a\neq b$, and let $\alpha_i,\ \alpha_j\in\Phi(\alpha,\beta,\gamma)$ with $\alpha_i<\alpha_j$. The radius of the circle inscribed in the curvilinear triangle bounded by $\alpha_i,\ \alpha_j$ and γ is*

$$\frac{ab(\mu(\alpha_i)-\mu(\alpha_j))}{b\mu(\alpha_i)-a\mu(\alpha_j)}.$$

Proof. If α_i and α_j are semicircles, then

$$\frac{ab(\mu(\alpha_i)-\mu(\alpha_j))}{b\mu(\alpha_i)-a\mu(\alpha_j)}=\frac{ab\left(\dfrac{a_i-a+b}{a_i}-\dfrac{a_j-a+b}{a_j}\right)}{b\cdot\dfrac{a_i-a+b}{a_i}-a\cdot\dfrac{a_j-a+b}{a_j}}=\frac{ab(a_j-a_i)}{a_ia_j-aa_i+ba_j}.$$

Hence the theorem follows from Lemma 1. If one of α_i, α_j is the line, the result follows from Lemma 2. $\square$

Similarly we have

Theorem 4. *Assume $a = b$, and let $\alpha_i,\ \alpha_j \in \Phi(\alpha, \beta, \gamma)$ with $\alpha_i < \alpha_j$. The radius of the circle inscribed in the curvilinear triangle bounded by α_i, α_j and γ is*

$$\frac{a^2(\mu(\alpha_j) - \mu(\alpha_i))}{a(\mu(\alpha_j) - \mu(\alpha_i)) - 1}.$$

The functions $x \mapsto \dfrac{ab(1-x)}{b-ax},\ a \neq b$ and $x \mapsto \dfrac{a^2x}{ax-1}, a > 0$ are injective. Therefore, we have

Corollary 5. *Let $\alpha_0,\ \alpha_1, \dots, \alpha_n \in \Phi(\alpha,\ \beta,\ \gamma)$ with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. The circles inscribed in the curvilinear triangle bounded by α_{i-1}, α_i and γ $(i = 1, 2, \dots n)$ are all congruent if and only if $\mu(\alpha_0),\ \mu(\alpha_1), \dots, \mu(\alpha_n)$ is a geometric sequence if $a \neq b$, or an arithmetic sequence if $a = b$.*

Theorem 6. *Let $\{\alpha_0 = \alpha,\ \alpha_1, \dots, \alpha_n = \beta,\ \gamma\}$ be an arbelos in n-aliquot parts. The common radius of the Archimedean circles in n-aliquot parts is*

$$\begin{cases} \dfrac{ab\left(b^{\frac{2}{n}} - a^{\frac{2}{n}}\right)}{b^{\frac{2}{n}+1} - a^{\frac{2}{n}+1}}, & \text{if } a \neq b, \\[2ex] \dfrac{2a}{n+2}, & \text{if } a = b. \end{cases}$$

Proof. First we consider the case $a \neq b$. We can assume $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ by renaming if necessary. The sequence $\dfrac{b}{a} = \mu(\alpha_0),\ \mu(\alpha_1), \dots, \mu(\alpha_n) = \dfrac{a}{b}$ is a geometric sequence by Corollary 5. If we write its common ratio as d, we have $\dfrac{a}{b} = d^n\left(\dfrac{b}{a}\right)$, and then $d = \left(\dfrac{a}{b}\right)^{\frac{2}{n}}$. By Theorem 3 the radius of the Archimedean circle is

$$\frac{ab(1-d)}{b-ad} = \frac{ab\left(1 - \left(\frac{a}{b}\right)^{\frac{2}{n}}\right)}{b - a\left(\frac{a}{b}\right)^{\frac{2}{n}}} = \frac{ab\left(b^{\frac{2}{n}} - a^{\frac{2}{n}}\right)}{b^{\frac{2}{n}+1} - a^{\frac{2}{n}+1}}.$$

Similarly we can get the second assertion. $\square$

Note that the second assertion is the limiting case of the first assertion when $b \to a$.

Theorem 7. *Let $\{\alpha_0 = \alpha,\ \alpha_1, \dots, \alpha_n = \beta,\ \gamma\}$ be an arbelos in n-aliquot parts with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. Then α_i is the line in $\Phi(\alpha, \beta, \gamma)$ if n is even and $i = \frac{n}{2}$.*

Otherwise it is a semicircle with radius

$$\begin{cases} \left| \dfrac{b^{\frac{2i}{n}-1}(a-b)}{a^{\frac{2i}{n}-1} - b^{\frac{2i}{n}-1}} \right|, & \text{if } a \neq b, \\[2ex] \left| \dfrac{na}{n-2i} \right|, & \text{if } a = b. \end{cases}$$

Proof. Suppose $a \neq b$. Since $\frac{b}{a} = \mu(\alpha_0),\ \mu(\alpha_1), \ldots, \mu(\alpha_n) = \frac{a}{b}$ is a geometric sequence with common ratio $\left(\frac{a}{b}\right)^{\frac{2}{n}}$, we have $\mu(\alpha_i) = \left(\frac{a}{b}\right)^{\frac{2i}{n}} \left(\frac{b}{a}\right) = \left(\frac{a}{b}\right)^{\frac{2i}{n}-1}$. If n is even and $i = \frac{n}{2}$, then $\mu(\alpha_i) = 1$ and α_i is the line. Otherwise, $\mu(\alpha_i) \neq 1$ and α_i is a semicircle. Let a_i be the x-coordinate of its center. The radius of α_i is $|a_i|$ and $\frac{a_i - a + b}{a_i} = \left(\frac{a}{b}\right)^{\frac{2i}{n}-1}$. From this, $a_i = \frac{b^{\frac{2i}{n}-1}(a-b)}{b^{\frac{2i}{n}-1} - a^{\frac{2i}{n}-1}}$.

The proof for the case $a = b$ is similar. □

3. Embedded patterns of the arbelos

Let $\{\alpha_0 = \alpha,\ \alpha_1, \ldots, \alpha_n = \beta,\ \gamma\}$ be an arbelos in n-aliquot parts with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. There exists a semicircle γ' which is tangent to all Archimedean circles externally. It is clearly concentric to γ. (If $n = 1$ we will take for γ' the semicircle concentric to γ and tangent to the Archimedean circle externally). Let α', β' be two semicircles in $y \geq 0$, tangent to $\alpha_i s$ at O and also tangent to γ'. We take α' in the region $x \geq 0$ and β' in the region $x \leq 0$. Let a' and b' be the radii of α' and β' respectively. Clearly α', β' are tangent externally at O, and γ' intersects the x-axis at $-2b'$ and $2a'$, and $\Phi(\alpha, \beta, \gamma) \subseteq \Phi(\alpha', \beta', \gamma')$. Moreover, for any $\alpha_i \in \Phi(\alpha, \beta, \gamma)$, $\mu(\alpha_i)$ considered in $\Phi(\alpha, \beta, \gamma)$ is equal to $\mu(\alpha_i)$ considered in $\Phi(\alpha', \beta', \gamma')$ since the centers of γ and γ' coincide.

Lemma 8. (a) *If* $a \neq b$, $\left(\frac{a'}{b'}\right)^n = \left(\frac{a}{b}\right)^{n+2}$.

(b) *If* $a = b$, $\frac{a'}{n} = \frac{a}{n+2}$.

Proof. If $a \neq b$ we have

$$a' = a - \frac{ab\left(a^{\frac{2}{n}} - b^{\frac{2}{n}}\right)}{a^{\frac{2}{n}+1} - b^{\frac{2}{n}+1}} = \frac{a^{\frac{2}{n}+1}(a-b)}{a^{\frac{2}{n}+1} - b^{\frac{2}{n}+1}},$$

$$b' = b - \frac{ab\left(a^{\frac{2}{n}} - b^{\frac{2}{n}}\right)}{a^{\frac{2}{n}+1} - b^{\frac{2}{n}+1}} = \frac{b^{\frac{2}{n}+1}(a-b)}{a^{\frac{2}{n}+1} - b^{\frac{2}{n}+1}},$$

by the definitions of a' and b'. Then the the first assertion follows. The second assertion follows similarly. □

Theorem 9. $\{\alpha', \alpha_0, \alpha_1, \ldots, \alpha_n, \beta', \gamma'\}$ *is an arbelos in* $(n+2)$*-aliquot parts.*

Proof. Let us assume $a \neq b$. By Lemma 8 and the proof of Theorem 6, $\mu(\alpha_0)$, $\mu(\alpha_1), \dots, \mu(\alpha_n)$ is a geometric sequence with common ratio $\left(\frac{a'}{b'}\right)^{\frac{2}{n+2}}$. Also by Lemma 8 we have

$$\frac{\mu(\alpha_0)}{\mu(\alpha')} = \frac{b}{a}\frac{a'}{b'} = \left(\frac{b'}{a'}\right)^{\frac{n}{n+2}} \frac{a'}{b'} = \left(\frac{a'}{b'}\right)^{\frac{2}{n+2}},$$

and

$$\frac{\mu(\beta')}{\mu(\alpha_n)} = \frac{a'}{b'}\frac{b}{a} = \frac{a'}{b'}\left(\frac{b'}{a'}\right)^{\frac{n}{n+2}} = \left(\frac{a'}{b'}\right)^{\frac{2}{n+2}}.$$

The case $a = b$ follows similarly. □

Let $\{\alpha,\ \beta,\ \gamma\}$ be an arbelos and all the semicircles be constructed in $y \geq 0$ such that the diameters lie on the x-axis. Let $\alpha_{-1} = \alpha$, $\alpha_1 = \beta$ and $\gamma_1 = \gamma$. If there exists an arbelos in $(2n-1)$-aliquot parts $\{\alpha_{-n}, \alpha_{-(n-1)}, \dots, \alpha_{-1}, \alpha_1, \dots, \alpha_n, \gamma_{2n-1}\}$ with $\alpha_{-n} < \alpha_{-(n-1)} < \cdots < \alpha_{-1} < \alpha_1 < \cdots < \alpha_n$, we shall construct an arbelos in $(2n+1)$-aliquot parts as follows.

Let γ_{2n+1} be the semicircle concentric to γ and tangent externally to all Archimedean circles of the above arbelos. This meets the x-axis at two points one of which is in the region $x > 0$ and the other in $x < 0$. We write the semicircle passing through O and the former point as $\alpha_{-(n+1)}$ and the semicircle passing through O and the latter point as α_{n+1}. Then $\{\alpha_{-(n+1)}, \alpha_{-n}, \dots, \alpha_{-1}, \alpha_1, \dots, \alpha_{n+1}, \gamma_{2n+1}\}$ is an arbelos in $(2n+1)$-aliquot parts by Theorem 9. Now we get the set of semicircles

$$\{\dots, \alpha_{-(n+1)}, \alpha_{-n}, \dots, \alpha_{-1}, \alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots \gamma_1, \gamma_3, \dots, \gamma_{2n-1} \dots\},$$

where $\{\alpha_{-n}, \dots, \alpha_{-1}, \alpha_1, \dots, \alpha_n, \gamma_{2n-1}\}$ form the arbelos in $(2n-1)$-aliquot parts for any positive integer n. We shall call the above configuration the *odd* pattern.

Theorem 10. *Let δ_{2n-1} be one of the Archimedean circles in*

$$\{\alpha_{-n},\ \alpha_{-(n-1)},\ \dots, \alpha_{-1},\ \alpha_1, \dots, \alpha_n\ , \gamma_{2n-1}\}.$$

Then the radii of α_{-n} and α_n are

$$\frac{a^{2n-1}(a-b)}{a^{2n-1}-b^{2n-1}} \quad \text{and} \quad \frac{b^{2n-1}(a-b)}{a^{2n-1}-b^{2n-1}},$$

and the radii of γ_{2n-1} and δ_{2n-1} are respectively

$$\frac{(a^{2n-1}+b^{2n-1})(a-b)}{a^{2n-1}-b^{2n-1}} \quad \text{and} \quad \frac{a^{2n-1}b^{2n-1}(a-b)(a^2-b^2)}{(a^{2n-1}-b^{2n-1})(a^{2n+1}-b^{2n+1})}.$$

Proof. Let $\overline{a_{-n}}$ and $\overline{a_n}$ be the radii of α_{-n} and α_n respectively. By Lemma 8 we have

$$\left(\frac{\overline{a_{-n}}}{\overline{a_n}}\right)^{\frac{1}{2n-1}} = \left(\frac{\overline{a_{-(n-1)}}}{\overline{a_{n-1}}}\right)^{\frac{1}{2n-3}} = \cdots = \frac{\overline{a_{-1}}}{\overline{a_1}} = \frac{a}{b}. \tag{3}$$

Since γ_{2n-1} and γ are concentric, we have

$$\overline{a_{-n}} - \overline{a_n} = a - b \ . \tag{4}$$

By (3) and (4) we have

$$\overline{a_{-n}} = \frac{a^{2n-1}(a-b)}{a^{2n-1} - b^{2n-1}} \ ,$$
$$\overline{a_n} = \frac{b^{2n-1}(a-b)}{a^{2n-1} - b^{2n-1}}.$$

It follows that the radius of γ_{2n-1} is

$$\overline{a_{-n}} + \overline{a_n} = \frac{(a^{2n-1} + b^{2n-1})(a-b)}{a^{2n-1} - b^{2n-1}},$$

and that of δ_{2n-1} is

$$\frac{(a^{2n-1} + b^{2n-1})(a-b)}{a^{2n-1} - b^{2n-1}} - \frac{(a^{2n+1} + b^{2n+1})(a-b)}{a^{2n+1} - b^{2n+1}}$$
$$= \frac{a^{2n-1}b^{2n-1}(a-b)(a^2-b^2)}{(a^{2n-1} - b^{2n-1})(a^{2n+1} - b^{2n+1})} \ .$$

□

As in the odd case, we can construct the *even* pattern of arbelos

$\{\ldots \beta_{-(n+1)},\ \beta_{-n},\ \ldots,\ \beta_{-1},\ \beta_0,\ \beta_1,\ \ldots,\ \beta_n,\ \beta_{n+1},\ \ldots,\ \gamma_2,\ \gamma_4,\ \ldots,\ \gamma_{2n} \ldots\}$
inductively by starting with an arbelos in 2-aliquot parts $\{\beta_{-1},\ \beta_0,\ \beta_1, \gamma_2\}$, where $\beta_{-1} = \alpha,\ \beta_1 = \beta$ and $\gamma_2 = \gamma$. By Theorem 9, $\{\beta_{-n}, \ldots, \beta_{-1},\ \beta_0,\ \beta_1, \ldots, \beta_n, \gamma_{2n}\}$ forms an arbelos in $2n$-aliquot parts for any positive integer n, and β_0 is the line by Theorem 7. Analogous to Theorem 10 we have

Theorem 11. *Let δ_{2n} be one of the Archimedean circles in*

$$\{\beta_{-n},\ \beta_{-(n-1)}, \ldots, \beta_{-1},\ \beta_0,\ \beta_1, \ldots, \beta_n\ , \gamma_{2n}\}.$$

The radii of β_{-n} and β_n are

$$\frac{a^n(a-b)}{a^n - b^n} \quad \textit{and} \quad \frac{b^n(a-b)}{a^n - b^n},$$

and the radii of γ_{2n} and δ_{2n} are respectively

$$\frac{(a^n + b^n)(a-b)}{a^n - b^n} \quad \text{and} \quad \frac{a^n b^n (a-b)^2}{(a^n - b^n)(a^{n+1} - b^{n+1})}.$$

Corollary 12. *Let c_n and d_n be the radii of γ_n and δ_n respecgtively.*

$$a_n = b_{2n-1},$$
$$a_{-n} = b_{-(2n-1)},$$
$$c_{2n-1} = c_{2(2n-1)},$$
$$d_{2n-1} = d_{4n-2} + d_{4n}.$$

Figure 3 shows the even pattern together with the odd pattern reflected in the x-axis. The trivial case of these patterns can be found in [2].

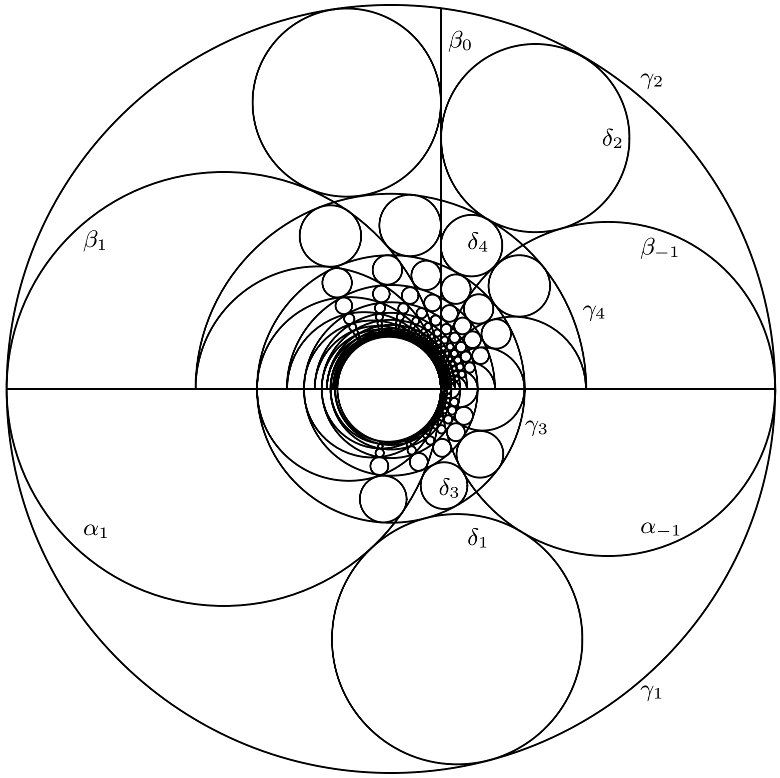


Figure 3

4. Some Applications

We give two applications here, with the same notations as in §3.

Theorem 13. *The external common tangent of β_n and β_{-n} touches γ_{4n} for any positive integer n.*

Proof. The distance between the external common tangents of β_n and β_{-n} and the center of γ_{2n} is $\dfrac{\overline{b_n}^2 + \overline{b_{-n}}^2}{\overline{b_n} + \overline{b_{-n}}}$ where $\overline{b_n}$ and $\overline{b_{-n}}$ are the radii of β_n and β_{-n}. By Theorem 11 this is equal to $\dfrac{(a-b)(a^{2n}+b^{2n})}{a^{2n}-b^{2n}}$, the radius of γ_{4n}. $\square$

Theorem 14. *Let BK_n be the circle orthogonal to α, β and δ_{2n-1}, and let AR_n be the inscribed circle of the curvilinear triangle bounded by β_n, β_0 and γ_{2n}. The circles BK_n and AR_n are congruent for every natural number n.*

Proof. Assume $a \neq b$. Since AR_n is the Archimedean circle of the arbelos in 2-aliquot parts $\{\beta_{-n},\ \beta_0,\ \beta_n,\ \gamma_{2n}\}$, the radius of AR_n is

$$\frac{\overline{b_n}\,\overline{b_{-n}}(\overline{b_n} - \overline{b_{-n}})}{\overline{b_n}^2 - \overline{b_{-n}}^2} = \frac{a^n b^n (a-b)}{a^{2n} - b^{2n}},$$

by Theorem 6 and Theorem 11.

On the other hand BK_n is the inscribed circle of the triangle bounded by the three centers of α, β, δ_{2n-1}. Since the length of three sides of the triangle are $a+d_{2n-1}$, $b+d_{2n-1}$, $a+b$, the radius of BK_n is

$$\sqrt{\frac{abd_{2n-1}}{a+b+d_{2n-1}}} = \frac{a^n b^n (a-b)}{a^{2n}-b^{2n}},$$

by Theorem 10. $\square$

This theorem is a generalization of Bankoff circle [1]. Bankoff's third circle corresponds to the case $n=1$ in this theorem.

References

[1] L. Bankoff, Are the twin circles of Archimedes really twins?, *Math. Magazine*, 47 (1974) 214–218.

[2] H. Okumura, Circles patterns arising from results in Japanese geometry, *Symmetry: Culture and Science*, 8 (1997) 4–23.

Hiroshi Okumura: Department of Information Engineering, Maebashi Institute of Technology, 460-1 Kamisadori Maebashi Gunma 371-0816, Japan

E-mail address: okumura@maebashi-it.ac.jp

Masayuki Watanabe: Department of Information Engineering, Maebashi Institute of Technology, 460-1 Kamisadori Maebashi Gunma 371-0816, Japan

E-mail address: watanabe@maebashi-it.ac.jp

Forum Geometricorum
Volume 5 (2005) 47–52.

FORUM GEOM
ISSN 1534-1178

On a Problem Regarding the n-Sectors of a Triangle

Bart De Bruyn

Abstract. Let Δ be a triangle with vertices A, B, C and angles $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$, $\gamma = \widehat{ACB}$. The $n-1$ lines through A which, together with the lines AB and AC, divide the angle α in $n \geq 2$ equal parts are called the n-sectors of Δ. In this paper we determine all triangles with the property that all three edges and all $3(n-1)$ n-sectors have rational lengths. We show that such triangles exist only if $n \in \{2, 3\}$.

1. Introduction

Let Δ be a triangle with vertices A, B, C and angles $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$, $\gamma = \widehat{ACB}$. The $n-1$ lines through A which, together with the lines AB and AC, divide the angle α in $n \geq 2$ equal parts are called the *n-sectors* of Δ. A triangle has $3(n-1)$ n-sectors. The 2-sectors and 3-sectors are also called *bisectors* and *trisectors*. In this paper we study triangles with the property that all three edges and all $3(n-1)$ n-sectors have rational lengths. We show that such triangles can exist only if $n = 2$ or 3. We also determine all triangles with the property that all edges and bisectors (trisectors) have rational lengths. In each of the cases $n = 2$ and $n = 3$, there are infinitely many nonsimilar triangles having that property.

In number theory, there are some open problems of the same type as the above-mentioned problem.

(i) Does there exists a *perfect cuboid*, i.e. a cuboid in which all 12 edges, all 12 face diagonals and all 4 body diagonals are rational? ([3, Problem D18]).

(ii) Does there exist a triangle with integer edges, medians and area? ([3, Problem D21]).

2. Some properties

An elementary proof of the following lemma can also be found in [2, p. 443].

Lemma 1. *The number* $\cos\frac{\pi}{n}$, $n \geq 2$, *is rational if and only if* $n = 2$ *or* $n = 3$.

Proof. Suppose that $\cos\frac{\pi}{n}$ is rational. Put

$$\zeta_{2n} = \cos\frac{2\pi}{2n} + i\sin\frac{2\pi}{2n},$$

Publication Date: March 29, 2005. Communicating Editor: J. Chris Fisher.

then ζ_{2n} is a zero of the polynomial $X^2 - (2 \cdot \cos\frac{\pi}{n}) \cdot X + 1 \in \mathbb{Q}[X]$. So, the minimal polynomial of ζ_{2n} over $\mathbb{Q}$ is of the first or second degree. On the other hand, we know that the minimal polynomial of ζ_{2n} over $\mathbb{Q}$ is the $2n$-th cyclotomic polynomial $\Phi_{2n}(x)$, see [4, Theorem 4.17]. The degree of $\Phi_{2n}(x)$ is $\phi(2n)$, where ϕ is the *Euler phi function*. We have $\phi(2n) = 2n \cdot \frac{p_1-1}{p_1} \cdot \frac{p_2-1}{p_2} \cdot \ldots \cdot \frac{p_k-1}{p_k}$, where $p_1, \ldots, p_k$ are the different prime numbers dividing $2n$. From $\phi(2n) \in \{1, 2\}$, it easily follows $n \in \{2, 3\}$. Obviously, $\cos\frac{\pi}{2}$ and $\cos\frac{\pi}{3}$ are rational. $\square$

Lemma 2. *For every $n \in \mathbb{N} \setminus \{0\}$, there exist polynomials $f_n(x), g_{n-1}(x) \in \mathbb{Q}[x]$ such that*
(i) $\deg(f_n) = n$, $f_n(x) = 2^{n-1}x^n + \cdots$ *and* $\cos(nx) = f_n(\cos x)$ *for every* $x \in \mathbb{R}$*;*
(ii) $\deg(g_{n-1}) = n-1$, $g_{n-1}(x) = 2^{n-1}x^{n-1} + \cdots$ *and* $\frac{\sin(nx)}{\sin x} = g_{n-1}(\cos x)$ *for every* $x \in \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$.

Proof. From $\cos x = \cos x$, $\frac{\sin x}{\sin x} = 1$,

$$\begin{aligned}
\cos(k+1)x &= \cos(kx)\cos x - \frac{\sin(kx)}{\sin x}(1-\cos^2 x), \\
\frac{\sin(k+1)x}{\sin x} &= \frac{\sin(kx)}{\sin x}\cos x + \cos(kx)
\end{aligned}$$

for $k \geq 1$, it follows that we should make the following choices for the polynomials:

$f_1(x) := x$, $g_0(x) := 1$;
$f_{k+1}(x) := f_k(x) \cdot x - g_{k-1}(x) \cdot (1-x^2)$ for every $k \geq 1$;
$g_k(x) := g_{k-1}(x) \cdot x + f_k(x)$ for every $k \geq 1$.

One easily verifies by induction that f_n and g_{n-1} ($n \geq 1$) have the claimed properties. $\square$

Lemma 3. *Let $n \in \mathbb{N} \setminus \{0\}$, $q \in \mathbb{Q}^+ \setminus \{0\}$ and $x_1, \ldots, x_n \in \mathbb{R}$. If*

$$\cos x_1,\ \sqrt{q} \cdot \sin x_1,\ \ldots,\ \cos x_n,\ \sqrt{q} \cdot \sin x_n$$

are rational, then so are $\cos(x_1 + \cdots + x_n)$ and $\sqrt{q} \cdot \sin(x_1 + \cdots + x_n)$.

Proof. This follows by induction from the following equations ($k \geq 1$).

$$\begin{aligned}
\cos(x_1 + \cdots + x_{k+1}) &= \cos(x_1 + \cdots + x_k) \cdot \cos(x_{k+1}) \\
&\quad - \frac{1}{q}\left(\sqrt{q} \cdot \sin(x_1 + \cdots + x_k)\right) \cdot \left(\sqrt{q} \cdot \sin(x_{k+1})\right); \\
\sqrt{q} \cdot \sin(x_1 + \cdots + x_{k+1}) &= \left(\sqrt{q} \cdot \sin(x_1 + \cdots + x_k)\right) \cdot \cos(x_{k+1}) \\
&\quad + \cos(x_1 + \cdots + x_k) \cdot \left(\sqrt{q} \cdot \sin(x_{k+1})\right).
\end{aligned}$$

$\square$

Lemma 4. *Let Δ be a triangle with vertices A, B and C. Put $a = |BC|$, $b = |AC|$, $c = |AB|$, $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$ and $\gamma = \widehat{BCA}$. Let $n \in \mathbb{N} \setminus \{0\}$ and suppose that $\cos(\frac{\alpha}{n})$, $\cos(\frac{\beta}{n})$ and $\cos(\frac{\gamma}{n})$ are rational. Then the following are equivalent:*

(i) $\frac{b}{a}$ *and* $\frac{c}{a}$ *are rational numbers.*
(ii) $\frac{\sin\frac{\beta}{n}}{\sin\frac{\alpha}{n}}$ *and* $\frac{\sin\frac{\gamma}{n}}{\sin\frac{\alpha}{n}}$ *are rational numbers.*

Proof. We have

$$\frac{b}{a} = \frac{\sin\beta}{\sin\alpha} = \frac{\sin\beta}{\sin\frac{\beta}{n}} \cdot \frac{\sin\frac{\alpha}{n}}{\sin\alpha} \cdot \frac{\sin\frac{\beta}{n}}{\sin\frac{\alpha}{n}}.$$

By Lemma 2, $\frac{\sin\beta}{\sin\frac{\beta}{n}} \cdot \frac{\sin\frac{\alpha}{n}}{\sin\alpha} \in \mathbb{Q}^+ \setminus \{0\}$. So, $\frac{b}{a}$ is rational if and only if $\frac{\sin\frac{\beta}{n}}{\sin\frac{\alpha}{n}}$ is rational. A similar remark holds for the fraction $\frac{c}{a}$. □

3. Necessary and sufficient conditions

Theorem 5. *Let $n \geq 2$ and $0 < \alpha, \beta, \gamma < \pi$ with $\alpha + \beta + \gamma = \pi$. There exists a triangle with angles α, β and γ all whose edges and n-sectors have rational lengths if and only if the following conditions hold:*

(1) $\cos\frac{\pi}{n} \in \mathbb{Q}$,
(2) $\cot\frac{\pi}{2n} \cdot \tan\frac{\alpha}{2n} \in \mathbb{Q}$,
(3) $\cot\frac{\pi}{2n} \cdot \tan\frac{\beta}{2n} \in \mathbb{Q}$.

Proof. (a) Let Δ be a triangle with the property that all edges and all n-sectors have rational lengths. Let A, B and C be the vertices of Δ. Put $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$ and $\gamma = \widehat{ACB}$. Let $A_0, \ldots, A_n$ be the vertices on the edge BC such that $A_0 = B$, $A_n = C$ and $\widehat{A_{i-1}AA_i} = \frac{\alpha}{n}$ for all $i \in \{1, \ldots, n\}$. Put $a_i = |A_{i-1}A_i|$ for every $i \in \{1, \ldots, n\}$. For every $i \in \{1, \ldots, n-1\}$, the line AA_i is a bisector of the triangle with vertices A_{i-1}, A and A_{i+1}. Hence, $\frac{a_i}{a_{i+1}} = \frac{|AA_{i-1}|}{|AA_{i+1}|} \in \mathbb{Q}$. Together with $a_1 + \ldots + a_n = |BC| \in \mathbb{Q}$, it follows that $a_i \in \mathbb{Q}$ for every $i \in \{1, \ldots, n\}$. The cosine rule in the triangle with vertices A, A_0 and A_1 gives

$$\cos\frac{\alpha}{n} = \frac{|AA_0^2| + |AA_1|^2 - a_1^2}{2 \cdot |AA_0| \cdot |AA_1|} \in \mathbb{Q}.$$

In a similar way one shows that $\cos\frac{\beta}{n}, \cos\frac{\gamma}{n} \in \mathbb{Q}$. Put $q := (1 - \cos^2\frac{\alpha}{n})^{-1}$. By Lemma 4, $\sqrt{q} \cdot \sin\frac{\alpha}{n}$, $\sqrt{q} \cdot \sin\frac{\beta}{n}$ and $\sqrt{q} \cdot \sin\frac{\gamma}{n}$ are rational. From Lemma 3, it follows that $\cos\frac{\pi}{n} \in \mathbb{Q}$ and $\sqrt{q} \cdot \sin\frac{\pi}{n} \in \mathbb{Q}$. Hence,

$$\cot\frac{\pi}{2n} \cdot \tan\frac{\alpha}{2n} = \frac{1 + \cos\frac{\pi}{n}}{\sqrt{q} \cdot \sin\frac{\pi}{n}} \cdot \frac{\sqrt{q} \cdot \sin\frac{\alpha}{n}}{1 + \cos\frac{\alpha}{n}} \in \mathbb{Q}.$$

Similarly, $\cot\frac{\pi}{2n} \cdot \tan\frac{\beta}{2n} \in \mathbb{Q}$ and $\cot\frac{\pi}{2n} \cdot \tan\frac{\gamma}{2n} \in \mathbb{Q}$.

(b) Conversely, suppose that $\cos\frac{\pi}{n} \in \mathbb{Q}$, $\cot\frac{\pi}{2n} \cdot \tan\frac{\alpha}{2n} \in \mathbb{Q}$ and $\cot\frac{\pi}{2n} \cdot \tan\frac{\beta}{2n} \in \mathbb{Q}$. Put $q := \sin^2\frac{\pi}{n} = 1 - \cos^2\frac{\pi}{n} \in \mathbb{Q}$. From $\sqrt{q} \cdot \cot\frac{\pi}{2n} = \sqrt{q} \cdot \frac{1+\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}} \in \mathbb{Q}$, it follows that $\sqrt{q} \cdot \tan\frac{\alpha}{2n} \in \mathbb{Q}$, $\sqrt{q} \cdot \tan\frac{\beta}{2n} \in \mathbb{Q}$, $\cos\frac{\alpha}{n} = \frac{1-\tan^2\frac{\alpha}{2n}}{1+\tan^2\frac{\alpha}{2n}} \in \mathbb{Q}$, $\cos\frac{\beta}{n} \in \mathbb{Q}$, $\sqrt{q} \cdot \sin\frac{\alpha}{n} = \frac{2\sqrt{q}\cdot\tan\frac{\alpha}{2n}}{1+\tan^2\frac{\alpha}{2n}} \in \mathbb{Q}$, $\sqrt{q} \cdot \sin\frac{\beta}{n} \in \mathbb{Q}$. By Lemma 3, also $\cos\frac{\gamma}{n}$, $\sqrt{q} \cdot \sin\frac{\gamma}{n} \in \mathbb{Q}$. Now, choose a triangle Δ with angles α, β and γ such that the edge opposite the angle α has rational length. By Lemma 4, it then follows that also the edges opposite to β and γ have rational lengths. Let A, B and C be the vertices of Δ such that $\widehat{BAC} = \alpha$, $\widehat{ABC} = \beta$ and $\widehat{ACB} = \gamma$. As before, let $A_0, \dots, A_n$ be vertices on the edge BC such that the $n+1$ lines AA_i, $i \in \{0, \dots, n\}$, divide the angle α in n equal parts. By the sine rule,

$$|AA_i| = \frac{|AB| \cdot \sin\beta}{\sin(\frac{i\alpha}{n} + \beta)}.$$

Now,

$$\frac{\sin(\frac{i\alpha}{n} + \beta)}{\sin\beta} = \frac{\sin\frac{i\alpha}{n}}{\sin\frac{\alpha}{n}} \cdot \frac{\sqrt{q} \cdot \sin\frac{\alpha}{n}}{\sqrt{q} \cdot \sin\frac{\beta}{n}} \cdot \frac{\sin\frac{\beta}{n}}{\sin\beta} \cdot \cos\beta + \cos\frac{i\alpha}{n}.$$

By Lemma 2, this number is rational. Hence $|AA_i| \in \mathbb{Q}$. By a similar reasoning it follows that the lengths of all other n-sectors are rational as well. $\square$

By Lemma 1 and Theorem 5 (1), we know that the problem can only have a solution in the case of bisectors or trisectors.

4. The case of bisectors

The bisector case has already been solved completely, see e.g. [1] or [5]. Here we present a complete solution based on Theorem 5. Without loss of generality, we may suppose that $\alpha \le \beta \le \gamma$. These conditions are equivalent with

$$0 < \alpha \le \frac{\pi}{3}, \tag{1}$$

$$\alpha \le \beta \le \frac{\pi}{2} - \frac{\alpha}{2}. \tag{2}$$

By Theorem 5, $q_\alpha := \tan\frac{\alpha}{4}$ and $q_\beta := \tan\frac{\beta}{4}$ are rational. Equation (1) implies $0 < q_\alpha \le \tan\frac{\pi}{12}$ and equation (2) implies $q_\alpha \le q_\beta \le x$, where $x := \tan(\frac{\pi}{8} - \frac{\alpha}{8})$. Now, $\frac{2x}{1-x^2} = \tan(\frac{\pi}{4} - \frac{\alpha}{4}) = \frac{1-q_\alpha}{1+q_\alpha}$ and hence $x = \frac{\sqrt{2+2q_\alpha^2}-1-q_\alpha}{1-q_\alpha}$. Summarizing, we have the following restrictions for $q_\alpha \in \mathbb{Q}$ and $q_\beta \in \mathbb{Q}$:

$$0 < q_\alpha \le \tan\frac{\pi}{12},$$
$$q_\alpha \le q_\beta \le \frac{\sqrt{2+2q_\alpha^2} - 1 - q_\alpha}{1 - q_\alpha}.$$

In Figure 1 we depict the area G corresponding with these inequalities. Every point in G with rational coordinates in G will give rise to a triangle all whose edges and bisectors have rational lengths. Two different points in G with rational coefficients correspond with nonsimilar triangles.

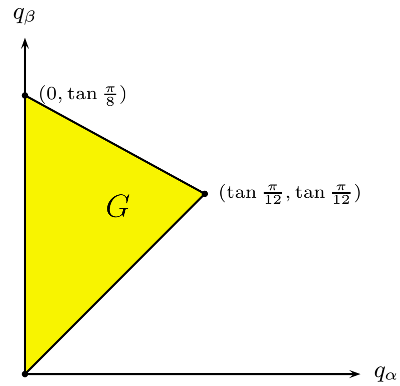


Figure 1

5. The case of trisectors

An infinite but incomplete class of solutions for the trisector case did also occur in the solution booklet of a mathematical competition in the Netherlands (universitaire wiskunde competitie, 1995). Here we present a complete solution based on Theorem 5. Again we may assume that $\alpha \leq \beta \leq \gamma$; so, equations (1) and (2) remain valid here. By Theorem 5, $q_\alpha := \sqrt{3} \cdot \tan\frac{\alpha}{6}$ and $q_\beta := \sqrt{3} \cdot \tan\frac{\beta}{6}$ are rational. As before, one can calculate the inequalities that need to be satisfied by $q_\alpha \in \mathbb{Q}$ and $q_\beta \in \mathbb{Q}$:

$$0 < q_\alpha \leq \sqrt{3} \cdot \tan\frac{\pi}{18},$$
$$q_\alpha \leq q_\beta \leq \frac{\sqrt{12 + 4q_\alpha^2} - 3 - q_\alpha}{1 - q_\alpha}.$$

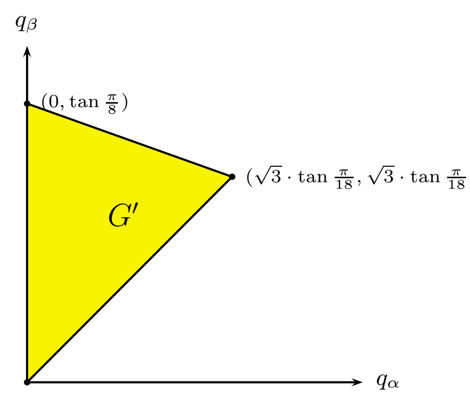


Figure 2

In Figure 2 we depict the area G' corresponding with these inequalities. Every point in G' with rational coordinates will give rise to a triangle all whose edges and trisectors have rational lengths. Two different points in G with rational coefficients correspond with nonsimilar triangles.

References

[1] W. E. Buker and E. P. Starke. Problem E418, *Amer. Math. Monthly*, 47 (1940) 240; solution, 48 (1941) 67–68.

[2] H. S. M. Coxeter. *Introduction to Geometry*. 2nd edition, John Wiley & Sons, New York, 1989.

[3] R. K. Guy. *Unsolved problems in number theory*. Problem books in Mathematics. Springer Verlag, New York, 2004.

[4] N. Jacobson. *Basic Algebra I*. Freeman, New York, 1985.

[5] D. L. Mackay and E. P. Starke. Problem E331, *Amer. Math. Monthly*, 45 (1938) 249; solution, 46 (1939) 172.

Bart De Bruyn: Department of Pure Mathematics and Computer Algebra, Ghent University, Galglaan 2, B-9000 Gent, Belgium

E-mail address: bdb@cage.ugent.be

Forum Geometricorum
Volume 5 (2005) 53–56.

FORUM GEOM
ISSN 1534-1178

A Simple Construction of a Triangle from its Centroid, Incenter, and a Vertex

Eric Danneels

Abstract. We give a simple ruler and compass construction of a triangle given its centroid, incenter, and one vertex. An analysis of the number of solutions is also given.

1. Construction

The ruler and compass construction of a triangle from its centroid, incenter, and one vertex was one of the unresolved cases in [3]. An analysis of this problem, including the number of solutions, was given in [1]. In this note we give a very simple construction of triangle ABC with given centroid G, incenter I, and vertex A. The construction depends on the following propositions. For another slightly different construction, see [2].

Proposition 1. *Given triangle ABC with Nagel point N, let D be the midpoint of BC. The lines ID and AN are parallel.*

Proof. The centroid G divides each of the segments AD and NI in the ratio $AG : GD = NG : GI = 2 : 1$. See Figure 1. □

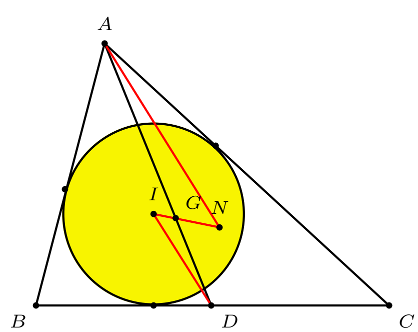


Figure 1

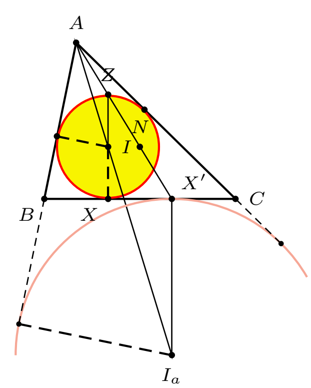


Figure 2

Proposition 2. *Let X be the point of tangency of the incircle with BC. The antipode of X on the circle with diameter ID is a point on AN.*

Publication Date: April 12, 2005. Communicating Editor: Paul Yiu.
The author wishes to thank Paul Yiu for his help in the preparation of this paper.

Proof. This follows from the fact that the antipode of X on the incircle lies on the segment AN. See Figure 2. $\square$

Construction. Given G, I, and A, extend AG to D such that $AG : GD = 2 : 1$. Construct the circle $\mathcal{C}$ with diameter ID, and the line $\mathcal{L}$ through A parallel parallel to ID.

Let Y be an intersection of the circle $\mathcal{C}$ and the line $\mathcal{L}$, and X the antipode of Y on $\mathcal{C}$ such that A is outside the circle $I(X)$. Construct the tangents from A to the circle $I(X)$. Their intersections with the line DX at the remaining vertices B and C of the required triangle. See Figure 3.

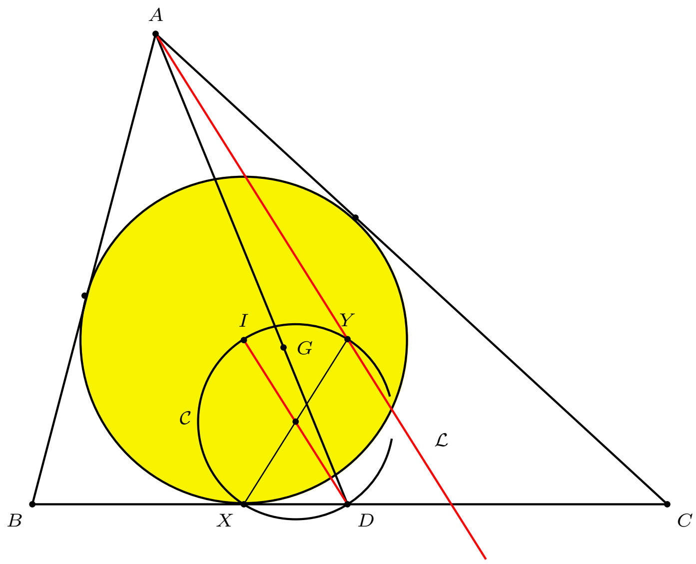


Figure 3

2. Number of solutions

We set up a Cartesian coordinate system such that $A = (0, 2k)$ and $I = (0, -k)$. If $G = (u, v)$, then $D = \frac{1}{2}(3G - A) = \left(\frac{3}{2}u,\ \frac{3}{2}v - k\right)$. The circle $\mathcal{C}$ with diameter ID has equation

$$2(x^2 + y^2) - 3ux - (3v - 4k)y + (2k^2 - 3kv) = 0$$

and the line $\mathcal{L}$ through A parallel to ID has slope $\frac{v}{u}$ and equation

$$vx - uy + 2ku = 0.$$

The line $\mathcal{L}$ and the circle $\mathcal{C}$ intersect at 0, 1, 2 real points according as

$$\Delta := (u^2 + v^2 - 4ku)(u^2 + v^2 + 4ku)$$

is negative, zero, or positive. Since $x^2 + y^2 \pm 4kx = 0$ represent the two circles of radii $2k$ tangent to each other externally and to the y-axis at $(0, 0)$, Δ is negative, zero, or positive according as G lies in the interior, on the boundary, or in the exterior of the union of the two circles.

The intersections of the circle and the line are the points

$$Y_\varepsilon = \left(\frac{3u(u^2+v^2-4kv-\varepsilon\sqrt{\Delta})}{4(u^2+v^2)}, \frac{8k(u^2+v^2)+3v(u^2+v^2-4kv-\varepsilon\sqrt{\Delta})}{4(u^2+v^2)}\right)$$

for $\varepsilon = \pm 1$. Their antipodes on $\mathcal{C}$ are the points

$$X_\varepsilon = \left(\frac{3u(u^2+v^2+4kv+\varepsilon\sqrt{\Delta})}{4(u^2+v^2)}, \frac{-16k(u^2+v^2)+3v(u^2+v^2+4kv+\varepsilon\sqrt{\Delta})}{4(u^2+v^2)}\right).$$

There is a triangle ABC tritangent to the circle $I(X_\varepsilon)$ and with DX_ε as a sideline if and only if the point A lies outside the circle $I(X_\varepsilon)$. Note that $IA = 3k$ and

$$IX_+^2 = \frac{9}{8}(u^2+v^2+\sqrt{\Delta}), \quad IX_-^2 = \frac{9}{8}(u^2+v^2-\sqrt{\Delta}).$$

From these, we make the following conclusions.

(i) If $u^2+v^2-8k^2 \geq \sqrt{\Delta}$, then A lies inside or on $I(X_-)$. In this case, there is no triangle.

(ii) If $-\sqrt{\Delta} \leq u^2+v^2-8k^2 < \sqrt{\Delta}$, then A lies outside $I(X_-)$ but not $I(X_+)$. There is exactly one triangle.

(iii) If $u^2+v^2-8k^2 < -\sqrt{\Delta}$, then A lies outside $I(X_+)$ (and also $I(X_-)$). There are in general two triangles.

It is easy to see that the condition $-\sqrt{\Delta} < u^2+v^2-8k^2 < \sqrt{\Delta}$ is equivalent to $(v-2k)(v+2k) > 0$, *i.e.*, $|v| > 2k$. We also note the following.

(i) When the line D_ε passes through A, the corresponding triangle degenerates. The condition for collinearity leads to

$$u(3u^2+3v^2-4kv \pm \sqrt{\Delta}) = 0.$$

Clearly, $u = 0$ gives the y-axis. The corresponding triangle is isosceles. On the other hand, the condition $3u^2+3v^2-4kv \pm \sqrt{\Delta} = 0$ leads to

$$(u^2+v^2)(u^2+v^2-3kv+2k^2) = 0,$$

i.e., (u,v) lying on the circle tangent to the circles $x^2+y^2 \pm 4kx = 0$ at $\left(\pm\frac{2k}{5}, \frac{6k}{5}\right)$ and the line $y = 2k$ at A.

(ii) If $v > 0$, the circle $I(X_\varepsilon)$, instead of being the incircle, is an excircle of the triangle. If G lies inside the region $ATOT'A$ bounded by the circular segments, one of the excircles is the A-excircle. Outside this region, the excircle is always a B/C-excircle.

From these we obtain the distribution of the position of G, summarized in Table 1 and depicted in Figure 4, for the various numbers of solutions of the construction problem. In Figure 4, the number of triangles is
0 if G in an unshaded region, on a dotted line, or at a solid point other than I,
1 if G is in a yellow region or on a solid red line,
2 if G is in a green region.

Table 1. Number N of non-degenerate triangles according to the location of G relative to A and I

N	Location of centroid $G(u,v)$
0	$(0,0),\ (\pm 2k, 2k)$; $\left(\pm\frac{2k}{5},\ \frac{6k}{5}\right)$; $v = 2k$; $\lvert u\rvert > 2k - \sqrt{4k^2 - v^2},\ -2k \le v < 2k$.
1	$u = 0,\ 0 < \lvert v\rvert < 2k$; $-2k < u < 2k,\ v = -2k$; $u = 2k - \sqrt{4k^2 - v^2},\ 0 < \lvert v\rvert < 2k$; $\lvert v\rvert > 2k$; $u^2 + v^2 - 3kv + 2k^2 = 0$ except $(0, 2k),\ \left(\pm\frac{2k}{5},\ \frac{6k}{5}\right)$.
2	$\lvert u\rvert < 2k - \sqrt{4k^2 - v^2},\ 0 < \lvert v\rvert < 2k$, but $u^2 + v^2 - 3kv + 2k^2 \neq 0$.

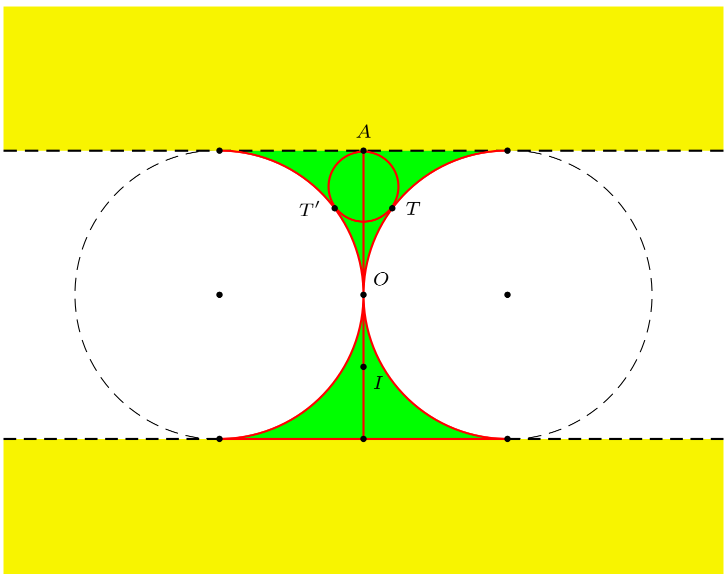


Figure 4

References

[1] J. Anglesio and V. Schindler, Problem 10719, *Amer. Math. Monthly*, 106 (1999) 264; solution, 107 (2000) 952–954.

[2] E. Danneels, Hyacinthos message 11103, March 22, 2005.

[3] W. Wernick, Triangle constructions with three located points, *Math. Mag.*, 55 (1982) 227–230.

Eric Danneels: Hubert d'Ydewallestraat 26, 8730 Beernem, Belgium

E-mail address: `eric.danneels@pandora.be`

Forum Geometricorum
Volume 5 (2005) 57–61.

FORUM GEOM
ISSN 1534-1178

Triangle-Conic Porism

Aad Goddijn and Floor van Lamoen

Abstract. We investigate, for a given triangle, inscribed triangles whose sides are tangent to a given conic.

Consider a triangle $A_1B_1C_1$ inscribed in ABC, and a conic $\mathcal{C}$ inscribed in $A_1B_1C_1$. We ask whether there are other inscribed triangles in ABC and tritangent to $\mathcal{C}$. Restricting to circles, Ton Lecluse wrote about this problem in [6]. See also [5]. He suggested after use of dynamic geometry software that in general there is a second triangle tritangent to $\mathcal{C}$ and inscribed in ABC. In this paper we answer Lecluse's question.

Proposition 1. *Let $A'B'C'$ be a variable triangle of which B' and C' lie on CA and AB respectively. If the sidelines of triangle $A'B'C'$ are tangent to a conic $\mathcal{C}$, then the locus of A' is either a conic or a line.*

Proof. Let XYZ be the points on $\mathcal{C}$ and where $C'A'$, $A'B'$, and $B'C'$ respectively meet $\mathcal{C}$. ZX is the polar (with respect to $\mathcal{C}$) of B', which passes through a fixed point P_B, the pole of CA. Similarly XY passes through a fixed point P_C. The mappings $Y \mapsto X$ and $X \mapsto Z$ are thus involutions on $\mathcal{C}$. Hence $Y \mapsto Z$ is a projectivity. That means that the lines YZ form a pencil of lines or envelope a conic according as $Y \mapsto Z$ is an involution or not. Consequently the poles of these lines, the points A', run through a line ℓ_A or a conic $\mathcal{C}_A$. □

Two degenerate triangles $A'B'C'$, corresponding to the tangents from A, arise as limit cases. Hence, when $Y \mapsto Z$ is an involution, the points U_1 and U_2 of contact of tangents from A to $\mathcal{C}$ are its fixed points, and $\ell_A = U_1U_2$ is the polar of A.

The conics $\mathcal{C}$ and $\mathcal{C}_A$ are tangent to each other in U_1 and U_2. We see that $\mathcal{C}$ and $\mathcal{C}_A$ generate a pencil, of which the pair of common tangents, and the polar of A (as double line) are the degenerate elements. In view of this we may consider the line ℓ_A as a conic $\mathcal{C}_A$ degenerated into a "double" line. We do so in the rest of this paper.

Publication Date: April 26, 2005. Communicating Editor: Paul Yiu.
The authors thank Ton Lecluse and Dick Klingens for their inspiring problem and correspondence.

Proposition 1 shows us that if there is one inscribed triangle tritangent to $\mathcal{C}$, there will be in general another such triangle. This answers Lecluse's question for the general case. But it turns out that the other cases lead to interesting configurations as well.

The number of intersections of $\mathcal{C}_A$ with BC gives the number of inscribed triangles tritangent to $\mathcal{C}$. There may be infinitely many, if $\mathcal{C}_A$ degenerates and contains BC. This implies that $BC = \ell_A$. By symmetry it is necessary that ABC is self-polar with respect to $\mathcal{C}$. Of course this applies also when the above A' runs through ℓ_A in the plane of the triangle bounded by AB, CA and ℓ_A.

There are two possibilities for $\mathcal{C}_A$ and BC to intersect in one "double" point. One is that $\mathcal{C}_A$ is nondegenerate and tangent to BC. In this case, by reasons of continuity, the point of tangency belongs to one triangle $A'B'C'$, and similar conics $\mathcal{C}_B$ and $\mathcal{C}_C$ are tangent to the corresponding side as well. The points of tangency form the cevian triangle of the perspector of $\mathcal{C}$.

This can be seen by considering the point M where U_1U_2 meets BC. The polar of M with respect to $\mathcal{C}$ passes through the pole of U_1U_2, and through the intersections of the polars of B and C, hence the pole of BC. So the polar ℓ_M of M is the A-cevian of the perspector[1] of $\mathcal{C}$. The point where U_1U_2 and ℓ_M meet is the harmonic conjugate of M with respect to U_1 and U_2. This all applies to $\mathcal{C}_A$ as well. In case $\mathcal{C}_A$ is tangent to BC, the point of tangency is the pole of BC, and is thus the trace of the perspector of $\mathcal{C}$.

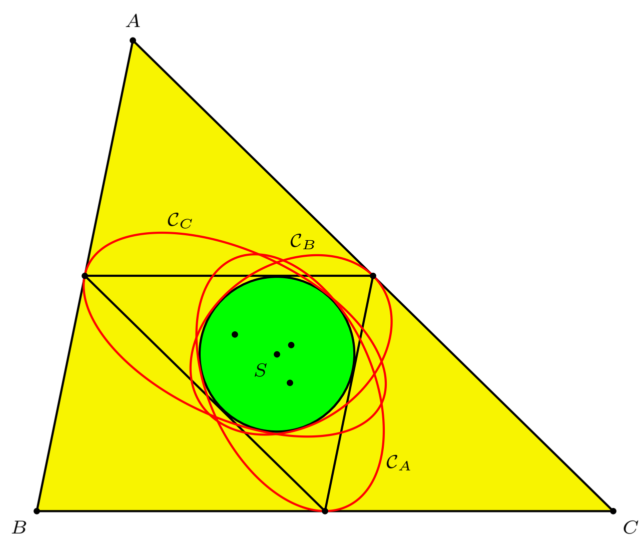


Figure 1

For example, if $\mathcal{C}$ is the incircle of the medial triangle, the conic $\mathcal{C}_A$ is tangent to BC at its midpoint, and contains the points $(s : s-b : b)$, $(s : c : s-c)$,

[1]By Chasles' theorem on polarity [1, 5.61], each triangle is perspective to its polar triangle. The perspector is called the perspector of the conic.

$((a+b+c)(b+c-a) : 2c(c+a-b) : b^2+3c^2-a^2+2ca)$ and $((a+b+c)(b+c-a) : 3b^2+c^2-a^2+2ab : 2b(a+b-c))$. It has center $(s : c+a : a+b)$. See Figure 1.

The other possibility for a double point is when $\mathcal{C}_A$ degenerates into ℓ_A. To investigate this case we prove the following proposition.

Proposition 2. *If $\mathcal{C}_A$ degenerates into a line, the triangle ABC is selfpolar with respect to each conic tangent to the sides of two cevian triangles. The cevian triangle of the trilinear pole of any tangent to such a conic is tritangent to this conic.*

Proof. Let P be a point and $A^PB^PC^P$ its anticevian triangle. ABC is a polar triangle with respect to each conic through $A^PB^PC^P$, as ABC are the diagonal points of the complete quadrilateral $PA^PB^PC^P$. Now consider a second anticevian triangle $A^QB^QC^Q$ of Q. The vertices of $A^PB^PC^P$ and $A^QB^QC^Q$ lie on a conic[2] $\mathcal{K}$. But we also know that triangle PB^PC^P is the anticevian triangle of A^P. So PB^PC^P and $A^QB^QC^Q$ lie on a conic as well, and having 5 common points this must be $\mathcal{K}$. We conclude that ABC is selfpolar with respect to $\mathcal{K}$.

Let R be a point on $\mathcal{K}$. AR intersects $\mathcal{K}$ in a second point R'. Let R_A be the intersection AR and BC, then R and R' are harmonic with respect to A and R_A. But that means that $R' = A^R$ is the $A-$vertex of the anti-cevian triangle of R. Consequently the anti-cevian triangle of R lies on $\mathcal{K}$. Proposition 2 is now proved by duality. $\square$

In the proof B^PC^P is the side of two anticevian triangles inscribed in $\mathcal{K}$ - by duality this means that the vertex of a cevian triangle tangent to $\mathcal{K}$ is a common vertex of two such cevian triangles. In the case of ℓ_A intersecting BC in a double point, clearly the two triangles are cevian triangles with respect to the triangle bounded by AB, AC and ℓ_A. Were they cevian triangles also with respect to ABC, then the four sidelines of these cevian triangles would form the dual of an anticevian triangle, and ABC would be selfpolar with respect to $\mathcal{C}$, and ℓ_A would be BC.

We conclude that two distinct triangles inscribed in ABC and circumscribing $\mathcal{C}$ cannot be cevian triangles.

In the case ABC is selfpolar with respect to $\mathcal{C}$, so that $\mathcal{C}_A$ degenerates into ℓ_A, not each point on ℓ_A belongs to (real) cevian triangles. On the other hand clearly infinitely many points on ℓ_A will lead to two cevian triangles tritangent to $\mathcal{C}$. The perpsectors run through a quartic, the tripoles of the tangents to $\mathcal{C}$.

Theorem 3. *Given a triangle ABC and a conic $\mathcal{C}$, the* triangle-conic poristic *triangles inscribed in ABC and tritangent to $\mathcal{C}$ are as follows.*

(i) *There are be no triangle-conic poristic triangle.*

(ii) *$\mathcal{C}$ is a conic inscribed in a cevian triangle, and ABC is not self-polar with respect to $\mathcal{C}$. In this case the cevian triangle is the only triangle-conic poristic triangle.*

[2] This follows from the dual of the well known theorem that two cevian triangles are circumscribed by and inscribed in a conic.

(iii) ABC *is self-polar with respect to* $\mathcal{C}$. *In this case there are infinitely many triangle-conic poristic triangles.*

(iv) *There are two distinct triangle-conic poristic triangles, which are not cevian triangles.*

Remarks. (1) In case of a conic with respect to which ABC is self-polar, instead of cevian triangles tritangent to $\mathcal{C}$, we should speak of cevian fourlines quadritangent to $\mathcal{C}$.

(2) When we investigate triangles inscribed in a conic and circumscribed to ABC we get similar results as Theorem 3, simply by duality.

In case $\mathcal{C}$ is a conic with respect to which ABC is selfpolar, we see that each tangent to $\mathcal{C}$ belongs to two cevian triangles tritangent to $\mathcal{C}$ and that each point on $\mathcal{C}$ belongs to two anticevian triangles inscribed in $\mathcal{C}$. In this case speak of *triangle-conic porism* and *conic-triangle porism* in extension of the well known Poncelet porism.

As an example, we consider the *nine-point circle triangles*, hence the medial and orthic triangles. We know that these circumscribe a conic $\mathcal{C}_N$, with respect to which ABC is selfpolar. By Proposition 2 we know that the perspectrices of the medial and orthic triangles are tangent to $\mathcal{C}_N$ as well, hence $\mathcal{C}_N$ must be a parabola tangent to the orthic axis. The barycentric equation of this parabola is

$$\frac{x^2}{a^2(b^2-c^2)}+\frac{y^2}{b^2(c^2-a^2)}+\frac{z^2}{c^2(a^2-b^2)}=0.$$

Its focus is X_{115} of [3, 4], its directrix the Brocard axis, and its axis is the Simson line of X_{98}. See Figure 2. The parabola contains the infinite point X_{512} and passes through X_{661}, X_{647} and X_{2519}. The Brianchon point of the parabola with respect to the medial triangle is $X_{670}(\mathrm{medial})$.

The perspectors of the tangent cevian triangles run through the quartic

$$a^2(b^2-c^2)y^2z^2+b^2(c^2-a^2)z^2x^2+c^2(a^2-b^2)x^2y^2=0,$$

which is the isotomic conjugate of the conic

$$a^2(b^2-c^2)x^2+b^2(c^2-a^2)y^2+c^2(a^2-b^2)z^2=0$$

through the vertices of the antimedial triangle, the centroid, and the isotomic conjugates of the incenter and the orthocenter.

This special case leads us to amusing consequences, to which we were pointed by [2]. The sides of every cevian triangle and its perspectrix are tangent to one parabola inscribed in the medial triangle. Consequently the *isotomic conjugates*[3] with respect to to the medial triangle of these are parallel.

In the dual case, we conclude for instance that the isotomic conjugates with respect to the antimedial triangle of the vertices and perspector D of any anticevian triangle are collinear with the centroid G. The line is GD', where D' is the barycentric square of D.

[3]The isotomic conjugate of a line ℓ with respect to a triangle is the line passing through the intercepts of ℓ with the sides reflected through the corresponding midpoints. In [3] this is referred to as *isotomic transversal*.

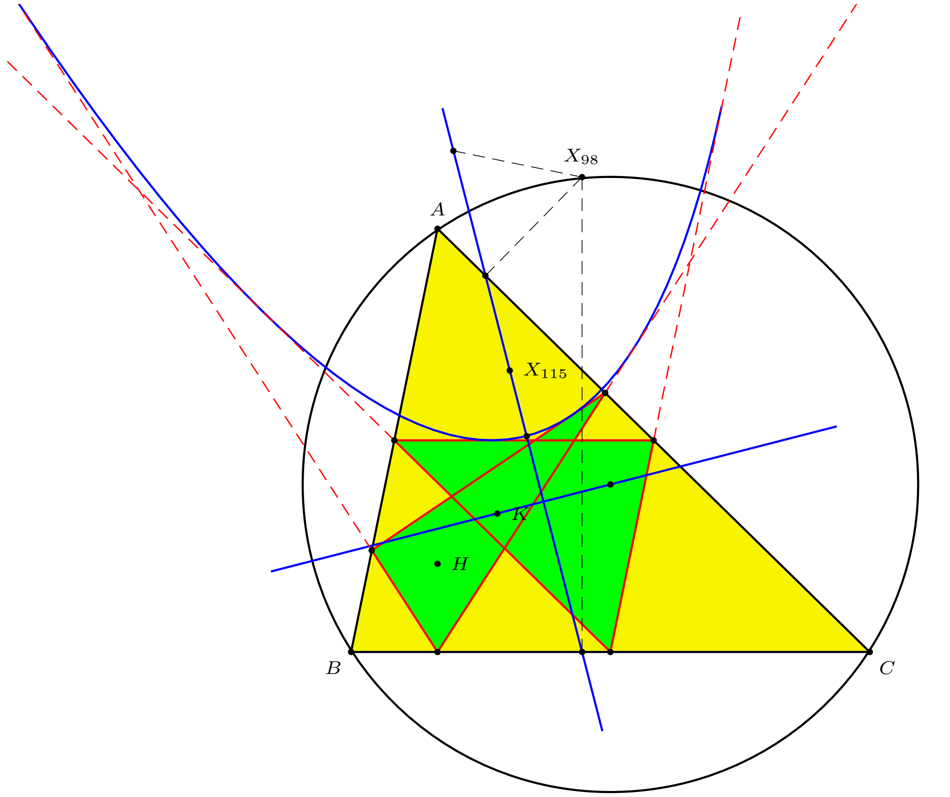


Figure 2.

References

[1] H.S.M. Coxeter, *The Real Projective Plane*, 3rd edition, Springer-Verlag, 1992.

[2] J.-P. Ehrmann and F. M. van Lamoen, A projective generalization of the Droz-Farny line theorem, *Forum Geom.*, 4 (2004) 225–227.

[3] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[5] D. Klingens, Zoek de driehoek, een vervolg, *Euclides*, 80-6 (2005) 334–338.

[6] T. Lecluse, Zoek de driehoek - Avonturen van een programmeur, *Euclides*, 80-5 (2005) 302–303.

Aad Goddijn: Universiteit Utrecht, Freudenthal instituut, Postbus 9432, 3506 GK Utrecht, The Netherlands

E-mail address: `A.Goddijn@fi.uu.nl`

Floor van Lamoen, St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands

E-mail address: `fvanlamoen@planet.nl`

Forum Geometricorum
Volume 5 (2005) 63–64.

FORUM GEOM
ISSN 1534-1178

A Maximal Property of Cyclic Quadrilaterals

Antreas Varverakis

Abstract. We give a very simple proof of the well known fact that among all quadrilaterals with given side lengths, the cyclic one has maximal area.

Among all quadrilaterals $ABCD$ be with given side lengths $AB = a$, $BC = b$, $CD = c$, $DA = d$, it is well known that the one with greatest area is the cyclic quadrilateral. All known proofs of this result make use of Brahmagupta formula. See, for example, [1, p.50]. In this note we give a very simple geometric proof.

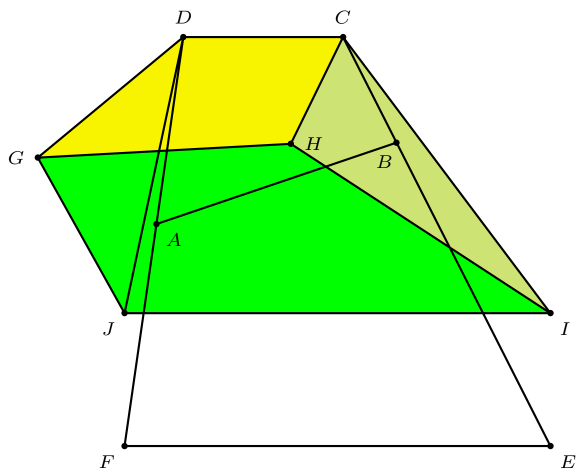


Figure 1

Let $ABCD$ be the cyclic quadrilateral and $GHCD$ an arbitrary one with the same side lengths: $GH = a$, $HC = b$, $CD = c$ and $DG = d$. Construct quadrilaterals $EFAB$ similar to $ABCD$ and $IJGH$ similar to $GHCD$ (in the same order of vertices). Note that
(i) FE is parallel to DC since $ABCD$ is cyclic and DAF, CBE are straight lines;
(ii) JI is also parallel to DC since

Publication Date: May 10, 2005. Communicating Editor: Paul Yiu.

$$\begin{aligned}
\angle CDJ + \angle DJI =&(\angle CDG - \angle JDG) + (\angle GJI - \angle GJD)\\
=&(\angle CDG - \angle JDG) + (\angle CHG - \angle GJD)\\
=&\angle CDG + \angle CHG - (\angle JDG + \angle GJD)\\
=&\angle CDG + \angle CHG - (180^\circ - \angle DGJ)\\
=&\angle CDG + \angle CHG + (\angle DGH + \angle HGJ) - 180^\circ\\
=&\angle CDG + \angle CHG + \angle DGH + \angle HCD - 180^\circ\\
=&180^\circ.
\end{aligned}$$

Since the ratios of similarity of the quadrilaterals are both $\frac{a}{c}$, the areas of $ABEF$ and $GHIJ$ are $\frac{a^2}{c^2}$ times those of $ABCD$ and $GHCD$ respectively. It is enough to prove that

$$\text{area}(DCEF) \geq \text{area}(DCHIJGD).$$

In fact, since $GD \cdot GJ = HC \cdot HI$ and $\angle DGJ = \angle CHI$, it follows that area(DGJ) = area(CHI), and we have

$$\text{area}(DCHIJGD) = \text{area}(DCHG) + \text{area}(GHIJ) = \text{area}(DCIJ).$$

Note that

$$\begin{aligned}
\overrightarrow{CD} \cdot \overrightarrow{DJ} =&\overrightarrow{CD} \cdot (\overrightarrow{DG} + \overrightarrow{GJ})\\
=&\overrightarrow{CD} \cdot \overrightarrow{DG} + \overrightarrow{CD} \cdot \overrightarrow{GJ}\\
=&\overrightarrow{CD} \cdot \overrightarrow{DG} + \frac{c^2}{a^2}\left(\overrightarrow{IJ} \cdot \overrightarrow{GJ}\right)\\
=&\overrightarrow{CD} \cdot \overrightarrow{DG} - \overrightarrow{CH} \cdot \overrightarrow{HG}\\
=&\frac{1}{2}\left(a^2 + b^2 - CG^2\right) - \frac{1}{2}\left(c^2 + d^2 - CG^2\right)\\
=&\frac{1}{2}(a^2 + b^2 - c^2 - d^2)
\end{aligned}$$

is independent of the position of J. This means that the line JF is perpendicular to DC; so is IE for a similar reason. The vector $\overrightarrow{DJ} = \overrightarrow{DG} + \overrightarrow{GJ}$ has a constant projection on $\overrightarrow{CD}$ (the same holds for $\overrightarrow{CI}$). We conclude that trapezium $DCEF$ has the greatest altitude among all these trapezia constructed the same way as $DCIJ$. Since all these trapezia have the same bases, $DCEF$ has the greatest area. This completes the proof that among quadrilaterals of given side lengths, the cyclic one has greatest area.

Reference

[1] N. D. Kazarinoff, *Geometric Inequalities*, Yale University, 1961.

Antreas Varverakis: Department of Mathematics, University of Crete, Crete, Greece
E-mail address: varver@math.uoc.gr

Forum Geometricorum
Volume 5 (2005) 65–74.

FORUM GEOM
ISSN 1534-1178

Some Brocard-like points of a triangle

Sadi Abu-Saymeh and Mowaffaq Hajja

Abstract. In this note, we prove that for every triangle ABC, there exists a unique interior point M the cevians AA', BB', and CC' through which have the property that $\angle AC'B' = \angle BA'C' = \angle CB'A'$, and a unique interior point M' the cevians AA', BB', and CC' through which have the property that $\angle AB'C' = \angle BC'A' = \angle CA'B'$. We study some properties of these Brocard-like points, and characterize those centers for which the angles $AC'B'$, $BA'C'$, and $CB'A'$ are linear forms in the angles A, B, and C of ABC.

1. Notations

Let ABC be a non-degenerate triangle, with angles A, B, and C. To every point P inside ABC, we associate, as shown in Figure 1, the following angles and lengths.

$$\begin{array}{lll}
\xi = \angle BAA', & \eta = \angle CBB', & \zeta = \angle ACC'; \\
\xi' = \angle CAA', & \eta' = \angle ABB', & \zeta' = \angle BCC'; \\
\alpha = \angle AC'B', & \beta = \angle BA'C', & \gamma = \angle CB'A'; \\
\alpha' = \angle AB'C', & \beta' = \angle BC'A', & \gamma' = \angle CA'B'; \\
x = BA', & y = CB', & z = AC'; \\
x' = A'C, & y' = B'A, & z' = C'B.
\end{array}$$

The well-known Brocard or Crelle-Brocard points are defined by the requirements $\xi = \eta = \zeta$ and $\xi' = \eta' = \zeta'$; see [11]. The angles ω and ω' that satisfy $\xi = \eta = \zeta = \omega$ and $\xi' = \eta' = \zeta' = \omega'$ are equal, and their common value is called the Brocard angle. The points known as Yff's analogues of the Brocard points are defined by the similar requirements $x = y = z$ and $x' = y' = z'$. These were introduced by Peter Yff in [12], and were so named by Clark Kimberling in a talk that later appeared as [8]. For simplicity, we shall refer to these points as *the Yff-Brocard points*.

2. The cevian Brocard points

In this note, we show that each of the requirements $\alpha = \beta = \gamma$ and $\alpha' = \beta' = \gamma'$ defines a unique interior point, and that the angles Ω and Ω' that satisfy $\alpha = \beta = \gamma = \Omega$ and $\alpha' = \beta' = \gamma' = \Omega'$ are equal. We shall call the resulting two points the first and second cevian Brocard points respectively, and the common value of Ω and Ω', the cevian Brocard angle of ABC.

Publication Date: May 24, 2005. Communicating Editor: Paul Yiu.
This work is supported by a research grant from Yarmouk University.

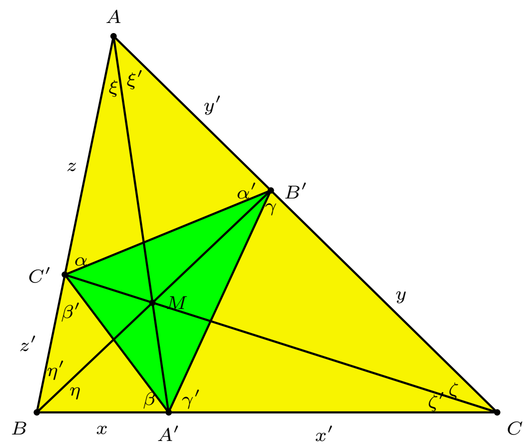


Figure 1.

We shall freely use the trigonometric forms

$$\sin\xi\sin\eta\sin\zeta = \sin\xi'\sin\eta'\sin\zeta' = \sin(A-\xi)\sin(B-\eta)\sin(C-\zeta)$$
$$\sin\alpha\sin\beta\sin\gamma = \sin\alpha'\ \sin\beta'\sin\gamma' = \sin(A+\alpha)\sin(B+\beta)\ \sin(C+\gamma)$$

of the cevian concurrence condition. We shall also freely use a theorem of Seebach stating that for any triangles ABC and UVW, there exists inside ABC a unique point P the cevians AA', BB', and CC' through which have the property that $(A', B', C') = (U, V, W)$, where A', B', and C' are the angles of $A'B'C'$ and U, V, and W are the angles of UVW; see [10] and [7].

Theorem 1. *For every triangle ABC, there exists a unique interior point M the cevians AA', BB', and CC' through which have the property that*

$$\angle AC'B' = \angle BA'C' = \angle CB'A' \ (= \Omega,\ \textit{say}), \tag{1}$$

and a unique interior point M' the cevians AA', BB', and CC' through which have the property that

$$\angle AB'C' = \angle BC'A' = \angle CA'B' \ (= \Omega',\ \textit{say}). \tag{2}$$

Also, the angles Ω and Ω' are equal and acute. See Figures 2A and 2B.

Proof. It is obvious that (1) is equivalent to the condition $(A', B', C') = (C, A, B)$, where A', B', and C' are the angles of the cevian triangle $A'B'C'$. Similarly, (2) is equivalent to the condition $(A', B', C') = (B, C, A)$. According to Seebach's theorem, the existence and uniqueness of M and M' follow by taking $(U, V, W) = (C, A, B)$ and $(U, V, W) = (B, C, A)$.

To prove that Ω is acute, observe that if Ω is obtuse, then the angles Ω, $A + \Omega$, $B + \Omega$, and $C + \Omega$ would all lie in the interval $[\pi/2, \pi]$ where the sine function is positive and decreasing. This would imply that

$$\sin^3\Omega \quad > \quad \sin(A+\Omega)\ \sin(B+\Omega)\ \sin(C+\Omega),$$

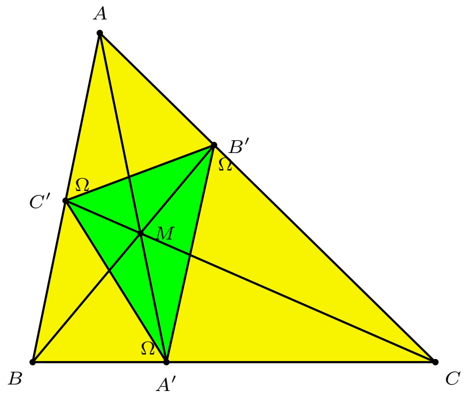


Figure 2A

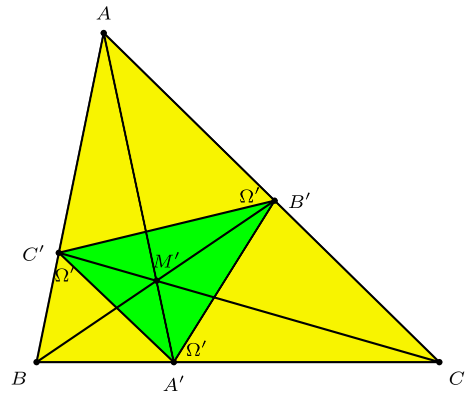


Figure 2B

contradicting the cevian concurrence condition

$$\sin^3\Omega \quad = \quad \sin(A+\Omega)\ \sin(B+\Omega)\ \sin(C+\Omega). \tag{3}$$

Thus Ω, and similarly Ω', are acute.

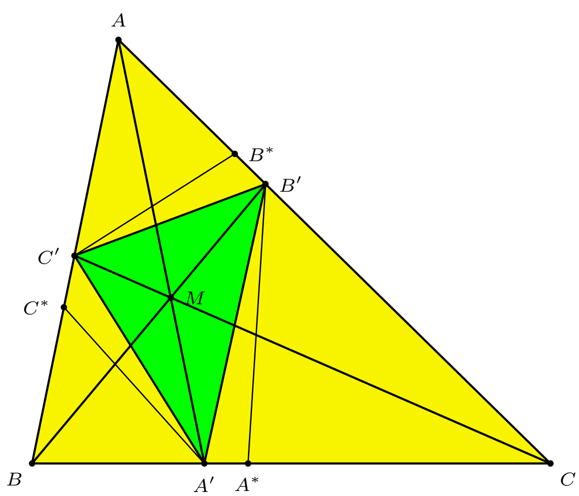


Figure 3.

It remains to prove that $\Omega' = \Omega$. Let $A'B'C'$ be the cevian triangle of M, and suppose that $\Omega' < \Omega$. Then there exist, as shown in Figure 3, points B^*, C^*, and A^* on the line segments $A'C$, $B'A$, and $C'B$, respectively, such that

$$\angle AC'B^* = \angle BA'C^* = \angle CB'A^* = \Omega'.$$

Then

$$\begin{aligned} 1 &= \frac{AB'}{B'C}\cdot\frac{CA'}{A'B}\cdot\frac{BC'}{C'A} > \frac{AB^*}{B'C}\cdot\frac{CA^*}{A'B}\cdot\frac{BC^*}{C'A} = \frac{AB^*}{AC'}\cdot\frac{CA^*}{CB'}\cdot\frac{BC^*}{BA'} \\ &= \frac{\sin\Omega'}{\sin(A+\Omega')}\cdot\frac{\sin\Omega'}{\sin(C+\Omega')}\cdot\frac{\sin\Omega'}{\sin(B+\Omega')}. \end{aligned}$$

This contradicts the cevian concurrence condition

$$\sin^3\Omega' \quad = \quad \sin(A+\Omega')\sin(B+\Omega')\sin(C+\Omega')$$

for M'. □

The points M and M' in Theorem 1 will be called the *first* and *second cevian Brocard points* and the common value of Ω and Ω' the *cevian Brocard angle*.

3. An alternative proof of Theorem 1

An alternative proof of Theorem 1 can be obtained by noting that the existence and uniqueness of M are equivalent to the existence and uniqueness of a positive solution $\Omega < \min\{\pi - A, \pi - B, \pi - C\}$ of (3). Letting $u = \sin\Omega$, $U = \cos\Omega$, and $T = U/u = \cot\Omega$, and setting

$$\begin{aligned} c_0 &= \sin A \sin B \sin C, \\ c_1 &= \cos A \sin B \sin C + \sin A \cos B \sin C + \sin A \sin B \cos C, \\ c_2 &= \cos A \cos B \sin C + \cos A \sin B \cos C + \sin A \cos B \cos C, \\ c_3 &= \cos A \cos B \cos C, \end{aligned}$$

(3) simplifies into

$$u^3 = c_0U^3 + c_1U^2u + c_2Uu^2 + c_3u^3. \tag{4}$$

Using the formulas

$$c_2 = c_0 \quad \text{and} \quad c_1 = c_3 + 1 \tag{5}$$

taken from [5, Formulas 674 and 675, page 165], this further simplifies into

$$\begin{aligned} u^3 &= c_0U^3 + (c_3+1)U^2u + c_0Uu^2 + c_3u^3 \\ &= c_0U(U^2+u^2) + c_3u(U^2+u^2) + U^2u \\ &= c_0U + c_3u + U^2u \\ &= u(c_0T + c_3 + U^2). \end{aligned}$$

Since $u^2 = \frac{1}{1+T^2}$ and $U^2 = \frac{T^2}{1+T^2}$, this in turn reduces to $f(T) = 0$, where

$$f(X) = c_0X^3 + (c_3+1)X^2 + c_0X + (c_3 - 1). \tag{6}$$

Arguing as in the proof of Theorem 1 that Ω must be acute, we restrict our search to the interval $\Omega \in [0, \pi/2]$, i.e., to $T \in [0, \infty)$. On this interval, f is clearly increasing. Also, $f(0) < 0$ and $f(\infty) > 0$. Therefore f has a unique zero in $[0, \infty)$. This proves the existence and uniqueness of M. A similar treatment of M' leads to the same f, proving that M' exists and is unique, and that $\Omega = \Omega'$.

This alternative proof of Theorem 1 has the advantage of exhibiting the defining polynomial of $\cot\Omega$, which is needed in proving Theorems 2 and 3.

4. The cevian Brocard angle

Theorem 2. *Let Ω be the cevian Brocard angle of triangle ABC.*
(i) $\cot\Omega$ *satisfies the polynomial f given in* (6), *where $c_0 = \sin A\sin B\sin C$ and $c_3 = \cos A\cos B\cos C$.*
(ii) $\Omega \le \pi/3$ *for all triangles.*
(iii) Ω *takes all values in* $(0, \pi/3]$.

Proof. (i) follows from the alternative proof of Theorem 1 given in the preceding section.

To prove (ii), it suffices to prove that $f(1/\sqrt{3}) \le 0$ for all triangles ABC. Let

$$G = f\left(\frac{1}{\sqrt{3}}\right) = \frac{4\sqrt{3}}{9}\sin A\sin B\sin C + \frac{4}{3}\cos A\cos B\cos C - \frac{2}{3}.$$

Then $G = 0$ if ABC is equilateral, and hence it is enough to prove that G attains its maximum at such a triangle. To see this, take a non-equilateral triangle ABC. Then we may assume that $A > B$ and $C < \pi/2$. If we replace ABC by the triangle whose angles are $(A+B)/2$, $(A+B)/2$, and C, then G increases. This follows from

$$2\sin A\sin B = \cos(A-B) - \cos(A+B) < 1 - \cos(A+B) = 2\sin^2\frac{A+B}{2},$$
$$2\cos A\cos B = \cos(A-B) + \cos(A+B) < 1 + \cos(A+B) = 2\cos^2\frac{A+B}{2}.$$

Thus G attains its maximal value, 0, at equilateral triangles, and hence $G \le 0$ for all triangles, as desired.

To prove (iii), we let $S = \tan\Omega = 1/T$ and we see that S is a zero of the polynomial $F(X) = c_0 + (c_3+1)X + c_0X^2 + (c_3-1)X^3$. The non-negative zero of F when ABC is degenerate, i.e., when $c_0 = 0$, is 0. By continuity of the zeros of polynomials, we conclude that $\tan\Omega$ can be made arbitrarily close to 0 by taking a triangle whose c_0 is close enough to 0. Note that $c_3 - 1$ is bounded away from zero since $c_3 \le 3\sqrt{3}/8$ for all triangles. □

Remarks. (1) Unlike the Brocard angle ω, the cevian Brocard angle Ω is not necessarily Euclidean constructible. To see this, take the triangle ABC with $A = \pi/2$, and $B = C = \pi/4$. Then $c_3 = 0$, $c_0 = 1/2$, and $2f(T) = T^3 + 2T^2 + T - 2$. This is irreducible over $\mathbb{Z}$ since none of ±1 and ±2 is a zero of f, and therefore it is the minimal polynomial of $\cot\ \Omega$. Since it is of degree 3, it follows that $\cot\ \Omega$, and hence the angle Ω, is not constructible.

(2) By the cevian concurrence condition, the Brocard angle ω is defined by

$$\sin^3\omega \quad = \quad \sin(A-\omega)\ \sin(B-\omega)\ \sin(C-\omega). \tag{7}$$

Letting $v = \sin\omega$, $V = \cos\omega$ and $t = \cot\omega$ as before, we obtain

$$v^3 \quad = \quad c_0V^3 - c_1V^2v + c_2Vv^2 - c_3v^3. \tag{8}$$

This reduces to the very simple form $g(t) = 0$, where

$$g(X) \quad = \quad c_0 X - c_3 - 1, \tag{9}$$

showing that

$$t = \cot \omega = \frac{1 + c_3}{c_0} = \frac{c_1}{c_0} = \cot A + \cot B + \cot C, \tag{10}$$

as is well known, and exhibiting the trivial constructibility of ω. This heavy contrast with the non-constructiblity of Ω is rather curious in view of the great formal similarity between (3) and (4) on the one hand and (7) and (8) on the other.

The next theorem shows that a triangle is completely determined, up to similarity, by its Brocard and cevian Brocard angles. This implies, in particular, that Ω and ω are independent of each other, since neither of them is sufficient for determining the shape of the triangle.

Theorem 3. *If two triangles have equal Brocard angles and equal cevian Brocard angles, then they are similar.*

Proof. Let ω and Ω be the Brocard and cevian Brocard angles of triangle ABC, and let $t = \cot \omega$ and $T = \cot \Omega$. From (10) it follows that $t = c_1/c_0$ and therefore $c_1 = tc_0$. Substituting this in (6), we see that $c_0(T + t)(T^2 + 1) = 2$, and therefore

$$c_0 = \frac{2}{(T + t)(T^2 + 1)}, \quad \text{and} \quad c_1 = \frac{2t}{(T + t)(T^2 + 1)}.$$

Letting s_1, s_2, and s_3 be the elementary symmetric polynomials in $\cot A$, $\cot B$, and $\cot C$, we see that

$$\begin{aligned} s_1 &= \cot A + \cot B + \cot C = t, \\ s_2 &= \cot A \cot B + \cot B \cot C + \cot C \cot A = \frac{c_2}{c_0} = 1, \\ s_3 &= \cot A \cot B \cot C = \frac{c_3}{c_1} = \frac{c_1 - 1}{c_1} = 1 - \frac{(T + t)(T^2 + 1)}{2t}. \end{aligned}$$

Since the angles of ABC are completely determined by their cotangents, which in turn are nothing but the zeros of $X^3 - s_1 X^2 + s_2 X - s_3$, it follows that the angles of ABC are determined by t and T, as claimed. $\square$

5. Some properties of the cevian Brocard points

It is easy to see that the first and second Brocard points coincide if and only if the triangle is equilateral. The same holds for the cevian Brocard points. The next theorem deals with the cases when a Brocard point and a cevian Brocard point coincide. We use the following simple theorem.

Theorem 4. *If the cevians AA', BB', and CC' through a point P inside triangle ABC have the property that two of the quadrilaterals $AC'PB'$, $BA'PC'$, $CB'PA'$, $ABA'B'$, $BCB'C'$, and $CAC'A'$ are cyclic, then P is the orthocenter of ABC. If, in addition, P is a Brocard point, then ABC is equilateral.*

Proof. The first part is nothing but [4, Theorem 4] and is easy to prove. The second part follows from $\omega = \pi/2 - A = \pi/2 - B = \pi/2 - C$. $\square$

Theorem 5. *If any of the Brocard points L and L' of triangle ABC coincides with any of its cevian Brocard points M and M', then ABC is equilateral.*

Proof. Let AA', BB', and CC' be the cevians through L, and let ω and Ω be the Brocard and cevian Brocard angles of ABC; see Figure 4A. By the exterior angle theorem, $\angle ALB' = \omega + (B - \omega) = B$. Similarly, $\angle BLC' = C$ and $\angle CLA' = A$.

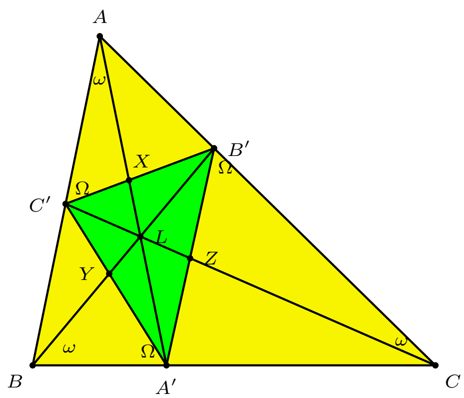


Figure 4A

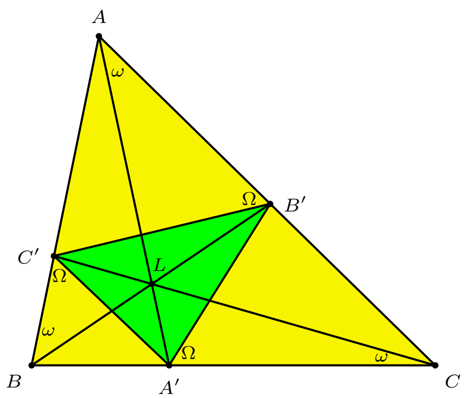


Figure 4B

Suppose that $L = M$. Then $(A', B', C') = (C, A, B)$. Referring to Figure 4A, let X, Y, and Z be the points where AA', BB', and CC' meet $B'C'$, $C'A'$, and $A'B'$, respectively. It follows from $\angle ALB' = B = C'$ and its iterates that the quadrilaterals $XC'YL$, $YA'ZL$, and $ZB'XL$ are cyclic. By Theorem 4, L is the orthocenter of $A'B'C'$. Therefore $\omega + \Omega = \pi/2$. Since $\omega \leq \pi/6$ and $\Omega \leq \pi/3$, it follows that $\omega = \pi/6$ and $\Omega = \pi/3$. Thus the Brocard and cevian Brocard angles of ABC coincide with those for an equilateral triangle. By Theorem 3, ABC is equilateral.

Suppose next that $L = M'$. Referring to Figure 4B, we see that $\angle AB'C' = \angle ACC' + \angle B'C'C$, and therefore $\angle B'C'C = \Omega - \omega$. Similarly $\angle C'A'A = \angle A'B'B = \Omega - \omega$. Therefore L is the second Brocard point of $A'B'C'$. Since $(A', B', C') = (B, C, A)$, it follows that ABC and $A'B'C'$ have the same Brocard angles. Therefore $\angle BAA' = \angle BB'A'$ and $ABA'B'$ is cyclic. The same holds for the quadrilaterals $BCB'C'$ and $CAC'A'$. By Theorem 4, ABC is equilateral. $\square$

The following theorem answers questions that are raised naturally in the proof of Theorem 5. It also restates Theorem 5 in terms of the Brocard points without reference to the cevian Brocard points.

Theorem 6. *Let L be the first Brocard point of ABC, and let AA', BB', and CC' be the cevians through L. Then L coincides with one of the two Brocard points N and N' of $A'B'C'$ if and only if ABC is equilateral. The same holds for the second Brocard point L'.*

Proof. Let the angles of $A'B'C'$ be denoted by A', B', and C'. The proof of Theorem 5 shows that the condition $L = N'$ is equivalent to $L = M'$, which in turn implies that ABC is equilateral. This leaves us with the case $L = N$. In this case, let ω and μ be the Brocard angles of ABC and $A'B'C'$, respectively, as shown in Figure 5. The exterior angle theorem shows that

$$A = \pi - \angle AC'B' - \angle AB'C' = \pi - (\mu + B - \omega) - (\omega + C' - \mu) = \pi - B - C'.$$

Thus $C = C'$. Similarly, $A = A'$ and $B = B'$. Therefore $\mu = \omega$, and the quadrilaterals $AC'LB'$ and $BA'LC'$ are cyclic. By Theorem 4, ABC is equilateral. $\square$

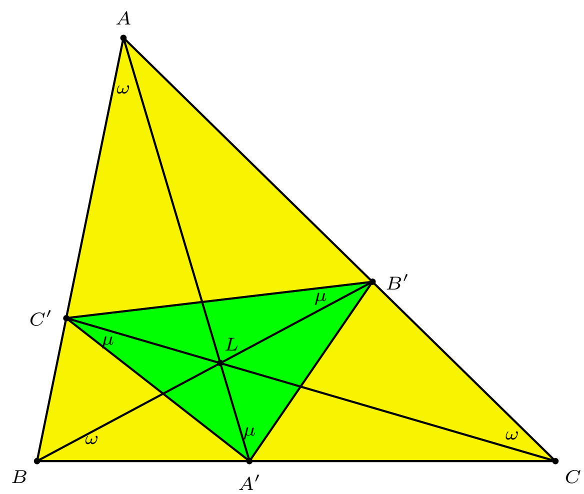


Figure 5

Remark. (3) It would be interesting to investigate whether the many inequalities involving the Brocard angle, such as Yff's inequality [1], have analogues for the cevian Brocard angles, and whether there are inequalities that involve both the Brocard and cevian Brocard angles. Similar questions can be asked about other properties of the Brocard points. For inequalities involving the Brocard angle, we refer the reader to [2] and [9, pp.329-333] and the references therein.

6. A characterization of some common triangle centers

We close with a theorem that complements Theorems 1 and 2 of [3].

Theorem 7. *The triangle centers for which the angles α, β, γ are linear forms in A, B, C are the centroid, the orthocenter, and the Gergonne point.*

Proof. Arguing as in Theorems 1 and 2 of [3], we see that α, β, γ are of the form

$$\alpha = \frac{\pi - A}{2} + t(B - C), \quad \beta = \frac{\pi - B}{2} + t(C - A), \quad \gamma = \frac{\pi - C}{2} + t(A - B).$$

In particular, $\alpha + \beta + \gamma = \pi$, and therefore

$$4 \sin\alpha \sin\beta \sin\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma;$$

see [5, Formula 681, p. 166]. Thus the Ceva's concurrence relation takes the form

$$\begin{aligned}&\sin\left(A-2t(B-C)\right)+\sin\left(B-2t(C-A)\right)+\sin\left(C-2t(A-B)\right)\\=&\sin\left(A+2t(B-C)\right)+\sin\left(B+2t(C-A)\right)+\sin\left(C+2t(A-B)\right),\end{aligned}$$

which reduces to

$$\cos A\sin(2t(B-C))+\cos B\sin(2t(C-A))+\cos C\sin(2t(A-B))=0.$$

Following word by word the way equation (5) of [3] was treated, we conclude that $t=-1/2$, $t=0$, or $t=1/2$.

If $t=0$, then $\alpha=(\pi-A)/2$, and therefore $\alpha=\alpha'$ and $AB'=AC'$. Thus A', B', and C' are the points of contact of the incircle, and the point of intersection of AA', BB', and CC' is the Gergonne point.

If $t=1/2$, then $(\alpha,\beta,\gamma)=(B,C,A)$, and $(A',B',C')=(A,B,C)$. This clearly corresponds to the centroid.

If $t=-1/2$, then $(\alpha,\beta,\gamma)=(C,A,B)$, and $(A',B',C')=(\pi-A,\pi-B,\pi-C)$. This clearly corresponds to the orthocenter. □

Remarks. (4) In establishing the parts pertaining to the centroid and the orthocenter in Theorem 7, we have used the uniqueness component of Seebach's theorem. Alternative proofs that do not use Seebach's theorem follow from [4, Theorems 4 and 7].

(5) In view of the proof of Theorem 7, it is worth mentioning that the proof of Theorem 2 of [3] can be simplified by noting that $\xi+\eta+\zeta=\pi/2$ and using the identity

$$1+4\sin\xi\sin\eta\sin\zeta=\cos 2\xi+\cos 2\eta+\cos 2\zeta$$

given in [5, Formula 678, p. 166].

(6) It is clear that the first and second cevian Brocard points of triangle ABC can be equivalently defined as the points whose cevian triangles $A'B'C'$ have the properties that $(A',B',C')=(C,A,B)$ and $(A',B',C')=(B,C,A)$, respectively. The point corresponding to the requirement that $(A',B',C')=(A,B,C)$ is the centroid; see [6] and [4, Theorem 7]. It would be interesting to explore the point defined by the condition $(A',B',C')=(A,C,B)$.

References

[1] F. F. Abi-Khuzam, Proof of Yff's conjecture on the Brocard angle of a triangle, *Elem. Math.*, 29 (1974) 141–142.

[2] F. F. Abi-Khuzam and A. B. Boghossian, Some recent geometric inequalities, *Amer. Math. Monthly*, 96 (1989) 576–589.

[3] S. Abu-Saymeh and M. Hajja, Triangle centers with linear intercepts and linear subangles, *Forum Geom.*, 5 (2005) 33–36.

[4] S. Abu-Saymeh and M. Hajja, In search of more triangle centres, to appear in *Internat. J. Math. Ed. Sci. Tech.*

[5] G. S. Carr, *Formulas and Theorems in Pure Mathematics*, 2nd edition, Chelsea, New York, 1970.

[6] M. Hajja, Problem 1711, *Math. Mag.*, 78 (2005) 68.

[7] M. Hajja, The arbitrariness of the cevian triangle, to appear in *Amer. Math. Monthly*.

[8] C. Kimberling, Central points and central lines in the plane of a triangle, *Math. Mag.*, 67 (1994) 163–187.
[9] D. S. Mitrinović, J. E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, 1989.
[10] K. Seebach, Ceva-Dreiecke, *Elem. Math.*, 42 (1987) 132–139.
[11] P. Yff, On the Brocard points of a triangle, *Amer. Math. Monthly*, 67 (1960) 520–525.
[12] P. Yff, An analogue of the Brocard points, *Amer. Math. Monthly*, 70 (1963) 495–501.

Sadi Abu-Saymeh: Department of Mathematics, Yarmouk University, Irbid, Jordan
E-mail address: `sade@yu.edu.jo`

Mowaffaq Hajja: Department of Mathematics, Yarmouk University, Irbid, Jordan
E-mail address: `mhajja@yu.edu.jo`

Forum Geometricorum
Volume 5 (2005) 75–96.

FORUM GEOM
ISSN 1534-1178

Elegant Geometric Constructions

Paul Yiu

Dedicated to Professor M. K. Siu

Abstract. With the availability of computer software on dynamic geometry, beautiful and accurate geometric diagrams can be drawn, edited, and organized efficiently on computer screens. This new technological capability stimulates the desire to strive for elegance in actual geometric constructions. The present paper advocates a closer examination of the geometric meaning of the algebraic expressions in the analysis of a construction problem to actually effect a construction as elegantly and efficiently as possible on the computer screen. We present a fantasia of euclidean constructions the analysis of which make use of elementary algebra and very basic knowledge of euclidean geometry, and focus on incorporating simple algebraic expressions into actual constructions using the Geometer's Sketchpad®.

After a half century of curriculum reforms, it is fair to say that mathematicians and educators have come full circle in recognizing the relevance of Euclidean geometry in the teaching and learning of mathematics. For example, in [15], J. E. McClure reasoned that "Euclidean geometry is the only mathematical subject that is really in a position to provide the grounds for its own axiomatic procedures". See also [19]. Apart from its traditional role as the training ground for logical reasoning, Euclidean geometry, with its construction problems, provides a stimulating milieu of learning mathematics *constructivistically*. One century ago, D. E. Smith [17, p.95] explained that the teaching of constructions using ruler and compass serves several purposes: "it excites [students'] interest, it guards against the slovenly figures that so often lead them to erroneous conclusions, it has a genuine value for the future artisan, and its shows that geometry is something besides mere theory". Around the same time, the British Mathematical Association [16] recommended teaching school geometry as two parallel courses of *Theorems* and *Constructions*. "The course of constructions should be regarded as a *practical*

Publication Date: June 18, 2005. Quest Editor: Ngai Ying Wong.

This paper also appears in N. Y. Wong et al (ed.), *Revisiting Mathematics Education in Hong Kong for the New Millennium*, pp.173–203, Hong Kong Association for Mathematics Education, 2005.

course, the constructions being accurately made with instruments, and no construction, or proof of a construction, should be deemed invalid by reason of its being different from that given in Euclid, or by reason of its being based on theorems which Euclid placed after it".

A good picture is worth more than a thousand words. This is especially true for students and teachers of geometry. With good illustrations, concepts and problems in geometry become transparent and more understandable. However, the difficulty of drawing good blackboard geometric sketches is well appreciated by every teacher of mathematics. It is also true that many interesting problems on constructions with ruler and compass are genuinely difficult and demand great insights for solution, as in the case of geometrical proofs. Like handling difficult problems in synthetic geometry with analytic geometry, one analyzes construction problems by the use of algebra. It is well known that historically analysis of such ancient construction problems as the trisection of an angle and the duplication of the cube gave rise to the modern algebraic concept of field extension. A geometric construction can be effected with ruler and compass if and only if the corresponding algebraic problem is reducible to a sequence of linear and quadratic equations with constructible coefficients. For all the strength and power of such algebraic analysis of geometric problems, it is often impractical to carry out detailed constructions with paper and pencil, so much so that in many cases one is forced to settle for mere constructibility. For example, Howard Eves, in his solution [6] of the problem of construction of a triangle given the lengths of a side and the median and angle bisector on the same side, made the following remark after proving constructibility.

> The devotee of the game of Euclidean constructions is not really interested in the actual mechanical construction of the sought triangle, but merely in the assurance that the construction is possible. To use a phrase of Jacob Steiner, the devotee performs his construction "simply by means of the tongue" rather than with actual instruments on paper.

Now, the availability in recent years of computer software on dynamic geometry has brought about a change of attitude. Beautiful and accurate geometric diagrams can be drawn, edited, and organized efficiently on computer screens. This new technological capability stimulates the desire to strive for elegance in actual geometric constructions. The present paper advocates a closer examination of the geometric meaning of the algebraic expressions in the analysis of a construction problem to actually effect a construction as elegantly and efficiently as possible on the computer screen. [1] We present a fantasia of euclidean constructions the analysis of which make use of elementary algebra and very basic knowledge of euclidean geometry. [2] We focus on incorporating simple algebraic expressions into actual constructions using the Geometer's Sketchpad®. The tremendous improvement

[1]See §6.1 for an explicit construction of the triangle above with a given side, median, and angle bisector.

[2]The Geometer's Sketchpad® files for the diagrams in this paper are available from the author's website http://www.math.fau.edu/yiu/Geometry.html.

on the economy of time and effort is hard to exaggerate. The most remarkable feature of the Geometer's Sketchpad® is the capability of customizing a tool folder to make constructions as efficiently as one would like. Common, basic constructions need only be performed once, and saved as tools for future use. We shall use the Geometer's Sketchpad® simply as ruler and compass, assuming a tool folder containing at least the following tools[3] for ready use:
(i) basic shapes such as equilateral triangle and square,
(ii) tangents to a circle from a given point,
(iii) circumcircle and incircle of a triangle.

Sitting in front of the computer screen trying to perform geometric constructions is a most ideal constructivistic learning environment: a student is to bring his geometric knowledge and algebraic skill to bear on natural, concrete but challenging problems, experimenting with various geometric interpretations of concrete algebraic expressions. Such analysis and explicit constructions provide a fruitful alternative to the traditional emphasis of the deductive method in the learning and teaching of geometry.

1. Some examples

We present a few examples of constructions whose elegance is suggested by an analysis a little more detailed than is necessary for constructibility or routine constructions. A number of constructions in this paper are based on diagrams in the interesting book [9]. We adopt the following notation for circles:
(i) $A(r)$ denotes the circle with center A, radius r;
(ii) $A(B)$ denotes the circle with center A, passing through the point B, and
(iii) (A) denotes a circle with center A and unspecified radius, but unambiguous in context.

1.1. Construct a regular octagon by cutting corners from a square.

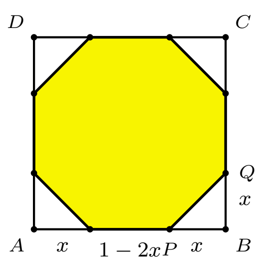


Figure 1A

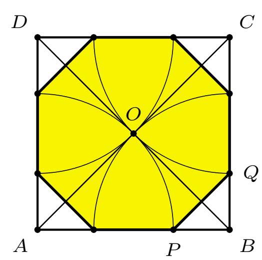


Figure 1B

Suppose an isosceles right triangle of (shorter) side x is to be cut from each corner of a unit square to make a regular octagon. See Figure 1A. A simple calculation shows that $x = 1 - \frac{\sqrt{2}}{2}$. This means $AP = 1 - x = \frac{\sqrt{2}}{2}$. The point P, and the

[3] A construction appearing in sans serif is assumed to be one readily performable with a customized tool.

other vertices, can be easily constructed by intersecting the sides of the square with quadrants of circles with centers at the vertices of the square and passing through the center O. See Figure 1B.

1.2. The centers A and B of two circles lie on the other circle. Construct a circle tangent to the line AB, to the circle (A) internally, and to the circle (B) externally.

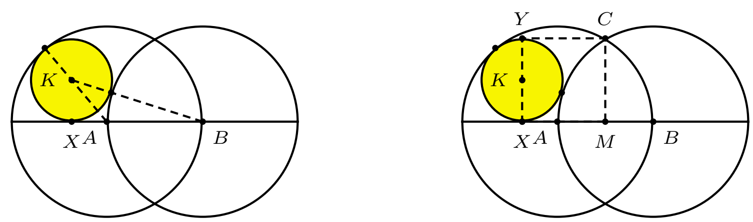


Figure 2A Figure 2B

Suppose $AB = a$. Let r = radius of the required circle (K), and $x = AX$, where X is the projection of the center K on the line AB. We have

$$(a+r)^2 = r^2 + (a+x)^2, \qquad (a-r)^2 = r^2 + x^2.$$

Subtraction gives $4ar = a^2 + 2ax$ or $x + \frac{a}{2} = 2r$. This means that in Figure 2B, $CMXY$ is a square, where M is the midpoint of AB. The circle can now be easily constructed by first erecting a square on CM.

1.3. *Equilateral triangle in a rectangle.* Given a rectangle $ABCD$, construct points P and Q on BC and CD respectively such that triangle APQ is equilateral.

Construction 1. *Construct* equilateral triangles CDX *and* BCY, *with* X *and* Y *inside the rectangle. Extend* AX *to intersect* BC *at* P *and* AY *to intersect* CD *at* Q.

The triangle APQ *is equilateral.* See Figure 3B.

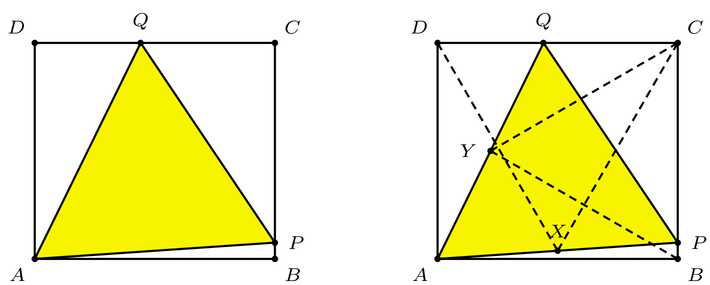


Figure 3A Figure 3B

This construction did not come from a lucky insight. It was found by an analysis. Let $AB = DC = a$, $BC = AD = b$. If $BP = y$, $DQ = x$ and APQ is equilateral, then a calculation shows that $x = 2a - \sqrt{3}b$ and $y = 2b - \sqrt{3}a$. From these expressions of x and y the above construction was devised.

1.4. *Partition of an equilateral triangle into 4 triangles with congruent incircles.* Given an equilateral triangle, construct three lines each through a vertex so that the incircles of the four triangles formed are congruent. See Figure 4A and [9, Problem 2.1.7] and [10, Problem 5.1.3], where it is shown that if each side of the equilateral triangle has length a, then the small circles all have radii $\frac{1}{8}(\sqrt{7}-\sqrt{3})a$. Here is a calculation that leads to a very easy construction of these lines.

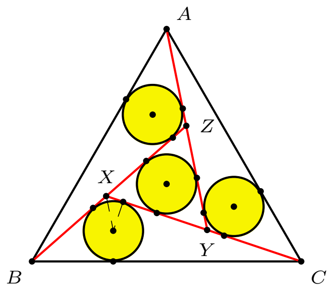


Figure 4A

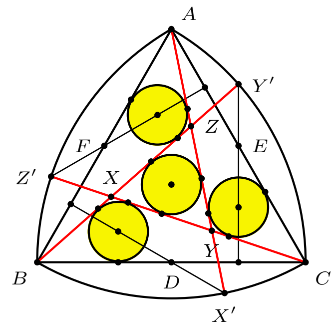


Figure 4B

In Figure 4A, let $CX = AY = BZ = a$ and $BX = CY = AZ = b$. The equilateral triangle XYZ has sidelength $a-b$ and inradius $\frac{\sqrt{3}}{6}(a-b)$. Since $\angle BXC = 120°$, $BC = \sqrt{a^2+ab+b^2}$, and the inradius of triangle BXC is

$$\frac{1}{2}(a+b-\sqrt{a^2+ab+b^2})\tan 60° = \frac{\sqrt{3}}{2}(a+b-\sqrt{a^2+ab+b^2}).$$

These two inradii are equal if and only if $3\sqrt{a^2+ab+b^2} = 2(a+2b)$. Applying the law of cosines to triangle XBC, we obtain

$$\cos XBC = \frac{(a^2+ab+b^2)+b^2-a^2}{2b\sqrt{a^2+ab+b^2}} = \frac{a+2b}{2\sqrt{a^2+ab+b^2}} = \frac{3}{4}.$$

In Figure 4B, Y' is the intersection of the arc $B(C)$ and the perpendicular from the midpoint E of CA to BC. The line BY' makes an angle $\arccos\frac{3}{4}$ with BC. The other two lines AX' and CZ' are similarly constructed. These lines bound the equilateral triangle XYZ, and the four incircles can be easily constructed. Their centers are simply the reflections of X' in D, Y' in E, and Z' in F.

2. Some basic constructions

2.1. *Geometric mean and the solution of quadratic equations.* The following constructions of the geometric mean of two lengths are well known.

Construction 2. (a) *Given two segments of length a, b, mark three points A, P, B on a line (P between A and B) such that $PA = a$ and $PB = b$. Describe a semicircle with AB as diameter, and let the perpendicular through P intersect the semicircle at Q. Then $PQ^2 = AP \cdot PB$, so that the length of PQ is the geometric mean of a and b.* See Figure 5A.

(b) *Given two segments of length $a < b$, mark three points P, A, B on a line such that $PA = a$, $PB = b$, and A, B are on the* same *side of P. Describe a semicircle with PB as diameter, and let the perpendicular through A intersect the semicircle at Q. Then $PQ^2 = PA \cdot PB$, so that the length of PQ is the geometric mean of a and b.* See Figure 5B.

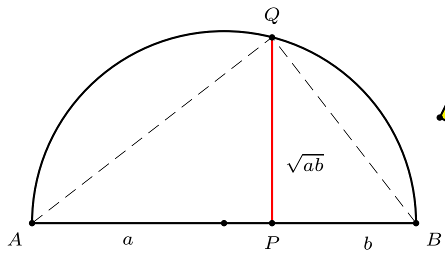


Figure 5A

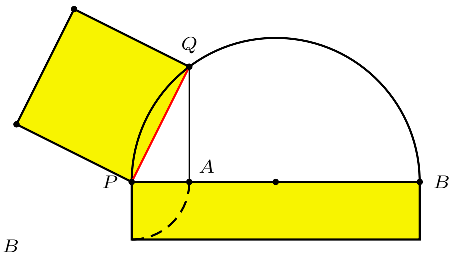


Figure 5B

More generally, a quadratic equation can be solved by applying the theorem of intersecting chords: *If a line through P intersects a circle $O(r)$ at X and Y, then the product $PX \cdot PY$ (of signed lengths) is equal to $OP^2 - r^2$.* Thus, if two chords AB and XY intersect at P, then $PA \cdot PB = PX \cdot PY$. See Figure 6A. In particular, if P is outside the circle, and if PT is a tangent to the circle, then $PT^2 = PX \cdot PY$ for any line intersecting the circle at X and Y. See Figure 6B.

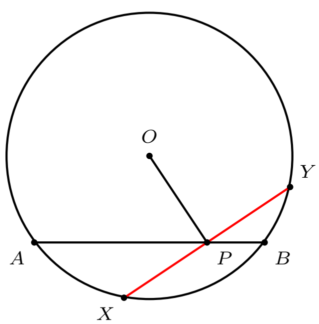


Figure 6A

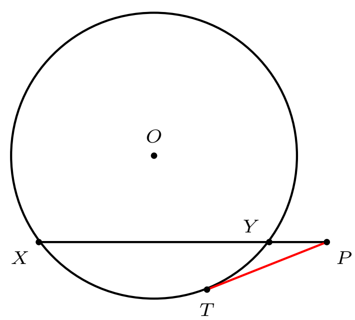


Figure 6B

A quadratic equation can be put in the form $x(x \pm a) = b^2$ or $x(a - x) = b^2$. In the latter case, for real solutions, we require $b \le \frac{a}{2}$. If we arrange a and b as the legs of a right triangle, then the positive roots of the equation can be easily constructed as in Figures 6C and 6D respectively.

The algebraic method of the solution of a quadratic equation by completing squares can be easily incorporated geometrically by using the Pythagorean theorem. We present an example.

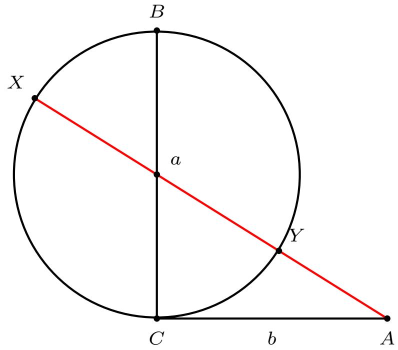


Figure 6C

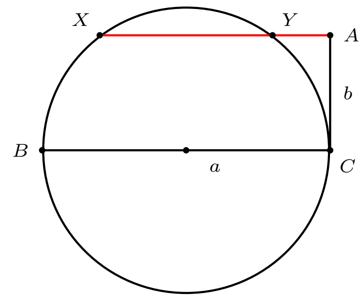


Figure 6D

2.1.1. Given a chord BC perpendicular to a diameter XY of circle (O), to construct a line through X which intersects the circle at A and BC at T such that AT has a given length t. Clearly, $t \leq YM$, where M is the midpoint of BC.

Let $AX = x$. Since $\angle CAX = \angle CYX = \angle TCX$, the line CX is tangent to the circle ACT. It follows from the theorem of intersecting chords that $x(x-t) = CX^2$. The method of completing squares leads to

$$x = \frac{t}{2} + \sqrt{CX^2 + \left(\frac{t}{2}\right)^2}.$$

This suggests the following construction.[4]

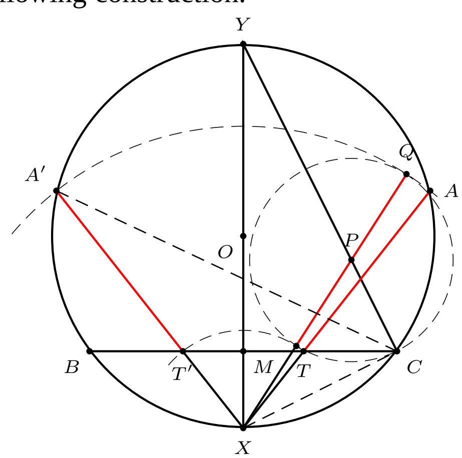


Figure 7

Construction 3. *On the segment CY, choose a point P such that $CP = \frac{t}{2}$. Extend XP to Q such that $PQ = PC$. Let A be an intersection of $X(Q)$ and (O). If the line XA intersects BC at T, then $AT = t$.* See Figure 7.

[4] This also solves the construction problem of triangle ABC with given angle A, the lengths a of its opposite side, and of the bisector of angle A.

2.2. *Harmonic mean and the equation* $\frac{1}{a} + \frac{1}{b} = \frac{1}{t}$. The harmonic mean of two quantities a and b is $\frac{2ab}{a+b}$. In a trapezoid of parallel sides a and b, the parallel through the intersection of the diagonals intercepts a segment whose length is the harmonic mean of a and b. See Figure 8A. We shall write this harmonic mean as $2t$, so that $\frac{1}{a} + \frac{1}{b} = \frac{1}{t}$. See Figure 8B.

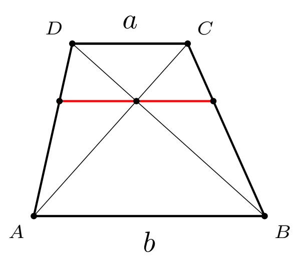


Figure 8A

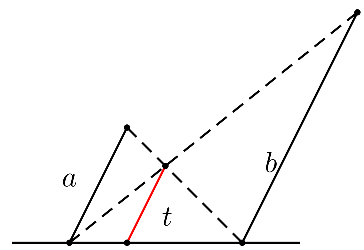


Figure 8B

Here is another construction of t, making use of the formula for the length of an angle bisector in a triangle. If $BC = a$, $AC = b$, then the angle bisector CZ has length

$$t_c = \frac{2ab}{a+b} \cos \frac{C}{2} = 2t \cos \frac{A}{2}.$$

The length t can therefore be constructed by completing the rhombus $CXZY$ (by constructing the perpendicular bisector of CZ to intersect BC at X and AC at Y). See Figure 9A. In particular, if the triangle contains a right angle, this trapezoid is a square. See Figure 9B.

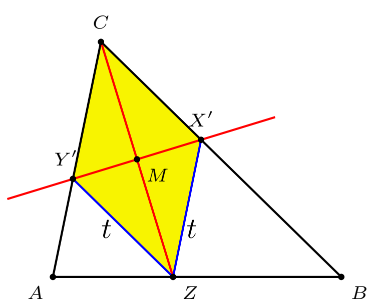


Figure 9A

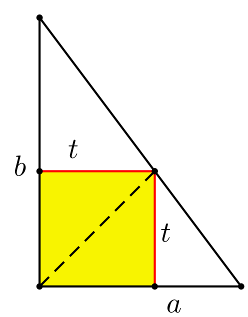


Figure 9B

3. The shoemaker's knife

3.1. *Archimedes' Theorem.* A shoemaker's knife (or arbelos) is the region obtained by cutting out from a semicircle with diameter AB the two smaller semicircles with diameters AP and PB. Let $AP = 2a$, $PB = 2b$, and the common tangent of the smaller semicircles intersect the large semicircle at Q. The following remarkable theorem is due to Archimedes. See [12].

Theorem 1 (Archimedes)**.** (1) *The two circles each tangent to PQ, the large semicircle and one of the smaller semicircles have equal radii $t = \frac{ab}{a+b}$.* See Figure 10A.

(2) *The circle tangent to each of the three semicircles has radius*

$$\rho = \frac{ab(a+b)}{a^2+ab+b^2}. \tag{1}$$

See Figure 10B.

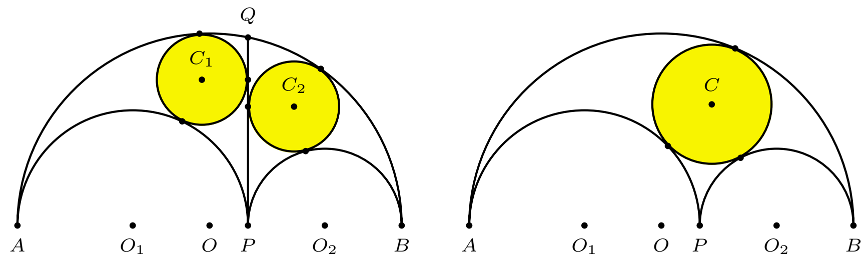


Figure 10A Figure 10B

Here is a simple construction of the Archimedean "twin circles". Let Q_1 and Q_2 be the "highest" points of the semicircles $O_1(a)$ and $O_2(b)$ respectively. The intersection $C_3 = O_1Q_2 \cap O_2Q_1$ is a point "above" P, and $C_3P = t = \frac{ab}{a+b}$.

Construction 4. *Construct the circle $P(C_3)$ to intersect the diameter AB at P_1 and P_2 (so that P_1 is on AP and P_2 is on PB).*

The center C_1 (respectively C_2) is the intersection of the circle $O_1(P_2)$ (respectively $O_2(P_1)$) and the perpendicular to AB at P_1 (respectively P_2). See Figure 11.

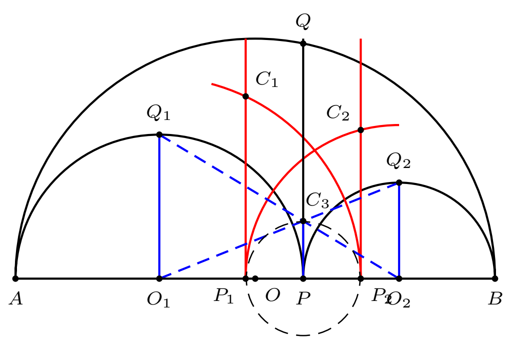


Figure 11

Theorem 2 (Bankoff [3])**.** *If the incircle $C(\rho)$ of the shoemaker's knife touches the smaller semicircles at X and Y, then the circle through the points P, X, Y has the same radius t as the Archimedean circles.* See Figure 12.

This gives a very simple construction of the incircle of the shoemaker's knife.

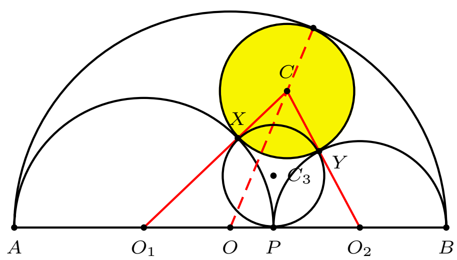


Figure 12

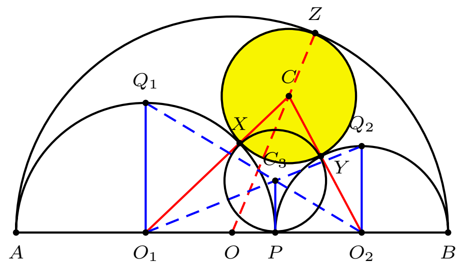


Figure 13

Construction 5. *Let* $X = C_3(P) \cap O_1(a)$, $Y = C_3(P) \cap O_2(b)$, *and* $C = O_1X \cap O_2Y$. *The circle* $C(X)$ *is the incircle of the shoemaker's knife. It touches the large semicircle at* $Z = OC \cap O(a + b)$. See Figure 13.

A rearrangement of (1) in the form

$$\frac{1}{a+b} + \frac{1}{\rho} = \frac{1}{t}$$

leads to another construction of the incircle (C) by directly locating the center and one point on the circle. See Figure 14.

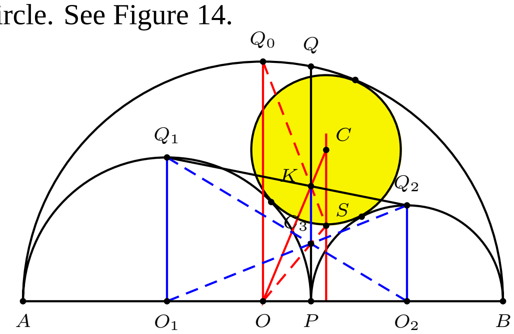


Figure 14

Construction 6. *Let* Q_0 *be the "highest" point of the semicircle* $O(a + b)$. *Construct*
(i) $K = Q_1Q_2 \cap PQ$,
(ii) $S = OC_3 \cap Q_0K$, *and*
(iii) *the perpendicular from* S *to* AB *to intersect the line* OK *at* C.
The circle $C(S)$ *is the incircle of the shoemaker's knife.*

3.2. *Other simple constructions of the incircle of the shoemaker's knife.* We give four more simple constructions of the incircle of the shoemaker's knife. The first is by Leon Bankoff [1]. The remaining three are by Peter Woo [21].

Construction 7 (Bankoff)**.** (1) *Construct the circle* $Q_1(A)$ *to intersect the semicircles* $O_2(b)$ *and* $O(a + b)$ *at* X *and* Z *respectively.*

(2) *Construct the circle* $Q_2(B)$ *to intersect the semicircles* $O_1(a)$ *and* $O(a + b)$ *at* Y *and the same point* Z *in* (1) *above.*

The circle through X, Y, Z *is the incircle of the shoemaker's knife.* See Figure 15.

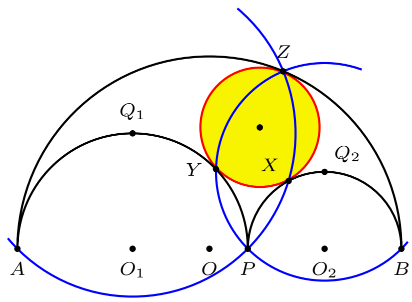


Figure 15

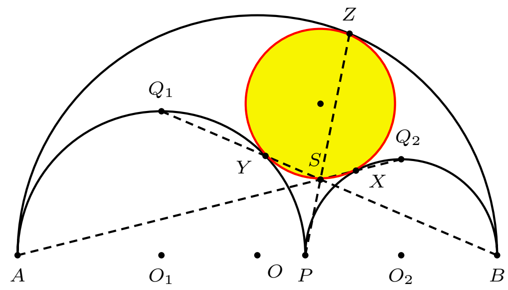


Figure 16

Construction 8 (Woo)**.** (1) *Construct the line AQ_2 to intersect the semicircle $O_2(b)$ at X.*

(2) *Construct the line BQ_1 to intersect the semicircle $O_1(a)$ at Y.*

(3) *Let $S = AQ_2 \cap BQ_1$. Construct the line PS to intersect the semicircle $O(a+b)$ at Z.*

The circle through X, Y, Z *is the incircle of the shoemaker's knife.* See Figure 16.

Construction 9 (Woo)**.** *Let M be the "lowest" point of the circle $O(a+b)$. Construct*

(i) *the circle $M(A)$ to intersect $O_1(a)$ at Y and $O_2(b)$ at X,*

(ii) *the line MP to intersect the semicircle $O(a+b)$ at Z.*

The circle through X, Y, Z *is the incircle of the shoemaker's knife.* See Figure 17.

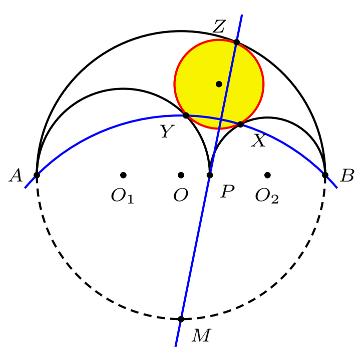


Figure 17

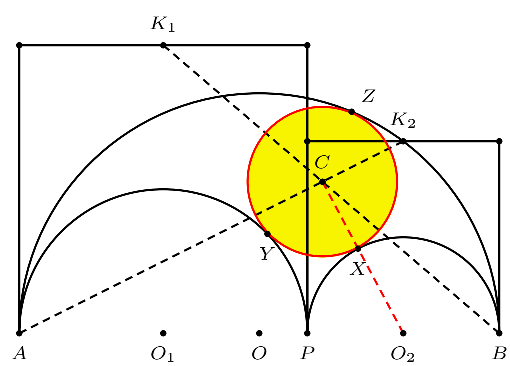


Figure 18

Construction 10 (Woo)**.** *Construct* squares *on AP and PB on the same side of the shoemaker knife. Let K_1 and K_2 be the midpoints of the opposite sides of AP and PB respectively. Let $C = AK_2 \cap BK_1$, and $X = CO_2 \cap O_2(b)$.*

The circle $C(X)$ is the incircle of the shoemaker's knife. See Figure 18.

4. Animation of bicentric polygons

A famous theorem of J. V. Poncelet states that if between two conics $\mathcal{C}_1$ and $\mathcal{C}_2$ there is a polygon of n sides with vertices on $\mathcal{C}_1$ and sides tangent to $\mathcal{C}_2$, then there is one such polygon of n sides with a vertex at an arbitrary point on $\mathcal{C}_1$. See, for example, [5]. For circles $\mathcal{C}_1$ and $\mathcal{C}_2$ and for $n = 3, 4$, we illustrate this theorem by constructing animation pictures based on simple metrical relations.

4.1. *Euler's formula.* Consider the construction of a triangle given its circumcenter O, incenter I and a vertex A. The circumcircle is $O(A)$. If the line AI intersects this circle again at X, then the vertices B and C are simply the intersections of the circles $X(I)$ and $O(A)$. See Figure 19A. This leads to the famous Euler formula

$$d^2 = R^2 - 2Rr, \tag{2}$$

where d is the distance between the circumcenter and the incenter.[5]

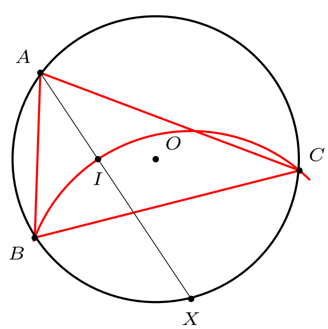


Figure 19A

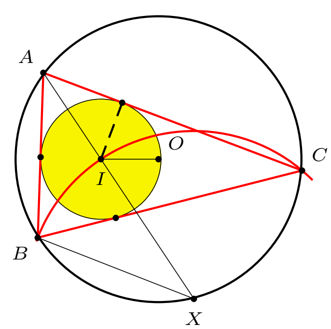


Figure 19B

4.1.1. Given a circle $O(R)$ and $r < \frac{R}{2}$, to construct a point I such that $O(R)$ and $I(r)$ are the circumcircle and incircle of a triangle.

Construction 11. *Let $P(r)$ be a circle tangent to (O) internally. Construct a line through O* tangent to the circle *$P(r)$ at a point I.*

The circle $I(r)$ is the incircle of triangles which have $O(R)$ as circumcircle. See Figure 20.

[5] *Proof*: If I is the incenter, then $AI = \frac{r}{\sin\frac{A}{2}}$ and $IX = IB = \frac{2R}{\sin\frac{A}{2}}$. See Figure 19B. The power of I with respect to the circumcircle is $d^2 - R^2 = IA \cdot IX = -r\sin\frac{A}{2} \cdot \frac{2R}{\sin\frac{A}{2}} = -2Rr$.

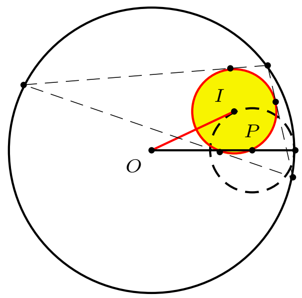


Figure 20

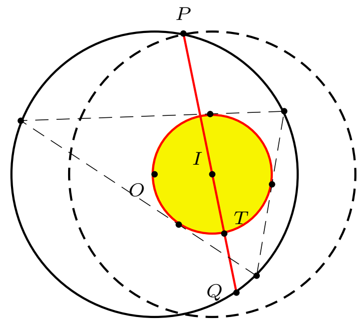


Figure 21

4.1.2. Given a circle $O(R)$ and a point I, to construct a circle $I(r)$ such that $O(R)$ and $I(r)$ are the circumcircle and incircle of a triangle.

Construction 12. *Construct the circle $I(R)$ to intersect $O(R)$ at a point P, and construct the line PI to intersect $O(R)$ again at Q. Let T be the midpoint of IQ.*

The circle $I(T)$ is the incircle of triangles which have $O(R)$ as circumcircle. See Figure 21.

4.1.3. Given a circle $I(r)$ and a point O, to construct a circle $O(R)$ which is the circumcircle of triangles with $I(r)$ as incircle. Since $R = r + \sqrt{r^2 + d^2}$ by the Euler formula (2), we have the following construction. See Figure 22.

Construction 13. *Let IP be a radius of $I(r)$ perpendicular to IO. Extend OP to a point A such that $PA = r$.*

The circle $O(A)$ is the circumcircle of triangles which have $I(r)$ as incircle.

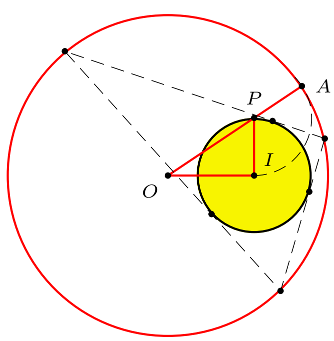


Figure 22

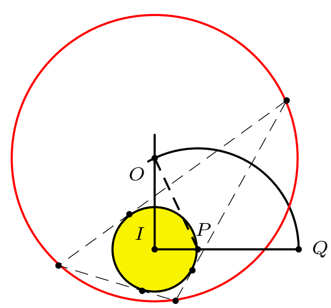


Figure 23

4.1.4. Given $I(r)$ and $R > 2r$, to construct a point O such that $O(R)$ is the circumcircle of triangles with $I(r)$ as incircle.

Construction 14. *Extend a radius IP to Q such that $IQ = R$. Construct the perpendicular to IP at I to intersect the circle $P(Q)$ at O.*

The circle $O(R)$ is the circumcircle of triangles which have $I(r)$ as incircle. See Figure 23.

4.2. *Bicentric quadrilaterals.* A bicentric quadrilateral is one which admits a circumcircle and an incircle. The construction of bicentric quadrilaterals is based on the Fuss formula

$$2r^2(R^2 + d^2) = (R^2 - d^2)^2, \tag{3}$$

where d is the distance between the circumcenter and incenter of the quadrilateral. See [7, §39].

4.2.1. Given a circle $O(R)$ and a point I, to construct a circle $I(r)$ such that $O(R)$ and $I(r)$ are the circumcircle and incircle of a quadrilateral.

The Fuss formula (3) can be rewritten as

$$\frac{1}{r^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}.$$

In this form it admits a very simple interpretation: r can be taken as the altitude on the hypotenuse of a right triangle whose shorter sides have lengths $R \pm d$. See Figure 24.

Construction 15. *Extend IO to intersect $O(R)$ at a point A. On the perpendicular to IA at I construct a point K such that $IK = R - d$. Construct the altitude IP of the right triange AIK.*

The circles $O(R)$ and $I(P)$ are the circumcircle and incircle of bicentric quadrilaterals.

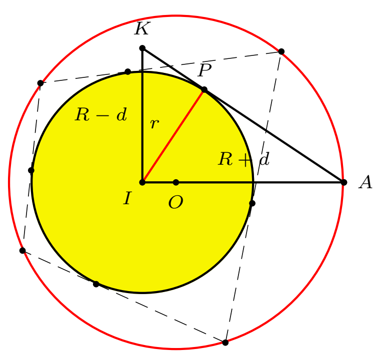


Figure 24

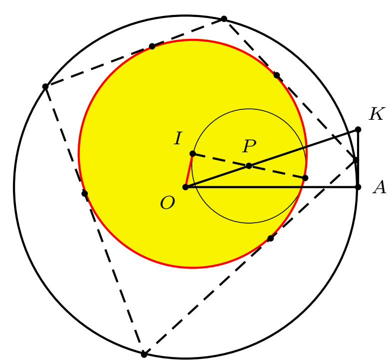


Figure 25

4.2.2. Given a circle $O(R)$ and a radius $r \leq \frac{R}{\sqrt{2}}$, to construct a point I such that $I(r)$ is the incircle of quadrilaterals inscribed in $O(R)$, we rewrite the Fuss formula (3) in the form

$$d^2 = \left(\sqrt{R^2 + \frac{r^2}{4}} - \frac{r}{2}\right)\left(\sqrt{R^2 + \frac{r^2}{4}} - \frac{3r}{2}\right).$$

This leads to the following construction. See Figure 25.

Construction 16. *Construct a right triangle OAK with a right angle at A, $OA = R$ and $AK = \frac{r}{2}$. On the hypotenuse OK choose a point P such that $KP = r$. Construct a* tangent from O to the circle $P(\frac{r}{2})$. *Let I be the point of tangency.*

The circles $O(R)$ and $I(r)$ are the circumcircle and incircle of bicentric quadrilaterals.

4.2.3. Given a circle $I(r)$ and a point O, to construct a circle (O) such that these two circles are respectively the incircle and circumcircle of a quadrilateral. Again, from the Fuss formula (3),

$$R^2 = \left(\sqrt{d^2 + \frac{r^2}{4}} + \frac{r}{2}\right)\left(\sqrt{d^2 + \frac{r^2}{4}} + \frac{3r}{2}\right).$$

Construction 17. *Let E be the midpoint of a radius IB perpendicular to OI. Extend the ray OE to a point F such that $EF = r$. Construct a* tangent OT to the circle $F\left(\frac{r}{2}\right)$. *Then OT is a circumradius.*

5. Some circle constructions

5.1. *Circles tangent to a chord at a given point.* Given a point P on a chord BC of a circle (O), there are two circles tangent to BC at P, and to (O) internally. The radii of these two circles are $\dfrac{BP \cdot PC}{2(R \pm h)}$, where h is the distance from O to BC. They can be constructed as follows.

Construction 18. *Let M be the midpoint of BC, and XY be the diameter perpendicular to BC. Construct*
(i) *the circle center P, radius MX to intersect the arc BXC at a point Q,*
(ii) *the line PQ to intersect the circle (O) at a point H,*
(iii) *the circle $P(H)$ to intersect the line perpendicular to BC at P at K* (*so that H and K are on the same side of BC*).

The circle with diameter PK is tangent to the circle (O). See Figure 26A.

Replacing X by Y in (i) *above we obtain the other circle tangent to BC at P and internally to (O).* See Figure 26B.

5.2. *Chain of circles tangent to a chord.* Given a circle (Q) tangent internally to a circle (O) and to a chord BC at a given point P, there are two neighbouring circles tangent to (O) and to the same chord. These can be constructed easily by observing that in Figure 27, the common tangent of the two circles cuts out a segment whose

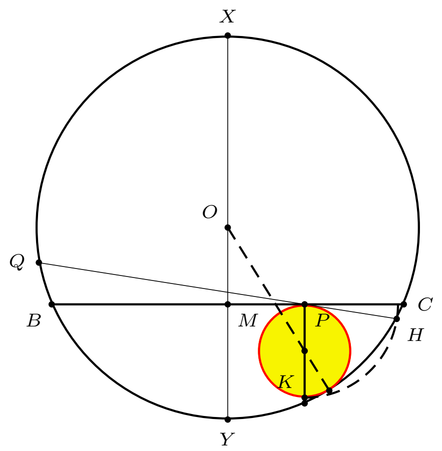


Figure 26A

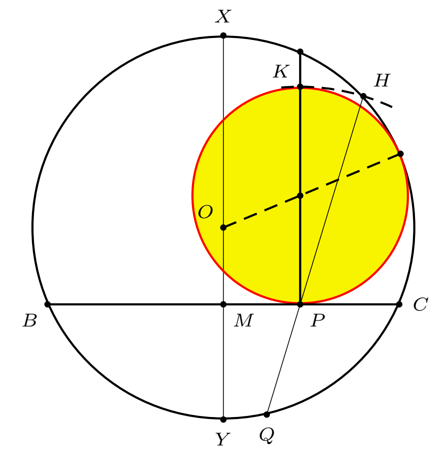


Figure 26B

midpoint is B. If (Q') is a neighbour of (Q), their common tangent passes through the midpoint M of the arc BC complementary to (Q). See Figure 28.

Construction 19. *Given a circle* (Q) *tangent to* (O) *and to the chord* BC*, construct*
(i) *the circle* $M(B)$ *to intersect* (Q) *at* T_1 *and* T_2*,* MT_1 *and* MT_2 *being tangents to* (Q)*,*
(ii) *the bisector of the angle between* MT_1 *and* BC *to intersect the line* QT_1 *at* Q_1*.*

The circle $Q_1(T_1)$ *is tangent to* (O) *and to* BC*.*

Replacing T_1 *by* T_2 *in* (ii) *we obtain* Q_2*. The circle* $Q_2(T_2)$ *is also tangent to* (O) *and* BC*.*

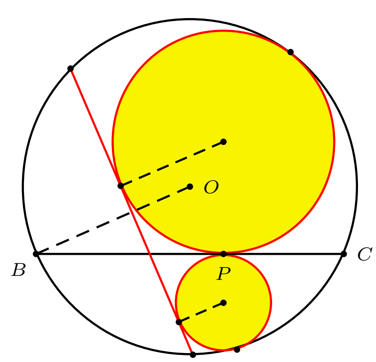


Figure 27

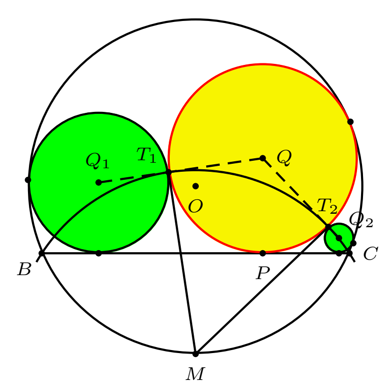


Figure 28

5.3. *Mixtilinear incircles.* Given a triangle ABC, we construct the circle tangent to the sides AB, AC, and also to the circumcircle internally. Leon Bankoff [4] called this the A- mixtilinear incircle of the triangle. Its center is clearly on the bisector of angle A. Its radius is $r\sec^2\frac{A}{2}$, where r is the inradius of the triangle. The mixtilinear incircle can be constructed as follows. See Figure 29.

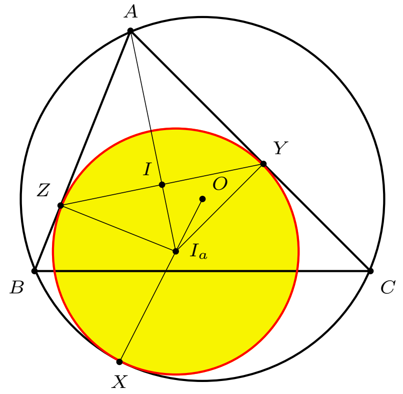


Figure 29

Construction 20 (Mixtilinear incircle)**.** *Let I be the* incenter *of triangle ABC. Construct*
(i) *the perpendicular to IA at I to intersect AC at Y,*
(ii) *the perpendicular to AY at Y to intersect the line AI at I_a.*
The circle $I_a(Y)$ is the A-mixtilinear incircle of ABC.

The other two mixtilinear incircles can be constructed in a similar way. For another construction, see [23].

5.4. *Ajima's construction.* The interesting book [10] by Fukagawa and Rigby contains a very useful formula which helps perform easily many constructions of inscribed circles which are otherwise quite difficult.

Theorem 3 (Ajima)**.** *Given triangles ABC with circumcircle (O) and a point P such that A and P are on the same side of BC, the circle tangent to the lines PB, PC, and to the circle (O) internally is the image of the incircle of triangle PBC under the homothety with center P and ratio $1+\tan\frac{A}{2}\tan\frac{BPC}{2}$.*

Construction 21 (Ajima)**.** *Given two points B and C on a circle (O) and an arbitrary point P, construct*
(i) *a point A on (O) on the same side of BC as P, (for example, by taking the midpoint M of BC, and intersecting the ray MP with the circle (O)),*
(ii) *the* incenter *I of triangle ABC,*
(iii) *the* incenter *I' of triangle PBC,*
(iv) *the perpendicular to $I'P$ at I' to intersect PC at Z.*
(v) *Rotate the ray ZI' about Z through an (oriented) angle equal to angle BAI to intersect the line AP at Q.*
Then the circle with center Q, tangent to the lines PB and PC, is also tangent to (O) internally. See Figure 30.

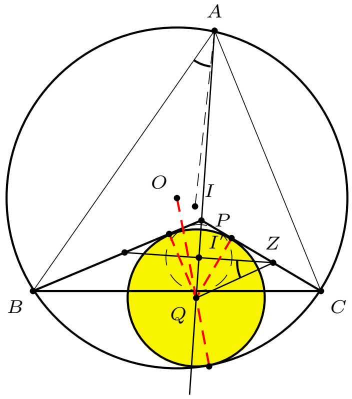


Figure 30

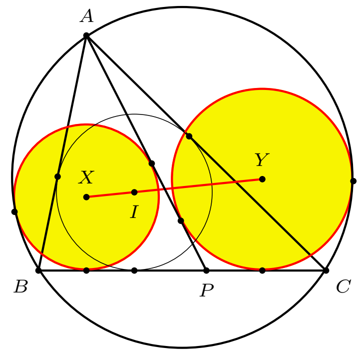


Figure 31

5.4.1. *Thébault's theorem.* With Ajima's construction, we can easily illustrate the famous Thébault theorem. See [18, 2] and Figure 31.

Theorem 4 (Thébault)**.** *Let P be a point on the side BC of triangle ABC. If the circles (X) and (Y) are tangent to AP, BC, and also internally to the circumcirle of the triangle, then the line XY passes through the incenter of the triangle.*

5.4.2. *Another example.* We construct an animation picture based on Figure 32 below. Given a segment AB and a point P, construct the squares $APX'X$ and $BPY'Y$ on the segments AP and BP. The locus of P for which A, B, X, Y are concyclic is the union of the perpendicular bisector of AB and the two quadrants of circles with A and B as endpoints. Consider P on one of these quadrants. The center of the circle $ABYX$ is the center of the other quadrant. Applying Ajima's construction to the triangle XAB and the point P, we easily obtain the circle tangent to AP, BP, and (O). Since $\angle APB = 135°$ and $\angle AXB = 45°$, the radius of this circle is twice the inradius of triangle APB.

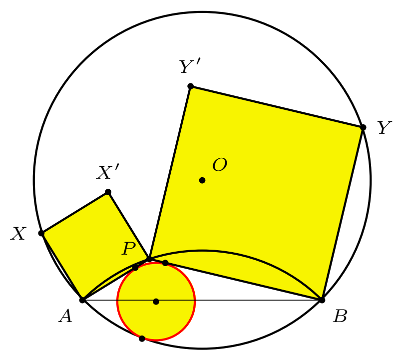


Figure 32

6. Some examples of triangle constructions

There is an extensive literature on construction problems of triangles with certain given elements such as angles, lengths, or specified points. Wernick [20] outlines a project of such with three given specific points. Lopes [14], on the other hand, treats extensively the construction problems with three given lengths such as sides, medians, bisectors, or others. We give three examples admitting elegant constructions. [6]

6.1. *Construction from a sidelength and the corresponding median and angle bisector.* Given the length $2a$ of a side of a triangle, and the lengths m and t of the median and the angle bisector on the same side, to construct the triangle. This is Problem 1054(a) of the *Mathematics Magazine* [6]. In his solution, Howard Eves denotes by z the distance between the midpoint and the foot of the angle bisector on the side $2a$, and obtains the equation

$$z^4 - (m^2 + t^2 + a^2)z^2 + a^2(m^2 - t^2) = 0,$$

from which he concludes constructibility (by ruler and compass). We devise a simple construction, assuming the data given in the form of a triangle $AM'T$ with $AT = t$, $AM' = m$ and $M'T = a$. See Figure 33. Writing $a^2 = m^2 + t^2 - 2tu$, and $z^2 = m^2 + t^2 - 2tw$, we simplify the above equation into

$$w(w - u) = \frac{1}{2}a^2. \tag{4}$$

Note that u is length of the projection of AM' on the line AT, and w is the length of the median AM on the bisector AT of the sought triangle ABC. The length w can be easily constructed, from this it is easy to complete the triangle ABC.

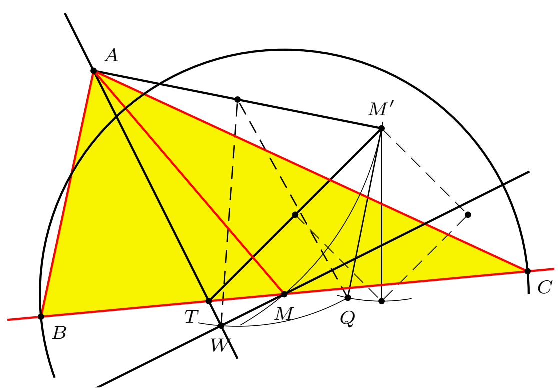


Figure 33

[6]Construction 3 (Figure 7) solves the construction problem of triangle ABC given angle A, side a, and the length t of the bisector of angle A. See Footnote 4.

Construction 22. (1) *On the perpendicular to AM' at M', choose a point Q such that $M'Q = \frac{M'T}{\sqrt{2}} = \frac{a}{\sqrt{2}}$.*

(2) *Construct the circle with center the midpoint of AM' to pass through Q and to intersect the line AT at W so that T and W are on the same side of A.* (*The length w of AW satisfies* (4) *above*).

(3) *Construct the perpendicular at W to AW to intersect the circle $A(M')$ at M.*

(4) *Construct the circle $M(a)$ to intersect the line MT at two points B and C.*

The triangle ABC has AT as bisector of angle A.

6.2. *Construction from an angle and the corresponding median and angle bisector*. This is Problem 1054(b) of the *Mathematics Magazine*. See [6]. It also appeared earlier as Problem E1375 of the *American Mathematical Monthly*. See [11]. We give a construction based on Thébault's solution.

Suppose the data are given in the form of a right triangle OAM, where $\angle AOM = A$ or $180° - A$, $\angle M = 90°$, $AM = m$, along with a point T on AM such that $AT = t$. See Figure 34.

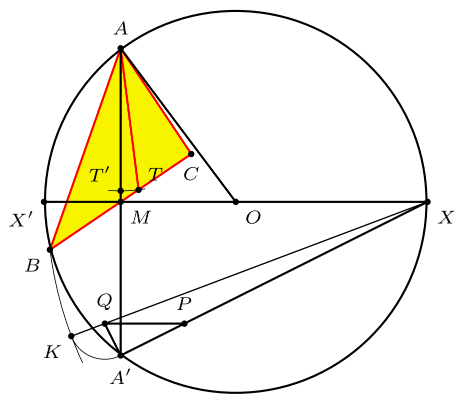


Figure 34

Construction 23. (1) *Construct the circle $O(A)$. Let A' be the mirror image of A in M. Construct the diameter XY perpendicular to AA', X the point for which $\angle AXA' = A$.*

(2) *On the segment $A'X$ choose a point P such that $A'P = \frac{t}{2}$. and construct the parallel through P to XY to intersect $A'Y$ at Q.*

(3) *Extend XQ to K such that $QK = QA'$.*

(4) *Construct a point B on $O(A)$ such that $XB = XK$, and its mirror image C in M.*

Triangle ABC has given angle A, median m and bisector t on the side BC.

6.3. *Construction from the incenter, orthocenter and one vertex.* This is one of the unsolved cases in Wernick [20]. See also [22]. Suppose we put the incenter I at the origin, $A = (a, b)$ and $H = (a, c)$ for $b > 0$. Let r be the inradius of the triangle.

A fairly straightforward calculation gives

$$r^2 - \frac{b-c}{2}r - \frac{1}{2}(a^2 + bc) = 0. \tag{5}$$

If M is the midpoint of IA and P the orthogonal projection of H on the line IA, then $\frac{1}{2}(a^2 + bc)$, being the dot product of IM and IH, is the (signed) product $IM \cdot IP$. Note that if angle AIH does not exceed a right angle, equation (5) admits a unique positive root. In the construction below we assume H closer than A to the perpendicular to AH through I.

Construction 24. *Given triangle AIH in which the angle AIH does not exceed a right angle, let M be the midpoint of IA, K the midpoint of AH, and P the orthogonal projection of H on the line IA.*
(1) *Construct the circle $\mathcal{C}$ through P, M and K. Let O be the center of $\mathcal{C}$ and Q the midpoint of PK.*
(2) *Construct a tangent from I to the circle $O(Q)$ intersecting $\mathcal{C}$ at T, with T farther from I than the point of tangency.*

The circle $I(T)$ is the incircle of the required triangle, which can be completed by constructing the tangents from A to $I(T)$, and the tangent perpendicular to AH through the "lowest" point of $I(T)$. See Figure 35.

If H is farther than A to the perpendicular from I to the line AH, the same construction applies, except that in (2) T is the intersection with $\mathcal{C}$ closer to I than the point of tangency.

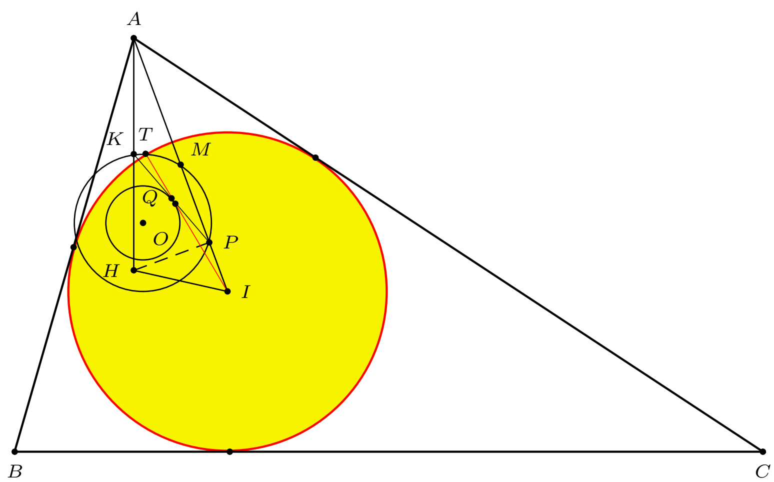


Figure 35

Remark. The construction of a triangle from its circumcircle, incenter, orthocenter was studied by Leonhard Euler [8], who reduced it to the problem of trisection of an angle. In Euler's time, the impossibility of angle trisection by ruler and compass was not yet confirmed.

References

[1] G. L. Alexanderson, A conversation with Leon Bankoff, *College Math. Journal*, 23 (1992) 98–117.
[2] J.-L. Ayme, Sawayama and Thébault's theorem, *Forum Geom.*, 3 (2003) 225–229.
[3] L. Bankoff, Are the twin circles of Archimedes really twin ?, *Math. Mag.* 47 (1974) 214–218.
[4] L. Bankoff, A mixtilinear adventure, *Crux Math.*, 9 (1983) 2–7.
[5] M. Berger, *Geometry II*, Springer-Verlag, 1987.
[6] J. C. Cherry and H. Eves, Problem 1054, *Math. Mag.*, 51 (1978) 305; solution, 53 (1980) 52–53.
[7] H. Dörrie, *100 Great Problems of Elementary Mathematics*, Dover, 1965.
[8] L. Euler, Variae demonstrationes geometriae, *Nova commentarii academiae scientiarum Petropolitanae*, 1 (1747/8), 49–66, also in *Opera Ommnia*, serie prima, vol.26, 15–32.
[9] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, Charles Babbage Research Centre, Winnipeg, 1989.
[10] H. Fukagawa and J. F. Rigby, *Traditional Japanese Mathematics Problems of the 18th and 19th Centuries*, SCT Press, Singapore, 2002.
[11] L. D. Goldstone, V. Thébault and R. Woods, Problem E 1375, *Amer. Math. Monthly*, 66 (1959) 513; solution, 67 (1960) 185–186.
[12] T. L. Heath, *The Works of Archimedes*, 1912, Dover reprint.
[13] D. Klanderman, Teaching and learning mathematics: the influence of constructivism, Chapter 12 of R. W. Howell and W. J. Bradley (ed.), *Mathematics in a Postmodern Age, A Christian Perspective*, pp.333–359, Wm. B Eerdmans, Grand Rapids, Michigan, 2001.
[14] L. Lopes, *Manuel de Construction de Triangles*, QED Texte, Québec, 1996.
[15] J. E. McClure, Start where they are: geometry as an introduction to proof, *Amer. Math. Monthly*, 107 (2000) 44–52.
[16] Report of the M. A. Committee on Geometry, *Math. Gazette*, 2 (1902) 167–172; reprinted in C. Pritchard (ed.) *The Changing Shape of Geometry, Celebrating a Century of Geometry and Geometry Teaching*, 529–536, Cambridge University Press, 2003.
[17] D. E. Smith, *The Teaching of Geometry*, 1911.
[18] V. Thébault, Problem 3887, *Amer. Math. Monthly*, 45 (1938) 482–483.
[19] Wang Yǒngjiàn, Shìtán píngmiàn jǐhé jiàoxué de zuōyòng yǔ dìweì, (On the function and role of the teaching of plane geometry), *Shuxue Tongbao*, 2004, Number 9, 23–24.
[20] W. Wernick, Triangle constructions with three located points, *Math. Mag.*, 55 (1982) 227–230.
[21] P. Woo, Simple constructions of the incircle of an arbelos, *Forum Geom.*, 1 (2001) 133–136.
[22] P. Yiu, Para-Euclidean teaching of Euclidean geometry, in M. K. Siu (ed.) *Restrospect and Outlook on Mathematics Education in Hong Kong, On the Occasion of the Retirement of Dr. Leung Kam Tim*, pp. 215–221, Hong Kong University Press, Hong Kong, 1995.
[23] P. Yiu, Mixtilinear incircles, *Amer. Math. Monthly*, 106 (1999) 952–955.
[24] Zheng yùxìn, Jiàngòu zhǔyì zhī shènsī (Careful consideration of constructivism), *Shuxue Tongbao*, 2004, Number 9, 18–22.

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida, 33431-0991, USA

E-mail address: yiu@fau.edu

Forum Geometricorum
Volume 5 (2005) 97–106.

FORUM GEOM
ISSN 1534-1178

Circles and Triangle Centers Associated with the Lucas Circles

Peter J. C. Moses

Abstract. The Lucas circles of a triangle are the three circles mutually tangent to each other externally, and each tangent internally to the circumcircle of the triangle at a vertex. In this paper we present some further interesting circles and triangle centers associated with the Lucas circles.

1. Introduction

In this paper we study circles and triangle centers associated with the three Lucas circles of a triangle. The Lucas circles of a triangle are the three circles mutually tangent to each other externally, and each tangent internally to the circumcircle of the triangle at a vertex.

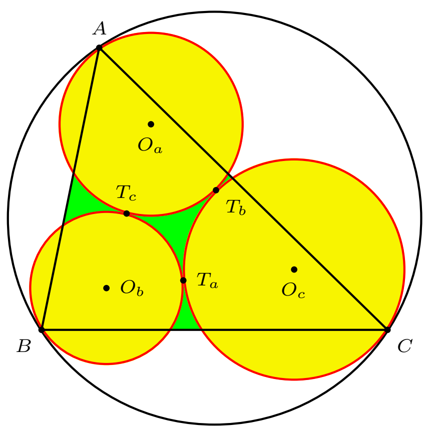


Figure 1

We work with homogeneous barycentric coordinates and make use of John H. Conway's notation in triangle geometry. The indexing of triangle centers follows Kimberling's *Encyclopedia of Triangle Centers* [2]. Many of the triangle centers in this paper are related to the Kiepert perspectors. We recall that given a triangle ABC, the Kiepert perspector $K(\theta)$ is the perspector of the triangle formed by the apices of similar isosceles triangles with base angles θ on the sides of ABC.

Publication Date: July 5, 2005. Communicating Editor: Paul Yiu.
The author thanks Clark Kimberling and Paul Yiu for their helps in the preparation of this paper.

In barycentric coordinates,

$$K(\theta) = \left(\frac{1}{S_A + S_\theta} : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta}\right).$$

Its isogonal conjugate is the point

$$K^*(\theta) = (a^2(S_A + S_\theta) : b^2(S_B + S_\theta) : c^2(S_C + S_\theta))$$

on the Brocard axis joining the circumcenter O and the symmedian point K.

2. The centers and points of tangency of the Lucas circles

The Lucas circles $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$ of triangle ABC are the images of the circumcircle under the homotheties with centers A, B, C, and ratios $\frac{S}{a^2+S}$, $\frac{S}{b^2+S}$, $\frac{S}{c^2+S}$ respectively. As such they have centers

$$\begin{aligned} O_a =&(a^2(S_A + 2S) : b^2 S_B : c^2 S_C), \\ O_b =&(a^2 S_A : b^2(S_B + 2S) : c^2 S_C), \\ O_c =&(a^2 S_A : b^2 S_B : c^2(S_C + 2S)), \end{aligned}$$

and equations

$$\mathcal{C}_A : \qquad a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2}{a^2+S} \cdot (x+y+z)\left(\frac{y}{b^2} + \frac{z}{c^2}\right) = 0,$$

$$\mathcal{C}_B : \qquad a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2}{b^2+S} \cdot (x+y+z)\left(\frac{z}{c^2} + \frac{x}{a^2}\right) = 0,$$

$$\mathcal{C}_C : \qquad a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2}{c^2+S} \cdot (x+y+z)\left(\frac{x}{a^2} + \frac{y}{b^2}\right) = 0.$$

The Lucas circles are mutually tangent to each other, externally, at

$$\begin{aligned} T_a = \mathcal{C}_B \cap \mathcal{C}_C =&(a^2 S_A : b^2(S_B + S) : c^2(S_C + S)), \\ T_b = \mathcal{C}_C \cap \mathcal{C}_A =&(a^2(S_A + S) : b^2 S_B : c^2(S_C + S)), \\ T_c = \mathcal{C}_A \cap \mathcal{C}_B =&(a^2(S_A + S) : b^2(S_B + S) : c^2 S_C). \end{aligned}$$

See Figure 1. These points of tangency form a triangle perspective with ABC at

$$K^*(\frac{\pi}{4}) = (a^2(S_A + S) : b^2(S_B + S) : c^2(S_C + C)),$$

which is X_{371} of [2].

By Desargues' theorem, the triangles $O_aO_bO_c$ and $T_aT_bT_c$ are perspective. Their perspector is clearly the Gergonne point of triangle $O_aO_bO_c$; it has coordinates

$$(a^2(3S_A + 2S) : b^2(3S_B + 2S) : c^2(3S_C + 2S)).$$

This is the point $K^*(\arctan \frac{3}{2})$.

The exsimilicenter (external center of similitude) of $\mathcal{C}_B$ and $\mathcal{C}_C$ is the point $(0 : b^2 : -c^2)$. Likewise, those of the pairs $\mathcal{C}_C$, $\mathcal{C}_A$ and $\mathcal{C}_A$, $\mathcal{C}_B$ are $(-a^2 : 0 : c^2)$ and $(a^2 - b^2 : 0)$. These three exsimilicenters all lie on the Lemoine axis,

$$\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} = 0.$$

Proposition 1. *The pedals of O_a on BC, O_b on CA, and O_c on AB form the cevian triangle of the Kiepert perspector $K(\arctan 2)$.*[1]

Proof. These pedals are the points $(0 : 2S_C + S : 2S_B + S)$, $(2S_C + S : 0 : 2S_A + S)$, and $(2S_B + S : 2S_A + S : 0)$. □

Proposition 2. *The pedals of T_a on BC, T_b on CA, and T_c on AB form the cevian triangle of the point $(a^2 + S : b^2 + S : c^2 + S)$.*

Proof. These pedals are the points $(0 : b^2 + S : c^2 + S)$, $(a^2 + S : 0 : c^2)$, and $(a^2 + S : b^2 + S : 0)$. □

3. The radical circle of the Lucas circles

From the equations of the Lucas circles, the radical center of these circles is the point $(x : y : z)$ satisfying

$$\frac{\frac{y}{b^2} + \frac{z}{c^2}}{a^2 + S} = \frac{\frac{z}{c^2} + \frac{x}{a^2}}{b^2 + S} = \frac{\frac{x}{a^2} + \frac{y}{b^2}}{c^2 + S}.$$

This means that $\left(\frac{x}{a^2} : \frac{y}{b^2} : \frac{z}{c^2}\right)$ is the anticomplement of $(a^2 + S : b^2 + S : c^2 + S)$, namely, $(2S_A + S : 2S_B + S : 2S_C + S)$, and the radical center is the point

$$K^*(\arctan 2) = (a^2(2S_A + S) : b^2(2S_B + S) : c^2(2S_C + S)) = X_{1151}$$

on the Brocard axis. Since the Lucas circles are tangent to each other, their radical circle is simply the circle through the tangent points T_a, T_b and T_c. It is also the incircle of triangle $O_aO_bO_c$. As such, it has radius $\frac{2S}{a^2+b^2+c^2+4S} \cdot R$, where R is the circumradius of triangle ABC. Its equation is

$$a^2yz + b^2zx + c^2xy - \frac{2a^2b^2c^2(x + y + z)}{a^2 + b^2 + c^2 + 4S}\left(\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2}\right).$$

4. The inner Soddy circle of the Lucas circles

There are two nonintersecting circles which are tangent to all three Lucas circles. These are the outer and inner Soddy circles of triangle $O_aO_bO_c$. Since the outer Soddy circle is the circumcircle of ABC, the inner Soddy circle is the inverse of this circumcircle with respect to the radical circle. Indeed the points of tangency are the inverses of A, B, C in the radical circle. They are simply the second

[1]This is X_{1131} of [2].

intersections of the lines AT with $\mathcal{C}_a$, BT with $\mathcal{C}_b$, and CT with $\mathcal{C}_c$, where $T = K^*(\arctan 2)$. These are the points

$$(a^2(4S_A + 3S) : 2b^2(2S_B + S) : 2c^2(2S_C + S)),$$
$$(2a^2(2S_A + S) : b^2(4S_B + 3S) : 2c^2(2S_C + S)),$$
$$(2a^2(2S_A + S) : 2b^2(2S_B + S) : c^2(4S_C + 3S)).$$

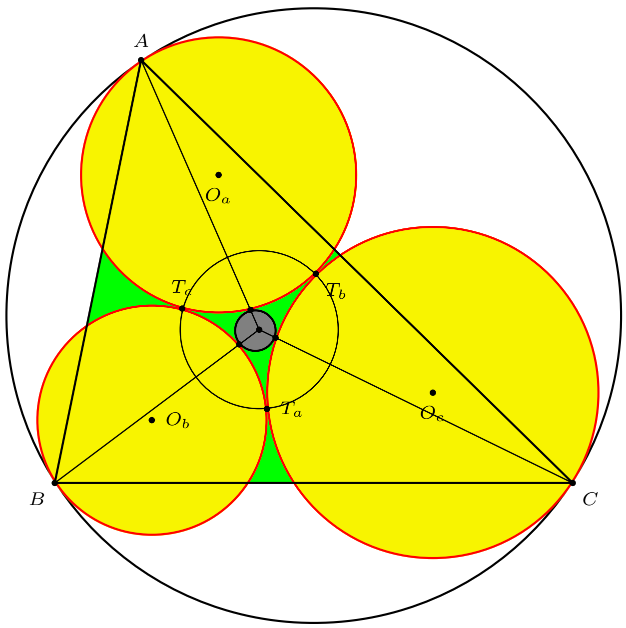


Figure 2

The circle through these points has center $K^*\left(\arctan \frac{7}{4}\right)$ and radius $\frac{2S\cdot R}{4(a^2+b^2+c^2)+14S}$. It has equation

$$a^2yz + b^2zx + c^2xy - \frac{4a^2b^2c^2(x+y+z)}{2(a^2+b^2+c^2)+7S}\left(\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2}\right) = 0.$$

Proposition 3. *The circumcircle, the radical circle, the inner Soddy circle, and the Brocard circles are coaxal, with the Lemoine axis as radical axis.*

The Brocard circle has equation

$$a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2(x+y+z)}{a^2+b^2+c^2}\left(\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2}\right) = 0.$$

The radical trace of these circles, namely, the intersection of the radical axis and the line of centers, is the point

$$(a^2(b^2 + c^2 - 2a^2) : \cdots : \cdots) = K^*(-\arctan(\frac{6S}{a^2+b^2+c^2})).$$

This is X_{187}, the inverse of K in the circumcircle.

5. The Schoute coaxal system

According to [5], the coaxal system of circles containing the circles in Proposition 3 is called the Schoute coaxal system. It has the two isodynamic points as limit points. Indeed, the circle with center X_{187} passing through the isodynamic point $X_{15} = K^*(\frac{\pi}{3})$ is the radical circle of these circles.

Proposition 4. *The circles of the Schoute coaxal system have centers $K^*(\theta)$ where $|\theta| \geq \frac{\pi}{3}$, and radius $\left|\frac{\sqrt{\tan^2\theta - 3S}}{2(S_\omega + S\cdot\tan\theta)}\right| \cdot R$. It has equation*

$$\mathcal{C}_{\rm s}(\theta): \qquad a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2(x+y+z)}{S_\omega + S\cdot\tan\theta}\left(\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2}\right) = 0.$$

Therefore, a circle with center $(a^2(pS_A + qS) : b^2(pS_B + qS) : c^2(pS_C + qS))$ and square radius $\frac{(p^2-3q^2)a^2b^2c^2}{(2pS+q(a^2+b^2+c^2))^2}$ is the circle $\mathcal{C}_{\rm s}(\arctan\frac{p}{q})$.

circle	$\mathcal{C}_{\rm s}(\theta)$with $\tan\theta =$
circumcircle	∞
Brocard circle	$\cot\omega$
Lemoine axis	$-\cot\omega$
radical circle of Lucas circles	2
inner Soddy circle of Lucas circles	$\frac{7}{4}$

$\theta = \frac{\pi}{3}$ yields the limit point X_{15}.

Proposition 5. *The inversive image of $\mathcal{C}_{\rm s}(\theta)$ in $\mathcal{C}_{\rm s}(\varphi)$ is the circle $\mathcal{C}_{\rm s}(\psi)$, where*

$$\tan\psi = \frac{\tan\theta(\tan^2\varphi + 3) - 6\tan\varphi}{2\tan\theta\tan\varphi - (\tan^2\varphi + 3)}.$$

Corollary 6. (a) *The inverse of $\mathcal{C}_{\rm s}(\theta)$ in the circumcircle is $\mathcal{C}_{\rm s}(-\theta)$.*

(b) *The inverse of the circumcircle in $\mathcal{C}_{\rm s}(\varphi)$ is the circle $\mathcal{C}_{\rm s}\left(\arctan\frac{\tan^2\varphi+3}{2\tan\varphi}\right)$.*

6. Three infinite families of circles

Let $A'B'C'$ be the circumcevian triangle of the symmedian point K, and $K' = K^*(\frac{\pi}{4})$. The line OA' intersects O_aK' at

$$O_1^a = (a^2(S_A - 2S) : b^2(S_B + 4S) : c^2(S_C + 4S)).$$

This is the center of the circle tangent to the B- and C-Lucas circles, and the circumcircle. It touches the circumcircle at K_0^a. We label this circle $\mathcal{C}_1^a$. The points of tangency with the B- and C-Lucas circles are

$$(a^2(S_A - S) : b^2(S_B + 3S) : c^2(S_C + 2S)),$$
$$(a^2(S_A - S) : b^2(S_B + 2S) : c^2(S_C + 3S))$$

respectively.

Similarly, there are circles $\mathcal{C}_1^b$ and $\mathcal{C}_1^c$ each tangent internally to the circumcircle and externally to two Lucas circles. The centers of the three circles $\mathcal{C}_1^a$, $\mathcal{C}_1^b$, $\mathcal{C}_1^c$ are perspective with ABC at $K^*(\arctan\frac{1}{4})$.

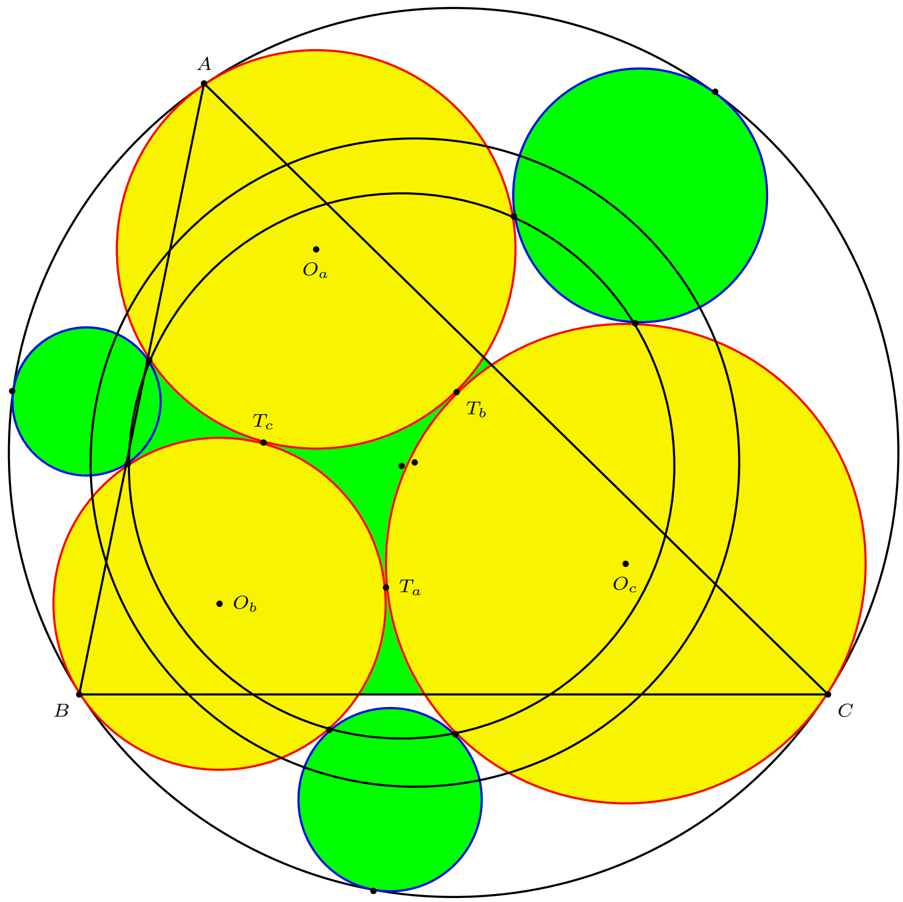


Figure 3

Remarks. (1) The 6 points of tangency with the Lucas circles lie on $\mathcal{C}_s(\arctan 4)$.
(2) The radical circle of these circles is $\mathcal{C}_s(\arctan 6)$. See Figure 3.

The Lucas circles lend themselves to the creation of more and more circle tangencies. There is, for example, an infinite sequence of circles $\mathcal{C}_n^a$ each tangent externally to the B- and C-Lucas circles, so that $\mathcal{C}_n^a$ touches $\mathcal{C}_{n-1}^a$ externally at T_n^a. (We treat $\mathcal{C}_0^a$ as the circumcircle of ABC so that $T_1^a = A'$.

$$\begin{aligned}O_n^a =&(a^2((2n^2-1)S_A-2nS):b^2((2n^2-1)S_B+2n(n+1)S)\\&\qquad\qquad\qquad\qquad\qquad\qquad :c^2((2n^2-1)S_C+2n(n+1)S)),\\T_n^a =&(a^2(2n(n-1)S_A-(2n-1)S):2nb^2((n-1)S_B+nS):2nc^2((n-1)S_C+nS)).\end{aligned}$$

The centers O_n^a of these circles lie on the hyperbola through O_a with foci O_b and O_c. It also contains O and T_a. This is the inner Soddy hyperbola of triangle $O_aO_bO_c$. The points of tangency T_n^a lie on the A-Apollonian circle.

Similarly, we have two analogous families of circles $\mathcal{C}_n^b$ and $\mathcal{C}_n^c$, respectively with centers O_n^b, O_n^c and points of tangency T_n^b, T_n^c.

Remarks. (1) The centers of $\mathcal{C}_n^a$, $\mathcal{C}_n^b$, $\mathcal{C}_n^c$ lie on the circle $\mathcal{C}_{\rm s}\left(\arctan\frac{4n^2-2n+1}{2n(n-1)}\right)$.

(2) The six points of tangency with the Lucas circles lie on the circle $\mathcal{C}_{\rm s}\left(\arctan\frac{2n^2+n+1}{n^2}\right)$.

(3) The radical circle of $\mathcal{C}_n^a$, $\mathcal{C}_n^b$, $\mathcal{C}_n^c$ is the circle $\mathcal{C}_{\rm s}\left(\arctan\frac{2n(2n+1)}{2n^2-1}\right)$.

Proposition 7. *The following pairs of triangles are perspective. The perspectors are all on the Brocard axis.*

Triangle	Triangle	Perspector $=K^*(\theta)$ with $\tan\theta=$
$O_n^aO_n^bO_n^c$	ABC	$\frac{2n^2-1}{2n(n+1)}$
$O_n^aO_n^bO_n^c$	$O_aO_bO_c$	$\frac{3n-1}{2n}$
$O_n^aO_n^bO_n^c$	$T_aT_bT_c$	$\frac{4n+1}{2n}$
$O_n^aO_n^bO_n^c$	circumcevian triangle of K	$\frac{6n^2-3}{2n(n-1)}$
$O_n^aO_n^bO_n^c$	$O_1^aO_1^bO_1^c$	$\frac{5n+3}{2n}$
$O_n^aO_n^bO_n^c$	$O_{n+1}^aO_{n+1}^bO_{n+1}^c$	$\frac{4n^2+6n+3}{2n(n+1)}$
$O_n^aO_n^bO_n^c$	$O_m^aO_m^bO_m^c$	$\frac{4mn+m+n+2}{2mn}$
$T_n^aT_n^bT_n^c$	ABC	$\frac{n-1}{n}$
$T_n^aT_n^bT_n^c$	$O_aO_bO_c$	$\frac{6n^2-2n-1}{4n^2}$
$T_n^aT_n^bT_n^c$	$T_aT_bT_c$	$\frac{4n-1}{2n-1}$
$T_n^aT_n^bT_n^c$	$T_m^aT_m^bT_m^c$	$\frac{4mn-m-n+1}{2mn-m-n}$

7. Centers of similitude

Since the Lucas radical circle, the inner Soddy circle and the circumcircle all belong to the Schoute family, their centers of similitude are all on the Brocard axis.

		Internal	External
inner Soddy circle	circumcircle	$K^*(\arctan 2)$	$K^*(\arctan\frac{3}{2})$
inner Soddy circle	radical circle	$K^*(\arctan\frac{9}{5})$	$K^*(\arctan\frac{5}{3})$

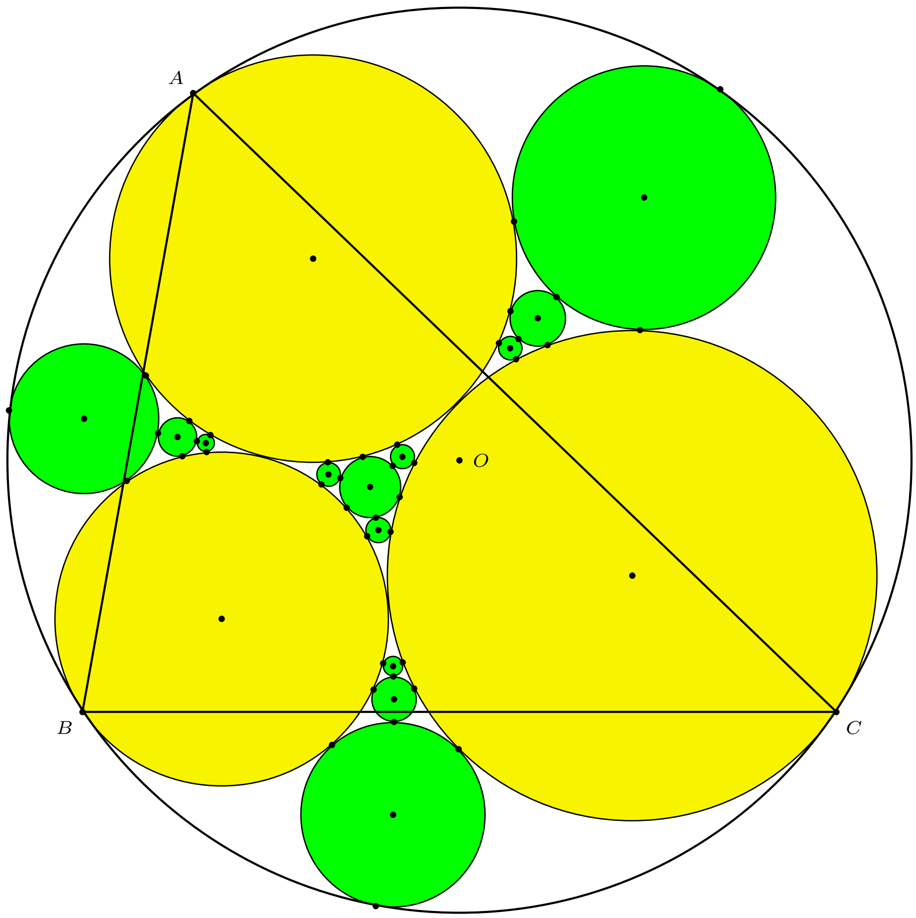


Figure 4

Proposition 8. (a) *The insimilicenters of the Lucas radical circle and the individual Lucas circles form a triangle perspective with ABC at $K^*(\arctan 3)$.*

(b) *The exsimilicenters of the Lucas radical circle and the individual Lucas circles form a triangle perspective with ABC at $K^*(\frac{\pi}{4})$.*

Proof. These insimilicenters are the points

$$\begin{array}{ccccc}(3a^2(S_A+S) & : & b^2(3S_B+S) & : & c^2(3S_C+S)),\\ (a^2(3S_A+S) & : & 3b^2(S_B+S) & : & c^2(3S_C+S)),\\ (a^2(3S_A+S) & : & b^2(3S_B+S) & : & 3c^2(S_C+S)).\end{array}$$

Likewise, the exsimilicenters are the points

$$\begin{array}{ccccc}(a^2(S_A-S) & : & b^2(S_B+S) & : & c^2(S_C+S)),\\ (a^2(S_A+S) & : & b^2(S_B-S) & : & c^2(S_C+S)),\\ (a^2(S_A+S) & : & b^2(S_B+S) & : & c^2(S_C-S)).\end{array}$$

□

8. Two conics

As explained in [1], the Lucas circles of a triangle are also associated with the inscribed squares of the triangle. We present two interesting conics associated with these inscribed squares. Given a triangle ABC, the A-inscribed square $X_1X_2X_3X_4$ has vertices

$$X_1 = (0 : S_C + S : S_B), \quad \text{and} \quad X_2 = (0 : S_C : S_B + S)$$

on the line BC and

$$X_3 = (a^2 : 0 : S) \quad \text{and} \quad X_4 = (a^2 : S : 0)$$

on AC and AB respectively. It has center $(a^2 : S_C + S : S_B + S)$. Similarly, the coordinates of the $B-$ and C-inscribed squares, and their centers, can be easily written down. It is clear that the centers of these squares form a triangle perspective with ABC at the Kiepert perspector

$$K(\frac{\pi}{4}) = \left(\frac{1}{S_A + S} : \frac{1}{S_B + S} : \frac{1}{S_C + S}\right).$$

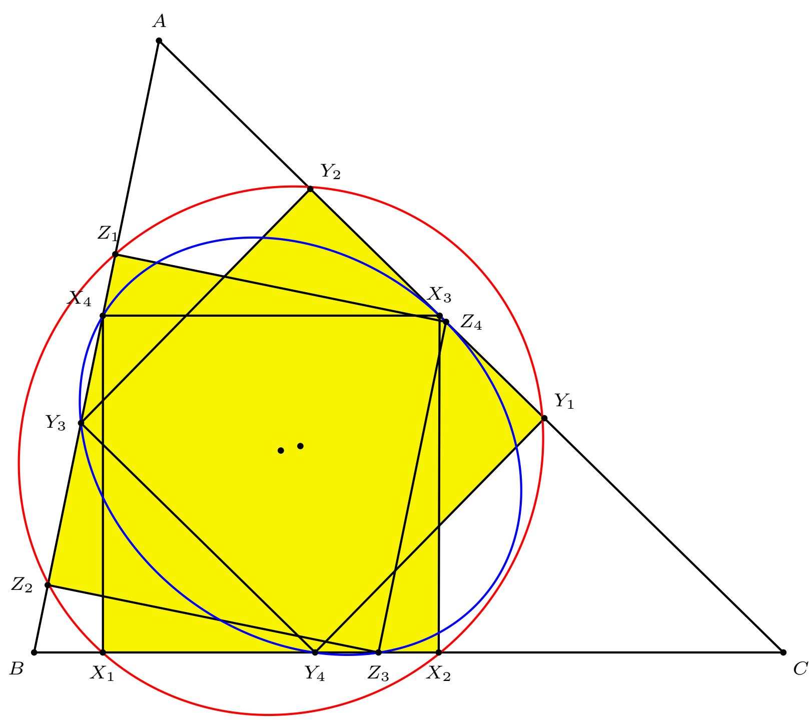


Figure 5.

Proposition 9. *The six points X_1, X_2, Y_1, Y_2, Z_1, Z_2 lie on the conic*

$$\sum_{\text{cyclic}}(a^2+S)^2yz=(x+y+z)\sum_{\text{cyclic}}S_A(S_A+S)x.$$

This conic has center $(a^2+S:b^2+S:c^2+S)$.

Proposition 10. *The six points X_3, X_4, Y_3, Y_4, Z_3, Z_4 lie on the conic*

$$\sum_{\text{cyclic}}\frac{a^2}{a^2+S}yz=\frac{a^2b^2c^2S(x+y+z)}{(a^2+S)(b^2+S)(c^2+S)}\left(\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}\right).$$

References

[1] A. P. Hatzipolakis and P. Yiu, The Lucas circles, *Amer. Math. Monthly*, 108 (2001) 444–446.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[3] W. Reyes, The Lucas circles and the Descartes formula, *Forum Geom.*, 3 (2003) 95–100.

[4] E. W. Weisstein, *Schoute Coaxal System*, from *MathWorld* – A Wolfram Web Resource, http://mathworld.wolfram.com/SchouteCoaxalSystem.html.

[5] P. Yiu, *Euclidean Geometry*, Florida Atlantic University Lecture Notes, 1998, available at http://www.math.fau.edu/yiu/Geometry,html.

[6] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University lecture notes, 2001, available at http://www.math.fau.edu/yiu/Geometry,html.

Peter J. C. Moses: Moparmatic Co., 1154 Evesham Road, Astwood Bank, Nr. Redditch Worcs. B96 6DT.

E-mail address: `mows@mopar.freeserve.co.uk`

Forum Geometricorum
Volume 5 (2005) 107–117.

FORUM GEOM
ISSN 1534-1178

On the Geometry of Equilateral Triangles

József Sándor

Dedicated to the memory of Angela Vasiu (1941-2005)

Abstract. By studying the distances of a point to the sides, respectively the vertices of an equilateral triangle, certain new identities and inequalities are deduced. Some inequalities for the elements of the Pompeiu triangle are also established.

1. Introduction

The equilateral (or regular) triangle has some special properties, generally not valid in an arbitrary triangle. Such surprising properties have been studied by many famous mathematicians, including Viviani, Gergonne, Leibnitz, Van Schooten, Toricelli, Pompeiu, Goormaghtigh, Morley, etc. ([2], [3], [4], [7]). Our aim in this paper is the study of certain identities and inequalities involving the distances of a point to the sides or the vertices of an equilateral triangle. For the sake of completeness, we shall recall some well-known results.

1.1. Let ABC be an equilateral triangle of side length $AB = BC = CA = l$, and height h. Let P be any point in the plane of the triangle. If O is the center of the triangle, then the Leibnitz relation (valid in fact for any triangle) implies that

$$\sum PA^2 = 3PO^2 + \sum OA^2. \tag{1}$$

Let $PO = d$ in what follows. Since in our case $OA = OB = OC = R = \dfrac{l\sqrt{3}}{3}$, we have $\sum OA^2 = l^2$, and (1) gives

$$\sum PA^2 = 3d^2 + l^2. \tag{2}$$

Therefore, $\sum PA^2$ = constant if and only if d = constant, *i.e.*, when P is on a circle with center O. For a proof by L. Moser via analytical geometry, see [12]. For a proof using Stewart's theorem, see [13].

1.2. Now, let P be in the interior of triangle ABC, and denote by p_a, p_b, p_c its distances from the sides. Viviani's theorem says that

$$\sum p_a = p_a + p_b + p_c = h = \frac{l\sqrt{3}}{2}.$$

Publication Date: July 20, 2005. Communicating Editor: Paul Yiu.
The author thanks the referee for some useful remarks, which have improved the presentation of the paper.

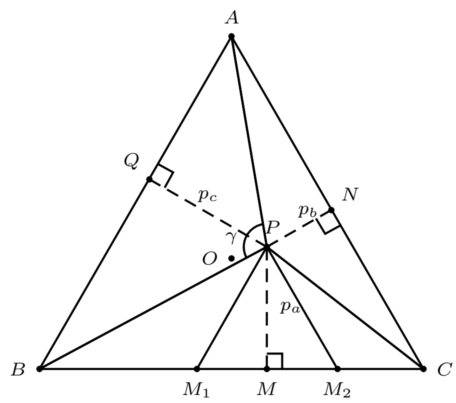


Figure 1

This follows by area considerations, since

$$S(BPC) + S(CPA) + S(APB) = S(ABC),$$

where S denotes area. Thus,

$$\sum p_a = \frac{l\sqrt{3}}{2}. \tag{3}$$

1.3. By Gergonne's theorem one has $\sum p_a^2 =$ constant, when P is on the circle of center O. For such related constants, see for example [13]. We shall obtain more general relations, by expressing $\sum p_a^2$ in terms of l and $d = OP$.

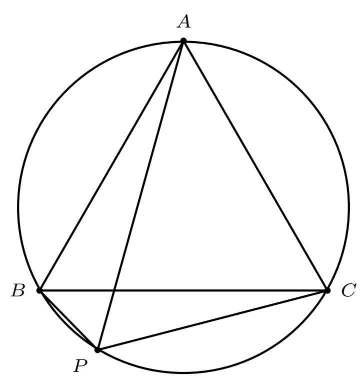


Figure 2

1.4. Another famous theorem, attributed to Pompeiu, states that for any point P in the plane of an equilateral triangle ABC, the distances PA, PB, PC can be the sides of a triangle ([9]-[10], [7], [12], [6]). (See also [1], [4], [11], [15], [16], where extensions of this theorem are considered, too.) This triangle is degenerate if P is on the circle circumscribed to ABC, since if for example P is on the interior or arc BC, then by Van Schooten's theorem,

$$PA = PA + PC. \tag{4}$$

Indeed, by Ptolemy's theorem on $ABPC$ one can write

$$PA \cdot BC = PC \cdot AB + PB \cdot AC,$$

so that $BC = AB = AC = l$ implies (4). For any other positions of P (*i.e.*, P **not** on this circle), by Ptolemy's inequality in quadrilaterals one obtains

$$PA < PB + PC, \quad PB < PA + PC, \text{ and } PC < PA + PB,$$

so that PA, PB, PC are the sides of a triangle. See [13] for many proofs. We shall call a triangle with sides PA, PB, PC a **Pompeiu triangle**. When P is in the interior, the Pompeiu triangle can be explicitly constructed. Indeed, by rotating the triangle ABP with center A through an angle of $60°$, one obtains a triangle $AB'C$ which is congruent to ABP. Then, since $AP = AB' = PB'$, $BP = CB'$, the Pompeiu triangle will be PCB'. Such a rotation will enable us also to compute the area of the Pompeiu triangle.

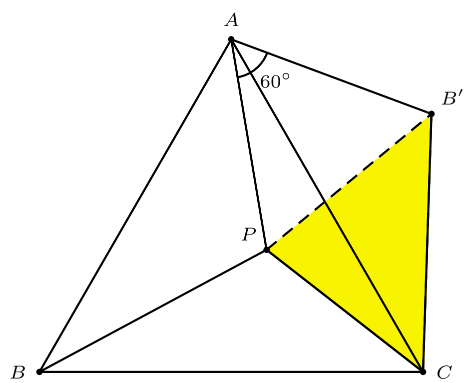


Figure 3

1.5. There exist many known inequalities for the distances of a point to the vertices of a triangle. For example, for any point P and any triangle ABC,

$$\sum PA \geq 6r, \tag{5}$$

where r is the radius of incircle (due to M. Schreiber (1935), see [7], [13]). Now, in our case $6r = l\sqrt{3}$, (5) gives

$$\sum PA \geq l\sqrt{3} \tag{6}$$

for any point P in the plane of equilateral triangle ABC. For an independent proof see [12, p.52]. This is based on the following idea: let M_1 be the midpoint of BC. By the triangle inequality one has $AP + PM_1 \geq AM_1$. Now, it is well known that

$PM_1 \le \dfrac{PB+PC}{2}$. From this, we get $l\sqrt{3} \le 2PA + PB + PC$, and by writing two similar relations, the relation (6) follows after addition. We note that already (2) implies $\sum PA^2 \ge l^2$, but (6) offers an improvement, since

$$\sum PA^2 \ge \frac{1}{3}\left(\sum PA\right)^2 \ge l^2 \tag{7}$$

by the classical inequality $x^2+y^2+z^2 \ge \dfrac{1}{3}(x+y+z)^2$. As in (7), equality holds in (6) when $P \equiv O$.

2. Identities for p_a, p_b, p_c

Our aim in this section is to deduce certain identities for the distances of an interior point to the sides of an equilateral triangle ABC.

Let P be in the interior of triangle ABC (see Figure 1). Let $PM \perp BC$, etc., where $PM = p_a$, etc. Let $PM_1 \| AB$, $PM_2 \| AC$. Then triangle PM_1M_2 is equilateral, giving $\overrightarrow{PM} = \dfrac{\overrightarrow{PM}_1 + \overrightarrow{PM}_2}{2}$. By writing two similar relations for $\overrightarrow{PQ}$ and $\overrightarrow{PN}$, and using $\overrightarrow{PO} = \dfrac{\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}}{3}$, one easily can deduce the following vectorial identity:

$$\overrightarrow{PM} + \overrightarrow{PN} + \overrightarrow{PQ} = \frac{3}{2}\overrightarrow{PO}. \tag{8}$$

Since $\overrightarrow{PM} \cdot \overrightarrow{PN} = PM \cdot PN \cdot \cos 120^\circ = -\dfrac{1}{2} PM \cdot PN$ (in the cyclic quadrilateral $CNPM$), by putting $PO = d$, one can deduce from (8)

$$\sum PM^2 + \frac{1}{2}\sum \overrightarrow{PM} \cdot \overrightarrow{PN} = \frac{9}{4}PO^2,$$

so that

$$\sum p_a^2 - \sum p_a p_b = \frac{9}{4}d^2. \tag{9}$$

For similar vectorial arguments, see [12]. On the other hand, from (3), we get

$$\sum p_a^2 + 2\sum p_a p_b = \frac{3l^2}{4}. \tag{10}$$

Solving the system (9), (10) one can deduce the following result.

Proposition 1.

$$\sum p_a^2 = \frac{l^2+6d^2}{4}, \tag{11}$$

$$\sum p_a p_b = \frac{l^2-3d^2}{4}. \tag{12}$$

There are many consequences of (11) and (12). First, $\sum p_a^2 =$ constant if and only if $d =$ constant, *i.e.*, P lying on a circle with center O. This is Gergonne's theorem. Similarly, (12) gives $\sum p_a \cdot p_b =$ constant if and only if $d =$ constant, *i.e.*, P again lying on a circle with center O. Another consequence of (11) and (12) is

$$\sum p_a p_b \le \frac{l^2}{4} \le \sum p_a^2. \tag{13}$$

An interesting connection between $\sum PA^2$ and $\sum p_a^2$ follows from (2) and (11):

$$\sum PA^2 = 2\sum p_a^2 + \frac{l^2}{2}. \tag{14}$$

3. Inequalities connecting p_a, p_b, p_c with PA, PB, PC

This section contains certain new inequalities for PA, p_a, etc. Among others, relation (18) offers an improvement of known results.

By the arithmetic-geometric mean inequality and (3), one has

$$p_a p_b p_c \le \left(\frac{p_a + p_b + p_c}{3}\right)^3 = \left(\frac{l\sqrt{3}}{6}\right)^3 = \frac{l^3\sqrt{3}}{72}.$$

Thus,

$$p_a p_b p_c \le \frac{l^3\sqrt{3}}{72} \tag{15}$$

for any interior point P of equilateral triangle ABC. This is an equality if and only only if $p_a = p_b = p_c$, *i.e.*, $P \equiv O$.

Now, let us denote $\alpha = mes(\sphericalangle BPC)$, etc. Writing the area of triangle BPC in two ways, we obtain

$$BP \cdot CP \cdot \sin\alpha = l \cdot p_a.$$

Similarly,

$$AP \cdot BP \cdot \sin\gamma = l \cdot p_c, \quad AP \cdot CP \cdot \sin\beta = l \cdot p_c.$$

By multiplying these three relations, we have

$$PA^2 \cdot PB^2 \cdot PC^2 = \frac{l^3 p_a p_b p_c}{\sin\alpha\sin\beta\sin\gamma}. \tag{16}$$

We now prove the following result.

Theorem 2. *For an interior point P of an equilateral triangle ABC, one has*

$$\prod PA^2 \ge \frac{8l^3}{3\sqrt{3}} \prod p_a \quad \textit{and} \quad \sum PA \cdot PB \ge l^2.$$

Proof. Let $f(x) = \ln \sin x$, $x \in (0, \pi)$. Since $f''(x) = -\dfrac{1}{\sin^2 x} < 0$, f is concave, and

$$f\left(\frac{\alpha+\beta+\gamma}{3}\right) \geq \frac{f(\alpha)+f(\beta)+f(\gamma)}{3},$$

giving

$$\prod \sin\alpha \leq \frac{3\sqrt{3}}{8}, \tag{17}$$

since $\dfrac{\alpha+\beta+\gamma}{3} = 120^\circ$ and $\sin 120^\circ = \dfrac{\sqrt{3}}{2}$. Thus, (16) implies

$$\prod PA^2 \geq \frac{8l^3}{3\sqrt{3}} \prod p_a. \tag{18}$$

We note that $\dfrac{8l^3}{3\sqrt{3}} \prod p_a \geq 64 \prod p_a^2$, since this is equivalent to $\prod p_a \leq \dfrac{l^3\sqrt{3}}{72}$, i.e. relation (15). Thus (18) improves the inequality

$$\prod PA \geq 8 \prod p_a \tag{19}$$

valid for any triangle (see [2, inequality 12.25], or [12, p.46], where a slightly improvement appears).

On the other hand, since $\dfrac{\alpha}{2} + \dfrac{\beta}{2} + \dfrac{\gamma}{2} = 180^\circ$, one has

$$\begin{aligned}
&\cos\alpha + \cos\beta + \cos\gamma + \frac{3}{2}\\
=&2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\cos^2\frac{\gamma}{2} + \frac{1}{2}\\
=&2\left(\cos^2\frac{\gamma}{2} - \cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + \frac{1}{4}\right)\\
=&2\left(\cos^2\frac{\gamma}{2} - \cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + \frac{1}{4}\cos^2\frac{\alpha-\beta}{2} + \frac{1}{4}\sin^2\frac{\alpha-\beta}{2}\right)\\
=&2\left[\left(\cos\frac{\gamma}{2} - \frac{1}{2}\cos\frac{\alpha-\beta}{2}\right)^2 + \frac{1}{4}\sin^2\frac{\alpha-\beta}{2}\right] \geq 0,
\end{aligned}$$

with equality only for $\alpha = \beta = \gamma = 120^\circ$. Thus:

$$\cos\alpha + \cos\beta + \cos\gamma \geq -\frac{3}{2} \tag{20}$$

for any α, β, γ satisfying $\alpha + \beta + \gamma = 360^\circ$.

Now, in triangle APB one has, by the law of cosines,

$$l^2 = PA^2 + PB^2 - 2PA \cdot PB \cdot \cos\gamma,$$

giving

$$\cos\gamma = \frac{PA^2 + PB^2 - l^2}{2PA \cdot PB}.$$

By writing two similar relations, one gets, by (20),

$$\frac{PA^2+PC^2-l^2}{2PA\cdot PC}+\frac{PB^2+PC^2-l^2}{2PB\cdot PC}+\frac{PA^2+PB^2-l^2}{2PA\cdot PB}+\frac{3}{2}\geq 0,$$

so that

$$\begin{aligned}&(PA^2\cdot PB+PB^2\cdot PA+PA\cdot PB\cdot PC)\\&+(PC^2\cdot PB+PB^2\cdot PC+PA\cdot PB\cdot PC)\\&+(PA^2\cdot PC+PC^2\cdot PA+PA\cdot PB\cdot PC)\\&-l^2(PA+PB+PC)\\&\geq 0.\end{aligned}$$

This can be rearranged as

$$(PA+PB+PC)\left(\sum PA\cdot PB-l^2\right)\geq 0,$$

and gives the inequality

$$\sum PA\cdot PB\geq l^2, \tag{21}$$

with equality when $P\equiv O$. □

4. The Pompeiu triangle

In this section, we deduce many relations connecting PA, PB, PC, etc by obtaining an identity for the area of Pompeiu triangle. In particular, a new proof of (21) will be given.

4.1. Let P be a point inside the equilateral triangle ABC (see Figure 3). The Pompeiu triangle $PB'C$ has the sides PA, PB, PC. Let R be the radius of circumcircle of this triangle. It is well known that $\sum PA^2\leq 9R^2$ (see [1, p.171], [6, p.52], [9, p.56]). By (2) we get

$$R^2\geq\frac{l^2+3d^2}{9}\geq\frac{l^2}{9}, \tag{22}$$

$$R\geq\frac{l}{3}, \tag{23}$$

with equality only for $d=0$, *i.e.*, $P\equiv O$. Inequality (23) can be proved also by the known relation $s\leq\frac{3R\sqrt{3}}{2}$, where s is the semi-perimeter of the triangle. Thus we obtain the following inequalities.

Proposition 3.

$$3R\sqrt{3}\geq\sum PA\geq l\sqrt{3}, \tag{24}$$

where the last inequality follows by (6).

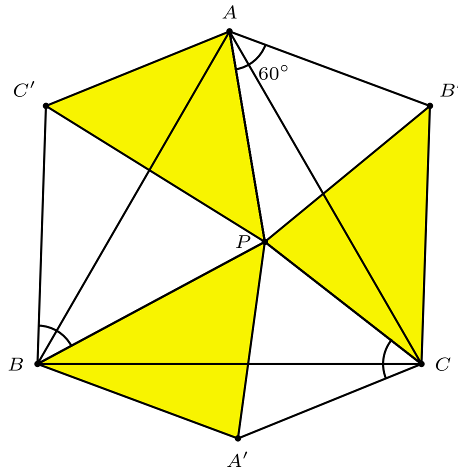


Figure 4

Now, in order to compute the area of the Pompeiu triangle, let us make two similar rotations as in Figure 3, *i.e.*, a rotation of angle $60°$ with center C of triangle APC, and another with center B of BPC. We shall obtain a hexagon (see Figure 4), $AB'CA'BC'$, where the Pompeiu triangles PBA', PAC', PCB' have equal area T. Since $\triangle APC \equiv \triangle BA'C$, $\triangle APB \equiv \triangle AB'C$, $\triangle AC'B \equiv \triangle BPC$, the area of hexagon $= 2\mathrm{Area}(ABC)$. But $\mathrm{Area}(APB') = \dfrac{AP^2\sqrt{3}}{4}$, APB' being an equilateral triangle. Therefore,

$$\frac{2l^2\sqrt{3}}{4} = 3T + \frac{PA^2\sqrt{3}}{4} + \frac{PB^2\sqrt{3}}{4} + \frac{PC^2\sqrt{3}}{4},$$

which by (2) implies

$$T = \frac{\sqrt{3}}{12}(l^2 - 3d^2). \tag{25}$$

Theorem 4. *The area of the Pompeiu triangle is given by relation (25).*

Corollary 5.

$$T \le \frac{\sqrt{3}}{12} l^2, \tag{26}$$

with equality when $d = 0$, i.e., *when* $P \equiv O$.

Now, since in any triangle of area T, and sides PA, PB, PC one has

$$2\sum PA \cdot PB - \sum PA^2 \ge 4\sqrt{3} \cdot T$$

(see for example [14], relation (8)), by (2) and (25) one can write

$$2\sum PA \cdot PB \ge 3d^2 + l^2 + l^2 - 3d^2 = 2l^2,$$

giving a new proof of (21).

Corollary 6.

$$\sum PA^2 \cdot PB^2 \geq \frac{1}{3}\left(\sum PA \cdot PB\right)^2 \geq \frac{l^4}{3}. \tag{27}$$

4.2. Note that in any triangle, $\sum PA^2 \cdot PB^2 \geq \frac{16}{9}S^2$, where $S = \mathrm{Area}(ABC)$ (see [13, pp.31-32]). In the case of equilateral triangles, (27) offers an improvement.

Since $r = \frac{T}{s}$, where s is the semi-perimeter and r the radius of inscribed circle to the Pompeiu triangle, by (6) and (26) one can write

$$r \leq \frac{\left(\frac{\sqrt{3}}{12}l^2\right)}{\left(\frac{l\sqrt{3}}{2}\right)} = \frac{l}{6}.$$

Thus, we obtain the following result.

Proposition 7. *For the radii r and R of the Pompeiu triangle one has*

$$r \leq \frac{l}{6} \leq \frac{R}{2}. \tag{28}$$

The last inequality holds true by (23). This gives an improvement of Euler's inequality $r \leq \frac{R}{2}$ for the Pompeiu triangle. Since $T = \frac{PA \cdot PB \cdot PC}{4R}$, and $r = \frac{T}{s}$, we get

$$PA \cdot PB \cdot PC = 2Rr(PA + PB + PC),$$

and the following result.

Proposition 8.

$$PA \cdot PB \cdot PC \geq \frac{2l^2 r\sqrt{3}}{3} \geq 4r^2 l\sqrt{3}. \tag{29}$$

The last inequality is the first one of (28). The following result is a counterpart of (29).

Proposition 9.

$$PA \cdot PB \cdot PC \leq \frac{\sqrt{3}l^2 R}{3}. \tag{30}$$

This follows by $T = \frac{PA \cdot PB \cdot PC}{4R}$ and (26).

4.3. The sides PA, PB, PC can be expressed also in terms of p_a, p_b, p_c. Since in triangle PNM (see Figure 1), $\sphericalangle NPM = 120°$, by the Law of cosines one has

$$MN^2 = PM^2 + PN^2 - 2PM \cdot PN \cdot \cos 120°.$$

On the other hand, in triangle NMC, $NM = PC \cdot \sin C$, PC being the diameter of circumscribed circle. Since $\sin C = \sin 60° = \frac{\sqrt{3}}{2}$, we have $MN = PC\frac{\sqrt{3}}{2}$, and the following result.

Proposition 10.

$$PC^2 = \frac{4}{3}(p_b^2 + p_a^2 + p_a p_b). \tag{31}$$

Similarly,

$$PA^2 = \frac{4}{3}(p_b^2 + p_c^2 + p_b p_c), \quad PB^2 = \frac{4}{3}(p_c^2 + p_a^2 + p_c p_a). \tag{32}$$

In theory, all elements of Pompeiu's triangle can be expressed in terms of p_a, p_b, p_c. We note that by (11) and (12) relation (2) can be proved again. By the arithmetic-geometric mean inequality, we have

$$\prod PA^2 \le \left(\frac{\sum PA^2}{3}\right)^3,$$

and the following result.

Theorem 11.

$$PA \cdot PB \cdot PC \le \left(\frac{l^2 + 3d^2}{3}\right)^{3/2}. \tag{33}$$

On the other hand, by the Pólya-Szegö inequality in a triangle (see [8], or [14]) one has

$$T \le \frac{\sqrt{3}}{4}(PA \cdot PB \cdot PC)^{2/3},$$

so by (25) one can write (using (12)):

Theorem 12.

$$PA \cdot PB \cdot PC \ge \left(\frac{l^2 - 3d^2}{3}\right)^{3/2} = \left(\frac{4\sum p_a p_b}{3}\right)^{3/2}. \tag{34}$$

4.4. Other inequalities may be deduced by noting that by (31),

$$(p_a + p_b)^2 \le PC^2 \le 2(p_a^2 + p_b^2).$$

Since $(\sqrt{x} + \sqrt{y} + \sqrt{z})^2 \le 3(x + y + z)$ applied to $x = p_a^2 + p_b^2$, etc., we get

$$\sum PA \le 4\sqrt{3} \cdot \sqrt{p_a^2 + p_b^2 + p_c^2},$$

i.e. by (11) we deduce the following inequality.

Theorem 13.

$$\sum PA \le \sqrt{3(l^2 + 6d^2)}. \tag{35}$$

This is related to (6). In fact, (6) and (35) imply that $\sum PA = l\sqrt{3}$ if and only if $d = 0$, *i.e.*, $P \equiv O$.

References

[1] N. G. Botea, Un relation entre nombres complexes et la généralization d'un théorème de géometrie élémentaire, *Bull. Math. Ph.*, VIIème Année (1935-1936), Nr. 1,2,3, 13–14.

[2] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969.

[3] H. S. M. Coxeter, *Introduction to Geometry*, John Wiley and Sons, 1961 (Hungarian Translation by Müszaki Ed., Budapest, 1973).

[4] M. Dincă, Extensions of Dimitrie Pompeiu's theorem (in Romanian), *Gazeta Matematică*, 84 (1979) nr. 10, 368–370 and 85 (1980) nr. 5, 198–199.

[5] T. Lalescu, *La Géométrie du Triangle*, Paris, 1937, 2nd ed., Ed. Jacques Gabay, reprint, Paris, 2003.

[6] N. N. Mihăileanu, *Utilization of Complex Numbers in Geometry* (Romanian), Ed. Tehnică, Bucureşti, 1968.

[7] D. S. Mitrinović, J. E. Pečarić and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Acad. Publ., 1989.

[8] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, II, Leipzig, 1925.

[9] D. Pompeiu, Une identité entre nombres complexes et un théorème de géometrie élémentaire, *Bull. Math. Phys. Ecole Polyt., Bucharest*, 6 (1936) 6–7.

[10] D. Pompeiu, *Opera matematică*, Ed. Academiei, Bucureşti, 1959.

[11] I. Pop, *Regarding Dimitrie Pompeiu's theorem on the equilateral triangle* (in Romanian), Gazeta Matematică, vol. XC, 3/1985, 70-73.

[12] J. Sándor, *Geometric Inequalities* (Hungarian), Ed. Dacia, Cluj, 1988.

[13] J. Sándor, *Geometric theorems, diophantine equations, and arithmetic functions*, American Research Press, Rehoboth, New Mexico, 2002.

[14] J. Sándor, On the cotangent inequality of a triangle, *Octogon Math. Mag.*, 12 (2004) no. 2, 738–740.

[15] J.-P. Sydler, Généralization d'un théorème de M. Pompeiu, *Elem. Math.*, 8 (1953) 136–138.

[16] V. Thébault, Sur un théorème de M. D. Pompeiu, *Bull. Math. Phys. École Polytechn. Bucarest*, 10 (1938-1939) 38–42.

József Sándor: Babeş-Bolyai University of Cluj, Romania
E-mail address: `jsandor@math.ubbcluj.ro, jjsandor@hotmail.com`

Forum Geometricorum
Volume 5 (2005) 119–126.

FORUM GEOM
ISSN 1534-1178

Construction of Brahmagupta n-gons

K. R. S. Sastry

Abstract. The Indian mathematician Brahmagupta's contributions to mathematics and astronomy are well known. His principle of adjoining Pythagorean triangles to construct general Heron triangles and cyclic quadrilaterals having integer sides, diagonals and area can be employed to appropriate Heron triangles themselves to construct any inscribable n-gon, $n \geq 3$, that has integer sides, diagonals and area. To do so we need a different description of Heron triangles by families that contain a common angle. In this paper we describe such a construction.

1. Introduction

A right angled triangle with rational sides is called a rational Pythagorean triangle. This has rational area. When these rationals are integers, it is called a Pythagorean triangle. More generally, an n-gon with rational sides, diagonals and area is called a rational Heron n-gon, $n \geq 3$. When these rationals are converted into integers by a suitable similarity transformation we obtain a Heron n-gon. If a Heron n-gon is cyclic, *i.e.*, inscribable in a circle then we obtain a Brahmagupta n-gon. In this journal and elsewhere a number of articles have appeared on various descriptions of Heron triangles and Brahmagupta quadrilaterals. Some of these are mentioned in the references. Hence we assume familiarity with the basic geometric and trigonometric results. Also, the knowledge of Pythagorean triples is assumed.

We may look upon the family of Pythagorean triangles as the particular family of Heron triangles that contain a right angle. This suggests that the complete set of Heron triangles may be described by families that contain a common Heron angle (A Heron angle has its sine and cosine rational). Once this is done we may look upon the Brahmagupta principle as follows: He took two Heron triangles ABC and $A'B'C'$ that have $\cos A + \cos A' = 0$ and ajoined them along a common side to describe Heron triangles. This enables us to generalize the Brahmagupta principle to members of appropriate families of Heron triangles to construct rational Brahmagupta n-gons, $n \geq 3$. A similarity transformation assures that these rationals can be rendered integers to obtain a Brahmagupta n-gon, $n \geq 3$.

Publication Date: August 4, 2005. Communicating Editor: Paul Yiu.

2. Description of Heron triangles by angle families

In the interest of clarity and simplicity we first take a numerical example and then give the general result [4]. Suppose that we desire the description of the family of Heron triangles ABC each member of which contains the common Heron angle given by $\cos A = \frac{3}{5}$. The cosine rule applied to a member of that family shows that the sides (a, b, c) are related by the equation

$$a^2 = b^2 + c^2 - \frac{6}{5}bc = \left(b - \frac{3}{5}c\right)^2 + \left(\frac{4}{5}c\right)^2.$$

Since a, b, c are natural numbers the triple a, $b - \frac{3}{5}c$, $\frac{4}{5}c$ must be a Pythagorean triple. That is to say

$$a = \lambda(u^2 + v^2), \quad b - \frac{3}{5}c = \lambda(u^2 - v^2), \quad \frac{4}{5}c = \lambda(2uv).$$

In the above, u, v are relatively prime natural numbers and $\lambda = 1, 2, 3, \ldots$. The least value of λ that makes c integral is 2. Hence we have the description

$$(a, b, c) = (2(u^2 + v^2),\ (u + 2v)(2u - v),\ 5uv), \quad (u, v) = 1,\ u > \frac{1}{2}v. \quad (1)$$

A similar procedure determines the Heron triangle family $A'B'C'$ that contains the supplementary angle of A, *i.e.*, $\cos A' = -\frac{3}{5}$:

$$(a, b, c) = (2(u^2 + v^2),\ (u - 2v)(2u + v),\ 5uv), \quad (u, v) = 1,\ u > 2v. \quad (2)$$

The reader is invited to check that the family (1) has $\cos A = \frac{3}{5}$ and that (2) has $\cos A' = -\frac{3}{5}$ independently of u and v.

More generally the Heron triangle family determining the common angle A given by $\cos A = \frac{p^2-q^2}{p^2+q^2}$ and the supplementary angle family generated by $\cos A' = -\frac{p^2-q^2}{p^2+q^2}$ are given respectively by

$$\begin{aligned}(a, b, c) = \quad & (pq(u^2 + v^2),\ (pu - qv)(qu + pv),\ (p^2 + q^2)uv), \\ & (u, v) = (p, q) = 1,\ u > \tfrac{q}{p}v \quad \text{and } p > q.\end{aligned} \quad (3)$$

$$\begin{aligned}(a', b', c') = \quad & (pq(u^2 + v^2),\ (pu + qv)(qu - pv),\ (p^2 + q^2)uv), \\ & (u, v) = (p, q) = 1,\ u > \tfrac{p}{q}v \quad \text{and } p > q.\end{aligned} \quad (4)$$

Areas of (3) and (4) are given by $\frac{1}{2}bc \sin A$ and $\frac{1}{2}b'c' \sin A'$ respectively. Notice that $p = 2$, $q = 1$ in (3) and (4) yield (1) and (2) and that $\angle BAC$ and $\angle B'A'C'$ are supplementary angles. Hence these triangles themselves can be adjoined when $u > \frac{p}{q}v$. The consequences are better understood by a numerical illustration:

$u = 5$, $v = 1$ in (1) and (2) yield $(a, b, c) = (52, 63, 25)$ and $(a', b', c') = (52, 33, 25)$. These can be adjoined along the common side 25. See Figure 1. The result is the isosceles triangle $(96, 52, 52)$ that reduces to $(24, 13, 13)$. As a matter of fact the families (1) and (2) or (3) and (4) may be adjoined likewise to describe the complete set of isosceles Heron triangles:

$$(a, b, c) = (2(u^2 - v^2),\ u^2 + v^2,\ u^2 + v^2), \quad u > v,\ (u, v) = 1. \tag{5}$$

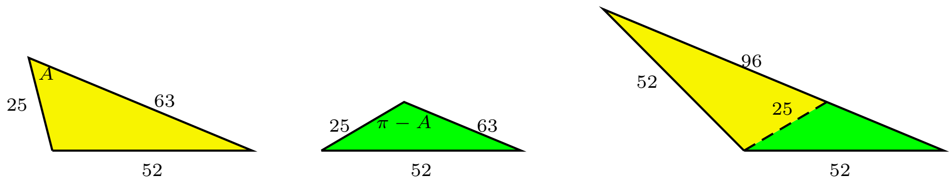


Figure 1

As mentioned in the beginning of this section, the general cases involve routine algebra so the details are left to the reader.

However, the families (1) and (2) or (3) and (4) may be adjoined in another way. This generates the complete set of Heron triangles. Again, we take a numerical illustration.

$u = 3$, $v = 2$ in (1) yields $(a, b, c) = (13, 14, 15)$ (after reduction by the gcd of (a, b, c)). Now we put different values for u, v in (2), say, $u = 4$, $v = 1$. This yields $(a', b', c') = (17, 9, 10)$. It should be remembered that we still have $\angle BAC + \angle B'A'C' = \pi$. As they are, triangles ABC and $A'B'C'$ cannot be adjoined. They must be enlarged suitably by similarity transformations to have $AB = A'B'$, and then adjoined. See Figure 2.

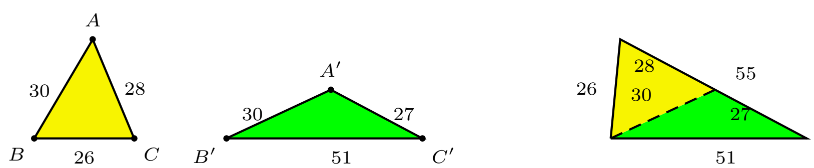


Figure 2

The result is the new Heron triangle $(55, 26, 51)$. More generally, if we put $u = u_1$, $v = v_1$ in (1) or(3) and $u = u_2$, $v = v_2$ in (2) or (4) and after applying the necessary similiarity transformations, the adjoin (after reduction by the gcd) yields

$$(a, b, c) = (u_1 v_1 (u_2^2 + v_2^2),\ (u_1^2 - v_1^2) u_2 v_2 + (u_2^2 - v_2^2) u_1 v_1,\ u_2 v_2 (u_1^2 + v_1^2). \tag{6}$$

This is the same description of Heron triangles that Euler and others obtained [1]. Now we easily see that Brahmagupta took the case of $p = q$ in (3) and (4).

In the next section we extend this remarkable adjoining idea to generate Brahmagupta n-gons, $n > 3$. At this point recall Ptolemy's theorem on convex cyclic quadrilaterals: *The product of the diagonals is equal to the sum of the products of the two pairs of opposite sides*. Here is an important observation: In a convex cyclic quadrilateral with sides a, b, c, d in order and diagonals e, f, Ptolemy's theorem, viz., $ef = ac + bd$ shows that if five of the preceding elements are rational then the sixth one is also rational.

3. Construction of Brahmagupta n-gons, $n > 3$

It is now clear that we can take any number of triangles, either all from one of the families or some from one family and some from the supplementary angle family and place them appropriately to construct a Brahmagupta n-gon. To convince the reader we do illustrate by numerical examples. We extensively deal with the case $n = 4$. This material is different from what has appeared in [5, 6]. The following table shows the primitive (a, b, c) and the suitably enlarged one, also denoted by (a, b, c). T_1 to T_6 are family (1) triangles, and T_7, T_8 are family (2) triangles. These triangles will be used in the illustrations to come later on.

Table 1: Heron triangles

	u	v	Primitive (a, b, c)	Enlarged (a, b, c)
T_1	3	1	$(4, 5, 3)$	$(340, 425, 255)$
T_2	4	1	$(17, 21, 10)$	$(340, 420, 200)$
T_3	5	3	$(68, 77, 75)$	$(340, 385, 375)$
T_4	7	6	$(85, 76, 105)$	$(340, 304, 420)$
T_5	9	2	$(85, 104, 45)$	$(340, 416, 180)$
T_6	13	1	$(68, 75, 13)$	$(340, 375, 65)$
T_7	4	1	$(17, 9, 10)$	$(340, 180, 200)$
T_8	13	1	$(340, 297, 65)$	$(340, 297, 65)$

The same or different Heron triangles can be adjoined in different ways. We first show this in the illustration of the case of quadrilaterals. Once the construction process is clear, the case of $n > 4$ would be analogous to that $n = 4$. Hence we just give one illustration of $n = 5$ and $n = 6$.

3.1. *Brahmagupta quadrilaterals.* The Brahmagupta quadrilateral can be generated in the following ways:

(i) A triangle taken from family (1) (respectively (3)) or family (2) (respectively (4), henceforth this is to be understood) adjoined with itself,

(ii) two different triangles taken from the same family adjoined,

(iii) one triangle taken from family (1) adjoined with a triangle from family (2).

Here are examples of each case.

Example 1. We take the primitive $(a, b, c) = (17, 21, 10)$, *i.e.*, T_2 and adjoin with itself (see Figure 3). Since $\angle CAD = \angle CBD$, $ABCD$ is cyclic. Ptolemy's theorem shows that $AB = \frac{341}{17}$ is rational. By enlarging the sides and diagonals 17 times each we get the Brahmagupta quadrilateral $ABCD$, in fact a trapezoid, with

$$AB = 341,\ BC = AD = 170,\ CD = 289,\ AC = BD = 357.$$

See Figure 3. Rather than calculating the actual area, we give an argument that shows that the area is integral. This is so general that it is applicable to other adjunctions to follow in our discussion.

Since $\angle BAC = \angle BDC$, $\angle ABD = \angle ACD$, and $\angle BAD = \angle BAC + \angle CAD$, $\angle BAD$ is also a Heron angle and that triangle ABD is Heron. (Note:

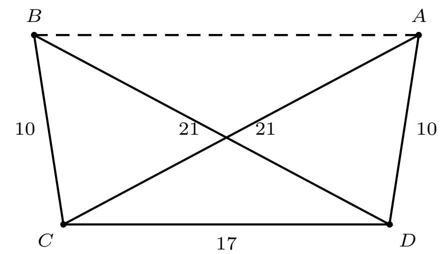


Figure 3

If α and β are Heron angles then $\alpha \pm \beta$ are also Heron angles. To see this consider $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$). $ABCD$ being the disjoint sum of the Heron triangles BCD and BDA, its area must be integral.

This particular adjunction can be done along any side, ***i.e.***, 17, 10, or 21. However, such a liberty is not enjoyed by the remaining constructions which involve adjunction of different Heron triangles. We leave it to the reader to figure out why.

Example 2. We adjoin the primitive triangles T_4, T_5 from Table 1. This can be done in two ways.

(i) Figure 4A illustrates one way. As in Example 1, $AB = \frac{1500}{17}$, so Figure 4A is enlarged 17 times. The area is integral (reasoned as above). Hence the resulting quadrilateral is Brahmagupta.

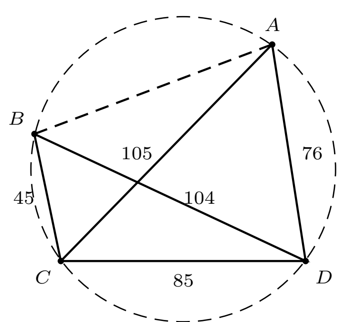


Figure 4A

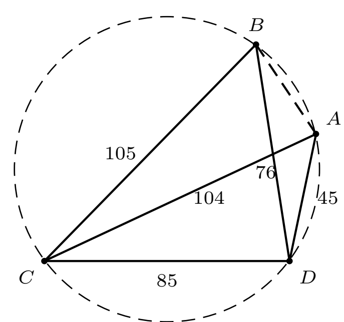


Figure 4B

(ii) Figure 4B illustrates the second adjunction in which the vertices of one base are in reverse order. In this case, $AB = \frac{187}{5}$ hence the figure needs only five times enlargement. Henceforth, we omit the argument to show that the area is integral.

Example 3. We adjoin the primitive triangles T_1 and T_7, which contain supplementary angles A and $\pi - A$. Here, too two ways are possible. In each case no enlargement is necessary. See Figures 5A and 5B.

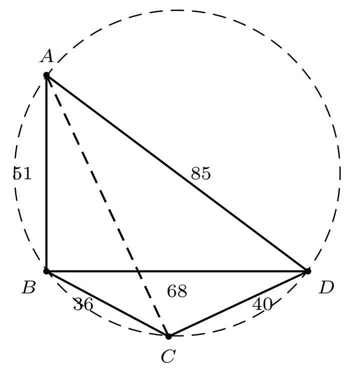


Figure 5A

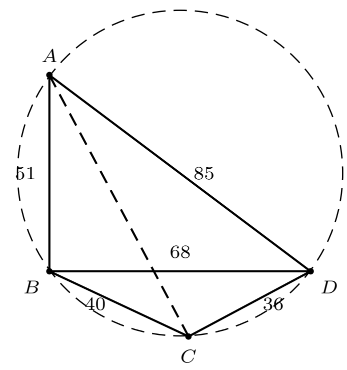


Figure 5B

3.2. *Brahmagupta pentagons.* To construct a Brahmagupta pentagon we need three Heron triangles, in general, taken either all from (1) or some from (1) and the rest from (2) in any combination. Here, too, one triangle can be used twice as in Example 1 above. Hence, a Brahmagupta pentagon can be constructed in more than two ways. We give just one illustration using the (enlarged) triangles T_3, T_4, and T_7. The reader is invited to play the adjuction game using these to consider all possibilities, *i.e.*, T_3, T_3, T_4; T_3, T_4, T_4; T_7, T_7, T_3 etc.

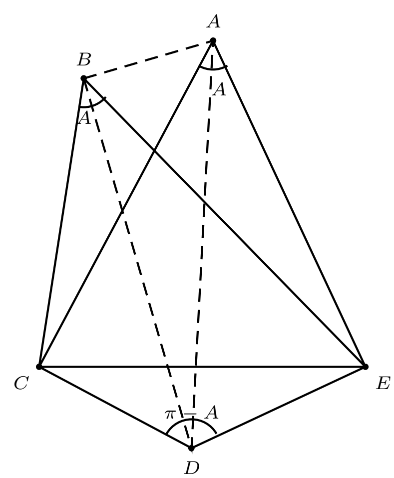


Figure 6

Figure 6 shows one Brahmagupta pentagon. It is easy to see that it must be cyclic. The side AB, the diagonals AD and BD are to be calculated. We apply Ptolemy's theorem successively to $ABCE$, $ACDE$ and $BCDE$. This yields

$$AB = \frac{2023}{17}, \quad AD = \frac{7215}{17}, \quad BD = \frac{6820}{17}.$$

The figure needs 17 times enlargement. The area $ABCDE$ must be integral because it is the disjoint sum of the Brahmagupta quadrilateral $ABCE$ and the Heron triangle ACD.

3.3. *Brahmagupta hexagons.* To construct a Brahmagupta hexagon it is now easy to see that we need at most four Heron triangles taken in any combination from the families (1) and (2). We use the four triangles T_2, T_3, T_5, T_8 to illustrate the hexagon in Figure 7. We leave the calculations to the reader.

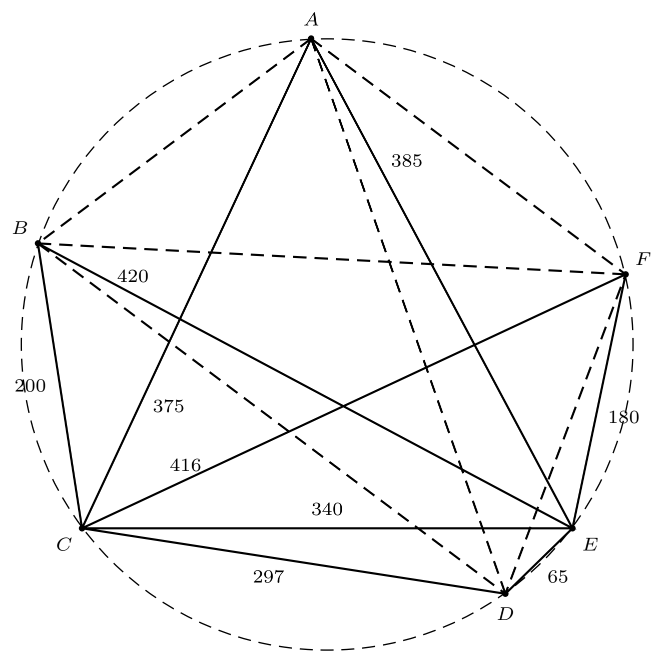


Figure 7

4. Conclusion

In principle the problem of determining Brahmagupta n-gons, $n > 3$, has been solved because all Heron triangle families have been determined by (3) and (4) (in fact by (3) alone). In general to construct a Brahmagupta n-gon, at most $n - 2$ Heron triangles taken in any combination from (3) and (4) are needed. They can be adjoined as described in this paper. We pose the following counting problem to the reader.

Given $n - 2$ Heron triangles, (i) all from a single family, or (ii) m from one Heron family and the remaining $n - m - 2$ from the supplementary angle family, how many Brahmagupta n-gons can be constructed?

It is now natural to conjecture that Heron triangles chosen from appropriate families adjoin to give Heron n-gons. To support this conjecture we give two Heron quadrilaterals generated in this way.

Example 4. From the $\cos\theta = \frac{3}{5}$ family, $7(5, 5, 6)$ and $6(4, 13, 15)$ adjoined with $(35, 53, 24)$ and $6(7, 15, 20)$ from the supplementary family (with $\cos\theta = -\frac{3}{5}$) to give $ABCD$ with

$$AB = 35,\ BC = 53,\ CD = 78,\ AD = 120,\ AC = 66,\ BD = 125,$$

and area 3300. See Figure 8A.

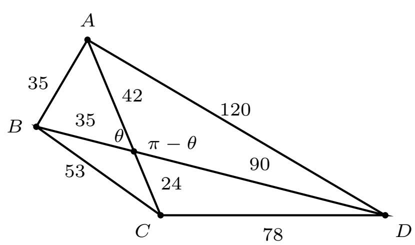


Figure 8A

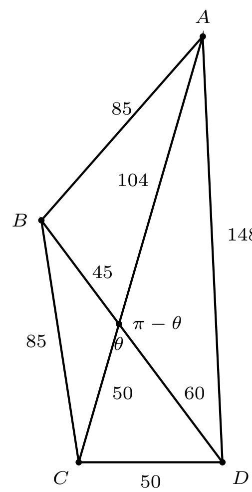


Figure 8B

Example 5. From the same families, the Heron triangles $10(5, 5, 6)$, $(85, 45, 104)$ with $5(17, 9, 10)$ and $4(37, 15, 26)$ to give a Heron quadrilateral $ABCD$ with

$$AB = 85,\ BC = 85,\ CD = 50,\ AD = 148,\ AC = 154,\ BD = 105,$$

and area 6468. See Figure 8B.

Now, the haunting question is: Which appropriate two members of the θ family adjoin with two appropriate members of the $\pi - \theta$ family to generate Heron quadrilaterals?

References

[1] L. E. Dickson, *History of the Theory of Numbers*, vol. II, Chelsea, New York, 1977; pp.165–224.
[2] C. Pritchard, Brahmagupta, *Math. Spectrum*, 28 (1995-96) 49–51.
[3] K. R. S. Sastry, Heron triangles, *Forum Geom.*, 1 (2001) 25–32.
[4] K. R. S. Sastry, Heron angles, *Mathematics and Computer Education*, 35 (2001) 51–60.
[5] K. R. S. Sastry, Brahmagupta quadrilaterals, *Forum Geom.*, 2 (2002) 167–174.
[6] K. R. S. Sastry, Brahmagupta quadrilaterals: A description, *Crux Math.*, 29 (2003) 39–42.

K. R. S. Sastry: Jeevan Sandhya, DoddaKalsandra Post, Raghuvana Halli, Bangalore, 560 062, India.

Forum Geometricorum
Volume 5 (2005) 127–132.

FORUM GEOM
ISSN 1534-1178

Another Proof of van Lamoen's Theorem and Its Converse

Nguyen Minh Ha

Abstract. We give a proof of Floor van Lamoen's theorem and its converse on the circumcenters of the cevasix configuration of a triangle using the notion of directed angle of two lines.

1. Introduction

Let P be a point in the plane of triangle ABC with traces A', B', C' on the sidelines BC, CA, AB respectively. We assume that P does not lie on any of the sidelines. According to Clark Kimberling [1], triangles PCB', $PC'B$, PAC', $PA'C$, PBA', $PB'A$ form the *cevasix configuration* of P. Several years ago, Floor van Lamoen discovered that when P is the centroid of triangle ABC, the six circumcenters of the cevasix configuration are concylic. This was posed as a problem in the *American Mathematical Monthly* and was solved in [2, 3]. In 2003, Alexei Myakishev and Peter Y. Woo [4] gave a proof for the converse, that is, if the six circumcenters of the cevasix configuration are concylic, then P is either the centroid or the orthocenter of the triangle.

In this note we give a new proof, which is quite different from those in [2, 3], of Floor van Lamoen's theorem and its converse, using the directed angle of two lines. Remarkably, both necessity part and sufficiency part in our proof are basically the same. The main results of van Lamoen, Myakishev and Woo are summarized in the following theorem.

Theorem. *Given a triangle ABC and a point P, the six circumcenters of the cevasix configuration of P are concyclic if and only if P is the centroid or the orthocenter of ABC.*

We shall assume the given triangle non-equilateral, and omit the easy case when ABC is equilateral. For convenience, we adopt the following notations used in [4].

Triangle	PCB'	$PC'B$	PAC'	$PA'C$	PBA'	$PB'A$
Notation	$\triangle(A_+)$	$\triangle(A_-)$	$\triangle(B_+)$	$\triangle(B_-)$	$\triangle(C_+)$	$\triangle(C_-)$
Circumcenter	A_+	A_-	B_+	B_-	C_+	C_-

It is easy to see that two of these triangles may possibly share a common circumcenter only when they share a common vertex of triangle ABC.

Publication Date: August 24, 2005. Communicating Editor: Floor van Lamoen.

The author thansk Le Chi Quang of Hanoi, Vietnam for his help in translation and preparation of the article.

2. Preliminary Results

Lemma 1. *Let P be a point not on the sidelines of triangle ABC, with traces B', C' on AC, AB respectively. The circumcenters of triangles APB' and APC' coincide if and only if P lies on the reflection of the circumcircle ABC in the line BC.*

The Proof of Lemma 1 is simple and can be found in [4]. We also omit the proof of the following easy lemma.

Lemma 2. *Given a triangle ABC and M, N on the line BC, we have*

$$\frac{\overline{BC}}{\overline{MN}} = \frac{S[ABC]}{S[AMN]},$$

where $\overline{BC}$ and $\overline{MN}$ denote the signed lengths of the line segments BC and MN, and $S[ABC]$, $S[AMN]$ the signed areas of triangle ABC, and AMN respectively.

Lemma 3. *Let P be a point not on the sidelines of triangle ABC, with traces A', B', C' on BC, AC, AB respectively, and K the second intersection of the circumcircles of triangles PCB' and $PC'B$. The line PK is a symmedian of triangle PBC if and only if A' is the midpoint of BC.*

Proof. Triangles $KB'B$ and KCC' are directly similar (see Figure 1). Therefore,

$$\frac{S[KB'B]}{S[KCC']} = \Big(\frac{\overline{B'B}}{\overline{CC'}}\Big)^2.$$

On the other hand, by Lemma 2 we have

$$\frac{S[KPB]}{S[KPC]} = \frac{\frac{\overline{PB}}{\overline{B'B}} \cdot S[KB'B]}{\frac{\overline{PC}}{\overline{CC'}} \cdot S[KCC']}.$$

Thus,

$$\frac{S[KPB]}{S[KPC]} = \frac{\overline{PB}}{\overline{PC}}.\frac{\overline{B'B}}{\overline{CC'}}.$$

It follows that PK is a symmedian line of triangle PBC, which is equivalent to the following

$$\frac{S[KPB]}{S[KPC]} = -\Big(\frac{\overline{PB}}{\overline{PC}}\Big)^2, \quad \frac{\overline{PB}.\overline{B'B}}{\overline{PC}.\overline{CC'}} = -\Big(\frac{\overline{PB}}{\overline{PC}}\Big)^2, \quad \frac{\overline{B'B}}{\overline{C'C}} = \frac{\overline{PB}}{\overline{PC}}.$$

The last equality is equivalent to $BC \parallel B'C'$, by Thales' theorem, or A' is the midpoint of BC, by Ceva's theorem. □

Remark. Since the lines BC' and CB' intersect at A, the circumcircles of triangles PCB' and $PC'B$ must intersect at two distinct points. This remark confirms the existence of the point K in Lemma 3.

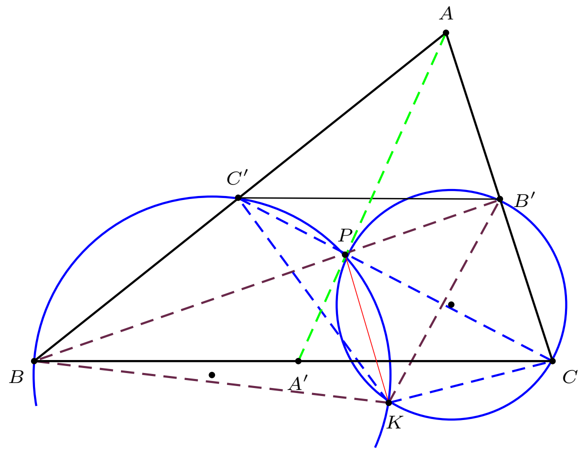


Figure 1

Lemma 4. *Given a triangle XYZ and pairs of points M, N on YZ, P, Q on ZX, and R, S on XY respectively. If the points in each of the quadruples P, Q, R, S; R, S, M, N; M, N, P, Q are concyclic, then all six points M, N, P, Q, R, S are concyclic.*

Proof. Suppose that (O_1), (O_2), (O_3) are the circles passing through the quadruples (P, Q, R, S), (R, S, M, N), and (M, N, P, Q) respectively. If O_1, O_2, O_3 are disctinct points, then YZ, ZX, XY are respectively the radical axis of pairs of circles $(O_2), (O_3)$; $(O_3), (O_1)$; $(O_1), (O_2)$. Hence, YZ, ZX, XY are concurrent, or parallel, or coincident, which is a contradiction. Therefore, two of the three points O_1, O_2, O_3 coincide. It follows that six points M, N, P, Q, R, S are concyclic. □

Remark. In Lemma 4, if $M = N$ and the circumcircles of triangles RSM, MPQ touch YZ at M, then the five points M, P, Q, R, S lie on the same circle that touches YZ at the same point M.

3. Proof of the main theorem

Suppose that perpendicular bisectors of AP, BP, CP bound a triangle XYZ. Evidently, the following pairs of points B_+, C_-; C_+, A_-; A_+, B_- lie on the lines YZ, ZX, XY respectively. Let H and K respectively be the feet of the perpendiculars from P on A_-A_+, B_-B_+ (see Figure 2).

Sufficiency part. If P is the orthocenter of triangle ABC, then $B_+ = C_-$; $C_+ = A_-$; $A_+ = B_-$. Obviously, the six points B_+, C_-, C_+, A_-, A_+, B_- lie on the same circle. If P is the centroid of triangle ABC, then no more than one of the three following possibilities happen: $B_+ = C_-$; $C_+ = A_-$; $A_+ = B_-$, by Lemma 1. Hence, we need to consider two cases.

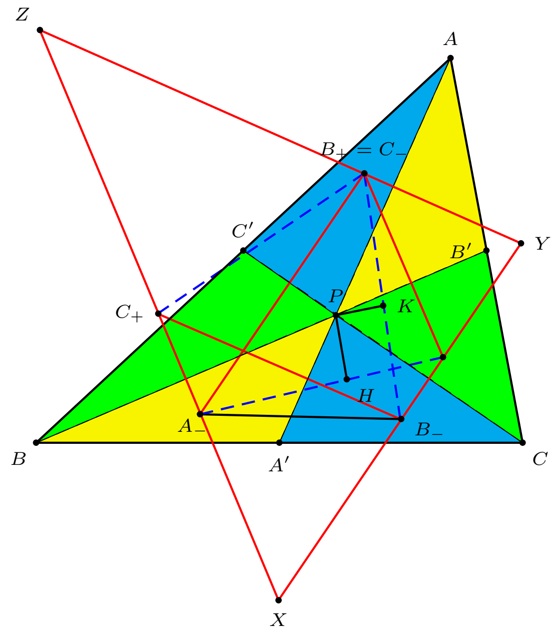


Figure 2

Case 1. Only one of three following possibilities occurs: $B_+ = C_-$, $C_+ = A_-$, $A_+ = B_-$.

Without loss of generality, we may assume that $B_+ = C_-$, $C_+ \neq A_-$ and $A_+ \neq B_-$ (see Figure 2). Since P is the centroid of triangle ABC, A' is the midpoint of the segment BC. By Lemma 3, we have

$$(PH, PB) = (PC, PA') \pmod{\pi}.$$

In addition, since A_-A_+, A_-C_+, B_-A_+, B_-C_+ are respectively perpendicular to PH, PB, PC, PA', we have

$$(A_-A_+, A_-C_+) \equiv (PH, PB) \pmod{\pi}.$$
$$(B_-A_+, B_-C_+) \equiv (PC, PA') \pmod{\pi}.$$

Thus, $(A_-A_+, A_-C_+) \equiv (B_-A_+, B_-C_+) \pmod{\pi}$, which implies that four points C_+, A_-, A_+, B_- are concyclic.

Similarly, we have

$$(PK, PC) = (PA, PB') \pmod{\pi}.$$

Moreover, since B_-B_+, B_-A_+, YZ, B_+A_+ are respectively perpendicular to PK, PC, PA, PB', we have

$$(B_-B_+, B_-A_+) \equiv (PK, PC) \pmod{\pi}.$$
$$(YZ, B_+A_+) \equiv (PA, PB') \pmod{\pi}.$$

Thus, $(B_-B_+, B_-A_+) \equiv (YZ, B_+A_+) \pmod{\pi}$, which implies that the circumcircle of triangle $B_+B_-A_+$ touches YZ at B_+.

The same reasoning also shows that the circumcircle of triangle $B_+C_+A_-$ touches YZ at B_+.

Therefore, the six points B_+, C_-, C_+, A_-, A_+, B_- lie on the same circle and this circle touches YZ at $B_+ = C_-$ by the remark following Lemma 4.

Case 2. None of the three following possibilities occurs: $B_+ = C_-; C_+ = A_-; A_+ = B_-$.

Similarly to case 1, each quadruple of points (C_+, A_-, A_+, B_-), (A_+, B_-, B_+, C_-), (B_+, C_-, C_+, A_-) are concyclic. Hence, by Lemma 4, the six points B_+, C_-, C_+, A_-, A_+, B_- are concyclic.

Necessity part. There are three cases.

Case 1. No less than two of the following possibilities occur: $B_+ = C_-$, $C_+ = A_-$, $A_+ = B_-$.

By Lemma 1, P is the orthocenter of triangle ABC.

Case 2. Only one of the following possibilites occurs: $B_+ = C_-$, $C_+ = A_-$, $A_+ = B_-$. We assume without loss of generality that $B_+ = C_-$, $C_+ \neq A_-$, $A_+ \neq B_-$.

Since the six points B_+, C_-, C_+, A_-, A_+, B_- are on the same circle, so are the four points C_+, A_-, A_+, B_-. It follows that

$$(A_-A_+, A_-C_+) \equiv (B_-A_+, B_-C_+) \pmod{\pi}.$$

Note that lines PH, PB, PC, PA' are respectively perpendicular to A_-A_+, A_-C_+, B_-A_+, B_-C_+. It follows that

$$(PH, PB) \equiv (A_-A_+, A_-C_+) \pmod{\pi}.$$
$$(PC, PA') \equiv (B_-A_+, B_-C_+) \pmod{\pi}.$$

Therefore, $(PH, PB) \equiv (PC, PA') \pmod{\pi}$. Consequently, A' is the midpoint of BC by Lemma 3.

On the other hand, it is evident that $B_+A_- \parallel B_-A_+$; $B_+A_+ \parallel C_+A_-$, and we note that each quadruple of points (B_+, A_-, B_-, A_+), (B_+, A_+, C_+, A_-) are concyclic. Therefore, we have $B_+B_- = A_+A_- = B_+C_+$. It follows that triangle $B_+B_-C_+$ is isosceles with $C_+B_+ = B_+B_-$. Note that YZ passes B_+ and is parallel to C_+B_-, so that we have YZ touches the circle passing six points $B_+ = C_-$, C_+, A_-, A_+, B_- at $B_+ = C_-$. It follows that

$$(B_-B_+, B_-A_+) \equiv (YZ, B_+A_+) \pmod{\pi}.$$

In addition, since PK, PC, PA, PB' are respectively perpendicular to B_-B_+, B_-A_+, YZ, B_+A_+, we have

$$(PK, PC) \equiv (B_-B_+, B_-A_+) \pmod{\pi}.$$
$$(PA, PB') \equiv (YZ, B_+A_+) \pmod{\pi}.$$

Thus, $(PK, PC) \equiv (PA, PB') \pmod{\pi}$. By Lemma 3, B' is the midpoint of CA. We conclude that P is the centroid of triangle ABC.

Case 3. None of the three following possibilities occur: $B_+ = C_-$, $C_+ = A_-$, $A_+ = B_-$.

Similarly to case 2, we can conclude that A', B' are respectively the midpoints of BC, CA. Thus, P is the centroid of triangle ABC.

This completes the proof of the main theorem.

References

[1] C. Kimberling Triangle centers and central triangles, *Congressus Numeratium*, 129 (1998), 1–285

[2] F. M. van Lamoen, Problem 10830, *Amer. Math. Monthly*, 2000 (107) 863; solution by the Monthly editors, 2002 (109) 396–397.

[3] K. Y. Li, Concyclic problems, *Mathematical Excalibur*, 6 (2001) Number 1, 1–2; available at http://www.math.ust.hk/excalibur.

[4] A. Myakishev and Peter Y. Woo, On the Circumcenters of Cevasix Configurations, *Forum Geom.*, 3 (2003) 57–63.

Nguyen Minh Ha: Faculty of Mathematics, Hanoi University of Education, Xuan Thuy, Hanoi, Vietnam

E-mail address: minhha27255@yahoo.com

Forum Geometricorum
Volume 5 (2005) 133–134.

FORUM GEOM
ISSN 1534-1178

Some More Archimedean Circles in the Arbelos

Frank Power

Abstract. We construct 4 circles in the arbelos which are congruent to the Archimedean twin circles.

Thomas Schoch [2] tells the remarkable story of his discovery in the 1970's of the many Archimedean circles in the arbelos (shoemaker's knife) that were eventually recorded in the paper [1]. In this note, we record four more Archimedean circles which were discovered in the summer of 1998, when the present author took a geometry course ([3]) with one of the authors of [1].

Consider an arbelos with inner semicircles $\mathcal{C}_1$ and $\mathcal{C}_2$ of radii a and b, and outer semicircle $\mathcal{C}$ of radius $a+b$. It is known the Archimedean circles have radius $t = \frac{ab}{a+b}$. Let Q_1 and Q_2 be the "highest" points of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively.

Theorem. *A circle tangent to $\mathcal{C}$ internally and to OQ_1 at Q_1 (or OQ_2 at Q_2) has radius $t = \frac{ab}{a+b}$.*

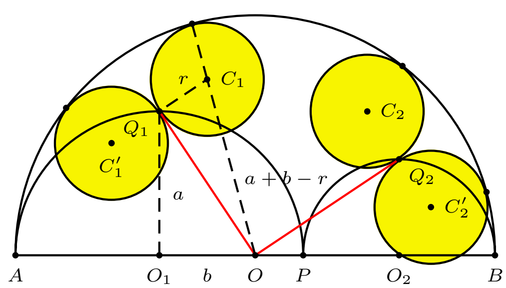


Figure 1

Proof. There are two such circles tangent at Q_1, namely, (C_1) and (C_1') in Figure 1. Consider one such circle (C_1) with radius r. Note that

$$OQ_1^2 = O_1Q_1^2 + OO_1^2 = a^2 + b^2.$$

It follows that

$$(a+b-r)^2 = (a^2+b^2) + r^2,$$

from which $r = \frac{ab}{a+b} = t$. The same calculation shows that (C_1') also has radius t, and similarly for the two circles at Q_2. □

Publication Date: September 2, 2005. Communicating Editor: Paul Yiu.

References

[1] C. W. Dodge, T. Schoch, P. Y. Woo and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.*, 72 (1999) 202–213.
[2] T. Schoch, Arbelos, http://www.retas.de/thomas/arbelos/arbelos.html.
[3] P. Yiu, *Euclidean Geometry*, available at http://www.math.fau.edu/Yiu/Geometry.html.

Frank Power: Atlantic Community High School, 2455 W. Atlantic Avenue, Delray Beach, Florida, 33445, USA
E-mail address: powerrf@palmbeach.k12.fl.us

Forum Geometricorum
Volume 5 (2005) 135–136.

FORUM GEOM
ISSN 1534-1178

Divison of a Segment in the Golden Section with Ruler and Rusty Compass

Kurt Hofstetter

Abstract. We give a simple 5-step division of a segment into golden section, using ruler and rusty compass.

In [1] we have given a 5-step division of a segment in the golden section with ruler and compass. We modify the construction by using a *rusty* compass, *i.e.*, one when set at a particular opening, is not permitted to change. For a point P and a segment AB, we denote by $P(AB)$ the circle with P as center and AB as radius.

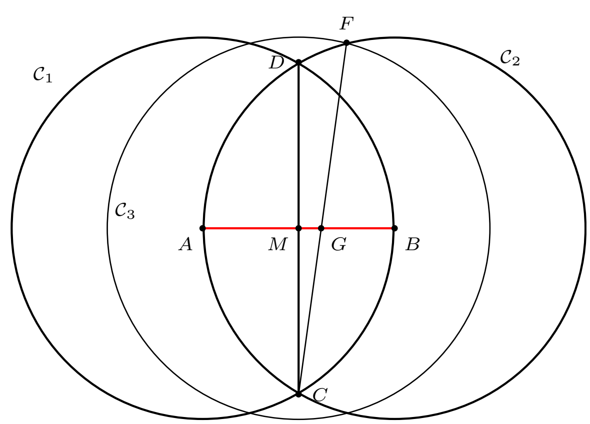


Figure 1

Construction. Given a segment AB, construct

(1) $\mathcal{C}_1 = A(AB)$,
(2) $\mathcal{C}_2 = B(AB)$, intersecting $\mathcal{C}_1$ at C and D,
(3) the line CD to intersect AB at its midpoint M,
(4) $\mathcal{C}_3 = M(AB)$ to intersect $\mathcal{C}_2$ at F (so that C and D are on opposite sides of AB),
(5) the segment CF to intersect AB at G.

The point G divides the segment AB in the golden section.

Publication Date: September 13, 2005. Communicating Editor: Paul Yiu.

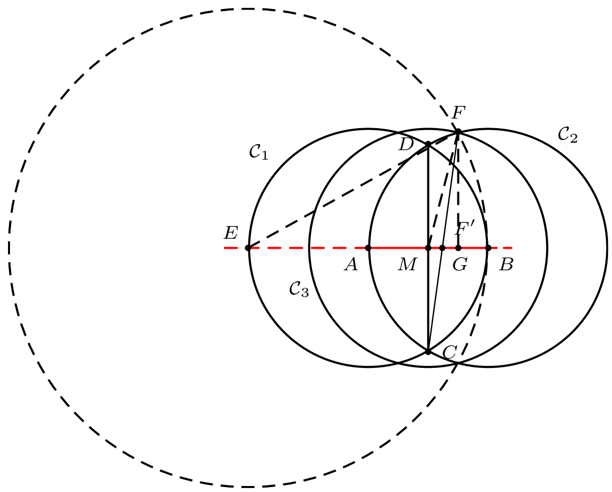


Figure 2

Proof. Extend BA to intersect $\mathcal{C}_1$ at E. According to [1], it is enough to show that $EF = 2 \cdot AB$. Let F' be the orthogonal projection of F on AB. It is the midpoint of MB. Without loss of generality, assume $AB = 4$, so that $MF' = F'B = 1$ and $EF' = 2 \cdot AB - F'B = 7$. Applying the Pythagorean theorem to the right triangles EFF' and MFF', we have

$$\begin{aligned} EF^2 =& EF'^2 + FF'^2 \\ =& EF'^2 + MF^2 - MF'^2 \\ =& 7^2 + 4^2 - 1^2 \\ =& 64. \end{aligned}$$

This shows that $EF = 8 = 2 \cdot AB$. $\square$

References

[1] K. Hofstetter, Another 5-step division of a segment in the golden section, *Forum Geom.*, 4 (2004) 21–22.

Kurt Hofstetter: Object Hofstetter, Media Art Studio, Langegasse 42/8c, A-1080 Vienna, Austria
E-mail address: pendel@sunpendulum.at

Forum Geometricorum
Volume 5 (2005) 137–141.

FORUM GEOM
ISSN 1534-1178

On an Erdős Inscribed Triangle Inequality

Ricardo M. Torrejón

Abstract. A comparison between the area of a triangle and that of an inscribed triangle is investigated. The result obtained extend a result of Aassila giving insight into an inequality of P. Erdős.

1. Introduction

Consider a triangle ABC divided into four smaller non-degenerate triangles, a central one $C_1A_1B_1$ inscribed in ABC and three others on the sides of this central triangle, as depicted in

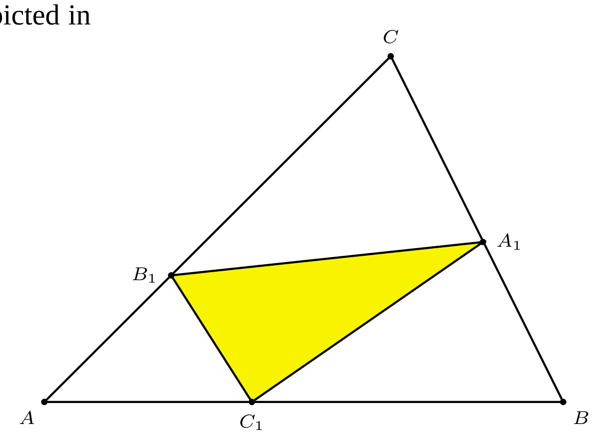


Figure 1

A question with a long history is that of comparing the area of ABC to that of the inscribed triangle $C_1A_1B_1$. In 1956, H. Debrunnner [5] proposed the inequality

$$\mathbf{area}\,(C_1A_1B_1) \geq \min\{\mathbf{area}\,(AC_1B_1),\ \mathbf{area}\,(C_1BA_1),\ \mathbf{area}\,(B_1A_1C)\}\,; \tag{1}$$

according to John Rainwater [7], this inequality originated with P. Erdős and was communicated by N. D. Kazarinoff and J. R. Isbell. However, Rainwater was more precise in stating that $C_1A_1B_1$ cannot have the smallest area of the four unless all four are equal with A_1, B_1, and C_1 the midpoints of the sides BC, CA, and AB.

A proof of (1) first appeared in A. Bager [2] and later in A. Bager [3] and P. H. Diananda [6]. Diananda's proof is particularly noteworthy; in addition to proving Erdős' inequality, it also shows that the stronger form of (1) holds

$$\mathbf{area}\,(C_1A_1B_1) \geq \sqrt{\mathbf{area}\,(AC_1B_1)\cdot\mathbf{area}\,(C_1BA_1)} \tag{2}$$

where, without loss of generality, it is assumed that

$$0 < \mathbf{area}\,(AC_1B_1) \leq \mathbf{area}\,(C_1BA_1) \leq \mathbf{area}\,(B_1A_1C)\,.$$

Publication Date: September 28, 2005. Communicating Editor: Paul Yiu.

The purpose of this paper is to show that a sharper inequality is possible when more care is placed in choosing the points $A_1,\ B_1$ and C_1. In so doing we extend Aassila's inequality [1]:

$$4 \cdot \mathbf{area}\,(A_1B_1C_1) \leq \mathbf{area}\,(ABC),$$

which is valid when these points are chosen so as to partition the perimeter of ABC into equal length segments. Our main result is

Theorem 1. *Let ABC be a triangle, and let $A_1,\ B_1,\ C_1$ be on $BC,\ CA,\ AB$, respectively, with none of $A_1,\ B_1,\ C_1$ coinciding with a vertex of ABC. If*

$$\frac{AB + BA_1}{AC + CA_1} = \frac{BC + CB_1}{AB + AB_1} = \frac{AC + AC_1}{BC + BC_1} = \alpha,$$

then

$$4 \cdot \mathbf{area}\,(A_1B_1C_1) \leq \mathbf{area}\,(ABC) + s^4 \left(\frac{\alpha - 1}{\alpha + 1}\right)^2 \cdot \mathbf{area}\,(ABC)^{-1}$$

where s is the semi-perimeter of ABC.

When $\alpha = 1$ we obtain Aassila's result.

Corollary 2 (Aassila [1])**.** *Let ABC be a triangle, and let $A_1,\ B_1,\ C_1$ be on $BC,\ CA,\ AB$, respectively, with none of $A_1,\ B_1,\ C_1$ coinciding with a vertex of ABC. If*

$$\begin{aligned} AB + BA_1 &= AC + CA_1, \\ BC + CB_1 &= AB + AB_1, \\ AC + AC_1 &= BC + BC_1, \end{aligned}$$

then

$$4 \cdot \mathbf{area}\,(A_1B_1C_1) \leq \mathbf{area}\,(ABC)\,.$$

2. Proof of Theorem 1

We shall make use of the following two lemmas.

Lemma 3 (Curry [4])**.** *For any triangle ABC, and standard notation,*

$$4\sqrt{3} \cdot \mathbf{area}\,(ABC) \leq \frac{9abc}{a + b + c}. \tag{3}$$

Equality holds if and only if $a = b = c$.

Lemma 4. *For any triangle ABC, and standard notation,*

$$\min\{a^2 + b^2 + c^2,\ ab + bc + ca\} \geq 4\sqrt{3} \cdot \mathbf{area}\,(ABC)\,. \tag{4}$$

To prove Theorem 1, we begin by computing the area of the corner triangle AC_1B_1:

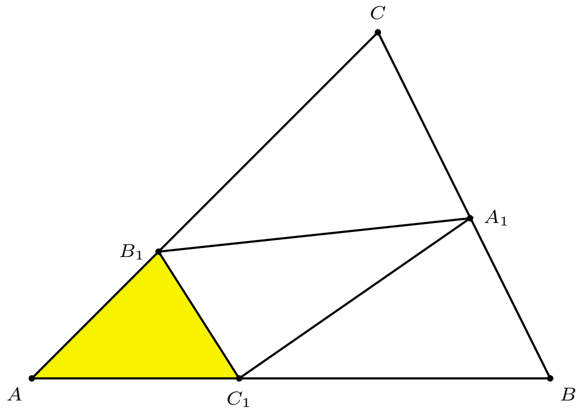


Figure 2

then

$$\begin{aligned}
\mathbf{area}\,(AC_1B_1) &= \frac{1}{2}AC_1 \cdot AB_1 \cdot \sin A \\
&= \frac{1}{2}AC_1 \cdot AB_1 \cdot \frac{2 \cdot \mathbf{area}\,(ABC)}{AB \cdot AC} \\
&= \frac{AC_1}{AB} \cdot \frac{AB_1}{AC} \cdot \mathbf{area}\,(ABC)\,.
\end{aligned}$$

For the semi-perimeter s of ABC we have

$$\begin{aligned}
2s &= AB + BC + AC \\
&= (AB + AB_1) + (BC + CB_1) \\
&= (\alpha + 1)(c + AB_1),
\end{aligned}$$

and

$$AB_1 = \frac{2}{\alpha + 1}s - c$$

where $c = AB$. Also,

$$\begin{aligned}
2s &= AB + BC + AC \\
&= (AC + AC_1) + (BC + BC_1) \\
&= \left(1 + \frac{1}{\alpha}\right)(AC + AC_1) \\
&= \frac{\alpha + 1}{\alpha}(b + AC_1),
\end{aligned}$$

and

$$AC_1 = \frac{2\alpha}{\alpha + 1}s - b$$

with $b = AC$. Hence

$$\mathbf{area}\,(AC_1B_1) = \frac{1}{bc}\left(\frac{2\alpha}{\alpha+1}s - b\right)\left(\frac{2}{\alpha+1}s - c\right)\cdot \mathbf{area}\,(ABC)\,. \qquad (5)$$

Similar computations yield

$$\mathbf{area}\,(C_1BA_1) = \frac{1}{ca}\left(\frac{2\alpha}{\alpha+1}s - c\right)\left(\frac{2}{\alpha+1}s - a\right)\cdot \mathbf{area}\,(ABC), \qquad (6)$$

and

$$\mathbf{area}\,(B_1A_1C) = \frac{1}{ab}\left(\frac{2\alpha}{\alpha+1}s - a\right)\left(\frac{2}{\alpha+1}s - b\right)\cdot \mathbf{area}\,(ABC)\,. \qquad (7)$$

From these formulae,

$$\begin{aligned}
&\mathbf{area}\,(A_1B_1C_1)\\
=&\,\mathbf{area}\,(ABC) - \mathbf{area}\,(AC_1B_1) - \mathbf{area}\,(C_1BA_1) - \mathbf{area}\,(B_1A_1C)\\
=&\left[1 - \frac{1}{bc}\left(\frac{2\alpha}{\alpha+1}s - b\right)\left(\frac{2}{\alpha+1}s - c\right) - \frac{1}{ca}\left(\frac{2\alpha}{\alpha+1}s - c\right)\left(\frac{2}{\alpha+1}s - a\right)\right.\\
&\quad\left. - \frac{1}{ab}\left(\frac{2\alpha}{\alpha+1}s - a\right)\left(\frac{2}{\alpha+1}s - b\right)\right]\cdot \mathbf{area}\,(ABC)\\
=&\frac{1}{abc}\left[\left(\frac{2}{\alpha+1}s - a\right)\left(\frac{2}{\alpha+1}s - b\right)\left(\frac{2}{\alpha+1}s - c\right)\right.\\
&\quad\left. + \left(\frac{2\alpha}{\alpha+1}s - a\right)\left(\frac{2\alpha}{\alpha+1}s - b\right)\left(\frac{2\alpha}{\alpha+1}s - c\right)\right]\cdot \mathbf{area}\,(ABC)\,.
\end{aligned}$$

But

$$\begin{aligned}
&\left(\frac{2}{\alpha+1}s - a\right)\left(\frac{2}{\alpha+1}s - b\right)\left(\frac{2}{\alpha+1}s - c\right)\\
+&\left(\frac{2\alpha}{\alpha+1}s - a\right)\left(\frac{2\alpha}{\alpha+1}s - b\right)\left(\frac{2\alpha}{\alpha+1}s - c\right)\\
=&2(s-a)(s-b)(s-c) + 2\left(\frac{\alpha-1}{\alpha+1}\right)^2 s^3\\
=&\frac{2}{s}\left[\mathbf{area}\,(ABC)\right]^2 + 2\left(\frac{\alpha-1}{\alpha+1}\right)^2 s^3.
\end{aligned}$$

Hence

$$\frac{abc\cdot s}{2}\cdot\mathbf{area}\,(A_1B_1C_1) = \left[\mathbf{area}\,(ABC)\right]^3 + s^4\cdot\left(\frac{\alpha-1}{\alpha+1}\right)^2\cdot\mathbf{area}\,(ABC)\,. \qquad (8)$$

From (3) and (4)

$$\begin{aligned}
\frac{abc\cdot s}{2} &\geq \frac{\sqrt{3}}{9}\cdot(a+b+c)^2\cdot\mathbf{area}\,(ABC)\\
&\geq \frac{\sqrt{3}}{9}[a^2+b^2+c^2+2(ab+bc+ca)]\cdot\mathbf{area}\,(ABC)\\
&\geq \frac{\sqrt{3}}{9}\cdot 12\sqrt{3}\cdot\mathbf{area}\,(ABC)^2\\
&\geq 4\cdot\mathbf{area}\,(ABC)^2.
\end{aligned}$$

Finally, from (8)

$$\begin{aligned}
&4\cdot\mathbf{area}\,(ABC)^2\cdot\mathbf{area}\,(A_1B_1C_1)\\
\leq\ &\frac{abc\cdot s}{2}\cdot\mathbf{area}\,(A_1B_1C_1)\\
\leq\ &[\mathbf{area}\,(ABC)]^3+s^4\cdot\left(\frac{\alpha-1}{\alpha+1}\right)^2\cdot\mathbf{area}\,(ABC)
\end{aligned}$$

and a division by $\mathbf{area}\,(ABC)^2$ produces

$$4\cdot\mathbf{area}\,(A_1B_1C_1)\leq\mathbf{area}\,(ABC)+s^4\cdot\left(\frac{\alpha-1}{\alpha+1}\right)^2\cdot[\mathbf{area}\,(ABC)]^{-1}$$

completing the proof of the theorem.

References

[1] M. Assila, Problem 1717, *Math. Mag.*, 78 (2005) 158.
[2] A. Bager, *Elem. Math.*, 12 (1957) 47.
[3] A. Bager, Solution to Problem 4908, *Amer. Math. Monthly*, 68 (1961) 386–387.
[4] T. R. Curry and L. Bankoff, Problem E 1861, *Amer. Math. Monthly*, 73 (1966) 199; solution 74 (1967) 724–725.
[5] H. Debrunner, Problem 260, *Elem. Math.*, 11 (1956) 20.
[6] P. H. Diananda, Solution to Problem 4908, *Amer. Math. Monthly*, 68 (1961) 386.
[7] J. Rainwater, A. Bager and P. H. Dianada, Problem 4908, *Amer. Math. Monthly*, 67 (1960) 479.

Ricardo M. Torrejón: Department of Mathematics, Texas State University | San Marcos, San Marcos, Texas 78666, USA

E-mail address: `rt04@txstate.edu`

Forum Geometricorum
Volume 5 (2005) 143–148.

FORUM GEOM
ISSN 1534-1178

Applications of Homogeneous Functions to Geometric Inequalities and Identities in the Euclidean Plane

Wladimir G. Boskoff and Bogdan D. Suceavă

Abstract. We study a class of geometric identities and inequalities that have a common pattern: they are generated by a homogeneous function. We show how to extend some of these homogeneous relations in the geometry of triangle. Then, we study the geometric configuration created by two intersecting lines and a pencil of n lines, where the repeated use of Menelaus's Theorem allows us to emphasize a result on homogeneous functions.

1. Introduction

The purpose of this note is to present an extension of a certain class of geometric identities or inequalities. The idea of this technique is inspired by the study of homogeneous polynomials and has the potential for additional applications besides the ones described here.

First of all, we recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called *homogeneous* if $f(tx_1, tx_2, ..., tx_n) = t^m f(x_1, x_2, ...x_n)$, for $t \in \mathbb{R} - \{0\}$ and $x_i \in \mathbb{R}$, $i = 1, ..., n$, $m, n \in \mathbb{N}$, $m \neq 0$, $n \geq 2$. The natural number m is called the degree of the homogeneous function f.

Remarks. 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous function. If for $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we have $f(x) \geq 0$, then $f(tx) \geq 0$, for $t > 0$. Furthermore, if m is an even natural number, $f(x) \geq 0$, yields $f(tx) \geq 0$ for any real number t.

2. Any $x > 0$ can be written as $x = \frac{a}{b}$, with $a, b \in (0, 1)$.

2. Application to the geometry of triangle

Consider the homogeneous function $f_\alpha : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f_\alpha(x_1, x_2, x_3) = \alpha x_1 x_2 x_3,$$

with $\alpha \in \mathbb{R} - \{0\}$. Denote by a, b, c the lengths of the sides of a triangle ABC, by R the circumradius and by $\triangle$ the area of this triangle. By the law of sines, we get

$$f_1(a, b, c) = f_1(a, b, 2R \sin C) = 2R f_1(a, b, \sin C) = 4R\triangle.$$

Thus, we obtain $abc = 4R\triangle$.

Publication Date: October 11, 2005. Communicating Editor: Paul Yiu.

Since $f_1(a,b,c) = 8R^3 f_1(\sin A, \sin B, \sin C)$, we get also the equality

$$\triangle = 2R^2 \sin A \sin B \sin C.$$

Heron's formula can be represented by the following setting. The function $f_{\sqrt{r}}(x_1, x_2, x_3)$ for $x_1 = \sqrt{s-a}$, $x_2 = \sqrt{s-b}$, $x_3 = \sqrt{s-c}$, yields

$$f_{\sqrt{s}}(\sqrt{s-a}, \sqrt{s-b}, \sqrt{s-c}) = \triangle.$$

Furthermore, using $\cot \frac{A}{2} = \frac{s-a}{r}$ and the similar equalities in B and C, we obtain

$$f_{\sqrt{s}}(\sqrt{s-a}, \sqrt{s-b}, \sqrt{s-c}) = r\sqrt{r} f_{\sqrt{s}} \left(\sqrt{\cot \frac{A}{2}}, \sqrt{\cot \frac{B}{2}}, \sqrt{\cot \frac{C}{2}} \right),$$

which yields

$$\triangle = r^2 \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

3. Homogeneous polynomials in $a^2, b^2, c^2, \triangle$ and their applications

Consider now a triangle ABC in the Euclidean plane, and denote by a, b, c the length of its sides and by $\triangle$ its area. We prove the following.

Proposition 1. *Let $p : \mathbb{R}^4 \to \mathbb{R}$ a homogeneous function with the property that $p(a^2, b^2, c^2, \triangle) \geq 0$, for any triangle in the Euclidean plane. Then for any $x > 0$ we have:*

$$p\left(xa^2, \frac{1}{x} b^2, c^2 + \left(1 - \frac{1}{x}\right)(xa^2 - b^2), \triangle \right) \geq 0. \qquad (1)$$

Proof. Consider $q(x) = \left(1 - \frac{1}{x}\right)(xa^2 - b^2)$, for $x > 0$. In the triangle ABC we consider A_1 and B_1 on the sides BC and AC, respectively, such that $CA_1 = \alpha a$, $BC = a$, $CB_1 = \beta b$, $AC = b$, with $\alpha, \beta \in (0,1)$. It results that the area of triangle CA_1B_1 is $\sigma[CA_1B_1] = \alpha\beta\triangle$. By the law of cosines we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

and therefore

$$A_1B_1^2 = \alpha\beta c^2 + (\alpha - \beta)(\alpha a^2 - \beta b^2).$$

Since the given inequality $p(a^2, b^2, c^2, \triangle) \geq 0$ takes place in any triangle, then it must take place also in the triangle CA_1B_1, thus

$$p(\alpha^2 a^2, \beta^2 b^2, \alpha\beta c^2 + (\alpha - \beta)(\alpha a^2 - \beta b^2), \alpha\beta\triangle) = 0.$$

Let us take now $t = \alpha\beta$, and $x = \frac{\alpha}{\beta}$, with $\alpha, \beta \in (0,1)$. For $x \in (0, \infty)$, we have

$$p\left(xa^2, \frac{1}{x} b^2, c^2 + q(x), \triangle \right) \geq 0.$$

□

Remark. In terms of identities, we state the following. Let $p : \mathbb{R}^4 \to \mathbb{R}$ a homogeneous function with the property that $p(a^2, b^2, c^2, \triangle) = 0$, for any triangle in the Euclidean plane. Then for any $x > 0$ we have

$$p\left(xa^2, \frac{1}{x}b^2, c^2 + \left(1 - \frac{1}{x}\right)(xa^2 - b^2), \triangle\right) = 0. \qquad (2)$$

The proof is similar to the proof of Proposition 1.

We present now a few applications of Proposition 1.

3.1. In any triangle ABC in the Euclidean plane, for any $x \in (0, \infty)$, we have

$$4\triangle \le \min\left[xa^2 + \frac{1}{x}b^2, xa^2 + c^2 + q(x), \frac{1}{x}b^2 + c^2 + q(x)\right].$$

To prove this inequality, it is sufficient to prove the statement for $x = 1$, then we apply Proposition 1. Let us assume, without losing any generality, that $a \ge b \ge c$. We also use $b^2 + c^2 \ge 2bc$, and $2bc \ge 2bc \sin A = 4\triangle$. Thus, $b^2 + c^2 \ge 4\triangle$, and this means

$$4\triangle \le \min(b^2 + c^2, a^2 + c^2, a^2 + b^2).$$

Applying this result in the triangle CA_1B_1, considered as in the proof of Proposition 1, we obtain the stated inequality.

3.2. Consider $q(x) = \left(1 - \frac{1}{x}\right)(xa^2 - b^2)$, for $x > 0$. Then in any triangle we have the inequality

$$a^2b^2[c^2 + q(x)] \ge \left(\frac{4\triangle}{3\sqrt{3}}\right)^3.$$

This results as a direct consequence of Carlitz' inequality

$$a^2b^2c^2 \ge \left(\frac{4\triangle}{3\sqrt{3}}\right)^3.$$

by applying Proposition 1.

3.3. It is known that in any triangle we have Hadwiger's inequality

$$a^2 + b^2 + c^2 \ge \triangle\sqrt{3}.$$

This inequality can be generalized for any $x \in (0, \infty)$ as follows

$$(2x - 1)a^2 + \left(\frac{2}{x} + 1\right)b^2 + c^2 \ge 4\triangle\sqrt{3}.$$

(This inequality appears in *Matematika v Shkole,* No. 5, 1989.)

Hadwiger's inequality can be proven by using the law of cosines to get

$$a^2 + b^2 + c^2 = 2(b^2 + c^2) - 2bc\cos A.$$

Then, keeping in mind that $2\triangle = bc\sin A$, we get

$$\begin{aligned} a^2+b^2+c^2-4\triangle\sqrt{3} =& 2(b^2+c^2-2bc\cos A-2bc\sqrt{3}\sin A) \\ =& 2\left(b^2+c^2-4bc\cos\left(\frac{\pi}{3}-A\right)\right) \\ \geq& 2\left(b^2+c^2-4bc\cos\frac{\pi}{3}\right) \\ =& 2(b-c)^2 \\ \geq& 0. \end{aligned}$$

The equality holds when $b=c$ and $A=\frac{\pi}{3}$, i.e. when triangle ABC is equilateral.

Applying Hadwiger's inequality to the triangle CA_1B_1 constructed in Proposition 1, we get

$$\alpha^2a^2+\beta^2b^2+\alpha\beta c^2+(\alpha-\beta)(\alpha a^2-\beta b^2)\geq 4\alpha\beta\triangle\sqrt{3}.$$

Dividing by $\alpha\beta$ and denoting, as before, $x=\frac{\alpha}{\beta}$, we obtain

$$xa^2+\frac{1}{x}b^2+c^2+q(x)\geq 4\triangle\sqrt{3}.$$

After grouping the factors, we get the inequality that we wanted to prove in the first place. $\square$

3.4. Consider Goldner's inequality

$$b^2c^2+c^2a^2+a^2b^2\geq 16\triangle^2.$$

This inequality can be extended by using the technique presented here to the following relation:

$$a^2b^2+\left(xa^2+\frac{1}{x}b^2\right)\left[c^2+\left(1-\frac{1}{x}\right)(xa^2-b^2)\right]\geq 16\triangle^2.$$

To remind here the proof of Goldner's inequality, we use an argument based on a consequence of Heron's formula:

$$2(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4)=16\triangle^2,$$

and the inequality

$$a^4+b^4+c^4\geq a^2b^2+a^2c^2+b^2c^2.$$

This proves Goldner's inequality. For its extension, we apply Goldner's inequality to triangle CA_1B_1, as in Proposition 1.

4. Menelaus' Theorem and homogeneous polynomials

In this section we prove the following result.

Proposition 2. *Let $p:\mathbb{R}^n\to\mathbb{R}$ be a homogeneous function of degree m, and consider n collinear points $A_1, A_2, ..., A_n$ lying on the line d. Let S be a point exterior to the line $\mathcal{L}$ and a secant $\mathcal{L}'$ whose intersection with each of the segments*

(SA_i) *is denoted* A_i', *with* $i = 1, ..., n$. *Denote by* K *the intersection point of* $\mathcal{L}$ *and* $\mathcal{L}'$. *Then,*

$$p(KA_1, KA_2, ..., KA_n) = 0$$

if and only if

$$p\left(\frac{A_1A_1'}{A_1'S}, \frac{A_2A_2'}{A_2'S}, ..., \frac{A_nA_n'}{A_n'S}\right) = 0.$$

Proof. Denote $a_i = \frac{A_iA_i'}{A_i'S}$, for $i = 1, ..., n$. Applying Menelaus' Theorem in each of the triangles $SA_1A_2, SA_2A_3, \ldots, SA_{n-1}A_n$ we have, for all $i = 1, ..., n-1$,

$$\frac{1}{a_i} \cdot \frac{A_iK}{A_{i+1}K} \cdot a_{i+1} = 1.$$

This yields

$$\frac{A_1K}{a_1} = \frac{A_2K}{a_2} = ... = \frac{A_nK}{a_n} = t,$$

where $t > 0$. The fact that $p(KA_1, KA_2, ..., KA_n) = 0$ is equivalent, by Remark 1, with

$$p(ta_1, ta_2, ..., ta_n) = 0,$$

or, furthermore

$$t^m p(a_1, a_2, ..., a_n) = 0.$$

Since $t > 0$, the conclusion follows immediately. □

Remark. 3. As in the case of Proposition 1, we can discuss this result in terms of inequalities. For example, the Proposition 2 is still true if we claim that

$$p(KA_1, KA_2, ..., KA_n) \geq 0$$

if and only if

$$p\left(\frac{A_1A_1'}{A_1'S}, \frac{A_2A_2'}{A_2'S}, ..., \frac{A_nA_n'}{A_n'S}\right) \geq 0.$$

We present now an application.

4.1. A line intersects the sides AC and BC and the median CM_0 of an arbitrary triangle in the points B_1, A_1, and M_3, respectively. Then,

$$\frac{1}{2}\left(\frac{AB_1}{B_1C} + \frac{BA_1}{A_1C}\right) = \frac{M_3M_0}{M_3C}, \tag{3}$$

$$\frac{M_3B_1}{M_3A_1} = \frac{KB_1}{KA_1} \cdot \frac{KB}{KA}. \tag{4}$$

Furthermore, (3) is still true if we apply to this configuration a projective transformation that maps K into ∞.

We use Proposition 2 to prove (3). Let $\{K\} = AB \cap A_1B_1$. Then, the relation we need to prove is equivalent to $KA + KB = 2KM_0$, which is obvious, since M_0 is the midpoint of (AB).

To prove (4), remark that the anharmonic ratios $[KM_3B_1A_1]$ and $[KM_0AB]$ are equal, since they are obtained by intersecting the pencil of lines CK, CA, CM_0, CB with the lines KA and KB. Therefore, we have

$$\frac{M_3B_1}{M_3A_1} : \frac{KB_1}{KA_1} = \frac{M_0A}{M_0B} : \frac{KA}{KB}.$$

Since $M_0A = M_0B$, we have

$$\frac{M_3B_1}{M_3A_1} = \frac{KB_1}{KA_1} \cdot \frac{KB}{KA}.$$

Finally, by mapping M into the point at infinity, the lines B_1A_1 and BA become parallel. By Thales Theorem, we have

$$\frac{B_1A}{B_1C} = \frac{BA_1}{A_1C} = \frac{M_3M_0}{M_3C},$$

therefore the relation is still true.

References

[1] O. Bottema, R. Ž. Djordjević, R.R. Janić, D. S. Mitrinović and P. M. Vasić, *Geometric Inequalities,* Wolters-Noordhoff Publ., Groningen 1968.

[2] B. Suceavă, *Use of homogeneous functions in the proof of some geometric inequalities or identities,* (in Romanian), Gazeta Matematică, 8-9(1990), 236-240.

Wladimir G. Boskoff: Department of Mathematics and Computer Science, University Ovidius, Constantza, Romania

E-mail address: boskoff@math.univ-ovidius.ro

Bogdan D. Suceavă: Department of Mathematics, California State University, Fullerton, CA 92835, U.S.A.

E-mail address: bsuceava@fullerton.edu

Forum Geometricorum
Volume 5 (2005) 149–164.

FORUM GEOM
ISSN 1534-1178

On the Complement of the Schiffler Point

Khoa Lu Nguyen

Abstract. Consider a triangle ABC with excircles (I_a), (I_b), (I_c), tangent to the nine-point circle respectively at F_a, F_b, F_c. Consider also the polars of A, B, C with respect to the corresponding excircles, bounding a triangle XYZ. We present, among other results, synthetic proofs of (i) the perspectivity of XYZ and $F_aF_bF_c$ at the complement of the Schiffler point of ABC, (ii) the concurrency at the same point of the radical axes of the nine-point circles of triangles I_aBC, I_bCA, and I_cAB.

1. Introduction

Consider a triangle ABC with excircles (I_a), (I_b), (I_c). It is well known that the nine-point circle (W) is tangent externally to the each of the excircles. Denote by F_a, F_b, and F_c the points of tangency. Consider also the polars of the vertices A with respect to (I_a), B with respect to (I_b), and C with respect to (I_c). These are the lines B_aC_a, C_bA_b, and A_cB_c joining the points of tangency of the excircles with the sidelines of triangle ABC. Let these polars bound a triangle XYZ. See Figure 1. Juan Carlos Salazar [12] has given the following interesting theorem.

Theorem 1 (Salazar)**.** *The triangles XYZ and $F_aF_bF_c$ are perspective at a point on the Euler line.*

Darij Grinberg [3] has identified the perspector as the triangle center X_{442} of [6], the complement of the Schiffler point. Recall that the Schiffler point S is the common point of the Euler lines of the four triangles IBC, ICA, IAB, and ABC, where I is the incenter of ABC. Denote by A', B', C' the midpoints of the sides BC, CA, AB respectively, so that $A'B'C'$ is the medial triangle of ABC, with incenter I' which is the complement of I. Grinberg suggested that the lines XF_a, YF_b and ZF_c are the Euler lines of triangles $I'B'C'$, $I'C'A'$ and $I'A'B'$ respectively. The present author, in [10], conjectured the following result.

Theorem 2. *The radical center of the nine-point circles of triangles I_aBC, I_bCA and I_cAB is a point on the Euler line of triangle ABC.*

Subsequently, Jean-Pierre Ehrmann [1] and Paul Yiu [13] pointed out that this radical center is the same point S', the complement of the Schiffler point S. In this paper, we present synthetic proofs of these results, along with a few more interesting results.

Publication Date: October 18, 2005. Communicating Editor: Paul Yiu.
The author is extremely grateful to Professor Paul Yiu for his helps in the preparation of this paper.

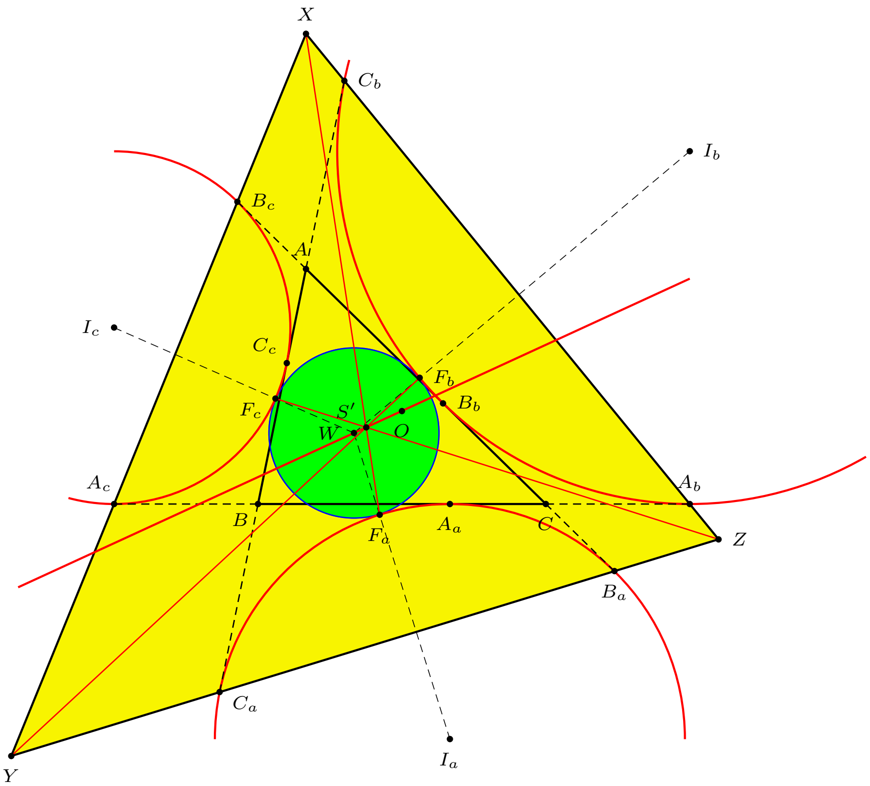


Figure 1.

2. Notations

a, b, c	Lengths of sides BC, CA, AB
R, r, s	Circumradius, inradius, semiperimeter
r_a, r_b, r_c	Exradii
O, G, W, H,	Circumcenter, centroid, nine-point center, orthocenter
I, F, S, M	Incenter, Feuerbach point, Schiffler point, Mittenpunkt
P'	Complement of P in triangle ABC
A', B', C'	Midpoints of BC, CA, AB
A_1, B_1, C_1	Points of tangency of incircle with BC, CA, AB
I_a, I_b, I_c	Excenters
F_a, F_b, F_c	Points of tangency of the nine-point circle with the excircles
A_a, B_a, C_a	Points of tangency of the A-excircle with the lines BC, CA, AB; similarly for A_b, B_b, C_b and A_c, B_c, C_c
W_a, W_b, W_c	Nine-point centers of I_aBC, I_bCA, I_cAB
M_a, M_b, M_c	Midpoints of AI_a, BI_b, CI_c
X	$A_bC_b \cap A_cB_c$; similarly for Y, Z
X_b, X_c	Orthogonal projections of B on CI_a and C on BI_a; similarly for Y_c, Y_a, Z_a, Z_b
J_a	Midpoint of arc BC of circumcircle not containing A; similarly for J_b, J_c
K_a	$A_bF_b \cap A_cF_c$; similarly for K_b, K_c

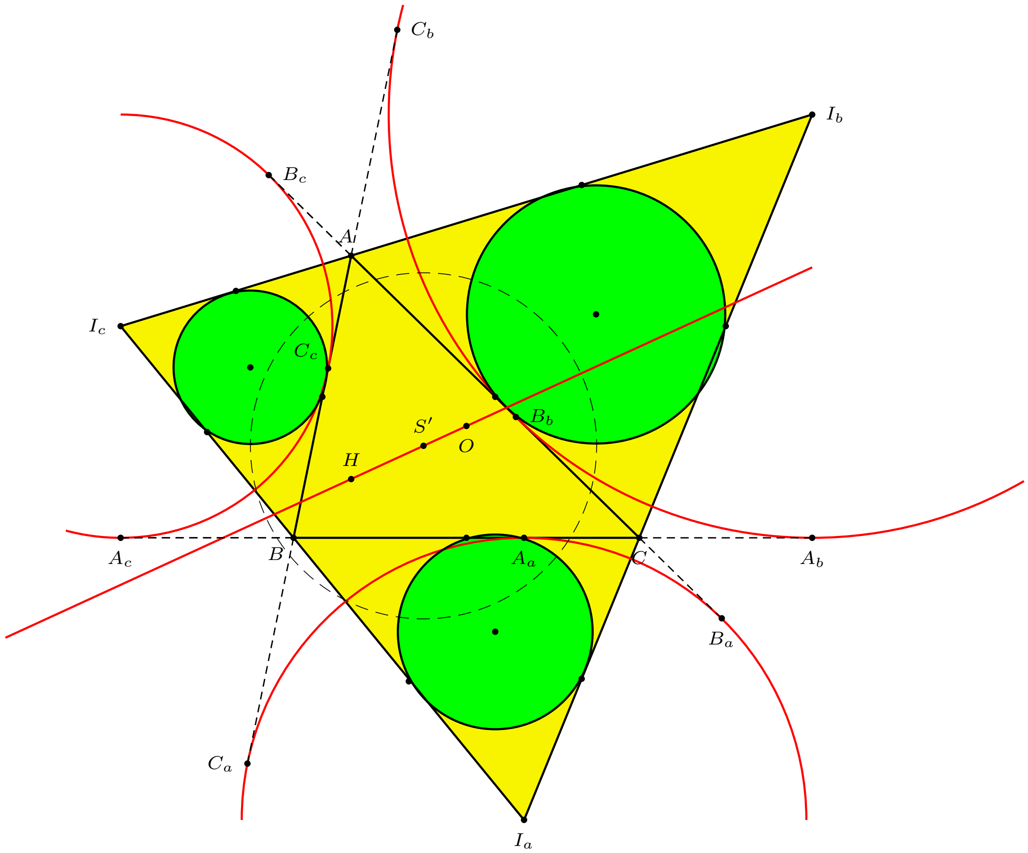


Figure 2.

3. Some preliminary results

We shall make use of the notion of directed angle between two lines. Given two lines a and b, the directed angle $(\mathsf{a},\ \mathsf{b})$ is the angle of counterclockwise rotation from a to b. It is defined modulo $180°$. We shall make use of the following basic properties of directed angles. For further properties of directed angles, see [7].

Lemma 3. (i) *For arbitrary lines* a, b, c,

$$(\mathsf{a},\ \mathsf{b}) + (\mathsf{b},\ \mathsf{c}) \equiv (\mathsf{a},\ \mathsf{c}) \bmod 180°.$$

(ii) *Four points* A, B, C, D *are concyclic if and only if* $(AC, CB) = (AD, DB)$.

Lemma 4. *Let* (O) *be a circle tangent externally to two circles* (O_a) *and* (O_b) *respectively at* A *and* B. *If* PQ *is a common external tangent of* (O_a) *and* (O_b), *then the quadrilateral* $APQB$ *is cyclic, and the lines* AP, BQ *intersect on the circle* (O).

Proof. Let PA intersect (O) at K. Since (O) and (O_a) touch each other externally at A, OK is parallel to O_aP. On the other hand, O_aP is also parallel to O_bQ as they are both perpendicular to the common tangent PQ. Therefore KO is parallel

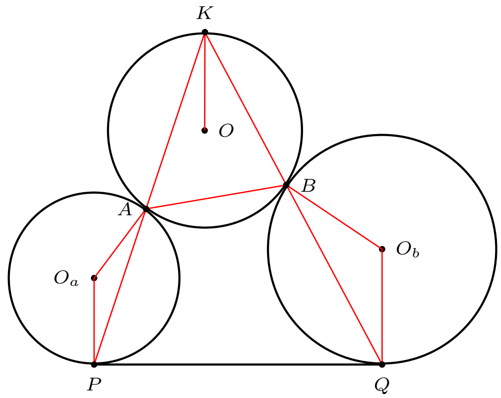


Figure 3

to O_bQ in the same direction. This implies that K, B, Q are collinear since (O_b) and (O) touch each other at externally at B. Therefore

$$(PQ, QB) = \frac{1}{2}(QO_b, O_bB) = \frac{1}{2}(KO, OB) = (KA, AB) = (PA, AB),$$

and $APQB$ is cyclic. □

We shall make use of the following results.

Lemma 5. *Let ABC be a triangle inscribed in a circle (O), and points M and N lying on AB and AC respectively. The quadrilateral $BNMC$ is cyclic if and only if MN is perpendicular to OA.*

Theorem 6. *The nine-point circles of ABC, I_aBC, I_aCA, and I_aAB intersect at the point F_a.*

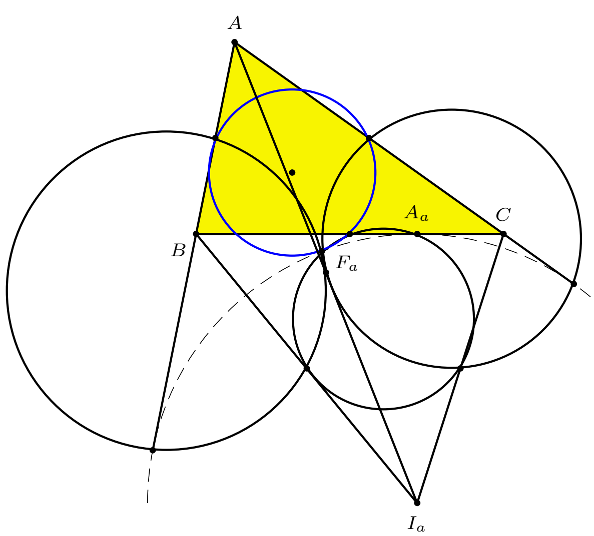


Figure 4.

Proposition 7. *The circle with diameter A_aM_a contains the point F_a.*

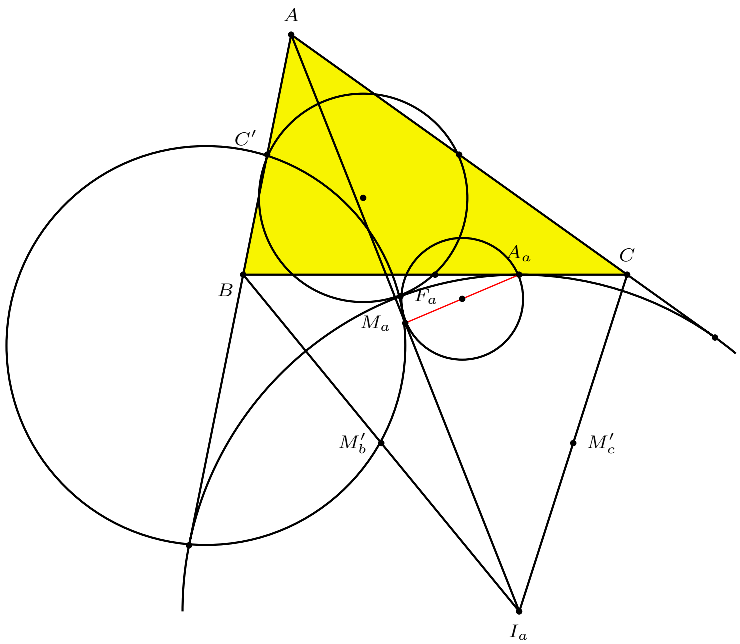


Figure 5.

Proof. Denote by M'_b and M'_c the midpoints of I_aB and I_aC respectively. The point F_a is common to the nine-point circles of I_aBC, I_aCA and I_aAB. See Figure 5. We show that $(A_aF_a, F_aM_a) = 90^\circ$.

$$\begin{aligned}(A_aF_a, F_aM_a) =&(A_aF_a, F_aM'_b) + (M'_bF_a, F_aM_a)\\ =&(A_aM'_c, M'_cM_b) + (M'_bC', C'M_a)\\ =&-(I_aM'_c, M'_cM'_b) - (BI_a, I_aA)\\ =&-((I_aC, BC) + (BI_a, I_aA)) = 90^\circ.\end{aligned}$$

□

4. Some properties of triangle XYZ

In this section we present some important properties of the triangle XYZ.

4.1. *Homothety with the excentral triangle.* Since YZ and I_bI_c are both perpendicular to the bisector of angle A, they are parallel. Similarly, ZX and XY are parallel to I_cI_a and I_aI_b respectively. The triangle XYZ is therefore homothetic to the excentral triangle $I_aI_bI_c$. See Figure 7. We shall determine the homothetic center in Theorem 11 below.

4.2. *Perspectivity with ABC.* Consider the orthogonal projections P and P' of A and X on the line BC. We have

$$A_cP : PA_b = (s-c) + c\cos B : (s-b) + b\cos C = s-b : s-c$$

by a straightforward calculation.

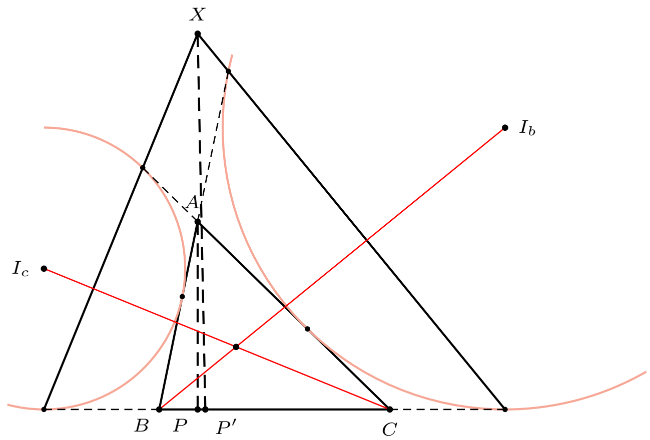


Figure 6.

On the other hand,

$$\begin{aligned} A_cP' : P'A_b =& \cot XA_cA_b : \cot XA_bA_c \\ =& \cot\left(90^\circ - \frac{C}{2}\right) : \cot\left(90^\circ - \frac{B}{2}\right) \\ =& \tan\frac{C}{2} : \tan\frac{B}{2} \\ =& \frac{1}{s-c} : \frac{1}{s-b} \\ =& s-b : s-c. \end{aligned}$$

It follows that P and P' are the same point. This shows that the line XA is perpendicular to BC and contains the orthocenter H of triangle ABC. The same is true for the lines YB and ZX. The triangles XYZ and ABC are perspective at H.

4.3. *The circumcircle of XYZ.* Applying the law of sines to triangle AXB_c, we have

$$XA = (s-b)\cdot\frac{\sin\left(90^\circ - \frac{C}{2}\right)}{\sin\frac{C}{2}} = (s-b)\cot\frac{C}{2} = r_a.$$

It follows that $HX = 2R\cos A + r_a = 2R + r$. See Figure 4. Similarly, $HY = HZ = 2R + r$. Therefore, triangle XYZ has circumcenter H and circumradius $2R + r$.

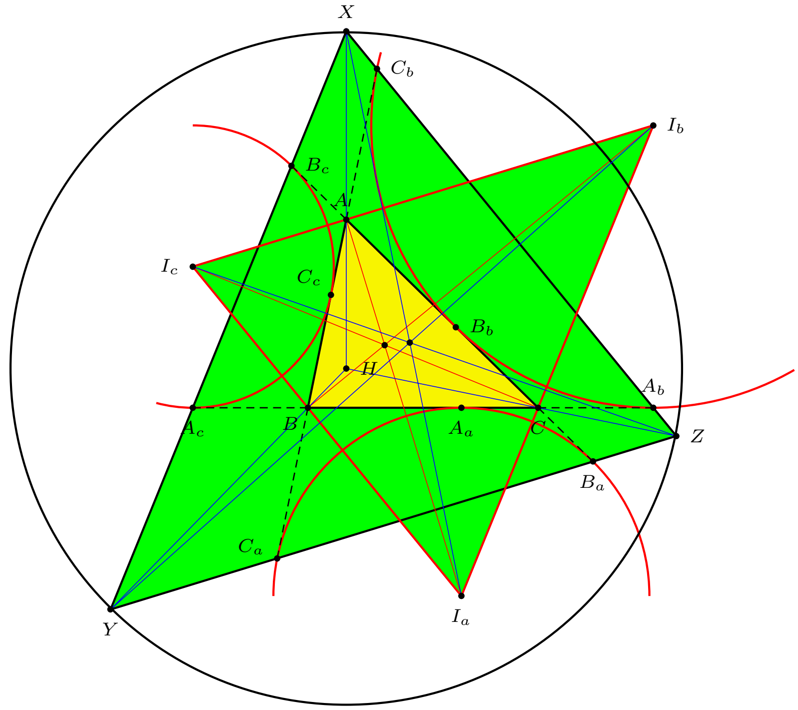


Figure 7.

5. The Taylor circle of the excentral triangle

Consider the excentral triangle $I_aI_bI_c$ with its orthic triangle ABC. The orthogonal projections Y_a and Z_a of A on I_aI_c and I_aI_b, Z_b and X_b of B on I_bI_c and I_aI_b, together with X_c and Y_c of C on I_bI_c and I_cI_a are on a circle called the Taylor circle of the excentral triangle. See Figure 8.

Proposition 8. *The points X_b, X_c lie on the line YZ.*

Proof. The collinearity of C_a, X_b, X_c follows from

$$\begin{aligned}(C_aX_b, X_bB) =&(C_aI_a, I_aB)\\ =&(C_aI_a, AB) + (AB, I_aB)\\ =&90^\circ + (I_aB, BC)\\ =&(X_cC, I_aB) + (I_aB, BC)\\ =&(X_cC, CB)\\ =&(X_cX_b, X_bB).\end{aligned}$$

Similarly, X_b is also on the line YZ, and Z_a, Z_b are on the line XY, Y_c, Y_a are on the line XZ. □

Proposition 9. *The line Y_aZ_a contains the midpoints B', C' of CA, AB, and is parallel to BC.*

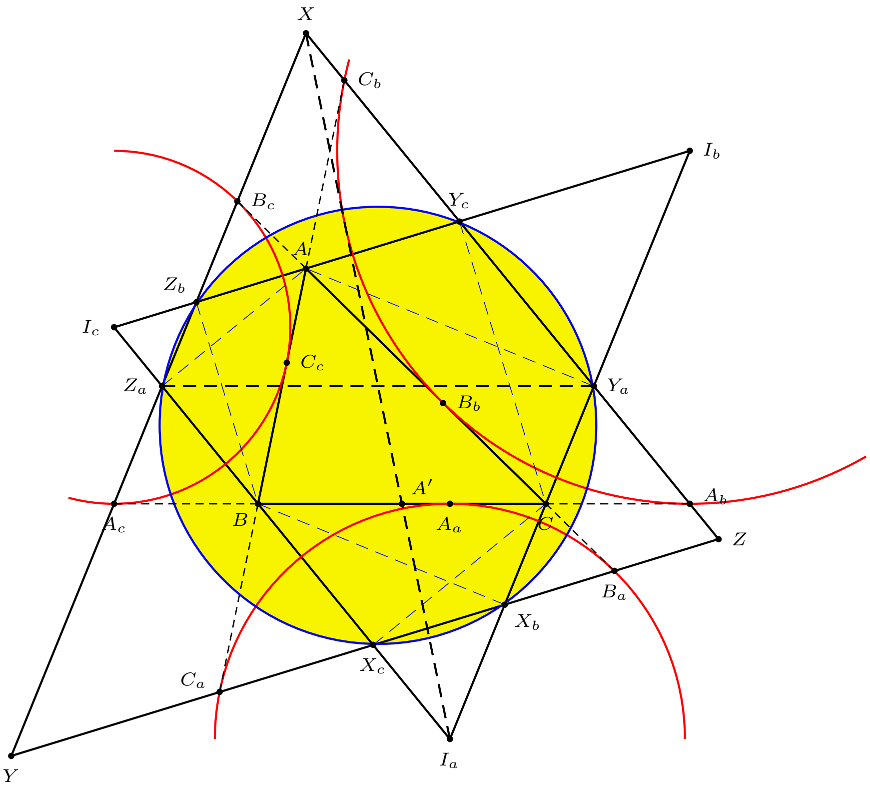


Figure 8.

Proof. Since A, Y_a, I_a, Z_a are concyclic,

$$(AY_a, Y_aZ_a) = (AI_a, I_aZ_a) = \frac{C}{2} = (CA, AY_a).$$

Therefore, the intersection of AC and Y_aZ_a is the circumcenter of the right triangle ACY_a, and is the midpoint B' of CA. Similarly, the intersection of AB and Y_aZ_a is the midpoint C' of AB. □

Proposition 10. *The line I_aX contains the midpoint A' of BC.*

Proof. Since the diagonals of the parallelogram $I_aY_aXZ_a$ bisect each other, the line I_aX passes through the midpoint of the segment Y_aZ_a. Since Y_aZ_a and BC are parallel, with B on I_aZ_a and C on I_aY_a, the same line I_aX also passes through the midpoint of the segment BC. □

Theorem 11. *The triangles XYZ and $I_aI_bI_c$ are homothetic at the Mittenpunkt M of triangle ABC, the ratio of homothety being $2R + r : -2R$.*

Proof. The lines I_aX, I_bY, I_cZ contain respectively the midpoints of A', B', C' of BC, CA, AB. They intersect at the common point of I_aA', I_bB', I_cC', the Mittenpunkt M of triangle ABC. This is the homothetic center of the triangles XYZ and $I_aI_bI_c$. The ratio of homothety of the two triangle is the same as the ratio of their circumradii. □

Theorem 12. *The Taylor circle of the excentral triangle is the radical circle of the excircles.*

Proof. The perpendicular bisector of Y_cZ_b is a line parallel to the bisector of angle A and passing through the midpoint A' of BC. This is the A'-bisector of the medial triangle $A'B'C'$. Similarly, the perpendicular bisectors of Z_aX_c and X_bY_a are the other two angle bisectors of the medial triangle. These three intersect at the incenter of the medial triangle, the Spieker center of ABC.

It is well known that $S_{\rm p}$ is also the center of the radical circle of the excircles. To show that the Taylor circle coincides with the radical circle, we show that they have equal radii. This follows easily from

$$I_aX_c \cdot I_aZ_a = \frac{r_a \sin\frac{A}{2}}{\cos\frac{C}{2}} \cdot I_aA\cos\frac{C}{2} = r_a \cdot I_aA\sin\frac{A}{2} = r_a^2.$$

□

6. Proofs of Theorems 1 and 2

We give a combined proof of the two theorems, by showing that the line XF_a is the radical axis of the nine-point circles (W_b) and (W_c) of triangles I_bCA and I_cAB. In fact, we shall identify some interesting points on this line to show that it is also the Euler line of triangle $I'B'C'$.

6.1. *XF_a as the radical axis of (W_b) and (W_c).*

Proposition 13. *X lies on the radical axis of the circles (W_b) and (W_c).*

Proof. By Theorem 12, $XZ_a \cdot XZ_b = XY_a \cdot XY_c$. Since Y_c, Y_a are on the nine-point circle (W_b) and Z_a, Z_b on the the circle (W_c), X lies on the radical axis of these two nine-point circles. □

Since AZ_a and AY_a are perpendicular to I_aI_c and I_aI_b, and $I_aI_bI_c$ and XYZ are homothetic, A is the orthocenter of triangle XY_aZ_a. It follows that X is the orthocenter of AY_aZ_a. Since $(AY_a, Y_aI_a) = (AZ_a, Z_aI_a) = 90^\circ$, the triangle AY_aZ_a has circumcenter the midpoint M_a of AI_a. It follows that XM_a is the Euler line of triangle AY_aZ_a.

Proposition 14. *M_a lies on the radical axis of the circles (W_b) and (W_c).*

Proof. Let M_b'' and M_c'' be the midpoints of AI_b and AI_c respectively. See Figure 9. Note that these lie on the nine-point circles (W_b) and (W_c) respectively. Since C, I_b, I_c, B are concyclic, we have $I_aB \cdot I_aI_c = I_aC \cdot I_aI_b$. Applying the homthety $\mathsf{h}(A, \frac{1}{2})$, we have the collinearity of M_a, C', M_c'', and of M_a, B', M_b'', Furthermore, $M_aC' \cdot M_aM_c'' = M_aB' \cdot M_aM_b''$. This shows that M_a lies on the radical axis of (W_b) and (W_c). □

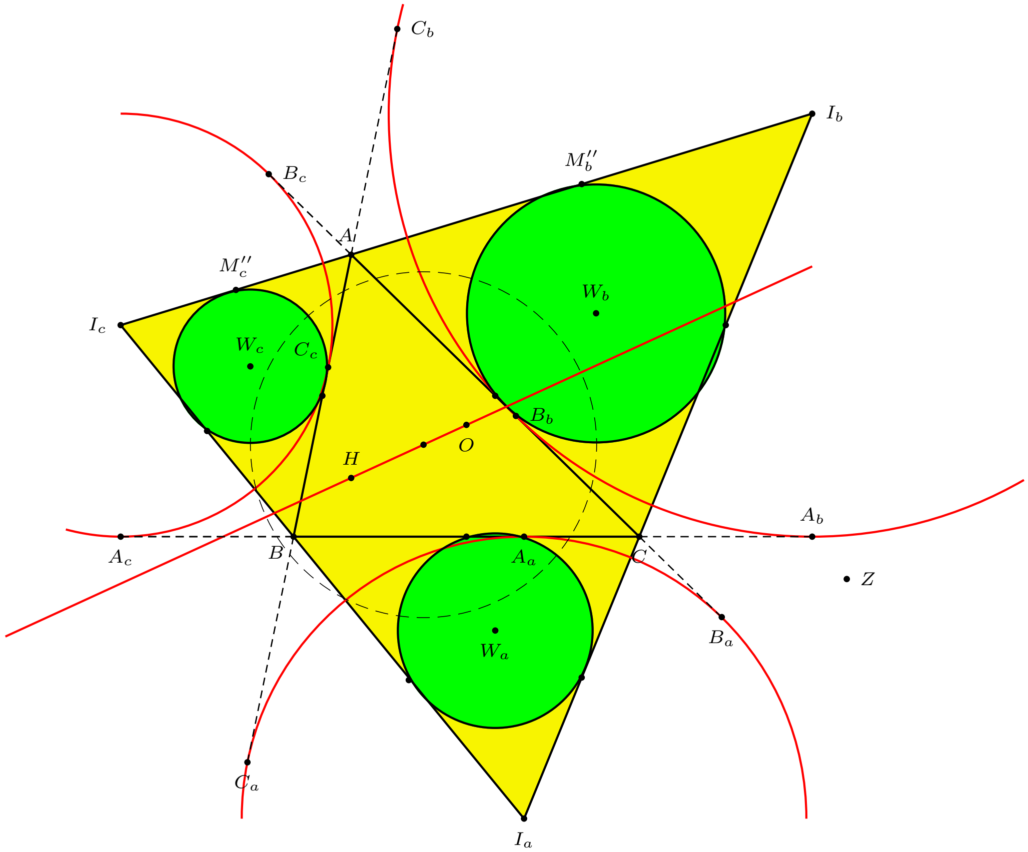


Figure 9.

Proposition 15. *X, F_a, and M_a are collinear.*

Proof. We prove that the Euler line of triangle AY_aZ_a contains the point F_a. The points X and M_a are respectively the orthocenter and circumcenter of the triangle.

Let A_a' be the antipode of A_a on the A-excircle. Since AX has length r_a and is perpendicular to BC, $XAA_a'I_a$ is a parallelogram. Therefore, XA_a' contains the midpoint M_a of AI_a.

By Proposition 7, $(A_aF_a, F_aM_a) = 90°$. Clearly, $(A_aF_a, F_aA_a') = 90°$. This means that F_a, M_a, and A_a' are collinear. The line containing them also contains X. □

Proposition 16. *XF_a is also the Euler line of triangle AY_aZ_a.*

Proof. The circumcenter of AY_aZ_a is clearly M_a. On the other hand, since A is the orthocenter of triangle XY_aZ_a, X is the orthocenter of triangle AY_aZ_a. Therefore the line XM_a, which also contains F_a, is the Euler line of triangle AY_aZ_a. □

6.2. *XF_a as the Euler line of triangle $I'B'C'$.*

Proposition 17. *M_a is the orthocenter of triangle $I'B'C'$.*

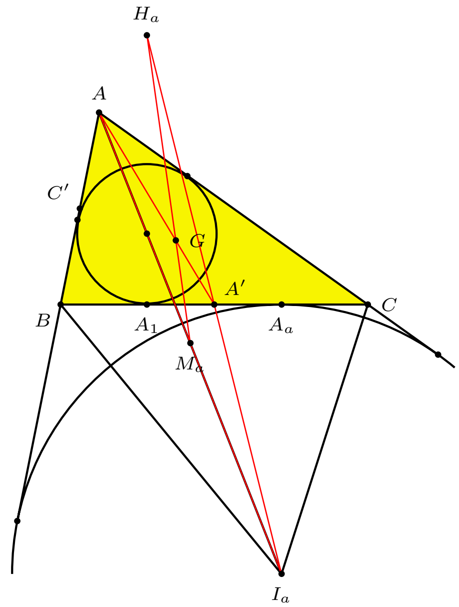


Figure 10.

Proof. Let H_a be the orthocenter of IBC. Since BH_a is perpendicular to IC, it is parallel to I_aC. Similarly, CH_a is parallel to I_aB. Thus, BH_aCI_a is a parallelogram, and A' is the midpoint of I_aH_a. Consider triangle AI_aH_a which has M_a and A' for the midpoints of two sides. The intersection of M_aH_a and AA' is the centroid of the triangle, which coincides with G. Furthermore,

$$GH_a : GM_a = GA : GA' = 2 : -1.$$

Hence, M_a is the orthocenter of $I'B'C'$. □

Proposition 18. *K_a is the circumcenter of $I'B'C'$.*

Proof. By Lemma 4, the points F_b, F_c, A_b and A_c are concyclic, and the lines A_bF_b and A_cF_c intersect at a point K_a on the nine-point circle, which is the midpoint of the arc $B'C'$ not containing A'. See Figure 11. The image of K_a under $\mathsf{h}(G, -2)$ is J_a, the circumcenter of IBC. It follows that K_a is the circumcenter of $I'B'C'$. □

Proposition 19. *K_a lies on the radical axis of (W_b) and (W_c).*

Proof. Let D and E be the second intersections of K_aF_b with (W_b) and K_aF_c with (W_c) respectively. We shall show that $K_aF_b \cdot K_aD = K_aF_c \cdot K_aE$.

Since A_c, F_c, F_b, A_b are concyclic, we have $K_aF_c \cdot K_aA_c = K_aF_b \cdot K_aA_b = k$, say. Note that

$$A_cE \cdot A_cF_c = A_cZ_a \cdot A_cZ_b = \frac{(s-a)^2 \sin\left(B + \frac{A}{2}\right)}{\tan\frac{B}{2}\cos\frac{A}{2}}.$$

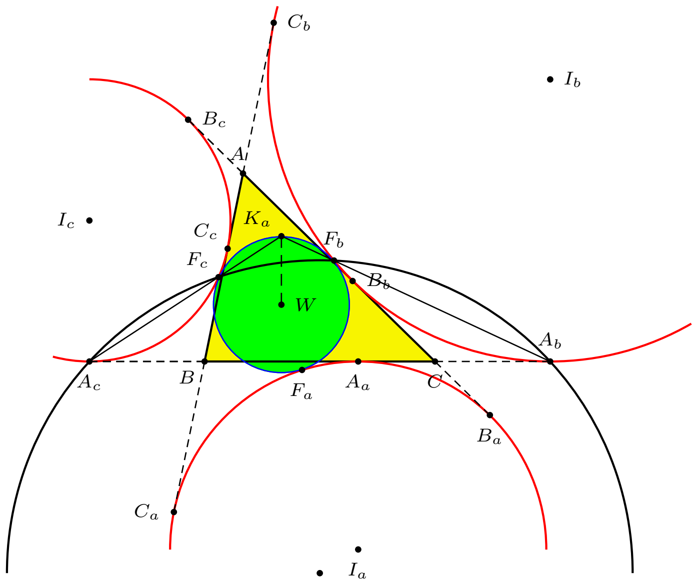


Figure 11.

Since (I_c) and (W) extouch at F_c, we have $\frac{K_aF_c}{A_cF_c} = -\frac{R}{2r_a}$. Therefore,

$$\begin{aligned}\frac{A_cE}{K_aA_c} =& \frac{K_aF_c}{A_cF_c} \cdot \frac{A_cE \cdot A_cF_c}{K_aF_c \cdot K_aA_c} \\ =& -\frac{R}{2r_a} \cdot \frac{(s-a)^2 \sin\left(B+\frac{A}{2}\right)}{k \cdot \tan\frac{B}{2}\cos\frac{A}{2}} \\ =& -\frac{R(s-a)^2 \sin\left(B+\frac{A}{2}\right)}{k \cdot s\tan\frac{B}{2}\tan\frac{C}{2}\cos\frac{A}{2}}.\end{aligned}$$

Similarly,

$$\frac{A_bD}{K_aA_b} = -\frac{R(s-a)^2 \sin\left(C+\frac{A}{2}\right)}{k \cdot s\tan\frac{B}{2}\tan\frac{C}{2}\cos\frac{A}{2}}.$$

Since $\sin\left(B+\frac{A}{2}\right) = \sin\left(C+\frac{A}{2}\right)$, it follows that $\frac{A_bD}{K_aA_b} = \frac{A_cE}{K_aA_c}$. Hence, DE is parallel to A_bA_c. From $K_aF_b \cdot K_aA_b = K_aF_c \cdot K_aA_c$, we have $K_aF_b \cdot K_aD = K_aF_c \cdot K_aE$. This shows that K_a lies on the radical axis of (W_b) and (W_c).

□

Corollary 20. *K_a lies on the line XF_a.*

6.3. *Proof of Theorems 1 and 2.* We have shown that the line XF_a is the radical axis of (W_b) and (W_c). Likewise, YF_b is that of (W_c), (W_a), and ZF_c that of (W_a), (W_b). It follows that the three lines are concurrent at the radical center of the three circles. This proves Theorem 1.

We have also shown that the line XF_a is the image of the Euler line of IBC under the homothety $\mathsf{h}(G, -\frac{1}{2})$; similarly for the lines YF_b and ZF_c. Since the Euler lines of IBC, ICA, and IAB intersect at the Schiffler point S on the Euler line of ABC, the lines XF_a, YF_b, ZF_c intersect at the complement of the Schiffler point S, also on the same Euler line. This proves Theorem 2.

7. Some further results

Theorem 21. *The six points Y, Z, A_b, A_c, F_b, F_c are concyclic.*

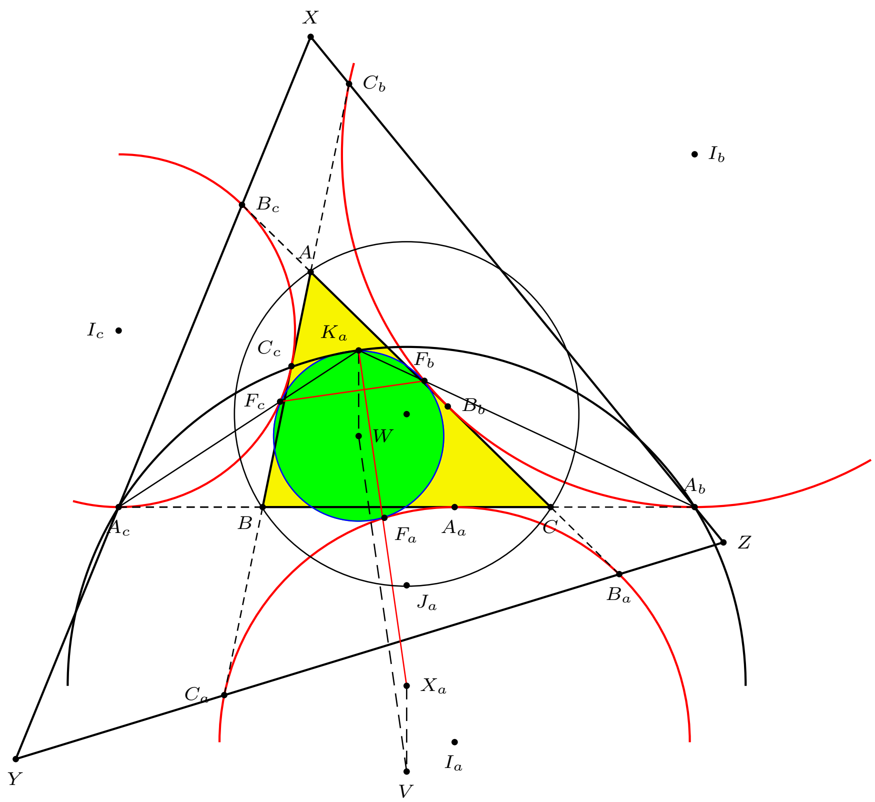


Figure 12.

Proof. (i) The points A_b, A_c, F_b, F_c are concyclic and the lines A_bF_b, A_cF_c meet at K_a. Let X_a be the circumcenter of $K_aA_bA_c$. Since F_b and F_c are points on K_aA_b and K_aA_c, and $F_bA_bA_cF_c$ is cyclic, it follows from Lemma 5 that K_aX_a is perpendicular to F_bF_c. Hence X_a is the intersection of the perpendicular from K_a to F_bF_c and the perpendicular bisector of BC. Since triangle $K_aA_bA_c$ is similar to $K_aF_cF_b$, and $A_bA_c = b + c$, its circumradius is

$$\frac{b+c}{F_bF_c} \cdot \frac{R}{2} = \frac{1}{2}\sqrt{(R+2r_b)(R+2r_c)}.$$

Here, we have made use of the formula

$$F_bF_c = \frac{b+c}{\sqrt{(R+2r_b)(R+2r_c)}} \cdot R$$

from [2].

(ii) A simple angle calculation shows that the points Y, Z, A_b, A_c are also concyclic. Its center is the intersection of the perpendicular bisectors of A_bA_c and YZ. The perpendicular bisector of A_bA_c is clearly the same as that of BC. Since YZ is parallel to I_bI_c, its perpendicular is the parallel through H (the circumcenter of XYZ) to the bisector of angle A.

(iii) Therefore, if this circumcenter is V, then $J_aV = AH = 2R\cos A$.

(iv) To show that the two circle $F_bA_bA_cF_c$ is the same as the circle in (ii), it is enough to show that V lies on the perpendicular bisector of F_bF_c. This is equivalent to showing that VW is perpendicular to F_bF_c. To prove this, we show that K_aWVX_a is a parallelogram. Applying the Pythagorean theorem to triangle $A'A_bX_a$, we have

$$\begin{aligned}
4A'X_a^2 =& (R+2r_b)(R+2r_c) - (b+c)^2 \\
=& R^2 + 4R(r_b+r_c) + 4r_br_c - (b+c)^2 \\
=& R^2 + 4R\cdot R(1+\cos A) + 4s(s-a) - (b+c)^2 \\
=& R^2(1+4(1+\cos A)) - a^2 \\
=& R^2(1+4(1+\cos A) - 4\sin^2 A) \\
=& R^2(1+2\cos A)^2.
\end{aligned}$$

This means that $A'X_a = \frac{R}{2}(1+2\cos A)$, and it follows that

$$\begin{aligned}
X_aV =& A'V - A'X_a = A'J + JV - A'X_a \\
=& R(1-\cos A) + 2R\cos A - \frac{R}{2}(1+2\cos A) \\
=& \frac{R}{2} = K_aW.
\end{aligned}$$

Therefore, VW, being parallel to K_aX_a, is perpendicular to F_bF_c. □

Denote by $\mathcal{C}_a$ the circle through these 6 points. Similarly define $\mathcal{C}_b$ and $\mathcal{C}_c$.

Corollary 22. *The radical center of the circles $\mathcal{C}_a$, $\mathcal{C}_b$, $\mathcal{C}_c$ is S'.*

Proof. The points X and F_a are common to the circles $\mathcal{C}_b$ and $\mathcal{C}_c$. The line XF_a is the radical axis of the two circles. Similarly the radical axes of the two other two pairs of circles are YF_b and ZF_c. The radical center is therefore S'. □

Proposition 23. *The line XA_a is perpendicular to YZ.*

Proof. With reference to Figure 8, note that

$$\begin{aligned}A_bY_a : A_bX =& A_bC \cdot \frac{\sin\left(C+\frac{A}{2}\right)}{\sin\frac{C}{2}} : A_bA_c \cdot \frac{\sin\frac{A+B}{2}}{\sin\frac{B+C}{2}}\\ =& A_bC : (b+c) \cdot \frac{\sin\frac{C}{2}\sin\frac{A+B}{2}}{\sin\left(C+\frac{A}{2}\right)\sin\frac{B+C}{2}}\\ =& A_bC : (b+c) \cdot \frac{\sin C}{\sin(C+A)+\sin C}\\ =& A_bC : c\\ =& A_bC : A_bA_a.\end{aligned}$$

This means that XA_a is parallel to Y_cC, which is perpendicularto I_bI_c and YZ. □

Corollary 24. *XYZ is perspective with the extouch triangle $A_aB_bC_c$, and the perspector is the orthocenter of XYZ.*

Remark. This is the triangle center X_{72} of [6].

Proposition 25. *The complement of the Schiffler point is the point S' which divides HW in the ratio*

$$HS' : S'W = 2(2R+r) : -R.$$

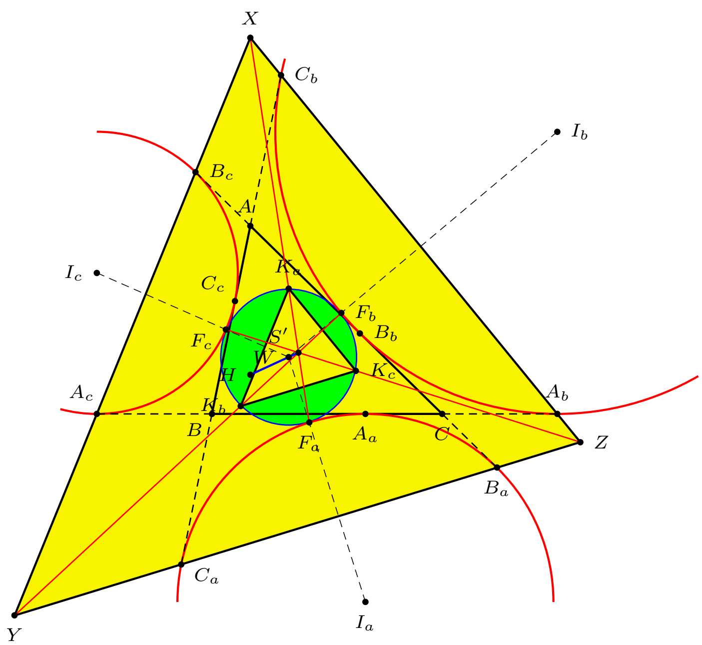


Figure 13.

Proof. We define K_b and K_c similarly as K_a. Since K_b and K_c are the midpoints of the arcs $C'A'$ and $A'B'$, K_bK_c is perpendicular to the A'-bisector of $A'B'C'$, and hence parallel to YZ. The triangle $K_aK_bK_c$ is homothetic to XYZ. The homothetic center is the common point of the lines XK_a, YK_b, and ZK_c, which are XF_a, YF_b, ZF_c. This is the complement of the Schiffler point. Since triangles $K_aK_bK_c$ and XYZ have circumcenters W, H, and circumradii $\frac{R}{2}$ and $2R+r$, this homothetic center S' divides the segment HW in the ratio given above. □

References

[1] J.-P. Ehrmann, Hyacinthos message 10564, October 1, 2004.
[2] L. Emelyanov and T. Emelyanova, A note on the Feuerbach point, *Forum Geom.*, 1 (2001) 121–124.
[3] D. Grinberg, Hyacinthos message 10342, August 31, 2004.
[4] D. Grinberg, Hyacinthos message 10562, October 1, 2004.
[5] D. Grinberg, Hyacinthos message 10587, October 3, 2004
[6] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[7] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.
[8] K. L. Nguyen, Hyacinthos message 10384, September, 5, 2004.
[9] K. L. Nguyen, Hyacinthos message 10520, September, 23, 2004.
[10] K. L. Nguyen, Hyacinthos message 10563, October 1, 2004.
[11] K. L. Nguyen, Hyacinthos message 10913, November 28, 2004.
[12] J. C. Salazar, Hyacinthos message 10323, August 29, 2004.
[13] P. Yiu, Hyacinthos message 10565, October 1, 2004.

Khoa Lu Nguyen: 806 Candler Dr, Houston, Texas, 77037-4122, USA
E-mail address: treegoner@yahoo.com

Forum Geometricorum
Volume 5 (2005) 165–171.

FORUM GEOM
ISSN 1534-1178

On the Existence of Triangles with Given Circumcircle, Incircle, and One Additional Element

Victor Oxman

Abstract. We give necesssary and sufficient conditions for the existence of poristic triangles with two given circles as circumcircle and incircle, and (1) a side length, (2) the semiperimeter (area), (3) an altitude, and (4) an angle bisector. We also consider the question of construction of such triangles.

1. Introduction

It is well known that the distance d between the circumcenter and incenter of a triangle is given by the formula:

$$d^2 = R^2 - 2Rr, \tag{1}$$

where R and r are respectively the circumradius and inradius of the triangle ([3, p.29]). Therefore, if we are given two circles on the plane, with radii R and r, $(R \geq 2r)$, a necessary condition for an existence of a triangle, for which the two circles will be the circumcircle and the incircle, is that the distance d between their centers satifies (1). From Poncelet's closure theorem it follows that this condition is also sufficient. Furthermore, each point on the circle with radius R may be one of the triangle vertex, *i.e.*, in general there are infinitely many such triangles. A natural question is on the existence and uniqueness of such a triangle if we specify one additional element. We shall consider this question when this additional element is one of the following: (1) a side length, (2) the semiperimeter (area), (3) an altitude, and (4) an angle bisector.

2. Main results

Throughout this paper, we consider two given circles $O(R)$ and $I(r)$ with distance d between their centers satisfying (1). Following [2], we shall call a triangle with circumcircle $O(R)$ and incircle $I(r)$ a poristic triangle.

Theorem 1. *Let a be a given positive number.* (1). *If $d \leq r$, i.e. $R \leq (\sqrt{2}+1)r$, then there is a unique poristic triangle ABC with $BC = a$ if and only if*

$$4r(2R - r - 2d) \leq a^2 \leq 4r(2R - r + 2d). \tag{2}$$

Publication Date: November 8, 2005. Communicating Editor: Paul Yiu.

(2). *If* $d > r$, *i.e.* $R > (\sqrt{2}+1)r$, *then there is a unique poristic triangle* ABC *with* $BC = a$ *if and only if*

$$4r(2R - r - 2d) \le a^2 < 4r(2R - r + 2d) \quad \textit{or} \quad a = 2R, \tag{3}$$

and there are two such triangles if and only if

$$4r(2R - r + 2d) \le a^2 < 4R^2. \tag{4}$$

Theorem 2. *Given* $s > 0$, *there is a unique poristic triangle with semiperimeter* s *if and only if*

$$\sqrt{R+r-d}(\sqrt{2R} + \sqrt{R-r+d}) \le s \le \sqrt{R+r+d}(\sqrt{2R} + \sqrt{R-r-d}). \tag{5}$$

Theorem 3. *Given* $h > 0$, *there is a unique poristic triangle with an altitude* h *if and only if*

$$R + r - d \le h \le R + r + d. \tag{6}$$

Theorem 4. *Given* $\ell > 0$, *there is a unique poristic triangle with an angle bisector* ℓ *if and only if*

$$R + r - d \le \ell \le R + r + d. \tag{7}$$

3. Proof of Theorem 1

3.1. *Case 1.* $d \le r$. The length of $BC = a$ attains its minimal value when the distance from O to BC is maximal, which is $d + r$. See Figure 1. Therefore,

$$a_{\min}^2 = 4r(2R - r - 2d).$$

Similarly, a attains its maximum when the distance from O to BC is minimal, *i.e.*, $r - d$. See Figure 2.

$$a_{\max}^2 = 4r(2R - r + 2d).$$

This shows that (2) is a necessary condition a to be a side of a poristic triangle.

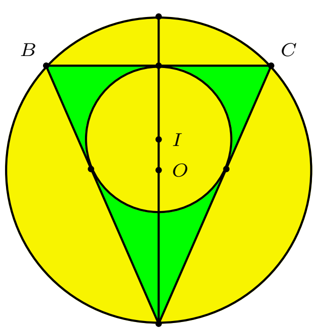


Figure 1

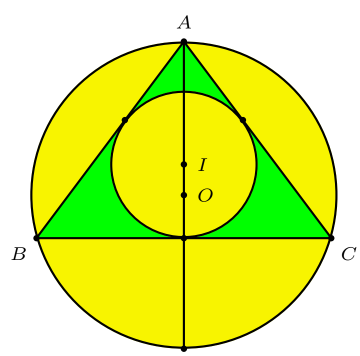


Figure2

We prove the sufficiency part by an explicit construction. If a satisfies (2), we construct the circle $O(R_1)$ with $R_1^2 = R^2 - \frac{a^2}{4}$, and a common tangent of this circle and $I(r)$. The segment of this tangent inside the circle $O(R)$ is a side of a poristic triangle with a side of length a. The third vertex is, by Poncelet's closure theorem, the intersection of the tangents from these endpoints to $I(r)$, and it lies on $O(R)$.

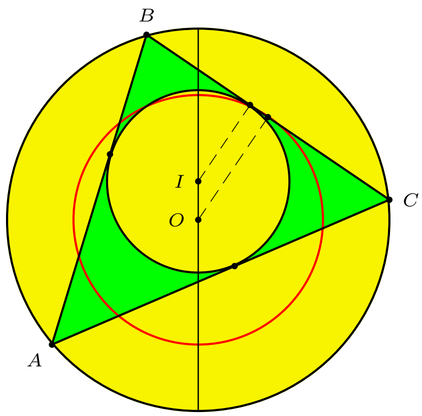


Figure 3

Remark. If $a \neq a_{\max}, a_{\min}$, we can construct two common tangents to the circles $O(R)$ and $I(r)$. These are both external common tangents and are symmetric with respect to the line OI. The resulting triangles are congruent.

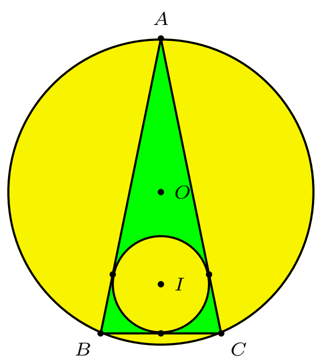


Figure 4

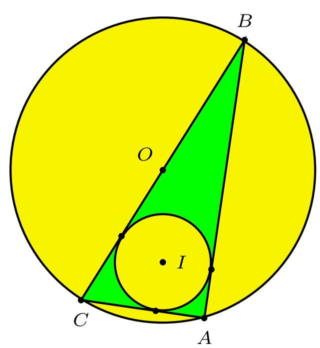


Figure 5

3.2. *Case 2.* $d > r$. In this case by the same way we have

$$a_{\min}^2 = 4r(2R - r - 2d).$$

See Figure 4. On the other hand, the maximum occurs when BC passes through the center O, *i.e.*, $a_{\max} = 2R$. See Figure 5.

For a given $a > 0$, we again construct the circle $O(R_1)$ with $R_1^2 = R^2 - \frac{a^2}{4}$. Chords of the circle (O) which are tangent to $O(R_1)$ have length a. If $R_1 > d - r$, the construction in §3.1 gives a poristic triangle with a side a. Therefore for

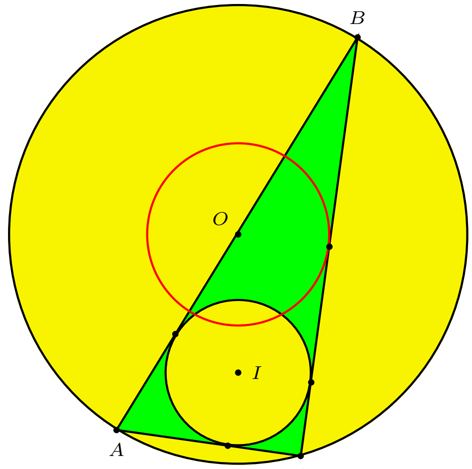


Figure 6

$4r(2R-r-2d) \leq a^2 < 4r(2R-r+2d)$, there is a unique poristic triangle with side a. See Figure 6. It is clear that this is also the case if $a = 2R$.

However, if $R_1 \leq d-r$, there are also internal common tangents of the circles $O(R_1)$ and $I(r)$. The internal common tangents give rise to an obtuse angled triangle. See Figures 7 and 8.

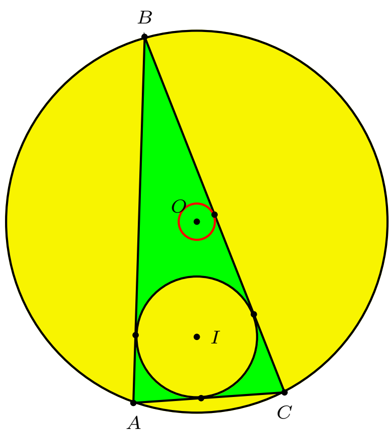


Figure 7

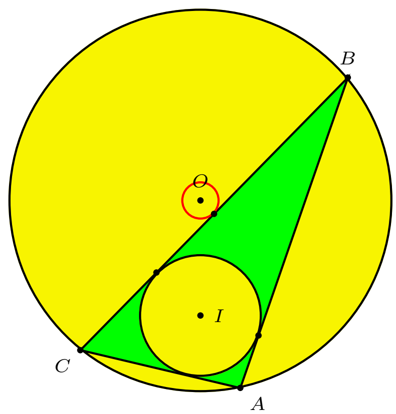


Figure 8

4. Proof of Theorem 2

Let $A_1B_1C_1$ and $A_2B_2C_2$ be the poristic triangles with A_1 and A_2 on the line OI. We assume $\angle A_1 \leq \angle A_2$. If $\angle A_1 = \angle A_2$, the triangle is equilateral and the statement of the theorem is trivial. We shall therefore assume $\angle A_1 < \angle A_2$. Consider an arbitrary poristic triangle ABC with semiperimeter s. According to

[4], s attains its maximum when the triangle coincides with $A_1B_1C_1$ and minimum when it coincides with $A_2B_2C_2$. Therefore,

$$\begin{aligned}
s_{\max} =& \sqrt{R^2-(r+d)^2}+\sqrt{R^2-(r+d)^2+(R+r+d)^2}\\
=& \sqrt{R+r+d}(\sqrt{2R}+\sqrt{R-r-d}),\\
s_{\min} =& \sqrt{R^2-(r-d)^2}+\sqrt{R^2-(r-d)^2+(R+r-d)^2}\\
=& \sqrt{R+r-d}(\sqrt{2R}+\sqrt{R-r+d}).
\end{aligned}$$

This proves (5).

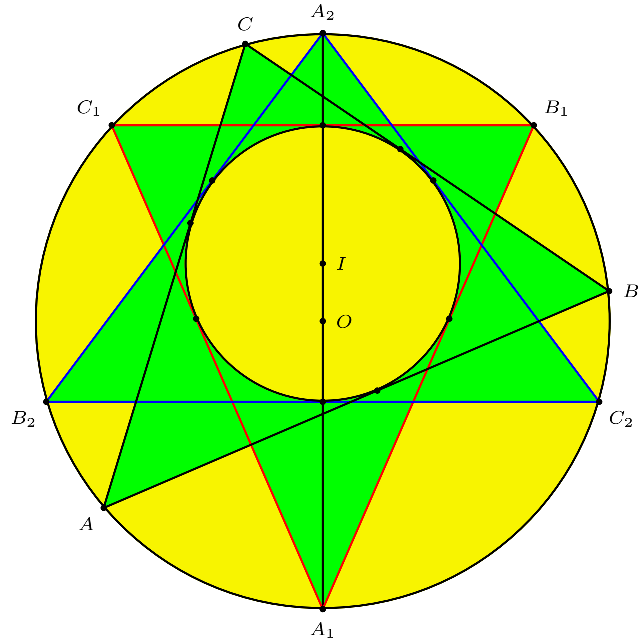


Figure 9

As A traverses a semicircle from position A_1 to A_2, the measure α of angle A is monotonically increasing from $\alpha_{\min} = \angle A_1$ to $\alpha_{\max} = \angle A_2$. For each $\alpha \in [\alpha_{\min},\ \alpha_{\max}]$,

$$s = s(\alpha) = \frac{r}{\tan\frac{\alpha}{2}} + 2R\sin\alpha.$$

Differentiating with respect to α, we have

$$s'(\alpha) = -\frac{r}{2\sin^2\frac{\alpha}{2}} + 2R\cos\alpha.$$

Clearly, $s'(\alpha) = 0$ if and only if $\sin^2\frac{\alpha}{2} = \frac{R\pm d}{4R}$. Since $\sin\frac{\alpha}{2} > 0$, there are two values of $\alpha \in (\alpha_{\min},\ \alpha_{\max})$ for which $s'(\alpha) = 0$. One of these is $\alpha_1 = \angle B_1$ for which $s(\alpha_1) = s_{\max}$ and the other is $\alpha_2 = \angle C_2$ for which $s(\alpha_2) = s_{\min}$.

Therefore for given real number $s > 0$ satisfying (5), there are three values of α (or two values if $s = s_{\min}$ or $s_{\max}$) for which $s(\alpha) = s$. These values are the values of the three angles of the same triangle that has semiperimeter s. So for such s the triangle is unique up to congruence.

Remark. Generally the ruler and compass construction of the triangle with given R, r and s is impossible. In fact, if $t = \tan\frac{\alpha}{2}$, then from $s = \frac{r}{\tan\frac{\alpha}{2}} + 2R\sin\alpha$ we have

$$st^3 - (4R+r)t^2 + st - r = 0.$$

The triangle is constructible if and only if t is constructible. It is known that the roots of a cubic equation with rational coefficients are constructible if and only if the equation has a rational root [1, p.16]. For $R = 4$, $r = 1$, $s = 8$ (such a triangle exists by Theorem 2) we have

$$8t^3 - 17t^2 + 8t - 1 = 0. \tag{8}$$

It is easy to see that it does not have rational roots. Therefore the roots of (8) are not constructible, and the triangle with given R, r, s is also not constructible.

5. Proof of Theorem 3

Let α be the measure of angle A.

$$h = \frac{2rs}{a} = \frac{\frac{2r^2}{\tan\frac{\alpha}{2}} + 4Rr\sin\alpha}{2R\sin\alpha} = \frac{r^2}{2R\sin^2\frac{\alpha}{2}} + 2r.$$

Since α is monotonically increasing (from $\alpha_{\min}$ to $\alpha_{\max}$ while vertex A moves from A_1 to A_2 along the arc A_1A_2, $h = h(\alpha)$ monotonically decreases from $h_{\max} = h(\alpha_{\min})$ to $h_{\min} = h(\alpha_{\max})$. Furthermore,

$$\begin{aligned} h_{\min} =& R + r - d, \\ h_{\max} =& R + r + d. \end{aligned}$$

This completes the proof of Theorem 3.

Remark. It is easy to construct the triangle by given R, r and h with the help of ruler and compass. Indeed, for a triangle ABC with given altitude $AH = h$ we have

$$AI^2 = \frac{r^2}{\sin^2\frac{\alpha}{2}} = 2R(h - 2r).$$

6. Proof of Theorem 4

The length of the bisector of angle A is given by

$$\ell = \frac{2bc\cos\frac{\alpha}{2}}{b+c}.$$

Since $R = \frac{abc}{4\triangle} = \frac{abc}{4rs}$, we have

$$\ell = \frac{\frac{8Rrs}{a}\cdot\cos\frac{\alpha}{2}}{2s-a} = \frac{r}{\sin\frac{\alpha}{2}} + \frac{2Rr\sin\frac{\alpha}{2}}{r + 2R\sin^2\frac{\alpha}{2}}.$$

Differentiating with respect to α, we have

$$\begin{aligned}\frac{\ell'(\alpha)}{r} &= -\frac{\cos\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2}} + \frac{R\cos\frac{\alpha}{2}(r-2R\sin^2\frac{\alpha}{2})}{(r+2R\sin^2\frac{\alpha}{2})^2} \\ &= -\frac{\cos\frac{\alpha}{2}(r^2+2Rr\sin^2\frac{\alpha}{2}+8R^2\sin^4\frac{\alpha}{2})}{2\sin^2\frac{\alpha}{2}(r+2R\sin^2\frac{\alpha}{2})^2} \\ &< 0.\end{aligned}$$

Therefore, $\ell(\alpha)$ monotonically decreases on $[\alpha_{\min}, \alpha_{\max}]$ from $\ell_{\max} = R + r + d$ to $\ell_{\min} = R + r - d$.

Remark. Generally the ruler and compass construction of the triangle with given R, r and ℓ is impossible. Indeed, if $t = \sin\frac{\alpha}{2}$, then

$$2R\ell t^3 - 4Rrt^2 + r\ell t - r^2 = 0.$$

For $R = 3$, $r = 1$ and $\ell = 5$ (such a triangle exists by Theorem 4), we have

$$30t^3 - 12t^2 + 5t - 1 = 0.$$

It can be easily checked that this equation doe not have a rational root. This shows that the ruler and compass construction of the triangle is not possible.

References

[1] B. Bold, *Famous Problems of Geometry and How to Solve Them*, Van Nostrand Reinhold, New York, 1969.

[2] E. Brisse, Perspective poristic triangles, *Forum Geom.*, 1 (2001) 9–16.

[3] H. S. M. Coxeter and S. Greitzer, *Geometry Revisited*, Math. Assoc. Amer., 1967.

[4] M. Radic, Extreme area of triangles in Poncelet's closure theorem, *Forum Geom.*, 4 (2004) 23–26.

Victor Oxman (Western Galilee College): Derech HaYam 191A, Haifa 34890, Israel

E-mail address: `voxman@013.net`

Forum Geometricorum
Volume 5 (2005) 173–180.

FORUM GEOM
ISSN 1534-1178

The Eppstein Centers and the Kenmotu Points

Eric Danneels

Abstract. The Kenmotu points of a triangle are triangle centers associated with squares each with a pair of opposite vertices on two sides of a triangle. Given a triangle ABC, we prove that the Kenmotu points of the intouch triangle are the same as the Eppstein centers associated with the Soddy circles of ABC.

1. Introduction

D. Eppstein [1] has discovered two interesting triangle centers associated with the Soddy circles of a triangle. Given a triangle ABC, construct three circles with centers at A, B, C, mutually tangent to each other externally at T_a, T_b, T_c respectively. These are indeed the points of tangency of the incircle of triangle ABC, and triangle $T_aT_bT_c$ is the intouch triangle of ABC. The inner (respectively outer) Soddy circle is the circle (S) (respectively (S')) tangent to each of these circles externally at S_a, S_b, S_c (respectively internally at S'_a, S'_b, S'_c).

Theorem 1 (Eppstein [1])**.** (1) *The lines T_aS_a, T_bS_b, and T_cS_c are concurrent at a point M.*
(2) *The lines $T_aS'_a$, $T_bS'_b$, and $T_cS'_c$ are concurrent at a point M'.*

See Figures 1 and 2. In [2], M and M' are the Eppstein centers X_{481} and X_{482}. Eppstein showed that these points are on the line joining the incenter I to the Gergonne point G_{e}. See Figure 1.

The Kenmotu points of a triangle, on the other hand, are associated with triads of congruent squares. Given a triangle ABC, the Kenmotu point K_{e} is the unique point such that there are congruent squares $K_{\mathrm{e}}B_cA_aC_b$, $K_{\mathrm{e}}C_aB_bA_c$, and $K_{\mathrm{e}}A_bC_cB_a$ with the same orientation as triangle ABC, and with A_b, A_c on BC, B_c, B_a on CA, and C_a, C_b on AB respectively. We call K_{e} the positive Kenmotu point. There is another triad of congruent squares with the opposite orientation as ABC, sharing a common vertex at the negative Kenmuto point K'_{e}. See Figure 3. These Kenmotu points lie on the Brocard axis of triangle ABC, which contains the circumcenter O and the symmedian point K.

The intouch triangle $T_aT_bT_c$ has circumcenter I and symmedian point G_{e}. It is immediately clear that the Kenmotu points of the intouch triangle lie on the same

Publication Date: November 15, 2005. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this paper.

line as do the Soddy and Eppstein centers of triangle ABC. The main result of this note is the following theorem.

Theorem 2. *The positive and negative Kenmotu points of the intouch triangle $T_aT_bT_c$ coincide with the Eppstein centers M and M'.*

We shall give two proofs of this theorem.

2. The Eppstein centers

According to [2], the coordinates of the Eppstein centers were determined by E. Brisse. [1] We shall work with homogeneous barycentric coordinates and make use of standard notations in triangle geometry. In particular, r_a, r_b, r_c denote the radii of the respective excircles, and S stands for twice the area of the triangle.

Theorem 3. *The homogeneous barycentric coordinates of the Eppstein centers are*
(1) $M = (a + 2r_a : b + 2r_b : c + 2r_c)$, *and*
(2) $M' = (a - 2r_a : b - 2r_b : c - 2r_c)$.

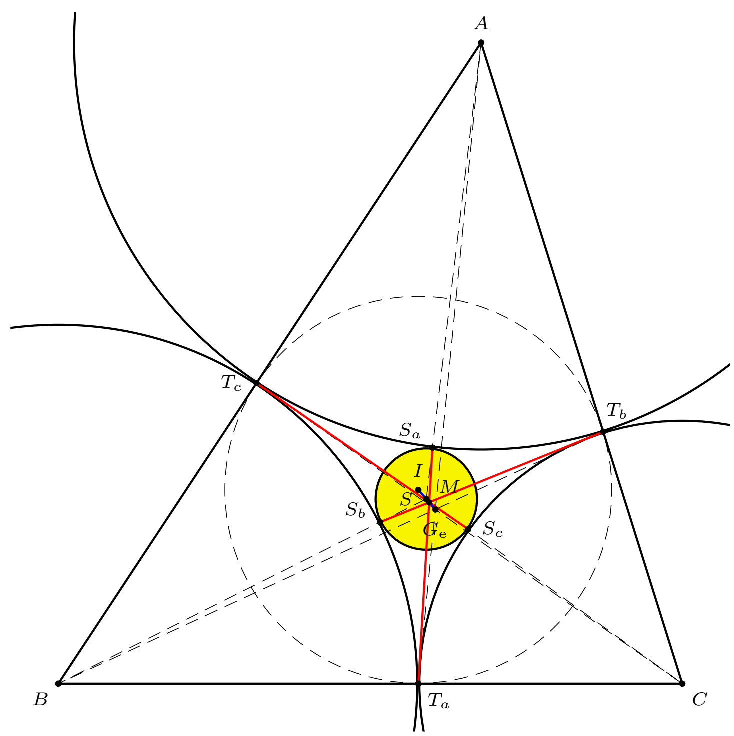


Figure 1. The Soddy center S and the Eppstein center M

[1]The coordinates of X_{481} and X_{482} in [2] (September 2005 edition) should be interchanged.

Remark. In [2], the Soddy centers appear as $X_{175} = S'$ and $X_{176} = S$. In homogeneous barycentric coordinates

$$S = (a + r_a : b + r_b : c + r_c),$$
$$S' = (a - r_a : b - r_b : c - r_c).$$

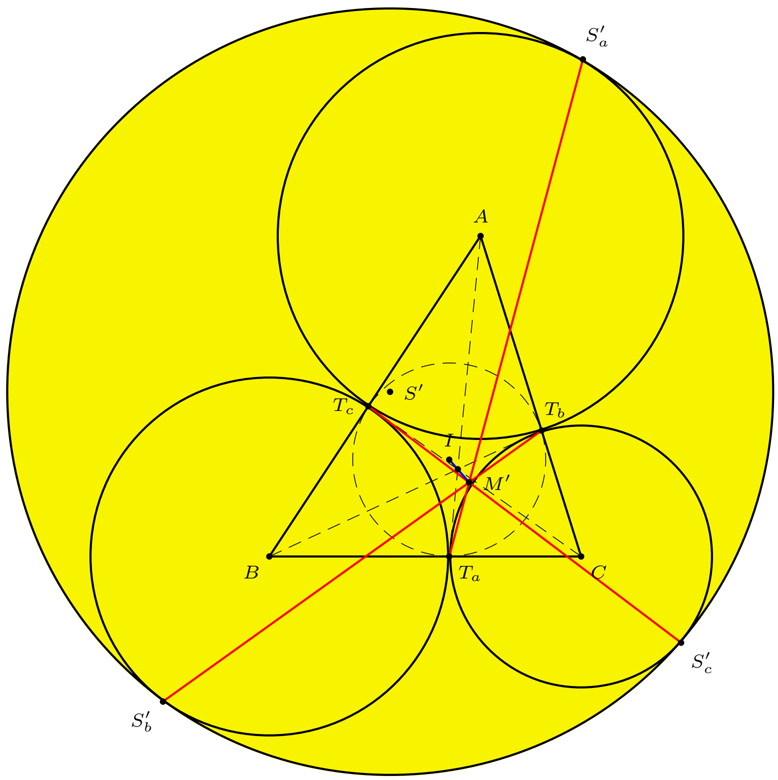


Figure 2. The Soddy center S' and the Eppstein center M'

3. The Kenmotu points

The Kenmotu points K_e and K'_e have homogeneous barycentric coordinates $(a^2(S_A \pm S) : b^2(S_B \pm S) : c^2(S_C \pm S))$. They are therefore points on the Brocard axis OK. See Figure 3.

Proposition 4. *The Kenmotu points K_e and K'_e divide the segment OK in the ratio*

$$OK_e : K_eK = a^2 + b^2 + c^2 : 2S,$$
$$OK'_e : K'_eK = a^2 + b^2 + c^2 : -2S.$$

Proof. A typical point on the Brocard axis has coordinates

$$K^*(\theta) = (a^2(S_A + S_\theta) : b^2(S_B + S_\theta) : c^2(S_C + S_\theta)).$$

It divides the segment OK in the ratio

$$OK^*(\theta) : K^*(\theta)K = (a^2 + b^2 + c^2)\sin\theta : 2S \cdot \cos\theta.$$

The Kenmotu points are the points $K_{\rm e}$ and $K_{\rm e}'$ are the points $K^*(\theta)$ for $\theta = \frac{\pi}{4}$ and $-\frac{\pi}{4}$ respectively. $\square$

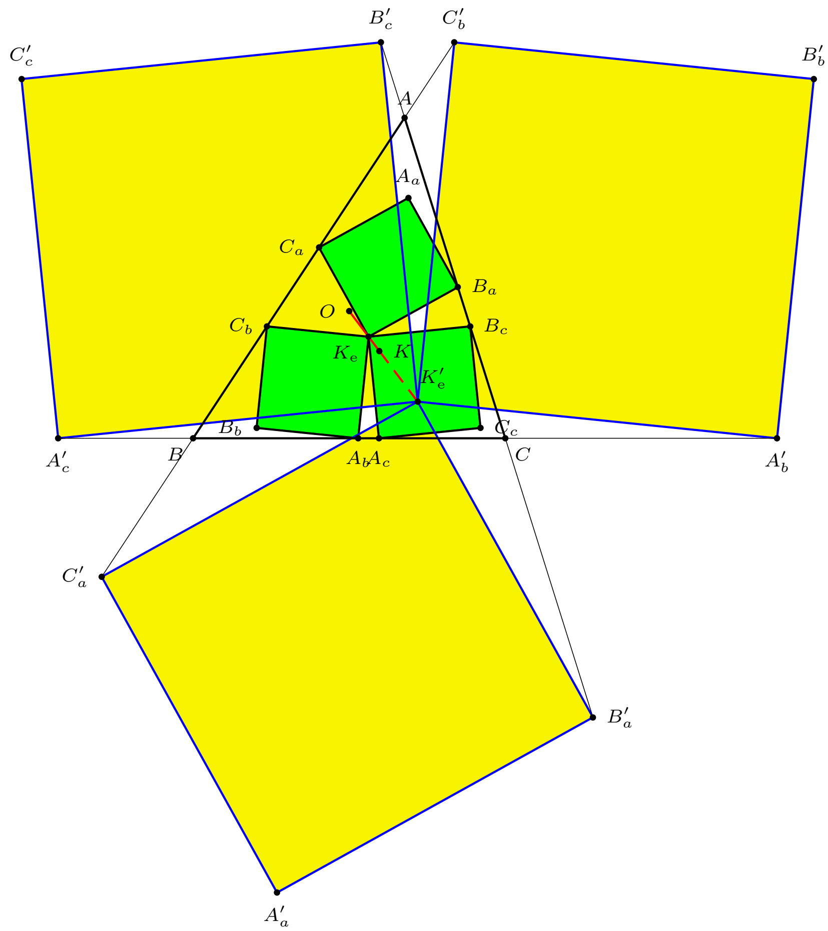


Figure 3. The Kenmotu points $K_{\rm e}$ and $K_{\rm e}'$

4. First proof of Theorem 2

We shall make use of the following results.

Lemma 5. (1) $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{s}{4R}$.
(2) $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = \frac{4R+r}{2R}$.
(3) $r_a + r_b + r_c = 4R + r$.

The intouch triangle $T_aT_bT_c$ has sidelengths

$$T_bT_c = 2r\cos\frac{A}{2}, \qquad T_cT_a = 2r\cos\frac{B}{2}, \qquad T_aT_b = 2r\cos\frac{C}{2}.$$

The area of the intouch triangle is

$$\frac{1}{2}\overline{S} = \frac{1}{2}T_cT_a \cdot T_aT_b \cdot \sin T_a = 2r^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 2r^2 \cdot \frac{s}{4R}.$$

On the other hand,

$$T_bT_c^2 + T_cT_a^2 + T_aT_b^2 = 4r^2\left(\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}\right) = \frac{2r^2(4R+r)}{R}.$$

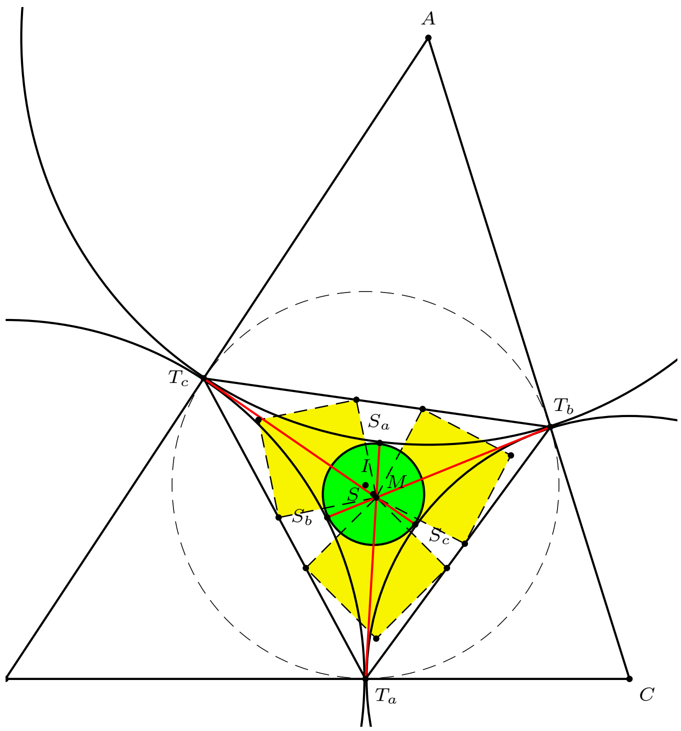


Figure 4. The positive Kenmotu point of the intouch triangle

By Proposition 4, the positive Kenmotu point $\overline{K}_{\rm e}$ of the intouch triangle divides the segment $IG_{\rm e}$ in the ratio

$$\begin{aligned} I\overline{K}_{\rm e} : \overline{K}_{\rm e}G_{\rm e} =& T_bT_c^2 + T_cT_a^2 + T_aT_b^2 : 2\overline{S} \\ =& 4R + r : s \\ =& r_a + r_b + r_c : s. \end{aligned}$$

It has absolute barycentric coordinates

$$\begin{aligned}\overline{K}_{\mathrm{e}} =&\frac{1}{s+r_a+r_b+r_c}\left(s\cdot I+(r_a+r_b+r_c)\cdot G_{\mathrm{e}}\right)\\ =&\frac{1}{s+r_a+r_b+r_c}\left(\frac{1}{2}(a,b,c)+(r_a,r_b,r_c)\right)\\ =&\frac{1}{2(s+r_a+r_b+r_c)}\cdot(a+2r_a,b+2r_b,c+2r_c).\end{aligned}$$

Therefore, $\overline{K}_{\mathrm{e}}$ has homogeneous barycentric coordinates $(a+2r_a : b+2r_b : +2r_c)$. By Theorem 3, it coincides with the Eppstein center M. See Figure 4.

Similar calculations show that the Eppstein center M' coincides with the negative Kenmotu point $\overline{K}'_{\mathrm{e}}$ of the intouch triangle. See Figure 5. The proof of Theorem 2 is now complete.

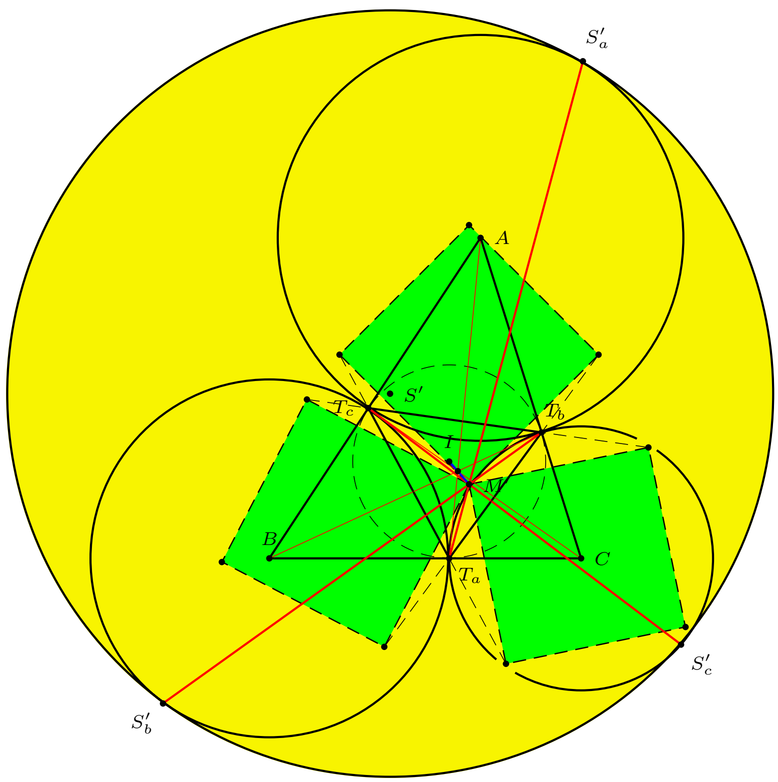


Figure 5. The negative Kenmotu point of the intouch triangle

5. Second proof of Theorem 2

Consider a point P with homogeneous barycentric coordinates $(u' : v' : w')$ with respect to the intouch triangle $T_aT_bT_c$. We determine its coordinates with respect to the triangle ABC. By the definition of barycentric coordinates, a system of three masses u', v' and w' at the points T_a, T_b and T_c will balance at P. The mass u' at T_a can be replaced by a mass $\frac{s-c}{a}\cdot u'$ at B and a mass $\frac{s-b}{a}\cdot u'$ at C. Similarly, the mass v' at T_b can be replaced by a mass $\frac{s-a}{b}\cdot v'$ at C and a mass $\frac{s-c}{b}\cdot v'$ at A, and the mass w' at T_c by a mass $\frac{s-b}{c}\cdot w'$ at A and a mass $\frac{s-a}{c}\cdot w'$ at B. The resulting mass at A is therefore

$$\frac{s-c}{b}\cdot v' + \frac{s-b}{c}\cdot w' = \frac{a(c(s-c)v'+b(s-b)w')}{abc}.$$

From similar expressions for the masses at B and C, we obtain

$$(a(c(s-c)v'+b(s-b)w') : b(a(s-a)w'+c(s-c)u') : c(b(s-b)u'+a(s-a)v'))$$

for the barycentric coordinates of P with respect to ABC.

The Kenmotu point K_{e} appears the triangle center X_{371} in [2]. For the Kenmotu point of the intouch triangle, we may take

$$\begin{aligned}
u' =& T_bT_c(\cos T_a + \sin T_a)\\
=& 2(s-a)\sin\frac{A}{2}\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right),\\
v' =& 2(s-b)\sin\frac{B}{2}\left(\sin\frac{B}{2}+\cos\frac{B}{2}\right),\\
w' =& 2(s-c)\sin\frac{C}{2}\left(\sin\frac{C}{2}+\cos\frac{C}{2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
u =& a(c(s-c)v'+b(s-b)w')\\
=& 2a(s-b)(s-c)\left(c\cdot\sin\frac{B}{2}\left(\sin\frac{B}{2}+\cos\frac{B}{2}\right)+b\cdot\sin\frac{C}{2}\left(\sin\frac{C}{2}+\cos\frac{C}{2}\right)\right)\\
=& 2a(s-b)(s-c)\left(c\sin^2\frac{B}{2}+b\sin^2\frac{C}{2}+c\cdot\frac{\sin B}{2}+b\cdot\frac{\sin C}{2}\right)\\
=& 2a(s-b)(s-c)\left(c\cdot\frac{(s-c)(s-a)}{ca}+b\cdot\frac{(s-a)(s-b)}{ab}+\frac{bc}{2R}\right)\\
=& 2(s-a)(s-b)(s-c)\left(a+\frac{abc}{2R(s-a)}\right)\\
=& 2(s-a)(s-b)(s-c)\left(a+\frac{S}{s-a}\right)\\
=& 2(s-a)(s-b)(s-c)(a+2r_a).
\end{aligned}$$

Similar expressions for v and w give

$$u:v:w = a+2r_a : b+2r_b : c+r_c,$$

which are the coordinates of the Eppstein center M.

References

[1] D. Eppstein, Tangent spheres and triangle centers, *Amer. Math. Monthly*, 108 (2001) 63–66.
[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Eric Danneels: Hubert d'Ydewallestraat 26, 8730 Beernem, Belgium
E-mail address: eric.danneels@pandora.be

Forum Geometricorum
Volume 5 (2005) 181–190.

FORUM GEOM
ISSN 1534-1178

Statics and the Moduli Space of Triangles

Geoff C. Smith

Abstract. The variance of a weighted collection of points is used to prove classical theorems of geometry concerning homogeneous quadratic functions of length (Apollonius, Feuerbach, Ptolemy, Stewart) and to deduce some of the theory of major triangle centers. We also show how a formula for the distance of the incenter to the reflection of the centroid in the nine-point center enables one to simplify Euler's method for the reconstruction of a triangle from its major centers. We also exhibit a connection between Poncelet's porism and the location of the incenter in the circle on diameter GH (the orthocentroidal or critical circle). The interior of this circle is the moduli (classification) space of triangles.

1. Introduction

There are some theorems of Euclidean geometry which have elegant proofs by means of mechanical principles. For example, if ABC is an acute triangle, one can ask which point P in the plane minimizes $AP + BP + CP$? The answer is the Fermat point, the place where $\angle APB = \angle BPC = \angle CPA = 2\pi/3$. The mechanical solution is to attach three pieces of inextensible massless string to P, and to dangle the three strings over frictionless pulleys at the vertices of the triangle, and attach the same mass to each string. Now hold the triangle flat and dangle the masses in a uniform gravitational field. The forces at P must balance so the angle equality is obtained, and the potential energy of the system is minimized when $AP + BP + CP$ is minimized.

In this article we will develop a geometric technique which involves a notion analogous to the moment of inertia of a mechanical system, but because of an averaging process, this notion is actually more akin to *variance* in statistics. The main result is well known to workers in the analysis of variance. The applications we give will (in the main) not yield new results, but rather give alternative proofs of classical results (Apollonius, Feuerbach, Stewart, Ptolemy) and make possible a systematic statical development of some of the theory of triangle centers. We will conclude with some remarks concerning the problem of reconstructing a triangle from O, G and I which will, we hope, shed more light on the constructions of Euler [3] and Guinand [4].

Publication Date: December 6, 2005. Communicating Editor: Paul Yiu.

I wish to thank Christopher Bradley of Bristol both for helping to rekindle my interest in Euclidean geometry, and for lending plausibility to some of the longer formulas in this article by means of computational experiments.

For geometrical background we recommend [1] and [2].

Definition. Let X and Y be non-empty finite subsets of an inner product space V. We have weight maps $m : X \longrightarrow \mathbb{R}$ and $n : Y \longrightarrow \mathbb{R}$ with the property that $M = \sum_x m(x) \neq 0 \neq \sum_{y \in Y} n(y) = N$. The *mean square distance* between these weighted sets is

$$d^2(X, m, Y, n) = \frac{1}{MN} \sum_{x \in X, y \in Y} m(x)n(y)||x - y||^2.$$

Let

$$\overline{x} = \frac{1}{M} \sum_{x \in X} m(x)x$$

be the centroid of X. Ignoring the distinction between $\overline{x}$ and $\{\overline{x}\}$, and assigning the weight 1 to $\overline{x}$, we put

$$\sigma^2(X, m) = d^2(X, m, \overline{x}, 1)$$

and call this the *variance* of X, m. In fact the non-zero weight assigned to $\overline{x}$ is immaterial since it cancels. When the weighting is clear in a particular context, mention of it may be suppressed. We will also be cavalier with the arguments of these functions for economy.

We call the main result the generalized parallel axis theorem (abbreviated to GPAT) because of its relationship to the corresponding result in mechanics.

Theorem 1 (GPAT)**.**

$$d^2(X, m, Y, n) = \sigma^2(X, m) + ||\overline{x} - \overline{y}||^2 + \sigma^2(Y, m).$$

Proof.

$$\begin{aligned}
&\frac{1}{MN} \sum_{x \in X, y \in Y} m(x)n(y)||x - y||^2 \\
=&\frac{1}{MN} \sum_{x \in X, y \in Y} m(x)n(y)||x - \overline{x} + \overline{x} - \overline{y} + \overline{y} - y||^2 \\
=&\frac{1}{MN} \sum_{x \in X, y \in Y} m(x)n(y)||x - \overline{x}||^2 + ||\overline{x} - \overline{y}||^2 + \frac{1}{MN} \sum_{x \in X, y \in Y} m(x)n(y)||y - \overline{y}||^2
\end{aligned}$$

since the averaging process makes the cross terms vanish. We are done. □

Corollary 2. $d^2(X, m, X, m) = 2\sigma^2(X, m)$.

Note that the averaging process ensures that scaling the weights of a given set does not alter mean square distances or variances.

The method of areal co-ordinates involves fixing a reference triangle ABC in the plane, and given a point P in its interior, assigning weights which are the areas of triangles: the weights $[PBC], [PCA]$ and $[PAC]$ are assigned to the points A, B and C respectively. The center of mass of $\{A, B, C\}$ with the given weights is P. With appropriate signed area conventions, this can be extended to define a co-ordinate system for the whole plane. If the weights are scaled by dividing by the area of $\triangle ABC$, then one obtains normalized areal co-ordinates; the co-ordinates of A are then $(1,0,0)$ for example. A similar arrangement works in Euclidean space of any dimension. The GPAT has much to say about these co-ordinate systems.

2. Applications

2.1. *Theorems of Apollonius and Stewart.* Let ABC be a triangle with corresponding sides of length a, b and c. A point D on the directed line CB is such that $CD = m$, $DB = n$ and these quantities may be negative. Let AD have length x. Weighting B with m and C with n, the center of mass of $\{B, C\}$ is at D and the variance of the weighted $\{B, C\}$ is $\sigma^2 = (mn^2 + nm^2)/(m+n) = mn$. The GPAT now asserts that

$$\frac{nb^2 + mc^2}{m+n} = 0 + x^2 + \sigma^2$$

or rather

$$nb^2 + mc^2 = (m+n)(x^2 + mn).$$

This is Stewart's theorem. If $m = n$ we deduce Apollonius's result that $b^2 + c^2 = 2(x^2 + (\frac{a}{2})^2)$.

2.2. *Ptolemy's Theorem.* Let A, B, C and D be four points in Euclidean 3-space. Consider the two sets $\{A, C\}$ and $\{B, D\}$ with weight 1 at each point. The GPAT asserts that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4t^2$$

where t is the distance between the midpoints of the line segments AC and BD. This may be familiar in the context that $t = 0$ and $ABCD$ is a parallelogram.

Recall that Ptolemy's theorem asserts that if $ABCD$ is a cyclic quadrilateral, then

$$AC \cdot BD = AB \cdot CD + BC \cdot DA.$$

We prove this as follows. Let the diagonals AC and BD meet at X. Now weight A, B, C and D so that the centers of mass of both $\{A, C\}$ and $\{B, D\}$ are at X. The GPAT now asserts that

$$\frac{XC \cdot AX^2 + AX \cdot XC^2}{AC} + \frac{XB \cdot DX^2 + DX \cdot BX^2}{BD}$$

$$= \frac{XC \cdot AB^2 \cdot XD + XC \cdot AD^2 \cdot BX + XA \cdot CB^2 \cdot XD + XA \cdot CD^2 \cdot XB}{AC \cdot BD}.$$

The left side of this equation tidies to $AX \cdot XC + BX \cdot XD$. One could regard this equation as a generalization of Ptolemy's theorem to quadrilaterals which are not necessarily cyclic.

Now we invoke cyclicity: $AX \cdot XC = BX \cdot XD = x$ by the intersecting chords theorem. Therefore $AC \cdot BD =$

$$\frac{XC \cdot AB^2 \cdot XD + XC \cdot AD^2 \cdot BX + XA \cdot CB^2 \cdot XD + XA \cdot CD^2 \cdot XB}{2x}.$$

However $AB/CD = BX/XC = AX/XD$ and $DA/BC = AX/BX = DX = CX$ (by similarity) so the right side of this equation is $AB \cdot CD + BC \cdot DA$ and Ptolemy's theorem is established.

2.3. *A geometric interpretation of* σ^2. Let ABC be a triangle with circumcenter O and incenter I and the usual side lengths a, b and c. We can arrange that the center of mass of $\{A, B, C\}$ is at I by placing weights a, b and c at A, B and C respectively. By calculating the mean square distance of this set of weighted triangle vertices to itself, we obtain the variance $\sigma_I^2 = \frac{abc}{a+b+c}$. However $abc/4R = [ABC]$, the area of the triangle, and $(a+b+c)r = 2[ABC]$ where R, r are the circumradius and inradius respectively. Therefore

$$\sigma_I^2 = 2Rr = \frac{abc}{a+b+c}. \tag{1}$$

Now calculate the mean square distance from O to the weighted triangle vertices both in the obvious way, and also by the GPAT to obtain Euler's result

$$OI^2 = R^2 - 2Rr. \tag{2}$$

Observation More generally suppose that a finite coplanar set of points Λ is concyclic, and is weighted to have center of mass at L, Let the center of the circle be at X and its radius be ρ. By the GPAT applied to X and the weighted set Λ we obtain

$$LX^2 = \rho^2 - \sigma^2(\Lambda, L)$$

so

$$\sigma^2(\Lambda, L) = \rho^2 - LX^2 = (\rho - LX)(\rho + LX).$$

Thus we conclude that $\sigma^2(\Lambda, L)$ is minus the *power of* L with respect to the circle.

2.4. *The Euler line.* Let ABC be a triangle with circumcenter O, centroid G and orthocenter H. These three points are collinear and this line is called the Euler line. It is easy to show that $OH = 3OG$. It is well known that

$$OH^2 = 9R^2 - (a^2 + b^2 + c^2). \tag{3}$$

We derive this formula using the GPAT. Assign unit weights to the vertices of triangle ABC. The center of mass will be at G the intersection of the medians. Calculate the mean square distance of this triangle to itself to obtain the variance σ_G^2 of this triple of points. By the GPAT we have

$$2\sigma_G^2 = \frac{2a^2 + 2b^2 + 2c^2}{9}$$

so $\sigma_G^2 = \frac{a^2+b^2+c^2}{9}$. Now calculate the mean square distance from O to this triangle with unit weight the sensible way, and also by the GPAT to obtain

$$R^2 = OG^2 + \sigma_G^2.$$

Multiply through by 9 and use the fact that $OH = 3OG$ to obtain (3).

2.5. *The Nine-point Circle.* Let ABC be a triangle. The nine-point circle of ABC is the circle which passes through the midpoints of the sides, the feet of the altitudes and the midpoints of the line segments joining the orthocenter H to each vertex. This circle has radius $R/2$ and is tangent to the inscribed circle of triangle ABC (they touch internally to the nine-point circle), and the three escribed circles (externally). We will prove this last result using the GPAT, and calculate the squares of the distances from I to important points on the Euler line.

Proposition 3. *Let p denote the perimeter of the triangle A, B, C. The distance between the incenter I and centroid G satisfies the following equation:*

$$IG^2 = \frac{p^2}{6} - \frac{5}{18}(a^2 + b^2 + c^2) - 4Rr. \tag{4}$$

Proof. Let $\triangle_G$ denote the triangle weighted 1 at each vertex and $\triangle_I$ denote the same triangle with weights attached to the vertices which are the lengths of the opposite sides. We apply the GPAT and a direct calculation:

$$d^2(\triangle_G, \triangle_I) = \sigma_G^2 + IG^2 + \sigma_I^2 = \frac{ab^2 + ba^2 + bc^2 + cb^2 + ca^2 + ac^2}{3(a+b+c)}$$

so

$$\begin{aligned}\frac{a^2+b^2+c^2}{9} + IG^2 + 2Rr &= \frac{(ab+bc+ca)(a+b+c) - 3abc}{3(a+b+c)} \\ &= \frac{ab+bc+ca}{3} - 2Rr.\end{aligned}$$

Therefore

$$4Rr + IG^2 + \frac{a^2+b^2+c^2}{9} = \frac{(a+b+c)^2}{6} - \frac{a^2+b^2+c^2}{6}.$$

This equation can be tidied into the required form.

Corollary Using Euler's inequality $R \geq 2r$ (which follows from $IO^2 \geq 0$) and the condition $|IG|^2 \geq 0$ we obtain that in any triangle we have

$$3p^2 \geq 5(a^2 + b^2 + c^2) + 144r^2$$

with equality exactly when $R = 2r$ and $I = G$. Thus the inequality becomes an equality if and only if the triangle is equilateral. $\square$

Theorem 4 (Feuerbach)**.** *The nine point circle of $\triangle ABC$ is internally tangent to the incircle.*

Proof. (outline) The radius of the nine point circle is $R/2$. The result will establish if we show that $|IN| = R/2 - r$. However, in $\triangle INO$ the point G is on the side NO and $NG : GO = 1 : 2$. We know $|IO|, |IG|, |NG|$ and $|GO|$, so Stewart's theorem and some algebra enable us to deduce the result.

Since $OG : GN = 2 : 1$ Stewart's theorem applies and we have

$$IG^2 + \frac{2}{9}ON^2 = \frac{2}{3}IN^2 = \frac{1}{3}IO^2.$$

Rearranging this becomes

$$IN^2 = \frac{3}{2}IG^2 + \frac{3}{4}OG^2 - \frac{1}{2}IO^2.$$

Now we aim to show that this expression is $(R/2 - r)^2$, or rather $R^2/4 - Rr + r^2$. We put in known values in terms of the side lengths, and perform algebraic manipulations, deploying Heron's formula where necessary. Feuerbach's theorem follows. □

It must be admitted that this calculation does little to illuminate Feuerbach's result. We will give a more conceptual statics proof shortly.

2.6. *The location of the incenter.*

Proposition 5. *The incenter of a non-equilateral triangle lies strictly in the interior of the circle on diameter GH.*

This was presumably known to Euler [5], and a stronger version of the result was proved in [4]. Given Feuerbach's theorem, this result almost proves itself. Let N be the nine-point center, the midpoint of the segment OH, Feuerbach's tangency result yields $IN = R/2 - r$. However $OI^2 = R^2 - 2Rr$ so $OI^2 - 4IN^2 = R^2 - 4Rr + 4r^2 - R^2 + 2Rr = 2r(R - 2r)$. However Euler's formula for OI yields $2r < R$ (with equality only for equilateral triangles). Therefore I lies in the interior of the circle of Apollonius consisting of points P such that $OP = 2NP$, which is precisely the circle on diameter GH as required.

We can verify this result by an explicit calculation. Let J be the center of the circle on diameter GH so $OG = GJ = JH$. Using Apollonius's theorem on $\triangle IHO$ we obtain

$$2IN^2 + 2(\frac{3}{2}OG)^2 = OI^2 + IH^2$$

which expands to reveal that

$$HI^2 = \frac{OH^2 - (R^2 - 4r^2)}{2}.$$

Now use Stewart's theorem on $\triangle IHO$ to calculate IJ^2. We have

$$IJ^2 + 2OG^2 = \frac{OI^2 \cdot OG + IH^2 \cdot 2OG}{OH}$$

which after simple manipulation yields that

$$IJ^2 = OG^2 - \frac{2r}{3}(R - 2r) < OG^2. \tag{5}$$

The formulas for the squares of the distances from I to important points on the Euler line can be quite unwieldy, and some care has been taken to calculate these quantities in such a way that the algebraic dependence between the triangle sides and r, R and OG is produces relatively straightforward expressions. More interesting relationship can be found; for example using Stewart's theorem on $\triangle INO$ with Cevian IG we obtain

$$6IG^2 + 3OG^2 = (3R - 2r)(R - 2r).$$

3. Areal co-ordinates and Feuerbach revisited

The use of areal or volumetric co-ordinates is a special but important case of weighted systems of points. The GPAT tells us about the change of co-ordinate frames: given two reference triangles $\triangle_1$ with vertices A, B, C and $\triangle_2$ with vertices A', B', C' and points P and Q in the plane. it is natural to consider the relationship between the areal co-ordinates of a point P in the first frame (x, y, z) and those of Q in the second (x', y', z'). We assume that co-ordinates are normalized. Now GPAT tells us that

$$d^2(\triangle_{1,P}, \triangle_{2,Q}) = \sigma^2_{1,P} + PQ^2 + \sigma^2_{1,Q}.$$

The resulting formulas can be read off. The recipe which determines the square of the distance between two points given in areal co-ordinates with respect to the same reference triangle is straightforward. Suppose that P has areal co-ordinates (p_1, p_2, p_3) and Q has co-ordinates (q_1, q_2, q_3). Let $(x, y, z) = (p_1, p_2, p_3) - (q_1, q_2, q_3)$ (subtraction of 3-tuples) and let $(u, v, w) = (yz, zx, xy)$ (the Cremona transformation) then we deduce that

$$PQ^2 = -(a^2, b^2, c^2) \cdot (u, v, w).$$

Here we are using the ordinary dot product of 3-tuples. Note that (a^2, b^2, c^2) viewed as an areal co-ordinate is the symmedian point, the isogonal conjugate of G. We do not know if this observation has any significance.

A another special situation arises when $\triangle_1$ and $\triangle_2$ have the same circumcircle (perhaps they are the same triangle) and points P and Q are both on the common circle. In this case $\sigma^2_{1,P} = 0 = \sigma^2_{2,Q}$ and

$$d^2(\triangle_{1,P}, \triangle_{2,Q}) = PQ^2.$$

In the context of areal co-ordinates, we are now in a position to revisit Feuerbach's theorem and give a more conceptual statics proof which yields an interesting corollary.

3.1. *Proof of Feuerbach's theorem.* To prove Feuerbach's theorem it suffices to show that the power of I with respect to the nine-point circle is $-r(R - r)$ or equivalently that $\widehat{\sigma}^2_I = r(R - r)$ where the hat indicates that we are using the medial triangle (with vertices the midpoints of the sides of $\triangle ABC$) as the triangle of reference. Now the medial triangle is obtained by rotating the original triangle about G through π, and scaling by 1/2. Let I' denote the incenter of the medial triangle with co-ordinates $(a/2, b/2, c/2)$. The co-ordinates of G are $(s/3, s/3, s/3)$. Now I', G, I are collinear and $I'G : GI = 1 : 2$. The co-ordinates of I are therefore $(s-a, s-b, s-c)$, Next we use cyc to indicate a sum over cyclic permutations of a, b and c, and *sym* a sum over all permutations. We calculate

$$\begin{aligned}\widehat{\sigma}_I^2 &= \sum_{cyc} \frac{(s-a)(s-b)c^2}{4s^2} \\ &= \frac{s^2 \sum_{cyc} a^2 - s \sum_{sym} a^2 b + 2abcs}{4s^2} \\ &= \frac{a^3+b^3+c^3}{4(a+b+c)} - \frac{\sum_{sym} a^2 b}{4(a+b+c)} + 2Rr.\end{aligned}$$

However by Heron's formula

$$r^2 = \frac{(b+c-a)(a+c-b)(a+b-c)}{4(a+b+c)}$$

so

$$rR - r^2 = \frac{2abc}{4(a+b+c)} + \frac{a^3+b^3+c^3}{4(a+b+c)} - \frac{\sum_{sym} a^2 b}{4(a+b+c)} + \frac{2abc}{4(a+b+c)} = \widehat{\sigma}_I^2$$

since $abc/(a+b+c) = 2Rr$.

Corollary 6. *The areal co-ordinates of I with respect to the medial triangle are $(s-a, s-b, s-c)$, perhaps better written $(\frac{s}{2}-\frac{a}{2}, \frac{s}{2}-\frac{b}{2}, \frac{s}{2}-\frac{c}{2})$. Therefore the incenter of the reference triangle is the Nagel point of the medial triangle.*

4. The Euler-Guinand problem

In 1765 Euler [3] recovered the sides lengths a, b and c of a non-equilateral triangle from the positions of O, G and I. At the time he did not have access to Feuerbach's formula for IN^2 nor our formula (5). This extra data enables us to make light of Euler's calculations. From (5) we have $r(2R-r)$ and combining with (2) we obtain first R/r and then both R and r. Now (3) yields $a^2+b^2+c^2$ and (4) gives $a+b+c$. Finally (1) yields abc. Thus the polynomial $\triangle(x) = (X-a)(X-b)(X-c)$ can be easily recovered from the positions of O, G and I. We call this the triangle polynomial This may be an irreducible rational cubic so the construction of a, b and c by ruler and compasses may not be possible.

The actual locations of A, B and C may be determined as follows. Note that this addresses the critical remark (3) of [5]. The circumcircle of $\triangle ABC$ is known since O and R are known. Now by the GPAT we obtain the well known formula

$$\frac{0^2+b^2+c^2}{3} = AG^2 + \frac{a^2+b^2+c^2}{9}$$

so

$$AG^2 = \frac{2b^2+2c^2-a^2}{9}$$

and similarly of BG^2 and CG^2. By intersecting circles of appropriate radii centered at G with the circumcircle, we recover at most two candidate locations for each point A, B and C. Now triangle ABC is one of at most $2^3 = 8$ triangles. These can be inspected to see which ones have correct O, G and I. Note that there is only one correct triangle since AG^2, AO^2 and AI^2 are all determined.

In fact every point in the interior of the circle on diameter GH other than the nine-point center N arises as a possible location of an incenter I [4]. We give a new derivation of this result addressing the same question as [4] and [5] but in a different way.

Given any value $k \in (0,1)$ there is a triangle such that $2r/R = k$. Choosing such a triangle, with circumradius R we observe that

$$\left(\frac{IO}{IN}\right)^2 = \frac{R^2 - 2Rr}{\left(\frac{R}{2} - r\right)^2}$$

so

$$\frac{IO}{IN} = 2\sqrt{\frac{R}{R-2r}}. \tag{6}$$

If O and N were fixed, this would force I to lie on a circle of Apollonius with defining ratio $2\sqrt{\frac{R}{R-2r}}$. In what follows we rescale our diagrams (when convenient) so that the distance ON is fixed, so the circle on diameter GH (the orthocentroidal or critical [4] circle) can be deemed to be of fixed diameter.

Consider the configuration of Poncelet's porism for triangle ABC. We draw the circumcircle with radius R and center O, and the incenter I internally tangent to triangle ABC at three points. Now move the point A to A' elsewhere on the circumcircle and generate a new triangle $A'B'C'$ with the same incircle. We move A to A' continuously and monotonically, and observe how the configuration changes; the quantities R and r do not change but in the scaled diagram the corresponding point I' moves continuously on the given circle of Apollonius. When A' reaches B the initial configuration is recovered. Consideration of the largest angle in the moving triangle $A'B'C'$ shows that until the initial configuration is regained, the triangles formed are pairwise dissimilar, so inside the scaled version of the circle on diameter GH, the point I' moves continuously on the circle of Apollonius in a monotonic fashion. Therefore I' makes exactly one rotation round the circle of Apollonius and A' moves to B. Thus all points on this circle of Apollonius arise as possible incenters, and since the defining constant of the circle is arbitrary, all points (other than N) in the interior of the scaled circle on diameter GH arise as possible locations for I and Guinand's result is obtained [4].

Letting the equilateral triangle correspond to N, the open disk becomes a moduli space for direct similarity types of triangle. The boundary makes sense if we allow triangles to have two sides parallel with included angle 0. Some caution should be exercised however. The angles of a triangle are not a continuous function of the side lengths when one of the side lengths approaches 0. Fix A and let B tend to C by spiraling in towards it. The point I in the moduli space will move enthusiastically round and round the disk, ever closer to the boundary.

Isosceles triangles live in the moduli space as the points on the distinguished (Euler line) diameter. If the unequal side is short, I is near H, but if it is long, I is near G.

References

[1] C. J. Bradley *Challenges in Geometry*, Oxford University Press, 2005.

[2] H. S. M. Coxeter and S. L. Greitzer *Geometry Revisited*, Math. Assoc. America, 1967.

[3] L. Euler, Solutio facili problematum quorundam geometricorum difficillimorum, *Novi Comm. Acad. Scie. Petropolitanae* 11 (1765); reprinted in *Opera omnia*a, serie prima, Vol. 26 (ed. by A. Speiser), (n.325) 139–157.

[4] A. P. Guinand, Tritangent centers and their triangles *Amer. Math. Monthly*, 91 (1984) 290-300.

[5] B. Scimemi Paper-folding and Euler's theorem revisited *Forum Geom.*, Vol 2 (2002) 93–104.

Geoff C. Smith: Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, England.

E-mail address: `G.C.Smith@bath.ac.uk`

Forum Geometricorum
Volume 5 (2005) 191–195.

FORUM GEOM
ISSN 1534-1178

A Gergonne Analogue of the Steiner - Lehmus Theorem

K. R. S. Sastry

Abstract. In this paper we prove an analogue of the famous Steiner - Lehmus theorem from the Gergonne cevian perspective.

1. Introduction

Can a theorem be both famous and infamous simultaneously? Certainly there is one such in Euclidean Geometry if the former is an indicator of a record number of correct proofs and the latter an indicator of a record number of incorrect ones. Most school students must have found it easy to prove the following: The angle bisectors of equal angles of a triangle are equal. However, not many can prove its converse theorem correctly:

Theorem 1 (Steiner-Lehmus)**.** *If two internal angle bisectors of a triangle are equal, then the triangle is isosceles.*

According to available history, in 1840 a Berlin professor named C. L. Lehmus (1780-1863) asked his contemporary Swiss geometer Jacob Steiner for a proof of Theorem 1. Steiner himself found a proof but published it in 1844. Lehmus proved it independently in 1850. The year 1842 found the first proof in print by a French mathematician [3]. Since then mathematicians and amateurs alike have been proving and re-proving the theorem. More than 80 correct proofs of the Steiner - Lehmus theorem are known. Even larger number of incorrect proofs have been offered. References [4, 5] provide extensive bibliographies on the Steiner - Lehmus theorem.

For completeness, we include a proof by M. Descube in 1880 below, recorded in [1, p.235]. The aim of this paper is to prove an analogous theorem in which we consider the equality of two Gergonne cevians. We offer two proofs of it and then consider an extension. Recall that a Gergonne cevian of a triangle is the line segment connecting a vertex to the point of contact of the opposite side with the incircle.

2. Proof of the Steiner - Lehmus theorem

Figure 1 shows the bisectors BE and CF of $\angle ABC$ and $\angle ACB$. We assume $BE = CF$. If $AB \neq AC$, let $AB < AC$, *i.e.*, $\angle ACB < \angle ABC$ or $\frac{C}{2} < \frac{B}{2}$. A

Publication Date: December 20, 2005. Communicating Editor: Paul Yiu.

comparison of triangles BEC with BFC shows that

$$CE > BF. \tag{1}$$

Complete the parallelogram $BFGE$. Since $EG = BF$, $\angle FGE = \frac{B}{2}$, $FG = BE = CF$ implying that $\angle FGC = \angle FCG$. But by assumption $\angle FGE = \frac{B}{2} > \angle FCE = \frac{C}{2}$. So $\angle EGC < \angle ECG$, and $CE < GE = BF$, contradicting (1).

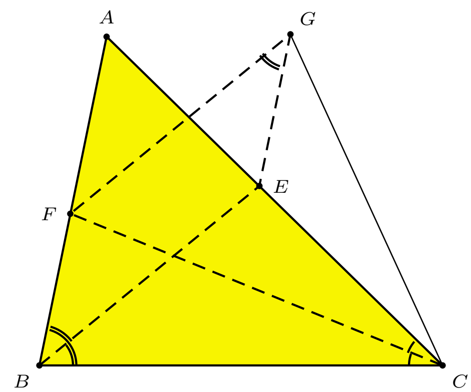


Figure 1.

Likewise, the assumption $AB > AC$ also leads to a contradiction. Hence, $AB = AC$ and $\triangle ABC$ must be isosceles.

3. The Gergonne analogue

We provide two proofs of Theorem 2 below. The first proof equates the expressions for the two Gergonne cevians to establish the result. The second one is modelled on the proof of the Steiner - Lehmus theorem in §2 above.

Theorem 2. *If two Gergonne cevians of a triangle are equal, then the triangle is isosceles.*

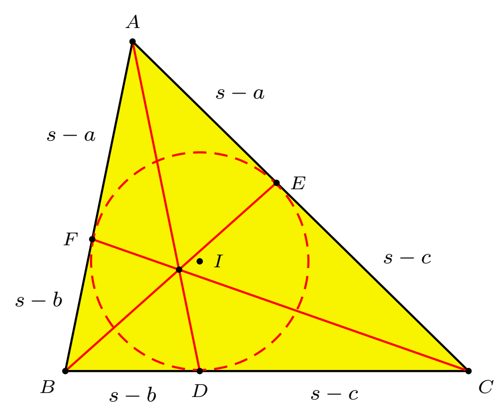


Figure 2.

3.1. *First proof.* Figure 2 shows the equal Gergonne cevians BE, CF of triangle ABC. We consider $\triangle ABE$, $\triangle ACF$ and apply the law of cosines:

$$\begin{aligned} BE^2 &= c^2 + (s-a)^2 - 2c(s-a)\cos A, \\ CF^2 &= b^2 + (s-a)^2 - 2b(s-a)\cos A. \end{aligned}$$

Equating the expressions for BE^2 and CF^2 we see that

$$2(b-c)(s-a)\cos A - (b^2 - c^2) = 0$$

or

$$(b-c)\left[\frac{(-a+b+c)(b^2+c^2-a^2)}{2bc} - (b+c)\right] = 0.$$

There are two cases to consider.

(i) $b - c = 0 \Rightarrow b = c$ and triangle ABC is isosceles.

(ii) $\frac{(-a+b+c)(b^2+c^2-a^2)}{2bc} - (b+c) = 0$. This can be put, after simplification, in the form

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) = 0.$$

This clearly is an impossibility by the triangle inequality.

Therefore (i) must hold and triangle ABC is isosceles.

3.2. *Second proof.* We employ the same construction as in Figure 1 for Theorem 1. Hence we do not repeat the description here for Figure 3.

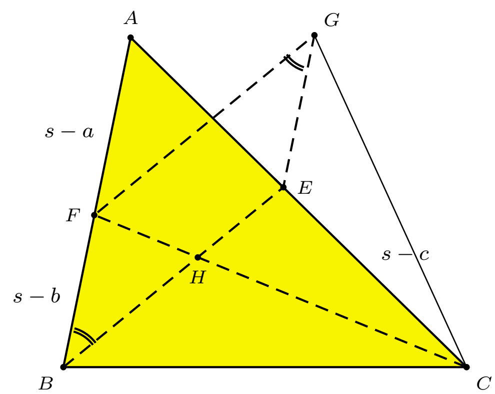


Figure 3.

If $AB \neq AC$, let $AB < AC$, *i.e.*, $c < b$, and $s - c > s - b$. As seen in the proof of Theorem 1, $\angle EBC > \angle FCB \Rightarrow CH > BH$. Since $CF = BE$, we have

$$FH < EH. \tag{2}$$

In triangles ABE and AFC, $AE = AF = s - a$, $BE = CF$ and by assumption $AB < AC$. Hence $\angle AEB < \angle AFC \Rightarrow \angle BEC > \angle BFC$ or

$$\angle HEC > \angle HFB. \tag{3}$$

Therefore, in triangles BFH and EHC, $\angle BHF = \angle EHC$ and from (3) we see that

$$\angle FBH > \angle HCE. \tag{4}$$

Triangle FGC is isosceles by construction, so $\angle FGC = \angle FCG$ or $\angle FGE + \angle EGC = \angle HCE + \angle ECG$. Because of (4) we see that $\angle EGC < \angle ECG$ or $EC < EG$, *i.e.*, $s - c < s - b \Rightarrow b < c$, contradicting the assumption.

Likewise the assumption $b > c$ would lead to a similar contradiction. Hence we must have $b = c$, and triangle ABC is isosceles.

4. An extension

Theorem 3 shows that the equality of the segments of two angle bisectors of a triangle intercepted by a Gergonne cevian itself implies that the triangle is isosceles.

Theorem 3. *The internal angle bisectors of the angles ABC and ACB of triangle ABC meet the Gergonne cevian AD at E and F respectively. If $BE = CF$, then triangle ABC is isosceles.*

Proof. We refer to Figure 4. If $AB \neq AC$, let $AB < AC$. Hence $b > c$, $s - b < s - c$ and E lies below F on AD. A simple calculation with the help of the angle bisector theorem shows that the Gergonne cevian AD lies to the left of the cevian that bisects $\angle BAC$ and hence that $\angle ADC$ is obtuse.

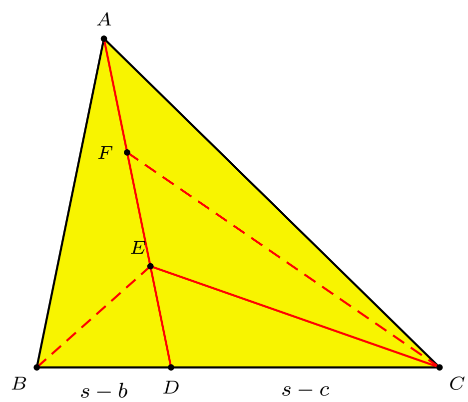


Figure 4.

By assumption, $\angle ABC > \angle ACB \Rightarrow \angle EBC > \angle FCD > \angle ECB$. Therefore,

$$CE > BE \quad \text{or} \quad CE > CF \tag{5}$$

because $BE = CF$. However, $\angle ADC = \angle EDC > \frac{\pi}{2}$ as mentioned above. Hence $\angle FEC = \angle EDC + \angle ECD > \frac{\pi}{2}$ and $\angle EFC < \frac{\pi}{2} \Rightarrow CE < CF$, contradicting (5).

Likewise, the assumption $AB > AC$ also leads to a contradiction. This means that triangle ABC must be isosceles. □

5. Conclusion

The reader is invited to consider other types of analogues or extensions of the Steiner - Lehmus theorem. To conclude the discussion, we pose two problems to the reader.

(1) The external angle bisectors of $\angle ABC$ and $\angle ACB$ meet the extension of the Gergonne cevian AD at the points E and F respectively. If $BE = CF$, prove or disprove that triangle ABC is isosceles.

(2) AD is an internal cevian of triangle ABC. The internal angle bisectors of $\angle ABC$ and $\angle ACB$ meet AD at E and F respectively. Determine a necesssary and sufficient condition so that $BE = CF$ implies that triangle ABC is isosceles.

References

[1] F. G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, Paris, 1991.
[2] D. C. Kay, Nearly the last comment on the Steiner – Lehmus theorem, *Crux Math.*, 3 (1977) 148–149.
[3] M. Lewin, On the Steiner - Lehmus theorem, *Math. Mag.*, 47 (1974) 87–89.
[4] L. Sauvé, The Steiner - Lehmus theorem, *Crux Math.*, 2 (1976) 19–24.
[5] C. W. Trigg, A bibliography of the Steiner - Lehmus theorem, *Crux Math.*, 2 (1976) 191–193.

K. R. S. Sastry: Jeevan Sandhya, DoddaKalsandra Post, Raghuvana Halli, Bangalore, 560 062, India.

Forum Geometricorum
Volume 5 (2005) 197–198.

FORUM GEOM
ISSN 1534-1178

Author Index

FORUM GEOMETRICORUM

A Journal on Classical Euclidean Geometry and Related Areas

published by

Department of Mathematical Sciences
Florida Atlantic University

FORUM GEOM

Volume 6
2006

http://forumgeom.fau.edu

ISSN 1534-1178

Editorial Board

Advisors:

John H. Conway	Princeton, New Jersey, USA
Julio Gonzalez Cabillon	Montevideo, Uruguay
Richard Guy	Calgary, Alberta, Canada
Clark Kimberling	Evansville, Indiana, USA
Kee Yuen Lam	Vancouver, British Columbia, Canada
Tsit Yuen Lam	Berkeley, California, USA
Fred Richman	Boca Raton, Florida, USA

Editor-in-chief:

Paul Yiu	Boca Raton, Florida, USA

Editors:

Clayton Dodge	Orono, Maine, USA
Roland Eddy	St. John’s, Newfoundland, Canada
Jean-Pierre Ehrmann	Paris, France
Chris Fisher	Regina, Saskatchewan, Canada
Rudolf Fritsch	Munich, Germany
Bernard Gibert	St Etiene, France
Antreas P. Hatzipolakis	Athens, Greece
Michael Lambrou	Crete, Greece
Floor van Lamoen	Goes, Netherlands
Fred Pui Fai Leung	Singapore, Singapore
Daniel B. Shapiro	Columbus, Ohio, USA
Steve Sigur	Atlanta, Georgia, USA
Man Keung Siu	Hong Kong, China
Peter Woo	La Mirada, California, USA

Technical Editors:

Yuandan Lin	Boca Raton, Florida, USA
Aaron Meyerowitz	Boca Raton, Florida, USA
Xiao-Dong Zhang	Boca Raton, Florida, USA

Consultants:

Frederick Hoffman	Boca Raton, Floirda, USA
Stephen Locke	Boca Raton, Florida, USA
Heinrich Niederhausen	Boca Raton, Florida, USA

Table of Contents

Forum Geometricorum
Volume 6 (2006) 1–16.

FORUM GEOM
ISSN 1534-1178

On Mixtilinear Incircles and Excircles

Khoa Lu Nguyen and Juan Carlos Salazar

Abstract. A mixtilinear incircle (respectively excircle) of a triangle is tangent to two sides and to the circumcircle internally (respectively externally). We study the configuration of the three mixtilinear incircles (respectively excircles). In particular, we give easy constructions of the circle (apart from the circumcircle) tangent the three mixtilinear incircles (respectively excircles). We also obtain a number of interesting triangle centers on the line joining the circumcenter and the incenter of a triangle.

1. Preliminaries

In this paper we study two triads of circles associated with a triangle, the mixtilinear incircles and the mixtilinear excircles. For an introduction to these circles, see [4] and §§2, 3 below. In this section we collect some important basic results used in this paper.

Proposition 1 (d'Alembert's Theorem [1])**.** *Let* $O_1(r_1)$*,* $O_2(r_2)$*,* $O_3(r_3)$ *be three circles with distinct centers. According as* $\varepsilon = +1$ *or* -1*, denote by* $A_{1\varepsilon}$*,* $A_{2\varepsilon}$*,* $A_{3\varepsilon}$ *respectively the insimilicenters or exsimilicenters of the pairs of circles* $((O_2),\ (O_3))$*,* $((O_3),\ (O_1))$*, and* $((O_1),\ (O_2))$*. For* $\varepsilon_i = \pm 1$*,* $i = 1, 2, 3$*, the points* $A_{1\varepsilon_1}$*,* $A_{2\varepsilon_2}$ *and* $A_{3\varepsilon_3}$ *are collinear if and only if* $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$. See Figure 1.

The insimilicenter and exsimilicenter of two circles are respectively their internal and external centers of similitude. In terms of one-dimensional barycentric coordinates, these are the points

$$\text{ins}(O_1(r_1),\ O_2(r_2)) = \frac{r_2 \cdot O_1 + r_1 \cdot O_2}{r_1 + r_2}, \tag{1}$$

$$\text{exs}(O_1(r_1),\ O_2(r_2)) = \frac{-r_2 \cdot O_1 + r_1 \cdot O_2}{r_1 - r_2}. \tag{2}$$

Proposition 2. *Let* $O_1(r_1)$*,* $O_2(r_2)$*,* $O_3(r_3)$ *be three circles with noncollinears centers. For* $\varepsilon = \pm 1$*, let* $O_\varepsilon(r_\varepsilon)$ *be the Apollonian circle tangent to the three circles, all externally or internally according as* $\varepsilon = +1$ *or* -1*. Then the Monge line containing the three exsimilicenters* $\text{exs}(O_2(r_2), O_3(r_3))$*,* $\text{exs}(O_3(r_3), O_1(r_1))$*, and* $\text{exs}(O_1(r_1), O_2(r_2))$ *is the radical axis of the Apollonian circles* (O_+) *and* (O_-)*.* See Figure 1.

Publication Date: January 18, 2006. Communicating Editor: Paul Yiu.
The authors thank Professor Yiu for his contribution to the last section of this paper.

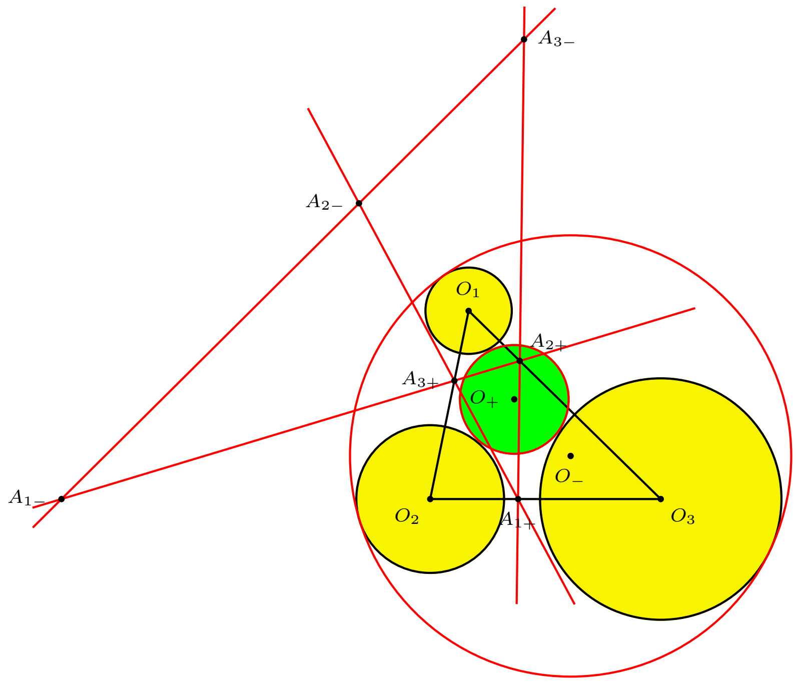


Figure 1.

Lemma 3. *Let BC be a chord of a circle $O(r)$. Let $O_1(r_1)$ be a circle that touches BC at E and intouches the circle (O) at D. The line DE passes through the midpoint A of the arc BC that does not contain the point D. Furthermore, $AD \cdot AE = AB^2 = AC^2$.*

Proposition 4. *The perspectrix of the circumcevian triangle of P is the polar of P with respect to the circumcircle.*

Let ABC be a triangle with circumcenter O and incenter I. For the circumcircle and the incircle,

$$\text{ins}((O),\ (I)) = \frac{r \cdot O + R \cdot I}{R + r} = X_{55},$$
$$\text{exs}((O),\ (I)) = \frac{-r \cdot O + R \cdot I}{R - r} = X_{56}.$$

in the notations of [3]. We also adopt the following notations.

A_0	point of tangency of incircle with BC
A_1	intersection of AI with the circumcircle
A_2	antipode of A_1 on the circumcircle

Similarly define B_0, B_1, B_2, C_0, C_1 and C_2. Note that
(i) $A_0B_0C_0$ is the intouch triangle of ABC,
(ii) $A_1B_1C_1$ is the circumcevian triangle of the incenter I,

(iii) $A_2B_2C_2$ is the medial triangle of the excentral triangle, *i.e.*, A_2 is the midpoint between the excenters I_b and I_c. It is also the midpoint of the arc BAC of the circumcircle.

2. Mixtilinear incircles

The A-mixtilinear incircle is the circle (O_a) that touches the rays AB and AC at C_a and B_a and the circumcircle (O) internally at X. See Figure 2. Define the B- and C-mixtilinear incircles (O_b) and (O_c) analogously, with points of tangency Y and Z with the circumcircle. See [4]. We begin with an alternative proof of the main result of [4].

Proposition 5. *The lines AX, BY, CZ are concurrent at* $\text{exs}((O),\ (I))$.

Proof. Since $A = \text{exs}((O_a),\ (I))$ and $X = \text{exs}((O),\ (O_a))$, the line AX passes through $\text{exs}((O),\ (I))$ by d'Alembert's Theorem. For the same reason, BY and CZ also pass through the same point. □

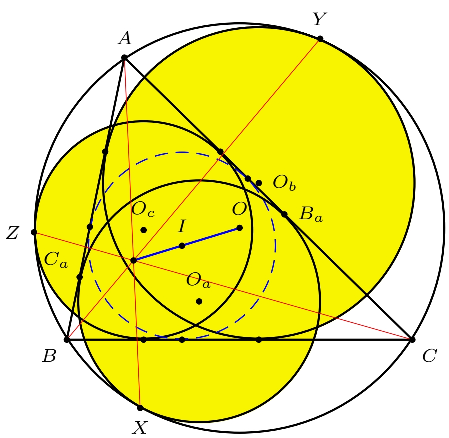


Figure 2

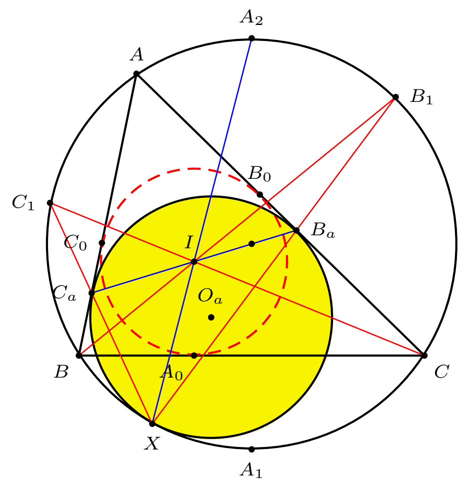


Figure 3

Lemma 6. (1) *I is the midpoint of B_aC_a.*
(2) *The A-mixtilinear incircle has radius* $r_a = \frac{r}{\cos^2 \frac{A}{2}}$.
(3) *XI bisects angle BXC.*
See Figure 3.

Consider the radical axis ℓ_a of the mixtilinear incircles (O_b) and (O_c).

Proposition 7. *The radical axis ℓ_a contains*
(1) *the midpoint A_1 of the arc BC of (O) not containing the vertex A,*
(2) *the midpoint M_a of IA_0, where A_0 is the point of tangency of the incircle with the side BC.*

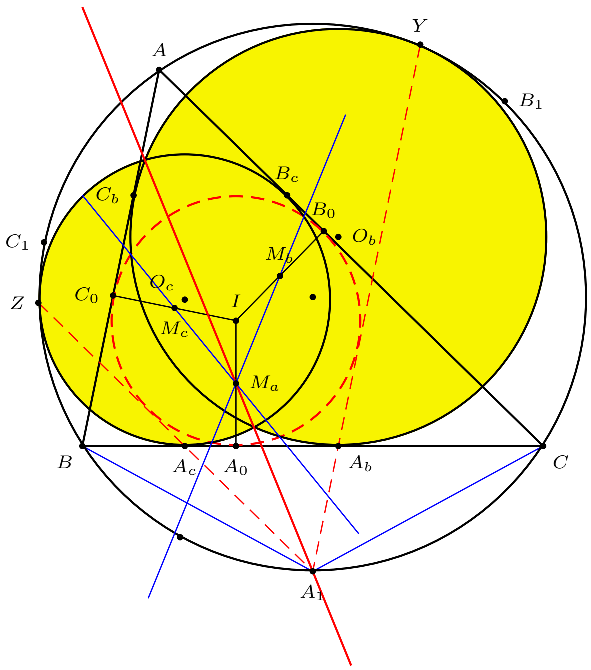


Figure 4.

Proof. (1) By Lemma 3, Z, A_c and A_1 are collinear, so are Y, A_b, A_1. Also, $A_1A_c \cdot A_1Z = A_1B^2 = A_1C^2 = A_1A_b \cdot A_1Y$. This shows that A_1 is on the radical axis of (O_b) and (O_c).

(2) Consider the incircle (I) and the B-mixtilinear incircle (O_b) with common ex-tangents BA and BC. Since the circle (I) touches BA and BC at C_0 and A_0, and the circle (O_b) touches the same two lines at C_b and A_b, the radical axis of these two circles is the line joining the midpoints of C_bC_0 and A_bA_0. Since A_b, I, C_b are collinear, the radical axis of (I) and (O_b) passes through the midpoints of IA_0 and IC_0. Similarly, the radical axis of (I) and (O_c) passes through the midpoints of IA_0 and IB_0. It follows that the midpoint of IA_0 is the common point of these two radical axes, and is therefore a point on the radical axis of (O_b) and (O_c). □

Theorem 8. *The radical center of (O_a), (O_b), (O_c) is the point J which divides OI in the ratio*

$$OJ : JI = 2R : -r.$$

Proof. By Proposition 7, the radical axis of (O_b) and (O_c) is the line A_1M_a. Let M_b and M_c be the midpoints of IB_0 and IC_0 respectively. Then the radical axes of (O_c) and (O_a) is the line B_1M_b, and that of (O_a) and (O_b) is the line C_1M_c. Note that the triangles $A_1B_1C_1$ and $M_aM_bM_c$ are directly homothetic. Since $A_1B_1C_1$ is inscribed in the circle $O(R)$ and $M_aM_bM_c$ in inscribed in the circle $I(\frac{r}{2})$, the homothetic center of the triangles is the point J which divides the segment OI in

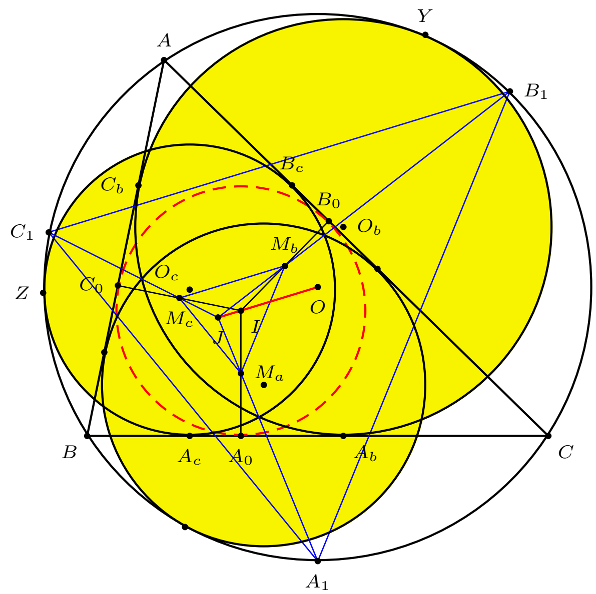


Figure 5.

the ratio

$$OJ : JI = R : -\frac{r}{2} = 2R : -r. \tag{3}$$

See Figure 5. □

Remark. Let T be the homothetic center of the excentral triangle $I_aI_bI_c$ and the intouch triangle $A_0B_0C_0$. This is the triangle center X_{57} in [3]. Since the excentral triangle has circumcenter I', the reflection of I in O,

$$OT : TI' = 2R : -r.$$

Comparison with (3) shows that J is the reflection of T in O.

3. The mixtilinear excircles

The mixtilinear excircles are defined analogously to the mixtilinear incircles, except that the tangencies with the circumcircle are external. The A-mixtilinear excircle (O'_a) can be easily constructed by noting that the polar of A passes through the excenter I_a; similarly for the other two mixtilinear excircles. See Figure 6.

Theorem 9. *If the mixtilinear excircles touch the circumcircle at X', Y', Z' respectively, the lines AX', BY', CZ' are concurrent at* $\mathrm{ins}((O), (I))$.

Theorem 10. *The radical center of the mixtilinear excircles is the reflection of J in O, where J is the radical center of the mixtilinear incircles.*

Proof. The polar of A with respect to (O'_a) passes through the excenter I_a. Similarly for the other two polars of B with respect to (O'_b) and C with respect to (O'_c).

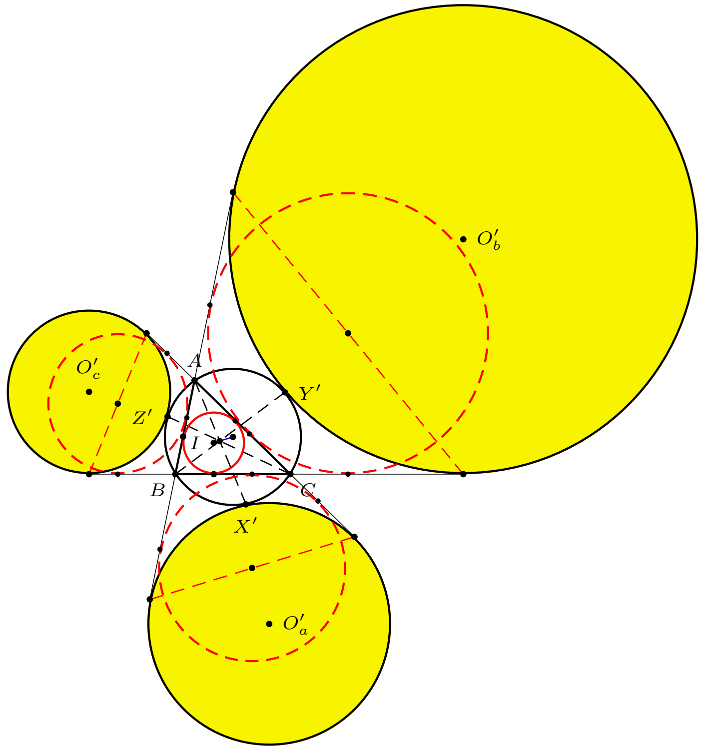


Figure 6.

Let $A_3B_3C_3$ be the triangle bounded by these three polars. Let A_4, B_4, C_4 be the midpoints of A_0A_3, B_0B_3, C_0C_3 respectively. See Figure 7.

Since $I_aI_bA_3I_c$ is a parallelogram, and A_2 is the midpoint of I_bI_c, it is also the midpoint of A_3I_a. Since B_3C_3 is parallel to I_bI_c (both being perpendicular to the bisector AA_1), I_a is the midpoint of B_3C_3. Similarly, I_b, I_c are the midpoints of C_3A_3 and A_3B_3, and the excentral triangle is the medial triangle of $A_3B_3C_3$. Note also that I is the circumcenter of $A_3B_3C_3$ (since it lies on the perpendicular bisectors of its three sides). This is homothetic to the intouch triangle $A_0B_0C_0$ at I, with ratio of homothety $-\frac{r}{4R}$.

If A_4 is the midpoint of A_0A_3, similarly for B_4 and C_4, then $A_4B_4C_4$ is homothetic to $A_3B_3C_3$ with ratio $\frac{4R-r}{4R}$.

We claim that $A_4B_4C_4$ is homothetic to $A_2B_2C_2$ at a point J', which is the radical center of the mixtilinear excircles. The ratio of homothety is clearly $\frac{4R-r}{R}$.

Consider the isosceles trapezoid $B_0C_0B_3C_3$. Since B_4 and C_4 are the midpoints of the diagonals B_0B_3 and C_0C_3, and B_3C_3 contains the points of tangency B_a, C_a of the circle (O'_a) with AC and AB, the line B_4C_4 also contains the midpoints of B_0B_a and C_0C_a, which are on the radical axis of (I) and (O'_a). This means that the line B_4C_4 is the radical axis of (I) and (O'_a).

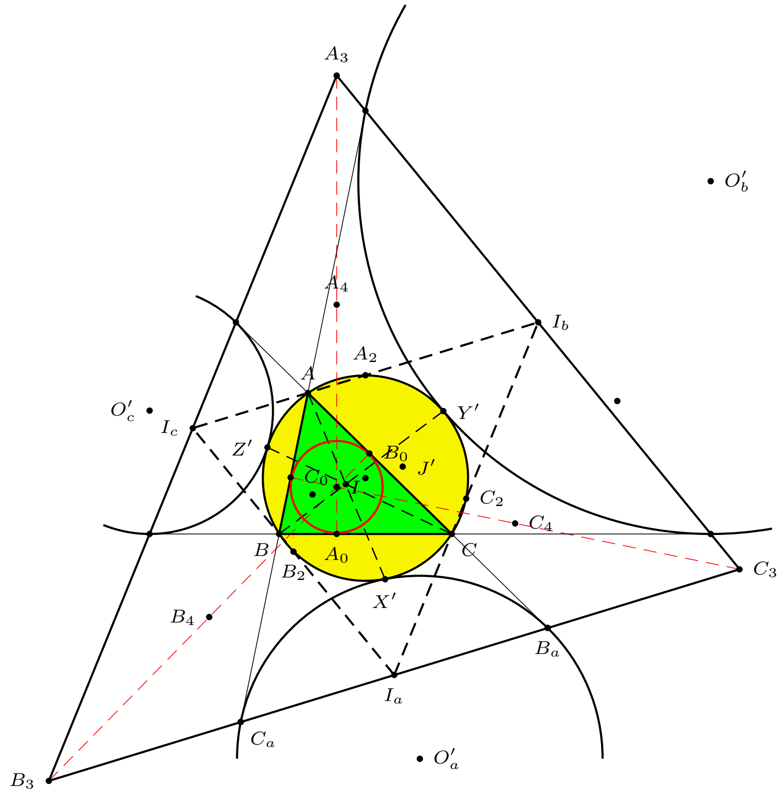


Figure 7.

It follows that A_4 is on the radical axis of (O'_b) and (O'_c). Clearly, A_2 also lies on the same radical axis. This means that the radical center of the mixtilinear excircles is the homothetic center of the triangles $A_2B_2C_2$ and $A_4B_4C_4$. Since these two triangles have circumcenters O and I, and circumradii R and $\frac{4R-r}{2}$, the homothetic center is the point J' which divides IO in the ratio

$$J'I : J'O = 4R - r : 2R. \tag{4}$$

Equivalently, $OJ' : J'I = -2R : 4R - r$. The reflection of J' in O divides OI in the ratio $2R : -r$. This is the radical center J of the mixtilinear incircles. $\square$

4. Apollonian circles

Consider the circle $O_5(r_5)$ tangent internally to the mixtilinear incircles at A_5, B_5, C_5 respectively. We call this the inner Apollonian circle of the mixtilinear incircles. It can be constructed from J since A_5 is the second intersection of the line JX with the A-mixtilinear incircle, and similarly for B_5 and C_5. See Figure 8. Theorem 11 below gives further details of this circle, and an easier construction.

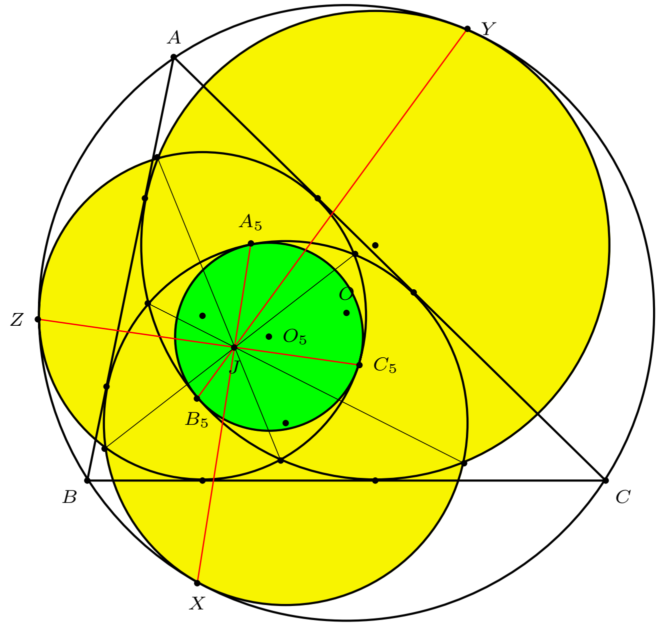


Figure 8.

Theorem 11. (1) *Triangles* $A_5B_5C_5$ *and* ABC *are perspective at* $\text{ins}((O),\ (I))$.

(2) *The inner Apollonian circle of the mixtilinear incircles has center* O_5 *dividing the segment* OI *in the ratio* $4R : r$ *and radius* $r_5 = \frac{3Rr}{4R+r}$.

Proof. (1) Let $P = \text{exs}((O_5), (I))$, and $Q_a = \text{exs}((O_5), (O'_a))$. The following triples of points are collinear by d'Alembert's Theorem:
(1) A, A_5, P from the circles (O_5), (I), (O_a);
(2) A, Q_a, A_5 from the circles (O_5), (O_a), (O'_a);
(3) A, Q_a, P from the circles (O_5), (I), (O'_a);
(4) A, X', $\text{ins}((O), (I))$ from the circles (O), (I), (O'_a).

See Figure 9. Therefore the lines AA_5 contains the points P and $\text{ins}((O), (I))$ (along with Q_a, X'). For the same reason, the lines BB_5 and CC_5 contain the same two points. It follows that P and $\text{ins}((O), (I))$ are the same point, which is common to AA_5, BB_5 and CC_5.

(2) Now we compute the radius r_5 of the circle (O_5). From Theorem 8, $OJ : JI = 2R : -r$. As $J = \text{exs}((O), (O_5))$, we have $OJ : JO_5 = R : -r_5$. It follows that $OJ : JI : JO_5 = 2R : -r : -2r_5$, and

$$\frac{OO_5}{OI} = \frac{2(R - r_5)}{2R - r}. \tag{5}$$

Since $P = \text{exs}((O_5), (I)) = \text{ins}((O), (I))$, it is also $\text{ins}((O), (O_5))$. Thus, $OP : PO_5 : PI = R : r_5 : r$, and

$$\frac{OO_5}{OI} = \frac{R + r_5}{R + r}. \tag{6}$$

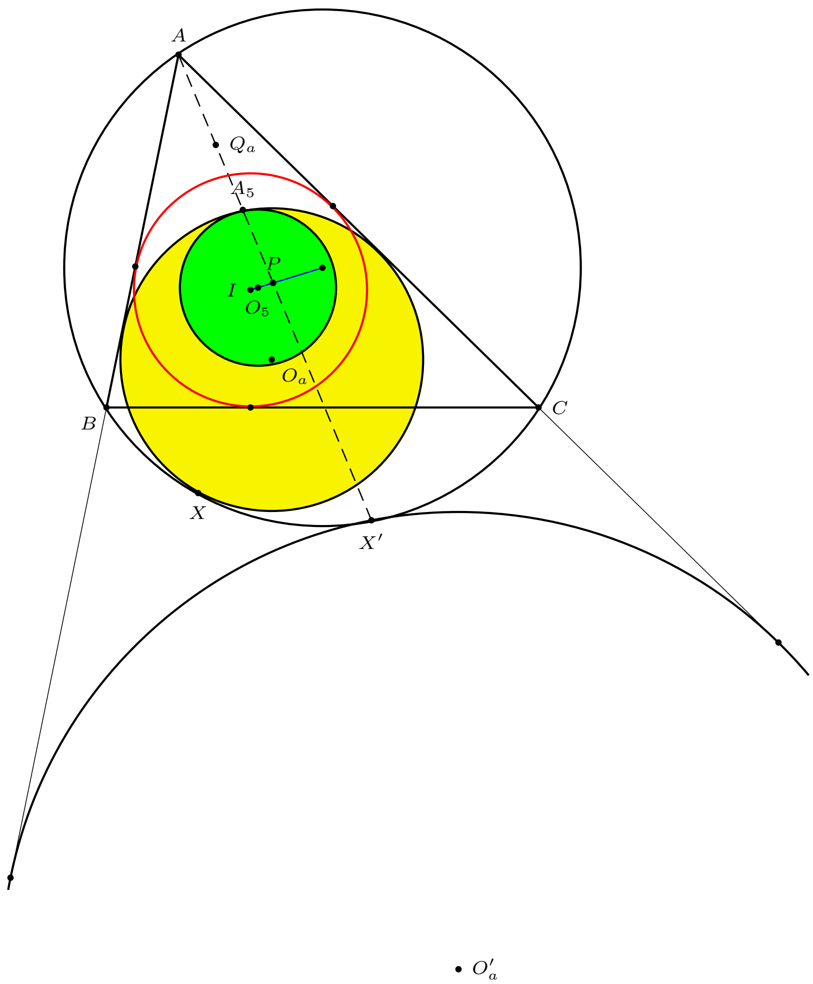


Figure 9.

Comparing (5) and (6), we easily obtain $r_5 = \frac{3Rr}{4R+r}$. Consequently, $\frac{OO_5}{OI} = \frac{4R}{4R+r}$ and O_5 divides OI in the ratio $OO_5 : O_5I = 4R : r$. $\square$

The outer Apollonian circle of the mixtilinear excircles can also be constructed easily. If the lines $J'X'$, $J'Y'$, $J'Z'$ intersect the mixtilinear excircles again at A_6, B_6, C_6 respectively, then the circle $A_6B_6C_6$ is tangent internally to each of the mixtilinear excircles. Theorem 12 below gives an easier construction without locating the radical center.

Theorem 12. (1) *Triangles $A_6B_6C_6$ and ABC are perspective at* $\mathrm{exs}((O),(I))$.

(2) *The outer Apollonian circle of the mixtilinear excircles has center O_6 dividing the segment OI in the ratio $-4R : 4R + r$ and radius $r_6 = \frac{R(4R-3r)}{r}$.*

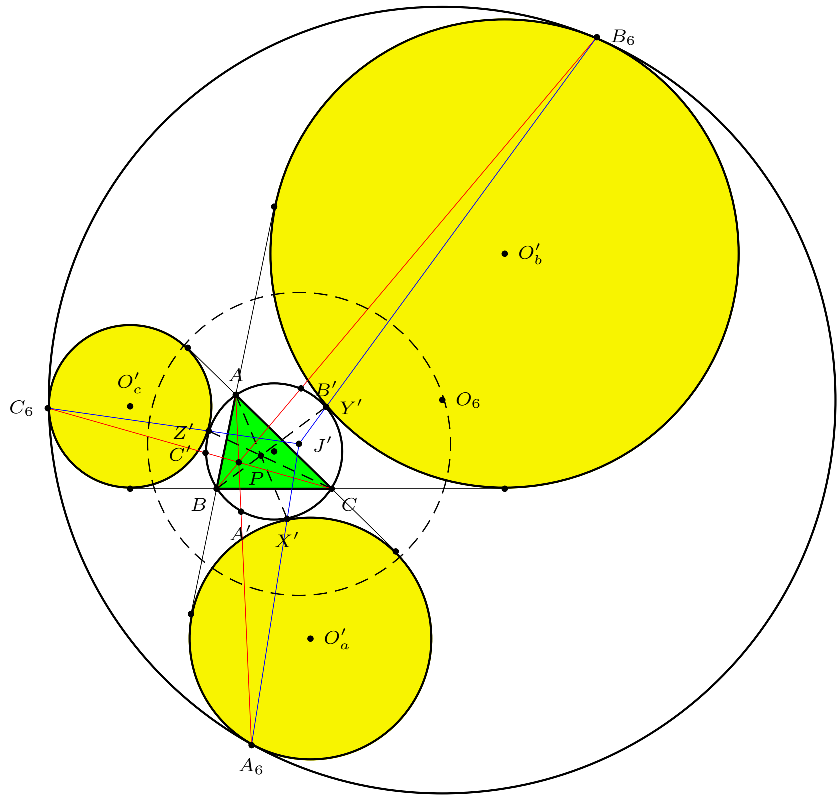


Figure 10.

Proof. (1) Since $A_6 = \text{exs}((O'_a), (O_6))$ and $A = \text{exs}((I), (O'_a))$, by d'Alembert's Theorem, the line AA_6 passes through $P = \text{exs}((O_6), (I))$. For the same reason BB_6 and CC_6 pass through the same point, the triangles $A_6B_6C_6$ and ABC are perspective at $P = \text{exs}((I), (O_6))$. See Figure 10.

By Proposition 2, AB, $X'Y'$, A_6B_6, and $O'_aO'_b$ concur at $S = \text{exs}((O'_a), (O'_b))$ on the radical axis of (O) and (O_6). Now: $SA \cdot SB = SX' \cdot SY' = SA_6 \cdot SB_6$. Let AA_6, BB_6, CC_6 intersect the circumcircle (O) at A', B', C' respectively. Since $SA \cdot SB = SA_6 \cdot SB_6$, ABA_6B_6 is cyclic. Since $\angle BAA_6 = \angle BB'A' = \angle BB_6A_6$, $A'B'$ is parallel to A_6B_6. Similarly, $B'C'$ and $C'A'$ are parallel to B_6C_6 and C_6A_6 respectively. Therefore the triangles $A'B'C'$ and $A_6B_6C_6$ are directly homothetic, and the center of homothety is $P = \text{exs}((O), (O_6))$.

Since $P = \text{exs}((O), (O_6)) = \text{exs}((I), (O_6))$, it is also $\text{exs}((O), (I))$, and $PI : PO = r : R$.

(2) Since $A_6 = \text{exs}((O_6), (O'_a))$ and $X' = \text{ins}((O'_a), (O))$, by d'Alembert's Theorem, the line A_6X' passes through $K = \text{ins}((O), (O_6))$. For the same reason, B_6Y' and C_6Z' pass through the same point K.

We claim that K is the radical center J' of the mixtilinear excircles. Since $SX' \cdot SY' = SA_6 \cdot SB_6$, we conclude that $X'A_6Y'B_6$ is cyclic, and $KX' \cdot KA_6 = KY' \cdot KB_6$. Also, $Y'B_6C_6Z'$ is cyclic, and $KY' \cdot KB_6 = KZ' \cdot KC_6$. It follows that

$$KX' \cdot KA_6 = KY' \cdot KB_6 = KZ' \cdot KC_6,$$

showing that $K = \text{ins}((O), (O_6))$ is the radical center J' of the mixtilinear excircles. Hence, $J'O : J'O_6 = R : -r_6$. Note also $PO : PO_6 = R : r_6$. Then, we have the following relations.

$$\begin{aligned} OJ' : IO =& 2R : 2R - r, \\ J'O_6 : IO =& 2r_6 : 2R - r, \\ OO_6 : IO =& r_6 - R : R - r. \end{aligned}$$

Since $OJ' + J'O_6 = OO_6$, we have

$$\frac{2R}{2R-r} + \frac{2r_6}{2R-r} = \frac{r_6 - R}{R-r}.$$

This gives: $r_6 = \frac{R(4R-3r)}{r}$. Since $K = \text{ins}((O), (O_6)) = \frac{r_6 \cdot O + R \cdot O_6}{R + r_6}$ and $J' = \frac{(4R-r)O - 2R \cdot I}{2R-r}$ are the same point, we obtain $O_6 = \frac{(4R+r)O - 4R \cdot I}{r}$. □

Remark. The radical circle of the mixtilinear excircles has center J' and radius $\frac{R}{2R-r}\sqrt{(4R+r)(4R-3r)}$.

Corollary 13. $IO_5 \cdot IO_6 = IO^2$.

5. The cyclocevian conjugate

Let P be a point in the plane of triangle ABC, with traces X, Y, Z on the sidelines BC, CA, AB respectively. Construct the circle through X, Y, Z. This circle intersects the sidelines BC, CA, AB again at points X', Y', Z'. A simple application of Ceva's Theorem shows that the AX', BY', CZ' are concurrent. The intersection point of these three lines is called the cyclocevian conjugate of P. See, for example, [2, p.226]. We denote this point by P°. Clearly, $(P^\circ)^\circ = P$. For example, the centroid and the orthocenters are cyclocevian conjugates, and Gergonne point is the cyclocevian conjugate of itself.

We prove two interesting locus theorems.

Theorem 14. *The locus of Q whose circumcevian triangle with respect to XYZ is perspective to $X'Y'Z'$ is the line PP°. For Q on PP°, the perspector is also on the same line.*

Proof. Let Q be a point on the line PP°. By Pascal's Theorem for the six points, X', B', X, Y', A', Y, the intersections of the lines $X'A'$ and $Y'B'$ lies on the line connecting Q to the intersection of XY' and $X'Y$, which according to Pappus' theorem (for Y, Z, Z' and C, Y, Y'), lies on PP°. Since Q lies on PP°, it follows that $X'A'$, $Y'B'$, and PP° are concurrent. Similarly, $A'B'C'$ and $X'Y'Z'$ are perspective at a point on PP°. The same reasoning shows that if $A'B'C'$ and $X'Y'Z'$ are perspective at a point S, then both Q and S lie on the line connecting the intersections $XY' \cap X'Y$ and $YZ' \cap Y'Z$, which is the line PP°. □

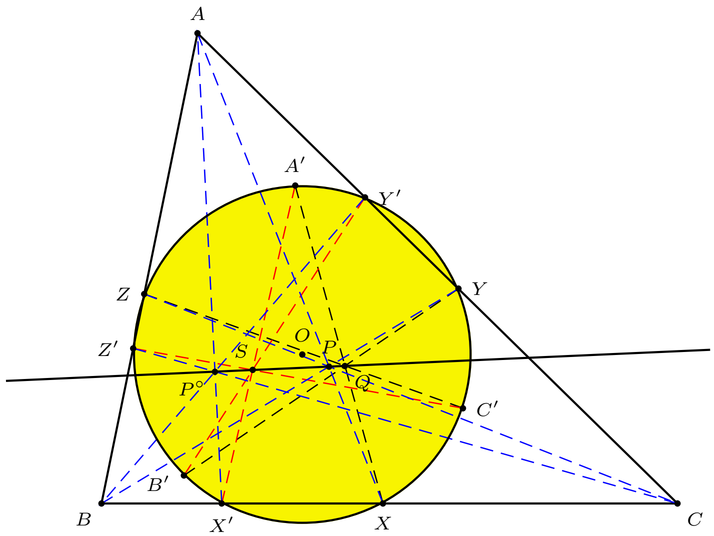


Figure 11.

For example, if $P = G$, then $P^{\circ} = H$. The line PP° is the Euler line. If $Q = O$, the circumcenter, then the circumcevian triangle of O (with respect to the medial triangle) is perspective with the orthic triangle at the nine-point center N.

Theorem 15. *The locus of Q whose circumcevian triangle with respect to XYZ is perspective to ABC is the line PP°.*

Proof. First note that

$$\begin{aligned}\frac{\sin Z'X'A'}{\sin A'X'Y'} =& \frac{\sin Z'XA'}{\sin A'YY'} = \frac{\sin Z'ZA'}{\sin ZAA'} \cdot \frac{\sin A'AY}{\sin A'YY'} \cdot \frac{\sin ZAA'}{\sin A'AY} \\ =& \frac{AA'}{ZA'} \cdot \frac{A'Y}{AA'} \cdot \frac{\sin BAX}{\sin XAC} = \frac{\sin YXA'}{\sin A'XZ} \cdot \frac{\sin BAA'}{\sin A'AC}.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{\sin Z'X'A'}{\sin A'X'Y'} \cdot \frac{\sin X'Y'B'}{\sin B'Y'Z'} \cdot \frac{\sin Y'Z'C'}{\sin C'Z'X'} \\ =& \left(\frac{\sin YXA'}{\sin A'XZ} \cdot \frac{\sin ZYB'}{\sin B'YX} \cdot \frac{\sin XZC'}{\sin C'ZY}\right)\left(\frac{\sin BAA'}{\sin A'AC} \cdot \frac{\sin ACC'}{\sin C'CB} \cdot \frac{\sin CBB'}{\sin B'BA}\right) \\ =& \frac{\sin BAA'}{\sin A'AC} \cdot \frac{\sin ACC'}{\sin C'CB} \cdot \frac{\sin CBB'}{\sin B'BA}.\end{aligned}$$

Therefore, $A'B'C'$ is perspective with ABC if and only if it is perspective with $X'Y'Z'$. By Theorem 14, the locus of Q is the line PP°. □

6. Some further results

We establish some further results on the mixtilinear incircles and excircles relating to points on the line OI.

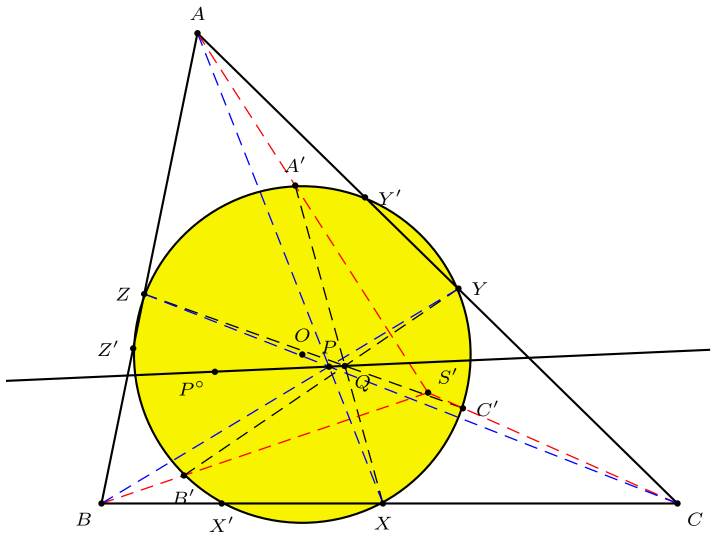


Figure 12.

Theorem 16. *The line OI is the locus of P whose circumcevian triangle with respect to $A_1B_1C_1$ is perspective with XYZ.*

Proof. We first show that $D = B_1Y \cap C_1Z$ lies on the line AA_1. Applying Pascal's Theorem to the six points Z, B_1, B_2, Y, C_1, C_2 on the circumcircle, the points $D = B_1Y \cap C_1Z$, $I = B_2Y \cap C_2Z$, and $B_1C_2 \cap B_2C_1$ are collinear. Since B_1C_2 and B_2C_1 are parallel to the bisector AA_1, it follows that D lies on AA_1. See Figure 13.

Now, if $E = C_1Z \cap A_1X$ and $F = A_1X \cap B_1Y$, the triangle DEF is perspective with $A_1B_1C_1$ at I. Equivalently, $A_1B_1C_1$ is the circumcevian triangle of I with respect to triangle DEF. Triangle XYZ is formed by the second intersections of the circumcircle of $A_1B_1C_1$ with the side lines of DEF. By Theorem 14, the locus of P whose circumcevian triangle with respect to $A_1B_1C_1$ is perspective with XYZ is a line through I. This is indeed the line IO, since O is one such point. (The circumcevian triangle of O with respect to $A_1B_1C_1$ is perspective with XYZ at I). □

Remark. If P divides OI in the ratio $OP : PI = t : 1-t$, then the perspector Q divides the same segment in the ratio $OQ : QI = (1+t)R : -2tr$. In particular, if $P = \text{ins}((O),\ (I))$, this perspector is T, the homothetic center of the excentral and intouch triangles.

Corollary 17. *The line OI is the locus of Q whose circumcevian triangle with respect to $A_1B_1C_1$ (or XYZ) is perspective with DEF.*

Proposition 18. *The triangle $A_2B_2C_2$ is perspective*
(1) *with XYZ at the incenter I,*
(2) *with $X'Y'Z'$ at the centroid of the excentral triangle.*

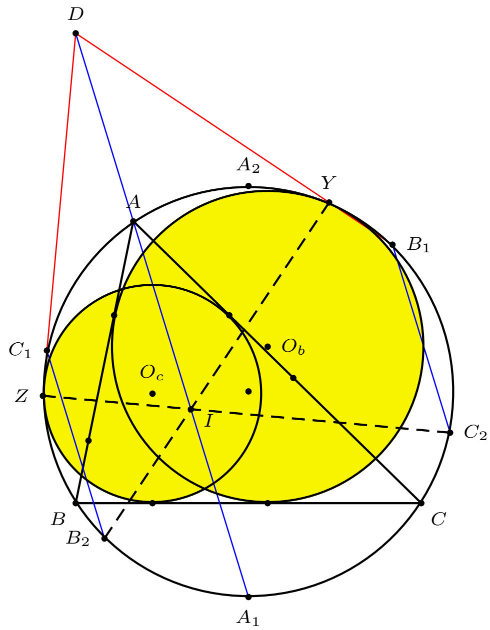


Figure 13.

Proof. (1) follows from Lemma 6(b).

(2) Referring to Figure 7, the excenter I_a is the midpoint B_aC_a. Therefore $X'I_a$ is a median of triangle $X'B_aC_a$, and it intersects B_2C_2 at its midpoint X''. Since $A_2B_2I_aC_2$ is a parallelogram, A_2, X', X'' and I_a are collinear. In other words, the line A_2X' contains a median, hence the centroid, of the excentral triangle. So do B_2Y' and C_2Z. □

Let A_7 be the second intersection of the circumcircle with the line ℓ_a, the radical axis of the mixtilinear incircles (O_b) and (O_c). Similarly define B_7 and C_7. See Figure 14.

Theorem 19. *The triangles $A_7B_7C_7$ and XYZ are perspective at a point on the line OI.*

Remark. This point divides OI in the ratio $4R - r : -4r$ and has homogeneous barycentric coordinates

$$\left(\frac{a(b+c-5a)}{b+c-a} : \frac{b(c+a-5b)}{c+a-b} : \frac{c(a+b-5c)}{a+b-c}\right).$$

7. Summary

We summarize the triangle centers on the OI-line associated with mixtilinear incircles and excircles by listing, for various values of t, the points which divide OI in the ratio $R : tr$. The last column gives the indexing of the triangle centers in [2, 3].

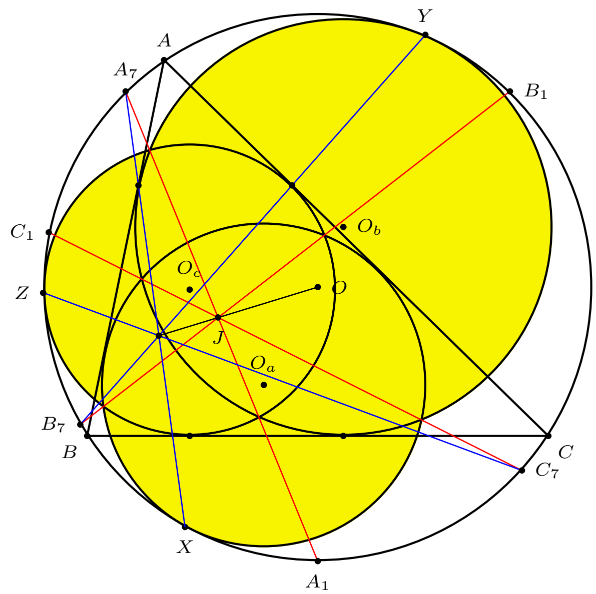


Figure 14.

t	first barycentric coordinate		X_n
1	$a^2(s-a)$	$\text{ins}((O),\ (I))$ perspector of ABC and $X'Y'Z'$ perspector of ABC and $A_5B_5C_5$	X_{55}
-1	$\frac{a^2}{s-a}$	$\text{exs}((O),\ (I))$ perspector of ABC and XYZ perspector of ABC and $A_6B_6C_6$	X_{56}
$-\frac{2R}{2R+r}$	$\frac{a}{s-a}$	homothetic center of excentral and intouch triangles	X_{57}
$-\frac{1}{2}$	$a^2(b^2+c^2-a^2-4bc)$	radical center of mixtilinear incircles	X_{999}
$\frac{1}{4}$	$a^2(b^2+c^2-a^2+8bc)$	center of Apollonian circle of mixtilinear incircles	
$-\frac{4R-r}{2r}$	$a^2f(a,b,c)$	radical center of mixtilinear excircles	
$-\frac{4R+r}{4r}$	$a^2g(a,b,c)$	center of Apollonian circle of mixtilinear excircles	
$-\frac{4R}{r}$	$a(3a^2-2a(b+c)-(b-c)^2)$	centroid of excentral triangle	X_{165}
$-\frac{4R}{4R-r}$	$\frac{a(b+c-5a)}{b+c-a}$	perspector of $A_7B_7C_7$ and XYZ	

The functions f and g are given by

$$\begin{aligned}
f(a,b,c) =& a^4 - 2a^3(b+c) + 10a^2bc + 2a(b+c)(b^2-4bc+c^2)\\
&- (b-c)^2(b^2+4bc+c^2),\\
g(a,b,c) =& a^5 - a^4(b+c) - 2a^3(b^2-bc+c^2) + 2a^2(b+c)(b^2-5bc+c^2)\\
&+ a(b^4 - 2b^3c + 18b^2c^2 - 2bc^3 + c^4) - (b-c)^2(b+c)(b^2-8bc+c^2).
\end{aligned}$$

References

[1] F. G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, Paris, 1991.
[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.
[3] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[4] P. Yiu, Mixtilinear incircles, *Amer. Math. Monthly*, 106 (1999) 952–955.
[5] P. Yiu, Mixtilinear incircles II, unpublished manuscript, 1998.

Khoa Lu Nguyen: 806 Candler Dr., Houston, Texas, 77037-4122, USA
E-mail address: `treegoner@yahoo.com`

Juan Carlos Salazar: Calle Maturín N$^{\rm o}$C 19, Urb. Mendoza, Puerto Ordaz 8015, Estado Bolívar, Venezuela
E-mail address: `caisersal@yahoo.com`

Forum Geometricorum
Volume 6 (2006) 17–23.

FORUM GEOM
ISSN 1534-1178

A Conic Associated with Euler Lines

Juan Rodríguez, Paula Manuel, and Paulo Semião

Abstract. We study the locus of a point C for which the Euler line of triangle ABC with given A and B has a given slope m. This is a conic through A and B, and with center (if it exists) at the midpoint of AB. The main properties of such an Euler conic are described. We also give a construction of a point C for which triangle ABC, with A and B fixed, has a prescribed Euler line.

1. The Euler conic

Given two points A and B and a real number m, we study the locus of a point C for which the Euler line of triangle ABC has slope m. We show that this locus is a conic through A and B. Without loss of generality, we assume a Cartesian coordinate system in which

$$A = (-1, 0) \quad \text{and} \quad B = (1, 0),$$

and write $C = (x, y)$. The centroid G and the orthocenter H of a triangle can be determined from the coordinates of its vertices. They are the points

$$G = \left(\frac{x}{3}, \frac{y}{3}\right) \quad \text{and} \quad H = \left(x, \frac{-x^2+1}{y}\right). \tag{1}$$

See, for example, [2]. The vector

$$\overrightarrow{GH} = \left(\frac{2x}{3}, \frac{-3x^2-y^2+3}{3y}\right), \tag{2}$$

is parallel to the Euler line. When the Euler line is not vertical, its slope is given by:

$$m = \frac{-3x^2-y^2+3}{2xy}, \quad x, y \neq 0.$$

Therefore, the coordinates of the vertex C satisfy the equation

$$3x^2 + 2mxy + y^2 = 3. \tag{3}$$

Publication Date: January 24, 2006. Communicating Editor: Paul Yiu.
The authors are extremely grateful to P. Yiu for his help in the preparation of this paper.

This clearly represents a conic. We call this the Euler conic associated with A, B and slope m. It clearly has center at the origin, the midpoint M of AB. Its axes are the eigenvectors of the matrix

$$\begin{pmatrix} 3 & m \\ m & 1 \end{pmatrix}.$$

The characteristic polynomial being $\lambda^2 - 4\lambda - (m^2 - 3)$, its eigenvectors with corresponding eigenvalues are as follows.

eigenvector	eigenvalue
$(\sqrt{m^2+1}+1,\ m)$	$2+\sqrt{m^2+1}$
$(\sqrt{m^2+1}-1,\ -m)$	$2-\sqrt{m^2+1}$

Thus, equation (3) can be rewritten in the form
(1) $(mx+y)^2 = 3$, if $m = \pm\sqrt{3}$, or
(2) $(2+\sqrt{m^2+1})(x\cos\alpha - y\sin\alpha)^2 + (2-\sqrt{m^2+1})(x\sin\alpha + y\cos\alpha)^2 = 3$, if $m \neq \pm\sqrt{3}$, where

$$\cos\alpha = \sqrt{\frac{\sqrt{m^2+1}+1}{2\sqrt{m^2+1}}}, \qquad \sin\alpha = \sqrt{\frac{\sqrt{m^2+1}-1}{2\sqrt{m^2+1}}}.$$

Remarks. (1) The pairs $(\pm 1, 0)$ are always solutions of (3) and correspond to the singular cases in which the vertex C coincides, respectively, with A or B, and consequently, it is not possible to define the triangle ABC.

(2) The pairs $\left(0, \pm\sqrt{3}\right)$ are also solutions of (3) and correspond to the trivial case when the triangle ABC is equilateral. In this case, the centroid, the orthocenter, and the circumcenter coincide.

2. Classification of the Euler conic

The Euler conic is an ellipse or a hyperbola according as $m^2 < 3$ or $m^2 > 3$. It degenerates into a pair of straight lines when $m^2 = 3$.

Proposition 1. *Suppose $m^2 < 3$. The Euler conic is an ellipse with eccentricity*

$$\varepsilon = \sqrt{\frac{2\sqrt{m^2+1}}{\sqrt{m^2+1}+2}}.$$

The foci are the points

$$\pm\left(-\operatorname{sgn}(m)\cdot\sqrt{\frac{3(\sqrt{m^2+1}-1)}{3-m^2}},\ \sqrt{\frac{3(\sqrt{m^2+1}+1)}{3-m^2}}\right),$$

where $\operatorname{sgn}(m) = +1, 0,$ *or* -1 *according as* $m >, =,$ *or* < 0.

Figure 1 shows the Euler ellipse for $m = \frac{3}{4}$, a triangle ABC with C on the ellipse, and its Euler line of slope m.

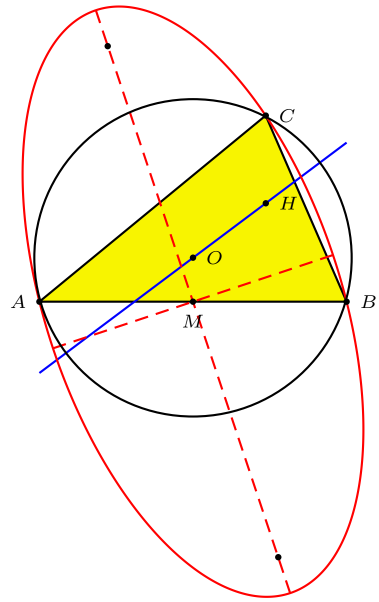


Figure 1

Proposition 2. *Suppose* $m^2 > 3$*. The Euler conic is a hyperbola with eccentricity*

$$\varepsilon = \sqrt{\frac{2\sqrt{m^2+1}}{\sqrt{m^2+1}-2}}.$$

The foci are the points

$$\pm\left(\operatorname{sgn}(m)\cdot\sqrt{\frac{3(\sqrt{m^2+1}+1)}{m^2-3}},\ \sqrt{\frac{3(\sqrt{m^2+1}-1)}{m^2-3}}\right),$$

where $\operatorname{sgn}(m) = +1$ *or* -1 *according as* $m > 0$ *or* < 0*. The asymptotes are the lines*

$$y = (-m \pm \sqrt{m^2-3})x.$$

Figure 2 shows the Euler hyperbola for $m = \frac{12}{5}$, a triangle ABC with C on the hyperbola, and its Euler line of slope m.

When $|m| = \sqrt{3}$, the Euler conic degenerates into a pair of parallel lines, whose equations are:

$$y = -mx \pm \sqrt{3}.$$

Examples of triangles for $m = \sqrt{3}$ are shown in Figures 3A and 3B, and for $m = -\sqrt{3}$ in Figures 4A and 4B.

Corollary 3. *The slope of the Euler line of the triangle* ABC *is*

$$m = \pm\sqrt{3},$$

if and only if, one of angles A *and* B *is* 60° *or* 120°.

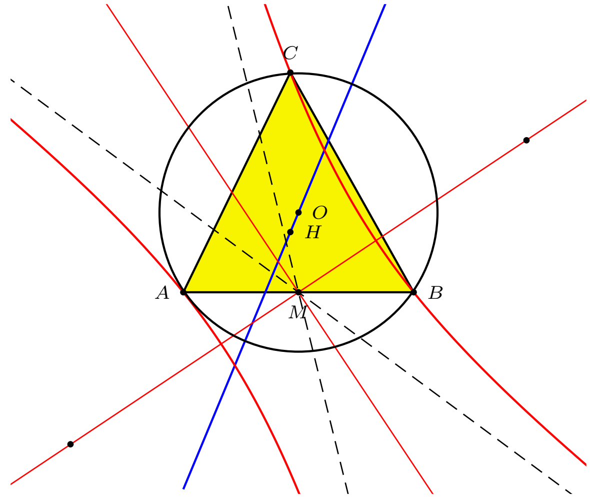


Figure 2

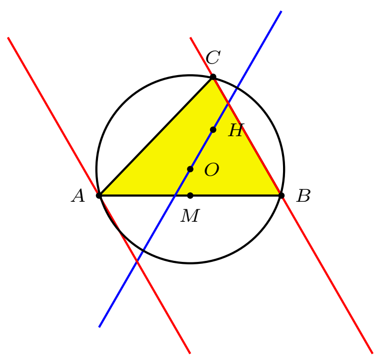


Figure 3A

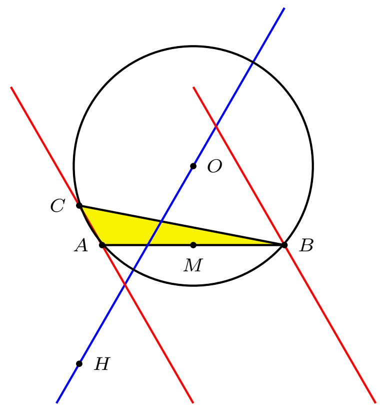


Figure 3B

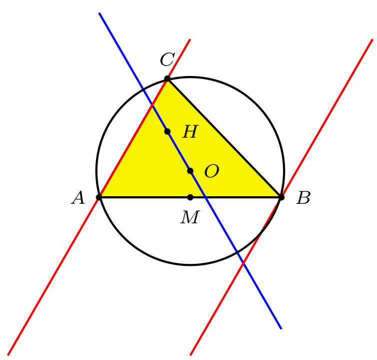


Figure 4A

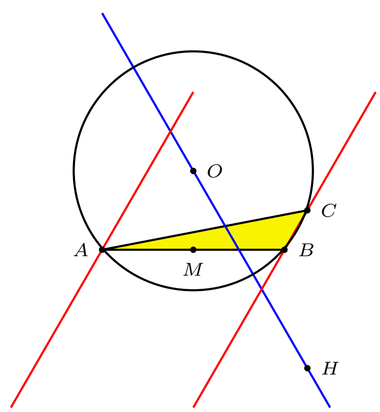


Figure 4B

3. Triangles with given Euler line

In this section we find the cartesian coordinates of the third vertex C in order that a given line be the Euler line of the triangle ABC with vertices $A = (-1, 0)$ and $B = (1, 0)$.

Lemma 4. *The Euler line of triangle ABC is perpendicular to AB if and only if $AB = AC$. In this case, the Euler line is the perpendicular bisector of AB.*

We shall henceforth assume that the Euler line is not perpendicular to AB. It therefore has an equation of the form

$$y = mx + k.$$

The circumcenter is the intersection of the Euler line with the line $x = 0$, the perpendicular bisector of AB. It is the point $O = (0, k)$. The circumcircle is

$$x^2 + (y - k)^2 = k^2 + 1$$

or

$$x^2 + y^2 - 2ky - 1 = 0. \tag{4}$$

Let M be the midpoint of AB; it is the origin of the Cartesian system. If G is the centroid, the vertex C is such that $MC : MG = 3 : 1$. Since G lies on the line $y = mx + k$, C lies on the line $y = mx + 3k$. It can therefore be constructed as the intersection of this line with the circle (4).

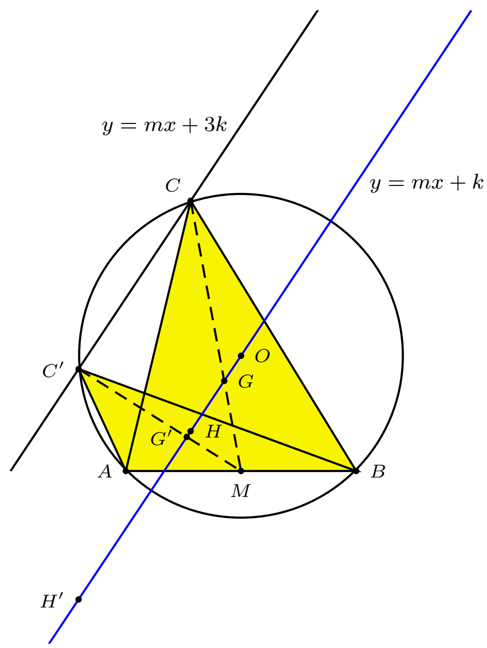


Figure 5

Proposition 5. *The number of points C for which triangle ABC has Euler line $y = mx + k$ is 0, 1, or 2 according as $(m^2 - 3)(k^2 + 1) <, =,$ or > -4.*

In the hyperbolic and degenerate cases $m^2 \geq 3$, there are always two such triangles. In the elliptic case, $m^2 < 3$. There are two such triangles if and only if $k^2 < \frac{m^2+1}{3-m^2}$.

Corollary 6. *For $m^2 < 3$ and $k = \pm\sqrt{\frac{m^2+1}{3-m^2}}$, there is a unique triangle ABC whose Euler line is the line $y = mx + k$. The lines $y = mx + 3k$ are tangent to the Euler ellipse* (3) *at the points*

$$\pm\left(\frac{-2m}{\sqrt{(m^2+1)(3-m^2)}}, \frac{3+m^2}{\sqrt{(m^2+1)(3-m^2)}}\right).$$

Figure 6 shows the configuration corresponding to $k = \sqrt{\frac{m^2+1}{3-m^2}}$. The other one can be obtained by reflection in M, the midpoint of AB.

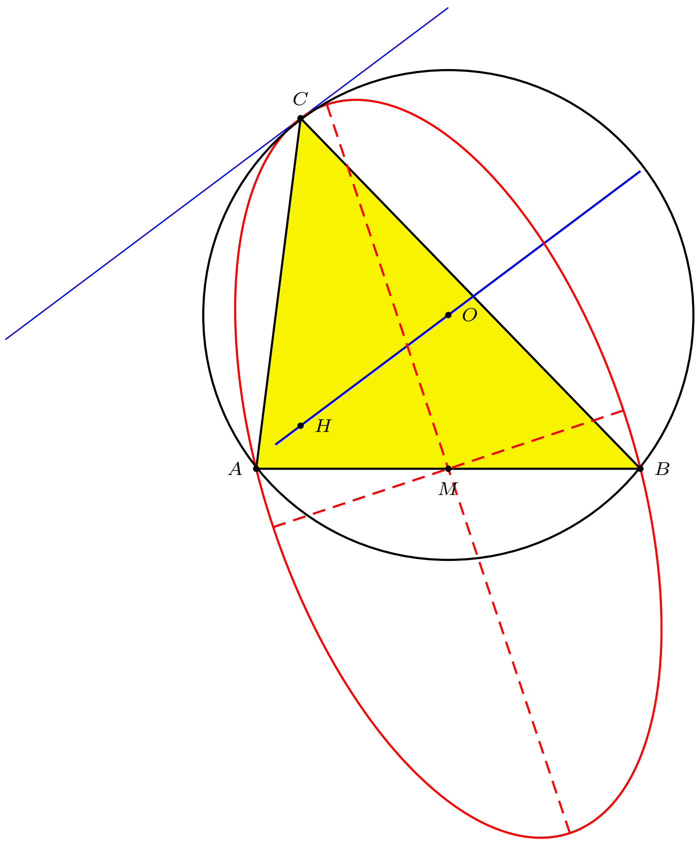


Figure 6

References

[1] A. Pogorelov. *Analytic Geometry*. Mir publishers, 1987.
[2] M. Postnikov. *Lectures in Geometry*. Nauka, 1994.

Juan Rodríguez: Dep. de Matemática, Univ. do Algarve, Fac. de Ciências e Tecnologia, Faro, Portugal
E-mail address: jsanchez@ualg.pt

Paula Manuel: Escola Secundária/3 Dr.ª Laura Ayres, Quarteira, Portugal
E-mail address: paula_manuel@sapo.pt

Paulo Semião: Dep. de Matemática, Univ. do Algarve, Fac. de Ciências e Tecnologia, Faro, Portugal
E-mail address: psemiao@ualg.pt

Forum Geometricorum
Volume 6 (2006) 25–28.

FORUM GEOM
ISSN 1534-1178

A Note on the Droz-Farny Theorem

Charles Thas

Abstract. We give a simple characterization of the Droz-Farny pairs of lines through a point of the plane.

In [3] J-P. Erhmann and F. van Lamoen prove a projective generalization of the Droz-Farny line theorem. They say that a pair of lines (l, l') is a *pair of DF-lines through a point P with respect to a given triangle ABC* if they intercept the line BC in the points X and X', CA in Y and Y', and AB in Z and Z' in such a way that the midpoints of the segments XX', YY', and ZZ' are collinear. They then prove that (l, l') is a pair of DF-lines if and only if l and l' are tangent lines of a parabola inscribed in ABC (see also [5]). Thus, the DF-lines through P are the pairs of conjugate lines in the involution $\mathcal{I}$ determined by the lines through P that are tangent to the parabolas of the pencil of parabolas inscribed in ABC. Through a general point P, there passes just one orthogonal pair of DF-lines with respect to ABC; call this pair the ODF-lines through P with respect to ABC.

Considering the tangent lines through P at the three degenerate inscribed parabolas of ABC, it also follows that (PA, line through P parallel with BC), (PB, line parallel through P with CA), and (PC, line parallel through P with AB), are three conjugate pairs of lines of the involution $\mathcal{I}$.

Recall that the medial triangle $A'B'C'$ of ABC is the triangle whose vertices are the midpoints of $BC, CA,$ and AB, and that the anticomplementary triangle $A''B''C''$ of ABC is the triangle whose medial triangle is ABC ([4]).

Theorem. *A pair (l, l') of lines is a pair of DF-lines through P with respect to ABC, if and only if (l, l') is a pair of conjugate diameters of the conic $\mathcal{C}_P$ with center P, circumscribed at the anticomplementary triangle $A''B''C''$ of ABC. In particular, the ODF-lines through P are the axes of this conic.*

Proof. Since A is the midpoint of $B''C''$, and $B''C''$ is parallel with BC, it follows immediately that PA, and the line through P, parallel with BC, are conjugate diameters of the conic $\mathcal{C}_P$. In the same way, PB and the line through P parallel with CA (and PC and the line trough P parallel with AB) are also conjugate

Publication Date: January 30, 2006. Communicating Editor: J. Chris Fisher.

diameters of $\mathcal{C}_P$. Since two pairs of corresponding lines determine an involution, this completes the proof. □

Remark that the orthocenter H of ABC is the center of the circumcircle of the anticomplementary triangle $A''B''C''$. Since any two orthogonal diameters of a circle are conjugate, we find by this special case the classical Droz-Farny theorem: Perpendicular lines through H are DF-pairs with respect to triangle ABC.

As a corollary of this theorem, we can characterize the axes of any circumscribed ellipse or hyperbola of ABC as the ODF-lines through its center with regard to the medial triangle $A'B'C'$ of ABC. And in the same way we can construct the axes of any circumscribed ellipse or hyperbola of any triangle, associated with ABC.

Examples

1. The Jerabek hyperbola of ABC (the isogonal conjugate of the Euler line of ABC) is the rectangular hyperbola through A, B, C, the orthocenter H, the circumcenter O and the Lemoine (or symmedian) point K of ABC, and its center is Kimberling center X_{125} with trilinear coordinates $(bc(b^2+c^2-a^2)(b^2-c^2)^2, .., ..)$, which is a point of the nine-point circle of ABC (the center of any circumscribed rectangular hyperbola is on the nine-point circle). The axes of this hyperbola are the ODF-lines through X_{125}, with respect to the medial triangle $A'B'C'$ of ABC.

2. The Kiepert hyperbola of ABC is the rectangular hyperbola through A, B, C, H, the centroid G of ABC, and through the Spieker center (the incenter of the medial triangle of ABC). It has center X_{115} with trilinear coordinates $(bc(b^2-c^2)^2, .., ..)$ on the nine-point circle. Its axes are the ODF-lines through X_{115} with respect to the medial triangle $A'B'C'$.

3. The Steiner ellipse of ABC is the circumscribed ellipse with center the centroid G of ABC. It is homothetic to (and has the same axes of) the Steiner ellipses of the medial triangle $A'B'C'$ and of the anticomplementary triangle $A''B''C''$ of ABC. These axes are the ODF-lines through G with respect to ABC (and to $A'B'C'$, and to $A''B''C''$).

4. The Feuerbach hyperbola is the rectangular hyperbola through A, B, C, H, the incenter I of ABC, the Mittenpunkt (the symmedian point of the excentral triangle $I_AI_BI_C$, where I_A, I_B, I_C are the excenters of ABC), with center the Feuerbach point F (at which the incircle and the nine-point circle are tangent; trilinear coordinates $(bc(b-c)^2(b+c-a), .., ..))$. Its axes are the ODF-lines through F, with respect to the medial triangle $A'B'C'$ of ABC.

5. The Stammler hyperbola of ABC has trilinear equation

$$(b^2-c^2)x_1^2 + (c^2-a^2)x_2^2 + (a^2-b^2)x_3^2 = 0.$$

It is the rectangular hyperbola through the incenter I, the excenters I_A, I_B, I_C, the circumcenter O, and the symmedian point K. It is also the Feuerbach hyperbola of the tangential triangle of ABC, and its center is the focus of the Kiepert parabola (inscribed parabola with directrix the Euler line of ABC), which is Kimberling center X_{110} with trilinear coordinates $(\frac{a}{b^2-c^2}, .., ..)$, on the circumcircle of ABC, which is the nine-point circle of the excentral triangle $I_AI_BI_C$. The axes of this Stammler hyperbola are the ODF-lines through X_{110}, with regard to the medial triangle of $I_AI_BI_C$.

Remark that center X_{110} is the fourth common point (apart from A, B, and C) of the conic through A, B, C, and with center the symmedian point K of ABC, which has trilinear equation

$$a(-a^2+b^2+c^2)x_2x_3 + b(a^2-b^2+c^2)x_3x_1 + c(a^2+b^2-c^2)x_1x_2 = 0,$$

and the circumcircle of ABC.

6. The conic with trilinear equation

$$a^2(b^2-c^2)x_1^2 + b^2(c^2-a^2)x_2^2 + c^2(a^2-b^2)x_3^2 = 0$$

is the rectangular hyperbola through the incenter I, through the excenters I_A, I_B, I_C, and through the centroid G of ABC. It is also circumscribed to the anticomplementary triangle $A''B''C''$ (recall that the trilinear coordinates of A'', B'', and C'' are $(-bc, ac, ab), (bc, -ac, ab)$, and $(bc, ac, -ab)$, respectively). Its center is the Steiner point X_{99} with trilinear coordinates $(\frac{bc}{b^2-c^2}, \frac{ca}{c^2-a^2}, \frac{ab}{a^2-b^2})$, a point of intersection of the Steiner ellipse and the circumcircle of ABC.

Remark that the circumcircle of ABC is the nine-point circle of $A''B''C''$ and also of $I_AI_BI_C$. It follows that the axes of this hyperbola, which is often called the Wallace or the Steiner hyperbola, are the ODF-lines through the Steiner point X_{99} with regard to ABC, and also with regard to the medial triangle of the excentral triangle $I_AI_BI_C$.

Remarks. (1) A biographical note on Arnold Droz-Farny can be found in [1].

(2) A generalization of the Droz-Farny theorem in the three-dimensional Euclidean space was given in an article by J. Bilo [2].

(3) Finally, we give a construction for the ODF-lines through a point P with respect to the triangle ABC, *i.e.*, for the orthogonal conjugate pair of lines through P of the involution $\mathcal{I}$ in the pencil of lines through P, determined by the conjugate pairs (PA, line l_a through P parallel to BC) and (PB, line l_b through P parallel to CA): intersect a circle $\mathcal{C}$ through P with these conjugate pairs:

$$\begin{array}{ll} \mathcal{C} \cap PA = Q, & \mathcal{C} \cap l_a = Q', \\ \mathcal{C} \cap PB = R, & \mathcal{C} \cap l_b = R', \end{array}$$

then (Q, Q') and (R, R') determine an involution $\mathcal{I}'$ on $\mathcal{C}$, with center $QQ' \cap RR' = T$. Each line through T intersect the circle $\mathcal{C}$ in two conjugate points of $\mathcal{I}'$. In particular, the line through T and through the center of $\mathcal{C}$ intersects $\mathcal{C}$ in two points S and S', such that PS and PS' are the orthogonal conjugate pair of lines of the involution $\mathcal{I}$.

References

[1] J-L. Ayme, A purely synthetic proof of the Droz-Farny line theorem, *Forum Geom.*, 4 (2004), 219 – 224.

[2] J. Bilo, Generalisations au tetraedre d'une demonstration du theoreme de Noyer-Droz-Farny, *Mathesis*, 56 (1947), 255 – 259.

[3] J-P. Ehrmann and F. M. van Lamoen, A projective generalization of the Droz-Farny line theorem, *Forum Geom.*, 4 (2004), 225 – 227.

[4] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

[5] C. Thas, On ellipses with center the Lemoine point and generalizations, *Nieuw Archief voor Wiskunde*, ser 4. 11 (1993), nr. 1. 1 – 7.

Charles Thas: Department of Pure Mathematics and Computer Algebra, University of Ghent, Krijgslaan 281 - S22, B-9000 Ghent, Belgium

E-mail address: pt@cage.ugent.be

Forum Geometricorum
Volume 6 (2006) 29–46.

FORUM GEOM
ISSN 1534-1178

On the Cyclic Complex of a Cyclic Quadrilateral

Paris Pamfilos

Abstract. To every cyclic quadrilateral corresponds naturally a complex of sixteen cyclic quadrilaterals. The radical axes of the various pairs of circumcircles, the various circumcenters and anticenters combine to interesting configurations. Here are studied some of these, considered to be basic in the study of the whole complex.

1. Introduction

Consider a generic convex cyclic quadrilateral $q = ABCD$. Here we consider a simple figure, resulting by constructing other quadrangles on the sides of q, similar to q. This construction was used in a recent simple proof, of the minimal area property of cyclic quadrilaterals, by Antreas Varverakis [1]. It seems though that the figure is interesting for its own. The principle is to construct the quadrilateral $q' = CDEF$, on a side of and similar to q, but with reversed orientation.

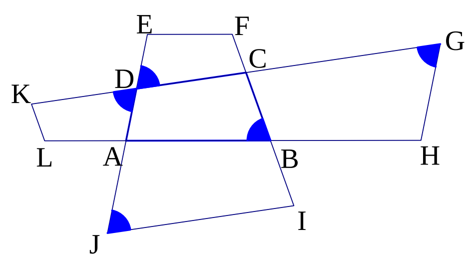


Figure 1. $CDEF$: top-flank of $ABCD$

The sides of the new align with those of the old. Besides, repeating the procedure three more times with the other sides of q, gives the previous basic Figure 1. For convenience I call the quadrilaterals: top-flank $t = CDEF$, right-flank $r = CGHB$, bottom-flank $b = BIJA$ and left-flank $l = DKLA$ of q respectively. In addition to these *main flanks*, there are some other flanks, created by the extensions of the sides of q and the extensions of sides of its four main flanks. Later create the *big flank* denoted below by q^*. To spare words, I drew in Figure 2 these sixteen quadrilaterals together with their names. All these quadrilaterals are cyclic and share the same angles with q. In general, though, only the main flanks are similar to q. More precisely, from their construction, flanks l, r are homothetic, t, b are also homothetic and two adjacent flanks, like r, t are antihomothetic with respect

Publication Date: February 6, 2006. Communicating Editor: Paul Yiu.

to their common vertex, here C. The symbol in braces, $\{rt\}$, will denote the circumcircle, the symbol in parentheses, (rt), will denote the circumcenter, and the symbol in brackets, $[rt]$, will denote the anticenter of the corresponding flank. Finally, a pair of symbols in parentheses, like (rt, rb), will denote the radical axis of the circumcircles of the corresponding flanks. I call this figure the *cyclic complex* associated to the cyclic quadrilateral.

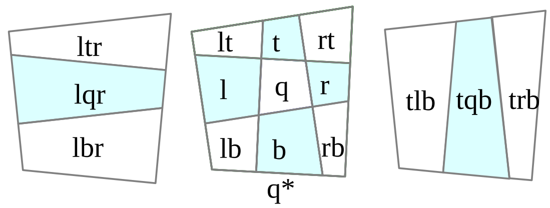


Figure 2. The cyclic complex of q

2. Radical Axes

By taking all pairs of circles, the total number of radical axes, involved, appears to be 120. Not all of them are different though. The various sides are radical axes of appropriate pairs of circles and there are lots of coincidences. For example the radical axes $(t, rt) = (q, r) = (b, rb) = (tqb, rqb)$ coincide with line BC. The same happens with every side out of the eight involved in the complex. Each side coincides with the radical axis of four pairs of circles of the complex. In order to study other identifications of radical axes we need the following:

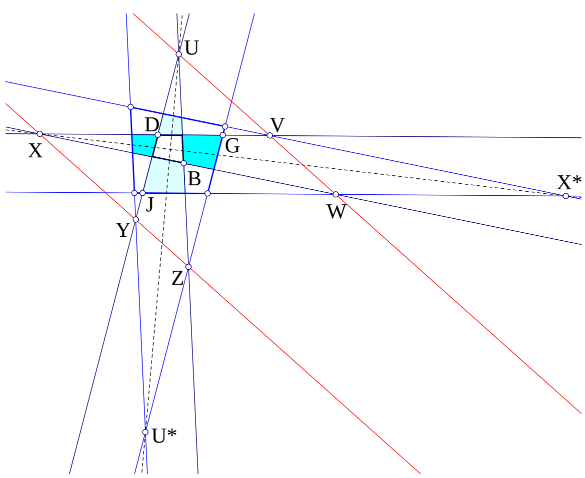


Figure 3. Intersections of lines of the complex

Lemma 1. *Referring to Figure 3, points X, U are intersections of opposite sides of q. U^*, X^* are intersections of opposite sides of $q*$. V, W, Y, Z are intersections of opposite sides of other flanks of the complex. Points X, Y, Z and U, V, W are aligned on two parallel lines.*

The proof is a trivial consequence of the similarity of opposite located main flanks. Thus, l, r are similar and their similarity center is X. Analogously t, b are similar and their similarity center is U. Besides triangles XDY, VDU are anti-homothetic with respect to D and triangles XGZ, WJU are similar to the previous two and also anti-homothetic with respect to B. This implies easily the properties of the lemma.

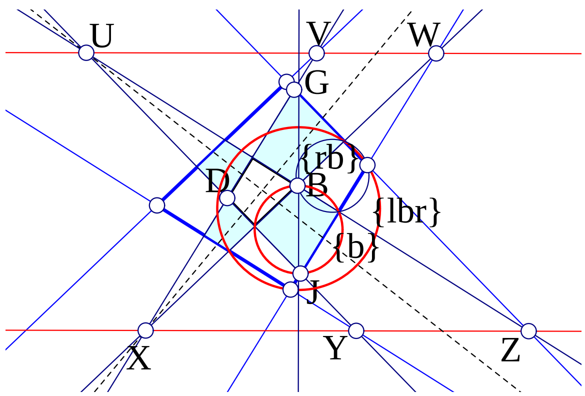


Figure 4. Other radical axes

Proposition 2. *The following lines are common radical axes of the circle pairs:*
(1) *Line XX^* coincides with* $(lt, lb) = (t, b) = (rt, rb) = (ltr, lbr)$.
(2) *Line UU^* coincides with* $(lt, rt) = (l, r) = (lb, rb) = (tlb, trb)$.
(3) *Line XYZ coincides with* $(lbr, b) = (lqr, q) = (ltr, t) = (q^*, tqb)$.
(4) *Line UVW coincides with* $(tlb, l) = (tqb, q) = (trb, r) = (q^*, lqr)$.

Referring to Figure 4, I show that line XYZ is identical with (lbr, b). Indeed, from the intersection of the two circles $\{lbr\}$ and $\{b\}$ with circle $\{rb\}$ we see that Z is on their radical axis. Similarly, from the intersection of these two circles with $\{lb\}$ we see that Y is on their radical axis. Hence line ZYX coincides with the radical axis (lbr, b). The other statements are proved analogously.

3. Centers

The centers of the cyclic complex form various parallelograms. The first of the next two figures shows the centers of the small flanks, and certain parallelograms created by them. Namely those that have sides the medial lines of the sides of the flanks. The second gives a panorama of all the sixteen centers together with a parallelogramic pattern created by them.

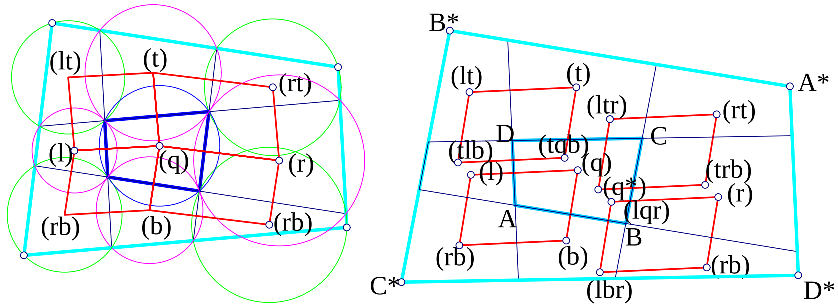


Figure 5. Panorama of centers of flanks

Proposition 3. *Referring to Figure 5, the centers of the flanks build equal parallelograms with parallel sides:* $(lt)(t)(tqb)(tlb)$, $(ltr)(rt)(trb)(q*)$, $(l)(q)(b)(rb)$ *and* $(lqr)(r)(rb)(lbr)$. *The parallelogramic pattern is symmetric with respect to the middle* M *of segment* $(q)(q*)$ *and the centers* $(tqb), (q), (q^*), (lqr)$ *are collinear.*

The proof of the various parallelities is a consequence of the coincidences of radical axes. For example, in the first figure, sides $(t)(rt)$, $(q)(r)$, $(b)(rb)$ are parallel because, all, are orthogonal to the corresponding radical axis, coinciding with line BC. In the second figure $(lt)(t), (tlb)(tqb), (l)(q), (rb)(b)$ are all orthogonal to AD. Similarly $(ltr)(rt)$, $(q*)(trb)$, $(lqr)(r)$, $(lbr)(rb)$ are orthogonal to A^*D^*. The parallelity of the other sides is proved analogously. The equality of the parallelograms results by considering other implied parallelograms, as, for example, $(rb)(b)(t)(lt)$, implying the equality of horizontal sides of the two left parallelograms. Since the labeling is arbitrary, any main flank can be considered to be the left flank of the complex, and the previous remarks imply that all parallelograms shown are equal. An impordant case, in the second figure, is that of the collinearity of the centers $(tqb), (q), (q^*), (lqr)$. Both lines $(tqb)(q)$ and $(q^*)(lqr)$ are orthogonal to the axis XYZ of the previous paragraph. $(tqb)(q^*)$ is orthogonal to the axis UVW, which, after lemma-1, is parallel to XYZ, hence the collinearity. In addition, from the parallelograms, follows that the lengths are equal: $|(tqb)(q)|{=}|(q^*)(lqr)|$. The symmetry about M is a simple consequence of the previous considerations.

There are other interesting quadrilaterals with vertices at the centers of the flanks, related directly to q. For example the next proposition relates the centers of the main flanks to the anticenter of the original quadrilateral q. Recall that the *anticenter* is the symmetric of the circumcenter with respect to the centroid of the quadrilateral. Characteristically it is the common intersection point of the orthogonals from the middles of the sides to their opposites. Some properties of the anticenter are discussed in Honsberger [2]. See also Court [3] and the miscelanea (remark after Proposition 11) below.

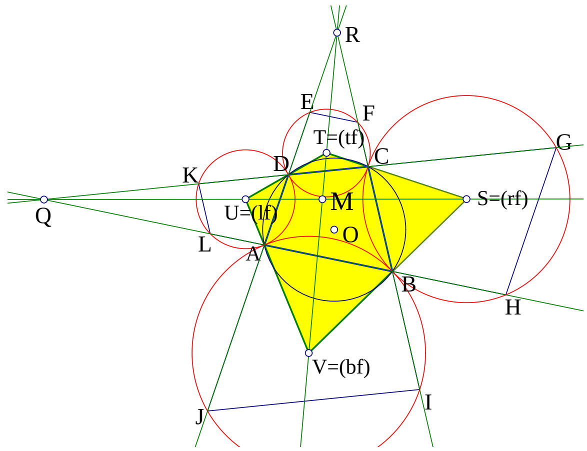


Figure 6. Anticenter through the flanks

Proposition 4. *Referring to Figure 6, the following properties are valid:*
(1) *The circumcircles of adjacent main flanks are tangent at the vertices of* q.
(2) *The intersection point of the diagonals* US, TV *of the quadrilateral* $TSUV$*, formed by the centers of the main flanks, coincides with the anticenter* M *of* q.
(3) *The intersection point of the diagonals of the quadrilateral formed by the anticenters of the main flanks coincides with the circumcenter* O *of* q.

The properties are immediate consequences of the definitions. (1) follows from the fact that triangles FTC and CSB are similar isosceli. To see that property (2) is valid, consider the parallelograms $p = OYMX$, $p^{**} = UXM^{**}Y$ and $p^* = SY^*M^*Y$, tightly related to the anticenters of q and its left and right flanks (Figure 7). X, Y, X^* and Y^* being the middles of the respective sides. One sees easily that triangles UXM and MYS are similar and points U, M, S are aligned. Thus the anticenter M of q lies on line US, passing through Q. Analogously it must lie also on the line joining the two other circumcenters. Thus, it coincides with their intersection.

The last assertion follows along the same arguments, from the similarity of parallelograms $UX^*M^{**}X$ and SYM^*Y^* of Figure 7. O is on the line M^*M^{**}, which is a diagonal of the quadrangle with vertices at the anticenters of the flanks.

Proposition 5. *Referring to the previous figure, the lines* QM *and* QO *are symmetric with respect to the bisector of angle* AQD*. The same is true for lines* QX *and* QY.

This is again obvious, since the trapezia $DAHG$ and $CBLK$ are similar and inversely oriented with respect to the sides of the angle AQD.

Remark. One could construct further flanks, left from the left and right from the right flank. Then repeat the procedure and continuing that way fill all the area of the angle AQD with flanks. All these having alternatively their anticenters and

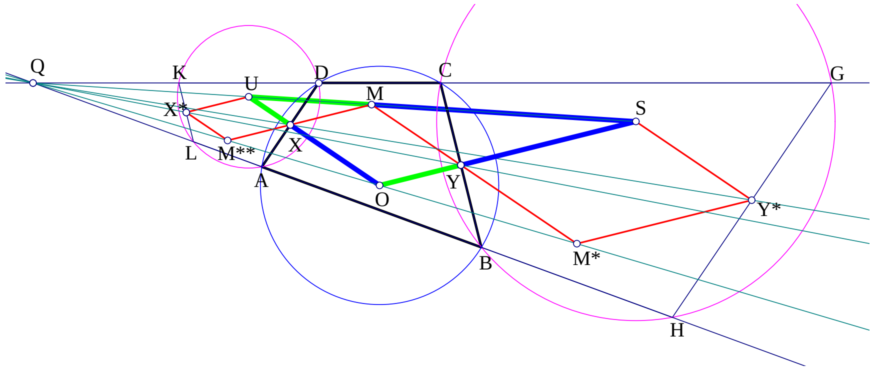


Figure 7. Anticenters M, M^*, M^{**} of the flanks

circumcenters on the two lines QO and QM and the centers of their sides on the two lines OX and OY.

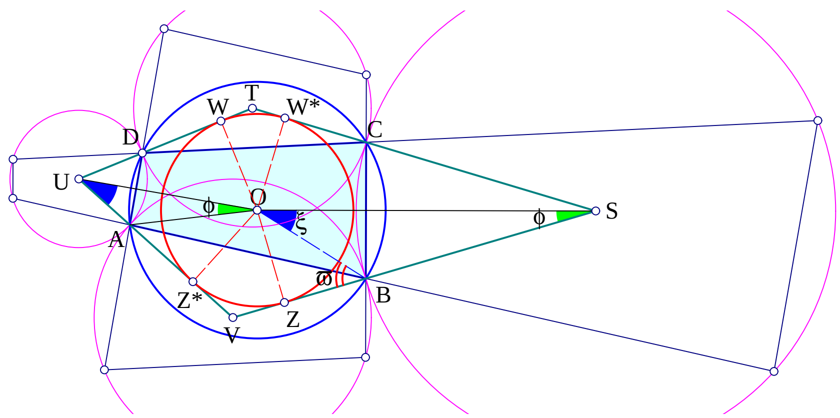


Figure 8. The circumscriptible quadrilateral $STUV$

Proposition 6. *The quadrilateral $STUV$, of the centers of the main flanks, is circumscriptible, its incenter coincides with the circumcenter O of q and its radius is $r \cdot sin(\phi + \xi)$. r being the circumcradius of q and $2\phi, 2\xi$ being the measures of two angles at O viewing two opposite sides of q.*

The proof follows immediately from the similarity of triangles UAO and OBS in Figure 8. The angle $\omega = \phi + \xi$, gives for $|OZ| = r \cdot sin(\omega)$. Z, Z^*, W, W^* being the projections of O on the sides of $STUV$. Analogous formulas hold for the other segments $|OZ^*| = |OW| = |OW^*| = r \cdot sin(\omega)$.

Remarks. (1) Referring to Figure 8, Z, Z^*, W, W^* are vertices of a cyclic quadrilateral q', whose sides are parallel to those of $ABCD$.

(2) The distances of the vertices of q and q' are equal: $|ZB| = |AZ^*| = |DW| = |CW^*| = r \cdot cos(\phi + \xi)$.

(3) Given an arbitrary circumscriptible quadrilateral $STUV$, one can construct the cyclic quadrangle $ABCD$, having centers of its flanks the vertices of $STUV$. Simply take on the sides of $STUV$ segments $|ZB| = |AZ^*| = |DW| = |CW^*|$ equal to the above measure. Then it is an easy exercise to show that the circles centered at S, U and passing from B, C and A, D respectively, define with their intersections on lines AB and CD the right and left flank of $ABCD$.

4. Anticenters

The anticenters of the cyclic complex form a parallelogramic pattern, similar to the previous one for the centers. The next figure gives a panoramic view of the sixteen anticenters (in blue), together with the centers (in red) and the centroids of flanks (white).

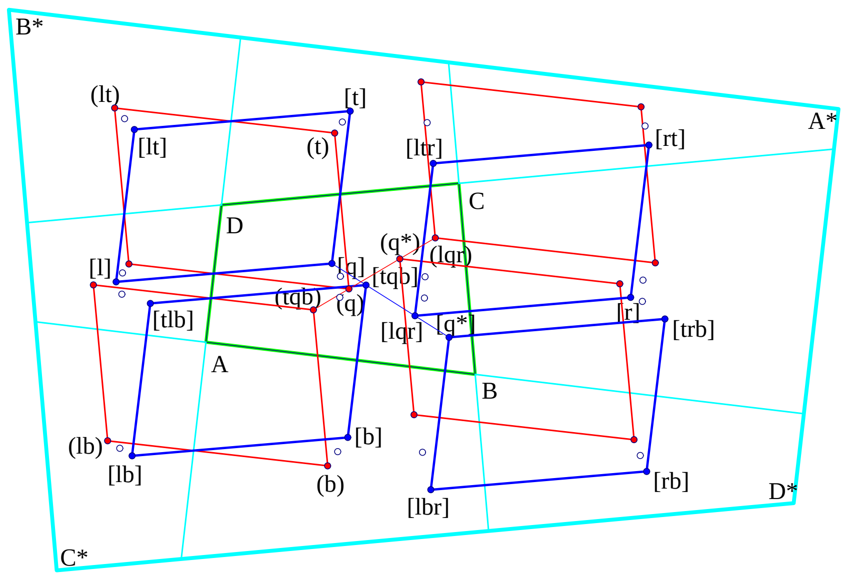


Figure 9. Anticenters of the cyclic complex

Proposition 7. *Referring to Figure 9, the anticenters of the flanks build equal parallelograms with parallel sides:* $[lt][t][q][l]$, $[tlb][tqb][b][lb]$, $[q*][trb][rb][lbr]$ *and* $[ltr][rt][r][lqr]$. *The parallelogramic pattern is symmetric with respect to the middle* M *of segment* $[q][q*]$ *and the anticenters* $[tqb], [q], [q^*], [lqr]$ *are collinear. Besides the angles of the parallelograms are the same with the corresponding of the parallelogramic pattern of the centers.*

The proof is similar to the one of Proposition 2. For example, segments $[lt][t]$, $[l][q]$, $[tlb][tqb]$, $[lb][b]$, $[ltr][rt]$, $[lqr][r]$, $[q*][trb]$, $[lbr][rb]$ are all parallel since they are orthogonal to BC or its parallel B^*C^*.

A similar argument shows that the other sides are also parallel and also proves the statement about the angles. To prove the equality of parallelograms one can

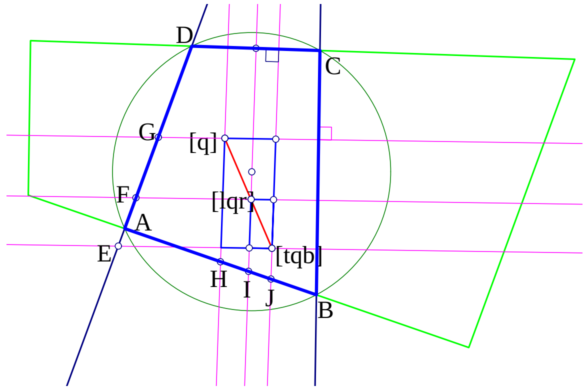


Figure 10. Collinearity of $[tqb]$, $[q]$, $[q^*]$, $[lqr]$

use again implied parallelograms, as, for example, $[lt][t][tqb][tlb]$, which shows the equality of horizontal sides of the two left parallelograms. The details can be completed as in Proposition 2. The only point where another kind of argument is needed is the collinearity assertion. For this, in view of the parallelities proven so far, it suffices to show that points $[q], [lqr], [tqb]$ are collinear. Figure 10 shows how this can be done. G, F, E, H, I, J are middles of sides of flanks, related to the definition of the three anticenters under consideration. It suffices to calculate the ratios and show that $|EF|/|EG| = |JI|/|JH|$. I omit the calculations.

5. Miscelanea

Here I will mention only a few consequences of the previous considerations and some supplementary properties of the complex, giving short hints for their proofs or simply figures that serve as hints.

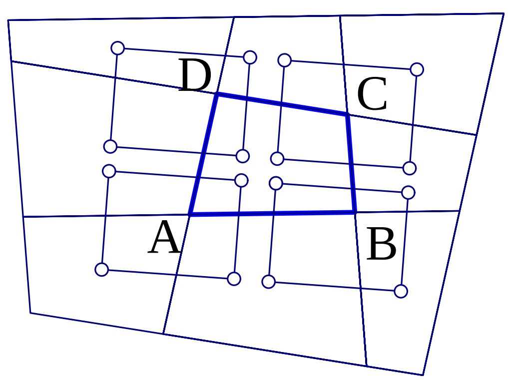


Figure 11. Barycenters of the flanks

Proposition 8. *The barycenters of the flanks build the pattern of equal parallelograms of Figure 11.*

Indeed, this is a consequence of the corresponding results for centers and anticenters of the flanks and the fact that *linear combinations* of parallelograms $c_i = (1-t)a_i + tb_i$, where a_i, b_j denote the vertices of parallelograms, are again parallelograms. Here $t = 1/2$, since the corresponding barycenter is the middle between center and anticenter. The equality of the parallelograms follows from the equality of corresponding parallelograms of centers and anticenters.

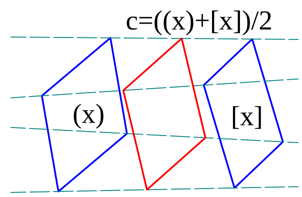


Figure 12. Linear combinations of parallelograms

Proposition 9. *Referring to Figure 13, the centers of the sides of the main flanks are aligned as shown and the corresponding lines intersect at the outer diagonal of q i.e. the line joining the intersection points of opposite sides of q.*

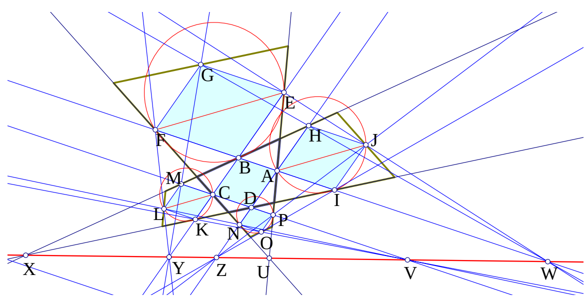


Figure 13. Lines of middles of main flanks

This is due to the fact that the main flanks are antihomothetic with respect to the vertices of q. Thus, the parallelograms of the main flanks are homothetic to each other and their homothety centers are aligned by three on a line. Later assertion can be reduced to the well known one for similarity centers of three circles, by considering the circumcircles of appropriate triangles, formed by parallel diagonals of the four parallelograms. The alignement of the four middles along the sides of $ABCD$ is due to the equality of angles of cyclic quadrilaterals shown in Figure 14.

Proposition 10. *Referring to Figure 15, the quadrilateral $(t)(r)(b)(l)$ of the centers of the main flanks is symmetric to the quadrilateral of the centers of the peripheral flanks $(tlb)(ltr)(trb)(lbr)$. The symmetry center is the middle of the line of $(q)(q^*)$.*

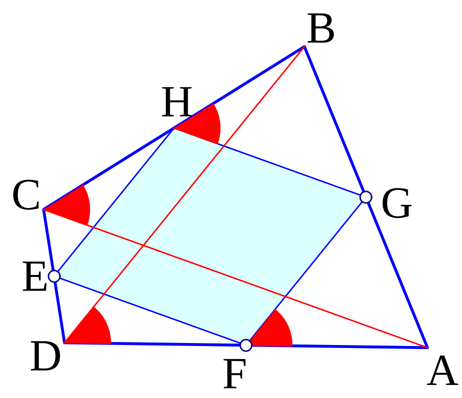


Figure 14. Equal angles in cyclic quadrilaterals

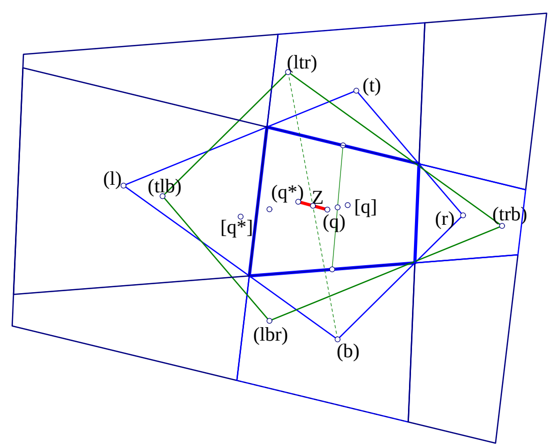


Figure 15. Symmetric quadrilaterals of centers

There is also the corresponding sort of dual for the anticenters, resulting by replacing the symbols (x) with $[x]$:

Proposition 11. *Referring to Figure 16, the quadrilateral* $[t][r][b][l]$ *of the anticenters of the main flanks is symmetric to the quadrilateral of the anticenters of the peripheral flanks* $[tlb][ltr][trb][lbr]$. *The symmetry center is the middle of the line of* $[q][q^*]$.

By the way, the symmetry of center and anticenter about the barycenter leads to a simple proof of the chararacteristic property of the anticenter. Indeed, consider the symmetric $A^*B^*C^*D^*$ of q with respect to the barycenter of q. The orthogonal from the middle of one side of q to the opposite one, to AD say, is also orthogonal to its symmetric A^*D^*, which is parallel to AD (Figure 17). Since A^*D^* is a chord

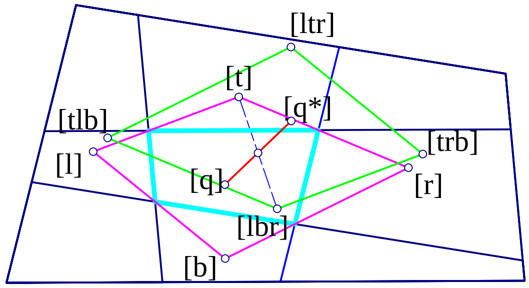


Figure 16. Symmetric quadrilaterals of anticenters

of the symmetric of the circumcircle, the orthogonal to its middle passes through the corresponding circumcenter, which is the anticenter.

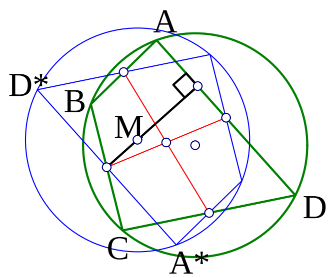


Figure 17. Anticenter's characteristic property

The following two propositions concern the radical axes of two particular pairs of circles of the complex:

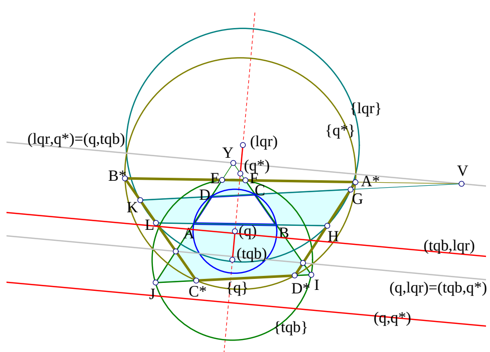


Figure 18. Harmonic bundle of radical axes

Proposition 12. *Referring to Figure 18, the radical axes* $(lqr, q^*) = (q, tqb)$ *and* $(q, lqr) = (tqb, q^*)$. *Besides the radical axes* (tqb, lqr) *and* (q, q^*) *are parallel to the previous two and define with them a harmonic bundle of parallel lines.*

Proposition 13. *Referring to Figure 19, the common tangent* (t, r) *is parallel to the radical axis* (tlb, lbr). *Analogous statements hold for the common tangents of the other pairs of adjacent main flanks.*

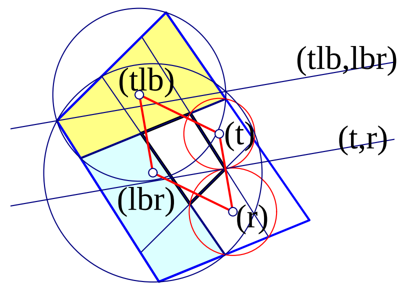


Figure 19. Common tangents of main flanks

6. Generalized complexes

There is a figure, similar to the cyclic complex, resulting in another context. Namely, when considering two arbitrary circles a, b and two other circles c, d tangent to the first two. This is shown in Figure 20. The figure generates a complex of quadrilaterals which I call a *generalized complex* of the cyclic quadrilateral. There are many similarities to the cyclic complex and one substantial difference, which prepares us for the discussion in the next paragraph. The similarities are:

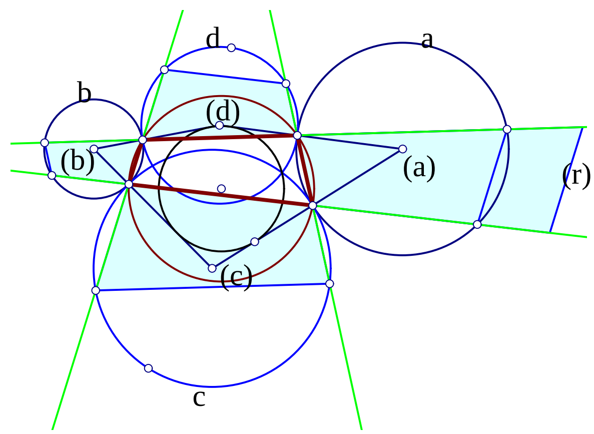


Figure 20. A complex similar to the cyclic one

(1) The points of tangency of the four circles define a cyclic quadrilateral q.
(2) The centers of the circles form a circumscriptible quadrilateral with center at the circumcenter of q.
(3) There are defined flanks, created by the other intersection points of the sides of q with the circles.
(4) Adjacent flanks are antihomothetic with homothety centers at the vertices of q.
(5) The same parallelogramic patterns appear for circumcenters, anticenters and barycenters.

Figure 21 depicts the parallelogrammic pattern for the circumcenters (in red) and the anticenters (in blue). Thus, the properties of the complex, discussed so far, could have been proved in this more general setting. The only difference is that the central cyclic quadrilateral q is not similar, in general, to the flanks, created in this way. In Figure 21, for example, the right cyclic-complex-flank r of q has been also constructed and it is different from the flank created by the general procedure.

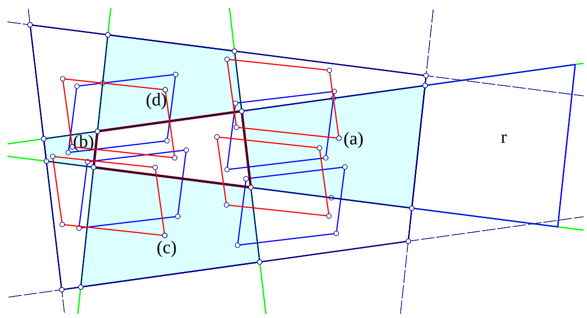


Figure 21. Circumcenters and anticenters of the general complex

Having a cyclic quadrilateral q, one could use the above remarks to construct infinite many generalized complexes (Figure 22) having q as their *central quadrilateral*. In fact, start with a point, F say, on the medial line of side AB of $q = ABCD$. Join it to B, extend FB and define its intersection point G with the medial of BC. Join G with C extend and define the intersection point H with the medial of CD. Finally, join H with D extend it and define I on the medial line of side DA. q being cyclic, implies that there are four circles centered, correspondingly, at points F, G, H and I, tangent at the vertices of q, hence defining the configuration of the previous remark.

From our discussion so far, it is clear, that the *cyclic complex* is a well defined complex, uniquely distinguished between the various generalized complexes, by the property of having its flanks similar to the original quadrilateral q.

7. The inverse problem

The inverse problem asks for the determination of q, departing from the big flank q^*. The answer is in the affirmative but, in general, it is not possible to construct q by elementary means. The following lemma deals with a completion of the figure handled in lemma-1.

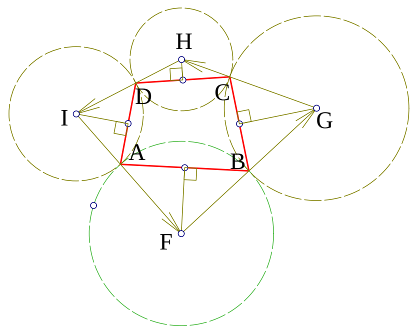


Figure 22. The generalized setting

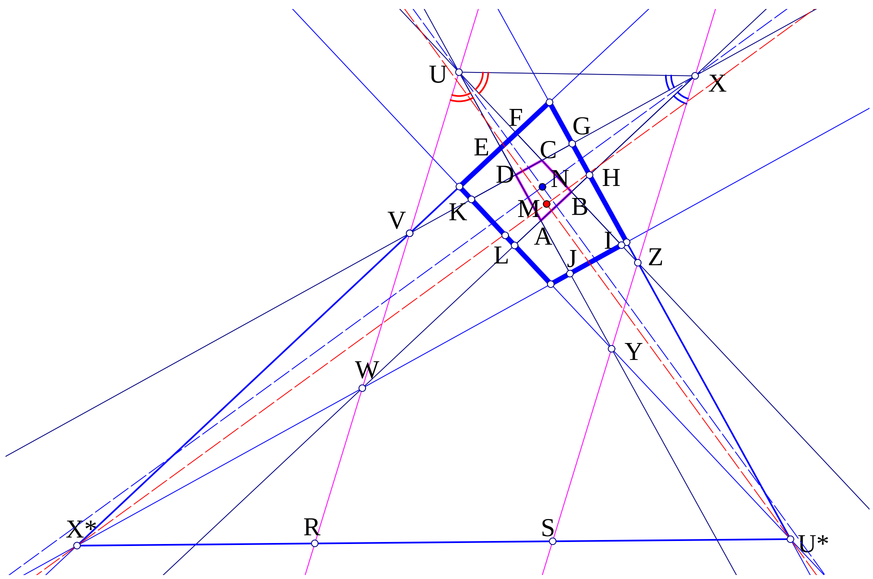


Figure 23. Lines XZY and UVW

Lemma 14. *Referring to Figure 23, lines XZY and UVW are defined by the intersection points of the opposite sides of main flanks of q. Line X^*U^* is defined by the intersection points of the opposite sides of q^*. The figure has the properties:*
(1) *Lines XZY and UVW are parallel and intersect line X^*U^* at points S, R trisecting segment X^*U^*.*
(2) *Triangles YKX, ZCX, UCV, UAX, UIV are similar.*
(3) *Angles $\widehat{VUD} = \widehat{CUX}$ and $\widehat{ZXH} = \widehat{GXU}$.*
(4) *The bisectors of angles $\widehat{VUX}, \widehat{UXZ}$ are respectively identical with those of $\widehat{DUC}, \widehat{DXA}$.*

(5) *The bisectors of the previous angles intersect orthogonaly and are parallel to the bisectors* X^*M, U^*M *of angles* $\widehat{VX^*W}$ *and* $\widehat{YU^*Z}$.

(1) is obvious, since lines UVW and XZY are diagonals of the parallelograms X^*WXV and U^*YUZ. (2) is also trivial since these triangles result from the extension of sides of similar quadrilaterals, namely q and its main flanks. (3) and (4) is a consequence of (2). The orthogonality of (5) is a general property of cyclic quadrilaterals and the parallelity is due to the fact that the angles mentioned are opposite in parallelograms.

The lemma suggests a solution of the inverse problem: Draw from points R, S two parallel lines, so that the parallelograms X^*WXV and U^*YUZ, with their sides intersections, create $ABCD$ with the required properties. Next proposition investigates a similar configuration for a general, not necessarily cyclic, quadrangle.

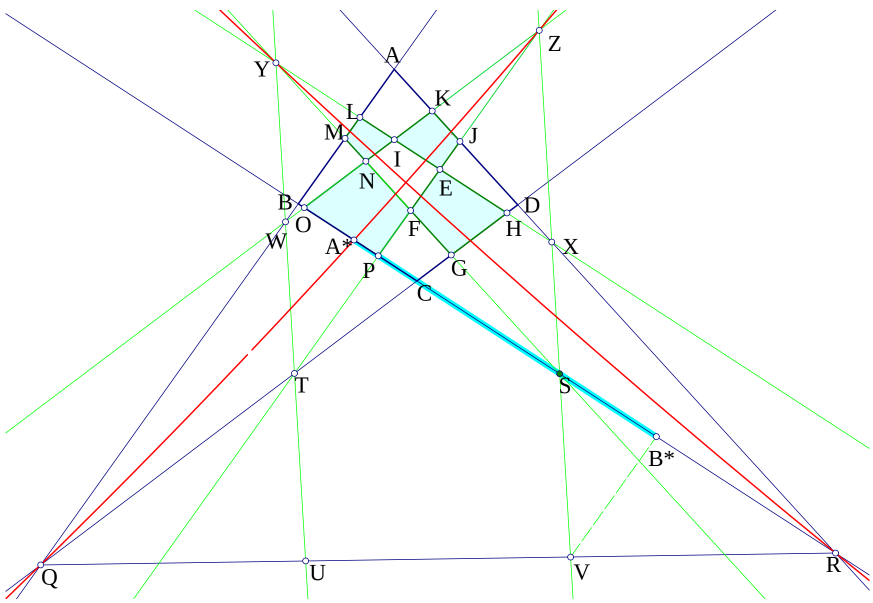


Figure 24. Inverse problem

Proposition 15. *Referring to Figure 24, consider a quadrilateral* $q^* = ABCD$ *and trisect the segmet* QR, *with end-points the intersections of opposite sides of* q^*. *From trisecting points* U, V *draw two arbitrary parallels* UY, VX *intersecting the sides of* q^* *at* W, T *and* S, X *respectively. Define the parallelograms* $QWZT, RSYX$ *and through their intersections and the intersections with* q^* *define the central quadrilateral* $q = EFNI$ *and its flanks* $FEHG, FPON, NMLI, IKJE$ *as shown.*

(1) *The central quadrilateral* q *has angles equal to* q^*. *The angles of the flanks are complementary to those of* q^*.

(2) *The flanks are always similar to each other, two adjacent being anti-homothetic*

with respect to their common vertex. (3) *There is a particular direction of the parallels, for which the corresponding central quadrilateral q has side-lengths-ratio $|EF|/|FN| = |ON|/|OP|$.*

In fact, (1) is trivial and (2) follows from (1) and an easy calculation of the ratios of the sides of the flanks. To prove (3) consider point S varying on side BC of q^*. Define the two parallels and in particular VX by joining V to S. Thus, the two parallels and the whole configuration, defined through them, becomes dependent on the location of point S on BC. For S varying on BC, a simple calculation shows that points Y, Z vary on two hyperbolas (red), the hyperbola containing Z intersecting BC at point A^*. Draw VB^* parallel to AB, B^* being the intersection point with BC. As point S moves from A^* towards B^* on segment A^*B^*, point Z moves on the hyperbola from A^* to infinity and the cross ratio $r(S) = \frac{|EF|}{|FN|} : \frac{|ON|}{|OP|}$ varies increasing continuously from 0 to infinity. Thus, by continuity it passes through 1.

Proposition 16. *Given a circular quadrilateral $q^* = A^*B^*C^*D^*$ there is another circular quadrilateral $q = ABCD$, whose cyclic complex has corresponding big flank the given one.*

The proof follows immediately by applying (3) of the previous proposition to the given cyclic quadrilateral q^*. In that case, the condition of the equality of ratios implies that the constructed by the proposition central quadrilateral q is similar to the main flanks. Thus the given q^* is identical with the big flank of q as required.

Remarks. (1) Figure 24 and the related Proposition 14 deserve some comments. First, they show a way to produce a complex out of any quadrilateral, not necessary a cyclic one. In particular, condition (3) of the aforementioned proposition suggests a unified approach for general quadrilaterals that produces the cyclic complex, when applied to cyclic quadrilaterals. The suggested procedure can be carried out as follows (Figure 25): (a) Start from the given general quadrilateral $q = ABCD$ and construct the first flank $ABFE$ using the restriction $|AE|/|EF| = |BC|/|AB| = k_1$. (b) Use appropriate anti-homotheties centered at the vertices of q to transplant the flank to the other sides of q. These are defined inductively. More precisely, having flank-1, use the anti-homothety $(B, \frac{|BC|}{|FB|})$ to construct flank-2 $BGHC$. Then repeat with analogous constructions for the two remaining flanks. It is easy to see that this procedure, applied to a cyclic quadrilateral, produces its cyclic complex, and this independently from the pair of adjacent sides of q, defining the ratio k_1. For general quadrilaterals though the complex depends on the initial choice of sides defining k_1. Thus defining $k_1 = |DC|/|BC|$ and starting with flank-2, constructed through the condition $|BG|/|GH| = k_1$ etc. we land, in general, to another complex, different from the previous one. In other words, the procedure has an element of arbitrariness, producing four complexes in general, depending on which pair of adjacent sides of q we start it.

(2) The second remark is about the results of Proposition 8, on the centroids or barycenters of the various flanks. They remain valid for the complexes defined

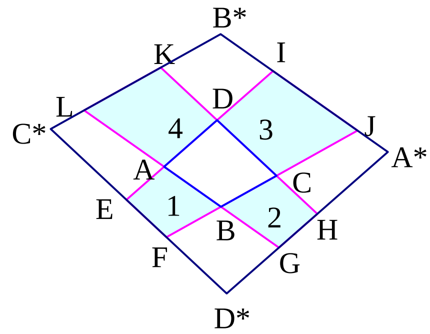


Figure 25. Flanks in arbitrary quadrilaterals

through the previously described procedure. The proof though has to be modified and given more generally, since circumcenters and anticenters are not available in the general case. The figure below shows the barycenters for a general complex, constructed with the procedure described in (1).

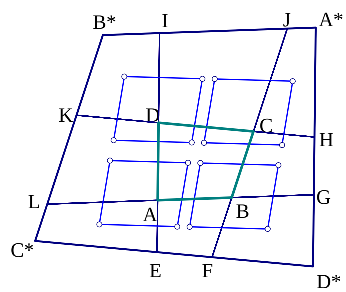


Figure 26. Barycenters for general complexes

An easy approach is to use vectors. Proposition-9, with some minor changes, can also be carried over to the general case. I leave the details as an exercise.
(3) Although Proposition 15 gives an answer to the existence of a sort of *soul* (q) of a given cyclic quadrilateral (q^*), a more elementary constuction of it is desirable. Proposition-14, in combination with the first remark, shows that even general quadrilaterals have *souls*.
(4) One is tempted to look after the soul of a soul, or, stepping inversely, the complex and corresponding big flank of the big flank etc.. Several questions arise in this context, such as (a) are there repetitions or periodicity, producing something similar to the original after a finite number of repetitions? (b) which are the limit points, for the sequence of souls?

References

[1] A. Varverakis, A maximal property of cyclic quadrilaterals, *Forum Geom.*, 5(2005), 63–64.
[2] R. Honsberger, *Episodes in Nineteenth and Twentieth century Euclidean Geometry*, Math. Assoc. Amer. 1994, p. 35.
[3] N. A. Court, *College Geometry*, Barnes & Noble 1970, p. 131.

Paris Pamfilos: Department of Mathematics, University of Crete, Crete, Greece
E-mail address: pamfilos@math.uoc.gr

Forum Geometricorum
Volume 6 (2006) 47–52.

FORUM GEOM
ISSN 1534-1178

Isocubics with Concurrent Normals

Bernard Gibert

Abstract. It is well known that the tangents at A, B, C to a pivotal isocubic concur. This paper studies the situation where the normals at the same points concur. The case of non-pivotal isocubics is also considered.

1. Pivotal isocubics

Consider a pivotal isocubic $\mathrm{p}\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$ with pole $\Omega = p : q : r$ and pivot P, *i.e.*, the locus of point M such as P, M and its $\Omega-$isoconjugate M^* are collinear. This has equation

$$ux(ry^2 - qz^2) + vy(pz^2 - rx^2) + wz(qx^2 - py^2) = 0.$$

It is well known that the tangents at A, B, C and P to $\mathrm{p}\mathcal{K}$, being respectively the lines $-\frac{v}{q}y + \frac{w}{r}z = 0$, $\frac{u}{p}x - \frac{w}{r}z = 0$, $-\frac{u}{p}x + \frac{v}{q}y = 0$, concur at $P^* = \frac{p}{u} : \frac{q}{v} : \frac{r}{w}$.[1] We characterize the pivotal cubics whose normals at the vertices A, B, C concur at a point. These normals are the lines

$$\begin{array}{ll} \mathsf{n}A: & (S_Arv + (S_A + S_B)qw)y + (S_Aqw + (S_C + S_A)rv)z = 0, \\ \mathsf{n}B: & (S_Bru + (S_A + S_B)pw)x + (S_Bpw + (S_B + S_C)ru)z = 0, \\ \mathsf{n}C: & (S_Cqu + (S_C + S_A)pv)x + (S_Cpv + (S_B + S_C)qu)y = 0. \end{array}$$

These three normals are concurrent if and only if

$$(pvw + qwu + ruv)(a^2qru + b^2rpv + c^2pqw) = 0.$$

Let us denote by $\mathcal{C}_\Omega$ the circumconic with perspector Ω, and by $\mathcal{L}_\Omega$ the line which is the $\Omega-$isoconjugate of the circumcircle.[2] These have barycentric equations

$$C_\Omega : \quad pyz + qzx + rxy = 0,$$

and

$$\mathcal{L}_\Omega : \quad \frac{a^2}{p}x + \frac{b^2}{q}y + \frac{c^2}{r}z = 0.$$

Publication Date: February 13, 2006. Communicating Editor: Paul Yiu.

[1]The tangent at P, namely, $u(rv^2 - qw^2)x + v(pw^2 - ru^2)y + w(qu^2 - pv^2)z = 0$, also passes through the same point.

[2]This line is also the trilinear polar of the isotomic conjugate of the isogonal conjugate of Ω.

Theorem 1. *The pivotal cubic* $\mathrm{p}\mathcal{K}(\Omega, P)$ *has normals at* A, B, C *concurrent if and only if*
(1) P *lies on* $\mathcal{C}_\Omega$, *equivalently,* P^* *lies on the line at infinity, or*
(2) P *lies on* $\mathcal{L}_\Omega$, *equivalently,* P^* *lies on the circumcircle.*

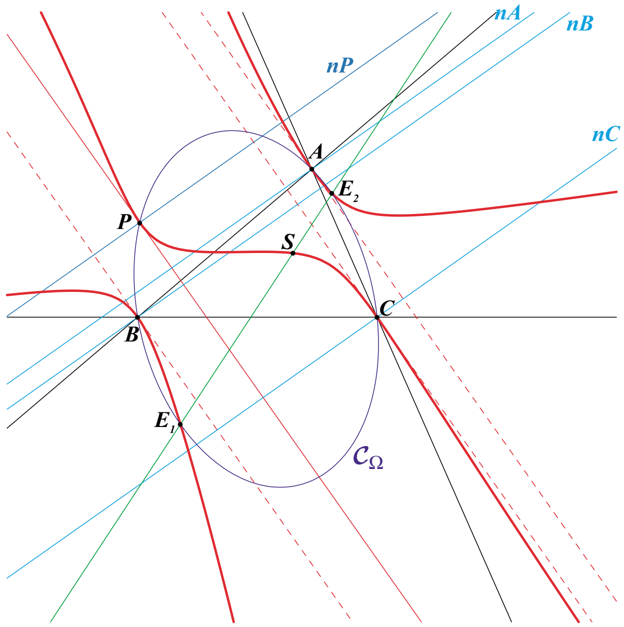


Figure 1. Theorem 1(1): $\mathrm{p}\mathcal{K}$ with concurring normals

More precisely, in (1), the tangents at A, B, C are parallel since P^* lies on the line at infinity. Hence the normals are also parallel and "concur" at X on the line at infinity. The cubic $\mathrm{p}\mathcal{K}$ meets $\mathcal{C}_\Omega$ at A, B, C, P and two other points E_1, E_2 lying on the polar line of P^* in $\mathcal{C}_\Omega$, *i.e.*, the conjugate diameter of the line PP^* in $\mathcal{C}_\Omega$. Obviously, the normal at P is parallel to these three normals. See Figure 1.

In (2), P^* lies on the circumcircle and the normals concur at X, antipode of P^* on the circumcircle. $\mathrm{p}\mathcal{K}$ passes through the (not always real) common points E_1, E_2 of $\mathcal{L}_\Omega$ and the circumcircle. These two points are isoconjugates. See Figure 2.

2. The orthopolar

The tangent $\mathtt{t}M$ at any non-singular point M to any curve is the polar line (or first polar) of M with respect to the curve and naturally the normal $\mathtt{n}M$ at M is the perpendicular at M to $\mathtt{t}M$. For any point M not necessarily on the curve, we define the *orthopolar* of M with respect to the curve as the perpendicular at M to the polar line of M.

In Theorem 1(1) above, we may ask whether there are other points on $\mathrm{p}\mathcal{K}$ such that the normal passes through X. We find that the locus of point Q such that the orthopolar of Q contains X is the union of the line at infinity and the circumconic

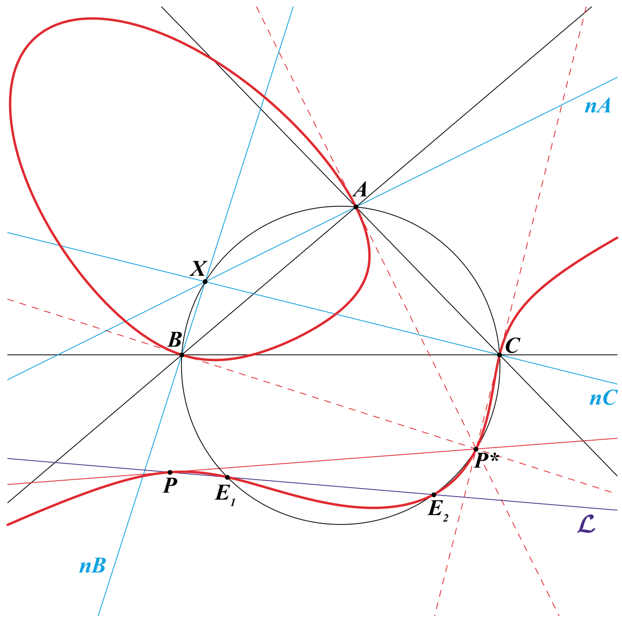


Figure 2. Theorem 1(2): p$\mathcal{K}$ with concurring normals

passing through P and P^*, the isoconjugate of the line PP^*. Hence, there are no other points on the cubic with normals passing through X.

In Theorem 1(2), the locus of point Q such that the orthopolar of Q contains X is now a circum-cubic (K) passing through P^* and therefore having six other (not necessarily real) common points with p$\mathcal{K}$. Figure 3 shows p$\mathcal{K}(X_2, X_{523})$ where four real normals are drawn from the Tarry point X_{98} to the curve.

3. Non-pivotal isocubics

Lemma 2. *Let M be a point and m its trilinear polar meeting the sidelines of ABC at U, V, W. The perpendiculars at A, B, C to the lines AU, BV, CW concur if and only if M lies on the Thomson cubic. The locus of the point of concurrence is the Darboux cubic.*

Let us now consider a non-pivotal isocubic n$\mathcal{K}$ with pole $\Omega = p : q : r$ and root[3] $P = u : v : w$. This cubic has equation :

$$ux(ry^2 + qz^2) + vy(pz^2 + rx^2) + wz(qx^2 + py^2) + kxyz = 0.$$

Denote by n$\mathcal{K}_0$ the corresponding cubic without xyz term, *i.e.*,

$$ux(ry^2 + qz^2) + vy(pz^2 + rx^2) + wz(qx^2 + py^2) = 0.$$

[3]An n$\mathcal{K}$ meets again the sidelines of triangle ABC at three collinear points U, V, W lying on the trilinear polar of the root.

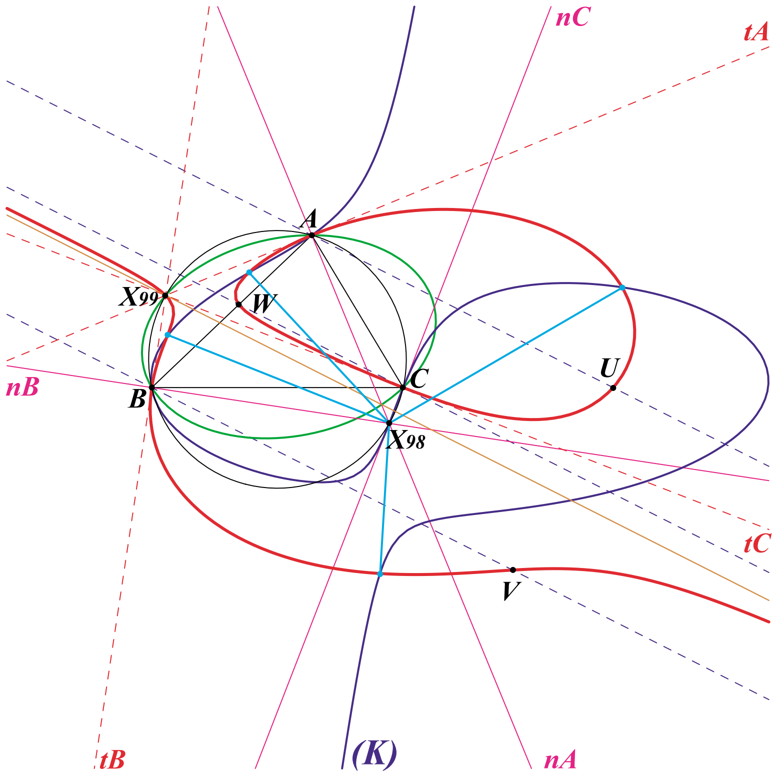


Figure 3. Theorem 1(2): Other normals to $\mathrm{p}\mathcal{K}$

It can easily be seen that the tangents $\mathtt{t}A$, $\mathtt{t}B$, $\mathtt{t}C$ do not depend of k and pass through the feet U', V', W' of the trilinear polar of P^* [4]. Hence it is enough to take the cubic $\mathsf{n}\mathcal{K}_0$ to study the normals at A, B, C.

Theorem 3. *The normals of* $\mathsf{n}\mathcal{K}_0$ *at* A, B, C *are concurrent if and only if*
(1) Ω *lies on the pivotal isocubic* $\mathrm{p}\mathcal{K}_1$ *with pole* $\Omega_1 = a^2u^2 : b^2v^2 : c^2w^2$ *and pivot* P, *or*
(2) P *lies on the pivotal isocubic* $\mathrm{p}\mathcal{K}_2$ *with pole* $\Omega_2 = \frac{p^2}{a^2} : \frac{q^2}{b^2} : \frac{r^2}{c^2}$ *and pivot* $P_2 = \frac{p}{a^2} : \frac{q}{b^2} : \frac{r}{c^2}$.

$\mathrm{p}\mathcal{K}_1$ is the $\mathrm{p}\mathcal{K}$ with pivot the root P of the $\mathsf{n}\mathcal{K}_0$ which is invariant in the isoconjugation which swaps P and the isogonal conjugate of the isotomic conjugate of P.

By Lemma 2, it is clear that $\mathrm{p}\mathcal{K}_2$ is the $\Omega-$isoconjugate of the Thomson cubic.

The following table gives a selection of such cubics $\mathrm{p}\mathcal{K}_2$. Each line of the table gives a selection of $\mathrm{n}\mathcal{K}_0(\Omega, X_i)$ with concurring normals at A, B, C.

[4] In other words, these tangents form a triangle perspective to ABC whose perspector is P^*. Its vertices are the harmonic associates of P^*.

Cubic	Ω	Ω_2	P_2	X_i on the curve for $i =$
$K034$	X_1	X_2	X_{75}	1, 2, 7, 8, 63, 75, 92, 280, 347, 1895
$K184$	X_2	X_{76}	X_{76}	2, 69, 75, 76, 85, 264, 312
$K099$	X_3	X_{394}	X_{69}	2, 3, 20, 63, 69, 77, 78, 271, 394
	X_4	X_{2052}	X_{264}	2, 4, 92, 253, 264, 273, 318, 342
	X_9	X_{346}	X_{312}	2, 8, 9, 78, 312, 318, 329, 346
	X_{25}	X_{2207}	X_4	4, 6, 19, 25, 33, 34, 64, 208, 393
$K175$	X_{31}	X_{32}	X_1	1, 6, 19, 31, 48, 55, 56, 204, 221, 2192
$K346$	X_{32}	X_{1501}	X_6	6, 25, 31, 32, 41, 184, 604, 2199
	X_{55}	X_{220}	X_8	1, 8, 9, 40, 55, 200, 219, 281
	X_{56}	X_{1407}	X_7	1, 7, 56, 57, 84, 222, 269, 278
	X_{57}	X_{279}	X_{85}	2, 7, 57, 77, 85, 189, 273, 279
	X_{58}	X_{593}	X_{86}	21, 27, 58, 81, 86, 285, 1014, 1790
	X_{75}	X_{1502}	X_{561}	75, 76, 304, 561, 1969

For example, all the isogonal n$\mathcal{K}_0$ with concurring normals must have their root on the Thomson cubic. Similarly, all the isotomic n$\mathcal{K}_0$ with concurring normals must have their root on $K184 = \mathrm{p}\mathcal{K}(X_{76}, X_{76})$.

Figure 4 shows n$\mathcal{K}_0(X_1, X_{75})$ with normals concurring at O. It is possible to draw from O six other (not necessarily all real) normals to the curve. The feet of these normals lie on another circum-cubic labeled (K) in the figure.

In the special case where the non-pivotal cubic is a singular cubic $\mathrm{c}\mathcal{K}$ with singularity F and root P, the normals at A, B, C concur at F if and only if F lies on the Darboux cubic. Furthermore, the locus of M whose orthopolar passes through F being also a nodal circumcubic with node F, there are two other points on $\mathrm{c}\mathcal{K}$ with normals passing through F. In Figure 5, $\mathrm{c}\mathcal{K}$ has singularity at O and its root is X_{394}. The corresponding nodal cubic passes through the points O, X_{25}, X_{1073}, X_{1384}, X_{1617}. The two other normals are labelled OR and OS.

References

[1] J.-P. Ehrmann and B. Gibert, *Special Isocubics in the Triangle Plane*, available at http://perso.wanadoo.fr/bernard.gibert/files/isocubics.html.

[2] B. Gibert, *Cubics in the Triangle Plane*, available at http://perso.wanadoo.fr/bernard.gibert/index.html .

[3] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France

E-mail address: bg42@wanadoo.fr

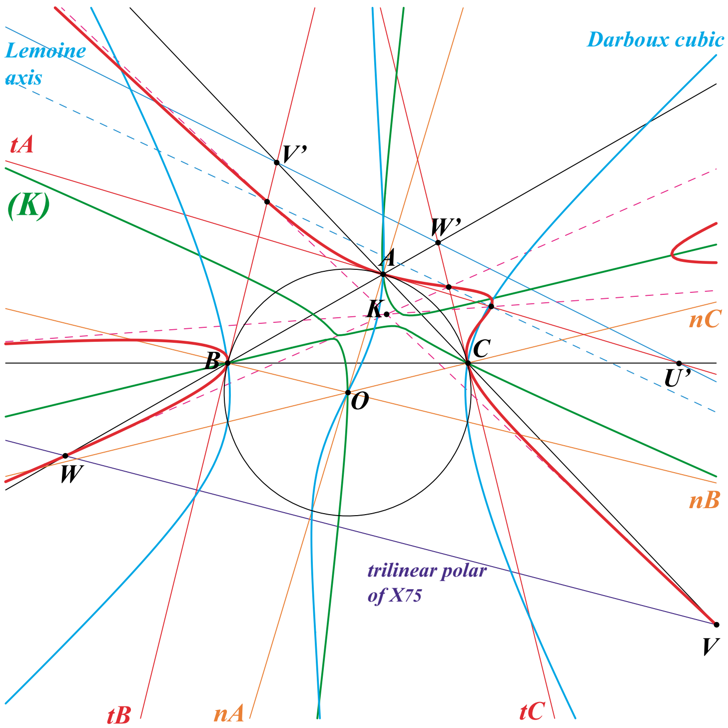


Figure 4. $\mathsf{n}\mathcal{K}_0(X_1, X_{75})$ with normals concurring at O

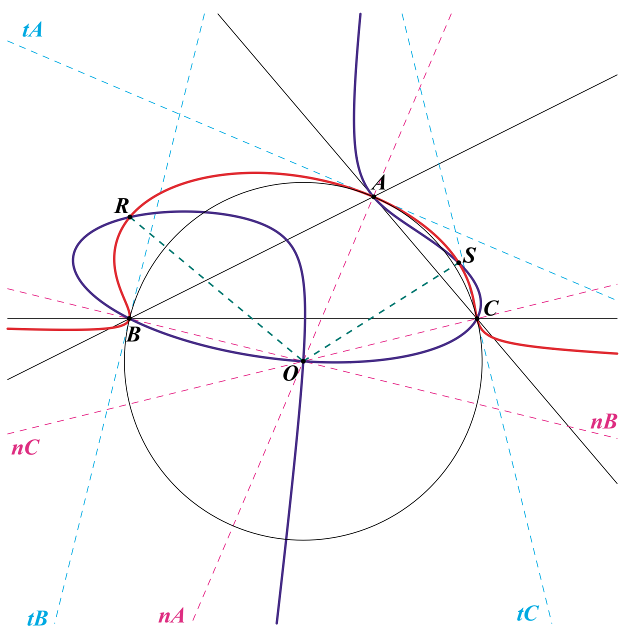


Figure 5. A $c\mathcal{K}$ with normals concurring at O

Forum Geometricorum
Volume 6 (2006) 53–55.

FORUM GEOM
ISSN 1534-1178

A Characterization of the Centroid Using June Lester's Shape Function

Mowaffaq Hajja and Margarita Spirova

Abstract. The notion of triangle shape is used to give another proof of the fact that if P is a point inside triangle ABC and if the cevian triangle of P is similar to ABC in the natural order, then P is the centroid.

Identifying the Euclidean plane with the plane of complex numbers, we define a (non-degenerate) triangle to be any ordered triple (A, B, C) of distinct complex numbers, and we write it as ABC if no ambiguity may arise. According to this definition, there are in general six different triangles having the same set of vertices. We say that triangles ABC and $A'B'C'$ are *similar* if

$$\|A - B\| : \|A' - B'\| = \|B - C\| : \|B' - C'\| = \|C - A\| : \|C' - A'\|.$$

By the SAS similarity theorem and by the geometric interpretation of the quotient of two complex numbers, this is equivalent to the requirement that

$$\frac{A-B}{A-C} = \frac{A'-B'}{A'-C'}.$$

June A. Lester called the quantity $\dfrac{A-B}{A-C}$ the *shape* of triangle ABC and she studied properties and applications of this shape function in great detail in [4], [5], and [6].

In this note, we use this shape function to prove that if P is a point inside triangle ABC, and if AA', BB', and CC' are the cevians through P, then triangles ABC and $A'B'C'$ are similar if and only if P is the centroid of ABC. This has already appeared as Theorem 7 in [1], where three different proofs are given, and as a problem in the Problem Section of the *Mathematics Magazine* [2]. A generalization to d-simplices for all d is being considered in [3].

Our proof is an easy consequence of two lemmas that may prove useful in other contexts.

Publication Date: February 21, 2006. Communicating Editor: Paul Yiu.
The first-named author is supported by a research grant from Yarmouk University.

Lemma 1. *Let ABC be a non-degenerate triangle, and let x, y, and z be real numbers such that*

$$x(A-B)^2 + y(B-C)^2 + z(C-A)^2 = 0. \tag{1}$$

Then either $x = y = z = 0$, or $xy + yz + zx > 0$.

Proof. Let $S = \frac{A-B}{A-C}$ be the shape of ABC. Dividing (1) by $(A-C)^2$, we obtain

$$(x+y)S^2 - 2yS + (y+z) = 0. \tag{2}$$

Since ABC is non-degenerate, S is not real. Thus if $x + y = 0$, then $y = 0$ and hence $x = y = z = 0$. Otherwise, $x + y \neq 0$ and the discriminant

$$4y^2 - 4(x+y)(y+z) = -(xy + yz + zx)$$

of (2) is negative, *i.e.*, $xy + yz + zx > 0$, as desired. □

Lemma 2. *Suppose that the cevians through an interior point P of a triangle divide the sides in the ratios $u : 1-u$, $v : 1-v$, and $w : 1-w$. Then*
(i) $uvw \leq \frac{1}{8}$, *with equality if and only if $u = v = w = \frac{1}{2}$, i.e., if and only if P is the centroid.*
(ii) $\left(u - \frac{1}{2}\right)\left(v - \frac{1}{2}\right) + \left(v - \frac{1}{2}\right)\left(w - \frac{1}{2}\right) + \left(w - \frac{1}{2}\right)\left(u - \frac{1}{2}\right) \leq 0$, *with equality if and only if $u = v = w = \frac{1}{2}$,* i.e., *if and only if P is the centroid.*

Proof. Let $uvw = p$. Then using the cevian condition $uvw = (1-u)(1-v)(1-w)$, we see that

$$\begin{aligned} p &= \sqrt{u(1-u)}\sqrt{v(1-v)}\sqrt{w(1-w)} \\ &\leq \frac{u+(1-u)}{2}\,\frac{v+(1-v)}{2}\,\frac{w+(1-w)}{2}, \text{ by the AM-GM inequality} \\ &= \frac{1}{8}, \end{aligned}$$

with equality if and only if $u = \frac{1}{2}$, $v = \frac{1}{2}$, and $w = \frac{1}{2}$. This proves (i).

To prove (ii), note that

$$\begin{aligned} &\left(u - \frac{1}{2}\right)\left(v - \frac{1}{2}\right) + \left(v - \frac{1}{2}\right)\left(w - \frac{1}{2}\right) + \left(w - \frac{1}{2}\right)\left(u - \frac{1}{2}\right) \\ &= (uv + vw + wu) - (u + v + w) + \frac{3}{4} \\ &= 2uvw - \frac{1}{4}, \end{aligned}$$

because $uvw = (1-u)(1-v)(1-w)$. Now use (i). □

We now use Lemmas 1 and 2 and the shape function to prove the main result.

Theorem 3. *Let AA', BB', and CC' be the cevians through an interior point P of triangle ABC. Then triangles ABC and $A'B'C'$ are similar if and only if P is the centroid of ABC.*

Proof. One direction being trivial, we assume that $A'B'C'$ and ABC are similar, and we prove that P is the centroid.

Suppose that the cevians AA', BB', and CC' through P divide the sides BC, CA, and AB in the ratios $u : 1-u$, $v : 1-v$, and $w : 1-w$, respectively. Since ABC and $A'B'C'$ are similar, it follows that they have equal shapes, *i.e.*,

$$\frac{A-B}{A-C} = \frac{A'-B'}{A'-C'}. \tag{3}$$

Substituting the values

$$A' = (1-u)B + uC, \quad B' = (1-v)C + vA, \quad C' = (1-w)A + wB$$

in (3) and simplifying, we obtain

$$\left(u - \frac{1}{2}\right)(A-B)^2 + \left(v - \frac{1}{2}\right)(B-C)^2 + \left(w - \frac{1}{2}\right)(C-A)^2 = 0.$$

By Lemma 1, either $u = v = w = \frac{1}{2}$, in which case P is the centroid, or

$$\left(u - \frac{1}{2}\right)\left(v - \frac{1}{2}\right) + \left(v - \frac{1}{2}\right)\left(w - \frac{1}{2}\right) + \left(w - \frac{1}{2}\right)\left(u - \frac{1}{2}\right) > 0,$$

in which case Lemma 2(ii) is contradicted. This completes the proof. □

References

[1] S. Abu-Saymeh and M. Hajja, In search of more triangle centres, *Internat. J. Math. Ed. Sci. Tech.*, 36 (2005) 889–912.
[2] M. Hajja, Problem 1711, *Math. Mag.* 78 (2005), 68.
[3] M. Hajja and H. Martini, Characterization of the centroid of a simplex, in preparation.
[4] J. A. Lester, Triangles I: Shapes, *Aequationes Math.*, 52 (1996), 30–54.
[5] J. A. Lester, Triangles II: Complex triangle coordinates, *Aequationes Math.* 52 (1996), 215–245.
[6] J. A. Lester, Triangles III: Complex triangle functions, *Aequationes Math.* 53 (1997), 4–35.

Mowaffaq Hajja: Mathematics Department, Yarmouk University, Irbid, Jordan.
E-mail address: mhajja@yu.edu.jo

Margarita Spirova: Faculty of Mathematics and Informatics, University of Sofia, 5 Yames Bourchier, 164 Sofia, Bulgaria.
E-mail address: spirova@fmi.uni-sofia.bg

Forum Geometricorum
Volume 6 (2006) 57–70.

FORUM GEOM
ISSN 1534-1178

The Locations of Triangle Centers

Christopher J. Bradley and Geoff C. Smith

Abstract. The orthocentroidal circle of a non-equilateral triangle has diameter GH where G is the centroid and H is the orthocenter. We show that the Fermat, Gergonne and symmedian points are confined to, and range freely over the interior disk punctured at its center. The Mittenpunkt is also confined to and ranges freely over another punctured disk, and the second Fermat point is confined to and ranges freely over the exterior of the orthocentroidal circle. We also show that the circumcenter, centroid and symmedian point determine the sides of the reference triangle ABC.

1. Introduction

All results concern non-equilateral non-degenerate triangles. The orthocentroidal circle $\mathcal{S}_{GH}$ has diameter GH, where G is the centroid and H is the orthocenter of triangle ABC. Euler showed [3] that O, G and I determine the sides a, b and c of triangle ABC. Here O denotes the circumcenter and I the incenter. Later Guinand [4] showed that I ranges freely over the open disk $\mathcal{D}_{GH}$ (the interior of $\mathcal{S}_{GH}$) punctured at the nine-point center N. This work involved showing that certain cubic equations have real roots. Recently Smith [9] showed that both results can be achieved in a straightforward way; that I can be anywhere in the punctured disk follows from Poncelet's porism, and a formula for IG^2 means that the position of I in $\mathcal{D}_{GH}$ enables one to write down a cubic polynomial which has the side lengths a, b and c as roots. As the triangle ABC varies, the Euler line may rotate and the distance GH may change. In order to say that I ranges freely over all points of this punctured open disk, it is helpful to rescale by insisting that the distance GH is constant; this can be readily achieved by dividing by the distance GH or OG as convenient. It is also helpful to imagine that the Euler line is fixed.

In this paper we are able to prove similar results for the symmedian (K), Fermat (F) and Gergonne ($G_{\rm e}$) points, using the same disk $\mathcal{D}_{GH}$ but punctured at its midpoint J rather than at the nine-point center N. We show that, O, G and K determine a, b and c. The Morleys [8] showed that O, G and the first Fermat point F determine the reference triangle by using complex numbers. We are not able to show that O, G and $G_{\rm e}$ determine a, b and c, but we conjecture that they do.

Since I, G, $S_{\rm p}$ and $N_{\rm a}$ are collinear and spaced in the ratio $2:1:3$ it follows from Guinand's theorem [4] that the Spieker center and Nagel point are confined

Publication Date: February 27, 2006. Communicating Editor: Paul Yiu.

to, and range freely over, certain punctured open disks, and each in conjunction with O and G determines the triangle's sides. Since G_e, G and M are collinear and spaced in the ratio $2:1$ it follows that M ranges freely over the open disk on diameter OG with its midpoint deleted. Thus we now know how each of the first ten of Kimberling's triangle centers [6] can vary with respect to the scaled Euler line.

Additionally we observe that the orthocentroidal circle forms part of a coaxal system of circles including the circumcircle, the nine-point circle and the polar circle of the triangle. We give an areal descriptions of the orthocentroidal circle. We show that the Feuerbach point must lie outside the circle $\mathcal{S}_{GH}$, a result foreshadowed by a recent internet announcement. This result, together with assertions that the symmedian and Gergonne points (and others) must lie in or outside the orthocentroidal disk were made in what amount to research announcements on the Yahoo message board Hyacinthos [5] on 27th and 29th November 2004 by M. R. Stevanovic, though his results do not yet seem to be in published form. Our results were found in March 2005 though we were unaware of Stevanovic's announcement at the time.

The two Brocard points enjoy the *Brocard exclusion principle*. If triangle ABC is not isosceles, exactly one of the Brocard points is in $\mathcal{D}_{GH}$. If it is isosceles, then both Brocard points lie on the circle $\mathcal{S}_{GH}$. This last result was also announced by Stevanovic.

The fact that the (first) Fermat point must lie in the punctured disk $\mathcal{D}_{GH}$ was established by Várilly [10] who wrote … *this suggests that the neighborhood of the Euler line may harbor more secrets than was previously known*. We offer this article as a verification of this remark.

We realize that some of the formulas in the subsequent analysis are a little daunting, and we have had recourse to the use of the computer algebra system DERIVE from time to time. We have also empirically verified our geometric formulas by testing them with the CABRI geometry package; when algebraic formulas and geometric reality co-incide to 9 decimal places it gives confidence that the formulas are correct. We recommend this technique to anyone with reason to doubt the algebra.

We suggest [1], [2] and [7] for general geometric background.

2. The orthocentroidal disk

This is the interior of the circle on diameter GH and a point X lies in the disk if and only if $\angle GXH > \frac{\pi}{2}$. It will lie on the boundary if and only if $\angle GXH = \frac{\pi}{2}$. These conditions may be combined to give

$$\overline{XG} \cdot \overline{XH} \leq 0, \tag{1}$$

with equality if and only if X is on the boundary.

In what follows we initially use Cartesian vectors with origin at the circumcenter O, with $\overline{OA} = \mathbf{x}, \overline{OB} = \mathbf{y}, \overline{OC} = \mathbf{z}$ and, taking the circumcircle to have radius 1, we have

$$|\mathbf{x}| = |\mathbf{y}| = |\mathbf{z}| = 1 \tag{2}$$

and

$$\mathbf{y}\cdot\mathbf{z} = \cos 2A = \frac{a^4+b^4+c^4-2a^2(b^2+c^2)}{2b^2c^2} \tag{3}$$

with similar expressions for $\mathbf{z}\cdot\mathbf{x}$ and $\mathbf{x}\cdot\mathbf{y}$ by cyclic change of a, b and c. This follows from $\cos 2A = 2\cos^2 A - 1$ and the cosine rule.

We take X to have position vector

$$\frac{u\mathbf{x}+v\mathbf{y}+w\mathbf{z}}{u+v+w},$$

so that the unnormalised areal co-ordinates of X are simply (u,v,w). Now

$$3\overline{XG} = \frac{((v+w-2u),(w+u-2v),(u+v-2w))}{u+v+w},$$

not as areals, but as components in the $\mathbf{x},\mathbf{y},\mathbf{z}$ frame and

$$\overline{XH} = \frac{(v+w,w+u,u+v)}{u+v+w}.$$

Multiplying by $(u+v+w)^2$ we find that condition (1) becomes

$$\begin{aligned}&\sum_{\text{cyclic}}\{(v+w-2u)(v+w)\}\\ &+\sum_{\text{cyclic}}[(w+u-2v)(u+v)+(w+u)(u+v-2w)]\mathbf{y}\cdot\mathbf{z}\le 0,\end{aligned}$$

where we have used (2). The sum is taken over cyclic changes. Next, simplifying and using (3), we obtain

$$\begin{aligned}&\sum_{\text{cyclic}} 2(u^2+v^2+w^2-vw-wu-uv)(a^2b^2c^2)\\ &+\sum_{\text{cyclic}}(u^2-v^2-w^2+vw)[a^2(a^4+b^4+c^4)-2a^4(b^2+c^2)].\end{aligned}$$

Dividing by $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$ the condition that $X(u,v,w)$ lies in the disk $\mathcal{D}_{GH}$ is

$$\begin{aligned}&(b^2+c^2-a^2)u^2+(c^2+a^2-b^2)v^2+(a^2+b^2-c^2)w^2\\ &-a^2vw-b^2wu-c^2uv<0\end{aligned} \tag{4}$$

and the equation of the circular boundary is

$$\begin{aligned}S_{GH}\equiv&(b^2+c^2-a^2)x^2+(c^2+a^2-b^2)y^2+(a^2+b^2-c^2)z^2\\ &-a^2yz-b^2zx-c^2xy=0.\end{aligned} \tag{5}$$

The polar circle has equation

$$S_P\equiv(b^2+c^2-a^2)x^2+(c^2+a^2-b^2)y^2+(a^2+b^2-c^2)z^2=0.$$

The circumcircle has equation

$$S_C\equiv a^2yz+b^2zx+c^2xy=0.$$

The nine-point circle has equation

$$S_N \equiv (b^2+c^2-a^2)x^2+(c^2+a^2-b^2)y^2+(a^2+b^2-c^2)z^2$$
$$-2a^2yz-2b^2zx-2c^2xy=0.$$

Evidently $S_{GH}-S_C=S_P$ and $S_N+2S_C=S_P$. We have established the following result.

Theorem 1. *The orthocentroidal circle forms part of a coaxal system of circles including the circumcircle, the nine-point circle and the polar circle of the triangle.*

It is possible to prove the next result by calculating that $JK<OG$ directly (recall that J is the midpoint of GH), but it is easier to use the equation of the orthocentroidal circle.

Theorem 2. *The symmedian point lies in the disc* $\mathcal{D}_{GH}$.

Proof. Substituting $u=a^2, v=b^2, w=c^2$ in the left hand side of equation (4) we get $a^4b^2+b^4c^2+c^4a^2+b^4a^2+c^4b^2+a^4c^2-3a^2b^2c^2-a^6-b^6-c^6$ and this quantity is negative for all real a,b,c except $a=b=c$. This follows from the well known inequality for non-negative l,m and n that

$$l^3+m^3+n^3+3lmn \geq \sum_{sym} l^2m$$

with equality if and only if $l=m=n$. □

We offer a second proof. The line AK with areal equation $c^2y=b^2z$ meets the circumcircle of ABC at D with co-ordinates $(-a^2,2b^2,2c^2)$, with similar expressions for points E and F by cyclic change. The reflection D' of D in BC has co-ordinates $(a^2,b^2+c^2-a^2,b^2+c^2-a^2)$ with similar expressions for E' and F'. It is easy to verify that these points lie on the orthocentroidal disk by substituting in (5) (the circle through D',E' and F' is the Hagge circle of K).

Let $\mathbf{d}',\mathbf{e}'$ and $\mathbf{f}'$ denote the vector positions D',E' and F' respectively. It is clear that

$$\mathbf{s}=(2b^2+2c^2-a^2)\mathbf{d}'+(2c^2+2a^2-b^2)\mathbf{e}'+(2a^2+2b^2-c^2)\mathbf{f}'$$

but $2b^2+2c^2-a^2=b^2+c^2+2bc\cos A\geq(b-c)^2>0$ and similar results by cyclic change. Hence relative to triangle $D'E'F'$ all three areal co-ordinates of K are positive so K is in the interior of triangle $D'E'F'$ and hence inside its circumcircle. We are done.

The incenter lies in $\mathcal{D}_{GH}$. Since IGN_{a} are collinear and $IG:GN_{\mathrm{a}}=1:2$ it follows that Nagel's point is outside the disk. However, it is instructive to verify these facts by substituting relevant areal co-ordinates into equation (5), and we invite the interested reader to do so.

Theorem 3. *One Brocard point lies in* $\mathcal{D}_{GH}$ *and the other lies outside* $\mathcal{S}_{GH}$*, or they both lie simultaneously on* $\mathcal{S}_{GH}$ (which happens if and only if the reference triangle is isosceles).

Proof. Let $f(u,v,w)$ denote the left hand side of equation (4). One Brocard point has unnormalised areal co-ordinates

$$(u,v,w) = (a^2b^2, b^2c^2, c^2a^2)$$

and the other has unnormalised areal co-ordinates

$$(p,q,r) = (a^2c^2, b^2a^2, c^2b^2),$$

but they have the same denominator when normalised. It follows that $f(u,v,w)$ and $f(p,q,r)$ are proportional to the powers of the Brocard points with respect to $\mathcal{S}_{GH}$ with the same constant of proportionality. If the sum of these powers is zero we shall have established the result. This is precisely what happens when the calculation is made. □

The fact that the Fermat point lies in the orthocentroidal disk was established recently [10] by Várilly.

Theorem 4. *Gergonne's point lies in the orthocentroidal disk* $\mathcal{D}_{GH}$.

Proof. Put $u = (c+a-b)(a+b-c)$, $v = (a+b-c)(b+c-a)$, $w = (b+c-a)(c+a-b)$ and the left hand side of (5) becomes

$$-18a^2b^2c^2 + \sum_{\text{cyclic}} \left(-a^5(b+c) + 4a^4(b^2-bc+b^2) - 6b^3c^3 + 5a^3(b^2c+bc^2)^2\right)$$

which we want to show is negative. This is not immediately recognisable as a known inequality, but performing the usual trick of putting $a = m+n$, $b = n+l$, $c = l+m$ where $l,m,n > 0$ we get the required inequality (after division by 8) to be

$$2(m^3n^3 + n^3l^3 + l^3m^3) > lmn\left(\sum_{sym} m^2n\right)$$

where the final sum is over all possible permutations and l,m,n not all equal. Now $l^3(m^3+n^3) > l^3(m^2n+mn^2)$ and adding two similar inequalities we are done. Equality holds if and only if $a = b = c$, which is excluded. □

3. The determination of the triangle sides.

3.1. *The symmedian point.* We will find a cubic polynomial which has roots a^2, b^2, c^2 given the positions of O, G and K.

The idea is to express the formulas for OK^2, GK^2 and JK^2 in terms of $u = a^2+b^2+c^2$, $v^2 = a^2b^2+b^2c^2+c^2a^2$ and $w^3 = a^2b^2c^2$.

We first note some equations which are the result of routine calculations.

$$\begin{aligned}16[ABC]^2 &= (a+b+c)(b+c-a)(c+a-b)(a+b-c)\\ &= \sum_{\text{cyclic}}(2a^2b^2 - a^4) = 4v^2 - u^2.\end{aligned}$$

It is well known that the circumradius R satisfies the equation $R = \frac{abc}{4[ABC]}$ so

$$\begin{aligned} R^2 =& \frac{a^2b^2c^2}{16[ABC]^2} = \frac{w^3}{(4v^2-u^2)}. \\ OG^2 =& \frac{1}{9a^2b^2c^2}\left[\left(\sum_{\text{cyclic}} a^6\right) + 3a^2b^2c^2 - \left(\sum_{sym} a^4b^2\right)\right] R^2 \\ =& \frac{u^3+9w^3-4uv^2}{9(4v^2-u^2)} = \frac{w^3}{(4v^2-u^2)} - \frac{u}{9} = R^2 - \frac{a^2+b^2+c^2}{9}. \end{aligned}$$

By areal calculations one may obtain the formulas

$$\begin{aligned} OK^2 =& \frac{4R^2\sum_{\text{cyclic}}\left(a^4-a^2b^2\right)}{(a^2+b^2+c^2)^2} = \frac{4w^3(u^2-3v^2)}{u^2(4v^2-u^2)}, \\ GK^2 =& \frac{\left(\sum_{\text{cyclic}} 3a^4(b^2+c^2)\right) - 15a^2b^2c^2 - \left(\sum_{\text{cyclic}} a^6\right)}{(a^2+b^2+c^2)^2} = \frac{6uv^2-u^3-27w^3}{9u^2}, \\ JK^2 =& OG^2\left(1 - \frac{48[ABC]^2}{(a^2+b^2+c^2)^2}\right) = \frac{4(u^3+9w^3-4uv^2)(u^2-3v^2)}{9u^2(4v^2-u^2)}. \end{aligned}$$

The full details of the last calculation will be given when justifying (14).

Note that

$$\frac{OK^2}{JK^2} = \frac{9w^3}{(u^3+9w^3-4uv^2)}$$

or

$$\frac{JK^2}{OK^2} = 1 - \frac{u(4v^2-u^2)}{9w^3}.$$

We simplify expressions by putting $u = p$, $4v^2 - u^2 = q$ and $w^3 = r$. We have

$$OG^2 = \frac{r}{q} - \frac{p}{9}. \tag{6}$$

Now $u^2 - 3v^2 = -\frac{3}{4}(4v^2-u^2) + \frac{1}{4}u^2 = -\frac{3}{4}q + \frac{1}{4}p^2$ so

$$OK^2 = 4r\frac{(\frac{1}{4}p^2 - \frac{3}{4}q)}{p^2q} = \frac{(p^2-3q)r}{p^2q} = \frac{r}{q} - \frac{3r}{p^2} = r\left(\frac{1}{q} - \frac{3}{p^2}\right) \tag{7}$$

Also $6v^2 - u^2 = \frac{3}{2}(4v^2-u^2) + \frac{1}{2}u^2 = \frac{3q}{2} + \frac{p^2}{2}$

$$GK^2 = \frac{p(3q/2 + p^2/2) - 27r}{9p^2} = \frac{p}{18} + \frac{q}{6p} - \frac{3r}{p^2} \tag{8}$$

$$\frac{OK^2}{JK^2} = 1 - \frac{pq}{9r} \tag{9}$$

We now have four quantities that are homogeneous of degree 1 in a^2, b^2 and c^2. These are $p, q/p, r/q, r/p^2 = x, y, z, s$ respectively, where $xs = r/p = yz$. We have (6) $OG^2 = z - x/9$, (7) $OK^2 = z - 3s$, (8) $GK^2 = x + 6y - 3s$ and (9) $\frac{OK^2}{JK^2} = 1 - x/(9z)$ or $9zOK^2 = (9z-x)JK^2$. Now u, v and w are known unambiguously and hence the equations determine a^2, b^2 and c^2 and therefore a, b and c.

3.2. *The Fermat point.* We assume that O, G and the (first) Fermat point are given. Then F determines and is determined by the second Fermat point F' since they are inverse in $\mathcal{S}_{GH}$. In pages 206-208 [8] the Morleys show that O, G and F determine triangle ABC using complex numbers.

4. Filling the disk

Following [9], we fix R and r, and consider the configuration of Poncelet's porism for triangles. This diagram contains a fixed circumcircle, a fixed incircle, and a variable triangle ABC which has the given circumcircle and incircle. Moving point A towards the original point B by sliding it round the circumcircle takes us continuously through a family of triangles which are pairwise not directly similar (by angle and orientation considerations) until A reaches the original B, when the starting triangle is recovered, save that its vertices have been relabelled. Moving through triangles by sliding A to B in this fashion we call a *Poncelet cycle*.

We will show shortly that for X the Fermat, Gergonne or symmedian point, passage through a Poncelet cycle takes X round a closed path arbitrarily close to the boundary of the orthocentroidal disk scaled to have constant diameter. By choosing the neighbourhood of the boundary sufficiently small, it follows that X has winding number 1 (with suitable orientation) with respect to J as we move through a Poncelet cycle.

We will show that when r approaches $R/2$ (as we approach the equilateral configuration) a Poncelet cycle will keep X arbitrarily close to, but never reaching, J in the scaled orthocentroidal disk. Moving the ratio r/R from close to 0 to close to $1/2$ induces a homotopy between the 'large' and 'small' closed paths. So the small path also has winding number 1 with respect to J. One might think it obvious that every point in the scaled punctured disk must arise as a possible X on a closed path intermediate between a path sufficiently close to the edge and a path sufficiently close to the deletion. There are technical difficulties for those who seek them, since we have not eliminated the possibility of exotic paths. However, a rigorous argument is available via complex analysis. Embed the scaled disk in the complex plane. Let γ be an anticlockwise path (i.e. winding number $+1$) near the boundary and δ be an anticlockwise path (winding number also 1) close to the puncture. Suppose (for contradiction) that the complex number z_0 represents a point between the wide path γ and the tight path δ which is not a possible location for X.

The function defined by $1/(z - z_0)$ is meromorphic in $\mathcal{D}_{GH}$ and is analytic save for a simple pole at z_0. However by our hypothesis we have a homotopy of paths from γ to δ which does not involve z_0 being on an intermediate path. Therefore

$$1 = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} = \frac{1}{2\pi i} \int_\delta \frac{dz}{z - z_0} = 0.$$

Thus $1 = 0$ and we have the required contradiction.

5. Close to the edge

The areal coordinates of the incenter and the symmedian point of triangle ABC are (a, b, c) and (a^2, b^2, c^2) respectively. We consider the mean square distance of the vertex set to itself, weighted once by (a, b, c) and once by (a^2, b^2, c^2). Note that $a, b, c > 0$ so $\sigma_I^2, \sigma_S^2 > 0$. The GPAT [9] asserts that

$$\sigma_I^2 + KI^2 + \sigma_K^2 = 2\frac{abc}{a+b+c}\frac{ab+bc+ca}{a^2+b^2+c^2} \leq 4Rr.$$

It follows that $SI < 2\sqrt{Rr}$.

In what follows we fix R and investigate what we can achieve by choosing r to be sufficiently small.

Since I lies in the critical disk we have $OH > OI$ so

$$OH^2 > OI^2 = R^2 - 2Rr.$$

By choosing $r < R/8$ say, we force $9GJ^2 = OH^2 > 3R^2/4$ so $GJ > R\sqrt{3}/6$. Now we have

$$\frac{KI}{GJ} < \frac{2\sqrt{Rr}}{R\sqrt{3}/6} = 4\sqrt{\frac{3r}{R}}. \tag{10}$$

For any $\varepsilon > 0$, there is $K_1 > 0$ so that if $0 < r < K_1$, then $\frac{KI}{GJ} < \frac{\varepsilon}{2}$. Observe that we are dividing by GJ to scale the orthocentroidal disk so that it has fixed radius.

Recall that a passage round a Poncelet cycle induces a path for I in the scaled critical disk which is a circle of Apollonius with defining points O and N with ratio $IO : IN = 2\sqrt{\frac{R}{R-2r}}$. It is clear from the theory of Apollonius circles that there is $K_2 > 0$ such that if $0 < r < K_2$, then $1 - \frac{IJ}{GJ} < \varepsilon/2$.

Now choosing r such that $0 < r < \min\{R/8, K_1, K_2\}$ we have

$$1 - \frac{IJ}{GJ} < \frac{\varepsilon}{2} \tag{11}$$

and $SI/GJ < \frac{\varepsilon}{2}$ so by the triangle inequality

$$\frac{IJ}{GJ} < \frac{KJ}{GJ} + \frac{\varepsilon}{2}. \tag{12}$$

Adding equations (11) and (12) and rearranging we deduce that

$$1 - \varepsilon < \frac{KJ}{GJ}.$$

This shows that for sufficiently small r, the path of K in the scaled critical disk (as the triangle moves through a Poncelet cycle) will be confined to a region at most ε from the boundary. Moreover, assuming that $\varepsilon < 1$ the winding number of K about J will increase by 1, because that it is what happens to I, and J moves in proximity to I.

A similar result holds for the Gergonne point G_{e}. This is the intersection of the Cevians joining triangle vertices to the opposite contact point of the incircle. The Gergonne point must therefore be inside the incircle.

As before we consider the case that R is fixed. Now $G_{\rm e}I < r$. We proceed as in the argument for the symmedian point. We get a new version of equation (10) which is

$$\frac{G_{\rm e}I}{GJ} < \frac{r}{R\sqrt{3}/6} = \frac{2r\sqrt{3}}{R} \tag{13}$$

For any $\varepsilon > 0$, there is a possibly new $K_1 > 0$ so that if $0 < r < K_1$, then $\frac{G_{\rm e}I}{GJ} < \frac{\varepsilon}{2}$. The rest of the argument proceeds unchanged.

6. Near the orthocentroidal center

6.1. *The symmedian point.* For the purposes of the following calculation only, we will normalize so that $R = 1$. We have

$$\overline{OK} = \frac{a^2\mathbf{x} + b^2\mathbf{y} + c^2\mathbf{z}}{a^2+b^2+c^2}$$

so

$$\begin{aligned}\overline{KJ} = &\sum_{\text{cyclic}}\left(\frac{2}{3} - \frac{a^2}{a^2+b^2+c^2}\right)\mathbf{x} = \frac{\sum_{\text{cyclic}}(2b^2+2c^2-a^2)\mathbf{x}}{3(a^2+b^2+c^2)}\\ =&l\mathbf{x} + m\mathbf{y} + n\mathbf{z}\end{aligned}$$

where l, m and n can be read off.

We have

$$\begin{aligned}a^2b^2c^2KJ^2 =&a^2b^2c^2\left(l^2+m^2+n^2+\sum_{\text{cyclic}}2mn\mathbf{y}\cdot\mathbf{z}\right)\\ =&(l^2+m^2+n^2)(a^2b^2c^2) + \sum_{\text{cyclic}} mn(a^2(a^4+b^4+c^4) - 2a^4(b^2+c^2))\\ =&\frac{4P_{10}}{9(a^2+b^2+c^2)^2}\end{aligned}$$

where

$$P_{10} = \sum_{\text{cyclic}} a^{10} - 2a^8(b^2+c^2) + a^6(b^4+4b^2c^2+c^4) - 3a^4b^4c^2.$$

Now

$$OG^2 = \frac{1}{9a^2b^2c^2}\left[\left(\sum_{\text{cyclic}} a^6\right) + 3a^2b^2c^2 - \left(\sum_{sym} a^4b^2)\right)\right]$$

so we define Q_6 by

$$OG^2 = \frac{Q_6}{9a^2b^2c^2}.$$

We have $9a^2b^2c^2OG^2 = Q_6$ and

$$9a^2b^2c^2KJ^2 = \frac{4P_{10}}{(a^2+b^2+c^2)^2}$$

so that

$$\frac{OG^2}{KJ^2} = \frac{Q_6(a^2+b^2+c^2)^2}{4P_{10}}.$$

Now a computer algebra (DERIVE) aided calculation reveals that

$$\frac{Q_6(a^2+b^2+c^2) - 4P_{10}}{3(a+b-c)(b+c-a)(c+a-b)(a+b+c)} = Q_6$$

It follows that

$$\frac{KJ^2}{OG^2} = 1 - \frac{48[ABC]^2}{(a^2+b^2+c^2)^2}. \tag{14}$$

Our convenient simplification that $R = 1$ can now be dropped, since the ratio on the left hand side of (14) is dimensionless. As the triangle approaches the equilateral, KJ/OG approaches 0. Therefore in the orthocentroidal disk scaled to have diameter 1, the symmedian point approaches the center J of the circle.

6.2. *The Gergonne point.* Fix the circumcircle of a variable triangle ABC. We consider that the case that r approaches $R/2$, so the triangle ABC approaches (but does not reach) the equilateral. Drop a perpendicular ID to BC.

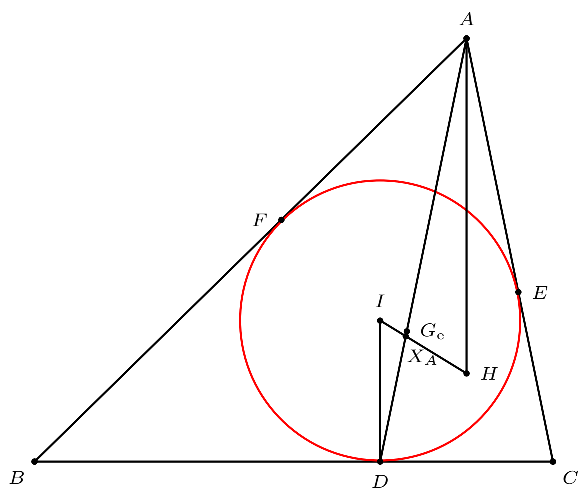


Figure 1

Let AD meet IH at X_A. Triangles IDX_A and HAX_A are similar. When r approaches $R/2$, H approaches O so HA approaches $OA = R$. It follows that $IX_A : X_AH$ approaches $1:2$. Similar results hold for corresponding points X_B and X_C.

If we rescale so that points O, G and H are fixed, the points X_A, X_B and X_C all converge to a point X on IH such that $IX : XH = 1 : 2$. Consider the three rays AX_A, BX_B and CX_C which meet at the Gergonne point of the triangle. As ABC approaches the equilateral, these three rays become more and more like the diagonals of a regular hexagon. In particular, if the points X_A, X_B and X_C arise

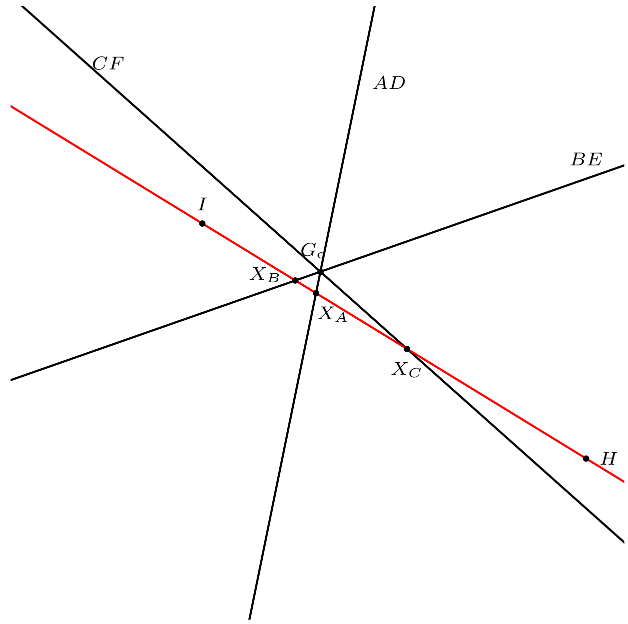


Figure 2

(without loss of generality) in that order on the directed line IH, then $\angle X_A G_e X_C$ is approaching $2\pi/3$, and so we may take this angle to be obtuse. Therefore G_e is inside the circle on diameter $X_A X_C$. Thus in the scaled diagram G_e approaches X, but I approaches N, so G_e approaches the point J which divides NH internally in the ratio $1:2$. Thus G_e converges to J, the center of the orthocentroidal circle.

Thus the symmedian and Gergonne points are confined to the orthocentroidal disk, make tight loops around its center J, as well as wide passages arbitrarily near its boundary (as moons of I). Neither of them can be at J for non-equilateral triangles (an easy exercise). By continuity we have proved the following result.

Theorem 5. *Each of the Gergonne and symmedian points are confined to, and range freely over the orthocentroidal disk punctured at its center.*

6.3. *The Fermat point.* An analysis of areal co-ordinates reveals that the Fermat point F lies on the line JK between J and K, and

$$\frac{JF}{FK} = \frac{a^2+b^2+c^2}{4\sqrt{3}[ABC]}. \tag{15}$$

From this it follows that as we approach the equilateral limit, F approaches the midpoint of JK. However we have shown that K approaches J in the scaled diagram, so F approaches J.

As r approaches 0 with R fixed, the area $[ABC]$ approaches 0, so in the scaled diagram F can be made arbitrarily close to K (uniformly). It follows that F performs closed paths arbitrarily close to the boundary.

Here is an outline of the areal algebra. The first normalized areal co-ordinate of H is
$$\frac{(c^2+a^2-b^2)(a^2+b^2-c^2)}{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}$$
and the other co-ordinates are obtained by cyclic changes. We suppress this remark in the rest of our explanation. Since J is the midpoint of GH the first areal co-ordinate of J is
$$\frac{a^4-2b^4-2c^4+b^2c^2+c^2a^2+a^2c^2}{3(a+b+c)(b+c-a)(c+a-b)(a+b-c)}.$$
One can now calculate the areal equation of the line JK as
$$\sum_{\text{cyclic}}(b^2-c^2)(a^2-b^2-bc-c^2)(a^2-b^2+bc-c^2)x=0$$

To calculate the areal co-ordinates of F we first assume that the reference triangle has each angle less that $2\pi/3$. In this case the rays AF, BF and CF meet at equal angles, and one can use trigonometry to obtain a formula for the areal co-ordinates which, when expressed in terms of the reference triangle sides, turns out to be correct for arbitrary triangles. One can either invoke the charlatan's *principle of permanence of algebraic form*, or analyze what happens when a reference angle exceeds $2\pi/3$. In the latter event, the trigonometry involves a sign change dependent on the region in which F lies, but the final formula for the co-ordinates remains unchanged. In such a case, of course, not all of the areal co-ordinates are positive.

The unnormalized first areal co-ordinate of F turns out to be
$$8\sqrt{3}a^2[ABC]+2a^4-4b^4-4c^4+2a^2b^2+2a^2c^2+8b^2c^2.$$

The first areal component K_x of K is
$$\frac{a^2}{a^2+b^2+c^2}$$
and the first component J_x of J is
$$\frac{a^4-2b^4-2c^4+a^2b^2+a^2c^2+4b^2c^2}{48[ABC]^2}.$$
Hence the first co-ordinate of F is proportional to
$$8\sqrt{3}K_x[ABC](a^2+b^2+c^2)+96J_x[ABC]^2.$$
This is linear in J_x and K_x, and the other co-ordinates are obtained by cyclic change. It follows that J, F, K are collinear (as is well known) but also that by the section theorem, JF/FK is given by (15).

The Fermat point cannot be at J in a non-equilateral triangle because the second Fermat point is inverse to the first in the orthocentroidal circle.

We have therefore established the following result.

Theorem 6. *Fermat's point is confined to, and ranges freely over the orthocentroidal disk punctured at its center and the second Fermat point ranges freely over the region external to $\mathcal{S}_{GH}$.*

7. The Feuerbach point

Let F_e denote the Feuerbach point.

Theorem 7. *The point F_e is always outside the orthocentroidal circle.*

Proof. Let J be the center of the orthocentroidal circle, and N be its nine-point center. In [9] it was established that

$$IJ^2 = OG^2 - \frac{2r}{3}(R - 2r).$$

We have $IN = R/2 - r$, $IF_e = r$ and $JN = OG/2$. We may apply Stewart's theorem to triangle JF_eN with Cevian JI to obtain

$$JF_e^2 = OG^2 \left(\frac{2R - r}{2R - 4r} \right) - \frac{rR}{6}. \tag{16}$$

This leaves the issue in doubt so we press on.

$$JF_e^2 = OG^2 + OG^2 \left(\frac{3r}{2R - 4r} \right) - \frac{rR}{6}.$$

Now I must lie in the orthocentroidal disk so $IO/3 < OG$ and therefore

$$JF_e^2 > OG^2 + \frac{3rR(R - 2r)}{18(R - 2r)} - \frac{rR}{6} = OG^2.$$

□

Corollary 8. *The positions of I and F_e reveal that the interior of the incircle intersects both $\mathcal{D}_{GH}$ and the region external to $\mathcal{S}_{GH}$ non-trivially.*

Acknowledgements. We thank J. F. Toland for an illuminating conversation about the use of complex analysis in homotopy arguments.

References

[1] C. J. Bradley *Challenges in Geometry*, OUP, 2005.
[2] H. S. M. Coxeter and S. L. Greitzer *Geometry Revisited*, Math. Assoc. America, 1967.
[3] L. Euler, Solutio facili problematum quorundam geometricorum difficillimorum, *Novi Comm. Acad. Scie. Petropolitanae* 11 (1765); reprinted in *Opera omnia*a, serie prima, Vol. 26 (ed. by A. Speiser), (n.325) 139-157.
[4] A. P. Guinand, Tritangent centers and their triangles *Amer. Math. Monthly*, 91 (1984) 290–300.
[5] http://www.groups.yahoo.com/group/Hyacinthos.
[6] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[7] T. Lalesco, *La Géométrie du Triangle*, Jacques Gabay, Paris 2e édition, reprinted 1987.
[8] F. Morley & F. V. Morley *Inversive Geometry* Chelsea, New York 1954.
[9] G. C. Smith, Statics and the moduli space of triangles, *Forum Geom.*, 5 (2005) 181–190.
[10] A. Várilly, Location of incenters and Fermat points in variable triangles, *Math. Mag.*, 7 (2001) 12–129.

Christopher J. Bradley: c/o Geoff C. Smith: Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, England

Geoff C. Smith: Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, England

E-mail address: `G.C.Smith@bath.ac.uk`

Forum Geometricorum
Volume 6 (2006) 71–77.

FORUM GEOM
ISSN 1534-1178

The Locations of the Brocard Points

Christopher J. Bradley and Geoff C. Smith

Abstract. We fix and scale to constant size the orthocentroidal disk of a variable non-equilateral triangle ABC. We show that the set of points of the plane which can be either type of Brocard point consists of the interior of the orthocentroidal disk. We give the locus of points which can arise as a Brocard point of specified type, and describe this region and its boundary in polar terms. We show that ABC is determined by the locations of the circumcenter, the centroid and the Brocard points. In some circumstances the location of one Brocard point will suffice.

1. Introduction

For geometric background we refer the reader to [1], [3] and [4]. In [2] and [5] we demonstrated that scaling the orthocentroidal circle (on diameter GH) to have fixed diameter and studying where other major triangle centers can lie relative to this circle is a fruitful exercise. We now address the Brocard points. We consider non-equilateral triangles ABC. We will have occasion to use polar co-ordinates with origin the circumcenter O. We use the Euler line as the reference ray, with OG of length 1. We will describe points, curves and regions by means of polar co-ordinates (r, θ). To fix ideas, the equation of the orthocentroidal circle with center $J = (2, 0)$ and radius 1 is

$$r^2 - 4r\cos\theta + 3 = 0. \tag{1}$$

This circle is enclosed by the curve defined by

$$r^2 - 2r(\cos\theta + 1) + 3 = 0. \tag{2}$$

Let Γ_1 denote the region enclosed by the closed curve defined by (2) for $\theta > 0$ and (1) for $\theta \leq 0$. Include the boundary when using (2) but exclude it when using (1). Delete the unique point Z in the interior which renders GJZ equilateral. (See Figure 1). Let Γ_2 be the reflection of Γ_1 in the Euler line. Let $\Gamma = \Gamma_1 \cup \Gamma_2$ so Γ consists of the set of points inside or on (2) for any θ, save that G and H are removed from the boundary and two points are deleted from the interior (the points Z such that GJZ is equilateral). It is easy to verify that if the points $(\sqrt{3}, \pm\pi/6)$ are restored to Γ, then it becomes convex, as do each of Γ_1 and Γ_2 if their deleted points are filled in.

2. The main theorem

Theorem. (a) *One Brocard point ranges freely over, and is confined to,* Γ_1*, and the other ranges freely over, and is confined to,* Γ_2.
(b) *The set of points which can be occupied by a Brocard point is* Γ.

Publication Date: February 28, 2006. Communicating Editor: Paul Yiu.

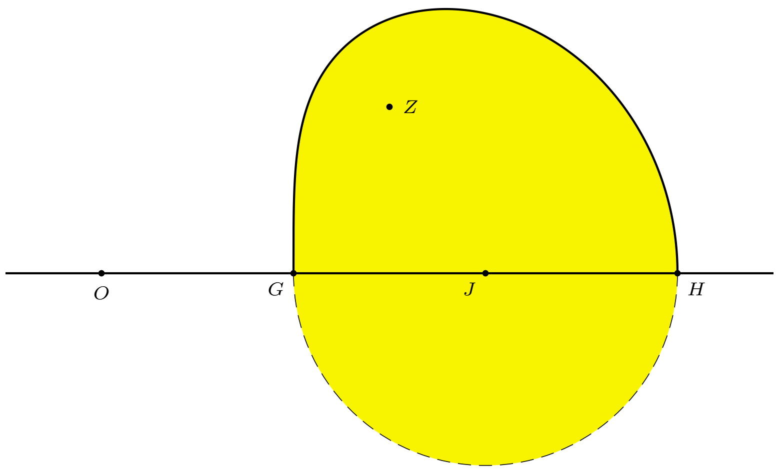


Figure 1

(c) *The points which can be inhabited by either Brocard point form the open orthocentroidal disk.*
(d) *The data consisting of O, G and one of the following items determines the sides of triangle ABC and which of the (generically) two possible orientations it has.*
(1) *the locations of the Brocard points without specifying which is which*;
(2) *the location of one Brocard point of specified type provided that it lies in the orthocentroidal disk*;
(3) *the location of one Brocard point of unspecified type outside the closed orthocentroidal disk together with the information that the other Brocard point lies on the same side or the other side of the Euler line*;
(4) *the location of a Brocard point of unspecified type on the orthocentroidal circle*.

Proof. First we gather some useful information. In [2] we established that

$$\frac{JK^2}{OG^2} = 1 - \frac{48[ABC]^2}{(a^2+b^2+c^2)^2} \tag{3}$$

where $[ABC]$ denotes the area of this triangle. It is well known [6] that

$$\cot\omega = \frac{(a^2+b^2+c^2)}{4[ABC]} \tag{4}$$

so

$$\frac{JK^2}{OG^2} = 1 - 3\tan^2\omega. \tag{5}$$

The sum of the powers of Ω and Ω' with respect to the orthocentroidal circle (with diameter GH) is 0. This result can be obtained by substituting the areal co-ordinates of these points into the areal equation of this circle [2].

$$J\Omega^2 + J\Omega'^2 = 2OG^2. \tag{6}$$

We can immediately conclude that $J\Omega$, $J\Omega' \leq OG\sqrt{2}$.

Now start to address the loci of the Brocard points. Observe that if we specify the locations of O, G and the symmedian point K (at a point within the orthocentroidal circle), then a triangle exists which gives rise to this configuration and its sides are determined [2]. In the subsequent discussion the points O and G will be fixed, and we will be able to conjure up triangles ABC with convenient properties by specifying the location of K. The angle α is just the directed angle $\angle KOG$ and ω can be read off from $JK^2/OG^2 = 1 - 3\tan\omega^2$.

We work with a non-equilateral triangle ABC. We adopt the convention, which seems to have majority support, that when standing at O and viewing K, the point Ω is diagonally to the left and Ω' diagonally to the right.

The Brocard or seven-point circle has diameter OK where K is the symmedian point, and the Brocard points are on this circle, and are mutual refections in the Brocard axis OK. It is well known that the Brocard angle manifests itself as

$$\omega = \angle KO\Omega = \angle KO\Omega'. \tag{7}$$

As the non-equilateral triangle ABC varies, we scale distances so that the length OG is 1 and we rotate as necessary so that the reference ray OG points in a fixed direction. Now let K be at an arbitrary point of the orthocentroidal disk with J deleted. Let $\angle KOJ = \alpha$, so $\angle JO\Omega' = \omega - \alpha$. Viewed as a directed angle the argument of Y in polar terms would be $\angle\Omega' OJ = \alpha - \omega$.

The positions of the Brocard points are determined by (5), (7) and the fact that they are on the Brocard circle. Let $r = O\Omega = O\Omega'$. By the cosine rule

$$J\Omega^2 = 4 + r^2 - 4r\cos(\omega + \alpha) \tag{8}$$

and

$$J\Omega'^2 = 4 + r^2 - 4r\cos(\omega - \alpha) \tag{9}$$

Now add equations (8) and (9) and use (6) so that

$$2OG^2 = 8 + 2r^2 - 8r\cos\omega\cos\alpha.$$

Recalling that the length OG is 1 we obtain

$$r^2 - 4r\cos\omega\cos\alpha + 3 = 0. \tag{10}$$

We focus on the Brocard point Ω with polar co-ordinates $(r, \alpha + \omega)$. The other Brocard point Ω' has co-ordinates $(r, \alpha - \omega)$, but reflection symmetry in the Euler line means that we need not study the region inhabited by Ω' separately.

Consider the possible locations of Ω for a specified $\alpha + \omega$. From (10) we see that its distance from the origin ranges over the interval

$$2\cos\omega\cos\alpha \pm \sqrt{4\cos^2\omega\cos^2\alpha - 3}.$$

This can be written

$$\cos(\alpha + \omega) + \cos(\alpha - \omega) \pm \sqrt{((\cos(\alpha + \omega) + \cos(\alpha - \omega))^2 - 3}. \tag{11}$$

Next suppose that $\alpha > 0$. This expression (11) is maximized when $\alpha = \omega$ and we use the plus sign. Since the product of the roots is 3, we see that the minimum distance also occurs when $\alpha = \omega$ and we use the minus sign.

Let $\theta = \angle \Omega OG$ for Ω on the boundary of the region under discussion, so $\theta = \omega + \alpha = 2\alpha = 2\omega$. Thus locus of the boundary when $\theta > 0$ is given by (10). Using standard trigonometric relations this transforms to

$$r^2 - 2r(\cos\theta + 1) + 3 = 0$$

for $\theta > 0$. This is equation (2). Note that it follows that Ω is on the boundary precisely when Ω' is on the Euler line because the argument of Ω' is $\alpha - \omega$.

Next suppose that $\alpha = 0$. Note that K cannot occupy J. When K is on the Euler line (so ABC is isosceles) equation (10) ensures that Ω is on the orthocentroidal circle. Also $0 < \omega < \pi/6$. Thus the points $(1, 0)$, $(\sqrt{3}, \pi/6)$ and $(3, 0)$ do not arise as possible locations for Ω. The endpoints of the GH interval are on the edge of our region. The more interesting exclusion is that of $(\sqrt{3}, \pi/6)$. We say that this is a *forbidden point* of Ω.

Now suppose that $\alpha < 0$. This time equation (11) tells another story. The expression is maximized (and minimized as before) when $\omega = 0$ which is illegal. In this region the boundary is not attained, and the point Ω' free to range on the axis side of the curve defined by

$$r^2 - 4r\cos\theta + 3 = 0$$

for $\theta = \alpha < 0$. Notice that this is the equation of the boundary of the orthocentroidal circle (1). The reflection of this last curve gives the unattained boundary of Ω' when $\theta > 0$, but it is easy to check that $r^2 - 2(\cos\theta + 1)r + 3 = 0$ encloses the relevant orthocentroidal semicircle and so is the envelope of the places which may be occupied by at least one Brocard point.

Moreover, our construction ensures that every point in Γ_1 arises a possible location for Ω.

We have proved (a). Then (b) follows by symmetry, and (c) is a formality.

Finally we address (d). Suppose that we are given O, G and a Brocard point X. Brocard points come in two flavours. If X is outside the open orthocentroidal disk on the side $\theta > 0$, then the Brocard point must be Ω, and if $\theta < 0$, then it must be Ω'. If X is in the disk, we need to be told which it is. Suppose without loss of generality we know the Brocard point is Ω. We know $J\Omega$ so from (6) we know $J\Omega'$. We also know $O\Omega' = O\Omega$, so by intersecting two circles we determine two candidates for the location of Ω'. Now, if Ω is outside the closed orthocentroidal disk then the two candidates for the location of Ω' are both inside the open orthocentroidal disk and we are stuck unless we know on which side of the Euler line Ω' can be found. If Ω is on the orthocentroidal circle then ABC is isosceles and the position of Ω' is known. If Ω is in the open orthocentroidal disk then only one of the two candidate positions for Ω' lies inside the set of points over which Ω' may range so the location of Ω' is determined. Now the Brocard circle and O are determined, so the antipodal point K to O is known. However, in [2] we showed that the triangle sides and its orientation may be recovered from O, G and K. We are done. $\square$

In Figure 2, we illustrate the loci of the Brocard points for triangles with various Brocard angles ω. These are enveloped by the curve (2). The region Γ_1 is shaded.

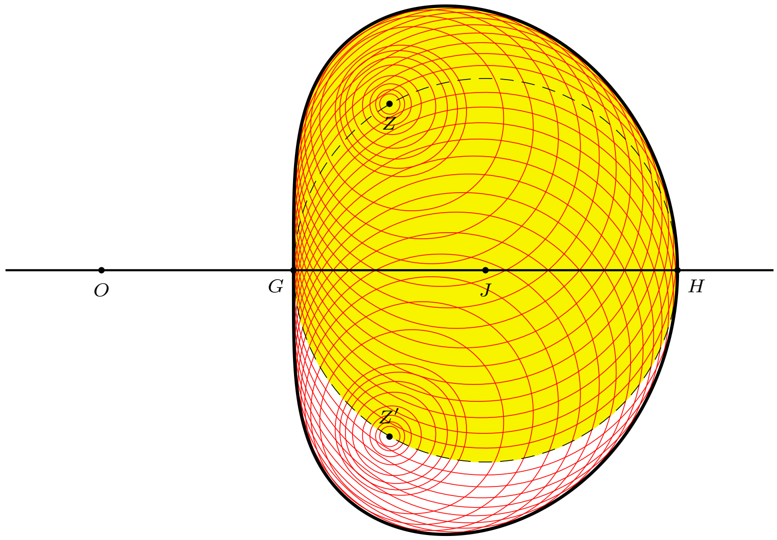


Figure 2

3. A qualitative description

We present an informal and loose qualitative description of the movements of Ω and Ω' as we steer K around the orthocentroidal disk. First consider K near G, with small positive argument. Both Brocard points are close to K, Ω just outside and Ω' just inside orthocentroidal circle. Now let K make one orbit, starting with positive arguments, just inside the circle. All the time ω stays small, and the two Brocard points nestle close to K in roughly the same configuration until K passes H at which point the Brocard points change roles; Ω' dives inside the circle and Ω moves outside. Though their paths cross, the Brocard points do not actually meet of course. For the second half of the passage of K just inside the circle it is Ω which is just inside the circle and Ω' which is just outside. When reaching the Euler line near G, the Brocard points park symmetrically on the circle with Ω having positive argument.

Now move K along the Euler line towards J; the Brocard points move round the circle, mutual reflections in the Euler line and Ω has positive argument. Triangle ABC is isosceles. As K approaches J each Brocard point approaches its forbidden point. Let K make a small swerve round J to rejoin the Euler line on the other side. Suppose that the swerve is on the side $\theta > 0$. In this case Ω swerves round its forbidden point outside the circle, and Ω' swerves inside, both points rejoining the circle almost immediately. Now K sails along the Euler line and the three points come close again together as K approaches H.

Next let K be at an arbitrary legal position on the Euler line. We fix OK and increase the argument of K. Both Brocard points also move in the same general direction; Ω leaves the orthocentroidal disk and heads towards the boundary, and

Ω' chases K. When K reaches a certain critical point, Ω reaches the boundary and at the same instant, Ω' crosses the Euler line. Now K keeps moving towards Ω followed by Ω', but Ω reverses direction and plunges back towards K. The three points come close together as K approaches the unattainable circle boundary. The process will unwrap as K reverses direction until it arrives back on the Euler line.

Another interesting tour which K can take is to move with positive arguments and starting near G along the path defined by the following equation:

$$r^2 - 4r\cos\theta + 3 + 3\tan^2\theta = 0 \tag{12}$$

This is the path of critical values which has Ω moving on the boundary of Γ_1 and Ω' on the Euler line. Now $S\Omega'O$ is a right angle so in this particular sweep the position of Ω' is the perpendicular projection on the Euler line of the position of K. We derive (12) as follows. Take (2) and express θ in terms of $\theta/2$ and multiply through by $\sec^2\frac{\theta}{2}$. Now $OK = r\sec\frac{\theta}{2}$. Relabel by replacing $r\sec\frac{\theta}{2}$ by r and then replacing the remaining occurence of $\frac{\theta}{2}$ by θ.

A final journey of note for K is obtained by fixing the Brocard angle ω. Then K is free to range over a circle with center J and radius KJ where $KJ^2 = 1 - 3\tan^2\omega$ because of (3). The direct similarity type of triangles $OK\Omega$ and $OK\Omega'$ will not change, so Ω and Ω' will each move round circles. As K takes this circular tour through the moduli space of directed similarity types of triangle, we make the same journey through triangles as when a triangle vertex takes a trip round a Neuberg circle.

4. Discussion

We can obtain an areal equation of the boundary of Γ using the fact that one Brocard point is on the boundary of Γ exactly when the other is on the Euler line. The description is therefore the union of two curves, but the points G and H must be removed by special fiat.

The equation of the Euler line is

$$(b^2+c^2-a^2)(b^2-c^2)x+(c^2+a^2-b^2)(c^2-a^2)y+(a^2+b^2-c^2)((a^2-c^2)z = 0.$$

The Brocard point (a^2b^2, b^2c^2, c^2a^2) lies on the Euler line if and only if

$$a^6c^2 + b^6a^2 + c^6b^2 = a^4b^4 + b^4c^4 + c^4a^4.$$

The locus of the other Brocard point $x = c^2a^2, y = a^2b^2, z = b^2c^2$ is then given by

$$x^3y^2 + y^3z^2 + z^3x^2 = xyz(x^2 + y^2 + z^2).$$

To get the other half of the boundary we must exchange the roles of Ω and Ω' and this yields

$$x^3z^2 + y^3x^2 + z^3y^2 = xyz(x^2 + y^2 + z^2)$$

so the locus is a quintic in areal co-ordinates.

Equation (10) exhibits an intriguing symmetry between $\alpha = \angle KOG$ and $\omega = \angle\Omega OS$ which we will now explain. Suppose that we are given the location Y of a Brocard point within the orthocentroidal circle, but not the information as to whether the Brocard point is Ω or Ω'. If this Brocard point is Ω, we call the location of the other Brocard point Ω_1 and the corresponding symmedian point K_1. On the other hand if the Brocard point at Y is Ω' we call the location of the other Brocard point Ω_2 and the corresponding symmedian point K_2. Let the respective Brocard angles be ω_1 and ω_2.

We have two Brocard circles, so $\angle K_2YO = \angle K_1YO = \pi/2$ and therefore Y lies on the line segment K_1K_2. Using equation (6) we conclude that the lengths $J\Omega_1$ and $J\Omega_2$ are equal. Also $O\Omega_1 = OY = O\Omega_2$. Therefore Ω_1 and Ω_2 are mutual reflections in the Euler line. Now

$$\angle\Omega_2O\Omega_1 = \angle\Omega_2OG + \angle GO\Omega_1 = 2\omega_1 + 2\omega_2.$$

However, the Euler line is the bisector of $\angle\Omega_2OG$ so $\angle K_2OG = \omega_1$ and $\angle GOK_1 = \omega_2$. Thus in the "ω, α" description of Ω_2 which follows from equation (10), we have $\omega = \omega_2$ and $\alpha = \omega_1$. However, exchanging the roles of left and right in the whole discussion (the way in which we have discriminated between the first and second Brocard points), the resulting description of Ω_1 would have $\omega = \omega_1$ and $\alpha = \omega_2$. The symmetry in (10) is explained.

There are a pair of triangles determines by by the quadruple

$$(O, G, \Omega, \Omega')$$

using the values (O, G, Ω_2, Y) and (O, G, Y, Ω_1) which are linked via their common Brocard point in the orthocentroidal disk. We anticipate that there may be interesting geometrical relationships between these pairs of non-isosceles triangles.

References

[1] C. J. Bradley *Challenges in Geometry*, OUP, 2005.
[2] C. J. Bradley and G. C. Smith The locations of triangle centers, *Forum Geom.*, 6 (2006) 57–70.
[3] H. S. M. Coxeter and S. L. Greitzer *Geometry Revisited*, Math. Assoc. America, 1967.
[4] T. Lalesco, *La Géométrie du Triangle*, Jacques Gabay, Paris 2e édition, reprinted 1987.
[5] G. C. Smith, Statics and the moduli space of triangles, *Forum Geom.*, 5 (2005) 181–190.
[6] E. W. Weisstein, "Brocard Points." From MathWorld–A Wolfram Web Resource.
`http://mathworld.wolfram.com/BrocardPoints.html`

Christopher J. Bradley: c/o Geoff C. Smith: Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, England

Geoff C. Smith: Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, England

E-mail address: `G.C.Smith@bath.ac.uk`

Forum Geometricorum
Volume 6 (2006) 79–96.

FORUM GEOM
ISSN 1534-1178

Archimedean Adventures

Floor van Lamoen

Abstract. We explore the arbelos to find more Archimedean circles and several infinite families of Archimedean circles. To define these families we study another shape introduced by Archimedes, the *salinon*.

1. Preliminaries

We consider an arbelos, consisting of three semicircles (O_1), (O_2) and (O), having radii r_1, r_2 and $r = r_1 + r_2$, mutually tangent in the points A, B and C as shown in Figure 1 below.

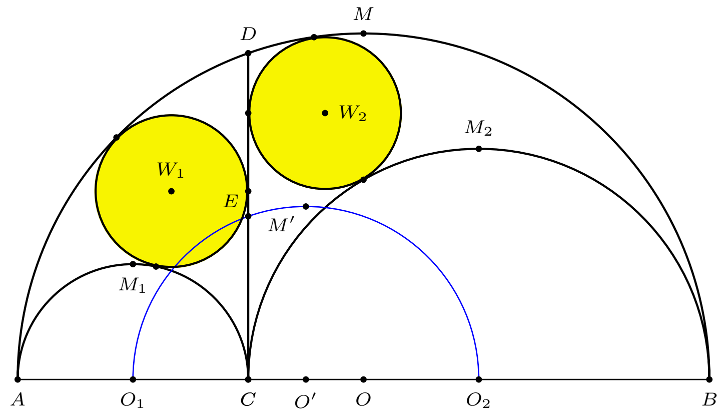


Figure 1. The arbelos with its Archimedean circles

It is well known that Archimedes has shown that the circles tangent to (O), (O_1) and CD and to (O), (O_2) and CD respectively are congruent. Their radius is $r_A = \frac{r_1 r_2}{r}$. See Figure 1. Thanks to [1, 2, 3] we know that these Archimedean twins are not only twins, but that there are many more Archimedean circles to be found in surprisingly beautiful ways. In their overview [4] Dodge et al have expanded the collection of Archimedean circles to huge proportions, 29 individual circles and an infinite family of Woo circles with centers on the Schoch line. Okumura and Watanabe [6] have added a family to the collection, which all pass through O, and have given new characterizations of the circles of Schoch and Woo.[1] Recently Power has added four Archimedean circles in a short note [7]. In this paper

Publication Date: March, 2006. Communicating Editor: Paul Yiu.

[1]The circles of Schoch and Woo often make use of tangent lines. The use of tangent lines in the arbelos, being a curvilinear triangle, may seem surprising, its relevance is immediately apparent when we realize that the common tangent of (O_1), (O_2), (O') and $C(2r_A)$ ($\mathcal{E}$ in [6]) and the common tangent of (O), (W_{21}) and the incircle (O_3) of the arbelos meet on AB.

we introduce some new Archimedean circles and some in infinite families. The Archimedean circles (W_n) are those appearing in [4]. New ones are labeled (K_n).

We adopt the following notations.

$P(r)$	circle with center P and radius r
$P(Q)$	circle with center P and passing through Q
(P)	circle with center P and radius clear from context
(PQ)	circle with diameter PQ
(PQR)	circle through P, Q, R

In the context of arbeloi, these are often interpreted as semicircles. Thus, an arbelos consists of three semicircles $(O_1) = (AC)$, $(O_2) = (CB)$, and $(O) = (AB)$ on the same side of the line AB, of radii r_1, r_2, and $r = r_1+r_2$. The common tangent at C to (O_1) and (O_2) meets (O) in D. We shall call the semicircle $(O') = (O_1O_2)$ (on the same side of AB) the *midway semicircle* of the arbelos.

It is convenient to introduce a cartesian coordinate system, with O as origin. Here are the coordinates of some basic points associated with the arbelos.

$A\ (-(r_1+r_2), 0)$	$B\ (r_1+r_2, 0)$	$C\ (r_1-r_2, 0)$
$O_1\ (-r_2, 0)$	$O_2\ (r_1, 0)$	
$M_1\ (-r_2,\ r_1)$	$M_2\ (r_1,\ r_2)$	$M\ (0, r_1+r_2)$
$O'\ \left(\frac{r_1-r_2}{2}, 0\right)$	$M'\ \left(\frac{r_1-r_2}{2},\ \frac{r_1+r_2}{2}\right)$	
$D\ (r_1-r_2,\ 2\sqrt{r_1r_2})$	$E\ (r_1-r_2,\ \sqrt{r_1r_2})$	

2. New Archimedean circles

2.1. (K_1) *and* (K_2). The Archimedean circle (W_8) has C as its center and is tangent to the tangents O_1K_2 and O_2K_1 to (O_1) and (O_2) respectively. By symmetry it is easy to find from these the Archimedean circles (K_1) and (K_2) tangent to CD, see Figure 2. A second characterization of the points K_1 and K_2, clearly equivalent, is that these are the points of intersection of (O_1) and (O_2) with (O'). We note that the tangent to $A(C)$ at the point of intersection of $A(C)$, (W_{13}) and (O) passes through B, and is also tangent to (K_1). A similar statement is true for the tangent to $B(C)$ at its intersection with (W_{14}) and (O).

To prove the correctness, let T be the perpendicular foot of K_1 on AB. Then, making use of the right triangle $O_1O_2K_1$, we find $O_1T = \frac{r_1^2}{r_1+r_2}$ and thus the radius of (K_1) is equal to $O_1C - O_1T = r_A$. By symmetry the radius of (K_2) equals r_A as well.

The circles (K_1) and (K_2) are closely related to (W_{13}) and (W_{14}), which are found by intersecting $A(C)$ and $B(C)$ with (O) and then taking the smallest circles through the respective points of intersection tangent to CD. The relation is clear when we realize that $A(C)$, $B(C)$ and (O) can be found by a homothety with center C, factor 2 applied to (O_1), (O_2) and (O').

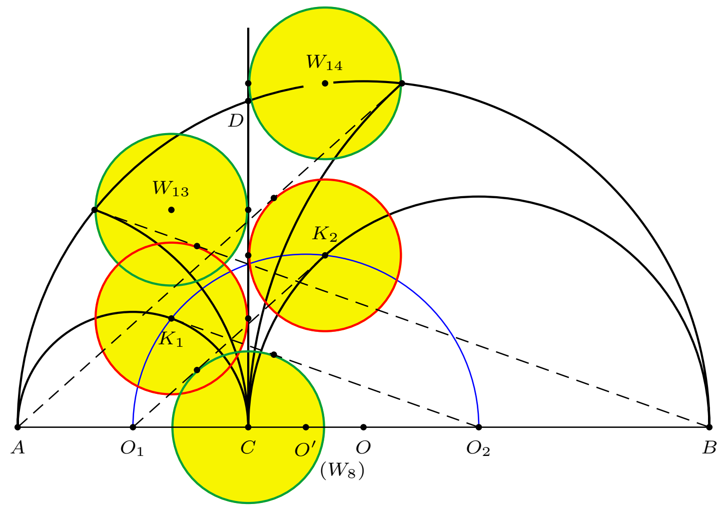


Figure 2. The Archimedean circles (K_1) and (K_2)

2.2. (K_3). In private correspondence [10] about (K_1) and (K_2), Paul Yiu has noted that the circle with center on CD tangent to (O) and (O') is Archimedean. Indeed, if x is the radius of this circle, then we have

$$(r-x)^2-(r_1-r_2)^2=\left(\frac{r}{2}+x\right)^2-\left(\frac{r_1-r_2}{2}\right)^2$$

and that yields $x = r_A$.

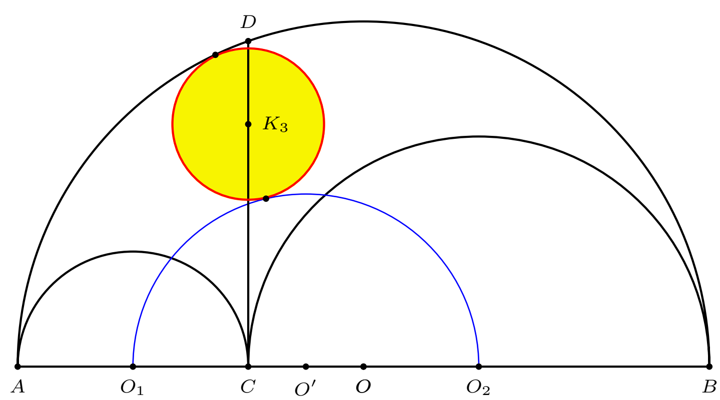


Figure 3. The Archimedean circle (K_3)

2.3. (K_4) *and* (K_5). There is an interesting similarity between (K_3) and Bankoff's triplet circle (W_3). Recall that this is the circle that passes through C and the points of tangency of (O_1) and (O_2) with the incircle (O_3) of the arbelos.[2] Just like (K_3), (W_3) has its center on CD and is tangent to (O'), a fact that seems to have been unnoted so far. The tangency can be shown by using Pythagoras' Theorem in triangle $CO'W_3$, yielding that $O'W_3 = \frac{r_1^2+r_2^2}{2r}$. This leaves for the length of the radius of (O') beyond W_3:

$$\frac{r}{2} - \frac{r_1^2 + r_2^2}{2r} = \frac{r_1 r_2}{r}.$$

This also gives us in a simple way the new Archimedean circle (K_4): the circle tangent to AB at O and to (O') is Archimedean by reflection of (W_3) through the perpendicular to AB in O'.

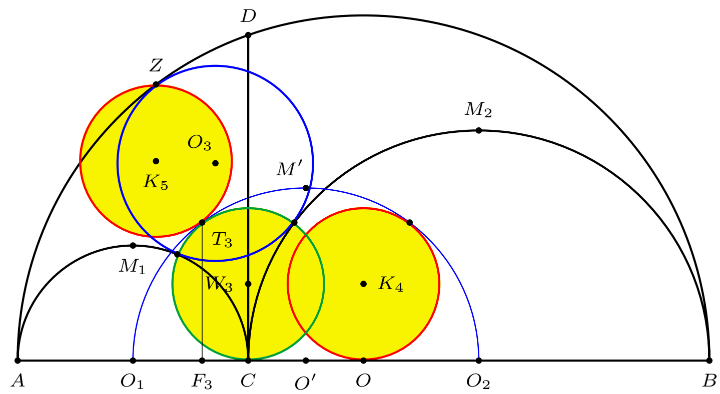


Figure 4. The Archimedean circles (K_4) and (K_5)

Let T_3 be the point of tangency of (O') and (W_3), and F_3 be the perpendicular foot of T_3 on AB. Then

$$O'F_3 = \frac{\frac{r}{2}}{\frac{r}{2} - r_A} \cdot O'C = \frac{(r_2 - r_1)r^2}{2(r_1^2 + r_2^2)},$$

and from this we see that $O_1F_3 : F_3O_2 = r_1^2 : r_2^2$, so that $O_1T_3 : T_3O_2 = r_1 : r_2$, and hence the angle bisector of $\angle O_1T_3O_2$ passes through C. If Z is the point of tangency of (O_3) and (O) then [9, Corollary 3] shows the same for the angle bisector of $\angle AZB$. But this means that the points C, Z and T_3 are collinear. This also gives us the circle (K_5) tangent to (O') and tangent to the parallel to AB through Z is Archimedean by reflection of (W_3) through T_3. See Figure 4.

There are many ways to find the interesting point T_3. Let me give two.

(1) Let M, M', M_1 and M_2 be the midpoints of the semicircular arcs of (O), (O'), (O_1) and (O_2) respectively. T_3 is the second intersection of (O') with line $M_1M'M_2$, apart from M'. This line $M_1M'M_2$ is an angle bisector of the angle formed by lines AB and the common tangent of (O), (O_3) and (W_{21}).

(2) The circle (AM_1O_2) intersects the semicircle (O_2) at Y, and the circle (BM_2O_1) intersects the semicircle (O_1) at X. X and Y are the points of tangency

[2]For simple constructions of (O_3), see [9, 11].

of (O_3) with (O_1) and (O_2) respectively. Now, each of these circles intersects the midway semicircle (O') at T_3. See Figure 5.

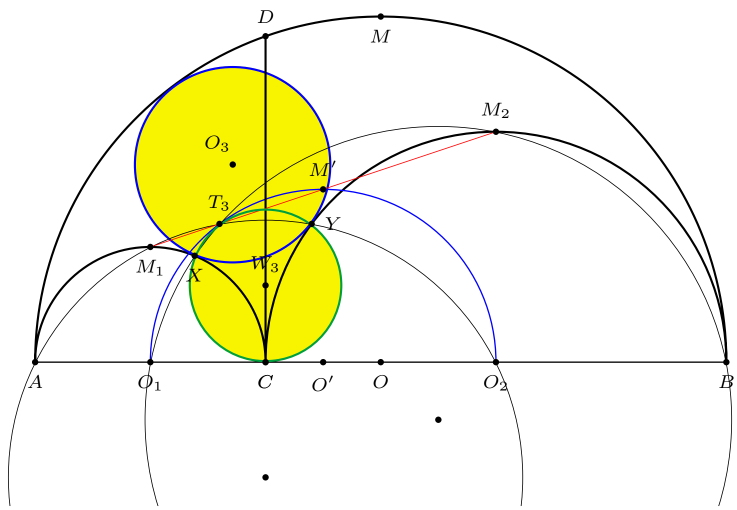


Figure 5. Construction of T_3

2.4. (K_6) *and* (K_7). Consider again the circles $A(C)$ and $B(C)$. The circle with center K_6 on the radical axis of $A(C)$ and (O) and tangent to $A(C)$ as well as (O') is the Archimedean circle (K_6). This circle is easily constructed by noticing that the common tangent of $A(C)$ and (K_6) passes through O_2. Let T_6 be the point of tangency, then K_6 is the intersection of the mentioned radical axis and AT_6. Similarly one finds (K_7). See Figure 6.

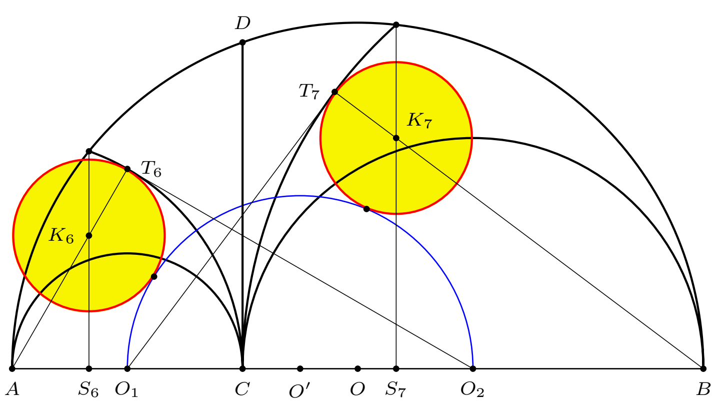


Figure 6. The Archimedean circles (K_6) and (K_7)

To prove that (K_6) is indeed Archimedean, let S_6 be the intersection of AB and the radical axis of $A(C)$ and (O), then $O'S_6 = 2r_A + \frac{r_2 - r_1}{2}$ and $AS_6 = 2(r_1 - r_A)$.

If x is the radius of (K_6), then

$$\left(\frac{r_1+r_2}{2}+x\right)^2-\left(2r_A+\frac{r_2-r_1}{2}\right)^2=(2r_1-x)^2-4(r_1-r_A)^2$$

which yields $x=r_A$.

For justification of the simple construction note that

$$\cos\angle S_6K_6A=\frac{AS_6}{AK_6}=\frac{2r_1-2r_A}{2r_1-r_A}=\frac{2r_1}{2r_1+r_2}=\frac{AT_6}{AO_2}=\cos\angle AO_2T_6.$$

2.5. (K_8) *and* (K_9). Let A' and B' be the reflections of C through A and B respectively. The circle with center on AB, tangent to the tangent from A' to (O_2) and to the radical axis of $A(C)$ and (O) is the circle (K_8). Similarly we find (K_9) from B' and (O_1) respectively.

Let x be the radius of (K_8). We have

$$\frac{4r_1+r_2}{r_2}=\frac{4r_1-2r_A-x}{x}$$

implying indeed that $x=r_A$. And (K_8) is Archimedean, while this follows for (K_9) as well by symmetry. See Figure 7.

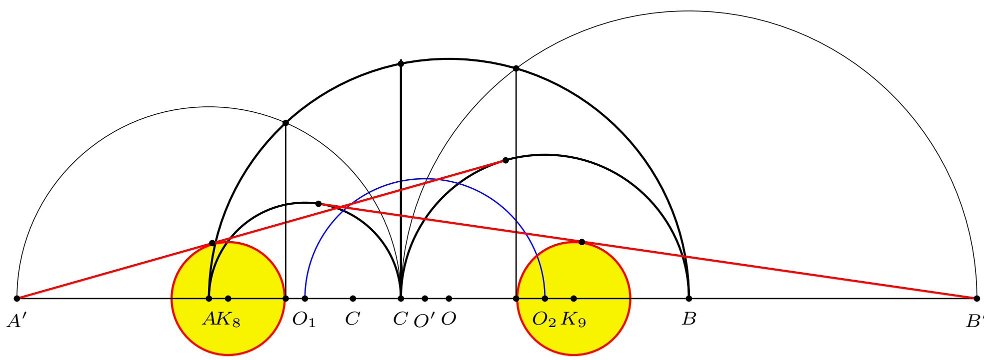


Figure 7. The Archimedean circles (K_8) and (K_9)

2.6. *Four more Archimedean circles from the midway semicircle.* The points M, M_1, M_2, O, C and the point of tangency of (O) and the incircle (O_3) are concyclic, the center of their circle, the *mid-arc circle* is M'. This circle (M') meets (O') in two points K_{10} and K_{11}. The circles with K_{10} and K_{11} as centers and tangent to AB are Archimedean. To see this note that the radius of (MM_1M_2) equals $\sqrt{\frac{r_1^2+r_2^2}{2}}$. Now we know the sides of triangle $O'M'K_{10}$. The altitude from K_{10} has length $\frac{\sqrt{(r_1^2+4r_1r_2+r_2^2)(r_1^2+r_2^2)}}{2r}$ and divides $O'M'$ indeed in segments of $\frac{r_1^2+r_2^2}{2r}$ and r_A. Of course by similar reasoning this holds for (K_{11}) as well.

Now the circle $M(C)$ meets (O) in two points. The smallest circles (K_{12}) and (K_{13}) through these points tangent to AB are Archimedean as well. This can be seen by applying homothety with center C with factor 2 to the points K_{10} and K_{11}.

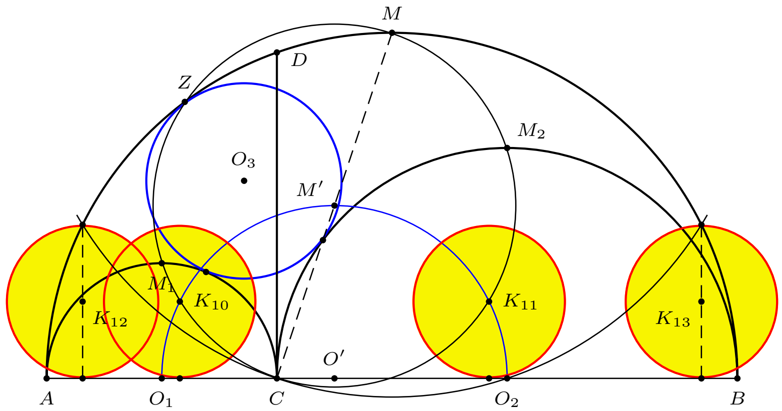


Figure 8. The Archimedean circles (K_{10}), (K_{11}), (K_{12}) and (K_{13})

The image points are the points where $M(C)$ meets (O) and are at distance $2r_A$ from AB. See Figure 8.

2.7. (K_{14}) *and* (K_{15}). It is easy to see that the semicircles $A(C)$, $B(C)$ and (O) are images of (O_1), (O_2) and (O') after homothety through C with factor 2. This shows that $A(C)$, $B(C)$ and (O) have a common tangent parallel to the common tangent d of (O_1), (O_2) and (O'). As a result, the circles (K_{14}) and (K_{15}) tangent internally to $A(C)$ and $B(C)$ respectively and both tangent to d at the opposite of (O_1), (O_2) and (O'), are Archimedean circles, just as is Bankoff's quadruplet circle (W_4).

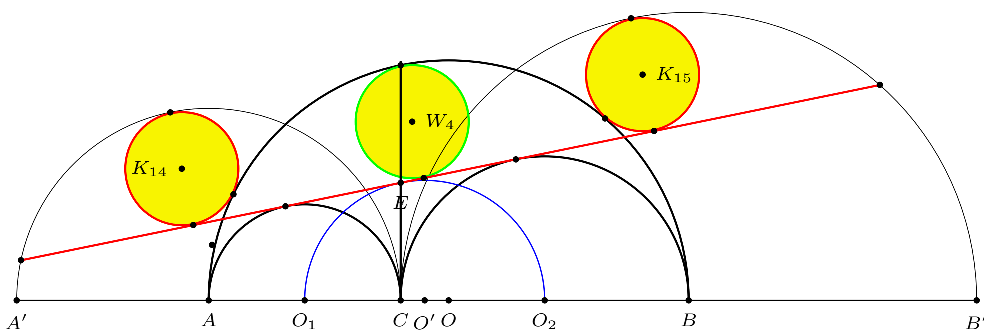


Figure 9. The Archimedean circles (K_{14}) and (K_{15})

An additional property of (K_{14}) and (K_{15}) is that these are tangent to (O) externally. To see this note that, using linearity, the distance from A to d equals $\frac{2r_1^2}{r}$, so $AK_{14} = \frac{r_1(2r_1+r_2)}{r}$. Let F_{14} be the perpendicular foot of K_{14} on AB. In triangle COD we see that $CD = 2\sqrt{r_1 r_2}$ and thus by similarity of COD and $F_{14}AK_{14}$

we have

$$K_{14}F_{14} = \frac{2\sqrt{r_1r_2}}{r}AK_{14} = \frac{2r1\sqrt{r_1r_2}(2r_1+r_2)}{r^2},$$
$$OF_{14} = r + \frac{r_2-r_1}{r}AK_{14} = \frac{r_2^3+4r_1r_2^2+4r_1^2r_2-r_1^3}{r^2},$$

and now we see that $K_{14}F_{14}^2 + OF_{14}^2 = (r+r_A)^2$. In the same way it is shown (K_{15}) is tangent to (O). See Figure 9. Note that O_1K_{15} passes through the point of tangency of d and (O_2), which also lies on $O_1(D)$. We leave details to the reader.

2.8. (K_{16}) *and* (K_{17}). Apply the homothety $\mathsf{h}(A,\lambda)$ to (O) and (O_1) to get the circles (Ω) and (Ω_1). Let $U(\rho)$ be the circle tangent to these two circles and to CD, and U' the perpendicular foot of U on AB. Then $|U'\Omega| = \lambda r - 2r_1 + \rho$ and $|U'\Omega_1| = (\lambda-2)r_1 + \rho$. Using the Pythagorean theorem in triangle $UU'\Omega$ and $UU'\Omega_1$ we find

$$(\lambda r_1+\rho)^2 - ((\lambda-2)r_1+\rho)^2 = (\lambda r-\rho)^2 - (\lambda r - 2r_1+\rho)^2.$$

This yields $\rho = r_A$.

By symmetry, this shows that the twin circles of Archimedes are members of a family of Archimedean twin circles tangent to CD. In particular, (W_6) and (W_7) of [4] are limiting members of this family. As special members of this family we add (K_{16}) as the circle tangent to $C(A)$, $B(A)$, and CD, and (K_{17}) tangent to $C(B)$, $A(B)$, and CD. See Figure 10.

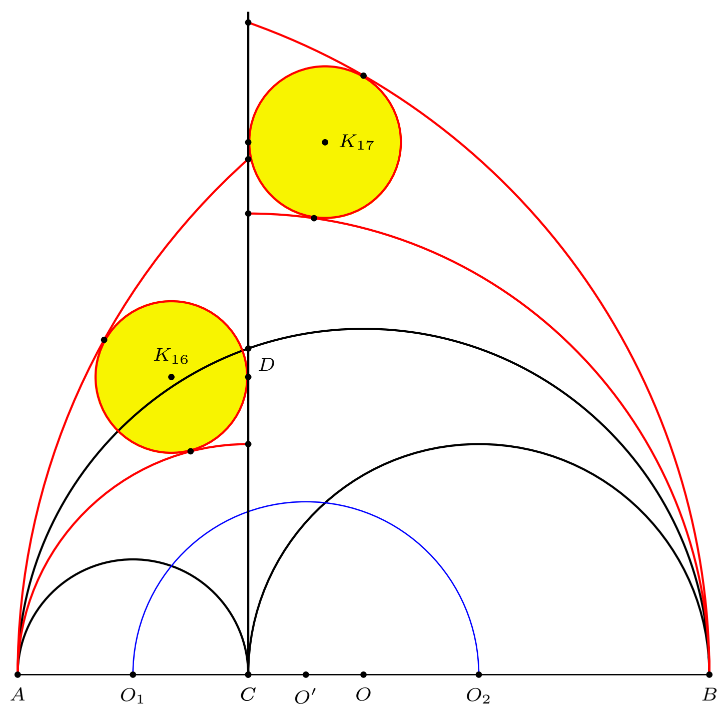


Figure 10. The Archimedean circles (K_{16}) and (K_{17})

3. Extending circles to families

There is a very simple way to turn each Archimedean circle into a member of an infinite Archimedean family, that is by attaching to (O) a semicircle (O') to be the two inner semicircles of a new arbelos. When (O'') is chosen smartly, this new arbelos gives Archimedean circles exactly of the same radii as Archimedean circles of the original arbelos. By repetition this yields infinite families of Archimedean circles. If r'' is the radius of (O''), then we must have

$$\frac{r_1 r_2}{r} = \frac{rr''}{r + r''}$$

which yields

$$r'' = \frac{rr_1 r_2}{r^2 - r_1 r_2},$$

surprisingly equal to r_3, the radius of the incircle (O_3) of the original arbelos, as derived in [4] or in generalized form in [6, Theorem 1]. See Figure 11.

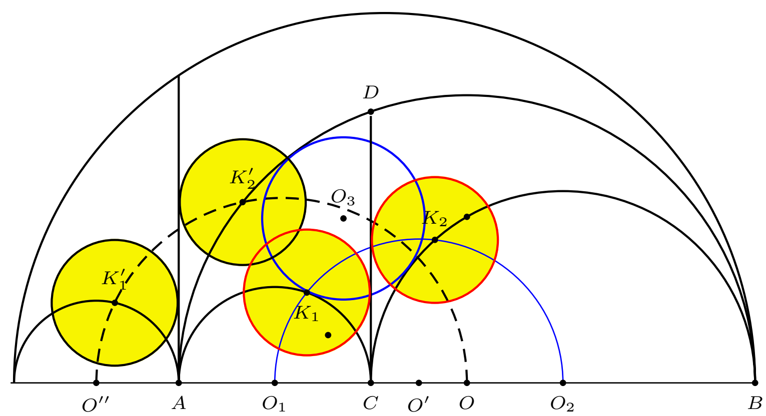


Figure 11

Now let $(O_1(\lambda))$ and $(O_2(\lambda))$ be semicircles with center on AB, passing through C and with radii λr_1 and λr_2 respectively. From the reasoning of §2.7 it is clear that the common tangent of $(O_1(\lambda))$ and $(O_2(\lambda))$ and the semicircles $(O_1(\lambda+1))$, $(O_2(\lambda + 1))$ and $(O_1(\lambda + 1)O_2(\lambda + 1))$ enclose Archimedean circles $(K_{14}(\lambda))$, $(K_{15}(\lambda))$ and $(W_4(\lambda))$. The result is that we have three families. By homothety the point of tangency of $(K_{14}(\lambda))$ and $(O_1(\lambda + 1))$ runs through a line through C, so that the centers $K_{14}(\lambda)$ run through a line as well. This line and a similar line containing the centers of $K_{15}(\lambda)$ are perpendicular. This is seen best by the well known observation that the two points of tangency of (O_1) and (O_2) with their common tangent together with C and D are the vertices of a rectangle, one of Bankoff's surprises [2]. Of course the centers $W_4(\lambda)$ lie on a line perpendicular to AB. See Figure 12.

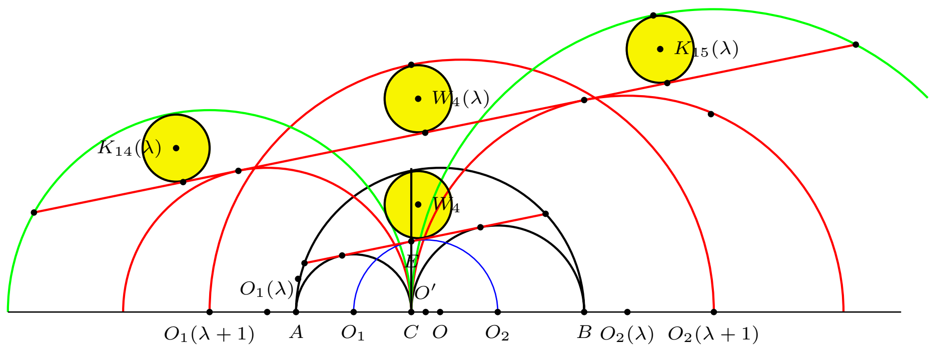


Figure 12

4. A family of salina

We consider another way to generalize the Archimedean circles in infinite families. Our method of generalization is to translate the two basic semicircles (O_1) and (O_2) and build upon them a (skew) *salinon*. We do this in such a way that to each arbelos there is a family of salina that accompanies it. The family of salina and the arbelos are to have common tangents.

This we do by starting with a point O'_t that divides O_1O_2 in the ratio $O_1O'_t : O'_tO_2 = t : 1-t$. We create a semicircle $(O'_t) = (O_{t,1}O_{t,2})$ with radius $r'_t = (1-t)r_1 + tr_2$, so that it is tangent to d. This tangent passes through E, see [6, Theorem 8], and meets AB in N, the external center of similitude of (O_1) and (O_2). Then we create semicircles $(O_{t,1})$ and $(O_{t,2})$ with radii r_1 and r_2 respectively. These two semicircles have a semicircular hull $(O_t) = (A_tB_t)$ and meet AB as second points in $C_{t,1}$ and $C_{t,2}$ respectively. Through these we draw a semicircle $(O_{t,4}) = (C_{t,2}C_{t,1})$ opposite to the other semicircles with respect to AB. Assume $r_1 < r_2$. $C_{t,1}$ is on the left of $C_{t,2}$ if and only if $t \geq \frac{1}{2}$.[3] In this case we call the region bounded by the 4 semicircles the t-salinon of the arbelos. See Figure 13.

Here are the coordinates of the various points.

O'_t $(tr_1 + (t-1)r_2, 0)$	
$O_{t,1}$ $((2t-1)r_1 - r_2, 0)$	$O_{t,2}$ $(r_1 + (2t-1)r_2, 0)$
A_t $((2t-2)r_1 - r_2, 0)$	B_t $(r_1 + 2tr_2, 0)$
$C_{t,1}$ $(2tr_1 - r_2, 0)$	$C_{t,2}$ $(r_1 + (2t-2)r_2, 0)$
O_t $\left(\left(t-\frac{1}{2}\right)r,\ 0\right)$	$O_{t,4}$ $\left((t+\frac{1}{2})r_1 + (t-1\frac{1}{2})r_2, 0\right)$
C_t $\left(\frac{r_1^2-r_2^2+(2t-1)r_1r_2}{r}, 0\right)$	

The radical axis of the circles

$$(O'_t): \qquad (x - tr_1 - (t-1)r_2)^2 + y^2 = ((1-t)r_1 + tr_2)^2$$

[3] If $t < \frac{1}{2}$ we can still find valid results by drawing $(O_{t,4})$ on the same side of AB as the other semicircles. In this paper we will not refer to these results, as the resulting figure is not really like Archimedes' salinon.

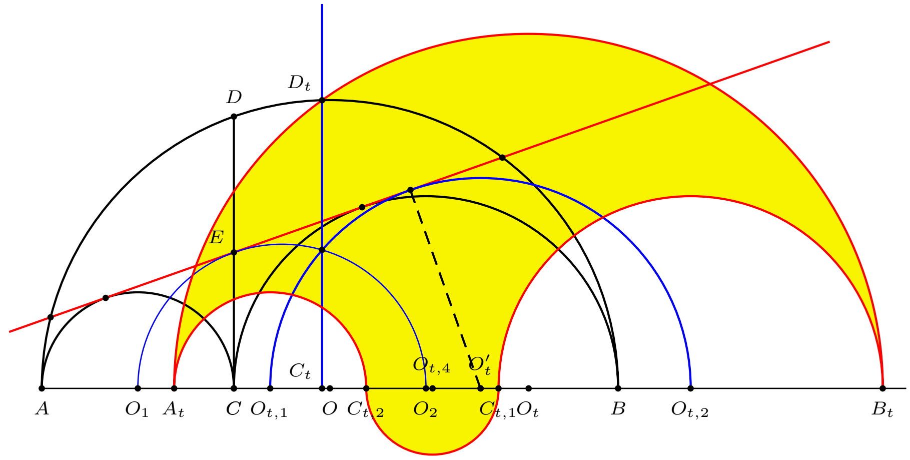


Figure 13. A t-salinon

and

$$(O'): \qquad \left(x - \frac{r_1 - r_2}{2}\right)^2 + y^2 = \frac{r^2}{4}$$

is the line

$$\ell_t: \qquad x = \frac{(2t-1)r_1r_2 + r_1^2 - r_2^2}{r}.$$

So is the radical axis of

$$(O): \qquad x^2 + y^2 = r^2$$

and

$$(O_t): \qquad \left(x - \left(t - \frac{1}{2}\right)r\right)^2 + y^2 = \left(\left(1\frac{1}{2} - t\right)r_1 + \left(t + \frac{1}{2}\right)r_2\right)^2.$$

On this common radical axis ℓ_t, we define points the points C_t on AB and D_t on (O). See Figure 13.

5. Archimedean circles in the t-salinon

5.1. *The twin circles of Archimedes.* We can generalize the well known Adam and Eve of the Archimedean circles to adjoint salina in the following way: The circles $W_{t,1}$ and $W_{t,2}$ tangent to both (O_t) and ℓ_t and to (O_1) and (O_2) respectively are Archimedean.

This can be proven with the above coordinates: The semicircles $O_t((1\frac{1}{2}-t)r_1 + (t+\frac{1}{2})r_2 - r_A))$ and $O_1(r_1 + r_A)$ intersect in the point

$$W_{t,1} \quad \left(\frac{(2t-2)r_1r_2 + r_1^2 - r_2^2}{r},\ r_A\sqrt{(3-2t)(2t+1+\frac{2r_1}{r_2})}\right),$$

which lies indeed r_A left of ℓ_t. Similarly we find for the intersection of $O_t((1\frac{1}{2} - t)r_1 + (t + \frac{1}{2})r_2 - r_A))$ and $O_2(r_2 + r_A)$

$$W_{t,2} \ \left(\frac{r_1^2 + 2r_1r_2t - r_2^2}{r},\ r_A\sqrt{(2t+1)(3-2t+\frac{2r_2}{r_1})} \right),$$

which lies r_A right of ℓ_t. See Figure 14.

These circles are real if and only if $\frac{1}{2} \leq t \leq \frac{3}{2}$.

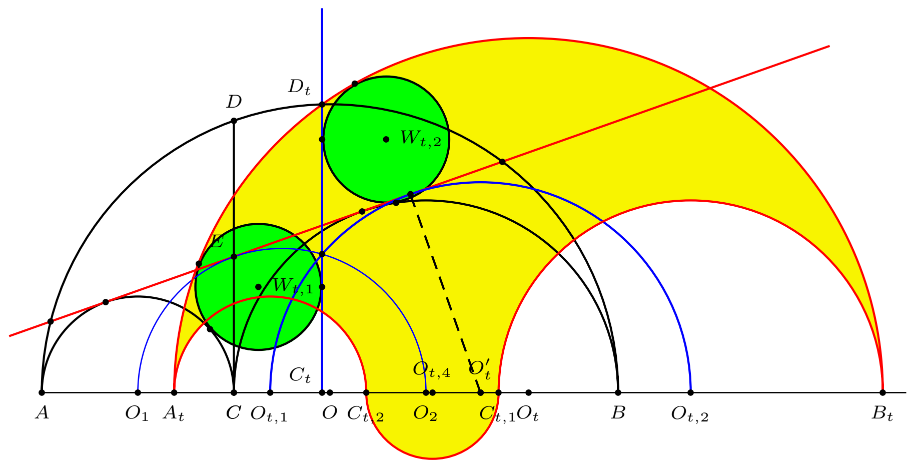


Figure 14. The Archimedean circles $(W_{t,1})$ and $(W_{t,2})$

Two properties of the twin circles of Archimedes can be generalized as well. Dodge et al [4] state that the circle $A(D)$ passes through the point of tangency of (O_2) and (W_2), while Wendijk [8] and d'Ignazio and Suppa in [5, p. 236] ask in a problem to show that the point N_2 where O_2W_2 meets CD lies on $O_2(A)$ (reworded). We can generalize these to

- the circle $A(D_t)$ passes through the point of tangency of $(W_{t,2})$ and (O_2);
- the point $N_{t,2}$ where $O_2W_{t,2}$ meets the perpendicular to AB through $C_{t,1}$ lies on $O_2(A)$.

Of course similar properties are found for $W_{t,1}$.

To verify this note that D_t has coordinates

$$D_t \ \left(\frac{r_1^2 - r_2^2 + (2t-1)r_1r_2}{r}, \frac{\sqrt{r_1r_2((3-2t)r_1 + 2r_2)(2r_1 + (2t+1)r_2)}}{r} \right),$$

while the point of contact R_2 of $(W_{t,2})$ and (O_2) is

$$\begin{aligned} & O_2 + \frac{r_2}{r_2 + r_A}(W_{t,2} - O_2) \\ = & \left(\frac{2r_1^2 - r_2^2 + 2tr_1r_2}{2r_1 + r_2}, \frac{\sqrt{(1+2t)r_1r_2^2((3-2t)r_1 + 2r_2)}}{2r_1 + r_2} \right). \end{aligned}$$

Straightforward verification now shows that

$$d(A, D_t)^2 = d(A, R_2)^2 = 2r_1(2r_1 + (1 + 2t)r_2).$$

Furthermore, we see that the point $N_{t,2} = O_2 + \frac{2r_1+r2}{r}(R_2 - O_2)$ has coordinates

$$N_{t,2} = \left(2tr_1 - r_2, \sqrt{(1 + 2t)r_1((3 - 2t)r_1 + 2r_2)}\right)$$

so that this point lies indeed on $O_2W_{t,2}$, on $O_2(A)$ and on the perpendicular to AB through $C_{t,1}$.

5.2. $(W_{t,3})$. Consider the circle through C tangent to (O'_t) and with center on C_tD_t. When u is the radius of this circle we have

$$\begin{aligned} (r't - u)^2 - O'tCt^2 &= u^2 - CCt^2 \\ ((1 - t)r_1 + tr_2 - u)^2 - (\frac{(t - 1)r_1^2 + tr_2^2}{r})^2 &= u^2 - ((2t - 1)r_A)^2 \end{aligned}$$

which yields $u = r_A$. This shows that this circle $(W_{t,3})$ is an Archimedean circle and generalizes the Bankoff triplet circle (W_3), using the tangency of (W_3) to (O') shown above. See Figure 15. These circles are real if and only if $\frac{1}{2} \le t \le 1$.

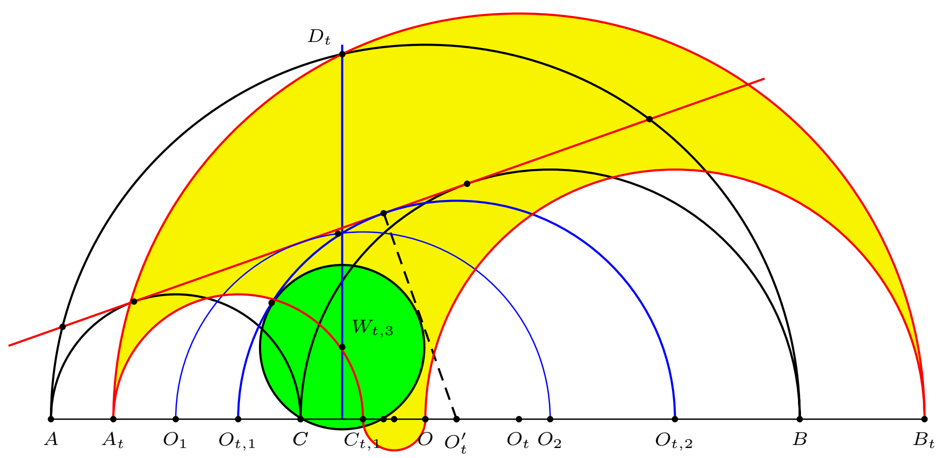


Figure 15. The Archimedean circle $(W_{t,3})$

5.3. $(W_{t,4})$. To generalize Bankoff's quadruplet circle (W_4) we start with a lemma.

Lemma 1. *Let (K) be a circle with center K and ℓ_1 and ℓ_2 be two tangents to (K) meeting in a point P. Let L be a point travelling through the line PK. When L travels linearly, then the radical axis of (K) and the circle (L) tangent to ℓ_1 and ℓ_2 moves linearly as well. The speed relative to the speed of L depends only on the angle of ℓ_1 and ℓ_2.*

Proof. Let P be the origin for Cartesian coordinates. Without loss of generality let ℓ_1 and ℓ_2 be lines making angles $\pm\phi$ with the $x-$axis and let $K(x_1, 0)$, $L(x_2, 0)$. With $v = \sin\phi$ the circles (K) and (L) are given by

$$\begin{aligned}(x-x_1)^2 + y^2 &= v^2 x_1^2,\\ (x-x_2)^2 + y^2 &= v^2 x_2^2.\end{aligned}$$

The radical axis of these is $x = \frac{1-v^2}{2}(x_1 + x_2) = \frac{1}{2}\cos^2\phi(x_1 + x_2)$. □

It is easy to check that the slope of the line through the midpoints of the semicircular arcs (O) and (O_t) is equal to $\frac{r_2 - r_1}{r}$ and thus equal to te slope of the line through M_1 and M_2. This shows that the common tangent of (O) and (O_t) is parallel to the common tangent of (O_1) and (O_2), i.e. d, also the common tangent of (O') and (O'_t). As a result of Lemma 1, it is now clear why the the radical axes of (O) and (O_t) and of (O') and (O'_t) coincide for all t. Another consequence is that the greatest circle tangent to d at the opposite side of (O_1) and (O_2) and to (O_t) internally is, just as the famous example of Bankoff's quadruplet circle (W_4), an Archimedean circle $(W_{t,4})$. See Figure 16. Of course this means that the circles (K_{12}) and (K_{13}) are found as members of the family $(W_{t,4})$.

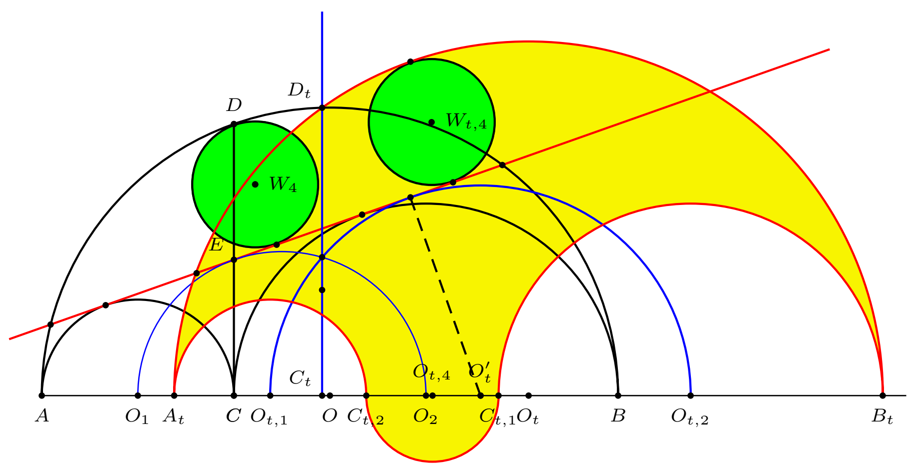


Figure 16. The Archimedean circle $(W_{t,4})$

We note that the the Archimedean circles $(W_{t,4})$ can be constructed easily in any (skew) salinon without seeing the salinon as adjoint to an arbelos and without reconstructing (parts of) this arbelos. To see this we note that N is external center of similitude of $(O_{t,1})$ and $(O_{t,2})$, and d is the tangent from N to (O'_t).

5.4. $(W_{t,6})$, $(W_{t,7})$, $(W_{t,13})$ *and* $(W_{t,14})$. Whereas (W_{13}), (W_{14}), (K_1) and (K_2) are defined in terms of intersections of (semi-)circles or radical axes, with Lemma 1 their generalizations are obvious: CD, (O) and (O') can be replaced by C_tD_t, (O_t) and (O'_t). The generalization of (W_6) and (W_7) as Archimedean circles with center on AB and tangent to $CtDt$ keep having some interest as well. The tangents from A to $(O_{t,2})$ and from B to $(O_{t,1})$ are tangent to $(W_{t,6})$ and $(W_{t,7})$ respectively. This is seen by straightforward calculation, which is left to the reader. As a result we have two stacks of three families. See Figure 17.

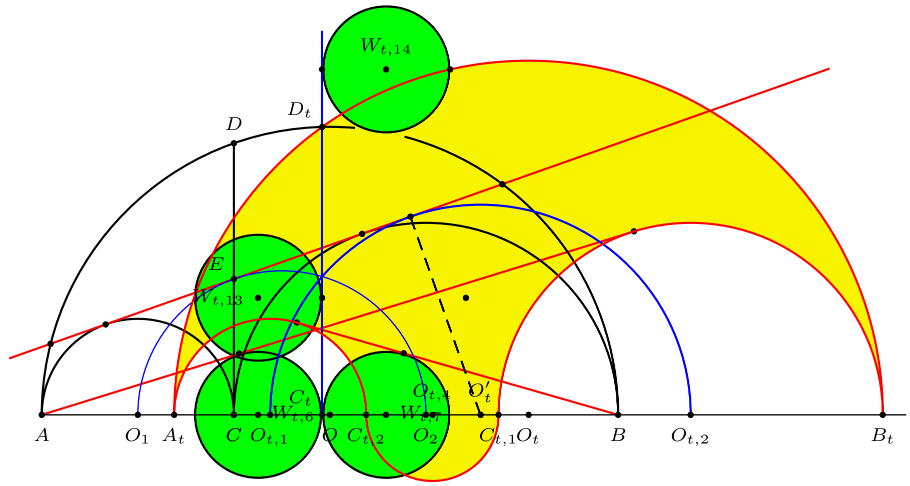


Figure 17. The Archimedean circles $(W_{t,6})$, $(W_{t,7})$, $(W_{t,13})$ and $(W_{t,14})$

We zoom in on the families $(W_{t,13})$ and $(W_{t,14})$. Let $T_{t,13}$ be the point where $(W_{t,13})$, $A(C)$ and (O_t) meet and similarly define $T_{t,14}$. We find that

$$T_{t,13} \quad \left(\frac{r_1^2+(2t-3)r_1r_2-r_2^2}{r}, \frac{r_1\sqrt{-r_2((8t-12)r_1+(4t^2-4t-3)r_2)}}{r}\right),$$
$$T_{t,14} \quad \left(\frac{r_1^2+(2t+1)r_1r_2-r_2^2}{r}, \frac{r_2\sqrt{-r_1((4t^2-4t-3)r_1-(8t+4)r_2)}}{r}\right).$$

The slope of the tangent to $A(C)$ in $T_{t,13}$ with respect to the $x-$axis is equal to

$$s_A = -\frac{x_A - x_{T_{t,13}}}{0 - y_{T_{t,13}}}$$

and the $x-$coordinate of the point R_t where this tangent meets AB is equal to

$$x_{R_t} = x_{T_{t,13}} - \frac{y_{T_{t,13}}}{s_A} = \frac{r(2r_1 + (1-2t)r_2)}{2r_1 + (2t-1)r_2}.$$

Similarly we find for the point L_t where the tangent to $B(C)$ in $T_{t,14}$ meets AB

$$x_{L_t} = \frac{r((2t-1)r_1 + 2r_2)}{(2t-1)r_1 - 2r_2}.$$

The coordinates of Z are given by

$$\left(\frac{r^2(r_1 - r_2)}{r_1^2 + r_2^2}, \frac{2rr_1r_2}{r_1^2 + r_2^2}\right).$$

By straightforward calculation we can now verify that $ZL_t^2 + ZR_t^2 = L_tR_t^2$ and we have a new characterization of the families $(W_{t,13})$ and $(W_{t,14})$.

Theorem 2. *Let $\mathcal{K}$ be a circle through Z with center on AB. This circle meets AB in two points L on the left hand side and R on the right hand side. Let the tangent to $A(C)$ through R meet $A(C)$ in P_1 and similarly find P_2 on $B(C)$. Let k be the line through the midpoint of P_1P_2 perpendicular to AB. Then the smallest circles through P_1 and P_2 tangent to k are Archimedean.*

Remark. Similar characterizations can be found for $(K_{t,1})$ and $(K_{t,2})$.

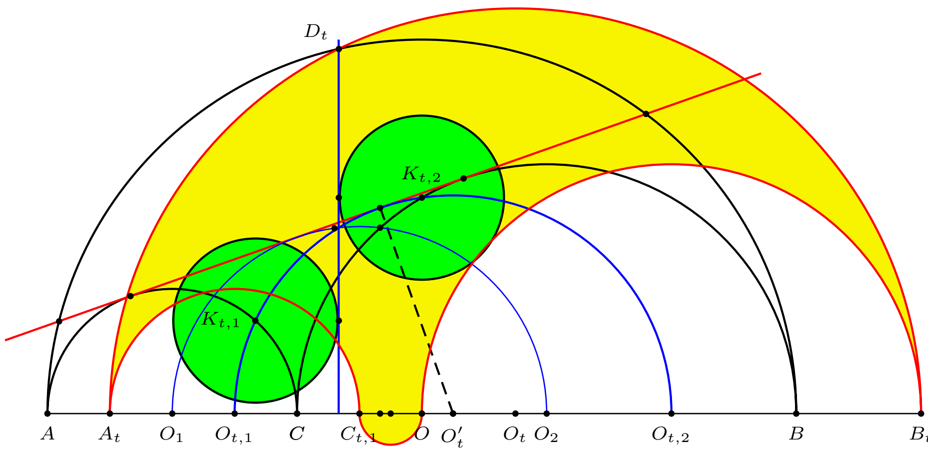


Figure 18. The Archimedean circles $(K_{t,1})$ and $(K_{t,2})$

5.5. *More corollaries.* We go back to Lemma 1. Since the distance between d and the common tangent of (O), (O_t), $A(C)$ and $B(C)$ is equal to $2r_A$, we notice that for instance $A(2r_1 - r_A)$ and $O'_t(r'_t + r_A)$ have a common tangent parallel to d as well. But that implies that we can use Lemma 1 on their intersection (and radical axis). This leads for instance to easy generalizations of (K_3) and (K_7). See Figure 19. The lemma can also help to generalize for instance (K_8) and (K_9), but then some more work has to be done. We leave this to the reader.

5.6. *Archimedean circles from the mid-arc circle of the salinon.* We end the adventures with a sole salinon. We note that the midpoints M_t, $M_{t,1}$, $M_{t,2}$ and $M_{t,4}$ of the semicircular arcs (O_t), $(O_{t,1})$, $(O_{t,2})$ and $(O_{t,4})$ are concyclic. More precisely they are vertices of a rectangle. Their circle, the mid-arc circle (O_5) of the salinon and the circle $(O_{t,1}O_{t,2})$ meet in two points, that are centers of Archimedean circles $(K_{t,10})$ and $(K_{t,11})$. (Of course the parameter t does not really play a role here, but for reasons of uniformity we still use it in the naming).

To verify this, denote by u the distance between $O_{t,1}$ and $O_{t,2}$. Then the distance between $M_{t,1}$ and $M_{t,2}$ equals $\sqrt{u^2 + (r_2 - r_1)^2}$ and the distance from O'' to AB

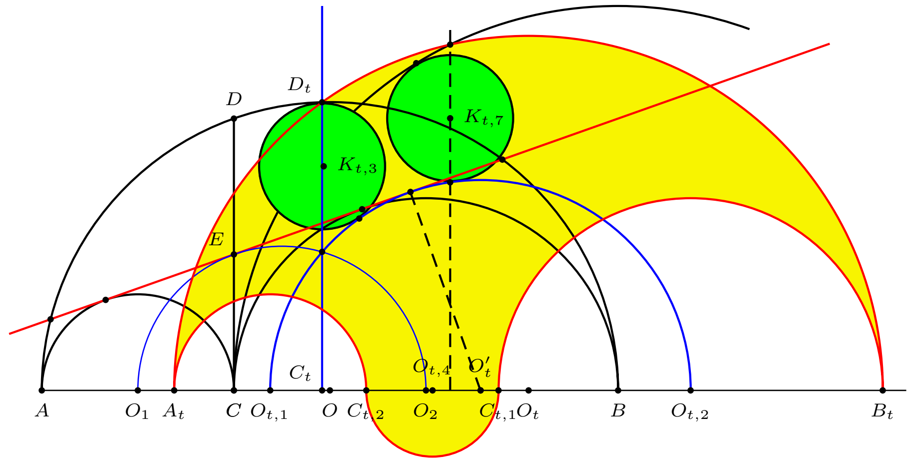


Figure 19. The Archimedean circles $(K_{t,3})$ and $(K_{t,7})$

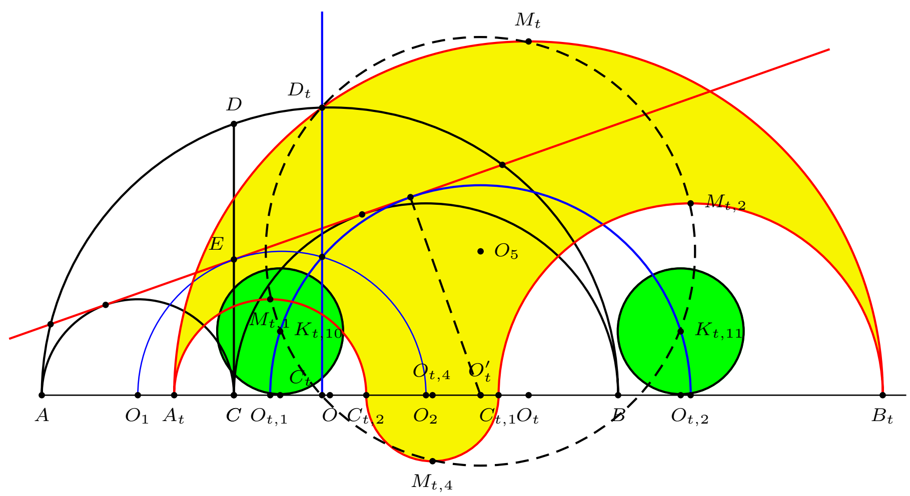


Figure 20. The Archimedean circles $(K_{t,10})$ and $(K_{t,11})$

equals $\frac{r_1+r_2}{2}$. If x is the distance from the intersections of $(O_{t,1}O_{t,2})$ and (O_5) to AB, then

$$x^2 - \frac{u^2}{4} = \left(\frac{r_1+r_2}{2} - x\right)^2 - \frac{u^2+(r_2-r_1)^2}{4}$$

which leads to $x = \frac{r_1 r_2}{r_1+r_2} = r_A$. We can generalize (K_{12}) and (K_{13}) in a similar way. To see this we note that the circle with center on $O_t M_t$ in the pencil generated by $(M_{t,4})$ and (O_5) intersects (O_t) in two points at a distance of $2r_A$ from AB. See Figure 21.

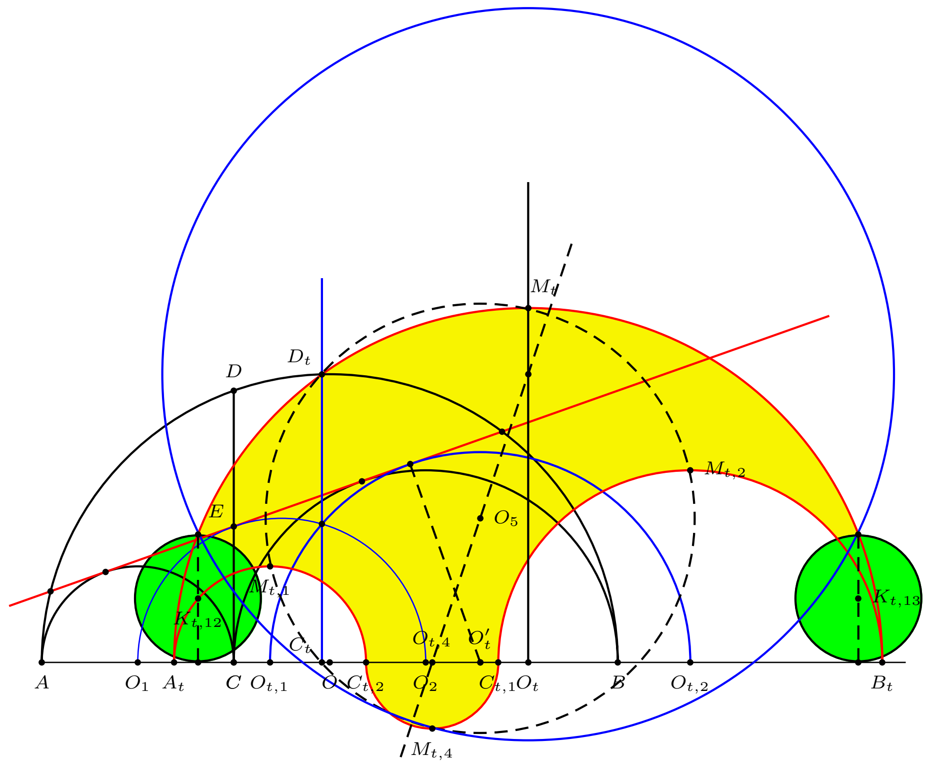


Figure 21. The Archimedean circles $(K_{t,12})$ and $(K_{t,13})$

References

[1] L. Bankoff, Are the twin circles of Archimedes really twins?, *Math. Mag.*, 47 (1974) 214–218.
[2] L. Bankoff, A mere coincidence, Mathematics Newsletter, Los Angeles City College, Nov. 1954; reprinted in *College Math. Journal*, 23 (1992) 106.
[3] L. Bankoff, The marvelous arbelos, in *The lighter Side of Mathematics*, 247–253, ed. R.K. Guy and R.E. Woodrow, Mathematical Association of America, 1994.
[4] C.W. Dodge, T. Schoch, P.Y. Woo and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.*, 72 (1999) 202–213.
[5] I. d'Ignazio and E. Suppa, *Il Problema Geometrico, dal Compasso al Cabri*, Interlinea Editrice, Téramo, 2001.
[6] H. Okumura and M. Watanabe, The Archimedean circles of Schoch and Woo, *Forum Geom.*, 4 (2004) 27–34.
[7] F. Power, Some more Archimedean circles in the arbelos, *Forum Geom.*, 5 (2005) 133–134.
[8] A. Wendijk and T. Hermann, Problem 10895, *Amer. Math. Monthly*, 108 (2001) 770; solution, 110 (2003) 63–64.
[9] P.Y. Woo, Simple constructions of the incircle of an arbelos, *Forum Geom.*, 1 (2001) 133–136.
[10] P. Yiu, Private correspondence, 1999.
[11] P. Yiu, Elegant geometric constructions, *Forum Geom.*, 5 (2005) 75–96.

Floor van Lamoen: St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands
E-mail address: `fvanlamoen@planet.nl`

Forum Geometricorum
Volume 6 (2006) 97–101.

FORUM GEOM
ISSN 1534-1178

Proof by Picture: Products and Reciprocals of Diagonal Length Ratios in the Regular Polygon

Anne Fontaine and Susan Hurley

Abstract. These "proofs by picture" link the geometry of the regular n-gon to formulae concerning the arithmetic of real cyclotomic fields. We illustrate the formula for the product of diagonal length ratios

$$r_h r_k = \sum_{i=1}^{\min[k,h,n-k,n-h]} r_{|k-h|+2i-1}.$$

and that for the reciprocal of diagonal length ratios when $\gcd(n,k)=1$,

$$\frac{1}{r_k} = \sum_{j=1}^{s} r_{k(2j-1)}, \qquad \text{where } s = \min\{j>0 : kj \equiv \pm 1 \bmod n\}.$$

1. Introduction

Consider a regular n-gon. Number the diagonals $d_1, d_2, \ldots, d_{n-1}$ (as shown in Figure 1.1 for $n=9$) including the sides of the polygon as d_1 and d_{n-1}. Although the length of d_i equals that of d_{n-i}, we shall use all $n-1$ subscripts since this simplifies our formulae concerning the diagonal lengths.

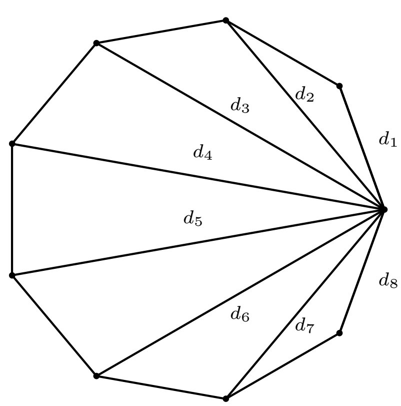


Figure 1.1

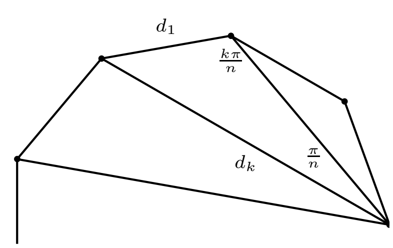


Figure 1.2

Ratios of the lengths of the diagonals are given by the law of sines. The ratio of the length of d_k to that of d_j is $\dfrac{\sin\frac{k\pi}{n}}{\sin\frac{j\pi}{n}}$. In particular, Figure 1.2 shows us that the

Publication Date: March 13, 2006. Communicating Editor: Paul Yiu.

ratio of the length of d_k to that of d_1 is $\dfrac{\sin\frac{k\pi}{n}}{\sin\frac{\pi}{n}}$. This ratio of sines will be denoted r_k. Note that if the length of the side $d_1 = 1$, r_k is simply the length of the k-th diagonal.

2. Products of diagonal length ratios

It is an exercise in the algebra of cyclotomic polynomials to show that

$$r_h r_k = \sum_{i=1}^{\min[k,h,n-k,n-h]} r_{|k-h|+2i-1}.$$

This formula appears in Steinbach [1] for the case where $h + k \leq n$. Steinbach names it the *diagonal product formula* and makes use of it to derive a number of interesting properties of the diagonal lengths of a regular polygon. It is not hard to extend the formula to cover all $n-1$ values of h and k.

In order to understand the geometry of the *diagonal product formula*, consider two regular n-gons with the side of the larger equal to some diagonal of the smaller. Denote the diagonal lengths by $\{s_i\}_{i=1,\,\ldots,n-1}$ for the smaller polygon and $\{l_i\}_{i=1,\,\ldots,n-1}$ for the larger, so that $l_1 = s_k$ for some k. In this case, since $r_k = \frac{s_k}{s_1} = \frac{l_1}{s_1}$ and $r_h = \frac{l_h}{l_1}$, the product $r_k r_h$ becomes $\frac{l_h}{s_1}$ and when we multiply through by s_1 the diagonal product formula becomes

$$l_h = \sum_{i=1}^{\min[k,h,n-k,n-h]} s_{|k-h|+2i-1}.$$

In other words, each of the larger diagonal lengths is expressible as a sum of the smaller ones.

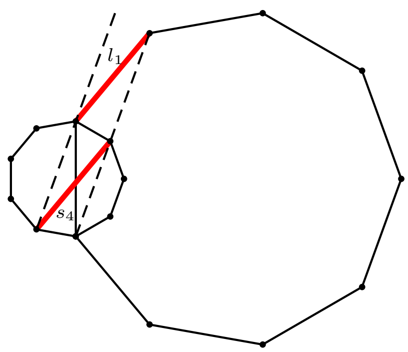


Figure 2.1

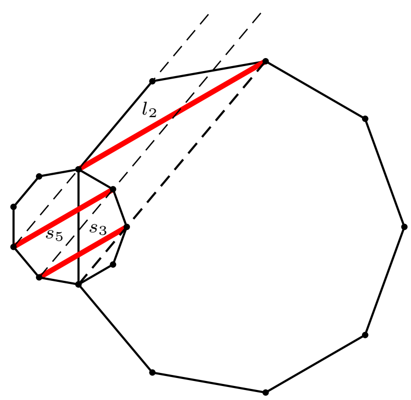


Figure 2.2

Figures 2.1-2.4 illustrate what happens when the nonagon is enlarged so that $l_1 = s_4$. The summation formula for the diagonals can be visualized by projecting the left edge of the larger polygon onto each of its other edges in turn. We observe from the first pair of nonagons that $l_1 = s_4 s_1 = s_4$, from the second that $l_2 =$

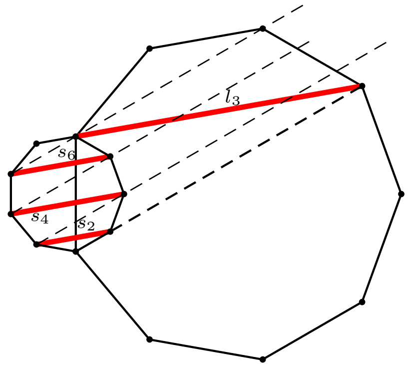


Figure 2.3

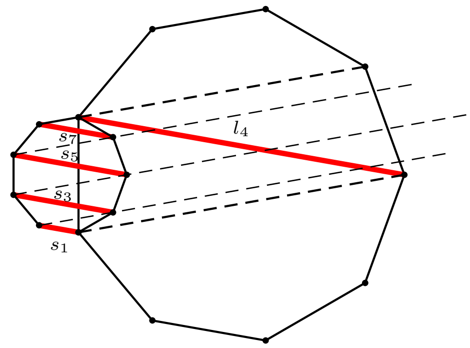


Figure 2.4

$s_4 s_2 = s_3 + s_5$, from the third that $l_3 = s_4 s_3 = s_2 + s_4 + s_6$, and from the last that $l_4 = s_4 s_4 = s_1 + s_3 + s_5 + s_7$. This is exactly what the *diagonal product formula* predicts.

When n and k are both even, the polygons do not have the same orientation, but the same strategy of projecting onto each side of the larger polygon in turn still works. Figure 3 shows the case $(n,k) = (6,2)$. The sums are

$$\begin{aligned} l_1 =& s_2 s_1 = s_2, \\ l_2 =& s_2 s_2 = s_1 + s_3, \\ l_3 =& s_2 s_3 = s_3 + s_4. \end{aligned}$$

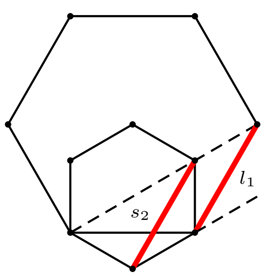


Figure 3.1

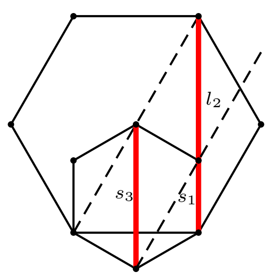


Figure 3.2

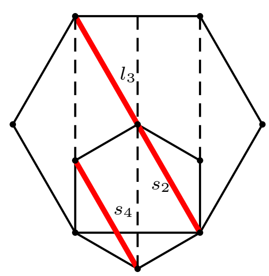


Figure 3.3

3. Reciprocal of diagonal length ratios

Provided $\gcd(k,n) = 1$, the diagonal length ratio r_k is a unit in the ring of integers of the real subfield of $\mathbf{Q}[\xi]$, ξ a primitive n-th root of unity. The set of ratios $\{r_i\}$ with $\gcd(i,n) = 1$ and $i \leq \frac{n}{2}$ forms a basis for this field [2]. Knowing that this was the case, we searched for a formula to express $\frac{1}{r_k}$ as an integral linear combination of diagonal length ratios. This time we found the picture first. We assume that the polygon has unit side, so that $r_i =$ the length of the i-th diagonal.

First note that $\frac{1}{r_k}$ is equal to the length of the line segment obtained when diagonal k intersects diagonal 2 as shown in Figure 4.1 for $(n,k) = (8,3)$. In order to express this length in terms of the r_i, $i = 1, \ldots, n-1$, notice that, as in Figure 4.2 for $(n,k) = (7,2)$, one can set off along diagonal k and zigzag back and forth, alternately parallel to diagonal k and in the vertical direction, until one arrives at a vertex adjoining the starting point. Summing the lengths of the diagonals parallel to diagonal k, with positive or negative sign according to the direction of travel, will give the desired reciprocal. For example follow the diagonals shown in Figure 4.2 to see that $\frac{1}{r_2} = r_2 + r_6 - r_4$.

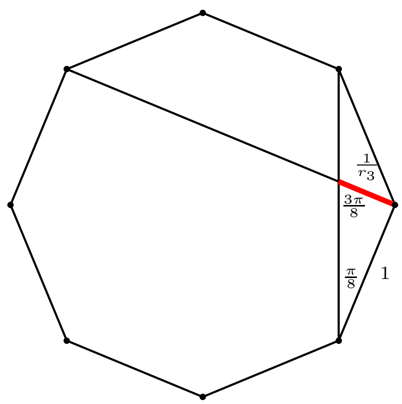


Figure 4.1

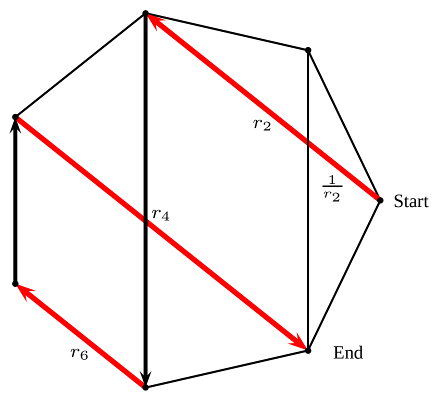


Figure 4.2

Following the procedure outlined above, the head of each directed diagonal used in the sum is k vertices farther around the polygon from the previous one. Also as we zigzag our way through the polygon, the positive contributions to the sum occur as we move in one direction with respect to diagonal k, while the negative contributions occur as we move the other way. In fact, allowing $r_i = \frac{\sin \frac{i\pi}{n}}{\sin \frac{\pi}{n}}$ to be defined for $i > n$, we realized that $r_i = -r_{2n-i}$ would have the correct sign to produce the simplest formula for the reciprocal, which is

$$\frac{1}{r_k} = \sum_{j=1}^{s} r_{k(2j-1)}, \qquad \text{where } s = \min\{j > 0 : kj \equiv \pm 1 \bmod n\}.$$

Once we had discovered this formula, we found it that it was a messy but routine exercise in cyclotomic polynomial algebra to verify its truth.

Although we are almost certain that these formulas for manipulating the diagonal length ratios must be in the classical literature, we have not been able to locate them, and would appreciate any lead in this regard.

References

[1] P. Steinbach, Golden fields: a case for the heptagon, *Math. Mag.*, 70 (1997) 22–31.
[2] L. C. Washington, *Introducion to Cyclotomic Fields*, 2nd edition, Springer-Verlag, New York, 1982.

Anne Fontaine: Hudson Valley Community College, Troy, New York, 12180, USA

Susan Hurley: Department of Mathematics, Siena College, Loudonville, New York, 12211, USA
E-mail address: HURLEY@siena.edu

Forum Geometricorum
Volume 6 (2006) 103–105.

FORUM GEOM
ISSN 1534-1178

A Generalization of Power's Archimedean Circles

Hiroshi Okumura and Masayuki Watanabe

Abstract. We generalize the Archimedean circles in an arbelos given by Frank Power.

Let three semicircles α, β and γ form an arbelos with inner semicircles α and β with diameters PA and PB respectively. Let a and b be the radii of the circles α and β. Circles with radii $t = \dfrac{ab}{a+b}$ are called Archimedean circles. Frank Power [2] has shown that for "highest" points Q_1 and Q_2 of α and β respectively, the circles touching γ and the line OQ_1 (respectively OQ_2) at Q_1 (respectively Q_2) are Archimedean (see Figure 1). We generalize these Archimedean circles.

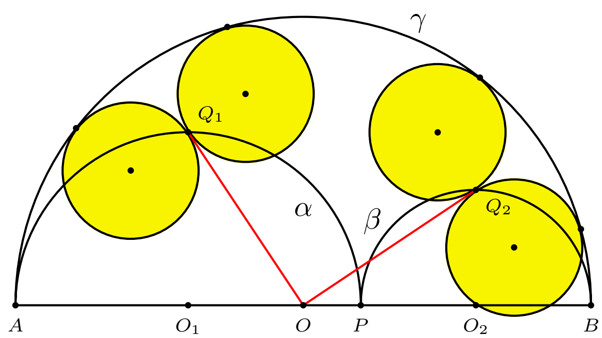


Figure 1

We denote the center of γ by O. Let Q be the intersection of the circle γ and the perpendicular of AB through P, and let δ be a circle touching γ at the point Q from the inside of γ. The radius of δ is expressed by $k(a+b)$ for a real number k satisfying $0 \leq k < 1$. The tangents of δ perpendicular to AB intersect α and β at points Q_1 and Q_2 respectively, and intersect the line AB at points P_1 and P_2 respectively (see Figures 2 and 3).

Theorem. (1) *The radii of the circles touching the circle γ and the line OQ_1 (respectively OQ_2) at the point Q_1 (respectively Q_2) are $2(1-k)t$.*
(2) *The circle touching the circles γ and α at points different from A and the line P_1Q_1 from the opposite side of B and the circle touching the circles γ and β at points different from B and the line P_2Q_2 from the opposite side of A are congruent with common radii $(1-k)t$.*

Publication Date: March 20, 2006. Communicating Editor: Paul Yiu.

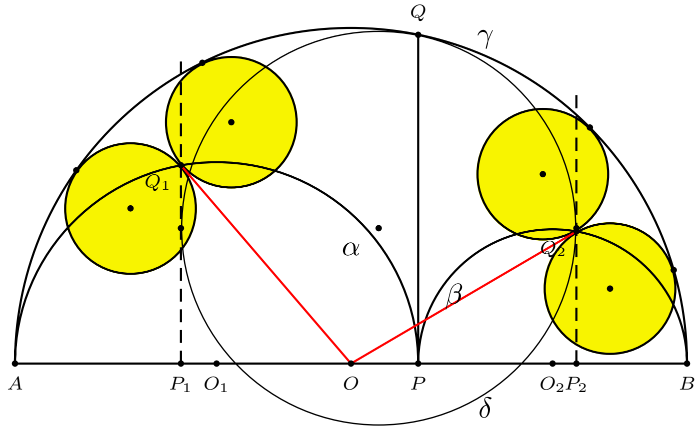


Figure 2

Proof. (1) Since $|PP_1| = 2ka$, $|OP_1| = (b-a) + 2ka$. While

$$|P_1Q_1|^2 = |PP_1||P_1A| = 2ka(2a - 2ka) = 4k(1-k)a^2.$$

Hence $|OQ_1|^2 = ((b-a)+2ka)^2 + 4k(1-k)a^2 = (a-b)^2 + 4kab$. Let x be the radius of one of the circles touching γ and the line OQ_1 at Q_1. From the right triangle formed by O, Q_1 and the center of this circle, we get

$$(a+b-x)^2 = x^2 + (a-b)^2 + 4kab$$

Solving the equation for x, we get $x = \frac{2(1-k)ab}{a+b} = 2(1-k)t$. The other case can be proved similarly.

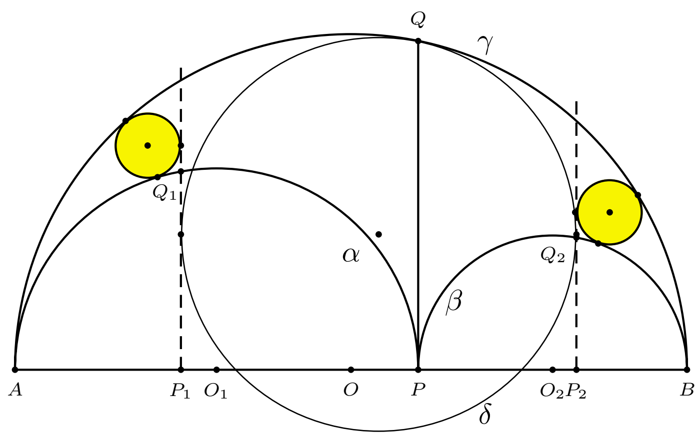


Figure 3

(2) The radius of the circle touching α externally and γ internally is proportional to the distance between the center of this circle and the radical axis of α and γ [1, p. 108]. Hence its radius is $(1-k)$ times of the radii of the twin circles of Archimedes. □

The Archimedean circles of Power are obtained when δ is the circle with a diameter OQ. The twin circles with the half the size of the Archimedean circles in [4] are also obtained in this case. The statement (2) is a generalization of the twin circles of Archimedes, which are obtained when δ is the point circle. In this case the points Q_1, Q_2 and P coincide, and we get the circle with radius $2t$ touching the line AB at P and the circle γ by (1) [3].

References

[1] J. L. Coolidge, *A treatise on the circle and the sphere*, Chelsea. New York, 1971 (reprint of 1916 edition).

[2] Frank Power, Some more Archimedean circles in the arbelos, *Forum Geom.*, 5 (2005) 133–134.

[3] H. Okumura and M. Watanabe, The twin circles of Archimedes in a skewed arbelos, *Forum Geom.*, 4 (2004) 229–251.

[4] H. Okumura and M. Watanabe, Non-Archimedean twin circles in the arbelos,(in Bulgarian), *Math. Plus*, 13 (2005) no.1, 60–62.

Hiroshi Okumura: Department of Information Engineering, Maebashi Institute of Technology, 460-1 Kamisadori Maebashi Gunma 371-0816, Japan

E-mail address: `okumura@maebashi-it.ac.jp`

Masayuki Watanabe: Department of Information Engineering, Maebashi Institute of Technology, 460-1 Kamisadori Maebashi Gunma 371-0816, Japan

E-mail address: `watanabe@maebashi-it.ac.jp`

Forum Geometricorum
Volume 6 (2006) 107–115.

FORUM GEOM
ISSN 1534-1178

Pseudo-Incircles

Stanley Rabinowitz

Abstract. This paper generalizes properties of mixtilinear incircles. Let (S) be any circle in the plane of triangle ABC. Suppose there are circles (S_a), (S_b), and (S_c) each tangent internally to (S); and (S_a) is inscribed in angle BAC (similarly for (S_b) and (S_c)). Let the points of tangency of (S_a), (S_b), and (S_c) with (S) be X, Y, and Z, respectively. Then it is shown that the lines AX, BY, and CZ meet in a point.

1. Introduction

A mixtilinear incircle of a triangle ABC is a circle tangent to two sides of the triangle and also internally tangent to the circumcircle of that triangle. In 1999, Paul Yiu discovered an interesting property of these mixtilinear incircles.

Proposition 1 (Yiu [8])**.** *If the points of contact of the mixtilinear incircles of $\triangle ABC$ with the circumcircle are X, Y, and Z, then the lines AX, BY, and CZ are concurrent* (Figure 1)*. The point of concurrence is the external center of similitude of the incircle and the circumcircle.* [1]

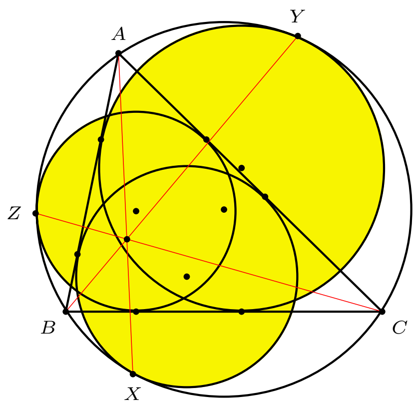


Figure 1

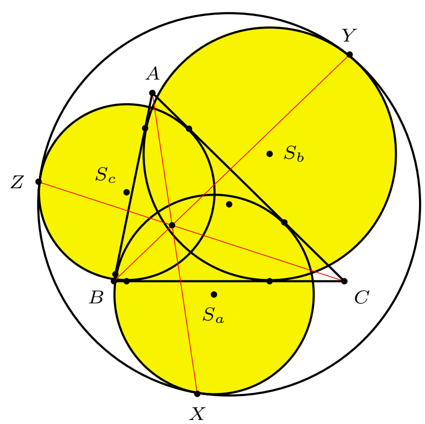


Figure 2

I wondered if there was anything special about the circumcircle in Proposition 1. After a little experimentation, I discovered that the result would remain true if the circumcircle was replaced with any circle in the plane of $\triangle ABC$.

Publication Date: March 26, 2006. Communicating Editor: Paul Yiu.

[1]This is the triangle center X_{56} in Kimberling's list [5].

Theorem 2. *Let (S) be any circle in the plane of $\triangle ABC$. Suppose that there are three circles, (S_a), (S_b), and (S_c), each internally* (*respectively externally*) *tangent to (S). Furthermore, suppose (S_a), (S_b), (S_c) are inscribed in $\angle BAC$, $\angle ABC$, $\angle ACB$ respectively* (Figure 2)*. Let the points of tangency of (S_a), (S_b), and (S_c) with (S) be X, Y, and Z, respectively. Then the lines AX, BY, and CZ are concurrent at a point P. The point P is the external* (*respectively internal*) *center of similitude of the incircle of $\triangle ABC$ and circle (S).*

When we say that a circle is *inscribed* in an angle ABC, we mean that the circle is tangent to the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

Definitions. Given a triangle and a circle, a *pseudo-incircle* of the triangle is a circle that is tangent to two sides of the given triangle and internally tangent to the given circle. A *pseudo-excircle* of the triangle is a circle that is tangent to two sides of the given triangle and externally tangent to the given circle.

There are many configurations that meet the requirements of Theorem 2. Figure 2 shows an example where (S) surrounds the triangle and the three circles are all internally tangent to (S). Figure 3a shows an example of pseudo-excircles where (S) lies inside the triangle. Figure 3b shows an example where (S) intersects the triangle. Figure 3c shows an example where (S) surrounds the triangle and the three circles are all externally tangent to (S).

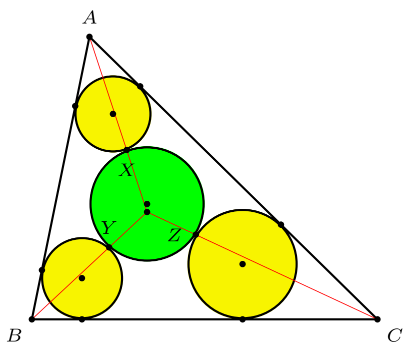


Figure 3a. Pseudo-excircles

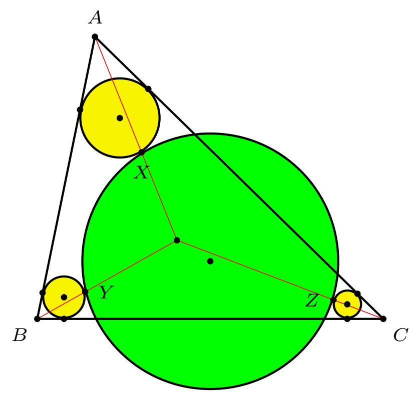


Figure 3b. Pseudo-excircles

There is another way of viewing Figure 3a. Instead of starting with the triangle and the circle (S), we can start with the other three circles. Then we get the following proposition.

Proposition 3. *Let there be given three circles in the plane, each external to the other two. Let triangle ABC be the triangle that circumscribes these three circles* (*that is, the three circles are inside the triangle and each side of the triangle is a common external tangent to two of the circles*)*. Let (S) be the circle that is externally tangent to all three circles. Let the points of tangency of (S) with the three circles be X, Y, and Z* (Figure 3a)*. Then lines AX, BY, and CZ are concurrent.*

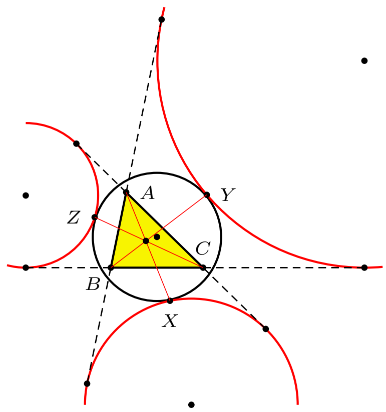


Figure 3c. Pseudo-excircles

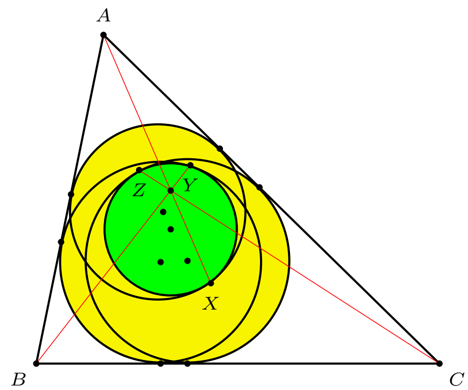


Figure 4. Pseudo-incircles

Figure 4 also shows pseudo-incircles (as in Figure 2), but in this case, the circle (S) lies inside the triangle. The other three circles are internally tangent to (S) and again, AX, BY, and CZ are concurrent. This too can be looked at from the point of view of the circles, giving the following proposition.

Proposition 4. *Let there be given three mutually intersecting circles in the plane. Let triangle ABC be the triangle that circumscribes these three circles* (Figure 4). *Let (S) be the circle that is internally tangent to all three circles. Let the points of tangency of (S) with the three circles be X, Y, and Z. Then lines AX, BY, and CZ are concurrent.*

We need the following result before proving Theorem 2. It is a generalization of Monge's three circle theorem ([7, p.1949]).

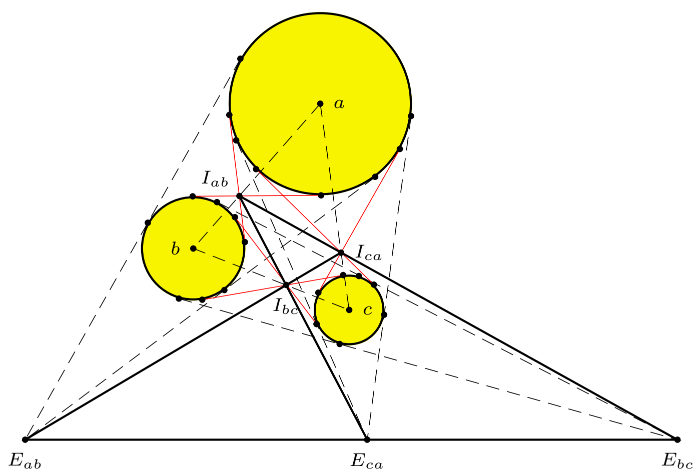


Figure 5. Six centers of similitude

Proposition 5 ([1, p.188], [2, p.151], [6]). *The six centers of similitude of three circles taken in pairs lie by threes on four straight lines* (Figure 5). *In particular, the three external centers of similitude are collinear; and any two internal centers of similitude are collinear with the third external one.*

2. Proof of Theorem 2

Figure 6 shows an example where the three circles are all externally tangent to (S), but the proof holds for the internally tangent case as well. Let I be the center of the incircle.

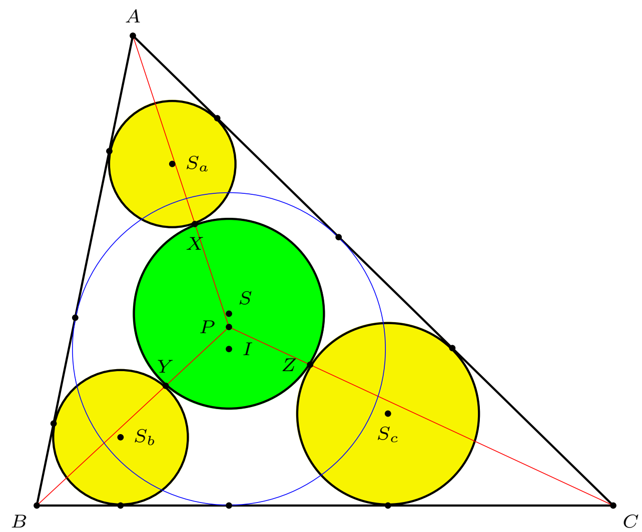


Figure 6

Consider the three circles (S), (I), and (S_c). The external common tangents of circles (J) and (S_c) are sides AC and BC, so C is their external center of similitude. Circles (S) and (S_c) are tangent externally (respectively internally), so their point of contact, Z, is their internal (respectively external) center of similitude. By Proposition 5, points C and Z are collinear with P, the internal (respectively external) center of similitude of circles (S) and (I). That is, line CZ passes through P. Similarly, lines AX and BY also pass through P.

Corollary 6. *In Theorem 2, the points S, P, and I are collinear* (Figure 6).

Proof. Since P is a center of similitude of circles (I) and (S), P must be collinear with the centers of the two circles. □

3. Special Cases

Theorem 2 holds for any circle, (S), in the plane of the triangle. We can get interesting special cases for particular circles. We have already seen a special case in Proposition 1, where (S) is the circumcircle of $\triangle ABC$.

3.1. *Mixtilinear excircles.*

Corollary 7 (Yiu [9])**.** *The circle tangent to sides AB and AC of $\triangle ABC$ and also externally tangent to the circumcircle of $\triangle ABC$ touches the circumcircle at point X. In a similar fashion, points Y and Z are determined* (Figure 7)*. Then the lines AX, BY, and CZ are concurrent. The point of concurrence is the internal center of similitude of the incircle and circumcircle of $\triangle ABC$.*[2]

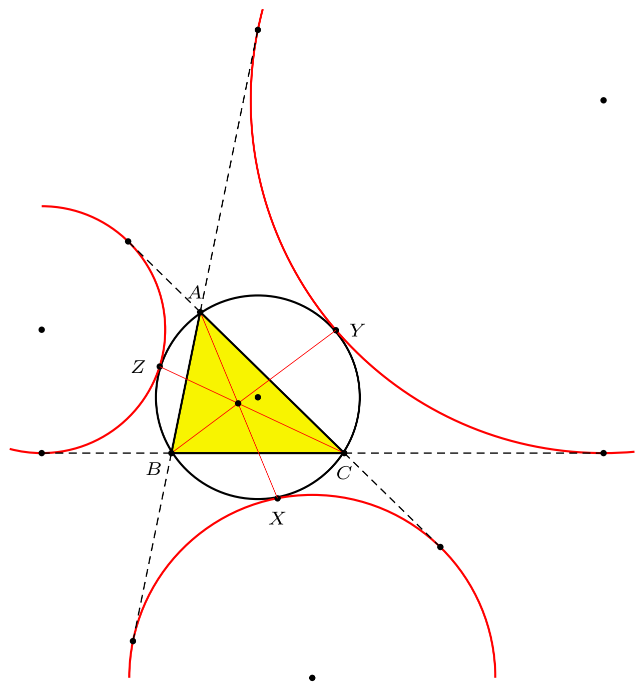


Figure 7. Mixtilinear excircles

3.2. *Malfatti circles.* Consider the Malfatti circles of a triangle ABC. These are three circles that are mutually externally tangent with each circle being tangent to two sides of the triangle.

Corollary 8. *Let (S) be the circle circumscribing the three Malfatti circles,* i.e., *internally tangent to each of them.* (Figure 8a)*. Let the points of tangency of (S) with the Malfatti circles be X, Y, and Z. Then AX, BY, and CZ are concurrent.*

Corollary 9. *Let (S) be the circle inscribed in the curvilinear triangle bounded by the three Malfatti circles of triangle ABC* (Figure 8b)*. Let the points of tangency of (S) with the Malfatti circles be X, Y, and Z. Then AX, BY, and CZ are concurrent.*

[2]This is the triangle center X_{55} in Kimberling's list [5]; see also [4, p.75].

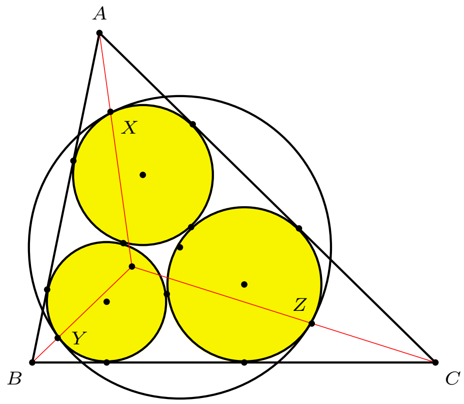


Figure 8a. Malfatti circumcircle

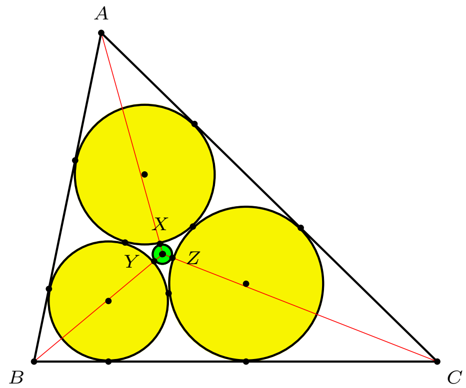


Figure 8b. Malfatti incircle

3.3. *Excircles.*

Corollary 10 (Kimberling [3])**.** *Let (S) be the circle circumscribing the three excircles of triangle ABC,* i.e.*, internally tangent to each of them. Let the points of tangency of (S) with the excircles be X, Y, and Z.* (Figure 9)*. Then AX, BY, and CZ are concurrent.*

The point of concurrence is known as the Apollonius point of the triangle. It is X_{181} in [5]. See also [4, p.102].

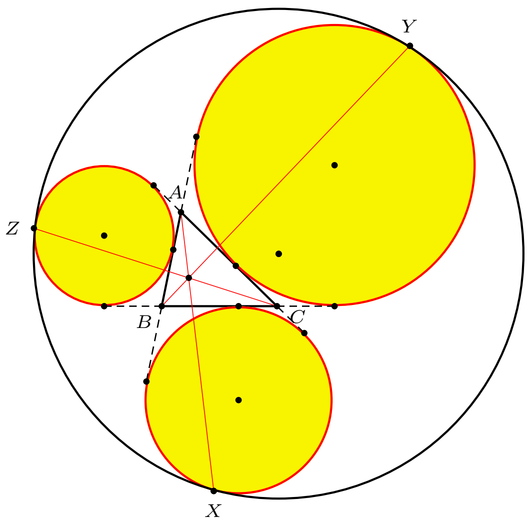


Figure 9. Excircles

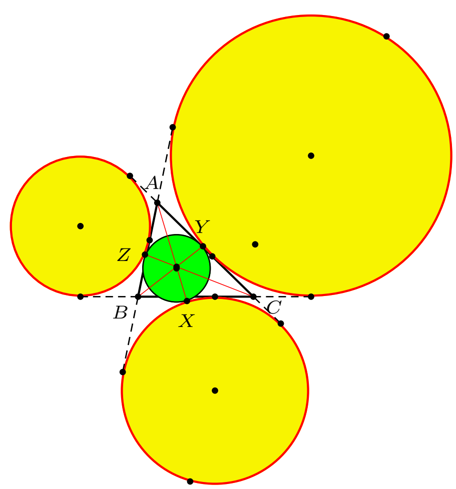


Figure 10. Nine-point circle

If we look at the circle externally tangent to the three excircles, we know by Feuerbach's Theorem, that this circle is the nine-point circle of $\triangle ABC$ (the circle that passes through the midpoints of the sides of the triangle).

Corollary 11 ([4, p.158]). *If the nine point circle of $\triangle ABC$ touches the excircles at points X, Y, and Z* (Figure 10)*, then AX, BY, and CZ are concurrent.*

4. Generalizations

In Theorem 2, we required that the three circles (S_a), (S_b), and (S_c) be inscribed in the angles of the triangle. In that case, lines AS_a, BS_b, and CS_c concur at the incenter of the triangle. We can use the exact same proof to handle the case where the three lines AS_a, BS_b, and CS_c meet at an excenter of the triangle. We get the following result.

Theorem 12. *Let (S) be any circle in the plane of $\triangle ABC$. Suppose that there are three circles, (S_a), (S_b), and (S_c), each tangent internally* (*respectively externally*) *to (S). Furthermore, suppose (S_a) is tangent to lines AB and AC; (S_b) is tangent to lines BC and BA; and (S_c) is tangent to lines CA and CB. Let the points of tangency of (S_a), (S_b), and (S_c) with (S) be X, Y, and Z, respectively. Suppose lines AS_a, BS_b, and CS_c meet at the point J, one of the excenters of $\triangle ABC$* (Figure 11)*. Furthermore, assume that sides BA and BC of the triangle are the external common tangents between excircle (J) and circle (S_a); similarly for circles (S_b), and (S_c). Then AX, BY, and CZ are concurrent at a point P. The point P is the external* (*respectively internal*) *center of similitude of circles (J) and (S). The points J, P, and S are collinear.*

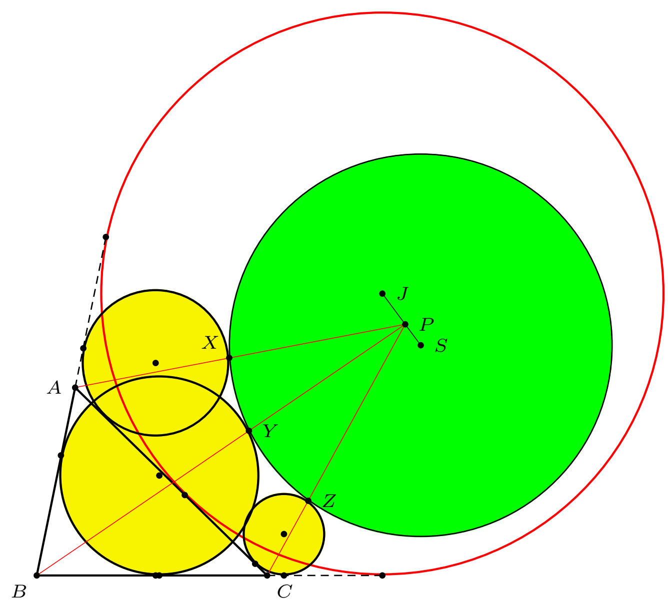


Figure 11. Pseudo-excircles

Figure 12 illustrates the case when (S) is tangent internally to each of (S_a), (S_b), (S_c).

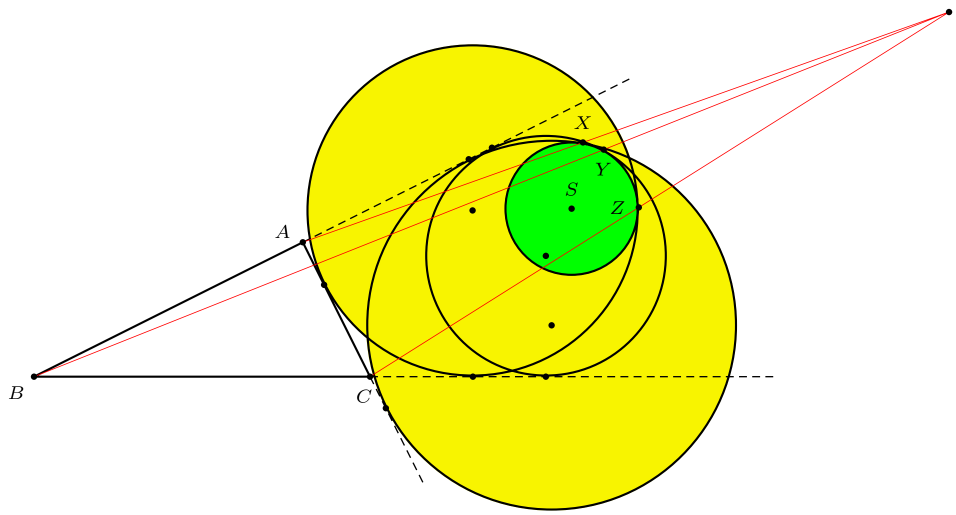


Figure 12. Pseudo-incircles

Examining the proof of Theorem 2, we note that at each vertex of the triangle, we have found that the line from that vertex to the point of contact of the circle inscribed in that angle and circle (S) passes through a fixed point (one of the centers of similitude of (S) and the incircle of the triangle). We note that the result and proof would be the same for a polygon provided that the polygon had an incircle. This gives us the following result.

Theorem 13. *Let $A_1A_2A_3\ldots A_n$ be a convex n-gon circumscribed about a circle (J). Let (S) be any circle in the interior of this n-gon. Suppose there are n circles, (S_1), (S_2), ..., (S_n) each tangent externally to (S) such that for $i = 1, 2, \ldots, n$, circle (S_i) is also inscribed in angle $A_{i-1}A_iA_{i+1}$ (where $A_0 = A_n$ and $A_{n+1} = A_1$). Let X_i be the point of tangency of circles (S_i) and (S)* (Figure 13). *Then the lines A_iX_i, $i = 1, 2, \ldots, n$ are concurrent. The point of concurrence is the internal center of similitude of the circles (S) and (J).*

In order to generalize Theorem 2 to three dimensions, we need to first note that Proposition 5 generalizes to 3 dimensions.

Theorem 14. *The six centers of similitude of three spheres taken in pairs lie by threes on four straight lines. In particular, the three external centers of similitude are collinear; and any two internal centers of similitude are collinear with the third external one.*

Proof. Consider the plane through the centers of the three spheres. This plane passes through all 6 centers of similitude. The plane cuts each sphere in a circle. Thus, on this plane, Proposition 5 applies, thus proving that the result holds for the spheres as well. □

Theorem 15. *Let $T = A_1A_2A_3A_4$ be a tetrahedron. Let (S) be any sphere in the interior of T* (*or let (S) be any sphere surrounding T*). *Suppose there are four*

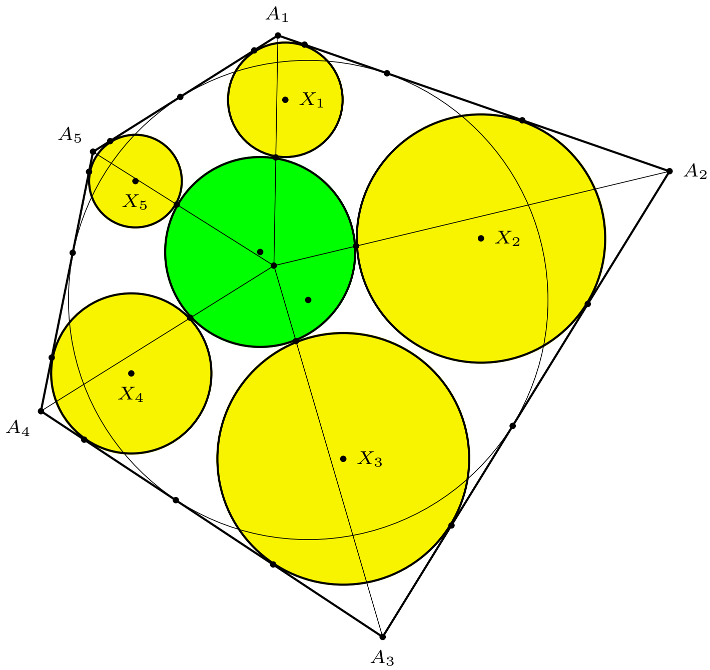


Figure 13. Pseudo-excircles in a pentagon

spheres, (S_1), (S_2), (S_3), *and* (S_4) *each tangent internally* (*respectively externally*) *to* (S) *such that sphere* (S_i) *is also inscribed in the trihedral angle at vertex* A_i. *Let* X_i *be the point of tangency of spheres* (S_i) *and* (S). *Then the lines* A_iX_i, $i = 1, 2, 3, 4$, *are concurrent. The point of concurrence is the external* (*respectively internal*) *center of similitude of sphere* (S) *and the sphere inscribed in* T.

The proof of this theorem is exactly the same as the proof of Theorem 2, replacing the reference to Proposition 5 by Theorem 14.

It is also clear that this result generalizes to E^n.

References

[1] N. Altshiller-Court, *College Geometry*, 2nd edition, Barnes & Noble. New York: 1952.
[2] R. A. Johnson, *Advanced Euclidean Geometry*, Dover Publications, New York: 1960.
[3] C. Kimberling, Problem 1091, *Crux Math.*, 11 (1985) 324.
[4] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.
[5] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[6] R. Walker, Monge's theorem in many dimensions, *Math. Gaz.*, 60 (1976) 185–188.
[7] E. W. Weisstein, *CRC Concise Encyclopedia of Mathematics*, 2nd edition. Chapman & Hall. Boca Raton: 2003.
[8] P. Yiu, Mixtilinear incircles, *Amer. Math. Monthly*, 106 (1999) 952–955.
[9] P. Yiu, Mixtilinear incircles, II, preprint, 1998.

Stanley Rabinowitz: MathPro Press, 10 Cliff Road, Chelmsford, Massachusetts, 01824. USA
E-mail address: `Stan.Rabinowitz@comcast.net`

Forum Geometricorum
Volume 6 (2006) 117–138.

FORUM GEOM
ISSN 1534-1178

Grassmann cubics and Desmic Structures

Wilson Stothers

Abstract. We show that each cubic of type $\mathrm{n}\mathcal{K}$ which is not of type $\mathrm{c}\mathcal{K}$ can be described as a Grassmann cubic. The geometry associates with each such cubic a cubic of type $\mathrm{p}\mathcal{K}$. We call this the parent cubic. On the other hand, each cubic of type $\mathrm{p}\mathcal{K}$ has infinitely many child cubics. The key is the existence of a desmic structure associated with parent and child. This extends work of Wolk by showing that, not only do (some) points of a desmic structure lie on a cubic, but also that they actually generate the cubic as a locus. Along the way, we meet many familiar cubics.

1. Introduction

In Hyacinthos, #3991 and follow-up, Ehrmann and others gave a geometrical description of cubics as loci. They showed that each cubic of type $\mathrm{n}\mathcal{K}_0$ has an associated sister of type $\mathrm{p}\mathcal{K}$, and that each cubic of type $\mathrm{p}\mathcal{K}$ has three sisters of type $\mathrm{n}\mathcal{K}_0$. Here, we show that each cubic of type $\mathrm{n}\mathcal{K}$ but not of type $\mathrm{c}\mathcal{K}$ has a parent of type $\mathrm{p}\mathcal{K}$, and every cubic of type $\mathrm{p}\mathcal{K}$ has infinitely many children, but just three of type $\mathrm{n}\mathcal{K}_0$. Our results do not appear to extend to cubics of type $\mathrm{c}\mathcal{K}$, so the geometry must be rather different. Throughout, we use barycentric coordinates. We write the coordinates of P as $p : q : r$. We are interested in isocubics, that is circumcubics which are invariant under an isoconjugation. The theory of isocubics is beautifully presented in [1]. There we learn that an isocubic has an equation of one of the following forms :

$$\begin{array}{ll} \mathrm{p}\mathcal{K}(W,R): & rx(wy^2 - vz^2) + sy(uz^2 - wx^2) + tz(vx^2 - uy^2) = 0. \\ \mathrm{n}\mathcal{K}(W,R,k): & rx(wy^2 + vz^2) + sy(uz^2 + wx^2) + tz(vx^2 + uy^2) + kxyz = 0. \end{array}$$

The point $R = r : s : t$ is known as the pivot of the cubic $\mathrm{p}\mathcal{K}$, and the root of the cubic $\mathrm{n}\mathcal{K}$ $W = u : v : w$ is the pole of the isoconjugation. This means that the isoconjuagte of $X = x : y : z$ is $\frac{u}{x} : \frac{v}{y} : \frac{w}{z}$. We may view W as the image of G under the isoconjugation. The constant k in the latter equation is determined by a point on the curve which is not on a sideline. Another interpretation of k appears below. One important subclass of type $\mathrm{n}\mathcal{K}$ occurs when $k = 0$. We have

$$\mathrm{n}\mathcal{K}_0(W,R): \qquad rx(wy^2 + vz^2) + sy(uz^2 + wx^2) + tz(vx^2 + uy^2) = 0.$$

Publication Date: April 3, 2006. Communicating Editor: Bernard Gibert.

Another subclass consists of the conico-pivotal isocubics. These are defined in terms of the root R and node $F = f : g : h$. The equation has the form

$$\mathrm{c}\mathcal{K}(\#F, R): \qquad rx(hy - gz)^2 + sy(fz - hx)^2 + tz(gx - fy)^2 = 0.$$

It is the $\mathrm{n}\mathcal{K}(W, R, k)$ with $W = f^2 : g^2 : h^2$, so F is a fixed point of the isoconjuation, and $k = -2(rgh + shf + tfg)$. We require a further concept from [1], that of the polar conic of R for $\mathrm{p}\mathcal{K}(W, R)$. In our notation, this has equation

$$\mathrm{p}\mathcal{C}(W, R): \qquad (vt^2 - ws^2)x^2 + (wr^2 - ut^2)y^2 + (us^2 - vr^2)z^2 = 0.$$

Notice that, $\mathrm{p}\mathcal{C}(W, R)$ contains the four fixed points of the isoconjugation, so is determined by one other point, such as R. Hence, as befits a polar object, if S is on $\mathrm{p}\mathcal{C}(W, R)$, then R is on $\mathrm{p}\mathcal{C}(W, S)$. Also, for fixed W, the class of cubics of type $\mathrm{p}\mathcal{K}$ splits into disjoint subclasses, each subclass having a common polar conic. We also remark that the equation for the polar conic vanishes identically when W is the barycentric square of R. In such a case, it is convenient to regard the class of $\mathrm{p}\mathcal{K}(W, R)$ as consisting of just $\mathrm{p}\mathcal{K}(W, R)$ itself.

Definition 1. (a) *For a point X with barycentrics $x : y : z$, the A-harmonic of X is the point X_A with barycentrics $-x : y : z$. The B- and C-harmonics, X_B and X_C, are defined analagously.*
(b) *The harmonic associate of the triangle $\triangle PQR$ is the triangle $\triangle P_A Q_B R_C$.*

Observe that $\triangle X_A X_B X_C$ is the anticevian triangle of X. Also, if X is on $\mathrm{p}\mathcal{C}(W, R)$ then X_A, X_B, X_C also lie on $\mathrm{p}\mathcal{C}(W, R)$. In the preamble to $X(2081)$ in [5] we meet Gibert's PK- and NK-transforms. These are defined in terms of isogonal conjugation. We shall need the general case for W-isoconjugation.

Definition 2. *Suppose that W is a fixed point, let P^* denote the W-isoconjugate of P.*
(a) *The PKW-transform of P is PK(W,P), the intersection of the tripolars of P and P^* (if $P \neq P^*$).*
(b) *The NKW-transform of P is $NK(W, P)$, the crosspoint of P and P^*.*

Note that $PK(W, P)$ is the perspector of the circumconic through P and P^*. $PK(W, P)$ occurs several times in our work, in apparently unrelated contexts. From these definitions, if $W = u : v : w$ and $P = p : q : r$, then

$$\begin{aligned} PK(W, P) &= pu(r^2v - q^2w) : qv(p^2w - r^2u) : rw(q^2u - p^2v), \\ NK(W, P) &= pu(r^2v + q^2w) : qv(p^2w + r^2u) : rw(q^2u + p^2v). \end{aligned}$$

Note that $PK(W, R)$ is undefined if $W = R^2$.

2. Grassmann cubics associated with desmic Structures

Definition 3. *Let $\Delta = \triangle ABC$ be the reference triangle. Let $\Delta' = \triangle A'B'C'$ where A' is not on BC, B' is not on CA and C' is not on AB.*
(a) $GP(\Delta') = \{P :$ *the triangle with vertices $A'P \cap BC$, $B'P \cap CA$, $C'P \cap AB$ is perspective with Δ*$\}$,
(b) $GN(\Delta') = \{P :\ A'P \cap BC,\ B'P \cap CA,\ C'P \cap AB$ *are collinear*$\}$.

We note that $GN(\Delta)$ is a special case of a Grassmann cubic. There are three examples in the current (November 2004) edition of [2]. The Darboux cubic $\mathrm{p}\mathcal{K}(K, X(20))$ is $K004$ in Gibert's catalogue. Under the properties listed is the fact that it is the locus of points whose pedal triangle is a cevian triangle. This is $GP(\triangle A'B'C')$, where the vertices are the infinite points on the altitudes. The corresponding GN is then the union of the line at infinity and the circumcircle, a degenerate cubic. In the discussion of the cubic $\mathrm{n}\mathcal{K}(K, X(5))$ - Gibert's $K216$ - it is mentioned that it is GN for a certain triangle, and that the corresponding GP is the Neuberg cubic, $\mathrm{p}\mathcal{K}(K, X(30))$ - Gibert's $K001$. As we shall see, there are other examples where suitable triangles are listed, and others where they can be identified.

Lemma 1. *If $\Delta' = \triangle A'B'C'$ has $A' = a_1 : a_2 : a_3$, $B' = b_1 : b_2 : b_3$, $C' = c_1 : c_2 : c_3$, then,*
(a) *$GP(\Delta')$ has equation*

$$(a_2x - a_1y)(b_3y - b_2z)(c_1z - c_3x) + (a_1z - a_3x)(b_1y - b_2x)(c_2z - c_3y) = 0,$$

(b) *$GN(\Delta')$ has equation*

$$(a_2x - a_1y)(b_3y - b_2z)(c_1z - c_3x) - (a_1z - a_3x)(b_1y - b_2x)(c_2z - c_3y) = 0.$$

(c) *Each locus contains A, B, C and A', B', C'.*

Proof. It is easy to see that, if $P = x : y : z$, then

$$\begin{aligned} A_P &= A'P \cap BC = 0 : a_2x - a_1y : a_3x - a_1z, \\ B_P &= B'P \cap CA = b_1y - b_2x : 0 : b_3y - b_2z, \\ C_P &= C'P \cap AB = c_1z - c_3x : c_2z - c_3y : 0. \end{aligned}$$

The condition in $GN(\Delta)$ is equivalent to the vanishing of the determinant of the coefficients of these points. This gives (b).

The condition in $GP(\Delta)$ is equivalent to the concurrence of AA_P, BB_P and CC_P. This is another determinant condition. The determinant is formed from the previous one by changing the sign of one entry in each row. This gives (a). Once we have the equations, (c) is clear. □

We will also require a condition for a triangle $\triangle A'B'C'$ to be perspective with $\triangle ABC$.

Lemma 2. *If $\Delta = \triangle A'B'C'$ has $A' = a_1 : a_2 : a_3$, $B' = b_1 : b_2 : b_3$, $C' = c_1 : c_2 : c_3$, then,*
(a) *$\triangle A'B'C'$ is perspective with $\triangle ABC$ if and only if $a_2b_3c_1 = a_3b_1c_2$.*
(b) *$\triangle A'B'C'$ is triply perspective with $\triangle ABC$ if and only if $a_1b_2c_3 = a_2b_3c_1 = a_3b_1c_2$.*

Proof. We observe that the perspectivity in (a) is equivalent to the concurrence of AA', BB' and CC'. The given equality expresses the condition for the intersection of AA' and BB' to lie on CC'. Part (b) follows by noting that $\triangle A'B'C'$ is triply perspective with $\triangle ABC$ if and only if each of $\triangle A'B'C'$, $\triangle B'C'A'$ and $\triangle C'A'B'$ is perspective with $\triangle ABC$. □

We observe that each of the equations in Lemma 1 has the form

$$x(f_1y^2 + f_2z^2) + y(g_1z^2 + g_2x^2) + z(h_1x^2 + h_2y^2) + kxyz = 0. \qquad (1)$$

This has the correct form for the cubic to be of type $\mathrm{p}\mathcal{K}$ or $\mathrm{n}\mathcal{K}$.

Theorem 3. *For a triangle $\Delta' = \triangle A'B'C'$,*
(a) *$GP(\Delta)$ is of type $\mathrm{p}\mathcal{K}$ if and only if Δ' is perspective with $\triangle ABC$.*
(b) *If Δ' is degenerate, then $GN(\Delta')$ is degenerate.*
Suppose that $GN(\Delta')$ is non-degenerate. Then
(c) *$GN(\Delta')$ is of type $\mathrm{n}\mathcal{K}$ if and only if Δ' is perspective with $\triangle ABC$.*
(d) *$GN(\Delta')$ is of type $\mathrm{n}\mathcal{K}_0$ if and only if Δ' is triply perspective with $\triangle ABC$.*

Proof. Suppose that $A' = a_1 : a_2 : a_3$, $B' = b_1 : b_2 : b_3$, $C' = c_1 : c_2 : c_3$, with $a_1b_2c_3 \neq 0$.

(a) We begin by observing that equation (1) gives a cubic of type $\mathrm{p}\mathcal{K}$ if and only if $f_1g_1f_1 + f_2g_2h_2 = 0$ and $k = 0$. The equation in Lemma 1(a) has k = 0 if and only if $a_2b_3c_1 = a_3b_1c_2$. By Lemma 2(a), this is the condition for the triangles to be perspective. A Maple calculation shows that the other condition for a $\mathrm{p}\mathcal{K}$ is satisfied if the triangles are perspective. This establishes (a).

(b) If Δ' is degenerate, then A', B' and C' lie on a line $\mathcal{L}$. For any P on $\mathcal{L}$, the intersection of PA' with BC lies on $\mathcal{L}$, as do those of PB' with CA, and PC' with AB. Thus, these intersections are collinear, so P is on $GN(\Delta')$. Now the locus contains the line $\mathcal{L}$, so that it must be degenerate.

Now suppose that the locus $GN(\Delta')$ is non-degenerate.

(c) Equation (1) gives a cubic of type $\mathrm{n}\mathcal{K}$ if and only if $f_1g_1f_1 - f_2g_2h_2 = 0$. The equation in Lemma 1(b) has this property if and only if

$$(a_2b_3c_1 - a_3b_1c_2)D = 0,$$

where D is the determinant of the matrix whose rows are the coordinates of A', B' and C'. Now, $D = 0$ if and only if A', B' and C' are collinear, so Δ' and hence $GN(\Delta')$ are degenerate. By Lemma 2(a), the other condition is equivalent to the perspectivity of the triangles.

(d) For a cubic of type $\mathrm{n}\mathcal{K}_0$, we require $a_2b_3c_1 - a_3b_1c_2 = 0$, as in (c). We also require that $k = 0$. From the equation in Lemma 1(b), $k = a_2b_3c_1 + a_3b_1c_2 - 2a_1b_2c_3$. These two conditions are equivalent to triple perspectivity by Lemma 2(b). $\square$

Our work so far leads us to consider triangles $\triangle A'B'C'$ perspective to $\triangle ABC$, with A' not on BC, B' not on CA, and C' not on AB. Looking at our loci, we are led to consider a further triangle, also perspective with $\triangle ABC$. This turns out to be the desmic mate of $\triangle A'B'C'$, so we are led to consider desmic structures which include the points A, B and C.

Theorem 4. *Suppose that $\triangle A'B'C'$ is perspective to $\triangle ABC$, with A' not on BC, B' not on CA, and C' not on AB. Let the perspector be $P_1 = p_{11} : p_{12} : p_{13}$.*

(a) *Suitably normalized, we have*

$$\begin{aligned} A' &= p_{21} : p_{12} : p_{13}, \\ B' &= p_{11} : p_{22} : p_{13}, \\ C' &= p_{11} : p_{12} : p_{23}. \end{aligned}$$

Let

$$\begin{aligned} W &= p_{11}p_{21} : p_{12}p_{22} : p_{13}p_{23}, \\ R &= p_{11} - p_{21} : p_{12} - p_{22} : p_{13} - p_{23}, \\ S &= p_{11} + p_{21} : p_{12} + p_{22} : p_{13} + p_{23}. \end{aligned}$$

(b) $GP(\triangle A'B'C') = \mathrm{p}\mathcal{K}(W, S)$,
(c) $GN(\triangle A'B'C') = \mathrm{n}\mathcal{K}(W, R, 2(p_{21}p_{22}p_{23} - p_{11}p_{12}p_{13}))$.

Let $A'' = p_{11} : p_{22} : p_{23}$, $B'' = p_{21} : p_{12} : p_{23}$, $C'' = p_{21} : p_{22} : p_{13}$. *These are the* W*-isoconjugates of* A', B' *and* C'.
(d) $\triangle A''B''C''$ *is perspective with* $\triangle ABC$, *with perspector* $P_2 = p_{21} : p_{22} : p_{23}$.
(e) $\triangle A''B''C''$ *is perspective with* $\triangle A'B'C'$, *with perspector* S.
(f) (P_1, P_2, R, S) *is a harmonic range.*
(g) $GP(\triangle A''B''C'') = GP(\triangle A'B'C')$. *The common locus includes* P_1, P_2 *and* S.
(h) $GN(\triangle A''B''C'') = GN(\triangle A'B'C')$. *The common locus includes the intersections of the tripolar of* R *with the sidelines.*
(i) $\triangle ABC$, $\triangle A'B'C'$ *and* $\triangle A''B''C''$ *have common perspectrix, the tripolar of* R.
(j) P_1 *and* P_2 *lie on* $\mathrm{p}\mathcal{K}(W, R)$.

Proof. (a) Since we have the perspector, the coordinates of the vertices A', B', C' must be as described.

(b),(c) These are simply verifications using the equations in Lemma 1.

(d) This follows at once from the coordinates of A'', B'', C''.

(e) This requires the calculations that the lines $A'A''$, $B'B''$ and $C'C''$ all pass through S.

(f) The coordinates of the points make this clear.

(g),(h) First, we note that, if we interchange the roles of P_1 and P_2, we get the same equations. The fact that the given points lie on the respective loci are simply verifications.

(i) Once again, this can be checked by calculation. We can also argue geometrically. Suppose that a point X on $B'C'$ lies on the locus. Then XB' and XC' are $B'C'$, so this must be the common line. Then XA' must meet BC on this line. But X is on $B'C'$, so $X = BC \cap B'C'$. If X also lies on $B''C''$, then $X = BC \cap B''C''$. This shows that we must have a common perspectrix. The identification of the perspectrix uses the fact that the cubic meets the each sideline of $\triangle ABC$ in just three points, two vertices and the intersection with the given tripolar.

(j) This is a routine verification. $\square$

In the notation of Theorem 4, we have a desmic structure with the twelve points described as vertices A, B, C, A', B', C', A'', B'', C'', perspectors P_1, P_2, S.

Many authors describe S as the desmon, and R as the harmon of the structure. Some refer to P_1 as the perspector, and P_2 as the coperspector. This description of a desmic structure with vertices including A, B, C is discussed by Barry Wolk in Hyacinthos #462. He observed that the twelve points all lie on $\mathrm{p}\mathcal{K}(W,S)$. What may be new is the fact that the other vertices may be used to generate this cubic as a locus, and the corresponding $\mathrm{n}\mathcal{K}$ as a Grassmann cubic.

Notice that the desmic structure in not determined by its perspectors. If we choose barycentrics for P_1, we need to scale P_2 so that the barycentrics of the desmon and harmon are, respectively, the sum and difference of those of P_1 and P_2. When P_2 is suitably scaled, we say that the perspectors are normalized. The normalization is determined by a single vertex, provided neither perspector is on a sideline.

There are two obvious questions.

(1) Is every cubic of type $\mathrm{p}\mathcal{K}$ a locus of type GP associated with a desmic structure?

(2) Is every cubic of type $\mathrm{n}\mathcal{K}$ a locus of type GN associated with a desmic structure?

The answer to (1) is that, with six exceptions, each point on a $\mathrm{p}\mathcal{K}$ is a perspector of a suitable desmic structure. The answer to (2) is more complicated. There is a class of $\mathrm{n}\mathcal{K}$ which do not possess a suitable desmic structure. This is the class of conico-pivotal isocubics. For each other cubic of type $\mathrm{n}\mathcal{K}$, there is a unique desmic structure.

Theorem 5. *Suppose that P is a point on $\mathrm{p}\mathcal{K}(W,S)$ which is not fixed by W-isoconjugation, and is not S or its W-isoconjugate. Then there is a unique desmic structure with vertices A, B, C and perspector $P_1 = P$ with locus $GP = \mathrm{p}\mathcal{K}(W,S)$.*

Proof. For brevity, we shall write X^* for the W-isoconjugate of a point X. As P is on $\mathrm{p}\mathcal{K}(W,S)$, S is on PP^*. Then $S = mP + nP^*$, for some constants m, n, with $mn \neq 0$. If $P = p : q : r$, $W = u : v : w$, put $A' = nu/p : mq : mr$, $B' = mp : nv/q : mr$, $C' = mp : mq : nw/r$. From Theorem 4, this has locus $GP(\triangle A'B'C') = \mathrm{p}\mathcal{K}(W,S)$. □

Note that the conditions on P are necessary to ensure that S can be expressed in the stated from. For the second question, we proceed in two stages. First, we show that a cubic of type $\mathrm{n}\mathcal{K}$ has at most one suitable desmic structure. This identifies the vertices of the structure. We then show that this choice does lead to a description of the cubic as a Grassmann cubic. The first result uses the idea of A-harmonics introduced in the Introduction.

Theorem 6. *Throughout, we use the notation of Theorem 4.*

(a) *The vertices of the desmic structure on $\mathrm{n}\mathcal{K}(W,R,k)$ are A, B, C and intersections of $\mathrm{n}\mathcal{K}(W,R,k)$ with the cubics $\mathrm{p}\mathcal{K}(W,R_A)$, $\mathrm{p}\mathcal{K}(W,R_B)$, $\mathrm{p}\mathcal{K}(W,R_C)$.*

(b) *$\mathrm{n}\mathcal{K}(W,R,k)$ and $\mathrm{p}\mathcal{K}(W,R_A)$ touch at B and C, intersect at A, at the intersections of the tripolar of R with AB and AC, and at two further points.*

(c) *If* $W = f^2 : g^2 : h^2$*, then* $\mathrm{c}\mathcal{K}(\#F, R)$ *and* $\mathrm{p}\mathcal{K}(W, R_A)$ *touch at* B *and* C*, and meet at* A*, at the intersections of the tripolar of* R *with* AB *and* AC*, and twice at* F*.*
(d) *If the final two points in* (b) *coincide, then* $\mathrm{n}\mathcal{K}(W, R, k) = \mathrm{c}\mathcal{K}(\#F, R)$*, where* F *is such that* $W = F^2$*.*
(e) *If either of the final points in* (b) *lie on a sideline, then* $k = 2ust/r$*,* $2vrt/s$ *or* $2wrs/t$*, where* $W = u : v : w$*,* $R = r : s : t$*.*

Proof. We observe that the vertices of the desmic structure can be derived from the normalized versions of the perspectors P_1 and P_2. The normalization is such that $R = P_1 - P_2$ and $S = P_1 + P_2$. Now consider the loci derived from the perspectors P_1 and $-P_2$. From Theorem 4, the cubics are $\mathrm{p}\mathcal{K}(W, R)$ and $\mathrm{n}\mathcal{K}(W, S, k')$, for some k'. The point $A' = p_{21} : p_{12} : p_{13}$ is on $\mathrm{n}\mathcal{K}(W, R, k)$, so that $A'_A = -p_{21} : p_{12} : p_{13}$ is on $\mathrm{p}\mathcal{K}(W, R)$. Then A' is on $\mathrm{p}\mathcal{K}(W, R_A)$ as only the first term is affected by the sign change, and this involves the product of the first coordinates. Thus (a) holds.

Part (b) is largely computational. Obviously the cubics meet at A, B and C. The tangents at B and C coincide. The intersections with the sidelines in each case include the stated meetings with the tripolar of R. Since two cubics have a total of nine meets, there are two unaccounted for. These are clearly W-isoconjugate.

For part (c), we use the result of (b) to get seven intersections. Then we need only verify that the cubics meet twice at F. Now F is clearly on $\mathrm{p}\mathcal{K}(W, R_A)$. But F is a double point on $\mathrm{c}\mathcal{K}(\#F, R)$, so there are two intersections here.

Part (d) relies on a Maple calculation. Solving the equations for $\mathrm{n}\mathcal{K}(W, R, k)$ and $\mathrm{p}\mathcal{K}(W, R_A)$ for $\{y, z\}$, we get the known points and the solutions of a quadratic. The discriminant vanishes precisely when $\mathrm{n}\mathcal{K}(W, R, k)$ is of type $\mathrm{c}\mathcal{K}$.

Part (e) uses the same computation. The quadratic equation in (d) has constant term zero precisely when $k = 2vrt/s$. Looking at other $\mathrm{p}\mathcal{K}(W, R_B)$, $\mathrm{p}\mathcal{K}(W, R_C)$ gives the other cases listed. □

To describe a cubic of type $\mathrm{n}\mathcal{K}$ as a Grassmann cubic as above, we require two perspectors interchanged by W-isoconjugation. Provided W is not on a sideline, we need six vertices not on a sideline. After Theorem 6, there are at most six candidates, with two on each of the associated $\mathrm{p}\mathcal{K}(W, R_X)$, $X = A, B, C$. Thus, there is at most one desmic structure defining the cubic. Also from Theorem 6, there is no structure if the cubic is a $\mathrm{c}\mathcal{K}(\#F, R)$, for then the "six" points and the perspectors are all F. We need to investigate the cases where either of the final solutions in Theorem 6(b) lie on a sideline. Rather than interrupt the general argument, we postpone the discussion of these cases to an Appendix. They still contain a unique structure which can be used to generate the cubics as Grassmann loci. The structure is a degenerate kind of desmic structure. We show that, in any other case, the six points do constitute a suitable desmic structure. In the proof, we assume that we can choose an intersection of $\mathrm{n}\mathcal{K}(W, R, k)$ and $\mathrm{p}\mathcal{K}(W, R_A)$ not on a sideline, so we need the discussion of the Appendix to tidy up the remaining cases.

Theorem 7. *If a cubic $\mathcal{C}$ is of type* $\mathrm{n}\mathcal{K}$*, but not of type* $\mathrm{c}\mathcal{K}$*, then there is a unique desmic structure which defines $\mathcal{C}$ as a Grassmann cubic.*

Proof. Suppose that $\mathcal{C} = \mathrm{n}\mathcal{K}(W, R, k)$ with $W = u : v : w$, and $R = r : s : t$. We require perspectors $P_1 = p_{11} : p_{12} : p_{13}$, and $P_2 = p_{21} : p_{22} : p_{23}$ such that $r : s : t = p_{11} - p_{21} : p_{12} - p_{22} : p_{13} - p_{23}$. This amounts to two linear equations which can be used to solve for p_{22} and p_{23} in terms of p_{11}, p_{12}, p_{13} and p_{21}. We also require that $u : v : w = p_{11}p_{21} : p_{12}p_{22} : p_{13}p_{23}$. This gives three relations in p_{11} and p_{21} in terms of p_{12} and p_{13}. These are consistent provided $A' = p_{21} : p_{12} : p_{13}$ is on $\mathrm{p}\mathcal{K}(W, R_A)$. This uses a Maple calculation. We can solve for p_{11} in terms of p_{12}, p_{13} and p_{21}, provided we do not have (after scaling) $p_{12} = -u/r$, $p_{13} = v/s$ and $p_{21} = w/t$. So far, we have shown that, if A' is on $\mathrm{p}\mathcal{K}(W, R_A)$, then we can reconstruct perspectors which give rise to some $\mathrm{n}\mathcal{K}(W, R, k')$. Provided that we can choose A' also on $\mathcal{C}$, but not on a sideline, then $k' = k$ directly, so we get $\mathcal{C}$ as a Grassmann cubic. As we saw in Theorem 6, there are just two such choices of A', and these are isoconjugate, so we have just one suitable desmic structure. We could equally use a point of intersection of C with $\mathrm{p}\mathcal{K}(W, R_B)$ or with $\mathrm{p}\mathcal{K}(W, R_C)$. It follows that there is a unique desmic structure unless $\mathcal{C}$ has the points $-u/r : v/s : w/t$, $u/r : -v/s : w/t$ and $u/r : v/s : -w/t$. But then $k = 2ust/r = 2vrt/s = 2wrs/t$, so that $W = R_2$, and $k = 2rst$. It follows that C is the degenerate cubic $(ty + sz)(rz + tx)(sx + ry) = 0$. It is easy to check that this is given as a Grassmann cubic by the degenerate desmic structure with $A' = A'' = -r : s : t$, and similarly for B', B'', S' and C''. This has perspectors, desmon and harmon equal to R. The GP locus is the whole plane. □

3. Parents and children

The reader will have noted the resemblance between the equations for the cubics $GP(\Delta')$ and $GN(\Delta')$. In [2, notes on K216], Gibert observes this for $K001$ and $K216$. He refers to $K216$ as a sister of $K001$. In Theorem 7, we saw that each cubic $\mathcal{C}$ of type $\mathrm{n}\mathcal{K}$ which is not of type $\mathrm{c}\mathcal{K}$ is the Grassmann cubic associated with a unique desmic structure, and hence with a unique cubic $\mathcal{C}'$ of type $\mathrm{p}\mathcal{K}$. We call the cubic $\mathcal{C}'$ the parent of $\mathcal{C}$. On the other hand, Theorem 5 shows that a cubic $\mathcal{C}$ of type $\mathrm{p}\mathcal{K}$ contains infinitely many desmic structures, each defining a cubic of type $\mathrm{n}\mathcal{K}$. We call each of these cubics a child of $\mathcal{C}$. Our first task is to describe the children of a cubic $\mathrm{p}\mathcal{K}(W, S)$. This involves the equation of the polar conic of S, see §1. Our calculations also give information on the parents of the family of cubics of type $\mathrm{n}\mathcal{K}$ with fixed pole and root.

Theorem 8. *Suppose that $\mathcal{C} = \mathrm{p}\mathcal{K}(W, S)$ with $W = u : v : w$, and $S = r : s : t$.*
(a) *Any child of $\mathcal{C}$ is of the form $\mathrm{n}\mathcal{K}(W, R, k)$, with R on*

$$\mathrm{p}\mathcal{C}(W, S) : \qquad (vt^2 - ws^2)x^2 + (wr^2 - ut^2)y^2 + (us^2 - vr^2)z^2 = 0.$$

(b) *If R is a point of $\mathrm{p}\mathcal{C}(W, S)$ which is not S and not fixed by W-isoconjugation, then there is a unique child of $\mathcal{C}$ of the form $\mathrm{n}\mathcal{K}(W, R, k)$.*
(c) *Any cubic $\mathrm{n}\mathcal{K}(W, R, k)$ which is not of type $\mathrm{c}\mathcal{K}$ has parent of the form $\mathrm{p}\mathcal{K}(W, S)$ with S on $\mathrm{p}\mathcal{C}(W, R)$.*

(d) *If* $\mathrm{n}\mathcal{K}(W,R,k)$ *has parent* $\mathrm{p}\mathcal{K}(W,S)$*, then the perspectors are the non-trivial intersections of* $\mathrm{p}\mathcal{K}(W,R)$ *and* $\mathrm{p}\mathcal{K}(W,S)$.

Proof. (a) We know from Theorem 7 that a child $\mathrm{n}\mathcal{K}(W,R,k)$ of $\mathcal{C}$ arises from a desmic structure. Suppose the perspectors are P_1 and P_2. From Theorem 4(f) (P_1,P_2,R,S) is a harmonic range. It follows that there are constants m and n with $P_1 = mR+nS$ and $P_2 = -mR+nS$. From Theorem 4(a), W is the barycentric product of P_1 and P_2. Suppose that $R = x:y:z$. Then we have

$$\frac{m^2x^2-n^2r^2}{u} = \frac{m^2y^2-n^2s^2}{v} = \frac{m^2z^2-n^2t^2}{w}.$$

If we eliminate m^2 and n^2 from these, we get $\mathrm{p}\mathcal{C}(W,S)=0$.

(b) Given such an R, we can reverse the process in (a) to obtain a suitable value for $(m/n)^2$. Choosing either root, we get the required perspectors.

(c) is really just the observation that S is on $\mathrm{p}\mathcal{C}(W,R)$ if and only if R is on $\mathrm{p}\mathcal{C}(W,S)$.

(d) In Theorem 4, we noted that the perspectors lie on $\mathrm{p}\mathcal{K}(W,S)$ and on $\mathrm{p}\mathcal{K}(W,R)$. Now these cubics meet at A, B, C and the four points fixed by W-isoconjugation. There must be just two other (non-trivial) intersections. □

Example 1. In terms of triangle centers, the most prolific parent seems to be the Neuberg cubic = $\mathrm{p}\mathcal{K}(K,X(30))$, Gibert's $K001$.

The polar cubic $\mathrm{p}\mathcal{C}(K,X(30))$ is mentioned in [5] in the discussion of its center, the Tixier point, X476. There, it is noted that it is a rectangular hyperbola passing through I, the excenters, and $X(30)$. Of course, being rectangular, the other infinite point must be $X(523)$. Using the information in [5], we see that its asymptotes pass through $X(74)$ and $X(110)$. The perspectors P_1, P_2 of desmic structures on $K001$ must be its isogonal pairs other than $\{X(30),\ X(74)\}$.

By Theorem 4(f), the root of the child cubic must be the mid-point of P_1 and P_2. The pair $\{O,\ H\}$ gives a cubic of the form $\mathrm{n}\mathcal{K}(K,X(5),k)$. The information in [2] identifies it as $K216$. The pair $\{X(13),\ X(15)\}$ gives a cubic of the form $\mathrm{n}\mathcal{K}(K,X(396),k)$. The pair $\{X(14),\ X(16)\}$ gives a cubic of the form $\mathrm{n}\mathcal{K}(K,X(395),k)$.

As noted above, $X(523)$ is on the polar conic, so we also have a child of the form $\mathrm{n}\mathcal{K}(K,X(523),k)$. Since $X(523)$ is not on the cubic, the perspectors must be at infinity. As they are isogonal conjugates, they must be the infinite circular points. These have already been noted as lying on $K001$. We now have additional centers on $\mathrm{p}\mathcal{C}(K,X(30))$, $X(5)$, $X(395)$, $X(396)$, as well as $X(1)$, $X(30)$, $X(523)$. We also have the harmonic associates of each of these points!

4. Roots and pivots

If we have a cubic $\mathrm{n}\mathcal{K}(W,R,k)$ defined by a desmic structure, then it has a parent cubic $\mathrm{p}\mathcal{K}(W,S)$. From Theorem 8(c), we know that S is on $\mathrm{p}\mathcal{C}(W,R)$. Since R is also on the conic, we can identify S from an equation for RS. Although the results were found by heavy computations, we can establish them quite simply by "guessing" the pole of RS with respect to $\mathrm{p}\mathcal{C}(W,R)$.

Theorem 9. *Suppose that $W = u : v : w$, $R = r : s : t$ and k are such that $\mathrm{n}\mathcal{K}(W, R, k)$ is defined by a desmic structure. Let $\mathrm{p}\mathcal{K}(W, S)$ be the parent of $\mathrm{n}\mathcal{K}(W, R, k)$.*
(a) *The line RS is the polar of $P = 2ust - kr : 2vtr - ks : 2wrs - kt$ with respect to $\mathrm{p}\mathcal{C}(W, R)$.*
(b) *The point S has first barycentric coordinate $4r(-r2vw + s2wu + t2uv) - 4kstu + k2r$.*

Proof. Suppose that the desmic structure has normalized perspectors $P_1 = f : g : h$ and $P_2 = f' : g' : h'$. Then we have

$$\begin{array}{lll} r = f - f', & s = g - g', & t = h - h'; \\ u = ff', & v = gg', & w = hh', \\ k = 2(f'g'h' - fgh). & & \end{array}$$

(a) The first barycentric of P is then

$$2(ff'(g - g')(h - h') - (f'g'h' - fgh)(f - f')) = 2(fg - f'g')(fh - f'h').$$

The coefficient of x^2 in $\mathrm{p}\mathcal{C}(W, R)$ is

$$vt^2 - ws^2 = (gh - g'h')(hg' - gh').$$

If we discard a symmetric factor the polar of P is the line P_1P_2, *i.e.*, the line RS.

(b) We know that $S = f + f' : g + g' : h + h'$. If we substitute the above values for r, s, t, u, v, w, k, the expression becomes $K(f + f')$, where K is symmetric in the variables. Thus, S is as stated. □

This result allows us to find the parent of a cubic, even if we cannot find the perspectors explicitly. It would also allow us to find the perspectors since these are the W-isoconjugate points on the line RS. Thus the perspectors arise as the intersections of a line and (circum)conic.

Example 2. The second Brocard cubic $\mathrm{n}\mathcal{K}_0(K, X(523))$, Gibert's $K018$. We cannot identify the perspectors of the desmic structure. They are complex. Theorem 9(b) gives the parent as $\mathrm{p}\mathcal{K}(K, X(5))$ - the Napolean cubic, and Gibert's $K005$.

Example 3. The kjp cubic $\mathrm{n}\mathcal{K}_0(K, K)$, Gibert's $K024$.

Theorem 9(b) gives the parent as $\mathrm{p}\mathcal{K}(K, O)$ - the McCay cubic and Gibert's $K003$. Theorem 9(a) gives RS as the Brocard axis. It follows that the perspectors of the desmic structure are the intersections of the Brocard axis with the Kiepert hyperbola.

Our next result identifies the children of a given $\mathrm{p}\mathcal{K}(W, R)$ which are of the form $\mathrm{n}\mathcal{K}_0(W, R)$. It turns out that the perspectors must lie on another cubic $\mathrm{n}\mathcal{K}_0(W, T)$. The root T is most neatly defined using the generalization of Gibert's PK-transform.

Theorem 10. *Suppose that $\{P, P^*\} \neq \{S, S^*\}$ are a pair of W-isoconjugates on $\mathrm{p}\mathcal{K}(W, S)$. Then they define a desmic structure with associated cubic of the form $\mathrm{n}\mathcal{K}_0(W, R)$ if and only if P, P^* lie on $\mathrm{n}\mathcal{K}_0(W, PK(W, S))$.*

Proof. Suppose that $P = x : y : z$, $S = r : s : t$, $W = u : v : w$. Then the point P^* is $u/x : v/y : w/z$. As P is on $\mathrm{p}\mathcal{K}(W, S)$, there exist constants m, n with $P + mP^* = nS$. Then the normalized forms for the perspectors of the desmic structure are P and mP^*, and $R = P - mP^*$. If we look at a pair of coordinates in the expression $P + mP^* = nS$, we get an expression for m in terms of two of x, y, z. These are

$$m_x = \frac{(yt - zt)yx}{ysw - ztv}, \quad m_y = \frac{(zr - xt)xz}{ztu - xrv}, \quad m_z = \frac{(xs - yr)xy}{xrv - ysu}.$$

We can now compute the k such that the cubic is $\mathrm{n}\mathcal{K}(W, R, k)$ as

$$2\left(\frac{m_x m_y m_z uvw}{xyz} - xyz\right).$$

We have an $\mathrm{n}\mathcal{K}_0$ if and only if this vanishes. Maple shows that this happens precisely when P (and hence P^*) is on the cubic $\mathrm{n}\mathcal{K}_0(W, T)$, where

$$T = ur(vt^2 - ws^2) : vs(wr^2 - ut^2) : wt(us^2 - vr^2) = PK(W, S).$$

$\square$

Note that we get an $\mathrm{n}\mathcal{K}_0$ when P, P^* lie on $\mathrm{p}\mathcal{K}(W, S)$ and $\mathrm{n}\mathcal{K}_0(W, PK(W, S))$. Then there are three pairs and three cubics. Also, $PK(W, S) = PK(W, S^*)$ - see §5 - so $\mathrm{p}\mathcal{K}(W, S)$ and $\mathrm{p}\mathcal{K}(W, S^*)$ give rise to the same $\mathrm{n}\mathcal{K}_0$.

Example 4. Applying Theorem 10 to the McCay cubic $\mathrm{p}\mathcal{K}(K, O)$ and the Orthocubic $\mathrm{p}\mathcal{K}(K, H)$ we get the second Brocard cubic $K019 = \mathrm{n}\mathcal{K}_0(K, X(647))$. In the former case, we can identify the perspectors of the desmic structures. From [2, K019], we know that the points of $K019$ are the foci of inconics with centers on the Brocard axis OK. Also, if P is on the McCay cubic, then PP^* passes through O. If PP^* is not OK, then the center is on PP^* and on OK, so must be O. This gives four of the intersections, two of which are real. Otherwise, P and P^* are the unique K-isoconjugates on OK. These are the intersections of the Brocard axis and the Kiepert hyperbola. Again these are complex. They give the cubic $\mathrm{n}\mathcal{K}_0(K, K)$ - see Example 3. In §5, we meet these last two points again.

Example 5. The Thomson cubic $\mathrm{p}\mathcal{K}(K, G)$ and the Grebe cubic $\mathrm{p}\mathcal{K}(K, K)$ give the cubic $\mathrm{n}\mathcal{K}_0(K, X(512))$. We will meet this cubic again in §5.

5. Desmic structures with triply perspective triangles

As we saw in Theorem 3, a cubic of type $\mathrm{n}\mathcal{K}_0$ arises from a desmic structure in which the triangles are triply perspective with the reference triangle $\triangle ABC$. We begin with a discussion of an obvious way of constructing such a structure. It turns out that almost all triply perspective structures arise in this way.

Lemma 11. *Suppose that $P = p : q : r$ and $Q = u : v : w$ are points with distinct cevians.*
(a) *Let*

$$\begin{array}{lll} A' = BP \cap CQ, & B' = CP \cap AQ, & C' = AP \cap BQ, \\ A'' = BQ \cap CP, & B'' = CQ \cap AP, & C'' = AQ \cap BP. \end{array}$$

Then the desmic structure with vertices A, B, C, A', B', C', A'', B'', C'', and perspectors $P_1 = \frac{1}{qw} : \frac{1}{ru} : \frac{1}{pv}$ and $P_2 = \frac{1}{rv} : \frac{1}{pw} : \frac{1}{qu}$ has triangles $\triangle ABC$, $\triangle A'B'C'$, $\triangle A''B''C''$ which are triply perspective.
(b) *Each desmic structure including the vertices A, B, C in which*
(i) *the triangles are triply perspective, and*
(ii) *the perspectors are distinct arises from a unique pair P, Q with $P \neq Q$, and perspectors normalized as in (a).*

Proof. (a) We begin by looking at the desmic structure which is derived from the given P_1 and P_2 as in Theorem 4. Thus, the first three vertices are obtained by replacing a coordinate of P_1 by the corresponding coordinate of P_2. This has the vertices named, for example the first vertex is $\frac{1}{rv} : \frac{1}{ru} : \frac{1}{pv}$. This is on BP and CQ, so is A'. It is easy to see that the triangles doubly perspective, with perspectors P and Q, and hence triply perspective.

(b) Suppose we are given such a desmic structure. Then the perspectors are $P_1 = f : g : h$ and $P_2 = f' : g' : h'$ with$fgh = f'g'h'$ (see Lemmata 2 and 3). We find points P and Q which give rise to these as in part (a). From the coordinates of P_1, we can solve for u, v and w in terms of p, q, r and the coordinates of P_1. Then, using two of the coordinates of P_2, we can find q and r in terms of p. The equality of the third coordinates follows from the condition $fgh = f'g'h'$. As the perspectors are distinct, $P \neq Q$. □

Note that from the normalized perpsectors, we can recover the vertices, even if some cevians coincide. For example, A' is $\frac{1}{rv} : \frac{1}{ru} : \frac{1}{pv}$.

Definition 4. *The desmic structure defined in Lemma 11 is denoted by $\mathcal{D}(P, Q)$.*

Theorem 12. *If P and Q are triangle centers with functions $p(a, b, c)$ and $q(a, b, c)$, then the desmic structure $\mathcal{D}(P, Q)$ has*
(a) *perspectors P_1, P_2 with functions $h(a, b, c) = \frac{1}{p(b,c,a)q(c,a,b)}$ and $h(a, c, b)$.*
(b) *$\{P_1,\ P_2\}$ is a bicentric pair.*
(c) *The vertices of the triangles are $[h(a, c, b), h(b, c, a), h(c, a, b)]$, and so on.*

The proof requires only the observation that, as P and Q are centers, $p(a, b, c) = p(a, c, b)$ and $q(a, b, c) = q(a, c, b)$.

We leave it as an exercise to the reader that, if $\{P, Q\}$ is a bicentric pair, then P_1, P_2 are centers. It follows that, if P, Q are centers and $\mathcal{D}(P, Q)$ has perspectors P_1, P_2, then $\mathcal{D}(P_1, P_2)$ has triangle centers P', Q' as perspectors. As further exercises, the reader may verify that P', Q' are the Q^2-isoconjugate of P and the P^2-isoconjugate of Q. The desmon of the second structure is the P-Hirst inverse of Q.

We can compute the equations of the cubics from the information in Lemma 11(a). These involve ideas introduced in our Definition 2.

Theorem 13. *Suppose that $P \neq Q$. The cubics associated with the desmic structure $\mathcal{D}(P, Q)$ are $\mathrm{p}\mathcal{K}(W, NK(W, P))$ and $\mathrm{n}\mathcal{K}_0(W, PK(W, P))$, where W is the isoconjugation which interchanges P and Q.*

Proof. We have the perspectors P_1 and P_2 of the desmic structure from Lemma 11(a). These give isoconjugation as that which interchanges P_1 and P_2 - see Theorem 4. Now observe that this also interchanges P and Q, so $Q = P^*$. From Theorem 4, we have coordinates for the desmon S and the harmon R in terms of those of P_1 and P_2. Using our formulae for P_1 and P_2, we get the stated values of S and R. $\square$

Definition 5. *Suppose that $P \neq Q$, and that $\mathcal{C} = \mathrm{n}\mathcal{K}_0(W, R)$ is associated with the desmic structure $\mathcal{D}(P, Q)$. Then*
(a) *P, Q are the cevian points for $\mathcal{C}$.*
(b) *The perspectors of $\mathcal{D}(P, Q)$ are the Grassmann points for $\mathcal{C}$.*

Theorem 14. *Suppose that $\mathcal{C} = \mathrm{n}\mathcal{K}_0(W, R)$ is not of type $\mathrm{c}\mathcal{K}$, and does not have $W = R^2$.*
(a) *The cevian points for $\mathcal{C}$ are the W-isoconjugate points on $\mathcal{T}(R^*)$. These are the intersections of $\mathcal{C}(R)$ and $\mathcal{T}(R^*)$.*
(b) *The Grassmann points for $\mathcal{C}$ are the W-isoconjugate points on the polar of $PK(W, R)$ with respect to $\mathcal{C}(R)$.*

Proof. From Theorems 3 and 7, we know that $\mathcal{C}$ is a Grassmann cubic associated with a desmic structure which has triply perspective triangles. If the perspectors of the structure coincide at X, then the equation shows that $R = X$ and $W = X^2$. But we assumed that $W \neq R^2$, so we do not have this case.

(a) From Theorem 13, this structure is $\mathcal{D}(P, Q)$ with $Q = P^*$, and $R = PK(W, P) = PK(W, Q)$. From a remark following Definition 2, P and Q lie on the conic $\mathcal{C}(R)$. Since they are W-isoconjugates, they also lie on the W-isoconjugate of $\mathcal{C}(R)$. This is $\mathcal{T}(R^*)$. The conic and line have just two intersections, so this gives precisely the pair $\{P, Q\}$. These are precisely the pair of W-isoconjugates on $\mathcal{T}(R^*)$.

(b) By Theorem 4, the Grassmann points are W-isoconjugate and lie on RS, where S is the desmon of the desmic structure. As $\mathcal{C}$ is an $\mathrm{n}\mathcal{K}_0$, Theorem 9 gives S as a point which we recognize as the pole of $\mathcal{T}(R^*)$ with respect to the conic $\mathcal{C}(R)$. Now R is the pole of $\mathcal{T}(R)$ for this conic, so RS is the polar of $\mathcal{T}(R) \cap \mathcal{T}(R^*) = PK(W, R)$. $\square$

In Theorem 14, we ignored cubics of type $\mathrm{n}\mathcal{K}(R^2, R, k)$. To make the algebra easier, we replace the constant k by $k'rst$, where $R = r : s : t$.

Theorem 15. *If $\mathcal{C} = \mathrm{n}\mathcal{K}(R^2, R, k'rst)$ is not of type $\mathrm{c}\mathcal{K}$, then $\mathcal{C}$ is the Grassmann cubic associated with the desmic structure having perspectors R and aR, where a is a root of*

$$x^2 + \left(1 + \frac{k'}{2}\right)x + 1 = 0.$$

When $k' = 2$, $a = -1$, the desmic structure and $\mathcal{C}$ are degenerate.

When $k' \neq 2, -6$, the corresponding $\mathrm{p}\mathcal{K}$ is $\mathrm{p}\mathcal{K}(R^2, R)$, which is the union of the R-cevians.

Proof. When we use Maple to solve the equations to identify A' and A'', we discover them as $r : as : at$, with a as above. This identifies the perspectors as R and aR. It is easy to verify that this choice leads to $\mathcal{C}$. Note that the two roots are inverse, so we get the same vertices from either choice. The quadratic has equal roots when $k' = 2$ or -6. The latter gives the $\mathrm{c}\mathcal{K}$ in the class. In this case, the "cubic" equation is identically zero as $a = 1$. In the former case, $a = -1$, so we do get the $\mathrm{n}\mathcal{K}$ as a Grassmann cubic, but the equation factorizes as $(ty + sz)(rz + tx)(sx + ry) = 0$. In the "desmic structure" $A' = A''$, $B' = B''$, $C' = C''$ as $a = -1$.

Note that here the cubic is not of type $\mathrm{c}\mathcal{K}$ as it contains three fixed points R_A, R_B, R_C of R^2-isoconjugation. When $k' \neq 2,\ -6$, the desmon is defined, and is again R. Finally, $\mathrm{p}\mathcal{K}(R^2, R)$ is $(ty - sz)(rz - tx)(sx - ry) = 0$. $\square$

Example 6. The third Brocard cubic, $\mathrm{n}\mathcal{K}_0(K, X(647))$, is $K019$ in Gibert's list. As it is of type $\mathrm{n}\mathcal{K}_0$, but not of type $\mathrm{c}\mathcal{K}$, Theorem 12(b) applies. The conic $\mathcal{C}(R)$ is the Jerabek hyperbola. Also, R^* is $X(648)$, so that the line $\mathcal{T}(R^*)$ is the Euler line. These intersect in (the K-isoconjugates) O and H. This gives some new points on the cubic:

$$\begin{array}{lll} A' = BO \cap CH, & B' = CO \cap AH, & C' = AO \cap BH, \\ A'' = BH \cap CO, & B'' = CH \cap AO, & C'' = AH \cap BO. \end{array}$$

Also, the cubic can be described as the Grassmann cubic $GN(\triangle A'B'C')$ or $GN(\triangle A''B''C'')$. The associated GP has the pivot $NK(K, O) = NK(K, H) = X(185)$. The cubic $\mathrm{p}\mathcal{K}(K, X(185))$ is not (yet) listed in [2], but we know that it has the points A', B', C', A'', B'', C'', the perspectors P_1 and P_2, as well as I, the excenters and $X(185)$.

Example 7. The first Brocard cubic, $\mathrm{n}\mathcal{K}_0(K, X(385))$, is $K017$ in Gibert's list. Here, the vertices A', B', C', A'', B'', C'' have already been identified. They are the vertices of the first and third Brocard triangles. The points P and Q are the Brocard points. The associated GP is $\mathrm{p}\mathcal{K}(K, X(384))$, the fourth Brocard cubic and $K020$ in Gibert's list. Again, Gibert's website shows the points on $K020$, together with $\{P_1,\ P_2\} = \{X(32),\ X(76)\}$.

Example 8. The cubics $\mathrm{p}\mathcal{K}(K, X(39))$ and $\mathrm{n}\mathcal{K}_0(K, X(512))$. These cubics do not appear in Gibert's list, but the associated desmic structure is well-known. Take $= G$, $Q = K$ in Lemma 11(a). Again we get isogonal cubics. The vertices of the structure are the intersections of medians and symmedians. From the proof of Lemma 11, P_1, P_2 are the Brocard points. This configuration is discussed in, for example, [4], but the proof that the triangles obtained from the intersections are perspective with $\triangle ABC$ uses special properties of G and K. Here, our Lemma 8(a) gives a very simple (geometric) proof of the general case. Here, $PK(K, G) = PK(K, K) = X(512)$, and $NK(K, G) = NK(K, K) = X(39)$.

The points P and Q for a cubic of type $\mathrm{n}\mathcal{K}_0$ are identified as the intersections of a line with a conic. Of course, it is possible that these have complex coordinates. This happens, for example, for the second Brocard cubic, $\mathrm{n}\mathcal{K}_0(K, X(523))$. There, the line is the Brocard axis and the conic is the Kiepert hyperbola. A Cabri sketch shows that these do not have real intersections. We have met these intersections in §4.

Example 9. The kjp cubic $\mathrm{n}\mathcal{K}_0(K, K)$ is $K024$ in Gibert's list. The desmic structure is $\mathcal{D}(P, Q)$, where P, Q are the intersections of the line at infinity and the circumcircle. These are the infinite circular points. We met this cubic in Example 3, where we identified the perspectors of the desmic structure as the intersections of the Brocard axis with the Kiepert hyperbola.

6. Harmonic associates and other cubics

In the Introduction, we introduced the idea of harmonic associates. This gives a pairing of our cubics. We begin with a result which relates desmic structures. This amplifies remarks made in the proof of Theorem 6.

Theorem 16. *Suppose that D is a desmic structure with normalized perspectors P_1, P_2, and cubics $\mathrm{n}\mathcal{K}(W, R, k)$, $\mathrm{p}\mathcal{K}(W, S)$. Then the desmic structure D' with normalized perspectors P_1, $-P_2$ has*
(a) *vertices the harmonic associates of those of D, and*
(b) *cubics $\mathrm{n}\mathcal{K}(W, S, k')$, $\mathrm{p}\mathcal{K}(W, R)$.*

Proof. We refer the reader to the proof of Theorem 6(a), which in turn uses the notation of Theorem 4(a). □

We refer to the cubic $\mathrm{n}\mathcal{K}(W, S, k')$ obtained in this way as the harmonic associate of $\mathrm{n}\mathcal{K}(W, R, k)$.

Corollary 17. *If $\mathrm{n}\mathcal{K}(W, R, k) = GN(\Delta)$, then $\mathrm{p}\mathcal{K}(W, R) = GP(\Delta')$, where Δ' is the harmonic associate of Δ.*

Example 10. The second Brocard cubic $\mathrm{n}\mathcal{K}(K, X(385)) = K017$ was discussed in Example 7. It is $GN(\Delta)$, where Δ is either the first or third Brocard triangle. From Corollary 17, the cubic $\mathrm{p}\mathcal{K}(K, X(385)) = K128$ is $GP(\Delta')$, where Δ' is the harmonic associate of either of these triangles. This gives us six new points on $K128$. The cubic $\mathrm{n}\mathcal{K}_0(K, X(512))$ was introduced in Example 8. It is $GN(\Delta)$, where Δ is formed from intersections of medians and symmedians. From the Corollary, the fifth Brocard cubic $\mathrm{p}\mathcal{K}(K, X(512)) = K021$ is $GP(\Delta')$. Now the Grassmann points for $\mathrm{n}\mathcal{K}_0(K, X(512))$ are the Brocard points, so these lie on $K021$, as do the vertices of Δ'.

Example 11. Let $\mathcal{C} = K216$ of [2]. This was mentioned in §3. It is of the form $\mathrm{n}\mathcal{K}(K, X(5), k)$ with parent $\mathrm{p}\mathcal{K}(K, X(30)) = K001$. From Theorem 16, the harmonic associate is of the form $\mathrm{n}\mathcal{K}(K, X(30), k')$ with parent $\mathrm{p}\mathcal{K}(K, X(5)) = K005$. Using Theorem 9(b), a calculation shows that $K067 = \mathrm{n}\mathcal{K}(K, X(30), k'')$ has parent $K005$. From Theorem 8(b), $K005$ has a unique child with root $X(30)$. Thus $K067$ is the harmonic associate of $K216$. This gives us six points on $K067$ as harmonic associates of the points identified in [2] as being on $K216$. We will give a geometrical description of these shortly.

Three of the vertices of the desmic structure for $K216$ are the reflections of the vertices A, B, C in BC, CA, AB. These give a triangle with perspector H. We can generalize these to a general X as the result of extending each X-cevian by its own length. If $X = x : y : z$, then, starting from A, we get the point $y+z : -2y : -2z$. The A-harmonic associate is $y+z : 2y : 2z$, the intersection of the H-cevian at A with the parallel to BC through G. This suggests the following definition.

Definition 6. *For $X = x : y : z$, the desmic structure $\mathcal{D}(X)$ is that with normalized perspectors $y+z : z+x : x+y$ and $-2x : -2y : -2z$.*

Note that the first perspector is the complement $\mathsf{c}X$ of X and the second is X. We can summarize our results on such structures as follows.

Theorem 18. *Suppose that $X = x : y : z$. The cubics associated with $\mathcal{D}(X)$ are $\mathrm{n}\mathcal{K}(W, R, k)$ and $\mathrm{p}\mathcal{K}(W, S)$, where*
(i) $W = x(y+z) : y(z+x) : z(x+y)$, *the center of the inconic with perspector X,*
(ii) $R = 2x+y+z : x+2y+z : x+y+2z$, *the mid-point of X and $\mathsf{c}X$,*
(iii) $S = -2x+y+z : x-2y+z : x+y-2z$, *the infinite point on GX.*

The harmonic associate of $\mathrm{n}\mathcal{K}(W, R, k)$ passes through G, the infinite point of $\mathcal{T}(X)$, and their W-isoconjugates.

Proof. The coordinates of W, R and S follow at once from those of the perspectors and Theorem 4. The final part needs an equation for the harmonic associate. This is given by Theorem 4. The fact that G and $x(y-z) : y(z-x) : z(x-y)$ lie on the cubic is a simple verification using Maple. □

For $X = H$, we get $K216$ and $K001$ and their harmonic associates $K067$ and $K005$. The desmic structures and the points given in Theorem 18 account for most of the known points on $K216$ and $K067$.

There is one further example in [2]. In the notes on $K022 = \mathrm{n}\mathcal{K}(O, X(524), k)$, it is observed that the cubic contains the vertices of the second Brocard triangle, and hence their O-isoconjugates. The latter are the the intersections of the $X(69)$-cevians with lines through G parallel to the corresponding sidelines. These are the harmonic associates of three vertices of $\mathcal{D}(X(69))$. The other perspector is $K = \mathsf{c}X(69)$. The mid-point of $X(69)$ and K is $X(141)$, so $K022$ is the harmonic associate of the cubic $\mathrm{n}\mathcal{K}(O, X(141), k')$ with parent $\mathrm{p}\mathcal{K}(O, X(524))$. These cubics contain the vertices of $\mathcal{D}(X(69))$, including the harmonic associates of the second Brocard triangle. Also, the parent of $K022$ is $\mathrm{p}\mathcal{K}(O, X(141))$.

In the Introduction, we mentioned that the Darboux cubic $\mathrm{p}\mathcal{K}(K, X(20))$ is $GP(\Delta)$, where Δ has vertices the infinite points on the altitudes. Of course, as Δ is degenerate, $GN(\Delta)$ degenerates. It is the union of the circumcircle and the line at infinity. The harmonic associate Δ' has vertices the mid-points of the altitudes, and this leads to an $\mathrm{n}\mathcal{K}(K, X(20), k)$, and its parent which is the Thomson cubic $\mathrm{p}\mathcal{K}(K, G) = K002$. This will follow from our next result. The fact that the mid-points lie on $K002$ is noted in [5], but now we know that those points can be used to generate $K002$ as a locus of type GP. We can replace the vertices of Δ or

Δ by their isogonal conjugates. In the case of Δ the isogonal points lie on the circumcircle. For any point $X = x : y : z$, the mid point of the cevian at A is $y + z : y : z$. We make the following definition.

Definition 7. *For $X = x : y : z$, the desmic structure $\mathcal{E}(X)$ is that with normalized perspectors $y + z : z + x : x + y$ and $x : y : z$.*

Theorem 19. *Suppose that $X = x : y : z$. Let Δ be the triangle $\triangle A'B'C'$ of $\mathcal{E}(X)$. Then $GN(\Delta) = \mathrm{n}\mathcal{K}(W, R, k)$ and $GP(\Delta) = \mathrm{p}\mathcal{K}(W, G)$, where*
$W = x(y + z) : y(z + x) : z(x + y)$, the complement of the isotomic conjugate of X,
$R = -x + y + z : x - y + z : x + y - z$, the anticomplement of X,
$k = 2((y + z)(z + x)(x + y) - xyz)$.

The harmonic associates are $GN(\Delta') = \mathrm{n}\mathcal{K}(W, G, k')$, which degenerates as $\mathcal{C}(W)$ and the line at infinity, and $GP(\Delta') = \mathrm{p}\mathcal{K}(W, R)$, which is a central cubic with center the complement of X.

Most of the result follow from the equations given by Theorem 4. The fact that $\mathrm{p}\mathcal{K}(W, R)$ is central is quite easy to check, but it is a known result. In [6], Yiu shows that the cubic defined by $GP(\Delta')$ has the given center. Yiu derives interesting geometry related to $\mathrm{p}\mathcal{K}(W, R)$, and these are summarized in [1, §3.1.3]. The case $X = H$ gives $W = K$, $R = X(20)$, so we get $GP(\Delta) = K002$, the Thomson cubic, and $GP(\Delta') = K004$, the Darboux cubic.

Remarks. (1) From [1, Theorem 3.1.2], we know that if $W \neq G$, there is a unique central $\mathrm{p}\mathcal{K}$ with pole W. After Theorem 19, this arises from the desmic structure $\mathcal{E}(X)$, where X is the isotomic conjugate of the anticomplement of W. The center is then the complement of X, and hence the G-Ceva conjugate of W. It is also the perspector of $\mathcal{E}(X)$ other than X. The pivot of the central $\mathrm{p}\mathcal{K}$ is then the anticomplement of X, and hence the anticomplement of the isotomic conjugate of the anticomplement of the pole.

(2) The cubic $\mathrm{p}\mathcal{K}(W, G)$ clearly contains G and W. From the previous remark, it contains the G-Ceva conjugate of W and its W-isoconjugate (the point X). It also includes the mid-points of the X-cevians and their W-isoconjugates. The last six are the vertices of a defining desmic structure. Finally, it includes the mid-points of the sides of $\triangle ABC$.

We have seen that there are several pairs of cubics of type $\mathrm{p}\mathcal{K}$ which are loci of type GP from harmonic associate triangles. We can describe when this is possible.

Theorem 20. *For a given W, suppose that R and S are distinct points, neither fixed by W-isoconjugation.*
(a) *There exist harmonic triangles Δ and Δ' with $\mathrm{p}\mathcal{K}(W, R) = GP(\Delta)$ and $\mathrm{p}\mathcal{K}(W, S) = GP(\Delta')$ if and only if $\mathrm{p}\mathcal{C}(W, R) = \mathrm{p}\mathcal{C}(W, S)$.*
(b) *If $\mathrm{p}\mathcal{C}(W, R) = \mathrm{p}\mathcal{C}(W, S)$, then the Grassmann points are*
(i) *the non-trivial intersections of $\mathrm{p}\mathcal{K}(W, R)$ and $\mathrm{p}\mathcal{K}(W, S)$,*
(ii) *the W-isoconjugate points on RS.*

Proof. (a) If $\mathrm{p}\mathcal{K}(W,R)$ and $\mathrm{p}\mathcal{K}(W,S)$ are loci of the given type, then $GN(\Delta') = \mathrm{n}\mathcal{K}(W,R,k)$ by Theorem 16. Thus this $\mathrm{n}\mathcal{K}$ has parent $\mathrm{p}\mathcal{K}(W,S)$. By Theorem 8, $\mathrm{p}\mathcal{C}(W,R) = \mathrm{p}\mathcal{C}(W,S)$. Now suppose that $\mathrm{p}\mathcal{C}(W,R) = \mathrm{p}\mathcal{C}(W,S)$. Then R is on $\mathrm{p}\mathcal{C}(W,S)$. By Theorem 8, there is a unique child of $\mathrm{p}\mathcal{K}(W,R)$ of the form $\mathrm{n}\mathcal{K}(W,S,k')$. From Theorem 7, $\mathrm{n}\mathcal{K}(W,S,k') = GN(\Delta)$ for a triangle Δ, and $GP(\Delta) = \mathrm{p}\mathcal{K}(W,R)$. By Theorem 16, $GP(\Delta') = \mathrm{p}\mathcal{K}(W,S)$.

(b) The Grassmann points are the same for Δ and Δ', and lie on both cubics. There are just two non-trivial points of intersection, so these are the Grassmann points. The Grassmann points are W-isoconjugate, and lie on RS, giving (ii). $\Box$

We already have some examples of pairs of this kind:
$K001$ and $K005$ with Grassmann points O, H. The desmic structures are those for $K067$ and $K216$.
$K002$ and $K004$ with Grassmann points O, H. The desmic structures appear above.
$K020$ and $K128$ with Grassmann points $X(32)$, $X(76)$. See the first of Example 10.

From Example 1, we found children of $K001$ with roots $X(395)$, $X(396)$, $X(523)$. We then have
$K001$ and $K129a = \mathrm{p}\mathcal{K}(K, X(395))$ with Grassmann points $X(14)$, $X(16)$;
$K001$ and $K129b = \mathrm{p}\mathcal{K}(K, X(396))$ with Grassmann points $X(13)$, $X(15)$;
$K001$ and $\mathrm{p}\mathcal{K}(K, X(523))$ with Grassmann points the infinite circular points.

In Example 3, we showed that $K024 = \mathrm{n}\mathcal{K}_0(K,K)$ is a child of $K003$. We therefore have $K003$ and $K102 = \mathrm{p}\mathcal{K}(K,K)$, the Grebe cubic. The Grassmann points are the intersection of the Brocard axis and the Kiepert hyperbola. These are complex.

7. Further examples

So far, almost all of our examples have been isogonal cubics. In this section, we look at some cubics with different poles. We have chosen examples where at least one of the cubics involved is in [2]. In the latest edition of [2], we have the class $CL041$. This includes cubics derived from $W = p : q : r$. In our notation, we have the cubic $\mathrm{n}\mathcal{K}_0(W,R)$, where $R = p^2 - qr : q^2 - pr : r^2 - pq$, the G-Hirst inverse of W. The parent is $\mathrm{p}\mathcal{K}(W,S)$, where $S = p^2 + qr : q^2 + pr : r^2 + pq$. The Grassmann points are the barycentric square and isotomic conjugate of W. The cevian points are the bicentric pair $1/r : 1/p : 1/q$ and $1/q : 1/r : 1/p$. Example 7 is the case where $W = K$, so that the Grassmann points are the centers $X(32)$, $X(76)$, and the cevian points are the Brocard points. Our first two examples come from $CL041$.

Example 12. If we put $W = X(1)$ in construction $CL041$, we get $\mathrm{n}\mathcal{K}_0(X(1), X(239))$ with parent $K132 = \mathrm{p}\mathcal{K}(X(1), X(894))$. The Grassmann points are K, $X(75)$, and the cevian points are those described in [3] as the Jerabek points. A harmonic associate of $K132$ is then $K323 = \mathrm{p}\mathcal{K}(X(1), X(239))$.

Example 13. If we put $W = H$ in construction of $CL041$, we get $\mathrm{n}\mathcal{K}_0(H, X(297))$ with parent $\mathrm{p}\mathcal{K}(H, S)$, where S is as given in the discussion of $CL041$. The Grassmann points are $X(393)$, $X(69)$, and the cevian points are those described in [3] as the cosine orthocenters.

Example 14. The shoemaker's cubics are $K070a = \mathrm{p}\mathcal{K}(H, X(1586))$ and $K070b$ = $\mathrm{p}\mathcal{K}(H, X(1585))$. As stated in [2], these meet in G and H. If we normalize G as $1 : 1 : 1$ and H as $\lambda \tan A : \lambda \tan B : \lambda \tan C$, with $\lambda = \pm 1$, we get an $\mathrm{n}\mathcal{K}(H, X(1585), k)$ with parent $K070a$, and an $\mathrm{n}\mathcal{K}(H, X(1585), k')$ with parent $K070b$. The Grassmann points are G and H. Note that $K070a$ and $K070b$ are therefore harmonic associates.

The next five examples arise from Theorem 19. Further examples may be obtained from the page on central $\mathrm{p}\mathcal{K}$ in [2].

Example 15. The complement of $X(1)$ is $X(10)$, the Spieker center, the anticomplement of $X(1)$ is $X(8)$, the Nagel point. The center of the inconic with perspector $X(1)$ is $X(37)$. If we apply Theorem 19 with $X = X(1)$, then we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(X(37), X(8), k)$ with parent $\mathrm{p}\mathcal{K}(X(37), G)$. The Grassmann points are $X(1)$ and $X(10)$. The cubic $\mathrm{p}\mathcal{K}(X(37), G)$ does not appear in the current [2], but contains $X(1)$, $X(2)$, $X(10)$ and $X(37)$. The harmonic associate of C degenerates as the line at infinity and the circumconic with perspector $X(37)$, and has parent $K033 = \mathrm{p}\mathcal{K}(X(37), X(8))$.

Example 16. The complement of $X(7)$, the Gergonne point, is $X(9)$, the Mittenpunkt, the anticomplement of $X(7)$ is $X(144)$. The center of the inconic with perspector $X(7)$ is $X(1)$. If we apply Theorem 19 with $X = X(7)$, then we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(X(1), X(144), k)$ with parent $\mathrm{p}\mathcal{K}(X(1), G)$. The Grassmann points are $X(7)$ and $X(9)$. The cubic $\mathrm{p}\mathcal{K}(X(1), G)$ does not appear in the current [2], but contains $X(1)$, $X(2)$, $X(7)$ and $X(9)$. The harmonic associate of $\mathcal{C}$ degenerates as the line at infinity and the circumconic with perspector $X(1)$, and has parent $K202 = \mathrm{p}\mathcal{K}(X(1), X(144))$.

Example 17. The complement of $X(8)$, the Nagel point, is $X(1)$, the incenter, the anticomplement of $X(8)$ is $X(145)$. The center of the inconic with perspector $X(1)$ is $X(9)$. If we apply Theorem 19 with $X = X(8)$, then we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(X(9), X(145), k)$ with parent $\mathrm{p}\mathcal{K}(X(9), G)$. The Grassmann points are $X(8)$ and $X(1)$. The cubic $\mathrm{p}\mathcal{K}(X(9), G)$ does not appear in the current [2], but contains $X(1)$, $X(2)$, $X(8)$ and $X(9)$. The harmonic associate of $\mathcal{C}$ degenerates as the line at infinity and the circumconic with perspector $X(9)$, and has parent $K201 = \mathrm{p}\mathcal{K}(X(9), X(145))$.

Example 18. The complement of $X(69)$ is K, the anticomplement of $X(69)$ is $X(193)$. The center of the inconic with perspector $X(69)$ is O. If we apply Theorem 19 with $X = X(69)$, then we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(O, X(193), k)$ with parent $K168 = \mathrm{p}\mathcal{K}(O, G)$. The Grassmann points are $X(69)$ and K. The harmonic associate of $\mathcal{C}$ degenerates as the line at infinity and the circumconic with perspector O, and has parent $\mathrm{p}\mathcal{K}(O, X(193))$.

Example 19. The complement of $X(66)$ is $X(206)$, the anticomplement is not in [5], but appears in [2] as $P161$ - see $K161$. The center of the inconic with perspector $X(66)$ is $X(32)$. From Theorem 19 with $X = X(66)$, we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(X(32), P161, k)$ with parent $K177 = \mathrm{p}\mathcal{K}(X(32), G)$. The Grassmann points are $X(66)$ and $X(206)$. The harmonic associate of C degenerates as the line at infinity and the circumconic with perspector $X(32)$, and has parent $K161 = \mathrm{p}\mathcal{K}(X(32), P161)$.

In Examples 3 and 9, we met the kjp cubic $K024 = \mathrm{n}\mathcal{K}_0(K, K)$. The parent is the McCay cubic $K003 = \mathrm{p}\mathcal{K}(K, O)$. It follows that the harmonic associate of $K024$ is of the form $\mathrm{n}\mathcal{K}(K, O, k)$, and that this has parent the Grebe cubic $K102 = \mathrm{p}\mathcal{K}(K, K)$. From Theorem 9(b), the general $\mathrm{n}\mathcal{K}_0(W, W)$ has parent $\mathrm{p}\mathcal{K}(W, V)$, where V is the G-Ceva conjugate of W. The harmonic associate will be of the form $\mathrm{n}\mathcal{K}(W, V, k)$, with parent $\mathrm{p}\mathcal{K}(W, W)$. This means that the class $CL007$, which contains cubics $\mathrm{p}\mathcal{K}(W, W)$, is related to the class $CL009$, which contains cubics $\mathrm{p}\mathcal{K}(W, V)$, and to the class $CL026$, which contains cubics $\mathrm{n}\mathcal{K}_0(W, W)$. We give four examples. In general, the Grassmann points and cevian points may be complex.

Example 20. As above, the cubic $\mathrm{n}\mathcal{K}_0(X(1), X(1))$ has parent $\mathrm{p}\mathcal{K}(X(1), X(9))$. The harmonic associate $\mathrm{n}\mathcal{K}(X(1), X(9), k)$ has parent $K101 = \mathrm{p}\mathcal{K}(X(1), X(1))$.

Example 21. As above, the cubic $\mathrm{n}\mathcal{K}_0(H, H)$ has parent $K159 = \mathrm{p}\mathcal{K}(H, X(1249))$. The harmonic associate $\mathrm{n}\mathcal{K}(H, X(1249), k)$ has parent $K181 = \mathrm{p}\mathcal{K}(H, H)$.

Example 22. The cubic $\mathrm{n}\mathcal{K}_0(X(9), X(9))$ has parent $K157 = \mathrm{p}\mathcal{K}(X(9), X(1))$. The harmonic associate $\mathrm{n}\mathcal{K}(X(9), X(1), k)$ has parent $\mathrm{p}\mathcal{K}(X(9), X(9))$.

Example 23. The cubic $\mathrm{n}\mathcal{K}_0(X(32), X(32))$ has parent $K160 = \mathrm{p}\mathcal{K}(X(32), X(206))$. The harmonic associate $\mathrm{n}\mathcal{K}(X(32), X(206), k)$ has parent $\mathrm{p}\mathcal{K}(X(32), X(32))$.

8. Gibert's theorem

In private correspondence, Bernard Gibert has noted a further characterization of the vertices of the desmic structures we have used.

Theorem 21 (Gibert)**.** *Suppose that P,and Q are two W-isoconjugate points on the cubic $\mathrm{p}\mathcal{K}(W, S)$. For X on $\mathrm{p}\mathcal{K}(W, S)$, let X^t be the tangential of X, and $\mathrm{p}\mathcal{C}(X)$ be the polar conic of X. Now $\mathrm{p}\mathcal{C}(P^t)$ meets $\mathrm{p}\mathcal{K}(W, S)$ at P^t (twice), at P, and at three other points A', B', C', and $\mathrm{p}\mathcal{C}(Q^t)$ meets $\mathrm{p}\mathcal{K}(W, S)$ at Q^t (twice), at Q, and at three other points A'', B'', C''. Then the points A', B', C', A'', B'', C'' are the vertices of a desmic structure with perspectors P and Q.*

This can be verified computationally.

Appendix

We observe that the cubic $\mathrm{n}\mathcal{K}(W, R, k)$ meets the sidelines of $\triangle ABC$ at A, B, C and at the intersections with $\mathcal{T}(R)$. This accounts for all three intersections of the cubic with each sideline. The calculation referred to in Theorem 6(e) shows that $\mathrm{n}\mathcal{K}(W, R, k)$ and $\mathrm{p}\mathcal{K}(W, R_A)$ touch at B, C, meet at A, and at the intersections of $\mathcal{T}(R)$ with AB and AC. On algebraic grounds, there are nine intersections, so in the generic case, there are two further intersections. We now look at the Maple results in detail. If we look at the equations for $\mathrm{n}\mathcal{K}(W, R, k)$ and $\mathrm{p}\mathcal{K}(W, R_A)$ and solve for y, z, then we get the expected solutions, and $z = ax$, $y = \frac{-2vx(t+ar)}{2aus+k}$, where a satisfies

$$2u(2vtr - ks)X^2 + (4uvt^2 + 4vwr^2 - 4uvs^2 - k^2)X + 2w(2vtr - ks) = 0.$$

We cannot have $x = 0, or y = z = 0$. Thus we must have $a = 0$, giving $z = 0$, or $a = -\frac{t}{r}$, giving $y = 0$. The equation for a has a nonzero root only when $k = \frac{2vrt}{s}$. If we put $-\frac{t}{r}$ in the equation for a, we get $k = \frac{2ust}{r}$ or $k = \frac{2wrs}{t}$. If we replace $\mathrm{p}\mathcal{K}(W, R_A)$ by $\mathrm{p}\mathcal{K}(W, R_B)$ or $\mathrm{p}\mathcal{K}(W, R_C)$, we clearly get the same results. This establishes the criteria set out in Theorem 6(e).

If we consider the case $k = \frac{2vtr}{s}$, the equation for a becomes $X = 0$, so we can regard the solutions as limits as a tends to 0 or ∞. The former leads to $r : -s : 0$, the latter to $0 : 0 : 1$. We will meet these again below. We now examine the locus $GN(\triangle A'B'C')$ where the vertex $A' = a_1 : a_2 : a_3$ lies on a sideline. If $a_1 = 0$, the equation for the locus has some zero coefficients. This cannot include the cubic $\mathrm{n}\mathcal{K}(W, R, k)$ unless it is the whole plane. Thus we cannot define a cubic as a locus in this case.

Guided by the above discussion, we now examine the case $a_3 = 0$. Let $B' = b_1 : b_2 : b_3$, $C' = c_1 : c_2 : c_3$. The condition for the locus to be an $\mathrm{n}\mathcal{K}$ becomes $a_2b_3c_1 = 0$. Taking $a_2 = 0$ or $b_3 = 0$ leads to an equation with zero coefficients, so we must have $c_1 = 0$. When we equate the coefficients of the equations for the locus and $\mathrm{n}\mathcal{K}(W, R, k)$ other than y^2z and xyz, we find a unique solution. After scaling this is $A' = r : -s : 0$, $B' = u/r : -v/s : w/t$, $C' = 0 : -s : t$. Then the locus is $\mathrm{n}\mathcal{K}(W, R, 2vtr/s)$. A little thought shows that for fixed W, R, there are only three such loci, giving $\mathrm{n}\mathcal{K}(W, R, k)$ with $k = 2ust/r$, $2vtr/s$, $2wrs/t$. From our earlier work, we know that there is no other way to express these cubics as loci of type GN.

We should expect to obtain a desmic stucture by taking W-isoconjugates of A', B', C'. If we write the isoconjugate of $X = x : y : z$ as $uyz : vzx : wxy$, then the isconjugates are $A'' = C$, $B'' = r : -s : t = R_B$, $C'' = A$. Then $\triangle ABC$ and $\triangle A'B'C'$ have perspector B, $\triangle ABC$ and $\triangle CR_BA$ have perspector $P = r : 0 : t$, $\triangle CR_BA$ and $\triangle A'B'C'$ have perspector R_B.

It is a moot point whether this should be termed a desmic structure. It satisfies the perspectivity conditions, but has only eight distinct points. If we replace B' by any point, we still get the same perspectors. If we allow this as a desmic, then Theorem 7 holds as stated. If not, we can either add these three cubics to the excluded list or reword in the weaker form.

Theorem 7′. *If a cubic $\mathcal{C}$ is of type $\mathrm{n}\mathcal{K}$, but not of type $\mathrm{c}\mathcal{K}$, then there is a triangle Δ with $\mathcal{C} = GN(\Delta)$, and at most two such triangles.*

To get R as the barycentric difference of perspectors, we need to scale B to $0 : -s : 0$. Then the sum is R_B. A check using Theorem 9(b) shows that the parent is indeed $\mathrm{p}\mathcal{K}(W, R_B) = GP(\triangle A'B'C')$. Replacing each coordinate of P in turn by the corresponding one from B, we get C, R_B, A, as expected. On the other hand, starting from B, we get A', $0 : 0 : 0$, C'. This reflects the fact that the other points do not determine B'. When we compute the equation of $GN(\triangle CR_BA)$, we find that all the coefficients are zero. Thus cubics of this kind are Grassmann cubics for only one triangle rather than the usual two.

References

[1] J-P. Ehrmann and B. Gibert, *Special Isocubics in the Triangle Plane*, available from http://perso.wanadoo.fr/bernard.gibert/downloads.html.

[2] B. Gibert, Cubics in the Triangle Plane, available from http://perso.wanadoo.fr/bernard.gibert/index.html.

[3] B. Gibert, Barycentric coordinates and Tucker cubics, available from http://perso.wanadoo.fr/bernard.gibert/Resources/tucker.pdf.

[4] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, New Mathematical Library 37, Math. Assoc. Amer. 1996.

[5] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[6] P. Yiu, The uses of homogeneous barycentric coordinates in euclidean plane geometry. *Int. J. Math. Educ. Sci. Technol.*, 31 (2000), 569–578.

Wilson Stothers: Department of Mathematics, University of Glasgow, Glasgow, UK
E-mail address: wws@maths.gla.ac.uk

Forum Geometricorum
Volume 6 (2006) 139–147.

FORUM GEOM
ISSN 1534-1178

Concurrent Medians of $(2n+1)$-gons

Matthew Hudelson

Abstract. We exhibit conditions that determine whether a set of $2n+1$ lines are the medians of a $(2n+1)$-sided polygon. We describe how to regard certain collections of sets of medians as a linear subspace of related collections of sets of lines, and as a consequence, we show that every set of $2n+1$ concurrent lines are the medians of some $(2n+1)$-sided polygon. Also, we derive conditions on $n+1$ points so that they can be consecutive vertices of a $(2n+1)$-sided polygon whose medians intersect at the origin. Each of these constructions demonstrates a procedure that generates $(2n+4)$-degree of freedom families of median-concurrent polygons. Furthermore, this number of degrees of freedom is maximal.

1. Motivation

It is well-known that the medians of a triangle intersect in a common point. We wish to explore which polygons in general have this property. Necessarily, such polygons must have an odd number of edges. One easy source of such polygons is to begin with a regular $(2n+1)$-gon centered at the origin and transform the vertices using an affine transformation. This exhausts the triangles as every triangle is an affine image of an origin-centered equilateral triangle. On the other hand, if we begin with either a regular pentagon or a regular pentagram, this process fails to exhaust the median-concurrent pentagons. Consider the pentagon with the sequence of vertices $v_0 = (0,1)$, $v_1 = (1,0)$, $v_2 = (2,1)$, $v_3 = (-2,-2)$, and $v_4 = (6,2)$, depicted in Figure 1.

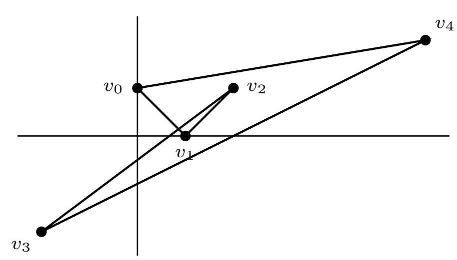


Figure 1. A non-affinely regular median concurrent pentagon

It is easy to check that for each i (all indices modulo 5), the line through v_i and $\frac{1}{2}(v_{i+2}+v_{i+3})$ contains the origin. Alternatively, it suffices to check that v_i and $v_{i+2}+v_{i+3}$ are scalar multiples. On the other hand, this pentagon has a single self-intersection whereas a regular pentagon has none and a regular pentagram has five,

Publication Date: April 10, 2006. Communicating Editor: Floor van Lamoen.

so this example cannot be the image under an affine map of a regular 5-gon. Thus, we seek alternative and more comprehensive means to construct median concurrent pentagons specifically and $(2n+1)$-gons in general.

We approach this problem by two different means. In the next section, we consider families of lines that serve as the medians of polygons and in the section afterwards, we begin with a collection of $n+1$ consecutive vertices and show how to "complete" the collection with the remaining n vertices; the result will also be a $(2n+1)$-gon whose medians intersect.

2. Families of polygons constructed by medians

2.1. *Oriented lines and the signed law of sines.* In this section, we will be working with oriented lines. Given a line ℓ' in $\mathbb{R}^2$, we associate with it a unit vector $\mathbf{v}$ that is parallel with ℓ'. The oriented line ℓ is defined as the pair $(\ell', \mathbf{v})$. Then given points A and B on ℓ, we can solve $\overrightarrow{AB} = t\mathbf{v}$ for t and say that t is the "signed length" from A to B along ℓ; this quantity is denoted $d_\ell(A, B)$. If $t > 0$, we will say that B is on the "positive side" of A along ℓ; if $t < 0$, we will say that B is on the "negative side" of A along ℓ. Switching the orientation of a line switches the sign of the signed length from one point to another on that line.

Let ℓ_1 and ℓ_2 be two non-parallel oriented lines and C be their intersection point. Let D_i be a point on the positive side of C along ℓ_i. The "signed angle" from ℓ_1 to ℓ_2, denoted θ_{12} is the angle whose magnitude (in the range $(0, \pi)$) is that of $\angle D_1 C D_2$ and whose sign is that of the cross product $\mathbf{v}_1 \times \mathbf{v}_2$, the vectors thought of as lying in the $z = 0$ plane of $\mathbb{R}^3$. The signed angle of two parallel lines with the same unit vector is 0, and with opposite unit vectors is π. Switching the orientation of a single line switches the sign of the signed angle between them.

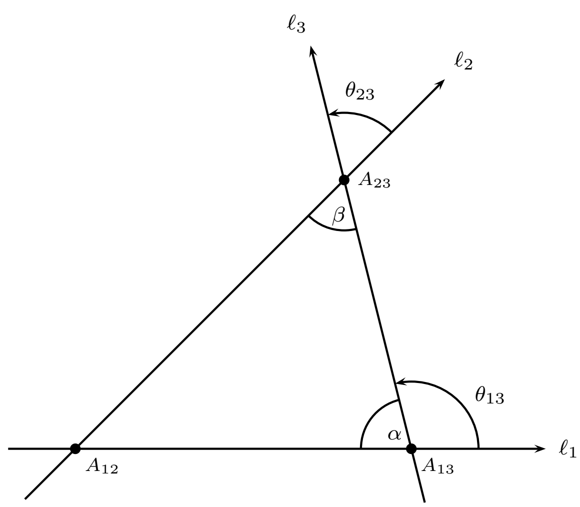


Figure 2. The signed law of sines

In Figure 2 we have three oriented lines ℓ_1, ℓ_2, ℓ_3. Point A_{ij} is at the intersection of ℓ_i and ℓ_j. As drawn, A_{13} is on the positive side of A_{12} along ℓ_1, A_{23} is on the positive side of A_{12} along ℓ_2, and A_{23} is on the positive side of A_{13} along ℓ_3. Letting $\alpha = \angle A_{23}A_{13}A_{12}$ and $\beta = \angle A_{12}A_{23}A_{13}$, we have $\sin\alpha = \sin\theta_{13}$ and $\sin\beta = \sin\theta_{23}$.

By the ordinary law of sines and the above comments about α and β, we have

$$\frac{d_{\ell_1}(A_{12}, A_{13})}{\sin\theta_{23}} = \frac{d_{\ell_2}(A_{12}, A_{23})}{\sin\theta_{13}}.$$

Note that if we switch the orientation of ℓ_1, then the numerator of the LHS and the denominator of the RHS change signs. Switching the orientation of ℓ_2 changes the signs of the numerator of the RHS and the denominator of the LHS. Switching the orientation of ℓ_3 changes the signs of both denominators. Any of these orientation switches leaves the LHS and RHS equal, and so the equation above is true for oriented lines and signed angles as well; we call this equation "the signed law of sines."

2.2. *Constructing polygons via medians.* Let $\ell_0, \ell_1, \ldots, \ell_{2n}$ be $2n + 1$ oriented lines, no two parallel, in $\mathbb{R}^2$. Let $B_{i,j}$ be the point of intersection of ℓ_i and ℓ_j, and let δ_{i+1} be $d_{\ell_{i+1}}(B_{i,i+1}, B_{i+1,i+2})$. Finally, choose a point A_0 on ℓ_0 and let $S_0 = d_{\ell_0}(B_{0,1}, A_0)$.

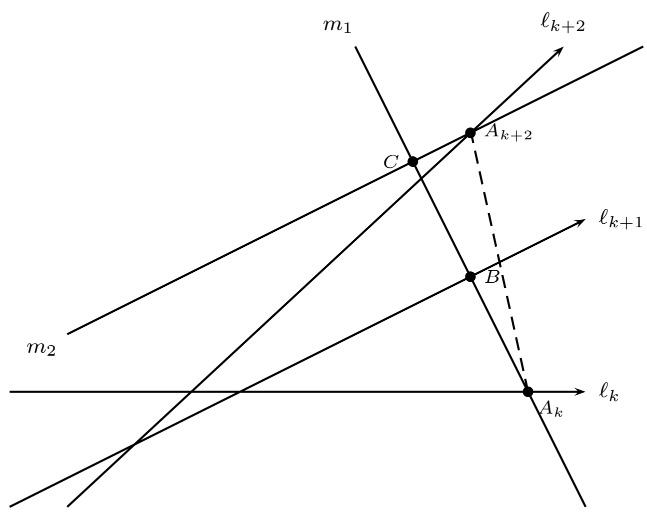


Figure 3. Constructing consecutive points via medians

Given a point A_k on ℓ_k, we construct the point A_{k+2} on ℓ_{k+2} (the indices of lines are modulo $2n + 1$) as follows and as depicted in Figure 3.

Construction 1. *Construct line m_1 through A_k and perpendicular to ℓ_{k+1}. Let point B be the intersection of line ℓ_{k+1} and m_1. Construct point C on line m_1 so that segments $\overline{A_kB}$ and $\overline{BC}$ are congruent. Construct line m_2 through C and perpendicular to m_1. Let point A_{k+2} be the intersection of lines m_2 and ℓ_{k+2}.*

We note that line ℓ_{k+1} intersects segment $\overline{A_k, A_{k+2}}$ at its midpoint.

We define $S_k = d_{\ell_k}(B_{k,k+1}, A_k)$ and θ_{ij} to be the signed angle subtended from ℓ_i to ℓ_j. Letting m be the line through A_k and A_{k+2} and $\varphi = \theta_{\ell_{k+1},m}$, we have by the signed law of sines,

$$\frac{x}{\sin\theta_{k,k+1}} = \frac{S_k}{\sin(\theta_{k,k+1}+\varphi)}$$

and

$$\frac{x}{\sin\theta_{k+1,k+2}} = \frac{S_{k+2}+\delta_{k+1}}{\sin(\theta_{k,k+1}+\varphi)}.$$

Eliminating x, we have

$$S_{k+2} = \frac{\sin\theta_{k,k+1}}{\sin\theta_{k+1,k+2}} S_k - \delta_{k+1}.$$

Let

$$g_k = \frac{\sin\theta_{k,k+1}}{\sin\theta_{k+1,k+2}}.$$

Then we have the recurrence

$$S_{k+2} = g_k S_k - \delta_{k+1}.$$

This leads to

$$\begin{aligned} S_{k+4} &= g_{k+2}S_{k+2} - \delta_{k+3} \\ &= g_{k+2}g_k S_k - g_{k+2}\delta_{k+1} - \delta_{k+3}, \end{aligned}$$

and

$$\begin{aligned} S_{k+6} &= g_{k+4}S_{k+4} - \delta_{k+5} \\ &= g_{k+4}g_{k+2}g_k S_k - g_{k+4}g_{k+2}\delta_{k+1} - g_{k+4}\delta_{k+3} - \delta_{k+5}. \end{aligned}$$

In general, if we define

$$h_{k,p,m} = g_{k+2p}g_{k+2p+2}g_{k+2p+4}\cdots g_{k+2(m-1)},$$

we have

$$S_{k+2m} = h_{k,0,m}S_k - h_{k,1,m}\delta_{k+1} - h_{k,2,m}\delta_{k+3} - \cdots - \delta_{k+2m-1}.$$

We are interested in the case when we begin with a point A_0 on ℓ_0 and eventually construct the point $A_{2(2n+1)}$ which will also be on line ℓ_0. When $k = 0$ and $m = 2n+1$, we have

$$S_{2(2n+1)} = h_{0,0,2n+1}S_0 - h_{0,1,2n+1}\delta_1 - h_{0,2,2n+1}\delta_3 - \cdots - \delta_{2(2n+1)-1}.$$

We notice first that

$$h_{0,0,2n+1} = \prod_{k=0}^{2n} g_{2k} = \prod_{k=0}^{2n} \frac{\sin\theta_{2k,2k+1}}{\sin\theta_{2k+1,2k+2}} = \frac{\prod_{k=0}^{2n}\sin\theta_{2k,2k+1}}{\prod_{k=0}^{2n}\sin\theta_{2k+1,2k+2}}.$$

Since the subscripts in the latter products are all modulo $2n+1$, it follows that the terms in the numerator are a permutation of those in the denominator. This means that $h_{0,0,2n+1} = 1$. The second observation is that

$$S_{2(2n+1)} = S_0 + \text{ a linear combination of the } h_{0,i,2n+1} \text{ values}.$$

The coefficients of this linear combination are the δ's. The nullspace of the h values will in fact be a codimension 1 subspace of the space of all possible choices of $(\delta_0, \delta_1, \dots, \delta_{2n})^T$. An immediate consequence of this is that if for all i we have $\delta_i = 0$ then $S_{4n+2} = S_0$ and so we have closed the polygon A_0, A_2, A_4, $\dots$, $A_{4n+2} = A_0$. We have shown

Proposition 1. *Any set of $2n + 1$ concurrent lines, no two parallel, in $\mathbb{R}^2$ are the medians of some $(2n + 1)$-gon.*

Consider choosing a family of $2n+1$ concurrent lines. Each line can be chosen by choosing a unit vector, the choice of each being a single degree of freedom (for instance, the angle that vector makes with the vector $(1, 0)^T$). Another degree of freedom is the choice of point A_0 on ℓ_0. Finally, there are two more degrees of freedom in the choice of the point of concurrency. This is a total of $2n + 4$ degrees of freedom for constructing $(2n + 1)$-gons with concurrent medians.

3. Families of median-concurrent polygons constructed by vertices

Suppose we have three points $(a, b), (c, d)$, and (e, f) in $\mathbb{R}^2$ such that $(a, b) \neq (-c, -d)$. We seek a fourth point (u, v) such that (u, v), $(a + c, b + d)$ and $(0, 0)$ are collinear, and (a, b), $(e + u, f + v)$ and $(0, 0)$ are also collinear.

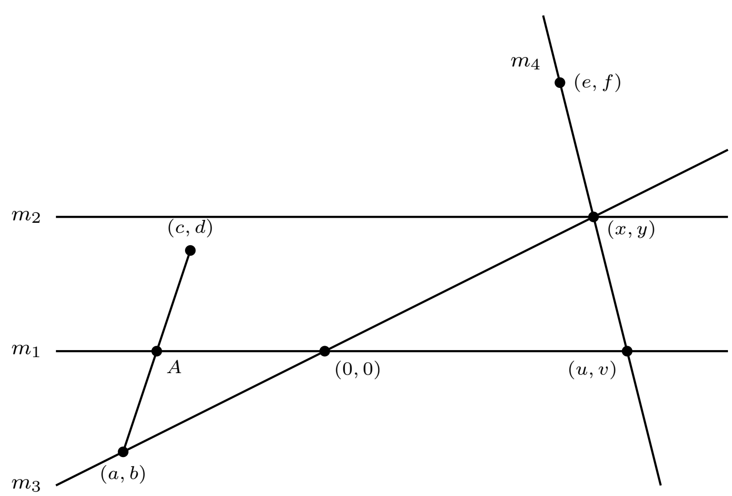


Figure 4. Constructing the fourth point

The point (u, v) can be constructed as follows:

Construction 2. *Let A be the midpoint of the segment joining (a, b) and (c, d) and m_1 be the line through A and the origin. Construct the line m_2 parallel to m_1 that is on the same side of, but half the distance from (e, f) as m_1. Let m_3 be the line through (a,b) and the origin, intersecting m_2 at (x, y), and let m_4 be the line through (e, f) and (x, y). The point (u, v) is the intersection of lines m_1 and m_4.*

It must be the case that $(u,v)=k(a+c,b+d)$ and also

$$u(b+d)=v(a+c)$$
$$b(e+u)=a(f+v).$$

Subtracting, we have

$$ud-be=vc-af$$

or

$$k(a+c)d-be=k(b+d)c-af.$$

Isolating k, we have

$$k=\frac{be-af}{ad-bc}.$$

Notice that the fourth point is uniquely determined by the other three, provided $ad-bc\neq 0$.

We formalize this in the following definition.

Definition 1. *Given point $A=(a,b), C=(c,d)$, and $E=(e,f)$, we define the point $F(A,C,E)$ by the formula*

$$F=F(A,C,E)=\frac{be-af}{ad-bc}(A+C).$$

This point satisfies the property that the lines $\overline{F,(A+C)}$ and $\overline{A,(E+F)}$ intersect at the origin.

Now, suppose we have $n+1$ points in $\mathbb{R}^2$, $x_i=(a_i,b_i)$, $0\le i\le n$, and suppose that no line joining two consecutive points contains the origin. Starting at $i=0$ we define $x_{i+n+1}=F(x_i,x_{i+1},x_{i+n})$. We address what happens in the sequence $x_0,x_1,x_2,\ldots$.

We can recast our definition of the point x_{i+n+1} using the following definition.

Definition 2. *For $0\le j,k$, $\Delta_{j,k}=a_jb_k-b_ja_k$.*

Armed with this, we have

$$x_{i+n+1}=\frac{\Delta_{i+n,i}}{\Delta_{i,i+1}}(x_i+x_{i+1}).$$

Also, we use induction to prove:

Proposition 2. *For all $j\ge 0$, $\Delta_{j+n,j}=\Delta_{n,0}$.*

Proof. The case $j=0$ is obvious. Suppose the result is true for $j=k$. Then

$$\begin{aligned}\Delta_{k+1+n,k+1}&=a_{k+1+n}b_{k+1}-b_{k+1+n}a_{k+1}\\&=\frac{\Delta_{k+n,k}}{\Delta_{k,k+1}}((a_k+a_{k+1})b_{k+1}-(b_k+b_{k+1})a_{k+1})\\&=\frac{\Delta_{k+n,k}}{\Delta_{k,k+1}}(a_kb_{k+1}-b_ka_{k+1})\\&=\Delta_{k+n,k}\\&=\Delta_{n,0}\end{aligned}$$

which completes the induction. □

As a consequence, we have

$$x_{i+n+1} = \frac{\Delta_{n,0}}{\Delta_{i,i+1}}(x_i + x_{i+1}).$$

We verify a useful property of $\Delta_{j,k}$:

Proposition 3. *For all* j, k, ℓ*,*

$$\Delta_{j,k}x_\ell + \Delta_{\ell,j}x_k = \Delta_{\ell,k}x_j.$$

Proof. We work component-by-component:

$$\begin{aligned}
\Delta_{j,k}a_\ell + \Delta_{\ell,j}a_k &= (a_jb_k - b_ja_k)a_\ell + (a_\ell b_j - b_\ell a_j)a_k \\
&= a_jb_ka_\ell - b_ja_ka_\ell + a\ell b_ja_k - b_\ell a_ja_k \\
&= (a_\ell b_k - b_\ell a_k)a_j \\
&= \Delta_{\ell,k}a_j,
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{j,k}b_\ell + \Delta_{\ell,j}b_k &= (a_jb_k - b_ja_k)b_\ell + (a_\ell b_j - b_\ell a_j)b_k \\
&= a_jb_kb_\ell - b_ja_kb_\ell + a\ell b_jb_k - b_\ell a_jb_k \\
&= (a_\ell b_k - b_\ell a_k)b_j \\
&= \Delta_{\ell,k}b_j.
\end{aligned}$$

□

We can now prove the following:

Proposition 4. *For all* $0 \leq i \leq 2n$*, there is a* k_i *such that* $x_{i-1} + x_i = k_ix_{n+i}$ (*all subscripts modulo* $2n+1$).

Proof. For $i = 0$, we calculate

$$\begin{aligned}
x_{2n} + x_0 &= \frac{\Delta_{n,0}}{\Delta_{n-1,n}}(x_{n-1} + x_n) + x_0 \\
&= \frac{1}{\Delta_{n-1,n}}(\Delta_{n,0}x_{n-1} + \Delta_{n,0}x_n + \Delta_{n-1,n}x_0) \\
&= \frac{1}{\Delta_{n-1,n}}((\Delta_{n,0}x_{n-1} + \Delta_{n-1,n}x_0) + \Delta_{n,0}x_n) \\
&= \frac{1}{\Delta_{n-1,n}}(\Delta_{n-1,0}x_n + \Delta_{n,0}x_n) \\
&= \frac{\Delta_{n-1,0} + \Delta_{n,0}}{\Delta_{n-1,n}}x_n
\end{aligned}$$

and so

$$k_0 = \frac{\Delta_{n-1,0} + \Delta_{n,0}}{\Delta_{n-1,n}}.$$

If $1 \leq i \leq n$, then by the definition of x_{n+i}, $k_i = \Delta_{i-1,i}/\Delta_{n,0}$.

To handle the case when $i = n+1$, we calculate

$$\begin{aligned} x_n + x_{n+1} &= x_n + \frac{\Delta_{n,0}}{\Delta_{0,1}}(x_0 + x_1) \\ &= \frac{1}{\Delta_{0,1}}(\Delta_{0,1}x_n + \Delta_{n,0}(x_0 + x_1)) \\ &= \frac{1}{\Delta_{0,1}}(\Delta_{0,1}x_n + \Delta_{n,0}x_1 + \Delta_{n,0}x_0) \\ &= \frac{1}{\Delta_{0,1}}(\Delta_{n,1}x_0 + \Delta_{n,0}x_0) \\ &= \frac{\Delta_{n,1} + \Delta_{n,0}}{\Delta_{0,1}}x_0 \end{aligned}$$

and so

$$k_{n+1} = \frac{\Delta_{n,1} + \Delta_{n,0}}{\Delta_{0,1}}.$$

For the values of i, $n+2 \le i \le 2n$, we set $m = i - n$ and we have, using the symbol $\delta = \Delta_{n,0}/(\Delta_{m-2,m-1}\Delta_{m-1,m})$,

$$\begin{aligned} x_{i-1} + x_i &= x_{n+m-1} + x_{n+m} \\ &= \frac{\Delta_{n,0}}{\Delta_{m-2,m-1}}(x_{m-2} + x_{m-1}) + \frac{\Delta_{n,0}}{\Delta_{m-1,m}}(x_{m-1} + x_m) \\ &= \delta(\Delta_{m-1,m}(x_{m-2} + x_{m-1}) + \Delta_{m-2,m-1}(x_{m-1} + x_m) \\ &= \delta((\Delta_{m-1,m}x_{m-2} + \Delta_{m-2,m-1}x_m) + (\Delta_{m-1,m} + \Delta_{m-2,m-1})x_{m-1}) \\ &= \delta(\Delta_{m-2,m} + \Delta_{m-1,m} + \Delta_{m-2,m-1})x_{m-1} \\ &= \delta(\Delta_{m-2,m} + \Delta_{m-1,m} + \Delta_{m-2,m-1})x_{i+n} \end{aligned}$$

noting that $m - 1 = i - n - 1 \equiv i + n$ modulo $2n+1$, and so for $n+2 \le i \le 2n$, we have

$$k_i = \frac{\Delta_{n,0}(\Delta_{i-n-2,i-n} + \Delta_{i-n-1,i-n} + \Delta_{i-n-2,i-n-1})}{\Delta_{i-n-2,i-n-1}\Delta_{i-n-1,i-n}}.$$

□

What this proposition says, geometrically, is that the points $x_{i-1} + x_i$, x_{i+n} and the origin are collinear. Alternatively, setting $i = j + n + 1$, we find that the points x_j, $x_{j+n} + x_{j+n+1}$ and the origin are collinear. But this means that $\frac{1}{2}(x_{j+n} + x_{j+n+1})$ is also on the same line, and so the line connecting x_j and the midpoint of the segment joining x_{j+n} and x_{j+n+1} contains the origin.

As a direct consequence, we obtain the following result:

Proposition 5. *Given any sequence of $n+1$ points, $x_0, x_1, \ldots, x_n$ such that the origin is not on any line $\overline{x_i, x_{i+1}}$ or $\overline{x_n, x_0}$, then these points are $n+1$ vertices in sequence of a unique $(2n+1)$-gon whose medians intersect in the origin.*

Here, we can choose $n+1$ points to determine a $(2n+1)$-gon whose medians intersect at the origin. Each point contributes two degrees of freedom for a total of

$2n+2$ degrees of freedom. Two more degrees of freedom are obtained for the point of concurrency, for a total of $(2n+4)$ degrees of freedom. This echoes the final result from the previous section. That we cannot obtain further degrees of freedom follows from the previous section as well. There, *any* set of $2n+1$ concurrent lines (in general position) produced a concurrent-median $(2n+1)$-gon. We cannot hope for more freedom than this.

Matthew Hudelson: Department of Mathematics, Washington State University, Pullman, Washington, 99164-3113, USA

E-mail address: hudelson@math.wsu.edu

Forum Geometricorum
Volume 6 (2006) 149–166.

FORUM GEOM
ISSN 1534-1178

On the Generating Motions and the Convexity of a Well-Known Curve in Hyperbolic Geometry

Dieter Ruoff

Abstract. In Euclidean geometry the vertices P of those angles $\angle APB$ of size α that pass through the endpoints A, B of a given segment trace the arc of a circle. In hyperbolic geometry on the other hand a set of equivalently defined points P determines a different kind of curve. In this paper the most basic property of the curve, its convexity, is established. No straight-forward proof could be found. The argument rests on a comparison of the rigid motions that map one of the angles $\angle APB$ into other ones.

1. Introduction

In the hyperbolic plane let AB be a segment and H one of the halfplanes with respect to the line through A and B. What will be established here is the convexity of the locus Ω of the point P which lies in H and which determines together with A and B an angle $\angle APB$ of a given fixed size. In Euclidean geometry this locus is well-known to be an arc of the circle through A and B whose center C determines the (oriented) angle $\angle ACB = 2 \cdot \angle APB$. In hyperbolic geometry, on the other hand, one obtains a wider, flatter curve (see Figure 1; [2, p.79, Exercise 4], [1], and also [6, Section 50], [7, Section 2]). The evidently greater complexity of the non-Euclidean version of this locus shows itself most clearly when one considers the (direct) motion that carries a defining angle $\angle APB$ into another defining angle $\angle AP'B$. Whereas in Euclidean geometry it has to be a rotation, it can in hyperbolic geometry also be a horocyclic rotation about an improper center, or, surprisingly, even a translation. For our convexity proof it appears to be practical to consider the given angle as fixed and the given segment as moving. Then, as will be shown in the *Main Lemma*, the relative position of the centers or axes of our motions can be described in a very simple fashion, with the sought-after convexity proof as an easy consequence. As to proving the Lemma itself, one has to take into account that the motions involved can be rotations, horocyclic rotations, or translations, and it seems that a distinction of cases is the only way to proceed. Still, it would be desirable if the possibility of an overarching but nonetheless elementary argument would be investigated further.

The fact of the convexity of our curve yields at least one often used by-product:

Theorem. *Let AB be a segment, H a halfplane with respect to the line through A and B, and ℓ a line which has points in common with H but avoids segment AB.*

Publication Date: April 17, 2006. Communicating Editor: Paul Yiu.

Many thanks to my colleague Dr. Chris Fisher for his careful reading of the manuscript and his helpful suggestions.

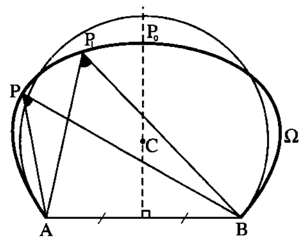


Figure 1

Then the point X, when running through ℓ in H, determines angles $\angle AXB$ that first monotonely increase, and thereafter monotonely decrease in size.

Our approach will be strictly axiomatic and elementary, based on Hilbert's axiom system of Bolyai-Lobachevskian geometry (see [3, Appendix III]). The application of Archimedes' axiom in particular is excluded. Beyond the initial concepts of hyperbolic plane geometry we will only rely on the facts about angle sum, defect, and area of polygons (see e.g. [2, 5, 6, 8]), and on the basic properties of isometries. To facilitate the reading of our presentation we precede it with a list of frequently used abbreviations.

1.1. *Abbreviations.*

1.1.1. $[A_1A_2\ldots A_h\ldots A_i\ldots A_k\ldots A_n]$ for an n-tuple of points with A_i *between* A_h and A_k for $1 \le h < i < k \le n$.

1.1.2. $AB, CD, \ldots$ for *segments*, and $(AB), (CD), \ldots$ for the related *open intervals* $AB - \{A, B\}$, $CD - \{C, D\}, \ldots$; $\overrightarrow{AB}, \overrightarrow{CD}, \ldots$ for the *rays* from A through B, from C through $D, \ldots$, and $\overrightarrow{(A)B}, \overrightarrow{(C)D}, \ldots$ for the related *halflines* $\overrightarrow{AB} - \{A\}$, $\overrightarrow{CD} - \{C\}, \ldots$; $\ell(AB), \ell(CD), \ldots$ for the *lines* through A and B, C and $D, \ldots$.

1.1.3. $a, b, c, \ldots$ are general abbreviations for lines, $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \ldots$ for rays in those lines, and $(\overrightarrow{a}), (\overrightarrow{b}), (\overrightarrow{c}), \ldots$ for the related halflines.

1.1.4. $H(a, B)$, where the point B is not on line a, for the *halfplane with respect to a which contains B*, and $\overline{H}(a, B)$ for the *halfplane with respect to a which does not contain B*. The improper ends of rays which enter halfplane H through a are considered as belonging to H.

1.1.5. perp(a, B) for the line which is *perpendicular* to a and incident with B; proj(S, ℓ) for the *orthogonal projection* of the point or pointset S to ℓ.

1.1.6. $ABC\underline{D}$ for the *Lambert quadrilateral* with right angles at A, B, C and an acute angle at D.

1.1.7. $\mathbf{R}$ for the size of a right angle.

1.1.8. $a \not\wedge b,\ \ a \not\wedge \overrightarrow{p}, \ldots$ for the *intersection point* of the lines a and b, of the line a and the ray $\overrightarrow{p}, \ldots$.

1.1.9. $\cdot, \cdot\cdot, \circ$ (in figures) for specific acute angles with $\cdot\cdot$ denoting a smaller angle than $\cdot$.

Remark. In the figures of Section 3, lines and metric are distorted to better exhibit the betweenness features.

2. Segments that join the legs of an angle

In this section we compile a number of facts about segments whose endpoints move along the legs of a given angle. All statements hold in Euclidean and hyperbolic geometry alike; the easy absolute proofs are for the most part left to the reader.

Let $\angle(\overrightarrow{a}, \overrightarrow{b})$ be an angle with vertex P, and $\mathcal{C}$ be the class of segments $A_\nu B_\nu$ of length s that have endpoint A_ν on leg $(\overrightarrow{a})$ and endpoint B_ν on leg $(\overrightarrow{b})$ of this angle, and satisfy the equivalent conditions

$$\text{(1a)} \quad \angle PA_\nu B_\nu \geq \angle PB_\nu A_\nu, \qquad\qquad \text{(1b)} \quad PA_\nu \leq PB_\nu,$$

(see Figures 2a, b). We will always draw $\overrightarrow{a}, \overrightarrow{b}$ as rays that are *directed downwards* and, to simplify expression, refer to P as the *summit* of $\angle(\overrightarrow{a}, \overrightarrow{b})$. As a result of (1a) the segments $A_\nu B_\nu$ are uniquely determined by their endpoints on $(\overrightarrow{a})$, and $\mathcal{C}$ can be generated by sliding downwards through the points on $(\overrightarrow{a})$ and finding the related points on $(\overrightarrow{b})$.

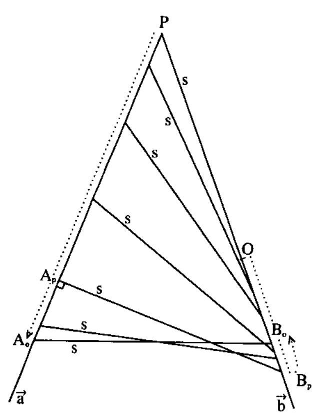


Figure 2a

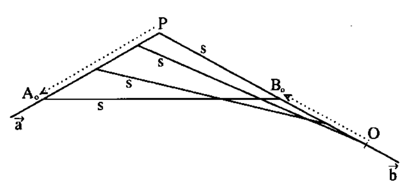


Figure 2b

It is easy to see that during this downwards movement $\angle PA_\nu B_\nu$ decreases and $\angle PB_\nu A_\nu$ increases in size. Due to (1a) the segment $A_0 B_0$ which satisfies $\angle PA_0 B_0 \equiv \angle PB_0 A_0,\ PA_0 \equiv PB_0$ is the lowest of class $\mathcal{C}$.

If $\angle(\overrightarrow{a}, \overrightarrow{b}) < \mathbf{R}$ then the class $\mathcal{C}$ contains a segment A_pB_p such that $\angle PA_pB_p = \mathbf{R}$. Note that when A_ν moves downwards from P to A_p, B_ν moves in tandem down from the point O s units below P to B_p, but that when A_ν moves on downwards from A_p to A_0, B_0 moves back upwards from B_p to B_0 (see Figure 2a). If $\angle(\overrightarrow{a}, \overrightarrow{b}) \geq \mathbf{R}$ no perpendicular line to $(\overrightarrow{a})$ meets $(\overrightarrow{b})$ and the points B_ν move invariably upwards when the points A_ν move downwards (see Figure 2b).

Now consider three segments $AB, A_1B_1, A_2B_2 \in \mathcal{C}$ whose endpoints on $(\overrightarrow{a})$ satisfy the order relation $[AA_1A_2P]$, and the direct motions that carry segment AB to segment A_1B_1 and to segment A_2B_2. These motions belong to the inverses of the ones described above and may carry B first downwards and then upwards. As a result there are seven conceivable situations as far as the order of the points B, B_1 and B_2 is concerned (see Figure 3):

(I)	$[B_2B_1BP]$,	(II)	$[B_1BP],\ B_2 = B_1$
(III)	$[B_1B_2BP]$,	(IV)	$[B_1BP],\ B_2 = B,$
(V)	$[B_1BB_2P]$,	(VI)	$[BB_2P],\ B_1 = B,$ and
(VII)	$[BB_1B_2P]$.		

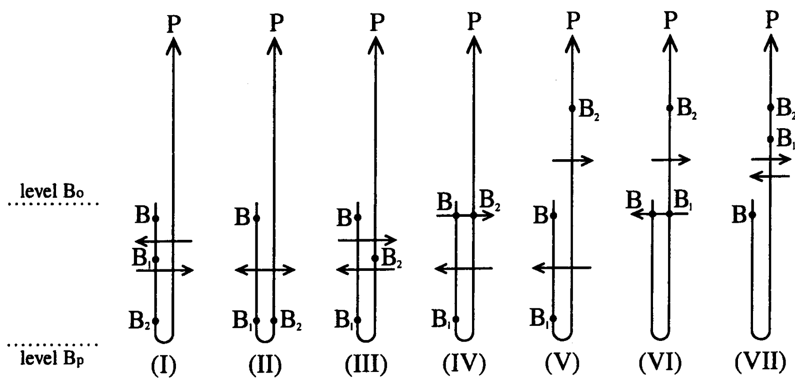


Figure 3

In case $\angle(\overrightarrow{a}, \overrightarrow{b}) \geq \mathbf{R}$ the point B moves solely downwards (see Figure 2b) and we find ourselves automatically in situation (I). On the other hand if $\angle(\overrightarrow{a}, \overrightarrow{b}) < \mathbf{R}$ and A lies on or above A_p both endpoints of segment AB move simultaneously upwards, first to A_1B_1 and then on to A_2B_2 (see Figure 2a); this means that we are dealing with situation (VII).

In Figure 3 the level of the midpoint N_1 of BB_1 is indicated by an arrow to the left, and the level of the midpoint N_2 of BB_2 by an arrow to the right. We recognize at once that we can use N_1 and N_2 instead of B_1 and B_2 to characterize the above seven situations. Set forth explicitly, a triple of segments $AB, A_1B_1, A_2B_2 \in \mathcal{C}$ with $[AA_1A_2P]$ can be classified according to the following conditions on the midpoints N_1, N_2 of BB_1, BB_2:

$$
(2)\quad \begin{array}{llll}
\text{(I)} & [N_2N_1BP], & \text{(II)} & [N_1BP], N_2 = N_1, \\
\text{(III)} & [N_1N_2BP], & \text{(IV)} & [N_1BP],\ N_2 = B, \\
\text{(V)} & [N_1BN_2P], & \text{(VI)} & [BN_2P],\ N_1 = B, \text{ and} \\
\text{(VII)} & [BN_1N_2P]. & &
\end{array}
$$

Note that always $N_1N_2 = \frac{1}{2}B_1B_2$. The midpoints M_1, M_2 of AA_1, AA_2 similarly satisfy $M_1M_2 = \frac{1}{2}A_1A_2$; here the direction $M_1 \to M_2$ like the direction $A_1 \to A_2$ points invariably upwards.

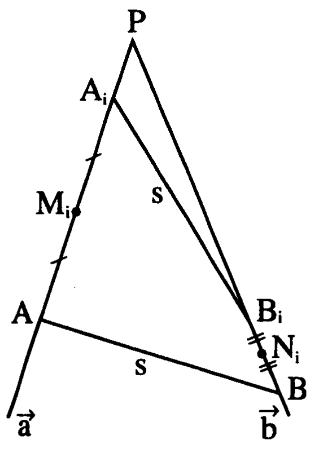


Figure 4

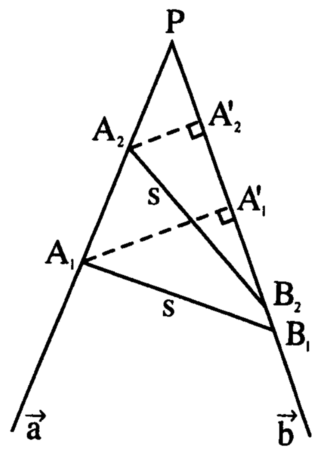


Figure 5

In closing this section we deduce two important inequalities which involve the points M_1, M_2, N_1 and N_2.

From $PA \leq PB,\ PA_i < PB_i$ (see (1b)) follows

$$(3) \qquad PM_i < PN_i \quad (i = 1, 2),$$

(see Figure 4). In addition, for situations (III) - (VII) in which B_2 lies above B_1 and N_2 above N_1 we can establish this. In the right triangles $\triangle A'_1A_1B_1$, $\triangle A'_2A_2B_2$ where $A'_1 = \text{proj}(A_1, b)$, $A'_2 = \text{proj}(A_2, b)$, $A'_1A_1 > A'_2A_2$, $A_1B_1 \equiv A_2B_2\ (= s)$, and as a result $A'_1B_1 < A'_2B_2$ (see Figure 5). So $A'_1A'_2 > B_1B_2$, and because $A_1A_2 > A'_1A'_2$, $A_1A_2 > B_1B_2$. Noting what was said above we therefore have:

(4) If N_2 lies above N_1 then $M_1M_2 > N_1N_2$.

3. The centers of two key segment motions

In this section we locate the centers of the segment motions described above. Our setting is the hyperbolic plane in which (as is well-known) three kinds of direct motions have to be considered. The Euclidean case could be subsumed with few modifications under the heading of rotations.

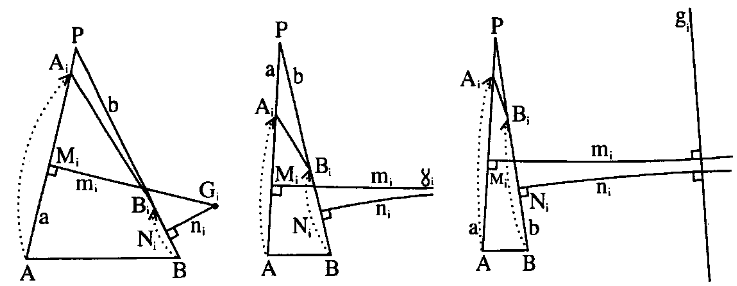


Figure 6

Let μ_i be the rigid, direct motion that carries the segment $AB \in \mathcal{C}$ onto the segment $A_iB_i \in \mathcal{C}$ $(i = 1, 2)$ where A_i lies above A, and let $m_i = \text{perp}(a, M_i)$, $n_i = \text{perp}(b, N_i)$. If lines m_i and n_i meet, μ_i is a *rotation* about their intersection point G_i, if they are boundary parallel, μ_i is an *improper* (horocyclic) *rotation* about their common end γ_i, and if they are hyperparallel, μ_i is a *translation* along their common perpendicular g_i (see e.g. [4, p. 455, Satz 13; Figure 6]). We call G_i, γ_i or g_i the *center* $[G_i]$ of the motion μ_i. For any point X disjoint from the center, $\ell(X[G_i])$ denotes the line joining X to the center of μ_i, namely $\ell(XG_i)$, $\ell(X\gamma_i)$, or $\text{perp}(g_i, X)$. The ray from X contained in this line and in the direction of $[G_i]$ will be referred to as the *ray* $\overrightarrow{X[G_i]}$ *from* X *towards the center of* μ_i; specifically for $X = P, M_i, N_i$ we define $\overrightarrow{p_i} = \overrightarrow{P[G_i]}$, $\overrightarrow{m_i} = \overrightarrow{M_i[G_i]}$ and (if it exists) $\overrightarrow{n_i} = \overrightarrow{N_i[G_i]}$.

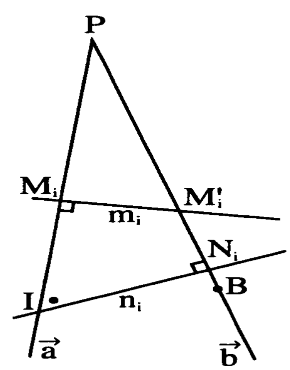


Figure 7

We now show that the center $[G_i]$ of motion μ_i must lie in $H(a, B)$.

If n_i does not intersect $(\overrightarrow{a})$ this is clear; if n_i meets a in a point I (see Figure 7) we verify the statement as follows. Segment PI as the hypothenuse of $\triangle PIN_i$ is larger than PN_i and so (see (3)) larger than PM_i. Consequently the angle $\angle PIN_i = \angle M_iIN_i$ is acute, which indicates that n_i when entering $H(a,B)$ at I, approaches m_i. As a result $[G_i]$ must lie in $H(a,B)$.

Some additional consequences are implied by the fact that the center $[G_i]$ of either motion μ_i is determined by a pair of perpendiculars m_i, n_i to lines a and b (see again Figure 6). If $[G_i] = \gamma_i$ is a common end of m_i, n_i and thus the center of a horocyclic rotation, it cannot be the end of ray $\overrightarrow{b}$. Similarly, if $[G_i] = g_i$ is the common perpendicular of m_i and n_i, and thus the translation axis, it is hyperparallel to both of the intersecting lines a, b and, as a result, has no point in common with either; furthermore, a and b, being connected, must belong to the same halfplane with respect to g_i. On the other hand, if $[G_i] = G_i$ is the common point of m_i and n_i, and thus the rotation center, it is indeed possible that it lies on $(\overrightarrow{b})$. The point G_i then is collinear with B and with its image B_i which means that for $B \neq B_i$ the rotation is a half-turn and G_i coincides with the midpoint N_i of BB_i; in addition G_i should be the midpoint M_i of AA_i which is impossible. So $B = B_i = N_i = G_i$; conversely, one establishes easily that if any two of the three points B, B_i, N_i coincide, μ_i is a rotation with center G_i equal to all three.

We now assume that our plane is furnished with an orientation (see [3, Section 20]), and that without loss of generality P lies to the left of ray $\overrightarrow{AB}$. This ray enters $H(a,B)$ at the point A of $(\overrightarrow{a})$ and $\overline{H}(b,A)$ at the point B of $(\overrightarrow{b})$. Also $\overrightarrow{m_i} = \overrightarrow{M_i[G_i]}$ enters $H(a,B)$ at a point of $(\overrightarrow{a})$ and so has P on its left hand side as well (see Figure 8). As to the ray $\overrightarrow{n_i} = \overrightarrow{N_i[G_i]}$ which (if existing, i.e. for $[G_i] \neq N_i$) originates at the point N_i of $(\overrightarrow{b})$, it has P on its left hand side if and only if it enters $\overline{H}(b,A)$, i.e. if and only if $[G_i]$ belongs to $\overline{H}(b,A)$.

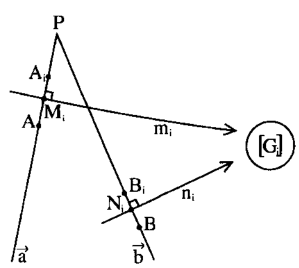


Figure 8a

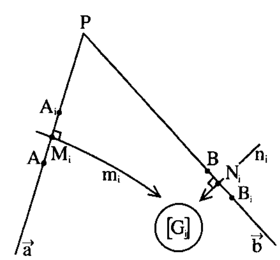


Figure 8b

Because the motion μ_i carries A across $\overrightarrow{m_i}$ to A_i on the side of P, A_i lies to the left and A to the right of $\overrightarrow{m_i}$. Being a direct motion, μ_i consequently also moves B (if $B \neq N_i$) from the right hand side of $\overrightarrow{n_i} = \overrightarrow{N_i[G_i]}$ to B_i on the left hand side which is the side of P iff $[G_i]$ belongs to $\overline{H}(b,A)$. In short, motion μ_i carries B upwards on $(\overrightarrow{b})$ iff $[G_i]$ lies in $\overline{H}(b,A)$.

We gather from the previous two paragraphs that

- $[G_i]$ belongs to $\overline{H}(b,A)$ if B_i and N_i lie above B on $(\overrightarrow{b})$,

- to $(\overrightarrow{b})$ if $B_i = N_i = B$, and
- to $H(b, A)$ if B_i and N_i lie below B on $(\overrightarrow{b})$.

Considering the motions μ_1, μ_2 again together we can tell in each of the seven situations listed in (2) where the two motion centers $[G_1], [G_2]$ (which both belong to $H(a, B)$) lie with respect to b. As we shall see, the relative positions of $[G_1], [G_2]$ can be described in a way that covers all seven situations: rotating ray $\overrightarrow{a} = \overrightarrow{PA}$ about P into $H(a, B)$ we always pass ray $\overrightarrow{P[G_1]}$ first, and ray $\overrightarrow{P[G_2]}$ second. More concisely,

MAIN LEMMA (ML). *Ray* $\overrightarrow{p_1} = \overrightarrow{P[G_1]}$ *always enters* $\angle(\overrightarrow{a}, \overrightarrow{p_2}) = \angle AP[G_2]$.

Proof. (The essential steps of the proof are outlined at the end.)

From (2) and the previous paragraph follows that $[G_1]$ lies in $H(b, A)$ in situations (I)-(V), on $(\overrightarrow{b})$ in situation (VI) and in $\overline{H}(b, A)$ in situation (VII); $[G_2]$ lies in $H(b, A)$ in situations (I)-(III), on $(\overrightarrow{b})$ in situation (IV) and in $\overline{H}(b, A)$ in situations (V)-(VII), (see Figure 9). As a result the Lemma follows trivially for situations (IV)-(VI). The other situations are more complex, and their proofs require that the nature of the motion centers $[G_i]$, $(i = 1, 2)$, which can be a point G_i, end γ_i or axis g_i be taken into account. Thus a pair of motion centers $[G_1], [G_2]$ can be equal to G_1, G_2; G_1, γ_2; G_1, g_2; γ_1, G_2; γ_1, γ_2; γ_1, g_2; g_1, G_2; g_1, γ_2; g_1, g_2.

The arguments to be presented are dependent on the mutual position of P, M_1, M_2 on $\overrightarrow{a}$ and of P, N_1, N_2 on $\overrightarrow{b}$, and are best followed through Figure 9.

We first consider situations (I)-(III) in which $\angle(\overrightarrow{a}, \overrightarrow{b})$ includes $(\overrightarrow{p_1})$ and $(\overrightarrow{p_2})$. To verify (ML) we have to show that $\overrightarrow{p_2}$ does not enter $\angle(\overrightarrow{a}, \overrightarrow{p_1})$, or equivalently that $\overrightarrow{p_1}$ does not enter $\angle(\overrightarrow{b}, \overrightarrow{p_2})$. (This assumes $\overrightarrow{p_1} \neq \overrightarrow{p_2}$ which either follows automatically or as an easy consequence of the arguments below.)

We begin with the special case that $\overrightarrow{p_1}$ meets m_2 in a point I. In this case statement (ML) holds if $\overrightarrow{p_2}$ does not intersect line m_2 at I or in a point between M_2 and I. Obviously this is so if $[G_2] = \gamma_2$ or g_2 because then $\overrightarrow{p_2}$ and m_2 do not intersect. If $[G_2] = G_2$, $\overrightarrow{p_2}$ and m_2 do intersect and we have to show that the intersection point, which is G_2, does not coincide with I or lie between M_2 and I. We first note that line n_1 does not intersect ray $\overrightarrow{p_1}$ in I or between I and P because the intersection point would have to be G_1 and so lie on m_1, a line entirely below m_2. As a consequence I, P, M_2, and, if it would lie between M_2 and I, also G_2, would all belong to the same halfplane with respect to n_1, namely $H(n_1, P)$. However this would entail that line n_2 which runs through G_2 would belong to this halfplane, which is not the case in situations (I) and (II). Thus we have established for those situations that $G_2 \neq I$, and $[M_2 G_2 I]$ does not hold, which means (ML) is true. We will present the proof of the same in situation (III) later on.

Due to the Axiom of Pasch the point I always exists if $\triangle PM_1[G_1]$ is a proper or improper triangle, i.e. if $[G_1] = G_1$, or γ_1. This means that we have so far proved (ML) for the cases G_1, γ_2; G_1, g_2; γ_1, γ_2; γ_1, g_2 and in addition for G_1, G_2; γ_1, G_2 in situations (I) and (II).

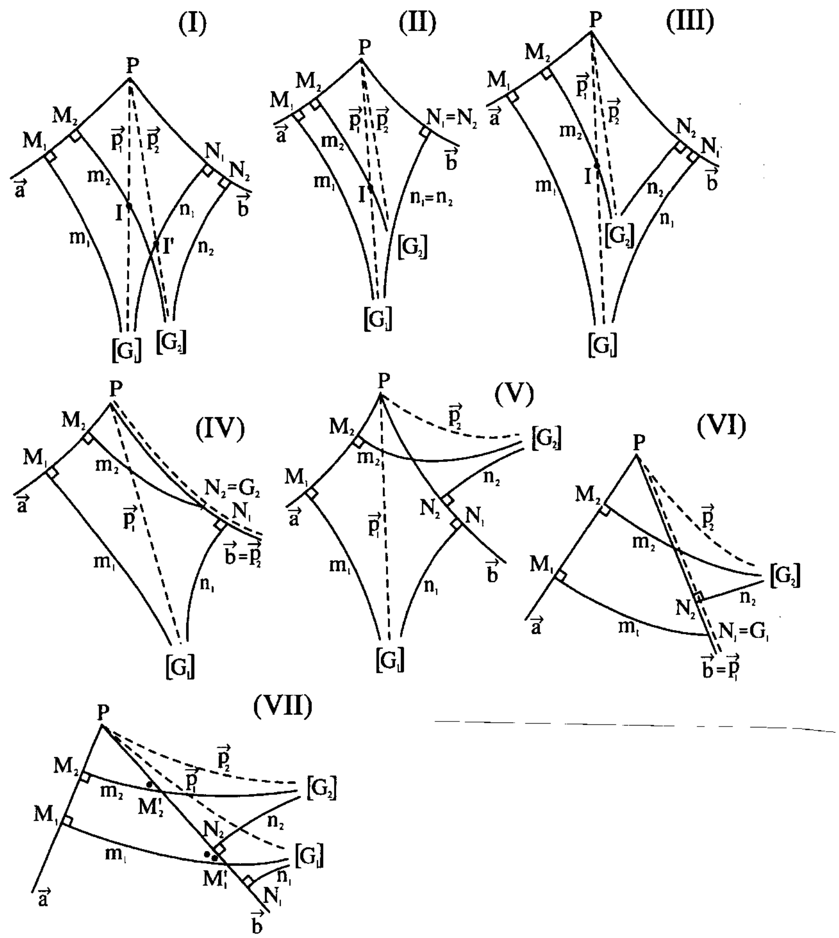


Figure 9

In the opposite special case that $\overrightarrow{p_2}$ meets n_1 in a point I' we can analogously show that for $[G_1] = \gamma_1$ or g_1 ray $\overrightarrow{p_1}$ does not enter $\angle(\overrightarrow{b}, \overrightarrow{p_2})$ and (ML) holds. In fact it is useful to mention here that this statement and its proof can be extended to include configurations in which $\overrightarrow{p_2}$ meets n_1 in an improper point ι'.

In situation (I) the point I' always exists if $[G_2] = G_2$ or γ_2 due to the Axiom of Pasch. In situation (II) with $n_1 = n_2$ I' exists for $[G_2] = G_2$ and ι' exists for $[G_2] = \gamma_2$ because in the first case $G_2 = I'$ and in the second case $\gamma_2 = \iota'$. This means that we have proved (ML) also for g_1, G_2; g_1, γ_2 in situations (I) and (II).

The proofs of the remaining cases, namely g_1, g_2 in situations (I)-(III), and G_1, G_2; γ_1, G_2; g_1, G_2; g_1, γ_2 in situation (III) require metric considerations and will be presented later.

Remark. Taking into account that we have already established (ML) in the case in which $\overrightarrow{p_1}$ and m_2 meet in a point I and $[G_2] = \gamma_2$ or g_2 we will assume when proving (ML) for g_1, g_2 and γ_1, g_2 that $\overrightarrow{p_1}$ and m_2 do not meet. At the same time, taking into account that we have already established (ML) in the case that $\overrightarrow{p_2}$ and n_1 meet in a point I' and $[G_1] = \gamma_1$ or g_1 we will assume that $\overrightarrow{p_2}$ and n_1 do not meet.

Turning to situation (VII) we observe that each of the rays $\overrightarrow{m_i}$ $(i = 1, 2)$ intersects $(\overrightarrow{b})$ in a point M_i' and approaches ray $\overrightarrow{n_i}$ in $\overline{H}(b, A)$, thus causing $\angle N_i M_i'[G_i]$ to be acute. Angle $\angle PM_i'M_i$ of the right triangle $\triangle PM_iM_i'$ is also acute with P above m_i, which means $\angle N_iM_i'[G_i]$ is its vertically opposite angle and N_i lies below m_i. As to the rays $\overrightarrow{p_i} = \overrightarrow{P[G_i]}$ they both enter $\overline{H}(b, A)$ at P which means that the angles $\angle(\overrightarrow{a}, \overrightarrow{p_i})$ have halfline $(\overrightarrow{b})$ in their interior.

If $\overrightarrow{p_2}$ does not intersect m_2, i.e. for $[G_2] = \gamma_2, g_2$, angle $\angle(\overrightarrow{a}, \overrightarrow{p_2})$ includes $(\overrightarrow{m_2}), (\overrightarrow{n_2})$ together with $(\overrightarrow{b})$. So, if in addition $[G_1] = G_1$ or γ_1, halfline $(\overrightarrow{p_1})$ crosses $(\overrightarrow{m_2})$ in order to meet $(\overrightarrow{m_1})$, i.e. runs in the interior of $\angle(\overrightarrow{a}, \overrightarrow{p_2})$. Lemma (ML) thus is fulfilled for G_1, γ_2; G_1, g_2; γ_1, γ_2; γ_1, g_2.

The remaining cases of (VII) depend on two metric properties. From $N_1N_2 < M_1M_2$ and $M_1M_2 = \text{proj}(M_1'M_2', a) < M_1'M_2'$ (see (4) and Figure 9, VII) follows $N_1N_2 < M_1'M_2'$ and so

$$N_1M_1' = N_1M_2' - M_1'M_2' < N_1M_2' - N_1N_2 = N_2M_2'. \tag{5}$$

In addition, from the fact that $\triangle PM_2'M_2$ has the smaller area (larger defect) than $\triangle PM_1'M_1$ follows $\angle PM_2'M_2 > \angle PM_1'M_1$, and so

$$\angle N_1M_1'[G_1] < \angle N_2M_2'[G_2]. \tag{6}$$

From (5) and (6) it is clear that if m_2 intersects or is boundary parallel to n_2 then m_1 must intersect n_1, i.e. that the cases $\gamma_1, G_2; g_1, G_2; g_1, \gamma_2$ cannot occur. Also, from (5) and (6) follows that if m_1, n_1 intersect in G_1 and m_2, n_2 intersect in G_2 then side N_1G_1 of $\triangle N_1M_1'G_1$ is shorter than side N_2G_2 of $\triangle N_2M_2'G_2$. This and $PN_1 > PN_2$ applied to $\triangle PN_1G_1, \triangle PN_2G_2$ implies $\angle(\overrightarrow{b}, \overrightarrow{p_1}) < \angle(\overrightarrow{b}, \overrightarrow{p_2})$, and so settles (ML) in the case of G_1, G_2.

The main case left is that of g_1, g_2, both in situation (VII) and situations (I) - (III). For use in the following we define $\text{proj}(M_i, g_i) = R_i$, $\text{proj}(N_i, g_i) = S_i$, $\text{proj}(P, g_i) = P_i$, and, assuming the points exist, $m_2 \curlywedge g_1 = U, n_2 \curlywedge g_1 = V, p_2 \between g_1 = W$.

If in situation (VII) (in which $\overrightarrow{p_2}$ lies above m_2, see Figure 10) the point W does not exist $\overrightarrow{n_1}$ lies with $(\overrightarrow{b})$, g_1 with $\overrightarrow{n_1}$ and $(\overrightarrow{p_1})$ with g_1 in the interior of $\angle(\overrightarrow{a}, \overrightarrow{p_2})$ thus fulfilling (ML). If W exists, line n_2 which runs between the lines n_1, m_2 and so avoids $\overrightarrow{p_2}$, enters quadrilateral PN_1S_1W and leaves it, defining V, between S_1 and W.

From (6) follows that Lambert quadrilateral $N_1S_1R_1\underline{M_1'}$ has the smaller angle sum and so the larger area than $N_2S_2R_2\underline{M_2'}$, which because of (5) requires that

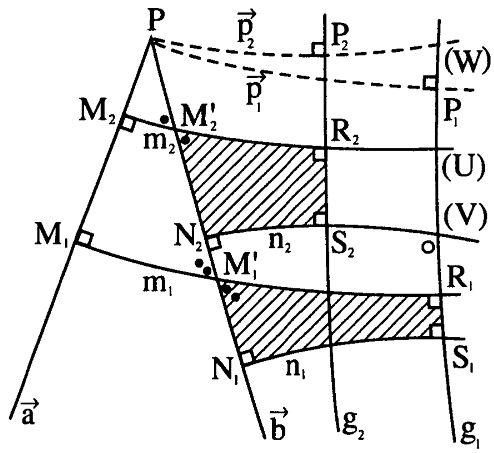


Figure 10

$N_1S_1 > N_2S_2$. As a result V, W on ray $\overrightarrow{S_1R_1}$ satisfy $[N_2S_2V]$, $[PP_2W]$ respectively. As $\angle S_1VN_2 = \angle S_1VS_2$ of $N_2N_1S_1\underline{V}$ is acute, $\angle V$ in P_2S_2VW is obtuse and $\angle P_2 + \angle S_2 + \angle V > 3\mathbf{R}$. This means that $\angle W = \angle PWV$ must be acute and identical with $\angle PWP_1$; consequently $(\overrightarrow{p_1}) = (\overrightarrow{PP_1})$ must lie with V, N_2 in the interior of $\angle(\overrightarrow{a}, \overrightarrow{p_2})$, again confirming (ML).

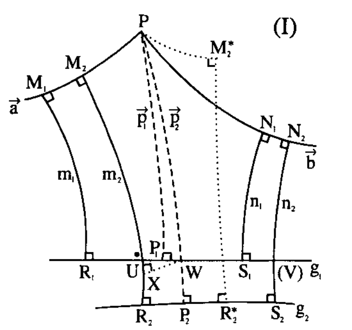


Figure 11a

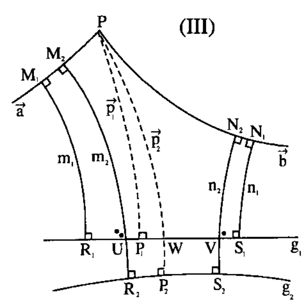


Figure 11b

Each of the Figures 11a, b relating to situations (I), (III) contains two pentagons $PM_iR_iS_iN_i$ $(i = 1, 2)$ with interior altitude PP_i. Adding the images M_i^*, R_i^* of M_i, R_i under reflection in PP_i (as illustrated for $i = 2$ in Figure 11a) we note that $P_iR_i^* < P_iS_i$ because otherwise we would have $PM_i \equiv PM_i^* \geq PN_i$ in contradiction to (3). Moreover $\angle P_iPM_i > \angle P_iPN_i$ as $\angle P_iPM_i \equiv \angle P_iPM_i^* \leq \angle P_iPN_i$ together with $P_iR_i^* < P_iS_i$ would imply that $PM_i^*R_i^*P_i$ would be a part polygon of $PN_iS_iP_i$ while not having a larger angle sum (i.e. smaller defect). So

$$(7a) \quad P_iR_i < P_iS_i, \quad (7b) \quad \angle P_iPM_i > \angle P_iPN_i, \quad (7c) \quad R_iM_i > S_iN_i.$$

In view of an earlier *Remark* we assume that the point U exists and that it satisfies $[R_1UP_1]$; together with $[R_1P_1S_1]$ this extends to $[R_1UP_1S_1]$. In situation (I) we can similarly assume that $\overrightarrow{p_2}$ and n_1 do not meet which means that the point W exists and that it satisfies $[UWS_1]$, a relation that can be extended to $[R_1UWS_1]$. In situation (III) we automatically have V such that $[UVS_1]$ and W such that $[UWV]$ is fulfilled, altogether therefore $[R_1UWVS_1]$.

In both situations m_2 and $\overrightarrow{UR_1}$ include an acute angle which coincides with the fourth angle $\angle R_1UM_2$ of Lambert quadrilateral $M_2M_1R_1\underline{U}$ and so lies on the upper side of g_1. It is congruent to the vertically opposite angle between m_2 and $\overrightarrow{UW}$ which thus lies on the lower side of g_1. In situation (III), for similar reasons, n_2 and $\overrightarrow{VW}$ include an acute angle which is congruent to $\angle N_2VS_1$ and lies on the lower side of g_1. As a result of all this in situation (III) the closest connection R_2S_2 between m_2 and n_2 lies below g_1, and so does the auxiliary point $X = \text{proj}(W, m_2)$ in situation (I).

Statement (ML) holds in both situations if $[UP_1W]$ is fulfilled i.e. if $P_1 = \text{proj}(P, g_1)$ belongs to leg $\overrightarrow{(W)U}$ of $\angle PWU$. We note that this is the case iff $\angle PWU$ is acute.

Now, if in situation (I) R_2, P_2 and S_2 lie below g_1 then the intersection point V of n_2 and g_1 exists and lies between N_2 and S_2, $\angle N_2VS_1$ is acute, $\angle S_2VS_1 = \angle S_2VW$ therefore obtuse and in quadrilateral $P_2S_2VW \quad \angle P_2 + \angle S_2 + \angle V > 3\mathbf{R}$; as a consequence $\angle W = \angle P_2WV$ is acute and so is its vertically opposite angle, $\angle PWU$. This, as we mentioned, proves (ML). If R_2P_2 lies below XW and ray $\overrightarrow{R_2P_2}$ intersects g_1 in a point Y, angle $\angle P_2WY$ in triangle $\triangle P_2WY$ is acute, which leads to the same conclusion. If $R_2P_2 = XW$ then $\angle PWU < \angle PWX = \angle PP_2R_2 = \mathbf{R}$. Finally, if R_2P_2 lies above XW, $\angle PWX$ as the fourth angle of Lambert quadrilateral $XR_2P_2\underline{W}$ is acute, and because $\angle PWU < \angle PWX, \ \angle PWU < \mathbf{R}$. This concludes the proof of (ML) in situation (I).

In situation (III) we have area $M_2M_1R_1\underline{U} > N_2N_1S_1\underline{V}$ because of (4), (7c). Consequently $\angle M_2UR_1 < \angle N_2VS_1$, and so $\angle WUR_2 < \angle WVS_2$ on the other side of g_1. If we also had $\angle P_2WU \leq \angle P_2WV$ then quadrilateral R_2P_2WU would have a smaller angle sum and larger defect than S_2P_2WV. At the same time (7a) and this angle inequality would imply that the former quadrilateral would fit into the latter, i.e. have the smaller area. Since this is contradictory $\angle P_2WU$ must be larger than the adjacent angle $\angle P_2WV$; as a result $\angle P_2WV < \mathbf{R}$ and vertically opposite, $\angle PWU < \mathbf{R}$ which establishes (ML) for g_1, g_2 in situation (III).

The proof of (ML) in situation (III) can be extended with only very minor changes to situation (II). Also closely related is the case of g_1, γ_2 in situation (III). If here, in addition to $\angle WU\gamma_2 < \angle WV\gamma_2$, the inequality $\angle\gamma_2WU \leq \angle\gamma_2WV$ were to hold then line m_2 would have to run farther away from line p_2 than line n_2 in contradiction to (3). An analogous argument applies to the case g_1, G_2 in situation (III) when G_2 lies below g_1. Note that if line m_2 intersects g_1 in a point U between

R_1 and P_1 rather than $\overrightarrow{p_1}$ in a point I between P and P_1 then G must lie below g_1 (see Figure 11b). This is so because according to (4) and (7c) the existence of a Lambert quadrilateral $M_2M_1R_1\underline{U}$ with $R_1U < R_1P_1$ implies the existence of $N_2N_1S_1\underline{V}$ with $S_1V < R_1U < R_1P_1$, and so due to (7a) with $S_1V < S_1P_1$; the point P_1 thus lies between U and V, and M_2 and N_2 meet below g_1.

To conclude the proof of (ML) we still have to settle the cases G_1, G_2; γ_1, G_2 and g_1, G_2 (this with $I = m_2 \curlywedge \overrightarrow{p_1}$ on or above P_1) of situation (III). We present here the last case (Figure 12b) which is easy and representative also for the proofs of the other two cases (Figure 12a).

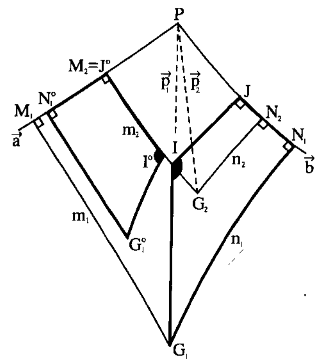


Figure 12a

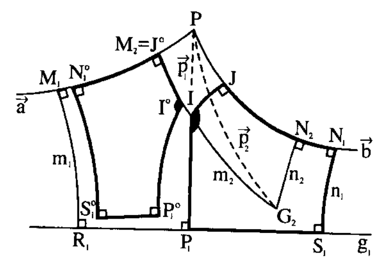


Figure 12b

Call $J = \text{proj}(I, b)$ and note that $\angle IPM_2 > \angle IPJ$ (7b) implies *(i)* $M_2I > JI$, and *(ii)* $\angle PIM_2 < \angle PIJ$, $\angle P_1IM_2 > \angle P_1IJ$. If G_2 would lie in $H(p_1, M_2)$ then the point $N_2 = \text{proj}(G_2, b)$ would determine a segment $N_1N_2 > N_1J$, and due to (4) the inequality *(iii)* $M_1M_2 > N_1J$ would result.

We now carry the pentagon $\mathcal{P}_b = JN_1S_1P_1I$ by an indirect motion to $\mathcal{P}_b^0 = J^0N_1^0S_1^0P_1^0I^0$ where $J_0 = M_2$, N_1^0 lies on $\overrightarrow{M_2M_1}$ and I^0 on $\overrightarrow{M_2I}$. Assuming that G_2 belongs to $H(p_1, M_2)$ we have according to *(i)*, *(iii)* that I^0 lies between M_2 and I, and N_1^0 between M_2 and M_1. Due to (7c) ray $\overrightarrow{S_1^0P_1^0}$ lies in the interior of $\angle M_1R_1P_1$, and due to *(ii)* halfline $\overrightarrow{(I^0)P^0}$ lies in the interior of $\angle P_1IM_2$ which implies that $\mathcal{P}_b^0$ is a proper part of polygon $\mathcal{P}_a = M_2M_1R_1P_1I$ in contradiction to the fact that $\mathcal{P}_b^0$ has the smaller angle sum, i.e. the larger defect. So G_2 and $\overrightarrow{p_2}$ do not lie in $\angle(\overrightarrow{a}, \overrightarrow{p_1})$ and the proof of (ML) is complete. $\square$

Summary of the Proof.
(1) Situations (IV) - (VI) are trivial.
(2) In situations (I) - (III), (ML) holds if $\overrightarrow{p_1} \curlywedge m_2 = I$ with $[G_2] \neq G_2$, and in situations (I), (II) also with $[G_2] = G_2$.
$\longrightarrow G_1, \gamma_2; G_1, g_2; \gamma_1, \gamma_2; \gamma_1, g_2$ of (I) - (III), $G_1, G_2; \gamma_1, G_2$ of (I), (II).
(3) In situations (I) - (III), (ML) holds if $\overrightarrow{p_2} \curlywedge n_1 = I'$ (ι') with $[G_1] \neq G_1$.

$\longrightarrow g_1, G_2; g_1, \gamma_2$ of (I), (II).
(4) In situation (VII) a direct comparison of $\triangle N_1 M_1'[G_1]$, $\triangle N_2 M_2'[G_2]$ reveals the relative position of $[G_1], [G_2]$ in all but one case.
$\longrightarrow$ all cases of (VII) except g_1, g_2.
(5) In situation (VII) the area comparison of $N_1 S_1 R_1 \underline{M_1'}$, $N_2 S_2 R_2 \underline{M_2'}$ helps to solve the remaining case.
$\longrightarrow g_1, g_2$ of (VII).
(6) In situations (I), (III) the area comparison of $PM_1 R_1 S_1 N_1$, $PM_2 R_2 S_2 N_2$ helps to solve the same case as in 5.
$\longrightarrow g_1, g_2$ of (I), (III).
(7) The arguments of 6. can be extended to three more cases.
$\longrightarrow g_1, g_2$ of (II); g_1, γ_2 of (III); g_1, G_2 of (III) for G_2 below g_1.
(8) The area comparison between a part polygon of $N_1 S_1 P_1 \underline{P}$, and one of $M_1 R_1 P_1 \underline{P}$, together with two similar comparisons, settle the remaining cases of (III).
$\longrightarrow g_1, G_2$ with G_2 above g_1; G_1, G_2; γ_1, G_2 of (III).

4. Reinterpretation and solution of the posed problem

In the following we formulate, re-formulate and prove a statement which essentially contains the convexity claim of Section 1. Subsequently we discuss the details which make the convexity proof complete.

Theorem 1. *Let AB be a fixed segment and P_2^-, P_1^- and P three points in the same halfplane with respect to the line through A and B such that*

$$\angle AP_2^- B \equiv \angle AP_1^- B \equiv \angle APB \tag{8}$$

and

$$\angle BAP_2^- > \angle BAP_1^- > \angle BAP \geq \angle ABP. \tag{9}$$

Then the line r which joins P_2^- and P separates the point P_1^- from the segment AB (see Figure 13a).

For the purpose of re-formulating this theorem we carry the points A, B, P_1^-, P and the line r of this configuration by a rigid, direct motion μ_1 into the points A_1, B_1, P, P_1 and the line r_1 respectively such that A_1 lies on $\overrightarrow{PA}$ and B_1 on $\overrightarrow{PB}$ (see Figure 13b). This allows us to substitute the following equivalent theorem for Theorem 1.

Theorem 2. *In the configuration of the points A, B, P, A_1, B_1, P_1 and the line r_1 as defined above, the line r_1 separates the point P from segment AB.*

Remark. Note that Theorem 1 amounts to the statement that the intersection point C_1^- of ray $\overrightarrow{AP_1^-}$ and line r lies between A and P_1^-, and Theorem 2 to the statement that the intersection point C_1 of $\overrightarrow{A_1 P}$ and r_1 (i.e. the image of C_1^- under the motion μ_1) lies between A_1 and P.

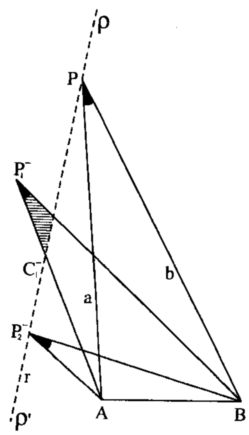


Figure 13a

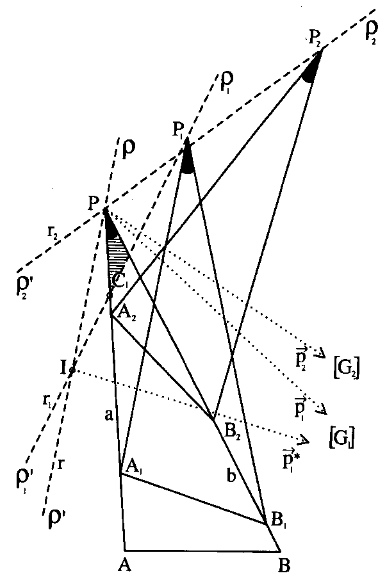


Figure 13b

Proof of Theorem 2. We first augment our configuration by the images of another rigid, direct motion μ_2 which carries the points A, B, P, P_2^- and the line r into A_2, B_2, P_2, P and r_2 respectively where A_2 lies on $\overrightarrow{PA}$ and B_2 on $\overrightarrow{PB}$. We note that because r joins P_2^- and P, r_2 joins P and P_2. From $\angle BAP_2^- > \angle BAP_1^-$ (see (9)) follows $\angle B_2A_2P = \mu_2(\angle BAP_2^-) > \mu_1(\angle BAP_1^-) = \angle B_1A_1P$ and so $[AA_1A_2P]$ according to Section 2. In the following we denote the ends of r by ρ and ρ' (with ρ' on the same side of $a = \ell(AP)$ as C_1^-) and their images on r_1 and r_2 by ρ_1, ρ_1' resp. ρ_2, ρ_2'. Since ρ' lies on the left (right) side of $\overrightarrow{AP_2^-}$ and of $\overrightarrow{AP_1^-}$ if and only if it lies on the left (right) side of $\overrightarrow{AP}$, and since $\overrightarrow{A_2P} = \mu_2(\overrightarrow{AP_2^-})$, $\overrightarrow{A_1P} = \mu_1(\overrightarrow{AP_1^-})$ and $\overrightarrow{AP}$ are equally directed, ρ_2', ρ_1' and ρ' lie together with C_1^- in $\overline{H}(a, B)$. We note that as an exterior angle of $\triangle AP_2^-C_1^-$, $\angle AP_2^-\rho' > \angle AC_1^-\rho'$, and that as an exterior angle of $\triangle AC_1^-P$, $\angle AC_1^-\rho' > \angle AP\rho'$. Applying μ_2 and μ_1 on the two sides of the first and μ_1 on the left hand side of the second inequality we obtain $\angle A_2P\rho_2' > \angle A_1C_1\rho_1'$ and $\angle A_1C_1\rho_1' > \angle AP\rho'$. The supplementary angles consequently satisfy

$$(10) \qquad \angle AP\rho > \angle AC_1\rho_1 > \angle AP\rho_2, \qquad \rho, \rho_1, \rho_2 \in H(a, B).$$

From (10) follows that ρ_2 lies on the same side of line $r = \ell(P\rho)$ as A, and (because ray $\overrightarrow{PA}$ does not enter $\angle \rho P \rho_2$) $\overrightarrow{PA}$ enters $\angle \rho' P \rho_2$.

At this point we augment our figure further by the rays $\overrightarrow{p_1}, \overrightarrow{p_2}$ which connect P to the centers $[G_1], [G_2]$ of the motions μ_1, μ_2 and, if r, r_1 have a point I in common, by the ray $\overrightarrow{p_1}^*$ connecting I to $[G_1]$. Because μ_2 maps r and ρ to r_2 and ρ_2, while μ_1 maps r and ρ to r_1 and ρ_1, the ray $\overrightarrow{p_2}$ is the bisector of angle $\angle\rho' P\rho_2$, and (if existing) the ray $\overrightarrow{p_1}^*$ is the bisector of angle $\angle\rho' I\rho_1$. Since $(\overrightarrow{p_2})$ lies together with ρ_2 in $H(a,B)$ (see Section 3) whereas ρ' lies in $\overline{H}(a,B)$, the ray $\overrightarrow{PA}$ enters $\angle\rho' P[G_2]$.

We now show by indirect proof that C_1 cannot lie on or above P on a.

For $C_1 = P$ (see Figure 14a) formula (10) reads: $\angle AP\rho > \angle AP\rho_1 > \angle AP\rho_2$, $\rho, \rho_1, \rho_2 \in H(a,B)$, and we can add to the sentence following (10) that also ρ_1 and A lie on the same side of r. Thus $\angle\rho' P\rho_1 = \angle\rho' PA + \angle AP\rho_1 > \angle\rho' PA + \angle AP\rho_2 = \angle\rho' P\rho_2$, and $\angle\rho' P[G_1] = \frac{1}{2}\angle\rho' P\rho_1 > \frac{1}{2}\angle\rho' P\rho_2 = \angle\rho' P[G_2]$. This means that $\overrightarrow{p_1}$ does not enter $\angle\rho' P[G_2]$ and so does not enter $\angle AP[G_2]$ in contradiction to Lemma (ML).

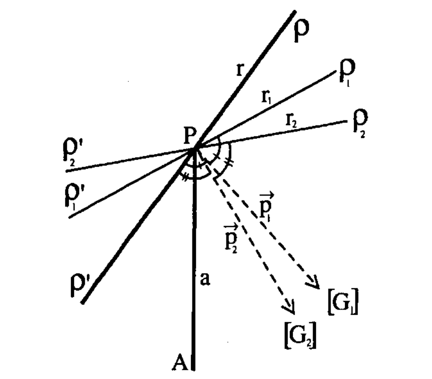


Figure 14a

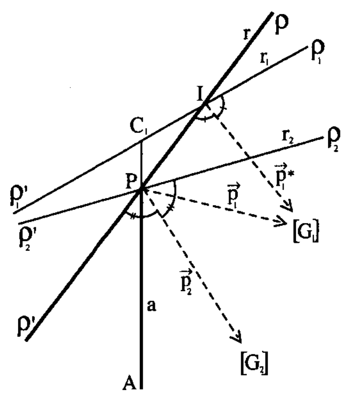


Figure 14b

If C_1 were to lie above P, on a, ray $\overrightarrow{P\rho}$ of r would, according to (10), approach ray $\overrightarrow{C_1\rho_1}$ when entering $H(a,B)$. This means $\overrightarrow{P\rho}$ and $\overrightarrow{P\rho_1}$ either have a point I or the ends ρ, ρ_1 in common, or $r = \ell(P\rho)$ and $r_1 = \ell(P\rho_1)$ share a perpendicular line whose intersection point with r lies in $H(a,B)$. Let us first assume that $\overrightarrow{P\rho}, \overrightarrow{P\rho_1}$ meet in I (see Figure 14b).

In this case line r intersects both segment C_1A and ray $\overrightarrow{C_1\rho_1}$ which means that A and ρ_1 lie on the same side of r. Note that $\angle\rho' I\rho_1$ is equal to the exterior angle $\angle C_1 I\rho$ of $\triangle PC_1I$ and so satisfies $\angle\rho' I\rho_1 > \angle PC_1I + \angle C_1PI$. Because $\angle PC_1I\ (= \angle AC_1\rho_1) > \angle AP\rho_2$ (see (10)) and because $\angle C_1PI \equiv \angle\rho' PA$ we have $\angle\rho' I\rho_1 > \angle AP\rho_2 + \angle\rho' PA = \angle\rho' P\rho_2$. The lower halves of the compared angles consequently satisfy $\angle\rho' I[G_1] > \angle\rho' P[G_2]$ which means that neither $\overrightarrow{p_1}^*$ nor the boundary parallel ray $\overrightarrow{p_1}$ would enter $\angle\rho' P[G_2]$, again in contradiction to Lemma (ML).

Should $\rho = \rho_1$ (see Figure 14c) then this common end is at the same time the center $[G_1]$ of motion μ_1. As a result the ray $\overrightarrow{p_1} = \overrightarrow{P\rho}$, and again fails to enter $\angle AP[G_2]$.

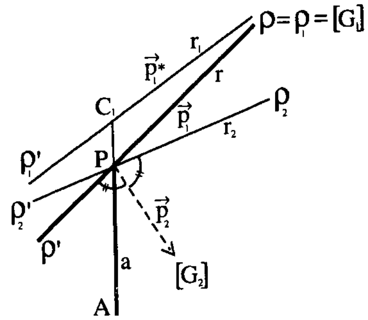


Figure 14c

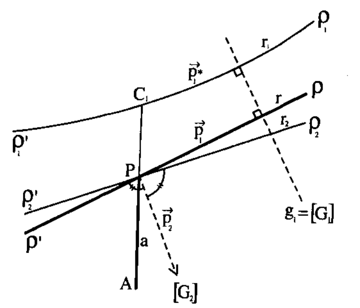


Figure 14d

Finally, if r and r_1 have the perpendicular line g_1 in common (see Figure 14d) then μ_1 which maps r, ρ to r_1, ρ_1 is a translation with axis g_1. As a result $\overrightarrow{p_1} = \overrightarrow{P[G_1]}$ must coincide with $\overrightarrow{P\rho}$ which means it does not enter $\angle\rho' P\rho_2$ and so does not enter $\angle\rho' P[G_2]$ and $\angle AP[G_2]$ in contradiction to Lemma (ML). This completes the proof of Theorem 2 and of Theorem 1.

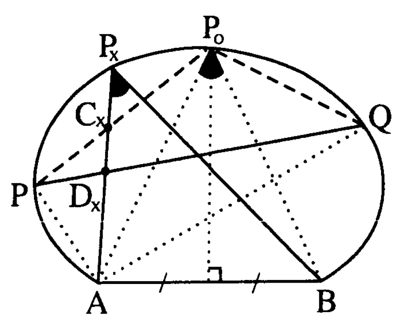


Figure 15

It should be noted that Theorem 1 contains the assumption (9), that P_2^-, P_1^-, P belong to the half-arc of our locus from A to the point P_0 on the perpendicular bisector of segment AB (see Figure 15)). By symmetry the Theorem also shows the convexity of the half-arc from B to P_0. In order to establish the convexity of the whole arc we need to confirm the additional fact that for P a point on the first half-arc and Q a point on the second, line $\ell(PQ)$ separates the points of arc $(\overset{\frown}{PQ})$ from those of segment AB. To do so we choose a point P_X, without loss of generality on half-arc $(\widehat{PP}_0)$, and establish that segment AP_X meets segment PQ between A and P_X. Obviously ray $\overrightarrow{AP_X}$, which enters $\angle PAQ$, meets PQ in a point D_X. Also, by Theorem 1 segment AP_X has a point C_X in common with segment P_0P, which means that our claim follows from $[AD_XC_X]$, a relation which is fulfilled if P_0, and so $(P_0P), (P_0Q)$ belong to $\overline{H}(PQ, A)$. This however is a consequence of the fact that P_0 has a greater distance from $\ell(AB)$ than P and Q, a fact of absolute geometry for which there are many easy proofs.

References

[1] L. Bitay, *Sur les angles inscrits dans un segment circulaire en géométrie hyperbolique*, Preprint Nr. 2, Seminar on Geometry, Research Seminars, Faculty of Mathematics, "Babes-Bolyai" University, Cluj-Napoca (1991).

[2] D. Gans, *An Introduction to Non-Euclidean Geometry*, Academic Press, Inc., Orlando, Florida, 1973.

[3] D. Hilbert, *Foundations of Geometry*, The Open Court Publishing Company, La Salle, Illinois, 1971.

[4] J. Hjelmslev, Neue Begründung der ebenen Geometrie, *Math. Ann.* **64**, 449-474 (1907).

[5] B. Klotzek and E. Quaisser, *Nichteuklidische Geometrie*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[6] O. Perron, *Nichteuklidische Elementargeometrie der Ebene*, B.G. Teubner Verlagsgesellschaft, Stuttgart, 1962.

[7] O. Perron, *Miszellen zur hyperbolischen Geometrie*, Bayerische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse, Sitzungsbericht 1964, 157–176.

[8] M. Simon, *Nichteuklidische Geometrie in elementarer Behandlung* (bearbeitet und herausgegeben von K. Fladt), B.G. Teubner, Leipzig, 1925.

Dieter Ruoff: Department of Mathematics and Statistics, University of Regina, Regina, Canada S4S 0A2

E-mail address: `ruoff@math.uregina.ca`

Forum Geometricorum
Volume 6 (2006) 167–169.

FORUM GEOM
ISSN 1534-1178

A Very Short and Simple Proof of "The Most Elementary Theorem" of Euclidean Geometry

Mowaffaq Hajja

Abstract. We give a very short and simple proof of the fact that if ABB' and $AC'C$ are straight lines with BC and $B'C'$ intersecting at D, then $AB+BD = AC' + C'D$ if and only if $AB' + B'D = AC + CD$. The "only if" part is attributed to Urquhart, and is referred to by Dan Pedoe as "the most elementary theorem of Euclidean geometry".

The theorem referred to in the title states that *if* ABB' *and* $AC'C$ *are straight lines with* BC *and* $B'C'$ *intersecting at* D *and if* $AB + BD = AC' + C'D$, *then* $AB' + B'D = AC + CD$; see Figure 1. The origin and some history of this theorem are discussed in [9], where Professor Pedoe attributes the theorem to

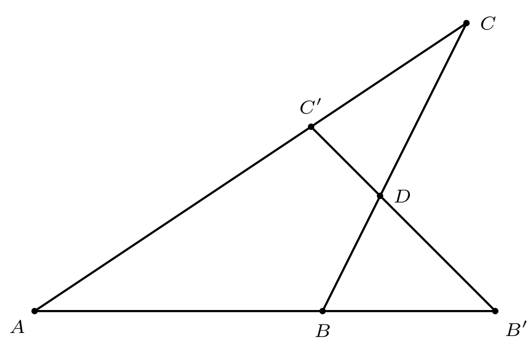


Figure 1

the late L. M. Urquhart (1902-1966) who *discovered it when considering some of the fundamental concepts of the theory of special relativity,* and where Professor Pedoe asserts that *the proof by purely geometric methods is not elementary*. Pedoe calls it *the most "elementary" theorem of Euclidean geometry* and gives variants and equivalent forms of the theorem and cites references where proofs can be found. Unaware of most of the existing proofs of this theorem (e.g., in [3], [4], [13], [14], [8], [10], [11] and [7, Problem 73, pages 23 and 128-129]), the author of this note has published a yet another proof in [5]. In view of all of this, it is interesting to know that De Morgan had published a proof of Urquhart's Theorem in 1841 and that Urquhart's Theorem may be viewed as a limiting case of a result due to Chasles that dates back to 1860; see [2] and [1].

Publication Date: April 24, 2006. Communicating Editor: Floor van Lamoen.
This work is supported by a research grant from Yarmouk University.

In this note, we give a much shorter proof based on a very simple and elegant lemma that Robert Breusch had designed for solving a 1961 MONTHLY problem. However, we make no claims that our proof meets the standards set by Professor Pedoe who hoped for a circle-free proof. Clearly our proof does not qualify since it rests heavily on properties of *circular* functions. Breusch's lemma [12] states that if $A_jB_jC_j$ $(j=1,2)$, are triangles with angles $A_j = 2\alpha_j$, $B_j = 2\beta_j$, $C_j = 2\gamma_j$, and if $B_1C_1 = B_2C_2$, then the perimeter $p(A_1B_1C_1)$ of $A_1B_1C_1$ is equal to or greater than the perimeter $p(A_2B_2C_2)$ of $A_2B_2C_2$ according as $\tan\beta_1 \tan\gamma_1$ is equal to or greater than $\tan\beta_2\tan\gamma_2$. This lemma follows immediately from the following sequence of simplifications, where we work with one of the triangles after dropping indices, and where we use the law of sines and the addition formulas for the sine and cosine functions.

$$\begin{aligned}\frac{p(ABC)}{BC} &= 1+\frac{AB+AC}{BC} = 1+\frac{\sin 2\gamma+\sin 2\beta}{\sin 2\alpha} = 1+\frac{\sin 2\gamma+\sin 2\beta}{\sin(2\gamma+2\beta)}\\ &= 1+\frac{2\sin(\gamma+\beta)\cos(\gamma-\beta)}{2\sin(\gamma+\beta)\cos(\gamma+\beta)} = 1+\frac{\cos\gamma\cos\beta+\sin\gamma\sin\beta}{\cos\gamma\cos\beta-\sin\gamma\sin\beta}\\ &= \frac{2\cos\gamma\cos\beta}{\cos\gamma\cos\beta-\sin\gamma\sin\beta} = \frac{2}{1-\tan\gamma\tan\beta}.\end{aligned}$$

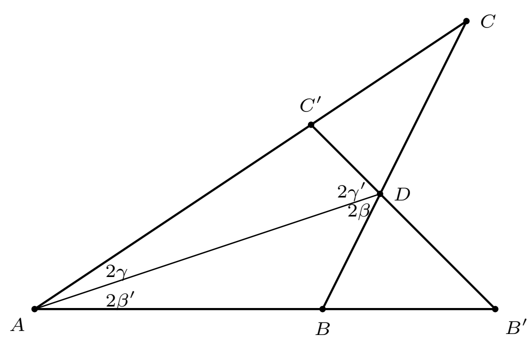


Figure 2

Urquhart's Theorem mentioned at the beginning of this note follows, together with its converse, immediately. Referring to Figure 1, and letting $\angle B'AD = 2\beta'$, $\angle CAD = 2\gamma$, $\angle BDA = 2\beta$, and $\angle C'DA = 2\gamma'$, as shown in Figure 2, we see from Breusch's Lemma that

$$\begin{aligned}p(AB'D) = p(ACD) &\Longleftrightarrow \tan\beta'\tan(90^\circ-\gamma') = \tan\gamma\tan(90^\circ-\beta)\\ &\Longleftrightarrow \tan\beta'\cot\gamma' = \tan\gamma\cot\beta\\ &\Longleftrightarrow \tan\beta'\tan\beta = \tan\gamma\tan\gamma'\\ &\Longleftrightarrow p(ABD) = p(AC'D),\end{aligned}$$

as desired.

The MONTHLY problem that Breusch's lemma was designed to solve appeared also as a conjecture in [6, page 78]. It states that if D, E, and F are points on the sides BC, CA, and AB, respectively, of a triangle ABC, then $p(DEF) \leq \min\{p(AFE), p(BDF), p(CED)\}$ if and only if D, E, and F are the midpoints of the respective sides, in which case the four perimeters are equal. In contrast with the analogous problem obtained by replacing perimeters by areas and the rich literature that this area version has generated, Breusch's solution of the perimeter version is essentially the only solution that the author was able to trace in the literature.

References

[1] M. A. B. Deakin, The provenance of Urquhart's theorem, *Aust. Math. Soc. Gazette*, 8 (1981), 26; addendum, *ibid.*, 9 (1982) 100.

[2] M. A. B. Deakin, Yet more on Urquhart's theorem, http://www.austms.org.au/Publ/Gazette/1997/Apr97/letters.html

[3] D. Eustice, Urquhart's theorem and the ellipse, *Crux Math.* (*Eureka*), 2 (1976) 132–133.

[4] H. Grossman, Urquhart's quadrilateral theorem, *The Mathematics Teacher*, 66 (1973) 643–644.

[5] M. Hajja, An elementary proof of the most "elementary" theorem of Euclidean Geometry, *J. Geometry Graphics*, 8 (2004) 17–22.

[6] N. D. Kazarinoff, *Geometric Inequalities*, New Mathematical Library 4, MAA, Washington, D. C., 1961.

[7] J. Konhauser, D. Velleman, and S. Wagon, *Which Way Did The Bicycle Go? ... and Other Intriguing Mathematical Mysteries*, Dolciani Mathematical Expositions 18, MAA, Washington D. C., 1996.

[8] L. Sauvé, On circumscribable quadrilaterals, *Crux Math.* (*Eureka*), 2 (1976) 63–67.

[9] D. Pedoe, The most "elementary" theorem of Euclidean geometry, *Math. Mag.*, 4 (1976) 40–42.

[10] D. Sokolowsky, Extensions of two theorems by Grossman, *Crux Math.* (*Eureka*), 2 (1976) 163–170.

[11] D. Sokolowsky, A 'no-circle' proof of Urquhart's theorem, *Crux Math.* (*Eureka*), 2 (1976) 133–134.

[12] E. Trost and R. Breusch Problem 4964, *Amer. Math. Monthly*, 68 (1961) 384; solution, *ibid.*, 69 (1962) 672–674.

[13] K. S. Williams, Pedoe's formulation of Urquhart's theorem, *Ontario Mathematics Gazette*, 15 (1976) 42–44.

[14] K. S. Williams, On Urquhart's elementary theorem of Euclidean geometry, *Crux Math.* (*Eureka*), 2 (1976) 108–109.

Mowaffaq Hajja: Department of Mathematics, Yarmouk University, Irbid, Jordan
E-mail address: mhajja@yu.edu.jo

Forum Geometricorum
Volume 6 (2006) 171–177.

FORUM GEOM
ISSN 1534-1178

The Orthic-of-Intouch and Intouch-of-Orthic Triangles

Sándor Kiss

Abstract. Barycentric coordinates are used to prove that the othic of intouch and intouch of orthic triangles are homothetic. Indeed, both triangles are homothetic to the reference triangle. Ratios and centers of homothety are found, and certain collinearities are proved.

1. Introduction

We consider a pair of triangles associated with a given triangle: the orthic triangle of the intouch triangle, and the intouch triangle of the orthic triangle. See Figure 1. Clark Kimberling [1, p. 274] asks if these two triangles are homothetic. We shall show that this is true if the given triangle is acute, and indeed each of them is homothetic to the reference triangle. In this paper, we adopt standard notations of triangle geometry, and denote the side lengths of triangle ABC by a, b, c. Let I denote the incenter, and the incircle (with inradius r) touching the sidelines BC, CA, AB at D, E, F respectively, so that DEF is the intouch triangle of ABC. Let H be the orthocenter of ABC, and let

$$D' = AH \cap BC, \quad E' = BH \cap CA, \quad F' = CH \cap AB,$$

so that $D'E'F'$ is the orthic triangle of ABC. We shall also denote by O the circumcenter of ABC and R the circumradius. In this paper we make use of homogeneous barycentric coordinates. Here are the coordinates of some basic triangle centers in the notations introduced by John H. Conway:

$$I = (a : b : c), \qquad H = \left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right) = (S_{BC} : S_{CA} : S_{AB}),$$
$$O = (a^2 S_A : b^2 S_B : c^2 S_C) = (S_A(S_B + S_C) : S_B(S_C + S_A) : S_C(S_A + S_B)),$$

where

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2},$$

and

$$S_{BC} = S_B \cdot S_C, \qquad S_{CA} = S_C \cdot S_A, \qquad S_{AB} = S_A \cdot S_B.$$

Publication Date: May 1, 2006. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this paper.

2. Two pairs of homothetic triangles

2.1. *Perspectivity of a cevian triangle and an anticevian triangle.* Let P and Q be arbitrary points not on any of the sidelines of triangle ABC. It is well known that the cevian triangle of $P = (u : v : w)$ is perspective with the anticevian triangle of $Q = (x : y : z)$ at

$$P/Q = \left(x\left(-\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right) : y\left(\frac{x}{u}-\frac{y}{v}+\frac{z}{w}\right) : z\left(\frac{x}{u}+\frac{y}{v}-\frac{z}{w}\right)\right).$$

See, for example, [3, §8.3].

2.2. *The intouch and the excentral triangles.* The intouch and the excentral triangles are homothetic since their corresponding sides are perpendicular to the respective angle bisectors of triangle ABC. The homothetic center is the point

$$\begin{aligned} P_1 =&(a(-a(s-a)+b(s-b)+c(s-c)) : b(a(s-a)-b(s-b)+c(s-c)) \\ &\qquad : c(a(s-a)+b(s-b)-c(s-c))) \\ =&(a(s-b)(s-c) : b(s-c)(s-a) : c(s-a)(s-b)) \\ =&\left(\frac{a}{s-a} : \frac{b}{s-b} : \frac{c}{s-c}\right). \end{aligned}$$

This is the triangle center X_{57} in [2].

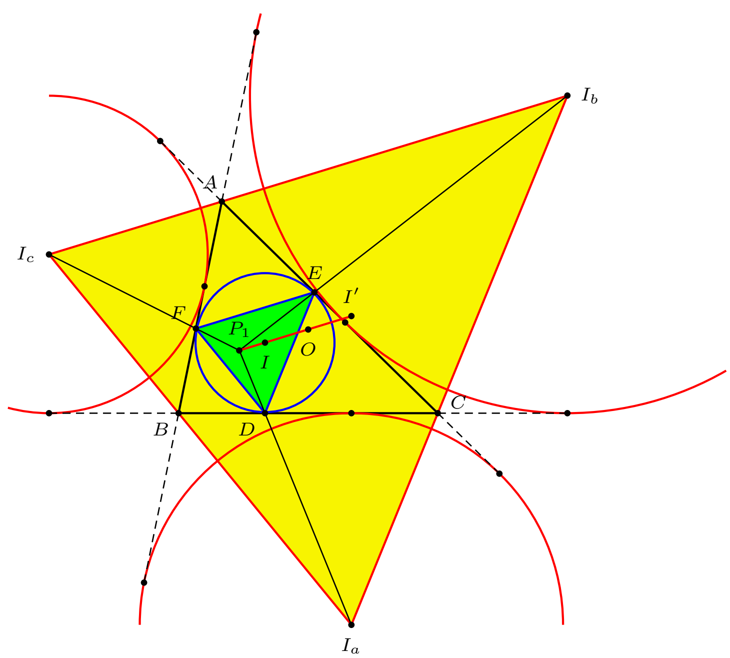


Figure 1.

2.3. *The orthic and the tangential triangle.* The orthic triangle and the tangential triangle are also homothetic since their corresponding sides are perpendicular to the respective circumradii of triangle ABC. The homothetic center is the point

$$\begin{aligned} P_2 = &\big(a^2(-a^2S_A + b^2S_B + c^2S_C) : b^2(-b^2S_B + c^2S_C + a^2S_A) \\ &\quad : c^2(-c^2S_C + a^2S_A + b^2S_B)\big) \\ =&(a^2S_{BC} : b^2S_{CA} : c^2S_{AB}) \\ =&\left(\frac{a^2}{S_A} : \frac{b^2}{S_B} : \frac{c^2}{S_C}\right). \end{aligned}$$

This is the triangle center X_{25} in [2].

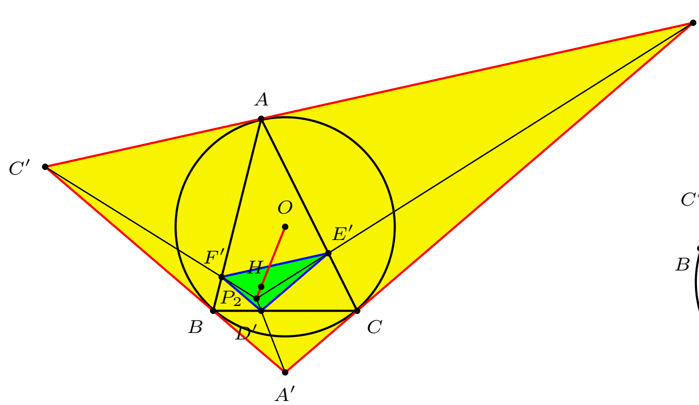


Figure 2A.

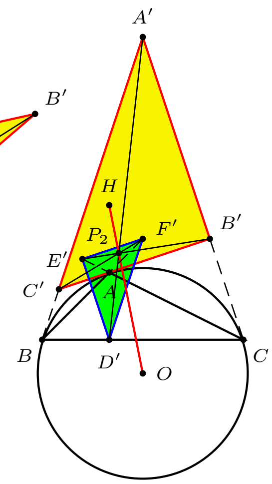


Figure 2B.

The ratio of homothety is positive or negative according as ABC is acute-angled and obtuse-angled. [1] See Figures 2A and 2B. When ABC is acute-angled, HD', HE' and HF' are the angle bisectors of the orthic triangle, and H is the incenter of the orthic triangle. If ABC is obtuse-angled, the incenter of the orthic triangle is the obtuse angle vertex.

3. The orthic-of-intouch triangle

The orthic-of-intouch triangle of ABC is the orthic triangle UVW of the intouch triangle DEF. Let h_1 be the homothety with center P_1, swapping D, E, F into U, V, W respectively. Consider an altitude DU of DEF. This is the image of the altitude I_aA of the excentral triangle under the homothety h_1. In particular, $U = \mathsf{h}_1(A)$. See Figure 3. Similarly, the same homothety maps B and C

[1]This ratio of homothety is $2\cos A\cos B\cos C$.

into V and W respectively. It follows that UVW is the image of ABC under the homothety h_1.

Since the circumcircle of UVW is the nine-point circle of DEF, it has radius $\frac{r}{2}$. It follows that the ratio of homothety is $\frac{r}{2R}$.

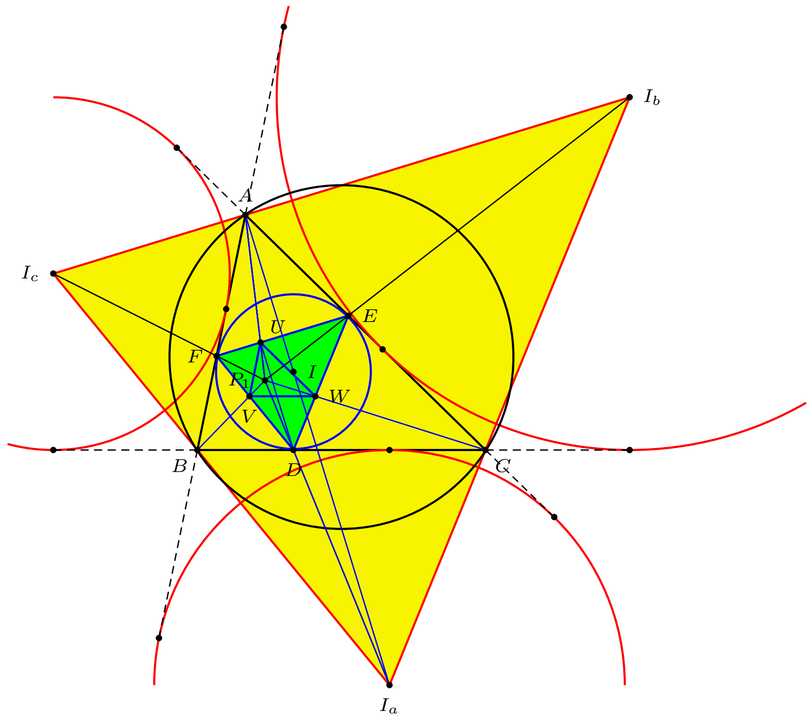


Figure 3.

Proposition 1. *The vertices of the orthic-of-intouch triangle are*

$$U = ((b+c)(s-b)(s-c) : b(s-c)(s-a) : c(s-a)(s-b)) = \left(\frac{b+c}{s-a} : \frac{b}{s-b} : \frac{c}{s-c}\right),$$

$$V = (a(s-b)(s-c) : (c+a)(s-c)(s-a) : c(s-a)(s-b)) = \left(\frac{a}{s-a} : \frac{c+a}{s-b} : \frac{c}{s-c}\right),$$

$$W = (a(s-b)(s-c) : b(s-c)(s-a) : (a+b)(s-a)(s-b)) = \left(\frac{a}{s-a} : \frac{b}{s-b} : \frac{a+b}{s-c}\right).$$

Proof. The intouch triangle DEF has vertices

$$D = (0 : s-c : s-b), \qquad E = (s-c : 0 : s-a), \qquad F = (s-b : s-a : 0).$$

The sidelines of the intouch triangle have equations

$$\begin{array}{ll} EF: & -(s-a)x + (s-b)y + (s-c)z = 0, \\ FD: & (s-a)x - (s-b)y + (s-c)z = 0, \\ DE: & (s-a)x + (s-b)y - (s-c)z = 0. \end{array}$$

The point U is the intersection of the lines AP_1 and EF. See Figure 3. The line AP_1 has equation

$$-c(s-b)y + b(s-c)z = 0.$$

Solving this with that of EF, we obtain the coordinates of U given above. Those of V and W are computed similarly. □

Corollary 2. *The equations of the sidelines of the orthic-of-intouch triangle are*

$$\begin{aligned}
VW: &\quad -s(s-a)x + (s-b)(s-c)y + (s-b)(s-c)z = 0,\\
WU: &\quad (s-c)(s-a)x - s(s-b)y + (s-c)(s-a)z = 0,\\
UV: &\quad (s-a)(s-b)x + (s-a)(s-b)y - s(s-c)z = 0.
\end{aligned}$$

4. The intouch-of-orthic triangle

Suppose triangle ABC is acute-angled, so that its orthic triangle $D'E'F'$ has incenter H, and is the image of the tangential triangle $A'B'C'$ under a homothety h_2 with center P_2. Consider the intouch triangle XYZ of $D'E'F'$. Under the homothety h_2, the segment $A'A$ is swapped into $D'X$. See Figure 4. In particular, $\mathsf{h}_2(A) = X$. For the same reason, $\mathsf{h}_2(B) = Y$ and $\mathsf{h}_2(C) = Z$. Therefore, the intouch-of-orthic triangle XYZ is homothetic to ABC under h_2.

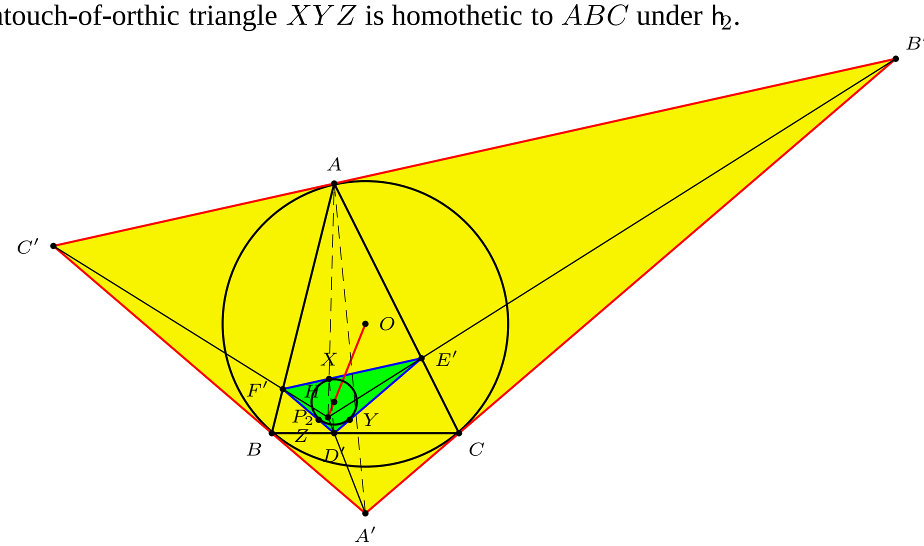


Figure 4

Proposition 3. *If ABC is acute angled, the vertices of the intouch-of-orthic triangle are*

$$\begin{aligned}
X =&((b^2+c^2)S_{BC} : b^2S_{CA} : c^2S_{AB}) = \left(\frac{b^2+c^2}{S_A} : \frac{b^2}{S_B} : \frac{c^2}{S_C}\right),\\
Y =&(a^2S_{BC} : (c^2+a^2)S_{CA} : c^2S_{AB}) = \left(\frac{a^2}{S_A} : \frac{c^2+a^2}{S_B} : \frac{c^2}{S_C}\right),\\
Z =&(a^2S_{BC} : b^2S_{CA} : (a^2+b^2)S_{AB}) = \left(\frac{a^2}{S_A} : \frac{b^2}{S_B} : \frac{a^2+b^2}{S_C}\right).
\end{aligned}$$

Proof. The orthic triangle $D'E'F'$ has vertices

$$D' = (0 : S_C : S_B), \qquad E' = (S_C : 0 : S_A), \qquad F' = (S_B : S_A : 0).$$

The sidelines of the orthic triangle have equations

$$\begin{array}{ll} E'F': & -S_A x + S_B y + S_C z = 0, \\ F'D': & S_A x - S_B y + S_C z = 0, \\ D'E': & S_A x + S_B y - S_C z = 0. \end{array}$$

The point X is the intersection of the lines AP_2 and $E'F'$. See Figure 4. The line AP_2 has equation

$$-c^2 S_B y + b^2 S_C z = 0.$$

Solving this with that of $E'F'$, we obtain the coordinates of U given above. Those of Y and Z are computed similarly. □

Corollary 4. *If ABC is acute-angled, the equations of the sidelines of the intouch-of-orthic triangle are*

$$\begin{array}{ll} YZ: & -S_A(S_A + S_B + S_C)x + S_{BC} y + S_{BC} z = 0, \\ ZX: & S_{CA} x - S_B(S_A + S_B + S_C)y + S_{CA} z = 0, \\ UV: & S_{AB} x + S_{AB} y - S_C(S_A + S_B + S_C)z = 0. \end{array}$$

5. Homothety of the intouch-of-orthic and orthic-of-intouch triangles

Proposition 5. *If triangle ABC is acute angled, then its intouch-of-orthic and orthic-of-intouch triangles are homothetic at the point*

$$Q = \left(\frac{a(a(b+c) - (b^2 + c^2))}{(s-a)S_A} : \frac{b(b(c+a) - (c^2 + a^2))}{(s-b)S_B} : \frac{c(c(a+b) - (a^2 + b^2))}{(s-c)S_C} \right).$$

Proof. The homothetic center is the intersection of the lines UX, VY, and WZ. See Figure 5. Making use of the coordinates given in Propositions 1 and 3, we obtain the equations of these lines as follows.

$$\begin{array}{ll} UX: & \quad bc(s-a)S_A(c(s-c)S_B - b(s-b)S_C)x \\ & +c(s-b)S_B((b^2+c^2)(s-a)S_C - (b+c)c(s-c)S_A)y \\ & +b(s-c)S_C(b(b+c)(s-b)S_A - (b^2+c^2)(s-a)S_B)z = 0, \\ VY: & \quad c(s-a)S_A(c(c+a)(s-c)S_B - (c^2+a^2)(s-b)S_C)x \\ & +ca(s-b)S_B(a(s-a)S_C - c(s-c)S_A)y \\ & +a(s-c)S_C((c^2+a^2)(s-b)S_A - (c+a)a(s-a)S_B)z = 0, \\ WZ: & \quad b(s-a)S_A((a^2+b^2)(s-c)S_B - (a+b)b(s-b)S_C)x \\ & +a(s-b)S_B(a(a+b)(s-a)S_C - (a^2+b^2)(s-c)S_A)y \\ & +ab(s-c)S_C(b(s-b)S_A - a(s-a)S_B)z = 0. \end{array}$$

It is routine to verify that Q lies on each of these lines. □

Remark. Q is the triangle center X_{1876} in [2].

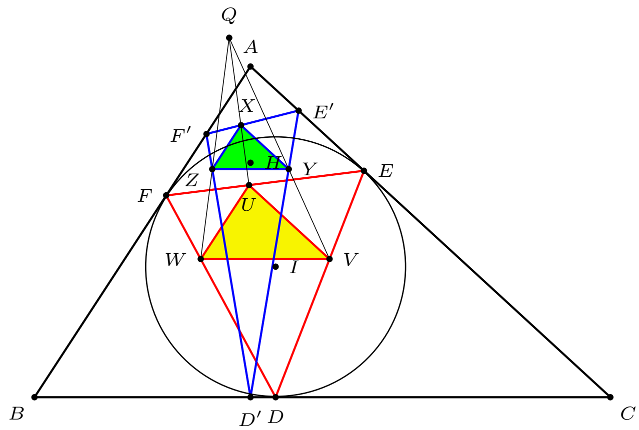


Figure 5

6. Collinearities

Because the circumcenter of XYZ is the orthocenter H of ABC, the center of homothety P_2 of ABC and XYZ lies on the Euler line OH of ABC. See Figure 4. We demonstrate a similar property for the point P_1, namely, that this point lies on the Euler line IF of DEF, where F is the circumcenter of UVW. Clearly, O, F, P_1 are collinear. Therefore, it suffices to prove that the points I, O, P_1 are collinear. This follows from

$$\begin{vmatrix} 1 & 1 & 1 \\ \cos A & \cos B & \cos C \\ (s-b)(s-c) & (s-c)(s-a) & (s-a)(s-b) \end{vmatrix} = 0,$$

which is quite easy to check. See Figure 1.

References

[1] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[3] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University lecture notes, 2001.

Sándor Kiss: 440095 Satu Mare, str. Avram Iancu, nr. 58, sc. A, ap. 8, Romania
E-mail address: kissandor@clicknet.ro

Forum Geometricorum
Volume 6 (2006) 179–180.

FORUM GEOM
ISSN 1534-1178

A 4-Step Construction of the Golden Ratio

Kurt Hofstetter

Abstract. We construct, in 4 steps using ruler and compass, three points two of the distances between which bear the golden ratio.

We present here a 4-step construction of the golden ratio using ruler and compass only. More precisely, we construct, in 4 steps using ruler and compass, three points with two distances bearing the golden ratio. It is fascinating to discover how simple the golden ratio appears. We denote by $P(Q)$ the circle with center P, passing through Q, and by $P(XY)$ that with center P and radius XY.

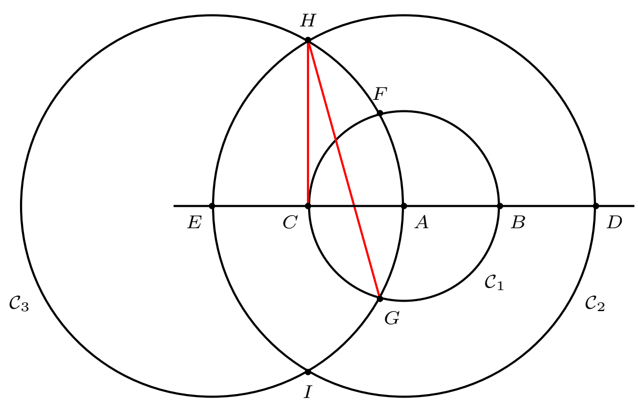


Figure 1

Construction. Given two points A and B, construct

(1) the circle $\mathcal{C}_1 = A(B)$,
(2) the line AB to intersect $\mathcal{C}_1$ again at C and extend it long enough to intersect
(3) the circle $\mathcal{C}_2 = A(BC)$ at D and E,
(4) the circle $\mathcal{C}_3 = E(BC)$ to intersect $\mathcal{C}_1$ at F and G, and $\mathcal{C}_2$ at H and I.

Then $\frac{GH}{CH} = \frac{\sqrt{5}+1}{2}$.

Proof. Without loss of generality let $AB = 1$, so that $BC = AE = AH = EH = 2$. Triangle AEH is equilateral. Let $\mathcal{C}_4 = H(A)$, intersecting $\mathcal{C}_1$ at J. By symmetry, AGJ is an equilateral triangle. Let $\mathcal{C}_5 = J(A) = J(AG) = J(AB)$, intersecting $\mathcal{C}_1$ at K. Finally, let $\mathcal{C}_6 = J(H) = J(BC)$. See Figure 2.

Publication Date: May 8, 2006. Communicating Editor: Paul Yiu.

With $\mathcal{C}_1$, $\mathcal{C}_5$, $\mathcal{C}_2$, $\mathcal{C}_6$, following [1], K divides GH in the golden section. It suffices to prove $CH = GK = \sqrt{3}$. This is clear for GK since the equilateral triangles AJG and AJK have sides of length 1. On the other hand, in the right triangle ACH, $AC = 1$ and $AH = 2$. By the Pythogorean theorem $CH = \sqrt{3}$. $\square$

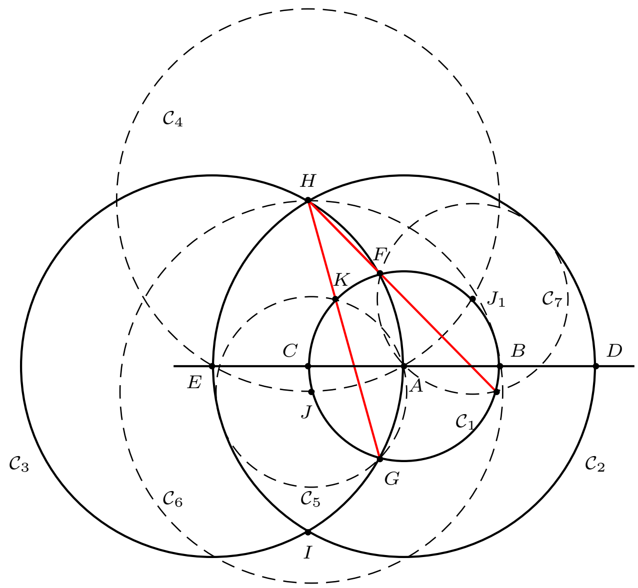


Figure 2

Remark. $\frac{CH}{FH} = \frac{GH}{CH} = \frac{\sqrt{5}+1}{2}$.

Proof. Since $CH^2 = GH \cdot KH$, it is enough to prove that $FH = KH$. Let $\mathcal{C}_4$ intersect $\mathcal{C}_1$ again at J_1. Consider the circle $\mathcal{C}_7 = J_1(A)$. By symmetry, F lies on $\mathcal{C}_7$ and $FH = KH$. $\square$

Reference

[1] K. Hofstetter, A simple construction of the golden section, *Forum Geom.*, 2 (2002) 65–66.

Kurt Hofstetter: Object Hofstetter, Media Art Studio, Langegasse 42/8c, A-1080 Vienna, Austria
E-mail address: `pendel@sunpendulum.at`

Forum Geometricorum
Volume 6 (2006) 181–185.

FORUM GEOM
ISSN 1534-1178

A Theorem by Giusto Bellavitis on a Class of Quadrilaterals

Eisso J. Atzema

Abstract. In this note we prove a theorem on quadrilaterals first published by the Italian mathematician Giusto Bellavitis in the 1850s, but that seems to have been overlooked since that time. Where Bellavitis used the functional equivalent of complex numbers to prove the result, we mostly rely on trigonometry. We also prove a converse of the theorem.

1. Introduction

Since antiquity, the properties of various special classes of quadrilaterals have been extensively studied. A class of quadrilaterals that appears to have been little studied is that of those quadrilaterals for which the products of the two pairs of opposite sides are equal. In case a quadrilateral ABCD is cyclic as well, $ABCD$ is usually referred to as a *harmonic* quadrilateral (see [2, pp.90–92], [3, pp.159–160]). Clearly, however, the class of all quadrilaterals $ABCD$ for which $AB \cdot CD = AD \cdot BC$ includes non-cyclic quadrilaterals as well. In particular, all kites are included. As far as we have able to ascertain, no name for this more general class of quadrilaterals has ever been proposed. For the sake of brevity, we will refer to the elements in this class as *balanced* quadrilaterals. In his *Sposizione del metodo delle equipollenze* of 1854, the Italian mathematician Giusto Bellavitis (1803-1880) proved a curious theorem on such balanced quadrilaterals that seems to have been forgotten.[1] In this note, we will give an elementary proof of the theorem. In addition, we will show how the converse of Bellavitis' theorem is (almost) true as well. Our proof of the first theorem is different from that of Bellavitis. The converse is not discussed by Bellavitis at all.

2. Bellavitis' Theorem

Let the lengths of the sides AB, BC, CD and DA of a (convex) quadrilateral $ABCD$ be denoted by a, b, c and d respectively. Similarly, the lengths of the quadrilateral's diagonals AC and BD will be denoted by e and f. Let E be the point of intersection of the two diagonals. The magnitude of $\angle DAB$ will be

Publication Date: May 15, 2006. Communicating Editor: Paul Yiu.

[1]Bellavitis' book is very hard to locate. We actually used Charles-Ange Laisant's 1874 translation into French [1], which is available on-line from the Bibliothèque Nationale. In this translation, the theorem is on p.26 as Corollary III of Bellavitis' derivation of Ptolemy's theorem.

referred to as α, with similar notations for the other angles of the quadrilateral. The magnitudes of $\angle DAC$, $\angle ADB$ etc will be denoted by α_B, δ_C and so on (see Figure 1). Finally, the magnitude of $\angle CED$ will be referred to as ϵ.

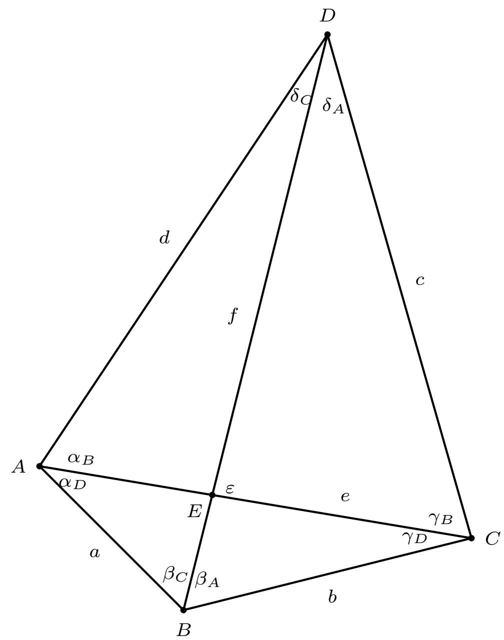


Figure 1. Quadrilateral Notations

With these notations, the following result can be proved.

Theorem 1 (Bellavitis, 1854)**.** *If a* (*convex*) *quadrilateral $ABCD$ is balanced, then*

$$\alpha_B + \beta_C + \gamma_D + \delta_A = \beta_A + \gamma_B + \delta_C + \alpha_D = 180^\circ.$$

Note that the convexity condition is a necessary one. The second equality sign does not hold for non-convex quadrilaterals. A trigonometric proof of Bellavitis' Theorem follows from the observation that by the law of sines for any balanced quadrilateral we have

$$\sin\gamma_B \cdot \sin\alpha_D = \sin\alpha_B \cdot \sin\gamma_D,$$

or

$$\cos(\gamma_B + \alpha_D) - \cos(\gamma_B - \alpha_D) = \cos(\alpha_B + \gamma_D) - \cos(\alpha_B - \gamma_D).$$

That is,

$$\cos(\gamma_B + \alpha_D) - \cos(\gamma_B - \alpha + \alpha_B) = \cos(\alpha_B + \gamma_D) - \cos(\alpha_B - \gamma + \gamma_B),$$

or

$$\cos(\gamma_B + \alpha_D) + \cos(\delta + \alpha) = \cos(\alpha_B + \gamma_D) + \cos(\delta + \gamma).$$

By cycling through, we also have

$$\cos(\delta_C + \beta_A) + \cos(\alpha + \beta) = \cos(\beta_C + \delta_A) + \cos(\alpha + \delta).$$

Since $\cos(\alpha + \beta) = \cos(\delta + \gamma)$, adding these two equations gives

$$\cos(\gamma_B + \alpha_D) + \cos(\delta_C + \beta_A) = \cos(\alpha_B + \gamma_D) + \cos(\beta_C + \delta_A),$$

or

$$\begin{aligned} &\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos\frac{1}{2}(\gamma_B + \alpha_D - \delta_C - \beta_A) \\ =&\cos\frac{1}{2}(\alpha_B + \beta_C + \gamma_D + \delta_A) \cdot \cos\frac{1}{2}(\alpha_B + \gamma_D - \beta_C - \delta_A). \end{aligned}$$

Now, note that

$$\gamma_B + \alpha_D - \delta_c - \beta_A = 360 - 2\epsilon - \delta - \beta$$

and, likewise

$$\alpha_B + \gamma_D - \beta_C - \delta_A = 2\epsilon - \beta - \delta.$$

Finally,

$$\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) + \frac{1}{2}(\alpha_B + \beta_C + \gamma_D + \delta_A) = 180°.$$

It follows that

$$\begin{aligned} &\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos(\epsilon + \frac{1}{2}(\beta + \delta)) \\ =& -\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos(\epsilon - \frac{1}{2}(\beta + \delta)), \end{aligned}$$

or

$$\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos(\epsilon) \cos\frac{1}{2}(\delta + \beta) = 0.$$

This almost concludes our proof. Clearly, if neither of the last two factors are equal to zero, the first factor has to be zero and we are done. The last factor, however, will be zero if and only if $ABCD$ is cyclic. It is easy to see that any such quadrilateral has the angle property of Bellavitis' theorem. Therefore, in the case that $ABCD$ is cyclic, Bellavitis' theorem is true. Consequently, we may assume that $ABCD$ is not cyclic and that the third term does not vanish. Likewise, the second factor only vanishes in case $ABCD$ is orthogonal. For such quadrilaterals, we know that $a^2 + c^2 = b^2 + d^2$. In combination with the initial condition $ac = bd$, this implies that each side has to be congruent to an adjacent side. In other words, $ABCD$ has to be a kite. Again, it is easy to see that in that case Bellavitis' theorem is true. We can safely assume that $ABCD$ is not a kite and that the second term does not vanish either. This proves Bellavitis' theorem.

3. The Converse to Bellavitis' Theorem

Now that we have proved Bellavitis' theorem, it is only natural to wonder for exactly which kinds of (convex) quadrilaterals the angle sums $\delta_C+\gamma_B+\beta_A+\alpha_D$ and $\alpha_A+\beta_C+\gamma_D+\delta_A$ are equal. Assuming that the two angle sums are equal and working our way backward from the preceding proof, we find that

$$\sin\gamma_B\cdot\sin\alpha_D+K=\sin\alpha_B\cdot\sin\gamma_D$$

for some K. Likewise,

$$\sin\delta_C\cdot\sin\beta_A=\sin\beta_C\cdot\sin\delta_A+K.$$

So,

$$\frac{\sin\gamma_B}{\sin\alpha_B}-\frac{\sin\gamma_D}{\sin\alpha_D}=-\frac{K}{\sin\alpha_B\cdot\sin\alpha_D}$$

and

$$\frac{\sin\delta_C}{\sin\beta_C}-\frac{\sin\delta_A}{\sin\beta_A}=\frac{K}{\sin\beta_A\cdot\sin\beta_C}$$

or

$$\frac{d}{c}-\frac{a}{b}=-\frac{K}{\sin\alpha_B\cdot\sin\alpha_D},\qquad \frac{a}{d}-\frac{b}{c}=\frac{K}{\sin\beta_A\cdot\sin\beta_C}.$$

If $K=0$, we have $bd=ac$ and $ABCD$ is balanced. If $K\neq 0$, it follows that

$$\frac{d}{b}=\frac{\sin\beta_A\cdot\sin\beta_C}{\sin\alpha_B\cdot\sin\alpha_D}.$$

Cycling through twice also gives us

$$\frac{b}{d}=\frac{\sin\delta_C\cdot\sin\delta_A}{\sin\gamma_D\cdot\sin\gamma_B}.$$

We find

$$\sin\beta_A\cdot\sin\beta_C\cdot\sin\delta_C\cdot\sin\delta_A=\sin\alpha_B\cdot\sin\alpha_D\cdot\sin\gamma_D\cdot\sin\gamma_B.$$

Division of each side by $abcd$ and grouping the factors in the numerators and denominators appropriately shows that this equation is equivalent to the equation

$$R_{ABC}\cdot R_{ADC}=R_{BAC}\cdot R_{BCD},$$

where R_{ABC} denotes the radius of the circumcircle to the triangle ABC etc. Now, the area of ABC is equal to both $abe/4R_{ABC}$ and $\frac{1}{2}e\cdot EB\cdot\sin\epsilon$ with similar expressions for ADC, BAC, and BCD. Consequently, the relation between the four circumradii can be rewritten to the form $EB\cdot EC=EA\cdot EC$. But this means that $ABCD$ has to be cyclic. We have the following result:

Theorem 2. *Any* (*convex*) *quadrilateral* $ABCD$ *for which*

$$\alpha_B+\beta_C+\gamma_D+\delta_A=\beta_A+\gamma_B+\delta_C+\alpha_D=180^\circ$$

is either cyclic or balanced.

4. Conclusion

We have not been able to find any references to Bellavitis' theorem other than in the *Sposizione*. Bellavitis was clearly mostly interested in the theorem because it allowed him to showcase the power of his method of equipollences.[2] Indeed, the *Sposizione* features a fair number of (minor) results on quadrilaterals that are proved using the method of equipollences. Most of these were definitely well-known at the time. This suggests that perhaps our particular result was reasonably well-known at the time as well. Alternatively, Bellavitis may have derived the theorem in one of the many papers that he published between 1833, when he first published on the method, and 1854. These earlier publications, however, are extremely hard to locate and we have not been able to consult any.[3] Whether the theorem originated with Bellavitis or not, it is not entirely surprising that this result seems to have been forgotten. The sums $\alpha_B+\beta_C+\gamma_D+\delta_A$ and $\beta_A+\gamma_B+\delta_C+\alpha_D$ do not usually show up in plane geometry. We do hope to finish up a paper shortly, however, in which these sums play a role as part of a generalization of Ptolemy's theorem to arbitrary (convex) quadrilaterals.

References

[1] G. Bellavitis, *Exposition de la méthode des equipollences* (traduit par C-A. Laisant) (Paris: Gauthier-Villars, 1874)

[2] W. Gallatly, *The Modern Geometry of the Triangle* (2nd ed.) (London: Hodgson, 1913)

[3] M. Simon, *Ueber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert*, Teubner, Leipzig, 1906 (= *Jahresberichte der Deutschen Mathematiker-Vereinigung. Ergänzungsbände*, B. I).

Eisso J. Atzema: Department of Mathematics, University of Maine, Orono, Maine 04469, USA
E-mail address: `atzema@math.umaine.edu`

[2] This method essentially amounted to a sometimes awkward mix of vector methods and the use of complex numbers in a purely geometrical disguise. In fact, for those interested in the use of complex numbers in plane geometry, it might be a worthwhile exercise to rework Bellavitis' equipollences proof of his theorem to one that uses complex numbers only. This should not be too hard.

[3] See the introduction of [1] for a list of references.

Forum Geometricorum
Volume 6 (2006) 187–190.

FORUM GEOM
ISSN 1534-1178

A Projectivity Characterized by the Pythagorean Relation

Wladimir G. Boskoff and Bogdan D. Suceavă

Abstract. We study an interesting configuration that gives an example of an elliptic projectivity characterized by the Pythagorean relation.

1. A Romanian Olympiad problem

It is known that any projectivity relating two ranges on one line with more than two invariant points is the identity transformation of the line onto itself. Depending on whether the number of invariant points is 0, 1, or 2 the projectivity would be called *elliptic*, *parabolic*, or *hyperbolic* (see Coxeter [1, p.45], or [3, pp.41-43]). This note will point out an interesting and unusual configuration that gives an example of projectivity characterized by a Pythagorean relation.

The configuration appears in a problem introduced in the National Olympiad 2001, in Romania, by Mircea Fianu. The statement of the problem is the following: *Consider the right isosceles triangle ABC and the points M, N on the hypothenuse BC in the order B, M, N, C such that $BM^2 + NC^2 = MN^2$. Prove that $\angle MAN = \frac{\pi}{4}$.*

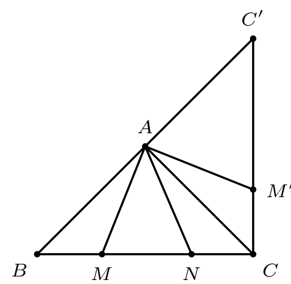


Figure 1

We present first an elementary solution for this problem. Consider a counterclockwise rotation around A of angle $\frac{\pi}{2}$. By applying this rotation, ΔABC becomes $\Delta ACC'$ (see Figure 1) and AM becomes AM'; thus, the angle $\angle MAM'$ is right. The equality $BM^2 + NC^2 = MN^2$ transforms into $CM'^2 + NC^2 = MN^2 = M'N^2$, since $\Delta CNM'$ is right in C. Therefore, $\Delta MAN \equiv \Delta NAM'$ (SSS case), and this means $\angle MAN \equiv \angle NAM'$. Since $\angle MAM' = \frac{\pi}{2}$, we get $\angle MAN = \frac{\pi}{4}$, which is what we wanted to prove.

We shall show that the metric relation introduced in the problem above, similar to the Pythagorean relation, is hiding an elliptic projectivity of focus A. Actually, this is what makes this problem and this geometric structure so special and deserving of our attention. First, we would like to recall a few facts of projective geometry.

Publication Date: May 22, 2006. Communicating Editor: Paul Yiu.

2. Projectivities

Let A, B, C, and D be four points, in this order, on the line $\mathcal{L}$ in the Euclidean plane. Consider a system of coordinates on $\mathcal{L}$ such that A, B, C, and D correspond to x_1, x_2, x_3, and x_4, respectively. The cross ratio of four ordered points A, B, C, D on $\mathcal{L}$, is by definition (see for example [5, p.248]):

$$(ABCD) = \frac{AC}{BC} \div \frac{AD}{BD} = \frac{x_3 - x_1}{x_3 - x_2} \div \frac{x_4 - x_1}{x_4 - x_2}. \tag{1}$$

This definition may be extended to a pencil consisting of four ordered lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$. By definition, the cross ratio of four ordered lines is the cross ratio determined by the points of intersection with a line $\mathcal{L}$. Therefore,

$$(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4) = \frac{A_1A_3}{A_2A_3} \div \frac{A_1A_4}{A_2A_4}$$

where $\{A_i\} = \mathcal{L} \cap \mathcal{L}_i$. The law of sines shows us that the above definition is independent on $\mathcal{L}$.

We call a projectivity on a line $\mathcal{L}$ a map $f : \mathcal{L} \to \mathcal{L}$ with the property that the cross ratio of any four points is preserved, that is

$$(A_1A_2A_3A_4) = (B_1B_2B_3B_4)$$

where $B_i = f(A_i)$, $i = 1, 2, 3, 4$. The points A_i and B_i are called homologous points of the projectivity on $\mathcal{L}$, and the relation $B_i = f(A_i)$ is denoted $A_i \to B_i$.

The following result is presented in many references (see for example [3], Theorem 4.12, p.34).

Theorem 1. *A projectivity on $\mathcal{L}$ is determined by three pairs of homologous points.*

A consequence of this theorem is that two projectivities which have three common pairs of homologous points must coincide. Actually, we will use this consequence in the proof we present below. In fact, the coordinates x and y of the homologous points under a projectivity are related by

$$y = \frac{mx + n}{px + q}, \ mq - np \neq 0,$$

where $m, n, p, q \in \mathbb{R}$.

In formula (1), it is possible that $(ABCD)$ takes the value -1, as for example in the case of the feet of interior and exterior bisectors associated to the side BD of a triangle MBD (see Figure 2).

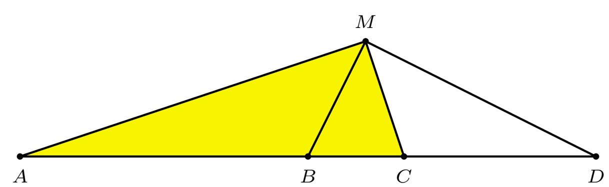


Figure 2

Observe that if C is the midpoint of the segment BD, then point A is not on the line determined by points B and D, since MA becomes parallel to BD. Indeed, for any C on the straight line BD there exist the point M in the plane (not necessarily unique) such that MC is the interior bisector of $\angle BMD$. The point A with the property $(ABCD) = -1$ can be found at the intersection between the external bisector of $\angle BMD$ and the straight line BD. In the particular case when C is the midpoint of BD, we have that ΔMBD is isosceles and the external bisector MA is parallel to BD. To extend the bijectivity of the projectivity presented above, we will say that the homologous of the point C is the point at infinity, denoted ∞, which we attach to the line d. We shall also accept the convention

$$\frac{\infty C}{\infty D} \div \frac{BC}{BD} = -1.$$

For our result, we need the following.

Lemma 2. *A moving angle with vertex in the fixed point A in the plane intersects a fixed line $\mathcal{L}$, A not on $\mathcal{L}$, in a pair of points related by a projectivity.*

Proof. As mentioned in the statement, let A be a fixed point and $\mathcal{L}$ a fixed line such that A is not on $\mathcal{L}$. Consider the rays h and k with origin in A, the moving angle $\angle hk$ with the vertex in A and of constant measure α. Denote by $\{M\} = h \cap \mathcal{L}$ and $\{N\} = k \cap \mathcal{L}$. We have to prove that $f : \mathcal{L} \to \mathcal{L}$ defined by $f(M) = N$ is a projectivity on the line $\mathcal{L}$ determined by the rotation of $\angle hk$. Consider four positions of the angle $\angle hk$, denoted consecutively $\angle h_1k_1$, $\angle h_2k_2$, $\angle h_3k_3$, $\angle h_4k_4$. Their intersections with the line $\mathcal{L}$ yield the points $M_1, N_1; M_2, N_2; M_3, N_3; M_4, N_4$, respectively. It is sufficient to prove that the cross ratio $[M_1M_2M_3M_4]$ and $[N_1N_2N_3N_4]$ are equal. The rotation of the moving angle $\angle hk$ yields, for the pencil of rays h_1, h_2, h_3, h_4 and k_1, k_2, k_3, k_4, respectively, the pairs of equal angles:

$$\begin{aligned}
\angle M_1AM_2 =& \angle N_1AN_2 = \beta_1,\\
\angle M_2AM_3 =& \angle N_2AN_3 = \beta_2,\\
\angle M_3AM_4 =& \angle N_3AN_4 = \beta_3.
\end{aligned}$$

By the law of sines we get that the two cross ratios are equal, both of them having the value

$$\frac{\sin(\beta_1 + \beta_2)}{\sin\beta_2} \div \frac{\sin(\beta_1 + \beta_2 + \beta_3)}{\sin(\beta_2 + \beta_3)}.$$

This proves the claim that f is a projectivity on $\mathcal{L}$ in which the homologous points are M and N. $\square$

3. A projective solution to Romanian Olympiad problem

With these preparations, we are ready to give a projective solution to the initial problem.

Consider a system of coordinates in which the vertices of the right isosceles triangle are $A(0, a)$, $B(-a, 0)$, and $C(a, 0)$. See Figure 3. We consider also $M(x, 0)$ and $N(y, 0)$. The relation $BM^2 + NC^2 = MN^2$ becomes

$$(x + a)^2 + (a - y)^2 = (y - x)^2,$$

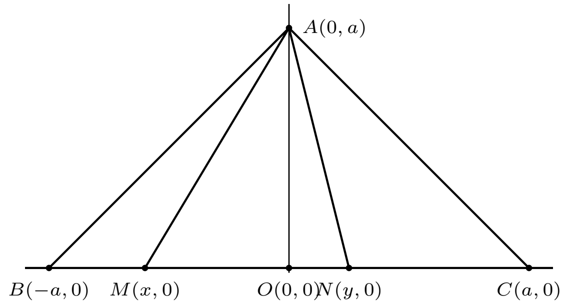


Figure 3

or, solving for y,

$$y = \frac{ax + a^2}{a - x}. \tag{2}$$

This is the equation of a projectivity on the line BC, represented by the homologous points $M \to N$.

Consider now another projectivity on BC determined by the rotation about A by $\frac{\pi}{4}$ (see Lemma 2). This projectivity is completely determined by three pairs of homologous points. First, we see that $B \to O$, since $\angle BAO = \frac{\pi}{4}$. We also have $O \to C$, since $\angle CAO = \frac{\pi}{4}$. Finally, $C \to \infty$, since $\angle CA\infty = \frac{\pi}{4}$.

On the other hand, $B \to O$, since by replacing the $x-$coordinate of B in (2) we get 0, i.e. the $x-$coordinate of O. Similarly, $0 \to a$ and $a \to \infty$ express that $O \to C$ and, respectively, $C \to \infty$. Since a projectivity is completely determined by a triple set of homologous points, the two projectivities must coincide. Therefore, the pair $M \to N$ has the property $\angle MAN = \frac{\pi}{4}$. $\square$

This concludes the proof and the geometric interpretation: the Pythagorean-like metric relation from the original problem reveals a projectivity, which makes this geometric structure remarkable. Furthermore, this solution shows that M and N can be anywhere on the line determined by the points B and C.

References

[1] H. S. M. Coxeter, *The Real Projective Plane,* Third edition, Springer-Verlag, 1992.
[2] H. S. M. Coxeter, *Non-Euclidean Geometry,* Sixth Edition, MAA, 1998.
[3] H. S. M. Coxeter, *Projective Geometry,* Second Edition, Springer-Verlag, 2003.
[4] N. Efimov, *Géométrie supérieure,* Ed. Mir, Moscow, 1981.
[5] M. J. Greenberg, *Euclidean and Non-Euclidean Geometries,* Freeman & Co., Third Edition, 1993.

Wladimir G. Boskoff: Department of Mathematics and Computer Science, University Ovidius, Bd. Mamaia 124 900527 Constantza, Romania
E-mail address: `boskoff@univ-ovidius.ro`

Bogdan D. Suceavă: Department of Mathematics, California State University, Fullerton, California 92834-6850, U.S.A.
E-mail address: `bsuceava@fullerton.edu`

Forum Geometricorum
Volume 6 (2006) 191–197.

FORUM GEOM
ISSN 1534-1178

The Feuerbach Point and Euler lines

Bogdan Suceavă and Paul Yiu

Abstract. Given a triangle, we construct three triangles associated its incircle whose Euler lines intersect on the Feuerbach point, the point of tangency of the incircle and the nine-point circle. By studying a generalization, we show that the Feuerbach point in the Euler reflection point of the intouch triangle, namely, the intersection of the reflections of the line joining the circumcenter and incenter in the sidelines of the intouch triangle.

1. A MONTHLY problem

Consider a triangle ABC with incenter I, the incircle touching the sides BC, CA, AB at D, E, F respectively. Let Y (respectively Z) be the intersection of DF (respectively DE) and the line through A parallel to BC. If E' and F' are the midpoints of DZ and DY, then the six points A, E, F, I, E', F' are on the same circle. This is Problem 10710 of the *American Mathematical Monthly* with slightly different notations. See [3].

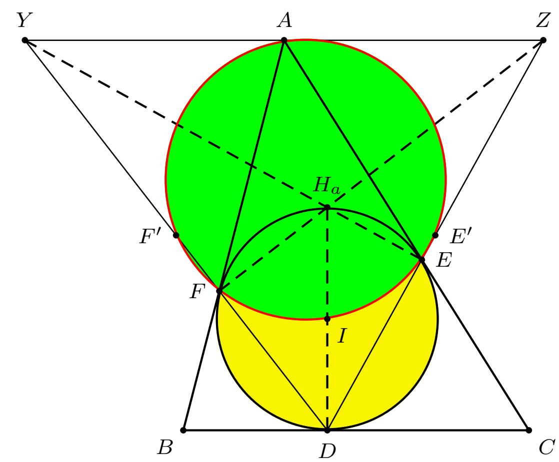


Figure 1. The triangle $\mathbf{T}_a$ and its orthocenter

Here is an alternative solution. The circle in question is indeed the nine-point circle of triangle DYZ. In Figure 1, $\angle AZE = \angle CDE = \angle CED = \angle AEZ$. Therefore $AZ = AE$. Similarly, $AY = AF$. It follows that $AY = AF = AE = AZ$, and A is the midpoint of YZ. The circle through A, E', F', the midpoints of the sides of triangle DYZ, is the nine-point circle of the triangle. Now, since $AY = AZ = AE$, the point E is the foot of the altitude on DZ. Similarly, F

Publication Date: June 4, 2006. Communicating Editor: Jean-Pierre Ehrmann.
The authors thank Jean-Pierre Ehrmann for his interesting remarks on the paper.

is the foot of the altitude on DY, and these two points are on the same nine-point circle. The intersection $H_a = EY \cap FZ$ is the orthocenter of triangle DYZ. Since $\angle H_aED = \angle H_aFD$ are right angles, H_a lies on the circle containing D, E, F, which is the incircle of triangle ABC, and has DH_a as a diameter. It follows that I, being the midpoint of the segment DH_a, is also on the nine-point circle. At the same time, note that H_a is the antipodal point of the D on the incircle of triangle ABC.

2. The Feuerbach point on an Euler line

The center of the nine-point circle of DYZ is the midpoint M of IA. The line MH_a is therefore the Euler line of triangle DYZ.

Theorem 1. *The Euler line of triangle DYZ contains the Feuerbach point of triangle ABC, the point of tangency of the incircle and the nine-point circle of the latter triangle.*

Proof. Let O, H, and N be respectively the circumcenter, orthocenter, and nine-point center of triangle ABC. It is well known that N is the midpoint of OH. Denote by ℓ the Euler line MH_a of triangle DYZ. We show that the parallel through N to the line IH_a intersects ℓ at a point N' such that $NN' = \frac{R}{2}$, where R is the circumradius of triangle ABC.

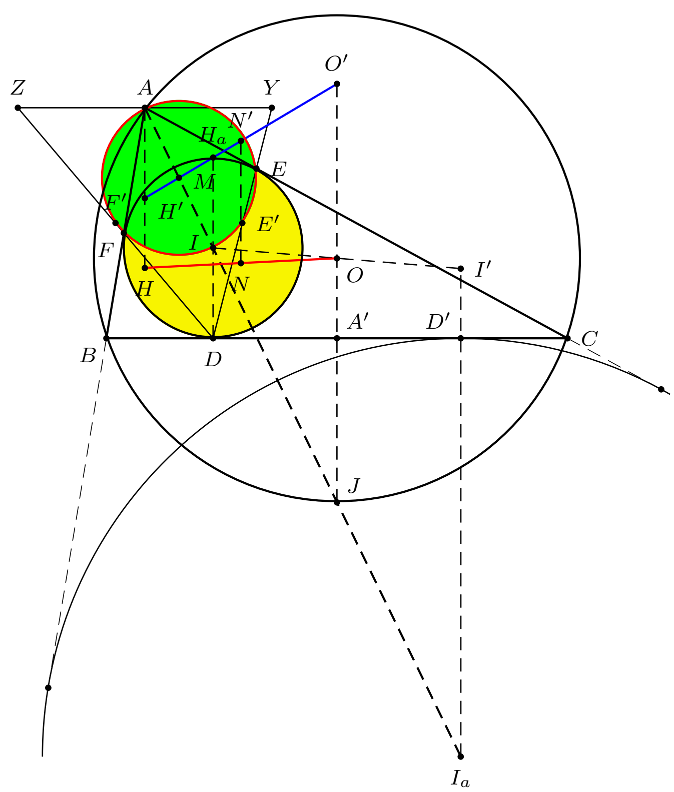


Figure 2. The Euler line of $\mathbf{T}_a$

Clearly, the line HA is parallel to IH_a. Since M is the midpoint of IA, AH intersects ℓ at a point H' such that $AH' = H_aI = r$, the inradius of triangle ABC. See Figure 2. Let the line through O parallel to IH_a intersect ℓ at O'.

If A' is the midpoint of BC, it is well known that $AH = 2 \cdot OA'$.

Consider the excircle (I_a) on the side BC, with radius r_a. The midpoint of II_a is also the midpoint J of the arc BC of the circumcircle (not containing the vertex A). Consider also the reflection I' of I in O, and the excircle (I_a). It is well known that $I'I_a$ passes through the point of tangency D' of (I_a) and BC. We first show that $JO' = r_a$:

$$JO' = \frac{JM}{IM} \cdot IH_a = \frac{I_aA}{IA} \cdot r = \frac{2r_a}{2r} \cdot r = r_a.$$

Since N is the midpoint of OH, and O that of II', we have

$$\begin{aligned}
2NN' =& HH' + OO' \\
=& (HA - H'A) + (JO' - R) \\
=& 2 \cdot A'O - r + r_a - R \\
=& DI + D'I' + r_a - (R + r) \\
=& r + (2R - r_a) + r_a - (R + r) \\
=& R.
\end{aligned}$$

This means that N' is a point on the nine-point circle of triangle ABC. Since NN' and IH_a are directly parallel, the lines $N'H_a$ and NI intersect at the external center of similitude of the nine-point circle and the incircle. It is well known that the two circles are tangent internally at the Feuerbach point F_e, which is their external center of similitude. See Figure 3. □

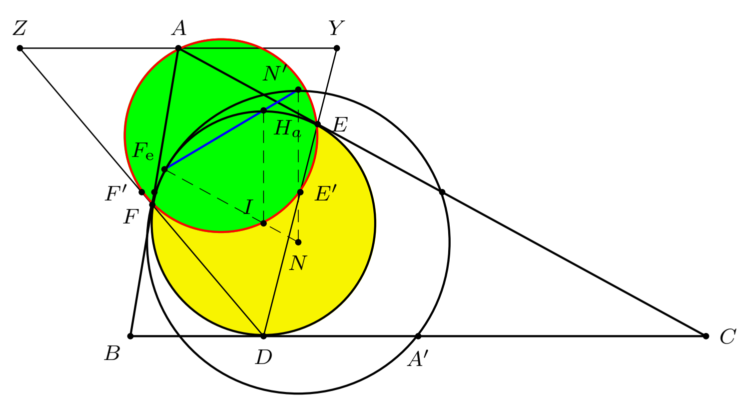


Figure 3. The Euler line of $\mathbf{T}_a$ passes through the Feuerbach point

Remark. Since DH_a is a diameter of the incircle, the Feuerbach point F_e is indeed the pedal of D on the Euler line of triangle DYZ.

Denote the triangle DYZ by $\mathbf{T}_a$. Analogous to $\mathbf{T}_a$, we can also construct the triangles $\mathbf{T}_b$ and $\mathbf{T}_c$ (containing respectively E with a side parallel to CA and F with a side parallel to AB). Theorem 1 also applies to these triangles.

Corollary 2. *The Feuerbach point is the common point of the Euler lines of the three triangles* $\mathbf{T}_a$, $\mathbf{T}_b$, *and* $\mathbf{T}_c$.

3. The excircle case

If, in the construction of $\mathbf{T}_a$, we replace the incircle by the A-excircle (I_a), we obtain another triangle $\mathbf{T}'_a$. More precisely, if the excircle (I_a) touches BC at D', and CA, AB at E', F' respectively, $\mathbf{T}'_a$ is the triangle DYZ bounded by the lines $D'E'$, $D'F'$, and the parallel through A to BC. The method in §2 leads to the following conclusions.

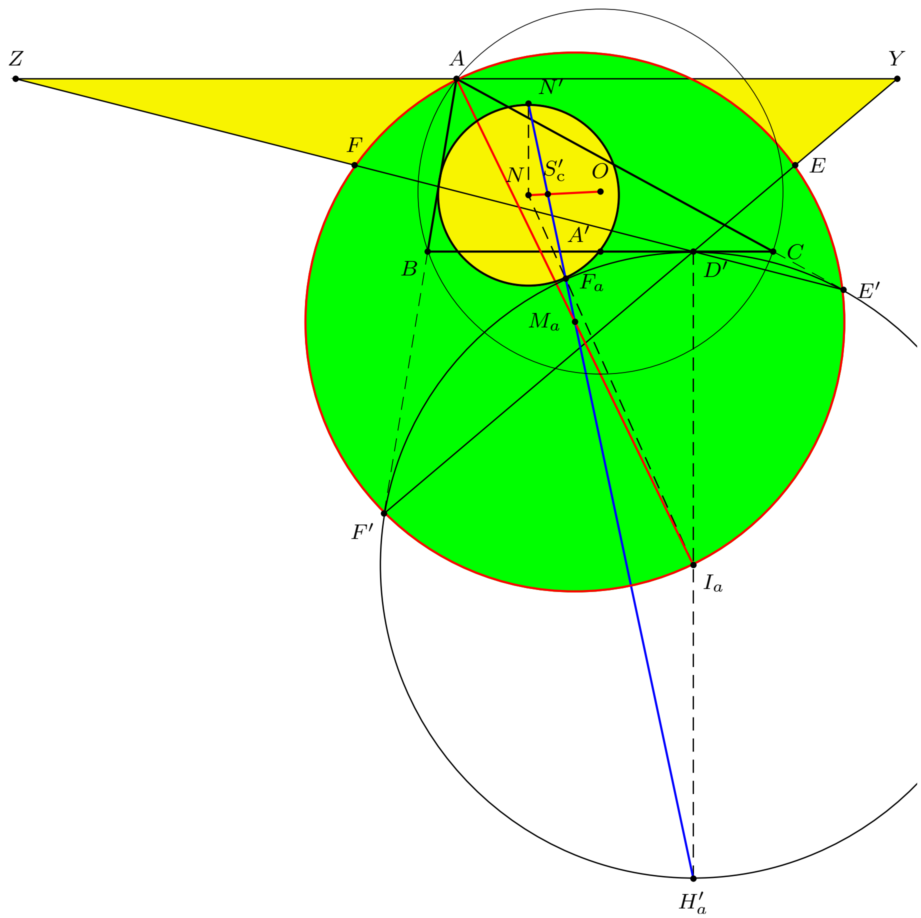


Figure 4. The Euler line of $\mathbf{T}'_a$ passes through $S'_c = X_{442}$

(1) The nine-point circle of $\mathbf{T}'_a$ contains the excenter I_a and the points E', F'; its center is the midpoint M_a of the segment AI_a.

(2) The orthocenter H'_a of $\mathbf{T}'_a$ is the antipode of D' on the excircle (I_a).

(3) The Euler line ℓ'_a of $\mathbf{T}'_a$ contains the point N'.

See Figure 4. Therefore, ℓ'_a also contains the internal center of similitude of the nine-point circle (N) and the excircle (I_a), which is the point of tangency F_a of these two circles. K. L. Nguyen [2] has recently studied the line containing F_a and M_a, and shown that it is the image of the Euler line of triangle IBC under the homothety $\mathsf{h} := \mathsf{h}\left(\mathsf{G}, -\frac{1}{2}\right)$. The same is true for the two analogous triangles $\mathbf{T}'_b$ and $\mathbf{T}'_c$. Their Euler lines are the images of the Euler lines of ICA and IAB under the same homothety. Recall that the Euler lines of triangles IBC, ICA, and IAB intersect at a point on the Euler line, the Schiffler point S_c, which is the triangle center X_{21} in [1]. From this we conclude that the Euler lines of $\mathbf{T}'_a$, $\mathbf{T}'_b$, $\mathbf{T}'_c$ concur at the image of S_c under the homothety h. This, again, is a point on the Euler line of triangle ABC. It appears in [1] as the triangle center X_{442}.

4. A generalization

The concurrency of the Euler lines of $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$, can be paraphrased as the perspectivity of the "midway triangle" of I with the triangle $H_aH_bH_c$. Here, H_a, H_b, H_c are the orthocenters of $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$ respectively. They are the antipodes of D, E, F on the incircle. More generally, every homothetic image of ABC in I is perspective with $H_aH_bH_c$. This is clearly equivalent to the following theorem.

Theorem 3. *Every homothetic image of ABC in I is perspective with the intouch triangle DEF.*

Proof. We work with homogeneous barycentric coordinates.

The image of ABC under the homothety $\mathsf{h}(I, t)$ has vertices

$$\begin{aligned}
A_t =&(a + t(b + c) : (1 - t)b : (1 - t)c),\\
B_t =&((1 - t)a : b + t(c + a) : (1 - t)c),\\
C_t =&((1 - t)a : (1 - t)b : c + t(a + b)).
\end{aligned}$$

On the other hand, the vertices of the intouch triangle are

$$D = (0 : s - c : s - b), \qquad E = (s - c : 0 : s - a), \qquad F = (s - b : s - a : 0).$$

The lines A_tD, B_tE, and C_tF have equations

$$\begin{array}{rcrcrcl}
(1-t)(b-c)(s-a)x & + & (s-b)(a+(b+c)t)y & - & (s-c)(a+(b+c)t)z & = & 0,\\
-(s-a)(b+(c+a)t)x & + & (1-t)(c-a)(s-b)y & + & (s-c)(b+(c+a)t)z & = & 0,\\
(s-a)(c+(a+b)t)x & - & (s-b)(c+(a+b)t)y & + & (1-t)(a-b)(s-c)z & = & 0.
\end{array}$$

These three lines intersect at the point

$$P_t = \left(\frac{(a + t(b + c))(b + c - a + 2at)}{b + c - a} : \cdots : \cdots\right).$$

□

Remark. More generally, for an arbitrary point P, every homothetic image of ABC in $P = (u : v : w)$ is perspective with the cevian triangle of the isotomic conjugate of the superior of P, namely, the point $\left(\frac{1}{v+w-u} : \frac{1}{w+u-v} : \frac{1}{u+v-w}\right)$. With $P = I$, we get the cevian triangle of the Gergonne point which is the intouch triangle.

Proposition 4. *The perspector of $A_tB_tC_t$ and $H_aH_bH_c$ is the reflection of P_{-t} in the incenter.*

It is clear that the perspector P_t traverses a conic Γ as t varies, since its coordinates are quadratic functions of t. The conic Γ clearly contains I and the Gergonne point, corresponding respectively to $t = 0$ and $t = 1$. Note also that $D = P_t$ for $t = -\frac{a}{b+c}$ or $-\frac{s-a}{a}$. Therefore, Γ contains D, and similarly, E and F. It is a cirumconic of the intouch triangle DEF. Now, as $t = \infty$, the line A_tD is parallel to the bisector of angle A, and is therefore perpendicular to EF. Simiarly, B_tE and C_tF are perpendicular to FD and DE respectively. The perspector P_∞ is therefore the orthocenter of triangle DEF, which is the triangle center X_{65} in [1]. It follows that Γ is a rectangular hyperbola. Since it contains also the circumcenter I of DEF, Γ is indeed the Jerabek hyperbola of the intouch triangle. Its center is the point

$$Q = \left(\frac{a(a^2(b+c) - 2a(b^2+c^2) + (b^3+c^3))}{b+c-a} : \cdots : \cdots\right).$$

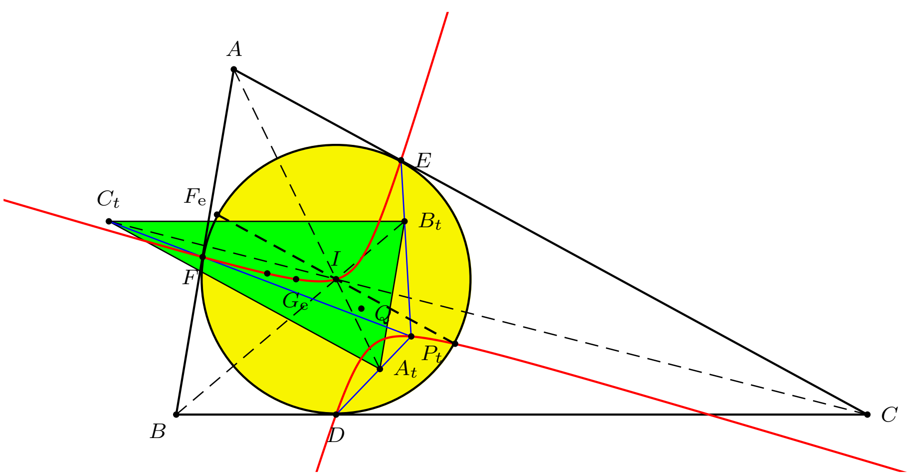


Figure 5. The Jerabak hyperbola of the intouch triangle

The reflection of Γ in the incenter is the conic Γ' which is the locus of the perspectors of $H_aH_bH_c$ and homothetic images of ABC in I.

Note that the fourth intersection of Γ with the incircle is the isogonal conjugate, with respect to the intouch triangle, of the infinite point of its Euler line. Its antipode on the incircle is therefore the Euler reflection point of the intouch triangle.

This must also be the perspector of $H_aH_bH_c$ (the antipode of DEF in the incircle) and a homothetic image of ABC. It must be the Feuerbach point on Γ'.

Theorem 5. *The Feuerbach point is the Euler reflection point of the intouch triangle. This means that the reflections of OI (the Euler line of the intouch triangle) concur at F.*

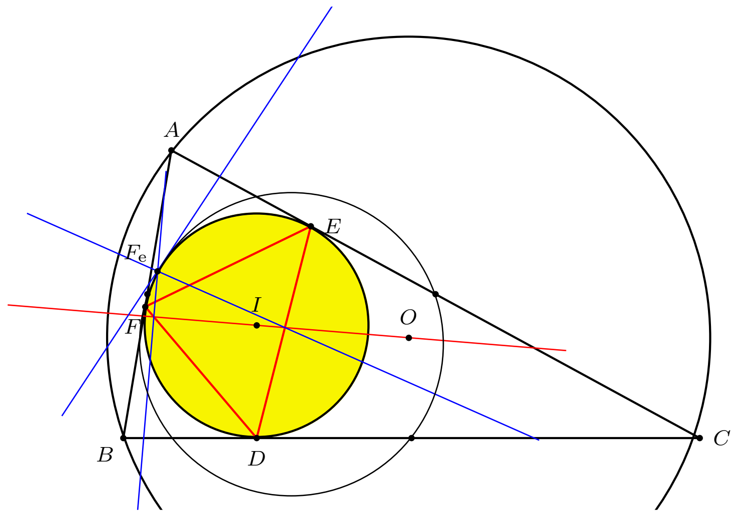


Figure 6. The Feuerbach point as the Euler reflection point of DEF

Remarks. (1) The fourth intersection of Γ with the incircle, being the antipode of the Feuerbach point, is the triangle cente X_{1317}. The conic Γ also contains X_n for the following values of n: 145, 224, and 1537. (Note: X_{145} is the reflection of the Nagel point in the incenter). These are the perspectors for the homothetic images of ABC with ratios $t = -1$, $-\frac{R}{R+r}$, and $-\frac{r}{2(R-r)}$ respectively.

(2) The hyperbola Γ' contains the following triangle centers apart from I and F_e: X_8 and X_{390} (which is the reflection of the Gergonne point in the incenter). These are the perspector for the homothetic images with ratio $+1$ and -1 respectively.

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[2] K. L. Nguyen, On the complement of the Schiffler point, *Forum Geom.* 5 (2005) 149–164.

[3] B. Suceavă and A. Sinefakopoulos, Problem 19710, *Amer. Math. Monthly*, 106 (1999) 68; solution, 107 (2000) 572–573.

Bogdan Suceavă: Department of Mathematics, California State University, Fullerton, CA 92834-6850, USA

E-mail address: `bsuceava@fullerton.edu`

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida 33431-0991, USA

E-mail address: `yiu@fau.edu`

Forum Geometricorum
Volume 6 (2006) 199–203.

FORUM GEOM
ISSN 1534-1178

A Simple Perspectivity

Eric Danneels

Abstract. We construct a simple perspectivity that is invariant under isotomic conjugation.

1. Introduction

In this note we consider a simple transformation of the plane of a given reference triangle ABC. Given a point P with cevian triangle XYZ, construct the parallels through B to XY and through C to XZ to intersect at A'; similarly define B' and C'. Construct

$$A^* = BB' \cap CC', \qquad B^* = CC' \cap AA', \qquad C^* = AA' \cap BB'.$$

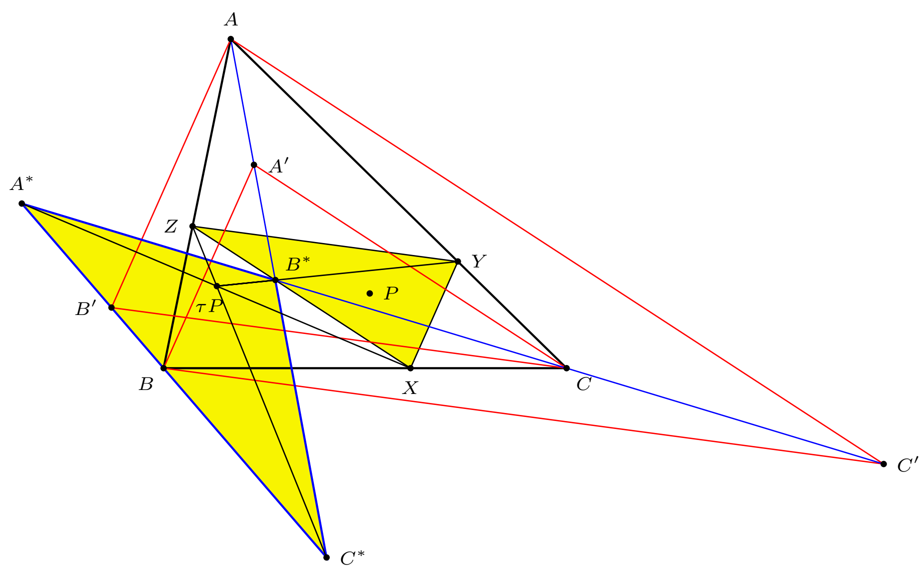


Figure 1

Proposition 1. *Triangle $A^*B^*C^*$ is the anticevian triangle of the infinite point $Q = (u(v-w) : v(w-u) : w(u-v))$ of the trilinear polar of P.*

We shall prove Proposition 1 in §2 below. As an anticevian triangle, $A^*B^*C^*$ is perspective with every cevian triangle. In particular, it is perspective with XYZ at the cevian quotient P/Q, which depends on P only. We write

$$\tau(P) := P/Q = (u(v-w)^2 : v(w-u)^2 : w(u-v)^2).$$

Publication Date: June 12, 2006. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this paper.

Let $P^\bullet$ denote the isotomic conjugate of P.

Proposition 2. *τ is invariant under isotomic conjugation: $\tau(P^\bullet) = \tau(P)$.*

Proposition 3. *$\tau(P)$ is*
(1) *the center of the circumconic through P and its isotomic conjugate $P^\bullet$,*
(2) *the perspector of the circum-hyperbola with asymptotes the trilinear polars of P and $P^\bullet$.*

Proof. (1) The circumconic through P and $P^\bullet$ has equation

$$\frac{u(v^2-w^2)}{x} + \frac{v(w^2-u^2)}{y} + \frac{w(u^2-v^2)}{z} = 0,$$

with perspector

$$P' = (u(v^2-w^2) : \ w(w^2-u^2) : \ w(u^2-v^2)). \tag{1}$$

Its center is the cevian quotient G/P'. This is $\tau(P)$.

(2) The pencil of hyperbolas with asymptotes the trilinears polars of P and $P^\bullet$ has equation

$$k(x+y+z)^2 + (ux+vy+cz)\left(\frac{x}{u} + \frac{y}{v} + \frac{z}{w}\right) = 0.$$

For $k = -1$, the hyperbola passes through A, B, C, and this circum-hyperbola has equation

$$\frac{u(v-w)^2}{x} + \frac{v(w-u)^2}{y} + \frac{w(u-v)^2}{z} = 0.$$

It has perspector $\tau(P)$, (and center P' given in (1) above). □

Remark. Wilson Stothers [2] has found that one asymptote of a circum-hyperbola determines the other. More precisely, if $ux + vy + wz = 0$ is an asymptote of a circum-hyperbola, then the other is $\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0$. This gives a stronger result than (2) above.

Here is a list of triangle centers with their images under τ. The labeling of triangle centers follows Kimberling [1].

P, $P^\bullet$	$\tau(P)$		P, $P^\bullet$	$\tau(P)$
X_1, X_{75}	X_{244}		$X_3,\ X_{264}$	X_{2972}
$X_4,\ X_{69}$	X_{125}		$X_7,\ X_8$	X_{11}
$X_{20},\ X_{253}$	X_{122}		$X_{30},\ X_{1494}$	X_{1650}
$X_{57},\ X_{312}$	X_{2170}		$X_{88},\ X_{88}^\bullet$	X_{2087}
$X_{94},\ X_{323}$	X_{2088}		$X_{98},\ X_{325}$	X_{868}
$X_{99},\ X_{523}$	X_{1649}		$X_{200},\ X_{1088}$	X_{2310}
$X_{519},\ X_{903}$	X_{1647}		$X_{524},\ X_{671}$	X_{1648}
$X_{536},\ X_{536}^\bullet$	X_{1646}		$X_{538},\ X_{538}^\bullet$	X_{1645}
$X_{694},\ X_{694}^\bullet$	X_{2086}		$X_{1022},\ X_{1022}^\bullet$	X_{1635}
$X_{1026},\ X_{1026}^\bullet$	X_{2254}		$X_{2394},\ X_{2407}$	X_{1637}
$X_{2395},\ X_{2396}$	X_{2491}		$X_{2398},\ X_{2400}$	X_{676}

2. Proof of Proposition 1

The line XY has equation $vwx+wuy-uvz=0$, and infinite point $(-u(v+w) : v(w+u) : w(u-v))$. The parallel through B to XY is the line

$$w(u-v)x+u(v+w)z=0.$$

Similarly, the parallel through C to XZ is the line

$$v(u-w)x+u(v+w)y=0.$$

These two lines intersect at

$$A' = (u(v+w) : v(w-u) : w(v-u)).$$

The two analogously defined points are

$$\begin{aligned} B' =&(u(w-v) : v(w+u) : w(u-v)), \\ C' =&(u(v-w) : v(u-w) : w(u+v)). \end{aligned}$$

Now the lines AA', BB', CC' intersect at the points

$$\begin{aligned} A^* =&BB' \cap CC' = (-u(v-w) : v(w-u) : w(u-v)), \\ B^* =&CC' \cap AA' = (u(v-w) : -v(w-u) : w(u-v)), \\ C^* =&AA' \cap BB' = (u(v-w) : v(w-u) : -w(u-v)). \end{aligned}$$

This is clearly the anticevian triangle of the point

$$Q = (u(v-w) : v(w-u) : w(u-v)) = \left(\frac{1}{v}-\frac{1}{w} : \frac{1}{w}-\frac{1}{u} : \frac{1}{u}-\frac{1}{v}\right),$$

which is the infinite point of the trilinear polar $\mathcal{L}$. This completes the proof of Proposition 1.

Remarks. (1) Here is an easy alternative construction of $A^*B^*C^*$. Construct the parallels through A, B, C to the trilinear polar $\mathcal{L}$, intersecting the sidelines BC, CA, AB at A_1, B_1, C_1 respectively. Then, A^*, B^*, C^* are the midpoints of the segments AA_1, BB_1, CC_1. See Figure 2.

(2) The equations of the sidelines of triangle $A^*B^*C^*$ are

$$\begin{aligned} B^*C^* &: \frac{y}{v(w-u)}+\frac{z}{w(u-v)}=0, \\ C^*A^* &: \frac{x}{u(v-w)}+\frac{z}{w(u-v)}=0, \\ A^*B^* &: \frac{x}{u(v-w)}+\frac{y}{v(w-u)}=0. \end{aligned}$$

Proposition 4. *The trilinear polar of $\tau(P)$ with respect to the cevian triangle of P passes through P.*

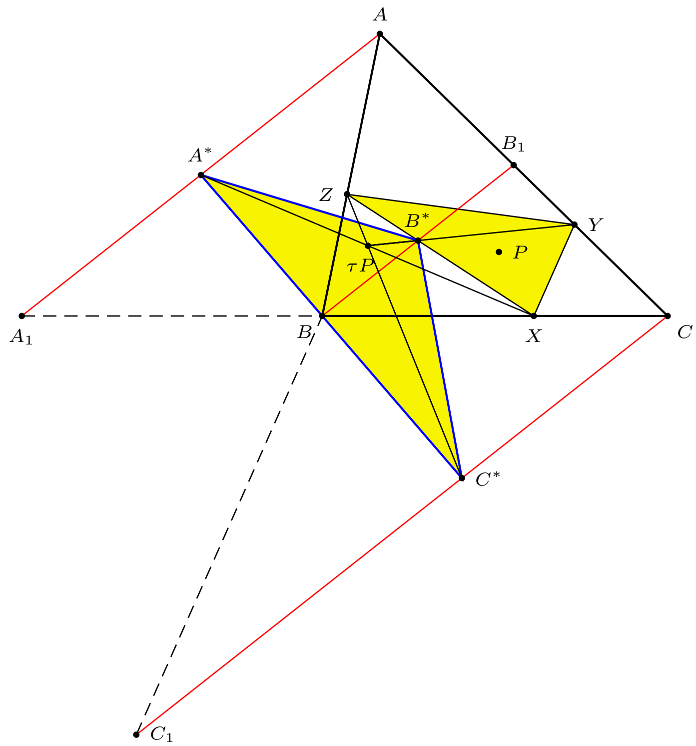


Figure 2

Proof. The trilinear polar of $\tau(P)$ with respect to XYZ is the perspectrix of the triangles XYZ and $A^*B^*C^*$. Now, the sidelines of these triangle intersect at the points

$$\begin{aligned} B^*C^* \cap YZ =&(u(v+w-2u) : \ v(w-u) : \ -w(u-v)), \\ C^*A^* \cap ZX =&(-u(v-w) : \ v(w+u-2v) : \ w(u-v)), \\ A^*B^* \cap XY =&(u(v-w) : \ -v(w-u) : \ w(u+v-2w)). \end{aligned}$$

The line through these three points has equation

$$\frac{v-w}{u}x + \frac{w-u}{v}y + \frac{u-v}{w}z = 0.$$

This clearly contains the point $P = (u : v : w)$. $\square$

3. Generalization

Since the construction in §1 is purely perspective we can replace the line at infinity by an arbitrary line $\ell : px + qy + rz = 0$. The parallel through B to XY becomes the line joining B to the intersection ℓ and XY, etc. The perspector becomes

$$\tau_\ell(P) = (u(qv - rw)^2 : \ v(rw - pu)^2 : \ w(pu - qv)^2).$$

Then $\tau_\ell(P^\ell) = \tau(P)$ where $P^\ell = \left(\frac{1}{p^2u} : \frac{1}{q^2v} : \frac{1}{r^2w}\right)$, and the following remain valid:

(1) $A^*B^*C^*$ is the anticevian triangle of $Q = \mathcal{L} \cap \ell$, where $\mathcal{L}$ is the trilinear polar of P.
(2) The perspectrix of XYZ and $A^*B^*C^*$ contains the point P.

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[2] W. Stothers, Hyacinthos message 8476, October 30, 2003.

Eric Danneels: Hubert d'Ydewallestraat 26, 8730 Beernem, Belgium
E-mail address: eric.danneels@pandora.be

Forum Geometricorum
Volume 6 (2006) 205–212.

FORUM GEOM
ISSN 1534-1178

Pedals on Circumradii and the Jerabek Center

Quang Tuan Bui

Abstract. Given a triangle ABC, beginning with the orthogonal projections of the vertices on the circumradii OA, OB, OC, we construct two triangles each with circumcircle tangent to the nine-point circle at the center of the Jerabek hyperbola.

1. Introduction

Given a triangle ABC, with circumcenter O, let A_b and A_c be the pedals (orthogonal projections) of the vertex A on the lines OB and OC respectively. Similarly, define B_c, B_a, C_a and C_b. In this paper we prove some interesting results on triangles associated with these pedals.

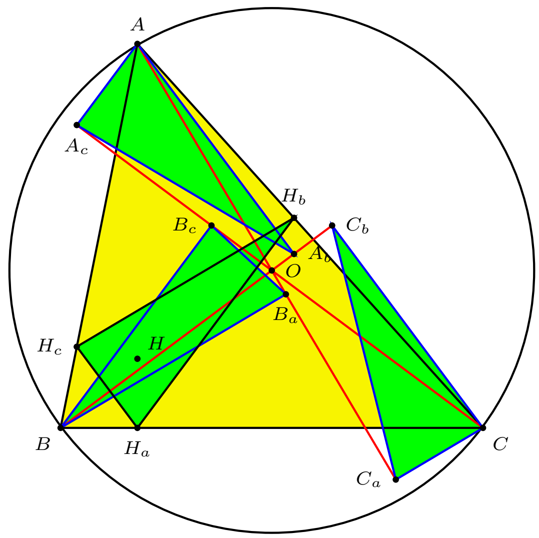


Figure 1

Theorem 1. *The triangles AA_bA_c, B_aBB_c and C_aC_bC are congruent to the orthic triangle $H_aH_bH_c$.* See Figure 1.

Publication Date: June 19, 2006. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu, Peter Moses, and other members of the Hyacinthos group for help and advice during the preparation of this paper.

Theorem 2. *The lines B_cC_b, C_aA_c and A_bB_a bound a triangle $\mathbf{T}_1$ homothetic to ABC. The circumcircle of $\mathbf{T}_1$ is tangent to the nine-point circle of ABC at the Jerabek center.*

Recall that the Jerabek center J is the center of the circum-hyperbola through the circumcenter O. This hyperbola is the isogonal conjugate of the Euler line. The Jerabek center J is the triangle center X_{125} in Kimberling's *Encyclopedia of Triangle Centers* [1].

Theorem 3. *The lines A_bA_c, B_cB_a and C_aC_b bound a triangle $\mathbf{T}_2$ whose circumcircle is tangent to the nine-point circle at the Jerabek center.*

Hence, the circumcircles of $\mathbf{T}_1$ and $\mathbf{T}_2$ are also tangent to each other at J. In this paper we work with homogeneous barycentric coordinates and adopt standard notations of triangle geometry. Basic results can be found in [2]. The Jerabek center J, for example, has coordinates

$$(S_A(S_B-S_C)^2 :\ S_B(S_C-S_A)^2 :\ S_C(S_A-S_B)^2). \tag{1}$$

The labeling of triangle centers, except for the common ones, follows [1].

Proposition 4. *The homogeneous barycentric coordinates of the pedals of the vertices of triangle ABC on the circumradii are as follows.*

$$\begin{aligned}
A_b =&(S_A(S_B+S_C) :\ S_C(S_C-S_A) :\ S_C(S_A+S_B)),\\
A_c =&(S_A(S_B+S_C) :\ S_B(S_C+S_A) :\ S_C(S_B-S_A));\\
B_c =&(S_A(S_B+S_C) :\ S_B(S_C+S_A) :\ S_A(S_A-S_B)),\\
B_a =&(S_A(S_C-S_B) :\ S_B(S_C+S_A) :\ S_C(S_A+S_B));\\
C_a =&(S_A(S_B-S_C) :\ S_B(S_C+S_A) :\ S_C(S_A+S_B)),\\
C_b =&S_A((S_B+S_C) :\ S_B(S_A-S_C) :\ S_C(S_A+S_B)).
\end{aligned}$$

Proof. We verify that the point

$$P=(S_A(S_B+S_C) :\ S_C(S_C-S_A) :\ S_C(S_A+S_B))$$

is the pedal A_b of A on the line OB. Since

$$\begin{aligned}
&(S_A(S_B+S_C),\ S_C(S_C-S_A),\ S_C(S_A+S_B))\\
=&(S_A(S_B+S_C),\ S_B(S_C+S_A),\ S_C(S_A+S_B))\\
&+(0, S_C(S_C-S_A)-S_B(S_C+S_A),\ 0),
\end{aligned}$$

this is a point on the line OB. The coordinate sum of P being $(S_B+S_C)(S_C+S_A)$, the infinite point of the line AP is

$$\begin{aligned}
&(S_A(S_B+S_C),\ S_C(S_C-S_A),\ S_C(S_A+S_B))-((S_C+S_A)(S_A+S_B),\ 0,\ 0)\\
=&S_C(-(S_B+S_C),\ (S_C-S_A),\ (S_A+S_B)).
\end{aligned}$$

The infinite point of OB is

$$\begin{aligned}
&(S_A(S_B+S_C),\ S_B(S_C+S_A),\ S_C(S_A+S_B))-(0,\ 2(S_{BC}+S_{CA}+S_{AB}),\ 0)\\
=&(S_A(S_B+S_C),\ -(S_{BC}+2S_{CA}+S_{AB}),\ S_C(S_A+S_B)).
\end{aligned}$$

By the theorem in [2, §4.5], the two lines AP and OB are perpendicular since

$$\begin{aligned}
&-S_A \cdot (S_B + S_C) \cdot S_A(S_B + S_C)\\
&-S_B \cdot (S_C - S_A) \cdot (S_{BC} + 2S_{CA} + S_{AB})\\
&+S_C \cdot (S_A + S_B) \cdot S_C(S_A + S_B)\\
&=0.
\end{aligned}$$

□

2. Proof of Theorem 1

Note that the points A_b and A_c lie on the circle with diameter OA, so do the midpoints of AC and AB. Therefore,

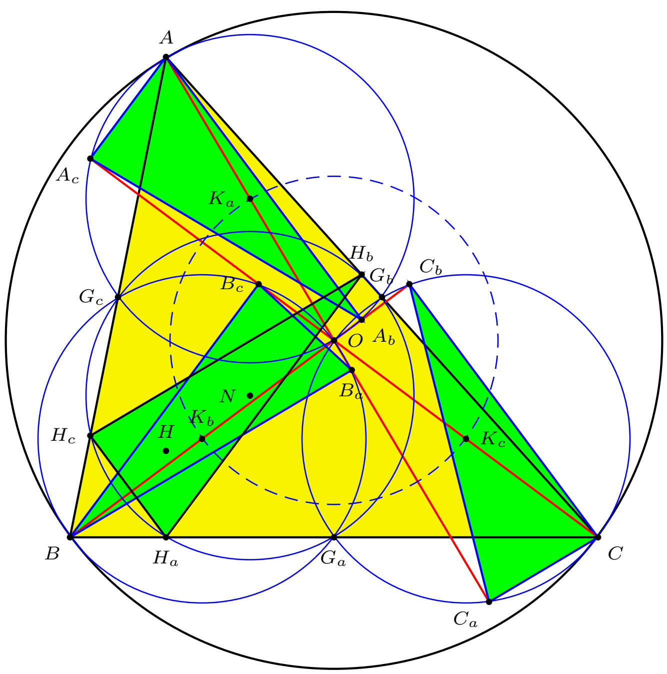


Figure 2

$$\begin{aligned}
\angle A_cAA_b =&\pi - \angle A_bOA_c = \pi - \angle BOC = \pi - 2A = \angle H_cH_aH_b,\\
\angle AA_bA_c =&\angle AOA_c = \pi - \angle COA = \pi - 2B = \angle H_aH_bH_c,\\
\angle A_bA_cA =&\angle A_bOA = \pi - \angle AOB = \pi - 2C = \angle H_bH_cH_a.
\end{aligned}$$

Therefore the angles in triangles AA_bA_c and $H_aH_bH_c$ are the same; similarly for triangles B_aBB_c and C_aC_bC. Since these four triangles have equal circumradii $\frac{R}{2}$, they are congruent. This completes the proof of Theorem 1.

Remarks. (1) The side lengths of these triangles are $a\cos A$, $b\cos B$, and $c\cos C$ respectively.

(2) If K_a, K_b, K_c are the midpoints of the circumradii OA, OB, OC, triangle $K_aK_bK_c$ is homothetic to
(i) ABC at O, with ratio of homothety $\frac{1}{2}$, and
(ii) the medial triangle $G_aG_bG_c$ at X_{140}, the nine-point center of the medial triangle, with ratio of homothety -1.

(3) The circles (K_b) and (K_c) intersect at the circumcenter O and the midpoint G_a of BC; similarly for the other two pairs (K_c), (K_a) and (K_a), (K_b). The midpoints G_a, G_b, G_c lie on the nine-point circle (N) of triangle ABC. See Figure 2.

3. The triangle $\mathbf{T}_1$

We now consider the triangle $\mathbf{T}_1$ bounded by the lines B_cC_b, C_aA_c, and A_bB_a.

Lemma 5. *The quadrilateral B_cC_bCB is an isosceles trapezoid.*

Proof. With reference to Figure 1, we have
(i) $\angle B_cBC = \frac{\pi}{2} - \angle OCB = \frac{\pi}{2} - \angle OBC = \angle C_bCB$,
(ii) $B_cB = C_bC$.
It follows that the quadrilateral B_cC_bCB is an isosceles trapezoid. □

Therefore, the lines B_cC_b and BC are parallel. Similarly, the lines C_aA_c and CA are parallel, as are A_bB_a and AB. The triangle $\mathbf{T}_1$ bounded by the lines B_cC_b, C_aA_c, A_bB_a is homothetic to triangle ABC, and also to the medial triangle $G_aG_bG_c$.

Proposition 6. *Triangle $\mathbf{T}_1$ is homothetic to*
(i) *ABC at the procircumcenter $(a^4S_A : b^4S_B : c^4S_C)$,* [1]
(ii) *the medial triangle $G_aG_bG_c$ at the Jerabek center J.*

Proof. The lines B_cC_b, C_aA_c, and A_bB_a have equations

$$\begin{aligned}
-(S_{AA}+S_{BC})x + S_A(S_B+S_C)y + S_A(S_B+S_C)z =&0,\\
S_B(S_C+S_A)x - (S_{BB}+S_{CA})y + S_B(S_C+S_A)z =&0,\\
S_C(S_A+S_B)x + S_C(S_A+S_B)y - (S_{CC}+S_{AB})z =&0.
\end{aligned}$$

From these, we obtain the coordinates of the vertices of $\mathbf{T}_1$:

$$\begin{aligned}
A_1 =&(S_A(S_B-S_C)^2 : S_B(S_C+S_A)^2 : S_C(S_A+S_B)^2),\\
B_1 =&(S_A(S_B+S_C)^2 : S_B(S_C-S_A)^2 : S_C(S_A+S_B)^2),\\
C_1 =&(S_A(S_B+S_C)^2 : S_B(S_C+S_A)^2 : S_C(S_A-S_B)^2).
\end{aligned}$$

From the coordinates of A_1, B_1, C_1, it is clear that the homothetic center of triangles $A_1B_1C_1$ and ABC is the point

$$(S_A(S_B+S_C)^2 : S_B(S_C+S_A)^2 : S_C(S_A+S_B)^2) = (a^4S_A : b^4S_B : c^4S_C).$$

[1]This is the triangle center X_{184} in [1].

For (ii), the equations of the lines G_aA_1, G_bB_1, G_cC_1 are respectively

$$\begin{aligned}(S_{AA}-S_{BC})x-S_A(S_B-S_C)y+S_A(S_B-S_C)z=&0,\\ S_B(S_C-S_A)x+(S_{BB}-S_{CA})y-S_B(S_C-S_A)z=&0,\\ -S_C(S_A-S_B)x+S_C(S_A-S_B)y+(S_{CC}-S_{AB})z=&0.\end{aligned}$$

It is routine to check that this contains the Jerabek center J whose coordinates are given in (1). □

4. Proof of Theorem 2

Theorem 2 is now an immediate consequence of Proposition 6(ii). Since the homothetic center J lies on the circumcircle of the medial triangle, it must also lie on the circumcircle of the other, and the two circumcircles are tangent at J.

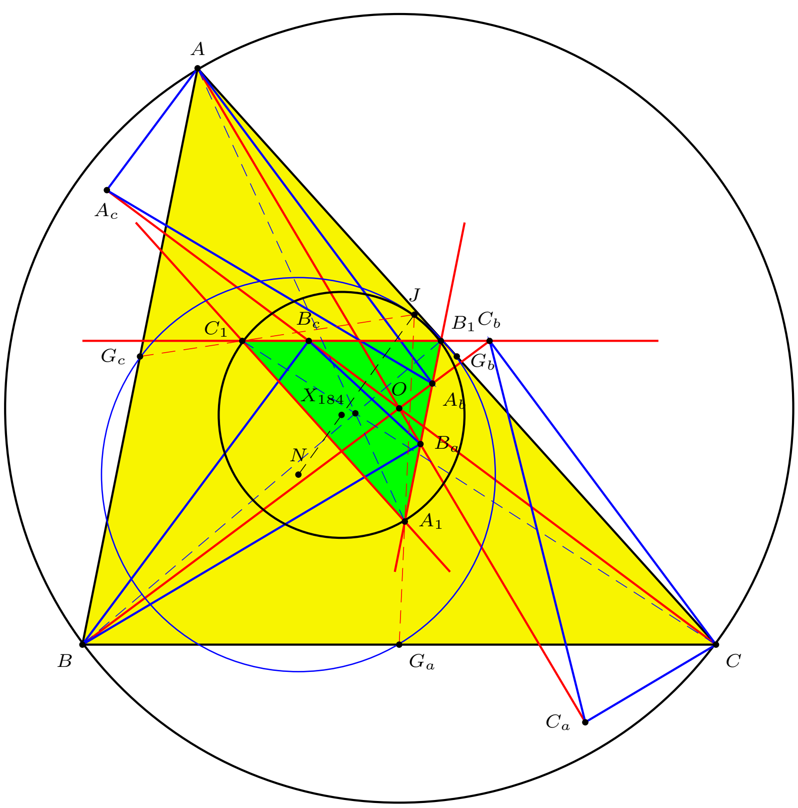


Figure 3

5. The triangle $\mathbf{T}_2$

From the coordinates of the pedals, we obtain the equations of the lines A_bA_c, B_cB_a, and C_aC_b:

$$\begin{aligned}
-2S_{BC}x + (S^2 - S_{BB})y + (S^2 - S_{CC})z =& 0,\\
(S^2 - S_{AA})x - 2S_{CA}y + (S^2 - S_{CC})z =& 0,\\
(S^2 - S_{AA})x + (S^2 - S_{BB})y - 2S_{AB}z =& 0.
\end{aligned}$$

From these, the vertices of triangle $\mathbf{T}_2$ are the points

$$\begin{aligned}
A_2 =& ((S_B - S_C)^2 : \ 3S_{AB} + S_{BC} + S_{CA} - S_{CC} : \ 3S_{CA} + S_{AB} + S_{BC} - S_{BB}),\\
B_2 =& (3S_{AB} + S_{BC} + S_{CA} - S_{CC} : \ (S_C - S_A)^2 : \ 3S_{BC} + S_{CA} + S_{AB} - S_{AA}),\\
C_2 =& (3S_{CA} + S_{AB} + SBC - S_{CC} : \ 3S_{BC} + S_{CA} + S_{AB} - S_{AA} : \ (S_A - S_B)^2).
\end{aligned}$$

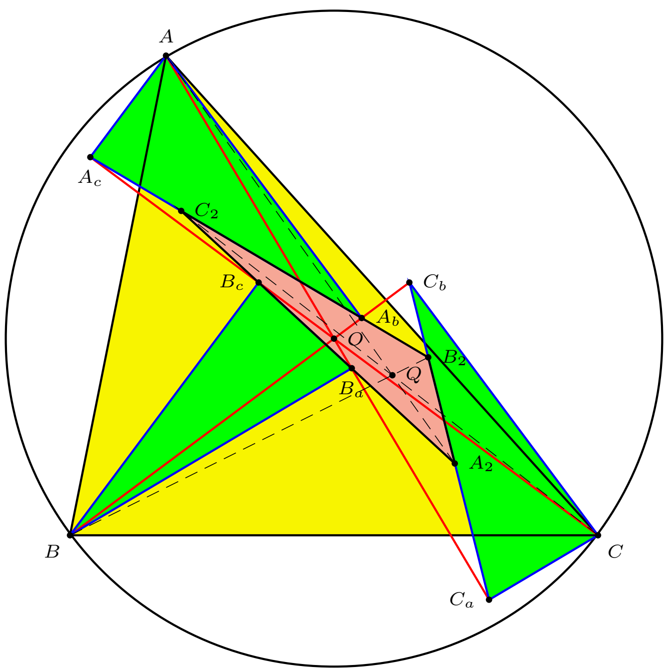


Figure 4

Proposition 7. *Triangles ABC and $A_2B_2C_2$ are perspective at*

$$Q = \left(\frac{1}{a^2b^2 + b^2c^2 + c^2a^2 - b^4 - c^4} : \ \frac{1}{a^2b^2 + b^2c^2 + c^2a^2 - c^4 - a^4} : \ \frac{1}{a^2b^2 + b^2c^2 + c^2a^2 - a^4 - b^4}\right).$$

Proof. From the coordinates of A_2, B_2, C_2 given above, $\mathbf{T_2}$ is perspective with ABC at

$$Q = \left(\frac{1}{3S_{BC}+S_{CA}+S_{AB}-S_{AA}} : \cdots : \cdots\right).$$

These are equivalent to those given above in terms of a, b, c. □

Remark. The triangle center Q does not appear in [1].

6. Proof of Theorem 3

It is easier to work with the image of triangle $\mathbf{T}_2$ under the homothety $\mathsf{h}(H, 2)$. The images of the vertices are

$$\begin{aligned} A_2' =& (S_A(S_B+S_C)(S_{BB}-4S_{BC}+S_{CC})+S_{BC}(S_B-S_C)^2 \\ &: (S_C+S_A)(S_A(S_B+S_C)(3S_B-S_C)+S_{BC}(S_B-S_C)) \\ &: (S_A+S_B)(S_A(S_B+S_C)(S_B-3S_C)+S_{BC}(S_B-S_C))), \end{aligned}$$

and B_2', C_2' whose coordinates are obtained by cyclic permutations of S_A, S_B, S_C. The circumcircle of $A_2'B_2'C_2'$ has equation

$$\begin{aligned} &8S^2 \cdot S_{ABC}((S_B+S_C)yz+(S_C+S_A)zx+(S_A+S_B)xy) \\ &+(x+y+z)\left(\sum_{\text{cyclic}}(S_A+S_B)(S_A+S_C)(S_{AB}+S_{CA}-2S_{BC})^2x\right)=0. \end{aligned}$$

To verify that this circle is tangent to the circumcircle

$$(S_B+S_C)yz+(S_C+S_A)zx+(S_A+S_B)xy=0,$$

it is enough to consider the pedal of the circumcenter O on the radical axis

$$\sum_{\text{cyclic}}(S_A+S_B)(S_A+S_C)(S_{AB}+S_{CA}-2S_{BC})^2x=0.$$

This is the point

$$Q' = \left(\frac{S_B+S_C}{S_{CA}+S_{AB}-2S_{BC}} : \frac{S_C+S_A}{S_{AB}+S_{BC}-2S_{CA}} : \frac{S_A+S_B}{S_{BC}+S_{CA}-2S_{AB}}\right),$$

which is clearly on the circumcircle, and also on the Jerabek hyperbola

$$\frac{S_A(S_{BB}-S_{CC})}{x}+\frac{S_B(S_{CC}-S_{AA})}{y}+\frac{S_C(S_{AA}-S_{BB})}{z}=0.$$

This shows that the circle $A_2'B_2'C_2'$ is tangent to the circumcircle at Q'. [2] Under the homothety $\mathsf{h}(H, 2)$, Q' is the image of the midpoint of HQ, which is the center of the Jerabek hyperbola. Under the inverse homothety, the circumcircle of $\mathbf{T}_2$ is tangent to the nine-point circle at J. This completes the proof of Theorem 3.

[2] Q' is the triangle center X_{74}

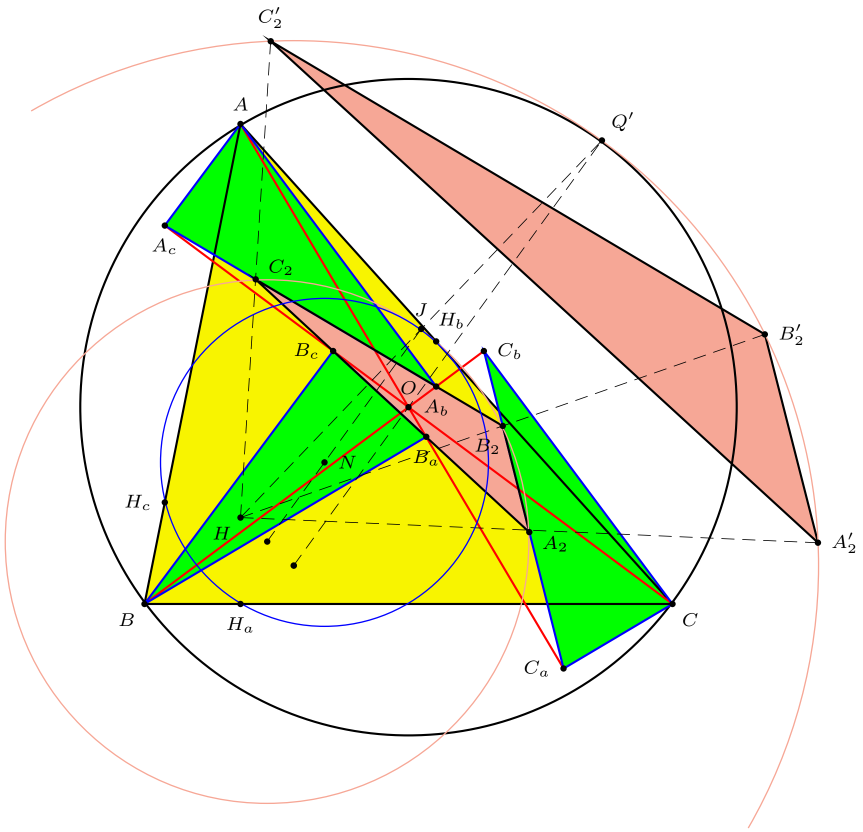


Figure 5

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[2] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University lecture notes, 2001.

Quang Tuan Bui: 45B, 296/86 by-street, Minh Khai Street, Hanoi, Vietnam

E-mail address: `bqtuan1962@yahoo.com`

Forum Geometricorum
Volume 6 (2006) 213–224.

FORUM GEOM
ISSN 1534-1178

Simmons Conics

Bernard Gibert

Abstract. We study the conics introduced by T. C. Simmons and generalize some of their properties.

1. Introduction

In [1, Tome 3, p.227], we find a definition of a conic called “ellipse de Simmons” with a reference to E. Vigarié series of papers [8] in 1887-1889. According to Vigarié, this “ellipse” was introduced by T. C. Simmons [7], and has foci the first isogonic center or Fermat point (X_{13} in [5]) and the first isodynamic point (X_{15} in [5]). The contacts of this “ellipse” with the sidelines of reference triangle ABC are the vertices of the cevian triangle of X_{13}. In other words, the perspector of this conic is one of its foci. The given trilinear equation is :

$$\sqrt{\alpha \sin\left(A+\frac{\pi}{3}\right)}+\sqrt{\beta \sin(\left(B+\frac{\pi}{3}\right)}+\sqrt{\gamma \sin\left(C+\frac{\pi}{3}\right)}=0.$$

It appears that this conic is not always an ellipse and, curiously, the corresponding conic with the other Fermat and isodynamic points is not mentioned in [1].

In this paper, working with barycentric coordinates, we generalize the study of inscribed conics and circumconics whose perspector is one focus.

2. Circumconics and inscribed conics

Let $P = (u : v : w)$ be any point in the plane of triangle ABC which does not lie on one sideline of ABC. Denote by $\mathcal{L}(P)$ its trilinear polar.

The locus of the trilinear pole of a line passing through P is a circumconic denoted by $\Gamma_c(P)$ and the envelope of trilinear polar of points of $\mathcal{L}(P)$ is an inscribed conic $\Gamma_i(P)$. In both cases, P is said to be the perspector of the conic and $\mathcal{L}(P)$ its perspectrix. Note that $\mathcal{L}(P)$ is the polar line of P in both conics.

The centers of $\Gamma_c(P)$ and $\Gamma_i(P)$ are

$$\begin{aligned}\Omega_c(P) =& (u(v+w-u)\ : v(w+u-v)\ : w(u+v-w)),\\ \Omega_i(P) =& (u(v+w)\ : v(w+u)\ : w(u+v))\end{aligned}$$

respectively. $\Omega_c(P)$ is also the perspector of the medial triangle and the anticevian triangle $A_P B_P C_P$ of P. $\Omega_i(P)$ is the complement of the isotomic conjugate of P.

Publication Date: June 26, 2006. Communicating Editor: Paul Yiu.

2.1. *Construction of the axes of* $\Gamma_c(P)$ *and* $\Gamma_i(P)$. Let X be the fourth intersection of the conic and the circumcircle (X is the trilinear pole of the line KP). The axes of $\Gamma_c(P)$ are the parallels at $\Omega_c(P)$ to the bisectors of the lines BC and AX. A similar construction in the cevian triangle $P_aP_bP_c$ of P gives the axes of $\Gamma_i(P)$.

2.2. *Construction of the foci of* $\Gamma_c(P)$ *and* $\Gamma_i(P)$. The line BC and its perpendicular at P_a meet one axis at two points. The circle with center $\Omega_i(P)$ which is orthogonal to the circle having diameter these two points meets the axis at the requested foci. A similar construction in the anticevian triangle of P gives the foci of $\Gamma_c(P)$.

3. Inscribed conics with focus at the perspector

Theorem 1. *There are two and only two non-degenerate inscribed conics whose perspector* P *is one focus : they are obtained when* P *is one of the isogonic centers.*

Proof. If P is one focus of $\Gamma_i(P)$, the other focus is the isogonal conjugate P^* of P and the center is the midpoint of PP^*. This center must be the isotomic conjugate of the anticomplement of P. A computation shows that P must lie on three circum-strophoids with singularity at one vertex of ABC. These strophoids are orthopivotal cubics as seen in [4, p.17]. They are the isogonal transforms of the three Apollonian circles which intersect at the two isodynamic points. Hence, the strophoids intersect at the isogonic centers. □

These conics will be called the (inscribed) *Simmons conics* denoted by $\mathcal{S}_{13} = \Gamma_i(X_{13})$ and $\mathcal{S}_{14} = \Gamma_i(X_{14})$.

Elements of the conics	$\mathcal{S}_{13}$	$\mathcal{S}_{14}$
perspector and focus	X_{13}	X_{14}
other real focus	X_{15}	X_{16}
center	X_{396}	X_{395}
focal axis	parallel to the Euler line	idem
non-focal axis	$\mathcal{L}(X_{14})$	$\mathcal{L}(X_{13})$
directrix	$\mathcal{L}(X_{13})$	$\mathcal{L}(X_{14})$
other directrix	$\mathcal{L}(X_{18})$	$\mathcal{L}(X_{17})$

Remark. The directrix associated to the perspector/focus in both Simmons conics is also the trilinear polar of this same perspector/focus. This will be generalized below.

Theorem 2. *The two (inscribed) Simmons conics generate a pencil of conics which contains the nine-point circle.*

The four (not always real) base points of the pencil form a quadrilateral inscribed in the nine point circle and whose diagonal triangle is the anticevian triangle of X_{523}, the infinite point of the perpendiculars to the Euler line. In Figure 1 we have four real base points on the nine point circle and on two parabolas $\mathcal{P}_1$ and $\mathcal{P}_2$.

Hence, all the conics of the pencil have axes with the same directions (parallel and perpendicular to the Euler line) and are centered on the rectangular hyperbola

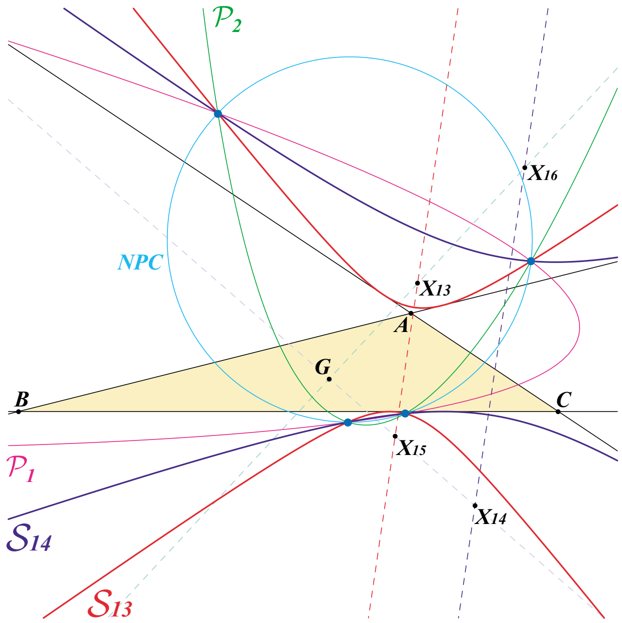


Figure 1. Simmons ponctual pencil of conics

which is the polar conic of X_{30} (point at infinity of the Euler line) in the Neuberg cubic. This hyperbola passes through the in/excenters, X_5, X_{30}, X_{395}, X_{396}, X_{523}, X_{1749} and is centered at X_{476} (Tixier point). See Figure 2. This is the diagonal conic with equation :

$$\sum_{\text{cyclic}} (b^2 - c^2)(4S_A^2 - b^2c^2)x^2 = 0.$$

It must also contain the vertices of the anticevian triangle of any of its points and, in particular, those of the diagonal triangle above. Note that the polar lines of any of its points in both Simmons inconics are parallel.

Theorem 3. *The two (inscribed) Simmons conics generate a tangential pencil of conics which contains the Steiner inellipse.*

Indeed, their centers X_{396} and X_{395} lie on the line GK. The locus of foci of all inconics with center on this line is the (second) Brocard cubic K018 which is $\mathrm{n}\mathcal{K}_0(K, X_{523})$ (See [3]). These conics must be tangent to the trilinear polar of the root X_{523} which is the line through the centers X_{115} and X_{125} of the Kiepert and Jerabek hyperbolas.

Another approach is the following. The fourth common tangent to two inconics is the trilinear polar of the intersection of the trilinear polars of the two perspectors. In the case of the Simmons inconics, the intersection is X_{523} at infinity (the perspector of the Kiepert hyperbola) hence the common tangent must be the trilinear polar of this point. In fact, more generally, any inconic with perspector on the

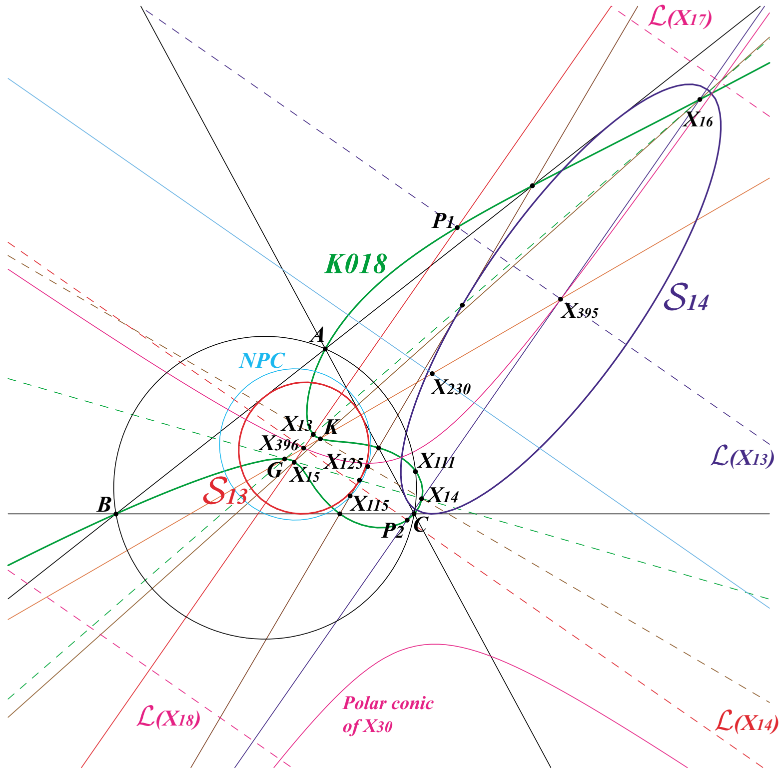


Figure 2. The two Simmons inconics $\mathcal{S}_{13}$ and $\mathcal{S}_{14}$

Kiepert hyperbola must be tangent to this same line (the perpector of each conic must lie on the Kiepert hyperbola since it is the isotomic conjugate of the anticomplement of the center of the conic). In particular, since G lies on the Kiepert hyperbola, the Steiner inellipse must also be tangent to this line. This is also the case of the inconic with center K, perspector H sometimes called K-ellipse (see [1]) although it is not always an ellipse.

Remarks. (1) The contacts of this common tangent with $\mathcal{S}_{13}$ and $\mathcal{S}_{14}$ lie on the lines through G and the corresponding perspector. See Figures 2 and 3.

(2) This line $X_{115}X_{125}$ meets the sidelines of ABC at three points on K018.

(3) The focal axes meet the non-focal axes at the vertices of a rectangle with center X_{230} on the orthic axis and on the line GK. These vertices are X_{396}, X_{395} and two other points P_1, P_2 on the cubic K018 and collinear with X_{111}, the singular focus of the cubic.

(4) The orthic axis is the mediator of the non-focal axes.

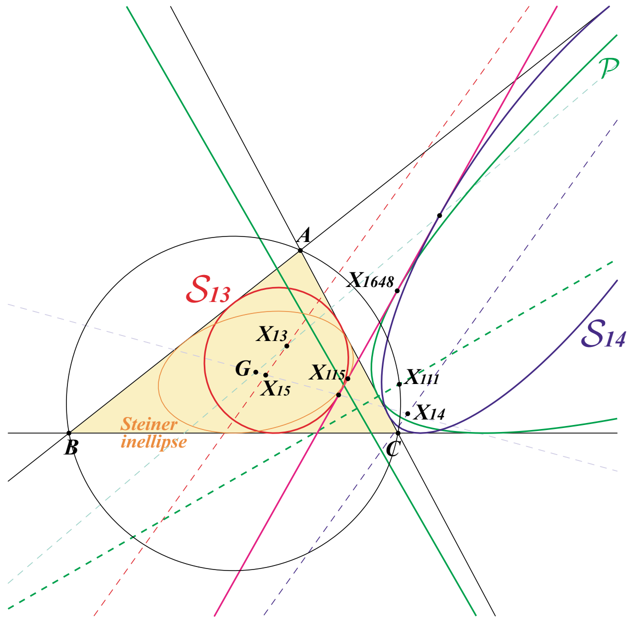


Figure 3. Simmons tangential pencil of conics

(5) The pencil contains one and only one parabola $\mathcal{P}$ we will call *the Simmons parabola*. This is the in-parabola with perspector X_{671} (on the Steiner ellipse), focus X_{111} (Parry point), touching the line $X_{115}X_{125}$ at X_{1648}.[1]

4. Circumconics with focus at the perspector

A circumconic with perspector P is inscribed in the anticevian triangle $P_aP_bP_c$ of P. In other words, it is the inconic with perspector P in $P_aP_bP_c$. Thus, P is a focus of the circumconic if and only if it is a Fermat point of $P_aP_bP_c$. According to a known result [2], it must then be a Fermat point of ABC. Hence,

Theorem 4. *There are two and only two non-degenerate circumconics whose perspector P is one focus : they are obtained when P is one of the isogonic centers.*

They will be called the *Simmons circumconics* denoted by $\Sigma_{13} = \Gamma_c(X_{13})$ and $\Sigma_{14} = \Gamma_c(X_{14})$. See Figure 4.

The fourth common point of these conics is X_{476} (Tixier point) on the circumcircle. The centers and other real foci are not mentioned in the current edition of [6] and their coordinates are rather complicated. The focal axes are those of the Simmons inconics.

---

[1]X_{1648} is the tripolar centroid of X_{523} i.e. the isobarycenter of the traces of the line $X_{115}X_{125}$. It lies on the line GK.

[2]The angular coordinates of a Fermat point of $P_aP_bP_c$ are the same when they are taken either with respect to $P_aP_bP_c$ or with respect to ABC.

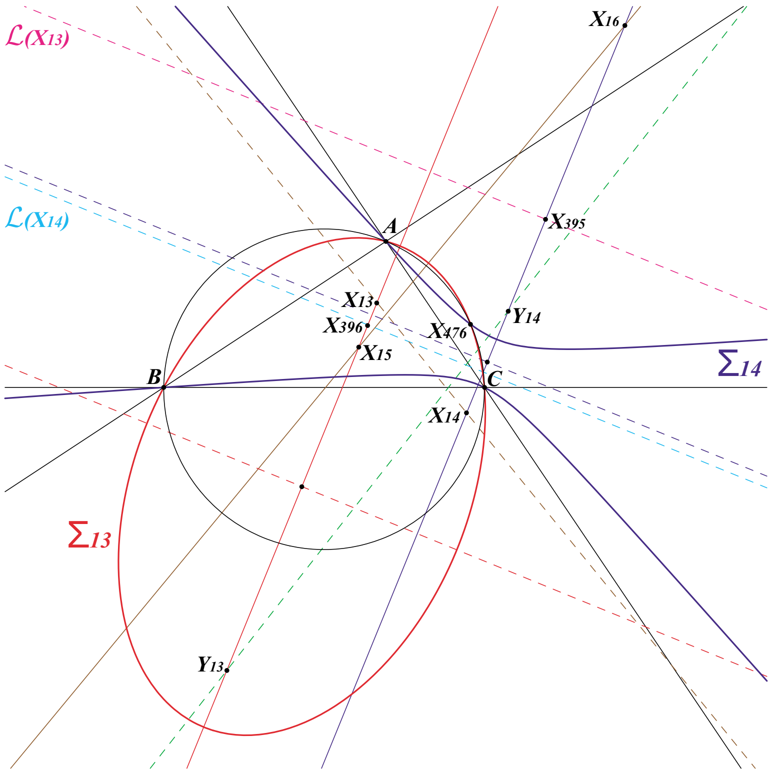


Figure 4. The two Simmons circumconics Σ_{13} and Σ_{14}

A digression: there are in general four circumconics with given focus F. Let $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ the circles passing through F with centers A, B, C. These circles have two by two six centers of homothety and these centers are three by three collinear on four lines. One of these lines is the trilinear polar $\mathcal{L}(Q)$ of the interior point $Q = \frac{1}{AF} : \frac{1}{BF} : \frac{1}{CF}$ and the remaining three are the sidelines of the cevian triangle of Q. These four lines are the directrices of the sought circumconics and their construction is therefore easy to realize. See Figure 5.

This shows that one can find six other circumconics with focus at a Fermat point but, in this case, this focus is not the perspector.

5. Some related loci

We now generalize some of the particularities of the Simmons inconics and present several higher degree curves which all contain the Fermat points.

5.1. *Directrices and trilinear polars.* We have seen that these Simmons inconics are quite remarkable in the sense that the directrix corresponding to the perspector/focus F (which is the polar line of F in the conic) is also the trilinear polar of F. The generalization gives the following

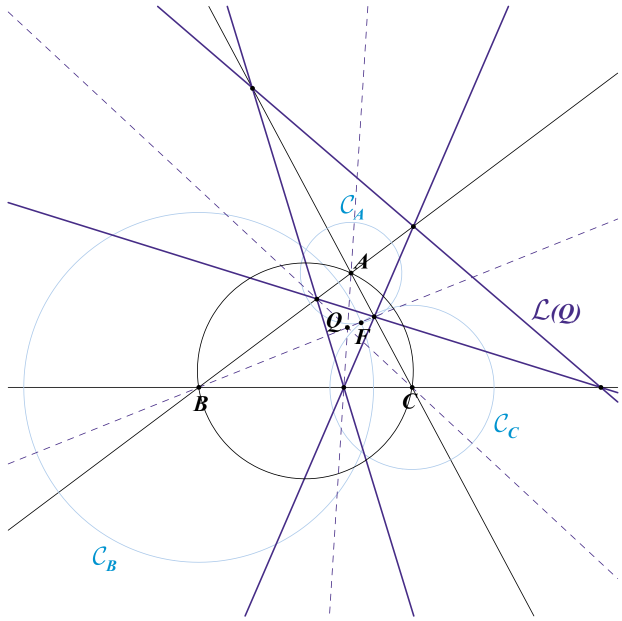


Figure 5. Directrices of circumconics with given focus

Theorem 5. *The locus of the focus F of the inconic such that the corresponding directrix is parallel to the trilinear polar of F is the Euler-Morley quintic Q003.*

Q003 is a very remarkable curve with equation

$$\sum_{\text{cyclic}} a^2(S_B\, y - S_C\, z)y^2z^2 = 0$$

which (at the time this paper is written) contains 70 points of the triangle plane. See [3] and [4].

In Figure 6, we have the inconic with focus F at one of the extraversions of X_{1156} (on the Euler-Morley quintic).

5.2. *Perspector lying on one axis.* The Simmons inconics (or circumconics) have their perspectors at a focus hence on an axis. More generally,

Theorem 6. *The locus of the perspector P of the inconic (or circumconic) such that P lies on one of its axis is the Stothers quintic Q012.*

The Stothers quintic Q012 has equation

$$\sum_{\text{cyclic}} a^2(y-z)(x^2-yz)yz = 0.$$

Q012 is also the locus of point M such that the circumconic and inconic with same perspector M have parallel axes, or equivalently such that the pencil of conics generated by these two conics contains a circle. See [3].

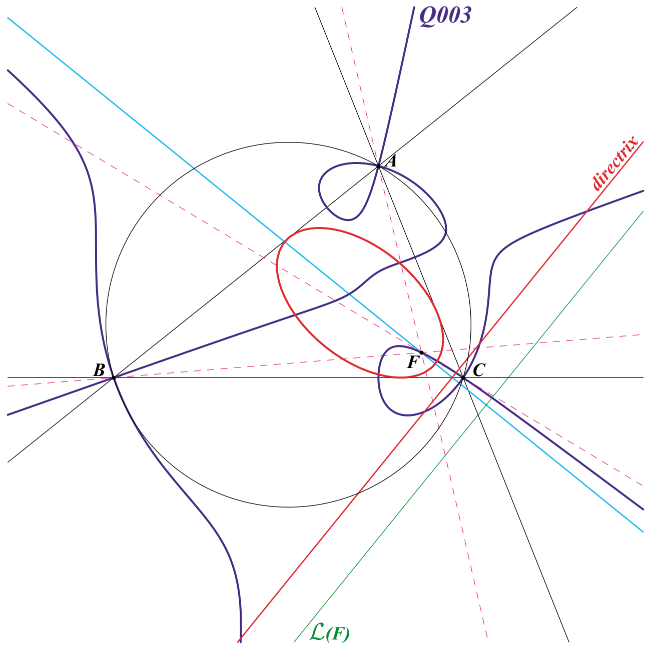


Figure 6. An inconic with directrix parallel to the trilinear polar of the focus

The center of the inconic in Theorem 6 must lie on the complement of the isotomic conjugate of Q012, another quartic with equation

$$\sum_{\text{cyclic}} a^2(y+z-x)(y-z)(y^2+z^2-xy-xz)=0.$$

In Figure 7, we have the inconic with perspector X_{673} (on the Stothers quintic) and center X_{3008}.

The center of the circumconic in Theorem 6 must lie on a septic which is the $G-$Ceva conjugate of Q012.

5.3. *Perspector lying on the focal axis.* The focus F, its isogonal conjugate F^* (the other focus), the center Ω (midpoint of FF^*) and the perspector P (the isotomic conjugate of the anticomplement of Ω) of the inconic may be seen as a special case of collinear points. More generally,

Theorem 7. *The locus of the focus F of the inconic such that F, F^* and P are collinear is the bicircular isogonal sextic* Q039.

Q039 is also the locus of point P whose pedal triangle has a Brocard line passing through P. See [3].

Remark. The locus of P such that the polar lines of P and its isogonal conjugate P^* in one of the Simmons inconics are parallel are the two isogonal pivotal cubics K129a and K129b.

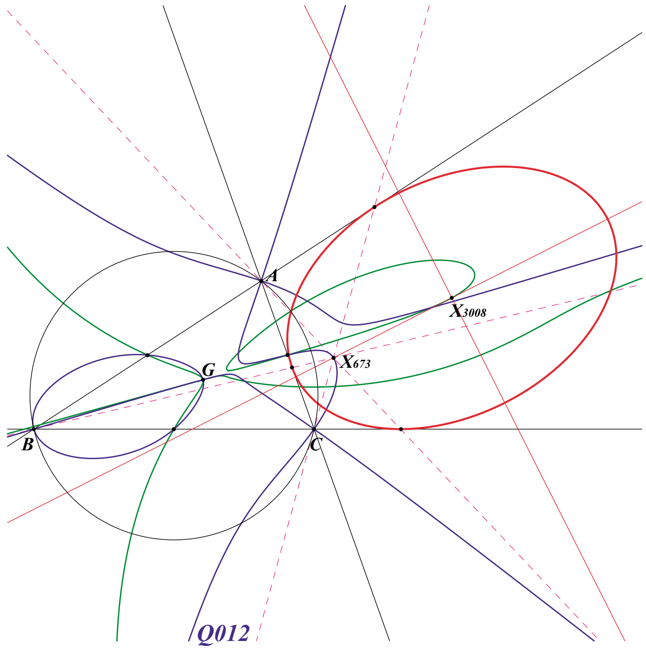


Figure 7. An inconic with perspector on one axis

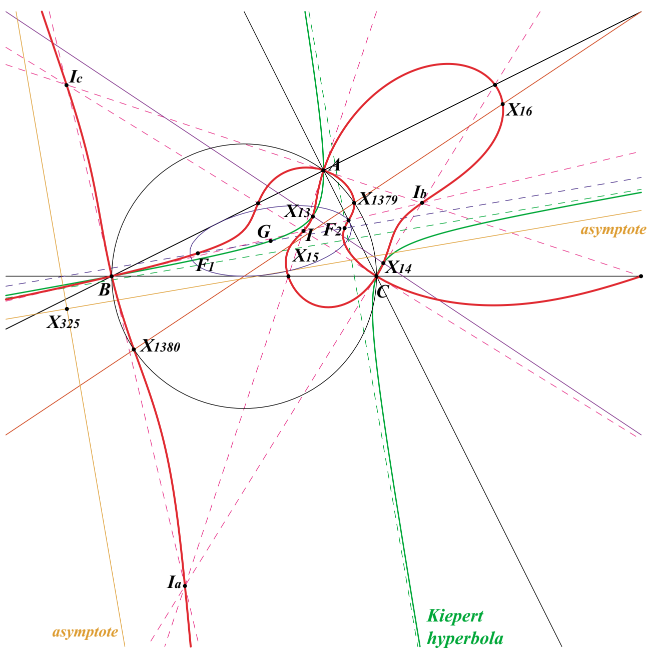


Figure 8. The bicircular isogonal sextic Q039

More precisely, with the conic $\mathcal{S}_{13}$ we obtain K129b = $\mathrm{p}\mathcal{K}(K, X_{396})$ and with the conic $\mathcal{S}_{14}$ we obtain K129a = $\mathrm{p}\mathcal{K}(K, X_{395})$. See [3].

6. Appendices

6.1. In his paper [7], T. C. Simmons has shown that the eccentricity of Σ_{13} is twice that of $\mathcal{S}_{13}$. This is also true for Σ_{14} and $\mathcal{S}_{14}$. The following table gives these eccentricities.

conic	eccentricity
$\mathcal{S}_{13}$	$\dfrac{1}{\sqrt{2(\cot\omega+\sqrt{3})}} \times \dfrac{OH}{\sqrt{\Delta}}$
$\mathcal{S}_{14}$	$\dfrac{1}{\sqrt{2(\cot\omega-\sqrt{3})}} \times \dfrac{OH}{\sqrt{\Delta}}$
Σ_{13}	$\dfrac{2}{\sqrt{2(\cot\omega+\sqrt{3})}} \times \dfrac{OH}{\sqrt{\Delta}}$
Σ_{14}	$\dfrac{2}{\sqrt{2(\cot\omega-\sqrt{3})}} \times \dfrac{OH}{\sqrt{\Delta}}$

where ω is the Brocard angle, Δ the area of ABC and OH the distance between O and H.

6.2. Since Σ_{13} and $\mathcal{S}_{13}$ (or Σ_{14} and $\mathcal{S}_{14}$) have the same focus and the same directrix, it is possible to find infinitely many homologies (perspectivities) transforming these two conics into concentric circles with center X_{13} (or X_{14}) and the radius of the first circle is twice that of the second circle.

The axis of such homology must be parallel to the directrix and its center must be the common focus. Furthermore, the homology must send the directrix to the line at infinity and, for example, must transform the point P_1 (or P_2, see remark 3 at the end of §3) into the infinite point X_{30} of the Euler line or the line $X_{13}X_{15}$.

Let Δ_1 and Δ_2 be the two lines with equations

$$\sum_{\text{cyclic}} (b^2+c^2-2a^2+\sqrt{a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2})\,x=0$$

and

$$\sum_{\text{cyclic}} (b^2+c^2-2a^2-\sqrt{a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2})\,x=0.$$

Δ_1 and Δ_2 are the tangents to the Steiner inellipse which are perpendicular to the Euler line. The contacts lie on the line GK and on the circle with center G passing through X_{115}, the center of the Kiepert hyperbola. Δ_1 and Δ_2 meet the Euler line at two points lying on the circle with center G passing through X_{125}, the center of the Jerabek hyperbola.

If we take one of these lines as an axis of homology, the two Simmons circumconics Σ_{13} and Σ_{14} are transformed into two circles Γ_{13} and Γ_{14} having the same radius. Obviously, the two Simmons inconics are also transformed into two circles having the same radius. See Figure 9.

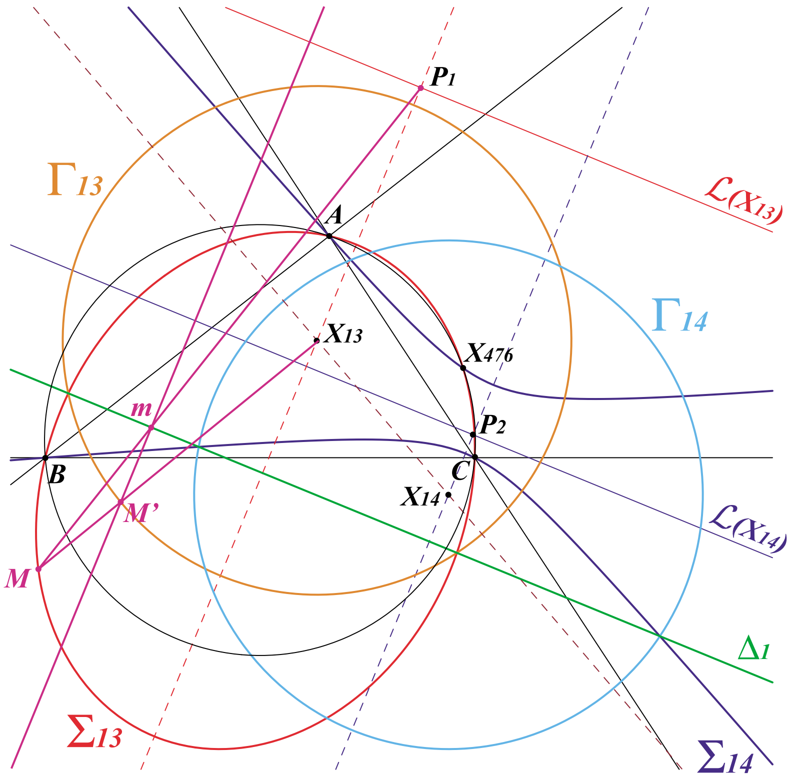


Figure 9. Homologies and circles

For any point M on Σ_{13}, the line MP_1 meets Δ_1 at m. The parallel to the Euler line at m meets the line MX_{13} at M' on Γ_{13}. A similar construction with M on Σ_{14} and P_2 instead of P_1 will give Γ_{14}.

References

[1] H. Brocard and T. Lemoyne, *Courbes Géométriques Remarquables*. Librairie Albert Blanchard, Paris, third edition, 1967.

[2] J.-P. Ehrmann and B. Gibert, *Special Isocubics in the Triangle Plane*, available at http://perso.wanadoo.fr/bernard.gibert/

[3] B. Gibert, *Cubics in the Triangle Plane*, available at http://perso.wanadoo.fr/bernard.gibert/

[4] B. Gibert, Orthocorrespondence and orthopivotal cubics, *Forum Geom*, 3 (2003) 1–27.

[5] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[6] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[7] T. C. Simmons, Recent Geometry, Part V of John J. Milne's *Companion to the Weekly Problem Papers*, 1888, McMillan, London.

[8] E. Vigarié, Géométrie du triangle, Étude bibliographique et terminologique, *Journal de Mathématiques Spéciales*, 3e série, 1 (1887) 34–43, 58–62, 77–82, 127–132, 154–157, 75–177, 199–203, 217–219, 248–250; 2 (1888) 9–13, 57–61, 102–04, 127–131, 182–185, 199–202, 242–244, 276–279; 3 (1889) 18–19, 27–30, 55–59, 83–86.

Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France
E-mail address: bg42@orange.fr

Forum Geometricorum
Volume 6 (2006) 225–227.

FORUM GEOM
ISSN 1534-1178

A Synthetic Proof and Generalization of Bellavitis' Theorem

Nikolaos Dergiades

Abstract. In this note we give a synthetic proof of Bellavitis' theorem and then generalizing this theorem, for not only convex quadrilaterals, we give a synthetic geometric proof for both theorems direct and converse, as Eisso Atzema proved, by trigonometry, for the convex case [1]. From this approach evolves clearly the connection between hypothesis and conclusion.

1. Bellavitis' theorem

Eisso J. Atzema has recently given a trigonometric proof of Bellavitis' theorem [1]. We present a synthetic proof here. Inside a convex quadrilateral $ABCD$, let the diagonal AC form with one pair of opposite sides angles w_1, w_3. Similarly let the angles inside the quadrilateral that the other diagonal BD forms with the remaining pair of opposite sides be w_2, w_4.

Theorem 1 (Bellavitis, 1854)**.** *If the side lengths of a convex quadrilateral $ABCD$ satisfy $AB \cdot CD = BC \cdot DA$, then $w_1 + w_2 + w_3 + w_4 = 180°$*

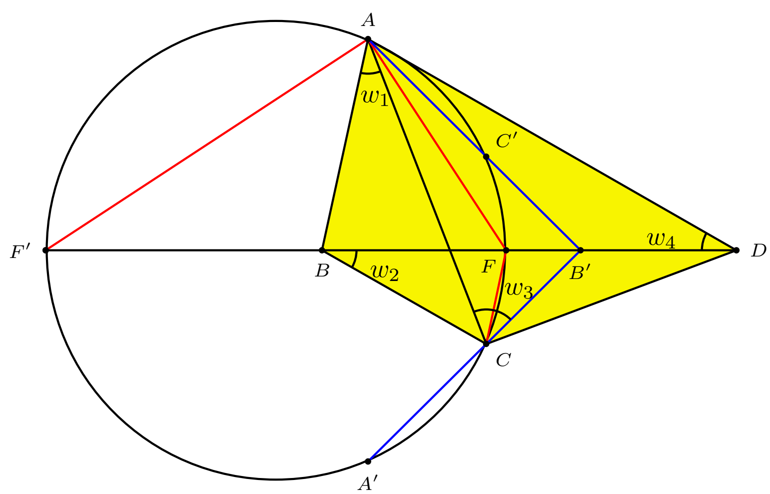


Figure 1

Proof. If $AB = AD$ then $BC = CD$ and AC is the perpendicular bisector of BD. Hence $ABCD$ is a kite, and it is obvious that $w_1 + w_2 + w_3 + w_4 = 180°$.

If $ABCD$ is not a kite, then from $AB \cdot CD = BC \cdot DA$, we have $\frac{AB}{AD} = \frac{CB}{CD}$. Hence, C lies on the A-Apollonius circle of triangle ABD. See Figure 1. This

Publication Date: September 18, 2006. Communicating Editor: Paul Yiu.

circle has diameter FF', where AF and AF' are the internal and external bisectors of angle BAD and CF is the bisector of angle BCD. The reflection of AC in AF meets the Apollonius circle at C'. Since arc CF = arc FC', the point C' is the reflection of C in BD. Similarly the reflection of AC in CF meets the Apollonius circle at A' that is the reflection of A in BD. Hence the lines AC', CA' are reflections of each other in BD and are met at a point B' on BD. So we have

$$w_2 + w_3 = w_2 + \angle BCB' = \angle CB'D = \angle AB'D \tag{1}$$
$$w_1 + w_4 = \angle B'AD + w_4 = \angle BB'A. \tag{2}$$

From (1) and (2) we get

$$w_1 + w_2 + w_3 + w_4 = \angle BB'A + \angle AB'D = 180^\circ.$$

□

2. A generalization

There is actually no need for $ABCD$ to be a convex quadrilateral. Since it is clear that $w_1 + w_2 + w_3 + w_4 = 180^\circ$ for a cyclic quadrilateral, we consider noncyclic quadrilaterals below. We make use of oriented angles and arcs. Denote by $\theta(XY, XZ)$ the oriented angle from XY to XZ. We continue to use the notation

$$w_1 = \theta(AB, AC), \qquad w_3 = \theta(CD, CA),$$
$$w_2 = \theta(BC, BD), \qquad w_4 = \theta(DA, DB).$$

Theorem 2. *In an arbitrary noncyclic quadrilateral $ABCD$, the side lengths satisfy the equality $AB \cdot CD = BC \cdot DA$ if and only if*

$$w_1 + w_2 + w_3 + w_4 = \pm 180^\circ.$$

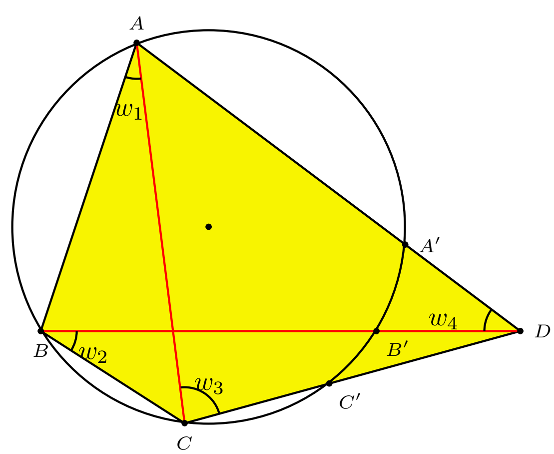


Figure 2

Proof. Since $ABCD$ is not a cyclic quadrilateral the lines DA, DB, DC meet the circumcircle of triangle ABC at the distinct points A', B', C'. The triangle $A'B'C'$ is the circumcevian triangle of D relative to ABC. Note that

$$\begin{aligned} 2w_1 =& \text{arc } BC, \\ 2w_2 =& \text{arc } CC' + \text{arc } C'B', \\ 2w_3 =& \text{arc } C'A, \\ 2w_4 =& \text{arc } AB + \text{arc} A'B'. \end{aligned}$$

From these, $w_1 + w_2 + w_3 + w_4 = \pm 180^\circ$ if and only if

$$(\text{arc } BC + \text{arc } CC' + \text{arc } C'A + \text{arc } AB) + \text{arc } C'B' + \text{arc } A'B' = \pm 360^\circ.$$

Since $\text{arc } BC + \text{arc } CC' + \text{arc } C'A + \text{arc } AB = \pm 360^\circ$, the above condition holds if and only if $\text{arc } C'B' = \text{arc } B'A'$. This means that the circumcevian triangle of D is isosceles, *i.e.*,

$$B'A' = B'C'. \tag{3}$$

It is well known that $A'B'C'$ is similar to with the pedal triangle $A''B''C''$ of D. See [2, §7.18] The condition (3) is equivalent to

$$B''A'' = B''C''.$$

This, in turn, is equivalent to the fact that D lies on the B-Apollonius circle of ABC because for a pedal triangle we know that

$$B''A'' = DC \cdot \sin C = B''C'' = DA \cdot \sin A$$

or

$$\frac{DC}{DA} = \frac{\sin A}{\sin C} = \frac{BC}{BA}.$$

From this we have $AB \cdot CD = BC \cdot DA$. □

References

[1] E. J. Atzema, A theorem by Giusto Bellavitis on a class of quadrilaterals, *Forum Geom.*, 6 (2006) 181–185.

[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece

E-mail address: ndergiades@yahoo.gr

Forum Geometricorum
Volume 6 (2006) 229–234.

FORUM GEOM
ISSN 1534-1178

On Some Theorems of Poncelet and Carnot

Huub P.M. van Kempen

Abstract. Some relations in a complete quadrilateral are derived. In connection with these relations some special conics related to the angular points and sides of the quadrilateral are discussed. A theorem of Carnot valid for a triangle is extended to a quadrilateral.

1. Introduction

The scope of Euclidean Geometry was substantially extended during the seventeenth century by the introduction of the discipline of Projective Geometry. Until then geometers were mainly concentrating on the *metric* (or *Euclidean*) properties in which the measure of distances and angles is emphasized. Projective Geometry has no distances, no angles, no circles and no parallelism but concentrates on the *descriptive* (or *projective*) properties. These properties have to do with the relative positional connection of the geometric elements in relation to each other; the properties are unaltered when the geometric figure is subjected to a projection.

Projective Geometry was started by the Grecian mathematician Pappus of Alexandria . After more than thirteen centuries it was continued by two Frenchmen, Desargues and his famous pupil Pascal. The latter one published in 1640 his well-known *Essay pour les coniques*. This short study contains the well-known *hexagrammum mysticum*, nowadays known as Pascal's Theorem. Meanwhile, the related subject of perspective had been studied by architects and artists (Leonardo da Vinci). The further development of Projective Geometry was about two hundred years later, mainly by a French group of mathematicians (Poncelet, Chasles, Carnot, Brianchon and others).

An important tool in Projective Geometry is a semi-algebraic instrument, called the *cross ratio*. This topic was introduced, independently of each other, by Möbius (1827) and Chasles (1829).

In this article we present an (almost forgotten) result of Poncelet [4], obtained in an alternative way and we derive some associated relations (Theorem 1). Furthermore, we extend a theorem by Carnot [1] from a triangle to a complete quadrilateral (Theorem 3).

Publication Date: September 25, 2006. Communicating Editor: Floor van Lamoen.
The author thanks Floor van Lamoen and the referee for their useful suggestions during the preparation of this paper.

We take as starting-point the theorems of Ceva and Pappus-Pascal. The first one is a close companion of the theorem of the Grecian mathematician Menelaus. In the analysis we will follow as much as possible the purist/synthetic approach. It will be shown that this approach leads to surprising results derived along unexpected lines.

2. Proof and extension of a Theorem by Poncelet

Theorem 1. *Let the diagonal points of a complete quadrilateral ABCD be P, Q and R. Let the intersections of PQ with AD and BC be H and F respectively and those of PR with CD and AB be G and E respectively* (*Figure 1*). *Then*

$$\frac{AE}{EB}\cdot\frac{BF}{FC}\cdot\frac{CG}{GD}\cdot\frac{DH}{HA}=1, \tag{1}$$

$$\frac{AP}{PC}\cdot\frac{CG}{GD}\cdot\frac{DP}{PB}\cdot\frac{BE}{EA}=1, \tag{2}$$

$$\frac{BP}{PD}\cdot\frac{DH}{HA}\cdot\frac{AP}{PC}\cdot\frac{CF}{FB}=1. \tag{3}$$

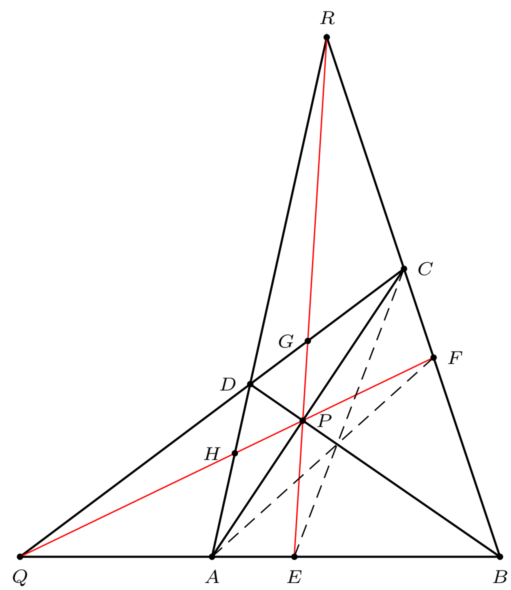


Figure 1

Proof. We apply the Pappus-Pascal theorem to the triples (Q, A, E) and (R, C, F) and find that in triangle ABC the lines AF, BP and CE are concurrent so that by Ceva's theorem

$$\frac{AE}{EB}\cdot\frac{BF}{FC}\cdot\frac{CP}{PA}=1. \tag{4}$$

Similarly with the triples (Q, G, C) and (R, H, A) we find

$$\frac{CG}{GD}\cdot\frac{DH}{HA}\cdot\frac{AP}{PC}=1. \tag{5}$$

Relation (1) immediately follows from (4) and (5). Again in the same way with triples (E, B, Q) and (H, D, R) we find that

$$\frac{AE}{EB} \cdot \frac{BP}{PD} \cdot \frac{DH}{HA} = 1. \tag{6}$$

(2) follows from (5) and (6), and (3) follows from (4) and (6). □

Poncelet [4] has derived relation (1) by using cross ratios.

We now consider a special case of Theorem 1, taking a convex quadrilateral $ABCD$ in which $AB + CD = BC + DA$, so that it is circumscriptable (Figure 2). Let E', F', G' and H' be the points of tangency of the incircle with AB, BC, CD and DA respectively. Clearly a relation similar to (1) holds:

$$\frac{AE'}{E'B} \cdot \frac{BF'}{F'C} \cdot \frac{CG'}{G'D} \cdot \frac{DH'}{H'A} = 1. \tag{7}$$

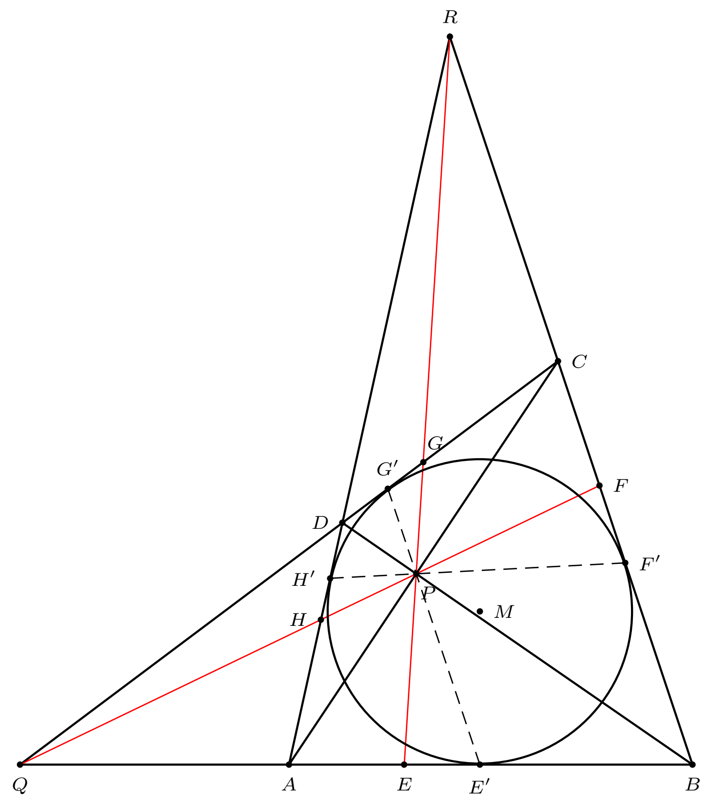


Figure 2

It is well known [5] that the point of intersection of $E'G'$ and $F'H'$ is P. This can be seen for a general quadrilateral with an inscribed conic from subsequent application of Brianchon's theorem to hexagons $AE'BCG'D$ and $BF'CDH'A$. See for instance [2, p.49]. This raises the questions whether or not a relation similar to (1) will hold. We will examine this problem by using Ceva's theorem.

3. Further Analysis

We start with a given quadrilateral $ABCD$ where E and G are arbitrary points on the lines AB and CD respectively. We then construct points F_1 and H_1 on BC and AD respectively such that (1) holds. We can do so by the following construction (Figure 3).

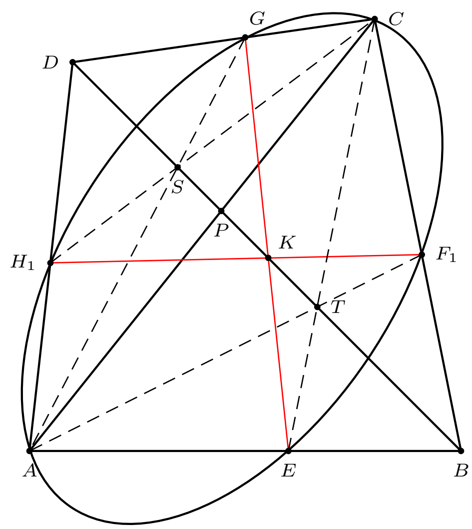


Figure 3

First we consider the triangles ABC and ADC. Let $T = CE \cap BD$ and $F_1 = BC \cap AT$. By Ceva's theorem we have in triangle ABC

$$\frac{AE}{EB} \cdot \frac{BF_1}{F_1C} \cdot \frac{CP}{PA} = 1. \tag{8}$$

Now if $S = AG \cap DB$ and $H_1 = AD \cap CS$, then Ceva's theorem applied to triangle ADC gives

$$\frac{CG}{GD} \cdot \frac{DH_1}{H_1A} \cdot \frac{AP}{PC} = 1. \tag{9}$$

By multiplication of (8) and (9) we find the desired equivalence of (1).

Theorem 2. *If in the quadrilateral $ABCD$ the points E, F_1, G and H_1 lie on AB, BC, CD and DA respectively such that $S = AG \cap CH_1$ and $T = AF_1 \cap CE$ lie on BD, then the points A, E, F_1, C, G and H_1 lie on a conic and $K = EG \cap F_1H_1$ lies on BD.*

Proof. Here we have to switch to the field of Projective Geometry. We will use the cross ratio of pencils in relation to the cross-ratio of ranges. These concepts are extensively described by Eves [3]. Now consider the two pencils (AH_1, AG, AF_1, AE) and (CH_1, CG, CF_1, CE) in Figure 3. We have the cross-ratio equality between ranges and pencils:

$$A(H_1, G; F_1, E) = (D, S; T, B) = (S, D; B, T) = C(H_1, G; F_1, E). \qquad (10)$$

From this equality we see that A, E, F_1, C, G and H_1 lie on a conic. Applying Pascal's theorem to the hexagon AF_1H_1CEG we find that the diagonal BD is the Pascal line and consequently the points S, K and T are collinear. □

By using triangle ABD and triangle CBD instead of triangle ABC and triangle ADC as above, we can also construct F_2 and H_2 such that relation (1) holds (Figure 4). Now we apply Theorem 2, finding that B, F_2, G, D, H_2 and E lie on a conic. Using Pascal's theorem for the hexagon $BGEDF_2H_2$ we find that $L = EG \cap F_2H_2$ lies on AC.

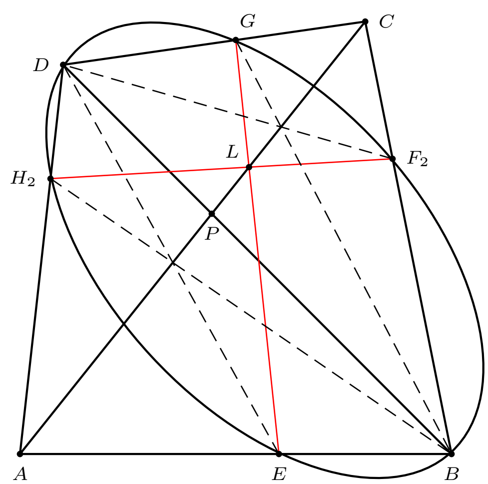


Figure 4

With the help of Theorem 2 we prove an extension of Carnot's theorem in [1] for a triangle to a complete quadrilateral.

Theorem 3. *If in the quadrilateral $ABCD$ the points E, F, G and H lie on AB, BC, CD and DA respectively and EG and FH concur in $P = AC \cap BD$, then (1) is satisfied if and only if there is a conic inscribed in quadrilateral $ABCD$, which touches its sides in the points E, F, G and H.*

Proof. Assume that relation (1) holds. By Theorem 2 we know that $BFGDHE$ and $AEFCGH$ lie on two conics. Let $V = DE \cap BH$ and $W = DF \cap BG$. First we apply Desargues' theorem to triangle GFC and triangle EHV (Figure 5).

The lines GE, FH and CV concur in P. This means that the intersection points of the corresponding sides are collinear. So $U = GF \cap EH$, $B = FC \cap HV$ and $D = GC \cap EV$ are collinear. Next, consider the unique conic Γ through E, F, G and H which is tangent to CD at G. We examine the direction of the tangent to Γ at the point H. Therefore we consider the hexagon $GGEHHF$. We find that

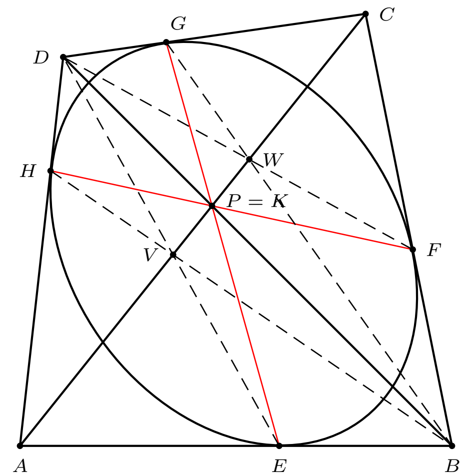


Figure 5

$GE \cap HF = P$ and $GF \cap EH = U$. Since both P and U are collinear with B and D, the line BD is the Pascal line. This means that the tangents to Γ at G and H intersect on BD, which implies that AD is the tangent to Γ at H.

In the same way we prove that the lines AB and BC are tangent to Γ at E and F respectively, which proves the sufficiency part.

Now assume that a conic is tangent to the sides of quadrilateral $ABCD$ at the points E, F, G and H. Note that of course EG and FH intersect in P, as stated earlier. With fixed E, F and G there is exactly one point H^* on AD such that the equivalent version of relation (1) holds. By the sufficiency part this leads to a conic tangent to the sides at E, F, G, and H^*. As these two conics have three double points in common, they must be the same conic. This leads to the conclusion that H and H^* are in fact the same point. This proves the necessity part. □

Applying Theorem 3 to the results of Theorem 2 we find

Corollary 4. *If in the quadrilateral $ABCD$ of Theorem 2 the lines EG and F_1H_1 concur in P, where $P = AC \cap BD$, then F_1H_1 of Figure 3 and F_2H_2 of Figure 4 coincide.*

References

[1] L. N. M. Carnot, *Essai sur la théorie des transversals*, Paris 1806.
[2] R. Deaux, *Compléments de Géométrie plane*, De Boeck, Brussels 1945.
[3] H. Eves, *A survey of Geometry*, Allun & Bacon, Boston 1972.
[4] J. V. Poncelet, *Traité des propriétés projectives des figures*, Bachelier, Paris 1822.
[5] P. Yiu, *Euclidean Geometry*, (1998), available at http://www.math.fau.edu/yiu/Geometry.html.

Huub P.M. van Kempen: Prins Mauritsplein 17, 2582 NC Den Haag, The Netherlands
E-mail address: hvkempen@versatel.nl

Forum Geometricorum
Volume 6 (2006) 235–240.

FORUM GEOM
ISSN 1534-1178

The Droz-Farny Theorem and Related Topics

Charles Thas

Abstract. At each point P of the Euclidean plane Π, not on the sidelines of a triangle $A_1A_2A_3$ of Π, there exists an involution in the pencil of lines through P, such that each pair of conjugate lines intersect the sides of $A_1A_2A_3$ in segments with collinear midpoints. If $P = H$, the orthocenter of $A_1A_2A_3$, this involution becomes the orthogonal involution (where orthogonal lines correspond) and we find the well-known Droz-Farny Theorem, which says that any two orthogonal lines through H intersect the sides of the triangle in segments with collinear midpoints. In this paper we investigate two closely related loci that have a strong connection with the Droz-Farny Theorem. Among examples of these loci we find the circumcirle of the anticomplementary triangle and the Steiner ellipse of that triangle.

1. The Droz-Farny Theorem

Many proofs can be found for the original Droz-Farny Theorem (for some recent proofs, see [1], [3]). The proof given in [3] (and [5]) probably is one of the shortest: Consider, in the Euclidean plane Π, the pencil $\mathcal{B}$ of parabola's with tangent lines the sides $a_1 = A_2A_3, a_2 = A_3A_1, a_3 = A_1A_2$ of $A_1A_2A_3$, and the line l at infinity. Let P be any point of Π, not on a sideline of $A_1A_2A_3$, and not on l, and consider the tangent lines r and r' through P to a non-degenerate parabola $\mathcal{P}$ of this pencil $\mathcal{B}$. A variable tangent line of $\mathcal{P}$ intersects r and r' in corresponding points of a projectivity (an affinity, i.e. the points at infinity of r and r' correspond), and from this it follows that the line connecting the midpoints of the segments determined by r and r' on a_1 and a_2, is a tangent line of $\mathcal{P}$, through the midpoint of the segment determined on a_3 by r and r'. Next, by the Sturm-Desargues Theorem, the tangent lines through P to a variable parabola of the pencil $\mathcal{B}$ are conjugate lines in an involution $\mathcal{I}$ of the pencil of lines through P, and this involution $\mathcal{I}$ contains in general just one orthogonal conjugate pair. In the following we call these orthogonal lines through P, the orthogonal Droz-Farny lines through P.

Remark that (PA_i, line through P parallel to a_i), $i = 1, 2, 3$ are the tangent lines through P of the degenerate parabola's of the pencil $\mathcal{B}$, and thus are conjugate pairs in the involution $\mathcal{I}$. From this it follows that in the case where $P = H$, the orthocenter of $A_1A_2A_3$, this involution becomes the orthogonal involution in the pencil of lines through H, and we find the Droz-Farny Theorem.

Publication Date: October 2, 2006. Communicating Editor: J. Chris Fisher.

Two other characterizations of the orthogonal Droz-Farny lines through P are obtained as follows: Let X and Y be the points at infinity of the orthogonal Droz-Farny lines through P. Since the two triangles $A_1A_2A_3$ and PXY are circumscribed triangles about a conic (a parabola of the pencil $\mathcal{B}$), their vertices are six points of a conic, namely the rectangular hyperbola through A_1, A_2, A_3, and P (and also through H, since any rectangular hyperbola through A_1, A_2, and A_3, passes through H). It follows that the orthogonal Droz-Farny lines through P are the lines through P which are parallel to the (orthogonal) asymptotes of this rectangular hyperbola through A_1, A_2, A_3, P, and H.

Next, since, if $P = H$, the involution $\mathcal{I}$ is the orthogonal involution, the directrix of any parabola of the pencil $\mathcal{B}$ passes through H, and the orthogonal Droz-Farny lines through any point P are the orthogonal tangent lines through P of the parabola, tangent to a_1, a_2, a_3, and with directrix PH.

2. The first locus

Let us recall some basic properties of trilinear (or normal) coordinates (see for instance [4]). Trilinear coordinates (x_1, x_2, x_3), with respect to a triangle A_1A_2A with side-lenghts l_1, l_2, l_3, of any point P of the Euclidean plane, are homogeneous projective coordinates, in the Euclidean plane, for which the vertices A_1, A_2, A_3 are the basepoints and the incenter I of the triangle the unit point. The line at infinity has in trilinear coordinates the equation $l_1x_1 + l_2x_2 + l_3x_3 = 0$. The centroid G of $A_1A_2A_3$ has trilinear coordinates $(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3})$, the orthocenter H is $(\frac{1}{\cos A_1}, \frac{1}{\cos A_2}, \frac{1}{\cos A_3})$, the circumcenter O is $(\cos A_1, \cos A_2, \cos A_3)$, the incenter I is $(1, 1, 1)$, and the Lemoine (or symmedian) point K is (l_1, l_2, l_3).

If X has trilinear coordinates (x_1, x_2, x_3) with respect to $A_1A_2A_3$, and if d_i is the "'signed"' distance from X to the side a_i (i.e. d_i is positive or negative, according as X lies on the same or opposite side of a_i as A_i), then, if F is the area of $A_1A_2A_3$, we have $d_i = \frac{2Fx_i}{l_1x_1+l_2x_2+l_3x_3}$, $i = 1, 2, 3$, and (d_1, d_2, d_3) are the *actual trilinear coordinates* of X with respect to $A_1A_2A_3$. Remark that $l_1d_1 + l_2d_2 + l_3d_3 = 2F$.

Our first locus is defined as follows ([5]):

Consider a fixed point P, not on a sideline of $A_1A_2A_3$, and not at infinity, with actual trilinear coordinates $(\delta_1, \delta_2, \delta_3)$ with respect to $A_1A_2A_3$, and suppose that s is a given real number and the set of points of the plane for which the distances d_i from (x_1, x_2, x_3) to a_i are connected by the equation

$$\frac{l_1}{\delta_1}d_1^2 + \frac{l_2}{\delta_2}d_2^2 + \frac{l_3}{\delta_3}d_3^2 = s. \tag{1}$$

Using $d_i = \frac{2F_ix_i}{l_1x_1+l_2x_2+l_3x_3}$, we see that the set is given by the equation

$$4F^2(\frac{l_1}{\delta_1}x_1^2 + \frac{l_2}{\delta_2}x_2^2 + \frac{l_3}{\delta_3}x_3^2) - s(l_1x_1 + l_2x_2 + l_3x_3)^2 = 0, \tag{2}$$

or, if we use general trilinear coordinates (p_1, p_2, p_3) of P:

$$2F(\frac{l_1}{p_1}x_1^2 + \frac{l_2}{p_2}x_2^2 + \frac{l_3}{p_3}x_3^2)(l_1p_1 + l_2p_2 + l_3p_3) - s(l_1x_1 + l_2x_2 + l_3x_3)^2 = 0. \quad (3)$$

We denote this conic by $\mathcal{K}(P, \Delta, s)$: it is the conic determined by (1) and (2), where $(\delta_1, \delta_2, \delta_3)$ are the actual trilinear coordinates of P with regard to $\Delta = A_1A_2A_3$, and also by (3), where (p_1, p_2, p_3) are any triple of trilinear coordinates of P with regard to Δ, and by the value of s. For P and Δ fixed and s allowed to vary, the conics $\mathcal{K}(\mathcal{P}, \Delta, s)$ belong to a pencil, and a straightforward calculation shows that all conics of this pencil have center P, and have the same points at infinity, which means that they have the same asymptotes and the same axes.

The conics $\mathcal{K}(P, \Delta, s)$ can be (homothetic) ellipses or hyperbola's: this depends on the location of P with regard to Δ, and again a straightforward calculation shows that we find ellipses or hyperbola's, according as the product $\delta_1\delta_2\delta_3 > 0$ or < 0.

Next, the *medial triangle* of $\Delta = A_1A_2A_3$ is the triangle whose vertices are the midpoints of the sides of Δ, and the *anticomplementary triangle* $A_1^{-1}A_2^{-1}A_3^{-1}$ of Δ is the triangle whose medial triangle is Δ. An easy calculation shows that the trilinear coordinates of the vertices A_1^{-1}, A_2^{-1}, and A_3^{-1} of this anticomplementary triangle are $(-l_2l_3, l_3l_1, l_1l_2)$, $(l_2l_3, -l_3l_1, l_1l_2)$, and $(l_2l_3, l_3l_1, -l_1l_2)$, respectively.

Lemma 1. *The locus $\mathcal{K}(P, \Delta, S)$ of the points for which the distances d_1, d_2, d_3 to the sides a_1, a_2, a_3 of $\Delta = A_1A_2A_3$ are connected by*

$$\frac{l_1}{\delta_1}d_1^2 + \frac{l_2}{\delta_2}d_2^2 + \frac{l_3}{\delta_3}d_3^2 = 4F^2(\frac{1}{l_1\delta_1} + \frac{1}{l_2\delta_2} + \frac{1}{l_3\delta_3}) \ = S,$$

where $(\delta_1, \delta_2, \delta_3)$ are the actual trilinear coordinates of a given point P, is the conic with center P, and circumscribed about the anticomplementary triangle $A_1^{-1}A_2^{-1}A_3^{-1}$ of Δ.

Proof. Substituting the coordinates $(-\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3})$, or $(\frac{1}{l_1}, -\frac{1}{l_2}, \frac{1}{l_3})$, or $(\frac{1}{l_1}, \frac{1}{l_2}, -\frac{1}{l_3})$ of A_1^{-1}, A_2^{-1}, and A_3^{-1}, in (2), we find immediately that

$$s = S = 4F^2(\frac{1}{l_1\delta_1} + \frac{1}{l_2\delta_2} + \frac{1}{l_3\delta_3}).$$

□

3. The second locus

We work again in the Euclidean plane Π, with trilinear coordinates with respect to $\Delta = A_1A_2A_3$. Assume that $P(p_1, p_2, p_3)$ is a point of Π, not at infinity and not on a sideline of Δ. We look for the locus of the points Q of Π, such that the points $Q_i = q_i \cap a_i,\ i = 1, 2, 3$, where q_i is the line through Q, parallel to PA_i, are collinear. This locus was the subject of the paper [2]. Since $l_1x_1 + l_2x_2 + l_3x_3 = 0$ is the equation of the line at infinity, the point at infinity of PA_1 has coordinates $(l_2p_2 + l_3p_3, -l_1p_2, -l_1p_3)$, and if we give Q the coordinates (x_1, x_2, x_3), we find after an easy calculation that Q_1 has coordinates $(0, l_1p_2x_1 +$

$x_2(l_2p_2+l_3p_3), x_1l_1p_3+x_3(l_2p_2+l_3p_3))$. In the same way, we find for the coordinates of Q_2, and of Q_3: $(x_1(l_1p_1+l_3p_3)+p_1x_2l_2, 0, p_3x_2l_2+x_3(l_1p_1+l_3p_3))$, and $(x_1(l_1p_1+l_2p_2)+x_3p_1l_3, x_2(l_1p_1+l_2p_2)+x_3p_2l_3, 0)$, respectively.

Next, after a rather long calculation, and deleting the singular part $l_1x_1+l_2x_2+l_3x_3=0$, the condition that Q_1, Q_2, and Q_3 are collinear, gives us the following equation for the locus of the point Q:

$$p_3(l_1p_1+l_2p_2)x_1x_2+p_1(l_2p_2+l_3p_3)x_2x_3+p_2(l_3p_3+l_1p_1)x_3x_1=0. \qquad (4)$$

This is our second locus, and we denote this conic, circumscribed about $A_1A_2A_3$, by $\mathcal{C}(P,\Delta)$, where $\Delta=A_1A_2A_3$, and where P is the point with trilinear coordinates (p_1,p_2,p_3) with regard to Δ.

Lemma 2. *The center M of $\mathcal{C}(P,\Delta)$ has trilinear coordinates*

$$(l_2l_3(l_2p_2+l_3p_3), l_3l_1(l_3p_3+l_1p_1), l_1l_2(l_1p_1+l_2p_2)).$$

It is the image $f(P)$, where f is the homothety with center G, the centroid of Δ, and homothetic ratio $-\frac{1}{2}$, or, in other words: $2\vec{GM}=-\vec{GP}$.

Proof. An easy calculation shows that the polar point of this point M with regard to the conic (4) is indeed the line at infinity, with equation $l_1x_1+l_2x_2+l_3x_3=0$. Moreover, if P_∞ is the point at infinity of the line PG, the equation $2\vec{GM}=-\vec{GP}$ is equivalent with the equality of the cross-ratio $(MPGP_\infty)$ to $-\frac{1}{2}$. Next, choose on the line PG homogeneous projective coordinates with basepoints $P(1,0)$ and $G(0,1)$, and give P_∞ coordinates (t_1,t_2) (thus $P_\infty=t_1P+t_2G$), then $(t_1,t_2)=(-3, l_1p_1+l_2p_2+l_3p_3)$ and the projective coordinates (t_1', t_2') of M follow from

$$(MPGP_\infty)=\frac{\begin{vmatrix} t_1' & t_2' \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} : \frac{\begin{vmatrix} t_1' & t_2' \\ -3 & l_1p_1+l_2p_2+l_3p_3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ -3 & l_1p_1+l_2p_2+l_3p_3 \end{vmatrix}}=-\frac{1}{2},$$

which gives $(t_1', t_2')=(-1, l_1p_1+l_2p_2+l_3p_3)$. ☐

Remark that the second part of the proof also follows from the connection between trilinears (x_1,x_2,x_3) for a point with respect to $A_1A_2A_3$ and trilinears (x_1', x_2', x_3') for the same point with respect to the medial triangle of $A_1A_2A_3$ (see [4, p.207]):

$$\begin{cases} x_1=l_2l_3(l_2x_2'+l_3x_3') \\ x_2=l_3l_1(l_3x_3'+l_1x_1') \\ x_3=l_1l_2(l_1x_1'+l_2x_2'). \end{cases}$$

4. The connection between the Droz-Farny -lines and the conics

Recall from §2 that

$$S=4F^2\left(\frac{1}{l_1\delta_1}+\frac{1}{l_2\delta_2}+\frac{1}{l_3\delta_3}\right)=2F(l_1p_1+l_2p_2+l_3p_3)\left(\frac{1}{l_1p_1}+\frac{1}{l_2p_2}+\frac{1}{l_3p_3}\right),$$

where F is the area of $\Delta = A_1A_2A_3$, (p_1, p_2, p_3) are trilinear coordinates of P with regard to $\Delta A_1A_2A_3$, and where $(\delta_1, \delta_2, \delta_3)$ are the actual trilinear coordinates of P with respect to this triangle.
Furthermore, in the foregoing section, f is the homothety with center G and homothetic ratio $-\frac{1}{2}$. Remark that $f^{-1}(\Delta)$ is the anticomplementary triangle Δ^{-1} of Δ. We have:

Theorem 3. (1) *The conics $\mathcal{K}(P, \Delta, S)$ and $\mathcal{C}(f^{-1}(P), f^{-1}(\Delta))$ coincide.*
(2) *The common axes of the conics $\mathcal{K}(P, \Delta, s),\ s \in R$, and of the conic $\mathcal{C}(f^{-1}(P), f^{-1}(\Delta))$ are the orthogonal Droz-Farny -lines through P, with regard to $\Delta = A_1A_2A_3$.*

Proof. (1) Because of Lemma 1 and 2, both conics have center P and are circumscribed about the complementary triangle $f^{-1}(\Delta)$ of $A_1A_2A_3$.

(2) For the conic with center P, circumscribed about the anticomplementary triangle of $A_1A_2A_3$, it is clear that (PA_i, line through P, parallel to a_i), $i = 1, 2, 3$, are conjugate diameters. And the result follows from section 1. □

5. Examples

5.1. If $P = H$, the orthocenter of $\Delta = A_1A_2A_3$, which is also the circumcenter of its anticomplementary triangle Δ^{-1}, the conics $\mathcal{K}(H, \Delta, s)$ are circles with center H, since any two orthogonal lines through H are axes of these conics. In particular, $\mathcal{K}(H, \Delta, S)$, where $S = 2F(\frac{\cos A_1}{l_1} + \frac{\cos A_2}{l_2} + \frac{\cos A_3}{l_3})(\frac{l_1}{\cos A_1} + \frac{l_2}{\cos A_2} + \frac{l_3}{\cos A_3})$, is the circumcircle of Δ^{-1} and it is the locus of the points for which the distances d_1, d_2, d_3 to the sides of Δ are related by

$$\begin{aligned}&(l_1 \cos A_1)d_1^2 + (l_2 \cos A_2)d_2^2 + (l_3 \cos A_3)d_3^2\\ =&4F^2(\frac{\cos A_1}{l_1} + \frac{\cos A_2}{l_2} + \frac{\cos A_3}{l_3})\\ =&2F^2\frac{l_1^2 + l_2^2 + l_3^2}{l_1l_2l_3},\end{aligned}$$

or equivalently,

$$l_1^2(l_2^2 + l_3^2 - l_1^2)d_1^2 + l_2^2(l_3^2 + l_1^2 - l_2^2)d_2^2 + l_3^2(l_1^2 + l_2^2 - l_3^2)d_3^2 = 4F^2(l_1^2 + l_2^2 + l_3^2).$$

Moreover, $\mathcal{C}(f^{-1}(H), \Delta^{-1})$ is also the circumcirle of Δ^{-1}, which is easily seen from the fact that this circumcircle is the locus of the points for which the feet of the perpendiculars to the sides of Δ^{-1} are collinear. Remark that $f^{-1}(H)$, the orthocenter of Δ^{-1}, is the de Longchamps point $X(20)$ of Δ.

5.2. If $P = K(l_1, l_2, l_3)$, the Lemoine point of $\Delta = A_1A_2A_3$, then $\mathcal{K}(K, \Delta, S)$, with

$$\begin{aligned}S =&2F(\frac{1}{l_1^2} + \frac{1}{l_2^2} + \frac{1}{l_3^2})(l_1^2 + l_2^2 + l_3^2)\\ =&2F^2\frac{(l_2^2l_3^2 + l_3^2l_1^2 + l_1^2l_2^2)(l_1^2 + l_2^2 + l_3^2)}{l_1^2l_2^2l_3^2},\end{aligned}$$

is the locus of the points for which the distances d_1, d_2, d_3 to the sides of Δ are related by $d_1^2 + d_2^2 + d_3^2 = 4F^2(\frac{1}{l_1^2} + \frac{1}{l_2^2} + \frac{1}{l_3^2})$, and it is the ellipse with center K, circumscribed about Δ^{-1}. Moreover, the locus $\mathcal{C}(f^{-1}(K), \Delta^{-1})$, where $f^{-1}(K)$ is the Lemoine point of Δ^{-1} (or $X(69)$ with coordinates

$$(l_2l_3(l_2^2 + l_3^2 - l_1^2), l_3l_1(l_3^2 + l_1^2 - l_2^2), l_1l_2(l_1^2 + l_2^2 - l_3^2)),$$

is the same ellipse. The axes of this ellipse are the orthogonal Droz-Farny lines through K with respect to Δ.

5.3. If $P = G(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3})$, the centroid of $\Delta = A_1A_2A_3$, then $\mathcal{K}(G, \Delta, S)$, with $S = 18F$, is the locus of the points for which the distances d_1, d_2, d_3 to the sides of Δ are related by $l_1^2d_1^2 + l_2^2d_2^2 + l_3^2d_3^2 = 12F^2$, and it is the ellipse with center G, circumscribed about Δ^{-1}, i.e., it is the Steiner ellipse of Δ^{-1}, since G is also the centroid of Δ^{-1}. The locus $\mathcal{C}(G, \Delta^{-1})$ is also this Steiner ellipse and its axes are the orthogonal Droz-Farny lines through G with respect to Δ.

5.4. If $P = I(1, 1, 1)$, the incenter of $\Delta = A_1A_2A_3$, then $\mathcal{K}(I, \Delta, S)$, with $S = 2F(l_1 + l_2 + l_3)(\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3})$, is the locus of the points for which the distances d_1, d_2, d_3 to the sides of Δ are related by $l_1d_1^2 + l_2d_2^2 + l_3d_3^2 = 4F^2(\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3})$, and it is the ellipse with center I, circumscribed about Δ^{-1}. Moreover the locus $\mathcal{C}(f^{-1}(I), \Delta^{-1})$, where $f^{-1}(I)$ is the incenter of Δ^{-1} (which is center $X(8)$ of Δ, the Nagel point with coordinates $(\frac{l_2+l_3-l_1}{l_1}, \frac{l_3+l_1-l_2}{l_2}, \frac{l_1+l_2-l_3}{l_3}))$ is the same ellipse. The axes of this ellipse are the orthogonal Droz-Farny lines through I with respect to Δ.

References

[1] J.-L. Ayme, A purely synthetic proof of the Droz-Farny line theorem, *Forum Geom.*, 4 (2004), 219–224.

[2] C.J. Bradley and J.T. Bradley, Countless Simson line configurations, *Math. Gazette*, 80 (1996) no. 488, 314–321.

[3] J.-P. Ehrmann and F.M. van Lamoen, A projective generalization of the Droz-Farny line theorem, *Forum Geom.*, 4 (2004), 225–227.

[4] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[5] C. Thas, On ellipses with center the Lemoine point and generalizations, *Nieuw Archief voor Wiskunde*, ser.4, 11 (1993), nr. 1, 1–7.

Charles Thas: Department of Pure Mathematics and Computer Algebra, Krijgslaan 281, S22, B-9000 Gent, Belgium

E-mail address: charles.thas@UGent.be

Forum Geometricorum
Volume 6 (2006) 241–246.

FORUM GEOM
ISSN 1534-1178

On Butterflies Inscribed in a Quadrilateral

Zvonko Čerin

Abstract. We explore a configuration consisting of two quadrilaterals which share the intersection of diagonals. We prove results analogous to the Sidney Kung's Butterfly Theorem for Quadrilaterals in [2].

1. The extended butterfly theorem for quadrilaterals

In this note we consider some properties of pairs of quadrilaterals $ABCD$ and $A'B'C'D'$ with A', B', C' and D' on lines AB, BC, CD, and DA respectively.

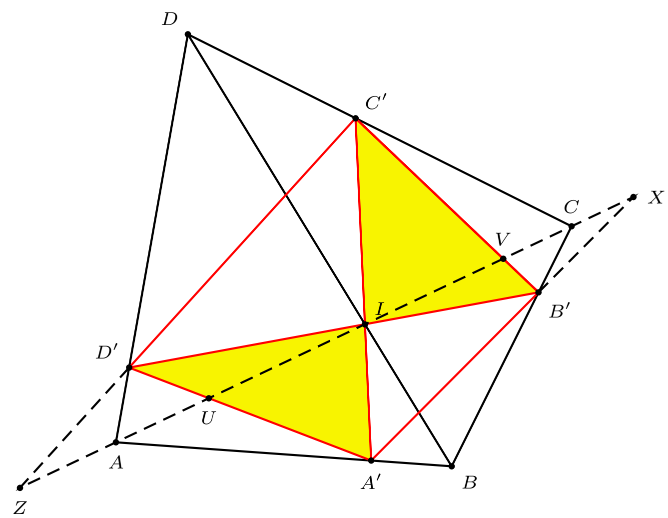


Figure 1

The segments AC, $A'C'$ and $B'D'$ are analogous to the three chords from the classical butterfly theorem (see [1] for an extensive overview of its many proofs and generalizations).

When the intersection of the lines $A'C'$ and $B'D'$ is the intersection I of the lines AC and BD, *i.e.*, when $ABCD$ and $A'B'C'D'$ share the same intersection of diagonals, in [2] the following equality, known as the Butterfly Theorem for Quadrilaterals, was established:

$$\frac{|AU|}{|UI|} \cdot \frac{|IV|}{|VC|} = \frac{|AI|}{|IC|}, \tag{1}$$

Publication Date: October 9, 2006. Communicating Editor: Paul Yiu.

where U and V are intersections of the line AC with the lines $D'A'$ and $B'C'$ (see Figure 1).

For the intersections $X = AC \cap A'B'$ and $Z = AC \cap C'D'$, in this situation, we have similar relations

$$\frac{|XA|}{|AI|} \cdot \frac{|IC|}{|CZ|} = \frac{|XI|}{|IZ|},$$

and

$$\frac{|XU|}{|UI|} \cdot \frac{|IV|}{|VZ|} = \frac{|XI|}{|IZ|}.$$

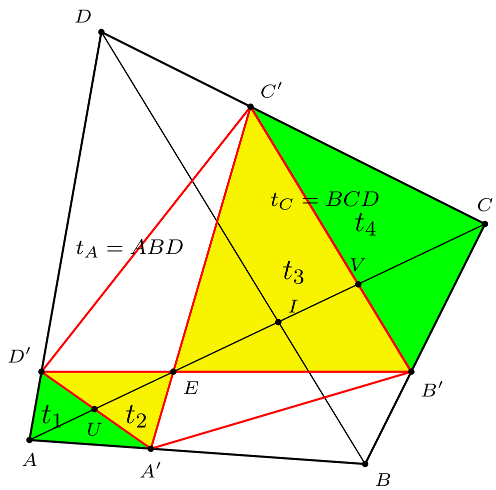


Figure 2

Our first result is the observation that (1) holds when the diagonals of $A'B'C'D'$ intersect at a point E on the diagonal AC of $ABCD$, not necessarily the intersection $I = AC \cap BD$ (see Figure 2).

Theorem 1. *Let $A'B'C'D'$ be an inscribed quadrilateral of $ABCD$, and $E = A'C' \cap B'C'$, $I = AC \cap BD$, $U = AC \cap D'A'$, $V = AC \cap B'C'$. If E lies on the line AC, then*

$$\frac{|AU|}{|UE|} \cdot \frac{|EV|}{|VC|} = \frac{|AI|}{|IC|}. \qquad (2)$$

Proof. We shall use analytic geometry of the plane. Without loss of generality we can assume that $A(0,0)$, $B(f,g)$, $C(1,0)$ and $D(p,q)$ for some real numbers f, g, p and q. The points $A'\left(\frac{fu}{u+1}, \frac{gu}{u+1}\right)$ and $B'\left(\frac{f+v}{v+1}, \frac{g}{v+1}\right)$ divide segments AB and BC in ratios u and v, which are real numbers different from -1. Let the rectangular coordinates of the point E be $(h,\ k)$. Then the vertices C' and D' are intersections of the lines CD and DA with the lines $A'E$ and $B'E$, respectively. Their coordinates are a bit more complicated. Next, we determine the points U and

V as intersections of the line AC with the lines $A'D'$ and $B'C'$ and with a small help from Maple V find that the difference

$$\begin{aligned}\mathcal{D} :=& \frac{|AU|^2}{|UE|^2} \cdot \frac{|EV|^2}{|VC|^2} - \frac{|AI|^2}{|IC|^2} \\ =& \frac{(fq-gp)^2 \cdot k \cdot P_5(h,k)}{Q_4(h,k) \cdot (g+fq-q-gp)^2 \cdot (guh+(u+1-fu)k-gu)^2},\end{aligned}$$

where $P_5(h,k)$ and $Q_4(h,k)$ are polynomials of degrees 5 and 4 respectively in variables h and k. Both are somewhat impractical to write down explicitly. However, since k is a factor in the numerator we see that when $k=0$, *i.e.*, when the point E is on the line AC, the difference $\mathcal{D}$ is zero so that the extended Butterfly Theorem for Quadrilaterals holds. □

Remark. There is a version of the above theorem where the points U and V are intersections $U = AE \cap A'D'$ and $V = CE \cap B'C'$. The difference $\mathcal{D}$ in this case is the quotient

$$\frac{-(fq-gp)^2 \cdot k \cdot P_1(h,k) \cdot P_2(h,k)}{(qh-pk)^2 \cdot (g+fq-q-gp)^2 \cdot (guh+(u+1-fu)k-gu)^2},$$

where

$$\begin{aligned}P_1(h,k) =& (u(q-g)+q(1+uv))h + (uv(1-p)+u(f-p)-p)k \\ & - u(qv-gp+fq), \\ P_2(h,k) =& 2qguh^2 + (gu-2ugp-2fqu+uq-vuq+q)\,hk - \\ & (p+uv+up+uf-pvu-2puf)\,k^2 - 2guqh + u\,(qv+gp+fq)\,k.\end{aligned}$$

Note that these are linear and quadratic polynomials in h and k. In other words, the extended Butterfly Theorem for Quadrilaterals holds not only for points E on the line AC but also when the point E is on a line through D (with the equation $P_1(h,k) = 0$) and on a conic through A, C and D (with the equation $P_2(h,k) = 0$).

Moreover, in this case we can easily prove the following converse of Theorem 1.

If the relation (2) holds when the points A' and B' divide segments AB and BC both in the ratio $1:3$, $1:1$ or $3:1$, then the point E lies on the line AC.

Indeed, if we substitute for $u = v = \frac{1}{3}$, 1 or 3 both into P_1 and P_2 we get three equations whose only common solutions in h and k are coordinates of points A, C, and D (which are definitely excluded as possible solutions).

2. A relation involving areas

Let us introduce shorter notation for six triangles in this configuration: $t_1 = D'AA'$, $t_2 = D'A'E$, $t_3 = EB'C'$, $t_4 = CC'B'$, $t_A = ABD$ and $t_C = CDB$ (see Figure 2).

Our next result shows that the above relationship also holds for areas of these triangles.

Theorem 2. (a) *If E lies on the line AC, then*

$$\frac{\mathtt{area}(t_1)}{\mathtt{area}(t_2)} \cdot \frac{\mathtt{area}(t_3)}{\mathtt{area}(t_4)} = \frac{\mathtt{area}(t_A)}{\mathtt{area}(t_C)}. \tag{3}$$

(b) *If the relation (3) holds when the points A' and B' divide AB and BC both in the ratio $1:2$ or $2:1$, then the point E lies on the line AC.*

Proof. If we keep the same assumptions and notation from the proof of Theorem 1, then

$$\frac{\mathtt{area}(t_1)}{\mathtt{area}(t_2)} \cdot \frac{\mathtt{area}(t_3)}{\mathtt{area}(t_4)} - \frac{\mathtt{area}(t_A)}{\mathtt{area}(t_C)}$$
$$= \frac{(fq-gp)\cdot k \cdot P_1(h,k)}{(qh-pk)\cdot(g+fq-q-gp)\cdot(guh+(u+1-fu)k-gu)}.$$

Hence, (a) is clearly true because $k=0$ when the point E is on the line AC.

On the other hand, for (b), when $u=\frac{1}{2}$ and $v=\frac{1}{2}$ then

$$\mathcal{E}_1 = 4\cdot P_1(h,k) = (2g-7q)\,h + (7p-2f-1)\,k + q - 2gp + 2fq,$$

while for $u=2$ and $v=2$ then

$$\mathcal{E}_2 = P_1(h,k) = (2g-7q)\,h + (7p-2f-4)\,k + 2fq + 4q - 2gp.$$

The only solution of the system

$$\mathcal{E}_1 = 0, \qquad \mathcal{E}_2 = 0$$

is $(h,h)=(p,q)$, *i.e.*, $E=D$. However, for this solution the triangle t_2 degenerates to a segment so that its area is zero which is unacceptable. □

3. Other relations

Note that the above theorem holds also for (lengths of) the altitudes $h(A,t_1)$, $h(E,t_2)$, $h(E,t_3)$, $h(C,t_4)$, $h(A,t_A)$ and $h(C,t_C)$ because (for example) $\mathtt{area}(t_1) = \frac{1}{2}h(A,t_1)\cdot|D'A'|$.

Let $G(t)$ denote the centroid of the triangle $t=ABC$, and $\varepsilon(A,t)$ the distance of $G(t)$ from the side BC opposite to the vertex A. Since $\varepsilon(A,t) = \frac{2\mathtt{area}(t)}{3|BC|}$ there is a version of the above theorem for the distances $\varepsilon(A,t_1)$, $\varepsilon(E,t_2)$, $\varepsilon(E,t_3)$, $\varepsilon(C,t_4)$, $\varepsilon(A,t_A)$ and $\varepsilon(C,t_C)$.

For a triangle t let $R(t)$ denote the radius of its circumcircle. The following theorem shows that the radii of circumcircles of the six triangles satisfy the same pattern without any restrictions on the point E.

Theorem 3. $\frac{R(t_1)}{R(t_2)} \cdot \frac{R(t_3)}{R(t_4)} = \frac{R(t_A)}{R(t_C)}$.

Proof. Let us keep again the same assumptions and notation from the proof of Theorem 1. Since $R(t) = \frac{\text{product of side lengths}}{4\mathtt{area}(t)}$, we see that the square of the circumradius of a triangle with vertices in the points (x,a), (y,b) and (z,c) is given by

$$\frac{[(y-z)^2+(b-c)^2]\cdot[(z-x)^2+(c-a)^2]\cdot[(x-y)^2+(a-b)^2]}{4(x(b-c)+y(c-a)+z(a-b))^2}.$$

Applying this formula we find $R(t_1)^2$, $R(t_2)^2$, $R(t_3)^2$, $R(t_4)^2$, $R(t_A)^2$ and $R(t_C)^2$. In a few seconds Maple V verifies that the difference $\frac{R(t_1)^2}{R(t_2)^2} \cdot \frac{R(t_3)^2}{R(t_4)^2} - \frac{R(t_A)^2}{R(t_C)^2}$ is equal to zero. $\square$

For a triangle t let $H(t)$ denote its orthocenter. In the next result we look at the distances of a particular vertex of the six triangles from its orthocenter. Again the pattern is independent from the position of the point E.

Theorem 4. $\frac{|AH(t_1)|}{|EH(t_2)|} \cdot \frac{|EH(t_3)|}{|CH(t_4)|} = \frac{|AH(t_A)|}{|CH(t_C)|}$.

Proof. This time one can see that $|AH(t)|^2$ for the triangle $t = ABC$ with the vertices in the points (x, a), (y, b) and (z, c) is given as

$$\frac{[(y-z)^2 + (b-c)^2] \cdot [x^2 + a^2 + yz - x(y+z) + bc - a(b+c)]^2}{(x(b-c) + y(c-a) + z(a-b))^2}.$$

Applying this formula we find $|AH(t_1)|^2$, $|EH(t_2)|^2$, $|EH(t_3)|^2$, $|CH(t_4)|^2$, $|AH(t_A)|^2$, and $|CH(t_C)|^2$. In a few seconds Maple V verifies that the difference $\frac{|AH(t_1)|^2}{|EH(t_2)|^2} \cdot \frac{|EH(t_3)|^2}{|CH(t_4)|^2} - \frac{|AH(t_A)|^2}{|CH(t_C)|^2}$ is equal to zero. $\square$

Let $O(t)$ be the circumcenter of the triangle $t = ABC$. Let $\delta(A, t)$ denote the distance of $O(t)$ from the side BC opposite to the vertex A. Since $|AH(t)| = 2\delta(A, t)$ there is a version of the above theorem for the distances $\delta(A, t_1)$, $\delta(E, t_2)$, $\delta(E, t_3)$, $\delta(C, t_4)$, $\delta(A, t_A)$ and $\delta(C, t_C)$.

References

[1] L. Bankoff, The Metamorphosis of the Butterfly Problem, *Math. Mag.*, 60 (1987) 195–210.
[2] S. Kung, A Butterfly Theorem for Quadrilaterals, *Math. Mag.*, 78 (2005) 314–316.

Kopernikova 7, 10020 Zagreb, Hrvatska, Europa
E-mail address: cerin@math.hr

Forum Geometricorum
Volume 6 (2006) 247–253.

FORUM GEOM
ISSN 1534-1178

On Triangles with Vertices on the Angle Bisectors

Eric Danneels

Abstract. We study interesting properties of triangles whose vertices are on the three angle bisectors of a given triangle. We show that such a triangle is perspective with the medial triangle if and only if it is perspective with the intouch triangle. We present several interesting examples with new triangle centers.

1. Introduction

Let ABC be a given triangle with incenter I. By an I-triangle we mean a triangle UVW whose vertices U, V, W are on the angle bisectors AI, BI, CI respectively. Such triangles are clearly perspective with ABC at the incenter I.

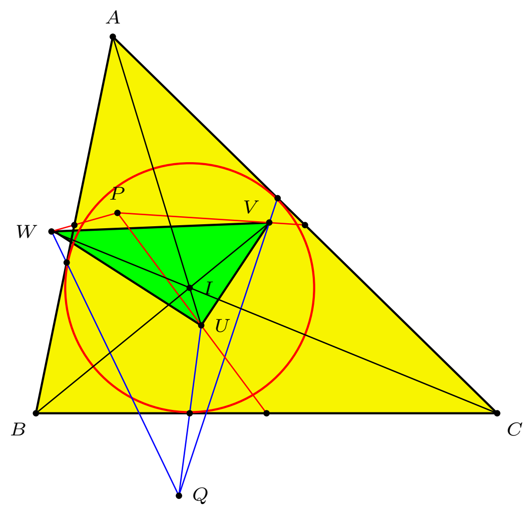


Figure 1.

Theorem 1. *An I-triangle is perspective with the medial triangle if and only if it is perspective with the intouch triangle.*

Proof. The homogeneous barycentric coordinates of the vertices of an I-triangle can be taken as

$$U = (u : b : c), \qquad V = (a : v : c), \qquad W = (a : b : w) \tag{1}$$

for some u, v, w. In each case, the condition for perspectivity is

$$F(u, v, w) := (b-c)vw+(c-a)wu+(a-b)uv+(a-b)(b-c)(c-a) = 0. \tag{2}$$

□

Publication Date: October 16, 2006. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this paper.

Let D, E, F be the midpoints of the sides BC, CA, AB of triangle ABC. If $P = (x : y : z)$ is the perspector of an I-triangle UVW with the medial triangle, then U is the intersection of the line DP with the bisector IA. It has coordinates

$$((b-c)x : b(y-z) : c(y-z)).$$

Similarly, the coordinates of V and W can be determined. The triangle UVW is perspective with the intouch triangle at

$$Q = \left(\frac{x(y+z-x)}{s-a} : \frac{y(z+x-y)}{s-b} : \frac{z(x+y-z)}{s-c}\right).$$

Conversely, if an I-triangle is perspective with the intouch triangle at $Q = (x : y : z)$, then it is perspective with the medial triangle at

$$P = ((s-a)x((s-b)y + (s-c)z - (s-a)x) : \cdots : \cdots).$$

Theorem 2. *Let UVW be an I-triangle perspective with the medial and the intouch triangles. If U_1, U_1, W_1 are the inversive images of U, V, W in the incircle, then $U_1V_1W_1$ is also an I-triangle perspective with the medial and intouch triangles.*

Proof. If the coordinates of U, V, W are as given in (1), then

$$U_1 = (u_1 : b : c), \qquad V_1 = (a : v_1 : c), \qquad W_1 = (a : b : w_1),$$

where

$$\begin{aligned} u_1 =& \frac{(a(b+c)-(b-c)^2)u - 2(s-a)(b-c)^2}{2(s-a)u - a(b+c) + (b-c)^2}, \\ v_1 =& \frac{(b(c+a)-(c-a)^2)v - 2(s-b)(c-a)^2}{2(s-b)v - b(c+a) + (c-a)^2}, \\ w_1 =& \frac{(c(a+b)-(a-b)^2)w - 2(s-c)(a-b)^2}{2(s-c)w - c(a+b) + (a-b)^2}. \end{aligned}$$

From these,

$$F(u_1, v_1, w_1) = \frac{64abc(s-a)(s-b)(s-c)}{\prod_{\text{cyclic}}(2(s-a)u - a(b+c) + (b-c)^2)} \cdot F(u, v, w) = 0.$$

It follows from (2) that $U_1V_1W_1$ is perspective to both the medial and the intouch triangles. $\square$

If an I-triangle UVW is perspective with the medial triangle at $(x : y : z)$, then $U_1V_1W_1$ is perspective with
(i) the medial triangle at

$$((y+z-x)((a(b+c)-(b-c)^2)x - (b+c-a)(b-c)(y-z)) : \cdots : \cdots),$$

(ii) the intouch triangle at

$$(a((a(b+c)-(b-c)^2)x - (b+c-a)(b-c)(y-z)) : \cdots : \cdots).$$

Theorem 3. *Let UVW be an I-triangle perspective with the medial and the intouch triangles. If U_2 (respectively V_2 and W_2) is the inversive image of U (respectively V and W) in the A- (respectively B- and C-) excircle, then $U_2V_2W_2$ is also an I-triangle perspective with the medial and intouch triangles.*

If an I-triangle UVW is perspective with the medial triangle at $(x : y : z)$, then $U_2V_2W_2$ is perspective with
(i) the medial triangle at

$$\big((s-a)(y+z-x)((a(b+c)+(b-c)^2)x+2s(b-c)(y-z)) : \cdots : \cdots\big),$$

(ii) the intouch triangle at

$$\left(\frac{a}{s-a}((a(b+c)+(b-c)^2)x+2s(b-c)(y-z)) : \cdots : \cdots\right).$$

2. Some interesting examples

We present some interesting examples of I-triangles perspective with both the medial and intouch triangles. The perspectors in these examples are new triangle centers not in the current edition of [1].

2.1. Let X_a, X_b, X_c be the inversive images of the excenters I_a, I_b, I_c in the incircle. We have $IX_a = \frac{r^2}{II_a}$ and $II_a = AI_a - AI = AI \cdot \left(\frac{s}{s-a} - 1\right) = \frac{a \cdot AI}{s-a}$. Hence,

$$\frac{AI}{IX_a} = \frac{a \cdot AI^2}{r^2(s-a)} = \frac{a}{sin^2(\frac{A}{2})} = \frac{abc}{(s-a)(s-b)(s-c)} = \frac{4R}{r},$$

and by symmetry $\frac{AI}{IX_a} = \frac{BI}{IX_b} = \frac{CI}{IX_c} = \frac{4R}{r}$. Therefore, triangles ABC and $X_aX_bX_c$ are homothetic with ratio $4R : -r$.

$$\begin{aligned}
X_a =&(a^2+b^2+c^2-2ab-2bc-2ca)(a,b,c)\\
&+(b+c-a)(c+a-b)(a+b-c)(1,0,0),\\
X_b =&(a^2+b^2+c^2-2ab-2bc-2ca)(a,b,c)\\
&+(b+c-a)(c+a-b)(a+b-c)(0,1,0),\\
X_c =&(a^2+b^2+c^2-2ab-2bc-2ca)(a,b,c)\\
&+(b+c-a)(c+a-b)(a+b-c)(0,0,1).
\end{aligned}$$

Proposition 4. *$X_aX_bX_c$ is an I-triangle perspective with*
(i) the medial triangle at

$$P_x = (a^2(b+c)+(b+c-2a)(b-c)^2 : \cdots : \cdots),$$

(ii) the intouch triangle at

$$Q_x = (a(b+c-a)(a^2(b+c)+(b+c-2a)(b-c)^2) : \cdots : \cdots).$$

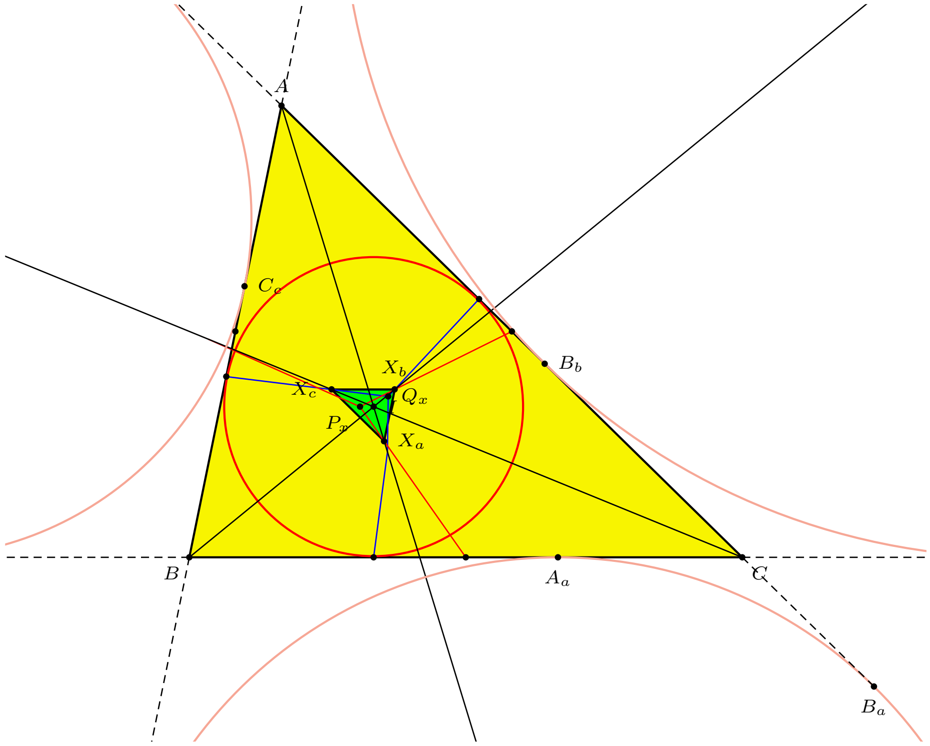


Figure 2.

2.2. Let Y_a, Y_b, Y_c be the inversive images of the incenter I with respect to the A-, B-, C-excircles.

$$\begin{aligned}
Y_a =&(a^2+b^2+c^2-2bc+2ca+2ab)(-a,b,c)\\
&+(a+b+c)(c+a-b)(a+b-c)(1,0,0),\\
Y_b =&(a^2+b^2+c^2+2bc-2ca+2ab)(a,-b,c)\\
&+(a+b+c)(a+b-c)(b+c-a)(0,1,0),\\
Y_c =&(a^2+b^2+c^2+2bc+2ca-2ab)(a,b,-c)\\
&+(a+b+c)(b+c-a)(c+a-b)(1,0,0).
\end{aligned}$$

Proposition 5. *$Y_aY_bY_c$ is an I-triangle perspective with*
(i) the medial triangle at

$$P_y = ((b+c-a)^2(a^2(b+c)+(2a+b+c)(b-c)^2) : \cdots : \cdots),$$

(ii) the intouch triangle at

$$Q_y = \left(\frac{a(a^2(b+c)+(2a+b+c)(b-c)^2)}{b+c-a} : \cdots : \cdots\right).$$

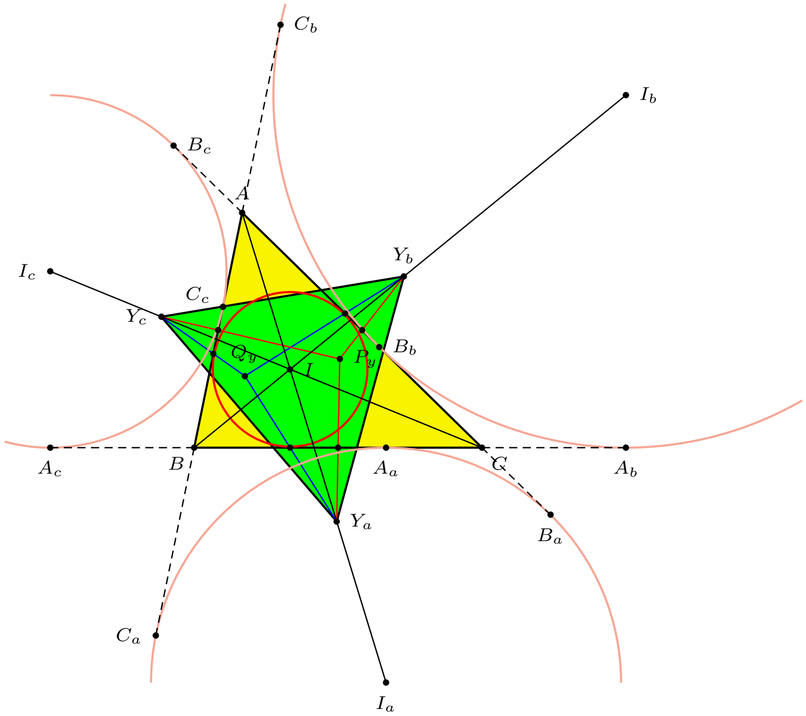


Figure 3.

2.3. Let V_a, V_b, V_c be the inversive images of X_a, X_b, X_c with respect to the A-, B-, C-excircles.

$$\begin{aligned}
V_a =&(3a^2 + (b-c)^2)(-a, b, c) + 2a(c+a-b)(a+b-c)(1, 0, 0),\\
V_b =&(3b^2 + (c-a)^2)(a, -b, c) + 2b(a+b-c)(b+c-a)(0, 1, 0),\\
V_c =&(3c^2 + (a-b)^2)(a, b, -c) + 2c(b+c-a)(c+a-b)(0, 0, 1).
\end{aligned}$$

Proposition 6. *$V_aV_bV_c$ is an I-triangle perspective with*
(i) the medial triangle at

$$P_v = (a(b+c-a)^3(a^2 + 3(b-c)^2) : \cdots : \cdots),$$

(ii) the intouch triangle at

$$Q_v = \left(\frac{a(a^2 + 3(b-c)^2)}{b+c-a} : \cdots : \cdots\right).$$

2.4. Let W_a, W_b, W_c be the inversive images of Y_a, Y_b, Y_c with respect to the incircle.

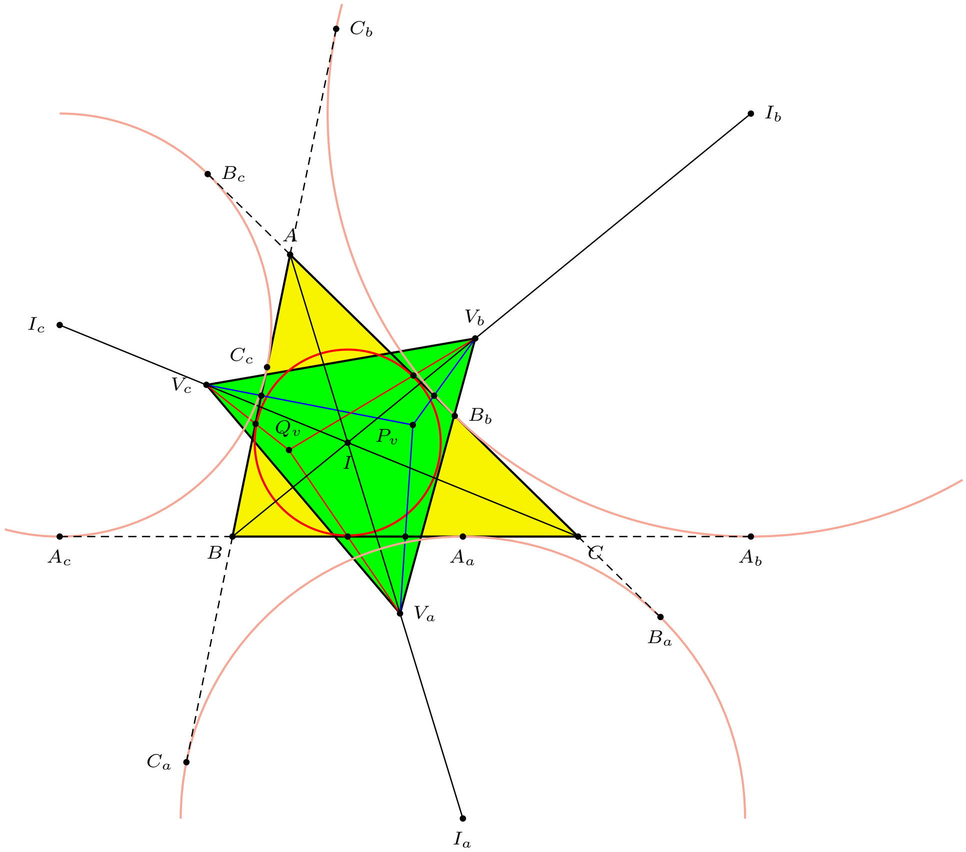


Figure 4.

$$
\begin{aligned}
W_a =&(3a^2 + (b-c)^2)(a,b,c) - 2a(c+a-b)(a+b-c)(1,0,0),\\
W_b =&(3b^2 + (c-a)^2)(a,b,c) - 2b(a+b-c)(b+c-a)(0,1,0),\\
W_c =&(3c^2 + (a-b)^2)(a,b,c) - 2c(b+c-a)(c+a-b)(0,0,1).
\end{aligned}
$$

Proposition 7. *$W_aW_bW_c$ is an I-triangle perspective with*
(i) the medial triangle at

$$P_w = (a(a^2 + 3(b-c)^2) : \cdots : \cdots),$$

(ii) the intouch triangle at

$$Q_w = (a(b+c-a)^2(a^2 + 3(b-c)^2) : \cdots : \cdots).$$

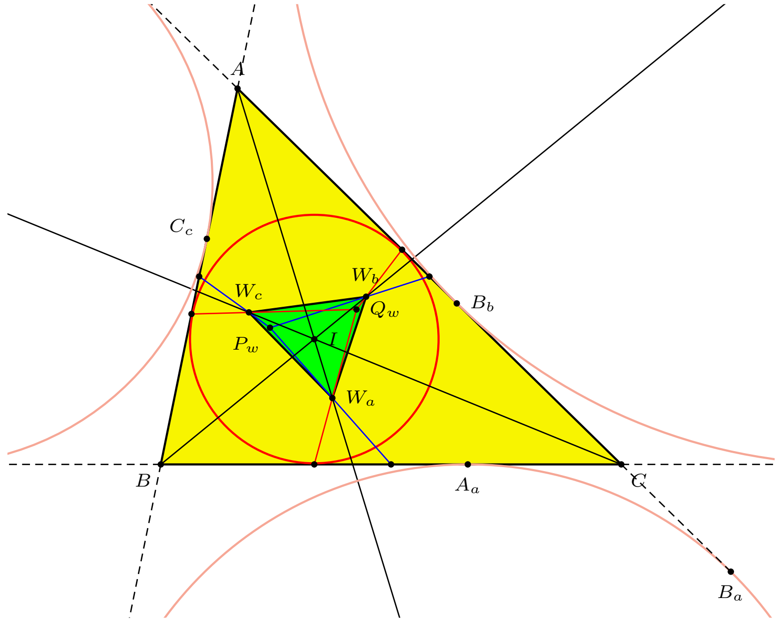


Figure 5.

Reference [1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Eric Danneels: Hubert d'Ydewallestraat 26, 8730 Beernem, Belgium
E-mail address: `eric.danneels@telenet.be`

Forum Geometricorum
Volume 6 (2006) 255–262.

FORUM GEOM
ISSN 1534-1178

Formulas Among Diagonals in the Regular Polygon and the Catalan Numbers

Matthew Hudelson

Abstract. We look at relationships among the lengths of diagonals in the regular polygon. Specifically, we derive formulas for all diagonals in terms of the shortest diagonals and other formulas in terms of the next-to-shortest diagonals, assuming unit side length. These formulas are independent of the number of sides of the regular polygon. We also show that the formulas in terms of the shortest diagonals involve the famous Catalan numbers.

1. Motivation

In [1], Fontaine and Hurley develop formulas that relate the diagonal lengths of a regular n-gon. Specifically, given a regular convex n-gon whose vertices are $P_0, P_1, \ldots, P_{n-1}$, define d_k as the distance between P_0 and P_k. Then the law of sines yields

$$\frac{d_k}{d_j} = \frac{\sin \frac{k\pi}{n}}{\sin \frac{j\pi}{n}}.$$

Defining

$$r_k = \frac{\sin \frac{k\pi}{n}}{\sin \frac{\pi}{n}},$$

the formulas given in [1] are

$$r_h r_k = \sum_{i=1}^{\min\{k,h,n-k,n-h\}} r_{|k-h|+2i-1}$$

and

$$\frac{1}{r_k} = \sum_{j=1}^{s} r_{k(2j-1)}$$

where $s = \min\{j > 0 : jk \equiv \pm 1 \text{ mod } n\}$.

Notice that for $1 \le k \le n-1$, $r_k = \frac{d_k}{d_1}$, but there is no *a priori* restriction on k in the definition of r_k. Thus, it would make perfect sense to consider $r_0 = 0$ and $r_{-k} = -r_k$ not to mention r_k for non-integer values of k as well. Also, the only restriction on n in the definition of r_n is that n not be zero or the reciprocal of an integer.

Publication Date: October 23, 2006. Communicating Editor: Paul Yiu.

2. Short proofs of r_k formulas

Using the identity $\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$, we can provide some short proofs of formulas equivalent, and perhaps simpler, to those in [1]:

Proposition 1. *For integers h and k,*

$$r_h r_k = \sum_{j=0}^{k-1} r_{h-k+2j+1}.$$

Proof. Letting h and k be integers, we have

$$\begin{aligned}
r_h r_k &= (\sin\frac{\pi}{n})^{-2} \sin\frac{h\pi}{n} \sin\frac{k\pi}{n} \\
&= (\sin\frac{\pi}{n})^{-2} \frac{1}{2}\left(\cos\frac{(h-k)\pi}{n} - \cos\frac{(h+k)\pi}{n}\right) \\
&= (\sin\frac{\pi}{n})^{-2}\left(\sum_{j=0}^{k-1}\frac{1}{2}\left(\cos\frac{(h-k+2j)\pi}{n} - \cos\frac{(h-k+2j+2)\pi}{n}\right)\right) \\
&= (\sin\frac{\pi}{n})^{-2}\sum_{j=0}^{k-1}\left(\sin\frac{(h-k+2j+1)\pi}{n}\sin\frac{\pi}{n}\right) \\
&= \sum_{j=0}^{k-1} r_{h-k+2j+1}.
\end{aligned}$$

The third equality holds since the sum telescopes. □

Note that we can switch the roles of h and k to arrive at the formula

$$r_k r_h = \sum_{j=0}^{h-1} r_{k-h+2j+1}.$$

To illustrate that this is not contradictory, consider the example when $k = 2$ and $h = 5$. From Proposition 1, we have

$$r_5 r_2 = r_4 + r_6.$$

Reversing the roles of h and k, we have

$$r_2 r_5 = r_{-2} + r_0 + r_2 + r_4 + r_6.$$

Recalling that $r_0 = 0$ and $r_{-j} = r_j$, we see that these two sums are in fact equal.

The reciprocal formula in [1] is proven almost as easily:

Proposition 2. *Given an integer k relatively prime to n,*

$$\frac{1}{r_k} = \sum_{j=1}^{s} r_{k(2j-1)}$$

where s is any (not necessarily the smallest) positive integer such that $ks \equiv \pm 1$ mod n.

Proof. Starting at the right-hand side, we have

$$\begin{aligned}
\sum_{j=1}^{s} r_{k(2j-1)} &= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\sum_{j=1}^{s}\sin\frac{k(2j-1)\pi}{n}\sin\frac{k\pi}{n}\\
&= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\sum_{j=1}^{s}\frac{1}{2}\left(\cos\frac{k(2j-2)\pi}{n}-\cos\frac{2jk\pi}{n}\right)\\
&= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\frac{1}{2}\left(\cos 0-\cos\frac{2sk\pi}{n}\right)\\
&= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\sin^2\frac{sk\pi}{n}\\
&= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\sin^2\frac{\pi}{n}\\
&= \left(\sin\frac{k\pi}{n}\right)^{-1}\sin\frac{\pi}{n}\\
&= \frac{1}{r_k}.
\end{aligned}$$

Here, the third equality follows from the telescoping sum, and the fifth follows from the definition of s. $\square$

3. From powers of r_2 to Catalan numbers

We use the special case of Proposition 1 when $h = 2$, namely

$$r_k r_2 = r_{k-1} + r_{k+1},$$

to develop formulas for powers of r_2.

Proposition 3. *For nonnegative integers m,*

$$r_2{}^m = \sum_{i=0}^{m}\binom{m}{i} r_{1-m+2i}.$$

Proof. We proceed by induction on m. When we have $m = 0$, the formula reduces to $r_2{}^0 = r_1$ and both sides equal 1. This establishes the basis step. For the inductive step, we assume the result for $m = n$ and begin with the sum on the right-hand side for $m = n + 1$.

$$
\begin{aligned}
\sum_{i=0}^{n+1}\binom{n+1}{i} r_{-n+2i} &= \sum_{i=0}^{n+1}\left(\binom{n}{i}+\binom{n}{i-1}\right) r_{-n+2i} \\
&= \left(\sum_{i=0}^{n+1}\binom{n}{i-1} r_{-n+2i}\right) + \left(\sum_{i=0}^{n+1}\binom{n}{i} r_{-n+2i}\right) \\
&= \left(\sum_{j=0}^{n}\binom{n}{j} r_{2-n+2j}\right) + \left(\sum_{i=0}^{n}\binom{n}{i} r_{-n+2i}\right) \\
&= \sum_{i=0}^{n}\binom{n}{i} (r_{-n+2i}+r_{2-n+2i}) \\
&= \sum_{i=0}^{n}\binom{n}{i} r_2 r_{1-n+2i} \\
&= r_2 {r_2}^n \\
&= {r_2}^{n+1}
\end{aligned}
$$

which completes the induction. The first equality uses the standard identity for binomial coefficients $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$.

The third equality is by means of the change of index $j = i-1$ and the fact that $\binom{n}{c} = 0$ if $c < 0$ or $c > n$.

The fifth equality is from Proposition 1 and the sixth is from the induction hypothesis. □

Now, we use Proposition 3 and the identity $r_{-k} = -r_k$ to consider an expression for ${r_2}^m$ as a linear combination of r_k's where $k > 0$, i.e., we wish to determine the coefficients α_k in the sum

$$
{r_2}^m = \sum_{k=1}^{m+1} \alpha_{m,k} r_k.
$$

From Proposition 2, the sum is known to end at $k = m+1$. In fact, we can determine α_k directly. One contribution occurs when $k = 1-m+2i$, or $i = \frac{1}{2}(m+k-1)$. A second contribution occurs when $-k = 1-m+2i$, or $i = \frac{1}{2}(m-k-1)$. Notice that if m and k have the same parity, there is no contribution to $\alpha_{m,k}$. Piecing this information together, we find

$$
\alpha_{m,k} = \begin{cases} \binom{m}{\frac{1}{2}(m+k-1)} - \binom{m}{\frac{1}{2}(m-k-1)}, & \text{if } m-k \text{ is odd;} \\ 0, & \text{if } m-k \text{ is even.} \end{cases}
$$

Notice that if $m-k$ is odd, then

$$
\binom{m}{\frac{1}{2}(m-k+1)} = \binom{m}{\frac{1}{2}(m+k-1)}.
$$

Therefore,

$$\alpha_{m,k} = \begin{cases} \dbinom{m}{\frac{1}{2}(m-k+1)} - \dbinom{m}{\frac{1}{2}(m-k-1)}, & \text{if } m+k \text{ is odd;} \\ 0, & \text{if } m+k \text{ is even.} \end{cases}$$

Intuitively, if we arrange the coefficients of the original formula in a table, indexed horizontally by $k \in \mathbb{Z}$ and vertically by $m \in \mathbb{N}$, then we obtain Pascal's triangle (with an extra row corresponding to $m = 0$ attached to the top):

	-2	-1	0	1	2	3	4	5	6
0	0	0	0	1	0	0	0	0	0
1	0	0	0	0	1	0	0	0	0
2	0	0	0	1	0	1	0	0	0
3	0	0	1	0	2	0	1	0	0
4	0	1	0	3	0	3	0	1	0
5	1	0	4	0	6	0	4	0	1

Next, if we subtract the column corresponding to $s = -k$ from that corresponding to $s = k$, we obtain the $\alpha_{m,k}$ array:

	1	2	3	4	5	6
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	1	0	1	0	0	0
3	0	2	0	1	0	0
4	2	0	3	0	1	0
5	0	5	0	4	0	1

As a special case of this formula, consider what happens when $m = 2p$ and $k = 1$. We have

$$\begin{aligned} \alpha_1 &= \binom{2p}{p} - \binom{2p}{p-1} \\ &= \frac{(2p)!}{p!p!} - \frac{(2p)!}{(p-1)!(p+1)!} \\ &= \frac{(2p)!}{p!p!}\left(1 - \frac{p}{p+1}\right) \\ &= \frac{1}{p+1}\binom{2p}{p} \end{aligned}$$

which is the closed form for the p^{th} Catalan number.

4. Inverse formulas, polynomials, and binomial coefficients

We have a formula for the powers of r_2 as linear combinations of the r_k values. We now derive the inverse relationship, writing the r_k values as linear combinations of powers of r_2. We start with Proposition 1: $r_2 r_k = r_{k-1} + r_{k+1}$.

We demonstrate that for natural numbers k, there exist polynomials $P_k(t)$ such that $r_k = P_k(r_2)$. We note that $P_0(t) = 0$ and $P_1(t) = 1$. Now assume the existence of $P_{k-1}(t)$ and $P_k(t)$. Then from the identity

$$r_{k+1} = r_2 r_k - r_{k-1}$$

we have

$$P_{k+1}(t) = tP_k(t) - P_{k-1}(t)$$

which establishes a second-order recurrence for the polynomials $P_k(t)$. Armed with this, we show for $k \geq 0$,

$$P_k(t) = \sum_i \binom{k-1-i}{i} (-1)^i t^{k-1-2i}.$$

By inspection, this holds for $k = 0$ as the binomial coefficients are all zero in the sum. Also, when $k = 1$, the $i = 0$ term is the only nonzero contributor to the sum. Therefore, it is immediate that this formula holds for $k = 0, 1$. Now, we use the recurrence to establish the induction. Given $k \geq 1$,

$$\begin{aligned}
P_{k+1}(t) &= tP_k(t) - P_{k-1}(t) \\
&= t\sum_i \binom{k-1-i}{i} (-1)^i t^{k-1-2i} - \sum_j \binom{k-2-j}{j} (-1)^j t^{k-2-2j} \\
&= \sum_i \binom{k-1-i}{i} (-1)^i t^{k-2i} - \sum_i \binom{k-1-i}{i-1} (-1)^{i-1} t^{k-2i} \\
&= \sum_i \binom{k-i}{i} (-1)^i t^{k-2i}
\end{aligned}$$

as desired. The third equality is obtained by replacing j with $i - 1$.

As a result, we obtain the desired formula for r_k in terms of powers of r_2:

Proposition 4.

$$r_k = \sum_i \binom{k-1-i}{i} (-1)^i {r_2}^{k-1-2i}.$$

Assembling these coefficients into an array similar to the $\alpha_{m,k}$ array in the previous section, we have

	0	1	2	3	4	5	6
1	1	0	0	0	0	0	0
2	0	1	0	0	0	0	0
3	-1	0	1	0	0	0	0
4	0	-2	0	1	0	0	0
5	1	0	-3	0	1	0	0
6	0	3	0	-4	0	1	0
7	-1	0	6	0	-5	0	1

This array displays the coefficient $\beta(k, m)$ in the formula

$$r_k = \sum_m \beta(k, m) r_2^m,$$

where k is the column number and m is the row number. An interesting observation is that this array and the $\alpha(m,k)$ array are inverses in the sense of "array multiplication":

$$\sum_i \alpha(m,i)\beta(i,p) = \begin{cases} 1, & m = p; \\ 0, & \text{otherwise.} \end{cases}$$

Also, the $k^{(th)}$ column of the $\beta(k,m)$ array can be generated using the generating function $x^k(1+x^2)^{-1-k}$. Using machinery in §5.1 of [2], this leads to the conclusion that the columns of the original $\alpha(m,k)$ array can be generated using an inverse function; in this case, the function that generates the $m^{(th)}$ column of the $\alpha(m,k)$ array is

$$x^{m-1}\left(\frac{1-\sqrt{1-4x^2}}{2x^2}\right)^m.$$

This game is similar but slightly more complicated in the case of r_3. Here, we use

$$r_3 r_k = r_{k+2} + r_k + r_{k-2}$$

which leads to

$$r_{k+2} = (r_3 - 1)r_k + r_{k-2}.$$

With this, we show that for $k \geq 0$, there are functions, but not necessarily polynomials, $Q_k(t)$ such that $r_k = Q_k(r_3)$. From the above identity, we have

$$Q_{k+2}(t) = (t-1)Q_k(t) - Q_{k-2}(t).$$

This establishes a fourth-order recurrence relation for the functions $Q_k(t)$ so determining the four functions Q_0, Q_1, Q_2, and Q_3 will establish the recurrence. By inspection, $Q_0(t) = 0$, $Q_1(t) = 1$, and $Q_3(t) = t$ so all that remains is to determine $Q_2(t)$. We have $r_3 = {r_2}^2 - 1$ from Proposition 4. Therefore, $r_2 = \sqrt{r_3+1}$ and so $Q_2(t) = \sqrt{t+1}$.

We now claim

Proposition 5. *For all natural numbers k,*

$$Q_{2k}(t) = \sqrt{t+1}\sum_i \binom{k-1-i}{i}(-1)^i(t-1)^{k-1-2i}$$
$$Q_{2k+1}(t) = \sum_i \binom{k-i}{i}(-1)^i(t-1)^{k-2i} + \sum_i \binom{k-1-i}{i}(-1)^i(t-1)^{k-1-2i}.$$

Proof. We proceed by induction on k. These are easily checked to match the functions Q_0, Q_1, Q_2, and Q_3 for $k = 0, 1$. For $k \geq 2$, we have

$$\begin{aligned} Q_{2k}(t) &= (t-1)Q_{2(k-1)}(t) - Q_{2(k-2)}(t) \\ &= \sqrt{t+1}\left((t-1)\sum_i \binom{k-2-i}{i}(-1)^i(t-1)^{k-2-2i}\right. \\ &\qquad \left. - \sum_j \binom{k-3-j}{j}(-1)^j(t-1)^{k-3-2j}\right) \\ &= \sqrt{t+1}\left(\sum_i \left(\binom{k-2-i}{i} + \binom{k-2-i}{i-1}\right)(-1)^i(t-1)^{k-1-2i}\right) \\ &= \sqrt{t+1}\left(\sum_i \binom{k-1-i}{i}(-1)^i(t-1)^{k-1-2i}\right) \end{aligned}$$

as desired. The third equality is obtained by replacing j with $i-1$. The proof for Q_{2k+1} is similar, treating each sum separately, and is omitted. □

As a corollary, we have

Corollary 6.

$$r_{2k} = \sqrt{r_3+1}\sum_i \binom{k-1-i}{i}(-1)^i(r_3-1)^{k-1-2i}$$

$$r_{2k+1} = \sum_i \binom{k-i}{i}(-1)^i(r_3-1)^{k-2i} + \sum_i \binom{k-1-i}{i}(-1)^i(r_3-1)^{k-1-2i}.$$

References

[1] A. Fontaine and S. Hurley, Proof by picture: Products and reciprocals of diagonal length ratios in the regular polygon, *Forum Geom.*, 6 (2006) 97–101.

[2] H. Wilf, *Generatingfunctionology*, Academic Press, 1994.

Department of Mathematics, Washington State University, Pullman, WA 99164-3113
E-mail address: hudelson@math.wsu.edu

Forum Geometricorum
Volume 6 (2006) 263–268.

FORUM GEOM
ISSN 1534-1178

A Note on the Barycentric Square Roots of Kiepert Perspectors

Khoa Lu Nguyen

Abstract. Let P be an interior point of a given triangle ABC. We prove that the orthocenter of the cevian triangle of the barycentric square root of P lies on the Euler line of ABC if and only if P lies on the Kiepert hyperbola.

1. Introduction

In a recent Mathlinks message, the present author proposed the following problem.

Theorem 1. *Given an acute triangle ABC with orthocenter H, the orthocenter H' of the cevian triangle of $\sqrt{H}$, the barycentric square root of H, lies on the Euler line of triangle ABC.*

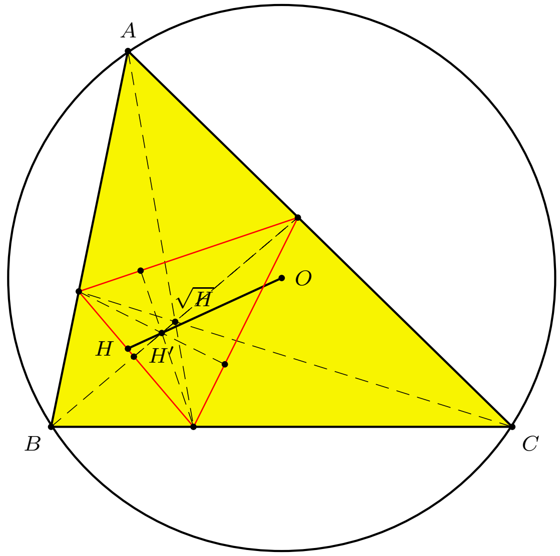


Figure 1.

Paul Yiu has subsequently discovered the following generalization.

Theorem 2. *The locus of point P for which the orthocenter of the cevian triangle of the barycentric square root $\sqrt{P}$ lies on the Euler line is the part of the Kiepert hyperbola which lies inside triangle ABC.*

Publication Date: October 30, 2006. Communicating Editor: Paul Yiu.

The author is grateful to Professor Yiu for his generalization of the problem and his help in the preparation of this paper.

The barycentric square root is defined only for interior points. This is the reason why we restrict to acute angled triangles in Theorem 1 and to the interior points on the Kiepert hyperola in Theorem 2. It is enough to prove Theorem 2.

2. Trilinear polars

Let $A'B'C'$ be the cevian triangle of P, and A_1, B_1, C_1 be respectively the intersections of $B'C'$ and BC, $C'A'$ and CA, $A'B'$ and AB. By Desargues' theorem, the three points A_1, B_1, C_1 lie on a line ℓ_P, the trilinear polar of P.

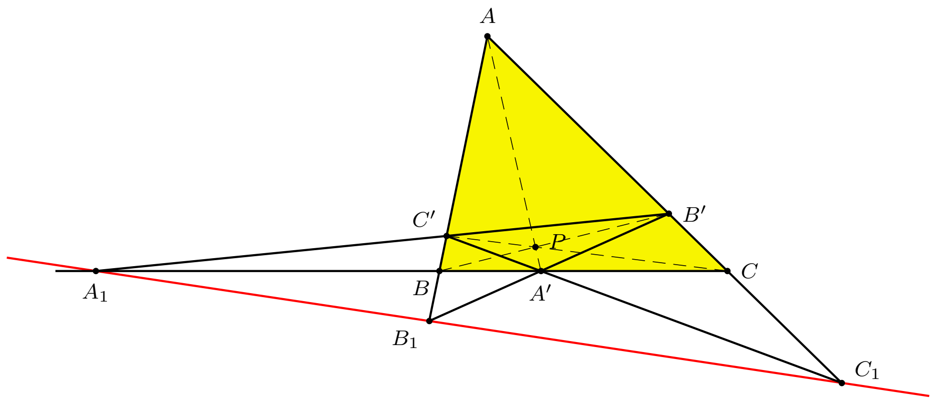


Figure 2.

If P has homogeneous barycentric coordinates $(u : v : w)$, then the trilinear polar is the line

$$\ell_P : \qquad \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.$$

For the orthocenter $H = (S_{BC} : S_{CA} : S_{AB})$, the trilinear polar

$$\ell_H : \qquad S_A x + S_B y + S_C z = 0.$$

is also called the orthic axis.

Proposition 3. *The orthic axis is perpendicular to the Euler line.*

This proposition is very well known. It follows easily, for example, from the fact that the orthic axis ℓ_H is the radical axis of the circumcircle and the nine-point circle. See, for example, [2, §§5.4,5].

The trilinear polar ℓ_P and the orthic axis ℓ_H intersect at the point

$$(u(S_B v - S_C w) : v(S_C w - S_A u) : w(S_A u - S_B v)).$$

In particular, ℓ_P and ℓ_H are parallel, i.e., their intersection is a point at infinity if and only if

$$u(S_B v - S_C w) + v(S_C w - S_A u) + w(S_A u - S_B v) = 0.$$

Equivalently,

$$(S_B - S_C)vw + (S_C - S_A)wu + (S_A - S_B)uv = 0. \tag{1}$$

Note that this equation defines the Kiepert hyperbola. Points on the Kiepert hyperbola are called Kiepert perspectors.

Proposition 4. *The trilinear polar ℓ_P is parallel to the orthic axis if and only if P is a Kiepert perspector.*

3. The barycentric square root of a point

Let P be a point inside triangle ABC, with homogeneous barycentric coordinates $(u : v : w)$. We may assume $u, v, w > 0$, and define the barycentric square root of P to be the point $\sqrt{P}$ with barycentric coordinates $(\sqrt{u} : \sqrt{v} : \sqrt{w})$.

Paul Yiu [2] has given the following construction of $\sqrt{P}$.

(1) Construct the circle $\mathcal{C}_A$ with BC as diameter.

(2) Construct the perpendicular to BC at the trace A' of P to intersect $\mathcal{C}_A$ at X'.

(3) Construct the bisector of angle $BX'C$ to intersect BC at X.

Then X is the trace of $\sqrt{P}$ on BC. Similar constructions on the other two sides give the traces Y and Z of $\sqrt{P}$ on CA and AB respectively. The barycentric square root $\sqrt{P}$ is the common point of AX, BY, CZ.

Proposition 5. *If the trilinear polar ℓ_P intersects BC at A_1, then*

$$A_1X^2 = A_1B \cdot A_1C.$$

Proof. Let M is the midpoint of BC. Since A_1, A' divide B, C harmonically, we have $MB^2 = MC^2 = MA_1 \cdot MA'$ (Newton's theorem). Thus, $MX'^2 = MA_1 \cdot MA'$. It follows that triangles $MX'A_1$ and $MA'X'$ are similar, and $\angle MX'A_1 = \angle MA'X' = 90°$. This means that A_1X' is tangent at X' to the circle with diameter BC. Hence, $A_1X'^2 = A_1B \cdot A_1C$.

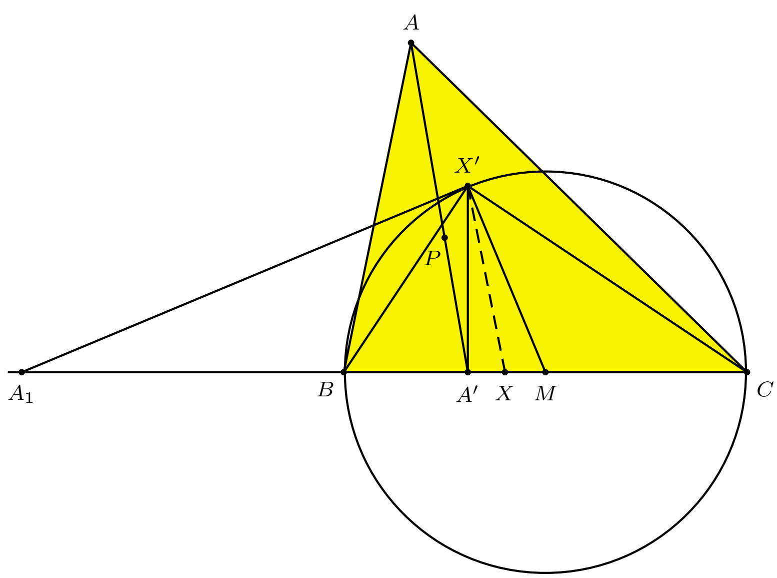


Figure 3.

To complete the proof it is enough to show that $A_1X = A_1X'$, *i.e.*, triangle A_1XX' is isosceles. This follows easly from

$$\begin{aligned}\angle A_1X'X =&\angle A_1X'B + \angle BX'X\\ =&\angle X'CB + \angle XX'C\\ =&\angle X'XA_1.\end{aligned}$$

□

Corollary 6. *If X_1 is the intersection of YZ and BC, then A_1 is the midpoint of XX_1.*

Proof. If X_1 is the intersection of YZ and BC, then X, X_1 divide B, C harmonically. The circle through X, X_1, and with center on BC is orthogonal to the circle $\mathcal{C}_A$. By Proposition 5, this has center A_1, which is therefore the midpoint of XX_1. □

4. Proof of Theorem 2

Let P be an interior point of triangle ABC, and XYZ the cevian triangle of its barycentric square root $\sqrt{P}$.

Proposition 7. *If H' is the orthocenter of XYZ, then the line OH' is perpendicular to the trilinear polar ℓ_P.*

Proof. Consider the orthic triangle DEF of XYZ. Since $DEXY$, $EFYZ$, and $FDZX$ are cyclic, and the common chords DX, EY, FZ intersect at H', H' is the radical center of the three circles, and

$$H'D \cdot H'X = H'E \cdot H'Y = H'F \cdot H'Z. \tag{2}$$

Consider the circles ξ_A, ξ_B, ξ_C, with diameters XX_1, YY_1, ZZ_1. These three circles are coaxial; they are the generalized Apollonian circles of the point $\sqrt{P}$. See [3]. As shown in the previous section, their centers are the points A_1, B_1, C_1 on the trilinear polar ℓ_P. See Figure 4.

Now, since D, E, F lie on the circles ξ_A, ξ_B, ξ_C respectively, it follows from (2) that H' has equal powers with respect to the three circles. It is therefore on the radical axis of the three circles.

We show that the circumcenter O of triangle ABC also has the same power with respect to these circles. Indeed, the power of O with respect to the circle ξ_A is

$$A_1O^2 - A_1X^2 = OA_1^2 - R^2 - A_1X^2 + R^2 = A_1B \cdot A_1C - A_1X^2 + R^2 = R^2$$

by Proposition 5. The same is true for the circles ξ_B and ξ_C. Therefore, O also lies on the radical axis of the three circles. It follows that the line OH' is the radical axis of the three circles, and is perpendicular to the line ℓ_P which contains their centers. □

The orthocenter H' of XYZ lies on the Euler line of triangle ABC if and only if the trilinear polar ℓ_P is parallel to the Euler line, and hence parallel to the orthic axis by Proposition 3. By Proposition 4, this is the case precisely when P lies on the Kiepert hyperbola. This completes the proof of Theorem 2.

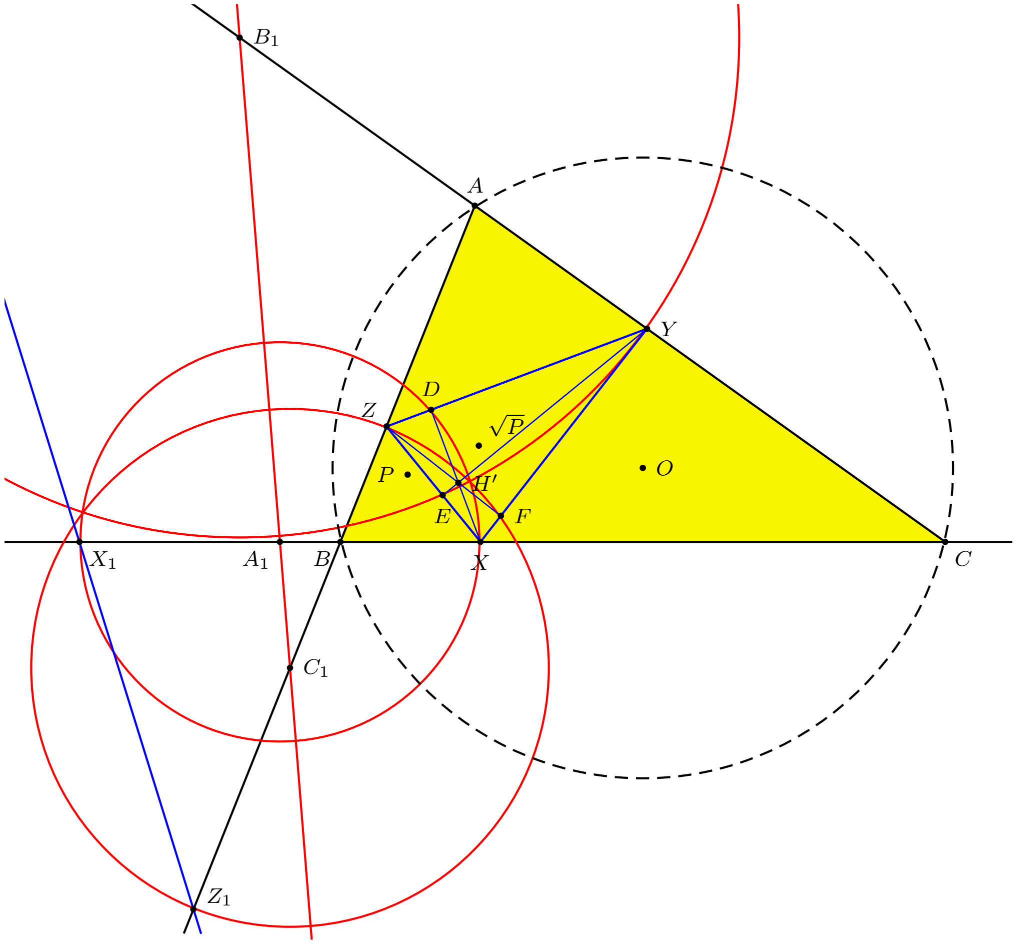


Figure 4.

Theorem 8. *The orthocenter of the cevian triangle of $\sqrt{P}$ lies on the Brocard axis if and only if P is an interior point on the Jerabek hyperbola.*

Proof. The Brocard axis OK is orthogonal to the Lemoine axis. The locus of points whose trilinear polars are parallel to the Brocard axis is the Jerabek hyperbola. □

5. Coordinates

In homogeneous barycentric coordinates, the orthocenter of the cevian triangle of $(u : v : w)$ is the point

$$\left(\left(S_B\left(\frac{1}{w}+\frac{1}{u}\right)+S_C\left(\frac{1}{u}+\frac{1}{v}\right)\right)\left(-S_A\left(\frac{1}{v}+\frac{1}{w}\right)^2+S_B\left(\frac{1}{u^2}-\frac{1}{w^2}\right)+S_C\left(\frac{1}{u^2}-\frac{1}{v^2}\right)\right)\right.$$
$$\left.:\cdots:\cdots\right).$$

Applying this to the square root of the orthocenter, with $(u^2 : v^2 : w^2) = \left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right)$, we obtain

$$\left(a^2 S_A \cdot \sqrt{S_{ABC}} + S_{BC}\sum_{\text{cyclic}} a^2\sqrt{S_A} : \cdots : \cdots\right),$$

which is the point H' in Theorem 1.

More generally, if P is the Kiepert perspector

$$K(\theta) = \left(\frac{1}{S_A+S_\theta} : \frac{1}{S_B+S_\theta} : \frac{1}{S_C+S_\theta}\right),$$

the orthocenter of the cevian triangle of $\sqrt{P}$ is the point

$$\left(a^2 S_A\sqrt{(S_A+S_\theta)(S_B+S_\theta)(S_C+S_\theta)}\right.$$
$$\left.+S_{BC}\sum_{\text{cyclic}} a^2\sqrt{S_A+S_\theta} + a^2 S_\theta \sum_{\text{cyclic}} S_A\sqrt{S_A+S_\theta} : \cdots : \cdots\right).$$

References

[1] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.
[2] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University lecture notes, 2001.
[3] P. Yiu, Generalized Apollonian circles, *Journal of Geometry and Graphics*, 8 (2004) 225–230.

Khoa Lu Nguyen: Massachusetts Institute of Technology, student, 77 Massachusetts Avenue, Cambridge, MA, 02139, USA
E-mail address: treegoner@yahoo.com

Forum Geometricorum
Volume 6 (2006) 269–284.

FORUM GEOM
ISSN 1534-1178

Translated Triangles Perspective to a Reference Triangle

Clark Kimberling

Abstract. Suppose A, B, C, D, E, F are points and L is a line other than the line at infinity. This work examines cases in which a translation $D'E'F'$ of DEF in the direction of L is perspective to ABC, in the sense that the lines AD', BE', CF' concur.

1. Introduction

In the transfigured plane of a triangle ABC, let L^∞ be the line at infinity and L a line other than L^∞. (To say "transfigured plane" means that the sidelengths a, b, c of triangle ABC are variables or indeterminates, and points are defined as functions of a, b, c, so that the "plane" of ABC is infinite dimensional.) Suppose that D, E, F are distinct points, none on L^∞, such that the set $\{A, B, C, D, E, F\}$ consists of at least five distinct points. We wish to translate triangle DEF in the direction of L and to discuss cases in which the translated triangle $D'E'F'$ is perspective to ABC, in the sense that the lines AD', BE', CF' concur. One of these cases is the limiting case that $D' = L \cap L^\infty$; call this point U, and note that $D' = E' = F' = U$.

Points and lines will be given (indeed, are *defined*) by homogeneous trilinear coordinates. The line L^∞ at infinity is given by $a\alpha + b\beta + c\gamma = 0$, and L, by an equation $l\alpha + m\beta + n\gamma = 0$, where $l : m : n$ is a point. Then the point $U = u : v : w$ is given by

$$u = bn - cm, \ v = cl - an, \ w = am - bl. \tag{1}$$

Write the vertices of DEF as

$$D = d_1 : e_1 : f_1, \qquad E = d_2 : e_2 : f_2, \qquad F = d_3 : e_3 : f_3,$$

and let

$$\delta = ad_1 + be_1 + cf_1, \qquad \epsilon = ad_2 + be_2 + cf_2, \qquad \varphi = ad_3 + be_3 + cf_3.$$

The hypothesis that none of D, E, F is on L^∞ implies that none of $\delta, \epsilon, \varphi$ is 0. The line L is given parametrically as the locus of point $D' = D_t = x_1 : y_1 : z_1$ by

$$x_1 = d_1 + \delta tu, \qquad y_1 = e_1 + \delta tv, \qquad z_1 = f_1 + \delta tw.$$

Publication Date: November 6, 2006. Communicating Editor: Paul Yiu.

The point E' traverses the line through E parallel to L, so that $E' = E_t = x_2 : y_2 : z_2$ is given by

$$x_2 = d_2 + \epsilon tu, \qquad y_2 = e_2 + \epsilon tv, \qquad z_2 = f_2 + \epsilon tw.$$

The point F' traverses the line through F parallel to L, so that $F' = F_t = x_3 : y_3 : z_3$ is given by

$$x_3 = d_3 + \varphi tu \quad y_3 = e_3 + \varphi tv \quad z_3 = f_3 + \varphi tw.$$

In these parameterizations, t represents a homogeneous function of a, b, c. The degree of homogeneity of t is that of $(x_1 - d_1)/(\delta u)$.

2. Two basic theorems

Theorem 1. *Suppose ABC and DEF are triangles such that $\{A, B, C, D, E, F\}$ consists of at least five distinct points. Suppose L is a line and $U = L \cap L^\infty$. As D_t traverses the line DU, the triangle $D_tE_tF_t$ of translation of DEF in the direction of L is either perspective to ABC for all t or else perspective to ABC for at most two values of t.*

Proof. The lines AD_t, BE_t, CF_t are given by the equations

$$-z_1\beta + y_1\gamma = 0, \quad z_2\alpha - x_2\gamma = 0, \quad -y_3\alpha + x_3\beta = 0,$$

respectively. Thus, the concurrence determinant,

$$\begin{vmatrix} 0 & -z_1 & y_1 \\ z_2 & 0 & -x_2 \\ -y_3 & x_3 & 0 \end{vmatrix} \tag{2}$$

is a polynomial P, formally of degree 2 in t:

$$P(t) = p_0 + p_1 t + p_2 t^2, \tag{3}$$

where

$$p_0 = d_3e_1f_2 - d_2e_3f_1, \tag{4}$$
$$p_1 = u(\varphi e_1 f_2 - \epsilon e_3 f_1) + v(\delta d_3 f_2 - \varphi d_2 f_1) + w(\epsilon e_1 d_3 - \delta e_3 d_2), \tag{5}$$
$$p_2 = \delta vw(\epsilon d_3 - \varphi d_2) + \epsilon wu(\varphi e_1 - \delta e_3) + \varphi uv(\delta f_2 - \epsilon f_1). \tag{6}$$

Thus, either p_0, p_1, p_0 are all zero, in which case $D_tE_tF_t$ is perspective to ABC for all t, or else $P(t)$ is zero for at most two values of t. □

If triangle DEF is homothetic to ABC, then $D_tE_tF_t$ is homothetic to ABC and hence perspective to ABC, for every t. This is well known in geometry. The geometric theorem, however, does not imply the "same" theorem in the more general setting of triangle algebra, in which the objects are defined in terms of variables or indeterminants. Specifically, perspectivity and parallelism (hence homothety) are defined by zero determinants. When such determinants are "symbolically zero", they are zero not only for Euclidean triangles, for which a, b, c are positive real numbers satisfying $(a > b + c,\ b > c + a,\ c > a + b)$ or $(a \geq b + c,\ b \geq c + a,$

$c \geq a + b)$, but also for a, b, c as indeterminates. Among geometric theorems that readily generalize to algebraic theorems are these:

If $L_1 \| L_2$ *and* $L_2 \| L_3$, *then* $L_1 \| L_3$.
If T_1 *is homothetic to* T_2 *and* T_2 *is homothetic to* T_3, *then* T_1 *is homothetic to* T_3.
If T_1 *is homothetic to* T_2, *then* T_1 *is perspective to* T_2.

Theorem 2. *Suppose* ABC *and* DEF *in Theorem 1 are homothetic. Then* $D_tE_tF_t$ *is perspective to* ABC *for all* t.

Proof. ABC and DEF are homothetic, and DEF and $D_tE_tF_t$ are homothetic. Therefore $D_tE_tF_t$ is homothetic to ABC, which implies that $D_tE_tF_t$ is perspective to ABC. □

It is of interest to express the coefficients p_0, p_1, p_2 more directly in terms of a, b, c and the coordinates of D, E, F. To that end, we shall use cofactors, as defined by the identity

$$\begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix}^{-1} = \frac{1}{\Delta} \begin{pmatrix} D_1 & D_2 & D_3 \\ E_1 & E_2 & E_3 \\ F_1 & F_2 & F_3 \end{pmatrix},$$

where

$$\Delta = \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix} = d_1D_1 + e_1E_1 + f_1F_1;$$

that is, $D_1 = e_2f_3 - f_2e_3$, etc. For example, in the case that ABC and DEF are homothetic, line EF is parallel to line BC, as defined by a zero determinant (e.g., [1], p. 29); likewise, the lines FD and CA are parallel, as are DE and AB. The zero determinants yield

$$bF_1 = cE_1, \qquad cD_2 = aF_2, \qquad aE_3 = bD_3. \tag{7}$$

These equations can be used to give a direct but somewhat tedious proof of Theorem 2; we digress to prove only that $p_0 = 0$. Let $\mathcal{L}$ and $\mathcal{R}$ denote the products of the left-hand sides and the right-hand sides in (7). Then $\mathcal{L} - \mathcal{R}$ factors as $abc\Psi\Delta$, where

$$\Psi = e_3d_2f_1 - e_1d_3f_2,$$

and $abc\Psi\Delta = 0$ by (7). It is understood that A, B, C are not collinear, so that D, E, F are not collinear. As the defining equation for collinearity of D, E, F is the determinant equation $\Delta = 0$, we have $\Delta \neq 0$. Therefore, $\Psi = 0$, so that $p_0 = 0$.

Next, substitute from (1) for u, v, w in (5) and (6), getting

$$p_1 = lp_{1l} + mp_{1m} + np_{1n} \qquad \text{and} \qquad p_2 = mnp_{2l} + nlp_{2m} + lm_{2n},$$

where

$$\begin{aligned}
p_{1l} =& b^2e_1F_1 + c^2f_1E_1 - abd_2F_2 - cad_3E_3,\\
p_{1m} =& c^2f_2D_2 + a^2d_2F_2 - bce_3D_3 - abe_1F_1,\\
p_{1n} =& a^2d_3E_3 + b^2e_3D_3 - caf_1E_1 - bcf_2D_2,
\end{aligned}$$

and

$$\begin{aligned}
p_{2l} =& 2a^2bc(e_1d_2f_3 - e_2d_3f_1) - ab^2e_3F_3 + ac^2f_2E_2\\
& - bc^2f_1D_1 + ba^2d_3F_3 - ca^2d_2E_2 + cb^2e_1D_1,\\
p_{2m} =& 2b^2ca(f_2e_3d_1 - f_3e_1d_2) - bc^2f_1D_1 + ba^2d_3F_3\\
& - ca^2d_2E_2 + cb^2e_1D_3 - ab^2e_3F_3 + ac^2f_2E_2,\\
p_{2n} =& 2c^2ab(e_3d_1e_2 - d_1f_2e_3) - ca^2d_2E_2 + cb^2e_1D_1\\
& - ab^2e_3F_3 + ac^2f_2E_1 - bc^2f_1D_1 + ba^2d_3F_3.
\end{aligned}$$

The task of expressing the coefficients p_0, p_1, p_2 more directly in terms of a, b, c and the coordinates of D, E, F is now completed.

3. Intersecting conics

We begin with a lemma proved in [7]; see also [2].

Lemma 3. *Suppose a point $P = p : q : r$ is given parametrically by*

$$\begin{aligned}
p =& p_1t^2 + q_1t + r_1,\\
q =& p_2t^2 + q_2t + r_2,\\
r =& p_3t^2 + q_3t + r_3,
\end{aligned}$$

where the matrix

$$M = \begin{pmatrix} p_1 & q_1 & r_1\\ p_2 & q_2 & r_2\\ p_3 & q_3 & r_3 \end{pmatrix}$$

is nonsingular with adjoint (cofactor) matrix

$$M^{\#} = \begin{pmatrix} P_1 & Q_1 & R_1\\ P_2 & Q_2 & R_2\\ P_3 & Q_3 & R_3 \end{pmatrix}.$$

Then P lies on this conic:

$$(Q_1\alpha + Q_2\beta + Q_3\gamma)^2 = (P_1\alpha + P_2\beta + P_3\gamma)(R_1\alpha + R_2\beta + R_3\gamma). \qquad (8)$$

In Theorem 4, we shall show that the point of concurrence of the lines AD_t, BE_t, CF_t is also the point of concurrence of three conics. Let

$$A_t = BE_t \cap CF_t, \quad B_t = CF_t \cap AD_t, \quad C_t = AD_t \cap BE_t.$$

Theorem 4. *If, in Theorem 1, the line L is not parallel to a sideline of triangle ABC, then the locus of each of the points A_t, B_t, C_t is a conic.*

Proof. The point $A_t = a_t : b_t : c_t$ is given by

$$a_t = x_2x_3, \qquad b_t = x_2y_3, \qquad c_t = z_2x_3,$$

so that

$$a_t = (d_2 + \epsilon tu)\,(d_3 + \varphi tu) = \epsilon\varphi u^2t^2 + (\varphi d_2u + \epsilon d_3u)t + d_2d_3, \tag{9}$$

$$b_t = (d_2 + \epsilon tu)\,(e_3 + \varphi tv) = \epsilon\varphi uvt^2 + (\varphi d_2v + \epsilon e_3u)t + d_2e_3, \tag{10}$$

$$c_t = (f_2 + \epsilon tw)\,(d_3 + \varphi tu) = \epsilon\varphi uwt^2 + (\varphi f_2u + \epsilon d_3w)t + f_2d_3. \tag{11}$$

By Lemma 3, the locus $\{A_t\}$ is a conic unless $\epsilon\varphi uvw = 0$, in which case u, v, or w must be zero. Consider the case $u = 0$; then $u : v : w = 0 : c : -b$, but this is the point in which line BC meets L^∞, contrary to the hypothesis. Likewise, the loci $\{B_t\}$ and $\{C_t\}$ are conics. Note that the conic $\{A_t\}$ passes through B and C, as indicated by Figure 1. [1] $\square$

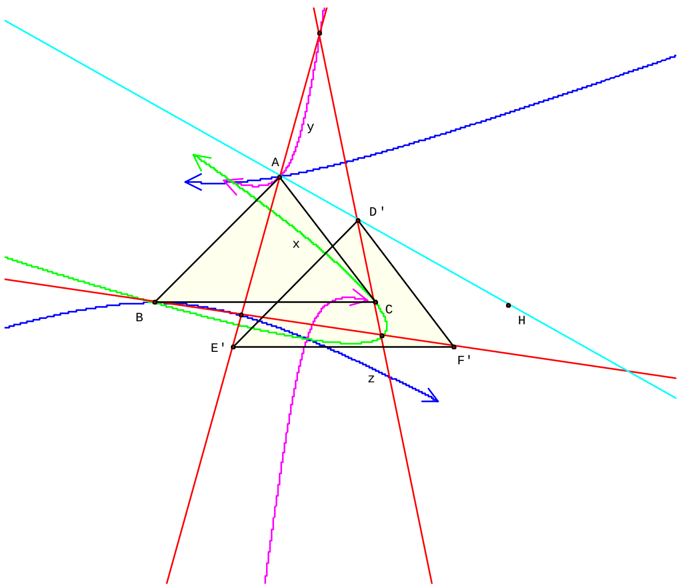


Figure 1. Intersecting conics

Lemma 3 shows how to write out equations of conics starting with a matrix M. As the lemma applies only to nonsingular M, we can, by factoring $|M|$, determine

[1]Figure 1 can be viewed dynamically using The Geometer's Sketchpad; see [6] for access. The choice of triangle DEF is given by the equations $D = C$, $E = A$, $F = B$. (The labels D, E, F are not shown.) Point H on line AD' is an independent point, and triangle $D'E'F'$ is a translation of DEF in the direction of line AH. Except for special cases, as D' traverses line AH, points x, y, z traverse conics as in Theorem 4, and the conics meet twice (with $x = y = z$), at the perspectors described in Theorem 1.

criteria for nonsingularity. In connection with $\{A_t\}$ and (9)-(11),

$$|M| = \begin{vmatrix} \epsilon\varphi u^2 & \varphi d_2 u + \epsilon d_3 u & d_2 d_3 \\ \epsilon\varphi uv & \varphi d_2 v + \epsilon e_3 u & d_2 e_3 \\ \epsilon\varphi wu & \varphi f_2 u + \epsilon d_3 w & f_2 d_3 \end{vmatrix}$$
$$= \epsilon\varphi u \left(uf_2 - wd_2\right)\left(vd_3 - ue_3\right)\left(be_3 d_2 - be_2 d_3 + cd_2 f_3 - cd_3 f_2\right).$$

By hypothesis, $\epsilon\varphi u \neq 0$. Also, $(uf_2 - wd_2)(vd_3 - ue_3) \neq 0$, as it is assumed that $E \neq U$ and $F \neq U$. Finally, the factor $be_3d_2 - be_2d_3 + cd_2f_3 - cd_3f_2$ is 0 if and only if line EF is parallel to line BC. In conclusion, if EF is not parallel to BC, and FD is not parallel to CA, and DE is not parallel to AB, then the three loci are conics and Theorem 3 applies.

4. Terminology and notation

The main theorem in this paper is Theorem 1. For various choices of DEF and L, the perspectivities indicated by Theorem 1 are of particular interest. Such choices are considered in Sections 3-6; they are, briefly, that DEF is a cevian triangle of a point, or an anticevian triangle, or a rotation of triangle ABC about its circumcenter. In order to describe the configurations, it will be helpful to adopt certain terms and notations.

Unless otherwise noted, the points $U = u : v : w$ and $P = p : q : r$ are arbitrary. If at least one of the products up, vq, wr is not zero, the product $U \cdot P$ is defined by the equation

$$U \cdot P = up : vq : wr.$$

The multiplicative inverse of P, defined if $pqr \neq 0$, is the isogonal conjugate of P, given by

$$P^{-1} = p^{-1} : q^{-1} : r^{-1}.$$

The quotient U/P is defined by

$$U/P = U \cdot P^{-1}.$$

The *isotomic conjugat*e of P is defined if $pqr \neq 0$ by the trilinears

$$a^{-2}p^{-1} : b^{-2}q^{-1} : c^{-2}r^{-1}.$$

Geometric definitions of isogonal and isotomic conjugates are given at *MathWorld* [8]. We shall also employ these terms and notations:

crossdifference of U and $P = CD(U,P) = rv - qw : pw - ru : qu - pv$,

crosssum of U and $P = CS(U,P) = rv + qw : pw + ru : qu + pv$,

crosspoint of U and $P = CP(U,P) = pu(rv + qw) : qv(pw + ru) : rw(qu + pv)$.

Geometric interpretations of these “cross operations” are given at [4].

Notation of the form X_i as in [3] will be used for certain special points, such as

$$\begin{aligned}\text{incenter} =& X_1 = 1 : 1 : 1 = \text{ the multiplicative identity},\\ \text{centroid} =& X_2 = 1/a : 1/b : 1/c,\\ \text{circumcenter} =& X_3 = \cos A : \cos B : \cos C,\\ \text{symmedian point} =& X_6 = a : b : c.\end{aligned}$$

Instead of the trigonometric trilinears for X_3, we shall sometimes use trilinears for X_3 expressed directly in terms of a, b, c. As $\cos A = (b^2 + c^2 - a^2)/(2bc)$, we shall use abbreviations:

$$a_1 = (b^2+c^2-a^2)/(2bc), \quad b_1 = (c^2+a^2-b^2)/(2ca), \quad c_1 = (a^2+b^2-c^2)/(2ab);$$

thus, $X_3 = a_1 : b_1 : c_1$.

The line X_2X_3 is the Euler line, and the line X_3X_6 is the Brocard axis. When working with lines algebraically, it is sometimes helpful to do so with reference to a parameter and the point in which the line meets L^∞. In the case of the Euler line, this point is

$$X_{30} = a_1 - 2b_1c_1 : b_1 - 2c_1a_1 : c_1 - 2a_1b_1,$$

and a parametric representation is given by $x : y : z = x(s) : y(s) : z(s)$, where

$$\begin{aligned}x(s) =& a_1 + s(a_1 - 2b_1c_1),\\ y(s) =& b_1 + s(b_1 - 2c_1a_1),\\ z(s) =& c_1 + s(c_1 - 2a_1b_1).\end{aligned}$$

The point X_{30} will be called the direction of the Euler line. More generally, for any line, its point of intersection with L^∞ will be called the *direction* of the line. The parameter s is not necessarily a numerical variable; rather, it is a function of a, b, c. In this paper, trilinears for any point are homogeneneous functions of a, b, c, all having the same degree of homogeneity; thus in a parametric expression of the form $p + su$, the degree of homogeneity of s is that of p minus that of u.

Two families of cubics will occur in the sequel. The cubic $\mathcal{Z}(U, P)$ is given by

$$(vqy - wrz)px^2 + (wrz - upx)qy^2 + (upx - vqy)rz^2 = 0,$$

and the cubic $\mathcal{ZC}(U, P)$, by

$$L(wy - vz)x^2 + M(uz - wx)y^2 + N(vx - uy)z^2 = 0.$$

For details on these and other families of cubics, see [5].

The remainder of this article is mostly about special translations. It will be helpful to introduce some related terminology. Suppose DEF is a triangle in the transfigured plane of ABC, and U is a direction (i.e., a point on L^∞). A triangle $D'E'F'$, other than DEF itself, such that $D'E'F'$ is a U-translation of DEF and $D'E'F'$ is perspective to ABC (in the sense that the lines AD', BE', CF' concur) will be called a *U-ppt of DEF*. The designation "ppt" means "proper perspective translation".

In view of Theorem 1, except for special cases, each DEF has, for every U, at most two U-ppt's. Thus, if DEF is perspective to ABC, as when DEF is a cevian triangle or an anticevian tiangle, there is "usually" just one ppt. That one ppt is of primary interest in the next three sections; especially in Case 5.4 and Case 6.4.

5. Translated cevian triangles

In this section, DEF is the cevian triangle of a point $X = x : y : z$; thus DEF is given as a matrix by

$$\begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix} = \begin{pmatrix} 0 & y & z \\ x & 0 & z \\ x & y & 0 \end{pmatrix},$$

and the perspectivity determinant (2) is given by $-t(\Delta_0 + t\Delta_1)$, where

$$\Delta_0 = \begin{vmatrix} ax^2 & by^2 & cz^2 \\ u & v & w \\ x & y & z \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} a^2ux^2 & b^2vy^2 & c^2wz^2 \\ u & v & w \\ x & y & z \end{vmatrix}.$$

In particular, the equations $\Delta_0 = 0$ and $\Delta_1 = 0$ represent cubics in x, y, z, specifically, $\mathcal{Z}(X_2, X_6 \cdot U^{-1})$ and $\mathcal{Z}(X_{75} \cdot U^{-1}, X_{31})$, respectively. We shall consider four cases:

Case 5.1: $\Delta_0 = 0$ and $\Delta_1 = 0$. In this case, $D_tE_tF_t$ is perspective to ABC for every t. Clearly this holds for $X = X_2$, for all U. Now for any given U, let X be the isotomic conjugate of U. Rows 1 and 3 of the determinant Δ_1 are equal, so that $\Delta_1 = 0$. Also,

$$\begin{aligned} \Delta_0 =& bcvw(b^2v^2 - c^2w^2) + cawu(c^2w^2 - a^2u^2) + abuv(a^2u^2 - b^2v^2) \\ =& -(bv - cw)(cw - au)(au - bv)(au + bv + cw) \\ =& 0. \end{aligned}$$

The cevian triangle DEF of X is not homothetic to ABC, yet $D_tE_tF_t$ is perspective to ABC for every t. Another such example is obtained by simply taking X to be U. Further results in Case 1 are given in Theorem 5.

Case 5.2: $\Delta_0 = 0$ and $\Delta_1 \neq 0$. For given U, the point $X = CP(X_2, U)$ satisfies $\Delta_0 = 0$ and $\Delta_1 \neq 0$. For quite a different example, let

$$U = X_{511} = \cos(A + \omega) : \cos(B + \omega) : \cos(C + \omega),$$

where ω denotes the Brocard angle. Then the cubic $\Delta_0 = 0$ passes through the following points, X_3 (the circumcenter), X_6 (the symmedian point), X_{297}, X_{325}, X_{694}, X_{2009}, and X_{2010}, none of which lies on the cubic $\Delta_1 = 0$. Other points on the cubic $\Delta_0 = 0$ are given at [5], where the cubic $\Delta_1 = 0$ is classified as $\mathcal{ZC}(511, L(30, 511))$.

Case 5.3: $\Delta_0 \neq 0$ and $\Delta_1 = 0$. In this case, for any U, there is no U-ppt. For example, take $U = X_{523}$. Then the cubic $\Delta_1 = 0$ passes through the two points in which the Euler line meets the circumcircle, these being X_{1113} and X_{1114}, and these points do not also lie on the cubic $\Delta_0 = 0$.

Case 5.4: $\Delta_0 \neq 0$ and $\Delta_1 \neq 0$. In this case, $D_tE_tF_t$ is perspective to ABC for $t = -\Delta_0/\Delta_1$. The perspector is the point $x_2x_3 : x_2y_3 : z_2x_3$, which, after cancellation of common factors, is the point $X' = x' : y' : z'$ given by

$$x' = \frac{by - cz}{(bv - cw)yz + ax(vz - wy)}, \tag{12}$$

$$y' = \frac{cz - ax}{(cw - au)zx + by(wx - uz)}, \tag{13}$$

$$z' = \frac{ax - by}{(au - bv)xy + cz(uy - vx)}. \tag{14}$$

Note that if X and U are triangle centers for which X' is a point, then X' is a triangle center. For the special case $U = X_{511}$, pairs X and X' are shown here:

X	4	7	54	68	69	99	183	190	385	401	668	670	671	903
X'	3	256	52	52	6	690	262	900	325	297	691	888	690	900

Returning to Case 5.1, in the subcase that X is the isotomic conjugate of U, it is natural to ask about the perspectors, and to find the following theorem.

Theorem 5. *Suppose U is any point on L^∞ but not on a sideline BC, CA, AB. Let X be the isotomic conjugate of U. The locus of the perspector P_t of triangles $D_tE_tF_t$ and ABC is a conic that passes through A, B, C, and the point*

$$X^2 = b^4c^4v^2w^2 : c^4a^4w^2u^2 : a^4b^4u^2v^2. \tag{15}$$

Proof. The perspector is the point $P_t = x_2x_3 : x_2y_3 : z_2x_3$. Substituting and simplifying give

$$\begin{aligned} P_t =& b^3c^3vw(bv - acuwt)(cw - abuvt) \\ & : c^3a^3wu(cw - bavut)(au - bcvwt) \\ & : a^3b^3uv(au - cbwvt)(bv - cawut). \end{aligned}$$

By Theorem 3, the locus of P_t is a conic. Clearly, P_t passes through A, B, C for $t = au/(bcvw), bv/(cawu), cw/(abuv)$, respectively, and P_0 is the point given by (15). See Figure 2. [2] □

An equation for the circumconic described in Theorem 5 is found from (8):

$$b^2c^2(b^2v^2 - c^2w^2)\beta\gamma + c^2a^2(c^2w^2 - a^2u^2)\gamma\alpha + a^2b^2(a^2u^2 - b^2v^2)\alpha\beta = 0.$$

Theorem 6. *Suppose X is the isotomic conjugate of a point U_1 on L^∞ but not on a sideline BC, CA, AB. Then the perspector X' in Case 5.4 is invariant of the point U. In fact, $X' = CD(X_6, U_1^{-1})$, and X' is on L^∞.*

[2] Figure 2 can be viewed dynamically using The Geometer's Sketchpad; see [6] for access. An arbitrary point U on L^∞ is given by $U = Au \cap L^\infty$, where u is an independent point; i.e., the user can vary u freely. The cevian triangle of U is def, the cevian triangle of the isotomic conjugate of U is DEF. Point D' is movable on line DU. Triangle $D'E'F'$ is thus a movable translation of DEF in the direction of U, and $D'E'F'$ stays perspective to ABC. The perspector P traverses a circumconic.

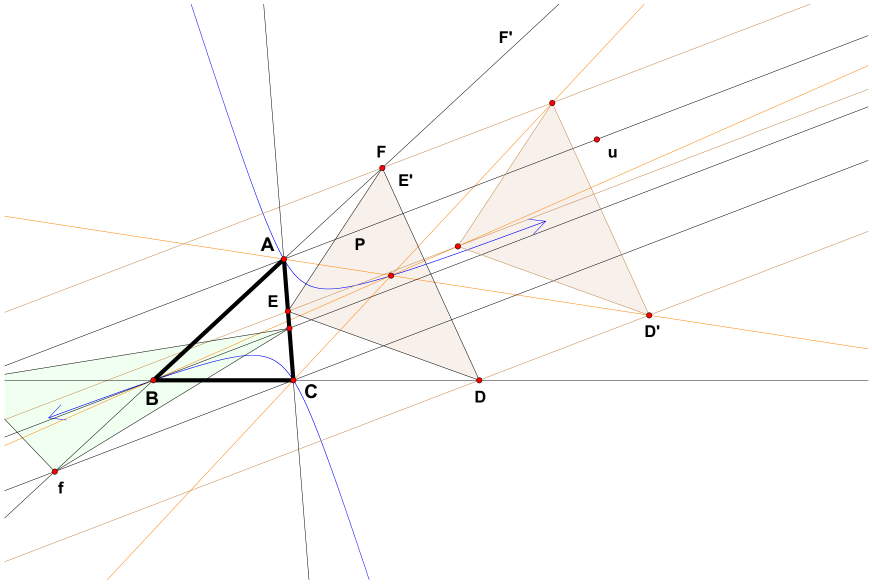


Figure 2. Cevian triangle and circumconic as in Theorem 5.

Proof. Write $X = x : y : z = b^2c^2v_1w_1 : c^2a^2w_1u_1 : a^2b^2u_1v_1$ where the point $U_1 = u_1 : v_1 : w_1$ is on L^∞ and $u_1v_1w_1 \neq 0$. Represent U_1 parametrically by

$$u_1 = (b-c)(1+bcs), \quad v_1 = (c-a)(1+cas), \quad w_1 = (a-b)(1+abs), \quad (16)$$

so that

$$(bv-cw)yz+ax(vz-wy) = \Lambda\left(bv-av-aw+cw+as(b^2v-abv+c^2w-acw)\right),$$

where

$$\Lambda = -a^3b^3c^3(b-c)(c-a)(a-b)(1+bcs)(1+cas)(1+abs).$$

Thus

$$(bv-cw)yz+ax(vz-wy) = \Lambda(-a(u+v+w)+as(a^2u+b^2v+c^2w)),$$

and by (12),

$$\begin{aligned} x' =& \frac{by-cz}{(bv-cw)yz+ax(vz-wy)} \\ =& \frac{(by-cz)/a}{\Lambda(-(u+v+w)+s(a^2u+b^2v+c^2w))}. \end{aligned}$$

Coordinates y' and z' are found in the same manner, and multiplying through by the common denominator gives

$$\begin{aligned} x' : y' : z' =& (by-cz)/a : (cz-ax)/b : (ax-by)/c \\ =& u_1(bv_1-cw_1) : v_1(cw_1-au_1) : w_1(au_1-bv_1) \\ =& CD(X_6, U_1^{-1}). \end{aligned}$$

Clearly, $ax'+by'+cz' = 0$, which is to say that the perspector X' is on L^∞. $\square$

Corollary 7. *As X traverses the Steiner circumellipse, the perspector X' traverses the line at infinity.*

Proof. The Steiner circumellipse is given by

$$bc\beta\gamma + ca\gamma\alpha + ab\alpha\beta = 0.$$

The corollary follows from the easy-to-verify fact that the isotomic conjugacy mapping carries the Steiner circumellipse to L^∞, to which Theorem 6 applies. □

Theorem 7 is exemplified by taking $X = X_{190}$; the isotomic conjugate of X is then X_{514}, for which the perspector is $X' = X_{900} = CD(X_6, X_{101})$. Other examples (X, X') are these: (X_{99}, X_{690}), (X_{668}, X_{891}), (X_{670}, X_{888}), (X_{671}, X_{690}), (X_{903}, X_{900}). These examples show that the mapping $X \to X'$ is not one-to-one.

6. Translated anticevian triangles

In this section, DEF is the anticevian triangle of a point $X = x : y : z$; given as a matrix by

$$\begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix} = \begin{pmatrix} -x & y & z \\ x & -y & z \\ x & y & -z \end{pmatrix}.$$

The perspectivity determinant (2) is given by $-2t(\Delta_0 + t\Delta_2)$, where

$$\Delta_0 = \begin{vmatrix} ax^2 & by^2 & cz^2 \\ u & v & w \\ x & y & z \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} ax^2 & by^2 & cz^2 \\ (bv+cw)u & (cw+au)v & (au+bv)w \\ x & y & z \end{vmatrix}.$$

The cubic $\Delta_0 = 0$ is already discussed in Section 3. The equation $\Delta_2 = 0$ represents the cubic $\mathcal{Z}(X_2/CS(X_6, U^{-1}))$. We consider four cases as in Section 3.

Case 6.1: $\Delta_0 = 0$ and $\Delta_2 = 0$. In this case, $D_tE_tF_t$ is perspective to ABC for every t. Clearly this holds for $X = X_2$, for all U. Now for any given U, the point $X = U$ is on both cubics. It is easy to prove that the point $CP(X_2, U)$ also lies on both cubics.

Case 6.2: $\Delta_0 = 0$ and $\Delta_2 \neq 0$. For given U, the isotomic conjugate X of U satisfies $\Delta_0 = 0$ and $\Delta_1 \neq 0$. For a diffferent example, let $U = X_{511}$; then the points listed for Case 5.2 in the cevian case are also points for which $\Delta_0 = 0$ and $\Delta_2 \neq 0$.

Case 6.3: $\Delta_0 \neq 0$ and $\Delta_2 = 0$. In this case, for any U, there is no ppt. For example, take $U = X_{523}$. Then the cubic $\Delta_2 = 0$ passes through the points in which the Brocard axis X_3X_6 meets the circumcircle these being X_{1312} and X_{1313}; these points do not also lie on the cubic $\Delta_0 = 0$.

Case 6.4: $\Delta_0 \neq 0$ and $\Delta_2 \neq 0$. In this case, $D_tE_tF_t$ is perspective to ABC for $t = -\Delta_0/\Delta_2$. The perspector is the point $x_2x_3 : x_2y_3 : z_2x_3$, which on cancellation of common factors, is the point $X' = x' : y' : z'$ given by

$$x' = \frac{by - cz}{bwy^2 - cvz^2 + (bv - cw)yz + ax(vz - wy)}, \tag{17}$$

$$y' = \frac{cz - ax}{cuz^2 - awx^2 + (cw - au)zx + by(wx - uz)}, \tag{18}$$

$$z' = \frac{ax - by}{avx^2 - buy^2 + (au - bv)xy + cz(uy - vx)}. \tag{19}$$

Theorem 8. *Suppose $X = CP(X_2, U_1)$, where U_1 is a point on L^∞ but not on a sideline BC, CA, AB. Then the perspector X' in Case 6.4 is invariant of the point U. In fact, $X' = CD(X_6, U_1^{-1})$, and X' lies on the circumconic given by*

$$au_1^2(bv_1 - cw_1)\beta\gamma + bv_1^2(cw_1 - au_1)\gamma\alpha + cw_1^2(au_1 - bv_1)\alpha\beta = 0. \tag{20}$$

Proof. Let $X = x : y : z = bu_1v_1 + cu_1w_1 : cv_1w_1 + av_1u_1 : aw_1u_1 + bw_1v_1$. Following the steps of the proof of Theorem 6, we have

$$\begin{aligned} & bwy^2 - cvz^2 + (bv - cw)yz + ax(vz - wy) \\ =& \widehat{\Lambda}(bv - av - aw + cw + as(b^2v - abv + c^2w - acw), \end{aligned}$$

where

$$\widehat{\Lambda} = 2abc(b - c)(c - a)(a - b)(1 + bcs)(1 + cas)(1 + abs).$$

Thus

$$bwy^2 - cvz^2 + (bv - cw)yz + ax(vz - wy) = \widehat{\Lambda}(-a(u + v + w) + as(a^2u + b^2v + c^2w)),$$

and by (17),

$$\begin{aligned} x' =& \frac{by - cz}{bwy^2 - cvz^2 + (bv - cw)yz + ax(vz - wy)} \\ =& \frac{(by - cz)/a}{\widehat{\Lambda}(-(u + v + w) + s(a^2u + b^2v + c^2w))}. \end{aligned}$$

Coordinates y' and z' are found in the same manner, and multiplying through by the common denominator gives

$$x' : y' : z' = CD(X_6, U_1^{-1}),$$

the same point as at the end of the proof of Theorem 6. It is easy to check that this point satisfies (20). □

Corollary 9. *As X traverses the Steiner inellipse, the perspector X' traverses the circumconic (20).*

Proof. The Steiner inellipse is given by

$$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 - 2bc\beta\gamma - 2ca\gamma\alpha - 2ab\alpha\beta = 0. \tag{21}$$

First, we note that, using (16), it is easy to show that if U_1 is on L^∞, then the point $X = x : y : z = CP(X_2, U_1)$ satisfies (21). Now, the mapping $U_1 \rightarrow$

$CP(X_2, U_1) = X$ is invertible; specifically, for given $X = x : y : z$ on the Steiner inellipse, the point $U_1 = u_1 : v_1 : w_1$ given by

$$u_1 : v_1 : w_1 = \frac{bc}{by + cz - ax} : \frac{ca}{cz + ax - by} : \frac{ab}{ax + by - cz}$$

is on L^∞, and Theorem 8 applies. $\square$

Corollary 9 is exemplified by taking $X = X_{1086}$, which is $CP(X_2, X_{514})$; the perspector is then $X' = X_{900} = CD(X_6, X_{101})$. Other examples (X, X') are these: (X_{115}, X_{690}), (X_{1015}, X_{891}), (X_{1084}, X_{888}), (X_{2482}, X_{690}). Note that the mapping $X \to X'$ is not one-to-one.

7. Translation along the Euler line

In this section, the perspectivity problem for both families, cevian and anticevian triangles, is discussed for translations in a single direction, namely the direction of the Euler line. Two points on the Euler line are the circumcenter, $a_1 : b_1 : c_1 = \cos A : \cos B : \cos C$ and

$$U = X_{30} = u : v : w = a_1 - 2b_1c_1 : b_1 - 2c_1a_1 : c_1 - 2a_1b_1,$$

the latter being the point in which the Euler line meets L^∞.

Theorem 10. *If X is the isotomic conjugate of a point X' on the Euler line other than X_2, then the perspector, in the case of the cevian triangle of X as given by (12)-(14), is X'.*

Proof. An arbitrary point X' on the Euler line is given parametrically by

$$a_1 + su : b_1 + sv : c_1 + sw,$$

and the isotomic conjugate $X = x : y : z$ by

$$a^{-2}(a_1 + su)^{-1} : b^{-2}(b_1 + sv)^{-1} : c^{-2}(c_1 + sw)^{-1}.$$

Substituting for x, y, z in (12) gives a product of several factors, of which exactly two involve s. The same holds for the results of substituting in (13) and (14). After canceling all common factors that do not contain s, the remaining coordinates for X' have a common factor $3s + 1$. This equals 0 for $s = -1/3$, for which $a_1 + su : b_1 + sv : c_1 + sw = X_2$. As $X' \neq X_2$, we can and do cancel $3s + 1$. The remaining coordinates are equivalent to those given just above for X'. $\square$

Theorem 11. *Suppose P is on the Euler line and $P \neq X_2$. Let $X = CP(X_2, P)$. Then the perspector X', in the case of the anticevian triangle of X, as given by (17)-(19), is the point P.*

Proof. Write

$$p = a_1 + su, \qquad q = b_1 + sv, \qquad r = c_1 + sw,$$

where $(u, v, w) = (a_1 - 2b_1c_1, b_1 - 2c_1a_1, c_1 - 2a_1b_1)$, so that the point $X = CP(X_2, P)$ is given by

$$x = p(bq + cr), \qquad y = q(cr + ap), \qquad z = r(ap + bq).$$

Substituting into (17)-(19) and factoring give expressions with several common factors. Canceling those, including the factor $3s+1$ which corresponds to the disallowed X_2, leaves trilinears for P. □

Theorem 12. *If P is on the circumcircle and $X = CS(X_6, P)$, then the perspector X', in the case of the anticevian triangle of X as given by (17)-(19), is the point $CD(X_6, P)$.*

Proof. Represent an arbitrary point $P = p : q : r$ on the circumcircle parametrically by

$$(p, q, r) = (\frac{1}{(b-c)(bc+s)}, \frac{1}{(c-a)(ca+s)}, \frac{1}{(a-b)(ab+s)}).$$

Then the point $X = CS(X_6, P)$ is given by

$$x = br + cq, \qquad y = cp + ar, \qquad z = aq + bp.$$

Substituting into (17)-(19) and factoring gives expressions with several common factors. Canceling those leaves trilinears for $CD(X_6, P)$. □

We conclude this section with a pair of examples. First, let $X = X_{618}$, the complement of the Fermat point (or 1st isogonic center), X_{13}. The perspector in the case of the anticevian triangle of X is the point X_{13}. Finally, let $X = X_{619}$, the complement of the 2nd isogonic center, X_{14}. The perspector in this case is the point X_{14}.

8. Translated rotated reference triangle

Let DEF be the rotation of ABC about the circumcenter of ABC. Let $U = u : v : w$ be a point on L^∞. In this section, we wish to translate DEF in the direction of line DU, seeking translations $D'E'F'$ that are perspective to ABC. Except for rotations of 0 and π, triangle DEF is not perspective to ABC, so that by Theorem 1, there are at most two perspective translations.

Yff's parameterization of the circumcircle ([1, p.39]) is used to express the rotation DEF of ABC counterclockwise with angle 2θ as follows:

$$\begin{aligned} D &= \csc\theta : \csc(C-\theta) : -\csc(B+\theta), \\ E &= -\csc(C+\theta) : \csc\theta : \csc(A-\theta), \\ F &= \csc(B-\theta) : -\csc(A+\theta) : \csc\theta. \end{aligned}$$

Let

$$\begin{aligned} r &= ((a+b+c)(b+c-a)(c+a-b)(a+b-c))^{1/2}/(2abc), \\ \theta_1 &= \sin\theta, \\ \theta_2 &= \cos\theta. \end{aligned}$$

Then the vertices D, E, F are given by the rows of the matrices

$$\begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix} = \begin{pmatrix} \theta_1^{-1} & (rc\theta_2 - c_1\theta_1)^{-1} & (rb\theta_2 + b_1\theta_1)^{-1} \\ (rc\theta_2 + c_1\theta_1)^{-1} & \theta_1^{-1} & (ra\theta_2 - a_1\theta_1)^{-1} \\ (rb\theta_2 - b_1\theta_1)^{-1} & (ra\theta_2 + a_1\theta_1)^{-1} & \theta_1^{-1} \end{pmatrix},$$

where

$$(a_1, b_1, c_1) = (\cos A, \cos B, \cos C)$$
$$= ((b^2 + c^2 - a^2)/(2bc), (c^2 + a^2 - b^2)/(2ca), (a^2 + b^2 - c^2)/(2ab)).$$

The perspectivity determinant (2) is factored using a computer. Only one of the factors involves t, and it is a polynomial $P(t)$ as in (3), with coefficients

$$p_0 = 4abc\theta_1^2,$$
$$p_1 = 4\theta_1^2 abc\,(au + bv + cw) = 0,$$
$$p_2 = (a + b - c)\,(a - b + c)\,(b - a + c)\,(a + b + c)\,(avw + buw + cuv)\,,$$

hence roots

$$\pm(\theta_1/r)(-abcs)^{-1/2}, \tag{22}$$

where $s = avw + buw + cuv$.

Conjecture. The perspectors given by (22) are a pair of antipodes on the circumcircle.

See Figure 3. [3] It would perhaps be of interest to study, for fixed H, the loci of D', E', F' as θ varies from 0 to π.

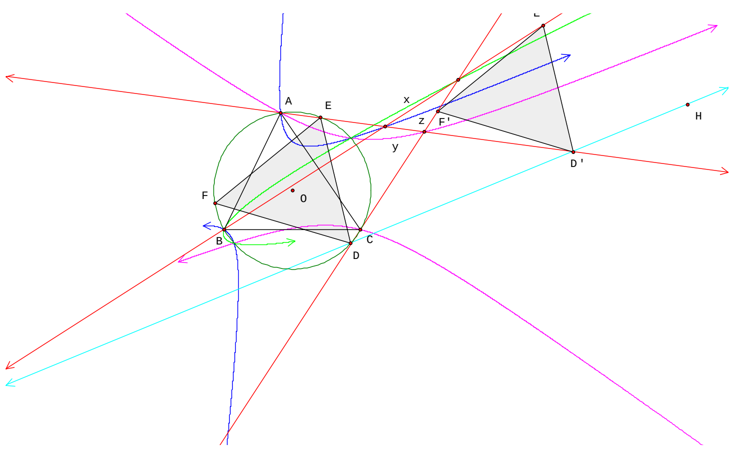


Figure 3. Translated rotation of ABC.

[3] Figure 3 can be viewed dynamically using The Geometer's Sketchpad; see [6] for access. Triangle DEF is a variable rotation of ABC about its circumcenter O. Independent point H determines line DH. Point D' is movable on line DH. Triangle $D'E'F'$ is thus a movable translation of DEF in the direction of DH. Three conics as in Theorem 4 meet in two points, which according to the Conjecture are a pair of antipodes on the circumcircle.

References

[1] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[2] C. Kimberling, Conics associated with a cevian nest, *Forum Geom.*, 1 (2001) 141–150.

[3] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000-, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[4] C. Kimberling, *Glossary*, 2002-, http://faculty.evansville.edu/ck6/encyclopedia/glossary.html.

[5] C. Kimberling, *Points on Cubics*, 2003-, http://faculty.evansville.edu/ck6/encyclopedia/Intro&Zcubics.html .

[6] C. Kimberling, *Translated Triangles*, 2005, http://faculty.evansville.edu/ck6/encyclopedia/TransTri.html .

[7] E. A. Maxwell, *General Homogeneous Coordinates*, Cambridge University Press, Cambridge, 1957.

[8] E. Weisstein, *MathWorld*, http://mathworld.wolfram.com/ .

Clark Kimberling: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, USA

E-mail address: ck6@evansville.edu

Forum Geometricorum
Volume 6 (2006) 285–288.

FORUM GEOM
ISSN 1534-1178

On the Derivative of a Vertex Polynomial

James L. Parish

Abstract. A geometric characterization of the critical points of a complex polynomial f is given, in the special case where the zeros of f are the vertices of a polygon affine-equivalent to a regular polygon.

1. Steiner Polygons

The relationship between the locations of the zeros of a complex polynomial f and those of its derivative has been extensively studied. The best-known theorem in this area is the Gauss-Lucas Theorem, that the zeros of f' lie in the convex hull of the zeros of f. The following theorem [1, p.93], due to Linfield, is also of interest:

Theorem 1. *Let $\lambda_j \in \mathbf{R}\backslash\{0\}$, $j = 1, \dots, k$, and let z_j, $j = 1, \dots, k$ be distinct complex numbers. Then the zeros of the rational function $R(z) := \sum_{j=1}^{k} \frac{\lambda_j}{z-z_j}$ are the foci of the curve of class $k-1$ which touches each of the $k(k-1)/2$ line segments $\overline{z_\mu, z_\nu}$ in a point dividing that line segment in the ratio $\lambda_\mu : \lambda_\nu$.*

Since $f' = f \cdot \sum_{j=1}^{k} \frac{1}{z-z_j}$, where the z_j are the zeros of f, Linfield's Theorem can be used to locate the zeros of f' which are not zeros of f.

In this paper, we will consider the case of a polynomial whose zeros form the vertices of a polygon which is affine-equivalent to a regular polygon; the zeros of the derivative can be geometrically characterized in a manner resembling Linfield's Theorem. First, let ζ be a primitive nth root of unity, for some $n \geq 3$. Define $G(\zeta)$ to be the n-gon whose vertices are $\zeta^0, \zeta^1, \dots, \zeta^{n-1}$.

Proposition 2. *Let $n \geq 3$, and let G be an n-gon with vertices $v_0, \dots, v_{n-1}$, no three of which are collinear. The following are equivalent.*

(1) *There is an ellipse which is tangent to the edges of G at their midpoints.*
(2) *G is affine-equivalent to $G(\zeta)$ for some primitive nth root of unity ζ.*
(3) *There is a primitive nth root of unity ζ and complex constants g, u, v such that $|u| \neq |v|$ and, for $k = 0, \dots, n-1$, $v_k = g + u\zeta^k + v\zeta^{-k}$.*

Proof. 1)$\Longrightarrow$2): Applying an affine transformation if necessary, we may assume that the ellipse is a circle centered at 0 and that $v_0 = 1$. Let m_0 be the midpoint of the edge v_0v_1. v_0v_1 is then perpendicular to $0m_0$, and v_0, v_1 are equidistant from m_0; it follows that the right triangles $0m_0v_0$ and $0m_0v_1$ are congruent, and in

Publication Date: November 13, 2006. Communicating Editor: Paul Yiu.

particular that v_1 also lies on the unit circle. Now let m_1 be the midpoint of v_1v_2; since m_0 and m_1 are equidistant from 0 and the triangles $0m_0v_1$, $0m_1v_1$ are right, they are congruent, and m_0, m_1 are equidistant from v_1. It follows that the edges v_0v_1 and v_1v_2 have the same length. Furthermore, the triangles $0v_0v_1$ and $0v_1v_2$ are congruent, whence $v_2 = v_1^2$. Similarly we obtain $v_k = v_1^k$ for all k, and in particular that $v_1^n = v_0 = 1$. $\zeta = v_1$ is a primitive nth root of unity since none of $v_0, \ldots, v_{n-1}$ coincide, and $G = G(\zeta)$.

2)$\Longrightarrow$1): $G(\zeta)$ has an ellipse – indeed, a circle – tangent to its edges at their midpoints; an affine transformation preserves this.

2)$\Longleftrightarrow$3): Any real-linear transformation of $\mathbf{C}$ can be put in the form $z \mapsto uz + v\overline{z}$ for some choice of u, v, and conversely; the transformation is invertible iff $|u| \neq |v|$. □

We will refer to an n-gon satisfying these conditions as a *Steiner n-gon*; when needed, we will say it *has root* ζ. The ellipse is its *Steiner inellipse*. (This is a generalization of the case $n = 3$; every triangle is a Steiner triangle.) The parameters g, u, v are its *Fourier coordinates*. Note that a Steiner n-gon is regular iff either u or v vanishes.

2. The Foci of the Steiner Inellipse

Now, let S_ζ be the set of Steiner n-gons with root ζ for which the constant g, above, is 0. We may use the Fourier coordinates u, v to identify it with an open subset of $\mathbf{C}^2$. Let Φ be the map taking the n-gon with vertices $v_0, v_1, \ldots, v_{n-1}$ to the n-gon with vertices $v_1, \ldots, v_{n-1}, v_0$. If f is a complex-valued function whose domain is a subset of S_ζ which is closed under Φ, write φf for $f \circ \Phi$. Note that $\varphi u = \zeta u$ and $\varphi v = \zeta^{-1} v$; this will prove useful. Note also that special points associated with n-gons may be identified with complex-valued functions on appropriate subsets of S_ζ.

We define several useful fields associated with S_ζ. First, let $F = \mathbf{C}(u, v, \overline{u}, \overline{v})$, where u, v are as in 3) of the above proposition. φ is an automorphism of F. Let $K = \mathbf{C}(x, y, \overline{x}, \overline{y})$ be an extension field of F satisfying $x^2 = u$, $y^2 = v$, $\overline{x}^2 = \overline{u}, \overline{y}^2 = \overline{v}$. Let θ be a fixed square root of ζ; we extend φ to K by setting $\varphi x = \theta x$, $\varphi y = \theta^{-1} y$, $\varphi \overline{x} = \theta^{-1} \overline{x}$, $\varphi \overline{y} = \theta \overline{y}$. Let K_0 be the fixed field of φ and K_1 the fixed field of φ^n. Elements of F may be regarded as complex-valued functions defined on dense open subsets of S_ζ. Functions corresponding to elements of K may only be defined locally; however, given $G \in S_\zeta$ such that $uv \neq 0$ and $f \in K_1$ defined at G, one may choose a small neighborhood U_0 of G which is disjoint from $\Phi^k(U_0)$, $k = 1, \ldots, n-1$ and on which neither u nor v vanish; f may then be defined on $U = \bigcup_{k=0}^{n-1} \Phi^k(U_0)$.

For the remainder of this section, G is a fixed Steiner n-gon with root ζ. The vertices of G are $v_0, \ldots, v_{n-1}$. We have the following.

Proposition 3. *The foci of the Steiner inellipse of G are located at $f_\pm = g \pm (\theta + \theta^{-1})xy$.*

Proof. Translating if necessary, we may assume that $g = 0$, i.e., $G \in S_\zeta$. Note first that $f_\pm \in K_0$. (This is to be expected, since the Steiner inellipse and its foci do not depend on the choice of initial vertex.) For $k = 0, \ldots, n-1$, let $m_k = (v_k + v_{k+1})/2$, the midpoint of the edge $v_k v_{k+1}$. Let $d_\pm$ be the distance from $f_\pm$ to m_0; we will first show that $d_+ + d_-$ is invariant under φ. (This will imply that the sum of the distances from $f_\pm$ to m_k is the same for all k.) Now, $m_0 = (v_0+v_1)/2 = ((1+\zeta)u+(1+\zeta^{-1})v)/2 = (\theta+\theta^{-1})(\theta x^2+\theta^{-1}y^2)/2$. Thus, $m_0 - f_+ = (\theta+\theta^{-1})(\theta x^2 - 2xy + \theta^{-1}y^2)/2 = (\zeta+1)(x-\theta^{-1}y)^2/2$. Hence $d_+ = |m_0 - f_+| = |\zeta+1|(x-\theta^{-1}y)(\overline{x}-\theta\overline{y})/2$. Similarly, $d_- = |\zeta+1|(x+\theta^{-1}y)(\overline{x}+\theta\overline{y})/2$, and so $d_+ + d_- = |\zeta+1|(x\overline{x}+y\overline{y})$, which is invariant under φ as claimed. This shows that there is an ellipse with foci $f_\pm$ passing through the midpoints of the edges of G. If $n \geq 5$, this is already enough to show that this ellipse is the Steiner inellipse; however, for $n = 3, 4$ it remains to show that this ellipse is tangent to the sides, or, equivalently, that the side $v_k v_{k+1}$ is the external bisector of the angle $\angle f_+ m_k f_-$. It suffices to show that $A_k = (m_k - v_k)(m_k - v_{k+1})$ is a positive multiple of $B_k = (m_k - f_+)(m_k - f_-)$. Now $A_0 = -(\zeta-1)^2(u-\zeta^{-1}v)^2/4$, and $B_0 = (\zeta+1)^2(x-\theta^{-1}y)^2(x+\theta^{-1}y)^2/4 = (\zeta+1)^2(u-\zeta^{-1}v)^2/4$; thus, $A_0/B_0 = -(\zeta-1)^2/(\zeta+1)^2 = -(\theta-\theta^{-1})^2/(\theta+\theta^{-1})^2$, which is evidently positive. This quantity is invariant under φ; hence A_k/B_k is also positive for all k. □

Corollary 4. *The Steiner inellipse of G is a circle iff G is similar to $G(\zeta)$.*

Proof. $f_+ = f_-$ iff $xy = 0$, i.e., iff one of u and v is zero. (Note that $\theta + \theta^{-1} \neq 0$.) □

Define the *vertex polynomial* $f_G(z)$ of G to be $\prod_{k=0}^{n-1}(z - v_k)$. We have the following.

Proposition 5. *The foci of the Steiner inellipse of G are critical points of f_G.*

Proof. Again, we may assume $G \in S_\zeta$. Since $f_G'/f_G = \sum_{k=0}^{n-1}(z-v_k)^{-1}$, it suffices to show that this sum vanishes at $f_\pm$. Now f_+ is invariant under φ, and $v_k = \varphi^k v_0$; hence $\sum_{k=0}^{n-1}(f_+ - v_k)^{-1} = \sum_{k=0}^{n-1}\varphi^k(f_+ - v_0)^{-1}$. $(f_+ - v_0)^{-1} = -\theta/((\theta y - x)(y - \theta x))$ Now let $g = \theta^2/((\theta^2-1)x(\theta y - x))$. Note that $g \in K_1$; that is, $\varphi^n g = g$. A straightforward calculation shows that $(f_+ - v_0)^{-1} = g - \varphi g$; therefore, $\sum_{k=0}^{n-1}\varphi^k(f_+ - v_0)^{-1} = \sum_{k=0}^{n-1}(\varphi^k g - \varphi^{k+1}g) = g - \varphi^n g = 0$, as desired. The proof that f_- is a critical point of f_G is similar. □

3. Holomorphs

Again, we let G be a Steiner n-gon with root ζ and vertices $v_0, \ldots, v_{n-1}$. For any integer m, we set $v_m = v_l$ where $l = 0, \ldots, n-1$ is congruent to m mod n. The following lemma is trivial.

Lemma 6. *Let $k = 1, \ldots, \lfloor n/2 \rfloor$. Then:*

(1) *If k is relatively prime to n, let G^k be the n-gon with vertices $v_0^k, \ldots, v_{n-1}^k$ given by $v_j^k = v_{jk}$. Then G^k is a Steiner n-gon with root ζ^k, and its Fourier coordinates are g, u, v.*

(2) *If* $d = \gcd(k, n)$ *is greater than* 1 *and less than* $n/2$*, set* $m = n/d$*. Then, for* $l = 0, \ldots, d-1$*, let* $G^{k,l}$ *be the* m*-gon with vertices* $v_0^{k,l}, \ldots, v_{m-1}^{k,l}$ *given by* $v_j^{k,l} = v_{kj+l}$*. Then, for each* l*,* $G^{k,l}$ *is a Steiner* m*-gon with root* ζ^k*, and the Fourier coordinates of* $G^{k,l}$ *are* $g, \zeta^l u, \zeta^{-l} v$*. The* $G^{k,l}$ *all have the same Steiner inellipse.*

(3) *If* $k = n/2$*, the line segments* $v_j v_{j+k}$ *all have midpoint* g*.*

In the three given cases, we will say k*-holomorph of* G to refer to G^k, the union of the m-gons $G^{k,l}$, or the union of the line segments $v_j v_{j+k}$. We extend the definition of *Steiner inellipse* to the k-holomorphs in Cases 2 and 3, meaning the common Steiner inellipse of the $G^{k,l}$ or the point g, respectively. The propositions of Section II clearly extend to Case 2; since the foci are critical points of the vertex polynomials of each of the $G^{k,l}$, they are also critical points of their product. In Case 3, taking g as a degenerate ellipse – indeed, circle – with focus at g, the propositions likewise extend; in this case, $\theta = \pm i$, so $\theta + \theta^{-1} = 0$, and the sole critical point of $(z - v_j)(z - v_{j+k})$ is $(v_j + v_{j+k})/2 = g$.

In Cases 1 and 2, it should be noted that the Steiner inellipse is a circle iff the Steiner inellipse of G itself is a circle – i.e., G is similar to $G(\zeta)$. It should also be noted that the vertex polynomials of the holomorphs of G are equal to f_G itself; hence they have the same critical points. Suppose that G is not similar to $G(\zeta)$. If n is odd, G has $(n-1)/2$ holomorphs, each with a noncircular Steiner inellipse and hence two distinct Steiner foci; these account for the $n-1$ critical points of f_G. If n is even, G has $(n-2)/2$ holomorphs in Cases 1 and 2, each with two distinct Steiner foci, and in addition the Case 3 holomorph, providing one more Steiner focus; again, these account for $n-1$ critical points of f_G. On the other hand, if G is similar to $G(\zeta)$, then $f_G = (z-g)^n - r^n$ for some real r; the Steiner foci of the holomorphs of G collapse together, and f_G has an $(n-1)$-fold critical point at g. We have proven the following.

Theorem 7. *If* G *is a Steiner* n*-gon, the critical points of* f_G *are the foci of the Steiner inellipses of the holomorphs of* G*, counted with multiplicities if* G *is regular. They are collinear, lying at the points* $g + (2\cos k\pi/n)xy$*, as* k *ranges from* 0 *to* $n-1$*.*

(For the last statement, note that $\cos(n-k)\pi/n = -\cos k\pi/n$.)

Reference

[1] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, Clarendon Press, Oxford, 2002.

James L. Parish:Department of Mathematics and Statistics, Southern Illinois University, Edwardsville, Edwardsville, IL USA 62026

E-mail address: jparish@siue.edu

Forum Geometricorum
Volume 6 (2006) 289–295.

FORUM GEOM
ISSN 1534-1178

On Two Remarkable Lines Related to a Quadrilateral

Alexei Myakishev

Abstract. We study the Euler line of an arbitrary quadrilateral and the Nagel line of a circumscriptible quadrilateral.

1. Introduction

Among the various lines related to a triangle the most popular are Euler and Nagel lines. Recall that the Euler line contains the orthocenter H, the centroid G, the circumcenter O and the nine-point center E, so that $HE : EG : GO = 3 : 1 : 2$. On the other hand, the Nagel line contains the Nagel point N, the centroid M, the incenter I and Spieker point S (which is the centroid of the perimeter of the triangle) so that $NS : SG : GI = 3 : 1 : 2$. The aim of this paper is to find some analogies of these lines for quadrilaterals.

It is well known that in a triangle, the following two notions of centroids coincide:
(i) the barycenter of the system of unit masses at the vertices,
(ii) the center of mass of the boundary and interior of the triangle.

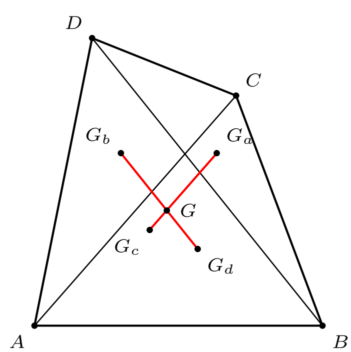


Figure 1.

But for quadrilaterals these are not necessarily the same. We shall show in this note, that to get some fruitful analogies for quadrilateral it is useful to consider the centroid G of quadrilateral as a whole figure. For a quadrilateral $ABCD$, this centroid G can be determined as follows. Let G_a, G_b, G_c, G_d be the centroids of triangles BCD, ACD, ABD, ABC respectively. The centroid G is the intersection of the lines G_aG_c and G_bG_d:

$$G = G_aG_c \cap G_bG_d.$$

See Figure 1.

Publication Date: November 20, 2006. Communicating Editor: Paul Yiu.

2. The Euler line of a quadrilateral

Given a quadrilateral $ABCD$, denote by O_a and H_a the circumcenter and the orthocenter respectively of triangle BCD, and similarly, O_b, H_b for triangle ACD, O_c, H_c for triangle ABD, and O_d, H_d for triangle ABC. Let

$$O = O_aO_c \cap O_bO_d,$$
$$H = H_aH_c \cap H_bH_d.$$

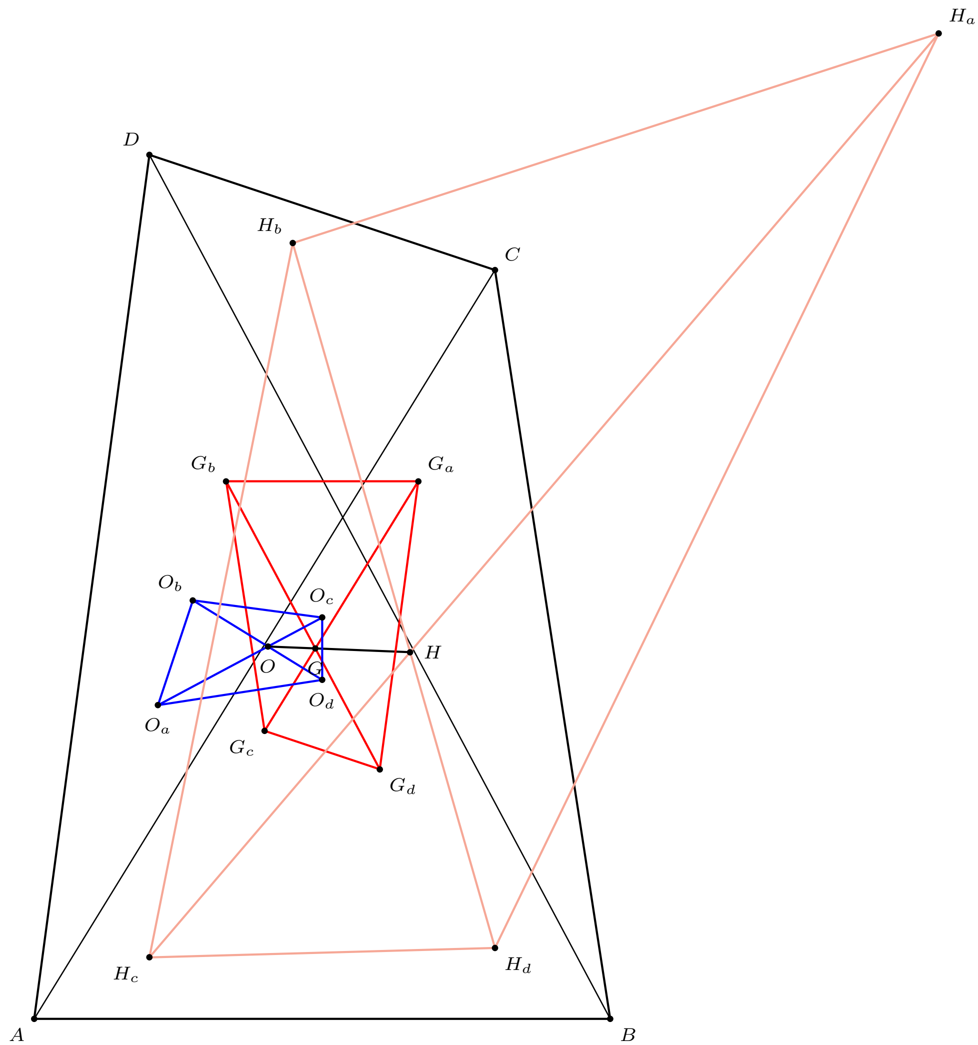


Figure 2

We shall call O the quasicircumcenter and H the quasiorthocenter of the quadrilateral $ABCD$. Clearly, the quasicircumcenter O is the intersection of perpendicular bisectors of the diagonals of $ABCD$. Therefore, if the quadrilateral is cyclic, then O is the center of its circumcircle. Figure 2 shows the three associated quadrilaterals $G_aG_bG_cG_d$, $O_aO_bO_cO_d$, and $H_aH_bH_cH_d$.

The following theorem was discovered by Jaroslav Ganin, (see [2]), and the idea of the proof was due to François Rideau [3].

Theorem 1. *In any arbitrary quadrilateral the quasiorthocenter H, the centroid G, and the quasicircumcenter O are collinear. Furthermore, $OH : HG = 3 : -2$.*

Proof. Consider three affine maps f_G, f_O and f_H transforming the triangle ABC onto triangle $G_aG_bG_c$, $O_aO_bO_c$, and $H_aH_bH_c$ respectively.

In the affine plane, write $D = xA + yB + zC$ with $x + y + z = 1$.

(i) Note that

$$\begin{aligned}
f_G(D) =& f_G(xA + yB + zC)\\
=& xG_a + yG_b + zG_c\\
=& \frac{1}{3}(x(B + C + D) + y(A + C + D) + z(A + B + D))\\
=& \frac{1}{3}((y + z)A + (z + x)B + (x + y)C + (x + y + z)D)\\
=& \frac{1}{3}((y + z)A + (z + x)B + (x + y)C + (xA + yB + zC))\\
=& \frac{1}{3}(x + y + z)(A + B + C)\\
=& G_d.
\end{aligned}$$

(ii) It is obvious that triangles ABC and $O_aO_bO_c$ are orthologic with centers D and O_d. See Figure 3. From Theorem 1 of [1], $f_O(D) = O_d$.

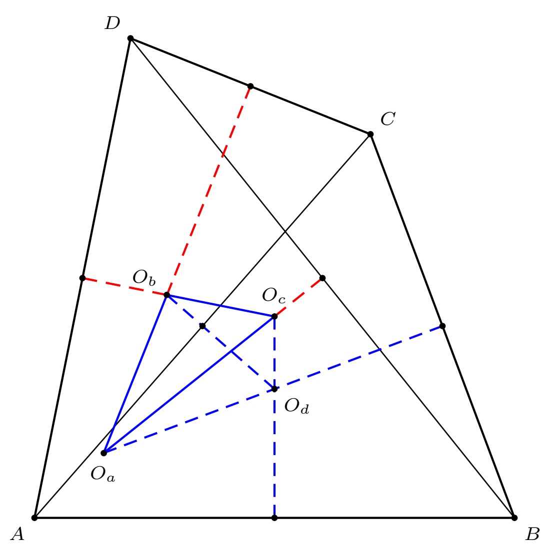


Figure 3

(iii) Since H_a divides O_aG_a in the ratio $O_aH_a : H_aG_a = 3 : -2$, and similarly for H_b and H_c, for $Q = A,\ B,\ C$, the point $f_H(Q)$ divides the segment $f_O(Q)f_G(Q)$ into the ratio $3 : -2$. It follows that for *every* point Q in the plane of ABC, $f_H(Q)$ divides $f_O(Q)f_G(Q)$ in the same ratio. In particular, $f_H(D)$ divides $f_O(D)f_G(D)$, namely, O_dG_d, in the ratio $3:-2$. This is clearly H_d. We have shown that $f_H(D) = H_d$.

(iv) Let $Q = AC \cap BD$. Applying the affine maps we have

$$
\begin{aligned}
f_G(Q) =& G_aG_c \cap G_bG_d = G, \\
f_O(Q) =& O_aO_c \cap O_bO_d = O, \\
f_H(Q) =& H_aH_c \cap H_bH_d = H.
\end{aligned}
$$

From this we conclude that H divides OG in the ratio $3:-2$. $\square$

Theorem 1 enables one to define the *Euler line* of a quadrilateral $ABCD$ as the line containing the centroid, the quasicircumcenter, and the quasiorthocenter. This line contains also the quasininepoint center E defined as follows. Let E_a, E_b, E_c, E_d be the nine-point centers of the triangles BCD, ACD, ABD, ABC respectively. We define the quasininepoint center to be the point $E = E_aE_c \cap E_bE_d$. The following theorem can be proved in a way similar to Theorem 1 above.

Theorem 2. *E is the midpoint of OH.*

3. The Nagel line of a circumscriptible quadrilateral

A quadrilateral is circumscriptible if it has an incircle. Let $ABCD$ be a circumscriptible quadrilateral with incenter I. Let T_1, T_2, T_3, T_4 be the points of tangency of the incircle with the sides AB, BC, CD and DA respectively. Let N_1 be the isotomic conjugate of T_1 with respect to the segment AB. Similarly define N_2, N_3, N_4 in the same way. We shall refer to the point $N := N_1N_3 \cap N_2N_4$ as the Nagel point of the circumscriptible quadrilateral. Note that both lines divide the perimeter of the quadrilateral into two equal parts.

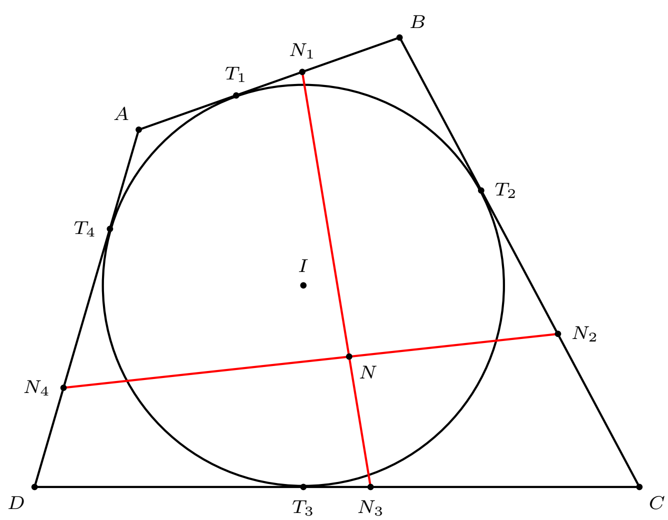


Figure 4.

In Theorem 6 below we shall show that N lies on the line joining I and G. In what follows we shall write

$$P = (x \cdot A,\ y \cdot B,\ z \cdot C,\ w \cdot D)$$

to mean that P is the barycenter of a system of masses x at A, y at B, z at C, and w at D. Clearly, x, y, z, w can be replaced by kx, ky, kz, kw for nonzero k without changing the point P. In Figure 4, assume that $AT_1 = AT_4 = p$, $BT_2 = BT_1 = q$, $CT_3 = CT_2 = r$, and $DT_4 = DT_3 = t$. Then by putting masses p at A, q at B, r at C, and t at D, we see that
(i) $N_1 = (p \cdot A,\ q \cdot B,\ 0 \cdot C,\ 0 \cdot D)$,
(ii) $N_3 = (0 \cdot A,\ 0 \cdot B,\ r \cdot C,\ t \cdot D)$, so that the barycenter $N = (p \cdot A,\ q \cdot B,\ r \cdot C,\ t \cdot D)$ is on the line N_1N_3. Similarly, it is also on the line N_2N_4 since
(iii) $N_2 = (0 \cdot A,\ q \cdot B,\ r \cdot C,\ 0 \cdot D)$, and
(iv) $N_4 = (p \cdot A,\ 0 \cdot B,\ 0 \cdot C,\ t \cdot D)$.

Therefore, we have established the first of the following three lemmas.

Lemma 3. $N = (p \cdot A,\ q \cdot B,\ r \cdot C,\ t \cdot D)$.

Lemma 4. $I = ((q + t)A,\ (p + r)B,\ (q + t)C,\ (p + r)D)$.

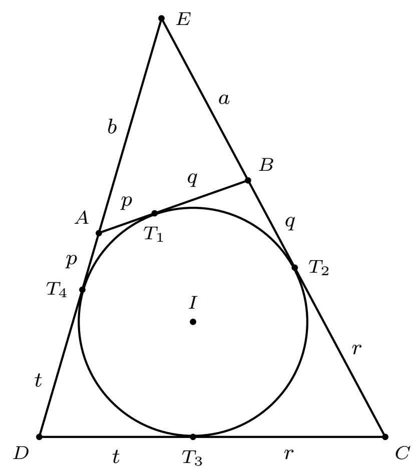


Figure 5.

Proof. Suppose the circumscriptible quadrilateral $ABCD$ has a pair of non-parallel sides AD and BC, which intersect at E. (If not, then $ABCD$ is a rhombus, $p = q = r = s$, and $I = G$; the result is trivial). Let $a = EB$ and $b = EA$.
(i) As the incenter of triangle EDC, $I = ((t + r)E,\ (a + q + r)D,\ (b + p + t)C)$.
(ii) As an excenter of triangle ABE, $I = ((p + q)E,\ -a \cdot A,\ -b \cdot B)$.

Note that $\frac{EC}{EB} = \frac{a+q+r}{a}$ and $\frac{ED}{EA} = \frac{b+p+t}{b}$, so that the system $(p+q+r+t)E$ is equivalent to the system $((a+q+r)B,\ -a \cdot C,\ (b+p+t)A,\ -b \cdot D)$. Therefore,

$$I = ((-a+b+p+t)A,\ (-b+a+q+r)B,\ (-a+b+p+t)C,\ (-b+a+q+r)D).$$

Since $b + p = a + q$, the result follows. $\square$

Lemma 5. $G = ((p+q+t)A,\ (p+q+r)B,\ (q+r+t)C,\ (p+r+t)D)$.

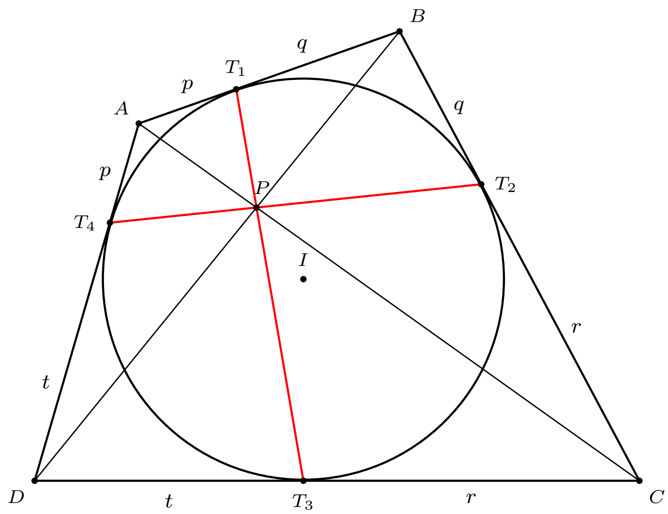


Figure 6.

Proof. Denote the point of intersection of the diagonals by P. Note that $\frac{AP}{CP} = \frac{p}{r}$ and $\frac{BP}{DP} = \frac{q}{t}$. Actually, according to one corollary of Brianchon's theorem, the lines T_1T_3 and T_2T_4 also pass through P. For another proof, see [4, pp.156–157]. Hence,

$$P = \left(\frac{1}{p} \cdot A,\ \frac{1}{q} \cdot B,\ \frac{1}{r} \cdot C,\ \frac{1}{t} \cdot D\right).$$

Consequently, $P = \left(\frac{1}{q} \cdot B,\ \frac{1}{t} \cdot D\right)$ and also $P = \left(\frac{1}{p} \cdot A,\ \frac{1}{r} \cdot C\right)$.

The quadrilateral $G_aG_bG_cG_d$ is homothetic to $ABCD$, with homothetic center $M = (1 \cdot A,\ 1 \cdot B,\ 1 \cdot C,\ 1 \cdot D)$ and ratio $-\frac{1}{3}$. Thus, $\frac{G_aG}{G_cG} = \frac{AP}{CP} = \frac{p}{r}$ and $\frac{G_bG}{G_dG} = \frac{BP}{DP} = \frac{q}{t}$. It follows that $G = (r \cdot G_a,\ p \cdot G_c) = (p \cdot A,\ (r+p)B, r \cdot C,\ (r+p)D)$ and $G = (t \cdot G_b,\ q \cdot G_d) = ((q+t)A,\ q \cdot B,\ (q+t)C,\ t \cdot D)$. To conclude the proof, it is enough to add up the corresponding masses. □

The following theorem follows easily from Lemmas 3, 4, 5.

Theorem 6. *For a circumscriptible quadrilateral, the Nagel point N, centroid G and incenter I are collinear. Furthermore, $NG : GI = 2 : 1$.*

See Figure 7.

Theorem 6 enables us to define the Nagel line of a circumscriptible quadrilateral. This line also contains the Spieker point of the quadrilateral, by which we mean the center of mass S of the perimeter of the quadrilateral, without assuming an incircle.

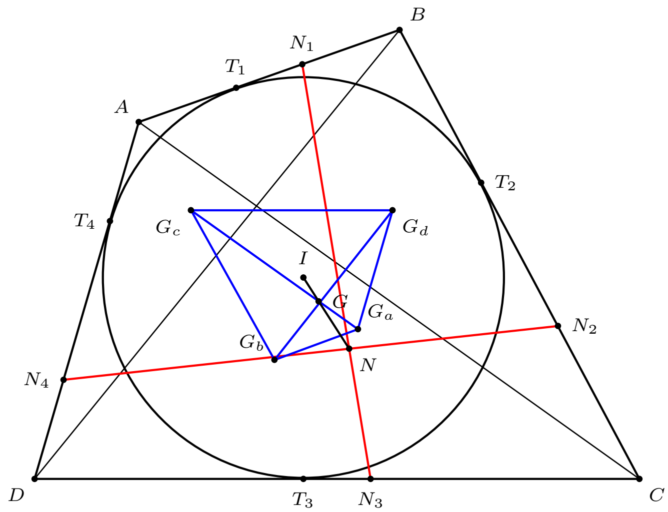


Figure 7.

Theorem 7. *For a circumscriptible quadrilateral, the Spieker point is the midpoint of the incenter and the Nagel point.*

Proof. With reference to Figure 6, each side of the circumscriptible quadrilateral is equivalent to a mass equal to its length located at each of its two vertices. Thus,

$$S = ((2p+q+t)A,\ (p+2q+r)B,\ (q+2r+t)C,\ (p+r+2t)D).$$

Splitting into two systems of equal total masses, we have

$$\begin{aligned} N =&(2pA,\ 2qB,\ 2rC,\ 2tD), \\ I =&((q+t)A,\ (p+r)B,\ (q+t)C,\ ((p+r)D). \end{aligned}$$

From this the result is clear. $\square$

References

[1] E. Danneels and N. Dergiades, A theorem on orthology centers, *Forum Geom.*, 4 (2004) 135–141.

[2] A. Myakishev, Hyacinthos message 12400, March 16, 2006.

[3] F. Rideau, Hyacinthos message 12402, March 16, 2006.

[4] P. Yiu, *Euclidean Geometry*, Florida Atlantic University Lecture Notes, 1998.

Alexei Myakishev: Smolnaia 61-2, 138, Moscow, Russia, 125445

E-mail address: `alex.geom@mtu-net.ru`

Forum Geometricorum
Volume 6 (2006) 297–300.

FORUM GEOM
ISSN 1534-1178

Intersecting Circles and their Inner Tangent Circle

Max M. Tran

Abstract. We derive the general equation for the radius of the inner tangent circle that is associated with three pairwise intersecting circles. We then look at three special cases of the equation.

It seems to the author that there should be one equation that gives the radius of the inner tangent circle inscribed in a triangular region bounded by either straight lines or circular arcs. As a step toward this goal of a single equation, consider three circles $\mathcal{C}_A, \mathcal{C}_B$ and $\mathcal{C}_C$ with radii α, β, γ respectively. $\mathcal{C}_A$ intersects $\mathcal{C}_B$ at an angle θ. $\mathcal{C}_B$ intersects $\mathcal{C}_C$ at an angle ρ. And $\mathcal{C}_C$ intersects $\mathcal{C}_A$ at an angle ϕ, with $0 \leqslant \theta, \rho, \phi \leqslant \pi$. We seek the radius of the circle $\mathcal{C}$, tangent externally to each of the given circles. See Figure 1. If the three intersecting cirlces were just touching instead, the inner tangent circle would be the inner Soddy circle. See [1]. The points of tangency of the inner tangent circle form the vertices of an inscribed triangle. We set up a coordinate system with the origin at the center of $\mathcal{C}$. See Figure 1.

Let the points of tangency A, B, C be represented by complex numbers of moduli R, the radius of $\mathcal{C}$. With these labels, the triangle ABC and the inscribed triangle is one and the same. Letting the lengths of the sides BC, CA, AB be a, b, c respectively, then

$$\| A - B \| = c \quad \text{and} \quad \langle A, B \rangle = R^2 - \frac{c^2}{2}. \tag{1}$$

Corresponding relations hold for the pairs B, C and C, A. With the above coordinate system, the centers of the circles $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$ are respectively $\frac{R+\alpha}{R}A$, $\frac{R+\beta}{R}B$, $\frac{R+\gamma}{R}C$.

The circles $\mathcal{C}_A$ and $\mathcal{C}_B$ intersect at angle θ if and only if

$$\left\| \frac{R+\alpha}{R}A - \frac{R+\beta}{R}B \right\| = \alpha^2 + \beta^2 + 2\alpha\beta\cos\theta.$$

By an application of (1) and the use of a half angle formula, the above can be shown to be equivalent to

$$c^2 = \frac{4R^2\alpha\beta\cos^2\frac{\theta}{2}}{(R+\alpha)(R+\beta)}.$$

Publication Date: November 27, 2006. Communicating Editor: Paul Yiu.

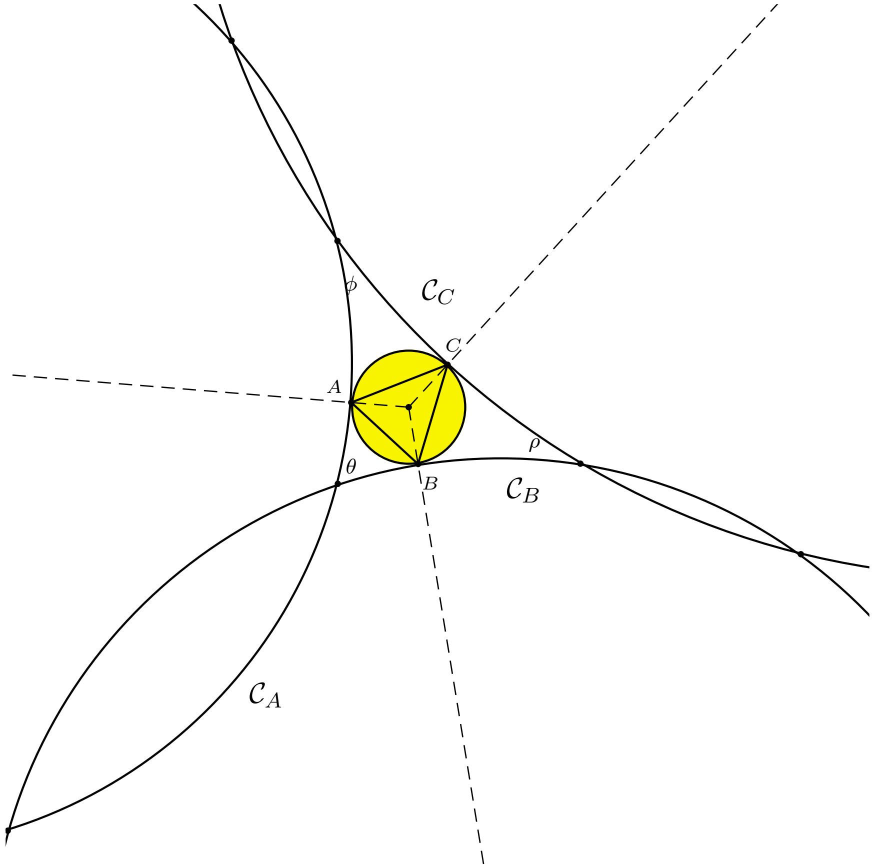


Figure 1

Thus the three circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ intersect each other at the given angles if and only if

$$\begin{aligned} a^2 &= \frac{4R^2\beta\gamma\cos^2\frac{\rho}{2}}{(R+\beta)(R+\gamma)}, \\ b^2 &= \frac{4R^2\alpha\gamma\cos^2\frac{\phi}{2}}{(R+\alpha)(R+\gamma)}, \\ c^2 &= \frac{4R^2\alpha\beta\cos^2\frac{\theta}{2}}{(R+\alpha)(R+\beta)}. \end{aligned} \tag{2}$$

These equations are then used to solve for R in terms of α, β, γ, θ, ϕ and ρ. In the first step of this process, we multiply the equations in (2) and take square root to obtain

$$abc = \frac{8\alpha\beta\gamma R^3\cos\frac{\theta}{2}\cos\frac{\phi}{2}\cos\frac{\rho}{2}}{(R+\alpha)(R+\beta)(R+\gamma)}. \tag{3}$$

Using (3) and (2) we obtain,

$$\begin{aligned}\frac{\alpha}{R+\alpha} &= \frac{bc\cos\frac{\rho}{2}}{2Ra\cos\frac{\theta}{2}\cos\frac{\phi}{2}},\\ \frac{\beta}{R+\beta} &= \frac{ac\cos\frac{\phi}{2}}{2Rb\cos\frac{\theta}{2}\cos\frac{\rho}{2}},\\ \frac{\gamma}{R+\gamma} &= \frac{ab\cos\frac{\theta}{2}}{2Rc\cos\frac{\rho}{2}\cos\frac{\phi}{2}}.\end{aligned} \tag{4}$$

The area, $\triangle$, of the inscribed triangle ABC is given by

$$\triangle = \frac{abc}{4R}. \tag{5}$$

Consequently, equations (4) and (5) lead to

$$\begin{aligned}a^2 &= \frac{(R+\alpha)\triangle\cos\frac{\rho}{2}}{\alpha\cos\frac{\theta}{2}\cos\frac{\phi}{2}},\\ b^2 &= \frac{(R+\beta)\triangle\cos\frac{\phi}{2}}{\beta\cos\frac{\theta}{2}\cos\frac{\rho}{2}},\\ c^2 &= \frac{(R+\gamma)\triangle\cos\frac{\theta}{2}}{\gamma\cos\frac{\rho}{2}\cos\frac{\phi}{2}}.\end{aligned} \tag{6}$$

Now, Heron's formula for the triangle ABC can be written in the form

$$16\triangle^2 = 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4.$$

Using the above equation together with equations (6) will enable us to get an equation for R in terms of the parameters of the intersecting circles. This process involves substituting the value of a^2, b^2, c^2 into Heron's formula, dividing by $\triangle^2$, and performing a lengthy algebraic manipulation to yield the equation:

$$\begin{aligned}0 = \frac{1}{R^2}\Big[&4\cos^2\frac{\theta}{2}\cos^2\frac{\rho}{2}\cos^2\frac{\phi}{2} + \cos^4\frac{\phi}{2} + \cos^4\frac{\rho}{2} + \cos^4\frac{\theta}{2}\\ &-2\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2} - 2\cos^2\frac{\theta}{2}\cos^2\frac{\rho}{2} - 2\cos^2\frac{\phi}{2}\cos^2\frac{\rho}{2}\Big]\\ -\frac{1}{R}\Big[&\frac{2\cos^2\frac{\rho}{2}}{\alpha}(\cos^2\frac{\theta}{2} + \cos^2\frac{\phi}{2} - \cos^2\frac{\rho}{2})\\ &+\frac{2\cos^2\frac{\phi}{2}}{\beta}(\cos^2\frac{\theta}{2} + \cos^2\frac{\rho}{2} - \cos^2\frac{\phi}{2})\\ &+\frac{2\cos^2\frac{\theta}{2}}{\gamma}(\cos^2\frac{\phi}{2} + \cos^2\frac{\rho}{2} - \cos^2\frac{\theta}{2})\Big]\\ &+\frac{\cos^4\frac{\rho}{2}}{\alpha^2} + \frac{\cos^4\frac{\phi}{2}}{\beta^2} + \frac{\cos^4\frac{\theta}{2}}{\gamma^2}\\ &-\frac{2\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}}{\beta\gamma} - \frac{2\cos^2\frac{\theta}{2}\cos^2\frac{\rho}{2}}{\alpha\gamma} - \frac{2\cos^2\frac{\phi}{2}\cos^2\frac{\rho}{2}}{\alpha\beta}.\end{aligned} \tag{7}$$

Although the equation can be formal solved in general, it is rather unwieldy. Let us consider some special cases.

When the three circles $\mathcal{C}_A, \mathcal{C}_B$ and $\mathcal{C}_C$ are mutually tangent, θ, ρ and ϕ equals zero, thus giving the equation:

$$0 = \frac{1}{R^2} - \frac{2}{R}\left[\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right] + \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{2}{\alpha\beta} - \frac{2}{\beta\gamma} - \frac{2}{\alpha\gamma}.$$

Solving for $\frac{1}{R}$ gives the standard Descartes formula for the Inner Soddy circle. See [2].

When $\mathcal{C}_C$ is a line tangent to $\mathcal{C}_A$ and $\mathcal{C}_B$, we have $\beta = \infty$ and $\theta = \rho = 0$, and equation (7) becomes

$$0 = \frac{1}{R^2}\left[\cos^4\frac{\phi}{2}\right] - \frac{2\cos^2\frac{\phi}{2}}{R}\left[\frac{1}{\alpha} + \frac{1}{\gamma}\right] + \left[\frac{1}{\alpha} - \frac{1}{\gamma}\right]^2.$$

Solving for 1/R, and using the fact that $\frac{1}{R} > \frac{1}{\alpha}$ and $\frac{1}{R} > \frac{1}{\gamma}$, gives the equation

$$\frac{1}{R} = \frac{1}{\cos^2\frac{\phi}{2}}\left[\frac{1}{\alpha} + \frac{1}{\gamma} + 2\sqrt{\frac{1}{\alpha\gamma}}\right].$$

When the circles $\mathcal{C}_A$ and $\mathcal{C}_C$ are lines that intersect at an angle $\phi > 0$ and are both tangent to the circle $\mathcal{C}_B$, we get a cone and equation (7) becomes

$$0 = \frac{1}{R^2}\left[\cos^4\frac{\phi}{2}\right] - \frac{2\cos^2\frac{\phi}{2}}{\gamma R}\left[2 - \cos^2\frac{\phi}{2}\right] + \left[\frac{\cos^2\frac{\phi}{2}}{\gamma}\right]^2.$$

After solving for 1/R, using some trigonometric identities and the fact that $\frac{1}{R} > \frac{1}{\gamma}$, we get the equation

$$\frac{1}{R} = \frac{2}{\gamma}\left[\frac{1 + \sin\frac{\phi}{2}}{1 - \sin\frac{\phi}{2}}\right]^2,$$

the same as obtained from working with the cone directly.

Unfortunately, equation (7) no longer gives any useful result when all three circles, $\mathcal{C}_A$, $\mathcal{C}_B$ and $\mathcal{C}_C$, becomes lines. The inner tangent circle in this case is just the inscribed circle in the triangle.

References

[1] H. S. M. Coxeter, *Introduction to Geometry, Second Edition,* John Wiley and Sons, New york 1969.

[2] W. Reyes, The Lucas Circles and the Descartes Formula, *Forum Geom.*, 3 (2003) 95–100.

Max M. Tran: Mathematics and Computer Science Department, Kingsborough Community College, 2001 Oriental Boulevard, Brooklyn, NY 11235-2398, USA

E-mail address: `mtran@kingsborough.edu`

Forum Geometricorum
Volume 6 (2006) 301–310.

FORUM GEOM
ISSN 1534-1178

Two Brahmagupta Problems

K. R. S. Sastry

Abstract. D. E. Smith reproduces two problems from Brahmagupta's work Kutakhādyaka (algebra) in his *History of Mathematics*, Volume 1. One of them involves a broken tree and the other a mountain journey. Normally such objects are represented by vertical line segments. However, it is every day experience that such objects need not be vertical. In this paper, we generalize these situations to include slanted positions and provide integer solutions to these problems.

1. Introduction

School textbooks on geometry and trigonometry contain problems about trees, poles, buildings, hills etc. to be solved using the Pythagorean theorem or trigonometric ratios. The assumption is that such objects are vertical. However, trees grow not only vertically (and tall offering a magestic look) but also assume slanted positions (thereby offering an elegant look). Buildings too need not be vertical in structure, for example the leaning tower of Pisa. Also, a distant planar view of a mountain is more like a scalene triangle than a right one. In this paper we regard the angles formed in such situations as having rational cosines. We solve the following Brahmagupta problems from [5] in the context of rational cosines triangles. In [4] these problems have been given Pythagorean solutions.

Problem 1. A bamboo 18 cubits high was broken by the wind. Its tip touched the ground 6 cubits from the root. Tell the lengths of the segments of the bamboo.

Problem 2. On the top of a certain hill live two ascetics. One of them being a wizard travels through the air. Springing from the summit of the mountain he ascends to a certain elevation and proceeds by an oblique descent diagonally to neighboring town. The other walking down the hill goes by land to the same town. Their journeys are equal. I desire to know the distance of the town from the hill and how high the wizard rose.

We omit the numerical data given in Problem 1 to extend it to an indeterminate one like the second so that an infinity of integer solutions can be found.

Publication Date: December 4, 2006. Communicating Editor: Paul Yiu.
The author thanks the referee for his suggestions.

2. Background material

An angle θ is a rational cosine angle if $\cos\theta$ is rational. If both $\cos\theta$ and $\sin\theta$ are rational, then θ is called a Heron angle. If the angles of a triangle are rational cosine (respectively Heron) angles, then the sides are rational in proportion, and they can be rendered integers, by after multiplication by the lcm of the denominators. Thus, in effect, we deal with triangles of integer sides. Given a rational cosine (respectively Heron) angle θ, it is possible to determine the infinite family of integer triangles (respectively Heron triangles) in which each member triangle contains θ. Our discussion depends on such families of triangles, and we give the following description.

2.1. *Integer triangle family containing a given rational cosine angle θ.* Let $\cos\theta = \lambda$ be a rational number. When θ is obtuse, λ is negative. Our discussion requires that $0 < \theta < \frac{\pi}{2}$ so we must have $0 < \lambda < 1$. Let ABC be a member triangle in which $\angle BAC = \theta$. Let $\angle ABC = \phi$ as shown in Figure 1.

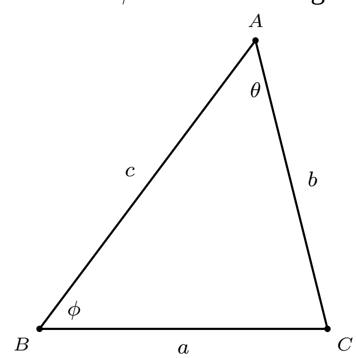


Figure 1

Applying the law of cosines to triangle ABC we have $a^2 = b^2 + c^2 - 2bc\lambda$, or

$$(c-a)(c+a) = b(2\lambda c - b).$$

By the triangle inequality $c - a < b$ so that

$$1 > \frac{c-a}{b} = \frac{2\lambda c - b}{c+a} = \frac{v}{u},$$

say, with $\gcd(u, v) = 1$. We then solve the resulting simultaneous equations

$$c - a = \frac{v}{u}a, \qquad c + a = \frac{u}{v}(2\lambda c - b)$$

for a, b, c in proportional values:

$$\frac{a}{u^2 - 2\lambda uv + v^2} = \frac{b}{2(\lambda u - v)} = \frac{c}{u^2 - v^2}.$$

We replace λ by a rational number $\frac{n}{m}$, $0 < \frac{n}{m} < 1$, and obtain a parametrization of triangles in the $\theta = \arccos\frac{n}{m}$ family:

$$(a, b, c) = (m(u^2 + v^2) - 2nuv,\ 2u(nu - mv),\ m(u^2 - v^2)), \quad \frac{u}{v} > \frac{m}{n}. \quad (\dagger)$$

It is routine to check that

$$\phi = \arccos \frac{mc - nb}{ma},$$

and that $\cos A = \frac{n}{m}$ independently of the parameters u, v of the family described in (†) above. Here are two particular integer triangle families.

(1) The $\frac{\pi}{3}$ integer family is given by (†) with $n = 1$, $m = 2$:

$$(a, b, c) = (u^2 - uv + v^2,\ u(u - 2v),\ u^2 - v^2), \quad u > 2v,\ \gcd(u, v) = 1. \tag{1}$$

It is common practice to list primitive solutions except under special circumstances. In (1) we have removed $\gcd(a, b, c) = 2$.

(2) When θ is a Heron angle, *i.e.*, $\cos\theta = \frac{p^2-q^2}{p^2+q^2}$ for integers p, q with $\gcd(p, q) = 1$, (†) describes a Heron triangle family. For example, with $p = 2$, $q = 1$, we have $\cos\theta = \frac{3}{5}$. Now with $n = 3$, $m = 5$, (†) yields

$$(a, b, c) = (5u^2 - 6uv + 5v^2,\ 2u(3u - 5v),\ 5(u^2 - v^2)), \quad \gcd(u, v) = 1. \tag{2}$$

This has area $\triangle = \frac{1}{2}bc\sin\theta = \frac{2}{5}bc$. We may put $u = 3$, $v = 1$ to obtain the specific Heron triangle $(a, b, c) = (4, 3, 5)$ that is Pythagorean. On the other hand, $u = 4$, $v = 1$ yields the non-Pythagorean triangle $(a, b, c) = (61, 56, 75)$ with area $\triangle = 1680$.

3. Generalization of the first problem

3.1. *Restatement.* Throughout this discussion an integer tree is one with the following properties.
(i) It has an integer length $AB = c$.
(ii) It makes a rational cosine angle ϕ with the horizontal.
(iii) When the wind breaks it at a point D the broken part $AD = d$ and the unbroken part $BD = e$ both have integer lengths.
(iv) The top A of the tree touches the ground at C at an integer length $BC = a$.
All the triangles in the configuration so formed have integer sidelengths. See Figure 2.

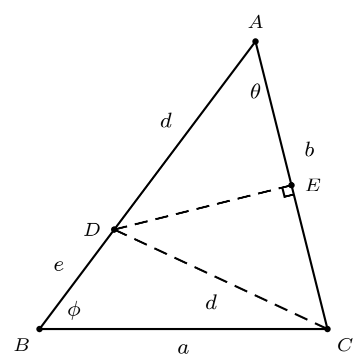


Figure 2

We note from triangle BDC, $BD + DC > BC$, *i.e.*, $AB > BC$. When $AB = BC$ the entire tree falls to the ground. Furthermore, when θ (and therefore

ϕ) is a Heron angle, AB is a Heron tree and Figure 2 represents a Heron triangle configuration.

In the original Problem 1, $c = 18$, $a = 6$ and ϕ is implicitly given (or assumed) to be $\frac{\pi}{2}$. In other words, these elements uniquely determine triangle ABC. Then the breaking point D on AB can be located as the intersection of ED, the perpendicular bisector of AC. Moreover, the present restatement of Problem 1, *i.e.*, the determination of the configuration of Figure 2, gives us an option to use either ϕ or θ as the rational cosine angle to determine triangle ABC and hence the various integer lengths $a, b, \dots, e$. We achieve this goal with the help of $(\dagger)$. Before dealing with the general solution of Problem 1, we consider some interesting examples.

3.2. *Examples.*

3.2.1. If heavy winds break an integer tree AB at D so that the resulting configuration is an isosceles triangle with $AB = AC$, then the length of the broken part is the cube of an integer.

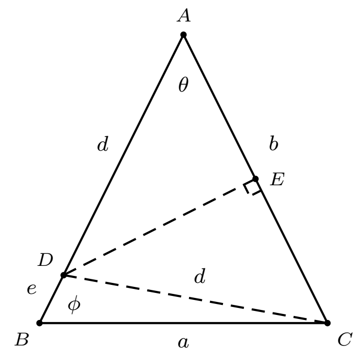


Figure 3

Proof. Suppose $AB = AC = \ell$ and $BC = m$ to begin with. From Figure 3, it follows that $\ell > m$, $\cos\theta = \frac{2\ell^2 - m^2}{2\ell^2}$, $\cos\phi = \frac{m}{2\ell}$, $AD = \frac{\ell}{2\cos\theta} = \frac{\ell^3}{2\ell^2 - m^2}$, $BD = \ell - \frac{\ell^3}{2\ell^2 - m^2} = \frac{\ell(\ell^2 - m^2)}{2\ell^2 - m^2}$. to obtain integer values we multiply each by $2\ell^2 - m^2$. In the notation of Figure 2, the solution is given by
(i) $c =$ the length of the tree $= \ell(2\ell^2 - m^2)$;
(ii) $d =$ the broken part $= \ell^3$, *an integer cube*;
(iii) $e =$ the unbroken part = $\ell(\ell^2 - m^2)$;
(iv) $a =$ the distance between the foot and top of the tree $= m(2\ell^2 - m^2)$;
(v) $\phi =$ the inclination of the tree with the ground $= \arccos\frac{m}{2\ell}$. □

In particular, if $\ell = p^2 + q^2$, and $m = 2(p^2 - q^2)$ for $(\sqrt{2} + 1)q > p > q$, then AB becomes a Heron tree broken by the wind. These yield

$$\begin{aligned} c = b =& 2(p^2+q^2)(2pq+p^2-q^2)(2pq-p^2+q^2), \\ d =& (p^2+q^2)^3, \\ e =& (p^2+q^2)(-p^2+3q^2)(3p^2-q^2), \\ a =& 4(p^2-q^2)(2pq+p^2-q^2)(2pq-p^2+q^2). \end{aligned}$$

For a numerical example, we put $p = 3$, $q = 2$. This gives a Heron tree of length 3094 broken by the wind into $d = 2197 = 13^3$ (an integer cube), and $e = 897$ to come down at $a = 2380$. The angle of inclination of the tree with the horizontal is $\phi = \arccos \frac{5}{13}$.

We leave the details of the following two examples to the reader as an exercise.

3.2.2. If the wind breaks an integer tree AB at D in such a way that $AC = BC$, then both the lengths of the tree and the broken part are perfect squares.

3.2.3. If the wind breaks an integer tree AB at D in such a way that $AD = DC = BC$, then the common length is a perfect square.

4. General solution of Problem 1

Ideally, the general solution of Problem 1 involves the use of integral triangles given by (†). For simplicity we first consider a special case of (†) in which $\theta = \frac{\pi}{3}$. The solution in this case is elegant. Then we simply state the general solution leaving the details to the reader.

4.1. *A particular case of Problem 1.* An integral tree AB is broken by the wind at D. The broken part DA comes down so that the top A of the tree touches the ground at C. If $\angle DAC = \frac{\pi}{3}$, determine parametric expressions for the various elements of the configuration formed.

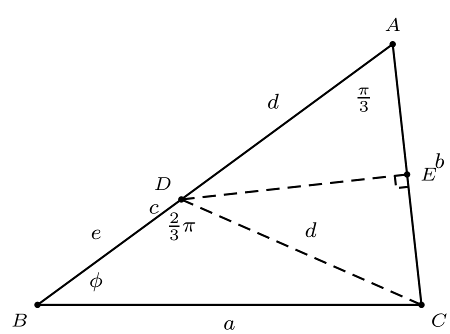


Figure 4

We refer to Figure 4. Since $\angle DAC = \frac{\pi}{3}$, triangle ADC is equilateral and $d = b = DC$. From (1), we have

$$\begin{aligned} a =& u^2 - uv + v^2, \\ d = b =& u(u - 2v), \\ c =& u^2 - v^2, \\ e =& c - d = v(2u - v), \\ \phi =& \arccos \frac{2c - b}{2a} = \arccos \frac{u^2 + 2uv - 2v^2}{2(u^2 - uv + v^2)}, \quad u > 2v. \end{aligned}$$

For a specific numerical example, we take $u = 5$, $v = 2$, and obtain a tree of length $c = 21$, broken into $d = b = 5$, $e = 16$ and $a = 19$, inclined at an angle $\phi = \arccos \frac{37}{38}$.

Remark. No tree in the $\frac{\pi}{3}$ family can be Heron because $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ is irrational.

Note also that in Figure 4, $\angle BDC = \frac{2}{3}\pi$. Hence the family of triangles

$$(a, b, e) = (u^2 - uv + v^2,\ u(u - 2v),\ v(2u - v))$$

contains the angle $\frac{2}{3}\pi$ in each member. For example, with $u = 5$, $v = 2$, we have $(a, b, c) = (19, 5, 16)$; $\cos A = -\frac{1}{2}$ and $\angle A = \frac{2}{3}\pi$.

4.2. *General solution of Problem 1.* Let $\cos \theta = \frac{n}{m}$. The corresponding integral trees have
(i) length $c = mn(u^2 - v^2)$,
(ii) broken part $d = mu(nu - mv)$,
(iii) unbroken part $e = mv(mu - nv)$,
(iv) distance between the foot and the top of the tree on the ground

$$a = n(m(u^2 + v^2) - 2nuv),$$

and
(v) the angle of inclination with the ground ϕ where

$$\cos \phi = \frac{(m^2 - 2n^2)u^2 + 2mnuv - m^2v^2}{ma}. \tag{3}$$

Remark. The solution in (3) yields the solution of the Heron tree problem broken by the wind when θ is a Heron angle, *i.e.*, when $\cos \theta = \frac{p^2 - q^2}{p^2 + q^2}$. Here is a numerical example. Suppose $p = 2$, $q = 1$. Then $\cos \theta = \frac{3}{5}$, *i.e.*, $n = 3$, $m = 5$. To break a specific Heron tree of this family, we put $u = 4$, $v = 1$. Then we find that $c = 225$, $d = 140$, $e = 85$, $a = 183$, and the angle of inclination $\phi = \arccos \frac{207}{305}$, a Heron angle.

5. The second problem

Brahmagupta's second problem does not need any restatement. It is an indeterminate one in its original form.

An integral mountain is one whose planar view is an integral triangle. If the angles of this integral triangle are Heron angles, then the plain view becomes a heron triangle. In such a case we have a Heron mountain. One interesting feature of the second problem is the pair of integral triangles that we are required to generate for its solution. Furthermore, it creates an amusing situation as we shall soon see – in a sense there is more wizardry!

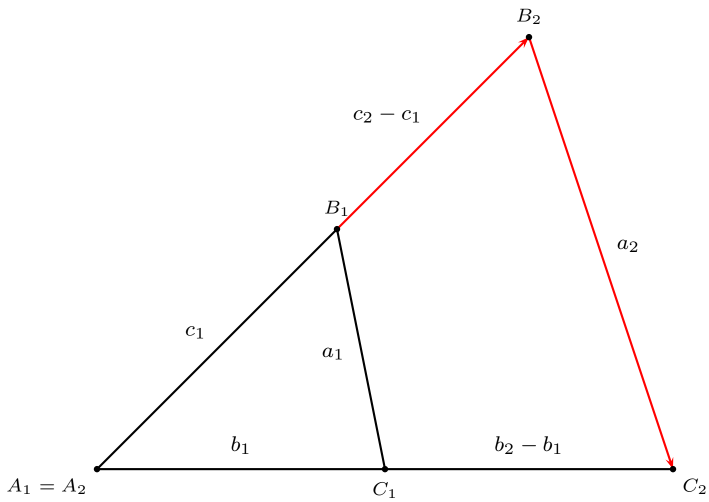


Figure 5

Figure 5 shows an integral mountain $A_1B_1C_1$. At B_1 live two ascetics. The wizard of them flies to B_2 along the direction of A_1B_1, and then reaches the town C_2. The other one walks along the path $B_1C_1C_2$. The hypothesis of the problem is $B_1B_2 + B_2C_2 = B_1C_1 + C_1C_2$, *i.e.*,

$$c_2 + a_2 - b_2 = c_1 + a_1 - b_1. \tag{4}$$

Hence the solution to Problem 2 lies in generating a pair of integral triangles $A_1B_1C_1$ and $A_2B_2C_2$, with
(i) $A_2 = A_1$,
(ii) $\angle B_1A_1C_1 = \angle B_2A_2C_2$,
(iii) $c_2 + a_2 - b_2 = c_1 + a_1 - b_1$. As the referee pointed out, together the two conditions above imply that triangles $A_1B_1C_1$ and $A_2B_2C_2$ have a common excircle opposite to the vertices C_1, C_2. Furthermore, we need integral answers to the questions
(i) the distance between the hill and the town $C_1C_2 = b_2 - b_1$,
(ii) the height the wizard rose, *i.e.*, the altitude $c_2 \sin A_1$ through B_2 of triangle $A_2B_2C_2$.

Now, if $c_2 \sin A_1$ is to be an integer, then $\sin A_1$ should necessarily be rational. Therefore the integral mountain must be a Heron mountain. We may now put $\cos\theta = \frac{p^2-q^2}{p^2+q^2}$, *i.e.*, $n = p^2 - q^2$, $m = p^2 + q^2$ in (†) to find the answers. As it turns out, the solution would not be elegant. Instead, we use the following description of the family of Heron triangles, each member triangle containing θ. This description has previously appeared in this journal [4] so we simply state the description.

Let $\cos A = \frac{p^2-q^2}{p^2+q^2}$. The Heron triangle family determining the common angle A is given by

$$\begin{aligned}(a, b, c) =&(pq(u^2+v^2),\ (pu-qv)(qu+pv),\ (p^2+q^2)uv),\\ &(u, v) = (p, q) = 1,\ p \geq 1 \text{ and } pu > qv.\end{aligned} \tag{5}$$

In particular, we note that
(i) $p = q \Rightarrow A = \frac{\pi}{2}$ and $(a, b, c) = (u^2+v^2,\ u^2-v^2,\ 2uv)$, and
(ii) $(p, q) = (u, v) \Rightarrow (a, b, c) = (u^2+v^2,\ 2(u^2-v^2),\ u^2+v^2)$
are respectively the Pythagorean triangle family and the isoceles Heron triangle family.

5.1. *The solution of Problem 2.* We continue to refer to Figure 5. Since $\angle B_1A_1C_1 = \angle B_2A_2C_2$, p and q remain the same in (5). This gives

$$\begin{aligned}(a_1, b_1, c_1) =&(pq(u_1^2+v_1^2),\ (pu_1-qv_1)(qu_1+pv_1),\ (p^2+q^2)u_1v_1),\\ (a_2, b_2, c_2) =&(pq(u_2^2+v_2^2),\ (pu_2-qv_2)(qu_2+pv_2),\ (p^2+q^2)u_2v_2).\end{aligned}$$

Next, $c_2 + a_2 - b_2 = c_1 + a_1 - b_1$ simplifies to

$$v_2(qu_2+pv_2) = v_1(qu_1+pv_1) = \lambda,\ \text{a constant}. \tag{6}$$

For given p, q, there are four variables u_1, v_1, u_2, v_2, and they generate an infinity of solutions satisfying the equation (6). We now obtain two particular, numerical solutions.

5.2. *Numerical examples.* (1) We put $p = 2$, $q = 1$, $u_1 = 3$, $v_1 = 2$ in (6). This gives $v_2(u_2+2v_2) = 14 = \lambda$. It is easy to verify that $u_2 = 12$, $v_2 = 1$ is a solution. Hence we have

$$(a_1, b_1, c_1) = (13, 14, 15) \quad \text{and} \quad (a_2, b_2, c_2) = (145, 161, 30).$$

Note that $\gcd(a_ib_i, c_i) = 2$, $i = 1, 2$, has been divided out. It is easy to verify that $c_i + a_i - b_i = 14$, $i = 1, 2$. The answers to the questions are
(i) the distance between the hill and th twon, $b_2 - b_1 = 147$.
(ii) The wizard rose to a height $c_2 \sin A_1 = 30 \times \frac{4}{5} = 24$.

In fact it is possible to give as many solutions (a_i, b_i, c_i) to (6) as we wish: we have just to take sufficiently large values for λ. This creates an amusing situation as we see below.

(2) Suppose $\lambda = 2 \times 3 \times 5 \times 7 = 210$. Then (6) becomes $v_i(u_i + 2v_i) = 210$. The indexing of the six solutions below is unconventional in the interest of Figure 6.

i	v_i	u_i	a_i	b_i	c_i
6	1	208	43265	43575	520
5	2	101	10205	10500	505
4	3	64	4105	4375	480
3	5	32	1049	1239	400
2	6	23	565	700	345
1	7	16	305	375	280

It is easy to check that for all six Heron triangles $c_i + a_i - b_i = 210$, $i = 1, 2, \ldots, 6$. From this we deduce that
(i) $B_iC_i + C_iC_{i+1} = B_iB_{i+1} + B_{i+1}C_{i+1}$ and
(ii) $B_iC_i + C_iC_j = B_iB_j + B_jC_j$, $i, j = 1, 2, 3, 4, 5$, $j > i$.

In other words, the two ascetics may choose to live at any of the places B_1, B_2, B_3, B_4, B_5. Then they may choose to reach any next town C_2, C_3, C_4, C_5, C_6. See Figure 6.

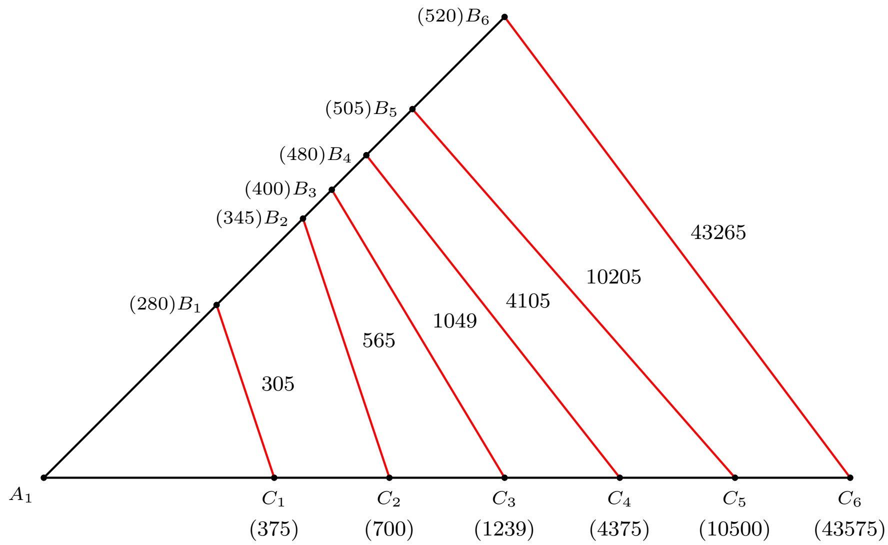


Figure 6

Another famous problem, ladders leaning against vertical walls, has been solved in the context of Heron triangles in [2].

References

[1] L. E. Dickson, *History of the Theory of Numbers*, volume II, Chelsea, New York, 1971; pp.165–224.
[2] K. R. S. Sastry, Heron ladders, *Math. Spectrum*, 34 (2001-2002) 61–64.
[3] K. R. S. Sastry, Construction of Brahmagupta n-gons, *Forum Geom.*, 5 (2005)119–126.
[4] K. R. S. Sastry, Brahmagupta's problems, Pythaogrean solutions, and Heron triangles, *Math. Spectrum*, 38 (2005-2006) 68–73.
[5] D. E. Smith, *History of Mathematics*, volume 1, Dover, 1958 pp.152–160.

K. R. S. Sastry: Jeevan Sandhya, DoddaKalsandra Post, Raghuvana Halli, Bangalore, 560 062, India.

Forum Geometricorum
Volume 6 (2006) 311–325.

FORUM GEOM
ISSN 1534-1178

Square Wreaths Around Hexagons

Floor van Lamoen

Abstract. We investigate the figures that arise when squares are attached to a triple of non-adjacent sides of a hexagon, and this procedure is repeated with alternating choice of the non-adjacent sides. As a special case we investigate the figure that starts with a triangle.

1. Square wreaths around hexagons

Consider a hexagon $\mathcal{H}_1 = H_{1,1}H_{2,1}H_{3,1}H_{4,1}H_{5,1}H_{6,1}$ with counterclockwise orientation. We attach squares externally on the sides $H_{1,1}H_{2,1}$, $H_{3,1}H_{4,1}$ and $H_{5,1}H_{6,1}$, to form a new hexagon $\mathcal{H}_2 = H_{1,2}H_{2,2}H_{3,2}H_{4,2}H_{5,2}H_{6,2}$. Following Nottrot, [8], we say we have made the first *square wreath* around $\mathcal{H}_1$. Then we attach externally squares to the sides $H_{6,2}H_{1,2}$, $H_{2,2}H_{3,2}$ and $H_{4,2}H_{5,2}$, to get a third hexagon $\mathcal{H}_3$, creating the second square wreath. We may repeat this operation to find a sequence of hexagons $\mathcal{H}_n = H_{1,n}H_{2,n}H_{3,n}H_{4,n}H_{5,n}H_{6,n}$ and square wreaths. See Figure 1.

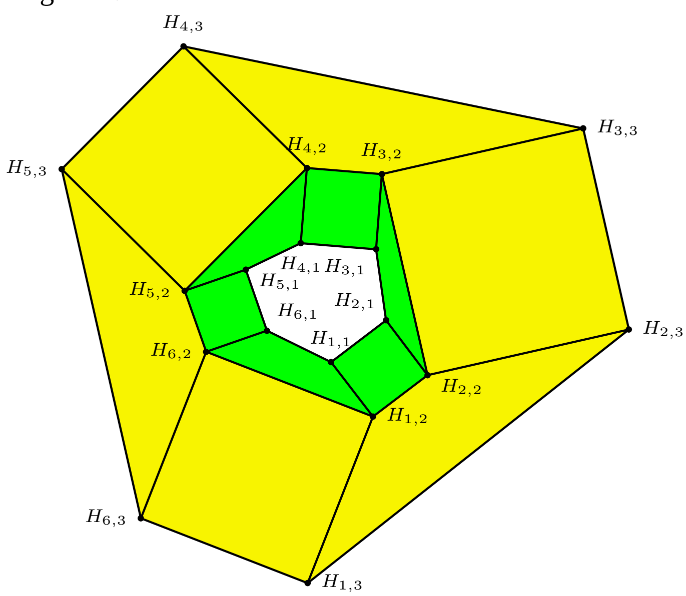


Figure 1

We introduce complex number coordinates, and abuse notations by identifying a point with its affix. Thus, we shall also regard $H_{m,n}$ as a complex number, the first subscript m taken modulo 6.

Publication Date: December 11, 2006. Communicating Editor: Paul Yiu.
The author wishes to thank Paul Yiu for his assistance in the preparation of this paper.

Assuming a standard orientation of the given hexagon $\mathcal{H}_1$ in the complex plane, we easily determine the vertices of the hexagons in the above iterations.

If n is even, then $H_{1,n}H_{2,n}$, $H_{3,n}H_{4,n}$ and $H_{5,n}H_{6,n}$ are the opposite sides of the squares erected on $H_{1,n-1}H_{2,n-1}$, $H_{3,n-1}H_{4,n-1}$ and $H_{5,n-1}H_{6,n-1}$ respectively. This means, for $k = 1, 2, 3$,

$$\begin{aligned} H_{2k-1,n} =& H_{2k-1,n-1} - i(H_{2k,n-1} - H_{2k-1,n-1}) \\ =& (1+i)H_{2k-1,n-1} - i \cdot H_{2k,n-1}, \end{aligned} \tag{1}$$

$$\begin{aligned} H_{2k,n} =& H_{2k,n-1} + i(H_{2k-1,n-1} - H_{2k,n-1}) \\ =& i \cdot H_{2k-1,n-1} + (1-i)H_{2k,n-1}. \end{aligned} \tag{2}$$

If n is odd, then $H_{2,n}H_{3,n}$, $H_{4,n}H_{5,n}$, $H_{6,n}H_{1,n}$ are the opposite sides of the squares erected on $H_{2,n-1}H_{3,n-1}$, $H_{4,n-1}H_{5,n-1}$ and $H_{6,n-1}H_{1,n-1}$ respectively. This means, for $k = 1, 2, 3$, reading first subscripts modulo 6, we have

$$\begin{aligned} H_{2k,n} =& H_{2k,n-1} - i(H_{2k+1,n-1} - H_{2k,n-1}) \\ =& (1+i)H_{2k,n-1} - i \cdot H_{2k+1,n-1}, \end{aligned} \tag{3}$$

$$\begin{aligned} H_{2k+1,n} =& H_{2k+1,n-1} + i(H_{2k,n-1} - H_{2k+1,n-1}) \\ =& i \cdot H_{2k,n-1} + (1-i)H_{2k+1,n-1}. \end{aligned} \tag{4}$$

The above recurrence relations (1, 2, 3, 4) may be combined into

$$\begin{aligned} H_{2k,n} =& (1+(-1)^n i)H_{2k,n-1} + (-1)^n i \cdot H_{2k+(-1)^{n+1},n-1}, \\ H_{2k+1,n} =& (1+(-1)^n i)H_{2k+1,n-1} + (-1)^{n+1} i \cdot H_{2k+1+(-1)^n,n-1}, \end{aligned}$$

or even more succintly,

$$H_{m,n} = (1+(-1)^n i)H_{m,n-1} + (-1)^{m+n} i \cdot H_{m+(-1)^{m+n+1},n-1}.$$

Proposition 1. *Triangles $H_{1,n}H_{3,n}H_{5,n}$ and $H_{1,n-2}H_{3,n-2}H_{5,n-2}$ have the same centroid, so do triangles $H_{2,n}H_{4,n}H_{6,n}$ and $H_{2,n-2}H_{4,n-2}H_{6,n-2}$.*

Proof. Applying the relations (1, 2, 3, 4) twice, we have

$$\begin{aligned} H_{1,n} =& -(1-i)H_{6,n-2} + 2H_{1,n-2} + (1-i)H_{2,n-2} - H_{3,n-2}, \\ H_{3,n} =& -(1-i)H_{2,n-2} + 2H_{3,n-2} + (1-i)H_{4,n-2} - H_{5,n-2}, \\ H_{5,n} =& -(1-i)H_{4,n-2} + 2H_{5,n-2} + (1-i)H_{6,n-2} - H_{1,n-2}. \end{aligned}$$

The triangle $H_{1,n}H_{3,n}H_{5,n}$ has centroid

$$\frac{1}{3}\left(H_{1,n} + H_{3,n} + H_{5,n}\right) = \frac{1}{3}\left(H_{1,n-2} + H_{3,n-2} + H_{5,n-2}\right),$$

which is the centroid of triangle $H_{1,n-2}H_{3,n-2}H_{5,n-2}$. The proof for the other pair is similar. □

Theorem 2. *For each $m = 1, 2, 3, 4, 5, 6$, the sequence of vertices $H_{m,n}$ satisfies the recurrence relation*

$$H_{m,n} = 6H_{m,n-2} - 6H_{m,n-4} + H_{m,n-6}. \tag{5}$$

Proof. By using the recurrence relations (1, 2, 3, 4), we have

$$\begin{aligned}
H_{1,2} =&(1+i)H_{1,1} - iH_{2,1},\\
H_{1,3} =&2H_{1,1} - (1+i)H_{2,1} - H_{5,1} + (1+i)H_{6,1},\\
H_{1,4} =&3(1+i)H_{1,1} - 4iH_{2,1} - (1+i)H_{3,1} + iH_{4,1} - (1+i)H_{5,1} + 2iH_{6,1},\\
H_{1,5} =&8H_{1,1} - 5(1+i)H_{2,1} - H_{3,1} - 6H_{5,1} + 5(1+i)H_{6,1},\\
H_{1,6} =&13(1+i)H_{1,1} - 18iH_{2,1} - 6(1+i)H_{3,1} + 6iH_{4,1} - 6(1+i)H_{5,1} + 11iH_{6,1},\\
H_{1,7} =&37H_{1,1} - 24(1+i)H_{2,1} - 6H_{3,1} - 30H_{5,1} + 24(1+i)H_{6,1}.
\end{aligned}$$

Elimination of $H_{m,1}$, $m = 2, 3, 4, 5, 6$, from these equations gives

$$H_{1,7} = 6H_{1,5} - 6H_{1,3} + H_{1,1}.$$

The same relations hold if we simultaneously increase each first subscript by 2, or each second subscript by 1. Thus, we have the recurrence relation (5) for $m = 1, 3, 5$. Similarly,

$$\begin{aligned}
H_{2,2} =&iH_{1,1} + (1-i)H_{2,1},\\
H_{2,3} =&-(1-i)H_{1,1} + 2H_{2,1} + (1-i)H_{3,1} - H_{4,1},\\
H_{2,4} =&4iH_{1,1} + 3(1-i)H_{2,1} - 2iH_{3,1} - (1-i)H_{4,1} - iH_{5,1} - (1-i)H_{6,1},\\
H_{2,5} =&-5(1-i)H_{1,1} + 8H_{2,1} + 5(1-i)H_{3,1} - 6H_{4,1} - H_{6,1},\\
H_{2,6} =&18iH_{1,1} + 13(1-i)H_{2,1} - 11iH_{3,1} - 6(1-i)H_{4,1} - 6iH_{5,1} - 6(1-i)H_{6,1},\\
H_{2,7} =&-24(1-i)H_{1,1} + 37H_{2,1} + 24(1-i)H_{3,1} - 30H_{4,1} - 6H_{6,1}.
\end{aligned}$$

Elimination of $H_{m,1}$, $m = 1, 3, 4, 5, 6$, from these equations gives

$$H_{2,7} - 6H_{2,5} + 6H_{2,3} - H_{2,1} = 0.$$

A similar reasoning shows that (5) holds for $m = 2, 4, 6$. □

2. Midpoint triangles

Let M_1, M_2, M_3 be the midpoints of $H_{4,1}H_{5,1}$, $H_{6,1}H_{1,1}$ and $H_{2,1}H_{3,1}$ and M_1', M_2', M_3' the midpoints of $H_{1,3}H_{2,3}$, $H_{3,3}H_{4,3}$ and $H_{5,3}H_{6,3}$ respectively. We have

$$\begin{aligned}
M_1' =&\frac{1}{2}\left(H_{1,3} + H_{2,3}\right)\\
=&\frac{1}{2}\left((1+i)H_{1,1} + (1-i)H_{2,1} + (1-i)H_{3,1} - H_{4,1} - H_{5,1} + (1+i)H_{6,1}\right)\\
=&-M_1 + (1+i)M_2 + (1-i)M_3.
\end{aligned}$$

Similarly,

$$\begin{aligned} M_2' =&(1-i)M_1 - M_2 + (1+i)M_3, \\ M_3' =&(1+i)M_1 + (1-i)M_2 - M_3. \end{aligned}$$

Proposition 3. *For a permutation (j,k,ℓ) of the integers $1,2,3$, the segments M_jM_k' and M_kM_j' are perpendicular to each other and equal in length, while $M_\ell M_\ell'$ is parallel to an angle bisector of M_jM_k' and M_kM_j', and is $\sqrt{2}$ times as long as each of these segments.*

Proof. From the above expressions for M_j', $j=1,2,3$, we have

$$\begin{aligned} M_2' - M_3 =&(1-i)M_1 - M_2 + i\cdot M_3, && (6)\\ M_3' - M_2 =&(1+i)M_1 - i\cdot M_2 - M_3, \\ =&i\left((1-i)M_1 - M_2 + iM_3\right), \\ =&i(M_2' - M_3); && (7)\\ M_1 - M_1' =&2M_1 - (1+i)M_2 - (1-i)M_3 \\ =&(M_2' - M_3) + (M_3' - M_2). && (8) \end{aligned}$$

From (6) and (7), M_2M_3' and M_3M_2' are perpendicular and have equal lengths. From (8), we conclude that $M_1'M_1$ is parallel to an angle bisector of M_2M_3' and M_3M_2', and is $\sqrt{2}$ times as long as each of these segments. The same results for $(k,\ell) = (3,1),\ (1,2)$ follow similarly. □

The midpoints of the segments M_jM_j', $j=1,2,3$, are the points

$$\begin{aligned} N_1 =&\frac{1}{2}\left((1+i)M_2 + (1-i)M_3\right), \\ N_2 =&\frac{1}{2}\left((1+i)M_3 + (1-i)M_1\right), \\ N_3 =&\frac{1}{2}\left((1+i)M_1 + (1-i)M_2\right). \end{aligned}$$

Note that

$$\begin{aligned} N_1 =&\frac{M_2+M_3}{2} + i\cdot\frac{M_2-M_3}{2}, \\ =&\frac{M_2'+M_3'}{2} - i\cdot\frac{M_2'-M_3'}{2}. \end{aligned}$$

Thus, N_1 is the center of the square constructed externally on the side M_2M_3 of triangle $M_1M_2M_3$, and also the center of the square constructed internally on $M_1'M_2'M_3'$. Similarly, for N_2 and N_3. From this we deduce the following corollaries. See Figure 2.

Corollary 4. *The triangles $M_1M_2M_3$ and $M_1'M_2'M_3'$ are perspective. The perspector is the outer Vecten point of $M_1M_2M_3$ and the inner Vecten point of $M_1'M_2'M_3'$.* [1]

Corollary 5. *The segments M_jN_ℓ and M_kN_ℓ are equal in length and are perpendicular. The same is true for $M_j'N_\ell$ and $M_k'N_\ell$.*

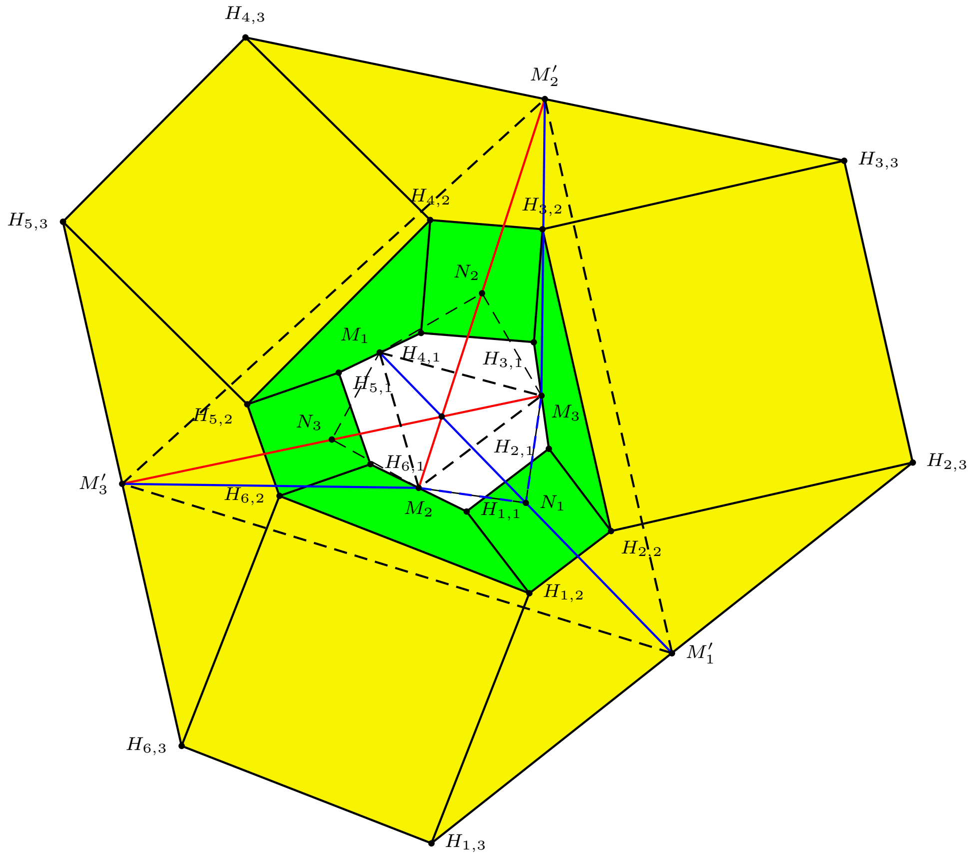


Figure 2

Let $M_1''M_2''M_3''$ be the desmic mate of $M_1M_2M_3$ and $M_1'M_2'M_3'$, *i.e.*, $M_1'' = M_2M_3' \cap M_3M_2'$ etc. By Proposition 3, $\angle M_2M_1''M_3$ is a right angle, and M_1'' lies on the circle with diameter M_2M_3. Since the bisector angle $M_2M_1''M_3$ is parallel to the line N_1M_1, $M_1''N_1$ is perpendicular to this latter line. See Figure 3.

Proposition 6. *If (j, k, ℓ) is a permutation of 1, 2, 3, the lines M_jM_k' and M_kM_j' and the line through N_ℓ perpendicular to $M_\ell M_\ell'$ are concurrent (at M_ℓ'').*

Corollary 7. *The circles $(M_jM_kN_\ell)$, $(M_j'M_k'N_\ell)$, $(M_\ell M_\ell''N_\ell)$ and $(M_\ell'M_\ell''N_\ell)$ are coaxial, so the midpoints of M_jM_k, $M_j'M_k'$, $M_\ell M_\ell''$ and $M_\ell'M_\ell''$ are collinear, the line being parallel to $M_\ell M_\ell'$.*

[1] The outer (respectively inner) Vecten point is the point X_{485} (respectively X_{486}) of [4].

Since the lines $M_jM'_j$, $j = 1, 2, 3$, concur at the outer Vecten point of triangle $M_1M_2M_3$, the intersection of the lines is the inferior (complement) of the outer Vecten point. [2] As such, it is the center of the circle through $N_1N_2N_3$ (see [4]).

Corollary 8. *The three lines joining the midpoints of $M_1M'_1$, $M_2M'_2$, $M_3M'_3$ are concurrent at the center of the circle through N_1, N_2, N_3, which also passes through M''_1, M'_2 and M''_3.*

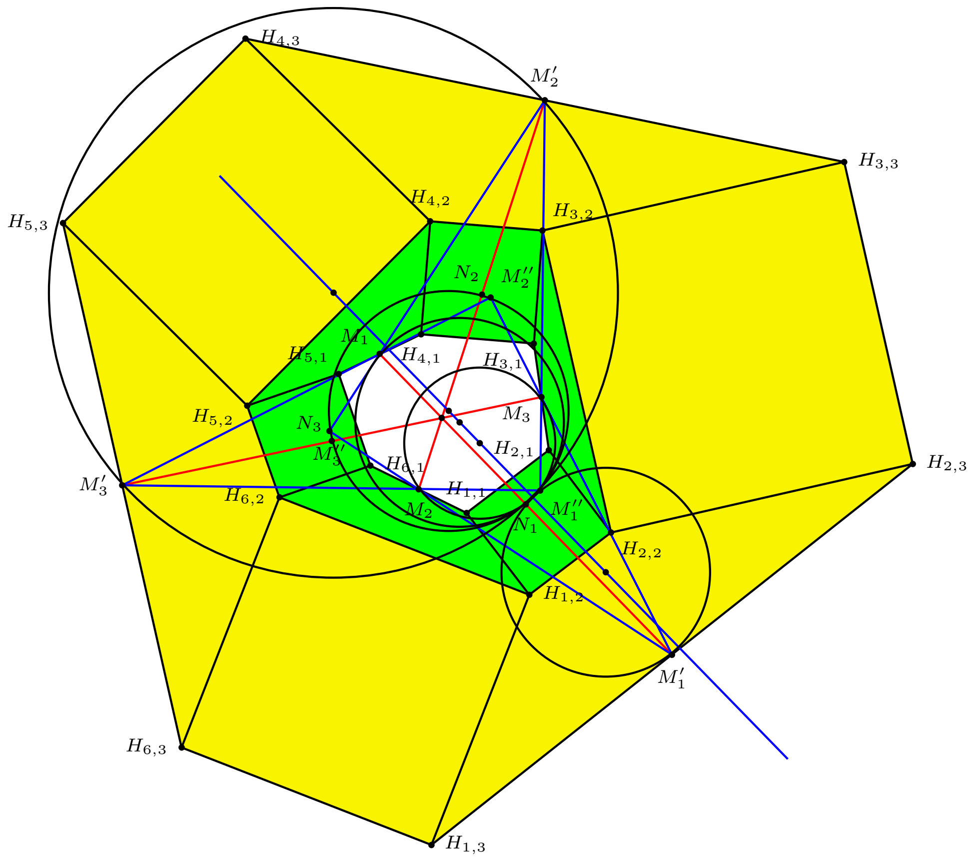


Figure 3

3. Starting with a triangle

An interesting special case occurs when the initial hexagon $\mathcal{H}_1$ degenerates into a triangle with

$$H_{1,1} = H_{6,1} = A, \qquad H_{2,1} = H_{3,1} = B, \qquad H_{4,1} = H_{5,1} = C.$$

This case has been studied before by Haight and Nottrot, who examined especially the side lengths and areas of the squares in each wreath. Under this assumption,

[2] X_{641} in [4].

between two consecutive hexagons $\mathcal{H}_n$ and $\mathcal{H}_{n+1}$ are three squares and three alternating trapezoids of equal areas. The trapzoids between $\mathcal{H}_1$ and $\mathcal{H}_2$ degenerate into triangles. The sides of the squares are parallel and perpendicular to the sides or to the medians of triangle ABC according as n is odd or even. We shall assume the sidelengths of triangle ABC to be a, b, c, and the median lengths m_a, m_b, m_c respectively.

The squares of the first wreath are attached to the triangle sides outwardly. [3] Haight [2] has computed the ratios of the sidelengths of the squares.

If $n = 2k - 1$, the squares have sidelengths a, b, c multiplied by $a_1(k)$, where

$$a_1(k) = 5a_1(k-1) - a_1(k-2),$$
$$a_1(1) = 1,\ a_1(2) = 4.$$

This is sequence A004253 in Sloane's *Online Encyclopedia of Integer sequences* [9]. This also means that

$$H_{m,2k} - H_{m,2k-1} = a_1(k)(H_{m,2} - H_{m,1}). \tag{9}$$

If $n = 2k$, the squares have sidelengths $2m_a$, $2m_b$, $2m_c$ multiplied by $a_2(k)$, where

$$a_2(k) = 5a_2(k-1) - a_2(k-2),$$
$$a_2(1) = 1,\ a_2(2) = 5.$$

This is sequence A004254 in [9]. This also means that

$$H_{m,2k+1} - H_{m,2k} = a_2(k)(H_{m,3} - H_{m,2}). \tag{10}$$

Proposition 9. *Each trapezoid in the wreath bordered by $\mathcal{H}_n$ and $\mathcal{H}_{n+1}$ has area $a_2(n) \cdot \triangle ABC$.*

Lemma 10. (1) $\sum_{j=1}^{k} a_1(j) = a_2(k)$.
(2) *The sums $a_3(k) = \sum_{j=1}^{k} a_2(j)$ satisfy the recurrence relation*

$$a_3(k) = 6a_3(k-1) - 6a_3(k-2) + a_3(k-3),$$
$$a_3(1) = 1,\ a_3(2) = 6,\ a_3(3) = 30.$$

The sequence $a_3(k)$ is essentially sequence A089817 in [9]. [4]

It follows from (9) and (10) that

$$\begin{aligned}
H_{m,2k} =& H_{m,1} + \sum_{j=1}^{k}(H_{m,2j} - H_{m,2j-1}) + \sum_{j=1}^{k-1}(H_{m,2j+1} - H_{m,2j}) \\
=& H_{m,1} + \left(\sum_{j=1}^{k} a_1(j)\right)(H_{m,2} - H_{m,1}) + \left(\sum_{j=1}^{k-1} a_2(j)\right)(H_{m,3} - H_{m,2}) \\
=& H_{m,1} + a_2(k)(H_{m,2} - H_{m,1}) + a_3(k-1)(H_{m,3} - H_{m,2}).
\end{aligned}$$

[3] Similar results as those in §§3, 4 can be found if these initial squares are constructed inwardly.

[4] Note that sequences a_1 and a_2 follow this third order recurrence relation as well.

Also,

$$\begin{aligned}
H_{m,2k+1} =& H_{m,1} + \sum_{j=1}^{k}(H_{m,2j} - H_{m,2j-1}) + \sum_{j=1}^{k}(H_{m,2j+1} - H_{m,2j}) \\
=& H_{m,1} + \left(\sum_{j=1}^{k} a_1(j)\right)(H_{m,2} - H_{m,1}) + \left(\sum_{j=1}^{k} a_2(j)\right)(H_{m,3} - H_{m,2}) \\
=& H_{m,1} + a_2(k)(H_{m,2} - H_{m,1}) + a_3(k)(H_{m,3} - H_{m,2})
\end{aligned}$$

Here is a table of the *absolute* barycentric coordinates (with respect to triangle ABC) of the initial values in the above recurrence relations.

m	$H_{m,1}$	$H_{m,2} - H_{m,1}$	$H_{m,3} - H_{m,2}$
1	$(1,0,0)$	$\frac{1}{S}(S_B, S_A, -c^2)$	$(2,-1,-1)$
2	$(0,1,0)$	$\frac{1}{S}(S_B, S_A, -c^2)$	$(-1,2,-1)$
3	$(0,1,0)$	$\frac{1}{S}(-a^2, S_C, S_B)$	$(-1,2,-1)$
4	$(0,0,1)$	$\frac{1}{S}(-a^2, S_C, S_B)$	$(-1,-1,2)$
5	$(0,0,1)$	$\frac{1}{S}(S_C, -b^2, S_A)$	$(-1,-1,2)$
6	$(1,0,0)$	$\frac{1}{S}(S_C, -b^2, S_A)$	$(2,-1,-1)$

From these we have the homogeneous barycentric coordinates

$$\begin{aligned}
H_{m,2k} =& H_{m,1} + a_2(k)(H_{m,2} - H_{m,1}) + a_3(k-1)(H_{m,3} - H_{m,2}), \\
H_{m,2k+1} =& H_{m,1} + a_2(k)(H_{m,2} - H_{m,1}) + a_3(k)(H_{m,3} - H_{m,2}).
\end{aligned}$$

These can be combined into a single relation

$$H_{m,n} = H_{m,1} + a_2(n')(H_{m,2} - H_{m,1}) + a_3(n'')(H_{m,3} - H_{m,2}),$$

in which $n' = \lfloor \frac{n}{2} \rfloor$ and $n'' = \lfloor \frac{n-1}{2} \rfloor$.

Here are the coordinates of the points $H_{m,n}$.

m	x	y	z
1	$(2a_3(n'')+1)S + a_2(n')S_B$	$-a_3(n'')S + a_2(n')S_A$	$-a_3(n'')S - a_2(n')c^2$
2	$-a_3(n'')S + a_2(n')S_B$	$(2a_3(n'')+1)S + a_2(n')S_A$	$-a_3(n'')S - a_2(n')c^2$
3	$-a_3(n'')S - a_2(n')a^2$	$(2a_3(n'')+1)S + a_2(n')S_C$	$-a_3(n'')S + a_2(n')S_B$
4	$-a_3(n'')S - a_2(n')a^2$	$-a_3(n'')S + a_2(n')S_C$	$(2a_3(n'')+1)S + a_2(n')S_B$
5	$-a_3(n'')S + a_2(n')S_C$	$-a_3(n'')S - a_2(n')b^2$	$(2a_3(n'')+1)S + a_2(n')S_A$
6	$(2a_3(n'')+1)S + a_2(n')S_C$	$-a_3(n'')S - a_2(n')b^2$	$-a_3(n'')S + a_2(n')S_A$

Note that the coordinate sum of each of the points in the above is equal to S.

Consider the midpoints of the following segments

segment	$H_{1,n}H_{2,n}$	$H_{2,n}H_{3,n}$	$H_{3,n}H_{4,n}$	$H_{4,n}H_{5,n}$	$H_{5,n}H_{6,n}$	$H_{6,n}H_{1,n}$
midpoint	$C_{1,n}$	$B_{2,n}$	$A_{1,n}$	$C_{2,n}$	$B_{1,n}$	$A_{2,n}$

For $j = 1, 2$, denote by $\mathcal{T}_{j,n}$ the triangle with vertices $A_{j,n}B_{j,n}C_{j,n}$.

$A_{1,n}$	$-2a_3(n'')S - 2a_2(n')a^2$	$(a_3(n'')+1)S + 2a_2(n')S_C$	$(a_3(n'')+1)S + 2a_2(n')S_B$
$B_{1,n}$	$(a_3(n'')+1)S + 2a_2(n')S_C$	$-2a_3(n'')S - 2a_2(n')b^2$	$(a_3(n'')+1)S + 2a_2(n')S_A$
$C_{1,n}$	$(a_3(n'')+1)S + 2a_2(n')S_B$	$(a_3(n'')+1)S + 2a_2(n')S_A$	$-2a_3(n'')S - 2a_2(n')c^2$
$A_{2,n}$	$-2(2a_3(n'')+1)S - a_2(n')a^2$	$2a_3(n'')S + a_2(n')S_C$	$2a_3(n'')S + a_2(n')S_B$
$B_{2,n}$	$2a_3(n'')S + a_2(n')S_C$	$-2(2a_3(n'')+1)S - a_2(n')b^2$	$2a_3(n'')S + a_2(n')S_A$
$C_{2,n}$	$2a_3(n'')S + a_2(n')S_B$	$2a_3(n'')S + a_2(n')S_A$	$-2(2a_3(n'')+1)S - a_2(n')c^2$

Proposition 11. *The triangles $\mathcal{T}_{1,n}$ and $\mathcal{T}_{2,n}$ are perspective.*

This is a special case of the following general result.

Theorem 12. *Every triangle of the form*

$$\begin{array}{lclcl} -2fS - ga^2 & : & (f+1)S + gS_C & : & (f+1)S + gS_B \\ (f+1)S + gS_C & : & -2fS - gb^2 & : & (f+1)S + gS_A \\ (f+1)S + gS_B & : & (f+1)S + gS_A & : & -2fS - gc^2 \end{array}$$

where f and g represent real numbers, is perspective with the reference triangle. Any two such triangles are perspective.

Proof. Clearly the triangle given above is perspective with ABC at the point

$$\left(\frac{1}{(f+1)S + gS_A} : \frac{1}{(f+1)S + gS_B} : \frac{1}{(f+1)S + gS_C}\right),$$

which is the Kiepert perspector $K(\phi)$ for $\phi = \cot^{-1}\frac{f+1}{g}$.

Consider a second triangle of the same form, with f and g replaced by p and q respectively. We simply give a description of this perspector P. This perspector is the centroid if and only if $(g-q) + 3(gp - fq) = 0$. Otherwise, the line joining this perspector to the centroid G intersects the Brocard axis at the point

$$Q = (a^2((g-q)S_A - (f-p)S) : b^2((g-q)S_B - (f-p)S) : c^2((g-q)S_C - (f-p)S)),$$

which is the isogonal conjugate of the Kiepert perspector $K\left(-\cot^{-1}\frac{f-p}{g-q}\right)$. The perspector P in question divides GQ in the ratio

$$\begin{aligned} &GP : GQ \\ =&((g-q) + 3(gp - fq))((g-q)S - (f-p)S_\omega) \\ &: (3(f-p)^2 + (g-q)^2)S + 2(f-p)(g-q)S_\omega. \end{aligned}$$

□

Note that $\mathcal{T}_{j,n}$ for $j \in \{1, 2\}$ and the Kiepert triangles $\mathcal{K}_\phi$[5] are of this form. Also the medial triangle of a triangle of this form is again of the same form. The perspectors of $\mathcal{T}_{j,n}$ and ABC lie on the Kiepert hyperbola. It is the Kiepert perspector $K(\phi_{j,n})$ where

$$\cot\phi_{1,n} = \frac{a_3(n'') + 1}{2a_2(n')},$$

[5] See for instance [6].

and

$$\cot \phi_{2,n} = \frac{2a_3(n'')}{a_2(n')}.$$

In particular the perspectors tend to limits when n tends to infinity. The perspectors and limits are given by

triangle	perspector K_ϕ with	limit for $k \to \infty$
$\mathcal{T}_{1,2k}$	$\phi_{1,2k} = \cot^{-1} \frac{a_3(k-1)+1}{2a_2(k)}$	$\phi_{1,\text{even}} = \cot^{-1} \frac{\sqrt{21}-3}{12}$
$\mathcal{T}_{1,2k+1}$	$\phi_{1,2k+1} = \cot^{-1} \frac{a_3(k)+1}{2a_2(k)}$	$\phi_{1,\text{odd}} = \cot^{-1} \frac{\sqrt{21}+3}{12}$
$\mathcal{T}_{2,2k}$	$\phi_{2,2k} = \cot^{-1} \frac{2a_3(k-1)}{a_2(k)}$	$\phi_{2,\text{even}} = \cot^{-1} \frac{\sqrt{21}-3}{3}$
$\mathcal{T}_{2,2k+1}$	$\phi_{2,2k+1} = \cot^{-1} \frac{2a_3(k)}{a_2(k)}$	$\phi_{2,\text{odd}} = \cot^{-1} \frac{\sqrt{21}+3}{3}$

Remarks. (1) $\mathcal{T}_{2,2}$ is the medial triangle of $\mathcal{T}_{1,3}$.

(2) The perspector of $\mathcal{T}_{2,2}$ and $\mathcal{T}_{2,3}$ is X_{591}.

(3) The perspector of $\mathcal{T}_{2,2}$ and $\mathcal{T}_{2,4}$ is the common circumcenter of $\mathcal{T}_{1,3}$ and $\mathcal{T}_{2,4}$.

Nottrot [8], on the other hand, has found that the sum of the areas of the squares between $\mathcal{H}_n$ and $\mathcal{H}_{n+1}$ as $a_4(n)(a^2+b^2+c^2)$, where wreaths.

$$a_4(n) = 4a_4(n-1) + 4a_4(n-2) - a_4(n-3),$$
$$a_4(1) = 1,\ a_4(2) = 3,\ a_4(3) = 16.$$

This is sequence A005386 in [9]. Note that $a_1(n) = a_4(n) + a_4(n-1)$ for $n \geq 2$.

4. Pairs of congruent triangles

Lemma 13. (a) *If $n \geq 2$ is even, then*

$$\overrightarrow{A_{1,n}A_{1,n+1}} = -\frac{1}{2}\overrightarrow{H_{6,n}H_{6,n+1}} = -\frac{1}{2}\overrightarrow{H_{1,n}H_{1,n+1}},$$
$$\overrightarrow{B_{1,n}B_{1,n+1}} = -\frac{1}{2}\overrightarrow{H_{2,n}H_{2,n+1}} = -\frac{1}{2}\overrightarrow{H_{3,n}H_{3,n+1}},$$
$$\overrightarrow{C_{1,n}C_{1,n+1}} = -\frac{1}{2}\overrightarrow{H_{4,n}H_{4,n+1}} = -\frac{1}{2}\overrightarrow{H_{5,n}H_{5,n+1}}.$$

(b) *If $n \geq 3$ is odd, then*

$$\overrightarrow{A_{2,n}A_{2,n+1}} = -\frac{1}{2}\overrightarrow{H_{3,n}H_{3,n+1}} = -\frac{1}{2}\overrightarrow{H_{4,n}H_{4,n+1}},$$
$$\overrightarrow{B_{2,n}B_{2,n+1}} = -\frac{1}{2}\overrightarrow{H_{5,n}H_{5,n+1}} = -\frac{1}{2}\overrightarrow{H_{6,n}H_{6,n+1}},$$
$$\overrightarrow{C_{2,n}C_{2,n+1}} = -\frac{1}{2}\overrightarrow{H_{1,n}H_{1,n+1}} = -\frac{1}{2}\overrightarrow{H_{2,n}H_{2,n+1}}.$$

Proof. Consider the case of $A_{1,n}A_{1,n+1}$ for even n. Translate the trapezoid $H_{5,n+1}H_{6,n+1}H_{6,n}H_{5,n}$ by the vector $\overrightarrow{H_{5,n+1}H_{4,n+1}}$ and the trapezoid $H_{2,n+1}H_{2,n}H_{1,n}H_{1,n+1}$ by the vector $\overrightarrow{H_{2,n+1}H_{3,n+1}}$. Together with the trapezoid $H_{3,n}H_{4,n}H_{4,n+1}H_{3,n+1}$, these images form two triangles $XH_{3,n+1}H_{4,n+1}$ and

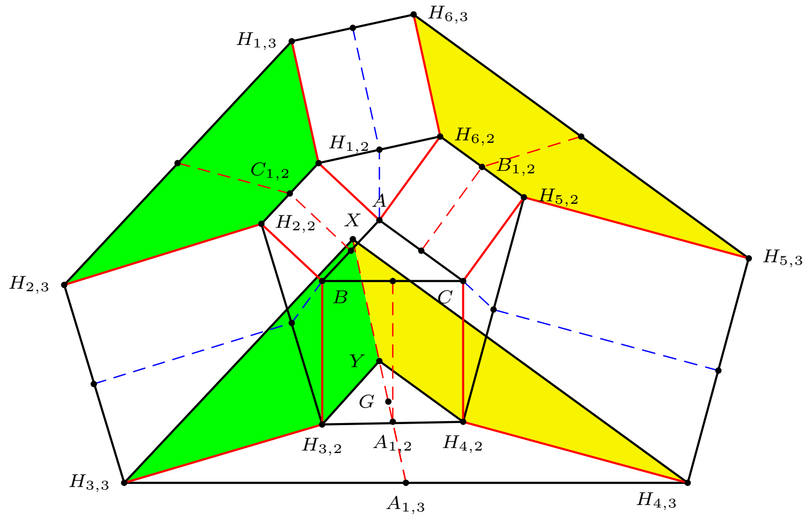


Figure 4.

$YH_{3,n}H_{4,n}$ homothetic at their common centroid G.[6] See Figure 4. It is clear that the points X, Y, $A_{1,n}$, $A_{1,n+1}$ all lie on a line through the centroid G. Furthermore, $\overrightarrow{A_{1,n}A_{1,n+1}} = \frac{1}{2}\overrightarrow{XY} = -\frac{1}{2}\overrightarrow{H_{1,n}H_{1,n+1}}$. The other cases follow similarly. $\square$

Proposition 14. (1) *If $n \geq 3$ is odd, the following pairs of triangles are congruent.*
(i) $H_{2,n}B_{2,n+1}C_{1,n-1}$ *and* $H_{5,n}B_{1,n-1}C_{2,n+1}$,
(ii) $H_{3,n}A_{1,n-1}B_{2,n+1}$ *and* $H_{6,n}A_{2,n+1}B_{1,n-1}$,
(iii) $H_{4,n}C_{2,n+1}A_{1,n-1}$ *and* $H_{1,n}C_{1,n-1}A_{2,n+1}$.
(2) *If $n \geq 2$ is even, the following pairs of triangles are congruent.*
(iv) $H_{2,n}B_{2,n-1}C_{1,n+1}$ *and* $H_{5,n}B_{1,n+1}C_{2,n-1}$,
(v) $H_{3,n}A_{1,n+1}B_{2,n-1}$ *and* $H_{6,n}A_{2,n-1}B_{1,n+1}$,
(vi) $H_{4,n}C_{2,n-1}A_{1,n+1}$ *and* $H_{1,n}C_{1,n+1}A_{2,n-1}$.

Proof. We consider the first of these cases. Let $n \geq 3$ be an odd number. Consider the triangles $H_{2,n}B_{2,n+1}C_{1,n-1}$ and $H_{5,n}B_{1,n-1}C_{2,n+1}$. We show that $H_{2,n}B_{2,n+1}$ and $H_{5,n}B_{1,n-1}$ are perpendicular to each other and equal in length, and the same for $H_{2,n}C_{1,n-1}$ and $H_{5,n}C_{2,n+1}$.

Consider the triangles $H_{2,n}B_{2,n}B_{2,n+1}$ and $B_{1,n-1}B_{1,n}H_{5,n}$. By Lemma 13, $B_{2,n}B_{2,n+1}$ is parallel to $H_{5,n}H_{5,n+1}$ and is half its length. It follows that $B_{2,n}B_{2,n+1}$ is perpendicular to and has the same length as $B_{1,n}H_{5,n}$. Similarly, $B_{2,n}H_{2,n}$ is perpendicular to and has the same length as $B_{1,n}B_{1,n-1}$. Therefore, the triangles $H_{2,n}B_{2,n}B_{2,n+1}$ and $B_{1,n-1}B_{1,n}H_{5,n}$ are congruent, and the segments $H_{2,n}B_{2,n+1}$

[6] As noted in the beginning of §3, the sides of the squares are parallel to and perpendicular to the sides of ABC or the medians of ABC according as n is odd or even. In the even case it is thus clear that the homothetic center is the centroid. In the odd case it can be seen as the lines perpendicular to the sides of ABC are parallel to the medians of the triangles in the first wreath (the flank triangles). The centroid and the orthocenter befriend each other. See [5].

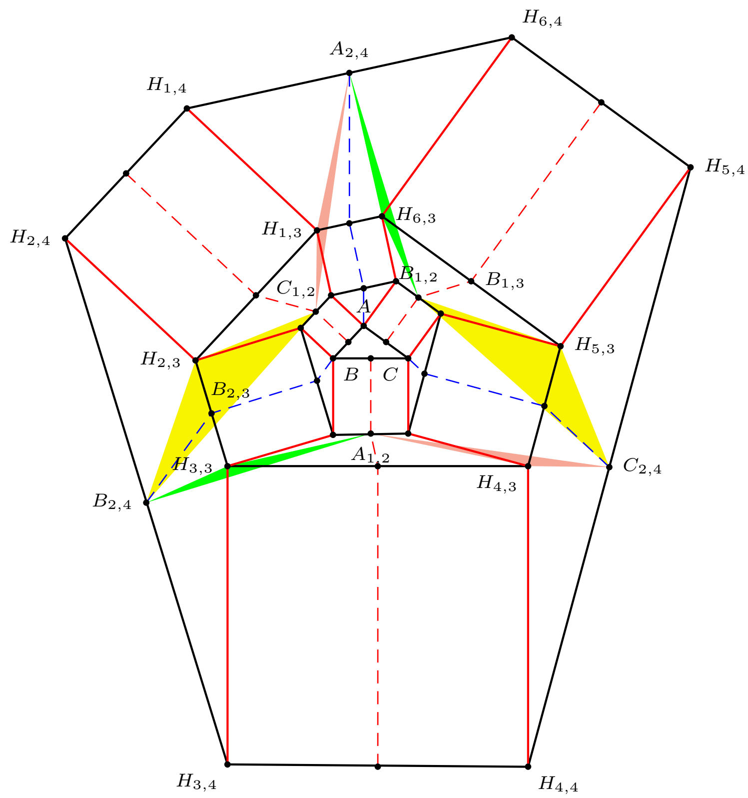


Figure 5.

and $B_{1,n-1}H_{5,n}$ are perpendicular and equal in length. See Figure 5. The same reasoning shows that the segments $H_{2,n}C_{1,n-1}$ and $H_{5,n}C_{2,n+1}$ are perpendicular and equal in length. Therefore, the triangles $H_{2,n}B_{2,n+1}C_{1,n-1}$ and $H_{5,n}B_{1,n-1}C_{2,n+1}$ are congruent.

The other cases can be proved similarly. □

Remark. The fact that the segments $B_{2,n+1}C_{1,n-1}$ and $B_{1,n-1}C_{2,n+1}$ are perpendicular and equal in length has been proved in Proposition 3 for square wreaths arising from an arbitrary hexagon.

5. A pair of Kiepert hyperbolas

The triangles $A_{1,3}H_{4,2}H_{3,2}$, $H_{5,2}B_{1,3}H_{6,2}$ and $H_{2,2}H_{1,2}C_{1,3}$ are congruent to ABC. The counterparts of a point P in these triangles are the points with the barycentric coordinates relative to these three triangles as P relative to ABC.

Theorem 15. *The locus of point P in ABC whose counterparts in the triangles $A_{1,3}H_{4,2}H_{3,2}$, $H_{5,2}B_{1,3}H_{6,2}$ and $H_{2,2}H_{1,2}C_{1,3}$ form a triangle $A'B'C'$ perspective to ABC is the union of the line at infinity and the rectangular hyperbola*

$$S\sum_{\text{cyclic}}(S_B-S_C)yz+(x+y+z)\left(\sum_{\text{cyclic}}(S_B-S_C)(S_A+S)x\right)=0. \quad (11)$$

The locus of the perspector is the union of the line at infinity and the Kiepert hyperbola of triangle ABC.

Proof. The counterparts of $P=(x:y:z)$ form a triangle perspective with ABC if and only if the parallels through A, B, C to $A_{1,3}P$, $B_{1,3}P$ and $C_{1,3}P$ are concurrent. These parallels have equations

$$\begin{aligned}
((S+S_B)(x+y+z)-Sz)Y-((S+S_C)(x+y+z)-Sy)Z&=0,\\
-((S+S_A)(x+y+z)-Sz)X+((S+S_C)(x+y+z)-Sx)Z&=0,\\
((S+S_A)(x+y+z)-Sy)X-((S+S_B)(x+y+z)-Sx)Y&=0.
\end{aligned}$$

They are concurrent if and only if

$$(x+y+z)\left(\sum_{cyclic}(b^2-c^2)((S_A+S)x^2-S_Ayz)\right)=0.$$

The locus therefore consists of the line at infinity and a conic. Rearranging the equation of the conic in the form (11), we see that it is homothetic to the Kiepert hyperbola.

For a point P on the locus (11), let $Q=(x:y:z)$ be the corresponding perspector. This means that the parallels through $A_{1,3}$, $B_{1,3}$, $C_{1,3}$ to AQ, BQ, CQ are concurrent. These parallels have equations

$$\begin{aligned}
((S+S_B)y-(S+S_C)z)X+((S+S_B)y-S_Cz)Y-((S+S_C)z-S_By)Z&=0,\\
-((S+S_A)x-S_Cz)X+((S+S_C)z-(S+S_A)x)Y+((S+S_C)z-S_Ax)Z&=0,\\
((S+S_A)x-S_By)X-((S+S_B)y-S_Ax)Y+((S+S_A)x-(S+S_B)y)Z&=0.
\end{aligned}$$

They are concurrent if and only if

$$(x+y+z)((S_B-S_C)yz+(S_C-S_A)zx+(S_A-S_B)xy)=0.$$

This means the perspector lies on the union of the line at infinity and the Kiepert hyperbola. □

Here are some examples of points on the locus (11) with the corresponding perspectors on the Kiepert hyperbola (see [1, 7]).

Q on Kiepert hyperbola	P on locus
$K\left(\arctan\frac{3}{2}\right)$	centroid
orthocenter	de Longchamps point
centroid	$G' = (-2S_A + a^2 + S : \cdots : \cdots)$
outer Vecten point	outer Vecten point
A	$A' = B_{1,3}H_{5,2} \cap C_{1,3}H_{2,2}$
B	$B' = C_{1,3}H_{1,2} \cap A_{1,3}H_{4,2}$
C	$C' = A_{1,3}H_{3,2} \cap B_{1,3}H_{6,2}$
A_0	$A_{1,3} = (-(S+S_B+S_C) : S+S_C : S+S_B)$
B_0	$B_{1,3} = (S+S_C : -(S+S_C+S_A) : S+S_A)$
C_0	$C_{1,3} = (S+S_B : S+S_A : -(S+S_A+S_B))$

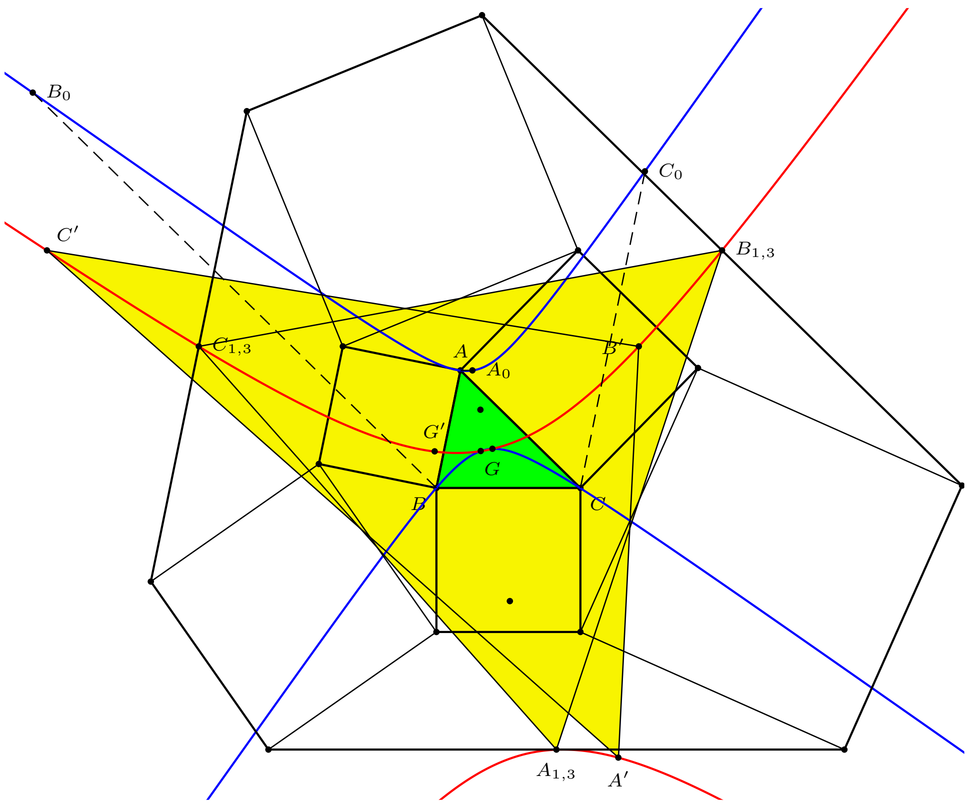


Figure 6.

Here, A_0 is the intersection of the Kiepert hyperbola with the parallel through A to BC; similarly for B_0 and C_0. Since the triangle $A_{1,3}B_{1,3}C_{1,3}$ has centroid G, the rectangular hyperbola (11) is the Kiepert hyperbola of triangle $A_{1,3}B_{1,3}C_{1,3}$. See Figure 6. We show that it is also the Kiepert hyperbola of triangle $A'B'C'$.

This follows from the fact that A', B', C' have coordinates

$$\begin{array}{llclcl} A' & -S-2S_A & : & S+S_A & : & S+S_A \\ B' & S+S_B & : & -S-2S_B & : & S+S_B \\ C' & S+S_C & : & S+S_C & : & -S-2S_C \end{array}$$

From these, the centroid of triangle $A'B'C'$ is the point G' in the above table. The rectangular hyperbola (11) is therefore the Kiepert hyperbola of triangle $A'B'C'$.

References

[1] B. Gibert, Hyacinthos message 13995, August 15, 2006.

[2] F. A. Haight, On a generalization of Pythagoras' theorem, in J.C. Butcher (ed.), *A Spectrum of Mathematics*, Auckland University Press (1971) 73–77.

[3] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[5] F. M. van Lamoen, Friendship of triangle centers, *Forum Geom.*, 1 (2001) 1–6.

[6] F. M. van Lamoen and P. Yiu, The Kiepert pencil of Kiepert hyperbolas, *Forum Geom.*, 1 (2001) 125–132.

[7] P. J. C. Moses, Hyacinthos message 13996, 15 Aug 2006.

[8] J. C. G. Nottrot, Vierkantenkransen rond een driehoek, *Pythagoras*, 14 (1975-1976) 77–81.

[9] N. J. A. Sloane, *Online Encyclopedia of Integer Sequences*, available at http://www.research.att.com/~njas/sequences/index.html

Floor van Lamoen: St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands

E-mail address: fvanlamoen@planet.nl

Forum Geometricorum
Volume 6 (2006) 327–334.

FORUM GEOM
ISSN 1534-1178

Some Geometric Constructions

Jean-Pierre Ehrmann

Abstract. We solve some problems of geometric construction. Some of them cannot be solved with ruler and compass only and require the drawing of a rectangular hyperbola: (i) construction of the Simson lines passing through a given point, (ii) construction of the lines with a given orthopole, and (iii) a problem of congruent incircles whose analysis leads to some remarkable properties of the internal Soddy center.

1. Simson lines through a given point

1.1. *Problem.* A triangle ABC and a point P are given, $P \neq H$, the orthocenter, and does not lie on the sidelines of the triangle. We want to construct the points of the circumcircle Γ of ABC whose Simson lines pass through P.

1.2. *Analysis.* We make use of the following results of Trajan Lalesco [3].

Proposition 1 (Lalesco)**.** *If the Simson lines of A', B', C' concur at P, then*
(a) *P is the midpoint of HH', where H' is the orthocenter of $A'B'C'$,*
(b) *for any point $M \in \Gamma$, the Simson lines $S(M)$ and $S'(M)$ of M with respect to ABC and $A'B'C'$ are parallel.*

Let h be the rectangular hyperbola through A, B, C, P. If the hyperbola h intersects Γ again at U, the center W of h is the midpoint of HU. Let h' be the rectangular hyperbola through A', B', C', U. The center W' of h' is the midpoint of $H'U$. Hence, by (a) above, $W' = T(W)$, where T is the translation by the vector $\overrightarrow{HP}$.

Let D, D' be the endpoints of the diameter of Γ perpendicular to the Simson line $S(U)$. The asymptotes of h are $S(D)$ and $S(D')$; as, by (b), $S(U)$ and $S'(U)$ are parallel, the asymptotes of h' are $S'(D)$ and $S'(D')$ and, by (b), they are parallel to the asymptotes of h.

It follows that T maps the asymptotes of h to the asymptotes of h'. Moreover, T maps $P \in h$ to $H' \in h'$. As a rectangular hyperbola is determined by a point and the asymptotes, it follows that $h' = T(h)$.

Construction 1. *Given a point $P \neq H$ and not on any of the sidelines of triangle ABC, let the rectangular hyperbola h through A, B, C, P intersect the circumcircle Γ again at U. Let h' be the image of h under the translation T by the vector*

Publication Date: December 18, 2006. Communicating Editor: Floor van Lamoen.

$\overrightarrow{HP}$. *Then* h' *passes through* U. *The other intersections of* h' *with* Γ *are the points whose Simson lines pass through* P. See Figure 1.

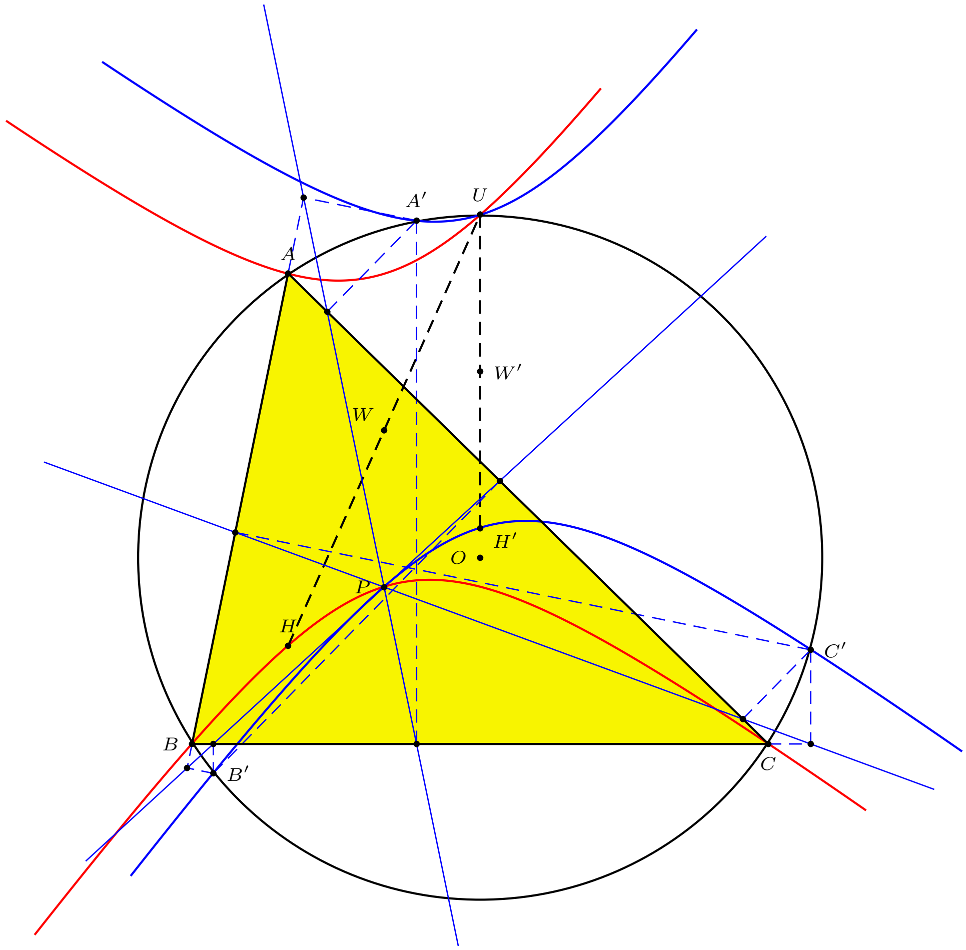


Figure 1.

1.3. *Orthopole.* The above construction leads to a construction of the lines whose orthopole is P. It is well known that, if M and N lie on the circumcircle, the orthopole of the line MN is the common point of the Simson lines of M and N (see [1]). Thus, if we have three real points A', B', C' whose Simson lines pass through P, the lines with orthopole P are the sidelines of $A'B'C'$.

Moreover, the orthopole of a line L lies on the directrix of the insribed parabola touching L (see [1, pp.241–242]). Thus, in any case and in order to avoid imaginary lines, we can proceed this way: for each point M whose Simson line passes through P, let Q be the isogonal conjugate of the infinite point of the direction orthogonal to HP. The line through Q parallel to the Simson line of M intersects the line HP at R. Then P is the orthopole of the perpendicular bisector of QR.

2. Two congruent incircles

2.1. *Problem.* Construct a point P inside ABC such that if B' and C' are the traces of P on AC and AB respectively, the quadrilateral $AB'PC'$ has an incircle congruent to the incircle of PBC.

2.2. *Analysis.* Let h_a be the hyperbola through A with foci B and C, and D_a the projection of the incenter I of triangle ABC upon the side BC.

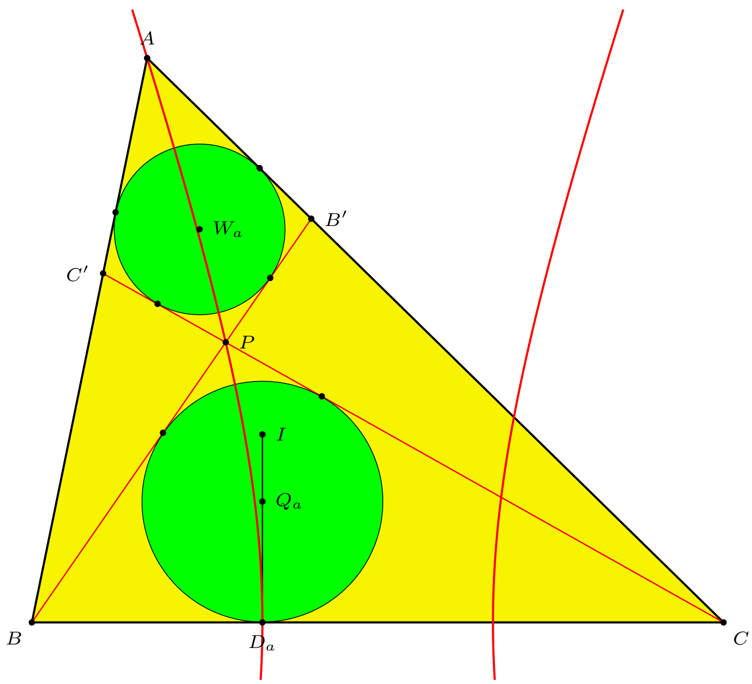


Figure 2.

Proposition 2. *Let P be a point inside ABC and Q_a the incenter of PBC. The following statements are equivalent.*
(a) $PB - PC = AB - AC$.
(b) *P lies on the open arc AD_a of h_a.*
(c) *The quadrilateral $AB'PC'$ has an incircle.*
(d) $IQ_a \perp BC$.
(e) *The incircles of PAB and PAC touch each other.*

Proof. (a) $\Longleftrightarrow$ (b). As $2BD_a = AB+BC-AC$ and $2CD_a = AC+BC-AB$, we have $BD_a - CD_a = AB - AC$ and D_a is the vertex of the branch of h_a through A.

(b)$\Longrightarrow$ (c). AI and PQ_a are the lines tangent to h_a respectively at A and P. If W_a is their common point, BW_a is a bisector of $\angle ABP$. Hence, W_a is equidistant from the four sides of the quadrilateral.

(c)$\Longrightarrow$ (b). If the incircle of $AB'PC'$ touches PB', PC', AC, AB respectively at U, U', V, V', we have $PB-PC = BU-CV = BV'-CU' = AB-AC$.

(a) $\Longleftrightarrow$ (d). If S_a is the projection of Q_a upon BC, we have $2BS_a = PB + BC - PC$. Hence, $PB - PC = AB - AC \Longleftrightarrow S_a = D_a \Longleftrightarrow IQ_a \perp BC$.

(a) $\Longleftrightarrow$ (e). If the incircles of PAC and PAB touch the line AP respectively at S_b and S_c, we have $2AS_b = AC + PA - PC$ and $2AS_c = AB + PA - PB$. Hence, $PB - PC = AB - AC \Longleftrightarrow S_b = S_c$. See Figure 3. □

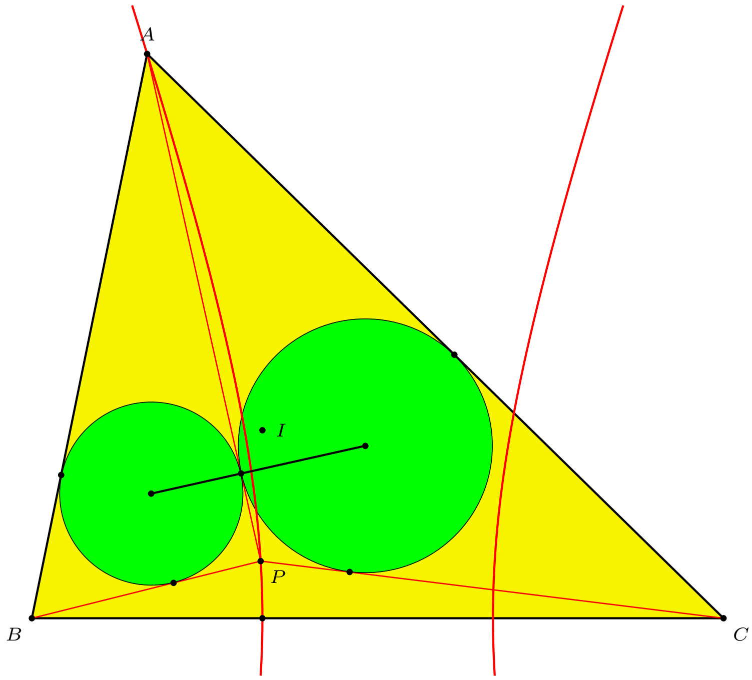


Figure 3.

Proposition 3. *When the conditions of Proposition 2 are satisfied, the following statements are equivalent.*
(a) *The incircles of PBC and $AB'PC'$ are congruent.*
(b) *P is the midpoint of W_aQ_a.*
(c) *W_aQ_a and AD_a are parallel.*
(d) *P lies on the line M_aI where M_a is the midpoint of BC.*

Proof. (a) $\Longleftrightarrow$ (b) is obvious.

Let's notice that, as I is the pole of AD_a with respect to h_a, M_aI is the conjugate diameter of the direction of AD_a with respect to h_a.

So (c) $\Longleftrightarrow$ (d) because W_aQ_a is the tangent to h_a at P.

As the line M_aI passes through the midpoint of AD_a, (b') $\Longleftrightarrow$ (c). □

Now, let us recall the classical construction of an hyperbola knowing the foci and a vertex: For any point M on the circle with center M_a passing through D_a, if L is the line perpendicular at M to BM, and N the reflection of B in M, L touches h_a at $L \cap CN$.

Construction 2. *The perpendicular from B to AD_a and the circle with center M_a passing through D_a have two common points. For one of them M, the perpendicular at M to BM will intersect M_aI at P and the lines D_aI and AI respectively at Q_a and W_a.* See Figure 4.

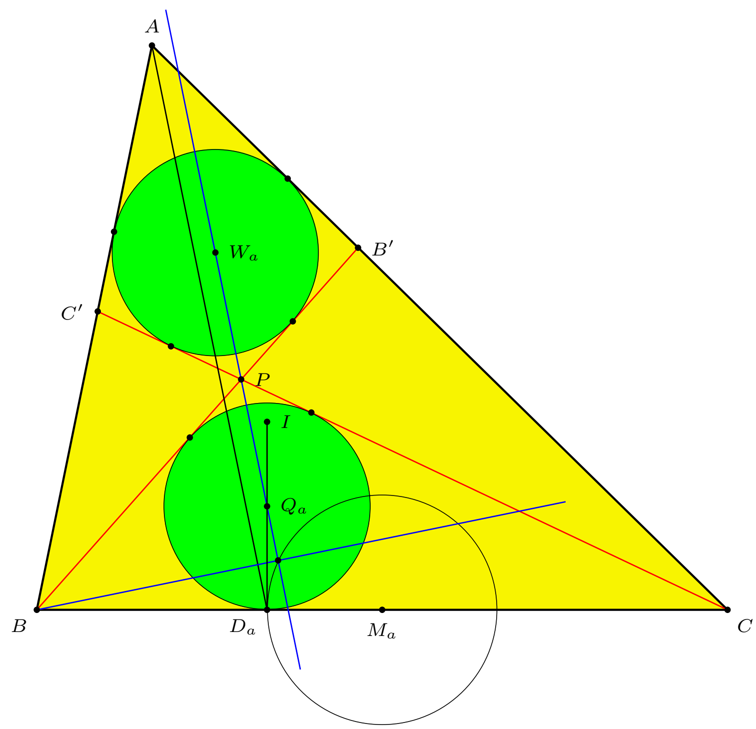


Figure 4.

Remark. We have already known that $PB - PC = c - b$. A further investigation leads to the following results.
(i) $PB + PC = \sqrt{as}$ where s is the semiperimeter of ABC.
(ii) The homogeneous barycentric coordinates of P, Q_a, W_a are as follows.

$$\begin{aligned} P: &\quad (a,\ b - s + \sqrt{as},\ c - s + \sqrt{as}) \\ Q_a: &\quad \left(a,\ b + 2(s-c)\sqrt{\frac{s}{a}},\ c + 2(s-b)\sqrt{\frac{s}{a}}\right) \\ W_a: &\quad (a + 2\sqrt{as},\ b,\ c) \end{aligned}$$

(iii) The common radius of the two incircles is $r_a\left(1 - \sqrt{\frac{a}{s}}\right)$, where r_a is the radius of the $A-$excircle.

3. The internal Soddy center

Let Δ, s, r, and R be respectively the area, the semiperimeter, the inradius, and the circumradius of triangle ABC.

The three circles $(A, s-a)$, $(B, s-b)$, $(C, s-c)$ touch each other. The internal Soddy circle is the circle tangent externally to each of these three circles. See Figure 5. Its center is $X(176)$ in [2] with barycentric coordinates

$$\left(a+\frac{\Delta}{s-a},\ b+\frac{\Delta}{s-b},\ c+\frac{\Delta}{s-c}\right)$$

and its radius is

$$\rho=\frac{\Delta}{2s+4R+r}.$$

See [2] for more details and references.

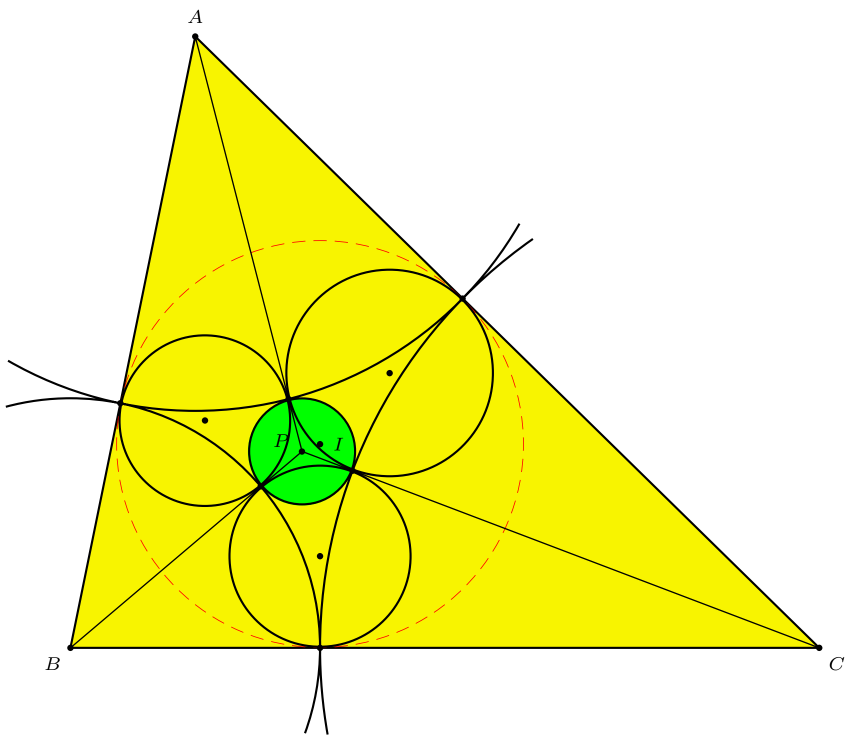


Figure 5.

Proposition 4. *The inner Soddy center $X(176)$ is the only point P inside ABC*
(a) *for which the incircles of PBC, PCA, PAB touch each other;*
(b) *with cevian triangle $A'B'C'$ for which each of the three quadrilaterals $AB'PC'$, $BC'PA'$, $CA'PB'$ have an incircle.* See Figure 6.

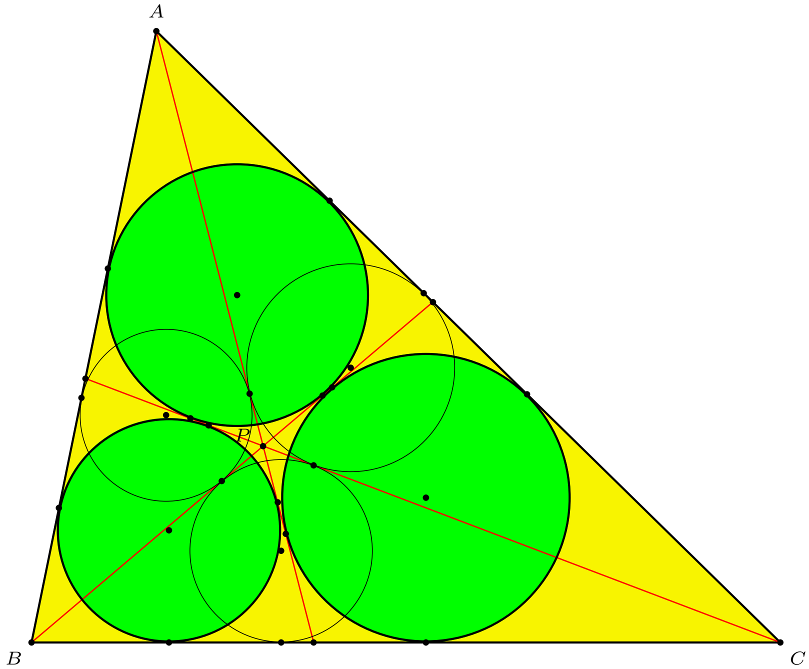


Figure 6.

Proof. Proposition 2 shows that the conditions in (a) and (b) are both equivalent to

$$PB - PC = c - b,\ PC - PA = a - c,\ PA - PB = b - a.$$

As $PA = \rho + s - a$, $PB = \rho + s - b$, $PC = \rho + s - c$, these conditions are satisfied for $P = X(176)$. Moreover, a point P inside ABC verifying these conditions must lie on the open arc AD_a of h_a and on the open arc BD_b of h_b and these arcs cannot have more than a common point. □

Remarks. (1) It follows from Proposition 2(d) that the contact points of the incircles of PBC, PCA, PAB with BC, CA, AB respectively are the same ones D_a, D_b, D_c than the contact points of incircle of ABC.[1]

(2) The incircles of PBC, PCA, PAB touch each other at the points where the internal Soddy circle touches the circles $(A, s-a)$, $(B, s-b)$, $(C, s-c)$.

(3) If Q_a is the incenter of PBC, and W_a the incenter of $AB'PC'$, we have $\dfrac{\overline{Q_aD_a}}{\overline{IQ_a}} = \dfrac{r_a}{a}$, and $\dfrac{\overline{W_aI}}{\overline{AW_a}} = \dfrac{r_a}{s}$, where r_a is the radius of the A-excircle.

(4) The four common tangents of the incircles of $BC'PA'$ and $CA'PB'$ are BC, Q_bQ_c, AP and D_aI.

(5) The lines AQ_a, BQ_b, CQ_c concur at

$$X(482) = \left(a + \frac{2\Delta}{s-a},\ b + \frac{2\Delta}{s-b},\ c + \frac{2\Delta}{s-c}\right).$$

[1] Thanks to François Rideau for this nice remark.

References

[1] J. W. Clawson, The complete quadrilateral, *Annals of Mathematics*, ser. 2, 20 (1919)? 232–261.
[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[3] T. Lalesco, *La geometrie du Triangle*, Paris Vuibert 1937; Jacques Gabay reprint 1987.

Jean-Pierre Ehrmann: 6, rue des Cailloux, 92110 - Clichy, France
E-mail address: Jean-Pierre.EHRMANN@wanadoo.fr

Forum Geometricorum
Volume 6 (2006) 335–342.

FORUM GEOM
ISSN 1534-1178

Hansen's Right Triangle Theorem, Its Converse and a Generalization

Amy Bell

Abstract. We generalize D. W. Hansen's theorem relating the inradius and exradii of a right triangle and its sides to an arbitrary triangle. Specifically, given a triangle, we find two quadruples of segments with equal sums and equal sums of squares. A strong converse of Hansen's theorem is also established.

1. Hansen's right triangle theorem

In an interesting article in *Mathematics Teacher*, D. W. Hansen [2] has found some remarkable identities associated with a right triangle. Let ABC be a triangle with a right angle at C, sidelengths a, b, c. It has an incircle of radius r, and three excircles of radii r_a, r_b, r_c.

Theorem 1 (Hansen)**.** (1) *The sum of the four radii is equal to the perimeter of the triangle:*

$$r_a + r_b + r_c + r = a + b + c.$$

(2) *The sum of the squares of the four radii is equal to the sum of the squares of the sides of the triangle:*

$$r_a^2 + r_b^2 + r_c^2 + r^2 = a^2 + b^2 + c^2.$$

We seek to generalize Hansen's theorem to an arbitrary triangle, by replacing a, b, c by appropriate quantities whose sum and sum of squares are respectively equal to those of r_a, r_b, r_c and r. Now, for a right triangle ABC with right angle vertex C, this latter vertex is the orthocenter of the triangle, which we generically denote by H. Note that

$$a = BH \quad \text{and} \quad b = AH.$$

On the other hand, the hypotenuse being a diameter of the circumcircle, $c = 2R$. Note also that $CH = 0$ since C and H coincide. This suggests that a possible generalization of Hansen's theorem is to replace the triple a, b, c by the quadruple AH, BH, CH and $2R$. Since $AH = 2R\cos A$ etc., one of the quantities AH, BH, CH is negative if the triangle contains an obtuse angle.

Publication Date: December 20, 2006. Communicating Editor: Floor van Lamoen.
The paper is a revision of part of the author's thesis for the degree of Master of Science in Teaching (Mathematics) at Florida Atlantic University, under the direction of Professor Paul Yiu. Thanks are also due to the referee for suggestions leading to improvements on the paper.

We shall establish the following theorem.

Theorem 2. *Let ABC be a triangle with orthocenter H and circumradius R.*
(1) $r_a + r_b + r_c + r = AH + BH + CH + 2R$;
(2) $r_a^2 + r_b^2 + r_c^2 + r^2 = AH^2 + BH^2 + CH^2 + (2R)^2$.

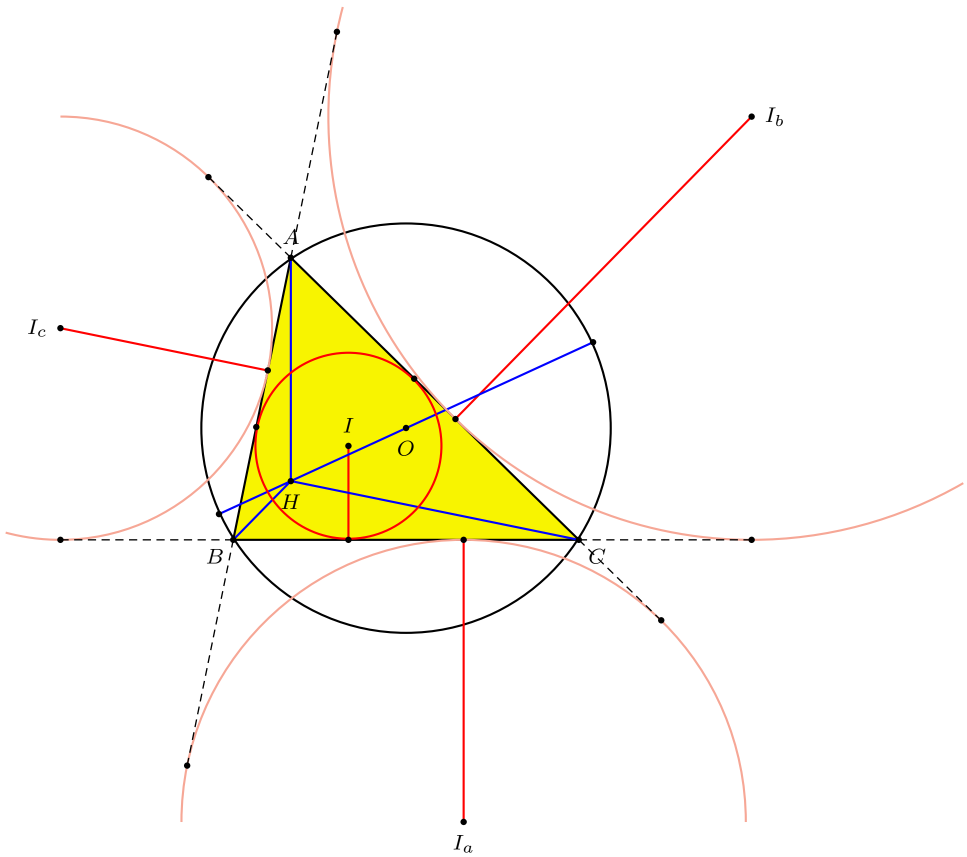


Figure 1. Two quadruples with equal sums and equal sums of squares

2. A characterization of right triangles in terms of inradius and exradii

Proposition 3. *The following statements for a triangle ABC are equivalent.*
(1) $r_c = s$.
(2) $r_a = s - b$.
(3) $r_b = s - a$.
(4) $r = s - c$.
(5) C *is a right angle.*

Proof. By the formulas for the exradii and the Heron formula, each of (1), (2), (3), (4) is equivalent to the condition

$$(s-a)(s-b) = s(s-c). \qquad (1)$$

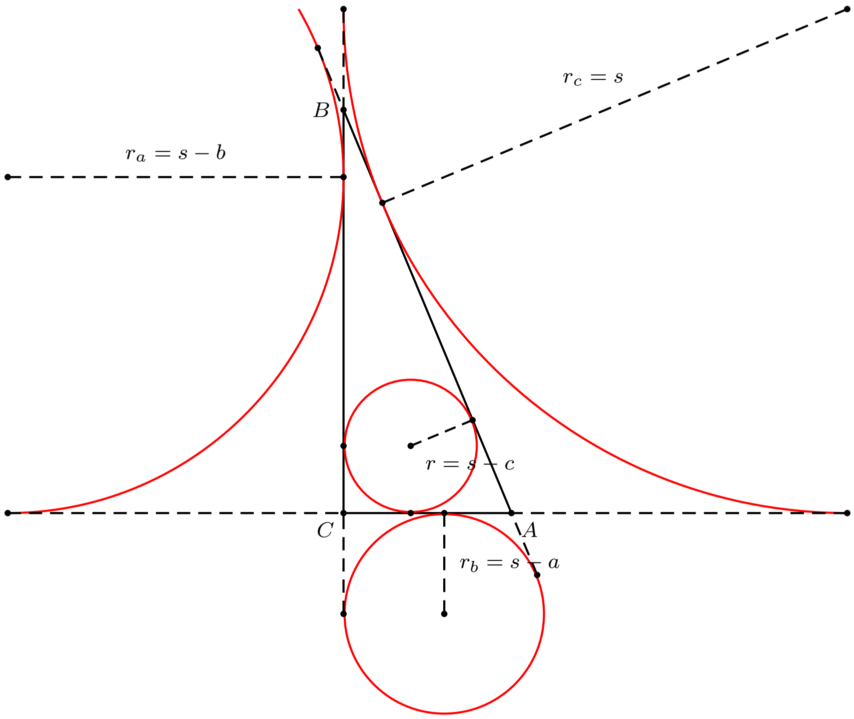


Figure 2. Inradius and exradii of a right triangle

Assuming (1), we have $s^2 - (a+b)s + ab = s^2 - cs$, $(a+b-c)s = ab$, $(a+b-c)(a+b+c) = 2ab$, $(a+b)^2 - c^2 = 2ab$, $a^2 + b^2 = c^2$. This shows that each of (1), (2), (3), (4) implies (5). The converse is clear. See Figure 2. $\square$

3. A formula relating the radii of the various circles

As a preparation for the proof of Theorem 2, we study the excircles in relation to the circumcircle and the incircle. We establish a basic result, Proposition 6, below. Lemma 4 and the statement of Proposition 6 can be found in [3, pp.185–193]. An outline proof of Proposition 5 can be found in [4, §2.4.1]. Propositions 5 and 6 can also be found in [5, §4.6.1]. [1] We present a unified detailed proof of these propositions here, simpler and more geometric than the trigonometric proofs outlined in [3].

Consider triangle ABC with its circumcircle (O). Let the bisector of angle A intersect the circumcircle at M. Clearly, M is the midpoint of the arc BMC. The line BM clearly contains the incenter I and the excenter I_a.

Lemma 4. $MB = MI = MI_a = MC$.

[1] The referee has pointed out that these results had been known earlier, and can be found, for example, in the nineteenth century work of John Casey [1].

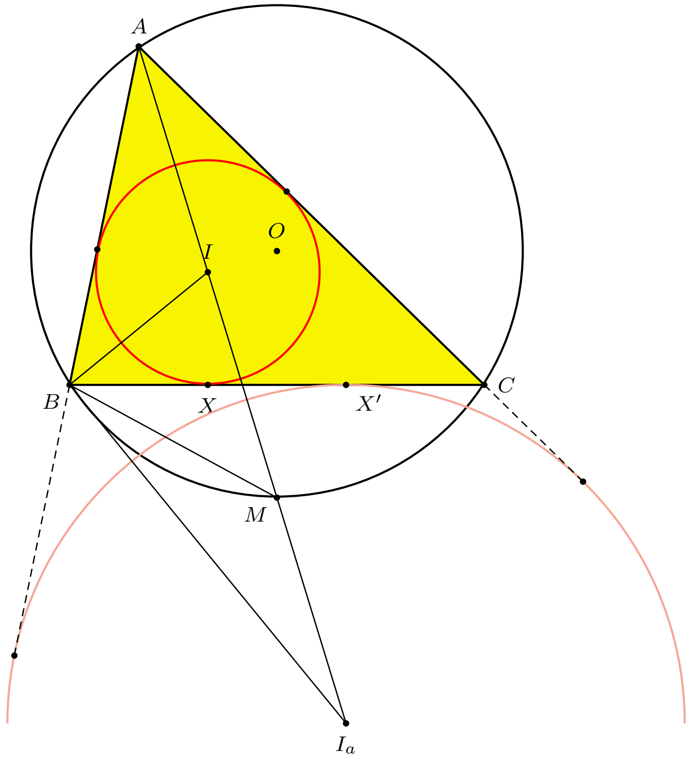


Figure 3. $r_a + r_b + r_c = 4R + r$

Proof. It is enough to prove that $MB = MI$. See Figure 3. This follows by an easy calculation of angles.
(i) $\angle IBI_a = 90^\circ$ since the two bisectors of angle B are perpendicular to each other.
(ii) The midpoint N of I_aI is the circumcenter of triangle IBI_a, so $NB = NI = NI_a$.
(iii) From the circle (IBI_a) we see $\angle BNA = \angle BNI = 2\angle BCI = \angle BCA$, but this means that N lies on the circumcircle (ABC) and thus coincides with M.
It follows that $MI_a = MB = MI$, and M is the midpoint of II_a.

The same reasoning shows that $MC = MI = MI_a$ as well. $\square$

Now, let I' be the intersection of the line IO and the perpendicular from I_a to BC. See Figure 4. Note that this latter line is parallel to OM. Since M is the midpoint of II_a, O is the midpoint of II'. It follows that I' is the reflection of I in O. Also, $I'I_a = 2 \cdot OM = 2R$. Similarly, $I'I_b = I'I_c = 2R$. We summarize this in the following proposition.

Proposition 5. *The circle through the three excenters has radius $2R$ and center I', the reflection of I in O.*

Remark. Proposition 5 also follows from the fact that the circumcircle is the nine point circle of triangle $I_aI_bI_c$, and I is the orthocenter of this triangle.

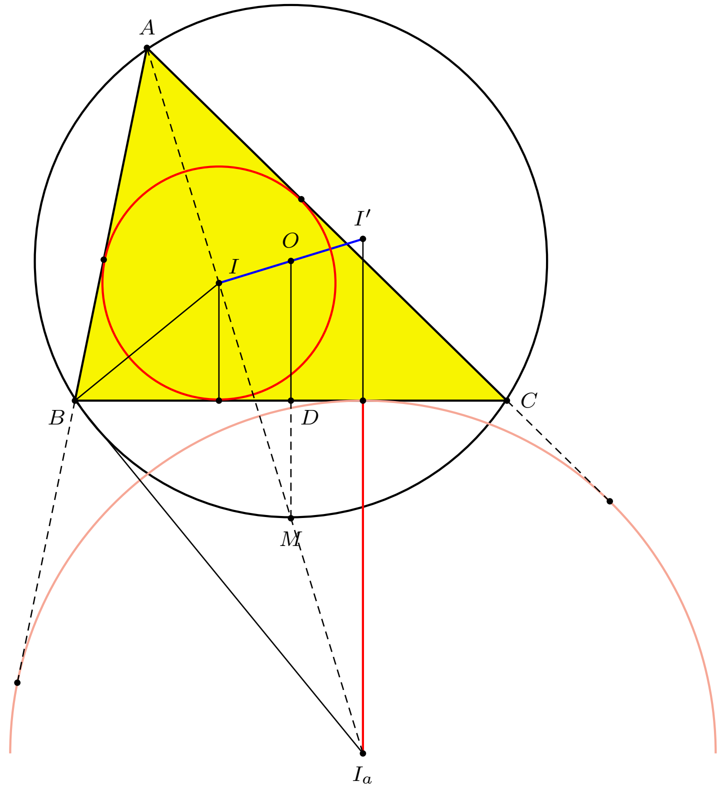


Figure 4. $I'I_a = 2R$

Proposition 6. $r_a + r_b + r_c = 4R + r$.

Proof. The line I_aI' intersects BC at the point X' of tangency with the excircle. Note that $I'X' = 2R - r_a$. Since O is the midpoint of II', we have $IX + I'X' = 2 \cdot OD$. From this, we have

$$2 \cdot OD = r + (2R - r_a). \tag{2}$$

Consider the excenters I_b and I_c. Since the angles I_bBI_c and I_bCI_c are both right angles, the four points I_b, I_c, B, C are on a circle, whose center is the midpoint N of I_bI_c. See Figure 5. The center N must lie on the perpendicular bisector of BC, which is the line OM. Therefore N is the antipodal point of M on the circumcircle, and we have $2ND = r_b + r_c$. Thus, $2(R + OD) = r_b + r_c$. From (2), we have $r_a + r_b + r_c = 4R + r$. $\square$

4. Proof of Theorem 2

We are now ready to prove Theorem 2.

(1) Since $AH = 2 \cdot OD$, by (2) we express this in terms of R, r and r_a; similarly for BH and CH:

$$AH = 2R + r - r_a, \qquad BH = 2R + r - r_b, \qquad CH = 2R + r - r_c.$$

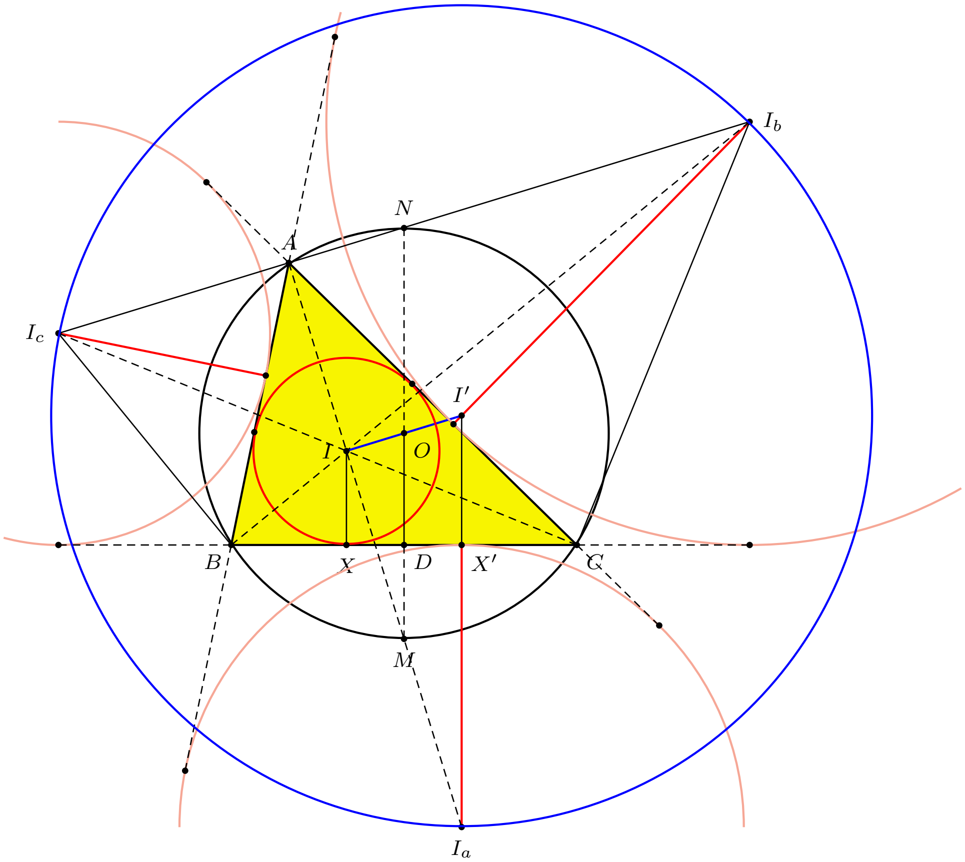


Figure 5. $r_a + r_b + r_c = 4R + r$

From these,

$$\begin{aligned} AH + BH + CH + 2R =& 8R + 3r - (r_a + r_b + r_c) \\ =& 2(4R + r) + r - (r_a + r_b + r_c) \\ =& 2(r_a + r_b + r_c) + r - (r_a + r_b + r_c) \\ =& r_a + r_b + r_c + r. \end{aligned}$$

(2) This follows from simple calculation making use of Proposition 6.

$$\begin{aligned} & AH^2 + BH^2 + CH^2 + (2R)^2 \\ =& (2R + r - r_a)^2 + (2R + r - r_b)^2 + (2R + r - r_c)^2 + 4R^2 \\ =& 3(2R + r)^2 - 2(2R + r)(r_a + r_b + r_c) + r_a^2 + r_b^2 + r_c^2 + 4R^2 \\ =& 3(2R + r)^2 - 2(2R + r)(4R + r) + 4R^2 + r_a^2 + r_b^2 + r_c^2 \\ =& r^2 + r_a^2 + r_b^2 + r_c^2. \end{aligned}$$

This completes the proof of Theorem 2.

5. Converse of Hansen's theorem

We prove a strong converse of Hansen's theorem (Theorem 10 below).

Proposition 7. *A triangle ABC satisfies*

$$r_a^2 + r_b^2 + r_c^2 + r^2 = a^2 + b^2 + c^2 \tag{3}$$

if and only if it contains a right angle.

Proof. Using $AH = 2R\cos A$ and $a = 2R\sin A$, and similar expressions for BH, CH, b, and c, we have

$$\begin{aligned}
&AH^2 + BH^2 + CH^2 + (2R)^2 - (a^2 + b^2 + c^2)\\
=&4R^2(\cos^2 A + \cos^2 B + \cos^2 C + 1 - \sin^2 A - \sin^2 B - \sin^2 C)\\
=&4R^2(2\cos^2 A + \cos 2B + \cos 2C)\\
=&8R^2(\cos^2 A + \cos(B + C)\cos(B - C))\\
=&-8R^2\cos A(\cos(B + C) + \cos(B - C))\\
=&-16R^2\cos A\cos B\cos C.
\end{aligned}$$

By Theorem 2(2), the condition (3) holds if and only if $AH^2 + BH^2 + CH^2 + (2R)^2 = a^2 + b^2 + c^2$. One of $\cos A$, $\cos B$, $\cos C$ must be zero from above. This means that triangle ABC contains a right angle. □

In the following lemma we collect some useful and well known results. They can be found more or less directly in [3].

Lemma 8. (1) $r_ar_b + r_br_c + r_cr_a = s^2$.
(2) $r_a^2 + r_b^2 + r_c^2 = (4R + r)^2 - 2s^2$.
(3) $ab + bc + ca = s^2 + (4R + r)r$.
(4) $a^2 + b^2 + c^2 = 2s^2 - 2(4R + r)r$.

Proof. (1) follows from the formulas for the exradii and the Heron formula.

$$\begin{aligned}
r_ar_b + r_br_c + r_cr_a =&\frac{\triangle^2}{(s-a)(s-b)} + \frac{\triangle^2}{(s-b)(s-c)} + \frac{\triangle^2}{(s-c)(s-a)}\\
=&s((s-c) + (s-a) + (s-b))\\
=&s^2.
\end{aligned}$$

From this (2) easily follows.

$$\begin{aligned}
r_a^2 + r_b^2 + r_c^2 =&(r_a + r_b + r_c)^2 - 2(r_ar_b + r_br_c + r_cr_a)\\
=&(4R + r)^2 - 2s^2.
\end{aligned}$$

Again, by Proposition 6,

$$
\begin{aligned}
&4R+r\\
=&r_a+r_b+r_c\\
=&\frac{\triangle}{s-a}+\frac{\triangle}{s-b}+\frac{\triangle}{s-c}\\
=&\frac{\triangle}{(s-a)(s-b)(s-c)}\left((s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)\right)\\
=&\frac{1}{r}\left(3s^2-2(a+b+c)s+(ab+bc+ca)\right)\\
=&\frac{1}{r}\left((ab+bc+ca)-s^2\right).
\end{aligned}
$$

An easy rearrangement gives (3).

(4) follows from (3) since $a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=4s^2-2(s^2+(4R+r)r)=2s^2-2(4R+r)r$. □

Proposition 9. *$r_a^2+r_b^2+r_c^2+r^2=a^2+b^2+c^2$ if and only if $2R+r=s$.*

Proof. By Lemma 8(2) and (4), $r_a^2+r_b^2+r_c^2+r^2=a^2+b^2+c^2$ if and only if $(4R+r)^2-2s^2+r^2=2s^2-2(4R+r)r$; $4s^2=(4R+r)^2+2(4R+r)r+r^2=(4R+2r)^2=4(2R+r)^2$; $s=2R+r$. □

Theorem 10. *The following statements for a triangle ABC are equivalent.*
(1) $r_a+r_b+r_c+r=a+b+c$.
(2) $r_a^2+r_b^2+r_c^2+r^2=a^2+b^2+c^2$.
(3) $R+2r=s$.
(4) *One of the angles is a right angle.*

Proof. (1) $\Longrightarrow$ (3): This follows easily from Proposition 6.

(3) $\Longleftrightarrow$ (2): Proposition 9 above.

(2) $\Longleftrightarrow$ (4): Proposition 7 above.

(4) $\Longrightarrow$ (1): Theorem 1 (1). □

References

[1] J. Casey, *A Sequel to the First Six Books of the Elements of Euclid*, 6th edition, 1888.

[2] D. W. Hansen, On inscribed and escribed circles of right triangles, circumscribed triangles, and the four-square, three-square problem, *Mathematics Teacher*, 96 (2003) 358–364.

[3] R. A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, 1960.

[4] P. Yiu, *Euclidean Geometry*, Florida Atlantic University Lecture Notes, 1998.

[5] P. Yiu, *Introduction to the Geometry of Triangle*, Florida Atlantic University Lecture Notes, 2001.

Amy Bell: Department of Mathematics, Broward Community College, North Campus, 1000 Cocunut Creek Boulevard, Coconut Creek, FL 33066, USA

E-mail address: abmath@earthlink.net

Forum Geometricorum
Volume 6 (2006) 343–357.

FORUM GEOM
ISSN 1534-1178

Some Constructions Related to the Kiepert Hyperbola

Paul Yiu

Abstract. Given a reference triangle and its Kiepert hyperbola $\mathcal{K}$, we study several construction problems related to the triangles which have $\mathcal{K}$ as their own Kiepert hyperbolas. Such triangles necessarily have their vertices on $\mathcal{K}$, and are called special Kiepert inscribed triangles. Among other results, we show that the family of special Kiepert inscribed triangles all with the same centroid G form part of a poristic family between $\mathcal{K}$ and an inscribed conic with center which is the inferior of the Kiepert center.

1. Special Kiepert inscribed triangles

Given a triangle ABC and its Kiepert hyperbola $\mathcal{K}$, consisting of the Kiepert perspectors

$$K(t) = \left(\frac{1}{S_A+t} : \frac{1}{S_B+t} : \frac{1}{S_C+t}\right), \qquad t \in \mathbb{R} \cup \{\infty\},$$

we study triangles with vertices on $\mathcal{K}$ having $\mathcal{K}$ as their own Kiepert hyperbolas. We shall work with homogeneous barycentric coordinates and make use of standard notations of triangle geometry as in [2]. Basic results on triangle geometry can be found in [3]. The Kiepert hyperbola has equation

$$K(x,y,z) := (S_B - S_C)yz + (S_C - S_A)zx + (S_A - S_B)xy = 0 \qquad (1)$$

in homogeneous barycentric coordinates. Its center, the Kiepert center

$$K_{\mathrm{i}} = ((S_B - S_C)^2 : (S_C - S_A)^2 : (S_A - S_B)^2),$$

lies on the Steiner inellipse. In this paper we shall mean by a Kiepert inscribed triangle one whose vertices are on the Kiepert hyperbola $\mathcal{K}$. If a Kiepert inscribed triangle is perspective with ABC, it is called the Kiepert cevian triangle of its perspector. Since the Kiepert hyperbola of a triangle can be characterized as the rectangular circum-hyperbola containing the centroid, our objects of interest are Kiepert inscribed triangles whose centroids are Kiepert perspectors. We shall assume the vertices to be finite points on $\mathcal{K}$, and call such triangles special Kiepert inscribed triangles. We shall make frequent use of the following notations.

Publication Date: December 28, 2006. Communicating Editor: Jean-Pierre Ehrmann.

$$
\begin{aligned}
P(t) &= ((S_B - S_C)(S_A + t) : (S_C - S_A)(S_B + t) : (S_A - S_B)(S_C + t)) \\
Q(t) &= ((S_B - S_C)^2(S_A + t) : (S_C - S_A)^2(S_B + t) : (S_A - S_B)^2(S_C + t)) \\
f_2 &= S_{AA} + S_{BB} + S_{CC} - S_{BC} - S_{CA} - S_{AB} \\
f_3 &= S_A(S_B - S_C)^2 + S_B(S_C - S_A)^2 + S_C(S_A - S_B)^2 \\
f_4 &= (S_{AA} - S_{BC})S_{BC} + (S_{BB} - S_{CA})S_{CA} + (S_{CC} - S_{AB})S_{AB} \\
g_3 &= (S_A - S_B)(S_B - S_C)(S_C - S_A)
\end{aligned}
$$

Here, $P(t)$ is a typical infinite point, and $Q(t)$ is a typical point on the tangent of the Steiner inellipse through $K_{\rm i}$. For $k = 2, 3, 4$, the function f_k, is a symmetric function in S_A, S_B, S_C of degree k.

Proposition 1. *The area of a triangle with vertices $K(t_i)$, $i = 1, 2, 3$, is*

$$\left|\frac{g_3(t_1-t_2)(t_2-t_3)(t_3-t_1)}{\prod(S^2+2(S_A+S_B+S_C)t_i+3t_i^2}\right| \cdot \triangle ABC.$$

Proposition 2. *A Kiepert inscribed triangle with vertices $K(t_i)$, $i = 1, 2, 3$, is special*, i.e., *with centroid on the Kiepert hyperbola, if and only if*

$$S^2 f_2' + (S_A + S_B + S_C) f_3' - 3f_4' = 0,$$

where f_2', f_3', f_4' are the functions f_2, f_3, f_4 with S_A, S_B, S_C replaced by t_1, t_2, t_3.

We shall make use of the following simple construction.

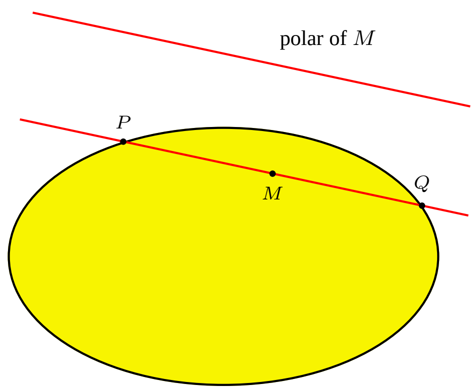


Figure 1. Construction of chord of conic with given midpoint

Construction 3. *Given a conic $\mathcal{C}$ and a point M, to construct the chord of $\mathcal{C}$ with M as midpoint, draw*
(i) *the polar of M with respect to $\mathcal{C}$,*
(ii) *the parallel through M to the line in* (i).
If the line in (ii) *intersects $\mathcal{C}$ at the two real points P and Q, then the midpoint of PQ is M.*

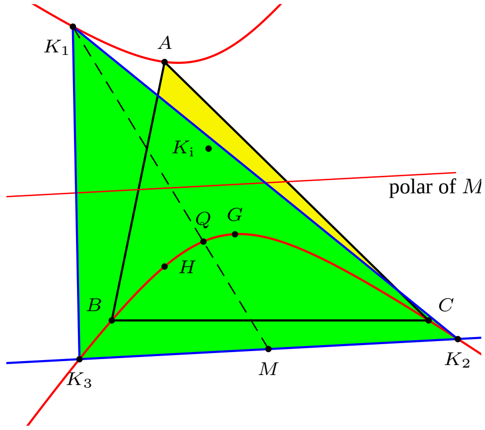


Figure 2. Construction of Kiepert inscribed triangle with prescribed centroid and one vertex

A simple application of Construction 3 gives a Kiepert inscribed triangle with prescribed centroid Q and one vertex K_1: simply take M to be the point dividing K_1Q in the ratio $K_1M : MQ = 3 : -1$. See Figure 2.

Here is an interesting family of Kiepert inscribed triangles with prescribed centroids on $\mathcal{K}$.

Construction 4. *Given a Kiepert perspector $K(t)$, construct*
(i) *K_1 on $\mathcal{K}$ and M such that the segment K_1M is trisected at K_i and $K(t)$,*
(ii) *the parallel through M to the tangent of $\mathcal{K}$ at $K(t)$,*
(iii) *the intersections K_2 and K_3 of $\mathcal{K}$ with the line in* (ii).
Then $K_1K_2K_3$ is a special Kiepert inscribed triangle with centroid $K(t)$. See Figure 3.

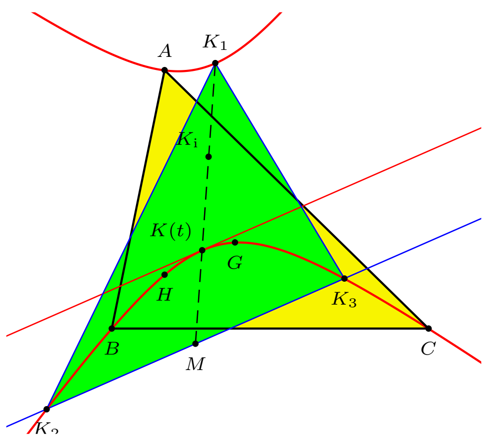


Figure 3. Kiepert inscribed triangle with centroid $K(t)$

It is interesting to note that the area of the Kiepert inscribed triangle is independent of t. It is $\frac{3\sqrt{3}}{2}|g_3|f_2^{-\frac{3}{2}}$ times that of triangle ABC. This result and many others in the present paper are obtained with the help of a computer algebra system.

2. Special Kiepert cevian triangles

Given a point $P = (u : v : w)$, the vertices of its Kiepert cevian triangle are

$$A_P = \left(\frac{-(S_B - S_C)vw}{(S_A - S_B)v + (S_C - S_A)w} : v : w\right),$$
$$B_P = \left(u : \frac{-(S_C - S_A)wu}{(S_B - S_C)w + (S_A - S_B)u} : w\right),$$
$$C_P = \left(u : v : \frac{-(S_A - S_B)uv}{(S_C - S_A)u + (S_B - S_C)v}\right).$$

These are Kiepert perspectors with parameters t_A, t_B, t_C given by

$$t_A = -\frac{S_Bv - S_Cw}{v - w}, \qquad t_B = -\frac{S_Cw - S_Au}{w - u}, \qquad t_C = -\frac{S_Au - S_Bv}{u - v}.$$

Clearly, if P is on the Kiepert hyperbola, the Kiepert cevian triangle $A_PB_PC_P$ degenerates into the point P.

Theorem 5. *The centroid of the Kiepert cevian triangle of P lies on the Kiepert hyperbola if and only if P is*
(i) *an infinite point, or*
(ii) *on the tangent at K_i to the Steiner inellipse.*

Proof. Let $P = (u : v : w)$ in homogeneous barycentric coordinates. Applying Proposition 2, we find that the centroid of $A_PB_PC_P$ lies on the Kiepert hyperbola if and only if

$$(u + v + w)K(u, v, w)^2L(u, v, w)P(u, v, w) = 0,$$

where

$$L(u, v, w) = \frac{u}{S_B - S_C} + \frac{v}{S_C - S_A} + \frac{w}{S_A - S_B},$$
$$P(u, v, w) = \prod((S_A - S_B)v^2 - 2(S_B - S_C)vw + (S_C - S_A)w^2).$$

The factors $u + v + w$ and $K(u, v, w)$ clearly define the line at infinity and the Kiepert hyperbola $\mathcal{K}$ respectively. On the other hand, the factor $L(u, v, w)$ defines the line

$$\frac{x}{S_B - S_C} + \frac{y}{S_C - S_A} + \frac{z}{S_A - S_B} = 0, \tag{2}$$

which is the tangent of the Steiner inellipse at K_i.

Each factor of $P(u, v, w)$ defines two points on a sideline of triangle ABC. If we set $(x, y, z) = (-(v + w), v, w)$ in (1), the equation reduces to $(S_A - S_B)v^2 - 2(S_B - S_C)vw + (S_C - S_A)w^2$. This shows that the two points on the line BC are the intercepts of lines through A parallel to the asymptotes of $\mathcal{K}$, and the corresponding Kiepert cevian triangles have vertices at infinite points. This is similarly the case for the other two factors of $P(u, v, w)$. □

Remark. Altogether, the six points defined by $P(u, v, w)$ above determine a conic with equation

$$G(x, y, z) = \sum \frac{x^2}{S_B - S_C} - \frac{2(S_B - S_C)yz}{(S_C - S_A)(S_A - S_B)} = 0.$$

Since

$$\begin{aligned} &g_3 \cdot G(x, y, z) \\ =& - f_2(x + y + z)^2 + \sum (S_B - S_C)^2 x^2 - 2(S_C - S_A)(S_A - S_B)yz, \end{aligned}$$

this conic is a translation of the inscribed conic

$$\sum (S_B - S_C)^2 x^2 - 2(S_C - S_A)(S_A - S_B)yz = 0,$$

which is the Kiepert parabola. See Figure 4.

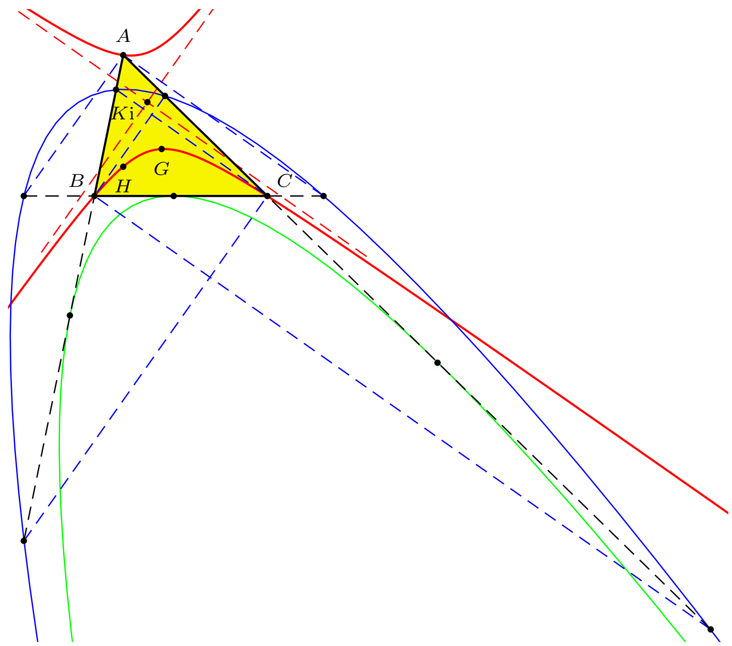


Figure 4. Translation of Kiepert parabola

3. Kiepert cevian triangles of infinite points

Consider a typical infinite point

$$P(t) = ((S_B - S_C)(S_A + t) : (S_C - S_A)(S_B + t) : (S_A - S_B)(S_C + t))$$

in homogeneous barycentric coordinates. It can be easily verified that $P(t)$ is the infinite point of perpendiculars to the line joining the Kiepert perspector $K(t)$ to the orthocenter H.[1] The Kiepert cevian triangle of $P(t)$ has vertices

[1]This is the line $\sum S_A(S_B - S_C)(S_A + t)x = 0$.

$$A(t)=\left(\frac{(S_B-S_C)(S_B+t)(S_C+t)}{S_B+S_C+2t}:\ (S_C-S_A)(S_B+t):\ (S_A-S_B)(S_C+t)\right),$$
$$B(t)=\left((S_B-S_C)(S_A+t):\ \frac{(S_C-S_A)(S_C+t)(S_A+t)}{S_C+S_A+2t}:\ (S_A-S_B)(S_C+t)\right),$$
$$C(t)=\left((S_B-S_C)(S_A+t):\ (S_C-S_A)(S_B+t):\ \frac{(S_A-S_B)(S_A+t)(S_B+t)}{S_C+S_A+2t}\right).$$

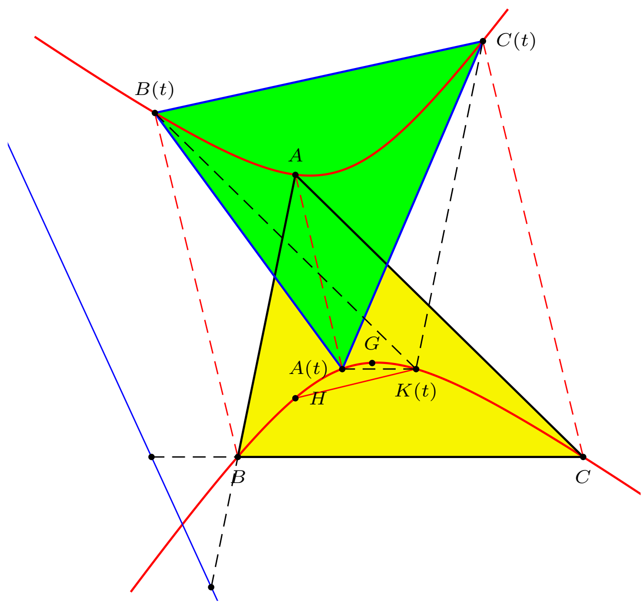


Figure 5. The Kiepert cevian triangle of $P(t)$ is the same as the Kiepert parallelian triangle of $K(t)$

It is also true that the line joining $A(t)$ to $K(t)$ is parallel to BC;[2] similarly for $B(t)$ and $C(t)$. Thus, we say that the Kiepert cevian triangle of the infinite point $P(t)$ is the same as the Kiepert parallelian triangle of the Kiepert perspector $K(t)$. See Figure 5. It is interesting to note that the area of triangle $A(t)B(t)C(t)$ is equal to that of triangle ABC, but the triangles have opposite orientations.

Now, the centroid of triangle $A(t)B(t)C(t)$ is the point

$$\left(\frac{S_B-S_C}{S_{AB}+S_{AC}-2S_{BC}-(S_B+S_C-2S_A)t}:\cdots:\cdots\right),$$

which, by Theorem 5, is a Kiepert perspector. It is $K(s)$ where s is given by

$$2f_2\cdot st+f_3\cdot(s+t)-2f_4=0. \tag{3}$$

Proposition 6. *Two distinct Kiepert perspectors have parameters satisfying* (3) *if and only if the line joining them is parallel to the orthic axis.*

[2] This is the line $-(S_A+t)(S_B+S_C+2t)x+(S_B+t)(S_C+t)(y+z)=0$.

Proof. The orthic axis $S_A x + S_B y + S_C z = 0$ has infinite point

$$P(\infty) = (S_B - S_C : S_C - S_A : S_A - S_B).$$

The line joining $K(s)$ and $K(t)$ is parallel to the orthic axis if and only if

$$\begin{vmatrix} \frac{1}{S_A+s} & \frac{1}{S_B+s} & \frac{1}{S_C+s} \\ \frac{1}{S_A+t} & \frac{1}{S_B+t} & \frac{1}{S_C+t} \\ S_B - S_C & S_C - S_A & S_A - S_B \end{vmatrix} = 0.$$

For $s \neq t$, this is the same condition as (3). □

This leads to the following construction.

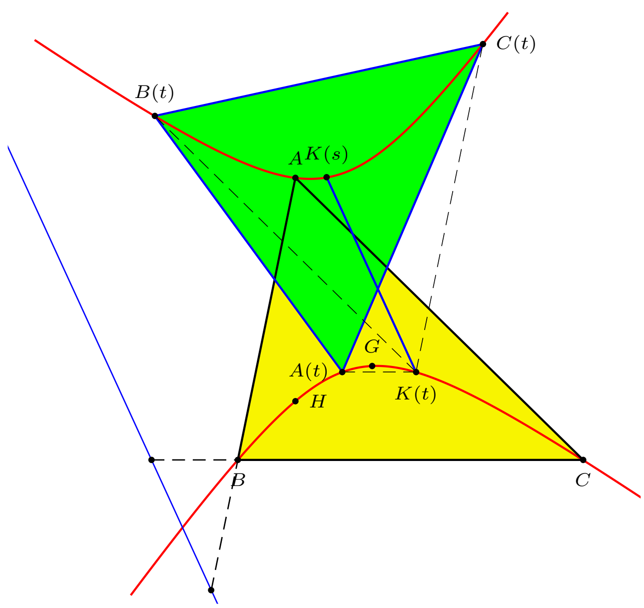


Figure 6. The Kiepert cevian triangle of $P(t)$ has centroid $K(s)$

Construction 7. *Given a Kiepert perspector $K(s)$, to construct a Kiepert cevian triangle with centroid $K(s)$, draw*
(i) *the parallel through $K(s)$ to the orthic axis to intersect the Kiepert hyperbola again at $K(t)$,*
(ii) *the parallels through $K(t)$ to the sidelines of the triangle to intersect $\mathcal{K}$ again at $A(t)$, $B(t)$, $C(t)$ respectively.*
Then, $A(t)B(t)C(t)$ has centroid $K(s)$. See Figure 6.

4. Special Kiepert inscribed triangles with common centroid G

We construct a family of Kiepert inscribed triangles with centroid G, the centroid of the reference triangle ABC. This can be easily accomplished with the help of Construction 3. Beginning with a Kiepert perspector $K_1 = K(t)$ and $Q = G$, we easily determine

$$M = ((S_A+t)(S_B+S_C+2t) : (S_B+t)(S_C+S_A+2t) : (S_C+t)(S_A+S_B+2t)).$$

The line through M parallel to its own polar with respect to $\mathcal{K}^3$ has equation

$$\frac{S_B - S_C}{S_A + t}x + \frac{S_C - S_A}{S_B + t}y + \frac{S_A - S_B}{S_C + t}z = 0. \tag{4}$$

As t varies, this line envelopes the conic

$$\begin{aligned}&(S_B - S_C)^4x^2 + (S_C - S_A)^4y^2 + (S_A - S_B)^4z^2\\&- 2(S_B - S_C)^2(S_C - S_A)^2xy - 2(S_C - S_A)^2(S_A - S_B)^2yz\\&- 2(S_A - S_B)^2(S_B - S_C)^2zx = 0,\end{aligned}$$

which is the inscribed ellipse $\mathcal{E}$ tangent to the sidelines of ABC at the traces of

$$\left(\frac{1}{(S_B - S_C)^2} : \frac{1}{(S_C - S_A)^2} : \frac{1}{(S_A - S_B)^2}\right),$$

and to the Kiepert hyperbola at G, and to the line (4) at the point

$$((S_A + t)^2 : (S_B + t)^2 : (S_C + t)^2).$$

It has center

$$((S_C-S_A)^2+(S_A-S_B)^2 : (S_A-S_B)^2+(S_B-S_C)^2 : (S_B-S_C)^2+(S_C-S_A)^2),$$

the inferior of the Kiepert center K_i. See Figure 7.

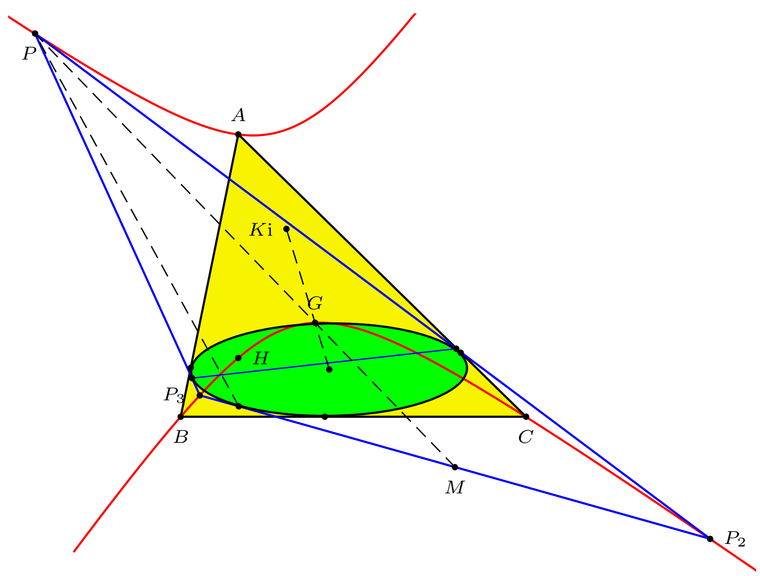


Figure 7. Poristic triangles with common centroid G

[3] The polar of M has equation $\sum(S_B - S_C)(S_{AA} - S^2 - 2(S_B + S_C)t - 2t^2)x = 0$ and has infinite point $((S_A + t)(S_A(S_B + S_C - 2t) - (S_B + S_C)(S_B - S_C + t)) : \cdots : \cdots)$.

Theorem 8. *A poristic triangle completed from a point on the Kiepert hyperbola outside the inscribed ellipse $\mathcal{E}$ (with center the inferior of $K_{\rm i}$) has its center at G and therefore has $\mathcal{K}$ as its Kiepert hyperbola.*

More generally, if we replace G by a Kiepert perspector $K_{\rm g}$, the envelope is a conic with center which divides $K_{\rm i}K_{\rm g}$ in the ratio $3 : -1$. It is an ellipse inscribed in the triangle in Construction 4.

5. A family of special Kiepert cevian triangles

5.1. *Triple perspectivity.* According to Theorem 5, there is a family of special Kiepert cevian triangles with perspectors on the line (2) which is the tangent of the Steiner inellipse at $K_{\rm i}$. Since this line also contains the Jerabek center

$$J_{\rm e} = (S_A(S_B - S_C)^2 : S_B(S_C - S_A)^2 : S_C(S_A - S_B)^2),$$

its points can be parametrized as

$$Q(t) = ((S_B - S_C)^2(S_A + t) : (S_C - S_A)^2(S_B + t) : (S_A - S_B)^2(S_C + t)).$$

The Kiepert cevian triangle of $Q(t)$ has vertices

$$A'(t) = \left(\frac{(S_C - S_A)(S_A - S_B)(S_B + t)(S_C + t)}{S_A + t} : (S_C - S_A)^2(S_B + t) : (S_A - S_B)^2(S_C + t)\right),$$
$$B'(t) = \left((S_B - S_C)^2(S_A + t) : \frac{(S_A - S_B)(S_B - S_C)(S_C + t)(S_A + t)}{S_B + t} : (S_A - S_B)^2(S_C + t)\right),$$
$$C'(t) = \left((S_B - S_C)^2(S_A + t) : (S_C - S_A)^2(S_B + t) : \frac{(S_B - S_C)(S_C - S_A)(S_A + t)(S_B + t)}{S_C + t}\right).$$

Theorem 9. *The Kiepert cevian triangle of $Q(t)$ is triply perspective to ABC. The three perspectors are collinear on the tangent of the Steiner inellipse at $K_{\rm i}$.*

Proof. The triangles $B'(t)C'(t)A'(t)$ and $C'(t)A'(t)B'(t)$ are each perspective to ABC, at the points

$$Q'(t) = \left(\frac{S_C + t}{S_C - S_A} : \frac{S_A + t}{S_A - S_B} : \frac{S_B + t}{S_B - S_C}\right),$$

and

$$Q''(t) = \left(\frac{S_B + t}{S_A - S_B} : \frac{S_C + t}{S_B - S_C} : \frac{S_A + t}{S_C - S_A}\right)$$

respectively. These two points are clearly on the line (2). $\square$

5.2. *Special Kiepert cevian triangles with the same area as ABC.* The area of triangle $A'(t)B'(t)C'(t)$ is

$$\frac{(f_2 \cdot t^2 + f_3 \cdot t - f_4)^3}{\prod (f_2 \cdot (S_A + t)^2 - (S_C - S_A)^2(S_A - S_B)^2)}$$

Among these, four have the same area as the reference triangle.

5.2.1. $t = \frac{S_A(S_B+S_C)-2S_{BC}}{S_B+S_C-2S_A}$. The points

$$Q(t) =(-2(S_B - S_C) : S_C - S_A : S_A - S_B),$$
$$Q'(t) =(S_B - S_C : -2(S_C - S_A) : S_A - S_B),$$
$$Q''(t) =(S_B - S_C : S_C - S_A : -2(S_A - S_B)),$$

give the Kiepert cevian triangle

$$A_1' =(-(S_B - S_C) : 2(S_C - S_A) : 2(S_A - S_B)),$$
$$B_1' =(2(S_B - S_C) : -(S_C - S_A) : 2(S_A - S_B)),$$
$$C_1' =(2(S_B - S_C) : 2(S_C - S_A) : -(S_A - S_B)).$$

This has centroid

$$K\left(-\frac{f_3}{2f_2}\right) = \left(\frac{S_B - S_C}{S_B + S_C - 2S_A} : \frac{S_C - S_A}{S_C + S_A - 2S_B} : \frac{S_A - S_B}{S_A + S_B - 2S_C}\right).$$

$A'(t)B'(t)C'(t)$ is also the Kiepert cevian triangle of the infinite point $P(\infty)$ (of the orthic axis). See Figure 8.

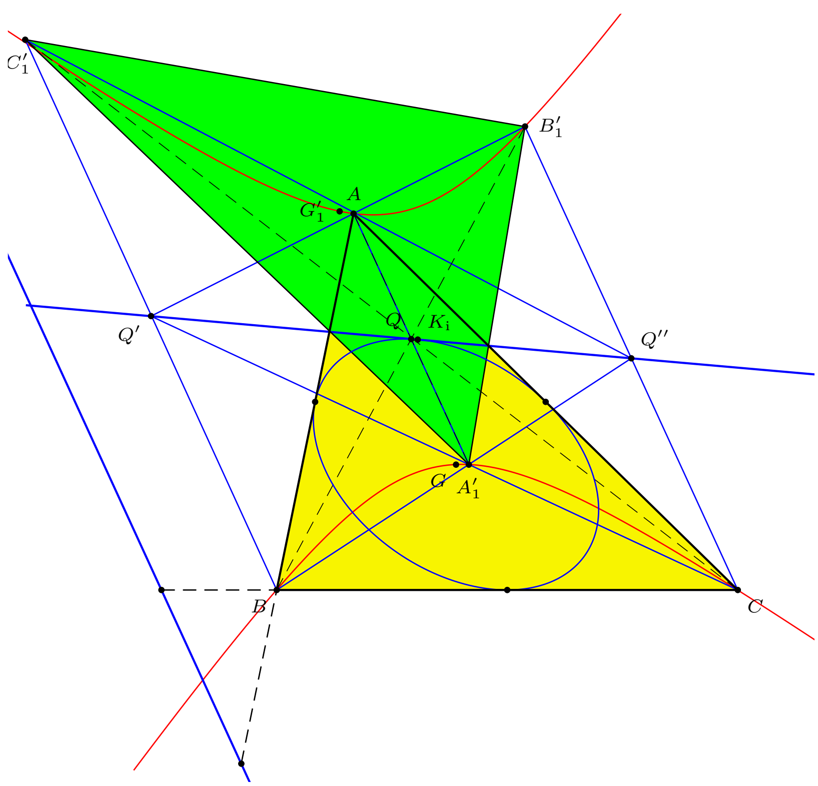


Figure 8. Oppositely oriented triangle triply perspective with ABC at three points on tangent at K_{i}

5.2.2. $t = \infty$. With the Kiepert center $K_{\rm i} = Q(\infty)$, we have the points

$$\begin{aligned} Q(\infty) =&((S_B - S_C)^2 : (S_C - S_A)^2 : (S_A - S_B)^2), \\ Q'(\infty) =& \left(\frac{1}{S_A - S_B} : \frac{1}{S_B - S_C} : \frac{1}{S_C - S_A}\right), \\ Q''(\infty) =& \left(\frac{1}{S_C - S_A} : \frac{1}{S_A - S_B} : \frac{1}{S_B - S_C}\right), \end{aligned}$$

The points $Q'(\infty)$ and $Q''(\infty)$ are the intersection with the parallels through B, C to the line joining A to the Steiner point $S_{\rm t} = \left(\frac{1}{S_B - S_C} : \frac{1}{S_C - S_A} : \frac{1}{S_A - S_B}\right)$. These points give the Kiepert cevian triangle which is the image of ABC under the homothety $\mathsf{h}(K_{\rm i}, -1)$:

$$\begin{aligned} A_2' =&((S_C - S_A)(S_A - S_B) : (S_C - S_A)^2 : (S_A - S_B)^2), \\ B_2' =&((S_B - S_C)^2 : (S_A - S_B)(S_B - S_C) : (S_A - S_B)^2), \\ C_2' =&((S_B - S_C)^2 : (S_C - S_A)^2 : (S_C - S_A)(S_B - S_C)), \end{aligned}$$

which has centroid

$$K\left(-\frac{S_A + S_B + S_C}{3}\right) = \left(\frac{1}{S_B + S_C - 2S_A} : \frac{1}{S_C + S_A - 2S_B} : \frac{1}{S_A + S_B - 2S_C}\right).$$

The points $Q'(t)$, $Q''(t)$ and G_2' are on the Steiner circum-ellipse. See Figure 9.

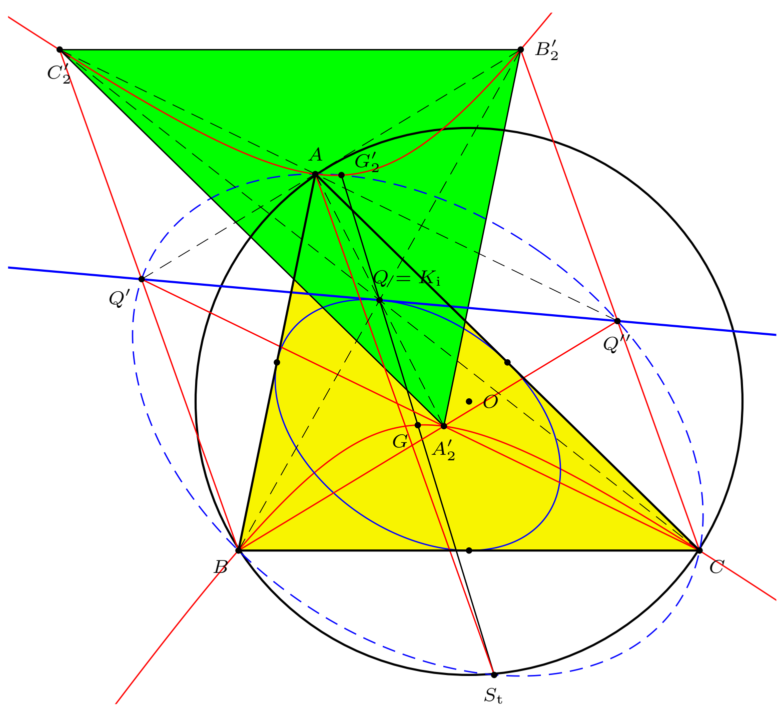


Figure 9. Oppositely congruent triangle triply perspective with ABC at three points on tangent at $K_{\rm i}$

5.2.3. $t = \frac{-f_3}{2f_2}$. $Q(t)$ is the infinite point of the line (2).

$$Q(t) = ((S_B - S_C)(S_B + S_C - 2S_A) : (S_C - S_A)(S_C + S_A - 2S_B) : (S_A - S_B)(S_A + S_B - 2S_C)),$$
$$Q'(t) = ((S_B - S_C)(S_A + S_B - 2S_C) : (S_C - S_A)(S_B + S_C - 2S_A) : (S_A - S_B)(S_C + S_A - 2S_B)),$$
$$Q''(t) = ((S_B - S_C)(S_C + S_A - 2S_B) : (S_C - S_A)(S_A + S_B - 2S_C) : (S_A - S_B)(S_B + S_C - 2S_A)).$$

These give the Kiepert cevian triangle

$$A_3' = \left(\frac{S_B - S_C}{S_B + S_C - 2S_A} : \frac{S_C - S_A}{S_A + S_B - 2S_C} : \frac{S_A - S_B}{S_C + S_A - 2S_B}\right),$$
$$B_3' = \left(\frac{S_B - S_C}{S_A + S_B - 2S_C} : \frac{S_C - S_A}{S_C + S_A - 2S_B} : \frac{S_A - S_B}{S_B + S_C - 2S_A}\right),$$
$$C_3' = \left(\frac{S_B - S_C}{S_C + S_A - 2S_B} : \frac{S_C - S_A}{S_B + S_C - 2S_A} : \frac{S_A - S_B}{S_A + S_B - 2S_C}\right),$$

with centroid

$$\left(\frac{S_B - S_C}{(S_B - S_C)^2 + 2(S_C - S_A)(S_A - S_B)} : \cdots : \cdots\right).$$

See Figure 10.

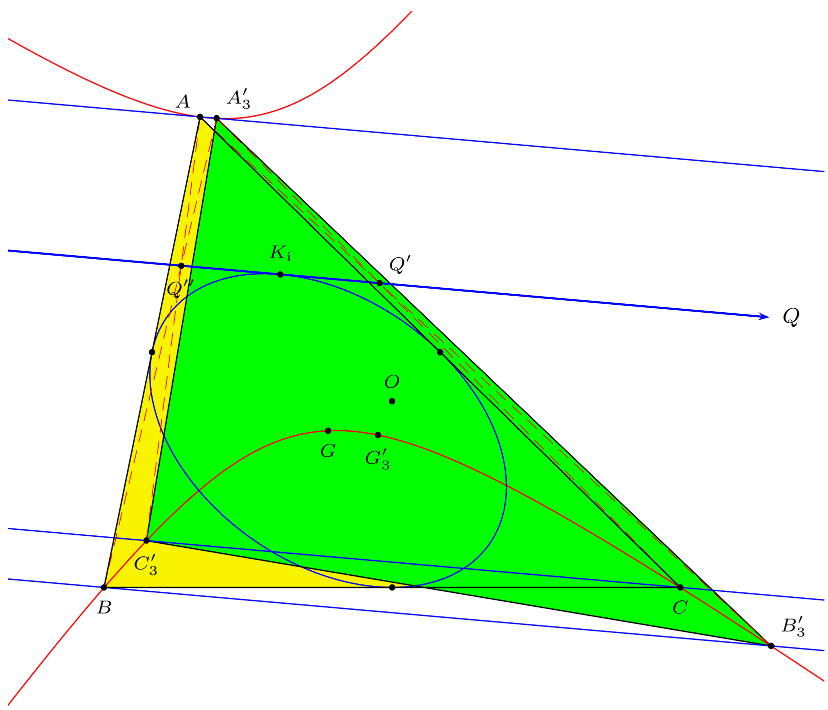


Figure 10. Triangle triply perspective with ABC (with the same orientation) at three points on tangent at K_i

5.2.4. $t = -S_A$. For $t = -S_A$, we have

$$\begin{aligned}Q(t) =&(0 : S_C - S_A : -(S_A - S_B)),\\ Q'(t) =&(-(S_B - S_C) : 0 : S_A - S_B),\\ Q''(t) =&(S_B - S_C : -(S_C - S_A) : 0).\end{aligned}$$

These points are the intercepts Q_a, Q_b, Q_c of the line (2) with the sidelines BC, CA, AB respectively. The lines AQ_a, BQ_b, CQ_c are the tangents to $\mathcal{K}$ at the vertices. The common Kiepert cevian triangle of Q_a, Q_b, Q_c is ABC oppositely oriented as ACB, CBA, BAC, triply perspective with ABC at Q_a, Q_b, Q_c respectively.

6. Special Kiepert inscribed triangles with two given vertices

Construction 10. *Given two points K_1 and K_2 on the Kiepert hyperbola $\mathcal{K}$, construct*
(i) *the midpoint M of K_1K_2,*
(ii) *the polar of M with respect to $\mathcal{K}$,*
(iii) *the reflection of the line K_1K_2 in the polar in* (ii).
If K_3 is a real intersection of $\mathcal{K}$ with the line in (iii), *then the Kiepert inscribed triangle $K_1K_2K_3$ has centroid on $\mathcal{K}$.* See Figure 11.

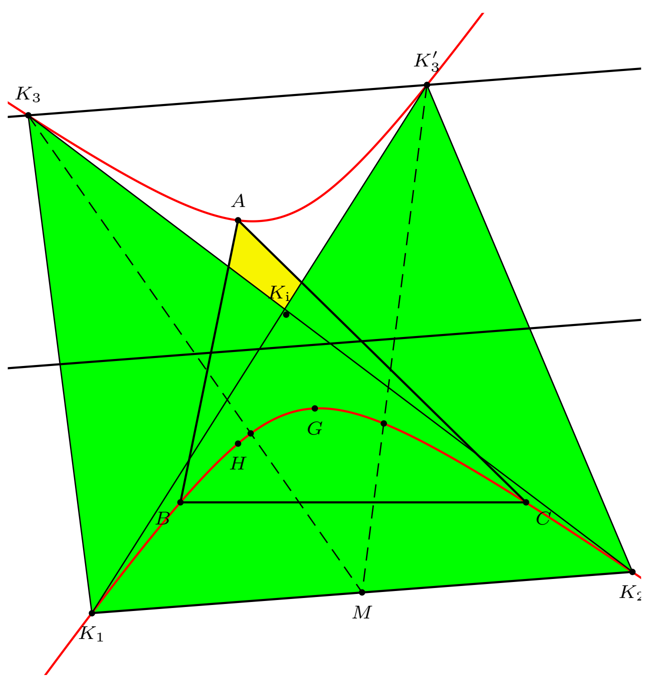


Figure 11. Construction of special Kiepert inscribed triangles given two vertices K_1, K_2

Proof. A point K_3 for which triangle $K_1K_2K_3$ has centroid on $\mathcal{K}$ clearly lies on the image of $\mathcal{K}$ under the homothety $\mathsf{h}(M, 3)$. It is therefore an intersection of $\mathcal{K}$ with this homothetic image. If $M = (u : v : w)$ in homogeneous barycentric coordinates, this homothetic conic has equation

$$\begin{aligned}&(u+v+w)^2K(x,y,z)\\ +&2(x+y+z)\left(\sum((S_B-S_C)vw+(S_C-S_A)(3u+w)w+(S_A-S_B)(3u+v)v)x\right)\\ =&0.\end{aligned}$$

The polar of M in $\mathcal{K}$ is the line

$$\sum((S_A - S_B)v + (S_C - S_A)w)x = 0. \qquad (5)$$

The parallel through M is the line

$$\sum(3(S_B - S_C)vw + (S_C - S_A)(u - w)w + (S_A - S_B)(u - v)v)x = 0. \qquad (6)$$

The reflection of (6) in (5) is the radical axis of $\mathcal{K}$ and its homothetic image above. $\square$

If there are two such real intersections K_3 and K_3', then the two triangles $K_1K_2K_3$ and $K_1K_2K_3'$ clearly have equal area. These two intersections coincide if the line in Construction 10 (iii) above is tangent to $\mathcal{K}$. This is the case when K_1K_2 is a tangent to the hyperbola

$$4f_2 \cdot K(x, y, z) - 3g_3 \cdot (x + y + z)^2 = 0,$$

which is the image of $\mathcal{K}$ under the homothety $\mathsf{h}(K_i, 2)$. See Figure 12.

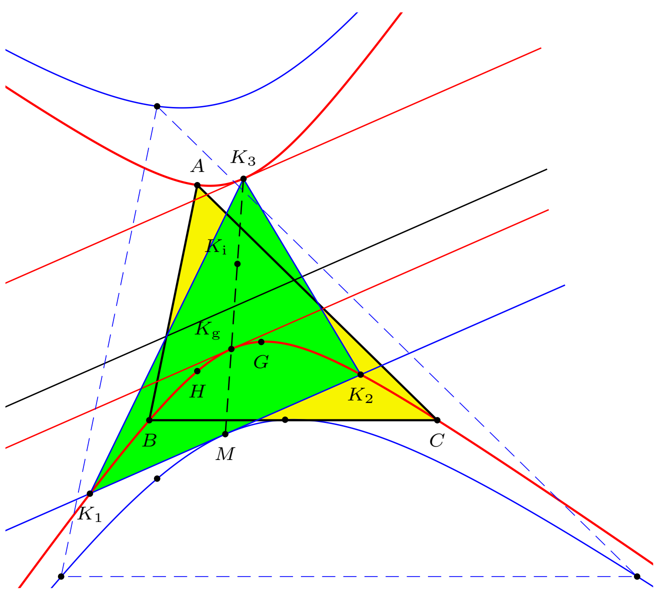


Figure 12. Family of special Kiepert inscribed triangles with K_1, K_2 uniquely determining K_3

The resulting family of special Kiepert inscribed triangles is the same family with centroid $K(t)$ and one vertex its antipode on $\mathcal{K}$, given in Construction 4.

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[2] F. M. van Lamoen and P. Yiu, The Kiepert pencil of Kiepert hyperbolas, *Forum Geom.*, 1 (2001) 125–132.

[3] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University lecture notes, 2001.

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida 33431-0991, USA

E-mail address: yiu@fau.edu

Forum Geometricorum
Volume 6 (2006) 359–360.

FORUM GEOM
ISSN 1534-1178

Author Index

FORUM GEOMETRICORUM

A Journal on Classical Euclidean Geometry and Related Areas

published by

Department of Mathematical Sciences
Florida Atlantic University

Volume 7
2007

http://forumgeom.fau.edu

ISSN 1534-1178

Editorial Board

Advisors:

John H. Conway	Princeton, New Jersey, USA
Julio Gonzalez Cabillon	Montevideo, Uruguay
Richard Guy	Calgary, Alberta, Canada
Clark Kimberling	Evansville, Indiana, USA
Kee Yuen Lam	Vancouver, British Columbia, Canada
Tsit Yuen Lam	Berkeley, California, USA
Fred Richman	Boca Raton, Florida, USA

Editor-in-chief:

Paul Yiu	Boca Raton, Florida, USA

Editors:

Clayton Dodge	Orono, Maine, USA
Roland Eddy	St. John’s, Newfoundland, Canada
Jean-Pierre Ehrmann	Paris, France
Chris Fisher	Regina, Saskatchewan, Canada
Rudolf Fritsch	Munich, Germany
Bernard Gibert	St Etiene, France
Antreas P. Hatzipolakis	Athens, Greece
Michael Lambrou	Crete, Greece
Floor van Lamoen	Goes, Netherlands
Fred Pui Fai Leung	Singapore, Singapore
Daniel B. Shapiro	Columbus, Ohio, USA
Steve Sigur	Atlanta, Georgia, USA
Man Keung Siu	Hong Kong, China
Peter Woo	La Mirada, California, USA

Technical Editors:

Yuandan Lin	Boca Raton, Florida, USA
Aaron Meyerowitz	Boca Raton, Florida, USA
Xiao-Dong Zhang	Boca Raton, Florida, USA

Consultants:

Frederick Hoffman	Boca Raton, Floirda, USA
Stephen Locke	Boca Raton, Florida, USA
Heinrich Niederhausen	Boca Raton, Florida, USA

Table of Contents

Forum Geometricorum
Volume 7 (2007) 1–9.

FORUM GEOM
ISSN 1534-1178

Euler's Triangle Determination Problem

Joseph Stern

Abstract. We give a simple proof of Euler's remarkable theorem that for a non-degenerate triangle, the set of points eligible to be the incenter is precisely the orthocentroidal disc, punctured at the nine-point center. The problem is handled algebraically with complex coordinates. In particular, we show how the vertices of the triangle may be determined from the roots of a complex cubic whose coefficients are functions of the classical centers.

1. Introduction

Consider the determination of a triangle from its centers.[1] What relations must be satisfied by points O, H, I so that a unique triangle will have these points as circumcenter, orthocenter, and incenter? In Euler's groundbreaking article [3], *Solutio facilis problematum quorundam geometricorum difficillimorum*, this intriguing question is answered synthetically, but without any comment on the geometric meaning of the solution.

Euler proved the existence of the required triangle by treating the lengths of the sides as zeros of a real cubic, the coefficients being functions of OI, OH, HI. He gave the following algebraic restriction on the distances to ensure that the cubic has three real zeros:

$$OI^2 < OH^2 - 2 \cdot HI^2 < 2 \cdot OI^2.$$

Though Euler did not remark on the geometric implications, his restriction was later proven equivalent to the simpler inequality

$$GI^2 + IH^2 < GH^2,$$

where G is the point that divides OH in the ratio $1 : 2$ (G is the centroid). This result was presented in a beautiful 1984 paper [4] by A. P. Guinand. Its geometric meaning is immediate: I must lie inside the circle on diameter GH. It also turns out that I cannot coincide with the midpoint of OH, which we denote by N (the nine-point center). The remarkable fact is that *all and only* points inside the circle and different from N are eligible to be the incenter. This region is often called the *orthocentroidal disc*, and we follow this convention.[2] Guinand considered the

Publication Date: January 8, 2007. Communicating Editor: Paul Yiu.
Dedicated to the tercentenary of Leonhard Euler.
[1]The phrase "determination of a triangle" is borrowed from [7].
[2]Conway discusses several properties of the orthocentroidal disc in [1].

cosines of the angles as zeros of a real cubic. He showed that this cubic has three real zeros with positive inverse cosines summing to π. Thus the angles are known, and the scale may be determined subsequently from OH. The problem received fresh consideration in 2002, when B. Scimemi [7] showed how to solve it using properties of the Kiepert focus, and again in 2005, when G. C. Smith [8] used statics to derive the solution.

The approach presented here uses complex coordinates. We show that the vertices of the required triangle may be computed from the roots of a certain complex cubic whose coefficients depend only upon the classical centers. This leads to a relatively simple proof.

2. Necessity of Guinand's Locus

Given a nonequilateral triangle, we show first that the incenter must lie within the orthocentroidal disc and must differ from the nine-point center. The equilateral triangle is uninteresting, since all the centers coincide.

Let $\triangle ABC$ be nonequilateral. As usual, we write O, H, I, G, N, R, r for the circumcenter, orthocenter, incenter, centroid, nine-point center, circumradius and inradius. Two formulas will feature very prominently in our discussion:

$$OI^2 = R(R-2r) \quad \text{and} \qquad NI = \tfrac{1}{2}(R-2r).$$

The first is due to Euler and the second to Feuerbach.[3] They jointly imply

$$OI > 2 \cdot NI,$$

provided the triangle is nonequilateral. Now given a segment PQ and a number $\lambda > 1$, the Apollonius Circle Theorem states that

(1) the equation $PX = \lambda \cdot QX$ describes a circle whose center lies on PQ, with P inside and Q outside;

(2) the inequality $PX > \lambda \cdot QX$ describes the interior of this circle (see [6]).

Thus the inequality $OI > 2 \cdot NI$ places I inside the circle $OX = 2 \cdot NX$, the center of which lies on the Euler line ON. Since G and H lie on the Euler line and satisfy the equation of the circle, GH is a diameter, and this circle turns out to be the orthocentroidal circle. Finally, the formulas of Euler and Feuerbach show that if $I = N$, then $O = I$. This means that the incircle and the circumcircle are *concentric*, forcing $\triangle ABC$ to be equilateral. Thus N is ineligible to be the incenter.

3. Complex Coordinates

Our aim now is to express the classical centers of $\triangle ABC$ as functions of A, B, C, regarded as complex numbers.[4] We are free to put $O = 0$, so that

$$|A| = |B| = |C| = R.$$

[3] Proofs of both theorems appear in [2].

[4] See [5] for a more extensive discussion of this approach.

The centroid is given by $3G = A+B+C$. The theory of the Euler line shows that $3G = 2O + H$, and since $O = 0$, we have

$$H = A + B + C.$$

Finally, it is clear that $2N = O + H = H$.

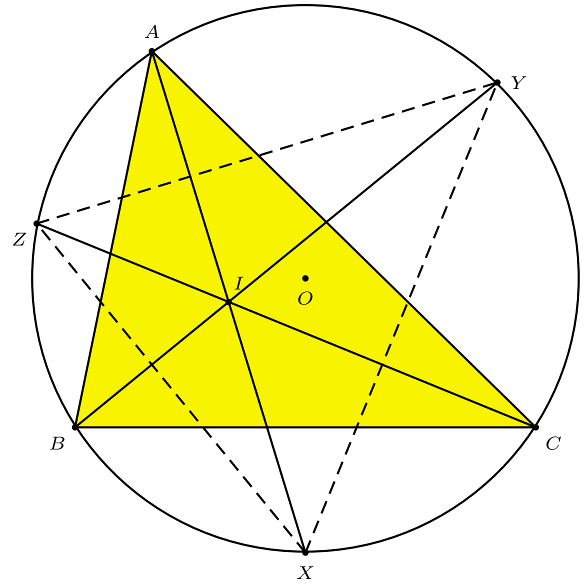


Figure 1.

To deal with the incenter, let X, Y, Z be the points at which the extended angle bisectors meet the circumcircle (Figure 1). It is not difficult to see that $AX \perp YZ$, $BY \perp ZX$ and $CZ \perp XY$. For instance, one angle between AX and YZ is the average of the minor arc from A to Z and the minor arc from X to Y. The first arc measures $\widehat{C}$, and the second, $\widehat{A} + \widehat{B}$. Thus the angle between AX and YZ is $\pi/2$. Evidently the angle bisectors of $\triangle ABC$ coincide with the altitudes of $\triangle XYZ$, and I is the orthocenter of $\triangle XYZ$. Since this triangle has circumcenter 0, its orthocenter is

$$I = X + Y + Z.$$

We now introduce complex square roots α, β, γ so that

$$\alpha^2 = A, \qquad \beta^2 = B, \qquad \gamma^2 = C.$$

There are two choices for each of α, β, γ. Observe that

$$|\beta\gamma| = R \qquad \text{and} \qquad \arg(\beta\gamma) = \tfrac{1}{2}(\arg B + \arg C),$$

so that $\pm\beta\gamma$ are the mid-arc points between B and C. It follows that $X = \pm\beta\gamma$, depending on our choice of signs. For reasons to be clarified later, we would like to arrange it so that

$$X = -\beta\gamma, \qquad Y = -\gamma\alpha, \qquad Z = -\alpha\beta.$$

These hold if α, β, γ are chosen so as to make $\triangle\alpha\beta\gamma$ *acute*, as we now show.

Let Γ denote the circle $|z| = \sqrt{R}$, on which α, β, γ must lie. Temporarily let α_1, α_2 be the two square roots of A, and β_1 a square root of B. Finally, let γ_1 be the square root of C on the side of $\alpha_1\alpha_2$ containing β_1 (Figure 2). Now $\triangle\alpha_i\beta_j\gamma_k$ is acute if and only if any two vertices are separated by the diameter of Γ through the remaining vertex. Otherwise one of its angles would be inscribed in a minor arc, rendering it obtuse. It follows that of all eight triangles $\triangle\alpha_i\beta_j\gamma_k$, only $\triangle\alpha_1\beta_2\gamma_1$ and $\triangle\alpha_2\beta_1\gamma_2$ are acute.

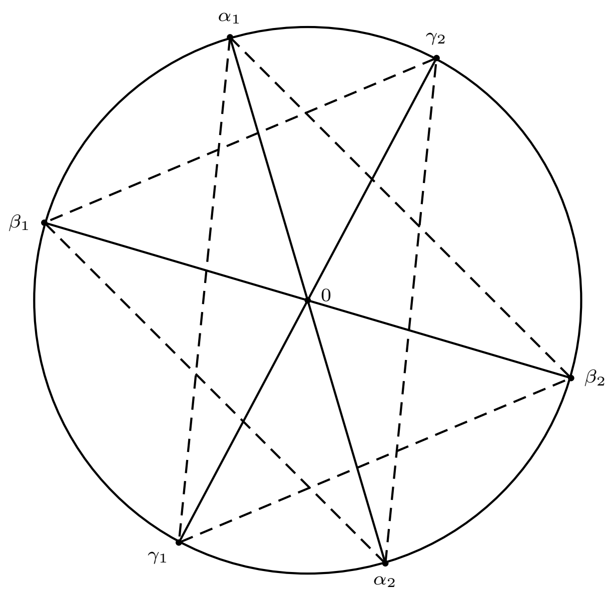


Figure 2.

Now let (α, β, γ) be either $(\alpha_1, \beta_2, \gamma_1)$ or $(\alpha_2, \beta_1, \gamma_2)$, so that $\triangle\alpha\beta\gamma$ is acute. Consider the stretch-rotation $z \mapsto \beta z$. This carries the diameter of Γ with endpoints $\pm\alpha$ to the diameter of $|z| = R$ with endpoints $\pm\alpha\beta$, one of which is Z. Now β and γ are separated by the diameter with endpoints $\pm\alpha$, and therefore B and $\beta\gamma$ are separated by the diameter with endpoints $\pm Z$. Thus to prove $X = -\beta\gamma$, we must only show that X and B are on the *same* side of the diameter with endpoints $\pm Z$. This will follow if the arc from Z to X passing through B is *minor* (Figure 3); but of course its measure is

$$\angle ZOB + \angle BOX = 2\angle ZCB + 2\angle BAX = \widehat{C} + \widehat{A} < \pi.$$

Hence $X = -\beta\gamma$. Similar arguments show that $Y = -\gamma\alpha$ and $Z = -\alpha\beta$.

To summarize, the incenter of $\triangle ABC$ may be expressed as

$$I = -(\beta\gamma + \gamma\alpha + \alpha\beta),$$

where α, β, γ are complex square roots of A, B, C for which $\triangle\alpha\beta\gamma$ is acute. Note that this expression is indifferent to the choice between $(\alpha_1, \beta_2, \gamma_1)$ and $(\alpha_2, \beta_1, \gamma_2)$, since each of these triples is the negative of the other.

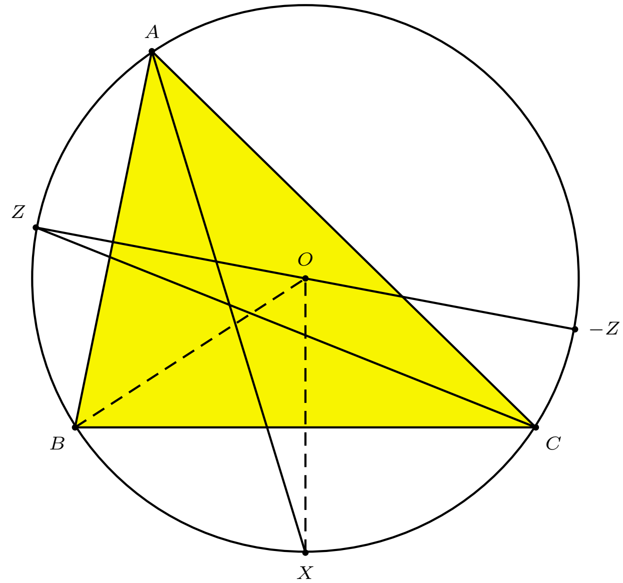


Figure 3.

4. Sufficiency of Guinand's Locus

Place O and H in the complex plane so that O lies at the origin. Define N and G as the points which divide OH internally in the ratios $1 : 1$ and $1 : 2$, respectively. Suppose that I is a point different from N selected from within the circle on diameter GH. Since $H - 2I = 2(N - I)$ is nonzero, we are free to scale coordinates so that $H - 2I = 1$. Let $u = |I|$. Guinand's inequality $OI > 2 \cdot NI$, which we write in complex coordinates as

$$|I| > 2|N - I|$$

now acquires the very simple form $u > 1$.

Consider the cubic equation

$$z^3 - z^2 - Iz + u^2 I = 0.$$

By the Fundamental Theorem of Algebra, this has three complex zeros α, β, γ. These turn out to be square roots of the required vertices. From the standard relations between zeros and coefficients, one has the important equations:

$$\alpha + \beta + \gamma = 1, \qquad \beta\gamma + \gamma\alpha + \alpha\beta = -I, \qquad \alpha\beta\gamma = -u^2 I.$$

Let us first show that the zeros lie on a circle centered at the origin. In fact,

$$|\alpha| = |\beta| = |\gamma| = u.$$

If z is a zero of the cubic, then $z^2(z - 1) = I(z - u^2)$. Taking moduli, we get

$$|z|^2|z - 1| = u|z - u^2|.$$

Squaring both sides and applying the rule $|w|^2 = w\bar{w}$, we find that

$$|z|^4(z - 1)(\bar{z} - 1) = u^2(z - u^2)(\bar{z} - u^2),$$

$$(|z|^6 - u^6) - (|z|^4 - u^4)(z + \bar{z}) + |z|^2(|z|^2 - u^2) = 0.$$

Assume for contradiction that a certain zero z has modulus $\neq u$. Then we may divide the last equation by the nonzero number $|z|^2 - u^2$, getting

$$|z|^4 + u^2|z|^2 + u^4 - (|z|^2 + u^2)(z + \bar{z}) + |z|^2 = 0,$$

or after a slight rearrangement,

$$(|z|^2 + u^2)(|z|^2 - (z + \bar{z})) + u^4 + |z|^2 = 0.$$

An elementary inequality of complex algebra says that

$$-1 \leq |z|^2 - (z + \bar{z}).$$

From this inequality and the above equation, we find that

$$(|z|^2 + u^2)(-1) + u^4 + |z|^2 \leq 0,$$

or after simplifying,

$$u^4 - u^2 \leq 0.$$

As this result is inconsistent with the hypothesis $u > 1$, we have proven that all the zeros of the cubic equation have modulus u.

Now *define* A, B, C by

$$A = \alpha^2, \qquad B = \beta^2, \qquad C = \gamma^2.$$

Clearly $|A| = |B| = |C| = u^2$. Since three points of a circle cannot be collinear, $\triangle ABC$ will be nondegenerate so long as A, B, C are distinct. Thus suppose for contradiction that $A = B$. It follows that $\alpha = \pm\beta$. If $\alpha = -\beta$, then $\gamma = \alpha+\beta+\gamma = 1$, yielding the falsehood $u = |\gamma| = 1$. The only remaining alternative is $\alpha = \beta$. In this case, $2\alpha + \gamma = 1$ and $\alpha(2\gamma + \alpha) = -I$, so that

$$|\alpha||2\gamma + \alpha| = |I|, \qquad \text{or} \qquad |2\gamma + \alpha| = 1.$$

Since $2\alpha + \gamma = 1$, one has $|2 - 3\alpha| = |2\gamma + \alpha| = 1$. Squaring this last result gives

$$4 - 6(\alpha + \bar{\alpha}) + 9|\alpha|^2 = 1, \qquad \text{or} \qquad 2(\alpha + \bar{\alpha}) = 1 + 3u^2.$$

Since $|\alpha + \bar{\alpha}| = 2|\mathrm{Re}(\alpha)| \leq 2|\alpha|$, we have $1 + 3u^2 \leq 4u$. Therefore the value of u is bounded between the zeros of the quadratic

$$3u^2 - 4u + 1 = (3u - 1)(u - 1),$$

yielding the falsehood $\frac{1}{3} \leq u \leq 1$. By this kind of reasoning, one shows that any two of A, B, C are distinct, and hence that $\triangle ABC$ is nondegenerate.

As in §3, since $\triangle ABC$ has circumcenter 0, its orthocenter is

$$\begin{aligned} A + B + C &= \alpha^2 + \beta^2 + \gamma^2 \\ &= (\alpha + \beta + \gamma)^2 - 2(\beta\gamma + \gamma\alpha + \alpha\beta) \\ &= 1 + 2I \\ &= H. \end{aligned}$$

Here we see the rationale for having chosen $I = -(\beta\gamma + \gamma\alpha + \alpha\beta)$.

Lastly we must show that the incenter of $\triangle ABC$ lies at I. It has already appeared that $I = -(\beta\gamma + \gamma\alpha + \alpha\beta)$. As in §3, exactly two of the eight possible triangles formed from square roots of A, B, C are acute, and these are mutual images under the map $z \mapsto -z$. Moreover, the incenter of $\triangle ABC$ is necessarily the value of the expression $-(z_2z_3 + z_3z_1 + z_1z_2)$ whenever $\triangle z_1z_2z_3$ is one of these two acute triangles. Thus to identify the incenter with I, we must only show that $\triangle\alpha\beta\gamma$ is acute.

Angle $\widehat{\alpha}$ is acute if and only if

$$|\beta - \gamma|^2 < |\alpha - \beta|^2 + |\alpha - \gamma|^2.$$

On applying the rule $|w|^2 = w\bar{w}$, this becomes

$$2u^2 - (\beta\bar{\gamma} + \bar{\beta}\gamma) < 4u^2 - (\alpha\bar{\beta} + \bar{\alpha}\beta + \alpha\bar{\gamma} + \bar{\alpha}\gamma),$$

$$\alpha\bar{\beta} + \bar{\alpha}\beta + \alpha\bar{\gamma} + \bar{\alpha}\gamma + \beta\bar{\gamma} + \bar{\beta}\gamma < 2(u^2 + \beta\bar{\gamma} + \bar{\beta}\gamma).$$

Here the left-hand side may be simplified considerably as

$$(\alpha + \beta + \gamma)(\bar{\alpha} + \bar{\beta} + \bar{\gamma}) - |\alpha|^2 - |\beta|^2 - |\gamma|^2 = 1 - 3u^2.$$

In a similar way, the right-hand side simplifies as

$$\begin{aligned} & 2u^2 + 2(\beta + \gamma)(\bar{\beta} + \bar{\gamma}) - 2|\beta|^2 - 2|\gamma|^2 \\ = \;& 2(1 - \alpha)(1 - \bar{\alpha}) - 2u^2 \\ = \;& 2(1 + |\alpha|^2 - \alpha - \bar{\alpha} - u^2) \\ = \;& 2 - 2(\alpha + \bar{\alpha}). \end{aligned}$$

To complete the proof that $\widehat{\alpha}$ is acute, it remains only to show that

$$2(\alpha + \bar{\alpha}) < 1 + 3u^2.$$

However, $2(\alpha + \bar{\alpha}) \leq 4|\alpha| = 4u$, and we have already seen that

$$4u < 1 + 3u^2,$$

since the opposite inequality yields the falsehood $\frac{1}{3} \leq u \leq 1$. Similar arguments establish that $\widehat{\beta}$ and $\widehat{\gamma}$ are acute.

To summarize, we have produced a nondegenerate triangle $\triangle ABC$ which has classical centers at the given points O, H, I. We now return to original notation and write $R = u^2$ for the circumradius of $\triangle ABC$.

5. Uniqueness

Suppose some other triangle $\triangle DEF$ has O, H, I as its classical centers. The formulas of Euler and Feuerbach presented in §2 have a simple but important consequence: If a triangle has O, N, I as circumcenter, nine-point center, and incenter, then its *circumdiameter* is OI^2/NI. This means that $\triangle ABC$ and $\triangle DEF$ share not only the same circumcenter, but also the same circumradius. It follows that $|D| = |E| = |F| = R$.

Since $\triangle DEF$ has circumcenter 0, its orthocenter H is equal to $D + E + F$. Choose square roots δ, ϵ, ζ of D, E, F so that the incenter I will satisfy

$$I = -(\epsilon\zeta + \zeta\delta + \delta\epsilon).$$

Then

$$\begin{aligned}(\delta+\epsilon+\zeta)^2 &= \delta^2+\epsilon^2+\zeta^2+2(\epsilon\zeta+\zeta\delta+\delta\epsilon)\\ &= D+E+F-2I\\ &= H-2I\\ &= 1.\end{aligned}$$

Since the map $z \mapsto -z$ leaves I invariant, but reverses the sign of $\delta+\epsilon+\zeta$, we may change the signs of δ, ϵ, ζ if necessary to make it so that

$$\delta+\epsilon+\zeta = 1.$$

Observe next that $|\delta\epsilon\zeta| = u^3 = |u^2 I|$. Thus we may write

$$\delta\epsilon\zeta = -\theta u^2 I, \qquad \text{where} \qquad |\theta| = 1.$$

The elementary symmetric functions of δ, ϵ, ζ are now

$$\delta+\epsilon+\zeta = 1, \qquad \epsilon\zeta+\zeta\delta+\delta\epsilon = -I, \qquad \delta\epsilon\zeta = -\theta u^2 I.$$

It follows that δ, ϵ, ζ are the roots of the cubic equation

$$z^3 - z^2 - Iz + \theta u^2 I = 0.$$

As in §4, we rearrange and take moduli of both sides to obtain

$$|z|^2|z-1| = u|z-\theta u|.$$

Squaring both sides of this result, we get

$$|z|^4(|z|^2 - z - \bar{z} + 1) = u^2(|z|^2 - u^2 z\bar{\theta} - u^2\bar{z}\theta + u^4).$$

Since all zeros of the cubic have modulus u, we may replace every occurrence of $|z|^2$ by u^2. This dramatically simplifies the equation, reducing it to

$$z + \bar{z} = z\bar{\theta} + \bar{z}\theta.$$

Substituting δ, ϵ, ζ here successively for z and adding the results, one finds that

$$2 = \bar{\theta} + \theta,$$

since

$$\delta+\epsilon+\zeta = \bar{\delta}+\bar{\epsilon}+\bar{\zeta} = 1.$$

It follows easily that $\theta = 1$. Evidently δ, ϵ, ζ are determined from the same cubic as α, β, γ. Therefore (D, E, F) is a permutation of (A, B, C), and the solution of the determination problem is unique.

References

[1] J. H. Conway, Hyacinthos messge, 7644, August 19, 2003.
[2] H. S. M. Coxeter and S. Greitzer, *Geometry Revisited*, Math. Assoc. of America, 1967.
[3] L. Euler, Solutio facilis problematum quorundam geometricorum difficillimorum, *Novi Comm. Acad. Scie. Petropolitanae* 11 (1765); reprinted in *Opera Omnia*, serie prima, vol. 26 (A. Speiser, ed.), n. 325, 139–157.
[4] A. Guinand, Euler lines, tritangent centers, and their triangles, *Amer. Math. Monthly*, 91 (1984) 290-300.
[5] L. Hahn, *Complex Numbers and Geometry*, Math. Assoc. of America, 1994.
[6] D. Pedoe, *Geometry: A Comprehensive Course*, Cambridge Univ. Press, 1970.

[7] B. Scimemi, Paper-folding and Euler's theorem revisited, *Forum Geom.*, 2 (2002) 93–104.
[8] G. C. Smith, Statics and the moduli space of triangles, *Forum Geom.*, 5 (2005) 181–190.

Joseph Stern: Department of Mathematics, Stuyvesant High School, 345 Chambers Street, New York, New York 10282, USA

E-mail address: `jstern@stuy.edu`

Forum Geometricorum
Volume 7 (2007) 11–17.

FORUM GEOM
ISSN 1534-1178

On a Porism Associated with the Euler and Droz-Farny Lines

Christopher J. Bradley, David Monk, and Geoff C. Smith

Abstract. The envelope of the Droz-Farny lines of a triangle is determined to be the inconic with foci at the circumcenter and orthocenter by using purely Euclidean means. The poristic triangles sharing this inconic and circumcircle have a common circumcenter, centroid and orthocenter.

1. Introduction

The triangle ABC has orthocenter H and circumcircle Σ. Suppose that a pair of perpendicular lines through H are drawn, then they meet the sides BC, CA, AB in pairs of points. The midpoints X, Y, Z of these pairs of points are known to be collinear on the Droz-Farny line [2]. The envelope of the Droz-Farny line is the inconic with foci at O and H, known recently as the Macbeath inconic, but once known as the Euler inconic [6]. We support the latter terminology because of its strong connection with the Euler line [3]. According to Goormaghtigh writing in [6] this envelope was first determined by Neuberg, and Goormaghtigh gives an extensive list of early articles related to the Droz-Farny line problem. We will not repeat the details since [6] is widely available through the archive service JSTOR.

We give a short determination of the Droz-Farny envelope using purely Euclidean means. Taken in conjunction with Ayme's recent proof [1] of the existence of the Droz-Farny line, this yields a completely Euclidean derivation of the envelope.

This envelope is the inconic of a porism consisting of triangles with a common Euler line and circumcircle. The sides of triangles in this porism arise as Droz-Farny lines of any one of the triangles in the porism. Conversely, if the orthocenter is interior to Σ, all Droz-Farny lines will arise as triangle sides.

2. The Droz-Farny envelope

Theorem. *Each Droz-Farny line of triangle ABC is the perpendicular bisector of a line segment joining the orthocenter H to a point on the circumcircle.*

Proof. Figure 1 may be useful. Let perpendicular lines l and l' through H meet BC, CA, AB at P and P', Q and Q', R and R' respectively and let X, Y, Z be the midpoints of PP', QQ', RR'.

The collinearity of X, Y, Z is the Droz-Farny theorem. Let K be the foot of the perpendicular from H to XYZ and produce HK to L with $HK = KL$. Now the circle HPP' has center X and $XH = XL$ so L lies on this circle. Let M, M' be the feet of the perpendiculars from L to l, l'. Note that $LMHM'$ is a rectangle

Publication Date: January 16, 2007. Communicating Editor: Paul Yiu.

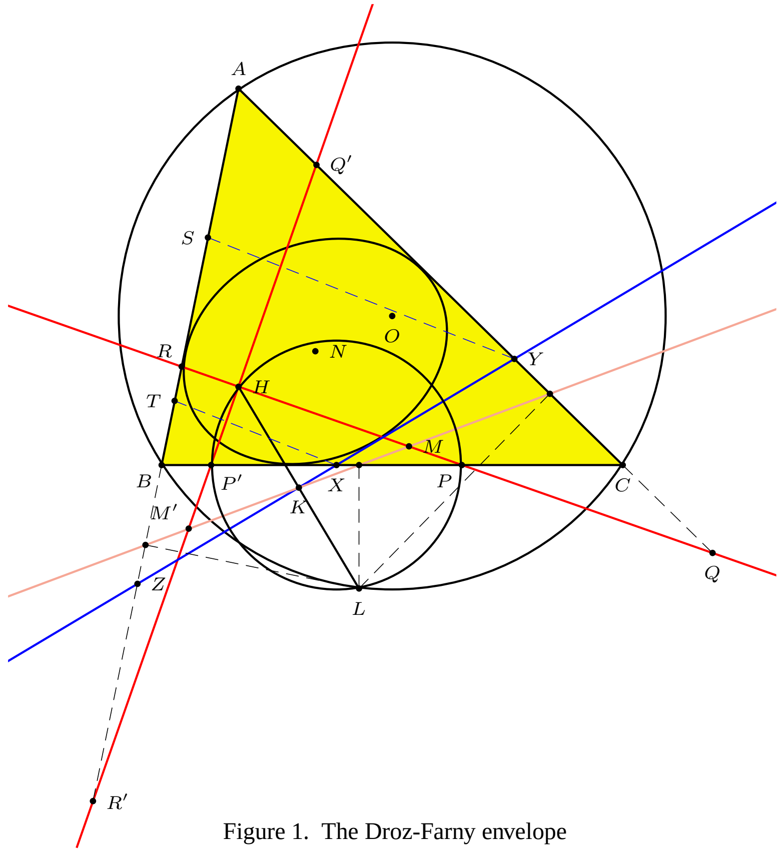


Figure 1. The Droz-Farny envelope

so K is on MM'. Then the foot of the perpendicular from L to the line PP' (*i.e.* BC) lies on MM' by the Wallace-Simson line property applied to the circumcircle of $PP'H$. Equally well, both perpendiculars dropped from L to AB and CA have feet on MM'. Hence L lies on circle ABC with MM' as its Wallace-Simson line. Therefore XYZ is a perpendicular bisector of a line segment joining H to a point on the circumcircle. □

Note that K lies on the nine-point circle of ABC. An expert in the theory of conics will recognize that the nine-point circle is the auxiliary circle of the Euler inconic of ABC with foci at the circumcenter and orthocenter, and for such a reader this article is substantially complete. The points X, Y, Z are collinear and the line XYZ is tangent to the conic inscribed in triangle ABC and having O, H as foci. The direction of the Droz-Farny line is a continuous function of the direction of the mutual perpendiculars; the argument of the Droz-Farny line against a reference axis increases monotonically as the perpendiculars rotate (say) anticlockwise through θ, with the position of the Droz-Farny line repeating itself as θ increases by $\frac{\pi}{2}$. By the intermediate value theorem, the envelope of the Droz-Farny lines is the whole Euler inconic.

We present a detailed discussion of this situation in §3 for the lay reader.

Incidentally, the fact that XY is a variable tangent to a conic of which BC, CA are fixed tangents mean that the correspondence $X \sim Y$ is a projectivity between the two lines. There is a neat way of setting up this map: let the perpendicular bisectors of AH, BH meet AB at S and T respectively. Then SY and TX are parallel. With a change of notation denote the lines BC, CA, AB, XYZ by a, b, c, d respectively; let e, f be the perpendicular bisectors of AH, BH. All these lines are tangents to the conic in question. Consider the Brianchon hexagon of lines a, b, c, d, e, f. The intersections ae, fb are at infinity so their join is the line at infinity. We have $ec = S$, $bd = Y$, $cf = T$, $da = X$. By Brianchon's theorem SY is parallel to XT.

3. The porism

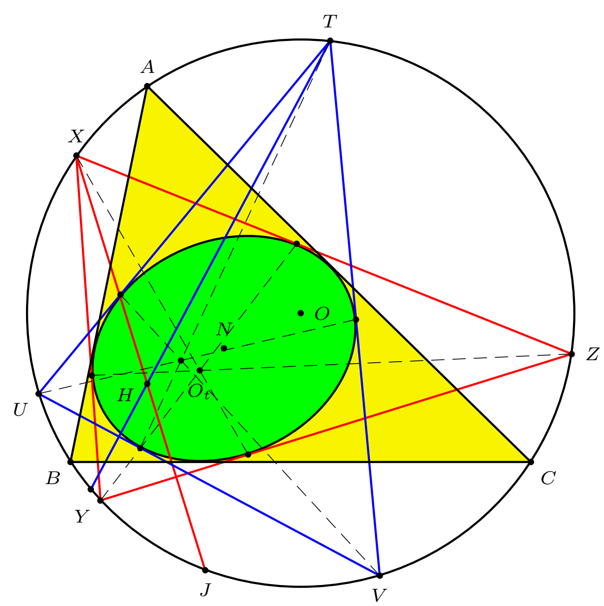


Figure 2. A porism associated with the Euler line

In a triangle with side lengths a, b and c, circumradius R and circumcenter O, the orthocenter H always lies in the interior of a circle center O and radius $3R$ since, as Euler showed, $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$.

We begin afresh. Suppose that we draw a circle Σ with center O and radius R in which is inscribed a non-right angled triangle ABC which has an orthocenter H, so $OH < 3R$ and H is not on Σ.

This H will serve as the orthocenter of infinitely many other triangles XYZ inscribed in the circle and a porism is obtained. We construct these triangles by choosing a point J on the circle. Next we draw the perpendicular bisector of HJ, and need this line to meet Σ again at Y and Z with XYZ anticlockwise. We can certainly arrange that the line and Σ meet by choosing X sufficiently close to A,

B or C. When this happens it follows from elementary considerations that triangle XYZ has orthocenter H, and is the only such triangle with circumcircle Σ and vertex X. In the event that H is inside the circumcircle (which happens precisely when triangle ABC is acute), then every point X on the circumcircle arises as a vertex of a triangle XYZ in the porism.

The construction may be repeated to create as many triangles ABC, TUV, PQR as we please, all inscribed in the circle and all having orthocenter H, as illustrated in Figure 2. Notice that the triangles in this porism have the same circumradius, circumcenter and orthocenter, so the sum of the squares of the side lengths of each triangle in the porism is the same.

We will show that all these triangles circumscribe a conic, with one axis of length R directed along the common Euler line, and with eccentricity $\frac{OH}{R}$. It follows that this inconic is an ellipse if H is chosen inside the circle, but a hyperbola if H is chosen outside.

Thus a porism arises which we call an *Euler line porism* since each triangle in the porism has the same circumcenter, centroid, nine-point center, orthocentroidal center, orthocenter *etc*. A triangle circumscribing a conic gives rise to a *Brianchon point* at the meet of the three Cevians which join each vertex to its opposite contact point.

We will show that the Brianchon point of a triangle in this porism is the isotomic conjugate O_t of the common circumcenter O.

In Figure 2 we pinpoint O_t for the triangle XYZ. The computer graphics system CABRI gives strong evidence for the conjecture that the locus of O_t, as one runs through the triangles of the porism, is a a subset of a conic.

It is possible to choose a point H at distance greater than $3R$ from O so there is no triangle inscribed in the circle which has orthocenter H and then there is no point J on the circle such that the perpendicular bisector of HJ cuts the circle.

The acute triangle case. See Figure 3. The construction is as follows. Draw AH, BH and CH to meet Σ at D, E and F. Draw DO, EO and FO to meet the sides at L, M, N. Let AO meet Σ at D^* and BC at L^*. Also let DO meet Σ at A^*. The points M^*, N^*, E^*, F^*, B^* and C^* are not shown but are similarly defined. Here A' is the midpoint of BC and the line through A' perpendicular to BC is shown.

3.1. *Proof of the porism.* Consider the ellipse defined as the locus of points P such that $HP + OP = R$, where R the circumradius of Σ. The triangle HLD is isosceles, so $HL + OL = LD + OL = R$; therefore L lies on the ellipse.

Now $\angle OLB = \angle CLD = \angle CLH$, because the line segment HD is bisected by the side BC. Therefore the ellipse is tangent to BC at L, and similarly at M and N. It follows that AL, BM, CN are concurrent at a point which will be identified shortly.

This ellipse depends only on O, H and R. It follows that if TUV is any triangle inscribed in Σ with center O, radius R and orthocenter H, then the ellipse touches the sides of TUV. The porism is established.

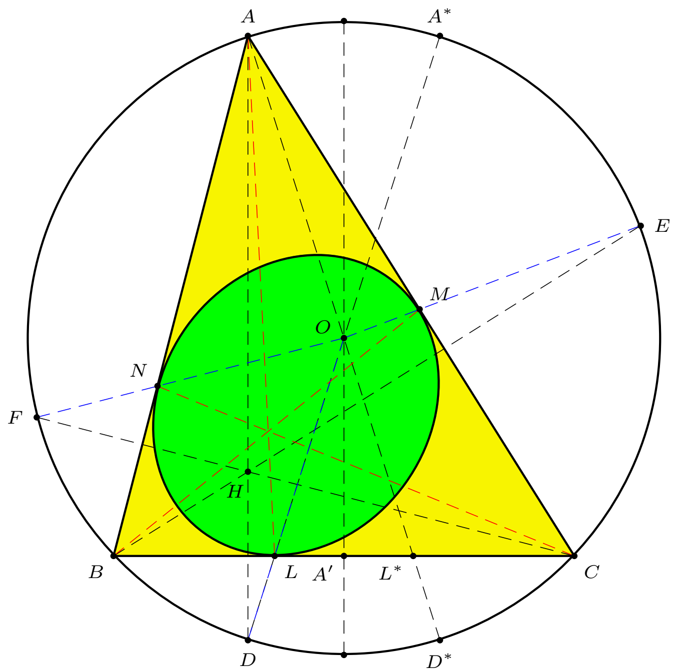


Figure 3. The inconic of the Euler line porism

Identification of the Brianchon point. This is the point of concurrence of AL, BM, CN. Since O and H are isogonal conjugates, it follows that D^* and A^* are reflections of D and A in the line which is the perpendicular bisector of BC. The same applies to B^*, C^*, E^* and F^* with respect to other perpendicular bisectors. Thus A^*D and AD^* are reflections of each other in the perpendicular bisector. Thus L^* is the reflection of L and thus $A'L = A'L^*$. Thus since AL^*, BM^*, CN^* are concurrent at O, the lines AL, BM and CN are concurrent at O_t, the isotomic conjugate of O.

The obtuse triangle case. Refer to Figure 4. Using the same notation as before, now consider the hyperbola defined as the locus of points P such that $|HP - OP| = R$. We now have $HL - OL = LD - OL = R$ so that L lies on the hyperbola.

Also $\angle A^*LB = \angle CLD = \angle HLC$, so the hyperbola touches BC at L, and the argument proceeds as before.

It is a routine matter to obtain the Cartesian equation of this inconic. Scaling so that $R = 1$ we may assume that O is at $(0, 0)$ and H at $(c, 0)$ where $0 \leq c < 3$ but $c \neq 1$.

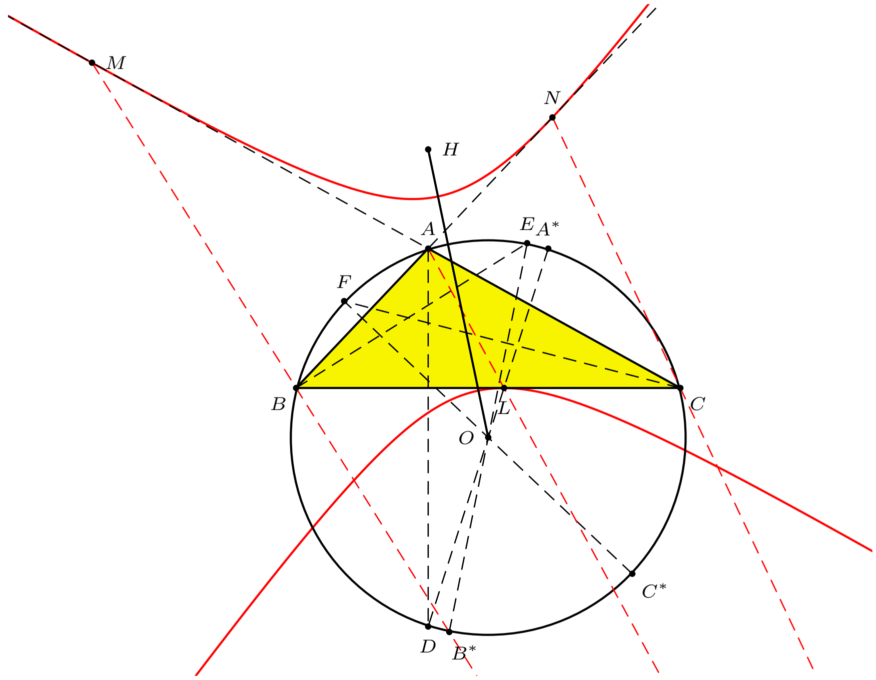


Figure 4. The Euler inconic can be a hyperbola

The inconic then has equation

$$4y^2 + (1 - c^2)(2x - c)^2 = (1 - c^2). \tag{1}$$

When $c < 1$, so H is internal to Σ, this represents an ellipse, but when $c > 1$ it represents a hyperbola. In all cases the center is at $(\frac{c}{2}, 0)$, which is the nine-point center.

One of the axes of the ellipse is the Euler line itself, whose equation is $y = 0$. We see from Equation (1) that the eccentricity of the inconic is $c = \frac{OH}{R}$ and of course its foci are at O and H. Not every tangent line to the inconic arises as a side of a triangle in the porism if H is outside Σ.

Areal analysis. One can also perform the geometric analysis of the envelope using areal co-ordinates, and we briefly report relevant equations for the reader interested in further areal work. Take ABC as triangle of reference and define $u = \cot B \cot C$, $v = \cot C \cot A$, $w = \cot A \cot B$ so that $H(u, v, w)$ and $O(v + w, w + u, u + v)$. This means that the isotomic conjugate O_t of O has co-ordinates

$$O_t \left(\frac{1}{v + w}, \frac{1}{w + u}, \frac{1}{u + v} \right).$$

The altitudes are AH, BH, CH with equations $wy = vz$, $uz = wx$, $vx = uy$ respectively.

The equation of the inconic is

$$(v+w)^2x^2+(w+u)^2y^2+(u+v)^2z^2-2(w+u)(u+v)yz \\ -2(u+v)(v+w)zx-2(v+w)(w+u)xy=0. \tag{2}$$

This curve can be parameterized by the formulas:

$$x=\frac{(1+q)^2}{v+w}, y=\frac{1}{w+u}, z=\frac{q^2}{u+v}, \tag{3}$$

where q has any real value (including infinity). The perpendicular lines l and l through H may be taken to pass through the points at infinity with co-ordinates $((1+t),-t,-1)$ and $((1+s),-s,-1)$ and then the Droz-Farny line has equation

$$-(sw+tw-2v)(2stw-sv-tv)x-(sw+tw+2(u+w))(2stw-sv-tv)y \\ +(sw+tw-2v)(2st(u+v)+sv+tv)z=0. \tag{4}$$

In Equation (4) for the midpoints X, Y, Z to be collinear we must take

$$s=-\frac{v(tw+u+w)}{w(t(u+v)+v)}. \tag{5}$$

If we now substitute Equation (3) into Equation (4) and use Equation (5), a discriminant test on the resulting quadratic equation with the help of DERIVE confirms the tangency for all values of t.

Incidentally, nowhere in this areal analysis do we use the precise values of u, v, w in terms of the angles A, B, C. Therefore we have a bonus theorem: if H is replaced by another point K, then given a line through K, there is always a second line through K (but not generally at right angles to it) so that XYZ is a straight line. As the line l rotates, l' also rotates (but not at the same rate). However the rotations of these lines is such that the variable points X, Y, Z remain collinear and the line XYZ also envelops a conic. This affine generalization of the Droz-Farny theorem was discovered independently by Charles Thas [5] in a paper published after the original submission of this article. We happily cede priority.

References

[1] J.-L. Ayme, A Purely Synthetic Proof of the Droz-Farny Line Theorem, *Forum Geom.*, 4 (2004) 219–224.

[2] A. Droz-Farny, Question 14111, *Educational Times*, 71 (1899) 89–90.

[3] L. Euler, Solutio facili problematum quorundam geometricorum difficillimorum, *Novi Comm. Acad. Scie. Petropolitanae*, 11 (1765); reprinted in *Opera omnia*, serie prima, Vol. 26 (ed. by A. Speiser), (n.325) 139-157.

[4] A. M. Macbeath, A Compactness Theorem for Affine Equivalence-Classes of Convex Regions, *Canad. J. Math.*, 3 (1951) 54–61.

[5] C. Thas, The Droz-Farny theorem and related topics, *Forum Geom.*, 6 (2006) 235–240.

[6] V. Thebault, J. R. Musselman and R. Goormaghtigh, Solution to problem 3758. *Amer. Math. Monthly*, 44 (1937) 668–672.

Christopher J. Bradley and David Monk, c/o Geoff C. Smith: Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, England

E-mail address: G.C.Smith@bath.ac.uk

Forum Geometricorum
Volume 7 (2007) 19–24.

FORUM GEOM
ISSN 1534-1178

The Edge-Tangent Sphere of a Circumscriptible Tetrahedron

Yu-Dong Wu and Zhi-Hua Zhang

Abstract. A tetrahedron is circumscriptible if there is a sphere tangent to each of its six edges. We prove that the radius ℓ of the edge-tangent sphere is at least $\sqrt{3}$ times the radius of its inscribed sphere. This settles affirmatively a problem posed by Z. C. Lin and H. F. Zhu. We also briefly examine the generalization into higher dimension, and pose an analogous problem for an $n-$dimensional simplex admitting a sphere tangent to each of its edges.

1. Introduction

Every tetrahedron has a circumscribed sphere passing through its four vertices and an inscribed sphere tangent to each of its four faces. A tetrahedron is said to be circumscriptible if there is a sphere tangent to each of its six edges (see [1, §§786–794]). We call this the edge-tangent sphere of the tetrahedron.

Let $\mathscr{P}$ denote a tetrahedron $P_0P_1P_2P_3$ with edge lengths $P_iP_j = a_{ij}$ for $0 \leq i < j \leq 3$. The following necessary and sufficient condition for a tetrahedron to admit an edge-tangent sphere can be found in [1, §§787, 790, 792]. See also [4, 6].

Theorem 1. *The following statement for a tetrahedron $\mathscr{P}$ are equivalent.*
(1) *$\mathscr{P}$ has an edge-tangent sphere.*
(2) $a_{01} + a_{23} = a_{02} + a_{13} = a_{03} + a_{12}$*;*
(3) *There exist $x_i > 0$, $i = 0, 1, 2, 3$, such that $a_{ij} = x_i + x_j$ for $0 \leq i < j \leq 3$.*

For $i = 0, 1, 2, 3$, x_i is the length of a tangent from P_i to the edge-tangent sphere of $\mathscr{P}$. Let ℓ denote the radius of this sphere.

Theorem 2. [1, §793] *The radius of the edge-tangent sphere of a circumscriptible tetrahedron of volume V is given by*

$$\ell = \frac{2x_0x_1x_2x_3}{3V}. \tag{1}$$

Lin and Zhu [4] have given the formula (1) in the form

$$\ell^2 = \frac{(2x_0x_1x_2x_3)^2}{2x_0x_1x_2x_3 \sum_{0\leq i<j\leq 3} x_ix_j - (x_1^2x_2^2x_3^2 + x_2^2x_3^2x_0^2 + x_3^2x_0^2x_1^2 + x_0^2x_1^2x_2^2)}. \tag{2}$$

Publication Date: January 22, 2007. Communicating Editor: Paul Yiu.
The authors would like to thank Prof. Han-Fang Zhang for his enthusiastic help.

The fact that this latter denominator is $(3V)^2$ follows from the formula for the volume of a tetrahedron in terms of its edges:

$$V^2 = \frac{1}{288}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & (x_0+x_1)^2 & (x_0+x_2)^2 & (x_0+x_3)^2 \\ 1 & (x_0+x_1)^2 & 0 & (x_1+x_2)^2 & (x_1+x_3)^2 \\ 1 & (x_0+x_2)^2 & (x_1+x_2)^2 & 0 & (x_2+x_3)^2 \\ 1 & (x_0+x_3)^2 & (x_1+x_3)^2 & (x_2+x_3)^2 & 0 \end{vmatrix}.$$

Lin and Zhu *op. cit.* obtained several inequalities for the edge-tangent sphere of $\mathscr{P}$. They also posed the problem of proving or disproving $\ell^2 \geq 3r^2$ for a circumscriptible tetrahedron. See also [2]. The main purpose of this paper is to settle this problem affirmatively.

Theorem 3. *For a circumscriptible tetrahedron with inradius r and edge-tangent sphere of radius ℓ, $\ell \geq \sqrt{3}r$.*

2. Two inequalities

Lemma 4. *If $x_i > 0$ for $0 \leq i \leq 3$, then*

$$\begin{aligned}&\left(\frac{x_1+x_2+x_3}{x_1x_2x_3}+\frac{x_2+x_3+x_0}{x_2x_3x_0}+\frac{x_3+x_0+x_1}{x_3x_0x_1}+\frac{x_0+x_1+x_2}{x_0x_1x_2}\right)\\ &\cdot\frac{4(x_0x_1x_2x_3)^2}{2x_0x_1x_2x_3\sum\limits_{0\leq i<j\leq 3}x_ix_j-(x_1^2x_2^2x_3^2+x_2^2x_3^2x_0^2+x_3^2x_0^2x_1^2+x_0^2x_1^2x_2^2)}\geq 6.\end{aligned} \tag{3}$$

Proof. From

$$\begin{aligned}x_0^2x_1^2(x_2-x_3)^2+x_0^2x_2^2(x_1-x_3)^2+x_0^2x_3^2(x_1-x_2)^2\\ +x_1^2x_2^2(x_0-x_3)^2+x_1^2x_3^2(x_0-x_2)^2+x_2^2x_3^2(x_0-x_1)^2\geq 0,\end{aligned}$$

we have

$$x_1^2x_2^2x_3^2+x_2^2x_3^2x_0^2+x_3^2x_0^2x_1^2+x_0^2x_1^2x_2^2\geq\frac{2}{3}x_0x_1x_2x_3\sum_{0\leq i<j\leq 3}x_ix_j,$$

and

$$\begin{aligned}&2x_0x_1x_2x_3\sum_{0\leq i<j\leq 3}x_ix_j-(x_1^2x_2^2x_3^2+x_2^2x_3^2x_0^2+x_3^2x_0^2x_1^2+x_0^2x_1^2x_2^2)\\ \leq&\frac{4}{3}x_0x_1x_2x_3\sum_{0\leq i<j\leq 3}x_ix_j,\end{aligned}$$

or

$$\begin{aligned}&\frac{4(x_0x_1x_2x_3)^2}{2x_0x_1x_2x_3\sum\limits_{0\leq i<j\leq 3}x_ix_j-(x_1^2x_2^2x_3^2+x_2^2x_3^2x_0^2+x_3^2x_0^2x_1^2+x_0^2x_1^2x_2^2)}\\ \geq&\frac{4(x_0x_1x_2x_3)^2}{\frac{4}{3}x_0x_1x_2x_3\sum\limits_{0\leq i<j\leq 3}x_ix_j}=\frac{3x_0x_1x_2x_3}{\sum\limits_{0\leq i<j\leq 3}x_ix_j}.\end{aligned} \tag{4}$$

On the other hand, it is easy to see that

$$\frac{x_1+x_2+x_3}{x_1x_2x_3}+\frac{x_2+x_3+x_0}{x_2x_3x_0}+\frac{x_3+x_0+x_1}{x_3x_0x_1}+\frac{x_0+x_1+x_2}{x_0x_1x_2}=\frac{2\sum\limits_{0\le i<j\le 3}x_ix_j}{x_0x_1x_2x_3}. \tag{5}$$

Inequality (3) follows immediately from (4) and (5). □

Corollary 5. *For a circumscriptible tetrahedron $\mathscr{P}$ with an edge-tangent sphere of radius ℓ, and faces with inradii r_0, r_1, r_2, r_3,*

$$\left(\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\right)\ell^2\ge 6.$$

Equality holds if and only if $\mathscr{P}$ is a regular tetrahedron.

Proof. From the famous Heron formula, the inradius of a triangle ABC of side-lengths $a=y+z$, $b=z+x$ and $c=x+y$ is given by

$$r^2=\frac{xyz}{x+y+z}.$$

Applying this to the four faces of $\mathscr{P}$, we see that the first factor on the left hand side of (3) is $\left(\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\right)$. Now the result follows from (2). □

Proposition 6. *Let $\mathscr{P}$ be a circumscriptible tetrahedron of volume V. If, for $i=0,1,2,3$, the opposite face of vertex P_i has area $\triangle_i$ and inradius r_i, then*

$$(\triangle_0+\triangle_1+\triangle_2+\triangle_3)^2\ge\frac{9V^2}{2}\left(\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\right). \tag{6}$$

Equality holds if and only if $\mathscr{P}$ is a regular tetrahedron.

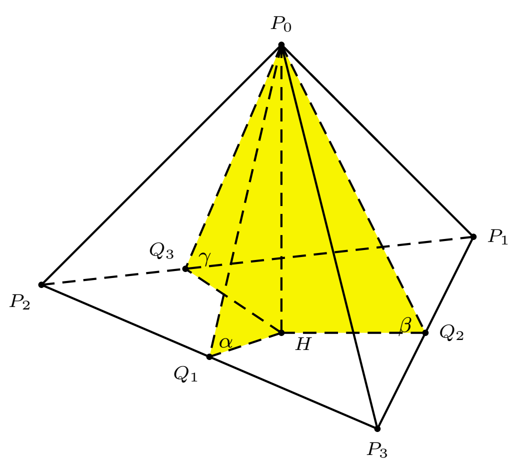


Figure 1.

Proof. Let α be the angle between the planes $P_0P_2P_3$ and $P_1P_2P_3$. If the perpendiculars from P_0 to the line P_2P_3 and to the plane $P_1P_2P_3$ intersect these at Q_1 and H respectively, then $\angle P_0QH = \alpha$. See Figure 1. Similarly, we have the angles β between the planes $P_0P_3P_1$ and $P_1P_2P_3$, and γ between $P_0P_1P_2$ and $P_1P_2P_3$. Note that

$$P_0H = P_0Q_1 \cdot \sin\alpha = P_0Q_2 \cdot \sin\beta = P_0Q_3 \cdot \sin\gamma.$$

Hence,

$$P_0H \cdot P_2P_3 = 2\triangle_1 \sin\alpha = 2\sqrt{(\triangle_1 + \triangle_1\cos\alpha)(\triangle_1 - \triangle_1\cos\alpha)}, \tag{7}$$

$$P_0H \cdot P_3P_1 = 2\triangle_2 \sin\beta = 2\sqrt{(\triangle_2 + \triangle_2\cos\beta)(\triangle_2 - \triangle_2\cos\beta)}, \tag{8}$$

$$P_0H \cdot P_1P_2 = 2\triangle_3 \sin\gamma = 2\sqrt{(\triangle_3 + \triangle_3\cos\gamma)(\triangle_3 - \triangle_3\cos\gamma)}. \tag{9}$$

From (7–9), together with $P_0H = \frac{3V}{\triangle_0}$ and $\frac{\triangle_0}{r_0} = \frac{1}{2}(P_1P_2 + P_2P_3 + P_3P_1)$, we have

$$\begin{aligned}\frac{3V}{r_0} =& \sqrt{(\triangle_1 + \triangle_1\cos\alpha)(\triangle_1 - \triangle_1\cos\alpha)} \\ &+ \sqrt{(\triangle_2 + \triangle_2\cos\beta)(\triangle_2 - \triangle_2\cos\beta)} \\ &+ \sqrt{(\triangle_3 + \triangle_3\cos\gamma)(\triangle_3 - \triangle_3\cos\gamma)}.\end{aligned} \tag{10}$$

Applying Cauchy's inequality and noting that

$$\triangle_0 = \triangle_1\cos\alpha + \triangle_2\cos\beta + \triangle_3\cos\gamma,$$

we have

$$\begin{aligned}\left(\frac{3V}{r_0}\right)^2 \le& (\triangle_1 + \triangle_1\cos\alpha + \triangle_2 + \triangle_2\cos\beta + \triangle_3 + \triangle_3\cos\gamma) \\ &\cdot (\triangle_1 - \triangle_1\cos\alpha + \triangle_2 - \triangle_2\cos\beta + \triangle_3 - \triangle_3\cos\gamma) \\ =& (\triangle_1 + \triangle_2 + \triangle_3 + \triangle_0)(\triangle_1 + \triangle_2 + \triangle_3 - \triangle_0) \\ =& (\triangle_1 + \triangle_2 + \triangle_3)^2 - \triangle_0^2,\end{aligned} \tag{11}$$

or

$$(\triangle_1 + \triangle_2 + \triangle_3)^2 - \triangle_0^2 \ge \left(\frac{3V}{r_0}\right)^2. \tag{12}$$

It is easy to see that equality in (12) holds if and only if

$$\frac{\triangle_1 + \triangle_1\cos\alpha}{\triangle_1 - \triangle_1\cos\alpha} = \frac{\triangle_2 + \triangle_2\cos\beta}{\triangle_2 - \triangle_2\cos\beta} = \frac{\triangle_3 + \triangle_3\cos\gamma}{\triangle_3 - \triangle_3\cos\gamma}.$$

Equivalently, $\cos\alpha = \cos\beta = \cos\gamma$, or $\alpha = \beta = \gamma$. Similarly, we have

$$(\triangle_2 + \triangle_3 + \triangle_0)^2 - \triangle_1^2 \ge \left(\frac{3\,V}{r_1}\right)^2, \tag{13}$$

$$(\triangle_3 + \triangle_0 + \triangle_1)^2 - \triangle_2^2 \ge \left(\frac{3\,V}{r_2}\right)^2, \tag{14}$$

$$(\triangle_0 + \triangle_1 + \triangle_2)^2 - \triangle_3^2 \ge \left(\frac{3\,V}{r_3}\right)^2. \tag{15}$$

Summing (12) to (15), we obtain the inequality (6), with equality precisely when all dihedral angles are equal, *i.e.*, when $\mathscr{P}$ is a regular tetrahedron. $\square$

Remark. Inequality (6) is obtained by X. Z. Yang in [5].

3. Proof of Theorem 3

Since $r = \frac{3V}{\triangle_0+\triangle_1+\triangle_2+\triangle_3}$, it follows from Proposition 6 and Corollary 5 that

$$\ell^2 \geq \frac{6}{\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}} \geq \frac{27V^2}{(\triangle_0+\triangle_1+\triangle_2+\triangle_3)^2} = 3r^2.$$

This completes the proof of Theorem 3.

4. A generalization with an open problem

As a generalization of the tetrahedron, we say that an $n-$dimensional simplex is circumscriptible if there is a sphere tangent to each of its edges. The following basic properties of a circumscriptible simplex can be found in [3].

Theorem 7. *Suppose the edge lengths of an n-simplex $\mathscr{P} = P_0P_1\cdots P_n$ are $P_iP_j = a_{ij}$ for $0 \leq i < j \leq n$. The n-simplex has an edge-tangent sphere if and only if there exist x_i, $i = 0, 1, \ldots, n$, satisfying $a_{ij} = x_i + x_j$ for $0 \leq i \neq j \leq n$. In this case, the radius of the edge-tangent sphere is given by*

$$\ell^2 = -\frac{D_1}{2D_2}, \tag{16}$$

where

$$D_1 = \begin{vmatrix} -2x_0^2 & 2x_0x_1 & \cdots & 2x_0x_{n-1} \\ 2x_0x_1 & -2x_1^2 & \cdots & 2x_1x_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 2x_0x_{n-1} & 2x_1x_{n-1} & \cdots & -2x_{n-1}^2 \end{vmatrix},$$

and

$$D_2 = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & \cdot & \cdots & \cdot \\ \vdots & \vdots & D_1 & \vdots \\ 1 & \cdot & \cdots & \cdot \end{vmatrix}.$$

We conclude this paper with an open problem: for a circumscriptible n-simplex with a circumscribed sphere of radius R, an inscribed sphere of radius r and an edge-tangent sphere of radius ℓ, prove or disprove that

$$R \geq \sqrt{\frac{2n}{n-1}}l \geq nr.$$

References

[1] N. Altshiller-Court, *Modern Pure Solid Geometry*, 2nd edition, Chelsea, 1964.
[2] J.-C. Kuang, *Chángyòng Bùděngshì* (*Applied Inequalities*), 3rd ed., Shandong Science and Technology Press, Jinan City, Shandong Province, China, 2004, 252.
[3] Z. C. Lin, The edge-tangent sphere of an n-simplex,(in Chinese) *Mathematics in Practice and Theory*, 4 (1995) 90–93.
[4] Z. C. Lin and H. F. Zhu, Research on the edge-tangent spheres of tetrahedra (in Chinese), *Geometric Inequalities in China*, pp.175–187; Jiangsu Educational Press, 1996,
[5] X. Z. Yang, On an inequality concerning tetrahedra, (in Chinese), *Study in High-School Mathematics*, 9 (2005) 25–26.
[6] Z. Yang, A necessary and sufficient for a tetrahedron to have an edge-tangent sphere, (in Chinese), *Hunan Mathematics Communication*, 6 (1985) 33-34.

Yu-Dong Wu: Xinchang High School, Xinchang, Zhejiang Province 312500, P. R. China.
E-mail address: zjxcwyd@tom.com

Zhi-Hua Zhang: Zixing Educational Research Section, Chenzhou, Hunan Province 423400, P. R. China
E-mail address: zxzh1234@163.com

Forum Geometricorum
Volume 7 (2007) 25–29.

FORUM GEOM
ISSN 1534-1178

A Stronger Triangle Inequality for Neutral Geometry

Melissa Baker and Robert C. Powers

Abstract. Bailey and Bannister [*College Math. Journal*, 28 (1997) 182–186] proved that a stronger triangle inequality holds in the Euclidean plane for all triangles having largest angle less than $\arctan(\frac{24}{7}) \approx 74^\circ$. We use hyperbolic trigonometry to show that a stronger triangle inequality holds in the hyperbolic plane for all triangles having largest angle less than or equal to 65.87°.

1. Introduction

One of the most fundamental results of neutral geometry is the triangle inequality. How can this cherished inequality be strengthened? Under certain restrictions, the sum of the lengths of two sides of a triangle is greater than the length of the remaining side plus the length of the altitude to this side.

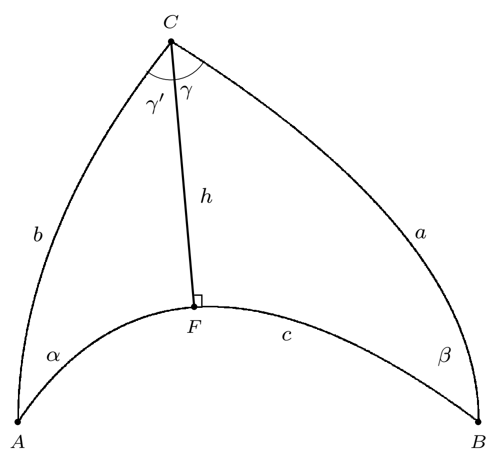


Figure 1. Strong triangle inequality $a + b > c + h$

Let ABC be a triangle belonging to neutral geometry (see Figure 1). Let a, b and c be the lengths of sides BC, AC and AB, respectively. Also, let α, β and γ denote the angles at A, B and C respectively. If we let F be the foot of the perpendicular from C onto side AB and if h is the length of the segment CF, when is it true that $a + b > c + h$? Since $a > h$ and $b > h$, this question is of interest only if c is the length of the longest side of ABC, or, equivalently, if γ is the the largest angle of ABC. With this notation, if the inequality $a + b > c + h$ holds where γ is the largest angle of the triangle ABC, we say that ABC satisfies the *strong triangle inequality*.

Publication Date: January 29, 2007. Communicating Editor: Paul Yiu.

The following result, due to Bailey and Bannister [1], explains what happens if the triangle ABC belongs to Euclidean geometry.

Theorem 1. *If ABC is a Euclidean triangle having largest angle $\gamma < \arctan(\frac{24}{7}) \approx 74°$, then ABC satisfies the strong triangle inequality.*

An elegant trigonometric proof of Theorem 1 can by found in [3]. It should be noted that the bound of $\arctan(\frac{24}{7})$ is the best possible since any isosceles Euclidean triangle with $\gamma = \arctan(\frac{24}{7})$ violates the strong triangle inequality.

The goal of this note is to extend the Bailey and Bannister result to neutral geometry. To get the appropriate bound for the extended result we need the function

$$f(\gamma) := -1 - \cos\gamma + \sin\gamma + \sin\frac{\gamma}{2}\sin\gamma. \tag{1}$$

Observe that $f'(\gamma) = \sin\gamma + \cos\gamma + \sin\frac{\gamma}{2}\cos\gamma + \frac{1}{2}\cos\frac{\gamma}{2}\sin\gamma > 0$ on the interval $\left[0, \frac{\pi}{2}\right]$. Therefore, $f(\gamma)$ is strictly monotone increasing on the interval $(0, \frac{\pi}{2})$. Since $f(0) = -2$, $f(\frac{\pi}{2}) = \frac{\sqrt{2}}{2}$, and f is continuous it follows that f has a unique root r in the interval $\left(0, \frac{\pi}{2}\right)$. In fact, r is approximately 1.15 (radians) or $65.87°$. See Figure 2.

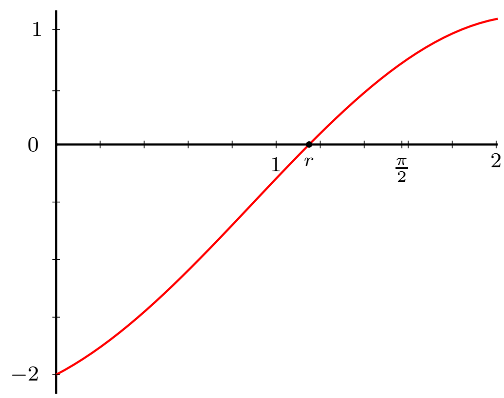


Figure 2. Graph of $f(\gamma)$

Theorem 2. *In neutral geometry a triangle ABC having largest angle γ satisfies the strong triangle inequality if $\gamma \leq r \approx 1.15$ radians or $65.87°$.*

The proof of Theorem 2 is based on the fact that a model of neutral geometry is isomorphic to either the Euclidean plane or a hyperbolic plane. Given Theorem 1, it is enough to establish our result for the case of hyperbolic geometry. Moreover, since the strong triangle inequality holds if and only if $ka + kb > kc + kh$ for any positive constant k, it is enough to assume that the distance scale in hyperbolic geometry is 1. An explanation about the distance scale k and how it is used in hyperbolic geometry can be found in [4].

2. Hyperbolic trigonometry

Recall that the hyperbolic sine and hyperbolic cosine functions are given by

$$\sinh x = \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2}.$$

The formulas needed to prove the main result are given below. First, there are the standard identities

$$\cosh^2 x - \sinh^2 x = 1 \tag{2}$$

and

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y. \tag{3}$$

If ABC is a hyperbolic triangle with a right angle at C, *i.e.*, $\gamma = \frac{\pi}{2}$, then

$$\sinh a = \sinh c \sin \alpha \tag{4}$$

and

$$\cosh a \sin \beta = \cos \alpha. \tag{5}$$

For any hyperbolic triangle ABC,

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma, \tag{6}$$

$$\frac{\sin \alpha}{\sinh a} = \frac{\sin \beta}{\sinh b} = \frac{\sin \gamma}{\sinh c}, \tag{7}$$

$$\cosh c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}. \tag{8}$$

See [2, Chapter 10] or [5, Chapter 8] for more details regarding (4 – 8).

3. Proof of Theorem 2

The strong triangle inequality $a + b > c + h$ holds if and only if $\cosh(a+b) > \cosh(c+h)$. Expanding both sides by the identity given in (3) we have

$$\cosh a \cosh b + \sinh a \sinh b > \cosh c \cosh h + \sinh c \sinh h,$$

$$\cosh c + \sinh a \sinh b \cos \gamma + \sinh a \sinh b > \cosh c \cosh h + \sinh c \sinh h, \text{by } (6)$$

$$\cosh c\,(1 - \cosh h) + \sinh a \sinh b\,(\cos \gamma + 1) - \sinh c \sinh h > 0.$$

Since ACF is a right triangle with the length of CF equal to h, it follows from (4) that $\sinh h = \sinh b \sin \alpha$. Applying (7), we have

$$\cosh c\,(1 - \cosh h) + \sinh a \sinh b\,(\cos \gamma + 1) - \frac{\sinh a}{\sin \alpha} \cdot \sin \gamma \sinh b \sin \alpha > 0,$$

$$\cosh c\,(1 - \cosh h) + \sinh a \sinh b\,(\cos \gamma + 1 - \sin \gamma) > 0,$$

$$\cosh c\left(1 - \cosh^2 h\right) + \sinh a \sinh b(1 + \cosh h)\,(\cos \gamma + 1 - \sin \gamma) > 0,$$

$$\cosh c\left(-\sinh^2 h\right) + \sinh a \sinh b(1 + \cosh h)\,(\cos \gamma + 1 - \sin \gamma) > 0, \quad \text{by } (2)$$

$$\cosh c\left(-\sinh^2 b \sin^2 \alpha\right) + \sinh a \sinh b(1 + \cosh h)(\cos \gamma + 1 - \sin \gamma) > 0.$$

Dividing both sides of the inequality by $\sinh b > 0$, we have

$$-\cosh c \sinh b \sin^2\alpha + \sinh a(1+\cosh h)(\cos\gamma + 1 - \sin\gamma) > 0.$$

By (7) and (8), we have

$$-\left(\frac{\cos\alpha\cos\beta+\cos\gamma}{\sin\alpha\sin\beta}\right)\frac{\sinh a\sin\beta}{\sin\alpha}\sin^2\alpha + \sinh a(1+\cosh h)(\cos\gamma+1-\sin\gamma) > 0.$$

Simplifying and dividing by $\sinh a > 0$, we have

$$-(\cos\alpha\cos\beta+\cos\gamma)\sinh a + \sinh a(1+\cosh h)(\cos\gamma+1-\sin\gamma) > 0,$$
$$-(\cos\alpha\cos\beta+\cos\gamma) + (1+\cosh h)(\cos\gamma+1-\sin\gamma) > 0, \qquad (9)$$

We have manipulated the original inequality into one involving the original angles, α, β, and γ, and the length of the altitude on AB. In the right triangle ACF, let γ' be the angle at C. We may assume $\gamma' \leq \frac{\gamma}{2}$ (otherwise we can work with the right triangle BCF). Applying (5) to triangle ACF gives $\cosh h = \frac{\cos\alpha}{\sin\gamma'}$. Now continuing with the inequality (9) we get

$$-(\cos\alpha\cos\beta+\cos\gamma) + \left(1+\frac{\cos\alpha}{\sin\gamma'}\right)(1+\cos\gamma-\sin\gamma) > 0$$

Multiplying both sides by $-\sin\gamma' < 0$, we have

$$\sin\gamma'(\cos\alpha\cos\beta+\cos\gamma) - \left(\sin\gamma'+\cos\alpha\right)(1+\cos\gamma-\sin\gamma) < 0,$$

Simplifying this and rearranging terms, we have

$$\cos\alpha\left(\sin\gamma'\cos\beta - 1 - \cos\gamma + \sin\gamma\right) + \sin\gamma'(\sin\gamma - 1) < 0. \qquad (10)$$

If $\sin\gamma'\cos\beta - 1 - \cos\alpha + \sin\alpha > 0$, then

$$\begin{aligned}
&\cos\alpha\left(\sin\gamma'\cos\beta - 1 - \cos\gamma + \sin\gamma\right) + \sin\gamma'(\sin\gamma - 1)\\
&< \sin\gamma' - 1 - \cos\gamma + \sin\gamma + \sin\gamma'(\sin\gamma - 1)\\
&= -1 - \cos\gamma + \sin\gamma + \sin\gamma'\sin\gamma\\
&\leq -1 - \cos\gamma + \sin\gamma + \sin\frac{\gamma}{2}\sin\gamma.
\end{aligned}$$

Note that this last expression is $f(\gamma)$ defined in (1). We have shown that

$$\cos\alpha(\sin\gamma'\cos\beta - 1 - \cos\gamma + \sin\gamma) + \sin\gamma'(\sin\gamma - 1) < \max\{0, f(\gamma)\}.$$

For $\gamma \leq r$, we have $f(\gamma) \leq 0$ and the strong triangle inequality holds.

This completes the proof of Theorem 2.

If $r < \gamma < \frac{\pi}{2}$, then $f(\gamma) > 0$. In this case, we can find an angle α such that $0 < \alpha < \frac{\pi}{2} - \frac{\gamma}{2}$ and

$$\cos\alpha\left(\sin\frac{\gamma}{2}\cos\alpha - 1 - \cos\alpha + \sin\alpha\right) + \sin\frac{\gamma}{2}(\sin\gamma - 1) > 0.$$

Since $\gamma + 2\alpha < \pi$ it follows from [5, Theorem 6.7] that there exists a hyperbolic triangle ABC with angles α, α, and γ. Our previous work shows that the triangle ABC satisfies the strong triangle inequality if and only if (10) holds. Consequently, $a + b > c + h$ provided $f(\gamma) \leq 0$. Therefore, the bound r given in Theorem 2 is the best possible.

References

[1] H. R. Bailey and R. Bannister, A stronger triangle inequality, *College Math. J.*, 28 (1997) 182–186.

[2] M. J. Greenberg, *Euclidean and Non-Euclidean Geometries, Development and History*, 3rd edition, W. H. Freeman and Company, New York, 1993.

[3] M. S. Klamkin, A sharp triangle inequality, *College Math. J.*, 29 (1998), 33.

[4] G. E. Martin, *The Foundations of Geometry and the Non-Euclidean Plane*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Berlin, 1982.

[5] S. Stahl, *The Poincaré Half-Plane, A Gateway to Modern Geometry*, Jones and Bartlett Publishers, Boston, MA, 1993.

Melissa Baker: Department of Mathematics, University of Louisville, Louisville, Kentucky 40292, USA

Robert C. Powers: Department of Mathematics, University of Louisville, Louisville, Kentucky 40292, USA

E-mail address: `rcpowe01@louisville.edu`

Forum Geometricorum
Volume 7 (2007) 31–32.

FORUM GEOM
ISSN 1534-1178

A Simple Construction of the Golden Ratio

Jingcheng Tong and Sidney Kung

Abstract. We construct the golden ratio by using an area bisector of a trapezoid.

Consider a trapezoid $PQRS$ with bases $PQ = b$, $RS = a$, $a < b$. Assume, in Figure 1, that the segment MN of length $\sqrt{\frac{a^2+b^2}{2}}$ is parallel to PQ. Then MN lies between the bases PQ and RS (see [1, p.57]). It is easy to show that MN bisects the area of the trapezoid. It is more interesting to note that M and N divide SP and RQ in the golden ratio if $b = 3a$. To see this, construct a segment SW parallel to RQ and let $V = MN \cap SW$. It is clear that

$$\frac{SM}{SP} = \frac{MV}{PW} = \frac{\sqrt{\frac{a^2+b^2}{2}} - a}{b-a} = \frac{\sqrt{5}-1}{2}$$

if $b = 3a$.

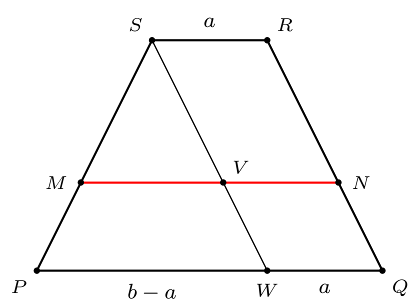


Figure 1

Based upon this result, we present the following simple division of a given segment AB in the golden ratio. Construct
(1) a trapezoid $ABCD$ with $AD//BC$ and $BC = 3 \cdot AD$,
(2) a right triangle BCD with a right angle at C and $CE = AD$,
(3) the midpoint F of BE and a point H on the perpendicular bisector of BE such that $FH = \frac{1}{2}BE$,
(4) a point I on BC such that $BI = BH$.

Complete a parallelogram $BIJG$ with J on DC and G on AB. See Figure 2. Then G divides AB in the golden ratio, *i.e.*, $\frac{AB}{AB} = \frac{\sqrt{5}-1}{2}$.

Publication Date: February 5, 2007. Communicating Editor: Paul Yiu.

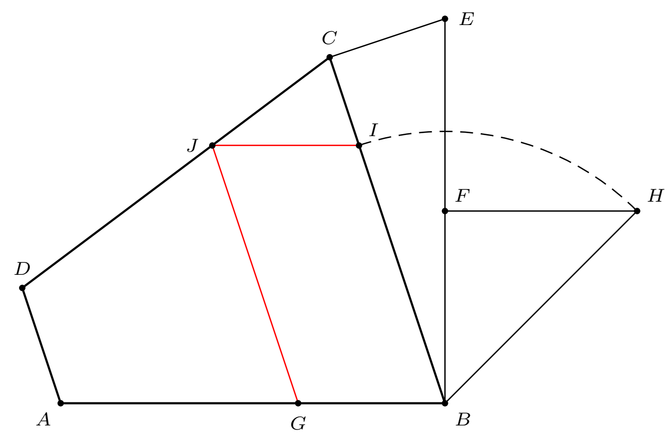


Figure 2

Proof. The trapezoid $ABCD$ has $AD = a$, $BC = b$ with $b = 3a$. The segment JG is parallel to the bases and

$$JG = BI = BH = \sqrt{2} \cdot \frac{\sqrt{a^2 + b^2}}{2} = \sqrt{\frac{a^2 + b^2}{2}}.$$

Therefore, $\frac{AG}{AB} = \frac{\sqrt{5}-1}{2}$. □

Reference

[1] R. B. Nelsen, *Proofs Without Words*, Mathematical Association of America, 1993.

Jingcheng Tong: Department of Mathematics, University of North Florida, Jacksonville, Florida, 32224, USA
E-mail address: `jtong@unf.edu`

Sidney Kung: 20488 Stevens Creek Blvd., #1411, Cupertino, California, 95014, USA
E-mail address: `sidneykung@yahoo.com`

Forum Geometricorum
Volume 7 (2007) 33–52.

FORUM GEOM
ISSN 1534-1178

The Method of Punctured Containers

Tom M. Apostol and Mamikon A. Mnatsakanian

Abstract. We introduce the method of punctured containers, which geometrically relates volumes and centroids of complicated solids to those of simpler punctured prismatic solids. This method goes to the heart of some of the basic properties of the sphere, and extends them in natural and significant ways to solids assembled from cylindrical wedges (Archimedean domes) and to more general solids, especially those with nonuniform densities.

1. Introduction

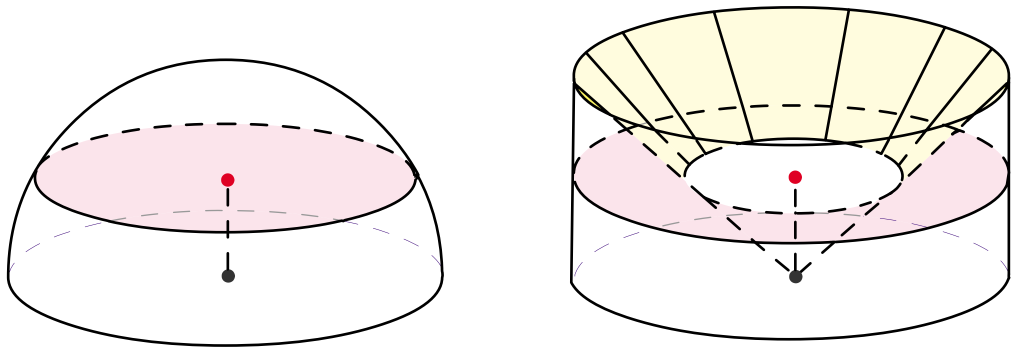
Archimedes (287-212 B.C.) is regarded as the greatest mathematician of ancient times because of his masterful and innovative treatment of a remarkable range of topics in both pure and applied mathematics. One landmark discovery is that the volume of a solid sphere is two-thirds the volume of its circumscribing cylinder, and that the surface area of the sphere is also two-thirds the total surface area of the same cylinder. Archimedes was so proud of this revelation that he wanted the sphere and circumscribing cylinder engraved on his tombstone. He discovered the volume ratio by balancing slices of the sphere against slices of a *larger* cylinder and cone, using centroids and the law of the lever, which he had also discovered.

Today we know that the volume ratio for the sphere and cylinder can be derived more simply by an elementary geometric method that Archimedes overlooked. It is illustrated in Figure 1. By symmetry it suffices to consider a hemisphere, as in Figure 1a, and its circumscribing cylindrical container. Figure 1b shows the cylinder with a solid cone removed. The punctured cylindrical container has exactly the same volume as the hemisphere, because every horizontal plane cuts the hemisphere and the punctured cylinder in cross sections of equal area. The cone's volume is one-third that of the cylinder, hence the hemisphere's volume is two-thirds that of the cylinder, which gives the Archimedes volume ratio for the sphere and its circumscribing cylinder.

This geometric method extends to more general solids we call Archimedean domes. They and their punctured prismatic containers are described below in Section 2. Any plane parallel to the equatorial base cuts such a dome and its punctured container in cross sections of equal area. This implies that two planes parallel to the base cut the dome and the punctured container in slices of equal volumes, equality of volumes being a consequence of the following:

Slicing principle. *Two solids have equal volumes if their horizontal cross sections taken at any height have equal areas.*

Publication Date: February 12, 2007. Communicating Editor: Xiao-Dong Zhang.



(a) (b)

Figure 1. (a) A hemisphere and (b) a punctured cylindrical container of equal volume.

This statement is often called Cavalieri's principle in honor of Bonaventura Cavalieri (1598-1647), who attempted to prove it for general solids. Archimedes used it sixteen centuries earlier for special solids, and he credits Eudoxus and Democritus for using it even earlier in their discovery of the volume of a cone. Cavalieri employed it to find volumes of many solids, and tried to establish the principle for general solids by applying Archimedes' method of exhaustion, but it was not demonstrated rigorously until integral calculus was developed in the 17th century. We prefer using the neutral and more descriptive term *slicing principle*.

To describe the slicing principle in the language of calculus, cut two solids by horizontal planes that produce cross sections of equal area $A(x)$ at an arbitrary height x above a fixed base. The integral $\int_{x_1}^{x_2} A(x)\,dx$ gives the volume of the portion of each solid cut by all horizontal planes as x varies over some interval $[x_1,\, x_2]$. Because the integrand $A(x)$ is the same for both solids, the corresponding volumes are also equal. We could just as well integrate any function $f(x, A(x))$, and the integral over the interval $[x_1,\, x_2]$ would be the same for both solids. For example, $\int_{x_1}^{x_2} xA(x)\,dx$ is the first moment of the area function over the interval $[x_1,\, x_2]$, and this integral divided by the volume gives the altitude of the *centroid* of the slice between the planes $x = x_1$ and $x = x_2$. Thus, not only are the volumes of these slices equal, but also the altitudes of their centroids are equal. Moreover, all moments $\int_{x_1}^{x_2} x^k A(x)\,dx$ with respect to the plane of the base are equal for both slices.

In [1; Theorem 6a] we showed that the lateral surface area of any slice of an Archimedean dome between two parallel planes is equal to the lateral surface area of the corresponding slice of the circumscribing (unpunctured) prism. This was deduced from the fact that Archimedean domes circumscribe hemispheres. It implies that the total surface area of a sphere is equal to the lateral surface area of its circumscribing cylinder which, in turn, is two-thirds the total surface area of the cylinder. The surface area ratio was discovered by Archimedes by a completely different method.

This paper extends our geometric method further, from Archimedean domes to more general solids. First we dilate an Archimedean dome in a vertical direction to produce a dome with elliptic profiles, then we replace its base by an arbitrary polygon, not necessarily convex. This leads naturally to domes with arbitrary curved bases. Such domes and their punctured prismatic containers have equal volumes and equal moments relative to the plane of the base because of the slicing principle, but if these domes do not circumscribe hemispheres the corresponding lateral surface areas will not be equal. This paper relaxes the requirement of equal surface areas and concentrates on solids having the same volume and moments as their punctured prismatic containers. We call such solids *reducible* and describe them in Section 3. Section 4 treats reducible domes and shells with polygonal bases, then Section 5 extends the results to domes with curved bases, and formulates reducibility in terms of mappings that preserve volumes and moments.

The full power of our method, which we call *the method of punctured containers*, is revealed by the treatment of nonuniform mass distributions in Section 6. Problems of calculating masses and centroids of nonuniform wedges, shells, and their slices with elliptic profiles, including those with cavities, are reduced to those of *simpler punctured prismatic containers*. Section 7 gives explicit formulas for volumes and centroids, and Section 8 reveals the surprising fact that uniform domes are reducible to their punctured containers if and only if they have elliptic profiles.

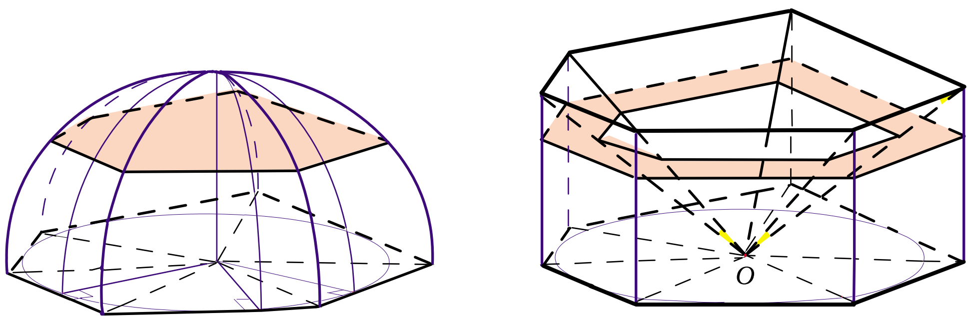
2. Archimedean domes

Archimedean domes are solids of the type shown in Figure 2a, formed by assembling portions of circular cylindrical wedges. Each dome circumscribes a hemisphere, and its horizontal base is a polygon, not necessarily regular, circumscribing the equator of the hemisphere. Cross sections cut by planes parallel to the base are similar polygons circumscribing the cross sections of the hemisphere. Figure 2b shows the dome's punctured prismatic container, a circumscribing prism, from which a pyramid with congruent polygonal base has been removed as indicated. The shaded regions in Figure 2 illustrate the fundamental relation between any Archimedean dome and its punctured prismatic container:

Each horizontal plane cuts both solids in cross sections of equal area.

Hence, by the slicing principle, any two horizontal planes cut both solids in slices of equal volume. Because the removed pyramid has volume one-third that of the unpunctured prism, we see that the volume of any Archimedean dome is two-thirds that of its punctured prismatic container.

We used the name “Archimedean dome” because of a special case considered by Archimedes. In his preface to The Method [3; Supplement, p. 12] Archimedes announced (without proof) that the volume of intersection of two congruent orthogonal circular cylinders is two-thirds the volume of the circumscribing cube. In [3; pp. 48-50], Zeuthen verifies this with the method of centroids and levers employed by Archimedes in treating the sphere. However, if we observe that half the solid of intersection is an Archimedean dome with a square base, and compare its volume with that of its punctured prismatic container, we immediately obtain the required two-thirds volume ratio.



(a) Archimedean dome (b) Punctured prismatic container

Figure 2. Each horizontal plane cuts the dome and its punctured prismatic container in cross sections of equal area.

As a limiting case, when the polygonal cross sections of an Archimedean dome become circles, and the punctured container becomes a circumscribing cylinder punctured by a cone, we obtain a purely geometric derivation of the Archimedes volume ratio for a sphere and cylinder.

When an Archimedean dome and its punctured container are *uniform* solids, made of material of the same constant density (mass per unit volume), the corresponding horizontal slices also have equal masses, and the center of mass of each slice lies at the same height above the base [1; Section 9].

3. Reducible solids

This paper extends the method of punctured containers by applying it first to general dome-like structures far removed from Archimedean domes, and then to domes with *nonuniform* mass distributions. The generality of the structures is demonstrated by the following examples.

Cut any Archimedean dome and its punctured container into horizontal slices and assign to each pair of slices the same constant density, which can differ from pair to pair. Because the masses are equal slice by slice, the total mass of the dome is equal to that of its punctured container, and the centers of mass are at the same height. Or, cut the dome and its punctured container into wedges by vertical half planes through the polar axis, and assign to each pair of wedges the same constant density, which can differ from pair to pair. Again, the masses are equal wedge by wedge, so the total mass of the dome is equal to that of its punctured container, and the centers of mass are at the same height. Or, imagine an Archimedean dome divided into thin concentric shell-like layers, like those of an onion, each assigned its own constant density, which can differ from layer to layer. The punctured container is correspondingly divided into coaxial prismatic layers, each assigned the same constant density as the corresponding shell layer. In this case the masses are equal shell by shell, so the total mass of the dome is equal to that of its punctured container, and again the centers of mass are at the same height. We are interested in a class of solids, with pyramidically punctured prismatic containers, that share the following property with Archimedean domes:

Definition. (Reducible solid) A solid is called reducible if an arbitrary horizontal slice of the solid and its punctured container have equal volumes, equal masses, and hence centers of mass at the same height above the base.

Every uniform Archimedean done is reducible, and in Section 5 we exhibit some nonuniform Archimedean domes that are reducible as well.

The method of punctured containers enables us to reduce both volume and mass calculations of domes to those of simpler prismatic solids, thus generalizing the profound volume relation between the sphere and cylinder discovered by Archimedes. Another famous result of Archimedes [3; Method, Proposition 6] states that the centroid of a uniform solid hemisphere divides its altitude in the ratio 5:3. Using the method of punctured containers we show that the same ratio holds for uniform Archimedean domes and other more general domes (Theorem 7), and we also extend this result to the center of mass of a more general class of nonuniform reducible domes (Theorem 8).

4. Polygonal elliptic domes and shells

To easily construct a more general class of reducible solids, start with any Archimedean dome, and dilate it and its punctured container in a vertical direction by the same scaling factor $\lambda > 0$. The circular cylindrical wedges in Figure 2a become elliptic cylindrical wedges, as typified by the example in Figure 3a. A circular arc of radius a is dilated into an elliptic arc with horizontal semi axis a and vertical semi axis λa. Dilation changes the altitude of the prismatic wedge from a to λa (Figure 3b). The punctured container is again a prism punctured by a pyramid.

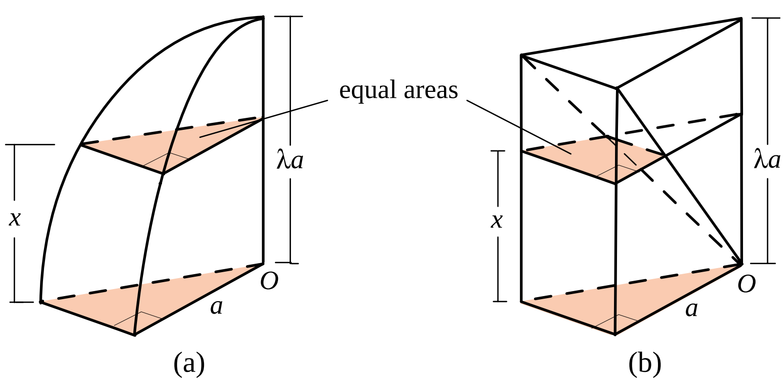


Figure 3. (a) Vertical dilation of a cylindrical wedge by a factor λ. (b) Its punctured prismatic container.

Each horizontal plane at a given height above the base cuts both the elliptic wedge and the corresponding punctured prismatic wedge in cross sections of equal area. Consequently, any two horizontal planes cut both solids in slices of equal volume.

If the elliptic and prismatic wedges have the same constant density, then they also have the same mass, and their centers of mass are at the same height above the base. In other words, we have:

Theorem 1. *Every uniform elliptic cylindrical wedge is reducible.*

Now assemble a finite collection of nonoverlapping elliptic cylindrical wedges with their horizontal semi axes, possibly of different lengths, in the same horizontal plane, but having a *common vertical semi axis*, which meets the base at a point O called the *center*. We assume the density of each component wedge is constant, although this constant may differ from component to component. For each wedge, the punctured circumscribing prismatic container with the same density is called its *prismatic counterpart*. The punctured containers assembled in the same manner produce the counterpart of the wedge assemblage. We call an assemblage *nonuniform* if some of its components can have different constant densities. This includes the special case of a *uniform* assemblage where all components have the same constant density. Because each wedge is reducible we obtain:

Corollary 1. *Any nonuniform assemblage of elliptic cylindrical wedges is reducible.*

Polygonal elliptic domes. Because the base of a finite assemblage is a polygon (a union of triangles with a common vertex O) we call the assemblage a *polygonal elliptic dome*. The polygonal base need not circumscribe a circle and it need not be convex. Corollary 1 gives us:

Corollary 2. *The volume of any polygonal elliptic dome is equal to the volume of its circumscribing punctured prismatic container, that is, two-thirds the volume of the unpunctured prismatic container, which, in turn, is the area of the base times the height.*

In the special limiting case when the equatorial polygonal base of the dome turns into an ellipse with center at O, the dome becomes half an ellipsoid, and the circumscribing prism becomes an elliptic cylinder. In this limiting case, Corollary 2 reduces to:

Corollary 3. *The volume of any ellipsoid is two-thirds that of its circumscribing elliptic cylinder.*

In particular, we have Archimedes' result for "spheroids" [3; Method, Proposition 3]:

Corollary 4. (Archimedes) *The volume of an ellipsoid of revolution is two-thirds that of its circumscribing circular cylinder.*

Polygonal elliptic shells. A *polygonal elliptic shell* is the solid between two concentric similar polygonal elliptic domes. From Theorem 1 we also obtain:

Theorem 2. *The following solids are reducible:*

(a) *Any uniform polygonal elliptic shell.*
(b) *Any wedge of a uniform polygonal elliptic shell.*
(c) *Any horizontal slice of a wedge of type* (b).
(d) *Any nonuniform assemblage of shells of type* (a).
(e) *Any nonuniform assemblage of wedges of type* (b).
(f) *Any nonuniform assemblage of slices of type* (c).

By using as building blocks horizontal slices of wedges cut from a polygonal elliptic shell, we can see intuitively how one might construct, from such building blocks, very general polygonal elliptic domes that are reducible and have more or less arbitrary mass distribution. By considering limiting cases of polygonal bases with many edges, and building blocks with very small side lengths, we can imagine elliptic shells and domes whose bases are more or less arbitrary plane regions, for example, elliptic, parabolic or hyperbolic segments.

The next section describes an explicit construction of general reducible domes with curvilinear bases.

5. General elliptic domes

Replace the polygonal base by any plane region bounded by a curve whose polar coordinates (r, θ) relative to a "center" O satisfy an equation $r = \rho(\theta)$, where ρ is a given piecewise continuous function, and θ varies over an interval of length 2π. Above this base we build an elliptic dome as follows. First, the altitude of the dome is a segment of fixed height $h > 0$ along the polar axis perpendicular to the base at O. We assume that each vertical half plane through the polar axis at angle θ cuts the surface of the dome along a quarter of an ellipse with horizontal semi axis $\rho(\theta)$ and the same vertical semi axis h, as in Figure 4a. The ellipse will be degenerate at points where $\rho(\theta) = 0$. Thus, an elliptic wedge is a special case of an elliptic dome.

When $\rho(\theta) > 0$, the cylindrical coordinates (r, θ, z) of points on the surface of the dome satisfy the equation of an ellipse:

$$\left(\frac{r}{\rho(\theta)}\right)^2 + \left(\frac{z}{h}\right)^2 = 1. \tag{1}$$

The dome is circumscribed by a cylindrical solid of altitude h whose base is congruent to that of the dome (Figure 4b). Incidentally, we use the term "cylindrical solid" with the understanding that the solid is a prism when the base is polygonal.

Each point (r', θ', z') on the lateral surface of the cylinder in Figure 4b is related to the corresponding point (r, θ, z) on the surface of the dome by the equations

$$\theta' = \theta, \quad z' = z, \quad r' = \rho(\theta).$$

From this cylindrical solid we remove a conical solid whose surface points have cylindrical coordinates (r'', θ, z), where $z/h = r''/\rho(\theta)$, or

$$r'' = z\rho(\theta)/h.$$

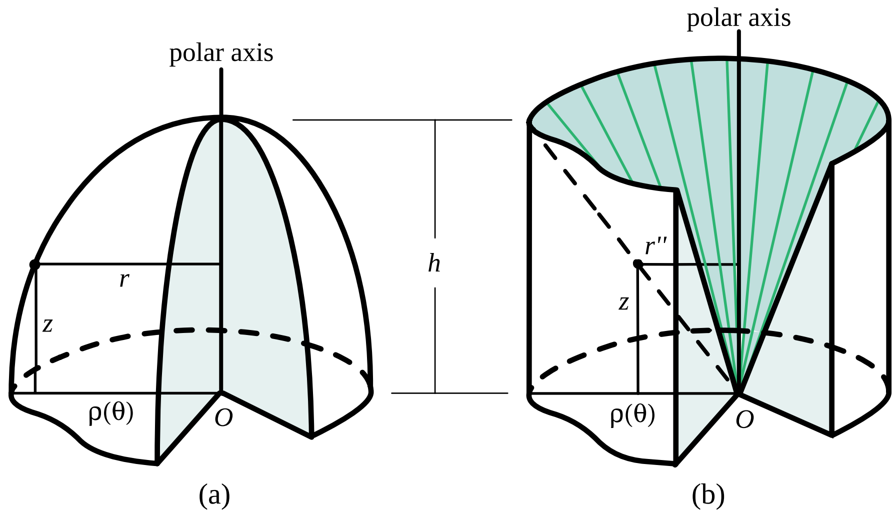


Figure 4. An elliptic dome (a), and its circumscribing punctured prismatic container (b).

When $z = h$, this becomes $r'' = \rho(\theta)$, so the base of the cone is congruent to the base of the elliptic dome. When the base is polygonal, the conical solid is a pyramid.

More reducible domes. The polar axis of an elliptic dome depends on the location of center O. For a given curvilinear base, we can move O to any point inside the base, or even to the boundary. Moving O will change the function $\rho(\theta)$ describing the boundary of the base, with a corresponding change in the shape of the ellipse determined by (1). Thus, this construction generates not one, but *infinitely many elliptic domes* with a given base. For any such dome, we can generate another family as follows: Imagine the dome and its prismatic counterpart made up of very thin horizontal layers, like two stacks of cards. Deform each solid by a horizontal translation and rotation of each horizontal layer. The shapes of the solids will change, but their cross-sectional areas will not change. In general, such a deformation may alter the shape of each ellipse on the surface to some other curve, and the deformed dome will no longer be elliptic. The same deformation applied to the prismatic counterpart will change the punctured container to a nonprismatic punctured counterpart. Nevertheless, all the results of this paper (with the exception of Theorem 11) will hold for such deformed solids and their counterparts.

However, if the deformation is a linear shearing that leaves the base fixed but translates each layer by a distance proportional to its distance from the base, then straight lines are mapped onto straight lines and the punctured prismatic solid is deformed into another prism punctured by a pyramid with the same base. The correspondingly sheared dome will be *elliptic* because each elliptic curve on the surface of the dome is deformed into an elliptic curve. To visualize a physical model of such a shearing, imagine a general elliptic dome and its counterpart sliced horizontally to form stacks of cards. Pierce each stack by a long pin along the polar axis, and let O be the point where the tip of the pin touches the base. Tilting the pin away from the vertical polar axis, keeping O fixed, results in horizontal linear shearing of the stacks and produces infinitely many elliptic domes, all with the same polygonal base. The prismatic containers are correspondingly tilted, and the domes are reducible.

Reducibility mapping. For a given general elliptic dome, we call the corresponding circumscribing punctured cylindrical solid *its punctured container*. Our goal is to show that *every uniform general elliptic dome is reducible*. This will be deduced from a more profound property, stated below in Theorem 3. It concerns a mapping that relates elliptic domes and their punctured containers.

To determine this mapping, regard the dome as a collection of layers of similar elliptic domes, like layers of an onion. Choose O as the center of similarity, and for each scaling factor $\mu \leq 1$, imagine a surface $E(\mu)$ such that a vertical half plane through the polar axis at angle θ intersects $E(\mu)$ along a quarter of an ellipse with semiaxes $\mu\rho(\theta)$ and μh. When $\rho(\theta) > 0,$ the coordinates r and z of points on this similar ellipse satisfy

$$\left(\frac{r}{\mu\rho(\theta)}\right)^2 + \left(\frac{z}{\mu h}\right)^2 = 1. \tag{2}$$

Regard the punctured container as a collection of coaxial layers of similar punctured cylindrical surfaces $C(\mu)$.

It is easy to relate the cylindrical coordinates $(r',\, \theta',\, z')$ of each point on $C(\mu)$ to the coordinates $(r,\, \theta,\, z)$ of the corresponding point on $E(\mu)$. First, we have

$$\theta' = \theta, \quad z' = z, \quad r' = \mu\rho(\theta). \tag{3}$$

From (2) we find $r^2 + z^2\rho\,(\theta)^2\,/h^2 = \mu^2\rho(\theta)^2$, hence (3) becomes

$$\theta' = \theta, \quad z' = z, \quad r' = \sqrt{r^2 + z^2\rho\,(\theta)^2\,/h^2}. \tag{4}$$

The three equations in (4), which are independent of μ, describe a *mapping* from each point $(r,\, \theta,\, z)$, not on the polar axis, of the solid elliptic dome to the corresponding point $(r',\, \theta',\, z')$ on its punctured container. On the polar axis, $r = 0$ and θ is undefined.

Using (2) in (4) we obtain (3), hence points on the ellipse described by (2) are mapped onto the vertical segment of length μh through the base point $(\mu\rho(\theta),\, \theta)$. It is helpful to think of the solid elliptic dome as made up of *elliptic fibers* emanating from the points on the base. Mapping (4) converts each elliptic fiber into a vertical fiber through the corresponding point on the base of the punctured container.

Preservation of volumes. Now we show that mapping (4) preserves volumes. The volume element in the $(r,\, \theta,\, z)$ system is given by $r\,dr\,d\theta\,dz$, while that in the $(r',\, \theta', z')$ system is $r'dr'd\theta'dz'$. From (4) we have

$$(r')^2 = r^2 + z^2\rho\,(\theta)^2\,/h^2$$

which, for fixed z and θ, gives $r'dr' = r dr$. From (4) we also have $d\theta' = d\theta$ and $dz' = dz$, so the volume elements are equal: $r\,dr\,d\theta\,dz = r'dr'd\theta'dz'$. This proves:

Theorem 3. *Mapping* (4)*, from a general elliptic dome to its punctured prismatic container, preserves volumes. In particular, every general uniform elliptic dome is reducible.*

As an immediate consequence of Theorem 3 we obtain:

Corollary 5. *The volume of a general elliptic dome is equal to the volume of its circumscribing punctured cylindrical container, that is, two-thirds the volume of the circumscribing unpunctured cylindrical container which, in turn, is simply the area of the base times the height.*

The same formulas show that for a fixed altitude z, we have $r\,dr\,d\theta = r'dr'd\theta'$. In other words, the mapping also preserves areas of horizontal cross sections cut from the elliptic dome and its punctured container. This also implies Corollary 5 because of the slicing principle.

Lambert's classical mapping as a special case. Our mapping (4) generalizes Lambert's classical mapping [2], which is effected by wrapping a tangent cylinder about the equator, and then projecting the surface of the sphere onto this cylinder by rays through the axis which are parallel to the equatorial plane. Lambert's mapping takes points on the spherical surface (not at the north or south pole) and maps them onto points on the lateral cylindrical surface in a way that preserves areas. For a solid sphere, our mapping (4) takes each point not on the polar axis and maps it onto a point of the punctured solid cylinder in a way that preserves volumes. Moreover, analysis of a thin shell (similar to that in [1; Section 6]) shows that (4) also preserves areas when the surface of an Archimedean dome is mapped onto the lateral surface of its prismatic container. Consequently, we have:

Theorem 4. *Mapping* (4)*, from the surface of an Archimedean dome onto the lateral surface of its prismatic container, preserves areas.*

In the limiting case when the Archimedean dome becomes a hemisphere we get:

Corollary 6. (Lambert) *Mapping* (4)*, from the surface of a sphere to the lateral surface of its tangent cylinder, preserves areas.*

If the hemisphere in this limiting case has radius a, it is easily verified that (4) reduces to Lambert's mapping: $\theta' = \theta, \;\; z' = z, \;\; r' = a.$

6. Nonuniform elliptic domes

Mapping (4) takes each point P of an elliptic dome and carries it onto a point P' of its punctured container. Imagine an arbitrary mass density assigned to P, and assign the same mass density to its image P'. If a set of points P fills out a portion of the dome of volume v and total mass m, say, then the image points P' fill out a solid, which we call the *counterpart*, having the same volume v and the same total mass m. This can be stated as an extension of Theorem 3:

Theorem 5. *Any portion of a general nonuniform elliptic dome is reducible.*

By analogy with Theorem 3, we can say that mapping (4) “with weights” also preserves masses.

Fiber-elliptic and shell-elliptic domes. Next we describe a special way of assigning variable mass density to the points of a general elliptic dome and its punctured container so that corresponding portions of the dome and its counterpart have the same mass. The structure of the dome as a collection of similar domes plays an essential role in this description.

First assign mass density $f(r,\theta)$ to each point (r,θ) on the base of the dome and of its cylindrical container. Consider the elliptic fiber that emanates from any point $(\mu\rho(\theta),\, \theta)$ on the base, and assign the same mass density $f(\mu\rho(\theta),\, \theta)$ to each point of this fiber. In other words, the mass density along the elliptic fiber has a constant value inherited from the point at which the fiber meets the base. Of course, the constant may differ from point to point on the base. The elliptic fiber maps into a vertical fiber in the punctured container (of length μh, where h is the altitude of the dome), and we assign the same mass density $f(\mu\rho(\theta),\, \theta)$ to each point on this vertical fiber. In this way we produce a nonuniform elliptic dome and its punctured container, each with variable mass density inherited from the base. We call such a dome *fiber-elliptic*. The punctured container with density assigned in this manner is called the *counterpart* of the dome. The volume element multiplied by mass density is the same for both the dome and its counterpart.

An important special case occurs when the assigned density is also constant along the base curve $r = \rho(\theta)$ and along each curve $r = \mu\rho(\theta)$ similar to the base curve, where the constant density depends only on μ. Then each elliptic surface $E(\mu)$ will have its own constant density. We call domes with this assignment of mass density *shell-elliptic*. For fiber-elliptic and shell-elliptic domes, horizontal slices cut from any portion of the dome and its counterpart have equal masses, and their centers of mass are at the same height above the base. Thus, as a consequence of Theorem 3 we have:

Corollary 7. (a) *Any portion of a fiber-elliptic dome is reducible.*

(b) *In particular, any portion of a shell-elliptic dome is reducible.*

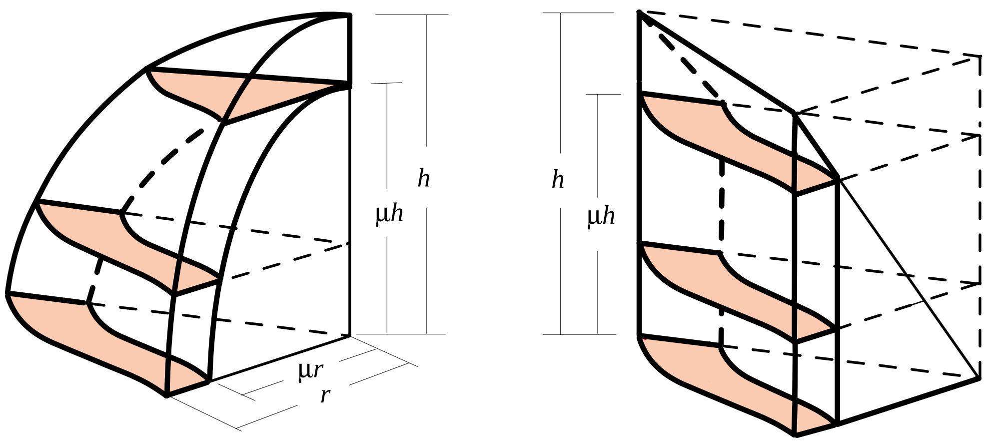
(c) *In particular, a sphere with spherically symmetric mass distribution is reducible.*

The reducibility properties of an elliptic dome also hold for the more general case in which we multiply the mass density $f(\mu\rho(\theta),\, \theta)$ by any function of z. Such change of density could be imposed, for example, by an external field (such as atmospheric density in a gravitational field that depends only on the height z). Consequently, not only are the volume and mass of any portion of this type of nonuniform elliptical dome equal to those of its counterpart, but the same is true for all moments with respect to the horizontal base.

Elliptic shells and cavities. Consider a general elliptic dome of altitude h, and denote its elliptic surface by $E(1)$. Scale $E(1)$ by a factor μ, where $0 < \mu < 1$, to produce a similar elliptic surface $E(\mu)$. The region between the two surfaces $E(\mu)$ and $E(1)$ is called an *elliptic shell*. It can be regarded as an elliptic dome with a cavity, or, equivalently, as a shell-elliptic dome with density 0 assigned to each point between $E(\mu)$ and the center.

Figure 5a shows an elliptic shell element, and Figure 5b shows its counterpart. Each base in the equatorial plane is bounded by portions of two curves with polar equations $r = \rho(\theta)$ and $r = \mu\rho(\theta)$, and two segments with $\theta = \theta_1$ and $\theta = \theta_2$. The shell element has two vertical plane faces, each consisting of a region between two similar ellipses. If μ is close to 1 and if θ_1 and θ_2 are nearly equal, the elliptic shell element can be thought of as a thin elliptic fiber, as was done earlier.

Consider a horizontal slice between two horizontal planes that cut both the inner and outer elliptic boundaries of the shell element. In other words, both planes are pierced by the cavity. The prismatic counterpart of this slice has horizontal cross sections congruent to the base, so its centroid lies *midway* between the two cutting planes. The same is true for the slice of the shell and for the center of mass of a slice cut from an assemblage of uniform elliptic shell elements, each with its own constant density.



(a) (b)

Figure 5. An elliptic shell element (a) and its counterpart (b).

In the same way, if we build a nonuniform shell-elliptic solid with a finite number of similar elliptic shells, each with its density inherited from the base, then any horizontal slice pierced by the cavity has its center of mass midway between the two horizontal cutting planes. Moreover, the following theorem holds for every such shell-elliptic wedge.

Theorem 6. *Any horizontal slice pierced by the cavity of a nonuniform shell-elliptic wedge has volume and mass equal, respectively, to those of its prismatic counterpart. Each volume and mass is independent of the height above the base and each is proportional to the thickness of the slice. Consequently, the center of mass of such a slice lies midway between the two cutting planes.*

Corollary 8. (Sphere with cavity) *Consider a spherically symmetric distribution of mass inside a solid sphere with a concentric cavity. Any slice between parallel* *planes pierced by the cavity has volume and mass proportional to the thickness of the slice, and is independent of the location of the slice.*

Corollary 8 implies that the one-dimensional vertical projection of the density is constant along the cavity. This simple result has profound consequences in tomography, which deals with the inverse problem of reconstructing spatial density distributions from a knowledge of their lower dimensional projections. Details of this application will appear elsewhere.

7. Formulas for volume and centroid

This section uses reducibility to give specific formulas for volumes and centroids of various building blocks of elliptic domes with an arbitrary curvilinear base.

Volume of a shell element. We begin with the simplest case. Cut a wedge from an elliptic dome of altitude h by two vertical half planes $\theta = \theta_1$ and $\theta = \theta_2$ through the polar axis, and then remove a similar wedge scaled by a factor μ, where $0 < \mu < 1$, as shown in Figure 5a. Assume the unpunctured cylindrical container in Figure 5b has volume V. By Corollary 5 the outer wedge has volume $2V/3$, and the similar inner wedge has volume $2\mu^3 V/3$, so the volume v of the shell element and its prismatic counterpart is the difference

$$v = \frac{2}{3}V(1-\mu^3). \tag{5}$$

Now $V = Ah$, where A is the area of the base of both the elliptic wedge and its container. The base of the elliptic shell element and its unpunctured container have area $B = A - \mu^2 A$, so $A = B/(1-\mu^2)$, $V = Bh/(1-\mu^2)$, and (5) can be written as

$$v = \frac{2}{3}Bh\frac{1-\mu^3}{1-\mu^2}. \tag{6}$$

Formula (6) also holds for the total volume of any assemblage of elliptic shell elements with a given h and μ, with B representing the total base area. The product Bh is the volume of the corresponding unpunctured cylindrical container of altitude h, so (6) gives us the formula

$$v_\mu(h) = \frac{2}{3}v_{cyl}\frac{1-\mu^3}{1-\mu^2}\,, \tag{7}$$

where $v_\mu(h)$ is the volume of the assemblage of elliptic shell elements and of the counterpart, and v_{cyl} is the volume of its *unpunctured* cylindrical container. When $\mu = 0$ in (7), the assemblage of elliptic wedges has volume $v_0(h) = 2v_{cyl}/3$, so we can write (7) in the form

$$v_\mu(h) = v_0(h)\frac{1-\mu^3}{1-\mu^2}, \tag{8}$$

where $v_0(h)$ is the volume of the outer dome of the assemblage and its counterpart. If μ approaches 1 the shell becomes very thin, the quotient $(1-\mu^3)/\left(1-\mu^2\right)$ approaches 3/2, and (7) shows that $v_\mu(h)$ approaches v_{cyl}. In other words, a very thin elliptic shell element has volume very nearly equal to that of its very thin unpunctured cylindrical container. An Archimedean shell has constant thickness equal to that of the prismatic container, so the lateral surface area of any assemblage of Archimedean wedges is equal to the lateral surface area of its prismatic container, a result derived in [1]. Note that this argument cannot be used to find the surface area of an nonspherical elliptic shell because it does not have constant thickness.

Next we derive a formula for the height of the centroid of any uniform elliptic wedge above the plane of its base.

Theorem 7. *Any uniform elliptic wedge or dome of altitude* h *has volume two-thirds that of its unpunctured prismatic container. Its centroid is located at height* c *above the plane of the base, where*

$$c = \frac{3}{8}h. \tag{9}$$

Proof. It suffices to prove (9) for the prismatic counterpart. For any prism of altitude h, the centroid is at a distance $h/2$ above the plane of the base. For a cone or pyramid with the same base and altitude it is known that the centroid is at a distance $3h/4$ from the vertex. To determine the height c of the centroid of a punctured prismatic container above the plane of the base, assume the unpunctured prismatic container has volume V and equate moments to get

$$c\left(\frac{2}{3}V\right) + \frac{3h}{4}\left(\frac{1}{3}V\right) = \frac{h}{2}V,$$

from which we find (9). By Theorem 5, the centroid of the inscribed elliptic wedge is also at height $3h/8$ above the base. The result is also true for any uniform elliptic dome formed as an assemblage of wedges. □

Equation (9) is equivalent to saying, in the style of Archimedes, that the centroid divides the altitude in the ratio 3:5.

Corollary 9. (a) *The centroid of a uniform Archimedean dome divides its altitude in the ratio* 3:5.

(b) (Archimedes) *The centroid of a uniform hemisphere divides its altitude in the ratio* 3:5.

Formula (9) is obviously true for the center of mass of any nonuniform assemblage of elliptic wedges of altitude h, each with its own constant density.

Centroid of a shell element. Now we can find, for any elliptic shell element, the height $c_\mu(h)$ of its centroid above the plane of its base. The volume and centroid results are summarized as follows:

Theorem 8. *Any nonuniform assemblage of elliptic shell elements with common altitude* h *and scaling factor* μ *has volume* $v_\mu(h)$ *given by* (8)*. The height* $c_\mu(h)$ *of the centroid above the plane of its base is given by*

$$c_\mu(h) = \frac{3}{8}h\frac{1-\mu^4}{1-\mu^3}. \tag{10}$$

Proof. Consider first a single uniform elliptic shell element. Again it suffices to do the calculation for the prismatic counterpart. The inner wedge has altitude $\mu h,$ so by (9) its centroid is at height $3\mu h/8$. The centroid of the outer wedge is at height $3h/8$. If the outer wedge has volume V_{outer}, the inner wedge has volume $\mu^3 V_{outer}$, and the shell element between them has volume $(1-\mu^3)V_{outer}$. Equating moments and canceling the common factor V_{outer} we find

$$\left(\frac{3}{8}\mu h\right)\mu^3 + c_\mu(h)(1-\mu^3) = \frac{3}{8}h,$$

from which we obtain (10). Formula (10) also holds for any nonuniform assemblage of elliptic shell elements with the same h and μ, each of constant density, although the density can differ from element to element. □

When $\mu = 0$, (10) gives $c_0(h) = 3h/8$.

When $\mu \to 1$, the shell becomes very thin and the limiting value of $c_\mu(h)$ in (10) is $h/2$. This also follows from Theorem 6 when the shell is very thin and the slice includes the entire dome. It is also consistent with Corollary 15 of [1], which states that the centroid of the surface area of an Archimedean dome is at the midpoint of its altitude.

Centroid of a slice of a wedge. More generally, we can determine the centroid of any slice of altitude z of a uniform elliptic wedge. By reducing this calculation to that of the prismatic counterpart, shown in Figure 6, the analysis becomes very simple. For clarity, the base in Figure 6 is shown as a triangle, but the same argument applies to a more general base like that in Figure 5. The slice in question is obtained from a prism of altitude z and volume $V(z) = \lambda V$, where V is the volume of the unpunctured prismatic container of altitude h, and $\lambda = z/h$. The centroid of the slice is at an altitude $z/2$ above the base. We remove from this slice a pyramidal portion of altitude z and volume $v(z) = \lambda^3 V/3$, whose centroid is at an altitude $3z/4$ above the base. The portion that remains has volume

$$V(z) - v(z) = \left(\lambda - \frac{1}{3}\lambda^3\right) V \tag{11}$$

and centroid at altitude $c(z)$ above the base. To determine $c(z)$, equate moments to obtain

$$\frac{3z}{4}v(z) + c(z)(V(z) - v(z)) = \frac{z}{2}V(z),$$

which gives

$$c(z) = \frac{\frac{z}{2}\,V(z) - \frac{3z}{4}v(z)}{V(z) - v(z)}.$$

Because $V(z) = \lambda V$, and $v(z) = \lambda^3 V/3$, we obtain the following theorem.

Theorem 9. *Any slice of altitude z cut from a uniform elliptic wedge of altitude h has volume given by* (11), *where $\lambda = z/h$ and V is the volume of the unpunctured prismatic container. The height $c(z)$ of the centroid is given by*

$$c(z) = \frac{3}{4}z\frac{2-\lambda^2}{3-\lambda^2}. \tag{12}$$

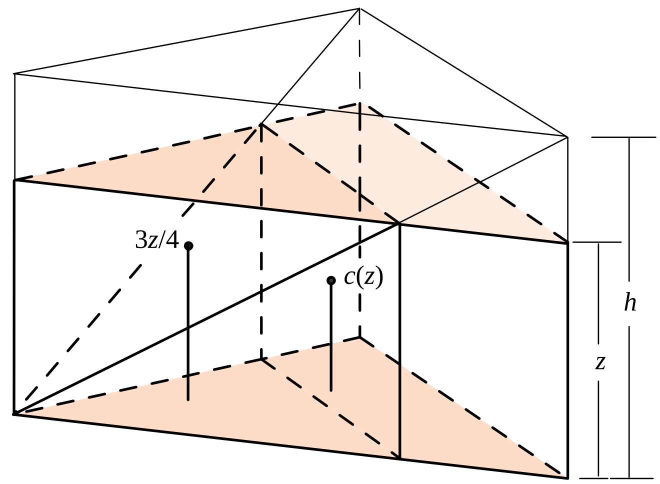


Figure 6. Calculating the centroid of a slice of altitude z cut from a wedge of altitude h.

When $z = h$ then $\lambda = 1$ and this reduces to (9). For small z the right member of (12) is asymptotic to $z/2$. This is reasonable because for small z the walls of the dome are nearly perpendicular to the plane of the equatorial base, so the dome is almost cylindrical near the base.

Centroid of a slice of a wedge shell element. There is a common generalization of (10) and (12). Cut a slice of altitude z from a shell element having altitude h and scaling factor μ, and let $c_\mu(z)$ denote the height of its centroid above the base. Again, we simplify the calculation of $c_\mu(z)$ by reducing it to that of its prismatic counterpart. The slice in question is obtained from an unpunctured prism of altitude z, whose centroid has altitude $z/2$ above the base. As in Theorem 9, let $\lambda = z/h$. If $\lambda \leq \mu$, the slice lies within the cavity, and the prismatic counterpart is the same unpunctured prism of altitude z, in which case we know from Theorem 6 that

$$c_\mu(z) = \frac{z}{2}, \quad (\lambda \leq \mu). \tag{13}$$

But if $\lambda \geq \mu$, the slice cuts the outer elliptic dome as shown in Figure 7a. In this case the counterpart slice has a slant face due to a piece removed by the puncturing pyramid, as indicated in Figure 7b.

Let V denote the volume of the unpunctured prismatic container of the outer dome. Then λV is the volume of the unpunctured prism of altitude z. Remove from this prism the puncturing pyramid of volume $\lambda^3 V/3$, leaving a solid whose volume is

$$V(z) = \lambda V - \frac{1}{3}\lambda^3 V, \quad (\lambda \geq \mu) \tag{14}$$

and whose centroid is at altitude $c(z)$ given by (12). This solid, in turn, is the union of the counterpart slice in question, and an adjacent pyramid with vertex O,

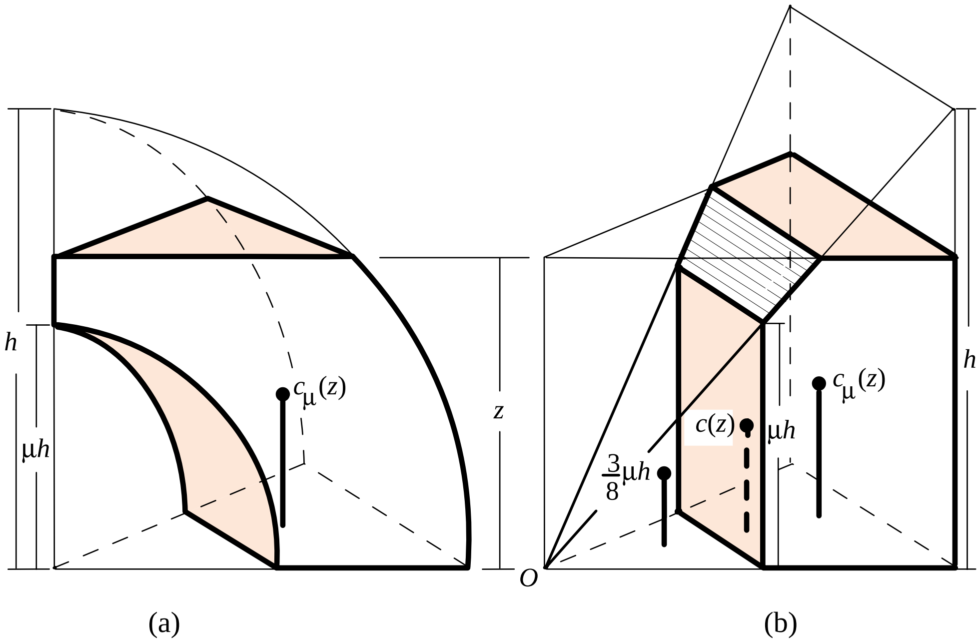


Figure 7. Determining the centroid of a slice of altitude $z \geq \mu h$ cut from an elliptic shell element.

altitude $\mu h,$ volume

$$v_\mu = \frac{2}{3}\mu^3 V, \tag{15}$$

and centroid at altitude $3\mu h/8$. The counterpart slice in question has volume

$$V(z) - v_\mu = \left(\lambda - \frac{1}{3}\lambda^3 - \frac{2}{3}\mu^3\right) V. \tag{16}$$

To find the altitude $c_\mu(z)$ of its centroid we equate moments and obtain

$$\left(\frac{3}{8}\mu h\right) v_\mu + c_\mu(z)(V(z) - v_\mu) = c(z)V(z),$$

from which we find

$$c_\mu(z) = \frac{c(z)V(z) - \left(\frac{3}{8}\mu h\right) v_\mu}{V(z) - v_\mu}.$$

Now we use (12), (14), (15)and (16). After some simplification we find the result

$$c_\mu(z) = \frac{3}{4}h \,\frac{\lambda^2(2-\lambda^2) - \mu^4}{\lambda(3-\lambda^2) - 2\mu^3} \quad (\lambda \geq \mu). \tag{17}$$

When $\lambda = \mu,$ (17) reduces to (13); when $\lambda = 1$ then $z = h$ and (17) reduces to (10); and when $\mu = 0$, (17) reduces to (12). The results are summarized by the following theorem.

Theorem 10. *Any horizontal slice of altitude $z \geq \mu h$ cut from a wedge shell element of altitude h and scaling factor μ has volume given by* (16)*, where $\lambda = z/h$. The altitude of its centroid above the base is given by* (17)*. In particular these formulas hold for any slice of a shell of an Archimedean, elliptic, or spherical dome.*

Note: Theorem 6 covers the case $z \leq \mu h$.

In deriving the formulas in this section we made no essential use of the fact that the shell elements are elliptic. The important fact is that each shell element is the region between two similar objects.

8. The necessity of elliptic profiles

We know that every horizontal plane cuts an elliptic dome and its punctured cylindrical container in cross sections of equal area. This section reveals the surprising fact that the elliptical shape of the dome is actually a consequence of this property.

Consider a dome of altitude h, and its punctured prismatic counterpart having a congruent base bounded by a curve satisfying a polar equation $r = \rho(\theta)$. Each vertical half plane through the polar axis at angle θ cuts the dome along a curve we call a *profile*, illustrated by the example in Figure 8a. This is like the elliptic dome in Figure 5a, except that we do not assume that the profiles are elliptic. Each profile passes through a point $(\rho(\theta),\, \theta)$ on the outer edge of the base. At altitude z above the base a point on the profile is at distance r from the polar axis, where r is a function of z that determines the shape of the profiles. We define a general profile dome to be one in which each horizontal cross section is similar to the base. Figure 8a shows a portion of a dome in which $\rho(\theta) > 0$. This portion is a wedge with two vertical plane faces that can be thought of as "walls" forming part of the boundary of the wedge.

Suppose that a horizontal plane at distance z above the base cuts a region of area $A(z)$ from the wedge and a region of area $B(z)$ from the punctured prism. We know that $A(0) = B(0)$. Now we assume that $A(z) = B(z)$ for some $z > 0$ and deduce that the point on the profile with polar coordinates (r, θ, z) satisfies the equation

$$\left(\frac{r}{\rho(\theta)}\right)^2 + \left(\frac{z}{h}\right)^2 = 1 \tag{18}$$

if $\rho(\theta) > 0$. In other words, the point on the profile at a height where the areas are equal lies on an ellipse with vertical semi axis of length h, and horizontal semi axis of length $\rho(\theta)$. Consequently, if $A(z) = B(z)$ for every z from 0 to h, the profile will fill out a quarter of an ellipse and the dome will necessarily be elliptic. Note that (18) implies that $r \to 0$ as $z \to h$.

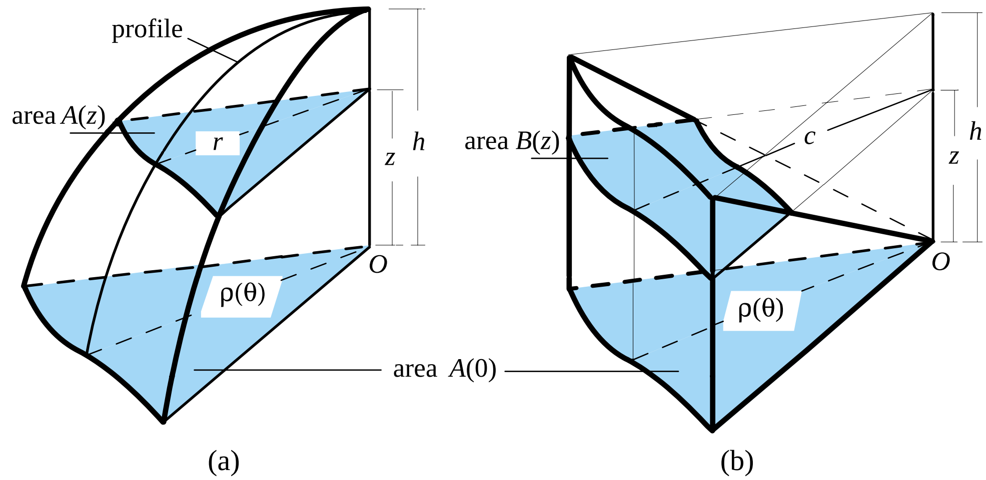


Figure 8. Determining the elliptic shape of the profiles as a consequence of the relation $A(z) = B(z)$.

To deduce (18), note that the horizontal cross section of area $A(z)$ in Figure 8a is similar to the base with similarity ratio $r/\rho(\theta)$, where $\rho(\theta)$ denotes the radial distance to the point where the profile intersects the base, and r is the length of the radial segment at height z. By similarity, $A(z) = (r/\rho(\theta))^2 A(0)$. In Figure 8b, area $B(z)$ is equal to $A(0)$ minus the area of a smaller similar region with similarity ratio $c/\rho(\theta)$, where c is the length of the parallel radial segment of the smaller similar region at height z. By similarity, $c/\rho(\theta) = z/h$, hence $B(z) = (1 - (z/h)^2)A(0)$. Equating this to $A(z)$ we find $(1 - (z/h)^2)A(0) = (r/\rho(\theta))^2 A(0)$, which gives (18). And, of course, we already know that (18) implies $A(z) = B(z)$ for every z. Thus we have proved:

Theorem 11. *Corresponding horizontal cross sections of a general profile uniform dome and its punctured prismatic counterpart have equal areas if, and only if, each profile is elliptic.*

As already remarked in Section 5, an elliptic dome can be deformed in such a way that areas of horizontal cross sections are preserved but the deformed dome no longer has elliptic profiles. At first glance, this may seem to contradict Theorem 11. However, such a deformation will distort the vertical walls; the dome will not satisfy the requirements of Theorem 11, and also the punctured counterpart will no longer be prismatic.

An immediate consequence of Theorem 11 is that any reducible general profile dome necessarily has elliptic profiles, because if all horizontal slices of such a dome and its counterpart have equal volumes then the cross sections must have equal areas. We have also verified that Theorem 11 can be extended to nonuniform general profile domes built from a finite number of general profile similar shells, each with its own constant density, under the condition that corresponding horizontal slices of the dome and its counterpart have equal masses, with no requirements on volumes or reducibility.

Concluding remarks. The original motivation for this research was to extend to more general solids classical properties which seemed to be unique to spheres and hemispheres. Initially an extension was given for Archimedean domes and a further extension was made by simply dilating these domes in a vertical direction. These extensions could also have been analyzed by using properties of inscribed spheroids.

A significant extension was made when we introduced polygonal elliptic domes whose bases could be arbitrary polygons, not necessarily circumscribing the circle. In this case there are no inscribed spheroids to aid in the analysis, but the method of punctured containers was applicable. This led naturally to general elliptic domes with arbitrary base, and the method of punctured containers was formulated in terms of mappings that preserve volumes.

But the real power of the method is revealed by the treatment of nonuniform mass distributions. Problems of determining volumes and centroids of elliptic wedges, shells, and their slices, including those with cavities, were reduced to those of simpler prismatic containers. Finally, we showed that domes with elliptic profiles are essentially the only ones that are reducible.

References

[1] T. M. Apostol and M. A. Mnatsakanian, A fresh look at the method of Archimedes, *Amer. Math. Monthly*, 111 (2004) 496–508.

[2] B. H. Brown, Conformal and equiareal world maps, *Amer. Math. Monthly*, 42 (1935) 212–223.

[3] T. L. Heath, *The Works of Archimedes*, Dover, New York, 1953.

Tom M. Apostol: *Project MATHEMATICS!* 253-37 Caltech, Pasadena, California, 91125 USA
E-mail address: `apostol@caltech.edu`

Mamikon A. Mnatsakanian: *Project MATHEMATICS!* 253-37 Caltech, Pasadena, California, 91125 USA
E-mail address: `mamikon@caltech.edu`

Forum Geometricorum
Volume 7 (2007) 53–65.

FORUM GEOM
ISSN 1534-1178

Midcircles and the Arbelos

Eric Danneels and Floor van Lamoen

Abstract. We begin with a study of inversions mapping one given circle into another. The results are applied to the famous configuration of an arbelos. In particular, we show how to construct three infinite Pappus chains associated with the arbelos.

1. Inversions swapping two circles

Given two circles $O_i(r_i)$, $i = 1, 2$, in the plane, we seek the inversions which transform one of them into the other. Set up a cartesian coordinate system such that for $i = 1,\ 2$, O_i is the point $(a_i,\ 0)$. The endpoints of the diameters of the circles on the x-axis are $(a_i \pm r_i,\ 0)$. Let $(a,\ 0)$ and Φ be the center and the power of inversion. This means, for an appropriate choice of $\varepsilon = \pm 1$,

$$(a_1 + \varepsilon \cdot r_1 - a)(a_2 + r_2 - a) = (a_1 - \varepsilon \cdot r_1 - a)(a_2 - r_2 - a) = \Phi.$$

Solving these equations we obtain

$$a = \frac{r_2 a_1 + \varepsilon \cdot r_1 a_2}{r_2 + \varepsilon \cdot r_1}, \tag{1}$$

$$\Phi = \frac{\varepsilon \cdot r_1 r_2((r_2 + \varepsilon \cdot r_1)^2 - (a_1 - a_2)^2)}{(r_2 + \varepsilon \cdot r_1)^2}. \tag{2}$$

From (1) it is clear that the center of inversion is a center of similitude of the two circles, internal or external according as $\varepsilon = +1$ or -1. The two circles of inversion, real or imaginary, are given by $(x-a)^2 + y^2 = \Phi$, or more explicitly,

$$r_2((x - a_1)^2 + y^2 - r_1^2) + \varepsilon \cdot r_1((x - a_2)^2 + y^2 - r_2^2) = 0. \tag{3}$$

They are members of the pencil of circles generated by the two given circles. Following Dixon [1, pp.86–88], we call these the *midcircles* $\mathcal{M}_\varepsilon$, $\varepsilon = \pm 1$, of the two given circles $O_i(r_i),\ i = 1, 2$. From (2) we conclude that
(i) the *internal* midcircle $\mathcal{M}_+$ is real if and only if $r_1 + r_2 > d$, the distance between the two centers, and
(ii) the *external* midcircle $\mathcal{M}_-$ is real if and only if $|r_1 - r_2| < d$.
In particular, if the two given circles intersect, then there are two real circles of inversion through their common points, with centers at the centers of similitudes. See Figure 1.

Lemma 1. *The image of the circle with center B, radius r, under inversion at a point A with power Φ is the circle of radius $\left|\frac{\Phi}{d^2 - r^2}\right| r$, and center dividing AB at the ratio $AP : PB = \Phi : d^2 - r^2 - \Phi$, where d is the distance between A and B.*

Publication Date: February 19, 2007. Communicating Editor: Paul Yiu.

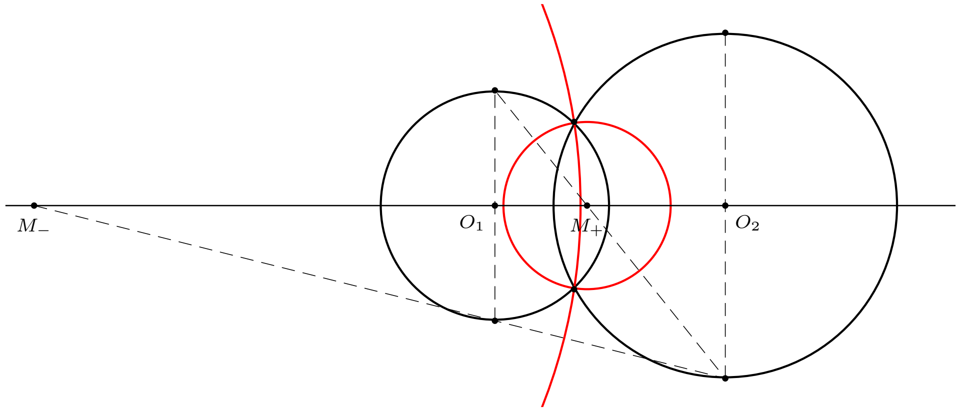


Figure 1.

2. A locus property of the midcircles

Proposition 2. *The locus of the center of inversion mapping two given circles $O_i(a_i)$, $i = 1, 2$, into two congruent circles is the union of their midcircles $\mathcal{M}_+$ and $\mathcal{M}_-$.*

Proof. Let $d(P, Q)$ denote the distance between two points P and Q. Suppose inversion in P with power Φ transforms the given circles into congruent circles. By Lemma 1,

$$\frac{d(P, O_1)^2 - r_1^2}{d(P, O_2)^2 - r_2^2} = \varepsilon \cdot \frac{r_1}{r_2} \tag{4}$$

for $\varepsilon = \pm 1$. If we set up a coordinate system so that $O_i = (a_i, 0)$ for $i = 1, 2$, $P = (x, y)$, then (4) reduces to (3), showing that the locus of P is the union of the midcircles $\mathcal{M}_+$ and $\mathcal{M}_-$. □

Corollary 3. *Given three circles, the common points of their midcircles taken by pairs are the centers of inversion that map the three given circles into three congruent circles.*

For $i, j = 1, 2, 3$, let $\mathcal{M}_{ij}$ be a midcircle of the circles $\mathcal{C}_i = O_i(R_i)$ and $\mathcal{C}_j = O_j(R_j)$. By Proposition 2 we have $\mathcal{M}_{ij} = R_j \cdot \mathcal{C}_i + \varepsilon_{ij} \cdot R_i \cdot \mathcal{C}_j$ with $\varepsilon_{ij} = \pm 1$. If we choose ε_{ij} to satisfy $\varepsilon_{12} \cdot \varepsilon_{23} \cdot \varepsilon_{31} = -1$, then the centers of $\mathcal{M}_{12}$, $\mathcal{M}_{23}$ and $\mathcal{M}_{31}$ are collinear. Since the radical center P of the triad $\mathcal{C}_i$, $i = 1, 2, 3$, has the same power with respect to these circles, they form a pencil and their common points X and Y are the poles of inversion mapping the circles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ into congruent circles.

The number of common points that are the poles of inversion mapping the circles $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ into a triple of congruent circles depends on the configuration of these circles.

(1) The maximal number is 8 and occurs when each pair of circles $\mathcal{C}_i$ and $\mathcal{C}_j$ have two distinct intersections. Of these 8 points, two correspond to

the three external midcircles while each pair of the remaining six points correspond to a combination of one external and two internal midcircles.

(2) The minimal number is 0. This occurs for instance when the circles belong to a pencil of circles without common points.

Corollary 4. *The locus of the centers of the circles that intersect three given circles at equal angles are* $0, 1, 2, 3$ *or* 4 *lines through their radical center* P *perpendicular to a line joining three of their centers of similitude.*

Proof. Let $\mathcal{C}_1 = A(R_1)$, $\mathcal{C}_2 = B(R_2)$, and $\mathcal{C}_3 = C(R_3)$ be the given circles. Consider three midcircles with collinear centers. If X is an intersection of these midcircles, reflection in the center line gives another common point Y. Consider an inversion τ with pole X that maps circle $\mathcal{C}_3$ to itself. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ become $\mathcal{C}_1' = A'(R_3)$ and $\mathcal{C}_2 = B'(R_3)$. If P' is the radical center of the circles $\mathcal{C}_1$, $\mathcal{C}_2'$ and $\mathcal{C}_3'$, then every circle $\mathcal{C} = P'(R)$ will intersect these 3 circles at equal angles. When we apply the inversion τ once again to the circles $\mathcal{C}_1'$, $\mathcal{C}_2'$, $\mathcal{C}_3$ and $\mathcal{C}$ we get the 3 original circles $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and a circle $\mathcal{C}'$ and since an inversion preserves angles circle $\mathcal{C}'$ will also intersect these original circles at equal angles.

The circles orthogonal to all circles $\mathcal{C}'$ are mapped by τ to lines through P'. This means that the circles orthogonal to $\mathcal{C}'$ all pass through the inversion pole X. By symmetry they also pass through Y, and thus form the pencil generated by the triple of midcircles we started with. The circles $\mathcal{C}'$ form therefore a pencil as well, and their centers lie on XY as X and Y are the limit-points of this pencil. $\square$

Remark. Not every point on the line leads to a real circle, and not every real circle leads to real intersections and real angles.

As an example we consider the A-, B- and C-Soddy circles of a triangle ABC. Recall that the A-Soddy circle of a triangle is the circle with center A and radius $s - a$, where s is the semiperimeter of triangle ABC. The area enclosed in the interior of ABC by the A-, B- and C-Soddy circles form a skewed arbelos, as defined in [5]. The circles $\mathcal{F}_\phi$ making equal angles to the A-, B- and C-Soddy circles form a pencil, their centers lie on the Soddy line of ABC, while the only real line of three centers of midcircles is the tripolar of the Gergonne point X_7. [1]

The points X and Y in the proof of Corollary 4 are the limit points of the pencil generated by $\mathcal{F}_\phi$. In barycentric coordinates, these points are, for $\varepsilon = \pm 1$,

$$(4R + r) \cdot X_7 + \varepsilon \cdot \sqrt{3}s \cdot I = \left(2r_a + \varepsilon \cdot \sqrt{3}a : 2r_b + \varepsilon \cdot \sqrt{3}b : 2r_c + \varepsilon \cdot \sqrt{3}c\right),$$

where R, r, r_a, r_b, r_c are the circumradius, inradius, and inradii. The midpoint of XY is the Fletcher-point X_{1323}. See Figure 2.

3. The Arbelos

Now consider an arbelos, consisting of two interior semicircles $O_1(r_1)$ [2] and $O_2(r_2)$ and an exterior semicircle $O(r) = O_0(r)$, $r = r_1 + r_2$. Their points of

[1] The numbering of triangle centers following numbering in [2, 3].

[2] We adopt notations as used in [4]: By (PQ) we denote the circle with diameter PQ, by $P(r)$ the circle with center P and radius r, while $P(Q)$ is the circle with center P through Q and (PQR)

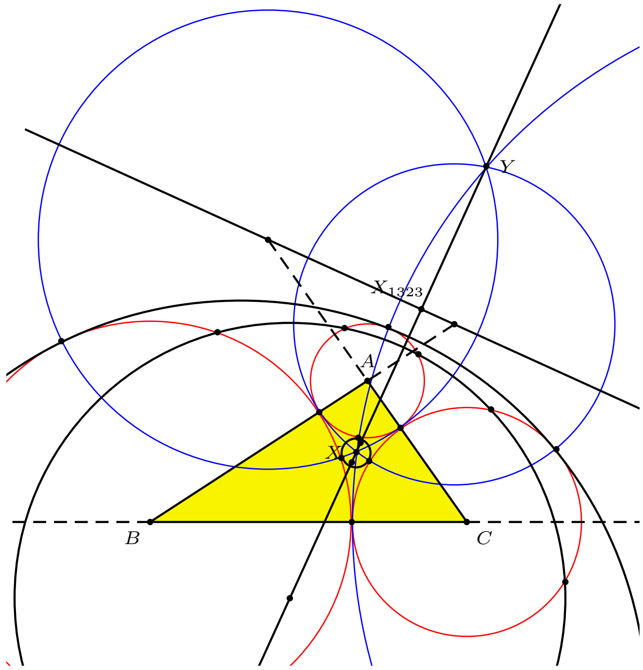


Figure 2.

tangency are A, B and C as indicated in Figure 3. The arbelos has an incircle (O'). For simple constructions of (O'), see [7, 8].

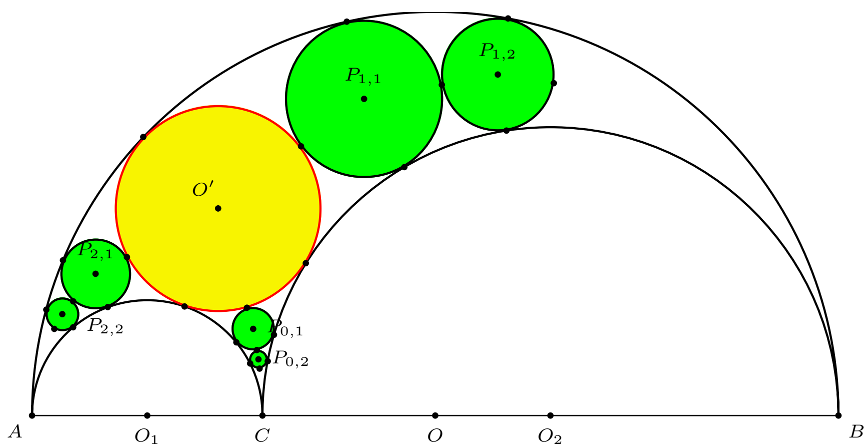


Figure 3.

We consider three Pappus chains $(\mathcal{P}_{i,n})$, $i = 0, 1, 2$. If (i, j, k) is a permutation of $(0, 1, 2)$, the Pappus chain $(\mathcal{P}_{i,n})$ is the sequence of circles tangent to both (O_j)

is the circle through P, Q and R. The circle (P) is the circle with center P, and radius clear from context.

and (O_k) defined recursively by
(i) $\mathcal{P}_{i,0} = (O')$, the incircle of the arbelos,
(ii) for $n \geq 1$, $\mathcal{P}_{i,n}$ is tangent to $\mathcal{P}_{i,n-1}$, (O_j) and (O_k),
(iii) for $n \geq 2$, $\mathcal{P}_{i,n}$ and $\mathcal{P}_{i,n-2}$ are distinct circles.

These Pappus chains are related to the centers of similitude of the circles of the arbelos. We denote by M_0 the external center of similitude of (O_1) and (O_2), and, for $i, j = 1, 2$, by M_i the internal center of similitude of (O) and (O_j). The midcircles are $M_0(C)$, $M_1(B)$ and $M_2(A)$. Each of the three midcircles leaves (O') and its reflection in AB invariant, so does each of the circles centered at A, B and C respectively and orthogonal to (O'). These six circles are thus members of a pencil, and O' lies on the radical axis of this pencil. Each of the latter three circles inverts two of the circles forming the arbelos to the tangents to (O') perpendicular to AB, and the third circle into one tangent to (O'). See Figure 4.

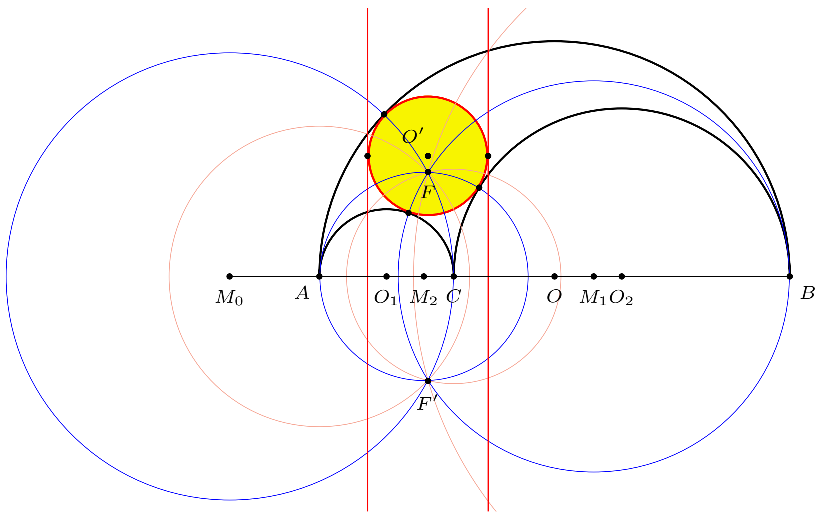


Figure 4.

We make a number of interesting observations pertaining to the construction of the Pappus chains. Denote by $P_{i,n}$ the center of the circle $\mathcal{P}_{i,n}$.

3.1. For $i = 0, 1, 2$, inversion in the midcircle (M_i) leaves $(P_{i,n})$ invariant. Consequently,

(1) the point of tangency of $(P_{i,n})$ and $(P_{i,n+1})$ lies on (M_i) and their common tangent passes through M_i;
(2) for every permuation (i, j, k) of $(0, 1, 2)$, the points of tangency of $(P_{i,n})$ with (O_j) and (O_k) are collinear with M_i. See Figure 5.

3.2. For every permutation (i, j, k) of $(0, 1, 2)$, inversion in (M_i) swaps $(P_{j,n})$ and $(P_{k,n})$. Hence,

(1) M_i, $P_{j,n}$ and $P_{k,n}$ are collinear;

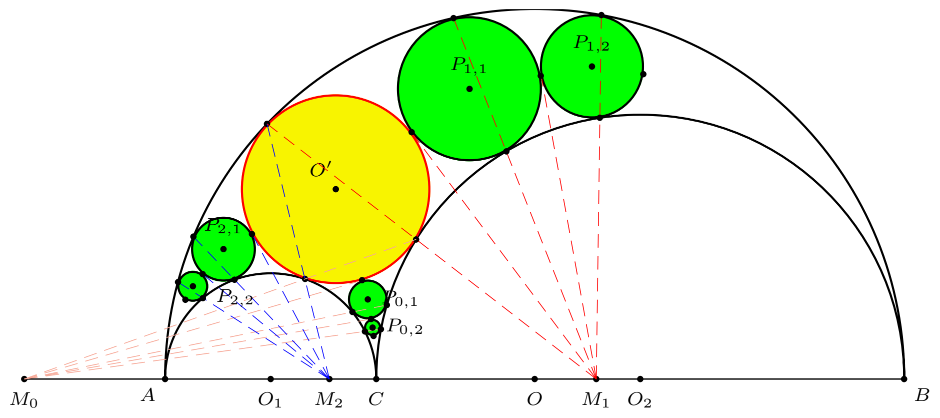


Figure 5.

(2) the points of tangency of $(P_{j,n})$ and $(P_{k,n})$ with (O_i) are collinear with M_i;

(3) the points of tangency of $(P_{j,n})$ with $(P_{j,n+1})$, and of $(P_{k,n})$ with $(P_{k,n+1})$ are collinear with M_i;

(4) the points of tangency of $(P_{j,n})$ with (O_k), and of $(P_{k,n})$ with (O_j) are collinear with M_i. See Figure 6.

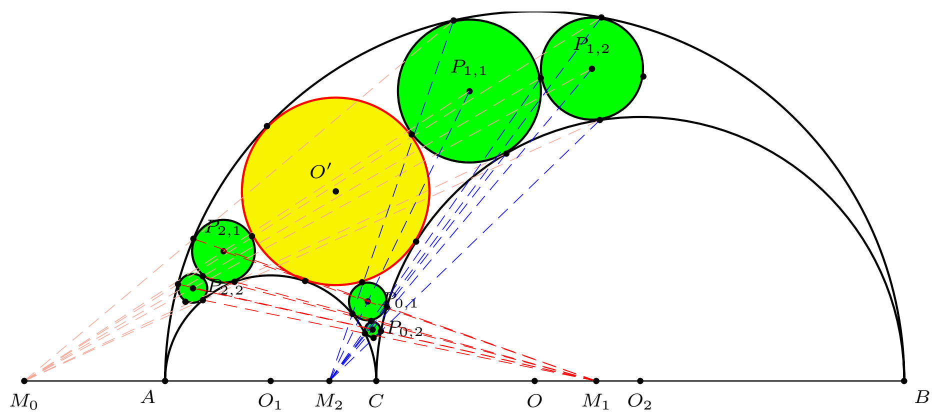


Figure 6.

3.3. Let (i, j, k) be a permutation of $(0, 1, 2)$. There is a circle $\mathcal{I}_i$ which inverts (O_j) and (O_k) respectively into the two tangents ℓ_1 and ℓ_2 of (O') perpendicular to AB. The Pappus chain $(P_{i,n})$ is inverted to a chain of congruent circles (Q_n) tangent to ℓ_1 and ℓ_2 as well, with $(Q_0) = (O')$. See Figure 7. The lines joining A to

(i) the point of tangency of (Q_n) with ℓ_1 (respectively ℓ_2) intersect $\mathcal{C}_0$ (respectively $\mathcal{C}_1$) at the points of tangency with $\mathcal{P}_{2,n}$,

(ii) the point of tangency of (Q_n) and (Q_{n-1}) intersect $\mathcal{M}_2$ at the point of tangency of $\mathcal{P}_{2,n}$ and $\mathcal{P}_{2,n-1}$.

From these points of tangency the circle $(P_{2,n})$ can be constructed.

Similarly, the lines joining B to

(iii) the point of tangency of (Q_n) with ℓ_1 (respectively ℓ_2) intersect $\mathcal{C}_2$ (respectively $\mathcal{C}_0$) at the points of tangency with $\mathcal{P}_{1,n}$,
(iv) the point of tangency of (Q_n) and (Q_{n-1}) intersect $\mathcal{M}_1$ at the point of tangency of $(P_{1,n})$ and $(P_{1,n-1})$.

From these points of tangency the circle $(P_{1,n})$ can be constructed.

Finally, the lines joining C to

(v) the point of tangency of (Q_n) with ℓ_i, $i = 1, 2$, intersect $\mathcal{C}_i$ at the points of tangency with $\mathcal{P}_{0,n}$,
(vi) the point of tangency of (Q_n) and (Q_{n-1}) intersect $\mathcal{M}_0$ at the point of tangency of $(P_{0,n})$ and $(P_{0,n-1})$.

From these points of tangency the circle $(P_{0,n})$ can be constructed.

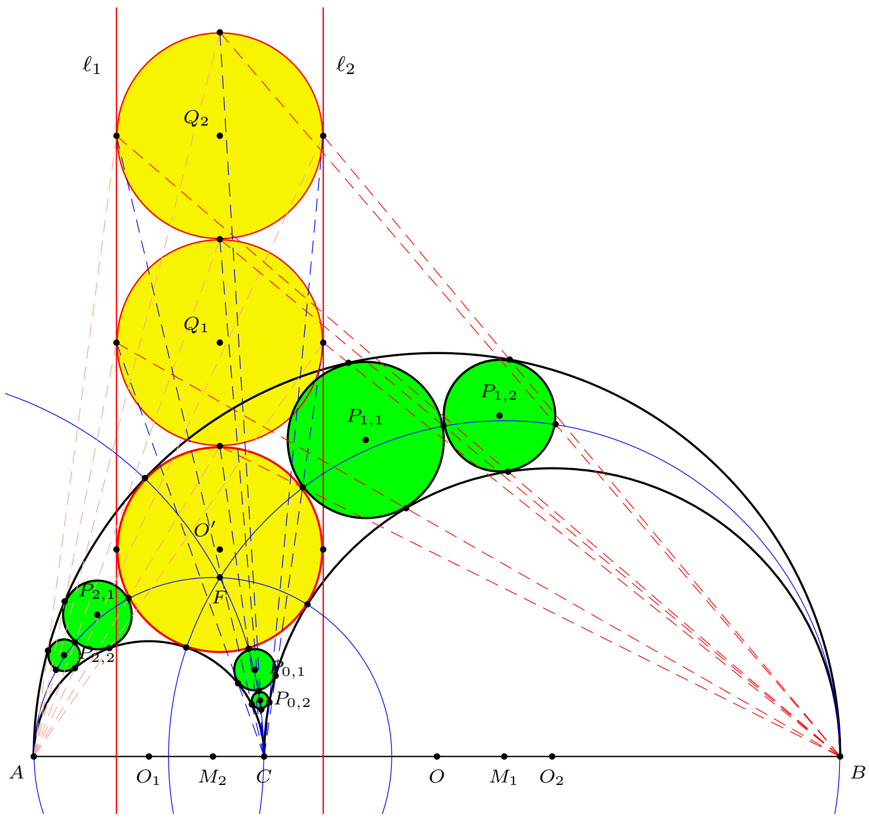


Figure 7.

3.4. Now consider the circle $\mathcal{K}_n$ through the points of tangency of $(P_{i,n})$ with (O_j) and (O_k) and orthogonal to $\mathcal{I}_i$. Then by inversion in $\mathcal{I}_i$ we see that $\mathcal{K}_n$ also passes through the points of tangency of (Q_n) with ℓ_1 and ℓ_2. Consequently the center K_n of $\mathcal{K}_n$ lies on the line through O' parallel to ℓ_1 and ℓ_2, which is the radical axis of the pencil of $\mathcal{I}_i$ and (M_i). By symmetry $\mathcal{K}_n$ passes through the points of tangency $(P_{i',n})$ with $(O_{j'})$ and $(O_{k'})$ for other permutations (i', j', k') of $(0, 1, 2)$ as well. The circle $\mathcal{K}_n$ thus passes through eight points of tangency, and all $\mathcal{K}_n$ are members of the same pencil.

With a similar reasoning the circle $\mathcal{L}_n = (L_n)$ tangent to $P_{i,n}$ and $P_{i,n+1}$ at their point of tangency as well as to (Q_n) and (Q_{n+1}) at their point of tangency, belongs to the same pencil as $\mathcal{K}_n$. See Figure 8.

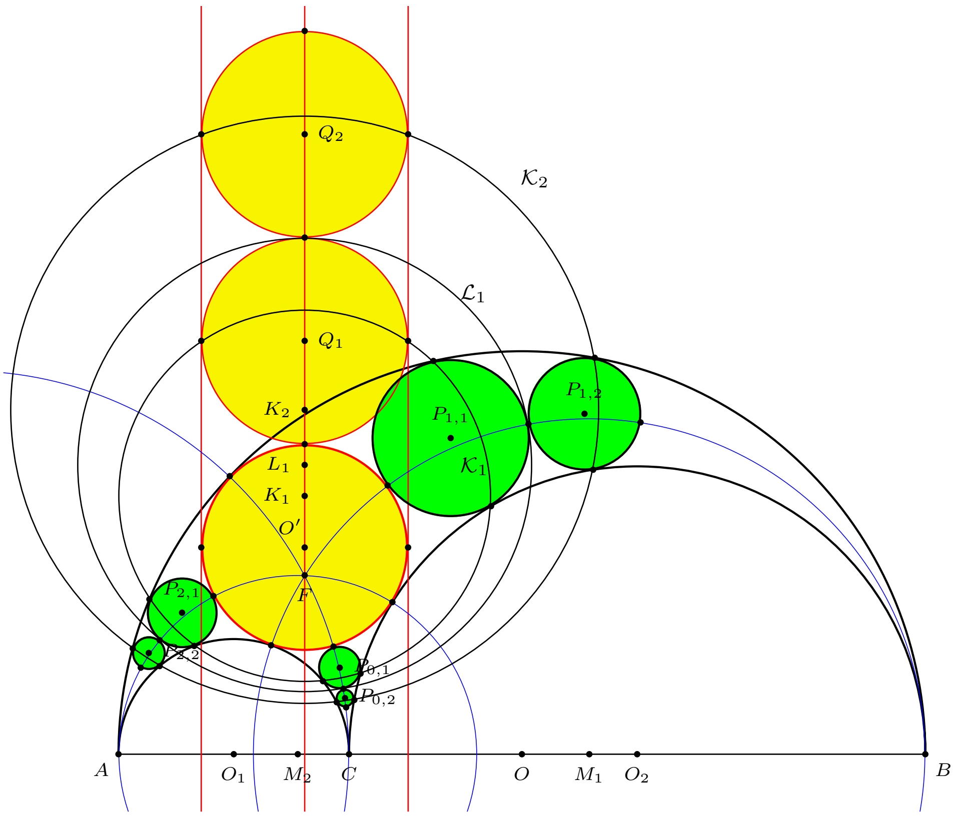


Figure 8.

The circles $\mathcal{K}_n$ and $\mathcal{L}_n$ make equal angles to the three arbelos semicircles (O), (O_1) and (O_2). In §5 we dive more deeply into circles making equal angles to three given circles.

4. λ-Archimedean circles

Recall that in the arbelos the twin circles of Archimedes have radius $r_A = \frac{r_1 r_2}{r}$. Circles congruent to these twin circles with relevant additional properties in the arbelos are called Archimedean.

Now let the homothety $\mathsf{h}(A, \mu)$ map O and O_1 to O' and O'_1. In [4] we have seen that the circle tangent to O' and O'_1 and to the line through C perpendicular to AB is Archimedean for any μ within obvious limitations. On the other hand from this we can conclude that when we apply the homothety $\mathsf{h}(A, \lambda)$ to the line through C perpendicular to AB, to find the line ℓ, then the circle tangent to ℓ, O and O' has radius λr_A. These circles are described in a different way in [6]. We call circles with radius λr_A and with additional relevant properties λ-Archimedean.

We can find a family of λ-Archimedean circles in a way similar to Bankoff's triplet circle. A proof showing that Bankoff's triplet circle is Archimedean uses the inversion in $A(B)$, that maps O and O_1 to two parallel lines perpendicular to AB, and (O_2) and the Pappus chain $(P_{2,n})$ to a chain of tangent circles enclosed by these two lines. The use of a homothety through A mapping Bankoff's triplet circle (W_3) to its inversive image shows that it is Archimedean. We can use this homothety as (W_3) circle is tangent to AB. This we know because (W_3) is invariant under inversion in (M_0), and thus intersects (M_0) orthogonally at C. In the same way we find λ-Archimedean circles.

Proposition 5. *For $i, j = 1, 2$, let $V_{i,n}$ be the point of tangency of (O_i) and $(P_{j,n})$. The circle $(CV_{1,n}V_{2,n})$ is $(n+1)$-Archimedean.*

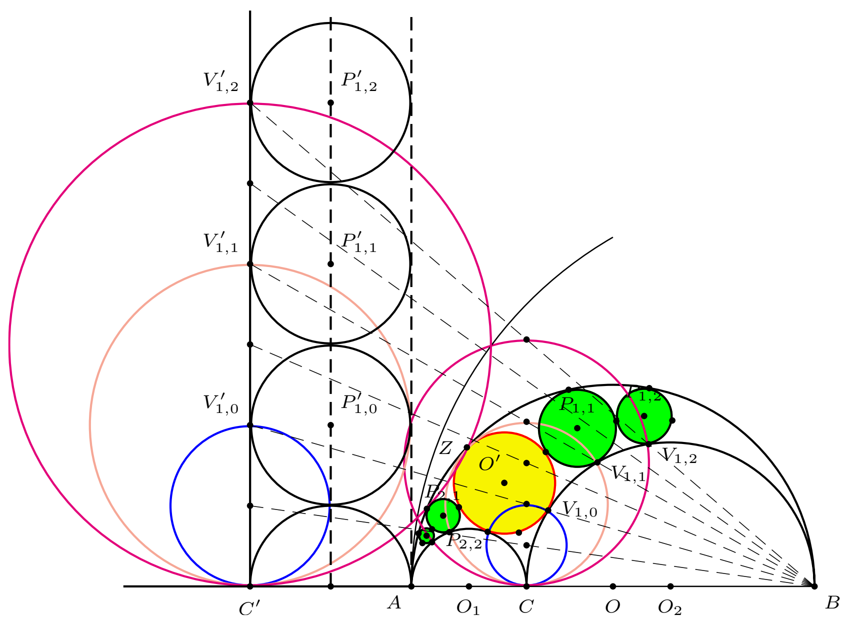


Figure 9.

A special circle of this family is $(L) = (CV_{1,1}V_{2,1})$, which tangent to (O) and (O') at their point of tangency Z, as can be easily seen from the figure after inversion. See Figure 9. We will meet again this circle in the final section.

Let $W_{1,n}$ be the point of tangency of $(P_{0,n})$ and (O_1). Similarly let $W_{2,n}$ be the point of tangency of $(P_{0,n})$ and (O_2). The circles $(CW_{1,n}W_{2,n})$ are invariant under inversion through (M_0), hence are tangent to AB. We may consider AB itself as preceding element of these circles, as we may consider (O) as $(P_{0,-1})$. Inversion through C maps $(P_{0,n})$ to a chain of tangent congruent circles tangent to two lines perpendicular to AB, and maps the circles $(CW_{1,n}W_{2,n})$ to equidistant lines parallel to AB and including AB. The diameters through C of $(CW_{1,n}W_{2,n})$ are thus, by inversion back of these equidistant lines, proportional to the harmonic sequence. See Figure 10.

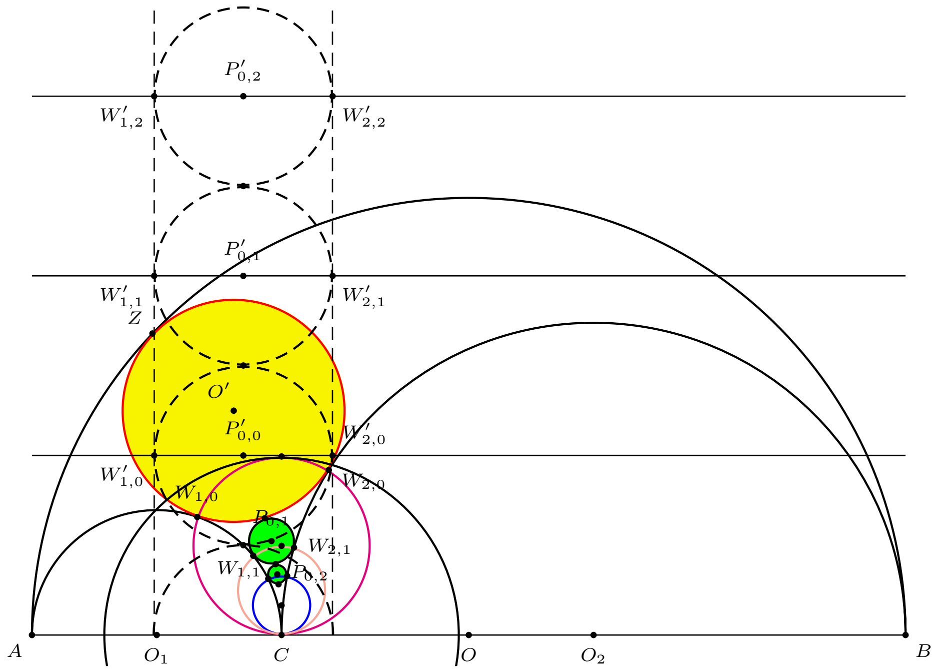


Figure 10.

Proposition 6. *The circle* $(CW_{1,n}W_{2,n})$ *is* $\frac{1}{n+1}$*-Archimedean.*

5. Inverting the arbelos to congruent circles

Let F_1 and F_2 be the intersection points of the midcircles (M_0), (M_1) and (M_2) of the arbelos. Inversion through F_i maps the circles (O), (O_1) and (O_2) to three congruent and pairwise tangent circles $(E_{i,0})$, $(E_{i,1})$ and $(E_{i,2})$. Triangle $E_{i,0}E_{i,1}E_{i,2}$ of course is equilateral, and stays homothetic independent of the power of inversion.

The inversion through F_i maps (M_0) to a straight line which we may consider as the midcircle of the two congruent circles $(E_{i,1})$ and $(E_{i,2})$. The center M'_0 of this degenerate midcircle we may consider at infinity. It follows that the line $F_iM_0 = F_iM'_0$ is parallel to the central $E_{i,1}E_{i,2}$ of these circles. Hence the lines through F_i parallel to the sides of $E_{i,1}E_{i,2}E_{i,3}$ pass through the points M_0, M_1 and M_2.

Now note that A, B, and C are mapped to the midpoints of triangle $E_{i,0}E_{i,1}E_{i,2}$, and the line AB thus to the incircle of $E_{i,0}E_{i,1}E_{i,2}$. The point F_i is thus on this circle, and from inscribed angles in this incircle we see that the directed angles (F_iA, F_iB), (F_iB, F_iC), (F_iC, F_iA) are congruent modulo π.

Proposition 7. *The points F_1 and F_2 are the Fermat-Torricelli points of degenerate triangles ABC and $M_0M_1M_2$.*

Let the diameter of (O') parallel AB meet (O') in G_1 and G_2 and Let G_1' and G_2' be their feet of the perpendicular altitudes on AB. From Pappus' theorem we know that $G_1G_2G_1'G_2'$ is a square. Construction 4 in [7] tells us that O' and its reflection through AB can be found as the Kiepert centers of base angles $\pm\arctan 2$. Multiplying all distances to AB by $\frac{\sqrt{3}}{2}$ implies that the points F_i form equilateral triangles with G_1' and G_2'. See Figure 11.

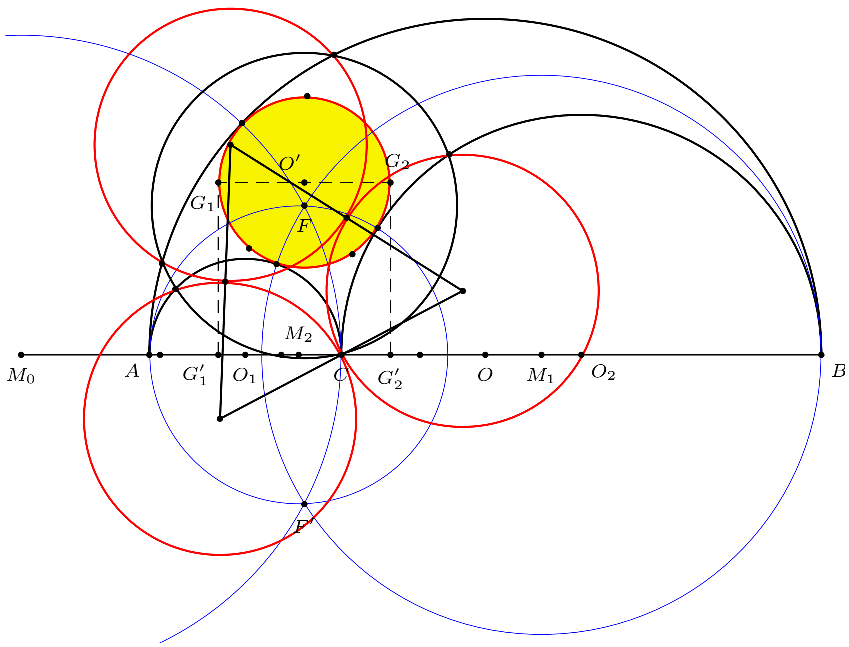


Figure 11.

A remarkable corollary of this and Proposition 7 is that the arbelos erected on $M_0M_1M_2$ shares its incircle with the original arbelos. See Figure 12.

Let F_1 be at the same side of ABC as the Arbelos semicircles. The inversion in $F_1(C)$ maps (O), (O_1) and (O_2) to three 2-Archimedean circles (E_0), (E_1) and (E_2), which can be shown with calculations, that we omit here. The 2-Archimedean circle (L) we met earlier meets (E_1) and (E_2) in their "highest" points H_1 and H_2 respectively. This leads to new Archimedean circles (E_1H_1) and (E_2H_2), which are tangent to Bankoff's triplet circle. Note that the points E_1, E_2, L, the point of tangency of (E_0) and (E_1) and the point of tangency of (E_0)

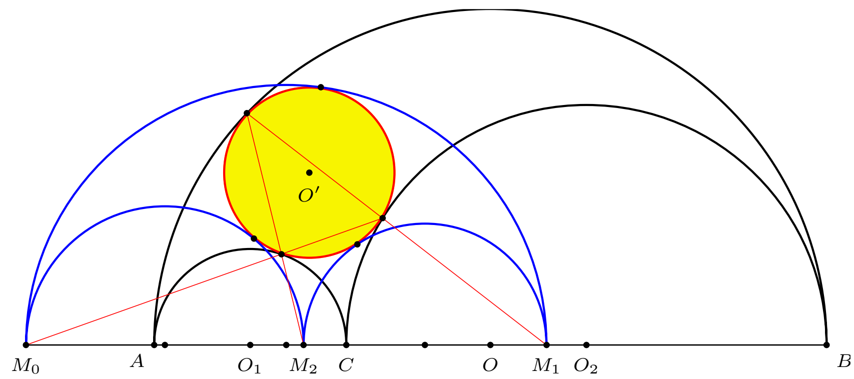


Figure 12.

and (E_2) lie on the 2-Archimedean circle with center C tangent to the common tangent of (O_1) and (O_2). See Figure 13.

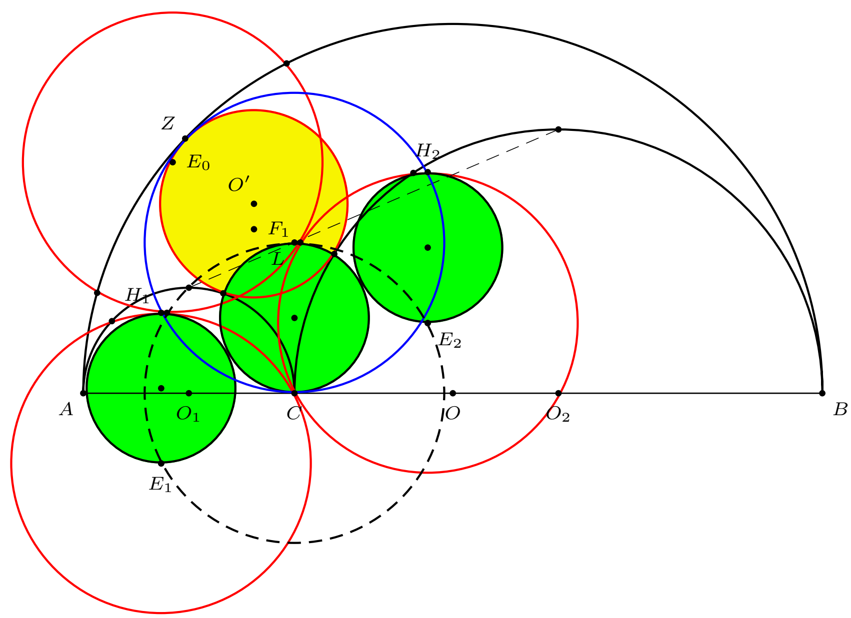


Figure 13.

References

[1] R. D. Dixon, *Mathographics*, Dover, 1991.

[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[3] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[4] F. M. van Lamoen, Archimedean adventures, *Forum Geom.*, 6 (2006) 79–96.
[5] H. Okumara and M. Watanabe, The twin circles of Archimedes in a skewed arbelos, *Forum Geom.*, 4 (2004) 229–251.
[6] H. Okumura and M. Watanabe, A generalization of Power's Archimedean circle, *Forum Geom.*, 6 (2006) 103–105.
[7] P. Y. Woo, Simple constructions of the incircle of an arbelos, *Forum Geom.*, 1 (2001) 133–136.
[8] P. Yiu, Elegant geometric constructions, *Forum Geom.*, 5 (2005) 75–96.

Eric Danneels: Hubert d'Ydewallestraat 26, 8730 Beernem, Belgium
E-mail address: `eric.danneels@pandora.be`

Floor van Lamoen: St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands
E-mail address: `fvanlamoen@planet.nl`

Forum Geometricorum
Volume 7 (2007) 67–72.

FORUM GEOM
ISSN 1534-1178

Ceva Collineations

Clark Kimberling

Abstract. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are lines. There exists a unique point U such that if $X \in \mathcal{L}_1$, then $X^{-1} \copyright U \in \mathcal{L}_2$, where X^{-1} denotes the isogonal conjugate of X and $X^{-1} \copyright U$ is the X^{-1}-Ceva conjugate of U. The mapping $X \mapsto X^{-1} \copyright U$ is the U-Ceva collineation. It maps every line onto a line and in particular maps $\mathcal{L}_1$ onto $\mathcal{L}_2$. Examples are given involving the line at infinity, the Euler line, and the Brocard axis. Collineations map cubics to cubics, and images of selected cubics under certain U-Ceva collineations are briefly considered.

1. Introduction

One of the great geometry books of the twentieth century states [1, p.221] that "Möbius's invention of homogeneous coordinates was one of the most far-reaching ideas in the history of mathematics". In triangle geometry, two systems of homogeneous coordinates are in common use: barycentric and trilinear. Trilinears are especially useful when the angle bisectors of a reference triangle ABC play a central role, as in this note.

Suppose that $X = x : y : z$ is a point. If at most one of x, y, z is 0, then the point

$$X^{-1} = yz : zx : xy$$

is the isogonal conjugate of X, and if none of x, y, z is 0, we can write

$$X^{-1} = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

A traditional construction for X^{-1} depends on interior angle bisectors: reflect line AX in the A-bisector, BX in the B-bisector, CX in the C-bisector; then the reflected lines concur in X^{-1}.

The triangle $A_X B_X C_X$ with vertices

$$A_X = AX \cap BC, \quad B_X = BX \cap CA, \quad C_X = CX \cap AB$$

is the *cevian triangle of* X, and

$$A_X = 0 : y : z, \quad B_X = x : 0 : z, \quad C_X = x : y : 0.$$

If $U = u : v : w$ is a point, then the triangle $A^U B^U C^U$ with vertices

$$A^U = -u : v : w, \quad B^U = u : -v : w, \quad C^U = u : v : -w$$

Publication Date: February 26, 2007. Communicating Editor: Paul Yiu.

is the *anticevian triangle* of U. The lines A_XA^U, B_XB^U, C_XC^U concur in the point

$$u(-uyz+vzx+wxy) : \ v(uyz-vzx+wxy) : \ w(uyz+vzx-wyz),$$

called the X-*Ceva conjugate* of U and denoted by XⓒU (see [2, p. 57]). It is easy to verify algebraically that Xⓒ$(X$ⓒ$U) = U$ and that if $P = p : q : r$ is a point, then the equation $P = X$ⓒU is equivalent to

$$\begin{aligned} X =&(ru+pw)(pv+qu) : (pv+qu)(qw+rv) : (qw+rv)(ru+pw) \\ =&\text{cevapoint}(P,U). \end{aligned} \tag{1}$$

A construction of cevapoint (P,U) is given in the Glossary of [3].

One more preliminary will be needed. A *circumconic* is a conic that passes through the vertices, A, B, C. Every point $P = p : q : r$, where $pqr \neq 0$, has its own circumconic, given by the equation $p\beta\gamma + q\gamma\alpha + r\alpha\beta = 0$; indeed, this curve is, loosely speaking, the isogonal conjugate of the line $p\alpha + q\beta + r\gamma = 0$, and the curve is an ellipse, parabola, or hyperbola according as the line meets the circumcircle in 0, 1, or 2 points. The circumcircle is the circumconic having equation $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$.

2. The Mapping $X \mapsto X^{-1}$ⓒU

In this section, we present first a lemma: that for given circumconic $\mathcal{P}$ and line $\mathcal{L}$, there is a point U such that the mapping $X \mapsto X$ⓒU takes each point X on $\mathcal{P}$ to a point on $\mathcal{L}$. The lemma easily implies the main theorem of the paper: that the mapping $X \mapsto X^{-1}$ⓒU takes each point of a certain line to $\mathcal{L}$.

Lemma 1. *Suppose $L = l : m : n$ and $P = p : q : r$ are points. Let $\mathcal{P}$ denote the circumconic $p\beta\gamma + q\gamma\alpha + r\alpha\beta = 0$ and $\mathcal{L}$ the line $l\alpha + m\beta + n\gamma = 0$. There exists a unique point U such that if $X \in \mathcal{P}$, then Xⓒ$U \in \mathcal{L}$. In fact,*

$$U = L^{-1}ⓒP = p(-lp+mq+nr) : q(lp-mq+nr) : r(lp+mq-nr).$$

Proof. We wish to solve the containment Xⓒ$U \in \mathcal{L}$ for U, given that $X \in \mathcal{P}$. That is, we seek $u : v : w$ such that

$$u(-uyz+vzx+wxy)l+v(uyz-vzx+wxy)m+w(uyz+vzx-wxy)n = 0, \tag{2}$$

given that $X = x : y : z$ is a point satisfying

$$pyz + qzx + qxy = 0. \tag{3}$$

Equation (2) is equivalent to

$$u(-ul+vm+wn)yz + v(ul-vm+wn)zx + w(ul+vm-wn)xy = 0, \tag{4}$$

so that, treating $x : y : z$ as a variable point, equations (3) and (4) represent the same circumconic. Consequently,

$$u(-lu+mv+nw)qr = v(lu-mv+nw)rp = w(lu+mv-nw)pq.$$

In order to solve for $u:v:w$, we assume, as a first of two cases, that p and q are not both 0. Then the equation

$$u(-lu+mv+nw)qr = v(lu-mv+nw)rp$$

gives

$$w = \frac{(mv-lu)\,(pv+qu)}{n(pv-qu)}. \tag{5}$$

Substituting for w in

$$u(-lu+mv+nw)qr - w(lu+mv-nw)pq = 0$$

gives

$$\frac{\left(mpqv - lpqu + nprv - nqru - lp^2v + mq^2u\right)(mv-lu)\,uv}{2nr(pv-qu)^2} = 0,$$

so that

$$u = \frac{(mq-lp+nr)\,pv}{q(lp-mq+nr)}. \tag{6}$$

Consequently, for given v, we have

$$u:v:w = \frac{(mq-lp+nr)\,pv}{q(lp-mq+nr)} : v : \frac{(mv-lu)\,(pv+qu)}{n(pv-qu)}.$$

Substituting for u from (6), canceling v, and simplifying lead to

$$u:v:w = p\,(-lp+mq+nr) : q\,(lp-mq+nr) : r\,(lp+mq-nr)\,,$$

so that $U = L^{-1}\copyright P$.

If, as the second case, we have $p = q = 0$, then $r \neq 0$ because $p:q:r$ is assumed to be a point. In this case, one can start with

$$u(-lu+mv+nw)qr = w(lu+mv-nw)pq$$

and solve for v (instead of w as in (5)) and continue as above to obtain $U = L^{-1}\copyright P$.

The method of proof shows that the point U is unique. $\square$

Theorem 2. *Suppose $\mathcal{L}_1$ is the line $l_1\alpha + m_1\beta + n_1\gamma = 0$ and $\mathcal{L}_2$ is the line $l_2\alpha + m_2\beta + n_2\gamma = 0$. There exists a unique point U such that if $X \in \mathcal{L}_1$, then $X^{-1}\copyright U \in \mathcal{L}_2$.*

Proof. The hypothesis that $X \in \mathcal{L}_1$ is equivalent to $X^{-1} \in \mathcal{P}$, the circumconic having equation $l_1\beta\gamma + m_1\gamma\alpha + n_1\alpha\beta = 0$. Therefore, the lemma applies to the circumconic $\mathcal{P}$ and the line $\mathcal{L}_2$. $\square$

We write the mapping $X \mapsto X^{-1}\copyright U$ as $\mathcal{C}_U(X) = X^{-1}\copyright U$ and call $\mathcal{C}_U$ the U-Ceva collineation. That $\mathcal{C}_U$ is indeed a collineation follows as in [4] from the linearity of x, y, z in the trilinears

$$\mathcal{C}_U(X) = u(-ux+vy+wz) :\ v(ux-vy+wz) :\ w(ux+vy-wz).$$

This collineation is determined by its action on the four points A, B, C, U^{-1}, with respective images A^U, B^U, C^U, U.

Regarding the surjectivity, or onto-ness, of $\mathcal{C}^U$, suppose F is a point on $\mathcal{L}_2$; then the equation $X^{-1}ⓒU = F$ has as solution

$$X = \text{cevapoint}(F, U))^{-1}.$$

3. Corollaries

Lemma 1 tells how to find U for given $\mathcal{L}$ and $\mathcal{P}$. Here, we tell how to find $\mathcal{L}$ from given $\mathcal{P}$ and U and how to find $\mathcal{P}$ from given U and $\mathcal{L}$.

Corollary 3. *Given a circumconic $\mathcal{P}$ and a point U, there exists a line $\mathcal{L}$ such that if $X \in \mathcal{P}$, then $XⓒU \in \mathcal{L}$.*

Proof. Assuming there is such a $\mathcal{L}$, we have the point $U = L^{-1}ⓒP$ as Theorem 2, so that $L^{-1} = \text{cevapoint}(U, P)$, and

$$L = (\text{cevapoint}(U, P))^{-1},$$

so that $\mathcal{L}$ is the line $(wq + vr)\alpha + (ur + wp)\beta + (vp + uq)\gamma = 0$. It is easy to check that if $X \in \mathcal{P}$, then $XⓒU \in \mathcal{L}$. □

Corollary 4. *Given a line $\mathcal{L}$ and a point U, there exists a circumconic $\mathcal{P}$ such that if $X \in \mathcal{P}$, then $XⓒU \in \mathcal{L}$.*

Proof. Assuming there is such a $\mathcal{L}$, we have the point $U = L^{-1}ⓒP$, and $P = L^{-1}ⓒU$, so that $\mathcal{P}$ is the circumconic

$$u(-ul + vm + wn)\beta\gamma + v(ul - vm + wn)\gamma\alpha + w(ul + vm - wn)\alpha\beta = 0.$$

It is easy to check that if $X \in \mathcal{P}$, then $XⓒU \in \mathcal{L}$. □

4. Examples

4.1. Let $L = P = 1 : 1 : 1$, so that $\mathcal{L}_1 = \mathcal{L}_2$ is the line $\alpha + \beta + \gamma = 1$. We find $U = 1 : 1 : 1$, so that

$$\mathcal{C}_U(X) = -x + y + z : \ x - y + z : \ x + y - z.$$

It is easy to check that $\mathcal{C}_U(X) = X$ for every X on the line $\alpha + \beta + \gamma = 1$, such as X_{44} and X_{513}. On the line X_1X_2 we have

$$\mathcal{C}_U(X) = X \text{ for } X \in \{X_1, X_{899}\},$$

so that $\mathcal{C}_U$ maps X_1X_2 onto itself; e.g., $\mathcal{C}_U(X_2) = X_{43}$, and $\mathcal{C}_U(X_{1201}) = X_8$, and $\mathcal{C}_U(X_8) = X_{972}$. On X_1X_6 we have fixed points X_1 and X_{44}, so that $\mathcal{C}_U$ maps the line X_1X_{44} to itself. Abbreviating $\mathcal{C}_U(X_i) = X_j$ as $X_i \mapsto X_j$, we have, among points on X_1X_{44},

$$X_{1100} \mapsto X_{37} \mapsto X_6 \mapsto X_9 \mapsto X_{1743}.$$

The Euler line, X_2X_3, is a link in a chain as indicated by

$$\cdots \mapsto X_{42}X_{65} \mapsto X_2X_3 \mapsto X_{43}X_{46} \mapsto \cdots$$

4.2. Let $L = L_1 = X_6 = a : b : c$, so that $\mathcal{L}_1$ is the line at infinity and $\mathcal{P}$ is the circumcircle. Let $\mathcal{L}_2$ be the Euler line, given by taking L_2 in the statement of the theorem to be

$$X_{647} = a(b^2-c^2)(b^2+c^2-a^2) : b(c^2-a^2)(c^2+a^2-b^2) : c(a^2-b^2)(a^2+b^2-c^2).$$

The Ceva collineation $\mathcal{C}_U$ that maps $\mathcal{L}_1$ onto $\mathcal{L}_2$ is given by

$$\begin{aligned} U = X_{523} =& a(b^2-c^2) : b(c^2-a^2) : c(a^2-b^2) \\ =& \sin(B-C) : \sin(C-A) : \sin(A-B), \end{aligned}$$

and we find

$$\begin{array}{lll} X_{512} \mapsto X_2, & X_{520} \mapsto X_4, & X_{523} \mapsto X_5, \\ X_{526} \mapsto X_{30}, & X_{2574} \mapsto X_{1312}, & X_{2575} \mapsto X_{1313}. \end{array}$$

The penultimate of these, namely $X_{2574} \mapsto X_{1312}$, is of particular interest, as $X_{2574} = X_{1113}^{-1}$, where X_{1113} is a point of intersection of the Euler line and the circumcircle and X_{1312} is a point of intersection of the Euler line and the nine-point circle; and similarly for $X_{2575} \mapsto X_{1313}$. The mapping $\mathcal{C}_U$ carries the Brocard axis, X_3X_6 onto the line $X_{115}X_{125}$, where X_{115} and X_{125} are the centers of the Kiepert and Jerabek hyperbolas, respectively.

4.3. Let $L_1 = X_{523}$, so that $\mathcal{L}_1$ is the Brocard axis, X_3X_6, and let $\mathcal{L}_2$ be the Euler line, X_2X_3. Then $U = X_6 = a : b : c$. The mapping of $\mathcal{L}_1$ to $\mathcal{L}_2$ is a link in a chain:

$$\cdots \mapsto X_2X_{39} \mapsto X_2X_6 \mapsto X_3X_6 \mapsto X_2X_3 \mapsto X_6X_{25} \mapsto X_3X_{66} \mapsto \cdots$$

4.4. Here, we reverse the roles played by the Brocard axis and Euler line in Example 3: let $\mathcal{L}_1$ be the Euler line and $\mathcal{L}_2$ be the Brocard axis. Then $U = X_{184} = a^2 \cos A : b^2 \cos B : c^2 \cos C$. A few images of the X_{184}-Ceva collineation are given here:

$$\begin{array}{lll} X_2 \mapsto X_{32}, & X_3 \mapsto X_{571}, & X_4 \mapsto X_{577}, \\ X_5 \mapsto X_6, & X_{30} \mapsto X_{50}, & X_{427} \mapsto X_3. \end{array}$$

4.5. Let $\mathcal{L}_1 = \mathcal{L}_2 =$ Brocard axis. Here,

$$U = X_5 = \cos(B-C) : \cos(C-A) : \cos(A-B),$$

the center of the nine-point circle, and

$$X_{389} \mapsto X_3 \mapsto X_{52} \quad \text{and} \quad X_{570} \mapsto X_6 \mapsto X_{216}.$$

4.6. Let $\mathcal{L}_1 = \mathcal{L}_2 =$ the line at infinity, $X_{30}X_{511}$. Here,

$$U = X_3 = \cos A : \cos B : \cos C,$$

the circumcenter. Among line-to-line images under X_3-collineation are these:

$$\begin{aligned} &X_4X_{51} \mapsto \text{Euler line} \mapsto X_3X_{49}, \\ &X_6X_{64} \mapsto X_4X_6 \mapsto \text{Brocard axis} \mapsto X_6X_{155}. \end{aligned}$$

5. Cubics

Collineations map cubics to cubics (e.g. [4, p. 23]). In particular, a U-Ceva collineation maps a cubic Λ that passes through the vertices A, B, C to a cubic $\mathcal{C}_U(\Lambda)$ that passes through the vertices A^U, B^U, C^U of the anticevian triangle of U.

5.1. Let $U = X_1$, as in §4.1, and let Λ be the Thompson cubic, $Z(X_2, X_1)$, with equation

$$bc\alpha(\beta^2 - \gamma^2) + ca\beta(\gamma^2 - \alpha^2) + ab\gamma(\alpha^2 - \beta^2) = 0.$$

Then $\mathcal{C}_U(\Lambda)$ circumscribes the excentral triangle, and for selected X_i on Λ, the image $\mathcal{C}_U(X_i)$ is as shown here:

X_i	1	2	3	4	6	9	57	223
$\mathcal{C}_U(X_i)$	1	43	46	1745	9	1743	165	1750

5.2. Let $U = X_1$, and let Λ be the cubic $Z(X_1, X_{75})$, with equation

$$\alpha(c^2\beta^2 - b^2\gamma^2) + \beta(a^2\gamma^2 - c^2\alpha^2) + \gamma(b^2\alpha^2 - a^2\beta^2) = 0.$$

For selected X_i on Λ, the image $\mathcal{C}_U(X_i)$ is as shown here:

X_i	1	6	19	31	48	55	56	204	221
$\mathcal{C}_U(X_i)$	1	9	610	63	19	57	40	2184	84

5.3. Let $U = X_6$, as in §4.3, and let Λ be the Thompson cubic. Then $\mathcal{C}_U(\Lambda)$ circumscribes the tangential triangle, and for selected X_i on Λ, the image $\mathcal{C}_U(X_i)$ is as shown here:

X_i	1	2	3	4	6	9	57	223	282	1073	1249
$\mathcal{C}_U(X_i)$	55	6	25	154	3	56	198	1436	1035	1033	64

References

[1] H. S. M. Coxeter, *Introduction to Geometry*, second edition, John Wiley & Sons, New York, 1969.

[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[3] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[4] C. Kimberling, Collineations, conjugacies, and cubics, *Forum Geom.*, 2 (2002) 21–32.

Clark Kimberling: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, USA

E-mail address: ck6@evansville.edu

Forum Geometricorum
Volume 7 (2007) 73–86.

FORUM GEOM
ISSN 1534-1178

Orthocycles, Bicentrics, and Orthodiagonals

Paris Pamfilos

Abstract. We study configurations involving a circle (orthocycle) intimately related to a cyclic quadrilateral. As an illustration of the usefulness of this circle we explore its connexions with bicentric (bicentrics) and orthodiagonal quadrilaterals (orthodiagonals) reviewing the more or less known facts and revealing some other properties of these classes of quadrilaterals.

1. Introduction

Consider a generic convex cyclic quadrilateral $q = ABCD$ inscribed in the circle $k(K, r)$ and having finite intersection points F, G of opposite sides. Line $e = FG$ is the polar of the intersection point E of the diagonals AC, BD. The circle c with diameter FG is orthogonal to k. Also, the midpoints X, Y of the diagonals and the center H of c are collinear. We call c the **orthocycle** of the cyclic quadrilateral q. Consider also the circle f with diameter EK. This is the

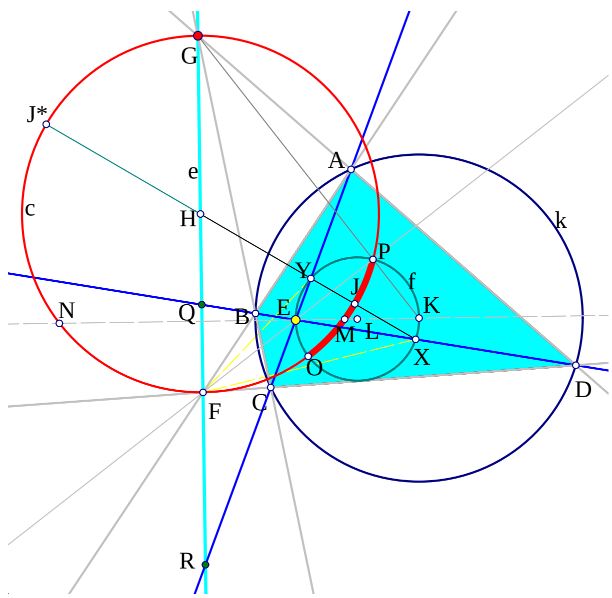


Figure 1. The orthocycle c of the cyclic quadrilateral $ABCD$

locus of the midpoints of chords of k passing through E. It is also the inverse of e with respect to k and is orthogonal to c. Thus, c belongs to the circle-bundle $\mathscr{C}'$, which is orthogonal to the bundle $\mathscr{C}(k, f)$ generated by k and f. The bundle $\mathscr{C}$ is of non intersecting type with limit points M, N, symmetric with respect to

Publication Date: March 12, 2007. Communicating Editor: Paul Yiu.

e, and $\mathscr{C}'$ is a bundle of intersecting type, all of whose members pass through M and N. If we fix the data (k, E, c), then all cyclic quadrilaterals q having these as *circumcircle, diagonals-intersection-point, orthocycle* respectively form a one-parameter family. A member q of this family is uniquely determined by a point J on the circular arc (OMP) of the orthocycle c. Thus the set of all q inscribed in the circle k and having diagonals through E is parameterized through pairs (c, J), c (the orthocycle) being a circle of bundle $\mathscr{C}'$ and J a point on the corresponding arc (OMP) intercepted on the orthocycle by f. In the following sections we consider these facts more closely and investigate (i) the bicentrics inscribed in k, and (ii) a certain 1-1 correspondence of cyclics to orthodiagonals in which the orthocycle plays an essential role.

Regarding the proofs of the statements made, everything (is or) follows immediately from standard, well known material. In fact, the statement on the polar relies on its usual construction from two intersecting chords ([3, p.103]). The statement on the collinearity follows from Newton's theorem on a complete quadrilateral ([3, p.62]). From the harmonic ratios appearing in complete quadrilaterals follows also that the intersection points Q, R of the diagonals with line e divide F, G harmonically. Consequently the circle with diameter QR is also orthogonal to c ([2, §1237]). The orthogonality of c, k follows from the fact that PF is the polar of G, which implies that P, G are inverse with respect to k. Besides, by measuring angles at P, circles f, c are shown to be orthogonal. The statement on the parametrization is analyzed in the following section.

The orthocycle gives a means to establish unity in apparently unrelated properties. For example the well known formula

$$\frac{1}{r^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}$$

is proved to be, essentially, a case of *Stewart's formula* (see next paragraph).

Furthermore, the orthogonality of c to f can be used to characterize the cyclics. To formulate the characterization we consider more general the *orthocycle* of a generic convex quadrilateral to be the circle on the diameter defined by the two intersection points of its pairs of opposite sides.

Proposition 1. *The quadrilateral $q = ABCD$ is cyclic if and only if its orthocycle c is orthogonal to the circle f passing through the midpoints X, Y of its diagonals and their intersection point E.*

Proof. If q is cyclic, then we have already seen that its orthocycle c belongs to the bundle $\mathscr{C}'$ which is orthogonal to the one generated by its circumcircle k and the circle f passing through the diagonal midpoints and their intersection point.

Conversely, if the orthocycle c and f intersect orthogonally, then Y and X are inverse with respect to c. Since the same is true with the intersection points R, Q of the diagonals of q with line FG (see Figure 2), there is a circle a passing through the four points X, Y, R and Q. Then we have

$$|ER| \cdot |EX| = |EY| \cdot |EQ|. \tag{1}$$

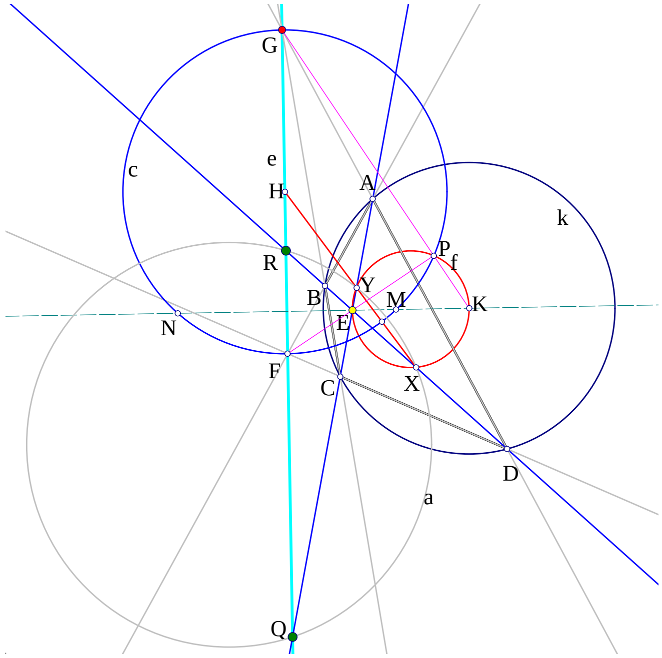


Figure 2. Cyclic characterization

But from the general properties of the complete quadrilaterals we have also that $(Q, E, C, A) = -1$ is a harmonic division, hence

$$|EC| \cdot |EA| = |EQ| \cdot |EY|. \tag{2}$$

Analogously, $(R, E, B, D) = -1$ implies

$$|EB| \cdot |ED| = |ER| \cdot |EX|. \tag{3}$$

Relations (1) to (3) imply that $|EB| \cdot |ED| = |EC| \cdot |EA|$, proving the proposition. $\square$

For a classical treatment of the properties discussed below see Chapter 10 of Paul Yiu's Geometry Notes [5]. Zaslavsky (see [6], [1]) uses the term *orthodiagonal* for a map between quadrangles and gives characterizations of cyclics in another context than the one discussed below.

2. Bicentrics

Denote by (k, E, c) the family of quadrilaterals characterized by these elements (*circumcircle*, *diagonal-intersection-point*, *orthocycle*) correspondingly. Referring to Figure 1 we have the following properties ([2, §§674, 675, 1276]).

Proposition 2. (1) *There is a 1-1 correspondance between the members of the family* (k, E, c) *and the points* J *of the open arc* (OMP) *of circle* c.

(2) *Let* X, Y *be the intersection points of* f *with line* HJ. X, Y *are the midpoints of the diagonals of* q *and are inverse with respect to* c.

(3) *Then FJ bisects angles AFD and XFY. Analogously, GJ bisects angles BGD and XGY.*

Proof. In fact, from the Introduction, it is plain that each member q of the family (k, E, c) defines a J as required. Conversely, a point J on arc (OMP) of c defines two intersection points X, Y of HJ with f, which are inverse with respect to c, since f and c are orthogonal. The chords EX and EY define the cyclic $q = ABCD$, having these as diagonals and X, Y as the midpoints of these diagonals. Consider the orthocycle c' of this q. By the analysis made in the Introduction, c' belongs to the bundle $\mathscr{C}'$ and is also orthogonal to the circle with diameter QR. Thus c' is uniquely defined by the chords XE, YE and must coincide with c. This proves (1).

(2) is already discussed in the Introduction.

(3) follows from the orthogonality of circles k, c. In fact, this implies that J, J^* divide X, Y harmonically. Then (FJ^*, FJ, FY, FX) is a harmonic bundle of lines and FJ^*, FJ are orthogonal. Hence, they bisect $\angle XFY$. They also bisect $\angle BFC$. This follows immediately from the similarity of triangles AFC and DFB. Analogous is the situation with the angles at G. □

Referring to figure 1, denote by $q(c)$ the particular quadrilateral of the family (k, E, c), constructed with the recipe of the previous proposition, for $J \equiv M$. The following two lemmas imply that $q(c)$ is bicentric.

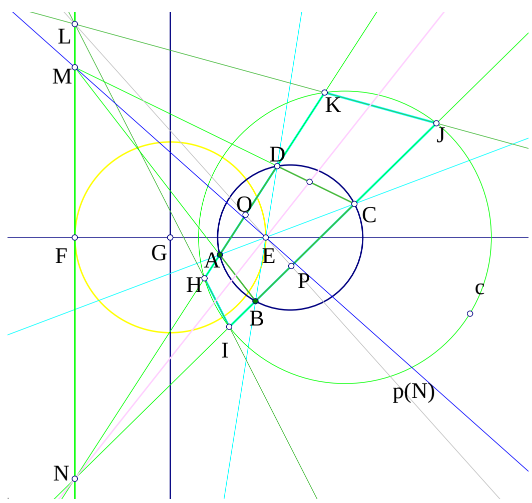


Figure 3. Bundle quadrilaterals

Lemma 3. *Consider a circle bundle of non intersecting type and two chords of a member circle passing through the limit point E of the bundle* (see Figure 3)*. The chords define a quadrilateral $q = ABCD$ having these as diagonals. Extend two opposite sides AD, BC until they intersect a second circle member c of the bundle. The intersection points form a quadrilateral $r = HIJK$. Then the intersection* *point L of the sides HI, JK lies on the polar MN of E with respect to a circle of the bundle (all circles c of the bundle have the same polar with respect to E).*

Indeed, N, M can be taken as the intersection points of opposite sides of q. Then N is on the polar of E, hence the polar $p(N)$ of N contains E. Consider the intersection points O, P of this polar with sides HK, IJ respectively. Then,
(a) these sides intersect at a point L lying on $p(N)$,
(b) L is also on line MN.

(a) follows from the standard theorem on cyclic quadrilaterals.

(b) follows from the fact that the quadruple of lines (NL, NH, NE, NI) at N is harmonic. But (NM, NH, NE, NI) is also harmonic, hence L is contained in line MN.

Lemma 4. *q(c) is bicentric.*

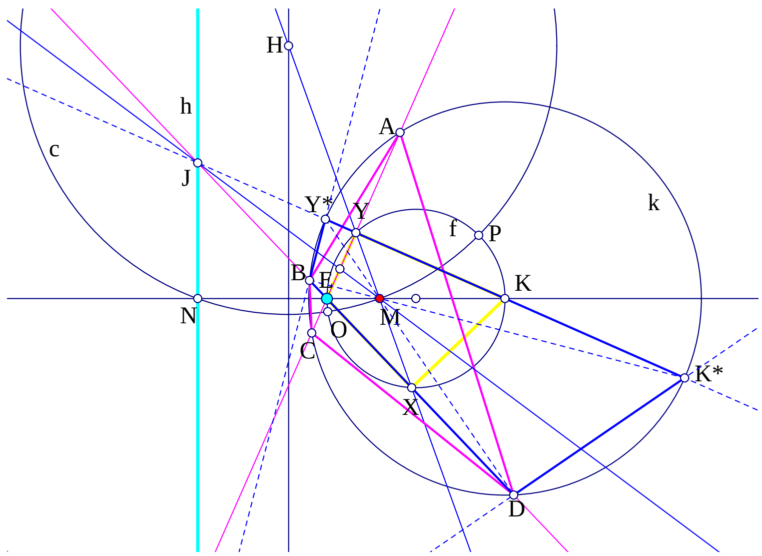


Figure 4. Bisectors of $q(c)$

Indeed, by Proposition 2 the bisectors of angles $\angle AGB, \angle BFC$ will intersect at M. It suffices to show that the bisectors of two opposite angles of $q(c)$ intersect also at M. Let us show that the bisector of angle $\angle ABC$ passes through M (Figure 4). We start with the quadrilateral $q_1 = EXKY$. Its diagonals intersect at M. According to the previous lemma the extensions of its sides will define a quadrilateral $q_2 = BDK^*Y^*$ inscribed in k and having its opposite sides intersecting on line h the common polar of E with respect to every member circle of the bundle I. This implies that the diagonals of q_2 intersect at the pole E of h. But BK^* joins B to the middle K^* of the arc (CK^*B), hence is the bisector of angle $\angle ABC$ and passes through M.

Proposition 5. (1) *There is a unique member $q = q(c) = ABCD$, of the family (k, E, c) which is bicentric. The corresponding J is the limit point M of bundle I contained in the circle k.*

(2) *There is a unique member $o = o(c) = A^*B^*C^*D^*$, of the family (k, E, c) which is orthodiagonal. The corresponding HJ passes through the center L of the circle f.*

(3) *For every bicentric the incenter M is on the line joining the intersection point of the diagonals with the circumcenter.*

(4) *For every bicentric the incenter M is on the line joining the midpoints of the diagonals.*

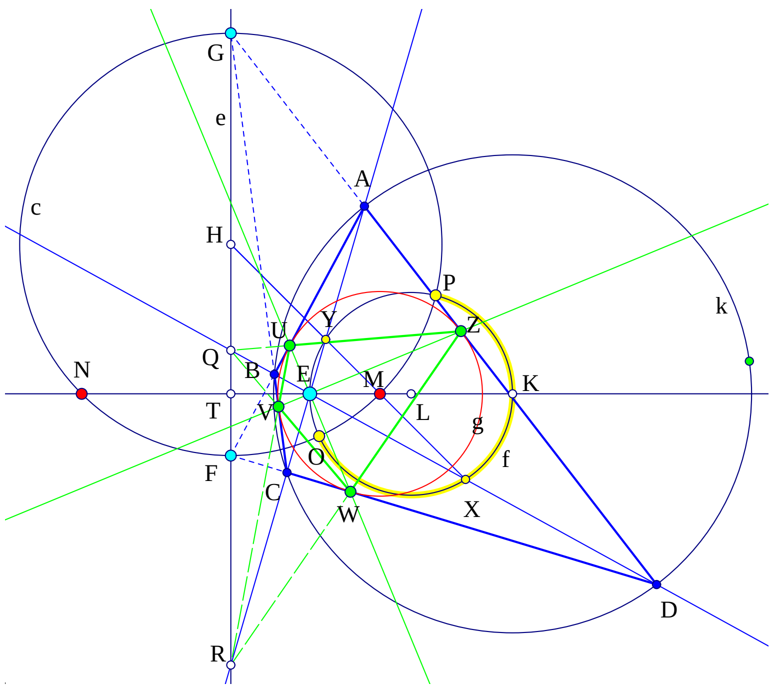


Figure 5. The bicentric member in (k, E, c)

Proof. In fact, by the previous lemmas we know that $q(c)$ is bicentric. To prove the uniqueness we assume that $q = ABCD$ is bicircular and consider the incircle g and the tangential quadrilateral $q' = UVWZ$. From Brianchon's theorem the diagonals of q' intersect also at E. Thus, the poles of the diagonals UW, VZ being correspondingly F, G, line e will be also polar of E with respect to g. In particular the center of g will be on line MN and the pairs of opposite sides of the tangential q' will intersect on e at points Q, R say. The diagonals of q pass through Q, R respectively. In fact, D being the pole of line WZ and B the pole of UV, BD is the polar of R with respect to g. By the standard construction of the polar it follows that Q is on BD. Analogously R is on AC. The center of g will be the intersection M' of the bisectors of angles BGA and BFC. By measuring the angles at M' we find easily that the bisectors form there a right angle. Thus, M' will be on the orthocycle c, hence, being also on line MN, it will coincide with M. In that case line HYX passes through M. This follows from proposition 1 which identifies the bisector of angle YFX with FM. This proves (1).

To prove (2) consider the quadrangle $s = EXKY$. If the diagonals intersect orthogonally then s is a rectangle. Consequently XY is a diameter of f and passes through L. The converse is also valid. If XY passes through L then s is a rectangle and o is orthodiagonal.

The other statements are immediate consequences. Notice that property 2 holds more generally for every circumscriptible quadrilateral ([2, §1614]). $\square$

Proposition 6. *Consider all tripples (k, E, c) with fixed k, E and c running through the members of the circle bundle $\mathscr{C}'$. Denote by $q(c)$ the bicentric member of the corresponding family (k, E, c) and by $q' = UVWZ$ the tangential quadrangle of $q(c)$* (see Figure 5)*. The following statements are consequences of the previous considerations:*

(1) *All tangential quadrilaterals $q' = UVWZ$ are orthodiagonal, the diagonals being each time parallel to the bisectors of angles $\angle BGA, \angle BFC$.*

(2) *The pairs of opposite sides or q' intersect at the points Q, R, which are the intersection points of the diagonals of $q(c)$ with e.*

(3) *The orthocycle c' of the tangential q' is the circle on the diameter QR and intersects the incircle g of $q(c)$ orthogonally. The radius r_g of the incircle satisfies $r_g^2 = |ME||MT|$.*

(4) *The bicentrics $\{q(c) : c \in \mathscr{C}'\}$ are precisely the inscribed in circle k and having their diagonals pass through E. They, all, have the same incircle g, depending only on k and E.*

(5) *The radii r_g of the inscribed circle g, r of circumscribed k, and the distance $d = |MK|$ of their centers satisfy the relation* $\dfrac{1}{r_g^2} = \dfrac{1}{(r+d)^2} + \dfrac{1}{(r-d)^2}$.

Proof. (1) follows from the fact that UV is orthogonal to the bisector FM of angle BFC. Analogously VZ is orthogonal to GM and FM, GM are orthogonal ([2, §674]).

(2) follows from the standard construction of the polar of E with respect to g. Thus e is also the polar of E with respect to the incircle g ([2, §1274]).

(3) follows also from (2) and the definition of the orthocycle. The relation for r_g is a consequence of the orthogonality of circles c', g.

(4) is a consequence of (3) and (5) is proved below by specializing to a particular bicentric $q(c)$ which is simultaneously orthodiagonal ([5, p.159]). Since the radius and the center of the incircle g is the same for all $q(c)$ this is legitimate. $\square$

Proposition 7. (1) *For fixed (k, E), the set of all bicentrics $\{q(c) : c \in \mathscr{C}'\}$ contains exactly one member which is simultaneously bicentric and orthodiagonal. It corresponds to the minimum circle of bundle $\mathscr{C}'$, is kite-shaped and symmetric with respect to MN* (see Figure 6).

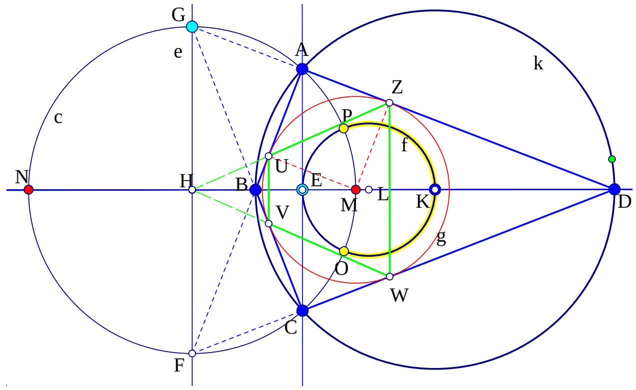


Figure 6. The orthodiagonal bicentric

(2) *All the bicentric orthodiagonals are constructed by reflecting an arbitrary right-angled triangle ABD on its hypotenuse* (see Figure 7). *The center of the incircle coincides with the trace E of the bisector with the hypotenuse, the length of this bisector w satisying $\frac{2}{w^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}$. Here R is the circumradius of ABD and $d = |EK|$ is the distance of the bisector's trace from the middle of BD.*

(3) *There is a particular bicentric orthodiagonal constructed directly from a regular octagon, with inradius $r = \frac{R}{\sqrt{2+\sqrt{2}}}$ and sides equal to w and $w + 2r$ respectively* (see Figure 8).

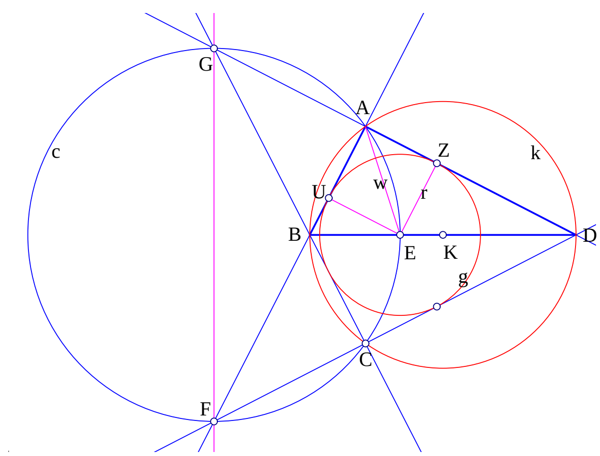


Figure 7. The general orthodiagonal bicentric

Proof. In fact, by applying the previous results to the tangential quadrilateral q' of $q(c)$, we know that the orthocycle of q' is orthogonal to the fixed incircle g and passes through two fixed points, depending only on (k, E) (the limit points of the corresponding circle bundle $\mathscr{C}'$ for the pair (g, E)). If the diagonals EQ and ER become orthogonal then E must be on the orthocycle of q' and this is possible only in the limiting position in which it coincides with line MN. Then the orthocycle of $q(c)$ has MN as diameter and this implies (1).

(2) follows immediately from (1). The formula is an application of Stewart's general formula (see [5, p.14]) on this particular configuration plus a simple calculation. The formula implies trivially the formula of the previous proposition, since all the bicirculars characterized by the fixed pair (k, E) have the same incircle g and from the square $EZAU$ (Figure 6) we have $w^2 = 2r^2$.

(3) is obvious and underlines the existence of a particular nice kite. □

Notice the necessary inequality between the distance $d = MK$ and the distance $d_1 = |EK|$ of circumcenter from the intersection point of diagonals: $2|MK| > |EK|$, holding for every bicentric ([6, p.44]) and being a consequence of a general propety of circle bundles of non intersecting type.

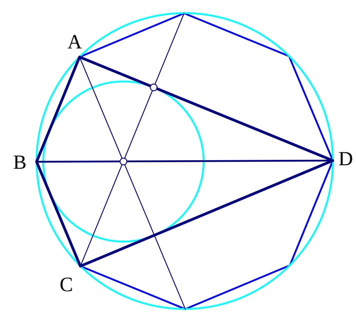


Figure 8. A distinguished kite

3. Circumscribed Quadrilateral

The following proposition give some well known properties of quadrilaterals circumscribed on circles by adding the ingredient of the orthocycle. For convenience we review here these properties and specialize in a subsequent proposition to the case of a bicentric circumscribed.

Proposition 8. *Consider the tangential quadrilateral $q' = QRST$ circumscribed on the circumcircle k of the cyclic quadrilateral $q = ABCD$* (Figure 9). *The following facts are true:*

(1) *The diagonals of q' and q intersect at the same point E.*

(2) *The pairs of opposite sides of q' and pairs of opposite sides of q intersect on the same line e, which is the polar of E with respect to the circumcircle k of q.*

(3) *The diameter UV of the orthocycle of q' is divided harmonically by the diameter FG of the orthocycle of q.*

(4) *The orthocycle of q' is orthogonal to the orthocycle of q.*

(5) *The diagonals of q' (respectively q) pass through the intersection points of opposite sides of q (respectively q').*

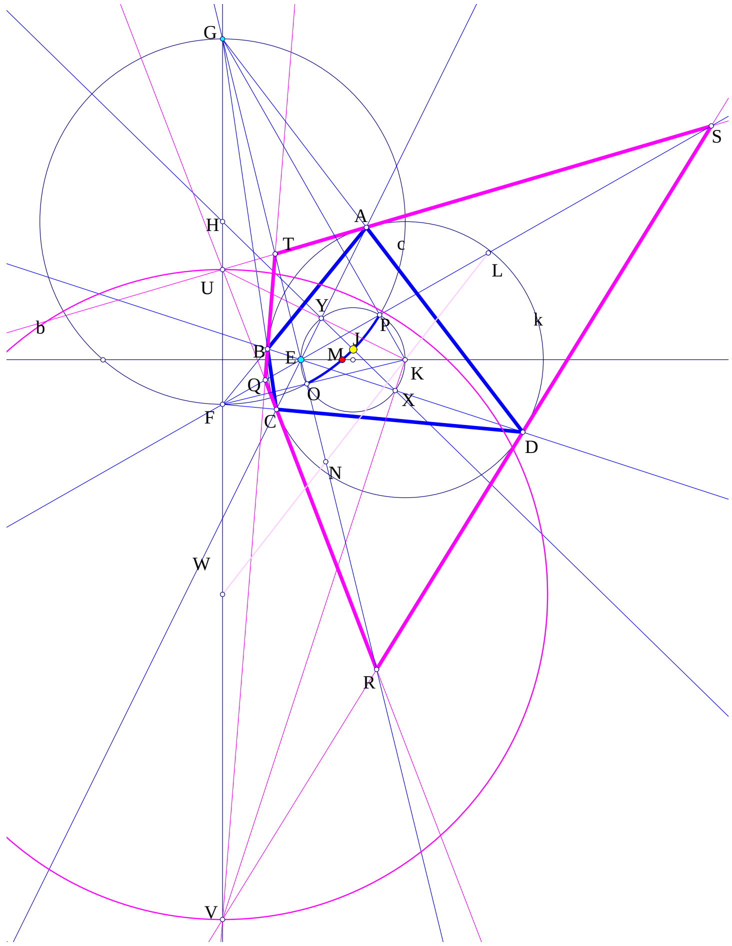


Figure 9. Circumscribed on cyclic

Proof. (1) is a consequence of Brianchon's theorem (see a simpler proof in [5, p.157]). Identify the polar of E with the diameter $e = FG$ of the orthocycle of q. The polar of F is PG and the polar of G is OF. T is the pole of AB which contains F. Hence the polar of F will pass through T, analogously it will pass through R. This proves (2) (see [2, §1275]) and (5).

To show (3) it suffices to see that lines (KP, KY, KO, KX) form a harmonic bundle. But the cross ratio of these four lines through K is independent of the location of K on the circle with diameter EK. Hence is the same with the cross ratio of lines (EP, EY, EO, EX) which is -1 by the general properties of complete quadrilaterals.

(4) is a consequence of (3). Note that the line LN joining the midpoints of the diagonals of q' passes through the center W of the orthocycle and the center of k ([2, §1614]). □

Proposition 9. *For each quadrangle of the family $q \in (k, E, c)$ construct the tangential quadrangle $q' = QRST$ of $q = ABCD$ (Figure 10). The following facts are true.*

(1) *There is exactly one $q_0 \in (k, E, c)$ whose corresponding tangential q' is cyclic. The corresponding line of diagonal midpoints of q' passes through the center K of circle k.*

(2) *The line of diagonal midpoints of q_0 is orthogonal to the corresponding line of diagonal midpoints of q'.*

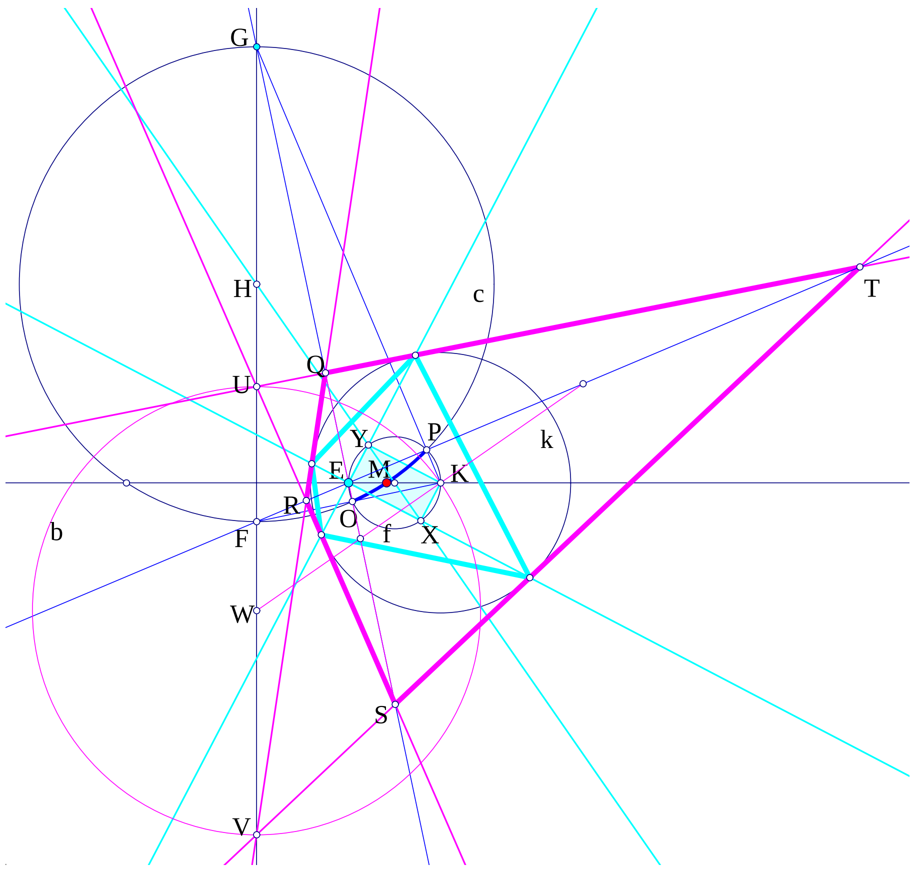


Figure 10. Bicentric circumscribed

Proof. q' being cyclic and circumscriptible it is bicentric. Hence the lines joining opposite contact points must be orthogonal and the orthocycle of q' passes through K. This follows from Proposition 2. Thus $p = EXKY$ is a rectangle, XY being a diameter of the circle f. Inversely, by Proposition 3, if q' is bicentric, then p is a rectangle and K is the limit point of the corresponding bundle I, and K is the center of the incircle. For the other statement notice that circle c being orthogonal simultaneously to circle k and b has its center on the radical axis of b and k. In the particular case of bicentric q', the angles WKK, XYK and EKY are equal and this implies that WL is then orthogonal to HXY which becomes the radical axis of k and b.

Note that the diagonals of all q' are the same and identical with the lines EF, EG which remain fixed for all members q of the family (k, E, c). Also combining this proposition and Proposition 3 we have (see [5, p.162]) that q' is cyclic, if and only if q is orthodiagonal. □

4. Orthodiagonals

The first part of the following proposition constructs an orthodiagonal from a cyclic. This is the inverse procedure of the well known one, which produces a cyclic by projecting the diagonals intersection point of an orthodiagonal to its sides ([4, vol. II p. 358], [6], [1]).

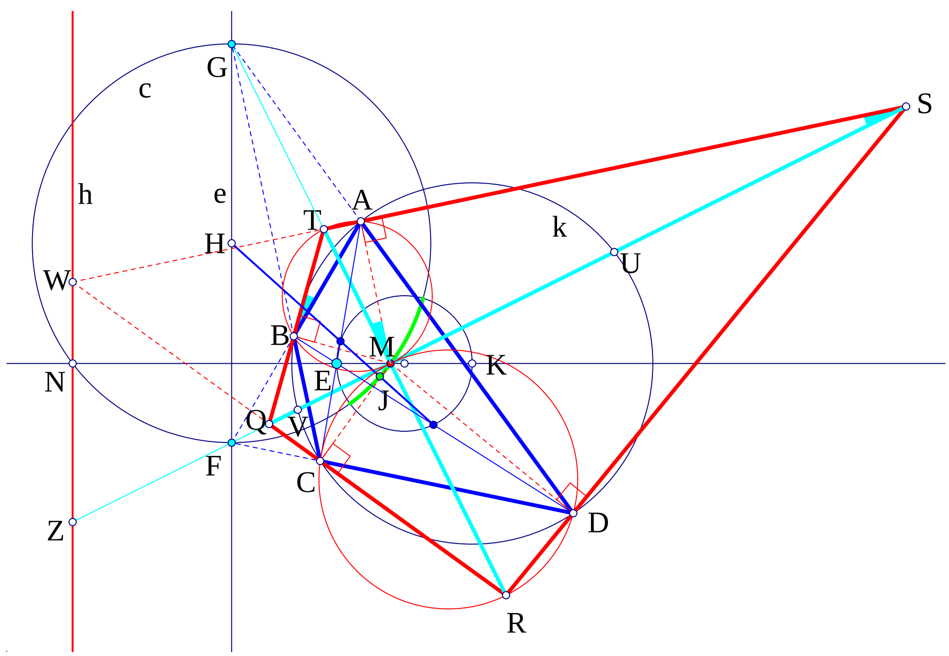


Figure 11. Orthodiagonal from cyclic

Proposition 10. (1) *For each cyclic quadrilateral* $q = ABCD$ *of the family* (k, E, c) *there is an orthodiagonal* $p = QRST$ *whose diagonals coincide with the sides of the right angled triangle* $t = FGM$, *defined by the limit point* M *of bundle* $\mathscr{C}$ *and the intersection points* F, G *of the pairs of opposite sides of* q. *The vertices of* q *are the projections of the intersection point* M *of the diagonals of* p *on its sides.*

(2) *The pairs of opposite sides of* p *intersect at points* W, W^* *on line* h *which is the common polar of all circles of bundle* I *with respect to its limit point* M.

(3) *The orthodiagonal p is cyclic if and only if the corresponding q is bicentric,* i.e., *point J is identical with M.*

(4) *The circumcircle of the orthodiagonal and cyclic p belongs to bundle I.*

Proof. Consider the lines orthogonal to MA, MB, MC, MD at the vertices of q (Figure 11). They build a quadrilateral. To show the statement on the diagonals consider the two resulting cyclic quadrilaterals $q_1 = MATB$ and $q_2 = MCRD$. Point F lies on the radical axis of their circumcircles since lines FBA, FCD are chords through F of circle k. Besides, for the same reason $|FB| \cdot |FA| = |FV| \cdot |FU| = |CF| \cdot |FD| = |FM|^2$. The last because circles c, k are orthogonal and M is the limit point of bundle I. From $|FB| \cdot |FA| = |FM|^2$ follows that line FM is tangent to the circumcircle of q_1. Analogously it is tangent to q_2 at M. Thus points G, T, M, R are collinear. Analogously points F, Q, M, S are collinear. This proves (1).

For (2), note that quadrangle $ABQS$ is cyclic, since $\angle TBA = \angle TMA = \angle MSA$. Thus $|FM|^2 = |FQ| \cdot |FS|$ and this implies that points M, Z divide harmonically Q, S, Z being the intersection point of h with the diagonal QS. Analogously the intersection point Z^* of h with the diagonal TR and M will divide T, R harmonically. Thus, by the characteristic property of the diagonals of a complete quadrilateral ZZ^* will be identical with line WW^*.

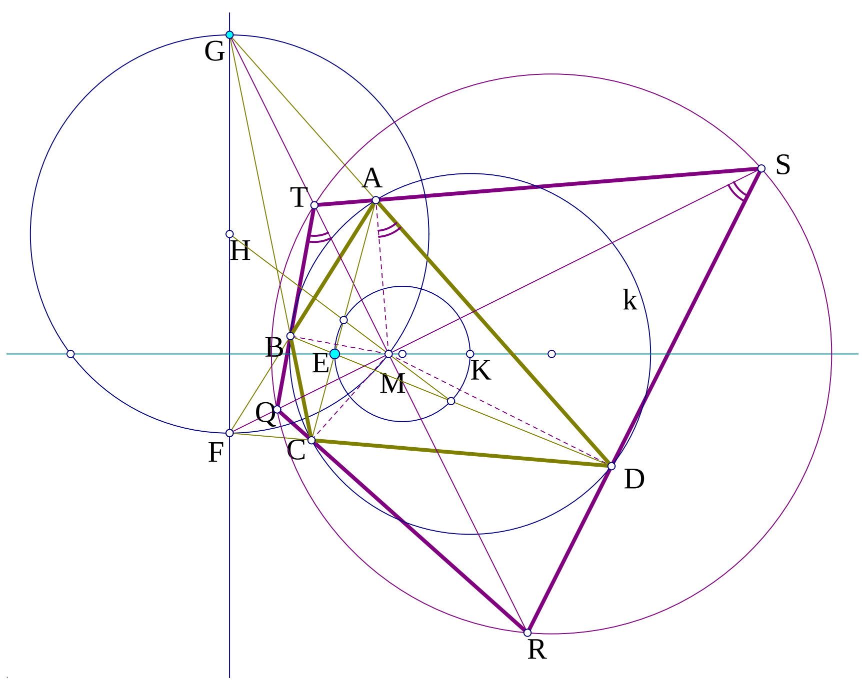


Figure 12. Orthodiagonal and cyclic

For (3), note that p is cyclic if and only if angles $\angle QTR = \angle QSR$ (Figure 12). By the definition of p this is equivalent to $\angle BAM = \angle MAD$, *i.e.*, AM being the bisector of angle A of q. Analogously MB, MC, MD must be bisectors of the corresponding angles of q.

For (4), note that the circumcenter of p must be on the line EM. This follows from the discussion in the first paragraph and the second statement. Indeed, the circumcenter must be on the line which is orthogonal from M to the diameter of the orthocycle of p. Besides the circle with center F and radius FM is a circle of bundle $\mathscr{C}'$ and, according to the proof of first statement, is orthogonal to this circumcircle. Thus the circumcircle of p, being orthogonal to two circles of bundle $\mathscr{C}'$, belongs to bundle $\mathscr{C}$. □

References

[1] A. Bogomolny, http://www.cut-the-knot.org/Curriculum/Geometry/BicentricQuadri.shtml.
[2] F. G.-M, *Exercices de Geometrie*, Cinquieme edition, Paris 1950.
[3] R. Johnson, *Modern Geometry*, Dover, New York 1960.
[4] J. Steiner, *Gesemeltes Werke*, I, II, Chelsea 1971.
[5] P. Yiu, *Notes on Euclidean Geometry*, 1998, available at http://www.math.fau.edu/yiu/Geometry.html.
[6] A. A. Zaslavsky, The Orthodiagonal Mapping of Quadrilaterals, *Kvant* nr. 4 (1998), 43–44 (in Russian), pdf available at: http://kvant.mccme.ru/1998/04.

Paris Pamfilos: Department of Mathematics, University of Crete, Crete, Greece
E-mail address: pamfilos@math.uoc.gr

Forum Geometricorum
Volume 7 (2007) 87–97.

FORUM GEOM
ISSN 1534-1178

Bicevian Tucker Circles

Bernard Gibert

Abstract. We prove that there are exactly ten bicevian Tucker circles and show several curves containing the Tucker bicevian perspectors.

1. Bicevian Tucker circles

The literature is abundant concerning Tucker circles. We simply recall that a Tucker circle is centered at T on the Brocard axis OK and meets the sidelines of ABC at six points A_b, A_c, B_c, B_a, C_a, C_b such that
(i) the lines X_yY_x are parallel to the sidelines of ABC,
(ii) the lines Y_xZ_x are antiparallel to the sidelines of ABC, *i.e.*, parallel to the sidelines of the orthic triangle $H_aH_bH_c$.

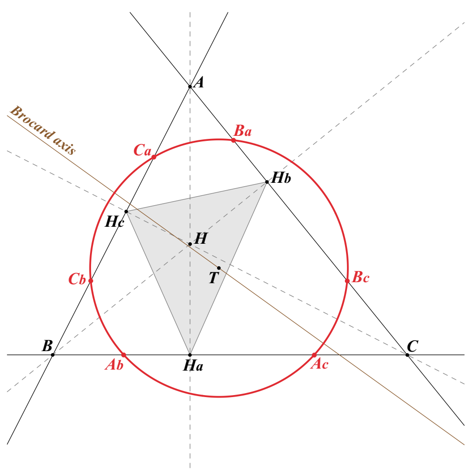


Figure 1. A Tucker circle

If T is defined by $\overrightarrow{OT} = t \cdot \overrightarrow{OK}$, we have

$$B_aC_a = C_bA_b = A_cB_c = \frac{2abc}{a^2+b^2+c^2}\,|t| = R|t|\tan\omega,$$

and the radius of the Tucker circle is

$$R_T = R\sqrt{(1-t)^2 + t^2\,\tan^2\omega}$$

where R is the circumradius and ω is the Brocard angle. See Figure 1.

Publication Date: March 19, 2007. Communicating Editor: Paul Yiu.

One obvious and trivial example consists of the circumcircle of ABC which we do not consider in the sequel. From now on, we assume that the six points are not all the vertices of ABC.

In this paper we characterize the *bicevian Tucker circles*, namely those for which a *Tucker triangle* formed by three of the six points (one on each sideline) is perspective to ABC. It is known that if a Tucker triangle is perspective to ABC, its companion triangle formed by the remaining three points is also perspective to ABC. The two perspectors are then said to be cyclocevian conjugates.

There are basically two kinds of Tucker triangles:
(i) those having one sideline parallel to a sideline of ABC: there are three pairs of such triangles e.g. $A_bB_cC_b$ and its companion $A_cB_aC_a$,
(ii) those not having one sideline parallel to a sideline of ABC: there is only one such pair namely $A_bB_cC_a$ and its companion $A_cB_aC_b$. These are the proper Tucker triangles of the literature.

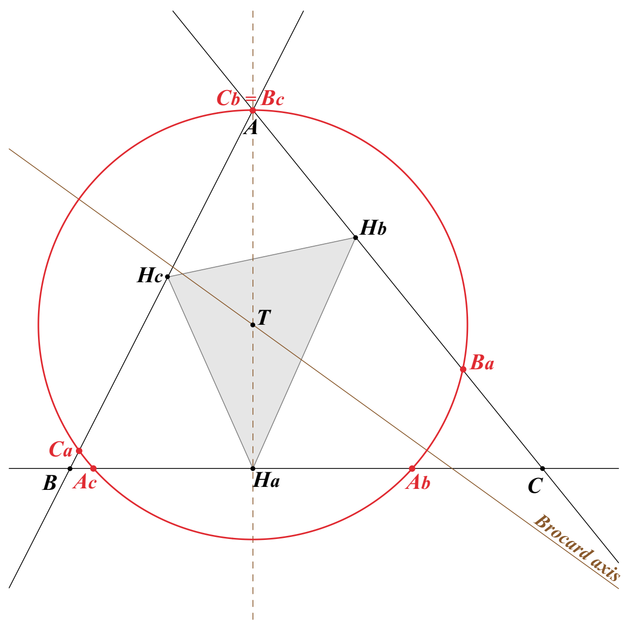


Figure 2. A Tucker circle through a vertex of ABC

In the former case, there are six bicevian Tucker circles which are obtained when T is the intersection of the Brocard axis with an altitude of ABC (which gives a Tucker circle passing through one vertex of ABC, see Figure 2) or with a perpendicular bisector of the medial triangle (which gives a Tucker circle passing through two midpoints of ABC, see Figure 3).

The latter case is more interesting but more difficult. Let us consider the Tucker triangle $A_bB_cC_a$ and denote by X_a the intersection of the lines BB_c and CC_a; define X_b and X_c similarly. Thus, ABC and $A_bB_cC_a$ are perspective (at X) if and only if the three lines AA_b, BB_c and CC_a are concurrent or equivalently the three

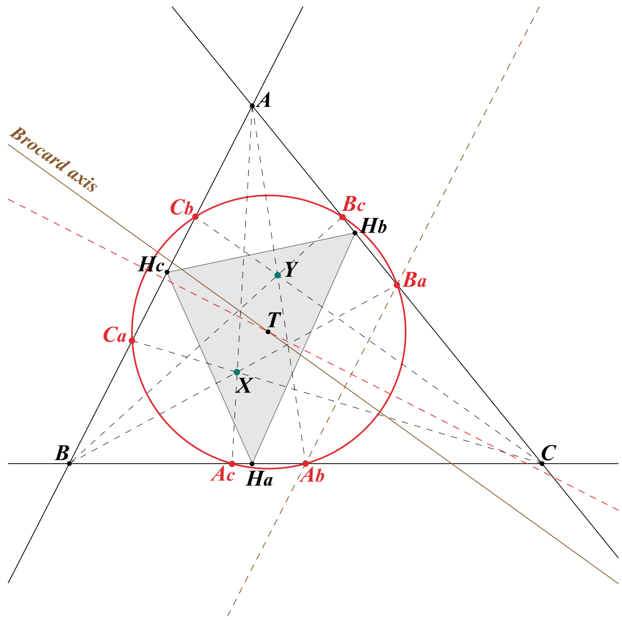


Figure 3. A Tucker circle through two midpoints of ABC

points X_a, X_b and X_c coincide. Consequently, the triangles ABC and $A_cB_aC_b$ are also perspective at Y, the cyclocevian conjugate of X.

Lemma 1. *When T traverses the Brocard axis, the locus of X_a is a conic γ_a.*

Proof. This can be obtained through easy calculation. Here is a synthetic proof. Consider the projections π_1 from the line AC onto the line BC in the direction of H_aH_b, and π_2 from the line BC onto the line AB in the direction of AC. Clearly, $\pi_2(\pi_1(B_c)) = \pi_2(A_c) = C_a$. Hence, the tranformation which associates the line BB_c to the line CC_a is a homography between the pencils of lines passing through B and C. It follows from the theorem of Chasles-Steiner that their intersection X_a must lie on a conic. □

This conic γ_a is easy to draw since it contains B, C, the anticomplement G_a of A, the intersection of the median AG and the symmedian CK and since the tangent at C is the line CA. Hence the perspector X we are seeking must lie on the three conics γ_a, γ_b, γ_c and Y must lie on three other similar conics γ'_a, γ'_b, γ'_c. See Figure 4.

Lemma 2. *γ_a, γ_b and γ_c have three common points X_i, $i = 1, 2, 3$, and one of them is always real.*

Proof. Indeed, γ_b and γ_c for example meet at A and three other points, one of them being necessarily real. On the other hand, it is clear that any point X lying on two conics must lie on the third one. □

This yields the following

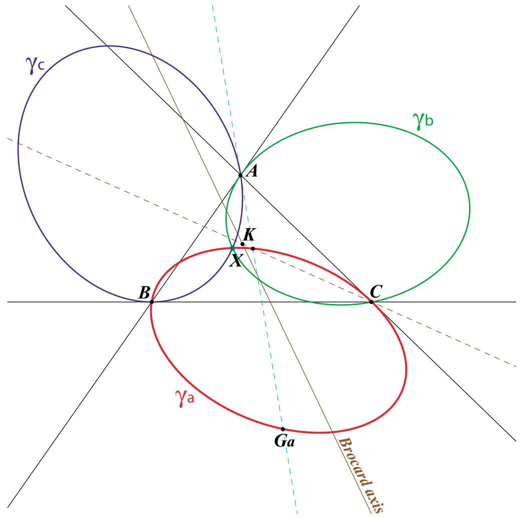


Figure 4. γ_a, γ_b and γ_c with only one real common point X

Theorem 3. *There are three* (*proper*) *bicevian Tucker circles and one of them is always real.*

2. Bicevian Tucker perspectors

The points X_i are not ruler and compass constructible since we need intersect two conics having only one known common point. For each X_i there is a corresponding Y_i which is its cyclocevian conjugate and the Tucker circle passes through the vertices of the cevian triangles of these two points. We call these six points X_i, Y_i the *Tucker bicevian perspectors*.

When X_i is known, it is easy to find the corresponding center T_i of the Tucker circle on the line OK : the perpendicular at T_i to the line H_bH_c meets AK at a point and the parallel through this point to H_bH_c meets the lines AB, AC at two points on the required circle. See Figure 5 where only one X is real and Figure 6 where all three points X_i are real.

We recall that the bicevian conic $\mathcal{C}(P,Q)$ is the conic passing through the vertices of the cevian triangles of P and Q. See [3] for general bicevian conics and their properties.

Theorem 4. *The three lines $\mathcal{L}_i$ passing through X_i, Y_i are parallel and perpendicular to the Brocard axis OK.*

Proof. We know (see [3]) that, for any bicevian conic $\mathcal{C}(P,Q)$, there is an inscribed conic bitangent to $\mathcal{C}(P,Q)$ at two points lying on the line PQ. On the other hand, any Tucker circle is bitangent to the Brocard ellipse and the line through the contacts is perpendicular to the Brocard axis. Hence, any bicevian Tucker circle must be tangent to the Brocard ellipse at two points lying on the line X_iY_i and this completes the proof. □

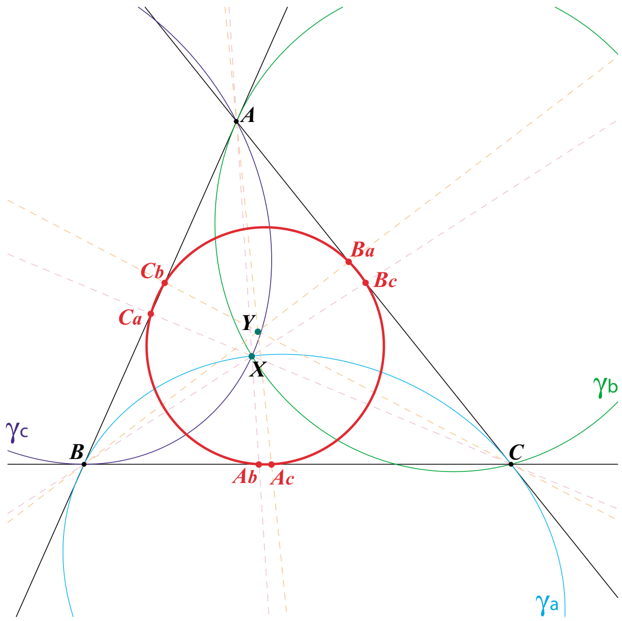


Figure 5. One real bicevian Tucker circle

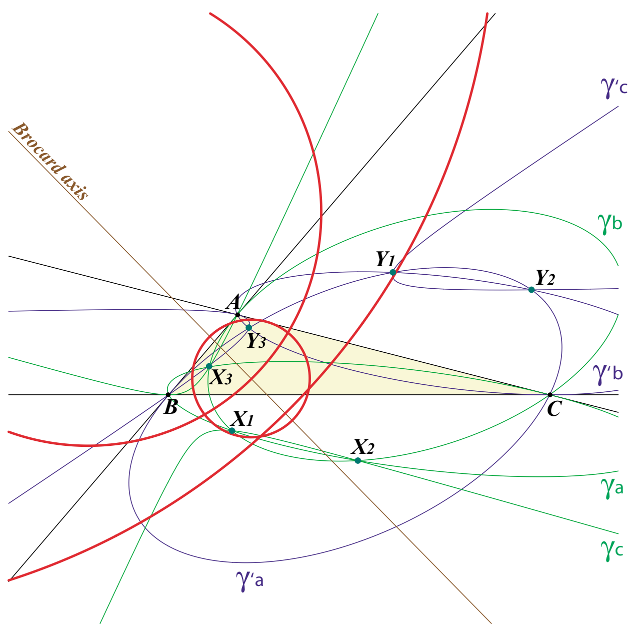


Figure 6. Three real bicevian Tucker circles

Corollary 5. *The two triangles $X_1X_2X_3$ and $Y_1Y_2Y_3$ are perspective at X_{512} and the axis of perspective is the line GK.*

Conversely, any bicevian conic $\mathcal{C}(P,Q)$ bitangent to the Brocard ellipse must verify $Q = K/P$. Such conic has its center on the Brocard line if and only if P lies either
(i) on $\mathrm{p}\mathcal{K}(X_{3051}, K)$ in which case the conic has always its center at the Brocard midpoint X_{39}, but the Tucker circle with center X_{39} is not a bicevian conic, or
(ii) on $\mathrm{p}\mathcal{K}(X_{669}, K)$ = K367 in [4].

This gives the following

Theorem 6. *The six Tucker bicevian perspectors X_i, Y_i lie on $\mathrm{p}\mathcal{K}(X_{669}, X_6)$, the pivotal cubic with pivot the Lemoine point K which is invariant in the isoconjugation swapping K and the infinite point X_{512} of the Lemoine axis.*

See Figure 7. We give another proof and more details on this cubic below.

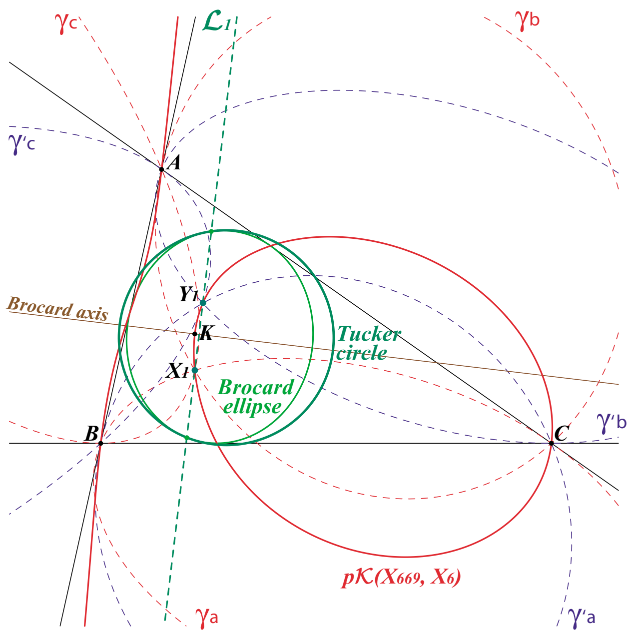


Figure 7. Bicevian Tucker circle and Brocard ellipse

3. Nets of conics associated with the Tucker bicevian perspectors

We now consider curves passing through the six Tucker bicevian perspectors X_i, Y_i. Recall that two of these points are always real and that all six points are two by two cyclocevian conjugates on three lines $\mathcal{L}_i$ perpendicular to the Brocard axis. We already know two nets of conics containing these points:
(i) the net $\mathcal{N}$ generated by γ_a, γ_b, γ_c which contain the points X_i, $i = 1, 2, 3$;
(ii) the net $\mathcal{N}'$ generated by γ'_a, γ'_b, γ'_c which contain the points Y_i, $i = 1, 2, 3$.

The equations of the conics are

$$\gamma_a : \qquad a^2 y(x+z) - b^2 x(x+y) = 0,$$

$$\gamma'_a: \qquad a^2z(x+y)-c^2x(x+z)=0;$$

the other equations are obtained through cyclic permutations.

Thus, for any point $P = u : v : w$ in the plane, a conic in $\mathcal{N}$ is

$$\mathcal{N}(P) = u\,\gamma_a + v\,\gamma_b + w\,\gamma_c;$$

similarly for $\mathcal{N}'(P)$. Clearly, $\mathcal{N}(A) = \gamma_a$, etc.

Proposition 7. *Each net of conics ($\mathcal{N}$ and $\mathcal{N}'$) contains one and only one circle. These circles Γ and Γ' contain X_{110}, the focus of the Kiepert parabola.*

These circles are

$$\Gamma: \quad \sum_{\text{cyclic}} b^2c^2(b^2-c^2)(a^2-b^2)x^2 + a^2(b^2-c^2)(c^4+a^2b^2-2a^2c^2)yz = 0$$

and

$$\Gamma': \quad \sum_{\text{cyclic}} b^2c^2(b^2-c^2)(c^2-a^2)x^2 - a^2(b^2-c^2)(b^4+a^2c^2-2a^2b^2)yz = 0.$$

In fact, $\Gamma = \mathcal{N}(P')$ and $\Gamma' = \mathcal{N}'(P'')$ where

$$\begin{aligned} P' =& \frac{c^2}{c^2-a^2} : \frac{a^2}{a^2-b^2} : \frac{b^2}{b^2-c^2}, \\ P'' =& \frac{b^2}{b^2-a^2} : \frac{c^2}{c^2-b^2} : \frac{a^2}{a^2-c^2}. \end{aligned}$$

These points lie on the trilinear polar of X_{523}, the line through the centers of the Kiepert and Jerabek hyperbolas and on the circum-conic with perspector X_{76}, which is the isotomic transform of the Lemoine axis. See Figure 8.

Proposition 8. *Each net of conics contains a pencil of rectangular hyperbolas. Each pencil contains one rectangular hyperbola passing through X_{110}.*

Note that these two rectangular hyperbolas have the same asymptotic directions which are those of the rectangular circum-hyperbola passing through X_{110}. See Figure 9.

4. Cubics associated with the Tucker bicevian perspectors

When P has coordinates which are linear in x, y, z, the curves $\mathcal{N}(P)$ and $\mathcal{N}'(P)$ are in general cubics but $\mathcal{N}(z : x : y)$ and $\mathcal{N}'(y : z : x)$ are degenerate. In other words, for any point $x : y : z$ of the plane, we (loosely) may write

$$z\,\gamma_a + x\,\gamma_b + y\,\gamma_c = 0$$

and

$$y\,\gamma'_a + z\,\gamma'_b + x\,\gamma'_c = 0.$$

We obtain two circum-cubics $\mathcal{K}(P)$ and $\mathcal{K}'(P)$ when P takes the form

$$P = q\,z - r\,y : r\,x - p\,z : p\,y - q\,x$$

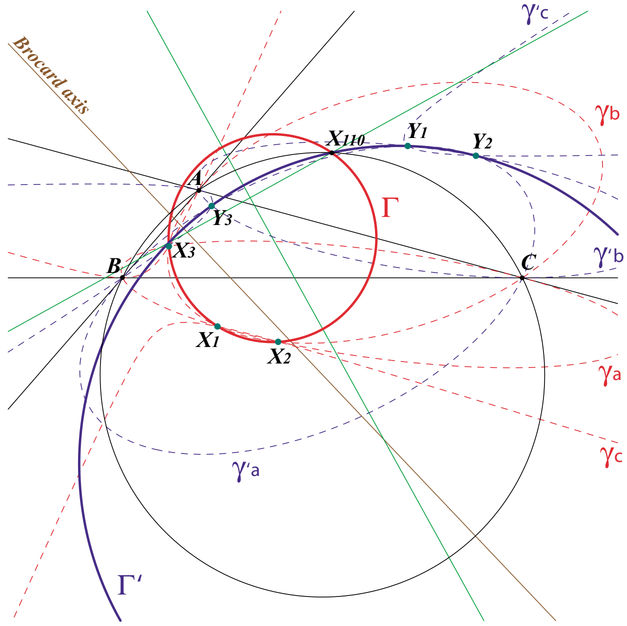


Figure 8. Circles through the Tucker bicevian perspectors

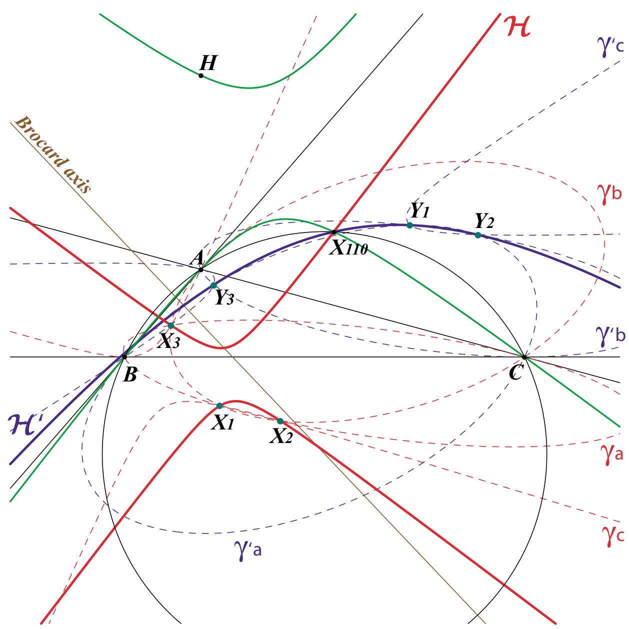


Figure 9. Rectangular hyperbolas through the Tucker bicevian perspectors

associated to the cevian lines of the point $Q = p : q : r$ and both cubics contain Q. Obviously, $\mathcal{K}(P)$ contains the points X_i and $\mathcal{K}'(P)$ contains the points Y_i.

For example, with $Q = G$, we obtain the two cubics $\mathcal{K}(G)$ and $\mathcal{K}'(G)$ passing through G and the vertices of the antimedial triangle $G_aG_bG_c$. See Figure 10.

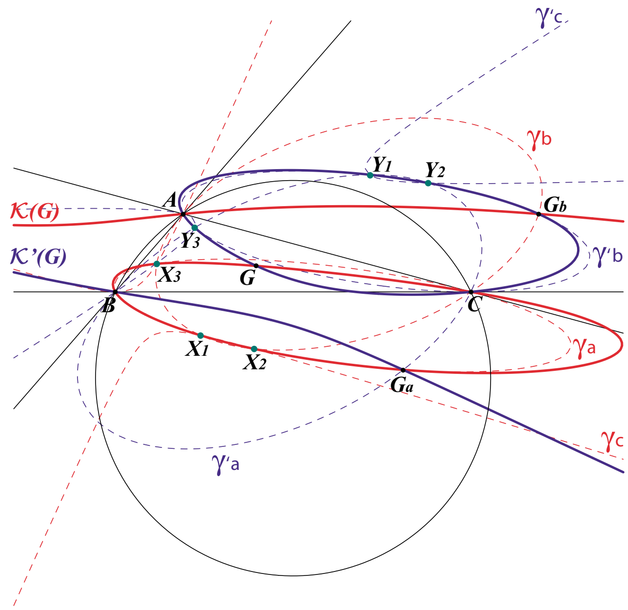


Figure 10. The two cubics $\mathcal{K}(G)$ and $\mathcal{K}'(G)$

These two cubics $\mathcal{K}(P)$ and $\mathcal{K}'(P)$ are isotomic pivotal cubics with pivots the bicentric companions (see [5, p.47] and [2]) of X_{523} respectively

$$X'_{523} = a^2 - b^2 : b^2 - c^2 : c^2 - a^2$$

and

$$X''_{523} = c^2 - a^2 : a^2 - b^2 : b^2 - c^2$$

both on the line at infinity. The two other points at infinity of the cubics are those of the Steiner ellipse.

4.1. *An alternative proof of Theorem 6.* We already know (Theorem 6) that the six Tucker bicevian perspectors X_i, Y_i lie on the cubic $\mathrm{p}\mathcal{K}(X_{669}, X_6)$. Here is an alternative proof. See Figure 11.

Proof. Let U, V, W be the traces of the perpendicular at G to the Brocard axis. We denote by Γ_a the decomposed cubic which is the union of the line AU and the conic γ_a. Γ_a contains the vertices of ABC and the points X_i. Γ_b and Γ_c are defined similarly and contain the same points.

The cubic $c^2\,\Gamma_a + a^2\,\Gamma_b + b^2\,\Gamma_c$ is another cubic through the same points since it belongs to the net of cubics. It is easy to verify that this latter cubic is $\mathrm{p}\mathcal{K}(X_{669}, X_6)$.

Now, if Γ'_a, Γ'_b, Γ'_c are defined likewise, the cubic $b^2\,\Gamma'_a + c^2\,\Gamma'_b + a^2\,\Gamma'_c$ is $\mathrm{p}\mathcal{K}(X_{669}, X_6)$ again and this shows that the six Tucker bicevian perspectors lie on the curve. □

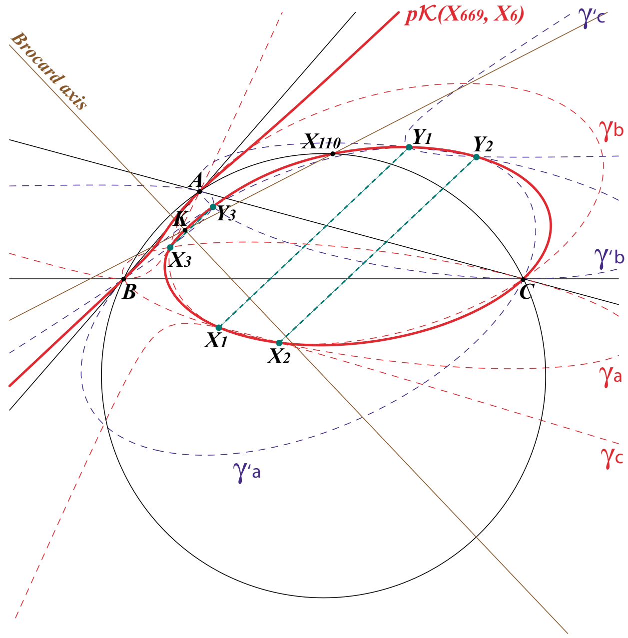


Figure 11. $\mathrm{p}\mathcal{K}(X_{669}, X_6)$ and the three lines $\mathcal{L}_i$

4.2. *More on the cubic* $\mathrm{p}\mathcal{K}(X_{669}, X_6)$. The cubic $\mathrm{p}\mathcal{K}(X_{669}, X_6)$ also contains K, X_{110}, X_{512}, X_{3124} and meets the sidelines of ABC at the feet of the symmedians. Note that the pole X_{669} is the barycentric product of K and X_{512}, the isopivot or secondary pivot (see [1], §1.4). This shows that, for any point M on the cubic, the point K/M (cevian quotient or Ceva conjugate) lies on the cubic and the line $M\ K/M$ contains X_{512} i.e. is perpendicular to the Brocard axis.

We can apply to the Tucker bicevian perspectors the usual group law on the cubic. For any two points P, Q on $\mathrm{p}\mathcal{K}(X_{669}, X_6)$ we define $P \oplus Q$ as the third intersection of the line through K and the third point on the line PQ.

For a permuation $i,\ j,\ k$ of $1,\ 2,\ 3$, we have

$$X_i \oplus X_j = Y_k, \qquad Y_i \oplus Y_j = X_k.$$

Furthermore, $X_i \oplus Y_i = K$. These properties are obvious since the pivot of the cubic is K and the secondary pivot is X_{512}.

The third point of $\mathrm{p}\mathcal{K}(X_{669}, X_6)$ on the line KX_{110} is $X_{3124} = a^2(b^2-c^2)^2 : b^2(c^2-a^2)^2 : c^2(a^2-b^2)^2$, the cevian quotient of K and X_{512} and the third point on the line $X_{110}X_{512}$ is the cevian quotient of K and X_{110}.

The infinite points of $\mathrm{p}\mathcal{K}(X_{669}, X_6)$ are X_{512} and two imaginary points, those of the bicevian ellipse $\mathcal{C}(G, K)$ or, equivalently, those of the circum-ellipse $\mathcal{C}(X_{39})$ with perspector X_{39} and center X_{141}.

The real asymptote is perpendicular to the Brocard axis and meets the curve at $X = K/X_{512}$, the third point on the line KX_{110} seen above. X also lies on the Brocard ellipse, on $\mathcal{C}(G, K)$. See Figure 12.

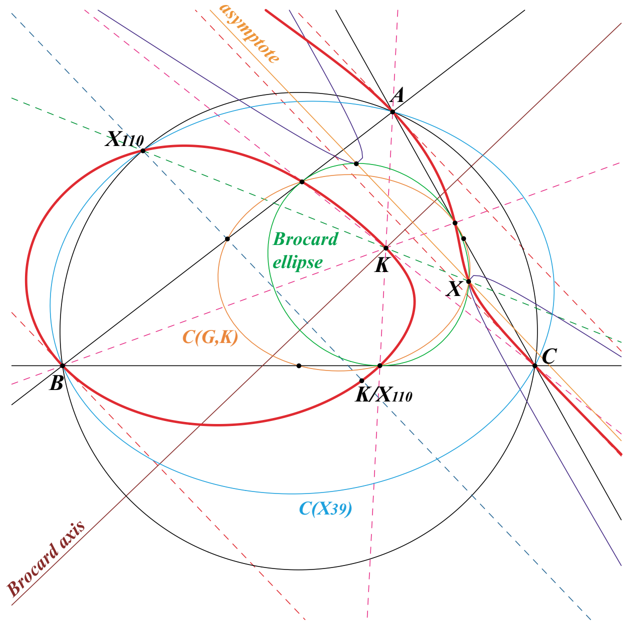


Figure 12. $K367 = \mathrm{p}\mathcal{K}(X_{669}, X_6)$

$\mathrm{p}\mathcal{K}(X_{669}, X_6)$ is the isogonal transform of $\mathrm{p}\mathcal{K}(X_{99}, X_{99})$, a member of the class CL007 in [4]. These are the $\mathrm{p}\mathcal{K}(W, W)$ cubics or parallel tripolars cubics.

References

[1] J.-P. Ehrmann and B. Gibert, Special Isocubics in the Triangle Plane, available at http://perso.orange.fr/bernard.gibert/

[2] B. Gibert, Barycentric coordinates and Tucker cubics, available at http://perso.orange.fr/bernard.gibert/

[3] B. Gibert, Bicevian Conics, available at http://perso.orange.fr/bernard.gibert/

[4] B. Gibert, *Cubics in the Triangle Plane*, available at http://perso.orange.fr/bernard.gibert/

[5] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[6] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France

E-mail address: `bg42@orange.fr`

Forum Geometricorum
Volume 7 (2007) 99–102.

FORUM GEOM
ISSN 1534-1178

A Visual Proof of the Erdős-Mordell Inequality

Claudi Alsina and Roger B. Nelsen

Abstract. We present a visual proof of a lemma that reduces the proof of the Erdős-Mordell inequality to elementary algebra.

In 1935, the following problem proposal appeared in the "Advanced Problems" section of the *American Mathematical Monthly* [5]:

> **3740**. *Proposed by Paul Erdős, The University, Manchester, England*.
> From a point O inside a given triangle ABC the perpendiculars OP, OQ, OR are drawn to its sides. Prove that
> $$OA + OB + OC \geq 2(OP + OQ + OR).$$

Trigonometric solutions by Mordell and Barrow appeared in [11]. The proofs, however, were not elementary. In fact, no "simple and elementary" proof of what had become known as the Erdős-Mordell theorem was known as late as 1956 [13]. Since then a variety of proofs have appeared, each one in some sense simpler or more elementary than the preceding ones. In 1957 Kazarinoff published a proof [7] based upon a theorem in Pappus of Alexandria's *Mathematical Collection*; and a year later Bankoff published a proof [2] using orthogonal projections and similar triangles. Proofs using area inequalities appeared in 1997 and 2004 [4, 9]. Proofs employing Ptolemy's theorem appeared in 1993 and 2001 [1, 10]. A trigonometric proof of a generalization of the inequality in 2001 [3], subsequently generalized in 2004 [6]. Many of these authors speak glowingly of this result, referring to it as a "beautiful inequality" [9], a "remarkable inequality" [12], "the famous Erdős-Mordell inequality" [4, 6, 10], and "the celebrated Erdős-Mordell inequality . . . a beautiful piece of elementary mathematics" [3].

In this short note we continue the progression towards simpler proofs. First we present a visual proof of a lemma that reduces the proof of the Erdős-Mordell inequality to elementary algebra. The lemma provides three inequalities relating the lengths of the sides of ABC and the distances from O to the vertices and to the sides. While the inequalities in the lemma are not new, we believe our proof of the lemma is. The proof uses nothing more sophisticated than elementary properties of triangles. In Figure 1(a) we see the triangle as described by Erdős, and in Figure

Publication Date: April 30, 2007. Communicating Editor: Paul Yiu.

1(b) we denote the lengths of relevant line segments by lower case letters, whose use will simplify the presentation to follow. In terms of that notation, the Erdős-Mordell inequality becomes

$$x + y + z \geq 2(p + q + r).$$

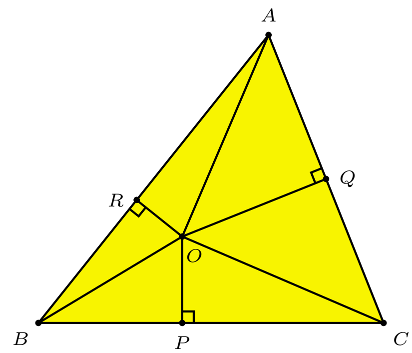


Figure 1(a)

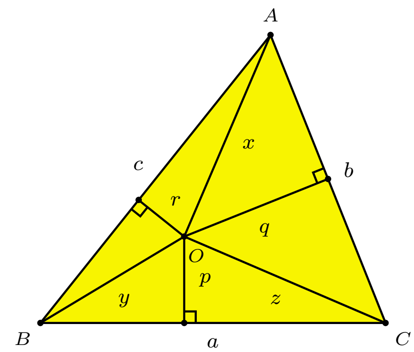


Figure 1(b)

In the proof of the lemma, we construct a trapezoid in Figure 2(b) from three triangles – one similar to ABC, the other two similar to two shaded triangles in Figure 2(a).

Lemma. *For the triangle* ABC *in Figure 1, we have* $ax \geq br + cq$, $by \geq ar + cp$, *and* $cz \geq aq + bp$.

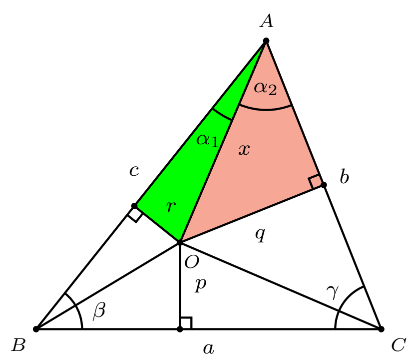


Figure 2(a)

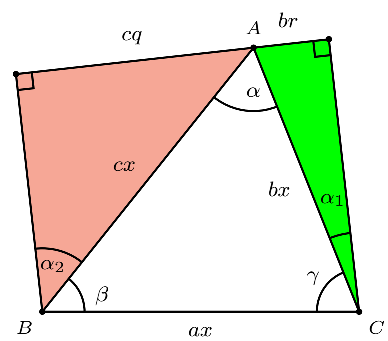


Figure 2(b)

Proof. See Figure 2 for a visual proof that $ax \geq br + cq$. The other two inequalities are established analogously. □

We should note before proceeding that the object in Figure 2(b) really is a trapezoid, since the three angles at the point where the three triangles meet measure $\frac{\pi}{2} - \alpha_2$, $\alpha = \alpha_1 + \alpha_2$, and $\frac{\pi}{2} - \alpha_1$, and thus sum to π.

We now prove

The Erdős-Mordell Inequality. *If O is a point within a triangle ABC whose distances to the vertices are x, y, and z, then*

$$x + y + z \geq 2(p + q + r).$$

Proof. From the lemma we have $x \geq \frac{b}{a}r + \frac{c}{a}q$, $y \geq \frac{a}{b}r + \frac{c}{b}p$, and $z \geq \frac{a}{c}q + \frac{b}{c}p$. Adding these three inequalities yields

$$x + y + z \geq \left(\frac{b}{c} + \frac{c}{b}\right)p + \left(\frac{c}{a} + \frac{a}{c}\right)q + \left(\frac{a}{b} + \frac{b}{a}\right)r. \tag{1}$$

But the arithmetic mean-geometric mean inequality insures that the coefficients of p, q, and r are each at least 2, from which the desired result follows. □

We conclude with several comments about the lemma and the Erdős-Mordell inequality and their relationships to other results.

1. The three inequalities in the lemma are equalities if and only if O is the center of the circumscribed circle of ABC. This follows from the observation that the trapezoid in Figure 2(b) is a rectangle if and only if $\beta + \alpha_2 = \frac{\pi}{2}$ and $\gamma + \alpha_1 = \frac{\pi}{2}$ (and similarly in the other two cases), so that $\angle AOQ = \beta = \angle COQ$. Hence the right triangles AOQ and COQ are congruent, and $x = z$. Similarly one can show that $x = y$. Hence, $x = y = z$ and O must be the circumcenter of ABC. The coefficients of p, q, and r in (1) are equal to 2 if and only if $a = b = c$. Consequently we have equality in the Erdős-Mordell inequality if and only if ABC is equilateral and O is its center.

2. How did Erdős come up with the inequality in his problem proposal? Kazarinoff [8] speculates that he generalized Euler's inequality: if $\overline{r}$ and $\overline{R}$ denote, respectively, the inradius and circumradius of ABC, then $\overline{R} \geq 2\overline{r}$. The Erdős-Mordell inequality implies Euler's inequality for acute triangles. Note that if we take O to be the circumcenter of ABC, then $3\overline{R} \geq 2(p + q + r)$. However, for *any* point O inside ABC, the quantity $p+q+r$ is somewhat surprisingly constant and equal to $\overline{R} + \overline{r}$, a result known as Carnot's theorem. Thus $3\overline{R} \geq 2(\overline{R} + \overline{r})$, or equivalently, $\overline{R} \geq 2\overline{r}$.

3. Many other inequalities relating x, y, and z to p, q, and r can be derived. For example, applying the arithmetic mean-geometric mean inequality to the right side of the inequalities in the lemma yields

$$ax \geq 2\sqrt{bcqr}, \quad by \geq 2\sqrt{carp}, \quad cz \geq 2\sqrt{abpq}.$$

Multiplying these three inequalities together and simplifying yields $xyz \geq 8pqr$. More such inequalities can be found in [8, 12].

4. A different proof of (1) appears in [4].

References

[1] A. Avez, A short proof of a theorem of Erdős and Mordell, *Amer. Math. Monthly*, 100 (1993) 60–62.

[2] L. Bankoff, An elementary proof of the Erdős-Mordell theorem, *Amer. Math. Monthly*, 65 (1958) 521.

[3] S. Dar and S. Gueron, A weighted Erdős-Mordell inequality, *Amer. Math. Monthly*, 108 (2001) 165–168.
[4] N. Dergiades, Signed distances and the Erdős-Mordell inequality, *Forum Geom.*, 4 (2004) 67–68.
[5] P. Erdős, Problem 3740, *Amer. Math. Monthly*, 42 (1935) 396.
[6] W. Janous, Further inequalities of Erdős-Mordell type, *Forum Geom.*, 4 (2004) 203–206.
[7] D. K. Kazarinoff, A simple proof of the Erdős-Mordell inequality for triangles, *Michigan Math. J.*, 4 (1957) 97–98.
[8] N. D. Kazarinoff, *Geometric Inequalities*, MAA, Washington, 1961.
[9] V. Komornik, A short proof of the Erdős-Mordell theorem, *Amer. Math. Monthly*, 104 (1997) 57–60.
[10] H. Lee, Another proof of the Erdős-Mordell theorem, *Forum Geom.*, 1 (2001) 7–8.
[11] L. J. Mordell and D. F. Barrow, Solution 3740, *Amer. Math. Monthly*, 44 (1937) 252–254.
[12] A. Oppenheim, The Erdős inequality and other inequalities for a triangle, *Amer. Math. Monthly*, 68 (1961) 226–230.
[13] G. Steensholt, Note on an elementary property of triangles, *Amer. Math. Monthly*, 63 (1956) 571–572.

Claudi Alsina: Secció de Matemàtiques, ETSAB, Universitat Politècnica de Catalunya, E-08028 Barcelona, Spain
E-mail address: `claudio.alsina@upc.edu`

Roger B. Nelsen: Department of Mathematical Sciences, Lewis & Clark College, Portland, Oregon 97219, USA
E-mail address: `nelsen@lclark.edu`

Forum Geometricorum
Volume 7 (2007) 103–106.

FORUM GEOM
ISSN 1534-1178

Construction of Triangle from a Vertex and the Feet of Two Angle Bisectors

Harold Connelly, Nikolaos Dergiades, and Jean-Pierre Ehrmann

Abstract. We give two simple constructions of a triangle given one vertex and the feet of two angle bisectors.

1. Construction from (A, T_a, T_b)

We present two simple solutions of the following construction problem (number 58) in the list compiled by W. Wernick [2]: Given three noncollinear points A, T_a and T_b, to construct a triangle ABC with T_a, T_b on BC, CA respectively such that AT_a and BT_b are bisectors of the triangle. L. E. Meyers [1] has indicated the constructibility of such a triangle. Let ℓ be the half line AT_b. Both solutions we present here make use of the reflection ℓ' of ℓ in AT_a. The vertex B necessarily lies on ℓ'. In what follows $P(Q)$ denotes the circle, center P, passing through the point Q.

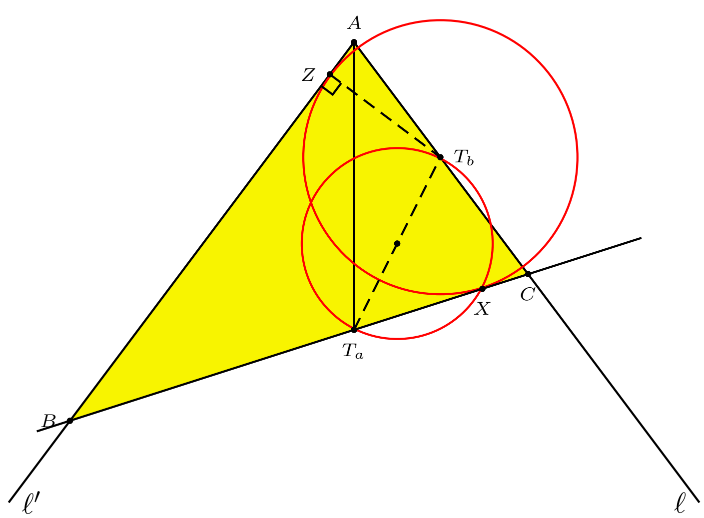


Figure 1

Construction 1. *Let Z be the pedal of T_b on ℓ'. Construct two circles, one $T_b(Z)$, and the other with T_aT_b as diameter. Let X be an intersection, if any, of these two circles. If the line XT_a intersects the half lines ℓ' at B and ℓ at C, then ABC is a desired triangle.*

Publication Date: May 7, 2007. Communicating Editor: Paul Yiu.

Construction 2. *Let P be an intersection, if any, of the circle $T_b(T_a)$ with the half line ℓ'. Construct the perpendicular bisector of PT_a. If this intersects ℓ' at a point B, and if the half line BT_a intersects ℓ at C, then ABC is a desired triangle.*

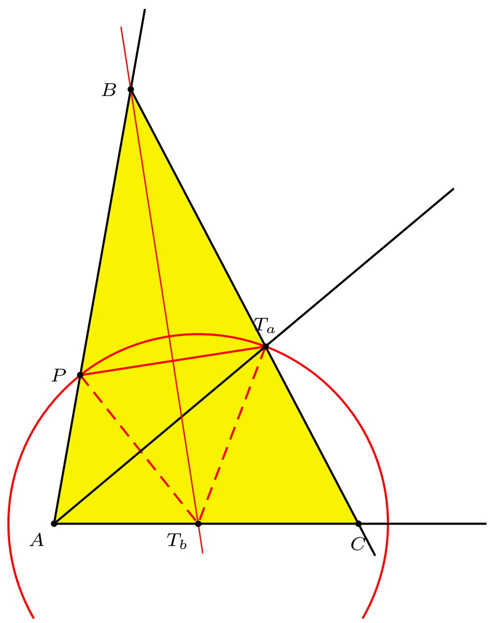


Figure 2

We study the number of solutions for various relative positions of A, T_a and T_b. Set up a polar coordinate system with A at the pole and T_b at $(1, 0)$. Suppose T_a has polar coordinates $(\rho,\ \theta)$ for $\rho > 0$ and $0 < \theta < \frac{\pi}{2}$. The half line ℓ' has polar angle 2θ. The circle $T_b(T_a)$ intersects ℓ' if the equation

$$\sigma^2 - 2\sigma\cos 2\theta = \rho^2 - 2\rho\cos\theta \qquad (1)$$

has a positive root σ. This is the case when

(i) $\rho > 2\cos\theta$, or

(ii) $\rho \le 2\cos\theta$, $\cos 2\theta > 0$ and $4\cos^2 2\theta + 4\rho(\rho - 2\cos\theta) \ge 0$. Equivalently, $\rho_+ \le \rho \le 2\cos\theta$, where

$$\rho_\pm = \cos\theta \pm \sqrt{\sin\theta\sin 3\theta}$$

are the roots of the equation $\rho^2 - 2\rho\cos\theta + \cos^2 2\theta = 0$ for $0 < \theta < \frac{\pi}{3}$.

Now, the perpendicular bisector of PT_a intersects the line ℓ' at the point B with polar coordinates $(\beta,\ 2\theta)$, where

$$\beta = \frac{\rho\cos\theta - \sigma\cos 2\theta}{\rho\cos\theta - \sigma}.$$

The requirement $\beta > 0$ is equivalent to $\sigma < \rho\cos\theta$. From (1), this is equivalent to $\rho < 4\cos\theta$.

For $0 < \theta < \frac{\pi}{3}$, let $P_\pm$ be the points with polar coordinates $(\rho_\pm,\ \theta)$. These points bound a closed curve $\mathscr{C}$ as shown in Figure 3. If T_a lies inside the curve $\mathscr{C}$, then the circle $T_b(T_a)$ does not intersect the half line ℓ'. We summarize the results with reference to Figure 3.

The construction problem of ABC from $(A,\ T_a,\ T_b)$ has
(1) a unique solution if T_a lies in the region between the two semicircles $\rho = 2\cos\theta$ and $\rho = 4\cos\theta$,
(2) two solutions if T_a lies between the semicircle $\rho = 2\cos\theta$ and the curve $\mathscr{C}$ for $\theta < \frac{\pi}{4}$.

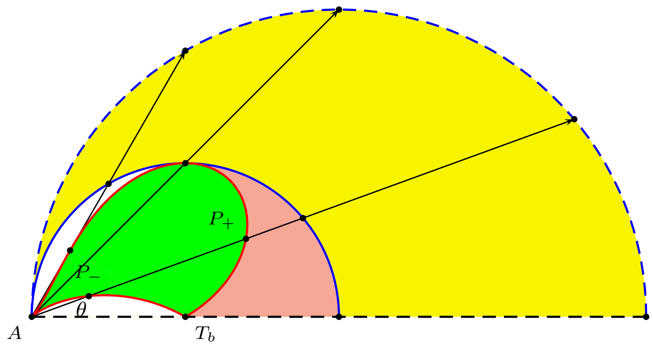


Figure 3.

2. Construction from $(A,\ T_b,\ T_c)$

The construction of triangle ABC from $(A,\ T_a,\ T_b)$ is Problem 60 in Wernick's list [2]. Wernick has indicated constructibility. We present two simple solutions.

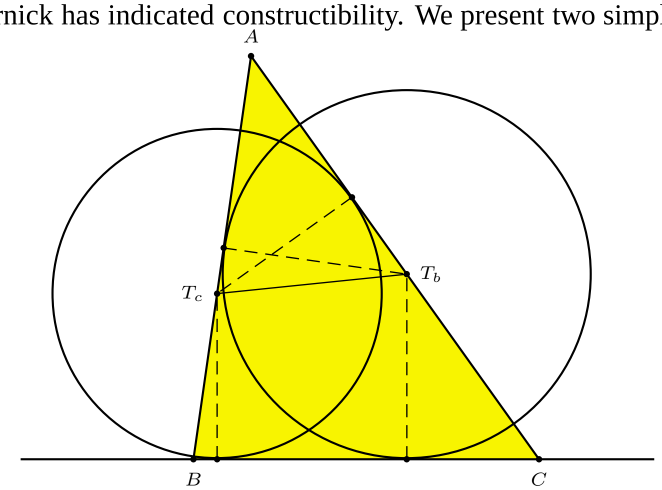


Figure 4.

Construction 3. *Given A, T_b, T_c, construct the circles with centers T_b and T_b, tangent to AT_c and AT_b respectively. The common tangent of these circles that lies opposite to A with respect to the line T_bT_c is the line BC of the required triangle ABC. The construction of the vertices B, C is obvious.*

Construction 4. *Given A, T_b, T_c, construct*
(i) *the circle through the three points*
(ii) *the bisector of angle T_bAT_c to intersect the circle at M,*
(iii) *the reflection M' of M in the line T_bT_c,*
(iv) *the circle $M'(T_b)$ to intersect the bisector at I (so that A and I are on opposite sides of T_bT_c),*
(v) *the half line T_bI to intersect the half line AT_c at B,*
(vi) *the half line T_cI to intersect the half line AT_b at C.*

ABC is the required triangle with incenter I.

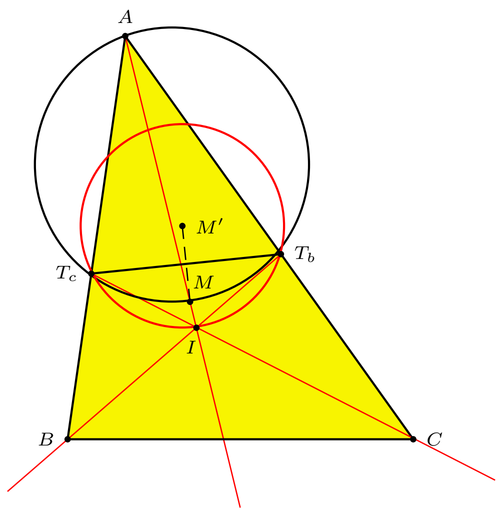


Figure 5.

References

[1] L. E. Meyers, Update on William Wernick's "triangle constructions with three located points", *Math. Mag.*, 69 (1996) 46–49.
[2] W. Wernick, Triangle constructions with three located points, *Math. Mag.*, 55 (1982) 227–230.

Harold Connelly: 102 Blowing Tree Drive, Georgetown, Kentucky 40324, USA
E-mail address: cherylandharold@roadrunner.com

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr

Jean-Pierre Ehrmann: 6, rue des Cailloux, 92110 - Clichy, France
E-mail address: Jean-Pierre.EHRMANN@wanadoo.fr

Forum Geometricorum
Volume 7 (2007) 107–109.

FORUM GEOM
ISSN 1534-1178

Three Pappus Chains Inside the Arbelos: Some Identities

Giovanni Lucca

Abstract. We consider the three different Pappus chains that can be constructed inside the arbelos and we deduce some identities involving the radii of the circles of n-th order and the incircle radius.

1. Introduction

The Pappus chain [1] is an infinite series of circles constructed starting from the Archimedean figure named arbelos (also said shoemaker knife) so that the generic circle $\mathcal{C}_i$, $(i = 1, 2, \ldots)$ of the chain is tangent to the the circles $\mathcal{C}_{i-1}$ and $\mathcal{C}_{i+1}$ and to two of the three semicircles $\mathcal{C}_{\mathrm{a}}$, $\mathcal{C}_{\mathrm{b}}$ and $\mathcal{C}_{\mathrm{r}}$ forming the arbelos. In a generic arbelos three different Pappus chains can be drawn (see Figure 1).

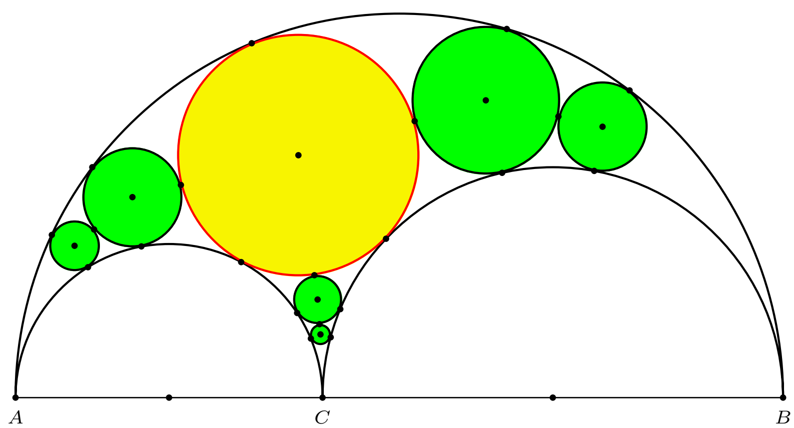


Figure 1.

In Figure 1, the diameter AC of the left semicircle $\mathcal{C}_{\mathrm{a}}$ is $2a$, the diameter CB of the right semicircle $\mathcal{C}_{\mathrm{b}}$ is $2b$, and the diameter AB of the outer semicircle $\mathcal{C}_{\mathrm{r}}$ is $2r$, $r = a + b$. The first circle $\mathcal{C}_1$ is common to all three chains and is named the incircle of the arbelos. By applying the circular inversion technique, it is possible to determine the center coordinates and radius of each chain; the radii are expressed by the formulas reported in Table I. The chain tending to point C is named Γ_{r}, the chain tending to point B is named Γ_{a} and the chain tending to point A is named Γ_{b}. As far as chains Γ_{a} and Γ_{b} are concerned, the expressions for the radii are given in [2] while for Γ_{r}, we give an inductive proof below.

Publication Date: June 4, 2007. Communicating Editor: Floor van Lamoen.

Table I: Radii of the circles forming the three Pappus chains

Chain	Γ_{r}	Γ_{a}	Γ_{b}
Radius of n-th circle	$\rho_{\mathrm{r}n} = \frac{rab}{n^2r^2-ab}$	$\rho_{\mathrm{a}n} = \frac{rab}{n^2a^2+rb}$	$\rho_{\mathrm{b}n} = \frac{rab}{n^2b^2+ra}$

For integers $n \geq 1$, consider the statement

$$P(n) \qquad \rho_{\mathrm{r}n} = \frac{rab}{n^2r^2 - ab}.$$

$P(1)$ is true since the first circle of the chain is the arbelos incircle having radius given by formula (3).

We show that $P(n) \Rightarrow P(n+1)$.

Let us consider the circles $\mathcal{C}_{\mathrm{rn}}$ and $\mathcal{C}_{\mathrm{rn+1}}$ in the chain Γ_{r}, together the inner semicircles $\mathcal{C}_{\mathrm{a}}$ and $\mathcal{C}_{\mathrm{b}}$ inside the arbelos. Applying Descartes' theorem we have

$$2(\varepsilon_{\mathrm{r}n}^2 + \varepsilon_{\mathrm{r}n+1}^2 + \varepsilon_{\mathrm{a}}^2 + \varepsilon_{\mathrm{b}}^2) = (\varepsilon_{\mathrm{r}n} + \varepsilon_{\mathrm{r}n+1} + \varepsilon_{\mathrm{a}} + \varepsilon_{\mathrm{b}})^2, \tag{1}$$

where $\varepsilon_{\mathrm{r}n}$, $\varepsilon_{\mathrm{r}n+1}$, ε_{a} and ε_{b} are the curvatures, *i.e.*, reciprocals of the radii of the circles. Rewriting this as

$$\varepsilon_{\mathrm{r}n+1}^2 - 2\varepsilon_{\mathrm{r}n+1}(\varepsilon_{\mathrm{r}n} + \varepsilon_{\mathrm{a}} + \varepsilon_{\mathrm{b}}) + \varepsilon_{\mathrm{r}n}^2 + \varepsilon_{\mathrm{a}}^2 + \varepsilon_{\mathrm{b}}^2 - 2(\varepsilon_{\mathrm{r}n}\varepsilon_{\mathrm{a}} + \varepsilon_{\mathrm{a}}\varepsilon_{\mathrm{b}} + \varepsilon_{\mathrm{b}}\varepsilon_{\mathrm{r}n}) = 0,$$

we have

$$\varepsilon_{\mathrm{r}n+1} = \varepsilon_{\mathrm{r}n} + \varepsilon_{\mathrm{a}} + \varepsilon_{\mathrm{b}} \pm 2\sqrt{\varepsilon_{\mathrm{r}n}\varepsilon_{\mathrm{a}} + \varepsilon_{\mathrm{a}}\varepsilon_{\mathrm{b}} + \varepsilon_{\mathrm{b}}\varepsilon_{\mathrm{r}n}}. \tag{2}$$

Substituting into (2) $\varepsilon_{\mathrm{a}} = \frac{1}{a}$, $\varepsilon_{\mathrm{b}} = \frac{1}{b}$ and $\varepsilon_{\mathrm{r}n} = \frac{rab}{n^2r^2-ab}$, we obtain, after a few steps of simple algebraic calculations,

$$\rho_{\mathrm{r}n+1} = \frac{1}{\varepsilon_{\mathrm{r}n+1}} = \frac{rab}{(n+1)^2r^2 - ab}.$$

This proves that $P(n) \Rightarrow P(n+1)$, and by induction, $P(n)$ is true for every integer $n \geq 1$.

2. Relationships among the n-th circles radii and incircle radius

For the following, it is useful to write explicitly the incircle radius ρ_{inc} that is given by:

$$\rho_{\mathrm{inc}} = \frac{rab}{a^2 + ab + b^2} \tag{3}$$

Formula (3) is directly obtained by each one of the three formulas for the radius in Table I for $n = 1$. It is useful too to write the square of the incircle radius that is:

$$\rho_{\mathrm{inc}}^2 = \frac{r^2a^2b^2}{a^4 + 2a^3b + 2a^2b^2 + 2ab^3 + b^4}. \tag{4}$$

We enunciate now the following proposition related to three different identities among the circles chains radii and the incircle radius.

Proposition. *Given a generic arbelos with its three Pappus chains, the following identities hold for each integer n:*

$$\rho_{\mathrm{inc}}\left(\frac{1}{\rho_{\mathrm{r}n}}+\frac{1}{\rho_{\mathrm{a}n}}+\frac{1}{\rho_{\mathrm{b}n}}\right)=2n^2+1, \tag{5}$$

$$\rho_{\mathrm{inc}}^2\left(\frac{1}{\rho_{\mathrm{r}n}^2}+\frac{1}{\rho_{\mathrm{a}n}^2}+\frac{1}{\rho_{\mathrm{b}n}^2}\right)=2n^4+1, \tag{6}$$

$$\rho_{\mathrm{inc}}^2\left(\frac{1}{\rho_{\mathrm{r}n}}\cdot\frac{1}{\rho_{\mathrm{a}n}}+\frac{1}{\rho_{\mathrm{a}n}}\cdot\frac{1}{\rho_{\mathrm{b}n}}+\frac{1}{\rho_{\mathrm{b}n}}\cdot\frac{1}{\rho_{\mathrm{r}n}}\right)=n^4+2n^2. \tag{7}$$

Proof. To demonstrate (5), one has to substitute in it the expression for the radius incircle given by (3) and the expressions for the radii of n-th circles chain given in Table I. Using the fact that $r=a+b$, one obtains

$$\frac{rab}{a^2+ab+b^2}\left(\frac{n^2r^2-ab}{rab}+\frac{n^2a^2+rb}{rab}+\frac{n^2b^2+ra}{rab}\right)=2n^2+1.$$

For (6), one has to substitute in it the expression for the square of the radius incircle given by (4) and to take the squares of the radii of n-th circles chain given in Table I. Using the fact that $r=a+b$, one obtains

$$\frac{r^2a^2b^2}{(a^2+ab+b^2)^2}\left(\left(\frac{n^2r^2-ab}{rab}\right)^2+\left(\frac{n^2a^2+rb}{rab}\right)^2+\left(\frac{n^2b^2+ra}{rab}\right)^2\right)=2n^4+1.$$

For (7), one has to substitute in it the expression for the square of the incircle radius given by (4) and the expressions for the radii of the n-th circles given in Table I. This leads to $\frac{r^2a^2b^2}{(a^2+ab+b^2)^2}\cdot\frac{D}{r^2a^2b^2}$, where

$$\begin{aligned}D=&(n^2r^2-ab)(n^2a^2+rb)+(n^2a^2+rb)(n^2b^2+ra)+(n^2b^2+ra)(n^2r^2-ab)\\=&(n^4+2n^2)(a^2+ab+b^2)^2,\end{aligned}$$

by using the fact that $r=a+b$. Finally, this leads to (7). □

3. Conclusion

Considering the three Pappus chains that can be drawn inside a generic arbelos, some identities involving the incircle radius and the n-th circles chain radii have been shown. All these identities generate sequences of integers.

References

[1] F. M. van Lamoen and E. W. Weisstein, Pappus Chain, MathWorld-A Wolfram Web Resource, http://mathworld.wolfram.com/PappusChain.html

[2] L. Bankoff, The golden arbelos, *Scripta Math.*, 21 (1955) 70–76.

Giovanni Lucca: Via Corvi 20, 29100 Piacenza, Italy

E-mail address: vanni_lucca@inwind.it

Forum Geometricorum
Volume 7 (2007) 111–113.

FORUM GEOM
ISSN 1534-1178

Some Powerian Pairs in the Arbelos

Floor van Lamoen

Abstract. Frank Power has presented two pairs of Archimedean circles in the arbelos. In each case the two Archimedean circles are tangent to each other and tangent to a given circle. We give some more of these Powerian pairs.

1. Introduction

We consider an arbelos with greater semicircle (O) of radius r and smaller semicircle (O_1) and (O_2) of radii r_1 and r_2 respectively. The semicircles (O_1) and (O) meet in A, (O_2) and (O) in B, (O_1) and (O_2) in C and the line through C perpendicular to AB meets (O) in D. Beginning with Leon Bankoff [1], a number of interesting circles congruent to the Archimedean twin circles has been found associated with the arbelos. These have radii $\frac{r_1 r_2}{r}$. See [2]. Frank Power [5] has presented two pairs of Archimedean circles in the Arbelos with a definition unlike the other known ones given for instance in [2, 3, 4].[1]

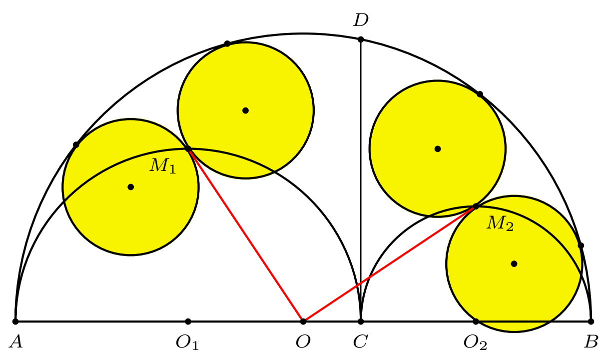


Figure 1

Proposition 1 (Power [5]). *Let M_1 and M_2 be the 'highest' points of (O_1) and (O_2) respectively. Then the pairs of congruent circles tangent to (O) and tangent to each other at M_1 and M_2 respectively, are pairs of Archimedean circles.*

To pairs of Archimedean circles tangent to a given circle and to each other at a given point we will give the name *Powerian pairs*.

Publication Date: June 12, 2007. Communicating Editor: Paul Yiu.

[1]The pair of Archimedean circles (A_{5a}) and (A_{5b}), with numbering as in [4], qualifies for what we will later in the paper refer to as *Powerian pair*, as they are tangent to each other at C and to the circular hull of Archimedes' twin circles. This however is not how they were originally defined.

2. Three double Powerian pairs

2.1. Let M be the midpoint of CD. Consider the endpoints U_1 and U_2 of the diameter of (CD) perpendicular to OM.

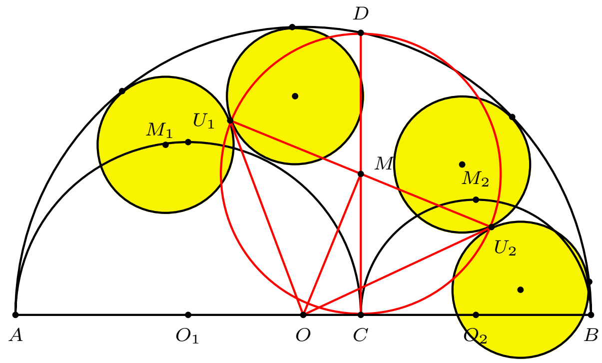


Figure 2

Note that $OC^2 = (r_1 - r_2)^2$ and as $CD = 2\sqrt{r_1 r_2}$ that $OD^2 = r_1^2 - r_1 r_2 + r_2^2$ and $OU_1^2 = r_1^2 + r_2^2$.

Now consider the pairs of congruent circles tangent to each other at U_1 and U_2 and tangent to (O). The radii ρ of these circles satisfy

$$(r_1 + r_2 - \rho)^2 = OU_1^2 + \rho^2$$

from which we see that $\rho = \frac{r_1 r_2}{r}$. This pair is thus Powerian. By symmetry the other pair is Powerian as well.

2.2. Let T_1 and T_2 be the points of tangency of the common tangent of (O_1) and (O_2) not through C. Now consider the midpoint O' of O_1O_2, also the center of the semicircle (O_1O_2), which is tangent to segment T_1T_2 at its midpoint.

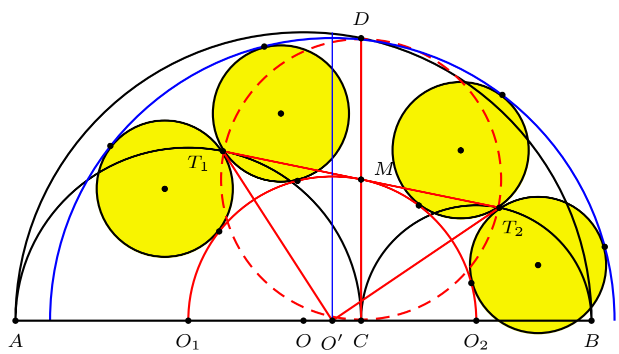


Figure 3

As $T_1T_2 = 2\sqrt{r_1 r_2}$ we see that $O'T_1^2 = \left(\frac{r_1+r_2}{2}\right)^2 + r_1 r_2$. Now consider the pairs of congruent circles tangent to each other at T_1 and tangent to (O_1O_2). The radii ρ of these circles satisfy

$$\left(\frac{r_1+r_2}{2}\right)+\rho)^2-\rho^2=O'T_1^2$$

from which we see that $\rho=\frac{r_1r_2}{r}$ and this pair is Powerian. By symmetry the pair of congruent circles tangent to each other at T_2 and to (O_1O_2) is Powerian.

Remark: These pairs are also tangent to the circle with center O' through the point where the Schoch line meets (O).

2.3. Note that $AD=2\sqrt{rr_1}$, hence

$$AT_1=\frac{r_1}{r}AD=\frac{2r_1\sqrt{r_1}}{\sqrt{r}}.$$

Now consider the pair of congruent circles tangent to each other at T_1 and to the circle with center A through C. The radii of these circles satisfy

$$AT_1^2+\rho^2=(2r_1-\rho)^2$$

from which we see that $\rho=\frac{r_1r_2}{r}$ and this pair is Powerian. In the same way the pair of congruent circles tangent to each other at T_2 and to the circle with center B through C is Powerian.

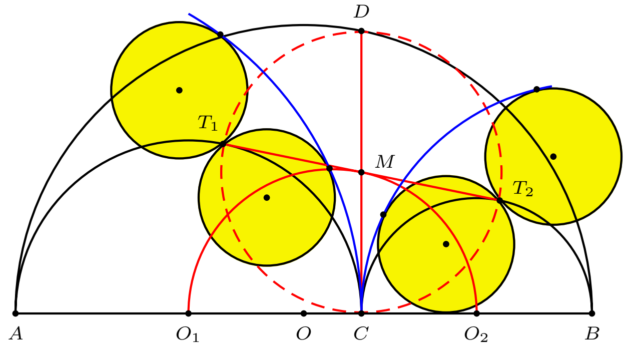


Figure 4

References

[1] L. Bankoff, Are the Archimedean circles really twin?, *Math. Mag.*, (19) .
[2] C. W. Dodge, T. Schoch, P. Y. Woo and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.*, 72 (1999) 202–213.
[3] F. M. van Lamoen, Archimedean Adventures, *Forum Geom.*, 6 (2006) 79–96.
[4] F. M. van Lamoen, *Online catalogue of Archimedean Circles*, available at http://home.planet.nl/ lamoen/wiskunde/Arbelos/Catalogue.htm
[5] F. Power, Some more Archimedean circles in the Arbelos, *Forum Geom.*, 5 (2005) 133–134.

Floor van Lamoen: St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands
E-mail address: fvanlamoen@planet.nl

Forum Geometricorum
Volume 7 (2007) 115–120.

FORUM GEOM
ISSN 1534-1178

The Arbelos and Nine-Point Circles

Quang Tuan Bui

Abstract. We construct some new Archimedean circles in an arbelos in connection with the nine-point circles of some appropriate triangles. We also construct two new pairs of Archimedes circles analogous to those of Frank Power, and one pair of Archimedean circles related to the tangents of the arbelos.

1. Introduction

We consider an arbelos consisting of three semicircles (O_1), (O_2), (O), with points of tangency A, B, P. Denote by r_1, r_2 the radii of (O_1), (O_2) respectively. Archimedes has shown that the two circles, each tangent to (O), the common tangent PQ of (O_1), (O_2), and one of (O_1), (O_2), have congruent radius $r = \frac{r_1 r_2}{r_1+r_2}$. See [1, 2]. Let C be a point on the half line PQ such that $PC = h$. We consider the nine-point circle (N) of triangle ABC. This clearly passes through O, the midpoint of AB, and P, the altitude foot of C on AB. Let AC intersect (O_1) again at A', and BC intersect (O_2) again at B'. Let $O_{\rm e}$ and H be the circumcenter and orthocenter of triangle ABC. Note that C and H are on opposite sides of the semicircular arc (O), and the triangles ABC and ABH have the same nine-point circle. We shall therefore assume C beyond the point Q on the half line PQ. See Figure 1. In this paper the labeling of knowing Archimedean circles follows [2].

2. Archimedean circles with centers on the nine-point circle

Let the perpendicular bisector of AB cut (N) at O and $M_{\rm e}$, and the altitude CP cut (N) at P and $M_{\rm h}$. See Figure 1.

2.1. It is easy to show that $POM_{\rm e}M_{\rm h}$ is a rectangle so $M_{\rm e}$ is the reflection of P in N. Because $O_{\rm e}$ is also the reflection of H in N, $HPO_{\rm e}M_{\rm e}$ is a parallelogram, and we have

$$O_{\rm e}M_{\rm e} = PH. \tag{1}$$

Furthermore, from the similarity of triangles HPB and APC, we have $\frac{PH}{PB} = \frac{PA}{PC}$. Hence,

$$PH = \frac{4r_1r_2}{h}. \tag{2}$$

Publication Date: June 18, 2007. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this paper.

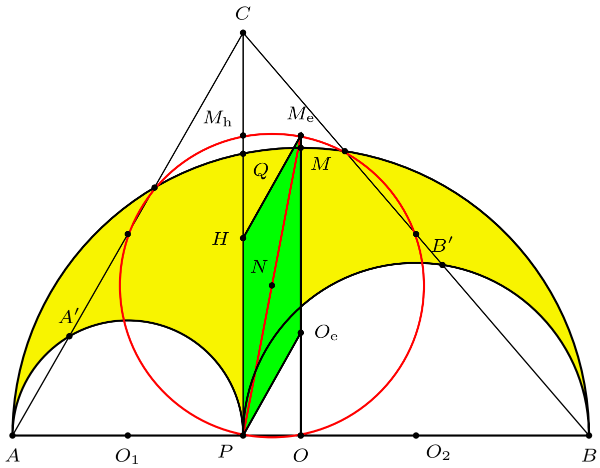


Figure 1.

2.2. Since C is beyond Q on the half line PQ, the intersection F of $A'O_1$ and $B'O_2$ is a point F below the arbelos. Denote by (I) the incircle of triangle FO_1O_2. See Figure 2. The line IO_1 bisects both angles O_2O_1F and $A'O_1A$. Because $O_1A' = O_1A$, IO_1 is perpendicular to AC, and therefore is parallel to BH. Similarly, IO_2 is parallel to AH. From these, two triangles AHB and O_2IO_1 are homothetic with ratio $\frac{AB}{O_2O_1} = 2$. It is easy to show that O is the touch point of (I) with AB and that the inradius is

$$IO = \frac{1}{2} \cdot PH. \tag{3}$$

In fact, if F' is the reflection of F in the midpoint of O_1O_2 then O_1FO_2F' is a parallelogram and the circle (PH) (with PH as diameter) is the incircle of $F'O_1O_2$. It is the reflection of (I) in midpoint of O_1O_2.

2.3. Now we apply these results to the arbelos. From (2), $\frac{1}{2} \cdot PH = \frac{2r_1r_2}{h} =$ Archimedean radius $\frac{r_1r_2}{r_1+r_2}$ if and only if

$$CP = h = 2(r_1 + r_2) = AB.$$

In this case, point C and the orthocenter H of ABC are easy constructed and the circle with diameter PH is the Bankoff triplet circle (W_3). From this we can also construct also the incircle of the arbelos. In this case $F' =$ incenter of the arbelos. From (3) we can show that when $CP = AB$, the incircle of FO_1O_2 is also Archimedean. See Figure 3.

Let M be the intersection of OO_e and the semicircle (O), *i.e.*, the highest point of (O). When $CP = h = 2(r_1 + r_2) = AB$,

$$OO_e = M_hH = M_hC = \frac{h - PH}{2} = (r_1 + r_2) - \frac{r_1r_2}{r_1 + r_2}.$$

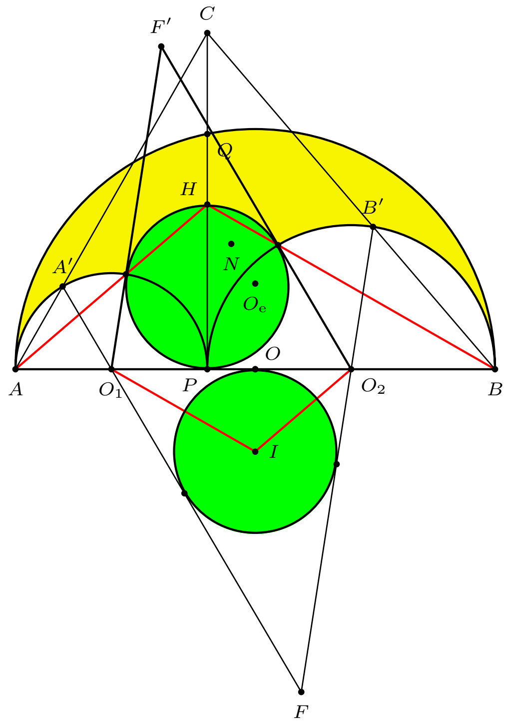


Figure 2.

Therefore,

$$O_{\mathrm{e}}M = (r_1+r_2) - \left(r_1+r_2-\frac{r_1r_2}{r_1+r_2}\right) = \frac{r_1r_2}{r_1+r_2}.$$

From (1), $O_{\mathrm{e}}M_{\mathrm{e}} = PH = \frac{2r_1r_2}{r_1+r_2}$. This means that M is the midpoint of $O_{\mathrm{e}}M_{\mathrm{e}}$, or the two circles centered at O_{e} and M_{e} and touching (O) at M are also Archimedean circles. See Figure 3.

We summarize the results as follows.

Proposition 1. *In the arbelos (O_1), (O_2), (O), if C is any point on the half line PQ beyond Q and H is orthocenter of ABC, then the circle (PH) is Archimedean if and only if $CP = AB = 2(r_1+r_2)$. In this case, we have the following results.*
(1). *The orthocenter H of ABC is the intersection point of Bankoff triplet circle (W_3) with PQ (other than P).*
(2). *The incircle of triangle FO_1O_2 is an Archimedean circle touching AB at O; it is reflection of (W_3) in the midpoint of O_1O_2.*
(3). *The circle centered at circumcenter O_{e} of ABC and touching (O) at its highest point M is an Archimedean circle. This circle is (W_{20}).*
(4). *The circle centered on nine point circle of ABC and touching (O) at M is an Archimedean circle; it is the reflection of (W_{20}) in M.*
(5). *The reflection F' of F in midpoint of O_1O_2 is the incenter of the arbelos.*

Remarks. (a) The Archimedean circles in (2) and (4) above are new.

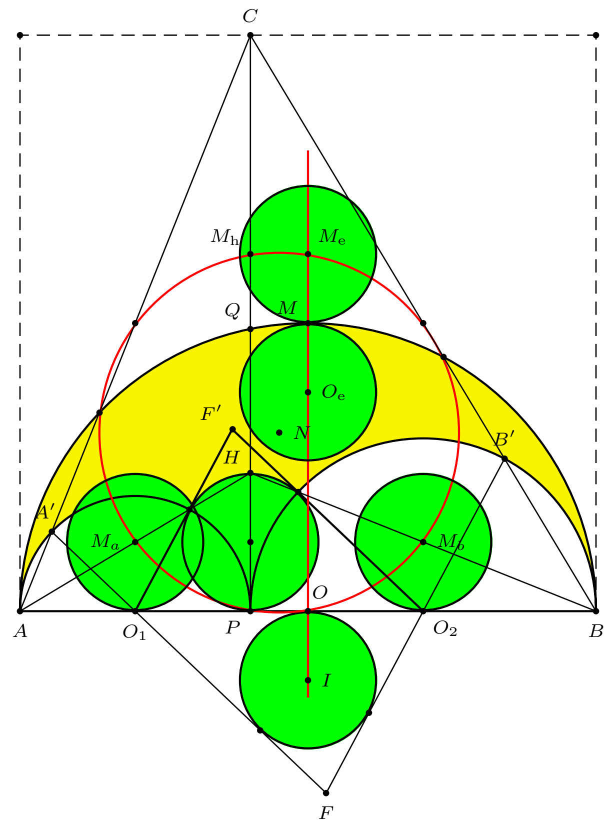


Figure 3.

(b) There are two more obvious Archimedean circles with centers on the nine-point circle. These are (M_a) and (M_b), where M_a and M_b are the midpoints of AH and BH respectively. See Figure 3.

(c) The midpoints M_a, M_b of HA, HB are on nine point circle of ABC and are two vertices of Eulerian triangle of ABC. Two circles centered at M_a, M_b and touch AB at O_1, O_2 respectively are congruent with (W_3) so they are also Archimedean circles (see [2]).

3. Two new pairs of Archimedean circles

If T is a point such that $OT^2 = r_1^2 + r_2^2$, then there is a pair of Archimedean circles mutually tangent at T, and each tangent internally to (O). Frank Power [5]. constructed two such pairs with $T = M_1$, M_2, the highest points of (O_1) and (O_2) respectively. Allowing tangency with other circles, Floor van Lamoen [4] called such a pair Powerian. We construct two new Powerian pairs.

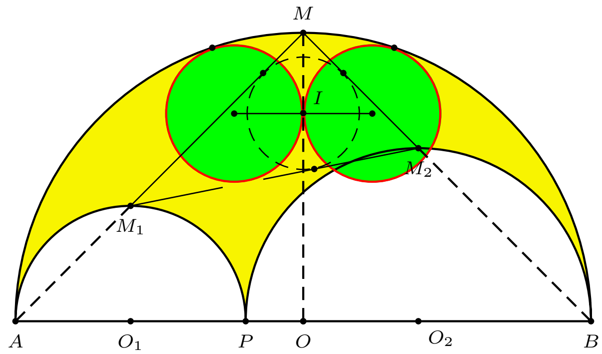


Figure 4.

3.1. The triangle MM_1M_2 has $MM_1 = \sqrt{2} \cdot r_2$, $MM_2 = \sqrt{2} \cdot r_1$, and a right angle at M. Its incenter is the point I on OM such that

$$MI = \sqrt{2} \cdot \frac{1}{2}(MM_1 + MM_2 - M_1M_2) = (r_1 + r_2) - \sqrt{r_1^2 + r_2^2}.$$

Therefore, $OI^2 = r_1^2 + r_2^2$, and we have a Powerian pair. See Figure 4.

3.2. Consider also the semicircles (T_1) and (T_2) with diameters AO_2 and BO_1. The intersection J of (T_1) and (T_2) satisfies

$$OJ^2 = OP^2 + PJ^2 = (r_1 - r_2)^2 + 2r_1r_2 = r_1^2 + r_2^2.$$

Therefore, we have another Powerian pair. See Figure 5.

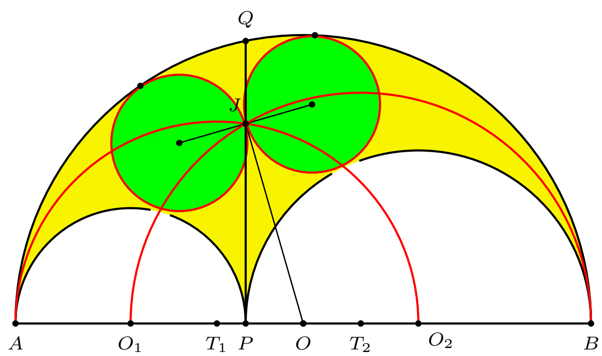


Figure 5.

4. Two Archimedean circles related to the tangents of the arbelos

We give two more Archimedean circles related to the tangents of the arbelos.

Let $\mathcal{L}$ be the tangent of (O) at Q, and Q_1, Q_2 the orthogonal projections of O_1, O_2 on $\mathcal{L}$. The lines O_1Q_1 and O_2Q_2 intersect the semicircles (O_1) and (O_2) at R_1 and R_2 respectively. Note that R_1R_2 is a common tangent of the semicircles (O_1) and (O_2). The circles (N_1), (N_2) with diameters Q_1R_1 and Q_2R_2 are Archimedean. Indeed, if (W_6) and (W_7) are the two Archimedean circles through

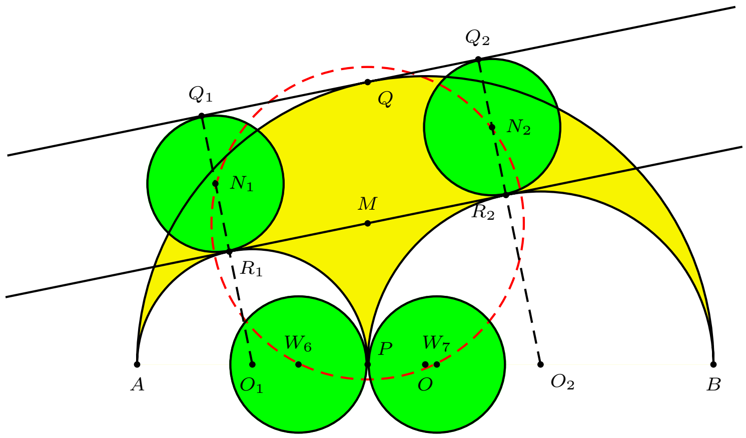


Figure 6.

P with centers on AB (see [2]), then N_1, N_2, W_6, W_7 lie on the same circle with center the midpoint M of PQ. See Figure 6. We leave the details to the reader.

References

[1] L. Bankoff, Are the Archimedean circles really twin?, *Math. Mag.*, 47 (1974) 214–218.

[2] C. W. Dodge, T. Schoch, P. Y. Woo and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.*, 72 (1999) 202–213.

[3] F. M. van Lamoen, *Online catalogue of Archimedean Circles*, available at http://home.planet.nl/ lamoen/wiskunde/Arbelos/Catalogue.htm

[4] F. M. van Lamoen, Some more Powerian pairs in the arbelos, *Forum Geom.*, 7 (2007) 111–113.

[5] F. Power, Some more Archimedean circles in the Arbelos, *Forum Geom.*, 5 (2005) 133–134.

Quang Tuan Bui: 45B, 296/86 by-street, Minh Khai Street, Hanoi, Vietnam

E-mail address: bqtuan1962@yahoo.com

Forum Geometricorum
Volume 7 (2007) 121–123.

FORUM GEOM
ISSN 1534-1178

Characterizations of an Infinite Set of Archimedean Circles

Hiroshi Okumura and Masayuki Watanabe

Abstract. For an arbelos with the two inner circles touching at a point O, we give necessary and sufficient conditions that a circle passing through O is Archimedean.

Consider an arbelos with two inner circles α and β with radii a and b respectively touching externally at a point O. A circle of radius $r_A = ab/(a+b)$ is called Archimedean. In [3], we have constructed three infinite sets of Archimedean circles. One of these consists of circles passing through the point O. In this note we give some characterizations of Archimedean circles passing through O. We set up a rectangular coordinate system with origin O and the positive x-axis along a diameter OA of α (see Figure 1).

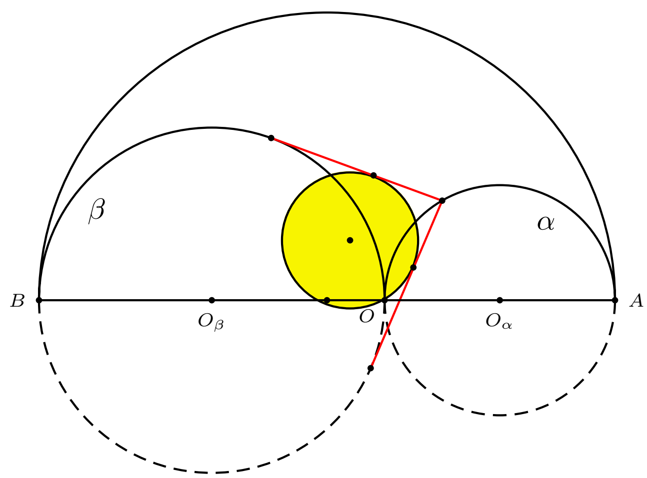


Figure 1

Theorem 1. *A circle through O (not tangent internally to β) is Archimedean if and only if its external common tangents with β intersect at a point on α.*

Proof. Consider a circle δ with radius $r \neq b$ and center $(r\cos\theta, r\sin\theta)$ for some real number θ with $\cos\theta \neq -1$. The intersection of the common external tangents

Publication Date: July 2, 2007. Communicating Editor: Paul Yiu.

of β and δ is the external center of similitude of the two circles, which divides the segment joining their centers externally in the ratio $b : r$. This is the point

$$\left(\frac{br(1+\cos\theta)}{b-r}, \frac{br\sin\theta}{b-r}\right). \tag{1}$$

The theorem follows from

$$\left(\frac{br(1+\cos\theta)}{b-r} - a\right)^2 + \left(\frac{br\sin\theta}{b-r}\right)^2 - a^2 = \frac{2br(a+b)(1+\cos\theta)}{(b-r)^2}(r - r_A).$$

□

Let O_α and O_β be the centers of the circles α and β respectively.

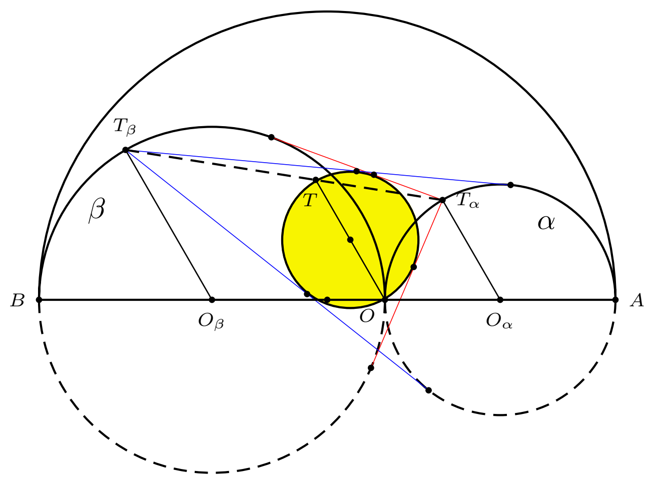


Figure 2

Corollary 2. *Let δ be an Archimedean circle with a diameter OT, and T_α the intersection of the external common tangents of the circles δ and β; similarly define T_β.*
(i) *The vectors $\overrightarrow{OT}$ and $\overrightarrow{O_\alpha T_\alpha}$ are parallel with the same direction.*
(ii) *The point T divides the segment $T_\alpha T_\beta$ internally in the ratio $a : b$.*

Proof. We describe the center of δ by $(r_A\cos\theta, r_A\sin\theta)$ for some real number θ (see Figure 2). Then the point T_α is described by

$$\left(\frac{br_A(1+\cos\theta)}{b-r_A}, \frac{br_A\sin\theta}{b-r_A}\right) = (a(1+\cos\theta), a\sin\theta)$$

by (1). This implies $\overrightarrow{O_\alpha T_\alpha} = a(\cos\theta, \sin\theta)$. (ii) is obtained directly, since T_β is expressed by $(b(-1+\cos\theta), b\sin\theta)$. □

In Theorem 1, we exclude the Archimedean circle which touches β internally at the point O. But this corollary holds even if the circle δ touches β internally. If δ is the Bankoff circle touching the line OA at the origin O [1], then T_α is the highest point on α. If δ is the Archimedean circle touching β externally at the point O, then T_α obviously coincides with the point A. This fact is referred in [2] using the circle labeled W_6. Another notable Archimedean circle passing through O is that having center on the Schoch line $x = \frac{b-a}{b+a} r_A$, which is labeled as U_0 in [2]. We have showed that the intersection of the external common tangents of β and this circle is the intersection of the line $x = 2r_A$ and the circle α [3].

By the uniqueness of the figure, we get the following characterizations of the Archimedean circles passing through the point O.

Corollary 3. *Let δ be a circle with a diameter OT, and let T_α and T_β be points on α and β respectively such that $\overrightarrow{O_\alpha T_\alpha}$ and $\overrightarrow{O_\beta T_\beta}$ are parallel to $\overrightarrow{OT}$ with the same direction.* (i) *The circle δ is Archimedean if and only if the points T divides the line segment $T_\alpha T_\beta$ internally in the ratio $a : b$.* (ii) *If the center of δ does not lie on the line OA, then δ is Archimedean if and only if the three points T_α, T_β and T are collinear.*

The statement (i) in this corollary also holds when δ touches β internally.

References

[1] L. Bankoff, Are the twin circles of Archimedes really twins?, Math. Mag., **47** (1974) 134-137.

[2] C. W. Dodge, T. Schoch, P. Y. Woo, and P. Yiu, Those ubiquitous Archimedean circles, Math. Mag., **72** (1999) 202-213.

[3] H. Okumura and M. Watanabe, The Archimedean circles of Schoch and Woo, Forum Geom., **4** (2004) 27-34.

Hiroshi Okumura: Department of Life Science and Information, Maebashi Institute of Technology, 460-1 Kamisadori Maebashi Gunma 371-0816, Japan

E-mail address: `okumura@maebashi-it.ac.jp`

Masayuki Watanabe: Department of Integrated Design Engineering, Maebashi Institute of Technology, 460-1 Kamisadori Maebashi Gunma 371-0816, Japan

E-mail address: `watanabe@maebashi-it.ac.jp`

Forum Geometricorum
Volume 7 (2007) 125–128.

FORUM GEOM
ISSN 1534-1178

Remarks on Woo's Archimedean Circles

Hiroshi Okumura and Masayuki Watanabe

Abstract. The property of Woo's Archimedean circles does not hold only for Archimedean circles but circles with any radii. The exceptional case of this has a close connection to Archimedean circles.

1. Introduction

Let A and B be points with coordinates $(2a, 0)$ and $(-2b, 0)$ on the x-axis with the origin O and positive real numbers a and b. Let α, β and γ be semicircles forming an arbelos with diameters OA, OB and AB respectively. We follow the notations in [4]. For a real number n, let $\alpha(n)$ and $\beta(n)$ be the semicircles in the upper half-plane with centers $(n, 0)$ and $(-n, 0)$ respectively and passing through the origin O. A circle with radius $r = \frac{ab}{a+b}$ is called an Archimedean circle. Thomas Schoch has found that the circle touching the circles $\alpha(2a)$ and $\beta(2b)$ externally and γ internally is Archimedean [2] (see Figure 1). Peter Woo called the Schoch line the one passing through the center of this circle and perpendicular to the x-axis, and found that the circle U_n touching the circles $\alpha(na)$ and $\beta(nb)$ externally with center on the Schoch line is Archimedean for a nonnegative real number n. In this note we consider the property of Woo's Archimedean circles in a general way.

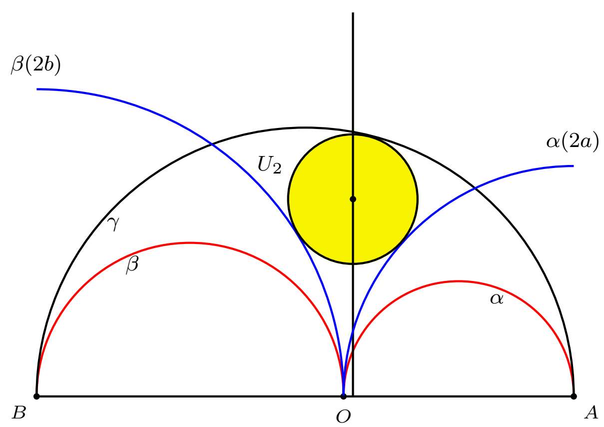


Figure 1.

Publication Date: September 4, 2007. Communicating Editor: Paul Yiu.

2. A generalization of Woo's Archimedean circles

We show that the property of Woo's Archimedean circles does not only hold for Archimedean circles. Indeed circles with any radii can be obtained in a similar way. We say that a circle touches $\alpha(na)$ *appropriately* if they touch externally (respectively internally) for a positive (respectively negative) number n. If one of the two circles is a point circle and lies on the other, we also say that the circle touches $\alpha(na)$ appropriately. The same notion of appropriate tangency applies to $\beta(nb)$.

Theorem 1. *Let s and t be nonzero real numbers such that $tb \pm sa \neq 0$. If there is a circle of radius ρ touching the circles $\alpha(nsa)$ and $\beta(ntb)$ appropriately for a real number n, then its center lies on the line*

$$x = \frac{tb - sa}{tb + sa}\rho. \tag{1}$$

Proof. Consider the center (x, y) of the circle with radius ρ touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately. The distance between (x, y) and the centers of $\alpha(nsa)$ and $\beta(ntb)$ are $|\rho + nsa|$ and $|\rho + ntb|$ respectively. Therefore by the Pythagorean theorem,

$$y^2 = (\rho + nsa)^2 - (x - nsa)^2 = (\rho + ntb)^2 - (x + ntb)^2.$$

Solving the equations, we get (1) above. □

For a real number k different from 0 and $\pm\rho$, we can choose the real numbers s and t so that (1) expresses the line $x = k$. Let us assume $st > 0$. Then the circles $\alpha(nsa)$ and $\beta(ntb)$ lie on opposite sides of the y-axis. If $sz > 0$ and $tz > 0$, there is always a circle of radius ρ touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately. If $ns < 0$ and $nt < 0$, such a circle exists when $-2n(sa + tb) \leq 2\rho$. Hence in the case $st > 0$, the tangent circle exists if $n(sa + tb) + \rho \geq 0$. Now let us assume $st < 0$. Then circles $\alpha(nsa)$ and $\beta(ntb)$ lie on the same side of the y-axis. The circle of radius ρ touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately exists if $-2n(sa + tb) \geq 2\rho$. Hence in the case $st < 0$, the tangent circle exists if $n(sa+tb)+\rho \leq 0$. In any case the center of the circle with radius ρ touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately is

$$\left(\frac{tb - sa}{tb + sa}\rho,\ \pm\frac{2\sqrt{nabst((sa + tb) + \rho)\rho}}{|sa + tb|}\right).$$

Therefore, for every point P not on the lines $x = 0, \pm\rho$, we can choose real numbers s, t and n so that the circle, center P, radius ρ, is touching $\alpha(nsa)$ and $\beta(tzb)$ appropriately.

The Schoch line is the line $x = \frac{b-a}{b+a}r$ (see [4]). Therefore Woo's Archimedean circles and the Schoch line are obtained when $s = t$ and $\rho = r$ in Theorem 1. If $st > 0$, then $-1 < \frac{tb-sa}{tb+sa} < 1$. Hence the line (1) lies in the region $-\rho < x < \rho$ in this case.

The external center of similitude of β and a circle with radius ρ and center on the line (1) lies on the line

$$x = \frac{2tb^2\rho}{(b-\rho)(sa+tb)}$$

by similarity. In particular, the external centers of similitude of Woo's Archimedean circles and β lie on the line $x = 2r$. See [4].

3. Circles with centers on the y-axis

We have excluded the cases $tb \pm sa \neq 0$ in Theorem 1. The case $tb + sa = 0$ is indeed trivial since the circles $\alpha(nsa)$ and $\beta(ntb)$ coincide. By Theorem 1, for $k \neq 0$, the circle touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately and with center on the line $x = k$ has radius $\frac{tb+sa}{tb-sa}k$. On the other hand, if $tb = sa$, the circles $\alpha(nsa)$ and $\beta(ntb)$ are congruent and lie on opposite sides of the y-axis, and the line (1) coincides with the y-axis. Therefore the radii of circles touching the two circles appropriately and having the center on this line cannot be determined uniquely.

We show that this exceptional case $(tb = sa)$ has a close connection with Archimedean circles. Since $\alpha(nsa)$ and $\beta(ntb)$ are congruent, we now define $\alpha[n] = \alpha(n(a+b))$ and $\beta[n] = \beta(n(a+b))$. The circles $\alpha[n]$ and $\beta[n]$ are congruent, and their radii are n times of the radius of γ. For two circles of radii ρ_1, ρ_2 and with distance d between their centers, consider their inclination [3] given by

$$\frac{\rho_1^2 + \rho_2^2 - d^2}{2\rho_1\rho_2}.$$

This is the cosine of the angle between the circles if they intersect, and is 0, $+1$, -1 according as they are orthogonal or tangent internally or externally.

Theorem 2. *If a circle $\mathcal{C}$ of radius ρ touches $\alpha[n]$ and $\beta[n]$ appropriately for a real number n, then the inclination of $\mathcal{C}$ and γ is $\frac{2r}{\rho} - n$.*

Proof. The square of the distance between the centers of the circles $\mathcal{C}$ and γ is $(\rho + n(a+b))^2 - (n(a+b))^2 + (a-b)^2$ by the Pythagorean theorem. Therefore their inclination is

$$\frac{\rho^2 + (a+b)^2 - (\rho + n(a+b))^2 + (n(a+b))^2 - (a-b)^2}{2\rho(a+b)} = \frac{2r}{\rho} - n.$$

□

Let k be a positive real number. The radius of a circle touching $\alpha[n]$ and $\beta[n]$ appropriately is kr if and only if the inclination of the circle and γ is $\frac{2}{k} - n$ for a real number n.

Corollary 3. *A circle touching $\alpha[n]$ and $\beta[n]$ appropriately for a real number n is Archimedean if and only if the inclination of this circle and γ is $2 - n$.*

This gives an infinite set of Archimedean circles δ_n with centers on the positive y-axis. The circle δ_n exists if $n \geq \frac{-r}{2(a+b)}$, and the maximal value of the inclination of γ and δ_n is $2 + \frac{r}{2(a+b)}$. The circle δ_1 touches γ internally, δ_2 is orthogonal to

γ, and δ_3 touches γ externally by the corollary (see Figure 2). The circle δ_0 is the Bankoff circle [1], whose inclination with γ is 2.

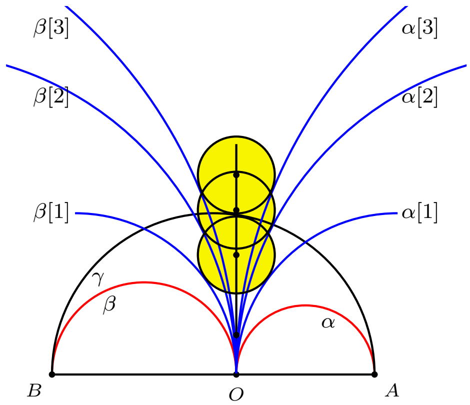


Figure 2

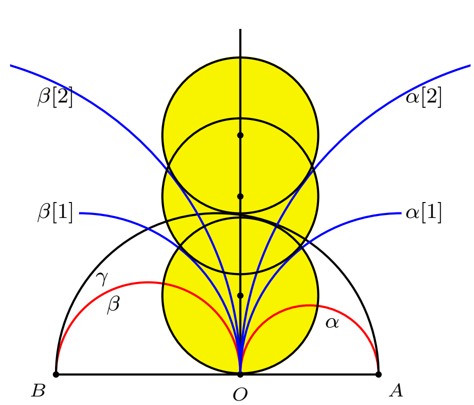


Figure 3

By the remark preceding Corollary 3 we can get circles with various radii and centers on the y-axis tangent or orthogonal to γ. Figure 3 shows some such examples. The three circles all have radii $2r$. One touches the degenerate circles $\alpha[0]$ and $\beta[0]$ (and the line AB) at O, and γ internally. A second circle touches $\alpha[1]$ and $\beta[1]$ externally and are orthogonal to γ. Finally, a third circle touches $\alpha[2]$, $\beta[2]$, and γ externally.

From [4], the center of the Woo circle U_n is the point

$$\left(\frac{b-a}{b+a}r,\ 2r\sqrt{n+\frac{r}{a+b}}\right).$$

The inclination of U_n and γ is $1+\frac{2(2-n)r}{a+b}$. This depends on the radii of α and β except the case $n=2$. In contrast to this, Corollary 3 shows that the inclination of δ_n and γ does not depend on the radii of α and β.

References

[1] L. Bankoff, Are the twin circles of Archimedes really twins?, *Math. Mag.*, 47 (1974) 134–137.

[2] C. W. Dodge, T. Schoch, P. Y. Woo, and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.*, 72 (1999) 202–213.

[3] J. G. Mauldon, Sets of equally inclined spheres, *Canad. J. Math.*, 14 (1962) 509–516.

[4] H. Okumura and M. Watanabe, The Archimedean circles of Schoch and Woo, *Forum Geom.*, 4 (2004) 27–34.

Hiroshi Okumura: Department of Life Science and Information, Maebashi Institute of Technology, 460-1 Kamisadori Maebashi Gunma 371-0816, Japan

E-mail address: `okumura@maebashi-it.ac.jp`

Masayuki Watanabe: Department of Integrated Design Engineering, Maebashi Institute of Technology, 460-1 Kamisadori Maebashi Gunma 371-0816, Japan

E-mail address: `watanabe@maebashi-it.ac.jp`

Forum Geometricorum
Volume 7 (2007) 129–135.

FORUM GEOM
ISSN 1534-1178

Heronian Triangles Whose Areas Are Integer Multiples of Their Perimeters

Lubomir Markov

Abstract. We present an improved algorithm for finding all solutions to Goehl's problem $A = mP$ for triangles, *i.e.*, the problem of finding all Heronian triangles whose area (A) is an integer multiple (m) of the perimeter (P). The new algorithm does not involve elimination of extraneous rational triangles, and is a true extension of Goehl's original method.

1. Introduction and main result

In a recent paper [3], we presented a solution to the problem of finding all Heronian triangles (triangles with integer sides and area) for which the area A is a multiple m of the perimeter P, where $m \in \mathbb{N}$. The problem was introduced by Goehl [2] and is of interest because although its solution is exceedingly simple in the special case of right triangles, the general case remained unsolved for about 20 years despite considerable effort. It is also remarkable and somewhat contrary to intuition that for each m there are only finitely many triangles with the property $A = mP$; for instance, the triangles $(6, 8, 10)$, $(5, 12, 13)$, $(6, 25, 29)$, $(7, 15, 20)$ and $(9, 10, 17)$ are the only ones whose area equals their perimeter (the case $m = 1$). Reproducing Goehl's solution to the problem in the special case of right triangles is a simple matter: Suppose that a and b are the legs of a right triangle and $c = \sqrt{a^2 + b^2}$ is the hypotenuse. Setting the area equal to a multiple m of the perimeter and manipulating, one immediately obtains the identities $8m^2 = (a - 4m)(b - 4m)$ and $c = a + b - 4m$. These allow us to determine a, b and c after finding all possible factorizations of the left-hand side of the form $8m^2 = d_1 \cdot d_2$ and matching d_1 and d_2 with $(a - 4m)$ and $(b - 4m)$, respectively; restricting d_1 to those integers that do not exceed $\sqrt{8m^2} = 2\sqrt{2}m$ assures $a < b$ and avoids repetitions. We state Goehl's result in the following form:

Publication Date: September 10, 2007. Communicating Editor: Paul Yiu.

The author expresses his sincerest thanks to Dr. John Goehl, Jr., for the meticulous care with which he reviewed this paper, and for the inspirational enthusiasm for integer number theory he has conveyed to him. The author also wishes to thank Barry University for granting him a sabbatical leave during the fall semester of 2006.

Theorem 1. *For a given m, the right-triangle solutions (a, b, c) to the problem $A = mP$ are determined from the relations*

$$8m^2 = (a - 4m)(b - 4m), \tag{1}$$

$$c = a + b - 4m. \tag{2}$$

Each factorization

$$8m^2 = d_1 \cdot d_2, \tag{3}$$

where

$$d_1 \leq \lfloor 2m\sqrt{2} \rfloor, \tag{4}$$

generates a solution triangle with sides given by the formulas

$$\begin{cases} a = d_1 + 4m, \\ b = d_2 + 4m, \\ c = d_1 + d_2 + 4m. \end{cases} \tag{5}$$

Our paper [3] extended Goehl's result to general triangles, but the solution involved extraneous rational triangles, which then had to be eliminated. The aim of this work is to present a radical simplification of our previous solution, which does not introduce extraneous triangles and is a direct generalization of Goehl's method. Our main goal is to prove the following theorem:

Theorem 2. *For a given m, all solutions (a, b, c) to the problem $A = mP$ are determined as follows: Find all divisors u of $2m$; for each u, find all numbers v relatively prime to u and such that $1 \leq v \leq \lfloor \sqrt{3}u \rfloor$; to each pair u and v, there correspond a factorization identity*

$$4m^2(u^2 + v^2) = \left[v\left(a - \frac{2m}{u}v\right) - 2mu \right] \left[v\left(b - \frac{2m}{u}v\right) - 2mu \right], \tag{6}$$

and a relation

$$c = a + b - \frac{4mv}{u}. \tag{7}$$

Each factorization

$$4m^2(u^2 + v^2) = \delta_1 \cdot \ \delta_2, \tag{8}$$

where

$$\delta_1 \leq \left\lfloor 2m\sqrt{u^2 + v^2} \right\rfloor \tag{9}$$

and only those factors δ_1, δ_2 for which $v \,\big|\, \delta_1 + 2mu$ and $v \,\big|\, \delta_2 + 2mu$ are considered, generates a solution triangle with sides given by the formulas

$$\begin{cases} a = \frac{\delta_1 + 2mu}{v} + \frac{2mv}{u}, \\ b = \frac{\delta_2 + 2mu}{v} + \frac{2mv}{u}, \\ c = \frac{\delta_1 + \delta_2 + 4mu}{v}. \end{cases} \tag{10}$$

Furthermore, for each fixed u, one concludes from the corresponding v's that
(1) *the obtuse-triangle solutions are obtained exactly when $v < u$;*
(2) *the acute-triangle solutions are obtained exactly when $u < v \leq \lfloor \sqrt{3}u \rfloor$, with*

the further restriction $\dfrac{2m}{u}(v^2-u^2) \le \delta_1 \le \left\lfloor 2m\sqrt{u^2+v^2} \right\rfloor$;
(3) *the right-triangle solutions are obtained exactly when* $u = v = 1$.

Note that Theorem 1 is a special case of Theorem 2 and that the substitution $u = v = 1$ transforms relations (6) through (10) into relations (1) through (5), respectively.

2. Summary of preliminary facts

Let A be the area and P the perimeter of a triangle with sides a, b, c, with the agreement that c shall always denote the largest side. Our problem (we call it $A = mP$ for short) is to find all Heronian triangles whose area equals an integer multiple m of the perimeter. We state all preliminaries as a sequence of lemmas whose proofs can either be easily reproduced by the reader, or can be found (except for Lemma 5) in [3].

First we note that Heron's formula

$$4A = \sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$$

and simple trigonometry easily imply the following lemma:

Lemma 3. *Assume that the triple* (a, b, c) *solves the problem* $A = mP$.
(1) $a + b - c$ *is an even integer.*
(2) $a + b - c < 4m\sqrt{3}$.
(3) *The resulting triangle is* $\begin{Bmatrix} obtuse \\ acute \\ right \end{Bmatrix}$ *if and only if* $a+b-c \begin{Bmatrix} < 4m \\ > 4m \\ = 4m \end{Bmatrix}$.

Next, we need a crucial rearrangement of Heron's formula:

Lemma 4. *The following doubly-Pythagorean form of Heron's formula holds:*

$$\left[c^2 - (a^2+b^2)\right]^2 + (4A)^2 = (2ab)^2. \tag{11}$$

This representation allows the problem $A = mP$ to be reduced to a problem about Pythagorean triples; for our purposes, a Pythagorean triple (x, y, z) shall consist of nonnegative integers such that z (the "hypotenuse") shall always represent the largest number, whereas x and y (the "legs") need not appear in any particular order. The following parametric representation of *primitive* Pythagorean triples (*i.e.*, such that the components do not have a common factor greater than 1) is the only preliminary statement not proved in [3]; a self-contained proof can be found in [1]:

Lemma 5. *Depending on whether the first leg* x *is odd or even, every primitive Pythagorean triple* (x, y, z) *is uniquely expressed as* $(u^2-v^2, 2uv, u^2+v^2)$ *where* u *and* v *are relatively prime of opposite parity, or* $\left(\dfrac{u^2-v^2}{2}, uv, \dfrac{u^2+v^2}{2}\right)$ *where* u *and* v *are relatively prime and odd.*

A combination of Lemmas 4 and 5 easily yields

Lemma 6. *For a fixed m, solving the problem $A = mP$ is equivalent to determining all integer a, b, c that satisfy the equation*

$$\left[c^2 - (a^2 + b^2)\right]^2 + [4m(a + b + c)]^2 = (2ab)^2, \tag{12}$$

or equivalently, to solving in positive integers the following system of three equations in six unknowns:

$$\begin{cases} \pm\left[c^2 - (a^2 + b^2)\right] &= k(u^2 - v^2); \\ 4m(a + b + c) &= 2kuv; \\ 2ab &= k(u^2 + v^2). \end{cases} \tag{13}$$

It is easy to see that the first equation in (13) can be interpreted as follows.

Lemma 7. *Assume that, corresponding to certain values of u and v, there is a triple (a, b, c) which solves the problem $A = mP$. Then the triangle (a, b, c) is $\begin{Bmatrix} obtuse \\ acute \\ right \end{Bmatrix}$ if and only if $\begin{Bmatrix} u > v \\ u < v \\ u = v = 1 \end{Bmatrix}$.*

3. Proof of Theorem 2

Let us first investigate the case of an obtuse triangle (the case $u > v$); thus, the system (13) is $c^2 - (a^2 + b^2) = k(u^2 - v^2),\ 4m(a + b + c) = 2kuv,\ 2ab = k(u^2 + v^2)$. For completeness, we reproduce the crucial proof of the main factorization identity from [3] (equation (17) below), which in essence solves the problem $A = mP$. Indeed, from the first and the third equations in (13) we get $(a + b)^2 - c^2 = 2kv^2$, and after factoring the left-hand side and using the second equation we get $a + b - c = \dfrac{4mv}{u}$. This implies that u must divide $2m$ because $a + b - c$ is even, and u, v are relatively prime. Combining the last relation with $a + b + c = \dfrac{kuv}{2m}$ and solving the resulting system yields

$$b + a = \frac{ku^2v + 8m^2v}{4mu}, \quad c = \frac{ku^2v - 8m^2v}{4mu}.$$

Similarly, adding the first and second equations and rearranging terms gives $(a - b)^2 = c^2 - 2ku^2$. Let us assume for a moment that $b \geq a$; then we have $b - a = \sqrt{c^2 - 2ku^2}$, and it is clear that the radicand must be a square. Put $Q = \dfrac{2m}{u}$ and substitute it in the expressions for c, $b + a$ and $b - a$. After simplification, one gets

$$c = \frac{kv - 2Q^2v}{2Q}, \ \ b + a = \frac{kv + 2Q^2v}{2Q}, \ b - a = \frac{1}{2Q}\sqrt{\left(kv - 2Q^2v\right)^2 - 32km^2}, \tag{14}$$

where the radicand must be a square. Put $\left(kv - 2Q^2v\right)^2 - 32km^2 = X^2$, and get

$$c = \frac{kv - 2Q^2v}{2Q}, \qquad b + a = \frac{kv + 2Q^2v}{2Q}, \qquad b - a = \frac{1}{2Q}X. \tag{15}$$

On the other hand, consider $\left(kv-2Q^2v\right)^2-32km^2=X^2$ as an equation in the variables X and k. Expanding the square and rearranging yields

$$k^2v^2-4k(v^2Q^2+8m^2)+4Q^4v^2=X^2.$$

The last equation is a Diophantine equation solvable by factoring: subtract the quantity $\left(kv-\dfrac{2(v^2Q^2+8m^2)}{v}\right)^2$ from both sides, simplify and rearrange terms; the result is

$$\left[2(v^2Q^2+8m^2)\right]^2-\left(2v^2Q^2\right)^2=\left(v^2k-2v^2Q^2-16m^2\right)^2-(Xv)^2. \quad (16)$$

In (16), factor both sides, substitute $Q=\dfrac{2m}{u}$ and simplify. This gives

$$\left(\frac{16m^2}{u}\right)^2(u^2+v^2)=\left[v^2(k-2Q^2)-16m^2-Xv\right]\left[v^2(k-2Q^2)-16m^2+Xv\right] \quad (17)$$

which is the main factorization identity mentioned above.

Now, the new idea is to eliminate k and X in (17), using (15) and the crucial fact that $a+b-c=\dfrac{4mv}{u}$. Indeed, from (15) we immediately obtain

$$X=2Q(b-a), \quad k=\frac{2Qc+2Q^2v}{v}, \quad (18)$$

which we substitute in (17) and simplify to get

$$16m^2(u^2+v^2)=\left[v(c-b+a)-4mu\right]\left[v(c+b-a)-4mu\right]. \quad (19)$$

In the last relation, substitute $c=a+b-\dfrac{4mv}{u}$ and simplify again. The result is

$$4m^2(u^2+v^2)=\left[v\left(a-\frac{2m}{u}v\right)-2mu\right]\left[v\left(b-\frac{2m}{u}v\right)-2mu\right],$$

which is exactly (6). This identity allows us to find sides a and b by directly matching factors of the left-hand side to respective quantities on the right; then c will be determined from $c=a+b-\dfrac{4mv}{u}$. Suppose $4m^2(u^2+v^2)=\delta_1\cdot\delta_2$. Since we want $\delta_1=v\left(a-\dfrac{2m}{u}v\right)-2mu$, it is clear that for a to be an integer, we necessarily must have $v \mid \delta_1+2mu$. Similarly, the requirement $v \mid \delta_2+2mu$ will ensure that b is an integer. Imposing these additional restrictions will produce *only* the integer solutions to the problem. Furthermore, choosing $\delta_1\le\delta_2$ (or equivalently, $\delta_1\le 2m\sqrt{u^2+v^2}$) will guarantee that $a\le b$.

Next, solve

$$\delta_1=v\left(a-\frac{2m}{u}v\right)-2mu, \quad \delta_2=v\left(a-\frac{2m}{u}v\right)-2mu$$

for a and b, express c in terms of them and thus obtain formulas for the sides:

$$\begin{cases} a = \dfrac{\delta_1 + 2mu}{v} + \dfrac{2mv}{u}, \\ b = \dfrac{\delta_2 + 2mu}{v} + \dfrac{2mv}{u}, \\ c = \dfrac{\delta_1 + \delta_2 + 4mu}{v}; \end{cases}$$

these are exactly the formulas (10). To ensure $c \geq b$, we solve the inequality

$$\frac{\delta_1 + \delta_2 + 4mu}{v} \geq \frac{\delta_2 + 2mu}{v} + \frac{2mv}{u}$$

and obtain, after simplification,

$$\delta_1 \geq \frac{2m}{u}(v^2 - u^2). \tag{20}$$

The last relation will always be true if $u > v$, and thus the proof of the obtuse-case part of the theorem is concluded. Now, consider the acute case; i.e., the case $v > u$. The first equation in (13) is again $c^2 - (a^2 + b^2) = k(u^2 - v^2)$ (both sides are negative), and all the above derivations continue to hold true; it is now crucial to use the important bound $a + b - c < 4m\sqrt{3}$ which, combined with $a + b - c = \dfrac{4mv}{u}$, implies that $u < v < \sqrt{3}u$. The only difference from the obtuse case is that the bound (20) does not hold automatically; now it must be imposed to avoid repetitions and guarantee that $b \leq c$. Since the right-triangle case is obviously incorporated in the theorem, the proof is complete.

4. An example

We again examine the case $m = 2$ (cf. [3]). Let $m = 2$ in the algorithm suggested by Theorem 2; then $2m = 4$ and thus u could be 4, 2 or 1. For each u, determine the corresponding v's:

(A) $u = 4 \Rightarrow v = 1, 3;\ 5$
(B) $u = 2 \Rightarrow v = 1;\ 3$
(C) $u = 1 \Rightarrow v = 1.$

Now observe how the case $u = 4,\ v = 5$ has to be discarded since we have $4m^2(u^2 + v^2) = 656 = 2^4 \cdot 41,\ 9 \leq \delta_1 \leq 25$, the only factor in that range is 16, and it must be thrown out because $v = 5$ does not divide $\delta_1 + 2mu = 32$. The working factorizations are shown in the table below.

u	v	type of triangle	δ_1 range	$4m^2(u^2+v^2)$	$\delta_1 \cdot \delta_2$	(a,b,c)
4	1	obtuse	$\delta_1 \leq 16$	272	$1 \cdot 272$	$(18, 289, 305)$
					$2 \cdot 136$	$(19, 153, 170)$
					$4 \cdot 68$	$(21, 85, 104)$
					$8 \cdot 34$	$(25, 51, 74)$
					$16 \cdot 17$	$(33, 34, 65)$
4	3	obtuse	$\delta_1 \leq 20$	400	$2 \cdot 200$	$(9, 75, 78)$
					$5 \cdot 80$	$(10, 35, 39)$
					$8 \cdot 50$	$(11, 25, 30)$
					$20 \cdot 20$	$(15, 15, 24)$
2	1	obtuse	$\delta_1 \leq 8$	80	$1 \cdot 80$	$(11, 90, 97)$
					$2 \cdot 40$	$(12, 50, 58)$
					$4 \cdot 20$	$(14, 30, 40)$
					$5 \cdot 16$	$(15, 26, 37)$
					$8 \cdot 10$	$(18, 20, 34)$
2	3	acute	$10 \leq \delta_1 \leq 14$	208	$13 \cdot 16$	$(13, 14, 15)$
1	1	right	$\delta_1 \leq 5$	32	$1 \cdot 32$	$(9, 40, 41)$
					$2 \cdot 16$	$(10, 24, 26)$
					$4 \cdot 8$	$(12, 16, 20)$

References

[1] R. Beauregard and E. Suryanarayan, Pythagorean triples: the hyperbolic view, *College Math. Journal*, 27 (1996) 170–181.

[2] J. Goehl, Area $= k$(perimeter), *Math. Teacher*, 76 (1985) 330–332.

[3] L. Markov, Pythagorean triples and the problem $A = mP$ for triangles, *Math. Mag.*, 79 (2006) 114–121.

Lubomir Markov: Department of Mathematics and Computer Science, Barry University, 11300 NE Second Avenue, Miami Shores, FL 33161

E-mail address: `lmarkov@mail.barry.edu`

Forum Geometricorum
Volume 7 (2007) 137–146.

FORUM GEOM
ISSN 1534-1178

Coincidence of Centers for Scalene Triangles

Sadi Abu-Saymeh and Mowaffaq Hajja

Abstract. A *center function* is a function $\mathcal{Z}$ that assigns to every triangle T in a Euclidean plane $\mathbf{E}$ a point $\mathcal{Z}(T)$ in $\mathbf{E}$ in a manner that is symmetric and that respects isometries and dilations. A family $\mathbf{F}$ of center functions is said to be *complete* if for every scalene triangle ABC and every point P in its plane, there is $\mathcal{Z} \in \mathbf{F}$ such that $\mathcal{Z}(ABC) = P$. It is said to be *separating* if no two center functions in $\mathbf{F}$ coincide for any scalene triangle. In this note, we give simple examples of complete separating families of continuous triangle center functions. Regarding the impression that no two different center functions can coincide on a scalene triangle, we show that for every center function $\mathcal{Z}$ and every scalene triangle T, there is another center function $\mathcal{Z}'$, of a simple type, such that $\mathcal{Z}(T) = \mathcal{Z}'(T)$.

1. Introduction

Exercise 1 of [33, p. 37] states that if any two of the four classical centers coincide for a triangle, then it is equilateral. This can be seen by proving each of the 6 substatements involved, as is done for example in [26, pp. 78–79], and it also follows from more interesting considerations as described in Remark 5 below. The statement is still true if one adds the Gergonne, the Nagel, and the Fermat-Torricelli centers to the list. Here again, one proves each of the relevant 21 substatements; see [15], where variants of these 21 substatements are proved. If one wishes to extend the above statement to include the hundreds of centers catalogued in Kimberling's encyclopaedic work [25], then one must be prepared to test the tens of thousands of relevant substatements. This raises the question whether it is possible to design a definition of the term *triangle center* that encompasses the well-known centers and that allows one to prove in one stroke that no two centers coincide for a scalene triangle. We do not attempt to answer this expectedly very difficult question. Instead, we adhere to the standard definition of what a center is, and we look at maximal families of centers within which no two centers coincide for a scalene triangle.

In Section 2, we review the standard definition of triangle centers and introduce the necessary terminology pertaining to them. Sections 3 and 4 are independent.

Publication Date: September 17, 2007. Communicating Editor: Paul Yiu.
This work is supported by a research grant from Yarmouk University.
We would like to thank the referee who, besides being responsible for Remark 5, has made many valuable suggestions that helped improve the paper.

In Section 3, we examine the family of polynomial centers of degree 1. Noting the similarity between the line that these centers form and the Euler line, we digress to discuss issues related to these two lines. In Section 4, we exhibit maximal families of continuous, in fact polynomial, centers within which no two centers coincide for a scalene triangle. We also show that for every scalene triangle T and for every center function $\mathcal{Z}$, there is another center function of a fairly simple type that coincides with $\mathcal{Z}$ on T.

2. Terminology

By a *non-degenerate* triangle ABC, we mean an ordered triple (A, B, C) of non-collinear points in a fixed Euclidean plane $\mathbf{E}$. Non-degenerate triangles form a subset of $\mathbf{E}^3$ that we denote by $\mathbf{T}$. For a subset $\mathbf{U}$ of $\mathbf{T}$, the set of triples $(a, b, c) \in \mathbf{R}^3$ that occur as the side-lengths of a triangle in $\mathbf{U}$ is denoted by $\mathbf{U}_0$. Thus

$$\begin{aligned}
\mathbf{U}_0 &= \{(a, b, c) \in \mathbf{R}^3 : a, b, c \text{ are the side-lengths of some triangle } ABC \text{ in } \mathbf{U}\},\\
\mathbf{T}_0 &= \{(a, b, c) \in \mathbf{R}^3 : 0 < a < b + c,\ 0 < b < c + a,\ 0 < c < a + b\}.
\end{aligned}$$

In the spirit of [23] – [25], a *symmetric triangle center function* (or simply, a *center function*, or a *center*) is defined as a function that assigns to every triangle in $\mathbf{T}$ (or more generally in some subset $\mathbf{U}$ of $\mathbf{T}$) a point in its plane in a manner that is symmetric and that respects isometries and dilations. Writing $\mathcal{Z}(A, B, C)$ as a barycentric combination of the position vectors A, B, and C, and letting a, b, and c denote the side-lengths of ABC in the standard order, we see that a center function $\mathcal{Z}$ on $\mathbf{U}$ is of the form

$$\mathcal{Z}(A, B, C) = f(a, b, c)A + f(c, a, b)B + f(b, c, a)C, \tag{1}$$

where f is a real-valued function on $\mathbf{U}_0$ having the following properties:

$$f(a, b, c) = f(a, c, b), \tag{2}$$
$$f(a, b, c) + f(b, c, a) + f(c, a, b) = 1, \tag{3}$$
$$f(\lambda a, \lambda b, \lambda c) = f(a, b, c)\ \forall\ \lambda > 0. \tag{4}$$

Here, we have treated the points in our plane $\mathbf{E}$ as position vectors relative to a fixed but arbitrary origin. We will refer to the center $\mathcal{Z}$ defined by (1) as *the center function defined by* f without referring explicitly to (1). The function f may be an explicit function of other elements of the triangle (such as its angles) that are themselves functions of a, b and c.

Also, we will always assume that the domain $\mathbf{U}$ of $\mathcal{Z}$ is closed under permutations, isometries and dilations, and has non-empty interior. In other words, we assume that $\mathbf{U}_0$ is closed under permutations and multiplication by a positive number, and that it has a non-empty interior.

According to this definition of a center $\mathcal{Z}$, one need only define $\mathcal{Z}$ on the similarity classes of triangles. On the other hand, the values that $\mathcal{Z}$ assigns to two triangles in different similarity classes are completely independent of each other. To reflect more faithfully our intuitive picture of centers, one must impose the condition that a center function be continuous. Thus a center function $\mathcal{Z}$ on $\mathbf{U}$ is called *continuous* if it is defined by a function f that is continuous on $\mathbf{U}_0$. If f can be chosen to be a rational function, then $\mathcal{Z}$ is called a *polynomial center function*. Since two rational functions cannot coincide on a non-empty open set, it follows that the rational function that defines a polynomial center function is unique. Also, a rational function $f(x, y, z)$ that satisfies (4) is necessarily of the form $f = g/h$, where g and h are d-forms, i.e., homogeneous polynomials of the same degree d. If $d = 1$, f is called a *projective linear function*. *Projective quadratic functions* correspond to $d = 2$, and so on. Thus a polynomial center $\mathcal{Z}$ is a center defined by a projective function.

A family $\mathbf{F}$ of center functions on $\mathbf{U}$ is said to be *separating* if no two elements in $\mathbf{F}$ coincide on any scalene triangle. It is said to be *complete* if for every scalene triangle T in $\mathbf{U}$, $\{\mathcal{Z}(T) : \mathcal{Z} \in \mathbf{F}\}$ is all of $\mathbf{E}$. The assumption that T is scalene is necessary here. In fact, if a triangle $T = ABC$ is such that $AB = AC$, then $\{\mathcal{Z}(T) : \mathcal{Z} \in \mathbf{F}\}$ will be contained in the line that bisects angle A, being a line of symmetry of ABC, and thus cannot cover $\mathbf{E}$.

3. Polynomial centers of degree 1

We start by characterizing the simplest polynomial center functions, i.e., those defined by projective linear functions. We note the similarity between the line these centers form and the Euler line and we discuss issues related to these two lines.

Theorem 1. *A projective linear function $f(x, y, z)$ satisfies (2), (3), and (4) if and only if*

$$f(x, y, z) = \frac{(1 - 2t)x + t(y + z)}{x + y + z} \tag{5}$$

for some t. If $\mathcal{S}_t$ is the center function defined by (5) (and(1)), then $\mathcal{S}_0$, $\mathcal{S}_{1/3}$, $\mathcal{S}_{1/2}$, and $\mathcal{S}_1$ are the incenter, centroid, Spieker center, and Nagel center, respectively. Also, the centers $\{\mathcal{S}_t(ABC) : t \in \mathbf{R}\}$ of a non-equilateral triangle ABC in $\mathbf{T}$ form the straight line whose trilinear equation is

$$a(b - c)\alpha + b(c - a)\beta + c(a - b)\gamma = 0.$$

Furthermore, the distance $|\mathcal{S}_t\mathcal{S}_u|$ between $\mathcal{S}_t$ and $\mathcal{S}_u$ is given by

$$|\mathcal{S}_t\mathcal{S}_u| = \frac{|t - u|\sqrt{H}}{a + b + c}, \tag{6}$$

where

$$\begin{aligned} H &= (-a + b + c)(a - b + c)(a + b - c) + (a + b)(b + c)(c + a) - 9abc \\ &= -(a^3 + b^3 + c^3) + 2(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2) - 9abc. \end{aligned} \tag{7}$$

Proof. Let $f(x, y, z) = L_0/M_0$, where L_0 and M_0 are linear forms in x, y, and z, and suppose that f satisfies (2), (3), and (4). Let σ be the cycle $(x\ y\ z)$, and let $L_i = \sigma^i(L_0)$ and $M_i = \sigma^i(M_0)$. Since f satisfies (3), it follows that $L_0M_1M_2 + L_1M_0M_2 + L_2M_0M_1 - M_0M_1M_2$ vanishes on $\mathbf{U}_0$ and hence vanishes identically. Thus M_0 divides $L_0M_1M_2$. If M_0 divides L_0, then f is a constant, and hence of the desired form, with $t = 1/3$. If M_0 divides M_1, then it follows easily that M_1 is a constant multiple of M_0 and that M_0 is a constant multiple of $x + y + z$. The same holds if M_0 divides M_2. Finally, we use (3) and (4) to see that L_0 is of the desired form.

Let $\mathcal{S}_t$ be as given. The barycentric coordinates of $\mathcal{S}_t(ABC)$ are given by

$$f(a,b,c) : f(b,c,a) : f(c,a,b)$$

and therefore the trilinear coordinates $\alpha : \beta : \gamma$ of $\mathcal{S}_t(ABC)$ are given by

$$\begin{aligned}\alpha a : \beta b : \gamma c &= (1-2t)a + t(b+c) : (1-2t)b + t(c+a) : (1-2t)c + t(a+b)\\ &= a + t(b+c-2a) : b + t(c+a-2b) : c + t(a+b-2c)\end{aligned}$$

Therefore there exists non-zero λ such that

$$\lambda\alpha a - a = t(b+c-2a),\ \lambda\beta b - b = t(c+a-2b),\ \lambda\gamma c - c = t(a+b-2c).$$

It is clear that the value $t = 0$ corresponds to the incenter. Thus we assume $t \neq 0$. Eliminating t, we obtain

$$\begin{aligned}(\lambda\alpha - 1)a(c+a-2b) &= (\lambda\beta - 1)b(b+c-2a),\\ (\lambda\beta - 1)b(a+b-2c) &= (\lambda\gamma - 1)c(c+a-2b).\end{aligned}$$

Eliminating λ and simplifying, we obtain

$$(a - 2b + c)\,[a(b-c)\alpha + b(c-a)\beta + c(a-b)\gamma] \;=\; 0.$$

Dividing by $a - 2b + c$, we get the desired equation.

Finally, the last statement follows after routine, though tedious, calculations. We simply note that the actual trilinear coordinates of $\mathcal{S}_t$ are given by

$$\frac{2K((1-2t)a + t(b+c))}{a(a+b+c)} : \frac{2K((1-2t)b + t(c+a))}{b(a+b+c)} : \frac{2K((1-2t)c + t(a+b))}{c(a+b+c)},$$

where K is the area of the triangle, and we use the fact that the distance $|PP'|$ between the points P and P' whose actual trilinear coordinates are $\alpha : \beta : \gamma$ and $\alpha' : \beta' : \gamma'$ is given by

$$|PP'| = \frac{1}{2K}\sqrt{-abc[a(\beta-\beta')(\gamma-\gamma') + b(\gamma-\gamma')(\alpha-\alpha') + c(\alpha-\alpha')(\beta-\beta')]};$$

see [25, Theorem 1B, p. 31]. □

4. The Euler-like line $L(\mathcal{I}, \mathcal{G})$

The straight line $\{\mathcal{S}_t : t \in \mathbf{R}\}$ in Theorem 1 is the first central line in the list of [25, p. 128], where it is denoted by $L(1, 2, 8, 10)$. The notation $L(1, 2, 8, 10)$ reflects the fact that it passes through the centers catalogued in [25] as X_1, X_2, X_8, and X_{10}. These are the incenter, centroid, Nagel center, and Spieker center, and they correspond in $\{\mathcal{S}_t : t \in \mathbf{R}\}$ to the values $t = 0, 1/3, 1$, and $1/2$, respectively. We shall denote this line by $L(\mathcal{I}, \mathcal{G})$ to indicate that it is the line joining the incenter $\mathcal{I}$ and the centroid $\mathcal{G}$. Letting $\mathcal{O}$ be the circumcenter, the line $L(\mathcal{G}, \mathcal{O})$ is then nothing but the Euler line. In this section, we survey similarities between these lines. For the third line $L(\mathcal{O}, \mathcal{I})$ and a natural context in which it occurs, we refer the reader to [17].

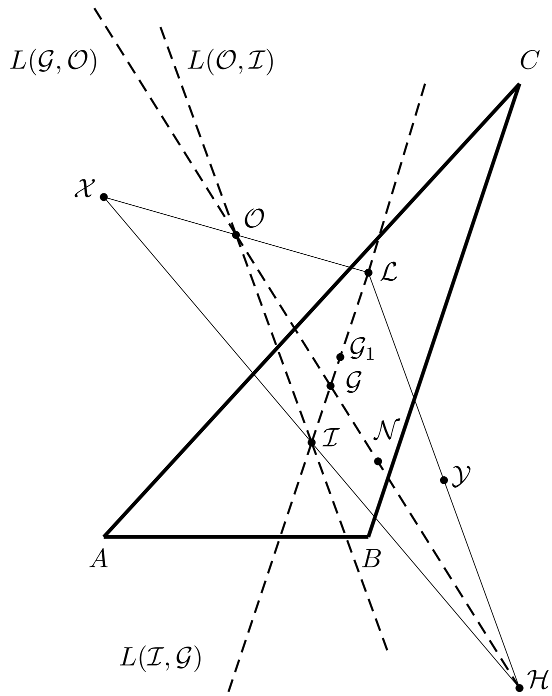


Figure 1

It follows from (6) that the Spieker center $\mathcal{G}_1 = \mathcal{S}_{1/2}$ and the centroid $\mathcal{G} = \mathcal{S}_{1/3}$ of a triangle are at distances in the ratio $3 : 2$ from the incenter $\mathcal{I} = \mathcal{S}_0$. This is shown in Figure 1 which is taken from [17]. The collinearity of $\mathcal{G}_1$, $\mathcal{I}$, and $\mathcal{G}$ and the ratio $3 : 2$ are highlighted in [6, pp. 137–138] and [27], and they also appear in [22, pp. 225–227] and the first row in [25, Table 5.5, p. 143]. In spite of this, we feel that these elegant facts and the striking similarity between the line $L(\mathcal{I},\mathcal{G})$ and the Euler line $L(\mathcal{O},\mathcal{G})$ deserve to be better known. Unaware of the aforementioned references, the authors of [3] rediscovered the collinearity of the incenter, the Spieker center, and the centroid and the ratio 3 : 2, and they proved, in Theorems 6 and 7, that the same thing holds for any polygon that admits an incircle, i.e., a circle that touches the sides of the polygon internally. Here, the centroid of a polygon is the center of mass of a lamina of uniform density that is laid on the polygon, the Spieker center is the centroid of wires of uniform density placed on the sides, and the incenter is the center of the incircle. Later, the same authors, again unaware of [8, p. 69], rediscovered (in [4]) similar properties of $L(\mathcal{I},\mathcal{G})$ in dimension 3 and made interesting generalizations to solids admitting inspheres. For a deeper explanation of the similarity between the Euler line and its rival $L(\mathcal{I},\mathcal{G})$ and for affine and other generalizations, see [29] and [28].

We should also mention that the special case of (6) pertaining to the distance between the incenter and the centroid appeared in [7]. Also, the fact that the Spieker center $\mathcal{G}_1$ is the midpoint of the segment joining the incenter $\mathcal{I}$ and the Nagel point $\mathcal{L}$ is the subject matter of [12], [30], and [31]. In each of these references, $\mathcal{L}$ (respectively, $\mathcal{G}_1$) is described as the point of intersection of the lines that bisect the perimeter and that pass through the vertices (respectively, the midpoints of the sides). It is not apparent that the authors of these references are aware that $\mathcal{L}$ and $\mathcal{G}_1$ are the Nagel and Spieker centers. For the interesting part that $\mathcal{G}_1$ is indeed the Spieker center, see [5] and [20, pp. 1–14]. One may also expect that the Euler line and the line $L(\mathcal{I},\mathcal{G})$ cannot coincide unless the triangle is isosceles. This is indeed so, as is proved in [21, Problem 4, Section 11, pp. 142–144]. It also follows from the fact that the area of the triangle $\mathcal{GOI}$ is given by the elegant formula

$$[\mathcal{GOI}] = \left|\frac{s(b-c)(c-a)(a-b)}{24K}\right|,$$

where s is the semiperimeter and K the area of ABC; see [34, Exercise 5.7].

We also note that the Euler line consists of the centers $\mathcal{T}_t$ defined by the function

$$g = \frac{(1-2t)\tan A + t(\tan B + \tan C)}{\tan A + \tan B + \tan C} \tag{8}$$

obtained from f of (5) by replacing a, b, and c by $\tan A$, $\tan B$, and $\tan C$, respectively. Then $\mathcal{T}_0$, $\mathcal{T}_{1/3}$, $\mathcal{T}_{1/2}$, and $\mathcal{T}_1$ are nothing but the circumcenter, centroid, the center of the nine-point circle, and the orthocenter, respectively. The distance $|\mathcal{T}_t\mathcal{T}_u|$ between $\mathcal{T}_t$ and $\mathcal{T}_u$ is given by

$$|\mathcal{T}_t\mathcal{T}_u| = \frac{|t-u|\sqrt{H^*}}{a+b+c},$$

where H^* is obtained from H in (7) by replacing a, b, and c with $\tan A$, $\tan B$, and $\tan C$, respectively. Letting K be the area of the triangle with side-lengths a, b, and c, and using the identity $\tan A = 4K/(b^2+c^2-a^2)$ and its iterates, H^* reduces to a rational function in a, b, and c. In view of the formula $144K^2r^2 = E$ given in [32], where

$$E = a^2b^2c^2 - (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2), \tag{9}$$

and where r is the distance between the circumcenter $\mathcal{T}_0$ and the centroid $\mathcal{T}_{1/3}$, H^* is expected to simplify into

$$H^* = \frac{(a+b+c)^2E}{16K^2},$$

where E is as given in (9), and where $16K^2$ is given by Heron's formula

$$16K^2 = 2(a^2b^2+b^2c^2+c^2a^2) - (a^4+b^4+c^4). \tag{10}$$

Referring to Figure 1, let $\mathcal{X}$ be the point where the lines $\mathcal{LO}$ and $\mathcal{HI}$ meet, and let $\mathcal{Y}$ be the midpoint of $\mathcal{HL}$. Then the Euler line and the line $L(\mathcal{I},\mathcal{G})$ are medians of both triangles $\mathcal{XHL}$ and $\mathcal{OIY}$. The points $\mathcal{X}$ and $\mathcal{Y}$ do not seem to be catalogued in [25]. Also, of the many lines that can be formed in Figure 1, the line $\mathcal{IN}$ is catalogued in [25] as the line joining $\mathcal{I}$, $\mathcal{N}$, and the Feuerbach point. As for distances between various points in Figure 1, formulas for the distances $\mathcal{IN}$, $\mathcal{IO}$, $\mathcal{IH}$, and $\mathcal{OH}$ can be found in [9, pp. 6–7]. The first two are quite well-known and they are associated with Euler, Steiner, Chapple and Feuerbach. Also, formulas for the distances $\mathcal{GI}$ and $\mathcal{GO}$ appeared in [7] and [32], as mentioned earlier. These formulas, as well as other formulas for distances between several other pairs of centers, had already been found by Euler [35, Section XIB, pp. 88–90].

5. Complete separating families of polynomial centers

In the next theorem, we exhibit a complete separating family of polynomial center functions that contains the functions used to define the line $L(\mathcal{I}, \mathcal{G})$ encountered in Theorem 1.

Theorem 2. *Let ABC be a scalene triangle and let V be any point in its plane. Then there exist unique real numbers t and v such that V is the center of ABC with respect to the center function $\mathcal{Q}_{t,v}$ defined by the projective quadratic function f given by*

$$f(x,y,z) \;=\; \frac{(1-2t)x^2 + t(y^2+z^2) + 2(1-v)yz + vx(y+z)}{(x+y+z)^2}. \tag{11}$$

Consequently, the family $\mathbf{F} = \{\mathcal{Q}_{t,v} : t, v \in \mathbf{R}\}$ is a complete separating family. Also, $\mathbf{F}$ contains the line $L(\mathcal{I}, \mathcal{G})$ described in Theorem 1.

Proof. Clearly f satisfies the conditions (2), (3), and (4). Since V is in the plane of ABC, it follows that $V = \xi A + \eta B + \zeta C$ for some ξ, η, and ζ with $\xi + \eta + \zeta = 1$. Let a, b, and c be the side-lengths of ABC as usual. The system $f(a,b,c) = \xi$, $f(b,c,a) = \eta$, $f(c,a,b) = \zeta$ of equations is equivalent to the system

$$\begin{aligned}(b^2+c^2-2a^2)t + (-2bc+ca+ab)v &= \xi(a+b+c)^2 - a^2 - 2bc,\\ (a^2+b^2-2c^2)t + (-2ab+bc+ca)v &= \zeta(a+b+c)^2 - c^2 - 2ab.\end{aligned}$$

The existence of a (unique) solution (t, v) to this system now follows from the fact that its determinant $-3(a-b)(b-c)(c-a)(a+b+c)$ is not zero.

The last statement follows from the observation that if $v = 1 - t$, then the expression of $f(x,y,z)$ in (11) reduces to the projective linear function $f(x,y,z)$ given in (5). $\square$

Remarks. (1) According to [25, p. 46], the Fermat-Torricelli point is not a polynomial center. Therefore it does not belong to the family $\mathbf{F}$ defined in Theorem 2. Also, the circumcenter, the orthocenter, and the Gergonne point do not belong to $\mathbf{F}$, although they are polynomial centers. In fact, these centers are defined by the functions f given by

$$\frac{x^2(y^2+z^2-x^2)}{16K^2},\ \frac{y^2+z^2-x^2}{x^2+y^2+z^2},\ \frac{(x-y+z)(x+y-z)}{2(xy+yz+zx)-(x^2+y^2+z^2)},$$

respectively, where K is the area of the triangle whose side-lengths are x, y, and z, and is given by Heron's formula as in (10); see [24, pp. 172–173].

(2) One may replace the denominator of f in (11) by an arbitrary symmetric quadratic form that does not vanish on any point in $\mathbf{T}_0$, and obtain a different

separating complete family of center functions. Thus if we replace f by the similar function

$$g(x,y,z) \;=\; \frac{(-1-2t)x^2+t(y^2+z^2)+2vyz+(1-v)x(y+z)}{2(xy+yz+zx)-(x^2+y^2+z^2)},$$

then we would obtain a complete separating family **G** of center functions that contains the centroid, the Gergonne center and the Mittenpunkt, but not any of the other well known traditional centers. Here the Mittenpunkt is the center defined by the function

$$g(x,y,z)=\frac{xy+xz-x^2}{2(xy+yz+zx)-(x^2+y^2+z^2)}.$$

(3) It is clear that complete families are maximal separating families. However, it is not clear whether the converse is true. It also follows from Zorn's Lemma that every separating family of center functions can be imbedded in a maximal separating family. Thus the seven centers mentioned at the beginning of this note belong to some maximal separating family of centers. The question is whether such a family can be defined in a natural way.

The next theorem shows that pairs of center functions that coincide on scalene triangles exist in abundance. However, it does not answer the question whether such a pair can be chosen from the hundreds of centers that are catalogued in [25]. In case this is not possible, the question arises whether this is due to certain intrinsic properties of the centers in [25].

Theorem 3. *Let $\mathcal{Z}$ be a center function, and let ABC be any scalene triangle in the domain of $\mathcal{Z}$. Then there exists another center function $\mathcal{Z}'$ defined by a projective function f such that $\mathcal{Z}(A,B,C)=\mathcal{Z}'(A,B,C)$.*

Moreover, if $\mathcal{Z}$ is not the centroid, then f can be chosen to be quadratic. If $\mathcal{Z}$ is the centroid, then f can be chosen to be quartic.

Proof. Let **F** and **G** be the families of centers defined in Theorem 2 and in Remark 4. Clearly, the centroid is the only center function that these two families have in common.

If $\mathcal{Z}\notin \mathbf{F}$, then we use Theorem 2 to produce the center $\mathcal{Z}'=\mathcal{Z}_{t,v}$ for which $\mathcal{Z}_{t,v}(A,B,C)=\mathcal{Z}(A,B,C)$, and we take $\mathcal{Z}'=\mathcal{Z}_{t,v}$. If $\mathcal{Z}\notin \mathbf{G}$, then we argue similarly as indicated in Remark 2 to produce the desired center function.

It remains to deal with the case when $\mathcal{Z}$ is the centroid. In this case, we let $f(x,y,z)=g(x,y,z)/h(x,y,z)$, where

$$\begin{aligned}
h(x,y,z) &= (x^4+y^4+z^4)+(x^3y+y^3z+z^3x+x^3z+y^3x+z^3y)\\
&\quad +(x^2y^2+y^2z^2+z^2x^2)\\
g(x,y,z) &= (1-2t)x^4+t(y^4+z^4)+vx^3(y+z)+wx(y^3+z^3)\\
&\quad +(1-v-w)x(y^3+z^3)+sx^2(y^2+z^2)+(1-2s)y^2z^2,
\end{aligned}$$

and we consider the equations

$$f(a,b,c)=f(b,c,a)=f(c,a,b)=\frac{1}{3}.$$

These are linear equations in the variables t, v, w, and s that have an obvious solution $(t,v,w,s)=(1/3,1/3,1/3,1/3)$. Hence they have infinitely many other solutions. Choose any of these solutions and let $\mathscr{Z}$ be the center defined by the function f that corresponds to that choice. Then for the given triangle ABC, $\mathscr{Z}$ is the centroid, as desired. □

Remarks. (4) The question that underlies this paper is whether two centers can coincide for a scalene triangle. The analogous question, for higher dimensional simplices, of how much regularity is implied by the coincidence of two or more centers has led to various interesting results in [18], [19], [10], [11], and [16].

(5, due to the referee) Let $\mathcal{O}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{I}$ be the circumcenter, centroid, orthocenter, and incenter of a non-equilateral triangle. Euler's theorem states that $\mathcal{O}$, $\mathcal{G}$, and $\mathcal{H}$ are collinear with $\mathcal{OG}:\mathcal{GH}=1:2$. A theorem of Guinand in [13] shows that $\mathcal{I}$ ranges freely over the interior of the centroidal disk (with diameter $\mathcal{GH}$) punctured at the nine-point center $\mathcal{N}$. It follows that no two of the centers $\mathcal{O}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{I}$ coincide for a non-equilateral triangle, thus providing a proof, other than case by case chasing, of the very first statement made in the introduction.

References

[1] S. Abu-Saymeh and M. Hajja, Some Brocard-like points of a triangle, *Forum Geom.*, 5 (2005) 65–74.

[2] S. Abu-Saymeh and M. Hajja, In search of more triangle centers: a source of classroom projects in Euclidean geometry, *Internat. J. Math. Ed. Sci. Tech.*, 36 (2005) 889–912.

[3] T. M. Apostol and M. A. Mnatsakanian, Figures circumscribing circles, *Amer. Math. Monthly*, 111 (2004) 851–863.

[4] T. M. Apostol and M. A. Mnatsakanian, Solids circumscribing spheres, *Amer. Math. Monthly*, 113 (2006) 521–540.

[5] D. Avishalom, The perimeter bisectors of triangles, *Math. Mag.*, 36 (1963) 60–62.

[6] C. J. Bradley, *Challenges in Geometry*, Oxford Univ. Press, New York, 2005.

[7] C. Çosmiţă, Problem E415, *Amer. Math. Monthly*, 47 (1940) 175; solution, *ibid*, 47 (1940) 712.

[8] P. Couderc and A. Balliccioni, *Premier Livre du Tétraèdre*, Gauthier-Villars, Paris, 1935.

[9] H. S. M. Coxeter, The Lehmus inequality, *Aequationes Math.*, 28 (1985) 4–12.

[10] A. L. Edmonds, M. Hajja, and H. Martini, Coincidences of simplex centers and related facial structures, *Beitr. Algebra Geom.*, 46 (2005) 491–512.

[11] A. L. Edmonds, M. Hajja, and H. Martini, Orthocentric simplices and their centers, *Results Math.*, 47 (2005) 266–295.

[12] J. R. Goggins, Perimeter bisectors, *Math. Gaz.*, **70** (1986) 133–134.

[13] A. P. Guinand, Tritangent centers and their triangles, *Amer. Math. Monthly*, 91 (1984) 290–300.

[14] A. P. Guinand, Incenters and excenters viewed from the Euler line, *Math. Mag.* **58** (1985), 89–92.

[15] M. Hajja, Triangle centres: some questions in Euclidean geometry, *Internat. J. Math. Ed. Sci. Tech.*, 32 (2001), 21–36.

[16] M. Hajja, Coincidences of centers of edge-incentric, or balloon, simplices, *Results Math.*, 49 (2006) 237–263.

[17] M. Hajja and M. Spirova, A new line associated with the triangle, to appear in *Elem. Math.*

[18] M. Hajja and P. Walker, Equifacial tetrahedra, *Internat. J. Math. Ed. Sci. Tech.*, 32 (2001) 501–508.
[19] M. Hajja and P. Walker, Equifaciality of tetrahedra whose incenter and Fermat-Torricelli center coincide, *J. Geometry Graphics*, 9 (2005) 37–41.
[20] R. Honsberger, *Episodes in the Nineteenth and Twentieth Century Euclidean Geometry*, Math. Assoc. America, 1995.
[21] R. Honsberger, *Mathematical Delights*, Math. Assoc. America, 2004.
[22] R. A. Johnson, *Advanced Euclidean Geometry*, Dover, NY, 1960.
[23] C. Kimberling, Triangle centers as functions, *Rocky Mountain J. Math.*, 23 (1993) 1269–1286.
[24] C. Kimberling, Central points and central lines in the plane of a triangle, *Math. Mag.*, 67 (1994) 163–187.
[25] C. Kimberling, Triangle centers and central triangles, *Congr. Numer.*, 129 (1998) 1–285.
[26] A. Liu, *Hungarian Problem Book III*, Math. Assoc. America, 2001.
[27] M. Longuet-Higgins, A fourfold point of concurrence lying on the Euler line of a triangle, *Math. Intelligencer*, 22:1 (2000), 54–59.
[28] V. Pambuccian, Euclidean geometry problems rephrased in terms of midpoints and point reflections, *Elem. Math.*, 60 (2005) 19–24.
[29] E. Snapper, An affine generalization of the Euler line, *Amer. Math. Monthly*, 88 (1981) 196–198.
[30] M. S. Sloyan, The intersection points of perimeter bisectors, *Math. Mag.*, 36 (1963) 312–313.
[31] P. D. Thomas, Problem 468, *Math. Mag.* **35** (1962), 55; solution, *ibid*, 35 (1962) 251–252; comment by N. A. Court, *ibid*, 36 (1963) 141–142.
[32] P. Walker, Problem 10948, *Amer. Math. Monthly*, 109 (2002) 476; solution, *ibid*, 111 (2004) 65–66.
[33] P. Yiu, *Notes on Euclidean Geometry*, 1998, http://www.math.fau.edu/Yiu/Geometry.html.
[34] P. Yiu, *Advanced Euclidean Geometry*, 1992, http://www.math.fau.edu/Yiu/Geometry.html.
[35] P. Yiu, *Elementary Mathematical Works of Leonhard Euler*, 1999, http://www.math.fau.edu/Yiu/Geometry.html.

Sadi Abu-Saymeh: Mathematics Department, Yarmouk University, Irbid, Jordan.
E-mail address: `sade@yu.edu.jo , ssaymeh@yahoo.com`

Mowaffaq Hajja: Mathematics Department, Yarmouk University, Irbid, Jordan.
E-mail address: `mhajja@yu.edu.jo , mowhajja@yahoo.com`

Forum Geometricorum
Volume 7 (2007) 147–149.

FORUM GEOM
ISSN 1534-1178

On the Diagonals of a Cyclic Quadrilateral

Claudi Alsina and Roger B. Nelsen

Abstract. We present visual proofs of two lemmas that reduce the proofs of expressions for the lengths of the diagonals and the area of a cyclic quadrilateral in terms of the lengths of its sides to elementary algebra.

The purpose of this short note is to give a new proof of the following well-known results of Brahmagupta and Parameśhvara [4, 5].

Theorem. If a, b, c, d denote the lengths of the sides; p, q the lengths of the diagonals, R the circumradius, and Q the area of a cyclic quadrilateral, then

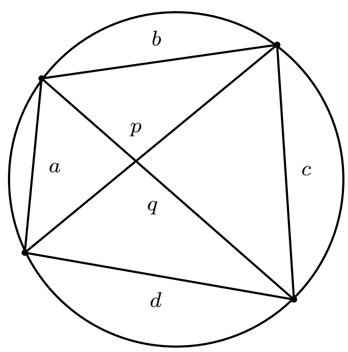


Figure 1

$$p=\sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}, \qquad q=\sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}},$$

and

$$Q=\frac{1}{4R}\sqrt{(ab+cd)(ac+bd)(ad+bc)}.$$

We begin with visual proofs of two lemmas, which will reduce the proof of the theorem to elementary algebra. Lemma 1 is the well-known relationship for the area of a triangle in terms of its circumradius and three side lengths; and Lemma 2 expresses the ratio of the diagonals of a cyclic quadrilateral in terms of the lengths of the sides.

Lemma 1. *If a, b, c denote the lengths of the sides, R the circumradius, and K the area of a triangle, then $K=\frac{abc}{4R}$.*

Publication Date: September 24, 2007. Communicating Editor: Paul Yiu.

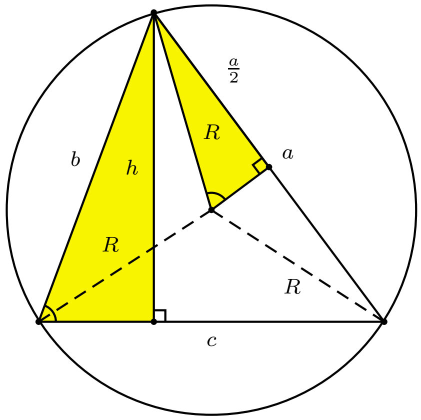


Figure 2

Proof. From Figure 2,

$$\frac{h}{b} = \frac{\frac{a}{2}}{R} \Rightarrow h = \frac{ab}{2R} \Rightarrow K = \frac{1}{2}hc = \frac{abc}{4R}.$$

□

Lemma 2 ([2])**.** *Under the hypotheses of the Theorem,* $\frac{p}{q} = \frac{ad+bc}{ab+cd}$.

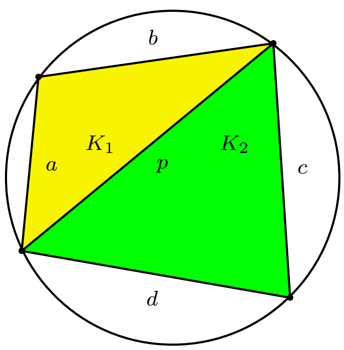


Figure 3

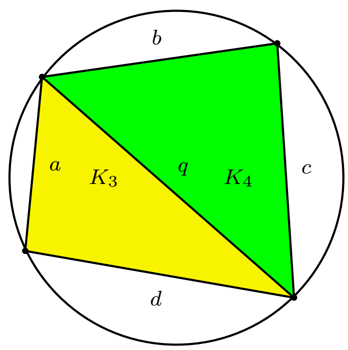


Figure 4

Proof. From Figures 3 and 4 respectively,

$$\begin{aligned} Q =& K_1 + K_2 = \frac{pab}{4R} + \frac{pcd}{4R} = \frac{p(ab+cd)}{4R}, \\ Q =& K_3 + K_4 = \frac{qad}{4R} + \frac{qbc}{4R} = \frac{q(ad+bc)}{4R}. \end{aligned}$$

Therefore,

$$p(ab+cd) = q(ad+bc),$$
$$\frac{p}{q} = \frac{ad+bc}{ab+cd}.$$

□

In the proof of our theorem, we use Lemma 2 and Ptolemy's theorem: Under the hypotheses of our theorem,

$$pq = ac+bd.$$

For proofs of Ptolemy's theorem, see [1, 3].

Proof of the Theorem.

$$p^2 = pq \cdot \frac{p}{q} = \frac{(ac+bd)(ad+bc)}{ab+cd},$$
$$q^2 = pq \cdot \frac{q}{p} = \frac{(ac+bd)(ab+cd)}{ad+bc};$$
$$Q^2 = \frac{pq(ab+cd)(ad+bc)}{(4R)^2} = \frac{(ac+bd)(ab+cd)(ad+bc)}{(4R)^2}.$$

References

[1] C. Alsina and R. B. Nelsen, *Math Made Visual: Creating Images for Understanding Mathematics*, Math. Assoc. America, 2006.

[2] A. Bogomolny, Diagonals in a cyclic quadrilateral, from Interactive Mathematics Miscellany and Puzzles, http://www.cut-the-knot.org/triangle/InscribedQuadri.shtml

[3] A. Bogomolny. Ptolemy's theorem, from Interactive Mathematics Miscellany and Puzzles, http://www.cut-the-knot.org/proofs/ptolemy.shtml

[4] R. C. Gupta, Parameśhvara's rule for the circumradius of a cyclic quadrilateral, *Historia Math.*, 4 (1977), 67–74.

[5] K. R. S. Sastry, Brahmagupta quadrilaterals, *Forum Geom.*, 2 (2002), 167–173.

Claudi Alsina: Secció de Matemàtiques, ETSAB, Universitat Politècnica de Catalunya, E-08028 Barcelona, Spain

E-mail address: claudio.alsina@upc.edu

Roger B. Nelsen: Department of Mathematical Sciences, Lewis & Clark College, Portland, Oregon 97219, USA

E-mail address: nelsen@lclark.edu

Forum Geometricorum
Volume 7 (2007) 151–158.

FORUM GEOM
ISSN 1534-1178

Some Triangle Centers Associated with the Excircles

Tibor Dosa

Abstract. We construct a few new triangle centers associated with the excircles of a triangle.

1. Introduction

Consider a triangle ABC with its excircles. We study a triad of extouch triangles and construct some new triangle centers associated with them. By the A-extouch triangle, we mean the triangle with vertices the points of tangency of the A-excircle with the sidelines of ABC. This is triangle $A_aB_aC_a$ in Figure 1. Similarly, the B- and C-extouch triangles are respectively $A_bB_bC_b$, and $A_cB_cC_c$. Consider also the incircles of these extouch triangles, with centers I_1, I_2, I_3 respectively, and points of tangency X of (I_1) with B_aC_a, Y of (I_2) with C_bA_b, and Z of (I_3) with A_cB_c.

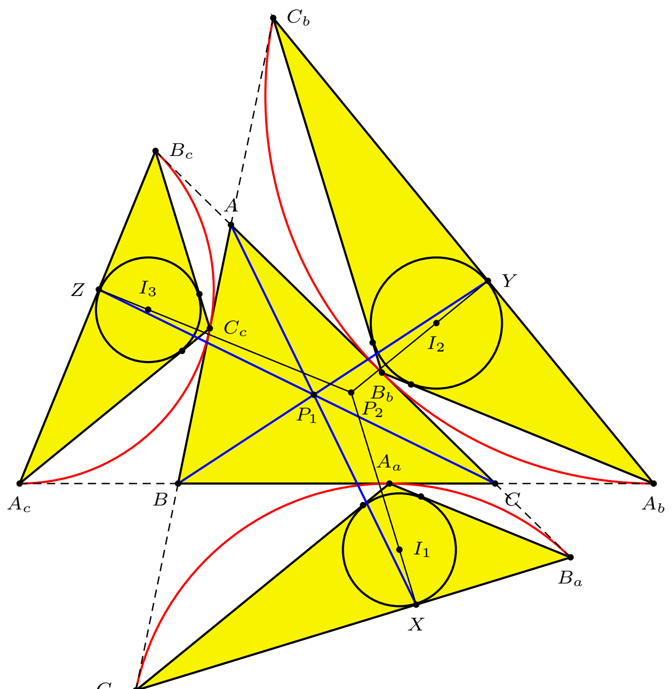


Figure 1.

In this paper, we adopt the usual notations of triangle geometry as in [3] and work with homogeneous barycentric coordinates with reference to triangle ABC.

Publication Date: October 1, 2007. Communicating Editor: Paul Yiu.

Theorem 1. (1) *The lines AX, BY, CZ are concurrent at*

$$P_1 = \left(\cos\frac{A}{2}\cos^2\frac{A}{4} \ :\cos\frac{B}{2}\cos^2\frac{B}{4} \ :\cos\frac{C}{2}\cos^2\frac{C}{4}\ \right).$$

(2) *The lines I_1X, I_2Y, I_3Z are concurrent at*

$$\begin{aligned} P_2 = \Bigg(a\left(1-\cos\frac{B}{2}-\cos\frac{C}{2}\right)+(b+c)\cos\frac{A}{2} \\ :b\left(1-\cos\frac{C}{2}-\cos\frac{A}{2}\right)+(c+a)\cos\frac{B}{2} \\ :c\left(1-\cos\frac{A}{2}-\cos\frac{B}{2}\right)+(a+b)\cos\frac{C}{2}\Bigg).\end{aligned}$$

2. Some preliminary results

Let s and R be the semiperimeter and circumradius respectively of triangle ABC. The following homogeneous barycentric coordinates are well known.

$$\begin{array}{lll} A_a = (0:s-b:s-c), & B_a = (-(s-b):0:s), & C_a = (-(s-c):s:0); \\ A_b = (0:-(s-a):s), & B_b = (s-a:0:s-c), & C_b = (s:-(s-c):0); \\ A_c = (0:s:-(s-c)), & B_c = (s:0:-(s-a)), & C_c = (s-a:s-b:0). \end{array}$$

The lengths of the sides of the A-extouch triangle are as follows:

$$B_aC_a = 2s\cdot\sin\frac{A}{2}, \quad C_aA_a = 2(s-c)\cos\frac{B}{2}, \quad A_aB_a = 2(s-b)\cos\frac{C}{2}. \quad (1)$$

Lemma 2.

$$\begin{aligned} s &= 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}, \\ s-a &= 4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, \\ s-b &= 4R\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}, \\ s-c &= 4R\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}. \end{aligned}$$

We omit the proof of this lemma. It follows easily from, for example, [1, §293].

3. Proof of Theorem 1

$$\begin{aligned}
B_aX =&\frac{1}{2}(B_aC_a - A_aC_a + A_aB_a)\\
=&s\cdot\sin\frac{A}{2} - (s-c)\cos\frac{B}{2} + (s-b)\cos\frac{C}{2}\\
=&4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left(\cos\frac{A}{2} - \sin\frac{B}{2} + \sin\frac{C}{2}\right)\\
=&4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left(\sin\frac{B+C}{2} - \sin\frac{B}{2} + \sin\frac{C}{2}\right)\\
=&4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left(2\sin\frac{B+C}{4}\cos\frac{B+C}{4} - 2\sin\frac{B-C}{4}\cos\frac{B+C}{4}\right)\\
=&16R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{B+C}{4}\cdot\cos\frac{B}{4}\sin\frac{C}{4}.
\end{aligned}$$

Similarly, $XC_a = 16R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{B+C}{4}\cdot\sin\frac{B}{4}\cos\frac{C}{4}$. The point X therefore divides B_aC_a in the ratio

$$B_aX : XC_a = \cos\frac{B}{4}\sin\frac{C}{4} : \sin\frac{B}{4}\cos\frac{C}{4}.$$

This allows us to compute its absolute barycentric coordinate in terms of B_a and C_a. Note that

$$B_a = \frac{\left(-\sin\frac{A}{2}\sin\frac{C}{2},\ 0,\ \cos\frac{A}{2}\cos\frac{C}{2}\right)}{\sin\frac{B}{2}}, \qquad C_a = \frac{\left(-\sin\frac{A}{2}\sin\frac{B}{2},\ \cos\frac{A}{2}\cos\frac{B}{2},\ 0\right)}{\sin\frac{C}{2}}.$$

From these we have

$$\begin{aligned}
X =&\frac{\sin\frac{B}{4}\cos\frac{C}{4}\cdot B_a + \cos\frac{B}{4}\sin\frac{C}{4}\cdot C_a}{\sin\frac{B+C}{4}}\\
=&\frac{\sin\frac{B}{4}\cos\frac{C}{4}\cdot\frac{\left(-\sin\frac{A}{2}\sin\frac{C}{2},0,\cos\frac{A}{2}\cos\frac{C}{2}\right)}{\sin\frac{B}{2}} + \cos\frac{B}{4}\sin\frac{C}{4}\cdot\frac{\left(-\sin\frac{A}{2}\sin\frac{B}{2},\cos\frac{A}{2}\cos\frac{B}{2},\ 0\right)}{\sin\frac{C}{2}}}{\sin\frac{B+C}{4}}\\
=&\frac{\cos\frac{C}{4}\cdot\frac{\left(-\sin\frac{A}{2}\sin\frac{C}{2},0,\cos\frac{A}{2}\cos\frac{C}{2}\right)}{2\cos\frac{B}{4}} + \cos\frac{B}{4}\cdot\frac{\left(-\sin\frac{A}{2}\sin\frac{B}{2},\cos\frac{A}{2}\cos\frac{B}{2},\ 0\right)}{2\cos\frac{C}{4}}}{\sin\frac{B+C}{4}}\\
=&\frac{\cos^2\frac{C}{4}\left(-\sin\frac{A}{2}\sin\frac{C}{2},0,\cos\frac{A}{2}\cos\frac{C}{2}\right) + \cos^2\frac{B}{4}\left(-\sin\frac{A}{2}\sin\frac{B}{2},\cos\frac{A}{2}\cos\frac{B}{2},\ 0\right)}{2\cos\frac{B}{4}\cos\frac{C}{4}\sin\frac{B+C}{4}}\\
=&\frac{\left(-\sin\frac{A}{2}\left(\sin\frac{B}{2}\cos^2\frac{B}{4} + \sin\frac{C}{2}\cos^2\frac{C}{4}\right),\cos\frac{A}{2}\cos\frac{B}{2}\cos^2\frac{B}{4},\ \cos\frac{A}{2}\cos\frac{C}{2}\cos^2\frac{C}{4}\right)}{2\cos\frac{B}{4}\cos\frac{C}{4}\sin\frac{B+C}{4}}.
\end{aligned}$$

From this we obtain the homogeneous barycentric coordinates of X, and those of Y and Z by cyclic permutations of A, B, C:

$$\begin{aligned}
X &= \left(-\sin\frac{A}{2}\left(\sin\frac{B}{2}\cos^2\frac{B}{4}+\sin\frac{C}{2}\cos^2\frac{C}{4}\right) : \cos\frac{A}{2}\cos\frac{B}{2}\cos^2\frac{B}{4} : \cos\frac{A}{2}\cos\frac{C}{2}\cos^2\frac{C}{4}\right),\\
Y &= \left(\cos\frac{B}{2}\cos\frac{A}{2}\cos^2\frac{A}{4} : -\sin\frac{B}{2}\left(\sin\frac{C}{2}\cos^2\frac{C}{4}+\sin\frac{A}{2}\cos^2\frac{A}{4}\right) : \cos\frac{B}{2}\cos\frac{C}{2}\cos^2\frac{C}{4}\right),\\
Z &= \left(\cos\frac{C}{2}\cos\frac{A}{2}\cos^2\frac{A}{4} : \cos\frac{C}{2}\cos\frac{B}{2}\cos^2\frac{B}{4} : -\sin\frac{C}{2}\left(\sin\frac{A}{2}\cos^2\frac{A}{4}+\sin\frac{B}{2}\cos^2\frac{B}{4}\right)\right).
\end{aligned}$$

Equivalently,

$$\begin{aligned}
X &= \left(-\tan\frac{A}{2}\left(\sin\frac{B}{2}\cos^2\frac{B}{4}+\sin\frac{C}{2}\cos^2\frac{C}{4}\right) : \cos\frac{B}{2}\cos^2\frac{B}{4} : \cos\frac{C}{2}\cos^2\frac{C}{4}\right),\\
Y &= \left(\cos\frac{A}{2}\cos^2\frac{A}{4} : -\tan\frac{B}{2}\left(\sin\frac{C}{2}\cos^2\frac{C}{4}+\sin\frac{A}{2}\cos^2\frac{A}{4}\right) : \cos\frac{C}{2}\cos^2\frac{C}{4}\right),\\
Z &= \left(\cos\frac{A}{2}\cos^2\frac{A}{4} : \cos\frac{B}{2}\cos^2\frac{B}{4} : -\tan\frac{C}{2}\left(\sin\frac{A}{2}\cos^2\frac{A}{4}+\sin\frac{B}{2}\cos^2\frac{B}{4}\right)\right).
\end{aligned}$$

It is clear that the lines AX, BY, CZ intersect at a point P_1 with coordinates

$$\left(\cos\frac{A}{2}\cos^2\frac{A}{4} : \cos\frac{B}{2}\cos^2\frac{B}{4} : \cos\frac{C}{2}\cos^2\frac{C}{4}\right).$$

This completes the proof of Theorem 1(1).

For (2), note that the line I_1X is parallel to the bisector of angle A. Its has barycentric equation

$$\begin{vmatrix} -\sin\frac{A}{2}\left(\sin\frac{B}{2}\cos^2\frac{B}{4}+\sin\frac{C}{2}\cos^2\frac{C}{4}\right) & \cos\frac{A}{2}\cos\frac{B}{2}\cos^2\frac{B}{4} & \cos\frac{A}{2}\cos\frac{C}{2}\cos^2\frac{C}{4}\\ -(b+c) & b & c\\ x & y & z \end{vmatrix} = 0.$$

A routine calculation, making use of the fact that the sum of the entries in the first row is $\sin\frac{C}{2}\cos^2\frac{B}{4}+\sin\frac{B}{2}\cos^2\frac{C}{4}$, gives

$$-(x+y+z)\left(b\cos\frac{C}{2}-c\cos\frac{B}{2}\right)+bz-cy=0.$$

Similarly, the lines I_2Y and I_3Z have equations

$$\begin{aligned}
-(x+y+z)\left(c\cos\frac{A}{2}-a\cos\frac{C}{2}\right)+cx-az &=0,\\
-(x+y+z)\left(a\cos\frac{B}{2}-b\cos\frac{A}{2}\right)+ay-bx &=0.
\end{aligned}$$

These three lines intersect at

$$P_2 = \left(a\left(1 - \cos\frac{B}{2} - \cos\frac{C}{2}\right) + (b+c)\cos\frac{A}{2} \right.$$
$$: b\left(1 - \cos\frac{C}{2} - \cos\frac{A}{2}\right) + (c+a)\cos\frac{B}{2}$$
$$\left. : c\left(1 - \cos\frac{A}{2} - \cos\frac{B}{2}\right) + (a+b)\cos\frac{C}{2} \right).$$

This completes the proof of Theorem 1(2).

Remark. The barycentric coordinates of the incenter I_1 of the A-extouch triangle are

$$\left(-\sin\frac{A}{2}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right) : \cos\frac{B}{2}\left(\sin\frac{C}{2}+\cos\frac{A}{2}\right) : \cos\frac{C}{2}\left(\cos\frac{A}{2}+\sin\frac{B}{2}\right)\right).$$

4. Some collinearities

The homogeneous barycentric coordinates of P_1 can be rewritten as

$$\left(\cos^2\frac{A}{2}+\cos\frac{A}{2} \ : \cos^2\frac{B}{2}+\cos\frac{B}{2} \ : \cos^2\frac{C}{2}+\cos\frac{C}{2}\right).$$

From this it is clear that the point P_1 lies on the line joining the two points with coordinates $\left(\cos^2\frac{A}{2} : \cos^2\frac{B}{2} : \cos^2\frac{C}{2}\right)$ and $\left(\cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}\right)$. We briefly recall their definitions.

(i) The point $M = \left(\cos^2\frac{A}{2} : \cos^2\frac{B}{2} : \cos^2\frac{C}{2}\right) = (a(s-a) : b(s-b) : c(s-c))$ is the Mittenpunkt. It is the perspector of the excentral triangle and the medial triangle. It is the triangle center X_9 of [2].

(ii) The point $Q = \left(\cos\frac{A}{2} \ : \cos\frac{B}{2} \ : \cos\frac{C}{2}\right)$ appears as X_{188} in [2], and is named the second mid-arc point. Here is an explicit description. Consider the anticomplementary triangle $A'B'C'$ of ABC, with its incircle (I'). If the segments $I'A'$, $I'B'$, $I'C'$ intersect the incircle (I') at A'', B'', C'', then the lines AA'', BB'', CC'' are concurrent at Q. See Figure 2.

Proposition 3. (1) *The point P_1 lies on the line MQ.*
(2) *The point P_2 lies on the line joining the incenter to Q.*

Proof. We need only prove (2). This is clear from

$$P_2 = \left(1-\cos\frac{A}{2}-\cos\frac{B}{2}-\cos\frac{C}{2}\right)I + \left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)Q.$$

In fact,

$$P_2 = I + \left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)\overrightarrow{IQ}. \qquad (2)$$

□

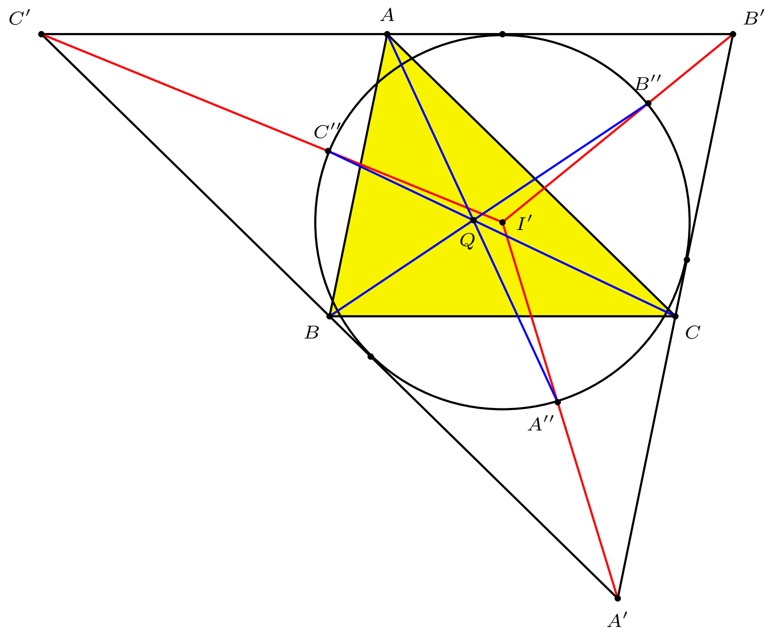


Figure 2.

5. The excircles of the extouch triangles

Consider the excircle of triangle $A_aB_aC_a$ tangent to the side B_aC_a at X'. It is clear that X' and X are symmetric with respect to the midpoint of B_aC_a. Since triangle AB_aC_a is isosceles, the lines AX' and AX are isogonal with respect to AB_a and AC_a. As such, they are isogonal with respect to AB and AC. Likewise, if we consider the excircle of $A_bB_bC_b$ tangent to C_bA_b at Y', and that of $A_cB_cC_c$ tangent to A_cB_c at Z', then the lines AX', BY', CZ', being respectively isogonal to AX, BY, CZ, intersect at the isogonal conjugate of P_1.

Proposition 4. *The barycentric coordinates of P_1^* are*

$$\left(\cos\frac{A}{2}\sin^2\frac{A}{4} \;:\cos\frac{B}{2}\sin^2\frac{B}{4} \;:\cos\frac{C}{2}\sin^2\frac{C}{4}\right).$$

Proof. This follows from

$$\begin{aligned}
P_1^* &= \left(\frac{\sin^2 A}{\cos\frac{A}{2}\cos^2\frac{A}{4}} \;:\frac{\sin^2 B}{\cos\frac{B}{2}\cos^2\frac{B}{4}} \;:\frac{\sin^2 C}{\cos\frac{C}{2}\cos^2\frac{C}{4}}\right)\\
&= \left(\frac{\sin^2\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{4}} \;:\frac{\sin^2\frac{B}{2}\cos\frac{B}{2}}{\cos^2\frac{B}{4}} \;:\frac{\sin^2\frac{C}{2}\cos\frac{C}{2}}{\cos^2\frac{C}{4}}\right)\\
&= \left(\cos\frac{A}{2}\sin^2\frac{A}{4} \;:\cos\frac{B}{2}\sin^2\frac{B}{4} \;:\cos\frac{C}{2}\sin^2\frac{C}{4}\right).:
\end{aligned}$$

□

Corollary 5. *The points P_1, P_1^* and Q are collinear.*

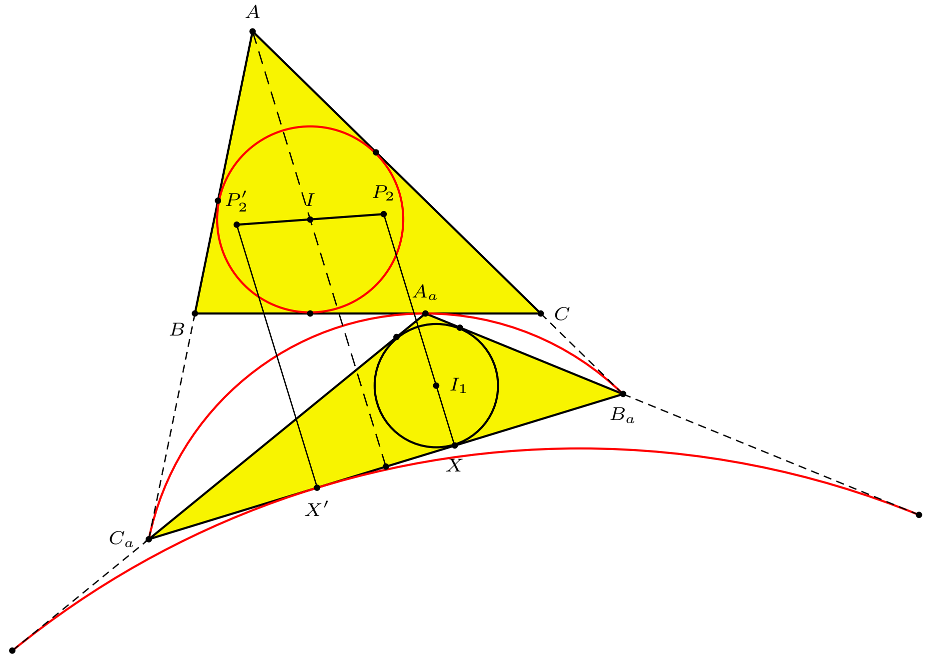


Figure 3.

Proposition 6. *The perpendiculars to B_aC_a at X', to C_bA_b at Y', and to A_cB_c at Z' are concurrent at the reflection of P_2 in I, which is the point*

$$\begin{aligned}P_2' =& a\left(1+\cos\frac{B}{2}+\cos\frac{C}{2}\right)-(b+c)\cos\frac{A}{2}\\ &:b\left(1+\cos\frac{C}{2}+\cos\frac{A}{2}\right)-(c+a)\cos\frac{B}{2}\\ &:c\left(1+\cos\frac{A}{2}+\cos\frac{B}{2}\right)-(a+b)\cos\frac{C}{2}.\end{aligned}$$

Proof. Let P_2' be the reflection of P_2 in I. Since X and X' are symmetric in the midpoint of B_aC_a, and P_2X is perpendicular to B_aC_a, it follows that $P_2'X'$ is also perpendicular to B_aC_a. The same reasoning shows that $P_2'Y'$ and $P_2'Z'$ are perpendicular to C_bA_b and A_cB_c respectively. It follows from (2) that

$$P_2' = I - \left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)\overrightarrow{IQ}.$$

From this, we easily obtain the homogeneous barycentric coordinates as given above. $\square$

We conclude this paper with the construction of another triangle center. It is known that the perpendiculars from A_a to B_aC_a, B_b to C_bA_b, and C_c to A_cB_c

intersect at

$$P_3 = ((b+c)\cos A\ :(c+a)\cos B\ :(a+b)\cos C). \tag{3}$$

This is the triangle center X_{72} in [2].

If we let X_0, Y_0, Z_0 be these pedals, then it is also known that AX_0, BY_0, CZ_0 intersect at the Mittenpunkt X_9. Now, let X_1, Y_1, Z_1 be the reflections of X_0, Y_0, Z_0 in the midpoints of B_aC_a, C_bA_b, A_cB_c respectively. The lines AX_1, BY_1, CZ_1 clearly intersect at the reflection of X_{72} in I. This is the point

$$P_3' = ((b+c)\cos A - 2a\ :(c+a)\cos B - 2b\ :(a+b)\cos C - 2c)\,.$$

These coordinates are particularly simple since the sum of the coordinates of P_3 given in (3) is $a+b+c$.

The triangle centers P_1, P_1^*, P_2, P_2' and P_3' do not appear in [2].

References

[1] R. A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, 1960.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[3] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University lecture notes, 2001, available at http://www.math.fau.edu/Yiu/Geometry.html.

Tibor Dosa: 83098 Brannenburg Tannenweg 7, Germany

E-mail address: dosa.tibor@t-onine.de

Forum Geometricorum
Volume 7 (2007) 159–168.

FORUM GEOM
ISSN 1534-1178

Fixed Points and Fixed Lines of Ceva Collineations

Clark Kimberling

Abstract. In the plane of a triangle ABC, the U-Ceva collineation maps points to points and lines to lines. If U is a triangle center other than the incenter, then the U-Ceva collineation has three distinct fixed points F_1, F_2, F_3 and three distinct fixed lines F_2F_3, F_3F_1, F_1F_2, these being the trilinear polars of F_1, F_2, F_3. When U is the circumcenter, the fixed points are the symmedian point and the isogonal conjugates of the points in which the Euler line intersects the circumcircle.

1. Introduction

This note is a sequel to [3], in which the notion of a U-Ceva collineation is introduced. In this introduction, we briefly summarize the main results of [3].

We use homogeneous trilinear coordinates and denote the isogonal conjugate of a point X by X^{-1}. The X-Ceva conjugate of $U = u : v : w$ and $X = x : y : z$ is given by

$$X \copyright U = u(-uyz + vzx + wxy) : v(uyz - vzx + wxy) : w(uyz + vzx - wxy),$$

and if $P = p : q : r$ is a point, then the equation $P = X \copyright U$ is equivalent to

$$\begin{aligned} X &= (ru + pw)(pv + qu) : (pv + qu)(qw + rv) : (qw + rv)(ru + pw) \qquad (1) \\ &= \text{cevapoint}(P, U). \end{aligned}$$

If $\mathcal{L}_1$ is a line $l_1\alpha + m_1\beta + n_1\gamma = 0$ and $\mathcal{L}_2$ is a line $l_2\alpha + m_2\beta + n_2\gamma = 0$, then there exists a unique point U such that if $X \in \mathcal{L}_1$, then $X^{-1} \copyright U \in \mathcal{L}_2$, and the mapping $X \to X^{-1} \copyright U$ is surjective. This mapping is written as $\mathcal{C}_U(X) = X^{-1} \copyright U$, and $\mathcal{C}_U$ is called the U-Ceva collineation. Explicitly,

$$\mathcal{C}_U(X) = u(-ux + vy + wz) : v(ux - vy + wz) : w(ux + vy - wz).$$

The inverse mapping is given by

$$\begin{aligned} \mathcal{C}_U^{-1}(X) &= wy + vz : uz + wx : vx + uy \\ &= (\text{cevapoint}(X, U))^{-1}. \end{aligned}$$

The collineation $\mathcal{C}_U$ maps the vertices A, B, C to the vertices of the anticevian triangle of U and maps U^{-1} to U. The collineation $\mathcal{C}_U^{-1}$ maps A, B, C to the vertices of the cevian triangle of U^{-1} and maps U to U^{-1}.

Publication Date: October 15, 2007. Communicating Editor: Paul Yiu.

2. Fixed points

The fixed points of the $\mathcal{C}_U$-collineation are also the fixed points of the inverse collineation, $\mathcal{C}_U^{-1}$. In this section, we seek all points X satisfying $\mathcal{C}_U^{-1}(X) = X$; *i.e.*, we wish to solve the equation

$$\mathcal{C}_U^{-1}(X) = \begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = MX$$

for the vector X. Writing $(M - tI)X = 0$, where I denotes the 3×3 identity matrix, we have the characteristic equation $\det(M - tI) = 0$ of M, which can be written

$$\begin{vmatrix} -t & w & v \\ w & -t & u \\ v & u & -t \end{vmatrix} = 0.$$

Expanding the determinant gives

$$t^3 - gt - h = 0, \tag{2}$$

where $g = u^2+v^2+w^2$ and $h = 2uvw$. Now suppose t is a root, *i.e.*, an eigenvalue of M. The equation $(M - tI)X = 0$ is equivalent to the system

$$\begin{aligned} -tx + wy + vz &= 0 \\ wx - ty + uz &= 0 \\ vx + uy - tz &= 0. \end{aligned}$$

For any z, the first two of the three equations can be written as

$$\begin{pmatrix} -t & w \\ w & -t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -vz \\ -uz \end{pmatrix},$$

and if $t^2 \neq w^2$, then

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -t & w \\ w & -t \end{pmatrix}^{-1} \begin{pmatrix} -vz \\ -uz \end{pmatrix} \\ &= \frac{1}{t^2 - w^2} \begin{pmatrix} tvz + uwz \\ tuz + vwz \end{pmatrix}, \end{aligned}$$

Thus, for each z,

$$x = \frac{1}{t^2 - w^2}(tvz + uwz) \quad \text{and} \quad y = \frac{1}{t^2 - w^2}(tuz + vwz),$$

so that

$$x : y = tv + uw : tu + vw \quad \text{and} \quad \frac{y}{z} = \frac{1}{t^2 - w^2}(tu + vw),$$

and $x : y : z$ is as shown in (6) below.

Continuing with the case $t^2 \neq w^2$, let $f(t)$ be the polynomial in (2), and let

$$r = \sqrt{(u^2 + v^2 + w^2)/3},$$

so that

$$f(-r) = -2uvw + \frac{2}{3}(u^2 + v^2 + w^2)r; \tag{3}$$
$$f(r) = -2uvw - \frac{2}{3}(u^2 + v^2 + w^2)r.$$

Clearly, $f(r) < 0$. To see that $f(-r) \geq 0$, we shall use the inequality of the geometric and arithmetic means, stated here for $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$:

$$(x_1x_2x_3)^{1/3} \leq \frac{x_1 + x_2 + x_3}{3}. \tag{4}$$

Taking $x_1 = u^2$, $x_2 = v^2$, $x_3 = w^2$ gives

$$27u^2v^2w^2 \leq \left(u^2 + v^2 + w^2\right)^3,$$

or equivalently,

$$3uvw \leq \left(u^2 + v^2 + w^2\right) r,$$

so that by (3), we have $f(-r) \geq 0$. We consider two cases: $f(-r) > 0$ and $f(r) = 0$. In the first case, there is a root t in the interval $(-\infty, -r)$. Since $f(0) < 0$, there is a root in $(-r, 0)$, and since $f(r) < 0$, there is a root in (r, ∞). For each of the three roots, or eigenvalues, there is an eigenvector, or point X, such that $\mathcal{C}_U^{-1}(X) = X$.

In the second case, that $f(r) = 0$, we have $(u^2 + v^2 + w^2)r = -3uvw$, so that $(u^2 + v^2 + w^2)^3 = 27u^2v^2w^2$, which implies that equality holds in (4). This is known to occur if and only if if $x_1 = x_2 = x_3$, or equivalently, $u^2 = v^2 = w^2$, which is to say that U is the incenter or one of the excenters; *i.e.*, that U is a member of the set

$$\{1 : 1 : 1, \quad -1 : 1 : 1, \quad 1 : -1 : 1, \quad 1 : 1 : -1\}. \tag{5}$$

We consider this case further in Examples 1 and 2 below, and summarize the rest of this section as a theorem.

Theorem 1. *Suppose U is not one of the four points in* (4)*, that t is a root of* (2)*, and that $t^2 \neq w^2$. Then the point*

$$X = tv + uw : tu + vw : t^2 - w^2 \tag{6}$$

is a fixed point of $\mathcal{C}_U^{-1}$, hence also a fixed point of C_U. There are three distinct roots t, hence three distinct fixed points X.

3. Examples

As a first example, we address the possibility that the hypothesis $t^2 \neq w^2$ in Theorem 1 does not hold.

Example 1. $U = 1 : 1 : 1$. The characteristic polynomial is

$$\begin{vmatrix} -t & 1 & 1 \\ 1 & -t & 1 \\ 1 & 1 & -t \end{vmatrix} = (2 - t)(t + 1)^2.$$

We have two cases: $t = 2$ and $t = -1$. For $t = 2$, we easily find the fixed point $1 : 1 : 1$. For $t = -1$, the method of proof of Theorem 1 does not apply because $t^2 = w^2$. Instead, the system to be solved degenerates to the single equation $z = -x - y$. The solutions, all fixed points, are many; for example, let $f : g : h$ be any point, and let

$$x = g - h, \quad y = h - f, \quad z = f - g$$

(e.g., $x : y : z = b - c : c - a : a - b$, which is the triangle center[1] X_{512}.). Geometrically, $x : y : z$ are coefficients of the line joining $1 : 1 : 1$ and $f : g : h$.

Example 2. $U = -1 : 1 : 1$, the A-excenter. The characteristic polynomial is

$$\begin{vmatrix} -t & 1 & 1 \\ -1 & -t & 1 \\ -1 & 1 & -t \end{vmatrix} = -(t+1)\left(t^2 - t + 2\right).$$

For $t = -1$, we find that every point on the line $x + y + z = 0$ is a fixed point. If $t^2 - t + 2 = 0$, then $t = (1 \pm \sqrt{-7})/2$, and the (nonreal) fixed point is $1 : 1 : t - 1$. Similar results are obtained for $U \in \{1 : -1 : 1, 1 : 1 : -1\}$.

Example 3. $U = \cos A : \cos B : \cos C$. It can be checked using a computer algebra system that X_6, X_{2574}, and X_{2575} are fixed points. The first of these corresponds to the eigenvalue $t = 1$, as shown here:

$$\begin{aligned} x : y : z = tv + uw : tu + vw : t^2 - w^2 \\ &= \cos B + \cos A \cos C : \cos A + \cos B \cos C : 1 - \cos^2 C \\ &= \sin A \sin C : \sin B \sin C : \sin C \sin C \\ &= \sin A : \sin B : \sin C \\ &= X_6. \end{aligned}$$

See also Example 6.

Example 4. $U = a(b^2 + c^2) : b(c^2 + a^2) : c(a^2 + b^2) = X_{39}$. The three roots of $t^3 - gt - h = 0$ are

$$-2abc, \quad abc - \sqrt{3a^2b^2c^2 + S(2,4)}, \quad abc + \sqrt{3a^2b^2c^2 + S(2,4)},$$

where

$$S(2,4) = a^2b^4 + a^4b^2 + a^2c^4 + a^4c^2 + b^2c^4 + b^4c^2.$$

The solution $t = -2abc$ easily leads to the fixed point

$$X_{512} = (b^2 - c^2)/a : (c^2 - a^2)/b : (a^2 - b^2)/c.$$

[1] We use the indexing of triangle centers in the *Encyclopedia of Triangle Centers* [3].

Example 5. For arbitrary real n, let $u = \cos nA$, $v = \cos nB$, $w = \cos nC$. A fixed point is $X = \sin nA : \sin nB : \sin nC$, as shown here:

$$\begin{aligned}\mathcal{C}_U^{-1}(X) =\ & \sin nB \cos nC + \sin nC \cos nB \\ &: \sin nC \cos nA + \sin nA \cos nC \\ &: \sin nA \cos nB + \sin nB \cos nA \\ =\ & \sin(nB + nC) : \sin(nC + nA) : \sin(nA + nB) \\ =\ & \sin nA : \sin nB : \sin nC.\end{aligned}$$

4. Images of lines

Let $\mathcal{L}$ be the line $l\alpha + m\beta + n\gamma = 0$ and let L the point[2] $l : m : n$. We shall determine coefficients of the line $\mathcal{C}_U^{-1}(\mathcal{L})$. Two points on $\mathcal{L}$ are

$$P = cm - bn : an - cl : bl - am \quad \text{and} \quad Q = m - n : n - l : l - m$$

Their images on $\mathcal{C}_U^{-1}(\mathcal{L})$ are given by

$$P' = \mathcal{C}_U^{-1}(P) = \begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix} \begin{pmatrix} cm - bn \\ an - cl \\ bl - am \end{pmatrix},$$
$$Q' = \mathcal{C}_U^{-1}(Q) = \begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix} \begin{pmatrix} m - n \\ n - l \\ l - m \end{pmatrix}.$$

We expand these products and use the resulting trilinears as rows 2 and 3 of the following determinant:

$$\begin{aligned}&\begin{vmatrix} \alpha & \beta & \gamma \\ w(an - cl) + v(bl - am) & w(cm - bn) + u(bl - am) & v(cm - bn) + u(an - cl) \\ w(n - l) + v(l - m) & w(m - n) + u(l - m) & v(m - n) + u(n - l) \end{vmatrix} \\ =\ & -\left((b - c)l + (c - a)m + (a - b)n\right) \\ &\cdot \left(u\left(-ul + vm + wn\right)\alpha + b\left(ul - vm + wn\right)\beta + c\left(ul + vm - wn\right)\gamma\right).\end{aligned}$$

If the first factor is not 0, then the required line $\mathcal{C}_U^{-1}(\mathcal{L})$ is given by

$$u\left(-ul + vm + wn\right)\alpha + v\left(ul - vm + wn\right)\beta + w\left(ul + vm - wn\right)\gamma = 0, \quad (7)$$

of which the coefficients are the trilinears of the point

$$L^{-1} \copyright U = u\left(-ul + vm + wn\right) : v\left(ul - vm + wn\right) : w\left(ul + vm - wn\right).$$

Even if the first factor is 0, the points P' and Q' are easily checked to lie on the line (7).

[2]Geometrically, $\mathcal{L}$ is the trilinear polar of L^{-1}. However, the methods in this paper are algebraic rather than geometric, and the results extend beyond the boundaries of euclidean geometry. For example, in this paper, a, b, c are unrestricted positive real numbers; *i.e.*, they need not be sidelengths of a triangle.

The same method shows that the coefficients of the line $\mathcal{C}_U(\mathcal{L})$ are the trilinears of $(\text{cevapoint}(L,U))^{-1}$; that is, $\mathcal{C}_U(\mathcal{L})$ is the line

$$(wm+vn)\,\alpha+(un+wl)\,\beta+w\,(vl+um)\,\gamma=0.$$

5. Fixed lines

The line $\mathcal{L}$ is a fixed line of $\mathcal{C}_U$ (and of $\mathcal{C}_U^{-1}$) if $\mathcal{C}_U(\mathcal{L})=\mathcal{L}$, that is, if

$$(\text{cevapoint}(U,L))^{-1}=L,$$

or, equivalently,

$$\begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix}\begin{pmatrix} l \\ m \\ n \end{pmatrix}=\begin{pmatrix} l \\ m \\ n \end{pmatrix}.$$

This is the same equation as already solved (with L in place of X) in Section 2. For each of the three roots of (2), there is an eigenvector, or point L, and hence a line $\mathcal{L}$, such that $\mathcal{C}_U(\mathcal{L})=\mathcal{L}$, and we have the following theorem.

Theorem 2. *The mapping C_U has three distinct fixed lines, corresponding to the three distinct real roots of $f(t)$ in* (2). *For each root t, the corresponding fixed line $l\alpha+m\beta+n\gamma=0$ is given by*

$$l:m:n=tv+uw:tu+vw:t^2-w^2. \tag{8}$$

6. Iterations and convergence

In this section we examine sequences

$$X,\ \ \mathcal{C}_U^{-1}(X),\ \ \mathcal{C}_U^{-1}(\mathcal{C}_U^{-1}(X)),\ldots \tag{9}$$

of iterates. If X is a fixed point of $\mathcal{C}_U^{-1}$, then the sequence is simply $X,X,X,...$; otherwise, with exceptions to be recognized, the sequence converges to a fixed point. We begin with the case that X lies on a fixed line, so that all the points in (9) lie on that same line. Let the two fixed points on the fixed line be

$$F_1=f_1:g_1:h_1 \quad \text{and} \quad F_2=f_2:g_2:h_2.$$

Then for X on the line F_1F_2, we have

$$X=f_1+tf_2:g_1+tg_2:h_1+th_2$$

for some function t homogeneous in a,b,c, and we wish to show that (9) converges to F_1 or F_2. As a first step,

$$\mathcal{C}_U^{-1}(X)=\begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix}\begin{pmatrix} f_1+tf_2 \\ g_1+tg_2 \\ h_1+th_2 \end{pmatrix}=\begin{pmatrix} wg_1+vh_1+t(wg_2+vh_2) \\ wf_1+uh_1+t(wf_2+uh_2) \\ vf_1+ug_1+t(vf_2+ug_2) \end{pmatrix}.$$

For $i = 1, 2$, because $f_i : g_i : h_i$ is fixed by $\mathcal{C}_U^{-1}$, there exists a homogeneous function t_i such that

$$\begin{aligned} wg_i + vh_i &= t_i f_i, \\ wf_i + vh_i &= t_i g_i, \\ vf_i + ug_i &= t_i h_i, \end{aligned}$$

so that

$$\mathcal{C}_U^{-1}(X) = \begin{pmatrix} t_1 f_1 + t_2 t f_2 \\ t_1 g_1 + t_2 t g_2 \\ t_1 h_1 + t_2 t h_2 \end{pmatrix} = t_1 \begin{pmatrix} f_1 + \dfrac{t_2}{t_1} t f_2 \\ g_1 + \dfrac{t_2}{t_1} t g_2 \\ h_1 + \dfrac{t_2}{t_1} t h_2 \end{pmatrix}.$$

Applying $\mathcal{C}_U^{-1}$ again thus gives

$$\mathcal{C}_U^{-2}(X) = \begin{pmatrix} f_1 + \dfrac{t_4}{t_3}\dfrac{t_2}{t_1} t f_2 \\ g_1 + \dfrac{t_4}{t_3}\dfrac{t_2}{t_1} t g_2 \\ h_1 + \dfrac{t_4}{t_3}\dfrac{t_2}{t_1} t h_2 \end{pmatrix},$$

where t_3 and t_4 satisfy

$$\begin{pmatrix} wg_1 + vh_1 + \dfrac{t_2}{t_1} t(wg_2 + vh_2) \\ wf_1 + uh_1 + \dfrac{t_2}{t_1} t(wf_2 + uh_2) \\ vf_1 + ug_1 + \dfrac{t_2}{t_1} t(vf_2 + ug_2) \end{pmatrix} = \begin{pmatrix} t_3 f_1 + t_4 \dfrac{t_2}{t_1} t f_2 \\ t_3 g_1 + t_4 \dfrac{t_2}{t_1} t f_2 \\ t_3 h_1 + t_4 \dfrac{t_2}{t_1} t f_2 \end{pmatrix} = t_3 \begin{pmatrix} f_1 + \dfrac{t_4}{t_3}\dfrac{t_2}{t_1} t f_2 \\ g_1 + \dfrac{t_4}{t_3}\dfrac{t_2}{t_1} t g_2 \\ h_1 + \dfrac{t_4}{t_3}\dfrac{t_2}{t_1} t h_2 \end{pmatrix}.$$

Now

$$\begin{aligned} t_1 &= \frac{wg_1 + vh_1}{f_1} = \frac{wf_1 + uh_1}{g_1} = \frac{vf_1 + ug_1}{h_1}, \\ t_2 &= \frac{wg_2 + vh_2}{f_2} = \frac{wf_2 + uh_2}{g_2} = \frac{vf_2 + ug_2}{h_2}, \\ t_3 &= \frac{wg_1 + vh_1}{f_1} = \frac{wf_1 + uh_1}{g_1} = \frac{vf_1 + ug_1}{h_1} = t_1, \\ t_4 &= \frac{w(\frac{t_2}{t_1})g_2 + v(\frac{t_2}{t_1})h_2}{(\frac{t_2}{t_1})f_2} = \frac{w(\frac{t_2}{t_1})f_2 + u(\frac{t_2}{t_1})h_2}{(\frac{t_2}{t_1})g_2} = \frac{v(\frac{t_2}{t_1})f_2 + u(\frac{t_2}{t_1})g_2}{(\frac{t_2}{t_1})h_2} = t_2. \end{aligned}$$

Consequently,

$$\mathcal{C}_U^{-2}(X) = \begin{pmatrix} f_1 + (\dfrac{t_2}{t_1})^2 t f_2 \\ g_1 + (\dfrac{t_2}{t_1})^2 t g_2 \\ h_1 + (\dfrac{t_2}{t_1})^2 t h_2 \end{pmatrix},$$

and, by induction,

$$\mathcal{C}_U^{-n}(X) = \begin{pmatrix} f_1 + (\frac{t_2}{t_1})^n t f_2 \\ g_1 + (\frac{t_2}{t_1})^n t g_2 \\ h_1 + (\frac{t_2}{t_1})^n t h_2 \end{pmatrix}. \tag{10}$$

Regarding the quotient $\frac{t_2}{t_1}$ in (10), if $\frac{t_2}{t_1} = 1$ then $\mathcal{C}_U^{-n}(X)$ is invariant of n, which is to say that X is a fixed point. If $\frac{t_2}{t_1} = -1$, then $\mathcal{C}_U^{-2}(X) = X$, which is to say that X is a fixed point of the collineation $\mathcal{C}_U^{-2}$. If $\left|\frac{t_2}{t_1}\right| \neq 1$, we call the line F_1F_2 a *regular fixed line*, and in this case, by (10), $\lim_{n\to\infty} \mathcal{C}_U^{-n}(X)$ is F_1 or F_2, according as $\left|\frac{t_2}{t_1}\right| < 1$ or $\left|\frac{t_2}{t_1}\right| > 1$. We summarize these findings as Lemma 3.

Lemma 3. *If X lies on a regular fixed line of $\mathcal{C}_U^{-1}$ (or equivalently, a regular fixed line of $\mathcal{C}_U$), then the sequence of points $\mathcal{C}_U^{-n}(X)$ (or equivalently, the points $\mathcal{C}_U^{n}(X)$) converges to a fixed point of $\mathcal{C}_U^{-1}$ (and of $\mathcal{C}_U$).*

Next, suppose that P is an arbitrary point in the plane of ABC. We shall show that $\mathcal{C}_U^{-n}(P)$ converges to a fixed point. Let F_1, F_2, F_3 be distinct fixed points. Define

$$P_2 = PF_2 \cap F_1F_3, \quad P_3 = PF_3 \cap F_1F_2 \quad P^{(0)} = \mathcal{C}_U^{-1}(P);$$
$$P_2^{(n)} = \mathcal{C}^{-n}(P_2) \text{ and } P_3^{(n)} = \mathcal{C}^{-n}(P_3) \text{ for } n = 1, 2, 3, \ldots$$

The collineation $\mathcal{C}_U^{-1}$ maps the line F_2P to the line $F_2P^{(0)}$, which is also the line $F_2P_2^{(1)}$ because F_2, P, P_2 are collinear; likewise, $\mathcal{C}_U^{-1}$ maps the line F_3P to the line $F_3P_3^{(1)}$. Consequently,

$$P^{(0)} = F_2P_2^{(1)} \cap F_3P_3^{(1)},$$

and by induction,

$$\mathcal{C}_U^{-n}(P) = F_2P_2^{(n)} \cap F_3P_3^{(n)}. \tag{10}$$

By Lemma 3,

$$\lim_{n\to\infty} \mathcal{C}_U^{-n}(P_2) \quad \text{and} \quad \lim_{n\to\infty} \mathcal{C}_U^{-n}(P_3)$$

are fixed points, so that by (10),

$$\lim_{n\to\infty} \mathcal{C}_U^{-n}(P)$$

must also be a fixed point. This completes a proof of the following theorem.

Theorem 4. *Suppose that the fixed lines of $\mathcal{C}_U^{-1}$ (or, equivalently, of $\mathcal{C}_U$) are regular. Then for every point X, the sequence of points $\mathcal{C}_U^{-n}(X)$ (or equivalently, the sequence $\mathcal{C}_U^{n}(X)$) converges to a fixed point of $\mathcal{C}_U^{-1}$(and of $\mathcal{C}_U$).*

Example 6. Extending Example 3, the three fixed lines, X_6X_{2574}, X_6X_{2575}, $X_{2574}X_{2575}$ are regular. The points X_{2574} and X_{2575} are the isogonal conjugates of the points X_{1113} and X_{1114} in which the Euler line intersects the circumcircle. Thus, the line $X_{2574}X_{2575}$ is the line at infinity. Because X_{1113} and X_{1114} are antipodal points on the circumcircle, the lines X_6X_{2574} are X_6X_{2575} are perpendicular (proof indicated at (x) below).

While visiting the author in February, 2007, Peter Moses analyzed the configuration in Example 6. His findings are given here.

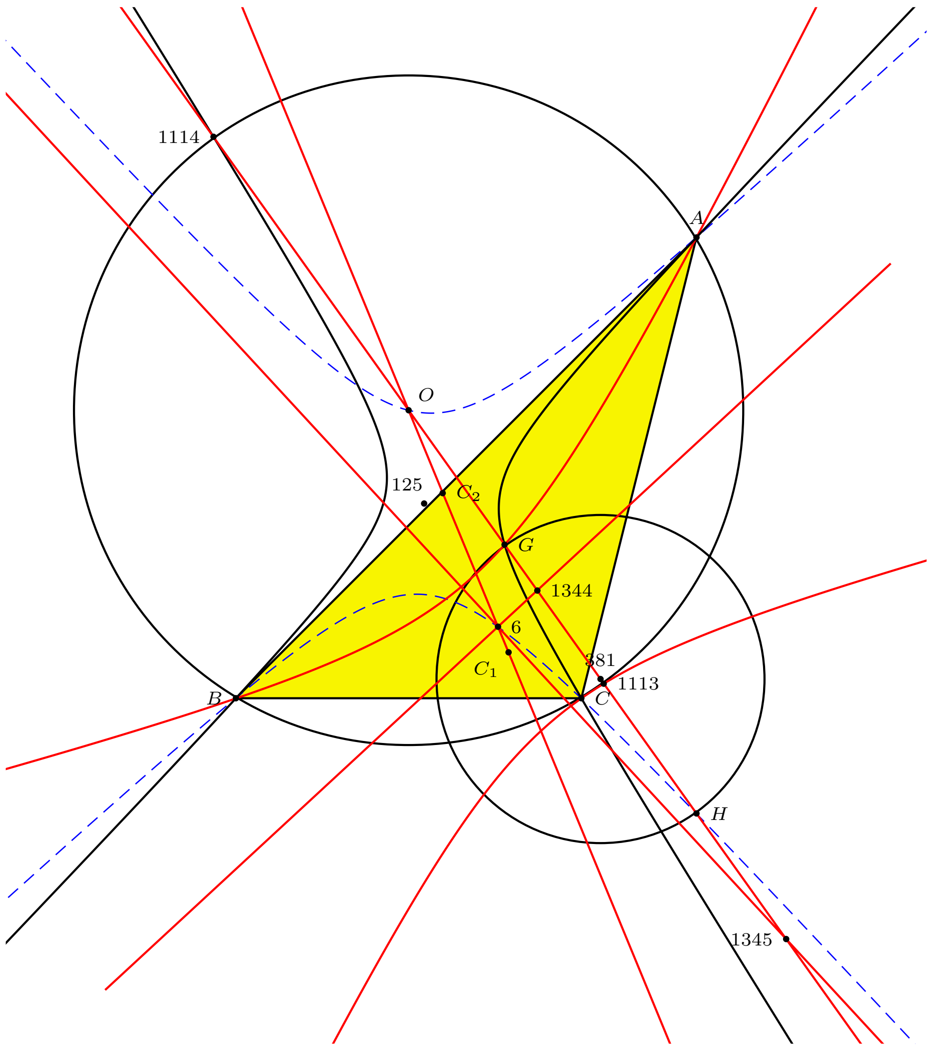


Figure 1.

(i) A point on line X_6X_{2574} is X_{1344}; a point on X_6X_{2575} is X_{1345}.

(ii) Segment GH (in Figure 1) is the diameter of the orthocentroidal circle, with center X_{381}. The points X_{1344} and X_{1345} are the internal and external centers of similitude of the orthocentroidal circle and the circumcircle.

(iii) Line GH, the Euler line, passes through the points

$$O,\ X_{1113},\ X_{1114},\ X_{1344},\ X_{1345}.$$

(iv) X_{125} is the center of the Jerabek hyperbola, which is the isogonal conjugate of the Euler line. (As isogonal conjugacy is a function, one may speak of its image when applied to lines as well as individual points).

(v) The line through X_{125} parallel to line X_6X_{1344} is the Simson line of X_{1114}, and the line through X_{125} parallel to line X_6X_{1345} is the Simson line of X_{1113}.

(vi) Hyperbola $ABCGX_{1113}$, with center C_1, is the isogonal conjugate of the $\mathcal{C}_U$-fixed line X_6X_{2574}, and hyperbola $ABCGX_{1114}$, with center C_2, is the isogonal conjugate of the $\mathcal{C}_U$-fixed line X_6X_{2575}.

(vii) C_1 is the barycentric square of X_{2575}, and C_2 is the barycentric square of X_{2574}.

(viii) The perspectors of the hyperbolas $ABCGX_{1113}$ and $ABCGX_{1114}$ are X_{2575} and X_{2574}, respectively. The fact that these perspectors are at infinity implies that the two conics, $ABCGX_{1113}$ and $ABCGX_{1114}$, are indeed hyperbolas.

(ix) The midpoint of the points C_1 and C_2 is the point $X_3X_6 \cap X_2X_{647}$.

(x) Line X_6X_{2574} is parallel to the Simson line of X_{1114}, and line X_6X_{2575} is parallel to the Simson line of X_{1113}. The two Simson lines are perpendicular ([1, p. 207]), so that the $\mathcal{C}_U$-fixed lines X_6X_{2574} and X_6X_{2575} are perpendicular.

(xi) The circle that passes through the points X_6, X_{1344}, and X_{1345} also passes through the point X_{2453}, which is the reflection of X_6 in the Euler line. This circle is a member of the coaxal family of the circumcircle, the nine-point circle, and the orthocentroidal circle.

References

[1] R. A. Johnson, *Modern Geometry*, Houghton Mifflin, Boston, 1929; Dover reprint, 1960.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[3] C. Kimberling, Ceva Collineations, *Forum Geom.*, 7 (2007) 67–72.

Clark Kimberling: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, USA

E-mail address: ck6@evansville.edu

Forum Geometricorum
Volume 7 (2007) 169–180.

FORUM GEOM
ISSN 1534-1178

On a Product of Two Points Induced by Their Cevian Triangles

Cosmin Pohoata and Paul Yiu

Abstract. The intersections of the corresponding sidelines of the cevian triangles of two points P_0 and P_1 form the anticevian triangle of a point $T(P_0, P_1)$. We prove a number of interesting results relating the pair of inscribed conics with perspectors (Brianchon points) P_0 and P_1, in particular, a simple description of the fourth common tangent of the conics. We also show that the corresponding sides of the cevian triangles of points are concurrent if and only if the points lie on a circumconic. A characterization is given of circumconics whose centers lie on the cevian circumcircles of points on them (Brianchon-Poncelet theorem). We also construct a number of new triangle centers with very simple coordinates.

1. Introduction

A famous problem in triangle geometry [8] asks to show that the corresponding sidelines of the orthic triangle, the intouch triangle, and the cevian triangle of the incenter are concurrent.

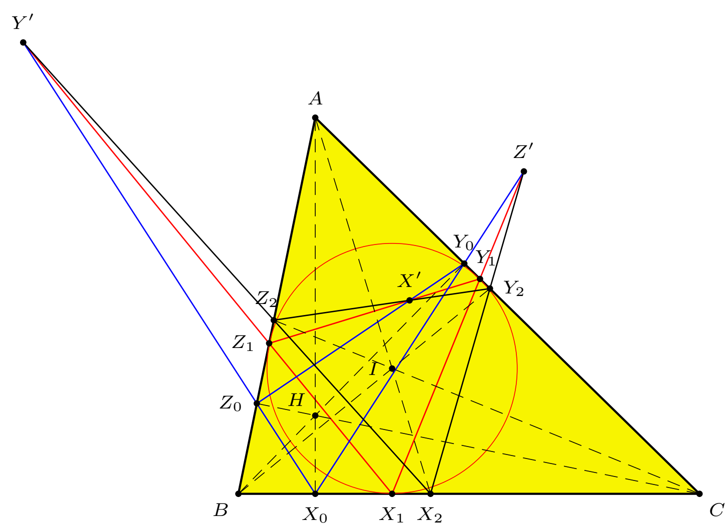


Figure 1.

Given a triangle ABC with orthic triangle $X_0Y_0Z_0$ and intouch triangle $X_1Y_1Z_1$, let

$$X' = Y_0Z_0 \cap Y_1Z_1, \qquad Y' = Z_0X_0 \cap Z_1X_1, \qquad Z' = X_0Y_0 \cap X_1Y_1.$$

Publication Date: November 14, 2007. Communicating Editor: Jean-Pierre Ehrmann.
We thank Jean-Pierre Ehrmann for his excellent comments leading to improvements of this paper, especially in pointing us to the classic references of Brianchon-Poncelet and Gergonne.

Emelyanov and Emelyanova [2] have proved the following interesting theorem. If XYZ is an inscribed triangle (with X, Y, Z on the sidelines BC, CA, AB respectively, and Y' on XZ and Z' on XY), then the circle through X, Y, Z also passes through the Feuerbach point, the point of tangency of the incircle with the nine-point circle of triangle ABC.

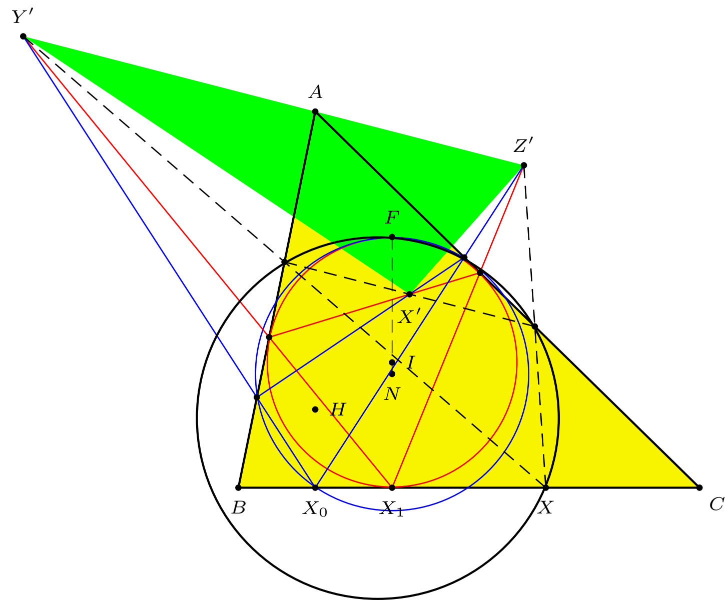


Figure 2.

In this note we study a general situation which reveals more of the nature of these theorems. By showing that the intersections of the corresponding sidelines of the cevian triangles of two points P_0 and P_1 form the anticevian triangle of a point $T(P_0,\ P_1)$, we prove a number of interesting results relating the pair of inscribed conics with perspectors (Brianchon points) P_0 and P_1. Proposition 5 below shows that the corresponding sidelines of the cevian triangles of three points are concurrent if and only if the three points lie on a circumconic. We characterize such circumconics whose centers lie on the cevian circumcircles of points on them (Proposition 9).

We shall work with homogeneous barycentric coordinates with reference to triangle ABC, and make use of standard notations of triangle geometry. A basic reference is [10]. Except for the commonest ones, triangle centers are labeled according to [7].

2. A product induced by two cevian triangles

Let $P_0 = (u_0 : v_0 : w_0)$ and $P_1 = (u_1 : v_1 : w_1)$ be two given points, with cevian triangles $X_0Y_0Z_0$ and $X_1Y_1Z_1$ respectively. The intersections

$$X' = Y_0Z_0 \cap Y_1Z_1, \qquad Y' = Z_0X_0 \cap Z_1X_1, \qquad Z' = X_0Y_0 \cap X_1Y_1$$

are the vertices of the anticevian triangle of a point with homogeneous barycentric coordinates

$$\left(u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right):\ v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right):\ w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)\right) \tag{1}$$

$$=\left(u_1\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right):\ v_1\left(\frac{w_1}{w_0}-\frac{u_1}{u_0}\right):\ w_1\left(\frac{u_1}{u_0}-\frac{v_1}{v_0}\right)\right). \tag{2}$$

That these two sets of coordinates should represent the same point is quite clear geometrically. They define a product of P_0 and P_1 which clearly lies on the trilinear polars of P_0 and P_1. This product is therefore the intersection of the trilinear polars of P_0 and P_1. We denote this product by $T(P_0,\ P_1)$.

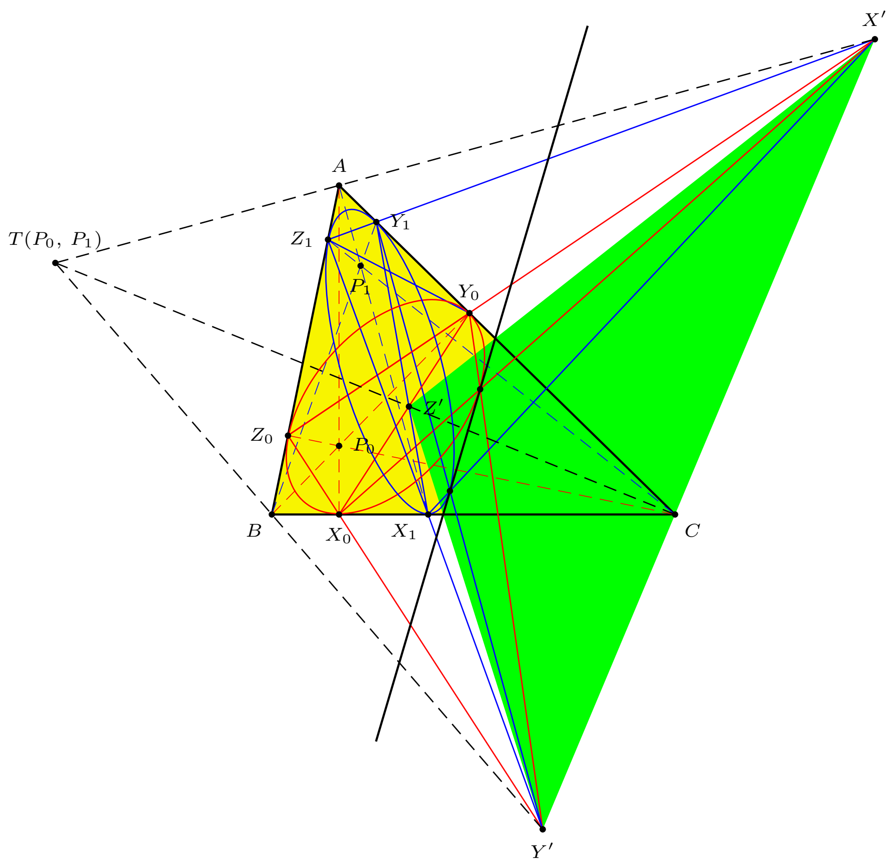


Figure 3.

The point $T(P_0,\ P_1)$ is also the perspector of the circumconic through P_0 and P_1. In particular, if P_0 and P_1 are both on the circumcircle, then $T(P_0,\ P_1)=K$, the symmedian point.

Proposition 1. *Triangle $X'Y'Z'$ is perspective to*
(i) *triangle $X_0Y_0Z_0$ at the point*

$$P_0/(T(P_0,\ P_1))=\left(u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)^2:v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)^2:w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)^2\right),$$

(ii) *triangle $X_1Y_1Z_1$ at the point*

$$P_1/(T(P_0,\ P_1))=\left(u_1\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)^2:v_1\left(\frac{w_1}{w_0}-\frac{u_1}{u_0}\right)^2:w_1\left(\frac{u_1}{u_0}-\frac{v_1}{v_0}\right)^2\right).$$

Proof. Since $X'Y'Z'$ is an anticevian triangle, the perspectivity is clear in each case by the cevian nest theorem (see [10, §8.3] and [4, p.165, Supp. Exercise 7]). The perspectors are the cevian quotients $P_0/(T(P_0,\ P_1))$ and $P_1/(T(P_0,\ P_1))$. We need only consider the first case.

$$\begin{aligned}
&P_0/(T(P_0,\ P_1))\\
=&\left(u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)\left(-\frac{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}{u_0}+\frac{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}{v_0}+\frac{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}{w_0}\right)\right.\\
&:v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)\left(\frac{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}{u_0}-\frac{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}{v_0}+\frac{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}{w_0}\right)\\
&\left.:w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)\left(\frac{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}{u_0}+\frac{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}{v_0}-\frac{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}{w_0}\right)\right)\\
=&\left(u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)^2:\ v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)^2:\ w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)^2\right).
\end{aligned}$$

□

Remark. See Proposition 12 for another triangle whose sidelines contain the points X', Y', Z'.

The conic with perspector P_0 has equation

$$\frac{x^2}{u_0^2}+\frac{y^2}{v_0^2}+\frac{z^2}{w_0^2}-\frac{2yz}{v_0w_0}-\frac{2zx}{w_0u_0}-\frac{2xy}{u_0v_0}=0,$$

and each point on the conic is of the form $(u_0p^2:v_0q^2:w_0r^2)$ for $p+q+r=0$. From this it is clear that $P_0/(T(P_0,\ P_1))$ lies on the inscribed conic with perspector P_0. Similarly, $P_1/(T(P_0,\ P_1))$ lies on that with perspector P_1.

Proposition 2. *The line joining $P_0/(T(P_0,\ P_1))$ and $P_1/(T(P_0,\ P_1))$ is the trilinear polar of $T(P_0,\ P_1)$ with respect to triangle ABC. It is also the (fourth) common tangent of the two inscribed conics with perspectors P_0 and P_1.* (See Figure 3).

Proof. For the first part it is enough to verify that $P_0/(T(P_0,\ P_1))$ lies on the said trilinear polar:

$$\frac{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)^2}{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}+\frac{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)^2}{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}+\frac{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)^2}{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}=0.$$

Note that the coordinates of $T(P_0,\ P_1)$ are given by both (1) and (2). Interchanging the subscripts 0's and 1's shows that the trilinear polar of $T(P_0,\ P_1)$ also contains the point $P_1/(T(P_0,\ P_1))$.

The inscribed conic with perspector P_0 is represented by the matrix

$$\begin{pmatrix}\frac{1}{u_0^2} & \frac{-1}{u_0v_0} & \frac{-1}{u_0w_0}\\ \frac{-1}{u_0v_0} & \frac{1}{v_0^2} & \frac{-1}{v_0w_0}\\ \frac{-1}{u_0w_0} & \frac{-1}{v_0w_0} & \frac{1}{w_0^2}\end{pmatrix}.$$

The tangent at the point $P_0/(T(P_0,\ P_1))$ has line coordinates

$$\begin{pmatrix}\frac{1}{u_0^2} & \frac{-1}{u_0v_0} & \frac{-1}{u_0w_0}\\ \frac{-1}{u_0v_0} & \frac{1}{v_0^2} & \frac{-1}{v_0w_0}\\ \frac{-1}{u_0w_0} & \frac{-1}{v_0w_0} & \frac{1}{w_0^2}\end{pmatrix}\begin{pmatrix}u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)^2\\ v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)^2\\ w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)^2\end{pmatrix}=\begin{pmatrix}\frac{2}{u_0}\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)\\ \frac{2}{v_0}\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)\\ \frac{2}{w_0}\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right).\end{pmatrix}$$

This is the line

$$\frac{x}{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}+\frac{y}{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}+\frac{z}{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}=0,$$

which is the trilinear polar of $T(P_0,\ P_1)$. Interchanging the subscripts 0's and 1's, we note that the same line is also the tangent at the point $P_1/(T(P_0,\ P_1))$of the inscribed conics with perspector P_1. It is therefore the common tangent of the two conics. □

Proposition 3. *The triangle $X'Y'Z'$ is self polar with respect to each of the inscribed conics with perspectors P_0 and P_1.*

Proof. Since $X_1Y_1Z_1$ is a cevian triangle and $X'Y'Z'$ is an anticevian triangle with respect to ABC, we have

$$(Y'Z_0,Y'A,Y'Y_0,Y'C)=(Y'Z_0,Y'A,Y'Y_0,Y'X')=-1.$$

Therefore, Y' lies on the polar of X' with respect to the inscribed conic with perspector P_0. Similarly, Z' also lies on the polar of X'. It follows that $Y'Z'$ is the polar of X'. For the same reason, $Z'X'$ and $X'Y'$ are the polars of Y' and Z' respectively. This shows that triangle $X'Y'Z'$ is self-polar with respect to the inscribed conic with perspector P_0. The same is true with respect to the inscribed conic with perspector P_1. □

In the case of the incircle (with $P_0 = X_7$), we have the following interesting result.

Corollary 4. *For an arbitrary point Q, the anticevian triangle $X_7 * Q$ has orthocenter I.*

We present some examples of $T(P_0, P_1)$.

	G	O	H	K	Ge	N_a	E
I	X_{513}	X_{652}	X_{650}	X_{649}	X_{650}	X_{650}	X_{2245}
G		X_{520}	X_{523}	X_{512}	X_{514}	X_{522}	X_{511}
O			X_{647}	X_{647}			
H				X_{647}	X_{650}	X_{650}	X_{3003}
K					X_{665}	X_{187}	
G_e						X_{650}	X_{3002}

Remarks. (1) X_{3002} is the intersection of the Brocard axis and the trilinear polar of the Gergonne point. It has coordinates

$$(a^2(a^3(b^2+c^2)-a^2(b+c)(b-c)^2-a(b^4+c^4)+(b+c)(b-c)^2(b^2+c^2)) : \cdots : \cdots).$$

(2) X_{3003} is the intersection of the orthic and Brocard axes. It has coordinates

$$(a^2(a^4(b^2+c^2)-2a^2(b^4-b^2c^2+c^4)+(b^2-c^2)^2(b^2+c^2)) : \cdots : \cdots).$$

The center of the rectangular hyperbola through E is X_{113}, the inferior of X_{74} on the the circumcircle.

Here are some examples of cevian products with very simple coordinates. They do not appear in the current edition of [7].

P_0	P_1	first barycentric coordinate of $T(P_0, P_1)$
G	X_9	$(a(b-c)(b+c-a)^2$
G	X_{56}	$(a^2(b-c)(a(b+c)+b^2+c^2)$
O	X_{21}	$(a^3(b-c)(b+c-a)(b^2+c^2-a^2)^2$
O	X_{55}	$(a^3(b-c)(b+c-a)^2(b^2+c^2-a^2)$
O	X_{56}	$(a^3(b-c)(b^2+c^2-a^2)$
K	N_a	$(a(b-c)(b+c-a)(a(b+c)+b^2+c^2)$
K	X_{99}	$(a^2(a^2(b^2+c^2)-2b^2c^2)$
G_e	X_{56}	$\frac{a^2(b^2-c^2)}{b+c-a}$
G_e	X_{57}	$\frac{a(b-c)}{b+c-a}$
N_a	X_{55}	$(a^2(b^2-c^2)(b+c-a)$
X_{21}	X_{55}	$(a^3(b-c)(b+c-a)$
X_{56}	X_{57}	$\frac{a^2(b-c)}{b+c-a}$

Remark. $T(X_{21}, X_{55}) = T(X_{21}, X_{56}) = T(X_{55}, X_{56})$.

3. Inscribed triangles which circumscribe a given anticevian triangle

Proposition 5. *Let P be a given point with anticevian triangle $X'Y'Z'$. If XYZ is an inscribed triangle of ABC* (*with X, Y, Z on the sidelines BC, CA, AB respectively*) *such that X', Y', Z' lie on the lines YZ, ZX, XY respectively,* i.e.,

$X'Y'Z'$ is an inscribed triangle of XYZ, then XYZ is the cevian triangle of a point on the circumconic with perspector P.

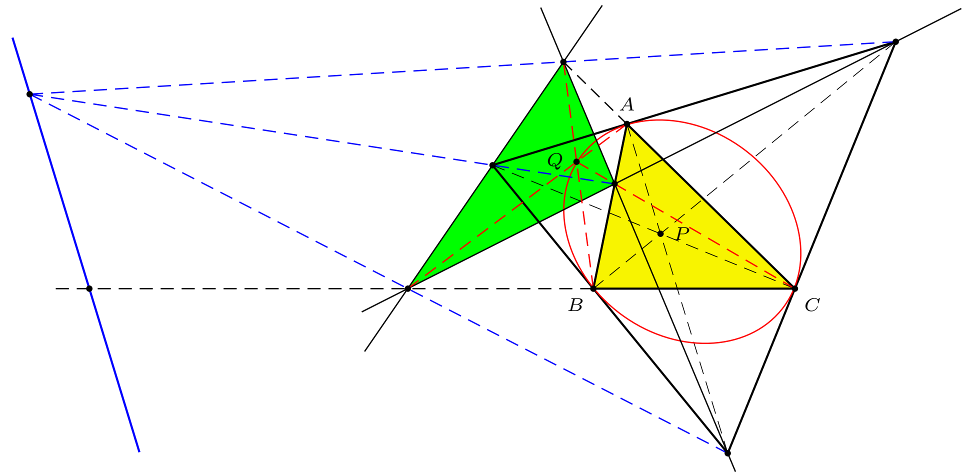


Figure 4.

Proof. Let $P = (u : v : w)$ so that

$$X' = (-u : v : w), \qquad Y' = (u : -v : w), \qquad Z' = (u : v : -w).$$

Since XYZ is an inscribed triangle of ABC,

$$X = (0 : t_1 : 1), \quad Y = (1 : 0 : t_2), \quad Z = (t_3 : 1 : 0),$$

for real numbers t_1, t_2, t_3. Here we assume that X, Y, Z do not coincide with the vertices of ABC. Since the lines YZ, ZX, XY contain the points respectively, we have

$$\begin{aligned} t_2u + t_2t_3v + w &= 0, \\ u + t_3v + t_3t_1w &= 0, \\ t_1t_2u + v + t_1w &= 0. \end{aligned}$$

From these,

$$0 = \begin{vmatrix} t_2 & t_2t_3 & 1 \\ 1 & t_3 & t_3t_1 \\ t_1t_2 & 1 & t_1 \end{vmatrix} = (t_1t_2t_3 - 1)^2.$$

It follows from the Ceva theorem that the lines AX, BY, CZ are concurrent. The inscribed triangle XYZ is the cevian triangle of a point $(p : q : r)$. The three collinearity conditions all reduce to

$$uqr + vrp + wpq = 0.$$

This means that $(p : q : r)$ lies on the circumconic with perspector $(u : v : w)$. $\square$

Proposition 6. *The locus of the perspector of the anticevian triangle of P and the cevian triangle of a point Q on the circumconic with perspector P is the trilinear polar of P.*

Proof. Let $Q = (u : v : w)$ be a point on the circumconic. The perspector is the cevian quotient

$$\left(p\left(-\frac{p}{u}+\frac{q}{v}+\frac{r}{w}\right) : \ q\left(\frac{p}{u}-\frac{q}{v}+\frac{r}{w}\right) : r\left(\frac{p}{u}+\frac{q}{v}-\frac{r}{w}\right)\right)$$
$$= \ (p(-pvw+qwu+ruv) : \ q(pvw-qwu+ruv) : \ r(pvw+qwu-ruv)) \,.$$

Since $pvw+qwu+ruv = 0$, this simplifies into $(p^2vw : q^2wu : r^2uv)$, which clearly lies on the line $\frac{x}{p}+\frac{y}{q}+\frac{z}{r} = 0$, the trilinear polar of P. □

4. Brianchon-Poncelet theorem

For $P_0 = H$, the orthocenter, and $P_1 = X_7$, the Gergonne point, we have $T(P_0, P_1) = X_{650}$. The circumconic through P_0 and P_1 is

$$a(b-c)(b+c-a)yz + b(c-a)(c+a-b)zx + c(a-b)(a+b-c)xy = 0,$$

the Feuerbach conic, which is the isogonal conjugate of the line OI, and has center at the Feuerbach point

$$X_{11} = ((b-c)^2(b+c-a) : \ (c-a)^2(c+a-b) : \ (a-b)^2(a+b-c)).$$

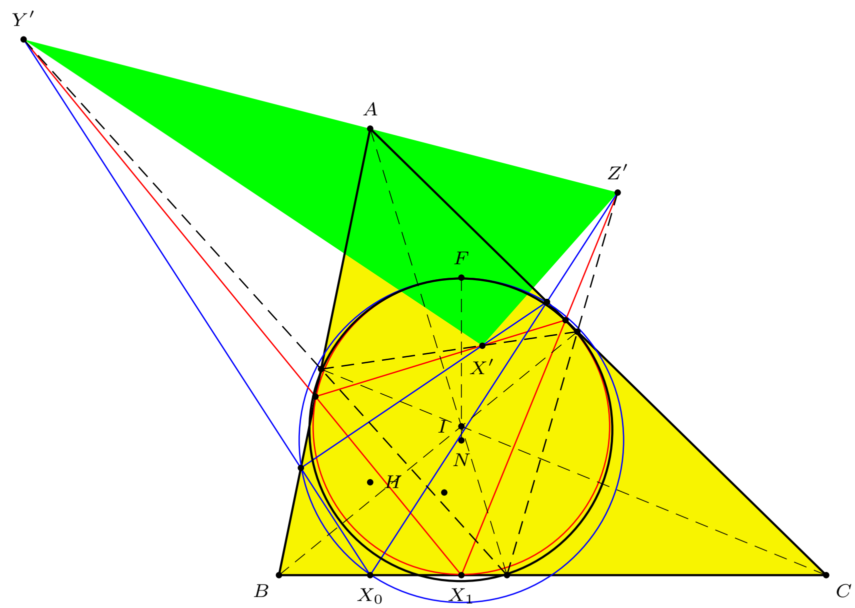


Figure 5.

The theorem of Emelyanov and Emelyanova therefore can be generalized as follows: *the the cevian circumcircle of a point on the Feuerbach hyperbola contains the Feuerbach point*. This in turn is a special case of a celebrated theorem of Brianchon and Poncelet in 1821.

Theorem 7 (Brianchon-Poncelet [1])**.** *Given a point P, the cevian circumcircle of an arbitrary point on the rectangular circum-hyperbola through P (and the orthocenter H) contains the center of the hyperbola which is on the nine-point circle of the reference triangle.*

At the end of their paper Brianchon and Poncelet made a remarkable conjecture about the locus of the centers of conics through four given points. This was subsequently proved by J. D. Gergonne [6].

Theorem 8 (Brianchon-Poncelet-Gergonne)**.** *The locus of the centers of conics through four given points in general positions in a plane is a conic through*
(i) the midpoints of the six segments joining them and
(ii) the intersections of the three pairs of lines joining them two by two.

Proposition 9. *The cevian circumcircle of a point on a nondegenerate circumconic contains the center of the conic if and only if the conic is a rectangular hyperbola.*

Proof. (a) The sufficiency part follows from Theorem 7.

(b) For the converse, consider a nondegenerate conic through A, B, C, P whose center W lies on the cevian circumcircle of P. The locus of centers of conics through A, B, C, P is, by Theorem 8, a conic $\mathcal{C}$ through the traces of P on the sidelines of triangle ABC. The four common points of $\mathcal{C}$ and the cevian circumcircle of P are the traces of P and W. By (a), the cevian circumcircle of P contains the center of the rectangular circum-hyperbola through P, which must coincide with W. Therefore the conic in question in rectangular. □

Since the Feuerbach hyperbola contain the incenter I, we have the following result. See Figure 5.

Corollary 10. *The cevian circumcircle of the incenter contains the Feuerbach point.*

Applying Brianchon-Poncelet theorem to the Kiepert perspectors, we obtain the following interesting result.

Corollary 11. *Given triangle ABC, construct on the sides similar isosceles triangles BCX', CAY', and ABZ'. Let AX', BY', CZ' intersect BC, CA, AB at X, Y, Z respectively. The circle through X, Y, Z also contains the center X_{115} of the Kiepert hyperbola, which is also the midpoint between the two Fermat points.*

5. Second tangents to an inscribed conic from the traces of a point

Consider an inscribed conic $\mathcal{C}_0$ with Brianchon point $P_0 = (u_0 : v_0 : w_0)$, so that its equation is

$$\left(\frac{x}{u_0}\right)^2 + \left(\frac{y}{v_0}\right)^2 + \left(\frac{z}{w_0}\right)^2 - \frac{2yz}{v_0w_0} - \frac{2zx}{w_0u_0} - \frac{2xy}{u_0v_0} = 0.$$

Let $P_1 = (u_1 : v_1 : w_1)$ be a given point with cevian triangle $X_1Y_1Z_1$. The sidelines of triangle ABC are tangents from X_1, Y_1, Z_1 to the conic $\mathcal{C}_0$. From each of these points there is a second tangent to the conic. J.-P. Ehrmann [5] has computed the second points of tangency X_2, Y_2, Z_2, and concluded that the triangle $X_2Y_2Z_2$ is perspective with ABC at the point

$$\left(\frac{u_1^2}{u_0} : \frac{v_1^2}{v_0} : \frac{w_1^2}{w_0}\right).$$

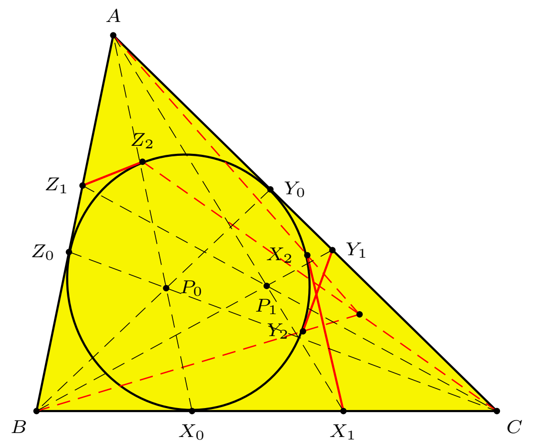


Figure 6.

More precisely, the coordinates of X_2, Y_2, Z_2 are as follows.

$$X_2 = \left(u_0\left(\frac{v_1}{v_0} - \frac{w_1}{w_0}\right)^2 : \frac{v_1^2}{v_0} : \frac{{w_1}^2}{w_0}\right),$$

$$Y_2 = \left(\frac{u_1^2}{u_0} : v_0\left(\frac{w_1}{w_0} - \frac{v_1}{v_0}\right)^2 : \frac{{w_1}^2}{w_0}\right),$$

$$Z_2 = \left(\frac{u_1^2}{u_0} : \frac{v_1^2}{v_0} : w_0\left(\frac{u_1}{u_0} - \frac{v_1}{v_0}\right)^2\right).$$

Proposition 12. *The lines Y_0Z_0, Y_1Z_1, Y_2Z_2 are concurrent; similarly for the triples Z_0X_0, Z_1X_1, Z_2X_2 and X_0Y_0, X_1Y_1, X_2Y_2.*

Proof. The line Y_2Z_2 has equation

$$u_1\left(\frac{x}{u_0}\left(-\frac{u_1}{u_0} + \frac{v_1}{v_0} + \frac{w_1}{w_0}\right) + \frac{y}{v_0}\left(\frac{u_1}{u_0} - \frac{v_1}{v_0} + \frac{w_1}{w_0}\right) + \frac{z}{w_0}\left(\frac{u_1}{u_0} + \frac{v_1}{v_0} - \frac{w_1}{w_0}\right)\right)$$
$$- \frac{2v_1w_1x}{v_0w_0} = 0.$$

With

$$(x : y : z) = \left(-u_1\left(\frac{v_1}{v_0} - \frac{w_1}{w_0}\right) : v_1\left(\frac{w_1}{w_0} - \frac{u_1}{u_0}\right) : w_1\left(\frac{u_1}{u_0} - \frac{v_1}{v_0}\right)\right),$$

we have, apart from a factor u_1,

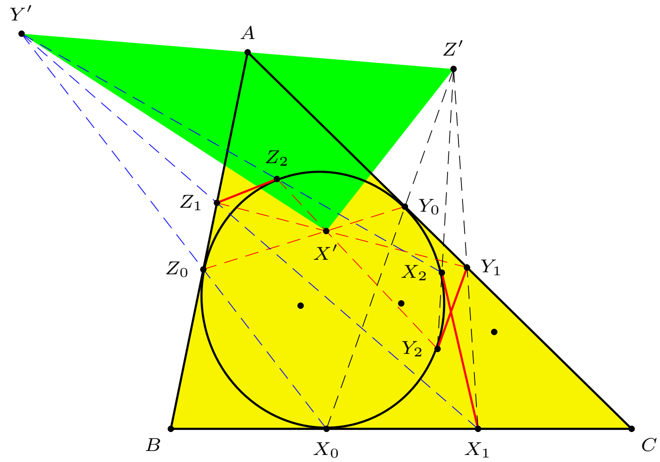


Figure 7.

$$
\begin{aligned}
& -\frac{u_1}{u_0}\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)\left(-\frac{u_1}{u_0}+\frac{v_1}{v_0}+\frac{w_1}{w_0}\right)+\frac{v_1}{v_0}\left(\frac{w_1}{w_0}-\frac{u_1}{u_0}\right)\left(\frac{u_1}{u_0}-\frac{v_1}{v_0}+\frac{w_1}{w_0}\right) \\
& +\frac{w_1}{w_0}\left(\frac{u_1}{u_0}-\frac{v_1}{v_0}\right)\left(\frac{u_1}{u_0}+\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)+\frac{2v_1w_1}{v_0w_0}\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right) \\
= & -\frac{2v_1w_1}{v_0w_0}\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)+\frac{2v_1w_1}{v_0w_0}\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right) \\
= & \, 0.
\end{aligned}
$$

This shows that the line Y_2Z_2 contains the point $X' = Y_0Z_0 \cap Y_1Z_1$. □

We conclude with some examples of the triangle centers from the inscribed conics with given perspectors P_0 and P_1. In the table below,

$$Q_{0,1} = \left(\frac{u_0^2}{u_1} : \frac{v_0^2}{v_1} : \frac{w_0^2}{w_1}\right) \quad \text{and} \quad Q_{1,0} = \left(\frac{u_1^2}{u_0} : \frac{v_1^2}{v_0} : \frac{w_1^2}{w_0}\right).$$

P_0	P_1	$T(P_0, P_1)$	$P_0/(T(P_0, P_1))$	$P_1/(T(P_0, P_1))$	$Q_{0,1}$	$Q_{1,0}$
G	H	X_{523}	X_{125}	X_{115}	X_{69}	X_{393}
G	K	X_{512}	Q_1	X_{1084}	X_{76}	X_{32}
G	G_e	X_{514}	X_{11}	X_{1086}	X_8	X_{279}
G	N_a	X_{522}	X_{11}	X_{1146}	X_7	X_{346}
H	K	X_{647}	Q_2	Q_3	X_{2052}	X_{184}
H	G_e	X_{650}	X_{3022}	Q_4	X_{1857}	Q_5
H	N_a	X_{650}	Q_6	Q_4	X_{1118}	X_{1265}
K	G_e	X_{665}	Q_7	Q_8	X_{2175}	Q_9
G_e	N_a	X_{650}	Q_6	X_{3022}	X_{479}	Q_{10}

The new triangle centers Q_i have simple coordinates given below.

Q_1	$a^2(b^2-c^2)^2$
Q_2	$a^2(b^2-c^2)^2(b^2+c^2-a^2)^2$
Q_3	$a^4(b^2-c^2)^2(b^2+c^2-a^2)^3$
Q_4	$a^2(b-c)^2(b+c-a)^2(b^2+c^2-a^2)$
Q_5	$\frac{b^2+c^2-a^2}{(b+c-a)^2}$
Q_6	$a^2(b-c)^2(b+c-a)$
Q_7	$a^4(b-c)^2(b+c-a)(a(b+c)-(b^2+c^2))^2$
Q_8	$a^2(b-c)^2(a(b+c)-(b^2+c^2))^2$
Q_9	$\frac{1}{a^2(b+c-a)^2}$
Q_{10}	$(b+c-a)^3$

References

[1] Ch. J. Brianchon and J. V. Poncelet, Géométrie des courbes. Recherches sur la détermination d'une hyperbole équilatère, au moyen de quatre conditions données, *Annales de Mathématiques pures et appliquées*, 11 (1820-21) 205–220.

[2] L. A. Emelyanov, T. L. Emelyanova, Semejstvo Feuerbacha, *Matematicheskoe Prosveshjenie*, 2002.

[3] L.A. Emelyanov, T.L. Emelyanova, A Note on the Feuerbach Point, *Forum Geom.*, 1 (2001) 121–124.

[4] N. Altshiller-Court, *College Geometry*, 1952; Dover reprint, 2007.

[5] J.-P. Ehrmann, Hyacinthos message 6966, April 14, 2003.

[6] J. D. Gergonne, Questions ŕsolues. Solution du premier des problèmes de géométrie proposés à la page 228 de ce volume, *Annales de Mathématiques pures et appliquées*, 11 (1820-21) 379–400.

[7] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[8] A. Pelletier, M. M. Young and G. A. Yanosik, Problem 3440, *Amer. Math. Monthly*, 37 (1930) 316; solution, 38 (1931) 177–178.

[9] C. Pohoata, Asupra unei concurente remarcabile, in apparition, *Gazeta Matematica*, 2007.

[10] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001.

Cosmin Pohoata: 13 Pridvorului Street, Bucharest, Romania 010014
E-mail address: pohoata_cosmin2000@yahoo.com

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida 33431-0991, USA
E-mail address: yiu@fau.edu

Forum Geometricorum
Volume 7 (2007) 181–185.

FORUM GEOM
ISSN 1534-1178

Steinhaus' Problem on Partition of a Triangle

Apoloniusz Tyszka

Abstract. H. Steinhaus has asked whether inside each acute triangle there is a point from which perpendiculars to the sides divide the triangle into three parts of equal areas. We present two solutions of Steinhaus' problem.

The n-dimensional case of Theorem 1 below was proved in [6], see also [2] and [4, Theorem 2.1, p. 152]. For an earlier mass-partition version of Theorem 1, for bounded convex masses in $\mathbb{R}^n$ and $r_1 = r_2 = ... = r_{n+1}$, see [7].

Theorem 1 (Kuratowski-Steinhaus)**.** *Let $\boldsymbol{T} \subseteq \mathbb{R}^2$ be a bounded measurable set, and let $|\boldsymbol{T}|$ be the measure of $\boldsymbol{T}$. Let α_1, α_2, α_3 be the angles determined by three rays emanating from a point, and let $\alpha_1 < \pi$, $\alpha_2 < \pi$, $\alpha_3 < \pi$. Let r_1, r_2, r_3 be nonnegative numbers such that $r_1 + r_2 + r_3 = |\boldsymbol{T}|$. Then there exists a translation $\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ such that $|\lambda(\boldsymbol{T}) \cap \alpha_1| = r_1$, $|\lambda(\boldsymbol{T}) \cap \alpha_2| = r_2$, $|\lambda(\boldsymbol{T}) \cap \alpha_3| = r_3$.*

H. Steinhaus asked ([10], [11]) whether *inside* each acute triangle there is a point from which perpendiculars to the sides divide the triangle into three parts with equal areas. Long and elementary solutions of Steinhaus' problem appeared in [8, pp. 101–104], [9, pp. 103–105], [12, pp. 133–138] and [13]. For some acute triangles with rational coordinates of vertices, the point solving Steinhaus' problem is not constructible with ruler and compass alone, see [15]. Following article [14], we will present two solutions of Steinhaus' problem.

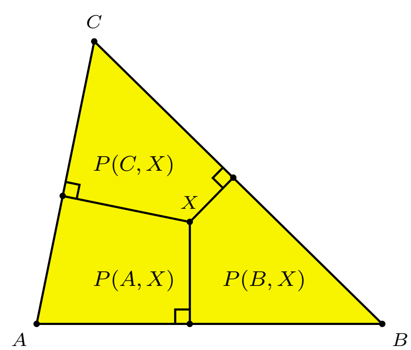


Figure 1

For $X \in \triangle ABC$, we denote by $P(A, X)$, $P(B, X)$, $P(C, X)$ the areas of the quadrangles containing vertices A, B, C respectively (see Figure 1). The areas

Publication Date: November 19, 2007. Communicating Editor: Paul Yiu.

$P(A,X)$, $P(B,X)$, $P(C,X)$ are continuous functions of X in the triangle ABC. The function

$$f(X) = \min\{P(A,X), P(B,X), P(C,X)\}$$

is also continuous. By Weierstrass' theorem f attains a maximum in triangle ABC, *i.e.*, there exists $X_0 \in \triangle ABC$ such that $f(X) \le f(X_0)$ for all $X \in \triangle ABC$.

Lemma 2. *For a point X lying on a side of an acute triangle, the area at the opposite vertex is greater than one of the remaining two areas.*

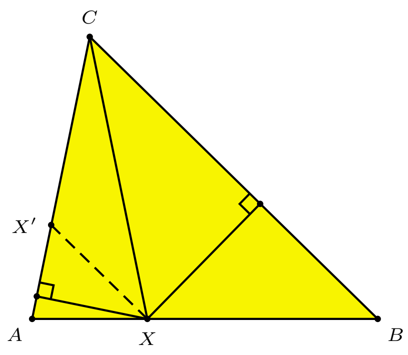


Figure 2

Proof. Without loss of generality, we may assume that $X \in \overline{AB}$ and $|AX| \le |BX|$, see Figure 2. Straight line XX' parallel to straight line BC cuts the triangle AXX' greater than $P(A,X)$ (as the angle ACB is acute), but not greater than the triangle CXX' because $|AX'| < \frac{|AC|}{2} < |X'C|$. Hence $P(A,X) < |\triangle AXX'| \le |\triangle CXX'| < P(C,X)$. □

Theorem 3. *If a triangle ABC is acute and f attains a maximum at X_0, then $P(A,X_0) = P(B,X_0) = P(C,X_0) = \frac{|\triangle ABC|}{3}$.*

Proof. $f(A) = f(B) = f(C) = 0$, and 0 is not a maximum of f. Therefore X_0 is not a vertex of the triangle ABC. Let us assume that $f(X_0) = P(A,X_0)$. By Lemma 2, $X_0 \notin \overline{BC}$. Suppose, on the contrary, that some of the other areas, let's say $P(C,X_0)$, is greater than $P(A,X_0)$.

Case 1: $X_0 \notin \overline{AC}$. When shifting X_0 from the segment $\overline{AB}$ by appropriately small ε and perpendicularly to the segment $\overline{AB}$ (see Figure 3), we receive $P(C,X)$ further greater than $f(X_0)$ and at the same time $P(A,X) > P(A,X_0)$ and $P(B,X) > P(B,X_0)$. Hence $f(X) > f(X_0)$, a contradiction.

Case 2: $X_0 \in \overline{AC} \setminus \{A,C\}$. By Lemma 2,

$$\begin{aligned} P(B,X_0) &> \min\{P(A,X_0), P(C,X_0)\} \\ &\ge \min\{P(A,X_0), P(B,X_0), P(C,X_0)\} \\ &= f(X_0). \end{aligned}$$

When shifting X_0 from the segment $\overline{AC}$ by appropriately small ε and perpendicularly to the segment $\overline{AC}$ (see Figure 4), we receive $P(B,X)$ further greater than

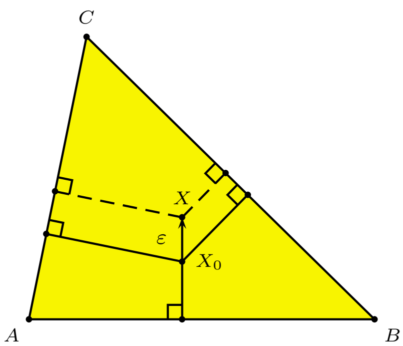


Figure 3

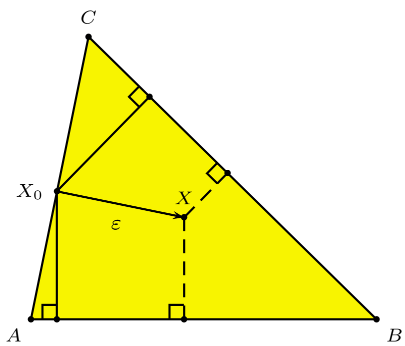


Figure 4

$f(X_0)$ and at the same time $P(A, X) > P(A, X_0)$ and $P(C, X) > P(C, X_0)$. Hence $f(X) > f(X_0)$, a contradiction. $\square$

For each acute triangle ABC there is a unique $X_0 \in \triangle ABC$ such that $P(A, X_0) = P(B, X_0) = P(C, X_0) = \frac{|\triangle ABC|}{3}$. Indeed, if $X \neq X_0$ then X lies in some of the quadrangles determined by X_0. Let us say that X lies in the quadrangle with vertex A (see Figure 5). Then $P(A, X) < P(A, X_0) = \frac{|\triangle ABC|}{3}$.

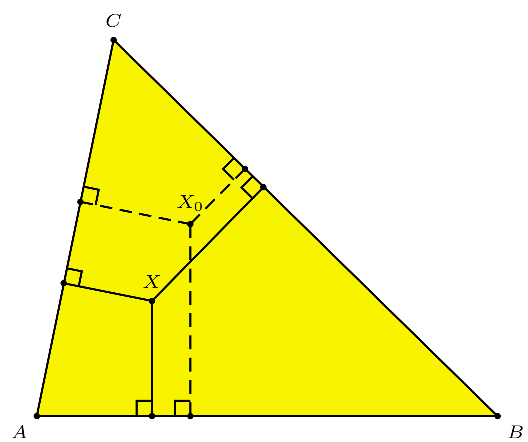


Figure 5.

The sets $R_A = \{X \in \triangle ABC : P(A, X) = f(X)\}$, $R_B = \{X \in \triangle ABC : P(B, X) = f(X)\}$ and $R_C = \{X \in \triangle ABC : P(C, X) = f(X)\}$ are closed and cover the triangle ABC. Assume that the triangle ABC is acute. By Lemma 2, $R_A \cap \overline{BC} = \emptyset$, $R_B \cap \overline{AC} = \emptyset$, and $R_C \cap \overline{AB} = \emptyset$. The theorem proved in [5] guarantees that $R_A \cap R_B \cap R_C \neq \emptyset$, see also [4, item D4, p. 101] and [1, item 2.23, p. 162]. Any point belonging to $R_A \cap R_B \cap R_C$ lies inside the triangle ABC and determines the partition of the triangle ABC into three parts with equal areas.

The above proof remains valid for all right triangles, because the hypothesis of Lemma 2 holds for all right triangles. For each triangle the following statements are true.

(1) There is a unique point in the plane which determines the partition of the triangle into three equal areas.
(2) The point of partition into three equal areas lies inside the triangle if and only if the hypothesis of Lemma 2 holds for the triangle.
(3) The point of partition into three equal areas lies inside the triangle if and only if the maximum of f on the boundary of the triangle is smaller than the maximum of f on the whole triangle. For each acute or right triangle ABC, the maximum of f on the boundary does not exceed $\frac{|\triangle ABC|}{4}$.
(4) The point of partition into three equal areas lies inside the triangle, if the triangle has two angles in the interval $\left(\arctan \frac{1}{\sqrt{2}}, \frac{\pi}{2}\right]$. This condition holds for each acute or right triangle.
(5) If the point of partition into three equal areas lies inside the triangle, then it is a partition into quadrangles.

Assume now $C > \frac{\pi}{2}$. The point of partition into three equal areas lies inside the triangle if and only if

$$\sqrt{(1+\tan^2 A)\tan B} + \sqrt{(1+\tan^2 B)\tan A} > \sqrt{3(\tan A + \tan B)}.$$

If, on the other hand,

$$\sqrt{(1+\tan^2 A)\tan B} + \sqrt{(1+\tan^2 B)\tan A} = \sqrt{3(\tan A + \tan B)},$$

then the unique $X_0 \in \overline{AB}$ such that

$$|AX_0| = \sqrt{\frac{(1+\tan^2 A)\tan B}{3(\tan A + \tan B)}}|AB|, \qquad |BX_0| = \sqrt{\frac{(1+\tan^2 B)\tan A}{3(\tan A + \tan B)}}|AB|$$

determines the partition of the triangle ABC into three equal areas. It is a partition into a triangle with vertex A, and a triangle with vertex B, and a quadrangle. Finally, when

$$\sqrt{(1+\tan^2 A)\tan B} + \sqrt{(1+\tan^2 B)\tan A} < \sqrt{3(\tan A + \tan B)}, \qquad (*)$$

there is a straight line a perpendicular to the segment $\overline{AC}$ which cuts from the triangle ABC a figure with the area $\frac{|\triangle ABC|}{3}$ (see Figure 6). There is a straight line b perpendicular to the segment $\overline{BC}$ which cuts from the triangle ABC a figure with the area $\frac{|\triangle ABC|}{3}$. By $(*)$, the intersection point of the straight lines a and b lies outside the triangle ABC. This point determines the partition of the triangle ABC into three equal areas.

J.-P. Ehrmann [3] has subsequently found a constructive solution of a generalization of Steinhaus' problem of partitioning a given triangle into three quadrangles with prescribed proportions.

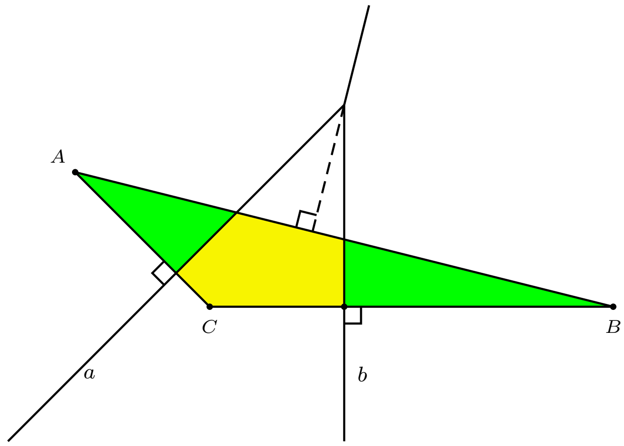


Figure 6.

References

[1] P. S. Alexandrov, *Combinatorial topology*, Dover Publications, Mineola, NY, 1998.
[2] K. Borsuk, An application of the theorem on antipodes to the measure theory, *Bull. Acad. Polon. Sci.*, Cl. III 1 (1953), pp. 87–90.
[3] J.-P. Ehrmann, Constructive solution of a generalization of Steinhaus' problem on partition of a triangle, *Forum Geom.*, 187–190.
[4] A. Granas and J. Dugundji, *Fixed point theory*, Springer-Verlag, New York, 2003.
[5] B. Knaster, K. Kuratowski and S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes fur n-dimensionale Simplexe, *Fund. Math.*, 14 (1929), pp.132–137; reprinted in: K. Kuratowski, *Selected papers*, PWN (Polish Scientific Publishers), Warsaw, 1988, pp.332–337, S. Mazurkiewicz, *Travaux de topologie et ses applications*, PWN (Éditions Scientifiques de Pologne), Warsaw, 1969, pp.192–197.
[6] K. Kuratowski and H. Steinhaus, Une application géométrique du théoréme de Brouwer sur les points invariants, *Bull. Acad. Polon. Sci.*, Cl. III 1 (1953), pp.83–86; reprinted in: K. Kuratowski, *Selected papers*, PWN (Polish Scientific Publishers), Warsaw, 1988, pp. 520–523, H. Steinhaus, *Selected papers*, PWN (Polish Scientific Publishers), Warsaw, 1985, pp. 636–639.
[7] F. Levi, Die Drittelungskurve, *Math. Z.*, 31 (1930), 339–345.
[8] E. Piegat, *Yet 105 problems of Hugo Steinhaus* (in Polish), Oficyna Wydawnicza GiS, Wrocław, 2000.
[9] E. Piegat, *Known and unknown problems of Hugo Steinhaus* (in Polish), Oficyna Wydawnicza GiS, Wrocław, 2005.
[10] H. Steinhaus, Problem No. 132 (in Polish), *Roczniki Polskiego Towarzystwa Matematycznego* (*Annales Societatis Mathematicae Polonae*), Seria II, Wiadomości Matematyczne 9 (1966) 99.
[11] H. Steinhaus, Problem No. 779 (in Polish), *Matematyka*, 19 (1966) 92.
[12] H. Steinhaus, *Problems and Reflections* (in Russian), Mir, Moscow, 1974.
[13] W. Stojda, A solution of Problem No. 779 (in Polish), *Matematyka*, 21 (1968) 267–273.
[14] A. Tyszka, A solution of a problem of Steinhaus (in Polish), *Matematyka*, 49 (1996) 3–5.
[15] A. Tyszka, Steinhaus' problem cannot be solved with ruler and compass alone (in Polish), *Matematyka* 49 (1996) 238–240.

Apoloniusz Tyszka: Technical Faculty, Hugo Kołłątaj University, Balicka 116B, 30-149 Kraków, Poland

E-mail address: `rttyszka@cyf-kr.edu.pl`

Forum Geometricorum
Volume 7 (2007) 187–190.

FORUM GEOM
ISSN 1534-1178

Constructive Solution of a Generalization of Steinhaus' Problem on Partition of a Triangle

Jean-Pierre Ehrmann

Abstract. We present a constructive solution to a generalization of Hugo Steinhaus' problem of partitioning a given triangle, by dropping perpendiculars from an interior point, into three quadrilaterals whose areas are in prescribed proportions.

1. Generalized Steinhaus problem

Given an acute angled triangle ABC, Steinhaus' problem asks a point P in its interior with pedals P_a, P_b, P_c on BC, CA, AB such that the quadrilaterals AP_bPP_c, BP_cP_a, and CP_aPP_b have equal areas. See [3] and the bibliographic information therein. A. Tyszka [2] has also shown that Steinhaus' problem is in general not soluble by ruler-and-compass. We present a simple constructive solution (using conics) of a generalization of Steinhaus' problem. In this note, the area of a polygon $\mathscr{P}$ will be denoted by $\Delta(\mathscr{P})$. In particular, $\Delta = \Delta(ABC)$. Thus, given three positive real numbers u, v, w, we look for the point(s) P such that
(1) P is inside ABC and P_a, P_b, P_c lie respectively in the segments BC, CA, AB,
(2) $\Delta(AP_bPP_c) : \Delta(BP_cPP_a) : \Delta(CP_aPP_b) = u : v : w$.

We do not require the triangle to be acute-angled.

Lemma 1. *Consider a point P inside the angular sector bounded by the half-lines AB and AC, with projections P_b and P_c on AC and AB respectively. For a positive real number k, $\Delta(AP_bPP_c) = k \cdot \Delta(ABC)$ if and only if P lies on the rectangular hyperbola with center A, focal axis the internal bisector AI, and semi-major axis $\sqrt{kbc}$.*

Proof. We take A for pole and the bisector AI for polar axis; let $(\rho,\ \theta)$ be the polar coordinates of P. As $AP_b = \rho\cos\left(\frac{A}{2} - \theta\right)$ and $PP_b = \rho\sin\left(\frac{A}{2} - \theta\right)$, we have $\Delta(APP_b) = \frac{1}{2}\rho^2\sin(A - 2\theta)$. Similarly, $\Delta(AP_cP) = \frac{1}{2}\rho^2\sin(A + 2\theta)$. Hence the quadrilateral AP_bPP_c has area $\frac{1}{2}\rho^2\sin A\cos 2\theta$. Therefore,

$$\Delta(AP_bPP_c) = k \cdot \Delta(ABC) \Longleftrightarrow \rho^2\cos 2\theta = \frac{2k \cdot \Delta(ABC)}{\sin A} = kbc.$$

□

Publication Date: November 21, 2007. Communicating Editor: Paul Yiu.

Theorem 2. *Let U be the point with barycentric coordinates $(u : v : w)$ and M_1, M_2, M_3 be the antipodes on the circumcircle Γ of ABC of the points whose Simson lines pass through U and P the incenter of the triangle $M_1M_2M_3$. If P verifies* (1)*, then P is the unique solution of our problem. Otherwise, the generalized Steinhaus problem has no solution.*

Remarks. (a) Of course, if ABC is acute angled, and P inside ABC, then (1) will be verified.

(b) As U lies inside the Steiner deltoid, there exist three real Simson lines through U; so M_1, M_2, M_3 are real and distinct.

(c) Let h_A be the rectangular hyperbola with center A, focal axis AI, and semi-major axis $\sqrt{\dfrac{u}{u+v+w}} \cdot bc$, and define rectangular hyperbolas h_B and h_C analogously.

If P verifies (1), it will verify (2) if and only if $P \in h_A \cap h_B$. In this case, $P \in h_C$, and the solutions of our problem are the common points of h_A, h_B, h_C verifying (1).

(d) The four common points P_1, P_2, P_3, P_4 (real or imaginary) of the rectangular hyperbolae h_A, h_B, h_C form an orthocentric system. As h_A, h_B, h_C are centered respectively at A, B, C, any conic through P_1, P_2, P_3, P_4 is a rectangular hyperbola with center on Γ. As the vertices of the diagonal triangle of this orthocentric system are the centers of the degenerate conics through P_1, P_2, P_3, P_4 , they lie on Γ.

(e) We will see later that P_1, P_2, P_3, P_4 are always real.

2. Proof of Theorem 2

If P has homogeneous barycentric coordinates $(x : y : z)$ with reference to triangle ABC, then

$$(x+y+z)^2\Delta(APP_b) = y\left(z + \frac{b^2+c^2-a^2}{2b^2}y\right)\Delta,$$
$$(x+y+z)^2\Delta(AP_cP) = z\left(y + \frac{b^2+c^2-a^2}{2c^2}z\right)\Delta,$$

where $\Delta = \Delta(ABC)$. Hence the barycentric equation of h_A is

$$h_A(x,y,z) := \frac{b^2+c^2-a^2}{2}\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) + 2\,y\,z - \frac{u}{u+v+w}\,(x+y+z)^2 = 0.$$

We get h_B and h_C by cyclically permuting a, b, c; u, v, w; x, y, z.

If $M = (x : y : z)$ is a vertex of the diagonal triangle of $P_1P_2P_3P_4$, it has the same polar line (the opposite side) with respect to the three conics h_A, h_B, h_C. Hence,

$$\frac{\partial h_B}{\partial y}\frac{\partial h_C}{\partial z} - \frac{\partial h_B}{\partial z}\frac{\partial h_C}{\partial y} = \frac{\partial h_C}{\partial z}\frac{\partial h_A}{\partial x} - \frac{\partial h_C}{\partial x}\frac{\partial h_A}{\partial z} = \frac{\partial h_A}{\partial x}\frac{\partial h_B}{\partial y} - \frac{\partial h_A}{\partial y}\frac{\partial h_B}{\partial x} = 0.$$

Let N be the reflection of M in the circumcenter O; $N_aN_bN_c$ the pedal triangle of N. Clearly, N_a, N_b, N_c are the reflections of the vertices of the pedal triangle of M in the midpoints of the corresponding sides of ABC. Now, N_b and N_c have coordinates

$$(b^2 + c^2 - a^2)y + 2b^2 z : \ 0 : \ (a^2 + b^2 - c^2)y + 2b^2 x$$

and

$$(b^2 + c^2 - a^2)z + 2c^2 y : \ (c^2 + a^2 - b^2)z + 2c^2 x : \ 0$$

respectively. A straightforward computation shows that

$$\det[N_b,\ N_c,\ U] = b^2 c^2 \,(u + v + w) \left(\frac{\partial h_B}{\partial y} \frac{\partial h_C}{\partial z} - \frac{\partial h_B}{\partial z} \frac{\partial h_C}{\partial y} \right) = 0.$$

Similarly, $\det[N_c,\ N_a,\ U] = \det[N_a,\ N_b,\ U] = 0$. It follows that N lies on the circumcircle (we knew that already by Remark (d)), and the Simson line of N passes through U.

Hence, $M_1M_2M_3$ is the diagonal triangle of the orthocentric system $P_1P_2P_3P_4$, which means that $P_1P_2P_3P_4$ are real and are the incenter and the three excenters of $M_1M_2M_3$.

As the three excenters of a triangle lie outside his circumcircle, the incenter of $M_1M_2M_3$ is the only common point of h_A, h_B, h_C inside Γ. This completes the proof of Theorem 2.

3. Constructions

In [1], the author has given a construction of the points on the circumcircle whose Simson line pass through a given point. Let U^- and U^+ be the complement and the anticomplement of U, *i.e.*, the images of U under the homotheties $\mathsf{h}\left(G, -\frac{1}{2}\right)$ and $\mathsf{h}(G, -2)$ respectively. Since

$$(\text{Reflection in } O) \circ (\text{Translation by } \overrightarrow{HU}) = \text{Reflection in } U^-,$$

if h_0 is the reflection in U^- of the rectangular circumhyperbola through U, and M_4 the antipode of U^+ on h_0, then M_1, M_2, M_3, M_4 are the four common points of h_0 and the circumcircle.

In the case $u = v = w = 1$, h_0 is the reflection in the centroid G of the Kiepert hyperbola of ABC. It intersects the circumcircle Γ at M_1, M_2, M_3 and the Steiner point of ABC. See Figure 1.

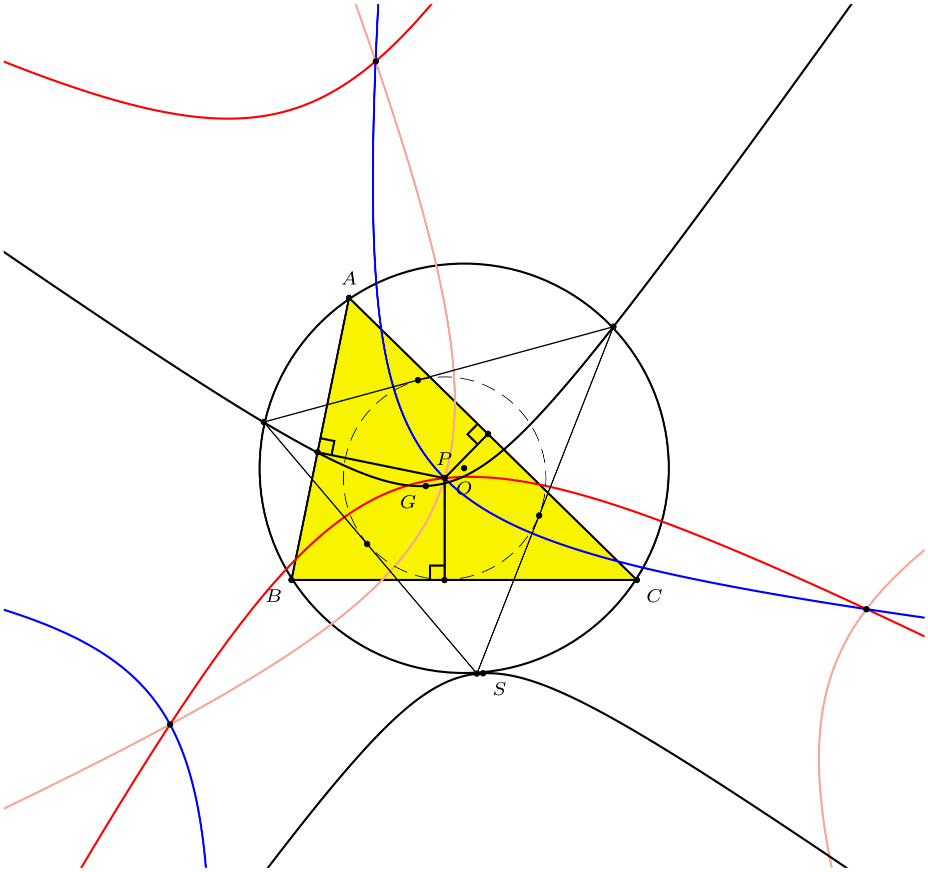


Figure 1.

References

[1] J.-P. Ehrmann, Some geometric constructions, *Forum Geom*, 6 (2006) 327–334.
[2] A. Tyszka, Steinhaus' problem cannot be solved with ruler and compass alone (in Polish), *Matematyka* 49 (1996) 238–240.
[3] A. Tyszka, Steinhaus' problem on partition of a triangle, *Forum Geom.*, 7 (2007) 181–185.

Jean-Pierre Ehrmann: 6, rue des Cailloux, 92110 - Clichy, France
E-mail address: Jean-Pierre.EHRMANN@wanadoo.fr

Forum Geometricorum
Volume 7 (2007) 191–197.

FORUM GEOM
ISSN 1534-1178

The Soddy Circles

Nikolaos Dergiades

Abstract. Given three circles externally tangent to each other, we investigate the construction of the two so called Soddy circles, that are tangent to the given three circles. From this construction we get easily the formulas of the radii and the barycentric coordinates of Soddy centers relative to the triangle ABC that has vertices the centers of the three given circles.

1. Construction of Soddy circles

In the general Apollonius problem it is known that, given three arbitrary circles with noncollinear centers, there are at most 8 circles tangent to each of them. In the special case when three given circles are tangent externally to each other, there are only two such circles. These are called the inner and outer Soddy circles respectively of the given circles. Let the mutually externally tangent circles be $\mathscr{C}_a(A, r_1)$, $\mathscr{C}_b(B, r_2)$, $\mathscr{C}_c(C, r_3)$, and A_1, B_1, C_1 be their tangency points (see Figure 1).

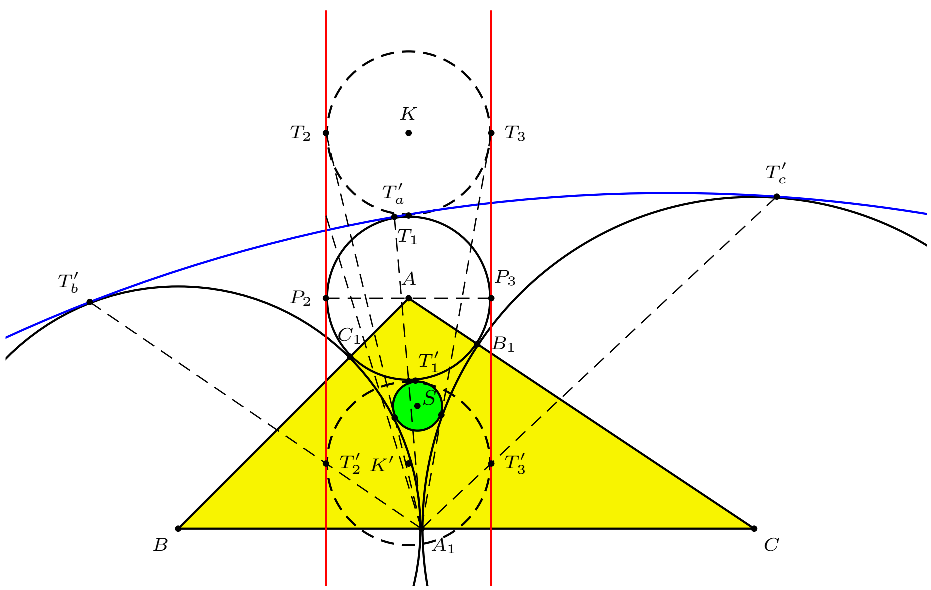


Figure 1.

Consider the inversion τ with pole A_1 that maps $\mathscr{C}_a$ to $\mathscr{C}_a$. This also maps the circles $\mathscr{C}_b$, $\mathscr{C}_c$ to the two lines perpendicular to BC and tangent to $\mathscr{C}_a$ at the points P_2, P_3 where P_2P_3 is parallel from A to BC. The only circles tangent to $\mathscr{C}_a$ and to the above lines are the circles $K(T_1)$, $K'(T'_1)$ where T_1, T'_1 are lying on $\mathscr{C}_a$ and

Publication Date: November 26, 2007. Communicating Editor: Paul Yiu.

the A-altitude of ABC. These circles are the images, in the above inversion, of the Soddy circles we are trying to construct. Since the circle $K(T_1)$ must be the inverse of the inner Soddy circle, the lines A_1T_1, A_1T_2, A_1T_3, $(P_2T_2 = P_3T_3 = P_2P_3)$ meet $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ at the points T_a, T_b, T_c respectively, that are the tangency points of the inner Soddy circle. Hence the lines BT_b and CT_c give the center S of the inner Soddy circle. Similarly the lines $A_1T'_1$, $A_1T'_2$, $A_1T'_3$, $(P_2T'_2 = P_3T'_3 = P_2P_3)$, meet $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ at the points T'_a, T'_b, T'_c respectively, that are the tangency points of the outer Soddy circle. Triangles $T_aT_bT_c$, $T'_aT'_bT'_c$ are the inner and outer Soddy triangles. A construction by the so called Soddy hyperbolas can be found in [5, §12.4.2].

2. The radii of Soddy circles

If the sidelengths of ABC are a, b, c, and $s = \frac{1}{2}(a+b+c)$, then

$$\begin{array}{lll} a = r_2 + r_3, & b = r_3 + r_1, & c = r_1 + r_2; \\ r_1 = s - a, & r_2 = s - b, & r_3 = s - c. \end{array}$$

If $\triangle$ is the area of ABC, then $\triangle = \sqrt{r_1r_2r_3(r_1+r_2+r_3)}$. The A-altitude of ABC is $AD = h_a = \frac{2\triangle}{a}$, and the inradius is $r = \frac{\triangle}{r_1+r_2+r_3}$.

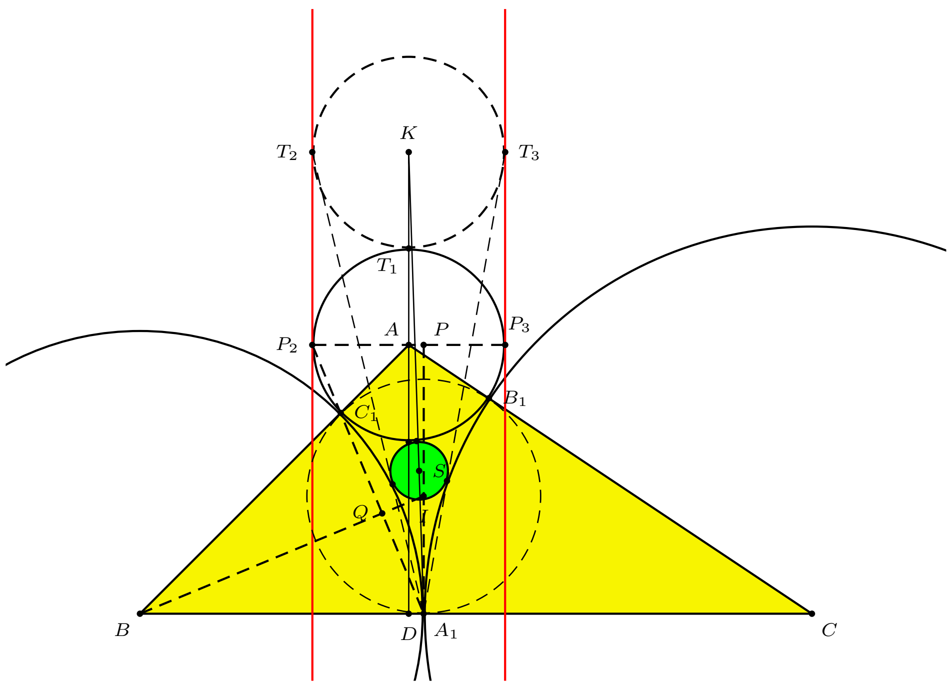


Figure 2.

The points A_1, B_1, C_1 are the points of tangency of the incircle $I(r)$ of ABC with the sidelines. If A_1P is perpendicular to P_2P_3 and IB meets A_1C_1 at Q, then the inversion τ maps C_1 to P_2, and the quadrilateral IQP_2P is cyclic (see Figure 2). The power of the inversion is

$$d^2 = A_1C_1 \cdot A_1P_2 = 2A_1Q \cdot A_1P_2 = 2A_1I \cdot A_1P = 2rh_a = \frac{4r_1r_2r_3}{r_2+r_3}. \tag{1}$$

2.1. *Inner Soddy circle.* Since the inner Soddy circle is the inverse of the circle $K(r_1)$, its radius is given by

$$x = \frac{d^2}{A_1K^2 - r_1^2} \cdot r_1. \tag{2}$$

In triangle A_1AK, $A_1K^2 - A_1A^2 = 2AK \cdot T_1D = 4r_1(r_1 + h_a)$. Hence,

$$A_1K^2 - r_1^2 = A_1A^2 - r_1^2 + 4r_1(r_1 + h_a) = d^2 + 4r_1(r_1 + h_a),$$

and from (1), (2),

$$x = \frac{r_1r_2r_3}{r_2r_3 + r_3r_1 + r_1r_2 + 2\triangle}. \tag{3}$$

Here is an alternative expression for x. If r_a, r_b, r_c are the exradii of triangle ABC, and R its circumradius, it is well known that

$$r_a + r_b + r_c = 4R + r.$$

See, for example, [4, §2.4.1]. Now also that $r_1r_a = r_2r_b = r_3r_c = \triangle$. Therefore,

$$\begin{aligned} x =& \frac{r_1r_2r_3}{r_2r_3 + r_3r_1 + r_1r_2 + 2\triangle} \\ =& \frac{\triangle}{\frac{\triangle}{r_1} + \frac{\triangle}{r_2} + \frac{\triangle}{r_3} + 2 \cdot \frac{\triangle^2}{r_1r_2r_3}} \\ =& \frac{\triangle}{r_a + r_b + r_c + 2(r_1 + r_2 + r_3)} \\ =& \frac{\triangle}{4R + r + 2s}. \end{aligned} \tag{4}$$

As a special case, if $r_1 \to \infty$, then the circle $\mathscr{C}_a$ tends to a common tangent of $\mathscr{C}_b$, $\mathscr{C}_c$, and

$$\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{r_2}} + \frac{1}{\sqrt{r_3}}. \tag{5}$$

In this case the outer Soddy circle degenerates into the common tangent of $\mathscr{C}_b$ and $\mathscr{C}_c$.

2.2. *Outer Soddy circle.* If $\mathscr{C}_a$ is the smallest of the three circles $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ and is greater than the circle of (5), *i.e.*, $\frac{1}{\sqrt{r_1}} < \frac{1}{\sqrt{r_2}} + \frac{1}{\sqrt{r_3}}$, then the outer Soddy circle is internally tangent to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$. Otherwise, the outer Soddy circle is externally tangent to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$.

Since the outer Soddy circle is the inverse of the circle $K'(r_1)$, its radius is given by

$$x' = \frac{d^2}{A_1K'^2 - r_1^2} \cdot r_1. \tag{6}$$

This is a signed radius and is negative when A_1 is inside the circle $K'(r_1)$ or when the outer Soddy circle is tangent internally to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$. In triangle A_1AK', $A_1A^2 - A_1K'^2 = 2AK' \cdot T_1'D = 4r_1(h_a - r_1)$, and from (6),

$$x' = \frac{r_1r_2r_3}{r_1r_2 + r_2r_3 + r_3r_1 - 2\triangle}. \tag{7}$$

Analogous to (4) we also have

$$x' = \frac{\triangle}{4R + r - 2s}. \tag{8}$$

Hence this radius is negative, equivalently, the outer Soddy circle is tangent internally to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$, when $4R + r < 2s$. From (4) and (8), we have

$$\frac{1}{x} - \frac{1}{x'} = \frac{2s}{\triangle} = \frac{4}{r}.$$

If $4R + r = 2s$, then $x = \frac{r}{4}$.

3. The barycentric coordinates of Soddy centers

3.1. *The Inner Soddy center.* If d_1 is the distance of the inner Soddy circle center S from BC, then since A_1 is the center of similitute of the inner Soddy circle and the circle $K(r_1)$ we have $\frac{d_1}{KD} = \frac{x}{r_1}$, or

$$d_1 = \frac{x(2r_1 + h_a)}{r_1} = 2x\left(1 + \frac{h_a}{2r_1}\right) = 2x\left(1 + \frac{\triangle}{a(s-a)}\right).$$

Similarly we obtain the distances d_2, d_3 from S to the sides CA and AB respectively. Hence the homogeneous barycentric coordinates of S are

$$(ad_1 : bd_2 : cd_3) = \left(a + \frac{\triangle}{s-a} : b + \frac{\triangle}{s-b} : c + \frac{\triangle}{s-c}\right).$$

The inner Soddy center S appears in [3] as the triangle center X_{176}, also called the equal detour point. It is obvious that for the Inner Soddy center S, the "detour" of triangle SBC is

$$SB + SC - BC = (x + r_2) + (x + r_3) - (r_2 + r_3) = 2x.$$

Similarly the triangles SCA and SAB also have detours $2x$. Hence the three incircles of triangles SBC, SCA, SAB are tangent to each other and their three tangency point A_2, B_2, C_2 are the points T_a, T_b, T_c on the inner Soddy circle [1] since $SA_2 = SB_2 = SC_2 = x$. See Figure 3.

Working with absolute barycentric coordinates, we have

$$\begin{aligned} S &= \frac{\left(a + \frac{\triangle}{s-a}\right)A + \left(b + \frac{\triangle}{s-b}\right)B + \left(c + \frac{\triangle}{s-c}\right)C}{a + \frac{\triangle}{s-a} + b + \frac{\triangle}{s-b} + c + \frac{\triangle}{s-c}} \\ &= \frac{(a+b+c)I + \triangle\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right)G_{\mathrm{e}}}{\frac{\triangle}{x}}, \end{aligned} \tag{9}$$

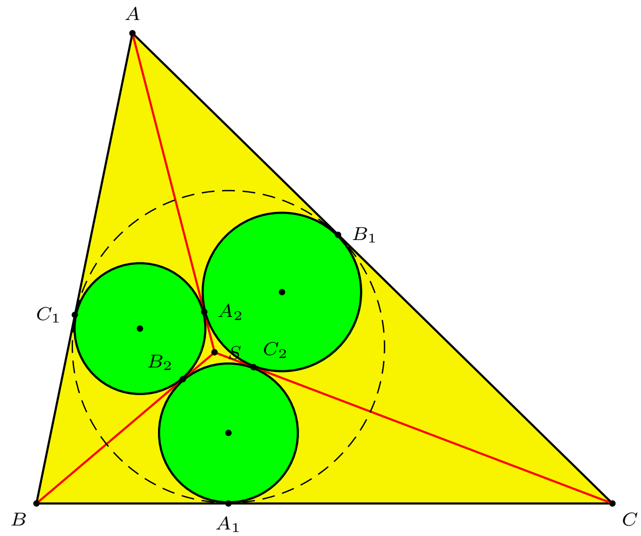


Figure 3.

where $G_e = \left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c}\right)$ is the Gergonne point. Hence, the inner Soddy center S lies on the line connecting the incenter I and G_e. This explains whey IG_e is called the Soddy line. Indeed, S divides IG_e in the ratio

$$IS : SG_e = r_a + r_b + r_c : a + b + c = 4R + r : 2s.$$

3.2. *The outer Soddy center.* If d_1' is the distance of the outer Soddy circle center S' from BC, then since A_1 is the center of similitute of the outer Soddy circle and the circle $K'(r_1)$, a similar calculation referring to Figure 1 shows that

$$d_1' = -2x\left(1 - \frac{\triangle}{a(s-a)}\right).$$

Similarly, we have the distances d_2' and d_3' from S' to CA and AB respectively. The homogeneous barycentric coordinates of S' are

$$(ad_1' : bd_2' : cd_3') = \left(a - \frac{\triangle}{s-a} : b - \frac{\triangle}{s-b} : c - \frac{\triangle}{s-c}\right).$$

This is the triangle center X_{175} of [3], called the isoperimetric point. It is obvious that if the outer Soddy circle is tangent internally to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ or $4R+r < 2s$, then the perimeter of triangle $S'BC$ is

$$S'B + S'C + BC = (x' - r_2) + (x' - r_3) + (r_2 + r_3) = 2x'.$$

Similarly the perimeters of triangles $S'CA$ and $S'AB$ are also $2x'$. Therefore the S'-excircles of triangles $S'BC$, $S'CA$, $S'AB$ are tangent to each other at the tangency points T_a', T_b', T_c' of the outer Soddy circle with $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$.

If the outer Soddy circle is tangent externally to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$, equivalently, $4R + r > 2s$, then the triangles $S'BC$, $S'CA$, $S'AB$ have equal detours $2x'$ because for triangle $S'BC$,

$$S'B + S'C - BC = (x' + r_2) + (x' + r_3) - (r_2 + r_3) = 2x',$$

and similarly for the other two triangles. In this case, S' is second equal detour point. Analogous to (9), we have

$$S' = \frac{(a+b+c)I - \triangle\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right) G_\mathrm{e}}{\frac{\triangle}{x'}}. \tag{10}$$

A comparison of (9) and (10) shows that S and S' are harmonic conjugates with respect to IG_e.

4. The barycentric equations of Soddy circles

We find the barycentric equation of the inner Soddy circle in the form

$$a^2yz + b^2zx + c^2xy - (x+y+z)(p_1x + p_2y + p_3z) = 0,$$

where p_1, p_2, p_3 are the powers of A, B, C with respect to the circle. See [5, Proposition 7.2.3]. It is easy to see that

$$\begin{aligned} p_1 =& r_1(r_1 + 2x) = (s-a)(s-a+2x), \\ p_2 =& r_2(r_2 + 2x) = (s-b)(s-b+2x), \\ p_3 =& r_3(r_3 + 2x) = (s-c)(s-c+2x). \end{aligned}$$

Similarly, the barycentric equation of the outer Soddy circle is

$$a^2yz + b^2zx + c^2xy - (x+y+z)(q_1x + q_2y + q_3z) = 0,$$

where

$$\begin{aligned} q_1 =& (s-a)(s-a+2x'), \\ q_2 =& (s-b)(s-b+2x'), \\ q_3 =& (s-c)(s-c+2x'), \end{aligned}$$

where x' is the *signed* radius of the circle given by (8), treated as negative when $2s > 4R + r$.

5. The Soddy triangles and the Eppstein points

The incenter I of ABC is the radical center of the circles $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$. The inversion with respect to the incircle leaves each of $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ invariant and swaps the inner and outer Soddy circles. In particular, it interchanges the points of tangency T_a and T'_a; similarly, T_b and T'_b, T_c and T'_c. The Soddy triangles $T_aT_bT_c$ and $T'_aT'_bT'_c$ are clearly perspective at the incenter I. They are also perspective with ABC, at S and S' respectively. Since $AT_a : T_aS = r_1 : x$, we have, $T_a = \frac{xA + r_1S}{x + r_1}$. In homogeneous barycentric coordinates,

$$T_a = \left(a + \frac{2\triangle}{r_1} :\ b + \frac{\triangle}{r_2} :\ c + \frac{\triangle}{r_3}\right).$$

Since the intouch point A_1 has coordinates $\left(0 : \frac{1}{r_2} : \frac{1}{r_3}\right)$, the line T_aA_1 clearly contains the point

$$E = \left(a + \frac{2\triangle}{r_1} : b + \frac{2\triangle}{r_2} : c + \frac{2\triangle}{r_3}\right).$$

Similarly, the lines T_bB_1 and T_cC_1 also contain the same point E, which is therefore the perspector of the triangles $T_aT_bT_c$ and the intouch triangle. This is the Eppstein pont X_{481} in [3]. See also [2]. It is clear that E also lies on the Soddy line. See Figure 4.

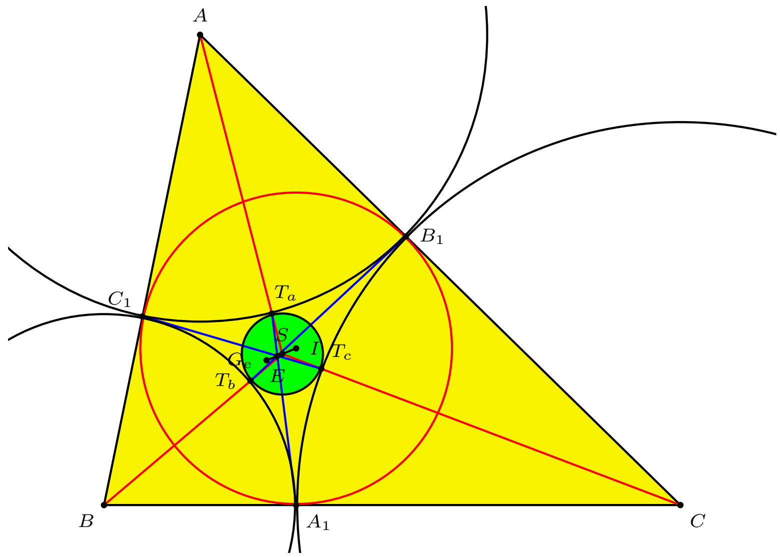


Figure 4.

The triangle $T'_aT'_bT'_c$ is also perspective with the intouch triangle, at a point

$$E' = \left(a - \frac{2\triangle}{r_1} : b - \frac{2\triangle}{r_2} : c - \frac{2\triangle}{r_3}\right),$$

on the Soddy line, dividing with E the segment IG_e harmonically. This is the second Eppstein point X_{482} of [3].

References

[1] J.-P. Ehrmann, Some geometric constructions, *Forum Geom.* 6 (2006) 327–334.

[2] D. Eppstein, Tangent spheres and triangle centers, *Amer. Math. Monthly*, 108 (2001) 63–66.

[3] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[4] P. Yiu, *Euclidean Geometry*, Florida Atlantic University Lecture Notes, 1998.

[5] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece

E-mail address: ndergiades@yahoo.gr

Forum Geometricorum
Volume 7 (2007) 199–206.

FORUM GEOM
ISSN 1534-1178

Cyclic Quadrilaterals with Prescribed Varignon Parallelogram

Michel Bataille

Abstract. We prove that the vertices of a given parallelogram $\mathcal{P}$ are the midpoints of the sides of infinitely many *cyclic* quadrilaterals and show how to construct such quadrilaterals. Then we discuss some of their properties and identify related loci. Lastly, the cases when $\mathcal{P}$ is a rectangle or a rhombus are examined.

1. Introduction

The following well-known theorem of elementary geometry, attributed to the French mathematician Pierre Varignon (1654-1722), was published in 1731: if A, B, C, D are four points in the plane, the respective midpoints P, Q, R, S of AB, BC, CD, DA are the vertices of a parallelogram. We will say that $PQRS$ is the Varignon parallelogram of $ABCD$, in short $PQRS = \mathcal{V}(ABCD)$. In a converse way, given a parallelogram $\mathcal{P}$, there exist infinitely many quadrilaterals $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. In §2, we offer a quick review of this general result, introducing the diagonal midpoints of $ABCD$ which are of constant use afterwards. The primary result of this paper, namely that infinitely many of these quadrilaterals $ABCD$ are cyclic, is proved in §3 and the proof leads naturally to a construction of such quadrilaterals. Further results, including a simpler construction, are established in §4, all centering on a rectangular hyperbola determined by $\mathcal{P}$. Finally, §5 is devoted to particular results that hold if $\mathcal{P}$ is either a rectangle or a rhombus.

In what follows, $\mathcal{P} = PQRS$ denotes a parallelogram whose vertices are not collinear. The whole work takes place in the plane of $\mathcal{P}$.

2. Quadrilaterals $ABCD$ with $\mathcal{P} = \mathcal{V}(ABCD)$

The construction of a quadrilateral $ABCD$ satisfying $\mathcal{P} = \mathcal{V}(ABCD)$ is usually presented as follows: start with an arbitrary point A and construct successively the symmetric B of A about P, the symmetric C of B about Q and the symmetric D of C about R (see Figure 1). Because $\mathcal{P}$ is a parallelogram, A is automatically the symmetric of D about S and $ABCD$ is a solution (see [1, 2]).

Let M, M' be the midpoints of the diagonals of $ABCD$ (in brief, the diagonal midpoints of $ABCD$) and let O be the center of $\mathcal{P}$. Since $4O = 2P + 2R = A + B + C + D = 2M + 2M'$, the midpoint of MM' is O. This simple property allows another construction of $ABCD$ from $\mathcal{P}$ that will be preferred in the next sections: start with two points M, M' symmetric about O; then obtain A, C such that $\overrightarrow{AM} = \overrightarrow{PQ} = \overrightarrow{MC}$ and B, D such that $\overrightarrow{BM'} = \overrightarrow{QR} = \overrightarrow{M'D}$. Exchanging the roles of M, M' provides another solution $A'B'C'D'$ with the same set $\{M, M'\}$ of diagonal

Publication Date: December 3, 2007. Communicating Editor: Paul Yiu.

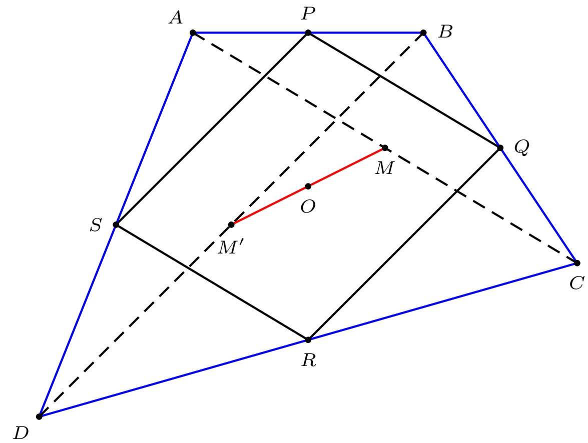


Figure 1

midpoints (see Figure 2). Clearly, $ABCD$ and $A'B'C'D'$ are symmetrical about O.

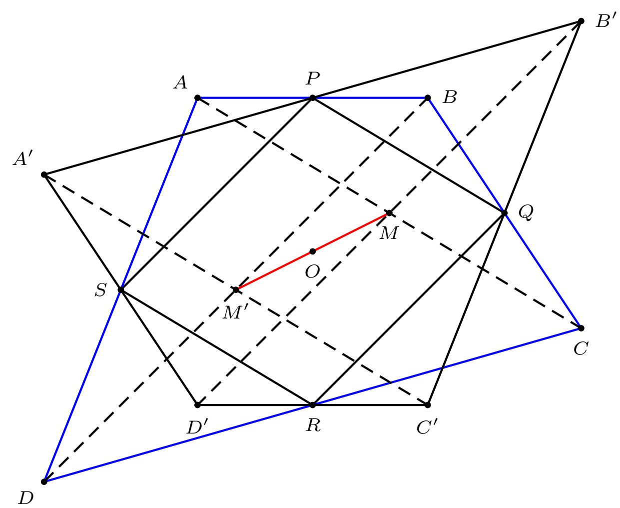


Figure 2

3. Cyclic quadrilaterals $ABCD$ with $\mathcal{P} = \mathcal{V}(ABCD)$

The previous section has brought out the role of diagonal midpoints when looking for quadrilaterals $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. We characterize the diagonal midpoints of *cyclic* solutions and show how to construct them from $\mathcal{P}$, obtaining the following theorem.

Theorem 1. *Given $\mathcal{P}$, there exist infinitely many cyclic quadrilaterals $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. Such quadrilaterals can be constructed from $\mathcal{P}$ by ruler and compass.*

Proof. Consider a Cartesian system with origin at O and x-axis parallel to PQ (see Figure 3). The affix of a point Z is denoted by z. For example, $q-p$ is a real number.

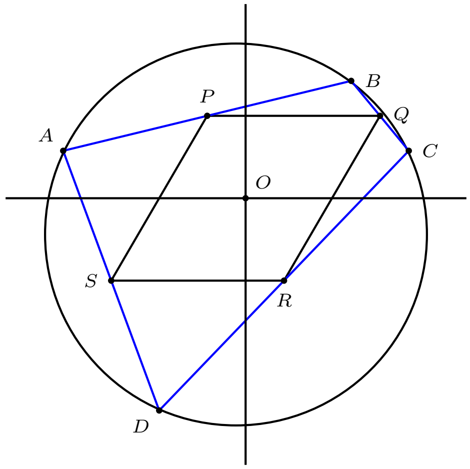


Figure 3

Let $ABCD$ be such that $\mathcal{P} = \mathcal{V}(ABCD)$. Then $A \neq C,\ B \neq D$ and the quadrilateral $ABCD$ is cyclic if and only if the cross-ratio $\rho = \frac{d-a}{d-b} \cdot \frac{c-b}{c-a}$ is a real number. With $b = 2p - a,\ c = 2q - 2p + a,\ d = -2q - a$ and allowing for $q - p \in \mathbb{R}$, the calculation of ρ yields the condition:

$$(q - p + a)^2 = p^2 + \lambda(p + q)$$

for some real number λ. Thus, $ABCD$ is cyclic if and only if the affixes $m,\ m' = -m$ of its diagonal midpoints M, M' are the square roots of a complex number of the form $p^2 + \lambda(p + q)$, where $\lambda \in \mathbb{R}$. Clearly, distinct values λ_1, λ_2 for λ lead to corresponding disjoint sets $\{M_1, M_1'\}, \{M_2, M_2'\}$ of diagonal midpoints, hence to distinct solutions for cyclic quadrilaterals. It follows that our problem has infinitely many solutions.

Consider P_2 with affix p^2 and choose a point K on the line through P_2 parallel to QR. The affix k of K is of the form $p^2 + \lambda(p+q)$ with $\lambda \in \mathbb{R}$. The construction of the corresponding pair M, M' is straightforward and achieved in Figure 4 where for the sake of simplification we take OP as the unit of length: M, M' are on the angle bisector of $\angle xOK$ and $OM = OM' = \sqrt{OK}$ (we skip the classical construction of the square root of a given length). $\square$

Exchanging the roles of M and M' (as in §2) evidently gives a solution inscribed in the symmetric of the circle $(ABCD)$ about O. In §4, we will indicate a different construction of suitable diagonal midpoints $M,\ M'$.

4. The rectangular hyperbola $\mathcal{H}(\mathcal{P})$

With the aim of obtaining the diagonal midpoints M, M' more directly, it seems interesting to identify their locus as the real number λ varies. This brings to light

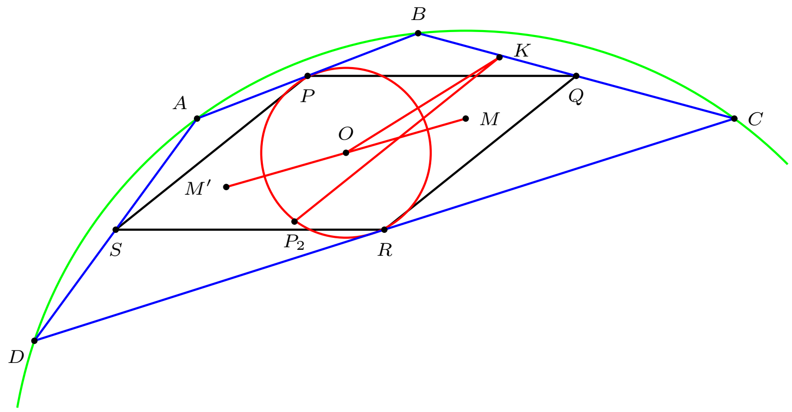


Figure 4

an unexpected hyperbola which will also provide more results about our quadrilaterals.

Theorem 2. *Consider the cyclic quadrilaterals $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. If $\mathcal{P}$ is not a rhombus, the locus of their diagonal midpoints is the rectangular hyperbola $\mathcal{H}(\mathcal{P})$ with the same center O as $\mathcal{P}$, passing through the vertices P, Q, R, S of $\mathcal{P}$. If $\mathcal{P}$ is a rhombus, the locus is the pair of diagonals of $\mathcal{P}$.*

Proof. We use the same system of axes as in the preceding section and continue to suppose that $OP = 1$. We denote by θ the directed angle $\angle(\overrightarrow{SR}, \overrightarrow{SP})$ that is, $\theta = \arg(p+q)$. Note that $\sin\theta \neq 0$. Let $m = x + iy$ with $x, y \in \mathbb{R}$. From $m^2 = p^2 + \lambda(p+q)$, we obtain $(x+iy)^2 = e^{2it} + \lambda\mu e^{i\theta}$ where $t = \arg(p)$ and $\mu = |p+q|$ and we readily deduce:

$$x^2 - y^2 = \cos 2t + \lambda\mu\cos\theta, \qquad 2xy = \sin 2t + \lambda\mu\sin\theta.$$

The elimination of λ shows that the locus of M (and of M' as well) is the curve $\mathcal{C}$ with equation

$$x^2 - y^2 - 2(\cot\theta)xy + \nu = 0, \tag{1}$$

where $\nu = \cot\theta\sin 2t - \cos 2t = \frac{\sin(2t-\theta)}{\sin\theta}$. Thus, when $\nu \neq 0$, $\mathcal{C}$ is a rectangular hyperbola centered at O with asymptotes

$$(\ell) \qquad y = x\tan(\theta/2),$$

and

$$(\ell') \qquad y = -x\cot(\theta/2),$$

and $\mathcal{C}$ degenerates into these two lines if $\nu = 0$ (we shall soon see that the latter occurs if and only if $\mathcal{P}$ is a rhombus). Note that (ℓ) and (ℓ') are the axes of symmetry of the medians of $\mathcal{P}$. An easy calculation shows that the coordinates $x_P = \cos t$, $y_P = \sin t$ of P satisfy (1), meaning that $P \in \mathcal{C}$. As for Q, the coordinates are $x_Q = \mu\cos\theta - \cos t$, $y_Q = \mu\sin\theta - \sin t$, but observing that

$y_Q = y_P$, we find $x_Q = 2\sin t \cot\theta - \cos t$, $y_Q = \sin t$. Again, x_Q, y_Q satisfy (1) and Q is a point of $\mathcal{C}$ as well. Thus, the parallelogram $\mathcal{P}$ is inscribed in $\mathcal{C}$. It follows that $\nu = 0$ if and only if (ℓ) and (ℓ') are the diagonals of $\mathcal{P}$. Since (ℓ) and (ℓ') are perpendicular, the situation occurs if $\mathcal{P}$ is a rhombus and only in that case. Otherwise, $\mathcal{C}$ is the rectangular hyperbola $\mathcal{H}(\mathcal{P})$, as defined in the statement of the theorem (see Figure 5). □

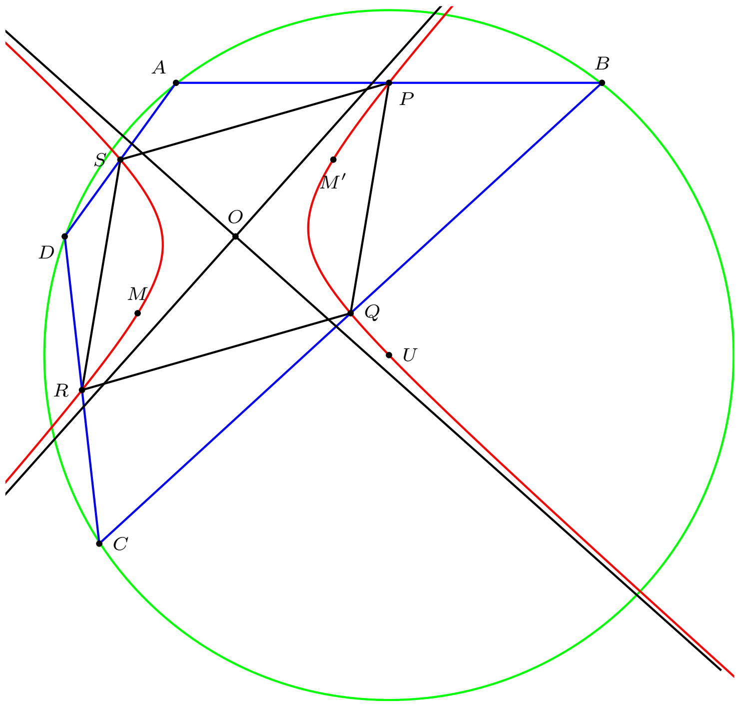


Figure 5

Figure 5 shows the center U of the circle through A, B, C, D as a point of $\mathcal{H}(\mathcal{P})$. This is no coincidence! Being the circumcenter of ΔABC, U is also the orthocenter of its median triangle MPQ. Since the latter is inscribed in $\mathcal{H}(\mathcal{P})$, a well-known property of the rectangular hyperbola ensures that its orthocenter is on $\mathcal{H}(\mathcal{P})$ as well. Conversely, any point U of $\mathcal{H}(\mathcal{P})$ can be obtained in this way by taking for M the orthocenter of ΔUPQ. We have proved:

Theorem 3. *If $\mathcal{P}$ is not a rhombus, $\mathcal{H}(\mathcal{P})$ is the locus of the circumcenter of a cyclic quadrilateral $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$.*

Of course, if $\mathcal{P}$ is a rhombus, the locus is the pair of diagonals of $\mathcal{P}$.

As another consequence of Theorem 2, we give a construction of a pair M, M' of diagonal midpoints simpler than the one in §3: through a vertex of $\mathcal{P}$, say Q, draw a line intersecting (ℓ) and (ℓ') at W and W'. As is well-known, the symmetric M of Q about the midpoint of WW' is on $\mathcal{H}(\mathcal{P})$. This point M and its symmetric M' about O provide a suitable pair. In addition, the orthocenter of ΔMPQ is the center U of the circumcircle of $ABCD$ (see Figure 6).

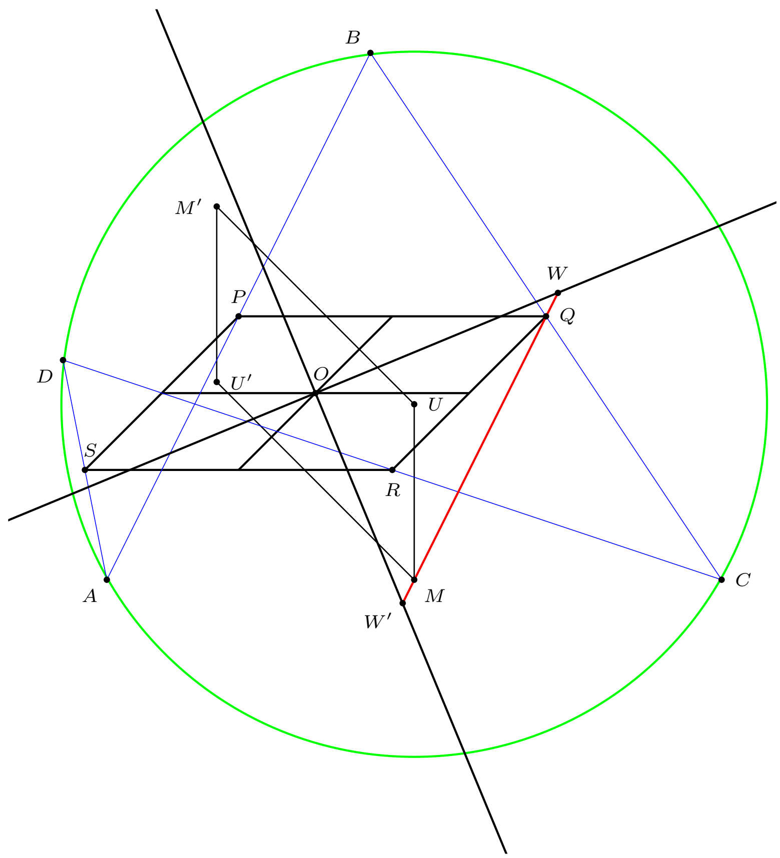


Figure 6

We shall end this section with a remark about the circumcenter U' of the quadrilateral $A'B'C'D'$ which shares the diagonal midpoints M, M' of $ABCD$ (as seen in §2). Clearly, $UMU'M'$ is a parallelogram with center O, inscribed in $\mathcal{H}(\mathcal{P})$ (Figure 6). Since UM and UM' are respectively perpendicular to PQ and PS, the directed angles of lines $\angle(UM, UM')$ and $\angle(PQ, PS)$ are equal (modulo π). Thus, $UMU'M'$ and $\mathcal{P}$ are equiangular.

5. Special cases

First, suppose that $\mathcal{P}$ is a rectangle and consider a cyclic quadrilateral $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. From the final remark of the previous section, $UMU'M'$ is a rectangle and since UM is perpendicular to PQ, the sides of $UMU'M'$ are parallel to those of $\mathcal{P}$. Recalling that M is on AC and M' on BD, we conclude that

U' is the point of intersection of the (perpendicular) diagonals of $ABCD$. Now, suppose that AC intersects PS at A_1, QR at C_1 and that BD intersects PQ at B_1, RS at D_1 (see Figure 7). Obviously, A_1, B_1, C_1 and D_1 are the midpoints of $U'A$, $U'B$, $U'C$ and $U'D$, so that A_1, B_1, C_1, D_1 are on the circle image of $(ABCD)$ under the homothety with center U' and ratio $\frac{1}{2}$. Since $\overrightarrow{U'O} = \frac{1}{2}\overrightarrow{U'U}$, the center of this circle $(A_1B_1C_1D_1)$ is just the center O of $\mathcal{P}$.

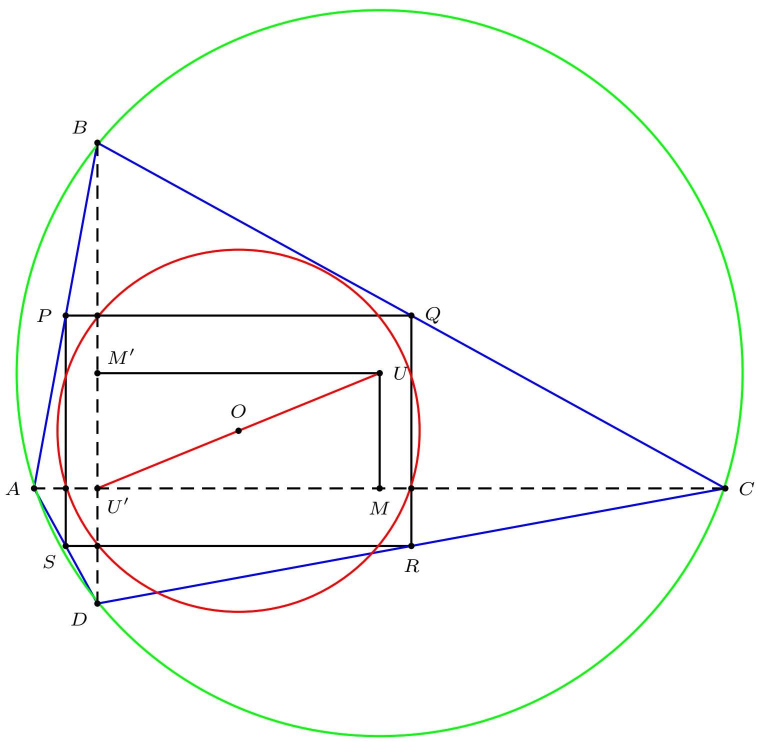


Figure 7

Conversely, draw any circle with center O intersecting the lines SP at A_1, A'_1, PQ at B_1, B'_1, QR at C_1, C'_1 and RS at D_1, D'_1, the notations being chosen so that A_1C_1, $A'_1C'_1$ are parallel to PQ and B_1D_1, $B'_1D'_1$ are parallel to QR. If $U' = A_1C_1 \cap B_1D_1$, then the image $ABCD$ of $A_1B_1C_1D_1$ under the homothety with center U' and ratio 2 is cyclic and satisfies $\mathcal{P} = \mathcal{V}(ABCD)$. For instance, because $U'A_1PB_1$ is a rectangle, P is the image of the midpoint of A_1B_1 and as such, is the midpoint of AB. The companion solution $A'B'C'D'$ is similarly obtained from $A'_1B'_1C'_1D'_1$.

Thus, in the case when $\mathcal{P}$ is a rectangle, a very quick construction provides suitable quadrilaterals $ABCD$. As a corollary of the analysis above, we have the following property that can also be proved directly:

Theorem 4. *If A, B, C, D are on a circle with center U and AC is perpendicular to BD at U', then the midpoint of UU' is the center of the rectangle $\mathcal{V}(ABCD)$.*

We conclude with a brief comment on the case when $\mathcal{P}$ is a rhombus. Remarking that if $\mathcal{P} = \mathcal{V}(ABCD)$, then $AC = 2PQ = 2QR = BD$, we see that any cyclic solution for $ABCD$ must be an isosceles trapezoid (possibly a self-crossing one). Conversely, if $ABCD$ is an isosceles trapezoid, then it is cyclic and $\mathcal{V}(ABCD)$ is a rhombus. The construction of a solution $ABCD$ from $\mathcal{P}$ simply follows from the choice of two points M, M' as diagonal midpoints of $ABCD$ on either diagonal of $\mathcal{P}$ (see Figure 8).

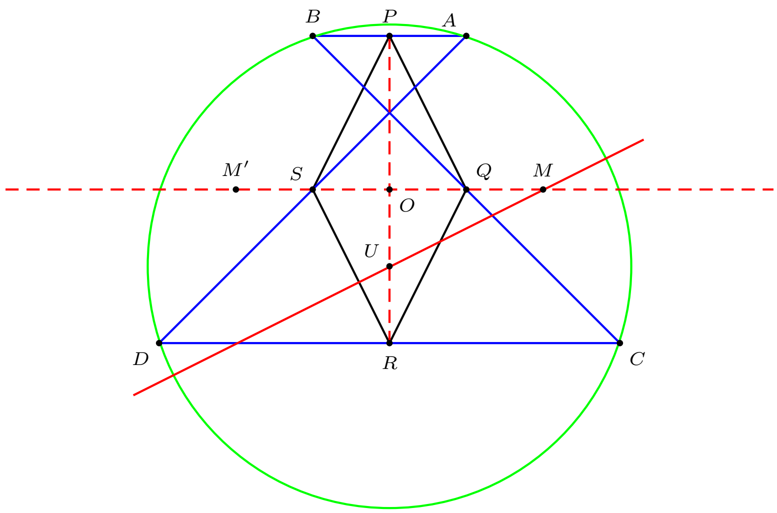


Figure 8

References

[1] M. Bataille, Solution to Problem 1, Croatian National Mathematics Competition, Additional competition for selection of the IMO Team, 1996, *Crux Mathematicorum*, 27 (2001) 502–503.

[2] M. Blanchard, Varignon aurait 336 ans!, *Mathématiques et Pédagogie*, 78 (1990) 39–48.

Michel Bataille: 12 rue Sainte-Catherine, 76000 Rouen, France
E-mail address: michelbataille@wanadoo.fr

Forum Geometricorum
Volume 7 (2007) 207–210.

FORUM GEOM
ISSN 1534-1178

Another Verification of Fagnano's Theorem

Finbarr Holland

Abstract. We present a trigonometrical proof of Fagnano's theorem which states that, among all inscribed triangles in a given acute-angled triangle, the feet of its altitudes are the vertices of the one with the least perimeter.

1. Introduction

At the outset, and to avoid ambiguity, we fix the following terminology. Let ABC be any triangle. The feet of its altitudes are the vertices of what we call its *orthic* triangle, and, if X, Y, and Z, respectively, are interior points of the sides AB, BC, and CA, respectively, we call the triangle XYZ an *inscribed triangle* of ABC.

In 1775, Fagnano proved the following theorem.

Theorem 1. *Suppose ABC is an acute-angled triangle. Of all inscribed triangles in ABC, its orthic triangle has the smallest perimeter.*

Not surprisingly, over the years this beautiful result has attracted the attentions of many mathematicians, and there are several proofs known of it [1]. Fagnano himself apparently used differential calculus to prove it, though, by modern standards, it seems to me that this is far from being a routine exercise. Perhaps the most appealing proofs of the theorem are those based on the Reflection Principle, and two of these, in particular, due independently to L. Fejér and H. A. Schwarz, have made their appearance in several books aimed at general audiences [2], [3], [4], [6]. A proof based on vector calculus appeared recently [5]. The purpose of this note is to offer one based on trigonometry.

Theorem 2. *Let ABC be any triangle, with $a = |BC|, b = |CA|, c = |AB|$, and area Δ. If XYZ is inscribed in ABC, then*

$$|XY| + |YZ| + |ZX| \geq \frac{8\Delta^2}{abc}. \tag{1}$$

Equality holds in (1) *if and only if ABC is acute-angled; and then only if XYZ is its orthic triangle. If ABC is right-angled* (*respectively, obtuse-angled*), *and C*

Publication Date: December 5, 2007. Communicating Editor: Paul Yiu.
The author is grateful to the referee for his helpful remarks.

is the right-angle (*respectively, the obtuse-angle*)*, then an inequality stronger than (1) holds, viz.,*

$$|XY| + |YZ| + |ZX| > 2h_c, \tag{2}$$

where h_c denotes the length of the altitude from C; and, in either case, this estimate is best possible.

2. Proof of Theorem 2

Let XYZ be a triangle inscribed in ABC. Let $x = |BX|, y = |CY|, z = |AZ|$. Then $0 < x < a$, $0 < y < b$, $0 < z < c$. By applying the Cosine Rule in the triangle ZBX we have

$$\begin{aligned}
|ZX|^2 &= (c-z)^2 + x^2 - 2x(c-z)\cos B \\
&= (c-z)^2 + x^2 + 2x(c-z)\cos(A+C) \\
&= (x\cos A + (c-z)\cos C)^2 + (x\sin A - (c-z)\sin C)^2.
\end{aligned}$$

Hence,

$$|ZX| \geq |x\cos A + (c-z)\cos C|,$$

with equality if and only if $x\sin A = (c-z)\sin C$, *i.e.*, if and only if

$$ax + cz = c^2, \tag{3}$$

by the Sine Rule. Similarly,

$$|XY| \geq |y\cos B + (a-x)\cos A|,$$

with equality if and only if

$$ax + by = a^2. \tag{4}$$

And

$$|YZ| \geq |z\cos C + (b-y)\cos B|,$$

with equality if and only if

$$by + cz = b^2. \tag{5}$$

Thus, by the triangle inequality for real numbers,

$$\begin{aligned}
&|XY| + |YZ| + |ZX| \\
&\geq |y\cos B + (a-x)\cos A| + |z\cos C + (b-y)\cos B| + |x\cos A + (c-z)\cos C| \\
&\geq |y\cos B + (a-x)\cos A + z\cos C + (b-y)\cos B + x\cos A + (c-z)\cos C| \\
&= |a\cos A + b\cos B + c\cos C| \\
&= \frac{|a^2(b^2+c^2-a^2) + b^2(c^2+a^2-b^2) + c^2(a^2+b^2-c^2)|}{2abc} \\
&= \frac{8\Delta^2}{abc}.
\end{aligned}$$

This proves (1). Moreover, there is equality here if and only if equations (3), (4), and (5) hold, and the expressions

$$u = x\cos A + (c - z)\cos C,$$
$$v = y\cos B + (a - x)\cos A,$$
$$w = z\cos C + (b - y)\cos B,$$

are either all non-negative or all non-positive. Now it is easy to verify that the system of equations (3), (4), and (5), has a unique solution given by

$$x = c\cos B,\ \ y = a\cos C,\ \ z = b\cos A,$$

in which case

$$u = b\cos B,\ v = c\cos C,\ w = a\cos A.$$

Thus, in this case, at most one of u, v, w can be non-positive. But, if one of u, v, w is zero, then one of x, y, z must be zero, which is not possible. It follows that

$$|XY| + |YZ| + |ZX| > \frac{8\Delta^2}{abc},$$

unless ABC is acute-angled, and XYZ is its orthic triangle. If ABC is acute-angled, then $\frac{8\Delta^2}{abc}$ is the perimeter of its orthic triangle, in which case we recover Fagnano's theorem, equality being attained in (1) when and only when XYZ is the orthic triangle.

Turning now to the case when ABC is not acute-angled, suppose first that C is a right-angle. Then

$$|XY| + |YZ| + |ZX| > \frac{8\Delta^2}{abc} = \frac{4\Delta}{c} = 2h_c,$$

and so (2) holds in this case. Next, if C is an obtuse-angle, denote by D and E, respectively, the points of intersection of the side AB and the lines through C that are perpendicular to the sides BC and CA, respectively. Then Z is an interior point of one of the line segments $[B, D]$ and $[E, A]$. Suppose, for definiteness, that Z is an interior point of $[B, D]$. If Y' is the point of intersection of $[X, Y]$ and $[C, D]$, then

$$\begin{aligned}|XY| + |YZ| + |ZX| &= |XY'| + |Y'Y| + |YZ| + |ZX| \\ &> |XY'| + |Y'Z| + |ZX| \\ &> 2h_c,\end{aligned}$$

since the triangle $XY'Z$ is inscribed in the right-angled triangle BCD. A similar argument works if Z is an interior point of $[E, A]$. Hence, (2) also holds if C is obtuse.

That (2) is stronger than (1), for a non acute-angled triangle, follows from the fact that, in any triangle ABC,

$$\frac{4\Delta^2}{abc} = \frac{2\Delta\sin C}{c} = a\sin B\sin C \le a\sin B = h_c.$$

It remains to prove that inequality (2) cannot be improved when the angle C is right or obtuse. To see this, let Z be the foot of the perpendicular from C to AB, and $0 < \varepsilon < 1$. Choose Y on CA so that $|CY| = \varepsilon b$, and X on BC so that XY is parallel to AB. Then, as $\varepsilon \to 0^+$, both X and Y converge to C, and so

$$\lim_{\varepsilon \to 0^+} (|XY| + |YZ| + |ZX|) = |CC| + |CZ| + |ZC| = 2|CZ| = 2h_c.$$

This finishes the proof.

References

[1] A. Bogomolny, Fagnano's Problem: What is it? http://www.cut-the-knot.org/Curriculum/Geometry/Fagnano.shtml

[2] H. S. M. Coxeter, *Introduction to Geometry*, Wiley, 1969.

[3] R. Courant and H. Robbins, *What is Mathematics?: An Elementary approach to Ideas and Methods*, Oxford Univesity Press, 1941.

[4] N. D. Kazarinoff, *Geometric Inequalities*, Random House, New York, 1961.

[5] M. H. Nguyen, Another proof of Fagnano's inequality, *Forum Geom.*, 4 (2004) 199–201.

[6] H. Rademacher and O. Toeplitz, *The Enjoyment of Mathematics*, Princeton University Press, 1957.

Finbarr Holland: Mathematics Department, University College, Cork, Ireland

E-mail address: f.holland@ucc.ie

Forum Geometricorum
Volume 7 (2007) 211–229.

FORUM GEOM
ISSN 1534-1178

How Pivotal Isocubics Intersect the Circumcircle

Bernard Gibert

Abstract. Given the pivotal isocubic $\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$, we seek its common points with the circumcircle and we also study the tangents at these points.

1. Introduction

A pivotal cubic $\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$ with pole Ω, pivot P, is the locus of point M such that P, M and its Ω-isoconjugate M^* are collinear. It is also the locus of point M such that P^* (the isopivot or secondary pivot), M and the cevian quotient P/M are collinear. See [2] for more information. [1] The isocubic $\mathcal{K}$ meets the circumcircle $(\mathcal{O})$ of the reference triangle ABC at its vertices and three other points Q_1, Q_2, Q_3, one of them being always real. This paper is devoted to a study of these points and special emphasis on their tangents.

2. Isogonal pivotal cubics

We first consider the case where the pivotal isocubic $\mathcal{K} = \mathrm{p}\mathcal{K}(X_6, P)$ is isogonal with pole the Lemoine point K.

2.1. *Circular isogonal cubics.* When the pivot P lies at infinity, $\mathcal{K}$ contains the two circular points at infinity. Hence it is a circular cubic of the class **CL035** in [3], and has only one real intersection with $(\mathcal{O})$. This is the isogonal conjugate P^* of the pivot.

The tangent at P is the real asymptote PP^* of the cubic and the isotropic tangents meet at the singular focus F of the circular cubic. F is the antipode of P^* on $(\mathcal{O})$.

The pair P and P^* are the foci of an inscribed conic, which is a parabola with focal axis PP^*. When P traverses the line at infinity, this axis envelopes the deltoid $\mathcal{H}_3$ tritangent to $(\mathcal{O})$ at the vertices of the circumtangential triangle. The contact of the deltoid with this axis is the reflection in P^* of the second intersection of PP^* with the circumcircle. See Figure 1 with the Neuberg cubic **K001** and the Brocard cubic **K021**. For example, with the Neuberg cubic, $P^* = X_{74}$, the second point on the axis is X_{476}, the contact is the reflection of X_{476} in X_{74}.

Publication Date: December 10, 2007. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this paper.

[1] Most of the cubics cited here are now available on the web-site http://perso.orange.fr/bernard.gibert/index.html, where they are referenced under a catalogue number of the form **Knnn**.

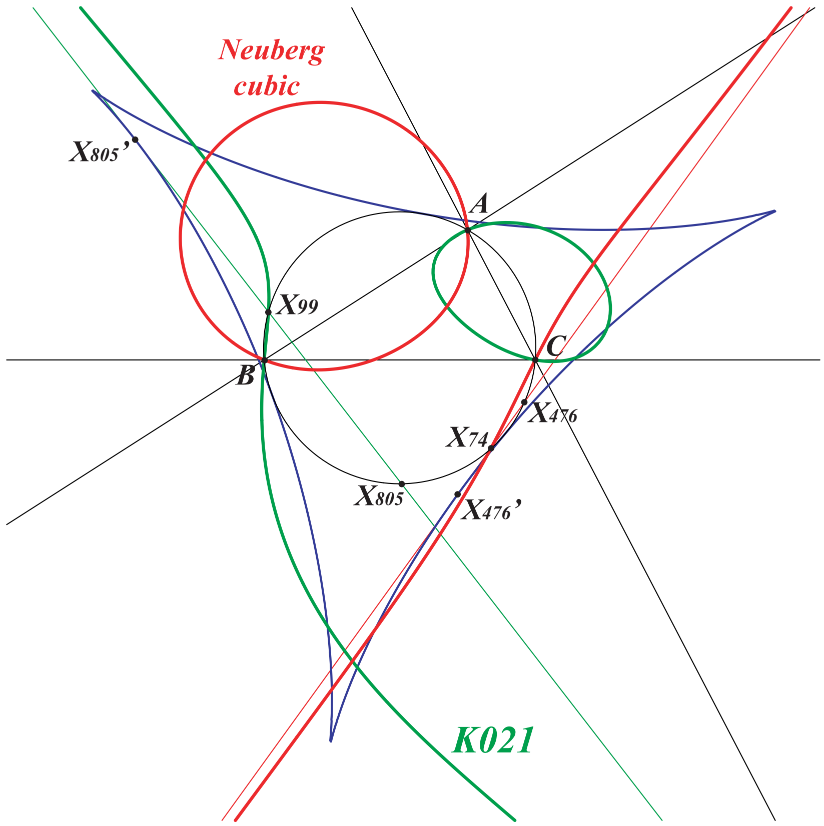


Figure 1. Isogonal circular cubic with pivot at infinity

2.2. *Isogonal cubics with pivot on the circumcircle.* When P lies on $(\mathcal{O})$, the remaining two intersections Q_1, Q_2 are antipodes on $(\mathcal{O})$. They lie on the perpendicular at O to the line PP^* or the parallel at O to the Simson line of P. The isocubic $\mathcal{K}$ has three real asymptotes:
(i) One is the parallel at P/P^* (cevian quotient) to the line PP^*.
(ii) The two others are perpendicular and can be obtained as follows. Reflect P in Q_1, Q_2 to get S_1, S_2 and draw the parallels at S_1^*, S_2^* to the lines PQ_1, PQ_2. These asymptotes meet at X on the line OP. Note that the tangent to the cubic at Q_1, Q_2 are the lines $Q_1S_1^*, Q_2S_2^*$. See Figure 2.

2.3. *The general case.* In both cases above, the orthocenter of the triangle formed by the points Q_1, Q_2, Q_3 is the pivot P of the cubic, although this triangle is not a proper triangle in the former case and a right triangle in the latter case. More generally, we have the following

Theorem 1. *For any point P, the isogonal cubic $\mathcal{K} = \mathrm{p}\mathcal{K}(X_6, P)$ meets the circumcircle at A, B, C and three other points Q_1, Q_2, Q_3 such that P is the orthocenter of the triangle $Q_1Q_2Q_3$.*

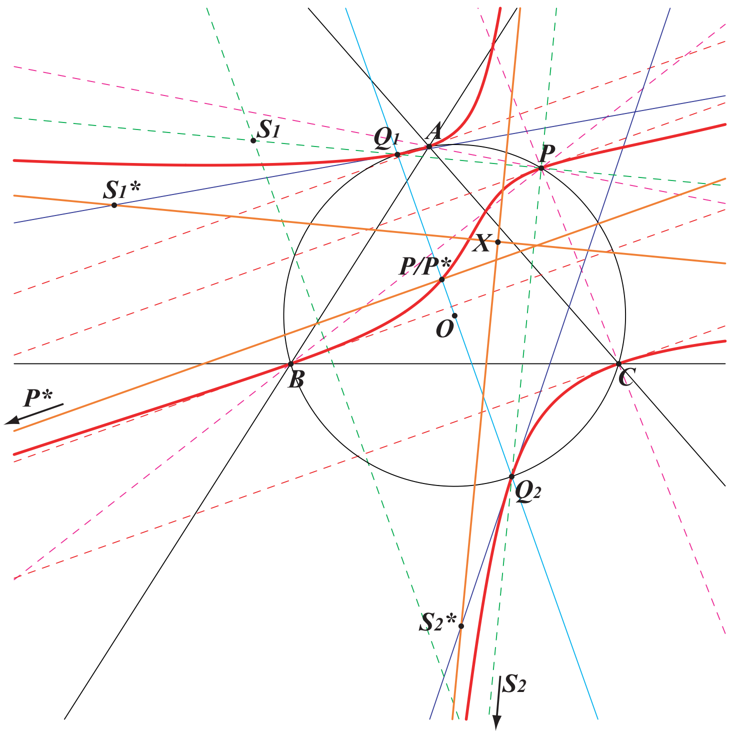


Figure 2. Isogonal cubic with pivot on the circumcircle

Proof. The lines $Q_1Q_1^*$, $Q_2Q_2^*$, $Q_3Q_3^*$ pass through the pivot P and are parallel to the asymptotes of the cubic. Since they are the axes of three inscribed parabolas, they must be tangent to the deltoid $\mathcal{H}_3$, the anticomplement of the Steiner deltoid. This deltoid is a bicircular quartic of class 3. Hence, for a given P, there are only three tangents (at least one of which is real) to the deltoid passing through P.

According to a known result, Q_1 must be the antipode on $(\mathcal{O})$ of Q_1', the isogonal conjugate of the infinite point of the line Q_2Q_3. The Simson lines of Q_1', Q_2, Q_3 are concurrent. Hence, the axes are also concurrent at P. But the Simson line of Q_1 is parallel to Q_2Q_3. Hence $Q_1Q_1^*$ is an altitude of $Q_1Q_2Q_3$. This completes the proof. See Figure 3. □

Remark. These points Q_1, Q_2, Q_3 are not necessarily all real nor distinct. In [1], H. M. Cundy and C. F. Parry have shown that this depends of the position of P with respect to $\mathcal{H}_3$. More precisely, these points are all real if and only if P lies strictly inside $\mathcal{H}_3$. One only is real when P lies outside $\mathcal{H}_3$. This leaves a special case when P lies on $\mathcal{H}_3$. See §2.4.

Recall that the contacts of the deltoid $\mathcal{H}_3$ with the line $PQ_1Q_1^*$ is the reflection in Q_1 of the second intersection of the circumcircle and the line $PQ_1Q_1^*$. Consequently, every conic passing through P, Q_1, Q_2, Q_3 is a rectangular hyperbola and all these hyperbolas form a pencil $\mathcal{F}$ of rectangular hyperbolas.

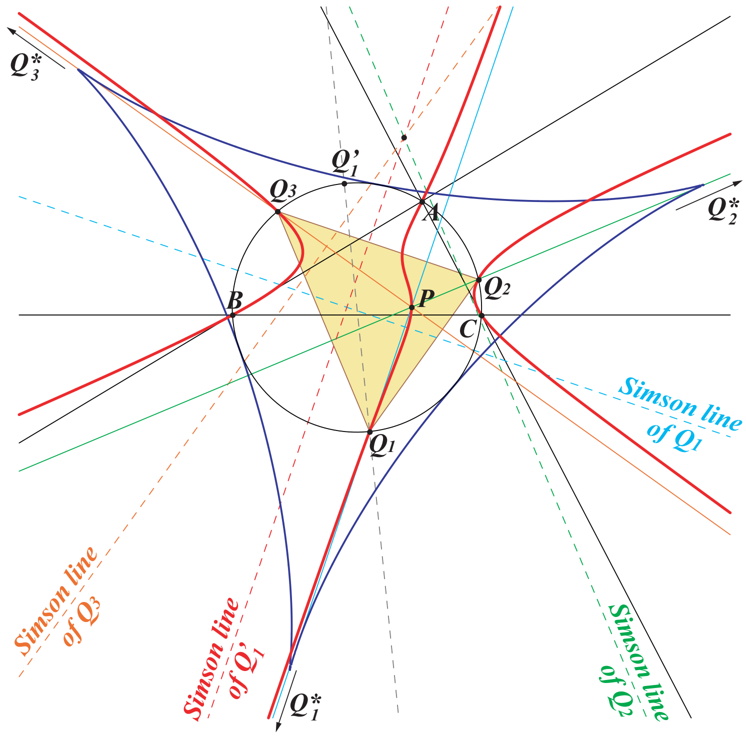


Figure 3. The deltoid $\mathcal{H}_3$ and the points Q_1, Q_2, Q_3

Let $\mathcal{D}$ be the diagonal rectangular hyperbola which contains the four in/excenters of ABC, P^*, and P/P^*. Its center is $\Omega_{\mathcal{D}}$. Note that the tangent at P^* to $\mathcal{D}$ contains P and the tangent at P/P^* to $\mathcal{D}$ contains P. In other words, the polar line of P in $\mathcal{D}$ is the line through P^* and P/P^*.

The pencil $\mathcal{F}$ contains the hyperbola $\mathcal{H}$ passing through P, P^*, P/P^* and $\Omega_{\mathcal{D}}$ having the same asymptotic directions as $\mathcal{D}$. The center of $\mathcal{H}$ is the midpoint of P and $\Omega_{\mathcal{D}}$. This gives an easy conic construction of the points Q_1, Q_2, Q_3 when P is given. See Figure 4. The pencil $\mathcal{F}$ contains another very simple rectangular hyperbola $\mathcal{H}'$, which is the homothetic of the polar conic of P in $\mathcal{K}$ under $\mathsf{h}\left(P, \frac{1}{2}\right)$. Since this polar conic is the diagonal conic passing through the in/excenters and P, $\mathcal{H}'$ contains P and the four midpoints of the segments joining P to the in/excenters.

Corollary 2. *The isocubic $\mathcal{K}$ contains the projections R_1, R_2, R_3 of P^* on the sidelines of $Q_1Q_2Q_3$. These three points lie on the bicevian conic $\mathcal{C}(G, P)$.* [2]

Proof. Let R_1 be the third point of $\mathcal{K}$ on the line Q_2Q_3. The following table shows the collinearity relations of nine points on $\mathcal{K}$ and proves that P^*, R_1 and Q_1^* are collinear.

---

[2]This is the conic through the vertices of the cevian triangles of G and P. This is the P-Ceva conjugate of the line at infinity.

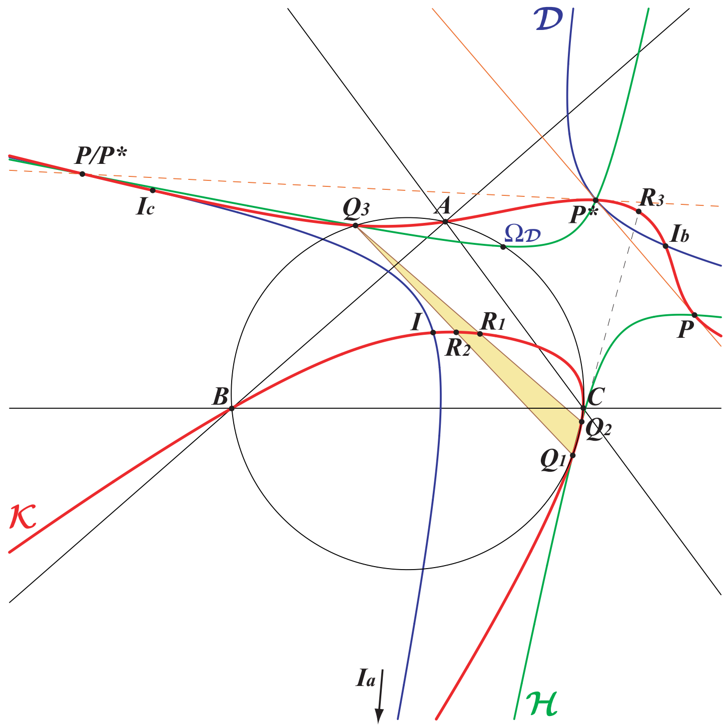


Figure 4. The hyperbolas $\mathcal{H}$ and $\mathcal{D}$

P	P	P^*	$\leftarrow$ P^* is the tangential of P
Q_2	Q_3	R_1	$\leftarrow$ definition of R_1
Q_2^*	Q_3^*	Q_1^*	$\leftarrow$ these three points lie at infinity

This shows that, for $i = 1, 2, 3$, the points P^*, R_i and Q_i^* are collinear and, since P, Q_i and Q_i^* are also collinear, the lines PQ_i and P^*R_i are parallel. It follows from Theorem 1 that R_i is the projection of P^* onto the line R_jR_k.

Recall that P^* is the secondary pivot of $\mathcal{K}$ hence, for any point M on $\mathcal{K}$, the points P^*, M and P/M (cevian quotient) are three collinear points on $\mathcal{K}$. Consequently, $R_i = P/Q_i^*$ and, since Q_i^* lies at infinity, R_i is a point on $\mathcal{C}(G, P)$. □

Corollary 3. *The lines* $Q_iR_i^*$, $i = 1, 2, 3$, *pass through the cevian quotient* P/P^*.

Proof. This is obvious from the following table.

P^*	P^*	P	$\leftarrow$ P/P^* is the tangential of P^*
P	Q_1^*	Q_1	$\leftarrow$ $Q_1Q_1^*$ must contain the pivot P
P	R_1	R_1^*	$\leftarrow$ $R_1R_1^*$ must contain the pivot P

Recall that P^* is the tangential of P (first column). The second column is the corollary above. □

Corollary 4. *Let S_1, S_2, S_3 be the reflections of P in Q_1, Q_2, Q_3 respectively. The asymptotes of $\mathcal{K}$ are the parallel at S_i^* to the lines PQ_i or P^*R_i.*

Proof. These points S_i lie on the polar conic of the pivot P since they are the harmonic conjugate of P with respect to Q_i and Q_i^*. The construction of the asymptotes derives from [2, §1.4.4]. □

Theorem 5. *The inconic $\mathcal{I}(P)$ concentric with $\mathcal{C}(G,P)$* [3] *is also inscribed in the triangle $Q_1Q_2Q_3$ and in the triangle formed by the Simson lines of Q_1, Q_2, Q_3.*

Proof. Since the triangles ABC and $Q_1Q_2Q_3$ are inscribed in the circumcircle, there must be a conic inscribed in both triangles. The rest is mere calculation. □

In [4, §29, p.88], A. Haarbleicher remarks that the triangle ABC and the reflection of $Q_1Q_2Q_3$ in O circumscribe the same parabola. These two parabolas are obviously symmetric about O. Their directrices are the line through H and the reflection P' of P in O in the former case, and its reflection in O in the latter case. The foci are the isogonal conjugates of the infinite points of these directrices and its reflection about O.

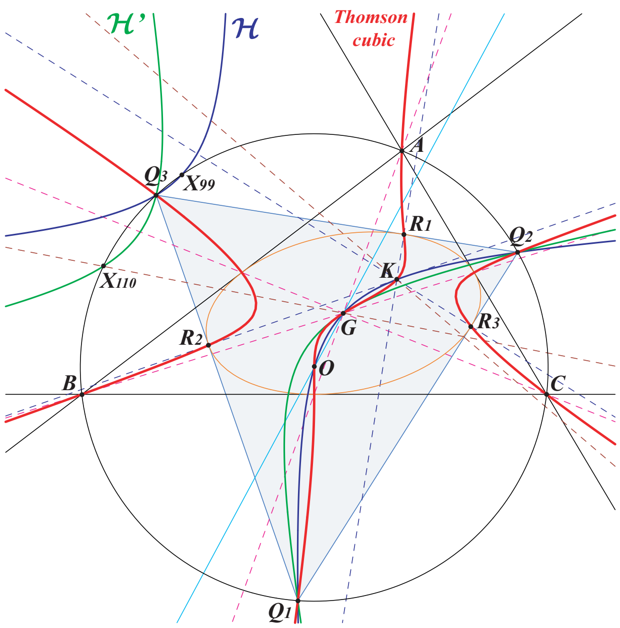


Figure 5. Thomson cubic

[3]This center is the complement of the complement of P, *i.e.*, the homothetic of P under $\mathsf{h}(G, \frac{1}{4})$. Note that these two conics $\mathcal{I}(P)$ and $\mathcal{C}(G,P)$ are bitangent at two points on the line GP. When $P = G$, they coincide since they both are the Steiner in-ellipse.

For example, Figures 5 and 6 show the case $P = G$. Note, in particular,
– $\mathcal{K}$ is the Thomson cubic,
– $\mathcal{D}$ is the Steiner (or Don Wallace) hyperbola,
– $\mathcal{H}$ contains X_2, X_3, X_6, X_{110}, X_{154}, X_{354}, X_{392}, X_{1201}, X_{2574}, X_{2575},
– $\mathcal{H}'$ contains X_2, X_{99}, X_{376}, X_{551},
– the inconic $\mathcal{I}(P)$ and the bicevian conic $\mathcal{C}(G, P)$ are the Steiner in-ellipse,
– the two parabolas are the Kiepert parabola and its reflection in O.

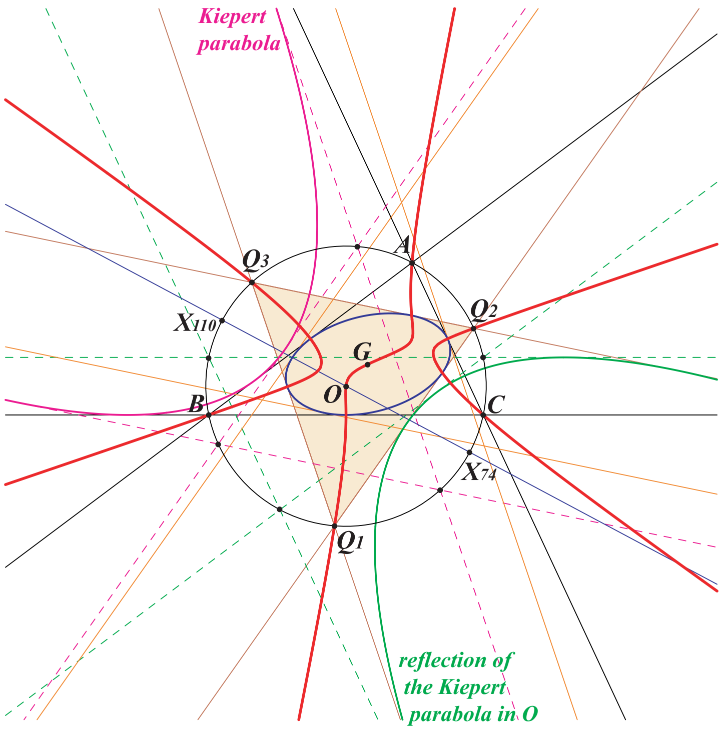


Figure 6. The Thomson cubic and the two parabolas

More generally, any $\mathrm{p}\mathcal{K}(X_6, P)$ with pivot P on the Euler line is obviously associated to the same two parabolas. In other words, any cubic of the Euler pencil meets the circumcircle at three (not always real) points Q_1, Q_2, Q_3 such that the reflection of the Kiepert parabola in O is inscribed in the triangle $Q_1Q_2Q_3$ and in the circumcevian triangle of O.

In particular, taking $P = O$, we obtain the McCay cubic and this shows that the reflection of the Kiepert parabola in O is inscribed in the circumnormal triangle.

Another interesting case is $\mathrm{p}\mathcal{K}(X_6, X_{145})$ in Figure 7 since the incircle is inscribed in the triangle $Q_1Q_2Q_3$.

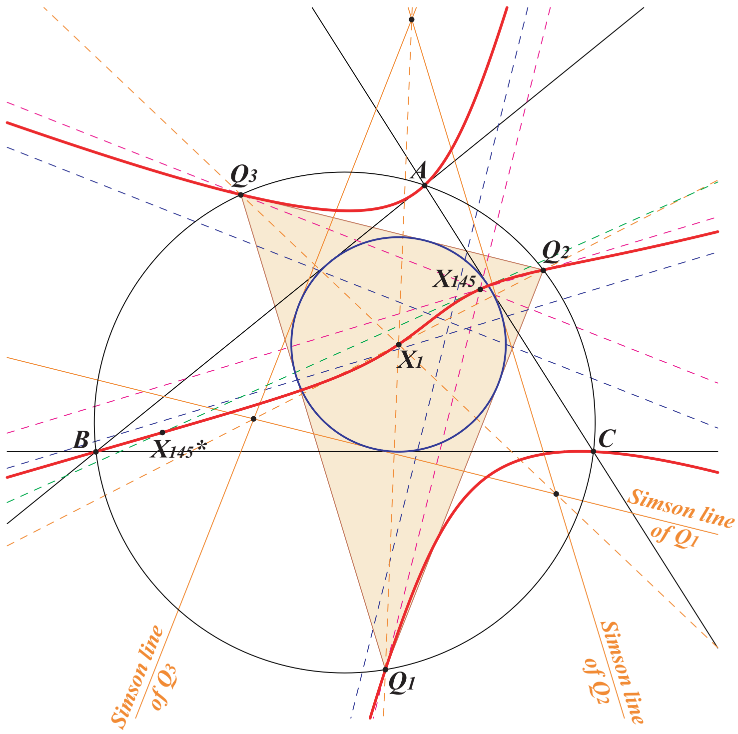


Figure 7. $\mathrm{p}\mathcal{K}(X_6, X_{145})$

2.4. *Isogonal pivotal cubics tangent to the circumcircle.* In this section, we take P on $\mathcal{H}_3$ so that $\mathcal{K}$ has a multiple point at infinity.

Here is a special case. $\mathcal{H}_3$ is tangent to the six bisectors of ABC. If we take the bisector AI, the contact P is the reflection of A in the second intersection A_i of AI with the circumcircle. The corresponding cubic $\mathcal{K}$ is the union of the bisector AI and the conic passing through B, C, the excenters I_b and I_c, A_i, the antipode of A on the circumcircle.

Let us now take M on the circle $\mathcal{C}_H$ with center H, radius $2R$ and let us denote by $\mathcal{T}_M$ the tangent at M to $\mathcal{C}_H$. The orthopole P of $\mathcal{T}_M$ with respect to the antimedial triangle is a point on $\mathcal{H}_3$.

The corresponding cubic $\mathcal{K}$ meets $(\mathcal{O})$ at P_1 (double) and P_3. The common tangent at P_1 to $\mathcal{K}$ and $(\mathcal{O})$ is parallel to $\mathcal{T}_M$. Note that P_1 lies on the Simson line $\mathcal{S}_P$ of P with respect to the antimedial triangle.

The perpendicular at P_1 to $\mathcal{S}_P$ meets $(\mathcal{O})$ again at P_3 which is the antipode on $(\mathcal{O})$ of the second intersection Q_3 of $\mathcal{S}_P$ and $(\mathcal{O})$. The Simson line of P_3 is parallel to $\mathcal{T}_M$.

It follows that $\mathcal{K}$ has a triple common point with $(\mathcal{O})$ if and only if P_1 and Q_3 are antipodes on $(\mathcal{O})$ i.e. if and only if $\mathcal{S}_P$ passes through O. This gives the following theorem.

Theorem 6. *There are exactly three isogonal pivotal cubics which are tridents.*

Their pivots are the cusps of the deltoid $\mathcal{H}_3$. The triple contacts with $(\mathcal{O})$ are the vertices of the circumnormal triangle.[4] These points are obviously inflexion points and the inflexional tangent is parallel to a sideline of the Morley triangle. See Figure 8.

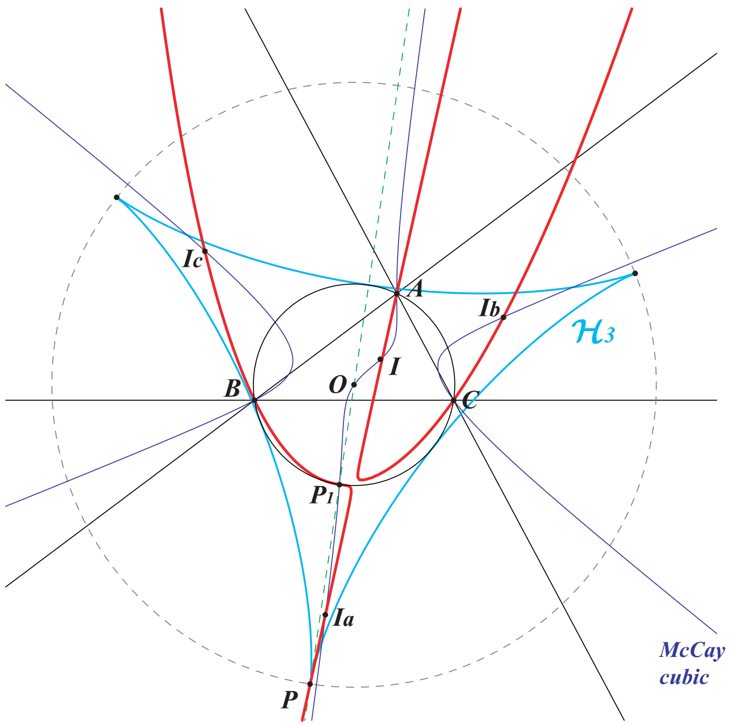


Figure 8. Isogonal pivotal trident

2.5. *Tangents at* Q_1, Q_2, Q_3. We know that the tangents at A, B, C to any pivotal cubic concur at P^*. This is not necessarily true for those at Q_1, Q_2, Q_3.

Theorem 7. *The tangents at* Q_1, Q_2, Q_3 *to the isogonal cubic* $\mathrm{p}\mathcal{K}(X_6, P)$ *concur if and only if* P *lies on the quintic* **Q063** *with equation*

$$\sum_{\text{cyclic}} a^2y^2z^2\left(S_C(x+y) - S_B(x+z)\right) = 0.$$

Q063 is a circular quintic with singular focus X_{376}, the reflection of G in O. It has three real asymptotes parallel to those of the Thomson cubic and concurrent at G.

[4]These three points are the common points of the circumcircle and the McCay cubic apart A, B, C.

A, B, C are nodes and the fifth points on the sidelines of ABC are the vertices A', B', C' of the pedal triangle of X_{20}, the de Longchamps point. The tangents at these points pass through X_{20} and meet the corresponding bisectors at six points on the curve. See Figure 9.

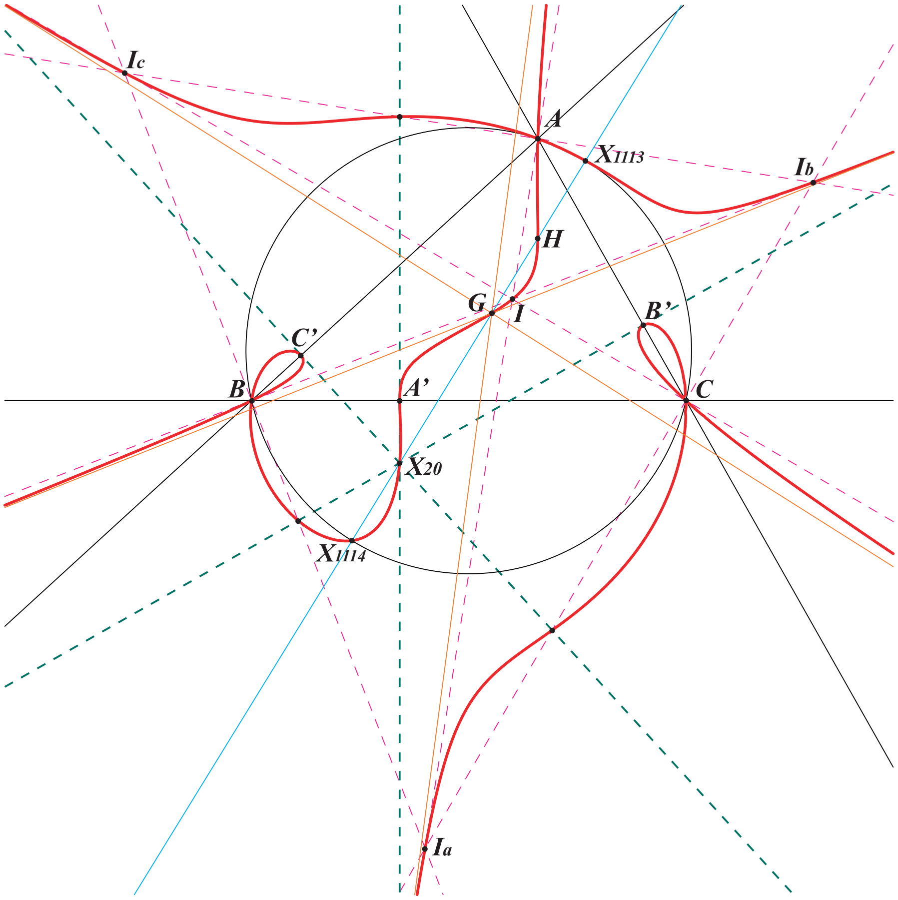


Figure 9. The quintic **Q063**

Q063 contains I, the excenters, G, H, X_{20}, X_{1113}, X_{1114}. Hence, for the Thomson cubic, the orthocubic, and the Darboux cubic, the tangents at Q_1, Q_2, Q_3 concur. The intersection of these tangents are X_{25} for the orthocubic, and X_{1498} for the Darboux cubic. For the Thomson cubic, this is an unknown point[5] in the current edition of ETC on the line GX_{1350}.

[5]This has first barycentric coordinate

$$a^2(3S_A^2 + 2a^2S_A + 5b^2c^2).$$

3. Non-isogonal pivotal cubics

We now consider a non-isogonal pivotal cubic $\mathcal{K}$ with pole $\omega \neq K$ and pivot π.

We recall that π^* is the $\omega-$isoconjugate of π and that π/π^* is the cevian quotient of π and π^*, these three points lying on the cubic.

3.1. *Circular cubics.* In this special case, two of the points, say Q_2 and Q_3, are the circular points at infinity. This gives already five common points of the cubic on the circumcircle and the sixth point Q_1 must be real.

The isoconjugation with pole ω swaps the pivot π and the isopivot π^* which must be the inverse (in the circumcircle of ABC) of the isogonal conjugate of π. In this case, the cubic contains the point T, isogonal conjugate of the complement of π. This gives the following

Theorem 8. *A non isogonal circular pivotal cubic $\mathcal{K}$ meets the circumcircle at A, B, C, the circular points at infinity and another (real) point Q_1 which is the second intersection of the line through T and π/π^* with the circle passing through π, π^* and π/π^*.*

Example: The Droussent cubic **K008**. This is the only circular isotomic pivotal cubic. See Figure 10.

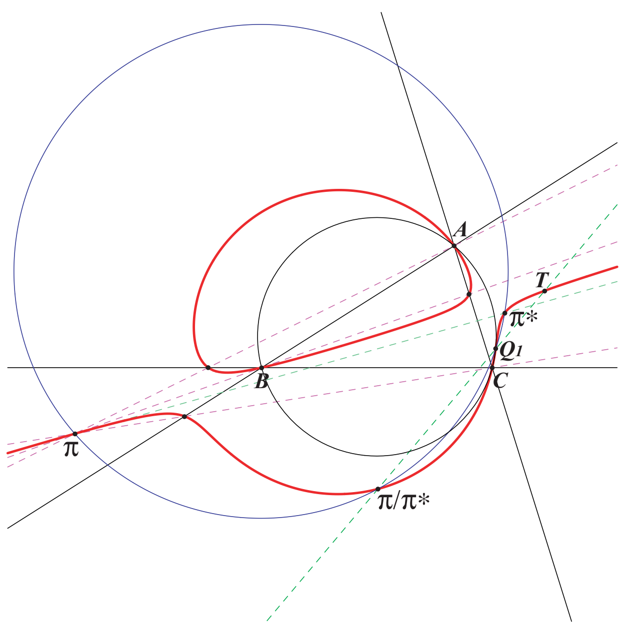


Figure 10. The Droussent cubic **K008**

The points π, π^*, T, Q_1 are X_{316}, X_{67}, X_{671}, X_{2373} respectively. The point π/π^* is not mentioned in the current edition of [6].

Note that when $\pi = H$, there are infinitely many circular pivotal cubics with pivot H, with isopivot π^* at infinity. These cubics are the isogonal circular pivotal cubics with respect to the orthic triangle. They have their singular focus F on the nine point circle and their pole ω on the orthic axis. The isoconjugate H^* of H is the point at infinity of the cubic. The intersection with their real asymptote is X, the antipode of F on the nine point circle and, in this case, $X = \pi/\pi^*$. This asymptote envelopes the Steiner deltoid $\mathcal{H}_3$. The sixth point Q_1 on the circumcircle is the orthoassociate of X, i.e. the inverse of X in the polar circle.

Example: The Neuberg orthic cubic **K050**. This is the Neuberg cubic of the orthic triangle. See [3].

3.2. *General theorems for non circular cubics.*

Theorem 9. *$\mathcal{K}$ meets the circumcircle at A, B, C and three other points Q_1, Q_2, Q_3 (one at least is real) lying on a same conic passing through π, π^* and π/π^*.*

Note that this conic meets the circumcircle again at the isogonal conjugate of the infinite point of the trilinear polar of the isoconjugate of ω under the isoconjugation with fixed point π.

With $\omega = p : q : r$ and $\pi = u : v : w$, this conic has equation

$$\sum_{\text{cyclic}} p^2v^2w^2(c^2y+b^2z)(wy-vz)+qru^2x(vw(c^2v-b^2w)x+u(b^2w^2y-c^2v^2z)) = 0,$$

and the point on the circumcircle is :

$$\frac{a^2}{u^2(rv^2-qw^2)} : \frac{b^2}{v^2(pw^2-ru^2)} : \frac{c^2}{w^2(qu^2-pv^2)}$$

Theorem 10. *The conic inscribed in triangles ABC and $Q_1Q_2Q_3$ is that with perspector the cevian product of π and $tg\omega$, the isotomic of the isogonal of ω.*

3.3. *Relation with isogonal pivotal cubics.*

Theorem 11. *$\mathcal{K}$ meets the circumcircle at the same points as the isogonal pivotal cubic with pivot $P = u : v : w$ if and only if its pole ω lie on the cubic $\mathcal{K}_{\text{pole}}$ with equation*

$$\sum_{\text{cyclic}} (v+w)(c^4y-b^4z)\,\frac{x^2}{a^2} - \left(\sum_{\text{cyclic}} (b^2-c^2)u\right) xyz = 0$$
$$\iff \sum_{\text{cyclic}} a^2u(c^2y-b^2z)(-a^4yz+b^4zx+c^4xy) = 0.$$

In other words, for any point ω on $\mathcal{K}_{\text{pole}}$, there is a pivotal cubic with pole ω meeting the circumcircle at the same points as the isogonal pivotal cubic with pivot $P = u : v : w$.

$\mathcal{K}_{\text{pole}}$ is a circum-cubic passing through K, the vertices of the cevian triangle of gc$P$, the isogonal conjugate of the complement of P. The tangents at A, B, C are the cevians of X_{32}.

The second equation above clearly shows that all these cubics belong to a same net of circum-cubics passing through K having the same tangents at A, B, C.

This net can be generated by three decomposed cubics, one of them being the union of the symmedian AK and the circum-conic with perspector the A-harmonic associate of X_{32}.

For example, with $P = H$, $\mathcal{K}_{\text{pole}}$ is a nodal cubic with node K and nodal tangents parallel to the asymptotes of the Jerabek hyperbola. It contains X_6, X_{66}, X_{193}, X_{393}, X_{571}, X_{608}, X_{1974}, X_{2911} which are the poles of cubics meeting the circumcircle at the same points as the orthocubic **K006**.

Theorem 12. *$\mathcal{K}$ meets the circumcircle at the same points as the isogonal pivotal cubic with pivot $P = u : v : w$ if and only if its pivot π lie on the cubic $\mathcal{K}_{\text{pivot}}$ with equation*

$$\sum_{\text{cyclic}} (v+w)(c^4y - b^4z)\, x^2 + \left(\sum_{\text{cyclic}} (b^2 - c^2)u\right) xyz = 0.$$

In other words, for any point π on $\mathcal{K}_{\text{pivot}}$, there is a pivotal cubic with pivot π meeting the circumcircle at the same points as the isogonal pivotal cubic with pivot $P = u : v : w$.

$\mathcal{K}_{\text{pivot}}$ is a circum-cubic tangent at A, B, C to the symmedians. It passes through P, the points on the circumcircle and on the isogonal pivotal cubic with pivot P, the infinite points of the isogonal pivotal cubic with pivot the complement of P, the vertices of the cevian triangle of tc$P$, the isotomic conjugate of the complement of P.

Following the example above, with $P = H$, $\mathcal{K}_{\text{pivot}}$ is also a nodal cubic with node H and nodal tangents parallel to the asymptotes of the Jerabek hyperbola. It contains X_3, X_4, X_8, X_{76}, X_{847} which are the pivots of cubics meeting the circumcircle at the same points as the orthocubic, three of them being $\mathrm{p}\mathcal{K}(X_{193}, X_{76})$, $\mathrm{p}\mathcal{K}(X_{571}, X_3)$ and $\mathrm{p}\mathcal{K}(X_{2911}, X_8)$.

Remark. Adding up the equations of $\mathcal{K}_{\text{pole}}$ and $\mathcal{K}_{\text{pivot}}$ shows that these two cubics generate a pencil containing the $\mathrm{p}\mathcal{K}$ with pole the X_{32}-isoconjugate of cP, pivot the X_{39}-isoconjugate of cP and isopivot X_{251}.

For example, with $P = X_{69}$, this cubic is $\mathrm{p}\mathcal{K}(X_6, X_{141})$. The nine common points of all the cubics of the pencil are A, B, C, K, X_{1169} and the four foci of the inscribed ellipse with center X_{141}, perspector X_{76}.

3.4. *Pivotal $\mathcal{K}_{\text{pole}}$ and $\mathcal{K}_{\text{pivot}}$.* The equations of $\mathcal{K}_{\text{pole}}$ and $\mathcal{K}_{\text{pivot}}$ clearly show that these two cubics are pivotal cubics if and only if P lies on the line GK. This gives the two following corollaries.

Corollary 13. *When P lies on the line GK, $\mathcal{K}_{\text{pole}}$ is a pivotal cubic and contains K, X_{25}, X_{32}. Its pivot is gc$P$ (on the circum-conic through G and K) and its isopivot is X_{32}. Its pole is the barycentric product of X_{32} and gc$P$. It lies on the circum-conic through X_{32} and X_{251}.*

All these cubics belong to a same pencil of pivotal cubics. Furthermore, $\mathcal{K}_{\text{pole}}$ contains the cevian quotients of the pivot gc$P$ and K, X_{25}, X_{32}. Each of these points is the third point of the cubic on the corresponding sideline of the triangle with vertices K, X_{25}, X_{32}. In particular, X_{25} gives the point gt$P$.

Table 1 shows a selection of these cubics.

P	$\mathcal{K}_{\text{pole}}$ contains K, X_{25}, X_{32} and	cubic
X_2	X_{31}, X_{41}, X_{184}, X_{604}, X_{2199}	**K346**
X_{69}	X_2, X_3, X_{66}, X_{206}, X_{1676}, X_{1677}	**K177**
X_{81}	X_{1169}, X_{1333}, X_{2194}, X_{2206}	
X_{86}	X_{58}, X_{1171}	
X_{193}	X_{1974}, X_{3053}	
X_{298}	X_{15}, X_{2981}	
X_{323}	X_{50}, X_{1495}	
X_{325}	X_{511}, X_{2987}	
X_{385}	X_{1691}, X_{1976}	
X_{394}	X_{154}, X_{577}	
X_{491}	X_{372}, X_{589}	
X_{492}	X_{371}, X_{588}	
X_{524}	X_{111}, X_{187}	
X_{1270}	X_{493}, X_{1151}	
X_{1271}	X_{494}, X_{1152}	
X_{1654}	X_{42}, X_{1918}, X_{2200}	
X_{1992}	X_{1383}, X_{1384}	
X_{1994}	X_{51}, X_{2965}	
X_{2895}	X_{37}, X_{213}, X_{228}, X_{1030}	
atX_{1916}	X_{237}, X_{384}, X_{385}, X_{694}, X_{733}, X_{904}, X_{1911}, X_{2076}, X_{3051}	

Table 1. $\mathcal{K}_{\text{pole}}$ with P on the line GK.

Remark. atX_{1916} is the anticomplement of the isotomic conjugate of X_{1916}.

Corollary 14. *When P lies on the line GK, $\mathcal{K}_{\text{pivot}}$ contains P, G, H, K. Its pole is gc$P$ (on the cubic) and its pivot is tc$P$ on the Kiepert hyperbola.*

All these cubics also belong to a same pencil of pivotal cubics.

Table 2 shows a selection of these cubics.

We remark that $\mathcal{K}_{\text{pole}}$ is the isogonal of the isotomic transform of $\mathcal{K}_{\text{pivot}}$ but this correspondence is not generally true for the pivot π and the pole ω. To be more precise, for π on $\mathcal{K}_{\text{pivot}}$, the pole ω on $\mathcal{K}_{\text{pole}}$ is the Ceva-conjugate of gc$P$ and gtπ.

From the two corollaries above, we see that, given an isogonal pivotal cubic $\mathcal{K}$ with pivot on the line GK, we can always find two cubics with poles X_{25}, X_{32} and three cubics with pivots G, H, K sharing the same points on the circumcircle as $\mathcal{K}$. Obviously, there are other such cubics but their pole and pivot both depend of P. In particular, we have $\mathrm{p}\mathcal{K}(\text{gc}P, \text{tc}P)$ and $\mathrm{p}\mathcal{K}(O\times \text{gc}P, \text{gc}P)$.

We illustrate this with $P = G$ (and gc$P = K$) in which case $\mathcal{K}_{\text{pivot}}$ is the Thomson cubic **K002** and $\mathcal{K}_{\text{pole}}$ is **K346**. For π and ω chosen accordingly on these

P	$\mathcal{K}_{\text{pivot}}$ contains X_2, X_4, X_6 and	cubic
X_2	X_1, X_3, X_9, X_{57}, X_{223}, X_{282}, X_{1073}, X_{1249}	**K002**
X_6	X_{83}, X_{251}, X_{1176}	
X_{69}	X_{22}, X_{69}, X_{76}, X_{1670}, X_{1671}	**K141**
X_{81}	X_{21}, X_{58}, X_{81}, X_{572}, X_{961}, X_{1169}, X_{1220}, X_{1798}, X_{2298}	**K379**
X_{86}	X_{86}, X_{1126}, X_{1171}	
X_{193}	X_{25}, X_{193}, X_{371}, X_{372}, X_{2362}	**K233**
X_{298}	X_{298}, X_{2981}	
X_{323}	X_{30}, X_{323}, X_{2986}	
X_{325}	X_{325}, X_{2065}, X_{2987}	
X_{385}	X_{98}, X_{237}, X_{248}, X_{385}, X_{1687}, X_{1688}, X_{1976}	**K380**
X_{394}	X_{20}, X_{394}, X_{801}	
X_{491}	X_{491}, X_{589}	
X_{492}	X_{492}, X_{588}	
X_{524}	X_{23}, X_{111}, X_{524}, X_{671}, X_{895}	**K273**
X_{1270}	X_{493}, X_{1270}	
X_{1271}	X_{494}, X_{1271}	
X_{1611}	X_{439}, X_{1611}	
X_{1654}	X_{10}, X_{42}, X_{71}, X_{199}, X_{1654}	
X_{1992}	X_{598}, X_{1383}, X_{1992}, X_{1995}	**K283**
X_{1993}	X_{54}, X_{275}, X_{1993}	
X_{1994}	X_5, X_{1166}, X_{1994}	
X_{2287}	X_{1817}, X_{2287}	
X_{2895}	X_{37}, X_{72}, X_{321}, X_{2895}, X_{2915}	
X_{3051}	X_{384}, X_{3051}	
atX_{1916}	X_{39}, X_{256}, X_{291}, X_{511}, X_{694}, X_{1432}, X_{1916}	**K354**

Table 2. $\mathcal{K}_{\text{pivot}}$ with P on the line GK

cubics, we obtain a family of pivotal cubics meeting the circumcircle at the same points as the Thomson cubic. See Table 3 and Figure 11.

With $P = X_{69}$ (isotomic conjugate of H), we obtain several interesting cubics related to the centroid G=gc$P$, the circumcenter O=gt$P$. $\mathcal{K}_{\text{pole}}$ is **K177**, $\mathcal{K}_{\text{pivot}}$ is **K141** and the cubics p$\mathcal{K}(X_2, X_{76})$ = **K141**, p$\mathcal{K}(X_3, X_2)$ = **K168**, p$\mathcal{K}(X_6, X_{69})$ = **K169**, p$\mathcal{K}(X_{32}, X_{22})$ = **K174**, p$\mathcal{K}(X_{206}, X_6)$ have the same common points on the circumcircle.

π	ω (X_i or SEARCH)	cubic or X_i on the cubic
X_1	X_{41}	X_1, X_6, X_9, X_{55}, X_{259}
X_2	X_6	**K002**
X_3	X_{32}	**K172**
X_4	0.1732184721703	X_4, X_6, X_{20}, X_{25}, X_{154}, X_{1249}
X_6	X_{184}	**K167**
X_9	X_{31}	X_1, X_6, X_9, X_{56}, X_{84}, X_{165}, X_{198}, X_{365}
X_{57}	X_{2199}	X_6, X_{40}, X_{56}, X_{57}, X_{198}, X_{223}
X_{223}	X_{604}	X_6, X_{57}, X_{223}, X_{266}, X_{1035}, X_{1436}
X_{282}	0.3666241407629	X_6, X_{282}, X_{1035}, X_{1436}, X_{1490}
X_{1073}	0.6990940852287	X_6, X_{64}, X_{1033}, X_{1073}, X_{1498}
X_{1249}	X_{25}	X_4, X_6, X_{64}, X_{1033}, X_{1249}

Table 3. Thomson cubic **K002** and some related cubics

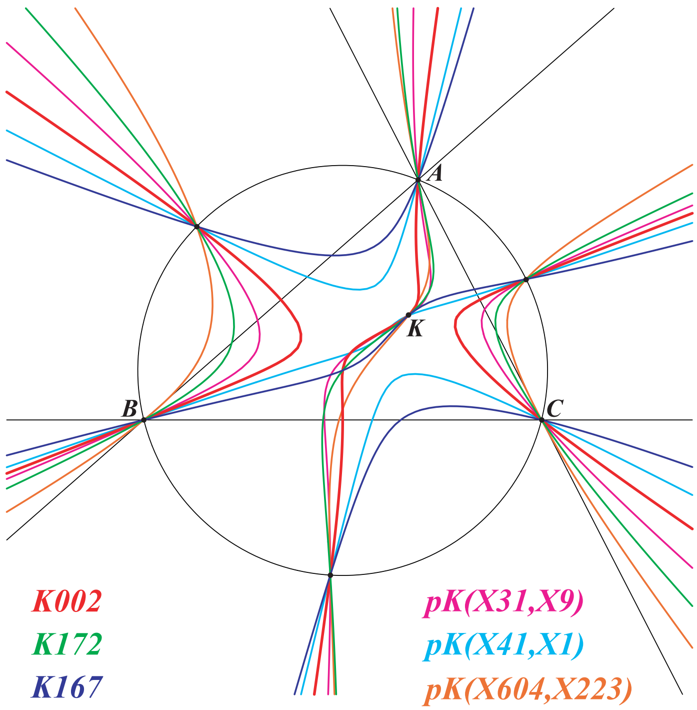


Figure 11. Thomson cubic **K002** and some related cubics

4. Non isogonal pivotal cubics and concurrent tangents

We now generalize Theorem 7 for any pivotal cubic with pole $\Omega = p : q : r$ and pivot $P = u : v : w$, meeting the circumcircle at A, B, C and three other points Q_1, Q_2, Q_3. We obtain the two following theorems.

Theorem 15. *For a given pole Ω, the tangents at Q_1, Q_2, Q_3 to the pivotal cubic with pole Ω are concurrent if and only if its pivot P lies on the quintic $\mathcal{Q}(\Omega)$.*

Remark. $\mathcal{Q}(\Omega)$ contains the following points:
– A, B, C which are nodes,
– the square roots of Ω,
– tgΩ, the Ω-isoconjugate of K,
– the vertices of the cevian triangle of $Z = \left(\frac{(c^4pq+b^4rp-a^4qr)p}{a^2} : \cdots : \cdots\right)$, the isoconjugate of the crossconjugate of K and tgΩ in the isoconjugation with fixed point tgΩ,
– the common points of the circumcircle and the trilinear polar Δ_1 of tgΩ,
– the common points of the circumcircle and the line Δ_2 passing through tgΩ and the cross-conjugate of K and tgΩ.

Theorem 16. *For a given pivot P, the tangents at Q_1, Q_2, Q_3 to the pivotal cubic with pivot P are concurrent if and only if its pole Ω lies on the quintic $\mathcal{Q}'(P)$.*

Remark. $\mathcal{Q}'(P)$ contains the following points:
– the barycentric product $P \times K$,
– A, B, C which are nodes, the tangents being the cevian lines of X_{32} and the sidelines of the anticevian triangle of $P \times K$,
– the barycentric square P^2 of P and the vertices of its cevian triangle, the tangent at P^2 passing through $P \times K$.

5. Equilateral triangles

The McCay cubic meets the circumcircle at A, B, C and three other points N_a, N_b, N_c which are the vertices of an equilateral triangle. In this section, we characterize all the pivotal cubics $\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$ having the same property.

We know that the isogonal conjugates of three such points N_a, N_b, N_c are the infinite points of an equilateral cubic (a $\mathcal{K}_{60}$, see [2]) and that the isogonal transform of $\mathcal{K}$ is another pivotal cubic $\mathcal{K}' = \mathrm{p}\mathcal{K}(\Omega', P')$ with pole Ω' the X_{32}-isocongate of Ω, with pivot P' the barycentric product of P and the isogonal conjugate of Ω. Hence $\mathcal{K}$ meets the circumcircle at the vertices of an equilateral triangle if and only if $\mathcal{K}'$ is a $\mathrm{p}\mathcal{K}_{60}$.

Following [2, §6.2], we obtain the following theorem.

Theorem 17. *For a given pole Ω or a given pivot P, there is one and only one pivotal cubic $\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$ meeting the circumcircle at the vertices of an equilateral triangle.*

With $\Omega = K$ (or $P = O$) we obviously obtain the McCay cubic and the equilateral triangle is the circumnormal triangle. More generally, a $\mathrm{p}\mathcal{K}$ meets the circumcircle at the vertices of circumnormal triangle if and only if its pole Ω lies on the circum-cubic **K378** passing through K, the vertices of the cevian triangle of the Kosnita point X_{54}, the isogonal conjugates of X_{324}, X_{343}. The tangents at A, B, C are the cevians of X_{32}. The cubic is tangent at K to the Brocard axis and K is a flex on the cubic. See [3] and Figure 12.

The locus of pivots of these same cubics is **K361**. See [3] and Figure 13.

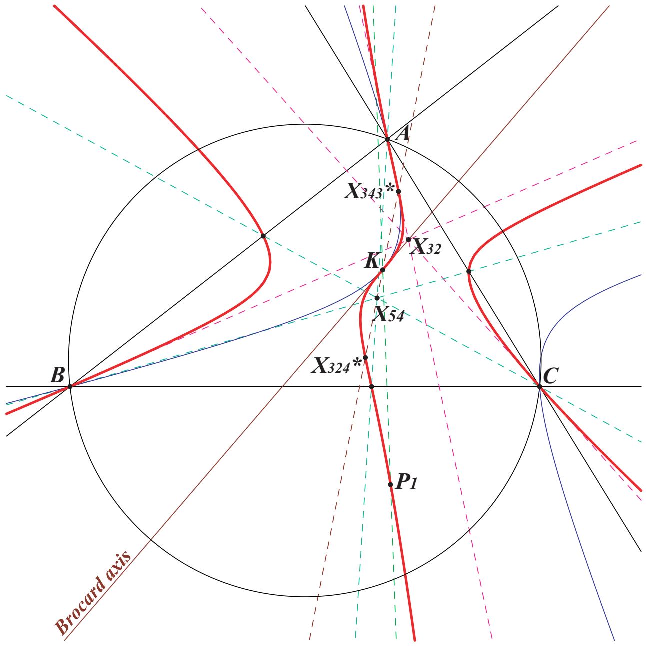


Figure 12. **K378**, the locus of poles of circumnormal p$\mathcal{K}$s

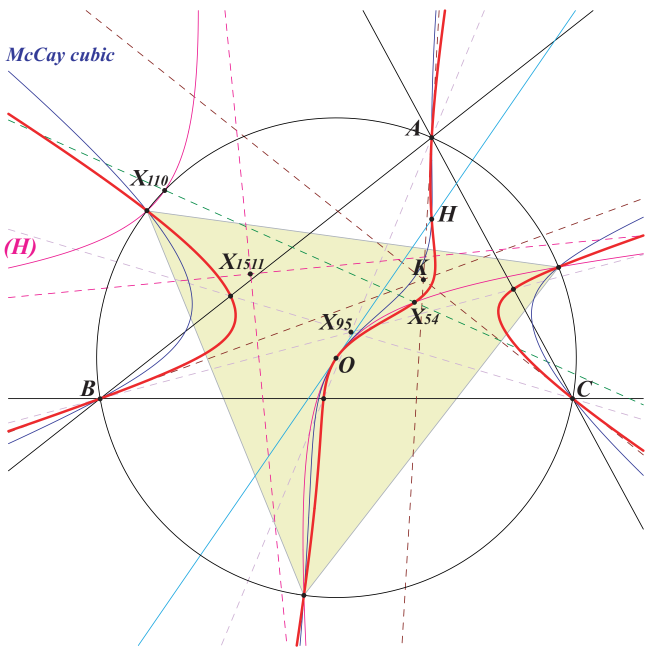


Figure 13. **K361**, the locus of pivots of circumnormal p$\mathcal{K}$s

References

[1] H. M. Cundy and C. F. Parry, Some cubic curves associated with a triangle, *Journal of Geometry*, 53 (1995) 41–66.
[2] J.-P. Ehrmann and B. Gibert, *Special Isocubics in the Triangle Plane*, available at http://perso.orange.fr/bernard.gibert/
[3] B. Gibert, *Cubics in the Triangle Plane*, available at http://perso.orange.fr/bernard.gibert/
[4] A. Haarbleicher, Cubiques auto-inverses isogonales par rapport à un triangle, *Annales de la faculté des sciences de Toulouse, 4ème série*, 4 (1940) 65–96.
[5] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.
[6] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France
E-mail address: bg42@orange.fr

Forum Geometricorum
Volume 7 (2007) 231–247.

FORUM GEOM
ISSN 1534-1178

On a Construction of Hagge

Christopher J. Bradley and Geoff C. Smith

Abstract. In 1907 Hagge constructed a circle associated with each cevian point P of triangle ABC. If P is on the circumcircle this circle degenerates to a straight line through the orthocenter which is parallel to the Wallace-Simson line of P. We give a new proof of Hagge's result by a method based on reflections. We introduce an axis associated with the construction, and (via an areal analysis) a conic which generalizes the nine-point circle. The precise locus of the orthocenter in a Brocard porism is identified by using Hagge's theorem as a tool. Other natural loci associated with Hagge's construction are discussed.

1. Introduction

One hundred years ago, Karl Hagge wrote an article in *Zeitschrift für Mathematischen und Naturwissenschaftliche Unterricht* entitled (in loose translation) "The Fuhrmann and Brocard circles as special cases of a general circle construction" [5]. In this paper he managed to find an elegant extension of the Wallace-Simson theorem when the generating point is not on the circumcircle. Instead of creating a line, one makes a circle through seven important points. In §2 we give a new proof of the correctness of Hagge's construction, extend and apply the idea in various ways. As a tribute to Hagge's beautiful insight, we present this work as a centenary celebration. Note that the name Hagge is also associated with other circles [6], but here we refer only to the construction just described. Here we present new synthetic arguments to justify Hagge's construction, but the first author has also performed detailed areal calculations which provide an algebraic alternative in [2].

The triangle ABC has circumcircle Γ, circumcenter O and orthocenter H. See Figure 1. Choose P a point in the plane of ABC. The cevian lines AP, BP, CP meet Γ again at D, E and F respectively. Reflect D in BC to a point U, E in CA to a point V and F in AB to a point W. Let UP meet AH at X, VP meet BH at Y and WP meet CH at Z. Hagge proved that there is a circle passing through X, Y, Z, U, V, W and H [5, 7]. See Figure 1. Our purpose is to amplify this observation.

Hagge explicitly notes [5] the similarities between ABC and XYZ, between DEF and UVW, and the fact that both pairs of triangles ABC, DEF and XYZ, UVW are in perspective through P. There is an indirect similarity which carries the points $ABCDEFP$ to $XYZUVWP$.

Peiser [8] later proved that the center $h(P)$ of this Hagge circle is the rotation through π about the nine-point center of ABC of the isogonal conjugate P^* of P. His proof was by complex numbers, but we have found a direct proof by classical

Publication Date: December 18, 2007. Communicating Editor: Paul Yiu.

We thank the editor Paul Yiu for very helpful suggestions which improved the development of Section 5.

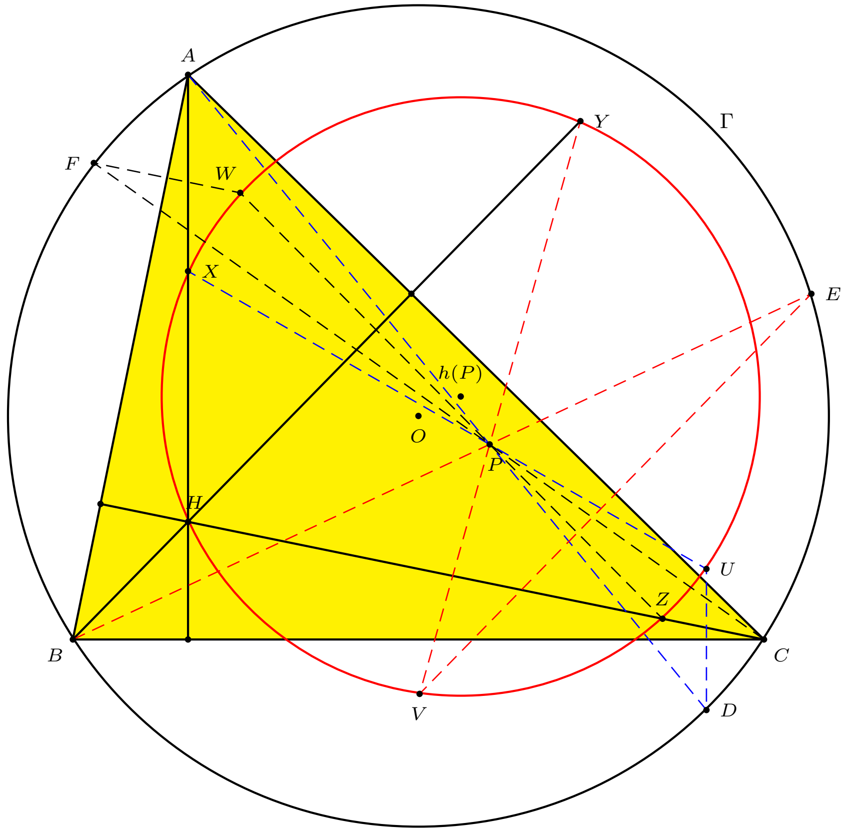


Figure 1. The Hagge construction

means [4]. In our proof of the validity of Hagge's construction we work directly with the center of the circle, whereas Hagge worked with the point at the far end of the diameter through H. This gives us the advantage of being able to study the distribution of points on a Hagge circle by means of reflections in lines through its center, a device which was not available with the original approach.

The point P^* is collinear with G and T, the far end of the diameter from H. The vector argument which justifies this is given at the start of §5.1. Indeed, we show that $P^*G : GT = 1 : 2$.

There are many important special cases. Here are some examples, but Hagge [5] listed even more.

(i) When $P = K$, the symmedian point, the Hagge circle is the orthocentroidal circle. [1]

(ii) When $P = I$, the incenter, the Hagge circle is the Fuhrmann circle.

(iii) When $P = O$, the circumcenter, the Hagge circle and the circumcircle are concentric.

---

[1]In [5] Hagge associates the name Böklen with the study of this circle (there were two geometers with this name active at around that time), and refers the reader to a work of Prof Dr Lieber, possibly H. Lieber who wrote extensively on advanced elementary mathematics in the *fin de siècle*.

(iv) When $P = H$, the orthocenter, the Hagge circle degenerates to the point H.

(v) The circumcenter is the orthocenter of the medial triangle, and the Brocard circle on diameter OK arises as a Hagge circle of the medial triangle with respect to the centroid G of ABC.

Note that UH is the doubled Wallace-Simson line of D, by which we mean the enlargement of the Wallace-Simson line with scale factor 2 from center D. Similarly VH and WH are the doubled Wallace-Simson lines of E and F. Now it is well known that the angle between two Wallace-Simson lines is half the angle subtended at O by the generating points. This applies equally well to doubled Wallace-Simson lines. A careful analysis (taking care to distinguish between angles and their supplements) will yield the angles between UH, VH and WH, from which it can be deduced that UVW is indirectly similar to DEF. We will not explain the details but rather we present a robust argument for Proposition 2 which does not rely on scrupulous bookkeeping.

Incidentally, if P is on Γ, then the Hagge circle degenerates to the doubled Wallace-Simson line of P. For the rest of this paper, we make the explicit assumption that P is not on Γ. The work described in the rest of this introduction is not foreshadowed in [5]. Since $ABCDEFP$ is similar to $XYZUVWP$, it follows that ABC is indirectly similar to XYZ and the similarity sends DEF to UVW. The point P turns out to be the unique fixed point of this similarity. This similarity must carry a distinguished point H^+ on Γ to H. We will give a geometric recipe for locating H^+ in Proposition 3.

This process admits of extension both inwards and outwards. One may construct the Hagge circle of XYZ with respect to P, or find the triangle RST so that the Hagge circle of RST with respect to P is Γ (with ABC playing the former role of XYZ). The composition of two of these indirect similarities is an enlargement with positive scale factor from P.

Proposition 2 sheds light on some of our earlier work [3]. Let G be the centroid, K the symmedian point, and ω the Brocard angle of triangle ABC. Also, let J be the center of the orthocentroidal circle (the circle on diameter GH). We have long been intrigued by the fact that $\dfrac{OK^2}{R^2} = \dfrac{JK^2}{JG^2}$ since areal algebra can be used to show that each quantity is $1-3\tan^2\omega$. In §3.3 we will explain how the similarity is a geometric explanation of this suggestive algebraic coincidence. In [3] we showed how to construct the sides of (non-equilateral) triangle ABC given only the data O, G, K. The method was based on finding a cubic which had a^2, b^2, c^2 as roots. We will present an improved algebraic explanation in §3.2.

We show in Proposition 4 that there is a point F which when used as a cevian point, generates the same Hagge circle for every triangle in a Brocard porism. Thus the locus of the orthocenter in a Brocard porism must be confined to a circle. We describe its center and radius. We also exhibit a point which gives rise to a fixed Hagge circle with respect to the medial triangles, as the reference triangle ranges over a Brocard porism.

We make more observations about Hagge's configuration. Given the large number of points lying on conics (circles), it is not surprising that Pascal's hexagon theorem comes into play. Let VW meet AH at L, WU meet BH at M, and UV meets CH at N. In §4 we will show that $LMNP$ are collinear, and we introduce the term Hagge axis for this line.

In §5 we will exhibit a *midpoint conic* which passes through six points associated with the Hagge construction. In special case (iv), when $P = H$, this conic is the nine-point circle of ABC. Drawings lead us to conjecture that the center of the midpoint conic is N.

In §6 we study some natural loci associated with Hagge's construction.

2. The Hagge Similarity

We first locate the center of the Hagge circle, but not, as Peiser [8] did, by using complex numbers. A more leisurely exposition of the next result appears in [4].

Proposition 1. *Given a point P in the plane of triangle ABC, the center $h(P)$ of the Hagge circle associated with P is the point such the nine-point center N is the midpoint of $h(P)P^*$ where P^* denotes the isogonal conjugate of P.*

Proof. Let AP meet the circumcircle at D, and reflect D in BC to the point U. The line UH is the doubled Simson line of D, and the reflections of D in the other two sides are also on this line. The isogonal conjugate of D is well known to be the point at infinity in the direction parallel to AP^* . (This is the degenerate case of the result that if D' is not on the circumcircle, then the isogonal conjugate of D' is the center of the circumcircle of the triangle with vertices the reflections of D' in the sides of ABC).

Thus $UH \perp AP^*$. To finish the proof it suffices to show that if OU' is the rotation through π of UH about N, then AP^* is the perpendicular bisector of OU'. However, $AO = R$ so it is enough to show that $AU' = R$. Let A' denote the rotation through π of A about N. From the theory of the nine-point circle it follows that A' is also the reflection of O in BC. Therefore $OUDA'$ is an isosceles trapezium with $OA'//UD$. Therefore $AU' = A'U = OD = R$. □

We are now in a position to prove what we call the Hagge similarity which is the essence of the construction [5].

Proposition 2. *The triangle ABC has circumcircle Γ, circumcenter O and orthocenter H. Choose a point P in the plane of ABC other than A, B, C. The cevian lines AP, BP, CP meet Γ again at D, E, F respectively. Reflect D in BC to a point U, E in CA to a point V and F in AB to a point W. Let UP meet AH at X, VP meet BH at Y and WP meet CH at Z. The points $XYZUVWH$ are concyclic, and there is an indirect similarity carrying $ABCDEFP$ to $XYZUVWP$.*

Discussion. The strategy of the proof is as follows. We consider six lines meeting at a point. Any point of the plane will have reflections in the six lines which are concyclic. The angles between the lines will be arranged so that there is an indirect similarity carrying $ABCDEF$ to the reflections of H in the six lines. The location of the point of concurrency of the six lines will be chosen so that the relevant six reflections of H are $UVWX_1Y_1Z_1$ where X_1, Y_1 and Z_1 are to be determined, but are placed on the appropriate altitudes so that they are candidates to become X, Y and Z respectively. The similarity then ensures that UVW and $X_1Y_1Z_1$ are in perspective from a point P'. Finally we show that $P = P'$, and it follows immediately that $X = X_1, Y = Y_1$ and $Z = Z_1$. We rely on the fact that we know where to make the six lines cross, thanks to Proposition 1. This is not the proof given in [5].

Proof of Proposition 2. Let $\angle DAC = a_1$ and $\angle BAD = a_2$. Similarly we define b_1, b_2, c_1 and c_2. We deduce that the angles subtended by A, F, B, D, D and E at O as shown in Figure 2.

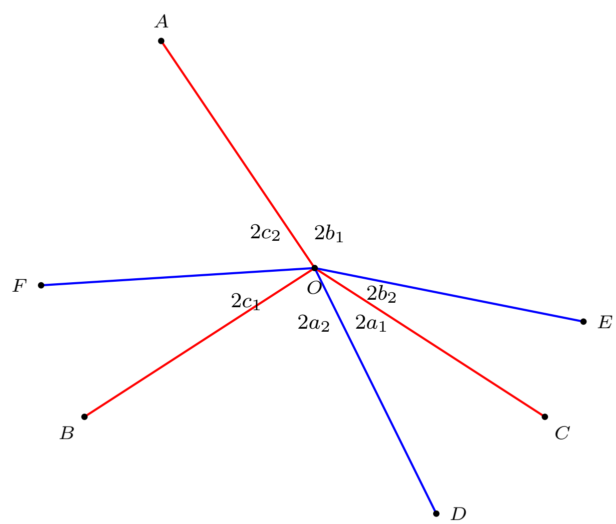


Figure 2. Angles subtended at the circumcenter of ABC

By Proposition 1, $h(P)$ is on the perpendicular bisector of UH which is parallel to AP^* (and similar results by cyclic change).

Draw three lines through $h(P)$ which are parallel to the sides of ABC and three more lines which are parallel to AP^* , BP^* and CP^* . See Figure 3.

Let X_1, Y_1 and Z_1 be the reflections of H in the lines parallel to BC, CA and AB respectively. Also U, V and W are the reflections of H in the lines parallel to AP^*, BP^* and CP^*. Thus $X_1Y_1Z_1UVW$ are all points on the Hagge circle. The angles between the lines are as shown, and the consequences for the six reflections of H are that $X_1Y_1Z_1UVW$ is a collection of points which are indirectly similar to $ABCDEF$. It is not necessary to know the location of H in Figure 3 to deduce this result. Just compare Figures 2 and 4. The point is that $\angle X_1h(P)V = \angle EOA$.

A similar argument works for each adjacent pair of vertices in the cyclic list $X_1VZ_1UY_1W$ and an indirect similarity is established. Let this similarity carrying

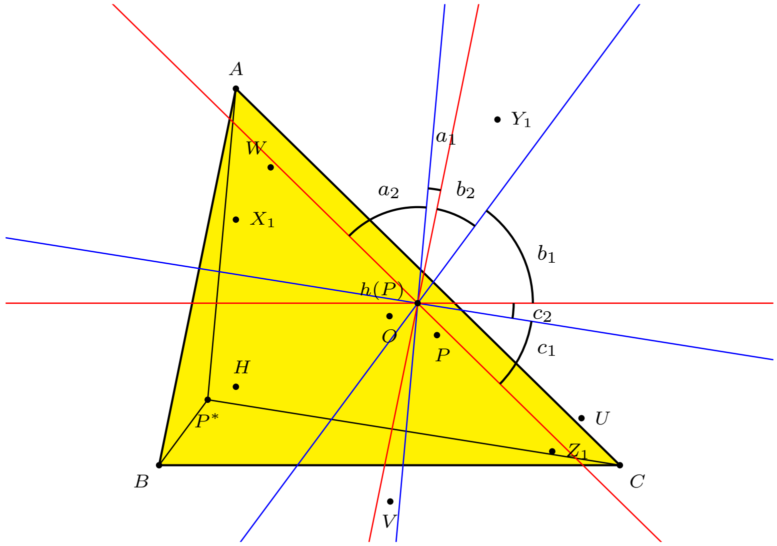


Figure 3. Reflections of the orthocenter

$ABCDEF$ to $X_1Y_1Z_1UVW$ be κ. It remains to show that $\kappa(P) = P$ (for then it will follow immediately that $X_1 = X$, $Y_1 = Y$ and $Z_1 = Z$).

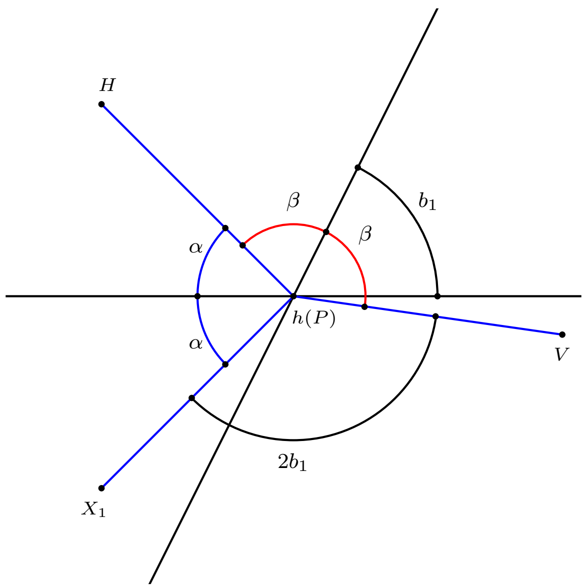


Figure 4. Two reflections of H

Now $X_1Y_1Z_1$ is similar to ABC, and the vertices of $X_1Y_1Z_1$ are on the altitudes of ABC. Also UVW is similar to DEF, and the lines X_1U, Y_1V and Z_1W are concurrent at a point P_1. Consider the directed line segments AD and X_1U which meet at Q. The lines AX_1 and UD are parallel so AX_1Q and DUQ are similar triangles, so in terms of lengths, $AQ : QD = X_1Q : QU$. Since κ carries AD to X_1U, it follows that Q is a fixed point of κ. Now if κ had at least two fixed points, then it would have a line of fixed points, and would be a reflection in that line. However κ takes DEF to UVW, to this line would have to be BC, CA and AB. This is absurd, so Q is the unique fixed point of κ. By cyclic change Q is on AD, BE and CF so $Q = P$. Also Q is on X_1U, Y_1V and Z_1W so $Q = P_1$. Thus X_1U, Y_1V and Z_1W concur at P. Therefore $X_1 = X$, $Y_1 = Y$ and $Z_1 = Z$. □

Proposition 3. *The similarity of Proposition 2 applied to ABC, P carries a point H^+ on Γ to H. The same result applied to XYZ, P carries H to the orthocenter H^- of XYZ. We may construct H^+ by drawing the ray PH^- to meet Γ at H^+.*

Proof. The similarity associated with ABC and P is expressible as: reflect in PA, scale by a factor of λ from P, and rotate about P through a certain angle. Note that if we repeat the process, constructing a similarity using the XYZ as the reference triangle, but still with cevian point P, the resulting similarity will be expressible as: reflect in XP, scale by a factor of λ from P, and rotate about P through a certain angle. Since $XYZP$ is indirectly similar to $ABCP$, the angles through which the rotation takes place are equal and opposite. The effect of composing the two similarities will be an enlargement with center P and (positive) scale factor λ^2. □

Thus in a natural example one would expect the point H^+ to be a natural point. Drawings indicate that when we consider the Brocard circle, H^+ is the Tarry point.

3. Implications for the Symmedian Point and Brocard geometry

3.1. *Standard formulas.* We first give a summary of useful formulas which can be found or derived from many sources, including Wolfram Mathworld [11]. The variables have their usual meanings.

$$abc = 4R\triangle, \tag{1}$$
$$a^2 + b^2 + c^2 = 4\triangle \cot\omega, \tag{2}$$
$$a^2b^2 + b^2c^2 + c^2a^2 = 4\triangle^2 \csc^2\omega, \tag{3}$$
$$a^4 + b^4 + c^4 = 8\triangle^2(\csc^2\omega - 2), \tag{4}$$

where (3) can be derived from the formula

$$\begin{aligned} R_B &= \frac{abc\sqrt{a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2}}{4(a^2 + b^2 + c^2)\triangle} \\ &= \frac{R\sqrt{1 - 4\sin^2\omega}}{2\cos\omega} \end{aligned}$$

for the radius R_B of the Brocard circle given in [11]. The square of the distance between the Brocard points was determined by Shail [9]:

$$\Omega\Omega'^2 = 4R^2 \sin^2\omega(1 - 4\sin^2\omega) \tag{5}$$

which in turn is an economical way of expressing

$$\frac{a^2b^2c^2(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)}{(a^2b^2+b^2c^2+c^2a^2)^2}.$$

We will use these formulas in impending algebraic manipulations.

3.2. *The symmedian point.* Let G be the centroid, K the symmedian point, and ω be the Brocard angle of triangle ABC. Also let J be the center of the orthocentroidal circle (the circle on diameter GH). It is an intriguing fact that

$$\frac{OK^2}{R^2} = \frac{JK^2}{JG^2} \tag{6}$$

since one can calculate that each quantity is $1 - 3\tan^2\omega$. The similarity of Proposition 2 explains this suggestive algebraic coincidence via the following paragraph.

We first elaborate on Remark (v) of §1. Let h_{med} denote the function which assigns to a point P the center $h_{\text{med}}(P)$ of the Hagge circle associated with P when the triangle of reference is the medial triangle. The medial triangle is the enlargement of ABC from G with scale factor $-\frac{1}{2}$. Let K_{med} be the symmedian point of the medial triangle. Now K_{med}, G, K are collinear and $K_{\text{med}}G : GK = 1 : 2 = QG : GN$, where Q is the midpoint of ON. Thus, triangle GNK and GQK_{med} are similar and Q is the nine-point center of the medial triangle. By [8], $h\text{med}(G)$ is the reflection in Q of K_{med}. But the line $Qh_{\text{med}}(G)$ is parallel to NK and Q is the midpoint of ON. Therefore, $h_{\text{med}}(G)$ is the midpoint of OK, and so is the center of the Brocard circle of ABC. The similarity of Proposition 2 and the one between the reference and medial triangle, serve to explain (6).

3.3. *The Brocard porism.* A Brocard porism is obtained in the following way. Take a triangle ABC and its circumcircle. Draw cevian lines through the symmedian point. There is a unique conic (the Brocard ellipse) which is tangent to the sides where the cevians cuts the sides. The Brocard points are the foci of the ellipse. There are infinitely many triangle with this circumcircle and this inconic. Indeed, every point of the circumcircle arises as a vertex of a unique such triangle.

These poristic triangles have the same circumcenter, symmedian point, Brocard points and Brocard angle. For each of them, the inconic is their Brocard ellipse. Any geometrical feature of the triangle which can be expressed exclusively in terms of R, ω and the locations of O and K will give rise to a conserved quantity among the poristic triangles.

This point of view also allows an improved version of the algebraic proof that a, b and c are determined by O, G and K [3]. Because of the ratios on the Euler line, the orthocenter H and the orthocentroidal center are determined. Now Equation (6) determines R and angle ω. However, $9R^2 - (a^2+b^2+c^2) = OH^2$ so $a^2+b^2+c^2$ is determined. Also the area $\triangle$ of ABC is determined by (2). Now (1) means abc and so $a^2b^2c^2$ is determined. Also, (3) determines $a^2b^2+b^2c^2+c^2a^2$. Thus the polynomial $(X-a^2)(X-b^2)(X-c^2)$ is determined and so the sides of the triangle can be deduced.

As we move through triangles in a Brocard porism using a fixed cevian point P, the Hagge circles of the triangles vary in general, but if P is chosen appropriately, the Hagge circle if each triangle in the porism is the same.

Proposition 4. *Let F be the fourth power point* [2] *of a triangle in a Brocard porism, so that it has areal coordinates (a^4, b^4, c^4). The fourth power point F is the same point for all triangles in the porism. Moreover, when $P = F$, the Hagge circle of each triangle is the same.*

Proof. Our plan is to show that the point $h(F)$ is the same for all triangles in the porism, and then to show that the distance $h(F)H$ is also constant (though the orthocenters H vary). Recall that the nine-point center is the midpoint of O and H, and of F^* and $h(P)$. Thus there is a (variable) parallelogram $Oh(F)HF^*$ which will prove very useful.

The fourth power point F is well known to lie on the Brocard axis where the tangents to the Brocard circle at Ω and Ω' meet. Thus F is the same point for all triangles in the Brocard porism. The isogonal conjugate of F (incidentally the isotomic conjugate of the symmedian point) is $F^* = K_t = \left(\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}\right)$.

In any triangle OK is parallel to F^*H. To see this, note that OK has equation

$$b^2c^2(b^2 - c^2)x + c^2a^2(c^2 - a^2)y + a^2b^2(a^2 - b^2)z = 0.$$

Also F^*H has equation

$$\sum_{\text{cyclic}} b^2c^2(b^2 - c^2)(b^2 + c^2 - a^2)x = 0.$$

These equations are linearly dependent with $x + y + z = 0$ and hence the lines are parallel. (DERIVE confirms that the 3×3 determinant vanishes). In a Hagge circle with $P = F$, $P^* = F^*$ and $F^*Hh(F)O$ is a parallelogram. Thus OK is parallel to F^*H and because of the parallelogram, $h(F)$ is a (possibly variable) point on the Brocard axis OK.

Next we show that the point $h(F)$ us a common point for the poristic triangles. The first component of the normalized coordinates of F^* and H are

$$F^*_x = \frac{b^2c^2}{a^2b^2 + b^2c^2 + c^2a^2}$$

and

$$H_x = \frac{(a^2 + b^2 - c^2)(c^2 + a^2 - b^2)}{16\triangle^2}$$

where $\triangle$ is the area of the triangle in question. The components of the displacement F^*H are therefore

$$\frac{a^2 + b^2 + c^2}{16\triangle^2}(a^2b^2 + b^2c^2 + c^2a^2)(x, y, z)$$

[2] Geometers who speak trilinear rather than areal are apt to call F the third power point for obvious reasons.

where $x = a^2(a^2b^2 + a^2c^2 - b^4 - c^4)$, with y and z found by cyclic change of a, b, c. Using the areal distance formula this provides

$$F^*H^2 = \frac{a^2b^2c^2(a^2+b^2+c^2)^2(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)}{16\triangle^2(a^2b^2+b^2c^2+c^2a^2)^2}.$$

Using the formulas of §3.1 we see that

$$Oh(F) = F^*H = 2R\cos\omega\sqrt{1-4\sin^2\omega}$$

is constant for the poristic triangles. The point O is fixed so there are just two candidates for the location of $h(F)$ on the common Brocard axis. By continuity $h(F)$ cannot move between these places and so $h(F)$ is a fixed point.

To finish this analysis we must show that the distance $h(F)H$ is constant for the poristic triangles. This distance is the same as F^*O by the parallelogram. If a point X has good areal coordinates, it is often easy to find a formula for OX^2 using the generalized parallel axis theorem [10] because $OX^2 = R^2 - \sigma_X^2$ and σ_X^2 denotes the mean square distance of the triangle vertices from themselves, given that they carry weights which are the corresponding areal coordinates of X.

In our case $F^* = (a^{-2},\ b^{-2},\ c^{-2})$, so

$$\begin{aligned}\sigma_{F^*}^2 &= \frac{1}{(a^{-2}+b^{-2}+c^{-2})^2}(a^2b^{-2}c^{-2} + a^{-2}b^2c^{-2} + a^{-2}b^{-2}c^2)\\ &= \frac{a^2b^2c^2}{a^2b^2+b^2c^2+c^2a^2}(a^4+b^4+c^4).\end{aligned}$$

This can be tidied using the standard formulas to show that $F^*O = R(1-4\sin^2\omega)$. The distance $Hh(F) = F^*O$ is constant for the poristic triangles and $h(F)$ is a fixed point, so the Hagge circle associated with F is the same for all the poristic triangles. $\square$

Corollary 5. *In a Brocard porism, as the poristic triangles vary, the locus of their orthocenters is contained in a circle with their common center $h(F)$ on the Brocard axis, where F is the* (areal) *fourth power point of the triangles. The radius of this circle is $R(1-4\sin^2\omega)$.*

In fact there is a direct method to show that the locus of H in the Brocard porism is a subset of a circle, but this approach reveals neither center nor radius. We have already observed that $\dfrac{JK^2}{JG^2} = 1 - 3\tan^2\omega$ so for triangles in a Brocard porism (with common O and K) we have $\dfrac{JK^2}{JO^2} = \dfrac{1-3\tan^2\omega}{4}$ is constant. So as you consider the various triangle in the porism, J is constrained to move on a circle of Apollonius with center some point on the fixed line OK. Now the vector **OH** is $\frac{2}{3}$**OJ**, so H is constrained to move on a circle with its center M on the line OK. In fact H can occupy any position on this circle but we do not need this result (which follows from K ranging over a circle center J for triangles in a Brocard porism [3]).

There is a point which, when used as P for the Hagge construction using medial triangles, gives rise to a common Hagge circle as we range over reference triangles in a Brocard porism. We use dashes to indicate the names of points with respect to the medial triangle $A'B'C'$ of a poristic triangle ABC. We now know that F is a common point for the porism, so the distance OF is fixed. Since O is fixed in the Brocard porism and the locus of H is a circle, it follows that the locus of N is a circle with center half way between O and the center of the locus of H.

Proposition 6. *Let P be the center of the Brocard ellipse (the midpoint of the segment joining the Brocard points of ABC). When the Hagge construction is made for the medial triangle $A'B'C'$ using this point P, then for each ABC in the porism, the Hagge circle is the same.*

Proof. If the areal coordinates of a point are (l, m, n) with respect to ABC, then the areal coordinates of this point with respect to the medial triangle are $(m + n - l, n + l - m, l + n - m)$. The reference areals of P are $(a^2(b^2 + c^2), b^2(c^2 + a^2), c^2(a^2 + b^2))$ so the medial areals are (b^2c^2, c^2a^2, a^2b^2). The medial areals of the medial isogonal conjugate $P^\dagger$ of P are (a^4, b^4, c^4). Now the similarity carrying ABC to $A'B'C'$ takes O to N and F to $P^\dagger$. Thus in terms of distance $OF = 2P^\dagger N$ and moreover OF is parallel to $P^\dagger N$. Now, $OP^\dagger Nh'(P)$ is a parallelogram with center the nine-point center of the medial triangle and $h'(P)$ is the center of the medial Hagge circle. It follows that $h'(P)$ lies on OK at the midpoint of OF. Therefore all triangles in the Brocard porism give rise to a Hagge circle of P (with respect to the medial triangle) which is the circle diameter OF. $\square$

Incidentally, P is the center of the locus of N in the Brocard porism. To see this, note that N is the midpoint of OH, so it suffices to show that $OP = PX$ where X is the center of the locus of H in the Brocard cycle (given that P is on the Brocard axis of ABC). However, it is well known that $OP = R\sqrt{1 - 4\sin^2\omega}$ and in Proposition 4 we showed that $OX = 2R\cos\omega\sqrt{1 - 4\sin^2\omega}$. We must eliminate the possibility that X and P are on different sides of O. If this happened, there would be at least one triangle for which $\angle HOK = \pi$. However, K is confined to the orthocentroidal disk [3] so this is impossible.

4. The Hagge axis

Proposition 7. *In the Hagge configuration, let VW meet AH at L, WU meet BH at M and UV meet CH and N. Then the points L, M, N and P are collinear.*

We prove the following more general result. In order to apply it, the letters should be interpreted in the usual manner for the Hagge configuration, and Σ should be taken as the Hagge circle.

Proposition 8. *Let three points X, Y and Z lie on a conic Σ and let l_1, l_2, l_3 be three chords XH, YH, ZH all passing through a point H on Σ. Suppose further that P is any point in the plane of Σ, and let XP, YP, ZP meet Σ again at U, V and W respectively. Now, let VW meet l_1 at L, WU meet l_2 at M, UV meet l_3 at N. Then LMN is a straight line passing through P.*

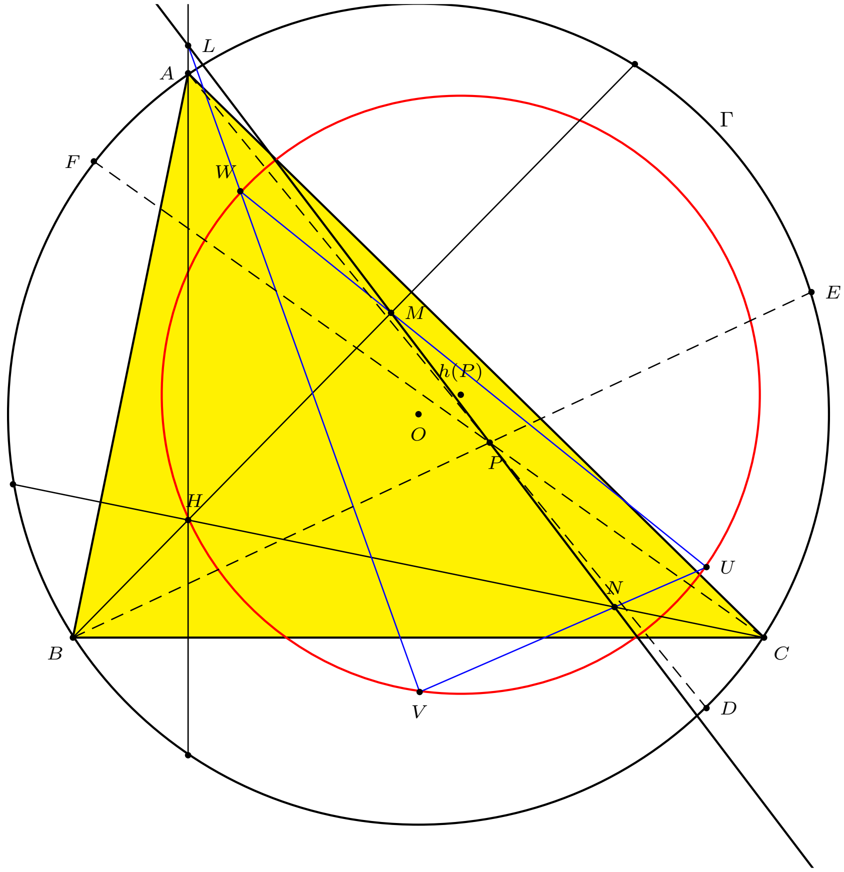


Figure 5. The Hagge axis LMN

Proof. Consider the hexagon $HYVUWZ$ inscribed in Σ. Apply Pascal's hexagon theorem. It follows that M, P, N are collinear. By taking another hexagon N, P, L are collinear. $\square$

5. The Hagge configuration and associated Conics

In this section we give an analysis of the Hagge configuration using barycentric (areal) coordinates. This is both an enterprise in its own right, serving to confirm the earlier synthetic work, but also reveals the existence of an interesting sequence of conics. In what follows ABC is the reference triangle and we take P to have homogeneous barycentric coordinates (u, v, w). The algebra computer package DERIVE is used throughout the calculations.

5.1. *The Hagge circle and the Hagge axis* . The equation of AP is $wy = vz$. This meets the circumcircle, with equation $a^2yz+b^2zx+c^2xy = 0$, at the point D with coordinates $(-a^2vw, v(b^2w + c^2v), w(b^2w + c^2v))$. Note that the sum of these coordinates is $-a^2vw + v(b^2w + c^2v) + w(b^2w + c^2v))$. We now want to find the coordinates of $U(l,m,n)$, the reflection of D in the side BC. It is convenient to take the normalization of D to be the same as that of U so that

$$l+m+n=-a^2vw+v(b^2w+c^2v)+w(b^2w+c^2v)). \tag{7}$$

In order that the midpoint of UD lies on BC the requirement is that $l=a^2vw$. There is also the condition that the displacements $BC(0,-1,1)$ and $UD(-a^2vw-l, v(b^2w+c^2v)-m, w(b^2w+c^2v)-n)$ should be at right angles. The condition for perpendicular displacements may be found in [1, p.180]. When these conditions are taken into account we find the coordinates of U are

$$(l,m,n)=(a^2vw, v(c^2(v+w)-a^2w), w(b^2(v+w)-a^2v)). \tag{8}$$

The coordinates of E, F, V, W can be obtained from those of D, U by cyclic permutations of a, b, c and u, v, w.

The Hagge circle is the circle through U, V, W and its equation, which may be obtained by standard means, is

$$\begin{aligned}&(a^2vw+b^2wu+c^2uv)(a^2yz+b^2zx+c^2xy)\\ -&(x+y+z)(a^2(b^2+c^2-a^2)vwx+b^2(c^2+a^2-b^2)wuy+c^2(a^2+b^2-c^2)uvz)\\ =&0.\end{aligned} \tag{9}$$

It may now be checked that this circle has the characteristic property of a Hagge circle that it passes through H, whose coordinates are

$$\left(\frac{1}{b^2+c^2-a^2}, \frac{1}{c^2+a^2-b^2}, \frac{1}{a^2+b^2-c^2}\right).$$

Now the equation of AH is $(c^2+a^2-b^2)y=(a^2+b^2-c^2)z$ and this meets the Hagge circle with Equation (9) again at the point X with coordinates $(-a^2vw+b^2wu+c^2uv, (a^2+b^2-c^2)vw, (c^2+a^2-b^2)vw)$. The coordinates of Y, Z can be obtained from those of X by cyclic permutations of a, b, c and u, v, w.

Proposition 9. *XU, YV, ZW are concurrent at P.*

This has already been proved in Proposition 2, but may be verified by checking that when the coordinates of X, U, P are placed as entries in the rows of a 3×3 determinant, then this determinant vanishes. This shows that X, U, P are collinear as are Y, V, P and Z, W, P.

If the equation of a conic is $lx^2+my^2+nz^2+2fyz+2gzx+2hxy=0$, then the first coordinate of its center is $(mn-gm-hn-f^2+fg+hf)$ and other coordinates are obtained by cyclic change of letters. This is because it is the pole of the line at infinity. The x-coordinate of the center $h(P)$ of the Hagge circle is therefore $-a^4(b^2+c^2-a^2)vw+(a^2(b^2+c^2)-(b^2-c^2)^2)(b^2wu+c^2uv)$ with y- and z-coordinates following by cyclic permutations of a, b, c and u, v, w.

In §4 we introduced the Hagge axis and we now deduce its equation. The lines VW and AH meet at the point L with coordinates

$$\begin{aligned}&(u(a^2(b^2w(u+v)(w+u-v)+c^2v(w+u)(u+v-w))+b^4w(u+v)(v+w-u)\\&\quad -b^2c^2(u^2(v+w)+u(v^2+w^2)+2vw(v+w))+c^4v(w+u)(v+w-u)),\\&vw(a^2+b^2-c^2)(a^2(u+v)(w+u)-u(b^2(u+v)+c^2(w+u))),\\&vw(c^2+a^2-b^2)(a^2(u+v)(w+u)-u(b^2(u+v)+c^2(w+u)))).\end{aligned}$$

The coordinates of M and N follow by cyclic permutations of a, b, c and u, v, w. From these we obtain the equation of the Hagge axis LMN as

$$\sum_{\text{cyclic}} vw(a^2(u+v)(w+u)-u(b^2(u+v)+c^2(w+u)))(a^2(v-w)-(b^2-c^2)(v+w))x = 0. \tag{10}$$

It may now be verified that this line passes through P.

5.2. *The midpoint Hagge conic* . We now obtain a dividend from the areal analysis in §5.1. The midpoints in question are those of AX, BY, CZ, DU, EV, FW and in Figure 6 these points are labeled X_1, Y_1, Z_1, U_1, V_1, W_1. This notation is not to be confused with the now discarded notation X_1, Y_1 and Z_1 of Proposition 2. We now show these six points lie on a conic.

Proposition 10. *The points X_1, Y_1, Z_1, U_1, V_1, W_1 lie on a conic (the Hagge midpoint conic).*

Their coordinates are easily obtained and are

$$\begin{aligned}&X_1\ (2u(b^2w+c^2v), vw(a^2+b^2-c^2), vw(c^2+a^2-b^2)),\\&U_1\ (0, v(2c^2v+w(b^2+c^2-a^2)), w(2b^2w+v(b^2+c^2-a^2))),\end{aligned}$$

with coordinates of Y_1, Z_1, V_1, W_1 following by cyclic change of letters. It may now be checked that these six points lie on the conic with equation

$$\begin{aligned}&4(a^2vw+b^2wu+c^2uv)\left(\sum_{\text{cyclic}} u^2(-a^2vw+b^2(v+w)w+c^2v(v+w))yz\right)\\&-(x+y+z)\left(\sum_{\text{cyclic}} v^2w^2((a^2+b^2-c^2)u+2a^2v)((c^2+a^2-b^2)u+2a^2w)x\right)=0.\end{aligned} \tag{11}$$

Following the same method as before for the center, we find that its coordinates are $(u(b^2w+c^2v), v(c^2u+a^2w), w(a^2v+b^2u))$.

Proposition 11. *U_1, X_1, P are collinear.*

This is proved by checking that when the coordinates of X_1, U_1, P are placed as entries in the rows of a 3×3 determinant, then this determinant vanishes. This shows that X_1, U_1, P are collinear as are Y_1, V_1, P and Z_1, W_1, P.

Proposition 12. *The center of the Hagge midpoint conic is the midpoint of $Oh(P)$. It divides P^*G in the ratio $3 : -1$.*

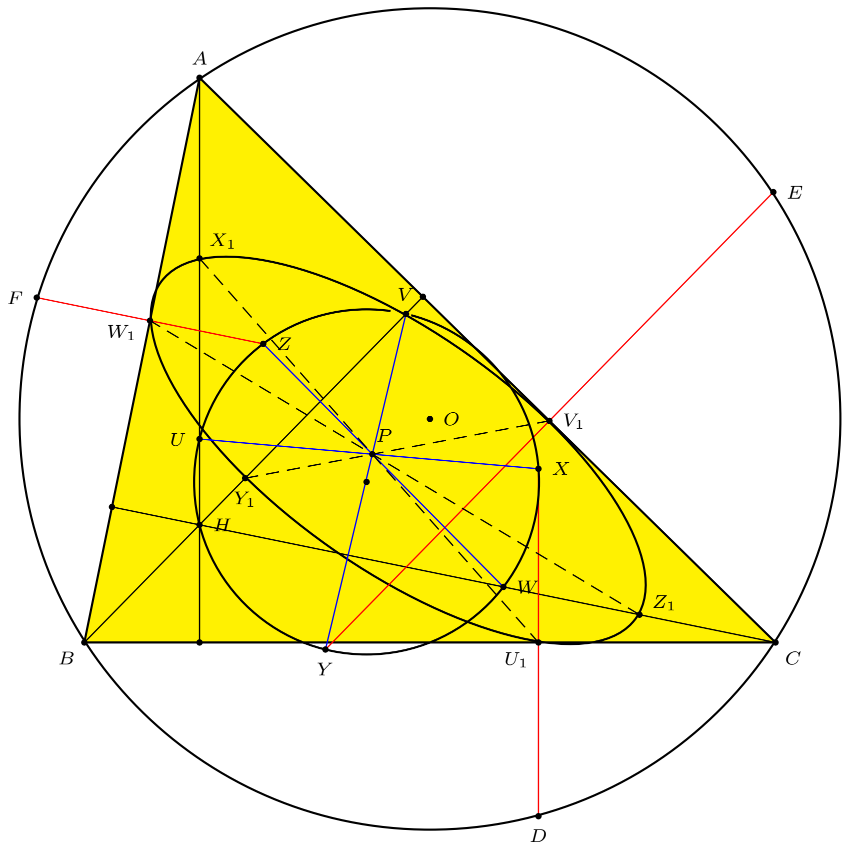


Figure 6.

The proof is straightforward and is left to the reader.

In similar fashion to above we define the six points X_k, Y_k, Z_k, U_k, V_k, W_k that divide the six lines AX, BY, CZ, DU, EV, FW respectively in the ratio $k:1$ (k real and $\neq 1$).

Proposition 13. *The six points X_k, Y_k, Z_k, U_k, V_k, W_k lie on a conic and the centers of these conics, for all values of k, lie on the line $Oh(P)$ and divide it in the ratio $k:1$.*

This proposition was originally conjectured by us on the basis of drawings by the geometry software package CABRI and we are grateful to the Editor for confirming the conjecture to be correct. We have rechecked his calculation and for the record the coordinates of X_k and U_k are

$$((1-k)a^2vw+(1+k)u(b^2w+c^2v), k(a^2+b^2-c^2)vw, k(c^2+a^2-b^2)vw),$$

and

$$(-a^2(1-k)vw, v((1+k)c^2v+(b^2+kc^2-ka^2)w), w((1+k)b^2w+(c^2+kb^2-ka^2)v)),$$

respectively. The conic involved has center with coordinates

$$((a^2(b^2+c^2-a^2)(a^2vw+b^2wu+c^2uv)$$
$$+k(-a^4(b^2+c^2-a^2)vw+(a^2(b^2+c^2)-(b^2-c^2)^2)(u(b^2w+c^2v)),$$
$$\cdots,\cdots).$$

Proposition 14. *U_k, X_k, P are collinear.*

The proof is by the same method as for Proposition 11.

6. Loci of Haggi circle centers

The Macbeath conic of ABC is the inconic with foci at the circumcenter O and the orthocenter H. The center of this conic is N, the nine-point center.

Proposition 15. *The locus of centers of those Hagge circles which are tangent to the circumcircle is the Macbeath conic.*

Proof. We address the elliptical case (see Figure 7) when ABC is acute and H is inside the circumcircle of radius R. The major axis of the Macbeath ellipse Σ is well known to have length R. Suppose that P is a point of the plane. Now $h(P)$ is on Σ if and only if $Oh(P)+h(P)H=R$, but $h(P)H$ is the radius of the Hagge circle, so this condition holds if and only if the Hagge circle is internally tangent to the circumcircle. Note that $h(P)$ is on Σ if and only if P^* is on Σ, and as P^* moves continuously round Σ, the Hagge circle moves around the inside of the circumcircle. The point P moved around the 'deltoid' shape as shown in Figure 7.

The case where ABC is obtuse and the Macbeath conic is a hyperbola is very similar. The associated Hagge circles are externally tangent to the circumcircle. □

Proposition 16. *The locus of centers of those Hagge circles which cut the circumcircle at diametrically opposite points is a straight line perpendicular to the Euler line.*

Proof. Let ABC have circumcenter O and orthocenter H. Choose H' on HO produced so that $HO\cdot OH'=R^2$ where R is the circumradius of ABC. Now if X, Y are diametrically opposite points on S (but not on the Euler line), then the circumcircle S' of XYH is of interest. By the converse of the power of a point theorem, H' lies on each S'. These circles S' form an intersecting coaxal system through H and H' and their centers lie on the perpendicular bisector of HH'. □

References

[1] C. J. Bradley, *Challenges in Geometry*, OUP, 2005.
[2] C. J. Bradley, *The Algebra of Geometry*, Highperception 2007.
[3] C. J. Bradley and G. C. Smith, The locations of triangle centers, *Forum Geom.*, 6 (2006) 57–70.
[4] C. J. Bradley and G. C. Smith, Hagge circles and isogonal conjugation, *Math. Gazette*, 91 (2007) 202–207.
[5] K. Hagge, Der Fuhrmannsche Kreis und der Brocardsche Kreis als Sonderfälle eines allgemeineren Kreises, *Zeitschrift für Math. Unterricht*, 38 (1907) 257–269.

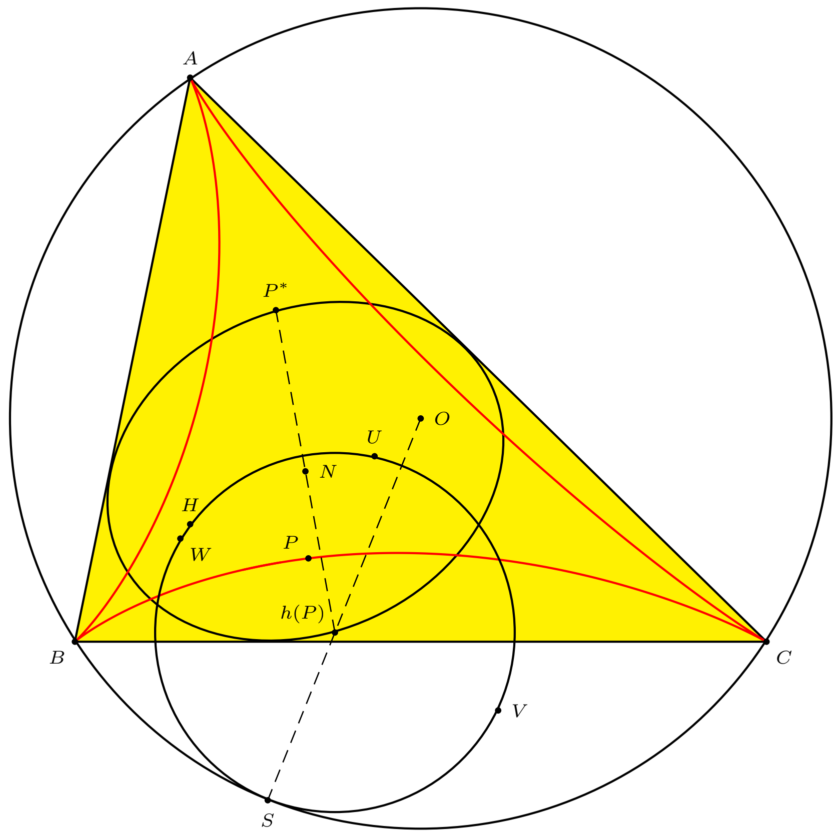


Figure 7.

[6] K. Hagge, Zur Theorie de Lemoineschen Kreise, *Zeitschrift für Math. Unterricht* 39 (1908) 337–341.

[7] R. A. Johnson, *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*, Boston, 1929; reprinted as *Advanced Euclidean Geometry*, Dover, 2007.

[8] A. M. Peiser, The Hagge circle of a triangle, *Amer. Math. Monthly*, 49 (1942) 524–527.

[9] R. Shail, Some properties of Brocard points, *Math. Gazette*. 80 (1996) 485–491.

[10] G. C. Smith, Statics and the moduli space of triangles, *Forum Geom.*, 5 (2005) 181–190.

[11] Wolfram Mathworld, http://mathworld.wolfram.com/

Christopher J. Bradley: c/o Geoff C. Smith, Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, England

Geoff C. Smith: Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, England

E-mail address: G.C.Smith@bath.ac.uk

Forum Geometricorum
Volume 7 (2007) 249–250.

FORUM GEOM
ISSN 1534-1178

Author Index

FORUM GEOMETRICORUM

A Journal on Classical Euclidean Geometry and Related Areas

published by

Department of Mathematical Sciences
Florida Atlantic University

FORUM GEOM

Volume 8
2008

http://forumgeom.fau.edu

ISSN 1534-1178

Editorial Board

Advisors:

John H. Conway	Princeton, New Jersey, USA
Julio Gonzalez Cabillon	Montevideo, Uruguay
Richard Guy	Calgary, Alberta, Canada
Clark Kimberling	Evansville, Indiana, USA
Kee Yuen Lam	Vancouver, British Columbia, Canada
Tsit Yuen Lam	Berkeley, California, USA
Fred Richman	Boca Raton, Florida, USA

Editor-in-chief:

Paul Yiu	Boca Raton, Florida, USA

Editors:

Nikolaos Dergiades	Thessaloniki, Greece
Clayton Dodge	Orono, Maine, USA
Roland Eddy	St. John’s, Newfoundland, Canada
Jean-Pierre Ehrmann	Paris, France
Chris Fisher	Regina, Saskatchewan, Canada
Rudolf Fritsch	Munich, Germany
Bernard Gibert	St Etiene, France
Antreas P. Hatzipolakis	Athens, Greece
Michael Lambrou	Crete, Greece
Floor van Lamoen	Goes, Netherlands
Fred Pui Fai Leung	Singapore, Singapore
Daniel B. Shapiro	Columbus, Ohio, USA
Steve Sigur	Atlanta, Georgia, USA
Man Keung Siu	Hong Kong, China
Peter Woo	La Mirada, California, USA
Li Zhou	Winter Haven, Florida, USA

Technical Editors:

Yuandan Lin	Boca Raton, Florida, USA
Aaron Meyerowitz	Boca Raton, Florida, USA
Xiao-Dong Zhang	Boca Raton, Florida, USA

Consultants:

Frederick Hoffman	Boca Raton, Floirda, USA
Stephen Locke	Boca Raton, Florida, USA
Heinrich Niederhausen	Boca Raton, Florida, USA

Table of Contents

Forum Geometricorum
Volume 8 (2008) 1–5.

FORUM GEOM
ISSN 1534-1178

On an Affine Variant of a Steinhaus Problem

Jean-Pierre Ehrmann

Abstract. Given a triangle ABC and three positive real numbers u, v, w, we prove that there exists a unique point P in the interior of the triangle, with cevian triangle $P_aP_bP_c$, such that the areas of the three quadrilaterals PP_bAP_c, PP_cBP_a , PP_aCP_b are in the ratio $u : v : w$. We locate P as an intersection of three hyperbolas.

In this note we study a variation of the theme of [2], a generalization of a problem initiated by H. Steinhaus on partition of a triangle (see [1]). Given a triangle ABC with interior $\mathscr{T}$, and a point $P \in \mathcal{T}$ with cevian triangle $P_aP_bP_c$, we denote by $\Delta_A(P)$, $\Delta_B(P)$, $\Delta_C(P)$ the areas of the oriented quadrilaterals PP_bAP_c, PP_cBP_a, PP_aCP_b. In this note we prove that given three arbitrary positive real numbers u, v, w, there exists a unique point $P \in \mathscr{T}$ such that

$$\Delta_A(P) : \Delta_B(P) : \Delta_C(P) = u : v : w.$$

To this end, we define

$$f(P) = \Delta_A(P) : \Delta_B(P) : \Delta_C(P).$$

This is the point of $\mathscr{T}$ such that

$$\Delta[BCf(P)] = \Delta_A(P), \quad \Delta[CAf(P)] = \Delta_B(P), \quad \Delta[ABf(P)] = \Delta_C(P).$$

Lemma 1. *If P has homogeneous barycentric coordinates $x : y : z$ with reference to triangle ABC, then*

$$f(P) = \frac{(y+z)(2x+y+z)}{x} : \frac{(z+x)(2y+z+x)}{y} : \frac{(x+y)(x+y+2z)}{z}.$$

Proof. If $P = x : y : z$, we have

$$\overrightarrow{AP_c} = \frac{y\overrightarrow{AB}}{x+y}, \quad \overrightarrow{AP} = \frac{y\,\overrightarrow{AB} + z\,\overrightarrow{AC}}{x+y+z}, \quad \overrightarrow{AP_b} = \frac{z\overrightarrow{AC}}{x+z},$$

so that

$$\Delta_a(P) = \Delta(AP_cP) + \Delta(APP_b) = \frac{y\,z}{x+y+z}\left(\frac{1}{x+y} + \frac{1}{x+z}\right)\Delta(ABC).$$

Publication Date: January 7, 2008. Communicating Editor: Paul Yiu.

By cyclic permutations of x, y, z, we get the values of $\Delta_B(P)$ and $\Delta_C(P)$, and the result follows. $\square$

We shall prove that $f: \mathscr{T} \to \mathscr{T}$ is a bijection. We adopt the following notations.

(i) G_a, G_b, G_c are the vertices of the anticomplementary triangle. They are the images A, B, C under the homothety $\mathsf{h}(G, -2)$, G being the centroid of ABC.

(ii) P^* denotes the isotomic conjugate of P with respect to ABC. Its traces P_a^*, P_b^*, P_c^* on the sidelines of ABC are the reflections of P_a, P_b, P_c with respect to the midpoint of the corresponding side.

(iii) $[L]_\infty$ denotes the infinite point of a line L.

Proposition 2. *Let $P = x : y : z$ and $U = u : v : w$. The lines G_aP and P_a^*U are parallel if and only if P lies on the hyperbola $\mathscr{H}_{a,U}$ through A, G_a, U_a^*, the reflection of U_b^* in C and the reflection of U_c^* in B.*

Proof. As $P_a^* = 0 : z : y$ and $[G_aP]_\infty = -(2x+y+z) : z+x : x+y$, the lines G_aP and P_a^*U are parallel if and only if

$$\begin{aligned} h_{a,U}(P) &:= \det([G_aP]_\infty,\ P_a^*,\ U) \\ &= \begin{vmatrix} -(2x+y+z) & z+x & x+y \\ 0 & z & y \\ u & v & w \end{vmatrix} \\ &= x((u+v)y-(w+u)z)+(x+y+z)(vy-wz) \\ &= 0. \end{aligned}$$

It is clear that $h_{a,U}(P) = 0$ defines a conic $\mathscr{H}_{a,U}$ through $A = 1 : 0 : 0$, and the infinite points of the lines $x = 0$ and $(u+v)y - (w+u)z = 0$. These are the lines BC and G_aU. It is also easy to check that it contains the points $G_a = -1 : 1 : 1$, $U_a^* = 0 : w : v$, and

$$\begin{aligned} U_{bc}^* &:= -w : 0 : u+2w, \\ U_{cb}^* &:= -v : u+2v : 0. \end{aligned}$$

These latter two are respectively the reflections of U_b^* in C and U_c^* in B. The conic $\mathscr{H}_{a,U}$ is a hyperbola since the four points A, G_a, U_{bc}^* and U_{cb}^* do not fall on two lines. $\square$

By cyclic permutations of coordinates, we obtain two hyperbolae $\mathscr{H}_{b,U}$ and $\mathscr{H}_{c,U}$ defined by

$$\begin{aligned} h_{b,U}(P) &:= \det([G_bP]_\infty,\ P_b^*,\ U) = 0, \\ h_{c,U}(P) &:= \det([G_cP]_\infty,\ P_c^*,\ U) = 0. \end{aligned}$$

It is easy to check that if $U = f(P)$, then

$$h_{a,U}(P) = h_{b,U}(P) = h_{c,U}(P) = 0.$$

From this we obtain a very easy construction of the point $f(P)$.

Corollary 3. *The point $f(P)$ is the intersection of the lines through P_a^*, P_b^* and P_c^* parallel to G_aP, G_bP, G_cP respectively.* See Figure 1.

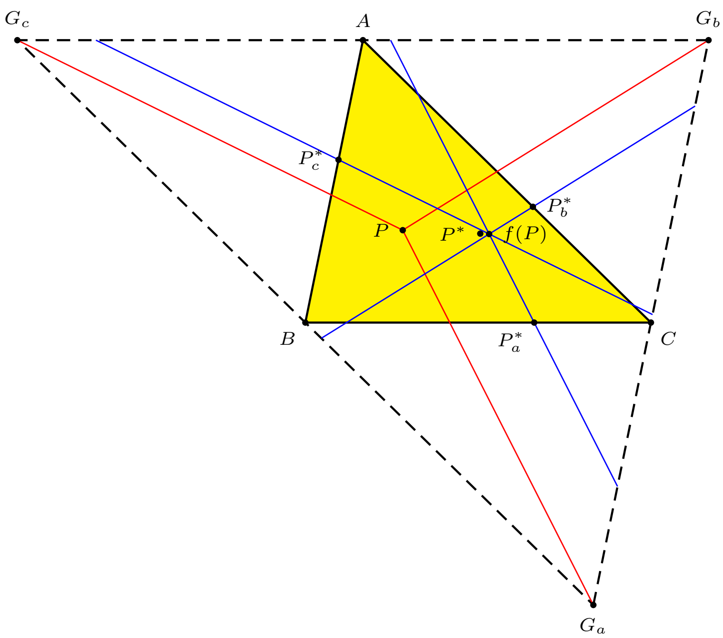


Figure 1.

Proof. The lines G_aP, G_bP, G_cP are parallel to $P_a^* f(P)$, $P_b^* f(P)$, $P_c^* f(P)$ respectively. □

Remarks. (1) $\mathscr{H}_{a,U}$ degenerates if and only if $v = w$, *i.e.*, when U lies on the median AG. In this case, $\mathscr{H}_{a,U}$ is the union of the median AG and of a line parallel to BC.

(2) P, P^*, $f(P)$ are collinear.

(3) As $h_{a,U}(P) + h_{b,U}(P) + h_{c,U}(P) = 0$, the three hyperbolae $\mathscr{H}_{a,U}$, $\mathscr{H}_{b,U}$, $\mathscr{H}_{c,U}$ are members of a pencil of conics. If $U \in \mathscr{T}$, the points P for which $f(P) = U$ are their common points lying in $\mathscr{T}$.

Lemma 4. *If $U \in \mathscr{T}$, $\mathscr{H}_{a,U}$ and $\mathscr{H}_{b,U}$ have a real common point in $\mathscr{T}$ and a real common point in $\mathscr{T}_A$, reflection in A of the open angular sector bounded by the half lines AB and AC.*

Proof. Using the fact that $\mathscr{H}_{a,U}$ passes through $[BC]_\infty$, we can cut $\mathscr{H}_{a,U}$ by lines parallel to BC to get a rational parametrization of $\mathscr{H}_{a,U}$. More precisely, let B_t and C_t be the images of B and C under the homothety $\mathsf{h}(A, 1-t)$. The point

$$(1-\mu)B_t + \mu C_t = t : (1-\mu)(1-t) : \mu(1-t)$$

lies on $\mathscr{H}_{a,U}$ if and only if

$$\mu = \mu_t = \frac{v + t(u+v)}{v + w + t(2u + v + w)}.$$

Let $P(t) = (1 - \mu_t)B_t + \mu_t C_t$. It has homogeneous barycentric coordinates

$$t((v + w) + t(2u + v + w)) : (1 - t)(w + t(w + u)) : (1 - t)(v + t(u + v)).$$

with coordinate sum is $(v + w) + t(2u + v + w)$.

If $t \geq 0$, we have $0 < \mu_t < 1$. It follows that, for $0 < t < 1$, $P(t) \in \mathscr{T}$ and for $t > 1$, $P(t) \in \mathscr{T}_A$. Consider

$$\varphi(t) := \frac{h_{b,U}(P(t))}{(u + v + w)((v + w) + t(2u + v + w))^2}.$$

More explicitly,

$$\varphi(t) = \frac{2(u + v)(u + w)(u + v + w)t^4 + \text{lower degree terms of } t}{(u + v + w)(v + w + t(2u + v + w))^2}.$$

Clearly, $\varphi(0) = \frac{2vw}{(v+w)(u+v+w)} > 0$ and $\varphi(1) = -\frac{u}{u+v+w} < 0$. Note also that $\varphi(+\infty) = +\infty$. As φ is continuous for $t \geq 0$, the result follows. $\square$

Theorem 5. *If $U \in \mathscr{T}$, the three hyperbolas $\mathscr{H}_{a,U}$, $\mathscr{H}_{b,U}$, $\mathscr{H}_{c,U}$ have four distinct real common points, exactly one of which lies in $\mathscr{T}$. This point is the only point $P \in \mathscr{T}$ satisfying $f(P) = U$.*

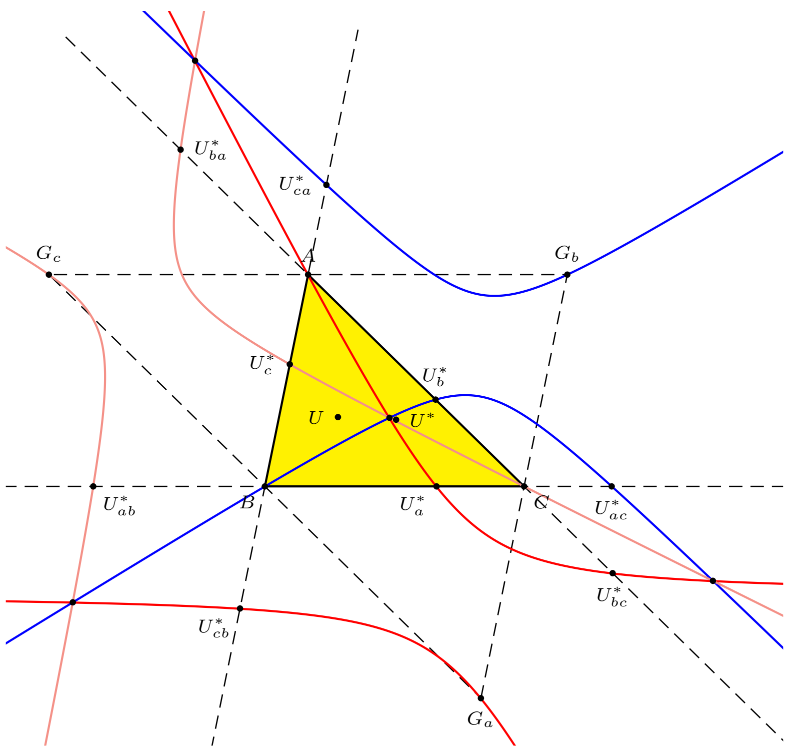


Figure 2.

Proof. In a similar way as in Lemma 4, we can see that $\mathscr{H}_{b,U}$ and $\mathscr{H}_{c,U}$ have a common point in $\mathscr{T}$ and a real common point in $\mathscr{T}_B$ and that $\mathscr{H}_{c,U}$ and $\mathscr{H}_{a,U}$ have a real common point in $\mathscr{T}$ and a real common point in $\mathscr{T}_B$. As the four sets $\mathscr{T}$, T_A, T_B, T_C pairwise have empty intersection, it follows that $\mathscr{H}_{a,U}$, $\mathscr{H}_{b,U}$, $\mathscr{H}_{c,U}$ have four real common points, one in each of $\mathscr{T}$, $\mathscr{T}_A$, $\mathscr{T}_B$ and $\mathscr{T}_C$. See Figure 2. □

Remark. (4) If $U \in \mathscr{T}$, the points P such that

$$\Delta(AP_cP) + \Delta(APP_b) : \Delta(BP_aP) + \Delta(BPP_c) : \Delta(CP_bP) + \Delta(CPP_a) = u : v : w$$

are the four common points of $\mathscr{H}_{a,U}$, $\mathscr{H}_{b,U}$ and $\mathscr{H}_{c,U}$.

Remark (2) shows that $f^{-1}(U)$ lies on the isotomic cubic with pivot U. Clearly, $f(G) = f^{-1}(G) = G$.

References

[1] A. Tyszka, Steinhaus' problem on partition of a triangle, *Forum Geom.*, 7(2007) 181–185.

[2] J.-P. Ehrmann, Constructive solution of a generalization of Steinhaus' problem on partition of a triangle, *Forum Geom.*, 7 (2007) 187–190.

Jean-Pierre Ehrmann: 6, rue des Cailloux, 92110 - Clichy, France

E-mail address: `Jean-Pierre.EHRMANN@wanadoo.fr`

Forum Geometricorum
Volume 8 (2008) 7–12.

FORUM GEOM
ISSN 1534-1178

Two Triads of Congruent Circles from Reflections

Quang Tuan Bui

Abstract. Given a triangle, we construct two triads of congruent circles through the vertices, one associated with reflections in the altitudes, and the other reflections in the angle bisectors.

1. Reflections in the altitudes

Given triangle ABC with orthocenter H, let B_a and C_a be the reflections of B and C in the line AH. These are points on the sideline BC so that $\mathbf{BC_a} = \mathbf{CB_a}$. Similarly, consider the reflections C_b, A_b of C, A respectively in the line BH, and A_c, B_c of A, B in the line CH.

Theorem 1. *The circles AC_bB_c, BA_cC_a, and CB_aA_b are congruent.*

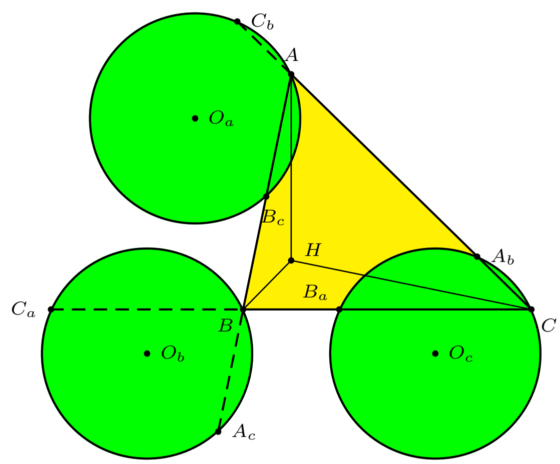


Figure 1.

Proof. Let O be the circumcenter of triangle ABC, and X its reflection in the A-altitude. This is the circumcenter of triangle AB_aC_a, the reflection of triangle ABC in its A-altitude. See Figure 2. It follows that H lies on the perpendicular bisector of OX, and $HX = OH$. Similarly, if Y and Z are the reflections of O in the lines BH and CH respectively, then $HY = HZ = OH$. It follows that O, X, Y, Z are concyclic, and H is the center of the circle containing them. See Figure 3.

Publication Date: January 14, 2008. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this paper.

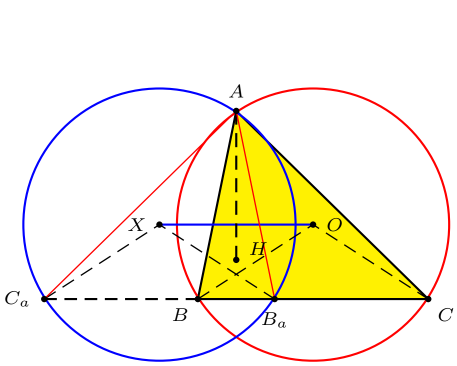


Figure 2

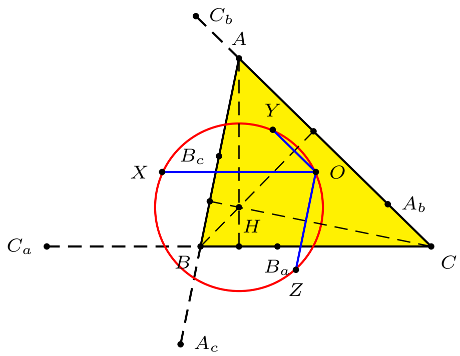


Figure 3

Let O be the circumcenter of triangle ABC. Note the equalities of vectors

$$\mathbf{OX} = \mathbf{BC_a} = \mathbf{CB_a},$$
$$\mathbf{OY} = \mathbf{CA_b} = \mathbf{AC_b},$$
$$\mathbf{OZ} = \mathbf{AB_c} = \mathbf{BA_c}.$$

The three triangles AC_bB_c, BA_cC_a, and CB_aA_b are the translations of OYZ by **OA**, OZX by **OB**, and OXY by **OC** respectively.

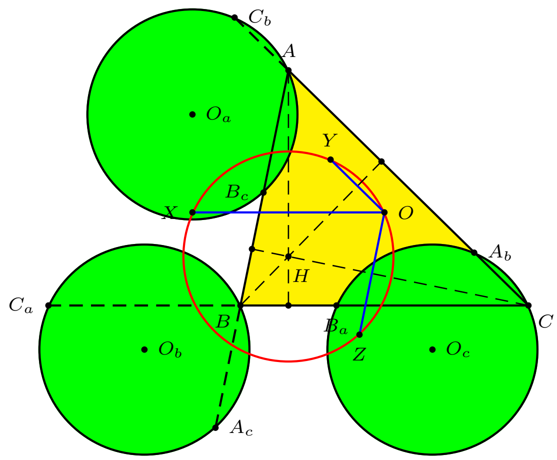


Figure 4.

Therefore, the circumcircles of the three triangles are all congruent and have radius OH. Their centers are the translations of H by the three vectors. $\square$

2. Reflections in the angle bisectors

Let I be the incenter of triangle ABC. Consider the reflections of the vertices in the angle bisectors: B'_a, C'_a of B, C in AI, C'_b, A'_b of C, A in BI, and A'_c, B'_c of A, B in CI. See Figure 5.

Theorem 2. *The circles $AC'_bB'_c$, $BA'_cC'_a$, and $CB'_aA'_b$ are congruent.*

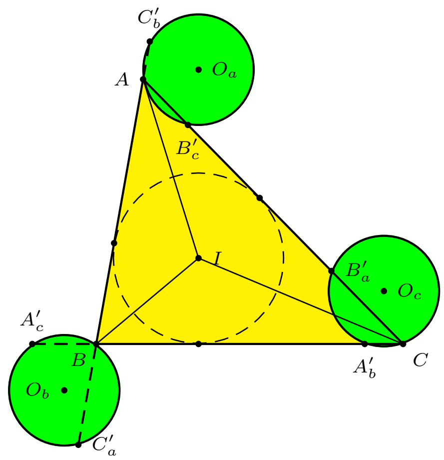


Figure 5.

Proof. Consider the reflections B''_c, C''_b of B'_c, C'_b in AI, C''_a, A''_c of C'_a, A'_c in BI, and A''_b, B''_a of A'_b, B'_a in CI. See Figure 6.

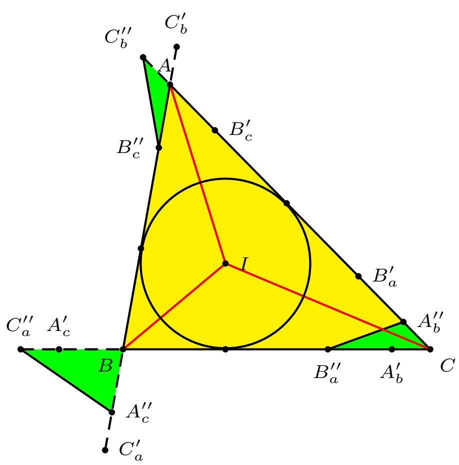


Figure 6

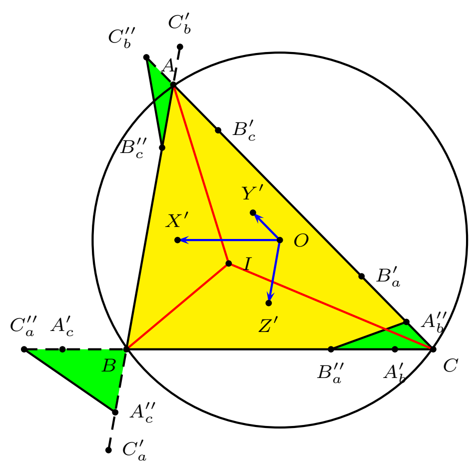


Figure 7

Note the equalities of vectors

$$\mathbf{BC''_a} = \mathbf{CB''_a}, \qquad \mathbf{CA''_b} = \mathbf{AC''_b}, \qquad \mathbf{AB''_c} = \mathbf{BA''_c}.$$

With the circumcenter O of triangle ABC, these define points X', Y', Z' such that

$$\mathbf{OX'} = \mathbf{BC''_a} = \mathbf{CB''_a},$$
$$\mathbf{OY'} = \mathbf{CA''_b} = \mathbf{AC''_b},$$
$$\mathbf{OZ'} = \mathbf{AB''_c} = \mathbf{BA''_c}.$$

The triangles $AC''_bB''_c$, $BA''_cC''_a$ and $CB''_aA''_b$ are the translations of $OY'Z'$, $OZ'X'$ and $OX'Y'$ by the vectors $\mathbf{OA}$, $\mathbf{OB}$ and $\mathbf{OC}$ respectively. See Figure 7.

Note, in Figure 8, that $OX'C''_aC$ is a symmetric trapezoid and $IC''_a = IC'_a = IC$. It follows that triangles IC''_aX' and ICO are congruent, and $IX' = IO$. Similarly, $IY' = IO$ and $IZ' = IO$. This means that the four points O, X', Y', Z' are on a circle center I. See Figure 9. The circumcenters O''_a, O''_b, O''_c of the triangles $AC''_bB''_c$, $BA''_cC''_a$ and $CB''_aA''_b$ are the translations of I by these vectors. These circumcircles are congruent to the circle $I(O)$.

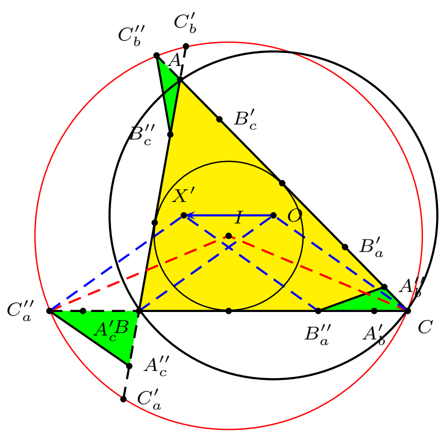


Figure 8

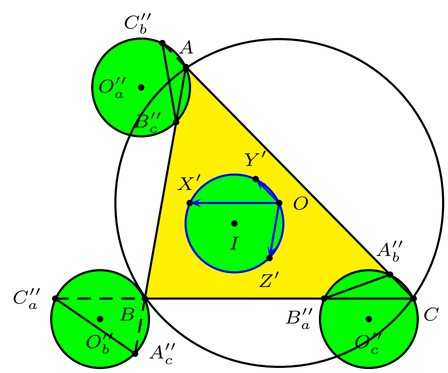


Figure 9

The segments AO''_a, BO''_b and CO''_c are parallel and equal in lengths. The triangles $AC'_bB'_c$, $BA'_cC'_a$ and $CB'_aA'_b$ are the reflections of $AC''_bB''_c$, $BA''_cC''_a$ and $CB''_aA''_b$ in the respective angle bisectors. See Figure 10. It follows that their circumcircles are all congruent to $I(O)$. □

Let O'_a, O'_b, O'_c be the circumcenters of triangles $AC'_bB'_c$, $BA'_cC'_a$ and $CB'_aA'_b$ respectively. The lines AO'_a and AO''_a are symmetric with respect to the bisector of angle A. Since AO''_a, BO''_b and CO''_c are parallel to the line OI, the reflections in the angle bisectors concur at the isogonal conjugate of the infinite point of OI. This is a point P on the circumcircle. It is the triangle center X_{104} in [1].

Finally, since $IO''_a = IO''_b = IO''_c$, we also have $IO'_a = IO'_b = IO'_c$. The 6 circumcenters all lie on the circle, center I, radius R.

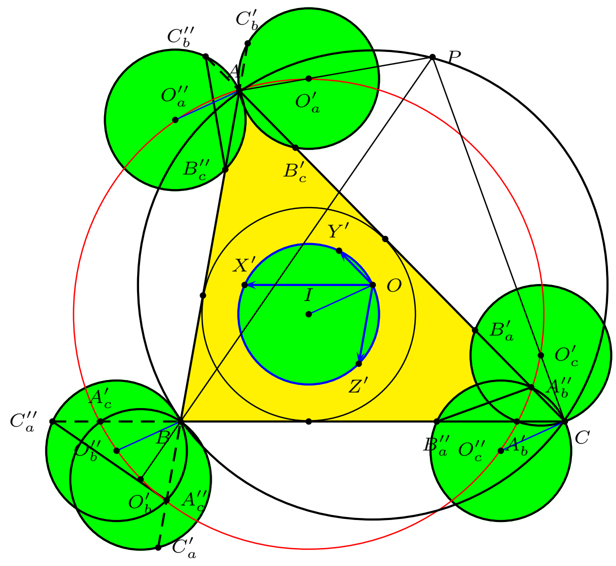


Figure 10.

To conclude this note, we establish an interesting property of the centers of the circles in Theorem 2.

Proposition 3. *The lines $O_a'I$, $O_b'I$ and $O_c'I$ are perpendicular to BC, CA and AB respectively.*

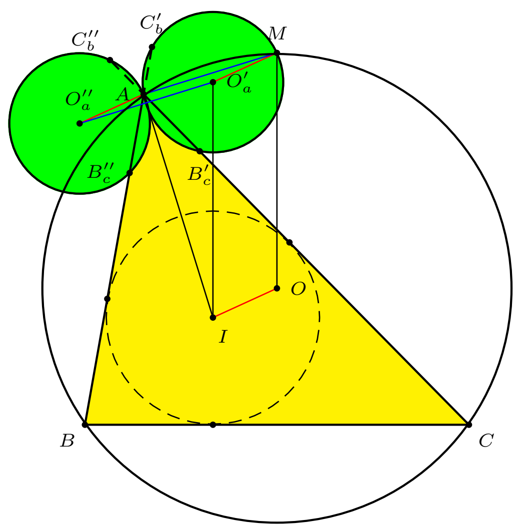


Figure 11.

Proof. It is enough to prove that for the line O'_aI. The other two cases are similar.

Let M be the intersection (other than A) of the circle (O'_a) with the circumcircle of triangle ABC. Since $IO'_a = OM$ (circumradius) and $O'_aM = IO$, O'_aMOI is a parallelogram. This means that $\mathbf{O'_aM} = \mathbf{IO} = \mathbf{O''_aA}$, and $AMO'_aO''_a$ is also a parallelogram. From this we conclude that AM, being parallel to $O''_aO'_a$, is perpendicular to the bisector AI. Thus, M is the midpoint of the arc BAC, and MO is perpendicular to BC. Since $\mathbf{O'_aI} = \mathbf{MO}$, the line O'_aI is also perpendicular to BC. □

Since the six circles (O'_a) and (O''_a) etc are congruent (with common radius OI) and their centers are all at a distance R from I, it is clear that there are two circles, center I, tangent to all these circles. These two circles are tangent to the circumcircle, the point of tangency being the intersection of the circumcircle with the line OI. These are the triangle centers X_{1381} and X_{1382} of [1].

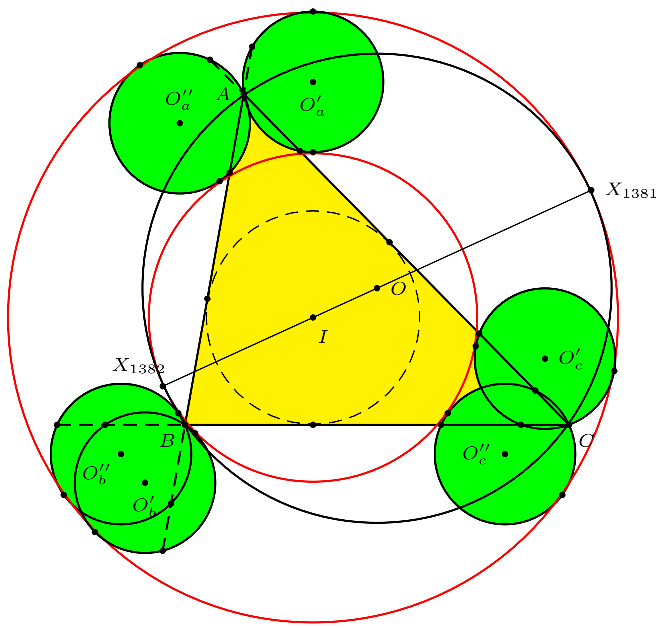


Figure 12.

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Quang Tuan Bui: 45B, 296/86 by-street, Minh Khai Street, Hanoi, Vietnam
E-mail address: bqtuan1962@yahoo.com

Forum Geometricorum
Volume 8 (2008) 13–25.

FORUM GEOM
ISSN 1534-1178

Angles, Area, and Perimeter Caught in a Cubic

George Baloglou and Michel Helfgott

Abstract. The main goal of this paper is to establish sharp bounds for the angles and for the side ratios of any triangle of known area and perimeter. Our work is also related to the well known isoperimetric inequality.

1. Isosceles triangles sharing area and perimeter

Suppose we wish to determine all isosceles triangles, if any, of area 3 and perimeter 10 – a problem that is a bit harder than the corresponding well known problem for rectangles!

Let x be the length of the base and y the length of the two equal sides, $x < 2y$. Then the height of the isosceles triangles we wish to determine is equal to $\sqrt{y^2 - \frac{x^2}{4}}$. Thus $x+2y = 10$ while $\frac{x}{2}\sqrt{y^2 - \frac{x^2}{4}} = 3$. Hence $\frac{x}{2}\sqrt{\left(5 - \frac{x}{2}\right)^2 - \frac{x^2}{4}} = 3$, which leads to $5x^3 - 25x^2 + 36 = 0$. The positive roots of this cubic are $x_1 \approx 1.4177$ and $x_2 \approx 4.6698$, so that $y_1 \approx 4.2911$ and $y_2 \approx 2.6651$. Thus there are just two isosceles triangles of area 3 and perimeter 10 (see Figure 1).

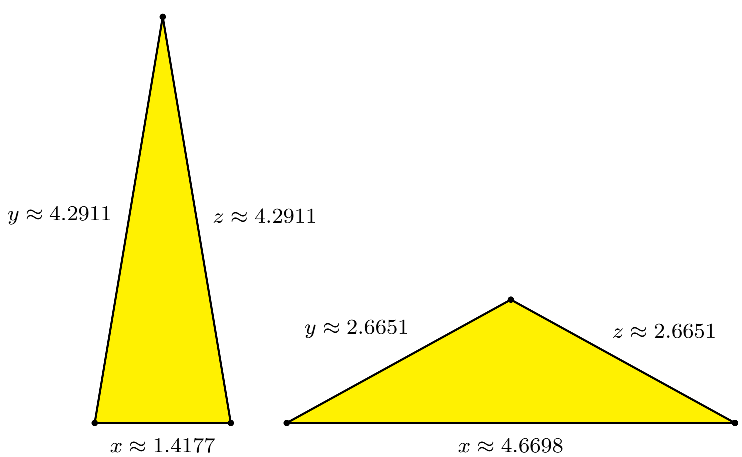


Figure 1. The two isosceles triangles of area 3 and perimeter 10

Are there always isosceles triangles of area A and perimeter P? A complete answer is provided by the following lemma and theorem.

Lemma 1. *Let x be the base of an isosceles triangle with given area A and perimeter P. Then*

$$2Px^3 - P^2x^2 + 16A^2 = 0. \tag{1}$$

Publication Date: January 22, 2008. Communicating Editor: Paul Yiu.

Proof. Working as in the above special case, we obtain $y = \frac{P-x}{2}$ and $\frac{x}{2}\sqrt{y^2 - \frac{x^2}{4}} = A$; substituting the former condition into the latter, we arrive at (1). □

Theorem 2. *There are exactly two distinct isosceles triangles of area A and perimeter P if and only if $P^2 > 12\sqrt{3}A$. There is exactly one if and only if $P^2 = 12\sqrt{3}A$ and the triangle is equilateral. The vertex angles $\phi_1 < \phi_2$ of these two isosceles triangles also satisfy $\phi_1 < \frac{\pi}{3} < \phi_2$.*

Proof. Let $f(x)$ be the cubic in (1). We first show that it has at most two distinct positive roots. Indeed the existence of three distinct positive roots would yield, by Rolle's theorem, two distinct positive roots for $f'(x) = 6Px^2 - 2P^2x$; but the roots of $f'(x)$ are $x = \frac{P}{3}$ and $x = 0$.

Notice now that $f''(x) = 12Px - 2P^2$, hence $f''(0) = -2P^2 < 0$ and $f''(\frac{P}{3}) = 2P^2 > 0$. So f has a positive local maximum of $16A^2$ at $x = 0$ and a local minimum at $x = \frac{P}{3}$ (Figure 2). It is clear that f has two distinct positive roots $x_1 < \frac{P}{3} < x_2$ if and only if $f(\frac{P}{3}) < 0$; but $f(\frac{P}{3}) = -\frac{P^4}{27} + 16A^2$, so $f(\frac{P}{3}) < 0$ is equivalent to $P^2 > 12\sqrt{3}A$.

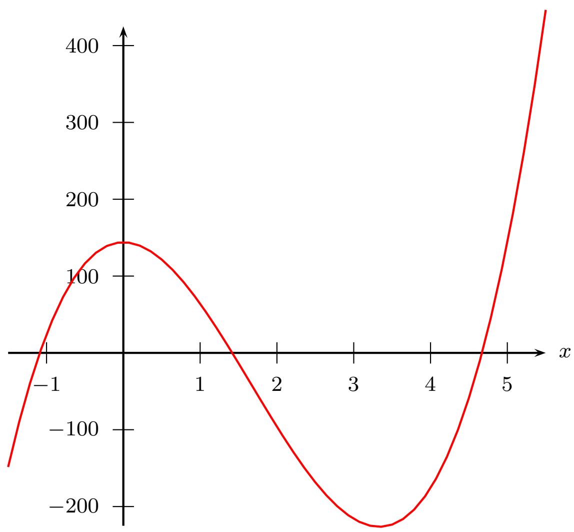


Figure 2. $2Px^3 - P^2x^2 + 16A^2$ for $A = 3$ and $P = 10$

Moreover, $f(\frac{P}{3}) = 0$ if and only if $P^2 = 12\sqrt{3}A$, implying that $f(x) = 0$ has precisely one ('tangential') positive solution if and only if $P^2 = 12\sqrt{3}A$. As it turns out, the cubic is then equivalent to $(3x - P)^2(6x + P) = 0$, and its unique positive solution corresponds to the equilateral triangle of side $\frac{P}{3}$.

As also noticed in [1], the vertex angles ϕ_1 and ϕ_2 of the two isosceles triangles of area A and perimeter P (that correspond to the positive roots x_1 and x_2 of (1)) do satisfy the inequalities $\phi_1 < \frac{\pi}{3} < \phi_2$. These inequalities follow from $x_1 < \frac{P}{3} < x_2$ since, in every triangle, the greater angle is opposite the greater side: indeed in every isosceles triangle of perimeter P, base x, vertex angle ϕ, and sides

$y = z$, the inequality $x < \frac{P}{3}$ implies $y = z > \frac{P}{3}$, so that $y = z > x$; therefore $\frac{\pi-\phi}{2} > \phi$, thus $\phi < \frac{\pi}{3}$. In a similar fashion one can prove that $x > \frac{P}{3}$ implies $\phi > \frac{\pi}{3}$. □

Remark. That the cubic in (1) can have at most two distinct positive roots may also be derived algebraically. Indeed, the existence of three distinct positive roots x_1, x_2, x_3 would imply that the cubic may be written as $c(x-x_1)(x-x_2)(x-x_3)$, with $c(x_1x_2 + x_2x_3 + x_3x_1)$ being the *positive* coefficient of the first power of x. That would contradict the fact that the cubic being analyzed has zero as the coefficient of the first power of x.

2. The isoperimetric inequality for arbitrary triangles

We have just seen that the inequality $P^2 \geq 12\sqrt{3}A$ holds for every isosceles triangle, with equality precisely when the triangle is equilateral. We will prove next that this *isoperimetric* inequality ([5, p.85], [3, p.42]) holds for every triangle.

First we notice that for every scalene triangle BCD, there exists an isosceles triangle ECD with BE parallel to CD (see Figure 3). Let ℓ be the line through B parallel to CD and F be the symmetric reflection of D with respect to ℓ. Let E and G be the points of ℓ on CF and DF, respectively. Clearly, $EG\|CD$ and $|FG| = |DG|$ imply $|FE| = |CE|$. Moreover, triangles FGE and DGE are congruent by symmetry, therefore $|FE| = |DE|$. We conclude that triangle ECD is isosceles with $|CE| = |DE|$.

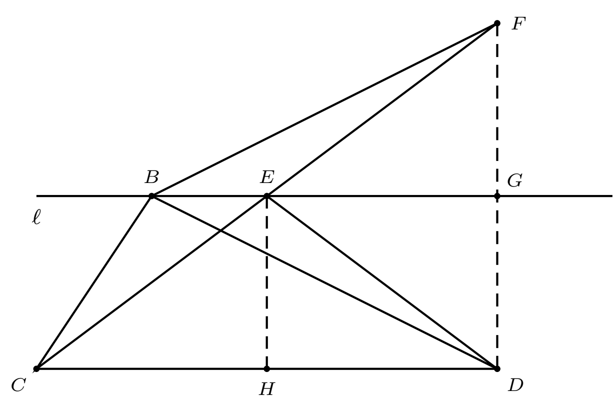


Figure 3. Reduction to the case of an isosceles triangle

It follows immediately from $BE\|CD$ that ΔECD and ΔBCD have equal areas. Less obviously, the perimeter of ΔECD is *smaller* than that of ΔBCD : $|CD|+|DE|+|EC| = |CD|+|FE|+|EC| = |CD|+|FC| < |CD|+|FB|+|BC| = |CD| + |DB| + |BC|$, with the last equality following from symmetry and the congruency of ΔFGB and ΔDGB.

So, given an arbitrary scalene triangle BCD of area A and perimeter P, there exists an isosceles triangle ECD of area A and perimeter $Q < P$. Since $Q^2 \geq$

$12\sqrt{3}A$, it follows that $P^2 > 12\sqrt{3}A$, so the isoperimetric inequality for triangles has been proven.

We invite the reader to use this geometrical technique to derive the isoperimetric inequality for quadrilaterals ($P^2 \geq 16A$ for every quadrilateral of area A and perimeter P), and possibly for other n-gons as well.

It should be mentioned here that the standard proof of the isoperimetric inequality for triangles (see for example [2, p.88]) relies on Heron's area formula (which we essentially derive later through a generalization of (1) for arbitrary triangles) and the arithmetic-geometric-mean inequalilty.

3. Newton's parametrization

Turning now to our main goal, namely the relations among a triangle's area, perimeter, and angles, we first find an expression for the sides of a triangle in terms of its area, perimeter, and *one* angle. To achieve this, we simply generalize Newton's derivation of the formula $x = \frac{P}{2} - \frac{2A}{P}$, expressing a right triangle's hypotenuse in terms of its area and perimeter; this work appeared in Newton's *Universal Arithmetick*, *Resolution of Geometrical Questions*, Problem III, p. 57 ([6, p.103]).

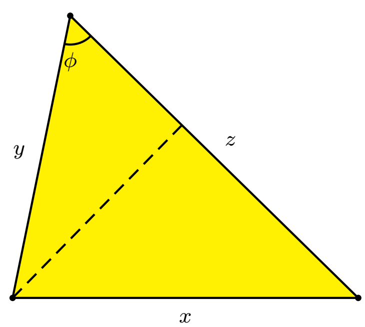


Figure 4. Toward 'Newton's parametrization'

Observe (as in Figure 4) that $A = \frac{1}{2}zy\sin\phi$, so $y^2 = Py - xy - \frac{2A}{\sin\phi}$; moreover, the law of cosines yields $y^2 = Px + Py - xy + \frac{2A\cos\phi}{\sin\phi} - \frac{P^2}{2}$. It follows that

$$x = x(\phi) = \frac{P}{2} - \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right), \tag{2}$$

extending Newton's formula for $0 < \phi < \pi$. Of course we need to have $\frac{P^2}{A} > 4\left(\frac{1+\cos\phi}{\sin\phi}\right)$ for x to be positive, so we need the condition $s(\phi) > 0$, where

$$s(\phi) = \frac{P^2\sin\phi}{4(1+\cos\phi)} - A. \tag{3}$$

Once x is determined, y and z are easily determined via $yz = \frac{2A}{\sin\phi}$ and $y + z = \frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)$: they are the roots of the quadratic $t^2 - \left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)t + \frac{2A}{\sin\phi} = 0$, provided that $h(\phi) \geq 0$, where

$$h(\phi) = \left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 - \frac{8A}{\sin\phi} \tag{4}$$

is the discriminant; that is, $y = y(\phi)$ and $z = z(\phi)$ are given by

$$z,\ y = \left(\frac{P}{4} + \frac{A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right) \pm \frac{1}{2}\sqrt{\left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 - \frac{8A}{\sin\phi}}. \tag{5}$$

Putting everything together, and observing that x, y, z as defined in (2) and (5) above do satisfy the triangle inequality and are the sides of a triangle of area A and perimeter P, we arrive at the following result.

Theorem 3. *The pair of conditions $s(\phi) > 0$ and $h(\phi) \geq 0$, where $s(\phi) = \frac{P^2\sin\phi}{4(1+\cos\phi)} - A$ and $h(\phi) = \left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 - \frac{8A}{\sin\phi}$, is equivalent to the existence of a triangle of area A, perimeter P, sides $x(\phi), y(\phi), z(\phi)$ as given in* (2), (5) *above, and angle ϕ between the sides y, z; that triangle is isosceles with vertex angle ϕ if and only if $h(\phi) = 0$.*

Figures 5 and 6 below offer visualizations of the three sides' parametrizations by the angle ϕ and of the two functions essential for the 'triangle conditions' of Theorem 3, respectively.

The 'vertical' intersections of $y(\phi)$ and $z(\phi)$ with each other in Figure 5 occur at $\phi \approx 0.33166 \approx 19.003°$ and $\phi \approx 2.13543 \approx 122.351°$: those are the positive roots of $h(\phi) = 0$, which are none other than the vertex angles of the two isosceles triangles in Figure 1. There are also intersections of $x(\phi)$ with $z(\phi)$ at $\phi \approx 1.40485 \approx 80.492°$ and of $x(\phi)$ with $y(\phi)$ at $\phi \approx 0.50305 \approx 28.822°$; which are again associated, via side renaming as needed and with ϕ being a *base* angle, with the isosceles triangles of Figure 1.

As we see in Figure 6, s and h cannot be simultaneously positive outside the interval defined by the two largest roots of h ($\phi \approx 0.33166$ and $\phi \approx 2.13543$): this fact remains true for arbitrary A and P and is going to be of central importance in what follows.

4. Angles 'bounded' by area and perimeter

We are ready to state and prove our first main result.

Theorem 4. *In every non-equilateral triangle of area A and perimeter P every angle ϕ must satisfy the inequality $\phi_1 \leq \phi \leq \phi_2$, where $\phi_1 < \frac{\pi}{3} < \phi_2$ are the vertex angles of the two isosceles triangles of area A and perimeter P; specifically,*

$$\arccos\left(\frac{P^2 - 2Px_1 - x_1^2}{P^2 - 2Px_1 + x_1^2}\right) \leq \phi \leq \arccos\left(\frac{P^2 - 2Px_2 - x_2^2}{P^2 - 2Px_2 + x_2^2}\right),$$

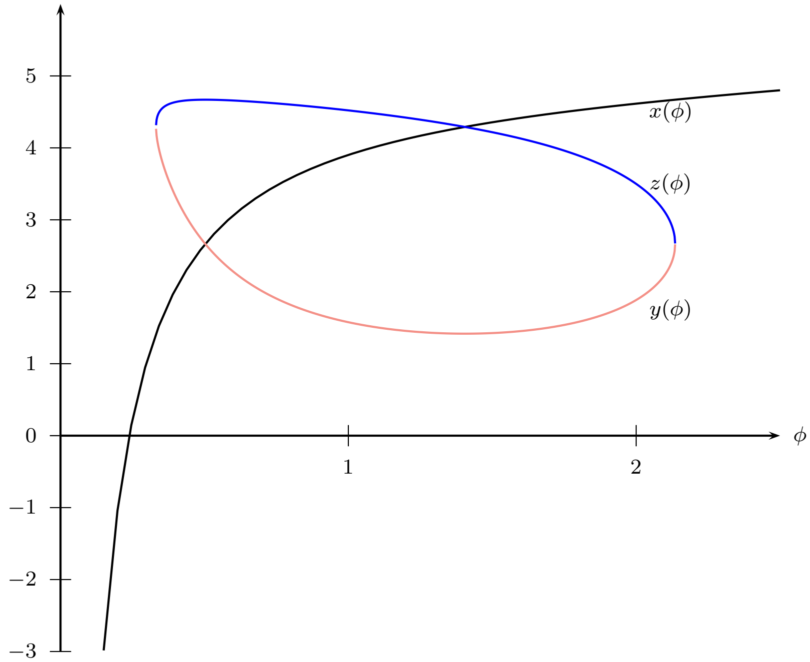


Figure 5. The triangle's three sides parametrized by ϕ for $19.003^\circ = 0.33166 \leq \phi \leq 2.13543 = 122.351^\circ$ at $A = 3$, $P = 10$

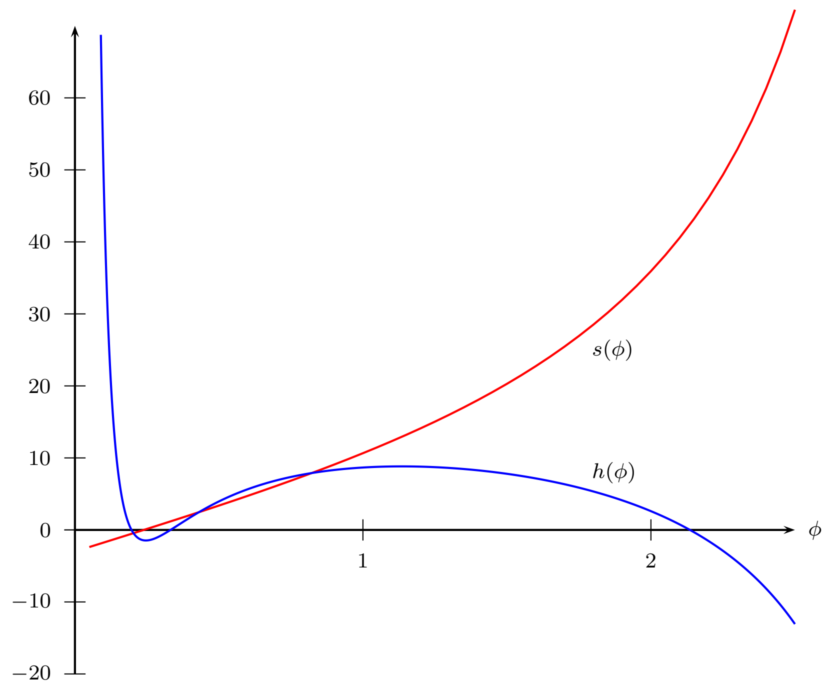


Figure 6. $s(\phi)$ and $h(\phi)$ for $0.1 \leq \phi \leq 2.3$ at $A = 3$, $P = 10$

where $x_1 < \frac{P}{3} < x_2$ *are the positive roots of* $2Px^3 - P^2x^2 + 16A^2 = 0$.

Proof. As we have seen in Lemma 1, the cubic (1) yields the base x of each of the two isosceles triangles of area A and perimeter P; and the formula above for the vertex angle ϕ of an isosceles triangle follows from $x^2 = 2y^2 - 2y^2 \cos\phi$ (law of cosines) and $y = \frac{P-x}{2}$.

So it suffices to show that the inequality $\phi_1 \le \phi \le \phi_2$ is equivalent to the pair of conditions $s(\phi) > 0$ and $h(\phi) \ge 0$, where $s(\phi)$ and $h(\phi)$ are defined as in Theorem 3; for this, we need four lemmas.

Lemma 5. *For some* ψ *in* $(0, \phi_1)$, $s(\psi) = 0$.

Proof. Notice that $\lim_{\phi\to 0^+} s(\phi) = -A < 0$. On the other hand, the existence of an isosceles triangle with vertex angle ϕ_1 guarantees that $s(\phi_1) > 0$ (Theorem 3). By the continuity of s on $(0, \pi)$, there must exist ψ such that $0 < \psi < \phi_1$ and $s(\psi) = 0$. □

Lemma 6. *The function* s *is strictly increasing on* $(0, \pi)$ *and, for* $\phi \ge \phi_1$, $s(\phi) > 0$.

Proof. Since the derivative $s'(\phi) = \frac{P^2}{4(1+\cos\phi)}$ is positive on $(0, \pi)$, s is strictly increasing; it follows that $s(\phi) \ge s(\phi_1) > 0$ for $\phi \ge \phi_1$. □

Lemma 7. *For* $\phi > \phi_2$, $h(\phi) < 0$.

Proof. Recall that $h(\phi) = \left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 - \frac{8A}{\sin\phi}$. By L'Hospital's rule, we have $\lim_{\phi\to\pi} \frac{1+\cos\phi}{\sin\phi} = \lim_{\phi\to\pi} \frac{-\sin\phi}{\cos\phi} = 0$; it follows that $\lim_{\phi\to\pi^-} h(\phi) = \frac{P^2}{4} - \lim_{\phi\to\pi^-} \frac{8A}{\sin\phi} = -\infty$. Suppose $h(\phi) \ge 0$ for some $\phi > \phi_2$. Then $h(\phi_3) = 0$ for some $\phi_3 > \phi_2$ because h is continuous on $(0, \pi)$ and $\lim_{\phi\to\pi^-} h(\phi) = -\infty$. At the same time, $s(\phi_3) > 0$ (Lemma 6). Then by Theorem 3, there exists a third isosceles triangle of area A and perimeter P, which is impossible. □

Lemma 8. *There is no* ϕ *in* $(0, \pi)$ *for which* $h(\phi) = h'(\phi) = 0$.

Proof. Suppose $h(\phi) = h'(\phi) = 0$ for some ϕ in $(0, \pi)$. It follows that

$$\left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 = \frac{8A}{\sin\phi} \quad \text{and} \quad \frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right) = \frac{2P\cos\phi}{1+\cos\phi}.$$

Squaring the latter and dividing it by the former expression we get $P^2 = \frac{2A(1+\cos\phi)^2}{\sin\phi\cos^2\phi}$. Substituting this expression for P^2 into $\left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 = \frac{8A}{\sin\phi}$ we arrive at the equation $\frac{A(1+\cos\phi)^2}{2\sin\phi\cos^2\phi} + \frac{2A(1+\cos\phi)}{\sin\phi} + \frac{2A\cos^2\phi}{\sin\phi} = \frac{8A}{\sin\phi}$, which reduces to $(\cos\phi - 1)(2\cos\phi - 1)(2\cos^2\phi + 5\cos\phi + 1) = 0$. The only roots in $(0, \pi)$ are given by $\phi = \frac{\pi}{3}$ and $\phi = \arccos\left(\frac{-5+\sqrt{17}}{4}\right)$. It is easy to see that $h'(\phi) < 0$ for $\phi > \frac{\pi}{2}$, so $\arccos\left(\frac{-5+\sqrt{17}}{4}\right)$ is an extraneous solution. Moreover, $\phi = \frac{\pi}{3}$ turns

$P^2 = \frac{2A(1+\cos\phi)^2}{\sin\phi\cos^2\phi}$ into $P^2 = 12\sqrt{3}A$, contradicting the fact that the given triangle was assumed to be non-equilateral. We conclude that $h(\phi) = h'(\phi) = 0$ is impossible. $\square$

Completing the proof of Theorem 4.
Claim(a) *For $\phi_1 \leq \phi \leq \phi_2$, $s(\phi) > 0$ and $h(\phi) \geq 0$, with $h(\phi) > 0$ for $\phi_1 < \phi < \phi_2$.*

Recall from Lemma 6 that $s(\phi) > 0$ for $\phi \geq \phi_1$. So it remains to establish $h(\phi) \geq 0$ for $\phi_1 \leq \phi \leq \phi_2$. We will argue by contradiction.

Of course $h(\phi_1) = h(\phi_2) = 0$. Notice that $h(\phi) = 0$ for $\phi_1 < \phi < \phi_2$ is impossible for this would imply (by Theorem 3) the existence of a third isosceles triangle of area A and perimeter P. If $h(\phi_3) < 0$ for some ϕ_3 strictly between ϕ_1 and ϕ_2 then continuity of h, together with the impossibility of $h(\phi) = 0$ for $\phi_1 < \phi < \phi_2$, implies $h(\phi) < 0$ for *all* angles strictly between ϕ_1 and ϕ_2. But we already know from Lemma 7 that $h(\phi) < 0$ for all angles greater than ϕ_2. It follows that h has a local maximum at $\phi = \phi_2$, so that $h(\phi_2) = h'(\phi_2) = 0$, contradicting Lemma 8.

Recalling the statement immediately before Lemma 5, we see that the proof of Theorem 4 will be completed by establishing
Claim(b) *At least one of the conditions $s(\phi) > 0$ and $h(\phi) \geq 0$ fails when either $\phi < \phi_1$ or $\phi > \phi_2$.*

Of course the failure of $h(\phi) \geq 0$ for $\phi > \phi_2$ has been established in Lemma 7, so we only need to show either $s(\phi) \leq 0$ or $h(\phi) < 0$ for $\phi < \phi_1$.

Lemma 5 asserts that there exists ψ in $(0, \pi)$ such that $\psi < \phi_1$ and $s(\psi) = 0$. Consider now an arbitrary $\phi < \phi_1$. If $\phi \leq \psi$ then by Lemma 6 $s(\phi) \leq s(\psi) = 0$, so we only need to pay attention to the possibility $\phi_1 > \phi > \psi$ and $s(\phi) > 0$. In that case we show below that $h(\phi) < 0$, arguing by contradiction.

The failure of $h(\phi) < 0$ implies, in the presence of $s(\phi) > 0$, that $h(\phi) > 0$: indeed $h(\phi) = 0$ and $s(\phi) > 0$ would yield a third isosceles triangle of area A and perimeter P, again by Theorem 3. The same argument applies in fact to all angles between ψ and ϕ_1. But we have already established through Claim(a) the strict positivity of h for all angles between ϕ_1 and ϕ_2. We conclude that h has a local minimum at $\phi = \phi_1$, so that $h(\phi_1) = h'(\phi_1) = 0$, contradicting Lemma 8. This completes the proof of Theorem 4.

Having completed the proof of Theorem 4, let us provide an example: the bases of the two isosceles triangles of area 3 and perimeter 10 (Figure 1) have already been computed as the positive roots of the cubic $5x^3 - 25x^2 + 36 = 0$; it follows then that *all* angles of *every* triangle of area 3 and perimeter 10 must be between about $19.003°$ and $122.351°$, the angles shown in Figure 5.

Remark. It can be shown that ϕ_1 and ϕ_2 are the two largest roots of

$$(P^2\sin\phi + 4A + 4A\cos\phi)^2 - 32P^2A\sin\phi = 0$$

in $(0, \pi)$, and that they also satisfy the equation

$$\sin\phi_2 \left(1 + \sin\frac{\phi_1}{2}\right)^2 = \sin\phi_1 \left(1 + \sin\frac{\phi_2}{2}\right)^2.$$

5. Heron's curve

Theorem 4 establishes bounds for the angles of every triangle of given area and perimeter; appealing to the law of sines, we see that it also yields bounds for the ratio of any two sides. Determining *sharp* bounds for side ratios relies on some machinery we develop next.

Instead of looking for isosceles triangles $(z = y)$ of area A and perimeter P, let us now look for triangles of area A and perimeter P where two sides have ratio r ($\frac{z}{y} = r$); without loss of generality, we may assume $r > 1$. (Observe here - as in fact noticed through Figure 5 and related discussion - that $r > 1$ does not rule out the possibilities $x = z$ (with $r \approx 3.0268$ at $A = 3$, $P = 10$) or $x = y$ (with $r \approx 1.7522$ at $A = 3$, $P = 10$).) Extending the procedure of Lemma 1 to arbitrary triangles, from $y^2 - x_1^2 = r^2y^2 - x_2^2$ and $x = x_2 \pm x_1$ (Figure 7) we find that

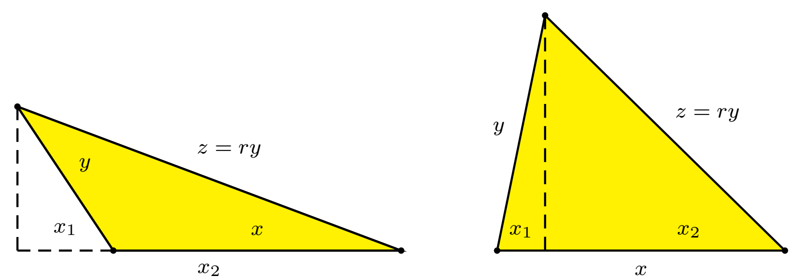


Figure 7. The case of an arbitrary triangle

$x_1 = \pm\frac{(1-r^2)y^2+x^2}{2x}$. In view of $\frac{x}{2}\sqrt{y^2 - x_1^2} = A$ and $y = \frac{P-x}{r+1}$, further algebraic manipulation leads to an equation that generalizes the isosceles triangle's cubic (1):

$$8rPx^3 + 4(r^2-3r+1)P^2x^2 - 4(1-r)^2P^3x + (1-r)^2P^4 + 16(1+r)^2A^2 = 0. \quad (6)$$

Appealing to Rolle's theorem as in the case of the isosceles triangle, we see that this cubic cannot have more than two positive roots. Indeed one of the derivative's roots, $\left(\frac{-(r^2-3r+1)-\sqrt{r^4-r^2+1}}{6r}\right)P$, is negative since $|r^2 - 3r + 1| < \sqrt{r^4 - r^2 + 1}$ for $r > 1$.

Unlike the case of the isosceles triangle, however, the isoperimetric inequality $P^2 > 12\sqrt{3}A$ does not guarantee the existence of two positive roots. So there can be *at most* two triangles of area A and perimeter P satisfying the condition $\frac{z}{y} = r > 1$.

Setting $x = P - y - z$ and $r = \frac{z}{y}$ in the cubic (6) leads to

$$P^4 - 4P^3(y+z) + 4P^2(y^2 + 3yz + z^2) - 8Pyz(y+z) + 16A^2 = 0, \quad (7)$$

which can be shown to be equivalent to Heron's area formula. The graph of this curve for $A = 3$ and $P = 10$ (Figure 8) illustrates the fact established above by (6): for every pair of A and P, there can be at most two triangles of area A and perimeter P satisfying $\frac{z}{y} = r > 1$. Indeed, the three unbounded regions shown in Figure 8 correspond to $x < 0$ (first quadrant), $y < 0$ (second quadrant), and $z < 0$ (fourth quadrant), hence it is only the boundary of the bounded region that corresponds to triangles of area 3 and perimeter 10; clearly, this boundary that we call *Heron's curve* (Figure 9) may be intersected by any line at most twice.

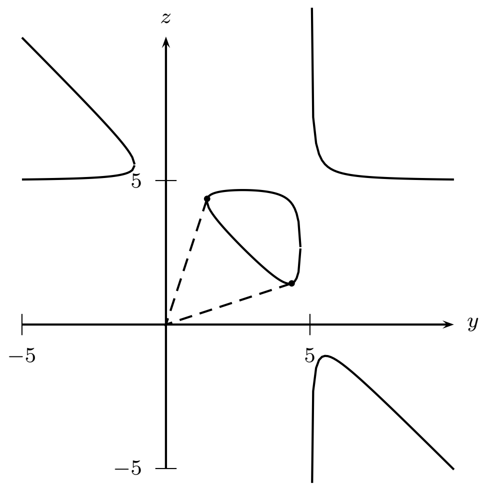


Figure 8. Graph of (7) for $A = 3$ and $P = 10$

Rather predictably, in view of its symmetry about $z = y$, the triangles corresponding to Heron's curve's intersections with (for example) $z = 2y$ and $z = \frac{y}{2}$ (see Figure 9) are mirror images of each other (about the third side x's perpendicular bisector); so it suffices to restrict our computations to $r > 1$, sticking to our initial assumption. These triangles are found by first solving the cubic (6) when $r = 2$ and are approximately $\{3.0077, 2.3307, 4.6615\}$ and $\{4.5977, 1.8007, 3.6015\}$; they are associated with parametrizing angles of about $33.529°$ and $112.315°$, respectively.

6. Side ratios 'bounded' by area and perimeter

We present now the following companion to Theorem 4.

Theorem 9. *In every non-equilateral triangle of area A and perimeter P, the ratio r of any two sides must satisfy the inequality $r_1 \le r \le r_2$, where $r_1 < 1 < r_2$, $r_1 r_2 = 1$ are the positive roots of the sextic*

$$32P^4A^2(2r^6 - 3r^4 - 3r^2 + 2) - P^8r^2(r-1)^2 + 6912A^4r^2(r+1)^2 = 0. \quad (8)$$

Proof. Figures 8 and 9 (and the discussion preceding them) make it clear that not all lines $z = ry$ intersect Heron's curve: such intersections (corresponding to triangles of area A and perimeter P satisfying $\frac{z}{y} = r$) occur only at $r = 1$ and a varying

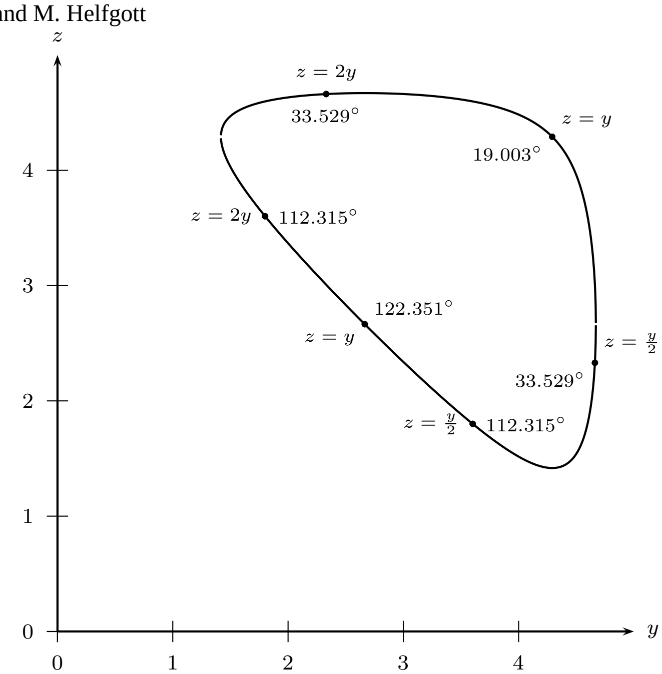


Figure 9. Heron's curve for $A = 3$ and $P = 10$

interval around it depending on A and P by way of (6). To establish sharp bounds for such 'intersecting' r, we observe that these bounds are none other than the slopes of the lines *tangent* to Heron's curve; in the familiar case $A = 3$, $P = 10$, these tangent lines are shown in Figure 8. But a line $z = ry$ is tangent to Heron's curve if and only if there is precisely one triangle of area A and perimeter P satisfying $\frac{z}{y} = r$; that is, if and only if the cubic (6) has a double root.

It is well known (see for example [4, p.91]) that the cubic $ax^3 + bx^2 + cx + d$ has a double root if and only if

$$b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd = 0.$$

(The reader may arrive at this 'tangential' condition independently, arguing as in the proof of Theorem 2.) So we may conclude that the slopes of the two lines tangent to Heron's curve and passing through the origin are the positive roots of the polynomial $S(r) = -64P^2(r+1)^2Q(r)$, where $Q(r)$ is the sixth degree polynomial in (8).

It may not be obvious but Q, and therefore S as well, must have precisely two positive roots, as they ought to. This relies on the following facts (which imply a total of *four* real roots for Q): the leading coefficient of Q is positive and its highest power is even, so $\lim_{r\to\pm\infty} Q(r) = +\infty$; $Q(-1) = -4P^8 - 64P^4A^2 < 0$; $Q(0) = 64P^4A^2 > 0$; $Q(1) = -64A^2(P^4 - (12\sqrt{3})^2A^2) < 0$; $Q(\frac{1}{r}) = \frac{Q(r)}{r^6}$ for $r \neq 0$, so that r is a root of Q if and only if $\frac{1}{r}$ is. □

In the familiar example of $A = 3$ and $P = 10$, the two positive roots of S are $r_1 \approx 0.3273$ and $r_2 \approx 3.0551$. As pointed out above, these two roots are inverses

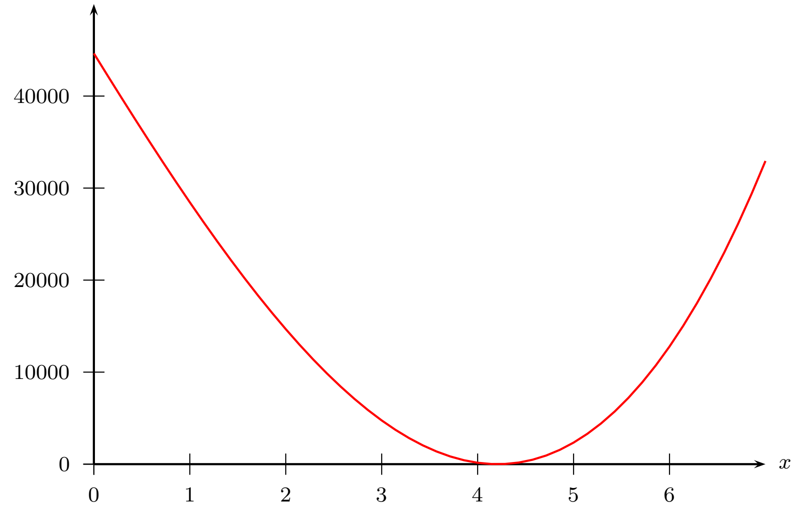


Figure 10. Graph of (6) for $A = 3$, $P = 10$, and $r \approx 3.0551$

of each other: this is geometrically justified by the fact that the two roots are the slopes of the two tangent lines in Figure 8, which are of course mirror images of each other about the diagonal $z = y$. Moreover, r_1 and r_2 lead to the *same* (modulo a factor) cubic in (6).

We conclude that the side ratios of every triangle of area 3 and perimeter 10 must be between approximately 0.3273 and 3.0551. To obtain the unique (modulo reflection) triangle of area 3 and perimeter 10 where these ratios are realized, we need to determine its third side x. It is the double root of the cubic (6) for r equal to approximately 3.0551 (Figure 10). It turns out that x equals approximately 4.2048.

The triangle is now fully determined through $y \approx \frac{10-4.2048}{3.0551+1} \approx 1.4291$ and $z \approx 3.0551 \times 1.4291 \approx 4.366$ (upper 'corner' in Figure 9). The angle-parameter (between sides y and z) at that 'corner' is now easy to find as $\arccos\left(\frac{y^2+z^2-x^2}{2yz}\right) \approx 74.079°$. The triangle obtained, approximately $\{4.2048, 4.3661, 1.4291\}$ (see Figure 11), is the furthest possible from being isosceles - or rather the furthest possible from being equilateral! - among all triangles of area 3 and perimeter 10.

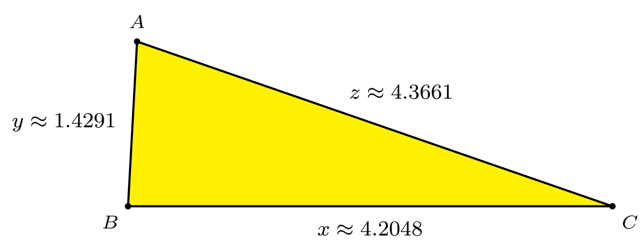


Figure 11. The unique extreme-side-ratio triangle of area 3 and perimeter 10

Our findings are confirmed in Figure 12 by a graph of $\frac{z(\phi)}{y(\phi)}$, where $z(\phi)$ and $y(\phi)$ are the Newton parametrizations of sides z and y in (5). That graph shows a maximum value of about 3.055 for $\frac{z(\phi)}{y(\phi)}$ with ϕ approximately equal to $1.293 \approx 74.08°$:

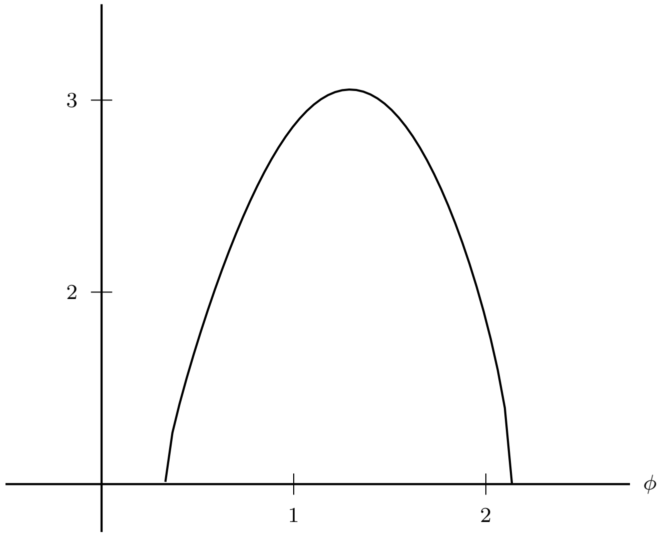


Figure 12. $\frac{z(\phi)}{y(\phi)}$ for $19.003° \approx 0.33166 \leq \phi \leq 2.13543 \approx 122.351°$, $A = 3$ and $P = 10$

References

[1] M. Barabash, A non-trivial counterexample in elementary geometry, *College Math. J.*, 5 (2005) 397–400.
[2] E. Beckenbach and R. Bellman, *An Introduction to Inequalities*, Random House, 1961.
[3] F. Cajori, *History of Mathematics* (4th ed.), Chelsea, 1985.
[4] C. C. MacDuffee, *Theory of Equations*, Wiley, 1954.
[5] I. Niven, *Maxima and Minima without Calculus*, Mathematical Association of America, 1981.
[6] D. T. Whiteside (Ed.), *The Mathematical Works of Isaac Newton* (Vol. 2), Cambridge University Press, 1964.

George Baloglou: Mathematics Department, State University of New York at Oswego, Oswego, New York 13126, USA
E-mail address: `baloglou@oswego.edu`

Michel Helfgott: Mathematics Department, East Tennessee State University, Johnson City, Tennessee 37614, USA
E-mail address: `helfgott@etsu.edu`

Forum Geometricorum
Volume 8 (2008) 27–37.

FORUM GEOM
ISSN 1534-1178

Kronecker's Approximation Theorem and a Sequence of Triangles

Panagiotis T. Krasopoulos

Abstract. We investigate the dynamic behavior of the sequence of nested triangles with a fixed division ratio on their sides. We prove a result concerning a special case that was not examined in [1]. We also provide an answer to an open problem posed in [3].

1. Introduction

The dynamic behavior of a sequence of polygons is an intriguing research area and many articles have been devoted to it (see e.g. [1], [2], [3] and the references therein). The questions that arise about these sequences are mainly two. The first one is about the existence of a limiting point of the sequence. The second one is about the dynamic behavior of the shapes of the polygons that belong to the sequence. Thus, it is possible to find a limiting shape, periodical shapes or an even more complicated behavior. In this article we are interesting for the sequence of triangles with a fixed division ratio on their sides. Let $A_0B_0C_0$ be an initial triangle and let the points A_1 on B_0C_0, B_1 on A_0C_0 and C_1 on A_0B_0 such that:

$$\frac{B_0A_1}{A_1C_0} = \frac{C_0B_1}{B_1A_0} = \frac{A_0C_1}{C_1B_0} = \frac{t}{1-t},$$

where t is a fixed real number in $(0, 1)$. Thus, the next triangle of the sequence is $A_1B_1C_1$. By using the fixed division ratio $t : (1 - t)$ we produce the members of the sequence consecutively (see Figure 1 where $t = \frac{1}{3}$).

In [1] a more complicated sequence of triangles is investigated thoroughly. The author uses complex analysis and so the vertices of a triangle can be defined by three complex numbers A_n, B_n, C_n on the complex plane. The basic iterative process that is studied in [1] has the following matrix form:

$$V_n = TV_{n-1}, \tag{1}$$

where $V_n = \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix}$, $T = \begin{pmatrix} 0 & 1-t & t \\ t & 0 & 1-t \\ 1-t & t & 0 \end{pmatrix}$ is a circulant matrix and V_0 is a given initial triangle. Note that in [1] t is considered generally as a complex number. We stress also that throughout the article we ignore the scaling factor $1/r_n$ that appears at the above iteration in [1]. This factor does not affect the shape of the triangles. As an exceptional case in Section 5 in [1], it is studied the above sequence with t a real number in $(0, 1)$. This is exactly the sequence that

Publication Date: February 4, 2007. Communicating Editor: Paul Yiu.

The author is indebted to the anonymous referee for the valuable suggestions and comments which helped to improve this work.

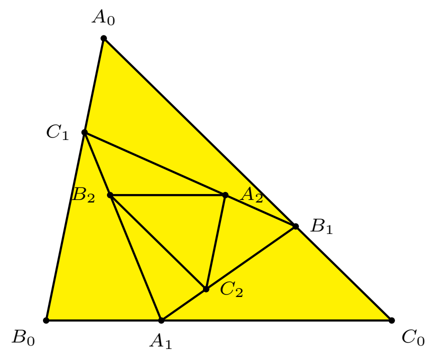


Figure 1.

we described previously and we study in this article. From now on we call this sequence the FDRS (*i.e.*, Fixed Division Ratio Sequence). Concerning the FDRS the author in [1] proved that if

$$t = \frac{1}{2} + \frac{1}{2\sqrt{3}} \tan(a\pi), \tag{2}$$

and a is a rational number, then the FDRS is periodic with respect to the shapes of the triangles. Apparently the same result is proved in [3] (although the proof is left as an exercise). At first sight the formula for the periodicity in [3] seems quite different from (2), but after some algebraic calculations it can be shown that is indeed the same. In [3] it is also proved, that the limiting point of the FDRS is the centroid of the initial triangle $A_0B_0C_0$. Obviously, this is a direct result from the recurrence (1) since it holds $A_{n+1} + B_{n+1} + C_{n+1} = A_n + B_n + C_n$, which means that all the triangles of the FDRS have the same centroid.

In this article we are interested in the behavior of the shapes of the triangles in the FDRS. Particularly, we examine the case when a in (2) is an irrational number. This case was not examined in [1] and [3]. Throughout the article we use the same nomenclature as in [1] and our results are an addendum to [1].

2. Preliminary results

In this Section we will repeat the formulation and the basic results from [1] and we will present some significant remarks. We use the recurrence (1) which is the FDRS as it represented on the complex plane. Without loss of generality as in [1], we can consider that the centroid of the initial triangle $A_0B_0C_0$ is at the origin (*i.e.*, $A_0 + B_0 + C_0 = 0$). This is legitimate since it is just a translation of the centroid to the origin and it does not affect the shapes of the triangles of the FDRS. By using results from circulant matrix theory in [1], it is proved that

$$V_n = T^n V_0 = s_1 \lambda_1^n F_{3,1} + s_2 \lambda_2^n F_{3,2} \tag{3}$$

where $F_{3,1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}$ and $F_{3,2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \omega^2 \\ \omega^4 \end{pmatrix}$ are columns of the 3×3 Fourier matrix $F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix}$. Moreover, $\lambda_j = (1-t)\omega^j + t\omega^{2j}$, $j = 0, 1, 2$ are the eigenvalues of T and $s = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix}$ such that $F_3 s = V_0$. We also consider $\omega = e^{i2\pi/3}$, $\eta = e^{i\pi/3}$ and as $\overline{x}$ we denote the conjugate of x. The following function $z : \mathbb{C}^3 \to \mathbb{C}$ is also defined in [1]:

$$z(V_n) = \frac{C_n - A_n}{B_n - A_n}. \tag{4}$$

This is a very useful function. First, it signifies the orientation of the triangle on the complex plane. Thus, if $\arg(z(V_n)) > 0$ (< 0) the triangle is positively (negatively) oriented (see Figure 2). Note also the angle $\widehat{A}_n$ of the triangle $A_n B_n C_n$ is equal to $\arg(z(V_n))$, so $\widehat{A}_n$ can be regarded as positive or negative.

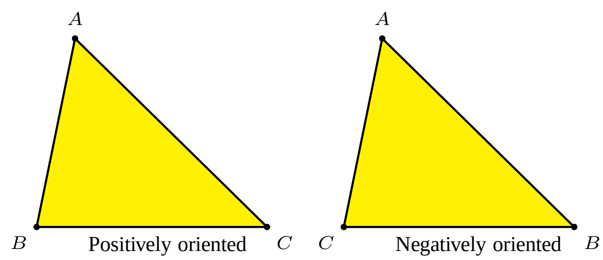


Figure 2.

Function $z(V_n)$ also signifies the ratio of the sides b_n, c_n since $|z(V_n)| = \frac{b_n}{c_n}$. If for instance we have that $|z(V_n)| = 1$, the triangle is isosceles ($b_n = c_n$). If additionally we have $\arg(z(V_n)) = \pi/3$ or $\arg(z(V_n)) = -\pi/3$ then the triangle is equilateral. From this observation we have the following Proposition:

Proposition 1. *A triangle $A_n B_n C_n$ on the complex plane is equilateral if and only if $z(V_n) = \eta$ (positively oriented) or $z(V_n) = \overline{\eta}$ (negatively oriented).*

All these facts stress the importance of function (4). It is apparent that the shape of a triangle on the complex plane is determined completely by function (4). Now, let us assume that the initial triangle $A_0 B_0 C_0$ of the FDRS is not degenerate (*i.e.*, two or three vertices do not coincide and the vertices are not collinear). Moreover, let us assume that $A_0 B_0 C_0$ is not equilateral (*i.e.*, $z(V_0) \neq \eta$ and $z(V_0) \neq \overline{\eta}$), because if it was equilateral then all members of the FDRS would be equilateral triangles. Let us next present two significant definitions and notations.

Firstly, after some algebraic calculations we define the following ratio:

$$\frac{s_2}{s_1} = \frac{B_0 - \omega A_0}{\omega^2 A_0 - B_0} = re^{i\rho}, \tag{5}$$

where $r = |\frac{s_2}{s_1}|$ and $\rho = \arg(\frac{s_2}{s_1})$. Note that (5) holds because we have considered $A_0 + B_0 + C_0 = 0$.

Secondly, from the eigenvalues λ_1 and λ_2 we can get the following definitions

$$\frac{\lambda_2}{\lambda_1} = e^{i\theta}, \quad and \quad \theta = 2\arctan(\sqrt{3}(2t-1)). \tag{6}$$

If we let $\theta = 2\pi a$ in the above equation we get directly equation (2). Now, we can consider the following cases:

(1) $\theta = 0$. In this case we have $t = 1/2$ and all the members of the FDRS are similar to $A_0B_0C_0$.
(2) $\theta = 2k\pi/m$. This case is studied in [1] where $a = k/m$ is rational. We have a periodical behavior and if $(k,m) = 1$ the period is equal to m (otherwise it is smaller than m).
(3) $\theta = 2a\pi$, where a is irrational. This is the case that we study in this article.

In what follows we prove a number of important facts about the FDRS.

Firstly, we note that it holds $s_1 \neq 0$ and $s_2 \neq 0$. This is a straightforward result from the equality $z(V_0) = \frac{s_1\eta + s_2}{s_1 + s_2\eta}$ (see [1]) and from the assumption that $z(V_n) \neq \eta$ and $z(V_n) \neq \overline{\eta}$.

Our next aim is to prove that $r \neq 1$. Let $A_0 = a_1 + ia_2$ and $B_0 = b_1 + ib_2$ and assume that $r = 1$ or equivalently $|B_0 - \omega A_0| = |\omega^2 A_0 - B_0|$. After some algebraic calculations we find $a_1 b_2 = a_2 b_1$, which means that the determinant $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$ and so the vectors $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ are linearly dependent. Thus, $A_0 = \lambda B_0$ where λ is real and $\lambda \neq 0$, $\lambda \neq 1$. Now from (4) we get

$$z(V_0) = \frac{C_0 - A_0}{B_0 - A_0} = \frac{-B_0 - 2A_0}{B_0 - A_0} = -\frac{1 + 2\lambda}{1 - \lambda} \in R.$$

Thus, $\arg(z(V_0)) = 0$ or $\arg(z(V_0)) = \pi$ which is impossible since the initial triangle is not degenerate. Consequently, it holds $r \neq 1$.

Next, we examine the case $r < 1$. From (3) and (6) we have

$$V_n = \lambda_1^n (s_1 F_{3,1} + s_2 e^{in\theta} F_{3,2}).$$

By using the above equation and (5), equation (4) becomes

$$z(V_n) = \frac{s_1\eta + s_2 e^{in\theta}}{s_1 + s_2\eta e^{in\theta}} = \eta \frac{1 + re^{i(\varphi_n - \pi/3)}}{1 + re^{i(\varphi_n + \pi/3)}},$$

where $\varphi_n = n\theta + \rho$. From the above equation we get directly that:

$$\begin{aligned} \arg(z(V_n)) = \; & \widehat{A}_n = \Phi(\varphi_n, r) = \\ = \; & \frac{\pi}{3} + \arctan\frac{r\sin(\varphi_n - \pi/3)}{1 + r\cos(\varphi_n - \pi/3)} - \arctan\frac{r\sin(\varphi_n + \pi/3)}{1 + r\cos(\varphi_n + \pi/3)}, \end{aligned} \tag{7}$$

and

$$|z(V_n)| = \frac{b_n}{c_n} = \mu(\varphi_n, r) = \tag{8}$$
$$= \sqrt{\frac{(1 + r\cos(\varphi_n - \pi/3))^2 + r^2\sin^2(\varphi_n - \pi/3)}{(1 + r\cos(\varphi_n + \pi/3))^2 + r^2\sin^2(\varphi_n + \pi/3)}}.$$

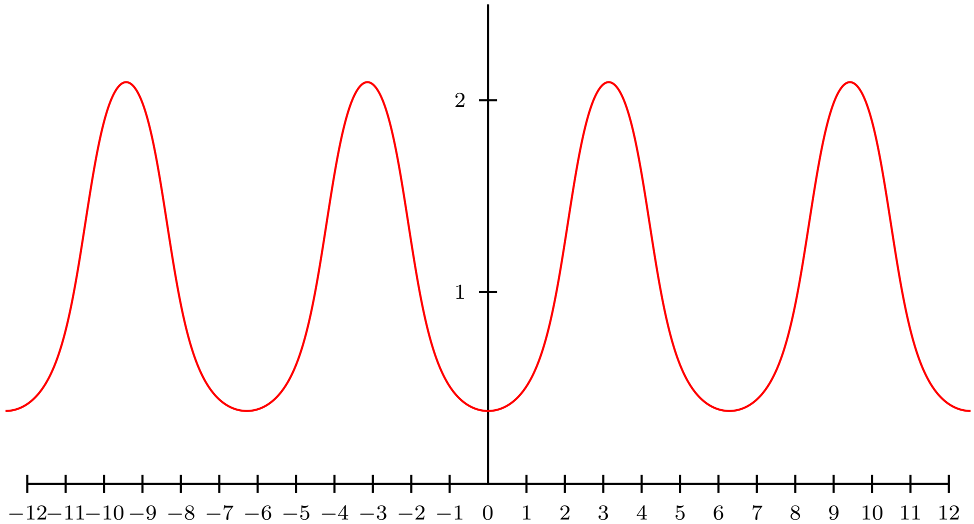


Figure 3(a)

Observe that in (7) and (8) functions $\Phi(\varphi, r)$ and $\mu(\varphi, r)$ are defined respectively. We also define $\Phi(\varphi) = \Phi(\varphi, r)$ and $\mu(\varphi) = \mu(\varphi, r)$. Function $\Phi(\varphi)$ is even (*i.e.*, $\Phi(\varphi) = \Phi(-\varphi)$) and periodic with period 2π (see Figure 3(a) where $r = 0.5$). The minima of $\Phi(\varphi)$ appear at $\varphi = 0, \pm 2\pi, \pm 4\pi, \ldots$ and the maxima at $\varphi = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$. Thus, $\arg(z(V_n)) = \Phi(\varphi_n, r) \in [m_1, m_2]$ where

$$m_1 = \Phi(0, r) = \frac{\pi}{3} - 2\arctan\frac{r\sqrt{3}}{2 + r}, \qquad m_2 = \Phi(\pi, r) = \frac{\pi}{3} + 2\arctan\frac{r\sqrt{3}}{2 - r}.$$

In Figure 3(b), where function Φ is depicted for different values of r, we can observe that the interval $[m_1, m_2]$ decreases as $r \to 0^+$ and increases as $r \to 1^-$. In every case since $r \in (0, 1)$ we find that $[m_1, m_2] \subset (0, \pi)$, which also means that the triangles of the FDRS are positively oriented.

Concerning function $\mu(\varphi)$ we have the following properties: $\mu(k\pi) = 1$ where k is integer, $\mu(-\varphi) = 1/\mu(\varphi)$ and $\mu(\varphi)$ is periodic with period 2π. Figure 3(c) depicts function $\mu(\varphi)$ in $[-4\pi, 4\pi]$ and $r = 0.5$.

Remark. Let us present a fact that we will need in Section 3. Let $r < 1$, since a similar argument applies for $r > 1$. Recall that function $\Phi(\varphi)$ is not injective (one-to-one) and so its inverse can not be determined uniquely. For an angle $\tilde{\theta} \in [m_1, m_2]$ (*i.e.*, $\tilde{\theta}$ belongs to the range of Φ), we want to find the elements $\tilde{\varphi}_m$

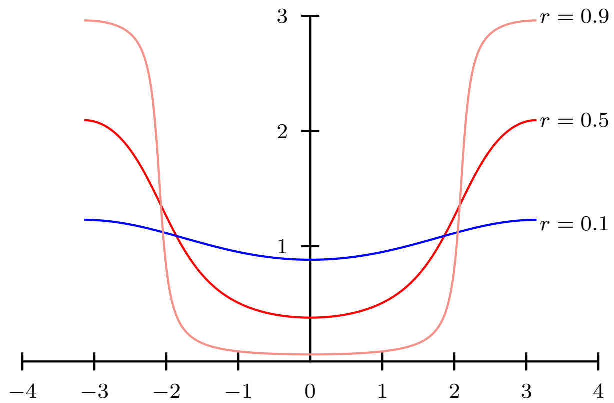


Figure 3(b)

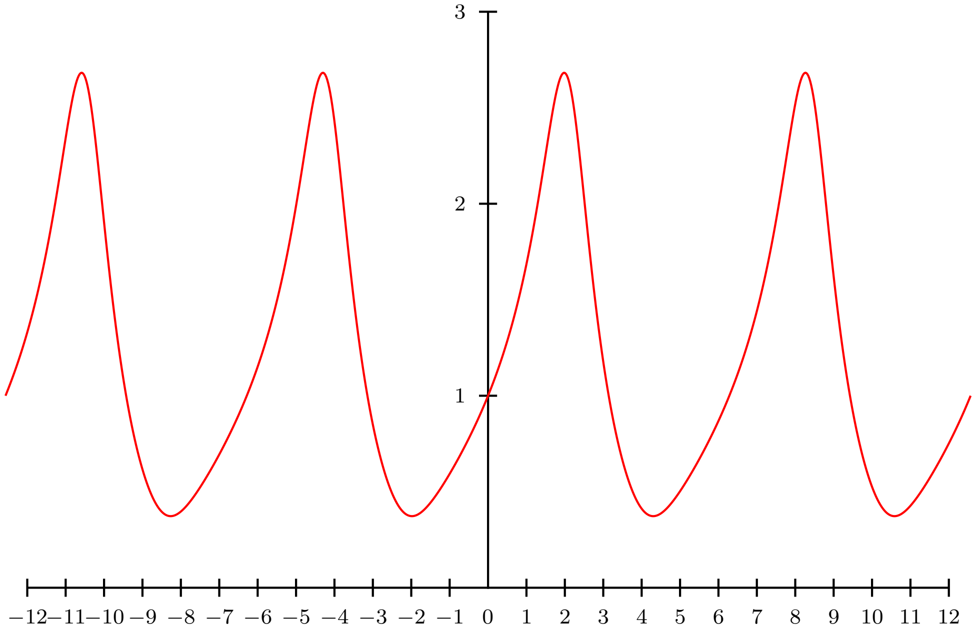


Figure 3(c)

which have the same image $\tilde{\theta}$ (*i.e.*, $\Phi(\tilde{\varphi}_m) = \tilde{\theta}$). Since $\Phi(\varphi)$ is periodic with period 2π, the elements $\tilde{\varphi}_m$ have the form: $2k\pi \pm \varphi_a(\tilde{\theta})$ (k is integer), where as $\varphi_a(\tilde{\theta})$ we define the minimum element $\tilde{\varphi}_m$ such that $\tilde{\varphi}_m \geq 0$ (see Figure 4). Apparently, $\varphi_a(\tilde{\theta}) \in [0, \pi]$ and it holds that $\Phi(2k\pi \pm \varphi_a(\tilde{\theta})) = \tilde{\theta}$ (*i.e.*, all the elements $2k\pi \pm \varphi_a(\tilde{\theta})$ have the same image $\tilde{\theta}$). Figure 4 depicts this characteristic of function $\Phi(\varphi)$.

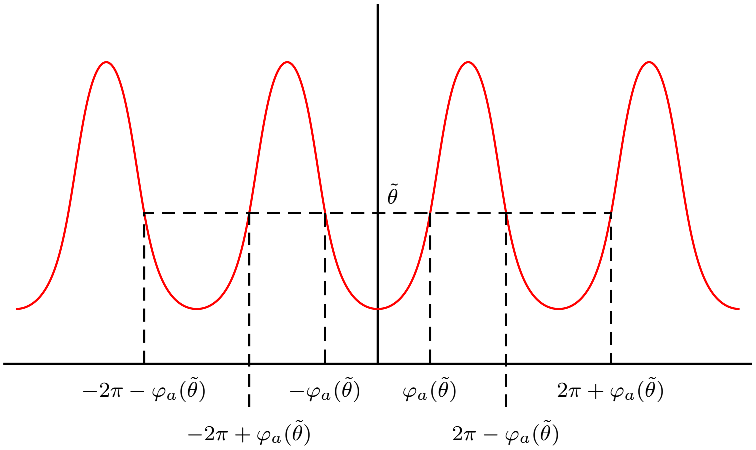


Figure 4.

Now, for the case $|\frac{s_2}{s_1}| = r > 1$ we can use the inverse ratios $\frac{s_1}{s_2} = \frac{1}{r}e^{-i\rho}$, $\frac{\lambda_1}{\lambda_2} = e^{-i\theta}$ and have that

$$z(V_n) = \frac{s_1\eta + s_2 e^{in\theta}}{s_1 + s_2\eta e^{in\theta}} = \overline{\eta}\frac{1+\frac{1}{r}e^{i(-\varphi_n+\pi/3)}}{1+\frac{1}{r}e^{i(-\varphi_n-\pi/3)}},$$

where again $\varphi_n = n\theta + \rho$. From the above we have as before:

$$\arg(z(V_n)) = \widehat{A}_n = -\Phi(-\varphi_n, 1/r) = \tag{9}$$
$$= -\frac{\pi}{3} + \arctan\frac{\frac{1}{r}\sin(-\varphi_n+\pi/3)}{1+\frac{1}{r}\cos(-\varphi_n+\pi/3)} - \arctan\frac{\frac{1}{r}\sin(-\varphi_n-\pi/3)}{1+\frac{1}{r}\cos(-\varphi_n-\pi/3)},$$

and

$$|z(V_n)| = \frac{b_n}{c_n} = \frac{1}{\mu(-\varphi_n, 1/r)} = \tag{10}$$
$$= \sqrt{\frac{(1+\frac{1}{r}\cos(-\varphi_n+\pi/3))^2 + \frac{1}{r^2}\sin^2(-\varphi_n+\pi/3)}{(1+\frac{1}{r}\cos(-\varphi_n-\pi/3))^2 + \frac{1}{r^2}\sin^2(-\varphi_n-\pi/3)}}.$$

It is now obvious that equations (7), (8) and equations (9), (10) signify similar triangles with different orientations provided of course that φ_n and r are common. When $r > 1$ the triangles of the FDRS are negatively oriented. Using similar arguments as before we can prove easily that $\arg(z(V_n)) = \widehat{A}_n \in [\overline{m}_1, \overline{m}_2]$ where

$$\overline{m}_1 = -\Phi(-\pi, 1/r) = -\frac{\pi}{3} - 2\arctan\frac{\frac{1}{r}\sqrt{3}}{2-\frac{1}{r}},$$

$$\overline{m}_2 = -\Phi(0, 1/r) = -\frac{\pi}{3} + 2\arctan\frac{\frac{1}{r}\sqrt{3}}{2+\frac{1}{r}}.$$

Thus, for any $r > 1$ we have $[\overline{m}_1, \overline{m}_2] \subset (-\pi, 0)$. The interval $[\overline{m}_1, \overline{m}_2]$ increases as $r \to 1^+$ and decreases as $r \to +\infty$. In the next Section we apply Kronecker's Approximation Theorem in order to get our main result for the FDRS when a in (2) is an irrational number.

3. Application of Kronecker's approximation theorem

First we present Kronecker's Approximation Theorem (see e.g. [4]).

Kronecker's approximation theorem *If ω is a given irrational number, then the sequence of numbers $\{n\omega\}$, where $\{x\} = x - \lfloor x \rfloor$, is dense in the unit interval. Explicitly, given any $\overline{p}$, $0 \leq \overline{p} \leq 1$, and given any $\epsilon > 0$, there exists a positive integer k such that $|\{k\omega\} - \overline{p}| < \epsilon$.*

We know that $\varphi_n = n\theta + \rho = 2\pi a n + \rho$ and recall that a is irrational and ρ is a function of A_0, B_0, so it is fixed. From Kronecker's Approximation Theorem we know that a member of the sequence $\{na\} = na - \lfloor na \rfloor$ will be arbitrarily close to any given $\overline{p} \in [0, 1]$. Similarly, a member of the sequence $2\pi\{na\} = \varphi_n - 2\pi \lfloor na \rfloor - \rho$ will be arbitrarily close to the angle $\overline{\theta} = 2\pi\overline{p} \in [0, 2\pi]$. Thus, a member of the sequence φ_n will be arbitrarily close to the angle $\overline{\theta} + 2\pi \lfloor na \rfloor + \rho$. Let us now define the sequence of angles φ_n on the unit circle. The quantity $2\pi \lfloor na \rfloor$ defines complete rotations on the unit circle and can be eliminated. This implies that a member of the sequence φ_n will be arbitrarily close to the angle $\overline{\theta} + \rho$ on the unit circle. If additionally, we imagine the unit circle to rotate by $-\rho$, we get that a member of the sequence φ_n will be arbitrarily close to the angle $\overline{\theta} = 2\pi\overline{p}$ on the unit circle. Since this holds for any given $\overline{p} \in [0, 1]$, we conclude that a member of the sequence φ_n will be arbitrarily close to any given angle $\overline{\theta} \in [0, 2\pi]$ on the unit circle. This important fact will be used in the proof of the next Theorem which is the main result of this article. Note that the Theorem uses the notation that has already been presented.

Theorem 2. *Let A_0, B_0, C_0 be complex numbers which define an initial non-degenerate and non-equilateral triangle on the complex plane such that its centroid is at the origin (*i.e., *$A_0 + B_0 + C_0 = 0$). Suppose we apply the FDRS with $t = \frac{1}{2} + \frac{1}{2\sqrt{3}} \tan(a\pi)$ where a is an irrational number. Let $\epsilon_1 > 0$ and $\epsilon_2 > 0$. We have the following cases:*

(1) *If $r = |\frac{s_2}{s_1}| < 1$ (positively oriented triangles), choose a $\widetilde{\theta} \in [m_1, m_2] \subset (0, \pi)$. Then there is a member of the FDRS $A_k B_k C_k$ such that:*

$$|\widehat{A}_k - \widetilde{\theta}| < \epsilon_1,$$

and

$$either \quad \left| \frac{b_k}{c_k} - \mu(\varphi_a(\widetilde{\theta}), r) \right| < \epsilon_2 \quad or \quad \left| \frac{b_k}{c_k} - \mu^{-1}(\varphi_a(\widetilde{\theta}), r) \right| < \epsilon_2.$$

(2) *If $r = |\frac{s_2}{s_1}| > 1$ (negatively oriented triangles), choose a $\widetilde{\theta} \in [\overline{m}_1, \overline{m}_2] \subset (-\pi, 0)$. Then there is a member of the FDRS $A_k B_k C_k$ such that:*

$$|\widehat{A}_k - \widetilde{\theta}| < \epsilon_1,$$

and

$$either \quad \left|\frac{b_k}{c_k} - \mu(\varphi_a(\widetilde{\theta}), 1/r)\right| < \epsilon_2 \quad or \quad \left|\frac{b_k}{c_k} - \mu^{-1}(\varphi_a(\widetilde{\theta}), 1/r)\right| < \epsilon_2.$$

Proof: Let $r < 1$, we have seen that there is φ_k which is arbitrarily close to any given angle on the unit circle. Since function $\Phi(\varphi_n)$ is continuous with respect to φ_n, it is apparent that $\widehat{A}_k = \Phi(\varphi_k)$ can be arbitrarily close to a $\widetilde{\theta}$ chosen from the interval $[m_1, m_2]$ (the range of $\Phi(\varphi_n)$). This proves that $|\widehat{A}_k - \widetilde{\theta}| < \epsilon_1$. Since $\widehat{A}_k = \Phi(\varphi_k)$ can be arbitrarily close to $\widetilde{\theta}$, from Remark we conclude that φ_k will be arbitrarily close to an element of the form $2k\pi \pm \varphi_a(\tilde{\theta})$ (see Figure 4). Since we have considered that φ_k can be defined on the unit circle, we have that φ_k will be arbitrarily close either to $\varphi_a(\tilde{\theta})$ or to $2\pi - \varphi_a(\tilde{\theta})$ which are both defined in $[0, 2\pi]$. Observe that function $\mu(\varphi_n, r)$ is continuous with respect to φ_n and so from equation (8) we get that the ratio $\frac{b_k}{c_k} = \mu(\varphi_k, r)$ will be arbitrarily close either to $\mu(\varphi_a(\widetilde{\theta}), r)$ or to $\mu(2\pi - \varphi_a(\widetilde{\theta}), r) = \mu(-\varphi_a(\widetilde{\theta}), r) = \mu^{-1}(\varphi_a(\widetilde{\theta}), r)$ (recall the properties of function μ). This proves that either $\left|\frac{b_k}{c_k} - \mu(\varphi_a(\widetilde{\theta}), r)\right| < \epsilon_2$ or $\left|\frac{b_k}{c_k} - \mu^{-1}(\varphi_a(\widetilde{\theta}), r)\right| < \epsilon_2$. The case $r > 1$ can be treated analogously. This completes the proof. $\square$

Concerning Theorem 2 we stress that ϵ_1 and ϵ_2 can be chosen independently. This is true since from the Kronecker's Approximation Theorem we can always find a φ_k as close as we want to a given $\widetilde{\theta}$. This implies that the angle $\widehat{A}_k$ can be as close as we want to $\widetilde{\theta}$, and so the ratio $\frac{b_k}{c_k}$ will be as close as we want either to $\mu(\varphi_a(\widetilde{\theta}), r)$ or to $\mu^{-1}(\varphi_a(\widetilde{\theta}), r)$. Ultimately, a φ_k will satisfy both inequalities no matter how small ϵ_1 and ϵ_2 are.

Although Theorem 2 and the analysis so far seem quite complicated, they have some interesting consequences. In what follows we consider that t is fixed and a is an irrational number as in Theorem 2.

We proved that there will be a member of the FDRS with an angle $\widehat{A}_k$ that will be arbitrarily close to any given $\widetilde{\theta} \in [m_1, m_2]$ or $\widetilde{\theta} \in [\overline{m}_1, \overline{m}_2]$. This means that the countable set of the angles $\widehat{A}_n$ (*i.e.*, $\{\widehat{A}_0, \widehat{A}_1, \ldots\}$) is dense in $[m_1, m_2]$ or in $[\overline{m}_1, \overline{m}_2]$. Also by choosing ϵ_1, ϵ_2 as small as we want, we expect that some members $A_kB_kC_k$ of the FDRS will have their shapes as follows: $\widehat{A}_k \simeq \widetilde{\theta}$ and either $\frac{b_k}{c_k} \simeq \mu(\varphi_a(\widetilde{\theta}), r)$ or $\frac{b_k}{c_k} \simeq \mu^{-1}(\varphi_a(\widetilde{\theta}), r)$.

Let us now find if there is a member of the FDRS that is arbitrarily close to an equilateral triangle. If this was true then $\frac{b_k}{c_k}$ should be arbitrarily close to the unity. Thus from Theorem 2 (assume that $r < 1$ since for $r > 1$ the same argument applies), $\mu(\varphi_a(\widetilde{\theta}), r) = 1$ and from Section 2 we know that $\varphi_a(\widetilde{\theta}) = 0$ or $\varphi_a(\widetilde{\theta}) = \pi$. From these equalities we get $\widetilde{\theta} = m_1$ or $\widetilde{\theta} = m_2$. It should also hold that $\widetilde{\theta} = \pi/3$ (positively oriented equilateral triangle). So, it should be $m_1 = \pi/3 \Rightarrow r = 0$ or $m_2 = \pi/3 \Rightarrow r = 0$. Obviously, $r = 0$ is impossible. Consequently, for a specific $r > 0$ all the members of the FDRS will have at least a constant discrepancy from the shape of an equilateral triangle. This discrepancy can not be further decreased for a fixed $r > 0$, it can only be reduced if we chose another $r > 0$ closer to zero.

Let an isosceles triangle with $b = c$ and $\widehat{A} = \widetilde{\theta} < \pi/3$ be given. We want to find the value of $r < 1$ that will give a member of the FDRS arbitrarily close to the isosceles triangle. In the previous paragraph we show that for this case it holds $\widetilde{\theta} = m_1$ or $\widetilde{\theta} = m_2$. Let $\widetilde{\theta} = m_1$ and we have

$$\widetilde{\theta} = m_1 \Longleftrightarrow 2\arctan\frac{r\sqrt{3}}{2+r} = \frac{\pi}{3} - \widetilde{\theta} \Longleftrightarrow r = \frac{2\tan(\frac{\pi}{6} - \frac{\widetilde{\theta}}{2})}{\sqrt{3} - \tan(\frac{\pi}{6} - \frac{\widetilde{\theta}}{2})}.$$

The above formula gives the value of r for which a member of the FDRS would be arbitrarily close to the isosceles triangle with $\widehat{A} = \widetilde{\theta} < \pi/3$. The corresponding formula for an isosceles triangle with $b = c$ and a given $\widehat{A} = \widetilde{\theta} > \pi/3$ is

$$\widetilde{\theta} = m_2 \Longleftrightarrow 2\arctan\frac{r\sqrt{3}}{2-r} = \widetilde{\theta} - \frac{\pi}{3} \Longleftrightarrow r = \frac{2\tan(\frac{\widetilde{\theta}}{2} - \frac{\pi}{6})}{\sqrt{3} + \tan(\frac{\widetilde{\theta}}{2} - \frac{\pi}{6})}.$$

In the next Section we offer a simple geometric presentation of the FDRS, we examine closer the significance of the parameters r and φ_n and we answer a question posed in [3].

4. Geometric interpretations and final remarks

We have seen that equation (3) is the solution of the recurrence (1) provided that $A_0 + B_0 + C_0 = 0$. We can rewrite (3) as follows:

$$V_n = \frac{s_1\lambda_1^n}{\sqrt{3}}\left[\begin{pmatrix}1\\ \omega\\ \omega^2\end{pmatrix} + \frac{s_2}{s_1}\left(\frac{\lambda_2}{\lambda_1}\right)^n\begin{pmatrix}1\\ \omega^2\\ \omega\end{pmatrix}\right].$$

In this article we are interested in the shapes of the triangles. The complex number $\frac{s_1\lambda_1^n}{\sqrt{3}}$ at the above equation signifies a scaling factor and a rotation of the triangle V_n, and so it does not affect its shape. This means that we can define the shapes of the triangles of the FDRS simply as

$$S_n = P + re^{i\varphi_n}N, \tag{11}$$

where $P = \begin{pmatrix}1\\ \omega\\ \omega^2\end{pmatrix}$, $N = \begin{pmatrix}1\\ \omega^2\\ \omega\end{pmatrix}$ and r, φ_n as in Section 2. We stress that the triangles V_n and S_n have the same shape (*i.e.*, they are similar and they have the same orientation). Note also that P is a positively oriented equilateral triangle inscribed in the unit circle and N is a negatively oriented equilateral triangle inscribed in the unit circle (1, ω, ω^2 are the third roots of unity). It can be seen now that every member of the FDRS on the complex plane is represented as the sum of two equilateral triangles: P and $re^{i\varphi_n}N$. It is now obvious that the parameter r is the circumradius and the parameter φ_n is the angle of rotation of the equilateral triangle rN at the nth iteration. Thus, the parameters r and φ_n determine completely the contribution of the negatively oriented triangle in (11).

Let us next consider an open problem that is posed in [3]. The authors of [3] asked to find all values of the division ratio $t \in (0,1)$ for which the FDRS is divergent in shape. From the analysis so far, we have seen that the division ratio t can be given by equation (2). Equation (2) defines a function $t = t(a)$ which is one-to-one and for $a \in (-\frac{1}{3}, \frac{1}{3})$ its range is $(0,1)$. Thus, we can describe the behavior of the members of the FDRS with respect to t, by using equation (2). Similar to the analysis of Section 2 we have the following cases:

(1) $a = 0$. Equation (2) implies $t = \frac{1}{2}$. In this case all the members of the FDRS are similar to $A_0B_0C_0$ and the sequence is convergent in shape.
(2) $a \neq 0$ is a rational number in $(-\frac{1}{3}, \frac{1}{3})$ and t is given by (2). The FDRS is periodic in shape.
(3) a is an irrational number in $(-\frac{1}{3}, \frac{1}{3})$ and t is given by (2). From the analysis of Section 3 we conclude that the FDRS is neither convergent nor periodic in shape.

Thus, only when $t = \frac{1}{2}$ we have that the FDRS is convergent in shape. The second case above gives the values of t for which the FDRS is periodic in shape. The last case is described by Theorem 2 and the behavior of the FDRS is rather complex since it is neither convergent nor periodic in shape.

It is clear that only the change of an a rational to an a irrational in (2) is enough to produce a complicated dynamic behavior of the FDRS. We believe that only results of qualitative character like Theorem 2 can be used to describe this sequence of triangles. However, it would be interesting if one could prove another result (e.g. a statistical result), for the behavior of the FDRS when a is an irrational number.

References

[1] B. Ziv, Napoleon-like configurations and sequences of triangles, *Forum Geom.*, 2 (2002) 115–128.
[2] L.R. Hitt and X.-M. Zhang, Dynamic geometry of polygons, *Elem. Math.*, 56 (2001) 21–37.
[3] D. Ismailescu and J. Jacobs, On sequences of nested triangles, *Period. Math. Hungar.*, 53 (2006) 169–184.
[4] E.W. Weisstein, Kronecker's approximation theorem, *Mathworld*, `http://mathworld.wolfram.com/KroneckersApproximationTheorem.html`

Panagiotis T. Krasopoulos: Skra 59, 176 73 Kallithea, Athens, Greece
E-mail address: `pankras@in.gr`

Forum Geometricorum
Volume 8 (2008) 39–42.

FORUM GEOM
ISSN 1534-1178

A Short Trigonometric Proof of the Steiner-Lehmus Theorem

Mowaffaq Hajja

Abstract. We give a short trigonometric proof of the Steiner-Lehmus theorem.

The well known Steiner-Lehmus theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles. Unlike its trivial converse, this challenging statement has attracted a lot of attention since 1840, when Professor Lehmus of Berlin wrote to Sturm asking for a purely geometrical proof. Proofs by Rougevain, Steiner, and Lehmus himself appeared in the following few years. Since then, a great number of people, including several renowned mathematicians, took interest in the problem, resulting in as many as 80 different proofs. Extensive histories are given in [14], [15], [16], and [21], and biographies and lists of references can be found in [33], [37], and [19]. More references will be referred to later when we discuss generalizations and variations of the theorem.

In this note, we present a new trigonometric proof of the theorem. Compared with the existing proofs, such as the one given in [17, pp. 194–196], it is also short and simple. It runs as follows.

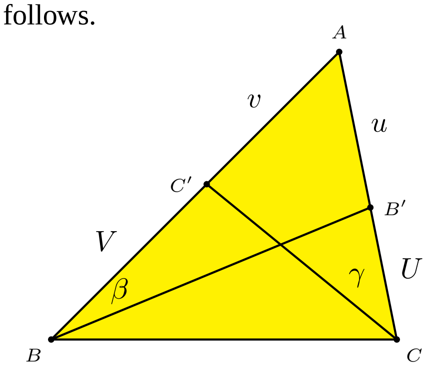


Figure 1

Let BB' and CC' be the respective internal angle bisectors of angles B and C in triangle ABC, and let a, b and c denote the sidelengths in the standard order. As shown in Figure 1, we set

$$B = 2\beta,\ C = 2\gamma, \quad u = AB',\ U = B'C, \quad v = AC',\ V = C'B.$$

Publication Date: February 18, 2008. Communicating Editor: Floor van Lamoen.

This work is supported by a research grant from Yarmouk University.

The author would like to thank the referee for suggestions that improved the exposition and for drawing his attention to references [4], [19], [33], and [37].

We shall see that the assumptions $BB' = CC'$ and $C > B$ (and hence $c > b$) lead to the contradiction that

$$\frac{b}{u} < \frac{c}{v}, \ \frac{b}{u} > \frac{c}{v}. \tag{1}$$

Geometrically, this means that the line $B'C'$ intersects both rays BC and CB.

To achieve (1), we use the law of sines, the angle bisector theorem, and the identity $\sin 2\theta = 2\sin\theta\cos\theta$ to obtain

$$\frac{b}{u} - \frac{c}{v} = \frac{u+U}{u} - \frac{v+V}{v} = \frac{U}{u} - \frac{V}{v} = \frac{a}{c} - \frac{a}{b} < 0, \tag{2}$$

$$\begin{aligned}\frac{b}{u} \div \frac{c}{v} &= \frac{b}{c}\,\frac{v}{u} = \frac{\sin B}{\sin C}\,\frac{v}{u} = \frac{2\cos\beta\sin\beta}{2\cos\gamma\sin\gamma}\,\frac{v}{u} = \frac{\cos\beta}{\cos\gamma}\,\frac{\sin\beta}{u}\,\frac{v}{\sin\gamma}\\ &= \frac{\cos\beta}{\cos\gamma}\,\frac{\sin A}{BB'}\,\frac{CC'}{\sin A} = \frac{\cos\beta}{\cos\gamma} > 1.\end{aligned} \tag{3}$$

Clearly (2) and (3) lead to the contradiction (1).

No new proofs of the Steiner-Lehmus theorem seem to have appeared in the past several decades, and attention has been focused on generalizations, variations, and certain foundational issues. Instead of taking angle bisectors, one may take r-sectors, i.e., cevians that divide the angles internally in the ratio $r : 1 - r$ for $r \in (0, 1)$. Then the result still holds; see [35], [15, X, p. 311], [36], and more recently, [5], [2], and [10]. In fact, the result still holds in absolute (or neutral) geometry; see [15, X, p. 311] and the references therein, and more recently [6, Exercise 7, p. 9; solution, p. 420] and [19, Exercise 15, p. 119]. One may also consider external angle bisectors. Then one sees that the equality of two external angle bisectors (and similarly the equality of one internal and one external angle bisectors) does not imply isoscelessness. This is considered in [16], [22], [23], and more recently in [11]; see also [30] and the references therein. The situation in spherical geometry was also considered by Steiner; see [16, IX, p. 310].

Variations on the Steiner-Lehmus theme have become popular in the past few decades with much of the contribution due to the late C. F. Parry. Here, one starts with a center P of triangle ABC, not necessarily the incenter, and lets the cevians AA', BB', CC' through P intersect the circumcircle of ABC at A^*, B^*, C^*, respectively. The classical Steiner-Lehmus theorem deals with the case when P is the incenter and considers the assumption $BB' = CC'$. One may start with any center and consider any of the assumptions $BB' = CC'$, $BB^* = CC^*$, $A'B' = A'C'$, $A^*B^* = A^*C^*$, etc. Such variations and others have appeared in [27], [28], [29], [34], [3], [12], [32], [31], [1], and [26, Problem 4, p. 31], and are surveyed in [13]. Some of these variations have been investigated in higher dimensions in [7] and interesting results were obtained. However, the generalization of the classical Steiner-Lehmus theorem to higher dimensions remains open: We still do not know what degree of regularity a d-simplex must enjoy so that two or even all the internal angle bisectors of the corner angles are equal. This problem is raised at the end of [7].

The existing proofs of the Steiner-Lehmus theorem are all indirect (many being proofs by contradiction or *reductio ad absurdum*) or use theorems that do not have direct proofs. The question, first posed by Sylvester in [36], whether there is a direct proof of the Steiner-Lehmus theorem is still open, and Sylvester's conjecture (and semi-proof) that no such proof exists seems to be commonly accepted; see the refutation made in [20] of the allegedly direct proof given in [24], and compare to [8], where we are asked on p. 58 (Problem 16) to *give a direct proof of the Steiner-Lehmus theorem*, and where such a *a proof* is given on p. 390 using Stewart's theorem. An interesting forum discussion can also be visited at [9]. We would like here to raise the question whether one can provide a direct proof of the following weaker version of the Steiner-Lehmus theorem: *If the three internal angle bisectors of the angles of a triangle are equal, then the triangle is equilateral.*

References

[1] S. Abu-Saymeh, M. Hajja, and H. A. ShahAli, A variation on the Steiner-Lehmus theme, preprint.

[2] R. Barbara, A quick proof of a generalised Steiner-Lehmus theorem, *Math. Gaz.*, 81 (1997) 450–451.

[3] S. Bilir and N. Ömür, A remark on *Steiner-Lehmus and the automedian triangle*, *Math. Gaz.*, 88 (2004) 134–136.

[4] O. Bottema, Verscheidenheden XVII: De driehoek met twee gelijke bissectrices, in *Verscheidenheden*, 15–18, Groningen, 1978.

[5] F. Chorlton, A generalisation of the Steiner-Lehmus theorem, *Math. Gaz.*, 69 (1985) 215–216.

[6] H. S. M. Coxeter, *Introduction to Geometry*, John Wiley and Sons, Inc., New York, 1969.

[7] A. L. Edmonds, M. Hajja, and H. Martini, Coincidences of simplex centers and related facial structures, *Beitr. Algebra Geom.*, 46 (2005) 491–512.

[8] H. Eves, *A Survey of Geometry*, Allyn and Bacon, Inc., Boston, 1978.

[9] R. Guy, Hyacinthos message 1410, September 12, 2000.

[10] M. Hajja, An analytical proof of the generalized Steiner-Lehmus theorem, *Math. Gaz.*, 83 (1999) 493–495.

[11] M. Hajja, Other versions of the Steiner-Lehmus theorem, *Amer. Math. Monthly*, 108 (2001) 760–767.

[12] M. Hajja, Problem 1704, *Math. Mag.*, 77 (2004) 320; solution, ibid., 78 (2005) 326–327.

[13] M. Hajja, C. F. Parry's variations on the Steiner-Lehmus theme, preprint.

[14] A. Henderson, A classic problem in Euclidean geometry, *J. Elisha Mitchell Soc.*, (1937), 246–281.

[15] A. Henderson, The Lehmus-Steiner-Terquem problem in global survey, *Scripta Mathematica*, 21 (1955) 223–232.

[16] A. Henderson, The Lehmus-Steiner-Terquem problem in global survey, *Scripta Mathematica*, 21 (1955) 309–312.

[17] R. Honsberger, *In Polya's Footsteps*, Dolciani Math. Expositions, No. 19, Math. Assoc. America, Washington, D. C., 1997.

[18] D. C. Kay, *College Geometry*, Holt, Rinehart and Winston, Inc., New York, 1969.

[19] D. C. Kay, Nearly the last comment on the Steiner-Lehmus theorem, *Crux Math.*, 3 (1977) 148–149.

[20] M. Lewin, On the Steiner-Lehmus theorem, *Math. Mag.*, 47 (1974) 87–89.

[21] J. S. Mackay, History of a theorem in elementary geometry, *Edinb. Math. Soc. Proc.*, 20 (1902), 18–22.

[22] D. L. MacKay, The pseudo-isosceles triangle, *School Science and Math.*, 464–468.

[23] D. L. MacKay, Problem E312, *Amer. Math. Monthly*, 45 (1938) 47; solution, ibid., 45 (1938) 629–630.

[24] J. V. Maleševič, A direct proof of the Steiner-Lehmus theorem, *Math. Mag.*, 43 (1970) 101–103.

[25] J. A. McBride, The equal internal bisectors theorem, *Proc. Edinburgh Math. Soc. Edinburgh Math. Notes*, 33 (1943) 1–13.
[26] W. Mientka, *Mathematical Olympiads 1996–1997: Olympiad Problems from Around the World*, American Mathematical Competitions, available at `http://www.unl.edu/amc/a-activities/a4-for-students/problemtext/mc96-97-01feb.pdf`.
[27] C. F. Parry, A variation on the Steiner-Lehmus theme, *Math. Gaz.*, 62 (1978) 89–94.
[28] C. F. Parry, Steiner-Lehmus and the automedian triangle, *Math. Gaz.*, 75 (1991) 151–154.
[29] C. F. Parry, Steiner-Lehmus and the Feuerbach triangle, *Math. Gaz.*, 79 (1995) 275–285.
[30] K. R. S. Sastry, Problem 862, *Math. Mag.*, 56 (1973); solution, ibid., 57 (1974) 52–53.
[31] K. R. S. Sastry, Problem 967, *Math. Mag.*, 49 (1976) 43; solution, ibid., 50 (1977) 167.
[32] K. R. S. Sastry, A Gergonne analogue of the Steiner-Lehmus theorem, *Forum Geom.*, 5 (2005) 191–195.
[33] L. Sauvé, The Steiner-Lehmus theorem, *Crux Math.*, 2 (1976) 19–24.
[34] J. A. Scott, Steiner-Lehmus revisited, *Math. Gaz.*, 87 (2003) 561–562.
[35] J. Steiner, The solution in Crelle's J. XXVIII (1844), (reproduced in Steiner's Gesammelte Werke II 323-b (1882)).
[36] J. J. Sylvester, On a simple geometric problem illustrating a conjectured principle in the theory of geometrical method, *Phil. Magazine*, 4 (1852), reproduced in *Collected Works*, volume I, 392–395.
[37] C. W. Trigg, A bibliography of the Steiner-Lehmus theorem, *Crux Math.*, 2 (1976) 191–193.

Mowaffaq Hajja: Mathematics Department, Yarmouk University, Irbid, Jordan
E-mail address: `mhajja@yu.edu.jo, mowhajja@yahoo.com`

Forum Geometricorum
Volume 8 (2008) 43–48.

FORUM GEOM
ISSN 1534-1178

On the Parry Reflection Point

Cosmin Pohoata

Abstract. We give a synthetic proof of C. F. Parry's theorem that the reflections in the sidelines of a triangle of three parallel lines through the vertices are concurrent if and only if they are parallel to the Euler line, the point of concurrency being the Parry reflection point. We also show that the Parry reflection point is common to a triad of circles associated with the tangential triangle and the triangle of reflections (of the vertices in their opposite sides). A dual result is also given.

1. The Parry reflection point

Theorem 1 (Parry). *Suppose triangle ABC has circumcenter O and orthocenter H. Parallel lines α, β, γ are drawn through the vertices A, B, C, respectively. Let α', β', γ' be the reflections of α, β, γ in the sides BC, CA, AB, respectively. These reflections are concurrent if and only if α, β, γ are parallel to the Euler line OH. In this case, their point of concurrency P is the reflection of O in E, the Euler reflection point.*

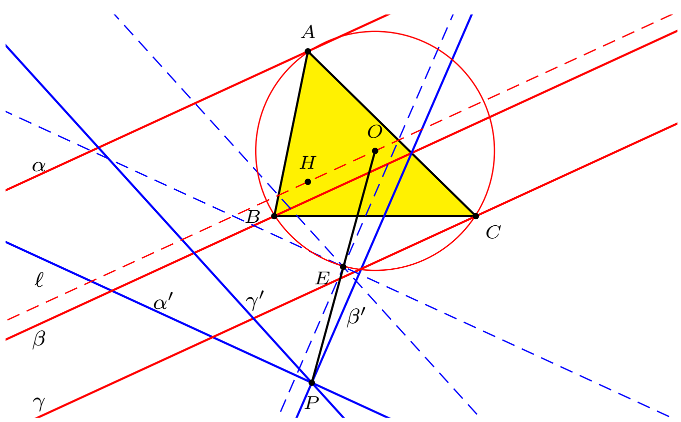


Figure 1.

We give a synthetic proof of this beautiful theorem below. C. F. Parry proposed this as a problem in the AMERICAN MATHEMATICAL MONTHLY, which was subsequently solved by R. L. Young using complex coordinates [6]. The point P in question is called the Parry reflection point. It appears as the triangle center X_{399}

Publication Date: February 25, 2008. Communicating Editor: Paul Yiu.

The author thanks an anonymous referee for suggestions leading to improvement of the paper, especially on the proof of Theorem 3.

in [5]. The Euler reflection point E, on the other hand, is the point on the circumcircle which is the point of concurrency of the reflections of the Euler line in the sidelines. See Figure 1. It appears as X_{110} in [5]. The existence of E is justified by another elegant result on reflections of lines, which we use to deduce Theorem 1.

Theorem 2 (Collings)**.** *Let ℓ be a line in the plane of a triangle ABC. Its reflections in the sidelines BC, CA, AB are concurrent if and only if ℓ passes through the orthocenter H of ABC. In this case, their point of concurrency lies on the circumcircle.*

Synthetic proofs of Theorem 2 can be found in [1] and [3].

We denote by A', B', C' the reflections of A, B, C in their opposite sides, and by $A_tB_tC_t$ the tangential triangle of ABC.

Theorem 3. *The circumcircles of triangles $A_tB'C'$, $B_tC'A'$ and $C_tA'B'$ are concurrent at Parry's reflection point P.* See Figure 2.

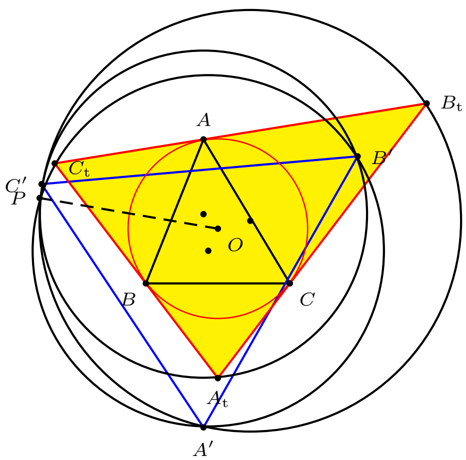


Figure 2

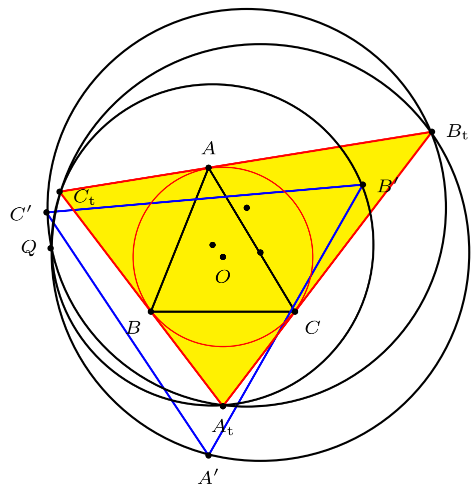


Figure 3

Theorem 4. *The circumcircles of triangles $A'B_tC_t$, $B'C_tA_t$ and $C'A_tB_t$ have a common point Q.* See Figure 3.

2. Proof of Theorem 1

Let $A_1B_1C_1$ be the image of ABC under the homothety $\mathsf{h}(O, 2)$. The orthocenter H_1 of $A_1B_1C_1$ is the reflection of O in H, and is on the Euler line of triangle ABC.

Consider the line ℓ through H parallel to the given lines α, β, γ. Let M be the midpoint of BC, and $M_1 = \mathsf{h}(O, 2)(M)$ on the line B_1C_1. The line AH intersects

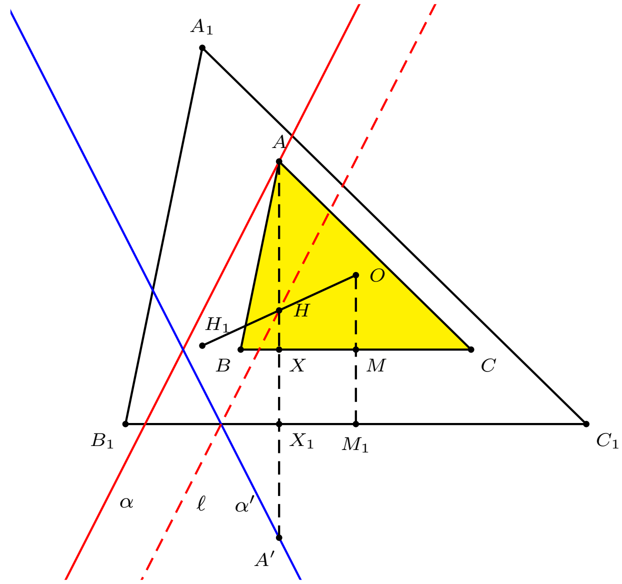


Figure 4.

BC and B_1C_1 at X and X_1 respectively. Note that the reflection of H in B_1C_1 is the reflection D of A in BC since $AH = 2 \cdot OM$ and

$$\begin{aligned} HA' =& AA' - AH = 2(AX - OM) = 2(AH + HX - OM) \\ =& 2(HX + OM) = 2(HX + XX_1) = 2HX_1. \end{aligned}$$

Therefore, α' coincides with the reflection of ℓ in the sides B_1C_1. Similarly, β' and γ' coincide with the reflections of ℓ in C_1A_1 and A_1B_1. By Theorem 2, the lines α', β', γ' are concurrent if and only if ℓ passes through the orthocenter H_1. Since H also lies on ℓ, this is the case when ℓ is the Euler line of triangle ABC, which is also the Euler line of triangle $A_1B_1C_1$. In this case, the point of concurrency is the Euler reflection point of $A_1B_1C_1$, which is the image of E under the homothety $\mathsf{h}(O, 2)$.

3. Proof of Theorem 3

We shall make use of the notion of directed angles $(\ell_1, \ \ell_2)$ between two lines ℓ_1 and ℓ_2 as the angle of rotation (defined modulo π) that will bring ℓ_1 to ℓ_2 in the same orientation as ABC. For the basic properties of directed angles, see [4, §§16–19].

Let α, β, γ be lines through the vertices A, B, C, respectively parallel to the Euler line. By Theorem 1, their reflections α', β', γ' in the sides BC, CA, AB pass through the Parry reflection point P.

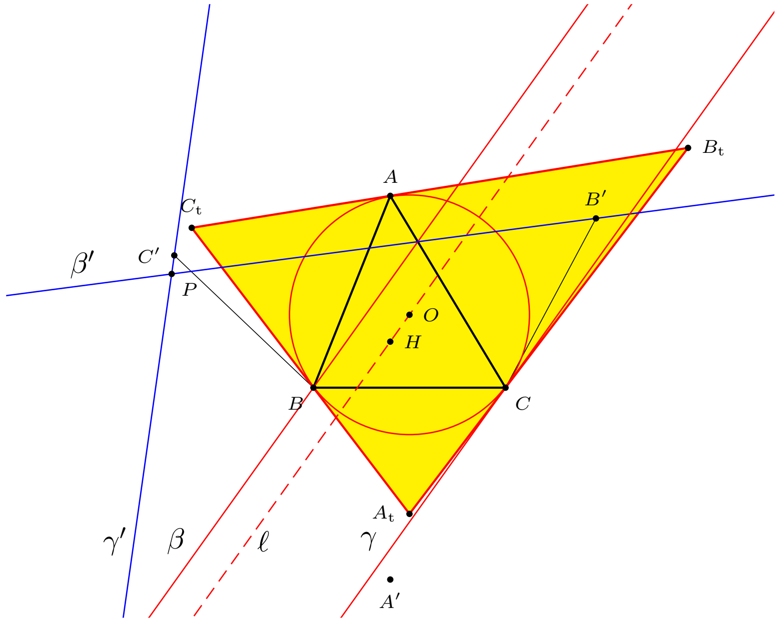


Figure 5

Now, since α, β, γ are parallel,

$$\begin{aligned}
(PB',\ PC') &= (\beta', \gamma') \\
&= (\beta', BC) + (BC, \gamma') \\
&= -(\beta', B'C) - (BC', \gamma') \quad \text{because of symmetry in } AC \\
&= (B'C,\ \beta) + (\gamma,\ BC') \\
&= (B'C,\ \beta) + (\beta,\ BC') \\
&= (B'C,\ BC') \\
&= (B'C,\ AC) + (AC,\ BC') \\
&= (AC,\ BC) + (AC,\ BC') \quad \text{because of symmetry in } AC \\
&= (AC,\ AB) + (AB,\ BC) + (AC,\ AB) + (AB,\ BC') \\
&= 2(AC,\ AB) \quad \text{because of symmetry in } AB \\
&= (OC,\ OB) \\
&= (A_tC,\ A_tB).
\end{aligned}$$

Since $A_tB = A_tC$ and $BC' = BC = B'C$, we conclude that the triangles A_tBC' and A_tCB' are directly congruent. Hence, $(A_tB',\ A_tC') = (A_tC,\ A_tB)$. This gives $(PB',\ PC') = (A_tB',\ A_tC')$, and the points P, A_t, B', C' are concyclic. The circle $A_tB'C'$ contains the Parry reflection point, so do the circles $B_tC'A'$ and $C_tA'B'$.

4. Proof of Theorem 4

Invert with respect to the Parry point P. By Theorem 3, the circles $A_tB'C'$, $B_tC'A'$, $C_tA'B'$ are inverted into the three lines bounding triangle $A'^*B'^*C'^*$. Here, A'^*, B'^*, C'^* are the inversive images of A', B', C' respectively. Since the points A_t^*, B_t^*, C_t^* lie on the lines $B'^*C'^*$, $C'^*A'^*$, $A'^*B'^*$, respectively, by Miquel's theorem, the circumcircles of triangles $A_t^*B'^*C'^*$, $B_t^*C'^*A'^*$, $C_t^*A'^*B'^*$ have a common point; so do their inversive images, the circles $A_tB'C'$, $B_tC'A'$, $C_tA'B'$. This completes the proof of Theorem 4.

The homogenous barycentric coordinates of their point of concurrency Q were computed by Javier Francisco Garcia Capitán [2] with the aid of Mathematica.

Added in proof. After the completion of this paper, we have found that the points P and Q are concyclic with the circumcenter O and the orthocenter H. See Figure 6. Paul Yiu has confirmed this by computing the coordinates of the center of the circle of these four points:

$$\begin{aligned}&(a^2(b^2-c^2)(a^8(b^2+c^2)-a^6(4b^4+3b^2c^2+4c^4)+2a^4(b^2+c^2)(3b^4-2b^2c^2+3c^4)\\&\quad -a^2(4b^8-b^6c^2+b^4c^4-b^2c^6+4c^8)+(b^2-c^2)^2(b^2+c^2)(b^4+c^4)))\\&:\cdots:\cdots),\end{aligned}$$

where the second and third coordinates are obtained by cyclic permutations of a, b, c.

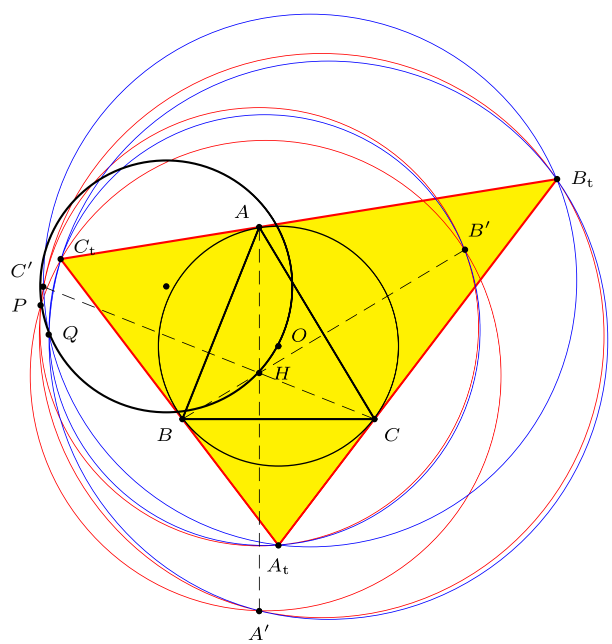


Figure 6.

For completeness, we record the coordinates of Q given by Garcia Capitán:

$$Q = (a^2 \sum_{k=0}^{10} a^{2(10-k)} f_{2k,a}(b,c) : b^2 \sum_{k=0}^{10} b^{2(10-k)} f_{2k,b}(c,a) : c^2 \sum_{k=0}^{10} c^{2(10-k)} f_{2k,c}(a,b)),$$

where

$$\begin{aligned}
f_{0,a}(b,c) &= 1,\\
f_{2,a}(b,c) &= -6(b^2+c^2)\\
f_{4,a}(b,c) &= 2(7b^4+12b^2c^2+7c^4),\\
f_{6,a}(b,c) &= -2(b^2+c^2)(7b^4+10b^2c^2+7c^4),\\
f_{8,a}(b,c) &= b^2c^2(18b^4+25b^2c^2+18c^4),\\
f_{10,a}(b,c) &= (b^2+c^2)(14b^8-15b^6c^2+8b^4c^4-15b^2c^6+14c^8),\\
f_{12,a}(b,c) &= -14b^{12}+b^{10}c^2+5b^8c^4-2b^6c^6+5b^4c^8+b^2c^{10}-14c^{12},\\
f_{14,a}(b,c) &= (b^2-c^2)^2(b^2+c^2)(6b^8+2b^6c^2+5b^4c^4+2b^2c^6+6c^8),\\
f_{16,a}(b,c) &= -(b^2-c^2)^2(b+c)^2(b^{12}-2b^{10}c^2-b^8c^4-6b^6c^6-b^4c^8-2b^2c^{10}+c^{12}),\\
f_{18,a}(b,c) &= -b^2c^2(b^2-c^2)^4(b^2+c^2)(3b^4+b^2c^2+3c^4),\\
f_{20,a}(b,c) &= b^2c^2(b^2-c^2)^6(b^2+c^2)^2.
\end{aligned}$$

References

[1] S. N. Collings, Reflections on a triangle, part 1, *Math. Gazette*, 57 (1973) 291–293.
[2] J. F. Garcia Capitán, Hyacinthos message 15827, November 19, 2007.
[3] D. Grinberg, Anti-Steiner points with respect to a triangle, available at `http://de.geocities.com/darij_grinberg`.
[4] R. A. Johnson, *Advanced Euclidean Geometry*, 1929, Dover reprint 2007.
[5] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.
[6] C. F. Parry and R. L. Young, Problem 10637, *Amer. Math. Monthly*, 106 (1999) 779–780; solution, ibid.
[7] C. Pohoata, Hyacinthos message 15825, November 18, 2007.

Cosmin Pohoata: 13 Pridvorului Street, Bucharest, Romania 010014
E-mail address: `pohoata_cosmin2000@yahoo.com`

Forum Geometricorum
Volume 8 (2008) 49–59.

FORUM GEOM
ISSN 1534-1178

Construction of Malfatti Squares

Floor van Lamoen and Paul Yiu

Abstract. We give a very simple construction of the Malfatti squares of a triangle, and study the condition when all three Malfatti squares are inside the given triangle. We also give an extension to the case of rectangles.

1. Introduction

The Malfatti squares of a triangle are the three squares each with two adjacent vertices on two sides of the triangle and the two remaining adjacent vertices from those of a triangle in its interior. We borrow this terminology from [3] (see also [1, p.48]) where the lengths of the sides of the Malfatti squares are stated. In Figure 1, the Malfatti squares of triangle ABC are $B'C'Z_aY_a$, $C'A'X_bZ_b$ and $A'B'Y_cX_c$. We shall call $A'B'C'$ the Malfatti triangle of ABC, and present a simple construction of $A'B'C'$ from a few common triangle centers of ABC. Specifically, we shall make use of the isogonal conjugate of the Vecten point of ABC. [1] This is a point on the Brocard axis, the line joining the circumcenter O and the symmedian point K.

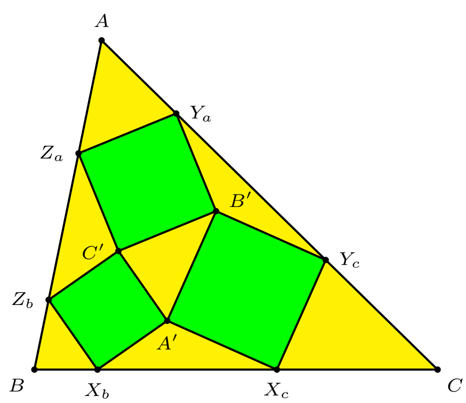


Figure 1.

Theorem 1. *Let P be the isogonal conjugate of the Vecten point of triangle ABC. The vertices of the Malfatti triangle are the intersections of the lines joining the centroid G to the pedals of the symmedian point K and the corresponding vertices to the pedals of P on the opposite sides of ABC.* See Figure 2.

Publication Date: March 10, 2008. Communicating Editor: Jean-Pierre Ehrmann.

[1]The Vecten point of a triangle is the perspector of the (triangle whose vertices are) the centers of the squares erected externally on the sides. It appears as X_{485} in [6]. See Figure 6. Its isogonal conjugate appears as X_{371}, and is also called the Kenmotu point. It is associated with the construction of a triad of congruent squares in a triangle. In [5, p.268] the expression for the edge length of the squares should be reduced by a factor $\sqrt{2}$. A correct expression appears in [6] and [2, p.94].

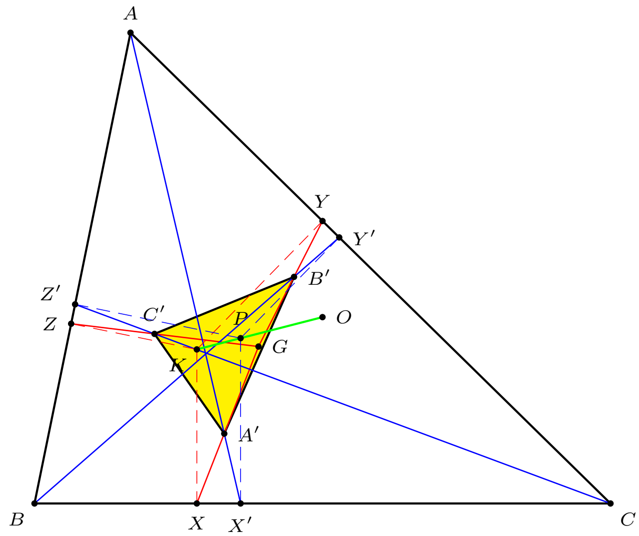


Figure 2.

2. Notations

We adopt the following notations. For a triangle of sidelengths a, b, c, let S denote *twice* the area of the triangle, and

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2}.$$

These satisfy

$$S_BS_C + S_CS_A + S_AS_B = S^2.$$

More generally, for an arbitrary angle θ, $S_\theta = S \cdot \cot\theta$. In particular,

$$S_A + S_B + S_C = \frac{a^2+b^2+c^2}{2} = S_\omega,$$

where ω is the Brocard angle of triangle ABC.

3. The triangle of medians

Given a triangle ABC with sidelengths a, b, c, let m_a, m_b, m_c denote the lengths of the medians. By the Apollonius theorem, these are given by

$$\begin{aligned} m_a^2 =& \frac{1}{4}(2b^2+2c^2-a^2),\\ m_b^2 =& \frac{1}{4}(2c^2+2a^2-b^2),\\ m_c^2 =& \frac{1}{4}(2a^2+2b^2-c^2). \end{aligned} \tag{1}$$

There is a triangle whose sidelengths are m_a, m_b, m_c. See Figure 3A. We call this the triangle of medians of ABC. The following useful lemma can be easily established.

Lemma 2. *Two applications of the triangle of medians construction gives a similar triangle of similarity factor* $\frac{3}{4}$. See Figure 3B.

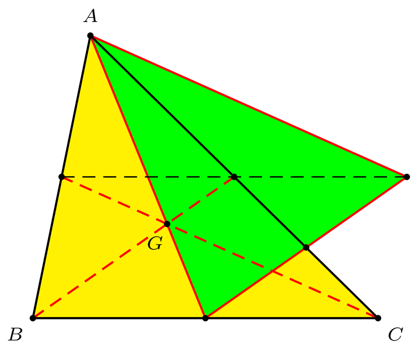


Figure 3A

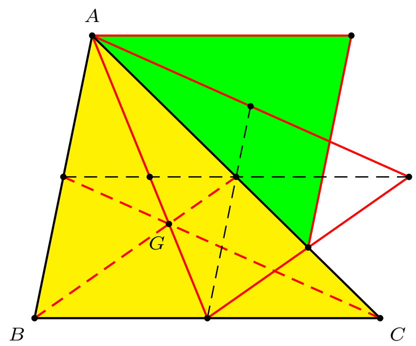


Figure 3B

We present an interesting example of a triangle similar to the triangle of medians which is useful for the construction of the Malfatti triangle.

Lemma 3. *The pedal triangle of the symmedian point is similar to the triangle of medians, the similarity factor being* $\tan\omega$.

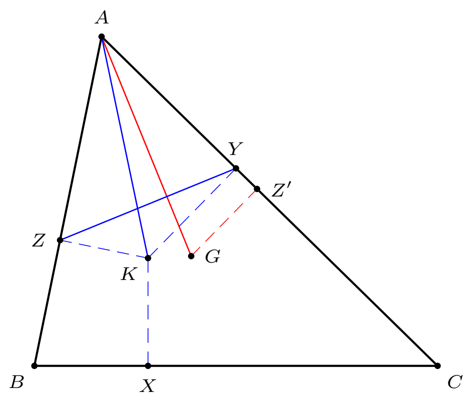


Figure 4

Proof. Since $S = bc\sin A$, the distance from the centroid G to AC is clearly $\frac{S}{3b}$. That from the symmedian point K to AB is

$$\frac{c^2}{a^2+b^2+c^2} \cdot \frac{S}{c} = \frac{S}{2S_\omega} \cdot c.$$

Since K and G are isogonal conjugates,

$$AK = AG \cdot \frac{\frac{S}{2S_\omega} \cdot c}{\frac{S}{3b}} = \frac{2}{3} m_a \cdot \frac{3bc}{2S_\omega} = \frac{bc}{S_\omega} \cdot m_a.$$

This is a diameter of the circle through A, K, and its pedals on AB and AC. It follows that the distance between the two pedals is

$$\frac{bc}{S_\omega}\cdot m_a\cdot \sin A=\frac{S}{S_\omega}\cdot m_a=\tan\omega\cdot m_a.$$

From this, it is clear that the pedal triangle is similar to the triangle of medians, the similarity factor being $\tan\omega$. □

Remark. The triangle of medians of ABC has the same Brocard angle as ABC.

Proposition 4. *Let P be a point with pedal triangle XYZ in ABC. The lines through A, B, C perpendicular to the sides YZ, ZX, XY concur at the isogonal conjugate of P.*

We shall also make use of the following characterization of the symmedian (Lemoine) point of a triangle.

Theorem 5 (Lemoine)**.** *The symmedian point is the unique point which is the centroid of its own pedal triangle.*

4. Proof of Theorem 1

Consider a triangle ABC with its Malfatti squares. Complete the parallelogram $A'B'A^*C'$. See Figure 5. Note that triangles $A'X_bX_c$ and $C'A'A^*$ are congruent. Therefore, $A'A^*$ is perpendicular to BC. Note that this line contains the centroid G' of triangle $A'B'C'$. Similarly, if we complete parallelograms $B'C'B^*A'$ and $C'A'C^*B'$, the lines $B'B^*$ and $C'C^*$ contain G' and are perpendicular to CA and AB respectively.

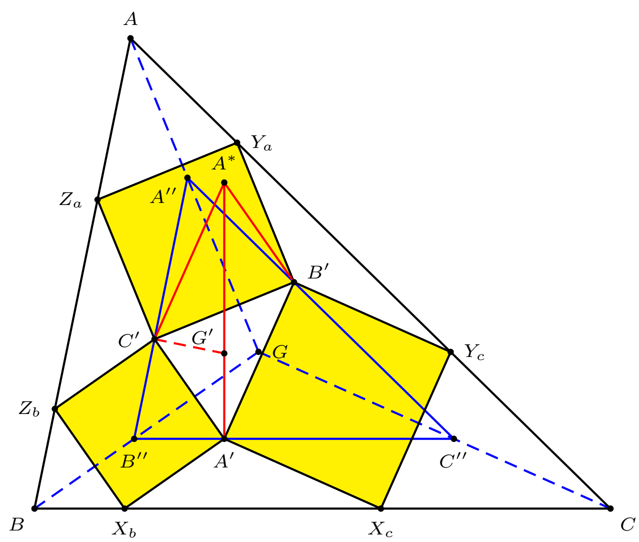


Figure 5.

Consider $A'B'C'$ as the pedal triangle of G' in a triangle $A''B''C''$ homothetic to ABC. By Lemoine's theorem, G' is the symmedian point of $A''B''C''$. Since

$A''B''C''$ is homothetic to ABC, $A'B'C'$ is homothetic to the pedal triangle of the symmedian point K of ABC.

In Figure 5, triangle $A''B'C'$ is the image of AY_aZ_a under the translation by the vector $\mathbf{Y_aB'} = \mathbf{Z_aC'} = \mathbf{AA''}$. This means that the line AA'' is perpendicular to $B'C'$, and to the A-side of the pedal triangle of K. Similarly, BB'' and CC'' are perpendicular to B- and C-sides of the same pedal triangle. By Proposition 4, the lines AA'', BB'', CC'' concur at the isogonal conjugate of K. This means that triangles $A''B''C''$ and ABC are homothetic at the centroid G of triangle ABC, and the sides of the Malfatti squares are parallel and perpendicular to the corresponding medians.

Denote by λ the homothetic ratio of $A''B''C''$ and ABC . This is also the homothetic ratio of the Malfatti triangle $A'B'C'$ and the pedal triangle of K. In Figure 5, $BX_b + X_cC = B''C'' = \lambda a$. Also, by Lemmas 2 and 3,

$$\begin{aligned} X_bX_c = A'A^* =& \ 2\lambda \cdot A\text{−median of pedal triangle of } K \\ &= 2\lambda \cdot \tan\omega \cdot A\text{−median of triangle of medians of } ABC \\ &= 2\lambda \cdot \tan\omega \cdot \frac{3}{4}a = \ \frac{3}{2}\lambda \cdot \tan\omega \cdot a. \end{aligned}$$

Since $BX_b + X_bX_c + X_cC = BX$, we have $\lambda(1 + \frac{3}{2}\tan\omega) = 1$ and

$$\lambda = \frac{2}{2 + 3\tan\omega} = \frac{2S_\omega}{3S + 2S_\omega}.$$

Let $\mathsf{h}(G, \lambda)$ be the homothety with center G and ratio λ. Since G' is the symmedian point of $A''B''C''$,

$$G' = \mathsf{h}(G, \lambda) = \lambda K + (1 - \lambda)G = \frac{1}{3S + 2S_\omega}(3S \cdot G + 2S_\omega \cdot K).$$

It has homogeneous barycentric coordinates $(a^2 + S : b^2 + S : c^2 + S)$. [2]

To compute the coordinates of the vertices of the Malfatti triangle, we make use of the pedals of the symmedian point K on the sidelines. The pedal on BC is the point

$$X = \frac{1}{2S_\omega}((S_A + 2S_C)B + (S_A + 2S_B)C).$$

A' is the point dividing the segment GX in the ratio $GA' : A'X = S_\omega : 3S$.

$$\begin{aligned} A' =& \frac{1}{3S + 2S_\omega}(3S \cdot G + 2S_\omega \cdot X) \\ =& \frac{1}{3S + 2S_\omega}\left(S \cdot A + (S + S_A + 2S_C)B + (S + S_A + 2S_B)C\right). \end{aligned}$$

[2]For a construction of G', see Proposition 6.

Similarly, we have B' and C'. In homogeneous barycentric coordinates, these are

$$\begin{aligned}A' =&(S : S + S_A + 2S_C : S + S_A + 2S_B),\\B' =&(S + S_B + 2S_C : S : S + S_B + 2S_A),\\C' =&(S + S_C + 2S_B : S + S_C + 2S_A : S).\end{aligned}$$

The lines AA', BB', CC' intersect the sidelines BC, CA, AB respectively at the points

$$\begin{aligned}X' =&(0 : S + S_A + 2S_C : S + S_A + 2S_B),\\Y' =&(S + S_B + 2S_C : 0 : S + S_B + 2S_A),\\Z' =&(S + S_C + 2S_B : S + S_C + 2S_A : 0).\end{aligned} \tag{2}$$

We show that these three intersections are the pedals of a specific point

$$P = (a^2(S_A + S) : b^2(S_B + S) : c^2(S_C + S)).$$

In absolute barycentric coordinates,

$$P = \frac{1}{2S(S + S_\omega)}\left((a^2(S_A + S)A + b^2(S_B + S)B + c^2(S_C + S)C\right).$$

The infinite point of the perpendiculars to BC being $-a^2 \cdot A + S_C \cdot B + S_B \cdot C$, the perpendicular from P to BC contains the point

$$\begin{aligned}&P + \frac{S_A + S}{2S(S + S_\omega)}(-a^2 \cdot A + S_C \cdot B + S_B \cdot C)\\=\ &\frac{1}{2S(S + S_\omega)}\left((b^2(S_B + S) + S_C(S_A + S))B + (c^2(S_C + S) + S_B(S_A + S))C\right)\\=\ &\frac{1}{2(S + S_\omega)}\left((S + S_A + 2S_C)B + (S + S_A + 2S_B)C\right).\end{aligned}$$

This is the point X' whose homogeneous coordinates are given in (2) above. Similarly, the pedals of P on the other two lines CA and AB are the points Y' and Z' respectively.

These lead to a simple construction of the vertex A', as the intersection of the lines GX and the line joining A to the pedal of P on BC. This completes the proof of Theorem 1.

Remark. Apart from A', B', C', the vertices of the Malfatti squares on the sidelines are

$$\begin{aligned}X_b &= (0 : 3S + S_A + 2S_C : S_A + 2S_B), & X_c &= (0 : S_A + 2S_C : 3S + S_A + 2S_B),\\Y_c &= (S_B + 2S_C : 0 : 3S + 2S_A + S_B), & Y_a &= (3S + S_B + 2S_C : 0 : 2S_A + S_B),\\Z_a &= (3S + 2S_B + S_C : 2S_A + S_C : 0), & Z_b &= (2S_B + S_C : 3S + 2S_A + S_C : 0).\end{aligned}$$

5. An alternative construction

The vertices of the Malfatti triangle $A'B'C'$ are the intersections of the perpendiculars from G' to the sidelines of triangle ABC with the corresponding lines joining G to the pedals of K on the sidelines. A simple construction of G' would lead to the Malfatti triangle easily. Note that G' divides GK in the ratio

$$GG' : G'K = a^2 + b^2 + c^2 : 3S.$$

On the other hand, the point P is the isogonal conjugate of the Vecten point

$$V = \left(\frac{1}{S_A + S} : \frac{1}{S_B + S} : \frac{1}{S_C + S}\right).$$

As such, it can be easily constructed, as the intersection of the perpendiculars from A, B, C to the corresponding sides of the pedal triangles of V. See Figure 6. It is a point on the Brocard axis, dividing OK in the ratio

$$OP : PK = a^2 + b^2 + c^2 : 2S.$$

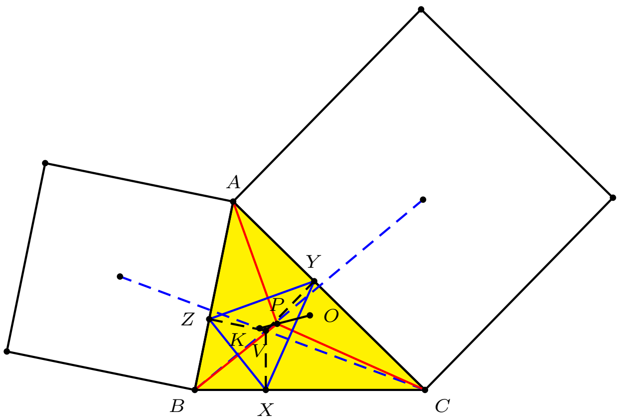


Figure 6.

This leads to a simple construction of the point G'.

Proposition 6. *G' is the intersection of GK with HP, where H is the orthocenter of triangle ABC.*

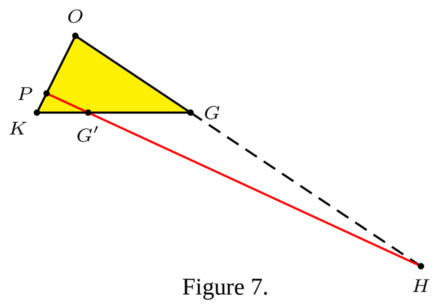


Figure 7.

Proof. Apply Menelaus' theorem to triangle OGK with transversal HP, noting that $OH : HG = 3 : -2$. See Figure 7. $\square$

6. Some observations

6.1. *Malfatti squares not in the interior of given triangle.* Sokolowsky [3] mentions the possibility that the Malfatti squares need not be contained in the triangle. Jean-Pierre Ehrmann pointed out that even the Malfatti triangle may have a vertex outside the triangle. Figure 8 shows an example in which both B' and Y_a are outside the triangle.

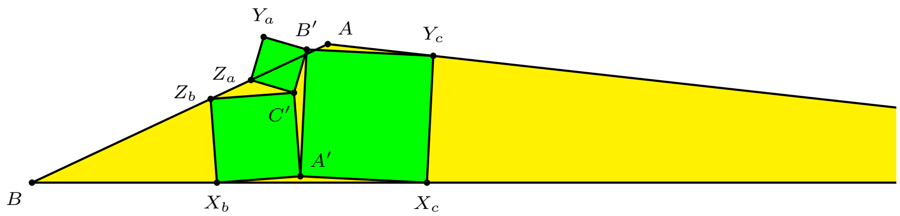


Figure 8.

Proposition 7. *At most one of the vertices the Malfatti triangle and at most one of the vertices of the Malfatti squares on the sidelines can be outside the triangle.*

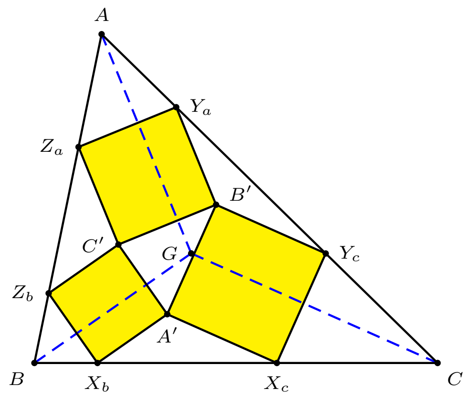


Figure 9.

Proof. If Y_a lies outside triangle ABC, then $\angle AZ_aC' < \frac{\pi}{2}$, and $\angle Z_bZ_aC' > \frac{\pi}{2}$. Since Z_aC' is parallel to AG, $\angle BAG = \angle Z_bZ_aC'$ is obtuse. Under the same hypothesis, if B' and C are on opposite sides of AB, then $\angle AZ_aC' < \frac{\pi}{4}$, and $\angle BAG > \frac{3\pi}{4}$.

Similarly, if any of Z_a, Z_b, X_b, X_c, Y_c lies outside the triangle, then correspondingly, $\angle CAG$, $\angle CBG$, $\angle ABG$, $\angle ACG$, $\angle BCG$ is obtuse. Since at most one of these angles can be obtuse, at most one of the six vertices on the sides and at most one of A', B', C' can be outside triangle ABC. $\square$

6.2. *A locus problem.* François Rideau [8] asked, given B and C, for the locus of A for which the Malfatti squares of triangle ABC are in the interior of the triangle. Here is a simple solution. Let M be the midpoint of BC, P the reflection of C in B, and Q that of B in C. Consider the circles with diameters PB, BM, MC, CQ, and the perpendiculars ℓ_P and ℓ_Q to BC at P and Q. See Figure 10.

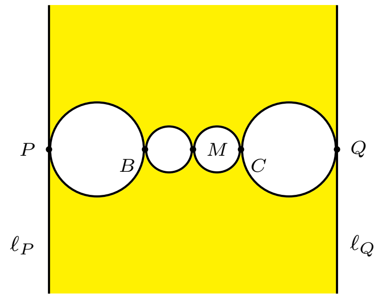


Figure 10.

For an arbitrary point A, consider ABC with centroid G.
(i) $\angle ABG$ is obtuse if A is inside the circle with diameter PB;
(ii) $\angle BAG$ is obtuse if A is inside the circle with diameter BM;
(iii) $\angle CAG$ is obtuse if A is inside the circle with diameter MC;
(iv) $\angle ACG$ is obtuse if A is inside the circle with diameter CQ;
(v) $\angle CBG$ is obtuse if A is on the side of ℓ_P opposite to the circles;
(vi) $\angle BCG$ is obtuse if A is on the side of ℓ_Q opposite to the circles.

Therefore, the locus of A for which the Malfatti squares of triangle ABC are in the interior of the triangle is the region between the lines ℓ_P and ℓ_Q with the four disks excised.

A similar reasoning shows that the locus of A for which the vertices A', B', C' of the Malfatti triangle of ABC are in the interior of triangle ABC is the shaded region in Figure 11.

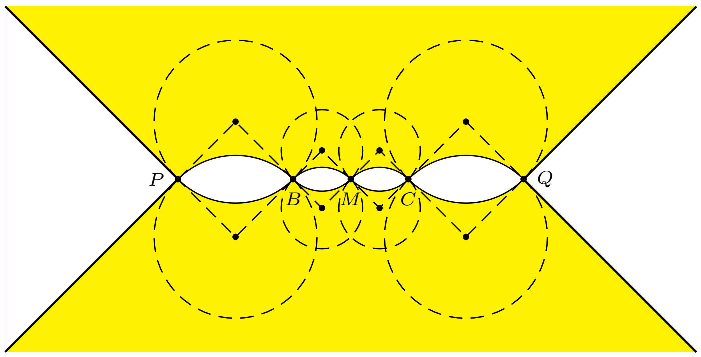


Figure 11.

7. Generalization

We present a generalization of Theorem 1 in which the Malfatti squares are replaced by rectangles of a specified shape. We say that a rectangle constructed on a side of triangle ABC has shape θ if its center is the apex of the isosceles triangle constructed on that side with base angle θ. We assume $0 < \theta < \frac{\pi}{2}$ so that the apex is on the opposite side of the corresponding vertex of the triangle. It is well known that for a given θ, the centers of the three rectangles of shape θ erected on the sides are perspective with ABC at the Kiepert perspector

$$K(\theta) = \left(\frac{1}{S_A + S_\theta} : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta}\right).$$

The isogonal conjugate of $K(\theta)$ is the point

$$K^*(\theta) = (a^2(S_A + S_\theta) : b^2(S_B + S_\theta) : c^2(S_C + S_\theta))$$

on the Brocard axis dividing the segment OK in the ratio $\tan\omega\tan\theta : 1$.

Theorem 8. *For a given θ, let $A(\theta)$ be the intersection of the lines joining* (i) *the centroid G to the pedal of the symmedian point K on BC,* (ii) *the vertex A to the pedal of $K^*(\theta)$ on BC. Analogously construct points $B(\theta)$ and $C(\theta)$. Construct rectangles of shape θ on the sides of $A(\theta)B(\theta)C(\theta)$. The remaining vertices of these rectangles lie on the sidelines of triangle ABC.*

Figure 12 illustrates the case of the isodynamic point J. The Malfatti rectangles $B'C'Z_aY_a$, $C'A'X_bZ_b$ and $A'B'Y_cX_c$ have shape $\frac{\pi}{3}$, *i.e.*, lengths and widths in the ratio $\sqrt{3} : 1$.

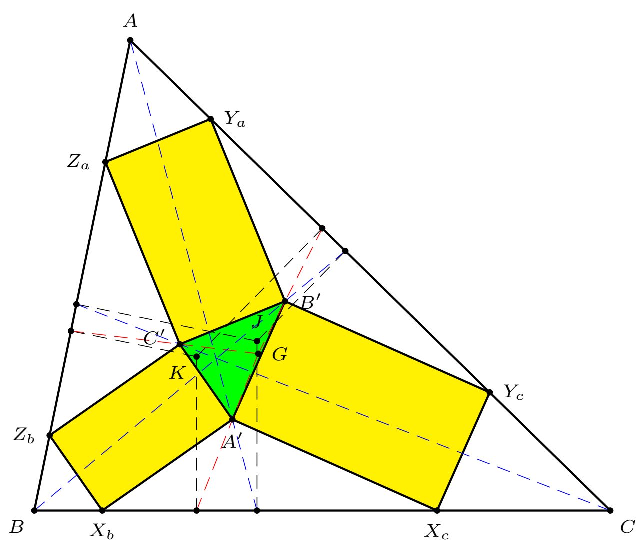


Figure 12.

The same reasoning in §6 shows that exactly one of the six vertices on the sidelines is outside the triangle if and only if a median makes an obtuse angle with an adjacent side. If this angle exceeds $\frac{\pi}{2} + \theta$, the corresponding vertex of Malfatti triangle is also outside ABC.

References

[1] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, Charles Babbage Research Centre, Winnipeg, 1989.

[2] H. Fukagawa and J. F. Rigby, *Traditional Japanese Mathematics Problems of the 18th and 19th Centuries*, SCT Press, Singapore, 2002.

[3] H. Fukagawa and D. Sokolowsky, Problem 1013, *Crux Math.*, 11 (1985) 50; solution, ibid., 12 (1986) 119–125, 181–182.

[4] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint, 2007.

[5] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

[6] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[7] F. M. van Lamoen, Friendship among triangle centers, *Forum Geom.*, 1 (2001) 1 – 6.

[8] F. Rideau, Hyacinthos message 15961, December 28, 2007.

Floor van Lamoen: Ostrea Lyceum, Bergweg 4, 4461 NB Goes, The Netherlands
E-mail address: `fvanlamoen@planet.nl`

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida, 33431-0991, USA
E-mail address: `yiu@fau.edu`

Forum Geometricorum
Volume 8 (2008) 61–62.

FORUM GEOM
ISSN 1534-1178

A Simple Ruler and Rusty Compass Construction of the Regular Pentagon

Kurt Hofstetter

Abstract. We construct in 13 steps a regular pentagon with given sidelength using a ruler and rusty compass.

Suppose a line segment AB has been divided in the golden ratio at a point G. Figure 1 shows the construction of the vertices of a regular pentagon with four circles of radii equal to AB. Thus, let $\mathcal{C}_1 = A(AB)$, $\mathcal{C}_2 = B(AB)$, $\mathcal{C}_4 = G(AB)$, intersecting the half line AB at P_1, and $\mathcal{C}_5 = P_1(AB)$. Then, with $P_2 = \mathcal{C}_1 \cap \mathcal{C}_5$, $P_4 = \mathcal{C}_1 \cap \mathcal{C}_4$, and $P_5 = \mathcal{C}_2 \cap \mathcal{C}_5$. Since the radii of the circles involved are equal, this construction can be performed with a ruler and a rusty compass. We claim that the pentagon $P_1P_2AP_4P_5$ is regular.

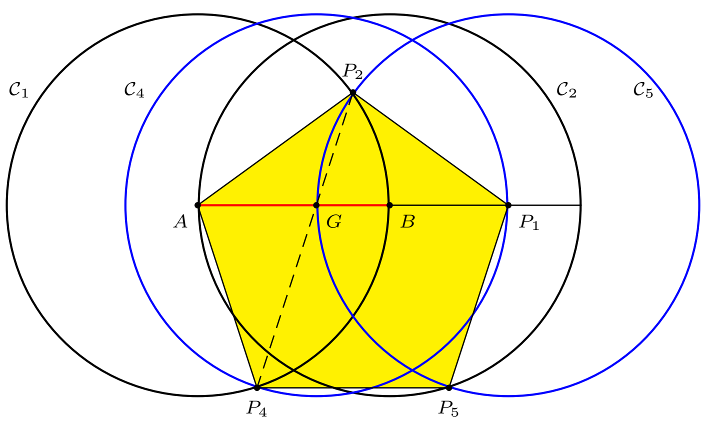


Figure 1

Here is a simple proof. Assume unit length for the segment AB. Let $\phi := \frac{\sqrt{5}+1}{2}$ be the golden ratio. It is well known that $AG = \frac{1}{\phi} = \phi - 1$. Now, $AP_1 = (\phi - 1) + 1 = \phi$. Therefore, the isosceles triangle AP_1P_2 consists of two sides and a diagonal of a regular pentagon. In particular, $\angle P_2AP_1 = 36^\circ$ and $\angle AP_2P_1 = 108^\circ$. On the other hand, triangle AGP_4 is also isosceles with sides in the proportions $1 : 1 : \frac{1}{\phi} = \phi : \phi : 1$. It consists of two diagonals and a side of a regular pentagon. In particular, $\angle GAP_4 = 72^\circ$ and $\angle AP_4G = 36^\circ$. From these, $\angle P_2AP_4 = 36^\circ + 72^\circ = 108^\circ$.

Publication Date: March 24, 2008. Communicating Editor: Paul Yiu.

Now, triangles AP_4P_2 and $P_2AP_1$are congruent. It follows that $\angle AP_2P_4 = 36°$, and P_2, G, P_4 are collinear.

By symmetry, we also have $\angle P_2P_1P_5 = 108°$.

In the pentagon $P_1P_2AP_4P_5$, since the angles at P_1, P_2, A are all $108°$, those at P_4 and P_5 are also $108°$. On the other hand, since the circles $\mathcal{C}_2$ and $\mathcal{C}_5$ are the translations of $\mathcal{C}_1$ and $\mathcal{C}_4$ by the vector $\overrightarrow{AB}$, P_4P_5 has unit length. This shows that the pentagon $P_1P_2AP_4P_5$ is regular.

Now, using a rusty compass (set at a radius equal to AB) we have constructed in [1] the point G in 5 steps, which include the circles $\mathcal{C}_1$ and $\mathcal{C}_2$. (In Figure 2, M is the midpoint of AB, $\mathcal{C}_3 = M(AB)$ intersects $\mathcal{C}_2$ at E on the opposite side of C; $G = CE \cap AB$). It follows that the vertices of the regular pentagon $P_1P_2AP_4P_5$ can be constructed in $5 + 3 = 8$ steps. The pentagon can be completed in 5 more steps by filling in the sides.

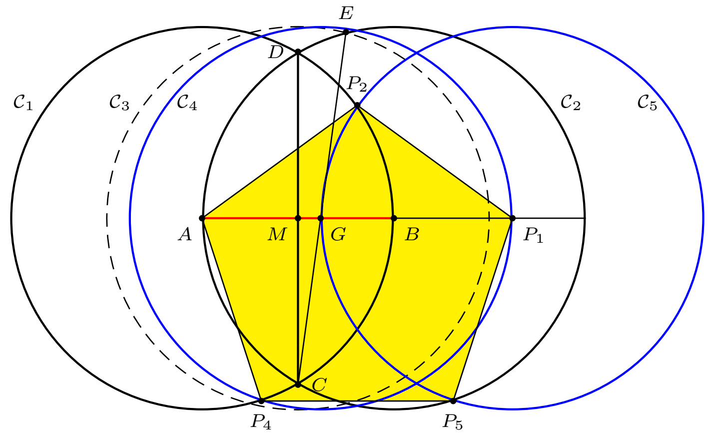


Figure 2

References

[1] K. Hofstetter, Divison of a segment in the golden section with ruler and rusty compass, *Forum Geom.*, 5 (2005) 135–136.

Kurt Hofstetter: Object Hofstetter, Media Art Studio, Langegasse 42/8c, A-1080 Vienna, Austria
E-mail address: hofstetter@sunpendulum.at

Forum Geometricorum
Volume 8 (2008) 63–72.

FORUM GEOM
ISSN 1534-1178

Haruki's Lemma and a Related Locus Problem

Yaroslav Bezverkhnyev

Abstract. In this paper we investigate the nature of the constant in Haruki's Lemma and study a related locus problem.

1. Introduction

In his papers [2, 3], Ross Honsberger mentions a remarkably beautiful lemma that he accredits to Professor Hiroshi Haruki. The beauty and mystery of Haruki's lemma is in its apparent simplicity.

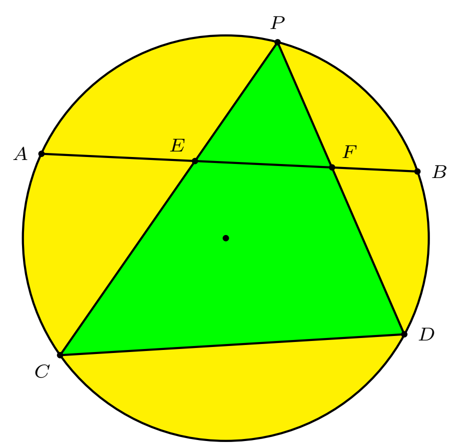


Figure 1. Haruki's lemma: $\frac{AE \cdot BF}{EF}$ = constant.

Lemma 1 (Haruki)**.** *Given two nonintersecting chords AB and CD in a circle and a variable point P on the arc AB remote from points C and D, let E and F be the intersections of chords PC, AB, and of PD, AB respectively. The value of $\dfrac{AE \cdot BF}{EF}$ does not depend on the position of P.*

A very intriguing statement indeed. It should be duly noted that Haruki's Lemma leads to an easy proof of the Butterfly Theorem; see [2], [3, pp.135–140]. The nature of the constant, however, remains unclear. By looking at it in more detail we shall discover some interesting results.

Publication Date: April 7, 2008. Communicating Editor: Paul Yiu.
The author wishes to thank Paul Yiu for his suggestions leading to improvement of the paper and Gene Foxwell for his help in obtaining some of the reference materials.

2. Proof of Haruki's lemma

A good interactive visualisation and proof of Haruki's lemma can be found in [1]. Here we present the proof essentially as it appeared in [3]. The proof is quite ingenious and relies on the fact that the angle $\angle CPD$ is constant.

We begin by constructing a circumcirlce of triangle PED and define point G to be the intersection of this cirumcircle with the line AB. Note that $\angle EGD = \angle EPD$ as they are subtended by the same chord ED of the circumcircle of $\triangle PED$ and so these angles remain constant as P varies on the arc AB. Hence, for all positions of P, $\angle EGD$ remains fixed and, therefore, point G remains fixed on the line AB (See Figure 2). So $BG = constant$.

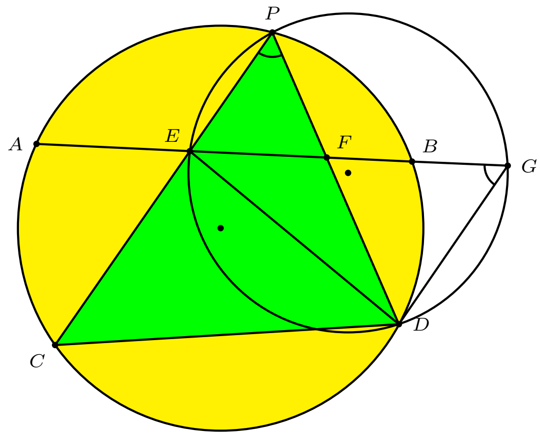


Figure 2. Point G is a fixed point on line AB.

Now, by applying the intersecting chords theorem to PD and AG in the two circles, we obtain the following:

$$AF \cdot FB = PF \cdot FD,$$
$$EF \cdot FG = PF \cdot FD.$$

From these, $(AE + EF) \cdot FB = EF \cdot (FB + BG)$, and $AE \cdot FB = EF \cdot BG$. Therefore, we have obtained $\dfrac{AE \cdot BF}{EF} = BG$, a constant. This completes the proof of Lemma 1.

Note that in the proof we could have used the circumcircle around $\triangle PFC$ instead of the one around $\triangle PED$.

3. An extension of Haruki's lemma

Haruki has apparently found the constant. However, finding it raises additional questions. Why is the ratio of distances that are bound to the circle (through points A, B, C, D, P) expressed by a constant that involves a point lying *outside* the circle? We explore the setup in Lemma 1. Consider an inversion with center P and

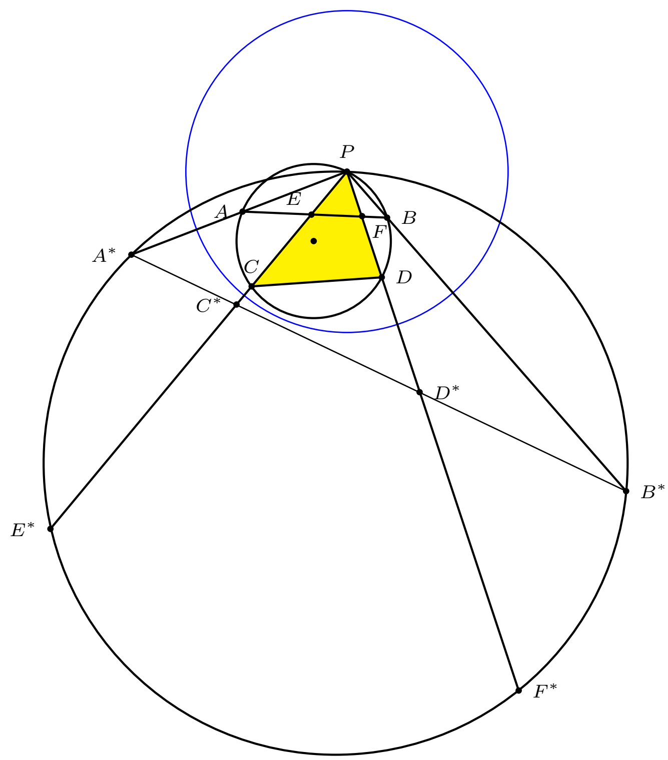


Figure 3. Applying inversion with center P

radius r that is bigger than the diameter of the circumcircle of $ABDC$ (See Figure 3).

Recall two basic facts about an inversion:

(a) It maps a line not through the center of inversion into a circle that goes through the center of inversion and vice versa.

(b) It maps the line that goes through the center of inversion into the same line.

Knowing these two facts, we can perform an inversion on the setup in Figure 1, the results of which are shown in Figure 3. We can see that the segments A^*E^*, B^*F^* and E^*F^* have taken the place of the segments AC, BD, CD. We shall use this hint to deduce the following extension of Haruki's Lemma.

Lemma 2. *Given two nonintersecting chords AB and CD in a circle and a variable point P on the arc AB remote from points C and D, let E and F be the intersections of chords PC, AB, and of PD, AB respectively. The following equalities hold:*

$$\frac{AE \cdot BF}{EF} = \frac{AC \cdot BD}{CD}, \tag{1}$$

$$\frac{AF \cdot BE}{EF} = \frac{AD \cdot BC}{CD}. \tag{2}$$

Proof. (1) Following the notation and proof of Lemma 1, we have $\frac{AE \cdot BF}{EF} = BG$. It remains to show that $BG = \frac{AC \cdot BD}{CD}$, or, equivalently,

$$\frac{BG}{BD} = \frac{AC}{CD}. \tag{3}$$

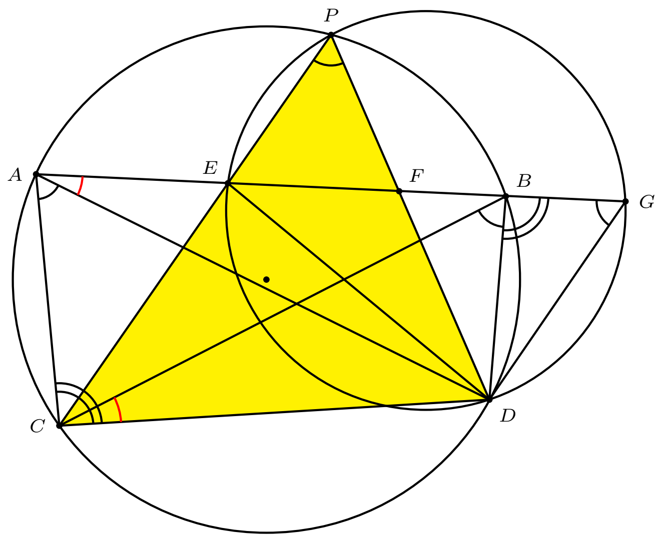


Figure 4. Triangles ACD and GBD are similar, as are AGD and CBD

Note that in Figure 4, $\angle CAD = \angle CPD = \angle EPD = \angle EGD$. Since $ABDC$ is a cyclic quadrilateral, we have $\angle ACD = \angle DBG$. This means that the triangles ACD and GBD are similar, thus yielding (3), and therefore (1).

For (2) we note that $\angle DCB = \angle DAB$. Also $\angle CBD = \angle CPD = \angle EPD = \angle EGD$, thus we get $\triangle AGD \sim \triangle CBD$ yielding:

$$\frac{AG}{AD} = \frac{BC}{CD} \Rightarrow AG = \frac{AD \cdot BC}{CD}.$$

However, $AF \cdot BE = (AE + EF) \cdot (EF + BF) = AE \cdot BF + AB \cdot EF$. We obtain, by using Lemma 1,

$$\frac{AF \cdot BE}{EF} = \frac{AE \cdot BF}{EF} + AB = BG + AB = AG = \frac{AD \cdot BC}{CD}.$$

□

Note that by switching the position of points C and D we effectively switch points E and F, thus equations (1) and (2) are equivalent. It may seem surprising; but the statement of Lemma 2 holds even for intersecting chords AB and CD and for any point P on the circle for which the points E and F are defined.

Theorem 3. *Given two distinct chords AB and CD in a circle and a point P on that circle distinct from A and B, let E and F be the intersections of the line AB with the lines PC and PD respectively. The equalities* (1) *and* (2) *hold.*

We leave the proof to the reader as an exercise. All that is necessary is to consider the different cases for the relative positions of A, B, C, D, P and to apply the ideas in the proofs of Lemmas 1 and 2, i.e. finding the point G as the intersection of the circumcircle of either $\triangle PED$ or $\triangle PFC$ with AB and then looking for similar triangles. Note that the point P may coincide with either C or D. In this case, by the line PC or PD we would mean the tangent to the circle at C or D.

4. A locus problem

Theorem 3 settles the case when points A, B, C, D lie on a circle. But what happens when points A, B, C, D do not belong to the same circle? Can we still find points P that will satisfy equation (1) or (2)? This gives rise to the following locus problem.

Problem. Given the points A, B, C, D find the locus $\mathcal{L}_1$ (respectively $\mathcal{L}_2$) of all points P that satisfy (1) (respectively (2)), where points E and F are the intersections of lines PC and AB, PD and AB respectively.

To investigate the loci $\mathcal{L}_1$ and $\mathcal{L}_2$, we begin with a result about the possibility of a point P belonging to both $\mathcal{L}_1$ and $\mathcal{L}_2$.

Lemma 4. *If there is a point P satisfying both* (1) *and* (2)*, then A, B, C, D are concyclic.*

Proof. First of all, points A, B, E, F are collinear, hence, they satisfy Euler's distribution theorem (See [4, p.3] and [5]), *i.e.*, if A, B, E, F are in this order, then, $AF \cdot BE + AB \cdot EF = AE \cdot BF$. Dividing through by EF, we obtain

$$\frac{AF \cdot BE}{EF} + AB = \frac{AE \cdot BF}{EF},$$

and so, by the fact that point P satisfies equations (1) and (2), we have:

$$\frac{AD \cdot BC}{CD} + AB = \frac{AC \cdot BD}{CD}.$$

Now multiplying by CD yields

$$AD \cdot BC + AB \cdot CD = AC \cdot BD,$$

which, by Ptolemy's inequality (See [6]), means that points A, B, C, D are concyclic with points A, C separating points B, D on the circle. The relative positions of A, B, E, F will influence the relative positions of points A, B, C, D on the circle. Similar argument can be applied to establish the validity of the statement of this lemma no matter the position of points A, B, E, F. □

This lemma is interesting in the way it ties the "linear" Euler's equality, Ptolemy's inequality together with the extension of Haruki's lemma.

5. Barycentric coordinates

In order to find the loci $\mathcal{L}_1$ and $\mathcal{L}_2$ for the general position of points A, B, C, D we make use of the notion of homogeneous barycentric coordinates. Given a reference triangle ABC, any three numbers x, y, z proportional to the signed areas of oriented triangles PBC, PCA, PAB form a set of *homogeneous barycentric coordinates* of P, written as $(x : y : z)$.

With reference to triangle ABC, the absolute barycentric coordinates of the vertices are obviously $A(1 : 0 : 0)$, $B(0 : 1 : 0)$ and $C(0 : 0 : 1)$. We shall make use of the following basic property of barycentric coordinates.

Lemma 5. *Let P be point with homogeneous barycentric coordinates $(x : y : z)$ with reference to triangle ABC. The line AP intersects BC at a point X with coordinates $(0 : y : z)$, which divides BC in the ratio $BX : XC = z : y$. Similarly, BP intersects CA at $Y(x : 0 : z)$ such that $CY : YA = x : z$ and CP intersects AB at $Z(x : y : 0)$ such that $AZ : ZB = y : x$.*

Assume that D and P have barycentric coordinates $D(u : v : w)$ and $P(x : y : z)$. It is our aim to compute the coordinates of points E and F.

When there is no danger of confusion, we shall represent a line $p\alpha+q\beta+r\gamma = 0$ by $(p : q : r)$. The intersection of two lines $(p : q : r)$ and $(s : t : u)$ is the point

$$\left(\begin{vmatrix} q & r \\ t & u \end{vmatrix} : \begin{vmatrix} r & p \\ u & s \end{vmatrix} : \begin{vmatrix} p & q \\ s & t \end{vmatrix} \right).$$

This same expression also gives the line through the two points with homogeneous barycentric coordinates $(p : q : r)$ and $(s : t : u)$.

6. Solution of the locus problem

From the above formula we compute the coordinates of the lines AB, PC and PD:

Line	Coordinates
AB	$(0 : 0 : 1)$
PC	$(-y : x : 0)$
PD	$\left(\begin{vmatrix} y & z \\ v & w \end{vmatrix} : \begin{vmatrix} z & x \\ w & u \end{vmatrix} : \begin{vmatrix} x & y \\ u & v \end{vmatrix} \right)$

From these we obtain the coordinates of E and F:

$$E\left(\begin{vmatrix} x & 0 \\ 0 & 1 \end{vmatrix} : \begin{vmatrix} 0 & -y \\ 1 & 0 \end{vmatrix} : \begin{vmatrix} -y & x \\ 0 & 0 \end{vmatrix} \right) = (x : y : 0),$$
$$F\left(\begin{vmatrix} z & x \\ w & u \end{vmatrix} : \begin{vmatrix} z & y \\ w & v \end{vmatrix} : 0 \right) = (uz - wx : vz - wy : 0).$$

Assume $BC = a$, $CA = b$, and $AB = c$. Also, $AD = a'$, $BD = b'$, and $CD = c'$. These are also fixed quantities. From the coordinates of E and F, we obtain, by Lemma 5, the following *signed* lengths:

$$AE = \tfrac{y}{x+y} \cdot c, \qquad EB = \tfrac{x}{x+y} \cdot c;$$
$$AF = \tfrac{vz-wy}{z(u+v)-w(x+y)} \cdot c, \quad FB = \tfrac{uz-wx}{z(u+v)-w(x+y)} \cdot c.$$

Consequently,

$$EF = EB - FB = \frac{z(vx - uy)}{(x + y)(z(u + v) - w(x + y))} \cdot c.$$

Now we determine the loci $\mathcal{L}_1$ and $\mathcal{L}_2$.

Theorem 6. *Given the points A, B, C, D and a point P, define points E and F as the intersections of lines PC and AB, PD and AB respectively.*
(a) *The locus $\mathcal{L}_1$ of points P satisfying* (1) *is the union of two circumconics of $ABCD$ given by the equations*

$$(cc' + \varepsilon bb')uyz - \varepsilon bb'vzx - cc'wxy = 0, \quad \varepsilon = \pm 1. \tag{4}$$

(b) *The locus $\mathcal{L}_2$ of points P satisfying* (2) *is the union of two circumconics of $ABCD$ given by the equations*

$$\varepsilon aa'uyz + (cc' - \varepsilon aa')vzx - cc'wxy = 0, \quad \varepsilon = \pm 1. \tag{5}$$

Proof. In terms of signed lengths, (1) and (2) should be interpreted as $AE \cdot BF \cdot CD = \varepsilon \cdot AC \cdot BD \cdot EF$ and $AF \cdot BE \cdot CD = \varepsilon \cdot AD \cdot BC \cdot EF$ for $\varepsilon = \pm 1$. The results follow from direct substitutions. It is easy to see that the conics represented by (4) and (5) all contain the points A, B, C, D, with barycentric coordinates $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$, $(u : v : w)$ respectively. □

7. Constructions

Theorem 6 tells us that the loci in question are each a union of two conics, each containing the four given points A, B, C, D. In order to construct these conics, we would need to find a fifth point on each of them. The following proposition helps with this problem.

Proposition 7. *The four intersections of the bisectors of angles ABD, ACD, and the four intersections of the bisectors of angles CAB and CDB are points on $\mathcal{L}_1$.*

Proof. First of all, it is routine to verify that for $P = (x : y : z)$, we have

$$[AEP] \cdot [BFP] \cdot [CDP] = [ACP] \cdot [BDP] \cdot [EFP], \tag{6}$$

where $[XYZ]$ denotes the signed area of the oriented triangle XYZ. Let d_{XY} be the distance from P to the line XY. In terms of distances, the relation (6) becomes

$$(AE \cdot d_{AE})(BF \cdot d_{BF})(CD \cdot d_{CD}) = (AC \cdot d_{AC})(BD \cdot d_{BD})(EF \cdot d_{EF}).$$

From this it is clear that (1) is equivalent to

$$d_{AE} \cdot d_{BF} \cdot d_{CD} = d_{AC} \cdot d_{BD} \cdot d_{EF}. \tag{7}$$

Since AE, BF, EF are the same line AB, this condition can be rewritten as

$$d_{AB} \cdot d_{CD} = d_{AC} \cdot d_{BD}. \tag{8}$$

If P is an intersection of the bisectors of angles ABD and ACD, then $d_{AB} = d_{BD}$ and $d_{AC} = d_{CD}$. On the other hand, if P is an intersection of the bisectors of angles CAB and CDB, then $d_{AC} = d_{AB}$ and $d_{CD} = d_{BD}$. In both cases, (7) is satisfied, showing that P is a point on the locus $\mathcal{L}_1$. □

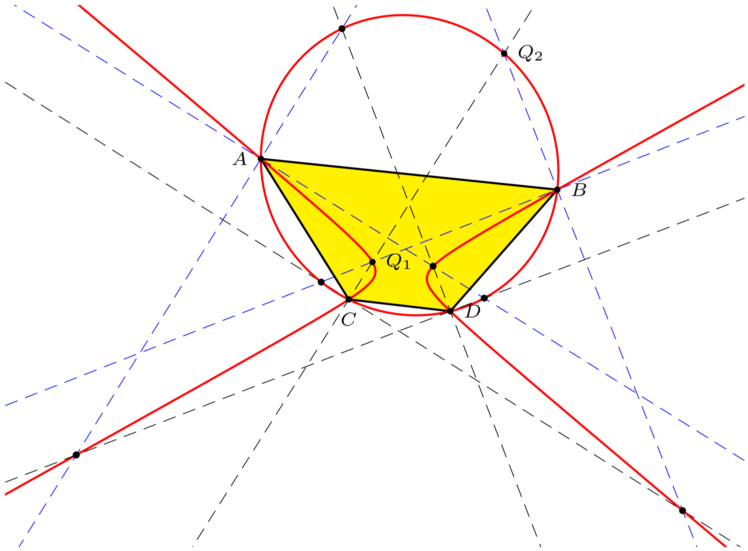


Figure 5. The locus $\mathcal{L}_1$

Let Q_1 be the intersection of the internal bisectors of angles ABD and ACD, and Q_2 as the intersection of the external bisector of angle ABD and the internal bisector of angle ACD. See Figure 5. Since Q_1, Q_2 and C are collinear, the points Q_1 and Q_2 must lie on distinct conics of $\mathcal{L}_1$.

Similarly, the locus $\mathcal{L}_2$ also contains the four intersections of the bisectors of angles BAD and BCD, and the four from angles ABC and ADC. Let Q_3 be the intersection of the internal bisectors and Q_4 the intersection of the internal bisector of BAD and the external bisector of angle BCD. See Figure 6. The points Q_3 and Q_4 are on different conics of $\mathcal{L}_2$.

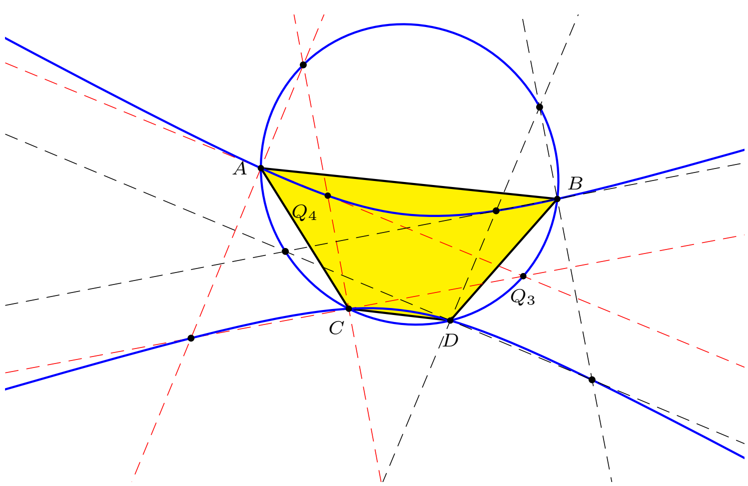


Figure 6. The locus $\mathcal{L}_2$

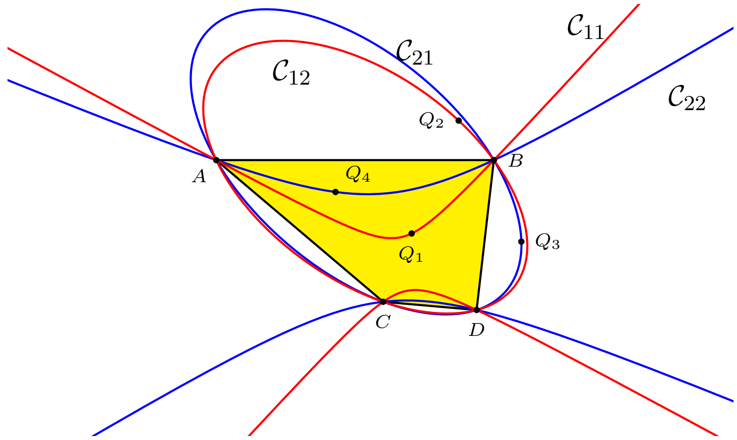


Figure 7.

Figure 7 shows the four conics, with $\mathcal{C}_{1,1}$, $\mathcal{C}_{1,2}$ forming $\mathcal{L}_1$ and $\mathcal{C}_{2,1}$, $\mathcal{C}_{2,2}$ forming $\mathcal{L}_2$.

Corollary 8. (a) *When points A, B, C, D all belong to the same circle $\mathcal{C}$, then one of the conics from $\mathcal{L}_1$ and one from $\mathcal{L}_2$ coincide with $\mathcal{C}$.*

(b) *If for some point P, (1) and (2) are both satisfied, then the points A, B, C, D, P are concyclic.*

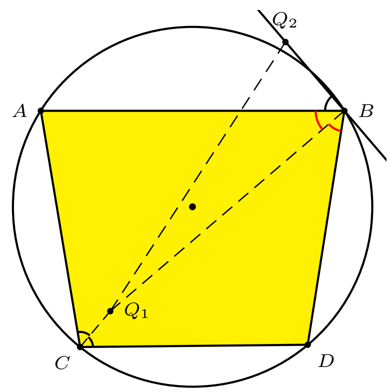


Figure 8.

Proof. (a) Assume Q_1 not on the circle $\mathcal{C}$. Suppose we have the situation as in Figure 8. In other cases the reasoning is similar. It is easy to see that $\angle ABQ_2 = \angle ACQ_2$ as Q_2 belongs to the external bisector of the angle ABD. This means that the points A, B, C and Q_2 are concyclic. But Q_2 lies on one of the conics from $\mathcal{L}_1$, therefore, this conic is actually a circle. Similarly, one can show that one of the conics from $\mathcal{L}_2$ coincides with $\mathcal{C}$. This proves (a).

(b) follows directly from (a) and Lemma 4. $\square$

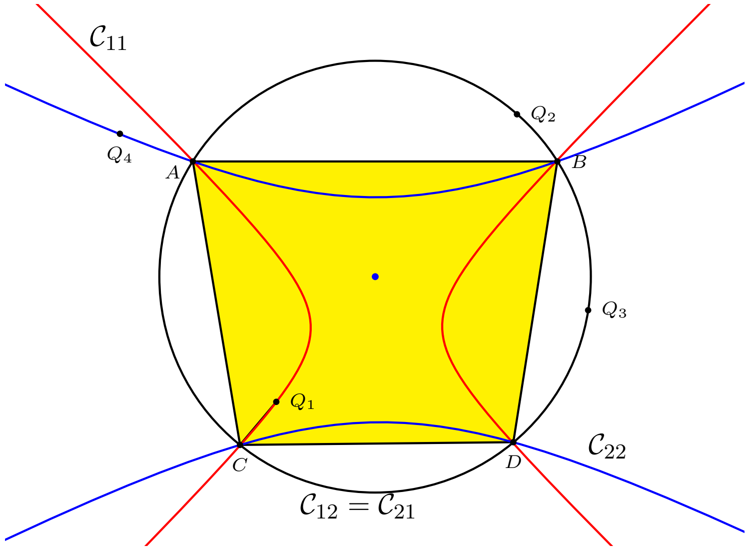


Figure 9. Loci $\mathcal{L}_1$ and $\mathcal{L}_2$ for cyclic quadrilateral $ABDC$

Theorem 3 together with Lemma 4 and part (b) of Corollary 8 provide us with the criteria for five points A, B, C, D and P to be concyclic. The case when $ABCD$ is a cyclic quadrilateral is depicted in Figure 9.

References

[1] A. Bogomolny, Cut The Knot, `http://www.cut-the-knot.org/Curriculum/Geometry/Haruki.shtml`

[2] R. Honsberger, The Butterfly Problem and Other Delicacies from the Noble Art of Euclidean Geometry I, *TYCMJ*, 14 (1983) 2 – 7.

[3] R. Honsberger, *Mathematical Diamonds*, Dolciani Math. Expositions No. 26, Math. Assoc. Amer., 2003.

[4] R. A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, 2007.

[5] E. W. Weisstein, Euler's Distribution Theorem, MathWorld - A Wolfram Web Resource, `http://mathworld.wolfram.com/EulersDistributionTheorem.html`

[6] E. W. Weisstein, Ptolemy Inequality, MathWorld - A Wolfram Web Resource, `http://mathworld.wolfram.com/PtolemyInequality.html`

Yaroslav Bezverkhnyev: Main Post Office, P/O Box 29A, 88000 Uzhgorod, Transcarpathia, Ukraine

E-mail address: `slavab59@yahoo.ca`

Forum Geometricorum
Volume 8 (2008) 73–76.

FORUM GEOM
ISSN 1534-1178

An Inequality Involving the Angle Bisectors and an Interior Point of a Triangle

Wei-Dong Jiang

Abstract. We establish a new weighted geometric inequality involving the lengths of the angle bisectors and the radii of three circles through an interior point of a triangle. From this, several interesting geometric inequalities are derived.

1. Introduction

Throughout this paper we consider a triangle ABC with sidelengths a, b, c, circumradius R, and inradius r. Denote by w_a, w_b, w_c the lengths of the bisectors of angles A, B, C. Let P be an interior point. Denote by R_a, R_b, R_c the radii of the circles PBC, PCA, PAB respectively. Liu [2] has conjectured the inequality

$$\frac{w_a}{R_b+R_c}+\frac{w_b}{R_c+R_a}+\frac{w_c}{R_a+R_b}\leq\frac{9}{4}. \tag{1}$$

We prove a stronger inequality in Theorem 1 below, which include the

$$\frac{w_a}{\sqrt{R_bR_c}}+\frac{w_b}{\sqrt{R_cR_a}}+\frac{w_c}{\sqrt{R_aR_b}}\leq\frac{9}{2}. \tag{2}$$

Theorem 1. *For an interior point P and positive real numbers x, y, z, we have*

$$\frac{xw_a}{\sqrt{R_bR_c}}+\frac{yw_b}{\sqrt{R_cR_a}}+\frac{zw_c}{\sqrt{R_aR_b}}\leq\sqrt{2+\frac{r}{2R}}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right). \tag{3}$$

Equality holds if and only if the triangle ABC is equilateral, P its center, and $x=y=z$.

We shall make use of the following lemma.

Lemma 2. *For arbitrary nonzero real numbers $x,\ y,\ z$,*

$$x^2\sin^2A+y^2\sin^2B+z^2\sin^2C\leq\frac{1}{4}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right)^2. \tag{4}$$

Equality holds if and only if $x^2 : y^2 : z^2 = \frac{1}{a^2(b^2+c^2-a^2)} : \frac{1}{b^2(c^2+a^2-b^2)} : \frac{1}{c^2(a^2+b^2-c^2)}$.

Publication Date: April 16, 2008. Communicating Editor: Paul Yiu.
The author is grateful to Professor Paul Yiu for his suggestions for the improvement of this paper.

Proof. We make use of Kooi's inequality [1, Inequality 14.1]: for real numbers λ_1, λ_2, λ_3 with $\lambda_1+\lambda_2+\lambda_3\neq 0$,

$$(\lambda_1+\lambda_2+\lambda_3)^2R^2\geq\lambda_2\lambda_3a^2+\lambda_3\lambda_1b^2+\lambda_1\lambda_2c^2;$$

equality holds if and only if the point with homogeneous barycentric coordinates $(\lambda_1:\lambda_2:\lambda_3)$ with reference to triangle ABC is the circumcenter of the triangle. Now, with $\lambda_1=\frac{yz}{x}$, $\lambda_2=\frac{zx}{y}$, $\lambda_3=\frac{xy}{z}$, the result follows from the law of sines: $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$. □

2. Proof of Theorem 1

The length of the bisector of angle A is given by $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$. Clearly, $w_a\leq\sqrt{bc}\cos\frac{A}{2}$; equality holds if and only if $b=c$.

Let $\angle BPC=\alpha$, $\angle CPA=\beta$ and $\angle APB=\gamma$. Obviously, $0<\alpha,\ \beta,\ \gamma<\pi$ and $\alpha+\beta+\gamma=2\pi$. By the law of sines, $b=2R_b\sin\beta$, $c=2R_c\sin\gamma$. We have

$$\begin{aligned}\frac{w_a}{\sqrt{R_bR_c}}&\leq\sqrt{\frac{bc}{R_bR_c}}\cdot\cos\frac{A}{2}\\&=2\sqrt{\sin\beta\sin\gamma}\cdot\cos\frac{A}{2}\\&\leq(\sin\beta+\sin\gamma)\cos\frac{A}{2}\\&=2\sin\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{A}{2}\\&\leq 2\sin\frac{\alpha}{2}\cos\frac{A}{2}.\end{aligned}$$

Equality holds if and only if $b=c$ and $\beta=\gamma$. Similarly, $\frac{w_b}{\sqrt{R_cR_a}}\leq 2\sin\frac{\beta}{2}\cos\frac{B}{2}$ and $\frac{w_c}{\sqrt{R_aR_b}}\leq 2\sin\frac{\gamma}{2}\cos\frac{C}{2}$ with analogous conditions for equality. Therefore, for $x,\ y,\ z>0$,

$$\begin{aligned}&\frac{xw_a}{\sqrt{R_bR_c}}+\frac{yw_b}{\sqrt{R_cR_a}}+\frac{zw_c}{\sqrt{R_aR_b}}\\&\leq 2x\sin\frac{\alpha}{2}\cos\frac{A}{2}+2y\sin\frac{\beta}{2}\cos\frac{B}{2}+2z\sin\frac{\gamma}{2}\cos\frac{C}{2} &(5)\\&\leq 2\sqrt{\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\left(x^2\sin^2\frac{\alpha}{2}+y^2\sin^2\frac{\beta}{2}+z^2\sin^2\frac{\gamma}{2}\right)} &(6)\\&\leq\sqrt{2+\frac{r}{2R}}\cdot\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right) &(7)\end{aligned}$$

Here, the inequality in (6) follows from the Cauchy-Schwarz inequality. On the other hand, the inequality in (7) follows from the identity

$$\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+\frac{r}{2R},$$

and application of Lemma 2 to a triangle with angles $\frac{\alpha}{2}$, $\frac{\beta}{2}$, $\frac{\gamma}{2}$. Equality holds in (5) holds if and only if $a = b = c$ and $\alpha = \beta = \gamma$. This means that ABC is equilateral and P is its center. Finally, by Lemma 2 again, equality holds in (7) if and only if $x^2 : y^2 : z^2 = 1 : 1 : 1$, *i.e.*, $x = y = z$. This completes the proof of Theorem 1.

3. Some applications

With $x = y = z$ in Theorem 1, we have

$$\frac{w_a}{\sqrt{R_bR_c}} + \frac{w_b}{\sqrt{R_cR_a}} + \frac{w_c}{\sqrt{R_aR_b}} \le 3\sqrt{2 + \frac{r}{2R}}.$$

By Euler's famous inequality $R \ge 2r$, we have (2).

Since $\sqrt{R_bR_c} \le \frac{1}{2}(R_b + R_c)$, $\sqrt{R_cR_a} \le \frac{1}{2}(R_c + R_a)$, $\sqrt{R_aR_b} \le \frac{1}{2}(R_a + R_b)$, we obtain from Theorem 1,

$$\frac{xw_a}{R_b + R_c} + \frac{yw_b}{R_c + R_a} + \frac{zw_c}{R_a + R_b} \le \frac{1}{2}\sqrt{2 + \frac{r}{2R}}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right). \tag{8}$$

With $x = y = z$, we have

$$\frac{w_a}{R_b + R_c} + \frac{w_b}{R_c + R_a} + \frac{w_c}{R_a + R_b} \le \frac{3}{2}\sqrt{2 + \frac{r}{2R}}.$$

Liu's inequality (1) follows from $R \ge 2r$.

Again, from Euler's inequality, we immediately conclude from Theorem 1 that

$$\frac{xw_a}{\sqrt{R_bR_c}} + \frac{yw_b}{\sqrt{R_cR_a}} + \frac{zw_c}{\sqrt{R_aR_b}} \le \frac{3}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right). \tag{9}$$

Corollary 3. *For an interior point P and positive real numbers x, y, z, we have*

$$x^2R_a + y^2R_b + z^2R_c \ge \frac{2}{3}(yzw_a + zxw_b + xyw_c).$$

Equality holds if and only if the triangle ABC is equilateral, P its center, and $x = y = z$.

Proof. Replace in (9) x, y, z respectively by $yz\sqrt{R_bR_c}$, $zx\sqrt{R_cR_a}$, $xy\sqrt{R_aR_b}$. □

In particular, with $x = y = z = 1$, we have

$$R_a + R_b + R_c \ge \frac{2}{3}(w_a + w_b + w_c);$$

equality holds if and only if the triangle is equilateral and P its center.

Corollary 4. *For an interior point P in a triangle ABC, $R_aR_bR_c \ge \frac{64}{27}w_aw_bw_c$. Equality holds if and only if ABC is equilateral and P its center.*

Proof. This follows from (9) by putting $x = y = z$ and applying the AM-GM inequality. □

References

[1] O. Bottema et al, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.
[2] J. Liu, A hundred unsolved triangle inequality problems, in *Geometric Inequalities in China* (in Chinese), Jiangsu Education Press, Nanjing, 1996.

Wei-Dong Jiang: Department of Information Engineering, Weihai Vocational College, Weihai, Shandong Province 264210, P. R. China
E-mail address: jackjwd@163.com

Forum Geometricorum
Volume 8 (2008) 77–95.

FORUM GEOM
ISSN 1534-1178

Cubics Related to Coaxial Circles

Bernard Gibert

Abstract. This note generalizes a result of Paul Yiu on a locus associated with a triad of coaxial circles. We present an interesting family of cubics with many properties similar to those of pivotal cubics. It is also an opportunity to show how different ways of writing the equation of a cubic lead to various geometric properties of the curve.

1. Introduction

In his Hyacinthos message [7], Paul Yiu encountered the cubic **K360** as the locus of point P (in the plane of a given triangle ABC) with cevian triangle XYZ such that the three circles $AA'X$, $BB'Y$, $CC'Z$ are coaxial. Here $A'B'C'$ is the circumcevian triangle of X_{56}, the external center of similitude of the circumcircle and incircle. See Figure 1. It is natural to study the coaxiality of the circles when $A'B'C'$ is the circumcevian triangle of a given point Q.

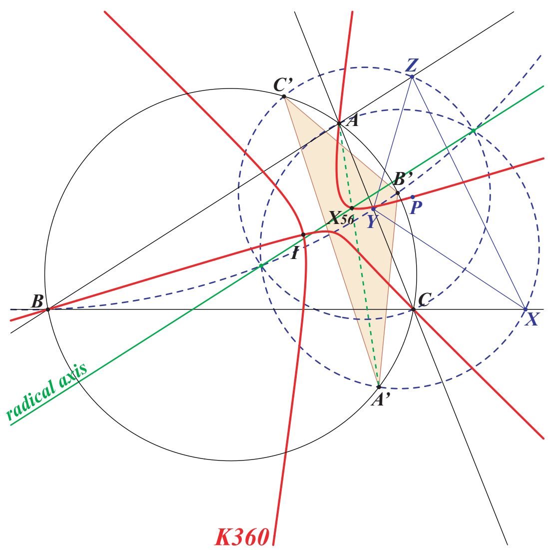


Figure 1. **K360** and coaxial circles

Throughout this note, we work with homogeneous barycentric coordinates with reference to triangle ABC, and adopt the following notations:

Publication Date: April 21, 2008. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this paper.

gX the isogonal conjugate of X
tX the isotomic conjugate of X
cX the complement of X
aX the anticomplement of X
tg$X$ the isotomic conjugate of the isogonal conjugate of X

2. Preliminaries

Let $Q = p : q : r$ be a fixed point with circumcevian triangle $A'B'C'$ and P a variable point with cevian triangle $P_aP_bP_c$. Denote by $\mathcal{C}_A$ the circumcircle of triangle $AA'P_a$ and define $\mathcal{C}_B, \mathcal{C}_C$ in the same way.

Lemma 1. *The radical center of the circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ is the point Q.*

Proof. The radical center of the circumcircle $\mathcal{C}$ of triangle ABC and $\mathcal{C}_B, \mathcal{C}_C$ must be Q. Indeed, it must be the intersection of BB' (the radical axis of $\mathcal{C}$ and $\mathcal{C}_B$) and CC' (the radical axis of $\mathcal{C}$ and $\mathcal{C}_C$). Hence the radical axis of $\mathcal{C}_B, \mathcal{C}_C$ contains Q. □

These three radical axes are in general distinct lines. For some choices of P, however, these circles are coaxial. For example, if $P = Q$, then the three circles degenerate into the cevian lines of Q and we regard these as infinite circles with radical axis the line at infinity. Another trivial case is when P is one of the vertices A, B, C, since two circles coincide with $\mathcal{C}$ and the third circle is not defined.

Lemma 2. *Let H be the orthocenter of triangle ABC. For any point $Q \neq H$ and $P = H$, the circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ are coaxial with radical axis HQ.*

Proof. When $P = H$, the cevian triangle of P is the orthic triangle $H_aH_bH_c$. The inversion with respect to the polar circle swaps A, B, C and H_a, H_b, H_c respectively. Hence the products of signed distances $HA \cdot HH_a$, $HB \cdot HH_b$, $HC \cdot HH_c$ are equal but, since they represent the power of H with respect to the circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$, H must be on their radical axes which turns out to be the line HQ. If $Q = H$, the property is a simple consequence of the lemma above. □

3. The cubic $\mathcal{K}(Q)$ and its construction

Theorem 3. *In general, the locus of P for which the circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ are coaxial is a circumcubic $\mathcal{K}(Q)$ passing through H, Q and several other remarkable points. This cubic is tangent at A, B, C to the symmedians of triangle ABC.*

This is obtained through direct and easy calculation. It is sufficient to write that the radical circle of $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$ degenerates into the line at infinity and another line which is obviously the common radical axis of the circles. This calculation gives several equivalent forms of the barycentric equation of $\mathcal{K}(Q)$. In §§4 – 9 below, we explore these various forms, deriving essential geometric properties and identifying interesting points of the cubic. For now we examine the simplest of all these:

$$\sum_{\text{cyclic}} b^2c^2\, p\, x\,(y+z)(ry-qz) = 0 \iff \sum_{\text{cyclic}} \frac{x(y+z)}{a^2}\left(\frac{y}{q}-\frac{z}{r}\right) = 0. \quad (1)$$

It is clear that $\mathcal{K}(Q)$ contains A, B, C, Q and the vertices A_1, B_1, C_1 of the cevian triangle of $\mathbf{tg}Q = \frac{p}{a^2} : \frac{q}{b^2} : \frac{r}{c^2}$. Indeed, when we take $x = 0$ in equation (1) we obtain $(b^2ry - c^2qz)yz = 0$.

$\mathcal{K}(Q)$ also contains $\mathbf{ag}Q$. Indeed, if we write $\mathbf{ag}Q = u : v : w$ then $v + w = \dfrac{a^2}{p}$, etc, since this is the complement of $\mathbf{ag}Q$ i.e. $\mathbf{g}Q$. The second form of equation (1) obviously gives $\displaystyle\sum_{\text{cyclic}} \frac{u}{p}\left(\frac{v}{q} - \frac{w}{r}\right) = 0.$

Finally, it is easy to verify that $\mathcal{K}(Q)$ is tangent at A, B, C to the symmedians of triangle ABC. Indeed, when b^2z is replaced by c^2y in (1), the polynomial factorizes by y^2.

3.1. *Construction.* Given Q, denote by S be the second intersection of the Euler line with the rectangular circumhyperbola $\mathcal{H}_Q$ through Q.

Let $\mathcal{H}'_Q$ be the rectangular hyperbola passing through O, Q, S and with asymptotes parallel to those of $\mathcal{H}_Q$.

A variable line L_Q through Q meets $\mathcal{H}'_Q$ at a point Q'.

L_Q meets the rectangular circumhyperbola through $\mathbf{g}Q'$ (the isogonal transform of the line OQ') at two points M, M' of $\mathcal{K}(Q)$ collinear with Q.

Note that Q is the coresidual of A, B, C, H in $\mathcal{K}(Q)$ and that $\mathbf{ag}Q$ is the coresidual of A, B, C, Q in $\mathcal{K}(Q)$. Thus, the line through $\mathbf{ag}Q$ and M meets again the circumconic through Q and M at another point on $\mathcal{K}(Q)$.

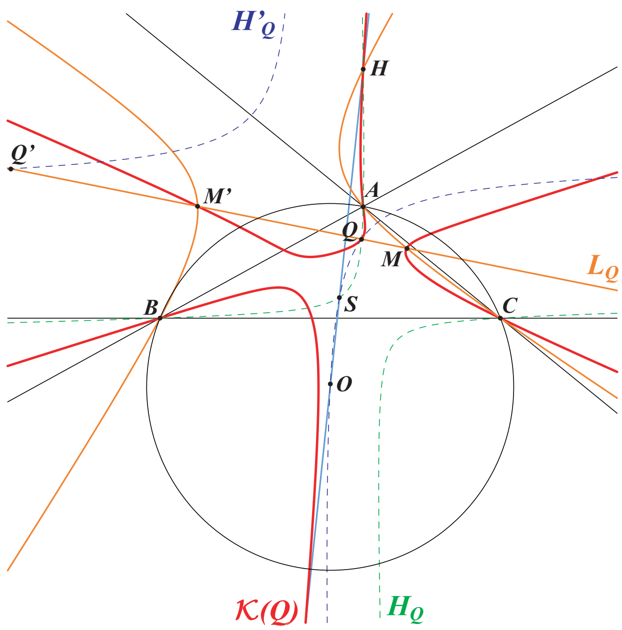


Figure 2. Construction of $\mathcal{K}(Q)$

4. Intersections with the circumcircle and the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$

Proposition 4. *$\mathcal{K}(Q)$ intersects the circumcircle at the same points as the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$ with pivot $\mathbf{ag}Q$.*

Proof. The equation of $\mathcal{K}(Q)$ can be written in the form

$$\begin{aligned} &\sum_{\text{cyclic}} (-a^2qr + b^2rp + c^2pq)\, x\, (c^2y^2 - b^2z^2) \\ &+ (a^2yz + b^2zx + c^2xy) \sum_{\text{cyclic}} p\, (c^2q - b^2r)\, x = 0. \end{aligned} \tag{2}$$

Any point common to $\mathcal{K}(Q)$ and the circumcircle also lies on the cubic

$$\sum_{\text{cyclic}} (-a^2qr + b^2pr + c^2pq)\, x\, (c^2y^2 - b^2z^2) = 0, \tag{3}$$

which is the pivotal isogonal circumcubic $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$. □

The two cubics $\mathcal{K}(Q)$ and $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$ must have three other common points on the line passing through G and $\mathbf{ag}Q$. One of them is $\mathbf{ag}Q$ and the two other points E_1, E_2 are not always real points. Indeed, the equation of this line is

$$\sum_{\text{cyclic}} p(c^2q - b^2r)x = 0.$$

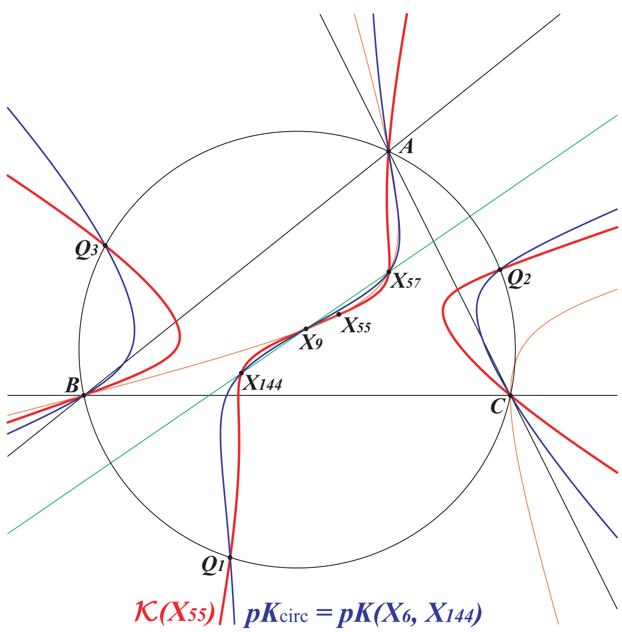


Figure 3. $\mathcal{K}(Q)$ and $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$ when $Q = X_{55}$

These points E_1, E_2 are the intersections of the line passing through G, **g**Q, $\mathbf{ag}Q$ with the circumconic $ABCKQ$ which is its isogonal conjugate. It follows that these points are the last common points of $\mathcal{K}(Q)$ and the Thomson cubic **K002**.

Figure 3 shows these cubics when $Q = X_{55}$, the isogonal conjugate of the Gergonne point X_7. Here, the points E_1, E_2 are X_9, X_{57} and $\mathbf{ag}Q$ is X_{144}.

Thus, $\mathcal{K}(Q)$ meets the circumcircle at A, B, C with concurrent tangents at K and three other points Q_1, Q_2, Q_3 (one of them is always real). Following [4], **ag**Q must be the orthocenter of triangle $Q_1Q_2Q_3$.

4.1. *Construction of the points* Q_1, Q_2, Q_3. The construction of these points again follows a construction of [4] : the rectangular hyperbola having the same asymptotic directions as those of $ABCHQ$ and passing through Q, $\mathbf{ag}Q$, the antipode Z on the circumcircle of the isogonal conjugate Z' of the infinite point of the line O**g**Q meets the circumcircle at Z and Q_1, Q_2, Q_3. Note that Z' is the fourth point of $ABCHQ$ on the circumcircle. The sixth common point of the hyperbola and $\mathcal{K}(Q)$ is the second intersection Q' of the line $H\mathbf{ag}Q$ with both hyperbolas. It is the tangential of Q in $\mathcal{K}(Q)$. It is also the second intersection of the line ZZ' with both hyperbolas. See Figure 4.

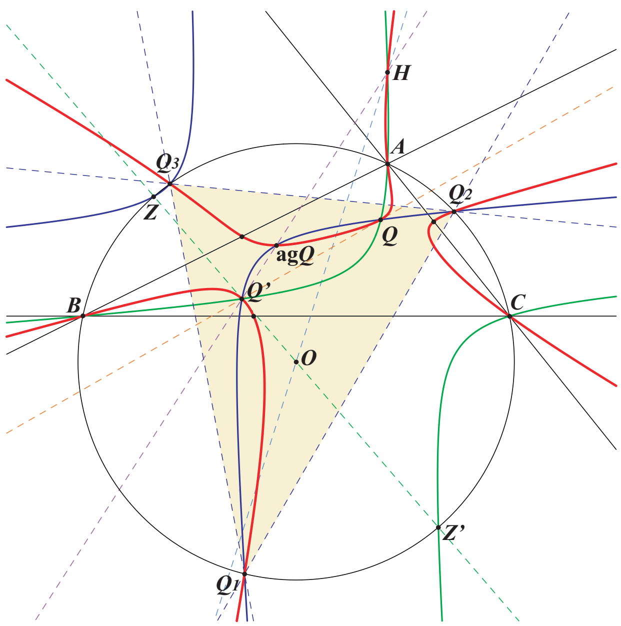


Figure 4. Construction of the points Q_1, Q_2, Q_3

These points Q_1, Q_2, Q_3 have several properties related with Simson lines obtained by manipulation of third degree polynomials. They derive from classical properties of triples of points on the circumcircle of ABC having concurring Simson lines.

Theorem 5. *The points Q_1, Q_2, Q_3 are the antipodes on the circumcircle of the three points Q'_1, Q'_2, Q'_3 whose Simson lines pass through* $\mathbf{g}Q$.

It follows that Q_1, Q_2, Q_3 are three real distinct points if and only if $\mathbf{g}Q$ lies inside the Steiner deltoid $\mathcal{H}_3$.

Theorem 6. *The Simson lines of Q_1, Q_2, Q_3 are tangent to the inconic $\mathcal{I}(Q)$ with perspector* $\mathbf{tg}Q$ *and center* $\mathbf{cg}Q$. *They form a triangle $S_1S_2S_3$ perspective at* $\mathbf{cg}Q$ *to $Q_1Q_2Q_3$.*

S_1 is the common point of the Simson lines of Q'_1, Q_2, Q_3. These points S_1, S_2, S_3 are the reflections of Q_1, Q_2, Q_3 in $\mathbf{cg}Q$. See Figure 5.

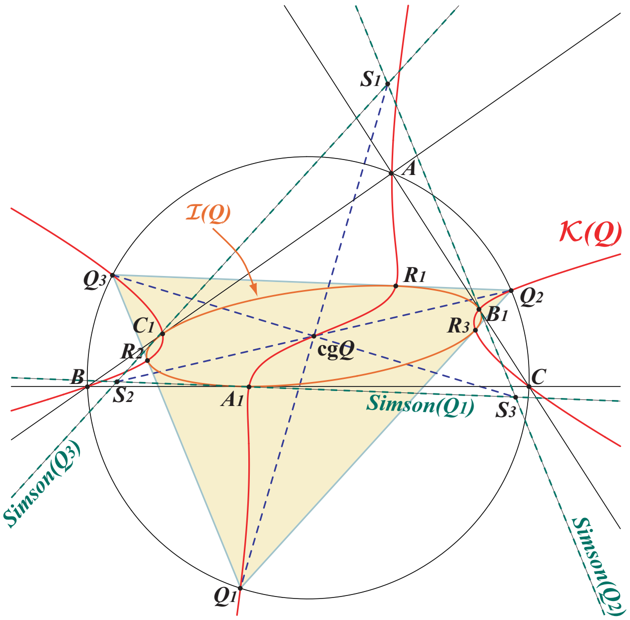


Figure 5. $\mathcal{K}(Q)$ and Simson lines

Another computation involving symmetric functions of the roots of a third degree polynomial gives

Theorem 7. *$\mathcal{K}(Q)$ meets the circumcircle at A, B, C with tangents concurring at the Lemoine point K of ABC and three other points Q_1, Q_2, Q_3 where the tangents are also concurrent at the Lemoine point of $Q_1Q_2Q_3$.*

This generalizes the property already encountered in a family of pivotal cubics seen in [4, §4]. Since the two triangles ABC and $Q_1Q_2Q_3$ are inscribed in the circumcircle, there must be a conic inscribed in both triangles. This gives

Theorem 8. *The inconic $\mathcal{I}(Q)$ with perspector* $\mathbf{tg}Q$ *is inscribed in the two triangles ABC and $Q_1Q_2Q_3$. It is also inscribed in the triangle formed by the Simson lines of Q_1, Q_2, Q_3.*

$\mathcal{K}(Q)$ meets $\mathcal{I}(Q)$ at six points which are the contacts of $\mathcal{I}(Q)$ with the sidelines of the two triangles. Three of them are the vertices A_1, B_1, C_1 of the cevian triangle of $\mathbf{tg}Q$ in ABC. The other points R_1, R_2, R_3 are the intersections of the sidelines of $Q_1Q_2Q_3$ with the cevian lines of H in $S_1S_2S_3$. In other words, $R_1 = HS_1 \cap Q_2Q_3$, etc. See Figure 5. Note that the reflections of R_1, R_2, R_3 in the center **cg**Q of $\mathcal{I}(Q)$ are the contacts T_1, T_2, T_3 of the Simson lines of Q_1, Q_2, Q_3 with $\mathcal{I}(Q)$.

5. Infinite points on $\mathcal{K}(Q)$ and intersection with $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$

Proposition 9. *$\mathcal{K}(Q)$ meets the line at infinity at the same points as the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$ with pivot* **g**Q.

Proof. This follows by writing the equation of $\mathcal{K}(Q)$ in the form

$$\sum_{\text{cyclic}} a^2\, qr\, x\,(c^2y^2 - b^2z^2) + (x+y+z) \sum_{\text{cyclic}} a^2\, p\,(c^2q - b^2r)\, yz = 0. \tag{4}$$

Any infinite point on $\mathcal{K}(Q)$ is also a point on the cubic

$$\sum_{\text{cyclic}} a^2\, qr\, x\,(c^2y^2 - b^2z^2) = 0 \iff \sum_{\text{cyclic}} \frac{x}{p}\left(\frac{y^2}{b^2} - \frac{z^2}{c^2}\right) = 0, \tag{5}$$

which is the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$ with pivot **g**Q. □

The six other common points of $\mathcal{K}(Q)$ and $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$ lie on the circumhyperbola through Q and K. They are A, B, C, Q and the two points E_1, E_2. Figure 6 shows these cubics when $Q = X_{55}$ thus **g**Q is the Gergonne point X_7. Recall that the points E_1, E_2 are X_9, X_{57}.

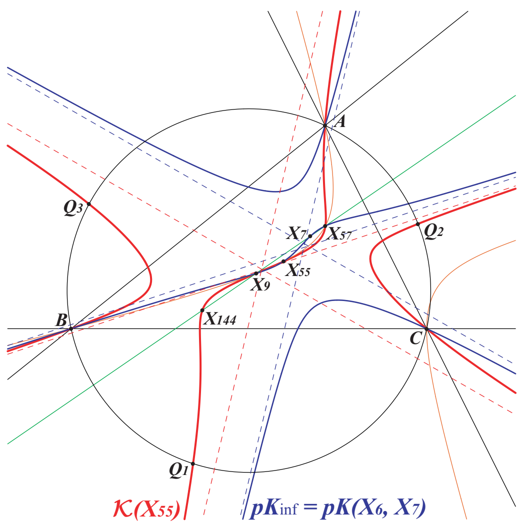


Figure 6. $\mathcal{K}(Q)$ and $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$ when $Q = X_{55}$

6. $\mathcal{K}(Q)$ and the inconic with center cg$Q$

Proposition 10. *The cubic $\mathcal{K}(Q)$ contains the four foci of the inconic with center* **cg**Q *and perspector* $\mathbf{tg}Q$.

Proof. This follows by writing the equation of $\mathcal{K}(Q)$ in the form

$$\sum_{\text{cyclic}} px(c^2q - b^2r)(c^2y^2 + b^2z^2) - 2\left(\sum_{\text{cyclic}} a^2(b^2 - c^2)qr\right) xyz \\ - \sum_{\text{cyclic}} px(c^2q + b^2r)(c^2y^2 - b^2z^2) = 0. \tag{6}$$

Indeed,

$$\sum_{\text{cyclic}} px(c^2q - b^2r)(c^2y^2 + b^2z^2) - 2\left(\sum_{\text{cyclic}} a^2(b^2 - c^2)qr\right) xyz = 0 \tag{7}$$

is the equation of the non-pivotal isogonal circular cubic $n\mathcal{K}_6(Q)$ which is the locus of foci of inconics with center on the line through G, **cg**Q and

$$\sum_{\text{cyclic}} px(c^2q + b^2r)(c^2y^2 - b^2z^2) = 0 \tag{8}$$

is the equation of the pivotal isogonal cubic p$\mathcal{K}_6(Q)$ with pivot **cg**Q. The two cubics $\mathcal{K}(Q)$ and p$\mathcal{K}_6(Q)$ obviously contain the above mentioned foci. □

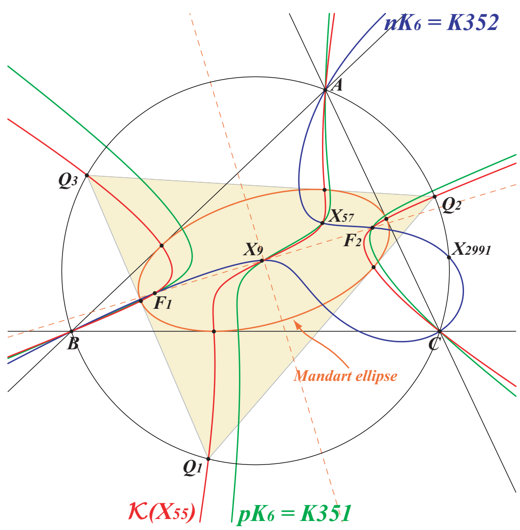


Figure 7. $\mathcal{K}(Q)$ and the related cubics $n\mathcal{K}_6(Q)$, p$\mathcal{K}_6(Q)$ when $Q = X_{55}$

These two cubics generate a pencil of cubics containing $\mathcal{K}(Q)$. Note that p$\mathcal{K}_6(Q)$ is a member of the pencil of isogonal pivotal cubics generated by p$\mathcal{K}_{\text{inf}}(Q)$ and

$\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$. The root of $n\mathcal{K}_6(Q)$ is the infinite point of the trilinear polar of $\mathbf{tg}Q$. Figure 7 shows these cubics when $Q = X_{55}$. The inscribed conic is the Mandart ellipse.

In the example above, $\mathcal{K}(Q)$ contains the center **cg**Q of the inconic $\mathcal{I}(Q)$ but this is not generally true. We have

Theorem 11. *$\mathcal{K}(Q)$ contains the center* **cg**Q *of $\mathcal{I}(Q)$ if and only if Q lies on the cubic* **K172** $= \mathrm{p}\mathcal{K}(X_{32}, X_3)$.

Since we know that $\mathcal{K}(Q)$ contains the perspector **tg**Q of this same inconic when it is a pivotal cubic, it follows that there are only two cubics $\mathcal{K}(Q)$ passing through the foci, the center, the perspector of $\mathcal{I}(Q)$ and its contacts with the sidelines of ABC. These cubics are obtained when
(i) $Q = X_6 : \mathcal{K}(X_6)$ is the Thomson cubic **K002** and $\mathcal{I}(Q)$ is the Steiner inscribed ellipse,
(ii) $Q = X_{25} : \mathcal{K}(X_{25})$ is **K233** $= \mathrm{p}\mathcal{K}(X_{25}, X_4)$.

In the latter case, **cg**$Q = X_6$, **tg**$Q = X_4$, **ag**$Q = X_{193}$, $\mathcal{I}(Q)$ is the K-ellipse, [1] the infinite points are those of **K169** = $\mathrm{p}\mathcal{K}(X_6, X_{69})$, the points on the circumcircle are those of $\mathrm{p}\mathcal{K}(X_6, X_{193})$. See Figure 8.

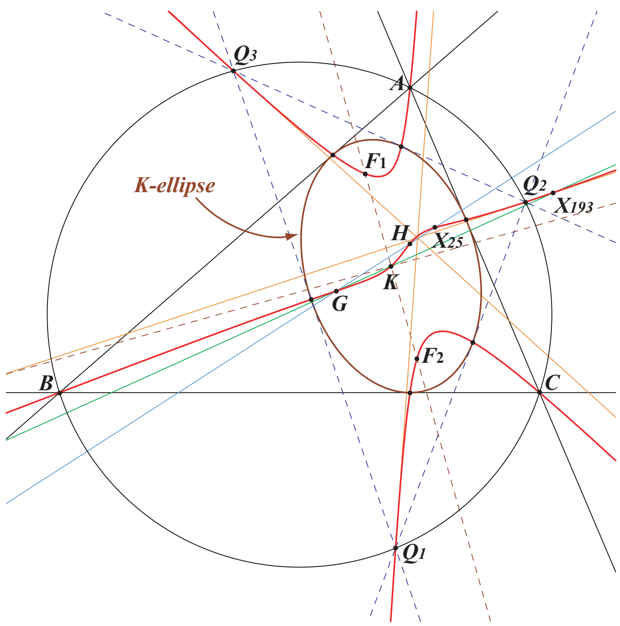


Figure 8. $\mathcal{K}(X_{25})$ and the related K-ellipse

[1]The $K-$ellipse is actually an ellipse only when triangle ABC is acute angled.

7. $\mathcal{K}(Q)$ and the Steiner ellipse

Proposition 12. *The cubic $\mathcal{K}(Q)$ meets the Steiner ellipse at the same points as* $\mathrm{p}\mathcal{K}(\mathbf{tg}Q,\ Q)$.

Proof. This follows by writing the equation of $\mathcal{K}(Q)$ in the form

$$\sum_{\text{cyclic}} a^2\, p\, x\, (b^2ry^2 - c^2qz^2) + (xy + yz + zx) \sum_{\text{cyclic}} a^2\, (b^2 - c^2)\, qr\, x = 0. \quad (9)$$

Indeed,

$$\sum_{\text{cyclic}} a^2\, p\, x\, (b^2ry^2 - c^2qz^2) = 0 \iff \sum_{\text{cyclic}} x \left(\frac{y^2}{c^2q} - \frac{z^2}{b^2r} \right) = 0 \quad (10)$$

is the equation of the pivotal cubic $\mathrm{p}\mathcal{K}(\mathbf{tg}Q,\ Q)$. □

Note that $\sum_{\text{cyclic}} a^2\, (b^2 - c^2)\, qr\, x = 0$ is the equation of the line $Q\mathbf{tg}Q$. This will be construed in the next paragraph.

8. $\mathcal{K}(Q)$ and rectangular hyperbolas

Let $P = u : v : w$ be a given point and let $\mathcal{H}(P)$, $\mathcal{H}(\mathbf{g}P)$ be the two rectangular circum-hyperbolas passing through P, $\mathbf{g}P$ respectively. These have equations

$$\sum_{\text{cyclic}} u(S_Bv - S_cw)yz = 0 \quad \text{and} \quad \sum_{\text{cyclic}} \left(\frac{S_Bw}{c^2} - \frac{S_Cv}{b^2} \right) yz = 0.$$

P must not lie on the McCay cubic in order to have two distinct hyperbolas. Indeed, $\mathbf{g}P$ lies on $\mathcal{H}(P)$ if and only if P lies on the line $O\mathbf{g}P$ i.e. P and $\mathbf{g}P$ are two isogonal conjugate points collinear with O.

Let $\mathcal{L}(Q)$ and $\mathcal{L}'(Q)$ be the two lines passing through Q with equations

$$\sum_{\text{cyclic}} a^2\, (vr(qx - py) - wq(rx - pz)) = 0$$

and

$$\sum_{\text{cyclic}} b^2c^2pu(v + w)(ry - qz) = 0.$$

These lines $\mathcal{L}(Q)$ and $\mathcal{L}'(Q)$ can be construed as the trilinear polars of the $Q-$isoconjugates of the infinite points of the polars of P and $\mathbf{g}P$ in the circumcircle.

The equation of $\mathcal{K}(Q)$ can be written in the form

$$\begin{aligned} &\left(\sum_{\text{cyclic}} u(S_Bv - S_cw)yz \right) \left(\sum_{\text{cyclic}} a^2\, (vr(qx - py) - wq(rx - pz)) \right) \\ &= \left(\sum_{\text{cyclic}} \left(\frac{S_Bw}{c^2} - \frac{S_Cv}{b^2} \right) yz \right) \left(\sum_{\text{cyclic}} b^2c^2pu(v + w)(ry - qz) \right) \end{aligned} \quad (11)$$

which will be loosely written under the form :

$$\mathcal{H}(P) \cdot \mathcal{L}(Q) = \mathcal{H}(\mathbf{g}P) \cdot \mathcal{L}'(Q).$$

If we recall that $\mathcal{K}(Q)$ and $\mathcal{H}(P)$ have already four common points namely A, B, C, H and that $\mathcal{K}(Q)$, $\mathcal{L}(Q)$ and $\mathcal{L}'(Q)$ all contain Q, then we have

Corollary 13. *$\mathcal{K}(Q)$ meets $\mathcal{H}(P)$ again at two points on the line $\mathcal{L}'(Q)$ and $\mathcal{H}(\mathbf{g}P)$ again at two points on the line $\mathcal{L}(Q)$.*

For example, with $P = G$, $\mathcal{H}(P)$ is the Kiepert hyperbola and $\mathcal{L}'(Q)$ is the line $Q\mathbf{gt}Q$, $\mathcal{H}(\mathbf{g}P)$ is the Jerabek hyperbola and $\mathcal{L}(Q)$ is the line $Q\mathbf{tg}Q$.

9. Further representations of $\mathcal{K}(Q)$

Proposition 14. *For varying Q, the cubics $\mathcal{K}(Q)$ form a net of cubics.*

Proof. This follows by writing the equation of $\mathcal{K}(Q)$ in the form

$$\begin{aligned} &\sum_{\text{cyclic}} a^2qr\,x\left(c^2y(x+z) - b^2z(x+y)\right) = 0 \\ \iff &\sum_{\text{cyclic}} a^2qr\,x\left(x(c^2y - b^2z) - (b^2-c^2)yz\right) = 0. \end{aligned} \tag{12}$$

The equation $c^2y(x+z) - b^2z(x+y) = 0$ is that of the rectangular circumhyperbola $\mathcal{H}_A$ tangent at A to the symmedian AK. Its center is the midpoint of BC. Its sixth common point with $\mathcal{K}(Q)$ is the intersection of the lines AQ and $A_1\mathbf{ag}Q$. Thus the net is generated by the three decomposed cubics which are the union of a sideline of ABC and the corresponding hyperbola such as $\mathcal{H}_A$. $\square$

Proposition 15. *$\mathcal{K}(Q)$ is a pivotal cubic $\mathrm{p}\mathcal{K}(Q)$ if and only if Q lies on the circumhyperbola $\mathcal{H}$ passing through G and K.*

Proof. We write the equation of $\mathcal{K}(Q)$ in the form

$$\sum_{\text{cyclic}} b^2c^2\,p\,x\,(ry^2 - qz^2) + \left(\sum_{\text{cyclic}} a^2(b^2-c^2)qr\right) xyz = 0. \tag{13}$$

Recall that $\mathcal{K}(Q)$ meets the sidelines of triangle ABC again at the vertices of the cevian triangle of $\mathbf{tg}Q$. Thus, the cubic is a pivotal cubic is and only if the term in xyz vanishes. It is now sufficient to observe that the equation of the hyperbola $\mathcal{H}$ is $\displaystyle\sum_{\text{cyclic}} a^2(b^2-c^2)yz = 0$. $\square$

See a more detailed study of these $\mathrm{p}\mathcal{K}(Q)$ in §10.1.

Proposition 16. *The cubic $\mathcal{K}(Q)$ belongs to another pencil of similar cubics generated by another pivotal cubic and another isogonal non-pivotal cubic.*

Proof.

$$\sum_{\text{cyclic}} px(c^2q-b^2r)(c^2y^2+b^2z^2)-\left(\sum_{\text{cyclic}} a^2(b^2-c^2)qr\right)xyz \\ +\sum_{\text{cyclic}} a^4qr(y-z)yz=0. \tag{14}$$

Indeed,

$$\sum_{\text{cyclic}} px(c^2q-b^2r)(c^2y^2+b^2z^2)-\left(\sum_{\text{cyclic}} a^2(b^2-c^2)qr\right)xyz=0 \tag{15}$$

is the equation of the non-pivotal isogonal cubic $n\mathcal{K}_7(Q)$ with root the infinite point of the trilinear polar of $\mathbf{tg}Q$ again and

$$\sum_{\text{cyclic}} a^4qr(y-z)yz=0 \tag{16}$$

is the equation of the pivotal cubic $\mathrm{p}\mathcal{K}_7(Q)$ with pivot the centroid G and pole the $X_{32}-$isoconjugate of Q i.e. the point **gtg**Q. □

The cubics $n\mathcal{K}_6(Q)$ and $n\mathcal{K}_7(Q)$ obviously coincide when Q lies on the circumhyperbola $\mathcal{H}$ passing through G and K. Figure 9 shows these cubics when $Q=X_{55}$.

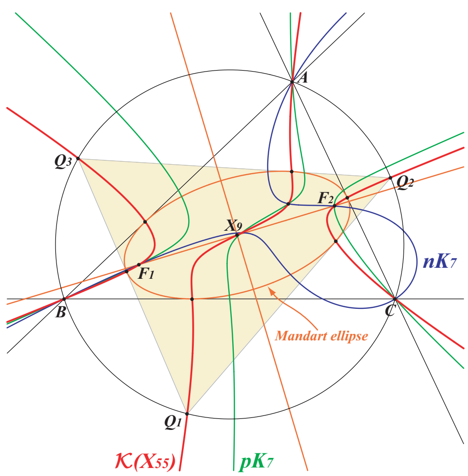


Figure 9. $\mathcal{K}(Q)$ and the related cubics $n\mathcal{K}_7(Q)$, $\mathrm{p}\mathcal{K}_7(Q)$ when $Q=X_{55}$

10. Special cubics $\mathcal{K}(Q)$

10.1. *Pivotal cubics* $\mathrm{p}\mathcal{K}(Q)$. Recall that for any point Q on the circumhyperbola $\mathcal{H}$ passing through G and K the cubic $\mathcal{K}(Q)$ becomes a pivotal cubic with pole Q and pivot $\mathbf{tg}Q$ on the Kiepert hyperbola. In this case, $\mathcal{K}(Q)$ has equation :

$$\sum_{\text{cyclic}} b^2c^2\, p\, x\,(ry^2 - qz^2) = 0 \iff \sum_{\text{cyclic}} \frac{x}{a^2}\left(\frac{y^2}{q} - \frac{z^2}{r}\right) = 0 \qquad (17)$$

The isopivot (secondary pivot) is clearly the Lemoine point K since the tangents at A, B, C are the symmedians. The points $\mathbf{g}Q$ and $\mathbf{ag}Q$ lie on the line GK namely the tangent at G to the Kiepert hyperbola.

These cubics form a pencil of pivotal cubics passing through A, B, C, G, H, K and tangent to the symmedians. Recall that they have the remarkable property to intersect the circumcircle at three other points Q_1, Q_2, Q_3 with concurrent tangents such that $\mathbf{ag}Q$ is the orthocenter of $Q_1Q_2Q_3$. See [4] for further informations.

This pencil is generated by the Thomson cubic **K002** (the only isogonal cubic) and by **K141** (the only isotomic cubic). See **CL043** in [2] for a selection of other cubics of the pencil among them **K273**, the only circular cubic, and **K233** seen above.

10.2. *Circular cubics* $\mathcal{K}(Q)$. We have seen that $\mathcal{K}(Q)$ meets the line at infinity at the same points as the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$ with pivot $\mathbf{g}Q$. It easily follows that $\mathcal{K}(Q)$ is a circular cubic if and only if $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$ is itself a circular cubic therefore if and only if $\mathbf{g}Q$ lies at infinity hence Q must lie on the circumcircle $\mathcal{C}$. Thus, we have :

Theorem 17. *For any point Q on the circumcircle, $\mathcal{K}(Q)$ is a circular cubic with singular focus on the circle with center O and radius $2R$. The tangent at Q always passes through O.*

The real asymptote envelopes a deltoid, the homothetic of the Steiner deltoid under $h(G, 4)$. See Figure 10.

For example, **K273** (obtained for $Q = X_{111}$, the Parry point) and **K306** (obtained for $Q = X_{759}$) are two cubics of this type in [2]. See also the bottom of the page **CL035** in [2].

10.3. *Lemoine generalized cubics* $\mathcal{K}(Q)$. A necessary (but not sufficient) condition to obtain a Lemoine generalized cubic $\mathcal{K}(Q)$ is that the cevian triangle of $\mathbf{tg}Q$ must be a pedal triangle. Hence, $\mathbf{tg}Q$ must be a point on the Lucas cubic **K007** therefore Q must be on its isogonal transform **K172**.

The only identified points that give a Lemoine generalized cubic are H and X_{56}.

$\mathcal{K}(H)$ is **K028**, the third Musselman cubic. It is also the only cubic with asymptotes making 60° angles with one another i.e. the only equilateral cubic of this type.

$\mathcal{K}(X_{56})$ is **K360**, at the origin of this note. See Figure 11.

The conic inscribed in the triangles ABC and $Q_1Q_2Q_3$ is the incircle of ABC since $\mathbf{tg}X_{56}$ is the Gergonne point X_7. $Q_1Q_2Q_3$ is a poristic triangle.

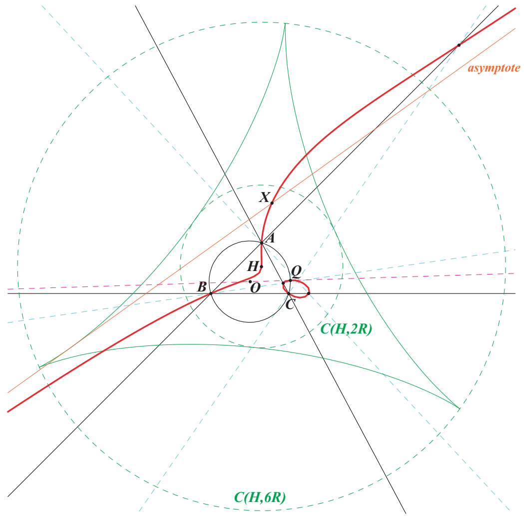


Figure 10. Circular cubics $\mathcal{K}(Q)$ and deltoid

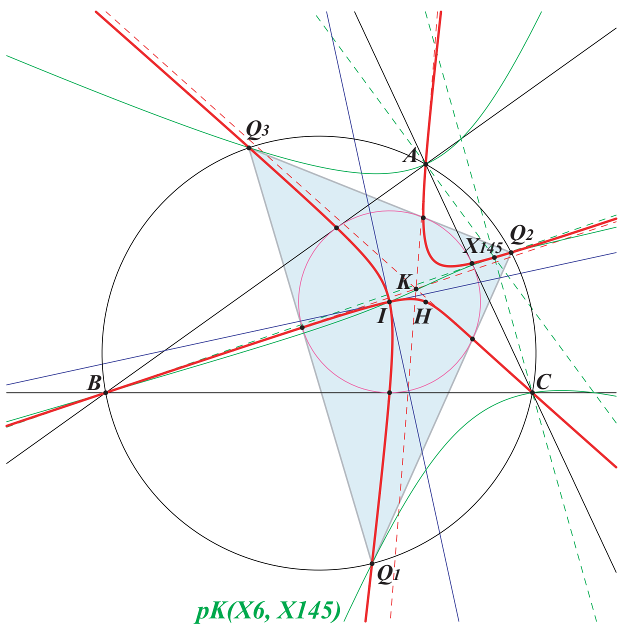


Figure 11. The Lemoine generalized cubic $\mathcal{K}(X_{56})$ = **K360**

10.4. $\mathcal{K}(X_{32})$. $\mathcal{K}(X_{32})$ has the remarkable property to have its six tangents at its common points with the circumcircle concurrent at the Lemoine point K. It follows that the triangles ABC and $Q_1Q_2Q_3$ have the same Lemoine point and the same Brocard axis. The polar conic of K is therefore the circumcircle.

The satellite conic of the circumcircle is the Brocard ellipse whose real foci Ω_1, Ω_2 (Brocard points) lie on the cubic. See Figure 12.

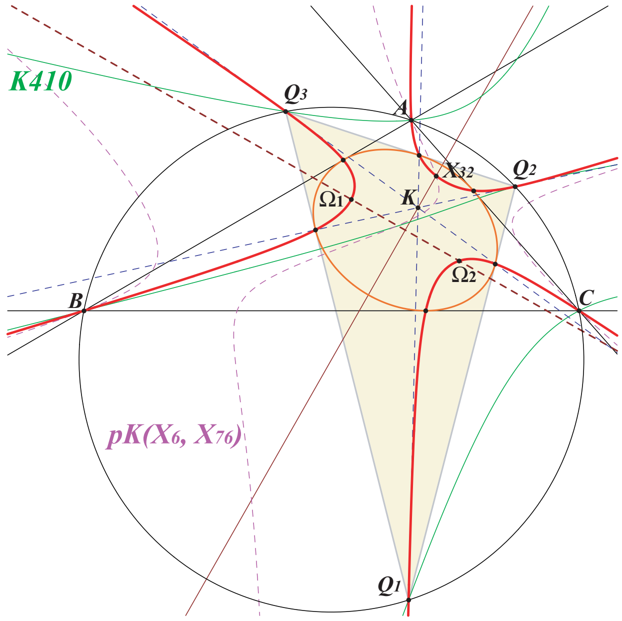


Figure 12. The cubic $\mathcal{K}(X_{32})$

Remark. $\mathcal{K}(X_{32})$ belongs to a pencil of circum-cubics having the same property to meet the circumcircle at six points A, B, C, Q_1, Q_2, Q_3 with tangents concurring at K hence the polar conic of K is always the circumcircle.

The cubic of the pencil passing through the given point $P = u : v : w$ has an equation of the form

$$\sum_{\text{cyclic}} a^4 vwyz \left((c^2v - b^2w)x - (u(c^2y - b^2z)\right) = 0,$$

which shows that the pencil is generated by three decomposed cubics, one of them being the union of the sidelines AB, AC and the line joining P to the feet K_a of the $A-$symmedian, the other two similarly. Each cubic meets the Brocard ellipse at six points which are the tangentials of the six points above. Three of them are K_a, K_b, K_c and the other points are the contacts of the Brocard ellipse with the sidelines of $Q_1Q_2Q_3$.

10.5. $\mathcal{K}(X_{54})$. $\mathcal{K}(X_{54})$ = **K361** is the only cubic of the family meeting the circumcircle at the vertices of an equilateral triangle $Q_1Q_2Q_3$ namely the circumnormal triangle. The tangents at these points concur at O. **K361** is the isogonal transform of **K026**, the (first) Musselman cubic and the locus of pivots of pivotal cubics that pass through the vertices of the circumnormal triangle. See Figure 13 and further details in [2].

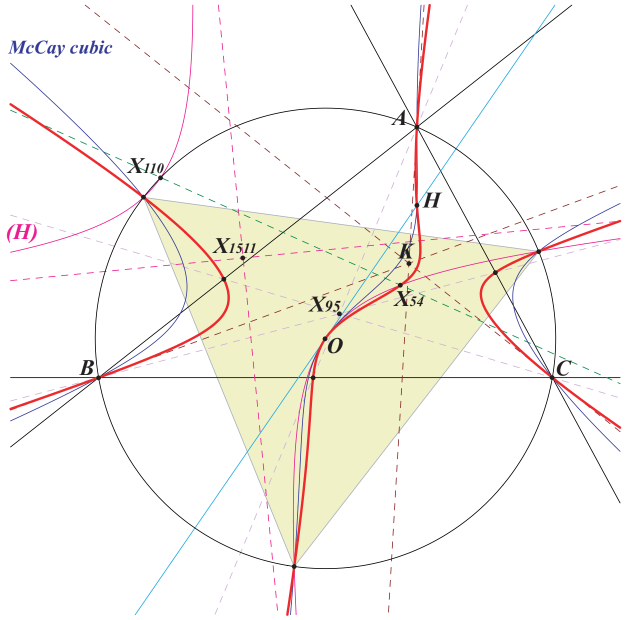


Figure 13. The cubic $\mathcal{K}(X_{54}) =$ **K361**

10.6. *$\mathcal{K}(Q)$ with concurring asymptotes.* $\mathcal{K}(Q)$ has three (not necessarily all real) concurring asymptotes if and only if Q lies on a circumcubic passing through O, H, X_{140}. This latter cubic is a $\mathcal{K}^+_{60}$ i.e. it has three real concurring asymptotes making $60°$ angles with one another. These are the parallels at X_{547} (the midpoint of X_2, X_5) to those of the McCay cubic **K003**. The cubic meets the cirumcircle at the same points as $\mathrm{p}\mathcal{K}(X_6, X_{140})$ where X_{140} is the midpoint of X_3, X_5. See Figure 14.

The two cubics $\mathcal{K}(H) =$ **K028** and $\mathcal{K}(X_{140})$ have concurring asymptotes but their common point is not on the curve. These are $\mathcal{K}^+$ cubics.

On the contrary, $\mathcal{K}(X_3)$ is a central cubic and the asymptotes meet at O on the curve. It is said to be a $\mathcal{K}^{++}$ cubic. See Figure 15.

11. Isogonal transform of $\mathcal{K}(Q)$

Under isogonal conjugation with respect to ABC, $\mathcal{K}(Q)$ is transformed into another circum-cubic **g**$\mathcal{K}(Q)$ meeting $\mathcal{K}(Q)$ again at the four foci of $\mathcal{I}(Q)$ and at the two points E_1, E_2 intersections of the line G**ag**Q with the conic $ABCKQ$.

Thus, $\mathcal{K}(Q)$ and **g**$\mathcal{K}(Q)$ have nine known common points. When they are distinct i.e. when Q is not K i.e. when $\mathcal{K}(Q)$ is not the Thomson cubic, they generate a pencil of cubics which contains $\mathrm{p}\mathcal{K}(X_6,$ **cg**$Q)$.

It is easy to verify that **g**$\mathcal{K}(Q)$

(i) contains the circumcenter O, **g**Q, the midpoints of ABC,

(ii) is tangent at A, B, C to the cevian lines of the $X_{32}-$isoconjugate of Q i.e. the point **gtg**Q,

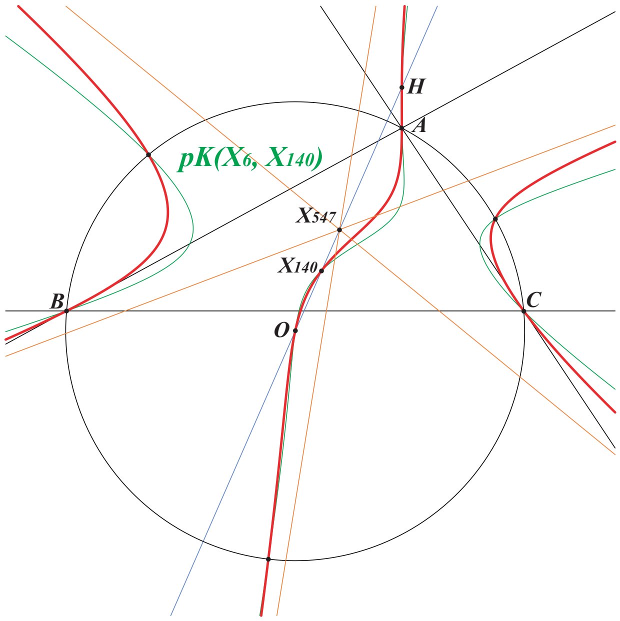


Figure 14. The cubic $\mathcal{K}^+_{60}$

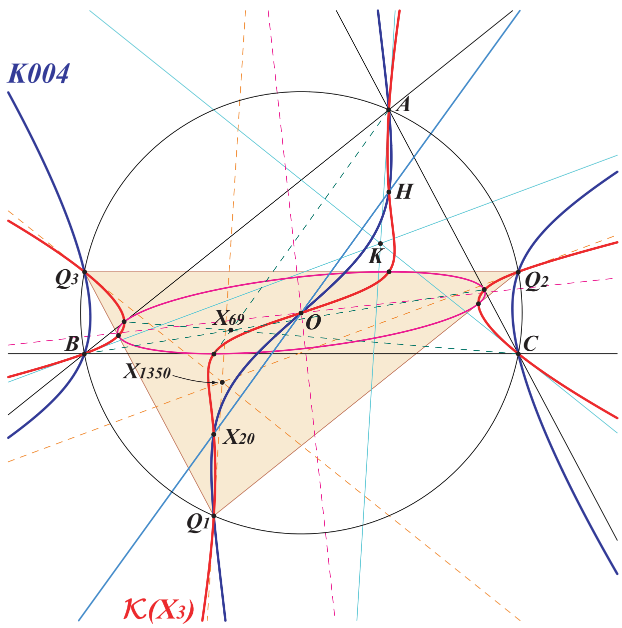


Figure 15. The cubic $\mathcal{K}(X_3)$

(iii) meets the circumcircle at the same points as $\mathrm{p}\mathcal{K}(X_6, \mathbf{g}Q)$ hence the orthocenter of the triangle $O_1O_2O_3$ formed by these points is $\mathbf{g}Q$; following a result of [4],

the inconic with perspector **tcg**Q is inscribed in ABC and $O_1O_2O_3$,
(iv) has the same asymptotic directions as $\mathrm{p}\mathcal{K}(X_6, \mathbf{ag}Q)$.

Except the case $Q = K$, **g**$\mathcal{K}(Q)$ cannot be a cubic of type $\mathcal{K}(Q)$.

The tangents to **g**$\mathcal{K}(Q)$ at A, B, C are still concurrent (at **gtg**Q) but in general, the tangents at the other intersections of **g**$\mathcal{K}(Q)$ with the circumcircle are not now concurrent unless Q lies on a circular circum-quartic which is the isogonal transform of **Q063**. This quartic contains X_1, X_3, X_6, X_{64}, X_{2574}, X_{2575}, the excenters.

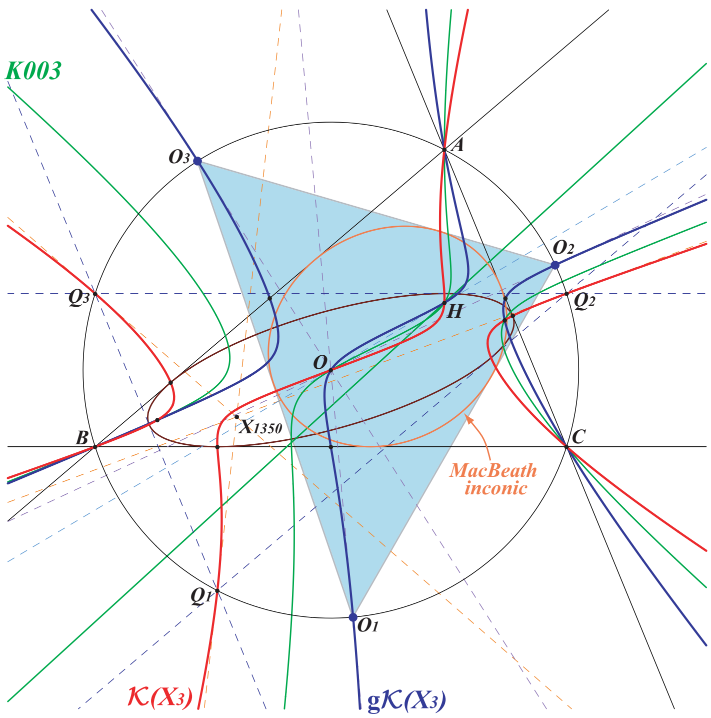


Figure 16. $\mathcal{K}(X_3)$, **g**$\mathcal{K}(X_3)$ and **K003**

Figure 16 presents $\mathcal{K}(X_3)$ and **g**$\mathcal{K}(X_3)$. These two cubics generate a pencil which contains the McCay cubic **K003** and the Euler isogonal focal cubic **K187**. The nine common points of these four cubics are A, B, C, O, H and the four foci of the inscribed conic with center O.

g$\mathcal{K}(X_3)$ meets the circumcircle at the same points O_1, O_2, O_3 as the Orthocubic **K006** and the triangles ABC, $O_1O_2O_3$ share the same orthocenter H therefore the same Euler line. The tangents at O_1, O_2, O_3 concur at O and those at A, B, C concur at X_{25}. The MacBeath inconic (with center X_5, foci O and H) is inscribed in ABC and $O_1O_2O_3$.

g$\mathcal{K}(X_3)$ meets the line at infinity at the same points as the Darboux cubic **K004**. Hence, its three asymptotes are parallel to the altitudes of ABC.

References

[1] J. P. Ehrmann and B. Gibert, *Special Isocubics in the Triangle Plane*, available at `http://perso.orange.fr/bernard.gibert/`

[2] B. Gibert, *Cubics in the Triangle Plane*, available at `http://perso.orange.fr/bernard.gibert/`

[3] B. Gibert, The Lemoine Cubic and its Generalizations, *Forum Geom.*, 2 (2002) 47–63.

[4] B. Gibert, How Pivotal Isocubics intersect the Circumcircle, *Forum Geom.*, 7 (2007) 211–229.

[5] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[6] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[7] P. Yiu, Hyacinthos message 16044, January 18, 2008.

Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France

E-mail address: `bg42@orange.fr`

Forum Geometricorum
Volume 8 (2008) 97–98.

FORUM GEOM
ISSN 1534-1178

A Short Proof of Lemoine's Theorem

Cosmin Pohoata

Abstract. We give a short proof of Lemoine's theorem that the Lemoine point of a triangle is the unique point which is the centroid of its own pedal triangle.

Lemoine's theorem states that the Lemoine (symmedian) point of a triangle is the unique point which is the centroid of its own pedal triangle. A proof of the fact that the Lemoine point has this property can be found in Honsberger [4, p.72]. The uniqueness part was conjectured by Clark Kimberling in the very first Hyacinthos message [6], and was subsequently confirmed by computations by Barry Wolk [7], Jean-Pierre Ehrmann [2], and Paul Yiu [8, §4.6.2]. Darij Grinberg [3] has given a synthetic proof. In this note we give a short proof by applying two elegant results on orthologic triangles.

Lemma 1. *If P is a point in plane of triangle ABC, with pedal triangle $A'B'C'$, then the perpendiculars from A to $B'C'$, from B to $C'A'$, from C to $A'B'$ are concurrent at Q, the isogonal conjugate of P.*

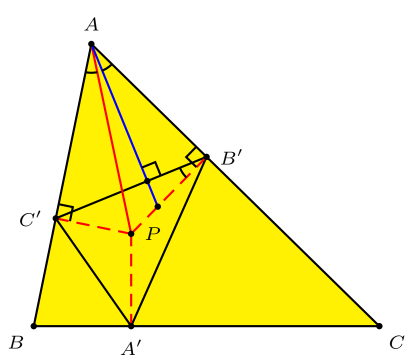


Figure 1

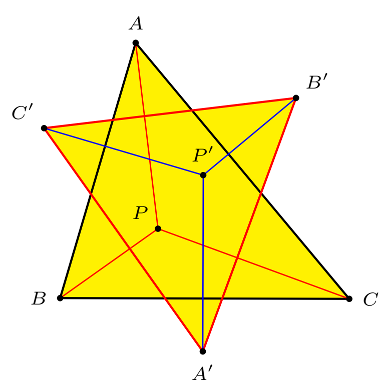


Figure 2

This is quite well-known. See, for example, [5, Theorem 237]. Figure 1 shows that AP and the perpendicular from A to $B'C'$ are isogonal with reference to A. From this Lemma 1 follows. The next beautiful result, illustrated in Figure 2, is the main subject of [1].

Publication Date: April 23, 2008. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this note.

Theorem 2 (Daneels and Dergiades)**.** *If triangles ABC and $A'B'C'$ are orthologic with centers P, P', with the perpendiculars from A, B, C to $B'C'$, $C'A'$, $A'B'$ intersecting at P and those from A', B', C' to BC, CA, AB intersecting at P', then the barycentric coordinates of P with reference to ABC are equal to the barycentric coordinates of P' with reference to $A'B'C'$.*

Now we prove Lemoine's theorem.

Let K be the Lemoine (symmedian) point of triangle ABC, and $A'B'C'$ its pedal triangle. According to Lemma 1, the perpendiculars from A to $B'C'$, from B to $C'A'$, from C to $A'B'$ are concurrent at the centroid G of ABC. Now since ABC and $A'B'C'$ are orthologic, with G as one of the orthology centers, by Theorem 2, the perpendiculars from A' to BC, from B' to CA, from C' to AB are concurrent at the centroid G' of $A'B'C'$. Hence, the symmedian point K coincides with the centroid of its pedal triangle.

Conversely, let P a point with pedal triangle $A'B'C'$, and suppose P is the centroid of $A'B'C'$; it has homogeneous barycentric coordinates $(1 : 1 : 1)$ with reference to $A'B'C'$. Since ABC and $A'B'C'$ are orthologic, by Theorem 2, we have that the perpendiculars from A to $B'C'$, from B to $C'A'$, from C to $A'B'$ are concurrent at a point Q with homogeneous barycentric coordinates $(1 : 1 : 1)$ with reference to ABC. This is the centroid G. By Lemma 1, this is also the isogonal conjugate of P. This shows that $P = K$, the Lemoine (symmedian) point.

This completes the proof of Lemoine's theorem.

References

[1] E. Danneels and N. Dergiades, A theorem on orthology centers, *Forum Geom.*, 4 (2004) 135–141.

[2] J.-P. Ehrmann, Hyacinthos message 95, January 8, 2000.

[3] D. Grinberg, New Proof of the Symmedian Point to be the centroid of its pedal triangle, and the Converse, available at `http://de.geocities.com/darij_grinberg`.

[4] R. Honsberger, *Episodes of 19th and 20th Century Euclidean Geometry*, Math. Assoc. America, 1995.

[5] R. A. Johnson, *Advanced Euclidean Geometry*, Dover reprint 2007.

[6] C. Kimberling, Hyacinthos message 1, December 22, 1999.

[7] B. Wolk, Hyacinthos message 19, December 27, 1999.

[8] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001.

Cosmin Pohoata: 13 Pridvorului Street, Bucharest, Romania 010014
E-mail address: `pohoata_cosmin2000@yahoo.com`

Forum Geometricorum
Volume 8 (2008) 99–101.

FORUM GEOM
ISSN 1534-1178

Means as Chords

Francisco Javier García Capitán

Abstract. On the circumcircle of a right triangle, we display chords whose lengths are the quadratic, arithmetic, geometric, and harmonic means of the two shorter sides.

Given two positive numbers a and b, the inequalities among their various means

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

are well known. In order, these are the harmonic, geometric, arithmetic, and quadratic means of a and b. Nelsen [1] has presented several few geometric proofs (without words). In the same spirit, we exhibit these various means as chords of a circle constructed from two segments of lengths a and b.

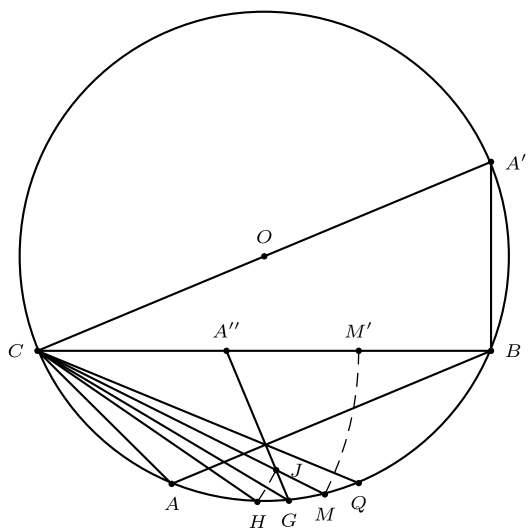


Figure 1

We shall assume $a \leq b$, and begin with a right triangle $A'BC$ with $A'B = a$, $CB = b$ and a right angle at B. Construct
(1) the circumcircle of the triangle (with center at the midpoint O of CA',

Publication Date: April 28, 2008. Communicating Editor: Paul Yiu.
The author is grateful to Professor Paul Yiu for his suggestions for the improvement of this paper, and dedicates it to the memory of our friend Juan Carlos Salazar.

(2) points A'' on the segment CB and and A the circle respectively such that $CA = CA'' = A'B$ and A, A' are on opposite sides of CB,
(3) the bisector AQ of angle ACB with Q on the circle (O),
(4) the midpoint M' of $A''B$ and the point M on the arc CAB with $CM = CM'$,
(5) the perpendicular from A'' to AB to intersect CM at J and the arc CAB at G,
(6) the point H on the arc CAB such that $CH = CJ$.

Proposition 1. *For the two segments CA and CB,*
(1) *CQ is the quadratic mean,*
(2) *CM is the arithmetic mean,*
(3) *CG is the geometric mean,*
(4) *CH is the harmonic mean.*

Proof. Note that the circle has radius $\frac{1}{2}\sqrt{a^2+b^2}$.

(1) Since $CA = A'B$, $CA'BA$ is an isosceles trapezoid, with AB parallel to CA'. Since CQ is the bisector of angle ACB, Q is the midpoint of the arc CAB, and OQ is perpendicular to AB. Hence, the radii OQ and OC are perpendicular to each other, and $CQ = \sqrt{2}\cdot OC = \frac{a^2+b^2}{2}$. This shows that CQ is the quadratic mean of a and b.

(2) $CM = CM' = \frac{1}{2}(a+b)$ is the arithmetic mean of a and b.

(3) Let $A''G$ intersect OA' at L. See Figure 2. Since CA' is parallel to AB, LG is perpendicular to CA'. From the similarity of the right triangles $CA''L$ and $CA'B$, we have $\frac{CL}{CA''} = \frac{CB}{CA'}$. In the right triangle $CA'G$, we have $CG^2 = CL\cdot CA' = CA''\cdot CB = ab$. This shows that CG is the geometric mean of a and b.

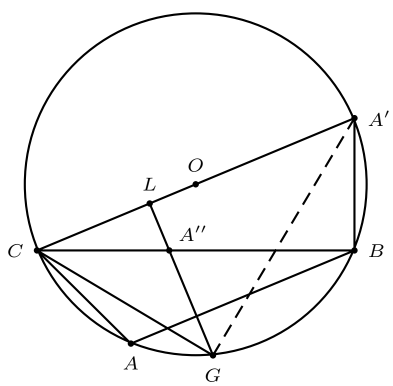


Figure 2

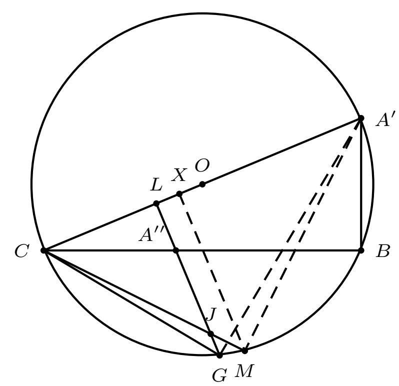


Figure 3

(4) Let the perpendicular from M to CA' intersect the latter at X. See Figure 3. From the similarity of triangles CLJ and CXM, we have

$$CJ = CM\cdot\frac{CL}{CX} = CM\cdot\frac{CL\cdot CA'}{CX\cdot CA'} = CM\cdot\frac{CG^2}{CM^2} = \frac{CG^2}{CM} = \frac{2ab}{a+b}.$$

This shows that $CH = CJ$ is the harmonic mean of a and b. $\square$

We conclude with an interesting concurrency.

Proposition 2. *The lines AB, CQ, and $A''G$ are concurrent.*

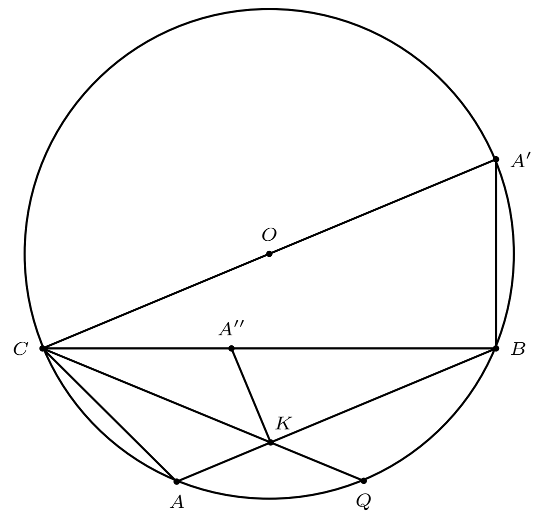


Figure 4

Proof. Let the bisector CQ of angle ACB intersect AB at K. See Figure 4. Clearly, the triangles ACK and $A''CK$ are congruent. Now,

$$\begin{aligned}\angle CA''K = \angle CAK = \angle CAB & \\ = 180^\circ - \angle CA'B & \qquad (C, A, B, A' \text{ concyclic}) \\ = \angle ABA' & \qquad (AB \text{ parallel to } CA') \\ = \angle ABC + 90^\circ. &\end{aligned}$$

It follows that $\angle A''KB = 90^\circ$, and $A''K$ is perpendicular to AB. This shows that K lies on $A''G$. □

Reference

[1] R. B. Nelsen, *Proofs Without Words*, MAA, 1994.

Francisco Javier García Capitán: Departamento de Matemáticas, I.E.S. Alvarez Cubero, Avda. Presidente Alcalá-Zamora, s/n, 14800 Priego de Córdoba, Córdoba, Spain

E-mail address: garciacapitan@gmail.com

Forum Geometricorum
Volume 8 (2008) 103–106.

FORUM GEOM
ISSN 1534-1178

A Condition for a Circumscriptible Quadrilateral to be Cyclic

Mowaffaq Hajja

Abstract. We give a short proof of a characterization, given by M. Radić et al, of convex quadrilaterals that admit both an incircle and a circumcircle.

A convex quadrilateral is said to be *cyclic* if it admits a circumcircle (*i.e.*, a circle that passes through the vertices); it is said to be *circumscriptible* if it admits an incircle (*i.e.*, a circle that touches the sides internally). A quadrilateral is *bicentric* if it is both cyclic and circumscriptible. For basic properties of these quadrilaterals, see [7, Chapter 10, pp. 146–170]. One of the two main theorems in [5], namely Theorem 1 (p. 35), can be stated as follows:

Theorem. *Let $ABCD$ be a circumscriptible quadrilateral with diagonals AC and BD of lengths u and v respectively. Let a, b, c, and d be the lengths of the tangents from the vertices A, B, C, and D* (see Figure 1). *The quadrilateral $ABCD$ is cyclic if and only if $\frac{u}{v} = \frac{a+c}{b+d}$.*

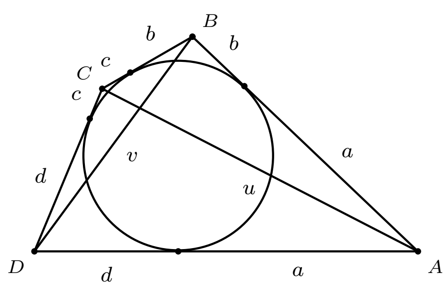


Figure 1

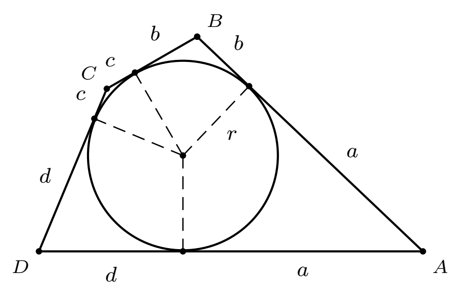


Figure 2

In this note, we give a proof that is much simpler than the one given in [5]. Our proof actually follows immediately from the three very simple lemmas below, all under the same hypothesis of the Theorem. Lemma 1 appeared as a problem in the MONTHLY [6] and Lemma 2 appeared in the solution of a quickie in the MAGAZINE [3], but we give proofs for the reader's convenience. Lemma 3 uses Lemma 2 and gives formulas for the lengths of the diagonals of a circumscriptible quadrilateral counterpart to those for cyclic quadrilaterals as given in [1], [7, § 10.2, p. 148], and other standard textbooks.

Publication Date: May 1, 2008. Communicating Editor: Paul Yiu.

The author would like to thank Yarmouk University for supporting this work and Mr. Esam Darabseh for drawing the figures.

Lemma 1. *$ABCD$ is cyclic if and only if $ac = bd$.*

Proof. Let $ABCD$ be any convex quadrilateral, not necessarily admitting an incircle, and let its vertex angles be $2A$, $2B$, $2C$, and $2D$. Then A, B, C, and D are acute, and $A + B + C + D = 180°$. We shall show that

$$ABCD \text{ is cyclic} \quad \Leftrightarrow \quad \tan A \tan C = \tan B \tan D. \tag{1}$$

If $ABCD$ is cyclic, then $A+C = B+D = 90°$, and $\tan A \tan C = \tan B \tan D$, each being equal to 1. Conversely, if $ABCD$ is not cyclic, then one may assume that $A + C > 90°$ and $B + D < 90°$. From

$$0 > \tan(A + C) = \frac{\tan A + \tan C}{1 - \tan A \tan C}$$

and the fact that A and C are acute, we conclude that $\tan A \tan C > 1$. Similarly $\tan B \tan D < 1$, and therefore $\tan A \tan C \neq \tan B \tan D$. This proves (1).

The result follows by applying (1) to the given quadrilateral, and using $\tan A = r/a$, etc., where r is the radius of the incircle (as shown in Figure 2). □

Lemma 2. *The radius r of the incircle is given by*

$$r^2 = \frac{bcd + acd + abd + abc}{a + b + c + d}. \tag{2}$$

Proof. Again, let the vertex angles of $ABCD$ be $2A$, $2B$, $2C$, and $2D$, and let

$$\alpha = \tan A,\ \beta = \tan B,\ \gamma = \tan C,\ \delta = \tan D.$$

Let $\varepsilon_1 = \sum \alpha$, $\varepsilon_2 = \sum \alpha\beta$, $\varepsilon_3 = \sum \alpha\beta\gamma$, and $\varepsilon_4 = \alpha\beta\gamma\delta$ be the elementary symmetric polynomials in α, β, γ, and δ. By [4, § 125, p. 132], we have

$$\tan(A + B + C + D) = \frac{\varepsilon_1 - \varepsilon_3}{1 - \varepsilon_2 + \varepsilon_4}.$$

Since $A+B+C+D = 180°$, it follows that $\tan(A+B+C+D) = 0$ and hence $\varepsilon_1 = \varepsilon_3$, *i.e.*,

$$\frac{r}{a} + \frac{r}{b} + \frac{r}{c} + \frac{r}{d} = \frac{r^3}{bcd} + \frac{r^3}{acd} + \frac{r^3}{abd} + \frac{r^3}{abc},$$

and (2) follows. □

Lemma 3.

$$u^2 = \frac{a + c}{b + d}\left((a + c)(b + d) + 4bd\right), \quad \textit{and} \quad v^2 = \frac{b + d}{a + c}\left((a + c)(b + d) + 4ac\right).$$

Proof. Again, let the vertex angles of $ABCD$ be $2A$, $2B$, $2C$, and $2D$. Then

$$\begin{aligned}
\cos 2A &= \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{a^2 - r^2}{a^2 + r^2} \\
&= \frac{a^2(a + b + c + d) - (bcd + acd + abd + abc)}{a^2(a + b + c + d) + (bcd + acd + abd + abc)}, \text{ by (2)} \\
&= \frac{a^2(a + b + c + d) - (bcd + acd + abd + abc)}{(a + b)(a + c)(a + d))}.
\end{aligned}$$

Therefore

$$\begin{aligned} v^2 &= (a+b)^2 + (a+d)^2 - 2(a+b)(a+d)\cos 2A \\ &= (a+b)^2 + (a+d)^2 - 2\frac{a^2(a+b+c+d) - (bcd+acd+abd+abc)}{a+c} \\ &= \frac{b+d}{c+a}\left((a+c)(b+d) + 4ac\right). \end{aligned}$$

A similar formula holds for u. □

Proof of the main theorem. Using Lemmas 1 and 3 we see that

$$\begin{aligned} ABCD \text{ is cyclic} &\iff ac = bd, \text{ by Lemma 1} \\ &\iff (a+c)(b+d) + 4bd = (a+c)(b+d) + 4ac \\ &\iff \frac{u^2}{v^2} = \left(\frac{c+a}{b+d}\right)^2, \text{ by Lemma 3} \\ &\iff \frac{u}{v} = \frac{c+a}{b+d}, \end{aligned}$$

as desired. This completes the proof of the main theorem.

Remarks. (1) As mentioned earlier, Theorem 1 is one of the two main theorems in [5]. The other theorem is similar and deals with those quadrilaterals that admit an *excircle*. Note that the terms *chordal* and *tangential* are used in that paper to describe what we referred to as *cyclic* and *circumscriptible* quadrilaterals.

(2) Let $A_1 \dots A_n$ be circumscriptible n-gon and let $B_1, \dots, B_n$ be the points where the incircle touches the sides $A_1A_2, \dots, A_nA_1$. Let $|A_iB_i| = a_i$ for $i = 1, \dots, n$. Theorem 2 states that if $n = 4$, then the polygon is cyclic if and only if $a_1a_3 = a_2a_4$. One wonders whether a similar criterion holds for $n > 4$.

(3) It is proved in [2] that if $a_1, \dots, a_n$ are any positive numbers, then there exists a unique circumscriptible n-gon $A_1 \dots A_n$ such that the points $B_1, \dots, B_n$ where the incircle touches the sides $A_1A_2, \dots, A_nA_1$ have the property $|A_iB_i| = a_i$ for $i = 1, \dots, n$. Thus one can, in principle, express all the elements of the circumscriptible polygon in terms of the parameters $a_1, \dots, a_n$. Instances of this, when $n = 4$, are found in Lemms 2 and 3 where the inradius r and the lengths of the diagonals are so expressed. When $n > 4$, one can prove that r^2 is the unique positive zero of the polynomial

$$\sigma_{n-1} - r^2\sigma_{n-3} + r^4\sigma_{n-5} - \dots \dots = 0,$$

where $\sigma_1, \dots, \sigma_n$ are the elementary symmetric polynomials in $a_1, \dots, a_n$, and where $a_1, \dots, a_n$ are as given in Remark 2. This is obtained in the same way we obtained (2) using the the formula

$$\tan(A_1 + \dots + A_n) = \frac{\varepsilon_1 - \varepsilon_3 + \varepsilon_5 - \dots}{1 - \varepsilon_2 + \varepsilon_4 - \dots},$$

where $\varepsilon_1, \dots, \varepsilon_n$ are the elementary symmetric polynomials in $\tan A_1, \dots, \tan A_n$, and where $A_1, \dots, A_n$ are half the vertex angles of the polygon.

References

[1] C. Alsina and R. B. Nelson, On the diagonals of a cyclic quadrilateral, *Forum Geom.*, 7 (2007) 147–149.

[2] D. E. Gurarie and R. Holzsager, Problem 10303, *Amer. Math. Monthy*, 100 (1993) 401; solution, *ibid.*, 101 (1994) 1019–1020.

[3] J. P. Hoyt, Quickie Q 694, *Math. Mag.*, 57 (1984) 239; solution, *ibid.*, 57 (1984) 242.

[4] S. L. Loney, Plane Trigonometry, S. Chand & Company Ltd, New Delhi, 1996.

[5] M. Radić, Z. Kaliman, and V. Kadum, A condition that a tangential quadrilateral is also a chordal one, *Math. Commun.*, 12 (2007) 33–52.

[6] A. Sinefakupoulos, Problem 10804, *Amer. Math. Monthy*, 107 (2000) 462; solution, *ibid.*, 108 (2001) 378.

[7] P. Yiu, *Euclidean Geometry*, Florida Atlantic Univesity Lecture Notes, 1998, available at `http://www.math.fau.edu/Yiu/Geometry.html`.

Mowaffaq Hajja: Mathematics Department, Yarmouk University, Irbid, Jordan
E-mail address: `mhajja@yu.edu.jo, mowhajja@yahoo.com`

Forum Geometricorum
Volume 8 (2008) 107–120.

FORUM GEOM
ISSN 1534-1178

Periodic Billiard Trajectories in Polyhedra

Nicolas Bedaride

Abstract. We consider the billiard map inside a polyhedron. We give a condition for the stability of the periodic trajectories. We apply this result to the case of the tetrahedron. We deduce the existence of an open set of tetrahedra which have a periodic orbit of length four (generalization of Fagnano's orbit for triangles), moreover we can study completely the orbit of points along this coding.

1. Introduction

We consider the billiard problem inside polyhedron. We start with a point of the boundary of the polyhedron and we move along a straight line until we reach the boundary, where there is reflection according to the mirror law. A famous example of a periodic trajectory is Fagnano's orbit: we consider an acute triangle and the foot points of the altitudes. Those points form a billiard trajectory which is periodic [1].

For the polygons some results are known. For example we know that there exists a periodic orbit in all rational polygons (the angles are rational multiples of π), and recently Schwartz has proved in [8] the existence of a periodic billiard orbit in every obtuse triangle with angle less than 100 degrees . A good survey of what is known about periodic orbits can be found in the article [4] by Gal'perin, Stëpin and Vorobets or in the book of Masur, Tabachnikov [6]. In this article they define the notion of stability: They consider the trajectories which remain periodic if we perturb the polygon. They find a combinatorial rule which characterize the stable periodic words. Moreover they find some results about periodic orbits in obtuse triangles.

The study of the periodic orbits has also been done by famous physicists. Indeed Glashow and Mittag prove that the billiard inside a triangle is equivalent to the system of three balls on a ring, [5]. Some others results can be found in the article of Ruijgrok and Rabouw [7]. In the polyhedral case much less is known. The result on the existence of periodic orbit in a rational polygon can be generalized, but it is less important, because the rational polyhedra are not dense in the set of polyhedra. There is no other general result, the only result concerns the example of the tetrahedron. Stenman [10] shows that a periodic word of length four exists in a regular tetrahedron.

Publication Date: May 19, 2008. Communicating Editor: Paul Yiu.

The aim of this paper is to find Fagnano's orbit in a regular tetrahedron and to obtain a rule for the stability of periodic words in polyhedra. This allows us to obtain a periodic orbit in each tetrahedron in a neighborhood of the regular one. Moreover we give examples which prove that the trajectory is not periodic in all tetrahedra, and we find bounds for the size of the neighborhood. In the last section we answer a question of Gal'perin, Krüger, Troubetzkoy [3] by an example of periodic word v with non periodic points inside its beam.

2. Statement of results

The definitions are given in the following sections as appropriate. In Section 4 we prove the following result. Consider a periodic billiard orbit coded by the word v. In §4.2, we derive a certain isometry S_v from the combinatorics of the path.

Theorem 1. *Let P be a polyhedron and v the prefix of a periodic word of period $|v|$ in P. If the period is an even number, and S_v is different from the identity, then v is stable. If the period is odd, then the word is stable if and only if S_v is constant as a function of P.*

In Section 5 we prove

Theorem 2. *Assume the billiard map inside the tetrahedron is coded by a, b, c, d.*

(1) *The word $abcd$ is periodic for all the tetrahedra in a neighborhood of the regular one.* (This orbit will be referred to as Fagnano's orbit).

(2) *In any right tetrahedron Fagnano's orbit does not exist. There exists an open set of obtuse tetrahedron where Fagnano's orbit does not exist.*

The last section of this article is devoted to the study of the first return map of the billiard trajectory.

3. Background

3.1. *Isometries.* We recall some usual facts about affine isometries of $\mathbb{R}^3$. A general reference is [1].

To an affine isometry a, we can associate an affine map f and a vector u such that: f has a fixed point or is equal to the identity, and such that $a = t_u \circ f = f \circ t_u$ where t_u is the translation of vector u. Then f can be seen as an element of the orthogonal group $O_3(\mathbb{R})$.

Definition. First assume that f belongs to $O_3(+)$, and is not equal to the identity. If u is not an eigenvector of f, then a is called an affine rotation. The axis of a is the set of invariants points. If u is an eigenvector of f, a is called a screw motion. In this case the axis of a is the axis of the affine rotation.

If f, in $O_2(-)$ or $O_3(-)$, is a reflection and u is an eigenvector of f with eigenvalue 1, then a is called a glide reflection.

We recall Rodrigue's formula which gives the axis and the angle of the rotation product of two rotations. It can be done by the following method.

Lemma 3 ([2]). *We assume that the two rotations are not equal to the identity, or to a rotation of angle π. Let θ and u be the angle and axis of the first rotation, and denote by t the vector $\tan\frac{\theta}{2}\cdot u$ and t' the associated vector for the second rotation. Then the product of the two rotations is given by the vector t'' such that*

$$t'' = \frac{1}{1 - t\cdot t'}(t + t' + t \wedge t').$$

3.2. *Combinatorics.* Let $\mathcal{A}$ be a finite set called the alphabet. By a language L over $\mathcal{A}$ we mean always a factorial extendable language. A language is a collection of sets $(L_n)_{n\geq 0}$ where the only element of L_0 is the empty word, and each L_n consists of words of the form $a_1a_2\dots a_n$ where $a_i \in \mathcal{A}$ such that
(i) for each $v \in L_n$ there exist $a, b \in \mathcal{A}$ with $av, vb \in L_{n+1}$, and
(ii) for all $v \in L_{n+1}$, if $v = au = u'b$ with $a, b \in \mathcal{A}$, then $u, u' \in L_n$.

If $v = a_1a_2\dots a_n$ is a word, then for all $i \leq n$, the word $a_1\dots a_i$ is called a prefix of v.

4. Polyhedral billiard

4.1. *Definition.* We consider the billiard map T inside a polyhedron P. Let $X \subset \partial P \times \mathbb{PR}^3$ consist of (m, θ) for which $m + \mathbb{R}^*\theta$ does not intersect ∂P on an edge. The map T is defined by the rule

$$T(m, \theta) = (m', \theta')$$

if and only if mm' is collinear with θ, where $\theta' = S\theta$ and S is the linear reflection over the face which contains m'.

We identify $\mathbb{PR}^3$ with the unit vectors of $\mathbb{R}^3$ in the preceding definition.

4.2. *Coding.* We code the trajectory by the letters from a finite alphabet where we associate a letter to each face.

We call s_i the reflection in the face i, S_i the linear reflection in this face. If we start with a point of direction θ which has a trajectory of coding $v = v_0\cdots v_{n-1}$ the image of θ is: $S_{v_{n-1}}\dots S_{v_1}\theta$. Indeed the trajectory of the point first meets the face v_1, then the face v_2 etc.

If it is a periodic orbit, it meets the face v_0 after the face v_{n-1} and we have: $S_{v_0}S_{v_{n-1}}\dots S_{v_1}\theta = \theta = S_v\theta$, S_v is the product of the S_i, and s_v the product of the s_i.

We recall a result of [3]: the word v is the prefix of a periodic word of period $|v|$ if and only if there exists a point whose orbit is periodic and has v as coding.

Remark. If a point is periodic, the initial direction is an eigenvector of the map S_v with eigenvalue 1. It implies that in $\mathbb{R}^3$, for a periodic word of odd period, S is a reflection.

Definition. Let v be a finite word. The beam associated to v is the set of (m, θ) where m is in the face v_o (respectively edge), θ a vector of $\mathbb{R}^3$ (respectively $\mathbb{R}^2$), such that the orbit of (m, θ) has a coding which begins with v. We denote it σ_v.

A vector u of $\mathbb{R}^3$ (respectively $\mathbb{R}^2$) is admissible for v, with base point m, if there exists a point m in the face (edge) v_0 such that (m, u) belongs to the beam of v.

Lemma 4. *Let s be an isometry of $\mathbb{R}^3$ not equal to a translation. Let S be the associated linear map and u the vector of translation. Assume s is either a screw motion or a glide reflection. Then the points n which satisfy $\overrightarrow{ns(n)} \in \mathbb{R}u$, are either on the axis of s (if S is a rotation), or on the plane of reflection. In this case the vector $\overrightarrow{ns(n)}$ is the vector of the glide reflection.*

Proof. We call θ the eigenspace of S related to the eigenvalue one. We have $s(n) = s(o) + S\overrightarrow{on}$ where o, the origin of the base will be chosen later. Elementary geometry yields $\overrightarrow{ns(n)} = (S - Id)X + Y$ (where $X = \overrightarrow{on}$, $Y = \overrightarrow{os(o)}$) is inside the space θ.

The map s has no fixed point by assumption, thus $\overrightarrow{ns(n)}$ is nonzero. The condition gives that $(S - Id)X + Y$ is an eigenvector of S associated to the eigenvalue one. Thus,

$$\begin{aligned} S((S - Id)X + Y) &= (S - Id)X + Y, \\ (S - Id)^2 X &= -(S - I)Y. \end{aligned} \tag{1}$$

We consider first the case $\det S > 0$. We choose o on the axis of s. Then θ is a line, we call the direction of the line by the same name. Since $\det S > 0$ we have $S \in O_3(+)$ and thus in an appropriate basis S has the form $\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$, where R is a matrix of rotation of $\mathbb{R}^2$. The equation (1) is equivalent to

$$(R - Id)^2 X' = -(R - Id)Y',$$

where X' is the vector of $\mathbb{R}^2$ such that $X = \begin{pmatrix} X' \\ x \end{pmatrix}$ in this basis. Furthermore, since S is a screw motion with axis $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in these coordinates, Y has the following coordinates $\begin{pmatrix} Y' \\ y \end{pmatrix}$ where $Y' = 0$. Since $S \neq Id$, $R - Id$ is invertible and thus $X' = 0$. Thus the vectors X solutions of this equation are collinear with the axis.

Consider now the case $detS < 0$. By assumption S is a reflection, it implies that the eigenspace related to one is a plane. We will solve (1), keeping the notation $X = \begin{pmatrix} X' \\ x \end{pmatrix}$ and $Y = \begin{pmatrix} Y' \\ y \end{pmatrix}$.

We may assume that o is on the plane of reflection. Moreover we can choose the coordinates such that that this plane is orthogonal to the line $\mathbb{R}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. It implies that $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, and $y = 0$. The equation (1) becomes $4x = 0$. It implies that X is on the plane of reflection. Since s is a glide reflection, the last point becomes obvious. $\square$

Proposition 5. *Let P a polyhedron, the following properties are equivalent.*
(1) *A word v is the prefix of a periodic word with period $|v|$.*
(2) *There exists $m \in v_0$ such that $\overrightarrow{s_v(m)m}$ is admissible with base point m for vv_0, and $\theta = \overrightarrow{s_v(m)m}$ is such that $S\theta = \theta$.*

Remark. Assume $|v|$ is even. In the polygonal case the matrix S_v can only be the identity, thus s_v is a translation. We see by unfolding that s_v can not have a fixed point, thus in the polyhedral case s_v is either a translation or a screw motion or a glide reflection. If we do not assume the admissibility in condition (2) it is not equivalent to condition (1) as can be seen in a obtuse triangle, or a right prism above the obtuse triangle and the word abc.

Proof of Proposition 5. First we claim the following fact. The vector connecting $T^{|v|}(m, \theta)$ to $s_v(m)$ is parallel to the direction of $T^{|v|}(m, \theta)$. For $|v| = 1$ if the billiard trajectory goes from (m, θ) to (m', θ') without reflection between, then the direction θ' is parallel to $\overrightarrow{s(m)m'}$, where s is the reflection over the face of m' (see Figure 1). Thus the claim follows combining this observation with an induction argument.

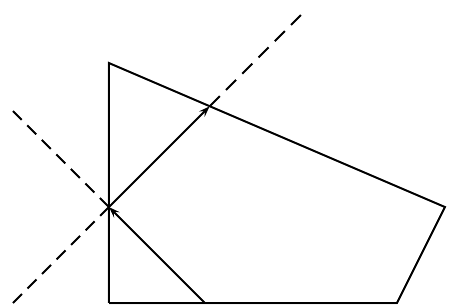


Figure 1. Billiard orbit and the associated map

Next assume (1). Then there exists (m, θ) periodic. We deduce that $S\theta = \theta$, moreover this direction is admissible. Then the claim implies that $\overrightarrow{s_v(m)m} = \theta$ and thus is admissible for vv_0.
Finally assume (2). First we consider the case where $S \neq Id$. Lemma 4 implies that m is on the axis of s if $|v|$ is even, otherwise on the plane of reflection. If $|v|$ is even then $\theta = \overrightarrow{s(m)m}$ is collinear to the axis of the screw motion. Since we have assumed $\overrightarrow{s_v(m)m}$ admissible we deduce that θ is admissible with base point m. If $|v|$ is odd then Lemma 4 implies that θ is the direction of the glide. The hypothesis implies that θ is admissible for v.

Now we prove that (m, θ) is a periodic trajectory. We consider the image $T^{|v|}(m, \theta)$. We denote this point (p, θ'). We have by hypothesis that p is in v_0. The above claim implies that $\overrightarrow{s_v(m)p}$ is parallel to the direction θ'. The equation $S\theta = \theta$ gives $\theta' = \theta$. Thus we have $\overrightarrow{s_v(m)m}$ is parallel to $\overrightarrow{s_v(m)p}$, since we do not consider direction included in a face of a polyhedron this implies $p = m$. Thus (m, θ) is a periodic point.

If $S = Id$, then s is a translation of vector $\overrightarrow{s_v(m)m} = u$. The vector u is admissible. Then we consider a point m on the face v_0 which is admissible. Then we show that (m, u) is a periodic point by the same argument related to the claim. □

Thus we have a new proof of the following result of [3].

Theorem 6. *Let v be a periodic word of even length. The set of periodic points in the face v_0 with code v and length $|v|$ can have two shapes. Either it is an open set or it is a point.*

If v is a periodic word of odd length, then the set of periodic points in the face v_0 with code v and period $|v|$ is a segment.

Proof. Let Π be a face of the polyhedron, and let $m \in \Pi$ be the starting point for a periodic billiard path. The first return map to m is an isometry of $\mathbb{R}^3$ that fixes both m and the direction u of the periodic billiard path.

Assume first $|v|$ is odd. Then the first return map is a reflection since it fixes a point. Then it fixes a plane Π'. Note that $u \in \Pi'$, and that the intersection $\Pi \cap \Pi'$ is a segment. Points in this segment sufficiently near m have a periodic orbit just as the one starting at v.

Assume now $|v|$ is even, we will use Proposition 5. If S_v is the identity, then the periodic points are the points such that the coding of the billiard orbit in the direction of the translation begins with v, otherwise there is a single point, at the intersection of the axis of s and v_0. However the set of points with code v is still an open set. □

Note that our proof gives an algorithm to locate this set in the face. We will use it in Section 6.

5. Stability

First of all we define the topology on the set of polyhedra with k vertices. As in the polygonal case we identify this set with $\mathbb{R}^{3(k-2)}$. But we remark the following fact. Consider a polyhedron P such that a face of P is not a triangle. Then we can find a perturbation of P, as small as we want, such that the new polyhedron has a different combinatorial type (*i.e.*, the numbers of vertices, edges and faces are different). In this case consider a triangulation of each face which does not add new vertices. Consider the set of all such triangulations of all faces. There are finitely many such triangulations. Each can be considered as a combinatorial type of the given polyhedron. Let $B(P, \varepsilon)$ be the ball of radius ε in $\mathbb{R}^{3(k-2)}$ of polyhedra Q. If P has a single combinatorial type, ε is chosen so small that all Q in the ball have the same combinatorial type. If P has several combinatorial types, then ε is taken so small that all Q have one of those combinatorial type. The definition of stability is now analogous to the definition in polygons. On the other hand, let v be a periodic word in P and g a piecewise similarity. Consider the polyhedron $g(P)$, and the same coding as in P. If v exists in $g(P)$ it is always a periodic word in $g(P)$. We note that the notion of periodicity only depends on the normal vectors to the planes of the faces.

Theorem 7. *Let P be a polyhedron and v the prefix of a periodic word of period $|v|$ in P.*
(1) *If the period is even, and S_v is different from the identity, then v is stable.*
(2) *If the period is odd, then the word is stable if and only if S_v is constant as a function of P.*

Remark. The second point has no equivalence in dimension two, since each element of $O(2,-)$ is a reflection. It is not the case for $O(3,-)$.

Proof of Theorem 7. First consider the case of period even. The matrix $S = S_v$ is not the identity, and $\theta = \theta_v$ is the eigenvector associated to the eigenvalue one. First note that by continuity v persists for sufficiently small perturbations of the polyhedron. Fix a perturbation and let $B = S_v^Q$ be the resulting rotation for the new polyhedron Q. We will prove that the eigenvalue of S is a continuous function of P. We take the reflections which appears in v two by two. The product of two of those reflections is a rotation. We only consider the rotations different of the identity. The axes of the rotations are continuous map as function of P since they are at the intersection of two faces. Then Rodrigue's formula implies that the axes of the rotation, product of two of those rotations, are continuous maps of the polyhedron, under the assumption that the rotation is not the identity (because t must be of non-zero norm). Since S^P is not equal to Id, there exists a neighborhood of P where $S^Q \neq Id$. It implies that the axis of S^P is a continuous function of P. Thus the two eigenvectors of B, S are near if B is sufficiently close to S. The direction θ was admissible for v, we know that the beam of v is an open set of the phase space [3], so we have for Q sufficiently close to P that α (the real eigenvector of B) is admissible for the same word. Moreover the foot points are not far from the initial points because they are on the axis of the isometries. Thus the perturbated word is periodic by Proposition 5.

If the length of v is odd, then Remark 4.2 implies that S is a reflection. We have two cases: either S_v is constant, or not. If it is not a constant function, then in any neighborhood there exists a polyhedron Q such that S_v^Q is different from a reflection. Then the periodic trajectory can not exist in Q. If S_v is constant, then it is always a reflection, and a similar argument to the even case shows that the plane of reflection of S is a continuous map of P. It completes the proof of Theorem 7. $\square$

Corollary 8. (1) *All the words of odd length are stable in a polygon.*
(2) *Consider a periodic billiard path in a right prism. Then its projection inside the polygonal basis is a billiard path. We denote the coding of the projected trajectory*

as the projected word. Assume that the projected word is not stable in the polygonal basis. Then the word is unstable.
(3) *All the words in the cube are unstable.*

Proof. (1) was already mentioned in [4]. The proof is the same as that of Theorem 7. Indeed is $|v|$ is odd then s has a real eigenvector, and we can apply the proof.

For (2) we begin with the period two trajectory which hits the top and the bottom of the prism. It is clearly unstable, for example we can change one face and keep the other. Let v be any other periodic word, and w the word corresponding to the projection of v to the base of the prism assumed to be unstable. We perturb a vertical face of the prism such that this face contains an edge which appears in the coding of w. The word v can not be periodic in this polyhedron by unstability of w.

For (3), let v be a periodic word, by preceding point its projection on each coordinate plane must be stable. But an easy computation shows that no word is stable in the square. □

We remark that the two and three dimensional cases are different for the periodic trajectories of odd length. They are all stable in one case, and all unstable in the second. Recently Vorobets has shown that if $S_v = Id$ then the word is not stable [11].

6. Tetrahedron

In the following two Sections we prove the following result.

Theorem 9. *Assume the billiard map inside the tetrahedron is coded by a, b, c, d.*
(1) *The word $abcd$ is periodic for all the tetrahedra in a neighborhood of the regular one.*
(2) *In any right tetrahedron Fagnano's orbit does not exist. There exists an open set of obtuse tetrahedron where Fagnano's orbit does not exist.*

Remark. Steinhaus in his book [9], cites Conway for a proof that $abcd$ is periodic in all tetrahedra, but our theorem gives a counter example. Moreover our proof gives an **algorithm** which find the coordinates of the periodic point, when it exists.

For the definition of obtuse tetrahedron, see Section 7.

We consider a regular tetrahedron. We can construct a periodic trajectory of length four, which is the generalization of Fagnano's orbit. To do this we introduce the appropriate coding (see Figure 2 in which the letter a is opposite to the vertex A, etc).

Lemma 10. *Let $ABCD$ be a regular tetrahedron, with the natural coding. If v is the word $adcb$, there exists a direction θ, there exists an unique point m such that (m, θ) is periodic and has v as prefix of its coding. Moreover m is on the altitude of the triangle BCD which starts at C.*

Remark. If we consider the word v^n, the preceding point m is the unique periodic point for v^n. Indeed the map s_{v^n} has the same axis as s_v, and we use Proposition 5.

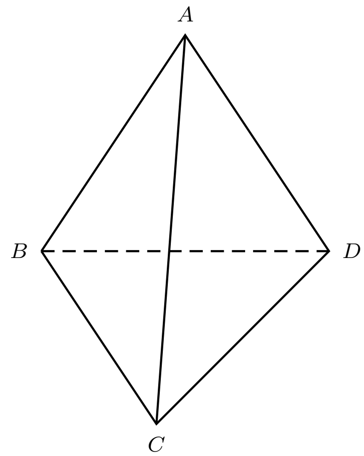


Figure 2. Coding of a tetrahedron

We use the following coordinates for two reasons. First these coordinates were used by Ruijgrok and Rabouw [7]. Secondly with these coordinates the matrix S_v has rational entries, and the computations seems more simples.

Proof. The lemma has already been proved in [10], but we rewrite it in a different form with the help of Proposition 5.

We have $S_v = S_a.S_b.S_c.S_d = R_{DC}.R_{AB}$ where R_{DC} is the linear rotation of axis DC, it is a product of the two reflections. We compute the real eigenvector of S_v, and we obtain the point m at the intersection of the axis of s and the face BCD. We consider an orthonormal base of $\mathbb{R}^3$ such that the points have the following coordinates (see [10]):

$$A = \frac{\sqrt{2}}{4}\begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \quad D = \frac{\sqrt{2}}{4}\begin{pmatrix}1\\-1\\1\end{pmatrix}, \quad C = \frac{\sqrt{2}}{4}\begin{pmatrix}1\\1\\-1\end{pmatrix}, \quad B = \frac{\sqrt{2}}{4}\begin{pmatrix}-1\\1\\1\end{pmatrix}.$$

The matrices of S_a, S_d, S_c, S_b are:

$$\frac{1}{3}\begin{pmatrix}1&-2&-2\\-2&1&-2\\-2&-2&1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1&2&2\\2&1&-2\\2&-2&1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1&-2&2\\-2&1&2\\2&2&1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1&2&-2\\2&1&2\\-2&2&1\end{pmatrix}.$$

From these we obtain

$$S = S_aS_bS_cS_d = \frac{1}{81}\begin{pmatrix}-79&-8&16\\8&49&64\\-16&64&-47\end{pmatrix}.$$

This has a real eigenvector $u = \frac{1}{\sqrt{5}}\begin{pmatrix}0\\2\\1\end{pmatrix}$. Now we compute the vector N such that $s(X) = SX + N$. To do this we use the relation $s(A) = s_a(A)$. s_a is the product of S_a and a translation of vector v. We obtain

$$v = \frac{\sqrt{2}}{6}\begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad s(A) = \frac{5\sqrt{2}}{12}\begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad N = \frac{\sqrt{2}}{81}\begin{pmatrix}16\\64\\34\end{pmatrix}.$$

We see that s is a screw motion. Finally we find the point at the intersection of the axis and the face a. The points of the axis verify the equation

$$SX + N = X + \lambda u.$$

where X are the coordinates of the point of the axis, and λ is a real number. The point m is on the face a if we have the dot product

$$\overrightarrow{Cm} \cdot (\overrightarrow{CB} \wedge \overrightarrow{CD}) = 0.$$

So X is the root of the system made by those two equations. The last equation gives $x + y + z = \frac{\sqrt{2}}{4}$. We obtain

$$m = \frac{\sqrt{2}}{20}\begin{pmatrix}2\\2\\1\end{pmatrix}.$$

We remark that $\overrightarrow{Cm} \cdot \overrightarrow{DB} = 0$ which proves that m is on the altitude of the triangle BCD. □

In fact there are six periodic trajectories of length four, one for each of the word

$$abcd,\ abdc,\ acbd,\ acdb,\ adbc,\ adcb.$$

The six orbits come in pairs which are related by the natural involution of direction reversal. Now we can ask the same question in a non regular tetrahedron. Applying Theorem 7 yield the first part of Theorem 9.

Now the natural question is to characterize the tetrahedron which contains this periodic word.

7. Stability for the tetrahedron

A tetrahedron is acute if and only if in each face the orthogonal projection of the other vertex is inside the triangle. It is a right tetrahedron if and only if there exists a vertex, where the three triangles are right triangles.

Proof of second part of Theorem 9. We consider a tetrahedron $ABCD$ with vertices

$$A = (0,0,0) \quad B = (a,0,0) \quad C = (0,b,0) \quad D = (0,0,1).$$

We study the word $v = abcd$. We have $S = S_a * S_d * S_c * S_b$. Since

$$S_b = \begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad S_c = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad S_d = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix},$$

we obtain $S = -S_a$. Thus S has 1 for eigenvalue, and the associated eigenvector is the normal vector to the plane a. We remark that $s(A) = s_a(A)$. The fact that

$S = -S_a$ implies that S is a rotation of angle π, thus s is the product of a rotation of angle π and a translation.

Consider the plane which contains A and orthogonal to the axis of S, let O the point of intersection. Then S is a rotation of angle π, thus O is the middle of $[AE]$, where E is given by $S(\overrightarrow{OE}) = \overrightarrow{OA}$. It implies that the middle M of the edge $[As(A)]$ is on the axis of s, see Figure 3.

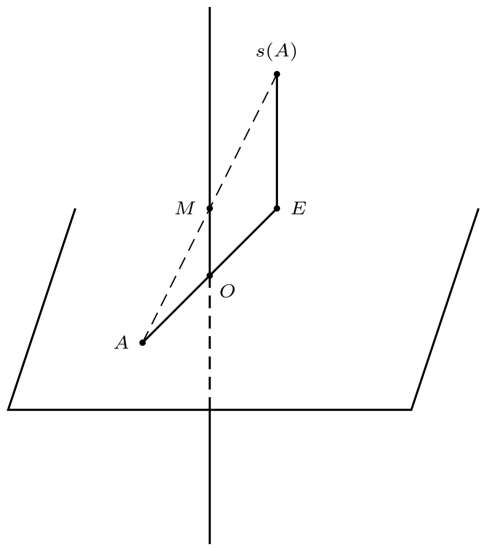


Figure 3. Screw motion associated to the word abcd

Clearly m is a point in the side ABC. If v is periodic then applying Proposition 5 yields that M is the base point of the periodic trajectory. Moreover, since the direction of the periodic trajectory is the normal vector to the plane a, we deduce that A is on the trajectory. So the periodic trajectory cannot exist.

Now we prove the second part of the theorem. We give an example of obtuse tetrahedron where Fagnano's orbit does not exist.

In this example the point on the initial face, which must be periodic see Proposition 5, is not in the interior of the triangle.

We consider the tetrahedron $ABCD$ with vertices

$$A(0,0,0), \quad B(2,0,0), \quad C(1,1,0), \quad D(3,2,1).$$

We study the word $v = abcd$. We obtain the matrix of S_v

$$\begin{pmatrix} \frac{1}{33} & \frac{8}{33} & \frac{32}{33} \\ \frac{104}{165} & \frac{-25}{33} & \frac{28}{165} \\ \frac{128}{165} & \frac{20}{33} & \frac{-29}{165} \end{pmatrix}.$$

Now s is the map $SX+N$ where $N=\begin{pmatrix}\frac{4}{11}\\ \frac{4}{11}\\ -\frac{12}{11}\end{pmatrix}$. S has eigenvector $u=\begin{pmatrix}\frac{9}{8}\\ \frac{1}{2}\\ 1\end{pmatrix}$.

Now s is a screw motion and we find the point at the intersection of the axis of s and the face a by solving the system

$$Sm+N=m+\lambda u \tag{2}$$

$$\overrightarrow{Bm}\cdot n=0. \tag{3}$$

This is equivalent to the system

$$\begin{pmatrix}S-Id & -u\\ n^t & 0\end{pmatrix}\begin{pmatrix}m\\ \lambda\end{pmatrix}=\begin{pmatrix}-N\\ 2\end{pmatrix}.$$

where $n=\begin{pmatrix}1\\ 1\\ -3\end{pmatrix}$ is the normal vector to the face BCD.

We obtain the matrix

$$\begin{pmatrix}-\frac{32}{33} & \frac{8}{33} & \frac{32}{33} & -\frac{9}{8}\\ \frac{104}{165} & -\frac{58}{33} & \frac{28}{165} & -\frac{1}{2}\\ \frac{128}{165} & \frac{20}{33} & -\frac{194}{165} & -1\\ 1 & 1 & -3 & 0\end{pmatrix}$$

and

$$m=\begin{pmatrix}\frac{22}{161}\\ \frac{6}{23}\\ -\frac{86}{161}\end{pmatrix}.$$

But this point is not inside BCD. Moreover we see that this point is not on the altitude at BD which passes through C.

The tetrahedron is obtuse, due to the triangle ABD. The triangle BCD is acute, and the axis of s does not cut this face in the interior of the triangle.

Moreover we obtain that there exists a neighborhood of this tetrahedron, where Fagnano's word is not periodic. Indeed in a neighborhood the point m can not be in the interior of $ABCD$.

Remark. We can remark that our proof gives a criterion for the existence of a periodic billiard path of this type. One computes the axis of the screw motion, and finds if it intersects the relevant faces.

For a generic tetrahedron we can use it to know if there exists a Fagnano's orbit. But we have not find a good system of coordinates where the computations are easy. Thus we are not able to caracterize the tetrahedra with a Fagnano's orbit.

8. First return map

In this section we use the preceding example to study a related problem for periodic billiard paths. We answer to a question of Gal'perin, Krüger and Troubetzkoy [3] by an example of periodic word v with non periodic points inside its beam.

We consider the word $v = (abcd)^\infty$ and the set σ_v. The projection of this set on the face a is an open set. Each point in this open set return to the face a after three reflections. We study this return map and the set $\pi_a(\sigma_v)$. We consider the same basis as in Section 6. Moreover, in the face a we consider the following basis

$$\begin{pmatrix} \frac{\sqrt{2}}{4} \\ 0 \\ 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Theorem 11. *In the regular tetrahedron, consider the word $v = (abcd)^\infty$. Then the set $\pi_a(\sigma_v)$ is an open set. There exists only one point in this set with a periodic billiard orbit.*

The theorem of [3] explains that some such cases could appear, but there were no example before this result.

Theorem 11 means that for all point in $\pi_a(\sigma_v)$, except one, the billiard orbit is coded by a periodic word, but it is never a periodic trajectory. For the proof we make use of the following lemma.

Lemma 12. *In the regular tetrahedron, consider the word $v = (abcd)^\infty$. The first return map r on $\pi_a(\sigma_v)$ is given by*

$$r\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + B,$$

where

$$A = \frac{1}{81}\begin{pmatrix} -83 & 28 \\ -12 & -75 \end{pmatrix}, \qquad B = \frac{1}{81}\begin{pmatrix} -15 \\ 9 \end{pmatrix}.$$

The set $\pi_a(\sigma_v)$ is the interior of the biggest ellipse of center m related to the matrix A.

Proof. If m is a point of the face a, the calculation in Section 6 shows that

$$r(m) = \frac{1}{81}\begin{pmatrix} -79x - 8y + 16\sqrt{2} \\ 66x - 21y + 42z + \frac{3\sqrt{2}}{2} \\ 13x + 29y - 58z\frac{33\sqrt{2}}{12} \end{pmatrix}$$

Now we compute m and rm in the basis of the face a. We obtain the matrices A, B. $\square$

Proof of Theorem 11. We can verify that the periodic point $\frac{\sqrt{2}}{20}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ is fixed by r. Indeed in this basis, it becomes $\frac{\sqrt{2}}{20}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Now the orbit of a point under r

is contained on an ellipse related to the matrix A. This shows that the set $\pi_a(\sigma_v)$ is the biggest ellipse included in the triangle. And an obvious computation shows that only one point is fixed by r.

References

[1] M. Berger, *Géométrie, 2*, Espaces euclidiens, triangles, cercles et sphères, CEDIC, Paris, 1977.

[2] M. Berger, *Géométrie, 3*, Convexes et polytopes, polyèdres réguliers, aires et volumes, CEDIC, Paris, 1977.

[3] G. Gal'perin, T. Krüger, and S. Troubetzkoy, Local instability of orbits in polygonal and polyhedral billiards, *Comm. Math. Phys.*, 169 (1995) 463–473.

[4] G. Gal'perin, A. M. Stëpin, and Ya. B. Vorobets, Periodic billiard trajectories in polygons: generation mechanisms, *Uspekhi Mat. Nauk*, 47 (1992) 9–74, 207.

[5] S. L. Glashow and L. Mittag, Three rods on a ring and the triangular billiard, *J. Statist. Phys.*, 87 (1997) 937–941.

[6] H.Masur and S. Tabachnikov, Rational billiards and flat structures, in *Handbook of Dynamical Systems*, volume 1A, 1015–1089, North-Holland, Amsterdam, 2002.

[7] F. Rabouw and Th. W. Ruijgrok, Three particles on a ring, *Phys. A*, 109 (1981) 500–516.

[8] R. Schwartz, Obtuse triangular billiards II: 100 degrees worth of periodic trajectories, preprint, 2005.

[9] H. Steinhaus, *One hundred problems in elementary mathematics*, Basic Books Inc. Publishers, New York, 1964.

[10] F. Stenman, Periodic orbits in a tetrahedral mirror, *Soc. Sci. Fenn. Comment. Phys.-Math.*, 45 (1975) 103–110.

[11] Ya. B. Vorobets, Periodic orbit in polygon, personal communication, 2006.

Nicolas Bedaride: Fédération de recherche des unités de mathématiques de Marseille, Laboratoire d'analyse, topologie et probabilités, UMR 6632 , Avenue Escadrille Normandie Niemen 13397 Marseille cedex 20, France

E-mail address: `nicolas.bedaride@univ-cezanne.fr`

Forum Geometricorum
Volume 8 (2008) 121–130.

FORUM GEOM
ISSN 1534-1178

On the Centroids of Polygons and Polyhedra

Maria Flavia Mammana, Biagio Micale, and Mario Pennisi

Abstract. In this paper we introduce the centroid of any finite set of points of the space and we find some general properties of centroids. These properties are then applied to different types of polygons and polyhedra.

1. Introduction

In elementary geometry the centroid of a figure in the plane or space (triangle, quadrilateral, tetrahedron, $\dots$) is introduced as the common point of some elements of the figure (medians or bimedians), once it has been proved that these elements are indeed concurrent. The proofs are appealing and have their own beauty in the spirit of Euclidean geometry. But they are different from figure to figure, and often use auxiliary elements. For example, the centroid of a triangle is defined as the common point of its three medians, after proving that they are concurrent. It is usually proved considering, as an auxiliary figure, the Varignon parallelogram of the quadrilateral whose vertices are the vertices of the triangle and the common point to two medians ([3, p. 10]). We can also define the centroid of a tetrahedron after proving that the four medians of the tetrahedron are concurrent (Commandino's Theorem, [1, p.57]). A natural question is: is it possible to characterize the properties of centroids of geometric figures with one unique and systematic method? In this paper we introduce the centroid of a finite set of points of the space, called a system, and find some of its general properties. These properties are then applied to different types of polygons and polyhedra. Then it is possible to obtain, in a simple and immediate way, old and new results of elementary geometry. At the end of the paper we introduce the notion of an extended system. This allows us to find some unexpected and charming properties of some figures, highlighting the great potential of the method that is used.

2. Systems and centroids

Throughout this paper, the ambient space is either a plane or a 3-dimensional space. Let $\mathcal{S}$ be a set of n points of the space. We call this an n-system or a system of order n. Let $\mathcal{S}'$ be a nonempty subset of S of k points, that we call a k-subsystem of $\mathcal{S}$ or a subsystem of order k of $\mathcal{S}$. There are $\binom{n}{k}$ different subsystems of order k. We say that two subsystems $\mathcal{S}'$ and $\mathcal{S}''$ of an n-system $\mathcal{S}$ are *complementary* if

Publication Date: June 2, 2008. Communicating Editor: Paul Yiu.

$\mathcal{S}' \cup S'' = S$ and $\mathcal{S}' \cap S'' = \emptyset$. We also say that $\mathcal{S}'$ is complementary to $\mathcal{S}''$ and $\mathcal{S}''$ is complementary to S'. If $\mathcal{S}'$ is a k-subsystem, $\mathcal{S}''$ is an $(n-k)$-subsystem. Let A_i, $i = 1, 2, \ldots, n$, be the points of an n-system S and $\mathbf{x}_i$ be the position vector of A_i with respect to a fixed point P. We call the *centroid* of $\mathcal{S}$ the point C whose position vector with respect to P is

$$\mathbf{x} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i.$$

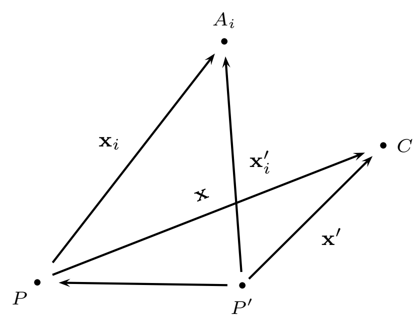


Figure 1

The point C does not depend on P. In fact, let P' be another point of the space and $\mathbf{x}'_i$ be the position vector of A_i with respect to P'. Since $\mathbf{x}'_i = \mathbf{x}_i + \overrightarrow{P'P}$, we have

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}'_i = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i + \overrightarrow{P'P}.$$

Every subsystem of $\mathcal{S}$ has its own centroid. The centroid of a 1-subsystem $\{A_i\}$ is A_i. The centroid of a 2-subsystem $\{A_i,\ A_j\}$ is the midpoint of the segment A_iA_j.

Let $\mathcal{S}'$ be a k-subsystem of $\mathcal{S}$ and C' its centroid. Let $\mathcal{S}''$ be the subsystem of $\mathcal{S}$ complementary to $\mathcal{S}'$ and C'' its centroid. We call the segment $C'C''$ the *median* of $\mathcal{S}$ relative to S'. The median relative to $\mathcal{S}''$ coincides with the one relative to $\mathcal{S}'$.

Let $\mathcal{S}$ be an n-system and C its centroid.

Theorem 1. *The medians of $\mathcal{S}$ are concurrent in C. Moreover, C divides the median $C'C''$ relative to a k-subsystem $\mathcal{S}'$ of $\mathcal{S}$ into two parts such that:*

$$\frac{C'C}{CC''} = \frac{n-k}{k}. \qquad (*)$$

Proof. In fact, let $\mathbf{v}$, $\mathbf{v}'$, $\mathbf{v}''$ the position vectors of C, C', C'' respectively. It is easy to prove that

$$\mathbf{v} - \mathbf{v}' = \frac{n-k}{k}(\mathbf{v}'' - \mathbf{v}).$$

This relation means that $\overrightarrow{C'C} = \frac{n-k}{k}\overrightarrow{CC''}$. Hence, C, C', C'' are collinear and (*) holds. $\square$

Here are some interesting consequences of Theorem 1.

Corollary 2. *The system of centroids of the $k-$subsystems of $\mathcal{S}$ is the image of the system of centroids of the $(n-k)$-subsystems of $\mathcal{S}$ in the dilatation with ratio $-\frac{n-k}{k}$ and center C. In this dilatation the centroid of a k-subsystem is the image of the centroid of its complementary.*

Corollary 3. *The segment $C_1'C_2'$ that joins the centroids of two k-subsystems $\mathcal{S}_1'$, $\mathcal{S}_2'$ of $\mathcal{S}$ is parallel to the segment $C_1''C_2''$ that joins the centroids of the $(n-k)$-subsystems complementary to $\mathcal{S}_1'$, $\mathcal{S}_2'$. Moreover,*

$$\frac{C_1'C_2'}{C_1''C_2''} = \frac{n-k}{k}.$$

Corollary 4. *If $n = 2k$, C is the center of symmetry of the system of centroids of the k-subsystems of $\mathcal{S}$. Moreover, the segment $C_1'C_2'$ that joins the centroids of two k-subsystems S_1', $\mathcal{S}_2'$ of $\mathcal{S}$ is parallel and equal to the segment $C_1''C_2''$ that joins the centroids of the k-subsystems complementary to $\mathcal{S}_1'$, S_2'.*

We conclude this section by the following theorem which is easily verified.

Theorem 5. *The centroid C of $\mathcal{S}$ is also the centroid of the system of centroids of the k-subsystems of $\mathcal{S}$.*

3. Applications

We propose here some applications to polygons and polyhedra. Let $\mathcal{P}$ be a polygon or a polyhedron. We associate with it the system $\mathcal{S}$ whose points are the vertices of $\mathcal{P}$.

3.1. *Triangles.* Let $\mathcal{T}$ be a triangle, with associated system $\mathcal{S}$ and centroid C. The 1-subsystems of $\mathcal{S}$ detect the vertices of $\mathcal{T}$, the 2-subsystems detect the sides. The centroids of the 2-subsystems of $\mathcal{S}$ are the midpoints of the sides of $\mathcal{T}$ and detect the medial triangle of $\mathcal{T}$. The medians of $\mathcal{S}$ are the medians of $\mathcal{T}$.

As a consequence of Theorem 1, we have

Proposition 6 ([3, p.10], [4, p.8])**.** *The three medians of a triangle all pass through one point which divides each median into two segments in the ratio* $2:1$.

It follows that the centroid of $\mathcal{T}$ coincides with the centroid C of $\mathcal{S}$.

From Theorem 5 and Corollary 2, we deduce

Proposition 7 ([4, p.18], [5, p.11])**.** *A triangle $\mathcal{T}$ and its medial triangle have the same centroid C. Moreover, the medial triangle is the image of $\mathcal{T}$ in the dilatation with ratio $-\frac{1}{2}$ and center C.* See Figure 2.

Corollary 3 yields

Proposition 8 ([4, p.53])**.** *The segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long as that third side.*

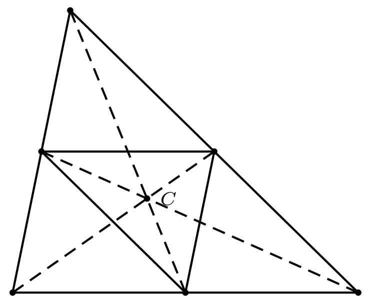


Figure 2.

3.2. *Quadrilaterals.* Let $A_1A_2A_3A_4$ be a quadrilateral which we denote by $\mathcal{Q}$. Let $\mathcal{S}$ be the system associated with $\mathcal{Q}$ and C its centroid. The 1-subsystems of $\mathcal{S}$ detect the vertices of $\mathcal{Q}$, the 2-subsystems detect the sides and the diagonals, the 3-subsystems detect the sub-triangles of $\mathcal{Q}$. The centroids of the 2-subsystems of $\mathcal{S}$ are the midpoints of the sides and of the diagonals of $\mathcal{Q}$. The centroids of the 3-subsystems are the centroids C_1, C_2, C_3, C_4 of the triangles $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$ respectively. We call $C_1C_2C_3C_4$ the quadrilateral of centroids and denote it by $\mathcal{Q}_c$ ([6]). The medians of $\mathcal{S}$ relative to the 2-subsystems are the *bimedians* of $\mathcal{Q}$ and the segment that joins the midpoints of the diagonals of $\mathcal{Q}$. The medians of $\mathcal{S}$ relative to the 1-subsystems are the segments A_iC_i, $i = 1, 2, 3, 4$.

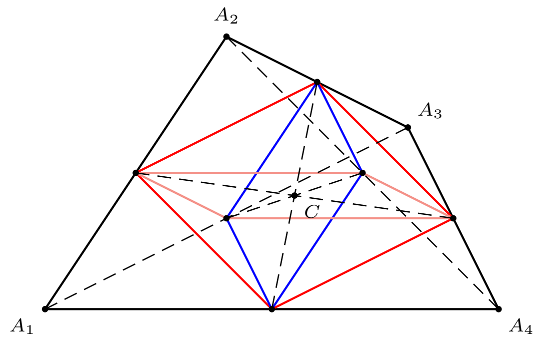


Figure 3

From Theorem 1 it follows that

Proposition 9 ([4, p.54])**.** *The bimedians of a quadrilateral and the segment joining the midpoints of the diagonals are concurrent and bisect one another.* See Figure 3.

Thus, the centroid of the quadrilateral $\mathcal{Q}$, *i.e.*, the intersection point of the bimedians, coincides with the centroid C of $\mathcal{S}$. From Corollary 4, we obtain

Proposition 10 ([4, p.53])**.** *The quadrilateral whose vertices are the midpoints of the sides of a quadrilateral is a parallelogram* (Varignon's Theorem). *Moreover, the quadrilateral whose vertices are the midpoints of the diagonals and of two opposite sides of a quadrilateral is a parallelogram.*

Thus, three parallelograms are naturally associated with a quadrilateral. These have the same centroid, which, by Theorem 1, coincides with the centroid of the quadrilateral.

Theorem 5 and Corollary 2 then imply

Proposition 11 ([6])**.** *The quadrilaterals $\mathcal{Q}$ and $\mathcal{Q}_c$ have the same centroid C. Moreover, $\mathcal{Q}_c$ is the image of $\mathcal{Q}$ in the dilatation with ratio $-\frac{1}{3}$ and center C.* See Figure 4.

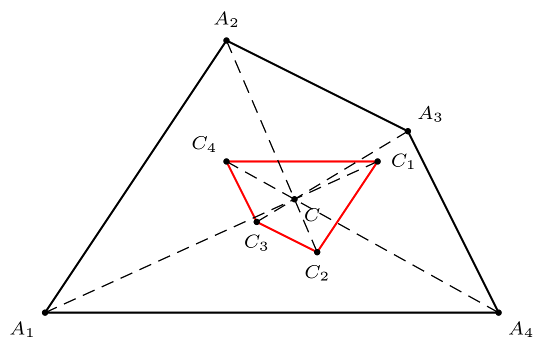


Figure 4

Some of these properties, with appropriate changes, hold also for polygons with more than four edges. For example, from Theorem 1 it follows that

Proposition 12. *The five segments that join the midpoint of a side of a pentagon with the centroid of the triangle whose vertices are the remaining vertices and the five segments that join a vertex of a pentagon with the centroid of the quadrilateral whose vertices are the remaining vertices are all concurrent in a point C that divides the first five segments in the ratio 3:2 and the other five in the ratio 4:1.*

The point C is the centroid of the system $\mathcal{S}$ associated with the pentagon. C will also be called the centroid of the pentagon.

3.3. *Tetrahedra.* Let $\mathcal{T}$ be a tetrahedron. Let $\mathcal{S}$ be the system associated with $\mathcal{T}$ and C its centroid. The subsystem of $\mathcal{S}$ of order 1, 2, and 3 detect the vertices, the edges and the faces of $\mathcal{T}$, respectively. The centroids of the 2-subsystems are the midpoints of the edges. Those of the 3-subsystems are the centroids of the faces of $\mathcal{T}$, which detect the medial tetrahedron of $\mathcal{T}$. The medians of $\mathcal{S}$ relative to the 2-subsystems are the bimedians of $\mathcal{T}$, *i.e.*, the segments that join the midpoints of two opposite sides. The medians of $\mathcal{S}$ relative to the 1-subsystems are the medians of $\mathcal{T}$, *i.e.*, the segments that join one vertex of $\mathcal{T}$ with the centroid of the opposite face.

From Theorem 1 follows Commandino's Theorem:

Proposition 13 ([1, p.57]). *The four medians of a tetrahedron meet in a point which divides each median in the ratio* $1:3$*.* See Figure 5.

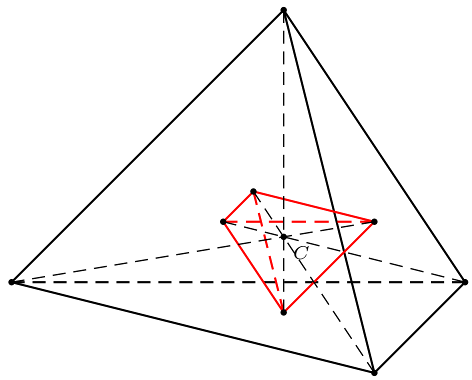


Figure 5

It follows that the centroid of the tetrahedron $\mathcal{T}$, intersection point of the medians, coincides with the centroid C of $\mathcal{S}$. From Theorem 5 and from Corollary 2 it follows that

Proposition 14 ([1, p.59]). *A tetrahedron $\mathcal{T}$ and its medial tetrahedron have the same centroid C. Moreover the medial tetrahedron is the image of $\mathcal{T}$ in the dilatation with ratio $-\frac{1}{3}$ and center C. The faces and the edges of the medial tetrahedron of a tetrahedron $\mathcal{T}$ are parallel to the faces and the edges of $\mathcal{T}$.*

Finally, Theorem 1 and Corollary 2 yield

Proposition 15 ([1, pp.54,58]). *The three bimedians of a tetrahedron are concurrent in the centroid of the tetrahedron and are bisected by it. Moreover, the midpoints of two pairs of opposite edges of tetrahedron are the vertices of a parallelogram.* See Figure 6.

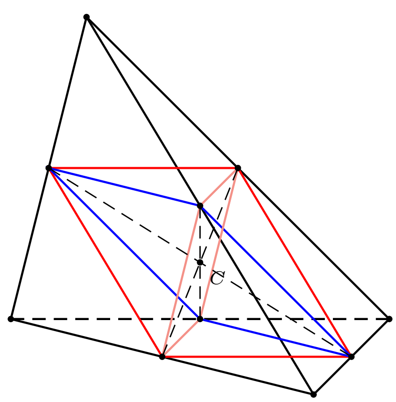


Figure 6

By using the theorems of the theory it is possible to find lots of interesting properties on polyhedra. For example, Corollary 4 gives

Proposition 16. *The centroids of the faces of an octahedron with triangular faces are the vertices of a parallelepiped. The centroids of the faces of a hexahedron with quadrangular faces are the vertices of an octahedron with triangular faces having a symmetry center C.* See Figures 7A and 7B.

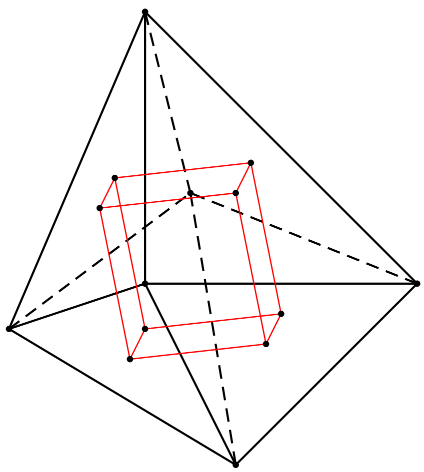


Figure 7A

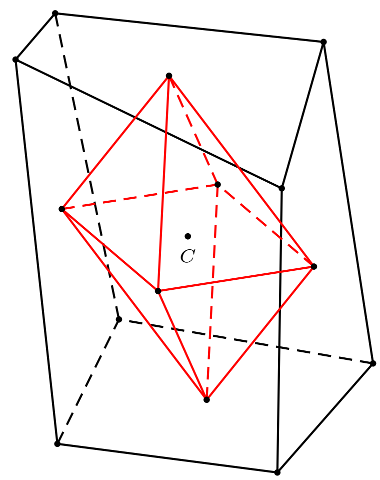


Figure 7B

The point C is the centroid of the system $\mathcal{S}$ associated with the hexahedron. This point is also called the centroid of the hexahedron.

4. Extended systems and applications

Let $\mathcal{S}$ be an n-system and h a fixed positive integer. Let H be a set of h points such that $\mathcal{S} \cap H = \emptyset$. We call h-extension of $\mathcal{S}$ the system $\mathcal{S}_H = \mathcal{S} \cup H$.

Let t be a fixed integer such that $1 \leq t < n$. Consider the system $\mathcal{C}_{H,t}$ of centroids of the subsystems of $\mathcal{S}_H$, of order $h+t$, that contain H. The complementary subsystems of these subsystems are the subsystems of $\mathcal{S}$ of order $n-t$ and we denote the system of their centroids by $\mathcal{C}'_{n-t}$.

Let us consider now two h-extensions of $\mathcal{S}$, $\mathcal{S}_{H_1}$ and $\mathcal{S}_{H_2}$, and let C_1 and C_2 be their centroids. Consider the systems $\mathcal{C}_{H_{1,t}}$ and $\mathcal{C}_{H_{2,t}}$, and the system $\mathcal{C}'_{n-t}$.

From Corollary 2 applied to the system $\mathcal{S}_{H_1}$ (respectively $\mathcal{S}_{H_2}$) it follows that $\mathcal{C}_{H_{1,t}}$ (respectively $\mathcal{C}_{H_{2,t}}$) is the image of $\mathcal{C}'_{n-t}$ in the dilatation with ratio $-\frac{n-t}{h+t}$ and center C_1 (respectively C_2).

Thus, we have

Theorem 17. *If $\mathcal{S}_{H_1}$ and $\mathcal{S}_{H_2}$ are two h-extension of $\mathcal{S}$, then the systems $\mathcal{C}_{H_{1,t}}$ and $\mathcal{C}_{H_{2,t}}$ are correspondent in a translation.*

It is easy to see that the vector of the translation transforming $\mathcal{C}_{H_{1,t}}$ into $\mathcal{C}_{H_{2,t}}$ is $\frac{n+h}{h+t}\overrightarrow{C_1C_2}$.

The following theorem is also of interest.

Theorem 18. *If $\mathcal{S}$ is an n-system, $\mathcal{S}_H$ is a 1-extension of $\mathcal{S}$, $\mathcal{S}_K$ is a $(n-1)$-extension of $\mathcal{S}$, then the systems $\mathcal{C}_{H,n-1}$ and $\mathcal{C}_{K,1}$ are correspondent is a half-turn.*

Proof. Let C and C_K be the centroids of $\mathcal{S}_H$ and K respectively. From Corollary 2 the system $\mathcal{C}_{H,n-1}$ is the image of the system $\mathcal{C}'_1 = \mathcal{S}$ in the dilatation with ratio $\frac{1}{n}$ and center C that is, $\mathcal{S}$ is the image of $\mathcal{C}_{H,n-1}$ in the dilatation with ratio $-n$ and center C.

Let $C' \in \mathcal{C}_{K,1}$ and suppose that C' is the centroid of the n-subsystem $\mathcal{S}' = K \cup \{A\}$ of $\mathcal{S}_K$, with $A \in \mathcal{S}$. From Theorem 1, C' lies on the median C_KA of $\mathcal{S}'$ and is such that $\frac{C_KC'}{C'A} = \frac{1}{n-1}$. It follows that $\frac{C_KC'}{C_KA} = \frac{1}{n}$, and $\mathcal{C}_{K,1}$ is the image of $\mathcal{S}$ in the dilatation with ratio $\frac{1}{n}$ and center C_K.

Since $\mathcal{S}$ is the image of $\mathcal{C}_{H,n-1}$ in the dilatation with ratio $-n$ and center C and $\mathcal{C}_{K,1}$ is the image of $\mathcal{S}$ in the dilatation with ratio $\frac{1}{n}$ and center C_K, then $\mathcal{C}_{H,n-1}$ and $\mathcal{C}_{K,1}$ are correspondent in a dilatation with ratio -1, *i.e.*, in a half-turn. □

It is easy to see that the center $\overline{C}$ of the half-turn is the point of the segment CC_K such that $\frac{C\overline{C}}{\overline{C}C_K} = \frac{n-1}{n+1}$.

Now, we offer some applications of Theorems 17 and 18.

4.1. *Triangles.* Let $\mathcal{T}$ be a triangle and $\mathcal{S}$ its associated system. Let S_H be a 1-extension of $\mathcal{S}$, with $H = \{P\}$, and $\mathcal{S}_K$ be a 2-extension of $\mathcal{S}$, with $K = \{P_1, P_2\}$. The points of the system $C_{H,2}$ are vertices of a triangle $\mathcal{T}_H$ and the points of the system $C_{K,1}$ are vertices of a triangle $\mathcal{T}_K$. Theorem 18 gives

Proposition 19. *The triangles $\mathcal{T}_H$ and $\mathcal{T}_K$ are correspondent in a half-turn.* See Figure 8.

Let $\{\mathcal{T}_H\}$ be the family of triangles $\mathcal{T}_H$ obtained by varying the point P and $\{\mathcal{T}_K\}$ be the family of triangles $\mathcal{T}_K$ obtained by varying the points P_1 and P_2.

From Theorem 17 the triangles of the family $\{\mathcal{T}_H\}$ are all congruent and have corresponding sides that are parallel. The same property also holds for the triangles of the family $\{\mathcal{T}_K\}$. On the other hand, each triangle $\mathcal{T}_H$ and each triangle $\mathcal{T}_K$ are correspondent in a half-turn, then:

Proposition 20. *The triangles of the family $\{\mathcal{T}_H\} \cup \{\mathcal{T}_K\}$ are all congruent and have corresponding sides that are parallel.*

4.2. *Quadrilaterals.* Let $\mathcal{Q}$ be a quadrilateral $A_1A_2A_3A_4$ and $\mathcal{S}$ its associated system. Let $\mathcal{S}_H$ be a 1-extension of $\mathcal{S}$, with $H = \{P\}$, and let C be its centroid.

Let us consider the subsystems $\{P,\ A_1,\ A_2\}$, $\{P,\ A_2,\ A_3\}$, $\{P,\ A_3,\ A_4\}$, $\{P,\ A_4,\ A_1\}$ of $\mathcal{S}_H$ and their centroids C_1, C_2, C_3, C_4 respectively, that are points of $C_{H,2}$. From Corollary 3 applied to the system $\mathcal{S}_H$, the segments C_1C_2, C_2C_3, C_3C_4, C_4C_1 are parallel to the sides of the Varignon parallelogram of $\mathcal{Q}$ respectively. Thus, $C_1C_2C_3C_4$ is a parallelogram, that we denote by $\mathcal{Q}_H$. Moreover,

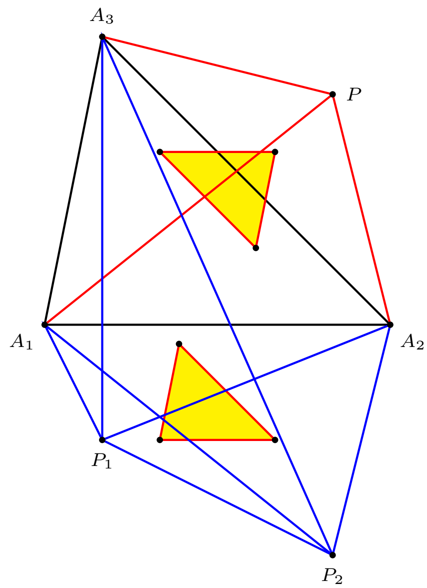


Figure 8.

from Corollary 2, $\mathcal{Q}_H$ is the image of the Varignon parallelogram of $\mathcal{Q}$ in the dilatation with ratio $-\frac{2}{3}$ and center C. In the case when P is the intersection point of the diagonals of Q, the existence of a dilatation between $\mathcal{Q}_H$ and the Varignon parallelogram of $\mathcal{Q}$ has already been proved ([2, p.424], [7, p.23]).

If we consider two 1-extensions of $\mathcal{S}$, the systems $\mathcal{C}_{H,2}$, for Theorem 17, are correspondent in a translation. Thus, if $\{\mathcal{Q}_H\}$ is the family of the parallelograms obtained as P varies, we obtain

Proposition 21. *The parallelograms of the family $\{\mathcal{Q}_H\}$ are all congruent and their corresponding sides are parallel.*

Moreover, taking P as the vertex of a pyramid with base $\mathcal{Q}$, we are led to

Proposition 22. *The centroids of the faces of a pyramid with a quadrangular base are vertices of the parallelogram that is the image to Varignon parallelogram of $\mathcal{Q}$ in the dilatation with ratio $-\frac{2}{3}$ and center C. Moreover, as P varies, the parallelograms whose vertices are the centroids of the faces are all congruent.* See Figure 9.

The point C is called the centroid of the pyramid.

References

[1] N. Altshiller - Court, *Modern Pure Solid Geometry*, Chelsea Publishing Company, New York, 1964.
[2] C. J. Bradley, Cyclic quadrilaterals, *Math. Gazette*, 88 (2004) 417–431.
[3] H. S. M. Coxeter, *Introduction to geometry*, John Wiley & Sons, Inc, New York, 1969.
[4] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, MAA, 1967.
[5] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, Math. Assoc. America, 1995.

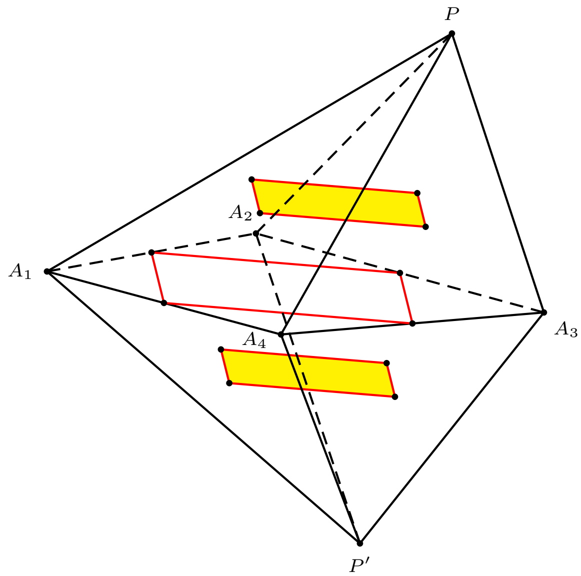


Figure 9

[6] M. F. Mammana and B. Micale, Quadrilaterals of triangle centers, to appear in *Math. Gazette*.

[7] M. F. Mammana and M. Pennisi, Analyse des situations problematiques concernant des quadrilatères: intuitions, conjectures, deductions, *Mathématique et Pédagogie*, 162 (2007) 20–33.

Maria Flavia Mammana: Department of Mathematics and Computer Science, University of Catania, Viale A. Doria 5, 95125, Catania, Italy
E-mail address: `fmammana@dmi.unict.it`

Biagio Micale: Department of Mathematics and Computer Science, University of Catania, Viale A. Doria 5, 95125, Catania, Italy
E-mail address: `micale@dmi.unict.it`

Mario Pennisi: Department of Mathematics and Computer Science, University of Catania, Viale A. Doria 5, 95125, Catania, Italy
E-mail address: `pennisi@dmi.unict.it`

Forum Geometricorum
Volume 8 (2008) 131–140.

FORUM GEOM
ISSN 1534-1178

Another Variation on the Steiner-Lehmus Theme

Sadi Abu-Saymeh, Mowaffaq Hajja, and Hassan Ali ShahAli

Abstract. Let the internal angle bisectors BB' and CC' of angles B and C of triangle ABC be extended to meet the circumcircle at B^* and C^*. The Steiner-Lehmus theorem states that if $BB' = CC'$, then $AB = AC$. In this article, we investigate those triangles for which $BB^* = CC^*$ and we address several issues that arise within this investigation.

1. Introduction

The celebrated Steiner-Lehmus theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles. In terms of triangle centers and cevians, it states that if two cevians through the *incenter* of a triangle are equal, then the triangle is isosceles. Variations on the theme can be obtained by replacing the incenter by any of the hundreds of centers known in the literature; see [6] and the website [7]. Other variations on this theme are obtained by letting the cevians of ABC through a center P meet the circumcircle of ABC at A^*, B^*, and C^* and asking whether the equality $BB^* = CC^*$ implies that $AB = AC$, where XY denotes the length of the line segment XY. This variation, together with several others, is investigated in [5] where it is proved that if P is the incenter, the orthocenter, or the Fermat-Torricelli point, then $BB^* = CC^*$ if and only if $AB = AC$ or $A = \frac{\pi}{3}$. When P is the centroid, the triangles for which $BB^* = CC^*$ are proved, in Theorem 9 below, to be the ones whose side lengths satisfy the relation $a^4 = b^4 - b^2c^2 + c^4$, a relation that has no geometric interpretation and cannot be fitted into a traditional geometry context such as Euclid's *Elements*.

Using geometric arguments, we show that if the centroid P of a scalene triangle ABC is such that $BB^* = CC^*$, then $\angle BAC$ must lie in the interval $[\frac{\pi}{3}, \frac{\pi}{2}]$ and that to every θ in $[\frac{\pi}{3}, \frac{\pi}{2}]$ there is essentially a unique scalene triangle with $\angle BAC = \theta$ and with $BB^* = CC^*$. The proof uses a generalization of Proposition 7 of Book III of Euclid's *Elements*, in brief Euclid III.7 [1], that deserves recording on its own.

Publication Date: June 16, 2008. Communicating Editor: Paul Yiu.

The first and second named authors are supported by a research grant from Yarmouk University and would like to express their thanks for this support. The authors would also like to thank the referee for his valuable remarks and for providing the construction given in Remark (2) at the end of this note, and to Mr. Essam Darabseh for drawing the figures.

[1]Throughout, the symbol Euclid $*.**$ designates Proposition $**$ of Book $*$ in Euclid's *Elements*.

2. Euclid III.7 and a generalization

Euclid III.7, not that well known, states that if Ω is a circle centered at O, if $M \neq O$ is a point inside Ω, and if the intersection of a ray MX with Ω is denoted by X', then
(i) the maximum value of MX' is attained when the ray MX passes through O and the minimum is attained when the ray MX is the opposite ray OM,
(ii) as the ray MX rotates from the position MO to the opposite position OM, the quantity MX' changes monotonically.

We restate this proposition in Theorem 1 as a preparation for the generalization that is made in Theorem 5.

Theorem 1 (Euclid III.7). *Let BC be a chord in a circle Ω, let M be the mid-point of BC, and let the line perpendicular to BC through M meet Ω at E and F. As a point P moves from E to F along the arc ECF of Ω, the length MP changes monotonically. It increases or decreases according as E is closer or farther than F from M.*

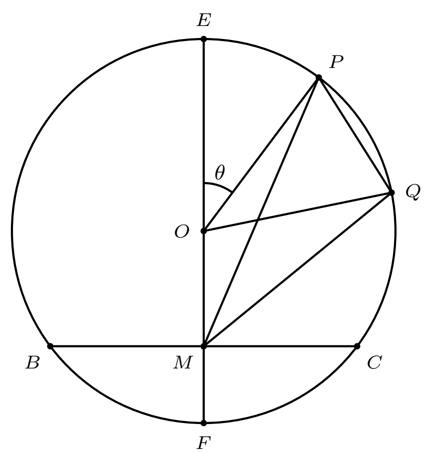


Figure 1.

Proof. Referring to Figure 1, we shall show that if $EM > MF$, *i.e.*, if the center O of Ω is between E and M, and if P and Q are any points on the arc ECF such that P is closer to E than Q, then $MP > MQ$. Under these assumptions,

$$\angle MQP > \angle OQP = \angle OPQ > \angle MPQ.$$

Thus $\angle MQP > \angle MPQ$, and therefore $MP > MQ$, as desired. □

Remark. The proof above uses the fairly simple-minded fact that in a triangle, the greater angle is subtended by the greater side. This is Euclid I.19. It is interesting that Euclid's proof uses the more sophisticated Euclid I.24. This theorem, referred to in [8, Theorem 6.3.9, page 140] as the *Open Mouth Theorem*, states that if triangles ABC and $A'B'C'$ are such that $AB = A'B'$, $AC = A'C'$, $\angle BAC > \angle B'A'C'$, then $BC > B'C'$. Quoting [8], this says that *the wider you open your* *mouth, the farther apart your lips are.* Although this follows immediately from the law of cosines, the intricate proofs given by Euclid and in [8] have the advantage of showing that the theorem is a theorem in neutral geometry.

Theorem 5 below generalizes Theorem 1. In fact Theorem 1 follows from Theorem 5 by taking BC to be a diameter of one of the circles Ω and Ω'. For the proof of Theorem 5, we need the following simple lemmas.

Lemma 2. *Let ABC be a triangle and let D and E be points on the sides AB and AC respectively* (see Figure 2). *Then $\frac{AD}{AB}$ is greater than, less than, or equal to $\frac{AE}{AC}$ according as $\angle ABC$ is greater than, less than, or equal to $\angle ADE$, respectively.*

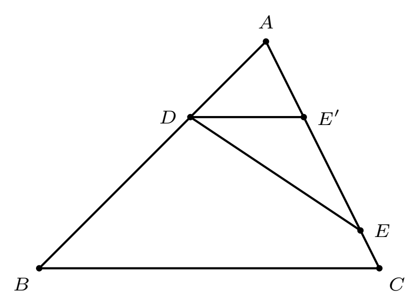


Figure 2

Proof. Let E' be the point on AC such that $\frac{AE'}{AC} = \frac{AD}{AB}$; *i.e.*, DE' is parallel to BC. If $\frac{AE}{AC} = \frac{AD}{AB}$, then $E' = E$ and $\angle ABC = \angle ADE$. If $\frac{AE}{AC} > \frac{AD}{AB}$, then E lies between E' and C, and $\angle ABC = \angle ADE' < \angle ADE$. Similarly for the case $\frac{AE}{AC} < \frac{AD}{AB}$. □

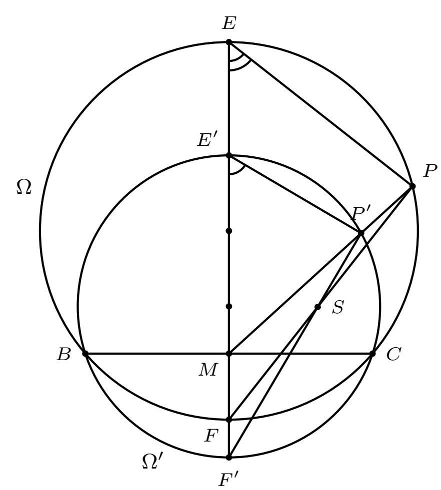


Figure 3

Lemma 3. *Two circles Ω and Ω' intersect at B and C, and the line perpendicular to BC through the midpoint M of BC meets Ω and Ω' at E and E', respectively, such that E' is inside Ω* (see Figure 3). *If P is any point on the arc ECF of Ω and if the ray MP meets Ω' at P', then $\frac{MP'}{MP} > \frac{ME'}{ME}$.*

Proof. Let S be the point of intersection of FP and $F'P'$. Since $\angle EPF = \frac{\pi}{2} = \angle E'P'F'$, it follows that $\angle ME'P' + \angle MF'P' = \frac{\pi}{2} = \angle MEP + \angle MFP$. But $\angle MFP > \angle MF'P'$, by the exterior angle theorem. Hence $\angle ME'P' > \angle MEP$. By Lemma 2, we have $\frac{MP'}{MP} > \frac{ME'}{ME}$, as desired. □

Lemma 4. *Let EBC be an isosceles triangle having $EB = EC$. Let M be the midpoint of BC and let E' be the circumcenter of EBC* (see Figure 4). *Then $\frac{ME'}{ME}$ is greater than, equal to, or less than $\frac{1}{3}$ according as $\angle BEC$ is less than, equal to, or greater than $\frac{\pi}{3}$, respectively.*

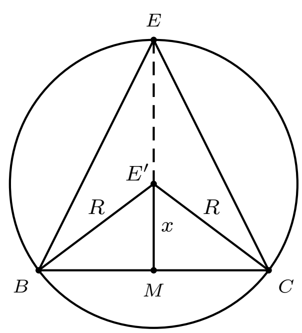


Figure 4.

Proof. Let $\theta = \angle BEC$, $x = ME'$, and let R be the circumradius of EBC. Then $\angle ME'C = \theta$ and

$$\frac{ME'}{ME} - \frac{1}{3} = \frac{x}{x+R} - \frac{1}{3} = \frac{R\cos\theta}{R\cos\theta + R} - \frac{1}{3} = \frac{2\cos\theta - 1}{3(\cos\theta + 1)}.$$

This is positive, zero, or negative according as $\cos\theta$ is greater than, equal to, or less than $\frac{1}{2}$. □

Theorem 5. *Two circles Ω and Ω' intersect at B and C and the line perpendicular to BC through the midpoint M of BC meets Ω at E and F and meets Ω' at E' and F'. For every point P on Ω, let P' be the point where the ray MP meets Ω'. As a point P moves from E to F along the arc ECF, the ratio $\frac{MP'}{MP}$ changes monotonically. It decreases or increases according as E' is inside or outside Ω.*

Proof. Referring to Figure 5, suppose that E' lies inside Ω and let P and Q be two points on the arc ECF of Ω such that P is closer to E than Q. we are to show that $\frac{MP'}{MP} < \frac{MQ'}{MQ}$.

Extend QM to meet Ω at U and Ω' at U'. Let T be the point of intersection of EU and $E'U'$. Since the quadrilaterals $EPQU$ and $E'P'Q'U'$ are cyclic, it

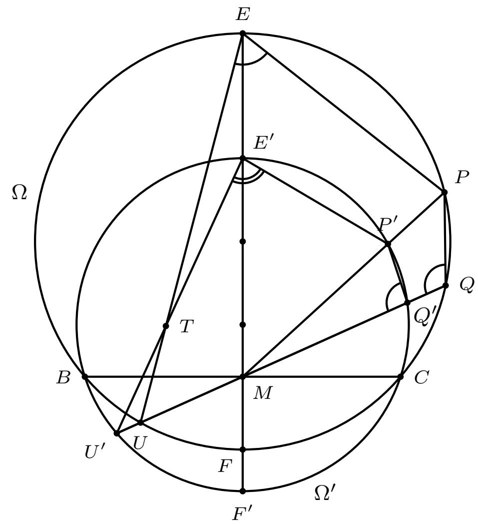


Figure 5

follows that

$$\angle UQP + \angle UEP = \pi = \angle U'Q'P' + \angle U'E'P'. \tag{1}$$

But

$$\begin{aligned} \angle U'E'P' &= \angle U'E'M + \angle ME'P' \\ &> \angle UEM + \angle ME'P' \text{ (by the exterior angle theorem)} \\ &> \angle UEM + \angle MEP \text{ (by Lemmas 3 and 2)} \\ &= \angle UEP. \end{aligned}$$

From this and (1) it follows that $\angle U'Q'P' > \angle UQP$. By Lemma 2, we conclude that $\frac{MP'}{MP} < \frac{MQ'}{MQ}$, as desired.

Note that if P is on the arc EC and Q is on the arc CF, then $\frac{MP'}{MP} < 1 < \frac{MQ'}{MQ}$. □

3. Conditions of equality of two chords through a given point

The next simple lemma exhibits the relation between two geometric properties of a point P inside a triangle ABC. It will be used in the proof of Theorem 9.

Lemma 6. *Let P be a point inside triangle ABC and let the rays BP and CP meet the circumcircle of ABC at B^* and C^* respectively* (see Figure 6). *Then*
(a) $BB^* = CC^*$ *if and only if* $PB = PC$ *or* $\angle BPC = 2\angle BAC$*;*
(b) $\angle BPC = 2\angle BAC \iff PB^* = PC \iff B^*C \parallel C^*B$.
Moreover, if P is the centroid, then
(c) $PB = PC \iff AB = AC \iff B^*C^* \parallel BC$.

Proof. (a) It is clear that

$$\begin{aligned} BB^* = CC^* &\iff \angle BAB^* = \angle CAC^* \text{ or } \angle BAB^* + \angle CAC^* = \pi \\ &\iff \angle CAB^* = \angle BAC^* \text{ or } \angle CAB^* + \angle BAC^* + 2\angle BAC = \pi \\ &\iff \angle CBB^* = \angle BCC^* \text{ or } \angle CBB^* + \angle BCC^* + 2\angle BAC = \pi \\ &\iff PB = PC \text{ or } \angle BPC = 2\angle BAC. \end{aligned}$$

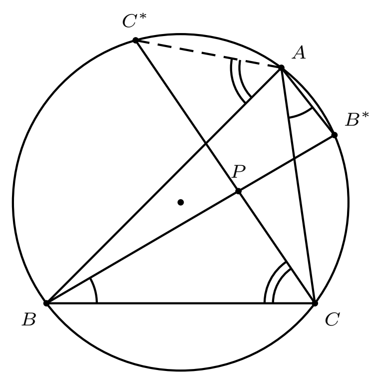


Figure 6.

(b) Also,

$$\begin{aligned} \angle BPC = 2\angle BAC &\iff \angle PB^*C + \angle PCB^* = 2\angle PB^*C \\ &\iff \angle PCB^* = \angle PB^*C \\ &\iff \angle PB^* = \angle PC. \end{aligned}$$

This proves the first part of (b). The implication $PB^* = PC \iff B^*C \parallel C^*B$ is easy.

(c) Let the lengths of the medians from B and C be β and γ, respectively. By Apollonius theorem, we have

$$\frac{b^2}{2} + 2\beta^2 = a^2 + c^2\ ,\ \frac{c^2}{2} + 2\gamma^2 = a^2 + b^2.$$

The rest follows from the facts that $PB = \frac{2\beta}{3}$ and $PC = \frac{2\gamma}{3}$. □

4. Chords of circumcircle through the centroid

In Theorem 7, we focus on triangles ABC whose centroid G has the property that $\angle BGC = 2\angle BAC$. Interest in this property stems from Lemma 6. Note that Part (i) provides a solution of the problem in [4].

Theorem 7. (i) *If ABC is a triangle whose centroid G has the property that $\angle BGC = 2\ \angle BAC$, then $\frac{\pi}{3} \leq \angle BAC < \frac{\pi}{2}$ with $\angle BAC = \frac{\pi}{3}$ if and only if ABC is equilateral.*

(ii) *If θ is any angle in the interval $(\frac{\pi}{3}, \frac{\pi}{2})$ and if BC is any line segment, then there is a triangle ABC, unique up to reflection about BC and about the perpendicular bisector of BC, having $\angle BAC = \theta$ and whose centroid G has the property $\angle BGC = 2\,\angle BAC$.*

Proof. (i) Let Ω be the circumcircle of ABC and let E' be its circumcenter. Let Ω' be the circumcircle of $E'BC$. Let M be the midpoint of BC and let the perpendicular bisector of BC meet Ω at E and F and meet Ω' at (E' and) F', where E is on the arc BAC of Ω (see Figure 7). Let $\angle BAC = \theta$, and let G be the centroid of ABC. Also, for every P on Ω, let P' be the point where the ray MP meets Ω'.

Suppose that $\angle BGC = 2\angle BAC$. Since $\angle BE'C = 2\angle BAC$, it follows that G lies on the arc $BE'C$ of Ω'. Also, G lies on the median AM of ABC. Therefore, G is the point A' where the ray MA meets Ω'. In particular, $\frac{MA'}{MA} = \frac{1}{3}$. As P moves from E to F along the arc ECF, the ratio $\frac{MP'}{MP}$ increases by Theorem 5. Therefore

$$\frac{ME'}{ME} \leq \frac{MA'}{MA} = \frac{1}{3}.$$

By Lemma 4, $\theta \geq \frac{\pi}{3}$, with equality if and only if $A = E$, or equivalently if and only if ABC is equilateral. The possibility that $\angle BAC \geq \frac{\pi}{2}$ is ruled out since it would lead to the contradiction $\angle BGC \geq \pi$.

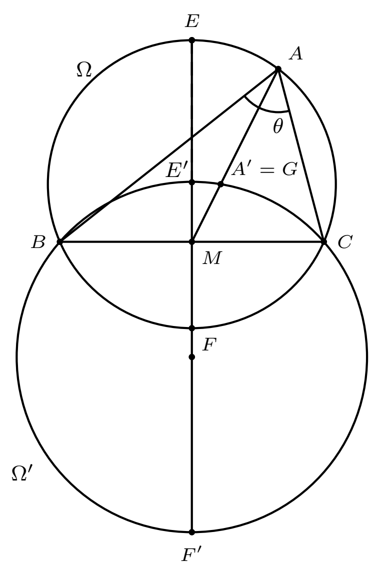


Figure 7

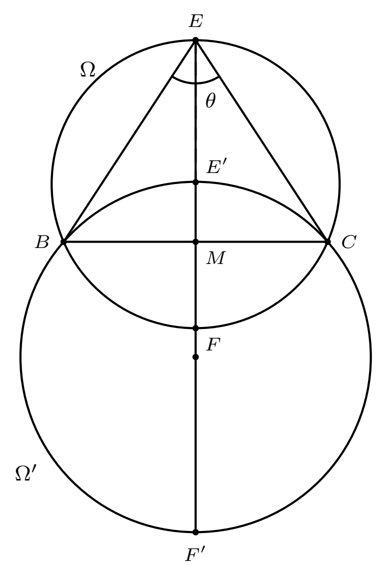


Figure 8

(ii) Suppose that θ is a given angle such that $\frac{\pi}{3} \leq \theta < \frac{\pi}{2}$ and that BC is a given segment. Let EBC be an isosceles triangle with $EB = EC$ and with $\angle BEC = \theta$. Let Ω be the circumcircle of EBC and let E' be its circumcenter. Let Ω' be the circumcircle of $E'BC$. Let M be the midpoint of BC and let the perpendicular bisector of BC meet Ω at (E and) F and meet Ω' at (E' and) F' (see Figure 8). For every P on Ω, we let P' be the point where the ray MP meets Ω'. Let $t = \frac{ME'}{ME}$. Since $\theta \geq \frac{\pi}{3}$, it follows from Lemma 4 that $t \leq \frac{1}{3}$. Also, $C' = C$ and $\frac{MC'}{MC} = 1$. Thus as P moves from E to C along one of the arcs EC of Ω, the ratio $\frac{MP'}{MP}$ increases from $t \leq \frac{1}{3}$ to 1. By continuity and the intermediate value theorem, there is a unique point A on that arc EC for which $\frac{MA'}{MA} = \frac{1}{3}$. If we think of MC as the x-axis and of ME as the y-axis, then the point A is the only point in the first quadrant for which ABC has the desired property. Points in the other quadrants are obtained by reflection about the x- and y-axes.

This is precisely the point A on the arc ECF for which A' is the centroid of ABC. This triangle ABC is the unique triangle (up to reflection about BC and about the perpendicular bisector of BC) whose vertex angle at A is θ and whose centroid G has the property that $\angle BGC = 2\angle BAC$. □

Theorem 9 characterizes those triangles whose centroid has the property $BB^* = CC^*$. For the proof, we need the following simple lemma.

Lemma 8. *Let ABC be a triangle with side-lengths a, b, and c* (in the standard order) *and with centroid G. Let the rays BG and CG meet the circumcircle of ABC at B^* and C^* respectively. Then*

$$BB^{*2} = \frac{(a^2+c^2)^2}{2a^2+2c^2-b^2}.$$

Proof. Let $m = BB'$, $x = BB^*$. By Apollonius' theorem, $m^2 = \frac{2(a^2+c^2)-b^2}{4}$. Since $BB'B^*$ and $AB'C$ are diagonals of a cyclic quadrilateral, $m(x-m) = \frac{b^2}{4}$. It follows that $mx = \frac{a^2+c^2}{2}$ and $x^2 = \frac{(a^2+c^2)^2}{4m^2} = \frac{(a^2+c^2)^2}{2a^2+2c^2-b^2}$. □

Theorem 9. *Let ABC be a triangle with side-lengths a, b, and c* (in the standard order) *and with centroid G. Let the rays BG and CG meet the circumcircle of ABC at B^* and C^*, respectively. If $b \neq c$, then the following are equivalent:*
(i) $BB^* = CC^*$,
(ii) $\angle BGC = 2\angle BAC$,
(iii) $a^4 = b^4 + c^4 - b^2c^2$.

Proof. Since $b \neq c$, it follows that $GB \neq GC$. By Lemma 6, (i) is equivalent to (ii). To see that (i) is equivalent to (iii), let $x = BB^*$, $y = CC^*$, and let $s = a^2 + b^2 + c^2$. By Lemma 8,

$$x^2 = \frac{(s-b^2)^2}{2s-3b^2}, \qquad y^2 = \frac{(s-c^2)^2}{2s-3c^2}.$$

Therefore

$$\begin{aligned}
x = y \iff & \frac{(s-b^2)^2}{2s-3b^2} = \frac{(s-c^2)^2}{2s-3c^2} \\
\iff & (s^2 - 2b^2 s + b^4)(2s - 3c^2) = (s^2 - 2c^2 s + c^4)(2s - 3b^2) \\
\iff & s^2(c^2 - b^2) - 2s(c^2 - b^2)(c^2 + b^2) + 3c^2 b^2 (c^2 - b^2) = 0 \\
\iff & s^2 - 2s(c^2 + b^2) + 3c^2 b^2 = 0 \qquad \text{(because } b \neq c) \\
\iff & (s - (c^2 + b^2))^2 = (c^2 + b^2)^2 - 3c^2 b^2 \\
\iff & a^4 = c^4 + b^4 - c^2 b^2,
\end{aligned}$$

as claimed. □

Remarks. (1) It follows from [1, Theorem 2.3.3., page 83] (or [9, page 20]) that the only positive solutions of the diophantine equation

$$a^4 + b^4 - a^2 b^2 = c^4 \tag{2}$$

are given by $a = b = c$. Thus there are no non-isosceles triangles ABC with integer side-lengths whose centroid G has the property $BB^* = CC^*$.

(2) A Euclidean construction, provided by a referee, of triangles ABC whose centroid has the property $BB^* = CC^*$. We start with any segment BC.

(i) Take any point A' on the major arc BA_0C of an equilateral triangle A_0BC.

(ii) Extend $A'C$ and $A'B$ to Y and Z respectively such that $CY = BZ = BC$.

(iii) Construct a circle with diameter $A'Z$ and the perpendicular at B to $A'Z$, intersecting the circle at B'

(iii′) Construct a circle with diameter $A'Y$ and the perpendicular at C to $A'Y$, intersecting the circle at C'

(iv) Construct the circles centered at B and C and passing through B' and C', respectively.

Letting A be a point of intersection of the two circles in (iv), one can verify that triangle ABC satisfies $BB^* = CC^*$.

(3) With reference to the previous remark and in view of Theorem 7(ii), one may ask whether one can construct a triangle ABC having the property $BB^* = CC^*$ and having preassigned side BC and angle A (in $[\frac{\pi}{3}, \frac{\pi}{2}]$). The answer is affirmative as seen below.

Without loss of generality, assume $BC = 1$. Let $b = AC$, $c = AB$, and $t = \cos A$. We are to show that b and c are constructible. These are defined by

$$b^4 + c^4 - b^2 c^2 = 1\ ,\ b^2 + c^2 = 2bct + 1.$$

Subtracting the square of the second from the first and simplifying, we obtain $bc = \frac{4t}{3-4t^2}$. Thus bc is constructible. Since $b^2 + c^2 = 2bct + 1$, it follows that $b^2 + c^2$ is constructible. Thus both b^2c^2 and $b^2 + c^2$ are constructible, and hence b^2 and c^2, being the zeros of $f(T) := T^2 - (b^2 + c^2)T + b^2c^2$, are constructible. This shows that b and c are constructible, as desired. The restriction $A \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$, *i.e.*, $t \in \left[0, \frac{1}{2}\right]$, guarantees that the zeros of $f(T)$ are real (and positive).

References

[1] T. Andreescu and D. Andrica, *An Introduction to Diophantine Equations*, GIL Publishing House, Zalau, Romania, 2002.
[2] Euclid, *The Elements*, Sir Thomas L. Heath, editor, Dover Publications, Inc., New York, 1956.
[3] Euclid's Elements, aleph0.clarku.edu/˜djoyce/mathhist/alexandria.html
[4] M. Hajja, Problem 1767, *Math. Mag.*, 80 (2007), 145; solution, ibid., 81 (2008), 137.
[5] M. Hajja, Cyril F. Parry's variations on the Steiner-Lehmus theme, Preprint.
[6] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.
[7] C. Kimberling, Encyclopaedia of Triangle Centers, http://faculty.evansville.edu/ck6/encyclopedia/ETC.html
[8] R. S. Millman and G. D. Parker, *Geometry – A Metric Approach with Models*, second edition, Springer-Verlag, New York, 1991.
[9] L. J. Mordell, *Diophantine Equations*, Academic Press, New York, 1969.
[10] B. M. Stewart, *Theory of Numbers*, second edition, The Macmillan Co., New York, 1964.

Sadi Abu-Saymeh: Mathematics Department, Yarmouk University, Irbid, Jordan
E-mail address: `sade@yu.edu.jo , ssaymeh@yahoo.com`

Mowaffaq Hajja: Mathematics Department, Yarmouk University, Irbid, Jordan
E-mail address: `mhajja@yu.edu.jo, mowhajja@yahoo.com`

Hassan Ali ShahAli: Fakultät für Mathematik und Physik, Leibniz Universität, Hannover, Welfengarten 1, 30167 Hannover, Germany

Forum Geometricorum
Volume 8 (2008) 141–145.

FORUM GEOM
ISSN 1534-1178

Haruki's Lemma for Conics

Yaroslav Bezverkhnyev

Abstract. We extend Haruki's lemma to conics.

1. Main results

In this paper we continue to explore Haruki's lemma introduced by Ross Honsberger in [2, 3]. In [1], we gave an extension of Haruki's lemma (Theorem 1 below) and studied a related locus problem, leading to certain interesting conics. [1]

Theorem 1 ([1, Lemma 2]). *Given two nonintersecting chords AB and CD in a circle and a variable point P on the arc AB remote from points C and D, let E and F be the intersections of chords PC, AB, and of PD, AB respectively. The following equalities hold:*

$$\frac{AE \cdot BF}{EF} = \frac{AC \cdot BD}{CD}, \tag{1}$$

$$\frac{AF \cdot BE}{EF} = \frac{AD \cdot BC}{CD}. \tag{2}$$

In this paper we generalize this result to conics.

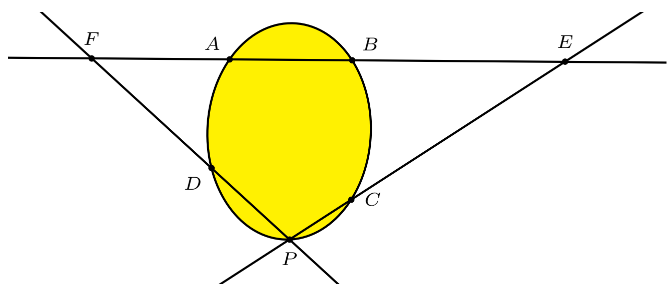


Figure 1.

Theorem 2. *Given a nondegenerate conic $\mathcal{C}$ with fixed points A, B, C, D on it, let P be a variable point distinct from A and B. Let E and F be the intersections of the lines PC, AB, and of PD, AB respectively. Then the ratios $\dfrac{AE \cdot BF}{EF}$ and $\dfrac{AF \cdot BE}{FE}$ are independent of the choice of P.*

Publication Date: June 23, 2008. Communicating Editor: Paul Yiu.

The author wishes to thank Paul Yiu for his invaluable additions and help with the preparation of the article.

[1]See Remark following the proof of Theorem 2 below.

It turns out that this result still holds when the points A and B coincide. In this case, we replace the line AB by the tangent to the conic at A. With a minor change of notations, we have the following result.

Theorem 3. *Given a nondegenerate conic $\mathcal{C}$ with fixed points A, B, C on it, let P be a variable point distinct from A. Let E and F be the intersections of the lines PB, PC with the tangent to the conic at A. Then the ratio $\dfrac{AE \cdot AF}{EF}$ is independent of the choice of P.*

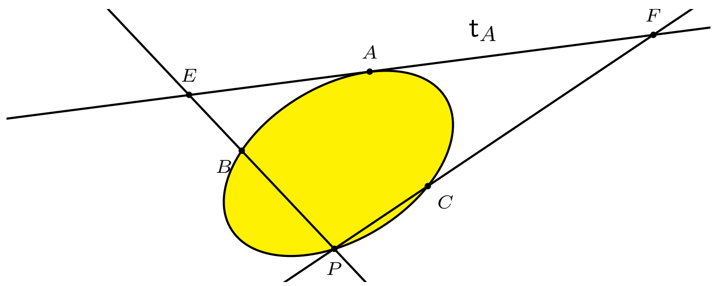


Figure 2

2. Proof of Theorem 2

We choose ABC as reference triangle. The nondegenerate conic $\mathcal{C}$ has equation of the form

$$fyz + gzx + hxy = 0 \tag{3}$$

for nonzero constants f, g, h. See Figure 1. Suppose D has homogeneous barycentric coordinates $(u : v : w)$, *i.e.*,

$$fvw + gwu + huv = 0. \tag{4}$$

Clearly, u, v, w are all nonzero. For an arbitrary point P with barycentric coordinates $(x : y : z)$, the coordinates of the intersections $E = AB \cap DC$ and $F = AB \cap PD$ can be easily determined:

$$E = (x : y : 0), \qquad F = (uz - wx : vz - wy : 0).$$

See [1, §6]. From these, we have the signed lengths of the various relevant segments:

$$\begin{aligned} AE &= \tfrac{y}{x+y} \cdot c, & EB &= \tfrac{x}{x+y} \cdot c, \\ AF &= \tfrac{vz-wy}{z(u+v)-w(x+y)} \cdot c, & FB &= \tfrac{uz-wx}{z(u+v)-w(x+y)} \cdot c, \\ EF &= \tfrac{z(vx-uy)}{(x+y)(z(u+v)-w(x+y))} \cdot c, \end{aligned}$$

where $c = AB$. It follows that $\dfrac{AE \cdot BF}{EF} = \dfrac{y(wx - uz)}{z(vx - uy)} \cdot c$. To calculate this fraction, note that from (4), we have $\frac{fw}{h} = -u(1 + k)$ for $k = \frac{gw}{hv}$. Now, from (3), we have

$$\begin{aligned}
&\frac{fw}{h}\cdot yz+\frac{gw}{h}\cdot zx+w\cdot xy=0,\\
&-u(1+k)yz+kvzx+wxy=0,\\
&y(wx-uz)+kz(vx-uy)=0.
\end{aligned}$$

Hence, $\dfrac{AE\cdot BF}{EF}=\dfrac{y(wx-uz)}{z(vx-uy)}\cdot c=-kc$, a constant.

A similar calculation gives $\dfrac{AF\cdot BE}{FE}=(1+k)c$, a constant. This completes the proof of the theorem.

Remark. Note that we have actually proved that

$$\frac{AE\cdot BF}{EF}=-\frac{gw}{hv}\cdot c\quad\text{and}\quad\frac{AF\cdot BE}{FE}=-\frac{fw}{hu}\cdot c.$$

In [1, Theorem 6], we have solved two loci problems in connection with Haruki's lemma. Denote, in Figure 1, $BC=a$, $CA=b$, $AB=c$, and $AD=a'$, $BD=b'$, $CD=c'$. The locus of points P satisfying (1) is the union of the two circumconics of $ABCD$

$$(cc'+\varepsilon bb')uyz-\varepsilon bb'vzx-cc'wxy=0,\quad \varepsilon=\pm1.$$

Now, with

$$f=(cc'+\varepsilon bb')u,\qquad g=-\varepsilon bb'v,\qquad h=-cc'w,$$

we have

$$\frac{AE\cdot BF}{EF}=-\frac{-\varepsilon bb'vw}{-cc'wv}\cdot c=-\varepsilon\cdot\frac{bb'}{c'}=\varepsilon\cdot\frac{AC\cdot BD}{CD}.$$

Similarly, the locus of points P satisfying (2) is the union of the two circumconics of $ABCD$

$$\varepsilon aa'uyz+(cc'-\varepsilon aa')vzx-cc'wxy=0,\quad \varepsilon=\pm1.$$

Now, with

$$f=\varepsilon aa'u,\qquad g=(cc'-\varepsilon aa')v,\qquad h=-cc'w,$$

we have

$$\frac{AF\cdot BE}{FE}=-\frac{fw}{hu}\cdot c=-\frac{\varepsilon aa'uw}{-cc'wu}\cdot c=\varepsilon\cdot\frac{aa'}{c'}=-\varepsilon\cdot\frac{AD\cdot BC}{DC}.$$

These confirm that Theorem 2 is consistent with Theorem 6 of [1].

3. Proof of Theorem 3

Again, we choose ABC as the reference triangle, and write the equation of the nondegenerate conic $\mathcal{C}$ in the form (3) with $fgh \neq 0$. The tangent at A is the line

$$\mathtt{t}_A: \qquad hy + gz = 0.$$

For an arbitrary point P with homogeneous barycentric coordinates $(x : y : z)$, the lines PB and PC intersect $\mathtt{t}_A$ respectively at

$$\begin{aligned} E =& (hx : -gz : hz), \\ F =& (gx : gy : -hy). \end{aligned}$$

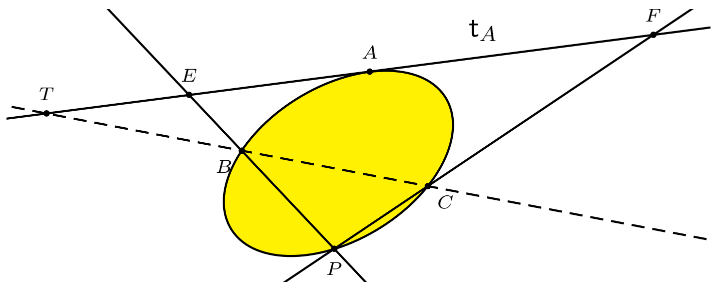


Figure 3

On the tangent line there is the point $T = (0 : -g : h)$, the intersection with the line BC. It is clearly possible to express the points E and F in terms of A and T. In fact, from

$$\begin{aligned} (hx, -gz, hz) =& hx(1, 0, 0) - z(0, g, -h), \\ (gx, gy, -hy) =& gx(1, 0, 0) + y(0, g, -h), \end{aligned}$$

we have, in absolute barycentric coordinates,

$$\begin{aligned} E =& \frac{hx}{hx - (g-h)z} \cdot A + \frac{-(g-h)z}{hx - (g-h)z} \cdot T, \\ F =& \frac{gx}{gx + (g-h)y} \cdot A + \frac{(g-h)y}{gx + (g-h)y} \cdot T. \end{aligned}$$

From these,

$$\frac{AE}{AT} = \frac{-(g-h)z}{hx - (g-h)z}, \qquad \frac{AF}{AT} = \frac{(g-h)y}{gx + (g-h)y}.$$

It follows that

$$\begin{aligned} \frac{EF}{AT} = \frac{AF - AE}{AT} &= \frac{(g-h)y}{gx + (g-h)y} + \frac{(g-h)z}{hx - (g-h)z} \\ &= \frac{(g-h)x(hy + gz)}{(gx + (g-h)y)(hx - (g-h)z)}. \end{aligned}$$

Therefore,

$$\frac{AE \cdot AF}{EF} = \frac{-(g-h)z \cdot (g-h)y}{(g-h)x(hy+gz)} \cdot AT = \frac{-(g-h)yz}{gzx+hxy} \cdot AT$$
$$= \frac{-(g-h)yz}{-fyz} \cdot AT = \frac{g-h}{f} \cdot AT.$$

This is independent of the choice of the point $P(x : y : z)$ on the conic. This completes the proof of Theorem 3.

References

[1] Y. Bezverkhynev, Haruki's lemma and a related locus problem, *Forum Geom.*, 8 (2008) 63–72.

[2] R. Honsberger, The Butterfly Problem and Other Delicacies from the Noble Art of Euclidean Geometry I, *TYCMJ*, 14 (1983) 2 – 7.

[3] R. Honsberger, *Mathematical Diamonds*, Dolciani Math. Expositions No. 26, Math. Assoc. Amer., 2003.

Yaroslav Bezverkhnyev: Main Post Office, P/O Box 29A, 88000 Uzhgorod, Transcarpathia, Ukraine

E-mail address: slavab59@yahoo.ca

Forum Geometricorum
Volume 8 (2008) 147–148.

FORUM GEOM
ISSN 1534-1178

A Simple Compass-Only Construction of the Regular Pentagon

Kurt Hofstetter

Abstract. In 7 steps we give a simple compass-only (Mascheroni) construction of the vertices of a regular pentagon .

In [1] we have given a simple 5-step compass-only (Mascheroni) construction of the golden section. Here we note that with two additional circles, it is possible to construct the vertices of a regular pentagon. As usual, we denote by $P(Q)$ the circle with center P and passing through Q.

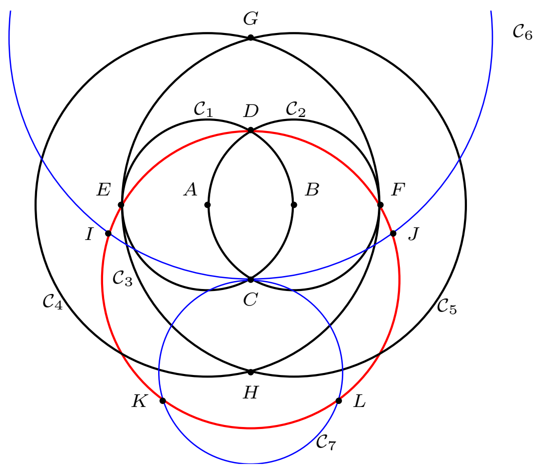


Figure 1

Construction 1. *Given two points A and B,*
(1) $\mathcal{C}_1 = A(B)$,
(2) $\mathcal{C}_2 = B(A)$ *to intersect* $\mathcal{C}_1$ *at* C *and* D,
(3) $\mathcal{C}_3 = C(D)$ *to intersect* $\mathcal{C}_1$ *at* E *and* $\mathcal{C}_2$ *at* F,
(4) $\mathcal{C}_4 = A(F)$,
(5) $\mathcal{C}_5 = B(E)$ *to intersect* $\mathcal{C}_4$ *at* G *and* H.
(6) $\mathcal{C}_6 = G(C)$ *to intersect* $\mathcal{C}_3$ *at* I *and* J,
(7) $\mathcal{C}_7 = H(C)$ *to intersect* $\mathcal{C}_3$ *at* K *and* L.

Then $DIKLJ$ is a regular pentagon.

Publication Date: August 8, 2008. Communicating Editor: Paul Yiu.

Proof. In [1] we have shown that the first five steps above lead to four collinear points C, D, G, H such that D divides CG, and C divides DH, in the golden section.

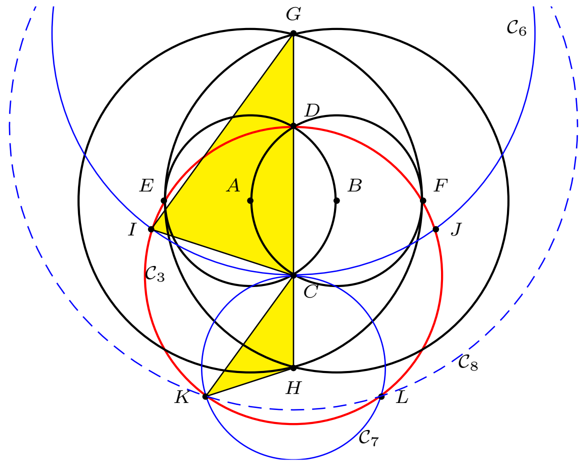


Figure 2

(i) This means that in the isosceles triangle GCI, $\frac{GC}{IC} = \frac{GC}{DC} = \phi$. The base angles are $72°$. Therefore, $\angle DCI = 72°$. By symmetry, $\angle DCJ = 72°$.

(ii) Also, in the isosceles triangle HCK, $\frac{KC}{CH} = \frac{DC}{CH} = \phi$. The base angles are $36°$. It follows that $\angle KCH = 36°$. By symmetry, $LCH = 36°$, and $KCL = 72°$.

(iii) Since C is on the line GH, $\angle ICK = 180° - \angle GCI - \angle KCH = 72°$. By symmetry, $\angle JCL = 72°$.

Therefore, the five points D, I, K, L, J are equally spaced on the circle $\mathcal{C}_3$. They form the vertices of a regular pentagon. □

Remark. The circle $\mathcal{C}_7$ can be replaced by $\mathcal{C}_8$ with center D and radius IJ. This intersects $\mathcal{C}_3$ at the same points K and L.

Reference

[1] K. Hofstetter, A simple construction of the golden section, *Forum Geom.*, 2 (2002) 65–66.

Kurt Hofstetter: Object Hofstetter, Media Art Studio, Langegasse 42/8c, A-1080 Vienna, Austria
E-mail address: hofstetter@sunpendulum.at

Forum Geometricorum
Volume 8 (2008) 149–150.

FORUM GEOM
ISSN 1534-1178

Two More Powerian Pairs in the Arbelos

Quang Tuan Bui

Abstract. We construct two more pairs of Archimedes circles analogous to those of Frank Power, in addition to those by Floor van Lamoen and the author.

Consider an arbelos with semicircles (O), (O_1), (O_2) with diameters AB, AC, BC as diameters respectively. Denote by r_1 and r_2 respectively the radii of (O_1) and (O_2), and D the intersection of (AB) with the perpendicular to AB at C. If P is a point such that $OP^2 = r_1^2 + r_2^2$, then the circles tangent to (O) and to OP at P are Archimedean. Examples were first given in Power [3], subsequently also in [1, 2].

We construct two more Powerian pairs.

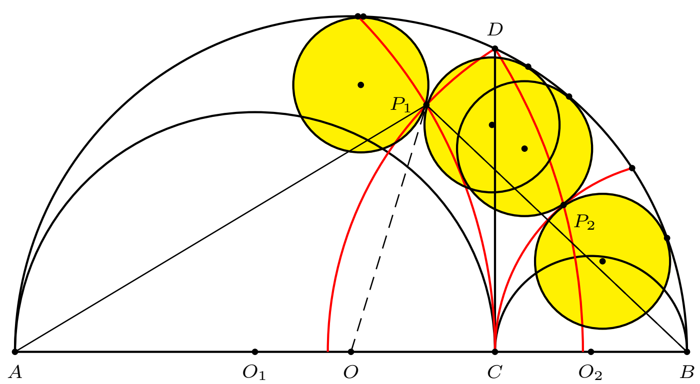


Figure 1

Let P_1 be the intersection of the circles $A(C)$ and $B(D)$. Consider OP_1 as a median of triangle P_1AB, we have, by Apollonius' theorem (see, for example, [4]),

$$\begin{aligned} OP_1^2 &= \frac{1}{2}\left(AP_1^2 + BP_1^2\right) - OA^2 \\ &= \frac{1}{2}\left((2r_1)^2 + 2r_2 \cdot 2(r_1 + r_2)\right) - (r_1 + r_2)^2 \\ &= r_1^2 + r_2^2. \end{aligned}$$

Similarly, for P_2 the intersection of $B(C)$ and $A(D)$, $OP_2^2 = r_1^2 + r_2^2$. Therefore, we have two Powerian pairs at P_1, P_2.

Publication Date: August 15, 2008. Communicating Editor: Paul Yiu.

References

[1] Q. T. Bui, The arbelos and nine-point circles, *Forum Geom.*, 7 (2007) 115–120.
[2] F. M. van Lamoen, Some more Powerian pairs in the arbelos, *Forum Geom.*, 7 (2007) 111–113.
[3] F. Power, Some more Archimedean circles in the arbelos, *Forum Geom.*, 5 (2005) 133–134.
[4] P. Yiu, *Euclidean Geometry*, Florida Atlantic University Lecture Notes, 1998, http://www.math.fau.edu/Yiu/Geometry.html.

Quang Tuan Bui: 45B, 296/86 by-street, Minh Khai Street, Hanoi, Vietnam
E-mail address: bqtuan1962@yahoo.com

Forum Geometricorum
Volume 8 (2008) 151–155.

FORUM GEOM
ISSN 1534-1178

On the Generalized Gergonne Point and Beyond

Miklós Hoffmann and Sonja Gorjanc

Abstract. In this paper we further extend the generalization of the concept of Gergonne point for circles concentric to the inscribed circle. Given a triangle $V_1V_2V_3$, a point I and three arbitrary directions q_1, q_2, q_3 from I, we find a distance $x = IQ_1 = IQ_2 = IQ_3$ along these directions, for which the three cevians V_iQ_i are concurrent. Types and number of solutions, which can be obtained by the common intersection points of three conics, are also discussed in detail.

1. Introduction

The Gergonne point is a well-known center of the triangle. It is the intersection of the three cevians defined by the touch points of the inscribed circle [3]. Konečný [1] has generalized this to circles concentric with the inscribed circle. Let $\mathcal{C}(I)$ be a circle with center I, the incenter of triangle $V_1V_2V_3$. Let Q_1, Q_2, Q_3 be the points of intersection of $\mathcal{C}(I)$ with the lines from I that are perpendicular to the sides V_2V_3, V_3V_1, V_1V_2 respectively. Then the lines V_iQ_i, $i = 1, 2, 3$, are concurrent (see Figure 1).

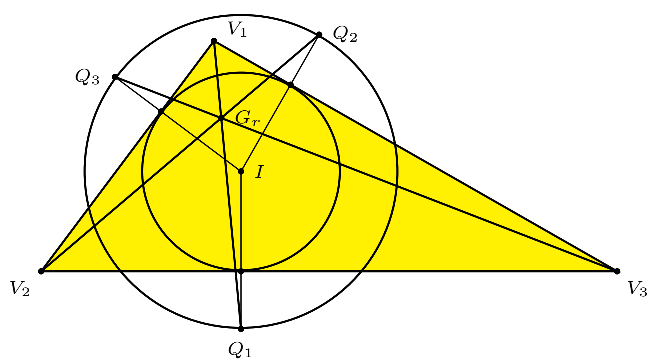


Figure 1. Lines V_iQ_i are also concurrent for circles concentric to the inscribed circle

The first question naturally arises: if the radius of the circle is altered, what will be the locus of the point G_r? Boyd and Raychowdhury [4] computed the convex coordinates of G_r, from which it is clear that the locus is a hyperbola.

Now instead of the inscribed circle consider an inscribed conic (see Figure 2). The lines V_iQ_i, $i = 1, 2, 3$, are still concurrent, at a point called the Brianchon point of the conic (c.f. [5]). There are infinitely many inscribed conics, thus the center

Publication Date: September 2, 2008. Communicating Editor: Paul Yiu.

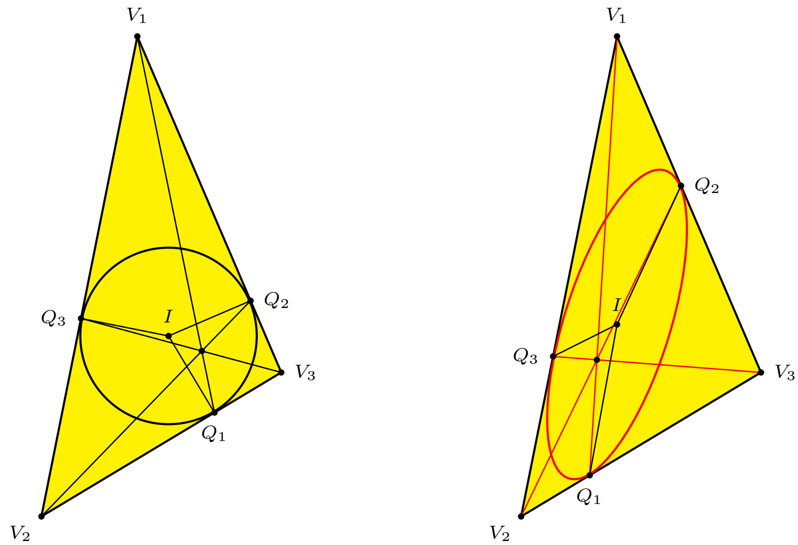


Figure 2. Inscribed ellipse and its directions IQ_i

I and directions q_i (corresponding to the line connecting I and Q_i) can be chosen in many ways, but not arbitrarily. Note that the center completely determines the inscribed conic and the points of tangency Q_1, Q_2, Q_3.

Using these directions we generalize the concept of concentric circles: given a triangle $V_1V_2V_3$ and an inscribed conic with center I and touch points Q_1, Q_2, Q_3, consider the three lines q_i connecting I and Q_i respectively. A circle with center I has to be found which meets the lines q_i at $\bar{Q}_i$ such that the lines $V_i\bar{Q}_i$, $i = 1, 2, 3$, are concurrent. In fact, as we will see in the next section, we do not have to restrict ourselves in terms of the position of the center and the given directions.

2. The general problem and its solution

The general problem can be formulated as follows: given a triangle $V_1V_2V_3$, a point I and three arbitrary directions q_i, find a distance $x = IQ_1 = IQ_2 = IQ_3$ along these directions, for which the three cevians V_iQ_i are concurrent. In general these lines will not meet in one point (see Figure 3): instead of one single center G we have three different intersection points G_{12}, G_{13} and G_{23}.

In the following theorem we will prove that altering the value x, the points G_{12},G_{13} and G_{23} will separately move on three conics. If there is a solution to our generalized problem, it would mean that these conics have to meet in one common point. It is easy to observe that each pair of conics have two common points at I and a vertex of the triangle. Here we prove that the other two intersection points can be common for all the three conics. Previously mentioned special cases are excluded from this point.

Theorem 1. *Let V_1, V_2, V_3 and I be four points in the plane in general positions. Let q_1, q_2, q_3 be three different oriented lines through I ($V_i \notin q_i$). There exist*

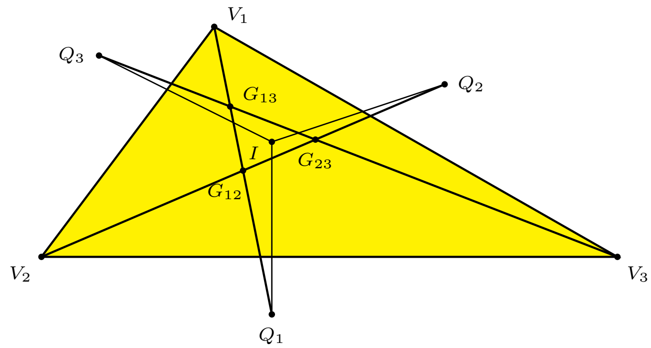


Figure 3. For arbitrary directions and distance, cevians V_iQ_i are generally not concurrent, but meet at three different points

at most two values $x \in \mathbb{R}\backslash\{0\}$ *such that for points* Q_i *along the lines* q_i *with* $IQ_1 = IQ_2 = IQ_3 = x$*, the lines* V_iQ_i *are concurrent.*

Proof. For a real number x and $i = 1, 2, 3$, let $Q_i(x)$ be a point on q_i for which $IQ_i(x) = x$. The correspondences $Q_i(x) \leftrightarrow Q_j(x)$ define perspectivities $(q_i) \overline{\overline{\wedge}} (q_j)$, $(i \neq j)$.

Now let $l_i(x)$ be the line connecting V_i and $Q_i(x)$. The correspondences $l_i(x) \leftrightarrow l_j(x)$ define projectivities $(V_i) \overline{\wedge} (V_j)$, $(i \neq j)$. The intersection points of corresponding lines of these projectivities lie on three conics:

$$\begin{aligned} (V_1) \overline{\wedge} (V_2) &\quad\Rightarrow\quad c_3 \\ (V_1) \overline{\wedge} (V_3) &\quad\Rightarrow\quad c_2 \\ (V_2) \overline{\wedge} (V_3) &\quad\Rightarrow\quad c_1. \end{aligned}$$

We find the intersection points of these conics. Since $Q_i(0) = I$, then $I \in c_i, (i = 1, 2, 3)$. $V_3 \in c_1 \cap c_2$, $V_2 \in c_1 \cap c_3$ and $V_1 \in c_2 \cap c_3$ also hold. Denote the other two intersection points of c_1 and c_2 by S_1 and S_2, *i.e.*,

$$c_2 \cap c_3 = \{I, V_1, S_1, S_2\}.$$

The points S_1 and S_2 can be real and distinct, real and identical, or imaginary in pair.

(i) If they are real and distinct, then for some x_1 and x_2,

$$\begin{aligned} S_1 = l_1(x_1) \cap l_2(x_1) = l_1(x_1) \cap l_3(x_1) &\quad\Rightarrow\quad S_1 = l_2(x_1) \cap l_3(x_1) \\ S_2 = l_1(x_2) \cap l_2(x_2) = l_1(x_2) \cap l_3(x_2) &\quad\Rightarrow\quad S_2 = l_2(x_2) \cap l_3(x_2) \end{aligned}$$

which immediately yields $S_1, S_2 \in c_1$ as well.

(ii) If they are identical, then for the unique x,

$$S = l_1(x) \cap l_2(x) = l_1(x) \cap l_3(x) \quad\Rightarrow\quad S = l_2(x) \cap l_3(x)$$

which yields $S \in c_1$.

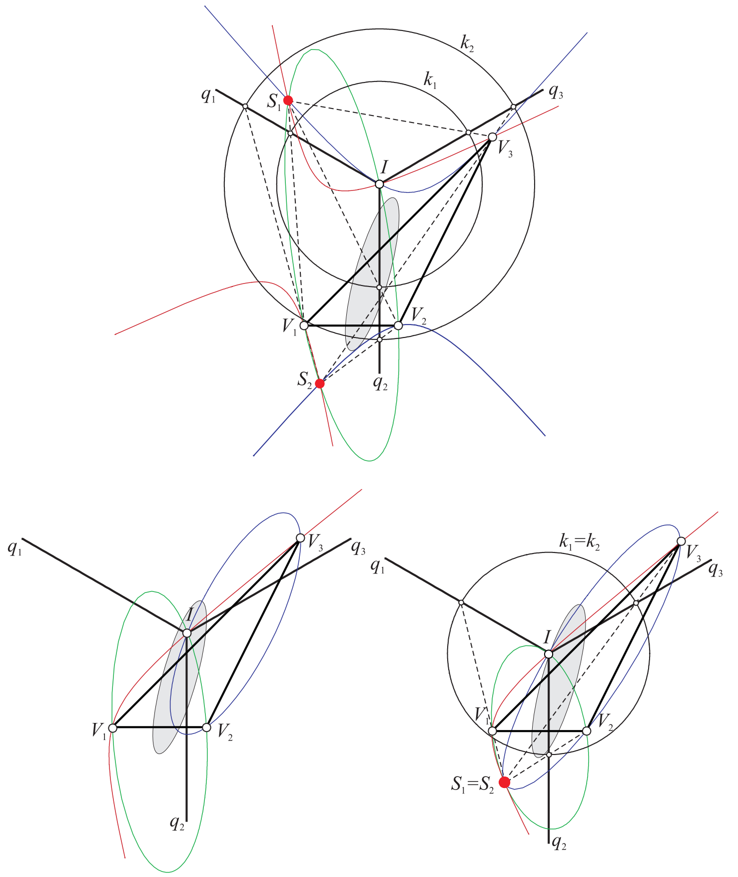


Figure 4. Given a triangle $V_1V_2V_2$ and directions $q_i (i = 1, 2, 3)$ there can be two different real solutions (upper figure), two coinciding solutions (bottom right) and two imaginary solutions (bottom left). Cevians are plotted by dashed lines. The type of solutions depends on the relative position of I to the shaded conic. The three conic paths of G_{12} (green), G_{13} (red) and G_{23} (blue) are also shown. (This figure is computed and plotted by the software *Mathematica*)

(iii) If the points S_1 and S_2 are the pair of imaginary points there are no real number x for which the lines V_iQ_i are concurrent. $\square$

Figure 4 shows the three different possibilities mentioned in the proof. If the triangle and the directions q_i, $i = 1, 2, 3$, are fixed, then the radius of the circle can be obtained by the solutions of a quadratic equation in which the only unknown is the point I. The type of the solutions depends on the discriminant, which is a quadratic function of I. This means that for every triangle and triple of directions there exists a conic which separates the possible positions of I in the following way: if I is outside the conic (discriminant > 0) then there are two different real solutions, if I is on the conic (discriminant $= 0$) then there are two coinciding real solutions, while if I is inside the conic (discriminant < 0) then there are two imaginary solutions. This conic is also shown in Figure 4.

Remarks. (1) Note that there are no further restrictions for the positions of the center and the directed lines. The center can even be outside the reference triangle.

(2) According to the projective principles in the proof, the statement remains valid if we replace the condition $IQ_1 = IQ_2 = IQ_3 = x$ with the more condition that the ratios of these lengths be fixed.

3. Further research

The conics c_i, $i = 1, 2, 3$, play a central role in the proof. The affine types of these conics however, can only be determined by analytical approach or by closer study the type of involutive pencils determined by cevians. It is also a topic of further research how the types of solutions depend on the ratios mentioned in Remark 2. The exact representation of the length of the radius by the given data can also be discussed analytically in a further study.

References

[1] V. Konečný, J. Heuver, and R. E. Pfiefer, Problem 1320 and solutions, *Math. Mag.*, 62 (1989) 137; 63 (1990) 130–131.

[2] B. Wojtowicz, Desargues' Configuration in a Special Layout, *Journal for Geometry and Graphics*, 7 (2003) 191–199.

[3] W. Gallatly, *The modern geometry of the triangle*, Hodgson Publisher, London, 1910.

[4] J. N. Boyd and P. N. Raychowdhury, P.N.: The Gergonne point generalized through convex coordinates, *Int. J. Math. Math. Sci.*, 22 (1999), 423–430.

[5] O. Veblen and J. W. Young, *Projective Geometry*, Boston, MA, Ginn, 1938.

Miklós Hoffmann: Institute of Mathematics and Computer Science, Károly Eszterházy College, Eger, Hungary

E-mail address: `hofi@ektf.hu`

Sonja Gorjanc: Department of Mathematics, Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia

Forum Geometricorum
Volume 8 (2008) 157–161.

FORUM GEOM
ISSN 1534-1178

Stronger Forms of the Steiner-Lehmus Theorem

Mowaffaq Hajja

Abstract. We give a short proof based on Breusch's lemma of a stronger form of the Steiner-Lehmus theorem, and we discuss other possible stronger forms.

1. A stronger form of Steiner-Lehmus Theorem

Let a, b, c, A, B, C denote, in the standard manner, the side lengths and angles of a triangle ABC. An elegant lemma that was designed by Robert Breusch for solving an interesting 1961 MONTHLY problem [4] states that

$$\frac{p(ABC)}{a} = \frac{2}{1 - \tan(B/2)\tan(C/2)}, \tag{1}$$

where $p(\dots)$ denotes the perimeter. Its simple proof is reproduced in [1], where it is used to give a very short proof of a theorem of Urquhart.

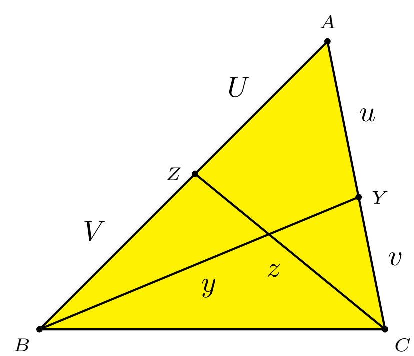


Figure 1

We now consider the Steiner-Lehmus configuration shown in Figure 1, where BY and CZ are the internal angle bisectors of angles B and C. Applying Breusch's lemma to triangles YBC and ZBC, we obtain

$$\frac{p(YBC)}{p(ZBC)} = \frac{1 - \tan(B/2)\tan(C/4)}{1 - \tan(B/4)\tan(C/2)}.$$

Publication Date: September 8, 2008. Communicating Editor: Nikolaos Dergiades.

The author would like to thank Yarmouk University for supporting this work, and Nikolaos Dergiades for his valuable comments and additions. Nikolaos is responsible for much of §2. Specifically, he is responsible for proving (8), which appeared as a conjecture in the first draft, and for strengthening the conjecture in (13) by testing more triangles and by introducing and evaluating the limit in (12).

If $c > b$, then

$$\tan\frac{C}{2}\ \tan\frac{B}{4} = \frac{2\tan(B/4)\tan(C/4)}{1-\tan^2(C/4)} > \frac{2\tan(B/4)\tan(C/4)}{1-\tan^2(B/4)} = \tan\frac{B}{2}\ \tan\frac{C}{4},$$

and therefore $p(YBC) > p(ZBC)$. Letting

$$|BY| = y\ ,\ |CZ| = z\ , |AZ| = U\ ,\ |ZB| = V\ ,\ |AY| = u\ ,\ |YC| = v,$$

we have proved the stronger form

$$c > b \Longleftrightarrow y + v > z + V \tag{2}$$

of the traditional Steiner-Lehmus theorem

$$c > b \Longleftrightarrow y > z. \tag{3}$$

To see that (2) is indeed stronger than (3), we need to show that $V > v$. By the angle bisector theorem, we have $\frac{V}{U} = \frac{a}{b}$. Therefore, $\frac{V}{V+U} = \frac{a}{a+b}$, and $V = \frac{ac}{a+b}$. A similar formula holds for v. Thus we have

$$V = \frac{ac}{a+b}\ ,\ v = \frac{ab}{a+c}, \tag{4}$$

and

$$V - v \quad = \quad \frac{ac}{a+b} - \frac{ab}{a+c} \ = \ \frac{a(c(a+c) - b(b+c))}{(a+b)(a+c)} \ = \ \frac{a(a+b+c)(c-b)}{(a+b)(a+c)}$$

Thus

$$c > b \Longrightarrow V > v, \tag{5}$$

and (2) is stronger than (3).

It follows from (4) that

$$U = \frac{bc}{a+b}\ ,\ u = \frac{bc}{a+c}, \tag{6}$$

and therefore

$$c > b \Longrightarrow U > u.$$

Thus the statement

$$c > b \Longrightarrow y + b > z + c \tag{7}$$

would be stronger, and more pleasant, than (2). Unfortunately, (7) is not true. In fact, a recent MONTHLY problem [3] states that if $a \geq c > b$, then the reverse inequality $y + b < z + c$ holds.

2. Additive stronger forms

2.1. One then wonders about the statement

$$c > b \Longrightarrow y + u > z + U. \tag{8}$$

This is also stronger than the classical form (3). In order to prove (8), since $c > b \Longrightarrow y > z$ and $U > u$, it is sufficient to prove that

$$\frac{y^2 - z^2}{U - u} > y + z,\ \text{or}\ \left(\frac{y^2 - z^2}{U - u}\right)^2 > (y+z)^2,\ \text{or}\ \left(\frac{y^2 - z^2}{U - u}\right)^2 > 2(y^2 + z^2),$$

or

$$\frac{a(a+b+c)(a^8+s_7a^7+\cdots+s_1a+s_0)}{b^2c^2(a+c)^2(a+b)^2}>0,$$

which is true because

$$\begin{array}{ll} s_7=3(b+c), & s_6=3(b^2+4bc+c^2), \\ s_5=b^2+16bc+16c^2, & s_4=bc(2b+5c)(5b+2c), \\ s_3=bc(b+c)(2b^2+17bc+2c^2), & s_2=5b^2c^2(b+c)^2, \\ s_1=b^2c^2(b+c)(b^2+c^2), & s_0=2b^3c^3(b-c)^2. \end{array}$$

Combining (8) with (2) would yield the form

$$c>b\Longrightarrow y+\frac{b}{2}>z+\frac{c}{2}$$

or

$$c>b\Longrightarrow y-z>\frac{1}{2}(c-b). \tag{9}$$

2.2. In all cases, it is interesting to investigate the best constant λ for which

$$c>b\Longrightarrow y-z\geq\lambda(c-b). \tag{10}$$

Similar questions can be asked about the best constants in

$$c>b\Longrightarrow y-z\geq\lambda(U-u),\quad c>b\Longrightarrow y-z\geq\lambda(V-v). \tag{11}$$

These may turn out to be quite easy given available computer packages such as BOTTEMA. For example, the stronger form $c>b\Longrightarrow y-z>0.8568(c-b)$ of (9) was verified for all triangles whose side lengths are integers less than 51. The minimum value 0.8568 of the fraction $\frac{y-z}{c-b}$, verified for all triangles whose side lengths are integers less than 51, is attained for $(a,b,c)=(48,37,38)$, and the minimum value 0.856762, verified for all triangles whose side lengths are integers less than 501, is attained for $(a,b,c)=(499,388,389)$. Hence one may cnjecture that the minimum value of $\frac{y-z}{c-b}$ is attained when c tends to b. Note that

$$\lim_{c\to b}\frac{y-z}{c-b}=\frac{\sqrt{(a+2b)b}(a^2+ab+2b^2)}{2b(a+b)^2}. \tag{12}$$

Let $f(x):=\frac{\sqrt{x+2}(x^2+x+2)}{2(x+1)^2}$, so that the above limit is $f(\frac{a}{b})$. Since

$$f'(x)=\frac{x^3+4x^2+x-10}{4(x+1)^3\sqrt{x+2}},$$

we conclude that the minimum is $f(q)=0.856762$, where $q=1.284277$ is the unique real zero of x^3+4x^2+x-10. Hence, we may conjecture that

$$c>b\Longleftrightarrow y-z>f(q)(c-b). \tag{13}$$

3. Multiplicative stronger forms

3.1. One may also wonder about possibilities such as

$$c > b \Longrightarrow yb > zc.$$

This is again false. In fact, it is proved in [5, Exercise 4, p. 15] that

$$y^2b^2 - z^2c^2 = \frac{abc(c-b)(a+b+c)^2(b^2-bc+c^2-a^2)}{(a+b)^2(a+c)^2}, \tag{14}$$

and therefore

$$c > b \Longrightarrow (yb > zc \Longleftrightarrow A < 60^\circ)\,. \tag{15}$$

However, it is direct to check that

$$y^2b - z^2c = \frac{abc(c-b)(a+b+c)(b^2+c^2+ab+ac)}{(a+b)^2(a+c)^2}, \tag{16}$$

showing that

$$c > b \Longleftrightarrow y^2b > z^2c, \tag{17}$$

yet another stronger form of the Steiner-Lehmus theorem (3).

3.2. Formulas (14) and (16) are derived from the formulas

$$y^2 = ac\left(1 - \left(\frac{b}{a+c}\right)^2\right), \quad z^2 = ab\left(1 - \left(\frac{c}{a+b}\right)^2\right). \tag{18}$$

These follow from Stewart's theorem using (6) and (4); see [5, Exercise 1, p. 15], where these are used to give a proof of the Steiner-Lehmus theorem via

$$y^2 - z^2 \quad = \quad \frac{(c-b)a(a+b+c)(a^2(a+b+c)+bc(b+c+3a))}{(a+b)^2(a+c)^2}.$$

Similarly one can prove the stronger forms

$$\begin{aligned} c > b &\quad\Longrightarrow\quad y^2u - z^2U > a(c-b), &(19)\\ c > b &\quad\Longrightarrow\quad y^2 - z^2 > V^2 - v^2 &(20)\end{aligned}$$

using

$$\begin{aligned} y^2u - z^2U &= \frac{(c-b)abc(a+b+c)Q_1}{(a+b)^3(a+c)^3},\\ (y^2+v^2) - (z^2+V^2) &= \frac{(c-b)a(a+b+c)Q_2}{(a+b)^2(a+c)^2}.\end{aligned}$$

where

$$\begin{aligned} Q_1 &= (a^3+2abc)(a+b+c) + bc(a^2+b^2+c^2),\\ Q_2 &= a^3+b^2c+c^2b+3abc-b^2a-c^2a.\end{aligned}$$

Here Q_2 can be seen to be positive by substituting $a = \beta+\gamma$, $b = \alpha+\gamma$, $c = \alpha+\beta$.

Remark. It is well known that the Steiner-Lehmus theorem (3) is valid in neutral (or absolute) geometry; see [2, p. 119]. One wonders whether the same is true of the stronger forms (2), (8), and the other possible forms discussed in §2.

References

[1] M. Hajja, A very short and simple proof of "the most elementary theorem" of Euclidean geometry, *Forum Geom.*, 6 (2006) 167–169.

[2] D. C.E Kay, *College Geometry*, Holt, Rinehart and Winston, Inc., New York, 1969.

[3] M. Tetiva, Problem 11337, *Amer. Math. Monthly*, 115 (2008) 71.

[4] E. Trost and R. Breusch, Problem 4964, *Amer. Math. Monthly*, 68 (1961) 384; solution by ibid, 69 (1962) 672–674.

[5] P. Yiu, *Euclidean Geometry*, Florida Atlantic University Lecture Notes, 1998, `http://www.math.fau.edu/Yiu/Geometry.html`.

Mowaffaq Hajja: Mathematics Department, Yarmouk University, Irbid, Jordan
E-mail address: `mhajja@yu.edu.jo, mowhajja@yahoo.com`

Forum Geometricorum
Volume 8 (2008) 163–166.

FORUM GEOM
ISSN 1534-1178

A New Proof of a Weighted Erdős-Mordell Type Inequality

Yu-Dong Wu

Dedicated to Miss Xiao-Ping Lü
on the occasion of the 24-th Teachers' Day

Abstract. In this short note, by making use of one of Liu's theorems and Cauchy-Schwarz Inequality, we solve a conjecture posed by Liu [3] and give a new proof of a weighted Erdős–Mordell type inequality. Some interesting corollaries are also given at the end.

1. Introduction and Main Results

Let P be an arbitrary point in the plane of triangle ABC. Denote by R_1, R_2, and R_3 the distances from P to the vertices A, B, and C, and r_1, r_2, and r_3 the signed distances from P to the sidelines BC, CA, and AB, respectively. The neat and famous inequality

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3), \tag{1}$$

conjectured by Paul Erdős in 1935, was first proved by L. J. Mordell and D. F. Barrow (see [2]). In 2005, Jian Liu [3] obtained a weighted Erdős-Mordell type inequality as follows.

Theorem 1. *For $x, y, z \in \mathbb{R}$,*

$$\begin{aligned} & x^2\sqrt{R_2 + R_3} + y^2\sqrt{R_3 + R_1} + z^2\sqrt{R_1 + R_2} \\ \geq & \sqrt{2}(yz\sqrt{r_2 + r_3} + zx\sqrt{r_3 + r_1} + xy\sqrt{r_1 + r_2}). \end{aligned} \tag{2}$$

Liu's proof, however, is quite complicated. We give a simple proof of Theorem 1 as a corollary of a more general result, also conjectured by Liu in [3].

Publication Date: Month, 2008. Communicating Editor: Li Zhou.
The author would like to thank Professor Li Zhou for valuable comments which helped improve the presentation, and Professor Zhi-Hua Zhang for his careful reading of this paper.

Theorem 2. *For $x, y, z \in \mathbb{R}$ and arbitrary positive real numbers u, v, w, we have*

$$\begin{aligned} & x^2\sqrt{v+w} + y^2\sqrt{w+u} + z^2\sqrt{u+v} \\ \geq & 2\left(yz\sqrt{u\sin\frac{A}{2}} + zx\sqrt{v\sin\frac{B}{2}} + xy\sqrt{w\sin\frac{C}{2}}\right). \end{aligned} \tag{3}$$

2. Preliminary Results

In order to prove our main results, we shall require the following two lemmas.

Lemma 3 ([4, 5])**.** *For $x, y, z \in \mathbb{R}$, $p_i \in (-\infty, 0)\bigcup(0, +\infty)$, and $q_i \in \mathbb{R}$ for $i = 1, 2, 3$, the quadratic inequality of three variables*

$$p_1x^2 + p_2y^2 + p_3z^2 \geq q_1yz + q_2zx + q_3xy$$

holds if and only if

$$\begin{cases} p_i > 0,\ i = 1,\ 2,\ 3; \\ 4p_2p_3 > q_1^2,\ 4p_3p_1 > q_2^2,\ 4p_1p_2 > q_3^2, \\ 4p_1p_2p_3 \geq p_1q_1^2 + p_2q_2^2 + p_3q_3^2 + q_1q_2q_3. \end{cases}$$

Lemma 4. *In $\triangle ABC$, we have*

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1.$$

Proof. This follows from the formula $\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$ and the known identity

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

□

3. Proof of Theorem 2

(1) For $u, v, w > 0$,

$$\begin{cases} \sqrt{v+w} > 0, \\ \sqrt{w+u} > 0, \\ \sqrt{u+v} > 0. \end{cases} \tag{4}$$

(2) From $\sin\frac{A}{2}, \sin\frac{B}{2}, \sin\frac{C}{2} \in (0,1)$, we easily get

$$\begin{cases} 4\sqrt{(w+u)(u+v)} > 4u > 4u\sin\frac{A}{2}, \\ 4\sqrt{(u+v)(v+w)} > 4v > 4v\sin\frac{B}{2}, \\ 4\sqrt{(v+w)(w+u)} > 4w > 4w\sin\frac{C}{2}. \end{cases} \tag{5}$$

By the Cauchy-Schwarz inequality and Lemma 4, we have

$$\begin{aligned}
&\left(u\sqrt{v+w}\sin\frac{A}{2}+v\sqrt{w+u}\sin\frac{B}{2}+w\sqrt{u+v}\sin\frac{C}{2}+\sqrt{2uvw}\sqrt{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\right)^2\\
\le&\left(u^2(v+w)+v^2(w+u)+w^2(u+v)+2uvw\right)\\
&\cdot\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\\
=&(u+v)(v+w)(w+u). \qquad (6)
\end{aligned}$$

From Lemma 3 and (4)–(6), we conclude that inequality (3) holds. The proof of Theorem 2 is complete.

4. Applications of Theorem 2

Proof of Theorem 1. If we take $u = R_1$, $v = R_2$, $w = R_3$ and with known inequalities (see [1])

$$2R_1\sin\frac{A}{2}\ge r_2+r_3,\quad 2R_2\sin\frac{B}{2}\ge r_3+r_1,\quad 2R_3\sin\frac{C}{2}\ge r_1+r_2,$$

we obtain Theorem 1 immediately. This completes the proof of Theorem 1.

Many further inequalities can be obtained from various substitutions for (u,v,w). Here are two examples.

Corollary 5. *For $\triangle ABC$ and real numbers x, y, z, we have*

$$\begin{aligned}
&x^2\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+y^2\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}+z^2\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}\\
\ge&2\left(yz\sin\frac{A}{2}+zx\sin\frac{B}{2}+xy\sin\frac{C}{2}\right).
\end{aligned}$$

Corollary 6. *For $\triangle ABC$ and real numbers x, y, z, we have*

$$\begin{aligned}
&x^2\sqrt{\csc\frac{B}{2}+\csc\frac{C}{2}}+y^2\sqrt{\csc\frac{C}{2}+\csc\frac{A}{2}}+z^2\sqrt{\csc\frac{A}{2}+\csc\frac{B}{2}}\\
\ge&2(yz+zx+xy).
\end{aligned}$$

Further inequalities can also be obtained from substitutions of (x,y,z) by geometric elements of $\triangle ABC$. The reader is invited to experiment with the possibilities.

References

[1] O. Bottema, R. Ž. Dordević, R. R. Janić and D. S. Mitrinović, *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969.

[2] P. Erdős, L. J. Mordell, and D. F. Barrow. Problem 3740, *Amer. Math. Monthly*, 42 (1935) 396; solutions, ibid., 44 (1937) 252–254.

[3] J. Liu, A new extension of the Erdős-Mordell's type inequality, (in Chinese), *Jilin Normal University Journal* (Natural Science Edition), 26-4 (2005) 8–11.

[4] J. Liu, Two theorems of three variables of quadratic type inequalities and applications, (in Chinese), *High School Mathematics* (Jiangsu), 5 (1996) 16–19.

[5] J. Liu, Two results and several conjectures of a kind of geometry inequalities, (in Chinese), *Journal of East China Jiaotong University*, 3 (2002) 89–94.

Yu-Dong Wu: Department of Mathematics, Zhejiang Xinchang High School, Shaoxing, Zhejiang 312500, P. R. China
E-mail address: yudongwu@foxmail.com, 39387088@qq.com

Forum Geometricorum
Volume 8 (2008) 167–169.

FORUM GEOM
ISSN 1534-1178

Another Compass-Only Construction of the Golden Section and of the Regular Pentagon

Michel Bataille

Abstract. We present a compass-only construction of the point dividing a *given* segment in the golden ratio. As a corollary, we obtain a very simple construction of a regular pentagon inscribed in a *given* circle.

Various constructions of the golden section and of the regular pentagon have already appeared in this journal. In particular, in $[1, 2]$, Kurt Hofstetter offers very interesting compass-only constructions that require only a small number of circles. However, the constructed divided segment and pentagon come into sight as fortunate outcomes of the completed figures and are not subject to any prior constraint. As a result, these constructions do not adjust easily to the usual cases when the segment to be divided or the circumcircle of the pentagon are given at the start. The purpose of this note is to propose direct, simple compass-only constructions adapted to such situations.

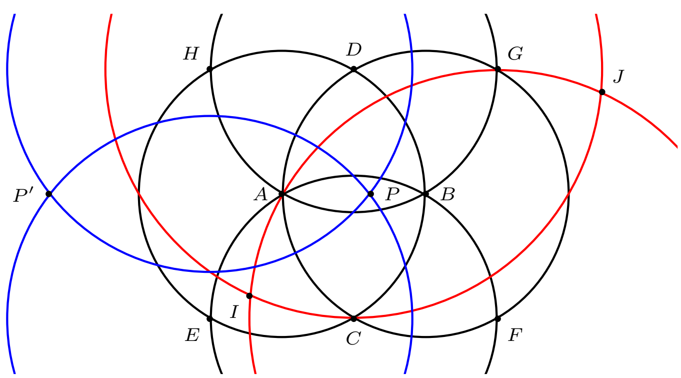


Figure 1

Construction 1. *Given two distinct points* A, B*, to obtain the point* P *of the line segment* AB *such that* $\frac{AP}{AB} = \frac{\sqrt{5}-1}{2}$*, construct*
(1) *with the same radius* AB*, the circles with centers* A *and* B*, to intersect at* C *and* D*,*
(2) *with the same radius* AB*, the circles with centers* C *and* D*, to intersect the two circles in* (1) *at* E, F, G, H (see Figure 1),

Publication Date: September 30, 2008. Communicating Editor: Paul Yiu.

(3) *with the same radius* DC*, the circles with centers* D *and* F*, to intersect at* I *and* J*,*
(4) *with the same radius* BI*, the circles with centers* E *and* H*.*
The points of intersection of these two circles are on the line AB*, and* P *is the one between* A *and* B*.*

Note that eight circles are needed, but if the line segment AB has been drawn, the number of circles drops to six, as it is easily checked. Note also that only three different radii are used.

Construction 2. *Given a point* B *on a circle* Γ *with center* A*, to obtain a regular pentagon inscribed in* Γ *with vertex* B*, construct*
(1) *the point* P *which divides* AB *in the golden section,*
(2) *the circle with center* P *and radius* AB*, to intersect* Γ *at* B_1 *and* B_4*,*
(3) *the circles* $B_1(B)$ *and* $B_4(B)$ *to intersect* Γ*, apart from* B*, at* B_2 *and* B_3 *respectively.*
The pentagon $BB_1B_2B_3B_4$ *is the desired one* (see Figure 2).

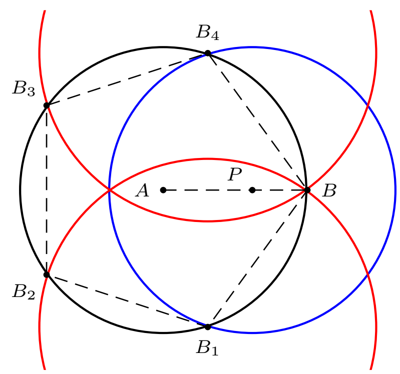


Figure 2

Proof of Construction 1. Let $a = AB$. Clearly, E, F (respectively F, D) are diametrically opposite on the circle with center C (respectively B) and radius a. It follows that EB is the perpendicular bisector of DF and since $IF = ID$, I is on the line EB. Therefore ΔIBF is right-angled at B, and $IB = a\sqrt{2}$ (since $IF = CD = a\sqrt{3}$ and $BF = a$). Now, the circles in (4) do intersect (since $HE = CD < 2BI$) and are symmetrical in the line AB, hence their intersections P, P' are certainly on this line. As for the relation $AP = \frac{\sqrt{5}-1}{2}AB$, it directly results from the following key property:

Let triangle BAE satisfy $AE = AB = a$ and $\angle BAE = 120^\circ$ and let P be on the side AB such that $EP = a\sqrt{2}$. Then $AP = \frac{\sqrt{5}-1}{2}\,a$ (see Figure 3).

Indeed, the law of cosines yields $PE^2 = AE^2 + AP^2 - 2AE \cdot AP \cdot \cos 120^\circ$ and this shows that AP is the positive solution to the quadratic $x^2 + ax - a^2 = 0$. Thus, $AP = \frac{\sqrt{5}-1}{2}\,a$. $\square$

Note that $AP' = \frac{\sqrt{5}+1}{2}\,a$ is readily obtained in a similar manner.

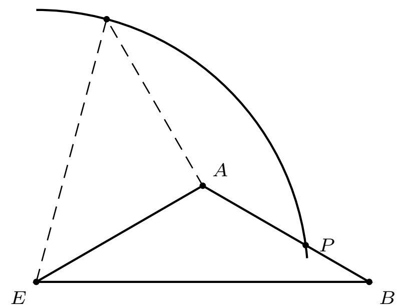


Figure 3

Proof of Construction 2. Since ΔAB_4P is isosceles with $B_4A = B_4P = a$, we have $\cos BAB_4 = \frac{1}{2}\frac{AP}{a} = \frac{\sqrt{5}-1}{4}$. Hence $\angle BAB_4 = 72°$ and the result immediately follows. □

As a final remark, Figure 3 and the property above lead to a quick construction of the golden section with ruler and compass.

References

[1] K. Hofstetter, A simple construction of the golden section, *Forum Geom.*, 2 (2002) 65–66.

[2] K. Hofstetter, A simple compass-only construction of the regular pentagon, *Forum Geom.*, 8 (2008) 147–148.

Michel Bataille: 12 rue Sainte-Catherine, 76000 Rouen, France
E-mail address: michelbataille@wanadoo.fr

Forum Geometricorum
Volume 8 (2008) 171–174.

FORUM GEOM
ISSN 1534-1178

Some Identities Arising From Inversion of Pappus Chains in an Arbelos

Giovanni Lucca

Abstract. We consider the inversive images, with respect to the incircle of an arbelos, of the three Pappus chains associated with the arbelos, and establish some identities connecting the radii of the circles involved.

In a previous work [1], we considered the three Pappus chains that can be drawn inside the arbelos and demonstrated some identities relating the radii of the circles in these chains. In Figure 1, the diameter AC of the left semicircle $\mathcal{C}_{\rm a}$ is $2a$, the diameter CB of the right semicircle $\mathcal{C}_{\rm b}$ is $2b$, and the diameter AB of the outer semicircle $\mathcal{C}_{\rm r}$ is $2r$, $r = a + b$. The first circle Γ_1 is common to all three chains and is the incircle of the arbelos.

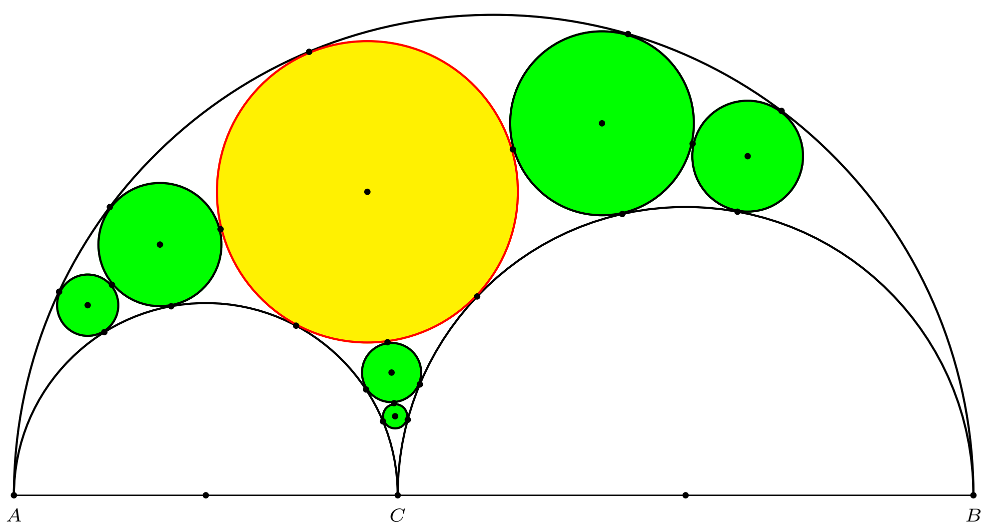


Figure 1. The Pappus chains in an arbelos

With reference to Figure 1, we denote by $\Gamma_{\rm r}$, $\Gamma_{\rm a}$ and $\Gamma_{\rm b}$ the chains converging to C, A, B respectively. Table 1 gives the coordinates of the centers and the radii of the circles in the chains, referring to a Cartesian reference system with origin at C and x-axis along AB.

Publication Date: October 13, 2008. Communicating Editor: Paul Yiu.

Table 1: Center coordinates and radii of the circles in the Pappus chains

Chain	Γ_{r}	Γ_{a}	Γ_{b}
Abscissa of n-th circle	$x_{\mathrm{r}n} = \frac{ab(a-b)}{n^2r^2-ab}$	$x_{\mathrm{a}n} = 2b - \frac{rb(r+b)}{n^2a^2+rb}$	$x_{\mathrm{b}n} = -2a + \frac{ra(r+a)}{n^2b^2+ra}$
Ordinate of n-th circle	$y_{\mathrm{r}n} = \frac{2nrab}{n^2r^2-ab}$	$y_{\mathrm{a}n} = \frac{2nrab}{n^2a^2+rb}$	$y_{\mathrm{b}n} = \frac{2nrab}{n^2b^2+ra}$
Radius of n-th circle	$\rho_{\mathrm{r}n} = \frac{rab}{n^2r^2-ab}$	$\rho_{\mathrm{a}n} = \frac{rab}{n^2a^2+rb}$	$\rho_{\mathrm{b}n} = \frac{rab}{n^2b^2+ra}$

The following proposition was established in [1].

Proposition 1. *Given a generic arbelos with its three Pappus chains, the following identities hold for each integer* n*:*

$$\rho_{\mathrm{inc}}\left(\frac{1}{\rho_{\mathrm{r}n}}+\frac{1}{\rho_{\mathrm{a}n}}+\frac{1}{\rho_{\mathrm{b}n}}\right)=2n^2+1, \tag{1}$$

$$\rho_{\mathrm{inc}}^2\left(\frac{1}{\rho_{\mathrm{r}n}^2}+\frac{1}{\rho_{\mathrm{a}n}^2}+\frac{1}{\rho_{\mathrm{b}n}^2}\right)=2n^4+1, \tag{2}$$

$$\rho_{\mathrm{inc}}^2\left(\frac{1}{\rho_{\mathrm{r}n}}\cdot\frac{1}{\rho_{\mathrm{a}n}}+\frac{1}{\rho_{\mathrm{a}n}}\cdot\frac{1}{\rho_{\mathrm{b}n}}+\frac{1}{\rho_{\mathrm{b}n}}\cdot\frac{1}{\rho_{\mathrm{r}n}}\right)=n^4+2n^2. \tag{3}$$

In particular, the center of the incircle of the arbelos is the point

$$(x_{\mathrm{inc}},\ y_{\mathrm{inc}})=\left(\frac{ab(a-b)}{a^2+ab+b^2},\ \frac{2ab(a+b)}{a^2+ab+b^2}\right).$$

Its radius is

$$\rho_{\mathrm{inc}}=\frac{ab(a+b)}{a^2+ab+b^2}.$$

We now consider the inversion of these three Pappus chains with respect to the incircle of arbelos. See Figure 2. For convenience, we record a useful formula, which can be found in [2], we use for the computation of the centers and radii of the inversive images of the circles in the Pappus chains.

Lemma 2. *With respect the circle of center* (x_0, y_0) *and radius* R_0*, the inversive image of the circle with center* $(x_C,\ y_C)$ *and radius* R *is the circle with center* $(x_C^{\mathrm{i}},\ y_C^{\mathrm{i}})$ *and radius* R^{i} *given by*

$$x_C^{\mathrm{i}}=x_0+\frac{R_0^2}{(x_C-x_0)^2+(y_C-y_0)^2-R^2}(x_C-x_0),$$

$$y_C^{\mathrm{i}}=y_0+\frac{R_0^2}{(x_C-x_0)^2+(y_C-y_0)^2-R^2}(y_C-y_0),$$

$$R^{\mathrm{i}}=\left|\frac{R_0^2}{(x_C-x_0)^2+(y_C-y_0)^2-R^2}\right|R.$$

We give in Table 2 the coordinates of the centers of the inversive images of the circles in the Pappus chains, and their radii.

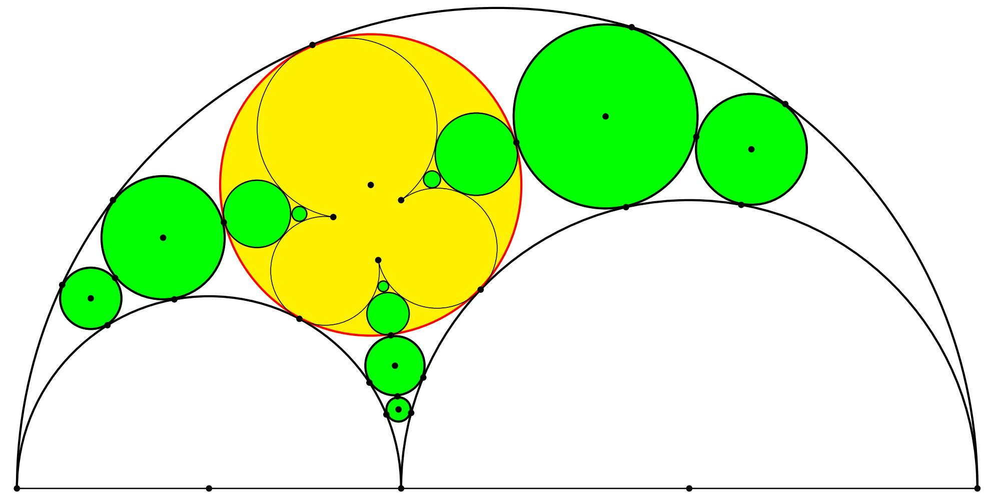


Figure 2. Inversive images of the Pappus chains

Table 2: Center coordinates and radii of inversive images of circles in the Pappus chains

	Inverted chain $\Gamma_{\mathrm{r}}^{\mathrm{i}}$
Abscissa of n-th circle	$x_{\mathrm{r}n}^{\mathrm{i}}=x_{\mathrm{inc}}+\frac{\rho_{\mathrm{inc}}^2(x_{\mathrm{r}n}-x_{\mathrm{inc}})}{(x_{\mathrm{r}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{r}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Ordinate of n-th circle	$y_{\mathrm{r}n}^{\mathrm{i}}=y_{\mathrm{inc}}+\frac{\rho_{\mathrm{inc}}^2(y_{\mathrm{r}n}-y_{\mathrm{inc}})}{(x_{\mathrm{r}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{r}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Radius of n-th circle	$\rho_{\mathrm{r}n}^{\mathrm{i}}=\left\lvert\frac{\rho_{\mathrm{inc}}^2}{(x_{\mathrm{r}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{r}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}\right\rvert\rho_{\mathrm{r}n}$
	Inverted chain $\Gamma_{\mathrm{a}}^{\mathrm{i}}$
Abscissa of n-th circle	$x_{\mathrm{a}n}^{\mathrm{i}}=x_{\mathrm{inc}}+\frac{\rho_{\mathrm{inc}}^2(x_{\mathrm{a}n}-x_{\mathrm{inc}})}{(x_{\mathrm{a}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{a}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Ordinate of n-th circle	$y_{\mathrm{a}n}^{\mathrm{i}}=y_{\mathrm{inc}}+\frac{\rho_{\mathrm{inc}}^2(y_{\mathrm{a}n}-y_{\mathrm{inc}})}{(x_{\mathrm{a}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{a}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Radius of n-th circle	$\rho_{\mathrm{a}n}^{\mathrm{i}}=\left\lvert\frac{\rho_{\mathrm{inc}}^2}{(x_{\mathrm{a}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{a}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}\right\rvert\rho_{\mathrm{a}n}$
	Inverted chain $\Gamma_{\mathrm{b}}^{\mathrm{i}}$
Abscissa of n-th circle	$x_{\mathrm{b}n}^{\mathrm{i}}=x_{\mathrm{inc}}+\frac{\rho_{\mathrm{inc}}^2(x_{\mathrm{b}n}-x_{\mathrm{inc}})}{(x_{\mathrm{b}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{b}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Ordinate of n-th circle	$y_{\mathrm{b}n}^{\mathrm{i}}=y_{\mathrm{inc}}+\frac{\rho_{\mathrm{inc}}^2(y_{\mathrm{b}n}-y_{\mathrm{inc}})}{(x_{\mathrm{b}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{b}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Radius of n-th circle	$\rho_{\mathrm{b}n}^{\mathrm{i}}=\left\lvert\frac{\rho_{\mathrm{inc}}^2}{(x_{\mathrm{b}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{b}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}\right\rvert\rho_{\mathrm{b}n}$

From these data, we can deduce some identities connecting the radii of these circles.

Theorem 3. *For the circles in the Pappus chains and their inversive images in the incircle, the following identities hold. For $n \geq 2$,*

$$\frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{r}n}} - \frac{\rho_{\text{inc}}}{\rho_{\text{r}n}} = \frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{a}n}} - \frac{\rho_{\text{inc}}}{\rho_{\text{a}n}} = \frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{b}n}} - \frac{\rho_{\text{inc}}}{\rho_{\text{b}n}} = 4n^2 - 8n + 2, \tag{4}$$

$$\frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{r}n}} + \frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{a}n}} + \frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{b}n}} = 14n^2 - 24n + 7, \tag{5}$$

$$\frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{r}n}} + \frac{\rho_{\text{inc}}}{\rho_{\text{a}n}} + \frac{\rho_{\text{inc}}}{\rho_{\text{b}n}} = \frac{\rho_{\text{inc}}}{\rho_{\text{r}n}} + \frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{a}n}} + \frac{\rho_{\text{inc}}}{\rho_{\text{b}n}} = \frac{\rho_{\text{inc}}}{\rho_{\text{r}n}} + \frac{\rho_{\text{inc}}}{\rho_{\text{a}n}} + \frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{b}n}} = 6n^2 - 8n + 3, \tag{6}$$

$$\frac{\rho_{\text{inc}}}{\rho_{\text{r}n}} + \frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{a}n}} + \frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{b}n}} = \frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{r}n}} + \frac{\rho_{\text{inc}}}{\rho_{\text{a}n}} + \frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{b}n}} = \frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{r}n}} + \frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{a}n}} + \frac{\rho_{\text{inc}}}{\rho_{\text{b}n}} = 10n^2 - 16n + 5, \tag{7}$$

$$\frac{\rho^2_{\text{inc}}}{\rho_{\text{r}n}\rho^{\text{i}}_{\text{r}n}} + \frac{\rho^2_{\text{inc}}}{\rho_{\text{a}n}\rho^{\text{i}}_{\text{a}n}} + \frac{\rho^2_{\text{inc}}}{\rho_{\text{b}n}\rho^{\text{i}}_{\text{b}n}} = 10n^4 - 16n^3 + 8n^2 - 8n + 3, \tag{8}$$

$$\frac{\rho^2_{\text{inc}}}{\rho^{\text{i}}_{\text{a}n}\rho^{\text{i}}_{\text{b}n}} + \frac{\rho^2_{\text{inc}}}{\rho^{\text{i}}_{\text{a}n}\rho^{\text{i}}_{\text{r}n}} + \frac{\rho^2_{\text{inc}}}{\rho^{\text{i}}_{\text{b}n}\rho^{\text{i}}_{\text{r}n}} = 65n^4 - 224n^3 + 258n^2 - 112n + 16, \tag{9}$$

$$\left(\frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{r}n}}\right)^2 + \left(\frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{a}n}}\right)^2 + \left(\frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{b}n}}\right)^2 = 66n^4 - 224n^3 + 256n^2 - 112n + 17. \tag{10}$$

From (9), (10) above, and also (2), (3) in Proposition 1, we have

$$\begin{aligned}
&\left(\frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{r}n}}\right)^2 + \left(\frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{a}n}}\right)^2 + \left(\frac{\rho_{\text{inc}}}{\rho^{\text{i}}_{\text{b}n}}\right)^2 - \left(\frac{\rho^2_{\text{inc}}}{\rho^{\text{i}}_{\text{a}n}\rho^{\text{i}}_{\text{b}n}} + \frac{\rho^2_{\text{inc}}}{\rho^{\text{i}}_{\text{a}n}\rho^{\text{i}}_{\text{r}n}} + \frac{\rho^2_{\text{inc}}}{\rho^{\text{i}}_{\text{b}n}\rho^{\text{i}}_{\text{r}n}}\right) \\
=& \left(\frac{\rho_{\text{inc}}}{\rho_{\text{r}n}}\right)^2 + \left(\frac{\rho_{\text{inc}}}{\rho_{\text{a}n}}\right)^2 + \left(\frac{\rho_{\text{inc}}}{\rho_{\text{b}n}}\right)^2 - \left(\frac{\rho^2_{\text{inc}}}{\rho_{\text{a}n}\rho_{\text{b}n}} + \frac{\rho^2_{\text{inc}}}{\rho_{\text{a}n}\rho_{\text{r}n}} + \frac{\rho^2_{\text{inc}}}{\rho_{\text{b}n}\rho_{\text{r}n}}\right) \\
=& (n^2 - 1)^2.
\end{aligned}$$

References

[1] G. Lucca, Three Pappus chains inside the arbelos: some identities, *Forum Geom.*, 7 (2008) 107–109.

[2] E. W. Weisstein, Inversion from MathWorld – A Wolfram Web Resource, `http://mathworld.wolfram.com/Inversion.html`.

Giovanni Lucca: Via Corvi 20, 29100 Piacenza, Italy
E-mail address: `vanni_lucca@inwind.it`

Forum Geometricorum
Volume 8 (2008) 175–182.

FORUM GEOM
ISSN 1534-1178

Second-Degree Involutory Symbolic Substitutions

Clark Kimberling

Abstract. Suppose a, b, c are algebraic indeterminates. The mapping $(a, b, c) \rightarrow (bc, ca, ab)$ is an example of a second-degree involutory symbolic substitution (SISS) which maps the transfigured plane of a triangle to itself. The main result is a classification of SISSs as four individual mappings and two families of mappings. The SISS $(a, b, c) \rightarrow (bc, ca, ab)$ maps the circumcircle onto the Steiner ellipse. This and other examples are considered.

1. Introduction

This article is a sequel to [2], in which symbolic substitutions are introduced. A brief summary follows. The symbols a, b, c are algebraic indeterminates over the field of complex numbers. Suppose α, β, γ are nonzero homogeneous algebraic functions of (a, b, c) :

$$\alpha(a, b, c), \beta(a, b, c), \gamma(a, b, c), \tag{1}$$

all of the same degree of homogeneity. Throughout this work, triples with notations such as $U = (u, v, w)$ and $X = (x, y, z)$ are understood to be as in (1), except that one or two (but not all three) of the coordinates can be 0. Triples (x, y, z) and (x', y', z') are *equivalent* if $xy' = yx'$ and $yz' = zy'$. The equivalence class containing any particular (x, y, z) is denoted by $x : y : z$ and is a *point*. The set of points is the *transfigured plane*, denoted by $\mathcal{P}$. A well known model of $\mathcal{P}$ is obtained by taking a, b, c to be sidelengths of a euclidean triangle ABC and taking $x : y : z$ to be the point whose directed distances [1] from the sidelines BC, CA, AB are respectively proportional to x, y, z.

A simple example of a symbolic substitution is indicated by

$$(a, b, c) \rightarrow (bc, ca, ab).$$

This means that a point

$$x : y : z = x(a, b, c) : y(a, b, c) : z(a, b, c) \tag{2}$$

Publication Date: October 21, 2008. Communicating Editor: Paul Yiu.

[1]The coordinates $x : y : z$ are *homogeneous trilinear coordinates*, or simply *trilinears*. The notation (x, y, z), in this paper, represents an ordinary ordered triple, as when x, y, z are actual directed distances or when (x, y, z) is the argument of a function. Unfortunately, the notation (x, y, z) has sometimes been used for homogeneous coordinates, so that, for example $(2x, 2y, 2z) = (x, y, z)$, which departs from ordinary ordered triple notation. On the other hand, using colons, we have $2x : 2y : 2z = x : y : z$.

maps to the point

$$x : y : z = x(bc, ca, ab) : y(bc, ca, ab) : z(bc, ca, ab), \tag{3}$$

so that $\mathcal{P}$ is mapped to itself. We are interested in the effects of such substitutions on various points and curves. Consider, for example the Thompson cubic, $\mathcal{Z}(X_2, X_1)$, given by the equation[2]

$$bc\alpha(\beta^2 - \gamma^2) + ca\beta(\gamma^2 - \alpha^2) + ab\gamma(\alpha^2 - \beta^2) = 0. \tag{4}$$

For each point (2) on (4), the point (3) is on the cubic $\mathcal{Z}(X_6, X_1)$, given by the equation

$$a\alpha(\beta^2 - \gamma^2) + b\beta(\gamma^2 - \alpha^2) + cc(\alpha^2 - \beta^2) = 0. \tag{5}$$

Letting $\mathcal{S}(X_i)$ denote the image of X_i under the substitution $(a, b, c) \to (bc, ca, ab)$, specific points on $\mathcal{Z}(X_2, X_1)$ map to points on $\mathcal{Z}(X_6, X_1)$ as shown in Table 1:

Table 1. From $\mathcal{Z}(X_2, X_1)$to $\mathcal{Z}(X_6, X_1)$

X_i on $\mathcal{Z}(X_2, X_1)$	X_1	X_2	X_3	X_4	X_6	X_9	X_{57}
$\mathcal{S}(X_i)$ on $\mathcal{Z}(X_6, X_1)$	X_1	X_6	X_{194}	X_{3224}	X_2	X_{43}	X_{87}

As suggested by Table 1, $\mathcal{S}(\mathcal{S}(X)) = X$ for every X, which is to say that $\mathcal{S}$ is involutory. The main purpose of this article is to find explicitly all second-degree involutory symbolic substitutions.

2. Terminology and Examples

A *polynomial triangle center* is a point U which has a representation

$$u(a, b, c) : v(a, b, c) : w(a, b, c),$$

where $u(a, b, c)$ is a polynomial in a, b, c and these conditions hold:

$$v(a, b, c) = \quad u(b, c, a); \tag{6}$$
$$w(a, b, c) = \quad u(c, a, b); \tag{7}$$
$$|u(a, c, b)| = \quad |u(a, b, c)| \,. \tag{8}$$

If $u(a, b, c)$ has degree 2, then U is a *second-degree triangle center*. A *second-degree symbolic substitution* is a transformation of $\mathcal{P}$ or some subsert thereof, with images in $\mathcal{P}$, given by a symbolic substitution of the form

$$(a, b, c) \longrightarrow (u(a, b, c), v(a, b, c), w(a, b, c))$$

for some second-degree triangle center U. The mapping (whether of polynomial form or not) is *involutory* if its compositional square is the identity; that is, if

$$u(u, v, w) : v(u, v, w) : w(u, v, w) = a : b : c,$$

[2]Triangle centers are indexed as in [1]: $X_1 =$ incenter, $X_2 =$ centroid, etc. The cubic $\mathcal{Z}(U, P)$ is defined as the set of points $\alpha : \beta : \gamma$ satisfying

$$up\alpha(q\beta^2 - r\gamma^2) + vq\beta(r\gamma^2 - p\alpha^2) + wr\gamma(p\alpha^2 - q\beta^2) = 0$$

where $U = u : v : w$ and $P = p : q : r$. Geometrically, $\mathcal{Z}(U, P)$ is the locus of $X = x : y : z$ such that the P-isoconjugate of X is on the line UX. The P-isoconjugate of X (and the X-isoconjugate of P) is the point $qryz : rpzx : pqxy$.

where

$$u = u(a, b, c),\ v = v(a, b, c),\ w = w(a, b, c).$$

Equivalently, a symbolic substitution $(a, b, c) \longrightarrow (u, v, w)$ is involutory if

$$u(u, v, w) = ta$$

for some function t of (a, b, c) that is symmetric in a, b, c. Henceforth we shall abbreviate "second-degree involutory symbolic substitution" as SISS. Following are four examples.

Example 1. The SISS

$$(a, b, c) \longrightarrow (bc, ca, ab) \tag{9}$$

gives

$$\begin{aligned} u(u, v, w) &= u(bc, ca, ab) \\ &= (bc)(ca) \\ &= ta, \end{aligned}$$

where $t = abc$.

Example 2. The SISS

$$(a, b, c) \longrightarrow (a^2 - bc, b^2 - ca, c^2 - ab) \tag{10}$$

gives

$$\begin{aligned} u(u, v, w) &= u(a^2 - bc, b^2 - ca, c^2 - ab) \\ &= (a^2 - bc)^2 - (b^2 - ca)(c^2 - ab) \\ &= ta, \end{aligned}$$

where

$$t = (a + b + c)\,(a^2 + b^2 + c^2 - bc - ca - ab).$$

Note that (10) is meaningless for $a = b = c$. As a, b, c, are indeterminates, however, such cases do not require additional writing, just as, when one writes "$\tan\theta$" where θ is a variable, it is understood that $\theta \neq \frac{\pi}{2}$.

Example 3. The SISS

$$(a, b, c) \longrightarrow (b^2 + c^2 - ab - ac, c^2 + a^2 - bc - ba, a^2 + b^2 - ca - cb) \tag{11}$$

gives

$$u(u, v, w) = ta,$$

where

$$t = 2\,(a + b + c)\,(a^2 + b^2 + c^2 - bc - ca - ab).$$

Example 4. The SISS

$$(a, b, c) \longrightarrow (a(a - b - c), b(b - c - a), c(c - a - b)) \tag{12}$$

gives

$$u(u, v, w) = ta,$$

where

$$t = (a-b-c)\,(b-c-a)(c-a-b).$$

3. Main result

Theorem. *In addition to the four SISSs (9)-(12), there are two families of SISSs given below by (17) and (18), and there is no other SISS.*

Proof. Equations (6) –(8) and the requirement that u be a polynomial of degree 2 imply that u is expressible in one of these two forms:

$$u = ja^2 + k(b^2+c^2) + lbc + ma(b+c) \tag{13A}$$

$$u = (b-c)(ja + k(b+c)) \tag{14}$$

for some complex numbers j, k, l, m. The proof will be given in two parts, depending on (13A) and (14).

Part 1: u given by (13A). In this case,

$$v = jb^2 + k(c^2+a^2) + lca + mb(c+a), \tag{13B}$$

$$w = jc^2 + k(a^2+b^2) + lab + mc(a+b). \tag{13C}$$

Let $P = u(u,v,w)$. We wish to find all j,k,l,m for which P factors as ta, where t is symmetric in a,b,c. The polynomial P can be written as $aQ + R$, where Q and R are polynomials and the R is invariant of a. In order to have $P = ta$, the coefficients j,k,l,m must make $R(a,b,c)$ identically 0. We have

$$R = (b^4+c^4)S_1 + 2bc(b^2+c^2)S_2 + b^2c^2S_3,$$

where

$$S_1 = jkl + jkm + k^3 + jk^2 + j^2k + k^2m,$$

$$S_2 = jkl + jkm + jlm + klm + km^2 + k^2m,$$

$$S_3 = 2jkm + 6jk^2 + jl^2 + j^2l + k^2l + 2km^2 + 2k^2m + 3lm^2.$$

Thus, we seek j,k,l,m for which $S_1 = S_2 = S_3 = 0$.

Case 1: $j = 0$. Here,
$S_1 = (k+m)\,k^2$, so that $k = 0$ or $k = -m$.
$S_2 = mk\,(k+l+m)$, so that $m = 0$ or $k = 0$ or $k+l+m = 0$.
$S_3 = k^2l + 2km^2 + 2k^2m + 3lm^2$.

Subcase 1.1: $j = 0$ and $k = 0$. Here, $S_2 = 0$, $S_3 = 3lm^2$, so that $l = 0$ or $m = 0$ but not both. If $l = 0$ and $m \neq 0$, then by (13A-C),

$$P = mu(v+w) = -m^3a\,(ab+ac+2bc)\,(b+c)\,,$$

not of the required form aQ where Q is symmetric in a,b,c. On the other hand, if $m = 0$ and $l \neq 0$, then $P = lvw = l^3a^2bc$, so that, on putting $l = 1$, we have $(u,v,w) = (bc,ab,ca)$, as in (9).

Subcase 1.2: $j = 0$ and $k = -m \neq 0$. Here, with $S_2 = 0$, $k \neq 0$, $m \neq 0$, and $k + l + m = 0$, we have $l = 0$, and (13A-C) give

$$P = k(v^2 + w^2) - ku(v + w) = 2a\,(a + b + c)\left(a^2 + b^2 + c^2 - bc - ca - ab\right)k^3,$$

so that taking $(j, k, l, m) = (0, 1, 0, -1)$ gives the SISS (11).

Case 2: $k = 0$. Here, $S_1 = 0$, $S_2 = jlm$, and $S_3 = l\left(jl + j^2 + 3m^2\right)$.

Subcase 2.1: $k = 0$ and $j = 0$. Here, since $S_3 = 0$, we have $3lm^2 = 0$. If $l = 0$, then

$$u = ma(b + c), v = mb(c + a), w = mc(a + b),$$
$$P = mu(v + w) = -(ab + ac + 2bc)(b + c)\,am^3,$$

not of the required form aQ. On the other hand, if $m = 0$, then

$$u = lbc, \quad v = lca, \quad w = lab,$$

so that taking $(j, k, l, m) = (0, 0, 1, 0)$ gives the SISS (9).

Subcase 2.2: $k = 0$ and $l = 0$. Here, $S_2 = S_3 = 0$, and (13A-C) give

$$\begin{aligned} P &= ju^2 + mu(v + w) \\ &= a\,(aj + bm + cm) \\ &\quad \cdot \left(abjm + acjm + abm^2 + acm^2 + 2bcm^2 + b^2jm + c^2jm + a^2j^2\right). \end{aligned}$$

In order for P to have the form aQ with Q symmetric in a, b, c, the factor

$$\left(abjm + acjm + abm^2 + acm^2 + 2bcm^2 + b^2jm + c^2jm + a^2j^2\right)$$

must factor as

$$(bj + cm + am)(cj + am + bm).$$

The identity

$$\begin{aligned} &\left(abjm + acjm + abm^2 + acm^2 + 2bcm^2 + b^2jm + c^2jm + a^2j^2\right) \\ &- (bj + cm + am)(cj + am + bm) \\ =\ &(m - j)(j + m)\left(bc - a^2\right) \end{aligned}$$

shows that this factorization occurs if and only if $j = m$ or $j = -m$. If $j = m$, then

$$u = a^2 + a(b + c), \quad v = b^2 + b(c + a), \quad w = c^2 + c(a + b),$$

leading to $(j, k, l, m) = (1, 0, 0, 1)$, but this is simply the identity substitution $(a, b, c) \to (a, b, c)$, not an SISS.

On the other hand, if $j = -m$, then

$$P = a\,(aj + bm + cm)\,(bj + cm + am)(cj + am + bm),$$

so that for $(j, k, l, m) = (1, 0, 0, -1)$, we have the SISS (12).

Subcase 2.3: $k = 0$ and $m = 0$. Here,

$$\begin{aligned} P &= ju^2 + lvw \\ &= a^4 j^3 + a^2 bcl^3 + ab^3 jl^2 + ac^3 jl^2 + 2a^2 bcj^2 l + b^2 c^2 jl^2 + b^2 c^2 j^2 l, \end{aligned}$$

which has the form aQ only if $b^2c^2jl^2 + b^2c^2j^2l = 0$, which means that $jl(j+l) = 0$. If $j = 0$ or $l = 0$, we have solutions already found. If $j = -l$, then

$$\begin{aligned} P &= ju^2 - jvw \\ &= l^3 a\,(a+b+c)\left(ab+ac+bc-a^2-b^2-c^2\right), \end{aligned}$$

giving $(j,k,l,m) = (1,0,-1,0)$, the SISS (10).

Case 3: $l = 0$. Here,

$$\begin{aligned} S_1 &= \left(jk + jm + km + j^2 + k^2\right) k, \\ S_2 &= 2mk\,(j+k+m)\,, \\ S_3 &= 2k\left(3jk + jm + km + m^2\right). \end{aligned}$$

Subcase 3.1: $l = 0$, $m = 0$, $S_1 = \left(jk + j^2 + k^2\right) k$, $S_2 = 0$, and $S_3 = 6jk^2$. Since $S_3 = 0$, we have $j = 0$ or $k = 0$, already covered.

Subcase 3.2: $l = 0$, and either $j = 0$ or $k = 0$, already covered.

Case 4: $m = 0$. Here, $S_1 = k\left(jk + jl + j^2 + k^2\right)$, $S_2 = 2jkl$, and $S_3 = \left(6jk^2 + jl^2 + j^2l + k^2l\right)$. Since $S_2 = 0$, we must have $j = 0$ or $k = 0$ or $l = 0$. All of these possibilities are already covered.

Case 5: none of j,k,l,m is 0. Here,

$$\begin{aligned} S_1 &= k\left(jk + jl + jm + km + j^2 + k^2\right), \\ S_2 &= jkl + jkm + jlm + klm + km^2 + k^2 m, \\ S_3 &= 2jkm + 6jk^2 + jl^2 + j^2 l + k^2 l + 2km^2 + 2k^2 m + 3lm^2. \end{aligned}$$

As $j \neq 0$ and $k \neq 0$, the requirement that $S_1 = 0$ gives

$$l = -\frac{jk + jm + km + j^2 + k^2}{j}. \tag{15}$$

Substitute l into the expression for S_2 and factor, getting

$$S_2 = -\frac{(k+m)(j+m)(jk+j^2+k^2)}{j} = 0. \tag{16}$$

Subcase 5.1: $m = -j$. Here, $l = -\frac{k^2}{j}$. This implies $S_1 = S_2 = S_3 = 0$ and

$$P = \frac{a(ak+(a-b-c)j)(bk+(b-c-a)j)(ck+(c-a-b)j)(k-j)^3}{j^3},$$

which is of the form aQ with Q symmetric in a, b, c. Because of homogeneity, we can without loss of generality take $(j, k, l, m) = (1, k, -k^2, -1)$, where $k \notin \{0, 1, -2\}$. This leaves a family of SISSs:

$$(a, b, c) \to (u, v, w), \tag{17}$$

where

$$\begin{aligned}
u &= a^2 + k(b^2 + c^2) - k^2bc - a(b + c),\\
v &= b^2 + k(c^2 + a^2) - k^2ca - b(c + a),\\
w &= c^2 + k(a^2 + b^2) - k^2ab - c(a + b),\\
P &= a(k - 1)^3(a - b - c + ak)(b - a - c + bk)(c - b - a + ck).
\end{aligned}$$

Note that for $k = -2$, we have $u = (a + b + c)(a - 2b - 2c)$, so that the involutory substitution

$$(a, b, c) \to (a - 2b - 2c, b - 2c - 2a, c - 2a - 2b)$$

is actually of first-degree, not second. (It is easy to check that for

$$u = a + mb + mc,$$

the only values of m for which the substitution $(a, b, c) \to (u, v, w)$ is involutory are 0 and -2.)

Subcase 5.2: $m = -k$. Here, $l = -j$. This implies $S_1 = S_2 = S_3 = 0$ and

$$P = a(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)(j + 2k)(k - j)^2,$$

which is of the form aQ with Q symmetric in a, b, c. Thus, if $j \neq k$ and $j \neq -2k$, we take $(j, k, l, m) = (j, k, -j, -k)$ and have a family of SISSs:

$$(a, b, c) \to (u, v, w) \tag{18}$$

where

$$\begin{aligned}
u &= a^2j + b^2k + c^2k - bcj - abk - ack,\\
v &= b^2j + c^2k + a^2k - caj - bck - bak,\\
w &= c^2j + a^2k + b^2k - abj - cak - cbk.
\end{aligned}$$

Note that $u = (a^2 - bc)j + (b^2 + c^2 - ab - ac)k$, a linear combination of second-degree polynomials appearing in (10) and (11).

Subcase 5.3: Equation (16) leaves one more subcase: $jk + j^2 + k^2 = 0$. This and (15) give $l = \frac{(j+k)k}{j}$, implying $S_1 = (k + m)(j + k)k$. Since $k \neq 0$ and $j + k \neq 0$ (because $l \neq 0$), we have $S_1 = 0$ only if $m = -k$, already covered in subcase 5.2.

Part 2: u given by (14). In this case,

$$P = (aj + bk + ck)(b - c)(j - k)\left(2bcj - acj - abj - 2a^2k + b^2k + c^2k\right),$$

which is not, for any (j, k, l, m), of the form aQ where Q is symmetric in a, b, c. $\square$

4. Mappings by symbolic substitutions

To summarize from [2], a symbolic substitution $\mathcal{S}$ maps points to points, lines to lines, conics to conics, cubics to cubics, circumconics to circumconics, and inconics to inconics. Regarding cubics, $\mathcal{S}$ maps each cubic $\mathcal{Z}(U,P)$ to the cubic $\mathcal{Z}(\mathcal{S}(U),\mathcal{S}(P))$ and each cubic $\mathcal{ZP}(U,P)$ to the cubic $\mathcal{ZP}(\mathcal{S}(U),\mathcal{S}(P))$. Symbolic substitutions thus have in common with projections and collineations various incidence properties and degree-preserving properties. On the other hand, symbolic substitutions are fundamentally different from strictly geometric transformations: given an ordinary 2-dimensional triangle ABC and a point $X = x(a,b,c) : y(a,b,c) : z(a,b,c)$ there seems no opportunity to apply geometric methods for describing the image-point

$$\mathcal{S}(X) = x' : y' : z' = x(bc,ca,ab) : y(bc,ca,ab) : z(bc,ca,ab).$$

Algebraically, however, it is clear if X lies on the circumcircle, which is to say that X is on the locus $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$, and if $\mathcal{S}$ is the symbolic substitution in (9), then $\mathcal{S}(X)$ satisfies $bcy'z' + caz'x' + abx'y' = 0$, which is to say that $\mathcal{S}(X)$ lies on the Steiner ellipse, $bc\beta\gamma + ca\gamma\alpha + ab\alpha\beta = 0$.

Table 2. From circumcircle Γ to Steiner ellipse $\mathbb{E}$

X_i on Γ	X_{98}	X_{99}	X_{100}	X_{101}	X_{105}	X_{106}	X_{110}	X_{111}
$\mathcal{S}(X_i)$ on $\mathbb{E}$	X_{3225}	X_{99}	X_{190}	X_{668}	X_{3226}	X_{3227}	X_{670}	X_{3228}

As a final example, note that the point $X_{101} = b-c : c-a : a-b$ is a fixed point of the SISS (10), as verified here:

$$b - c \to b^2 - ca - (c^2 - ab) = (a+b+c)(b-c).$$

Consequently, the line X_1X_6, given by the equation

$$(b-c)\alpha + (c-a)\beta + (a-b)\gamma = 0,$$

is left fixed by the SISS $\mathcal{S}$ in (10), as typified by Table 3.

Table 3. From X_1X_6 to X_1X_6

X_i on X_1X_6	X_1	X_6	X_9	X_{37}	X_{44}	X_{238}
$\mathcal{S}(X_i)$ on X_1X_6	X_1	X_{238}	X_{1757}	X_{518}	X_{44}	X_6

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[2] C. Kimberling, Symbolic substitutions in the transfigured plane of a triangle, *Aequationes Mathematicae*, 73 (2007) 156–171.

Clark Kimberling: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, USA

E-mail address: ck6@evansville.edu

Forum Geometricorum
Volume 8 (2008) 183–196.

FORUM GEOM
ISSN 1534-1178

On the Nagel Line and a Prolific Polar Triangle

Jan Vonk

Abstract. For a given triangle ABC, the polar triangle of the medial triangle with respect to the incircle is shown to have as its vertices the orthocenters of triangles AIB, BIC and AIC. We prove results which relate this polar triangle to the Nagel line and, eventually, to the Feuerbach point.

1. A prolific triangle

In a triangle ABC we construct a triad of circles $\mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c$ that are orthogonal to the incircle Γ of the triangle, with their centers at the midpoints D, E, F of the sides BC, AC, AB. These circles pass through the points of tangency X, Y, Z of the incircle with the respective sides. We denote by ℓ_a (respectively ℓ_b, ℓ_c) the radical axis of Γ and $\mathcal{C}_a$ (respectively $\mathcal{C}_b$, $\mathcal{C}_c$), and examine the triangle $A^*B^*C^*$ bounded by these lines (see Figure 1). J.-P. Ehrmann [1] has shown that this triangle has the same area as triangle ABC.

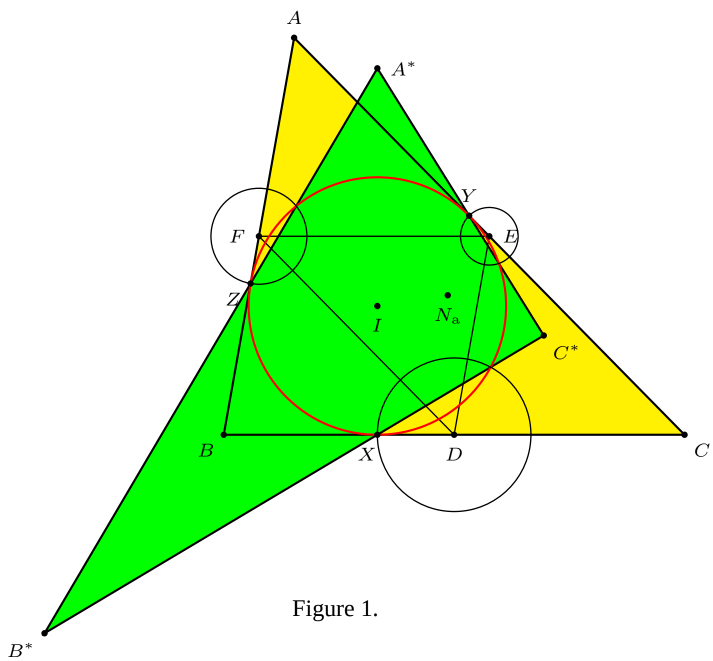


Figure 1.

Lemma 1. *The triangle $A^*B^*C^*$ is the polar triangle of the medial triangle DEF of triangle ABC with respect to Γ.*

Publication Date: November 26, 2008. Communicating Editor: J. Chris Fisher.
The author thanks Chris Fisher, Charles Thas and Paul Yiu for their help in the preparation of this paper.

Proof. Because $\mathcal{C}_a$ is orthogonal to Γ, the line ℓ_a is the polar of D with respect to Γ. Similarly, ℓ_b and ℓ_c are the polars of E and F with respect to the same circle. $\square$

Note that Lemma 1 implies that triangles $A^*B^*C^*$ and XYZ are perspective with center I: $A^*I \perp EF$ because EF is the polar line of A^* with respect to Γ. Because $EF \parallel BC$ and $BC \perp XI$, the assertion follows.

Lemma 2. *The lines XY, BI, EF, and AC^* are concurrent at a point of $\mathcal{C}_b$, as are the lines YZ, BI, DE, and AB^** (see Figure 2).

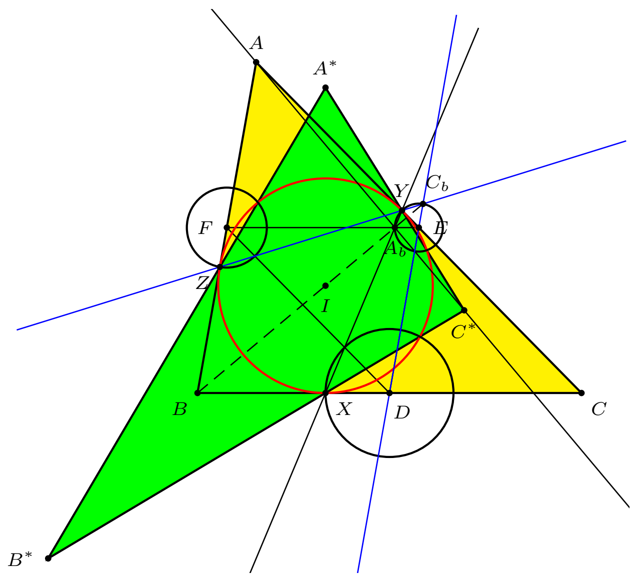


Figure 2.

Proof. Let A_b as the point on EF, on the same side of F as E, so that $FA_b = FA$.

(i) Because $FA = FA_b = FB$, the points A, A_b and B all lie on a circle with center F. This implies that $\angle ABC = \angle AFA_b = 2\angle ABA_b$, yielding $\angle ABI = \angle ABA_b$. This shows that A_b lies on BI.

(ii) Because $YC = \frac{1}{2}(AC + CB - BA) = EC + EF - FA$, we have

$$EY = YC - EC = EF - FA = FE - FA_b = EA_b,$$

showing that A_b lies on $\mathcal{C}_b$. Also, noting that $CX = CY$, we have $\dfrac{EY}{CY} = \dfrac{EA_b}{CX}$. This implies that triangles EYA_b and CYX are isosceles and similar. From this we deduce that A_b lies on XY.

A similar argument shows that DE, BI, YZ are concurrent at a point C_b on the circle $\mathcal{C}_b$. We will use this to prove the last part of this lemma.

(iii) Because YZ and DE are the polar lines of A and C^* with respect to Γ, AC^* is the polar line of C_b, which also lies on BI. This implies that $AC^* \perp BI$, so the intersection of AC^* and BI lies on the circle with diameter AB. We have shown that A_b lies on this circle, and on BI, so A_b also lies on AC^*.

Similarly, C_b also lies on the line AB^*. □

Note that the points A_b and C_b are the orthogonal projections of A and C on BI. Analogous statements can be made of quadruples of lines intersecting on the circles $\mathcal{C}_a$ and $\mathcal{C}_c$. Reference to this configuration can be found, for example, in a problem on the $2002 - 2003$ Hungarian Mathematical Olympiad. A solution and further references can be found in *Crux Mathematicorum with Mathematical Mayhem*, 33 (2007) 415–416.

We are now ready for our first theorem, conjectured in 2002 by D. Grinberg [2].

Theorem 3. *The points A^*, B^*, and C^* are the respective orthocenters of triangles BIC, CIA, and AIB.*

Proof. Because the point A_b lies on the polar lines of A^* and C with respect to Γ, we know that $A^*C \perp BI$. Combining this with the fact that $A^*I \perp BC$ we conclude that A^* is indeed the orthocenter of triangle BIC. □

Theorem 4. *The medial triangle DEF is perspective with triangle $A^*B^*C^*$, at the Mittenpunkt M_t* [1] *of triangle ABC* (see Figure 3).

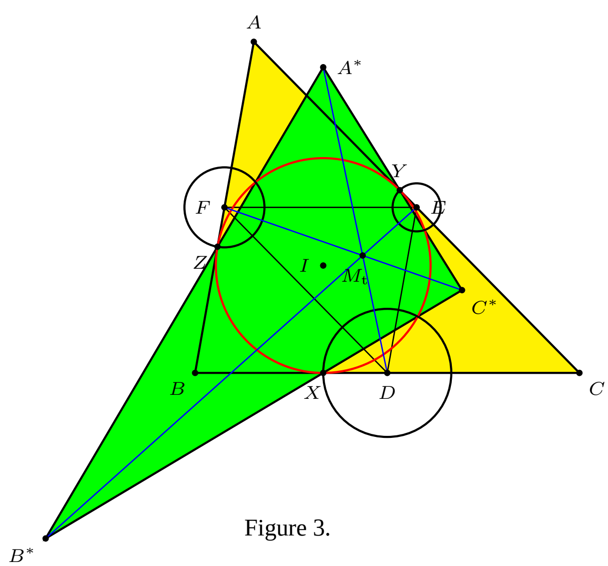


Figure 3.

Proof. Because A^*C is perpendicular to BI, it is parallel to the external bisector of angle B. A similar argument holds for BA^*, so we conclude that A^*BI_aC is a parallelogram. It follows that A^*, D, and I_a are collinear. This shows that M_t lies on I_aD, and similar arguments show that M_t lies on the lines I_bE and I_cF. □

We already know that triangle $A^*B^*C^*$ and triangle XYZ are perspective at the incenter I. By proving Theorem 4, we have in fact found two additional triangles that are perspective with triangle $A^*B^*C^*$: the medial triangle DEF and the

[1]The Mittenpunkt (called $X(9)$ in [4]) is the point of concurrency of the lines joining D to the excenter I_a, E to the excenter I_b, and C to the excenter I_c. It is also the symmedian point of the excentral triangle $I_aI_bI_c$.

excentral triangle $I_aI_bI_c$, both with center M_{t}. This is however just a taste of the many special properties of triangle $A^*B^*C^*$, which will be treated throughout the rest of this paper.

Theorem 3 shows that B, C, A^*, I are four points that form an orthocentric system. A consequence of this is that I is the orthocenter of triangles A^*BC, AB^*C, ABC^*. In the following theorem we prove a similar result that will produce an unexpected point.

Theorem 5. *The Nagel point N_{a} of triangle ABC is the common orthocenter of triangles AB^*C^*, A^*BC^*, A^*B^*C.*

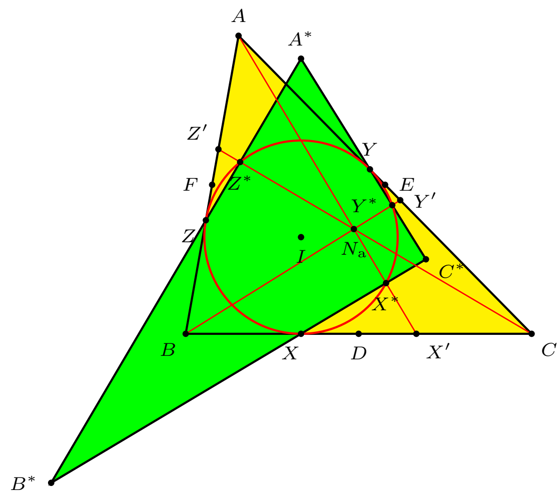


Figure 4.

Proof. Consider the homothety $\zeta := \mathsf{h}(D, -1)$. [2] This carries A into the vertex A' of the anticomplementary triangle $A'B'C'$ of ABC. It follows from Theorem 4 that $\zeta(A^*) = I_a$. This implies that $A'A^*$ is the bisector of $\angle BA'C$.

The Nagel line is the line IG joining the incenter and the centroid. It is so named because it also contains the Nagel point N_{a}. Since $2IG = GN_{\mathrm{a}}$, the Nagel point N_{a} is the incenter of the anticomplementary triangle. This implies that $A'N_{\mathrm{a}}$ is the bisector of $\angle BA'C$. Hence, ζ carries A^*N_{a} into AI, so A^*N_{a} and AI are parallel. From this, $A^*N_{\mathrm{a}} \perp CB^*$.

Similarly, we deduce that $B^*N_{\mathrm{a}} \perp CA^*$, so N_{a} is the orthocenter of triangle A^*B^*C. □

The next theorem was proved by J.-P. Ehrmann in [1] using barycentric coordinates. We present a synthetic proof here.

Theorem 6 (Ehrmann)**.** *The centroid G^* of triangle $A^*B^*C^*$ is the point dividing IH in the ratio $IG^* : G^*H = 2 : 1$.*

[2] A homothety with center P and factor k is denoted by $\mathsf{h}(P, k)$.

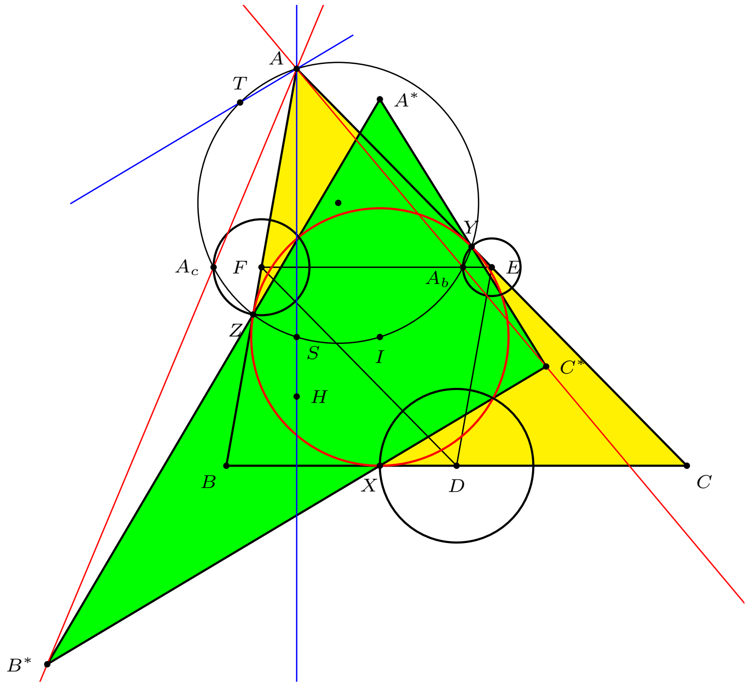


Figure 5.

Proof. The four points A, A_b, I, A_c all lie on a circle with diameter IA, which we will call $\mathcal{C}'_a$. Let H be the orthocenter of triangle ABC, and S the (second) intersection of $\mathcal{C}'_a$ with the altitude AH. Construct also the parallel AT to B^*C^* through A to intersect the circle at T (see Figure 5).

Denote by R_b and R_c the circumradii of triangles AIC and AIB respectively. Because C^* is the orthocenter of triangle AIB, we can write $AC^* = R_c \cdot \cos\frac{A}{2}$, and similarly for AB^*. Using this and the property $B^*C^* \parallel AT$, we have

$$\frac{\sin TAA_b}{\sin TAA_c} = \frac{\sin AC^*B^*}{\sin AB^*C^*} = \frac{AB^*}{AC^*} = \frac{R_b}{R_c} = \frac{\sin\frac{B}{2}}{\sin\frac{C}{2}} = \frac{IC}{IB}.$$

The points A_b, A_c are on EF according to Lemma 2, so triangle IA_bA_c and triangle IBC are similar. This implies $\dfrac{IC}{IB} = \dfrac{IA_c}{IA_b}$.

In any triangle, the orthocenter and circumcenter are known to be each other's isogonal conjugates. Applying this to triangle AA_bA_c, we find that $\angle SAA_b = \angle A_cAI$. We can now see that $\dfrac{SA_b}{SA_c} = \dfrac{IA_c}{IA_b}$.

Combining these results, we obtain

$$\frac{SA_b}{SA_c} = \frac{IA_c}{IA_b} = \frac{IC}{IB} = \frac{\sin TAA_b}{\sin TAA_c} = \frac{TA_b}{TA_c}.$$

This proves that $TA_c \cdot SA_b = SA_c \cdot TA_b$, so TA_cSA_b is a harmonic quadrilateral. It follows that AC^*, AB^* divide AH, AT harmonically. Because $AT \parallel B^*C^*$, we know that AH must pass through the midpoint of B^*C^*.

Let us call D^* the midpoint of B^*C^*, and consider the homothety $\xi = \mathsf{h}(G^*, -2)$. Because ξ takes D^* to B^* while $AH \parallel A^*X$, we know that ξ takes AH to A^*X. Similar arguments applied to B and B^* establish that ξ takes H to I. □

2. Two more triads of circles

Consider again the orthogonal projections A_b, A_c of A on the bisectors BI and CI. It is clear that the circle $\mathcal{C}'_a$ with diameter IA in Theorem 6 contains the points Y and Z as well. Similarly, we consider the circles $\mathcal{C}'_b$ and $\mathcal{C}'_c$ with diameters IB and IC (see Figure 6). It is easy to determine the intersections of the circles from the two triads $\mathcal{C}_a$, $\mathcal{C}_b$, $\mathcal{C}_c$, and $\mathcal{C}'_a$, $\mathcal{C}'_b$, $\mathcal{C}'_c$, which we summarize in the following table.

Table 1. Intersections of circles

	$\mathcal{C}'_a$	$\mathcal{C}'_b$	$\mathcal{C}'_c$
$\mathcal{C}_a$		$X,\ B_a$	$X,\ C_a$
$\mathcal{C}_b$	$Y,\ A_b$		$Y,\ X_b$
$\mathcal{C}_c$	$Z,\ A_c$	$Z,\ B_c$	

Now we introduce another triad of circles.

Let X^* (respectively Y^*, Z^*) be the intersection of Γ with $\mathcal{C}_a$ (respectively $\mathcal{C}_b$, $\mathcal{C}_c$) different from X (respectively Y, Z). Consider also the orthogonal projections A^*_b and A^*_c of A^* onto B^*N_{a} and C^*N_{a}, and similarly defined B^*_a, B^*_c, C^*_a, C^*_b.

Lemma 7. *The six points A^*, A^*_b, A^*_c, Y^*, Z^*, and N_{a} all lie on the circle with diameter A^*N_{a}* (see Figure 6).

Proof. The points A^*_b and A^*_c lie on the circle with diameter A^*N_{a} by definition.

We know that the Nagel point and the Gergonne point are isotomic conjugates, so if we call Y' the intersection of BN_{a} and AC, it follows that $YE = Y'E$. Therefore, Y' lies on $\mathcal{C}_b$.

Clearly YY' is a diameter of $\mathcal{C}_b$. It follows from Theorem 5 that BN_a is perpendicular to A^*C^*, so their intersection point must lie on $\mathcal{C}_b$. Since Y^* is the intersection point of A^*C^* and $\mathcal{C}_b$ different from Y, it follows that Y^* lies on BN_{a}.

Combining the above results, we obtain that $N_{\mathrm{a}}Y^* \perp A^*Y^*$, so Y^* lies on the circle with diameter A^*N_{a}. A similar proof holds for Z^*. □

We will call this circle $\mathcal{C}^*_a$. Likewise, $\mathcal{C}^*_b$ and $\mathcal{C}^*_c$ are the ones with diameters B^*N_{a} and C^*N_{a}. Here are the intersections of the circles in the two triads $\mathcal{C}_a$, $\mathcal{C}_b$, $\mathcal{C}_c$, and $\mathcal{C}^*_a$, $\mathcal{C}^*_b$, $\mathcal{C}^*_c$.

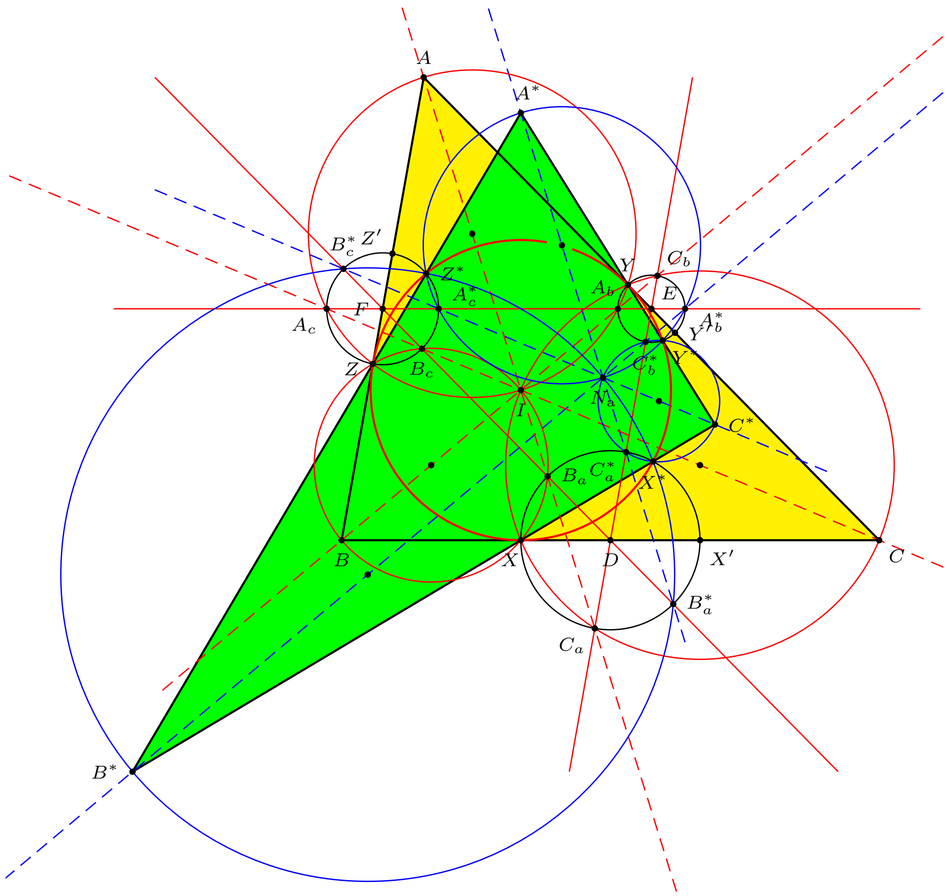


Figure 6.

Table 2. Intersections of circles

	$\mathcal{C}_a^*$	$\mathcal{C}_b^*$	$\mathcal{C}_c^*$
$\mathcal{C}_a$		$X^*,\ B_a^*$	$X^*,\ C_a^*$
$\mathcal{C}_b$	$Y^*,\ A_b^*$		$Y^*,\ X_b^*$
$\mathcal{C}_c$	$Z^*,\ A_c^*$	$Z^*,\ B_c^*$	

Lemma 8. *The circle* $\mathcal{C}_a^*$ *intersects* $\mathcal{C}_b$ *in the points* Y^* *and* A_b^*. *The point* A_b^* *lies on* EF (see Figure 7).

Proof. The point Y^* lies on $\mathcal{C}_b$ by definition, and on $\mathcal{C}_a^*$ by Lemma 7.

Consider the homothety $\phi := \mathsf{h}(E, -1)$. We already know that $\phi(AC^*) = CA^*$ and $\phi(BI) = B^*N_{\mathrm{a}}$. This shows that the intersection points are mapped onto each other, or $\phi(A_b) = A_b^*$. It follows that A_b^* lies on $\mathcal{C}_b$ and EF. $\square$

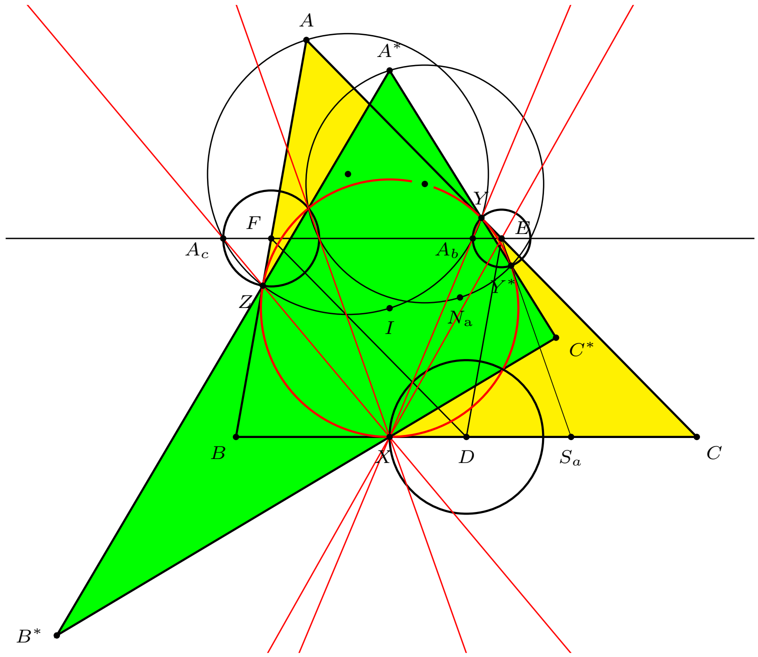


Figure 7.

The two triads of circles have some remarkable properties, strongly related to the Nagel line and eventually to the Feuerbach point. We will start with a property that may be helpful later on.

Theorem 9. *The point X has equal powers with respect to the circles $\mathcal{C}_b$, $\mathcal{C}_c$, $\mathcal{C}_a^*$, and $\mathcal{C}'_a$* (see Figure 7).

Proof. Let us call S_a the intersection of EY^* and BC, and S_b the intersection of XY^* and EF. Because EY^* is tangent to Γ, we have $S_aY^* = S_aX$. Because triangles XS_aY^* and S_bEY^* are similar, it follows that $EY^* = ES_b$. This implies that S_b lies on $\mathcal{C}_b$ so in fact S_b and A_b^* coincide. This shows that X lies on $Y^*A_b^*$. Similar arguments can be used to prove that X lies on $Z^*A_c^*$.

From Table 1, it follows that A_bY (respectively A_cZ) is the radical axis of the circles $\mathcal{C}'_a$ and $\mathcal{C}_b$ (respectively $\mathcal{C}_c$). Lemma 2 implies that X lies on both A_bY and A_cZ, so it is the radical center of $\mathcal{C}'_a$, $\mathcal{C}_b$ and $\mathcal{C}_c$.

From Lemma 8, it follows that $Y^*A_b^*$ (respectively $Z^*A_c^*$) is the radical axis of the circles $\mathcal{C}_b$ and $\mathcal{C}_a^*$ (respectively $\mathcal{C}_c$ and $\mathcal{C}_b^*$). We have just proved that X lies on both $Y^*A_b^*$ and $Z^*A_c^*$, so it is the radical center of $\mathcal{C}_a^*$, $\mathcal{C}_b$, and $\mathcal{C}_c$. The conclusion follows. $\square$

3. Some similitude centers and the Nagel line

Denote by U, V, W the intersections of the Nagel line IG with the lines EF, DF and DE respectively (see Figure 8).

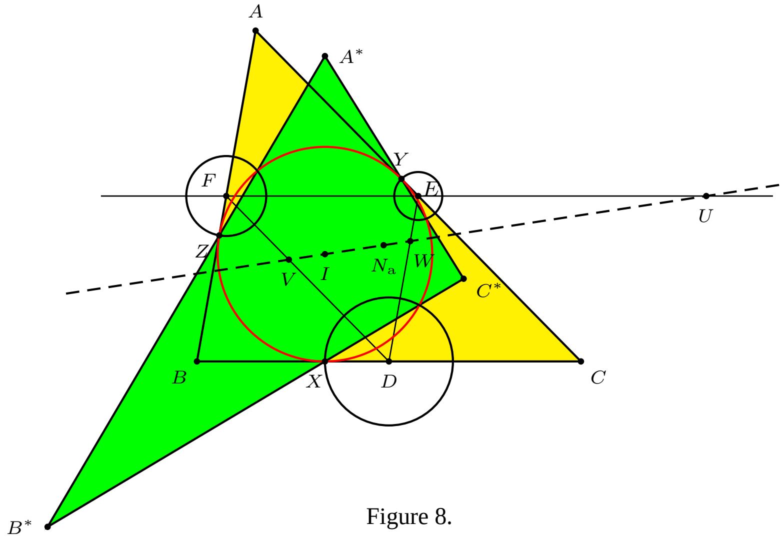


Figure 8.

Theorem 10. *The point U is a center of similitude of circles $\mathcal{C}'_a$ and $\mathcal{C}^*_a$. Likewise, V is a center of similitude of circles $\mathcal{C}'_b$ and $\mathcal{C}^*_b$, and W of $\mathcal{C}'_c$ and $\mathcal{C}^*_c$.*

Proof. We know from Lemma 2 and Theorem 5 that $A^*A^*_b \parallel AA_b$, and $AI \parallel A^*N_{\rm a}$, as well as $A^*_bN_{\rm a} \parallel A_bI$. Hence triangles triangle $A^*N_{\rm a}A^*_b$ and triangle AIA_b have parallel sides. It follows from Desargues' theorem that AA^*, $A_bA^*_b$, $IN_{\rm a}$ are concurrent. Clearly, the point of concurrency is a center of similitude of both triangles, and therefore also of their circumcircles, $\mathcal{C}^*_a$ and $\mathcal{C}_a$. This point of concurrency is the intersection point of EF and the Nagel line as shown above, so the theorem is proved. □

Theorem 11. *The point U is a center of similitude of circles $\mathcal{C}_b$ and $\mathcal{C}_c$. Likewise, V is a center of similitude of circles $\mathcal{C}_c$ and $\mathcal{C}_a$, and W of $\mathcal{C}_a$ and $\mathcal{C}_b$.*

Proof. By Theorem 10, we know that

$$\frac{A_bU}{A_cU} = \frac{A^*_bU}{A^*_cU}. \tag{1}$$

By Table 1 and Theorem 8, we know that A_b, A^*_c lie on $\mathcal{C}_c$ and A_b, A^*_b lie on $\mathcal{C}_b$. Knowing that U lies on EF, the line connecting the centers of $\mathcal{C}_b$ and $\mathcal{C}_c$, relation (1) now directly expresses that U is a center of similitude of $\mathcal{C}_b$ and $\mathcal{C}_c$. □

Depending on the shape of triangle ABC, the center of similitude of $\mathcal{C}_b$ and $\mathcal{C}_c$ which occurs in the above theorem could be either external or internal. Whichever it is, we will meet the other in the next theorem.

Theorem 12. *The lines BV and CW intersect at a point on EF. This point is the center of similitude different from U of $\mathcal{C}_b$ and $\mathcal{C}_c$* (see Figure 9).

Proof. Let us call U' the point of intersection of BV and EF. We have that $G = BE \cap CF$ and $V = DF \cap BU'$. By the theorem of Pappus-Pascal applied to the collinear triples E, U', F and C, D, B, the intersection of $U'C$ and DE must lie

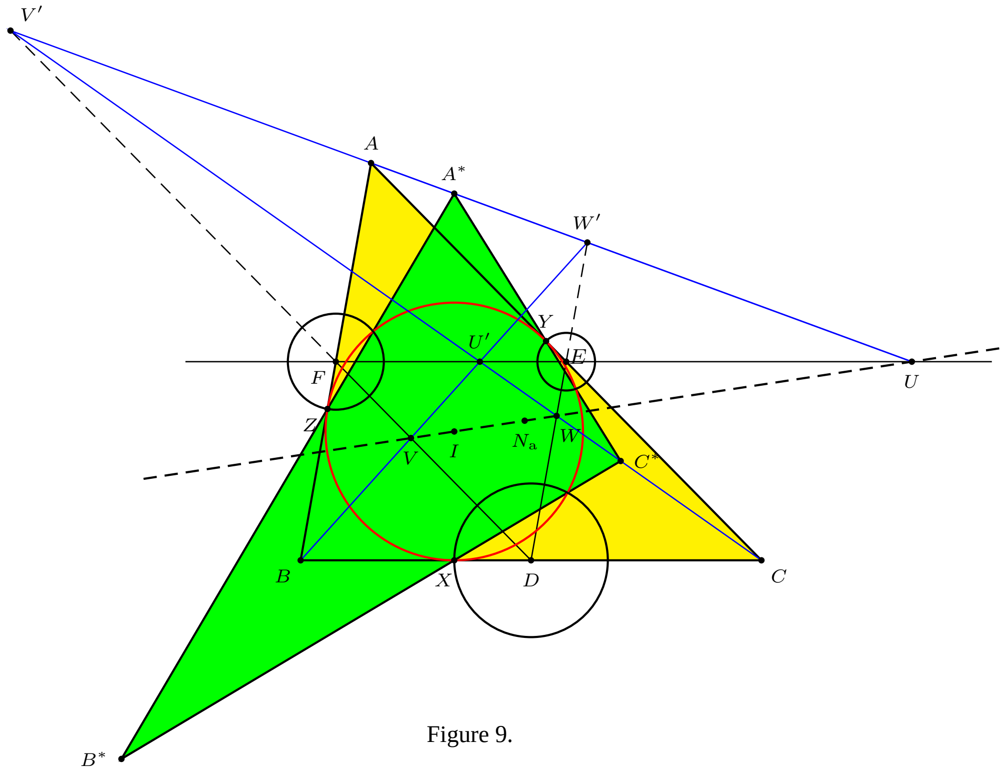


Figure 9.

on GV, and therefore, it must be W. It follows that BV and CW are concurrent in the point U' on EF.

By similarity of triangles, we have $\frac{DB}{DV} = \frac{FU'}{FV}$ and $\frac{DC}{DW} = \frac{EU'}{EW}$. This gives us:

$$\frac{WE}{WD} \cdot \frac{VD}{VF} \cdot \frac{U'F}{U'E} = \frac{EU'}{DC} \cdot \frac{DB}{FU'} \cdot \frac{U'F}{U'E} = \frac{DB}{DC} = -1.$$

Hence DU', EV, FW are concurrent by Ceva's theorem applied to triangle DEF. By Menelaus's theorem applied to the transversal WVU we obtain that U' is the harmonic conjugate of U with respect to E and F. Therefore, it is a center of similitude of $\mathcal{C}_b$ and $\mathcal{C}_c$. □

Let us call X'', Y'', Z'' the antipodes of X, Y, Z respectively on the incircle Γ.

Theorem 13. *The point X'' is the center of similitude different from U of circles $\mathcal{C}'_a$ and $\mathcal{C}^*_a$. Likewise, Y'' is a center of similitude of $\mathcal{C}'_b$ and $\mathcal{C}^*_b$, and Z'' one of $\mathcal{C}'_c$ and $\mathcal{C}^*_c$.*

Proof. We construct the line $l_{X''}$ which passes through X'' and is parallel to BC. The triangle bounded by $AC, AB, l_{X''}$ has Γ as its excircle opposite A. This implies that its Nagel point lies on AX'', and because it is homothetic to triangle ABC from center A, we have that X'' lies on AN_a. We have also proved that A^*,

I, X are collinear, so it follows that X'' lies on A^*I. Hence the intersection point of AN_{a} and A^*I is X'', a center of similitude of $\mathcal{C}_a$ and $\mathcal{C}_a^*$, different from U. □

Having classified all similitude centers of the pairs of circles $\mathcal{C}_a', \mathcal{C}_a^*$ and $\mathcal{C}_b, \mathcal{C}_c$ (and we obtain similar results for the other pairs of circles), we now establish a surprising concurrency. Not only does this involve hitherto inconspicuous points introduced at the beginning of §2, it also strongly relates the triangle $A^*B^*C^*$ to the Nagel line of ABC.

Theorem 14. *The triangles $A^*B^*C^*$ and $X^*Y^*Z^*$ are perspective at a point on the Nagel line* (see Figure 10).

Proof. Considering the powers of A^*, B^*, C^* with respect to the incircle Γ of triangle ABC, we have

$$A^*Z{\cdot}A^*Z^* = A^*Y^*{\cdot}A^*Y, \quad B^*X^*{\cdot}B^*X = B^*Z^*{\cdot}B^*Z, \quad C^*X{\cdot}C^*X^* = C^*Y{\cdot}C^*Y^*.$$

From these,

$$\begin{aligned}\frac{B^*X^*}{X^*C^*}\cdot\frac{C^*Y^*}{Y^*A^*}\cdot\frac{A^*Z^*}{Z^*B^*} &= \frac{B^*X^*}{Z^*B^*}\cdot\frac{C^*Y^*}{X^*C^*}\cdot\frac{A^*Z^*}{Y^*A^*}\\ &= \frac{B^*Z}{XB^*}\cdot\frac{C^*X}{YC^*}\cdot\frac{A^*Y}{ZA^*} = \frac{B^*Z}{ZA^*}\cdot\frac{C^*X}{XB^*}\cdot\frac{A^*Y}{YC^*} = 1\end{aligned}$$

since $A^*B^*C^*$ and XYZ are perspective. By Ceva's theorem, we conclude that $A^*B^*C^*$ and $X^*Y^*Z^*$ are perspective, *i.e.*, A^*X^*, B^*Y^*, C^*Z^* intersect at a point Q.

To prove that Q lies on the Nagel line, however, we have to go a considerable step further. First, note that $A_b^*Y^*ZA_c$ is a cyclic quadrilateral, because $XA_b^* \cdot XY^* = XA_c \cdot XZ$ using Theorem 9. We call N_c the point where DE meets ZY^* and working with directed angles we deduce that

$$\measuredangle ZY^*A_b^* = \measuredangle ZA_cU = \measuredangle N_cA_bU = \measuredangle N_cA_bA_b^* = \measuredangle N_cY^*A_b^*$$

We conclude that N_c, Y^*, Z and therefore also Z, Y^*, U are collinear. Similar proofs show that

$$U \in YZ^*,\ V \in XZ^*,\ V \in ZX^*, W \in XY^*,\ W \in YX^*.$$

If we construct the intersection points

$$J = FZ^* \cap BC \quad \text{and} \quad K = DX^* \cap AB,$$

we know that the pole of JK with respect to Γ is the intersection of XZ^* with X^*Z, which is V. The fact that JK is the polar line of V shows that B^* lies on JK, and that JK is perpendicular to the Nagel line.

Now we construct the points

$$O = EF \cap DX^*, \quad P = DE \cap FZ^*, \quad R = OD \cap FZ^*.$$

Recalling Lemma 1 and the definitions of X^* and Z^* following Lemma 3, we see that OP is the polar line of Q with respect to Γ. We also know by similarity of the triangles ORF and DRJ that $OR \cdot RJ = DR \cdot RF$. Likewise, we find by similarity of the triangles KFR and DPR that $RF \cdot DR = KR \cdot RP$. Combining these identities we get $OR \cdot RJ = KR \cdot RP$, and this proves that OP and JK are

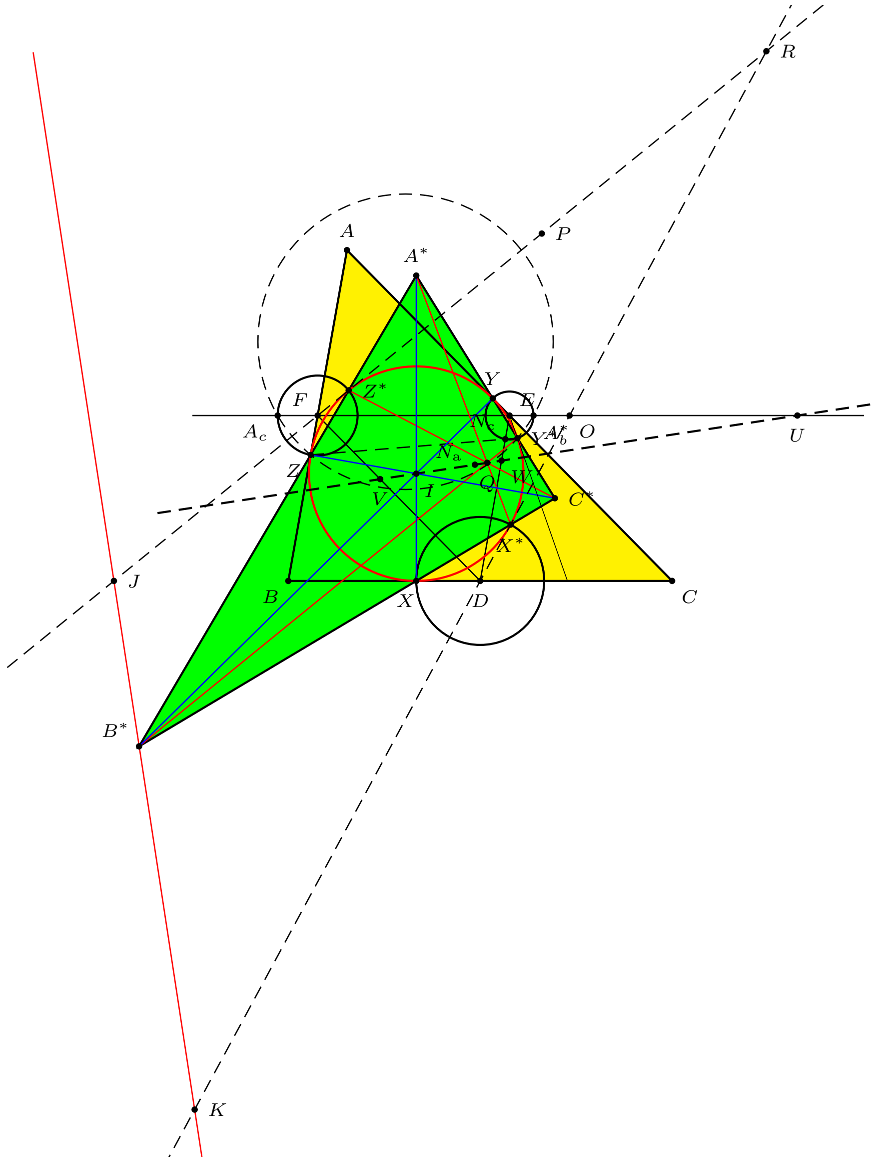


Figure 10.

parallel. Thus, OP is perpendicular to the Nagel line, whence its pole Q lies on the Nagel line. □

4. The Feuerbach point

Theorem 15. *The line connecting the centers of $\mathcal{C}'_a$ and $\mathcal{C}^*_a$ passes through the Feuerbach point of triangle ABC; so do the lines joining the centers of $\mathcal{C}'_b$, $\mathcal{C}^*_b$ and those of $\mathcal{C}'_c$, $\mathcal{C}^*_c$* (see Figure 11).

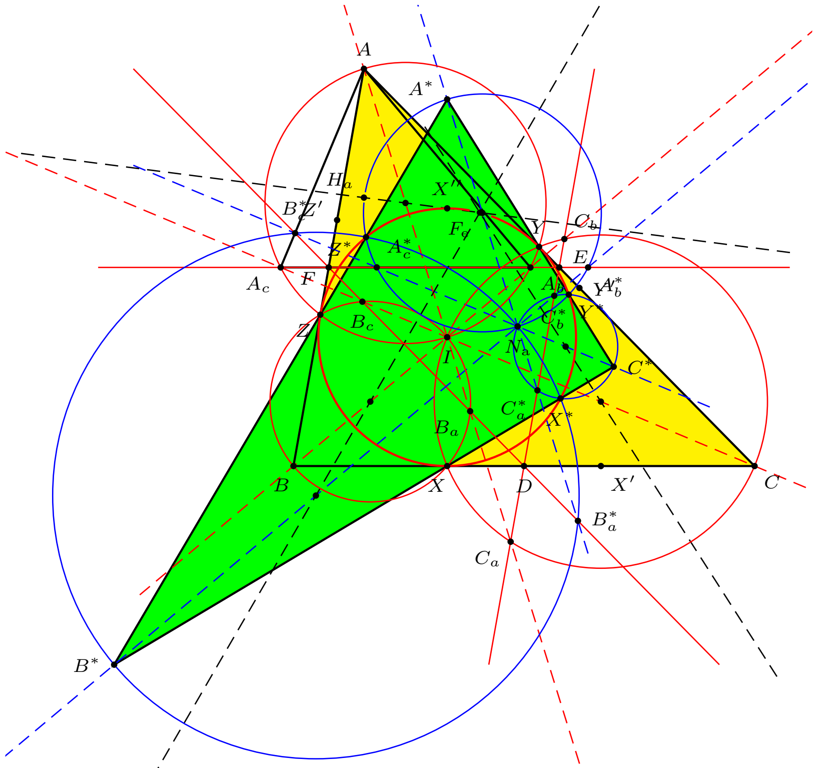


Figure 11.

Proof. Let us call H_a the orthocenter of triangle AA_bA_c. Since AI is the diameter of $\mathcal{C}'_a$ (as in the proof of Theorem 6), we have $AH_a = AI \cdot \cos A_bAA_c = AI \cdot \sin \frac{A}{2}$, where the last equality follows from $\frac{\pi}{2} - \frac{A}{2} = \angle BIC = \angle A_bIA_c = \pi - \angle A_bAA_c$. By observing triangle AIZ, for instance, and writing r for the inradius of triangle ABC we find that

$$AH_a = AI \cdot \sin \frac{A}{2} = r.$$

Now consider the homothety χ with factor -1 centered at the midpoint of AI (which is also the center of $\mathcal{C}'_a$). We have that $\chi(A) = I$ and $\chi(AH_a) = A^*I$. But we just proved that $AH_a = r = IX''$, so it follows that $\chi(H_a) = X''$. This shows that X'' lies on the Euler line of triangle AA_bA_c, so the line joining the centers of $\mathcal{C}'_a$ and $\mathcal{C}^*_a$ is exactly the Euler line of triangle AA_aA_b.

According to A. Hatzipolakis ([3]; see also [5]), the Euler line of triangle AA_bA_c passes through the Feuerbach point of triangle ABC. From this our conclusion follows immediately. □

In summary, the Euler line of triangle AA_bA_c and the Nagel line of triangle ABC intersect on EF. We will show that the circles $\mathcal{C}_a$, $\mathcal{C}^*_a$ have another amazing connection to the Feuerbach point.

Theorem 16. *The radical axis of $\mathcal{C}'_a$ and $\mathcal{C}^*_a$ passes through the Feuerbach point of triangle ABC; so do the radical axes of $\mathcal{C}'_b$, $\mathcal{C}^*_b$, and of $\mathcal{C}'_c$, $\mathcal{C}^*_c$* (see Figure 12).

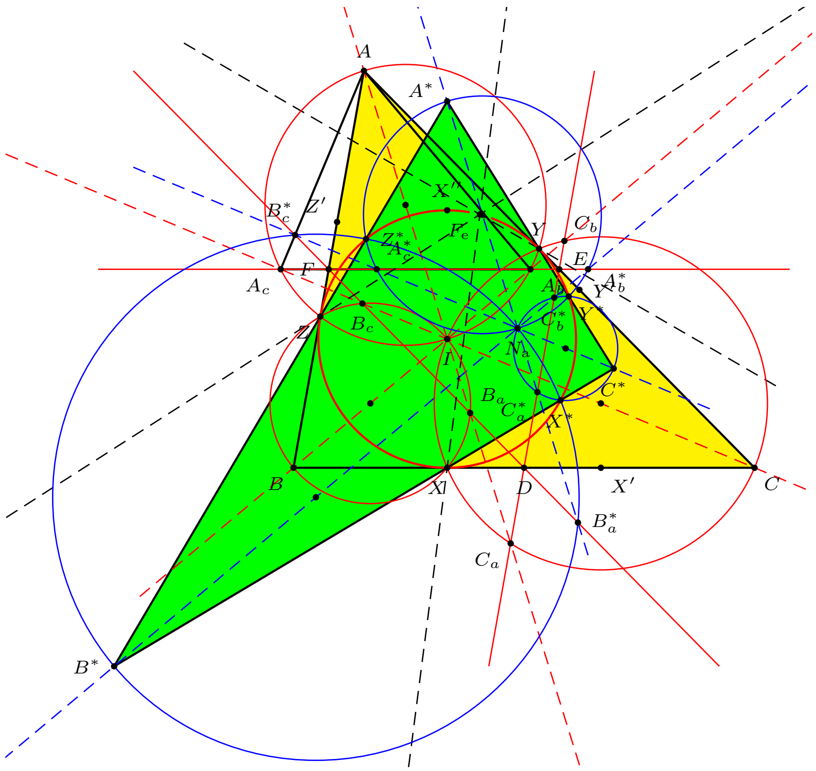


Figure 12.

Proof. Because the radical axis of two circles is perpendicular to the line joining the centers of the circles, the radical axis $\mathcal{R}_a$ of $\mathcal{C}'_a$ and $\mathcal{C}^*_a$ is perpendicular to the Euler line of triangle AA_bA_c. Since this Euler line contains X'', and $\mathcal{R}_a$ contains X (see Theorem 9), their intersection lies on Γ. This point is also the intersection point of the Euler line with Γ, different from X''. It is the Feuerbach point of ABC. □

References

[1] J. P. Ehrmann, Hyacinthos message 6130, December 10, 2002.
[2] D. Grinberg, Hyacinthos message 6194, December 21, 2002.
[3] A. Hatzipolakis, Hyacinthos message 10485, September 18, 2004.
[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.
[5] J. Vonk, The Feuerbach point and reflections of the Euler line, *Forum Geom.*, to appear.

Jan Vonk: Groenstraat 70, 9340 Lede, Belgium
E-mail address: `jan.vonk.jv@gmail.com`

Forum Geometricorum
Volume 8 (2008) 197–200.

FORUM GEOM
ISSN 1534-1178

A Purely Geometric Proof of the Uniqueness of a Triangle With Prescribed Angle Bisectors

Victor Oxman

Abstract. We give a purely geometric proof of triangle congruence on three angle bisectors without using trigonometry, analysis and the formulas for triangle angle bisector length.

It is known that three given positive numbers determine a unique triangle with the angle bisectors lengths equal to these numbers [1]. Therefore two triangles are congruent on three angle bisectors. In this note we give a pure geometric proof of this fact. We emphasize that the proof does not use trigonometry, analysis and the formulas for triangle angle bisector length, but only synthetic reasoning.

Lemma 1. *Suppose triangles ABC and $AB'C'$ have a common angle at A, and that the incircle of $AB'C'$ is not greater than the incircle of ABC. If $C' > C$, then the bisector of C' is less than the bisector of C.*

Proof. Let CF and $C'F'$ be the bisectors of angles C, C' of triangles ABC, $AB'C'$. Assuming $C' > C$, we shall prove that $C'F' < CF$.

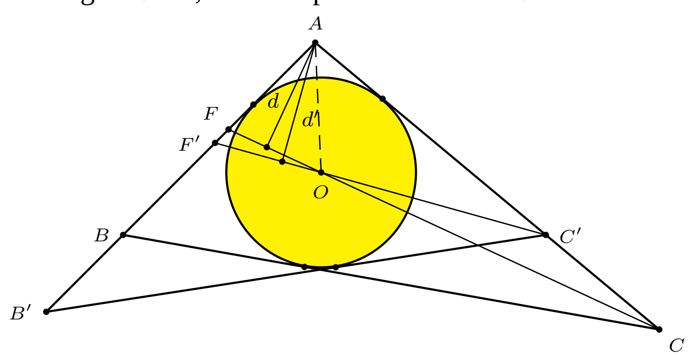


Figure 1.

Case 1. The triangles have equal incircles (see Figure 1). Without loss of generality assume $B > B'$ and the point C' between A and C. Let O be the center of the common incircle of the triangles. It is known that $OF < OC$ and $OF' < OC'$. Hence, in areas,

$$\triangle OFF' < \triangle OCC'. \tag{1}$$

Publication Date: December 1, 2008. Communicating Editor: Nikolaos Dergiades.
The author thanks Nikolaos Dergiades for his helps in the preparation of this paper.

Let d, d' be the distances of A from the bisectors CF, $C'F'$ respectively. Since $\angle AOF' = \angle OAC' + \angle AC'O = \frac{A+C'}{2} < 90°$, we have $\angle AOF < \angle AOF' < 90°$, and $d < d'$. Now, from (1), we have

$$\triangle OFF' + \triangle OC'AF < \triangle OCC' + OC'AF.$$

This gives $\triangle AF'C' < \triangle AFC$, or $\frac{1}{2}d' \cdot C'F' < \frac{1}{2}d \cdot CF$. Since $d < d'$, we have $C'F' < CF$.

Case 2. The incircle of $AB'C'$ is smaller than the incircle of ABC (see Figure 2). Since the incircle of $AB'C'$ is inside triangle ABC, we construct a tangent $B''C''$ parallel to BC that is closer to A than BC. Let $C''F''$ be the bisector of triangle $AB''C''$. We have $C''F''||CF$ and

$$C''F'' < CF. \tag{2}$$

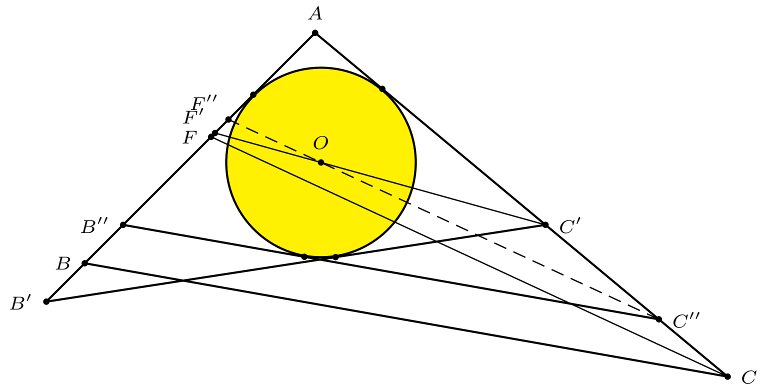


Figure 2.

Since $\angle AC''B'' = \angle ACB < \angle AC'B'$, from Case 1 we have

$$C'F' < C''F'' \tag{3}$$

From (2) and (3) we have $C'F' < CF$. □

Lemma 2. *Suppose triangles ABC and $AB'C'$ have a common angle at A, and a common angle bisector AD, the common angle not greater than any other angle of $AB'C'$. If $C' > C$, then the bisector of C' is less than the bisector of C.*

Proof. If the incirle of triangle $AB'C'$ is not greater than that of ABC, then the result follows from Lemma 1.

Assume the incircle of $AB'C'$ greater than the incircle of ABC (see Figure 3). The line BC cuts the incircle of $AB'C'$ incircle. Hence, the tangent from C to this incircle meets AB' at a point B'' between B and B'. Let CF, $C'F'$ be the bisectors of angles C, C' in triangles ABC and $AB'C'$ respectively. We shall prove that $C'F' < CF$.

Consider also the bisector CF'' in triangle $AB''C$. Since B is between A and B'', F is between A and F''. From lemma 1 we have

$$C'F' < CF'' \tag{4}$$

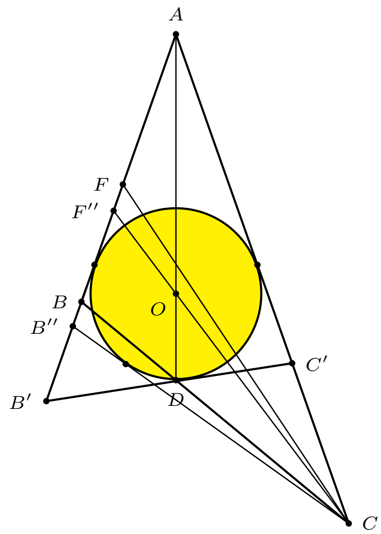


Figure 3.

Since $\angle CB''A > \angle C'B'A \geq \angle B'AC'$, we have $\angle CF''A > 90°$, and from triangle CFF''

$$CF'' < CF. \tag{5}$$

From (4) and (5) we conclude that $C'F' < CF$. □

Now we prove the main theorem of this note.

Theorem 3. *If three internal angle bisectors of triangle ABC are respectively equal to three internal angle bisectors of triangle $A'B'C'$, then the triangles are congruent.*

Proof. Denote the angle bisectors of ABC by AD, BE, CF and let $AD = A'D'$, $BE = B'E'$, $CF = C'F'$.

If for the angles of the triangles we have $A = A'$, $B = B'$, $C = C'$, then from the similarity of ABC with $A'B'C'$ and of ABD with $A'B'D'$ we conclude the congruence of ABC with $A'B'C'$.

Let A' be an angle that is not greater than any other angle of triangles $A'B'C'$ and ABC. We construct a triangle AB_1C_1 congruent to $A'B'C'$ that has AD as bisector of angle B_1AC_1.

If $A' = A$ and $C' > C$, then the triangles ABC and AB_1C_1 satisfy the conditions of Lemma 2. It follows that $C'F' < CF$, a contradiction.

If $A' < A$ and the lines AB_1, AC_1 meet BC at the points B_2, C_2 respectively, without loss of generality we assume C_1 between A and C_2, possibly coinciding with C_2 (see Figure 4). Suppose the bisector of angle AC_2B_2 meets AB_2 at F_2 and AB at F_3. Since triangles AB_1C_1 and AB_2C_2 satisfy the conditions of Lemma 2, we have

$$C'F' \leq C_2F_2 < C_2F_3. \tag{6}$$

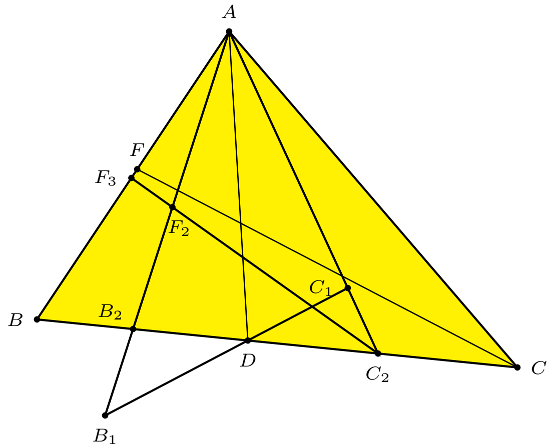


Figure 4.

The incircle of triangle ABC_2 is smaller than that of triangle ABC. Since $\angle AC_2B > \angle ACB$, by Lemma 1, $C_2F_3 < CF$ and from (6) we conclude $C'F' < CF$. This again is a contradiction. Hence, triangles ABC and $A'B'C'$ are congruent. $\square$

References

[1] P. Mironescu and L. Panaitopol, The existence of a triangle with prescribed angle bisector lengths, *Amer. Math. Monthly*, 101 (1994) 58–60.

Victor Oxman: Western Galilee College, P.O.B. 2125 Acre 24121 Israel
E-mail address: victor.oxman@gmail.com

Forum Geometricorum
Volume 8 (2008) 201–204.

FORUM GEOM
ISSN 1534-1178

An Elementary Proof of a Theorem by Emelyanov

Eisso J. Atzema

Abstract. In this note, we provide an alternative proof of a theorem by Lev Emelyanov stating that the Miquel point of any complete quadrilateral (in general position) lies on the nine-point circle of the triangle formed by the diagonals of that same complete quadrilateral.

1. Introduction and terminology

In their recent book on the geometry of conics, Akopyan and Zaslavsky prove a curious theorem by Lev Emelyanov on complete quadrilaterals. Their proof is very concise, but it does rely on the theory of conic sections, as presumably does Emelyanov's original proof. Indeed, it is the authors' contention that the theorem does not seem to allow for a "short and simple" proof without using the so-called inscribed parabola of the complete quadrilateral.[1] In this note, we will show that actually it is possible to avoid the use of conic sections and to give a proof that uses elementary means only. It is left to the reader to decide whether our proof is reasonably short and simple.

Recall that a *complete* quadrilateral is usually defined as the configuration of four given lines, no three of which are concurrent, and the six points at which they intersect each other. For this paper, we will also assume that no two of the lines are parallel. Without loss of generality, we can think of a complete quadrilateral as the configuration associated with a quadrilateral $ABCD$ in the traditional sense with no two sides parallel and no two vertices coinciding, together with the points $F = AD \cap BC$ and $G = AB \cap CD$. By abuse of notation, we will refer to a generic complete quadrilateral as a *complete* quadrilateral $\square ABCD$, where we will assume that none of the sides of $ABCD$ are parallel and no three are concurrent. [2] The lines AC, BD and FG are known as the *diagonals* of $\square ABCD$. Let $AC \cap BD$ be denoted by E_{FG} and so on. Then, the triangle $\triangle E_{AC}E_{BD}E_{FG}$ formed by the diagonals of $\square ABCD$ is usually referred to as the *diagonal triangle* of $\square ABCD$ (see Figure 2). With these notations, we are now ready to prove Emelyanov's Theorem.

Publication Date: December 3, 2008. Communicating Editor: Paul Yiu.

[1] See [1, pp.110–111] for both the proof (which relies on two propositions proved earlier) and the authors' contention.

[2] Thus, for any quadrilateral $ABCD$ with F and G as above, $\square ABCD$, $\square AFCG$, and $\square BGDF$ and so on, all denote the same configuration.

2. Emelyanov's Theorem

We will prove Emelyanov's Theorem as a corollary to a slightly more general result. For this we first need the following lemma (see Figure 1).

Lemma 1. *For any complete quadrilateral* $\Box ABCD$ (as defined above), *let* F_{BC} *be the unique point on* AD *such that* $F_{BC}E_{FG}$ *is parallel to* BC *and let* F_{DA}, G_{AB} *and* G_{CD} *be defined similarly. Finally, let* F_G *and* G_F *be the midpoints of* FE_{FG} *and* GE_{FG}*, respectively. Then* F_{BC}, F_{DA}, G_{AB}, G_{CD} *all four lie on the line* F_GG_F.

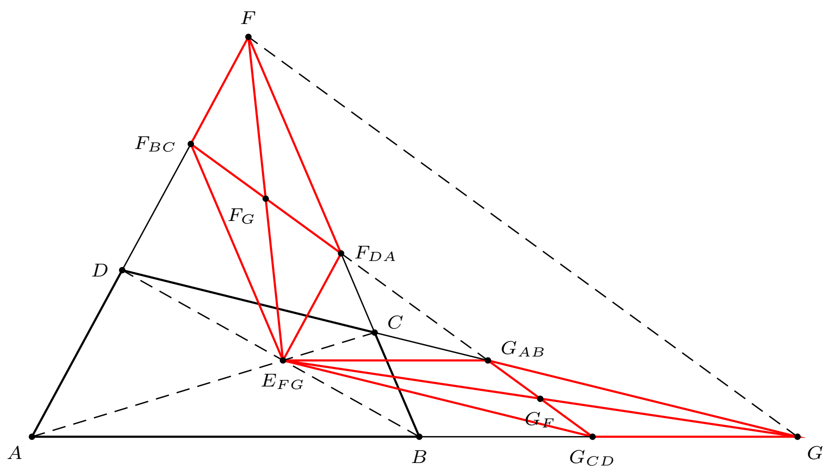


Figure 1. Collinearity of F_{BC}, F_{DA}, G_{AB}, G_{CD} and of F_G, G_F

Proof. Note that by the harmonic property of quadrilaterals, the sides DA and BC are harmonically separated by FE_{FG} and FG. Therefore, the points F_{DA} and F_{BC} are harmonically separated by the points of intersection $FE_{FG} \cap F_{DA}F_{BC}$ and $FG \cap F_{DA}F_{BC}$. By the construction of F_{DA} and F_{BC}, $E_{FG}F_{DA}FF_{BC}$ is a parallelogram and therefore $FE_{FG} \cap F_{DA}F_{BC}$ coincides with F_G. As F_G is also the midpoint of $F_{DA}F_{BC}$, it follows that $FG \cap F_{DA}F_{BC}$ has to be the point at infinity of $F_{DA}F_{BC}$. In other words, FG and $F_{DA}F_{BC}$ are parallel. As F_GG_F is parallel to FG as well and F_G also lies on $F_{DA}F_{BC}$, it follows that $F_{DA}F_{BC}$ and F_GG_F coincide. By the same argument, $G_{AB}G_{CD}$ coincides with F_GG_F as well. It follows that the six points are collinear. $\Box$

Corollary 2. *With the notation introduced above, the directed ratios* $\dfrac{F_{BC}D}{F_{BC}A}$ *and* $\dfrac{F_{DA}C}{F_{DA}B}$ *are equal, as are the ratios* $\dfrac{G_{CD}A}{F_{CD}B}$ *and* $\dfrac{G_{AB}D}{F_{AB}C}$.

Proof. It suffices to prove the first part of the statement. Note that by construction the ratio $\dfrac{F_{BC}D}{F_{BC}A}$ is equal to the cross ratio $[E_{FG}D, E_{FG}A; E_{FG}F_{BC}, E_{FG}F_{DA}]$ of the lines $E_{FG}D$, $E_{FG}A$, $E_{FG}F_{BC}$, and $E_{FG}F_{DA}$. Similarly, the ratio $\dfrac{F_{DA}C}{F_{DA}B}$

equals the cross ratio $[E_{FG}C, E_{FG}B; E_{FG}F_{DA}, E_{FG}F_{BC}]$. As ED is parallel to EB, while EA is parallel to EC, the two cross ratios are equal. Therefore, the two ratios are equal as well. $\square$

We are now ready to derive our main result. We start with a lemma about Miquel points, which we prefer to associate to a complete quadrilateral $\square ABCD$, rather than to $ABCD$.

Lemma 3. *For any quadrilateral* $ABCD$ (with its sides in general position), *the Miquel points of* $\square ABF_{DA}F_{BC}$ *and* $\square CDF_{BC}F_{DA}$ *both coincide with the Miquel point* M *of* $\square ABCD$.

Proof. Let M be constructed as the second point of intersection (other than F) of the circumcircles of $\triangle FAB$ and $\triangle FCD$. By Corollary 2, the ratio of the power of F_{BC} with respect to the circumcircle of $\triangle FCD$ and the power of F_{BC} with respect to the circumcircle of $\triangle FAB$ equals the ratio of the power of F_{DA} with respect to the same two circles. This means that F_{BC} and F_{DA} lie on the same circle of the coaxal system generated by the circumcircles of $\triangle FCD$ and $\triangle FAB$. In other words, F, F_{BC}, F_{DA} and M are co-cyclic. Since M lies on both the circumcircle of $\triangle F_{BC}F_{DA}F$ and the circumcircle of $\triangle FAB$, it follows that M is also the Miquel point of $\square ABF_{DA}F_{BC}$. By a similar argument, M is the Miquel point of $\square CDF_{BC}F_{DA}$ as well. $\square$

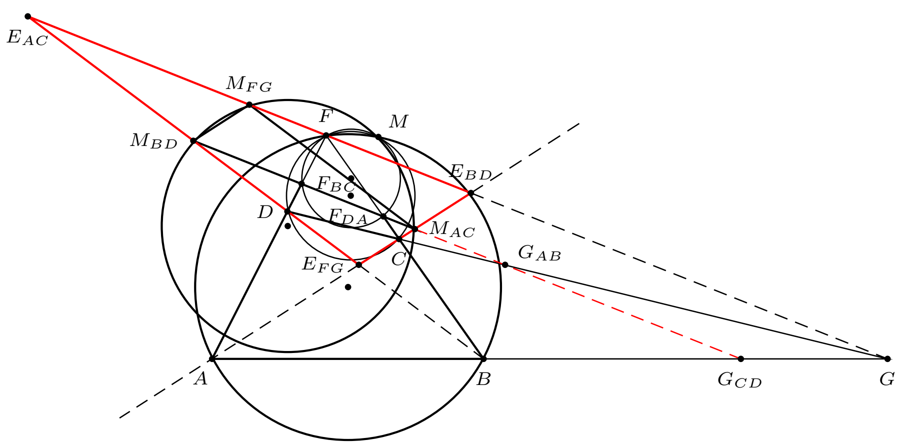


Figure 2. Coincidence of Miquel points

Corollary 4. *For any quadrilateral* $ABCD$ (with sides in general position), *the (orthogonal) projection of* M *on* $F_{BC}F_{DA}$ *lies on the pedal line of* $\square ABCD$.

Proof. By Lemma 3 and the properties of Miquel points, the (orthogonal) projection of M on $F_{BC}F_{DA}$ is collinear with the (orthogonal) projections of M on AB, BF_{DA} and $F_{BC}A$, i.e. its projections on AB, BC, and DA. But for $ABCD$ in general position, the latter points do not all three coincide. As they also lie on the pedal line of $\square ABCD$, they therefore define the pedal line and the (orthogonal) projection of M on $F_{BC}F_{DA}$ has to lie on it. $\square$

Now, let M_{AC} be the midpoint of $E_{BD}E_{FG}$ and so on. Clearly, $M_{AC}M_{BD}$ coincides with $F_{BC}F_{DA}$. Furthermore, Corollary 4 applies to the quadrilaterals $AFCG$ and $BFDG$ as well. Since $\square AFCG$ and $\square BFDG$ coincide with $\square ABCD$, their Miquel points also coincide. These observations immediately lead to our main result.

Theorem 5. *For any quadrilateral $ABCD$* (with sides in general position), *the (orthogonal) projections of the Miquel point M of $\square ABCD$ on the sides of the triangle $\triangle M_{AC}M_{BD}M_{FG}$ all three lie on the pedal line of $\square ABCD$.*

Emelyanov's Theorem follows from Theorem 5 as a corollary.

Corollary 6 (Emelyanov)**.** *For any quadrilateral $ABCD$* (with sides in general position), *the Miquel point M of $\square ABCD$ lies on the nine-point circle of the diagonal triangle $\triangle E_{AC}E_{BD}E_{FG}$ of $\square ABCD$.*

Proof. Since the (orthogonal) projections of M on the sides of $\triangle M_{AC}M_{BD}M_{FG}$ are collinear, M has to lie on the circumcircle of $\triangle M_{AC}M_{BD}M_{FG}$. But this is the same as saying that M lies on the nine-point circle of $\triangle E_{AC}E_{BD}E_{FG}$. $\square$

3. Conclusion

In this note we derived an elementary proof of Emelyanov's Theorem as stated in [**?**] from a more general result. At this point, it is unclear to us whether this Theorem 5 may have any other implications than Emelyanov's Theorem, but it was not our goal to look for such implications. Similarly, we could have shortened our proof a little bit by noting that Corollary 2 implies that $F_{BC}F_{DA}$ is a tangent line to the unique inscribed parabola of $\square ABCD$. The same parabola therefore is also the inscribed parabola to $\square ABF_{DA}F_{BC}$ and $\square CDF_{BC}F_{DA}$. Since the focal point of the parabola inscribing a complete quadrilateral is the Miquel point of the same, Lemma 3 immediately follows. As stated in the introduction, however, our goal was to provide a proof of the theorem without using the theory of conic sections.

Reference

[1] A. V. Akopyan and A. A. Zaslavsky, *Geometry of Conics*, *Mathematical World*, Vol. 26, Amer. Math. Soc. 2007.

Eisso J. Atzema: Department of Mathematics, University of Maine, Orono, Maine 04469, USA
E-mail address: atzema@math.umaine.edu

Forum Geometricorum
Volume 8 (2008) 205–208.

FORUM GEOM
ISSN 1534-1178

A Generalization of Thébault's Theorem on the Concurrency of Three Euler Lines

Shao-Cheng Liu

Abstract. We prove a generalization of Victor Thébault's theorem that if $H_aH_bH_c$ is the orthic triangle of ABC, then the Euler lines of triangles AH_cH_b, BH_aH_c, and CH_bH_a are concurrent at the center of the Jerabek hyperbola which is the isogonal tranform of the Euler line.

In this note we generalize a theorem of Victor Thébault's as given in [1, Theorem 1]. Given a triangle ABC with orthic triangle $H_aH_bH_c$, the Euler lines of the triangles AH_bH_c, BH_cH_a, and CH_aH_b are concurrent at a point on the nine-point circle, which is the center of the Jerabek hyperbola, the isogonal transform of the Euler line of triangle ABC.

Since triangle AH_cH_b is similar to ABC, it is the reflection in the bisector of angle A of a triangle AB_aC_a, which is a homothetic image of ABC. Let P be a triangle center of triangle ABC. Its counterpart in AH_cH_b is the point P_a constructed as the reflection in the bisector of angle A of the point on AP which is the intersection of the parallels to BP, CP through C_a, B_a respectively (see Figure 1).

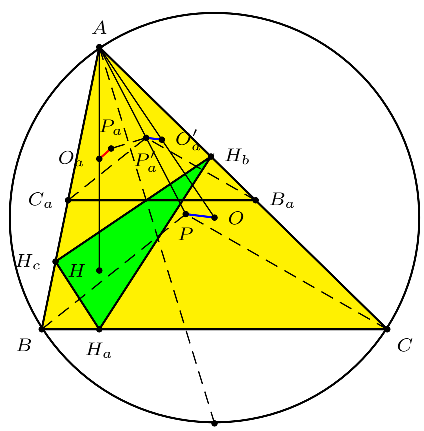


Figure 1.

Note that the circumcenter O_a of triangle AH_cH_b is the midpoint of AH. It is also the reflection (in the bisector of angle A) of the circumcenter O'_a of triangle AB_aC_a. The line O_aP_a is the reflection of $O'_aP'_a$ in the bisector of angle A.

Publication Date: December 5, 2008. Communicating Editor: Paul Yiu.

Here is an alternative description of the line O_aP_a that leads to an interesting result. Consider the line ℓ'_a through A parallel to OP, and its reflection ℓ_a in the bisector of angle A. It is well known that ℓ_a intersects the circumcircle at a point Q' which is the isogonal conjugate of the infinite point of OP. Now, the line O_aP_a is clearly the image of ℓ_a under the homothety $\mathsf{h}(H, \frac{1}{2})$. As such, it contains the midpoint Q of the segment HQ'.

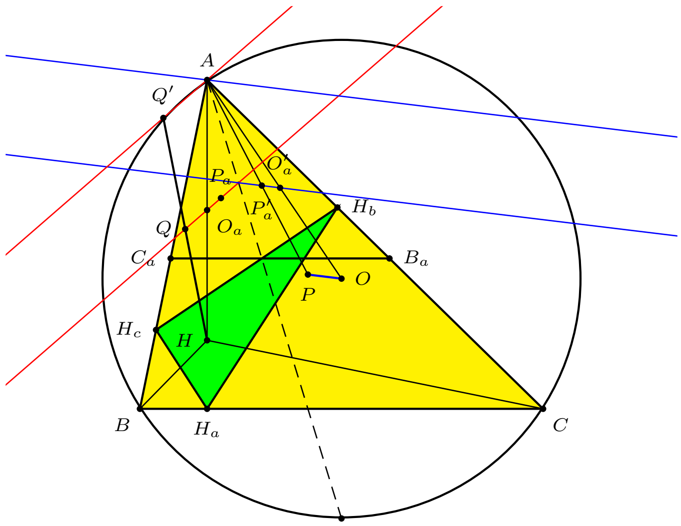


Figure 2.

The above reasoning applies to the lines O_bP_b and O_cP_c as well. The reflections of the parallels to OP through B and C in the respective angle bisectors intersect the circumcircle of ABC at the same point Q', which is the isogonal conjugate of the infinite point of OP (see Figure 3). Therefore, the lines O_bP_b and O_cP_c also contain the same point Q, which is the image of the Q' under the homothety $\mathsf{h}(H, \frac{1}{2})$. As such, it lies on the nine-point circle of triangle BAC. It is well known (see [3]) that Q is the center of the rectangular circum-hyperbola which is the isogonal transform of the line OP.

We summarize this in the following theorem.

Theorem. *Let P be a triangle center of triangle ABC. If P_a, P_b, P_c are the corresponding triangle centers in triangles AH_cH_b, BH_aH_c, CH_bH_a respectively, the lines O_aP_a, O_bP_b, O_cP_c intersect at a point Q on the nine-point circle of ABC, which is the center of the rectangular circumhyperbola which is the isogonal transform of the line OP.*

Thébault's theorem is the case when P is the orthocenter.

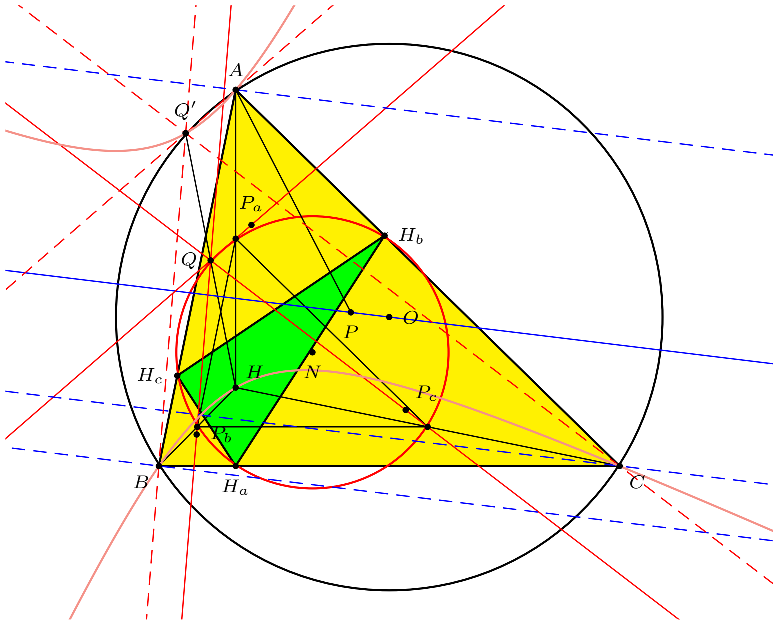


Figure 3.

We conclude with a record of coordinates. Suppose P has homogeneous barycentric coordinates $(u : v : w)$ in reference to triangle ABC. The line O_aP_a, O_bP_b, O_cP_c intersect at the point

$$\begin{aligned} Q = \big(&((b^2-c^2)u + a^2(v-w))(c^2(a^2+b^2-c^2)v - b^2(c^2+a^2-b^2)w) \\ :&\,(c^2-a^2)v + b^2(w-u))(a^2(b^2+c^2-a^2)w - c^2(a^2+b^2-c^2)u) \\ :&\,(a^2-b^2)w + c^2(u-v))(b^2(c^2+a^2-b^2)u - a^2(b^2+c^2-a^2)v)\big) \end{aligned}$$

on the nine-point circle, which is the center of the rectangular hyperbola through A, B, C, H and

$$\begin{aligned} Q' = \Bigg(&\frac{a^2}{((b^2-c^2)^2 - a^2(b^2+c^2))u + a^2(b^2+c^2-a^2)(v+w)} \\ &: \frac{b^2}{((c^2-a^2)^2 - b^2(c^2+a^2))v + b^2(c^2+a^2-b^2)(w+u)} \\ &: \frac{c^2}{((a^2-b^2)^2 - c^2(a^2+b^2))w + c^2(a^2+b^2-c^2)(u+v)}\Bigg). \end{aligned}$$

on the circumcircle. Here are some examples. The labeling of triangle centers follows [2].

P	Q on nine-point circle	Q' on circumcircle
Orthocenter X_4	Jerabek center X_{125}	X_{74}
Symmedian point X_6	Kiepert center X_{115}	X_{98}
Incenter X_1	Feuerbach point X_{11}	X_{104}
Nagel point X_8	X_{3259}	X_{953}
Spieker center X_{10}	X_{124}	X_{102}
X_{66}	X_{127}	X_{1297}
Steiner point X_{99}	X_{2679}	X_{2698}

References

[1] N. Dergiades and P. Yiu, Antiparallels and Concurrent Euler Lines, *Forum Geom.*, 4(2004) 1–20.
[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.
[3] P. Yiu, *Introduction to the Geometry of the Triangles*, Florida Atlantic University Lecture Notes, 2001.

Shao-Cheng Liu: 2F., No.8, Alley 9, Lane 22, Wende Rd., 11475 Taipei, Taiwan
E-mail address: `liu471119@yahoo.com.tw`

Forum Geometricorum
Volume 8 (2008) xx.

FORUM GEOM
ISSN 1534-1178

Author Index

FORUM GEOMETRICORUM

A Journal on Classical Euclidean Geometry and Related Areas

published by

Department of Mathematical Sciences
Florida Atlantic University

Volume 9

2009

http://forumgeom.fau.edu

ISSN 1534-1178

Editorial Board

Advisors:

John H. Conway	Princeton, New Jersey, USA
Julio Gonzalez Cabillon	Montevideo, Uruguay
Richard Guy	Calgary, Alberta, Canada
Clark Kimberling	Evansville, Indiana, USA
Kee Yuen Lam	Vancouver, British Columbia, Canada
Tsit Yuen Lam	Berkeley, California, USA
Fred Richman	Boca Raton, Florida, USA

Editor-in-chief:

Paul Yiu	Boca Raton, Florida, USA

Editors:

Nikolaos Dergiades	Thessaloniki, Greece
Clayton Dodge	Orono, Maine, USA
Roland Eddy	St. John's, Newfoundland, Canada
Jean-Pierre Ehrmann	Paris, France
Chris Fisher	Regina, Saskatchewan, Canada
Rudolf Fritsch	Munich, Germany
Bernard Gibert	St Etiene, France
Antreas P. Hatzipolakis	Athens, Greece
Michael Lambrou	Crete, Greece
Floor van Lamoen	Goes, Netherlands
Fred Pui Fai Leung	Singapore, Singapore
Daniel B. Shapiro	Columbus, Ohio, USA
Man Keung Siu	Hong Kong, China
Peter Woo	La Mirada, California, USA
Li Zhou	Winter Haven, Florida, USA

Technical Editors:

Yuandan Lin	Boca Raton, Florida, USA
Aaron Meyerowitz	Boca Raton, Florida, USA
Xiao-Dong Zhang	Boca Raton, Florida, USA

Consultants:

Frederick Hoffman	Boca Raton, Floirda, USA
Stephen Locke	Boca Raton, Florida, USA
Heinrich Niederhausen	Boca Raton, Florida, USA

Table of Contents

Forum Geometricorum
Volume 9 (2009) 1–17.

FORUM GEOM
ISSN 1534-1178

On n-Sections and Reciprocal Quadrilaterals

Eisso J. Atzema

Abstract. We introduce the notion of an n-section and reformulate a number of standard Euclidean results regarding angles in terms of 2-sections (with proof). Using 6-sections, we define the notion of reciprocal (complete) quadrangles and derive some properties of such quadrangles.

1. Introduction

While classical geometry is still admired as a model for mathematical reasoning, it is only fair to admit that following through an argument in Euclidean geometry in its full generality can be rather cumbersome. More often than not, a discussion of all manner of special cases is required. Specifically, Euclid's notion of an angle is highly unsatisfactory. With the rise of projective geometry in the 19th century, some of these issues (such as the role of points at infinity) were addressed. The need to resolve any of the difficulties connected with the notion of an angle was simply obviated by (largely) avoiding any direct appeal to the concept. By the end of the 19th century, as projective geometry and metric geometry aligned again and vectorial methods became commonplace, classical geometry saw the formal introduction of the notion of *orientation*. In the case of the concept of an angle, this led to the notion of a *directed (oriented, sensed)* angle. In France, the (elite) high school teacher and textbook author Louis Gérard was an early champion of this notion, as was Jacques Hadamard (1865-1963); in the USA, Roger Arthur Johnson (1890-1954) called for the use of such angles in classical geometry in two papers published in 1917.[1] Today, while the notion of a directed angle certainly has found its place in classical geometry research and teaching, it has by no means supplanted the traditional notion of an angle. Many college geometry textbooks still ignore the notion of orientation altogether.

In this paper we will use a notion very closely related to that of a directed angle. This notion was introduced by the Australian mathematician David Kennedy Picken (1879-1956) as the *complete angle* in 1922. Five years later and again in 1947, the New Zealand mathematician Henry George Forder (1889-1981) picked

Publication Date: January 26, 2009. Communicating Editor: Paul Yiu.

This paper is an extended version of a presentation with the same title at the Invited Paper Session: Classical Euclidean Geometry in MathFest, July 31–August 2, 2008 Madison, Wisconsin, USA.

[1]See [5], as well as [6] and [7]; Johnson also consistently uses directed angles in his textbook [8].

up on the idea, preferring the term *cross*.[2] Essentially, where we can look upon an angle as the configuration of two rays departing from the same point, the cross is the configuration of two intersecting lines. Here, we will refer to a *cross* as a 2-section and consider it a special case of the more general notion of an n-section.

We first define these n-sections and establish ground rules for their manipulation. Using these rules, we derive a number of classical results on angles in terms of 2-sections. In the process, to overcome some of the difficulties that Picken and Forder ran into, we will also bring the theory of circular inversion into the mix. After that, we will focus on 6-sections formed by the six sides of a complete quadrangle. This will lead us to the introduction of the reciprocal to a complete quadrangle as first introduced by James Clerk Maxwell (1831-1879). We conclude this paper by studying some of the properties of reciprocal quadrangles.

2. The notion of an n-section

In the Euclidean plane, let $\{l\}$ denote the equivalence class of all lines parallel to the line l. We will refer to $\{l\}$ as the *direction* of l. Now consider the ordered set of directions of a set of lines $l_1, ..., l_n$ ($n \geq 2$). We refer to such a set as an n-section (of lines), which we will write as $\{\ell_1, ..., \ell_n\}$.[3]. Clearly, any n-section is an equivalence class of all lines $m_1, ..., m_n$ each parallel to the corresponding of $l_1, ..., l_n$. Therefore, we can think of any n-section as represented by n lines all meeting in one point. Also note that any n-section corresponds to a configuration of points on the line at infinity.

We say that two n-sections $\{l_1, ..., l_n\}$ and $\{m_1, ..., m_n\}$ are directly congruent if for any representation of the two sections by means of concurrent lines there is a rotation combined with a translation that maps each line of the one representation onto the corresponding line of the other. We write $\{l_1, ..., l_n\} \cong_D \{m_1, ..., m_n\}$. If in addition a reflection is required, $\{l_1, ..., l_n\}$ is said to be *inversely* congruent to $\{m_1, ..., m_n\}$, which we write as $\{l_1, ..., l_n\} \cong_I \{m_1, ..., m_n\}$.

Generally, no two n-sections can be both directly and inversely congruent to each other. Particularly, as a rule, an n-section is not inversely congruent to itself. A notable exception is formed by the 2-sections. Clearly, a 2-section formed by two parallel lines is inversely as well as directly congruent to itself. We will refer to such a 2-section as trivial. Any non-trivial 2-section that is inversely congruent to itself is called *perpendicular* and its two directions are said to be perpendicular to each other. We will just assume here that for every direction there always is exactly one direction perpendicular to it.[4]

No other n-sections can be both directly and inversely congruent, except for such n-sections which only consist of pairs of lines that either all parallel or are perpendicular. We will generally ignore such sections.

---

[2]See [3] (pp.120-121+151-154), [4], [16], and [17]. The term *cross* seems to have been coined by Edward Hope Neville in [14]. Forder may actually have also used crosses in his two geometry textbooks from 1930 and 1931, but we have not been able to locate copies of these.

[3]We adapt this notation from [15].

[4]A proof using SAS is fairly straightforward.

The following basic principles for the manipulation of n-sections apply. We would like to insist here that these principles are just working rules and not axioms (in particular they are not independent) and serve the purpose of providing a shorthand for frequent arguments more than anything else.

Principle 1 (Congruency)**.** *Two n-sections are congruent if and only if all corresponding sub-sections are congruent, where the congruencies are either all direct or all inverse.*

Principle 2 (Transfer)**.** *For any three directions $\{a\}$, $\{b\}$, and $\{c\}$, there is exactly one direction $\{d\}$ such that $\{a,b\} \cong_D \{c,d\}$.*

Principle 3 (Chain Rule)**.** *If $\{a,b\} \cong \{a',b'\}$ and $\{b,c\} \cong \{b',c'\}$ then $\{a,c\} \cong \{a',c'\}$, where the congruencies are either all direct or all inverse.*

Principle 4 (Rotation)**.** *Two n-sections $\{a_1,...,a_n\}$ and $\{b_1,...,b_n\}$ are directly congruent if and only if all $\{a_i,b_i\}$ ($1 \leq i \leq n$) are directly congruent.*

Principle 5 (Reflection)**.** *Two n-sections $\{a_1,...,a_n\}$ and $\{b_1,...,b_n\}$ are inversely congruent if and only if there is a direction $\{c\}$ such that $\{a_i,c\} \cong_I \{b_i,c\}$ for all $1 \leq i \leq n$.*

Most of the usual triangle similarity tests are still valid (up to orientation) if we replace the notion of an angle by that of a cross or 2-section, except for Side-Cross-Side (SCS). Since we cannot make any assumptions about the orientation on an arbitrary line, SCS is ambiguous in terms of sections in that a 2-section with a length on each of its legs, (generally) determines two non-congruent triangles. The only situation in which SCS holds true (up to orientation) is for perpendicular sections. Since we are in the Euclidean plane, the *Dilation Principle* applies to any 2-section as well: For any triangle $\triangle ABC$ with P on CA and Q on CB, $\triangle PQC$ is directly similar to $\triangle ABC$ if and only if $\overline{CP}/\overline{CA} = \overline{CQ}/\overline{CB}$, where $\overline{CA}$ and so on denote *directed* lengths.

Once again, note that we do not propose to use the n-sections to completely replace the notion of an angle. The notion of n-sections just provides a uniform way to discuss the large number of problems in geometry that are really about configurations of lines rather than configurations of rays. Starting from our definition of a perpendicular section, for instance, the basic principles suffice to give a formal proof that all perpendicular sections are congruent. In other words, they suffice to prove that all perpendicular lines are made equal. Essentially this proof streamlines the standard proof (first given by Hilbert). Let $\{a,a'\}$ be a perpendicular section and let $\{b\}$ be arbitrary direction. Now, let $\{b'\}$ be such that (i) $\{a,b\} \cong_D \{a',b'\}$ (BP 2). Then, since $\{a,a'\} \cong_D \{a',a\}$, also $\{a',b\} \cong_D \{a,b'\}$ or (ii) $\{b',a\} \cong_D \{b,a'\}$ (BP 3). Combining (i) and (ii), it follows that $\{b,b'\} \cong_D \{b',b\}$ (BP 3). In other words, $\{b'\}$ is perpendicular to $\{b\}$. Finally, by BP 4, $\{b,b'\} \cong_D \{a,a'\}$.

The same rules also naturally allow for the introduction of both angle bisectors to an angle and do not distinguish the two. Indeed, note that the "symmetry" direction $\{c\}$ in BP 5 is not unique. If $\{a_i,c\} \cong_I \{b_i,c\}$, then the same is true for the direction $\{c'\}$ perpendicular to $\{c\}$ by BP 3. Conversely, for any direction $\{d\}$ such that $\{a_i, d\} \cong_I \{b_i, d\}$, it follows that $\{d, c\} \cong_D \{c, d\}$ by BP 3. In other words, $\{c, d\}$ is a perpendicular section. We will refer to the perpendicular section $\{c, c'\}$ as the *symmetry section* of the inversely congruent n-sections $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_n\}$. In the case of the inversely congruent systems $\{a_1, a_2\}$ and $\{a_2, a_1\}$, we speak of the symmetry section of the 2-section. Obviously, the directions of the latter section are those of the angle bisectors of the angle formed by any two rays on any two lines representing $\{a_1, a_2\}$.

Using the notion of a symmetry section, we can now easily prove Thales' Theorem (as it is known in the Anglo-Saxon world).

Theorem 6 (Thales)**.** *For any three distinct points A, B and C, the line AC is perpendicular to BC if and only if C lies on the unique circle with diameter AB.*

Proof. Let O be the center of the circle with diameter AB. Since both $\triangle AOC$ and $\triangle BOC$ are isosceles, the two line of the symmetry section of $\{AB, OC\}$ are each perpendicular to one of AC and BC. Consequently, by BP 4 (Rotation), $\{AC, BC\}$ is congruent to the symmetry section, *i.e.*, AC and BC are perpendicular. Conversely, let A', B', C' be the midpoints of BC, CA, AB, respectively. Then, by dilation, $C'A'$ and $C'B'$ are parallel to CA and CB, respectively. It follows that $C'A'CB'$ is a rectangle and therefore $|B'A'| = |C'C|$, but by dilation $|B'A'| = |AC'| = |BC'|$, *i.e.*, C lies on the unique circle with diameter AB. $\square$

3. Circular inversion

To allow further comparison of n-sections, we need the equivalent of a number of the circle theorems from Book III of Euclid's *Elements*. It is easy to see how to state any of these theorems in terms of 2-sections. As Picken remarks, however, really satisfactory proofs (in terms of 2-sections) are not so obvious and probably impossible if we do not want to use rays and angles at all. Be that as it may, we can still largely avoid directly using angles.[5] In this paper we will have recourse to the notion of *circular inversion*, which allows for reasonably smooth derivations. This transformation of (most of) the affine plane is defined with respect to a given circle with radius r and center O. For any point P of the plane other than O, its image under inversion with respect to O and the circle of radius r is defined as the unique point P' such that $\overline{OP} \cdot \overline{OP'} = r^2$ (where $\overline{OP}$ and so on denote *directed* lengths). Note that by construction circular inversion is a closed (and bijective) operation on the affine plane (excluding O). Also, if A' and B' are the images of A and B under a circular inversion with respect to a point O, then by construction $\triangle A'B'O$ is inversely similar to $\triangle ABO$. The following fundamental lemma applies.

Lemma 7. *Let O be the center of a circular inversion. Then, under this inversion (i) any circle not passing through O is mapped onto a circle not passing through O, (ii) any line not passing through O is mapped onto a circle passing through O and vice versa (with the point at infinity of the line corresponding to O), (iii) any*

[5] See [16], p.190 and [4], p.231. Forder is right to claim that the difficulty lies with the lack of an ordering for crosses and that directed angles need to be used at some point.

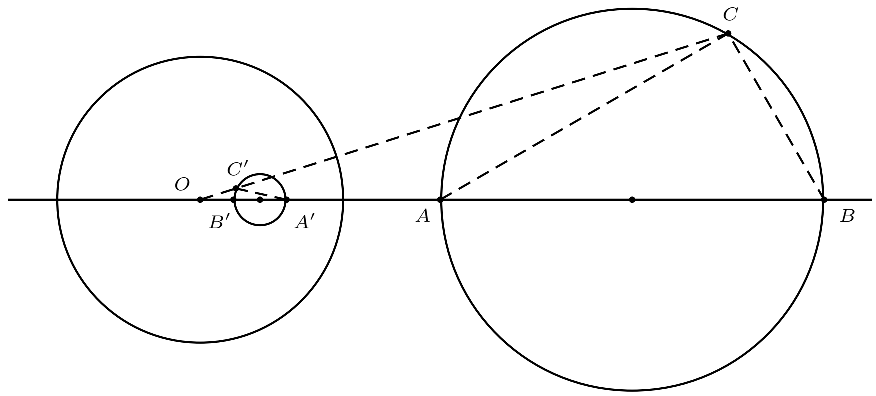


Figure 1. Circular Inversion of a Circle

line passing through O is mapped onto itself (with the point at infinity of the line again corresponding to O).

Proof. Starting with (i), draw the line connecting O with the center of the circle not passing through O and let the points of intersection of this line with the latter circle be A and B (see Figure 1). Then, by Theorem 6 (Thales), the lines AC and BC are perpendicular. Let A', B' and C' be the images of A, B, and C under the inversion. By the previous lemma $\{OA, AC, CO\}$ is indirectly congruent to $\{OC', C'A', A'O\}$. Likewise $\{OB, BC, CO\} \cong_I \{OC', C'B', B'O\}$. Since OA, OB, OA', and OB' coincide, it follows that $\{OB, BC, CA, CO\}$ is inversely congruent to $\{OC', C'B', C'A', B'O\}$. Therefore $\{BC, CA\}$ is indirectly congruent to $\{C'B', C'A'\}$. Consequently, $C'A'$ and $C'B'$ are perpendicular as well. This means that C' lies on the circle that has the segment $A'B'$ for a diameter. The second statement is proved in a similar way, while the third statement is immediate. □

We can now prove the following theorem, which is essentially a rewording in the language of sections of Propositions 21 and 22 from Book III of Euclid's *Elements* (with a trivial extension).

Theorem 8 (Equal Angle)**.** *For four points on either a circle or a straight line, let X, Y, Z, W be any permutation of A, B, C, D. Then, any 2-section $X\{Y, Z\}$ is directly congruent to the 2-section $W\{Y, Z\}$ and the sections are either trivial (in case the points are collinear) or non-trivial (in case the points are co-cyclic). Conversely, any four (distinct) points A, B, C, D for which there is a permutation X, Y, Z, W such that $X\{Y, Z\}$ and $W\{Y, Z\}$ are directly congruent either are co-cyclic (in case the sections are non-trivial) or collinear (in case the two sections are trivial).*

Proof. It suffices to prove both statements for one permutation of A, B, C, D. Assume that A, B, C, D are co-cyclic or collinear. Let B', C' and D' denote the images of B, C and D, respectively, under circular inversion with respect to A. Then, $\{DA, DC\} \cong_I \{C'A, D'C'\}$ and $\{BA, BC\} \cong_I \{C'A, B'C'\}$. Since by

Lemma 7, $B'C'$ coincides with $D'C'$, it follows that $\{DA, DC\} \cong_D \{BA, BC\}$. Conversely, assume that $\{DA, DC\} \cong_D \{BA, BC\}$. Then, $\{C'A, D'C'\} \cong_D \{C'A, B'C'\}$, *i.e.*, B', C' and D' are collinear. By Lemma 7 again, if the two 2-sections are non-trivial, A, B, C, D are co-cyclic. If not, the four points are collinear. □

Corollary 9. *Let A, B, C, D be any four co-cyclic points with $E = AC \cap BD$. Then the product of directed lengths $\overline{AE} \cdot \overline{CE}$ equals the product $\overline{BE} \cdot \overline{DE}$.*

Proof. Let A' and B' be the images under inversion of A and B with respect to E (and a circle of radius r). Then $\triangle A'B'E$ and $\triangle CDE$ are directly similar with two legs in common. Therefore $\overline{CE}/\overline{A'E} = \overline{DE}/\overline{B'E}$ or $\overline{CE} \cdot \overline{AE}/r^2 = \overline{DE} \cdot \overline{BE}/r^2$. □

For the sake of completeness, although we will not use it in this paper, we end with a sometimes quite useful reformulation of Propositions 20 and 32 from Book III of Euclid's *Elements*.

Lemma 10 (Bow, String and Arrow)**.** *For any triangle $\triangle ABC$, let C' be the midpoint of AB and let O be the circumcenter of the triangle and let $T_{AB,C}$ denote the tangent line to the circumcircle of $\triangle ABC$ at C. Then $C\{B, A\}$ is directly congruent to (i) both $O\{C', A\}$ and $O\{B, C'\}$ and (ii) $\{BA, T_{CB,A}\}$ and $\{T_{CA,B}, AB\}$.*

Proof. It suffices to prove the first statements of (i) and (ii). Let A' be the midpoint of BC. Since OC' is perpendicular to AB and OA' is perpendicular to BC, it follows that $\square C'OA'B$ is cyclic and therefore that $A'\{B, C'\} \cong_D O\{B, C'\}$. But $A'C'$ is parallel to CA and therefore $A'\{B, C'\} \cong_D C\{B, A\}$ as well, which proves the first statement of (i). As for (ii), since BA is perpendicular to OC' and $T_{CB,A}$ is perpendicular to OA, it follows that $\{BA, T_{CB,A}\}$ is directly congruent to $O\{C', A\}$ by BP 4 (Rotation). Since $O\{C', A\}$ is directly congruent to $C\{B, A\}$, the first statement of (ii) follows. □

The preceding results provide a workable framework for the application of n-sections to a great many problems in plane geometry involving configurations of circles and lines (as opposed to rays). The well-known group of circle theorems usually attributed to Steiner and Miquel as well as most theorems associated with the Wallace line are particularly amenable to the use of n-sections. Examples can be found in [16], [17], and [5].

4. 6-sections and complete quadrangles

So far we have essentially only used 2-sections and 3-sections. Note how any 3-section (with distinct directions) always corresponds to a unique class of directly similar triangles. Clearly, there is no such correspondence for 4-sections. To determine a quadrilateral, we need the direction of at least one of its diagonals as well. Therefore, it makes sense to consider the 6-sections and their connection to the so-called *complete quadrangles* $\boxtimes ABCD$, *i.e.*, all configurations of four points (with no three collinear) and the six lines passing through each two of them. Clearly any $\boxtimes ABCD$ defines a 6-section. Conversely, not every 6-section can be represented by the six sides of a complete quadrangle. In order to see under what condition a 6-section originates from a complete quadrangle, we need a little bit of projective geometry.

Any two n-sections are said to be *in perspective* or to form a *perspectivity* if for a representation of each of the sections by concurrent lines the points of intersection of the corresponding lines are collinear. Two sections are said to be *projective* if a representation by concurrent lines of the one section can be obtained from a similar representation of the other as a sequence of perspectivities. It can be shown that any two sections that are congruent are also projective. In the case of 2-sections and 3-sections all are actually projective. As for 4-sections, the projectivity of two sections is determined by their so-called *cross ratio*. Every 4-section $\{\ell_1, ..., \ell_4\}$ has an associated cross ratio $[\ell_1, ..., \ell_4]$. If $\underline{A}$ denotes the pencil of lines passing through A, represent the lines of any section by lines $\ell_i \in \underline{A}$. If ℓ_i has an equation $\mathcal{L}_i = 0$, we can write $\mathcal{L}_3$ as $\lambda_{31}\mathcal{L}_1 + \lambda_{32}\mathcal{L}_2$ and $\mathcal{L}_4$ as $\lambda_{41}\mathcal{L}_1 + \lambda_{42}\mathcal{L}_2$. We now (unambiguously) define the cross ratio $[\ell_1, \ldots, \ell_4]$ as the quotient $(\lambda_{31}/\lambda_{32}) : (\lambda_{41}/\lambda_{42})$. From this definition of a cross ratio it follows that its value does not change when the first pair of elements and the second pair are switched or when the elements within each pair are swapped. Note that for any two 3-sections $\{l_1, l_2, l_3\}$ and $\{m_1, m_2, m_3\}$ (with $\{l_1, l_2, l_3\}$ and $\{m_1, m_2, m_3\}$ each formed by three distinct directions), the cross ratio defines a bijective map φ between any two pencils $\underline{A}$ and $\underline{B}$, by choosing the l_i in $\underline{A}$ and the m_i in $\underline{B}$ and defining the image $\varphi(l)$ of any line $l \in \underline{A}$ as the line of $\underline{B}$ such that $[l_1, l_2; l_3, l]$ equals $[m_1, m_2; m_3, \varphi(l)]$. The map φ is called a projective map (of the pencil). It can be shown that any projective map can be obtained as a projectivity and vice versa. Therefore, two 4-sections are projective if and only if their corresponding cross ratios are equal. By the duality of projective geometry, all of the preceding applies to the points of a line instead of the lines of a pencil as well. Moreover, for any four points L_1, L_2, L_3, L_4 on a line ℓ and a point L_0 outside ℓ, the cross ratio $[L_1, L_2; L_3, L_4]$ is equal to $[L_0L_1, L_0L_2; L_0L_3, L_0L_4]$. By the latter property, we can associate any projective map defined by two sections of lines with a projective map from the line at infinity to itself.

The notion of a projective map can be extended to the projective plan where any such map φ maps any line to a straight line and the restriction of φ to a line and its image line is a projective map. Where a projective map between two lines is defined by two triples of (non-coinciding) points, a projective map between two planes requires two sets of four points, no three of which can be collinear. In other words, any two quadrilaterals define a projective map. Finally, we define an *involution* as a projective map which is its own inverse. In the case of an involution of a line or pencil, any two distinct pairs of elements (with the elements within each pair possibly coinciding) fully determine the map.

We can now formulate the following result.

Theorem 11. *An arbitrary* 6*-section* $\{l_1, l_2, m_1, m_2, n_1, n_2\}$ *can be formed from the sides of a complete quadrangle* $\boxtimes ABCD$ *(such that* l_1*,* l_2 *and so on are pairs* *of opposite sides) if and only if the three pairs of opposite sides can be rearranged such that* $\{l_1, l_2\}$ *is non-trivial and* $[l_1, l_2; m_1, n_2]$ *equals* $[l_1, l_2; n_1, m_2]$.

Proof. Since any two quadrilaterals determine a projective map, every complete quadrangle is projective to the configuration of a rectangle and its diagonals. Therefore the diagonal points of a complete quadrangle are never collinear and every quadrangle in the affine plane has at least one pair of opposite sides which are not parallel. Without loss of generality, we may assume that $\{l_1, l_2\}$ corresponds to this pair of opposite sides. Let $\underline{A}$, $\underline{B}$ denote the pencils of lines through A and B respectively. Now define a map φ from $\underline{A}$ to $\underline{B}$ by assigning the line AX to BX for all X on a line ℓ not passing through A or B. It is easily verified that φ is a projectivity, which assigns AB to itself and the line of $\underline{A}$ parallel to L to the corresponding parallel line of $\underline{B}$. Therefore, if C and D are distinct points on ℓ, the cross ratio $[AB, CD; AC, AD]$ equals the cross ratio $[AB, CD, BC, BD]$. Conversely, for any 6-section $\{l_1, l_2, m_1, m_2, n_1, n_2\}$ such that $[l_1, l_2; m_1, n_2]$ equals $[l_1, l_2; n_1, m_2]$, we can choose A and B such that AB is parallel to l_1 and let D be the point of intersection of the line of $\underline{A}$ parallel to m_1 and the line of $\underline{B}$ parallel to n_1. Likewise, let C be the point of intersection of the line of $\underline{A}$ parallel to m_2 and the line through D parallel to l_2. Then, since $\{l_1, l_2\}$ is non-trivial, the line BC has to be parallel to n_2. $\square$

Note that the previous theorem is a projective version of Ceva's Theorem determining the concurrency of transversals in a triangle and the usual expression of that theorem can be readily derived from the condition above. We now have the following corollary.

Corollary 12. *For any complete quadrangle* $\boxtimes ABCD$*, there is an involution that pairs the points of intersection of its opposite sides with the line at infinity.*

Proof. Without loss of generality, we may assume that $\{AB, CD\}$ is non-trivial. Let $L_1 = AB \cap \ell_\infty$ and so on. Then $[L_1, L_2, M_1, N_2] = [L_1, L_2, N_1, M_2]$. Now let φ be the involution of ℓ_∞ determined by pairing L_1 with L_2 and M_1 with M_2. Then $[L_1, L_2, M_1, N_2]$ equals $[L_2, L_1, M_2, \varphi(N_2)]$. Since the former expression is also equal to $[L_2, L_1, M_2, N_1]$ (and L_1, L_2 and M_2 are distinct), it follows that $\varphi(N_2) = N_1$. In other words, the involution pairs N_1 and N_2 as well. $\square$

In case $\boxtimes ABCD$ is a trapezoid, the point on the line at infinity corresponding to the parallel sides is a fixed point of the involution; in case the complete quadrangle is a parallelogram, the two points corresponding to the two pairs of parallel sides both are fixed points.

In the language of classical projective geometry, we say that a 6-section formed by the sides of any complete quadrangle defines an involution of six lines pairing the opposite sides of the quadrangle. Note that this statement implies what is known as Desargues' Theorem, which states that any complete quadrangle defines an involution (of points) on any line not passing through any of its vertices that pairs the points of intersection of that line with the opposite sides of the quadrangle. For this reason, we will say that any 6-section satisfying the condition of Theorem 11 is *Desarguesian*.

Corollary 13. *Any Desarguesian* 6*-section is associated with two similarity classes of quadrilaterals (which may coincide).*

Proof. Let the 6-section be denoted by $\{l_1, l_2, m_1, m_2, n_1, n_2\}$. If the cross ratio $[l_1, l_2; m_1, n_2]$ equals $[l_1, l_2; n_1, m_2]$, then the cross ratio $[l_2, l_1; m_2, n_1]$ also equals $[l_2, l_1; n_2, m_1]$. Whereas the quadrilateral constructed from the first equality contains a triangle formed by the lines l_2, m_2, n_2, while l_1, m_1, n_1 meet in one point, this is reversed for the quadrilateral formed from the second equality. Since $\{l_1, m_1, n_1\}$ and $\{l_2, m_2, n_2\}$ are not necessarily congruent, the two quadrilaterals will be different (but may coincide in some cases). □

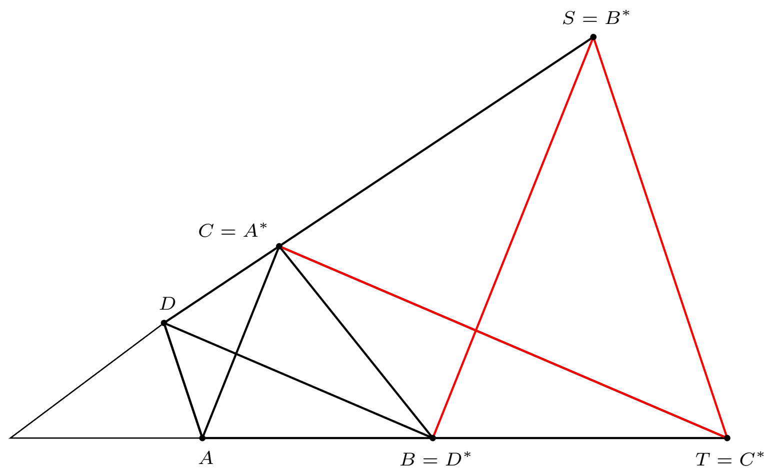


Figure 2. Constructing $\boxtimes A^*B^*C^*D^*$

If the complete quadrangle $\boxtimes ABCD$ is one of the two quadrangles forming a given 6-section, we can easily construct the other quadrangle $\boxtimes A^*B^*C^*D^*$. Indeed, let $\boxtimes ABCD$ be as in Figure 2. Then, draw the line through B parallel to AC, meeting CD in S. Likewise, draw the line through C parallel to BD meeting AB in T. Then, by construction ST is a parallel to AD and all the opposite sides of $\boxtimes ABCD$ are parallel to a pair of opposite sides of $\boxtimes BCST$. The two quadrangles, however, are generally not similar. Alternatively, we can consider the quadrangle formed by the circumcenters of the four circles circumscribing the four triangle formed by A, B, C, D. For this quadrangle, all three pairs of opposite sides are parallel to a pair of opposite sides of the original quadrangle. Again, it is easy to see that this quadrilateral is generally not similar to the original one. The latter construction was first systematically studied by Maxwell in [10] and [11], in which he referred to the quadrilateral of circle centers as a reciprocal figure. For this reason, we will refer to the two complete quadrangles associated with a Desarguesian 6-section as *reciprocal* quadrangles.

Relabeling the vertices of the preceding quadrangles as indicated in Figure 2, we will formally define two complete quadrilaterals $\boxtimes ABCD$ and $\boxtimes A^*B^*C^*D^*$ as directly/inversely reciprocal if and only if

$$\{AB, CD, AC, BD, DA, BC\} \cong \{C^*D^*, A^*B^*, B^*D^*, A^*C^*, B^*C^*, D^*A^*\},$$

where the congruence is either direct or inverse. From this definition, we immediately derive the following two corollaries.

Corollary 14. *A complete quadrangle is directly reciprocal to itself if and only if it is orthocentric.*

Proof. Since for any complete quadrangle directly reciprocal to itself all three 2-sections of opposite sides have to be both directly and inversely congruent, it follows that all opposite sides are perpendicular to each other. In other words, every vertex is the orthocenter of the triangle formed by the other three vertices, which is what orthocentric means. $\square$

Corollary 15. *A complete quadrangle is inversely reciprocal to itself if and only if it is cyclic.*

Proof. Let $\boxtimes ABCD$ denote the complete quadrangle. Then, if $\boxtimes ABCD$ is inversely reciprocal to itself, $\{AB, AC\}$ has to be inversely congruent to $\{CD, BD\}$ or $A\{B, C\} \cong_D D\{B, C\}$. But this means that $\boxtimes ABCD$ is cyclic. The converse readily follows. $\square$

Because of the preceding corollaries, when studying the relations between reciprocal quadrangles, we can often just assume that a complete quadrangle is neither orthocentric nor cyclic. Also, as a special case, note that if a complete quadrangle $\boxtimes ABCD$ has a pair of parallel opposite sides, then its reciprocal is directly congruent to $\boxtimes BADC$. For this reason, it is usually fine to assume that $\boxtimes ABCD$ does not have any parallel sides either.

Maxwell's application of his reciprocal figures to the study of statics contributed to the development of a heavily geometrical approach to that field (know as *graphostatics*) which ultimately made projective geometry a required course at many engineering schools until well into the 20th century. At the same time, the idea of "reciprocation" was largely ignored within the classical geometry community. This only changed in the 1890s, when (probably not entirely independently of Maxwell) Joseph Jean Baptiste Neuberg (1840-1826) reintroduced the concept of reciprocation under the name of *metapolarity*. This notion, however, seems to have been quickly eclipsed by the related notion of *orthology* that was introduced by Émile Michel Hyacinthe Lemoine (1840-1912) and others as a tool to study triangles. In this context, consider a triangle $\triangle ABC$ and a point P in the plane of the triangle. Now, construct a new triangle $\triangle A'B'C'$ such that each of its sides is perpendicular to the corresponding side of $\{CP, AP, BP\}$. In this new triangle, construct transversals each perpendicular to the corresponding line of $\triangle ABC$. Then, these three transversals will meet in a new point P'. The triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be *orthologic* with poles P and P'. Clearly, for any two orthologic triangles $\triangle ABC$ and $\triangle A'B'C'$ with poles P and P', $\boxtimes ABCP$ and $\boxtimes A'B'C'P'$ are reciprocal quadrangles. Conversely, for any two reciprocal quadrangles $\boxtimes ABCD$ and $\boxtimes A^*B^*C^*D^*$, $\triangle ABC$ and $\triangle A^*B^*C^*$ are orthologic with poles D and D^* (up to a rotation), and similarly for the three other pairs of triangles contained in the two quadrangles. It is in the form of some variation of orthology that the notion of reciprocation is best known today.[6]

A nice illustration of the use of reciprocal quadrangles (or orthology, in this case) is the following problem from a recent International Math Olympiad Training Camp.[7]

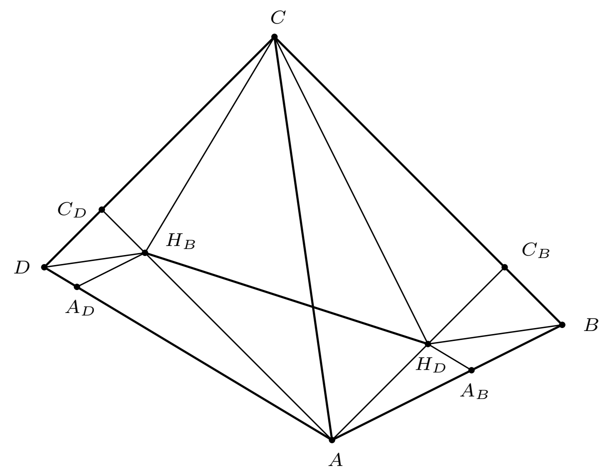


Figure 3. $\boxtimes ABCD$ with H_B and H_D

Problem (IMOTC 2005)**.** Let $ABCD$ be a quadrilateral, and H_D the orthocenter of triangle $\triangle ABC$. The parallels to the lines AD and CD through the point H_D meet the lines AB and BC at the points C_B and A_B, respectively. Prove that the perpendicular to the line C_BA_B through the point H_D passes through the orthocenter H_B of triangle $\triangle ACD$.

Solution. The proposition still has to be true if we switch the role of B and D. Now note that the complete quadrangles $\boxtimes H_BC_DDA_D$ and $\boxtimes BC_BH_DA_B$ have five parallel corresponding sides. Therefore, they are similar. Moreover, five of the sides of the complete quadrangle $\boxtimes H_BCH_DA$ are perpendicular to the opposite of the corresponding sides of $\boxtimes H_BC_DDA_D$ and $\boxtimes BC_BH_DB$. We conclude that $\boxtimes H_BCH_DA$ is directly reciprocal to $\boxtimes H_BC_DDA_D$ and $\boxtimes BC_BH_DA_B$. Consequently, its sixth side H_BH_D is perpendicular to A_DC_D and A_BC_B.

5. Some relations between reciprocal quadrangles

In order to study the relations between reciprocal quadrangles, we note yet another way to generate a reciprocal to a given complete quadrangle. In fact,

[6] On metapolar quadrangles, see e.g. [12] and [13] or (more accessibly) Neuberg's notes to [18] (p.458). On orthology, see [9]. In 1827, well before Lemoine (and Maxwell), Steiner had also outlined the idea of orthology (see [19], p.287, Problem 54), but nobody seems to have picked up on the idea at the time. Around 1900, the Spanish mathematician Juan Jacobo Durán Loriga (1854-1911) extended the notion of orthology to that of *isogonology*, which concept was completely equivalent to reciprocation. Durán-Loriga's work, however, met with the same fate as Neuberg's metapolarity.

[7] See [2] and the references there.

let $\boxtimes ABCD$ be a complete quadrangle with diagonal points $E = AC \cap BD$, $F = BC \cap DA$, $G = AB \cap CD$, with A, B, C, D, and, F in the affine plane. Now let A^* be the image of D under circular inversion with respect to F (see Figure 4). Likewise let D^* be the image of A under the same inversion. Similarly B^* is the image of C and C^* is the image of B. Then, using the properties of inversion it is easily verified that $\boxtimes A^*B^*C^*D^*$ is inversely reciprocal to $\boxtimes ABCD$. We can use this construction to derive the following two lemmata.

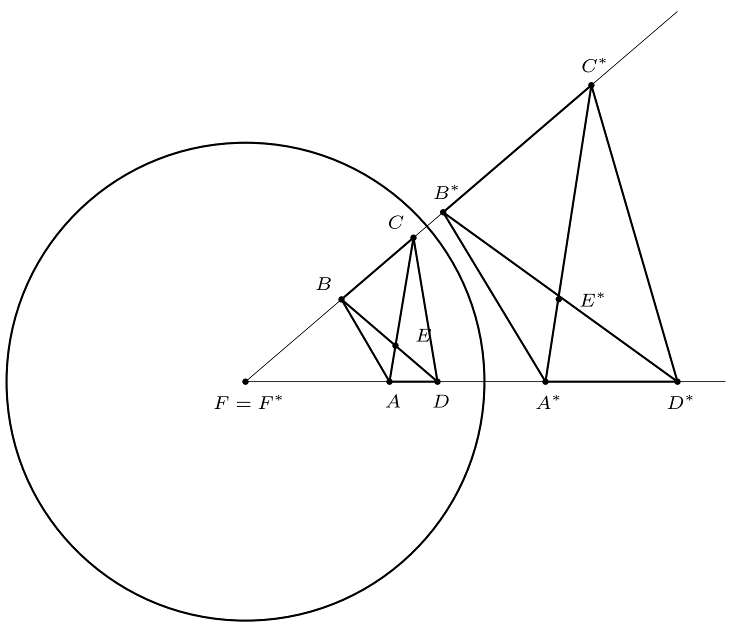


Figure 4. Constructing $\boxtimes A^*B^*C^*D^*$ by Inversion

Lemma 16 (Invariance of Ratios). *Let $\boxtimes ABCD$ and $\boxtimes A^*B^*C^*D^*$ be a pair of (affine) reciprocal quadrangles and diagonal points E, F, G and E^*, F^*, G^*, respectively. Moreover, let X, Y, and Z be any collinear triple of two vertices and a diagonal point of $\boxtimes ABCD$ with X^*, Y^*, Z^* the corresponding triple of $\boxtimes A^*B^*C^*D^*$. Then*

$$\frac{\overline{XY}}{\overline{YZ}} = \frac{\overline{X^*Y^*}}{\overline{Y^*Z^*}},$$

where $\overline{XY}$ denotes the directed length of the line segment XY and so on.

Proof. The statement is trivial for any diagonal point on ℓ_∞. Without loss of generality, let us assume that the diagonal point F is in the affine plane. It now suffices to prove the statement for B, C and F. Under inversion with respect to F and a circle of radius r, we find that $\overline{B^*F^*} = r^2/\overline{CF}$ and $\overline{C^*F^*} = r^2/\overline{BF}$. The statement of the lemma now immediately follows. $\square$

Lemma 17 (Maxwell)**.** *Let $\boxtimes ABCD$ and $\boxtimes A^*B^*C^*D^*$ be a pair of reciprocal quadrangles. Then*

$$\frac{|AB||CD|}{|A^*B^*||C^*D^*|} = \frac{|AC||BD|}{|A^*C^*||B^*D^*|} = \frac{|AD||CB|}{|A^*D^*||C^*B^*|},$$

where $|AB|$ denotes the absolute length of the segment AB and so on.

Proof. Assume again that the point F is in the affine plane. Under inversion with respect to F and a circle of radius r, we find $|A^*D^*| = |r^2/|FD| - r^2/|FA|| = r^2|AD|/(|FA||FD|)$ and $|B^*C^*| = |r^2/|FC| - r^2/|FB|| = r^2|BC|/(|FB||FD|)$. Similarly, $|A^*B^*| = r^2|AB|/(|FA||FB|)$ and $|C^*D^*| = r^2|CD|/(|FC||FD|)$, while $|A^*C^*| = r^2|AC|/(|FA||FC|)$ and $|B^*D^*| = r^2|BD|/(|FB||FD|)$. Combining these expressions shows the equality of the three expressions. □

Note that for any three collinear points, the ratio $|AC|/|BC|$ equals the cross ratio $[A, B, C, I_{AB}]$, where I_{AB} denotes the point at infinity of the line AB. Now, for any pair of reciprocal quadrangles $\boxtimes ABCD$ and $\boxtimes A^*B^*C^*D^*$, let φ be the unique projective map sending A to A^* and so on. Then, φ maps the line AB to the line A^*B^* and $[A, B, G, I_{AB}] = [A^*, B^*, G^*, \varphi(I_{AB})]$ (where $G = AB \cap CD$). By Lemma 16, $[A, B, G, I_{AB}]$ also equals $[A^*, B^*, G^*, I_{A^*B^*}]$. Therefore, since G is distinct from A and B, φ maps I_{AB} to $I_{A^*B^*}$. Likewise, the points at infinity of BC and CA are mapped to the points at infinity of B^*C^* and C^*A^*, respectively. But then, φ must map the whole line at infinity onto itself. Therefore, any map defined by "reciprocation" of a complete quadrangle is an *affine* map. Conversely, any affine map can be modeled by a reciprocation of a complete quadrangle (which we may assume not to have any parallel sides). To see this, we first need another lemma.

Lemma 18. *For a given triangle $\triangle ABC$ and any non-trivial 3-section $\{l, m, n\}$ not inversely congruent to $\{BC, CA, AB\}$ there is exactly one point D in the plane of $\triangle ABC$ (and not on the sides of $\triangle ABC$) such that $\{AD, BD, CD\}$ is directly congruent to $\{l, m, n\}$. In case $\{BC, CA, AB\} \cong_I \{l, m, n\}$, $\{AD, BD, CD\}$ will be directly congruent to $\{l, m, n\}$ for any point D on the circumcircle of $\triangle ABC$.*

Proof. Without loss of generality, we may assume that l, m, and n are concurrent at a point Q. Let a point L be a fixed point on l and let M be a variable point on m. Now construct a triangle $\triangle LMN$ directly similar to $\triangle ABC$. Then, the locus of N as M moves along m is a straight line as N is obtained from M by a fixed dilation followed by a rotation over a fixed angle. Therefore, this locus will intersect n in exactly one point as long as $\{AC, AB\}$ is not directly congruent to $\{n, m\}$. The point D we are looking for now has the same position with respect to $\triangle ABC$ as has Q with respect to $\triangle LMN$. If the two 2-sections are directly congruent, we can repeat the process starting with M or N. This means that we cannot find a point D as stated in the lemma using the procedure above only if $\triangle ABC$ is inversely congruent to $\{l, m, n\}$. But if the latter is the case, we can take any point D on the circumcircle of $\triangle ABC$ by Cor. 15. □

As an aside, note that for $\{l, m, n\}$ directly congruent to either $\{AB, BC, CA\}$ or $\{CA, AB, BC\}$, this construction also guarantees the existence of the two so-called Brocard points Ω^+ and Ω^- of $\triangle ABC$. Moreover, it is easily checked that $\boxtimes ABC\Omega^+$ and $\boxtimes BCA\Omega^-$ are reciprocal quadrangles. This explains the congruence of the two Brocard angles. We are now ready to prove the following theorem.

Theorem 19. *A projective map of the plane is affine if and only if it can be obtained by reciprocation of a complete quadrangle $\boxtimes ABCD$ with no parallel sides. Any such map reverses orientation if $\boxtimes ABCD$ is convex and retains orientation when not. The map is Euclidean if and only if $\boxtimes ABCD$ is orthocentric (in which case the map retains orientation) or cyclic (in which case the map reverses orientation).*

Proof. We already proved the if-part above. For an affine map, consider a triangle $\triangle ABC$ and its image $\triangle A^*B^*C^*$. By the previous lemma there is at least one point D (not on the sides of $\triangle ABC$) such that $\{AD, BD, CD\}$ is directly congruent to $\{B^*C^*, C^*A^*, A^*B^*\}$. The reciprocation of $\boxtimes ABCD$ that A maps to $\triangle ABC$ maps to $\triangle A^*B^*C^*$, then, must be the affine map. The connection between convexity of $\boxtimes ABCD$ follows from the various constructions (and relabeling) of a reciprocal quadrangle. The last statement follows immediately. In case $\boxtimes ABCD$ has parallel opposite sides, note that the affine map (after a rotation aligning one pair of parallel sides with their images) induces a map on the line at infinity with either one or two fixed points (if not just a translation combined with a dilation), corresponding to a glide or a dilation in two different directions. This means that if we choose the sides of $\triangle ABC$ such that they are not parallel to the directions represented by the fixed points on the line at infinity, no opposite sides of $\boxtimes ABCD$ will be parallel. $\square$

Finally, note that if a complete quadrangle $\boxtimes ABCD$ is cyclic, then its reciprocal $\boxtimes A^*B^*C^*D^*$ is as well. Likewise, by Lemma 17, if for a complete quadrangle the product of the lengths of a pair of opposite sides equals that of the lengths of another pair, the same is true for the corresponding pairs of its reciprocal. More surprisingly perhaps, reciprocation also retains inscribability, *i.e.*, if $\square ABCD$ has an incircle, then so has $\square A^*B^*C^*D^*$. To see this, we can use the following generalization of a standard result.

Lemma 20 (Generalized Ptolemy)**.** *For any six points A, B, C, D, P, and Q in the (affine) plane*

$$\begin{aligned} &|\triangle PAB||\triangle QCD| + |\triangle PCD||\triangle QAB| \\ &+ |\triangle PAD||\triangle QBC| + |\triangle PBC||\triangle QAD| \\ &= |\triangle PAC||\triangle QBD| + |\triangle PBD||\triangle QAC|. \end{aligned}$$

Proof. We represent the points A, B, C, D, P, and Q by vectors $\vec{a} = (a_1, a_2, 1)$ and so on. Now consider the vectors $(\vec{a} \oplus \vec{a})^T$, ..., $(\vec{d} \oplus \vec{d})^T$, as well as the vectors $(\vec{p} \oplus i\vec{p})^T$ and $(i\vec{q} \oplus \vec{q})^T$. Then clearly, the 6×6-determinant formed by these six vectors equals zero. If we now evaluate this determinant as the sum of the signed product of every 3×3-determinant contained in the three first rows and its complementary 3×3-determinant in the three bottom rows, we obtain exactly the identity of the lemma. $\square$

Note that the imaginary numbers are necessary to ensure that no two of the products automatically cancel against each other. Also, note that this result really is about octahedrons in 3-space and can immediately be extended to their analogs in any dimension. Ptolemy's Theorem follows by letting P and Q coincide and assuming this point is on the circumcircle of $\boxtimes ABCD$.

Corollary 21. *For any complete quadrangle $\boxtimes ABCD$ and $E = AC \cap BD$ and a point P both in the (affine) plane of the quadrangle,*

$$\begin{aligned} &|\triangle PDA| \cdot |\triangle EBC| + |\triangle PBC| \cdot |\triangle EDA| \\ &= |\triangle PCD| \cdot |\triangle EAB| + |\triangle PAB| \cdot |\triangle ECD|, \end{aligned}$$

where E is the point of intersection of AC with BD.

Proof. Let Q coincide with E. $\square$

Now, let $\boxtimes ABCD$ be convex. Then $E = AC \cap BD$ is in the affine plane and we can obtain $\boxtimes A^*B^*C^*D^*$ by circular inversion with respect to E. Also, $\square A^*B^*C^*D^*$ is convex by Theorem 19. Therefore, the equality of $|A^*B^*| + |C^*D^*|$ and $|D^*A^*| + |B^*C^*|$ is both necessary and sufficient for the quadrangle to be inscribable. By the properties of inversion, this condition is equivalent to the condition

$$\frac{|AB|}{|EA||EB|} + \frac{|CD|}{|EC||ED|} = \frac{|DA|}{|ED||EA|} + \frac{|BC|}{|EB||EC|},$$

or

$$DA \cdot |\triangle EBC| + BC \cdot |\triangle EDA| = CD \cdot |\triangle EAB| + BA \cdot |\triangle ECD|.$$

If $\square ABCD$ is inscribable, this condition can also be written in the form

$$|\triangle IDA| \cdot |\triangle EBC| + |\triangle IBC| \cdot |\triangle EDA| = |\triangle ICD| \cdot |\triangle EAB| + |\triangle IAB| \cdot |\triangle ECD|.$$

But this equality is true by Cor. 21. We conclude that if $\square ABCD$ is inscribable, then so is $\square A^*B^*C^*D^*$.

Alternatively, we can use a curious result that received some on-line attention in recent years, but which is probably considerably older.

Theorem 22. *For any convex quadrilateral $\square ABCD$ with $E = AC \cap BD$, let I_{AB} be the incenter of $\triangle EAB$ and so on. Then $\boxtimes I_{AB}I_{BC}I_{CD}I_{DA}$ is cyclic if and only if $\square ABCD$ is inscribable.*

Proof. See [1] and the references there. The convexity requirement might not be necessary. $\square$

Let us assume again that $\square ABCD$ is inscribable. This means that the quadrangle is convex and that $E = AC \cap BD$ is in the affine plane. Also, note that $E = I_AI_C \cap I_BI_D$. Therefore, $\overline{EI_{AB}} \cdot \overline{EI_{CD}}$ equals $\overline{EI_{BC}} \cdot \overline{EI_{DA}}$ by Theorem 22. Now, let $\boxtimes A^*B^*C^*D^*$ be a reciprocal of $\boxtimes ABCD$ obtained by circular inversion with respect to E and a circle with radius r. As we assumed that $\boxtimes ABCD$ is convex, so is $\boxtimes A^*B^*C^*D^*$ by Theorem 19. Since $\triangle EA^*B^*$ is inversely similar to $\triangle ECD$ while $|A^*B^*| = r^2|BD|/(|EB||ED|)$, it follows that $\overline{EI_{A^*B^*}} = r^2\overline{EI_{CD}}/(|EC||ED|)$ and so on. Consequently, $\overline{EI_{A^*B^*}} \cdot \overline{EI_{C^*D^*}} = \overline{EI_{B^*C^*}} \cdot \overline{EI_{D^*A^*}}$. Therefore $\boxtimes I_{A^*B^*}I_{B^*C^*}I_{C^*D^*}I_{D^*A^*}$ is cyclic and $\square A^*B^*C^*D^*$ is inscribable by Theorem 22 again.

As a third proof, it is relatively straightforward to actually construct a reciprocal $\boxtimes A^*B^*C^*D^*$ with its sides tangent to the incircle of $\square ABCD$. More generally, this approach proves that the existence of any tangent circle to a quadrangle implies the existence of one for its reciprocal. This construction can actually be looked upon as a special case of yet another way to construct reciprocal quadrangles. The proof of the validity of this more general construction, however, seems to require a property of reciprocal quadrangles that we have not touched upon in this paper. We plan to discuss this property (and the specific construction of reciprocal quadrangles that follows from it) in a future paper.

6. Conclusions

In this paper we outlined how in many cases the concept of an angle can be replaced by the more rigorous notion of an n-section. Other than the increased rigor, one advantage of n-sections over angles is that reasoning with the former is somewhat more similar to the kind of reasoning one might see in other parts of mathematics, particularly in algebra. Although perhaps a little bit of an overstatement, Picken did have a point when he claimed that his paper did not have diagrams because they were “quite unnecessary.”[8] Also, the formalism of n-sections provides a natural framework in which to study geometrical problems involving multiple lines and their respective inclinations. As such, it both provides a clearer description of known procedures and is bound to lead to questions that the use of the notion of angles would not naturally give rise to. As a case in point, we showed how the notion of n-section suggests both a natural description of the procedure involving orthologic triangles in the form of the notion of reciprocal quadrangles and give rise to the question what properties of a complete quadrangle are retained under the “reciprocation” of quadrangles.

At the same time, the fact that the “reciprocation” of quadrangles does not favor any of the vertices of the figures involved comes at a cost. Indeed, its use does not naturally give rise to certain types of questions that the use of orthologic triangles does lead to. For instance, it is hard to see how an exclusive emphasis on the notion of reciprocal quadrangles could ever lead to the study of antipedal triangles and similar constructions. In short, the notion of reciprocal quadrangles should be seen as a general notion underlying the use of orthologic triangles and not as a replacement of the latter.

[8] See [16], p.188.

References

[1] A. Bogomolny, Cyclic incenters, *Interactive Mathematics Miscellany and Puzzles* = http://www.cut-the-knot.org/Curriculum/Geometry/CyclicIncenters.shtml.

[2] A. Bogomolny, Orthologic triangles in a quadrilateral, *Interactive Mathematics Miscellany and Puzzles* = http://www.cut-the-knot.org/Curriculum/Geometry/OrthologicQuadri.shtml.

[3] H. G. Forder, *Foundations of Euclidean Geometry*, Cambridge: CUP, 1927, also reprinted by Dover without any changes in 1958.

[4] H. G. Forder, The cross and the foundations of Euclidean Geometry, *Mathematical Gazette* 31:296 (Oct., 1947) 227–233.

[5] M. Guillerault, L'affermissement des démonstrations en géométrie élémentaire au tournant du XXème siècle, *Preuve. International Newsletter on the Teaching and Learning of Mathematical Proof* (Sept./Oct., 1998) = http://www.lettredelapreuve.it/Newsletter/980910.html.

[6] R. A. Johnson, Directed angles in elementary geometry, *Amer. Math. Monthly*, 24 (1917) 101–105.

[7] R. A. Johnson, Directed angles and inversion, with a proof of Schoute's Theorem, *Amer. Math. Monthly*, 24(1917) 313–317.

[8] R. A. Johnson, *Modern Geometry - An Elementary Treatise on the Geometry of the Triangle and the Circle*, New York: Houghton Mifflin, 1929; reprinted as *Advanced Euclidean Geometry*, Dover, 2007.

[9] E. Lemoine, Sur les triangles orthologiques et sur divers sujets de la géométrie du triangle, *Compte-Rendu de la 19ième Séance de l'Association Française pour l'Avancement des Sciences. Deuxième Partie: Notes et Mémoires*, (Paris, 1891) 111–140 (lu 9 août, 1890).

[10] J. C. Maxwell, On reciprocal figures and diagrams of force, *Philosophical Magazine* 27 (1864) 250–261.

[11] J. C. Maxwell, On reciprocal figures, frames and diagrams of force, *Transactions of the Royal Society of Edinburgh* 26 (1872) 1–40.

[12] J. J. B. Neuberg and A. Gob, Sur les foyers de Steiner d'un triangle, *Compte-Rendu de la 18ième Séance de l'Association Française pour l'Avancement des Sciences. Deuxième Partie: Notes et Mémoires*, (Paris, 1890) 179- -196 (lu 12 août 1889).

[13] J. J. B. Neuberg, Sur les quadrangles complets, *Mathesis*, 2 (1891) 33–35,67–70,81–82,189–195.

[14] E. H. Neville, Complete angle or cross?, *Mathematical Gazette* 11:163 (March, 1923) 275.

[15] D. Pedoe, *Geometry. A Comprehensive Course*, Cambridge University Press, 1970; also available as Dover pocket.

[16] D. K. Picken, The complete angle and geometrical generality, *Mathematical Gazette* 11:161 (Dec., 1922), 188–193.

[17] D. K. Picken, Euclidean geometry of angle, *Proceedings of the London Mathematical Society* (2) 23(1924) 45–55.

[18] E. Rouché et C. de Camberousse, *Traité de Géométrie*, 7th edition, Paris: Gauthier-Villars, 1900.

[19] J. Steiner, Aufgaben und Lehrsätze, erstere aufzulösen, letztere zu beweisen, *Journal für die reine und angewandte Mathematik* 2 (1827) 287–292 (problems 54–65).

Eisso J. Atzema: Department of Mathematics, University of Maine, Orono, Maine 04469, USA
E-mail address: atzema@math.umaine.edu

Forum Geometricorum
Volume 9 (2009) 19–26.

FORUM GEOM
ISSN 1534-1178

The Lost Daughters of Gergonne

Steve Butler

Abstract. Given a triangle center we can draw line segments from each vertex through the triangle center to the opposite side, this splits the triangle into six smaller triangles called daughters. Consider the following problem: Given a triangle S and a rule for finding a center find a triangle T, if possible, so that one of the daughters of T, when using the rule is S. We look at this problem for the incenter, median and Gergonne point.

1. Introduction

Joseph-Diaz Gergonne (1771–1859) was a famous French geometer who founded the *Annales de Gergonne*, the first purely mathematical journal. He served for a time in the army, was the chair of astronomy at the University of Montpellier, and to the best of our knowledge never misplaced a single daughter [3].

The "daughters" that we will be looking at come from triangle subdivision. Namely, for any well defined triangle center in the interior of the triangle one can draw line segments (or Cevians) connecting each vertex through the triangle center to the opposite edge. These line segments then subdivide the original triangle into six daughter triangles.

Given a triangle and a point it is easy to find the daughter triangles. We are interested in going the opposite direction.

Problem. Given a triangle S and a well defined rule for finding a triangle center; construct, if possible, a triangle T so that S is a daughter triangle of T for the given triangle center.

For instance suppose that we use the *incenter* as our triangle center (which can be found by taking the intersection of the angle bisectors). Then if we represent the angles of the triangle T by the triple (A, B, C) it easy to see that one daughter will have angles $\left(\frac{A}{2}, \frac{A}{2} + \frac{B}{2}, \frac{B}{2} + C\right)$, all the other daughter triangles are found by permuting A, B and C. Since this is a linear transformation this can be easily inverted. So if S has angles a, b and c then the possible candidates for T are $(2a, 2b-2a, c-b+a)$, along with any permutation of a, b and c. It is easy to show that if the triangle S is not equilateral or an isoceles triangle with largest angle $\geq 90^\circ$ then there is at least one non-degenerate T for S.

Publication Date: February 9, 2009. Communicating Editor: Paul Yiu.
This work was done with support of an NSF Mathematical Sciences Postdoctoral Fellowship.

We could also use the *centroid* which for a triangle in the complex plane with vertices at 0, Z and W is $\frac{Z+W}{3}$. In particular if $[0, Z, W]$ is the location of the vertices of T then $\left[0, \frac{Z}{2}, \frac{Z+W}{3}\right]$ is the location of the vertices of a daughter of T. This map is easily inverted, let S be $[0, z, w]$ then we can choose T to be $[0, 2z, 3w - 2z]$. So, for example, if S is an equilateral triangle then we should choose T to be a triangle similar to one with side lengths 2, $\sqrt{7}$ and $\sqrt{13}$.

In this note we will be focusing on the case when our triangle center is the *Gergonne point*, which is found by the intersection of the line segments connecting the vertices of the triangle to the point of tangency of the incircle on the opposite edges (see Figures 2-5 for examples).

Unlike centroids where every triangle is a possible daughter, or incenters where all but $(60°, 60°, 60°)$, $(45°, 45°, 90°)$ and obtuse isoceles are daughters, there are many triangles which cannot be a Gergonne daughter. We call such triangles *the lost daughters of Gergonne*.

To see this pictorially if we again represent triangles as triples (A, B, C) of the angles, then each "oriented" triangle (up to similarity) is represented by a point in P, where P is the intersection of the plane $A + B + C = 180°$ with the positive orthant (see [1, 2, 5, 6] for previous applications of P). Note that P is an equilateral triangle where the points on the edges are degenerate triangles with an angle of $0°$ and the vertices are $(180°, 0°, 0°)$, $(0°, 180°, 0°)$, $(0°, 0°, 180°)$; the center of the triangle is $(60°, 60°, 60°)$. In Figure 1 we have plotted the location of the possible Gergonne daughters in P, the large white regions are the lost daughters.

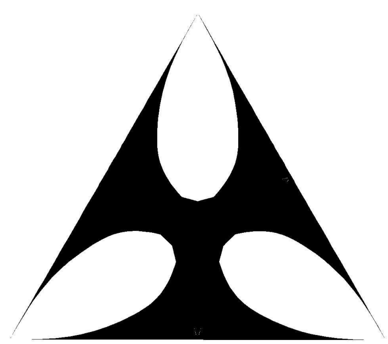


Figure 1. The possible Gergonne daughters in P.

2. Constructing T

We start by putting the triangle S into a standard position by putting one vertex at $(-1, 0)$ (with associated angle α), another vertex at $(0, 0)$ (with associated angle β) and the final vertex in the upper half plane. We now want to find (if possible) a triangle T which produces this Gergonne daughter in such a way that $(-1, 0)$ is a vertex and $(0, 0)$ is on an edge of T (see Figure 2). Since $(0, 0)$ will correspond to a point of tangency of the incircle we see that the incircle must be centered at $(0, t)$ with radius t for some positive t. Our method will be to solve for t in terms of α

and β. We will see that some values of α and β have no valid t, while others can have one or two.

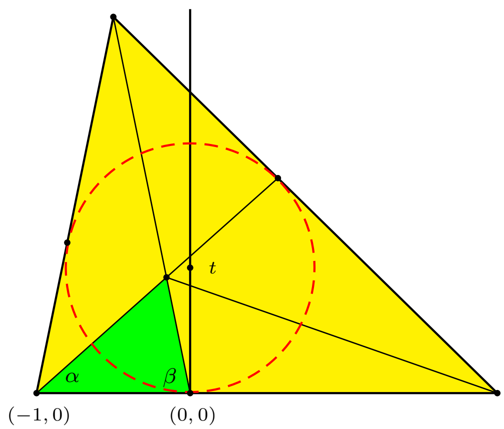


Figure 2. A triangle in standard position.

Since the point of tangency of the incircle to the edge opposite $(-1,0)$ must occur in the first quadrant, we immediately have that the angle α is acute and we will implicitly assume that in our calculations.

2.1. *The case* $\beta = 90°$. We begin by considering the special case $\beta = 90°$. In this setting it is easy to see that T must be an isosceles triangle of the form shown in Figure 3.

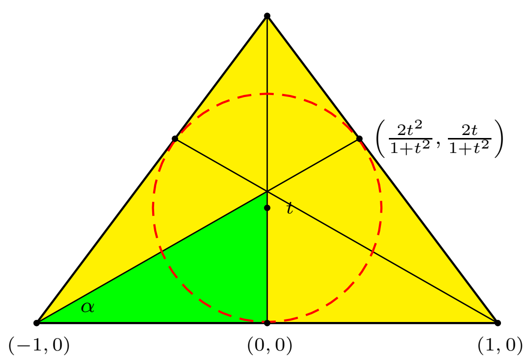


Figure 3. The $\beta = 90°$ case.

The important part of Figure 3 is the location of the point $\left(\frac{2t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$. There are several ways to find this point. Ours will be to find the slope of the tangent line, then once this is found the point of tangency can easily be found. The key tool is the following lemma.

Lemma 1. *The slope m of the lines that pass through the point (p,q) and are tangent to the circle $x^2+(y-t)^2=t^2$ satisfy*

$$m^2 + \frac{2p(t-q)}{p^2-t^2}m + \frac{q^2-2qt}{p^2-t^2} = 0. \tag{1}$$

Proof. In order for the line $y = m(x-p)+q$ to be tangent to the circle $x^2+(y-t)^2 = t^2$ the minimum distance between the line and $(0,t)$ must be t. Since the minimum distance between $(0,t)$ and the line $y = mx+(q-pm)$ is given by the formula $\frac{|t+pm-q|}{\sqrt{m^2+1}}$, we must have

$$t^2 = \left(\frac{|t+pm-q|}{\sqrt{m^2+1}}\right)^2.$$

Simplifying this relationship gives (1). □

Applying this with $(p,q) = (1,0)$ we have that the slopes must satisfy,

$$m^2 + \frac{2t}{1-t^2}m = 0.$$

We already know the solution $m = 0$, so the slope of the tangent line is $\frac{-2t}{1-t^2}$. Some simple algebra now gives us the point of tangency. We also have that the top vertex is located at $\left(0, \frac{2t}{1-t^2}\right)$.

Using the newly found point we must have

$$\tan\alpha = \frac{\frac{2t}{1+t^2}}{\frac{2t^2}{1+t^2}+1} = \frac{2t}{1+3t^2},$$

which rearranges to

$$3(\tan\alpha)t^2 - 2t + \tan\alpha = 0, \text{ so that } t = \frac{1 \pm \sqrt{1-3\tan^2\alpha}}{3\tan\alpha}.$$

There are two restrictions. First, t must be real, and so we have $0 < \tan\alpha \le \frac{\sqrt{3}}{3}$, or $0 < \alpha \le 30°$. Second, $t < 1$ (if $t \ge 1$ then the triangle cannot close up), and so we need

$$\frac{1+\sqrt{1-3\tan^2\alpha}}{3\tan\alpha} < 1 \text{ which reduces to } \tan\alpha > \frac{1}{2},$$

so for this root of t we need to have $\alpha > \arctan(1/2) \approx 26.565°$.

Theorem 2. *For $\beta = 90°$ and α given for a triangle S in standard position then*
(i) *if $\alpha > 30°$ there is no T which produces S;*
(ii) *if $\alpha = 30°$ then the T which produces S is an equilateral triangle;*
(iii) *if $\arctan\frac{1}{2} < \alpha < 30°$ then there are two triangles T which produce S, these correspond to the two roots $t = \frac{1\pm\sqrt{1-3\tan^2\alpha}}{3\tan\alpha}$;*
(iv) *if $\alpha \le \arctan\frac{1}{2}$ then there is one triangle T which produces S, this corresponds to the root $t = \frac{1-\sqrt{1-3\tan^2\alpha}}{3\tan\alpha}$.*

An example of the case when there can be two T is shown in Figure 4 for $\alpha = 29.85°$.

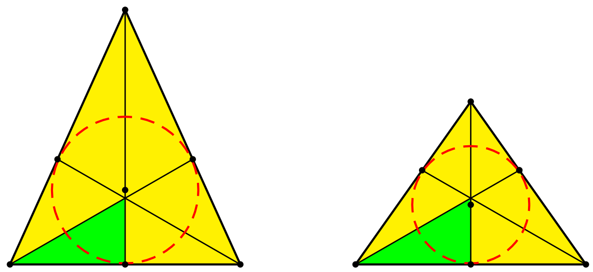


Figure 4. An example of a fixed S in standard position with two possible T.

2.2. *The case* $\beta \neq 90°$. Our approach is the same as in the previous case where we find the point of tangency opposite the vertex at $(-1, 0)$ and then use a slope condition to restrict t. The only difference now is that finding the point takes a few more steps.

To start we can apply Lemma 1 with $(p, q) = (-1, 0)$ and see that the slope of the line tangent to the circle is $\frac{2t}{1-t^2}$. The top vertex of T is then the intersection of the lines

$$y = \frac{2t}{1-t^2}(x+1) \text{ and } y = -(\tan\beta)x.$$

Solving for the point of intersection the top vertex is located at

$$(p^*, q^*) = \left(\frac{-2t}{2t + (1-t^2)\tan\beta}, \frac{2t\tan\beta}{2t + (1-t^2)\tan\beta}\right). \tag{2}$$

We can again apply Lemma 1 with (p^*, q^*) from (2), along with the fact that one of the two slopes is $\frac{2t}{1-t^2}$ to see that the slope of the edge opposite $(-1, 0)$ is

$$m^* = \frac{2\tan\beta(t\tan\beta - 2)}{t^2\tan^2\beta - 4t\tan\beta + 4 - \tan^2\beta}.$$

It now is a simple matter to check that the point of tangency is

$$(x^*, y^*) = \left(\frac{(m^*)^2p^* + tm^* - q^*m^*}{(m^*)^2+1}, \frac{(m^*)^2t - p^*m^* + q^*}{(m^*)^2+1}\right).$$

We can also find that the x-intercept of the line, which will correspond to the final vertex of the triangle, is located at $\big(t\tan\beta/(t\tan\beta - 2), 0\big)$.

So as before we must have

$$\begin{aligned}\tan\alpha &= \frac{y^*}{x^*+1} = \frac{(m^*)^2t - p^*m^* + q^*}{(m^*)^2p^* + tm^* - q^*m^* + (m^*)^2 + 1} \\ &= \frac{2t\tan^2\beta}{3t^2\tan^2\beta - 8t\tan\beta + \tan^2\beta + 4}.\end{aligned}$$

Which can be rearranged to give

$$(3\tan\alpha\tan^2\beta)t^2 - (2\tan^2\beta + 8\tan\alpha\tan\beta)t + (\tan\alpha\tan^2\beta + 4\tan\alpha) = 0.$$

Finally giving

$$t = \frac{\tan\beta + 4\tan\alpha \pm \sqrt{\tan^2\beta + 8\tan\alpha\tan\beta + 4\tan^2\alpha - 3\tan^2\alpha\tan^2\beta}}{3\tan\alpha\tan\beta}. \tag{3}$$

Theorem 3. *For $\beta \neq 90^\circ$ and α given there are at most two triangles T which can produce S in standard position. These triangles T have vertices located at*

$$(-1,0), \left(\frac{-2t}{2t+(1-t^2)\tan\beta}, \frac{2t\tan\beta}{2t+(1-t^2)\tan\beta}\right), \text{ and } \left(\frac{t\tan\beta}{t\tan\beta-2}, 0\right),$$

where t satisfies (3). *Further, we must have that t is positive and satisfies*

$$\frac{2}{\tan\beta} < t < \frac{1+\sec\beta}{\tan\beta}.$$

Proof. The only thing left to prove are the bounds. For the upper bound, we must have that the second vertex is in the top half plane and so we need

$$\frac{2t\tan\beta}{2t+(1-t^2)\tan\beta} > 0.$$

If $\tan\beta > 0$ then we need

$$2t + (1-t^2)\tan\beta > 0 \text{ or } (\tan\beta)t^2 - 2t - \tan\beta < 0.$$

This is an upward facing parabola with negative y-intercept and so we need that t is less than the largest root, i.e.,

$$t < \frac{2+\sqrt{4+4\tan^2\beta}}{2\tan\beta} = \frac{1+\sec\beta}{\tan\beta}.$$

The case for $\tan\beta < 0$ is handled similarly.

For the lower bound we must have that the x-coordinate of the third vertex is positive. If $\tan\beta < 0$ this is trivially satisfied. If $\tan\beta > 0$ then we need $t\tan\beta - 2 > 0$ giving the bound. □

As an example, if we let $\alpha = \beta = 45^\circ$, then (3) gives $t = \frac{5\pm\sqrt{10}}{3} \approx 0.6125$, or 2.7207. But neither of these satisfy $2 < t < 1+\sqrt{2}$, so there is no T for this S. Combined with Theorem 2 this shows that $(45^\circ, 45^\circ, 90^\circ)$ is a lost daughter of Gergonne.

On the other hand if we let $\alpha = \beta = 60^\circ$ then (3) gives $t = \frac{\sqrt{3}}{3}, \frac{7\sqrt{3}}{9}$. The value $\frac{\sqrt{3}}{3}$ falls outside the range of allowable t, but the other one does fall in the range. The resulting triangle is shown in Figure 5 and has side lengths $\frac{19}{5}$, 8 and $\frac{49}{5}$.

3. Concluding comments

We now have a way given a triangle S to construct, if possible, a triangle T so that S is a Gergonne daughter of T. Using this it is possible to characterize triangles which are not Gergonne daughters. One can then look at what triangles are not Gergonne granddaughters (i.e., triangles which can be formed by repeating the subdivision rule on the daughters). Figure 6 shows the location of the Gergonne

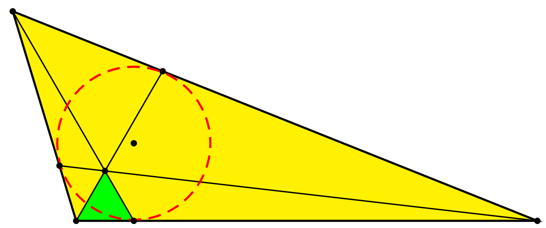


Figure 5. The unique triangle which has an equilateral triangle as a Gergonne daughter.

granddaughters in P. It can be shown the triangle in Figure 5 is not a Gergonne daughter, so that the equilateral triangle is not a Gergonne granddaughter.

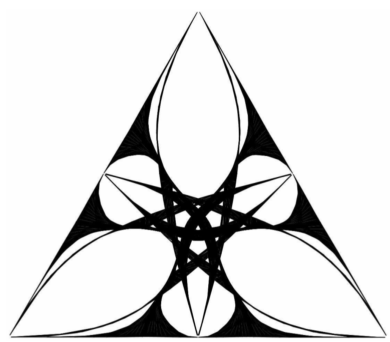


Figure 6. The possible Gergonne granddaughters in P.

One interesting problem is to find what triangles (up to similarity) can occur if we repeat the subdivision rule arbitrarily many times (see [2])? One example of this would be any triangle which is similar to one of its Gergonne daughters. Do any such triangles exist? (For the incenter there are only two such triangles, $(36°, 72°, 72°)$ and $(40°, 60°, 80°)$; for the centroid there is none.)

Besides the incenter, centroid and Gergonne point there are many other possible center points to consider (see the Encyclopedia of Triangle Centers [4] for a complete listing of well known center points, along with many others). One interesting point would be the Lemoine point, which can have up to *three* triangles T for a triangle S in standard position (as compared to 2 for the Gergonne point and 1 for the centroid).

References

[1] J. C. Alexander, The symbolic dynamics of pedal triangles, *Math. Mag.*, 66 (1993) 147–158.

[2] S. Butler and R. Graham, Iterated triangle partitions, preprint.

[3] *Gergonne, Joseph Diaz*, Complete Dictionary of Scientific Biography, Vol. 5, Charles Scribner's Sons, Detroit, 2008, 367–369.

[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[5] J. Kingston and J. Synge, The sequence of pedal triangles, *Amer. Math. Monthly*, 95 (1988) 609–620.

[6] P. Lax, The ergodic character of sequences of pedal triangles, *Amer. Math. Monthly*, 97 (1990) 377–381.

Steve Butler: Department of Mathematics, UCLA, Los Angeles, California 90095, USA
E-mail address: `butler@math.ucla.edu`

Forum Geometricorum
Volume 9 (2009) 27–39.

FORUM GEOM
ISSN 1534-1178

Mappings Associated with Vertex Triangles

Clark Kimberling

Abstract. Methods of linear algebra are applied to triangle geometry. The vertex triangle of distinct circumcevian triangles is proved to be perspective to the reference triangle ABC, and similar results hold for three other classes of vertex triangles. Homogeneous coordinates of the perspectors define four mappings $\mathcal{M}_i$ on pairs of points (U, X). Many triangles homothetic to ABC are examined, and properties of the four mappings are presented. In particular, $\mathcal{M}_i(U, X) = \mathcal{M}_i(X, U)$ for $i = 1, 2, 3, 4$, and $\mathcal{M}_1(U, \mathcal{M}_1(U, X)) = X$; for this reason, $\mathcal{M}_1(U, X)$ is given the name *U-vertex conjugate of X*. In the introduction of this work, *point* is defined algebraically as a homogeneous function of three variables. Subsequent definitions and methods include symbolic substitutions, which are strictly algebraic rather than geometric, but which have far-reaching geometric implications.

1. Introduction

In [1], H. S. M. Coxeter proved a number of geometric results using methods of linear algebra and homogenous trilinear coordinates. However, the fundamental notions of triangle geometry, such as point and line in [1] are of the traditional geometric sort. In the present paper, we begin with an algebraic definition of point.

Suppose a, b, c are variables (or indeterminates) over the field of complex numbers and that x, y, z are homogeneous algebraic functions of (a, b, c) :

$$x = x(a, b, c), \quad y = y(a, b, c), \quad z = z(a, b, c),$$

all of the same degree of homogeneity and not all identically zero. Triples (x, y, z) and (x_1, y_1, z_1) are *equivalent* if $xy_1 = yx_1$ and $yz_1 = zy_1$. The equivalence class containing any particular (x, y, z) is denoted by $x : y : z$ and is a *point.* Let

$$A = 1 : 0 : 0, \quad B = 0 : 1 : 0, \quad C = 0 : 0 : 1.$$

These three points define the *reference triangle* ABC. The set of all points is the *transfigured plane,* as in [6]. If we assign to a, b, c numerical values which are the sidelengths of a euclidean triangle, then $x : y : z$ are homogeneous coordinates (e.g., trilinear or barycentric) as in traditional geometry, and points as defined just above are then points in the plane of a euclidean triangle ABC.

Possibly the earliest treatment of triangle-related points as functions rather than two-dimensional points appears in [3]; in [3]–[9], points-as-functions methods

Publication Date: February 23, 2009. Communicating Editor: Paul Yiu.

lead to problems whose meanings and solutions are nongeometric but which have geometric consequences. Perhaps the most striking are symbolic substitutions [6]–[8], the latter typified by substituting bc, ca, ab for a, b, c respectively. To see the nongeometric character of this substitution, one can easily find values of a, b, c that are sidelengths of a euclidean triangle but bc, ca, ab are not – and yet, this substitution and others have deep geometric consequences, as they preserve collinearity, tangency, and algebraic degree of loci. (In §6, the symbolic substitution $(a, b, c) \to (bc, ca, ab)$ is again considered.)

Having started with an algebraic definition of "point" as in [3], we now use it as a basis for defining other *algebraic* objects. A *line* is a set of points $x : y : z$ such that $lx+my+nz = 0$ for some point $l : m : n$; in particular, the line of two points $p : q : r$ and $u : v : w$ is given by

$$\begin{vmatrix} x & y & z \\ p & q & r \\ u & v & w \end{vmatrix} = 0.$$

A *triangle* is a set of three points. Harmonic conjugacy, isogonal conjugacy, and classes of curves are likewise defined by algebraic equations that are familiar in the literature of geometry (e.g. [1], [5], [10], [12], and many nineteenth-century works), where they occur as consequences of geometric foundations, not as definitions The same is true for other relationships, such as concurrence of lines, collinearity of points, perspectivity of triangles, similarity, and homothety.

So far in this discussion, coordinates have been general homogeneous. In traditional triangle geometry, two specific systems of homogeneous coordinates are common: barycentric and trilinear. In order to define special points and curves, we shall use their traditional trilinear representations. Listed here are a few examples: the centroid of ABC is *defined* as the point $1/a : 1/b : 1/c$; the line $\mathcal{L}^\infty$ at infinity, as $ax+by+cz = 0$. The isogonal conjugate of a point $x : y : z$ satisfying $xyz \neq 0$ is defined as the point $1/x : 1/y : 1/z$ and denoted by X^{-1}, and the circumcircle Γ is defined by $ayz + bzx + cxy = 0$, this being the set of isogonal conjugates of points on $\mathcal{L}^\infty$. Of course, we may illustrate definitions and relationships by evaluating a, b, c numerically—and then all the algebraic objects become geometric objects. (On the other hand, if, for example, $(a, b, c) = (6, 2, 3)$, then the algebraic objects remain intact even though there is no triangle with sidelengths $6, 2, 3$.)

Next, we define four classes of triangles. Suppose $X = x : y : z$ is a point not on a sideline of ABC; *i.e.*, $xyz \neq 0$. Let

$$\begin{aligned} A_1 &= AX \cap BC = 0 : y : z \\ B_1 &= BX \cap CA = x : 0 : z \\ C_1 &= CX \cap AB = x : y : 0. \end{aligned}$$

The triangle $A_1B_1C_1$ is the *cevian triangle* of X. Let A_2 be the point, other than A, in which the line AX meets Γ. Define B_2 and C_2 cyclically. The triangle $A_2B_2C_2$ is the *circumcevian triangle* of X, as indicated in Figure 1.

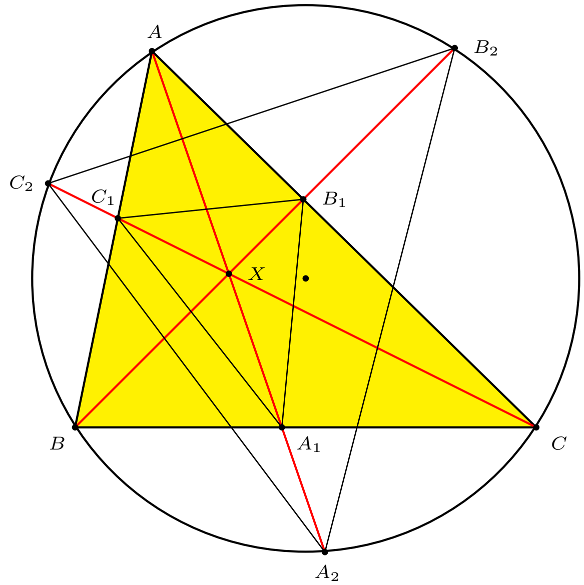


Figure 1.

Let A_3 be the $\{A, A_1\}$-harmonic conjugate of X (*i.e.*, $A_3 = -x : y : z$), and define B_3 and C_3 cyclically. Then $A_3B_3C_3$ is the *anticevian triangle* of X. Let

$$A' = BC \cap B_1C_1, \quad B' = CA \cap C_1A_1, \quad C' = AB \cap A_1B_1,$$

so that $A' = \{B, C\}$-harmonic conjugate of A_1 (*i.e.*, $A_1 = 0 : y : -z$), etc. The lines AA', BB', CC' are the *anticevians* of X, and the points

$$A_4 = AA' \cap \Gamma, \qquad B_4 = BB' \cap \Gamma, \qquad C_4 = CC' \cap \Gamma,$$

as in Figure 2, are the vertices of the *circum-anticevian triangle*, $A_4B_4C_4$, of X.

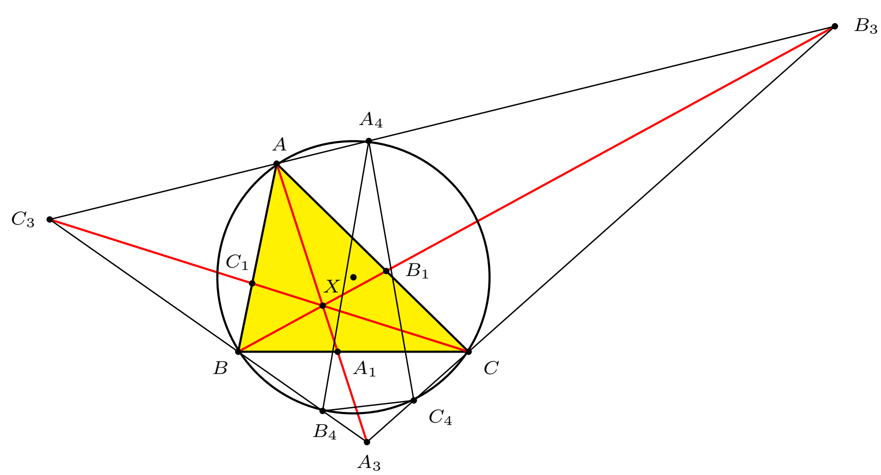


Figure 2.

With these four classes of triangles in mind, suppose $T = DEF$ and $T' = D'E'F'$ are triangles. The *vertex triangle* of T and T' is formed by the lines DD', EE', FF' as in Figure 3. Note that T and T' are perspective if and only if their vertex triangle is a single point.

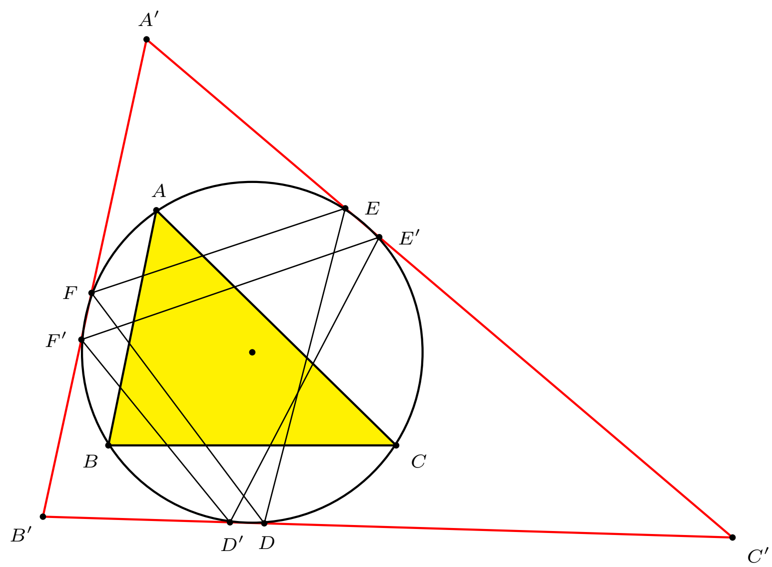


Figure 3.

2. The first mapping $\mathcal{M}_1$

Theorem 1. *The vertex triangle of distinct circumcevian triangles is perspective to ABC.*

Proof. Let $A'B'C'$ be the circumcevian triangle of $X = x : y : z$, and let $A''B''C''$ be the circumcevian triangle of $U = u : v : w$. The former can be represented as a matrix (e.g. [5, p.201]), as follows:

$$\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \begin{pmatrix} -ayz & (cy+bz)y & (bz+cy)z \\ (cx+az)x & -bzx & (az+cx)z \\ (bx+ay)x & (ay+bx)y & -cxy \end{pmatrix},$$

and likewise for $A''B''C''$ using vertices $u_i : v_i : w_i$ in place of $x_i : y_i : z_i$. Lines $A'A'', B'B'', C'C''$ are given by equations $x_i\alpha + y_i\beta + z_i\gamma = 0$ for $i = 4, 5, 6$, where

$$\begin{pmatrix} x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \end{pmatrix} = \begin{pmatrix} y_1w_1 - z_1v_1 & z_1u_1 - x_1w_1 & x_1v_1 - y_1u_1 \\ y_2w_2 - z_2v_2 & z_2u_2 - x_2w_2 & x_2v_2 - y_2u_2 \\ y_3w_3 - z_3v_3 & z_3u_3 - x_3w_3 & x_3v_3 - y_3u_3 \end{pmatrix},$$

so that the vertex triangle is given by

$$\begin{aligned}\begin{pmatrix}A'''\\B'''\\C'''\end{pmatrix} &= \begin{pmatrix}x_7 & y_7 & z_7\\x_8 & y_8 & z_8\\x_9 & y_9 & z_9\end{pmatrix}\\ &= \begin{pmatrix}y_5z_6 - z_5y_6 & z_5x_6 - x_5z_6 & x_5y_6 - y_5x_6\\y_6z_4 - z_6y_4 & z_6x_4 - x_6z_4 & x_6y_4 - y_6x_4\\y_4z_5 - z_4y_5 & z_4x_5 - x_4z_5 & x_4y_5 - y_4x_5\end{pmatrix}. \qquad (1)\end{aligned}$$

The line AA''' thus has equation $0\alpha + z_7\beta - y_7\gamma = 0$, and equations for the lines BB''' and CC''' are obtained cyclically. The three lines concur if

$$\begin{vmatrix}0 & z_7 & -y_7\\-z_8 & 0 & x_8\\y_9 & -x_9 & 0\end{vmatrix} = 0,$$

and this is found (by computer) to be true. □

In connection with Theorem 1, the perspector is the point $P = x_8x_9 : x_8y_9 : z_8x_9$. After canceling long common factors, we obtain

$$\begin{aligned}P = & \; a/(a^2vwyz - ux(bw + cv)(bz + cy))\\ & : b/(b^2wuzx - vy(cu + aw)(cx + az))\\ & : c/(c^2uvxy - wz(av + bu)(ay + bx)). \qquad (2)\end{aligned}$$

The right-hand side of (2) defines the first mapping, $\mathcal{M}_1(U, X)$. If U and X are triangle centers (defined algebraically, for example, in [3], [5], [11]), then so is $\mathcal{M}_1(U, X)$. It can be easily shown that $\mathcal{M}_1(U, X)$ is an involution; that is, $\mathcal{M}_1(\mathcal{M}_1(U, X)) = X$. In view of this property, we call $\mathcal{M}_1(U, X)$ the *U-vertex conjugate* of X. For example, the incenter-vertex conjugate of the circumcenter is the isogonal conjugate of the Bevan point; *i.e.*, $\mathcal{M}_1(X_1, X_3) = X_{84}$. The indexing of named triangle centers, such as X_{84}, is given in the *Encyclopedia of Triangle Centers* [9].

Vertex-conjugacy shares the *iso-* property with another kind of conjugacy called isoconjugacy; viz., the U-vertex-conjugate of X is the same as the X-vertex-conjugate of U. (The U-isoconjugate of X is the point $vwyz : wuzx : uvxy$; see the Glossary at [9].)

Other properties of $\mathcal{M}_1$ are given in §9 and in Gibert's work [2] on cubics associated with vertex conjugates.

3. The second mapping $\mathcal{M}_2$

Theorem 2. *The vertex triangle of distinct circum-anticevian triangles is perspective to ABC.*

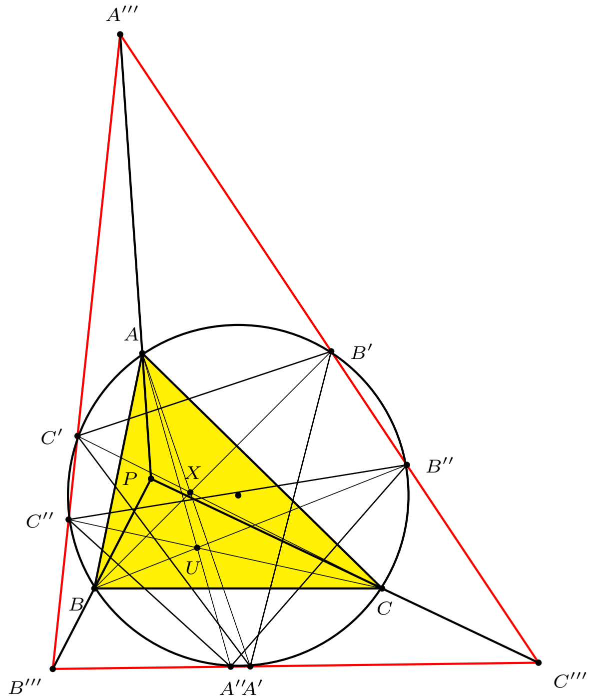


Figure 4.

Proof. The method of §2 applies, starting with

$$\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

$$= \begin{pmatrix} ayz & (cy-bz)y & (bz-cy)z \\ (cx-az)x & bzx & (az-cx)z \\ (bx-ay)x & (ay-bx)y & cxy \end{pmatrix},$$

and likewise for $A''B''C''$. □

The perspector is given by

$$P = p : q : r = \frac{a}{f(a,b,c,x,y,z)} : \frac{b}{f(b,c,a,y,z,a)} : \frac{c}{f(c,a,b,z,a,b))}, \qquad (3)$$

where

$$f(a,b,c,s,y,z) = a^2 vwyz - xu(bw-cv)(bz-cy),$$

and we define $\mathcal{M}_2(U,X) = P$ as in (3).

As this mapping is not involutory, we wish to solve the equation $P = \mathcal{M}_2(U,X)$ for X. The system to be solved, and the solution, are given by

$$\begin{pmatrix} g_1 & h_1 & k_1 \\ g_2 & h_2 & k_2 \\ g_3 & h_3 & k_3 \end{pmatrix} \begin{pmatrix} 1/x \\ 1/y \\ 1/z \end{pmatrix} = \begin{pmatrix} a/p \\ b/q \\ c/r \end{pmatrix}$$

and

$$\begin{pmatrix} 1/x \\ 1/y \\ 1/z \end{pmatrix} = \begin{pmatrix} g_1 & h_1 & k_1 \\ g_2 & h_2 & k_2 \\ g_3 & h_3 & k_3 \end{pmatrix}^{-1} \begin{pmatrix} a/p \\ b/q \\ c/r \end{pmatrix},$$

where

$$\begin{pmatrix} g_1 & h_1 & k_1 \\ g_2 & h_2 & k_2 \\ g_3 & h_3 & k_3 \end{pmatrix} = \begin{pmatrix} a^2vw & -bu(bw-cv) & cu(bw-cv) \\ av(cu-aw) & b^2wu & -cv(cu-aw) \\ -aw(av-bu) & bw(av-bu) & k_3 = c^2uv \end{pmatrix}.$$

Again, long factors cancel, leaving

$$X = x : y : z = g(a,b,c) : g(b,c,a) : g(c,a,b),$$

where

$$g(a,b,c) = \frac{a}{a^3qrv^2w^2 - a^2G_2 + aG_1 + G_0},$$

where

$$\begin{aligned} G_0 &= u^2p\,(bw-cv)^2\,(br+cq)\,, \\ G_1 &= uvwp\,(br-cq)\,(cv-bw)\,, \\ G_2 &= uvwrq\,(bw+cv)\,. \end{aligned}$$

4. The third mapping $\mathcal{M}_3$

Given a point $X = x : y : z$, we introduce a triangle $A'B'C'$ as follows:

$$\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \begin{pmatrix} ayz & (cy+bz)y & (bz+cy)z \\ (cx+az)x & bzx & (az+cx)z \\ (bx+ay)x & (ay+bx)y & cxy \end{pmatrix},$$

and likewise for $A''B''C''$ in terms of $u : v : w$. The method of §2 shows that ABC is perspective to the vertex triangle of $A'B'C'$ and $A''B''C''$. The perspector is given by

$$\begin{aligned} \mathcal{M}_3(U,X) = a/(a^2vwyz + xu(bw+cv)(bz+cy)) \\ : b/(b^2wuzx + yv(cu+aw)(cx+az)) \\ : c/(c^2uvxy + zw(av+bu)(ay+bx)). \end{aligned} \tag{4}$$

A formula for inversion is found as in §3: if $\mathcal{M}_3(U,X) = P = p : q : r$, then X is the point $g(a,b,c) : g(b,c,a) : g(c,a,b)$, where

$$g(a,b,c) = \frac{a}{a^3qrv^2w^2 + a^2G_2 - aG_1 - G_0},$$

where

$$\begin{aligned} G_0 &= u^2p\left(b^2w^2 - c^2v^2\right)(br-cq)\,, \\ G_1 &= uvwp\,(br+cq)\,(bw+cv)\,, \\ G_2 &= uvwrq\,(bw+cv)\,. \end{aligned}$$

Geometrically, A' is the $\{A, \widehat{A}\}$-harmonic conjugate of $\widetilde{A}$, where $\widehat{A}\widehat{B}\widehat{C}$ and $\widetilde{A}\widetilde{B}\widetilde{C}$ are the cocevian and circumcevian triangles of X, respectively. (The vertices of the cocevian triangle of X are $\widehat{A} = 0 : z : -y$, $\widehat{B} = z : 0 : x$, $\widehat{C} = y : -x : 0$. The point $\widehat{A}$ is the $\{B, C\}$-harmonic conjugate of the A-vertex of the cevian triangle of U. The triangle $\widehat{A}\widehat{B}\widehat{C}$ is degenerate, as its vertices are collinear.)

5. The fourth mapping $\mathcal{M}_4$

For given $X = x : y : z$, define a triangle $A'B'C'$ by

$$\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \begin{pmatrix} -ayz & (cy-bz)y & (bz-cy)z \\ (cx-az)x & -bzx & (az-cx)z \\ (bx-ay)x & (ay-bx)y & -cxy \end{pmatrix}.$$

Again, ABC is perspective to the vertex triangle of $A'B'C'$ and the triangle $A''B''C''$ similarly defined from U. The perspector is given by

$$\begin{aligned} \mathcal{M}_4(U, X) &= a/(a^2vwyz + xu(bw-cv)(bz-cy)) \\ &: b/(b^2wuzx + yv(cu-aw)(cx-az)) \\ &: c/(c^2uvxy + zw(av-bu)(ay-bx)). \end{aligned} \tag{5}$$

A formula for inversion is found as for §3: if $\mathcal{M}_4(U, X) = P = p : q : r$, then X is the point $g(a, b, c) : g(b, c, a) : g(c, a, b)$, where

$$g(a, b, c) = \frac{a}{(avw - buw - cuv)\left(a^2qrvw + up\,(cq - br)\,(bw - cv)\right)}.$$

Geometrically, A' is the $\{A, \widehat{A}\}$-harmonic conjugate of $\widetilde{A}$, where $\widehat{A}\widehat{B}\widehat{C}$ and $\widetilde{A}\widetilde{B}\widetilde{C}$ are the cevian and circum-anticevian triangles of X, respectively.

6. Summary and extensions

To summarize §§2–5, vertex triangles associated with circumcevian and circum-anticevian triangles are perspective to the reference triangle ABC, and all four perspectors, given by (2-5), are representable by the following form for first trilinear coordinate:

$$\frac{a}{\frac{a^2}{ux} \pm \left(\frac{b}{v} \pm \frac{c}{w}\right)\left(\frac{b}{y} \pm \frac{c}{z}\right)}; \tag{6}$$

here, the three $\pm$ signs are limited to $-++$, $---$, $+++$, and $+--$, which correspond in order to the four mappings $\mathcal{M}_i(U, X)$.

Regarding each perspector $P = \mathcal{M}_i(U, X)$, formulas for the inverse mapping of X, for given U, have been given, and in the case of the first mapping, the transformation is involutory. The representation (6) shows that $\mathcal{M}_i(U, X) = \mathcal{M}_i(X, U)$ for each i, which is to say that $\mathcal{M}_i(U, X)$ can be viewed as a commutative binary operation. There are many interesting examples regarding the four mappings; some of them are given in §9.

For all four configurations, define $\mathcal{M}_i(U, U)$ by putting $x : y : z = u : v : w$ in (2)–(5), and note that (6) gives the perspector in these cases. In Figure 3, taking $X = U$ corresponds to moving E', F', G' to E, F, G so that in the limit the lines $B'C', C'A', A'B'$ are tangent to Γ. It would be of interest to know the set of triangle centers X for which there exists a triangle center U such that $\mathcal{M}_i(U, U) = X$.

The symbolic substitution

$$(a, b, c) \to (bc, ca, ab) \tag{7}$$

transforms the transfigured plane onto itself, as (7) maps each point $X = x : y : z = x(a, b, c) : y(a, b, c) : z(a, b, c)$ to the point

$$X' = x' : y' : z' = x(bc, ca, ab) : y(bc, ca, ab) : z(ca, ab, bc).$$

The circumcircle, as the locus of X satisfying $ayz + bzx + cxy = 0$, maps to the Steiner circumellipse [10], which is the locus of $x' : y' : z'$ satisfying

$$bcy'z' + caz'x' + abx'y' = 0.$$

Circumcevian triangles map to perspective triangles as in Theorem 1, and perspectors are given by applying (7) to (2). The substitution (7) likewise applies to the developments in §§3–5. Analogous (geometric) results spring from other (non-geometric) symbolic substitutions, such as $(a, b, c) \to (b + c, c + a, a + b)$ and $(a, b, c) \to (a^2, b^2, c^2)$.

7. Homothetic triangles

We return now to the vertex-triangles introduced in §§2–5 and establish that if U and X are a pair of isogonal conjugates, then their vertex triangle is homothetic to ABC.

Theorem 3. *Suppose X is a point not on a sideline of triangle ABC. Then the vertex triangle of the circumcevian triangles of X and X^{-1} is homothetic to ABC, and likewise for the pairs of vertex triangles in §§3–5.*

Proof. In accord with the definition of isogonal conjugate, trilinears for $U = X^{-1}$ are given by $u = yz$, $v = zx$, $w = xy$, so that

$$u : v : w = x^{-1} : y^{-1} : z^{-1}.$$

In the notation of §2, the vertex triangle (1) is given by its A-vertex $x_7 : y_7 : z_7$, where

$$\begin{aligned} x_7 =& \, abc(x^2 + y^2)(x^2 + z^2) + (a^2(bz + cy) + bc(by + cz))x^3 \\ & + a(a^2 + b^2 + c^2)x^2yz + (bcyz(bz + cy) + a^2yz(by + cz))x, \\ y_7 =& - bxz(ab(x^2 + y^2) + (a^2 + b^2 - c^2)xy), \\ z_7 =& - cxy(ac(x^2 + z^2) + (a^2 - b^2 + c^2)xz). \end{aligned}$$

Writing out the coordinates $x_8 : y_8 : z_8$ and $x_9 : y_9 : z_9$, we then evaluate the determinant for parallelism of sideline BC and the A-side of the vertex triangle:

$$\begin{vmatrix} a & b & c \\ 1 & 0 & 0 \\ y_8z_9 - z_8y_9 & z_8x_9 - x_8z_9 & x_8y_9 - y_8x_9 \end{vmatrix} = 0.$$

Likewise, the other two pairs of sides are parallel, so that the vertex triangle of the two circumcevian triangles is now proved homothetic to ABC.

The same procedure shows that the vertex triangles in §§3–5, when $U = X^{-1}$, are all homothetic to ABC (hence homothetic to one another, as well as the medial triangle, the anticomplementary triangle, and the Euler triangle). □

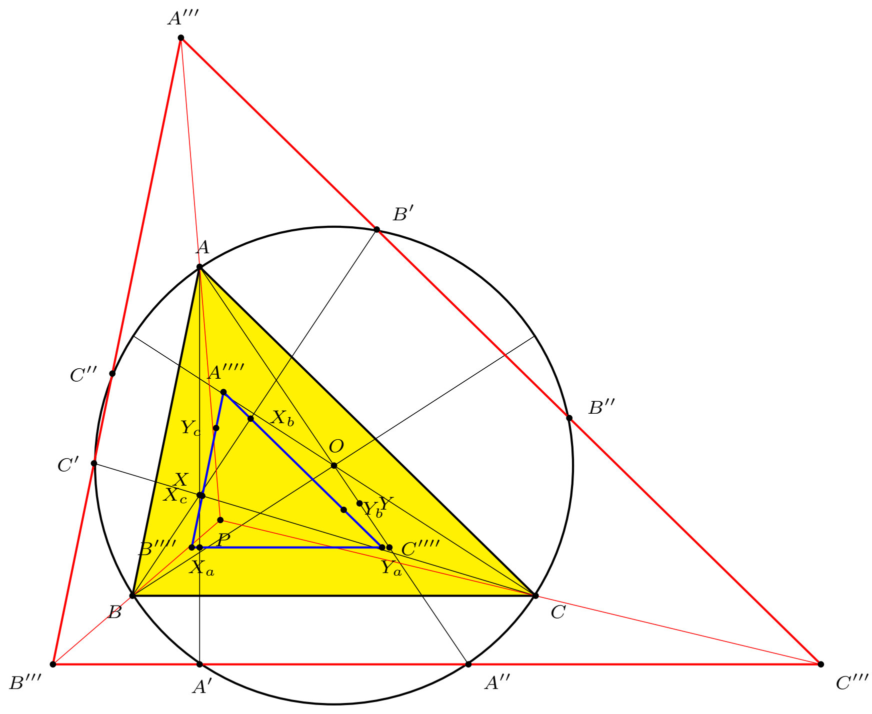


Figure 5.

Substituting into (6) gives a compact expression for the four classes of homothetic centers (*i.e.*, perspectors), given by the following first trilinear:

$$\frac{ayz}{(a^2 \pm (b^2 + c^2))yz \pm bc(y^2 + z^2)},$$

from which it is clear that the homothetic centers for X and X^{-1} are identical.

8. More Homotheties

Let $\mathcal{C}(X)$ denote the circumcevian triangle of a point X, and let O denote the circumcenter, as in Figure 6.

Theorem 4. *Suppose U is a point not on a sideline of triangle ABC. The vertex triangle of $\mathcal{C}(U)$ and $\mathcal{C}(O)$ is homothetic to the pedal triangle of U^{-1}.*

Proof. The vertex triangle $A'''B'''C'''$ of $C(U)$ and $C(O)$ is given by (1), using $U = u : v : w$ and

$$x = a(b^2 + c^2 - a^2), \quad y = b(c^2 + a^2 - b^2), \quad z = c(a^2 + b^2 - c^2).$$

Trilinears for A''' as initially computed include many factors. Canceling those common to all three trilinears leaves

$$x_7 = 4abc(avw + bwu + cuv) + u^2(a + b + c)(-a + b + c)(a - b + c)(a + b - c),$$
$$y_7 = uv(a^2 - b^2 + c^2)(b^2 - a^2 + c^2) - 2cw(a^2 + b^2 - c^2)(av + bu),$$
$$z_7 = uw(a^2 - c^2 + b^2)(c^2 - a^2 + b^2) - 2bv(a^2 + c^2 - b^2)(aw + cu),$$

and x_8, y_8, z_8 and x_9, y_9, z_9 are obtained from x_7, y_7, z_7 by cyclic permutations of a, b, c and u, v, w.

Since $U^{-1} = vw : wu : uv$, the vertices of the pedal triangle of U^{-1} are given ([5, p.186]) by

$$\begin{pmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{pmatrix} = \begin{pmatrix} 0 & w(u + vc_1) & v(u + wb_1) \\ w(v + uc_1) & 0 & u(v + wa_1) \\ v(w + ub_1) & u(w + va_1) & 0 \end{pmatrix},$$

where

$$(a_1, b_1, c_1) = (a(b^2 + c^2 - a^2), b(c^2 + a^2 - b^2), c(a^2 + b^2 - c^2)).$$

Side $B'''C'''$ of the vertex triangle is parallel to the corresponding sideline of the pedal triangle if the determinant

$$\begin{vmatrix} a & b & c \\ g_2h_3 - h_2g_3 & h_2f_3 - f_2h_3 & f_2g_3 - g_2f_3 \\ y_8z_9 - z_8y_9 & z_8x_9 - x_8z_9 & x_8y_9 - y_8x_9 \end{vmatrix} \tag{8}$$

equals 0. It is helpful to factor the polynomials in row 3 and cancel common factors. That and putting $f_1 = g_2 = h_3 = 0$ lead to the following determinant which is a factor of (8):

$$\begin{vmatrix} a & b & c \\ -h_2g_3 & h_2f_3 & f_2g_3 \\ 2a(bw + cv) & w(a^2 + b^2 - c^2) & v(a^2 - b^2 + c^2) \end{vmatrix}.$$

This determinant indeed equals 0. The parallelism of the other matching pairs of sides now follows cyclically. □

9. Properties of the four mappings

This section consists of properties of the mappings $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, $\mathcal{M}_4$ introduced in §§2–5. Proofs are readily given by use of well known formulas. In several cases, a computer is needed because of very lengthy trilinears. Throughout, it is assumed that neither U nor X lies on a sideline of ABC.

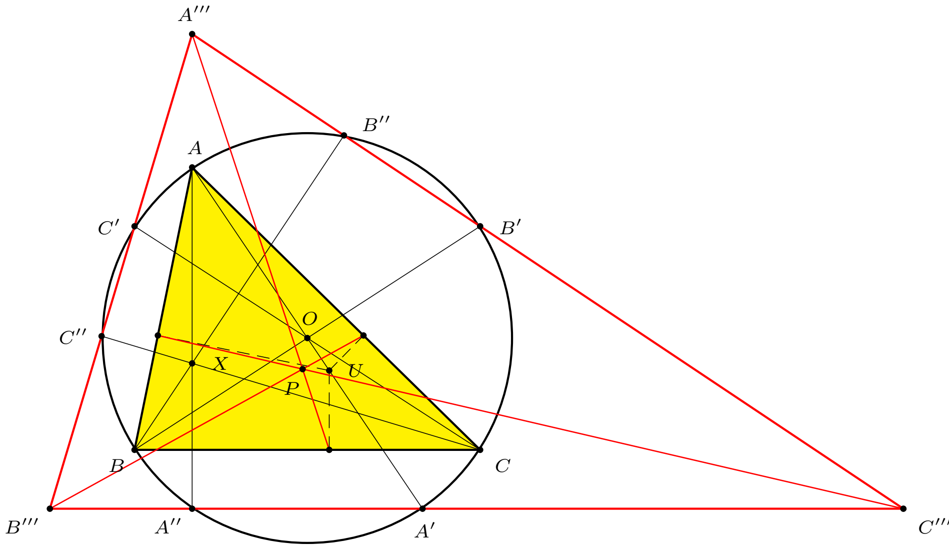


Figure 6.

1a. If $U \in \Gamma$, then $\mathcal{M}_1(U, X) = U$.

1b. If $X \notin \Gamma$, then

$$\mathcal{M}_1(X, X) = \frac{a}{ayz - bzx - cxy} : \frac{b}{bzx - cxy - ayz} : \frac{c}{cxy - ayz - bzx}.$$

If $U = \mathcal{M}_1(X, X)$, then

$$X = \frac{avw}{bw + cv} : \frac{bwu}{cu + aw} : \frac{cuv}{av + bu}.$$

1c. If X is the 1st Saragossa point of U, then $\mathcal{M}_1(U, X) = X$. (The 1st Saragossa point is the point

$$\frac{a}{bzx + cxy} : \frac{b}{cxy + ayz} : \frac{c}{ayz + bzx},$$

discussed at [9] just before X_{1166}.

1d. Suppose U is on the line at infinity, and let U^* be the isogonal conjugate of the antipode of the isogonal conjugate of U. Let L be the line X_3U^*. Then $\mathcal{M}_1(U, U^*) = X_3$, and if $X \in L$, then $\mathcal{M}_1(U, X)$ is the inverse-in-Γ of X.

1e. $\mathcal{M}_1$ maps the Darboux cubic to itself. (See [2] for a discussion of cubics associated with $\mathcal{M}_1$.

2a. $\mathcal{M}_2(X_6, X) = X$.

2b. $\mathcal{M}_2(X, X) = X$-Ceva conjugate of X_6.

2c. Let L be the line UX_6 and let L' be the line UU^c, where $U^c = \mathcal{M}_2(U, U)$. If $X \in L$, then $\mathcal{M}_2(U, X) \in L'$.

3a. $\mathcal{M}_3(X_6, X) = X$.

3b. If $X \in \Gamma$ and X is not on a sideline of ABC, then $\mathcal{M}_3(X, X)$ is the cevapoint X and X_6. (The cevapoint [9, Glossary] of points $P = p : q : r$ and $U = u : v : w$ is defined by trilinears

$$(pv + qu)(pw + ru) : (qw + rv)(qu + pv) : (ru + pw)(rv + qw).)$$

3c. If $U \in \Gamma$, then

$$\mathcal{M}_3(U, X) = \frac{u}{ayz - bzx - cxy} : \frac{v}{bzx - cxy - ayz} : \frac{w}{cxy - ayz - bzx},$$

which is the trilinear product $U \cdot \widehat{X}$, where $\widehat{X}$ is the X_2-isoconjugate of the X-Ceva conjugate of X_6.

4a. $\mathcal{M}_4(X_6, X) = X$.

4b. Suppose P is on the line at infinity (so that P^{-1} is on Γ). Let X be the cevapoint of X_6 and P. Then $\mathcal{M}_4(X_{251}, X) = P^{-1}$.

4c. Let $X^* = X \cdot \widehat{X}$, where $\widehat{X}$ is as in 3c. Then $\mathcal{M}_4(X, X^*) = X_6$.

References

[1] H. S. M. Coxeter, Some applications of trilinear coordinates, *Linear Algebra Appl.*, 226–228 (1995) 375–388.

[2] B. Gibert, Vertex conjugate cubics, available at `http://pagesperso-orange.fr/bernard.gibert/Classes/c1050.html`.

[3] C. Kimberling, Triangle centers as functions, *Rocky Mountain J. Math.*, 23 (1993) 1269–1286.

[4] C. Kimberling, Major centers of triangles, *Amer. Math. Monthly*, 104 (1997) 431–488.

[5] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[6] C. Kimberling, Symbolic substitutions in the transfigured plane of a triangle, *Aequationes Mathematicae*, 73 (2007) 156–171.

[7] C. Kimberling, Second-degree involutory symbolic substitutions, *Forum Geom.*, 8 (2008) 175–182.

[8] C. Kimberling, Cubics defined from symmetric functions, to appear in *Aequationes Mathematicae*.

[9] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[10] S. L. Loney, *The Elements of Coordinate Geometry*, Part II, Macmillan, London, 1957.

[11] E. Weisstein, *MathWorld*, `http://mathworld.wolfram.com`.

[12] P. Yiu, A tour of triangle geometry, `http://www.math.fau.edu/yiu/Geometry.html`.

Clark Kimberling: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, U.S.A.

E-mail address: `ck6@evansville.edu`

Forum Geometricorum
Volume 9 (2009) 41–46.

FORUM GEOM
ISSN 1534-1178

On Integer Relations Between the Area and Perimeter of Heron Triangles

Allan J. MacLeod

Abstract. We discuss the relationship $P^2 = nA$ for a triangle with integer sides, with perimeter P and area A, where n is an integer. We show that the problem reduces to finding rational points of infinite order in a family of elliptic curves. The geometry of the curves plays a crucial role in finding real triangles.

1. Introduction

In a recent paper, Markov [2] discusses the problem of solving $A = mP$, where A is the area and P is the perimeter of an integer-sided triangle, and m is an integer. This relation forces A to be integral and so the triangle is always a Heron triangle.

In many ways, this is not a proper question to ask, since this relation is not scale-invariant. Doubling the sides to a similar triangle changes the area/perimeter ratio by a factor of 2. Basically, we have unbalanced dimensions - area is measured in square-units, perimeter in units but m is a dimensionless quantity.

It would seem much better to look for relations between A and P^2, which is the purpose of this paper. Another argument in favour of this is that the recent paper of Baloglou and Helfgott [1], on perimeters and areas, has the main equations (1) to (8) all balanced in terms of units.

We assume the triangle has sides (a, b, c) with $P = a + b + c$ and $s = \frac{P}{2}$. Then the area is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{4}\sqrt{P(P-2a)(P-2b)(P-2c)}$$

so that it is easy to see that $A < P^2/4$. Thus, to look for an integer link, we should study $P^2 = nA$ with $n > 4$.

It is easy to show that this bound on n can be increased quite significantly. We have

$$\frac{P^4}{A^2} = 16\frac{(a+b+c)^3}{(a+b-c)(a+c-b)(b+c-a)} \tag{1}$$

and we can, without loss of generality, assume $a = 1$. Then the ratio in equation (1) is minimised when $b = 1, c = 1$. This is obvious from symmetry, but can be easily proven by finding derivatives. Thus $\frac{P^4}{A^2} \geq 432$ and so $P^2 \geq 12\sqrt{3}A$ for all triangles, so we need only consider $n \geq 21$.

Publication Date: March 9, 2009. Communicating Editor: Paul Yiu.

As an early example of a solution, the $(3, 4, 5)$ triangle has $P^2 = 144$ and $A = 6$ so $n = 24$.

To proceed, we consider the equation

$$16\frac{(a+b+c)^3}{(a+b-c)(a+c-b)(b+c-a)} = n^2. \tag{2}$$

2. Elliptic Curve Formulation

Firstly, it is clear that we can let a, b, c be rational numbers, since a rational-sided solution is easily scaled up to an integer one.

From equation (2), we have

$$(n^2+16)a^3 + (48-n^2)(b+c)a^2 - (b^2(n^2-48) - 2bc(n^2+48) + c^2(n^2-48))a$$
$$+ (b+c)(b^2(n^2+16) + 2bc(16-n^2) + c^2(n^2+16)) = 0.$$

This cubic is very difficult to deal with directly, but a considerable simplification occurs if we use $c = P - a - b$, giving

$$4n^2(P-2b)a^2 - 4n^2(2b^2 - 3bP + P^2)a + P(4b^2n^2 - 4bn^2P + P^2(n^2+16)) = 0. \tag{3}$$

For this quadratic to have rational roots, we must have the discriminant being a rational square. This means that there must be rational solutions of

$$d^2 = 4n^2b^4 - 4n^2Pb^3 + n^2P^2b^2 + 32P^3b - 16P^4$$

and, if we define $y = \frac{2nd}{P^2}$ and $x = \frac{2nb}{P}$, we have

$$y^2 = x^4 - 2nx^3 + n^2x^2 + 64nx - 64n^2. \tag{4}$$

A quartic in this form is birationally equivalent to an elliptic curve, see Mordell [3]. Using standard transformations and some algebraic manipulation, we find the equivalent curves are

$$E_n : v^2 = u^3 + n^2u^2 + 128n^2u + 4096n^2 = u^3 + n^2(u+64)^2 \tag{5}$$

with the backward transformation

$$\frac{b}{P} = \frac{n(u-64)+v}{4nu}. \tag{6}$$

Thus, from a suitable point (u, v) on E_n, we can find b and P from this relation. To find a and c, we use the quadratic for a, but written as

$$a^2 - (P-b)a + \frac{P(16P^2 + n^2(P-2b)^2)}{4n^2(P-2b)} = 0. \tag{7}$$

The sum of the roots of this quadratic is $P - b = a + c$, so the two roots give a and c.

But, we should be very careful to note that the analysis based on equation (2) is just about relations between numbers, which could be negative. Even if they are all positive, they may not form a real-life triangle - they do not satisfy the triangle inequalities. Thus we need extra conditions to give solutions, namely $0 < a, b, c < \frac{P}{2}$.

3. Properties of E_n

The curves E_n are clearly symmetric about the u-axis. If the right-hand-side cubic has 1 real root R, then the curve has a single infinite component for $u \geq R$. If, however, there are 3 real roots $R_1 < R_2 < R_3$, then E_n consists of an infinite component for $u \geq R_3$ and a closed component for $R_1 \leq u \leq R_2$, usually called the "egg".

Investigating with the standard formulae for cubic roots, we find 3 real roots if $n^2 > 432$ and 1 real root if $n^2 < 432$. Since we assume $n \geq 21$, there must be 3 real roots and so 2 components. Descartes' rule of signs shows that all roots are negative.

It is clear that $u = -64$ does not give a point on the curve, but $u = -172$ gives $v^2 = 16(729n^2 - 318028)$ which is positive if $N \geq 21$. Thus we have $R_1 < -172 < R_2 < -64 < R_3 < 0$.

The theory of rational points on elliptic curves is an enormously developed one. The rational points form a finitely-generated Abelian group with the addition operation being the standard secant/tangent method. This group of points is isomorphic to the group $T \oplus \mathbb{Z}^r$, where T is one of $\mathbb{Z}_m,\ m = 1, 2, \dots, 10, 12$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_m, m = 1, 2, 3, 4$, and r is the rank of the curve. T is known as the torsion-subgroup and consists of those points of finite order on the curve, including the point-at-infinity which is the identity of the group. Note that the form of E_n ensures that torsion points have integer coordinates by the Nagell-Lutz theorem, see Silverman and Tate [5].

We easily see the two points $(0, \pm 64n)$ and since they are points of inflexion of the curve, they have order 3. Thus T is one of $\mathbb{Z}_3, \mathbb{Z}_6, \mathbb{Z}_9, \mathbb{Z}_{12}, \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Some of these possibilities would require a point of order 2 which correspond to integer zeroes of the cubic. Numerical investigations show that only $n = 27$, for $n \leq 499$, has an integer zero (at $u = -576$). Further investigations show $\mathbb{Z}_3$ as being the only torsion subgroup to appear, for $n \leq 499$, apart from $\mathbb{Z}_6$ for $n = -27$. Thus we conjecture that, apart from $n = 27$, the group of rational points is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}^r$. The points of order 3 give $\frac{b}{P}$ undefined so we would need $r \geq 1$ to possibly have triangle solutions.

For $n = 27$, we find the point $H = (-144, 1296)$ of order 6, which gives the isosceles triangle $(5, 5, 8)$ with $P = 18$ and $A = 12$. In fact, all multiples of H lead to this solution or $\frac{b}{P}$ undefined.

4. Rank Calculations

There is, currently, no known guaranteed method to determine the rank r. We can estimate r very well, computationally, if we assume the Birch and Swinnerton-Dyer conjecture [6]. We performed the calculations using some home-grown software, with the Pari-gp package for the multiple-precision calculations. The results for $21 \leq n \leq 99$ are shown in Table 1.

TABLE 1
Ranks of $E_n, n = 21, \ldots, 99$

n	0	1	2	3	4	5	6	7	8	9
20+	–	1	0	0	1	0	1	0	1	0
30+	1	2	0	1	0	1	1	1	0	1
40+	0	0	2	2	0	1	0	2	0	0
50+	1	1	1	0	0	1	1	0	2	0
60+	1	1	1	2	1	0	1	1	0	0
70+	0	1	0	0	2	1	1	1	0	2
80+	0	1	0	1	0	1	1	0	2	0
90+	1	2	0	1	1	1	1	0	1	1

We can see that the curve from the $(3, 4, 5)$ triangle with $n = 24$ has rank 1, but the $(5, 5, 8)$ triangle has a curve with rank 0, showing that this is the only triangle giving $n = 27$.

For those curves with rank 1, a by-product of the Birch and Swinnerton-Dyer computations is an estimate for the height of the generator of the rational points of infinite order. The height essentially gives an idea of the sizes of the numerators and denominators of the u coordinates. The largest height encountered was 10.25 for $n = 83$ (the height normalisation used is the one used by Silverman [4]).

All the heights computed are small enough that we could compute the generators fairly easily, again using some simple software. From the generators, we derive the list of triangles in Table 2.

5. Geometry of $0 < \frac{b}{P} < \frac{1}{2}$

The sharp-eyed reader will have noticed that several values of n, which have positive rank in Table 1, do not give a triangle in Table 2, despite generators being found To explain this, we need to consider the geometric implications on (u, v) from the bounds $0 < \frac{b}{P} < \frac{1}{2}$, or

$$0 < \frac{n(u-64)+v}{4nu} < \frac{1}{2} \tag{8}$$

Consider first $u > 0$. Then $\frac{b}{P} > 0$ when $v > 64n - nu$. The line $v = 64n - nu$, only meets E_n at $u = 0$, and the negative gradient shows that $\frac{b}{P} > 0$ when we take points on the upper part of the curve. To have $\frac{b}{P} < \frac{1}{2}$, we need $v < nu + 64n$. The line $v = nu + 64n$ has an intersection of multiplicity 3 at $u = 0$, so never meets E_n again. Thus $v < nu + 64n$ only on the lower part of E_n for $u > 0$. Thus, we cannot have $0 < \frac{b}{P} < \frac{1}{2}$ for any points with $u > 0$.

Now consider $u < 0$. Then, for $\frac{b}{P} > 0$ we need $v < 64n - nu$. The negative gradient and single intersection show that this holds for all points on E_n with $u < 0$. For $\frac{b}{P} < \frac{1}{2}$, we need $v > nu + 64n$. This line goes through $(0, 64n)$ on the curve and crosses the u-axis when $u = -64$, which we saw earlier lies strictly between the egg and the infinite component. Since $(0, 64n)$ is the only intersection we must have the line above the infinite component of E_n when $u < 0$ but below the egg.

TABLE 2
Triangles for $21 \leq n \leq 99$

n	a	b	c
21	15	14	13
28	35	34	15
31	85	62	39
35	97	78	35
39	37	26	15
43	56498	31695	29197
47	4747	3563	1560
51	149	85	72
55	157	143	30
58	85	60	29
62	598052	343383	275935
66	65	34	33
75	74	51	25
77	1435	2283	902
81	26	25	3
88	979	740	261
93	2325	2290	221
98	2307410	2444091	255319

n	a	b	c
24	5	4	3
30	13	12	5
33	30	25	11
36	17	10	9
42	20	15	7
45	41	40	9
50	1018	707	375
52	5790	4675	1547
56	41	28	15
60	29	25	6
63	371	250	135
74	740	723	91
76	47575	43074	7163
79	1027	1158	185
85	250	221	39
91	1625	909	742
95	24093	29582	6175
99	97	90	11

Thus, $0 < \frac{b}{P} < \frac{1}{2}$ only on the egg where $u < -64$. The results in Table 2 come from generators satisfying this condition.

It might be thought that forming integer multiples of generators and possibly adding the torsion points could resolve this. This is not the case, due to the closed nature of the egg. If a line meets the egg and is not a tangent to the egg, then it enters the egg and must exit the egg. Thus any line has a double intersection with the egg.

So, if we add a point on the infinite component to either torsion point, also on the infinite component, we must have the third intersection on the infinite component. Similarly doubling a point on the infinite component must lead to a point on the infinite component. So, if no generator lies on the egg, there will never be a point on the egg, and so no real-life triangle will exist.

We can generate other triangles for a value of n by taking multiples of the generator. Using the same arguments as before, it is clear that a generator G on the egg has $2G$ on the infinite component but $3G$ must lie on the egg. So, for $n = 24$, the curve $E_{24} : v^2 = u^3 + 576u^2 + 73728u + 2359296$ has $G = (-384, 1536)$, hence $2G = (768, -29184)$ and $3G = \left(-\frac{2240}{9}, \frac{55808}{27}\right)$, which leads to the triangle $(287, 468, 505)$ where $P = 1260$ and $A = 66150$.

References

[1] G. Baloglou and M. Helfgott, Angles, area and perimeter caught in a cubic, *Forum. Geom.*, 8 (2008) 13–25.

[2] L. Markov, Heronian triangles whose areas are integer multiples of their perimeters, *Forum Geom.*, 7 (2007) 129–135.

[3] L. J. Mordell, *Diophantine Equations*, Academic Press, New York, 1968.
[4] J. H. Silverman, Computing rational points on rank 1 elliptic curves via L-series and canonical heights, *Math. Comp.* 68 (1999) 835–858.
[5] J. H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992.
[6] A. Wiles, *The Birch and Swinnerton-Dyer Conjecture*, available at the Clay Mathematics website `www.claymath.org`.

Allan J. MacLeod: Department of Mathematics and Statistics, University of the West of Scotland, High St., Paisley, Scotland. PA1 2BE.
E-mail address: `allan.macleod@uws.ac.uk`

Forum Geometricorum
Volume 9 (2009) 47–55.

FORUM GEOM
ISSN 1534-1178

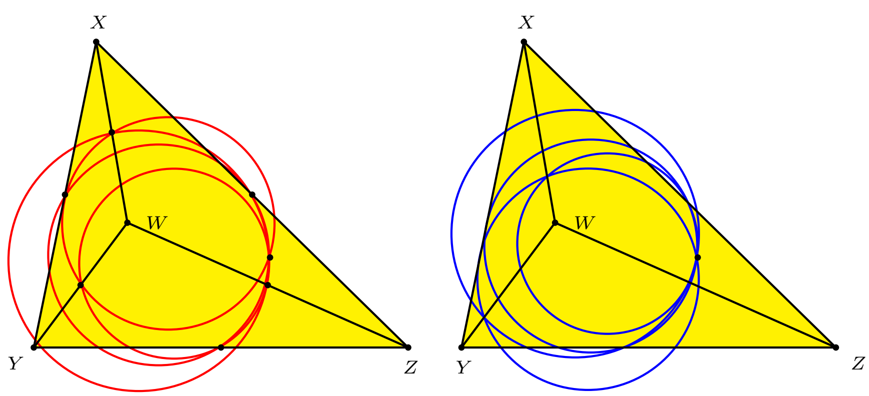
The Feuerbach Point and Reflections of the Euler Line

Jan Vonk

Abstract. We investigate some results related to the Feuerbach point, and use a theorem of Hatzipolakis to give synthetic proofs of the facts that the reflections of OI in the sidelines of the intouch and medial triangle all concur at the Feuerbach point. Finally we give some results on certain reflections of the Feuerbach point.

1. Poncelet point

We begin with a review of the Poncelet point of a quadruple of points W, X, Y, Z. This is the point of concurrency of
(i) the nine-point circles of triangles WXY, WXZ, XYZ, WYZ,
(ii) the four pedal circles of W, X, Y, Z with respect to XYZ, WYZ, WXZ, WXY respectively.



Nine-point circles

Pedal circles

Figure 1.

Basic properties of the Poncelet point can be found in [4]. Let I be the incenter of triangle ABC. The Poncelet point of I, A, B, C is the famous Feuerbach point $F_{\rm e}$, as we show in Theorem 1 below. In fact, we can find a lot more circles passing through $F_{\rm e}$, using the properties mentioned in [4].

Theorem 1. *The nine-point circles of triangles AIB, AIC, BIC are concurrent at the Feuerbach point $F_{\rm e}$ of triangle ABC.*

Publication Date: March 30, 2009. Communicating Editor: A. P. Hatzipolakis.
The author would like to thank Paul Yiu for his help in the preparation of this paper.

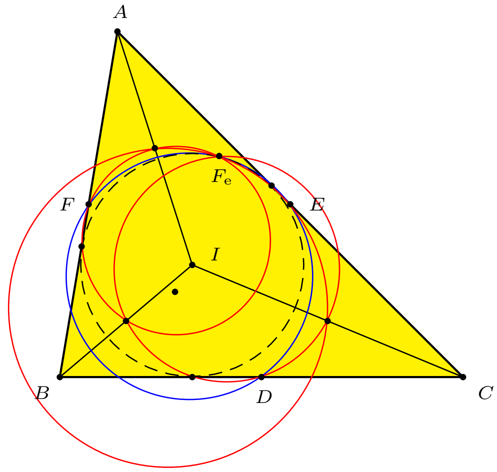


Figure 2.

Proof. The Poncelet point of A, B, C, I must lie on the pedal circle of I with respect to triangle ABC, and on the nine-point circle of triangle ABC (see Figure 1). Since these two circles have only the Feuerbach point F_e in common, it must be the Poncelet point of A, B, C, I. □

A second theorem, conjectured by Antreas Hatzipolakis, involves three curious triangles which turn out to have some very surprising and beautiful properties. We begin with an important lemma, appearing in [9] as Lemma 2 with a synthetic proof. The midpoints of BC, AC, AB are labeled D, E, F.

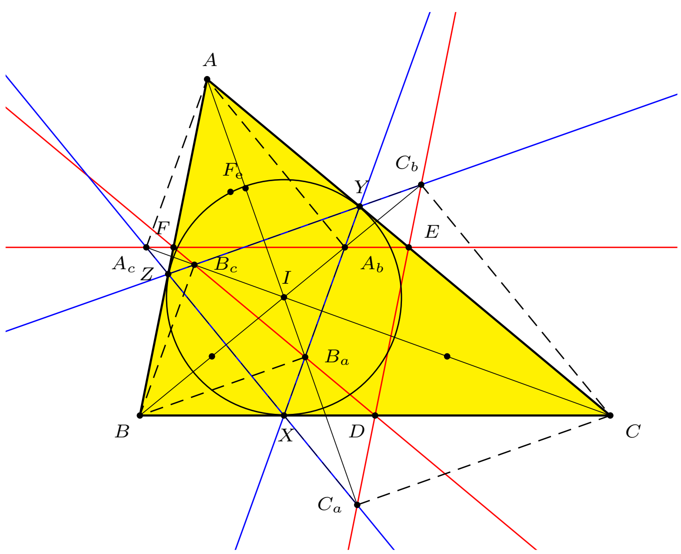


Figure 3.

We shall adopt the notations of [9]. Given a triangle ABC, let D, E, F be the midpoints of the sides BC, CA, AB, and X, Y, Z the points of tangency of the incircle with these sides. Let A_b and A_c be the orthogonal projections of A on the bisectors BI and CI respectively. Similarly define B_c, B_a, C_a, C_b (see Figure 3).

Lemma 2. (a) *A_b and A_c lie on EF.*
(b) *A_b lies on XY, A_c lies on XZ.*
Similar statements are true for B_a, B_c and C_a, C_b.

We are now ready for the second theorem, stated in [6]. An elementary proof was given by Khoa Lu Nguyen in [7]. We give a different proof, relying on the Kariya theorem (see [5]), which states that if X', Y', Z' are three points on IX, IY, IZ with $\frac{IX'}{IX} = \frac{IY'}{IY} = \frac{IZ'}{IZ} = k$, then the lines AX', BY', CZ' are concurrent. For $k = -2$, this point of concurrency is known to be X_{80}, the reflection of I in F_{e}.

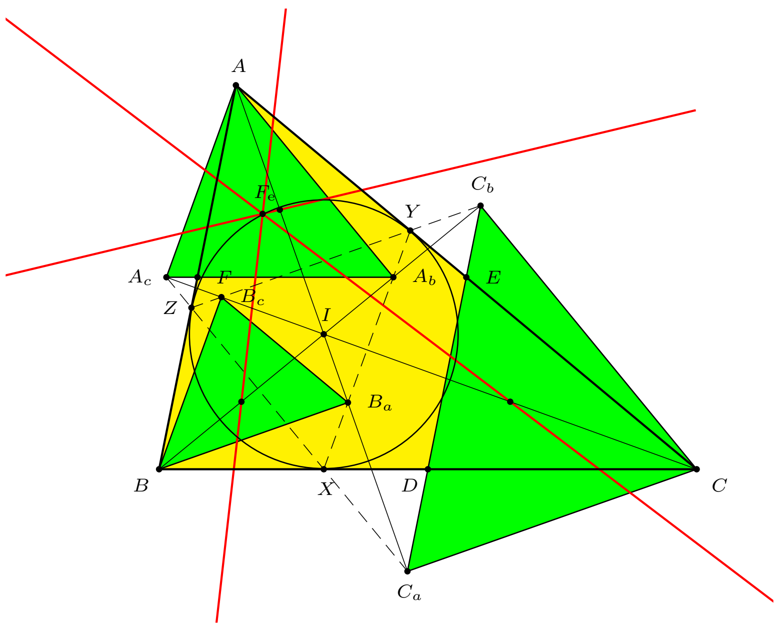


Figure 4.

Theorem 3 (Hatzipolakis)**.** *The Euler lines of triangles AA_bA_c, BB_aB_c, CC_aC_b are concurrent at F_{e}* (see Figure 4).

Proof. If X' is the antipode of X in the incircle, O_a the midpoint of A and I, H_a the orthocenter of triangle AA_bA_c, then clearly H_aO_a is the Euler line of triangle AA_bA_c. Also, $\angle A_bAA_c = \pi - \angle A_cIA_b = \frac{B+C}{2}$. Because AI is a diameter of the circumcircle of triangle AA_bA_c, it follows that $AH_a = AI \cdot \cos\frac{B+C}{2} = AI \cdot \sin\frac{A}{2} = r$, where r is the inradius of triangle ABC. Clearly, $IX' = r$, and it follows from Lemma 1 that $AH_a \parallel IX'$. Triangles AH_aO_a and $IX'O_a$ are congruent, and X' is the reflection of H_a in O_a. Hence X' lies on the Euler line of triangle AA_bA_c.

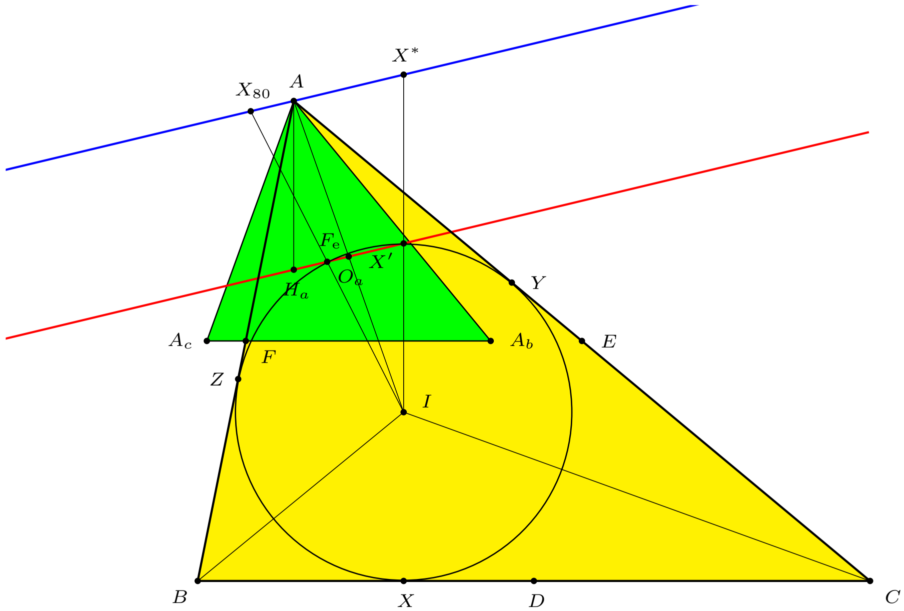


Figure 5.

If X^* is the reflection of I in X', we know by the Kariya theorem that A, X^*, and X_{80} are collinear. Now the homothety $\mathsf{h}(I, \frac{1}{2})$ takes A to O_a, X^* to X', and X_{80} to the Feuerbach point F_e. $\square$

We establish one more theorem on the Feuerbach point. An equivalent formulation was posed as a problem in [10].

Theorem 4. *If X'', Y'', Z'' are the reflections of X, Y, Z in AI, BI, CI, then the lines DX'', EY'', FZ'' concur at the Feuerbach point F_e* (see Figure 6).

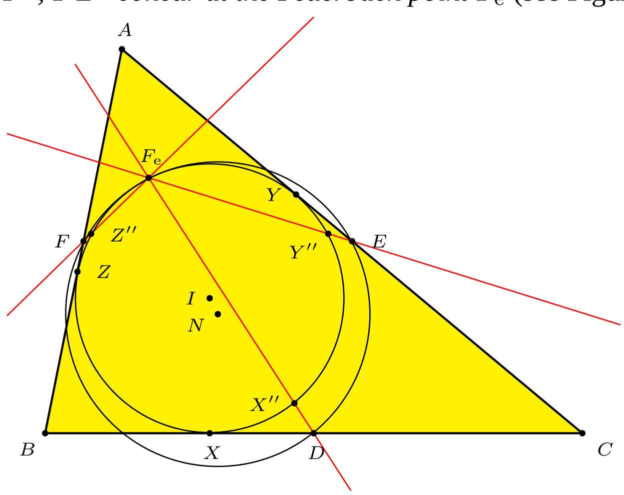


Figure 6.

Proof. We show that the line DX'' contains the Feuerbach point F_e. The same reasoning will apply to EY'' and FZ'' as well.

Clearly, X'' lies on the incircle. If we call N the nine-point center of triangle ABC, then the theorem will follow from $IX'' \parallel ND$ since F_e is the external center of similitude of the incircle and nine-point circle of triangle ABC. Now, because $IX \parallel AH$, and because O and H are isogonal conjugates, $IX'' \parallel AO$. Furthermore, the homothety $\mathsf{h}(G, -2)$ takes D to A and N to O. This proves that $ND \parallel AO$. It follows that $IX'' \parallel ND$. □

2. The Euler reflection point

The following theorem was stated by Paul Yiu in [11], and proved by barycentric calculation in [8]. We give a synthetic proof of this result.

Theorem 5. *The reflections of OI in the sidelines of the intouch triangle DEF are concurrent at the Feuerbach point of triangle ABC* (see Figure 7).

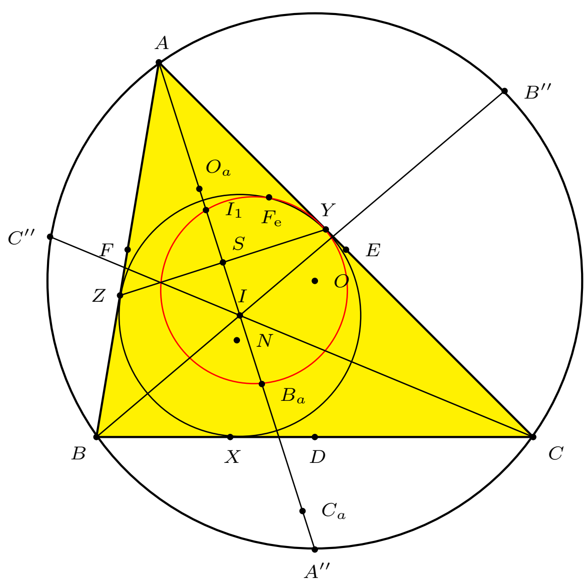


Figure 7.

Proof. Let us call I_1 the reflection of I in YZ. By Theorem 1, the nine-point circle of triangle AIC, which clearly passes through Y, O_a, C_a, also passes through F_e. If S is the intersection of YZ and AI, then clearly A is the inverse of S with respect to the incircle. Because $2 \cdot IO_a = IA$ and $2 \cdot IS = II_1$, it follows that O_a is the inverse of I_1 with respect to the incircle. Because C_a lies on XZ, its polar line must pass through B and be perpendicular to AI. This shows that B_a is the inverse of C_a with respect to the incircle.

Now invert the nine-point circle of triangle AIC with respect to the incircle of triangle ABC. This circle can never pass through I since $\angle AIC > \frac{\pi}{2}$, so the image is a circle. This shows that $YI_1F_eB_a$ is a cyclic quadrilateral, so it follows that $\angle F_eI_1B_a = \angle F_eYX = \angle F_eX'X$.

If we call $A''B''C''$ the circumcevian triangle of I, then we notice that $\angle AA_bA_c = \angle AIA_c = \angle A''IC$. Now, it is well known that $A''C = A''I$, so it follows that $\angle AA_bA_c = \angle ICA'' = \angle C''B''A''$. Similar arguments show that triangle AA_bA_c and triangle $A''B''C''$ are inversely similar.

As we have pointed out before as a consequence of Lemma 2, AH_a and IX' are parallel. By Theorem 3, F_eX' is the Euler line of triangle AA_bA_c. Therefor, $\angle F_eX'X = \angle O_aX'X = \angle O_aH_aA$. We know that triangle AA_bA_c is inversely similar to triangle $A''B''C''$. Since O and I are the circumcenter and orthocenter of triangle $A''B''C''$, it follows that $\angle O_aH_aA = \angle A''IO = \angle OIA$.

We conclude that $\angle F_eI_1S = \angle F_eI_1B_a = \angle F_eYX = \angle F_eX'X = \angle AIO = \angle SIO$. This shows that the reflection of OI in EF passes through F_e. Similar arguments for the reflections of OI in XY and XZ complete the proof. □

A very similar result is stated in the following theorem. We give a synthetic proof, similar to the proof of the last theorem in many ways. First, we will need another lemma.

Lemma 6. *The vertices of the polar triangle of DEF with respect to the incircle are the orthocenters of triangles BIC, AIC, AIB. Furthermore, they are the reflections of the excenters in the respective midpoints of the sides.*

This triangle is the main subject of [9], in which a synthetic proof can be found.

Theorem 7. *The reflections of OI in the sidelines of the medial triangle DEF are concurrent at the Feuerbach point of triangle ABC* (see Figure 8).

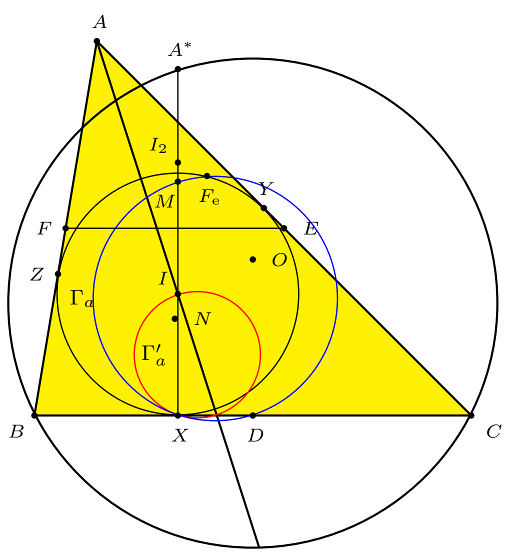


Figure 8.

Proof. Call I_2 the reflection of I in EF, and A^* the orthocenter of triangle BIC. The midpoint of I and A^* is called M. Using Lemma 6, we know that EF is the polar line of A^* with respect to the incircle. A similar argument as the one we gave in the proof of Theorem 5 shows that I_2 is the inverse of M with respect to the incircle.

Clearly, F_{e}, M, X, D all lie on the nine-point circle of triangle BIC. Call this circle Γ_a and call Γ'_a the circumcircle of triangle IXX''. Clearly, the center of Γ'_a is on AI. Because I is the orthocenter of triangle BA^*C, we have that the reflection of I in D is the antipode of A^* in the circumcircle of A^*BC. Call this point L'. Consider the homothety $\mathsf{h}(I, 2)$, MD is mapped, and hence is parallel, to A^*L'. We know that A^* is the reflection in D of the A-excenter of triangle ABC (see [9]), so A^*L' is also parallel to AI. It follows that AI and MD are parallel.

If we call T the intersection of AI and BC, then it is clear that T lies on Γ'_a. Because IT and MD are parallel diameters of two circles, there exists a homothety centered at X which maps Γ'_a to Γ_a. Because X lies on both circles, we now conclude that X is the point of tangency of Γ_a and Γ'_a. Inverting these two circles in the incircle, we see that XX'' is tangent to the circumcircle of $XF_{\mathrm{e}}I_2$.

Finally, $\angle MIO = \angle AIO + \angle MIA = \angle F_{\mathrm{e}}X'X + \angle IMD = \angle F_{\mathrm{e}}XD + \angle XF_{\mathrm{e}}D = \angle F_{\mathrm{e}}XD + \angle DXX'' = \angle F_{\mathrm{e}}XX'' = \angle F_{\mathrm{e}}I_2X$, where the last equation follows from the alternate segment theorem. This proves that I_2F_{e} is the reflection of OI in EF. Similar arguments for DF and DE prove the theorem. □

The following theorem gives new evidence for the strong correlation between the nature of the Feuerbach point and the Euler reflection point.

Theorem 8. *The three reflections of H_aO_a in the sidelines of triangle AA_bA_c and the line OI are concurrent at the reflection E_a of F_{e} in A_bA_c. Similar theorems hold for triangles BB_aB_c, CC_aC_b* (see Figure 9).

Proof. The 3 reflections of H_aO_a in the sidelines of triangle AA_bA_c are concurrent at the Euler reflection point of triangle AA_bA_c. We will first show that this point is the reflection of F_{e} in A_bA_c.

The circle with diameter XH_a clearly passes through A_b, A_c by definition of A_b, A_c. It also passes through F_{e}, since $H_aF_{\mathrm{e}} = X'F_{\mathrm{e}} \perp XF_{\mathrm{e}}$, so we conclude that F_{e}, A_c, X, A_b are concyclic. Because AA_cXA_b is a parallellogram, we see that the reflection in the midpoint of A_b and A_c of the circle through A_b, A_c, X, F_{e} is in fact the circumcircle of triangle AA_bA_c. We deduce that the reflection of F_{e} in A_bA_c lies on the circumcircle of triangle AA_bA_c. Since $F_{\mathrm{e}} \neq H_a$ lies on the Euler line of triangle AA_bA_c and E_a lies on the circumcircle of triangle AA_bA_c, we have proven that the reflection of F_{e} in A_bA_c is the Euler reflection point of triangle AA_bA_c.

By theorem 7, it immediately follows that E_a lies on OI. This completes the proof. □

We know that we can see E_a as an intersection point of the perpendicular to A_bA_c through F_{e} with the circumcircle of triangle AA_bA_c. This line intersects the

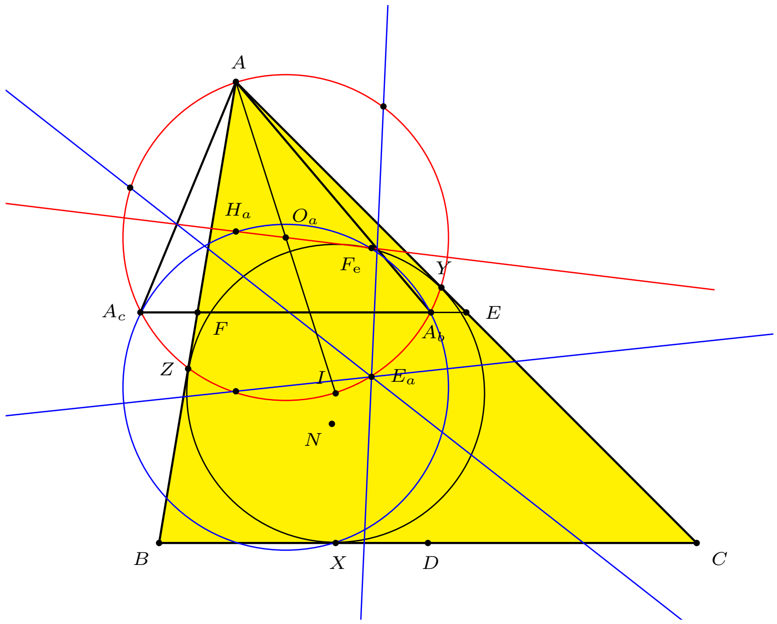


Figure 9.

circle in another point, which we will call U. Similarly define V and W on the circumcircles of triangles BB_aB_c and CC_aC_b.

Theorem 9. *The lines AU, BV, CW are concurrent at X_{80}, the reflection of I in F_{e}* (see Figure 10).

Proof. The previous theorem tells us that E_a lies on OI. It follows that $\angle E_aIO_a = \angle OIA$. In the proof of Theorem 5, we prove that $\angle AIO = \angle AH_aO_a$. Since F_eE_a and AH_a are parallel, we deduce that E_a, I, O_a and F_e are concyclic. If we call U' the intersection of E_aF_{e} and the line through A parallel to O_aF_{e}, then we have that $\angle E_aIA = \angle E_aF_{\mathrm{e}}O_a = \angle E_aU'A$. It follows that A, U', E_a, I are concyclic, so $U \equiv U'$.

Now consider a homothety centered at I with factor 2. Clearly, O_aF_{e} is mapped to a parallel line through A, which is shown to pass through U. The image of F_{e} however is X_{80}, so AU passes through X_{80}. Similar arguments for BV, CW complete the proof. $\square$

References

[1] J.-L. Ayme, Two similar triangles, *Mathlinks*,
`http://www.mathlinks.ro/viewtopic.php?t=219830`.

[2] J.-L. Ayme, Again with the Feuerbach's point, *Mathlinks*,
`http://www.mathlinks.ro/viewtopic.php?t=220032`.

[3] D. Grinberg, Hyacinthos message 10500, September 20, 2004.

[4] D. Grinberg, Poncelet points and antigonal conjugates, *Mathlinks*,
`http://www.mathlinks.ro/Forum/viewtopic.php?t=109112`.

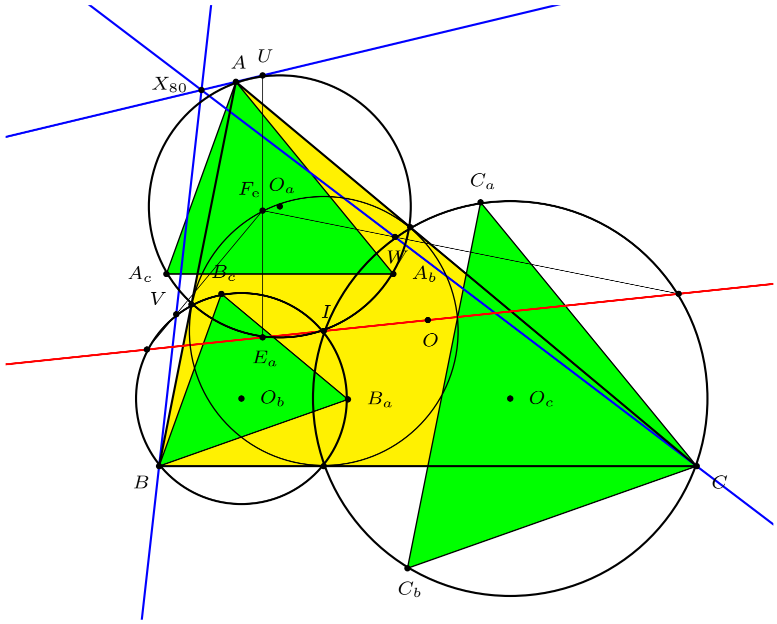


Figure 10.

[5] D. Grinberg, Hyacinthos message 10504, September 20, 2004.
[6] A. P. Hatzipolakis, Hyacinthos message 10485, September 18, 2004.
[7] K. L. Nguyen, Hyacinthos message 10913, November 28, 2004.
[8] B. Suceava and P. Yiu, the Feuerbach point and Euler lines, *Forum Geom.*, 6 (2006), 191–197.
[9] J. Vonk, On the Nagel line and a prolific polar triangle, *Forum Geom.*, 8 (2008) 183–197.
[10] J. Vonk, Problem O70, *Mathematical Reflections*, 6/2007.
[11] P. Yiu, Hyacinthos message 11652, October 18, 2005.

Jan Vonk: Groenstraat 70, 9340 Lede, Belgium
E-mail address: `jan.vonk.jv@gmail.com`

Forum Geometricorum
Volume 9 (2009) 57–80.

FORUM GEOM
ISSN 1534-1178

Rings of Squares Around Orthologic Triangles

Zvonko Čerin

Abstract. We explore some properties of the geometric configuration when a ring of six squares with the same orientation are erected on the segments BD, DC, CE, EA, AF and FB connecting the vertices of two orthologic triangles ABC and DEF. The special case when DEF is the pedal triangle of a variable point P with respect to the triangle ABC was studied earlier by Bottema [1], Deaux [5], Erhmann and Lamoen [4], and Sashalmi and Hoffmann [8]. We extend their results and discover several new properties of this interesting configuration.

1. Introduction – Bottema's Theorem

The orthogonal projections P_a, P_b and P_c of a point P onto the sidelines BC, CA and AB of the triangle ABC are vertices of its pedal triangle.

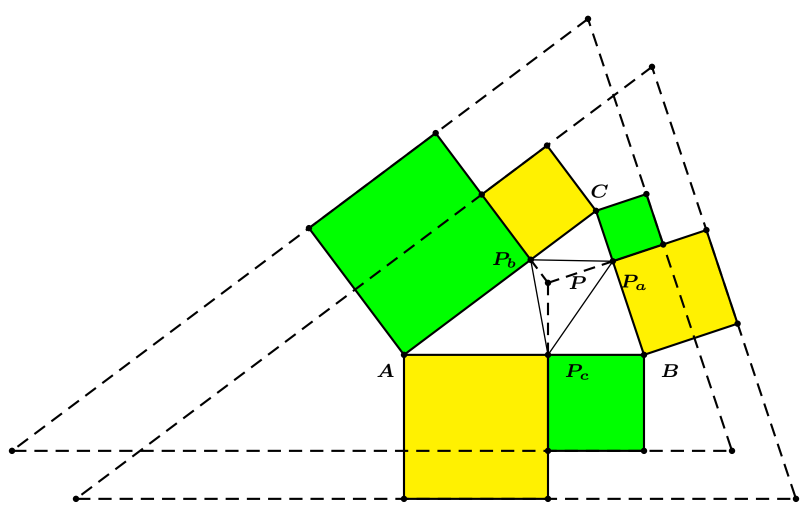


Figure 1. Bottema's Theorem on sums of areas of squares.

In [1], Bottema made the remarkable observation that

$$|BP_a|^2 + |CP_b|^2 + |AP_c|^2 = |P_aC|^2 + |P_bA|^2 + |P_cB|^2.$$

This equation has an interpretation in terms of area which is illustrated in Figure 1. Rather than using geometric squares, other similar figures may be used as in [8].

Publication Date: April 13, 2009. Communicating Editor: Paul Yiu.

Figure 1 also shows two congruent triangles homothetic with the triangle ABC that are studied in [4] and [8].

The primary purpose of this paper is to extend Bottema's Theorem (see Figure 2). The longer version of this paper is available at the author's Web home page `http://math.hr/~cerin/`. We thank the referee for many useful suggestions that improved greatly our exposition.

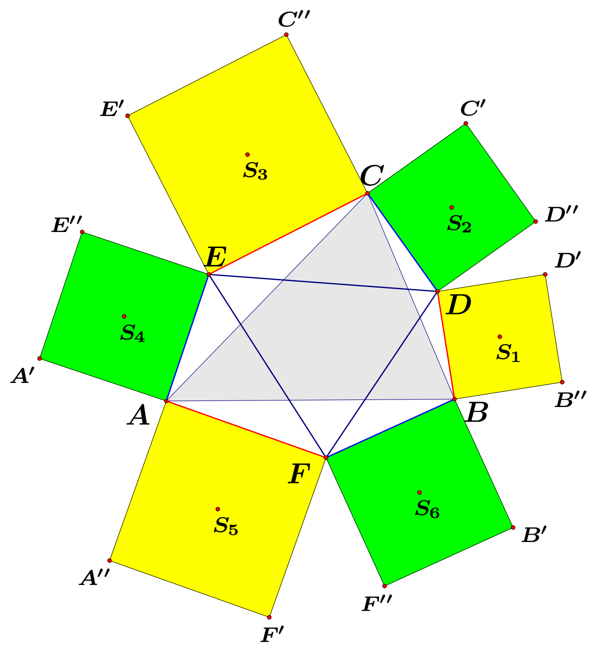


Figure 2. Notation for a ring of six squares around two triangles.

2. Connection with orthology

The origin of our generalization comes from asking if it is possible to replace the pedal triangle $P_aP_bP_c$ in Bottema's Theorem with some other triangles. In other words, if ABC and DEF are triangles in the plane, when will the following equality hold?

$$|BD|^2 + |CE|^2 + |AF|^2 = |DC|^2 + |EA|^2 + |FB|^2 \tag{1}$$

The straightforward analytic attempt to answer this question gives the following simple characterization of the equality (1).

Throughout, triangles will be non-degenerate.

Theorem 1. *The relation (1) holds for triangles ABC and DEF if and only if they are orthologic.*

Recall that triangles ABC and DEF are *orthologic* provided the perpendiculars at vertices of ABC onto sides EF, FD and DE of DEF are concurrent. The point of concurrence of these perpendiculars is denoted by $[ABC, DEF]$. It is

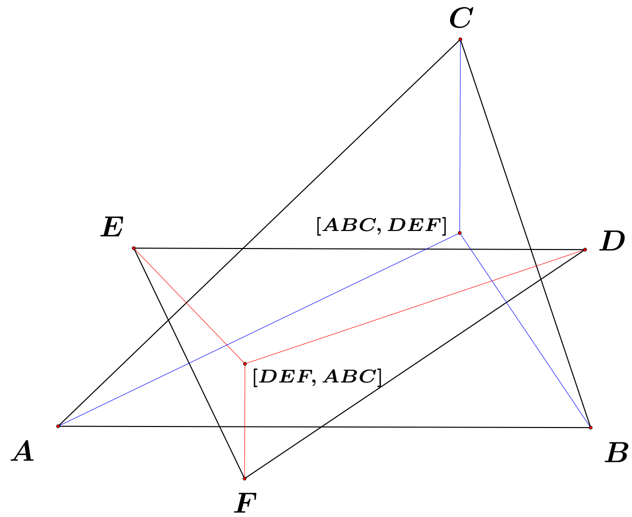


Figure 3. The triangles ABC and DEF are orthologic.

well-known that this relation is reflexive and symmetric. Hence, the perpendiculars from vertices of DEF onto the sides BC, CA, and AB are concurrent at the point $[DEF,\, ABC\,]$. These points are called the *first* and *second orthology centers* of the (orthologic) triangles ABC and DEF.

It is obvious that a triangle and the pedal triangle of any point are orthologic so that Theorem 1 extends Bottema's Theorem and the results in [8] (Theorem 3 and the first part of Theorem 5).

Proof. The proofs in this paper will all be analytic.

In the rectangular co-ordinate system in the plane, we shall assume throughout that $A(0,\,0)$, $B(1,\,0)$, $C(u,\,v)$, $D(d,\,\delta)$, $E(e,\,\varepsilon)$ and $F(f,\,\varphi)$ for real numbers $u,\ v,\ d,\ \delta,\ e,\ \varepsilon,\ f$ and φ. The lines will be treated as ordered triples of coefficients $(a,\,b,\,c)$ of their (linear) equations $a\,x + b\,y + c = 0$. Hence, the perpendiculars from the vertices of DEF onto the corresponding sidelines of ABC are $(u-1,\,v,\,d(1-u) - v\,\delta)$, $(u,\,v,\,-(u\,e + v\,\varepsilon))$ and $(1,\,0,\,-f)$. They will be concurrent provided the determinant $v\,\Delta = v((u-1)\,d - ue + f + v\,(\delta - \varepsilon))$ of the matrix from them as rows is equal to zero. In other words, $\Delta = 0$ is a necessary and sufficient condition for ABC and DEF to be orthologic.

On the other hand, the difference of the right and the left side of (1) is $2\,\Delta$ which clearly implies that (1) holds if and only if ABC and DEF are orthologic triangles. $\square$

3. The triangles $S_1S_3S_5$ and $S_2S_4S_6$

We continue our study of the ring of six squares with the Theorem 2 about two triangles associated with the configuration. Like Theorem 1, this theorem detects when two triangles are orthologic. Recall that $S_1, \ldots,\, S_6$ are the centers of the squares in Figure 2. Note that a similar result holds when the squares are folded inwards, and the proof is omitted.

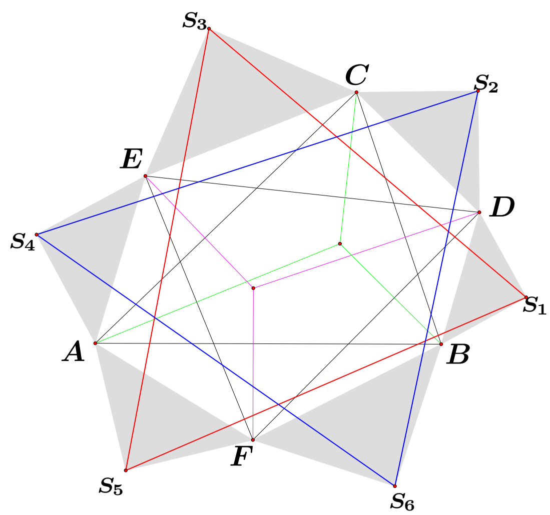


Figure 4. $|S_1S_3S_5| = |S_2S_4S_6|$ iff ABC and DEF are orthologic.

Theorem 2. *The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal area if and only if the triangles ABC and DEF are orthologic.*

Proof. The vertices V and U of the square $DEVU$ have co-ordinates $(e+\varepsilon-\delta,\, \varepsilon+d-e)$ and $(d+\varepsilon-\delta,\, \delta+d-e)$. From this we infer easily co-ordinates of all points in Figure 2. With the notation $u_+ = u+v$, $u_- = u-v$, $d_+ = d+\delta$, $d_- = d-\delta$, $e_+ = e+\varepsilon$, $e_- = e-\varepsilon$, $f_+ = f+\varphi$ and $f_- = f-\varphi$ they are the following.

$$A'(-\varepsilon,\, e), \qquad A''(\varphi,\, -f), \qquad B'(1-\varphi,\, f-1), \qquad B''(1+\delta,\, 1-d),$$
$$C'(u_+-\delta,\, u_-+d), \quad C''(u_-+\varepsilon,\, u_+-e), \quad D'(d_+,\, 1-d_-),$$
$$D''(d_-+v,\, d_+-u), \qquad E'(e_+-v,\, u-e_-), \qquad E''(e_-,\, e_+),$$
$$F'(f_+,\, -f_-), \quad F''(f_-,\, f_+-1), \quad S_1\left(\tfrac{1+d_+}{2},\, \tfrac{1-d_-}{2}\right), \quad S_2\left(\tfrac{d_-+u_+}{2},\, \tfrac{d_+-u_-}{2}\right),$$
$$S_3\left(\tfrac{u_-+e_+}{2},\, \tfrac{u_+-e_-}{2}\right), \quad S_4\left(\tfrac{e_-}{2},\, \tfrac{e_+}{2}\right), \quad S_5\left(\tfrac{f_+}{2},\, -\tfrac{f_-}{2}\right), \quad S_6\left(\tfrac{f_-+1}{2},\, \tfrac{f_+-1}{2}\right).$$

Let P^x and P^y be the x– and y– co-ordinates of the point P. Since the area $|DEF|$ is a half of the determinant of the matrix with the rows $(D^x,\, D^y,\, 1)$, $(E^x,\, E^y,\, 1)$ and $(F^x,\, F^y,\, 1)$, the difference $|S_2S_4S_6| - |S_1S_3S_5|$ is $\frac{\Delta}{4}$. We conclude that the triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal area if and only if the triangles ABC and DEF are orthologic. $\square$

4. The first family of pairs of triangles

The triangles $S_1S_3S_5$ and $S_2S_4S_6$ are just one pair from a whole family of triangle pairs which all have the same property with a single notable exception.

For any real number t different from -1 and 0, let $S_1^t, \ldots, S_6^t$ denote points that divide the segments AS_1, AS_2, BS_3, BS_4, CS_5 and CS_6 in the ratio $t:1$. Let $\rho(P, \theta)$ denote the rotation about the point P through an angle θ. Let G_σ and G_τ be the centroids of ABC and DEF.

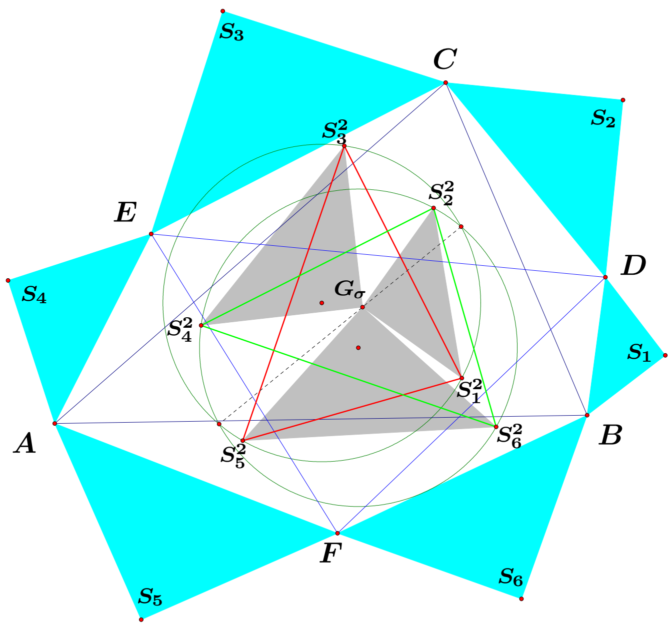


Figure 5. The triangles $S_1^2S_3^2S_5^2$ and $S_2^2S_4^2S_6^2$ are congruent.

The following result is curios (See Figure 5) because the particular value $t = 2$ gives a pair of congruent triangles regardless of the position of the triangles ABC and DEF.

Theorem 3. *The triangle $S_2^2S_4^2S_6^2$ is the image of the triangle $S_1^2S_3^2S_5^2$ under the rotation $\rho\left(G_\sigma, \frac{\pi}{2}\right)$. The radical axis of their circumcircles goes through the centroid G_σ.*

Proof. Since the point that divides the segment DE in the ratio $2:1$ has coordinates $\left(\frac{d+2\,e}{3}, \frac{\delta+2\,\varepsilon}{3}\right)$, it follows that

$$S_1^2\left(\frac{1+d_+}{3}, \frac{1-d_-}{3}\right) \quad \text{and} \quad S_2^2\left(\frac{d_-+u_+}{3}, \frac{d_+-u_-}{3}\right).$$

Since $G_\sigma\left(\frac{1+u}{3}, \frac{v}{3}\right)$, it is easy to check that S_2^2 is the vertex of a (negatively oriented) square on $G_\sigma S_1^2$. The arguments for the pairs $\left(S_3^2, S_4^2,\right)$ and $\left(S_5^2, S_6^2,\right)$ are analogous.

Finally, the proof of the claim about the radical axis starts with the observation that since the triangles $S_1S_3S_5$ and $S_2S_4S_6$ are congruent it suffices to show that $|G_\sigma O_{odd}|^2 = |G_\sigma O_{even}|^2$, where O_{odd} and O_{even} are their circumcenters. This routine task was accomplished with the assistance of a computer algebra system. $\square$

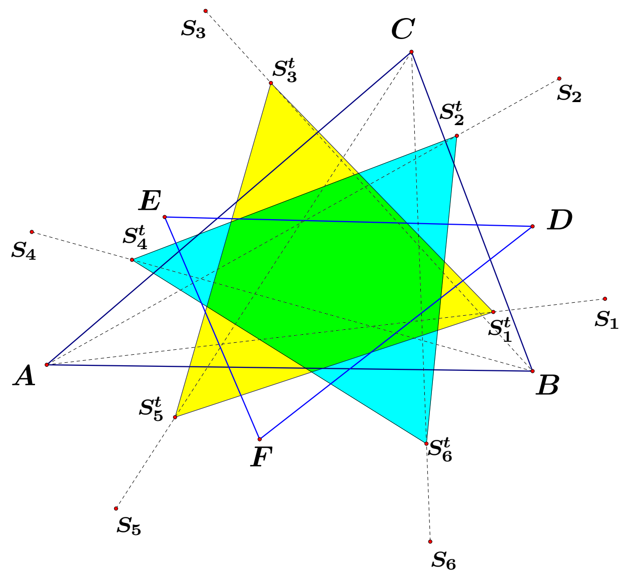


Figure 6. $|S_1^tS_3^tS_5^t| = |S_2^tS_4^tS_6^t|$ iff ABC and DEF are orthologic.

The following result resembles Theorem 2 (see Figure 6) and shows that each pair of triangles from the first family could be used to detect if the triangles ABC and DEF are orthologic.

Theorem 4. *For any real number t different from -1, 0 and 2, the triangles $S_1^tS_3^tS_5^t$ and $S_2^tS_4^tS_6^t$ have equal area if and only if the triangles ABC and DEF are orthologic.*

Proof. Since the point that divides the segment DE in the ratio $t:1$ has co-ordinates $\left(\frac{d+te}{t+1}, \frac{\delta+t\varepsilon}{t+1}\right)$, it follows that the points S_i^t have the co-ordinates

$$S_1^t\left(\tfrac{t(1+d_+)}{2(t+1)}, \tfrac{t(1-d_-)}{2(t+1)}\right),\ S_2^t\left(\tfrac{t(d_-+u_+)}{2(t+1)}, \tfrac{t(d_+-u_-)}{2(t+1)}\right),\ S_3^t\left(\tfrac{2+t(u_-+e_+)}{2(t+1)}, \tfrac{t(u_+-e_-)}{2(t+1)}\right),$$

$$S_4^t\left(\tfrac{2+t\,e_-}{2(t+1)}, \tfrac{t\,e_+}{2(t+1)}\right),\quad S_5^t\left(\tfrac{2\,u+t\,f_+}{2(t+1)}, \tfrac{2\,v-t\,f_-}{2(t+1)}\right),\quad S_6^t\left(\tfrac{2\,u+t(1+f_-)}{2(t+1)}, \tfrac{2\,v-t(1-f_+)}{2(t+1)}\right).$$

As in the proof of Theorem 2, we find that the difference of areas of the triangles $S_2^tS_4^tS_6^t$ and $S_1^tS_3^tS_5^t$ is $\frac{t(2-t)\Delta}{4(t+1)^2}$. Hence, for $t \neq -1,\ 0,\ 2$, the triangles $S_1^tS_3^tS_5^t$ and $S_2^tS_4^tS_6^t$ have equal area if and only if the triangles ABC and DEF are orthologic. $\square$

5. The second family of pairs of triangles

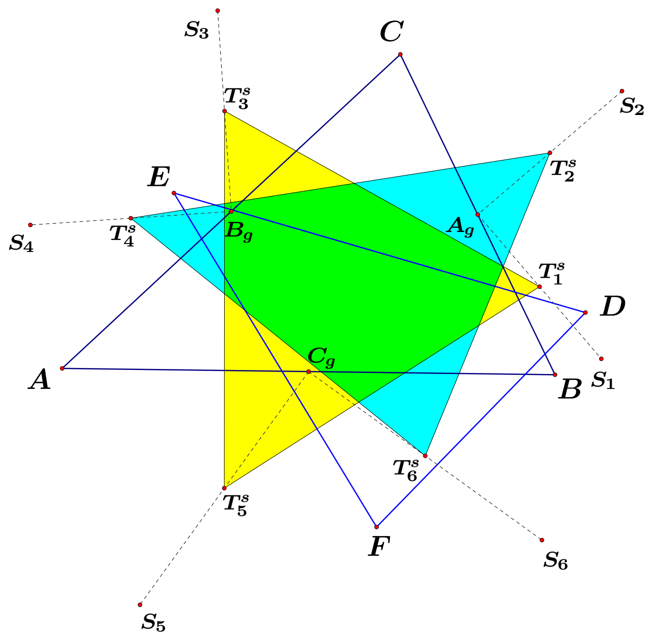


Figure 7. $|T_1^sT_3^sT_5^s| = |T_2^sT_4^sT_6^s|$ iff ABC and DEF are orthologic.

The first family of pairs of triangles was constructed on lines joining the centers of the squares with the vertices A, B and C. In order to get the second analogous family we shall use instead lines joining midpoints of sides with the centers of the squares (see Figure 7). A slight advantage of the second family is that it has no exceptional cases.

Let A_g, B_g and C_g denote the midpoints of the segments BC, CA and AB. For any real number s different from -1, let $T_1^s, \ldots, T_6^s$ denote points that divide the segments A_gS_1, A_gS_2, B_gS_3, B_gS_4, C_gS_5 and C_gS_6 in the ratio $s : 1$. Notice that $T_1^sT_2^sA_g$, $T_3^sT_4^sB_g$ and $T_5^sT_6^sC_g$ are isosceles triangles with the right angles at the vertices A_g, B_g and C_g.

Theorem 5. *For any real number s different from -1 and 0, the triangles $T_1^sT_3^sT_5^s$ and $T_2^sT_4^sT_6^s$ have equal area if and only if the triangles ABC and DEF are orthologic.*

Proof. As in the proof of Theorem 4, we find that the difference of areas of the triangles $T_1^sT_3^sT_5^s$ and $T_2^sT_4^sT_6^s$ is $\frac{s\,\Delta}{4(s+1)}$. Hence, for $s \neq -1,\ 0$, the triangles $T_1^tT_3^tT_5^t$ and $T_2^tT_4^tT_6^t$ have equal area if and only if the triangles ABC and DEF are orthologic. $\square$

6. The third family of pairs of triangles

When we look for reasons why the previous two families served our purpose of detecting orthology it is clear that the vertices of a triangle homothetic with ABC should be used. This leads us to consider a family of pairs of triangles that depend on two real parameters and a point (the center of homothety).

For any real numbers s and t different from -1 and any point P the points X, Y and Z divide the segments PA, PB and PC in the ratio $s:1$ while the points $U_i^{(s,t)}$ for $i = 1, \dots, 6$ divide the segments XS_1, XS_2, YS_3, YS_4, ZS_5 and ZS_6 in the ratio $t:1$.

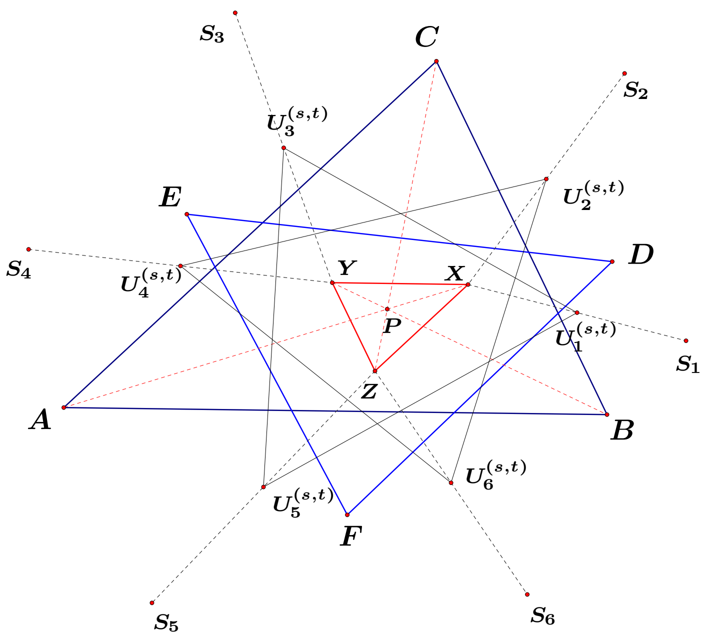


Figure 8. $|U_1^{(s,t)}U_3^{(s,t)}U_5^{(s,t)}| = |U_2^{(s,t)}U_4^{(s,t)}U_6^{(s,t)}|$ iff ABC and DEF are orthologic.

The above results (Theorems 4 and 5) are special cases of the following theorem (see Figure 8).

Theorem 6. *For any point P and any real numbers $s \neq -1$ and $t \neq -1, \frac{2\,s}{s+1}$, the triangles $U_1^{(s,t)}U_3^{(s,t)}U_5^{(s,t)}$ and $U_2^{(s,t)}U_4^{(s,t)}U_6^{(s,t)}$ have equal areas if and only if the triangles ABC and DEF are orthologic.*

The proof is routine. See that of Theorem 4.

7. The triangles $A_0B_0C_0$ and $D_0E_0F_0$

In this section we shall see that the midpoints of the sides of the hexagon $S_1S_2S_3S_4S_5S_6$ also have some interesting properties.

Let A_0, B_0, C_0, D_0, E_0 and F_0 be the midpoints of the segments S_1S_2, S_3S_4, S_5S_6, S_4S_5, S_6S_1 and S_2S_3. Notice that the triangles $A_0B_0C_0$ and $D_0E_0F_0$ have as centroid the midpoint of the segment $G_\sigma G_\tau$.

Recall that triangles ABC and XYZ are *homologic* provided the lines AX, BY, and CZ are concurrent. In stead of homologic many authors use *perspective*.

Theorem 7. (a) *The triangles ABC and $A_0B_0C_0$ are orthologic if and only if the triangles ABC and DEF are orthologic.*

(b) *The triangles DEF and $D_0E_0F_0$ are orthologic if and only if the triangles ABC and DEF are orthologic.*

(c) *If the triangles ABC and DEF are orthologic, then the triangles $A_0B_0C_0$ and $D_0E_0F_0$ are homologic.*

Proof. Let $D_1(d_1,\, \delta_1)$, $E_1(e_1,\, \varepsilon_1)$ and $F_1(f_1,\, \varphi_1)$. Recall from [2] that the triangles DEF and $D_1E_1F_1$ are orthologic if and only if $\Delta_0 = 0$, where

$$\Delta_0 = \Delta_0(DEF,\, D_1E_1F_1) = \begin{vmatrix} d & d_1 & 1 \\ e & e_1 & 1 \\ f & f_1 & 1 \end{vmatrix} + \begin{vmatrix} \delta & \delta_1 & 1 \\ \varepsilon & \varepsilon_1 & 1 \\ \varphi & \varphi_1 & 1 \end{vmatrix}.$$

Then (a) and (b) follow from the relations

$$\Delta_0(ABC,\, A_0B_0C_0) = -\frac{\Delta}{2} \quad \text{and} \quad \Delta_0(DEF,\, D_0E_0F_0) = \frac{\Delta}{2}.$$

The line DD_1 is $(\delta - \delta_1, d_1 - d, \delta_1\, d - d_1\, \delta)$, so that the triangles DEF and $D_1E_1F_1$ are homologic if and only if $\Gamma_0 = 0$, where

$$\Gamma_0 = \Gamma_0(DEF,\, D_1E_1F_1) = \begin{vmatrix} \delta - \delta_1 & d_1 - d & \delta_1\, d - d_1\, \delta \\ \varepsilon - \varepsilon_1 & e_1 - e & \varepsilon_1\, e - e_1\, \varepsilon \\ \varphi - \varphi_1 & f_1 - f & \varphi_1\, f - f_1\, \varphi \end{vmatrix}.$$

Part (c) follows from the observation that $\Gamma_0(A_0B_0C_0,\, D_0E_0F_0)$ contains Δ as a factor. □

8. Triangles from centroids

Let G_1, G_2, G_3 and G_4 denote the centroids of the triangles $G_{12A}G_{34B}G_{56C}$, $G_{12D}G_{34E}G_{56F}$, $G_{45A}G_{61B}G_{23C}$ and $G_{45D}G_{61E}G_{23F}$ where G_{12A}, G_{12D}, G_{34B}, G_{34E}, G_{56C}, G_{56F}, G_{45A}, G_{45D}, G_{61B}, G_{61E}, G_{23C} and G_{23F} are centroids of the triangles S_1S_2A, S_1S_2D, S_3S_4B, S_3S_4E, S_5S_6C, S_5S_6F, S_4S_5A, S_4S_5D, S_6S_1B, S_6S_1E, S_2S_3C and S_2S_3F.

Theorem 8. *The points G_1 and G_2 are the points G_3 and G_4 respectively. The points G_1 and G_2 divide the segments $G_\sigma G_\tau$ and $G_\tau G_\sigma$ in the ratio $1:2$.*

Proof. The centroids G_{12A}, G_{34B} and G_{56C} have the co-ordinates $\left(\frac{2\,d+1+v+u}{6},\, \frac{2\,\delta+1+v-u}{6}\right)$, $\left(\frac{2(e+1)+u-v}{6},\, \frac{2\,\varepsilon+u+v}{6}\right)$ and $\left(\frac{2(f+u)+1}{6},\, \frac{2(\varphi+v)-1}{6}\right)$. It follows that G_1 and G_2 have coordinates $\left(\frac{d+e+f+2(u+1)}{9},\, \frac{\delta+\varepsilon+\varphi+2\,v}{9}\right)$ and $\left(\frac{2(d+e+f)+u+1}{9},\, \frac{2(\delta+\varepsilon+\varphi)+v}{9}\right)$ respectively. It is now easy to check that $G_3 = G_1$ and $G_4 = G_2$.

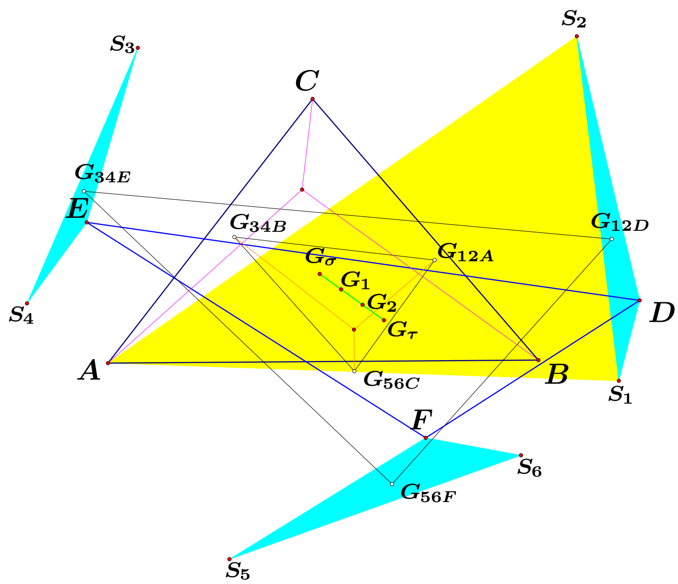


Figure 9. G_1 and G_2 divide $G_\sigma G_\tau$ in four equal parts and ABC is orthologic with $G_{12A}G_{34B}G_{56C}$ iff it is orthologic with DEF.

Let G_1' divide the segment $G_\sigma G_\tau$ in the ratio $1:2$. Since $G_\tau\left(\frac{d+e+f}{3}, \frac{\delta+\varepsilon+\varphi}{3}\right)$ and $G_\sigma\left(\frac{1+u}{3}, \frac{v}{3}\right)$, we have $(G_1')^x = \frac{2\,(G_\sigma)^x + (G_\tau)^x}{3} = \frac{(2+2\,u)+(d+e+f)}{9} = (G_1)^x$. Of course, in the same way we see that $(G_1')^y = (G_1)^y$ and that G_2 divides $G_\tau G_\sigma$ in the same ratio $1:2$. □

Theorem 9. *The following statements are equivalent:*

(a) *The triangles ABC and $G_{12A}G_{34B}G_{56C}$ are orthologic.*
(b) *The triangles ABC and $G_{12D}G_{34E}G_{56F}$ are orthologic.*
(c) *The triangles DEF and $G_{45A}G_{61B}G_{23C}$ are orthologic.*
(d) *The triangles DEF and $G_{45D}G_{61E}G_{23F}$ are orthologic.*
(e) *The triangles $G_{12A}G_{34B}G_{56C}$ and $G_{45A}G_{61B}G_{23C}$ are orthologic.*
(f) *The triangles $G_{12D}G_{34E}G_{56F}$ and $G_{45D}G_{61E}G_{23F}$ are orthologic.*
(g) *The triangles ABC and DEF are orthologic.*

Proof. The equivalence of (a) and (g) follows from the relation

$$\Delta_0(ABC,\, G_{12A}G_{34B}G_{56C}) = \frac{\Delta}{3}.$$

The equivalence of (g) with (b), (c), (d), (e) and (f) one can prove in the same way. □

9. Four triangles on vertices of squares

In this section we consider four triangles $A'B'C'$, $D'E'F'$, $A''B''C''$, $D''E''F''$ which have twelve outer vertices of the squares as vertices. The sum of areas of the first two is equal to the sum of areas of the last two. The same relation holds if we replace the word ”area” by the phrase ”sum of the squares of the sides”.

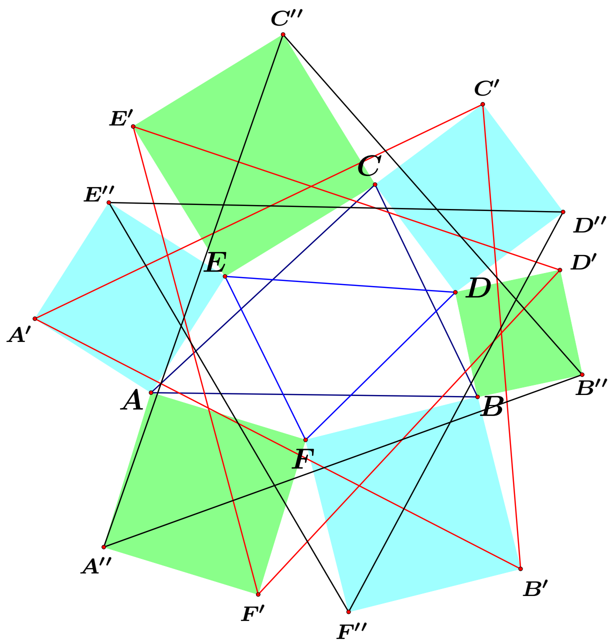


Figure 10. Four triangles $A'B'C'$, $D'E'F'$, $A''B''C''$ and $D''E''F''$.

For a triangle XYZ let $|XYZ|$ and $s_2(XYZ)$ denote its (oriented) area and the sum $|YZ|^2 + |ZX|^2 + |XY|^2$ of squares of lengths of its sides.

Theorem 10. (a) *The following equality for areas of triangles holds:*

$$|A'B'C'| + |D'E'F'| = |A''B''C''| + |D''E''F''|.$$

(b) *The following equality also holds:*

$$s_2(A'B'C') + s_2(D'E'F') = s_2(A''B''C'') + s_2(D''E''F'').$$

The proofs of both parts can be accomplished by a routine calculation.

Let A_1', B_1' and C_1' denote centers of squares of the same orientation built on the segments $B'C'$, $C'A'$ and $A'B'$. The points D_1', E_1', F_1', A_1'', B_1'', C_1'', D_1'', E_1'' and F_1'' are defined analogously. Notice that $(A'B'C', A_1'B_1'C_1')$, $(A''B''C'', A_1''B_1''C_1'')$, $(D'E'F', D_1'E_1'F_1')$ and $(D''E''F'', D_1''E_1''F_1'')$ are four pairs of both orthologic and homologic triangles.

The following theorem claims that the four triangles from these centers of squares retain the same property regarding sums of areas and sums of squares of lengths of sides.

Theorem 11. (a) *The following equality for areas of triangles holds:*

$$|A_1'B_1'C_1'| + |D_1'E_1'F_1'| = |A_1''B_1''C_1''| + |D_1''E_1''F_1''|.$$

(b) *The following equality also holds:*

$$s_2(A_1'B_1'C_1') + s_2(D_1'E_1'F_1') = s_2(A_1''B_1''C_1'') + s_2(D_1''E_1''F_1'').$$

The proofs of both parts can be accomplished by a routine calculation.

Notice that in the above theorem we can take instead of the centers any points that have the same position with respect to the squares erected on the sides of the triangles $A'B'C'$, $D'E'F'$, $A''B''C''$ and $D''E''F''$. Also, there are obvious extensions of the previous two theorems from two triangles to the statements about two n-gons for any integer $n > 3$.

Of course, it is possible to continue the above sequences of triangles and define for every integer $k \geq 0$ the triangles $A_k'B_k'C_k'$, $A_k''B_k''C_k''$, $D_k'E_k'F_k'$ and $D_k''E_k''F_k''$. The sequences start with $A'B'C'$, $A''B''C''$, $D'E'F'$ and $D''E''F''$. Each member is homologic, orthologic, and shares the centroid with all previous members and for each k an analogue of Theorem 11 is true.

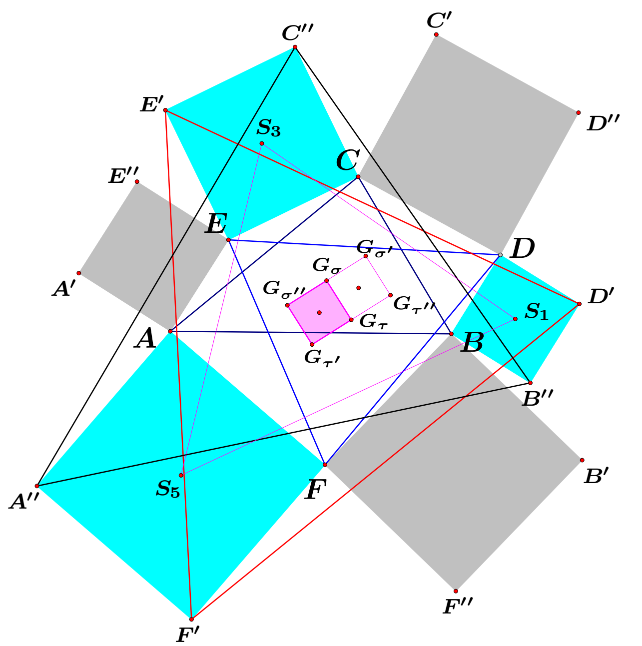


Figure 11. $G_\sigma G_\tau G_{\tau'} G_{\sigma''}$ and $G_\sigma G_\tau G_{\tau''} G_{\sigma'}$ are squares.

10. The centroids of the four triangles

Let $G_{\sigma'}$, $G_{\tau'}$, $G_{\sigma''}$, $G_{\tau''}$, G_o and G_e be shorter notation for the centroids $G_{A'B'C'}$, $G_{D'E'F'}$, $G_{A''B''C''}$, $G_{D''E''F''}$, $G_{S_1S_3S_5}$ and $G_{S_2S_4S_6}$. The following theorem shows that these centroids are the vertices of three squares associated with the ring of six squares.

Theorem 12. (a) *The centroids $G_{\sigma''}$, $G_{\tau'}$, G_τ and G_σ are vertices of a square.*

(b) *The centroids $G_{\sigma'}$ and $G_{\tau''}$ are reflections of the centroids $G_{\sigma''}$ and $G_{\tau'}$ in the line $G_\sigma G_\tau$. Hence, the centroids $G_{\tau''}$, $G_{\sigma'}$, G_σ and G_τ are also vertices of a square.*

(c) *The centroids G_e and G_o are the centers of the squares in* (a) *and (b), respectively. Hence, the centroids G_σ, G_e, G_τ and G_o are also vertices of a square.*

The proofs are routine.

11. Extension of Ehrmann–Lamoen results

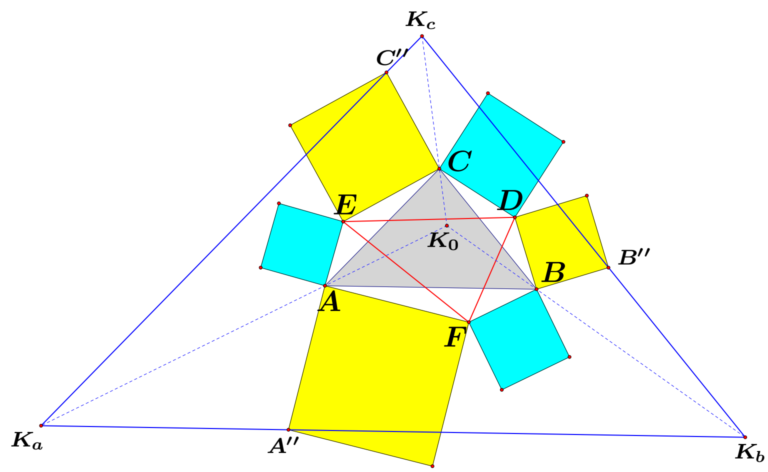


Figure 12. The triangle $K_aK_bK_c$ from parallels to BC, CA, AB through B'', C'', A'' is homothetic to ABC from the center K_0.

Let $K_aK_bK_c$ be a triangle from the intersections of parallels to the lines BC, CA and AB through the points B'', C'' and A''. Similarly, Let $K_aK_bK_c$ be a triangle from intersections of parallels to the lines BC, CA and AB through the points B'', C'' and A''. Similarly, the triangles $L_aL_bL_c$, $M_aM_bM_c$, $N_aN_bN_c$, $P_aP_bP_c$ and $Q_aQ_bQ_c$ are constructed in the same way through the triples of points (C', A', B'), (D'', E'', F''), (D', E', F'), (S_1, S_3, S_5) and (S_2, S_4, S_6), respectively. Some of these triangles have been considered in the case when the triangle DEF is the pedal triangle $P_aP_bP_c$ of the point P. Work has been done by Ehrmann and Lamoen in [4] and also by Hoffmann and Sashalmi in [8]. In this section we shall see that natural analogues of their results hold in more general situations.

Theorem 13. (a) *The triangles $K_aK_bK_c$, $L_aL_bL_c$, $M_aM_bM_c$, $N_aN_bN_c$, $P_aP_bP_c$ and $Q_aQ_bQ_c$ are each homothetic with the triangle ABC.*

(b) *The quadrangles $K_aL_aM_aN_a$, $K_bL_bM_bN_b$ and $K_cL_cM_cN_c$ are parallelograms.*

(c) *The centers J_a, J_b and J_c of these parallelograms are the vertices of a triangle that is also homothetic with the triangle ABC.*

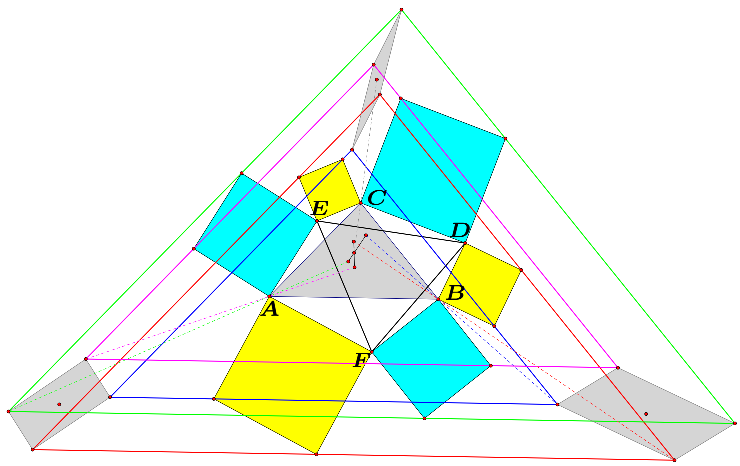


Figure 13. The triangles $K_aK_bK_c$, $L_aL_bL_c$, $M_aM_bM_c$ and $N_aN_bN_c$ together with three parallelograms.

Proof of parts (a) and (c) are routine while the simplest method to prove the part (b) is to show that the midpoints of the segments K_xM_x and L_xN_x coincide for $x = a,\, b,\, c$.

Let J_0, K_0, L_0, M_0, N_0, P_0 and Q_0 be centers of the above homotheties. Notice that J_0 is the intersection of the lines K_0M_0 and L_0N_0.

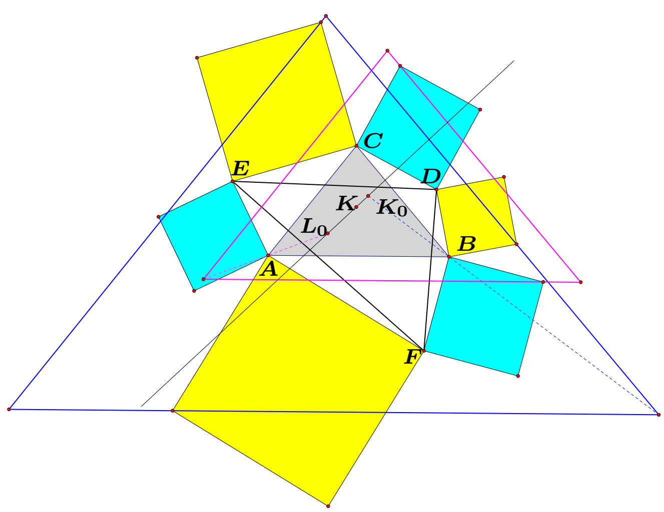


Figure 14. The line K_0L_0 goes through the symmedian point K of the triangle ABC.

Theorem 14. (a) *The symmedian point K of the triangle ABC lies on the line K_0L_0.*

(b) *The points P_0 and Q_0 coincide with the points N_0 and M_0.*

(c) *The equalities* $2 \cdot \overrightarrow{P_vQ_v} = \overrightarrow{K_vL_v}$ *hold for* $v = a,\, b,\, c$.

Proof. (a) It is straightforward to verify that the symmedian point with co-ordinates $\left(\frac{u^2+u+v^2}{2(u^2-u+v^2+1)}, \frac{v}{2(u^2-u+v^2+1)}\right)$ lies on the line K_0L_0.

(b) That the center P_0 coincides with the center N_0 follows easily from the fact that $(A,\, N_a,\, P_a)$ and $(B,\, N_b,\, P_b)$ are triples of collinear points.

(c) Since $Q_a^y = \frac{f+\varphi-1}{2}$, $P_a^y = \frac{\varphi-f}{2}$, $L_a^y = f-1$ and $K_a^y = -f$, we see that

$$2 \cdot (Q_a^y - P_a^y) = L_a^y - K_a^y.$$

Similarly, $2 \cdot (Q_a^x - P_a^x) = L_a^x - K_a^x$. This proves the equality $2 \cdot \overrightarrow{P_aQ_a} = \overrightarrow{K_aL_a}$. □

Theorem 15. *The triangles* $K_aK_bK_c$ *and* $L_aL_bL_c$ *are congruent if and only if the triangles* ABC *and* DEF *are orthologic.*

Proof. Since the triangles $K_aK_bK_c$ and $L_aL_bL_c$ are both homothetic to the triangle ABC, we conclude that they will be congruent if and only if $|K_aK_b| = |L_aL_b|$. Hence, the theorem follows from the equality

$$|K_aK_b|^2 - |L_aL_b|^2 = \frac{\left[(2u-1)^2 + (2v+1)^2 + 2\right]\,\Delta}{v^2}.$$

□

Let O and ω denote the circumcenter and the Brocard angle of the triangle ABC.

Theorem 16. *If the triangles* ABC *and* DEF *are orthologic then the following statements are true.*

(a) *The symmedian point* K *of the triangle* ABC *is the midpoint of the segment* K_0L_0.

(b) *The triangles* $M_aM_bM_c$ *and* $N_aN_bN_c$ *are congruent.*

(c) *The triangles* $P_aP_bP_c$ *and* $Q_aQ_bQ_c$ *are congruent.*

(d) *The common ratio of the homotheties of the triangles* $K_aK_bK_c$ *and* $L_aL_bL_c$ *with the triangle* ABC *is* $(1 + \cot\,\omega) : 1$.

(e) *The translations* $K_aK_bK_c \mapsto L_aL_bL_c$ *and* $N_aN_bN_c \mapsto M_aM_bM_c$ *are for the image of the vector* $2 \cdot \overrightarrow{O[DEF,\, ABC]}$ *under the rotation* $\rho\left(O,\, \frac{\pi}{2}\right)$.

(f) *The vector of the translation* $P_aP_bP_c \mapsto Q_aQ_bQ_c$ *is the image of the vector* $\overrightarrow{O[DEF,\, ABC]}$ *under the rotation* $\rho\left(O,\, \frac{\pi}{2}\right)$.

Proof. (a) Let $\xi = u^2 - u + v^2$. Let the triangles ABC and DEF be such that the centers K_0 and L_0 are well-defined. In other words, let $M,\, N \neq 0$, where $M,\, N = (u-1)d + v\delta - ue - v\varepsilon + f \pm (\xi + 1)$. Let Z_0 be the midpoint of the segment K_0L_0. Then $|Z_0K|^2 = \frac{\Delta^2\, P}{4(\xi+1)^2\, M^2\, N^2}$, where

$$P = \frac{Q\, S^2}{(\xi+u)^2(\xi+3u+1)^2} + \frac{4v^2(\xi+1)^2\, T^2}{(\xi+u)(\xi+3u+1)},$$

$$S = (ue + v\varepsilon)(\xi^2 + \xi - 3u(u-1)) + (\xi + u)$$
$$[(\xi+3u+1)((u-1)d + v\delta) + ((1-2\xi)u - \xi - 1)f - (\xi+1)(\xi+u-1)],$$

$Q = \xi^2 + (4u+1)\xi + u(3u+1)$ and $T = ue + v\varepsilon + (\xi + u)(f-1)$. Hence, if the triangles ABC and DEF are orthologic (i. e., $\Delta = 0$), then $K = Z_0$. The converse is not true because the factors S and T can be simultaneously equal to zero. For example, this happens for the points $A(0,0)$, $B(1,0)$, $C\left(\frac{1}{3},\, 1\right)$, $D(2,5)$, $E\left(4,\, -\frac{32}{9}\right)$ and $F(3,\, -1)$. An interesting problem is to give geometric description for the conditions $S = 0$ and $T = 0$.

(b) This follows from the equality

$$|N_aN_b|^2 - |M_aM_b|^2 = \frac{4(vd + (1-u)\delta - ve + u\varepsilon - \varphi + \xi + 1)\,\Delta}{v^2}.$$

(c) This follows similarly from the equality

$$|P_aP_b|^2 - |Q_aQ_b|^2 = \frac{(vd + (1-u)\delta - ve + u\varepsilon - \varphi + \xi + v + 1)\,\Delta}{v^2}.$$

(d) The ratio $\frac{|K_aK_b|}{|AB|}$ is $\frac{|\Delta + u^2 - u + v^2 + v + 1|}{v}$. Hence, when the triangles ABC and DEF are orthologic, then $\Delta = 0$ and this ratio is

$$\frac{u^2 - u + v^2 + v + 1}{v} = 1 + \frac{|BC|^2 + |CA|^2 + |AB|^2}{4 \cdot |ABC|} = 1 + \cot\omega.$$

(e) The tip of the vector $\overrightarrow{K_aL_a}$ (translated to the origin) is at the point

$$V(\xi - 2(ue + v\varepsilon - uf),\, 2f - 1).$$

The intersection of the perpendiculars through the points D and E onto the side-lines BC and CA is the point

$$U\left((1-u)d - v\delta + ue + v\varepsilon,\, \frac{uv\delta + (u-1)(du - ue - v\varepsilon)}{v}\right).$$

When the triangles ABC and DEF are orthologic this point will be the second orthology center $[DEF,\, ABC]$. Since the circumcenter O has the co-ordinates $\left(\frac{1}{2},\, \frac{\xi}{2v}\right)$, the tip of the vector $2 \cdot \overrightarrow{OU}$ is at the point

$$W^*\left(2((1-u)d - v\delta + ue + v\varepsilon) - 1,\, \frac{2((u-1)(ud - ue - v\varepsilon) + uv\delta) - \xi}{v}\right).$$

Its rotation about the circumcenter by $\frac{\pi}{2}$ has the tip at $W(-(W^*)^y,\, (W^*)^x)$. The relations $U^x - W^x = \frac{2u\Delta}{v}$ and $U^y - W^y = 2\Delta$ now confirm that the claim (e) holds.

(f) The proof for this part is similar to the proof of the part (e). □

12. New results for the pedal triangle

Let a, b, c and S denote the lengths of sides and the area of the triangle ABC. In this section we shall assume that DEF is the pedal triangle of the point P with respect to ABC. Our goal is to present several new properties of Bottema's original configuration. It is particularly useful for the characterizations of the Brocard axis.

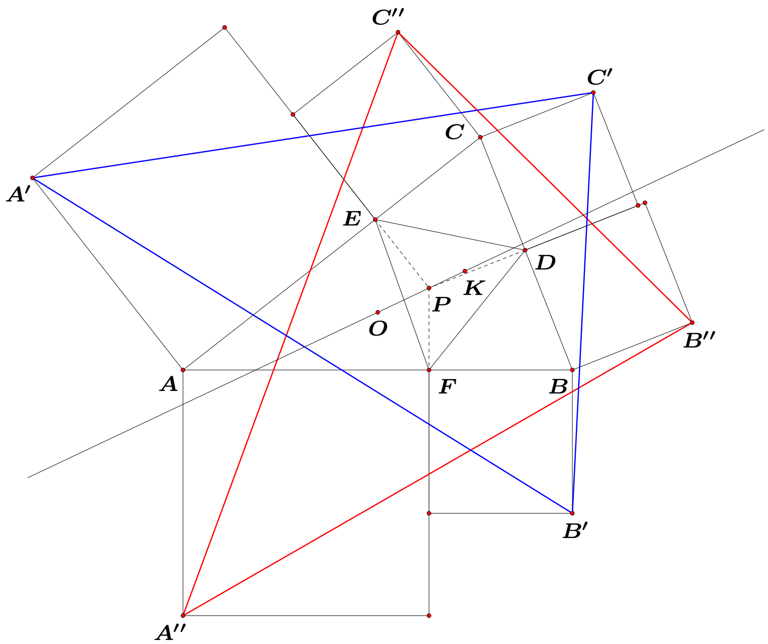


Figure 15. $s_2(A'B'C') = s_2(A''B''C'')$ iff P is on the Brocard axis.

Theorem 17. *There is a unique central point P with the property that the triangles $S_1S_3S_5$ and $S_2S_4S_6$ are congruent. The first trilinear co-ordinate of this point P is $a((b^2+c^2+2\,S)a^2 - b^4 - c^4 - 2\,S(b^2+c^2))$. It lies on the Brocard axis and divides the segment OK in the ratio $(-\cot\omega) : (1+\cot\omega)$ and is also the image of K under the homothety $h(O, -\cot\omega)$.*

Proof. Let $P(p,\, q)$. The orthogonal projections P_a, P_b and P_c of the point P onto the sidelines BC, CA and AB have the co-ordinates

$$\left(\frac{(u-1)^2p + v(u-1)q + v^2}{\xi - u + 1},\, \frac{v((u-1)p + vq - u + 1)}{\xi - u + 1}\right),$$

$\left(\frac{u(up+vq)}{\xi+u},\, \frac{v(up+vq)}{\xi+u}\right)$ and $(p,\, 0)$.

Since the triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal area, it is easy to prove using the Heron formula that they will be congruent if and only if two of their corresponding sides have equal length. In other words, we must find the solution of the equations

$$|S_3S_5|^2 - |S_4S_6|^2 = \frac{v\xi p}{\xi+u} - \frac{v^2 q}{\xi+u} + \frac{\xi+u-1}{2} = 0,$$

$$|S_5S_1|^2 - |S_6S_2|^2 = \frac{v(\xi\, p + v\, q)}{\xi - u + 1} - \frac{\xi^2 - (2(u-v)-1)\,\xi + u\,(u-1)}{2(\xi - u + 1)} = 0.$$

As this is a linear system it is clear that there is only one solution. The required point is $P\left(\frac{1-2\,u+v}{2v},\ \frac{\xi^2+(v+1)\xi-v^2}{2v^2}\right)$. Let $s = -\frac{1}{1+\frac{v}{\xi+1}} = \frac{-\cot\omega}{1+\cot\omega}$. The point P divides the segment OK in the ratio $s:1$, where $O\left(\frac{1}{2},\ \frac{\xi}{2v}\right)$ and $K\left(\frac{\xi+2u}{2(\xi+1)},\ \frac{v}{2(\xi+1)}\right)$. $\square$

Theorem 18. *The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have the same centroid if and only if the point P is the circumcenter of the triangle ABC.*

Proof. We get $|G_oG_e|^2 = \frac{M^2+N^2}{9(\xi-u+1)(\xi+u)(1+4\xi)}$, with

$$M = 3\xi(2u-1)p + v(1+4\xi)q - \xi(2\xi+3u-1)$$

and $N = v(1+\xi)(2p-1)$. Hence, $G_o = G_e$ if and only if $N = 0$ and $M = 0$. In other words, the centroids of the triangles $S_1S_3S_5$ and $S_2S_4S_6$ coincide if and only if $p = \frac{1}{2}$ and $q = \frac{\xi}{2v}$ (i. e., if and only if the point P is the circumcenter O of the triangle ABC). $\square$

Recall that the Brocard axis of the triangle ABC is the line joining its circumcenter with the symmedian point.

Let s be a real number different from 0 and -1. Let the points A_s, B_s and C_s divide the segments BD, CE and AF in the ratio $s:1$ and let the points D_s, E_s and F_s divide the segments DC, EA and FB in the ratio $1:s$.

Theorem 19. *For the pedal triangle DEF of a point P with respect to the triangle ABC the following statements are equivalent:*

(a) *The triangles $A_0B_0C_0$ and $D_0E_0F_0$ are orthologic.*

(b) *The triangles ABC and $G_{45A}G_{61B}G_{23C}$ are orthologic.*

(c) *The triangles ABC and $G_{45D}G_{61E}G_{23F}$ are orthologic.*

(d) *The triangles $G_{12A}G_{34B}G_{56C}$ and $G_{45D}G_{61E}G_{23F}$ are orthologic.*

(e) *The triangles $G_{12D}G_{34E}G_{56F}$ and $G_{45A}G_{61B}G_{23C}$ are orthologic.*

(f) *The triangles $A'B'C'$ and $A''B''C''$ have the same area.*

(g) *The triangles $A'B'C'$ and $A''B''C''$ have the same sums of squares of lengths of sides.*

(h) *The triangles $D'E'F'$ and $D''E''F''$ have the same area.*

(i) *The triangles $D'E'F'$ and $D''E''F''$ have the same sums of squares of lengths of sides.*

(j) *The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal sums of squares of lengths of sides.*

(k) *For any real number $t \neq -1,\ 0,\ 2$, the triangles $S_1^tS_3^tS_5^t$ and $S_2^tS_4^tS_6^t$ have equal sums of squares of lengths of sides.*

(l) *For any real number $s \neq -1,\ 0$, the triangles $T_1^sT_3^sT_5^s$ and $T_2^sT_4^sT_6^s$ have equal sums of squares of lengths of sides.*

(m) *The triangles $A_sB_sC_s$ and $D_sE_sF_s$ have the same area.*

(n) *The point P lies on the Brocard axis of the triangle ABC.*

Proof. (a) The orthology criterion $\Delta_0(A_0B_0C_0,\ D_0E_0F_0)$ is equal to the quotient $\frac{-v\,M}{8(\xi+u)(\xi-u+1)}$, with M the following linear polynomial in p and q.

$$M = 2\,\left(\xi^2+\xi-v^2\right)p + 2\,v\,(2\,u-1)\,q - (\xi+u)\,(\xi+u-1)\,.$$

In fact, $M=0$ is the equation of the Brocard axis because the co-ordinates $\left(\frac{1}{2},\ \frac{\xi}{2v}\right)$ and $\left(\frac{\xi+2u}{2(\xi+1)},\ \frac{v}{2(\xi+1)}\right)$ of the circumcenter O and the symmedian point K satisfy this equation. Hence, the statements (a) and (n) are equivalent.

(f) It follows from the equality $|A''B''C''| - |A'B'C'| = \frac{v\,M}{2(\xi+u)(\xi-u+1)}$ that the statements (f) and (n) are equivalent.

(i) It follows from the equality $s_2(D'E'F') - s_2(D''E''F'') = \frac{v\,M}{2(\xi+u)(\xi-u+1)}$ that the statements (i) and (n) are equivalent. □

It is well-known that $\cot\omega = \frac{a^2+b^2+c^2}{4S}$ so that we shall assume that the degenerate triangles do not have well-defined Brocard angle. It follows that the statement ”The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal Brocard angles” could be added to the list of the previous theorem provided we exclude the points for which the triangles $S_1S_3S_5$ and $S_2S_4S_6$ are degenerate. The following result explains when this happens. Let $K_{-\omega}$ denote the point described in Theorem 17.

Theorem 20. *The following statements are equivalent:*

(a) *The points S_1, S_3 and S_5 are collinear.*

(b) *The points S_2, S_4 and S_6 are collinear.*

(c) *The point P is on the circle with the center $K_{-\omega}$ and the radius equal to the circumradius R of the triangle ABC times the number $\sqrt{(1+\cot\omega)^2+1}$.*

Proof. Let M be the following quadratic polynomial in p and q:

$$v^2(p^2+q^2) + v\,(2\,u-w)\,p - \left(\xi^2+w\,\xi-v^2\right)q - (\xi+u)\,(\xi-u+w)\,,$$

where $w=v+1$. The points S_1, S_3 and S_5 are collinear if and only if

$$0 = \begin{vmatrix} S_1^x & S_1^y & 1 \\ S_3^x & S_3^y & 1 \\ S_5^x & S_5^y & 1 \end{vmatrix} = \frac{v\,M}{2(u-1-\xi)(u+\xi)}.$$

The equivalence of (a) and (c) follows from the fact that $M=0$ is the equation of the circle described in (c). Indeed, we see directly that the co-ordinates of its center are $\left(\frac{w-2u}{v},\ \frac{\xi^2+w\,\xi-v^2}{2v^2}\right)$ so that this center is the point $K_{-\omega}$ while the square of its radius is $\frac{(\xi-u+1)(\xi+u)\left((\xi+w)^2+v^2\right)}{4v^4} = \frac{(\xi-u+1)(\xi+u)}{4v^2}\cdot\frac{(\xi+w)^2+v^2}{v^2} = R^2\cdot\beta^2$, where β is equal to the number $\sqrt{(1+\cot\omega)^2+1}$ because $\cot\omega = \frac{\xi+1}{v}$.

The equivalence of (b) and (c) is proved in the same way. □

Theorem 21. *The triangles $A_0B_0C_0$ and $D_0E_0F_0$ always have different sums of squares of lengths of sides.*

Proof. The difference $s_2(A_0B_0C_0) - s_2(D_0E_0F_0)$ is equal to $\frac{3v^3\,N}{4(\xi-u+1)(u+\xi)}$, where N denotes the following quadratic polynomial in variables p and q:

$$\left(p-\frac{1}{2}\right)^2+\left(q-\frac{\xi}{2v}\right)^2+\frac{3(\xi-u+1)(\xi+u)}{4v^2}.$$

However, this polynomial has no real roots. □

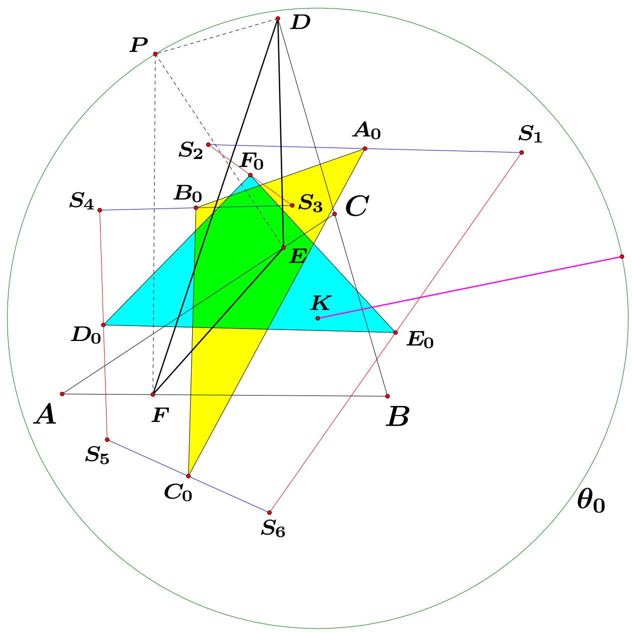


Figure 16. $|A_0B_0C_0| = |D_0E_0F_0|$ iff P is on the circle θ_0.

Theorem 22. *The triangles $A_0B_0C_0$ and $D_0E_0F_0$ have the same areas if and only if the point P lies on the circle θ_0 with the center at the symmedian point K of the triangle ABC and the radius $R\sqrt{4-3\tan^2\omega}$, where R and ω have their usual meanings associated with triangle ABC.*

Proof. The difference $|D_0E_0F_0| - |A_0B_0C_0|$ is equal to the quotient $\frac{v^2\,\zeta^2\,M}{16\,\mu\,(\zeta-u)}$, where $\zeta = \xi+1$, $\mu = \xi+u$ and M denotes the following quadratic polynomial in variables p and q:

$$\left(p-\frac{\mu+u}{2\,\zeta}\right)^2+\left(q-\frac{v}{2\,\zeta}\right)^2-\frac{\mu\,(\zeta-u)\,\left(4\,\zeta^2-3\,v^2\right)}{4\,\zeta^2\,v^2}.$$

The third term is clearly equal to $-R^2(4-3\tan^2\omega)$. Hence, $M=0$ is the equation of the circle whose center is the symmedian point of the triangle ABC with the co-ordinates $\left(\frac{\mu+u}{2\,\zeta},\,\frac{v}{2\,\zeta}\right)$ and the radius $R\sqrt{4-3\tan^2\omega}$. □

Let A^*, B^*, C^*, D^*, E^* and F^* denote the midpoints of the segments $A'A''$, $B'B''$, $C'C''$, $D'D''$, $E'E''$ and $F'F''$. Notice that the points A^*, B^*, C^*, D^*, E^* and F^* are the centers of squares built on the segments S_4S_5, S_6S_1, S_2S_3, S_1S_2, S_3S_4 and S_5S_6, respectively. Also, the triangles $A^*B^*C^*$ and $D^*E^*F^*$ share the centroids with the triangles ABC and DEF.

Notice that the lines AA^*, BB^* and CC^* intersect in the isogonal conjugate of the point P with respect to the triangle ABC.

Theorem 23. *The triangles $A^*B^*C^*$ and $D^*E^*F^*$ have the same sums of squares of lengths of sides if and only if the point P lies on the circle θ_0.*

Proof. The proof is almost identical to the proof of the previous theorem since the difference $s_2(D^*E^*F^*) - s_2(A^*B^*C^*)$ is equal to $\frac{v^2(\xi+1)^2\,M}{2(\xi-u+1)(\xi+u)}$. □

Theorem 24. *For any point P the triangles $A^*B^*C^*$ and $D^*E^*F^*$ always have different areas.*

Proof. The proof is similar to the proof of Theorem 21 since the difference $|D^*E^*F^*| - |A^*B^*C^*|$ is equal to $\frac{v^3\,N}{8(\xi-u+1)(\xi+u)}$. □

13. New results for the antipedal triangle

Recall that the antipedal triangle $P_a^*P_b^*P_c^*$ of a point P not on the side lines of the triangle ABC has as vertices the intersections of the perpendiculars erected at A, B and C to PA, PB and PC respectively. Note that the triangle $P_a^*P_b^*P_c^*$ is orthologic with the triangle ABC so that Bottema's Theorem also holds for antipedal triangles.

Our final result is an analogue of Theorem 19 for the antipedal triangle of a point. It gives a nice connection of a Bottema configuration with the Kiepert hyperbola (i. e., the rectangular hyperbola which passes through the vertices, the centroid and the orthocenter [3]).

In the next theorem we shall assume that DEF is the antipedal triangle of the point P with respect to ABC. Of course, the point P must not be on the side lines BC, CA and AB.

Theorem 25. *The following statements are equivalent:*

(a) *The triangles $A_0B_0C_0$ and $D_0E_0F_0$ are orthologic.*

(b) *The triangles ABC and $G_{45A}G_{61B}G_{23C}$ are orthologic.*

(c) *The triangles ABC and $G_{45D}G_{61E}G_{23F}$ are orthologic.*

(d) *The triangles $G_{12A}G_{34B}G_{56C}$ and $G_{45D}G_{61E}G_{23F}$ are orthologic.*

(e) *The triangles $G_{12D}G_{34E}G_{56F}$ and $G_{45A}G_{61B}G_{23C}$ are orthologic.*

(f) *The triangles $A'B'C'$ and $A''B''C''$ have the same area.*

(g) *The triangles $A'B'C'$ and $A''B''C''$ have the same sums of squares of lengths of sides.*

(h) *The triangles $D'E'F'$ and $D''E''F''$ have the same area.*

(i) *The triangles $D'E'F'$ and $D''E''F''$ have the same sums of squares of lengths of sides.*

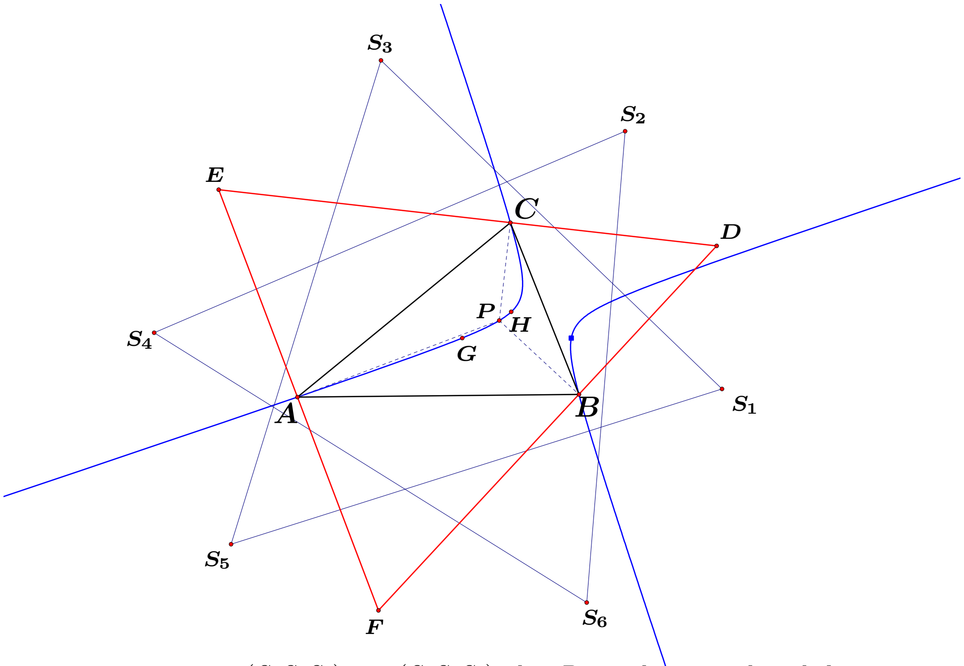


Figure 17. $s_2(S_1S_3S_5) = s_2(S_2S_4S_6)$ when P is on the Kiepert hyperbola.

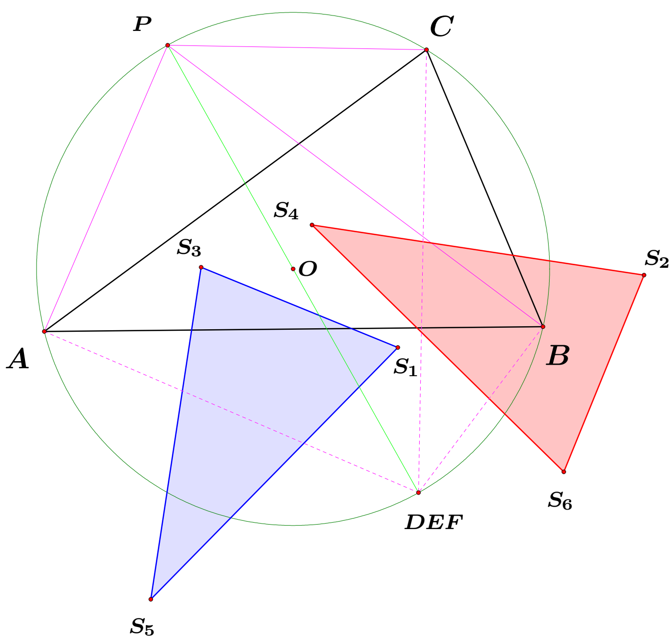


Figure 18. $s_2(S_1S_3S_5) = s_2(S_2S_4S_6)$ also when P is on the circumcircle.

(j) *The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal sums of squares of lengths of sides.*

(k) *For any real number $t \neq -1,\, 0,\, 2$, the triangles $S_1^t S_3^t S_5^t$ and $S_2^t S_4^t S_6^t$ have equal sums of squares of lengths of sides.*

(l) *For any real number $s \neq -1,\, 0$, the triangles $T_1^s T_3^s T_5^s$ and $T_2^s T_4^s T_6^s$ have equal sums of squares of lengths of sides.*

(m) *The triangles $A_s B_s C_s$ and $D_s E_s F_s$ have the same area.*

(n) *The point P lies either on the Kiepert hyperbola of the triangle ABC or on its circumcircle.*

Proof. (g) $s_2(A''B''C'') - s_2(A'B'C') = \frac{2v\,M\,N}{q(vp-uq)(v(p-1)-(u-1)q)}$, with

$$M = \left(p - \frac{1}{2}\right)^2 + \left(q - \frac{\xi}{2v}\right)^2 - \frac{\xi^2 + v^2}{4v^2},$$

$$N = v\,(2u-1)\left(p^2 - q^2 - p\right) - 2\,\left(u^2 - u - v^2 + 1\right)pq + \left(u^2 + u - v^2\right)q.$$

In fact, $M = 0$ is the equation of the circumcircle of the triangle ABC while $N = 0$ is the equation of its Kiepert hyperbola because the co-ordinates of the vertices A, B and C and the co-ordinates $\left(u,\, \frac{u(1-u)}{v}\right)$ and $\left(\frac{u+1}{3},\, \frac{v}{3}\right)$ of the orthocenter H and the centroid G satisfy this equation. Hence, the statements (g) and (n) are equivalent.

(j) It follows from the equality

$$s_2(S_2S_4S_6) - s_2(S_1S_3S_5) = \frac{v\,M\,N}{q(vp-uq)(v(p-1)-(u-1)q)}$$

that the statements (j) and (n) are equivalent.

(m) It follows from the equality

$$|D_sE_sF_s| - |A_sB_sC_s| = \frac{s\,v\,M\,N}{2(s+1)^2\,q(vp-uq)(v(p-1)-(u-1)q)}$$

that the statements (m) and (n) are equivalent. □

Of course, as in the case of the pedal triangles, we can add the statement ”The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal Brocard angles.” to the list in Theorem 25 but the points on the circle described in Theorem 20 must be excluded from consideration.

Notice that when the point P is on the circumcircle of ABC then much more could be said about the properties of the six squares built on segments BD, DC, CE, EA, AF and FB. A considerable simplification arises from the fact that the antipedal triangle DEF reduces to the antipodal point Q of the point P. For example, the triangles $S_1S_3S_5$ and $S_2S_4S_6$ are the images under the rotations $\rho(U,\, \frac{\pi}{4})$ and $\rho(V,\, -\frac{\pi}{4})$ of the triangle $A_\diamond B_\diamond C_\diamond = h(O,\, \frac{\sqrt{2}}{2})(ABC)$ (the image of ABC under the homothety with the circumcenter O as the center and the factor $\frac{\sqrt{2}}{2}$). The points U and V are constructed as follows.

Let the circumcircle $\sigma_\diamond$ of the triangle $A_\diamond B_\diamond C_\diamond$ intersect the segment OQ in the point R, let ℓ be the perpendicular bisector of the segment QR and let T be the midpoint of the segment OQ. Then the point U is the intersection of the line ℓ with $\rho(T, \frac{\pi}{4})(PQ)$ while the point V is the intersection of the line ℓ with $\rho(T, -\frac{\pi}{4})(PQ)$.

References

[1] O. Bottema, *De Elementaire Meetkunde van het Platte Vlak*, P. Noordhoff, Groningen- Batavia, 1938.

[2] Z. Čerin, Hyperbolas, orthology, and antipedal triangles, *Glasnik Mat.*, 33 (1998) 143 – 160.

[3] R. H. Eddy and R. Fritsch, The conics of Ludwig Kiepert: a comprehensive lesson in the geometry of the triangle, *Math. Mag.*, 67 (1994) 188 – 205.

[4] J.-P. Ehrmann and F. M. van Lamoen, Some Similarities Associated with Pedals, *Forum Geom.*, 2 (2002) 163 – 166.

[5] R. Deaux, *Compléments de Géométrie Plane*, A. de Boeck, Brussels, 1945.

[6] C. Kimberling, Central points and central lines in the plane of a triangle, *Math. Mag.*, 67 (1994) 163 – 187.

[7] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[8] É. Sashalmi and M. Hoffmann, Generalizations of Bottema's theorem on pedal points, *Acta Acad. Paed. Agriensis, Sectio Mathematicae*, 31 (2004) 25 – 31.

Zvonko Čerin:Kopernikova 7, 10010 Zagreb, CROATIA
E-mail address: `cerin@math.hr`

Forum Geometricorum
Volume 9 (2009) 81–98.

FORUM GEOM
ISSN 1534-1178

On the Newton Line of a Quadrilateral

Paris Pamfilos

Abstract. We introduce the idea of the conjugate polygon of a point relative to another polygon and examine the closing property of polygons inscribed in others and having sides parallel to a conjugate polygon. Specializing for quadrangles we prove a characterization of their Newton line related to the possibility to inscribe a quadrangle having its sides parallel to the sides of a conjugate one.

1. Introduction

Given two quadrangles $a = A_1A_2A_3A_4$ and $b = B_1B_2B_3B_4$ one can ask whether it is possible to inscribe in the first a quadrangle $c = C_1C_2C_3C_4$ having its sides parallel to corresponding sides of the second. It is also of importance to know how many solutions to the problem exist and which is their structure. The corresponding problem for triangles is easy to solve, well known and has relations to pivoting around a pivot-point of which there are twelve in the generic case ([9, p. 297], [8, p. 109]). Here I discuss the case of quadrangles and in some extend the case of arbitrary polygons. While in the triangle case the inscribed one is *similar* to a given triangle, for quadrangles and more general polygons this is no more possible. I start the discussion by examining properties of polygons inscribed in others to reveal some general facts. In this frame it is natural to introduce the class of *conjugate polygons* with respect to a point, which generalize the idea of the *precevian* triangle, having for vertices the *harmonic associates* of a point [12, p.100]. Then I discuss some properties of them, which in the case of quadrangles relate the inscription-problem to the Newton line of their associated *complete* quadrilateral (in this sense I speak of the *Newton line of the quadrangle* [13, p.169], [6, p.76], [3, p.69], [4], [7]). After this preparatory discussion I turn to the examination of the case of quadrangles and prove a characteristic property of their Newton line (§5, Theorems 11, 14).

2. Periodic polygon with respect to another

Consider two closed polygons $a = A_1 \cdots A_n$ and $b = B_1 \cdots B_n$ and pick a point C_1 on side A_1A_2 of the first. From this draw a parallel to side B_1B_2 of the second polygon until it hits side A_2A_3 to a point C_2 (see Figure 1). Continue in this way picking points C_i on the sides of the first polygon so that C_iC_{i+1} is parallel to side B_iB_{i+1} of the second polygon (indices $i > n$ are reduced modulo n if corresponding points X_i are not defined). In the last step draw a parallel to B_nB_1 from C_n until it hits the initial side A_1A_2 at a point C_{n+1}. I call polygon $c = C_1 \cdots C_{n+1}$ *parallel to* b *inscribed in* a *and starting at* C_1. In general polygon c is not closed. It can even have self-intersections and/or some side(s) degenerate to

Publication Date: April 20, 2009. Communicating Editor: Paul Yiu.

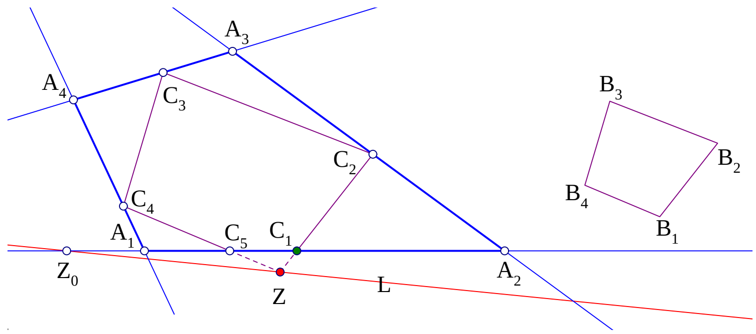


Figure 1. Inscribing a polygon

points (identical with vertices of a). One can though create a corresponding closed polygon by extending segment C_nC_{n+1} until to hit C_1C_2 at a point Z. Polygon $ZC_2\cdots C_n$ has sides parallel to corresponding sides of $B_1\cdots B_n$. Obviously triangle $C_{n+1}ZC_1$ has fixed angles and remains similar to itself if the place of the starting point C_1 changes on A_1A_2. Besides one can easily see that the function expressing the coordinate y of C_{n+1} in terms of the coordinate x of C_1 is a linear one $y = ax + b$. This implies that point Z moves on a fixed line L ([10, Tome 2, p. 10]) as point C_1 changes its position on line A_1A_2 (see Figure 1). This in turn shows that there is, in general, a unique place for C_1 on side A_1A_2 such that points C_{n+1}, C_1 coincide and thus define a *closed* polygon $C_1\cdots C_n$ inscribed in the first polygon and having its sides parallel to corresponding sides of the second. This place for C_1 is of course the intersection point Z_0 of line L with side A_1A_2. In the exceptional case in which L is parallel to A_1A_2 there is no such polygon. By the way notice that, for obvious reasons, in the case of triangles line L passes through the vertex opposite to side A_1A_2.

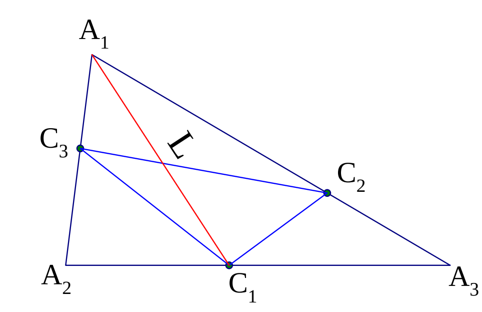


Figure 2. The triangle case

This example shows that the answer to next question is not in general in the affirmative. The question is: Under which conditions for the two polygons is line L identical with side A_1A_2, so that the above procedure produces always closed polygons $C_1\cdots C_n$? If this is the case then I say that polygon $B_1\cdots B_n$ is *periodic* with respect to $A_1 \cdots A_n$. Below it will be shown that this condition is independent of the side A_1A_2 selected. If it is satisfied by starting points C_1 on this side and drawing a parallel to B_1B_2 then it is satisfied also by picking the starting point C_i on side A_iA_{i+1}, drawing a parallel to B_iB_{i+1} and continuing in this way.

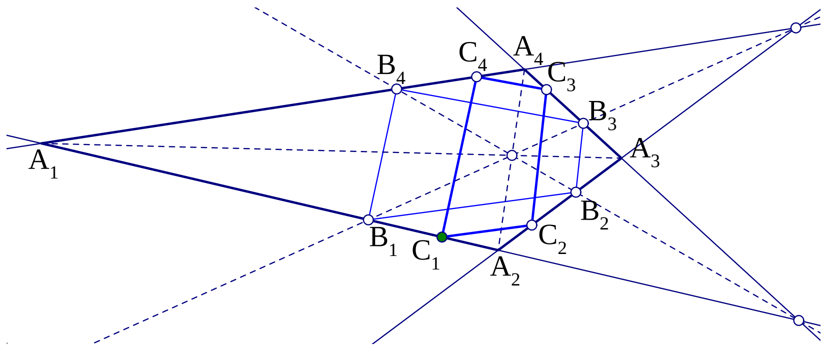


Figure 3. $B_1B_2B_3B_4$ periodic with respect to $A_1A_2A_3A_4$

There are actually plenty of examples of pairs of polygons satisfying the periodicity condition. For instance take an arbitrary quadrangle $A_1A_2A_3A_4$ and consider its $dual$ quadrangle $B_1B_2B_3B_4$, created through the intersections of its sides with the lines joining the intersection of its diagonals with the two intersection points of its pairs of opposite sides (see Figure 3). For every point C_1 on A_1A_2 the procedure described above closes and defines a quadrangle $C_1C_2C_3C_4$ inscribed in $A_1A_2A_3A_4$ and having its sides parallel to $B_1B_2B_3B_4$. This will be shown to be a consequence of Theorem 11 in combination with Proposition 16. It should be noticed though that periodicity, as defined here, is a relation depending on the *ordered* sets of vertices of two polygons. $B_1 \cdots B_n$ can be periodic with respect to $A_1 \cdots A_n$ but $B_2 \cdots B_nB_1$ not. Figure 4 displays such an example.

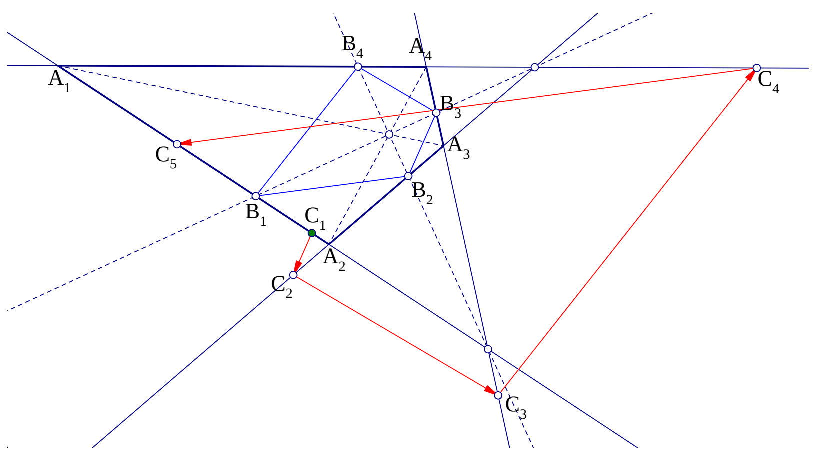


Figure 4. $B_2B_3B_4B_1$ not periodic with respect to $A_1A_2A_3A_4$

To handle the question in a systematic way I introduce some structure into the problem, which obviously is affinely invariant ([1], [2, vol.I, pp.32–66], [5]). I will consider the correspondence $C_1 \longmapsto C_{n+1}$ as the restriction on line A_1A_2 of a globally defined affine transformation G_1 and investigate the properties of this map. Figure 5 shows how transformation G_1 is constructed. It is the composition of *affine reflections* F_i ([5, p. 203]). The affine reflection F_i has its *axis* along A_iY_i which is the harmonic conjugate line of A_iX_i with respect to the two adjacent sides $A_{i-1}A_i$, A_iA_{i+1} at A_i. Its *conjugate direction* is A_iX_i which is parallel to side $B_{i-1}B_i$. By its definition map F_i corresponds to each point X the point Y such that the line-segment XY is parallel to the conjugate direction A_iX_i and has its middle on the axis A_iY_i of the map.

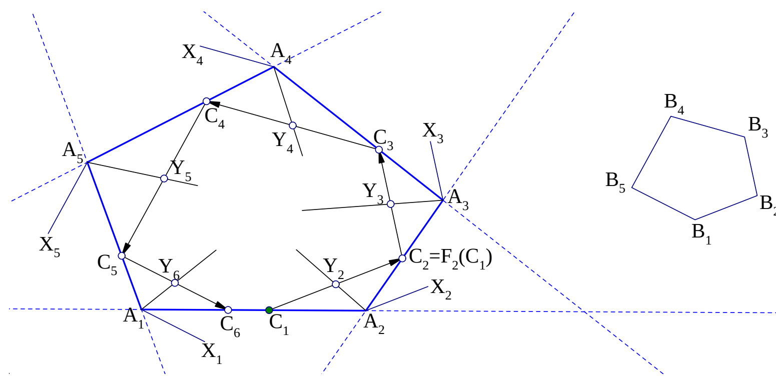


Figure 5. An affine transformation

The map $G_1 = F_1 \circ F_n \circ F_{n-1} \circ \cdots \circ F_2$ is a globally defined affine transformation, which on line A_1A_2 coincides with correspondence $C_1 \longmapsto C_{n+1}$. I call it *the first recycler of* b *in* a. Line A_1A_2 remains invariant by G_1 as a whole and each solution to our problem having $C_1 = C_{n+1}$ represents a fixed point of G_1. Thus, if there are more than one solutions, then line A_1A_2 will remain pointwise fixed under G_1. Assume now that G_1 leaves line A_1A_2 pointwise fixed. Then it is either an affine reflection or a *shear* ([5, p.203]) or it is the identity map, since these are the only affine transformations fixing a whole line and having determinant ± 1. Since G_1 is a product of affine reflections, its kind depends only on the number n of sides of the polygon. Thus for n even it is a shear or the identity map and for n odd it is an affine reflection. For n even it is shown by examples that both cases can happen: map G_1 can be a shear as well as the identity. In the second case I call $B_1 \cdots B_n$ *strongly periodic* with respect to $A_1 \cdots A_n$. The strongly periodic case delivers closed polygons $D_1 \cdots D_n$ with sides parallel to those of $B_1 \cdots B_n$ and the position of D_1 can be arbitrary. To construct such polygons start with an arbitrary point D_1 of the plane and define $D_2 = F_2(D_1), D_3 = F_3(D_2), \cdots, D_n = F_n(D_{n-1})$. The previous example of the dual of a quadrangle is a strongly periodic one (see Figure 6).

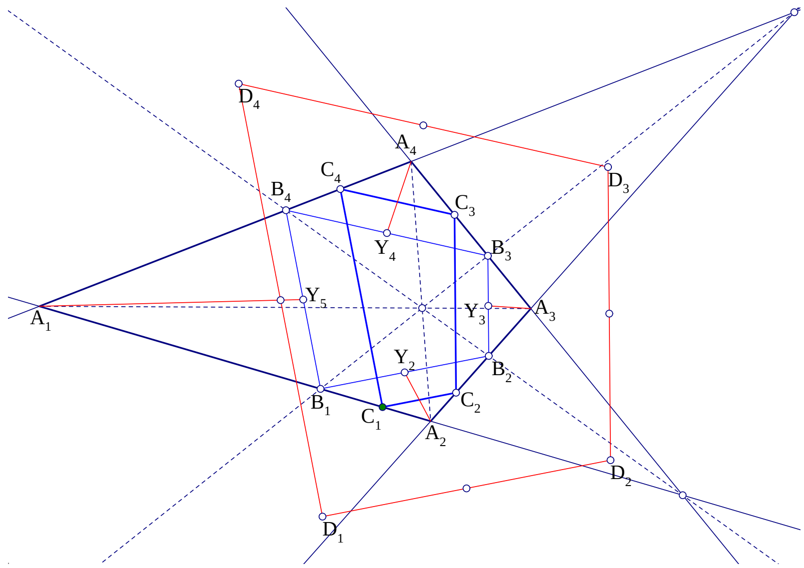


Figure 6. Strongly periodic case

Another case delivering many strongly periodic examples is that of a square $A_1A_2A_3A_4$ and the inscribed in it quadrangle $B_1B_2B_3B_4$, resulting by projecting an arbitrary point X on the sides of the square (see Figure 7).

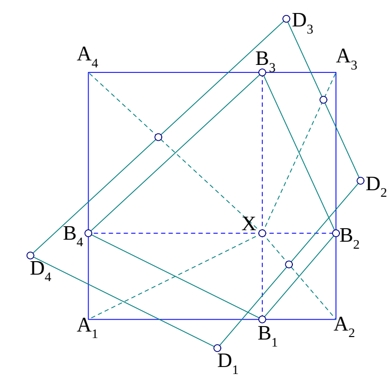


Figure 7. Strongly periodic case II

Analogously to G_1 one can define the affine map $G_2 = F_2 \circ F_1 \circ F_n \circ F_{n-1} \circ \cdots \circ F_3$, which I call *second recycler of* b *in* a. This does the same work in constructing a polygon $D_2 \cdots D_n D_1$ inscribed in $A_1 \cdots A_n$ and with sides parallel to those of $B_2 \cdots B_n B_1$ but now the starting point D_2 is to be taken on side A_2A_3, whereas the sides will be parallel successively to $B_2B_3, B_3B_4, \cdots$. Analogously are defined the affine maps $G_i, i = 3, \cdots, n$ (*i-th recycler of* b *in* a). It follows immediately from their definition that G_i are conjugate to each other. Obviously, since the F_i are involutive, we have $G_2 = F_2 \circ G_1 \circ F_2$ and more general $G_k = F_k \circ G_{k-1} \circ F_k$.

Thus, if there is a fixed point X_1 of G_1 on side A_1A_2, then $X_2 = F_2(X_1)$ will be a fixed point of G_2 on A_2A_3 and more general $X_k = F_k \circ \cdots \circ F_2(X_1)$ will be a fixed point of G_k on side A_kA_{k+1}. Corresponding property will be also valid in the case A_1A_2 remains pointwise fixed under G_1. Then every side A_kA_{k+1} will remain fixed under the corresponding G_k. The discussion so far is summarized in the following proposition.

Proposition 1. (1) *Given two closed polygons $a = A_1 \cdots A_n$ and $b = B_1 \cdots B_n$ there is in the generic case only one closed polygon $c = C_1 \cdots C_n$ having its vertex C_i on side A_iA_{i+1} and its sides C_iC_{i+1} parallel to B_iB_{i+1} for $i = 1, \cdots, n$. If there are two such polygons then there are infinite many and their corresponding point C_1 can be an arbitrary point of A_1A_2. In this case b is called periodic with respect to a.*

(2) *Using the sides of polygons a and b one can construct an affine transformation G_1 leaving invariant the side A_1A_2 and having the property: b is periodic with respect to a precisely when G_1 leaves side A_1A_2 pointwise fixed.*

(3) *In the periodic case, if n is odd then G_1 is an affine reflection with axis (mirror) line A_1A_2 and if n is even then it is a shear with axis A_1A_2 or the identity map. In the last case b is called strongly periodic with respect to a.*

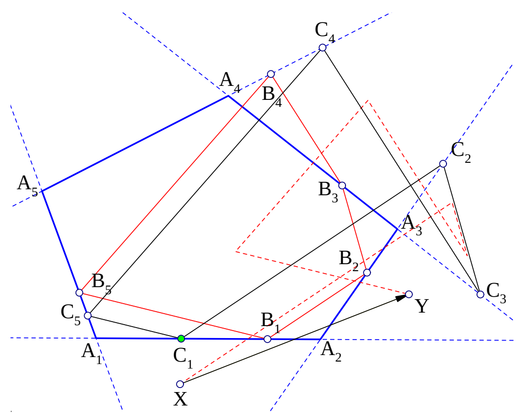


Figure 8. Periodic pentagons

Figure 8 shows a periodic case for $n = 5$. The figure shows also a typical pair $Y = G_1(X)$ of points related by the affine reflection G_1 resulting in this case.

3. Conjugate polygon

Given a closed polygon $a = A_1 \cdots A_n$ and a point P not lying on the side-lines of a, consider for each $i = 1, \dots, n$ the harmonic conjugate line A_iX_i of line A_iP with respect to the two adjacent sides of a at A_i. The polygon $b = B_1 \cdots B_n$ having sides these lines is called *conjugate of a with respect to P*. The definition generalizes the idea of the *precevian triangle* of a triangle $a = A_1A_2A_3$ with respect to a point P, which is the triangle $B_1B_2B_3$ having vertices the *harmonic associates* B_i of P with respect to a ([12, p.100]).

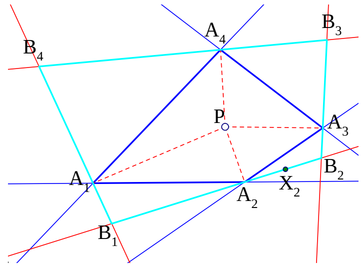


Figure 9. Conjugate quadrangle with respect to P

Proposition 2. *Given a closed polygon $a = A_1 \cdots A_n$ with n odd and a point P not lying on its side-lines, let $b = B_1 \cdots B_n$ be the conjugate polygon of a with respect to P. Then the transformation G_1 is an affine reflection the axis of which passes through P and its conjugate direction is that of line A_1A_2.*

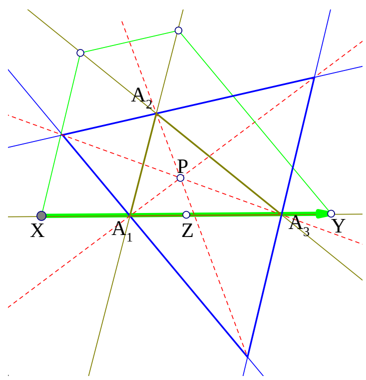


Figure 10. G_1 is an affine reflection

That point P remains fixed under G_1 is obvious, since G_1 is a composition of affine reflections all of whose axes pass through P. From this, using the preservation of proportions by affinities and the invariance of A_1A_2 follows also the that the parallels to A_1A_2 remain also invariant under G_1. Let us introduce coordinates (x, y) with origin at P and x-axis parallel to A_1A_2. Then G_1 has a representation of the form $\{x' = ax + by, y' = y\}$. Since its determinant is -1 it follows that $a = -1$. Thus, on every line $y = y_0$ parallel to A_1A_2 the transformation acts through $x' = -x + by_0 \Leftrightarrow x' + x = by_0$, showing that the action on line $y = y_0$ is a point symmetry at point Z with coordinates $(by_0/2, y_0)$, which remains also fixed by G_1 (see Figure 10). Then the whole line PZ remains fixed by G_1, thus showing it to be an affine reflection as claimed. The previous proposition completely solves the initial problem of inscription for conjugate polygons with n sides and n odd. In fact, as noticed at the beginning, such an inscription possibility corresponds to a fixed point of the map G_1 and this has a unique such point on A_1A_2. Thus we have next corollary.

Corollary 3. *If $b = B_1 \cdots B_n$ is the conjugate of the closed polygon $a = A_1 \cdots A_n$ with respect to a point P not lying on its side-lines and n is odd, then there is exactly one closed polygon $C_1 \cdots C_n$ with $C_i \in A_iA_{i+1}$ for every $i = 1, \cdots, n$ and sides parallel to corresponding sides of b. In particular, for n odd there are no periodic conjugate polygons.*

The analogous property for conjugate polygons and n even is expressed by the following proposition.

Proposition 4. *Given a closed polygon $a = A_1 \cdots A_n$ with n even and a point P not lying on its side-lines, let $b = B_1 \cdots B_n$ be the conjugate polygon of a with respect to P. Then the transformation G_1 either is a shear the axis of which is the parallel to side A_1A_2 through P, or it is the identity map.*

The proof, up to minor changes, is the same with the previous one, so I omit it. The analogous corollary distinguishes now two cases, the second corresponding to G_1 being the identity. Periodicity and strong periodicity coincide when n is even and when b is the conjugate of a with respect to some point.

Corollary 5. *If $b = B_1 \cdots B_n$ is the conjugate of the closed polygon $a = A_1 \cdots A_n$ with respect to a point P not lying on its side-lines and n is even, then there is either no closed polygon $C_1 \cdots C_n$ with $C_i \in A_iA_{i+1}$ for every $i = 1, \cdots, n$ and sides parallel to corresponding sides of b, or b is strongly periodic with respect to a.*

Remark. Notice that the existence of even one fixed point not lying on the parallel to A_1A_2 through P (the axis of the shear) imply that G_1 is the identity or equivalently, the corresponding conjugate polygon is strongly periodic.

The next propositions deal with some properties of conjugate polygons needed, in the case of quadrangles, in relating the periodicity to the Newton's line.

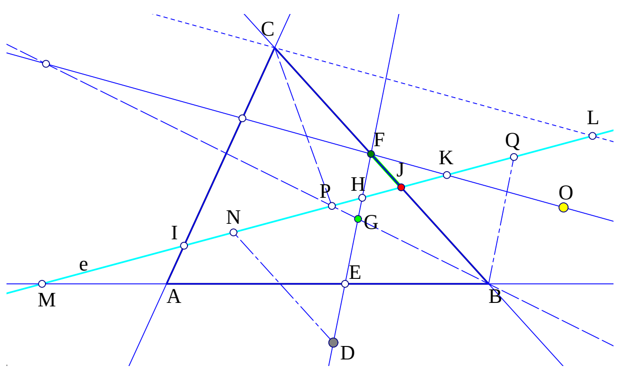


Figure 11. Fixed point O

Lemma 6. *Let $\{ABC, D, e\}$ be correspondingly a triangle, a point and a line. Consider a variable line through D intersecting sides AB, BC correspondingly at points E, F. Let G be the middle of EF and P the intersection point of lines*

e and BG. Let further CL be the harmonic conjugate of line CP with respect to CA, CB. Then the parallel to CL from F passes through a fixed point O.

To prove the lemma introduce affine coordinates with axes along lines $\{BC, e\}$ and origin at J, where $I = e \cap CA, J = e \cap CB$ (see Figure 11). The points on line e are: $M = e \cap AB, N = e \cap (||BC, D), H = e \cap DE, Q = e \cap (||DE, B)$, where the symbol $(||XY, Z)$ means: *the parallel to XY from Z*. Denote abscissas/ordinates by the small letters corresponding to labels of points, with the exceptions of $a = DN$, the abscissa x of F and the ordinate y of K. The following relations are easily deduced.

$$h = \frac{hx}{x+a}, \quad q = b\frac{h}{x}, \quad p = \frac{mq}{2q-m}, \quad l = \frac{pi}{2p-i}, \quad y = \frac{lx}{c}.$$

Successive substitutions produce a homographic relation between variables x, y:

$$p_1 x + p_2 y + p_3 xy = 0,$$

with constants (p_1, p_2, p_3), which is equivalent to the fact that line FK passes through point O with coordinates $(-\frac{p_2}{p_3}, -\frac{p_1}{p_3})$.

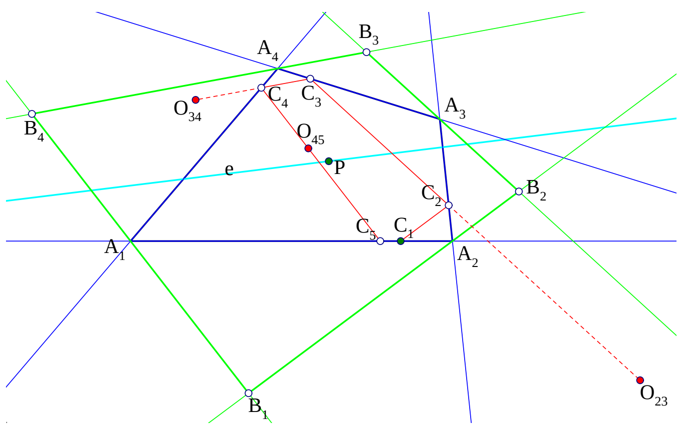


Figure 12. Sides through fixed points

Lemma 7. *Let $\{A_1 \cdots A_n, C_1, e\}$ be correspondingly a closed polygon, a point on side A_1A_2 and a line. Consider a point P varying on line e and the corresponding conjugate polygon $b = B_1 \cdots B_n$. Construct the parallel to b polygon $c = C_1 \cdots C_{n+1}$ starting at C_1. As P varies on e, every side of polygon c passes through a corresponding fixed point.*

The proof results by inductively applying the previous lemma to each side of c, starting with side C_1C_2, which by assumption passes through C_1 (see Figure 12). Next prove that side C_2C_3 passes through a point O_{23} by applying previous lemma to the triangle with sides A_1A_2, A_2A_3, A_3A_4 and by taking C_1 to play the role of D in the lemma. Then apply the lemma to the triangle with sides A_2A_3, A_3A_4, A_4A_5 taking for D the fixed point O_{23} of the previous step. There results a fixed point

O_{34} through which passes side C_3C_4. The induction continues in the obvious way, using in each step the fixed point obtained in the previous step, thereby completing the proof.

Lemma 8. *Let $\{A_1 \cdots A_n, C_1, e\}$ be correspondingly a closed polygon, a point on side A_1A_2 and a line. Consider a point P varying on line e, the corresponding conjugate polygon $b = B_1 \cdots B_n$ and the corresponding parallel to b polygon $c = C_1 \cdots C_{n+1}$ starting at C_1. Then the correspondence $P \longmapsto C_{n+1}$ is either constant or a projective one from line e onto line A_1A_2.*

Assume that the correspondence is not a constant one. Proceed then by applying the previous lemma and using the fixed points $O_{23}, O_{34}, \cdots$ through which pass the sides of the inscribed polygons c as P varies on line e. It is easily shown inductively that correspondences $f_1 : P \mapsto C_2,\ f_2 : P \mapsto C_3,\ \ldots, f_n : P \mapsto C_{n+1}$ are projective maps between lines. That f_1 is a projectivity is a trivial calculation. Map f_2 is the composition of f_1 and the perspectivity between lines A_3A_2, A_3A_4 from O_{23}, hence also projective. Map f_3 is the composition of f_2 and the perspectivity between lines A_4A_3, A_4A_5 from O_{34}, hence also projective. The proof is completed by the obvious induction.

4. The case of parallelograms

The only quadrangles not possessing a Newton line are the parallelograms. For these though the periodicity question is easy to answer. Next two propositions show that parallelograms are characterized by the strong periodicity of their conjugates with respect to *every* point not lying on their side-lines.

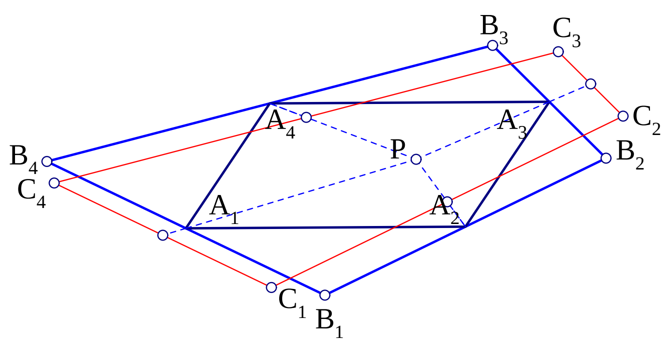


Figure 13. Parallelograms and periodicity

Proposition 9. *For every parallelogram $a = A_1A_2A_3A_4$ and every point P not lying on its side-lines the corresponding conjugate quadrangle $b = B_1B_2B_3B_4$ is strongly periodic.*

The proposition (see Figure 13) is equivalent to the property of the corresponding first recycler G_1 to be the identity. To prove this it suffices to show that G_1 fixes a point not lying on the parallel to A_1A_2 through P (see the remark after corollary 5 of previous paragraph). In the case of parallelograms however it is easily seen that a is the parallelogram of the middles of the sides of the conjugates b.

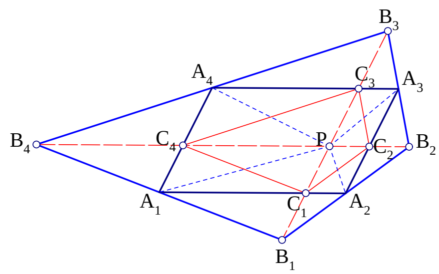


Figure 14. C_1 fixed by G_1

In fact, let $b = B_1B_2B_3B_4$ be the conjugate of a with respect to P and consider the intersection points $C_1, C_2, \cdots$ of the sides $A_1A_2, A_2A_3, \cdots$ of the parallelogram correspondingly with lines $PB_1, PB_2, \ldots$ (see Figure 14). The bundles of lines $A_1(B_1, P, C_1, A_4)$ at A_1 and $A_2(B_1, P, C_1, A_3)$ at A_2 are harmonic by the definition of b. Besides their three first rays intercept on line PB_1 correspondingly the same three points B_1, P, C_1 hence the fourth harmonic of these three points is the intersection point of their fourth rays A_1A_4, A_2A_3, which is the point at infinity. Consequently C_1 is the middle of PB_1. The analogous property for C_2, C_3, C_4 implies that quadrangle $c = C_1C_2C_3C_4$ has its sides parallel to those of b and consequently lines PA_i are the medians of triangles $PB_{i-1}B_i$. Thus point B_1 is a fixed point of G_1 not lying on its axis, consequently G_1 is the identity.

Proposition 10. *If for every point P not lying on the side-lines of the quadrangle $a = A_1A_2A_3A_4$ the corresponding conjugate quadrangle $b = B_1B_2B_3B_4$ is strongly periodic, then a is a parallelogram.*

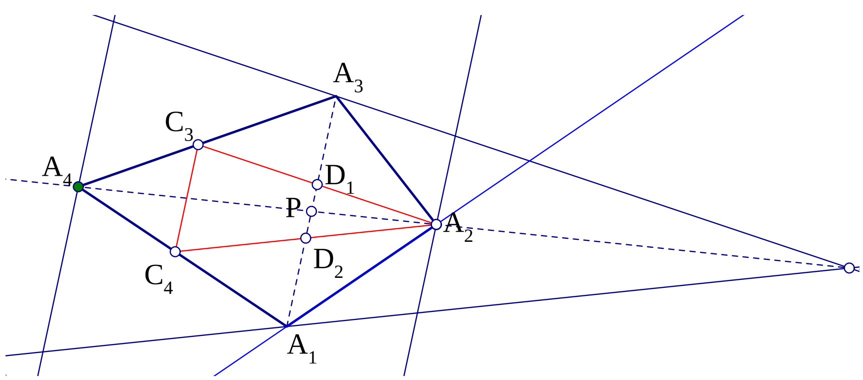


Figure 15. Parallelogram characterization

This is seen by taking for P the intersection point of the diagonals of the quadrangle. Consider then the parallel to b polygon starting at A_2. By assumption this must be closed, thus defining a triangle $A_2C_3C_4$ (see Figure 15). The middles D_1, D_2 of the sides of the triangle are by definition on the diagonal A_1A_3, which is parallel to C_3C_4. Thus the diagonal A_1A_3 is parallel to the conjugate direction of the other diagonal A_2A_4, consequently P is the middle of A_1A_3. Working in the same way with side A_2A_3 and the recycler G_2 it is seen that P is also the middle of A_2A_4, hence the quadrangle is a parallelogram.

5. A property of the Newton line

By the convention made above the *Newton line* of a quadrangle, which is not a parallelogram, is the line passing through the middles of the diagonals of the associated *complete* quadrilateral. In this paragraph I assume that the quadrangle of reference is not a parallelogram, thus has a Newton line. The points of this line are then characterized by having their corresponding conjugate quadrangle strongly periodic.

Theorem 11. *Given a non-parallelogramic quadrangle $a = A_1A_2A_3A_4$ and a point P on its Newton-line, the corresponding conjugate quadrangle $b = B_1B_2B_3B_4$ with respect to P is strongly periodic.*

Before starting the proof I supply two lemmata which reduce the periodicity condition to a simpler geometric condition that can be easily expressed in projective coordinates.

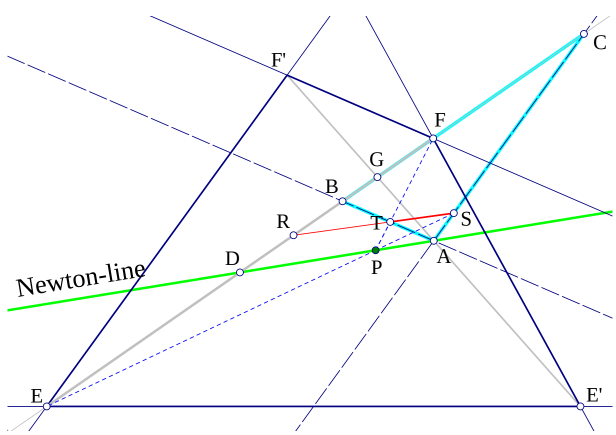


Figure 16. A fixed point

Lemma 12. *Let $a = EE'FF'$ be a quadrangle with diagonals $EF, E'F'$ and corresponding middles on them D, A. Draw from A parallels AB, AC correspondingly to sides $FF', F'E$ intersecting the diagonal EF correspondingly at points B, C. For every point P on the Newton-line AD of the quadrangle lines PE, PF*

intersect correspondingly lines AC, AB at points S, T. Line ST intersects the diagonal EF always at the same point R, which is the harmonic conjugate of the intersection point G of the diagonals with respect to B, C.

The proof is carried out using barycentric coordinates with respect to triangle ABC. Then points D, E, F, ... on line BC are represented using the corresponding small letters for parameters $D = B + dC$, $E = B + eC$, $F = B + fC$, ... (see Figure 16). In addition P is represented through a parameter p in $P = D + pA$. First we calculate E', F' in terms of these parameters:

$$E' = (f + g + 2fg)A - fB - (fg)C,$$
$$F' = (g - f)A + fB + fgC.$$

Then the coordinates of S, T are easily shown to be:

$$S = pA + (d - e)C,$$
$$T = (pf)A + (f - d)B.$$

From these the intersection point R of line ST with BC is seen to be:

$$R = (d - f)B + (f(d - e))C.$$

This shows that R is independent of the value of parameter p hence the same for all points P on the Newton-line. Some more work is needed to verify the claim about its precise location on line AB. For this the parallelism EF' to AC and the fact that D is the middle of EF are proved to be correspondingly equivalent to the two conditions:

$$g = \frac{f(1 + e)}{1 + f}, \quad d = \frac{f(e + 1) + e(f + 1)}{(e + 1) + (f + 1)}.$$

These imply in turn the equation

$$g = \frac{f(e - d)}{d - f},$$

which is easily shown to translate to the fact that R is the harmonic conjugate of G with respect to B, C.

Lemma 13. *Let $a = ABCD$ be a quadrangle with diagonals AC, BD and corresponding middles on them M, N. Draw from M parallels ME, MF correspondingly to sides AB, AD intersecting the diagonal BD at points E, F. Let P be a point of the Newton-line MN and S, T correspondingly the intersections of line-pairs $(PB, ME), (PD, MF)$. The conjugate quadrangle of P is periodic precisely when the harmonic conjugate of AP with respect to AB, AD is parallel to ST.*

In fact, consider the transformation $G_1 = F_4 \circ F_3 \circ F_2 \circ F_1$ composed by the affine reflections with corresponding axes PC, PD, PA, PB. By the discussion in the previous paragraph, the periodicity of the conjugate quadrilateral to P is equivalent to G_1 being the identity. Since G_1 is a shear and acts on BC in general as a translation by a vector $\mathbf{v}$ to show that $\mathbf{v} = \mathbf{0}$ it suffices to show that it fixes

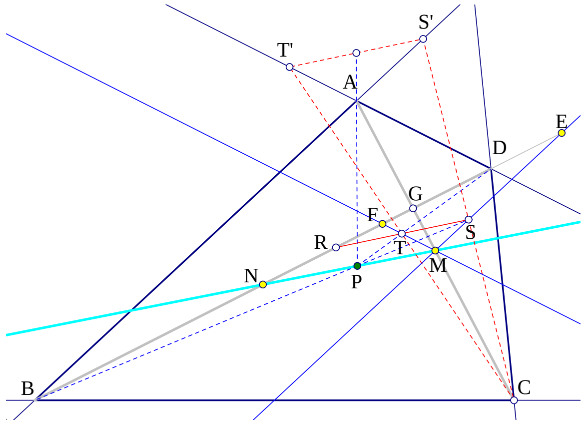


Figure 17. Equivalent problem

an arbitrary point on BC. This criterion applied to point C means that for $T' = F_2(C), S' = F_3(T')$ point $C' = F_4(S')$ is identical with C (see Figure 17). Since T, S are the middles of CT', CS', this implies the lemma.

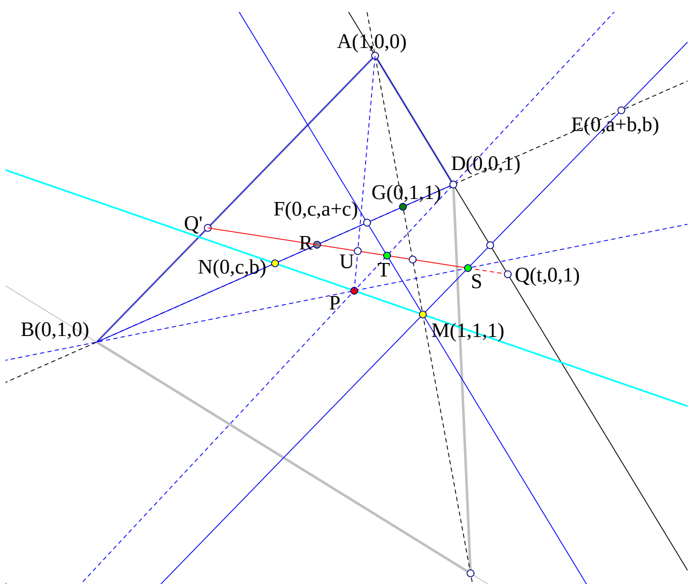


Figure 18. Representation in coordinates

Proof of the theorem: Because of the lemmata 12 and 13 one can consider the variable point P not as an independent point varying on the Newton-line MN but as a construct resulting by varying a line through R which is the harmonic conjugate of the intersection point G of the diagonals with respect to E, F. Such a line intersects the parallels ME, MF to sides AB, AD at S, T and determines P as intersection of lines BS, DT. Consider the coordinates defined by the projective basis (see Figure 18) $\{A(1,0,0), B(0,1,0), D(0,0,1), M(1,1,1)\}$. Assume further that the line at infinity is represented by an equation in the form

$$ax + by + cz = 0.$$

Then all relevant points and lines of the figure can be expressed in terms of the constants (a, b, c). In particular

$$ax - (a + c)y + cz = 0, \quad ax + by - (a + b)z = 0, \quad (c - b)x + by - cz = 0,$$

are the equations of lines MF, ME and the Newton-line MN. Point R has coordinates $(0, a', b')$, where $a' = (c - a - b)$, $b' = (a + c - b)$. Assume further that the parametrization of a line through R is done by a point $Q(t, 0, 1)$ on line AD. This gives for line RQ the equation $RQ : a'x + (tb')y - (ta')z = 0$. Point S has coordinates (a'', b'', c'') where $a'' = t(a'b-b'(a+b))$, $b'' = a'(a+b)-taa'$, $c'' = a'b - tab'$. This gives for P the coordinates $(ba'', cc'' - a''(c - b), bc'')$ and the coordinates of the intersection point U of PA with RQ can be shown to be $U = c''Q - (c + at)Q'$, where $Q'(tb', -a', 0)$ is the intersection point of AB and RQ. From these follows easily that U is the middle of QQ' showing the claim according to Lemma 13.

Theorem 14. *For a non-parallelogramic quadrangle $a = A_1A_2A_3A_4$ only the points P on its Newton-line have the corresponding conjugate quadrangle $b = B_1B_2B_3B_4$ strongly periodic.*

The previous theorem guarantees that all points of the Newton line have a strongly periodic corresponding conjugate polygon b. Assume now that there is an additional point P_0, not on the Newton line, which has also a strongly periodic corresponding conjugate polygon. In addition fix a point C_1 on A_1A_2. Take then a point P_1 on the Newton line and consider line $e = P_0P_1$. By Lemma 8 the correspondence $f : e \longrightarrow A_1A_2$ sending to each point $P \in e$ the end-point C_{n+1} of the polygon parallel to the conjugate b of a with respect to P starting at a fixed point C_1 is either a constant or a projective map. Since f takes for two points P_0, P_1 the same value (namely $f(P_0) = f(P_1) = C_1$) this map is constant. Hence the whole line e consists of points having corresponding conjugate polygon strongly periodic. This implies that any point of the plane has the same property. In fact, for an arbitrary point Q consider a line e_Q passing through Q and intersecting e and the Newton line at two points Q_0 and Q_1. By the same reasoning as before we conclude that all points of line e_Q have corresponding conjugate polygons strongly periodic, hence Q has the same property. By Proposition 10 of the preceding paragraph it follows that the quadrangle must be a parallelogram, hence a contradiction to the hypothesis for the quadrangle.

6. The dual quadrangle

In this paragraph I consider a non-parallelogramic quadrangle $a = A_1A_2A_3A_4$ and its *dual* quadrangle $b = B_1B_2B_3B_4$, whose vertices are the intersections of the sides of the quadrangle with the lines joining the intersection point of its diagonals with the intersection points of its two pairs of opposite sides. After a preparatory lemma, Proposition 16 shows that b is the conjugate polygon with respect to an appropriate point on the Newton line, hence b is strongly periodic.

Lemma 15. *Let $a = A_1A_2A_3A_4$ be a quadrangle and with diagonals intersecting at E. Let also $\{F, G\}$ be the two other diagonal points of its associated complete quadrilateral. Let also $b = B_1B_2B_3B_4$ be the dual quadrangle of a.*
(1) *Line EG intersects the parallel A_4N to the side B_1B_4 of b at its middle M.*
(2) *Side B_1B_2 of b intersects the segment MN at its middle O.*

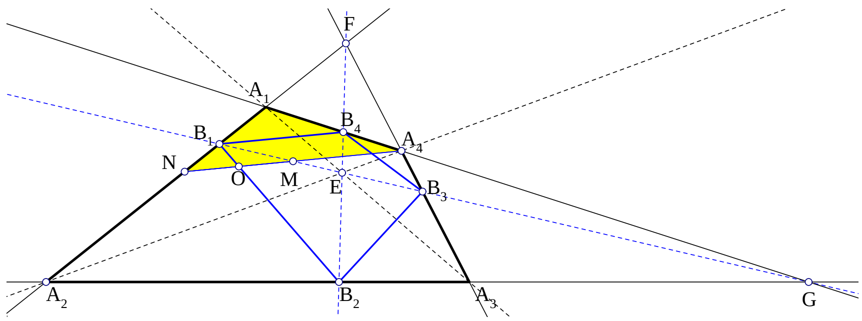


Figure 19. Dual property

$MN/MA_4 = 1$, since Menelaus theorem applied to triangle A_1NA_4 with secant line B_1B_3G gives $(B_1N/B_1A_1)(MA_4/MN)(GA_1/GA_4) = 1$. But $B_1N/B_1A_1 = B_4A_4/B_4A_1 = GA_4/GA_1$. Later equality because (B_4, G) are harmonic conjugate to (A_1, A_4). Also $ON/OM = 1$, since the bundle $B_1(B_2, B_4, E, F)$ is harmonic. Thus the parallel NM to line B_1B_4 of the bundle is divided in two equal parts by the other three rays of the bundle.

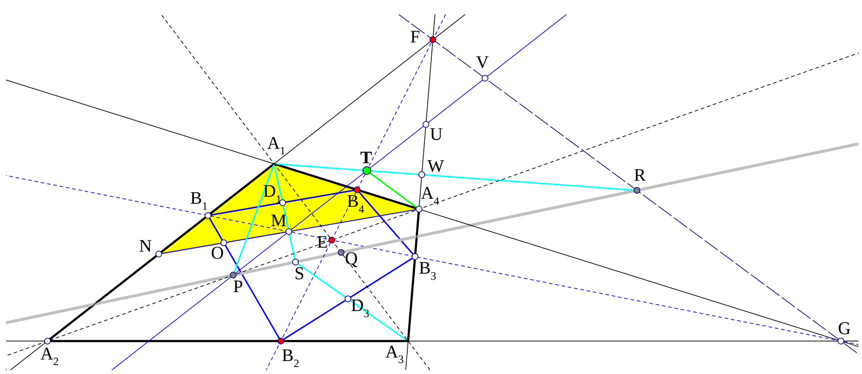


Figure 20. Dual is strongly periodic

Proposition 16. *Let $a = A_1A_2A_3A_4$ be a quadrangle and with diagonals intersecting at E. Let also $\{F, G\}$ be the two other diagonal points of its corresponding complete quadrilateral and $\{P, Q, R\}$ the middles of the diagonals $\{A_2A_4, A_1A_3, FG\}$ contained in the Newton line of the quadrilateral. Let $b =$*

$B_1B_2B_3B_4$ be the dual quadrangle of a
(1) *The four medians $\{A_1D_1, A_2D_2, A_3D_3, A_4D_4\}$ of triangles $\{A_1B_1B_4$, $A_2B_2B_1$, $A_3B_3B_2$, $A_4B_4B_3\}$ respectively meet at a point S on the Newton line.*
(2) *S is the harmonic conjugate of the diagonal middle R with respect to the two others (P, Q).*

Start with the intersection point T of diagonal B_2B_4 with line A_1R (see Figure 20). Draw from T line TV parallel to side A_1A_2 intersecting side A_3A_4 at U. Since the bundle $F(V, T, U, A_1)$ is harmonic and TV is parallel to ray FA_1 of it point U is the middle of TV. Since $A_4(A_1, W, T, R)$ is a harmonic bundle and R is the middle of FG, its ray A_4T is parallel to FG. It follows that A_4TFV is a parallelogram. Thus U is the middle of A_4F, hence the initial parallel TV to line A_1A_2 passes through the middles of segments having one end-point at A_4 and the other on line A_1A_2. Among them it passes through the middles of $\{A_1A_4, A_4N, A_4A_2\}$ the last being P the middle of the diagonal A_2A_4. Extend the median A_1D_1 of triangle $A_1B_1B_4$ to intersect the Newton line at S. Bundle $A_1(P, Q, S, R)$ is harmonic. In fact, using Lemma 15 it is seen that it has the same traces on line TV with those of the harmonic bundle $E(P, A_1, M, T)$. Thus S is the harmonic conjugate of R with respect to (P, Q).

Remarks. (1) Poncelet in a preliminary chapter [10, Tome I, p. 308] to his celebrated *porism* (see [2, Vol. II, pp. 203–209] for a modern exposition) examined the idea of *variable* polygons $b = B_1 \cdots B_n$ having *all but one* of their vertices on fixed lines (sides of another polygon) and restricted by having their sides to pass through corresponding fixed points $E_1, \cdots, E_n$. Maclaurin had previously shown that in the case of triangles ($n = 3$) the free vertex describes a conic([11, p. 248]). This generalizes to polygons with arbitrary many sides. If the fixed points through which pass the variable sides are *collinear* then the free vertex describes a line ([10, Tome 2, p. 10]). This is the case here, since the fixed points are the points on the line at infinity determining the directions of the sides of the inscribed polygons.

(2) In fact one could formulate the problem handled here in a somewhat more general frame. Namely consider polygons inscribed in a fixed polygon $a = A_1 \cdots A_n$ and having their sides passing through corresponding fixed *collinear* points. This case though can be reduced to the one studied here by a projectivity f sending the line carrying the fixed points to the line at infinity. The more general problem lives of course in the projective plane. In this frame the affine reflections F_i, considered above, are replaced by *harmonic homologies* ([5, p. 248]). The center of each F_i is the corresponding fixed point E_i through which passes a side B_iB_{i+1} of the variable polygon. The axis of the homology is the polar of this fixed point with respect to the side-pair $(A_{i_1}A_i, A_iA_{i+1})$ of the fixed polygon. The definitions of periodicity and the related results proved here transfer to this more general frame without difficulty.

(3) Though I am speaking all the time about a quadrangle, the property proved in §5 essentially characterizes the associated *complete quadrilateral*. If a point P has a periodic conjugate with respect to one, out of the three, quadrangles embedded

in the complete quadrilateral then it has the same property also with respect to the other two quadrangles embedded in the quadrilateral.

References

[1] M. Audin, *Geometry*, Springer, Berlin, 1998, pp. 7–42.
[2] M. Berger, *Geometry*, 2 volumes, Springer, Berlin, 1980.
[3] J. Casey, *A Treatise on the Analytical Geometry of the Point, Line, Circle and Conic Sections*, Hodges Figgis, Dublin, 1893.
[4] J. W. Clawson, The Complete Quadrilateral, *Ann. Math.*, 20 (1919) 232–261.
[5] H. S. M. Coxeter, *Introduction to Geometry*, 2nd ed., John Wiley, New York, 1969.
[6] R. Deltheil and D. Caire, *Geometrie et Complements*, Editions Gabay, Paris, 1989.
[7] J.-P. Ehrmann, Steiner's Theorems on the Complete Quadrilateral, *Forum Geom.*, 4 (2004) 35–52.
[8] W. Gallatly, *The modern Geometry of the Triangle*, Francis Hodgson, London.
[9] R. A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, 2007.
[10] J. V. Poncelet, *Applications d' analyse et de geometrie*, Gauthier-Villars 1864 (1964).
[11] G. Salmon, *A treatise on conic sections*, London 1855, Longmans.
[12] P. Yiu, *Introduction to the Geometry of the Triangle*, 2001, available at `http://math.fau.edu/yiu/GeometryNotes020402.pdf`.
[13] P. Yiu, *Notes on Euclidean Geometry*, 1998, available at `http://math.fau.edu/yiu/EuclideanGeometryNotes.pdf`.

Paris Pamfilos: Department of Mathematics, University of Crete, 71409 Herakleion, Greece
E-mail address: `pamfilos@math.uoc.gr`

Forum Geometricorum
Volume 9 (2009) 99–107.

FORUM GEOM
ISSN 1534-1178

Folding a Square to Identify Two Adjacent Sides

Cristinel Mortici

Abstract. The purpose of this paper is to establish some properties that appear in a square cut by two rays at 45 degrees passing through a vertex of the square. Elementary proofs and other interesting comments are provided.

1. A simple problem and a reformulation

The starting point of this work is the following problem from [3], partially discussed in [4]. [1]

Proposition 1. *Two points M and N on the hypotenuse BD of the isosceles, right-angled triangle ABD, with M between B and N, define an angle $\measuredangle MAN = 45^\circ$ if and only if $BM^2 + ND^2 = MN^2$* (see Figure 1).

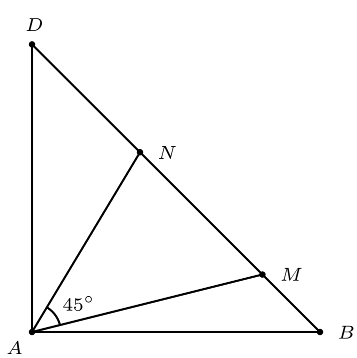


Figure 1

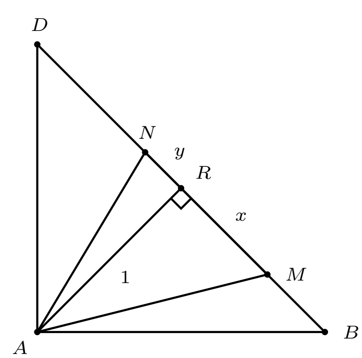


Figure 3

Proof. Let R be the midpoint of BD so that $AR = BR = BR$, and AR is an altitude of triangle ABD. We assume $AR = 1$ and denote $RM = x$, $RN = y$ (see Figure 3). Note that

$$\tan(\measuredangle MAN) = \tan(\measuredangle MAR + \measuredangle NAR) = \frac{x+y}{1-xy} = 1.$$

It follows that $\measuredangle MAN = 45^\circ$ if and only if $x + y = 1 - xy$. On the other hand, $BM^2 + ND^2 = MN^2$ if and only if $(1-x)^2 + (1-y)^2 = (x+y)^2$. Equivalently, $x + y = 1 - xy$, the same condition for $\measuredangle MAN = 45^\circ$. $\square$

Publication Date: April 27, 2009. Communicating Editor: Paul Yiu.

The author thanks Paul Yiu and an anonymous referee for their helps in improvement of this paper.

[1] This problem (erroneously attributed to another author in [1]) was considered by Boskoff and Suceavă as an example of an elliptic projectivity characterized by the Pythagorean relation.

This necessary and sufficient condition assumes new, interesting forms if we consider the isosceles right triangle as a half-square, and fold the adjacent sides AB and AC along the lines AM and AN. Without loss of generality we assume $AB = AC = 1$.

Theorem 2. *Let $ABCD$ be a unit square. Two half-lines through A meet the diagonal BD at M and N, and the sides BC, CD at M and P and Q respectively* (see Figure 2). *Assume $AP \not\equiv AQ$.*

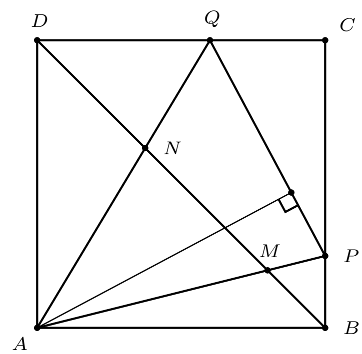


Figure 2

The following statements are equivalent:
(i) $\measuredangle PAQ = 45^\circ$.
(ii) $MN^2 = BM^2 + ND^2$.
(iii) *The perimeter of triangle CPQ is equal to* 2.
(iv) $PQ = BP + QD$.
(v) *The distance from A to line PQ is equal to* 1.
(vi) *The area of triangle AMN is half of the area of triangle APQ.*
(vii) $PQ = \sqrt{2} \cdot MN$.
(viii) $PQ^2 = 2(BM^2 + ND^2)$.
(ix) *The line passing through A and $MQ \cap NP$ is perpendicular on PQ.*
(x) $AN = NP$.
(xi) $AM = MQ$.

Remark. In the excluded case $AP = AQ$, statement (ix) does not imply the other statements.

Proof of Theorem 2. With Cartesian coordinates $A(0,\ 0)$, $B(1,\ 0)$, $C(1,\ 1)$, $D(0,\ 1)$ and $P(1,\ a)$, $Q(b,\ 1)$ for some distinct $a, b \in (0,\ 1)$, we have $M(\frac{1}{1+a},\ \frac{a}{1+a})$ and $N(\frac{b}{1+b},\ \frac{1}{1+b})$. Then (i)-(xi) are each equivalent to

$$a + b + ab = 1. \tag{1}$$

This is clear from the following, which are obtained from routine calculations.
(i): $\tan \measuredangle PAQ = 1 - \frac{a+b+ab-1}{a+b}$.
(ii): $MN^2 - BM^2 - ND^2 = -\frac{2(a+b+ab-1)}{(b+1)(a+1)}$.

(iii, iv): $(PQ - BP - QD) + 2 = CP + PQ + QC = 2 - \frac{2(a+b+ab-1)}{a+b+\sqrt{(1-a)^2+(1-b)^2}}$.

(v): $\mathrm{dist}(A, PQ) = 1 + \frac{(1-a)(1-b)(a+b+ab-1)}{(1-ab+\sqrt{(1-a)^2+(1-a)^2})\sqrt{(1-a)^2+(1-b)^2}}$.

(vi): $\frac{\mathrm{area}[AMN]}{\mathrm{area}[APQ]} = \frac{1}{2} + \frac{a+b+ab-1}{2(1+a)(1+b)}$.

(vii): $PQ^2 - 2MN^2 = (ab + a + b - 1) \cdot \frac{(a+b)(a-b)^2+(ab^3+a^3b+a^2+b^2+2-6ab)}{(1+a)^2(1+b)^2}$.

(viii): $PQ^2 - 2(BM^2 + ND^2) = (a + b + ab - 1) \cdot f(a, b)$, where

$$f(a,b) := \frac{-4a - 4b - ab^2 - a^2b + ab^3 + a^3b - 10ab + a^2 + a^3 + b^2 + b^3 - 2}{(a+1)^2(b+1)^2}.$$

(ix): If O is the intersection of PN and QN, then

$$m_{AO}m_{PQ} = -1 + \frac{(a-b)(a+b+ab-1)}{b(1-b)(a+1)}.$$

(x): $AN^2 - NP^2 = (a + b + ab - 1) \cdot \frac{1-a}{1+b}$.

(xi) $AM^2 - MQ^2 = (a + b + ab - 1) \cdot \frac{1-b}{1+a}$.

The expression (vii) is indeed equivalent with (1), if we take into account that

$$\frac{a^2 + b^2 + a^3b + ab^3 + 1 + 1}{6} > \sqrt[6]{a^2 \cdot b^2 \cdot a^3b \cdot ab^3 \cdot 1 \cdot 1} = ab.$$

For (viii), we prove that the $f(a, b) < 0$ for $a, b \in [0, 1]$. This is because, regarded as a function of $a \in [0, 1]$, $f''(a) = 6a + 6ab + 2(1 - b) > 0$. Since $f(0) < 0$ and $f(1) < 0$, we conclude that $f(a) < 0$ for $a \in [0, 1]$. □

2. A simple geometric proof of (i) ⇔ (ii)

Statement (ii) clearly suggests a right triangle with sides congruent to BM, ND and MN. One way to do this is indicated in Figure 5, where M' is chosen such that the segment DM' is perpendicular to BD and is congruent to BM. Under the hypothesis (ii), we have $M'N = MN$. Moreover, $\Delta AMB \equiv \Delta AM'D$, and $\measuredangle MAM' = 90°$. It also follows that the triangles AMN and $AM'N$ have three pairs of equal corresponding sides, and are congruent. From this, $\measuredangle MAN = \measuredangle NAM' = 45°$. This shows that (ii) $\implies$ (i).

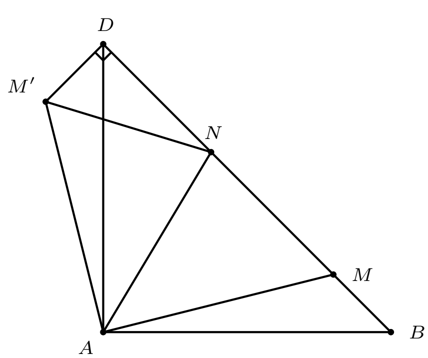


Figure 5

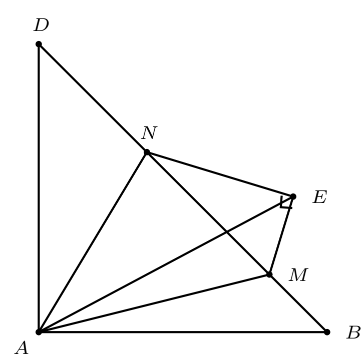


Figure 6

Another idea is to build an auxiliary right triangle with the hypotenuse MN, whose legs have lengths equal to BM and ND. This is based on the simple idea of folding the half-square ABD along AM and AN to identify the adjacent sides AB and AC. Let E be the reflection of B in the line AM (see Figure 6). Note that $BM = ME$. Assuming $\measuredangle MAN = 45°$, we see that E is also the reflection of D in the line AN. Now the triangles AMB and AME are congruent, so are the triangles ANE and AND. Thus, $\measuredangle MEN = \measuredangle MEA + \measuredangle NEA = \measuredangle MBA + \measuredangle NDA = 45° + 45° = 90°$. By the Pythagorean theorem, $MN^2 = ME^2 + EN^2 = BM^2 + ND^2$. This shows that (i) $\implies$ (ii).

3. A generalization

V. Proizolov has given in [6] the following nice result illustrating the beauty of the configuration of Theorem 2.

Proposition 3. *If M and N are points inside a square $ABCD$ such that $\measuredangle MAN = \measuredangle MCN = 45°$, then $MN^2 = BM^2 + ND^2$* (see Figure 8).

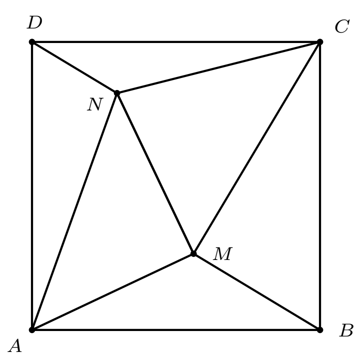


Figure 8

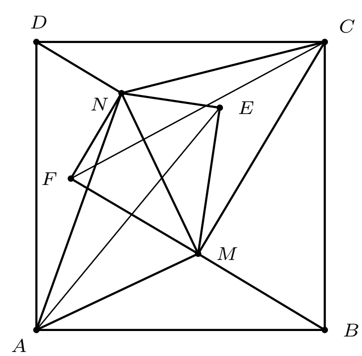


Figure 8A

This situation can be viewed as a surprising extension from the case of triangle ABD in Figure 6 is distorted into the polygon $ABMND$. In fact, by considering the symmetric of triangle ABD with respect to hypotenuse BD in Figure 1, a particular case of Proposition 3 is obtained. This analogy carries over to the general case. More precisely, we try to use the auxiliary construction from Figure 6, namely to consider the point E such that the triangles ANE and AND are symmetric and also the triangles AME and AMB are symmetric.

Let F be analogue defined, starting from the vertex C (see Figure 8A).

It follows that $\measuredangle MEN + \measuredangle MFN = 180°$, as the sum of the angles $\measuredangle B$ and $\measuredangle D$ of the square. But the triangles MEN and MFN are congruent, so $\measuredangle MEN = \measuredangle MFN = 90°$. The conclusion follows now from Pythagorean theorem applied in triangle MEN.

4. Rotation of the square

We show how to use the above auxiliary constructions to establish further interesting results. Complete the right triangle ABD from Figure 5 to an entire square $ABCD$. Triangle ADM' is obtained by rotating triangle ABM about A, through $90°$. This fact suggests us to make a clockwise rotation with center A of the entire figure to obtain the square $ADST$ (see Figure 9).

Denote the points corresponding to M, N, P, Q by M', N', P', Q' respectively. Assume that $\measuredangle PAQ = 45°$, or equivalently, $MN^2 = MB^2 + ND^2$.

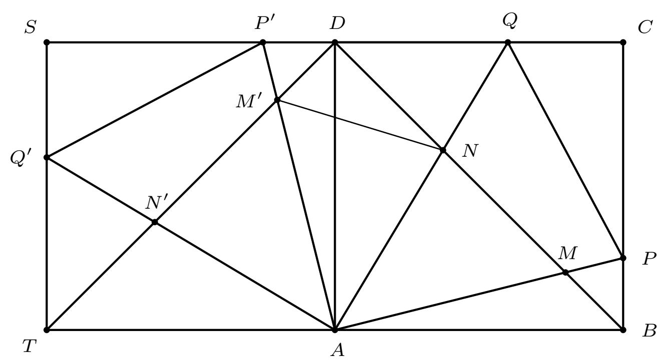


Figure 9

From $\Delta APQ \equiv \Delta AP'Q$ it follows that $PQ = P'Q$. If $AB = 1$, then

$$2 = SC = SP' + P'Q + QC = CP + PQ + QC$$

and we obtained the implication (i) $\Longrightarrow$ (iii).

The converse (iii) $\Longrightarrow$ (i) was first stated by A. B. Hodulev in [2].

5. Secants, tangents and lines external to a circle

We begin this section with an interesting question. Assuming $ABCD$ a unit square, how can we construct points P, Q such that the perimeter of triangle PQC is equal to 2? As we have already seen, one method is to make $\measuredangle PAQ = 45°$. Alternatively, note that the perimeter of triangle PQC is equal to 2 if and only if $PQ = BP + DQ$. This characterization allows us to construct points P, Q on the sides with the required property.

If we draw the arc with center A, passing through B and D, then every tangent line meeting the circle at T and the sides at P and Q determines the triangle ΔPQC of perimeter 2, because $PT = PB$ and $QT = QD$ (see Figure 10).

Moreover, if PQ does not meet the arc, then the length of PQ is less than the parallel tangent $P'Q'$ to the circle (see Figure 11). Consequently, if a segment PQ does not meet the circle, then $\measuredangle PAQ < 45°$. On the other hand, if PQ meets the circle twice, then $\measuredangle PAQ > 45°$.

We summarize these in the following theorem.

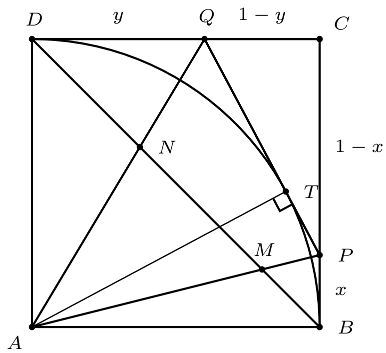


Figure 10

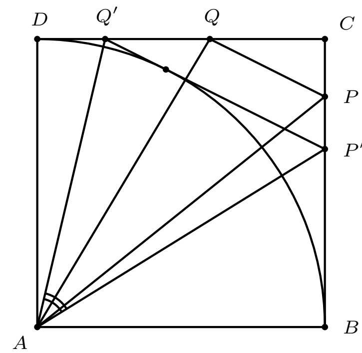


Figure 11

Theorem 4. *Let $ABCD$ be a unit square, and P, Q be points on the sides BC and CD respectively. Consider the quadrant ω of the circle with center A, passing through B and D.*
(a) $\measuredangle PAQ = 45°$ *if and only if PQ is tangent to ω. Equivalently, the perimeter of triangle PQC is equal to 2.*
(b) $\measuredangle PAQ > 45°$ *if and only if PQ intersects ω at two points. Equivalently, the perimeter of triangle PQC is greater than 2.*
(c) $\measuredangle PAQ < 45°$ *if and only if PQ is exterior to ω. Equivalently, the perimeter of triangle PQC is less than 2.*

6. Comparison of areas

The implication (i) $\Longrightarrow$ (vi) was first discovered by Z. G. Gotman in [1].

In Figure 12 below, observe that the quadrilaterals $ABPN$ and $ADQM$ are cyclic, respectively because $\measuredangle NAP = \measuredangle NBP$ and $\measuredangle MAQ = \measuredangle MDQ$.

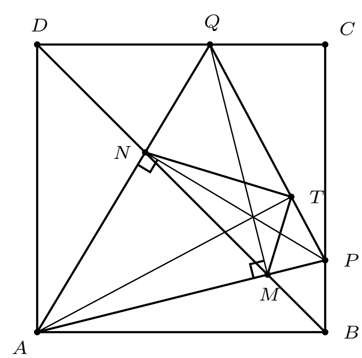


Figure 12

Consequently, AMQ and ANP are isosceles right-angled triangles. Hence,

$$\frac{S_{AMN}}{S_{APQ}} = \frac{AM \cdot AN}{AP \cdot AQ} = \frac{AM}{AQ} \cdot \frac{AN}{AP} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}.$$

Now we establish the implication (i) $\Longrightarrow$ (ix).

In triangle ΔAPQ, QM and PN are altitudes, so the radius AT from Figure 10 is in fact the third altitude of the triangle ΔAPQ.

We can continue with the identifications, making use of the congruences $\Delta APB \equiv \Delta APT$ and $\Delta AQD \equiv \Delta AQT$. We deduce that $TM = MB$ and $TN = ND$. It follows that

$$MN^2 = MT^2 + TN^2 = BM^2 + ND^2.$$

Remark. The point E from Figure 6, coinciding with the point T from Figure 12, is more interesting than we have initially thought. It lies on the circumcircle of the given triangle ABD.

7. Two pairs of congruent segments

The implications (i) $\Longrightarrow$ (x) and (xi) follow from the fact that ANP and AMQ are isosceles right-angled triangles.

For the converses, let us assume by way of contradiction that $\measuredangle MAN_1 = 45°$, with N_1 in BD, distinct from N. Then $AN_1 = N_1P$. As we have also $AN = NP$, it follows that NN_1 and consequently BD is the perpendicular bisector of AP, which is absurd.

8. Concluding remarks

Now let us return for a short time to the opposite angles drawn in Figure 8. It is the moment to celebrate the contribution of V. Proizvolov which proves in [5] the following nice result.

Proposition 5. *If M and N are points inside a square $ABCD$ such that $\measuredangle MAN = \measuredangle MCN = 45°$, then*

$$S_{MCN} + S_{MAB} + S_{NAD} = S_{MAN} + S_{MBC} + S_{NCD}.$$

Having at hand the previous construction from Figure 8A (where F is defined by the conditions $\Delta CND \equiv \Delta CNF$ and $\Delta CMB \equiv \Delta CMF$), we have

$$S_{MCN} + S_{MAB} + S_{NAD} = S_{MCN} + S_{AMEN} = S_{AMCN} + S_{MEN}.$$

Similarly, $S_{MAN}+S_{MBC}+S_{NCD} = S_{AMCN}+S_{MFN}$ and the conclusion follows from the congruence of the triangles MEN and MFN.

We mention for example that the idea of folding a square as in Figure 6 leads to new results under weaker hypotheses. Indeed, if we consider that piece of paper as an isosceles triangle, not necessarily right-angled, then similar results hold. Thus, if triangle ABD is isosceles, then in triangle MEN, the angle $\measuredangle MEN$ is the sum of angles $\measuredangle ABD$ and $\measuredangle ADB$. Consequently, by applying the law of cosines to triangle MEN, we obtain the following extension of Proposition 1.

Proposition 6. *Let M and N be two points on side BD of the isosceles triangle ABD such that the angle $\measuredangle MAN = \frac{1}{2}\measuredangle BAD$. Then*

$$MB^2 - MN^2 + DN^2 = -2MB \cdot DN \cos A.$$

Another interesting extension is the following problem proposed by the author at the 5th Selection Test of the Romanian Team participating at 44th IMO Japan 2003.

Problem. Find the angles of a rhombus $ABCD$ with $AB = 1$ given that on sides CD (CB) *there exist points* P, *respective* Q *such that the angle* $\measuredangle PAQ = \frac{1}{2}\measuredangle BAD$ and the perimeter of triangle CPQ is equal to 2.

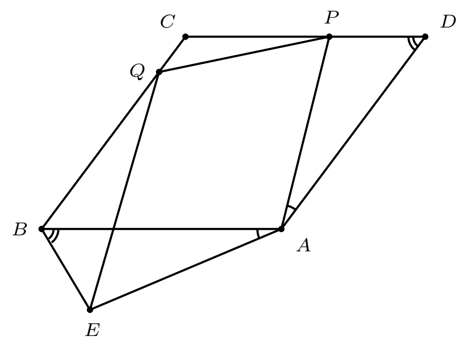


Figure 13

Let E be as in Figure 13 such that $\Delta APD \equiv \Delta AEB$. In fact we rotate triangle APD about A and what it is interesting for us is that $PQ = QE$ and $PD = BE$. Now, the equality $PQ = PD + QB$ can be written as $QE = BE + QB$, so the points Q, B, E are collinear.

This is possible only when $ABCD$ is square.

Finally, we consider replacing the square in Theorem 2 by a rhombus. Proposition 7 below was proposed by the author as a problem for the 12th Edition of the Clock-Tower School Competition, Râmnicu Vâlcea, Romania, 2009, then given at the first selection test for the Romanian team participating at the Junior Balkan Mathematical Olympiad, Neptun-Constanta, April, 15-th, 2009.

Proposition 7. *Let $ABCD$ be a rhombus. Two rays through A meet the diagonal BD at M, N, and the sides BC and CD at P, Q respectively* (see Figure 14). *Then $AN = NP$ if and only if $AM = MQ$.*

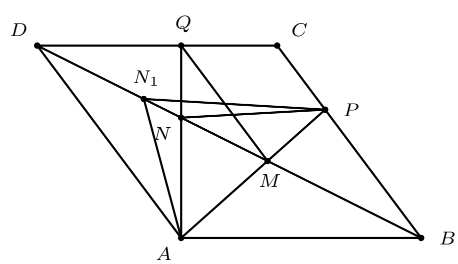


Figure 14

Proof. The key idea is that the statements $AN = NP$ and $AM = MQ$ are equivalent to $\measuredangle PAQ = \frac{1}{2}\measuredangle ABC$.

First, if $\measuredangle PAQ = \frac{1}{2}\measuredangle ABC$, then $\measuredangle NAP = \measuredangle NBP$, and the quadrilateral $ABPN$ is cyclic. As $\measuredangle ABN = \measuredangle PBN$, we have $AN = NP$.

For the converse, we consider N_1 on BD such that $\measuredangle PAN_1 = \frac{1}{2}\measuredangle BAD$. As above, we get $AN_1 = N_1P$. But $AN = NP$ so that BD must be the perpendicular bisector of the segment AP. This is absurd. $\square$

References

[1] W. G. Boskoff and B. Suceavă, A projectivity characterized by the Pythagorean relation, *Forum Geom.*, 6 (2006) 187–190.
[2] Z. G. Gotman, Problem M976, *Kvant*, 4 (1986) 31.
[3] A. B. Hodulev, Problem M851, *Kvant*, 3 (1984) 37.
[4] C. Mortici, Selection Tests of the Romanian Team participating at the 4th Junior Balkan Mathematical Olympiad. Problem 3, *Gazeta Matematică*, 7/8 (2000) 282–283.
[5] C. Mortici, Interesting properties of the secants at 45 degrees in the unit square, *Revista de Matematică din Timişoara*, 4 (2003) 3–7.
[6] V. Proizvolov, Problem M1767, *Kvant*, 2 (2001) 19.
[7] V. Proizvolov, Problem M1803, *Kvant*, 4 (2002) 18.

Cristinel Mortici: Valahia University of Târgovişte, Department of Mathematics, Bd. Unirii 18, 130082 Târgovişte, Romania
E-mail address: cmortici@valahia.ro

Forum Geometricorum
Volume 9 (2009) 109–112.

FORUM GEOM
ISSN 1534-1178

An Extension of Triangle Constructions from Located Points

Harold Connelly

Abstract. W. Wernick has tabulated 139 triangle construction problems using a list of sixteen points associated with the triangle. We add four points to his list and find an additional 140 construction problems.

William Wernick [3] and Leroy Meyers [2] discussed the problem of constructing a triangle with ruler and compass given the location of three points associated with the triangle. Wernick tabulated all the significantly distinct problems that could be formed from the following list of sixteen points:

$A,\ B,\ C$	Three vertices
$M_a,\ M_b,\ M_c$	Three midpoints of the sides
$H_a,\ H_b,\ H_c$	Three feet of the altitudes
$T_a,\ T_b,\ T_c$	Three feet of the internal angle bisectors
$G,\ H,\ I,\ O$	The centroid, orthocenter, incenter and circumcenter

Wernick found 139 triples that could be made from these points. They can be divided into the following four distinct types:

R – Redundant. Given the location of two of the points of the triple, the location of the third point is determined. An example would be: A, B, M_c.

L – Locus Restricted. Given the location of two points, the third must lie on a certain locus. Example: A, B, O.

S – Solvable. Known ruler and compass solutions exist for these triples.

U – Unsolvable. By using algebraic means, it is possible to prove that no ruler and compass solution exists for these triples. Example: O, H, I; see [1, 4].

To extend the work of Wernick and Meyers, we add the following four points to their list:

$E_a,\ E_b,\ E_c$	Three Euler points, which are the midpoints between the vertices and the orthocenter
N	The center of the nine-point circle.

Tabulated below, along with their types, are all of the 140 significantly distinct triples that can be formed by adding our new points to the original sixteen. Problems that remain unresolved as to type are left blank. In keeping with the spirit

Publication Date: May 11, 2009. Communicating Editor: Paul Yiu.

of Wernick's article, we have listed all of the possible combinations of points that are significantly distinct, even though many of them are easily converted, using redundancies, to problems in Wernick's list. We point out that although many of the problems are quite simple, a few provide a fine challenge. Our favorites include $A,\ E_b,\ G$ (Problem 17) and $E_a,\ E_b,\ O$ (Problem 50).

1. A, B, E_a	**S**	36. A, M_a, N	**S**	71. E_a, H, T_b	**U**	106. E_a, M_b, T_c	
2. A, B, E_c	**S**	37. A, M_b, N	**S**	72. E_a, H_a, H_b	**S**	107. E_a, N, O	**S**
3. A, B, N	**S**	38. A, N, O	**S**	73. E_a, H_a, I	**S**	108. E_a, N, T_a	
4. A, E_a, E_b	**S**	39. A, N, T_a		74. E_a, H_a, M_a	**L**	109. E_a, N, T_b	
5. A, E_a, G	**S**	40. A, N, T_b		75. E_a, H_a, M_b	**S**	110. E_a, O, T_a	
6. A, E_a, H	**R**	41. E_a, E_b, E_c	**S**	76. E_a, H_a, N	**L**	111. E_a, O, T_b	
7. A, E_a, H_a	**L**	42. E_a, E_b, G	**S**	77. E_a, H_a, O	**S**	112. E_a, T_a, T_b	
8. A, E_a, H_b	**L**	43. E_a, E_b, H	**S**	78. E_a, H_a, T_a	**L**	113. E_a, T_b, T_c	
9. A, E_a, I	**S**	44. E_a, E_b, H_a	**S**	79. E_a, H_a, T_b		114. G, H, N	**R**
10. A, E_a, M_a	**S**	45. E_a, E_b, H_c	**S**	80. E_a, H_b, H_c	**L**	115. G, H_a, N	**S**
11. A, E_a, M_b	**S**	46. E_a, E_b, I	**U**	81. E_a, H_b, I		116. G, I, N	**U**
12. A, E_a, N	**S**	47. E_a, E_b, M_a	**L**	82. E_a, H_b, M_a	**L**	117. G, M_a, N	**S**
13. A, E_a, O	**S**	48. E_a, E_b, M_c	**S**	83. E_a, H_b, M_b	**S**	118. G, N, O	**R**
14. A, E_a, T_a	**S**	49. E_a, E_b, N	**L**	84. E_a, H_b, M_c	**S**	119. G, N, T_a	**U**
15. A, E_a, T_b	**U**	50. E_a, E_b, O	**S**	85. E_a, H_b, N	**L**	120. H, H_a, N	**S**
16. A, E_b, E_c	**S**	51. E_a, E_b, T_a		86. E_a, H_b, O	**S**	121. H, I, N	**U**
17. A, E_b, G	**S**	52. E_a, E_b, T_c	**U**	87. E_a, H_b, T_a		122. H, M_a, N	**S**
18. A, E_b, H	**S**	53. E_a, G, H	**S**	88. E_a, H_b, T_b	**U**	123. H, N, O	**R**
19. A, E_b, H_a	**S**	54. E_a, G, H_a	**S**	89. E_a, H_b, T_c		124. H, N, T_a	**U**
20. A, E_b, H_b	**L**	55. E_a, G, H_b	**S**	90. E_a, I, M_a	**S**	125. H_a, H_b, N	**L**
21. A, E_b, H_c	**S**	56. E_a, G, I		91. E_a, I, M_b		126. H_a, I, N	**S**
22. A, E_b, I		57. E_a, G, M_a	**S**	92. E_a, I, N	**S**	127. H_a, M_a, N	**L**
23. A, E_b, M_a	**S**	58. E_a, G, M_b	**S**	93. E_a, I, O		128. H_a, M_b, N	**L**
24. A, E_b, M_b	**S**	59. E_a, G, N	**S**	94. E_a, I, T_a		129. H_a, N, O	**S**
25. A, E_b, M_c	**S**	60. E_a, G, O	**S**	95. E_a, I, T_b		130. H_a, N, T_a	
26. A, E_b, N	**S**	61. E_a, G, T_a		96. E_a, M_a, M_b	**L**	131. H_a, N, T_b	
27. A, E_b, O	**S**	62. E_a, G, T_b		97. E_a, M_a, N	**R**	132. I, M_a, N	**S**
28. A, E_b, T_a		63. E_a, H, H_a	**L**	98. E_a, M_a, O	**S**	133. I, N, O	**U**
29. A, E_b, T_b		64. E_a, H, H_b	**L**	99. E_a, M_a, T_a		134. I, N, T_a	
30. A, E_b, T_c		65. E_a, H, I	**S**	100. E_a, M_a, T_b		135. M_a, M_b, N	**L**
31. A, G, N	**S**	66. E_a, H, M_a	**S**	101. E_a, M_b, M_c	**S**	136. M_a, N, O	**S**
32. A, H, N	**S**	67. E_a, H, M_b	**S**	102. E_a, M_b, N	**L**	137. M_a, N, T_a	
33. A, H_a, N	**S**	68. E_a, H, N	**S**	103. E_a, M_b, O	**S**	138. M_a, N, T_b	
34. A, H_b, N	**S**	69. E_a, H, O	**S**	104. E_a, M_b, T_a		139. N, O, T_a	**U**
35. A, I, N		70. E_a, H, T_a	**S**	105. E_a, M_b, T_b		140. N, T_a, T_b	

Many of the problems in our list can readily be converted to one in Wernick's list. Here are those by the application of a redundancy.

Problem	5	7	8	9	10	11	13	14	15	31	32	38
Wernick	40	45	50	57	24	33	16	55	56	16	16	16

Problem	53	63	64	65	66	67	69	70	71			
Wernick	40	45	50	57	24	33	16	55	56			

Problem	115	116	117	119	120	121	122	124	129	133	136	139
Wernick	75	80	66	79	75	80	66	79	75	80	66	79

A few solutions follow.

Problem 41. Given points E_a, E_b, E_c.
Solution. The orthocenter of triangle $E_aE_bE_c$ is also the orthocenter, H, of triangle ABC. Since E_a is the midpoint of AH, A can be found. Similarly, B and C.

Problem 50. Given points E_a, E_b, O.
Solution. Let P and Q be the midpoints of E_aO and E_bO, respectively. Let R be the reflection of P through Q. The line through E_b, perpendicular to E_aE_b, intersects the circle with diameter OR at M_a. The circumcircle, with center O and radius E_aM_a, intersects M_aR at B and C. The line through E_a perpendicular to BC intersects the circumcircle at A. There are in general two solutions.

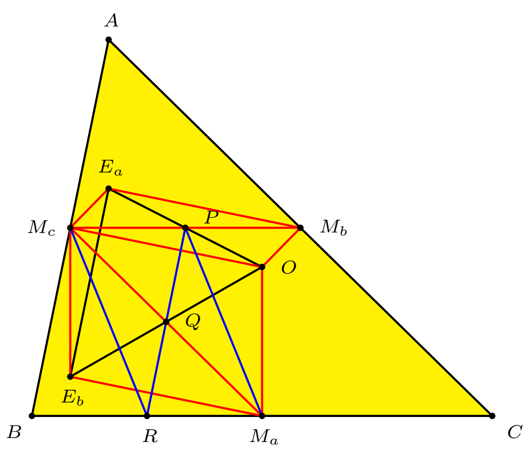


Figure 1.

Proof. In parallelogram $OM_aE_bM_c$, since diagonals bisect each other, Q is the midpoint of M_aM_c (see Figure 1). Similarly, P is the midpoint of M_bM_c. Since Q is also the midpoint of PR, PM_aRM_c is also a parallelogram and R must lie on BC. Therefore, the circle with diameter OR is a locus for M_a. Since M_aE_b is perpendicular to E_aE_b, the line through E_b perpendicular to PQ is a second locus for M_a. □

Problem 72. Given points E_a, H_a, H_b.
Solution. The line through Ha perpendicular to the line E_aH_a is the side BC. All three given points lie on the nine-point circle, so it can be found. The second intersection of the nine-point circle with BC gives M_a. The circle with M_a as

center and passing through H_b intersects the side BC at B and C. Finally, CH_b intersects E_aH_a at A.

Problem 103. Given points E_a, M_b, O.
Solution. The line through M_b, perpendicular to M_bO is AC. Reflecting AC through E_a, then dilating this line with O as center and ratio $\frac{1}{2}$ and finally intersecting this new line with the perpendicular bisector of E_aM_b gives N. Reflecting O through N gives H. E_aH intersects AC at A. The circumcircle, with center O passing through A, intersects AC again at C. The perpendicular from H to AC intersects the circumcircle at B.

References

[1] J. Stern, Euler' triangle determination problem, *Forum Geom.*, 7 (2007) 1–9.
[2] L. F. Meyers, Update on William Wernick's "Triangle constructions with three located points", *Math. Mag.*, 69 (1996) 46–49.
[3] W. Wernick, Triangle constructions with three located points, *Math. Mag.*, 55 (1982) 227–230.
[4] P. Yiu, Conic solution of Euler's triangle determination problem, *Journal for Geometry and Graphics*, 12 (2008) 75–80.

Harold Connelly: 102 Blowing Tree Drive, Georgetown, Kentucky 40324, USA
E-mail address: cherylandharold@roadrunner.com

Forum Geometricorum
Volume 9 (2009) 113–118.

FORUM GEOM
ISSN 1534-1178

Characterizations of a Tangential Quadrilateral

Nicuşor Minculete

Abstract. In this paper we will present several relations about the tangential quadrilaterals; among these, we have that the quadrilateral $ABCD$ is tangential if and only if the following equality

$$\frac{1}{d(O,AB)} + \frac{1}{d(O,CD)} = \frac{1}{d(O,BC)} + \frac{1}{d(O,DA)}$$

holds, where O is the point where the diagonals of convex quadrilateral $ABCD$ meet. This is equivalent to Wu's Theorem.

A tangential quadrilateral is a convex quadrilateral whose sides all tangent to a circle inscribed in the quadrilateral. [1] In a tangential quadrilateral, the four angle bisectors meet at the center of the inscribed circle. Conversely, a convex quadrilateral in which the four angle bisectors meet at a point must be tangential. A necessary and sufficient condition for a convex quadrilateral to be tangential is that its two pairs of opposite sides have equal sums (see [1, 2, 4]). In [5], Marius Iosifescu proved that a convex quadrilateral $ABCD$ is tangential if and only if

$$\tan\frac{x}{2} \cdot \tan\frac{z}{2} = \tan\frac{y}{2} \cdot \tan\frac{w}{2},$$

where x, y, z, w are the measures of the angles ABD, ADB, BDC, and DBC respectively (see Figure 1). In [3], Wu Wei Chao gave another characterization of tangential quadrilaterals. The two diagonals of any convex quadrilateral divide the quadrilateral into four triangles. Let r_1, r_2, r_3, r_4, in cyclic order, denote the radii of the circles inscribed in each of these triangles (see Figure 2). Wu found that the quadrilateral is tangential if and only if

$$\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}.$$

In this paper we find another characterization (Theorem 1 below) of tangential quadrilaterals. This new characterization is shown to be equivalent to Wu's condition and others (Proposition 2).

Consider a convex quadrilateral $ABCD$ with diagonals AC and BD intersecting at O. Denote the lengths of the sides AB, BC, CD, DA by a, b, c, d respectively.

Publication Date: May 26, 2009. Communicating Editor: Paul Yiu.
The author thanks an anonymous referee for his opinions leading to an improvement of a prior verions of this paper.

[1]Tangential quadrilateral are also known as circumscriptible quadrilaterals, see [2, p.135].

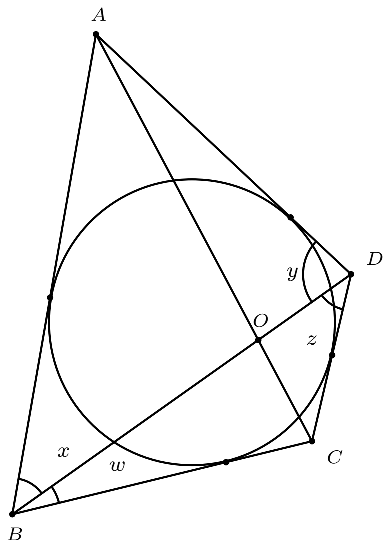


Figure 1

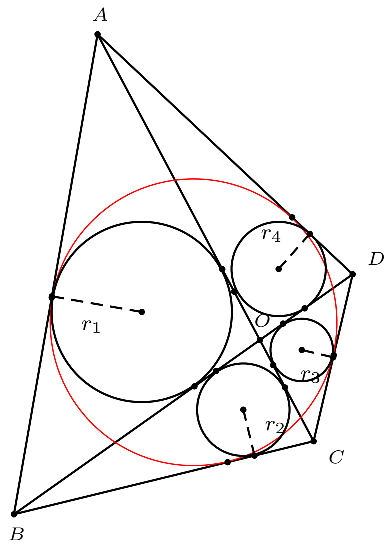


Figure 2

Theorem 1. *A convex quadrilateral $ABCD$ with diagonals intersecting at O is tangential if and only if*

$$\frac{1}{d(O,AB)}+\frac{1}{d(O,CD)}=\frac{1}{d(O,BC)}+\frac{1}{d(O,DA)}, \tag{1}$$

where $d(O,AB)$ is the distance from O to the line AB etc.

Proof. We first express (1) is an alternative form. Consider the projections M, N, P and Q of O on the sides AB, BC, CD, DA respectively.

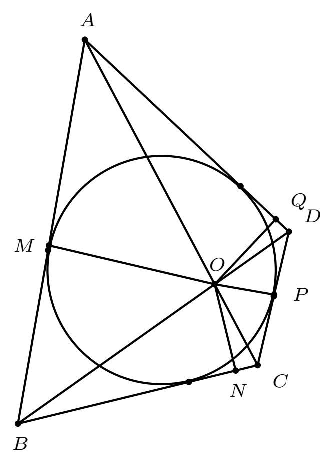


Figure 3

Let $AB = a$, $BC = b$, $CD = c$, $DA = d$. It is easy to see

$$\frac{OM}{d(C,AB)} = \frac{AO}{AC} = \frac{OQ}{d(C,AD)},$$
$$\frac{OM}{d(D,AB)} = \frac{BO}{BD} = \frac{ON}{d(D,BC)},$$
$$\frac{ON}{d(A,BC)} = \frac{OC}{AC} = \frac{OP}{d(A,DC)}.$$

This means

$$\frac{OM}{b\sin B} = \frac{OQ}{c\sin D}, \quad \frac{OM}{d\sin A} = \frac{ON}{c\sin C}, \quad \frac{ON}{a\sin B} = \frac{OP}{d\sin D}.$$

The relation (1) becomes

$$\frac{1}{OM} + \frac{1}{OP} = \frac{1}{ON} + \frac{1}{OQ},$$

which is equivalent to

$$1 + \frac{OM}{OP} = \frac{OM}{ON} + \frac{OM}{OQ},$$

or

$$1 + \frac{a\sin A\sin B}{c\sin C\sin D} = \frac{d\sin A}{c\sin C} + \frac{b\sin B}{c\sin D}.$$

Therefore (1) is equivalent to

$$a\sin A\sin B + c\sin C\sin D = b\sin B\sin C + d\sin D\sin A. \tag{2}$$

Now we show that $ABCD$ is tangential if and only if (2) holds.

($\Rightarrow$) If the quadrilateral $ABCD$ is tangential, then there is a circle inscribed in the quadrilateral. Let r be the radius of this circle. Then

$$a = r\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right), \qquad b = r\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right),$$
$$c = r\left(\cot\frac{C}{2} + \cot\frac{D}{2}\right), \qquad d = r\left(\cot\frac{D}{2} + \cot\frac{A}{2}\right).$$

Hence,

$$\begin{aligned}
a\sin A\sin B &= r\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\cdot 4\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\\
&= 4r\left(\cos\frac{A}{2}\sin\frac{B}{2}+\cos\frac{B}{2}\sin\frac{A}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\\
&= 4r\sin\frac{A+B}{2}\cos\frac{A}{2}\cos\frac{B}{2}\\
&= 4r\sin\frac{C+D}{2}\cos\frac{A}{2}\cos\frac{B}{2}\\
&= 4r\left(\cos\frac{D}{2}\sin\frac{C}{2}+\cos\frac{C}{2}\sin\frac{D}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\\
&= 4r\left(\tan\frac{C}{2}+\tan\frac{D}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{D}{2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
b\sin B\sin C &= 4r\left(\tan\frac{D}{2}+\tan\frac{A}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{D}{2},\\
c\sin C\sin D &= 4r\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{D}{2},\\
d\sin D\sin A &= 4r\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{D}{2}.
\end{aligned}$$

From these relations it is clear that (2) holds.

($\Leftarrow$) We assume (2) and $ABCD$ not tangential. From these we shall deduce a contradiction.

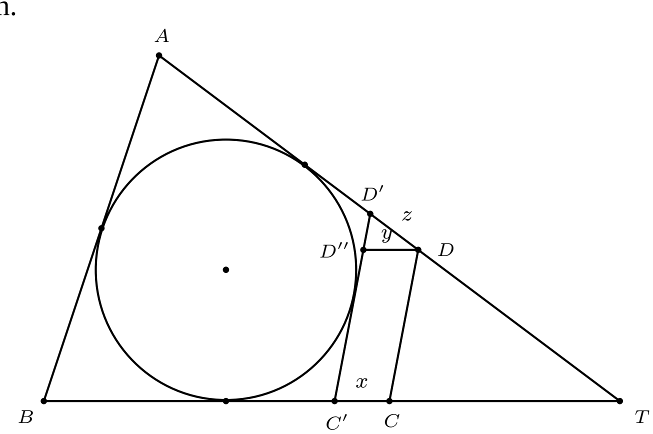


Figure 4.

Case 1. Suppose the opposite sides of $ABCD$ are not parallel.

Let T be the intersection of the lines AD and BC. Consider the incircle of triangle ABT (see Figure 4). Construct a parallel to the side DC which is tangent to the circle, meeting the sides BC and DA at C' and D' respectively. Let $BC' =$

b', $C'D' = c'$, $D'A = d'$, $C'C = x$, $D''D' = y$, and $D'D = z$, and where D'' is the point on $C'D'$ such that $C'CDD''$ is a parallelogram. Note that

$$b = b' + x, \quad c = c' - y, \quad d = d' + z.$$

Since the quadrilateral $ABC'D'$ is tangential, we have

$$a \sin A \sin B + c' \sin C \sin D = b' \sin B \sin C + d' \sin D \sin A. \tag{3}$$

Comparison of (2) and (3) gives

$$a \sin A \sin B + c \sin C \sin D = b \sin B \sin C + d \sin D \sin A,$$

we have

$$-y \sin C \sin D = x \sin B \sin C + z \sin D \sin A.$$

This is a contradiction since x, y, z all have the same sign, [2] and the trigonometric ratios are all positive.

Case 2. Now suppose $ABCD$ has a pair of parallel sides, say AD and BC. Consider the circle tangent to the sides AB, BC and DA (see Figure 5).

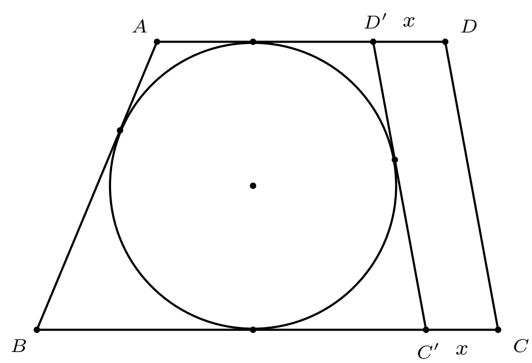


Figure 5.

Construct a parallel to DC, tangent to the circle, and intersecting BC, DA at C' and D' respectively. Let $C'C = D'D = x$, $BC' = b'$, and $D'A = d'$. [3] Clearly, $b' = b - x$, $d = d' + x$, and $C'D' = CD = c$. Since the quadrilateral $ABC'D'$ is tangential, we have

$$a \sin A \sin B + c \sin C \sin D = b' \sin B \sin C + d' \sin D \sin A. \tag{4}$$

Comparing this with (2), we have $x(\sin B \sin C + \sin D \sin A) = 0$. Since $x \neq 0$, $\sin A = \sin B$ and $\sin C = \sin D$, this reduces to $2 \sin A \sin C = 0$, a contradiction. □

Proposition 2. *Let O be the point where the diagonals of the convex quadrilateral $ABCD$ meet and r_1, r_2, r_3, and r_4 respectively the radii of the circles inscribed in the triangles AOB, BOC, COD and DOA respectively. The following statements are equivalent:*

[2]In Figure 4, the circle does not intersect the side CD. In case it does, we treat x, y, z as negative.

[3]Again, if the circle intersects CD, then x is regarded as negative.

(a) $\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}$.
(b) $\frac{1}{d(O,AB)} + \frac{1}{d(O,CD)} = \frac{1}{d(O,BC)} + \frac{1}{d(O,DA)}$.
(c) $\frac{a}{\Delta AOB} + \frac{c}{\Delta COD} = \frac{b}{\Delta BOC} + \frac{d}{\Delta DOA}$.
(d) $a \cdot \Delta COD + c \cdot \Delta AOB = b \cdot \Delta DOA + d \cdot \Delta BOC$.
(e) $a \cdot OC \cdot OD + c \cdot OA \cdot OB = b \cdot OA \cdot OD + d \cdot OB \cdot OC$.

Proof. (a) $\Leftrightarrow$ (b). The inradius of a triangle is related to the altitudes by the simple relation

$$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}.$$

Applying this to the four triangles AOB, BOC, COD, and DOA, we have

$$\begin{aligned}
\frac{1}{r_1} &= \frac{1}{d(O,AB)} + \frac{1}{d(A,BD)} + \frac{1}{d(B,AC)},\\
\frac{1}{r_2} &= \frac{1}{d(O,BC)} + \frac{1}{d(C,BD)} + \frac{1}{d(B,AC)},\\
\frac{1}{r_3} &= \frac{1}{d(O,CD)} + \frac{1}{d(C,BD)} + \frac{1}{d(D,AC)},\\
\frac{1}{r_4} &= \frac{1}{d(O,DA)} + \frac{1}{d(A,BD)} + \frac{1}{d(D,AC)}.
\end{aligned}$$

From these the equivalence of (a) and (b) is clear.

(b) $\Leftrightarrow$ (c) is clear from the fact that $\frac{1}{d(O,AB)} = \frac{a}{a \cdot d(O,AB)} = \frac{a}{2\Delta AOB}$ etc.

The equivalence of (c), (d) and (e) follows from follows from

$$\Delta AOB = \frac{1}{2} \cdot OA \cdot OB \cdot \sin\varphi$$

etc., where φ is the angle between the diagonals. Note that

$$\Delta AOB \cdot \Delta COD = \Delta BOC \cdot \Delta DOA = \frac{1}{4} \cdot OA \cdot OB \cdot OC \cdot OD \cdot \sin^2\varphi.$$

□

References

[1] T. Andreescu and B. Enescu, *Mathematical Olympiad Treasures*, Birkhäuser, Boston, 2004.
[2] N. Altshiller-Court, *College Geometry*, second edition, Barnes and Noble, 1952; Dover reprint, 2007.
[3] W. C. Wu and P. Simeonov, Problem 10698, *Amer. Math. Monthly*, 105 (1998) 995; solution, 107 (2000) 657–658.
[4] P. Yiu, Euclidean Geometry, Florida Atlantic University Lecture Notes, 1998, available at `http://math.fau.edu/Yiu/Geometry.html`.
[5] M. Iosifescu, Problem 1421, *The Mathematical Gazette*, (in Romanian) no.11, 1954.

Nicuşor Minculete: "Dimitrie Cantemir" University of Braşov, Department of REI, Str. Bisericii Române nr. 107, Braşov, Romania
E-mail address: `minculeten@yahoo.com`

Forum Geometricorum
Volume 9 (2009) 119–123.

FORUM GEOM
ISSN 1534-1178

A Note on the Anticomplements of the Fermat Points

Cosmin Pohoata

Abstract. We show that each of the anticomplements of the Fermat points is common to a triad of circles involving the triangle of reflection. We also generate two new triangle centers as common points to two other triads of circles. Finally, we present several circles passing through these new centers and the anticomplements of the Fermat points.

1. Introduction

The Fermat points $F_\pm$ are the common points of the lines joining the vertices of a triangle $\mathbf{T}$ to the apices of the equilateral triangles erected on the corresponding sides. They are also known as the isogonic centers (see [2, pp.107, 151]) and are among the basic triangle centers. In [4], they appear as the triangle centers X_{13} and X_{14}. Not much, however, is known about their anticomplements, which are the points $P_\pm$ which divide $F_\pm G$ in the ratio $F_\pm G : GP_\pm = 1 : 2$.

Given triangle $\mathbf{T}$ with vertices A, B, C,
(i) let A', B', C' be the reflections of the vertices A, B, C in the respective opposite sides, and
(ii) for $\varepsilon = \pm 1$, let A_ε, B_ε, C_ε be the apices of the equilateral triangles erected on the sides BC, CA, AB of triangle ABC respectively, on opposite or the same sides of the vertices according as $\varepsilon = 1$ or -1 (see Figures 1A and 1B).

Theorem 1. *For $\varepsilon = \pm 1$, the circumcircles of triangles $A'B_\varepsilon C_\varepsilon$, $B'C_\varepsilon A_\varepsilon$, $C'A_\varepsilon B_\varepsilon$ are concurrent at the anticomplement $P_{-\varepsilon}$ of the Fermat point $F_{-\varepsilon}$.*

2. Proof of Theorem 1

For $\varepsilon = \pm 1$, let $O_{a,\varepsilon}$ be the center of the equilateral triangle $A_\varepsilon BC$; similarly for $O_{b,\varepsilon}$ and $O_{c,\varepsilon}$.

(1) We first note that $O_{a,-\varepsilon}$ is the center of the circle through A', B_ε, and C_ε. Rotating triangle $O_{a,\varepsilon}AB$ through B by an angle $\varepsilon \cdot \frac{\pi}{3}$, we obtain triangle $O_{a,-\varepsilon}C_\varepsilon B$. Therefore, the triangles are congruent and $O_{a,-\varepsilon}C_\varepsilon = O_{a,\varepsilon}A$. Similarly, $O_{a,-\varepsilon}B_\varepsilon = O_{a,\varepsilon}A$. Clearly, $O_{a,\varepsilon}A = O_{a,-\varepsilon}A'$. It follows that $O_{a,-\varepsilon}$ is the center of the circle through A', B_ε and C_ε. Figures 1A and 1B illustrate the cases $\varepsilon = +1$ and $\varepsilon = -1$ respectively.

(2) Let $A_1B_1C_1$ be the anticomplementary triangle of ABC. Since AA_1 and A_+A_- have a common midpoint, $AA_+A_1A_-$ is a parallelogram. The lines $AA_{-\varepsilon}$ and A_1A_ε are parallel. Since A_1 is the anticomplement of A, it follows that the line A_1A_ε is the anticomplement of the line $AA_{-\varepsilon}$. Similarly, B_1B_ε and C_1C_ε are the anticomplements of the lines $BB_{-\varepsilon}$ and $CC_{-\varepsilon}$. Since $AA_{-\varepsilon}$, $BB_{-\varepsilon}$, $CC_{-\varepsilon}$

Publication Date: June 16, 2009. Communicating Editor: Paul Yiu.

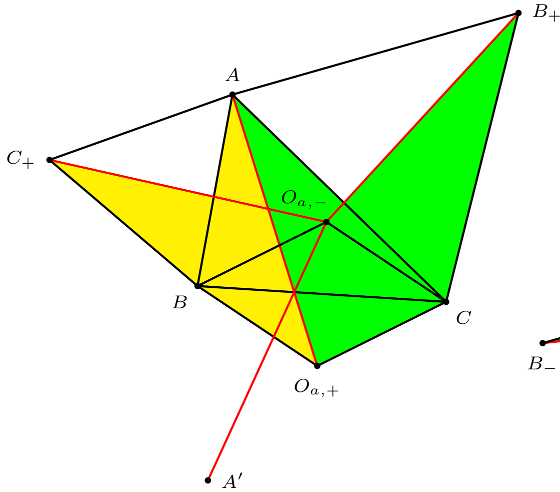


Figure 1A

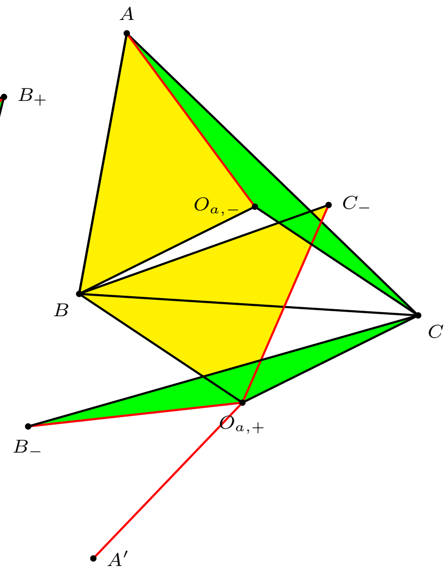


Figure 1B

concur at $F_{-\varepsilon}$, it follows that A_1A_ε, B_1B_ε, C_1C_ε concur at the anticomplement of $F_{-\varepsilon}$. This is the point $P_{-\varepsilon}$.

(3) $A_1B_1C_1$ is also the ε-Fermat triangle of $A_{-\varepsilon}B_{-\varepsilon}C_{-\varepsilon}$.

(i) Triangles $AB_\varepsilon C_\varepsilon$ and $CB_\varepsilon A_1$ are congruent, since $AB_\varepsilon = CB_\varepsilon$, $AC_\varepsilon = AB = CA_1$, and each of the angles $B_\varepsilon AC_\varepsilon$ and $B_\varepsilon CA_1$ is $\min\left(A + \frac{2\pi}{3},\ B + C + \frac{\pi}{3}\right)$. It follows that $B_\varepsilon C_\varepsilon = B_\varepsilon A_1$.

(ii) Triangles $AB_\varepsilon C_\varepsilon$ and BA_1C_ε are also congruent for the same reason, and we have $B_\varepsilon C_\varepsilon = A_1C_\varepsilon$.

It follows that triangle $A_1B_\varepsilon C_\varepsilon$ is equilateral, and $\angle C_\varepsilon A_1B_\varepsilon = \frac{\pi}{3}$.

(4) Because $P_{-\varepsilon}$ is the second Fermat point of $A_1B_1C_1$, we may assume $\angle C_\varepsilon P_{-\varepsilon}B_\varepsilon = \frac{\pi}{3}$. Therefore, $P_{-\varepsilon}$ lies on the circumcircle of $A_1B_\varepsilon C_\varepsilon$, which is the same as that of $A'B_\varepsilon C_\varepsilon$. On the other hand, since the quadrilateral $AA_+A_1A_-$ is a parallelogram (the diagonals AA_1 and A_-A_+ have a common midpoint D, the midpoint of segment BC), the anticomplement of the line AA_- coincides with A_1A_+. It now follows that the lines A_1A_+, B_1B_+, C_1C_+ are concurrent at the anticomplement P_- of the second Fermat points, and furthermore, $\angle C_+P_-B_+ = \frac{\pi}{3}$. Since

$$\begin{aligned}
\angle C_+AB_+ &= 2\pi - (\angle BAC_+ + \angle CAB + \angle B_+AC)\\
&= \frac{4\pi}{3} - \angle CAB\\
&= \frac{\pi}{3} + \angle ABC + \angle BCA\\
&= \angle ABC_+ + \angle ABC + \angle CBM\\
&= \angle C_+BM,
\end{aligned}$$

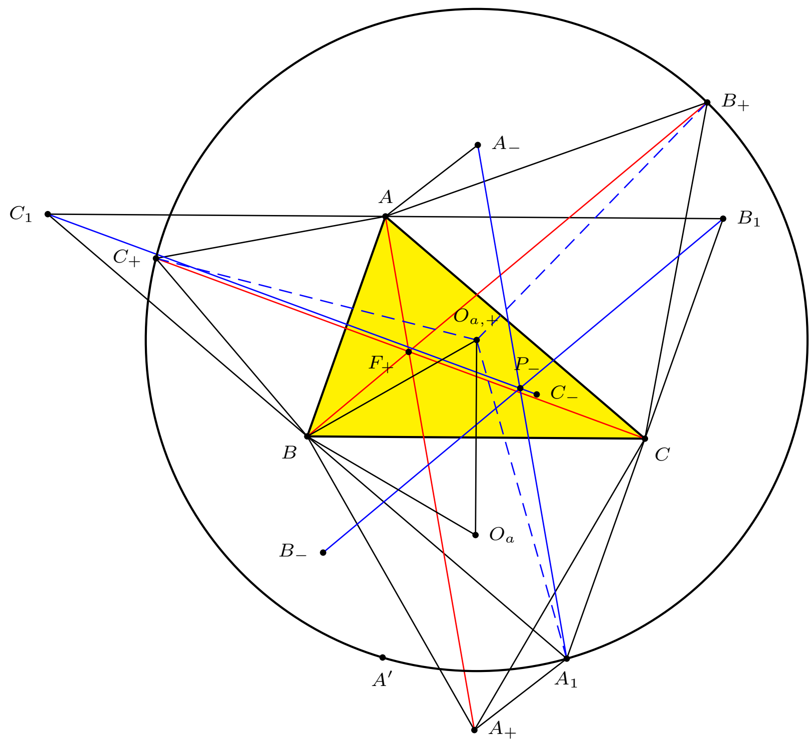


Figure 2

it follows that the triangles C_+AB_+ and C_+BM are congruent. Likewise, $\angle C_+AB_+ = \angle MCB_+$, and so the triangles C_+AB_+ and MCB_+ are congruent. Therefore, the triangle C_+MB_+ is equilateral, and thus $\angle C_+MB_+ = \frac{\pi}{3}$. Combining this with $\angle C_+P_-B_+ = 60°$, yields that the quadrilateral $MP_-B_+C_+$ is cyclic, and since $A'MB_+C_+$ is also cyclic, we conclude that the anticomplement P_- of the second Fermat point F_- lies on the circumcircle of triangle $A'B_+C_+$. Similarly, P_- lies on the circumcircles of triangles $B'C_+A_+$, and $C'A_+B_+$, respectively. This completes the proof of Theorem 1.

3. Two new triangle centers

By using the same method as in [6], we generate two other concurrent triads of circles.

Theorem 2. *For $\varepsilon = \pm 1$, the circumcircles of the triangles $A_\varepsilon B'C'$, $B_\varepsilon C'A'$, $C_\varepsilon A'B'$ are concurrent.*

Proof. Consider the inversion Ψ with respect to the anticomplement of the second Fermat point. According to Theorem 1, the images of the circumcircles of triangles $A'B_+C_+$, $B'C_+A_+$, $C'A_+B_+$ are three lines which bound a triangle $A'_+B'_+C'_+$, where A'_+, B'_+, C'_+ are the images of A_+, B_+, and C_+, respectively. Since the images A'', B'', C'' of A', B', C' under Ψ lie on the sidelines $B'_+C'_+$, $C'_+A'_+$, $A'_+B'_+$, respectively, by Miquel's theorem, we conclude that the circumcircles of triangles $A'_+B''C''$, $B'_+C''A''$, $C'_+A''B''$ are concurrent. Thus, the circumcircles of triangles $A_+B'C'$, $B_+C'A'$, $C_+A'B'$ are also concurrent (see Figure 3).

Similarly, inverting with respect to the anticomplement of the first Fermat point, by Miquel's theorem, one can deduce that the circumcircles of triangles $A_-B'C'$, $B_-C'A'$, $C_-A'B'$ are concurrent. □

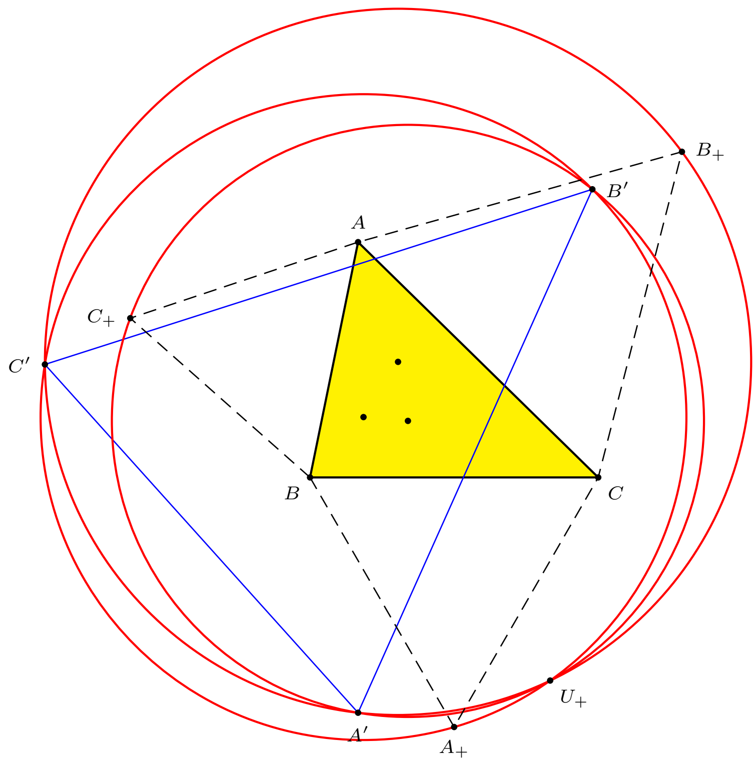


Figure 3.

Javier Francisco Garcia Capitan has kindly communicated that their points of concurrency do not appear in [4]. We will further denote these points by U_+, and U_-, respectively. We name these centers U_+, U_- the *inversive associates* of the anticomplements P_+, P_- of the Fermat points.

4. Circles around $P_\pm$ and their inversive associates

We denote by O, H the circumcenter, and orthocenter of triangle ABC. Let J_+, J_- be respectively the inner and outer isodynamic points of the triangle. Though the last two are known in literature as the common two points of the Apollonius circles, L. Evans [1] gives a direct relation between them and the Napoleonic configuration, defining them as the perspectors of the triangle of reflections $A'B'C'$

with each of the Fermat triangles $A_+B_+C_+$, and $A_-B_-C_-$. They appear as X_{15}, X_{16} in [4].

Furthermore, let W_+, W_- be the Wernau points of triangle ABC. These points are known as the common points of the folloing triads of circles: AB_+C_+, BC_+A_+, CA_+B_+, and respectively AB_-C_-, BC_-A_-, CA_-B_- [3]. According to the above terminology, W_+, W_- are the inversive associates of the Fermat points F_+, and F_-. They appear as X_{1337} and X_{1338} in [4].

We conclude with a list of concyclic quadruples involving these triangle centers. The first one is an immediate consequence of the famous Lester circle theorem [5]. The other results have been verified with the aid of Mathematica.

Theorem 3. *The following quadruples of points are concyclic:*
(i) P_+, P_-, O, H;
(ii) P_+, P_-, F_+, J_+;
(ii′) P_+, P_-, F_-, J_-;
(iii) P_+, U_+, F_+, O;
(iii′) P_-, U_-, F_-, O;
(iv) P_+, U_-, F_+, W_+;
(iv′) P_-, U_+, F_-, W_-;
(v) U_+, J_+, W_+, W_-;
(v′) U_-, J_-, W_+, W_-.

References

[1] L. Evans, Some configurations of triangle centers, *Forum Geom.*, 3 (2003) 49–56.
[2] W. Gallatly, *The Modern Geometry of the Triangle*, 2nd ed., London: Hodgson, 1913.
[3] D. Grinberg, Hyacinthos messages 6874, 6881, 6882, April, 2003.
[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.
[5] J. Lester, Triangles III: Complex Triangle Functions, *Aequationes Math.*, 53 (1997) 4–35.
[6] C. Pohoata, On the Parry reflection point, *Forum Geom.*, 8 (2008) 43–48.

Cosmin Pohoata: 13 Pridvorului Street, Bucharest, Romania 010014
E-mail address: `pohoata_cosmin2000@yahoo.com`

Forum Geometricorum
Volume 9 (2009) 125–148.

FORUM GEOM
ISSN 1534-1178

Heptagonal Triangles and Their Companions

Paul Yiu

Abstract. A heptagonal triangle is a non-isosceles triangle formed by three vertices of a regular heptagon. Its angles are $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$. As such, there is a unique choice of a companion heptagonal triangle formed by three of the remaining four vertices. Given a heptagonal triangle, we display a number of interesting companion pairs of heptagonal triangles on its nine-point circle and Brocard circle. Among other results on the geometry of the heptagonal triangle, we prove that the circumcenter and the Fermat points of a heptagonal triangle form an equilateral triangle. The proof is an interesting application of Lester's theorem that the Fermat points, the circumcenter and the nine-point center of a triangle are concyclic.

1. The heptagonal triangle $\mathbf{T}$ and its companion

A heptagonal triangle $\mathbf{T}$ is one with angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$. Its vertices are three vertices of a regular heptagon inscribed in its circumcircle. Among the remaining four vertices of the heptagon, there is a unique choice of three which form another (congruent) heptagonal triangle $\mathbf{T}'$. We call this the companion of $\mathbf{T}$, and the seventh vertex of the regular heptagon the residual vertex of $\mathbf{T}$ and $\mathbf{T}'$ (see Figure 1). In this paper we work with complex number coordinates, and take the unit circle

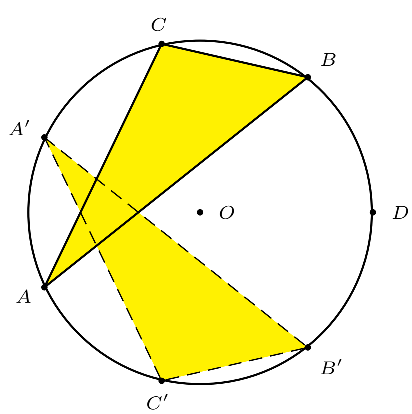


Figure 1. A heptagonal triangle and its companion

in the complex plane for the circumcircle of $\mathbf{T}$. By putting the residual vertex D at 1, we label the vertices of $\mathbf{T}$ by

$$A = \zeta^4, \qquad B = \zeta, \qquad C = \zeta^2,$$

Publication Date: June 22, 2009. Communicating Editor: Antreas P. Hatzipolakis.

The author thanks the Editor for suggesting the configuration studied in §8, leading to, among other results, Theorem 20 on six circles concurring at the Feuerbach point of the heptagonal triangle.

and those of $\mathbf{T}'$ by

$$A' = \zeta^3, \qquad B' = \zeta^6, \qquad C' = \zeta^5,$$

where $\zeta := \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ is a primitive 7-th root of unity.

We study the triangle geometry of $\mathbf{T}$, some common triangle centers, lines, circles and conics associated with it. We show that the Simson lines of A', B', C' with respect to $\mathbf{T}$ are concurrent (Theorem 4). We find a number of interesting companion pairs of heptagonal triangles associated with $\mathbf{T}$. For example, the medial triangle and the orthic triangle of $\mathbf{T}$ form such a pair on the nine-point circle (Theorem 5), and the residual vertex is a point on the circumcircle of $\mathbf{T}$. It is indeed the Euler reflection point of $\mathbf{T}$. In the final section we prove that the circumcenter and the Fermat points form an equilateral triangle (Theorem 22). The present paper can be regarded as a continuation of Bankoff-Garfunkel [1].

2. Preliminaries

2.1. *Some simple coordinates.* Clearly, the circumcenter O of $\mathbf{T}$ has coordinate 0, and the centroid is the point $G = \frac{1}{3}(\zeta + \zeta^2 + \zeta^4)$. Since the orthocenter H and the nine-point center N are points (on the Euler line) satisfying

$$OG : GN : NH = 2 : 1 : 3,$$

we have

$$\begin{aligned} H &= \zeta + \zeta^2 + \zeta^4, \\ N &= \frac{1}{2}(\zeta + \zeta^2 + \zeta^4). \end{aligned} \tag{1}$$

This reasoning applies to any triangle with vertices on the unit circle. The bisectors of angles A, B, C of $\mathbf{T}$ intersect the circumcircle at $-C'$, A', B' respectively. These form a triangle whose orthocenter is the incenter I of $\mathbf{T}$ (see Figure 2). This latter is therefore the point

$$I = \zeta^3 - \zeta^5 + \zeta^6. \tag{2}$$

Similarly, the external bisectors of angles A, B, C intersect the circumcircle at C', $-A'$, $-B'$ respectively. Identifying the excenters of $\mathbf{T}$ as orthocenters of triangles with vertices on the unit circle, we have

$$\begin{aligned} I_a &= -(\zeta^3 + \zeta^5 + \zeta^6), \\ I_b &= \zeta^3 + \zeta^5 - \zeta^6, \\ I_c &= -\zeta^3 + \zeta^5 + \zeta^6. \end{aligned} \tag{3}$$

Figure 2 shows the tritangent circles of the heptagonal triangle $\mathbf{T}$.

2.2. *Representation of a companion pair*. Making use of the simple fact that the complex number coordinates of vertices of a regular heptagon can be obtained from any one of them by multiplications by $\zeta, \ldots, \zeta^6$, we shall display a companion pair of heptagonal triangle by listing coordinates of the center, the residual vertex and the vertices of the two heptagonal triangles, as follows.

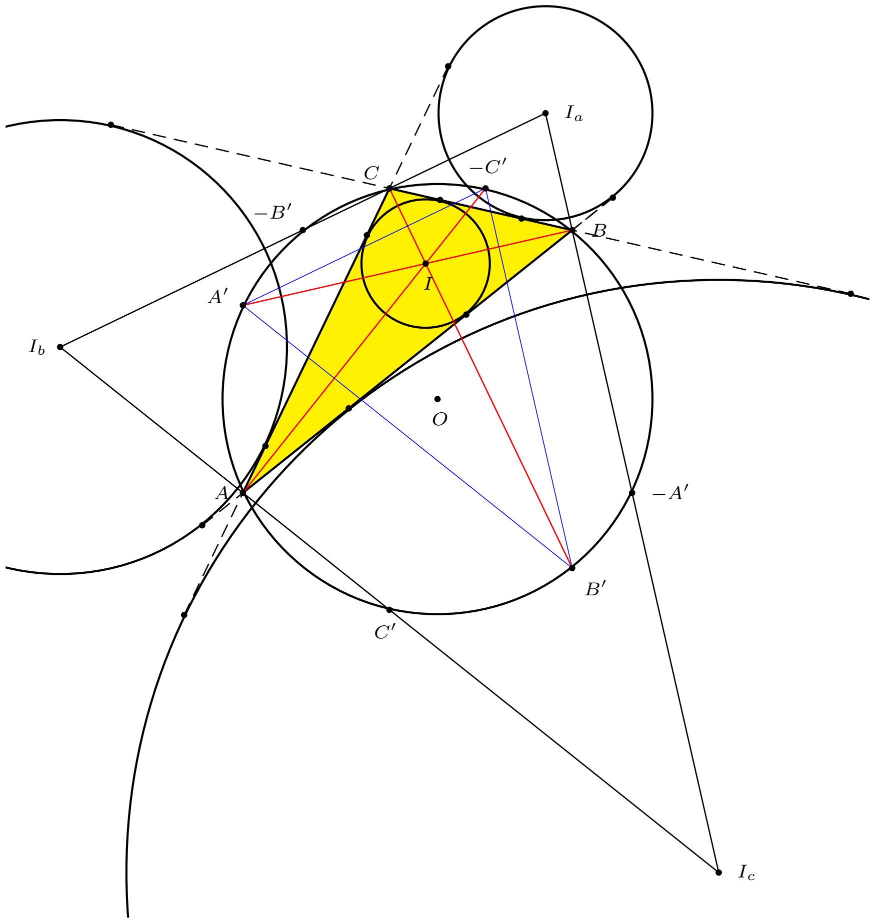


Figure 2. The tritangent centers

Center:	P
Residual vertex:	Q

Rotation	Vertices	Rotation	Vertices
ζ^4	$P+\zeta^4(Q-P)$	ζ^3	$P+\zeta^3(Q-P)$
ζ	$P+\zeta(Q-P)$	ζ^6	$P+\zeta^6(Q-P)$
ζ^2	$P+\zeta^2(Q-P)$	ζ^5	$P+\zeta^5(Q-P)$

2.3. While we shall mostly work in the cyclotomic field $\mathbb{Q}(\zeta)$, [1] the complex number coordinates of points we consider in this paper are *real* linear combinations of ζ^k for $0 \leq k \leq 6$, (the vertices of the regular heptagon on the circumcircle of

[1]See Corollary 23 for an exception.

$\mathbf{T}$). The real coefficients involved are rational combinations of

$$c_1 = \frac{\zeta + \zeta^6}{2} = \cos\frac{2\pi}{7}, \quad c_2 = \frac{\zeta^2 + \zeta^5}{2} = \cos\frac{4\pi}{7}, \quad c_3 = \frac{\zeta^3 + \zeta^4}{2} = \cos\frac{6\pi}{7}.$$

Note that $c_1 > 0$ and $c_2, c_3 < 0$. An expression of a complex number z as a real linear combination of ζ^4, ζ, ζ^2 (with sum of coefficients equal to 1) actually gives the absolute barycentric coordinate of the point z with reference to the heptagonal triangle $\mathbf{T}$. For example,

$$\begin{array}{rcrcrcr}
\zeta^3 & = & -2c_2 \cdot \zeta^4 & + & 2c_2 \cdot \zeta & + & 1 \cdot \zeta^2, \\
\zeta^5 & = & 2c_1 \cdot \zeta^4 & + & 1 \cdot \zeta & - & 2c_1 \cdot \zeta^2, \\
\zeta^6 & = & 1 \cdot \zeta^4 & - & 2c_3 \cdot \zeta & + & 2c_3 \cdot \zeta^2, \\
1 & = & -2c_2 \cdot \zeta^4 & - & 2c_3 \cdot \zeta & - & 2c_1 \cdot \zeta^2.
\end{array}$$

We shall make frequent uses of the important result.

Lemma 1 (Gauss). $1 + 2(\zeta + \zeta^2 + \zeta^4) = \sqrt{7}i$.

Proof. Although this can be directly verified, it is actually a special case of Gauss' famous theorem that if $\zeta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ for an odd integer n, then

$$\sum_{k=0}^{n-1} \zeta^{k^2} = \begin{cases} \sqrt{n} & \text{if } n \equiv 1 \pmod 4, \\ \sqrt{n}i & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

For a proof, see [2, pp.75–76]. $\square$

2.4. *Reflections and pedals.*

Lemma 2. *If α, β, γ are unit complex numbers, the reflection of γ in the line joining α and β is $\gamma' = \alpha + \beta - \alpha\beta\overline{\gamma}$.*

Proof. As points in the complex plane, γ' has equal distances from α and β as γ does. This is clear from

$$\begin{aligned} \gamma' - \alpha &= \beta(1 - \alpha\overline{\gamma}) = \beta\overline{\gamma}(\gamma - \alpha), \\ \gamma' - \beta &= \alpha(1 - \beta\overline{\gamma}) = \alpha\overline{\gamma}(\gamma - \beta). \end{aligned}$$

$\square$

Corollary 3. (1) *The reflection of ζ^k in the line joining ζ^i and ζ^j is $\zeta^i + \zeta^j - \zeta^{i+j-k}$.*
(2) *The pedal* (orthogonal projection) *of ζ^k on the line joining ζ^i and ζ^j is*

$$\frac{1}{2}(\zeta^i + \zeta^j + \zeta^k - \zeta^{i+j-k}).$$

(3) *The reflections of A in BC, B in CA, and C in AB are the points*

$$\begin{aligned} A^* &= \zeta + \zeta^2 - \zeta^6, \\ B^* &= \zeta^2 + \zeta^4 - \zeta^5, \\ C^* &= \zeta - \zeta^3 + \zeta^4. \end{aligned} \tag{4}$$

3. Concurrent Simson lines

The Simson line of a point on the circumcircle of a triangle is the line containing the pedals of the point on the sidelines of the triangle.

Theorem 4. *The Simson lines of A', B', C' with respect to the heptagonal triangle $\mathbf{T}$ are concurrent.*

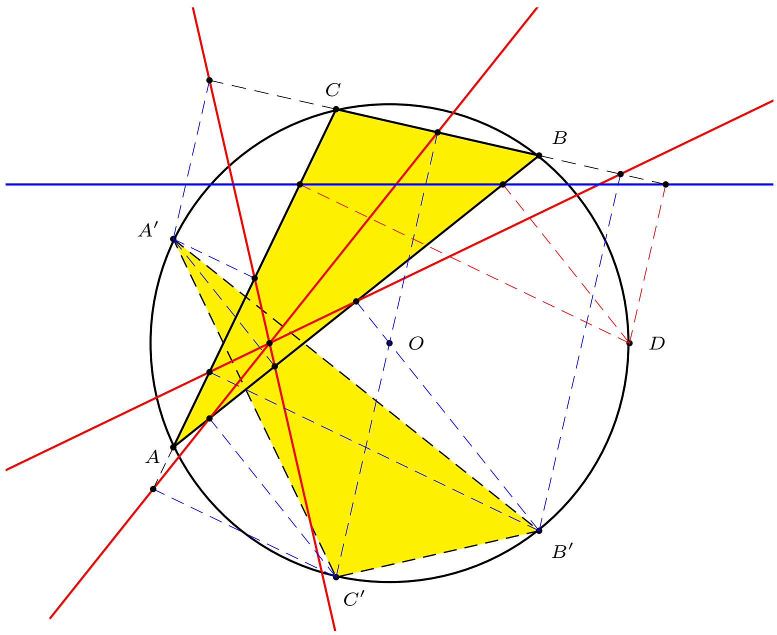


Figure 3. Simson lines

Proof. The pedals of A on BC is the midpoint A' of AA^*; similarly for those of B on CA and C on AB. We tabulate the coordinates of the pedals of A', B', C' on the sidelines BC, CA, AB respectively. These are easily calculated using Corollary 3.

	BC	CA	AB
A'	$\frac{1}{2}(-1+\zeta+\zeta^2+\zeta^3)$	$\frac{1}{2}(\zeta^2+\zeta^4)$	$\frac{1}{2}(\zeta-\zeta^2+\zeta^3+\zeta^4)$
B'	$\frac{1}{2}(\zeta+\zeta^2-\zeta^4+\zeta^6)$	$\frac{1}{2}(-1+\zeta^2+\zeta^4+\zeta^6)$	$\frac{1}{2}(\zeta+\zeta^4)$
C'	$\frac{1}{2}(\zeta+\zeta^2)$	$\frac{1}{2}(-\zeta+\zeta^2+\zeta^4+\zeta^5)$	$\frac{1}{2}(-1+\zeta+\zeta^4+\zeta^5)$

We check that the Simson lines of A', B', C' all contain the point $-\frac{1}{2}$. For these, it is enough to show that the complex numbers

$$(\zeta+\zeta^2+\zeta^3)(\overline{1+\zeta^2+\zeta^4}), \quad (\zeta^2+\zeta^4+\zeta^6)(\overline{1+\zeta+\zeta^4}), \quad (\zeta+\zeta^4+\zeta^5)(\overline{1+\zeta+\zeta^2})$$

are real. These are indeed $\zeta+\zeta^6$, $\zeta^2+\zeta^5$, $\zeta^3+\zeta^4$ respectively. $\square$

Remark. The Simson line of D, on the other hand, is parallel to OD (see Figure 3). This is because the complex number coordinates of the pedals of D, namely,

$$\frac{1+\zeta+\zeta^2-\zeta^3}{2}, \quad \frac{1+\zeta^2+\zeta^4-\zeta^6}{2}, \quad \frac{1+\zeta+\zeta^4-\zeta^5}{2},$$

all have the same imaginary part $\frac{1}{4}(\zeta-\zeta^6+\zeta^2-\zeta^5-\zeta^3+\zeta^4)$.

4. The nine-point circle

4.1. *A companion pair of heptagonal triangles on the nine-point circle.* As is well known, the nine-point circle is the circle through the vertices of the medial triangle and of the orthic triangle. The medial triangle of $\mathbf{T}$ clearly is heptagonal. It is known that $\mathbf{T}$ is the only obtuse triangle with orthic triangle similar to itself. [2] The medial and orthic triangles of $\mathbf{T}$ are therefore congruent. It turns out that they are companions.

Theorem 5. *The medial triangle and the orthic triangle of* $\mathbf{T}$ *are companion heptagonal triangles on the nine-point circle of* $\mathbf{T}$. *The residual vertex is the Euler reflection point* E (on the circumcircle of $\mathbf{T}$).

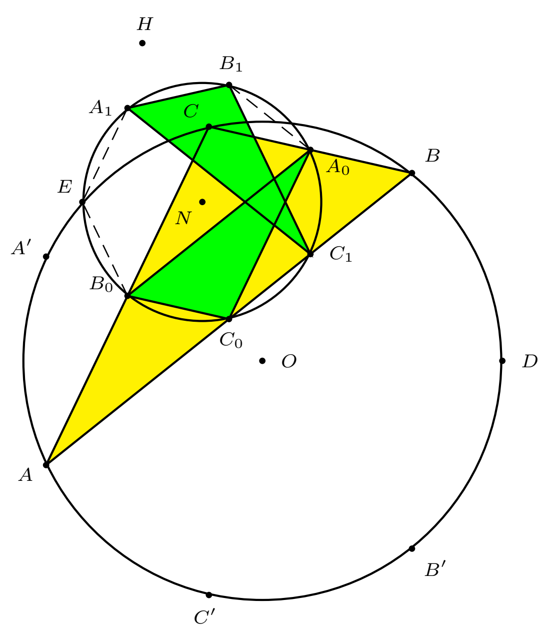


Figure 4. A companion pair on the nine-point circle

[2]If the angles of an obtuse angled triangle are $\alpha \leq \beta < \gamma$, those of its orthic triangle are 2α, 2β, and $2\gamma-\pi$. The two triangles are similar if and only if $\alpha = 2\gamma-\pi$, $\beta = 2\alpha$ and $\gamma = 2\beta$. From these, $\alpha = \frac{\pi}{7}$, $\beta = \frac{2\pi}{7}$ and $\gamma = \frac{4\pi}{7}$. This shows that the triangle is heptagonal. The equilateral triangle is the only acute angled triangle similar to its own orthic triangle.

Proof. (1) The companionship of the medial and orthic triangles on the nine-point circle is clear from the table below.

Center:	$N = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4)$
Residual vertex:	$E = \frac{1}{2}(-1 + \zeta + \zeta^2 + \zeta^4)$

Rotation	Medial triangle	Rotation	Orthic triangle
ζ^4	$A_0 = \frac{1}{2}(\zeta + \zeta^2)$	ζ^3	$C_1 = \frac{1}{2}(\zeta + \zeta^2 - \zeta^3 + \zeta^4)$
ζ	$B_0 = \frac{1}{2}(\zeta^2 + \zeta^4)$	ζ^6	$A_1 = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4 - \zeta^6)$
ζ^2	$C_0 = \frac{1}{2}(\zeta + \zeta^4)$	ζ^5	$B_1 = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4 - \zeta^5)$

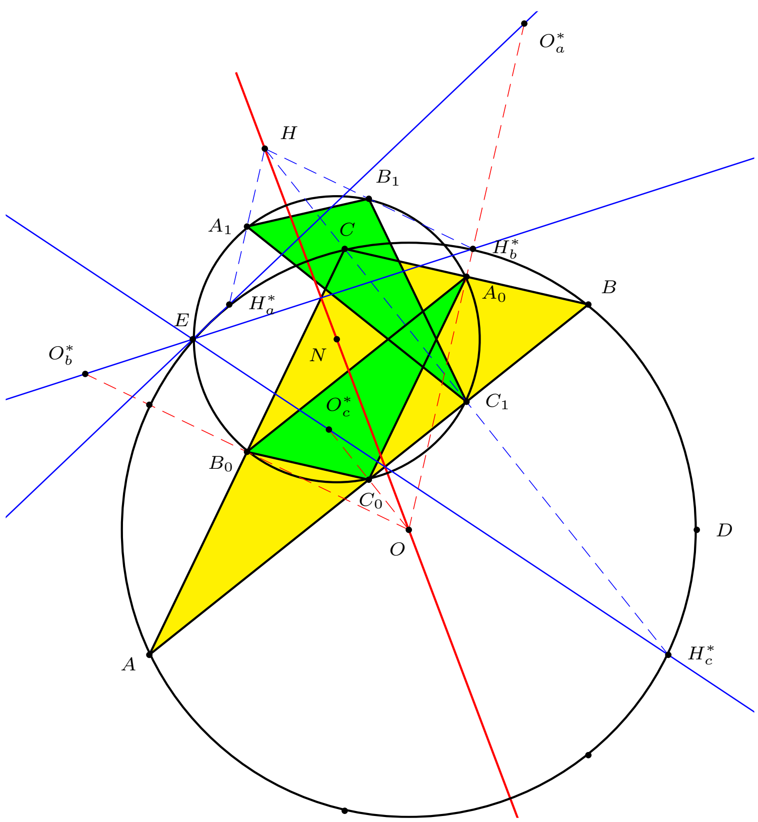


Figure 5. The Euler reflection point of $\mathbf{T}$

(2) We show that E is a point on the reflection of the Euler line in each of the sidelines of $\mathbf{T}$. In the table below, the reflections of O are computed from the simple fact that OBO_a^*C, OCO_b^*A, OAO_c^*B are rhombi. On the other hand, the reflections of H in the sidelines can be determined from the fact that HH_a^* and AA^* have the same midpoint, so do HH_b^* and BB^*, HH_c^* and CC^*. The various expressions for E given in the rightmost column can be routinely verified.

Line	Reflection of O	Reflection of H	$E=$
BC	$O_a^* = \zeta + \zeta^2$	$H_a^* = -\zeta^6$	$(-2c_1 - c_2 - c_3)O_a^* + (-c_2 - c_3)H_a^*$
CA	$O_b^* = \zeta^2 + \zeta^4$	$H_b^* = -\zeta^5$	$(-c_1 - 2c_2 - c_3)O_b^* + (-c_1 - c_3)H_b^*$
AB	$O_c^* = \zeta + \zeta^4$	$H_c^* = -\zeta^3$	$(-c_1 - c_2 - 2c_3)O_c^* + (-c_1 - c_2)H_c^*$

Thus, E, being the common point of the reflections of the Euler line of $\mathbf{T}$ in its sidelines, is the Euler reflection point of $\mathbf{T}$, and lies on the circumcircle of $\mathbf{T}$. $\square$

4.2. *The second intersection of the nine-point circle and the circumcircle.*

Lemma 6. *The distance between the nine-point center N and the A-excenter I_a is equal to the circumradius of the heptagonal triangle $\mathbf{T}$.*

Proof. Note that $I_a - N = \frac{2+\zeta+\zeta^2+\zeta^4}{2} = \frac{3+1+2(\zeta+\zeta^2+\zeta^4)}{4} = \frac{3+\sqrt{7}i}{4}$ is a unit complex number. $\square$

This simple result has a number of interesting consequences.

Proposition 7. (1) *The midpoint F_a of NI_a is the point of tangency of the nine-point circle and the A-excircle.*
(2) *The A-excircle is congruent to the nine-point circle.*
(3) *F_a lies on the circumcircle.*

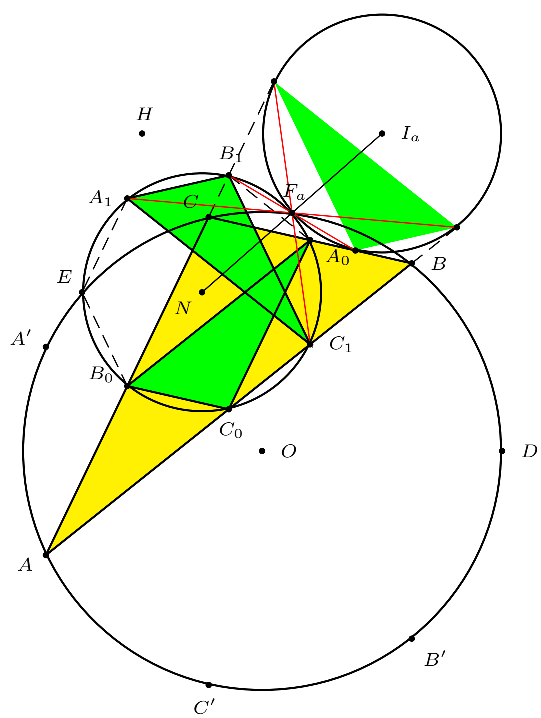


Figure 6. The A-Feuerbach point of $\mathbf{T}$

Proof. (1) By the Feuerbach theorem, the nine-point circle is tangent externally to each of the excircles. Since $NI_a = R$, the circumradius, and the nine-point circle has radius $\frac{1}{2}R$, the point of tangency with the A-excircle is the midpoint of NI_a, *i.e.*,

$$F_a = \frac{I_a + N}{2} = \frac{2 + 3(\zeta + \zeta^2 + \zeta^4)}{4}. \tag{5}$$

This proves (1).

(2) It also follows that the radius of the A-excircle is $\frac{1}{2}R$, and the A-excircle is congruent to the nine-point circle.

(3) Note that $F_a = \frac{1+3+6(\zeta+\zeta^2+\zeta^4)}{8} = \frac{1+3\sqrt{7}i}{8}$ is a unit complex number. □

Remark. The reflection of the orthic triangle in F_a is the A-extouch triangle, since the points of tangency are

$$-(\zeta^3 + \zeta^5 + \zeta^6) + \frac{\zeta^3}{2}, \qquad -(\zeta^3 + \zeta^5 + \zeta^6) + \frac{\zeta^5}{2}, \qquad -(\zeta^3 + \zeta^5 + \zeta^6) + \frac{\zeta^6}{2}$$

(see Figure 6).

4.3. *Another companion pair on the nine-point circle.*

Center:	$N = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4)$
Residual vertex:	$F_a = \frac{1}{4}(2 + 3(\zeta + \zeta^2 + \zeta^4))$

Rot.	Feuerbach triangle	Rot.	Companion
ζ^3	$F_b = \frac{1}{4}(\zeta + \zeta^2 + \zeta^3 + 2\zeta^4 - \zeta^6)$	ζ^4	$F'_a = \frac{1}{4}(3\zeta + 2\zeta^2 + 4\zeta^4 + \zeta^5 + \zeta^6)$
ζ^6	$F_\mathrm{e} = \frac{1}{4}(2\zeta + \zeta^2 + \zeta^4 - \zeta^5 + \zeta^6)$	ζ	$F'_b = \frac{1}{4}(4\zeta + 3\zeta^2 + \zeta^3 + 2\zeta^4 + \zeta^5)$
ζ^5	$F_c = \frac{1}{4}(\zeta + 2\zeta^2 - \zeta^3 + \zeta^4 + \zeta^5)$	ζ^2	$F'_c = \frac{1}{4}(2\zeta + 4\zeta^2 + \zeta^3 + 3\zeta^4 + \zeta^6)$

Proposition 8. *F_e, F_a, F_b, F_c are the points of tangency of the nine-point circle with the incircle and the A-, B-, C-excircles respectively* (see Figure 7).

Proof. We have already seen that $F_a = \frac{1}{2} \cdot N + \frac{1}{2} \cdot I_a$. It is enough to show that the points F_e, F_b, F_c lie on the lines NI, NI_b, NI_c respectively:

$$\begin{aligned} F_\mathrm{e} &= -(c_1 - c_3) \cdot N + (-c_1 - 2c_2 - 3c_3) \cdot I, \\ F_b &= (c_2 - c_3) \cdot N + (-2c_1 - 3c_2 - c_3) \cdot I_b, \\ F_c &= (c_1 - c_2) \cdot N + (-3c_1 - c_2 - 2c_3) \cdot I_c. \end{aligned}$$

□

Proposition 9. *The vertices F'_a, F'_b, F'_c of the companion of $F_bF_\mathrm{e}F_c$ are the second intersections of the nine-point circle with the lines joining F_a to A, B, C respectively.*

Proof.

$$\begin{aligned} F'_a &= -2c_2 \cdot F_a - 2(c_1 + c_3)A, \\ F'_b &= -2c_3 \cdot F_a - 2(c_1 + c_2)B, \\ F'_c &= -2c_1 \cdot F_a - 2(c_2 + c_3)C. \end{aligned}$$

□

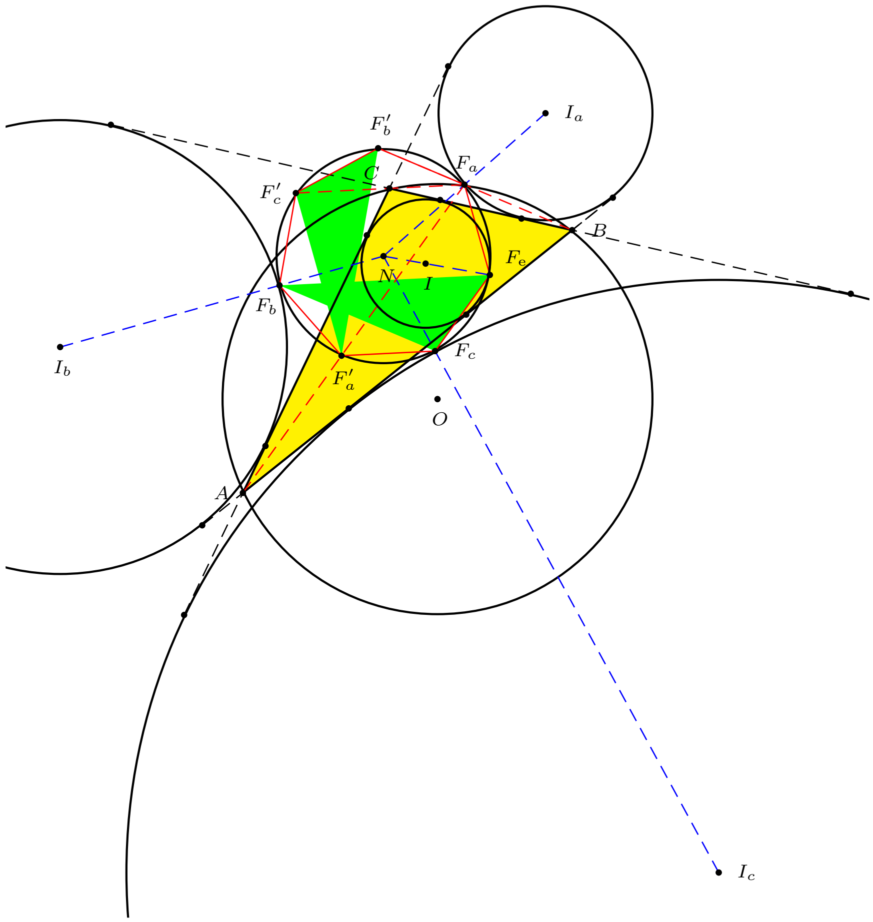


Figure 7. Another companion pair on the nine-point circle

5. The residual vertex as a Kiepert perspector

Theorem 10. *D is a Kiepert perspector of the heptagonal triangle ABC.*

Proof. What this means is that there are similar isosceles triangles $A''BC$, $B''CA$, $C''AB$ with the same orientation such that the lines AA'', BB'', CC'' all pass through the point D. Let A'' be the intersection of the lines AD and $A'B'$, B'' that of BD and $B'C'$, and C'' that of CD and $C'A'$ (see Figure 8). Note that $AC'B'A''$, $BAC'B''$, and $A'B'CC''$ are all parallelograms. From these,

$$A'' = \zeta^4 - \zeta^5 + \zeta^6,$$
$$B'' = \zeta - \zeta^3 + \zeta^5,$$
$$C'' = \zeta^2 + \zeta^3 - \zeta^6.$$

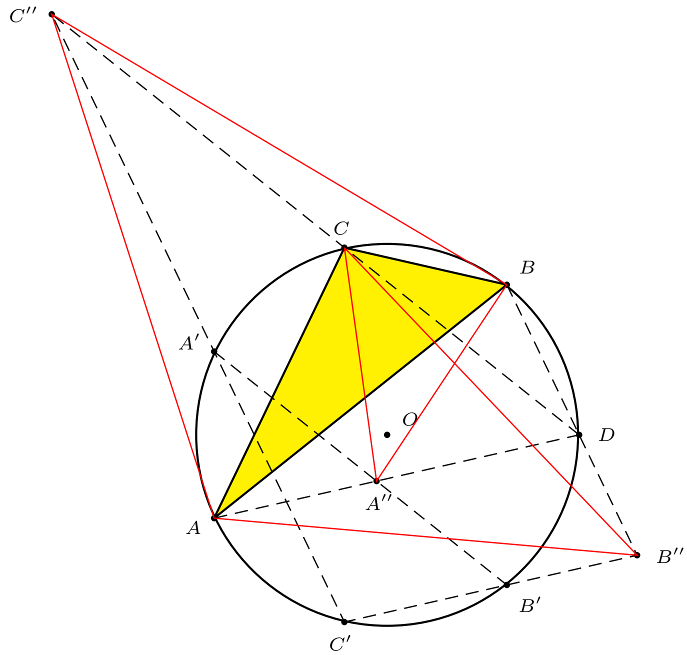


Figure 8. D as a Kiepert perspector of **T**

It is clear that the lines AA'', BB'' and CC'' all contain the point D. The coordinates of A'', B'', C'' can be rewritten as

$$\begin{aligned}A'' =& \frac{\zeta^2+\zeta}{2} + \frac{\zeta^2-\zeta}{2}\cdot(1+2(\zeta+\zeta^2+\zeta^4)),\\ B'' =& \frac{\zeta^4+\zeta^2}{2} + \frac{\zeta^4-\zeta^2}{2}\cdot(1+2(\zeta+\zeta^2+\zeta^4)),\\ C'' =& \frac{\zeta+\zeta^4}{2} + \frac{\zeta-\zeta^4}{2}\cdot(1+2(\zeta+\zeta^2+\zeta^4)).\end{aligned}$$

Since $1+2(\zeta+\zeta^2+\zeta^4) = \sqrt{7}i$ (Gauss sum), these expressions show that the three isosceles triangles all have base angles $\arctan\sqrt{7}$. Thus, the triangles $A''BC$, $B''CA$, $C''AB$ are similar isosceles triangles of the same orientation. From these we conclude that D is a point on the Kiepert hyperbola. $\square$

Corollary 11. *The center of the Kiepert hyperbola is the point*

$$K_{\rm i} = -\frac{1}{2}(\zeta^3+\zeta^5+\zeta^6). \tag{6}$$

Proof. Since D is the intersection of the Kiepert hyperbola and the circumcircle, the center of the Kiepert hyperbola is the midpoint of DH, where H is the orthocenter of triangle ABC (see Figure 9). This has coordinate as given in (6) above. $\square$

Remark. $K_{\rm i}$ is also the midpoint of OI_a.

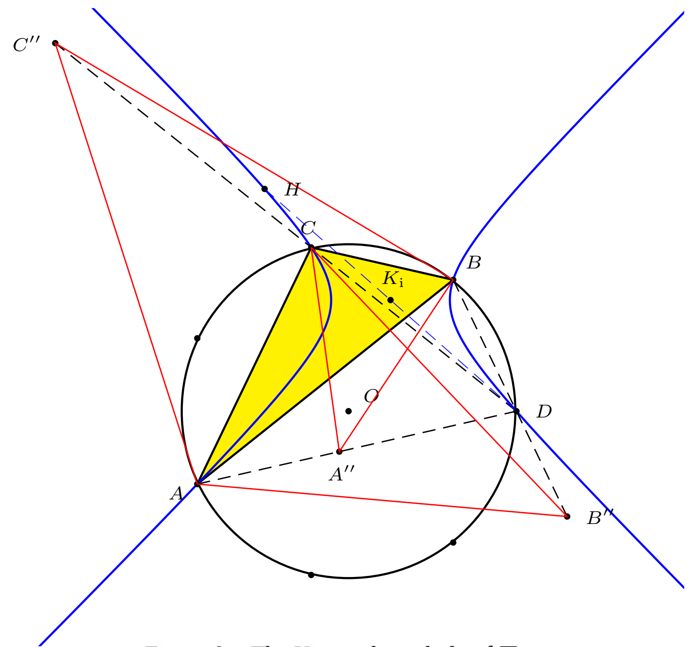


Figure 9. The Kiepert hyperbola of $\mathbf{T}$

Since $X = -1$ is antipodal to the Kiepert perspector $D = 1$ on the circumcircle, it is the Steiner point of $\mathbf{T}$, which is the fourth intersection of the Steiner ellipse with the circumcircle. The Steiner ellipse also passes through the circumcenter, the A-excenter, and the midpoint of HG. The tangents at I_a and X pass through H, and that at O passes through $Y = \frac{1}{2}(1 - (\zeta^3 + \zeta^5 + \zeta^6))$ on the circumcircle such that $OXNY$ is a parallelogram (see Lemma 21).

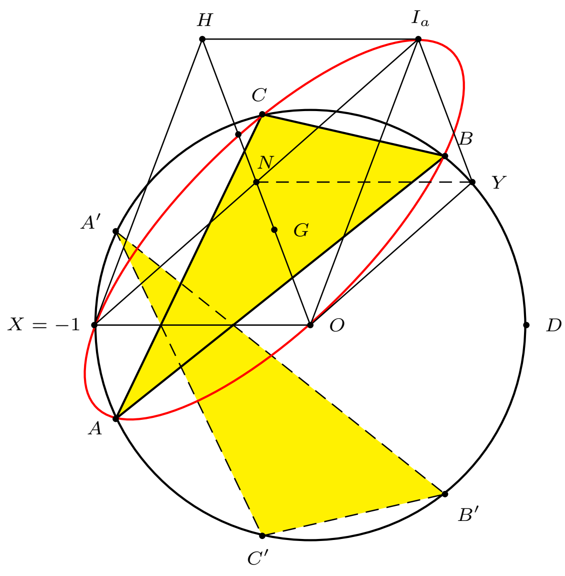


Figure 10. The Steiner ellipse of $\mathbf{T}$

6. The Brocard circle

6.1. *The Brocard points.*

Proposition 12 (Bankoff and Garfunkel)**.** *The nine-point center N is the first Brocard point.*

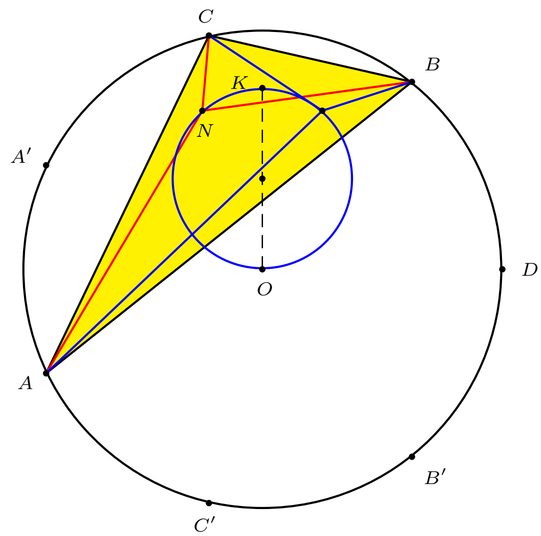


Figure 11. The Brocard points of the heptagonal triangle $\mathbf{T}$

Proof. The relations

$$\frac{1}{2}(\zeta+\zeta^2+\zeta^4)-\zeta^4 = \frac{(-2c_1-3c_2-2c_3)(4+\zeta+\zeta^2+\zeta^4)}{7}\cdot(\zeta-\zeta^4),$$
$$\frac{1}{2}(\zeta+\zeta^2+\zeta^4)-\zeta = \frac{(-2c_1-2c_2-3c_3)(4+\zeta+\zeta^2+\zeta^4)}{7}\cdot(\zeta^2-\zeta),$$
$$\frac{1}{2}(\zeta+\zeta^2+\zeta^4)-\zeta^2 = \frac{(-3c_1-2c_2-2c_3)(4+\zeta+\zeta^2+\zeta^4)}{7}\cdot(\zeta^4-\zeta^2)$$

show that the lines NA, NB, NC are obtained by rotations of BA, CB, AC through the same angle (which is necessarily the Brocard angle ω). This shows that the nine-point center N is the first Brocard point of the heptagonal triangle $\mathbf{T}$. □

Remark. It follows that $4+\zeta+\zeta^2+\zeta^4=\sqrt{14}(\cos\omega+i\sin\omega)$.

Proposition 13. *The symmedian point K has coordinate* $\frac{2(1+2(\zeta+\zeta^2+\zeta^4))}{7}=\frac{2i}{\sqrt{7}}$.

Proof. It is known that on the Brocard circle with diameter OK, $\angle NOK = -\omega$. From this,

$$\begin{aligned}K &= \frac{1}{\cos\omega}(\cos\omega - i\sin\omega)\cdot N\\ &= \left(1-\frac{i}{\sqrt{7}}\right)\cdot N\\ &= \frac{2(4+\zeta^3+\zeta^5+\zeta^6)}{7}\cdot\frac{\zeta+\zeta^2+\zeta^4}{2}\\ &= \frac{2}{7}(1+2(\zeta+\zeta^2+\zeta^4))\\ &= \frac{2i}{\sqrt{7}}\end{aligned}$$

by Lemma 1. □

Corollary 14. *The second Brocard point is the Kiepert center K_{i}.*

Proof. By Proposition 13, the Brocard axis OK is along the imaginary axis. Now, the second Brocard point, being the reflection of N in OK, is simply $-\frac{1}{2}(\zeta^3+\zeta^5+\zeta^6)$. This, according to Corollary 11, is the Kiepert center K_{i}. □

Since OD is along the real axis, it is tangent to the Brocard circle.

6.2. *A companion pair on the Brocard circle.*

Center:	$\frac{1}{7}(1+2(\zeta+\zeta^2+\zeta^4))$
Residual vertex:	$O=0$

Rot.	First Brocard triangle	Rot.	Companion
ζ^3	$A_{-\omega}=\frac{1}{7}(-4c_1-2c_2-8c_3)\cdot(-\zeta^5)$	ζ^4	$\frac{1}{7}(-4c_1-2c_2-8c_3)\cdot\zeta^2$
ζ^6	$B_{-\omega}=\frac{1}{7}(-8c_1-4c_2-2c_3)\cdot(-\zeta^3)$	ζ	$\frac{1}{7}(-8c_1-4c_2-2c_3)\cdot\zeta^4$
ζ^5	$C_{-\omega}=\frac{1}{7}(-2c_1-8c_2-4c_3)\cdot(-\zeta^6)$	ζ^2	$\frac{1}{7}(-2c_1-8c_2-4c_3)\cdot\zeta$

Since $-\zeta^5$ is the midpoint of the minor arc joining ζ and ζ^2, the coordinate of the point labeled $A_{-\omega}$ shows that this point lies on the perpendicular bisector of BC. Similarly, $B_{-\omega}$ and $C_{-\omega}$ lie on the perpendicular bisectors of CA and AB respectively. Since these points on the Brocard circle, they are the vertices of the first Brocard triangle.

The vertices of the companion are the second intersections of the Brocard circle with and the lines joining O to C, A, B respectively.

Proposition 15. *The first Brocard triangle is perspective with ABC at the point $-\frac{1}{2}$ (see Figure 12).*

Proof.

$$\begin{aligned}-\frac{1}{2} &= (-3c_1-2c_2-2c_3)\cdot A_{-\omega}+c_1\cdot\zeta^4,\\ &= (-2c_1-3c_2-2c_3)\cdot B_{-\omega}+c_2\cdot\zeta,\\ &= (-2c_1-2c_2-3c_3)\cdot C_{-\omega}+c_3\cdot\zeta^2.\end{aligned}$$

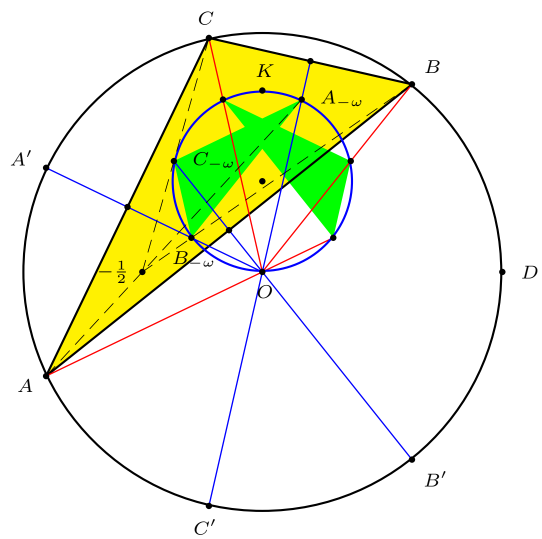


Figure 12. A regular heptagon on the Brocard circle

□

7. A companion of the triangle of reflections

We have computed the coordinates of the vertices of the triangle of reflections $A^*B^*C^*$ in (4). It is interesting to note that this is also a heptagonal triangle, and its circumcenter coincides with I_a. The residual vertex is the reflection of O in I_a.

Center:	$I_a = -(\zeta^3 + \zeta^5 + \zeta^6)$
Residual vertex:	$\overline{D} = -2(\zeta^3 + \zeta^5 + \zeta^6)$

Rotation	Triangle of reflections	Rotation	Companion
ζ^4	$A^* = \zeta + \zeta^2 - \zeta^6$	ζ^3	$\overline{B} = 1 + \zeta^4 - \zeta^6$
ζ	$B^* = \zeta^2 + \zeta^4 - \zeta^5$	ζ^6	$\overline{C} = 1 + \zeta - \zeta^5$
ζ^2	$C^* = \zeta - \zeta^3 + \zeta^4$	ζ^5	$\overline{A} = 1 + \zeta^2 - \zeta^3$

The companion has vertices on the sides of triangle ABC,

$$\overline{A} = (1 + 2c_1)\zeta - 2c_1 \cdot \zeta^2;$$
$$\overline{B} = (1 + 2c_2)\zeta^2 - 2c_2 \cdot \zeta^4;$$
$$\overline{C} = (1 + 2c_3)\zeta^4 - 2c_3 \cdot \zeta.$$

It is also perspective with $\mathbf{T}$. Indeed, the lines $A\overline{A}$, $B\overline{B}$, $C\overline{C}$ are all perpendicular to the Euler line, since the complex numbers

$$\frac{1 + \zeta^2 - \zeta^3 - \zeta^4}{\zeta + \zeta^2 + \zeta^4}, \quad \frac{1 + \zeta^4 - \zeta^6 - \zeta}{\zeta + \zeta^2 + \zeta^4}, \quad \frac{1 + \zeta - \zeta^5 - \zeta^2}{\zeta + \zeta^2 + \zeta^4}$$

are all imaginary, being respectively $-\sqrt{2}(\zeta^2 - \zeta^5)$, $\sqrt{2}(\zeta^3 - \zeta^4)$, $-\sqrt{2}(\zeta - \zeta^6)$.

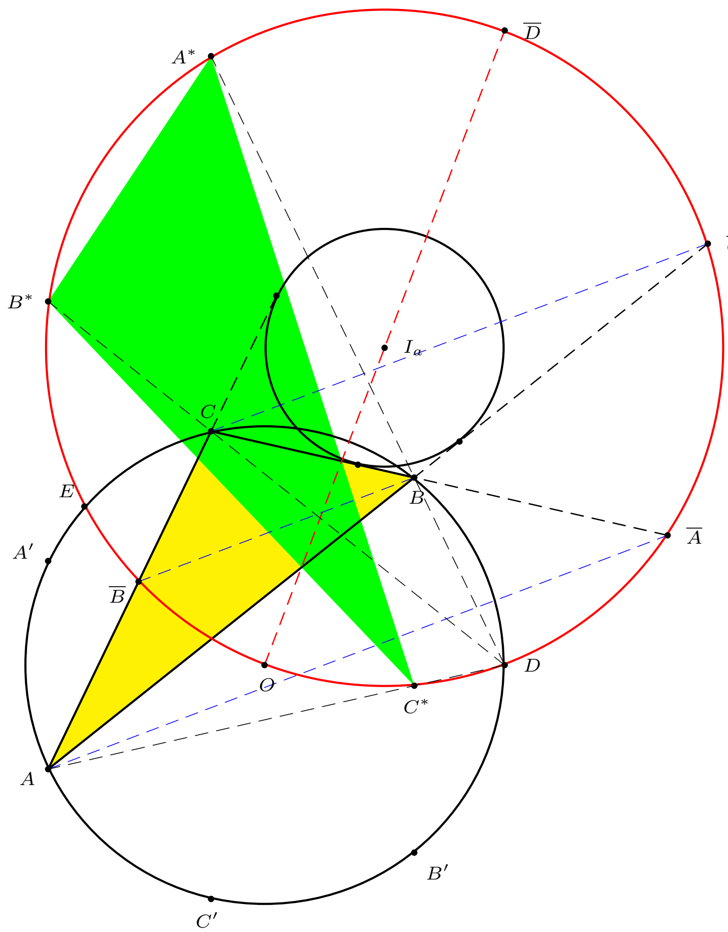


Figure 13. The triangle of reflections of $\mathbf{T}$

Proposition 16. *The triangle of reflections $A^*B^*C^*$ is triply perspective with* $\mathbf{T}$.

Proof. The triangle of reflection $A^*B^*C^*$ is clearly perspective with ABC at the orthocenter H. Since A^*C, B^*A, C^*B are all parallel (to the imaginary axis), the two triangles are triply perspective ([3, Theorem 381]). In other words, $A^*B^*C^*$ is also perspective with BCA. In fact, the perspector is the residual vertex D:

$$\begin{aligned} A^* &= -(1+2c_1)\cdot 1 + (2+2c_1)\zeta, \\ B^* &= -(1+2c_2)\cdot 1 + (2+2c_2)\zeta^2, \\ C^* &= -(1+2c_3)\cdot 1 + (2+2c_3)\zeta^4. \end{aligned}$$

□

Remark. The circumcircle of the triangle of reflections also contains the circumcenter O, the Euler reflection point E, and the residual vertex D.

8. A partition of T by the bisectors

Let $A_IB_IC_I$ be the cevian triangle of the incenter I of the heptagonal triangle $\mathbf{T} = ABC$. It is easy to see that triangles BCI, ACC_I and BB_IC are also heptagonal. Each of these is the image of the heptagonal triangle ABC under an affine mapping of the form $w = \alpha z + \beta$ or $w = \alpha\overline{z} + \beta$, according as the triangles have the same or different orientations. Note that the image triangle has circumcenter β and circumradius $|\alpha|$.

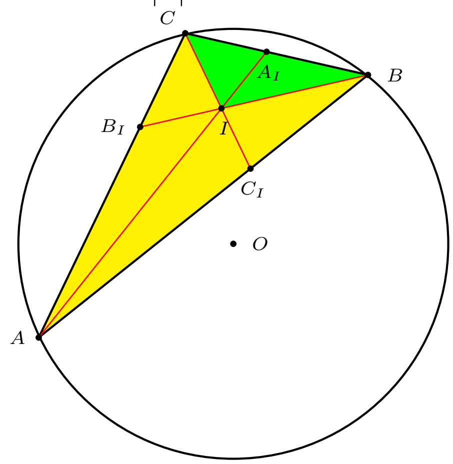


Figure 14. Partition of $\mathbf{T}$ by angle bisectors

Each of these mappings is determined by the images of two vertices. For example, since ABC and BCI have the same orientation, the mapping $f_1(z) = \alpha z + \beta$ is determined by the images $f_1(A) = B$ and $f_1(B) = C$; similarly for the mappings f_2 and f_3.

Affine mapping	A	B	C
$f_1(z) = (\zeta + \zeta^4)z - \zeta^5$	B	C	I
$f_2(z) = (1 + \zeta + \zeta^3 + \zeta^4)\overline{z} - (1 + \zeta^3 + \zeta^6)$	A	C	C_I
$f_3(z) = (1 + \zeta^2 + \zeta^4 + \zeta^5)\overline{z} - (1 + \zeta^3 + \zeta^5)$	B	B_I	C

Thus, we have

$$\begin{aligned} I =& \ f_1(C) = \zeta^3 - \zeta^5 + \zeta^6, \\ C_I =& \ f_2(C) = -1 + \zeta + \zeta^2 - \zeta^3 + \zeta^5, \\ B_I =& \ f_3(B) = -1 + \zeta + \zeta^4 - \zeta^5 + \zeta^6. \end{aligned}$$

Note also that from $f_2(A_I) = I$, it follows that

$$A_I = 1 + \zeta^2 - \zeta^3 + \zeta^4 - \zeta^6.$$

Remark. The affine mapping that associates a heptagonal triangle with circumcenter c and residual vertex d to its companion is given by

$$w = \frac{d - c}{\overline{d} - \overline{c}} \cdot \overline{z} + \frac{\overline{d}c - \overline{c}d}{\overline{d} - \overline{c}}.$$

8.1. *Four concurrent lines.* A simple application of the mapping f_1 yields the following result on the concurrency of four lines.

Proposition 17. *The orthocenter of the heptagonal triangle BCI lies on the line OC and the perpendicular from C_I to AC.*

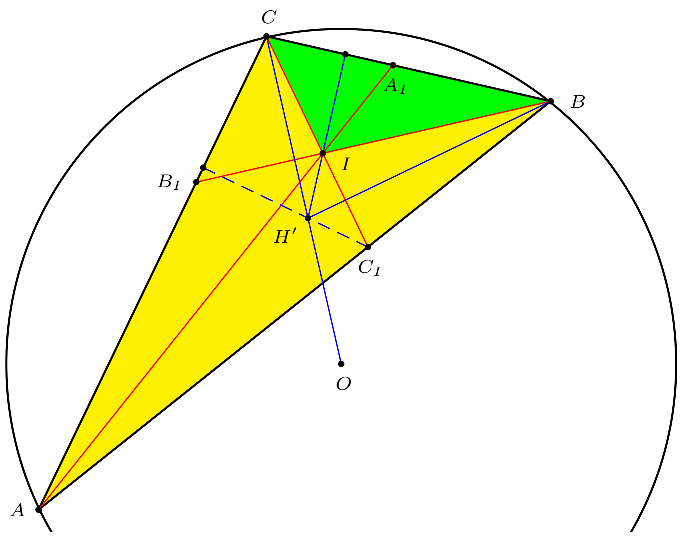


Figure 15. Four concurrent altitudes

Proof. Since ABC has orthocenter $H = \zeta + \zeta^2 + \zeta^4$, the orthocenter of triangle BCI is the point

$$H' = f_1(H) = -(1 + \zeta^4) = -(\zeta^2 + \zeta^5)\zeta^2.$$

This expression shows that H' lies on the radius OC. Now, the vector $H'C_I$ is given by

$$\begin{aligned} C_I - H' = & \ (-1 + \zeta + \zeta^2 - \zeta^3 + \zeta^5) + (1 + \zeta^4) \\ = & \ \zeta + \zeta^2 - \zeta^3 + \zeta^4 + \zeta^5. \end{aligned}$$

On the other hand, the vector AC is given by $\zeta^2 - \zeta^4$. To check that $H'C_I$ is perpendicular to AC, we need only note that

$$(\zeta + \zeta^2 - \zeta^3 + \zeta^4 + \zeta^5)\overline{(\zeta^2 - \zeta^4)} = -2(\zeta - \zeta^6) + (\zeta^2 - \zeta^5) + (\zeta^3 - \zeta^4)$$

is purely imaginary. □

Remark. Similarly, the orthocenter of ACC_I lies on the C-altiude of ABC, and that of BB_IC on the B-altitude.

8.2. *Systems of concurrent circles.*

Proposition 18. *The nine-point circles of ACC_I and* (the isosceles triangle) $B'A'C$ *are tangent internally at the midpoint of* $B'C$.

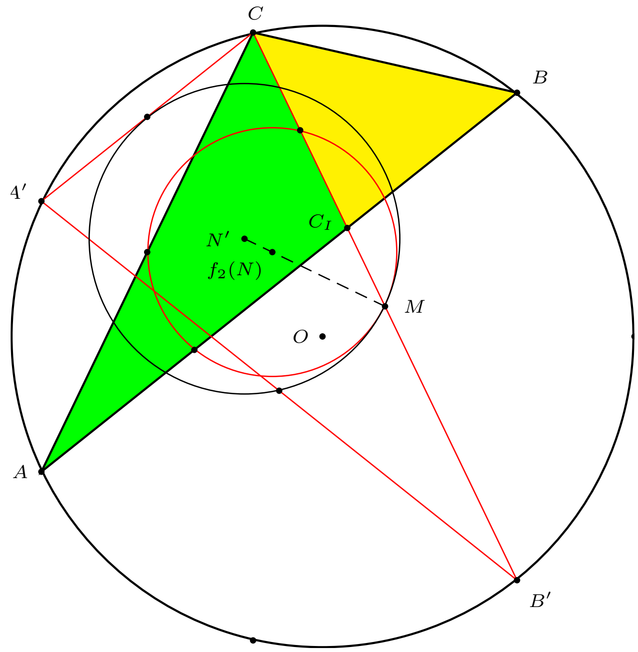


Figure 16. Two tangent nine-point circles

Proof. The nine-point circle of the isosceles triangle $B'A'C$ clearly contains the midpoint M of $B'C$. Since triangle $AB'C$ is also isosceles, the perpendicular from A to $B'C$ passes through M. This means that M lies on the nine-point circle of triangle ACC_I. We show that the two circles are indeed tangent at M.

The nine-point center of ACC_I is the point

$$f_2(N) = \frac{1}{2}(2\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^6).$$

On the other hand, the nine-point center of the isosceles triangle $B'A'C$ is the point

$$N' = \frac{1}{2}(\zeta^2 + \zeta^3 + \zeta^6).$$

Since

$$M = \frac{\zeta^2 + \zeta^6}{2} = (1 - 2c_2 - 4c_3)f_2(N) + (2c_2 + 4c_3)N'$$

as can be verified directly, we conclude that the two circles are tangent internally. □

Theorem 19. *The following circles have a common point.*
(i) *the circumcircle of* ACC_I,
(ii) *the nine-point circle of* ACC_I,
(iii) *the* A*-excircle of* ACC_I,
(iv) *the nine-point circle of* BB_IC.

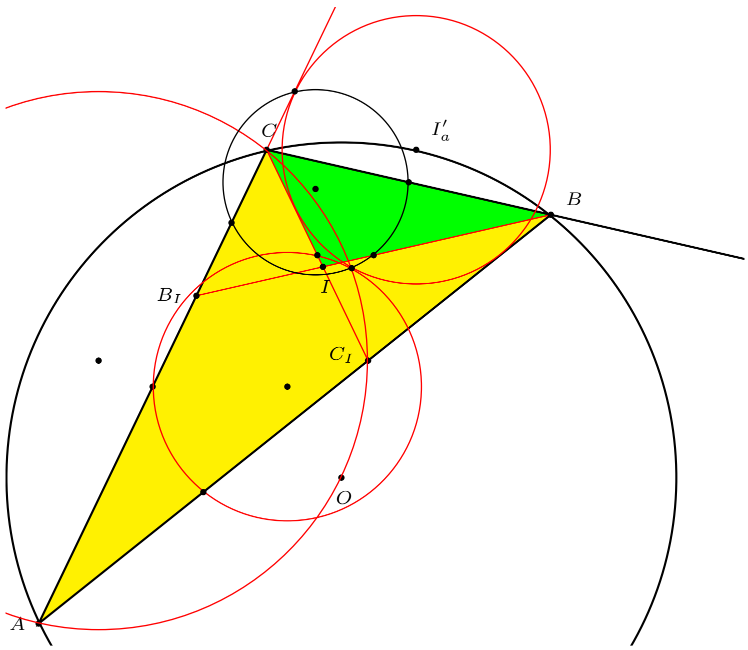


Figure 17. Four concurrent circles

Proof. By Proposition 7(3), the first three circles concur at the A-Feuerbach point of triangle ACC_I, which is the point

$$f_2(F_a) = \frac{1}{4}(\zeta + 2\zeta^2 + \zeta^4 - \zeta^5 + \zeta^6).$$

It is enough to verify that this point lies on the nine-point circle of BB_IC, which has center

$$f_3\left(\frac{\zeta + \zeta^2 + \zeta^4}{2}\right) = \frac{2\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^6}{2},$$

and square radius

$$\frac{1}{4}|1 + \zeta^2 + \zeta^4 + \zeta^5|^2 = -\frac{1}{4}\left(3(\zeta + \zeta^6) + (\zeta^2 + \zeta^5) + 2(\zeta^3 + \zeta^4)\right).$$

This is exactly the square distance between $f_2(F_a)$ and the center, as is directly verified. This shows that $f_2(F_a)$ indeed lies on the nine-point circle of BB_IC. □

Theorem 20. *Each of the following circles contains the Feuerbach point F_e of* $\mathbf{T}$:
(i) *the nine-point circle of* $\mathbf{T}$,
(ii) *the incircle of* $\mathbf{T}$,
(iii) *the nine-point circle of the heptagonal triangle* BCI,
(iv) *the* C*-excircle of* BCI,
(v) *the* A*-excircle of the heptagonal triangle* ACC_I,
(vi) *the incircle of the isosceles triangle* BIC_I.

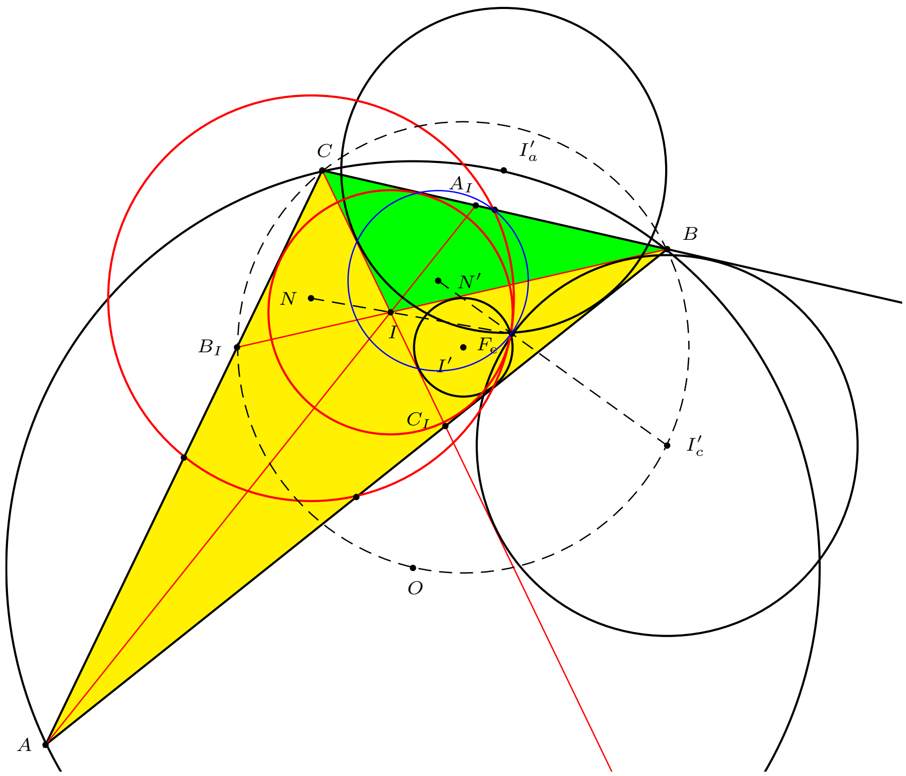


Figure 18. Six circles concurrent at the Feuerbach point of $\mathbf{T}$

Proof. It is well known that the nine-point circle and the incircle of $\mathbf{T}$ are tangent to each other internally at the Feuerbach point F_e. It is enough to verify that this point lies on each of the remaining four circles.

(iii) and (iv) The C-excircle of BCI is the image of the B-excircle of ABC under the affine mapping f_1. It is therefore enough to check that $f_1(F_b) = F_e$:

$$\begin{aligned} f_1(F_b) = &\ \frac{1}{4}(\zeta+\zeta^4)(\zeta+\zeta^2+\zeta^3+2\zeta^4-\zeta^6)-\zeta^5 \\ = &\ \frac{1}{4}(2\zeta+\zeta^2+\zeta^4-\zeta^5+\zeta^6) = F_e. \end{aligned}$$

(v) The heptagonal triangle ACC_I is the image of ABC under the mapping f_2. It can be verified directly that $W = -\frac{1}{4}(\zeta - \zeta^2 + 3\zeta^3 + 3\zeta^5) - \zeta^6$ is the point for which $f_2(W) = F_{\rm e}$. The square distance of W from the A-excenter $I_a = -(\zeta^3 + \zeta^5 + \zeta^6)$ is the square norm of $W - I_a = \frac{1}{4}(-\zeta + \zeta^2 + \zeta^3 + \zeta^5)$. An easy calculation shows that this is

$$\frac{1}{16}(-\zeta + \zeta^2 + \zeta^3 + \zeta^5)(\zeta^2 + \zeta^4 + \zeta^5 - \zeta^6) = \frac{1}{4} = r_a^2.$$

It follows that, under the mapping f_2, $F_{\rm e}$ lies on the A-excircle of ACC_I.

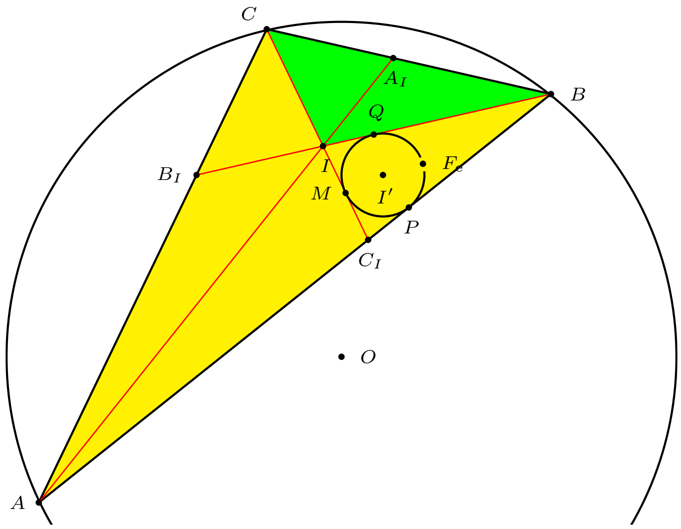


Figure 19. The incircle of an isosceles triangle

(vi) Since C_IBC and ICB_I are isosceles triangles, the perpendicular bisectors of BC and CB_I are the bisectors of angles IC_IB and C_IIB respectively. It follows that the incenter of the isosceles triangle BIC_I coincides with the circumcenter of triangle BB_IC, which is the point $I' = -(1 + \zeta^3 + \zeta^5)$ from the affine mapping f_3. This incircle touches the side IC_I at its midpoint M, the side IB at the midpoint Q of BB_I, and the side BC_I at the orthogonal projection P of C on AB (see Figure 19). A simple calculation shows that $\angle PMQ = \frac{3\pi}{7}$. To show that $F_{\rm e}$ lies on the same circle, we need only verify that $\angle PF_{\rm e}Q = \frac{4\pi}{7}$. To this end, we first determine some complex number coordinates:

$$P = \frac{1}{2}(\zeta + \zeta^2 - \zeta^3 + \zeta^4),$$

$$Q = \frac{1}{2}(-1 + 2\zeta + \zeta^4 - \zeta^5 + \zeta^6).$$

Now, with $F_{\rm e} = \frac{1}{4}(2\zeta + \zeta^2 + \zeta^4 - \zeta^5 + \zeta^6)$, we have

$$Q - F_{\rm e} = (\zeta^4 + \zeta^6)(P - F_{\rm e}).$$

From the expression $\zeta^4 + \zeta^6 = \zeta^{-2}(\zeta + \zeta^6)$, we conclude that indeed $\angle PF_{\rm e}Q = \frac{4\pi}{7}$. $\square$

9. A theorem on the Fermat points

Lemma 21. *The perpendicular bisector of the segment ON is the line containing $X = -1$ and $Y = \frac{1}{2}(1 - (\zeta^3 + \zeta^5 + \zeta^6))$.*

Proof. (1) Complete the parallelogram OI_aHX, then

$$X = O + H - I_a = (\zeta + \zeta^2 + \zeta^4) + (\zeta^3 + \zeta^5 + \zeta^6) = -1$$

is a point on the circumcircle. Note that N is the midpoint of I_aX. Thus, $NX = NI_a = R = OX$. This shows that X is on the bisector of ON.

(2) Complete the parallelogram ONI_aY, with $Y = O + I_a - N$. Explicitly, $Y = \frac{1}{2}(1 - (\zeta^3 + \zeta^5 + \zeta^6))$. But we also have

$$X+Y = (O+H-I_a)+(O+I_a-N) = (2\cdot N-I_a)+(O+I_a-N) = O+N.$$

This means that $OXNY$ is a rhombus, and $NY = OY$.

From (1) and (2), XY is the perpendicular bisector of ON. □

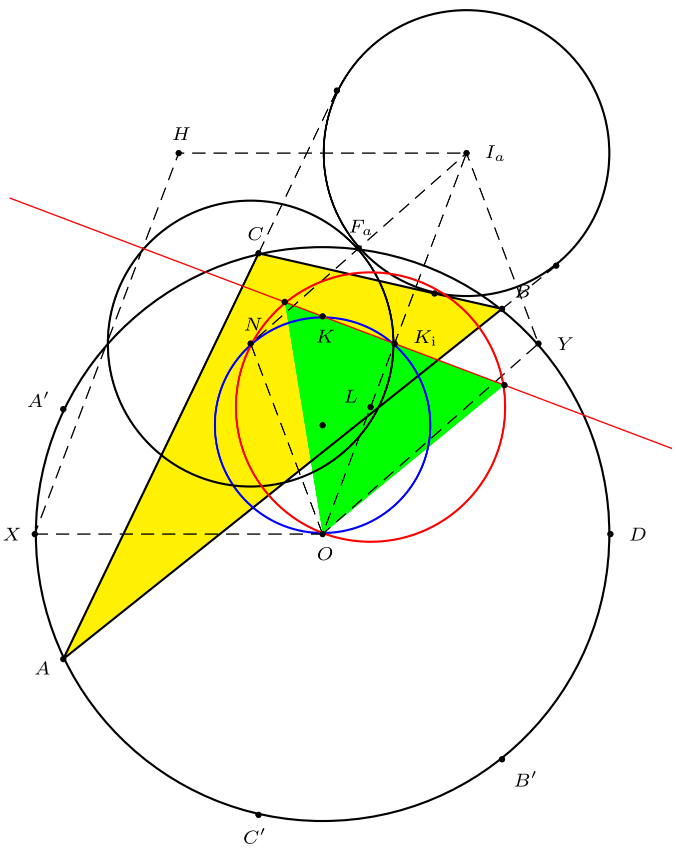


Figure 20. The circumcenter and the Fermat points form an equilateral triangle

Theorem 22. *The circumcenter and the Fermat points of the heptagonal triangle* **T** *form an equilateral triangle.*

Proof. (1) Consider the circle through O, with center at the point

$$L := -\frac{1}{3}(\zeta^3 + \zeta^5 + \zeta^6).$$

This is the center of the equilateral triangle with O as a vertex and $K_{\rm i} = -\frac{1}{2}(\zeta^3 + \zeta^5 + \zeta^6)$ the midpoint of the opposite side. See Figure 20.

(2) With X and Y in Lemma 21, it is easy to check that $L = \frac{1}{3}(X + 2Y)$. This means that L lies on the perpendicular bisector of ON.

(3) Since $K_{\rm i}$ is on the Brocard circle (with diameter OK), $OK_{\rm i}$ is perpendicular to the line $KK_{\rm i}$. It is well known that the line $KK_{\rm i}$ contains the Fermat points. [3] Indeed, $K_{\rm i}$ is the midpoint of the Fermat points. This means that L is lies on the perpendicular bisector of the Fermat points.

(4) By a well known theorem of Lester (see, for example, [5]), the Fermat points, the circumcenter, and the nine-point center are concyclic. The center of the circle containing them is necessarily L, and this circle coincides with the circle constructed in (1). The side of the equilateral triangle opposite to O is the segment joining the Fermat points. $\square$

Corollary 23. *The Fermat points of the heptagonal triangle* $\mathbf{T}$ *are the points*

$$F_+ = \frac{1}{3}(\lambda + 2\lambda^2)(\zeta^3 + \zeta^5 + \zeta^6),$$
$$F_- = \frac{1}{3}(\lambda^2 + 2\lambda)(\zeta^3 + \zeta^5 + \zeta^6),$$

where $\lambda = \frac{1}{2}(-1 + \sqrt{3}i)$ *and* $\lambda^2 = \frac{1}{2}(-1 - \sqrt{3}i)$ *are the imaginary cube roots of unity.*

Remarks. (1) The triangle with vertices I_a and the Fermat points is also equilateral.

(2) Since $OI_a = \sqrt{2}R$, each side of the equilateral triangle has length $\sqrt{\frac{2}{3}}R$.

(3) The Lester circle is congruent to the orthocentroidal circle, which has HG as a diameter.

(4) The Brocard axis OK is tangent to the A-excircle at the midpoint of I_aH.

References

[1] L. Bankoff and J. Garfunkel, The heptagonal triangle, *Math. Mag.*, 46 (1973) 7–19.
[2] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, 2nd edition, 1990, Springer-Verlag.
[3] R. A. Johnson, *Advanced Euclidean Geometry*, 1929, Dover reprint 2007.
[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.
[5] P. Yiu, The circles of Lester, Evans, Parry, and their generalizations, preprint.

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, 777 Glades Road, Boca Raton, Florida 33431-0991, USA
E-mail address: `yiu@fau.edu`

[3] The line joining the Fermat points contains K and $K_{\rm i}$.

Forum Geometricorum
Volume 9 (2009) 149–154.

FORUM GEOM
ISSN 1534-1178

The Symmedian Point and Concurrent Antiparallel Images

Shao-Cheng Liu

Abstract. In this note, we study the condition for concurrency of the GP lines of the three triangles determined by three vertices of a reference triangle and six vertices of the second Lemoine circle. Here G is the centroid and P is arbitrary triangle center different from G. We also study the condition for the images of a line in the three triangles bounded by the antiparallels through a given point to be concurrent.

1. Antiparallels through the symmedian point

Given a triangle ABC with symmedian point K, we consider the three triangles AB_aC_a, A_bBC_b, and A_cB_cC bounded by the three lines ℓ_a, ℓ_b, ℓ_c antiparallel through K to the sides BC, CA, AB respectively (see Figure 1). It is well known [4] that the 6 intercepts of these antiparallels with the sidelines are on a circle with center K. In other words, K is the common midpoint of the segments B_aC_a, C_bA_b and A_cB_c. The circle is called the second Lemoine circle.

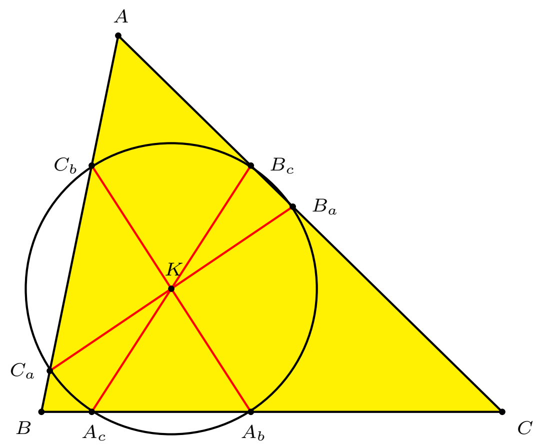


Figure 1.

Triangle AB_aC_a is similar to ABC, because it is the reflection in the bisector of angle A of a triangle which is a homothetic image of ABC. For an arbitrary triangle center P of ABC, denote by P_a the corresponding center in triangle AB_aC_a; similarly, P_b and P_c in triangles A_bBC_b and A_cB_cC.

Publication Date: June 29, 2009. Communicating Editor: Paul Yiu.

Now let P be distinct from the centroid G. Consider the line through A parallel to GP. Its reflection in the bisector of angle A intersects the circumcircle at a point Q', which is the isogonal conjugate of the infinite point of GP. So, the line G_aP_a is the image of AQ' under the homothety $\mathsf{h}\left(K, \frac{1}{3}\right)$, and it passes through a trisection point of the segment KQ' (see Figure 2).

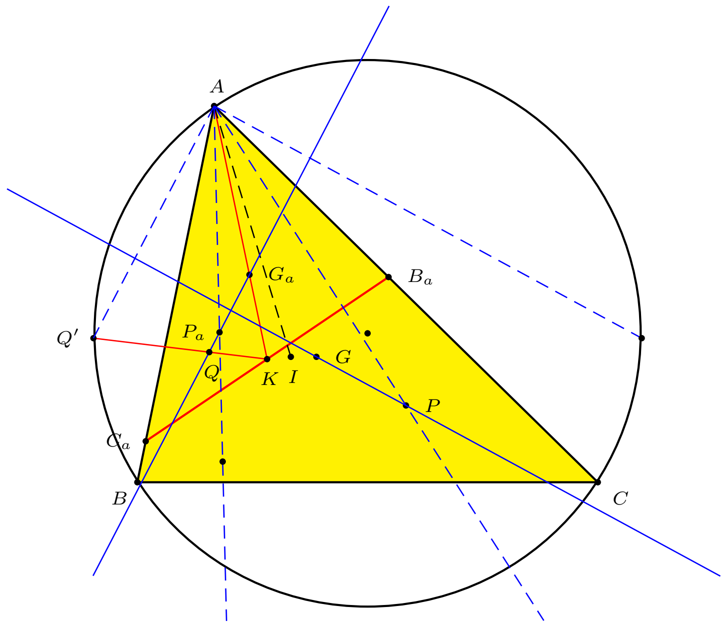


Figure 2.

In a similar manner, the reflections of the parallels to GP through B and C in the respective angle bisectors intersect the circumcircle at the same point Q'. Hence, the lines G_bP_b and G_cP_c also pass through the point Q, which is the image of Q' under the homothety $\mathsf{h}\left(K, \frac{1}{3}\right)$. It is clear that the point Q lies on the circumcircle of triangle $G_aG_bG_c$ (see Figure 3). We summarize this in the following theorem.

Theorem 1. *Let P be a triangle center of ABC, and P_a, P_b, P_c the corresponding centers in triangles AB_aC_a, BC_bA_b, CA_cB_c, which have centroids G_a, G_b, G_c respectively. The lines G_aP_a, G_bP_b, G_cP_c intersect at a point Q on the circumcircle of triangle $G_aG_bG_c$.*

Here we use homogeneous barycentric coordinates. Suppose $P = (u : v : w)$ with reference to triangle ABC.

(i) The isogonal conjugate of the infinite point of the line GP is the point

$$Q' = \left(\frac{a^2}{-2u+v+w} : \frac{b^2}{u-2v+w} : \frac{c^2}{u+v-2w}\right)$$

on the circumcircle.

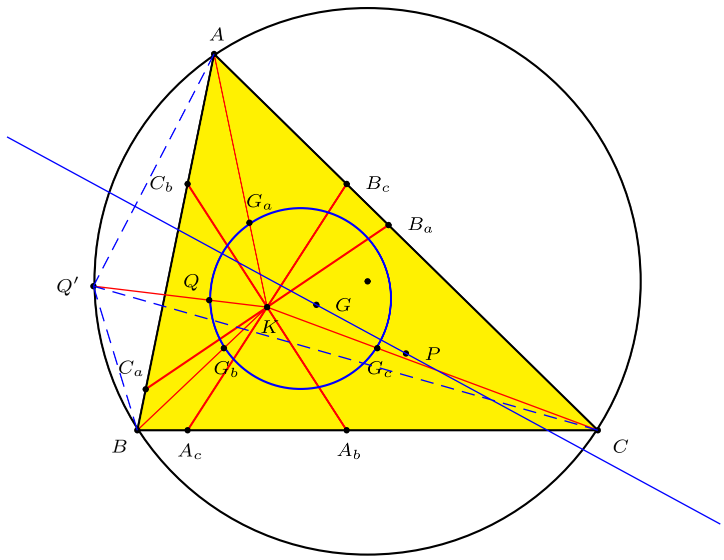


Figure 3.

(ii) The lines G_aP_a, G_bP_b, G_cP_c intersect at the point

$$Q = \left(\frac{a^2}{v+w-2u} \left(a^2 + \frac{b^2(w-u)}{w+u-2v} + \frac{c^2(v-u)}{u+v-2w} \right) : \cdots : \cdots \right).$$

which divides KQ' in the ratio $KQ : QQ' = 1 : 2$.

2. A generalization

More generally, given a point $T = (x : y : z)$, we consider the triangles intercepted by the antiparallels through T. These are the triangles AB_aC_a, A_bBC_b and A_cB_cC with coordinates (see [1, §3]):

$$\begin{aligned}
B_a &= (b^2x + (b^2 - c^2)y : 0 : c^2y + b^2z),\\
C_a &= (c^2x - (b^2 - c^2)z : c^2y + b^2z : 0),\\
C_b &= (a^2z + c^2x : c^2y + (c^2 - a^2)z : 0),\\
A_b &= (0 : a^2y - (c^2 - a^2)x : a^2z + c^2x),\\
A_c &= (0 : b^2x + a^2y : a^2z + (a^2 - b^2)x),\\
B_c &= (b^2x + a^2y : 0 : b^2z - (a^2 - b^2)y).
\end{aligned}$$

Now, for a point P with coordinates $(u : v : w)$ with reference to triangle ABC, the one with the same coordinates with reference to triangle AB_aC_a is

$$\begin{aligned}
P_a = \big(& b^2c^2(x+y+z)u + c^2(b^2x + (b^2 - c^2)y)v + b^2(c^2x - (b^2 - c^2)z)w :\\
& b^2(c^2y + b^2z)w : c^2(c^2y + b^2z)v\big).
\end{aligned}$$

By putting $u = v = w = 1$, we obtain the coordinates of the centroid

$$G_a = \left(3b^2c^2x + c^2(2b^2 - c^2)y - b^2(b^2 - 2c^2)z) : b^2(c^2y + b^2z) : c^2(c^2y + b^2z)\right)$$

of AB_aC_a. The equation of the line G_aP_a is

$$\begin{aligned}
&(c^2y + b^2z)(v - w)\mathbb{X} \\
&+ (c^2(x + y + z)u + (-2c^2x - c^2y + (b^2 - 2c^2)z)v + (c^2x - (b^2 - c^2)z)w)\mathbb{Y} \\
&- (b^2(x + y + z)u + (b^2x + (b^2 - c^2)y)v - (2b^2x + (2b^2 - c^2)y + b^2z)w)\mathbb{Z} \\
&= 0.
\end{aligned}$$

By cyclically replacing (a, b, c), (u, v, w), (x, y, z), and $(\mathbb{X}, \mathbb{Y}, \mathbb{Z})$ respectively by (b, c, a), (v, w, u), (y, z, x), and $(\mathbb{Y}, \mathbb{Z}, \mathbb{X})$, we obtain the equation of the line G_bP_b. One more applications gives the equation of G_cP_c.

Proposition 2. *The three lines G_aP_a, G_bP_b, G_cP_c are concurrent if and only if*

$$f(u, v, w)(x + y + z)^2 \left(b^2c^2(v - w)x + c^2a^2(w - u)y + a^2b^2(u - v)z\right) = 0,$$

where

$$f(u, v, w) = \sum_{\text{cyclic}} \left((2b^2 + 2c^2 - a^2)u^2 + (b^2 + c^2 - 5a^2)vw\right).$$

Computing the distance between G and P, we obtain

$$f(u, v, w) = 9(u + v + w)^2 \cdot GP^2.$$

This is nonzero for $P \neq G$. From this we obtain the following theorem.

Theorem 3. *For a fixed point $P = (u : v : w)$, the locus of a point T for which the GP-lines of triangles AB_aC_a, A_bBC_b, and A_cB_cC are concurrent is the line*

$$b^2c^2(v - w)\mathbb{X} + c^2a^2(w - u)\mathbb{Y} + a^2b^2(u - v)\mathbb{Z} = 0.$$

Remarks. (1) The line clearly contains the symmedian point K and the point $(a^2u : b^2v : c^2w)$, which is the isogonal conjugate of the isotomic conjugate of P.

(2) The locus of the point of concurrency is the line

$$\sum_{\text{cyclic}} b^2c^2(v - w)((c^2 + a^2 - b^2)(u - v)^2 + (a^2 + b^2 - c^2)(u - w)^2)\mathbb{X} = 0.$$

This line contains the points

$$\left(\frac{a^2}{(c^2 + a^2 - b^2)(u - v)^2 + (a^2 + b^2 - c^2)(u - w)^2} : \cdots : \cdots\right)$$

and

$$\left(\frac{a^2u}{(c^2 + a^2 - b^2)(u - v)^2 + (a^2 + b^2 - c^2)(u - w)^2} : \cdots : \cdots\right).$$

Theorem 4. *For a fixed point $T = (x : y : z)$, the locus of a point P for which the GP-lines of triangles AB_aC_a, A_bBC_b, and A_cB_cC are concurrent is the line*

$$\left(\frac{y}{b^2} - \frac{z}{c^2}\right)\mathbb{X} + \left(\frac{z}{c^2} - \frac{x}{a^2}\right)\mathbb{Y} + \left(\frac{x}{a^2} - \frac{y}{b^2}\right)\mathbb{Z} = 0.$$

Remark. This is the line containing the centroid G and the point $\left(\frac{x}{a^2} : \frac{y}{b^2} : \frac{z}{c^2}\right)$.

More generally, given a point $T = (x : y : z)$, we study the condition for which the images of the line

$$\mathcal{L} : \qquad u\mathbb{X} + v\mathbb{Y} + w\mathbb{Z} = 0$$

in the three triangles AB_aC_a, A_bBC_b and A_cB_cC are concurrent. Now, the image of the line $\mathcal{L}$ in AB_aC_a is the line

$$\begin{aligned} &- u(c^2y + b^2z)\mathbb{X} + ((c^2x - (b^2 - c^2)z)u - c^2(x + y + z)w)\mathbb{Y} \\ &+ ((b^2x + (b^2 - c^2)y)u - b^2(x + y + z)v)\mathbb{Z} = 0. \end{aligned}$$

Similarly, we write down the equations of the images in A_bBC_b and A_cB_cC. The three lines are concurrent if and only if

$$\begin{aligned} &((b^2 + c^2 - a^2)(v - w)^2 + (c^2 + a^2 - b^2)(w - u)^2 + (a^2 + b^2 - c^2)(u - v)^2) \\ &\cdot (x + y + z)^2 \left(\sum_{\text{cyclic}} u \cdot a^2(c^2y + b^2z)\right) = 0. \end{aligned}$$

Since the first two factors are nonzero for nonzero (u, v, w) and (x, y, z), we obtain the following result.

Theorem 5. *Given $T = (x : y : z)$, the antiparallel images of a line are concurrent if and only if the line contains the point*

$$T' = \left(\frac{y}{b^2} + \frac{z}{c^2} : \frac{z}{c^2} + \frac{x}{a^2} : \frac{x}{a^2} + \frac{y}{b^2}\right).$$

Here are some examples of correspondence:

T	T'	T	T'	T	T'
X_1	X_{37}	X_{19}	X_{1214}	X_{69}	X_{1196}
X_2	X_{39}	X_{20}	X_{800}	X_{99}	X_{1084}
X_3	X_6	X_{40}	X_{1108}	X_{100}	X_{1015}
X_4	X_{216}	X_{55}	X_1	X_{110}	X_{115}
X_5	X_{570}	X_{56}	X_9	X_{111}	X_{2482}
X_6	X_2	X_{57}	X_{1212}	X_{887}	X_{888}

References

[1] N. Dergiades and P. Yiu, Antiparallels and Concurrent Euler Lines, *Forum Geom.*, 4(2004) 1–20.
[2] C. Kimberling, Encyclopedia of Triangle Centers, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.
[3] P. Yiu, *Introduction to the Geometry of the Triangles*, Florida Atlantic University Lecture Notes, 2001, available at `http://math.fau.edu/Yiu/Geometry.html`.

[4] P. Yiu, A Tour of Triangle Geometry (2004), available at
http://math.fau.edu/Yiu/Geometry.html.

Shao-Cheng Liu: 2F., No.8, Alley 9, Lane 22, Wende Rd., 11475 Taipei, Taiwan
E-mail address: liu471119@yahoo.com.tw

Forum Geometricorum
Volume 9 (2009) 155–160.

FORUM GEOM
ISSN 1534-1178

On Trigonometric Proofs of the Steiner-Lehmus Theorem

Róbert Oláh-Gál and József Sándor

Dedicated to the memory of Professor Ferenc Radó (*1921-1990*)

Abstract. We offer a survey of some lesser known or new trigonometric proofs of the Steiner-Lehmus theorem. A new proof of a recent refined variant is also given.

1. Introduction

The famous Steiner-Lehmus theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles. For a recent survey of the Steiner-Lehmus theorem, see M. Hajja [8]. From the bibliography of [8] one can find many methods of proof, purely geometric, or trigonometric, of this theorem. Our aim in this note is to add some new references, and to draw attention to some little or unknown proofs, especially trigonometric ones. We shall also include a new trigonometric proof of a refined version of the Steiner-Lehmus theorem, published recently [9].

First, we want to point out some classical geometric proofs published in 1967 by A. Froda [4], attributed to W. T. Williams and G. T. Savage. Another interesting proof by A. Froda appears in his book [5] (see also the book of the second author [15]). Another purely geometric proof was published in 1973 by M. K. Sathya Narayama [16]. Other papers are by K. Seydel and C. Newman [17], or the more recent papers by D. Beran [1] or D. Rüthing [13]. None of the recent extensive surveys connected with the Steiner-Lehmus theorem mentions the use of complex numbers in the proof. Such a method appears in the paper by C. I. Lubin [11] in 1959.

Trigonometric proofs of Euclidean theorems have gained additional importance after the appearance of Ungar's book [18]. In this book, the author develops a kind of trigonometry that serves Hyperbolic Geometry in the same way our ordinary trigonometry does Euclidean Geometry. He calls it Gyrotrigonometry and proves that the ordinary trigonometric identities have counterparts in that trigonometry.

Publication Date: July 6, 2009. Communicating Editor: Paul Yiu.

The authors thank Professor D. Plachky for a reprint of [9] and Professor A. Furdek for providing a copy of [11]. They are also indebted to the referee for his many remarks and suggestions which helped improve the presentation of the paper, and for pointing out the new references [6, 7, 10, 18].

Consequently, he takes certain trigonometrical proofs of Euclidean theorems and shows that these proofs, hence also the corresponding theorems, remain valid in Hyperbolic Geometry. In this context, he includes the trigonometric proofs of the Urquhart and the Steiner-Lehmus theorems that appeared in [7] and [8]. Related to the question, first posed by Sylvester (also mentioned in [8]), whether there is a direct proof of the Steiner-Lehmus theorem, recently J. H. Conway (see [2]) has given an intriguing argument that there is no such proof. However, the validity of Conway's argument is debatable since a claim of the non-existence of a direct proof should be formulated in a more precise manner using, for example, the language of intuitionistic logic.

2. Trigonometric proofs of the Steiner-Lehmus theorem

2.1. Perhaps one of the shortest trigonometric proofs of the Steiner-Lehmus theorem one can find in a forgotten paper (written in Romanian) in 1916 by V. Cristescu [3]. Let BB' and CC' denote two angle bisectors of the triangle ABC (see Fig. 1). By using the law of sines in triangle $BB'C$, one gets

$$\frac{BB'}{\sin C} = \frac{BC}{\sin\left(C + \frac{B}{2}\right)}.$$

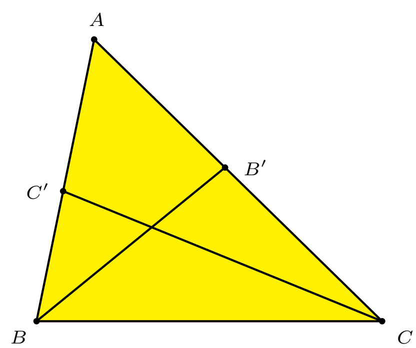


Figure 1.

As $C + \frac{B}{2} = C + \frac{180^\circ - C - A}{2} = 90^\circ - \frac{A-C}{2}$, one has

$$BB' = a \cdot \frac{\sin C}{\cos \frac{A-C}{2}}.$$

Similarly,

$$CC' = a \cdot \frac{\sin B}{\cos \frac{A-B}{2}}.$$

Assuming $BB' = CC'$, and using the identities $\sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2}$, and $\sin \frac{C}{2} = \cos \frac{A+B}{2}$, $\sin \frac{B}{2} = \cos \frac{A+C}{2}$, we have

$$\cos \frac{C}{2} \cdot \cos \frac{A+B}{2} \cos \frac{A-B}{2} = \cos \frac{B}{2} \cos \frac{A+C}{2} \cos \frac{A-C}{2}. \qquad (1)$$

Now from the identity

$$\cos(x+y)\cdot\cos(x-y) = \cos^2 x + \cos^2 y - 1,$$

relation (1) becomes

$$\cos\frac{C}{2}\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-1\right) = \cos\frac{B}{2}\left(\cos^2\frac{A}{2}+\cos^2\frac{C}{2}-1\right).$$

This simplifies into

$$\left(\cos\frac{B}{2}-\cos\frac{C}{2}\right)\left(\sin^2\frac{A}{2}+\cos\frac{B}{2}\cos\frac{C}{2}\right)=0.$$

As the second paranthesis of (6) is strictly positive, this implies $\cos\frac{B}{2}-\cos\frac{C}{2}=0$, so $B=C$.

2.2. In 2000, respectively 2001, the German mathematicians D. Plachky [12] and D. Rüthing [14] have given other trigonometric proofs of the Steiner-Lehmus theorem, based on area considerations. We present here the method by Plachky. Denote the angles at B and C respectively by β and γ, and the angle bisectors BB' and AA' by w_b and w_a (see Figure 2).

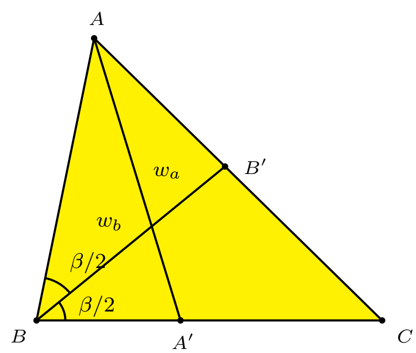


Figure 2.

By using the trigonometric form $\frac{1}{2}ab\sin\gamma$ of the area of triangle ABC, and decomposing the initial triangle in two triangles, we get

$$\frac{1}{2}aw_\beta\sin\frac{\beta}{2}+\frac{1}{2}cw_\beta\sin\frac{\beta}{2}=\frac{1}{2}bw_\alpha\sin\frac{\alpha}{2}+\frac{1}{2}cw_\alpha\sin\frac{\alpha}{2}.$$

By the law of sines we have

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin(\pi-(\alpha+\beta))}{c},$$

so assuming $w_\alpha = w_\beta$, we obtain

$$\frac{c\sin\alpha}{\sin(\alpha+\beta)}\sin\frac{\beta}{2}+c\sin\frac{\beta}{2}=\frac{c\sin\beta}{\sin(\alpha+\beta)}\sin\frac{\alpha}{2}+c\sin\frac{\alpha}{2},$$

or

$$\sin(\alpha+\beta)\left(\sin\frac{\alpha}{2}-\sin\frac{\beta}{2}\right)+\sin\frac{\alpha}{2}\sin\beta-\sin\alpha\sin\frac{\beta}{2}=0. \tag{2}$$

Writing $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ *etc*, and using the formulae

$$\sin u - \sin v = 2\sin\frac{u-v}{2}\cos\frac{u+v}{2}, \tag{3}$$

$$\cos u - \cos v = -2\sin\frac{u-v}{2}\sin\frac{u+v}{2}, \tag{4}$$

we rewrite (2) as

$$2\sin\frac{\alpha-\beta}{4}\left(\sin(\alpha+\beta)\cos\frac{\alpha+\beta}{4} + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\alpha+\beta}{2}\right) = 0.$$

Since $\alpha+\beta < \pi$, the expression inside the parenthesis is strictly positive. It follows that $\alpha = \beta$.

2.3. The following trigonometric proof seems to be much simpler. It can found in [10, pp. 194-196]. According to Honsberger, this proof was rediscovered by M. Hajja who later came across in it some obscure Russian book. The authors rediscovered again this proof, and wish to thank the referee for this information. Writing the area of triangle ABC in two different ways (using triangles ABB' and $BB'C$) we get immediately

$$w_b = \frac{2ac}{a+c}\cos\frac{\beta}{2}. \tag{5}$$

Similarly,

$$w_a = \frac{2bc}{b+c}\cos\frac{\alpha}{2}. \tag{6}$$

Suppose now that, $a > b$. Then $\alpha > \beta$, so $\frac{\alpha}{2} > \frac{\beta}{2}$. As $\frac{\alpha}{2}$, $\frac{\beta}{2} \in (0, \frac{\pi}{2})$, one gets $\cos\frac{\alpha}{2} < \cos\frac{\beta}{2}$. Also, $\frac{bc}{b+c} < \frac{ac}{a+c}$ is equivalent to $b < a$. Thus (5) and (6) imply $w_a > w_b$. This is indeed a proof of the Steiner-Lehmus theorem, as supposing $w_a = w_b$ and letting $a > b$, we would lead to the contradiction $w_a > w_b$, a contradiction; similarly with $a < b$.

For another trigonometric proof of a generalized form of the theorem, we refer the reader to [6].

3. A new trigonometric proof of a refined version

Recently, M. Hajja [9] proved the following stronger version of the Steiner-Lehmus theorem. Let BY and CZ be the angle bisectors and let $BY = y$, $CZ = z$, $YC = v$, $BZ = V$ (see Figure 3).

Then

$$c > b \Rightarrow y + v > z + V. \tag{7}$$

As $V = \frac{ac}{a+b}$, $v = \frac{ab}{a+c}$, it is immediate that $c > b \Rightarrow V > v$. Thus, assuming $c > b$, and using (7) we get $y > z$, i.e. the Steiner-Lehmus theorem (see §2.3). In [9], the proof of (7) made use of a nice lemma by R. Breusch. We offer here a new trigonometric proof of (7), based only on the law of sines, and simple trigonometric facts.

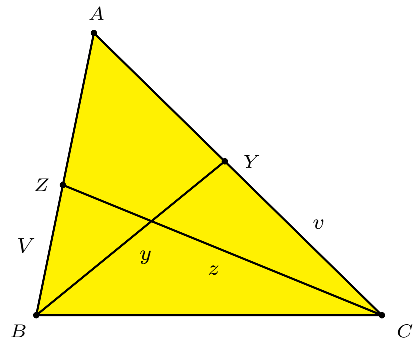


Figure 3.

In triangle BCY one can write

$$\frac{a}{\sin\left(C+\frac{B}{2}\right)}=\frac{CY}{\sin\frac{B}{2}}=\frac{BY}{\sin C},$$

so

$$\frac{y+v}{\sin C+\sin\frac{B}{2}}=\frac{a}{\sin\left(C+\frac{B}{2}\right)},$$

implying

$$y+v=\frac{a\left(\sin C+\sin\frac{B}{2}\right)}{\sin\left(C+\frac{B}{2}\right)}. \tag{8}$$

Similarly,

$$z+V=\frac{a\left(\sin B+\sin\frac{C}{2}\right)}{\sin\left(B+\frac{C}{2}\right)}. \tag{9}$$

Assume now that $y+v>z+V$. Applying $\sin u+\sin v=2\sin\frac{u+v}{2}\cos\frac{u-v}{2}$ and using the facts that $\cos\left(\frac{C}{2}+\frac{B}{4}\right)>0$, $\cos\left(\frac{B}{2}+\frac{C}{4}\right)>0$, after simplification, from (8) and (9) we get the inequality

$$\cos\left(\frac{C}{2}-\frac{B}{4}\right)\cos\left(\frac{B}{2}+\frac{C}{4}\right)>\cos\left(\frac{B}{2}-\frac{C}{4}\right)\cos\left(\frac{C}{2}+\frac{B}{4}\right).$$

Using $2\cos u\cos v=\cos\frac{u+v}{2}+\cos\frac{u-v}{2}$, this implies

$$\cos\left(\frac{3C}{4}+\frac{B}{4}\right)+\cos\left(\frac{C}{4}-\frac{3B}{4}\right)>\cos\left(\frac{3B}{4}+\frac{C}{4}\right)+\cos\left(\frac{B}{4}-\frac{3C}{4}\right),$$

or

$$\cos\left(\frac{3C}{4}+\frac{B}{4}\right)-\cos\left(\frac{3B}{4}+\frac{C}{4}\right)>\cos\left(\frac{B}{4}-\frac{3C}{4}\right)-\cos\left(\frac{C}{4}-\frac{3B}{4}\right).$$

Now applying (4), we get

$$-\sin\frac{B}{2}\sin\frac{3C}{2}>-\sin\frac{C}{2}\sin\frac{3B}{2}. \tag{10}$$

By $\sin 3u=3\sin u-4\sin^3 u$ we get immediately from (10) that

$$-3+4\sin^2\frac{C}{2}>-3+4\sin^2\frac{B}{2}. \tag{11}$$

Since the function $x \mapsto \sin^2 x$ is strictly increasing in $x \in \left(0, \frac{\pi}{2}\right)$, the inequality (11) is equivalent to $C > B$. We have actually shown that $y+v > z+V \Leftrightarrow C > B$, as desired.

References

[1] D. Beran, SSA and the Steiner-Lehmus theorem, *Math. Teacher*, 85:5 (1992) 381–383.

[2] J. H. Conway, `http://www.mathematik.uni-bielefeld.de/sillke/PUZZLES/steiner-lehmus`.

[3] V. Cristescu, On a problem of geometry (Romanian), *Gazeta Matematică* (*Bucureşti*), 2 (1916) 35–37.

[4] A. Froda, Nontrivial aspects of certain questions of euclidean geometry (Romanian), *Gazeta Matematică*, 72:1 (1967) 1–7.

[5] A. Froda, *Errors and paradoxes in Mathematics* (Romanian), Ed. Enc. Rom., Bucureşti, 1971.

[6] M. Hajja, An analytical proof of the generalized Steiner-Lehmus theorem, *Math.Gaz.* 83 (1999) 493–495.

[7] M. Hajja, A very short and simple proof of the most elementary theorem in Euclidean geometry, *Forum Geom.*, 6 (2006) 167–169.

[8] M. Hajja, A short trigonometric proof of the Steiner-Lehmus theorem, *Forum Geom.*, 8 (2008) 34–42.

[9] M. Hajja, Stronger forms of the Steiner-Lehmus theorem, *Forum Geom.*, 8 (2008) 157–161.

[10] R. Honsberger, *In Pólya's Footsteps – Miscellaneous Problems and Essays*, Dolciani Math. Exp., No. 19, MMA, Washington, D.C., 1997.

[11] C. I. Lubin, The theorem of Lehmus and complex numbers, *Scripta Math.*, 24 (1959) 137–140.

[12] D. Plachky, Ein direkter und elementar-trigonometrischer Beweis des Satzes von Steiner-Lehmus und trigonometrisch-geometrische Lösung einer Variante, *Preprints Universität Münster, Angewandte Mathematik und Informatik*, 22/2000, S.5.

[13] D. Rüthing, Ein einfacher elementar geometrischer Beweis des Satzes von Steiner-Lehmus, *Praxis Math. Heft*, 4/2000, S.176.

[14] D. Rüthing, Ein direkter trigonometrischer Beweis des Satzes von Steiner-Lehmus, *Wurzel*, 7 (2001) 160–163.

[15] J. Sándor, *Geometric inequalities* (Hungarian), Ed. Dacia, Cluj, 1988.

[16] M. K. Sathya Narayama, A proof of the Steiner-Lehmus theorem, *Math. Student*, 41:4 (1973) 395–396.

[17] K. Seidel and C. Newman, The Steiner-Lehmus theorem as a challenge problem, *The Two-Year College Math. J.*, 14 (1983) 72–75.

[18] A. A. Ungar, *A Gyrovector Space Approach to Hyperbolic Geometry*, Morgan & Claypool Publ., 2008.

Róbert Oláh-Gál: Department of Mathematics and Informatics, Babeş–Bolyai University, Extension of Miercurea–Ciuc, Str. Topliţa Nr. 20. Miercurea–Ciuc, Romania
E-mail address: `olah_gal@topnet.ro`

József Sándor: Department of Mathematics and Informatics, Babeş–Bolyai University, Extension of Miercurea–Ciuc, Str. Topliţa Nr. 20. Miercurea–Ciuc, Romania
E-mail address: `jsandor@math.ubbcluj.ro`

Forum Geometricorum
Volume 9 (2009) 161–163.

FORUM GEOM
ISSN 1534-1178

An Angle Bisector Parallel Applied to Triangle Construction

Harold Connelly and Beata Randrianantoanina

Abstract. We prove a theorem describing a line parallel to an angle bisector of a triangle and passing through the midpoint of the opposite side. We then apply this theorem to the solution of four triangle construction problems.

Consider the triangle ABC with angle bisector AT_a, altitude AH_a, midpoint M_a of side BC and Euler point E_a (see Figure 1). Let the circle with center at E_a and passing through M_a intersect AH_a at P. Draw the line M_aP. We prove the following theorem.

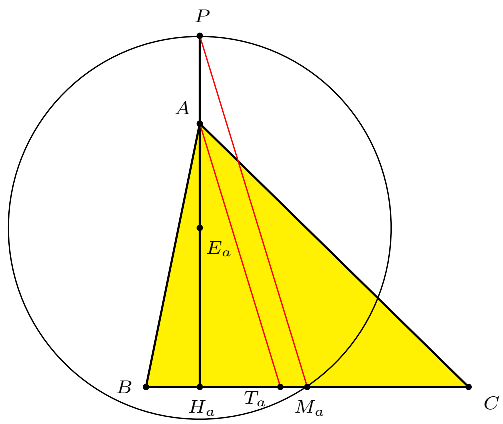


Figure 1.

Theorem 1. *In any triangle ABC with H_a not coinciding with M_a, the line M_aP is parallel to the angle bisector AT_a.*

Proof. Let O be the circumcenter of ABC (see Figure 2). The perpendicular bisector M_aO and the angle bisector AT_a intersect the circumcircle (O) at S. Let the midpoint of E_aO be R, and reflect the entire figure through R. Let the reflection of ABC be $A'B'C'$. Since E_aM_a is equal to the circumradius, the circle $E_a(M_a)$ is the reflection of (O) and is the circumcircle of $A'B'C'$. Since segments AE_a and M_aO are equal and parallel, A is the reflection of M_a and is therefore the midpoint of $B'C'$. Thus, AH_a is the perpendicular bisector of $B'C'$. Finally, AH_a intersects circle $E_a(M_a)$ at P, therefore M_aP is the bisector of angle $B'A'C'$ and parallel to AT_a. □

Publication Date: July 13, 2009. Communicating Editor: Paul Yiu.

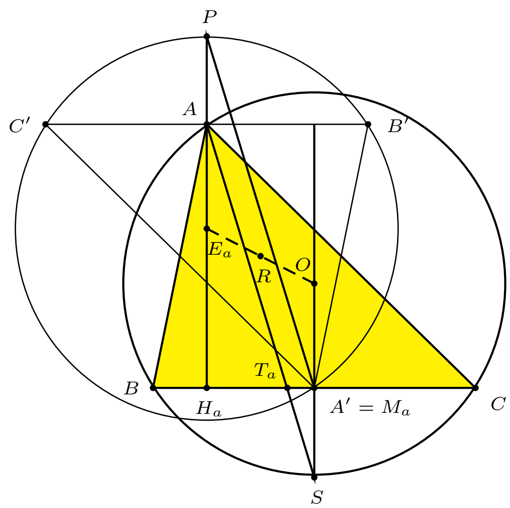


Figure 2.

Remark. For the case where H_a and M_a coincide, triangle ABC is isosceles (with apex at A) or equilateral. The lines M_aP and AT_a will coincide.

Wernick [3] presented 139 problems of constructing a triangle with ruler and compass given the location of three points associated with the triangle and chosen from a list of sixteen points. See Meyers [2] for updates on the status of the problems from this list. Connelly [1] extended this work by adding four more points to the list and 140 additional problems, many of which were designated as unresolved. We now apply Theorem 1 to solve four of these previously unresolved problems. The problem numbers are those given by Connelly.

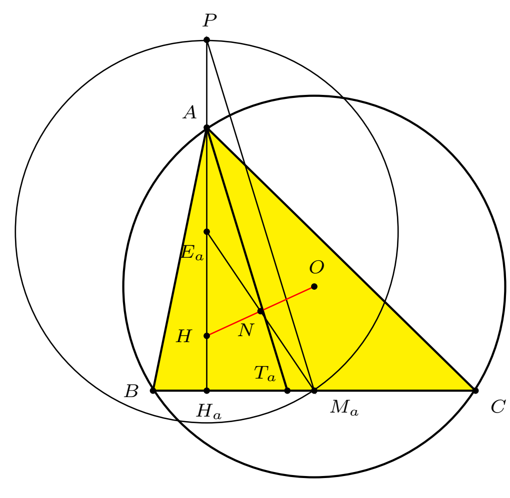


Figure 3.

Problem 99. Given E_a, M_a and T_a construct triangle ABC.
Solution. Draw line M_aT_a containing the side BC and then the altitude E_aH_a

to this side (see Figure 3). The circle with center E_a and passing through M_a intersects the altitude at P. Draw M_aP. By Theorem 1, the line through T_a parallel to M_aP intersects the altitude at A. Reflect A through E_a to get the orthocenter H. The midpoint of E_aM_a is the nine-point center N. Reflect H through N to obtain O. Draw the circumcircle through A, intersecting M_aT_a at B and C.

Number of Solutions. Depending on the relative positions of the three points, there are two solutions, no solution or an infinite number of solutions. We start by locating E_a and M_a. Then the segment E_aM_a is a diameter of the nine-point circle (N). Since, for any triangle, angle $E_aT_aM_a$ must be greater than $90°$, T_a must be inside (N), or coincide with M_a, to have a valid solution. For the case with T_a inside (N), we have two solutions since the circle $E_a(M_a)$ intersects the altitude twice and each intersection leads to a distinct solution. If the three points are collinear, the two triangles are congruent reflections of each other through the line. If T_a is outside or on (N), except at M_a, there is no solution. If T_a coincides with M_a, there are an infinite number of solutions. In this case, the vertex A can be chosen anywhere on the open segment $M_aM'_a$ (where M'_a is the reflection of M_a in E_a), and there is a resultant isosceles triangle.

Problem 108. Given E_a, N and T_a construct triangle ABC.

Problem 137. Given M_a, N and T_a construct triangle ABC.
Solution. Since N is the midpoint of E_aM_a, both of these problems reduce to Problem 99.

Problem 130. Given H_a, N and T_a construct triangle ABC.
Solution. The nine-point circle, with center N and passing through H_a, intersects line H_aT_a again at M_a, also reducing this problem to Problem 99.

Related to these, the solutions of the following two problems are locus restricted:
(i) Problem 78: given E_a, H_a, T_a;
(ii) Problem 99 in Wernick's list [3]: given M_a, H_a, T_a.

References

[1] H. Connelly, An extension of triangle construction from located points, *Forum Geom.*, 9 (2009) 109–112.
[2] L. F. Meyers, Update on William Wernick's "Triangle constructions with three located points", *Math. Mag.*, 69 (1996) 46–49.
[3] W. Wernick, Triangle constructions with three located points, *Math. Mag.*, 55 (1982) 227–230.

Harold Connelly: 102 Blowing Tree Drive, Georgetown, Kentucky 40324, USA
E-mail address: cherylandharold@roadrunner.com

Beata Randrianantoanina: Department of Mathematics, Miami University, Oxford, Ohio 45056, USA
E-mail address: randrib@muohio.edu

Forum Geometricorum
Volume 9 (2009) 165–171.

FORUM GEOM
ISSN 1534-1178

A Family of Quartics Associated with a Triangle

Peter Yff

Abstract. It is known [1, p.115] that the envelope of the family of pedal lines (Simson or Wallace lines) of a triangle ABC is Steiner's deltoid, a three-cusped hypocycloid that is concentric with the nine-point circle of ABC and touches it at three points. Also known [2, p.249] is that the nine-point circle is the locus of the intersection point of two perpendicular pedal lines. This paper considers a generalization in which two pedal lines form any acute angle θ. It is found that the locus of their intersection point, for any value of θ, is a quartic curve with the same axes of symmetry as the deltoid. Moreover, the deltoid is the envelope of the family of quartics. Finally, it is shown that all of these quartics, as well as the deltoid and the nine-point circle, may be simultaneously generated by points on a circular disk rolling on the inside of a fixed circle.

1. Sketching the loci

Consider two pedal lines of triangle ABC which intersect and form an angle θ. It is required to find the locus of the intersection point for all such pairs of pedal lines for any fixed value of θ. There are infinitely many loci as θ varies between 0 and $\frac{\pi}{2}$. By plotting points, some of the loci are sketched in Figure 1. These include the cases $\theta = \frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{5\pi}{12}$, and $\frac{\pi}{2}$, the curves have been colored. As $\theta \to 0$, the locus approaches Steiner's deltoid. It will be shown later that in general the locus is a quartic curve. As $\theta \to \frac{\pi}{2}$, the quartic merges into two coincident circles (the nine-point circle). Otherwise each curve has three double points, which seem to merge into a triple point when $\theta = \frac{\pi}{3}$. This case resembles the familiar trefoil, or "three-leaved rose" of polar coordinates.

2. A conjecture

Figure 1 seems to suggest that all of the loci might be generated simultaneously by points on a circular disk that rolls inside a fixed circle concentric with the nine-point circle. For example, the deltoid could be generated by a point on the circumference of the disk, provided that the radius of the disk is one third that of the circle. The other curves might be hypotrochoids generated by interior points of the disk. However, this fails because, for example, there is no generating point for the nine-point circle.

Publication Date: July 27, 2009. Communicating Editor: Paul Yiu.

Another possible approach is given by Zwikker [2, pp.248–249], who shows that the same hypocycloid of three cusps may be generated when the radius of the rolling circle is two thirds of the radius of the fixed circle. In this case the deltoid is generated in the opposite sense, and two circuits of the rolling circle are required to generate the entire curve. Simultaneously the nine-point circle is generated by the center of the rolling disk. It is now necessary to prove that every locus in the family is generated by a point on the rolling disk.

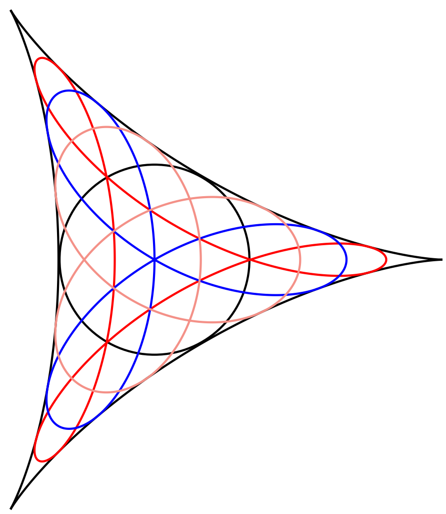


Figure 1

3. Partial proof of the conjecture

In Figure 2 the nine-point circle is placed with its center at the origin of an xy-plane. Its radius is $\frac{R}{2}$, R being the radius of the circumcircle of ABC. The radius of the fixed circle, also with center at the origin, is $\frac{3R}{2}$. The rolling disk has radius R, and initially it is placed so that it is touching the fixed circle at a cusp of the deltoid. Let the x-axis pass through this point of tangency. The center of the rolling disk is designated by Q, so that $OQ = \frac{R}{2}$. ST is a diameter of the rolling disk, with T initially at its starting point $\left(\frac{3R}{2}, 0\right)$. Let $P = \left(\frac{R}{2} + u,\ 0\right)$ be any point on the radius $QT (0 \leq u \leq R)$. Then, as the disk rotates clockwise about its center, it rolls counterclockwise along the circumference of the fixed circle, and the locus of P is represented parametrically by

$$
\begin{aligned}
x &= \frac{R}{2}\cos t + u\cos\frac{t}{2},\\
y &= \frac{R}{2}\sin t - u\sin\frac{t}{2}.
\end{aligned} \tag{1}
$$

In these equations u is the parameter of the family of hypotrochoids, while t is the running parameter on each curve. When $u = 0$, the locus is the nine-point circle $x^2 + y^2 = \frac{R^2}{4}$. When $u = R$, the parametric equations become

$$x = \frac{R}{2}\left(\cos t + 2\cos\frac{t}{2}\right),$$
$$y = \frac{R}{2}\left(\sin t - 2\sin\frac{t}{2}\right), \tag{2}$$

which are well known to represent a deltoid.

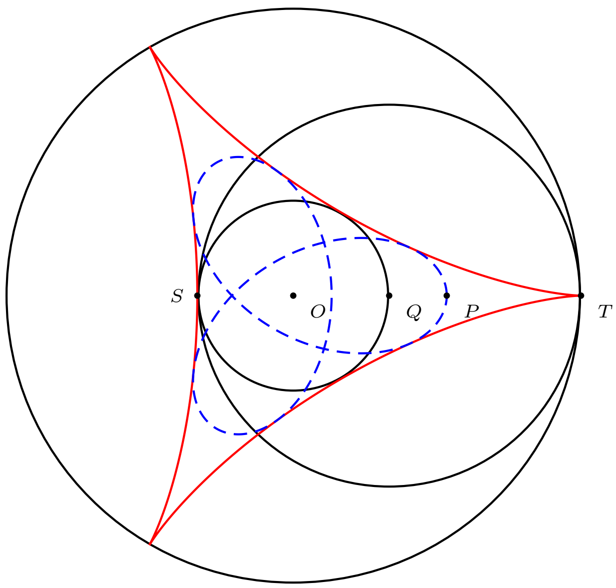


Figure 2

One further example is the case $u = \frac{R}{2}$, for which

$$x = \frac{R}{2}\left(\cos t + \cos\frac{t}{2}\right) = R\cos\frac{3t}{4}\cos\frac{t}{4},$$
$$y = \frac{R}{2}\left(\sin t - \sin\frac{t}{2}\right) = R\cos\frac{3t}{4}\sin\frac{t}{4}. \tag{3}$$

These equations represent a trefoil, for which the standard equation in polar coordinates is

$$r = a\cos 3\theta,$$

from which $x = a\cos 3\theta\cos\theta$ and $y = a\cos 3\theta\sin\theta$. This result is identical with (3) when $t = 4R$ and $R = a$. Hence $u = \frac{R}{2}$ gives a trefoil (see Figure 3).

The foregoing is not a complete proof of the conjecture, because it is necessary to establish a connection with the loci of Figure 1. These are the curves generated by the intersection points of pedals lines that form a constant angle.

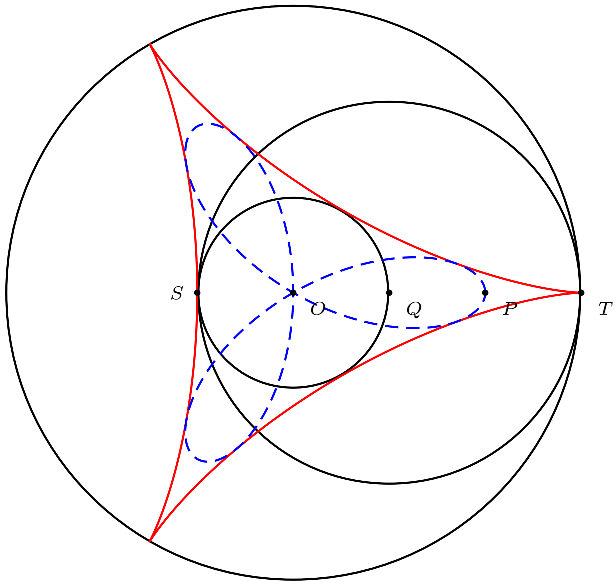


Figure 3

4. A family of quartics

By means of elementary algebra and trigonometric identities, the parameter t may be eliminated from equations (1) to obtain

$$(4R^2(x^2+y^2)+24u^2Rx+8u^4+2u^2R62-T^4)^2$$
$$=4u^2(4Rx+4u^2-R^2)^2(4Rx+u^2+2R^2). \qquad (4)$$

Thus, (1) is transformed into an equation of degree 4 in x and y. The only exceptional case is $u=0$, which reduces to $(4x^2+4y^2-R^2)^2=0$. This represents the nine-point circle, taken twice.

5. Envelope of the family

In order to find the envelope of (4), it is more practical to use the parametric form (1). The parameter u will be eliminated by using the partial differential equation

$$\frac{\partial x}{\partial t}\frac{\partial y}{\partial u}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial t},$$

or

$$-\left(\frac{R}{2}\sin t+\frac{u}{2}\sin\frac{t}{2}\right)\left(-\sin\frac{t}{2}\right)=\left(\cos\frac{t}{2}\right)\left(\frac{R}{2}\cos t-\frac{u}{2}\cos\frac{t}{2}\right).$$

This reduces to $u=R\cos\frac{3t}{2}$, and substitution in (1) results in the equations

$$x=\frac{R}{2}\left(2\cos t+\cos 2t\right),$$
$$y=\frac{R}{2}\left(2\sin t-\sin 2t\right). \qquad (5)$$

Replacing t by $-\frac{t}{2}$ transforms (5) to (2), showing that the envelope of (1) or (4) is the deltoid, which is itself a member of the family.

6. A "rolling" diameter

At the point given by (2) the slope of the deltoid is easily found to be $\tan\frac{t}{4}$. Hence the equation of the tangent line may be calculated to be

$$y-\frac{R}{2}\left(\sin t-2\sin\frac{t}{2}\right)=\tan\frac{t}{4}\left(x-\frac{R}{2}\left(\cos t+2\cos\frac{t}{2}\right)\right),$$

or

$$y=\tan\frac{t}{4}\left(x-\frac{R}{2}\left(1+2\cos\frac{t}{2}\right)\right). \tag{6}$$

Since the deltoid is a quartic curve, and since the point of tangency may be regarded as a double intersection with the tangent line (6), the tangent must meet the curve at two other points. Let $\left(\frac{R}{2}\left(\cos v+2\cos\frac{v}{2}\right),\ \frac{R}{2}\left(\sin v-2\sin\frac{v}{2}\right)\right)$ be any point on the curve, and substitute this for (x,y) in (6). The result is

$$\frac{R}{2}\left(\sin v-2\sin\frac{v}{2}\right)=\tan\frac{t}{4}\left(\frac{R}{2}\left(\cos v+2\cos\frac{v}{2}\right)-\frac{R}{2}\left(1+2\cos\frac{t}{2}\right)\right),$$

which becomes

$$\begin{aligned}&2\sin\frac{v}{2}\cos\frac{v}{2}-2\sin\frac{v}{2}\\ =\ &\tan\frac{t}{4}\left(\cos^2\frac{v}{2}-\sin^2\frac{v}{2}+2\cos\frac{v}{2}-1-2\cos\frac{t}{2}\right).\end{aligned} \tag{7}$$

In order to rewrite this as a homogeneous quartic equation, we make use of the identities

$$\begin{aligned}\sin\frac{v}{2}&=2\sin\frac{v}{4}\cos\frac{v}{4},\\ \cos\frac{v}{2}&=\cos^2\frac{v}{4}-\sin^2\frac{v}{4},\\ 1&=\sin^2\frac{v}{4}+\cos^2\frac{v}{4}.\end{aligned}$$

Then (7) becomes

$$\begin{aligned}&4\sin\frac{v}{4}\cos\frac{v}{4}\left(\cos^2\frac{v}{4}-\sin^2\frac{v}{4}\right)-4\sin\frac{v}{4}\cos\frac{v}{4}\left(\sin^2\frac{v}{4}+\cos^2\frac{v}{4}\right)\\ =\ &\tan\frac{t}{4}\left[\left(\cos^2\frac{v}{4}-\sin^2\frac{v}{4}\right)^2-\left(2\sin\frac{v}{4}\cos\frac{v}{4}\right)^2\right.\\ &+2\left(\cos^2\frac{v}{4}-\sin^2\frac{v}{4}\right)\left(\sin^2\frac{v}{4}+\cos^2\frac{v}{4}\right)-\left(\sin^2\frac{v}{4}+\cos^2\frac{v}{4}\right)^2\\ &\left.-2\cos\frac{t}{2}\left(\sin^2\frac{v}{4}+\cos^2\frac{v}{4}\right)^2\right].\end{aligned}$$

The terms are then arranged according to descending powers of $\sin\frac{v}{4}$ to obtain

$$\begin{aligned}&2\tan\frac{t}{4}\left(1+\cos\frac{t}{2}\right)\sin^4\frac{v}{4}-8\sin^3\frac{v}{4}\cos\frac{v}{4}\\ &+4\tan\frac{t}{4}\left(2+\cos\frac{t}{2}\right)\sin^2\frac{v}{4}\cos^2\frac{v}{4}-2\tan\frac{t}{4}\left(1-\cos\frac{t}{2}\right)\cos^4\frac{v}{4}=0.\end{aligned}$$

Dividing by $2\tan\frac{t}{4}\cos^4\frac{v}{4}$ and letting $V:=\tan\frac{v}{4}$ simplifies this to

$$\left(1+\cos\frac{t}{2}\right)V^4-4\cot\frac{t}{4}\cdot V^3+2\left(2+\cos\frac{t}{2}\right)V^2-\left(1-\cos\frac{t}{2}\right)=0.$$

Since the tangent line touches the deltoid where $v=t$, the quartic expression must contain the double factor $\left(V-\tan\frac{t}{4}\right)^2$. The factored result is

$$\left(1+\cos\frac{t}{2}\right)\left(V-\tan\frac{t}{4}\right)^2\left[V^2-2\cot\frac{t}{4}\cdot V-1\right]=0.$$

Hence the other solutions are found by solving

$$V^2-2\cot\frac{t}{4}\cdot V-1=0,$$

which yields $V=\cot\frac{t}{4}\pm\csc\frac{t}{4}=\cot\frac{t}{8}$ or $-\tan\frac{t}{8}$. Since $V=\tan\frac{v}{4}$, these may be expressed as $v=2\pi-\frac{t}{2}$ and $v=-\frac{t}{2}$ respectively. Because of periodicity there are other solutions to the quadratic equation, but geometrically there are only two, and the ones found here are distinct. The first one, substituted in (2), gives

$$(x,y)=\left(\frac{R}{2}\left(\cos\frac{t}{2}-2\cos\frac{t}{4}\right),\ \frac{R}{2}\left(-\sin\frac{t}{2}-2\sin\frac{t}{4}\right)\right).$$

Let this be the point T, shown in Figures 2 and 3. The point S at the other end of the diameter is given by the second solution $v=-\frac{t}{2}$:

$$S=\left(\frac{R}{2}\left(\cos\frac{t}{2}+2\cos\frac{t}{4}\right),\ \frac{R}{2}\left(-\sin\frac{t}{2}+2\sin\frac{t}{4}\right)\right).$$

The usual distance formula shows that the length of ST is $2R$. Moreover, the midpoint of ST is $\left(\frac{R}{2}\cos\frac{t}{2},\ -\frac{R}{2}\sin\frac{t}{2}\right)$, which is on the nine-point circle. Therefore it is the center of the rolling disk, and ST is a diameter. Since both S and T generate the deltoid, this confirms the fact that, for any line tangent to the deltoid, the segment within the curve is of constant length. See [2, p.249].

In order for the point T to trace one arch of the deltoid, the rolling disk travels through $\frac{4\pi}{3}$ radians on the fixed circle. Simultaneously the diameter ST rolls end over end to generate (as a tangent) the other two arches of the deltoid.

7. Proof of the conjecture

It remains to be shown that every locus defined by the intersection point of two pedal lines meeting at a fixed angle is a hypotrochoid defined by (1). Let one pedal line be given by (6), with slope $\tan\frac{t}{4}$. A second pedal line, forming the angle θ with the first, is obtained by replacing $\frac{t}{4}$ by $\frac{t}{4}+\theta$. (There is no need to include

$-\theta$, because this will be taken care of while t ranges over all of its values). The equation of the second pedal line will therefore be

$$y = \tan\left(\frac{t}{4} + \theta\right)\left[x - \frac{R}{2}\left(1 + 2\cos\left(\frac{t}{2} + 2\theta\right)\right)\right]. \tag{8}$$

Simultaneous solution of (6) and (7), after manipulation with trigonometrical identities, gives the result

$$\begin{aligned} x &= \frac{R}{2}\left[\cos(t + 2\theta) + 2\cos\theta\cos\left(\frac{t}{2} + \theta\right)\right], \\ y &= \frac{R}{2}\left[\sin(t + 2\theta) - 2\cos\theta\sin\left(\frac{t}{2} + \theta\right)\right]. \end{aligned} \tag{9}$$

Finally, replacing $t + 2\theta$ by t and $R\cos\theta$ by u, we transform (9) into

$$\begin{aligned} x &= \frac{R}{2}\cos t + u\cos\frac{t}{2}, \\ y &= \frac{R}{2}\sin t - u\sin\frac{t}{2}, \end{aligned}$$

precisely equal to (1), the parametric equations of the family of hypotrochoids. Thus the result is established.

Remark. The family of quartics contains loci which are outside the deltoid, but these correspond to values of $u > R$, in which case θ would be imaginary.

References

[1] H. S. M. Coxeter, *Introduction to Geoemtry*, Wiley, New York, 1951.
[2] C. Zwikker, *The Advanced Geometry of Plane Curves and Their Applications*, Dover, 1963.

Peter Yff: 10840 Cook Ave., Oak Lawn, Illinois 60453, USA
E-mail address: pjyff@aol.com

Forum Geometricorum
Volume 9 (2009) 173–179.

FORUM GEOM
ISSN 1534-1178

Circle Chains Inside a Circular Segment

Giovanni Lucca

Abstract. We consider a generic circles chain that can be drawn inside a circular segment and we show some geometric properties related to the chain itself. We also give recursive and non recursive formulas for calculating the centers coordinates and the radius of the circles.

1. Introduction

Consider a circle with diameter AB, center C, and a chord GH perpendicular to AB (see Figure 1). Point O is the intersection between the diameter and the chord. Inside the circular segment bounded by the chord GH and the arc GBH, it is possible to construct a doubly infinite chain of circles each tangent to the chord, and to its two immediate neighbors.

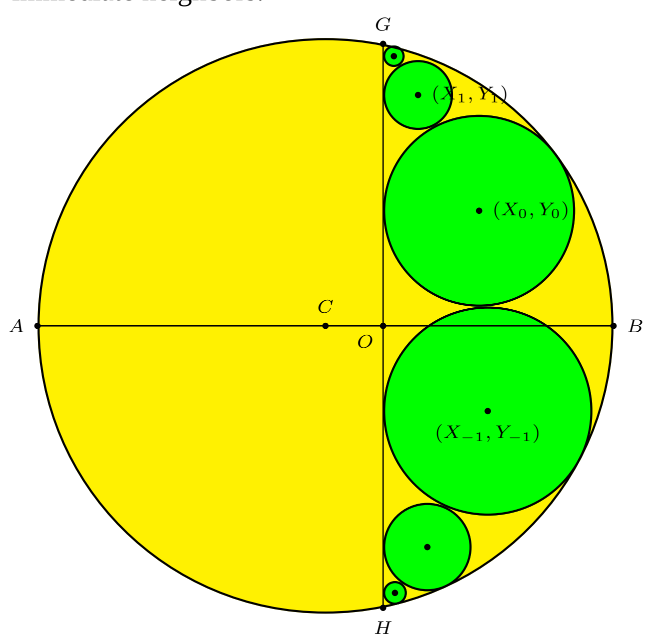


Figure 1. Circle chain inside a circular segment

Let $2(a+b)$ be the diameter of the circle and $2b$ the length of the segment OB. We set up a cartesian coordinate system with origin at O. Beginning with a circle with center (X_0, Y_0) and radius r_0 tangent to the chord GH and the arc GBH, we construct a doubly infinite chain of tangent circles, with centers (X_i, Y_i) and radius r_i for integer values of i, positive and negative.

Publication Date: August 31, 2009. Communicating Editor: Paul Yiu.
The author thanks Professor Paul Yiu for his help in improving this paper.

2. Some geometric properties of the chain

We first demonstrate some basic properties of the doubly infinite chain of circles.

Proposition 1. *The centers of the circles lie on the parabola with axis along AB, focus at C, and vertex the midpoint of OB.*

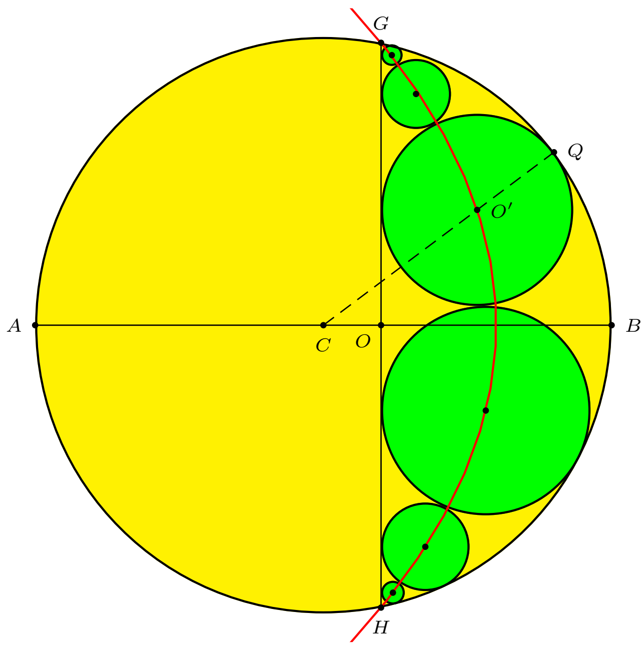


Figure 2. Centers of circles in chain on a parabola

Proof. Consider a circle of the chain with center $O'(x, y)$, radius r, tangent to the arc GBH at Q. Since the segment CQ contains O' (see Figure 2), we have, by taking into account that C has coordinates $(b - a, 0)$ and

$$\begin{aligned} CQ &= a + b, \\ CO' &= \sqrt{(x - b + a)^2 + y^2}, \\ O'Q &= r = x, \\ CO' &= CQ - O'Q. \end{aligned}$$

From these, we have

$$\sqrt{(x - b + a)^2 + y^2} = a + b - x,$$

which simplifies into

$$y^2 = -4a(x - b). \tag{1}$$

This clearly represents the parabola symmetric with respect to the x-axis, vertex $(b, 0)$, the midpoint of OB, and focus $(b - a, 0)$, which is the center C of the given circle. □

Proposition 2. *The points of tangency between consecutive circles of the chain lie on the circle with center A and radius AG.*

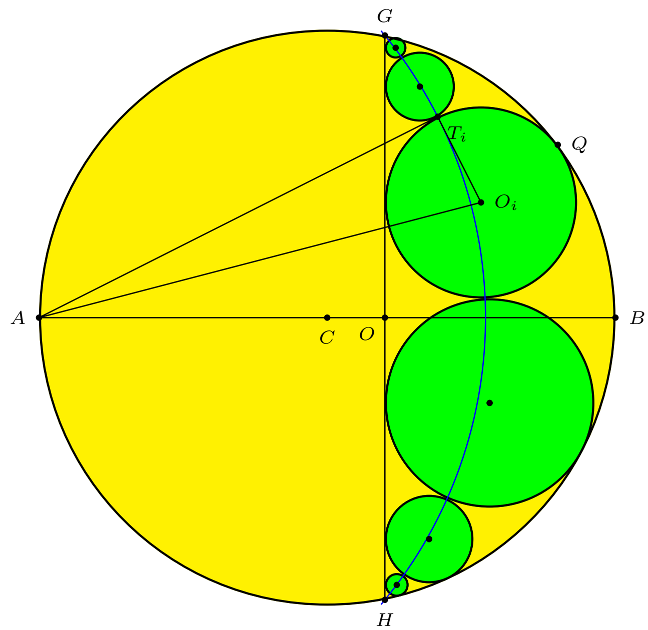


Figure 3. Points of tangency on a circular arc

Proof. Consider two neighboring circles with centers (X_i, Y_i), (X_{i+1}, Y_{i+1}), radii r_i, r_{i+1} respectively, tangent to each other at T_i (see Figure 3). By using Proposition 1 and noting that A has coordinates $(-2a, 0)$, we have

$$AO_i^2 = (X_i + 2a)^2 + Y_i^2 = \left(-\frac{Y_i^2}{4a} + b + 2a\right)^2 + Y_i^2,$$
$$r_i^2 = X_i^2 = \left(-\frac{Y_i^2}{4a} + b\right)^2.$$

Applying the Pythagorean theorem to the right triangle AO_iT_i, we have

$$AT_i^2 = AO_i^2 - r_i^2 = 4a(a+b) = AO \cdot AB = AG^2.$$

It follows that T_i lies on the circle with center A and radius AG. □

Proposition 3. *If a circle of the chain touches the chord GH at P and the arc GBH at Q, then the points A, P, Q are collinear.*

Proof. Suppose the circle has center O'. It touches GH at P and the arc GBH at Q (see Figure 4). Note that triangles CAQ and $O'PQ$ are isosceles triangles with $\angle ACQ = \angle PO'Q$. It follows that $\angle CQA = \angle O'QP$, and the points A, P, Q are collinear. □

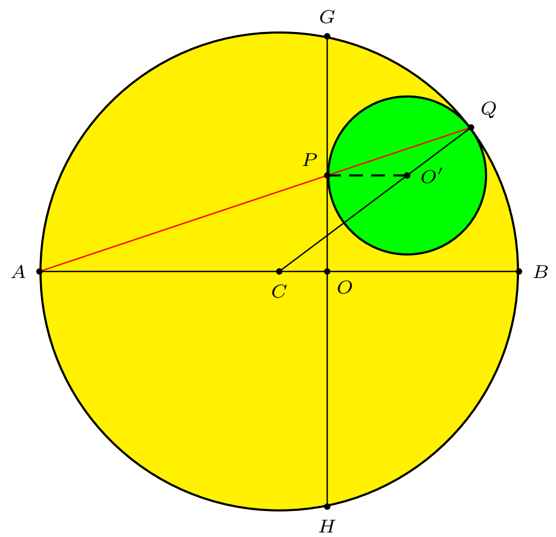


Figure 4. Line joining points of tangency

Remark. Proposition 3 gives an easy construction of the circle given any one of the points of tangency. The center of the circle is the intersection of the line CQ and the perpendicular to GH at P.

3. Coordinates of centers and radii

Figure 5 shows a right triangle $O_iO_{i-1}K_i$ with the centers O_{i-1} and O_i of two neighboring circles of the chain. Since these circles have radii $r_{i-1} = X_{i-1}$ and $r_i = X_i$ respectively, we have

$$\begin{aligned}(X_{i-1} - X_i)^2 + (Y_i - Y_{i-1})^2 &= (r_i + r_{i-1})^2 = (X_i + X_{i-1})^2, \\ (Y_i - Y_{i-1})^2 &= 4X_iX_{i-1}.\end{aligned}$$

Making use of (1), we rewrite this as

$$(Y_i - Y_{i-1})^2 = 4\left(b - \frac{Y_i^2}{4a}\right)\left(b - \frac{Y_{i-1}^2}{4a}\right),$$

or

$$\frac{4a(a+b) - Y_{i-1}^2}{4a^2} \cdot Y_i^2 - 2Y_{i-1}Y_i + \frac{(a+b)Y_{i-1}^2 - 4ab^2}{a} = 0. \tag{2}$$

If we index the circles in the chain in such a way that the ordinate Y_i increases with the index i, then from (2) we have

$$Y_i = \frac{2Y_{i-1} - \left(\frac{Y_{i-1}^2}{a} - 4b\right)\sqrt{1 + \frac{b}{a}}}{2\left(1 + \frac{b}{a} - \frac{Y_{i-1}^2}{4a^2}\right)}. \tag{3a}$$

This is a recursive formula that can be applied provided that the ordinate Y_0 of the first circle is known. Note that Y_0 must be chosen in the interval $(-2\sqrt{ab},\ 2\sqrt{ab})$.

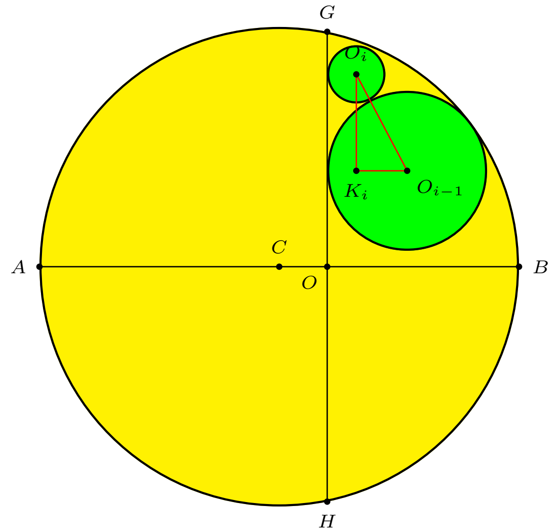


Figure 5. Construction for determination of recursive formula

Formulas for the abscissa of the centers and radii are immediately derived from (1), *i.e.*:

$$X_i = r_i = -\frac{Y_i^2}{4a} + b. \tag{4}$$

Now, it is possible to transform the recursion formula (**??**) into a continued fraction. In fact, after some simple algebraic steps (which we omit for brevity), we have

$$Y_i = 2a\left(\sqrt{1+\frac{b}{a}} - \frac{1}{\frac{Y_{i-1}}{2a} + \sqrt{1+\frac{b}{a}}}\right). \tag{5}$$

Defining

$$\alpha = 2\sqrt{1+\frac{b}{a}}, \quad \text{and} \quad Z_i = \frac{Y_i}{2a} - \sqrt{1+\frac{b}{a}} \tag{6}$$

for $i = 1,\ 2,\ \ldots$, we have

$$Z_i = -\frac{1}{\alpha + Z_{i-1}}.$$

Thus, for positive integer values of i,

$$Z_i = -\cfrac{1}{\alpha - \cfrac{1}{\alpha - \cfrac{1}{\ddots - \frac{1}{\alpha + Z_{0+}}}}},$$

where we have used Z_{0+} in place of Z_0

$$Z_{0+} = \frac{Y_0}{2a} - \sqrt{1+\frac{b}{a}}.$$

This is to distinguish from the extension of the chain by working the recursion (3a) backward: [1]

$$Y_{i-1} = \frac{2Y_i + \left(\frac{Y_i^2}{a} - 4b\right)\sqrt{1+\frac{b}{a}}}{2\left(1+\frac{b}{a}-\frac{Y_i^2}{4a^2}\right)}. \tag{3b}$$

Thus, for negative integer values of i, with

$$Z_{-i} = \frac{Y_{-i}}{2a} + \sqrt{1+\frac{b}{a}},$$

we have

$$Z_{-i} = -\cfrac{1}{-\alpha - \cfrac{1}{-\alpha - \cfrac{1}{\ddots - \frac{1}{-\alpha + Z_{0-}}}}},$$

where

$$Z_{0-} = \frac{Y_0}{2a} + \sqrt{1+\frac{b}{a}}.$$

It is possible to give nonrecursive formulas for calculating Y_i and Y_{-i}. For brevity, in the following, we shall consider only Y_i for positive integer indices because, as far as Y_{-i} is concerned, it is enough to change, in all the formulae involved, α into $-\alpha$, Z_i into Z_{-i}, and Z_{0+} into Z_{0-}. Starting from (5), and by considering its particular structure, one can write, for $i = 1,\ 2,\ 3,\ \ldots,$

$$Z_i = -\frac{Q_{i-1}(\alpha)}{Q_i(\alpha)}$$

where $Q_i(\alpha)$ are polynomials with integer coefficients. Here are the first ten of them.

$Q_0(\alpha)$	1
$Q_1(\alpha)$	$\alpha + Z_{0+}$
$Q_2(\alpha)$	$(\alpha^2 - 1) + \alpha Z_{0+}$
$Q_3(\alpha)$	$(\alpha^3 - 2\alpha) + (\alpha^2 - 1)Z_{0+}$
$Q_4(\alpha)$	$(\alpha^4 - 3\alpha^2 + 1) + (\alpha^3 - 2\alpha)Z_{0+}$
$Q_5(\alpha)$	$(\alpha^5 - 4\alpha^3 + 3\alpha) + (\alpha^4 - 3\alpha^2 + 1)Z_{0+}$
$Q_6(\alpha)$	$(\alpha^6 - 5\alpha^4 + 6\alpha^2 - 1) + (\alpha^5 - 4\alpha^3 + 3\alpha)Z_{0+}$
$Q_7(\alpha)$	$(\alpha^7 - 6\alpha^5 + 10\alpha^3 - 4\alpha) + (\alpha^6 - 5\alpha^4 + 6\alpha^2 - 1)Z_{0+}$
$Q_8(\alpha)$	$(\alpha^8 - 7\alpha^6 + 15\alpha^4 - 10\alpha^2 + 1) + (\alpha^7 - 6\alpha^5 + 10\alpha^3 - 4\alpha)Z_{0+}$
$Q_9(\alpha)$	$(\alpha^9 - 8\alpha^7 + 21\alpha^5 - 20\alpha^3 + 5\alpha) + (\alpha^8 - 7\alpha^6 + 15\alpha^4 - 10\alpha^2 + 1)Z_{0+}$

According to a fundamental property of continued fractions [1], these polynomials satisfy the second order linear recurrence

$$Q_i(\alpha) = \alpha Q_{i-1}(\alpha) - Q_{i-2}(\alpha). \tag{7}$$

[1] Equation (3b) can be obtained by solving equation (2) for Y_{i-1}.

We can further write

$$Q_i(\alpha) = P_i(\alpha) + P_{i-1}(\alpha)Z_{0+}, \tag{8}$$

for a sequence of simpler polynomials $P_i(\alpha)$, each either odd or even. In fact, from (7) and (8), we have

$$P_{i+2}(\alpha) = \alpha P_{i+1}(\alpha) - P_i(\alpha).$$

Explicitly,

$$P_i(\alpha) = \begin{cases} 1, & i = 0, \\ \sum_{k=0}^{\frac{i}{2}} (-1)^{\frac{i}{2}+k} \binom{\frac{i}{2}+k}{2k} \alpha^{2k}, & i = 2, 4, 6, \ldots, \\ \sum_{k=1}^{\frac{i+1}{2}} (-1)^{\frac{i+1}{2}+k} \binom{\frac{i-1}{2}+k}{2k-1} \alpha^{2k-1}, & i = 1, 3, 5, \ldots. \end{cases}$$

From (6), we have

$$Y_i = a\left(\alpha - 2\frac{Q_{i-1}(\alpha)}{Q_i(\alpha)}\right),$$

for $i = 1,\ 2,\ \ldots.$

References

[1] H. Davenport, *Higher Arithmetic*, 6-th edition, Cambridge University Press, 1992.

Giovanni Lucca: Via Corvi 20, 29100 Piacenza, Italy
E-mail address: vanni_lucca@inwind.it

Forum Geometricorum
Volume 9 (2009) 181–193.

FORUM GEOM
ISSN 1534-1178

On Three Circles

David Graham Searby

Abstract. The classical Three-Circle Problem of Apollonius requires the construction of a fourth circle tangent to three given circles in the Euclidean plane. For circles in general position this may admit as many as eight solutions or even no solutions at all. Clearly, an "experimental" approach is unlikely to solve the problem, but, surprisingly, it leads to a more general theorem. Here we consider the case of a chain of circles which, starting from an arbitrary point on one of the three given circles defines (uniquely, if one is careful) a tangent circle at this point and a tangency point on another of the given circles. Taking this new point as a base we construct a circle tangent to the second circle at this point and to the third circle, and repeat the construction cyclically. For any choice of the three starting circles, the tangency points are concyclic and the chain can contain at most six circles. The figure reveals unexpected connections with many classical theorems of projective geometry, and it admits the Three-Circle Problem of Apollonius as a particular case.

In the third century B.C., Apollonius of Perga proposed (and presumably solved, though the manuscript is now lost) the problem of constructing a fourth circle tangent to three given circles. A partial solution was found by Jean de la Viète around 1600, but here we shall make use of Gergonne's extremely elegant solution, which covers all cases. The closure theorem presented here is a generalization of this classical problem, and it reveals somewhat surprising connections with theorems of Monge, D'Alembert, Pascal, Brianchon, and Desargues.

Unless the three given circles are tangent at a common point, the Problem of Apollonius may have no solutions at all or it may have as many as eight – a Cartesian formulation would have to take into consideration the coordinates of the three centers as well as the three radii, and even after normalization we would still be left with an eighth degree polynomial. Algebraic and geometrical considerations lead us to consider points as circles with radius zero, and lines as circles with infinite radius. Inversion will, of course, permit us to eliminate lines altogether, however

Publication Date: September 8, 2009. Communicating Editor: J. Chris Fisher.

A biographical note is included at the end of the paper by the Editor, who also provides the annotations.

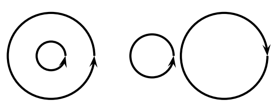
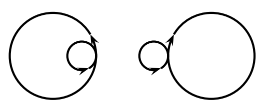
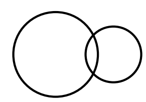
we must take into account the possibility of negative radii[1]. This apparent complication in reality allows us to define general parameters to describe the relationship between pairs of circles:

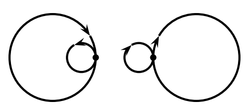
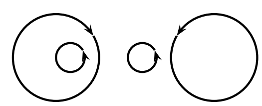
Notation and Definitions (*Circular Excess*) Let $\mathcal{C}_i = \mathcal{C}_i(x_i, y_i; r_i)$ be the circle with center (x_i, y_i) and radius r_i; define

$$e_i = x_i^2 + y_i^2 - r_i^2, \quad e_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2 - (r_i - r_j)^2, \quad \text{and} \quad \epsilon_{ij} = \frac{e_{ij}}{4 r_i r_j}.$$

The usefulness of the "excess" quantities e and ϵ will be evident from the following definitions.

Definition. We distinguish five types of relationships between pairs of circles $\mathcal{C}_i$ and $\mathcal{C}_j$ with nonzero radii, as illustrated in the accompanying table.

		
Nested: $\varepsilon_{ij} < 0$	Homogeneosuly tangent: $\varepsilon_{ij} = 0$	Intersecting: $0 < \varepsilon_{ij} < 1$

	
Oppositely tangent: $\varepsilon_{ij} = 1$	External: $\varepsilon_{ij} > 1$

These descriptions are preserved by inversion – specifically,

Theorem 1. (Inversive Invariants). *The parameter ϵ_{ij} is invariant under inversion in any circle whose center does not lie on either of the two given circles.*

Proof. The circle $\mathcal{C}_0(x_0, y_0; r_0)$ inverts $\mathcal{C}(x, y; r)$ to $\mathcal{C}'(x', y'; r')$, where if d is the Euclidean distance between the centers of $\mathcal{C}$ and $\mathcal{C}_0$, and $I_0 = \frac{r_0^2}{d^2 - r^2}$, we find

$$\begin{aligned} x' &= x_0 + I_0(x - x_0), \\ y' &= y_0 + I_0(y - y_0), \\ r' &= r I_0. \end{aligned}$$

[1]There are two common ways to interpret signed radii. They provide an orientation to the circles (as in [6]), so that $r > 0$ would indicate a counterclockwise orientation, $r < 0$ clockwise, and $r = 0$ an unoriented point. In the limit $r = \pm\infty$, and one obtains oriented lines. This seems to be Searby's interpretation. Alternatively, as in [11], one can assume a circle for which $r > 0$ to be a disk (that is, a circle with its interior), while $r < 0$ indicates a circle with its exterior; a line for which $r = \infty$ determines one half plane and $r = -\infty$ the other. This interpretation works especially well in the inversive plane (called the *circle plane* here) which, in the model that fits best with this paper, is the Euclidean plane extended by a single point at infinity that is incident with every line of the plane.

(See [5, p. 79]). Upon applying the formula for ϵ_{ij} to $\mathcal{C}_i'$ and $\mathcal{C}_j'$ then simplifying, we obtain the theorem. □

Theorem 1 permits us to work in a *circle plane* using Cartesian coordinates, the Euclidean definition of circles being extended to admit negative, infinite, and zero radii.

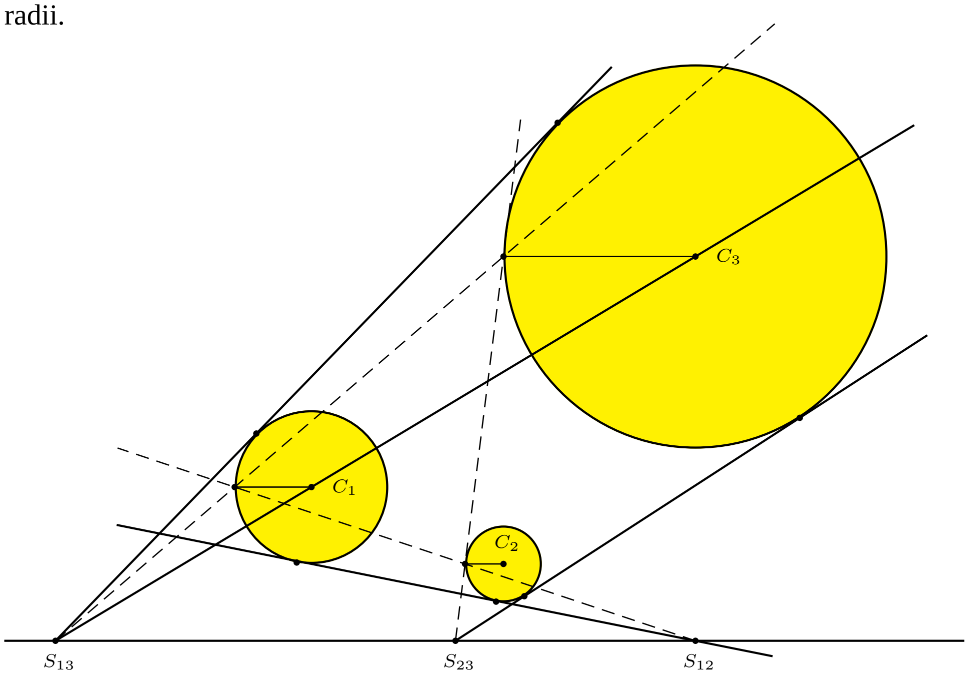


Figure 1. The centers of similitude S_{ij} of three circles lie on the axis of similitude.

Observation (*D'Alembert-Monge*). The centers of similitude S_{ij} of two circles $\mathcal{C}_i$ and $\mathcal{C}_j$ are the points on the line of centers where the common tangents (when they exist) intersect. In Cartesian coordinates we have [7, Art. 114, p.105]

$$S_{ij} = \left(\frac{r_i x_j - r_j x_i}{r_i - r_j}, \frac{r_i y_j - r_j y_i}{r_i - r_j} \right).$$

Note that if the radii are of the same sign these coordinates correspond to the *external* center of similitude; if the signs are opposite the center is *internal*. Moreover, three circles with signed radii generate three collinear points that lie on a line called the *axis of similitude* (or *Monge Line*) σ, whose equation is [7, Art.117, p.107]

$$\sigma = \begin{vmatrix} y_1 & y_2 & y_3 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix} x - \begin{vmatrix} x_1 & x_2 & x_3 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix} y = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ r_1 & r_2 & r_3 \end{vmatrix}.$$

As similar determinants appear frequently, we shall write them as Δ_{abc} if the rows are a_i, b_i, c_i; or simply Δ_{ab} should $c_i = 1$.

Lemma 2 (Second tangency point)**.** *If $P(x_0, y_0)$ is a point on a circle $\mathcal{C}_i$ while $\mathcal{C}_j$ is a second circle, then there exists exactly one circle $\mathcal{C}_a(x_a, y_a; r_a)$ that is homogeneously tangent to $\mathcal{C}_i$ at P and to $\mathcal{C}_j$ at some point $P'(x'_0, y'_0)$. Moreover $\mathcal{C}_a$ has parameters*

$$x_a = x_i + \frac{(x_0 - x_i)e_{ij}}{2f^0_{ij}}, \quad y_a = y_i + \frac{(y_0 - y_i)e_{ij}}{2f^0_{ij}}; \quad r_a = -r_i - \frac{(r_0 - r_i)e_{ij}}{2f^0_{ij}},$$

where

$$r_0 := 0 \quad \text{and} \quad f^0_{ij} := r_i r_j - (x_0 - x_i)(x_0 - x_j) - (y_0 - y_i)(y_0 - y_j);$$

and the coordinates of P' are

$$\begin{aligned} x'_0 = \ & x_i + \frac{r_i e_{0j}(x_j - x_i) + r_j e_{ij}(x_0 - x_i)}{r_i e_{0j} + r_j(e_{ij} - e_{0j})}, \\ y'_0 = \ & y_i + \frac{r_i e_{0j}(y_j - y_i) + r_j e_{ij}(y_0 - y_i)}{r_i e_{0j} + r_j(e_{ij} - e_{0j})}, \end{aligned}$$

where

$$e_{0j} = (x_0 - x_j)^2 + (y_0 - y_j)^2 - r_j^2$$

.

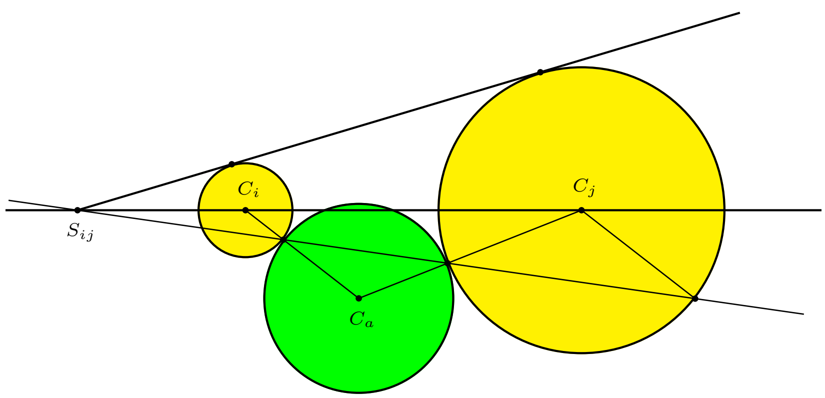


Figure 2. The second tangency point of Lemma 2.

Proof. (Outline)[2] The two tangency points P and P' are collinear with a center of similitude S_{ij}, which will be external or internal according as the radii have the same or different signs [7, Art. 117, p. 108]. It is then sufficient to find the intersections of $S_{ij}P$ with $\mathcal{C}_j$. One of the roots of the resulting quadratic equation

[2]The existence and uniqueness of $\mathcal{C}$ is immediate to anybody familiar with inversive geometry: inversion in a circle with center P sends P to infinity and $\mathcal{C}_i$ to an oriented line; the image of $\mathcal{C}_a$ under that inversion is then the unique parallel oriented line that is homogeneously tangent to the image of $\mathcal{C}_j$. Searby's intent here was to provide explicit parameters, which were especially useful to him for producing accurate figures in the days before the graphics programs that are now common. I, however, drew the figures using *Cinderella*. Searby did all calculations by hand, but they are too lengthy to include here; I confirmed the more involved formulas using *Mathematica*.

represents the point on $\mathcal{C}_j$ whose radius is parallel to that of P on $\mathcal{C}_i$; the other yields the coordinates of P', and the rest follows. □

We are now ready for the main theorem. The first part of the theorem – the closure of the chain of circles – was first proved by Tyrrell and Powell [10], having been conjectured earlier from a drawing.

Theorem 3 (Apollonius Closure)**.** *Let $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$ be three circles in the Circle Plane, and choose a point P_1 on $\mathcal{C}_1$. Define $\mathcal{C}_{12}$ to be the unique circle homogeneously tangent to $\mathcal{C}_1$ at P_1 and to $\mathcal{C}_2$, thus defining $P_2 \in \mathcal{C}_2$. Continue with $\mathcal{C}_{23}$ homogeneously tangent to $\mathcal{C}_2$ at P_2 and to $\mathcal{C}_3$ at P_3, then $\mathcal{C}_{34}$ homogeneously tangent to $\mathcal{C}_3$ at P_3 and to $\mathcal{C}_1$ at P_4, ..., and $\mathcal{C}_{67}$ homogeneously tangent to $\mathcal{C}_3$ at P_6 and to $\mathcal{C}_1$ at P_7. Then this chain closes with $\mathcal{C}_{78} = \mathcal{C}_{12}$ or, more simply, $P_7 = P_1$. Moreover, the points $P_1, \ldots, P_6$ are cyclic* (see Figure 3).

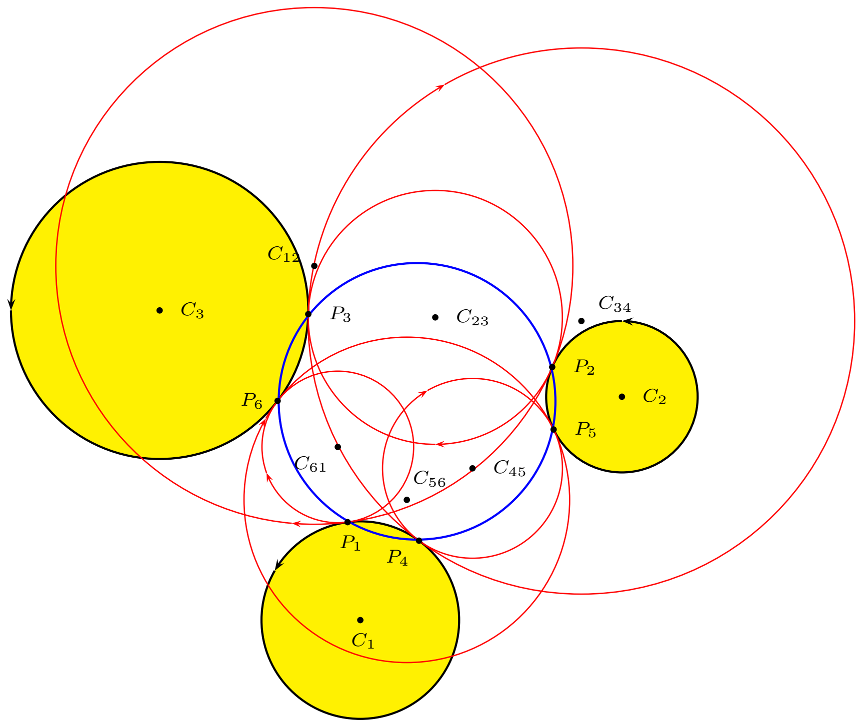


Figure 3. For $i = 1, 2$, and 3 the given circle $\mathcal{C}_i$ (in yellow) is homogeneously tangent at P_i to $\mathcal{C}_{i(i+1)}$ and $\mathcal{C}_{(i+5)i}$, and at P_{i+3} to $\mathcal{C}_{(i+3)(i+4)}$ and $\mathcal{C}_{(i+2)(i+3)}$ (where the subscripts $6 + \ell$ of $\mathcal{C}_{jk}$ are reduced to ℓ).

Proof.[3] We first show that four consecutive P_i's lie on a circle, taking P_1, P_2, P_3, P_4 as a typical example. See Figure 4.

[3]Rigby provides two proofs of this theorem in [6]. Searby independently rediscovered the result around 1987; he showed it to me at that time and I provided yet another proof in [3]. Searby's approach has the virtue of being entirely elementary.

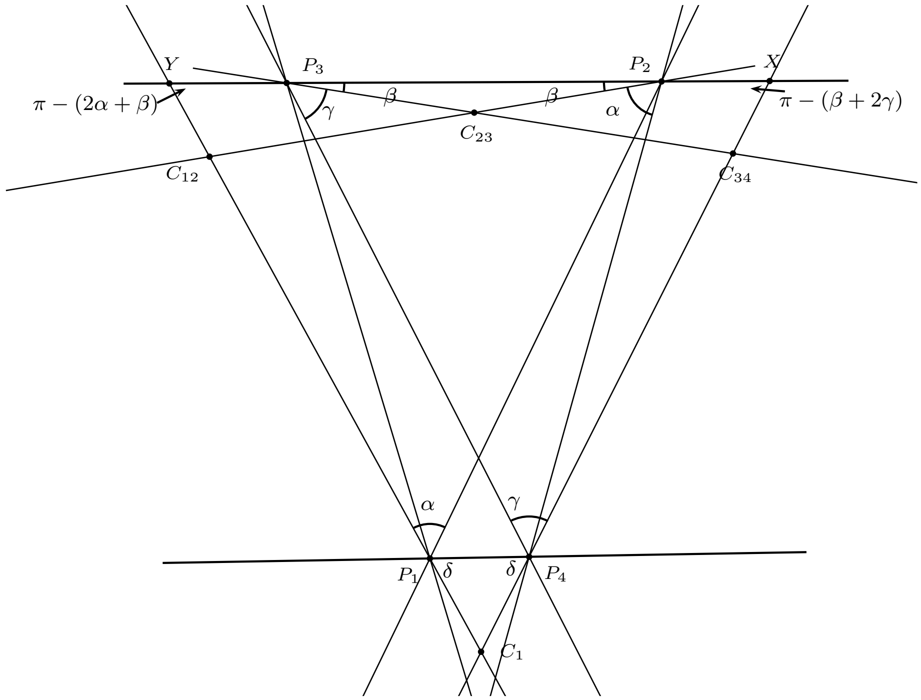


Figure 4. Proof of the Main Theorem 3

Special cases are avoided by using directed angles (so that $\angle ABC$ is the angle between 0 and π through which the line BA must be rotated counterclockwise about B to coincide with BC). Denote by C_i and C_{ij} the centers of the circles $\mathcal{C}_i$ (where $i = 1, 2, 3$) and $\mathcal{C}_{ij}$ (where $1 \leq i < j$). By hypothesis P_i is on the lines joining $C_{(i-1)i}$ to $C_{i(i+1)}$ and C_i to both $C_{i(i+1)}$ and $C_{(i-1)i}$, where we use the convention that $C_{3+k} = C_k$ as shown in Figure 4. In that figure we denote the base angles of the isosceles triangles $\triangle C_{i(i+1)}P_iP_{i+1}$ by α, β, and γ, while δ is the base angle of $\triangle C_1P_4P_1$. Consider $\triangle C_1XY$ formed by the lines $C_1P_4C_{34},\ P_2P_3$, and $C_{12}P_1C_1$. In $\triangle XP_3P_4$, $\angle P_4 = \gamma$ and $\angle P_3 = \angle P_2P_3C_{23} + \angle C_{23}P_3P_4 = \beta + \gamma$, whence $\angle X = \pi - (\beta + 2\gamma)$. In $\triangle YP_1P_2$, $\angle P_1 = \alpha$ and $\angle P_2 = \angle P_1P_2C_{12} + \angle C_{12}P_2P_3 = \alpha + \beta$, whence $\angle Y = \pi - (2\alpha + \beta)$. Consequently, $\angle C_1 = \pi - (\angle X + \angle Y) = 2(\alpha + \beta + \gamma) - \pi$. But in $\triangle C_1P_4P_1$, $\angle C_1 = \pi - 2\delta$; whence, $2(\alpha + \beta + \gamma) - \pi = \pi - 2\delta$, or

$$\alpha + \beta + \gamma + \delta = \pi.$$

Because $\angle P_2P_3P_4 = \beta + \gamma$ and $\angle P_2P_1P_4 = \alpha + \delta$, we conclude that these angles are equal and the points P_1, P_2, P_3, P_4 lie on a circle. By cyclically permuting the indices we deduce that P_5 and P_6 lie on that same circle, which proves the claim in the final statement of the theorem. This new circle already intersects $\mathcal{C}_1$ at P_1 and P_4, so that the sixth circle of the chain, namely the unique circle $\mathcal{C}_{67}$ that is homogeneously tangent to $\mathcal{C}_3$ at P_6 and to $\mathcal{C}_1$, would necessarily be tangent to $\mathcal{C}_1$ at P_1 or P_4. Should the tangency point be P_4, recalling that $\mathcal{C}_{34}$ is the unique circle homogeneously tangent to $\mathcal{C}_1$ at P_4 and to $\mathcal{C}_3$ at P_3, we would necessarily have $P_3 = P_6$. In that case we would necessarily have also $P_1 = P_4$ and $P_2 = P_5$, and the circle $P_1P_2P_3$ would be one of the pair of Apollonius Circles mentioned earlier. In any case, $P_7 = P_1$, and the sixth circle $\mathcal{C}_{67}$ touches $\mathcal{C}_1$ at P_1, closing the chain, as claimed. □

This Euclidean proof is quite general: if any of the circles were straight lines we could simply invert the figure in any appropriate circle to obtain a configuration of ten proper circles. We shall call the circle through the six tangency points the *six-point circle*, and denote it by $\mathcal{S}$. Note the symmetric relationship among the nine circles – any set of three non-tangent circles chosen from the circles of the configuration aside from $\mathcal{S}$ will generate the same figure. Indeed, the names of the circles can be arranged in an array

$$\begin{array}{c|ccc} & P_1 & P_5 & P_3 \\ \hline P_4 & C_1 & C_{45} & C_{34} \\ P_2 & C_{12} & C_2 & C_{23} \\ P_6 & C_{61} & C_{56} & C_3 \end{array}$$

so that the circles in any row or column homogeneously touch one another at the point that heads the row or column. Given the configuration of these nine circles without any labels, there are six ways to choose the initial three non-tangent circles. This observation should make clear that the closure of the chain is guaranteed even when the Apollonius Problem has no solution.

Observation (*Apollonius Axis*). The requirement that a circle $\mathcal{C}(x, y; r)$ be tangent to three circles $\mathcal{C}_i(x_i, y_i; r_i)$ yields a system of three quadratic equations which can be simplified to a linear equation in x and y, and which will be satisfied by the coordinates of the centers of two of the solutions of the Apollonius Problem. (The other six solutions are obtained by taking one of the radii to be negative.) We shall call the line through those two centers (whose points satisfy the resulting linear equation) the *Apollonius Axis* and denote it by α; its equation [7, Art. 118, pp.108-110] is

$$\alpha : x\Delta_{xr} + y\Delta_{yr} = \frac{\Delta_{er}}{2}.$$

Note that the two lines σ and α are perpendicular; they are defined even when the corresponding Apollonius Circles fail to exist (or, more precisely, are not real).

Observation (*Radical Center*). The locus of all points having the same power (that is, the square of the distance from the center minus the square of the radius) with respect to two circles is a straight line, the *radical axis* [7, Art. 106, 107, pp.98–99]:

$$\rho_{ij} : 2(x_i - x_j)x + 2(y_i - y_j)y = e_i - e_j.$$

The axes determined by three circles are concurrent at their ***radical center***

$$C_R = \left(\frac{\Delta_{ey}}{2\Delta_{xy}}, \frac{\Delta_{xe}}{2\Delta_{xy}} \right).$$

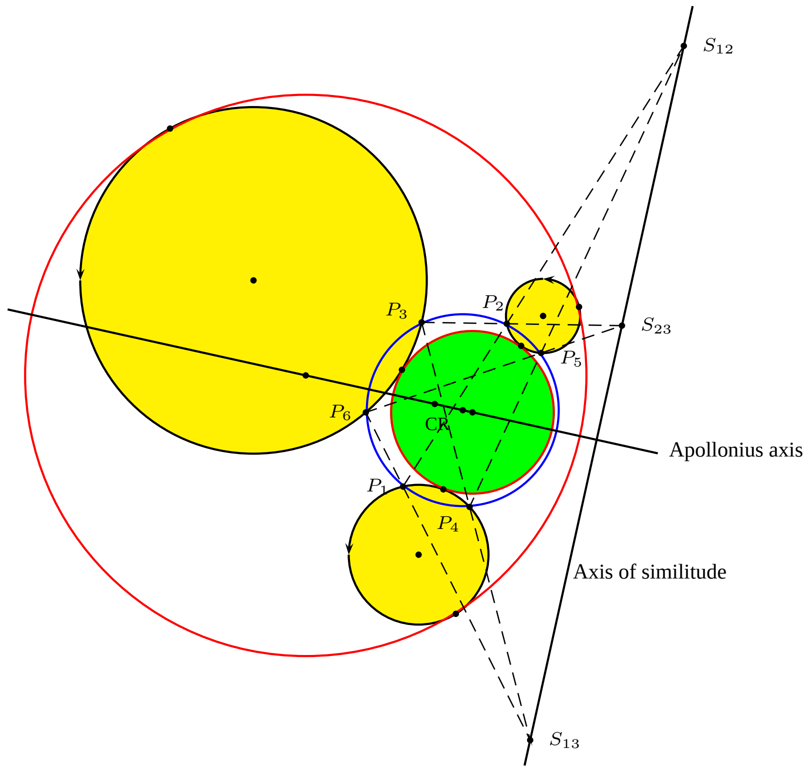


Figure 5. The Apollonius Axis of three oriented circles contains the centers of the two Apollonius circles, the radical center $\mathcal{C}_R$, and the centers S of all six-point circles. It is perpendicular to the axis of similitude.

This point is also known as the *Monge Point*, as it is the center of the circle, called the *Monge Circle*, that is orthogonal to all three given circles whenever such a circle exists. By substitution one sees that C_R lies on α. In summary,

Theorem 4 (Monge Circle)**.** *The Apollonius Axis α of three given circles is the line through their radical center C_R that is perpendicular to the axis of similitude σ; furthermore the Monge circle, if it exists, is a six-point circle that inverts the nine-circle configuration of Theorem 3 into itself.*

Theorem 5 (Centers of Six-Point Circles)**.** *For any three given non-tangent circles, as P_1 moves around $\mathcal{C}_1$ the locus of the center S of the corresponding six-point circle is either the entire Apollonius Axis α, the segment of α between the centers of the two Apollonius Circles (homogeneously tangent to all three of the given circles), or that segment's complement in α.*

Proof. Let $P_1 = (x_0, y_0)$. We saw (while finding the second tangency point) that the line P_1P_2 coincides with $S_{12}P_1$, which (by the formula for S_{12}) has gradient

$$\frac{r_1(y_0 - y_2) - r_2(y_0 - y_1)}{r_1(x_0 - x_2) - r_2(x_0 - x_1)};$$

because the perpendicular bisector β of P_1P_2 passes through C_{12}, its equation must therefore be

$$\beta:\ \frac{y-y_1-\frac{(y_0-y_1)e_{12}}{2f_{12}^0}}{x-x_1-\frac{(x_0-x_1)e_{12}}{2f_{12}^0}}=-\frac{r_1(x_0-x_2)-r_2(x_0-x_1)}{r_1(y_0-y_2)-r_2(y_0-y_1)}.$$

We can then use Cramer's rule to find the point where β intersects the Apollonius Axis α, which entails the arduous but rewarding calculation of the denominator,

$$(r_2-r_1)\begin{vmatrix} x_0 & x_1 & x_2 & x_3\\ y_0 & y_1 & y_2 & y_3\\ r_0 & r_1 & r_2 & r_3\\ 1 & 1 & 1 & 1\end{vmatrix}.$$

It requires only a little more effort to find the coordinates of the desired intersection point, which we claim to be S, namely

$$S\ =\ \left(\frac{1}{2}\frac{\begin{vmatrix} e_0 & e_1 & e_2 & e_3\\ y_0 & y_1 & y_2 & y_3\\ 0 & r_1 & r_2 & r_3\\ 1 & 1 & 1 & 1\end{vmatrix}}{\begin{vmatrix} x_0 & x_1 & x_2 & x_3\\ y_0 & y_1 & y_2 & y_3\\ 0 & r_1 & r_2 & r_3\\ 1 & 1 & 1 & 1\end{vmatrix}},\ -\frac{1}{2}\frac{\begin{vmatrix} e_0 & e_1 & e_2 & e_3\\ x_0 & x_1 & x_2 & x_3\\ 0 & r_1 & r_2 & r_3\\ 1 & 1 & 1 & 1\end{vmatrix}}{\begin{vmatrix} x_0 & x_1 & x_2 & x_3\\ y_0 & y_1 & y_2 & y_3\\ 0 & r_1 & r_2 & r_3\\ 1 & 1 & 1 & 1\end{vmatrix}}\right), \tag{1}$$

where we have used e_0 to stand for $x_0^2+y_0^2$ (with $r_0=0$). Of course, the same calculation could be applied to $\mathcal{C}_{61}$, and we would obtain the same point (1). In other words, the perpendicular bisectors of the chords of $\mathcal{S}$ formed by the tangency points of $\mathcal{C}_{12}$ with $\mathcal{C}_1$ and $\mathcal{C}_2$, and of $\mathcal{C}_{61}$ with $\mathcal{C}_1$ and $\mathcal{C}_3$, must intersect at S, which is necessarily the center of $\mathcal{S}$. As a byproduct of the way its coordinates were calculated, we must have S on α, as claimed. Finally, the Main Theorem guarantees the existence of S, and (1) shows that its coordinates are continuous functions of x_0 and y_0. Since a solution circle to the Problem of Apollonius is obviously a (degenerate) six-point circle, the second part of the theorem is also proved. □

Setting $S=(s_1,s_2)$ and rewriting the first coordinate of (1) as

$$\begin{vmatrix} e_0 & e_1 & e_2 & e_3\\ y_0 & y_1 & y_2 & y_3\\ 0 & r_1 & r_2 & r_3\\ 1 & 1 & 1 & 1\end{vmatrix}\ =\ 2s_1\begin{vmatrix} x_0 & x_1 & x_2 & x_3\\ y_0 & y_1 & y_2 & y_3\\ 0 & r_1 & r_2 & r_3\\ 1 & 1 & 1 & 1\end{vmatrix}, \tag{2}$$

we readily see that this is an equation of the form

$$x_0^2-2s_1x_0+y_0^2-2s_2y_0+\text{ (terms involving neither } x_0 \text{ nor } y_0\text{) }=0;$$

the only step that cannot be done in one's head is checking that the coefficient of y_0 necessarily equals the second coordinate of (1). In particular, we see that the point (x_0,y_0) satisfies the equation of a circle with center $S=(s_1,s_2)$. But, the unique circle with center S that passes through (x_0, y_0) is our six-point circle $\mathcal{S}$. Because both equation (2) and the corresponding equation using the second coordinate of (1) hold for any point P_1 in the plane, even if P_1 does not lie on $\mathcal{C}_1$, we see that $\mathcal{S}$ is part of a larger family of circles that cover the plane. We therefore deduce that

Theorem 6 (Six-point Pencil)**.** *The equations* (2) *represent the complete set of six-point circles, which is part of a pencil of circles whose radical axis is σ. When the pencil consists of intersecting circles, σ might itself be a six-point circle.*[4]

Proof. S_{12} has the same power[5], namely $\frac{e_{12}r_1r_2}{r_1^2-r_2^2}$, with respect to all circles tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$; but, for any point $P_1 \in \mathcal{C}_1$, the quantity $S_{12}P_1 \times S_{12}P_2$ is also the power of S_{12} with respect to the six-point circle determined by P_1. Since similar claims hold for S_{23} and S_{31}, it follows that σ (the line containing the centers of similitude) is the required radical axis. The rest follows quickly from the definitions. □

Since the tangency points P_1 and P_2 of $\mathcal{C}_{12}$ with $\mathcal{C}_1$ and $\mathcal{C}_2$ are collinear with S_{12}, and similarly for the other pairs, we see immediately that (as in Figure 6)

Theorem 7 (Pascal)**.** *The points where the six-point circle $\mathcal{S}$ meets the given circles form a Pascal hexagon $P_1P_2P_3P_4P_5P_6$ whose axis is the axis of similitude σ.*

Again, the pair of Apollonius circles deriving from Gergonne's construction and (if they are real) delimiting the pencil of Theorem 6 are special positions of the $\mathcal{C}_{ij}$, whence (as in Figure 7)

Theorem 8 (Gergonne-Desargues)**.** *For any given triple of circles, the six tangency points of a pair of Apollonius Circles, the three centers of similitude S_{ij}, and the radical center C_R are ten points of a Desargues Configuration.*

Proof. We should mention for completeness that by Gergonne's construction[6], the poles (x_i', y_i') of σ with respect to $\mathcal{C}_i$ are collinear with the radical center C_R and the tangency points of the two Apollonius Circles with $\mathcal{C}_i$. For those who prefer the use of coordinates,

$$x_i' = x_i + r_i \begin{vmatrix} y_1 & y_2 & y_3 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix}, \qquad y_i' = y_i + r_i \begin{vmatrix} x_1 & x_2 & x_3 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix},$$

[4]One easily sees that each six-point circle cuts the three given circles at equal angles. Salmon [7, Art. 118] derives the same conclusion as our Theorem 6 while determining the locus of the center of a circle cutting three given circles at equal angles.

[5]I wonder if Searby used the definition of *power* that he gave earlier (in the form $d^2 - r^2$), which seems quite awkward for the calculations needed here. The claim about the constant power of S_{12} is clear, however, without such a calculation: the circle, or circles, of inversion that interchange $\mathcal{C}_1$ with $\mathcal{C}_2$, called the *mid-circles* in [2, Sections 5.7 and 5.8] (see, especially, Exercise 5.8.1 on p.126), is the locus of points P such that two circles, tangent to both $\mathcal{C}_1$ and $\mathcal{C}_2$, are tangent to each other at P. The center of this mid-circle is S_{12}, and the square of its radius $S_{12}P$ is the power of S_{12} with respect to any of these common tangent circles.

[6]Details concerning Gergonne's construction can be found in many of the references that deal with the Problem of Apollonius such as [1, Section 10.11.1, p.318], [4, Section 1.10, pp.22–23], or [7, Art. 119 to 121, pp.110–113].

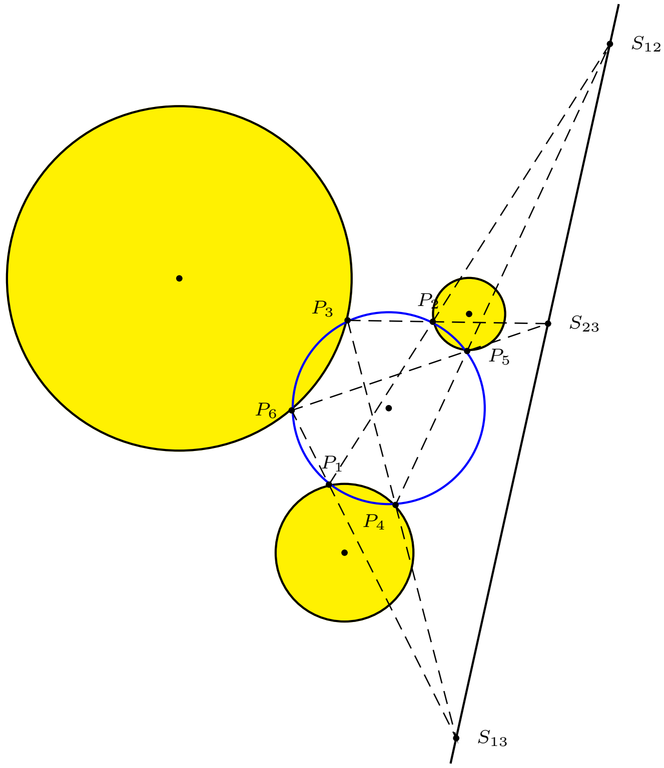


Figure 6. The points where the six-point circle $\mathcal{S}$ (in blue) meets the given circles (in yellow) form a Pascal hexagon whose axis is the axis of similitude

and the equation of the line joining C_R to the points where $\mathcal{C}_i$ is tangent to the Apollonius Circles is

$$(x-x_i)\begin{vmatrix} e_{1i} & e_{2i} & e_{3i} \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} + (y-y_i)\begin{vmatrix} e_{1i} & e_{2i} & e_{3i} \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} + r_i\begin{vmatrix} e_{1i} & e_{2i} & e_{3i} \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Gergonne's construction yields the tangency points in three pairs collinear with C_R, which is, consequently, the center of perspectivity of the triangles inscribed in the Apollonius Circles. The axis is clearly σ because, as with the circles $\mathcal{C}_{ij}$, an Apollonius circle is tangent to the given circles $\mathcal{C}_i$ and $\mathcal{C}_j$ at points whose joining line passes through S_{ij}. $\square$

Finally, on inverting the intersection point of σ and α in $\mathcal{S}$ and tracing the six tangent lines to $\mathcal{S}$ at the points P_i where it meets the given circles, after much routine algebra (which we leave to the reader)[7] we obtain

Theorem 9 (Brianchon)**.** *The inverse image of $\sigma \cap \alpha$ in $\mathcal{S}$ is the Brianchon Point of the hexagon circumscribing $\mathcal{S}$ and tangent to it at the six points where it intersects the given circles $\mathcal{C}_i$, taken in the order indicated by the labels.*

[7]There is no need for any calculation here: Theorem 9 is the projective dual of Theorem 7 – the polarity defined by $\mathcal{S}$ takes each point P_i to the line tangent there to $\mathcal{S}$, while (because $\sigma \perp \alpha$) it interchanges the axis of similitude σ with the inverse image of $\sigma \cap \alpha$ in $\mathcal{S}$.

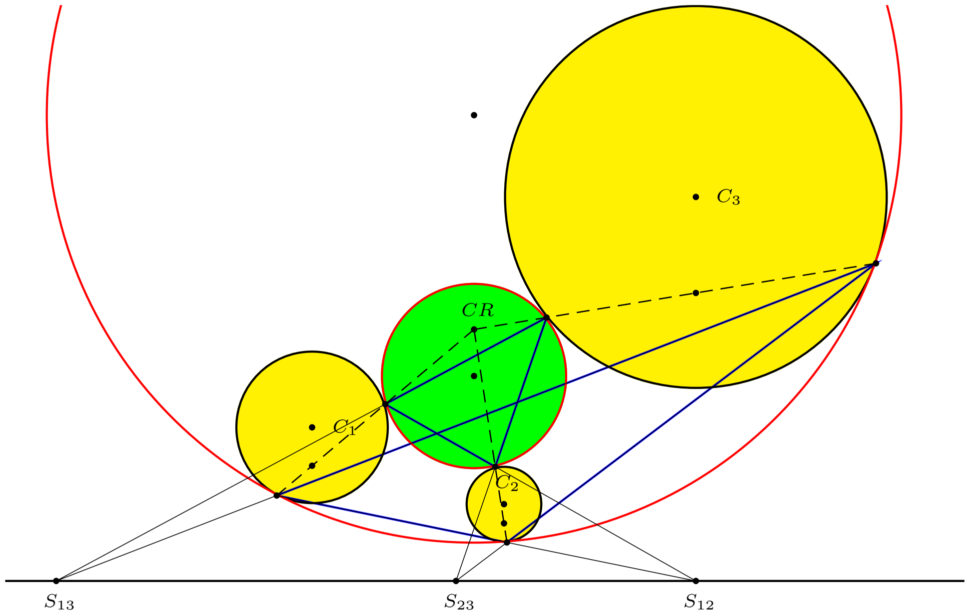


Figure 7. The triangles (shown in blue) whose vertices are the points where the Apollonius circles (red) are tangent to the three given oriented circles (yellow) are perspective from the radical center $\mathcal{C}_R$; the axis of the perspectivity is the axis of similitude of the given circles.

Conclusion. Uniting as it does the classical theorems of Monge, D'Alembert, Desargues, Pascal, and Brianchon together with the problem of Apollonius, we feel that this figure merits to be better known. The ubiquitous and extremely useful e and ϵ symbols take their name from the Einstein-Minkowski metric: in fact, the circle plane (or its three-dimensional analogue) is a vector space which, on substitution of the last coordinate (that is, the radius) by the imaginary distance cti (where $i^2 = -1$) yields interesting analogies with relativity theory. [8]

References

[1] M. Berger, *Geometry I*, Springer-Verlag, 1987.

[2] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Mathematical Association of America, New Mathematical Library No. 19, 1967.

[3] J. Ch. Fisher, Models and theorems of the classical circle planes, *Abh. Math. Sem. Univ. Hamburg*, 63 (1993) 245–264.

[4] D. Pedoe, *Circles*, Dover, 1979.

[5] D. Pedoe, *Geometry: A Comprehensive Course*, Dover,1988.

[6] J. F. Rigby, The geometry of cycles, and generalized Laguerre inversion, in *The Geometric Vein: The Coxeter Festschrift*, ed. by Chandler Davis et al., Springer-Verlag (1982), 355–378.

[7] G. Salmon, *A Treatise on Conic Sections*, 6th ed. Chelsea, 1954.

[8] D. G. Searby and J. W. Wamsley, Minimal presentations for certain metabelian groups. *Proc. Amer. Math. Soc.*, 32 (1972) 342–348.

[8]See, for example, [11, Sections 11 and 15] where there is a discussion of the Lorentz group and further references can be found.

[9] D. G. Searby, A note on the existence of the Pappus configuration in planes of order nine. *Boll. Un. Mat. Ital.*, (4) 10 (1974) 529–531.

[10] J.A. Tyrrel and M. T. Powell, A theorem in circle geometry, *Bull. London Math. Soc.*, 3 (1971) 70–74.

[11] J.B. Wilker, Inversive Geometry, in *The Geometric Vein: The Coxeter Festschrift*, ed. by Chandler Davis et al., Springer-Verlag (1982), 379–442.

David Graham Searby (1948 - 1998): David Searby was born on October 30, 1948 in Melbourne, Australia, and lived in both Melbourne and Adelaide while growing up. He graduated from Flinders University of South Australia with a Bachelor of Science degree (1970) and a Bachelor of Science Honours (1972). He began graduate work there under the supervision of John Wamsley, with whom he published his first paper [8]. He moved to Italy in 1974, and for three years was supported by a foreign-student scholarship at the University of Bologna, funded by the National Research Council (CNR) of Italy. He lived in Bologna for the remainder of his life; although he maintained a tenuous relationship with the university, he survived by giving private lessons in English and in mathematics. He was an effective and inspiring teacher (although he taught relatively little) and scholar – in addition to his mathematics, he knew every corner of Italy, its history and literature; he spoke flawless Italian and even mastered the local Bolognese dialect. He was driven by boundless curiosity and intellectual excitement, and loved to spend long evenings in local bars, sustained by soup, beer and ideas. His lifestyle, however, was unsustainable, and he died on August 19, 1998, just short of his 50th birthday. He left behind a box full of his notes and computations in no apparent order. One of his research interests concerned configurations in projective planes, both classical and finite. An early paper [9] on the existence of Pappus configurations in planes of order nine indicated the direction his research was to take: he found that in the Hughes Plane of order nine there exist triangles which fail to contain a Pappus configuration that has three points on each of its sides. Were coordinates introduced using such a triangle as the triangle of reference, the imposed algebraic structure would be nearly trivial. From this Searby speculated on the existence of finite projective planes whose order is not the power of a prime, and whose coordinates have so little structure that the plane could be discovered only by computer. He collected configuration theorems throughout most of his life with a goal toward finding configurations on which he could base a computer search. Unfortunately, he never had access to a suitable computer. Among his papers was the first draft of a monograph (in Italian) that brought together many of his elementary discoveries on configurations; it is highly readable, but a long way from being publishable. There was also the present paper, almost ready to submit for publication, which brings together several of his discoveries involving configuration theorems. It has been lightly edited by me; I added the footnotes, references, and figures. I would like to thank David's brother Michael and his friends David Glynn, Ann Powell, and David Tiley for their help in preparing the biographical note. (J. Chris Fisher)

Forum Geometricorum
Volume 9 (2009) 195–211.

FORUM GEOM
ISSN 1534-1178

Class Preserving Dissections of Convex Quadrilaterals

Dan Ismailescu and Adam Vojdany

Abstract. Given a convex quadrilateral Q having a certain property $\mathcal{P}$, we are interested in finding a dissection of Q into a finite number of smaller convex quadrilaterals, each of which has property $\mathcal{P}$ as well. In particular, we prove that every cyclic, orthodiagonal, or circumscribed quadrilateral can be dissected into cyclic, orthodiagonal, or circumscribed quadrilaterals, respectively. The problem becomes much more interesting if we restrict our study to a particular type of partition we call *grid dissection*.

1. Introduction

The following problem represents the starting point and the motivation of this paper.

Problem. Find all convex polygons which can be dissected into a finite number of pieces, each similar to the original one, but not necessarily congruent.

It is easy to see that all triangles and parallelograms have this property (see e.g. [1, 7]).

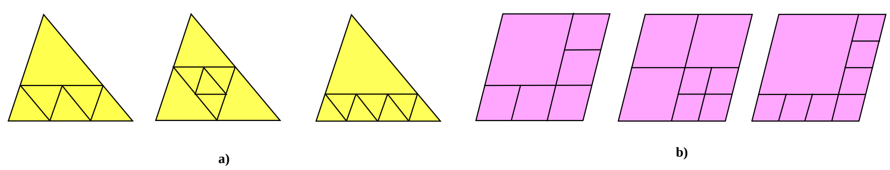


Figure 1. (a) Triangle dissection into similar triangles.
(b) Parallelogram dissection into similar parallelograms.

Indeed, every triangle can be partitioned into 6, 7 or 8 triangles, each similar to the initial one (see Figure 1 a). Simple inductive reasoning shows that for every $k \geq 6$, any triangle T can be dissected into k triangles similar to T. An analogous statement is true for parallelograms (see Figure 1 b). Are there any other polygons besides these two which have this property?

The origins of Problem 1 can be traced back to an early paper of Langford [10]. More then twenty years later, Golomb [8] studied the same problem without notable success. It was not until 1974 when the first significant results were published by Valette and Zamfirescu.

Theorem 1 (Valette and Zamfirescu, [13])**.** *Suppose a given convex polygon P can be dissected into four congruent tiles, each of which similar to P. Then P is either a triangle, a parallelogram or one of the three special trapezoids shown in Figure 2 below.*

Publication Date: September 14, 2009. Communicating Editor: Paul Yiu.

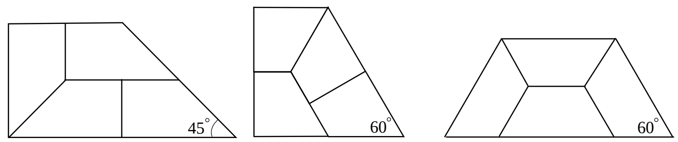


Figure 2. Trapezoids which can be partitioned into four congruent pieces.

Notice that the hypothesis of the above theorem is much more restrictive: the number of pieces must be exactly four and the small polygons must all be congruent to each other, not only similar. However, as of today, the convex polygons presented in figures 1 and 2 are the only known solutions to the more general problem 1.

From a result of Bleicher [2], it is impossible to dissect a convex n-gon (a convex polygon with n vertices) into a finite number of convex n-gons if $n \geq 6$. The same result was proved by Bernheim and Motzkin [3] using slightly different techniques.

Although any convex pentagon can be partitioned into any number $k \geq 6$ of convex pentagons, a recent paper by Ding, Schattschneider and Zamfirescu [4] shows that it is impossible to dissect a convex pentagon into similar replicas of itself.

Given the above observations, it follows that for solving problem 1 we can restrict ourselves to convex quadrilaterals. It is easy to prove that a necessary condition for a quadrilateral to admit a dissection into similar copies of itself is that the measures of its angles are linearly dependent over the integers. Actually, a stronger statement holds true: if the angles of a convex quadrilateral Q do not satisfy this dependence condition, then Q cannot be dissected into a finite number of smaller similar convex polygons which are not necessarily similar to Q (for a proof one may consult [9]). Nevertheless, in spite of all the above simplifications and renewed interest in the geometric dissection topic (see e. g. [6, 12, 15]), problem 1 remains open.

2. A Related Dissection Problem

Preserving similarity under dissection is difficult: although all triangles have this property, there are only a handful of known quadrilaterals satisfying this condition (parallelograms and some special trapezoids), while no n-gon can have this property if $n \geq 5$. In the sequel, we will try to examine what happens if we weaken the similarity requirement.

Problem. Suppose that a given polygon P has a certain property $\mathcal{C}$. Is it possible to dissect P into smaller polygons, each having property $\mathcal{C}$ as well?

For instance, suppose $\mathcal{C}$ means "convex polygon with n sides". As we have mentioned in the previous section, in this particular setting Problem 2 has a positive answer if $3 \leq n \leq 5$ and a negative answer for all $n \geq 6$. Before we proceed we need the following:

Definition. a) A convex quadrilateral is said to be *cyclic* if there exists a circle passing through all of its vertices.
b) A convex quadrilateral is said to be *orthodiagonal* if its diagonals are perpendicular.
c) A convex quadrilateral is said to be *circumscribed* if there exists a circle tangent to all of its sides.
d) A convex quadrilateral is said to be a *kite* if it is both orthodiagonal and circumscribed.

The following theorem provides characterizations for all of the quadrilaterals defined above and will be used several times throughout the remainder of the paper.

Theorem 2. *Let $ABCD$ be a convex quadrilateral.*
(a) *$ABCD$ is cyclic if and only if opposite angles are supplementary – say, $\angle A + \angle C = 180°$.*
(b) *$ABCD$ is orthodiagonal if and only if the sum of squares of two opposite sides is equal to the sum of the squares of the remaining opposite sides – that is, $AB^2 + CD^2 = AD^2 + BC^2$.*
(c) *$ABCD$ is circumscribed if and only if the two pairs of opposite sides have equal total lengths – that is, $AB + CD = AD + BC$.*
(d) *$ABCD$ is a kite if and only if (after an eventual relabeling) $AB = BC$ and $CD = DA$.*

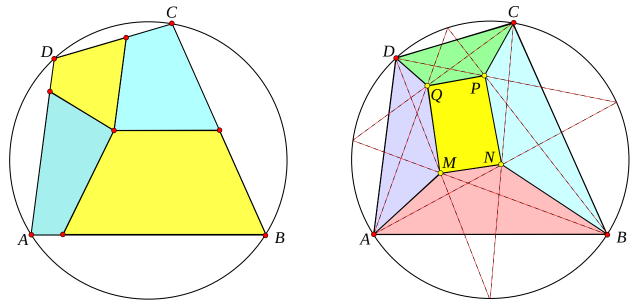
A comprehensive account regarding cyclic, orthodiagonal and circumscribed quadrilaterals and their properties, including proofs of the above theorem, can be found in the excellent collection of geometry notes [14]. An instance of Problem 2 we will investigate is the following:

Problem. Is it true that every cyclic, orthodiagonal or circumscribed quadrilateral can be dissected into cyclic, orthodiagonal or circumscribed quadrilaterals, respectively?

It has been shown in [1] and [11] that every cyclic quadrilateral can be dissected into **four** cyclic quadrilaterals two of which are isosceles trapezoids (see Figure 3 a).

Another result is that every cyclic quadrilateral can be dissected into **five** cyclic quadrilaterals one of which is a rectangle (see Figure 3 b). This dissection is based on the following property known as *The Japanese Theorem* (see [5]).

Theorem 3. *Let $ABCD$ be a cyclic quadrilateral and let M, N, P and Q be the incenters of triangles ABD, ABC, BCD and ACD, respectively. Then $MNPQ$ is a rectangle and quadrilaterals $AMNB$, $BNPC$, $CPQD$ and $DQMA$ are cyclic* (see Figure 3).

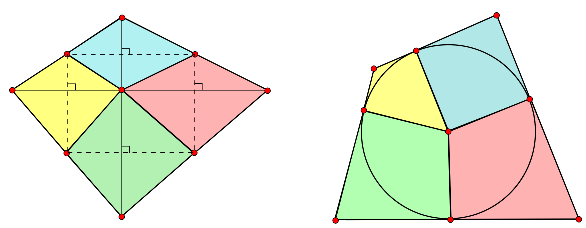


a) b)

Figure 3. (a) Cyclic quadrilateral = 2 isosceles trapezoids + 2 cyclic quadrilaterals.
(b) Cyclic quadrilateral = one rectangle + four cyclic quadrilaterals.

Since every isosceles trapezoid can be dissected into an arbitrary number of isosceles trapezoids, it follows that every cyclic quadrilateral can be dissected into k cyclic quadrilaterals, for every $k \geq 4$.

It is easy to dissect an orthodiagonal quadrilateral into four smaller orthodiagonal ones.



a) b)

Figure 4. (a) Orthodiagonal quadrilateral = four orthodiagonal quadrilaterals.
(b) Circumscribed quadrilateral = four circumscribed quadrilaterals.

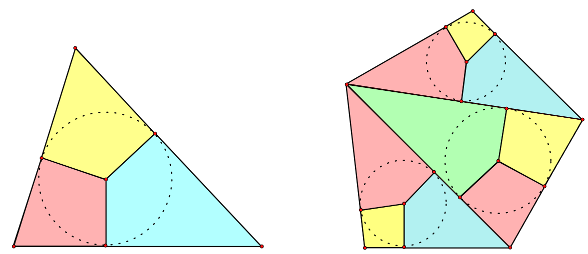
Consider for instance the quadrilaterals whose vertex set consists of one vertex of the initial quadrilateral, the midpoints of the sides from that vertex and the intersection point of the diagonals (see Figure 4 a). It is easy to prove that each of these quadrilaterals is orthodiagonal.

A circumscribed quadrilateral can be dissected into four quadrilaterals with the same property by simply taking the radii from the incenter to the tangency points (see Figure 4 b).

Actually, it is easy to show that each of these smaller quadrilaterals is not only circumscribed but cyclic and orthodiagonal as well.

The above discussion provides a positive answer to problem 2. In fact, much more is true.

Theorem 4 (Dissecting arbitrary polygons). *Every convex n-gon can be partitioned into $3\,(n-2)$ cyclic kites* (see Figure 5).



a) b)

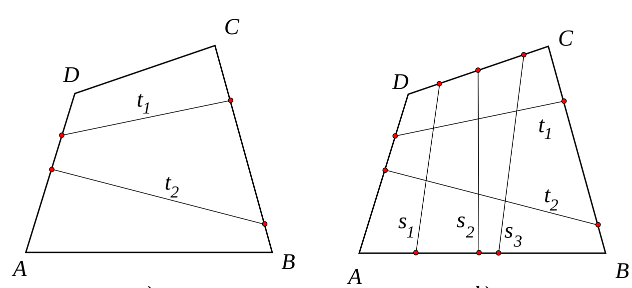
Figure 5. Triangle = three cyclic kites; Pentagon = nine cyclic kites.

Proof. Notice first that every triangle can be dissected into three cyclic kites by dropping the radii from the incenter to the tangency points (see Figure 5 a). Partition the given n-gon into triangles. For instance, one can do this by drawing all the diagonals from a certain vertex. We obtain a triangulation consisting of $n-2$ triangles. Dissect then each triangle into cyclic kites as indicated in Figure 5 b). □

3. Grid Dissections of Convex Quadrilaterals

We have seen that the construction used in theorem 4 renders problem 2 almost trivial. The problem becomes much more challenging if we do restrict the type of dissection we are allowed to use. We need the following

Definition. Let $ABCD$ be a convex quadrilateral and let m and n be two positive integers. Consider two sets of segments $\mathcal{S} = \{s_1, s_2, \ldots, s_{m-1}\}$ and $\mathcal{T} = \{t_1, t_2, \ldots, t_{n-1}\}$ with the following properties:
a) If $s \in \mathcal{S}$ then the endpoints of s belong to the sides AB and CD. Similarly, if $t \in \mathcal{T}$ then the endpoints of t belong to the sides AD and BC.
b) Every two segments in $\mathcal{S}$ are pairwise disjoint and the same is true for the segments in $\mathcal{T}$.
We then say that segments $s_1, s_2, \ldots, s_{m-1}, t_1, t_2, \ldots, t_{n-1}$ define an m-by-n *grid dissection* of $ABCD$ (see Figure 6).



a) b)

Figure 6. A 3-by-1 and a 3-by-4 grid dissection of a convex quadrilateral

The really interesting problem is the following:

Problem. Is it true that every cyclic, orthodiagonal or circumscribed quadrilateral can be partitioned into cyclic, orthodiagonal, or circumscribed quadrilaterals,respectively, **via a grid dissection?** Such dissections shall be referred to as *class preserving grid dissections,* or for short *CPG* dissections (or CPG partitions).

3.1. *Class Preserving Grid Dissections of Cyclic Quadrilaterals.* In this section we study whether cyclic quadrilaterals have class preserving grid dissections. We start with the following

Question. Under what circumstances does a cyclic quadrilateral admit a 2-by-1 grid dissection into cyclic quadrilaterals? What about a 2-by-2 grid dissection with the same property?

The answer can be readily obtained after a straightforward investigation of the sketches presented in Figure 7.

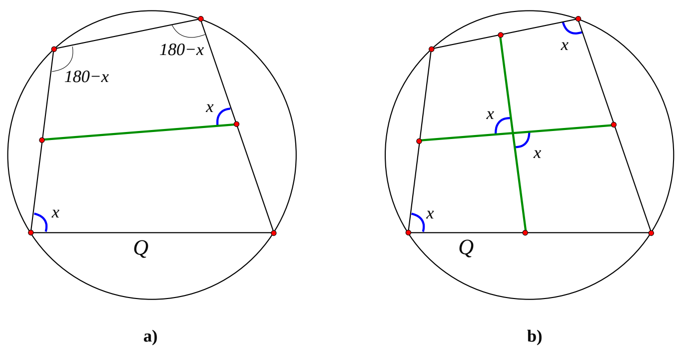


Figure 7. (a) 2-by-1 CPG dissection of cyclic Q exists iff $Q =$ trapezoid.
(b) 2-by-2 CPG dissection of cyclic Q exists iff $Q =$ rectangle.

A quick analysis of the angles reveals that a 2-by-1 CPG partition is possible if only if the initial cyclic quadrilateral is an isosceles trapezoid - see Figure 7 a). A similar reasoning leads to the conclusion that a 2-by-2 CPG partition exists if and only if the original quadrilateral is a rectangle – Figure 7 b). These observations can be easily extended to the following:

Theorem 5. *Suppose a cyclic quadrilateral Q has an m-by-n grid partition into mn cyclic quadrilaterals. Then:*

a) If m and n are both even, Q is necessarily a rectangle.

b) If m is odd and n is even, Q is necessarily an isosceles trapezoid.

We leave the easy proof for the reader. It remains to see what happens if both m and n are odd. The next two results show that in this case the situation is more complex.

Theorem 6 (A class of cyclic quadrilaterals which have 3-by-1 CPG dissections)**.** *Every cyclic quadrilateral all of whose angles are greater than* $\arccos \frac{\sqrt{5}-1}{2} \approx 51.83°$ *admits a* 3*-by-*1 *grid dissection into three cyclic quadrilaterals.*

Proof. If $ABCD$ is an isosceles trapezoid, then any two segments parallel to the bases will give the desired dissection. Otherwise, assume that $\angle D$ is the largest angle (a relabeling of the vertices may be needed). Since $\angle B + \angle D = \angle A + \angle C = 180^\circ$ it follows that $\angle B$ is the smallest angle of $ABCD$. We therefore have:

$$\angle B < \min\{\angle A,\ \angle C\} \le \max\{\angle A,\ \angle C\} < \angle D. \tag{1}$$

Denote the measures of the arcs $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$ and $\widehat{DA}$ on the circumcircle of $ABCD$ by $2a$, $2b$, $2c$ and $2d$ respectively (see Figure 8 a). Inequalities (1) imply that $c + d < \min\{b + c,\ a + d\} \le \max\{b + c,\ a + d\} < a + b$, that is, $c < a$ and $d < b$.

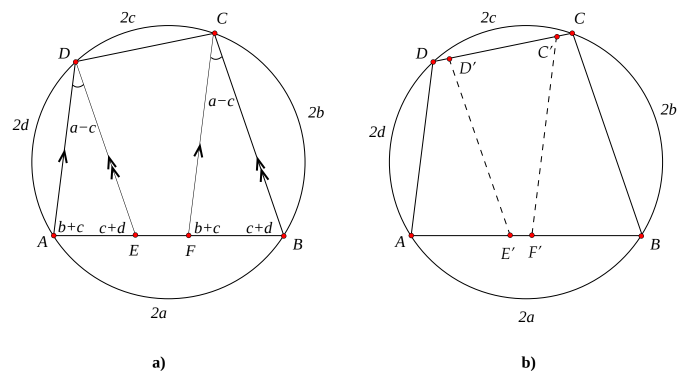


Figure 8. (a) $DE \parallel BC$, $CF \parallel AD$, E and F between A and B due to $c < a$.
(b) 3-by-1 grid dissection into cyclic quads if DE and CF do not intersect.

Through vertex D construct a segment $DE \parallel BC$ with E on line AB. Since $c < a$, point E is going to be between A and B. Similarly, through vertex C construct a segment $CF \parallel AD$ with F on AB. As above, since $c < a$, point F will lie between A and B.

If segments DE and CF do not intersect then a 3-by-1 grid dissection of $ABCD$ into cyclic quadrilaterals can be obtained in the following way:
Choose two points C' and D' on side CD, such that C' is close to C and D' is close to D. Construct $D'E' \parallel DE$ and $C'F' \parallel CF$ as shown in figure 8 b). Since segments DE and CF do not intersect it follows that for choices of C' and D' sufficiently close to C and D respectively, the segments $D'E'$ and $C'F'$ will not intersect. A quick verification shows that each of the three quadrilaterals into which $ABCD$ is dissected ($AE'D'D$, $D'E'F'C'$ and $C'F'BC$) is cyclic.

It follows that a sufficient condition for this grid dissection to exist is that points $A - E - F - B$ appear exactly in this order along side AB, or equivalently, $AE + BF < AB$.

The law of sines in triangle ADE gives that $AE \sin(c + d) = AD \sin(a - c)$ and since $AD = 2R \sin d$ we obtain

$$AE = \frac{2R \sin d\ \sin(a - c)}{\sin(c + d)}, \tag{2}$$

where R is the radius of the circumcircle of $ABCD$.

Similarly, using the law of sines in triangle BCF we have $BF\,\sin(b+c) = BC\,\sin(a-c)$ and since $BC = 2R\,\sin b$ it follows that

$$BF = \frac{2R\sin b\,\sin(a-c)}{\sin(b+c)}. \tag{3}$$

Using equations (2), (3) and the fact that $AB = 2R\,\sin a$, the desired inequality $AE + BF < AB$ becomes equivalent to

$$\begin{aligned}
&\frac{\sin d\,\sin(a-c)}{\sin(c+d)} + \frac{\sin b\,\sin(a-c)}{\sin(b+c)} < \sin a\\
\Leftrightarrow\ &\frac{\sin d}{\sin(c+d)} + \frac{\sin(b+c-c)}{\sin(b+c)} < \frac{\sin(a-c+c)}{\sin(a-c)}\\
\Leftrightarrow\ &\frac{\sin d}{\sin(c+d)} + \cos c - \sin c\,\cot(b+c) < \cos c + \sin c\,\cot(a-c),
\end{aligned}$$

and after using $a+b+c+d = 180^\circ$ and simplifying further,

$$AE + BF < AB \Leftrightarrow \sin(a-c)\,\sin(b+c)\,\sin d < \sin^2(c+d)\,\sin(c). \tag{4}$$

Recall that points E and F belong to AB as a result of the fact that $c < a$. A similar construction can be achieved using the fact that $d < b$.

Let $AG \parallel CD$ and $DH \parallel AB$ as shown in Figure 9 a). Since $d < b$, points G and H will necessarily belong to side BC. As in the earlier analysis, if segments AG and DH do not intersect, small parallel displacements of these segments will produce a 3-by-1 grid partition of $ABCD$ into 3 cyclic quadrilaterals: $ABG'A'$, $A'G'H'D''$ and $H'D''DC$ (see Figure 9 b).

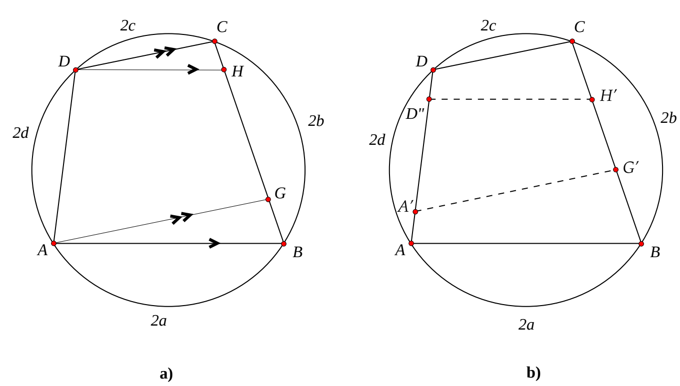


Figure 9. a) $AG \parallel CD$, $DH \parallel AB$, G and H between B and C since $d < b$
b) 3-by-1 CPG grid dissection if AG and DH do not intersect.

The sufficient condition for this construction to work is that points $B{-}G{-}H{-}C$ appear in this exact order along side BC, or equivalently, $BG + CH < BC$.

Using similar reasoning which led to relation (4) we obtain that

$$BG + CH < BC \Leftrightarrow \sin(b-d)\,\sin(a+d)\,\sin c < \sin^2(a+b)\,\sin d. \tag{5}$$

The problem thus reduces to proving that if $\min\{\angle A, \angle B, \angle C, \angle D\} > \arccos \frac{\sqrt{5}-1}{2}$ then at least one of the inequalities that appear in (4) and (5) will hold.
To this end, suppose none of these inequalities is true. We thus have:

$$\begin{aligned} \sin(a-c)\,\sin(b+c) &\geq \sin^2(c+d)\cdot\frac{\sin c}{\sin d} \qquad \text{and,} \\ \sin(b-d)\,\sin(a+d) &\geq \sin^2(a+b)\cdot\frac{\sin d}{\sin c}. \end{aligned}$$

Recall that $a+b+c+d = 180^\circ$ and therefore $\sin(a+d) = \sin(b+c)$ and $\sin(a+b) = \sin(c+d)$. Adding the above inequalities term by term we obtain

$$\begin{aligned} &\sin(b+c)\cdot(\sin(a-c)+\sin(b-d)) \geq \sin^2(c+d)\cdot\left(\frac{\sin c}{\sin d}+\frac{\sin c}{\sin d}\right) \\ \Rightarrow\ &\sin(b+c)\cdot 2\cdot\sin(90^\circ-c-d)\cdot\cos(90^\circ-b-d) \geq \sin^2(c+d)\cdot 2 \\ \Leftrightarrow\ &\sin(b+c)\cdot\sin(c+d)\cdot\cos(c+d) \geq \sin^2(c+d) \\ \Rightarrow\ &\cos(c+d) \geq 1-\cos^2(c+d) \\ \Rightarrow\ &\cos(c+d) = \cos(\angle B) \geq \frac{\sqrt{5}-1}{2}, \text{contradiction.} \end{aligned}$$

This completes the proof. Notice that the result is the best possible in the sense that $\arccos\frac{\sqrt{5}-1}{2} \approx 51.83^\circ$ cannot be replaced by a smaller value. Indeed, it is easy to check that a cyclic quad whose angles are $\arccos\frac{\sqrt{5}-1}{2}$, 90°, 90° and $180^\circ - \arccos\frac{\sqrt{5}-1}{2}$ does not have a 3-by-1 grid partition into cyclic quadrilaterals. $\square$

The following result can be obtained as a corollary of Theorem 6.

Theorem 7. (A class of cyclic quadrilaterals which have 3-by-3 grid dissections) *Let $ABCD$ be a cyclic quadrilateral such that the measure of each of the arcs $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$ and $\widehat{DA}$ determined by the vertices on the circumcircle is greater than 60°. Then $ABCD$ admits a 3-by-3 grid dissection into nine cyclic quadrilaterals.*

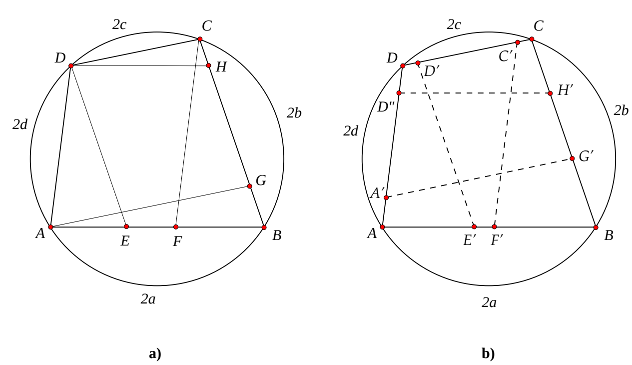


Figure 10. a) $DE \parallel BC$, $CF \parallel AD$, $AG \parallel CD$, $DH \parallel AB$
b) 3-by-3 grid dissection into nine cyclic quadrilaterals.

Proof. Notice that the condition regarding the arc measures is stronger than the requirement that all angles of $ABCD$ exceed 60°. We will use the same assumptions and notations as in Theorem 6. The idea is to overlay the two constructions in Theorem 6 (see Figure 10).

It is straightforward to check that each of the nine quadrilaterals shown in figure 10 b) is cyclic. The problem reduces to proving that $\min\{a,\, b,\, c,\, d\} > 30^\circ$ implies that both inequalities in (4) and (5) hold simultaneously. Due to symmetry it is sufficient to prove that (5) holds. Indeed,

$$\begin{aligned} BG + CH < BC &\Leftrightarrow\ \sin(b-d)\,\sin(b+c)\,\sin c < \sin^2(c+d)\,\sin d \\ &\Leftrightarrow\ \cos(c+d) - \cos(2b+c-d) < \frac{2\,\sin^2(c+d)\,\sin d}{\sin c} \\ &\Leftarrow\ \cos(c+d) + 1 < \frac{2\sin^2(c+d)\sin d}{\sin c} \\ &\Leftrightarrow\ 2\cos^2\frac{c+d}{2}\sin c < 8\sin^2\frac{c+d}{2}\cos^2\frac{c+d}{2}\sin d \\ &\Leftrightarrow\ \sin c < 2\,\sin d\cdot(1-\cos(c+d)) \\ &\Leftrightarrow\ \sin c < 2\,\sin d - 2\,\sin d\,\cos(c+d) \\ &\Leftrightarrow\ \sin c < 2\,\sin d + \sin c - \sin(c+2d)) \\ &\Leftrightarrow\ \sin(c+2d) < 2\sin d \\ &\Leftarrow\ 1 < 2\sin d. \end{aligned}$$

The last inequality holds true since we assumed $d > 30^\circ$. This completes the proof.

□

3.2. *Class Preserving Grid Dissections of Orthodiagonal Quadrilaterals.* It is easy to see that an orthodiagonal quadrilateral cannot have a 2-by-1 grid dissection into orthodiagonal quadrilaterals. Indeed, if say we attempt to dissect the quadrilateral $ABCD$ with a segment MP, where M is on AB and P is on CD, then the diagonals of $ADPM$ are forced to intersect in the interior of the right triangle AOD, preventing them from being perpendicular to each other (see Figure 11 a).

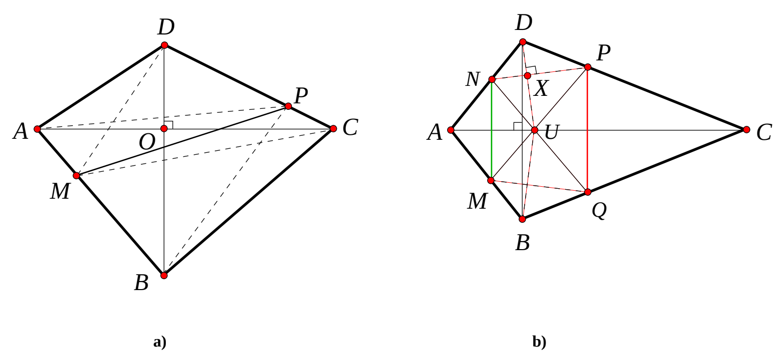


Figure 11. a) Orthodiagonal quadrilaterals have no 2-by-1 CPG dissections
b) A kite admits infinitely many 2-by-2 CPG dissections

The similar question concerning the existence of 2-by-2 CPG dissections turns out to be more difficult. We propose the following:

Conjecture 1. *An orthodiagonal quadrilateral has a* 2*-by-*2 *grid dissection into four orthodiagonal quadrilaterals if and only if it is a kite.*

The "only if" implication is easy to prove. We can show that every kite has infinitely many 2-by-2 CPG grid dissections. Indeed, let $ABCD$ be a kite ($AB = AD$ and $BC = CD$) and let $MN \parallel BD$ with M and N fixed points on sides AB and respectively AD. Consider then a variable segment $PQ \parallel BD$ as shown in figure 11 b). Denote $U = NQ \cap MP$; due to symmetry $U \in AC$. Consider the grid dissection generated by segments MP and NQ. Notice that quadrilaterals $ANUM$ and $CPUQ$ are orthodiagonal independent of the position of PQ. Also, quadrilaterals $DNUP$ and $BMUQ$ are congruent and therefore it is sufficient to have one of them be orthodiagonal.

Let point P slide along CD. If P is close to vertex C, it follows that Q and U are also close to C and therefore the measure of angle $\angle DXN$ is arbitrarily close to the measure of $\angle DCN$, which is acute. On the other hand, when P is close to vertex D, Q is close to B and the angle $\angle DXN$ becomes obtuse.

Since the measure of $\angle DXN$ depends continuously on the position of point P it follows that for some intermediate position of P on CD we will have $\angle DXN = 90°$. For this particular choice of P both $DNUP$ and $BMUQ$ are orthodiagonal. This proves the "only if" part of the conjecture.

Extensive experimentation with Geometer's Sketchpad strongly suggests the direct statement also holds true. We used MAPLE to verify the conjecture in several particular cases - for instance, the isosceles orthodiagonal trapezoid with base lengths of 1 and $\sqrt{7}$ and side lengths 2 does not admit a 2-by-2 dissection into orthodiagonal quadrilaterals.

3.3. *Class Preserving Grid Dissections of Circumscribed Quadrilaterals.* After the mostly negative results from the previous sections, we discovered the following surprising result.

Theorem 8. *Every circumscribed quadrilateral has a* 2*-by-*2 *grid dissection into four circumscribed quadrilaterals.*

Proof. (Sketch) This is in our opinion a really unexpected result. It appears to be new and the proof required significant amounts of inspiration and persistence. We approached the problem analytically and used MAPLE extensively to perform the symbolic computations. Still, the problem presented great challenges, as we will describe below.

Let $MNPQ$ be a circumscribed quadrilateral with incenter O. With no loss of generality suppose the incircle has unit radius. Let O_i, $1 \leq i \leq 4$ denote projections of O onto the sides as shown in Figure 12 a). Denote the angles $\angle O_4OO_1 = 2a$, $\angle O_1OO_2 = 2b$, $\angle O_2OO_3 = 2c$ and $\angle O_3OO_4 = 2d$. Clearly, $a + b + c + d = 180°$ and $\max\{a, b, c, d\} < 90°$. Consider a coordinate system centered at O such that the coordinates of O_4 are $(1, 0)$.

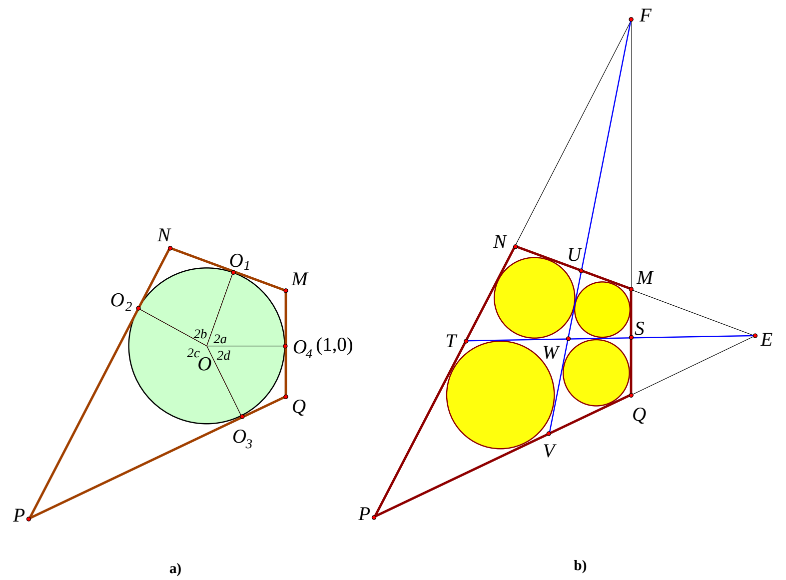


Figure 12. a) A circumscribed quadrilateral
b) Attempting a 2-by-2 CPG dissection with lines through E and F

We introduce some more notation: $\tan a = A$, $\tan b = B$, $\tan c = C$, $\tan d = D$. Notice that quantities A, B, C and D are not independent. Since $a+b+c+d = 180^\circ$ it follows that $A+B+C+D = ABC+ABD+ACD+BCD$. Moreover, since $\max\{a,\, b,\, c,\, d\} < 90^\circ$ we have that A, B, C and D are all positive.

It is now straightforward to express the coordinates of the vertices M, N, P and Q in terms of the tangent values A, B, C and D. Two of these vertices have simple coordinates: $M(1, A)$ and $Q(1, -D)$. The other two are

$$N\left(\frac{1-A^2-2AB}{1+A^2}, \frac{2A+B-A^2B}{1+A^2}\right) \quad \text{and} \quad P\left(\frac{1-D^2-2CD}{1+D^2}, \frac{C+2D-CD^2}{1+D^2}\right).$$

The crux of the proof lies in the following idea. Normally, we would look for four points (one on each side), which create the desired 2-by-2 grid partition. We would thus have four degrees of freedom (choosing the points) and four equations (the conditions that each of the smaller quadrilaterals formed is circumscribed).

However, the resulting algebraic system is extremely complicated. Trying to eliminate the unknowns one at a time leads to huge resultants which even MAPLE cannot handle.

Instead, we worked around this difficulty. Extend the sides of $MNPQ$ until they intersect at points E and F as shown in Figure 12 b). (Ignore the case when $MNPQ$ is a trapezoid for now). Now locate a point U on side MN and a point S on side QM such that when segments FU and ES are extended as in the figure, the four resulting quadrilaterals are all circumscribed. This reduces the number of variables from four to two and thus the system appears to be over-determined. However, extended investigations with Geometer's Sketchpad indicated that this construction is possible. At this point we start computing the coordinates of the newly introduced points. We have

$$E\left(\frac{1+AD}{1-AD}, \frac{A-D}{1-AD}\right) \quad \text{and} \quad F\left(1, \frac{A+B}{1-AB}\right).$$

Denote

$$m = \frac{MU}{MN} \qquad \text{and} \qquad q = \frac{QS}{QM}.$$

Clearly, the coordinates of U and V are rational functions on m, A, B, C and D while the coordinates of S and T depend in a similar manner on q, A, B, C and D. These expressions are quite complicated; for instance, each one of the coordinates of point T takes five full lines of MAPLE output. The situation is the same for the coordinates of point V.

Define the following quantities:

$$\begin{aligned} Z_1 &= MU + WS - WU - MS, \\ Z_2 &= NT + WU - WT - NU, \\ Z_3 &= PV + WT - WV - PT, \\ Z_4 &= QS + WV - WS - QV. \end{aligned}$$

By Theorem 2 b), a necessary and sufficient condition for the quadrilaterals $MUWS$, $NTWU$, $PVWT$ and $QSWV$ to be cyclic is that $Z_1 = Z_2 = Z_3 = Z_4 = 0$. Notice that

$$Z_1 + Z_2 + Z_3 + Z_4 = MU - NU + NT - PT + PV - QV + QS - MS \quad (6)$$

and

$$\begin{aligned} Z_1 - Z_2 + Z_3 - Z_4 &= MN - NP + PQ - QM + 2(WS + WT - WU - WV) \\ &= 2(ST - UV), \end{aligned} \quad (7)$$

the last equality is due to the fact that $MNPQ$ is circumscribed.

Since we want $Z_i = 0$ for every $1 \leq i \leq 4$, we need to have the right hand terms from (6) and (7) each equal to 0. In other words, **necessary** conditions for finding the desired grid dissection are

$$MU - NU + PV - QV = PT - NT + MS - QS \qquad \text{and} \qquad UV = ST. \quad (8)$$

There is a two-fold advantage we gain by reducing the number of equations from four to two: first, the system is significantly simpler and second, we avoid using point W - the common vertex of all four small quadrilaterals which is also the point with the most complicated coordinates.

System (8) has two equations and two unknowns - m and q - and it is small enough for MAPLE to handle. Still, after eliminating variable q, the resultant is a polynomial of degree 10 of m with polynomial functions of A, B, C and D as coefficients.

This polynomial can be factored and the value of m we are interested in is a root of a quadratic. Although m does not have a rational expression depending on A, B, C and D it can still be written in terms of $\sqrt{\sin a}$, $\sqrt{\cos a}$, $\ldots$, $\sqrt{\sin d}$, $\sqrt{\cos d}$.

Explicit Formulation of Theorem 8. Let $MNPQ$ be a circumscribed quadrilateral as described in figure 12. Denote

$$\begin{aligned} s_1 &= \sqrt{\sin a}, \quad s_2 = \sqrt{\sin b}, \quad s_3 = \sqrt{\sin c}, \quad s_4 = \sqrt{\sin d} \\ c_1 &= \sqrt{\cos a}, \quad c_2 = \sqrt{\cos b}, \quad c_3 = \sqrt{\cos c}, \quad c_4 = \sqrt{\cos d}. \end{aligned}$$

Define points $U \in MN$, $T \in NP$, $V \in TQ$ and $S \in QM$ such that

$$\frac{MU}{MN} = m, \quad \frac{NT}{NP} = n, \quad \frac{PV}{PQ} = p, \quad \frac{QS}{QM} = q$$

where

$$m = \frac{s_2 s_4 c_2^2 (s_4 s_1 (s_1^2 c_2^2 + s_2^2 c_1^2) + s_2 s_3)}{(s_1^2 c_2^2 + s_2^2 c_1^2)(s_1 s_2 s_4 + s_3 c_1^2)(s_1 s_2 s_3 + s_4 c_2^2)}, \tag{9}$$

$$n = \frac{s_3 s_1 c_3^2 (s_1 s_2 (s_2^2 c_3^2 + s_3^2 c_2^2) + s_3 s_4)}{(s_2^2 c_3^2 + s_3^2 c_2^2)(s_2 s_3 s_1 + s_4 c_2^2)(s_2 s_3 s_4 + s_1 c_3^2)}, \tag{10}$$

$$p = \frac{s_4 s_2 c_4^2 (s_2 s_3 (s_3^2 c_4^2 + s_4^2 c_3^2) + s_4 s_1)}{(s_3^2 c_4^2 + s_4^2 c_3^2)(s_3 s_4 s_2 + s_1 c_3^2)(s_3 s_4 s_1 + s_2 c_4^2)}, \tag{11}$$

$$q = \frac{s_1 s_3 c_1^2 (s_3 s_4 (s_4^2 c_1^2 + s_1^2 c_4^2) + s_1 s_2)}{(s_4^2 c_1^2 + s_1^2 c_4^2)(s_4 s_1 s_3 + s_2 c_4^2)(s_4 s_1 s_2 + s_3 c_1^2)}. \tag{12}$$

Denote $W = ST \cap UV$. Then, quadrilaterals $MUWS$, $NTWU$, $PVWT$ and $QSWV$ are all circumscribed (*i.e.*, $Z_1 = Z_2 = Z_3 = Z_4 = 0$).
Verifying these assertions was done in MAPLE. Recall that m and q were obtained as solutions of the system $Z_1 + Z_2 + Z_3 + Z_4 = 0$, $Z_1 - Z_2 + Z_3 - Z_4 = 0$. At this point it is not clear why for these choices of m, n, p and q we actually have $Z_i = 0$, for all $1 \le i \le 4$.

Using the expressions of m, n, p and q given above, we can write the coordinates of all points that appear in figure 12 in terms of s_i and c_i where $1 \le i \le 4$. We can then calculate the lengths of all the twelve segments which appear as sides of the smaller quadrilaterals.

For instance we obtain:

$$\begin{aligned} MU &= \frac{(s_1^3 s_4 c_2^2 + s_1 s_2^2 s_4 c_1^2 + s_2 s_3) s_2 s_4}{c_1^2 (s_1 s_2 s_3 + c_2^2 s_4)(s_1 s_2 s_4 + c_1^2 s_3)}, \\ NU &= \frac{(s_2^3 s_3 c_1^2 + s_1^2 s_2 s_3 c_2^2 + s_1 s_4) s_1 s_3}{c_2^2 (s_1 s_2 s_3 + c_2^2 s_4)(s_1 s_2 s_4 + c_1^2 s_3)}. \end{aligned}$$

and similar relations can be written for NT, PV, QS and PT, QV, MS by circular permutations of the expressions for MU and NU, respectively.

In the same way it can be verified that

$$\begin{aligned} &UV = ST \\ &= \frac{(s_1^2 c_4^2 + s_4^2 c_1^2)(s_1^2 s_4^2 + s_3^2 s_2^2 + 2 s_1 s_2 s_3 s_4 (s_1^2 c_2^2 + s_2^2 c_1^2))(s_3^2 c_1^2 + s_1^2 c_3^2 + 2 s_1 s_2 s_3 s_4)}{(s_1 s_2 s_3 + c_2^2 s_4)(s_2 s_3 s_4 + c_3^2 s_1)(s_3 s_4 s_1 + c_4^2 s_2)(s_1 s_2 s_4 + c_1^2 s_3)} \end{aligned}$$

and

$$UW = \frac{\lambda_r \cdot UV}{\lambda_r + \mu_r}, \quad VW = \frac{\mu_r \cdot UV}{\lambda_r + \mu_r}, \quad SW = \frac{\lambda_s \cdot ST}{\lambda_s + \mu_s}, \quad TW = \frac{\mu_s \cdot ST}{\lambda_s + \mu_s},$$

where

$$\begin{aligned}
\lambda_r &= (s_1^3 s_2 c_4^2 + s_1 s_2 s_4^2 c_1^2 + s_3 s_4)(s_2 s_3 s_4 + s_1 c_3^2)(s_3 s_4 s_1 + s_2 c_4^2) \\
\mu_r &= (s_3^3 s_4 c_2^2 + s_3 s_4 s_2^2 c_3^2 + s_1 s_2)(s_4 s_1 s_2 + s_3 c_1^2)(s_1 s_2 s_3 + s_4 c_2^2) \\
\lambda_s &= (s_4^3 s_1 c_3^2 + s_4 s_1 s_3^2 c_4^2 + s_2 s_3)(s_2 s_3 s_4 + s_1 c_3^2)(s_1 s_2 s_3 + s_4 c_2^2) \\
\mu_s &= (s_2^3 s_3 c_1^2 + s_2 s_3 s_1^2 c_2^2 + s_4 s_1)(s_4 s_1 s_2 + s_3 c_1^2)(s_3 s_4 s_1 + s_2 c_4^2).
\end{aligned}$$

Still, verifying that $Z_i = 0$ is not as simple as it may seem. The reason is that the quantities s_i and c_i are not independent. For instance we have $s_i^4 + c_i^4 = 1$, for all $1 \le i \le 4$. Also, since $a+b+c+d = 180°$ we have $\sin(a+b) = \sin(c+d)$ which translates to $s_1^2\, c_2^2 + s_2^2\, c_1^2 = s_3^2\, c_4^2 + s_4^2\, c_3^2$. Similarly, $\cos(a+b) = -\cos(c+d)$ which means $c_1^2\, c_2^2 - s_1^2\, s_2^2 = s_3^2\, s_4^2 - c_3^2\, c_4^2$. There are $4+3+3 = 10$ such side relations which have to be used to prove that two expressions which look different are in fact equal. MAPLE cannot do this directly.

For example, it is not at all obvious that the expressions of m, n, p and q defined above represent numbers from the interval $(0,\, 1)$. Since each expression is obtained via circular permutations from the preceding one it is enough to look at m.

Clearly, since $s_i > 0$ and $c_i > 0$ for all $1 \le i \le 4$ we have that $m > 0$. On the other hand, using the side relations we mentioned above we get that

$$1 - m = \frac{s_3 s_1 c_1^2 (s_2 s_3 (s_1^2 c_2^2 + s_2^2 c_1^2) + s_1 s_4)}{(s_1^2 c_2^2 + s_2^2 c_1^2)(s_1 s_2 s_4 + s_3 c_1^2)(s_1 s_2 s_3 + s_4 c_2^2)}.$$

Obviously, $1 - m > 0$ and therefore $0 < m < 1$.

As previously eluded the construction works in the case when $MNPQ$ is a trapezoid as well. In this case if $MN \parallel PQ$ then $UV \parallel MN$ too. In conclusion, it is quite tricky to check that the values of m, n, p and q given by equalities (9) - (12) imply that $Z_1 = Z_2 = Z_3 = Z_4 = 0$. The MAPLE file containing the complete verification of theorem 8 is about 15 pages long. On request, we would be happy to provide a copy. □

4. Conclusions and Directions of Future Research

In this paper we mainly investigated what types of geometric properties can be preserved when dissecting a convex quadrilateral. The original contributions are contained in section 3 in which we dealt exclusively with grid dissections. There are many very interesting questions which are left unanswered.

1. The results from Theorems 6 and 7 suggest that if a cyclic quadrilateral $ABCD$ has an m-by-n grid dissection into cyclic quadrilaterals with $m \cdot n$ a large odd integer, then $ABCD$ has to be “close” to a rectangle. It would be desirable to quantify this relationship.

2. Conjecture 1 implies that orthodiagonal quadrilaterals are “bad” when it comes to class preserving dissections. On the other hand, theorem 8 proves that circumscribed quadrilaterals are very well behaved in this respect. Why does this happen? After all, the characterization Theorem 2 b) and 2 c) suggest that these two properties are not radically different.

More precisely, let us define an α-quadrilateral to be a convex quadrilateral $ABCD$ with $AB^\alpha + CD^\alpha = BC^\alpha + AD^\alpha$, where α is a real number. Notice that for $\alpha = 1$ we get the circumscribed quadrilaterals and for $\alpha = 2$ the orthodiagonal ones. In particular, a kite is an α-quadrilateral for all values of α. The natural question is:

Problem. For which values of α does every α-quadrilateral have a 2-by-2 grid dissection into α-quadrilaterals?

3. Theorem 8 provided a constructive method for finding a grid dissection of any circumscribed quadrilateral into smaller circumscribed quadrilaterals. Can this construction be extended to a 4-by-4 class preserving grid dissection? Notice that extending the opposite sides of each one of the four small cyclic quadrilaterals which appear in Figure 12 we obtain the same pair of points, E and F. It is therefore tempting to verify whether iterating the procedure used for $MNPQ$ for each of these smaller quads would lead to a 4-by-4 grid dissection of $MNPQ$ into 16 cyclic quadrilaterals. Maybe even a 2^n-by-2^n grid dissection is possible. If true, it is desirable to first find a simpler way of proving Theorem 8.

References

[1] T. Andreescu and R. Gelca, *Mathematical Olympiad challenges*, Birkhäuser Boston, Inc., Boston, MA, 2000, 13–16.

[2] M. N. Bleicher, Decomposition of a k-gon by l-gons. *Mitteilungen aus dem Mathematischen Seminar Giessen*, 166 (1984) 1–16.

[3] B. Bernheim and Th. Motzkin, A criterion for divisibility of n-gons into k-gons. *Comment. Math. Helvetici*, 22 (1949) 93–102.

[4] R. Ding, D. Schattschneider and T. Zamfirescu, Tiling the pentagon, Selected papers in honor of Ludwig Danzer, *Discrete Mathematics*, 221 (2000) 113–124.

[5] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, The Charles Babbage Research Center, Winnipeg, 1989.

[6] G. N. Frederickson, Unexpected twists in geometric dissections. *Graphs Combin.*, 23 (2007) suppl. 1, 245–258.

[7] R. W. Freese, A. K. Miller, and Z. Usiskin, Can every triangle be divided into n triangles similar to it?, *Amer. Math. Monthly*, 77 (1970) 867–869.

[8] S. W. Golomb, Replicating figures in the plane *The Mathematical Gazette*, 48 (1964) 403–412.

[9] “Ponder This!”, IBM collection of monthly puzzles, December 2003, available at `domino.research.ibm.com/Comm/wwwr-ponder.nsf/Challenges/December2003.html`.

[10] D. C. Langford, Uses of a geometric puzzle. *The Mathematical Gazette*, 24 (1940) 209–211.

[11] M. Pimsner and S. Popa, *Problems in Elementary Geometry*, E.D.P. Bucharest, 1979.

[12] Z. J. Su and H. X. Gao, Dissections of regular polygons into similar triangles. *Dongbei Shida Xuebao*, 38 (2006) no. 3, 5–8.

[13] G. Valette and T. Zamfirescu, Les partages d’un polygone convexe en 4 polygones semblables au premier. (French) *Journal of Combinatorial Theory, Series B*, 16 (1974) 1–16.

[14] P. Yiu, *Notes on Euclidean Geometry*, Florida Atlantic University Lecture Notes, 1998; available at `www.math.fau.edu/yiu/EuclideanGeometryNotes.pdf`.

[15] A. Zak, Dissection of a triangle into similar triangles. *Discrete Comput. Geom.*, 34 (2005) 295–312.

Dan Ismailescu: Department of Mathematics, Hofstra University, Hempstead, New York 11549, USA

E-mail address: `matdpi@hofstra.edu`

Adam Vojdany: Paul D. Schreiber High School, 101 Campus Dr, Port Washington, New York 11050, USA

E-mail address: `avojdany1@yahoo.com`

Forum Geometricorum
Volume 9 (2009) 213–223.

FORUM GEOM
ISSN 1534-1178

On the Construction of Regular Polygons and Generalized Napoleon Vertices

Dimitris Vartziotis and Joachim Wipper

Abstract. An algebraic foundation for the derivation of geometric construction schemes transforming arbitrary polygons with n vertices into k-regular n-gons is given. It is based on circulant polygon transformations and the associated eigenpolygon decompositions leading to the definition of generalized Napoleon vertices. Geometric construction schemes are derived exemplarily for different choices of n and k.

1. Introduction

Because of its geometric appeal, there is a long, ongoing tradition in discovering geometric constructions of regular polygons, not only in a direct way, but also by transforming a given polygon with the same number of vertices [2, 6, 9, 10]. In the case of the latter, well known results are, for example, Napoleon's theorem constructing an equilateral triangle by taking the centroids of equilateral triangles erected on each side of an arbitrary initial triangle [5], or the results of Petr, Douglas, and Neumann constructing k-regular n-gons by $n-2$ iteratively applied transformation steps based on taking the apices of similar triangles [8, 3, 7]. Results like these have been obtained, for example, by geometric creativity, target-oriented constructions or by analyzing specific configurations using harmonic analysis.

In this paper the authors give an algebraic foundation which can be used in order to systematically derive geometric construction schemes for k-regular n-gons. Such a scheme is hinted in Figure 1 depicting the construction of a 1-regular pentagon (left) and a 2-regular pentagon (right) starting from the same initial polygon marked yellow. New vertex positions are obtained by adding scaled parallels and perpendiculars of polygon sides and diagonals. This is indicated by intermediate construction vertices whereas auxiliary construction lines have been omitted for the sake of clarity.

The algebraic foundation is derived by analyzing circulant polygon transformations and the associated Fourier basis leading to the definition of eigenpolygons. By choosing the associated eigenvalues with respect to the desired symmetric configuration and determining the related circulant matrix, this leads to an algebraic representation of the transformed vertices with respect to the initial vertices and the eigenvalues. Interpreting this algebraic representation geometrically yields the desired construction scheme.

Publication Date: September 21, 2009. Communicating Editor: Paul Yiu.

The authors would like to thank Bernd Scholz from TWT GmbH, Engineering Department, for pointing out a modified triangle transformation based on the results of [11] which attracted our interest on deriving geometric construction schemes from circulant polygon transformations.

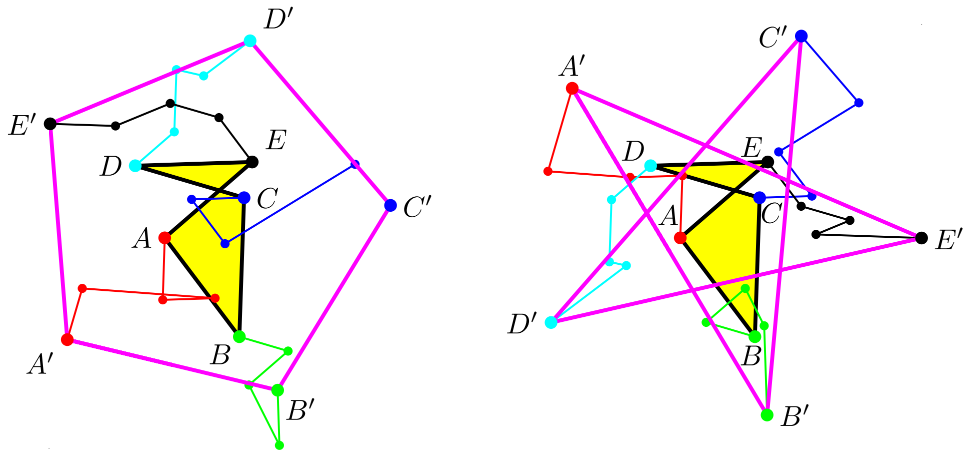


Figure 1. Construction of regular pentagons.

A special choice of parameters leads to a definition of generalized Napoleon vertices, which coincide with the vertices given by Napoleon's theorem in the case of $n = 3$. Geometric construction schemes based on such representations are derived for triangles, quadrilaterals, and pentagons.

2. Eigenpolygon decompositions

Let $z \in \mathbb{C}^n$ denote a polygon with n vertices z_k, $k \in \{0, \ldots, n-1\}$, in the complex plane using zero-based indexes. In order to obtain geometric constructions leading to regular polygons, linear transformations represented by complex circulant matrices $M \in \mathbb{C}^{n\times n}$ will be analyzed. That is, each row of M results from a cyclic shift of its preceding row, which reflects that new vertex positions are constructed in a similar fashion for all vertices.

The eigenvectors $f_k \in \mathbb{C}^n$, $k \in \{0, \ldots, n-1\}$, of circulant matrices are given by the columns of the Fourier matrix

$$F := \frac{1}{\sqrt{n}} \begin{pmatrix} r^{0\cdot 0} & \ldots & r^{0\cdot(n-1)} \\ \vdots & \ddots & \vdots \\ r^{(n-1)\cdot 0} & \ldots & r^{(n-1)\cdot(n-1)} \end{pmatrix},$$

where $r := \exp(2\pi \mathrm{i}/n)$ denotes the n-th complex root of unity [1]. Hence, the eigenvector $f_k = (1/\sqrt{n})(r^{0\cdot k}, r^{1\cdot k}, \ldots, r^{(n-1)\cdot k})^{\mathrm{t}}$ represents the k-th Fourier polygon obtained by successively connecting counterclockwise n times each k-th scaled root of unity starting by $r^0/\sqrt{n} = 1/\sqrt{n}$. This implies that f_k is a $\big(n/\gcd(n,k)\big)$-gon with vertex multiplicity $\gcd(n,k)$, where $\gcd(n,k)$ denotes the greatest common divisor of the two natural numbers n and k. In particular, f_0 degenerates to one vertex with multiplicity n, and f_1 as well as f_{n-1} are convex regular n-gons with opposite orientation. Due to its geometric configuration f_k is called k-*regular*, which will also be used in the case of similar polygons.

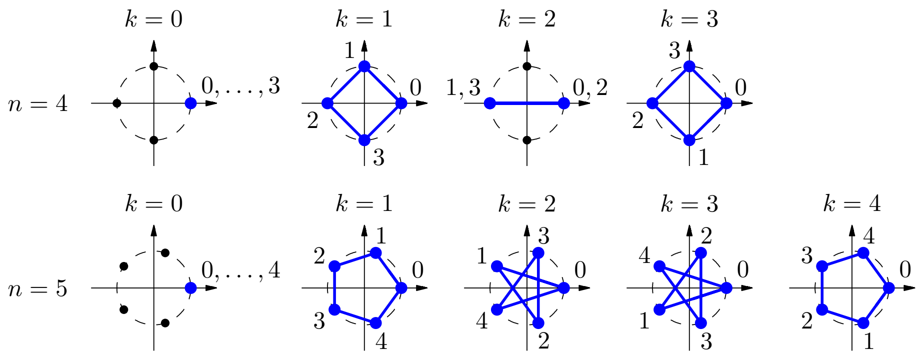


Figure 2. Fourier polygons f_k for $n \in \{4, 5\}$ and $k \in \{0, \ldots, n-1\}$.

Examples of Fourier polygons are depicted in Figure 2. In this, black markers indicate the scaled roots of unity lying on a circle with radius $1/\sqrt{n}$, whereas blue markers denote the vertices of the associated Fourier polygons. Also given is the vertex index or, in the case of multiple vertices, a comma separated list of indexes. If n is a prime number, all Fourier polygons except for $k = 0$ are regular n-gons as is shown in the case of $n = 5$. Otherwise reduced Fourier polygons occur as is depicted for $n = 4$ and $k = 2$.

Since F is a unitary matrix, the diagonalization of M based on the eigenvalues $\eta_k \in \mathbb{C}$, $k \in \{0, \ldots, n-1\}$, and the associated diagonal matrix $D = \operatorname{diag}(\eta_0, \ldots, \eta_{n-1})$ is given by $M = FDF^*$, where F^* denotes the conjugate transpose of F. The coefficients c_k in the representation of $z = \sum_{k=0}^{n-1} c_k f_k$ in terms of the Fourier basis are the entries of the vector $c = F^* z$ and lead to the following definition.

Definition. The k-th *eigenpolygon* of a polygon $z \in \mathbb{C}^n$ is given by

$$e_k := c_k f_k = \frac{c_k}{\sqrt{n}} \left(r^{0 \cdot k}, r^{1 \cdot k}, \ldots, r^{(n-1) \cdot k} \right)^{\mathrm{t}}, \tag{1}$$

where $c_k := (F^* z)_k$ and $k \in \{0, \ldots, n-1\}$.

Since e_k is f_k times a complex coefficient c_k representing a scaling and rotation depending on z, the symmetric properties of the Fourier polygons f_k are preserved. In particular, the coefficient $c_0 = (F^* z)_0 = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} z_k$ implies that $e_0 = \frac{1}{n} \left(\sum_{k=0}^{n-1} z_k \right) (1, \ldots, 1)^{\mathrm{t}}$ is n times the centroid of the initial polygon. This is also depicted in Figure 3 showing the eigenpolygon decomposition of two random polygons. In order to clarify the rotation and orientation of the eigenpolygons, the first three vertices are colored red, green, and blue.

Due to the representation of the transformed polygon

$$z' := Mz = M \left(\sum_{k=0}^{n-1} e_k \right) = \sum_{k=0}^{n-1} M e_k = \sum_{k=0}^{n-1} \eta_k e_k \tag{2}$$

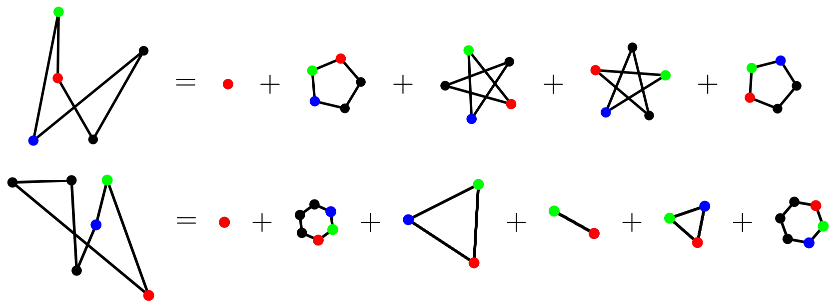


Figure 3. Eigenpolygon decomposition of a 5- and 6-gon.

applying the transformation M scales each eigenpolygon according to the associated eigenvalue $\eta_k \in \mathbb{C}$ of M. This is utilized by geometric construction schemes leading to scaled eigenpolygons. One is the Petr-Douglas-Neumann theorem [8, 3, 7] which is based on $n-2$ polygon transformations each consisting of taking the apices of similar isosceles triangles erected on the sides of the polygon. In each step a different apex angle taken from the set $\{k2\pi/n \,|\, k = 1, \ldots, n-1\}$ is used. The characteristic angles are chosen in such a way that an eigenvalue in the decomposition (2) becomes zero in each case. Since all transformation steps preserve the centroid, $n-2$ steps successively eliminate the associated eigenpolygons until one scaled eigenpolygon with preserved centroid remains. In the case of $n = 3$ this leads to the familiar Napoleon's theorem [5] in which one transformation step suffices to obtain a regular triangle.

3. Construction of regular polygons

The eigenpolygon decomposition presented in the previous section can be used to prove that specific geometric transformations result in regular polygons. Beyond that, it can also be used to find new geometric construction schemes leading to predefined symmetric configurations. This is done by an appropriate choice of the eigenvalues η_k and by interpreting the resulting transformation matrix $M = FDF^*$ geometrically.

3.1. *General case.* In this subsection, a specific choice of eigenvalues will be analyzed in order to derive transformations, which lead to k-regular polygons and additionally preserve the centroid. The latter implies $\eta_0 = 1$ since e_0 already represents the centroid. By choosing $\eta_j = 0$ for all $j \in \{1, \ldots, n-1\} \setminus \{k\}$ and $\eta_k \in \mathbb{C} \setminus \{0\}$, the transformation eliminates all eigenpolygons except the centroid e_0 and the designated eigenpolygon e_k which is scaled by the absolute value of η_k and rotated by the argument of η_k. This implies

$$\begin{aligned} M &= F \operatorname{diag}(1, 0, \ldots, 0, \eta_k, 0, \ldots, 0)\, F^* \\ &= F \operatorname{diag}(1, 0, \ldots, 0)\, F^* + \eta_k F \operatorname{diag}(0, \ldots, 0, 1, 0, \ldots, 0)\, F^* . \end{aligned} \tag{3}$$

Hence, M is a linear combination of matrices of the type $E_k := F I_k F^*$, where I_k denotes a matrix with the only nonzero entry $(I_k)_{k,k} = 1$. Taking into account that $(F)_{\mu,\nu} = r^{\mu\nu}/\sqrt{n}$ and $(F^*)_{\mu,\nu} = r^{-\mu\nu}/\sqrt{n}$, the matrix $I_k F^*$ has nonzero elements only in its k-th row, where $(I_k F^*)_{k,\nu} = r^{-k\nu}/\sqrt{n}$. Therefore, the ν-th column of $E_k = F I_k F^*$ consists of the k-th column of F scaled by $r^{-k\nu}/\sqrt{n}$, thus resulting in $(E_k)_{\mu,\nu} = (F)_{\mu,k} r^{-k\nu}/\sqrt{n} = r^{\mu k} r^{-k\nu}/n = r^{k(\mu-\nu)}/n$. This yields the representation

$$(M)_{\mu,\nu} = (E_0)_{\mu,\nu} + \eta_k (E_k)_{\mu,\nu} = \frac{1}{n}\left(1 + \eta_k r^{k(\mu-\nu)}\right),$$

since all entries of E_0 equal $1/n$. Hence, transforming an arbitrary polygon $z = (z_0, \ldots, z_{n-1})^{\mathrm{t}}$ results in the polygon $z' = Mz$ with vertices

$$z'_\mu = (Mz)_\mu = \sum_{\nu=0}^{n-1} \frac{1}{n}\left(1 + \eta_k r^{k(\mu-\nu)}\right) z_\nu,$$

where $\mu \in \{0, \ldots, n-1\}$. In the case of $\mu = \nu$ the weight of the associated summand is given by $\omega := (1 + \eta_k)/n$. Substituting this expression in the representation of z'_μ using $\eta_k = n\omega - 1$, hence $\omega \neq 1/n$, yields the decomposition

$$\begin{aligned} z'_\mu &= \sum_{\nu=0}^{n-1} \frac{1}{n}\left(1 + (n\omega - 1) r^{k(\mu-\nu)}\right) z_\nu \\ &= \underbrace{\frac{1}{n}\sum_{\nu=0}^{n-1}\left(1 - r^{k(\mu-\nu)}\right) z_\nu}_{=:u_\mu} + \omega \underbrace{\sum_{\nu=0}^{n-1} r^{k(\mu-\nu)} z_\nu}_{=:v_\mu} = u_\mu + \omega v_\mu \end{aligned} \tag{4}$$

of z'_μ into a geometric location u_μ not depending on ω, and a complex number v_μ, which can be interpreted as vector scaled by the parameter ω. It should also be noticed that due to the substitution u_μ does not depend on z_μ, since the associated coefficient becomes zero.

A particular choice is $\omega = 0$, which leads to $z'_\mu = u_\mu$. As will be seen in the next section, in the case of $n = 3$ this results in the configuration given by Napoleon's theorem, hence motivating the following definition.

Definition. For $n \geq 3$ let $z = (z_0, \ldots, z_{n-1})^{\mathrm{t}} \in \mathbb{C}^n$ denote an arbitrary polygon, and $k \in \{1, \ldots, n-1\}$. The vertices

$$u_\mu := \frac{1}{n}\sum_{\nu=0}^{n-1}\left(1 - r^{k(\mu-\nu)}\right) z_\nu, \qquad \mu \in \{0, \ldots, n-1\},$$

defining a k-regular n-gon are called *generalized Napoleon vertices*.

According to its construction, M acts like a filter on the polygon z removing all except the eigenpolygons e_0 and e_k. The transformation additionally weightens e_k by the eigenvalue $\eta_k \neq 0$. As a consequence, if e_k is not contained in the eigenpolygon decomposition of z, the resulting polygon $z' = Mz$ degenerates to the centroid e_0 of z.

The next step consists of giving a geometric interpretation of the algebraically derived entities u_0 and v_0 for specific choices of n, k, and ω resulting in geometric construction schemes to transform an arbitrary polygon into a k-regular polygon. Examples will be given in the next subsections.

3.2. *Transformation of triangles.* The general results obtained in the previous subsection will now be substantiated for the choice $n = 3$, $k = 1$. That is, a geometric construction is to be found, which transforms an arbitrary triangle into a counterclockwise oriented equilateral triangle with the same centroid. Due to the circulant structure, it suffices to derive a construction scheme for the first vertex of the polygon, which can be applied in a similar fashion to all other vertices.

In the case of $n = 3$ the root of unity is given by $r = \exp(\frac{2}{3}\pi \mathrm{i}) = \frac{1}{2}(-1+\mathrm{i}\sqrt{3})$. By using (4) in the case of $\mu = 0$, as well as the relations $r^{-1} = r^2 = \overline{r}$ and $r^{-2} = r$, this implies

$$\begin{aligned} u_0 &= \frac{1}{3}\sum_{\nu=0}^{2}(1 - r^{-\nu})z_\nu = \frac{1}{3}\left[\left(\frac{3}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}\right) z_1 + \left(\frac{3}{2} - \mathrm{i}\,\frac{\sqrt{3}}{2}\right) z_2\right] \\ &= \frac{1}{2}(z_1 + z_2) - \mathrm{i}\,\frac{1}{3}\,\frac{\sqrt{3}}{2}(z_2 - z_1) \end{aligned}$$

and

$$\begin{aligned} v_0 &= \sum_{\nu=0}^{2} r^{-\nu} z_\nu = z_0 + \left(-\frac{1}{2} - \mathrm{i}\,\frac{\sqrt{3}}{2}\right) z_1 + \left(-\frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}\right) z_2 \\ &= z_0 - \frac{1}{2}(z_1 + z_2) + \mathrm{i}\,\frac{\sqrt{3}}{2}(z_2 - z_1)\,. \end{aligned}$$

Thereby, the representations of u_0 and v_0 have been rearranged in order to give geometric interpretations as depicted in Figure 4.

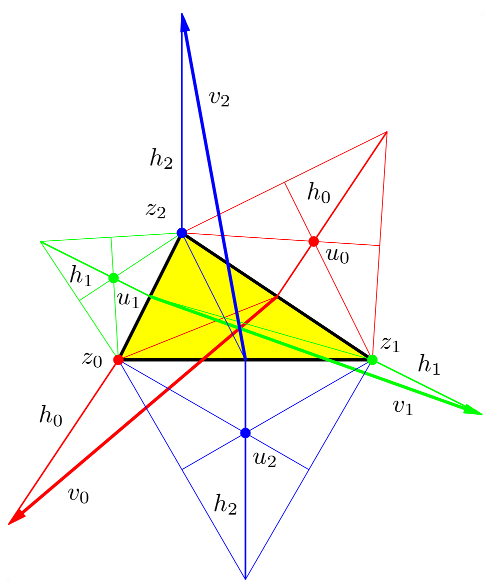


Figure 4. Napoleon vertices u_μ and directions v_μ in the case $n = 3$, $k = 1$.

Since multiplication by $-\mathrm{i}$ denotes a clockwise rotation by $\pi/2$, u_0 represents the centroid of a properly oriented equilateral triangle erected on the side z_1z_2. In a vectorial sense, v_0 represents the vector from the midpoint of the side z_1z_2 to z_0 added by the opposite directed height h_0 of the equilateral triangle erected on z_1z_2. Due to the circulant structure of M, the locations u_1, u_2 and the vectors v_1, v_2 can be constructed analogously. Using this geometric interpretation of the algebraically derived elements, the task is now to derive a construction scheme which combines the elements of the construction.

Algebraically, an obvious choice in the representation $z'_\mu = u_\mu + \omega v_\mu$ is $\omega = 0$, which leads to the familiar Napoleon configuration since in this case $z'_\mu = u_\mu$. Geometrically, an alternative construction is obtained by parallel translation of v_μ to u_μ. This is equivalent to the choice $\omega = 1$, hence $z'_0 = z_0 + \frac{\mathrm{i}}{\sqrt{3}}(z_2 - z_1)$. An according geometric construction scheme is depicted in Figure 5.

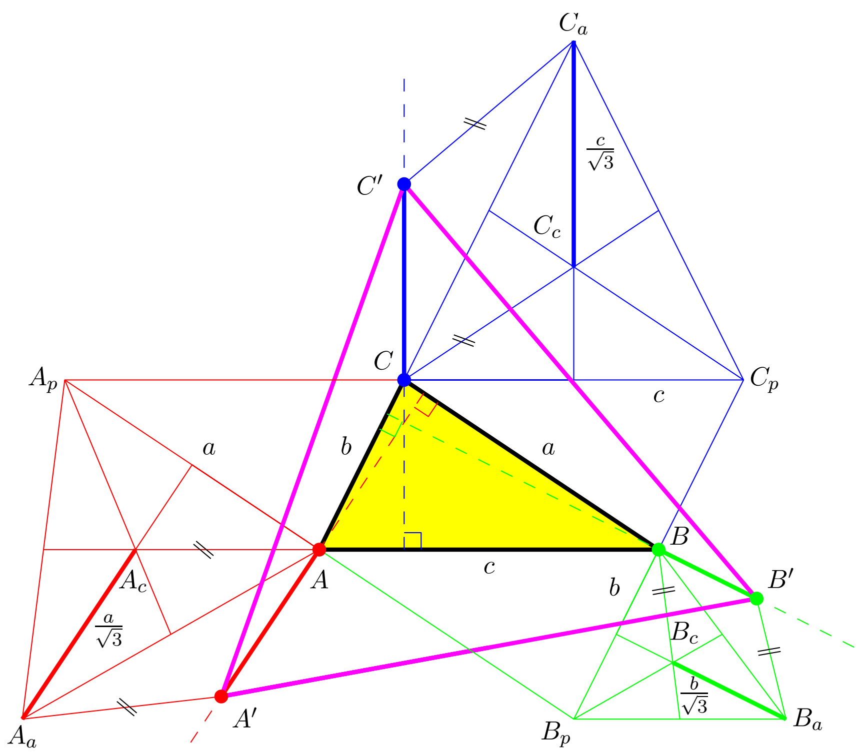


Figure 5. Construction of an equilateral triangle.

Thereby, the new position A' of A is derived as follows. First, the parallelogram $ABCA_p$ is constructed and an equilateral triangle is erected on AA_p. Since the distance from the centroid A_c to the apex A_a of this triangle is of the required length $a/\sqrt{3}$, where $a = |BC|$, one can transfer it by parallel translation to the vertical line on BC through A. The other vertices are constructed analogously as is also depicted in Figure 5.

According to the choice of parameters in the definition of M, the resulting triangle $A'B'C'$ is equilateral and oriented counterclockwise. A geometric proof is given by the fact that the triangle $A_pB_pC_p$ of the associated parallelogram vertices is similar to ABC with twice the side length. Due to their construction the new vertices A', B', and C' are the Napoleon vertices of $A_pB_pC_p$, hence $A'B'C'$ is equilateral. In particular, the midpoints of the sides of $A'B'C'$ yield the Napoleon triangle of ABC. Thus, A' can also be constructed by intersecting the line through A_p and A_c with the vertical line on BC through A.

3.3. *Transformation of quadrilaterals.* As a second example, the generalized Napoleon configuration in the case of $n = 4$, $k = 1$ is presented, that is $\omega = 0$ resulting in $z'_\mu = u_\mu$, $\mu \in \{0, \dots, 3\}$. Using $r = \mathrm{i}$ and the representation (4) implies

$$\begin{aligned} u_0 &= \frac{1}{4}\sum_{\nu=0}^{3}(1 - r^{-\nu})z_\nu = \frac{1}{4}\Big((1+\mathrm{i})z_1 + (1+1)z_2 + (1-\mathrm{i})z_3\Big) \\ &= z_1 + \frac{1}{2}(z_2 - z_1) + \frac{1}{4}(z_3 - z_1) - \mathrm{i}\,\frac{1}{4}(z_3 - z_1)\,, \end{aligned}$$

which leads to the construction scheme depicted in Figure 6.

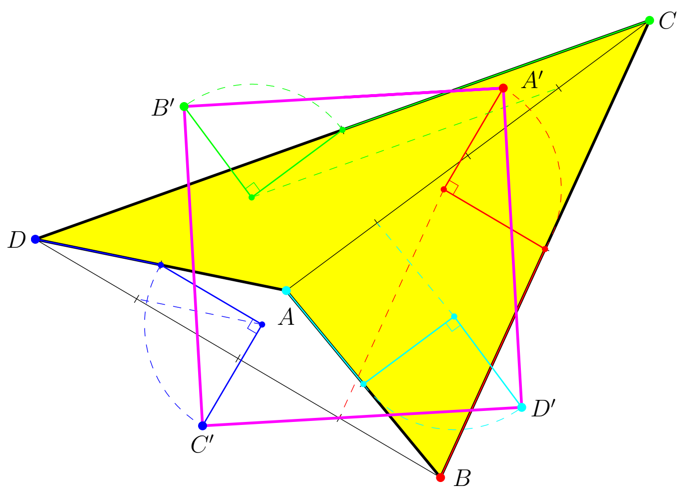


Figure 6. Construction of a regular quadrilateral.

As in the case of $n = 3$, the generalized Napoleon vertices can be constructed with the aid of scaled parallels and perpendiculars. Figure 6 depicts the intermediate vertices obtained by successively adding the summands given in the representation of u_μ from left to right. Parallels, as well as rotations by $\pi/2$ are marked by dashed lines. Diagonals, as well as subdivision markers are depicted by thin black lines.

3.4. *Transformation of pentagons.* In the case of $n = 5$, the root of unity is given by $r = (-1 + \sqrt{5})/4 + \mathrm{i}\sqrt{(5 + \sqrt{5})/8}$. The pentagon depicted on the left of Figure 1 has been transformed by using $k = 1$ and $\omega = 1$ resulting in

$$\begin{aligned} u_0 + v_0 = \; & z_0 + \frac{1}{\sqrt{5}}(z_1 - z_2) + \frac{1}{\sqrt{5}}(z_4 - z_3) \\ & - \mathrm{i}\,\frac{\sqrt{10 + 2\sqrt{5}}}{5}(z_1 - z_4) - \mathrm{i}\,\frac{\sqrt{10 - 2\sqrt{5}}}{5}(z_2 - z_3)\,. \end{aligned}$$

Due to the choice $k = 1$, z' is a regular convex pentagon. The same initial polygon transformed by using $k = 2$ and $\omega = 1$ resulting in

$$\begin{aligned} u_0 + v_0 = \; & z_0 + \frac{1}{\sqrt{5}}(z_2 - z_1) + \frac{1}{\sqrt{5}}(z_3 - z_4) \\ & - \mathrm{i}\,\frac{\sqrt{10 - 2\sqrt{5}}}{5}(z_1 - z_4) - \mathrm{i}\,\frac{\sqrt{10 + 2\sqrt{5}}}{5}(z_3 - z_2) \end{aligned}$$

is depicted on the right. Since $k = 2$ is not a divisor of $n = 5$, a star shaped nonconvex 2-regular polygon is constructed. Again, the representation also gives the intermediate constructed vertices based on scaled parallels and perpendiculars, which are marked by small markers. Thereby, auxiliary construction lines have been omitted in order to simplify the figure.

3.5. *Constructibility.* According to (4) the coefficients of the initial vertices z_μ in the representation of the new vertices z'_μ are given by $1 - r^{k(\mu-\nu)}$ and $\omega r^{k(\mu-\nu)}$ respectively. Using the polar form of the complex roots of unity, these involve the expressions $\cos(2\pi\xi/n)$ and $\sin(2\pi\xi/n)$, where $\xi \in \{0, \ldots, n-1\}$. Hence a compass and straightedge based construction scheme can only be derived if there exists a representation of these expressions and ω only using the constructible operations addition, subtraction, multiplication, division, complex conjugate, and square root.

Such representations are given exemplarily in the previous subsections for the cases $n \in \{3, 4, 5\}$. As is well known, Gauß proved in [4] that the regular polygon is constructible if n is a product of a power of two and any number of distinct Fermat prime numbers, that is numbers $F_m = 2^{(2^m)} + 1$ being prime. A proof of the necessity of this condition was given by Wantzel [13]. Thus, the first non constructible case using this scheme is given by $n = 7$. Nevertheless, there exists a neusis construction using a marked ruler to construct the associated regular heptagon.

4. Conclusion

A method of deriving construction schemes transforming arbitrary polygons into k-regular polygons has been presented. It is based on the theory of circulant matrices and the associated eigenpolygon decomposition. Following a converse approach, the polygon transformation matrix is defined by the choice of its eigenvalues representing the scaling and rotation parameters of the eigenpolygons. As has been shown for the special case of centroid preserving transformations leading to k-regular polygons, a general representation of the vertices of the new polygon can be derived in terms of the vertices of the initial polygon and an arbitrary transformation parameter ω. Furthermore, this leads to the definition of generalized Napoleon vertices, which are in the case of $n = 3$ identical to the vertices given by Napoleon's theorem.

In order to derive a new construction scheme, the number of vertices n and the regularity index k have to be chosen first. Since the remaining parameter ω has influence on the complexity of the geometric construction it should usually be chosen in order to minimize the number of construction steps. Finally giving a geometric interpretation of the algebraically derived representation of the new vertices is still a creative task. Examples for $n \in \{3, 4, 5\}$ demonstrate this procedure. Naturally, the problems in the construction of regular convex n-gons also apply in the presented scheme, since scaling factors of linear combinations of vertices have also to be constructible.

It is evident that construction schemes for arbitrary linear combinations of eigenpolygons leading to other symmetric configurations can be derived in a similar fashion. Furthermore, instead of setting specific eigenvalues to zero, causing the associated eigenpolygons to vanish, they could also be chosen in order to successively damp the associated eigenpolygons if the transformation is applied iteratively. This has been used by the authors to develop a new mesh smoothing scheme presented in [11, 12]. It is based on successively applying transformations to low quality mesh elements in order to regularize the polygonal element boundary iteratively. In this context transformations based on positive real valued eigenvalues are of particular interest, since they avoid the rotational effect known from other regularizing polygon transformations.

References

[1] P. J. Davis, *Circulant Matrices*, Chelsea Publishing, 2nd edition, 1994.

[2] J. Douglas, Geometry of polygons in the complex plane, *Journal of Mathematics and Physics*, Massachusetts Institute of Technology, 19(2):93–130, 1940.

[3] J. Douglas, On linear polygon transformations, *Bulletin of the American Mathematical Society*, 46(6):551–560, 1940.

[4] C. F. Gauß, *Disquisitiones Arithmeticae*, Gerhard Fleischer, 1801.

[5] H. Martini, On the theorem of Napoleon and related topics, *Mathematische Semesterberichte*, 43(1):47–64, 1996.

[6] D. Merriell, Further remarks on concentric polygons, *Amer. Math. Monthly*, 72 (1965) 960–965.

[7] B. H. Neumann, Some remarks on polygons, *Journal of the London Mathematical Society*, s1-16(4):230–245, 1941.

[8] K. Petr, Ein Satz über Vielecke, *Archiv der Mathematik und Physik*, 13:29–31, 1908.

[9] W. Schuster, Regularisierung von Polygonen, *Mathematische Semesterberichte*, 45(1):77–94, 1998.

[10] D. Vartziotis and M. Bataille, Problem 11328, *Amer. Math. Monthly*, 114 (2007) 925; solution, 116 (2009) 656.

[11] D. Vartziotis, T. Athanasiadis, I. Goudas, and J. Wipper, Mesh smoothing using the Geometric Element Transformation Method, *Computer Methods in Applied Mechanics and Engineering*, 197(45–48):3760–3767, 2008.

[12] D. Vartziotis and J. Wipper, The geometric element transformation method for mixed mesh smoothing, *Engineering with Computers*, 25(3):287–301, 2009.

[13] P. L. Wantzel, Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas, *Journal de Mathématiques Pures et Appliquées*, 1(2):366–372, 1837.

Dimitris Vartziotis: NIKI Ltd. Digital Engineering, Research Center, 205 Ethnikis Antistasis Street, 45500 Katsika, Ioannina, Greece
E-mail address: dimitris.vartziotis@nikitec.gr

Dimitris Vartziotis and Joachim Wipper: TWT GmbH Science & Innovation, Research Department, Bernhäuser Straße 40–42, 73765 Neuhausen, Germany

Forum Geometricorum
Volume 9 (2009) 225–228.

FORUM GEOM
ISSN 1534-1178

A Simple Barycentric Coordinates Formula

Nikolaos Dergiades

Abstract. We establish a simple formula for the barycentric coordinates with respect to a given triangle ABC of a point P specified by the oriented angles BPC, CPA and APB. Several applications are given.

We establish a simple formula for the homogeneous barycentric coordinates of a point with respect to a given triangle.

Theorem 1. *With reference to a given a triangle ABC, a point P specified by the oriented angles*

$$x = \measuredangle BPC, \quad y = \measuredangle CPA, \quad z = \measuredangle APB,$$

has homogeneous barycentric coordinates

$$\left(\frac{1}{\cot A - \cot x} : \frac{1}{\cot B - \cot y} : \frac{1}{\cot C - \cot z}\right). \tag{1}$$

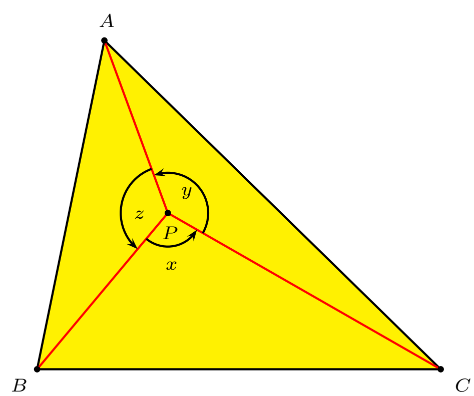


Figure 1.

Proof. Construct the circle through B, P, C, and let it intersect the line AP at A' (see Figure 2). Clearly, $\angle A'BC = \angle A'PC = \pi - \angle CPA = \pi - y$ and similarly, $\angle A'CB = \pi - z$. It follows from Conway's formula [5, §3.4.2] that in barycentric coordinates

$$A' = (-a^2 : S_C + S_{\pi-z} : S_B + S_{\pi-y}) = (-a^2 : S_C - S_z : S_B - S_y).$$

Publication Date: September 28, 2009. Communicating Editor: Paul Yiu.
The author thanks the editor for many improvements, especially in the proof of Corollary 3.

Similarly, the lines BP intersects the circle CPA at a point B', and CP intersects the circle APB at C' whose coordinates can be easily written down. These be reorganized as

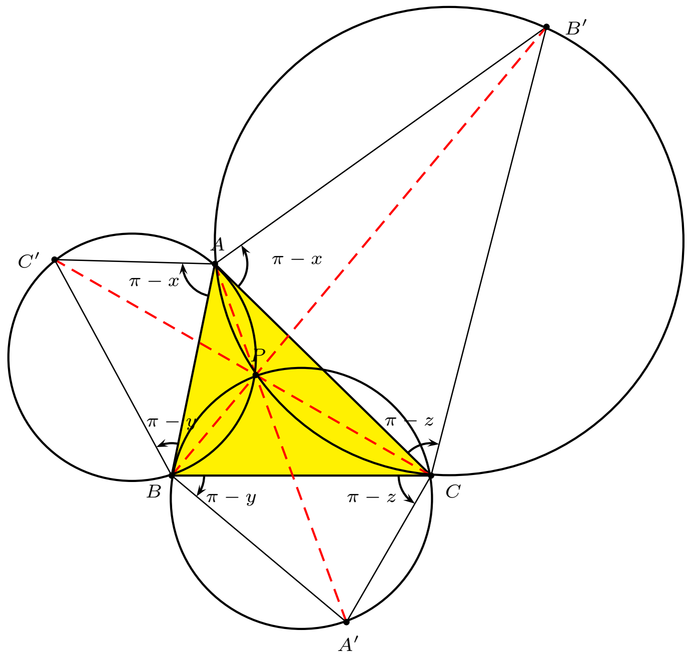


Figure 2.

$$A' = \left(-\frac{a^2}{(S_B - S_y)(S_C - S_z)} : \frac{1}{S_B - S_y} : \frac{1}{S_C - S_z}\right),$$
$$B' = \left(\frac{1}{S_A - S_x} : -\frac{b^2}{(S_C - S_z)(S_A - S_x)} : \frac{1}{S_C - S_z}\right),$$
$$C' = \left(\frac{1}{S_A - S_x} : \frac{1}{S_B - S_y} : -\frac{c^2}{(S_A - S_x)(S_B - S_y)}\right).$$

According the version of Ceva's theorem given in [5, §3.2.1], the lines AA', BB', CC' intersect at a point, which is clearly P, whose coordinates are

$$\left(\frac{1}{S_A - S_x} : \frac{1}{S_B - S_y} : \frac{1}{S_C - S_z}\right).$$

Since by definition $S_\theta = S \cdot \cot\theta$, this formula is clearly equivalent to (1). □

Remark. This note is a revision of [1]. Antreas Hatzipolakis has subsequently given a traditional trigonometric proof [3].

The usefulness of formula (1) is that it is invariant when we substitute x, y, z by directed angles.

Corollary 2 (Schaal)**.** *If for three points A', B', C' the directed angles $x = (A'B,\ A'C)$, $y = (B'C,\ B'A)$ and $z = (C'A,\ C'B)$ satisfy $x+y+z \equiv 0 \bmod \pi$, then the circumcircles of triangles $A'BC$, $B'CA$, $C'AB$ are concurrent at P.*

Proof. Referring to Figure 2, if the circumcircles of triangles $A'BC$ and $B'CA$ intersect at P, then from concyclicity,

$$\begin{aligned}(PB,\ PC) =&\ (A'B,\ A'C) = x,\\ (PC,\ PA) =&\ (B'C,\ B'A) = y.\end{aligned}$$

It follows that

$$(PA,\ PB) = (PA,\ PC) + (PC,\ PB) = -y - x \equiv z = (C'A,\ C'B) \bmod \pi,$$

and C', A, B, P are concyclic. Now, it is obvious that the barycentrics of P are given by (1). $\square$

For example, if the triangles $A'BC$, $B'CA$, $C'AB$ are equilateral on the exterior of triangle ABC, then $x = y = z = -\frac{\pi}{3}$, and $x + y + z \equiv 0 \bmod \pi$. By Corollary 2, we conclude that the circumcircles of these triangles are concurrent at

$$\begin{aligned}P =& \left(\frac{1}{\cot A - \cot\left(-\frac{\pi}{3}\right)} : \frac{1}{\cot B - \cot\left(-\frac{\pi}{3}\right)} : \frac{1}{\cot C - \cot\left(-\frac{\pi}{3}\right)}\right)\\ =& \left(\frac{1}{\cot A + \cot\left(\frac{\pi}{3}\right)} : \frac{1}{\cot B + \cot\left(\frac{\pi}{3}\right)} : \frac{1}{\cot C + \cot\left(-\frac{\pi}{3}\right)}\right).\end{aligned}$$

This is the first Fermat point, X_{13} of [4].

Corollary 3 (Hatzipolakis [2])**.** *Given a reference triangle ABC and two points P and Q, let R_a be the intersection of the reflections of the lines BP, CP in the lines BQ, CQ respectively* (see Figure 3). *Similarly define the points R_b and R_c. The circumcircles of triangles R_aBC, R_bCA, R_cAB are concurrent at a point*

$$f(P,Q) = \left(\frac{1}{\cot A - \cot(2x' - x)} : \frac{1}{\cot B - \cot(2y' - y)} : \frac{1}{\cot C - \cot(2z' - z)}\right), \tag{2}$$

where

$$\begin{aligned}x &= (PB,\ PC), & y &= (PC,\ PA), & z &= (PA,\ PB);\\ x' &= (QB,\ QC), & y' &= (QC,\ QA), & z' &= (QA,\ QB).\end{aligned} \tag{3}$$

Proof. Let $x'' = (R_aB,\ R_aC)$. Note that

$$\begin{aligned}x'' =&\ (R_aB,\ QB) + (QB,\ QC) + (QC,\ R_aC)\\ =&\ (QB,\ QC) + (R_aB,\ QB) + (QC,\ R_aC)\\ =&\ (QB,\ QC) + (QB,\ PB) + (PC,\ QC)\\ =&\ (QB,\ QC) + (QB,\ QC) - (PB,\ PC)\\ =&\ 2x' - x.\end{aligned}$$

Similarly, $y'' = (R_bC,\ R_bA) = 2y' - y$ and $z'' = (R_cA,\ R_cB) = 2z' - z$. Hence,

$$x'' + y'' + z'' \equiv 2(x' + y' + z') - (x + y + z) \equiv 0 \bmod \pi.$$

By Corollary 2, the circumcircles of triangles R_aBC, R_bCA, R_cAB are concurrent at the point $R = f(P,Q)$ given by (2). $\square$

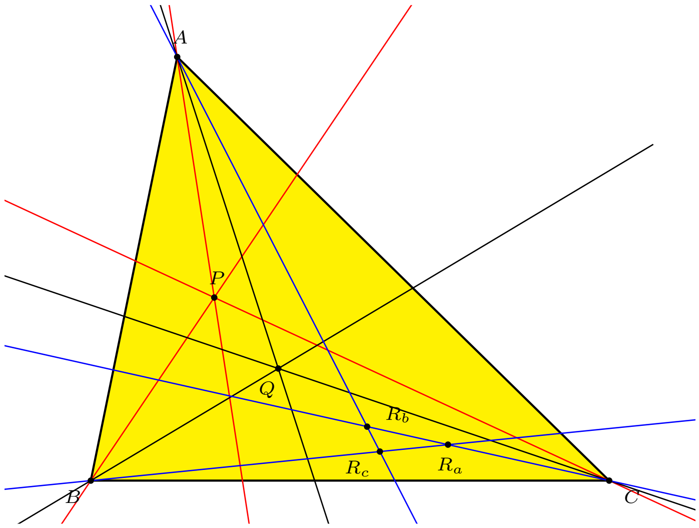


Figure 3.

Clearly, for the incenter I, $f(P, I) = P^*$, since $R_a = R_b = R_c = P^*$, the isogonal conjugate of P.

Corollary 4. *The mapping f preserves isogonal conjugation,* i.e.,

$$f^*(P, Q) = f(P^*, Q^*).$$

Proof. If the points P and Q are defined by the directed angles in (3), and $R = f(P, Q)$, $S = f(P^*,\ Q^*)$, then by Corollary 3, $(R^*B,\ R^*C) = A - (2x' - x)$ and

$$\begin{aligned}
(SB,\ SC) &\equiv\ 2(Q^*B,\ Q^*C) - (P^*B,\ P^*C)\\
&\equiv\ 2(A - x') - (A - x)\\
&\equiv\ A - (2x' - x)\\
&\equiv\ (R^*B,\ R^*C) \bmod \pi.
\end{aligned}$$

Similarly, $(SC, SA) \equiv (R^*C,\ R^*A)$ and $(SA,\ SB) \equiv (R^*A,\ R^*B) \bmod \pi$. Hence, $R^* = S$, or $f^*(P, Q) = f(P^*, Q^*)$. □

References

[1] N. Dergiades, Hyacinthos message 17892, June 20, 2009.
[2] A. P. Hatzipolakis, Hyacinthos message 17281, February 24, 2009.
[3] A. P. Hatzipolakis, Hyacinthos message 17893, June 20, 2009.
[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.
[5] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: `ndergiades@yahoo.gr`

Forum Geometricorum
Volume 9 (2009) 229–257.

FORUM GEOM
ISSN 1534-1178

Conic Homographies and Bitangent Pencils

Paris Pamfilos

Abstract. Conic homographies are homographies of the projective plane preserving a given conic. They are naturally associated with bitangent pencils of conics, which are pencils containing a double line. Here we study this connection and relate these pencils to various groups of homographies associated with a conic. A detailed analysis of the automorphisms of a given pencil specializes to the description of affinities preserving a conic. While the algebraic structure of the groups involved is simple, it seems that a geometric study of the various questions is lacking or has not been given much attention. In this respect the article reviews several well known results but also adds some new points of view and results, leading to a detailed description of the group of homographies preserving a bitangent pencil and, as a consequence, also the group of affinities preserving an affine conic.

1. Introduction

Deviating somewhat from the standard definition I call *bitangent* the pencils $\mathcal{P}$ of conics which are defined in the projective plane through equations of the form

$$\alpha c + \beta e^2 = 0.$$

Here $c(x, y, z) = 0$ and $e(x, y, z) = 0$ are the equations in *homogeneous coordinates* of a non-degenerate conic and a line and α, β are arbitrary, no simultaneously zero, real numbers. To be short I use the same symbol for the set and an equation representing it. Thus c denotes the set of points of a conic and $c = 0$ denotes an equation representing this set in some system of homogeneous coordinates. To denote bitangent pencils I use the letter $\mathcal{P}$ but also the more specific symbol (c, e). For any other member-conic c' of the pencil (c', e) represents the same pencil. I call line e and the pole E of e with respect to c respectively *invariant line* and *center* of the pencil. The intersection points $c \cap e$, if any, are called *fixed* or *base* points of the pencil. As is seen from the above equation, if such points exist, they lie on every member-conic of the pencil.

Traditionally the term *bitangent* is used only for pencils (c, e) for which line e either intersects c or is disjoint from it. This amounts to a second order (real or complex) contact between the members of the pencil, wherefore also the stem of the term. Pencils for which c and e^2 are tangent have a fourth order contact between their members and are classified under the name *superosculating* pencils

Publication Date: October 5, 2009. Communicating Editor: Paul Yiu.

([4, vol.II, p.188], [12, p.136]) or *penosculating* pencils ([9, p.268]). Here I take the liberty to incorporate this class of pencils into the bitangent ones, thus considering as a distinguished category the class of pencils which contain among their members a double line. This is done under the perspective of the tight relationship of conic homographies with bitangent pencils under this wider sense. An inspiring discussion in synthetic style on pencils of conics, which however, despite its wide extend, does not contain the relationship studied here, can be found in Steiner's lectures ([11, pp.224–430]).

Every homography[1] f of the plane preserving the conic c defines a bitangent pencil (c, e) to which conic c belongs as a member and to which f acts by preserving each and every member of the pencil. The pencil contains a double line e, which coincides with the *axis* of the homography. In this article I am mainly interested in the investigation of the geometric properties of four groups: $\mathcal{G}(c)$, $\mathcal{G}(c, e)$, $\mathcal{K}(c, e)$ and $\mathcal{A}(c)$, consisting respectively of homographies (i) preserving a conic c, (ii) preserving a pencil (c, e), (iii) permuting the members of a pencil, and (iv) preserving an affine conic. Last group is identical with a group of type $\mathcal{G}(c, e)$ in which line e is identified with the line at infinity. In Section 2 (*Conic homographies*) I review the well-known basic facts on homographies of conics stating them as propositions for easy reference. Their proofs can be found in the references given (especially [4, vol.II, Chapter 16], [12, Chapter VIII]). Section 3 (*Bitangent pencils*) is a short review on the classification of bitangent pencils. In Section 4 (*The isotropy at a point*) I examine the isotropy of actions of the groups referred above. In this, as well as in the subsequent sections, I supply the proofs of propositions for which I could not find a reference. Section 5 (*Automorphisms of pencils*) is dedicated to an analysis of the group $\mathcal{G}(c, e)$. Section 6 (*Bitangent flow*) comments on the vector-field point of view of a pencil and the characterization of its flow through a simple configuration on the invariant line. Section 7 (*The perspectivity group of a pencil*) contains a discussion on the group $\mathcal{K}(c, e)$ permuting the members of a pencil. Finally, Section 8 (*Conic affinities*) applies the results of the previous sections to the description of the group of affinities preserving an affine conic.

2. Conic homographies

Conic homographies are by definition restrictions on c of homographies of the plane that preserve a given conic c. One can define also such maps intrinsically, without considering their extension to the ambient plane. For this fix a point A on c and a line m and define the image $Y = f(X)$ of a point X by using its representation $f' = p \circ f \circ p^{-1}$ through the (stereographic) projection p of the conic onto line m centered at A.

[1] I use this term coming from my native language (greek) as an alternative equivalent to *projectivity*.

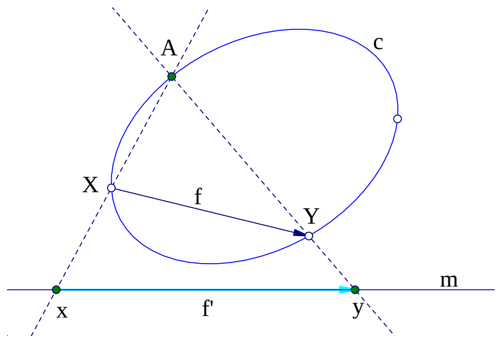


Figure 1. Conic homography

Homography f is the defined using a Moebius transformation ([10, p.40]) (see Figure 1)

$$y = f'(x) = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

It can be shown ([4, vol.II, p.179]) that the two definitions are equivalent. Depending on the kind of the question one can prefer the first definition, through the restriction of a global homography, or the second through the projection. Later point of view implies the following ([10, p.47]).

Proposition 1. *A conic homography on the conic c is completely determined by giving three points A, B, C on the conic and their images A', B', C'. In particular, if a conic homography fixes three points on c it is the identity.*

The two ways to define conic homographies on a conic c reflect to the representation of their group $\mathcal{G}(c)$. In the first case, since every conic can be brought in appropriate homogeneous coordinates to the form ([2, p.209])

$$x^2 + y^2 - z^2 = 0$$

their group is represented through the group preserving this quadratic form which is $O(2, 1)$. By describing homographies through Moebius transformations $\mathcal{G}(c)$ is represented with the group $PGL(2, \mathbb{R})$. The two representations are isomorphic but not *naturally* isomorphic ([4, vol.II, p.180]). An isomorphism between them can be established by fixing $A \in c$ and associating to each $f \in O(2, 1)$ the corresponding induced in m transformation $f' \in PGL(2, \mathbb{R})$ ([15, p.235]), in the way this was defined above through the stereographic projection from A onto some line m (see Figure 1).

Next basic property of conic homographies is the existence of their *homography axis* ([4, vol.II, p.178]).

Proposition 2. *Given a conic homography f of the conic c, for every pair of points A, B on c, lines AB' and BA', with $A' = f(A), B' = f(B)$ intersect on a fixed line e, the homography axis of f. The fixed points of f, if any, are the intersection points of c and e.*

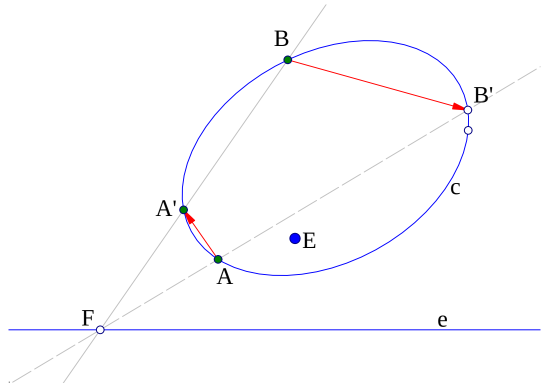


Figure 2. Homography axis e

Remark. This property implies (see Figure 2) an obvious geometric construction of the image B' of an arbitrary point B under the homography once we know the axis and a single point A and its image A' on the conic: Draw $A'B$ to find its intersection $F = A'B \cap e$ and from there draw line FA to find its intersection $B' = FA \cap c$.

Note that the existence of the axis is a consequence of the existence of at least a fixed point P for every homography f of the plane ([15, p.243]). If f preserves in addition a conic c, then it is easily shown that the polar e of P with respect to the conic must be invariant and coincides either with a tangent of the conic at a fixed point of f or coincides with the axis of f.

Next important property of conic homographies is the preservation of the whole bitangent family (c, e) generated by the conic c and the axis e of the homography. Here the viewpoint must be that of the restriction on c (and e) of a global homography of the plane.

Proposition 3. *Given a conic homography f of the conic c with homograpy axis e, the transformation f preserves every member $c' = \alpha c + \beta e^2$ of the pencil generated by c and the (double) line e. The pole E of the axis e with respect to c is a fixed point of the homography. It is also the pole of e with respect to every conic of the pencil. Line e is the axis of the conic homography induced by f on every member c' of the pencil.*

To prove the claims show first that line e is preserved by f (see Figure 3). For this take on c points A, B collinear with the pole E and consider their images $A' = f(A), B' = f(B)$. Since AB contains the pole of e, the pole Q of AB will be on line e. By Proposition 2 lines AB', BA' intersect at a point G of line e. It follows that the intersection point F of AA', BB' is also on e and that $A'B'$ passes through E. Hence the pole Q' of $A'B'$ will be on line e. Since homographies preserve polarity it must be $Q' = f(Q)$ and f preserves line e. From this follow easily all other claims of the proposition.

I call pencil $\mathcal{P} = (c, e)$ the *associated to f bitangent pencil*. I use also for E the name *center of the pencil* or/and *center of the conic homography f*.

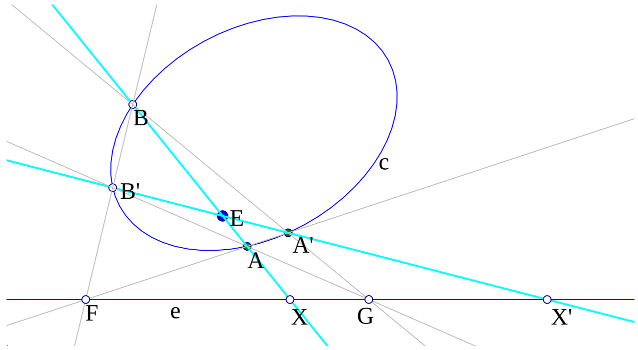


Figure 3. Invariance of axis e

Next deal with conic homographies is their distinction in *involutive* and non-involutive, i.e. homographies of period two and all others ([4, vol.II, p.179], [12, p.223]). Following proposition identifies involutive homographies with *harmonic homologies* (see Section 7) preserving a conic.

Proposition 4. *Every involutive conic homography f of the conic c fixes every point of its axis e. Inversely if it fixes its axis and $E \notin e$ it is involutive. Equivalently for each point $P \in c$ with $P' = f(P)$, line PP' passes through E the pole of the axis e of f. Point E is called in this case the center or Fregier point of the involution.*

Involutions are important because they can represent through their compositions every conic homography. The bitangent pencils (c, e) of interest, though, are those created by non-involutive conic homographies $f : c \to c$, and it will be seen that the automorphisms of such pencils consist of all homographies of the conic which commute with f. The following proposition clarifies the decomposition of every conic homography in two involutions ([4, vol.II, p.178], [12, p.224]).

Proposition 5. *Every conic homography f of a conic c can be represented as the product $f = I_2 \circ I_1$ of two involutions I_1, I_2. The centers of the involutions are necesserilly on the axis e of f. In addition the center of one of them may be any arbitrary point $P_1 \in e$ (not a fixed point of f), the center of the other $P_2 \in e$ is then uniquely determined.*

Following well known proposition signals also an important relation between a non-involutive conic homography and the associated to it bitangent pencil. I call the method suggested by this proposition the *tangential generation* of a non-involutive conic homography. It expresses for non-involutive homographies the counterpart of the property of involutive homographies to have all lines PP', with $(P \in c,\ P' = f(P))$, passing through a fixed point.

Proposition 6. *For every non-involutive conic homography f of a conic c and every point $P \in c$ and $P' = f(P)$ lines PP' envelope another conic c'. Conic c' is a member of the associated to f bitangent pencil. Inversely, given two member* *conics c, c' of a bitangent pencil the previous procedure defines a conic homography on c having its axis identical with the invariant line e of the bitangent pencil. Further the contact point Q' of line PP' with c' is the harmonic conjugate with respect to (P, P') of the intersection point Q of PP' with the axis e of f.*

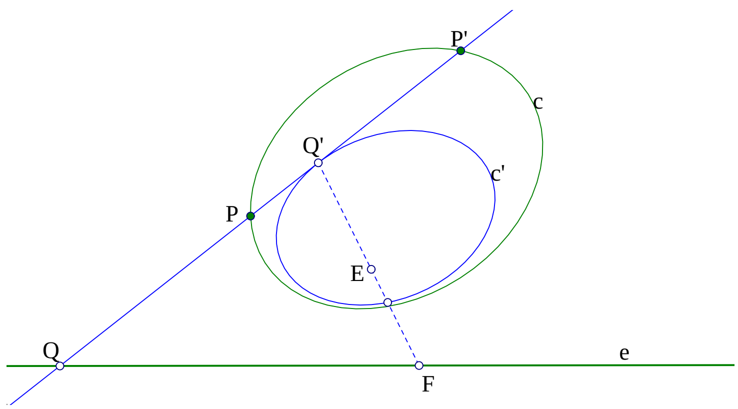


Figure 4. Tangential Generation

An elegant proof of these statements up to the last is implied by a proposition proved in [8, p.253], see also [4, vol.II, p.214] and [5, p.245]. Last statement follows from the fact that Q is the pole of line $Q'E$ (see Figure 4).

Propositions 5 and 6 allow a first description of the *automorphism group* $\mathcal{G}(c, e)$ of a given pencil (c, e) i.e. the group of homographies mapping every member-conic of the pencil onto itself. The group consists of homographies of two kinds. The first kind are the involutive homographies which are completely defined by giving their center on line e or their axis through E. The other homographies preserving the pencil are the non-involutive, which are compositions of pairs of involutions of the previous kind. Since we can put the center of one of the two involutions anywhere on e (except the intersection points of e and c), the homographies of this kind are parameterized by the location of their other center.

Before to look closer at these groups I digress for a short review of the classification of bitangent pencils and an associated naming convention for homographies.

3. Bitangent pencils

There are three cases of bitangent pencils in the real projective plane which are displayed in Figure 5. They are distinguished by the relative location of the invariant line and the conic generating the pencil.

Proposition 7. *Every bitangent pencil of conics is projectively equivalent to one generated by a fixed conic c and a fixed line e in one of the following three possible configurations.*

(I) *The line e non-intersecting the conic c (elliptic).*

(II) *The line e intersecting the conic c at two points (hyperbolic).*

(III) *The line e being tangent to the conic c (parabolic).*

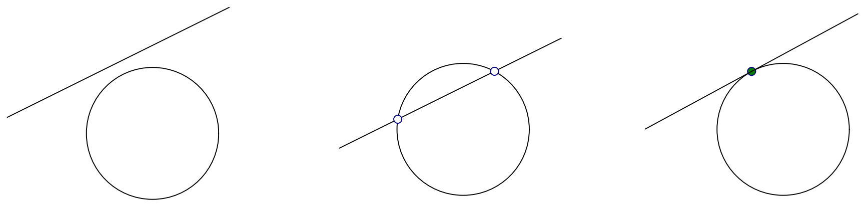

(I) (II) (III)

Figure 5. Bitangent pencils classification

The proof follows by reducing each case to a kind of *normal* form. For case (I) select a projective basis A, B, C making a *self-polar* triangle with respect to c. For this take A to be the pole of e with respect to c, take then B arbitrary on line e and define C to be the intersection of e and the polar p_B of B with respect to c. The triangle ABC thus defined is self-polar with respect to c and the equations of c and e take the form

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 0, \quad x = 0.$$

In this we can assume that $\alpha > 0$, $\beta > 0$ and $\gamma < 0$. Applying then a simple projective transformation we reduce the equations in the form

$$x^2 + y^2 - z^2 = 0, \quad x = 0.$$

For case (II) one can define a projective basis A, B, C for which the equation of c and e take respectively the form

$$x^2 - yz = 0, \quad x = 0.$$

For this it suffices to take for A the intersection of the two tangents t_B, t_C to the conic at the intersections B, C of the line e with the conic c and the *unit* point of the basis on the conic. The projective equivalence of two such systems is obvious. Finally a system of type (III) can be reduced to one of type (II) by selecting again an appropriate projective base A, B, C. For this take B to be the contact point of the line and the conic. Take then A to be an arbitrary point on the conic and define C to be the intersection point of the tangents t_A, t_B. This reduces again the equations to the form ([4, vol.II, p.188])

$$xy - z^2 = 0, \quad x = 0.$$

The projective equivalence of two such *normal forms* is again obvious.

Remark. The disctinction of the three cases of bitangent pencils leads to a natural distinction of the non-involutive homographies in four general classes. The first class consists of homographies preserving a conic, such that the associated bitangent pencil is elliptic. It is natural to call these homographies *elliptic*. Analogously homographies preserving a conic and such that the associated bitangent pencil is hyperbolic or parabolic can be called respectively *hyperbolic* or *parabolic*. All other non-involutive homographies, not falling in one of these categories (i.e. not preserving a conic), could be called *loxodromic*. Simple arguments related to the set of fixed points of an homography show easily that the four classes are disjoint. In addition since, by Proposition 2, the fixed points of a homography f preserving a conic are its intersection points with the respective homography axis e, we see that the three classes of non-involutive homographies preserving a conic are characterized by the number of their fixed points on the conic ([12, p.101], [15, p 243]). This naming convention of the first three classes conforms also with the traditional naming of the corresponding kinds of real Moebius transformations induced on the invariant line of the associated pencil ([10, p.68]).

4. The isotropy at a point

Next proposition describes the structure of the isotropy group $\mathcal{G}_{AB}(c, e)$ for a hyperbolic pencil (c, e) at each one of the two intersection points $\{A, B\} = c \cap e$.

Proposition 8. *Every homography preserving both, a conic c, an intersecting the conic line e, and fixing one (A say) of the two intersection points A, B of c and e belongs to a group $\mathcal{G}_{AB}(c, e)$ of homographies, which is isomorphic to the multiplicative group $\mathbb{R}^*$ and can be parameterized by the points of the two disjoint arcs into which c is divided by A, B.*

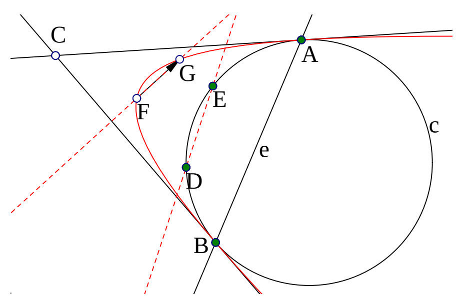


Figure 6. Isotropy of type IIb

Figure 6 illustrates the proof. Assume that homography f preserves both, the conic c, the line e, and also fixes A. Then it fixes also the other point B and also the pole C of line AB. Consequently f is uniquely determined by prescribing its value $f(D) = E \in c$ at a point $D \in c$. I denote this homography by f_{DE}. This map has a simple matrix representation in the projective basis $\{C, A, B, D\}$ in which conic c is represented by the equation $yz - x^2 = 0$ and line AB by $x = 0$, the unit point $D(1, 1, 1)$ being on the conic. In this basis and for $E \in c$ with coordinates (x, y, z) map f_{DE} is represented by non-zero multiples of the matrix

$$F_{DE} = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}.$$

This representation shows that $\mathcal{G}_{AB}(c, e)$ is isomorphic to the multiplicative group $\mathbb{R}^*$ which has two connected components. The group $\mathcal{G}_{AB}(c, e)$ is the union of two cosets $\mathcal{G}_1, \mathcal{G}_2$ corresponding to the two arcs on c, defined by the two points

A, B. The arc containing point D corresponds to subgroup $\mathcal{G}_1$, coinciding with the connected component containing the identity. The other arc defined by AB corresponds to the other connected component $\mathcal{G}_2$ of the group. For points E on the same arc with D the corresponding homography f_{DE} preserves the two arcs defined by A, B, whereas for points E on the other arc than the one containing D the corresponding homography f interchanges the two arcs.

Obviously point D can be any point of c different from A and B. Selecting another place for D and varying E generates the same group of homographies. Clearly also there is a symmetry in the roles of A, B and the group can be identified with the group of homographies preserving conic c and fixing both points A and B.

Remark. Note that there is a unique involution I_0 contained in $\mathcal{G}_{AB}(c, e)$. It is the one having axis AB and center C, obtained for the position of E for which line DE passes through C, the corresponding matrix being then the $diagonal(-1, 1, 1)$.

Following proposition deals with the isotropy of pencils (c, e) at *normal* points of the conic c, i.e. points different from its intersection point(s) with the invariant line e.

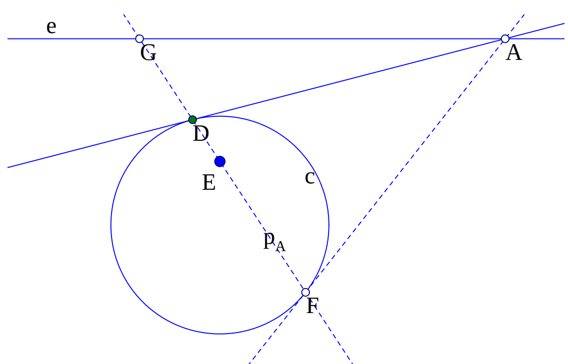


Figure 7. Isotropy at normal points

Proposition 9. *For every normal point D of the conic c the isotropy group $\mathcal{G}_D(c, e)$ is isomorphic to $\mathbb{Z}_2$. The different from the identity element of this group is the involution I_D with axis DE.*

For types (I) and (II) of pencils a proof is the following. Let the homography f preserve the conic c, the line e and fix point D. Then it preserves also the tangent t_D at D and consequently fixes also the intersection point A of this line with the axis e (see Figure 7). It is easily seen that the polar DF of A passes through the center E of the pencil and that f preserves DF. Thus the polar DF carries three points, which remain fixed under f. Since f has three fixed points on line DF it leaves the whole line fixed, hence it coincides with the involution with axis DF and center A.

For type (III) pencils the proof follows from the previous proposition. In fact, assuming $B = c \cap e$ and $A \in c, A \neq B$ an element f of the isotropy group $\mathcal{G}_A(c, e)$ fixes points A, B hence $f \in \mathcal{G}_{AB}$. But from all elements f of the last group only the involution I_B with axis AB preserves the members of the pencil (c, e). This is immediately seen by considering the decomposition of f in two involutions. Would f preserve the member-conics of the bitangent family (c, e) then, by Proposition 5, the centers of these involutions would be points of e but this is impossible for $f \in \mathcal{G}_{AB}$, since the involutions must in this case be centered on line AB.

A byproduct of the short investigation on the isotropy group $\mathcal{G}_{AB}$ of a hyperbolic pencil (c, e) is a couple of results concerning the orbits of $\mathcal{G}_{AB}$ on points of the plane, other than the fixed points A, B, C. To formulate it properly I adopt for triangle ABC the name of *invariant triangle*.

Proposition 10. *For every point F not lying on the conic c and not lying on the side-lines of the invariant triangle ABC the orbit $\mathcal{G}_{AB}F$ is the member conic c_F of the hyperbolic bitangent pencil (c, e) which passes through F.*

In fact, $\mathcal{G}_{AB}F \subset c_F$ since all $f \in \mathcal{G}_{AB}$ preserve the member-conics of the pencil (see Figure 6). By the continuity of the action the two sets must then be identical. The second result that comes as byproduct is the one suggested by Figure 8. In its formulation as well the formulation of next proposition I use the maps introduced in the course of the proof of Proposition 8.

Proposition 11. *For every point F not lying on the conic c and not lying on the side-lines of the invariant triangle ABC, the intersection point H of lines DE and FG, where $G = f_{DE}(F)$, as E varies on the conic c, describes a conic passing through points A, B, C, D and F.*

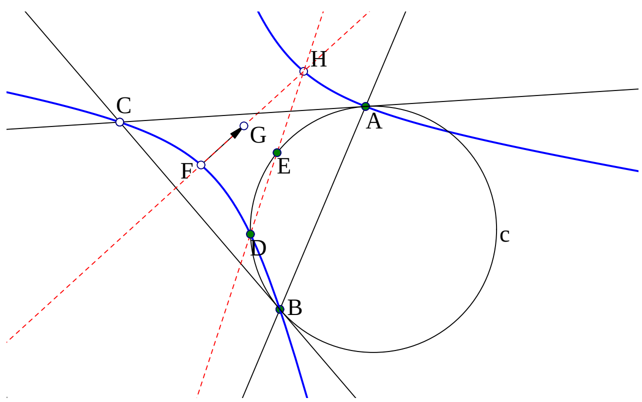


Figure 8. A triangle conic

To prove this consider the projective basis and the matrix representation of f_{DE} given above. It is easy to describe in this basis the map sending line DE to FG. Indeed let $E(x, y, z)$ be a point on the conic. Line DE has coefficients $(y - z, z - x, x - y)$. Thus, assuming F has coordinates (α, β, γ), its image will be described by the coordinates $(\alpha x, \beta y, \gamma z)$. The coefficients of the line FG will be then $(\beta\gamma(y-z), \gamma\alpha(z-x), \alpha\beta(x-y))$. Thus the correspondence of line FG to line DE will be described in terms of their coefficients by the projective transformation

$$(y-z, z-x, x-y) \mapsto (\,\beta\gamma(y-z),\ \gamma\alpha(z-x),\ \alpha\beta(x-y)\,).$$

The proposition is proved then by applying the *Chasles-Steiner* theorem, according to which the intersections of homologous lines of two pencils related by a homography describe a conic ([3, p.73], [4, vol.II, p.173]). According to this theorem the conic passes through the vertices of the pencils D, F. It is also easily seen that the conic passes through points A, B and C.

Proposition 12. *For every point F not lying on the conic c and not lying on the side-lines of the invariant triangle ABC, lines EG with $G = f_{DE}(F)$ as E varies on c envelope a conic which belongs to the bitangent pencil $(c, e = AB)$.*

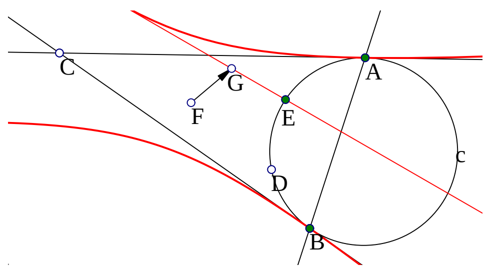


Figure 9. Bitangent member as envelope

The proof can be based on the dual of the argument of *Chasles-Steiner* ([3, p.89]), according to which the lines joining homologous points of a homographic relation between two ranges of points envelope a conic. Here lines EG (see Figure 9) join points (x, y, z) on the conic c with points $(\alpha x, \beta y, \gamma z)$ on the conic c_F, hence their coefficients are given by

$$(\,(\gamma-\beta)yz,\ (\alpha-\gamma)zx,\ (\beta-\alpha)xy\,).$$

Taking the traces of these lines on $x = 0$ and $y = 0$ we find that the corresponding coordinates $(0, y', z')$ and $(x'', 0, z'')$ satisfy an equation of the form $\tau'\tau'' = \kappa$, where $\tau' = y'/z',\ \tau'' = x''/z''$ and κ is a constant. Thus lines EG join points on $x = 0$ and $y = 0$ related by a homographic relation hence they envelope a conic. It is also easily seen that this conic passes through A, B and has there tangents CA, CB hence it belongs to the bitangent family.

Continuing the examination of possible isotropies, after the short digression on the three last propositions, I examine the isotropy group $\mathcal{G}_A(c, e)$ of a parabolic pencil (c, e), for which the axis e is tangent to the conic c at a point A. An element $f \in \mathcal{G}_A(c, e)$ may have A as its unique fixed point or may have an additional fixed point $B \neq A$.

An element $f \in \mathcal{G}_A(c, e)$ having A as a unique fixed point cannot leave invariant another line through A, since this would create a second fixed point on c. Also there is no other fixed point on the tangent e since this would also create another fixed point on c.

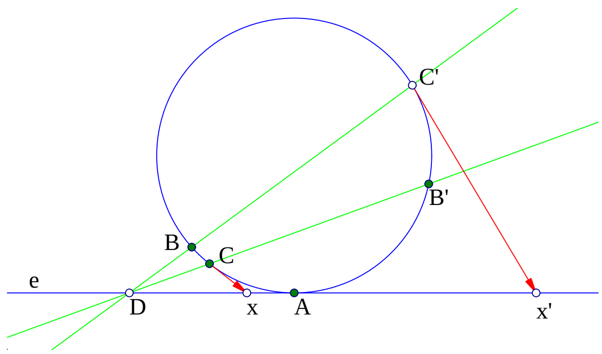


Figure 10. Parabolic isotropy

Proposition 13. *The group $\mathcal{G}_A^0$ including the identity and all homographies f, which preserve a conic c and have A as a unique fixed point, is isomorphic to the additive group $\mathbb{R}$. Every non-indetity homography in this group induces in the tangent e at A a parabolic transformation, which in line coordinates with origin at A is described by a function of the kind $x' = ax/(bx + a)$ or equivalently, by setting $d = b/a$, through the relation*

$$\frac{1}{x'} - \frac{1}{x} = d.$$

This function uniquely describes the conic homography from which it is induced in line e. All elements of this group are non-involutive.

In fact consider the induced Moebius transformation on line e with respect to coordinates with origin at A (see Figure 10). Since A is a fixed point this transformation will have the form $x' = ax/(bx + c)$. Since this is the only root of the equation $x(bx + c) = ax \Leftrightarrow bx^2 + (c - a)x = 0$, it must be $c = a$. Since for every point B other than A the tangents $t_B, t_{B'}$ where $B' = f(B)$ intersect line e at corresponding points $C, C' = f(C)$ the definition of f from its action on line e is complete and unique. The statement on the isomorphism results from the above representation of the transformation. The value $d = 0$ corresponds to the identity transformation. Every other value $d \in \mathbb{R}$ defines a unique parabolic transformation and the product of two such transformations corresponds to the sum $d + d'$ of these constants.

The group $\mathcal{G}_A$ of all homographies preserving a conic c and fixing a point A contains obviously the group $\mathcal{G}_A^0$. The other elements of this group will fix an additional point B on the conic. Consequently the group will be represented as a union $\mathcal{G}_A = \mathcal{G}_A^0 \cup_{B \neq A} \mathcal{G}_{AB}$. For another point C different from A and B the corresponding group $\mathcal{G}_{AC}$ is conjugate to $\mathcal{G}_{AB}$, by an element of the group $\mathcal{G}_A^0$. In fact, by the previous discussion there is a unique element $f \in \mathcal{G}_A^0$ mapping B to C. Then $Ad_f(\mathcal{G}_{AB}) = \mathcal{G}_{AC}$ i.e. every element $f_C \in \mathcal{G}_{AC}$ is represented as $f_C = f \circ f_B \circ f^{-1}$ with $f_B \in \mathcal{G}_{AB}$. These remarks lead to the following proposition.

Proposition 14. *The isotropy group $\mathcal{G}_A$ of conic homographies fixing a point A of the conic c is the semi-direct product of its subgroups $\mathcal{G}_A^0$ of all homographies f which preserve c and have the unique fixed point A on c and the subgroup $\mathcal{G}_{AB}$ of conic homographies which fix simultaneously A and another point $B \in c$ different from A.*

To prove this apply the criterion ([1, p.285]) by which such a decomposition of the group is a consequence of the following two properties: (i) Every element g of the group $\mathcal{G}_A$ is expressible in a unique way as a product $g = g_B \circ g_A$ with $g_A \in \mathcal{G}_A^0,\ g_B \in \mathcal{G}_{AB}$ and (ii) Group $\mathcal{G}_A^0$ is a normal subgroup of $\mathcal{G}_A$. Starting from property (ii) assume that $f \in \mathcal{G}_A$ has the form $f = g_B \circ g_A \circ g_B^{-1}$. Should f fix a point $C \in c$ different from A then it would be $g_A(g_B^{-1}(C)) = g_B^{-1}(C)$ i.e. $g_B^{-1}(C)$ would be a fixed point of g_A, hence $g_B^{-1}(C) = A$ which is impossible. To prove (i) show first that every element in $\mathcal{G}_A$ is expressible as a product $g = g_B \circ g_A$. This is clear if $g \in \mathcal{G}_A^0$ or $g \in \mathcal{G}_{AB}$. Assume then that g in addition to A fixes also the point $C \in c$ different from A. Then as remarked above g can be written in the form $g = g_A \circ g_B \circ g_A^{-1}$, hence $g = g_B \circ (g_B^{-1} \circ g_A \circ g_B \circ g_A^{-1})$ and the parenthesis is an element of $\mathcal{G}_A^0$. That such a representation is also unique follows trivially, since the equation $g_A \circ g_B = g'_A \circ g'_B$ would imply $g_A^{-1} \circ g'_A = g'_B \circ g_B^{-1}$ implying $g_A = g'_A$ and $g_B = g'_B$, since the two subgroups $\mathcal{G}_A^0$ and $\mathcal{G}_{AB}$ have in common only the identity element.

5. Automorphisms of pencils

In this section I examine the automorphism group $\mathcal{G}(c, e)$ of a pencil (c, e) and in particular the non-involutive automorphisms. Every such automorphism is a conic homography f of conic c preserving also the line e. Hence it induces on line e a homography which can be represented by a Moebius transformation

$$x' = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

Inversely, knowing the induced homography on line e from a non-involutive homography one can reconstruct the homography on every other member-conic c of the pencil. Figure 11 illustrates the construction of the image point $B' = f(B)$ by drawing the tangent t_B of c at B and finding its intersection C with e. The image $f(B)$ is found by taking the image point $C' = f(C)$ on e and drawing from there the tangents to c and selecting the appropriate contact point B' or B'' of the tangents from C'. The definition of the homography on c is unambiguous only for pencils of type (III). For the other two kinds of pencils one can construct two homographies f and f^*, which are related by the involution I_0 with center E and

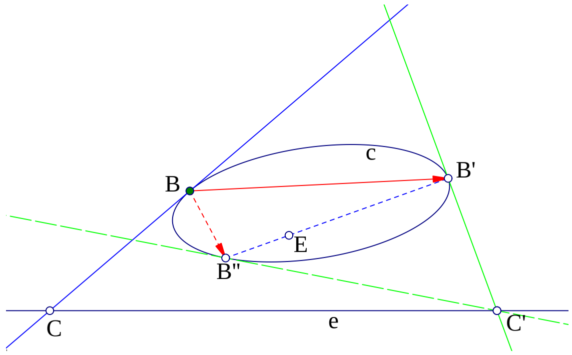


Figure 11. Using line e

axis e. The relation is $f^* = f \circ I_0 = I_0 \circ f$ (last equality is shown in Proposition 16).

Using this method one can easily answer the question of periodic conic homographies.

Proposition 15. *Only the elliptic bitangent pencils have homographies periodic of period $n > 2$. Inversely, if a conic homography is periodic, then it is elliptic.*

In fact, in the case of elliptic pencils, selecting the homography on e to be of the kind

$$x' = \frac{cos(\phi)x - sin(\phi)}{sin(\phi)x + cos(\phi)}, \quad \phi = \frac{2\pi}{n},$$

we define by the procedure described above an n-periodic homography preserving the pencil. For the cases of hyperbolic and parabolic pencils it is impossible to define a periodic homography with period $n > 2$. This because, for such pencils, every homography preserving them has to fix at least one point. If it fixes exactly one, then it is a parabolic homography, hence by Proposition 13 can not be periodic. If it fixes two points, then as we have seen in Proposition 8, the homography can be represented by a real *diagonal* matrix and this can not be periodic for $n > 2$. The inverse is shown by considering the associated bitangent pencil and applying the same arguments.

Since a general homography preserving a conic c can be written as the composition of two involutions, it is of interest to know the structure of the set of involutions preserving a given bitangent pencil. For non-parabolic pencils there is a particular involution I_0, namely the one having for axis the invariant line e of the family and for center E the pole of this line with respect to c.

If I is an arbitrary, other than I_0, involution preserving the bitangent family (c, e) then, since e is invariant by I, either its center Q is on line e or its axis coincides with e. Last case can be easily excluded by showing that the composition $f = I_0 \circ I$ is then an elation with axis e and drawing from this a contradiction. Consequently the axis $e_I = EF$ (see Figure 12) of the involution must pass through the pole E of e with respect to c. It follows that I commutes with I_0. A consequence of this is

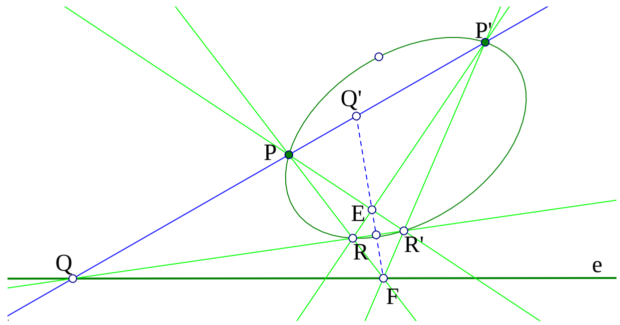


Figure 12. Involutive automorphisms

that $I' = I_0 \circ I$ is another involution the axis of which is line EQ and its center is F. Since by Proposition 5 every homography f preserving the bitangent pencil is a product of two involutions with centers on the axis e it follows that I_0 commutes with f. We arrive thus at the following.

Proposition 16. *The group $\mathcal{G}(c, e)$ of all homographies preserving a non-parabolic bitangent pencil is a subgroup of the group of homographies of the plane preserving line e, fixing the center E of the pencil and commuting with involution I_0.*

For the rest of the section I omit the reference to (c, e) and write simply $\mathcal{G}$ instead of $\mathcal{G}(c, e)$. Involution I_0 is a singularium and should be excluded from the set of all other involutions. It can be represented in infinite many ways as a product of involutions. In fact for any other involutive automorphism of the pencil I the involution $I' = I \circ I_0 = I_0 \circ I$ represents it as a product $I_0 = I \circ I'$. Counting it to the non-involutive automorphisms, it is easy to see that we can separate the group $\mathcal{G}$ into two disjoint sets. The set of non-involutive automorphisms $\mathcal{G}' \subset \mathcal{G}$ containing the identity and I_0 as particular elements, and the set $\mathcal{G}'' \subset \mathcal{G}$ of all other involutive automorphisms.

Proposition 17. *For non-parabolic pencils two involutions I, I' commute, if and only if their product is I_0. Further if the product of two involutions is an involution, then this involution is I_0. For parabolic pencils $I \circ I'$ is never commutative.*

For the first claim notice that $I' \circ I = I_0$ implies $I' = I_0 \circ I = I \circ I_0$. Last because every element of $\mathcal{G}$ commutes with I_0. Last equation implies $I \circ I' = I' \circ I$. Inversely, if last equation is valid it is readily seen that the two involutions have common fixed points on e and fix E hence their composition is $I' \circ I = I_0$. Next claim is a consequence of the previous, since $I' \circ I$ being involution implies $(I' \circ I) \circ (I' \circ I) = 1 \Rightarrow I' \circ I = I \circ I'$. Last claim is a consequence of the fact that $I \circ I'$ and $I' \circ I$ are inverse to each other and non-involutive, according to Proposition 13.

Proposition 18. *The automorphism group $\mathcal{G}$ of a pencil (c, e) is the union of two cosets $\mathcal{G} = \mathcal{G}' \cup \mathcal{G}''$. $\mathcal{G}'$ consists of the non-involutive automorphisms (and I_0 for non-parabolic pencils) and builds a subgroup of $\mathcal{G}$. $\mathcal{G}''$ consists of all involutive* *automorphisms of the pencil (which are different from I_0 for non-parabolic pencils) and builds a coset of $\mathcal{G}'$ in $\mathcal{G}$. Further it is $\mathcal{G}''\mathcal{G}'' \subset \mathcal{G}'$ and $\mathcal{G}'\mathcal{G}'' \subset \mathcal{G}''$.*

In fact, given an involutive $I \in \mathcal{G}''$ and a non-involutive $f \in \mathcal{G}'$, we can, according to Proposition 5, represent f as a product $f = I \circ I'$ using involution I and another involution I' completely determined by f. Then $I \circ f = I' \in \mathcal{G}''$. This shows $\mathcal{G}''\mathcal{G}' \subset \mathcal{G}''$. The inclusion $\mathcal{G}'\mathcal{G}' \subset \mathcal{G}'$ proving $\mathcal{G}'$ a subgroup of $\mathcal{G}$ is seen similarly. The other statements are equally trivial.

Regarding commutativity, we can easily see that the (co)set of involutions contains non-commuting elements in general ($I' \circ I$ is the inverse of $I \circ I'$), whereas the subgroup $\mathcal{G}'$ is always commutative. More precisely the following is true.

Proposition 19. *The subgroup $\mathcal{G}' \subset \mathcal{G}$ of non-involutive automorphisms of the bitangent pencil (c, e) is commutative.*

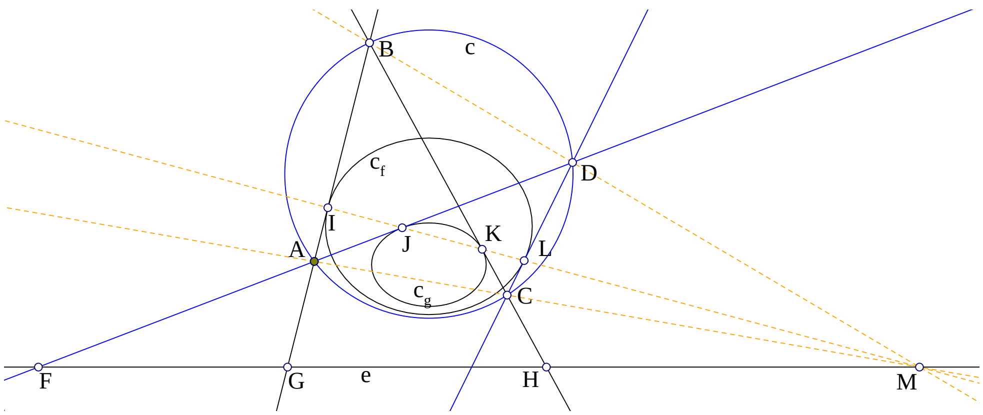


Figure 13. Commutativity for type I

The proof can be given on the basis of Figure 13, illustrating the case of elliptic pencils, the arguments though being valid also for the other types of pencils. In this figure the two products $f \circ g$ and $g \circ f$ of two non-involutive automorphisms of the pencil $f \in \mathcal{G}'$ and $g \in \mathcal{G}'$ are represented using the *tangential generation* of Proposition 6. For $A \in c$ point $B = f(A)$ has line AB tangent at I to a conic c_f of the pencil. Analogously $C = g(B)$ defines line BC tangent at K to a second conic c_g of the pencil. Let $D = g(A)$ and consequently AD be tangent at point J to c_g. It must be shown that $f(D) = C$ or equivalently that line DC is tangent at a point L to c_f. For this note first that lines $\{BD, IJ\}$ intersect at a point M on e. This happens because of the harmonic ratios $(A, B, G, I) = -1$ and $(A, D, F, J) = -1$. Similarly lines AC, IK intersect at a point M' of e. This follows again by the harmonic ratios $(B, A, I, G) = (B, C, K, H) = -1$. Hence $M' = M$ and consequently lines AC, BD intersect at M, hence according to Proposition 2, $C = f(D)$.

For hyperbolic pencils the result is also a consequence of the representation of these homographies through diagonal matrices, as in Proposition 8. For parabolic pencils the proof follows also directly from Proposition 13.

Note that for pencils (c, e) of type (II) for which c and e intersect at two points $\{A, B\}$, the involutions I_A, I_B with axes respectively BE, AE, do not belong to $\mathcal{G}$ but define through their composition $I_A \circ I_B = I_0$. This is noticed in Proposition 5 which represents every automorphism as the product of two involutions. It is though a case to be excluded in the following proposition, which results from Proposition 5 and the previous discussion.

Proposition 20. *If an automorphism $f \in \mathcal{G}$ of a pencil (c, e) is representable as a product of two involutions $f = I_2 \circ I_1$, then with the exception of $I_0 = I_A \circ I_B$ in the case of an hyperbolic pencil, in all other cases I_1 and I_2 are elements of $\mathcal{G}$.*

Regarding the transitivity of $\mathcal{G}(c, e)$ on the conics of the pencil, the following result can be easily proved.

Proposition 21. *(i) For elliptic pencils (c, e) each one of the cosets $\mathcal{G}'$, $\mathcal{G}''$ acts simply transitively on the points of the conic c.*
(ii) For hyperbolic pencils (c, e) each one of the cosets $\mathcal{G}'$, $\mathcal{G}''$ acts simply transitively on $c - \{A, B\}$, where $\{A, B\} = c \cap e$. All elements of $\mathcal{G}''$ interchange (A, B), whereas all elements of $\mathcal{G}'$ fix them.
(iii) For parabolic pencils each one of the cosets $\mathcal{G}'$, $\mathcal{G}''$ acts simply transitively on $c - \{A\}$ where $A = c \cap e$ and all of them fix point A.

6. Bitangent flow

Last proposition shows that every non-involutive conic homography f of a conic c is an element of a one-dimensional Lie group ([6, p.210], [13, p.82]) $\mathcal{G}$ acting on the projective plane . The invariant conic c is then a union of orbits of the action of this group. Group $\mathcal{G}$ is a subgroup of the Lie group $PGL(3, \mathbb{R})$ of all projectivities of the plane and contains a one-parameter group ([13, p.102]) of this group, which can be easily identified with the connected component of the subgroup $\mathcal{G}'$ containing the identity. Through the one-parameter group one can define a vector field on the plane, the integral curves of which are contained in the conics of the bitangent pencil associated to the non-involutive homography. Thus the bitangent pencil represents the flow of a vector field on the projective plane ([6, p.139], [14, p.292]). The fixed points correspond to the singularities of this vector field.

This point of view rises the problem of the determination of the simplest possible data needed in order to define such a flow on the plane. The answer (Proposition 26) to this problem lies in a certain involution on e related to the coset $\mathcal{G}''$ of the involutive automorphisms of the bitangent pencil. I start with non-parabolic pencils, characterized by the existence of the particular involution I_0.

Proposition 22. *For every non-parabolic pencil the correspondence $\mathcal{J} : Q \mapsto F$ between the centers of the involutions I and $I \circ I_0$ defines an involutive homography on line e. The fixed points of $\mathcal{J}$ coincide with the intersection points $\{A, B\} = c \cap e$.*

In fact considering the pencil E^* of lines through E it is easy to see that the correspondence $\mathcal{J} : F \mapsto Q$ (see Figure 14) is projective and has period two. The identification of the fixed points of $\mathcal{J}$ with $\{A, B\} = c \cap e$ is equally trivial.

Proposition 23. *The automorphism group $\mathcal{G}(c,e)$ of a non-parabolic pencil is uniquely determined by the triple*

$$(e, E, \mathcal{J})$$

consisting of a line e a point $E \notin e$ and an involutive homography on line e.

In fact $\mathcal{J}$ completely determines the involutive automorphisms I_Q of the pencil, since for each point Q on e point $F = \mathcal{J}(Q)$ defines the axis FE of the involution I_Q. The involutive automorphisms in turn, through their compositions, determine also the non involutive elements of the group.

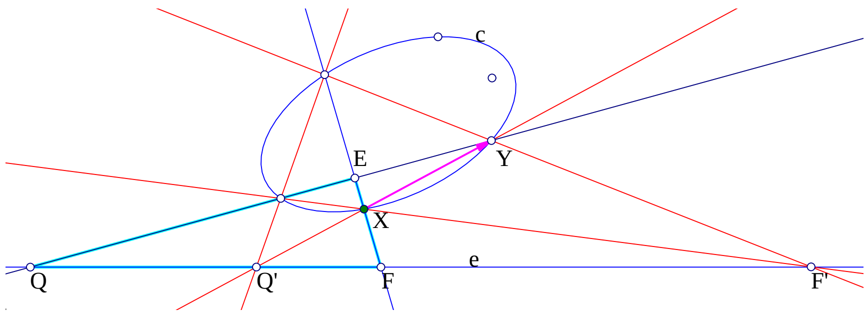


Figure 14. Quadrilateral in case I

Remark. For elliptic pencils involution $\mathcal{J}$ induces on every member-conic c of the pencil a correspondence of points $X \mapsto Y$ through its intersection with lines (EF, EQ) (see Figure 14). This defines an automorphism of the pencil of order 4 and through it infinite many convex quadrangles, each of which completely determines the pencil. Inversely, by the results of this section it will follow that for each convex quadrangle there is a well defined bitangent pencil having a member c circumscribed and a member c' inscribed in the quadrangle. Conic c is characterized by having its tangents at opposite vertices intersect on line e. Conic c' contacts the sides of the quadrangle at their intersections with lines $\{EI, EJ\}$ (see Figure 15). Note that for cyclic quadrilaterals in the euclidean plane the corresponding conic c does not coincide in general with their circumcircle. It is instead identical with the image of the circumcircle of the square under the unique projective map sending the vertices of the square to those of the given quadrilateral (e is the image under this map of the line at infinity).

Knowing the group $\mathcal{G}$ of its automorphisms, one would expect a complete reconstruction of the whole pencil, through the orbits $\mathcal{G}X$ of points X of the plane under the action of this group. Before to proceed to the proof of this property I modify slightly the point of view in order to encompass also parabolic pencils. For this consider the map $\mathcal{I} : e \mapsto E^*$ induced in the pencil E^* of lines emanating from E, the pole of the invariant line of the pencil. This map associates to every point $Q \in e$ the axis EF of the involution centered at Q. Obviously for non-parabolic pencils $\mathcal{I}$ determines $\mathcal{J}$ and vice versa. The first map though can be defined also

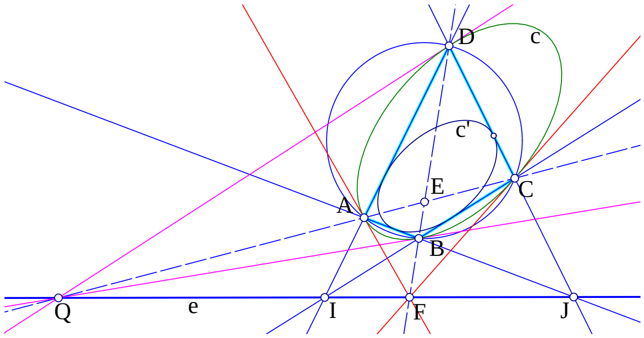


Figure 15. Circumcircle and circumconic

for parabolic pencils, since also in this case, for each point $Q \in e$ there is a unique line FQ representing the axis of the unique involutive automorphism of the pencil centered at Q. Following general fact is on the basis of the generation of the pencil through orbits.

Proposition 24. *Given a line e and a point E consider a projective map $\mathcal{I} : e \mapsto E^*$ of the line onto the pencil E^* of lines through E. Let e' denote the complement in e of the set $e'' = \{Q \in e : Q \in \mathcal{I}(Q)$. For every $Q \in e'$ denote the involution with center Q and axis $\mathcal{I}(Q)$. Then for every point $X \notin e$ of the plane the set $\{I_Q(X) : Q \in e'\} \cup e''$ is a conic.*

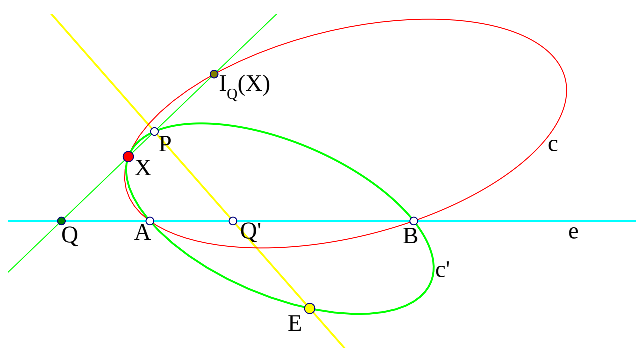


Figure 16. Orbits of involutions

In fact, by the Chasles-Steiner construction method of conics ([3, p.73]), lines XQ and $\mathcal{I}(Q)$ intersect at a point P describing a conic c', which passes through X and E. Every point $Q \in e''$ i.e. satisfying $Q \in \mathcal{I}(Q)$ coincides with a point of the intersection $c' \cap e$ and vice versa. Thus e'' has at most two points ($\{A, B\}$ in Figure 16).

The locus $\{I_Q(X) : Q \in e\}$ coincides then with the image c of the conic c', under the perspectivity p_X with center at X, axis the line e and homology coefficient $k = 1/2$.

Proposition 25. *The conics generated by the previous method belong to a bitangent pencil with axis e and center E if and only if they are invariant by all involutions I_Q for $Q \in e'$. The points in e'' are the fixed points of the pencil.*

The necessity of the condition is a consequence of Proposition 21. To prove the sufficiency assume that c is invariant under all $\{I_Q : Q \in e'\}$. Then for every $Q \in e'$ line $\mathcal{I}(Q)$ is the polar of Q with respect to c. Consequently line e is the polar of E with respect to c and, if $E \notin e$, the involution I_0 with axis e and center E leaves invariant c. Since the center of each involution from the pair $(I_Q,\, I_0)$ is on the axis of the other the two involutions commute and $I_Q \circ I_0$ defines an involution with center at the intersection $Q' = e \cap \mathcal{I}(Q)$ and axis the polar of this point with respect to c, which, by the previous arguments, coincides with $\mathcal{I}(Q')$. This implies that the map induced in line e by $\mathcal{J}' : Q \mapsto Q' = \mathcal{I}(Q) \cap e$ is an involution. Consider now the pencil (c, e). It is trivial to show that its member-conics coincide with the conics $\{I_Q(X) : Q \in e\}$ for $X \notin e$ and $\mathcal{J}'$ is identical with the involution $\mathcal{J}$ of the pencil. This completes the proof of the proposition for the case $E \notin e$.

The proof for the case $E \in e$ is analogous with minor modifications. In this case the assumption of the invariance of c under I_Q implies that line $\mathcal{I}(Q)$ is the polar of Q with respect to c. From this follows that c is tangent to e at E and $\mathcal{I}(E) = e$. Thus e'' contains the single element E. Then it is again trivial to show that the conics of the pencil (c, e) coincide with the conics $\{I_Q(X) : Q \in e\}$ for $X \notin e$.

The arguments in the previous proof show that non-parabolic pencils are completely determined by the involution $\mathcal{J}$ on line e, whereas parabolic pencils are completely defined by a projective map $\mathcal{I} : e \to E^*$ with the property $\mathcal{I}(E) = e$. Following proposition formulates these facts.

Proposition 26. *(i) Non-parabolic pencils correspond bijectively to triples $(e, E, \mathcal{J})$ consisting of a line e, a point $E \notin e$ and an involution $\mathcal{J} : e \to e$. The fixed points of the pencil coincide with the fixed points of $\mathcal{J}$.*
(ii) Parabolic pencils correspond bijectively to triples $(e, E, \mathcal{I})$ consisting of a line e, a point $E \in e$ and a projective map $\mathcal{I} : e \to E^*$ onto the pencil E^* of lines through E, such that $\mathcal{I}(E) = e$.

7. The perspectivity group of a pencil

Perspectivities are homographies of the plane fixing a line e, called the *axis* and leaving invariant every line through a point E, called the *center* of the perspectivity. If $E \in e$ then the perspectivity is called an *elation*, otherwise it is called *homology*. Tightly related to the group $\mathcal{G}$ of automorphisms of the pencil (c, e) is the group $\mathcal{K}$ of perspectivities, with center E the center of the pencil and axis the axis e of the pencil. As will be seen, this group acts on the pencil (c, e) by permuting its members. For non-parabolic pencils the perspectivities of this group are *homologies*, and for parabolic pencils the perspectivities are *elations*. The basic facts about perspectivities are summarized by the following three propositions ([12, p.72], [15, p.228], [7, p.247]).

Proposition 27. *Given a line e and three collinear points E, X, X', there is a unique perspectivity f with axis e and center E and $f(X) = X'$.*

Proposition 28. *For any perspectivity f with axis e and center E and two points (X, Y) with $(X' = f(X),\ Y' = f(Y))$, lines XY and $X'Y'$ intersect on e. For homologies the cross ratio $(X, X', E, X_e) = \kappa$, where $X_e = XX' \cap e$, is a constant κ called homology coefficient. Involutive homographies are homologies with $\kappa = -1$ and are called harmonic homologies.*

Proposition 29. *The set of homologies having in common the axis e and the center E builds a commutative group $\mathcal{K}$ which is isomorphic to the multiplicative group of real numbers.*

That the composition $h = g \circ f$ of two homologies with the previous characteristics is a homology follows directly from their definition. The homology coefficients multiply homomorphically $\kappa_h = \kappa_g \kappa_f$, this being a consequence of Proposition 27 and the well-known identity for cross ratios of five points (X, Y, Z, H) on a line d ([2, p.174])

$$(X, Y, E, H)(Y, Z, E, H)(Z, X, E, H) = 1,$$

where $H = d \cap e$. This implies also the commutativity.

Whereas the previous isomorphism is canonical, the following one, easily proved by using coordinates is not canonical. The usual way to realize it is to send e to infinity and have the elations conjugate to translations parallel to the direction determined by E ([4, vol.II, p.191]). The representation of the elation as a composition, given below follows directly from the definitions.

Proposition 30. *The set of all elations having in common the axis e and the center E builds a commutative group $\mathcal{K}$ which is isomorphic to the additive group of real numbers. Every elation f can be represented as a composition of two harmonic homologies $f = I_B \circ I_A$, which share with f the axis e and have their centers $\{A, B\}$ collinear with E. In this representation the center $A \notin e$ can be arbitrary, the other center B being then determined by f and lying on line AE.*

Returning to the pencil (c, e), the group $\mathcal{G}$ of its automorphisms and the corresponding group $\mathcal{K}$ of perspectivities, which are homologies in the non-parabolic case and elations in the parabolic, combine in the way shown by the following propositions.

Proposition 31. *For every bitangent pencil (c, e) the elements of $\mathcal{K}(c, e)$ commute with those of $\mathcal{G}(c, e)$.*

The proposition is easily proved first for involutive automorphisms of the pencil, characterized by having their centers Q on the perspectivity axis e and their axis q passing through the perspectivity center E. Figure 17 suggests the proof of the commutativity of such an involution f_Q with a homology f_E with center at E and axis the line e. Point $Y = f_E(X)$ satisfies the cross-ratio condition of the perspectivity $(X, Y, E, H) = \kappa$, where κ is the homology coefficient of the perspectivity.

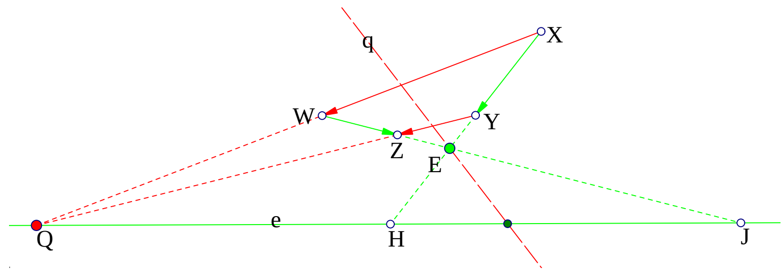


Figure 17. Homology commuting with involution

Then taking $Z = f_Q(Y)$ and the intersection W of line ZE with XQ it is readily seen that $f_Q(f_E(X)) = f_E(f_Q(X))$. Thus perspectivity f_E commutes with all involutive automorphisms of the pencil.

In the case of parabolic pencils, if EF is the axis of the involutive automorphism f_Q, $Q \in e$ of the pencil, according to Proposition 30, one can represent the elation f_E as a composition $I_B \circ I_A$ of two involutions with centers lying on EF and axis the invariant line e. Each of these involutions commutes then with f_Q, hence their composition will commute with f_Q too. Since the involutive automorphisms generate all automorphisms of the pencil it follows that f_E commutes with every automorphism of the pencil.

Proposition 32. *For non-hyperbolic pencils and every two member-conics (c, c') of the pencil there is a perspectivity with center at E and axis the line e, which maps c to c'. For hyperbolic pencils this is true if c and c' belong to the same connected component of the plane defined by lines (EA, EB), where $\{A, B\} = c \cap e$ are the base points of the pencil and E the center of the pencil.*

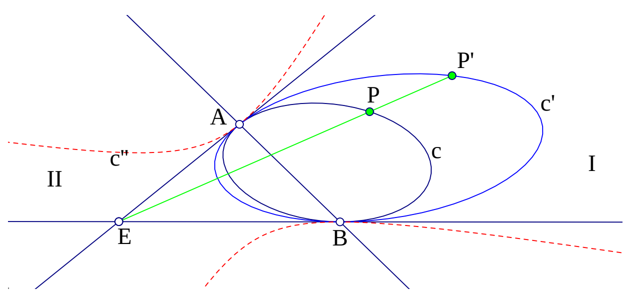


Figure 18. Perspectivity permuting member-conics

To prove the claim consider a line through E intersecting two conics of the pencil at points $P \in c$, $P' \in c'$ (see Figure 18). By Proposition 27 there is a perspectivity f mapping P to P'. By the previous proposition f commutes with all $g \in \mathcal{G}$ which can be used to map P to any other (than the base points of the pencil) point Q of c and point P' to $Q' \in c' \cap EQ$. This implies that $f(c) = c'$. The restriction for hyperbolic pencils is obviously necessary, since perspectivities leave invariant the lines through their center E.

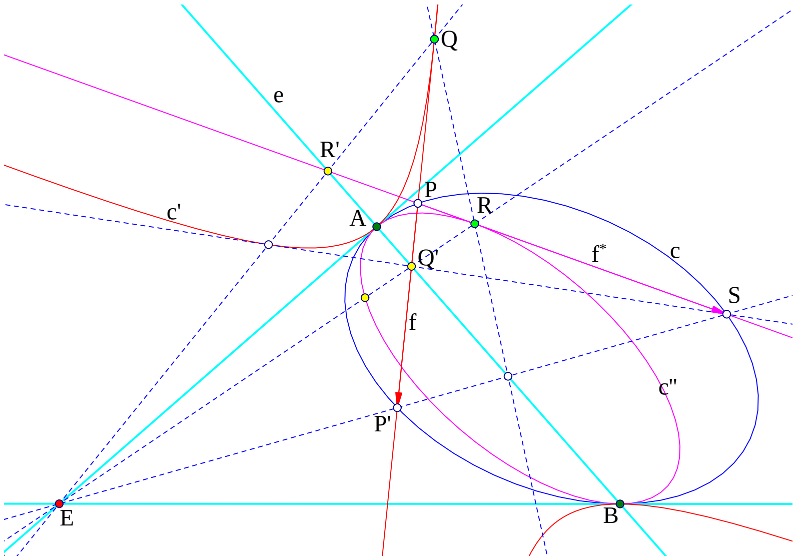


Figure 19. Conjugate member-conic

Proposition 33. *Let f be a non-involutive automorphism of the pencil (c, e) and c' be the member-conic determined by its tangential generation with respect to c(proposition-6). Then, for non-hyperbolic pencils there is a perspectivity $p_f \in \mathcal{K}$ mapping c to c'. This is true also for hyperbolic pencils provided f preserves the components of c cut out by e. Further p_f is independent of c.*

In fact, given $f \in \mathcal{G}$, according to *proposition-6*, there is a conic c' of the bitangent pencil such that lines PP', $P' = f(P)$ are tangent to c'. The exceptional case for pencils of type (II) occurs when f interchanges the two components cut out from c by the axis e. In this case conics c and c' are on different connected components of the plane defined by lines $\{EA, EB\}$. This is due to the fact (ibid) that the contact point Q of PP' with c' is the harmonic conjugate with respect to (P, P') of the intersection $Q' = PP' \cap e$. Figure 19 illustrates this case and shows that for such automorphisms the resulting automorphism $f^* = f \circ I_0 = I_0 \circ f$, mapping P to $S = f^*(P) = I_0(f(P)) = I_0(P')$, defines through its corresponding tangential generation a kind of *conjugate* conic c'' to c' with respect to c.

To come back to the proof, first claim follows from the previous proposition. Last claim means that if the pencil is represented through another member-conic d by the pair (d, e), and the tangential generation of f is determined by a conic d', then the corresponding p'_f mapping d to d' is identical to p_f. The property is indeed a trivial consequence of the commutativity between the members of the groups $\mathcal{G}$ and $\mathcal{K}$. To see this consider a point $P \in c$ and its image $P' = f(P) \in c$. Consider also the perspectivity $g \in \mathcal{K}$ sending c to d and let $Q = g(P), Q' = g(P')$. By the commutativity of f, g it is $f(Q) = f(g(P)) = g(f(P)) = g(P') = Q'$. Thus the envelope c' of lines PP' maps via g to the envelope d' of lines QQ'. Hence if $p_f(c) = c'$ and $p'_f(d) = d'$ then $d' = g(c') = g(p_f(c))$ implying $p'_f(g(c)) = g(p_f(c))$ and from this $p'_f = g \circ p_f \circ g^{-1} = p_f$ since g and p_f commute.

Remark. Given a bitangent pencil (c, e) the correspondence of p_f to f considered above is univalent only for parabolic pencils. Otherwise it is bivalent, since both p_f and $p_f \circ I_0 = I_0 \circ p_f$ do the same job. Even in the univalent case the correspondence is not a homomorphism, since it is trivially seen that f and $g = f^{-1}$ have $p_f = p_g$. This situation is reflected also in simple configurations as, for example, in the case of the bitangent pencil (c, e) of concentric circles with common center E, the invariant line e being the line at infinity.

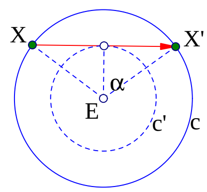


Figure 20. A case of $\mathcal{G}' \ni f \mapsto p_f \in \mathcal{K}$

In this case the rotation R_α by angle $\alpha \in (0, \pi)$ at E (see Figure 20), which is an element of the corresponding $\mathcal{G}'$, maps to the element $H_{cos(\frac{\alpha}{2})}$ of $\mathcal{K}$, which is the homothety with center E and ratio $cos(\frac{\alpha}{2})$.

8. Conic affinities

By identifying the invariant line e of a bitangent pencil (c, e) with the line at infinity all the results of the previous sections translate to properties of affine maps preserving affine conics ([2, p.184], [4, vol.II, p.146]). The automorphism group $\mathcal{G}$ of the pencil (c, e) becomes the group $\mathcal{A}$ of affinities preserving the conic c. Elliptic pencils correspond to *ellipses*, hyperbolic pencils correspond to *hyperbolas* and parabolic pencils to *parabolas*. The center E of the pencils becomes the *center* of the conic, for ellipses and hyperbolas, called collectively *central* conics. For these kinds of conics involution I_0 becomes the *symmetry* or *half turn* at the center of the conic. Every involution I other that I_0, becomes an *affine reflection* ([7, p.203]) with respect to the corresponding axis of the involution, which coincides with a *diameter* d of the conic. The center of the affine reflection I is a point at infinity defining the *conjugate* direction of lines XX' ($X' = f(X)$) of the reflection. This direction coincides with the one of the conjugate diameter to d. For an affine reflection I with diameter d the reflection $I \circ I_0$ is the reflection with respect to the *conjugate* diameter d' of the conic. Products of two affine reflexions are called *equiaffinities* ([7, p.208]) or *affine rotations*. For central conics the group $\mathcal{K}$ of perspectivities becomes the group of homotheties centered at E.

For parabolas the center of the pencil E is the contact point of the curve with the line at infinity. All affine reflections have in this case their axes passing through E

i.e. they are parallel to the direction defined by this point at infinity, which is also the contact point of the conic with e. The group $\mathcal{K}$ of elations in this case becomes the group of translations parallel to the direction defined by E.

In order to stress the differences between the three kinds of affine conics I translate the results of the previous paragraphs for each one separately.

By introducing an euclidean metric into the plane ([4, vol.I, p.200]) and taking for c the unit circle $c : x^2 + y^2 = 1$, the group of affinities of an ellipse becomes equal to the group of isometries of the circle. The subgroup $\mathcal{G}'$ equals then the group of rotations about the center of the circle and the coset $\mathcal{G}''$ equals the coset of reflections on diameters of the circle. I_0 is the symmetry at the center of the circle and the map $I \mapsto I \circ I_0$ sends the reflection on a diameter d to the reflection on the orthogonal diameter d' of the circle. An affine rotation is identified with an euclidean rotation and in particular a periodic affinity is identified with a periodic rotation. This and similar simple arguments lead to the following well-known results.

Proposition 34. (1) *The group $\mathcal{G}$ of affinities preserving the ellipse c is isomorphic to the rotation group of the plane.*
(2) *For each point $P \in c$ there is a unique conic affinity (different from identity) preserving c and fixing P. This is the affine reflection I_P on the diameter through P.*
(3) *For every $n > 2$ there is a unique cyclic group of n elements $\{f, f^2, ..., f^n = 1\} \subset \mathcal{G}'$ with f periodic of period n.*
(4) *For every affine rotation f of an ellipse c the corresponding axis e is the line at infinity and the center E is the center of the conic.*
(5) *The pencil (c, e) consists of the conics which are homothetic to c with respect to its center.*
(6) *Group $\mathcal{K}$ is identical with the group of homotheties with center at the center of the ellipse. To each affine rotation f of the conic corresponds a real number $r_f \in [0, 1]$ which is the homothety ratio of the element $p_f \in \mathcal{K} : c' = p_f(c)$, where c' realizes through its tangents the tangential generation of f.*

In the case of hyperbolas the groups differ slightly from the corresponding ones for ellipses in the connectedness of the cosets $\mathcal{G}'$, $\mathcal{G}''$ which now have two components. The existence of two components has a clear geometric meaning. The components result from the two disjoint parts into which is divided the axis e through its intersection points A, B with the conic c. Involutions I_P which have their center P in one of these parts have their axis non-intersecting the conic. These involutions are characterized in the affine plane by diameters non-intersecting the hyperbola. They represent affine reflections which have no fixed points on the hyperbola. The other connected component of the coset of affine reflections is characterized by the property of the corresponding diameters to intersect the hyperbola, thus defining two fixed points of the corresponding reflection.

Group $\mathcal{G}'$ is isomorphic to the multiplicative group $\mathbb{R}^*$ corresponding to $\mathcal{G}_{AB}$ of Proposition 8. This group is the disjoint union of the subgroup $\mathcal{G}'_0$ of affine *hyperbolic* rotations that preserve the components of the hyperbola and its coset $\mathcal{G}'_1 = I_0\mathcal{G}'_0$ of affine *hyperbolic crossed* rotations that interchange the two components of the hyperbola ([7, p.206]). By identifying c with the hyperbola $xy = 1$ one can describe the elements of $\mathcal{G}'_0$ through the affine maps $\{(x, y) \mapsto (\mu x, \frac{1}{\mu}y),\ \mu > 0\}$. The other component $\mathcal{G}'_1$ is then identified with the set of maps $\{(x, y) \mapsto (-\mu x, -\frac{1}{\mu}y),\ \mu > 0\}$. Following proposition summarizes the results.

Proposition 35. (1) *The group of affinities $\mathcal{G}$ of a given affine hyperbola c consists of affine reflections and affine rotations which are compositions of two reflections.*
(2) *The affine rotations build a commutative subgroup $\mathcal{G}' \subset \mathcal{G}$ and the affine reflections build the unique coset $\mathcal{G}'' \subset \mathcal{G}$ of this group. $\mathcal{G}'$ and $\mathcal{G}''$ are each homeomorphic to the pointed real line $\mathbb{R}^*$ and group $\mathcal{G}'$ is isomorphic to the multiplicative group $\mathbb{R}^*$.*
(3) *Group $\mathcal{G}' = \mathcal{G}'_0 \cup \mathcal{G}'_1$ has two components corresponding to rotations that preserve the components of the hyperbola and the others $\mathcal{G}'_1 = I_0\mathcal{G}'_0$, called crossed rotations, that interchange the two components. There are no periodic affinities preserving the hyperbola for a period $n > 2$.*
(4) *The coset of affine reflections of the hyperbola is the union $\mathcal{G}'' = \mathcal{G}''_0 \cup \mathcal{G}''_1$ of two components. Reflections $I \in \mathcal{G}''_0$ preserve hyperbola's components and have fixed points on them, whereas reflections $I \in \mathcal{G}''_1 = I_0\mathcal{G}''_0$ interchange the two components and have no fixed points.*
(5) *For each point $P \in c$ there is a unique conic affinity (different from identity) preserving c and fixing P. This is the affine reflection I_P on the diameter through P.*
(6) *Group $\mathcal{K}$ is identical with the group of homotheties with center at the center of the hyperbola.*
(7) *For each non-involutive affinity f of an hyperbola c preserving the components the tangential generation of f defines another hyperbola c' homothetic to c with respect to E. If f permutes the components of c then $f^* = f \circ I_0$ defines through tangential generation c' homothetic c. The homotety ratios for the two cases are correspondingly $r_f > 1$ and $r_{f^*} \in (0, 1)$.*

The case of affine parabolas demonstrates significant differences from ellipses and hyperbolas. Since the affinities preserving a parabola fix its point at infinity E, their group is isomorphic to the group $\mathcal{G}_E$ discussed in Proposition 14. This group contains the subgroup $\mathcal{G}^0_E$ of so-called *parabolic rotations*, which are products of two *parabolic reflections*. These are the only affine reflections preserving the parabola and their set is a coset of $\mathcal{G}^0_E$ in $\mathcal{G}_E$. The most important addition in the case of parabolas are the isotropy groups $\mathcal{G}_{EB}$ fixing the point at infinity E of the parabola and an additional point B of it. Figure 21 illustrates an example of such a group $\mathcal{G}_{EB}$ in which the parabola is described in an affine frame by the equation $y = x^2$ and point B is the origin of coordinates. In this example the group $\mathcal{G}_{EB}$ is represented by affine transformations of the form

$$(x, y) \mapsto (rx, r^2y),\ \ r \in \mathbb{R}^*.$$

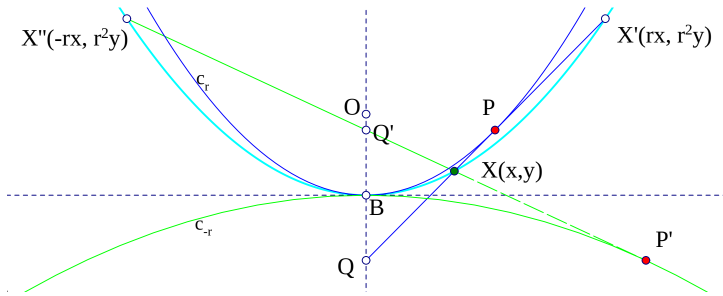


Figure 21. The group $\mathcal{G}_{EB}$

The figure displays also two other parabolas c_r, c_{-r}. These are the conics realizing the tangential generations (Proposition 6) for the corresponding affinities $(x, y) \mapsto (rx, r^2y)$ and $(x, y) \mapsto (-rx, r^2y)$ for $r > 0$. Note that, in addition to the unique affine reflection and the unique affine rotation sending a point X to another point X' and existing for affine conics of all kinds, there are for parabolas infinite more affinities doing the same job. In fact, in this case, by Proposition 14, for every two points (X, X') different from B there is precisely one element $f \in \mathcal{G}_{EB}$ mapping X to X'. This affinity preserves the parabola, fixes B and is neither an affine reflection nor an affine rotation.

Figure 22 shows the decomposition of the previous affinity into two involutions $f_r = I \circ I'$ with centers Q_1, Q_2 lying on the axis ($x = 0$) of the parabola. These are not affine since they do not preserve the line at infinity. They have though their axis parallel to the tangent d at B and their intersections with the axis R_i are symmetric to Q_i with respect to B. Thus, both of them map the line at infinity onto the tangent d at B, so that their composition leaves the line at infinity invariant. Since for another point $C \in c$ the corresponding group $\mathcal{G}_{EC} = Ad_f(\mathcal{G}_{EB})$ is conjugate to $\mathcal{G}_{EB}$ by an affine rotation $f \in \mathcal{G}_E^0$ the previous analysis transfers to the isotropy at C.

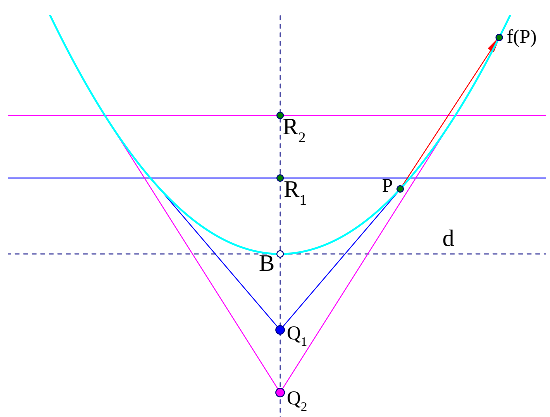


Figure 22. f_r as product of involutions

Another issue to be discussed when comparing the kinds of affine conics is that of area. Area in affine planes is defined up to a multiplicative constant ([4, vol.I, p.59]). To measure areas one fixes an affine frame, thus fixing simultaneously the *orientation* of the plane, and refers everything to this frame. Affinities preserving the area build a subgroup of the group of affinities of the plane, to which belong all affine rotations. Affine reflections reverse the sign of the area. Thus affine reflections and rotations, considered together build a group preserving the unsigned area. The analysis in the previous sections shows that affinities preserving an affine conic are automatically also unsigned-area-preserving in all cases with the exception of some types of affinities of parabolas. These affinities are the elements of the subgroups $\mathcal{G}_{EB}$ with the exception of the identity and the parabolic reflection I_B fixing B. Thus, while for all conics there are exactly two area preserving affinities mapping a point X to another point X' (an affine rotation and an affine reflection), for parabolas there is in addition a one-parameter infinity of area non-preserving affinities mapping X to X'.

Proposition 36. *In the following e denotes the line at infinity and E its unique common point with the parabola c.*
(1) *The group $\mathcal{G}$ of affinities preserving parabola c is the union $\mathcal{G} = \mathcal{G}_E^0 \cup_{B\in c} \mathcal{G}_{EB}$. This group is also the semidirect product of its subgroups $\mathcal{G}_E^0$ and $\mathcal{G}_{EB}$, the first containing all parabolic rotations and the second being the isotropy group at a point $B \in c$ of $\mathcal{G}$.*
(2) *$\mathcal{G}_E^0$ is the group of affine rotations, which are products of two affine reflections preserving the conic. This group is isomorphic to the additive group of real numbers. There are no periodic affinities preserving a parabola for a period $n > 2$.*
(3) *The set $\mathcal{G}''$ of all affine reflections preserving the parabola consists of affinities having their axis parallel to the axis of the parabola, which is the direction determined by its point at infinity E. This is a coset of the previous subgroup of $\mathcal{G}$ acting simply transitively on c.*
(4) *The group $\mathcal{G}_{EB}$ is isomorphic to the multiplicative group $\mathbb{R}^*$ and its elements, except the affine reflection $I_B \in \mathcal{G}_{EB}$, the axis of which passes through B, though they preserve c, are neither affine reflections nor affine rotations and do not preserve areas. This group acts simply transitively on $c - \{B\}$.*
(5) *For every pair of points $B \in c, C \in c$ there is a unique affine rotation $f \in \mathcal{G}_E^0$ such that $f(B) = C$. This element conjugates the corresponding isotropy groups: $Ad_f(\mathcal{G}_{EB}) = \mathcal{G}_{EC}$.*
(6) *Every coset of $\mathcal{G}_E^0$ intersects each subgroup $\mathcal{G}_{EB} \subset \mathcal{G}$ in exactly one element.*
(7) *For each affine rotation $f \in \mathcal{G}_E^0$ the tangential generation defines an element $p_f \in \mathcal{K}$. Last group coincides with the group of translations parallel to the axis of the parabola.*
(8) *For each element $f \in \mathcal{G}_{EB}$ the tangential generation defines a parabola which is a member of the bitangent pencil (c, EB). This pencil consists of all parabolas sharing with c the same axis and being tangent to c at B.*

References

[1] S. S. Abhyankar and C. Christensen, Semidirect products: $x \mapsto ax+b$ as a first example, *Math. Mag.*, 75 (2002) 284–289.

[2] M. Audin, *Geometry*, Springer, Berlin, 1998.

[3] H. F. Baker, *Plane Geometry*, Chelsea Publishing Company, New York, 1971.

[4] M. Berger, *Geometry*, 2 volumes, Springer, Berlin 1987.

[5] M. Berger, P. Pensu, J.-P. Berry, and X. Saint-Raymond, *Problems in Geometry*, Springer, Berlin 1984.

[6] F. Brickell and R. S. Clark, *Differentiable Manifolds*, Scott, Van Nostrand Reinhold, London, 1970.

[7] H. S. M. Coxeter, *Introduction to Geometry*, 2nd ed., John Wiley, New York, 1969.

[8] G. Salmon, *A treatise on conic sections*, Longmans, London 1855.

[9] S. H. Schot, Aberrancy: Geometry of the third derivative, *Math. Mag.*, 51 (1978) 259–275.

[10] H. Schwerdtfenger, *Geometry of complex numbers*, Dover 1979.

[11] J. Steiner, *Vorlesungen ueber Synthetische Geometrie II* (bearbeited von H. Schroeter), B. G. Teubner Verlag, Leipzig, 1867.

[12] O. Veblen and J. W. Young, *Projective Geometry I*, Ginn and Company, Boston 1938.

[13] F. W. Warner, *Foundations of Differential Geometry and Lie Groups*, Scott, Foresman and Company, New York, 1969.

[14] R. W. Wong, The orbits of a unimodular affine transformation, *College Math. Journal*, 31 (2000) 290–296.

[15] P. B. Yale, *Geometry and Symmetry*, Holden Day, San Francisco, 1968.

Paris Pamfilos: Department of Mathematics, University of Crete, Crete, Greece

E-mail address: `pamfilos@math.uoc.gr`

Forum Geometricorum
Volume 9 (2009) 259–270.

FORUM GEOM
ISSN 1534-1178

Some Triangle Centers Associated with the Tritangent Circles

Nikolaos Dergiades and Juan Carlos Salazar

Abstract. We investigate two interesting special cases of the classical Apollonius problem, and then apply these to the tritangent circles of a triangle to find pair of perspective (or homothetic) triangles. Some new triangle centers are constructed.

1. An interesting construction

We begin with a simple construction of a special case of the classical Apollonius problem. Given two circles $O(r)$, $O'(r')$ and an external tangent $\mathcal{L}$, to construct a circle $O_1(r_1)$ tangent to the circles and the line, with point of tangency X between A and A', those of (O), (O') and $\mathcal{L}$ (see Figure 1). A simple calculation shows that $AX = 2\sqrt{r_1 r}$ and $XA' = 2\sqrt{r_1 r'}$, so that $AX : XA' = \sqrt{r} : \sqrt{r'}$. The radius of the circle is

$$r_1 = \frac{1}{4}\left(\frac{AA'}{\sqrt{r}+\sqrt{r'}}\right)^2.$$

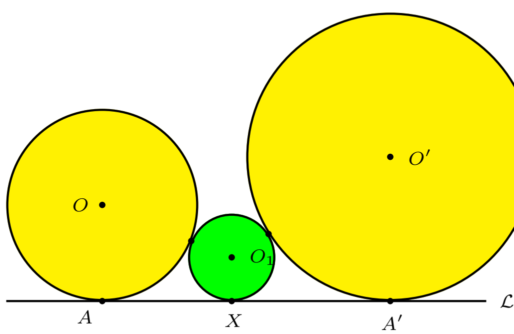


Figure 1

From this we design the following construction.

Construction 1. *On the line $\mathcal{L}$, choose two points P and Q be points on opposite sides of A such that $PA = r$ and $AQ = r'$. Construct the circle with diameter PQ to intersect the line OA at F such that O and F are on opposite sides of $\mathcal{L}$. The intersection of $O'F$ with $\mathcal{L}$ is the point X satisfying $AX : XA' = \sqrt{r} : \sqrt{r'}$. Let M be the midpoint of AX. The perpendiculars to OM at M, and to $\mathcal{L}$ at X intersect at the center O_1 of the required circle* (see Figure 2).

Publication Date: October 12, 2009. Communicating Editor: Paul Yiu.
Many thanks to the editor for many additions and especially for Construction 1.

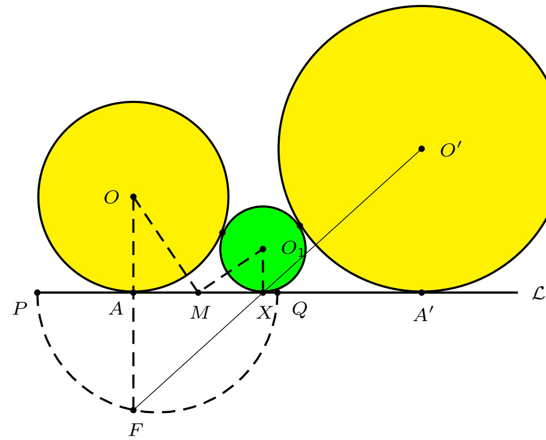


Figure 2

For a construction in the case when $\mathcal{L}$ is not necessarily tangent of (O) and (O'), see [1, Problem 471].

2. An application to the excircles of a triangle

We apply the above construction to the excircles of a triangle ABC. We adopt standard notations for a triangle, and work with homogeneous barycentric coordinates. The points of tangency of the excircles with the sidelines are as follows.

	BC	CA	AB
(I_a)	$A_a = (0 : s-b : s-c)$	$B_a = (-(s-b) : 0 : s)$	$C_a = (-(s-c) : s : 0)$
(I_b)	$A_b = (0 : -(s-a) : s)$	$B_b = (s-a : 0 : s-c)$	$C_b = (s : -(s-c) : 0)$
(I_c)	$A_c = (0 : s : -(s-a))$	$B_c = (s : 0 : -(s-c))$	$C_c = (s-a : s-b : 0)$

Consider the circle $O_1(X)$ tangent to the excircles $I_b(r_b)$ and $I_c(r_c)$, and to the line BC at a point X between A_c and A_b (see Figure 3).

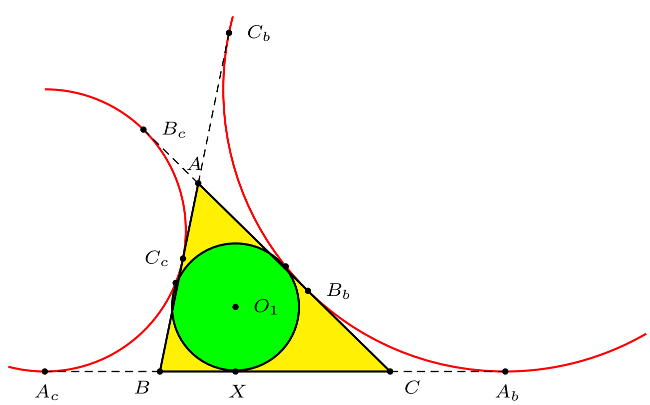


Figure 3

Lemma 2. *The point X has coordinates*

$$(0 : s\sqrt{s-c} - (s-a)\sqrt{s-b} : s\sqrt{s-b} - (s-a)\sqrt{s-c}).$$

Proof. If (O_1) is the circle tangent to (I_b), (I_c), and to BC at X between A_c and A_b, then $A_cX : XA_b = \sqrt{r_c} : \sqrt{r_b} = \sqrt{s-b} : \sqrt{s-c}$. Note that $A_cA_b = b + c$, so that

$$\begin{aligned} BX =& A_cX - A_cB \\ =& \frac{\sqrt{s-b}}{\sqrt{s-b}+\sqrt{s-c}} \cdot (b+c) - (s-a) \\ =& \frac{s\sqrt{s-b} - (s-a)\sqrt{s-c}}{\sqrt{s-b}+\sqrt{s-c}}. \end{aligned}$$

Similarly

$$XC = \frac{s\sqrt{s-c} - (s-a)\sqrt{s-b}}{\sqrt{s-b}+\sqrt{s-c}}.$$

It follows that the point X has coordinates given above. □

Similarly, there are circles $O_2(Y)$ and $O_3(Z)$ tangent to CA at Y and to AB at Z respectively, each also tangent to a pair of excircles. Their coordinates can be written down from those of X by cyclic permutations of a, b, c.

3. The triangle bounded by the polars of the vertices with respect to the excircles

Consider the triangle bounded by the polars of the vertices of ABC with respect to the corresponding excircles. The polar of A with respect to the excircle (I_a) is the line B_aC_a; similarly for the other two polars.

Lemma 3. *The polars of the vertices of ABC with respect to the corresponding excircles bound a triangle with vertices*

$$\begin{aligned} U =& (-a(b+c) : S_C : S_B), \\ V =& (S_C : -b(c+a) : S_A), \\ W =& (S_B : S_A : -c(a+b)). \end{aligned}$$

Proof. The polar of A with respect to the excircle (I_a) is the line B_aC_a, whose barycentric equation is

$$\begin{vmatrix} x & y & z \\ -(s-b) & 0 & s \\ -(s-c) & s & 0 \end{vmatrix} = 0,$$

or

$$sx + (s-c)y + (s-b)z = 0.$$

Similarly, the polars C_bA_b and A_cB_c have equations

$$\begin{aligned} (s-c)x + sy + (s-a)z =& 0, \\ (s-b)x + (s-a)y + sz =& 0. \end{aligned}$$

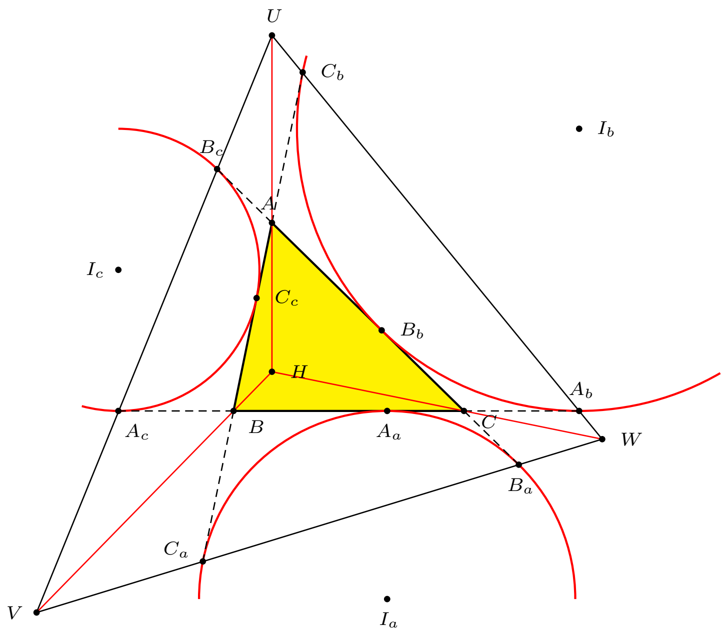


Figure 4

These intersect at the point

$$\begin{aligned}U =&(-a(2s-a) : ab - 2s(s-c) : ca - 2s(s-b))\\ =&(-2a(b+c) : a^2+b^2-c^2 : c^2+a^2-b^2)\\ =&(-a(b+c) : S_C : S_B).\end{aligned}$$

The coordinates of V and W can be obtained from these by cyclic permutations of a, b, c. □

Corollary 4. *Triangles UVW and ABC are*
(a) *perspective at the orthocenter H,*
(b) *orthologic with centers H and I respectively.*

Proposition 5. *The triangle UVW has circumcenter H and circumradius $2R+r$.*

Proof. Since H, B, V are collinear, HV is perpendicular to CA. Similarly, HW is perpendicular to AB. Since VW makes equal angles with CA and AB, it makes equal angles with HV and HW. This means $HV = HW$. For the same reason, $HU = HV$, and H is the circumcenter of UVW.

Applying the law of sines to triangle AUB_c, we have we have

$$AU = AB_c \cdot \frac{\sin\frac{180^\circ - C}{2}}{\sin\frac{C}{2}} = (s-b)\cot\frac{C}{2} = r_a.$$

The circumradius of UVW is $HU = HA + AU = 2R\cos A + r_a = 2R + r$, as a routine calculation shows. □

Proposition 6. *The triangle UVW and the intouch triangle DEF are homothetic at the point*

$$J = \left(\frac{b+c}{b+c-a} : \frac{c+a}{c+a-b} : \frac{a+b}{a+b-c}\right). \tag{1}$$

The ratio of homothety is $-\frac{2R+r}{r}$.

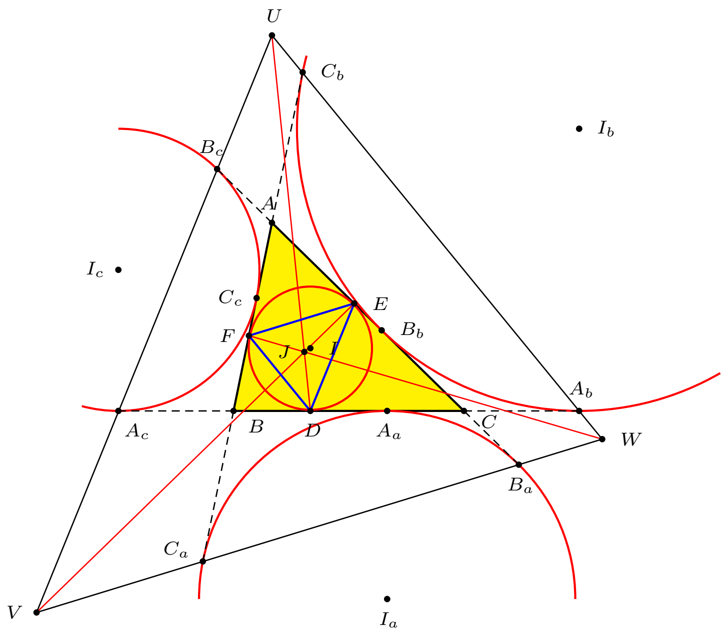


Figure 5

Proof. The homothety follows easily from the parallelism of VW and EF, and of WU, FD, and UV, DE. The homothetic center is the common point J of the lines DU, EV, and FW (see Figure 5). These lines have equations

$$\begin{aligned}
(b-c)(b+c-a)x + (b+c)(c+a-b)y - (b+c)(a+b-c)z =& 0,\\
-(c+a)(b+c-a)x + (c-a)(c+a-b)y + (c+a)(a+b-c)z =& 0,\\
(a+b)(b+c-a)x - (a+b)(c+a-b)y + (a-b)(a+b-c)z =& 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(b+c-a)x : (c+a-b)y : (a+b-c)z\\
=&\begin{vmatrix} c-a & c+a \\ -(a+b) & a-b \end{vmatrix} : \begin{vmatrix} c+a & -(c+a) \\ a+b & a-b \end{vmatrix} : \begin{vmatrix} -(c+a) & c-a \\ a+b & -(a+b) \end{vmatrix}\\
=&2a(b+c) : 2a(c+a) : 2a(a+b)\\
=&b+c : c+a : a+b.
\end{aligned}$$

The coordinates of the homothetic center J are therefore as in (1) above.

Since the triangles UVF and DEF have circumcircles $H(2R+r)$ and $I(r)$, the ratio of homothety is $-\frac{2R+r}{r}$. The homothetic center J divides IH in the ratio $IJ : JH = r : 2R + r$. □

Remark. The triangle center J appears as X_{226} in Kimberling's list [2].

4. Perspectivity of XYZ and UVW

Theorem 7. *Triangles XYZ and UVW are perspective at a point with coordinates*

$$\left(\frac{S_B}{\sqrt{s-c}}+\frac{S_C}{\sqrt{s-b}}-\frac{a(b+c)}{\sqrt{s-a}}:\frac{S_C}{\sqrt{s-a}}+\frac{S_A}{\sqrt{s-c}}-\frac{b(c+a)}{\sqrt{s-b}}:\frac{S_A}{\sqrt{s-b}}+\frac{S_B}{\sqrt{s-a}}-\frac{c(a+b)}{\sqrt{s-c}}\right).$$

Proof. With the coordinates of X and U from Lemmas 1 and 2, the line XU has equation

$$\begin{vmatrix} x & y & z \\ -a(b+c) & S_C & S_B \\ 0 & s\sqrt{s-c}-(s-a)\sqrt{s-b} & s\sqrt{s-b}-(s-a)\sqrt{s-c} \end{vmatrix}=0.$$

Since the coefficient of x is

$$\begin{aligned} & (s(S_B+S_C)-aS_B)\sqrt{s-b}-(s(S_B+S_C)-aS_C)\sqrt{s-c} \\ =& a((as-S_B)\sqrt{s-b}-(as-S_C)\sqrt{s-c}) \\ =& a(b+c)((s-c)\sqrt{s-b}-(s-b)\sqrt{s-c}). \end{aligned}$$

From this, we easily simplify the above equation as

$$\begin{aligned} & ((s-c)\sqrt{s-b}-(s-b)\sqrt{s-c})x \\ & +(s\sqrt{s-b}-(s-a)\sqrt{s-c})y+((s-a)\sqrt{s-b}-s\sqrt{s-c})z=0. \end{aligned}$$

With $u=\sqrt{s-a}$, $v=\sqrt{s-b}$, and $w=\sqrt{s-c}$, we rewrite this as

$$-vw(v-w)x+(v(u^2+v^2+w^2)-u^2w)y+(u^2v-w(u^2+v^2+w^2))z=0. \tag{2}$$

Similarly the equations of the lines VY, WZ are

$$(v^2w-u(u^2+v^2+w^2))x-wu(w-u)y+(w(u^2+v^2+w^2)-uv^2)z=0, \tag{3}$$

$$(u(u^2+v^2+w^2)-vw^2)x+(w^2u-v(u^2+v^2+w^2))y-uv(u-v)z=0. \tag{4}$$

It is clear that the sum of the coefficients of x (respectively y and z) in (2), (3) and (4) is zero. The system of equations therefore has a nontrivial solution. Solving them, we obtain the coordinates of the common point of the lines XU, YV, ZW as

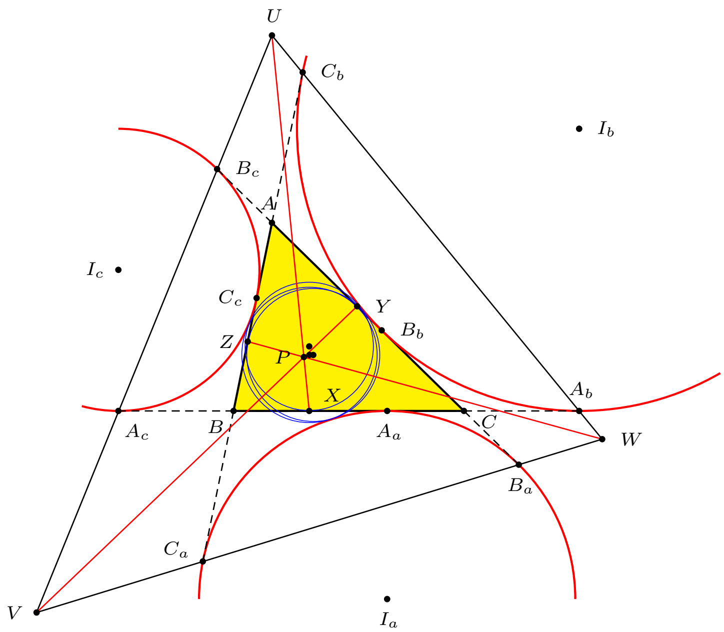


Figure 6

$$\begin{aligned}
&x : y : z \\
=&uv(v^2(u^2+v^2+w^2)-w^2u^2)+wu(w^2(u^2+v^2+w^2)-u^2v^2) \\
&-vw(v^2+w^2)(2u^2+v^2+w^2) \\
:&vw(w^2(u^2+v^2+w^2)-u^2v^2)+uv(u^2((u^2+v^2+w^2)-v^2w^2) \\
&-wu(w^2+u^2)(u^2+2v^2+w^2) \\
:&wu(u^2(u^2+v^2+w^2)-v^2w^2)+vw(v^2((u^2+v^2+w^2)-w^2u^2) \\
&-uv(u^2+v^2)(u^2+v^2+2w^2)
\end{aligned}$$

$$\begin{aligned}
=&\frac{(s-b)s-(s-c)(s-a)}{w}+\frac{(s-c)s-(s-a)(s-b)}{v}-\frac{a(b+c)}{u} \\
:&\frac{(s-c)s-(s-a)(s-b)}{u}+\frac{(s-a)s-(s-b)(s-c)}{w}-\frac{b(c+a)}{v} \\
:&\frac{(s-a)s-(s-b)(s-c)}{v}+\frac{(s-b)s-(s-c)(s-a)}{u}-\frac{c(a+b)}{w} \\
=&\frac{S_B}{w}+\frac{S_C}{v}-\frac{a(b+c)}{u}:\frac{S_C}{u}+\frac{S_A}{w}-\frac{b(c+a)}{v}:\frac{S_A}{v}+\frac{S_B}{u}-\frac{c(a+b)}{w}.
\end{aligned}$$

□

The triangle center constructed in Theorem 3 above does not appear in [2].

5. Another construction

Given three circles $O_i(r_i)$, $i = 1, 2, 3$, on one side of a line $\mathcal{L}$, tangent to the line, we construct a circle $O(r)$, tangent to each of these three circles externally.

For $i = 1, 2, 3$, let the circle $O_i(r_i)$ touch $\mathcal{L}$ at S_i and $O(r)$ at T_i. If the line S_1T_1 meets the circle (O) again at T, then the tangent to (O) at T is a line $\mathcal{L}'$ parallel to $\mathcal{L}$. Hence, T, T_2, S_2 are collinear; so are T, T_3, S_3. Since the line T_2T_3 is antiparallel to $\mathcal{L}'$ with respect to the lines TT_2 and TT_3, it is also antiparallel to $\mathcal{L}$ with respect to the lines TS_2 and TS_3, and the points T_2, T_3, S_3, S_2 are concyclic. From $TT_2 \cdot TS_2 = TT_3 \cdot TS_3$, we conclude that the point T lies on the radical axis of the circles $O_2(r_2)$ and $O_3(r_3)$, which is the perpendicular from the midpoint of S_2S_3 to the line O_2O_3. For the same reason, it also lies on the radical axis of the circles $O_3(r_3)$ and $O_1(r_1)$, which is the perpendicular from the midpoint of S_1S_3 to the line O_1O_3. Hence T is the radical center of the three given circles $O_i(r_i)$, $i = 1, 2, 3$, and the circle $T_1T_2T_3$ is the image of the line $\mathcal{L}$ under the inversion with center T and power $TT_1 \cdot TS_1$. From this, the required circle (O) can be constructed as follows.

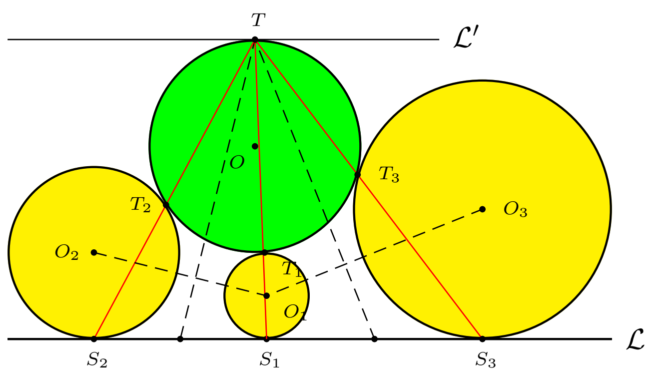


Figure 7

Construction 8. *Construct the perpendicular from the midpoint of S_1S_2 to O_1O_2, and from the midpoint of S_1S_3 to O_1O_3. Let T be the intersection of these two perpendiculars. For $i = 1, 2, 3$, let T_i be the intersection of the line TS_i with the circle (O_i). The required circle (O) is the one through T_1, T_2, T_3* (see Figure 7).

6. Circles tangent to the incircle and two excircles

We apply Construction 2 to obtain the circle tangent to the incircle (I) and the excircles (I_b) and (I_c). Let the incircle (I) touch the sides BC, CA, AB at D, E, F respectively.

Proposition 9. *The radical center of (I), (I_b), (I_c) is the point*

$$J_a = (b + c : c - a : b - a).$$

This is also the midpoint of the segment DU.

Proof. The radical axis of (I) and (I_b) is the line joining the midpoints of the segments DA_b and FC_b. These midpoints have coordinates $(0 : a-c : a+c)$ and $(c+a : c-a : 0)$. This line has equation

$$-(c-a)x + (c+a)y + (c-a)z = 0.$$

Similarly, the radical axis of (I) and (I_c) is the line

$$(a-b)x - (a-b)y + (a+b)z = 0.$$

The radical center J_a of the three circles is the intersection of these two radical axes. Its coordinates are as given above.

By Construction 2, J_a is the intersection of the lines through the midpoints of A_bD and A_cD perpendicular to II_b and II_c respectively. As such, it is the midpoint of DU. $\square$

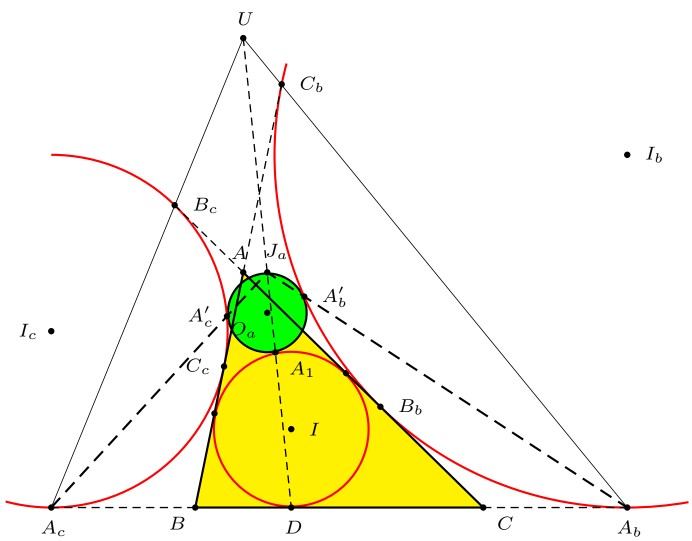


Figure 8.

The lines J_aD, J_aA_b and J_aA_c intersect the circles (I), (I_b) and (I_c) respectively again at

$$\begin{aligned}
A_1 =&((b+c)^2(s-b)(s-c) : c^2(s-a)(s-c) : b^2(s-a)(s-b)),\\
A'_b =&((b+c)^2 s(s-c) : -(ab-c(s-a))^2 : b^2 s(s-a)),\\
A'_c =&((b+c)^2 s(s-b) : c^2 s(s-a) : -(ca-b(s-a))^2).
\end{aligned}$$

The circle through these points is the one tangent to (I), (I_b), and (I_c) (see Figure 8). It has radius $\frac{a}{b+c} \cdot \frac{(s-a)^2+r_a^2}{4r_a}$.

In the same way, we have a circle (O_b) tangent to (I), (I_c), (I_a) respectively at B_1, B'_c, B'_a, and passing through the radical center J_b of these three circles, and another circle (O_c) tangent to (I), (I_a), (I_b) respectively at C_1, C'_a, C'_b, passing through the radical center J_c of the circles. J_b and J_c are respectively the midpoints of the segments EV and FW. The coordinates of J_b, J_c, B_1, C_1 are as follows.

$$\begin{aligned}
J_b =& (c-b : c+a : a-b),\\
J_c =& (b-c : a-c : a+b);\\
B_1 =& (a^2(s-b)(s-c) : (c+a)^2(s-c)(s-a) : c^2(s-a)(s-b)),\\
C_1 =& (a^2(s-b)(s-c) : b^2(s-c)(s-a) : (a+b)^2(s-a)(s-b)).
\end{aligned}$$

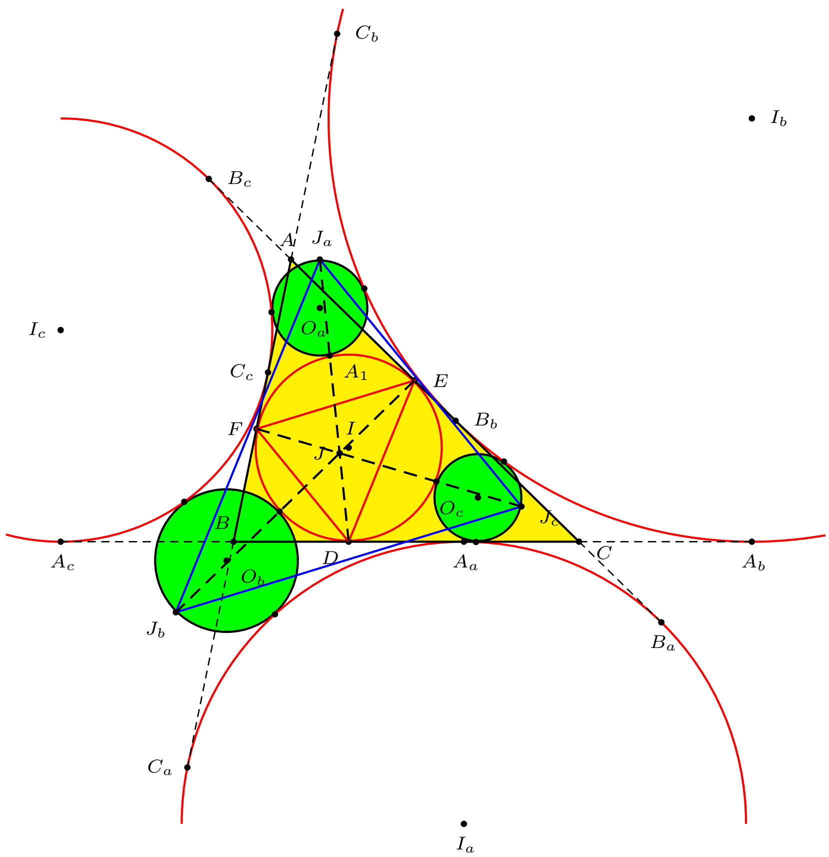


Figure 9.

Proposition 10. *The triangle* $J_aJ_bJ_c$ *is the image of the intouch triangle under the homothety* $\mathsf{h}(J, -\frac{R}{r})$.

Proof. Since UVW and DEF are homothetic at J, and J_a, J_b, J_c are the midpoints of DU, EV, FW respectively, it is clear that $J_aJ_bJ_c$ and DEF are also homothetic at the same J. Note that $J_bJ_c = \frac{1}{2}(VW - EF)$. The circumradius of $J_aJ_bJ_c$ is $\frac{1}{2}((2R + r) - r) = R$. The ratio of homothety of $J_aJ_bJ_c$ and DEF is $\frac{-R}{r}$. □

Corollary 11. *J is the radical center of the circles (O_a), (O_b), (O_c).*

Proof. Note that $JJ_a \cdot JA_1 = \frac{R}{r} \cdot DJ \cdot JA_1$. This is $\frac{R}{r}$ times the power of J with respect to the incircle. The same is true for $JJ_b \cdot JB_1$ and $JJ_c \cdot JC_1$. This shows that J is the radical center of the circles (O_a), (O_b), (O_c). $\square$

Since the incircle (I) is the inner Apollonius circle and the circumcircle (O_i), $i = 1, 2, 3$, it follows that $J_aJ_bJ_c$ is the outer Apollonius circle to the same three circles (see Figure 10). The center O' of the circle $J_aJ_bJ_c$ is the midpoint between the circumcenters of DEF and UVW, namely, the midpoint of IH. It is the triangle center X_{946} in [2].

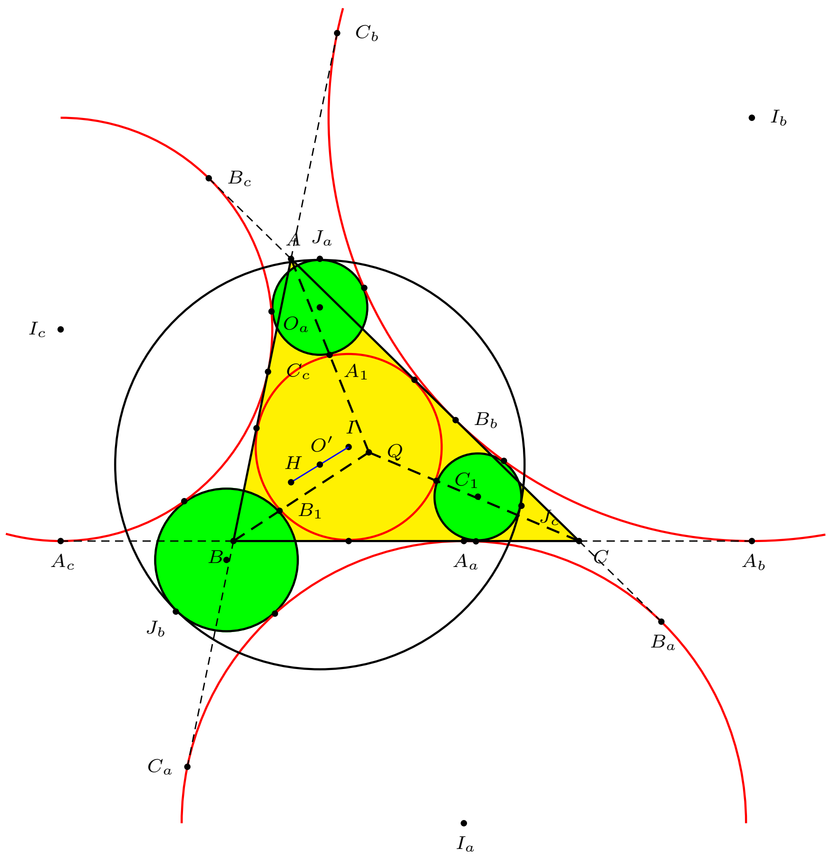


Figure 10.

Proposition 12. *$A_1B_1C_1$ is perspective with ABC at the point*

$$Q = \left(\frac{1}{a^2(s-a)} : \frac{1}{b^2(s-b)} : \frac{1}{c^2(s-c)}\right),$$

which is the isotomic conjugate of the insimilicenter of the circumcircle and the incircle.

This is clear from the coordinates of A_1, B_1, C_1. The perspector Q is the isotomic conjugate of the insimilicenter of the circumcircle and the incircle. It is not in the current listing in [2].

References

[1] F. G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, Paris, 1991.
[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr

Juan Carlos Salazar: Deceased

Forum Geometricorum
Volume 9 (2009) 271–274.

FORUM GEOM
ISSN 1534-1178

Ten Concurrent Euler Lines

Nikolai Ivanov Beluhov

Dedicated to Svetlozar Doichev

Abstract. Let F_1 and F_2 denote the Fermat-Toricelli points of a given triangle ABC. We prove that the Euler lines of the 10 triangles with vertices chosen from A, B, C, F_1, F_2 (three at a time) are concurrent at the centroid of triangle ABC.

Given a (positively oriented) triangle ABC, construct externally on its sides three equilateral triangles BCT_a, CAT_b, and ABT_c with centers N_a, N_b and N_c respectively (see Figure 1). As is well known, triangle $N_aN_bN_c$ is equilateral. We call this the first Napoleon triangle of ABC.

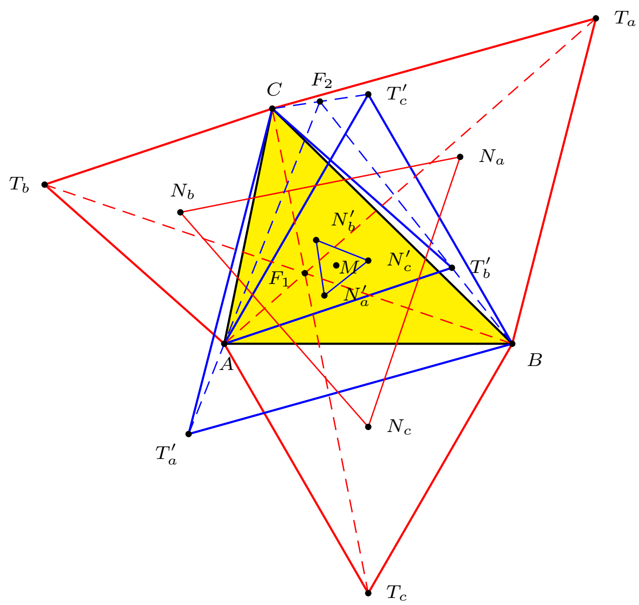


Figure 1.

The same construction performed internally gives equilateral triangles BCT'_a, CAT'_b and ABT'_c with centers N'_a, N'_b, and N'_c respectively, leading to the second Napoleon triangle $N'_aN'_bN'_c$. The centers of both Napoleon triangles coincide with the centroid M of triangle ABC.

Publication Date: October 19, 2009. Communicating Editor: Paul Yiu.

The lines AT_a, BT_b and CT_c make equal pairwise angles, and meet together with the circumcircles of triangles BCT_a, CAT_b, and ABT_c at the first Fermat-Toricelli point F_1. Denoting by $\angle XYZ$ the oriented angle $\angle(YX, YZ)$, we have $\angle AF_1B = \angle BF_1C = \angle CF_1A = 120°$. Analogously, the second Fermat-Toricelli point satisfies $\angle AF_2B = \angle BF_2C = \angle CF_2A = 60°$.

Clearly, the sides of the Napoleon triangles are the perpendicular bisectors of the segments joining their respective Fermat-Toricelli points with the vertices of triangle ABC (as these segments are the common chords of the circumcircles of the the equilateral triangles ABT_c, BCT_a, etc).

We prove the following interesting theorem.

Theorem *The Euler lines of the ten triangles with vertices from the set* $\{A,\ B,\ C,\ F_1,\ F_2\}$ *are concurrent at the centroid* M *of triangle* ABC.

Proof. We divide the ten triangles in three types:
(I): Triangle ABC by itself, for which the claim is trivial.
(II): The six triangles each with two vertices from the set $\{A, B, C\}$ and the remaining vertex one of the points F_1, F_2.
(III) The three triangles each with vertices F_1, F_2, and one from $\{A, B, C\}$.

For type (II), it is enough to consider triangle ABF_1. Let M_c be its centroid and M_C be the midpoint of the segment AB. Notice also that N_c is the circumcenter of triangle ABF_1 (see Figure 2).

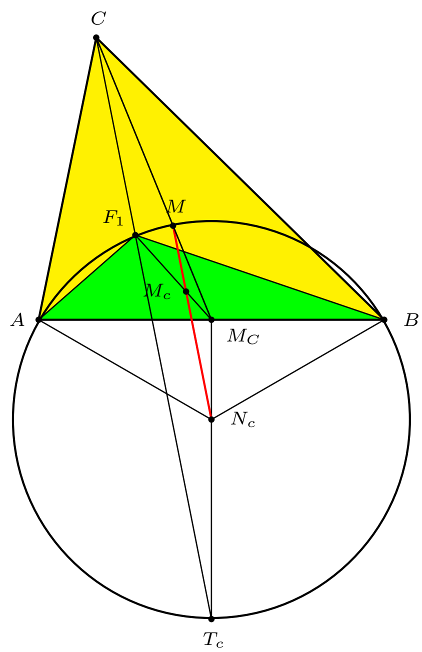


Figure 2.

Now, the points C, F_1 and T_c are collinear, and the points M, M_c and N_c divide the segments M_CC, M_CF_1 and M_CT_c in the same ratio $1:2$. Therefore, they are collinear, and the Euler line of triangle ABF contains M.

For type (III), it is enough to consider triangle CF_1F_2. Let M_c and O_c be its centroid and circumcenter. Let also M_C and M_F be the midpoints of AB and F_1F_2. Notice that O_c is the intersection of N_aN_b and $N'_aN'_b$ as perpendicular bisectors of F_1C and F_2C. Let also P be the intersection of N_bN_c and $N'_cN'_a$, and F' be the reflection of F_1 in M_C (see Figure 3).

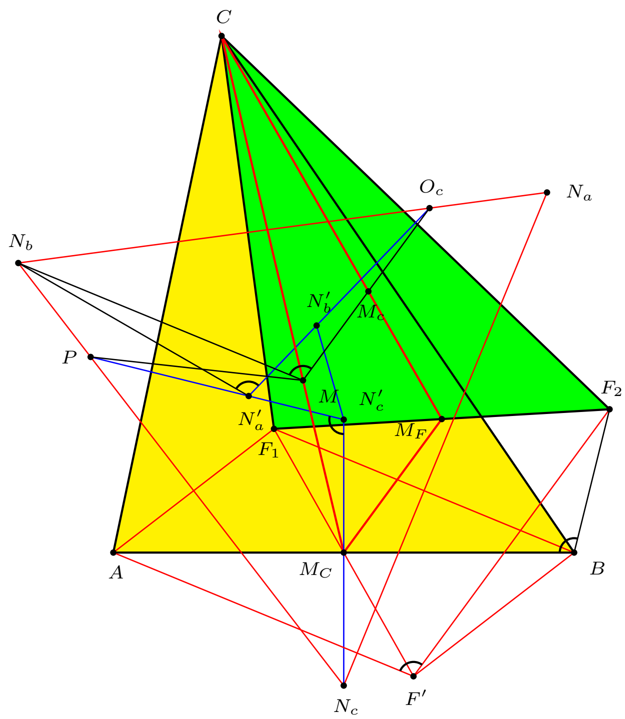


Figure 3.

The rotation of center M and angle 120° maps the lines N_aN_b and $N'_aN'_b$ into N_bN_c and $N'_cN'_a$ respectively. Therefore, it maps O_c to P, and $\angle O_cMP = 120^\circ$. Since $\angle O_cN'_aP = 120^\circ$, the four points O_c, M, N'_a, P are concyclic. The circle containing them also contains N_b since $\angle PN_bO_c = 60^\circ$. Therefore, $\angle O_cMN_b = \angle O_cN'_aN_b$.

The same rotation maps angle $O_cN'_aN_b$ onto angle PN'_cN_c, yielding $\angle O_cN'_aN_b = \angle PN'_cN_c$. Since $PN'_c \perp BF_2$ and $N_cN'_c \perp BA$, $\angle PN'_cN_c = \angle F_2BA$.

Since $\angle BF'A = \angle AF_1B = 120^\circ = 180^\circ - \angle AF_2B$, the quadrilateral AF_2BF' is also cyclic and $\angle F_2BA = \angle F_2F'A$. Thus, $\angle F_2F'A = \angle O_cMN_b$.

Now, $AF' \| F_1B \perp N_aN_c$ and $N_bM \perp N_aN_c$ yield $AF' \| N_bM$. This, together with $\angle F_2F'A = \angle O_cMN_b$, yields $F'F_2 \| MO_c$.

Notice now that the points M_c and M divide the segments CM_F and CM_C in ratio $1:2$, therefore $M_cM \| M_CM_F$. The same argument, applied to the segments F_1F_2 and F_1F' with ratio $1:1$, yields $M_CM_F \| F'F_2$.

In conclusion, we obtain $M_cM \| F'F_2 \| MO_c$. The collinearity of the points M_c, M and O_c follows. $\square$

References

[1] N. Beluhov, Sets of Euler lines, *Matematika +*, 2/2006.
[2] I. F. Sharygin, *Geometriya. Planimetriya*, Drofa, 2001.

Nikolai Ivanov Beluhov: ”Bulgarsko opalchenie” str. 5, Stara Zagora 6000, Bulgaria
E-mail address: `nbehulov@abv.bg`

Forum Geometricorum
Volume 9 (2009) xx–xx.

FORUM GEOM
ISSN 1534-1178

On the Possibility of Trigonometric Proofs of the Pythagorean Theorem

Jason Zimba

Abstract. The identity $\cos^2 x + \sin^2 x = 1$ can be derived independently of the Pythagorean theorem, despite common beliefs to the contrary.

1. Introduction

In a remarkable 1940 treatise entitled *The Pythagorean Proposition*, Elisha Scott Loomis (1852–1940) presented literally hundreds of distinct proofs of the Pythagorean theorem. Loomis provided both "algebraic proofs" that make use of similar triangles, as well as "geometric proofs" that make use of area reasoning. Notably, none of the proofs in Loomis's book were of a style one would be tempted to call "trigonometric". Indeed, toward the end of his book ([1, p.244]) Loomis asserted that all such proofs are circular:

> There are no trigonometric proofs [of the Pythagorean theorem], because all of the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean theorem; because of this theorem we say $\sin^2 A + \cos^2 A = 1$ etc.

Along the same lines but more recently, in the discussion page behind Wikipedia's Pythagorean theorem entry, one may read that a purported proof was once deleted from the entry because it "...depend[ed] on the veracity of the identity $\sin^2 x + \cos^2 x = 1$, which is the Pythagorean theorem ..." ([5]).

Another highly ranked Internet resource for the Pythagorean theorem is Cut-The-Knot.org, which lists dozens of interesting proofs ([2]). The site has a page devoted to fallacious proofs of the Pythagorean theorem. On this page it is again asserted that the identity $\cos^2 x + \sin^2 x = 1$ cannot be used to prove the Pythagorean theorem, because the identity "is based on the Pythagorean theorem, to start with" ([3]).

All of these quotations seem to reflect an implicit belief that the relation $\cos^2 x + \sin^2 x = 1$ cannot be derived independently of the Pythagorean theorem. For the record, this belief is false. We show in this article how to derive this identity independently of the Pythagorean theorem.

Publication Date: Month, 2009. Communicating Editor: Paul Yiu.
The author would like to thank an anonymous referee for a number of suggestions, both general and specific, which greatly improved the manuscript during the editorial process.

2. Sine and cosine of acute angles

We begin by defining the sine and cosine functions for positive acute angles, independently of the Pythagorean theorem, as ratios of sides of similar right triangles. Given $\alpha \in \left(0,\ \frac{\pi}{2}\right)$, let $\mathcal{R}_\alpha$ be the set of all right triangles containing an angle of measure α, and let $\mathbf{T}$ be one such triangle. Because the angle measures in $\mathbf{T}$ add up to π (see Euclid's *Elements*, I.32), [1] $\mathbf{T}$ must have angle measures $\frac{\pi}{2}$, $\frac{\pi}{2} - \alpha$ and α. The side opposite to the right angle is the longest side (see *Elements* I.19), called the hypotenuse of the right triangle; we denote its length by $H_\mathbf{T}$.

First consider the case $\alpha \neq \frac{\pi}{4}$. The three angle measures of $\mathbf{T}$ are distinct, so that the three side lengths are also distinct (see *Elements*, I.19). Let $A_\mathbf{T}$ denote the length of the side of $\mathbf{T}$ adjacent to the angle of measure α, and $O_\mathbf{T}$ the length of the opposite side. If $\mathbf{T}$ and $\mathbf{S}$ are any two triangles in $\mathcal{R}_\alpha$, then because $\mathbf{T}$ and $\mathbf{S}$ have angles of equal measures, corresponding side ratios in $\mathbf{S}$ and $\mathbf{T}$ are equal:

$$\frac{A_\mathbf{T}}{H_\mathbf{T}} = \frac{A_\mathbf{S}}{H_\mathbf{S}} \quad \text{and} \quad \frac{O_\mathbf{T}}{H_\mathbf{T}} = \frac{O_\mathbf{S}}{H_\mathbf{S}}$$

(see *Elements*, VI.4). Therefore, for $\alpha \neq \frac{\pi}{4}$ in the range $\left(0,\ \frac{\pi}{2}\right)$, we may define

$$\cos\alpha := \frac{A}{H} \quad \text{and} \quad \sin\alpha := \frac{O}{H},$$

where the ratios may be computed using any triangle in $\mathcal{R}_\alpha$. [2]

We next consider the case $\alpha = \frac{\pi}{4}$. Any right triangle containing an angle of measure $\frac{\pi}{4}$ must in fact have two angles of measure $\frac{\pi}{4}$ (see *Elements*, I.32), so its three angles have measures $\frac{\pi}{2}$, $\frac{\pi}{4}$ and $\frac{\pi}{4}$. Such a triangle is isosceles (see *Elements*, I.6), and therefore has only two distinct side lengths, H and L, where $H > L$ is the length of the side opposite to the right angle and L is the common length shared by the two other sides (see *Elements*, I.19). Because any two right triangles containing angle $\alpha = \frac{\pi}{4}$ have the same three angle measures, any two such triangles are similar and have the same ratio $\frac{L}{H}$ (see *Elements*, VI.4). Now therefore define

$$\cos\frac{\pi}{4} := \frac{L}{H} \quad \text{and} \quad \sin\frac{\pi}{4} := \frac{L}{H},$$

where again the ratios may be computed using any triangle in $\mathcal{R}_{\pi/4}$.

The ratios $\frac{A}{H}$, $\frac{O}{H}$, and $\frac{L}{H}$ are all strictly positive, for the simple reason that a triangle always has sides of positive length (at least in the simple conception of a triangle that operates here). These ratios are also all strictly less than unity, because H is the longest side (*Elements*, I.19 again). Altogether then, we have defined the

[1] The Pythagorean theorem is proved in Book I of the *Elements* as Proposition I.47, and the theorem is proved again in Book VI using similarity arguments as Proposition VI.31. References to the Elements should not be taken to mean that we are adopting a classical perspective on geometry. The references are only meant to reassure the reader that the annotated claims do not rely on the Pythagorean theorem (by showing that they precede the Pythagorean theorem in Euclid's exposition).

[2] We shall henceforth assume that for any $\alpha \in \left(0,\ \frac{\pi}{2}\right)$, there exists a right triangle containing an angle of measure α. The reader wishing to adopt a more cautious or classical viewpoint may replace the real interval $\left(0,\ \frac{\pi}{2}\right)$ everywhere throughout the paper by the set $\langle 0,\ \frac{\pi}{2}\rangle$ defined as the set of all $\alpha \in \left(0,\ \frac{\pi}{2}\right)$ for which there exists a right triangle containing an angle of measure α.

functions $\cos : \left(0, \frac{\pi}{2}\right) \to (0, 1)$ and $\sin : \left(0, \frac{\pi}{2}\right) \to (0, 1)$ independently of the Pythagorean theorem.

Because sine and cosine as defined above are independent of the Pythagorean theorem, any proof of the Pythagorean theorem may validly employ these functions. Indeed, Elements VI.8 very quickly leads to the Pythagorean theorem with the benefit of trigonometric notation. [3] However, our precise concern in this paper is to derive trigonometric identities, and to this we now turn.

3. Subtraction formulas

The sine and cosine functions defined above obey the following subtraction formulas, valid for all $\alpha,\ \beta \in \left(0, \frac{\pi}{2}\right)$ with $\alpha - \beta$ also in $\left(0, \frac{\pi}{2}\right)$:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta, \tag{1}$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{2}$$

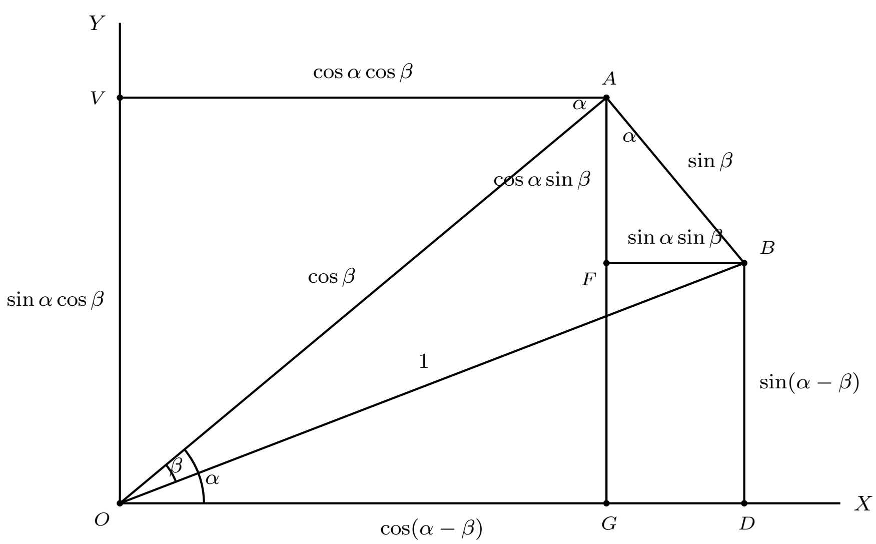


Figure 1.

The derivation of these formulas, as illustrated in Figure 1, is a textbook exercise. It is independent of the Pythagorean theorem, for although there are three hypotenuses OA, OB, and AB, their lengths are not calculated from the Pythagorean theorem, but rather from the sine and cosine we have just defined. Thus, assigning $OB = 1$, we have $OA = \cos\beta$ and $AB = \sin\beta$. The lengths of the horizontal and vertical segments are easily determined as indicated in Figure 1.

[3] See proof #6 in [2], specifically the observation attributed to R. M. Mentock.

4. The Pythagorean theorem from the subtraction formula

It is tempting to try to derive the identity $\cos^2 x + \sin^2 x = 1$ by setting $\alpha = \beta = x$ and $\cos 0 = 1$ in (1). [4] This would not be valid, however, because the domain of the cosine function does not include zero. But there is a way around this problem. Given any $x \in \left(0,\ \frac{\pi}{2}\right)$, let y be any number with $0 < y < x < \frac{\pi}{2}$. Then x, y, and $x - y$ are all in $\left(0,\ \frac{\pi}{2}\right)$. Therefore, applying (1) repeatedly, we have

$$\begin{aligned}\cos y &= \cos(x - (x - y)) \\ &= \cos x \cos(x - y) + \sin x \sin(x - y) \\ &= \cos x(\cos x \cos y + \sin x \sin y) + \sin x(\sin x \cos y - \cos x \sin y) \\ &= (\cos^2 x + \sin^2 x) \cos y.\end{aligned}$$

From this, $\cos^2 x + \sin^2 x = 1$.

5. Proving the Pythagorean theorem as a corollary

Because the foregoing proof is independent of the Pythagorean theorem, we may deduce the Pythagorean theorem as a corollary without risk of *petitio principii*. The identity $\cos^2 x + \sin^2 x = 1$ applied to a right triangle with legs a, b and hypotenuse c gives $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$, or $a^2 + b^2 = c^2$.

References

[1] E. S. Loomis, *The Pythagorean Proposition*, National Council of Teachers of Mathematics, 1968; available online at `http://eric.ed.gov/ERICDocs/data/ericdocs2sql/content_storage_01/0000019b/80/32/54/86.pdf`.

[2] `http://www.cut-the-knot.org/pythagoras/index.shtml`; last accessed 9/25/2009.

[3] `http://www.cut-the-knot.org/pythagoras/FalseProofs.shtml`; last accessed 9/25/2009.

[4] J. Versluys, *Zes en negentig bewijzen voor het Theorema van Pythagoras* (Ninety-Six Proofs of the Pythagorean Theorem), Amsterdam: A Versluys, 1914.

[5] `http://en.wikipedia.org/wiki/Talk:Pythagorean_theorem`; last accessed 9/25/2009.

Jason Zimba: Bennington College, 1 College Drive, Bennington, Vermont 05201, USA
E-mail address: `jzimba@gmail.com`

[4] A similar maneuver was attempted in 1914 by J. Versluys, who took $\alpha + \beta = \frac{\pi}{2}$ and $\sin \frac{\pi}{2} = 1$ in (2). Versluys cited Schur as the source of this idea (see [4, p.94]). A sign of trouble with the approach of Versluys/Schur is that the diagram typically used to derive the addition formula cannot be drawn for the case $\alpha + \beta = \frac{\pi}{2}$.

Forum Geometricorum
Volume 9 (2009) 279–280.

FORUM GEOM
ISSN 1534-1178

On the Construction of a Triangle from the Feet of Its Angle Bisectors

Alexey V. Ustinov

Abstract. We give simple examples of triangles not constructible by ruler and compass from the feet of its angle bisectors when the latter form a triangle with an angle of 60° or 120°.

Given a triangle ABC with sides a, b, c, we want to construct a triangle $A'B'C'$ such that that segments AA', BB' and CC' are its angle bisectors, internal or external. Restricted to internal bisectors, this is Problem 138 of Wernick's list [3] (see also [2]). Yiu [4] has given a conic solution of the problem. Implicit in this is the impossibility of a ruler-and-compass construction in general, though in the case of a right angled triangle, this is indeed possible ([4, §7]). The purpose of this note is to give simple examples of $A'B'C'$ not constructible from ABC by ruler-and-compass when the latter contains a 60° or 120° angle.

Following [4] we denote by $(x:y:z)$ the barycentric coordinates of the incenter of triangle $A'B'C'$ with respect to triangle ABC, when A, B, C are the feet of the internal angle bisectors, or an excenter when one of A, B, C is the foot of an internal bisector and the remaining two external. The vertices of triangle $A'B'C'$ have coordinates $(-x,y,z)$, $(x,-y,z)$, $(x,y,-z)$. These coordinates satisfy the following equations (see [4, §3]):

$$\begin{aligned}
-x(c^2y^2-b^2z^2)+yz((c^2+a^2-b^2)y-(a^2+b^2-c^2)z) &= 0,\\
-y(a^2z^2-c^2x^2)+xz((a^2+b^2-c^2)z-(b^2+c^2-a^2)x) &= 0, \qquad (1)\\
-z(b^2x^2-a^2y^2)+xy((b^2+c^2-a^2)x-(c^2+a^2-b^2)y) &= 0.
\end{aligned}$$

These three equations being dependent, it is enough to consider the last two. Elimination of z from these leads to a quartic equation in x and y. This fact already suggests the impossibility of a ruler-and-compass construction. However, this can be made precise if we put $c^2=a^2-ab+b^2$. In this case, angle C is 60° and we obtain, by writing $bx=t\cdot ay$, the following cubic equation in t:

Publication Date: November 2, 2009. Communicating Editor: Paul Yiu.
The research of the author was supported by Dynasty Foundation.
The author is grateful to V. Dubrovsky for the references [2, 3].

$$3(a-b)bt^3 - (a^2 - 4ab + b^2)t^2 + (a^2 - 4ab + b^2)t + 3a(a-b) = 0.$$

With $a = 8$, $b = 7$ (so that $c = \sqrt{57}$ and angle C is $60°$), this reduces to

$$7t^3 + 37t^2 - 37t + 8 = 0,$$

which is easily seen not to have rational roots. The roots of the cubic equation are not constructible by ruler and compass (see [1, Chapter 3]). Explicit solutions can be realized by taking $A = (7, 0)$, $B = (4, 4\sqrt{3})$, $C = (0, 0)$, with resulting $A'B'C'$ and the corresponding incenter (or excenter) exhibited in the table below.

t	$0.5492\cdots$	$0.3370\cdots$	$-6.1721\cdots$
A'	$(-0.3891, 6.8375)$	$(1.3112, 6.9711)$	$(5.8348, 0.7573)$
B'	$(1.4670, -25.7766)$	$(5.5301, 29.3999)$	$(7.6694, 0.9954)$
C'	$(8.5071, 7.0213)$	$(6.6557, 6.8857)$	$(6.3481, -0.9692)$
I	$(3.6999, 3.0537)$ incenter	$(3.7956, 3.9267)$ B' − excenter	$(3.7956, 3.9267)$ A' − excenter

On the other hand, if $c^2 = a^2 + ab + b^2$, the eliminant of z from (1) is also a cubic (in x and y) which, with the substitution $bx = t \cdot ay$, reduces to

$$3(a+b)bt^3 - (a^2 + 4ab + b^2)t^2 - (a^2 - 4ab + b^2)t + 3a(a+b) = 0.$$

With $a = 2$, $b = 1$ (so that $c = \sqrt{7}$ and angle C is $120°$), this reduces to

$$9t^3 - 13t^2 - 13t + 18 = 0,$$

with three irrational roots. Explicit solutions can be realized by taking $A = (1, 0)$, $B = (-1, \sqrt{3})$, $C = (0, 0)$, with resulting $A'B'C'$ and the corresponding excenter exhibited in the table below.

t	$1.0943\cdots$	$1.5382\cdots$	$-1.1881\cdots$
A'	$(0.6876, -0.3735)$	$(5.2374, -2.2253)$	$(0.0436, 0.0549)$
B'	$(-1.4112, 0.7665)$	$(1.2080, -0.5132)$	$(-0.0555, -0.0699)$
C'	$(0.1791, 0.2609)$	$(0.7473, 0.6234)$	$(0.1143, -0.0586)$
I	$(0.1791, 0.2609)$ C' − excenter	$(0.3863, 0.3222)$ A' − excenter	$(-0.1261, 0.0646)$ C' − excenter

References

[1] R. Courant and H. Robbins, *What is mathematics?* Oxford University Press, 1979.

[2] L. F. Meyers, Update on William Wernick's "Triangle constructions with three located points", *Math. Mag.*, 69 (1996) 46–49.

[3] W. Wernick, Triangle constructions with three located points, *Math. Mag.*, 55 (1982) 227–230.

[4] P. Yiu, Conic solution of a triangle from the feet of its angle bisectors, *Journal for Geometry and Graphics*, 12 (2008) 171–182.

Alexey V. Ustinov: Institute of Applied Mathematics, Dzerzhinskogo 54, Khabarovsk 680000, Russia

E-mail address: ustinov.alexey@gmail.com

Forum Geometricorum
Volume 9 (2009) 281–282.

FORUM GEOM
ISSN 1534-1178

Pythagorean Triangles with Square of Perimeter Equal to an Integer Multiple of Area

John F. Goehl, Jr.

Abstract. We determine all primitive Pythagorean triangles with square on perimeter equal to an integer multiple of its area.

Complete solutions can be found for several special cases of the problem of solving $P^2 = nA$, where P is the perimeter and A is the area of an integer-sided triangle, and n is an integer. The general problem is considered in a recent paper [1]. We consider the case of right triangles. Let the sides be a, b, and c, where c is the hypotenuse. We require

$$n = \frac{2(a+b+c)^2}{ab}.$$

By the homogeneity of the problem, it is enough to consider primitive Pythagorean triangles. It is well known that there are positive integers p and q, relatively prime and of different parity, such that

$$a = p^2 - q^2, \quad b = 2pq, \quad c = p^2 + q^2.$$

With these, $n = \frac{4p(p+q)}{q(p-q)} = \frac{4t(t+1)}{t-1}$, where $t = \frac{p}{q}$. Rewriting this as

$$4t^2 - (n-4)t + n = 0,$$

we obtain

$$t = \frac{(n-4) \pm d}{8},$$

where

$$d^2 = (n-4)^2 - 16n = (n-12)^2 - 128. \tag{1}$$

Since t is rational, d must be an integer (which we may assume positive). Equation (1) may be rewritten as

$$(n-12-d)(n-12+d) = 128 = 2^7.$$

Publication Date: November 16, 2009. Communicating Editor: Paul Yiu.

From this,

$$\begin{aligned} n - 12 - d &= 2^k, \\ n - 12 + d &= 2^{7-k}, \end{aligned}$$

for $k = 1, 2, 3$. We have

$$t = \frac{n-4+d}{8} = 2^{4-k} + 1 \quad \text{or} \quad t = \frac{n-4-d}{8} = \frac{2^k+8}{8}.$$

Since p and q are relatively prime integers of different parity, we exclude the cases when t is an odd integer. Thus, the primitive Pythagorean triangles solving $P^2 = nA$ are precisely those shown in the table below.

k	t	(p, q)	(a, b, c)	n	A	P
1	$\frac{5}{4}$	$(5, 4)$	$(9, 40, 41)$	45	180	90
2	$\frac{3}{2}$	$(3, 2)$	$(5, 12, 13)$	30	30	30
3	2	$(2, 1)$	$(3, 4, 5)$	24	6	12

Among these three solutions, only in the case of $(3, 4, 5)$ can the square on the perimeter be tessellated by n copies of the triangle.

References

[1] A. J. MacLeod, On integer relations between the area and perimeter of Heron triangles, *Forum Geom.*, 9 (2009) 41–46.

John F. Goehl, Jr.: Department of Physical Sciences, Barry University, 11300 NE Second Avenue, Miami Shores, Florida 33161, USA

E-mail address: jgoehl@mail.barry.edu

Forum Geometricorum
Volume 9 (2009) 283–290.

FORUM GEOM
ISSN 1534-1178

Trilinear Polars and Antiparallels

Shao-Cheng Liu

Abstract. We study the triangle bounded by the antiparallels to the sidelines of a given triangle ABC through the intercepts of the trilinear polar of a point P other than the centroid G. We show that this triangle is perspective with the reference triangle, and also study the condition of concurrency of the antiparallels. Finally, we also study the configuration of induced GP-lines and obtain an interesting conjugation of finite points other than G.

1. Perspector of a triangle bounded by antiparallels

We use the barycentric coordinates with respect to triangle ABC throughout. Let $P = (u : v : w)$ be a finite point in the plane of ABC, distinct from its centroid G. The trilinear polar of P is the line

$$\mathcal{L}: \qquad \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0,$$

which intersects the sidelines BC, CA, AB respectively at

$$P_a = (0 : v : -w), \quad P_b = (-u : 0 : w), \quad P_c = (u : -v : 0).$$

The lines through P_a, P_b, P_c antiparallel to the respective sidelines of ABC are

$$\begin{array}{ll}
\mathcal{L}_a: & (b^2w - c^2v)x + (b^2 - c^2)wy + (b^2 - c^2)vz = 0, \\
\mathcal{L}_b: & (c^2 - a^2)wx + (c^2u - a^2w)y + (c^2 - a^2)uz = 0, \\
\mathcal{L}_c: & (a^2 - b^2)vx + (a^2 - b^2)uy + (a^2v - b^2u)z = 0.
\end{array}$$

They bound a triangle with vertices

$$\begin{aligned}
A' = \ & (-a^2(a^2(u^2 - vw) + b^2u(w - u) - c^2u(u - v)) \\
& : (c^2 - a^2)(a^2v(w - u) + b^2u(v - w)) \\
& : (a^2 - b^2)(c^2u(v - w) + a^2w(u - v))), \\
B' = \ & ((b^2 - c^2)(a^2v(w - u) + b^2u(v - w)) \\
& : -b^2(b^2(v^2 - wu) + c^2v(u - v) - a^2v(v - w)) \\
& : (a^2 - b^2)(b^2w(u - v) + c^2v(w - u))), \\
C' = \ & ((b^2 - c^2)(c^2u(v - w) + a^2w(u - v)) \\
& : (c^2 - a^2)(b^2w(u - v) + c^2v(w - u)) \\
& : -c^2(c^2(w^2 - uv) + a^2w(v - w) - b^2w(w - u))).
\end{aligned}$$

Publication Date: November 30, 2009. Communicating Editor: Paul Yiu.

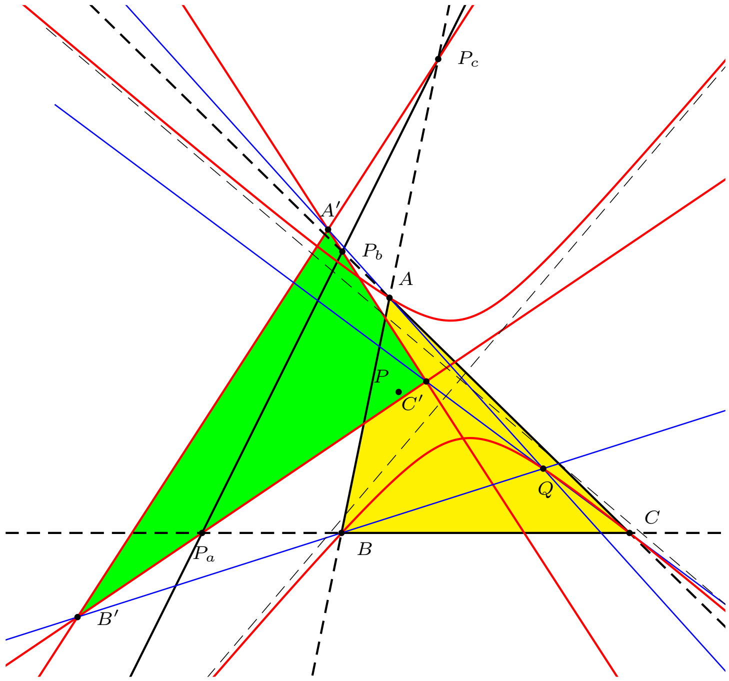


Figure 1. Perspector of triangle bounded by antiparallels

The lines AA', BB', CC' intersect at a point

$$Q=\left(\frac{b^2-c^2}{b^2w(u-v)+c^2v(w-u)}:\frac{c^2-a^2}{c^2u(v-w)+a^2w(u-v)}:\frac{a^2-b^2}{a^2v(w-u)+b^2u(v-w)}\right) \tag{1}$$

We show that Q is a point on the Jerabek hyperbola. The coordinates of Q in (1) can be rewritten as

$$Q=\left(\frac{a^2(b^2-c^2)}{\frac{\frac{1}{u}-\frac{1}{v}}{c^2}+\frac{\frac{1}{w}-\frac{1}{u}}{b^2}}:\frac{b^2(c^2-a^2)}{\frac{\frac{1}{v}-\frac{1}{w}}{a^2}+\frac{\frac{1}{u}-\frac{1}{v}}{c^2}}:\frac{c^2(a^2-b^2)}{\frac{\frac{1}{w}-\frac{1}{u}}{b^2}+\frac{\frac{1}{v}-\frac{1}{w}}{a^2}}\right).$$

If we also write this in the form $\left(\frac{a^2}{x}:\frac{b^2}{y}:\frac{c^2}{z}\right)$, then

$$\frac{\frac{1}{u}-\frac{1}{v}}{c^2}+\frac{\frac{1}{w}-\frac{1}{u}}{b^2}:\frac{\frac{1}{v}-\frac{1}{w}}{a^2}+\frac{\frac{1}{u}-\frac{1}{v}}{c^2}:\frac{\frac{1}{w}-\frac{1}{u}}{b^2}+\frac{\frac{1}{v}-\frac{1}{w}}{a^2}=(b^2-c^2)x:(c^2-a^2)y:(a^2-b^2)z.$$

$$\frac{\frac{1}{v}-\frac{1}{w}}{a^2}:\frac{\frac{1}{w}-\frac{1}{u}}{b^2}:\frac{\frac{1}{u}-\frac{1}{v}}{c^2} = -(b^2-c^2)x+(c^2-a^2)y+(a^2-b^2)z:\dots:\dots$$
$$\frac{1}{v}-\frac{1}{w}:\frac{1}{w}-\frac{1}{u}:\frac{1}{u}-\frac{1}{v} = a^2(-(b^2-c^2)x+(c^2-a^2)y+(a^2-b^2)z):\dots:\dots$$

Since $\left(\frac{1}{v}-\frac{1}{w}\right)+\left(\frac{1}{w}-\frac{1}{u}\right)+\left(\frac{1}{u}-\frac{1}{v}\right)=0$, we have

$$\begin{aligned}
0 = & \sum_{\text{cyclic}} a^2(-(b^2-c^2)x+(c^2-a^2)y+(a^2-b^2)z) \\
= & \sum_{\text{cyclic}} (-a^2(b^2-c^2)+b^2(b^2-c^2)+c^2(b^2-c^2))x \\
= & \sum_{\text{cyclic}} (b^2-c^2)(b^2+c^2-a^2)x.
\end{aligned}$$

This is the equation of the Euler line. It shows that the point Q lies on the Jerabek hyperbola. We summarize this in the following theorem, with a slight modification of (1).

Theorem 1. *Let $P=(u:v:w)$ be a point in the plane of triangle ABC, distinct from its centroid. The antiparallels through the intercepts of the trilinear polar of P bound a triangle perspective with ABC at a point*

$$Q(P)=\left(\frac{b^2-c^2}{b^2\left(\frac{1}{u}-\frac{1}{v}\right)+c^2\left(\frac{1}{w}-\frac{1}{u}\right)}:\dots:\dots\right)$$

on the Jerabek hyperbola.

Here are some examples.

P	X_1	X_3	X_4	X_6	X_9	X_{23}	X_{24}	X_{69}
$Q(P)$	X_{65}	X_{64}	X_4	X_6	X_{1903}	X_{1177}	X_3	X_{66}
P	X_{468}	X_{847}	X_{193}	X_{93}	X_{284}	X_{943}	X_{1167}	X_{186}
$Q(P)$	X_{67}	X_{68}	X_{69}	X_{70}	X_{71}	X_{72}	X_{73}	X_{74}

Table 1. The perspector $Q(P)$

Note that for the orthocenter $X_4=H$ and $X_6=K$, we have $Q(H)=H$ and $Q(K)=K$. In fact, for $P=H$, the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ bound the orthic triangle. On the other hand, for $P=K$, these lines bound the tangential triangle, anticevian triangle of K. We prove that these are the only points satisfying $Q(P)=P$.

Proposition 2. *The perspector $Q(P)$ coincides with P if and only if P is the orthocenter or the symmedian point.*

Proof. The perspector R coincides with P if and only if the lines AP, $\mathcal{L}_b$, $\mathcal{L}_c$ are concurrent, so are the triples BP, $\mathcal{L}_c$, $\mathcal{L}_a$ and CP, $\mathcal{L}_a$, $\mathcal{L}_b$. Now, AP, $\mathcal{L}_b$, $\mathcal{L}_c$ are concurrent if and only if

$$\begin{vmatrix} 0 & w & -v \\ (c^2-a^2)w & (c^2u-a^2w) & (c^2-a^2)u \\ (a^2-b^2)v & (a^2-b^2)u & (a^2v-b^2u) \end{vmatrix} = 0,$$

or

$$a^2(a^2-b^2)v^2w + a^2(c^2-a^2)vw^2 - b^2(c^2-a^2)w^2u - c^2(a^2-b^2)uv^2 = 0.$$

From the other two triples we obtain

$$-a^2(b^2-c^2)vw^2 + b^2(b^2-c^2)w^2u + b^2(a^2-b^2)wu^2 - c^2(a^2-b^2)u^2v = 0$$

and

$$-a^2(b^2-c^2)v^2w - b^2(c^2-a^2)wu^2 + c^2(c^2-a^2)u^2v + c^2(b^2-c^2)uv^2 = 0.$$

From the difference of the last two, we have, apart from a factor b^2-c^2,

$$u(b^2w^2-c^2v^2) + v(c^2u^2-a^2w^2) + w(a^2v^2-b^2u^2) = 0.$$

This shows that P lies on the Thomson cubic, the isogonal cubic with pivot the centroid G. The Thomson cubic is appears as K002 in Bernard Gibert's catalogue [2]. The same point, as a perspector, lies on the Jerabek hyperbola. Since the Thomson cubic is self-isogonal, its intersections with the Jerabek hyperbola are the isogonal conjugates of the intersections with the Euler line. From [2], P^* is either G, O or H. This means that P is K, H, or O. Table 1 eliminates the possiblility $P = O$, leaving H and K as the only points satisfying $Q(P) = P$. □

Proposition 3. *Let P be a point distinct from the centroid G, and Γ the circum-hyperbola containing G and P. If T traverses Γ, the antiparallels through the intercepts of the triliner polar of T bound a triangle perspective with ABC with the same perspector $Q(P)$ on the Jerabek hyperbola.*

Proof. The circum-hyperbola containing G and P is the isogonal transform of the line KP^*. If we write $P^* = (u : v : w)$, then a point T on Γ has coordinates $\left(\frac{a^2}{u+ta^2} : \frac{b^2}{v+tb^2} : \frac{c^2}{w+tc^2}\right)$ for some real number t. By Theorem 1, we have

$$\begin{aligned} Q(T) = & \left(\frac{b^2-c^2}{b^2\left(\frac{u+ta^2}{a^2} - \frac{v+tb^2}{b^2}\right) + c^2\left(\frac{w+tc^2}{c^2} - \frac{u+ta^2}{a^2}\right)} : \cdots : \cdots \right) \\ = & \left(\frac{b^2-c^2}{b^2\left(\frac{u}{a^2} - \frac{v}{b^2}\right) + c^2\left(\frac{w}{c^2} - \frac{u}{a^2}\right)} : \cdots : \cdots \right) \\ = & \ Q(P). \end{aligned}$$

□

2. Concurrency of antiparallels

Proposition 4. *The three lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ are concurrent if and only if*

$$-2(a^2-b^2)(b^2-c^2)(c^2-a^2)uvw+\sum_{\text{cyclic}} b^2c^2u((c^2+a^2-b^2)v^2-(a^2+b^2-c^2)w^2)=0. \tag{2}$$

Proof. The three lines are concurrent if and only if

$$\begin{vmatrix} b^2w-c^2v & (b^2-c^2)w & (b^2-c^2)v \\ (c^2-a^2)w & c^2u-a^2w & (c^2-a^2)u \\ (a^2-b^2)v & (a^2-b^2)u & a^2v-b^2u \end{vmatrix}=0.$$

□

For $P=X_{25}$ (the homothetic center of the orthic and tangential triangles), the trilinear polar is parallel to the Lemoine axis (the trilinear polar of K), and the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ concur at the symmedian point (see Figure 2).

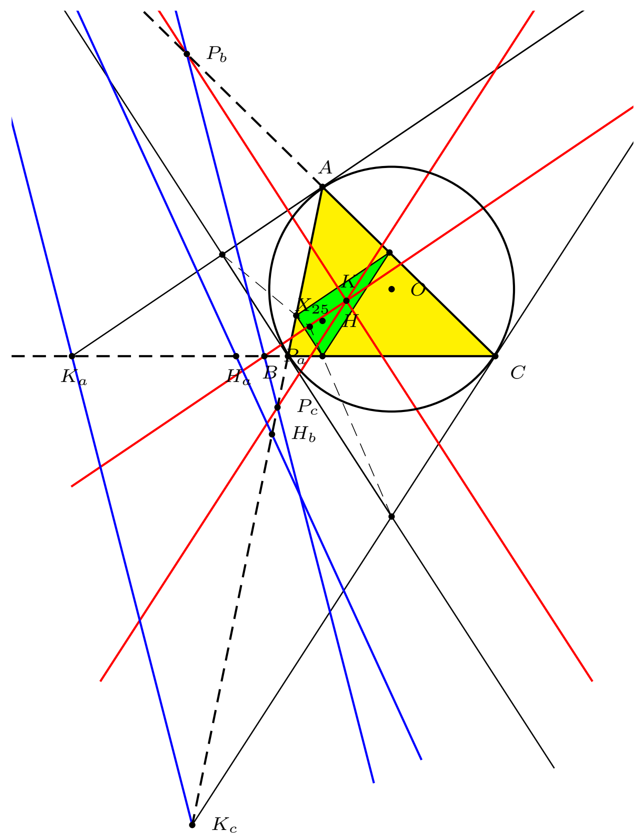


Figure 2. Antiparallels through the intercepts of X_{25}

The cubic defined by (2) can be parametrized as follows. If Q is the point $\left(\frac{a^2}{a^2(b^2+c^2-a^2)+t}:\dots:\dots\right)$ on the Jerabek hyperbola, then the antiparallels through the intercepts of the trilinear polar of

$$P_0(Q) = \left(\frac{a^2(b^2c^2+t)}{(b^2+c^2-a^2)(a^2(b^2+c^2-a^2)+t)} : \cdots : \cdots\right)$$

are concurrent at Q. On the other hand, given $P = (u : v : w)$, the antiparallels through the intercepts of the trilinear polars of

$$P_0 = \left(\frac{u(v-w)}{a^2(b^2+c^2-a^2)(b^2w(u-v)+c^2v(w-u))} : \cdots : \cdots\right)$$

are concurrent at $Q(P)$. Here are some examples.

P	X_1	X_3	X_4	X_6	X_{24}
Q	X_{65}	X_{64}	X_4	X_6	X_3
$P_0(Q)$	X_{278}	X_{1073}	X_{2052}	X_{25}	X_{1993}

Table 2. $P_0(Q)$ for Q on the Jerabek hyperbola

$$\begin{aligned}
P_0(X_{66}) =& \left(\frac{1}{a^2(b^4+c^4-a^4)} : \cdots : \cdots\right),\\
P_0(X_{69}) =& (b^2c^2(b^2+c^2-3a^2) : \cdots : \cdots),\\
P_0(X_{71}) =& (a^2(b+c-a)(a(bc+ca+ab)-(b^3+c^3)) : \cdots : \cdots),\\
P_0(X_{72}) =& (a^3-a^2(b+c)-a(b+c)^2+(b+c)(b^2+c^2) : \cdots : \cdots).
\end{aligned}$$

3. Triple of induced GP-lines

Let P be a point in the plane of triangle ABC, distinct from the centroid G, with trilinear polar intersecting BC, CA, AB respectively at P_a, P_b, P_c. Let the antiparallel to BC through P_a intersect CA and AB at B_a and C_a respectively; similarly define C_b, A_b, and A_c, B_c. These are the points

$$\begin{aligned}
B_a &= ((b^2-c^2)v : 0 : c^2v-b^2w), \quad & C_a &= ((b^2-c^2)w : c^2v-b^2w : 0);\\
A_b &= (0 : (c^2-a^2)u : a^2w-c^2u), \quad & C_b &= (a^2w-c^2u : (c^2-a^2)w : 0);\\
A_c &= (0 : b^2u-a^2v : (a^2-b^2)u), \quad & B_c &= (b^2u-a^2v : 0 : (a^2-b^2)v).
\end{aligned}$$

The triangles AB_aC_a, A_bBC_b, A_cB_cC are all similar to ABC. For every point T with reference to ABC, we can speak of the corresponding points in these triangles with the same homogeneous barycentric coordinates. Thus, the P-points in these triangles are

$$\begin{aligned}
P_A =& (b^2c^2(u+v+w)(v-w)-c^4v^2+b^4w^2 : b^2w(c^2v-b^2w) : c^2v(c^2v-b^2w)),\\
P_B =& (a^2w(a^2w-c^2u) : c^2a^2(u+v+w)(w-u)-a^4w^2+c^4u^2 : c^2u(a^2w-c^2u)),\\
P_C =& (a^2v(b^2u-a^2v) : b^2u(b^2u-a^2v) : a^2b^2(u+v+w)(u-v)-b^4u^2+a^4v^2).
\end{aligned}$$

On the other hand, the centroids of these triangles are the points

$$\begin{aligned}
G_A = &\ (2b^2c^2(v-w) - c^4v + b^4w : b^2(c^2v - b^2w) : c^2(c^2v - b^2w)),\\
G_B = &\ (a^2(a^2w - c^2u) : 2c^2a^2(w-u) - a^4w + c^4u : c^2(a^2w - c^2u)),\\
G_C = &\ (a^2(b^2u - a^2v) : b^2(b^2u - a^2v) : 2a^2b^2(u-v) - b^4u + a^4v).
\end{aligned}$$

We call G_AP_A, G_BP_B, G_CP_C the triple of GP-lines induced by antiparallels through the intercepts of the trilinear polar of P, or simply the triple of induced GP-lines.

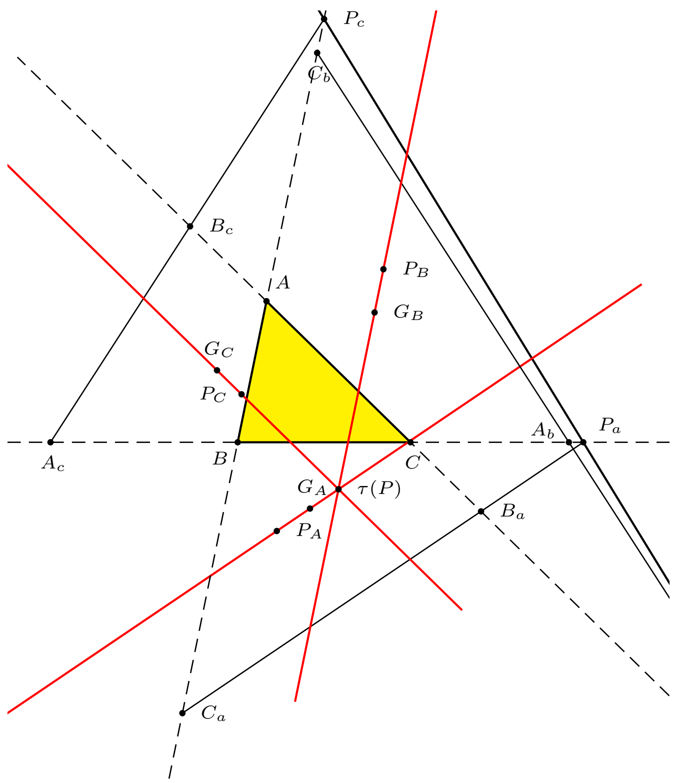


Figure 3. Triple of induced GP-lines

Theorem 5. *The triple of induced GP-lines are concurrent at*

$$\begin{aligned}
\tau(P) = &(-a^2(u^2 - v^2 + vw - w^2) + b^2u(u+v-2w) + c^2u(w+u-2v)\\
&: a^2v(u+v-2w) - b^2(v^2 - w^2 + wu - u^2) + c^2v(v+w-2u)\\
&: a^2w(w+u-2v) + b^2w(v+w-2u) - c^2(w^2 - u^2 + uv - v^2)).
\end{aligned}$$

Proof. The equations of the lines G_AP_A, G_BP_B, G_CP_C are

$$\begin{aligned}
(c^2v-b^2w)x+(c^2(w+u-v)-b^2w)y-(b^2(u+v-w)-c^2v)z=&\ 0,\\
-(c^2(v+w-u)-a^2w)x+(a^2w-c^2u)y+(a^2(u+v-w)-c^2u)z=&\ 0,\\
(b^2(v+w-u)-a^2v)x-(a^2(w+u-v)-b^2u)y+(b^2u-a^2v)z=&\ 0.
\end{aligned}$$

These three lines intersect at $\tau(P)$ given above. □

Remark. If T traverses the line GP, then $\tau(T)$ traverses the line $G\tau(P)$.

Note that the equations of induced GP-lines are invariant under the permutation $(x,y,z)\leftrightarrow(u,v,w)$, *i.e.*, these can be rewritten as

$$\begin{aligned}
(c^2y-b^2z)u+(c^2(z+x-y)-b^2z)v-(b^2(x+y-z)-c^2y)w=&\ 0,\\
-(c^2(y+z-x)-a^2z)u+(a^2z-c^2x)v+(a^2(x+y-z)-c^2x)w=&\ 0,\\
(b^2(y+z-x)-a^2y)u-(a^2(z+x-y)-b^2x)v+(b^2x-a^2y)w=&\ 0.
\end{aligned}$$

This means that the mapping τ is a conjugation of the finite points other than the centroid G.

Corollary 6. *The triple of induced GP-lines concur at Q if and only if the triple of induced GQ-lines concur at P.*

We conclude with a list of pairs of triangle centers conjugate under τ.

$X_1,\ X_{1054}$	$X_3,\ X_{110}$	$X_4,\ X_{125}$	$X_6,\ X_{111}$	X_{23}, X_{182}	X_{69}, X_{126}
X_{98}, X_{1316}	X_{100}, X_{1083}	$X_{184},\ X_{186}$	$X_{187},\ X_{353}$	$X_{352},\ X_{574}$	

Table 3. Pairs conjugate under τ

References

[1] N. Dergiades and P. Yiu, Antiparallels and concurrent Euler lines, *Forum Geom.*, 4 (2004) 1–20.
[2] B. Gibert, *Cubics in the Triangle Plane*, available at `http://pagesperso-orange.fr/bernard.gibert/index.html`.
[3] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.
[4] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001.

Shao-Cheng Liu: 2F., No.8, Alley 9, Lane 22, Wende Rd., 11475 Taipei, Taiwan
E-mail address: `liu471119@yahoo.com.tw`

Forum Geometricorum
Volume 9 (2009) 291–295.

FORUM GEOM
ISSN 1534-1178

A Sequence of Triangles and Geometric Inequalities

Dan Marinescu, Mihai Monea, Mihai Opincariu, and Marian Stroe

Abstract. We construct a sequence of triangles from a given one, and deduce a number of famous geometric inequalities.

1. A geometric construction

Throughout this paper we use standard notations of triangle geometry. Given a triangle ABC with sidelengths a, b, c, let s, R, r, and Δ denote the semiperimeter, circumradius, inradius, and area respectively. We begin with a simple geometric construction. Let H be the orthocenter of triangle ABC. Construct a circle, center H, radius $R' = \sqrt{2Rr}$ to intersect the half lines HA, HB, HC at A', B', C' respectively (see Figure 1).

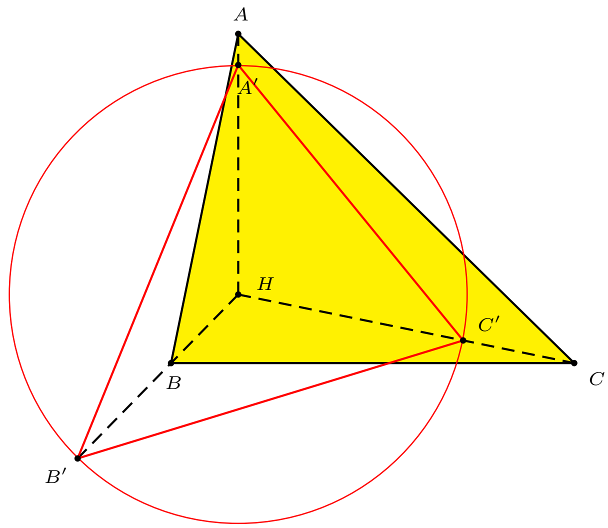


Figure 1.

If the triangle ABC has a right angle at A with altitude AD (D on the hypotenuse BC), we choose A' on the line AD such that A is between D and A'.

Lemma 1. *Triangle $A'B'C'$ has*
(a) *angle measures* $A' = \frac{\pi}{2} - \frac{A}{2}$, $B' = \frac{\pi}{2} - \frac{B}{2}$, $C' = \frac{\pi}{2} - \frac{C}{2}$,
(b) *sidelengths* $a' = \sqrt{a(b+c-a)}$, $b' = \sqrt{b(c+a-b)}$, $c' = \sqrt{c(a+b-c)}$, *and*
(c) *area* $\Delta' = \Delta$.

Publication Date: December 16, 2009. Communicating Editor: Paul Yiu.

Proof. (a) $\angle B'A'C' = \frac{1}{2}\angle B'HC' = \frac{1}{2}\angle BHC = \frac{\pi - A}{2}$; similarly for B' and C'.

(b) By the law of sines,

$$a' = 2R' \sin A' = 2\sqrt{2Rr} \cos \frac{A}{2} = 2\sqrt{2 \cdot \frac{abc}{4\Delta} \cdot \frac{\Delta}{s}} \cdot \sqrt{\frac{s(s-a)}{bc}} = \sqrt{a(b+c-a)};$$

similarly for b' and c'.

(c) Triangle $A'B'C'$ has area

$$\begin{aligned} \Delta' = &\ \frac{1}{2} b'c' \sin A' = \frac{1}{2} b'c' \cos \frac{A}{2} \\ = &\ \frac{1}{2}\sqrt{b(c+a-b)} \cdot \sqrt{c(a+b-c)} \cdot \sqrt{\frac{s(s-a)}{bc}} \\ = &\ \sqrt{s(s-a)(s-b)(s-c)} \\ = &\ \Delta. \end{aligned}$$

□

Proposition 2. (a) $a'^2 + b'^2 + c'^2 = a^2 + b^2 + c^2 - (b-c)^2 - (c-a)^2 - (a-b)^2$.
(b) $a'^2 + b'^2 + c'^2 \le a^2 + b^2 + c^2$.
(c) $a' + b' + c' \le a + b + c$.
(d) $\sin A' + \sin B' + \sin C' \ge \sin A + \sin B + \sin C$.
(e) $R' \le R$.
(f) $r' \ge r$.

In each case, equality holds if and only if ABC is equilateral.

Proof. (a) follows from Lemma 1(b); (b) follows from (a). For (c),

$$\begin{aligned} a' + b' + c' = &\ \sqrt{a(b+c-a)} + \sqrt{b(c+a-b)} + \sqrt{c(a+b-c)} \\ \le &\ \frac{b+c}{2} + \frac{c+a}{2} + \frac{a+b}{2} \\ = &\ a + b + c. \end{aligned}$$

For (d), we have

$$\begin{aligned} & \sin A + \sin B + \sin C \\ = & \frac{1}{2}(\sin B + \sin C + \sin C + \sin A + \sin A + \sin B) \\ = &\ \sin \frac{B+C}{2} \cos \frac{B-C}{2} + \sin \frac{C+A}{2} \cos \frac{C-A}{2} + \sin \frac{A+B}{2} \cos \frac{A-B}{2} \\ \le & \sin \frac{B+C}{2} + \sin \frac{C+A}{2} + \sin \frac{A+B}{2} \\ = & \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} \\ = & \sin A' + \sin B' + \sin C'. \end{aligned}$$

(e) $R' = \frac{a'+b'+c'}{2(\sin A' + \sin B' + \sin C')} \le \frac{a+b+c}{2(\sin A + \sin B + \sin C)} = R$.

(f) $r' = \frac{\Delta'}{s'} \ge \frac{\Delta}{s} = r$. □

Remark. The inequality $R' \leq R$ certainly follows from Euler's inequality $R \geq 2r$. From the direct proof of (e), Euler's inequality also follows (see Theorem 6(b) below).

2. A sequence of triangles

Beginning with a triangle ABC, we repeatedly apply the construction in §1 to obtain a sequence of triangles $(A_nB_nC_n)_{n\in\mathbb{N}}$ with $A_0B_0C_0 \equiv ABC$, and angle measures and sidelengths defined recursively by

$$\begin{aligned}
&A_{n+1} = \frac{\pi - A_n}{2}, \quad B_{n+1} = \frac{\pi - B_n}{2}, \quad C_{n+1} = \frac{\pi - C_n}{2};\\
&a_{n+1} = \sqrt{a_n(b_n + c_n - a_n)}, \quad b_{n+1} = \sqrt{b_n(c_n + a_n - a_n)},\\
&c_{n+1} = \sqrt{c_n(a_n + b_n - c_n)}.
\end{aligned}$$

Denote by s_n, R_n, r_n, Δ_n the semiperimeter, circumradius, inradius, and area of triangle $A_nB_nC_n$. Note that $\Delta_n = \Delta$ for every n.

Lemma 3. *The sequences* $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(C_n)_{n\in\mathbb{N}}$ *are convergent and*

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \frac{\pi}{3}.$$

Proof. It is enough to consider the sequence $(A_n)_{n\in\mathbb{N}}$. Rewrite the relation $A_{n+1} = \frac{\pi}{2} - \frac{A_n}{2}$ as

$$A_{n+1} - \frac{\pi}{3} = -\frac{1}{2}\left(A_n - \frac{\pi}{3}\right).$$

It follows that the sequence $\left(A_n - \frac{\pi}{3}\right)_{n\in\mathbb{N}}$ is a geometric sequence with common ratio $-\frac{1}{2}$. It converges to 0, giving $\lim_{n\to\infty} A_n = \frac{\pi}{3}$. $\square$

Proposition 4. *The sequence* $(R_n)_{n\in\mathbb{N}}$ *is convergent and* $\lim_{n\to\infty} R_n = \frac{2}{3}\sqrt{\sqrt{3}\Delta}$.

Proof. Since $R_n = \frac{a_nb_nc_n}{4\triangle_n} = \frac{8R_n^3 \sin A_n \sin B_n \sin C_n}{4\triangle_n}$, we have

$$R_n^2 = \frac{\triangle}{2\sin A_n \sin B_n \sin C_n}.$$

The result follows from Lemma 3. $\square$

Proposition 5. *The sequences* $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ *are convergent and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 2\sqrt{\frac{\triangle}{\sqrt{3}}}.$$

Proof. This follows from $a_n = 2R_n \sin A_n$, Lemma 3 and Proposition 4. $\square$

From these basic results we obtain a number of interesting convergent sequences. In each case, the increasing or decreasing property is clear from Proposition 2.

	Sequence		Limit	Reference
(a)	Δ_n	constant	Δ	Lem.1(c)
(b)	$\sin A_n + \sin B_n + \sin C_n$	increasing	$\frac{3\sqrt{3}}{2}$	Prop.2(d), Lem.3
(c)	R_n	decreasing	$\frac{2}{3}\sqrt{\sqrt{3}\Delta}$	Prop.2(e), 4
(d)	s_n	decreasing	$\sqrt{3\sqrt{3}\Delta}$	Prop.2(c), 4
(e)	r_n	increasing	$\frac{1}{3}\sqrt{\sqrt{3}\Delta}$	Prop.2(f)
(f)	$\frac{R_n}{r_n}$	decreasing	2	
(g)	$a_n^2 + b_n^2 + c_n^2$	decreasing	$4\sqrt{3}\Delta$	Prop.2(b), 5
(h)	$a_n^2 + b_n^2 + c_n^2 - (b_n - c_n)^2$ $-(c_n - a_n)^2 - (a_n - b_n)^2$	decreasing	$4\sqrt{3}\Delta$	Prop.2(a, b), 5

3. Geometric inequalities

The increasing or decreasing properties of these sequences, along with their limits, lead easily to a number of famous geometric inequalities [1, 3].

Theorem 6. *The following inequalities hold for an arbitrary angle ABC.*

(a) $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$.

(b) [Euler's inequality] $R \geq 2r$.

(c) [Weitzenböck inequality] $a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta$.

(d) [Hadwiger-Finsler inequality] $a^2+b^2+c^2-(b-c)^2-(c-a)^2-(a-b)^2 \geq 4\sqrt{3}\Delta$.

In each case, equality holds if and only if the triangle is equilateral.

Remark. Weitzenböck's inequality is usually proved as a consequence of the Hadwiger - Finsler's inequality ([2, 4]). Our proof shows that they are logically equivalent.

References

[1] O. Bottema, R. Z. Djordjević, R. R. Janić, D. S. Mitrinović, and P. M. Vasic, *Geometric Inequalities*, Wolters-Noordhoe Publishing, Groningen, 1969.

[2] P. von Finsler and H. Hadwiger, Einige Relationen im Dreieck, *Comment. Math. Helvetici*, 10 (1937) 316–326.

[3] D. S. Mitrinović, J. E. Peĉarić, and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publisher, 1989

[4] R. Weitzenböck, Uber eine Ungleichung in der Dreiecksgeometrie, *Math. Zeit.*, 5 (1919) 137–146.

Dan Marinescu: Colegiul Naţional “Iancu de Hunedoara”, Hunedoara, Str.l Libertatii No.2 Bl. 9 Ap.14, 331032, Romania
E-mail address: marinescuds@yahoo.com

Mihai Monea: Colegiul Naţional “Decebal” Deva, Str. Decebal Bl. 8 Ap.10, 330021, Romania
E-mail address: mihaimonea@yahoo.com

Mihai Opincariu: Colegiul Naţional “Avram Iancu” Brad, Str. 1 Iunie No.13 Bl. 15 Ap.14, 335200, Romania
E-mail address: opincariumihai@yahoo.com

Marian Stroe: Coleguil Economic “Emanoil Gojdu” Hunedoara, Str. Viorele No. 4, Bl 10 Ap.10, 331093, Romania
E-mail address: maricu_stroe@yahoo.com

Forum Geometricorum
Volume 9 (2009) 297–300.

FORUM GEOM
ISSN 1534-1178

Trilinear Polars of Brocardians

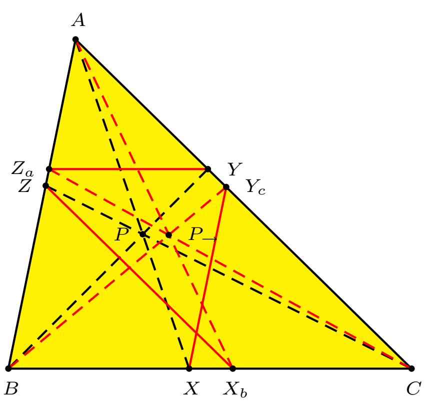
Francisco Javier García Capitán

Abstract. We study the trilinear polars of the Brocardians of a point, and investigate the condition for their orthogonality.

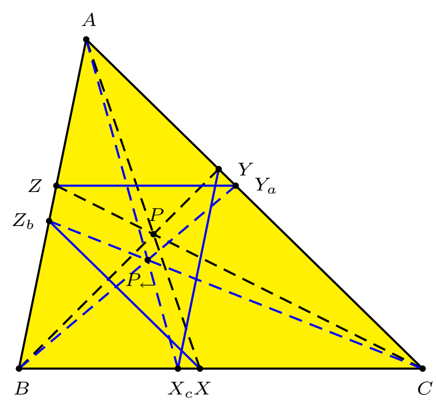
1. The Brocardians

Let P be a point not on any of the sidelines of triangle ABC, with homogeneous barycentric coordinates $(u : v : w)$ and cevian triangle XYZ. Construct the parallels of $\begin{array}{c}CA\\AB\\BC\end{array}$ through $\begin{array}{c}Z\\X\\Y\end{array}$ to intersect $\begin{array}{c}BC\\CA\\AB\end{array}$ at $\begin{array}{c}X_b\\Y_c\\Z_a\end{array}$ (see Figure 1(a)). The triangle $X_bY_cZ_a$ is perspective with ABC at the point

$$P_{\rightarrow} := \left(\frac{1}{w} : \frac{1}{u} : \frac{1}{v}\right).$$



1(a) The Brocardian $P_{\rightarrow}$



1(b) The Brocardian $P_{\leftarrow}$

Likewise, the parallels of $\begin{array}{c}AB\\BC\\CA\end{array}$ through $\begin{array}{c}Y\\Z\\X\end{array}$ intersect $\begin{array}{c}BC\\CA\\AB\end{array}$ at $\begin{array}{c}X_c\\Y_a\\Z_b\end{array}$ such that triangle $X_cY_aZ_b$ is perspective with ABC at

$$P_{\leftarrow} := \left(\frac{1}{v} : \frac{1}{w} : \frac{1}{u}\right)$$

(see Figure 1(b)). The points $P_{\rightarrow}$ and $P_{\leftarrow}$ are called the Brocardians of P (see [2, §8.4]). For example, the Brocardians of the symmedian point are the Brocard points $\Omega = \left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}\right)$ and $\Omega' = \left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right)$.

Publication Date: December 18, 2009. Communicating Editor: Paul Yiu.

2. Noncollinearity of P and its Brocardians

The points P, $P_{\rightarrow}$ and $P_{\leftarrow}$ are never collinear since

$$\begin{vmatrix} u & v & w \\ \frac{1}{w} & \frac{1}{u} & \frac{1}{v} \\ \frac{1}{v} & \frac{1}{w} & \frac{1}{u} \end{vmatrix} = \frac{u^2+v^2+w^2-vw-wu-uv}{uvw} \neq 0.$$

It is well known that the Brocard points are equidistant from the symmedian point. It follows that the pedal of K on the line $\Omega\Omega'$ is the midpoint of the segment $\Omega\Omega'$, the triangle center $X_{39} = (a^2(b^2+c^2) : b^2(c^2+a^2) : c^2(a^2+b^2))$ in [1].

Now, for the Gergonne point $G_{\mathrm{e}} = \left(\frac{1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c}\right)$, the Brocardians are the points $G_{\mathrm{e}\rightarrow} = (a+b-c : b+c-a : c+a-b)$ and $G_{\mathrm{e}\leftarrow} = (c+a-b : a+b-c : b+c-a)$. The midpoint of $G_{\mathrm{e}\rightarrow}G_{\mathrm{e}\leftarrow}$ is the incenter $I = (a : b : c)$. Indeed, I is the pedal of the Gergonne point on the line $G_{\mathrm{e}\rightarrow}G_{\mathrm{e}\leftarrow}$

$$(b^2+c^2-a(b+c))x + (c^2+a^2-b(c+a))y + (a^2+b^2-c(a+b))z = 0.$$

3. Trilinear polars of the Brocardians

The trilinear polars of the Brocardians of P are the lines

$\ell_{\rightarrow}$ $$wx + uy + vz = 0,$$

and

$\ell_{\leftarrow}$ $$vx + wy + uz = 0.$$

These lines intersect at the point

$$Q = (u^2 - vw : v^2 - wu : w^2 - uv).$$

Since

$$(u^2-vw,\ v^2-wu,\ w^2-uv) = (u+v+w)(u,\ v,\ w) - (vw+wu+uv)(1,\ 1,\ 1),$$

the point Q divides the segment GP in the ratio

$$GQ : QP = (u+v+w)^2 : -3(vw+wu+uv).$$

The point Q is never an infinite point since

$$u^2+v^2+w^2-vw-wu-uv \neq 0.$$

It follows that the trilinear polars $\ell_{\rightarrow}$ and $\ell_{\leftarrow}$ are never parallel.

4. Orthogonality of trilinear polars of Brocardians

The trilinear polars $\ell_{\rightarrow}$ and $\ell_{\leftarrow}$ have infinite points $(u-v : v-w : w-u)$ and $(w-u : u-v : v-w)$ respectively. They are orthogonal if and only if

$$S_A(u-v)(w-u) + S_B(v-w)(u-v) + S_C(w-u)(v-w) = 0 \qquad (1)$$

(see [2, §4.5]). Now, (1) defines a conic with center $G = (1 : 1 : 1)$ (see [2, §10.7.2]). Since the conic contains G, it is necessarily degenerate. Solving for the infinite points of the conic, we obtain the condition that the conic consists of a pair of real lines if and only if

$$S_{AA} + S_{BB} + S_{CC} - 2S_{BC} - 2S_{CA} - 2S_{AB} \geq 0.$$

Equivalently,

$$5(a^4 + b^4 + c^4) - 6(b^2c^2 + c^2a^2 + a^2b^2) \geq 0. \tag{2}$$

Here is a characterization of triangles satisfying condition (2). Given two points B and C with $BC = a$, we set up a Cartesian coordinates system such that $B = \left(-\frac{a}{2},\ 0\right)$ and $C = \left(\frac{a}{2},\ 0\right)$. If $A = (x, y)$, then

$$\begin{aligned} \left(x - \frac{a}{2}\right)^2 + y^2 =&\ b^2, \\ \left(x + \frac{a}{2}\right)^2 + y^2 =&\ c^2. \end{aligned}$$

With these, condition (2) becomes

$$(4x^2 + 4y^2 - 8ay + 3a^2)(4x^2 + 4y^2 + 8ay + 3a^2) \geq 0.$$

This is the exterior of the two circles, centers $(0,\ \pm a)$, radii $\frac{a}{2}$. Here is a simple example. If we require $C = \frac{\pi}{2}$, then $S_C = 0$ and the degenerate conic (1) is the union of the two lines $v - w = 0$ and $S_A(z - x) + S_B(y - z) = 0$. These are the C-median and the line GK_c, K_c being the C-trace of the symmedian point K. Figure 2 illustrates the trilinear polars of the Brocardians of a point P on GK_c

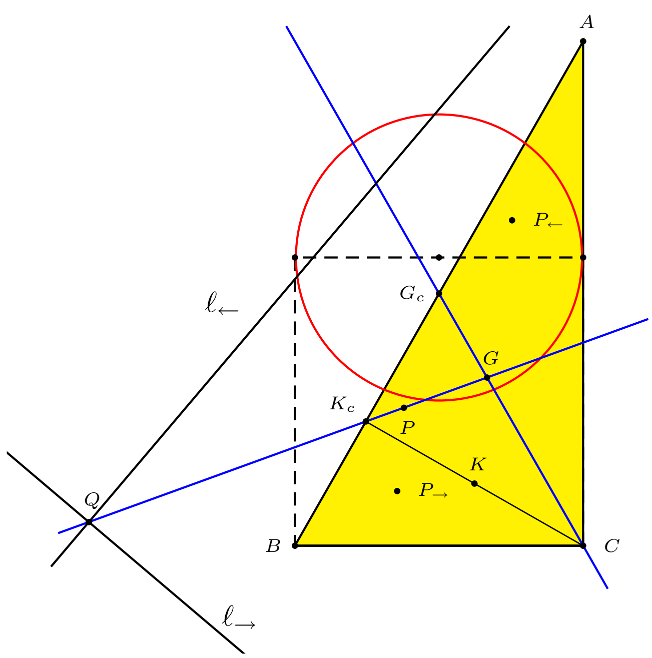


Figure 2.

On the other hand, for points A on the circumferences of the two circles, the triangle ABC has exactly one real line through the centroid G such that for every P on the line, the trilinear polars of the Brocardians intersect orthogonally (on the same line). It is enough to consider A on the circle $4(x^2+y^2)-8ay+3a^2=0$, with coordinates $\left(\frac{a}{2}\cos\theta,\ a+\frac{a}{2}\sin\theta\right)$. The center of triangle ABC is the point $G=\left(\frac{a}{6}\cos\theta,\ \frac{a}{6}(2+\sin\theta)\right)$. The line in question connects G to the fixed point $M=\left(0,\ \frac{a}{2}\right)$:

$$(1-\sin\theta)x+\cos\theta\left(y-\frac{a}{2}\right)=0.$$

The trilinear polars of the Brocardians of an arbitrary point P on this line are symmetric with respect to GM, and intersect orthogonally (see Figure 3).

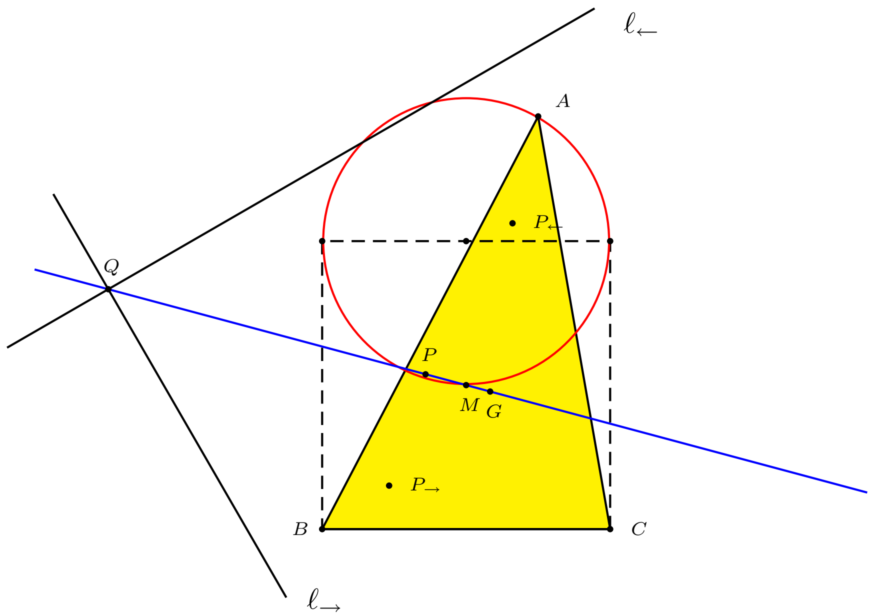


Figure 3.

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[2] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001.

Francisco Javier García Capitán: Departamento de Matemáticas, I.E.S. Alvarez Cubero, Avda. Presidente Alcalá-Zamora, s/n, 14800 Priego de Córdoba, Córdoba, Spain

E-mail address: `garciacapitan@gmail.com`

Forum Geometricorum
Volume 9 (2009) 301–348.

FORUM GEOM
ISSN 1534-1178

Reflections in Triangle Geometry

Antreas P. Hatzipolakis and Paul Yiu

On the 10th Anniversary of Hyacinthos

Abstract. This paper is a survey of results on reflections in triangle geometry. We work with homogeneous barycentric coordinates with reference to a given triangle ABC and establish various concurrency and perspectivity results related to triangles formed by reflections, in particular the reflection triangle $P^{(a)}P^{(b)}P^{(c)}$ of a point P in the sidelines of ABC, and the triangle of reflections $A^{(a)}B^{(b)}C^{(c)}$ of the vertices of ABC in their respective opposite sides. We also consider triads of concurrent circles related to these reflections. In this process, we obtain a number of interesting triangle centers with relatively simple coordinates. While most of these triangle centers have been catalogued in Kimberling's *Encyclopedia of Triangle Centers* [27] (ETC), there are a few interesting new ones. We give additional properties of known triangle centers related to reflections, and in a few cases, exhibit interesting correspondences of cubic curves catalogued in Gibert's *Catalogue of Triangle Cubics* [14] (CTC).

CONTENTS

Publication Date: December 22, 2009. Communicating Editor: Floor van Lamoen.

Notations. We adopt the usual notations of triangle geometry and work with homogeneous barycentric coordinates with reference to a given triangle ABC with sidelengths a, b, c and angle measures A, B, C. Occasionally, expressions for coordinates are simplified by using Conway's notation:

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-a^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2},$$

subject to $S_{AB} + S_{BC} + S_{CA} = S^2$, where S is twice the area of triangle ABC, and S_{BC} stands for $S_B S_C$ etc. The labeling of triangle centers follows ETC [27], except for the most basic and well known ones listed below. References to triangle cubics are made to Gibert's CTC [14].

G	X_2	centroid
H	X_4	orthocenter
E_∞	X_{30}	Euler infinity point
I	X_1	incenter
$N_{\rm a}$	X_8	Nagel point
K	X_6	symmedian point
$J_\pm$	$X_{15},\ X_{16}$	isodynamic points

O	X_3	circumcenter
N	X_5	nine point center
E	X_{110}	Euler reflection point
$G_{\rm e}$	X_7	Gergonne point
$F_{\rm e}$	X_{11}	Feuerbach point
$F_\pm$	X_{13}, X_{14}	Fermat points
W	X_{484}	first Evans perspector

P^*	isogonal conjugate of P
$P^\bullet$	isotomic conjugate of P
P^{-1}	inverse of P in circumcircle
P/Q	cevian quotient
$P_aP_bP_c$	cevian triangle of P
$P^aP^bP^c$	anticevian triangle of P
$P_{[a]}$	pedal of P on BC
$P^{(a)}$	reflection of P in BC
E_t	Point on Euler line dividing OH in the ratio $t : 1-t$
$\mathscr{C}(P,Q)$	Bicevian conic through the traces of P and Q on the sidelines

1. The reflection triangle

Let P be a point with the homogeneous barycentric coordinates $(u : v : w)$ in reference to triangle ABC. The reflections of P in the sidelines BC, CA, AB are the points

$$\begin{aligned} P^{(a)} &= (-a^2u : (a^2+b^2-c^2)u + a^2v : (c^2+a^2-b^2)u + a^2w), \\ P^{(b)} &= ((a^2+b^2-c^2)v + b^2u : -b^2v : (b^2+c^2-a^2)v + a^2w), \\ P^{(c)} &= ((c^2+a^2-b^2)w + b^2u : (b^2+c^2-a^2)w + c^2v : -c^2w). \end{aligned}$$

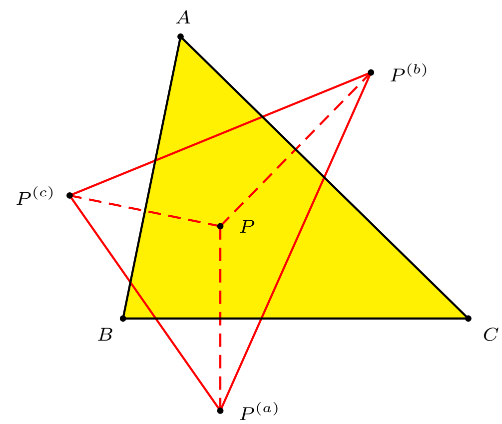


Figure 1. The reflection triangle

We call $P^{(a)}P^{(b)}P^{(c)}$ the reflection triangle of P (see Figure 1). Here are some examples.

(1) The reflection triangle of the circumcenter O is oppositely congruent to ABC at the midpoint of OH, which is the nine-point center N. This is the only reflection triangle congruent to ABC.

(2) The reflection triangle of H is inscribed in the circumcircle of ABC (see Remark (1) following Proposition 2 and Figure 3(b) below).

(3) The reflection triangle of N is homothetic at O to the triangle of reflections (see Propositioin 5 below).

Proposition 1. *The reflection triangle of P is*

(a) *right-angled if and only if P lies on one of the circles with centers $\begin{array}{c} K^a \\ K^b \\ K^c \end{array}$ passing through $\begin{array}{c} B,\ C \\ C,\ A \\ A,\ B \end{array}$ respectively,*

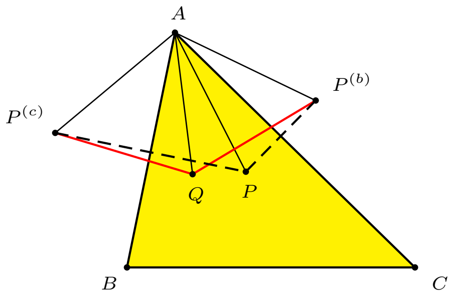
(b) *isosceles if and only if P is on one of the Apollonian circles, each with diameter the feet of the bisectors of an angle on its opposite side,*

(c) *equilateral if and only if P is one of the isodynamic points $J_{\pm}$,*

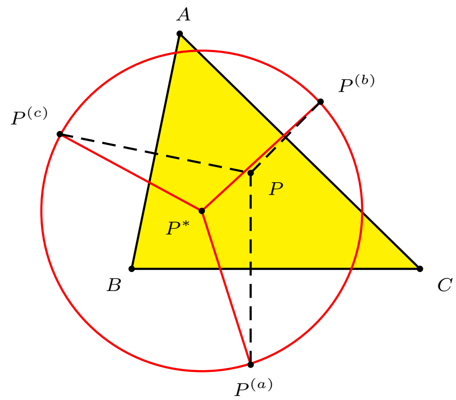
(d) *degenerate if and only if P lies on the circumcircle.*

1.1. *Circle of reflections.*

Proposition 2. *The circle $P^{(a)}P^{(b)}P^{(c)}$ has center P^*.*



(a) Reflections and isogonal lines



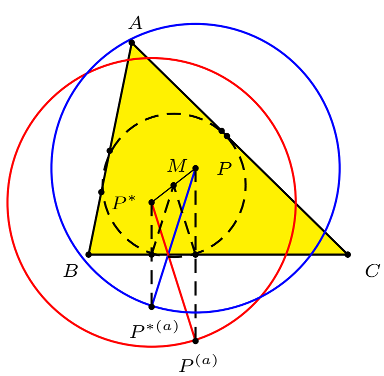
(b) circle of reflections

Figure 2. Circle of reflections of P with center P^*

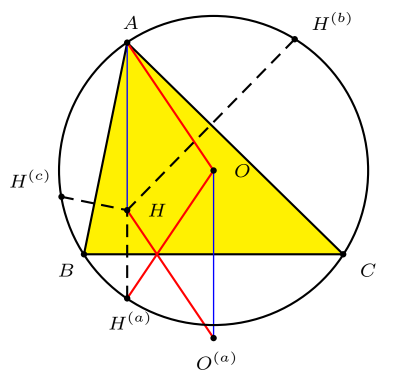
Proof. Let Q be a point on the line isogonal to AP with respect to angle A, *i.e.*, the lines AQ and AP are symmetric with respect to the bisector of angle BAC (see Figure 2(a)). Clearly, the triangles $AQP^{(b)}$ and $AQP^{(c)}$ are congruent, so that Q is equidistant from $P^{(b)}$ and $P^{(c)}$. For the same reason, any point on a line isogonal to BP is equidistant from $P^{(c)}$ and $P^{(a)}$. It follows that the isogonal conjugate P^* is equidistant from the three reflections $P^{(a)}$, $P^{(b)}$, $P^{(c)}$. □

This simple fact has a few interesting consequences.

(1) The circle through the reflections of P and the one through the reflections of P^* are congruent (see Figure 3(a)). In particular, the reflections of the orthocenter H lie on the circumcircle (see Figure 3(b)).



(a) Congruent circles of reflection



(b) Reflections of H on circumcircle

Figure 3. Congruence of circles of reflection of P and P^*

(2) The (six) pedals of P and P^* on the sidelines of triangle ABC are concyclic. The center of the common pedal circle is the midpoint of PP^* (see Figure 3(a)). For the isogonal pair O and H, this pedal circle is the nine-point circle.

1.2. *Line of reflections* .

Theorem 3. (a) *The reflections of P in the sidelines are collinear if and only if P lies on the circumcircle. In this case, the line containing the reflections passes through the orthocenter H.*
(b) *The reflections of a line ℓ in the sidelines are concurrent if and only if the line contains the orthocenter H. In this case, the point of concurrency lies on the circumcircle.*

Remarks. (1) Let P be a point on the circumcircle and ℓ a line through the orthocenter H. The reflections of P lies on ℓ if and only if the reflections of ℓ concur at P ([6, 29]). Figure 4 illustrates the case of the Euler line.

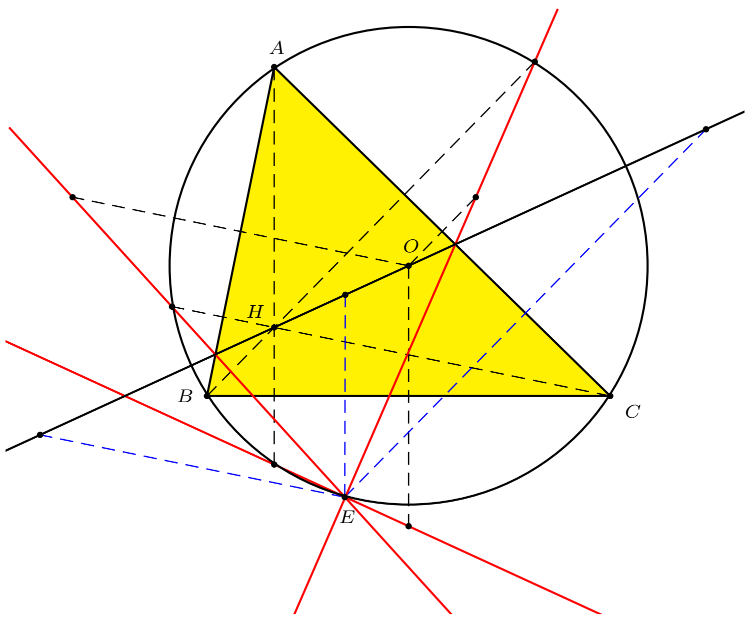


Figure 4. Euler line and Euler reflection point

(2) If $P = \left(\frac{a^2}{v-w} : \frac{b^2}{w-u} : \frac{c^2}{u-v}\right)$ is the isogonal conjugate of the infinite point of a line $ux + vy + wz = 0$, its line of reflections is

$$S_A(v-w)x + S_B(w-u)y + S_C(u-v)z = 0.$$

(3) Let ℓ be the line joining H to $P = (u : v : w)$. The reflections of ℓ in the sidelines of triangle ABC intersect at the point

$$r_0(P) := \left(\frac{a^2}{S_Bv - S_Cw} : \frac{b^2}{S_Cw - S_Au} : \frac{c^2}{S_Au - S_Bv}\right).$$

Clearly, $r_0(P_1) = r_0(P_2)$ if and only if P_1, P_2, H are collinear.

line HP	$r_0(P)$ = intersection of reflections
Euler line	$E = \left(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2}\right)$
HI	$X_{109} = \left(\frac{a^2}{(b-c)(b+c-a)} : \frac{b^2}{(c-a)(c+a-b)} : \frac{c^2}{(a-b)(a+b-c)}\right)$
HK	$X_{112} = \left(\frac{a^2}{(b^2-c^2)S_A} : \frac{b^2}{(c^2-a^2)S_B} : \frac{c^2}{(a^2-b^2)S_C}\right)$

Theorem 4 (Blanc [3]). *Let ℓ be a line through the circumcenter O of triangle ABC, intersecting the sidelines at X, Y, Z respectively. The circles with diameters AX, BY, CZ are coaxial with two common points and radical axis $\mathscr{L}$ containing the orthocenter H.*

(a) *One of the common points P lies on the nine-point circle, and is the center of the rectangular circum-hyperbola which is the isogonal conjugate of the line ℓ.*

(b) *The second common point Q lies on the circumcircle, and is the reflection of $r_0(P)$ in ℓ.*

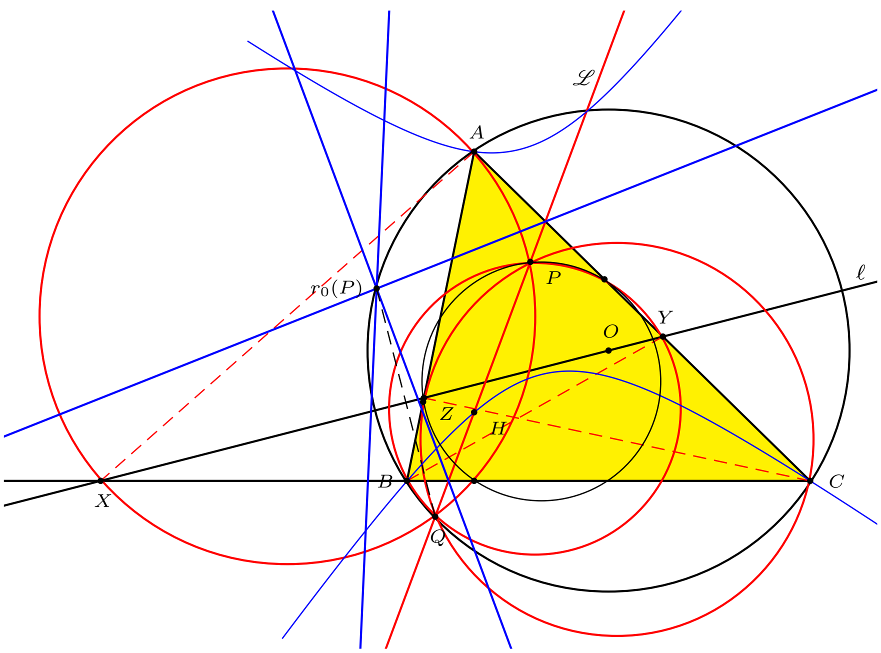


Figure 5. Blanc's theorem

Here are some examples.

Line ℓ	P	Q	$r_0(P)$
Euler line	X_{125}	$X_{476} = \left(\frac{1}{(S_B-S_C)(S^2-3S_{AA})} : \cdots : \cdots\right)$	E
Brocard axis	X_{115}	X_{112}	X_{2715}
OI	X_{11}	$X_{108} = \left(\frac{a}{(b-c)(b+c-a)S_A} : \cdots : \cdots\right)$	X_{2720}

1.3. *The triangle of reflections.* The reflections of the vertices of triangle ABC in their opposite sides are the points

$$\begin{aligned} A^{(a)} &= (-a^2 : a^2+b^2-c^2 : c^2+a^2-b^2), \\ B^{(b)} &= (a^2+b^2-c^2 : -b^2 : b^2+c^2-a^2), \\ C^{(c)} &= (c^2+a^2-b^2 : b^2+c^2-a^2 : -2c^2). \end{aligned}$$

We call triangle $A^{(a)}B^{(b)}C^{(c)}$ the triangle of reflections.

Proposition 5. *The triangle of reflections $A^{(a)}B^{(b)}C^{(c)}$ is the image of the reflection triangle of N under the homothety $\mathsf{h}(O,2)$.*

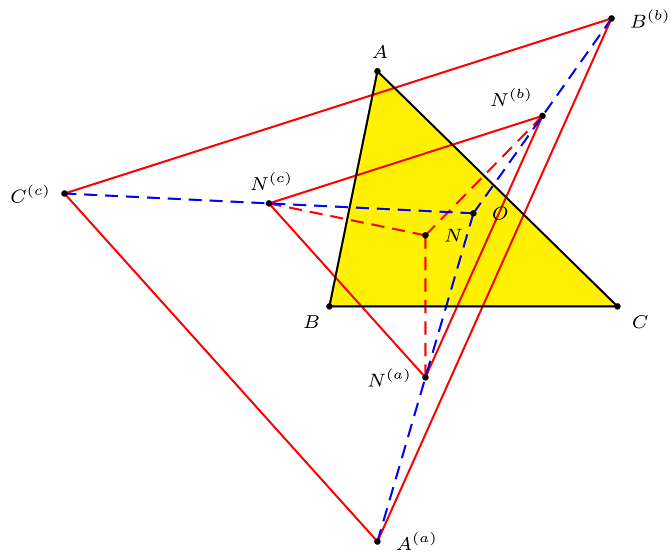


Figure 6. Homothety of triangle of reflections and reflection triangle of N

From this we conclude that
(1) the center of the circle $A^{(a)}B^{(b)}C^{(c)}$ is the point $\mathsf{h}(O,2)(N^*)$, the reflection of O in N^*, which appears as X_{195} in ETC, and
(2) the triangle of reflections is degenerate if and only if the nine-point center N lies on the circumcircle. Here is a simple construction of such a triangle (see Figure 7). Given a point N on a circle $O(R)$, construct

(a) the circle $N\left(\frac{R}{2}\right)$ and choose a point D on this circle, inside the given one (O),

(b) the perpendicular to OD at D to intersect (O) at B and C,

(c) the antipode X of D on the circle (N), and complete the parallelogram $ODXA$ (by translating X by the vector $\mathbf{DO}$).

Then triangle ABC has nine-point center N on its circumcircle. For further results, see [4, p.77], [18] or Proposition 21 below.

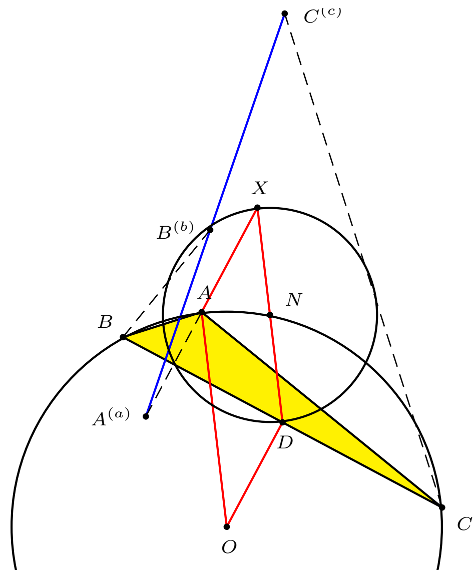


Figure 7. Triangle with degenerate triangle of reflections

2. Perspectivity of reflection triangle

2.1. *Perspectivity with anticevian and orthic triangles.*

Proposition 6 ([10])**.** *The reflection triangle of P is perspective with its anticevian triangle at the cevian quotient $Q = H/P$, which is also the isogonal conjugate of P in the orthic triangle.*

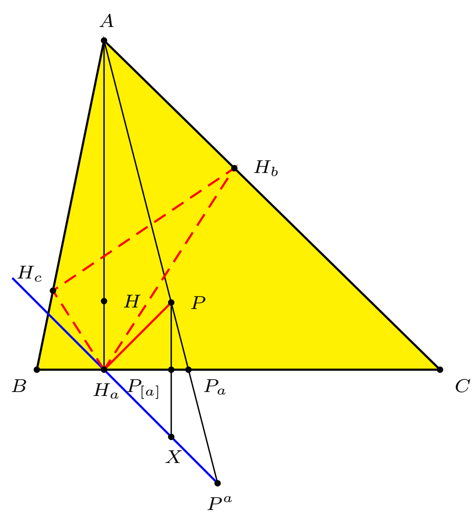


Figure 8. H_aP and $H_aP^{(a)}$ isogonal in orthic triangle

Proof. Let $P_aP_bP_c$ be the cevian triangle of P, and $P^aP^bP^c$ the anticevian triangle. Since P and P^a divide AP_a harmonically, we have $\frac{1}{AP^a}+\frac{1}{AP}=\frac{2}{AP_a}$. If the perpendicular from P to BC intersects the line P^aH_a at X, then

$$\frac{PX}{AH_a}=\frac{PP^a}{AP^a}=\frac{PP_a+P_aP^a}{AP^a}=\frac{PP_a}{AP^a}+\frac{PP_a}{AP}=\frac{2PP_a}{AP_a}=\frac{2PP_{[a]}}{AH_a}.$$

Therefore, $PX=2PP_{[a]}$, and $X=P^{(a)}$. This shows that $P^{(a)}$ lies on the line P^aH_a. Similarly, $P^{(b)}$ and $P^{(c)}$ lie on P^bH_b and P^cH_c respectively. Since the anticevian triangle of P and the orthic triangle are perspective at the cevian quotient H/P, these triangles are perspective with the reflection triangle $P^{(a)}P^{(b)}P^{(c)}$ at the same point.

The fact that $P^{(a)}$ lies on the line H_aP^a means that the lines H_aP^a and H_aP are isogonal lines with respect to the sides H_aH_b and H_aH_c of the orthic triangle; similarly for the pairs H_bP^b, H_bP and H_cP^c, H_cP. It follows that H/P and P are isogonal conjugates in the orthic triangle. □

If $P=(u:v:w)$ in homogeneous barycentric coordinates, then

$$H/P=(u(-S_Au+S_Bv+S_Cw):v(-S_Bv+S_Cw+S_Au):w(-S_Cw+S_Au+S_Bv)).$$

Here are some examples of $(P,\ H/P)$ pairs.

P	I	G	O	H	N	K
H/P	X_{46}	X_{193}	X_{155}	H	X_{52}	X_{25}

2.2. *Perspectivity with the reference triangle.*

Proposition 7. *The reflection triangle of a point P is perspective with ABC if and only if P lies on the Neuberg cubic*

$$\sum_{\text{cyclic}}(S_{AB}+S_{AC}-2S_{BC})u(c^2v^2-b^2w^2)=0. \tag{1}$$

As P traverses the Neuberg cubic, the locus of the perspector Q is the cubic

$$\sum_{\text{cyclic}}\frac{S_Ax}{S^2-3S_{AA}}\left(\frac{y^2}{S^2-3S_{CC}}-\frac{z^2}{S^2-3S_{BB}}\right)=0. \tag{2}$$

The first statement can be found in [30]. The cubic (1) is the famous Neuberg cubic, the isogonal cubic p$K(K,\ E_\infty)$ with pivot the Euler infinity point. It appears as K001 in CTC, where numerous locus properties of the Neuberg cubic can be found; see also [5]. The cubic (2), on the other hand, is the pivotal isocubic p$K(X_{1989},\ X_{265})$, and appears as K060. Given Q on the cubic (2), the point P on the Neuberg cubic can be constructed as the perspector of the cevian and reflection triangles of Q (see Figure 9). Here are some examples of (P,Q) with P on Neuberg cubic and perspector Q of the reflection triangle.

P	O	H	I	W	X_{1157}
Q	N	H	X_{79}	X_{80}	X_{1141}

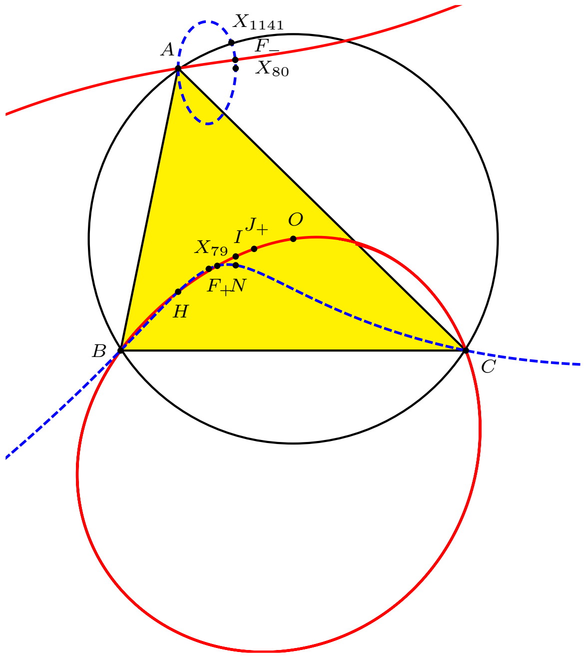


Figure 9. The Neuberg cubic and the cubic K060

Remarks. (1) $X_{79} = \left(\frac{1}{b^2+c^2-a^2+bc} : \frac{1}{c^2+a^2-b^2+ca} : \frac{1}{a^2+b^2-c^2+ab}\right)$ is also the perspector of the triangle formed by the three lines each joining the perpendicular feet of a trace of the incenter on the other two sides (see Figure 10).

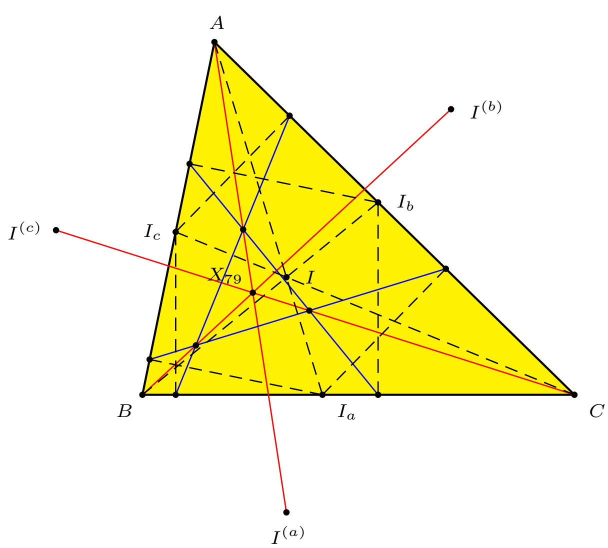


Figure 10. Perspector of reflection triangle of I

(2) For the pair (W, X_{80}),
(i) $W = X_{484} = (a(a^3+a^2(b+c)-a(b^2+bc+c^2)-(b+c)(b-c)^2) : \cdots : \cdots)$ is the first Evans perspector, the perspector of the triangle of reflections $A^{(a)}B^{(b)}C^{(c)}$ and the excentral triangle $I^aI^bI^c$ (see [45]),
(ii) $X_{80} = \left(\frac{1}{b^2+c^2-a^2-bc} : \cdots : \cdots\right)$ is the reflection conjugate of I (see §3 below).
(3) For the pair $(X_{1157},\ X_{1141})$,
(i) $X_{1157} = \left(\frac{a^2(a^6-3a^4(b^2+c^2)+a^2(3b^4-b^2c^2+3c^4)-(b^2-c^2)^2(b^2+c^2))}{a^2(b^2+c^2)-(b^2-c^2)^2} : \cdots : \cdots\right)$ is the inverse of N^* in the circumcircle,
(ii) $X_{1141} = \left(\frac{1}{(S^2+S_{BC})(S^2-3S_{AA})} : \cdots : \cdots\right)$ lies on the circumcircle.

The Neuberg cubic also contains the Fermat points and the isodynamic points. The perspectors of the reflection triangles of
(i) the Fermat points $F_\varepsilon = \left(\frac{1}{\sqrt{3}S_A+\varepsilon S} : \frac{1}{\sqrt{3}S_B+\varepsilon S} : \frac{1}{\sqrt{3}S_C+\varepsilon S}\right)$, $\varepsilon = \pm 1$, are

$$\left(\frac{(S_A+\varepsilon\sqrt{3}S)^2}{(\sqrt{3}S_A+\varepsilon S)^2} : \frac{(S_B+\varepsilon\sqrt{3}S)^2}{(\sqrt{3}S_B+\varepsilon S)^2} : \frac{(S_C+\varepsilon\sqrt{3}S)^2}{(\sqrt{3}S_C+\varepsilon S)^2}\right),$$

(ii) the isodynamic points $J_\varepsilon = \left(a^2(\sqrt{3}S_A+\varepsilon S) : b^2(\sqrt{3}S_B+\varepsilon S) : c^2(\sqrt{3}S_C+\varepsilon S)\right)$, $\varepsilon = \pm 1$, are

$$\left(\frac{1}{(S_A+\varepsilon\sqrt{3}S)(\sqrt{3}S_A+\varepsilon S)} : \frac{1}{(S_B+\varepsilon\sqrt{3}S)(\sqrt{3}S_B+\varepsilon S)} : \frac{1}{(S_C+\varepsilon\sqrt{3}S)(\sqrt{3}S_C+\varepsilon S)}\right).$$

The cubic (2) also contains the Fermat points. For these, the corresponding points on the Neuberg cubic are

$$\left(a^2(2(b^2+c^2-a^2)^3-5(b^2+c^2-a^2)b^2c^2-\varepsilon\cdot 2\sqrt{3}b^2c^2S) : \cdots : \cdots\right).$$

2.3. *Perspectivity with cevian triangle and the triangle of reflections.*

Proposition 8. *The reflection triangle of P is perspective with the triangle of reflections if and only if P lies on the cubic* (2)*. The locus of the perspector Q is the Neuberg cubic* (1)*.*

Proof. Note that $A^{(a)}$, $P^{(a)}$ and P_a are collinear, since they are the reflections of A, P and P_a in BC. Similarly, $B^{(b)}$, $P^{(b)}$, P_b are collinear, so are $C^{(c)}$, $P^{(c)}$, P_c. It follows that the reflection triangle of P is perspective with the triangle of reflections if and only if it is perspective with the cevian triangle of P. □

Remark. The correspondence (P, Q) in Proposition 8 is the inverse of the correspondence in Proposition 7 above.

2.4. *Perspectivity of triangle of reflections and anticevian triangles.*

Proposition 9. *The triangle of reflections is perspective to the anticevian triangle of P if and only if P lies on the Napoleon cubic,* i.e., *the isogonal cubic $pK(K, N)$*

$$\sum_{\text{cyclic}} (a^2(b^2+c^2)-(b^2-c^2)^2)u(c^2v^2-b^2w^2)=0. \tag{3}$$

The locus of the perspector Q is the Neuberg cubic (1).

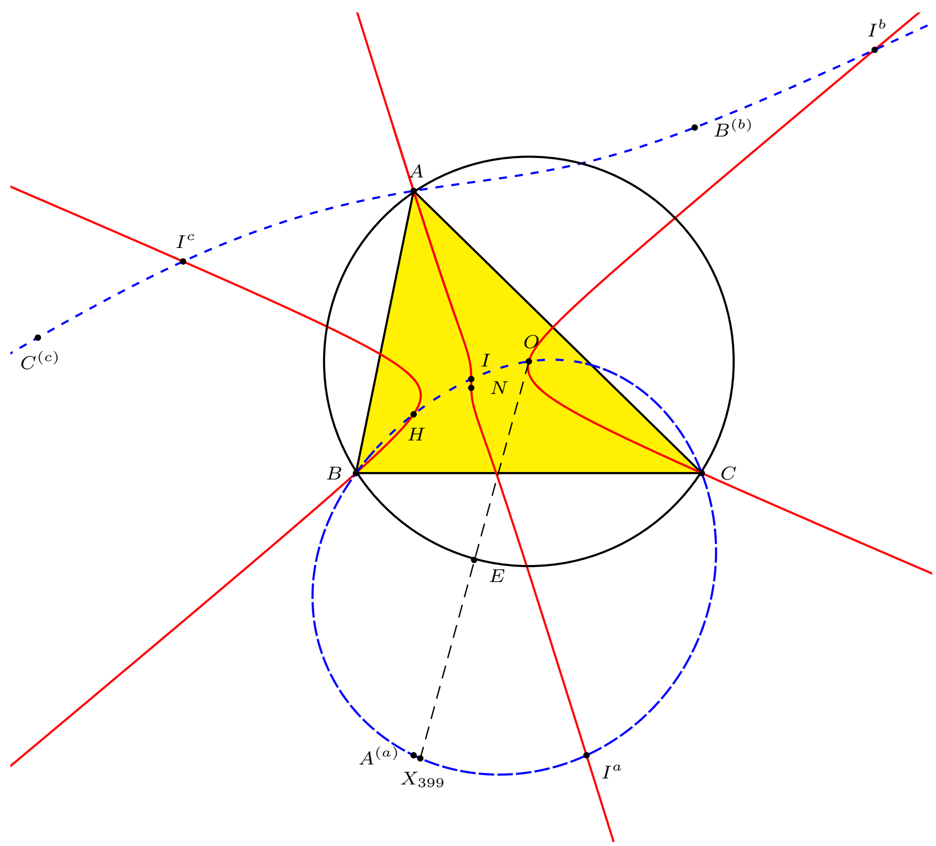


Figure 11. The Napoleon cubic and the Neuberg cubic

P	I	O	N	N^*	X_{195}
Q	W	X_{399}	E_∞	X_{1157}	O

Remarks. (1) For the case of O, the perspector is the Parry reflection point, the triangle center X_{399} which is the reflection of O in the Euler reflection point E. It is also the point of concurrency of reflections in sidelines of lines through vertices parallel to the Euler line (see [34, 35]). In other words, it is the perspector of the triangle of reflections and the cevian triangle of E_∞. The Euler line is the only direction for which these reflections are concurrent.

(2) N^* is the triangle center X_{54} in ETC, called the Kosnita point. It is also the perspector of the centers of the circles OBC, OCA, OAB (see Figure 12).

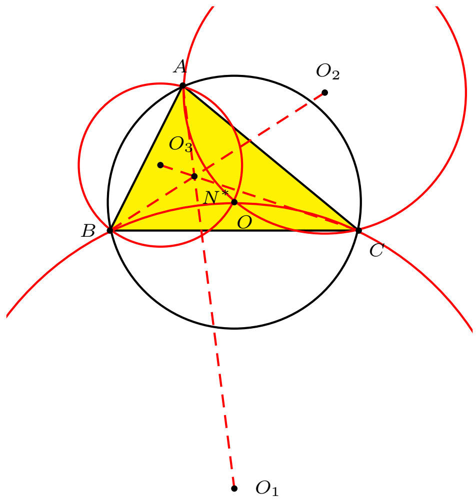


Figure 12. Perspectivity of the centers of the circles OBC, OCA, OAB

3. Reflection conjugates

Proposition 10. *The three circles $P^{(a)}BC$, $P^{(b)}CA$, and $P^{(c)}AB$ have a common point*

$$r_1(P) = \left(\frac{u}{(b^2+c^2-a^2)u(u+v+w)-(a^2vw+b^2wu+c^2uv)} : \cdots : \cdots\right). \tag{4}$$

It is easy to see that $r_1(P) = H$ if and only if P lies on the circumcircle. If $P \neq H$ and P does not lie on the circumcircle, we call $r_1(P)$ the reflection conjugate of P; it is the antipode of P in the rectangular circum-hyperbola $\mathscr{H}(P)$ through P (and the orthocenter H). It also lies on the circle of reflections $P^{(a)}P^{(b)}P^{(c)}$ (see Figure 13).

P	$r_1(P)$	midpoint	hyperbola
I	$X_{80} = \left(\frac{1}{b^2+c^2-a^2-bc} : \cdots : \cdots\right)$	X_{11}	Feuerbach
G	$X_{671} = \left(\frac{1}{b^2+c^2-2a^2} : \cdots : \cdots\right)$	X_{115}	Kiepert
O	$X_{265} = \left(\frac{S_A}{S^2-3S_{AA}} : \cdots : \cdots\right)$	X_{125}	Jerabek
K	$X_{67} = \left(\frac{1}{b^4+c^4-a^4-b^2c^2} : \cdots : \cdots\right)$	X_{125}	Jerabek
X_7	$X_{1156} = \left(\frac{a}{-2a^2+a(b+c)+(b-c)^2} : \cdots : \cdots\right)$	X_{11}	Feuerbach
X_8	$X_{1320} = \left(\frac{a(b+c-a)}{b+c-2a} : \cdots : \cdots\right)$	X_{11}	Feuerbach
X_{13}	X_{14}	X_{115}	Kiepert

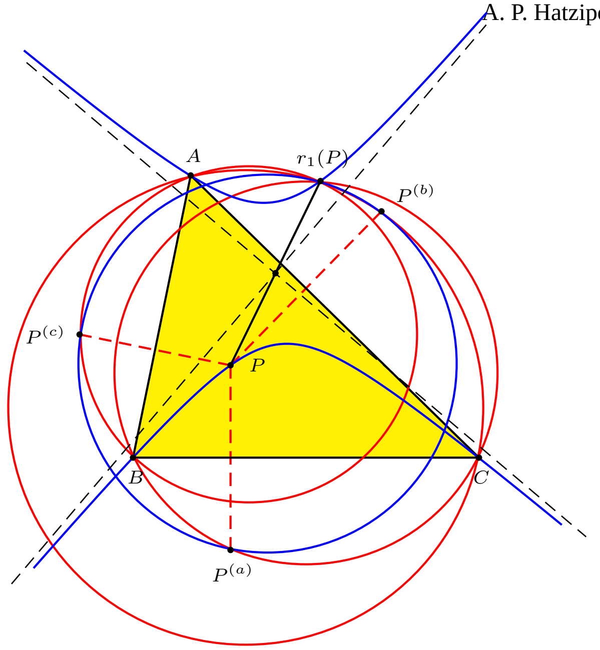


Figure 13. $r_1(P)$ and P are antipodal in $\mathscr{H}(P)$

Remark. $r_1(I) = X_{80}$ is also the perspector of the reflections of the excenters in the respective sidelines (see [42] and Figure 14). In §2.2, we have shown that $r_1(I)$ is the perspector of the reflection triangle of W.

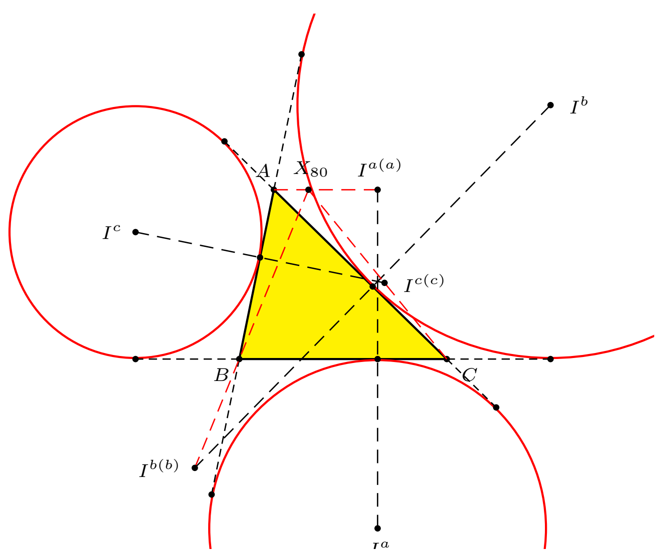


Figure 14. $r_1(I)$ as perspector of reflections of excenters

Proposition 11. *Let $P^{[a]}P^{[b]}P^{[c]}$ be the antipedal triangle of $P = (u : v : w)$. The reflections of the circles $P^{[a]}BC$ in BC, $P^{[b]}CA$ in CA and $P^{[c]}AB$ in AB all contain the reflection conjugate $r_1(P)$.*

Proof. Since B, P, C, and $P^{[a]}$ are concyclic, so are their reflections in the line BC. The circle $P^{[a]}BC$ is identical with the reflection of the circle $P^{(a)}BC$ in BC; similarly for the other two circles. The triad of circles therefore have $r_1(P)$ for a common point. □

Proposition 12. *Let $P_{[a]}P_{[b]}P_{[c]}$ be the pedal triangle of $P = (u : v : w)$. The reflections of the circles $AP_{[b]}P_{[c]}$ in $P_{[b]}P_{[c]}$, $BP_{[c]}P_{[a]}$ in $P_{[c]}P_{[a]}$, and $CP_{[a]}P_{[b]}$ in $P_{[a]}P_{[b]}$ have a common point*

$$\begin{aligned} r_2(P) = \; & (a^2(2a^2b^2c^2u + c^2((a^2+b^2-c^2)^2 - 2a^2b^2)v + b^2((c^2+a^2-b^2)^2 - 2c^2a^2)w) \\ & \cdot (b^2c^2u^2 - c^2(c^2-a^2)uv + b^2(a^2-b^2)uw - a^2(b^2+c^2-a^2)vw) : \cdots : \cdots). \end{aligned}$$

P	$r_2(P)$
G	$(a^2(b^4+c^4-a^4-b^2c^2)(a^4(b^2+c^2)-2a^2(b^4-b^2c^2+c^4)+(b^2+c^2)(b^2-c^2)^2)$ $: \cdots : \cdots)$
I	$(a(b^2+c^2-a^2-bc)(a^3(b+c)-a^2(b^2+c^2)-a(b+c)(b-c)^2+(b^2-c^2)^2)$ $: \cdots : \cdots)$
O	circles coincide with nine-point circle
H	$X_{1986} = \left(\frac{a^2((b^2+c^2-a^2)^2-b^2c^2)(a^4(b^2+c^2)-2a^2(b^4-b^2c^2+c^4)+(b^2+c^2)(b^2-c^2)^2)}{b^2+c^2-a^2} : \cdots : \cdots\right)$
X_{186}	X_{403}

Remarks. (1) For the case of $(H,\ X_{1986})$, see [22].

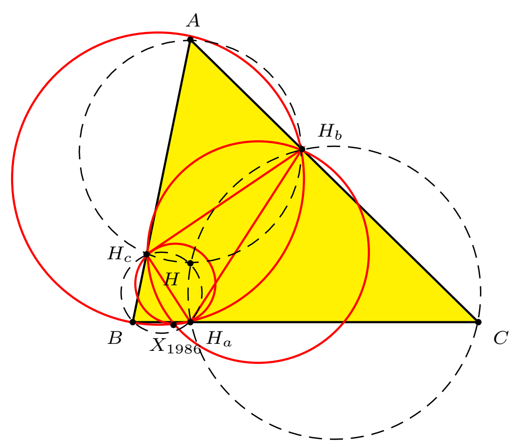


Figure 15. X_{1986} as the common point of reflections of circumcircles of residuals of orthic triangle

(2) For the pair $(X_{186},\ X_{403})$,
(i) X_{186} is the inverse of H in the circumcircle,
(ii) X_{403} is the inverse of H in the nine-point circle (see §4 below).

4. Inversion in the circumcircle

The inverse of P in the circumcircle is the point

$$\begin{aligned} P^{-1} = & \ (a^2(b^2c^2u^2 + b^2(a^2-b^2)wu + c^2(a^2-c^2)uv - a^2(b^2+c^2-a^2)vw) \\ : & \ \ b^2(c^2a^2v^2 + a^2(b^2-a^2)vw - b^2(c^2+a^2-b^2)wu + c^2(b^2-c^2)uv) \\ : & \ \ c^2(a^2b^2w^2 + a^2(c^2-a^2)vw + b^2(c^2-b^2)wu - c^2(a^2+b^2-c^2)uv)). \end{aligned}$$

4.1. *Bailey's theorem.*

Theorem 13 (Bailey [1, Theorem 5])**.** *The isogonal conjugates of P and $r_1(P)$ are inverse in the circumcircle.*

Proof. Let $P = (u : v : w)$, so that $P^* = (a^2vw : b^2wu : c^2uv)$. From the above formula,

$$\begin{aligned} (P^*)^{-1} = & \ \left(a^2vw(a^2vw + (a^2-b^2)uv + (a^2-c^2)wu - (b^2+c^2-a^2)u^2) : \cdots : \cdots\right) \\ = & \ \left(a^2vw(-(b^2+c^2-a^2)u(u+v+w) + a^2vw + b^2wu + c^2uv) : \cdots : \cdots\right). \end{aligned}$$

This clearly is the isogonal conjugate of $r_1(P)$ by a comparison with (4). $\square$

4.2. *The inverses of $A^{(a)}$, $B^{(b)}$, $C^{(c)}$.*

Proposition 14. *The inversive images of $A^{(a)}$, $B^{(b)}$, $C^{(c)}$ in the circumcircle are perspective with ABC at N^*.*

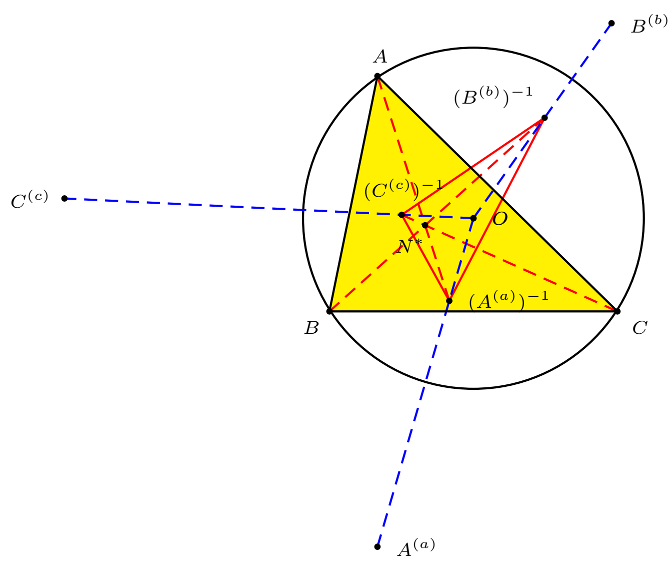


Figure 16. N^* as perspector of inverses of reflections of vertices in opposite sides

Proof. These inversive images are

$$\begin{aligned}
(A^{(a)})^{-1} &= (-a^2(S^2 - 3S_{AA}) : b^2(S^2 + S_{AB}) : c^2(S^2 + S_{CA})), \\
(B^{(b)})^{-1} &= (a^2(S^2 + S_{AB}) : -b^2(S^2 - 3S_{BB}) : c^2(S^2 + S_{BC})), \\
(C^{(c)})^{-1} &= (a^2(S^2 + S_{CA}) : b^2(S^2 + S_{BC}) : -c^2(S^2 - 3S_{CC})).
\end{aligned}$$

From these, the triangles ABC and $(A^{(a)})^{-1}(B^{(b)})^{-1}(C^{(c)})^{-1}$ are perspective at

$$N^* = \left(\frac{a^2}{S^2 + S_{BC}} : \frac{b^2}{S^2 + S_{CA}} : \frac{c^2}{S^2 + S_{AB}}\right).$$

□

Corollary 15 (Musselman [32]). *The circles $AOA^{(a)}$, $BOB^{(b)}$, $COC^{(c)}$ are coaxial with common points O and $(N^*)^{-1}$.*

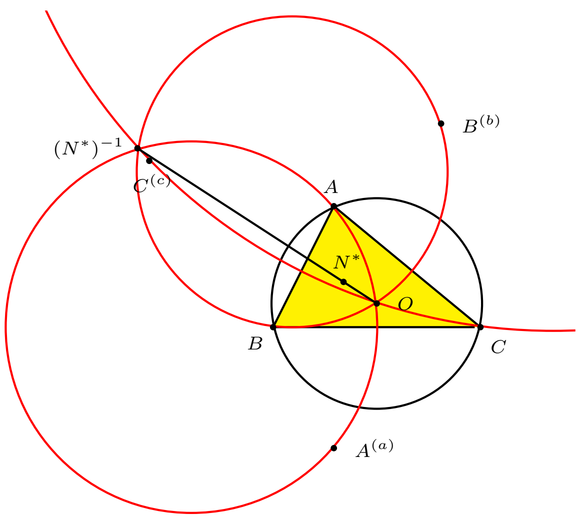


Figure 17. Coaxial circles $APA^{(a)}$, $BOB^{(a)}$, $COC^{(c)}$

Proof. Invert the configuration in Proposition 14 in the circumcircle. □

A generalization of Corollary 15 is the following.

Proposition 16 (van Lamoen [28]). *The circles $APA^{(a)}$, $BPB^{(b)}$ and $CPC^{(c)}$ are coaxial if and only if P lies on the Neuberg cubic.*

Remarks. (1) Another example is the pair (I, W).

(2) If P is a point on the Neuberg cubic, the second common point of the circles $APA^{(a)}$, $BPB^{(b)}$ and $CPC^{(c)}$ is also on the same cubic.

4.3. *Perspectivity of inverses of cevian and anticevian triangles.*

Proposition 17. *The inversive images of P_a, P_b, P_c in the circumcircle form a triangle perspective with ABC if and only if P lies on the circumcircle or the Euler line.*

(a) *If P lies on the circumcircle, the perspector is the isogonal conjugate of the inferior of P. The locus is the isogonal conjugate of the nine-point circle* (see Figure 18).

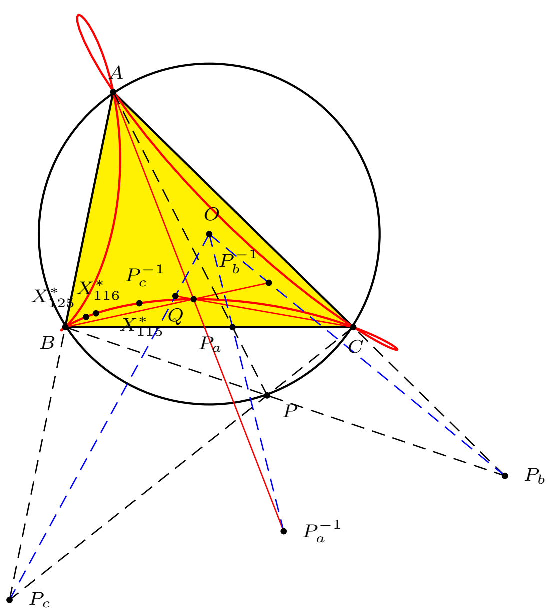


Figure 18. Isogonal conjugate of the nine-point circle

(b) *If P lies on the Euler line, the locus of the perspector is the bicevian conic through the traces of the isogonal conjugates of the Kiepert and Jerabek centers* (see Figure 19).

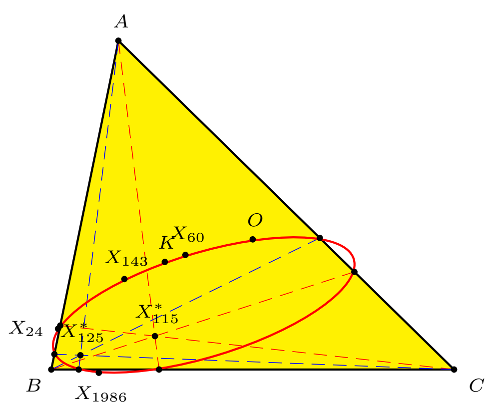


Figure 19. The bicevian conic $\mathscr{C}(X_{115}^*,\ X_{125}^*)$

The conic in Proposition 17(b) has equation

$$\sum_{\text{cyclic}} b^4c^4(b^2-c^2)^4(b^2+c^2-a^2)x^2 - 2a^6b^2c^2(c^2-a^2)^2(a^2-b^2)^2yz = 0.$$

P	O	G	H	N	X_{21}	H^{-1}
Q	O	K	X_{24}	X_{143}	X_{60}	X_{1986}

Remarks. (1) X_{21} is the Schiffler point, the intersection of the Euler lines of IBC, ICA, IAB (see [21]). Here is another property of X_{21} relating to reflections discovered by L. Emelyanov [11]. Let X be the reflection of the touch point of A-excircle in the line joining the other two touch points; similarly define Y and Z. The triangles ABC and XYZ are perspective at the Schiffler point (see Figure 20).

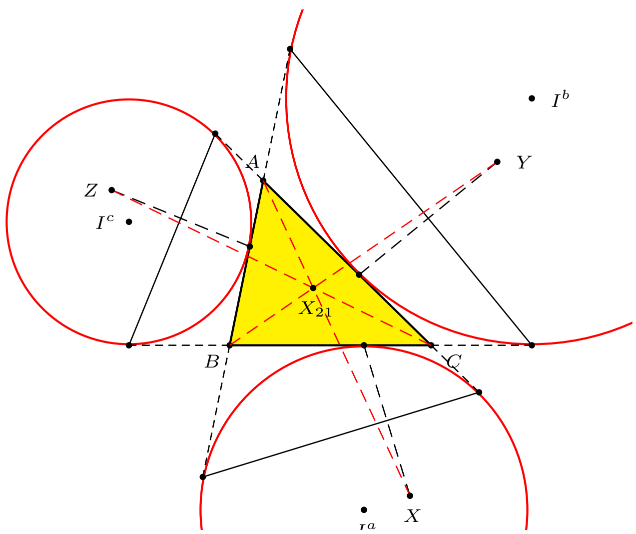


Figure 20. Schiffler point and reflections

(2) $X_{24} = \left(\frac{a^2(S_{AA}-S^2)}{S_A} : \frac{b^2(S_{BB}-S^2)}{S_B} : \frac{c^2(S_{CC}-S^2)}{S_C}\right)$ is the perspector of the orthic-of-orthic triangle (see [26]).

(3) X_{143} is the nine-point center of the orthic triangle.

(4) $X_{60} = \left(\frac{a^2(b+c-a)}{(b+c)^2} : \frac{b^2(c+a-b)}{(c+a)^2} : \frac{c^2(a+b-c)}{(a+b)^2}\right)$ is the isogonal conjugate of the outer Feuerbach point X_{12}.

Proposition 18. *The inversive images of P^a, P^b, P^c in the circumcircle form a triangle perspective with ABC if and only if P lies on*
(1) *the isogonal conjugate of the circle $S_Ax^2 + S_By^2 + S_Cz^2 = 0$, or*
(2) *the conic*

$$b^2c^2(b^2-c^2)x^2 + c^2a^2(c^2-a^2)y^2 + a^2b^2(a^2-b^2)z^2 = 0.$$

Remarks. (1) The circle $S_A x^2 + S_B y^2 + S_C z^2 = 0$ is real only when ABC contains an obtuse angle. In this case, it is the circle with center H orthogonal to the circumcircle.

(2) The conic in (2) is real only when ABC is acute. It has center E and is homothetic to the Jerabek hyperbola, with ratio $\sqrt{\frac{1}{2\cos A\cos B\cos C}}$.

5. Dual triads of concurrent circles

Proposition 19. *Let* $\begin{matrix} X, & Y, & Z \\ X', & Y' & Z' \end{matrix}$ *be two triads of points. The triad of circles* $XY'Z'$, $YZ'X'$ *and* $ZX'Y'$ *have a common point if and only if the triad of circles* $X'YZ$, $Y'ZX$ *and* $Z'XY$ *have a common point.*

Proof. Let Q be a common point of the triad of circles $XY'Z'$, $YZ'X'$, $ZX'Y'$. Inversion with respect to a circle, center Q transforms the six points X, Y, Z, X', Y', Z' into x, y, z, x', y', z' respectively. Note that $xy'z'$, $yz'x'$ and $zx'y'$ are lines bounding a triangle $x'y'z'$. By Miquel's theorem, the circles $x'yz$, $y'zx$ and $z'xy$ have a common point q'. Their inverses $X'YZ$, $Y'ZX$ and $Z'XY$ have the inverse Q' of q' as a common point. □

Proposition 20 (Musselman [31])**.** *The circles* $AP^{(b)}P^{(c)}$, $BP^{(c)}P^{(a)}$, $CP^{(a)}P^{(b)}$ *intersect at the point* $r_0(P)$ *on the circumcircle.*

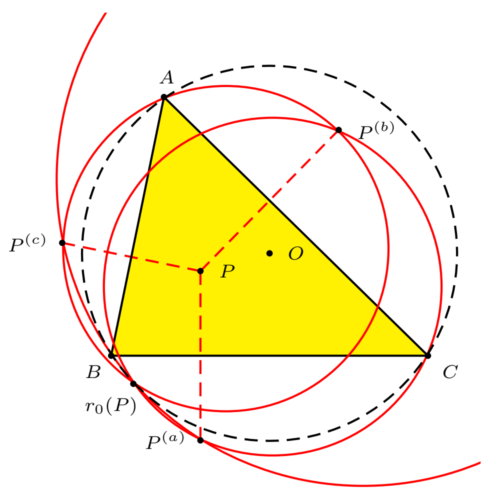


Figure 21. The circles $AP^{(b)}P^{(c)}$, $BP^{(c)}P^{(a)}$, $CP^{(a)}P^{(b)}$ intersect on the circumcircle

5.1. *Circles containing* $A^{(a)}$, $B^{(b)}$, $C^{(c)}$.

5.1.1. *The triad of circles* $AB^{(b)}C^{(c)}$, $A^{(a)}BC^{(c)}$, $A^{(a)}B^{(b)}C$. Since the circles $A^{(a)}BC$, $AB^{(b)}C$ and $ABC^{(c)}$ all contain the orthocenter H, it follows that that the circles $AB^{(b)}C^{(c)}$, $A^{(a)}BC^{(c)}$ and $A^{(a)}B^{(b)}C$ also have a common point. This is the point $X_{1157} = (N^*)^{-1}$ (see [41, 18]). The radical axes of the circumcircle with each of these circles bound the anticevian triangle of N^* (see Figure 22).

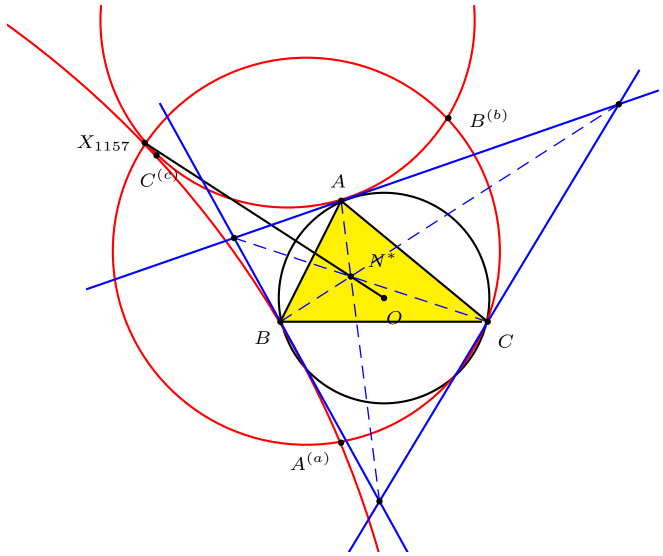


Figure 22. Concurrency of circles $AB^{(b)}C^{(c)}$, $A^{(a)}BC^{(c)}$, $A^{(a)}B^{(b)}C$

5.1.2. *The tangential triangle* . The circles $K^aB^{(b)}C^{(c)}$, $A^{(a)}K^bC^{(c)}$, $A^{(a)}B^{(b)}K^c$ have X_{399} the Parry reflection point as a common point. On the other hand, the circles $A^{(a)}K^bK^c$, $B^{(b)}K^cK^a$, $C^{(c)}K^aK^b$ are concurrent. (see [35]).

5.1.3. *The excentral triangle* . The circles $A^{(a)}I^bI^c$, $I^aB^{(b)}I^c$, $I^aI^bC^{(c)}$ also have the Parry reflection point X_{399} as a common point (see Figure 23).

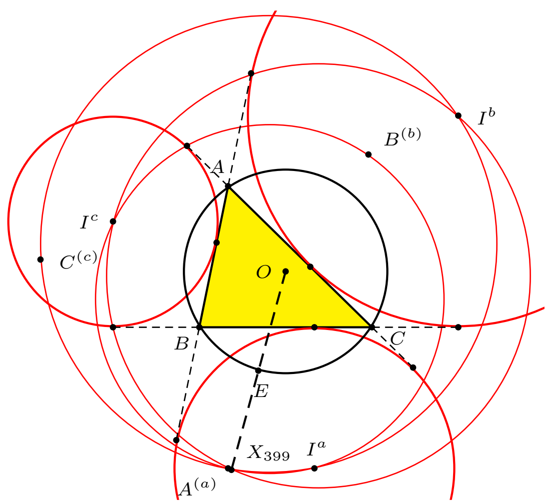


Figure 23. The Parry reflection point X_{399}

The Parry reflection point X_{399}, according to Evans [12], is also the common point of the circles $II^aA^{(a)}$, $II^bB^{(b)}$ and $II^cC^{(c)}$.

By Proposition 19, the circles $I^aB^{(b)}C^{(c)}$, $A^{(a)}I^bC^{(c)}$ and $A^{(a)}B^{(b)}I^c$ have a common point as well. Their centers are perspective with ABC at the point

$$(a(a^2(a+b+c)-a(b^2-bc+c^2)-(b+c)(b-c)^2):\cdots:\cdots)$$

on the OI line.

5.1.4. *Equilateral triangles on the sides .* For $\varepsilon = \pm 1$, let A_ε, B_ε, C_ε be the apices of the equilateral triangles erected on the sides BC, CA, AB of triangle ABC respectively, on opposite or the same sides of the vertices according as $\varepsilon = 1$ or -1. Now, for $\varepsilon = \pm 1$, the circles $A^{(a)}B_\varepsilon C_\varepsilon$, $B^{(b)}C_\varepsilon A_\varepsilon$, $C^{(c)}A_\varepsilon B_\varepsilon$ are concurrent at the superior of the Fermat point $F_{-\varepsilon}$ (see [36]).

5.1.5. *Degenerate triangle of reflections .*

Proposition 21 ([18, Theorem 4])**.** *Suppose the nine-point center N of triangle ABC lies on the circumcircle.*
(1) *The reflection triangle $A^{(a)}B^{(b)}C^{(c)}$ degenerates into a line $\mathcal{L}$.*
(2) *If X, Y, Z are the centers of the circles BOC, COA, AOB, the lines AX, BY, CZ are all perpendicular to $\mathcal{L}$.*
(3) *The circles $AOA^{(a)}$, $BOB^{(b)}$, $COC^{(c)}$ are mutually tangent at O. The line joining their centers is the parallel to $\mathcal{L}$ through O.*
(4) *The circles $AB^{(b)}C^{(c)}$, $BC^{(c)}A^{(a)}$, $CA^{(a)}B^{(b)}$ pass through O.*

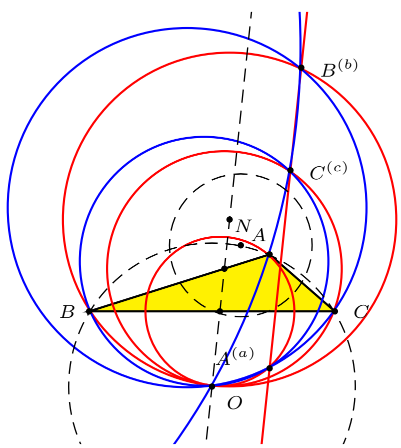


Figure 24. Triangle with degenerate triangle of reflections

5.2. *Reflections in a point.*

Proposition 22. *Given $P = (u : v : w)$, let X, Y, Z be the reflections of A, B, C in P.*

(a) *The circles AYZ, BZX, CXY have a common point a point*

$$r_3(P) = \left(\frac{1}{c^2v(w+u-v)-b^2w(u+v-w)}:\cdots:\cdots\right)$$

which is also the fourth intersection of the circumcircle and the circumconic with center P (see Figure 25).

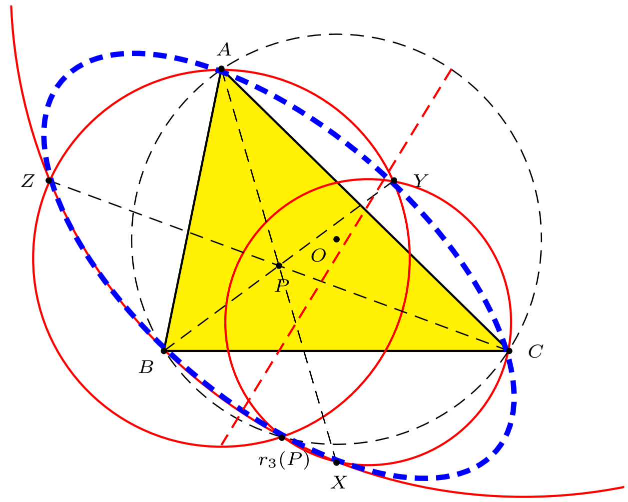


Figure 25. Circles AYZ, BZX, CXY through $r_3(P)$ on circumcircle and circumconic with center P

(b) *The circles* XBC, YCA *and* ZAB *intersect have a common point*

$$r_4(P) = \left(\frac{v+w-u}{2a^2vw-(v+w-u)(bw+cv)} : \cdots : \cdots \right)$$

which is the antipode of $r_3(P)$ *on the circumconic with center* P (see Figure 26). *It is also the reflection conjugate of the superior of* P.

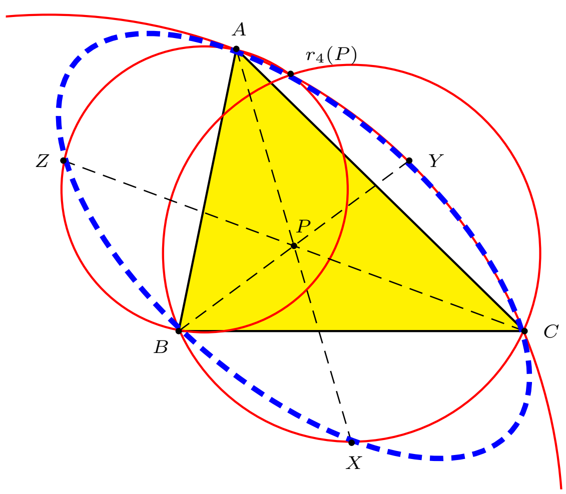


Figure 26. Circles XBC, YCA, ZAB through $r_4(P)$ circumconic with center P

(c) *For a given Q on the circumcircle, the locus of P for which $r_3(P) = Q$ is the bicevian conic $\mathscr{C}(G, Q)$.*

Here are some examples of $r_3(P)$ and $r_4(P)$.

P	G	I	N	K	X_9	X_{10}	X_{2482}	X_{214}	X_{1145}
$r_3(P)$	X_{99}	X_{100}	E	E	X_{100}	X_{100}	X_{99}	X_{100}	X_{100}
$r_4(P)$	$r_1(G)$	X_{1320}	$r_1(O)$	X_{895}	X_{1156}	X_{80}	G	I	N_a

6. Reflections and Miquel circles

6.1. *The reflection of I in O.* If X, Y, Z are the points of tangency of the excircles with the respective sides, the Miquel point of the circles AYZ, BZX, CXY is the reflection of I in O, which is X_{40} in ETC. It is also the circumcenter of the excentral triangle.

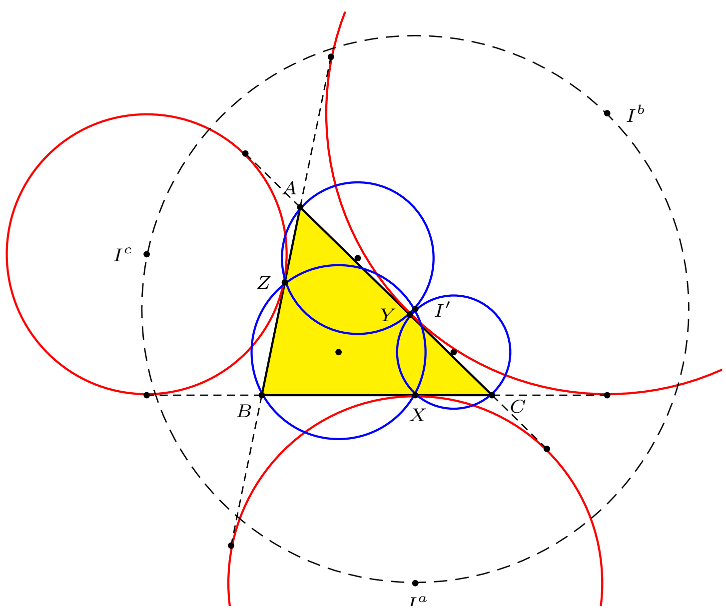


Figure 27. Reflection of I in O as a Miquel point

6.2. *Miquel circles.* For a real number t, we consider the triad of points

$$X_t = (0 : 1 - t : t), \quad Y_t = (t : 0 : 1 - t), \quad Z_t = (1 - t : t : 0)$$

on the sides of the reference triangle. The circles AY_tZ_t, BZ_tX_t and CX_tY_t intersect at the Miquel point

$$\begin{aligned} M_t = \ & \big(a^2(b^2t^2 + c^2(1-t)^2 - a^2t(1-t)) \\ & : \ b^2(c^2t^2 + a^2(1-t)^2 - b^2t(1-t)) \\ & : \ c^2(a^2t^2 + b^2(1-t)^2 - c^2t(1-t))\big) . \end{aligned}$$

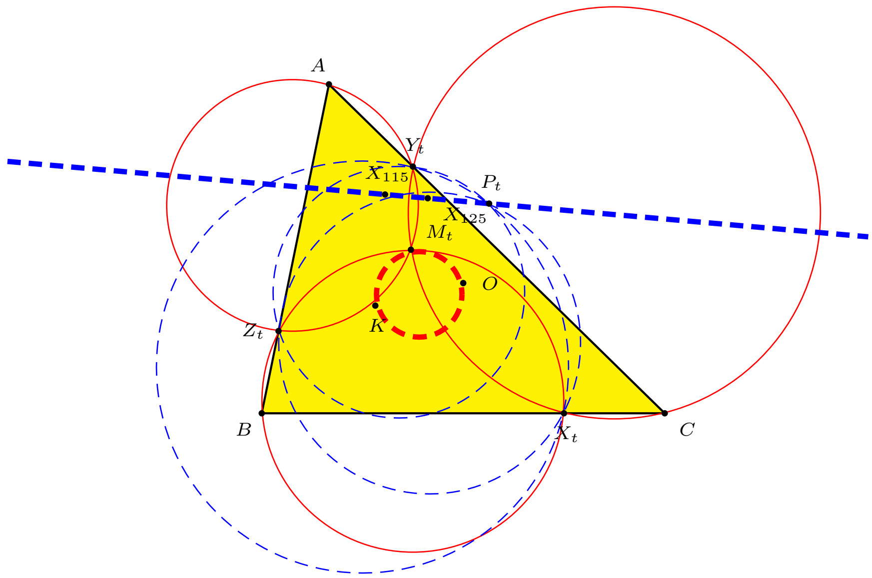


Figure 28. Miquel circles and their reflections

The locus of M_t is the Brocard circle with diameter OK, as is evident from the data in the table below; see Figure 28 and [37, 17].

t	M_t	P_t
0	$\Omega = \frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}$	$\frac{1}{c^2-a^2} : \frac{1}{a^2-b^2} : \frac{1}{b^2-c^2}$
$\frac{1}{2}$	O	$X_{115} = ((b^2-c^2)^2 : (c^2-a^2)^2 : (a^2-b^2)^2)$
1	$\Omega' = \frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}$	$\frac{1}{a^2-b^2} : \frac{1}{b^2-c^2} : \frac{1}{c^2-a^2}$
∞	K	$((b^2-c^2)(b^2+c^2-2a^2) : \cdots : \cdots)$
$\frac{a^2b^2-c^4}{(b^2-c^2)(a^2+b^2+c^2)}$	$B_1 = a^2 : c^2 : b^2$	$-(b^4-c^4) : b^2(c^2-a^2) : c^2(a^2-b^2)$
$\frac{c^2a^2-b^4}{(c^2-a^2)(a^2+b^2+c^2)}$	$B_2 = c^2 : b^2 : a^2$	$a^2(b^2-c^2) : -(c^4-a^4) : c^2(a^2-b^2)$
$\frac{a^2b^2-c^4}{(a^2-b^2)(a^2+b^2+c^2)}$	$B_3 = b^2 : a^2 : c^2$	$a^2(b^2-c^2) : b^2(c^2-a^2) : -(a^4-b^4)$

6.3. *Reflections of Miquel circles.* Let A_t, B_t, C_t be the reflections of A in Y_tZ_t, B in Z_tX_t, C in X_tY_t. The circles $A_tY_tZ_t$, $B_tZ_tX_t$ and $C_tX_tY_t$ also have a common point

$$\begin{aligned} P_t = \; & ((b^2-c^2)((c^2-a^2)t+(a^2-b^2)(1-t)) \\ & : (c^2-a^2)((a^2-b^2)t+(b^2-c^2)(1-t)) \\ & : (a^2-b^2)((b^2-c^2)t+(c^2-a^2)(1-t))). \end{aligned}$$

For $t=\frac{1}{2}$, all three reflections coincide with the nine-point circle. However, P_t approaches the Kiepert center $X_{115} = ((b^2-c^2)^2 : (c^2-a^2)^2 : (a^2-b^2)^2)$ as $t \to \frac{1}{2}$. The locus of P_t is the line

$$\frac{x}{b^2-c^2} + \frac{y}{c^2-a^2} + \frac{z}{a^2-b^2} = 0,$$

which clearly contains both the Kiepert center X_{115} and the Jerabek center X_{125} (see Figure 28). This line is the radical axis of the nine-point circle and the pedal circle of G. These two centers are the common points of the two circles (see Figure 29).

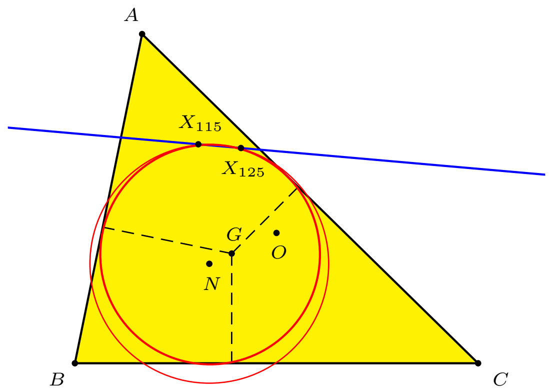


Figure 29. X_{115} and X_{125} as the intersections of nine-point circle and pedal circle of G

6.4. *Reflections of circles of anticevian residuals.* Consider points X^t, Y^t, Z^t such that A, B, C divide Y^tZ^t, Z^tX^t, X^tY^t respectively in the ratio $1-t:t$. Figure 30 shows the construction of these points from X_t, Y_t, Z_t and the midpoints of the sides. Explicitly,

$$\begin{aligned} X^t &= (-t(1-t):(1-t)^2:t^2), \\ Y^t &= (t^2:-t(1-t):(1-t)^2), \\ Z^t &= ((1-t)^2:t^2:-t(1-t)). \end{aligned}$$

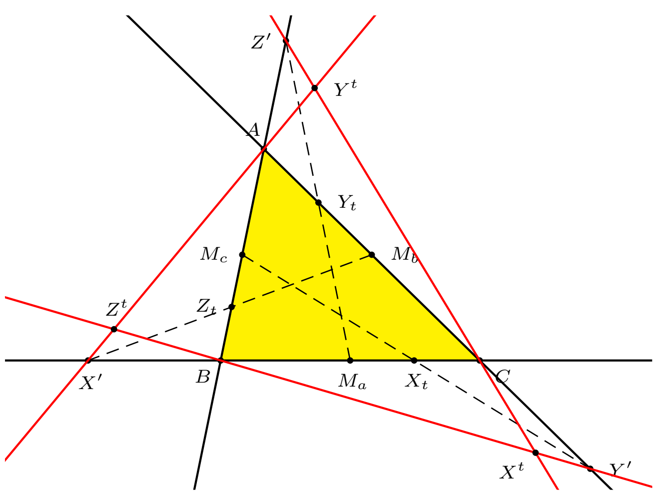


Figure 30. Construction of $X^tY^tZ^t$ from $X_tY_tZ_t$

The circles X^tBC, Y^tCA, Z^tAB intersect at the isogonal conjugate of M_t. The locus of the intersection is therefore the isogonal conjugate of the Brocard circle. On the other hand, the reflections of the circles X_tBC, Y_tCA, Z_tAB intersect at the point

$$\left(\frac{1}{(b^2+c^2-2a^2)t+(a^2-b^2)}:\frac{1}{(c^2+a^2-b^2)t+(b^2-c^2)}:\frac{1}{(a^2+b^2-c^2)t+(c^2-a^2)}\right),$$

which traverses the Steiner circum-ellipse.

7. Reflections of a point in various triangles

7.1. *Reflections in the medial triangle.* If $P = (u : v : w)$, the reflections in the sides of the medial triangle are

$$\begin{aligned} X' = \; & ((S_B+S_C)(v+w) : S_Bv - S_C(w-u) : S_Cw + S_B(u-v)), \\ Y' = \; & (S_Au + S_C(v-w) : (S_C+S_A)(w+u) : S_Cw - S_A(u-v)), \\ Z' = \; & (S_Au - S_B(v-w) : S_Bv + S_A(w-u) : (S_A+S_B)(u+v)). \end{aligned}$$

Proposition 23. *The reflection triangle of P in the medial triangle is perspective with ABC if and only if P lies on the Euler line or the nine-point circle of ABC.*

(a) *If P lies on the Euler line, the perspector traverses the Jerabek hyperbola.*

(b) *If P lies on the nine-point circle, the perspector is the infinite point which is the isogonal conjugate of the superior of P.*

Remarks. (1) If $P = E_t$, then the perspector $Q = E^*_{t'}$, where

$$t' = \frac{a^2b^2c^2(1-t)}{a^2b^2c^2(1-t) - 4S_{ABC}(1-2t)}.$$

P	G	O	H	N	X_{25}	X_{403}	X_{427}	X_{429}	X_{442}	E_∞
Q	$H^\bullet$	X_{68}	H	O	X_{66}	X_{74}	K	X_{65}	X_{72}	X_{265}

(2) For $P = G$, these reflections are the points

$$X' = (2a^2 : S_B : S_C), \quad Y' = (S_A : 2b^2 : S_C), \quad Z' = (S_A : S_B : 2c^2).$$

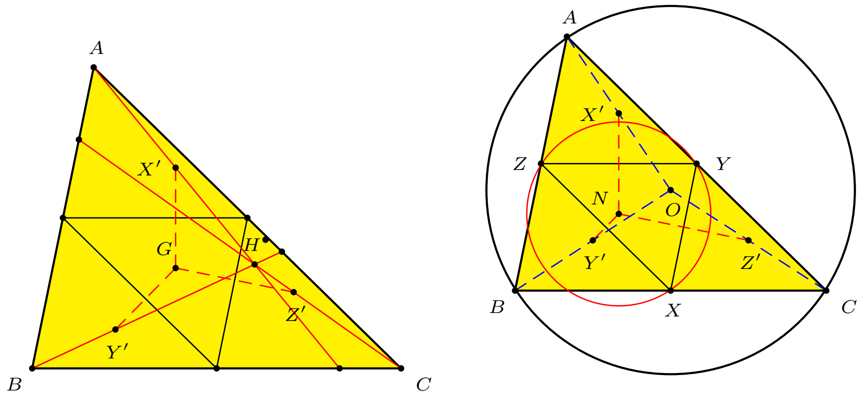
They are trisection points of the corresponding $H^\bullet-$cevian (see Figure 31(a)). The perspector of $X'Y'Z'$ is $X_{69} = H^\bullet$.

(3) If $P = N$, the circumcenter of the medial triangle, the circle through its reflections in the sides of the medial triangle is congruent to the nine-point circle and has center at the orthocenter of the medial triangle, which is the circumcenter O of triangle ABC. These reflections are therefore the midpoints of the circumradii OA, OB, OC (see Figure 31(b)).

(4) $X_{25} = \left(\frac{a^2}{b^2+c^2-a^2} : \frac{b^2}{c^2+a^2-b^2} : \frac{c^2}{a^2+b^2-c^2}\right)$ is the homothetic center of the tangential and orthic triangles. It is also the perspector of the tangential triangle and the reflection triangle of K. In fact,

$$A'X' : X'H_a = a^2 : S_A, \quad B'Y' : Y'H_b = b^2 : S_B, \quad C'Z' : Z'H_c = c^2 : S_C.$$

(5) $X_{427} = \left(\frac{b^2+c^2}{b^2+c^2-a^2} : \frac{c^2+a^2}{c^2+a^2-b^2} : \frac{a^2+b^2}{a^2+b^2-c^2}\right)$ is the inverse of X_{25} in the orthocentroidal circle. It is also the homothetic center of the orthic triangle and the



(a) Reflections of G (b) Reflections of N

Figure 31. Reflections in the medial triangle

triangle bounded by the tangents to the nine-point circle at the midpoints of the sidelines (see [7]).

(6) If P is on the nine-point circle, it is the inferior of a point P' on the circumcircle. In this case, the perspector Q is the infinite point which is the isogonal conjugate of P'. In particular, for the Jerabek center $J = X_{125}$ (which is the inferior of the Euler reflection point $E = X_{110}$), the reflections are the pedals of the vertices on the Euler line. The perspector is the infinite point of the perpendicular to the Euler line (see Figure 32).

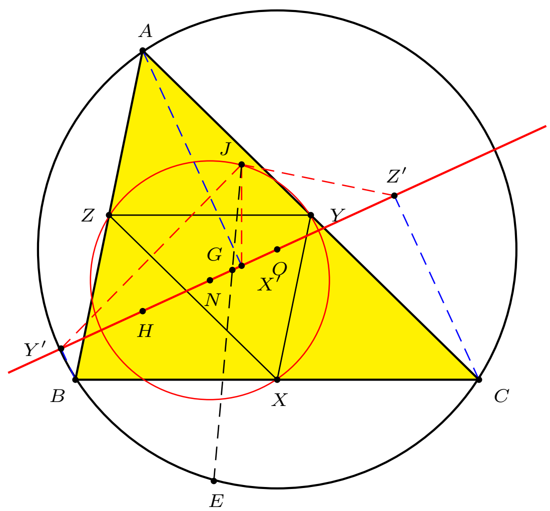


Figure 32. Reflections of Jerabek center in medial triangle

Proposition 24. *The reflections of AP, BP, CP in the respective sidelines of the medial triangle are concurrrent* (i.e., triangle $X'Y'Z'$ is perspective with the orthic triangle) *if and only if P lies on the Jerabek hyperbola of ABC. As P traverses the Jerabek hyperbola, the locus of the perspector is the Euler line* (see Figure 33).

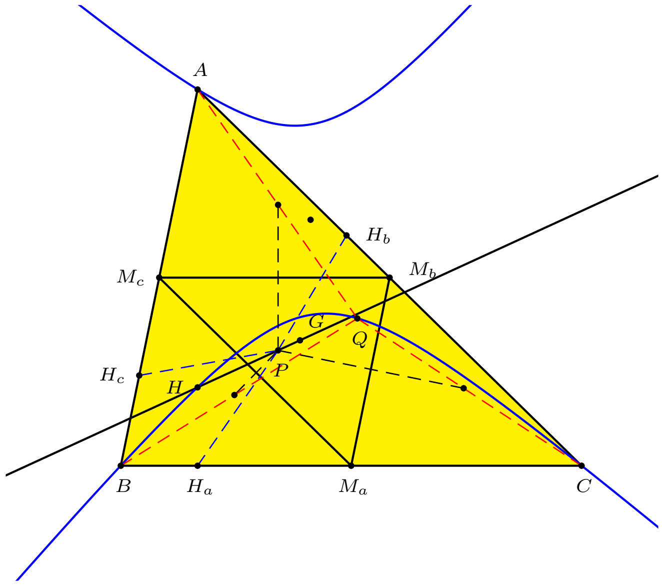


Figure 33. Reflections in medial triangle

Remark. The correspondence is the inverse of the correspondence in Proposition 23(a).

7.2. *Reflections in the orthic triangle.*

Proposition 25. *The reflection triangle of P in the orthic triangle $H_aH_bH_c$ is perspective with ABC if and only if P lies on the cubic*

$$\sum_{\text{cyclic}} \frac{u}{b^2+c^2-a^2}\left(f(c,a,b)v^2 - f(b,c,a)w^2\right) = 0. \tag{5}$$

where

$$f(a,b,c) = a^4(b^2+c^2) - 2a^2(b^4-b^2c^2+c^4) + (b^2+c^2)(b^2-c^2)^2.$$

The locus of the perspector Q is the cubic

$$\sum_{\text{cyclic}} \frac{a^2(S^2-3S_{AA})x}{b^2+c^2-a^2}\left(c^4(S^2-S_{CC})y^2 - b^4(S^2-S_{BB})z^2\right) = 0. \tag{6}$$

Remarks. (1) The cubic (5) is the isocubic p$K(X_{3003},\ H)$, labeled K339 in TCT.
(2) The cubic (6) is the isocubic p$K(X_{186},\ X_{571})$ (see Figure 34).
(3) Here are some correspondences of

P	H	O	X_{1986}
Q	X_{24}	O	X_{186}

The reflection triangle of H in the orthic triangle is homothetic to ABC at $X_{24} = \left(\frac{a^2(S_{AA}-S^2)}{S_A} : \frac{b^2(S_{BB}-S^2)}{S_B} : \frac{c^2(S_{CC}-S^2)}{S_C}\right)$.

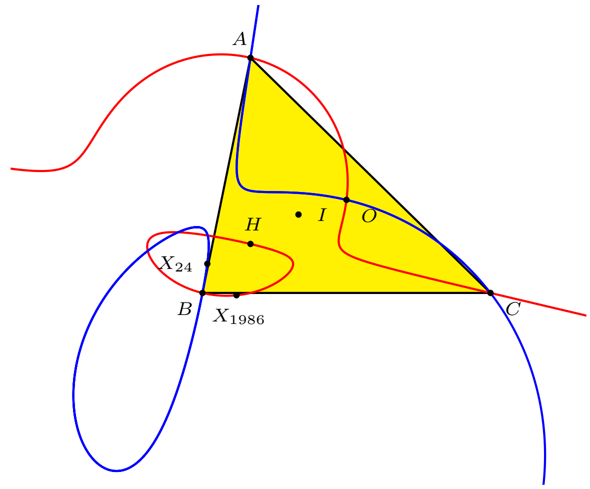


Figure 34. The cubics K339 and p$K(X_{186},\ X_{571})$

7.3. *Reflections in the pedal triangle.*

Proposition 26. *The reflection triangle of P in its pedal triangle are perspective with*
(a) ABC *if and only if P lies on the orthocubic cubic*

$$\sum_{\text{cyclic}} S_{BC}x(c^2y^2 - b^2z^2) = 0, \tag{7}$$

(b) *the pedal triangle if and only if P lies on the Neuberg cubic* (1).

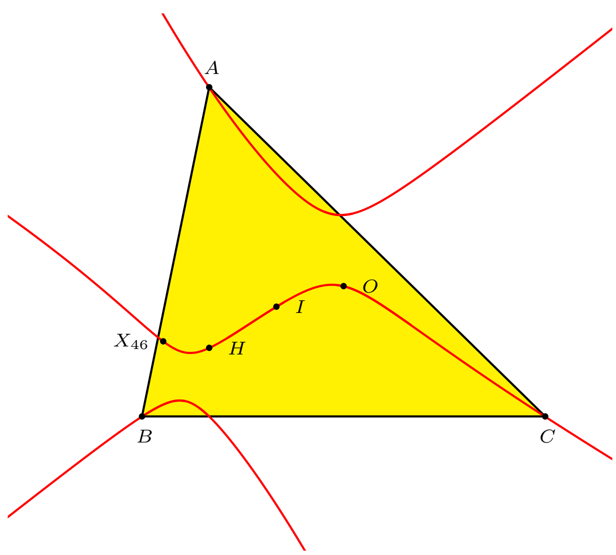


Figure 35. The orthocubic cubic

Remarks. (1) The orthocubic defined by (7) is the curve K006 in CTC.

(2) Both cubics contain the points I, O, H. Here are the corresponding perspectors.

P	I	O	H
perspector with ABC	I	X_{68}	X_{24}
perspector with pedal triangle	I	O	

The missing entry is the perspector of the orthic triangle and the reflection triangle of H in the orthic triangle; it is the triangle center

$$(a^2S_{BC}(3S^2 - S_{AA})(a^2b^2c^2 + 2S_A(S^2 + S_{BC})) : \cdots : \cdots).$$

7.4. *Reflections in the reflection triangle.*

Proposition 27. *The reflections of P in the sidelines of its reflection triangle are perspective with*
(a) ABC *if and only if P lies on the Napoleon cubic* (3).
(b) *the reflection triangle if and only if P lies on the Neuberg cubic* (1).

Remark. Both cubics contain the points I, O, H. Here are the corresponding perspectors.

P	I	O	H
perspector with ABC	I	X_{265}	X_{186}
perspector with reflection triangle	I	O	

The missing entry is the perspector of $H^{(a)}H^{(b)}H^{(c)}$ and the reflection triangle of H in $H^{(a)}H^{(b)}H^{(c)}$; it is the triangle center

$$(a^2S_{BC}(a^2b^2c^2(3S^2 - S_{AA}) + 8S_A(S^2 + S_{BC})(S^2 - S_{AA})) : \cdots : \cdots).$$

8. Reflections in lines

8.1. *Reflections in a line.*

Proposition 28. *Let ℓ be a line through the circumcenter O, and $A'B'C'$ be the reflection of ABC in ℓ. $A'B'C'$ is orthologic to ABC at the fourth intersection of the circumcircle and the rectangular circum-hyperbola which is the isogonal conjugate of ℓ* (see Figure 36).

Remarks. (1) By symmetry, if $A'B'C'$ is orthologic to ABC at Q, then ABC is orthologic to $A'B'C'$ at the reflection of Q in the line ℓ.

Line ℓ	Q	Q'
Euler line	$X_{74} = \left(\frac{a^2}{S^2-3S_{BC}} : \ : \ \right)$	X_{477}
Brocard axis	$X_{98} = \left(\frac{1}{S_{BC}-S_{AA}} : \cdots : \cdots\right)$	X_{2698}
OI	$X_{104} = \left(\frac{a}{a^2(b+c)-2abc-(b+c)(b-c)^2} : \cdots : \cdots\right)$	X_{953}

(2) The orthology is valid if ℓ is replaced by an arbitrary line.

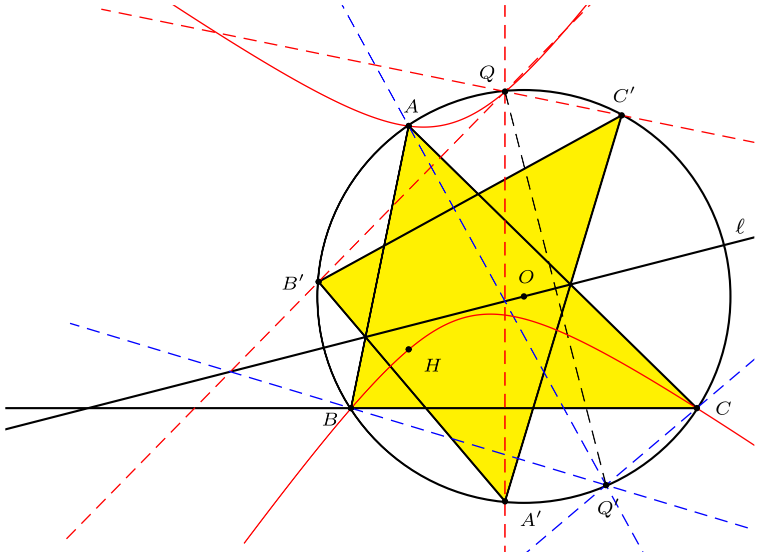


Figure 36. Orthology of triangles symmetric in ℓ

Proposition 29. *Let ℓ be a line through a given point P, and A', B', C' the reflections of A, B, C in ℓ. The lines $A'P$, $B'P$, $C'P$ intersect the sidelines BC, CA, AB respectively at X, Y, Z. The points X, Y, Z are collinear, and the line $\mathscr{L}$ containing them envelopes the inscribed conic with P as a focus* (see Figure 37).

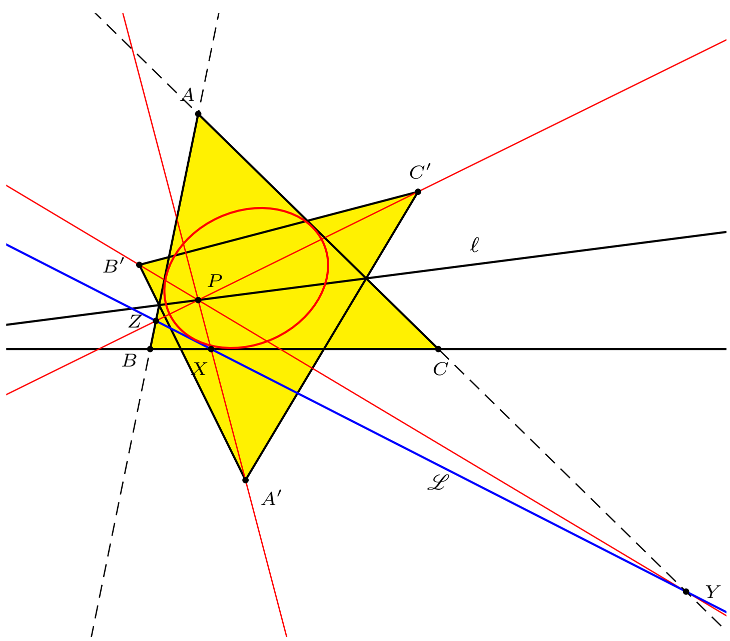


Figure 37. Line $\mathscr{L}$ induced by reflections in ℓ

Proof. Let ℓ be the line joining $P = (u : v : w)$ and $Q = (x : y : z)$. The line $\mathscr{L}$ containing X, Y, Z is

$$\sum_{\text{cyclic}} \frac{u\mathbb{X}}{(b^2u^2 + 2S_C uv + a^2v^2)(uz - wx)^2 - (a^2w^2 + 2S_B wu + c^2u^2)(vx - uy)^2} = 0,$$

equivalently with line coordinates

$$\left(\frac{u}{(b^2u^2 + 2S_C uv + a^2v^2)(uz - wx)^2 - (a^2w^2 + 2S_B wu + c^2u^2)(vx - uy)^2} : \cdots : \cdots\right).$$

Now, the inscribed conic $\mathscr{C}$ with a focus at $P = (u : v : w)$ has center the midpoint between P and P^* and perspector

$$\left(\frac{1}{u(c^2v^2 + 2S_A vw + b^2w^2)} : \frac{1}{v(a^2w^2 + 2S_B wu + c^2u^2)} : \frac{1}{w(b^2u^2 + 2S_C uv + a^2w^2)}\right).$$

Its dual conic is the circumconic

$$\sum_{\text{cyclic}} \frac{u(c^2v^2 + 2S_A vw + b^2w^2)}{\mathbb{X}} = 0,$$

which, as is easily verified, contains the line $\mathscr{L}$ (see [38, §10.6.4]). This means that $\mathscr{L}$ is tangent to the inscribed conic $\mathscr{C}$. □

Remarks. (1) For the collinearity of X, Y, Z, see [23].

(2) The line $\mathscr{L}$ touches the inscribed conic $\mathscr{C}$ at the point

$$\left(\frac{1}{u(c^2v^2 + 2S_A vw + b^2w^2)}\left(\frac{(uz - wx)^2}{a^2w^2 + 2S_B wu + c^2u^2} - \frac{(vx - uy)^2}{b^2u^2 + 2S_C uv + a^2v^2}\right)^2 : \cdots : \cdots\right).$$

(i) If $P = I$, then the line $\mathscr{L}$ is tangent to the incircle. For example, if ℓ is the OI-line, then $\mathscr{L}$ touches the incircle at

$$X_{3025} = (a^2(b-c)^2(b+c-a)(a^2 - b^2 + bc - c^2) : \cdots : \cdots).$$

(ii) If P is a point on the circumcircle, then the conic $\mathscr{C}$ is an inscribed parabola, with focus P and directrix the line of reflections of P (see §1.2). If we take ℓ to be the diameter OP, then the line $\mathscr{L}$ touches the parabola at the point

$$(a^4(b^2 - c^2)(S^2 - 3S_{AA})^2 : \cdots : \cdots).$$

(3) Let ℓ be the Euler line. The two lines $\mathscr{L}$ corresponding to O and H intersect at

$$X_{3258} = ((b^2 - c^2)^2(S^2 - 3S_{BC})(S^2 - 3S_{AA}) : \cdots : \cdots)$$

on the nine-point circle, the inferior of X_{476}, the reflection of E in the Euler line (see [15]). More generally, for isogonal conjugate points P and P^* on the Macay cubic K003, *i.e.*, p$K(K, O)$, the two corresponding lines $\mathscr{L}$ with respect to the line PP^* intersect at a point on the common pedal circle of P and P^*. For other results, see [24, 16].

8.2. *Reflections of lines in cevian triangle.*

Proposition 30 ([9]). *The reflection triangle of $P = (u : v : w)$ in the cevian triangle of P is perspective with ABC at*

$$r_5(P) = \left(u\left(-\frac{a^2}{u^2} + \frac{b^2}{v^2} + \frac{c^2}{w^2} + \frac{b^2+c^2-a^2}{vw}\right) : \cdots : \cdots\right). \tag{8}$$

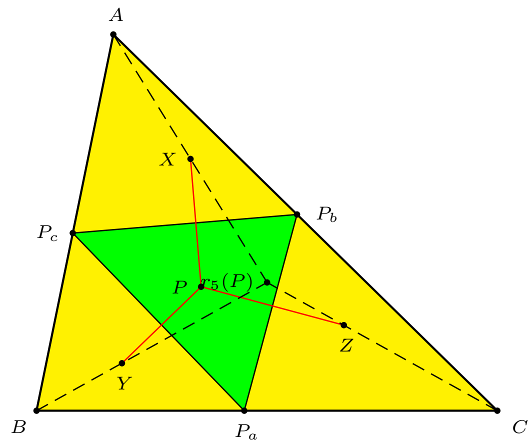


Figure 38. Reflections in sides of cevian triangle

Proof. Relative to the triangle $P_aP_bP_c$, the coordinates of P are $(v + w : w + u : u + v)$. Similarly, those of A, B, C are

$$(-(v+w) : w+u : u+v), \quad (v+w : -(w+u) : u+v), \quad (v+w : w+u : -(u+v)).$$

Triangle ABC is the anticevian triangle of P relative to $P_aP_bP_c$. The perspectivity of ABC and the reflection triangle of P in $P_aP_bP_c$ follows from Proposition 6.

The reflection of P in the line P_bP_c is the point

$$\begin{aligned} X = &\left(u\left(\frac{3a^2}{u^2} + \frac{b^2}{v^2} + \frac{c^2}{w^2} - \frac{b^2+c^2-a^2}{vw} + \frac{2(c^2+a^2-b^2)}{wu} + \frac{2(a^2+b^2-c^2)}{uv}\right)\right. \\ &\left. : v\left(\frac{a^2}{u^2} - \frac{b^2}{v^2} + \frac{c^2}{w^2} + \frac{c^2+a^2-b^2}{wu}\right) : \ w\left(\frac{a^2}{u^2} + \frac{b^2}{v^2} - \frac{c^2}{w^2} + \frac{a^2+b^2-c^2}{uv}\right)\right). \end{aligned}$$

Similarly, the coordinates of the reflections Y of P in P_cP_a, and Z of P in P_aP_b can be written down. From these, it is clear that the lines AX, BY, CZ intersect at the point with coordinates given in (8). □

The triangle XYZ is clearly orthologic with the cevian triangle $P_aP_bP_c$, since the perpendiculars from X to P_bP_c, Y to P_cP_a, and Z to P_aP_b intersect at P. It follows that the perpendiculars from P_a to YZ, P_b to ZX, and P_c to XY are also concurrent. The point of concurrency is

$$r_6(P) = \left(u\left(\frac{a^2}{u^2} + \frac{b^2}{v^2} + \frac{c^2}{w^2} + \frac{b^2+c^2-a^2}{vw}\right) : \cdots : \cdots\right).$$

In fact, P_a, P_b, P_c lie respectively on the perpendicular bisectors of YZ, ZX, XY. The point $r_6(P)$ is the center of the circle XYZ (see Figure 39). As such, it is the isogonal conjugate of P in its own cevian triangle.

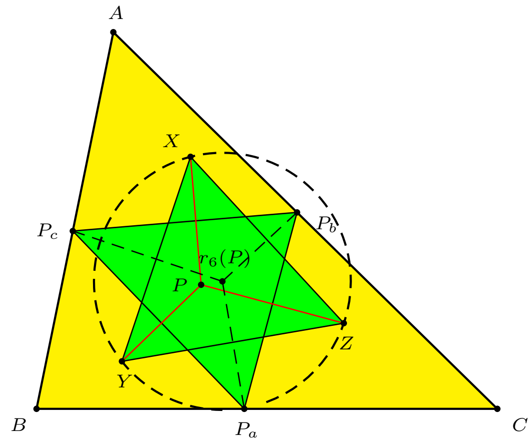


Figure 39. Circumcircle of reflections in cevian triangle

P	I	G	H	G_e	X_{99}	X_{100}	E
$r_5(P)$	X_{35}	$H^\bullet$	X_{24}	X_{57}	X^*_{115}	F^*_e	X^*_{125}
$r_6(P)$		X_{141}	H	X_{354}		X_{1618}	

Remarks. (1) In ETC, $r_5(P)$ is called the Orion transform of P.

(2) $X_{35} = (a^2(b^2+c^2-a^2+bc) : b^2(c^2+a^2-b^2+ca) : c^2(a^2+b^2-c^2+ab))$ divides OI in the ratio $R : 2r$. On the other hand,

$$r_6(I) = (a^2(b^2+c^2-a^2+3bc) : b^2(c^2+a^2-b^2+3ca) : c^2(a^2+b^2-c^2+3ab))$$

divides OI in the ratio $3R : 2r$ (see also Remark (3) following Proposition 31 below).

8.3. *Reflections of sidelines of cevian triangles.* Let P be a point with cevian triangle $P_aP_bP_c$. It is clear that the lines BC, P_bP_c, and their reflections in one another concur at a point on the trilinear polar of P (see Figure 40).

This is the same for line CA, P_cP_a and their reflections in one another; similarly for AB and P_aP_b. Therefore, the following four triangles are line-perspective at the trilinear polars of P:

(i) ABC,
(ii) the cevian triangle of P,
(iii) the triangle bounded by the reflections of P_bP_c in BC, P_cP_a in CA, P_aP_b in AB,
(iv) the triangle bounded by the reflections of BC in P_bP_c, CA in P_cP_a, AB in P_aP_b.

It follows that these triangles are also vertex-perspective (see [25, Theorems 374, 375]. Clearly if P is the centroid G, these triangles are all homothetic at G.

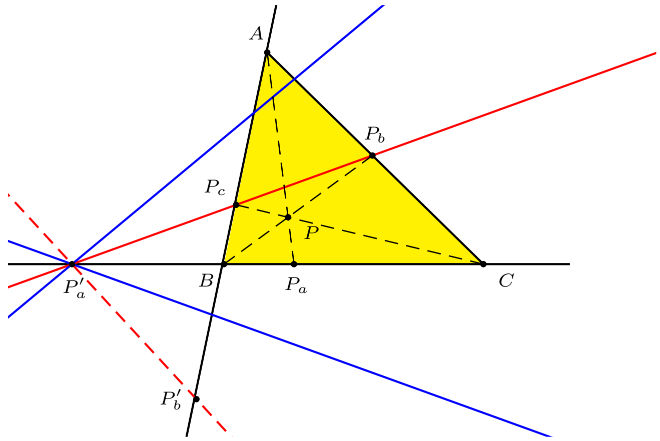


Figure 40. Reflections of sidelines of cevian triangle

Proposition 31. *Let $P_aP_bP_c$ be the cevian triangle of $P = (u : v : w)$.*
(a) *The reflections of P_bP_c in BC, P_cP_a in CA, and P_aP_b in AB bound a triangle perspective with ABC at*

$$r_7(P) = \left(\frac{a^2}{u((c^2+a^2-b^2)v + (a^2+b^2-c^2)w)} : \cdots : \cdots\right).$$

(b) *The reflections of BC in P_bP_c, CA in P_cP_a, and AB in P_aP_b bound a triangle perspective with ABC at*

$$r_8(P) = \left(\frac{a^2vw + u(S_Bv + S_Cw)}{-3a^2v^2w^2 + b^2w^2u^2 + c^2u^2v^2 - 2uvw(S_Au + S_Bv + S_Cw)} : \cdots : \cdots\right).$$

Here are some examples.

P	I	O	H	K	X_{19}	E	X_{393}
$r_7(P)$	X_{21}	X_{1105}	O	$N^\bullet$	X_{1444}	X_{925}	$H^\bullet$

Remarks. (1) The pair $(X_{19},\ X_{1444})$.
(i) $X_{19} = \left(\frac{a}{S_A} : \frac{b}{S_B} : \frac{c}{S_C}\right)$ is the Clawson point. It is the perspector of the triangle bounded by the common chords of the circumcircle with the excircles.
(ii) $X_{1444} = \left(\frac{aS_A}{b+c} : \frac{bS_B}{c+a} : \frac{cS_C}{a+b}\right)$ is the intersection of X_3X_{69} and X_7X_{21}.

(2) $X_{393} = \left(\frac{1}{S_{AA}} : \frac{1}{S_{BB}} : \frac{1}{S_{CC}}\right)$ is the barycentric square of the orthocenter. Let $H_aH_bH_c$ be the orthic triangle, and A_b, A_c the pedals of H_a on CA and AB respectively, and $A' = BA_c \cap CA_b$. Similarly define B' and C'. The lines AA', BB', CC' intersect at X_{393} (see [40]).

(3) The coordinates of $r_8(P)$ are too complicated to list here. For $P = I$, the incenter, note that
(i) $r_8(I) = X_{942} = (a(a^2(b+c) + 2abc - (b+c)(b-c)^2) : \cdots : \cdots)$, and
(ii) the reflections of BC in P_aP_b, CA in P_cP_a, and AB in P_aP_b form a triangle perspective with $P_aP_bP_c$ at $r_6(I)$ which divides OI in the ratio $3R : 2r$.

8.4. *Reflections of H in cevian lines.*

Proposition 32 (Musselman [33]). *Given a point P, let X, Y, Z be the reflections of the orthocenter H in the lines AP, BP, CP respectively. The circles APX, BPY, CPZ have a second common point*

$$r_9(P) = \left(\frac{1}{-2S^2vw + S_A(a^2vw + b^2wu + c^2uv)} : \cdots : \cdots\right).$$

Remark. $r_9(P)$ is also the second intersection of the rectangular circum-hyperbola $\mathscr{H}(P)$ (through H and P) with the circumcircle (see Figure 41).

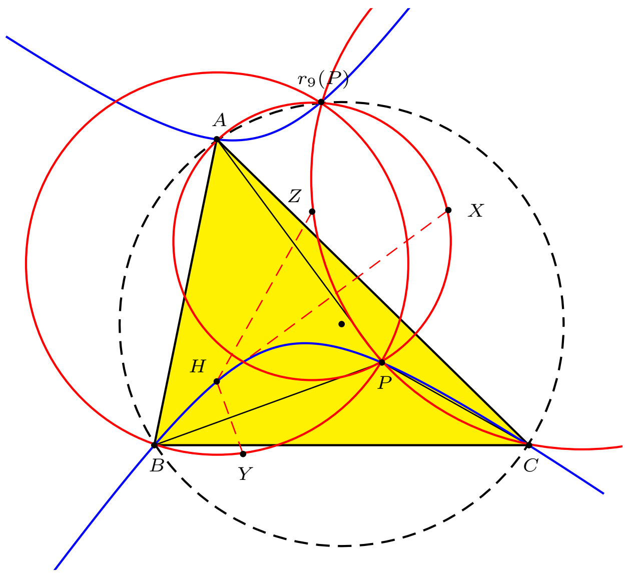


Figure 41. Triad of circles through reflections of H in three cevian lines

8.5. *Reflections in perpendicular bisectors.*

Proposition 33 ([8]). *Given a point P with reflections X, Y, Z in the perpendicular bisectors of BC, CA, AB respectively, the triangle XYZ is perspective with ABC if and only if P lies on the circumcircle or the Euler line.*

(a) *If P is on the circumcircle, the lines AX, BY, CZ are parallel. The perspector is the isogonal conjugate of P* (see Figure 42).

(b) *If $P = E_t$ on the Euler line, then the perspector is $E^*_{t'}$ on the Jerabek hyperbola, where*

$$t' = \frac{a^2b^2c^2(1+t)}{a^2b^2c^2(1+t) - (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)t}$$

(see Figure 43).

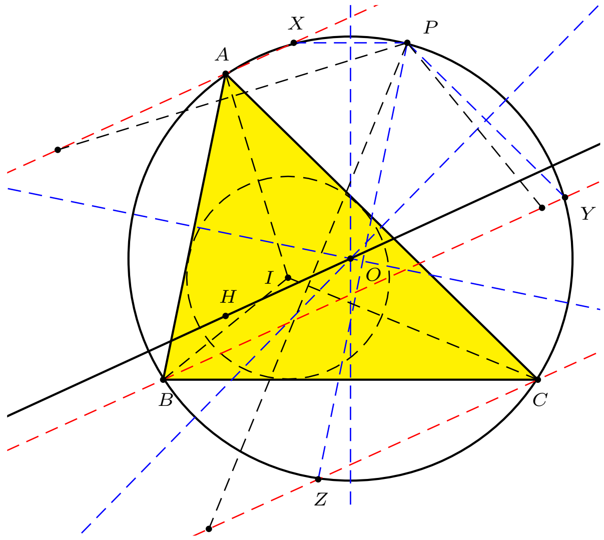


Figure 42. Reflections of P on circumcircle in perpendicular bisectors

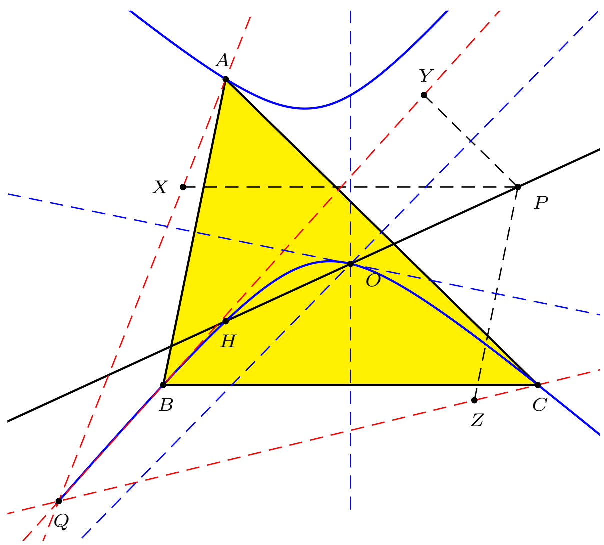


Figure 43. Reflections of P on Euler line in perpendicular bisectors

8.6. *Reflections in altitudes.* Let X, Y, Z be the reflections of P in the altitudes of triangle ABC. The lines AX, BY, CZ are concurrent (at a point Q) if and only if P lies on the reflection conjugate of the Euler line. The perspector lies on the same cubic curve (see Figure 44). This induces a conjugation on the cubic.

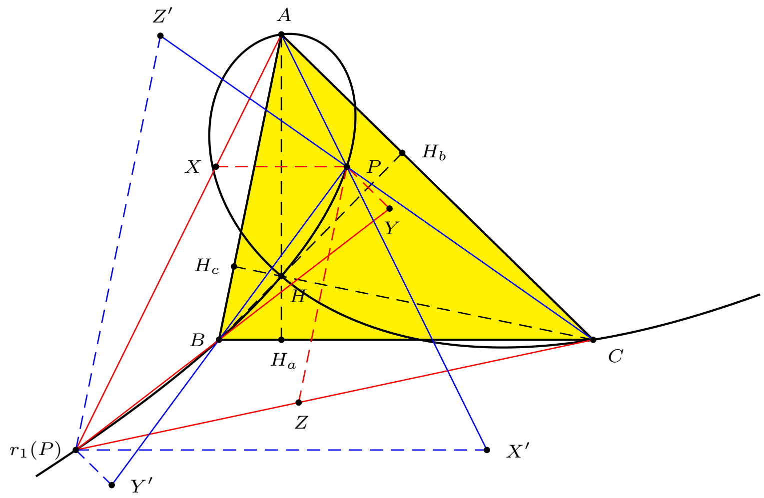


Figure 44. Reflections in altitudes and the reflection conjugate of the Euler line

Proposition 34. *The reflections of $r_1(E_t)$ in the altitudes are perspective with ABC at $r_1(E_{t'})$ if and only if*

$$tt' = \frac{a^2b^2c^2}{a^2b^2c^2 - (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}.$$

9. Reflections of lines in the cevian triangle of incenter

Let $I_aI_bI_c$ be the cevian triangle of I.

Proposition 35 ([20, 44])**.** *The reflections of I_bI_c in AI_a, I_cI_a in BI_b, and I_aI_b in CI_c bound a triangle perspective with ABC at*

$$X_{81} = \left(\frac{a}{b+c} : \frac{b}{c+a} : \frac{c}{a+b}\right)$$

(see Figure 45).

Proof. The equations of these reflection lines are

$$\begin{aligned} -bcx + c(c+a-b)y + b(a+b-c)z &= 0, \\ c(b+c-a)x - cay + a(a+b-c)z &= 0, \\ b(b+c-a)x + a(c+a-b)y - abz &= 0. \end{aligned}$$

The last two lines intersect at the point

$$(-a(b^2+c^2-a^2-bc) : b(a+b)(b+c-a) : c(c+a)(b+c-a)).$$

With the other two points, this form a triangle perspective with ABC at X_{81} with coordinates indicated above. □

Remark. X_{81} is also the homothetic center of ABC and the triangle bounded by the three lines each joining the perpendicular feet of a trace of an angle bisector on the other two angle bisectors ([39]).

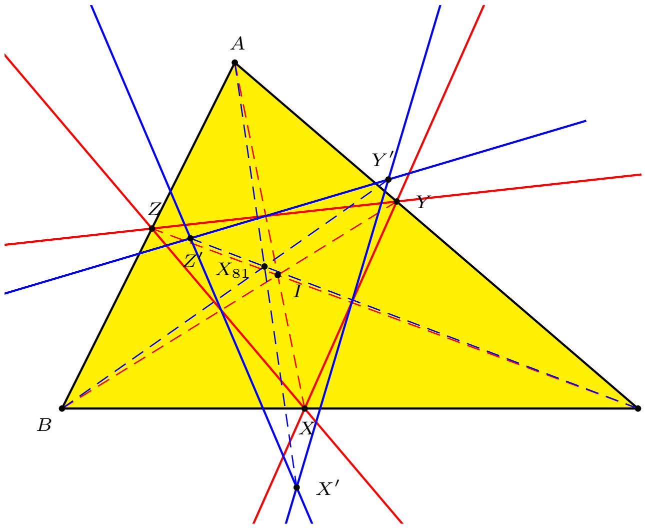


Figure 45. Reflections in the cevian triangle of incenter

Proposition 36. *The reflections of BC in AI_a, CA in BI_b, and AB in CI_c bound a triangle perspective with $I_aI_bI_c$ at*

$$X_{55} = (a^2(b+c-a) : b^2(c+a-b) : c^2(a+b-c)).$$

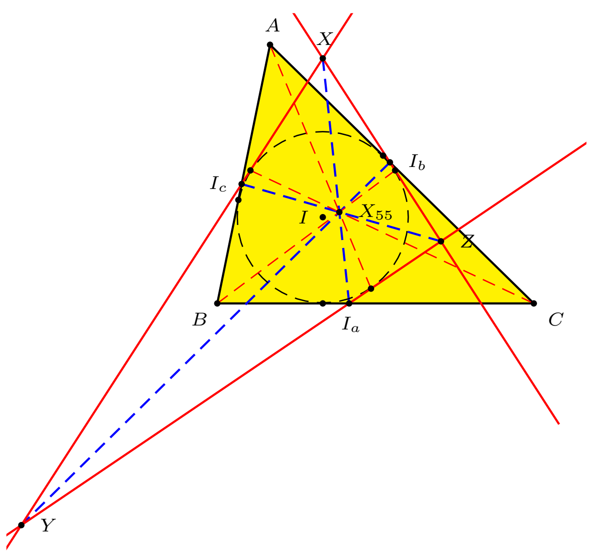


Figure 46. Reflections in angle bisectors

Proposition 37 ([43]). *The reflections of AI_a in I_bI_c, BI_b in I_cI_a, and CI_c in I_aI_b are concurrent at a point with coordinates*

$$\begin{aligned}&(a(a^6+a^5(b+c)-4a^4bc-a^3(b+c)(2b^2+bc+2c^2)\\&\quad -a^2(3b^4-b^2c^2+3c^4)+a(b+c)(b-c)^2(b^2+3bc+c^2)+2(b-c)^2(b+c)^4)\\&:\cdots:\cdots)\end{aligned}$$

(see Figure 47).

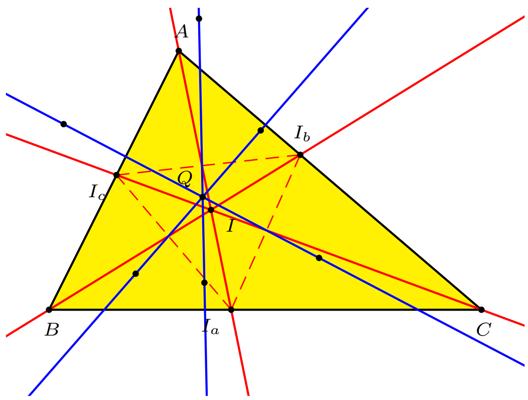


Figure 47. Reflections of angle bisectors in the sidelines of cevian triangle of incenter

10. Reflections in a triangle of feet of angle bisectors

Let P be a given point. Consider the bisectors of angles BPC, CPA, APB, intersecting the sides BC, CA, AB at D_a, D_b, D_c respectively (see Figure 48).

Proposition 38. *The reflections of the lines AP in D_bD_c, BP in D_cD_a, and CP in D_aD_b are concurrent.*

Proof. Denote by x, y, z the distances of P from A, B, C respectively. The point D_a divides BC in the ratio $y : z$ and has homogeneous barycentric coordinates $(0 : z : y)$. Similarly, $D_b = (z : 0 : x)$ and $D_c = (y : x : 0)$. These can be regarded as the traces of the isotomic conjugate of the point $(x : y : z)$. Therefore, we consider a more general situation. Given points $P = (u : v : w)$ and $Q = (x : y : z)$, let $D_aD_bD_c$ be the cevian triangle of $Q^\bullet$, the isotomic conjugate of Q. Under what condition are the reflections of the cevians AP, BP, CP in the lines D_bD_c, D_cD_a, D_aD_b concurrent?

The line D_bD_c being $-x\mathbb{X}+y\mathbb{Y}+z\mathbb{Z}=0$, the equation of the reflection of the cevian AP in D_bD_c is

$$\begin{aligned}&(-x((c^2+a^2-b^2)x-(b^2+c^2-a^2)y+2c^2z)v+x((a^2+b^2-c^2)x+2b^2y-(b^2+c^2-a^2)z)w)\mathbb{X}\\&+(y((c^2+a^2-b^2)x-(b^2+c^2-a^2)y+2c^2z)v+(a^2x^2-b^2y^2+c^2z^2+(c^2+a^2-b^2)zx)w)\mathbb{Y}\\&-((a^2x^2+b^2y^2-c^2z^2+(a^2+b^2-c^2)xy)v+z((a^2+b^2-c^2)x+2b^2y-(b^2+c^2-a^2)z)w)\mathbb{Z}\\&=0.\end{aligned}$$

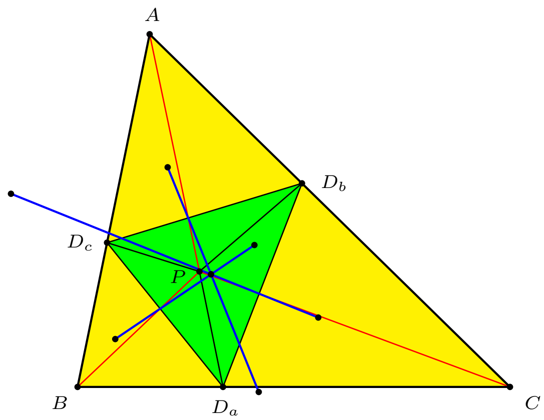


Figure 48. Reflections in a triangle of feet of angle bisectors

By permutating cyclically u, v, w; x, y, z; $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, we obtain the equations of the reflections of BP in D_cD_a and CP in D_aD_b. The condition for the concurrency of the three lines is $F = 0$, where F is a cubic form in u, v, w with coefficients which are sextic forms in x, y, z given in the table below.

term	coefficient
vw^2	$a^2zx(-a^2x^2+b^2y^2+c^2z^2+(b^2+c^2-a^2)yz)\cdot$ $(a^2x^2-3b^2y^2+c^2z^2+(b^2+c^2-a^2)yz+(c^2+a^2-b^2)zx-(a^2+b^2-c^2)xy)$
v^2w	$-a^2xy(-a^2x^2+b^2y^2+c^2z^2+(b^2+c^2-a^2)yz)\cdot$ $(a^2x^2+b^2y^2-3c^2z^2+(b^2+c^2-a^2)yz-(c^2+a^2-b^2)zx+(a^2+b^2-c^2)xy)$
wu^2	$b^2xy(a^2x^2-b^2y^2+c^2z^2+(c^2+a^2-b^2)zx)\cdot$ $(a^2x^2+b^2y^2-3c^2z^2-(b^2+c^2-a^2)yz+(c^2+a^2-b^2)zx+(a^2+b^2-c^2)xy)$
w^2u	$-b^2yz(a^2x^2-b^2y^2+c^2z^2+(c^2+a^2-b^2)zx)\cdot$ $(-3a^2x^2+b^2y^2+c^2z^2-(b^2+c^2-a^2)yz+(c^2+a^2-b^2)zx-(a^2+b^2-c^2)xy)$
uv^2	$c^2yz(a^2x^2+b^2y^2-c^2z^2+(a^2+b^2-c^2)xy)\cdot$ $(-3a^2x^2+b^2y^2+c^2z^2+(b^2+c^2-a^2)yz-(c^2+a^2-b^2)zx+(a^2+b^2-c^2)xy)$
u^2v	$-c^2zx(a^2x^2+b^2y^2-c^2z^2+(a^2+b^2-c^2)xy)\cdot$ $(a^2x^2-3b^2y^2+c^2z^2-(b^2+c^2-a^2)yz+(c^2+a^2-b^2)zx+(a^2+b^2-c^2)xy)$
uvw	$\sum_{\text{cyclic}} a^4u^5((a^2+b^2-c^2)v-(c^2+a^2-b^2)w)$ $+\sum_{\text{cyclic}} a^2u^4((a^2+b^2-c^2)^2v^2-(c^2+a^2-b^2)^2w^2)$ $+\sum_{\text{cyclic}} a^2u^3vw(((c^2-a^2)^2+3b^2(c^2+a^2)-4b^4)v$ $-((a^2-b^2)^2+3c^2(a^2+b^2)-4c^4)w)$

By substituting
x^2 by $c^2v^2+(b^2+c^2-a^2)vw+b^2w^2$,
y^2 by $a^2w^2+(c^2+a^2-b^2)wu+c^2u^2$, and
z^2 by $b^2u^2+(a^2+b^2-c^2)uv+a^2v^2$,
which are proportional to the squares of the distances AP, BP, CP respectively, with the help of a computer algebra system, we verify that $F = 0$. Therefore we conclude that the reflections of AP, BP, CP in the sidelines of $D_aD_bD_c$ do concur. $\square$

In the proof of Proposition 38, if we take $Q = G$, the centroid, this yields Proposition 24. On the other hand, if $Q = X_8$, the Nagel point, we have the following result.

Proposition 39. *The locus of P for which the reflections of the cevians AP, BP, CP in the respective sidelines of the intouch triangle is the union of the circumcircle and the line OI:*

$$\sum_{\text{cyclic}} bc(b-c)(b+c-a)\mathbb{X} = 0.$$

(a) *If P is on the circumcircle, the cevians are parallel, with infinite point the isogonal conjugate of P* (see Figure 49).

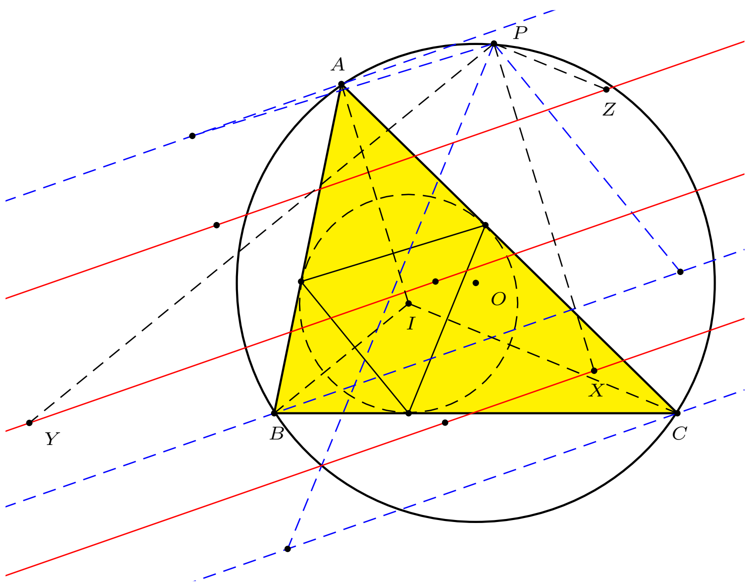


Figure 49. Reflections of cevians of P in the sidelines of the intouch triangle

(b) *If P is on the line OI, the point of concurrency traverses the conic*

$$\sum_{\text{cyclic}} (b-c)(b+c-a)^2x^2 + (b-c)(c+a-b)(a+b-c)yz = 0,$$

which is the Jerabek hyperbola of the intouch triangle (see Figure 50). *It has center*

$$(a(c+a-b)(a+b-c)(a^2(b+c) - 2a(b^2+c^2) + (b^3+c^3)) : \cdots : \cdots).$$

Finally, if we take $Q = \left(\frac{u}{a^2} : \frac{v}{b^2} : \frac{w}{c^2}\right)$ in the proof of Proposition 38, we obtain the following result.

Proposition 40. *Let $P_a^*P_b^*P_c^*$ be the cevian triangle of the isogonal conjugate of P. The reflections of AP in $P_b^*P_c^*$, BP in $P_c^*P_a^*$, CP in $P_a^*P_b^*$ are concurrent* (see Figure 51).

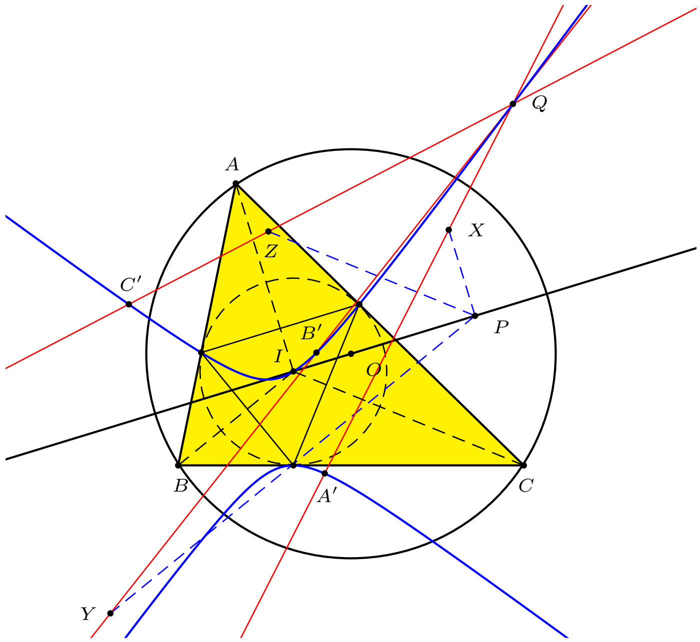


Figure 50. Reflections of cevians of P in the sidelines of the intouch triangle

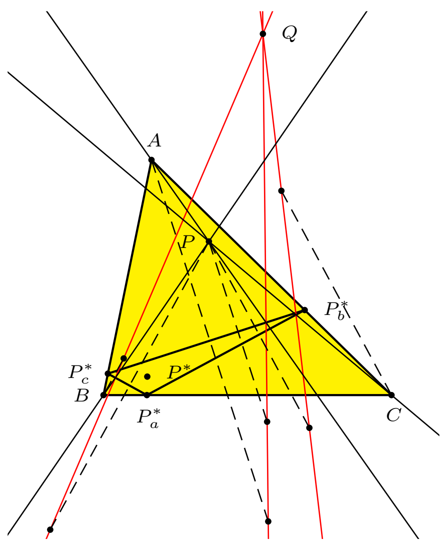


Figure 51. Reflections of cevians of P in cevian triangle of P^*

A special case is Proposition 37 above. For $P = X_3 = O$, the common point is $X_3 = O$. This is because the cevian triangle of $O^* = H$ is the orthic triangle, and the radii OA, OB, OC are perpendicular to the respective sides of the orthic triangle. Another example is $(P, Q) = (K,\ X_{427})$.

Synopsis

Triangle centers	References	**Triangle centers**	References
$F_{\rm e}$	Table following Thm. 4	X_{79}	Rmk (1) following Prop. 7
	Table following Prop. 10	X_{80}	Rmk (2) following Prop. 7
$F_{\pm}$	End of §2.2		Rmk following Thm. 10
	Table following Prop. 10		Table following Prop. 22
$J_{\pm}$	Prop. 1(c); end of §2.2	X_{81}	Prop. 35
Ω, Ω'	Table in §6.3	$X_{95} = N^{\bullet}$	Table following Prop. 31
E	Rmk (3) following Thm. 3; Figure 4	X_{98}	Table in Rmk (1) following Prop. 28
	Table following Thm. 4	X_{99}	Table following Prop. 22
	Rmk (1) following Prop. 9		Table in §8.2
	Table following Prop. 22	X_{100}	Table following Prop. 22
	Table in §8.2		Table in §8.2
	Table following Prop. 31	X_{104}	Table in Rmk (1) following Prop. 28
E_{∞}	Rmk following Prop. 7	X_{108}	Table following Thm. 4
	Rmk (2) following Prop. 9	X_{109}	Rmk (3) following Thm. 3
$W = X_{484}$	Rmk (2) following Prop. 7	X_{112}	Rmk (3) following Thm. 3
	Rmk following Prop. 16		Table following Thm. 4
N^*	Rmk (1) following Prop. 5	X_{115}	Table following Thm. 4
	Rmk (2) following Prop. 9		Table following Prop. 10
	Prop. 14		§§6.2, 6.3
X_{19}	Rmk (1) following Prop. 31	X_{125}	Table following Thm. 4
X_{21}	Rmk (1) following Prop. 17		Table following Prop. 10;
	Table following Prop. 31		§6.3
X_{24}	Rmk (2) following Prop. 17		Rmk (6) following Prop. 23
	Rmk (3) following Prop. 25	X_{141}	Table in §8.2
	Table in Rmk (2) following Prop. 12	X_{143}	Rmk (3) Prop. following 17
	Table in §8.2	$X_{155} = H/O$	Table following Prop. 6
$X_{25} = H/K$	Table following Prop. 6	X_{186}	Rmk (2) followingProp. 12
	Rmk (4) following Prop. 23		Rmk (2) following Prop. 25
X_{35}	Rmk (2) at the end of §8.2		Table in Rmk following Prop. 27
X_{40}	§6.1	$X_{193} = H/G$	Table following Prop. 6
$X_{46} = H/I$	Table following Prop. 6	X_{195}	Rmk (1) following Prop. 5
$X_{52} = H/N$	Table following Prop. 6		Rmk (2) following Prop. 9
$X_{55} = G_{\rm e}^*$	Prop. 36	X_{214}	Table following Prop. 22
X_{57}	Table in §8.2	$X_{249} = X_{115}^*$	Prop. 17(b); Table in §8.2
$X_{59} = F_{\rm e}^*$	Table in §8.2	$X_{250} = X_{125}^*$	Prop. 17(b); Table in §8.2
X_{60}	Rmk (4) following Prop. 17	$X_{265} = r_1(O)$	Table following Prop. 10
X_{65}	Table in Rmk (1) following Prop. 23		Tables following Prop. 22, 23, 27
X_{66}	Table in Rmk (1) following Prop. 23	X_{354}	Table in §8.2
X_{67}	Table following Prop. 10	X_{393}	Rmk (2) following Prop. 31
X_{68}	Table in Rmk (1) following Prop. 23	X_{399}	Rmk (1) following Prop. 9
	Table in Rmk (2) following Prop. 12		§5.1.2; §5.1.3
$X_{69} = H^{\bullet}$	Table in Rmk (1) following Prop. 23	X_{403}	Rmk (2) following Prop. 12
	Table in §8.2		Table in Rmk (1) Prop. 23
	Table following Prop. 31	X_{427}	Rmk (5) following Prop. 23
X_{72}	Table in Rmk (1) following Prop. 23		Rmk following Prop. 40
X_{74}	Table in Rmk (1) following Prop. 23	X_{429}	Table in Rmk (1) Prop. 23
	Table in Rmk (1) following Prop. 28	X_{442}	Table in Rmk (1) Prop. 23

Triangle centers	References
X_{476}	Table following Thm. 4
X_{477}	Table in Rmk (1) following Prop. 28
X_{571}	Rmk (2) following Prop. 25
X_{671}	Tables following Prop. 10, 22
X_{895}	Table following Prop. 22
X_{942}	Rmk (3) following Prop. 31
X_{925}	Table Prop. 31
X_{953}	Table in Rmk (1) following Prop. 28
X_{1105}	Table Prop. 31
X_{1141}	Rmk (3) following Prop. 7
X_{1145}	Table following Prop. 22
X_{1156}	Tables following Prop. 10, 22
X_{1157}	Rmk (3) following Prop. 7
$= (N^*)^{-1}$	Table following Prop. 9
	Corollary 15; §5.1.1
X_{1320}	Table following Prop. 10
	Table following Prop. 22
X_{1444}	Rmk (1) following Prop. 31
X_{1618}	Table in §8.2
X_{1986}	Rmk (1) following Prop. 12
	Table following Prop. 17
	Rmk (3) following Prop. 25
X_{2698}	Table in Rmk (1) following Prop. 28
X_{2715}	Table following Thm. 4
X_{2720}	Table following Thm. 4
X_{2482}	Table following Prop. 22
X_{3003}	Rmk (1) following Prop. 25
X_{3025}	Rmk (2) following Prop. 29
X_{3528}	Rmk (3) following Prop. 29
superiors of Fermat points	§5.1.4
new	End of §2.2
	Rmk (2) following Prop. 12
	§5.1.3
	Rmk following Prop. 27
	Rmk (2) following Prop. 29
	Rmk (2) following Prop. 30
	Prop. 37, 39

Reflection triangles	References
O	§1
H	Rmk (1) following Prop. 12; Rmk following Prop. 27
N	§1, Prop. 5
K	Rmk (4) following Prop. 23

Cevian triangles	References
G (medial)	§7.1
I (incentral)	Rmk (1) following Prop. 7; §9
H (orthic)	Prop. 6; §5.1.1, §7.2; Rmk (5) following Prop. 23

Anticevian triangles	References
I (excentral)	Figure 11; §5.1.3
K (tangential)	Prop. 1(a); §5.1.2; Rmk (4) following Prop. 23
N^*	§5.1.1

Lines	References
Euler line	Figure 4; Prop. 17, 23, 24, 33; Rmk (2) following Prop. 29
OI	Prop. 39

Circles	References
Circumcircle	Prop. 1(d); Thm. 3; Prop. 17, 33, 39
Incircle	Rmk (2) following Prop. 29
Nine-point circle	Rmk 2 Prop. 2; §6.3; Prop. 23
Apollonian circles	Prop. 1(b)
Brocard circle	§6.2
Pedal circle of G	§6.3
$P^{(a)}P^{(b)}P^{(c)}$	Prop. 2; Rmk following Prop. 10
Circles containing $A^{(a)}$, $B^{(b)}$, $C^{(c)}$	passim

Conics	References
Steiner circum-ellipse	§6.4
Jerabek hyperbola	Prop. 23, 24, 33
bicevian conic $\mathscr{C}(G, Q)$	Prop. 22
bicevian conic $\mathscr{C}(X_{115}^*,\ X_{125}^*)$	Prop. 17
Jerabek hyperbola of intouch triangle	Prop. 39
circumconic with center P	Prop. 22
Inscribed parabola with focus E	Rmk (2) following Prop. 29
rectangular circum-hyperbola through P	Rmk following Prop. 10; Rmk following Prop. 32
Inscribed conic with a given focus P	Prop. 29

Cubics	References
Neuberg cubic K001	Prop. 7, 8, 9, 16, 26, 27
Macay cubic K003	Rmk (3) following Prop. 29
Napoleon cubic K005	Prop. 9, 27
Orthocubic K006	Prop. 26
p$K(X_{1989},\ X_{265})$ = K060	Prop. 7, 8
p$K(X_{3003},\ H)$ = K339	Prop. 25
p$K(X_{186},\ X_{571})$	Prop. 25
Reflection conjugate of Euler line	§8.6

Quartics	References
Isogonal conjugate of nine-point circle	Prop. 17
Isogonal conjugate of Brocard circle	§6.4

Constructions	References
H/P	Prop. 6
$r_0(P)$	Rmk (3) following Thm. 3; Thm. 4; Prop. 20
$r_1(P)$	Prop. 10, Prop. 11
$r_2(P)$	Prop. 12
$r_3(P)$	Prop. 22
$r_4(P)$	Prop. 22
$r_5(P)$	Prop. 30
$r_6(P)$	§8.2
$r_7(P)$	Prop. 31
$r_8(P)$	Prop. 31
$r_9(P)$	Prop. 32

References

[1] D. M. Bailey, Some reflective geometry of the triangle, *Math. Mag.*, 33 (1960) 241–259.

[2] D. M. Bailey, *Topics from Triangle Geometry*, Princeton, Virginia, 1972.

[3] C. Blanc, *Nouvelles Annales de Mathématique*, Third series, 19 (1900) 573.

[4] O. Bottema, *Hoofdstukken uit de Elementaire Meetkunde*, 2nd ed. 1997, Epsilon Uitgaven, Utrecht; English translation: *Topics in Elementary Geometry* by R. Ernié, Springer, 2008.

[5] Z. Čerin, Locus properties of the Neuberg Cubic, *Journal of Geometry*, 63 (1998) 39–56.

[6] S. N. Collings, Reflections on a triangle, part 1, *Math. Gazette*, 57 (1973) 291–293.

[7] N.A. Court, H. E. Fettis, and , G. B. Charlesworth, Problem E1350, *Amer. Math. Monthly*, 66 (1959) 62; solution, *ibid*, 594–595.

[8] J.-P. Ehrmann, Hyacinthos message 2204, December 26, 2000.

[9] J.-P. Ehrmann, Hyacinthos message 7999, September 24, 2003.

[10] J.-P. Ehrmann, Hyacinthos message 8039, September 28, 2003.

[11] L. Emelyanov, Hyacinthos message 6694, March 12, 2003.

[12] L. Evans, Hyacinthos message 1461, September 21, 2000.

[13] L. Evans, Some configurations of triangle centers, *Forum Geom.*, 3 (2003) 49–56.

[14] B. Gibert, *Cubics in the Triangle Plane*, available at
`http://perso.orange.fr/bernard.gibert/`
[15] B. Gibert, Hyacinthos message 17212, February 11, 2009.
[16] B. Gibert, Hyacinthos message 17229, February 14, 2009.
[17] A. Goddijn, Tien punten op één cirkel, *Euclides* 77 (2002) 150–155.
[18] D. Grinberg, On the Kosnita point and the reflection triangle, *Forum Geom.*, 3 (2003) 105–111.
[19] A. P. Hatzipolakis, Hyacinthos message 10533, September 24, 2004.
[20] A. P. Hatzipolakis, Hyacinthos message 11505, September 1, 2005.
[21] A. P. Hatzipolakis, F. M. van Lamoen, B. Wolk, and P. Yiu, Concurrency of four Euler lines, *Forum Geom.*, 1 (2001) 59–68.
[22] A. P. Hatzipolakis, Hyacinthos messages 7868, September 12, 2003; 7876, September 13, 2003.
[23] A. P. Hatzipolakis, Hyacinthos message 17206, February 11, 2009.
[24] A. P. Hatzipolakis, Hyacinthos message 17227, February 14, 2009.
[25] R. A. Johnson, *Advanced Euclidean Geometry*, 1929, Dover reprint 2007.
[26] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.
[27] C. Kimberling, *Encyclopedia of Triangle Centers*, available at
`http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.
[28] F. M. van Lamoen, Hyacinthos message 6358, January 13, 2003.
[29] M. S. Longuet-Higgins, Reflections on a triangle, part 2, *Math. Gazette*, 57 (1973) 293–296.
[30] F. Morley and F. V. Morley, *Inversive Geometry*, Oxford, 1931.
[31] J. R. Musselman, On the line of images, *Amer. Math. Monthly*, 45 (1938) 421 – 430; correction, 46 (1939) 281.
[32] J. R. Musselman and R. Goormaghtigh, Advanced Problem 3928, *Amer. Math. Monthly*, 46 (1939) 601; solution, 48 (1941) 281–283.
[33] J. R. Messelman, Some loci connected with a triangle, *Amer. Math. Monthly*, 47 (1940) 354–361.
[34] C. F. Parry and R. L. Young, Problem 10637, *Amer. Math. Monthly*, 105 (1998) 68; solution, ibid., 106 (1999) 779–780.
[35] C. Pohoata, On the Parry reflection point, *Forum Geom.*, 8 (2008) 43–48.
[36] C. Pohoata, A note on the anticomplements of the Fermat points, *Forum Geom.*, 9 (2009) 113–116.
[37] P. Yiu, The Brocard circle as a locus of Miquel point, November, 1999.
[38] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001.
[39] P. Yiu, Hyacinthos message 1969, December 4, 2000.
[40] P. Yiu, Hyacinthos message 2001, December 8, 2000.
[41] P. Yiu, Hyacinthos message 4533, December 12, 2001.
[42] P. Yiu, A tour of triangle geometry, 2004.
[43] P. Yiu, Hyacinthos message 10143, July 23, 2004.
[44] P. Yiu, Hyacinthos message 11506, September 1, 2005.
[45] P. Yiu, The circles of Lester, Evans, Parry, and their generalizations, preprint.

Antreas P. Hatzipolakis: 81 Patmou Street, Athens 11144, Greece
E-mail address: `anopolis72@gmail.com`

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, 777 Glades Road, Boca Raton, Florida 33431-0991, USA
E-mail address: `yiu@fau.edu`

Forum Geometricorum
Volume 9 (2009) 349–350.

FORUM GEOM
ISSN 1534-1178

Author Index