The Lamoen Theorem on the Cross-Triangle / Darij Grinberg

Two triangles ABC and $A'B'C'$ are called perspective if the lines AA', BB' and CC' meet at one point. Then, this point is called **perspector** or **perspective center** of the two triangles. After the Desargues Theorem, this condition is equivalent to the condition that the intersections $X = BC \cap B'C'$, $Y = CA \cap C'A'$ and $Z = AB \cap A'B'$ lie on one line. Then, this line is called **perspectrix** or **perspective axis** of the two triangles (Fig. 1).

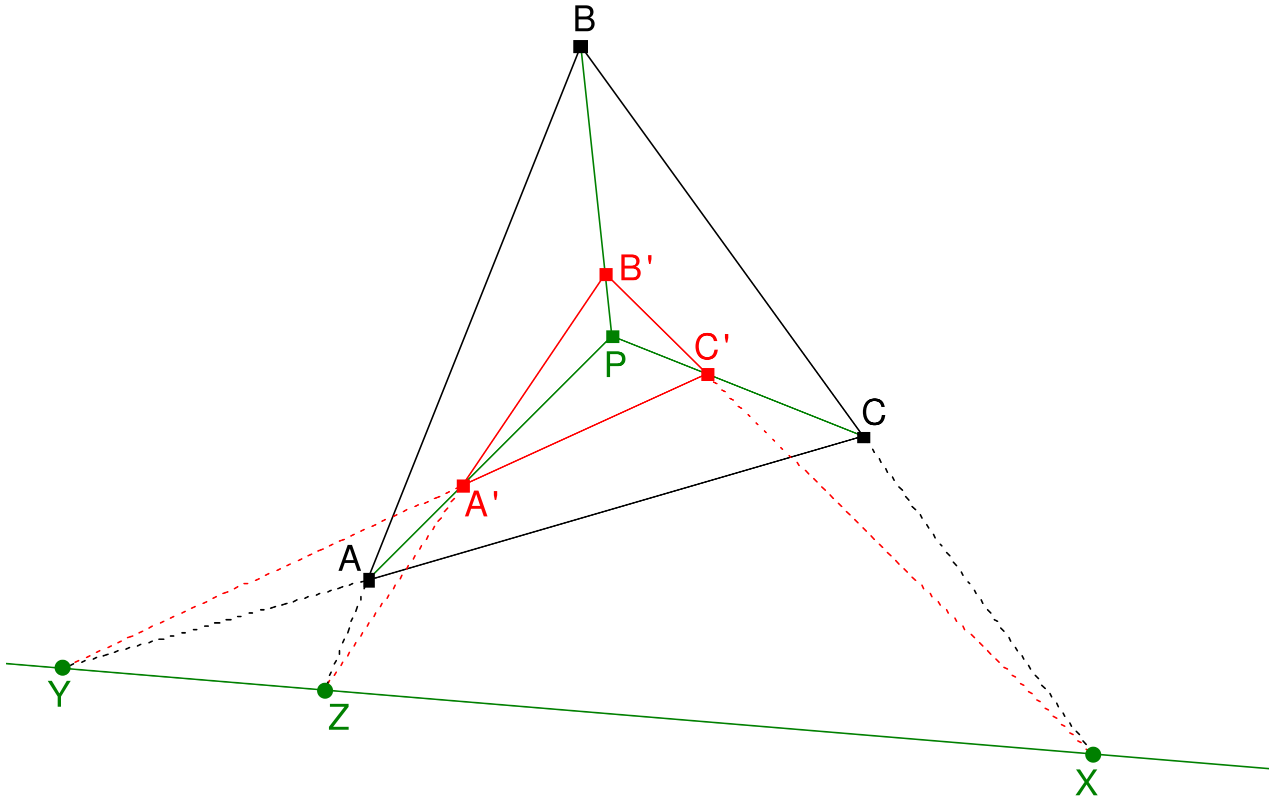

Fig. 1

We shall prove an important projective theorem, the **Lamoen Theorem on the cross-triangle**, found by Floor van Lamoen in 1997. It was established in [2] by stereometric observations (see also [1]), and it can be also easily verified using trilinear coordinates, but we will give a proof that only makes use of the Desargues Theorem. At first, we state the Lamoen Theorem in one of its forms:

Theorem 1. Let ABC and $A'B'C'$ be two perspective triangles. If we construct the points $A'' = BC' \cap B'C$, $B'' = CA' \cap C'A$ and $C'' = AB' \cap A'B$, the triangle $A''B''C''$ is called the **cross-triangle** of the triangles ABC and $A'B'C'$ (Fig. 2). Then, the following facts are true:

a) The triangles ABC, $A'B'C'$ and $A''B''C''$ are pairwise perspective (i. e. any two of them are perspective), and they have a common perspectrix (Fig. 3), i. e. the lines BC, $B'C'$ and $B''C''$ meet at one point, the lines CA, $C'A'$ and $C''A''$ meet at one point, and the lines AB, $A'B'$ and $A''B''$ meet at one point.

b) The pairwise perspectors of the triangles ABC, $A'B'C'$ and $A''B''C''$ are collinear.

Note. This theorem is also called **Desmic Theorem**. Triangles ABC, $A'B'C'$ and $A''B''C''$ are said to form a **desmic triple** of triangles.

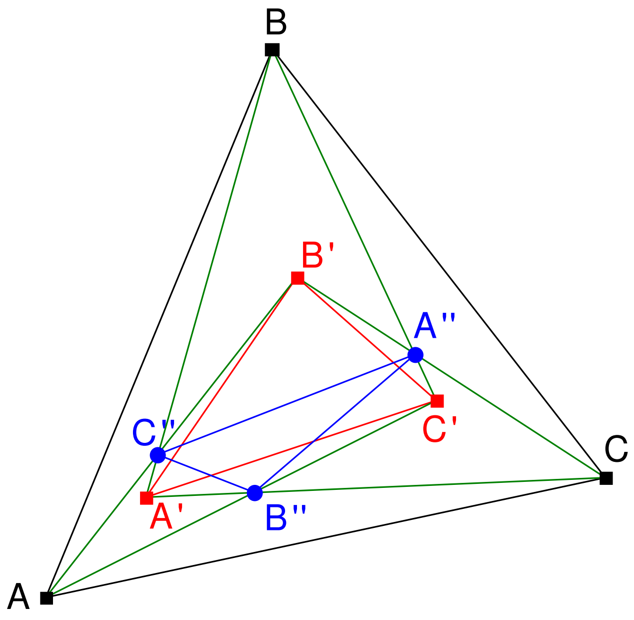

Fig. 2

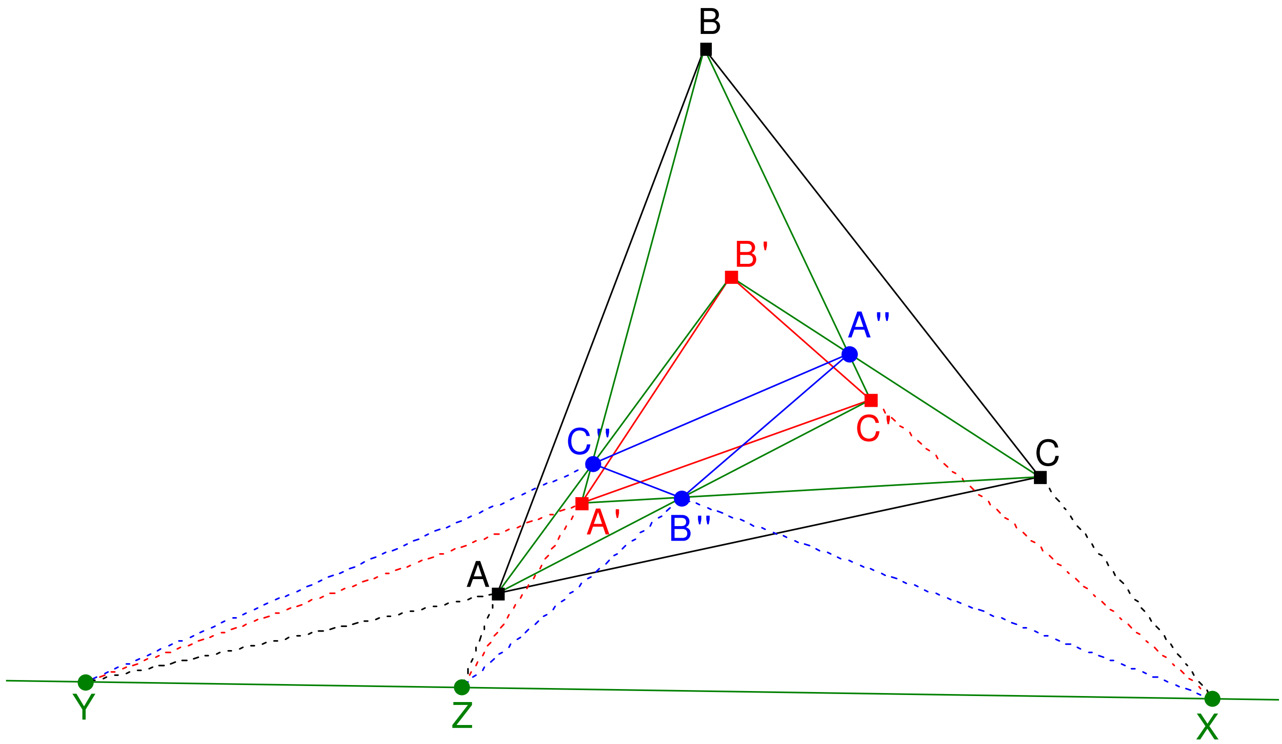

Fig. 3

For the *proof* of Theorem 1 (Fig. 4) we will use the Desargues Theorem.

a) Denote $X = BC \cap B'C'$, $Y = CA \cap C'A'$ and $Z = AB \cap A'B'$. Let the lines AA', BB', CC' intersect at P.

The triangles $AB'C'$ and $A'BC$ are perspective (since the lines AA', $B'B$ and $C'C$ meet at P). After the Desargues theorem, the intersections of respective sidelines, i. e. the points

$$AB' \cap A'B = C'', \qquad B'C' \cap BC = X, \qquad C'A \cap CA' = B''$$

are collinear. This yields that X lies on $B''C''$. Thus, the point X lies on all three lines BC, $B'C'$ and $B''C''$. Analogously, the point Y lies on CA, $C'A'$ and $C''A''$, and the point Z lies on AB, $A'B'$ and $A''B''$.

The points X, Y and Z lie on one line (after the Desargues Theorem, for the triangles ABC and $A'B'C'$ are perspective). Hence, this line XYZ is the common perspectrix of the triangles ABC, $A'B'C'$ and $A''B''C''$. Consequently, the three triangles are pairwise perspective.

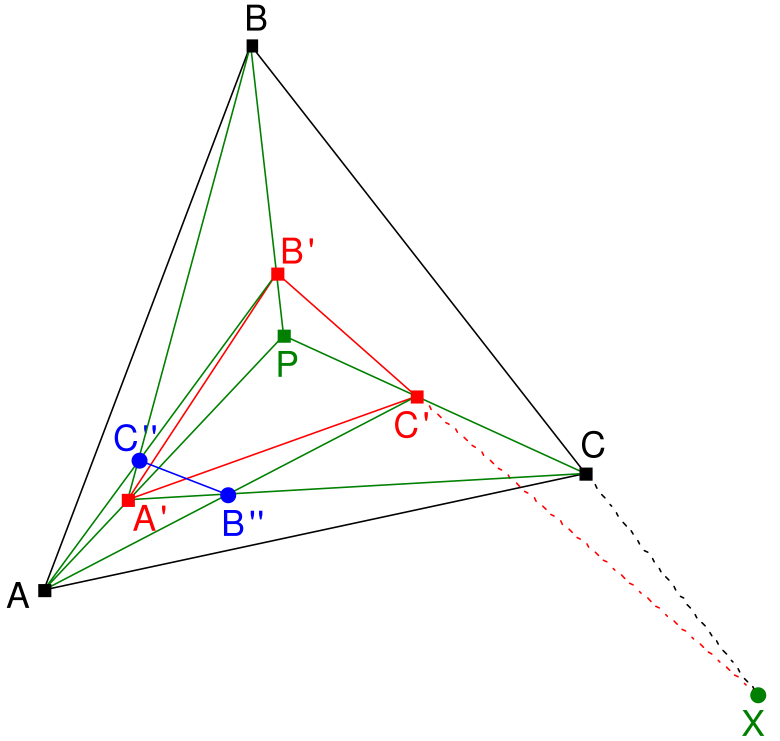

Fig. 4

b) We know that triangles ABC, $A'B'C'$ and $A''B''C''$ are pairwise perspective and have a common perspectrix. The pairwise perspectors are $P = AA' \cap BB' \cap CC'$, $Q = AA'' \cap BB'' \cap CC''$ (Fig. 5) and $R = A'A'' \cap B'B'' \cap C'C''$ (Fig. 6). We want to show that P, Q and R are collinear (Fig. 7).

This is a corollary of the following general fact:

Theorem 2. If three triangles ABC, $A'B'C'$ and $A''B''C''$ are perspective with a common perspectrix, then their pairwise perspectors are collinear.

It remains to establish this theorem.

Let the pairwise perspectors of the three triangles be $P = AA' \cap BB' \cap CC'$, $Q = AA'' \cap BB'' \cap CC''$ and $R = A'A'' \cap B'B'' \cap C'C''$.

Since the lines BC, $B'C'$ and $B''C''$ have a common point X (common perspectrix!), the triangles $BB'B''$ and $CC'C''$ are perspective. From the Desargues Theorem, this yields that the intersections $BB' \cap CC' = P$, $B'B'' \cap C'C'' = R$ and $B''B \cap C''C = Q$ are collinear, qed.

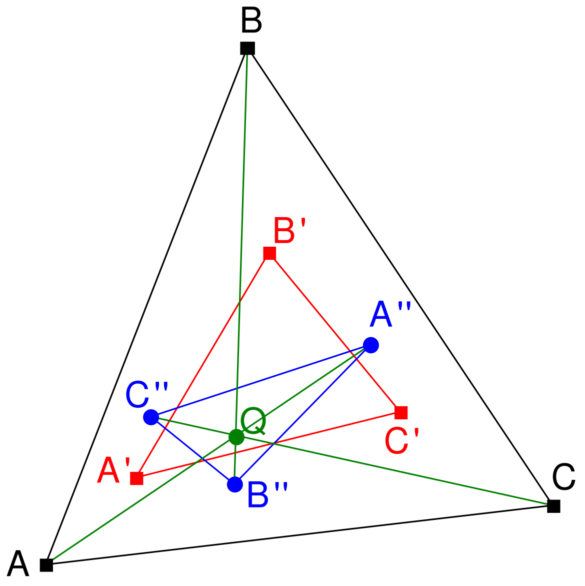

Fig. 5

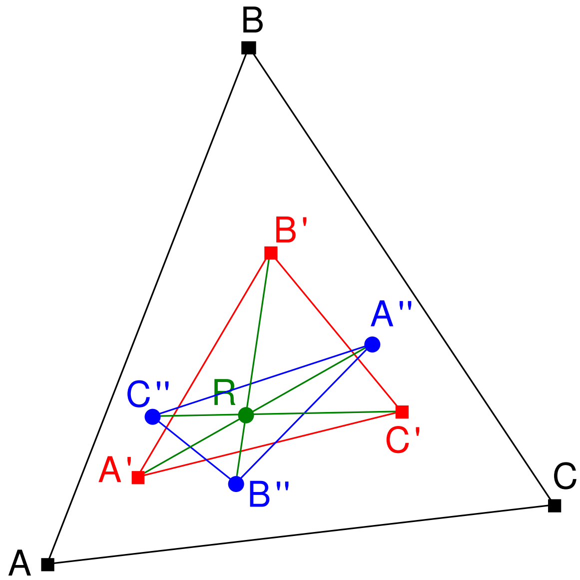

Fig. 6

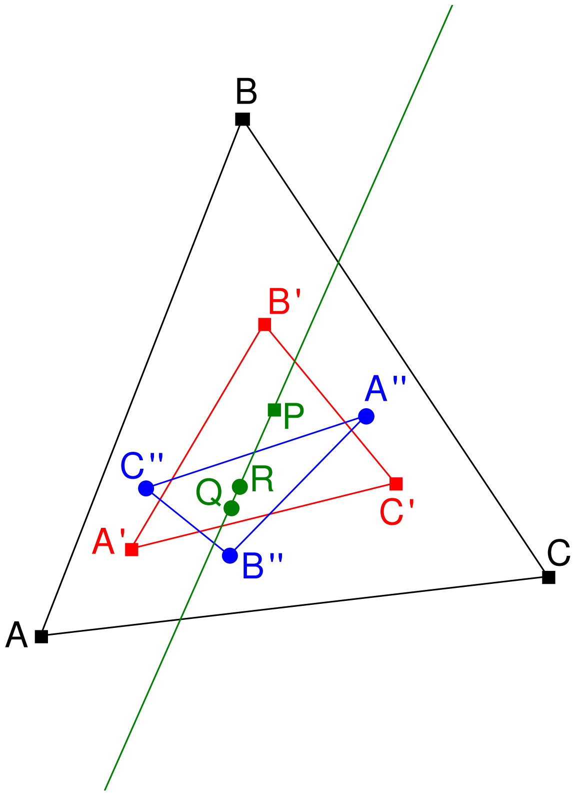

Fig. 7

This completes the proof of Theorem 1. Now we proceed with further properties.

Theorem 3. For the triangles ABC, $A'B'C'$, $A''B''C''$ from Theorem 1 and the perspectors $P = AA' \cap BB' \cap CC'$, $Q = AA'' \cap BB'' \cap CC''$ and $R = A'A'' \cap B'B'' \cap C'C''$, the following properties hold:

a) The lines PA'' and QA' meet at the harmonic conjugate X' of X with respect to the segment BC (Fig. 8).

b) The line AX' passes through the harmonic conjugate of R with respect to the segment PQ (Fig. 9).

Note. Part **b**) is a paraphrasing of Jean-Pierre Ehrmann's result in Hyacinthos message #7981. See also my Hyacinthos message #7985 and Floor van Lamoen's Hyacinthos message #8013.

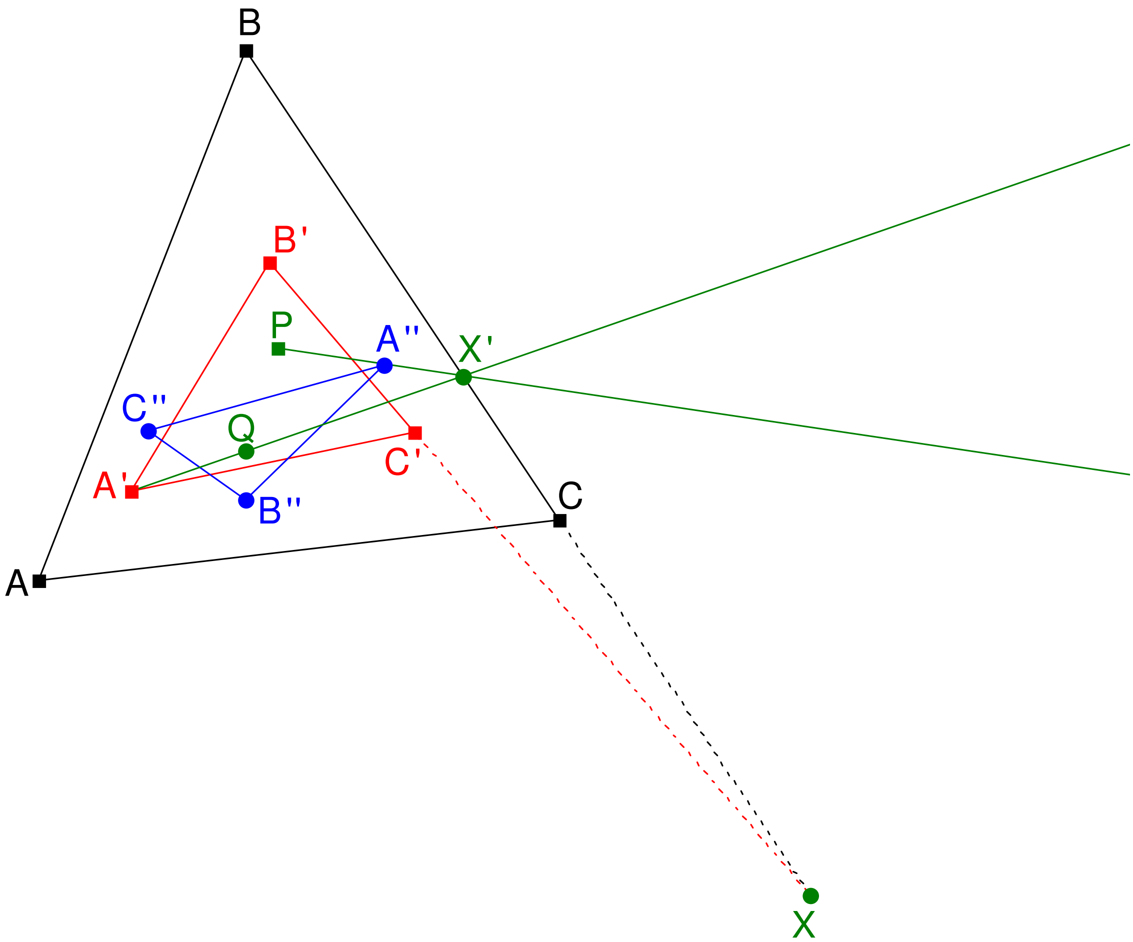

Fig. 8

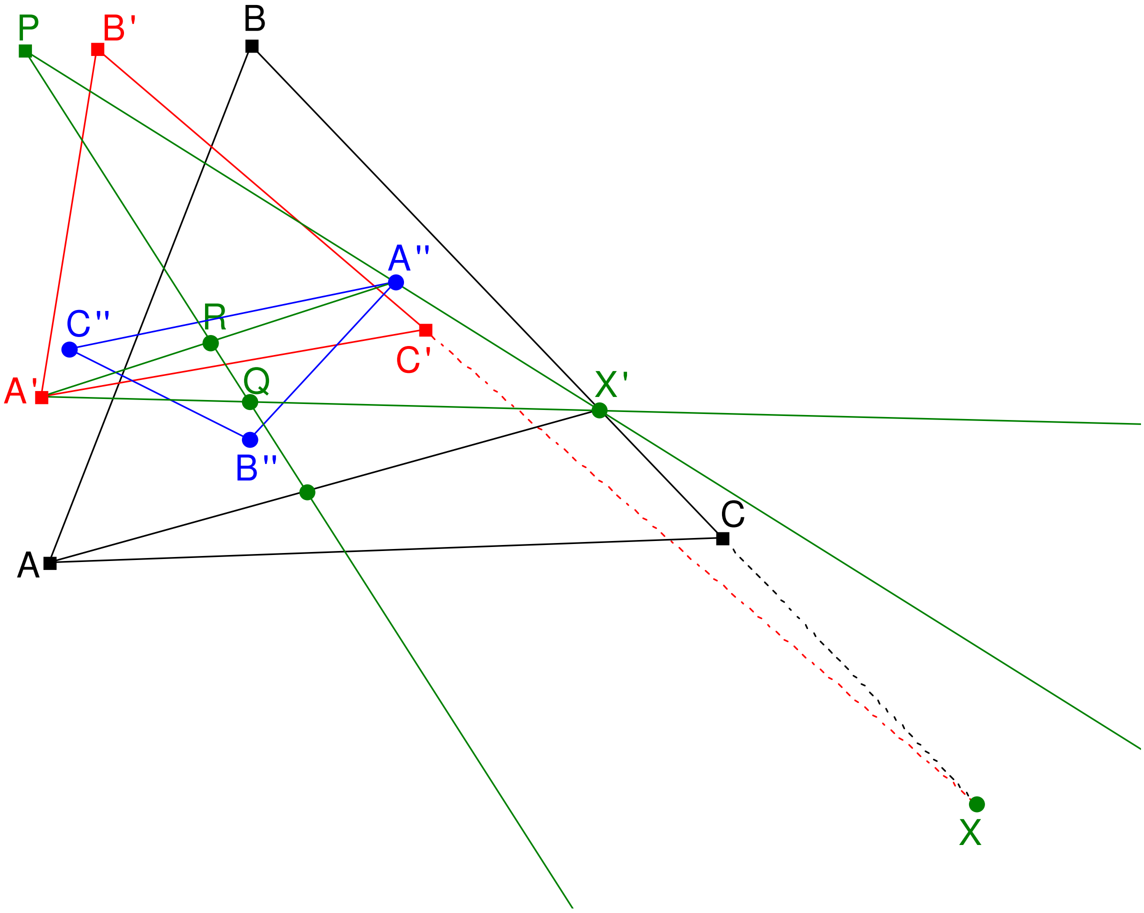

Fig. 9

Proof.

a) Let X' be the harmonic conjugate of X with respect to the segment BC. We have to show that this point X' lies on the lines PA'' and QA'.

(Fig. 10.) The collinear points B', C', X on the sides BP, CP, BC of triangle BPC yield by the Menelaos theorem

$$\frac{CC'}{C'P} \bullet \frac{PB'}{B'B} \bullet \frac{BX}{XC} = -1$$

(where segments are signed). Since X and X' are harmonic conjugates with respect to the segment BC, we have

$$\frac{BX'}{X'C} = -\frac{BX}{XC}, \qquad \text{and}$$

$$\frac{CC'}{C'P} \bullet \frac{PB'}{B'B} \bullet \frac{BX'}{X'C} = -\frac{CC'}{C'P} \bullet \frac{PB'}{B'B} \bullet \frac{BX}{XC} = -(-1) = 1.$$

The Ceva theorem (applied to triangle BPC) yields that the lines BC', CB', PX' concur. I. e., the line PX' passes through the point $BC' \cap CB' = A''$. In other words, the point X' lies on the line PA''.

Similarly, X' lies on the line QA'.

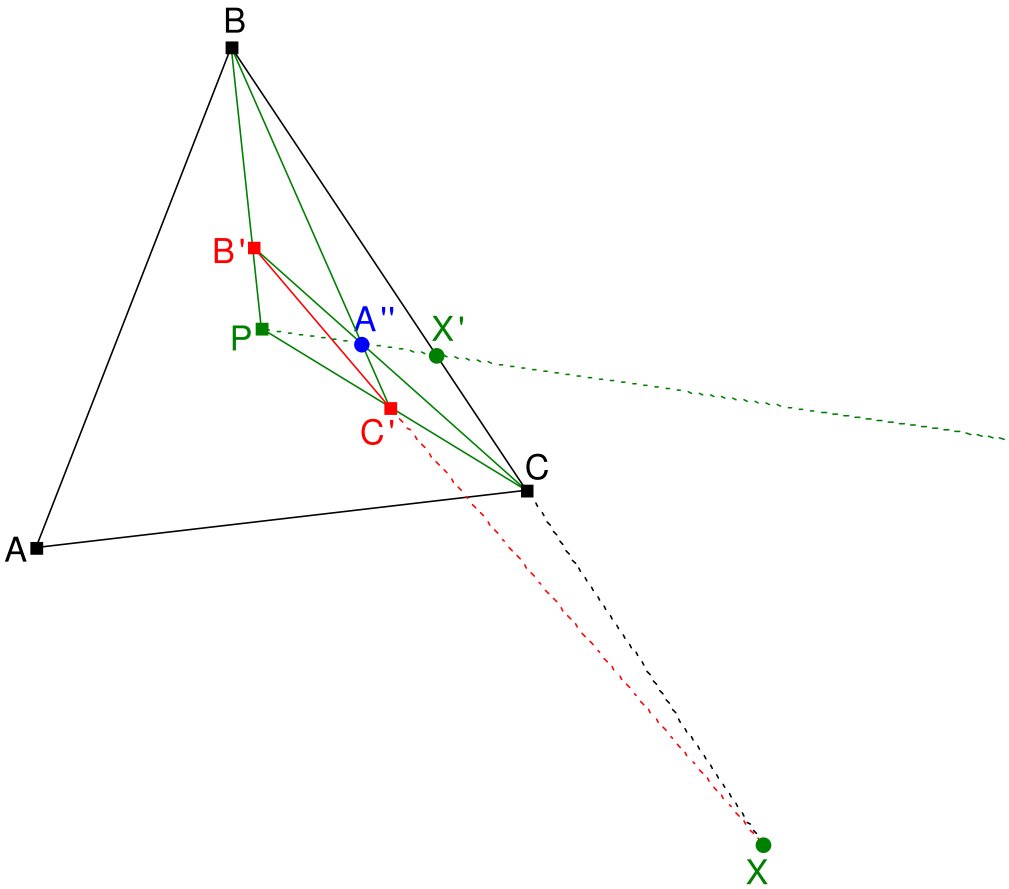

Fig. 10

Hence, the lines PA'' and QA' meet at the harmonic conjugate X' of X with respect to the segment BC. Theorem 3 **a**) is proven.

b) If the lines AX' and PQ meet at S, the Menelaos theorem for the triangle APQ and the collinear points A', A'', R gives

$$\frac{PR}{RQ} \bullet \frac{QA''}{A''A} \bullet \frac{AA'}{A'P} = -1.$$

The lines AS, PA'' and QA' meet at one point (namely X'). Hence, the Ceva theorem for triangle APQ shows

$$\frac{PS}{SQ} \bullet \frac{QA''}{A''A} \bullet \frac{AA'}{A'P} = 1.$$

Thus,

$$\frac{PR}{RQ} = -\frac{PS}{SQ}.$$

This indicates that the points R and S are harmonic conjugates with respect to the segment PQ. Consequently, the line AX' passes through the harmonic conjugate of R with respect to the segment PQ.

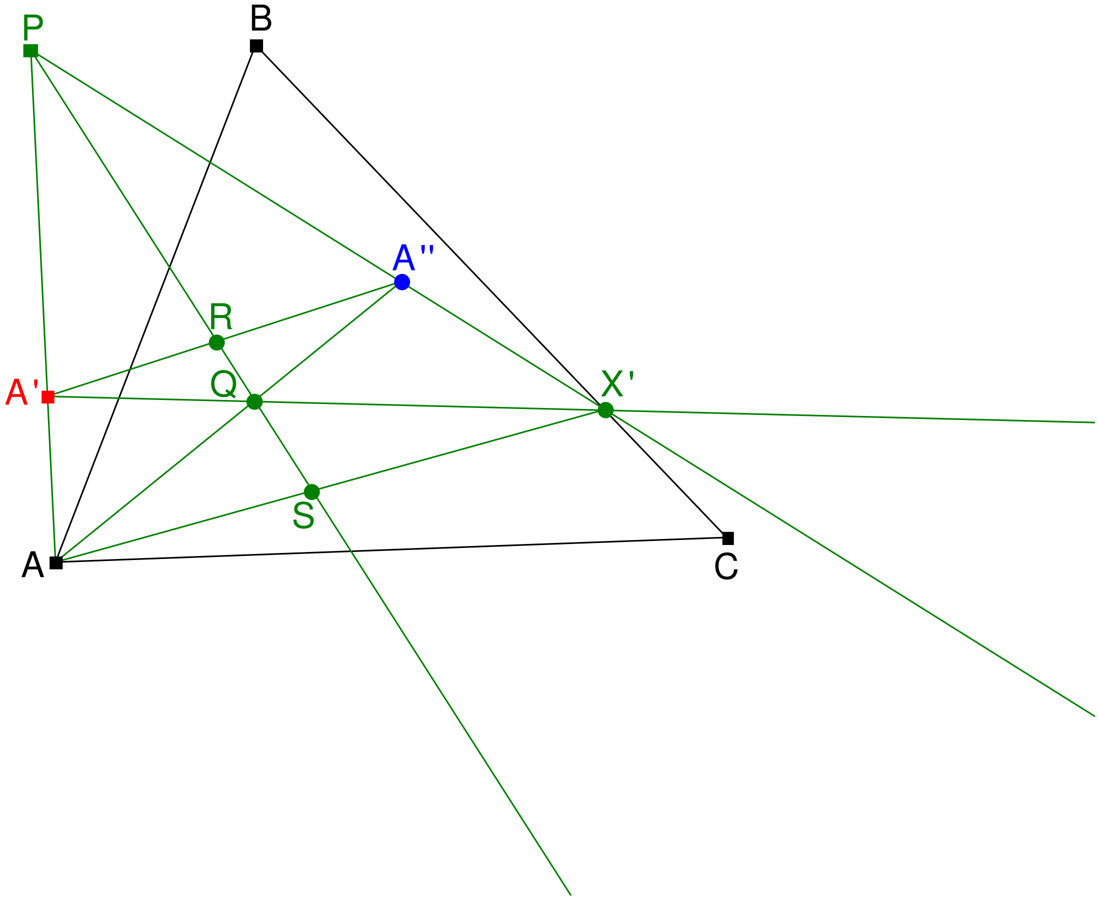

Fig. 11

In the proofs of Theorem 3 **a**) and **b**) above, the use of the Ceva and Menelaos theorems could be replaced by an application of the Theorem on the Complete Quadrilateral, but the latter result is less known.

References

[1] Floor van Lamoen: *The cross-triangle theorem*, `http://home.wxs.nl/~lamoen/wiskunde/cross.htm`

[2] Floor van Lamoen: *Bicentric triangles*, Nieuw Archief voor Wiskunde 17-3 (1999) pages 363-372.