Begonia points and coaxal circles / Darij Grinberg

We begin with a (new?) result from circle geometry.

Theorem 1. Let ABC be a triangle and P a point. Let A' be the point of intersection of the circle BCP and the line AP different from P, and similarly define B' and C'. Let X, Y, Z be the centers of the circles $B'C'P$, $C'A'P$, $A'B'P$. Then, the circles PAX, PBY, PCZ are coaxal, i. e. they have a common point different from P (or touch at P).

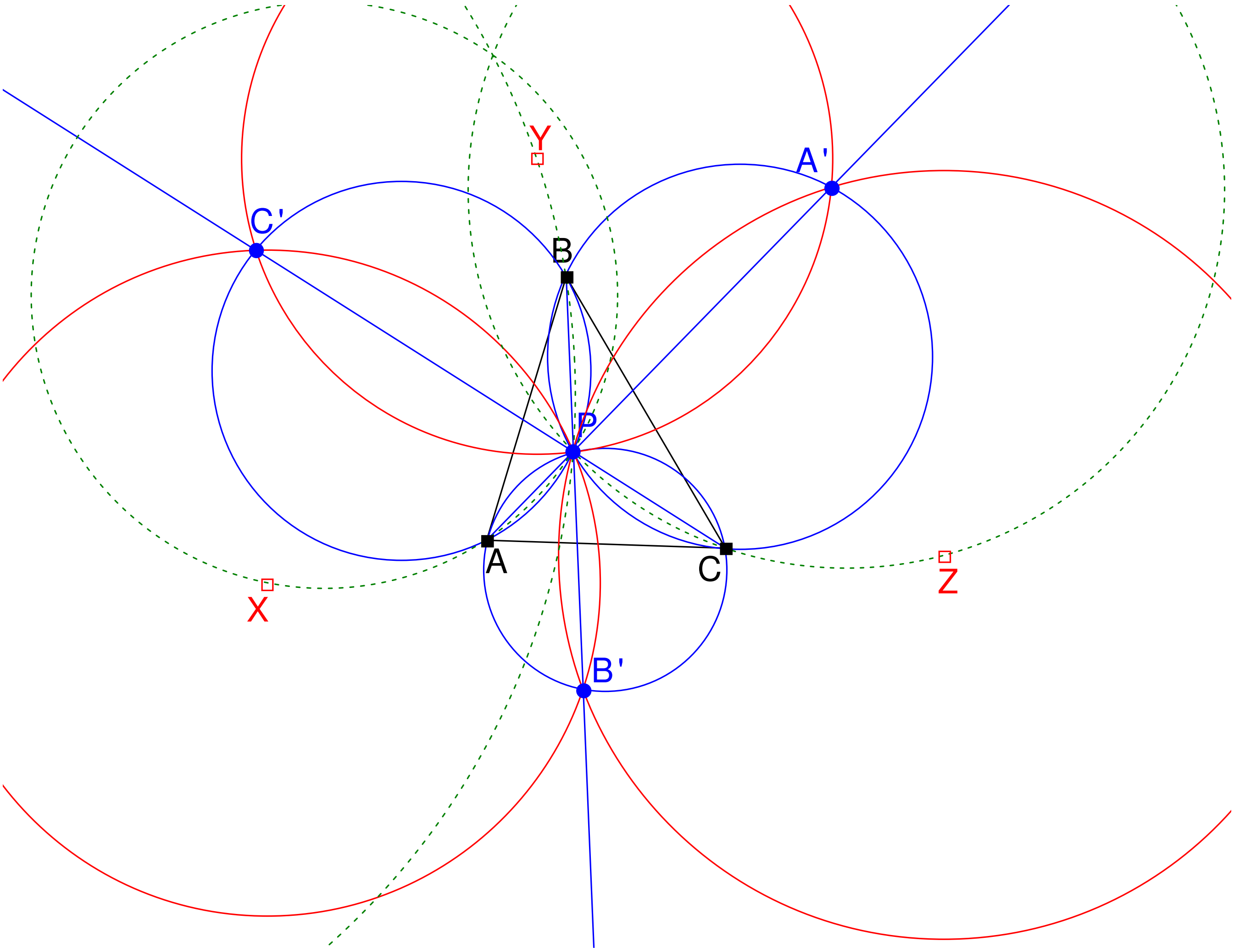

Fig. 1

Proof. The center of a circle lies on the perpendicular bisector of any chord. This fact will be used very often in the following proof.

The centers Y and Z of the circles $C'A'P$ and $A'B'P$ lie on the perpendicular bisector of PA'. Hence, the line YZ is the perpendicular bisector of PA'. Similarly, the lines ZX and XY are the perpendicular bisectors of PB' and PC', respectively.

Let X', Y', Z' be the centers of the circles BCP, CAP, ABP. We obtain by the same reasoning that the lines $Y'Z'$, $Z'X'$ and $X'Y'$ are the perpendicular bisectors of PA, PB, PC, respectively.

Being the perpendicular bisectors of PA and PA', the lines $Y'Z'$ and YZ must be parallel (in fact, the lines PA and PA' coincide). Analogously, $Z'X' \parallel ZX$ and $X'Y' \parallel XY$.

Since the circle BCP passes through A', its center X' lies on the perpendicular bisector of PA', i. e. on the line YZ. Equally, Y' lies on ZX and Z' lies on XY.

The parallelograms $XZ'X'Y'$ and $ZY'Z'X'$ yield $XY' = Z'X'$ and $ZY' = Z'X'$, so that $XY' = ZY'$, and Y' is the midpoint of the segment ZX. Similarly, X' is the midpoint of YZ and Z' is the midpoint of XY.

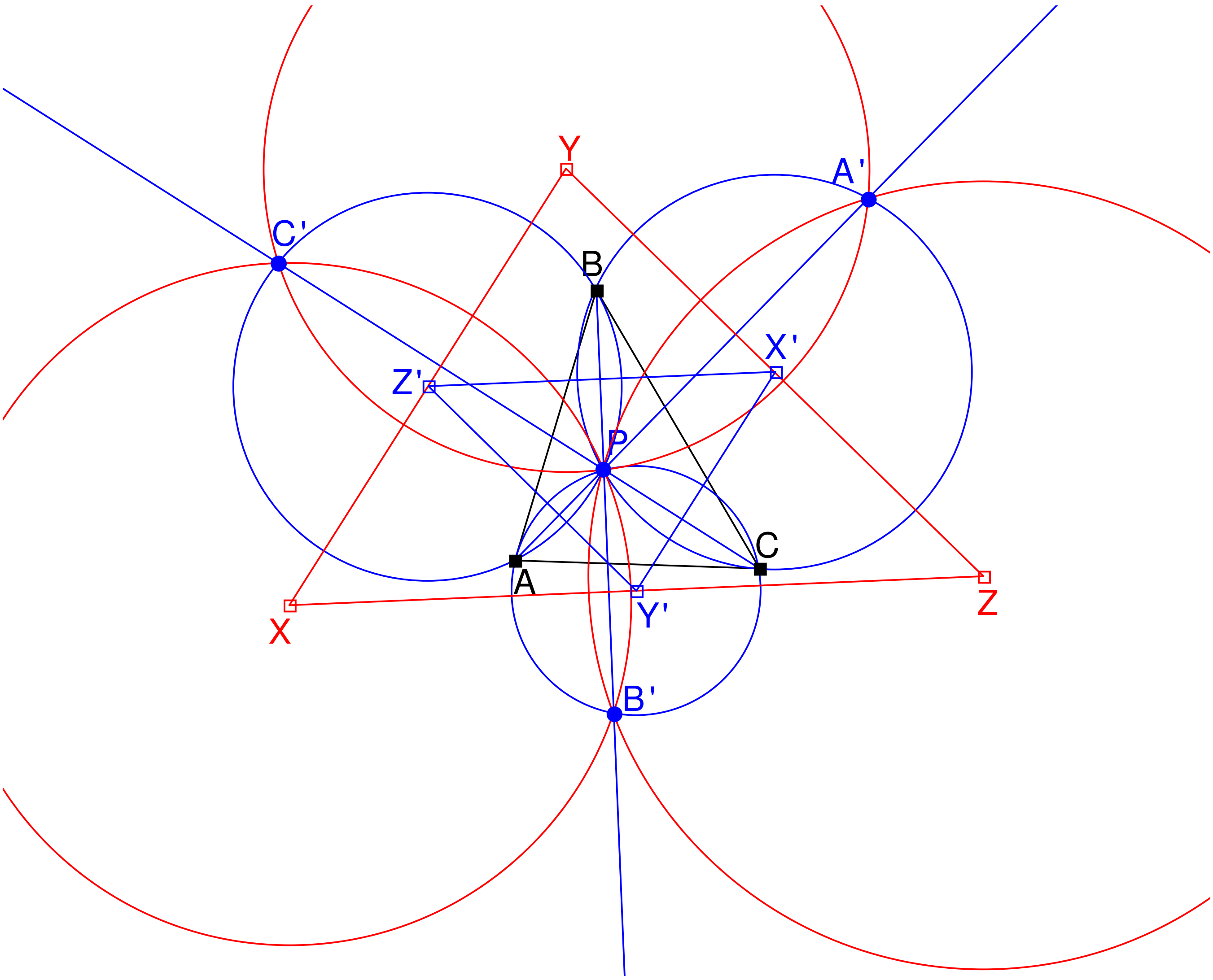

Fig. 2

Now let D, E, F the centers of the circles YZP, ZXP, XYP. By the same arguments as before, the lines EF, FD, DE will be the perpendicular bisectors of PX, PY, PZ.

Being the center of the circle PYZ, the point D also lies on the perpendicular bisector of YZ. Hence, the line DX' is the perpendicular bisector of YZ (remember that X' is the midpoint of YZ). Likewise, the lines EY' and FZ' are the perpendicular bisectors of ZX and XY. Hence, the lines DX', EY', FZ' concur. By the Desargues theorem, we infer that the points $EF \cap Y'Z'$, $FD \cap Z'X'$, $DE \cap X'Y'$ are collinear.

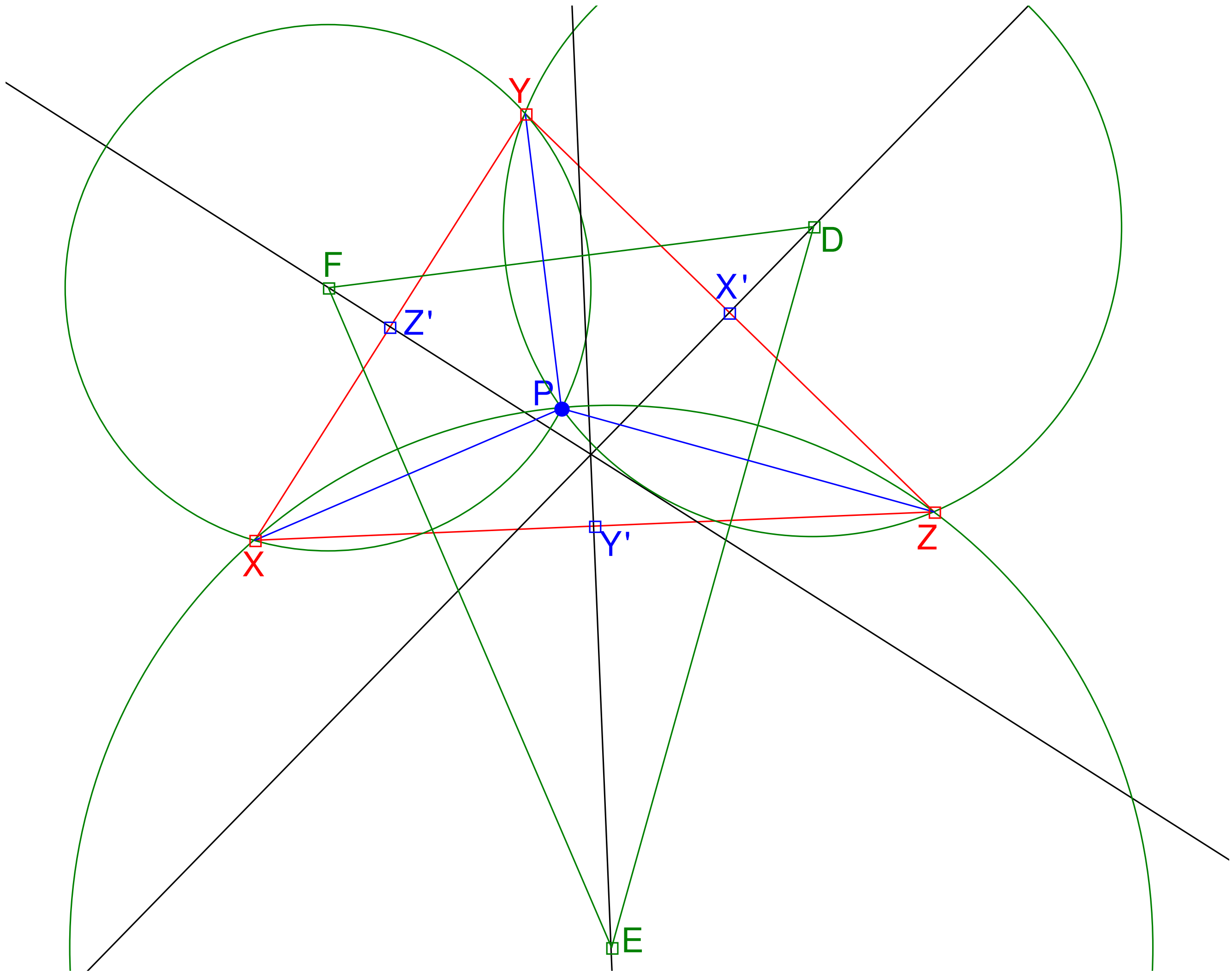

Fig. 3

Now, the line EF is the perpendicular bisector of PX, and the line $Y'Z'$ is the perpendicular bisector of PA. Hence, the point $EF \cap Y'Z'$ is the center of the circle PAX. Analogously, the two other points, $FD \cap Z'X'$ and $DE \cap X'Y'$, will be the centers of the circles PBY and PCZ. Thus, the centers of the circles PAX, PBY, PCZ are collinear. Now, having P as a common point, it follows that the circles are coaxal, proving Theorem 1.

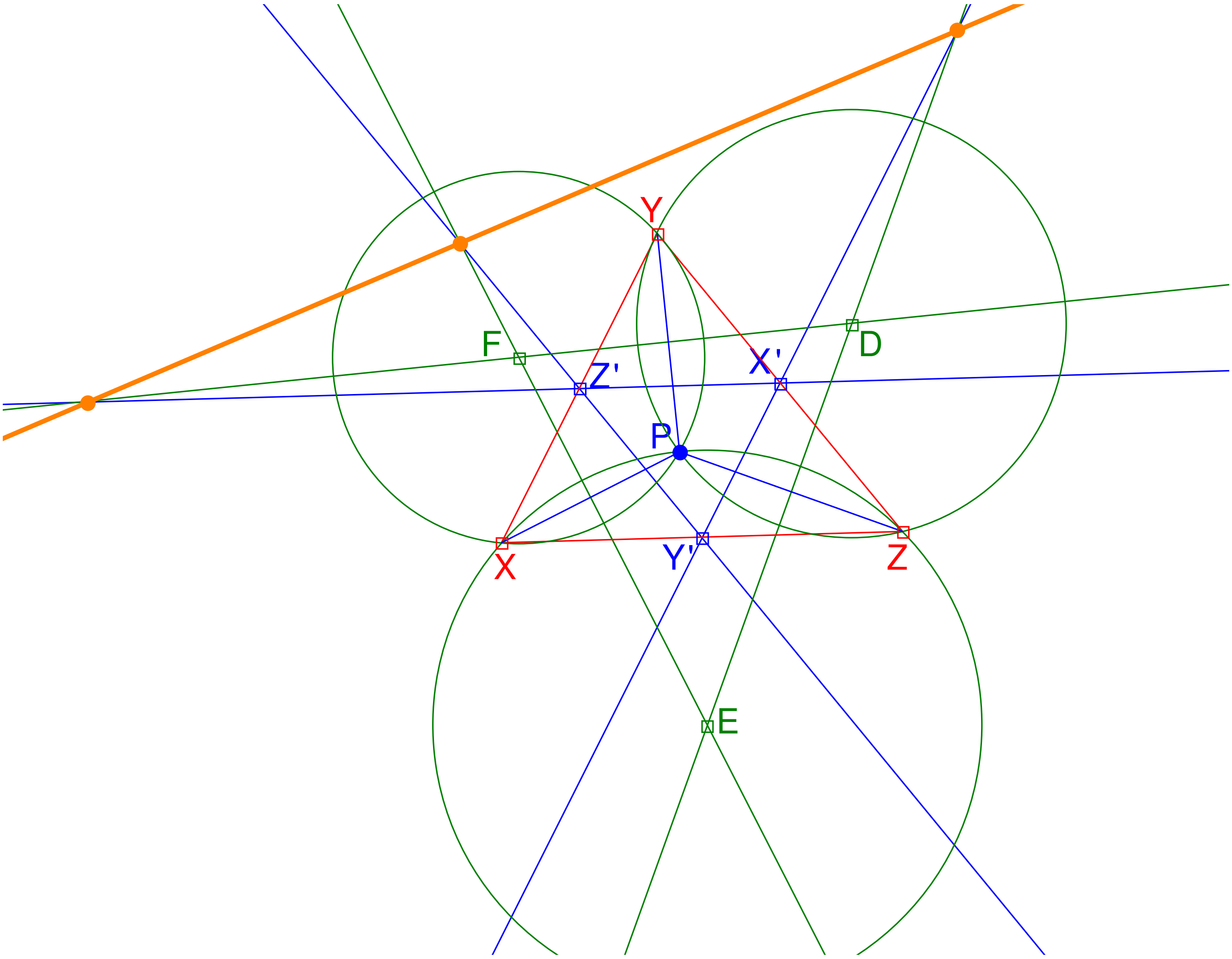

Fig. 4

Let's now establish a property of Inversion:

Lemma 2. If P' and Q' are the images of two points P and Q in an inversion with center O, then the reflection of O in the line PQ is mapped to the center of the circle $P'Q'O$.

Although this is a special case of a general result about orthogonal circles, we give an elementary *proof*.

Let R be the reflection of O in PQ, and R' the image of R in our inversion. We shall show that R' is the center of the circle $P'Q'O$.

If r is the radius of the inversion circle, $OP \bullet OP' = r^2$, $OQ \bullet OQ' = r^2$ and $OR \bullet OR' = r^2$. Thus, $OP \bullet OP' = OR \bullet OR'$, and $OP : OR = OR' : OP'$. This entails that triangles POR and $R'OP'$ are similar, and $\measuredangle R'P'O = \measuredangle PRO$. But as R is the reflection of O in PQ, we have $\measuredangle PRO = \measuredangle POR$, and thus $\measuredangle R'P'O = \measuredangle POR = \measuredangle P'OR'$. Consequently, triangle $P'R'O$ is isosceles, and $OR' = P'R'$. The same argument shows $OR' = Q'R'$. Thus, R' is the center of the circle $P'Q'O$, qed..

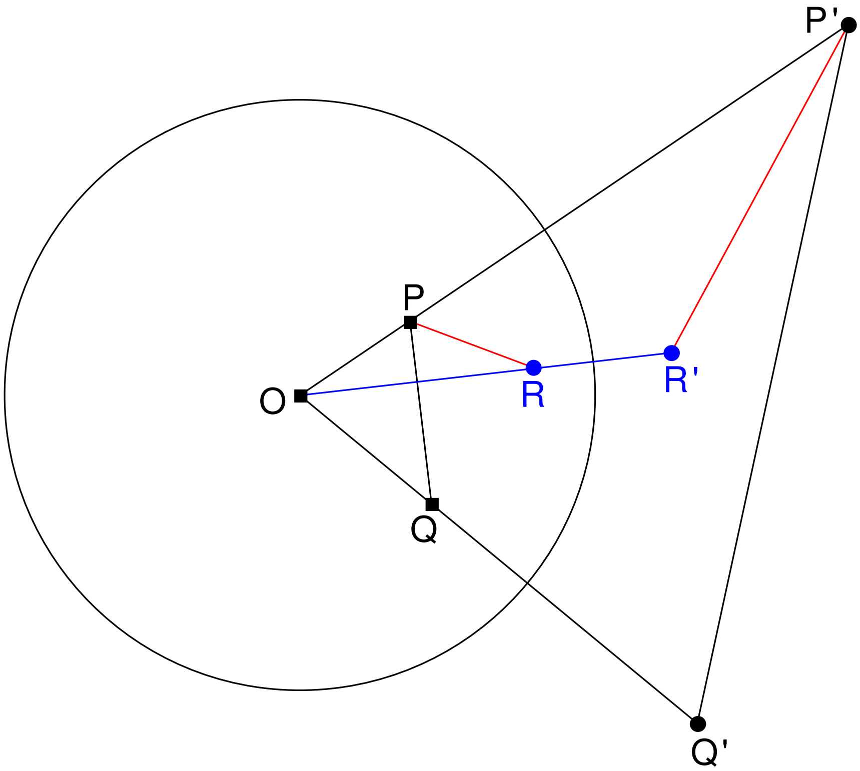

Fig. 5

Finally, we will prove the theorem of Jean-Pierre Ehrmann in Hyacinthos message #7999:

Theorem 3. Given a triangle ABC and a point P. Let $A'B'C'$ be the cevian triangle of P (i. e., the triangle formed by the points $A' = AP \cap BC$, $B' = BP \cap CA$ and $C' = CP \cap AB$), and X, Y, Z be the reflections of P in the lines $B'C'$, $C'A'$, $A'B'$, respectively. Then the lines AX, BY, CZ are concurrent.

Their common point is called **Begonia point** of P with respect to triangle ABC.

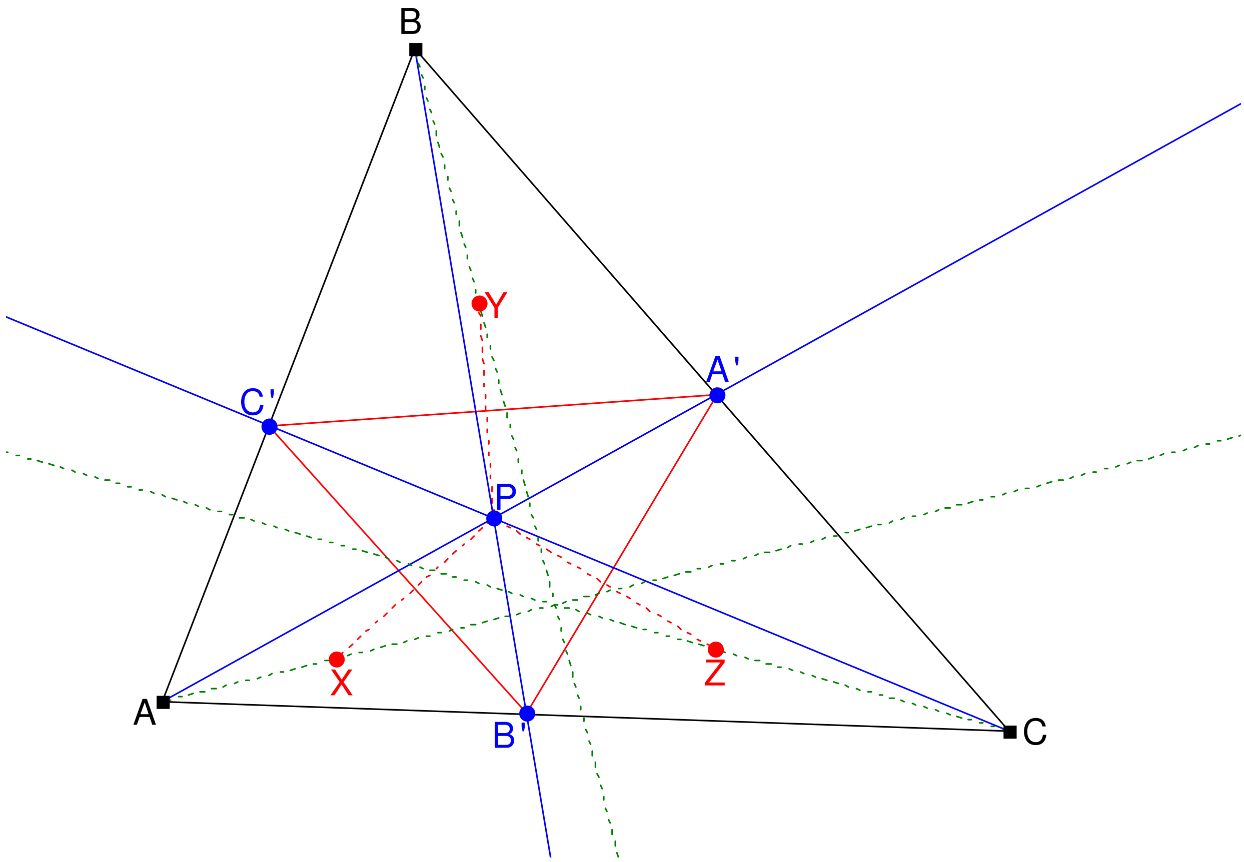

Fig. 6

Proof of Theorem 3. We apply an inversion with center P. Let a, b, c, a', b', c', x, y, z be the images of the points A, B, C, A', B', C', X, Y, Z. The image of the line BC is the circle bcP. The image of the line AP is itself, but we could better say it is the line aP. The point of intersection A' of the line AP and BC is mapped to the point of intersection a' of the line aP with the circle bcP (different from P). Analogously, we identify b' and c'.

After Lemma 2, the reflection X of P in $B'C'$ will be mapped to the center x of the circle $b'c'P$. Similarly, y and z are the centers of the circles $c'a'P$ and $a'b'P$.

Now, Theorem 1 shows that the circles Pax, Pby and Pcz have a common point different from P. Inverting back convinces that the lines AX, BY, CZ are concurrent, qed..

Another proof of Theorem 3, using some projective geometry arguments, was given by Jean-Pierre Ehrmann in Hyacinthos message #8039.