23^{rd}
Iranian Mathematical Olympiad
2005-2006

This collection of problems consists of the problems used for the National Mathematical Olympiad in Iran and Iranian Team Selection. Almost all the problems are proposed by Iranians. We hope that you enjoy these challenges. Special thanks are due to N. Ahmadi Pour Anari for his carefull typing the manuscript, and making some solutions.

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1 Problems

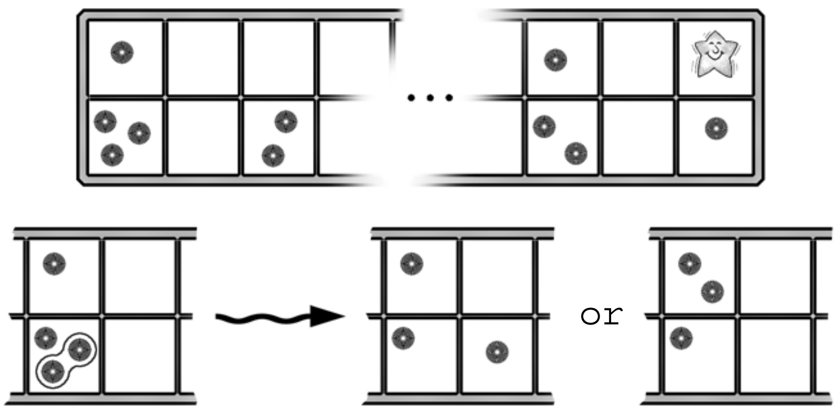
1.1 First Round

1. n is a positive integer and p is a prime number, such that $n \mid p-1$, $p \mid n^3-1$. Show that $4p-3$ is a perfect square.

2. In triangle ABC we have $\angle A = 60°$. Consider a variable point D on BC. Let O_1 be the circumcenter of ABD and O_2 be the circumcenter of ACD. Let M be the intersection of BO_1 and CO_2 and N be the circumcenter of DO_1O_2. Prove that MN passes through a fixed point.

3. We have 10^6 points in the space. If we consider the set of distances between these points, show that this set has at least 79 elements.

4. Consider a table of size $2 \times n$, with $2n$ unit cells. In some cells there are some (maybe more than one) coins. In each step we choose a cell with more than one coin, then we remove two coins, put one coin either in the upper cell, or in the right cell. Assume that in the begining, there are at least 2^n coins on the table. Show that we can treat such that there will be at least one coin in the last upper cell.



5. BC is the diameter of a circle. XY is a chord perpendicular to BC. Points P and M are chosen on XY and CY such that $CY \parallel PB$ and $CX \parallel MP$. Let K be the intersection of CX and PB. Prove that $PB \perp MK$.

6. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for all $x, y \in \mathbb{R}^+$ we have

$$(x+y)f(f(x)y) = x^2 f(f(x)+f(y))$$

($\mathbb{R}^+$ denotes the set of positive reals)

1.2 Second Round

1. Suppose that $P(x) \in \mathbb{Q}[x]$ is an irreducible polynomial, and $\deg P$ is an odd number. Also $Q(x), R(x)$ are rational polynomials with the property that $P(x) \mid Q(x)^2 + Q(x)R(x) + R(x)^2$. Show that $P(x)^2 \mid Q(x)^2 + Q(x)R(x) + R(x)^2$.

2. Suppose that H and O are the orthocenter and the circumcenter of triangle ABC. ω is the circumcircle of ABC. AO intersects ω at A_1. A_1H intersects ω at A' and A'' is the intersection point of ω and AH. We define points B', B'', C' and C'' similarly. Prove that $A'A''$, $B'B''$ and $C'C''$ are concurrent at a point on the Euler line of the triangle ABC.

3. a, b and c are nonnegative real numbers. Suppose that $\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}=2$, show that $ab+bc+ca \leq \frac{3}{2}$.

4. k is an integer. The sequence $\{a_n\}_{n=0}^{\infty}$ is defined as following,

$$a_0 = 0, a_1 = 1, a_n = 2ka_{n-1} - (k^2+1)a_{n-2}, \qquad n \geq 2$$

   and p is a prime number of the form $4m+3$. Prove that

   i) $a_{n+p^2-1} \equiv a_n \mod p$

   ii) $a_{n+p^3-p} \equiv a_n \mod p^2$

5. The sets $A_1, A_2, A_3, \ldots, A_{35}$ are given with the property $|A_i| = 27$ for $1 \leq i \leq 35$, such that the intersection of every three of them has exactly one element. Show that the intersection of $A_1, A_2, \ldots, A_{35}$ is nonempty.

6. Triangle ABC is given. L is a point on BC and M, N are on the extensions of AB, AC such that B is between M and A, C is between N and A, $2\angle AMC = \angle ALC$ and $2\angle ANB = \angle ALB$. Suppose that O is the circumcenter of AMN. Show that OL is perpendicular to BC.

1.3 Third Round

1. Let ABC be a triangle such that its circumcircle radius is equal to the radius of outer inscribed circle with respect to A. Suppose that the outer inscribed circle with respect to A touches BC, AC, AB at M, N, L respectively. Show that O (Center of circumcircle) is the orthocenter of MNL.

2. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that

$$\sum_{i,j=1}^{n} |x_i + x_j| \geq n \sum_{i=1}^{n} |x_i|$$

3. Let G be a tournament with its edges colored red or blue. Show that there exists a vertex v of G with the property that, for every other vertex u there is a mono-color directed path from v to u.

4. We have n points on the plane, no three on a line. A k-tuple of these points is called good if their convex hull contains no other point. Let c_k denote the number of good k-tuples. Show that the following sum is independent of the structure of points and only depends on n :

$$\sum_{i=3}^{n} (-1)^i c_i$$

5. Let n be a fixed natural number greater than one. Find all n tuples of natural pairwise distinct and coprime numbers like $a_1, a_2, \ldots, a_n$ such that for $1 \leq i \leq n$ we have

$$a_1 + a_2 + \ldots + a_n \mid a_1^i + a_2^i + \ldots + a_n^i$$

6. Suppose we have a simple polygon (that is it does not intersect itself, but not necessarily convex). Show that this polygon has a diagonal which is completely inside the polygon and divides the perimeter into two parts such that both parts have at least one third of the vertices of the polygon.

2 Solutions

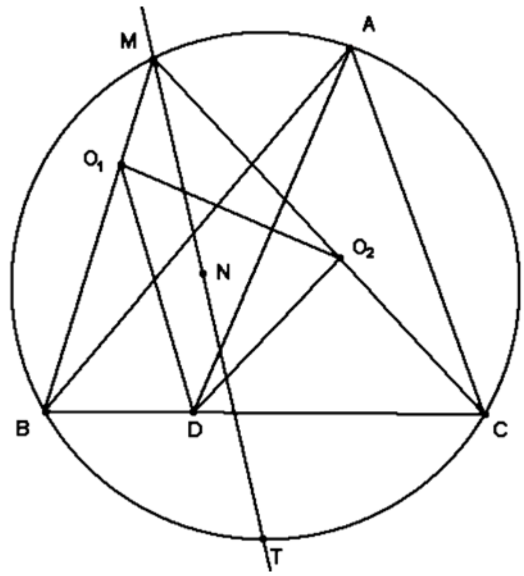
2.1 First Round

1. n is a positive integer and p is a prime number, such that $n \mid p-1$, $p \mid n^3-1$. Show that $4p-3$ is a perfect square.

   Solution. Since p is prime we have $p \mid n-1$ or $p \mid n^2+n+1$. The former is not true since $n|p-1$. So $p \mid n^2+n+1$, i.e. $n^2+n+1 = pt$, where $t \in \mathbb{N}$. The assumption $n \mid p-1$ implies $p = nk+1$ for some positive integer k, hence $n^2+n+1 = (nk+1)(nk'+1)$. If $k' > 0$, then the right hand side is greater which is a contradiction, so $k' = 0$. Now we conclude that $n^2+n+1 = p$, so $4p-3 = 4n^2+4n+1 = (2n+1)^2$.

2. In triangle ABC we have $\angle A = 60°$. Consider a variable point D on BC. Let O_1 be the circumcenter of ABD and O_2 be the circumcenter of ACD. Let M be the intersection of BO_1 and CO_2 and N be the circumcenter of DO_1O_2. Prove that MN passes through a fixed point.

   Solution. Let ω be the circumcircle of triangle ABC. We have $\angle O_1BC = 90° - \angle BAD$ and $\angle O_2CB = 90° - \angle CAD$, hence $\angle BMC = 180° - \angle O_1BC - \angle O_2CB = \angle BAD + \angle CAD = 60° = \angle BAC$. It follows that M is on ω. Furthermore $\angle O_1DO_2 = 180° - \angle O_1BC - \angle O_2CB = \angle BMC = 60°$, hence $\angle O_1NO_2 = 2\angle O_1DO_2 = 120°$.



   So the quadrilateral MO_1NO_2 is inscribed. Now we have $\angle NMO_1 = \angle O_1O_2N = 90° - \angle O_1DO_2 = 30°$, similarly we have $\angle NMO_2 = 30°$, so MN is the bisector of angle O_1MO_2. Because M is on the circumcenter, MN intersects ω in point T the midarc of the arc BC. Point T is obviously independent of D.

3. We have 10^6 points in the space. If we consider the set of distances between these points, show that this set has at least 79 elements.

   Solution. Consider three points A, B, C which are not on a line, if there wouldn't be such points we conclude all 10^6 points lie on a line, in this case there are at least $10^6 - 1$ distances.

   Now to every point A_i other than A, B, C we assign a triple (a, b, c), where a, b, c respectively denote the distance of A_i to A, B, C. We have $10^6 - 3$ triples. Now if the statement of the problem is not true, we have at most 78 distances. In this case the variety of triples is at most 78^3. So at least $\left\lceil \frac{10^6-3}{78^3} \right\rceil = 3$ of these triples are equal. But this is not true, since A, B, C are not collinear, and the intersection of three spheres has at most two points.

4. Consider a table of size $2 \times n$, with $2n$ unit cells. In some cells there are some (maybe more than one) coins. In each step we choose a cell with more than one coin, then we remove two coins, put one coin either in the upper cell, or in the right cell. Assume that in the begining, there are at least 2^n coins on the table. Show that we can treat such that there will be at least one coin in the last upper cell.

   Solution. First, we prove a lemma.

   Lemma. In a $1 \times n$ table containing at least 2^{n-1} coins we can treat such that the last cell is nonempty.

   Proof of lemma is by induction on n. If the last cell contains one coin, there is nothing to prove. Otherwise by induction we can treat such that $(n-1)$th cell contains two coins, now we are done.

   To each cell we assign a coordinate according to its row and column. If $(2, n)$ cell contains at least two coins, there is nothing to prove. If it is empty, by induction on n we can treat such that $(1, n-1)$ cell contains two coins, now with one more step the $(1, n)$ cell contains a coin as desired. Now we assume $(2, n)$ cell contains one coin. We have $2^n - 1$ coins in $2 \times (n-1)$ table. If the first row contains at least 2^{n-1} coins then by the lemma, we can have two coins in the $(1, n-1)$ cell. By a further step the solution is complete. Otherwise the second row contains at least 2^{n-1} coins, similarly we are done.

5. BC is the diameter of a circle. XY is a chord perpendicular to BC. Points P and M are chosen on XY and CY such that $CY \parallel PB$ and $CX \parallel MP$. Let K be the intersection of CX and PB. Prove that $PB \perp MK$.

   Solution. Since $BK \parallel CY$ we have $\angle KBC = \angle BCY = \angle BCK$, so $BK = CK$, also we have $\angle XPK = \angle XYC = \angle YXC$, so $KX = KP$, and also we have $\angle BKX = \angle YCX$, so the two triangles XKB and CMK are equal.

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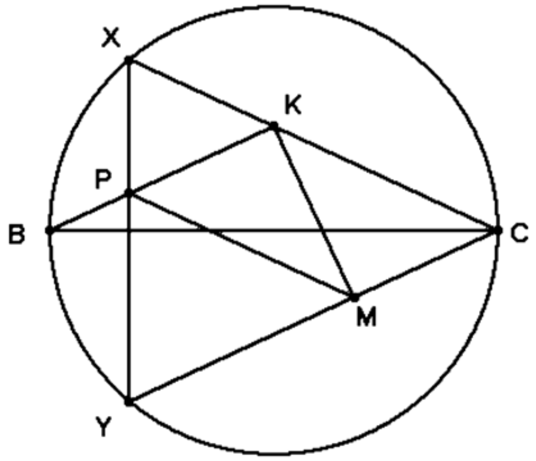


Because $PKCM$ is parallelogram, triangles PMK and KMC are equal, so are triangles XKB and PMK. But since BC is a diameter $\angle PKM = \angle BXK = 90°$.

6. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for all $x, y \in \mathbb{R}^+$ we have

$$(x+y)f(f(x)y) = x^2 f(f(x)+f(y)) \tag{1}$$

($\mathbb{R}^+$ denotes the set of positive reals)

Solution. First we show that f is injective. Assume that $f(a) = f(b)$. Set $y = 1$ in the equation (1), we get $\frac{x+1}{x^2} = \frac{f(f(x)+f(1))}{f(f(x))}$. We have $\frac{a+1}{a^2} = \frac{b+1}{b^2}$ which implies $a = b$. Now there exists a positive real x, such that $x + 1 = x^2$, so $f(f(x) + f(1)) = f(f(x))$, since f is injective we have $f(x) + f(1) = f(x)$ which implies $f(1) = 0$ which is a contradiction!

So there is no such f.

2.2 Second Round

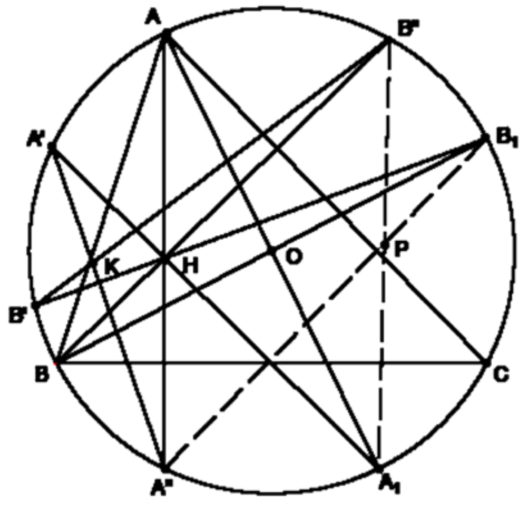
1. Suppose that $P(x) \in \mathbb{Q}[x]$ is an irreducible polynomial, and $\deg P$ is an odd number. Also $Q(x), R(x)$ are rational polynomials with the property that $P(x) \mid Q(x)^2 + Q(x)R(x) + R(x)^2$. Show that $P(x)^2 \mid Q(x)^2 + Q(x)R(x) + R(x)^2$.

   Solution. Since $\deg P$ is odd $P(x)$ has a real root α.

   We have $P(\alpha) = 0$ so $Q(\alpha)^2 + Q(\alpha)R(\alpha) + R(\alpha)^2 = 0$ so $Q(\alpha) = R(\alpha) = 0$. Since $P(x)$ is irreducible, it is the minimal polynomial of α, so $P(x) \mid Q(x)$ and $P(x) \mid R(x)$. Therefore $P(x)^2 \mid Q(x)^2 + Q(x)R(x) + R(x)^2$.

2. Suppose that H and O are the orthocenter and the circumcenter of triangle ABC. ω is the circumcircle of ABC. AO intersects ω at A_1. A_1H intersects ω at A' and A'' is the intersection point of ω and AH. We define points B', B'', C' and C'' similarly. Prove that $A'A''$, $B'B''$ and $C'C''$ are concurrent at a point on the Euler line of the triangle ABC.

   Solution. Suppose that C_1A', A_1C' meet each other at P. By Pascal theorem in hexagon $A'AA_1C'CC_1$ we conclude that H, O and P are collinear, and by this theorem in hexagon $A_1A''A'C_1C''C'$ we conclude that P, H and the intersection point of $C'C''$ and $A'A''$ are collinear.



   Since O, H and P are collinear, O, H and the intersection point of $A'A''$ and $C'C''$ are collinear. Similarly the intersection point of $A'A''$ and $B'B''$ lies on the line OH, so the lines $A'A''$, $B'B''$ and $C'C''$ are concurrent.

3. a, b and c are nonnegative real numbers. Suppose that $\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = 2$, show that $ab + bc + ca \leq \frac{3}{2}$.

Solution. From the assumption we have $\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}=1$. By Cauchy-Schwartz inequality we have

$$\left(\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}\right)\left(a^2+1+b^2+1+c^2+1\right)\geq(a+b+c)^2$$

$$\Longrightarrow a^2+b^2+c^2+3\geq a^2+b^2+c^2+2ab+2bc+2ca$$

so $ab+ac+bc\leq\frac{3}{2}$.

4. k is an integer. The sequence $\{a_n\}_{n=0}^{\infty}$ is defined as following,

$$a_0=0, a_1=1, a_n=2ka_{n-1}-(k^2+1)a_{n-2}, \qquad n\geq 2$$

and p is a prime number of the form $4m+3$. Prove that

i) $a_{n+p^2-1}\equiv a_n \mod p$

ii) $a_{n+p^3-p}\equiv a_n \mod p^2$

Solution. i) Because of its form, p is also a prime in $\mathbb{Z}[i]$ ($i=\sqrt{-1}$). Also $k+i$ and $k-i$ are not divisible by p. We use Lagrange theorem in the multiplicative group $\frac{\mathbb{Z}[i]}{p\mathbb{Z}[i]}-\{[0]\}$. This group has p^2-1 elements. So for every element a of this group we have $a^{p^2-1}=1$. On the other hand $a_n=A\alpha^n+B\beta^n$, where $A,B\in\frac{\mathbb{Z}[i]}{p\mathbb{Z}[i]}$ and α,β are the roots of the equation $n^2=2kn-k^2-1$, so $\{\alpha,\beta\}=\{k+i,k-i\}$. So

$$a_{n+p^2-1}\equiv A\alpha^{n+p^2-1}+B\beta^{n+p^2-1}\equiv A\alpha^n+B\beta^n\equiv a_n \mod p$$

ii) If $(a,p)=1$ we claim that $a^{p^3-p}\equiv 1 \mod p^2$. By (i) $a^{p^2-1}\equiv 1 \mod p$ so $p\mid a^{p^2-1}-1$, also $a^{p^3-p}-1=(a^{p^2-1}-1)(a^{{p^2-1}^{p-1}}+a^{{p^2-1}^{p-2}}+\ldots+1)$, and $a^{{p^2-1}^{p-1}}+a^{{p^2-1}^{p-2}}+\ldots+1\equiv 1+1+1+\ldots+1\equiv 0 \mod p$ so $p^2\mid a^{p^3-p}-1$. Now we have

$$a_{n+p^3-p}\equiv A\alpha^{n+p^3-p}+B\beta^{n+p^3-p}\equiv A\alpha^n+B\beta^n\equiv a_n \mod p^2$$

5. The sets $A_1,A_2,A_3,\ldots,A_{35}$ are given with the property $|A_i|=27$ for $1\leq i\leq 35$, such that the intersection of every three of them has exactly one element. Show that the intersection of $A_1,A_2,\ldots,A_{35}$ is nonempty.

Solution. We first prove a lemma.

Lemma. There is an element α which appears in at least 8 sets.

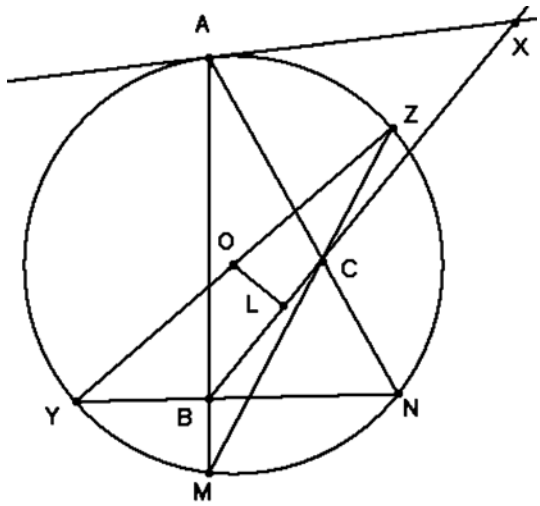
Proof : Let $C_i=A_i\cap A_{35}$ $1\leq i\leq 34$, then $|C_i\cap C_j|=|A_i\cap A_{35}\cap A_j\cap A_{35}|=1$ $i\neq j$. Let $C_i\cap C_j=\{x_{ij}\}$. The number of distinct pairs $\{i,j\}$ is $\binom{34}{2}$ and the number of x_{ij}'s is 21, hence by the pigeonhole principle, there is an $\alpha\in A_{35}$ which appears in at least $\lceil\frac{\binom{34}{2}}{27}\rceil=21$ $C_i\cap C_j$'s. Since

$\binom{7}{2} = 21$, α appears in at least 7 C_i's for $i = 1, \ldots, 34$. Since $\alpha \in A_{35}$, the proof of the lemma is complete.

Suppose the contrary. We show that every element α appears in at most 7 sets. We may assume that $\alpha \in A_1, \ldots, A_r$, $\alpha \notin A_{r+1}, \ldots, A_n$, and $r \geq 8$. For every pair i, j with $1 \leq i < j \leq r$, let $B_{ij} = A_i \cap A_j \cap A_{r+1}$. The B_{ij}'s are singleton, and do not contain α. Since B_{ij}'s are $\binom{8}{2} = 28$ sets and A_{r+1} has 27 elements, a couple of B_{ij}'s are equal. Suppose that $B_{pq} = B_{rs}$. Thus A_p, A_q, A_r and A_s have a common element other than α which is a contradiction!

6. Triangle ABC is given. L is a point on BC and M, N are on the extensions of AB, AC such that B is between M and A, C is between N and A, $2\angle AMC = \angle ALC$ and $2\angle ANB = \angle ALB$. Suppose that O is the circumcenter of AMN. Show that OL is perpendicular to BC.

   Solution. Suppose that MC and NB intersect the circumcircle at Z and Y respectively. Since $\angle AMC + \angle ANB = \frac{1}{2}(\angle ALB + \angle ALC) = 90°$, we conclude that Y, Z are endpoints of the diameter. Let X be the intersection of BC, YZ. Then by Pascal theorem in hexagon $ANYZMA$, we conclude that X lies on the tangent line to circumcircle of AMN at A.



   So it is sufficient to prove that $LOAX$ is an inscribed quadrilateral. We have $\angle AOX = \angle AOZ = 2\angle AMC = \angle ALC = \angle ALX$, so $LOAX$ is inscribed.

2.3 Third Round

1. Let ABC be a triangle such that its circumcircle radius is equal to the radius of outer inscribed circle with respect to A. Suppose that the outer inscribed circle with respect to A touches BC, AC, AB at M, N, L respectively. Show that O (Center of circumcircle) is the orthocenter of MNL.

   Solution. Let R be the radius of circumcircle. Let I_a be the center of the outer inscribed circle with respect to A. Then AI_a intersects the circumcircle at middle of the arc BC (the one without A). We denote this point by M'. Then we have $OM' \perp BC$ and $I_aM \perp BC$. Moreover $OM' = R = I_aM$, so $OM'I_aM$ is a parallelogram, hence $I_aM' || OM$. Since I_aM' lies on the line AI_a which is perpendicular to NL, we have $OM \perp NL$. Now BI_a intersects the circumcircle at middle of the arc AC (the one that contains C). This point is denoted by N'. Then we have $ON' \perp AC$ and $I_aN \perp AC$. Moreover $ON' = R = I_aN$ so $ON'I_aN$ is a parallelogram, hence $I_aN' || ON$. Since I_aN' lies on the line BI_a which is perpendicular to ML, we have $ON \perp ML$. One can similarly show that $(OL \perp MN)$.

2. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that

$$\sum_{i,j=1}^{n} |x_i + x_j| \geq n \sum_{i=1}^{n} |x_i|$$

   Solution. Let $f(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} |x_i + x_j|$ and $g(x_1, \ldots, x_n) = n \sum_{i=1}^{n} |x_i|$. We want to show that if we replace positive x_is by their arithmetic mean, then $f - g$ will not increase. It is clear that by doing this, g does not change. Suppose that $x_1, \ldots, x_k$ are the positive ones. Then the value of $\sum_{i,j=1}^{k} |x_i + x_j|$ also does not change. And also the value of $\sum_{i,j=k+1}^{n} |x_i + x_j|$ does not change. Now take $f_i(x_1, \ldots, x_k)$ to be $\sum_{j=1}^{k} |x_i + x_j|$ for $i > k$. Since the function $|x_i + x|$ is convex, by Jensen inequality, replacing $x_1, \ldots, x_k$ by their arithmetic mean does not increase the value of $\sum_{j=1}^{k} |x_i + x_j|$. We know that

$$f = \sum_{i,j=1}^{k} |x_i + x_j| + \sum_{i,j=k+1}^{n} |x_i + x_j| + 2 \sum_{i=k+1}^{n} f_i(x_1, \ldots, x_k)$$

   Therefore f will not increase either. So we may replace positive x_i's by their arithmetic mean. We may also do this for non-positive x_i's (by similar arguments). So we may assume that $x_1 = x_2 = \ldots = x_k = -a$ and $x_{k+1} = x_{k+2} = \ldots = x_n = b$ for some non-negative numbers a, b. Then we shall prove that

$$2k^2a + 2(n-k)^2b + 2k(n-k)|a-b| \geq kna + (n-k)nb$$

We may assume that $a \geq b$. Then the inequality becomes

$$(2k^2 + 2k(n-k) - kn)a + (2(n-k)^2 - 2k(n-k) - (n-k)n)b \geq 0$$

Simplifying the above inequality we arrive at

$$kna + (n^2 + 4k^2 - 5nk) \geq 0$$

Since $a \geq b$ we have

$$kna + (n^2 + 4k^2 - 5nk)b \geq (kn + n^2 + 4k^2 - 5nk)b = (n-2k)^2 b \geq 0$$

3. Let G be a tournament with its edges colored red or blue. Show that there exists a vertex v of G with the property that, for every other vertex u there is a mono-color directed path from v to u.

   Solution. We use induction on the number of vertices of G. We say v sees u red if there is a red-color path from v to u (We define seeing blue similarly). For $|G| = 2$ it's obvious. So suppose that $|G| > 2$. Let v be a verex of G. Removing v from G, there will be a vertex in the remaining tournament with the requested property. Denote this vertex by u. Suppose that the edge between u and v is directed from v (otherwise u has the requested property in G too). We may assume vu is red. If $v_1, \ldots, v_k$ are the vertices that u sees them blue, then in the tournoment composed of $v, v_1, \ldots, v_k$ there is a vertex with the requested property like w. If $w = v$ then v sees $v_1, \ldots, v_k$ mono-colored and it also sees other verices red (Because vu is red). Otherwise, $w \neq v$. If w sees v red, then w sees $v_1, \ldots, v_k, v$ mono-colored and sees the other vertices red. If w sees v blue, then u sees w blue and w sees v blue, so u sees v blue, and has the requested property in G. So in all cases we are done.

4. We have n points on the plane, no three on a line. A k-tuple of these points is called good if their convex hull contains no other point. Let c_k denote the number of good k-tuples. Show that the following sum is independent of the structure of points and only depends on n :

$$\sum_{i=3}^{n} (-1)^i c_i$$

   Solution. Let S be the set of our n points. We calculate the following sum in two ways

$$\sum_{k \geq 3, v_1, \ldots, v_k \in S} (-1)^k$$

First it does not depend on structure of S, so we may call it $f(n)$. Alternatively, calculate the sum where sum runs on all subsets whose convex hull is a fixed polygon like $v_1, \ldots, v_t$. If there are k points in the polygon $v_1, \ldots, v_t$ this sum will be

$$(-1)^t \sum_{i=0}^{k} (-1)^i \binom{k}{i}$$

which equals zero if $k \geq 1$ and $(-1)^t$ otherwise. So we may divide the terms in the first sum into parts, each being summed over the set of vertices with the same convex hull. Here the sum of each part equals 0 if the corresponding k-tuple is not good and $(-1)^v$ otherwise, where v is the number of vertices of the convex hull. So this sum is equal to

$$\sum_{i=3}^{n} (-1)^i c_i$$

So,

$$\sum_{i=3}^{n} (-1)^i c_i = f(n)$$

5. Let n be a fixed natural number greater than one. Find all n tuples of natural pairwise distinct and coprime numbers like $a_1, a_2, \ldots, a_n$ such that for $1 \leq i \leq n$ we have

$$a_1 + a_2 + \ldots + a_n \mid a_1^i + a_2^i + \ldots + a_n^i$$

Solution. Let $\sigma_k = a_1^k + a_2^k + \ldots + a_n^k$. We have $\sigma_1 \mid \sigma_1, \sigma_2, \ldots, \sigma_n$. We will show that $\sigma_1 \mid \sigma_i$ for $i > n$, too. Take $P(x) = (x - a_1)(x - a_2) \ldots (x - a_n) = x^n + c_{n-1}x^{n-1} + \ldots + c_0$. We have $P(a_1) = P(a_2) = \ldots = P(a_n) = 0$. Now take the following sum

$$0 = a_1 P(a_1) + a_2 P(a_2) + \ldots + a_n P(a_n) = \sigma_{n+1} + c_{n-1}\sigma_n + \ldots + c_0 \sigma_1$$

From above we conclude $\sigma_1 \mid \sigma_{n+1}$. Using similar arguments and induction we have $\sigma_1 \mid \sigma_{n+i}$. Let p^k be one of the prime powers in factorization of σ_1. Let m be a number greater than k and $\varphi(p^k) \mid m$, then we have $p^k \mid \sigma_m$. If one of a_i's is divisible by p then $p^k \mid a_i^m$. So by Euler theorem σ_m is equal to n or $n-1$ modulo p^k (depending on whether any a_i is divisible by p or not). So we have $p^k \mid n$ or $p^k \mid n-1$. So we have $\sigma_1 \mid n(n-1)$, and hence $\sigma_1 \leq n(n-1)$. Since a_i's are pairwise coprime and distinct, σ_1 is greater than or equal to sum of the first $n-1$ primes plus 1. (Because if one of

a_i's is not 1 replacing it by one of its prime divisors decreases σ_1). If $n > 5$ then we have $\sigma_1 \geq 1+2+3+5+7+11+13+15+17+\ldots+(2n-1)$ (9 is not included in the sum). So $\sigma_1 \geq n^2-9+2 = n^2-7$ and whenever $n > 7$ we have $n^2-7 > n^2-n = n(n-1)$, so when $n > 7$ there is no answer. For $n = 4, 6$ we have sum of the first n primes plus 1 is exactly $n(n-1)-1$, so we have no answer (If a_i's are $\{1, 2, 3, 5, \ldots\}$ they can not divide $n(n-1)$ and for any other set of a_i's their sum is at least two units greater, because replacing one a_i with one of its prime divisors will decrease it by at least 2). For $n = 5$ the sum is $n(n-1)-2$. So by similar arguments as above the only possible answer is $\{1, 3, 4, 5, 7\}$, which is not an answer because $\sigma_4 = 1+1+1+0+1 = 4(mod 5)$ and $5 \mid \sigma_1$. For $n = 7$ the sum is exactly $n(n-1)$, so the only possible answer is $\{1, 2, 3, 5, 7, 11, 13\}$, which is not an answer because $\sigma_6 = 1+1+1+1+0+1+1 = 6(mod 7)$ but $7 \mid \sigma_1$. For $n = 3$ again the sum is exactly $n(n-1)$, so the only possible solution is $\{1, 2, 3\}$, which is not an answer because $\sigma_2 = 1+1+0 = 2(mod 3)$ but $3 \mid \sigma_1$. For $n = 2$ the sum is greater than $n(n-1)$. So we have no solution at all.

6. Suppose we have a simple polygon (that is it does not intersect itself, but not necessarily convex). Show that this polygon has a diagonal which is completely inside the polygon and divides the perimeter into two parts such that both parts have at least one third of the vertices of the polygon.

   Solution. Let us decompose the polygon into triangles (it can be done by a famous theorem). If n is the number of vertices of polygon then there are exactly $n-2$ triangles (for example by double counting the sum of angles of polygons). Now let us create a graph whose vertices are triangles in our decomposition and two vertices are adjacent if and only if the two triangles have a common side. Observe that this graph has no cycles (otherwise our polygon would have holes in it or would have one of its vertices inside itself!). So the graph is a tree (it's connected too). Let $t = n-2$ be the number of its vertices. Now we claim that there is an edge, such that by cutting it, the two resulting connected components have at least $\frac{v-4}{3}$ of the vertices. These two components give us two arcs in the polygon, both have at least $\frac{n-2-4}{3}+2$ of vertices, that is $\frac{n}{3}$ of vertices, which finishes the proof. In order to prove our claim, observe that in our graph each vertex has at most three neighbors. Take one vertex like v. If we remove one outgoing edge of v then the component without v has either less than $\frac{v-4}{3}$ or more than $\frac{2v+4}{3}$ vertices (otherwise we are done). Now note that there is exactly one neighbor of v whose induced subtree has more than $\frac{2v+4}{3}$ vertices. Because if two neighbours have this property then the graph would have more than $2\frac{2v+4}{3} > v$ vertices! If none has this property, then the graph would have less than $3\frac{v-4}{3}+1 < v$ vertices! Denote this neighbor of v by $f(v)$.$(v, f(v)$ are adjacent in the graph)

   Now consider the sequence : $v, f(v), f(f(v)), \ldots$. This sequence is a walk in our graph, and because our graph is a tree, after some steps we should walk back the last edge we walked through. More specifically there is a vertex like v such that $f(f(v)) = v$. Now cutting the edge between v and $f(v)$ yields two components each having at least $\frac{2v+4}{3}$ vertices, so our graph has at least $2\frac{2v+4}{3} > v$ vertices, contradiction!