The Poncelet point and its applications

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Abstract

This paper will discuss properties and results that arise when studying a quite lesser known point in Geometry, the Poncelet point. The Poncelet point is very powerful when it comes to studying Projective geometry, but has suprising applications in Euclidean geometry as well. As such, this paper will be exploring this point in both a projective and euclidean light.

1 Definitions and Notation

We first give a Euclidean definition of the Poncelet point:

Theorem 1.1. *Let P be a point and consider a triangle ABC. Then the ninepoint circles of APB, APC, BPC, ABC concur at the Poncelet point with respect to $ABCP$.*

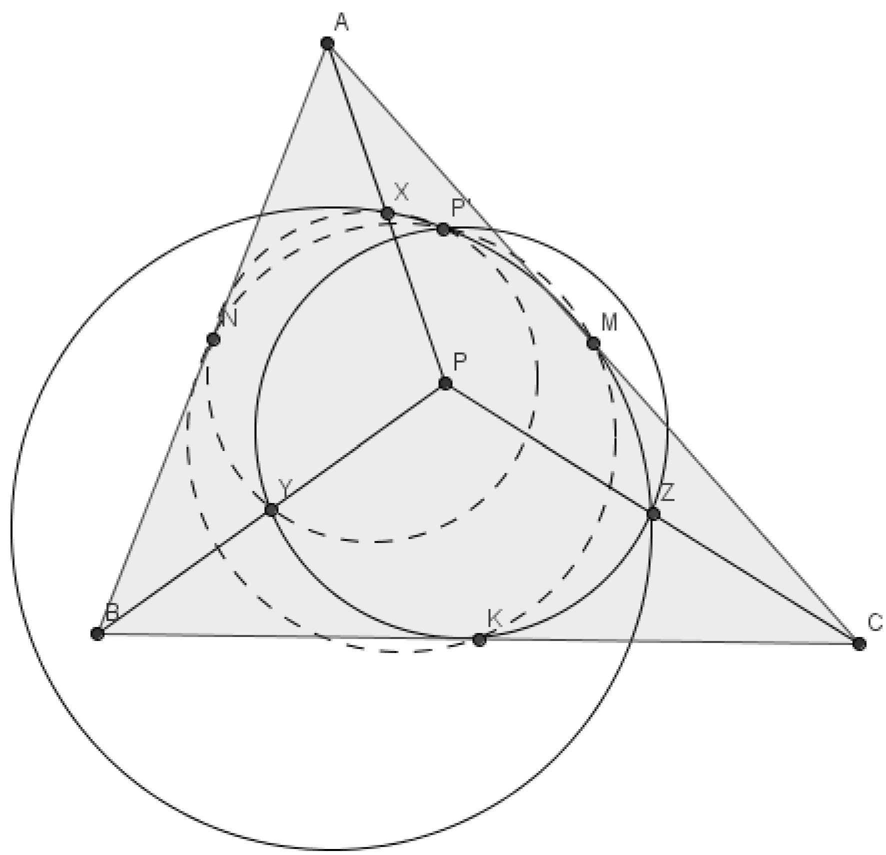

Figure 1: The Poncelet point P' with respect to $ABCP$

Proof. Let the medial triangle of $\triangle ABC$ be $\triangle KMN$ and the midpoints of PA, PB, PC be XYZ. Then if the intersection of the circumcircles of KYZ and MXZ be P', we find $\angle YP'Z = \angle BPC$ and $\angle XP'Z = 180 - \angle APC \implies \angle YP'Z = 180 - \angle BPA = 180 - \angle YNX$ and so $P' \in \odot YNX$. Similar arguments can be accomplished to show that P' lies on $\odot KMN$. □

Common notations that will be used throughout this article are as follows:

1. $\odot ABC$ will refer to the circle passing through ABC.

2. When speaking about a triangle ABC, we will generally omit using the symbol $\triangle$ except when the author believes it is necessary

3. For convenience, the abbreviation "wrt" will be used to mean "with respect to."

And thats it! It is expected that the reader is familiar with advanced geometric tools.

2 A subtle definition

Consider the following scenario:

Proposition 2.1. *Let ABC be a triangle, P be a point in its plane and P' be the Poncelet point of $ABCP$. Denote Z to be the reflection of P in P'.*

The definition above was given by Darij Grinberg in an article a few years ago, he defines this point as the *antigonal conjugate* of P wrt ABC. We will keep this notation throughout this article.

Theorem 2.2. *The antigonal conjugate of a point P wrt ABC is the isogonal conjugate of the inverse of the isogonal conjugate of P wrt ABC.*

The result is very suprising indeed. To prove this, we will prove the following:

Theorem 2.3. *The antigonal conjugate Z of P wrt ABC satisfies $\angle BZC = 180 - \angle BPC, \angle AZB = 180 - \angle APB, \angle AZC = 180 - \angle APC$.*

Proof. Let the reflection of P over the midpoint of BC be P_A, and let the circle through BCP_A be w_A. This circle is the image of the ninepoint circle of BCP under dilation throguh P with scale factor 2. If we define P_B, P_C, w_B, w_C similarly, then these circles are concurrent at the image of the poncelet point P' of P wrt ABC under this same dilation, which is the reflection of P through P'. But of course, $\angle BZC = 180 - \angle BPC = 180 - \angle BP_AC$. Applying a symmetrical argument, the result follows. □

Now we prove theorem 2.2.

Proof. We start off with a lemma:

Lemma 2.4. *Let P, Q be isogonal conjugates wrt ABC, then $\angle BQC + \angle BPC = 180 + A$*

Proof. Simple angle chasing gives $180 - \angle BPC + 180 - \angle BQC = 180 - A$ giving the result. However, do note that we can find similar relation for when P is outside of ABC, and this is $\angle BPC + \angle BQC = A$.

□

Using theorem 2.2, it suffices to show that the isogonal conjugate Q of P, the inverse Q' of Q is the isogonal conjugate of Z by finding $\angle BQ'C$. Using the above lemma, we will find $\angle BQ'C$ and then apply this lemma once more to yield a conclusion. Let O be the circumcentre of $\odot ABC$, we have $\angle OBQ = \angle OQ'B = \angle QBC - (90 - A)$ and similarly, $\angle OCQ = \angle OQ'C = \angle 90 - A - \angle QCB$ (depending on where P is). It follows $\angle BQ'C = \angle OQ'B - \angle OQ'C = (\angle QBC + \angle QCB) - 180 + 2A = 180 - \angle BQC - 180 + 2A = 2A - \angle BQC$. Lemma 2.4 then gives $\angle BQC = 180 + A - (\angle BPC)$, thus $\angle BQ'C = \angle BPC + A - 180$. The isogonal conjugate Z' of Q' wrt ABC will satisfy $\angle BZ'C = A - \angle BQ'C$ by lemma 2.4, which means $\angle BZ'C = 180 - \angle BPC$. By a symmetric argument, it follows that $Z = Z'$, and thus the proof is complete. $\square$

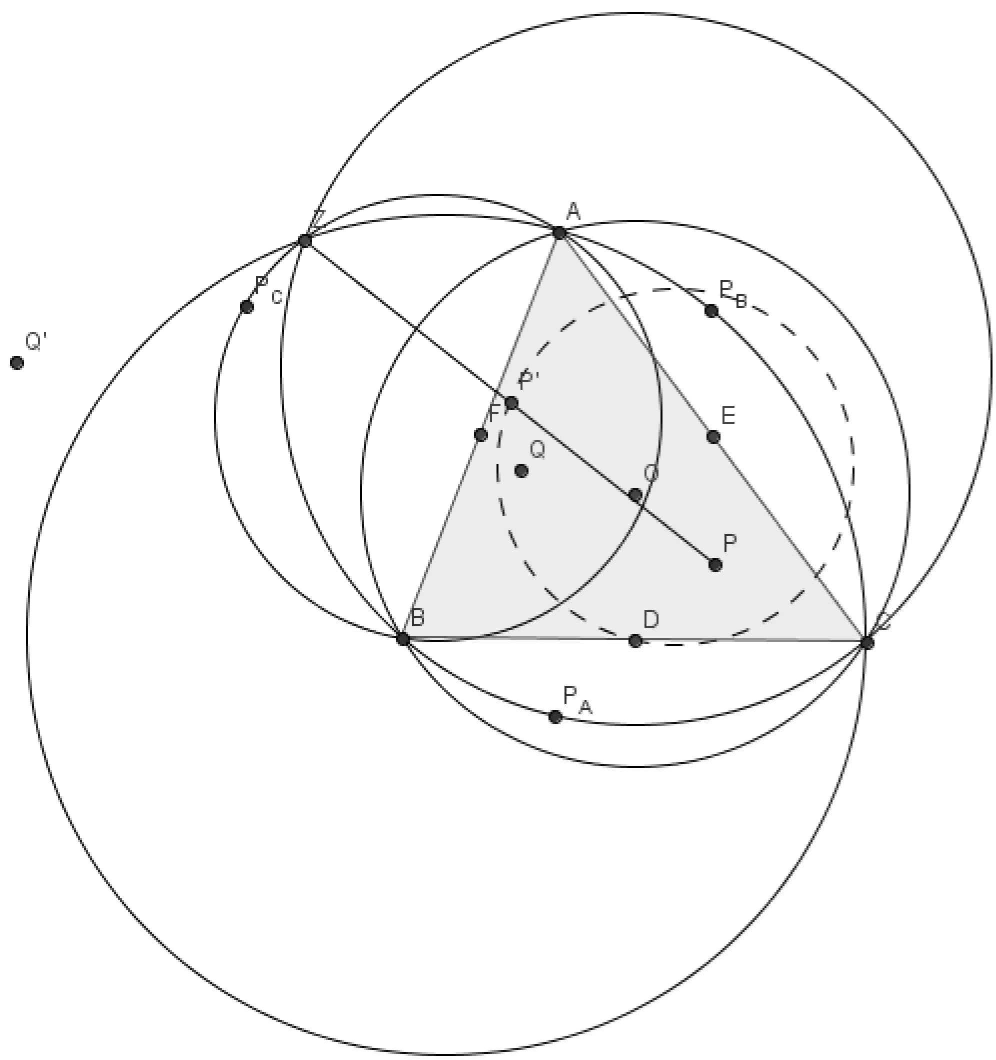

Figure 2: The antigonal conjugate of P, DEF is the medial triangle of ABC

This result yields very interesting corrolaries, which we will discuss in the next section.

3 What if P is on the circumcircle?

Theorem 3.1. *Consider a point P on $\odot ABC$ and let H_A be the orthocentre of BCP and define H_B, H_C, H_P similarly. $H_AH_BH_CH_P$ is the result of a reflection through P'.*

Proof. First, let us prove $H_AH_BH_CH_P$ and $ABCP$ are homothetic. Let M_A be the midpoint of BC, and consider the quadrilateral AH_PH_AP. $AH_P \parallel DH_A$ clearly and $AH_P = 2OM_A = PH_A$ (a well-known result), so AH_AH_PP is a parallelogram, and therefore, $H_AH_P \parallel AP$. Applying a symmetrical argument, it follows $ABCP, H_AH_BH_CHP_P$ are homothetic, and this homothetic centre is the midpoint of PH_P (by the virtue of the parallelogram), so it suffices to show this midpoint is P'.

We note by theorem 2.3 we have P' is the midpoint of PZ, Z being the antigonal conjugate of P wrt ABC. The isogonal conjugate of P wrt is the point at infinity, so the inverse is O, and the isogonal conjugate of O is H_P, which by theorem 2.2 is Z, so the midpoint of PH_P is P'. □

Corollary 3.2. *The Simson line ℓ of P passes through P'.*

Proposition 3.3. *Let $ABCP$ be a cyclic quadrilateral, let the centre be O. Let the midpoint of AB be M_{AB}, and define M_{BC}, M_{CP}, M_{AP} similarly. Let the perpendicular from M_{AB} to CP be ℓ_{AB} and define $\ell_{BC}, \ell_{CP}, \ell_{PA}$ similarly. Then $\ell_{AB}, \ell_{BC}, \ell_{CP}, \ell_{PA}$ concur at the Mathots point of $ABCP$.*

Proof. $M_{AB}M_{BC}M_{CP}M_{AP}$ is the Varignon paralleogram of $ABCP$, let the diagonals intersect at Q, and consider a half-turn about Q. This affine transformation maps $M_{AB} \mapsto M_{CP}$ and $M_{BC} \mapsto M_{AP}$, and because an affine transformation preserves parallelism, ℓ_{AB} is now the perpendicular bisector of CP, simlarly for the other lines, and so it follows these lines concur at O, therefore before the half-turn, they were concurrent at the reflection of O through Q. □

Theorem 3.4. *P' is the Mathots point of $ABCP$.*

Proof. It suffices to show, $M_{AB}P'M_{CD}O$ form a parallelogram. It suffice to prove $P'M_{AB} \perp CD$, and $P'M_{AB} = OM_{BC}$. Under a dilation with factor 2 about A, we have $P' \mapsto H_A$ and $M_{AB} \mapsto P$ by theorem 4, so then $P'M_{AB} \parallel PH_A \perp CD$ and $OM_{BC} = \dfrac{1}{2}AH_P = \dfrac{1}{2}PH_A = P'MAB$, and therefore the result follows. □

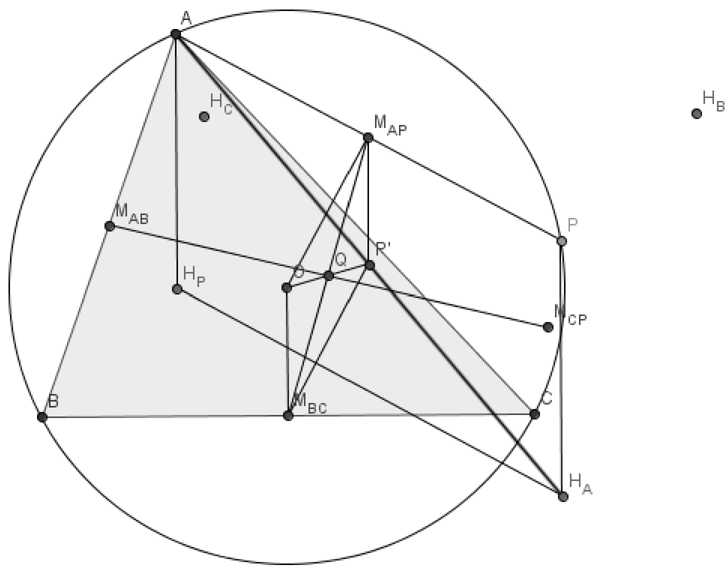

Figure 3: When P is on the circumcircle!

4 A step into the projective light

Proposition 4.1. *Let ABC be a triangle a P be a point. Then, the hyperbola passing through $ABCP$ and the orthocenter H of ABC has perpendicular asymptotes.*

Proof. Let the isogonal conjugate of D be D' wrt ABC and let O be the centre of ABC, then the hyperbola is the isogonal conjugate of line OD' wrt ABC. Let OD' intersect $\odot ABC$ at X, Y. The asymptotes pass through the isogonal conjugates of X, Y, and since $\angle XAY = 90$, the directions of the points at infinity of X, Y are perpendicular, meaning the hyperbola will be rectangular (equilateral). $\square$

Theorem 4.2. *Let ABC be a triangle and $\mathcal{H}$ be a rectangular hyperbola passing through ABC. Then the centres of $\mathcal{H}$ lie on the ninepoint circle of $\triangle ABC$.*

Proof. This is just theorem 3.1 in disguise! Let $\mathcal{H} \cap \odot ABC$ be P (other intersections). By definition, H_A, H_B, H_C, H_P lie on $\mathcal{H}$. We know that $H_AH_BH_CH_P$ and $ABCP$ are centrally symmetric about P' by theorem 3.1, so it follows P' is the centre of $\mathcal{H}$. $\square$

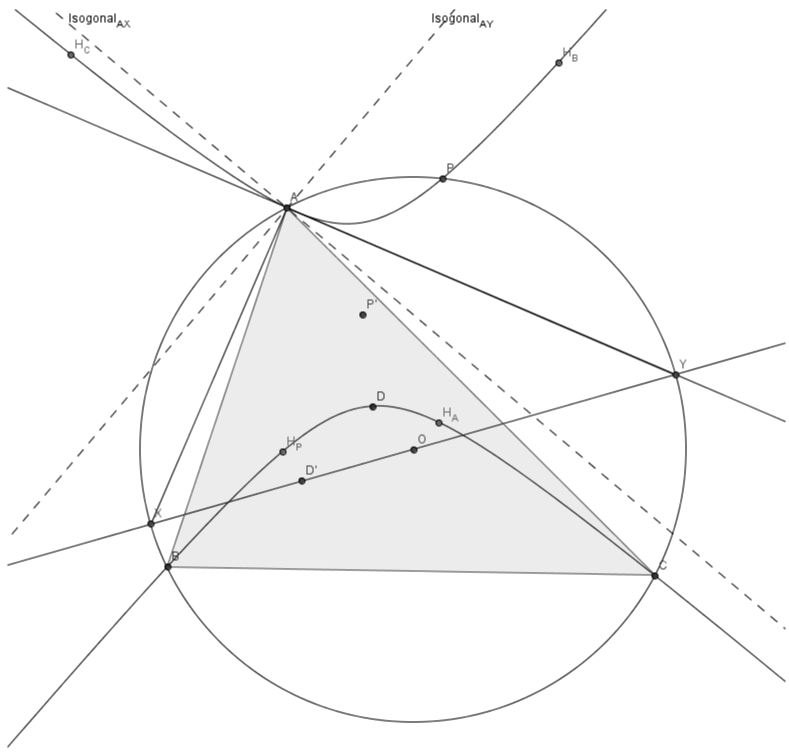

Figure 4: An introduction to rectangular hyperbolas

We now form an intersecting definition of a rectangular hyperbola.

Proposition 4.3. *Let B, C be two points, and consider lines ℓ_1, ℓ_2 that pass through B, C respectively. Suppose they rotate around B, C in the same velocity but in opposite directions. Then the intersections of ℓ_1, ℓ_2 create a rectangular hyperbola with B, C being anti-podal points.*

Proof. Let P, A be points on the rectangular hyperbola with diametre B, C. Let DEF be the medial triangle of APB and M be the centre of the hyperbola, which is the midpoint of BC. We have $\angle ABP = \angle DEF = \angle DMF = \angle ACP$ as desired. □

Corollary 4.4. *The antigonal conjugate Z of P is the antipode of P.*

Proof. Indeed, apply preposition 4.3. Due to theorem 2.3, $BPC = 180 - \angle BZC$ and similarly for AC, AB and so by definition, $ABCPZ$ lie on a rectangular hyperbola with P, Z being antipodal. This is unique.

□

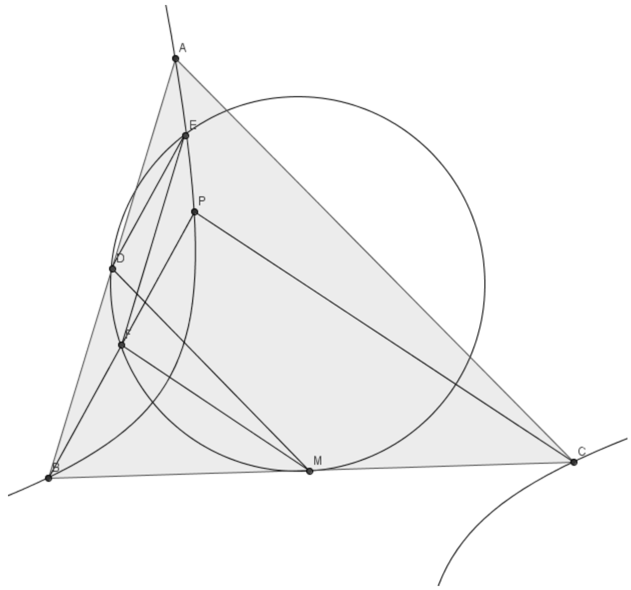

Figure 5: An interesting definition

Theorem 4.5. *Let ABC be a triangle, P be a point with its reflections over the midpoints of BC, AC, AB being P_A, P_B, P_C. Let Z be the antigonal conjugate of P wrt ABC. We have $Z, P_A, P_B, P_C, A, B, C$ lie on a conic.*

Proof. We first show $Z \in \odot P_A B P_B P_C$. Note that under a dilation about P with factor 2, the ninepoint circle of ABC goes to $\odot P_A P_B P_C$ and P' goes to Z, so the result follows. By definion, if the reflections of P over BC, AC, AB are P_1, P_2, P_3 respectively, then $P_1 P_2 P_3 Z$ are concyclic. Hence a direct relation is that the Poncelet point P' is contained within the pedal circle, because of a $\frac{1}{2}$ dilation about P.

Now, let $P_B Z \cap P_A C = X$ and $ZB \cap P_A P_C = Y, ZB \cap CP_A = M, ZP_A \cap BC = N$. Then the angle bisector of $\angle P_A NC$ is perpendicular to the angle bisector of $\angle BMC$. Since $P_C P_B \parallel BC$, if we let $ZP_B \cap P_C P_A = K$ then $P_C P_B$ is anti-parallel to ZP_A wrt $K \implies BC$ is anti-parallel to ZP_A wrt K, but ZP_A is anti parallel to BC wrt M so $ZP_A XY$ is cyclic. It follows $XY \parallel BC \parallel P_B P_C$ so by pascals theorem, Z is on the conic passing through $P_C P_B BC P_A$ and we can show it lies on $ABCP_C P_B P_A$ by a symmetric argument.

This also yields a proof that P, Z are antipodal. Denote the conic by $\mathcal{C}$ from the above. If we dilate by factor $\frac{1}{2}$ about P, let $\mathcal{C}$ go to $\mathcal{C}'$. This conic is through the medial triangle of ABC and the midpoints of AP, BP, CP. We also know that the Poncelet point P' lies on $\mathcal{C}'$ since P' the midpoint of PZ. But, the P' lies on the ninepoint centre of ABC, denote it as (N), and if we assume that the centre T of $\mathcal{H}$ and P' the Poncelet point are distinct, then $\mathcal{C}' \cap (N)$ has 5 distinct points. This is a contradiction, as their are at most 4, and thus $P' = T$, so the Poncelet point is the centre of $\mathcal{H}$.

□

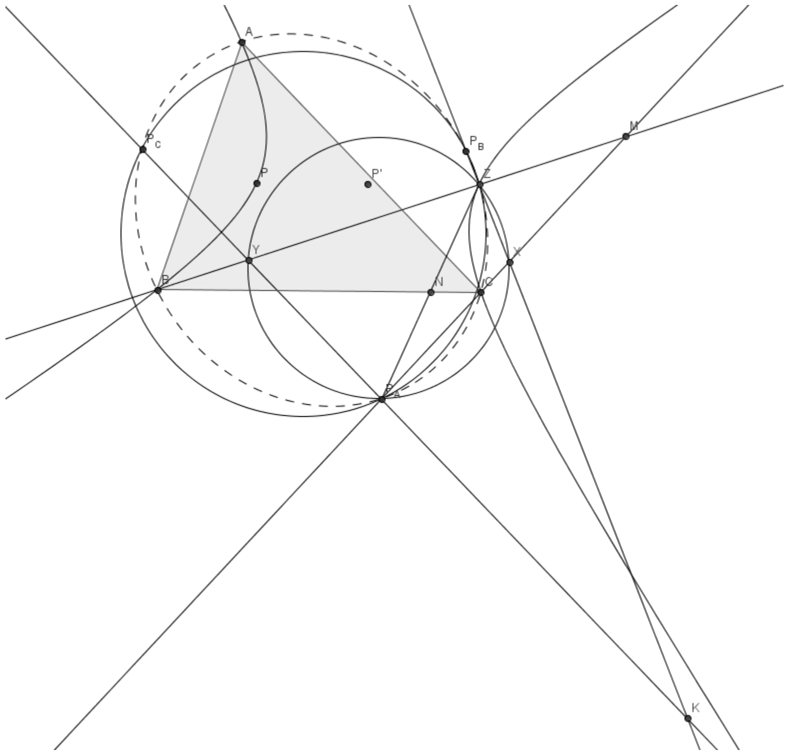

Figure 6: A suprising conic!

Theorem 4.6. *Let ℓ be a line passing through the circumcentre O of ABC. If P is a point moving on this line, then the pedal circle of P with respect to triangle ABC intersects the Nine-point circle of ABC at a fixed point.*

Proof. It is easy to verify that this theorem is just corollary of corollary 4.4 by taking the isogonal conjugate of ℓ. $\square$

Corollary 4.7. *If the lines ℓ intersects AB, AC and M, N, then the circles with diametres BN, CM intersect the nine-point circle at a point P.*

Corollary 4.8. *Let O be the circumcenter of triangle ABC. A line through O intersects sides AB and AC at M and N, respectively. Let S and R be the midpoints of BN and CM, respectively. Prove that $\angle ROS = \angle BAC$.*

Proof. Consider the isogonal conjugate of this line. It is a rectangular hyperbola so it follows that the pedal circles of M, N wrt ABC intersect the nine-point circle at P. Now, let $(S) \cap (R) = Q$ and the altitudes from B, C be E, F. By an easy angle chase (note $\angle CQP = \angle PFC$ etc...) we have Q lies on $\odot ABC$. It then follows that $\angle SOR = 180 - (180 - \angle BAC) = \angle BAC$. $\square$

Remark. This problem appeared in the Awesome Math Summer Program and was considered to be very hard. Solutions using pascal's theorem are also possible.

Corollary 4.9. *If F is the first fermat point, and F' is the second fermat point, then the midpoint of FF' lies on the nine-point circle.*

Proof. If $F = P$, then note that the isogonal conjugate of F is the first apolonius point, and the inverse is the second, so the isogonal conjugate of this point is F', so $F' = Z$, therefore the midpoint of FF' is the centre of the Kiepert Hyperbola, so is lies on the nine-point circle. □

5 A tangent

Theorem 5.1. *If we have a line ℓ passing through O that intersects the circumcircle at P, Q, then the asymptotes of the isogonal conjugate of OPQ are the Simson lines of P, Q.*

Proof. Let $\mathcal{H}$ be the isogonal conjugate of OPQ, and suppose the 4th intersection of $\mathcal{H}$ with $\odot ABC$ is T. Then T is the isogonal conjugate of the point at infinity of PQ. Let the midpoint of HT be T' and the midpoint of HP be P' and note that the Simson line of P contains P'. Notice that T is the antipode of H wrt $\mathcal{H}$, by theorem 4.2, so T' is the centre of $\mathcal{H}$. Since the Simson line of P is in the perpendicular direction of the isogonal conjugate of P, it passes through the isogonal conjugate of Q, meaning the Simson lines are in the proper directions and it suffice to show they passes through T'. To do this, it suffice to show the rate of change of $TP \parallel T'P'$ is rate of change of the Simson line, and this is left to the reader. To finish, note when P coincides with the antipode of A wrt $\odot ABC$ the result is clear. Hence as P runs along $\odot ABC$, $T'P'$ is the Simson line of P and apply a symmetrical argument for Q. □

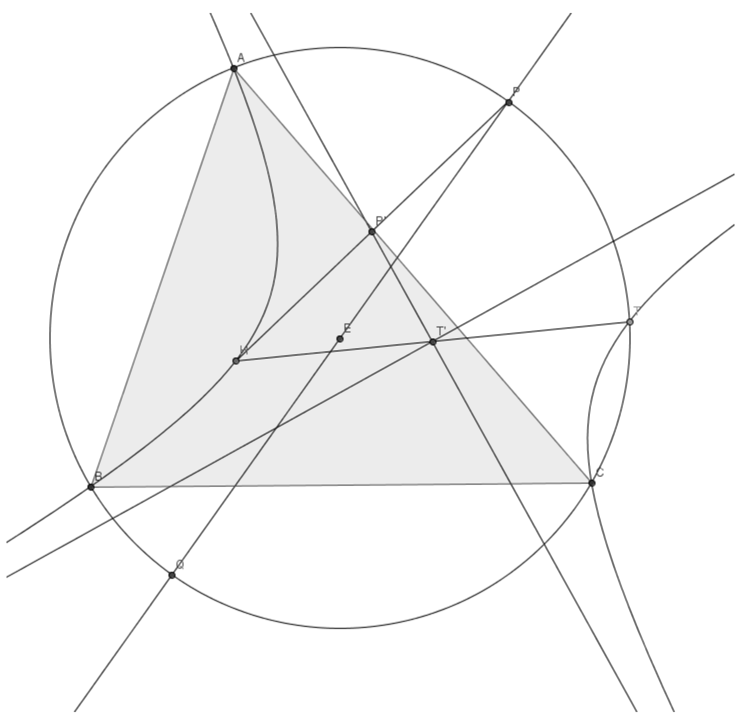

Figure 7: It's definitely "Simson" not "Simpson"!

Corollary 5.2. *There are three points on the circumcircle that have simson lines tangent to the ninepoint circle and that the Simson lines form an equilateral triangle.*

Proof. Consider a point P and its Simson line ℓ. Let the midpoint of HP be $P' \implies P' \in \ell$. It is easy to verify that if the change of the arc formed by P' as P moves along ℓ is x, then the simson line moves at an angle $\frac{x}{2}$. Hence, it takes $x = \frac{\pi}{3}$ to make $\frac{\pi}{2}$ net rotation. Hence, the angle formed by (NP', ℓ) cycles three times, at $\frac{\pi}{3}$ intervals, so the result follows trivially. $\square$

Remark. It has been discusses that this triangle is homothetic to the Morley triangle wrt ABC. This problem appeared in the Romanian TST.

Theorem 5.3. *Let ABC be a triangle, P, Q be points in ABC and let P', Q' be their isogonal conjugates. Prove that $PQ \cap P'Q' = X, PQ' \cap QP' = Y$ are isogonal conjugates.*

Proof. If we take the circumhyperbola passing through PQABC then we must show Y is on it. Consider the one-to-one transformation taking a line to a line, also taking $P \mapsto Q$ and keeping ABC fixed. We want to show $(PA, PB, PC, PY) = (QA, QB, QC, QY)$, which follows if this transformation takes $P' \mapsto Q'$. Indeed, note if P' goes to Z then $(AB, AC, AP, AP') = (AB, AC, AQ, AQ') = (AB, AC, AQ, AZ)$ since P goes to Q, so we have $Z \in AQ'$. By a symmetrical argument, it follows that $Z = Q'$, so P' indeed goes to Q' under this transformation. $\square$

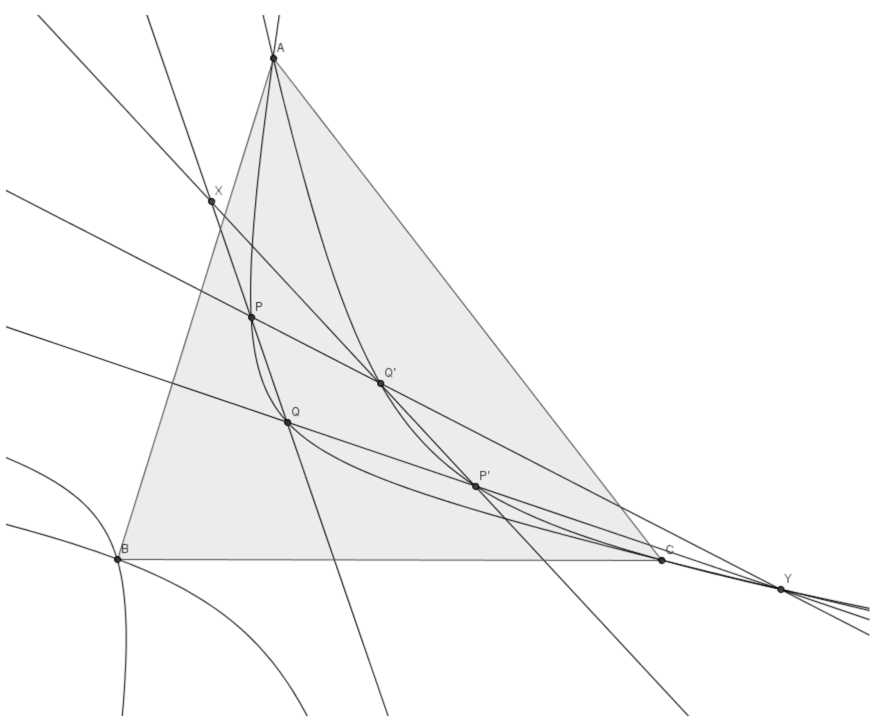

Figure 8: Isogonality of the Complete Quadrilateral!