Introduction to Vectors

K. Behrend

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Abstract

An introduction to vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. Lines and planes in $\mathbb{R}^3$. Linear dependence.

Contents

Introduction

These notes are in somewhat preliminary form. The content is not yet stable. Watch for updates.

1 Vectors

1.1 Plane vectors

Plane vectors (vectors in the plane) are pairs of real numbers. Here are some examples:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \begin{pmatrix} -\frac{5}{3} \\ 17.334 \end{pmatrix} \qquad \begin{pmatrix} \pi \\ -e \end{pmatrix}$$

where $\pi = 3.1415\ldots$ and $e = 2.7182\ldots$.

The vector $\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$ has *first component* the real number 2 and *second component* 3. The vector $\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$ is *not* equal to the vector $\left(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right)$.

All the above numbers $0, 1, 2, 3, -\frac{5}{3}, 17.334, \pi, -e$ are examples of real numbers. The set of all real numbers is $\mathbb{R}$. We write $\pi \in \mathbb{R}$ ('π is an element of $\mathbb{R}$'), if we want to say that π is a real number.

Notice that we always write vectors as columns:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \textit{not} \qquad \begin{pmatrix} 2 & 3 \end{pmatrix}$$

To emphasize this fact, we also call these vectors *column vectors*. The set of all vectors with two components is denoted $\mathbb{R}^2$.

Often we call the first component the x-component, and the second component the y-component.

1.2 Geometric meaning: position vectors

To every column vector we associate a point in the *Euclidean plane*, see Figure 1. To locate the point corresponding to the vector $\binom{2}{3}$, start at the origin (where the x-axis and the y-axis intersect), move over by 2 units into the x-direction (horizontally to the right) and by 3 units into the y-direction (vertically upwards). If one of the components is negative, go in the opposite direction. In Figure 1, we have plotted the points given by the vectors $\binom{2}{3}$, $\binom{0}{1}$, $\binom{\frac{5}{2}}{-\frac{1}{2}}$ and $\binom{-2.25}{-1}$.

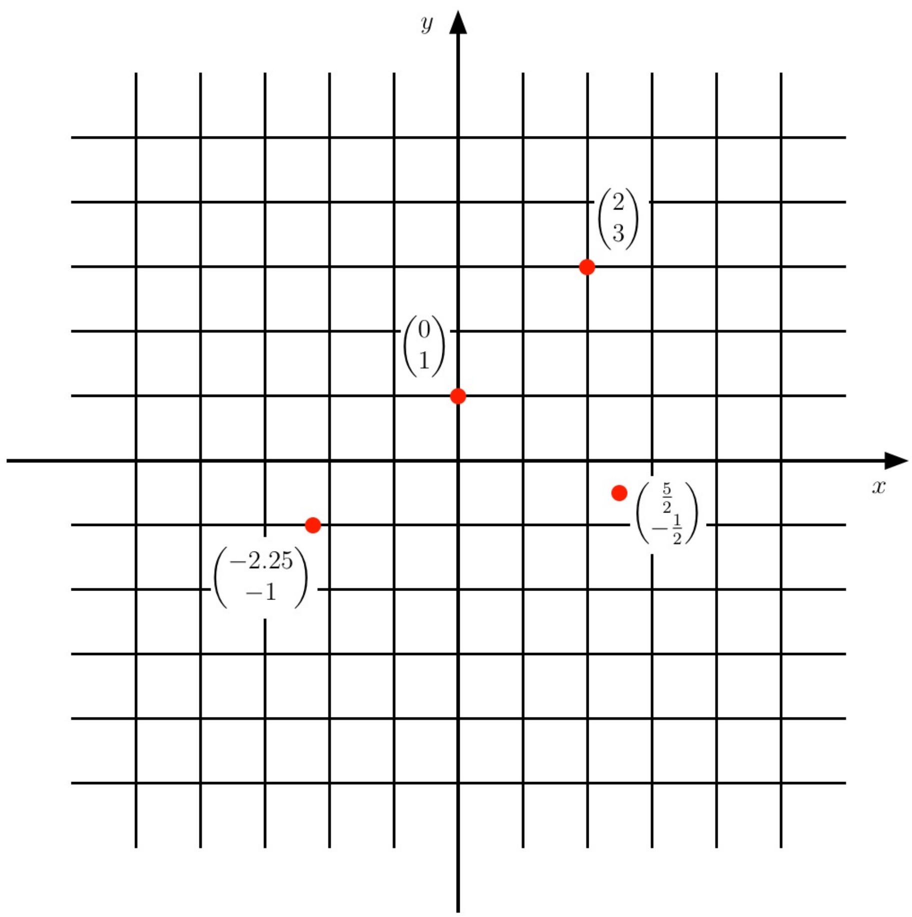


Figure 1: The points (in red) corresponding to a few column vectors

Associating points in the plane to column vectors assumes that we have chosen a *Cartesian coordinate system* in the plane. So it is better to speak of the *Cartesian plane*, instead of the Euclidean plane.

We can also go the other way: given a point in the plane, we associate to it a column vector as in Figure 2. Drop the perpendiculars (the dotted lines) onto the two axes to read off the x-coordinate and the y-coordinate of the point. In Figure 2, the point is called P, and its two coordinates are -3.5 and 2. The corresponding column vector is $\binom{-3.5}{2}$. Because the vector $\binom{-3.5}{2}$ tells us the position of the point P, we call $\binom{-3.5}{2}$ the *position vector* of P. The two *coordinates* of P make up the *components* of the position vector of P.

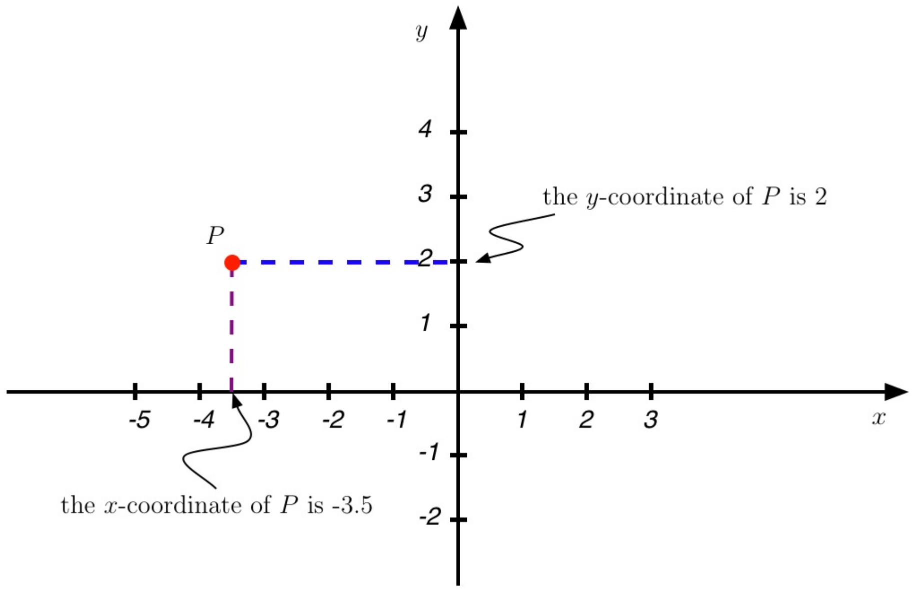


Figure 2: The point P (in red) and its two coordinates

So we can go both ways: from column vectors to points in the plane and from points in the plane to column vectors. We might as well think of column vectors and points in the plane as *the same thing*. Thus, the plane is now *identified* with the set of column vectors $\mathbb{R}^2$. *The plane is* $\mathbb{R}^2$.

1.3 Application

The idea that vectors with two components correspond to points in the plane is used in daily life. In Figure 3, a coordinate system is superimposed onto a map of Vancouver. Locations in East Vancouver correspond to vectors with positive x-coordinate. Locations on the West Side correspond to vectors with negative x-coordinate. The intersection of Cambie and Broadway has position vector $\binom{-5}{-9}$, because Cambie Street is the fifth street West of Ontario (which is on the y-axis) and Broadway is Ninth Avenue. The beginning of Kingsway has position vector $\binom{2}{-7}$.

The x-coordinate of a location in Vancouver is the number of blocks from Ontario Street in the easterly direction. The y-coordinate is the number of blocks from the (imaginary) 0th Avenue in the northerly direction.

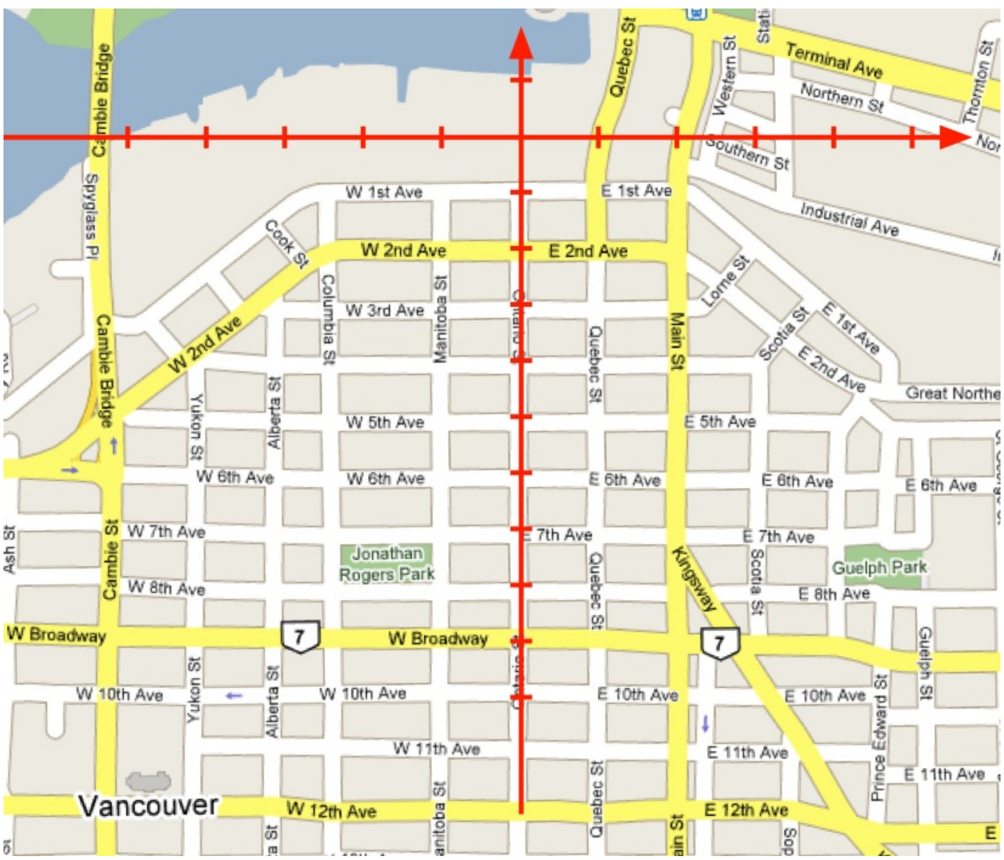


Figure 3: Locations in Vancouver correspond to plane vectors.
Note that units in x- and y-direction differ slightly: blocks are longer in East-West direction than in North-South direction.

If a house with position vector $\binom{x}{y}$ is located on an Avenue in East Vancouver, its house number will be $100x$. If it is on an Avenue on the West Side, its house number will be $-100x$. For example, a house half way between Alberta and Yukon, on the South side of 6th Avenue will have a position vector $\binom{-3.5}{-6}$. Its address will be 350 West 6th Ave, because its house number is $-100x = -100 \cdot (-3.5) = 350$.

On a global scale, we can associate position vectors to locations on the Earth. For this we pretend that the Earth is *flat*, see Figure 4. The location of Vancouver is marked in red. The position vector of Vancouver in the coordinate system of Figure 4 is $\left(\begin{smallmatrix}-8.21\\3.28\end{smallmatrix}\right)$. Unit length in Figure 4 corresponds to $15°$, so the x-coordinate of -8.21 corresponds to a longitude of

$$8.21 \cdot 15° = 123.15° \quad \text{West}$$

and the y-coordinate 3.28 corresponds to a latitude of

$$3.28 \cdot 15° = 49.2° \quad \text{North.}$$

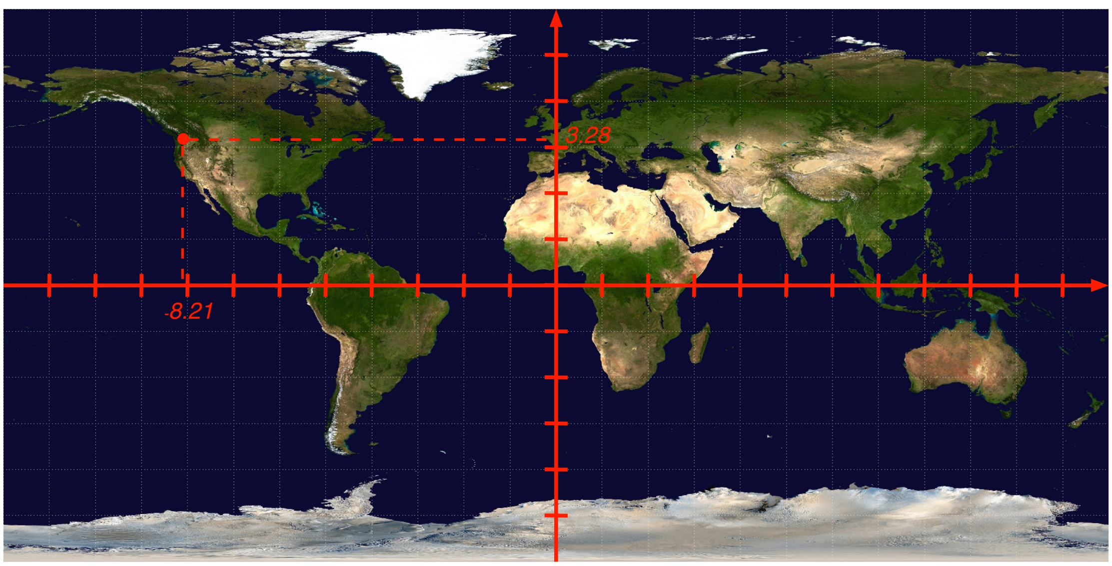


Figure 4: Flat Earth (actually an *equirectangular projection* of the earth). The x-axis measures longitude, in units of $15°$, the y-axis measure latitude, also in units of $15°$. Locations in the Western hemisphere have negative x-coordinate, locations in the Eastern hemisphere have positive x-coordinate.

1.4 Geometric meaning: displacement vectors

Another way to associate a geometric concept to a vector $\binom{x}{y}$ is that of *displacement vector*.

The vector $\binom{3}{2}$ moves, or *displaces*, everything 3 units to the right and 2 units up. We use *arrows* to draw movements (displacements). For example, the point with position vector $\binom{1}{1}$ gets moved to the point with position vector $\binom{4}{3}$.

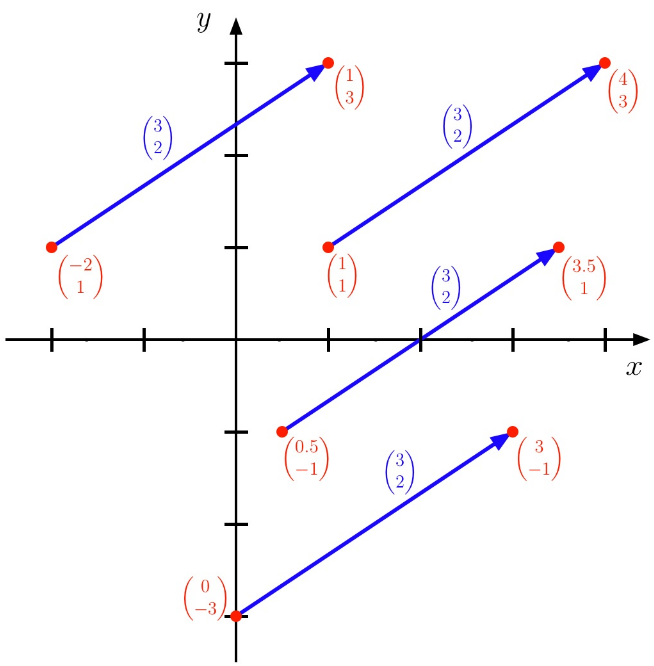


Figure 5: Position vectors are red dots, the displacement vector a blue arrow. All four blue arrows represent the *same* vector, even though they all have different beginning and end point.

This concept of displacement is the origin of the term vector. (The word *vector* derives from Latin, where it means *carrier*.)

As we see in Figure 5, we can move the displacement vector anywhere we want, it is always the same displacement vector. It displaces its beginning point onto its endpoint.

Of course, a displacement vector is determined by its beginning and end point. If the vector starts at A and ends at B, we sometimes use the notation $\overrightarrow{AB}$, see Figure 6.

If the position vector of A is $\binom{x}{y}$ and the position vector of B is $\binom{x'}{y'}$, then the displacement vector $\overrightarrow{AB}$ is $\binom{x'-x}{y'-y}$. For example (Figure 5), the displacement vector $\overrightarrow{\binom{1}{1}\binom{4}{3}}$ is equal to $\binom{4-1}{3-1} = \binom{3}{2}$.

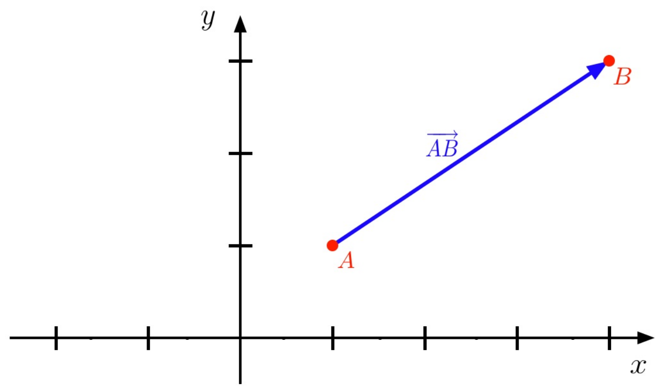


Figure 6: The displacement vector going from A to B.

$$\boxed{\overrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}} = \begin{pmatrix} x' - x \\ y' - y \end{pmatrix}}$$

Among all the different locations we can put the arrow $\binom{3}{2}$, there is one special one. If we put it at the origin, then the displacement vector $\binom{3}{2}$ points to the position vector $\binom{3}{2}$, see Figure 7.

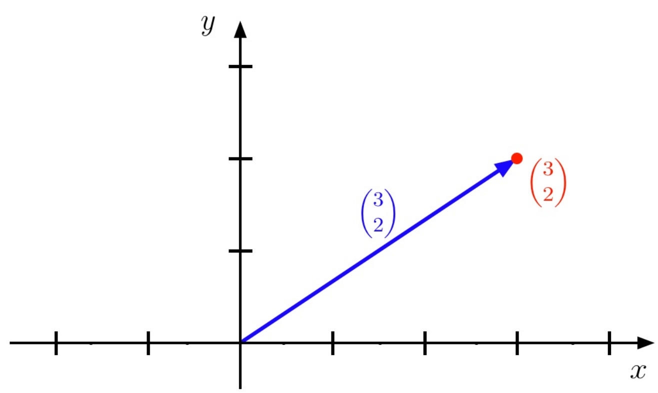


Figure 7: The position vector of a point is equal to the displacement vector from the origin to the point.

$$\boxed{\overrightarrow{OP} = P}$$

In this formula, O is the origin. The displacement vector moving the origin to the point P is the same vector as the position vector of the point P. Thus, usually, we will not make a difference between position vectors and displacement vectors. Unless otherwise specified, we move all displacement vectors to the origin, and then they become indistinguishable from position vectors.

Alternatively, this means that we represent position vectors by arrows originating at the origin. So, in the end, instead of drawing pictures such as Figure 7, where we can see the difference between the position vector $\binom{3}{2}$ (the red dot) and the displacement vector $\binom{3}{2}$ (the blue arrow), we will simply draw pictures such as Figure 8, where we can't tell the difference any more.

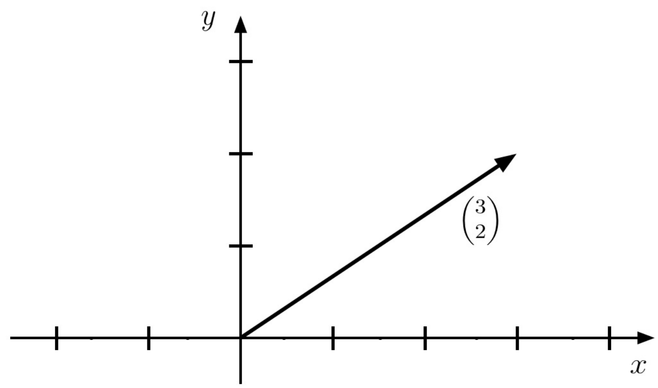


Figure 8: Position vector as displacement vector based at the origin.

But we should never forget that vectors have this dual nature of position and displacement vectors.

1.5 Exercises

The first three exercises refer to Figure 3.

Exercise 1.1 Find the position vector of the intersection of Granville and King Edward, referring to Figure 3. Find the position vector of the intersection of SE Marine Drive and Knight St.

Exercise 1.2 Find the address of a house on the South side of Broadway, half way between Willow and Heather.

Exercise 1.3 If a house with position vector $\vec{xy}$ is on a street, rather than an avenue, its house number is *not* $-100y$, as expected, but rather $1600 - 100y$. Find the address of a house on the East side of Cambie Street, half way between 13th and 14th Avenues.

The next exercise refers to Figure 4.

Exercise 1.4 Find longitude and latitude of the point on Earth with position vector $\binom{4}{2}$. Find the position vectors of Berlin, Tokyo and New York City, in the coordinate system of Figure 4.

Exercise 1.5 Suppose we are given points $A = \binom{2}{3}$, $B = \binom{-2}{5}$, $C = \binom{-1}{-2}$, find the displacement vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CB}$ and $\overrightarrow{AC}$.

2 Operations with vectors

2.1 Vector addition

A physical concept which is modelled very well using vectors is that of *velocity*. Velocity is closely related to displacement, in fact velocity is *displacement per unit time*. So velocity is a vector, and velocity vectors can be pictured in the same way as displacement vectors.

As usually is the case with applications to the physical world, we need to choose units. Let us use metres for distance and seconds for time.

Then a velocity vector of $\binom{3}{2}$ means that in one second we move 3 meters to the right and 2 meters up. If we start at A, then after one second we are at B, where B is obtained by displacing A by the displacement vector $\binom{3}{2}$.

Suppose a ship sails across the sea. The sailors on board the ship observe that the ship is moving relative to the water at a (constant) velocity of $\binom{3}{2}$. To find the actual velocity of the ship across the ocean floor, they will have to take into account the current of the sea. Suppose the sea moves relative to the ocean floor with a current, whose velocity vector is $\binom{-2}{1}$, see Figure 9.

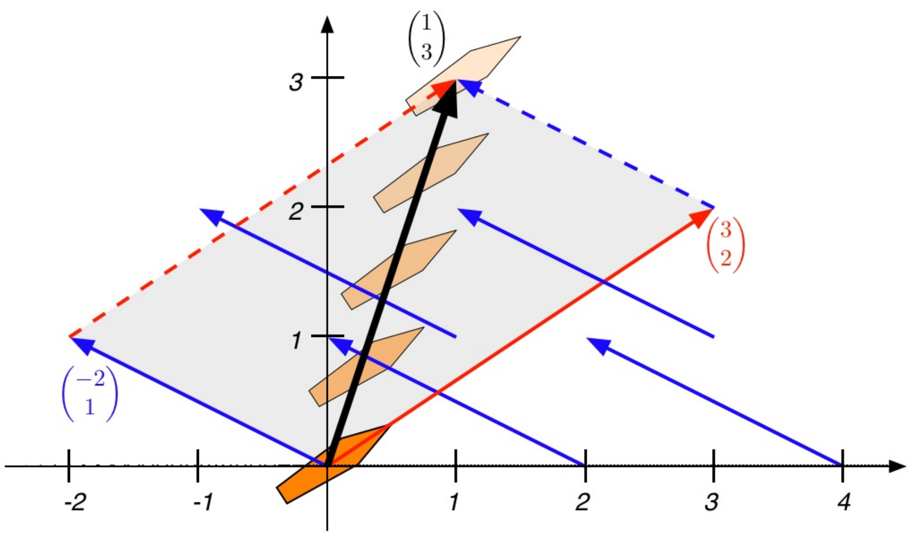


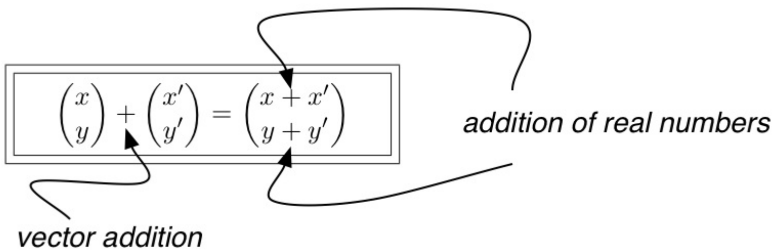
Figure 9: The net velocity across the ocean floor (black) is the vector sum of the velocity relative to the water (red) and the velocity of the current (blue). The movement of the ship during the unit time interval is also indicated.

The resulting velocity of the boat across the ocean floor is obtained as the *sum* of the two velocities (boat across water, and water across ocean floor).

$$\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix} + \begin{pmatrix}-2\\1\end{pmatrix}$$

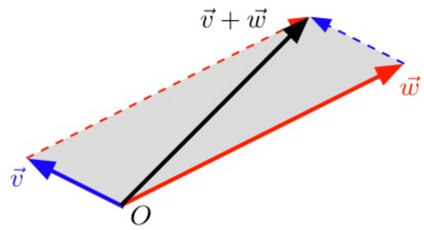
To compute the vector sum, simply add the components. Geometrically, the sum of the two vectors is the diagonal of the parallelogram spanned by the two individual vectors. This parallelogram is shaded in Figure 9.

Vector addition (sum) is an operation which takes two vectors as input and produces another vector as output. If the two input vectors are $\binom{3}{2}$ and $\binom{-2}{1}$, then the output vector is $\binom{1}{3}$. In general, if the input vectors are $\binom{x}{y}$ and $\binom{x'}{y'}$, then the output vector is $\binom{x+x'}{y+y'}$. The sum of the vectors $\vec{v}$ and $\vec{w}$ is denoted by $\vec{v}+\vec{w}$.

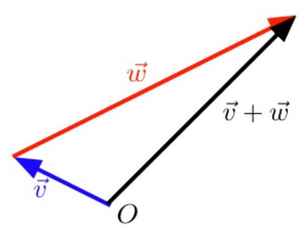


2.2 Tail to head addition

There are two ways to picture vector addition. First there is the parallelogram construction:

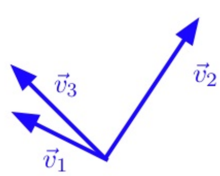


If we think in terms of displacement, then it may be more natural to first displace by the vector $\vec{v}$, and then displace by the vector $\vec{w}$. This means drawing the two displacement arrows head to tail:

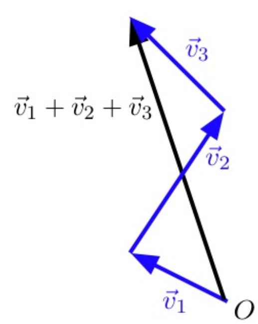


The result is, of course, the same.

The second point of view is particularly useful for adding more than two vectors. Say we wish to add the three vectors



Moving the displacement vectors tail to head we construct the sum:



2.3 Scalar multiplication

Say we move across the plane at a (constant) velocity $\vec{v}$. If we keep the direction, but move at twice the speed, the new velocity vector will be $2\vec{v}$, the result of multiplying the vector $\vec{v}$ by the real number 2. If we move with velocity $2\vec{v}$, we move twice as far in the x-direction every second, and twice as far in the y-direction every second. So if $\vec{v} = \binom{3}{2}$, then $2\vec{v} = \binom{6}{4}$.

$$\boxed{\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}}$$

multiplication of real numbers

scalar multiplication

Figure 10 shows various scalar multiples of the vector $\vec{v} = \binom{3}{2}$.

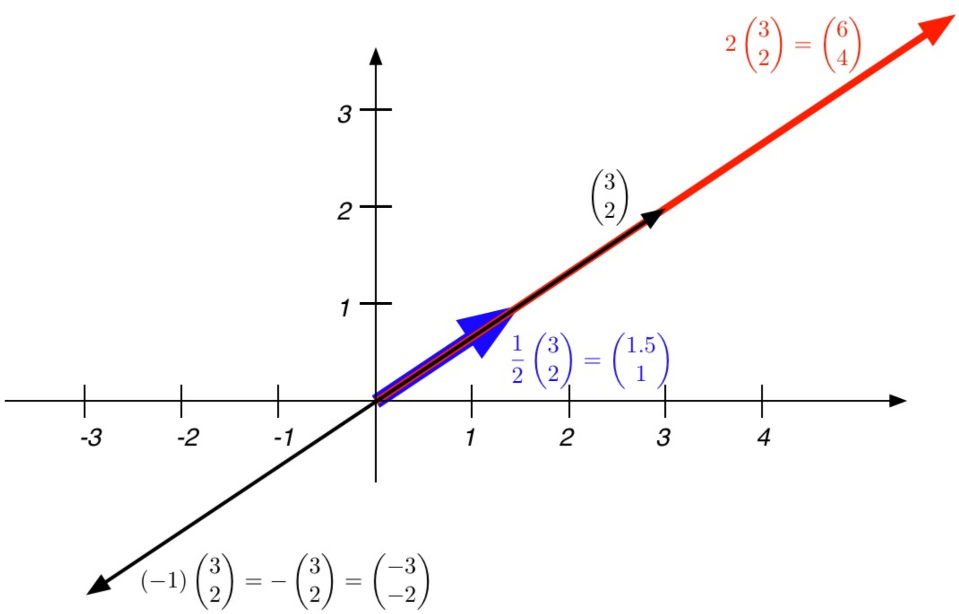


Figure 10: The original vector is in black, rescaling by factor 2 is in red and by factor $\frac{1}{2}$ in blue. Rescaling by negative numbers such as -1 reverses the direction of the vector.

Scalar multiplication is an operation which has two inputs: a vector such as $\binom{x}{y}$ and a real number such as α (the *scalar*). The output of this operation is a new vector.

2.4 Superpositions: combining scalar multiplication and vector addition.

Given vectors $\vec{v}$ and $\vec{w}$, and scalars α and β, we can form the *superposition*

$$\alpha\vec{v} + \beta\vec{w}$$

Figure 11 shows the superposition with $\alpha = \frac{1}{2}$ and $\beta = 2$.

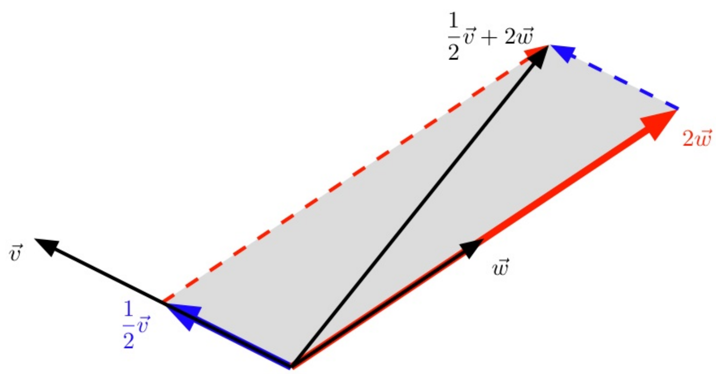


Figure 11: The vector $\vec{v}$ is rescaled by the factor $\frac{1}{2}$, giving the blue vector, the vector $\vec{w}$ is rescaled by the factor 2, giving the red vector. Then the blue and the red vector are added, giving the superposition $\frac{1}{2}\vec{v} + 2\vec{w}$, in black.

2.5 Subtraction

We get a particular case of a superposition if we take $\alpha = 1$ and $\beta = -1$.

$$\vec{v} + (-1)\vec{w} = \vec{v} - \vec{w}$$

This is *vector subtraction*. The vector $-\vec{w}$ points in the opposite direction as $\vec{w}$. We move $-\vec{w}$ to the head of $\vec{v}$, to construct $\vec{v} - \vec{w}$. This amounts to the same as moving $\vec{w}$ head to head with $\vec{v}$ (instead of tail to head), see Figure 12.

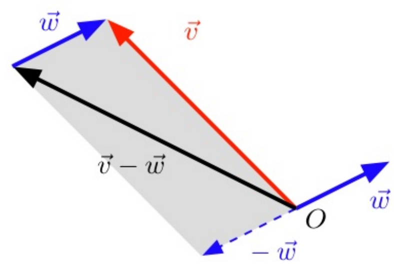


Figure 12: To subtract $\vec{w}$ from $\vec{v}$, move $\vec{w}$ head to head with $\vec{v}$. The tail end of $\vec{w}$ is then at $\vec{v} - \vec{w}$.

2.6 Dot product

The dot product of two vectors $\binom{x}{y}$ and $\binom{x'}{y'}$ is defined to be the real number $xx' + yy'$.

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} = xx' + yy'}$$

For example, the dot product of $\binom{3}{2.5}$ and $\binom{\frac{1}{3}}{-2}$ is -4:

$$\begin{pmatrix} 3 \\ 2.5 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{3} \\ -2 \end{pmatrix} = 3\frac{1}{3} + 2.5(-2) = 1 - 5 = -4$$

Thus the dot product is an operation which takes two vectors as inputs and has a real number as output.

The main reason for using the dot product is the following fact:

> Two vectors $\vec{v}$ and $\vec{w}$ are perpendicular (orthogonal) to each other, if and only if their dot product is zero: $\vec{v} \cdot \vec{w} = 0$

For example,

$$\begin{pmatrix} -3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = -3 \cdot 2 + 2 \cdot 3 = -6 + 6 = 0$$

and the vectors $\binom{-3}{2}$ and $\binom{2}{3}$ are perpendicular, as we can see in Figure 13.

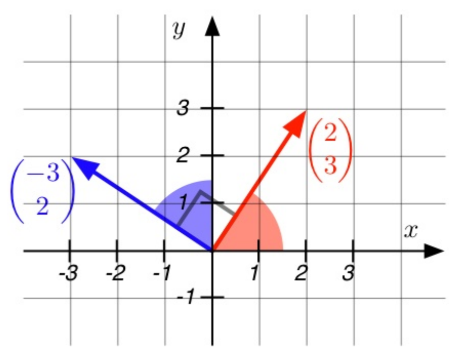


Figure 13: The red and the blue vector are orthogonal to each other, as you can see by looking at the two indicated angles.

Other vectors whose dot product with $\binom{-3}{2}$ vanishes, include $\binom{1}{1.5}$ or $\binom{-2}{-3}$, see Figure 14.

Let us find *all* vectors $\binom{x}{y}$, whose dot product with $\binom{-3}{2}$ is zero. The condition is

$$\begin{pmatrix} -3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{1}$$

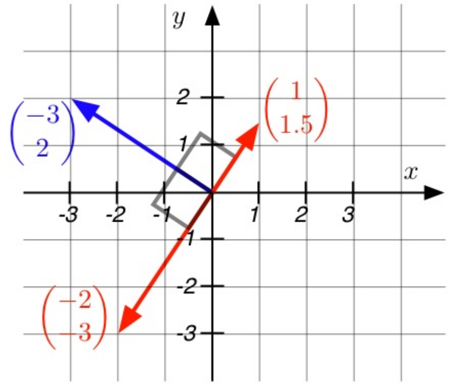


Figure 14: Both red vectors are orthogonal to the blue vector.

If we compute the dot product, we see that this is the same as

$$-3x + 2y = 0$$

Or

$$y = \frac{3}{2}x \tag{2}$$

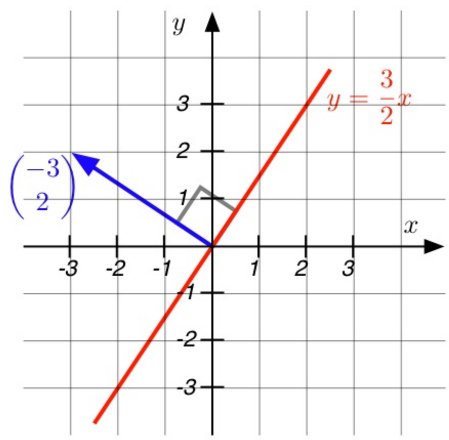


Figure 15: The points on the red line have position vectors whose dot product with the blue vector is zero.

In Figure 15, we have drawn the points $\binom{x}{y}$ satisfying Equation (2) in red. (These points form a line.) As we can see in the sketch, this red line is perpendicular to the vector $\binom{-3}{2}$. So we see that

> The vectors $\binom{x}{y}$, which satisfy the equation $\binom{-3}{2} \cdot \binom{x}{y} = 0$, form a line, and have position vectors perpendicular to $\binom{-3}{2}$.

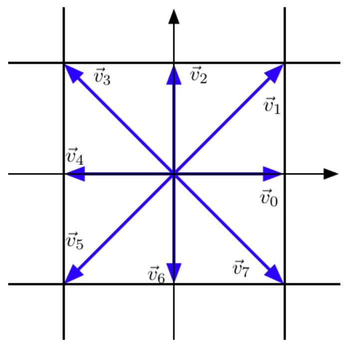
2.7 Exercises

Exercise 2.1 Consider the vectors $\vec{v} = \binom{1}{1}$ and $\vec{w} = \binom{-1}{1}$. Sketch the superpositions

$$(\tfrac{1}{2})^n \vec{v} + (\tfrac{3}{2})^n \vec{w}$$

for $n = -2, -1, 0, 1, 2$.

Exercise 2.2 Referring to the sketch



find the following superpositions:

$$\begin{aligned}
\vec{a} &= \vec{v}_0 + \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 + \vec{v}_5 + \vec{v}_6 + \vec{v}_7 \\
\vec{b} &= \vec{v}_1 + \vec{v}_3 - \vec{v}_2 \\
\vec{c} &= \vec{v}_0 + \vec{v}_3 + \vec{v}_5 \\
\vec{d} &= 2\vec{v}_7 - 4\vec{v}_6 + \vec{v}_5 \\
\vec{e} &= 2\vec{v}_1 + 2\vec{v}_3 - 3\vec{v}_2
\end{aligned}$$

In each case draw a sketch, where the vectors are moved tail to head, to justify your answers.

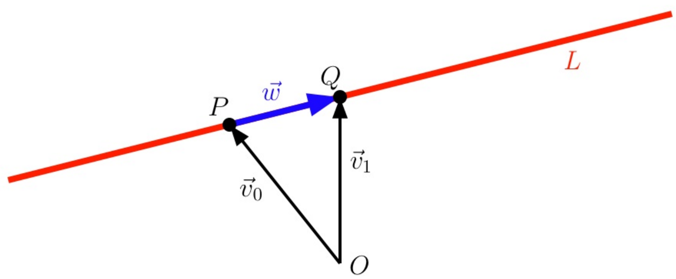
3 Lines in the plane

3.1 Parametric form of a line

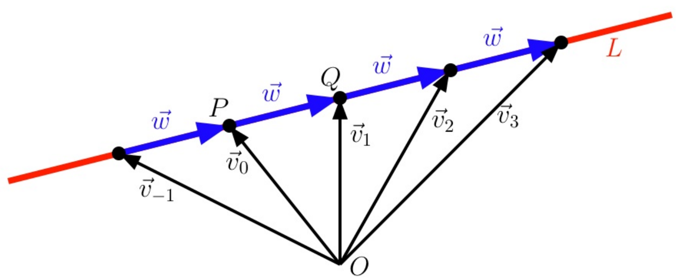
Suppose we have two distinct points P and Q in the plane. Then there is a unique line L passing through P and Q. We will construct some more points on L, besides P and Q.

Let us write $\vec{v}_0$ for the position vector of P and $\vec{v}_1$ for the position vector of Q and $\vec{w}$ for the displacement vector pointing from P to Q:

$$\vec{w} = \overrightarrow{PQ}$$



We keep laying off the vector $\vec{w}$ over and over again, all head to tail, to get more points on L:



So now we have a whole sequence of position vectors of points on L:

$$\ldots, \vec{v}_{-1}, \vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots$$

By looking at the sketch, we see that

$$\begin{aligned}
\vec{v}_1 &= \vec{v}_0 + \vec{w} \\
\vec{v}_2 &= \vec{v}_0 + \vec{w} + \vec{w} = \vec{v}_0 + 2\vec{w} \\
\vec{v}_3 &= \vec{v}_0 + \vec{w} + \vec{w} + \vec{w} = \vec{v}_0 + 3\vec{w} \\
\vec{v}_{-1} &= \vec{v}_0 - \vec{w}
\end{aligned}$$

In general, the point labelled by the integer n is given by

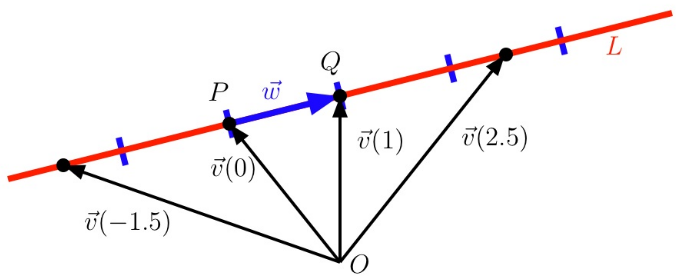
$$\vec{v}_n = \vec{v}_0 + n\vec{w}$$

If we replace the integer n by the real number t, we get *all* the points on the line L:

$$\boxed{\vec{v}(t) = \vec{v}_0 + t\vec{w} \qquad t \in \mathbb{R}} \tag{3}$$

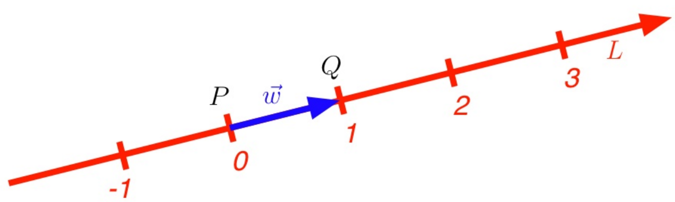
The sketch below contains the points with position vectors

$$\vec{v}(-1.5) = \vec{v}_0 - 1.5\vec{w} \qquad \vec{v}(0) = \vec{v}_0 \qquad \vec{v}(1) = \vec{v}_0 + \vec{w} \qquad \vec{v}(2.5) = \vec{v}_0 + 2.5\vec{w}$$



Equation (3) is a *parametric* equation for the line L. The real number $t \in \mathbb{R}$ is the *parameter*. For every parameter value we get a point on the line L. Think of t as time. As time passes, the point $\vec{v}(t)$ moves along the line L. At time $t = 0$, we pass through P, at time $t = 1$, we pass through Q.

The parametric form (3) of the line L turns the line L into a copy of the real number line:



As an example, suppose the points $P = \binom{3}{5}$ and $Q = \binom{2}{-3}$ are on the line L. Then the position vector $\vec{v}_0$ is $\binom{3}{5}$ and the displacement vector $\vec{w}$ is

$$\vec{w} = \overrightarrow{PQ} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2-3 \\ -3-5 \end{pmatrix} = \begin{pmatrix} -1 \\ -8 \end{pmatrix}$$

and so we get the following parametric form of L:

$$\vec{v}(t) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + t\begin{pmatrix} -1 \\ -8 \end{pmatrix} = \begin{pmatrix} 3-t \\ 5-8t \end{pmatrix} \qquad t \in \mathbb{R} \tag{4}$$

If we write $x(t)$ and $y(t)$ for the two components of $\vec{v}(t)$

$$\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

then we can rewrite the vector equation (4) as the two component equations

$$\begin{aligned} x(t) &= 3 - t \qquad t \in \mathbb{R} \\ y(t) &= 5 - 8t \qquad t \in \mathbb{R} \end{aligned}$$

Let us emphasize the fact that in the parametric equation (3), $\vec{v}_0$ is (the position vector of) a point on the line, and $\vec{w}$ is a (displacement vector) pointing *along* the line.

3.2 Lines given by equations

An equation such as

$$2x - 3y = 6 \tag{5}$$

defines a line. All the vectors $\binom{x}{y}$, whose two components x and y satisfy the equation (5) lie on a line. We can see this, for example, by solving for y

$$y = \frac{2}{3}x - 2$$

and graphing this function (Figure 16):

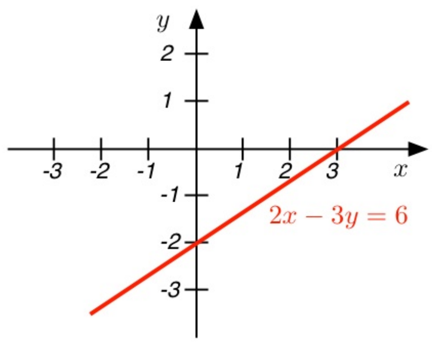


Figure 16: The line with slope $\frac{2}{3}$ and y-intercept -2

Giving a line by an equation such as (5) is somewhat implicit. A parametric form would be much more explicit: in some sense, a parametric equation is a list of all the points on the line. So how do we find a parametric equation for the line given by (5)?

One way would be to find two points P and Q on the line (for example, the x- and y-intercepts) and then follow the outline in 3.1.

But even simpler is to just use y as parameter. This means that $y = t$. Then we can find x in terms of t by solving (5) for x in terms of y:

$$x = 3 + \frac{3}{2}y$$

and then plugging in $y = t$ to get

$$x = 3 + \frac{3}{2}t$$

So we have now expressed both x and y in terms of the parameter t:

$$\begin{aligned} x(t) &= 3 + \frac{3}{2}t \\ y(t) &= t \end{aligned} \tag{6}$$

Which we can write as a single vector equation:

$$\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 3 + \frac{3}{2}t \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$

This is the parametric form corresponding to the two points $P = \vec{v}(0) = \binom{3}{0}$ and $Q = \vec{v}(1) = \binom{3}{0} + \binom{3/2}{1} = \binom{9/2}{1}$, see Figure 17. Note that because we have used

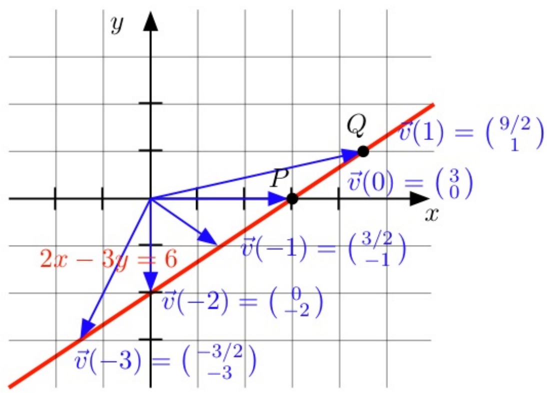


Figure 17: The position vectors of the points with parameter values $-3, -2, -1, 0, 1$ on the line $2x - 3y = 6$, using y as parameter.

y as parameter, the point P is on the x-axis (where $y = 0$), the point Q is on the horizontal line $y = 1$, the point $\vec{v}(-2)$ lies on the horizontal line $y = -2$, etc. The parametrization *identifies* the line $2x - 3y = 6$ with the y-axis, see Figure 18.

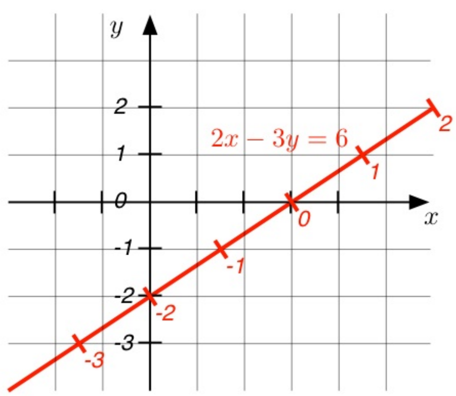


Figure 18: When we use y as parameter, the parameter values on the red line correspond to the numbers on the y-axis.

Remark 1 Using y as parameter does not always work: if the line has an equation such as

$$3y = 6$$

where x doesn't occur in the equation, we cannot solve for x in terms of y. In this case, we can use x as parameter. Solving for y "in terms of x" gives:

$$y = 2$$

and we get the parametric form

$$\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let us remark that the parametric form of the line $2x - 3y = 6$ is very far from unique: if we use x instead of y as a parameter, we get a different parametric form. We can use any two points P and Q to obtain a parametric form. So there are *infinitely many* parametric forms of this line.

3.3 Comparing equation and parametric form

Let us suppose a line L is given by the equation

$$2x - 3y = 6 \tag{7}$$

and that we also have a parametric form for L:

$$\vec{v}(t) = \vec{v}_0 + t\vec{w} \tag{8}$$

Our goal is figure out how these two representations of L relate to each other.

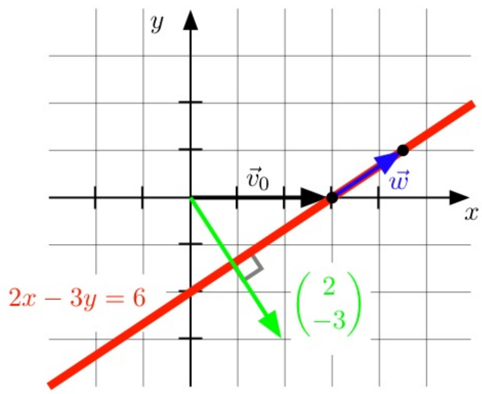


Figure 19: Graphic representation of Equation (11). The green and the blue vector are perpendicular.

We can rewrite the equation $2x - 3y = 6$ as

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 6 \tag{9}$$

All the points $\vec{v}(t)$ are supposed to lie on L, in particular the points $\vec{v}(0) = \vec{v}_0$ and $\vec{v}(1) = \vec{v}_0 + \vec{w}$ lie on L and hence satisfy Equation (9). For $\vec{v}(0)$, we get

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} \cdot \vec{v}_0 = 6 \tag{10}$$

and for $\vec{v}(1)$ we get

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} \cdot (\vec{v}_0 + \vec{w}) = 6$$

The dot product is distributive, so we get

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} \cdot \vec{v}_0 + \begin{pmatrix} 2 \\ -3 \end{pmatrix} \vec{w} = 6$$

Subtracting off (10), we get

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} \cdot \vec{w} = 0 \tag{11}$$

So the vector $\vec{w}$ (which points in the direction of L) is orthogonal to the vector $\binom{2}{-3}$, see Figure 19.

> The vector $\binom{2}{-3}$, which is extracted from the left hand side of the defining equation (7) is perpendicular to L. We say $\binom{2}{-3}$ is a *normal vector* to L.

We also conclude:

> The vector $\vec{v}_0$ satisfies the original equation (10), the vector $\vec{w}$ satisfies the *associated homogeneous equation* (11), in which the right hand side of the equation is replaced by 0.

3.4 Exercises

Exercise 3.1 Sketch the lines with equations

$$
\begin{aligned}
3x + y &= -3 \\
3x + y &= 0 \\
3x + y &= 3 \\
3x + y &= 6
\end{aligned}
$$

Exercise 3.2 Suppose a line passes through the points $\binom{1}{2}$ and $\binom{2}{3}$. Find a parametric equation for this line.

Exercise 3.3 Consider the line with equation

$$-5x + 17y = 23$$

Find the parametric equation using y as parameter, and the parametric equation using x as parameter.

Exercise 3.4 Consider the line with equation

$$4x + 2y = 8$$

Find the parametric equation such that $\vec{v}(0)$ is the x-intercept and $\vec{v}(1)$ is the y-intercept of this line.

Exercise 3.5 Find an equation for the line with parametric equation

$$\vec{v}(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad t \in \mathbb{R}$$

Determine if the point $\binom{-6}{-7}$ is on this line. If so, what value of t gives you this point?

4 Spatial vectors

Vectors in $\mathbb{R}^3$ have three components. Here are a few examples.

$$\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 1.2 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \in \mathbb{R}^3$$

The basic operations of addition, scalar multiplication and dot product are defined analogously to $\mathbb{R}^2$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \\ z + z' \end{pmatrix}$$

$$\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = xx' + yy' + zz'$$

Using a three-dimensional Cartesian coordinate system, position vectors describe locations in space, and displacement vectors describe movements in space. Two spatial vectors are orthogonal if and only if their dot product vanishes.

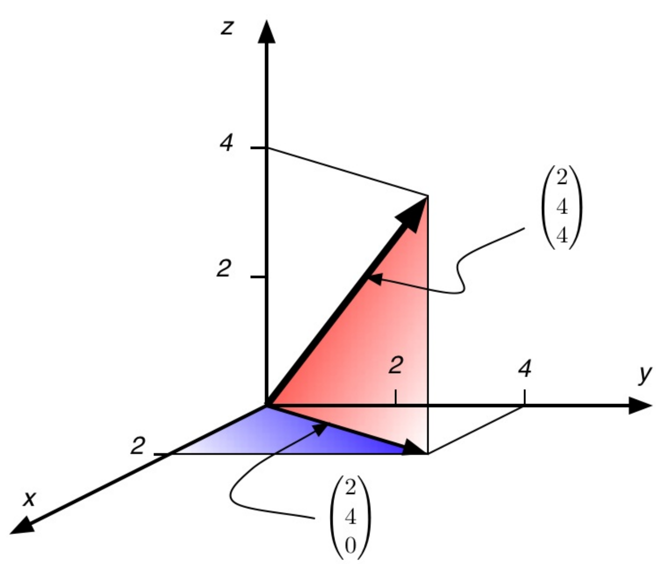


Figure 20: A three-dimensional coordinate system. The xy-plane is now horizontal, the z-component measures distance from the xy-plane, upwards. If we set the last component of a vector to 0, we obtain the *projection* of the vector into the xy-plane.

4.1 Planes in $\mathbb{R}^3$

Parametric equations for planes look like this

$$\boxed{\vec{v}(s,t) = \vec{v}_0 + s\vec{u} + t\vec{w} \qquad s,t \in \mathbb{R}} \tag{12}$$

In this equation, $\vec{v}_0$ is the position vector a point on the plane, and $\vec{u}$ and $\vec{w}$ are two displacement vectors inside the plane, see Figure 21. Note that there are two parameters, s and t. In Figure 21, we have indicated three points A, B, and C on

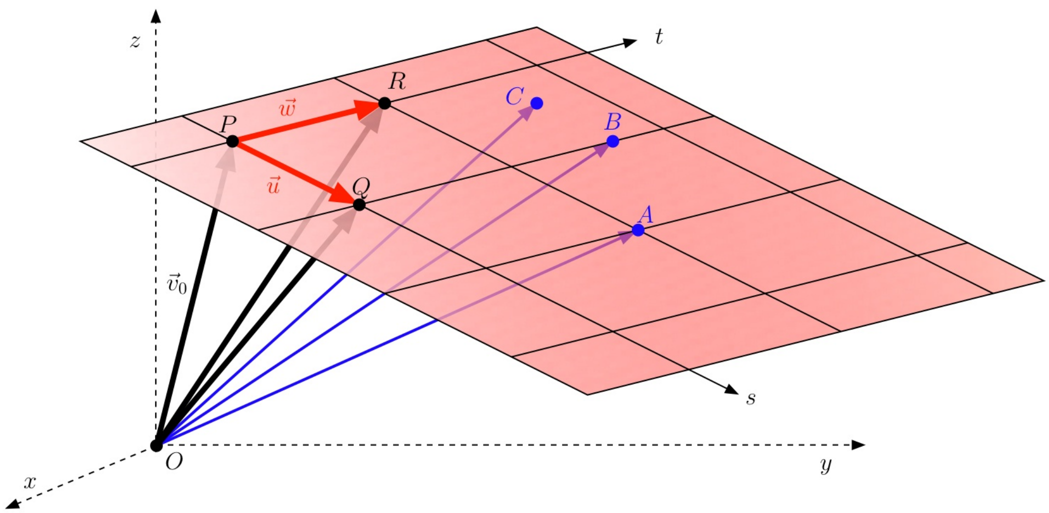


Figure 21: A plane in $\mathbb{R}^3$. The three points P, Q and R define the parametric equation for the plane. We have indicated three random points A, B, C on the plane in blue.

the plane. The position vector of the point A is

$$\vec{v}_0 + 2\vec{u} + \vec{w}$$

This corresponds in (12) to $s = 2$ and $t = 1$. So the position vector of A is $\vec{v}(2,1)$:

$$\vec{v}(2,1) = \vec{v}_0 + 2\vec{u} + \vec{w} = A$$

Similarly, we see that the parameter values corresponding to the point B are $s = 1$ and $t = \frac{5}{3}$:

$$\vec{v}(1, \tfrac{5}{3}) = \vec{v}_0 + \vec{u} + \tfrac{5}{3}\vec{w} = B$$

For C, the parameter values are $s = 0.4$ and $t = 1.67$.

$$\vec{v}(0.4, 1.67) = \vec{v}_0 + 0.4\,\vec{u} + 1.67\,\vec{w} = C$$

If we set the parameter $t = 0$, we get all vectors of the form

$$\vec{v}(s,0) = \vec{v}_0 + s\vec{u} \qquad s \in \mathbb{R}$$

This is the line through P and Q, which we can think of as the s-axis on our plane. Similarly, setting $s = 0$, we get the t-axis, which passes through P and R, and consists of all points

$$\vec{v}(0,t) = \vec{v}_0 + t\vec{w} \qquad t \in \mathbb{R}$$

Moreover, we have

$$P = \vec{v}(0,0)\,, \qquad Q = \vec{v}(1,0)\,, \qquad R = \vec{v}(0,1)$$

and

$$\vec{u} = \overrightarrow{PQ}\,, \qquad \vec{w} = \overrightarrow{PR}$$

The parametric equation (12) makes the plane in $\mathbb{R}^3$ look just like $\mathbb{R}^2$:

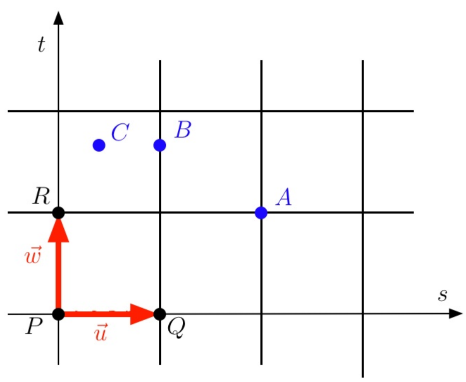


Figure 22: The two parameters s and t define coordinates inside the plane.

Remark 2 Equation (12) is only a valid parametric equation for a plane if the three points P, Q and R, (which give rise to the displacement vectors $\vec{u} = \overrightarrow{PQ}$ and $\vec{w} = \overrightarrow{PR}$) **do not lie on a line**. For three points to define a unique plane, they cannot be on the same line. For $\vec{v}_0 + s\vec{u} + t\vec{w}$ to be a parametric equation for a plane, we need $\vec{u}$ and $\vec{w}$ to be *linearly independent*. We'll explain what that means in the next section.

4.2 Special case: Span of two vectors, linear dependence for two vectors

Let $\vec{u}$ and $\vec{w}$ be two vectors in $\mathbb{R}^3$.

Definition 1 The **span** of $\vec{u}$ and $\vec{w}$, notation $\text{span}(\vec{u}, \vec{v})$, is the set of all vectors

$$s\vec{u} + t\vec{w} \qquad s, t \in \mathbb{R}$$

Any such expression $s\vec{u}+t\vec{w}$ is called **linear combination** of $\vec{u}$ and $\vec{w}$. So $\text{span}(\vec{u}, \vec{w})$ is the set of all linear combinations of $\vec{u}$ and $\vec{w}$.

Before, we called such expressions *superpositions*, but *linear combination* is the term more commonly used in mathematics.

For example, we have

$$\begin{aligned}\text{span}(\begin{pmatrix}1\\2\\4\end{pmatrix}, \begin{pmatrix}3\\-1\\2\end{pmatrix}) &= \{\text{all vectors} \quad s\begin{pmatrix}1\\2\\4\end{pmatrix} + t\begin{pmatrix}3\\-1\\2\end{pmatrix}, \quad \text{where } s, t \in \mathbb{R}\} \\ &= \{\text{all vectors} \quad \begin{pmatrix}s+3t\\2s-t\\4s+2t\end{pmatrix}, \quad \text{where } s, t \in \mathbb{R}\}\end{aligned}$$

If we write

$$\vec{v}(s,t) = s\vec{u} + t\vec{w} \qquad s, t \in \mathbb{R}$$

we get a special case of Equation (12), namely the case where $\vec{v}_0 = \vec{0}$. So $\vec{v}(s,t) = s\vec{u} + t\vec{w}$ looks like a parametric equation for a plane *passing through the origin*.

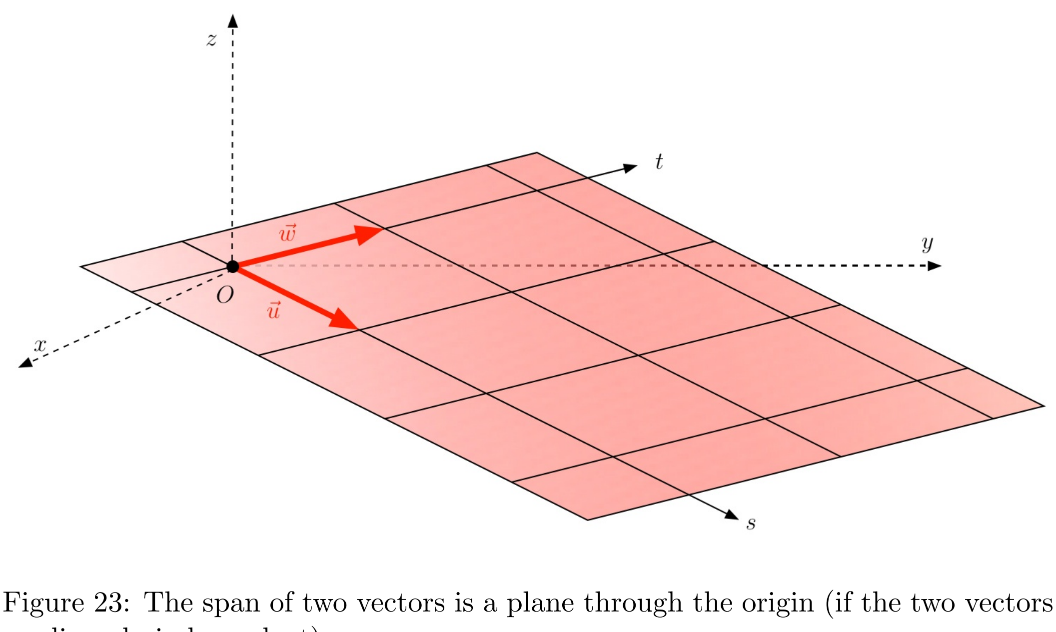


Figure 23: The span of two vectors is a plane through the origin (if the two vectors are linearly independent).

Let us look at another example:

$$\begin{aligned}
\text{span}(\begin{pmatrix}1\\2\\-1\end{pmatrix},\begin{pmatrix}-2\\-4\\2\end{pmatrix}) &= \{\text{all vectors} \quad s\begin{pmatrix}1\\2\\-1\end{pmatrix}+t\begin{pmatrix}-2\\-4\\2\end{pmatrix}, \quad \text{where } s,t\in\mathbb{R}\}\\
&= \{\text{all vectors} \quad \begin{pmatrix}s-2t\\2s-4t\\-s+2t\end{pmatrix}, \quad \text{where } s,t\in\mathbb{R}\}\\
&= \{\text{all vectors} \quad \begin{pmatrix}s-2t\\2(s-2t)\\-(s-2t)\end{pmatrix}, \quad \text{where } s,t\in\mathbb{R}\}\\
&= \{\text{all vectors} \quad (s-2t)\begin{pmatrix}1\\2\\-1\end{pmatrix}, \quad \text{where } s,t\in\mathbb{R}\}
\end{aligned}$$

In this case it turns out that every vector in $\text{span}(\left(\begin{smallmatrix}1\\2\\-1\end{smallmatrix}\right),\left(\begin{smallmatrix}-2\\-4\\2\end{smallmatrix}\right))$ is a scalar multiple of $\left(\begin{smallmatrix}1\\2\\-1\end{smallmatrix}\right)$. The scalar multiples of this single vector all lie on a line through the origin, the *line spanned by* the vector $\left(\begin{smallmatrix}1\\2\\-1\end{smallmatrix}\right)$ In fact, we have

$$\text{span}(\begin{pmatrix}1\\2\\-1\end{pmatrix},\begin{pmatrix}-2\\-4\\2\end{pmatrix}) = \text{span}(\begin{pmatrix}1\\2\\-1\end{pmatrix})$$

So in this case the span of the two vectors is not a plane, but only a line, as illustrated in Figure 24. This happened because the second vector was a scalar multiple of the

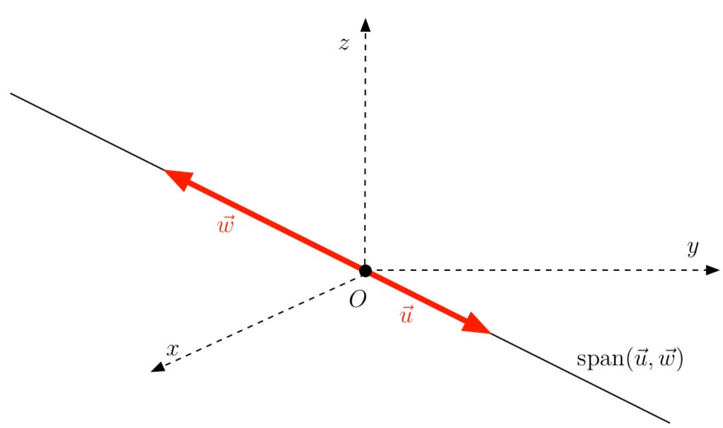


Figure 24: If the two vectors are linearly dependent, they span only a line through the origin.

first:

$$\begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Definition 2 If the two vectors are scalar multiples of each other, i.e., if $\vec{u} = \alpha\vec{w}$, or $\vec{w} = \beta\vec{u}$, they are said to be **linearly dependent**. If they are *not* scalar multiples of each other, they are said to be **linearly independent**.

Note that two vectors are linearly dependent if they point in the same or directly opposite directions. If they point in different directions (but not opposite directions) they are linearly independent.

If $\vec{u}, \vec{w} \in \mathbb{R}^3$, then there are three possibilities for $\text{span}(\vec{u}, \vec{w})$:

(i) $\text{span}(\vec{u}, \vec{w})$ consists of only the origin. This happens if both vectors $\vec{u}$ and $\vec{w}$ are the zero vector.

(ii) $\text{span}(\vec{u}, \vec{w})$ is a line through the origin. This happens when the two vectors are linearly dependent, and at least one of them is not just the zero vector.

(iii) $\text{span}(\vec{u}, \vec{w})$ is a plane through the origin. This happens when the two vectors are linearly independent.

Usually, when two vectors are linearly dependent, such as $\left(\begin{smallmatrix} 1 \\ 2 \\ -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -2 \\ -4 \\ 2 \end{smallmatrix}\right)$, the second one is a scalar multiple of the first:

$$\begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

and the first one is a scalar multiple of the second one, as well:

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = -\tfrac{1}{2} \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix}$$

But the vectors $\left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$ are linearly dependent, because the second one is a scalar multiple of the first:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

even though the first is *not* a scalar multiple of the second. There exists no real number $t \in \mathbb{R}$ such that

$$t \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

4.3 From equations to parametric forms

An equation such as

$$2x - 4y + 2z = 6 \tag{13}$$

defines a plane in $\mathbb{R}^3$. It is the plane formed by all points whose position vectors $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ satisfy (13). Since we can solve for x in terms of y and z, we can use y and z as parameters s and t. Solving for x gives

$$x = 3 + 2y - z \tag{14}$$

Setting $y = s$ and $z = t$, and plugging these values into (14), we obtain the parametric equations

$$\begin{aligned} x &= 3 + 2s - t \\ y &= s \\ z &= t \end{aligned}$$

which we can write in vector form

$$\vec{v}(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix} = \begin{pmatrix} 3 + 2s - t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

So in this case we have $\vec{v}_0 = \left(\begin{smallmatrix} 3 \\ 0 \\ 0 \end{smallmatrix}\right)$, $\vec{u} = \left(\begin{smallmatrix} 2 \\ 1 \\ 0 \end{smallmatrix}\right)$, and $\vec{w} = \left(\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right)$.

If x does not explicitly occur in the equation for the plane, we cannot solve for x, so we cannot use y and z as parameters. In this case solve either for y (and use x and z as parameters) or for z (and use x and y as parameters).

4.4 Normal vectors

Let us rewrite Equation (13) as

$$\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6$$

and let us call the plane defined by this equation E. So whenever $\vec{v}_0$ is the position vector of a point in E, then $\vec{v}_0$ satisfies the equation

$$\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \cdot \vec{v}_0 = 6$$

If $\vec{v}_1$ is the position vector of another point in E, then this vector also satisfies this equation:

$$\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \cdot \vec{v}_1 = 6$$

If we subtract these two equations from each other, we get:

$$\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \cdot \vec{v}_1 - \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \cdot \vec{v}_0 = 6 - 6 = 0$$

or, using distributive law for dot products

$$\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \cdot (\vec{v}_1 - \vec{v}_0) = 0$$

The vector $\vec{w} = \vec{v}_1 - \vec{v}_0$ is a displacement vector *along the plane.* So we see that all displacement vectors $\vec{w}$ along the plane satisfy the equation

$$\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \cdot \vec{w} = 0$$

In other words, the vector $\left(\begin{smallmatrix} 2 \\ -4 \\ 2 \end{smallmatrix}\right)$ is orthogonal to all vectors $\vec{w}$ pointing along the plane. Hence, $\left(\begin{smallmatrix} 2 \\ -4 \\ 2 \end{smallmatrix}\right)$ is orthogonal to the plane E. Such an orthogonal vector is also called a *normal vector* of the plane E. (See Figure 25.)

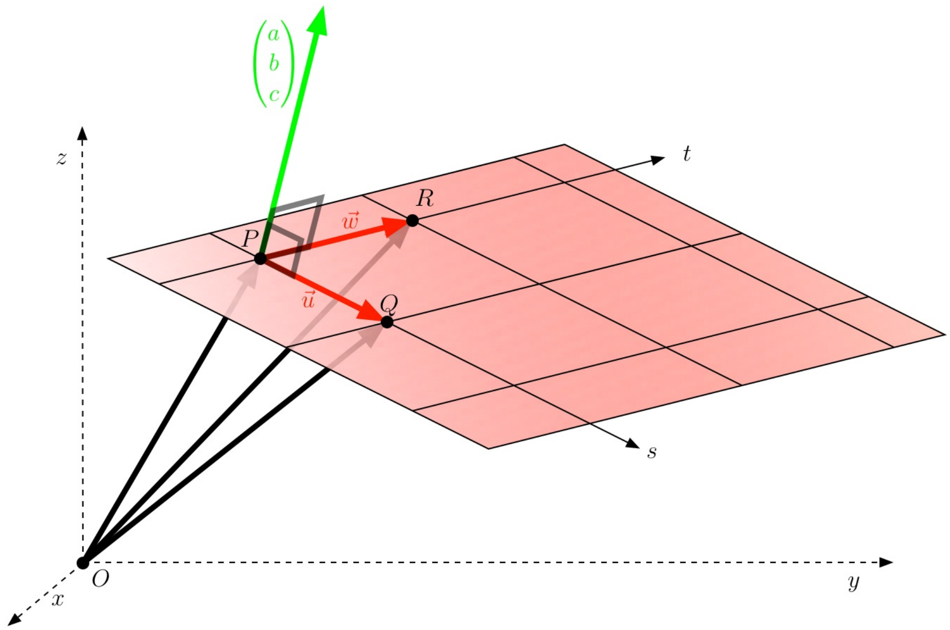


Figure 25: The normal vector to the plane is indicated in green. It is orthogonal to all displacement vectors such as $\vec{u}$ and $\vec{w}$, which point along the plane. There is no nice relation between the normal vector and the position vectors (in black) of points in the plane.

> If the equation of a plane E in $\mathbb{R}^3$ is
>
> $$ax + by + cz = d$$
>
> then the vector $\left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right)$ is orthogonal to E.
> All displacement vectors pointing along the plane satisfy the *associated homogeneous equation*
>
> $$ax + by + cz = 0$$

So if

$$\vec{v}(s,t) = \vec{v}_0 + s\vec{u} + t\vec{w} \qquad s,t \in \mathbb{R}$$

is a parametric equation for the plane E defined by the equation

$$ax + by + cz = d$$

then $\vec{v}_0$ satisfies the equation of E

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \vec{v}_0 = d$$

and $\vec{u}$, $\vec{w}$ satisfy the associated homogeneous equation

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \vec{u} = 0 \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \vec{w} = 0$$

Since the two planes

$$ax + by + cz = d \qquad \text{and} \qquad ax + by + cz = 0$$

have the same normal vector $\left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right)$, they are parallel to each other. In fact, $ax + by + cz = 0$ is the plane through the origin, which is parallel to $ax + by + cz = d$

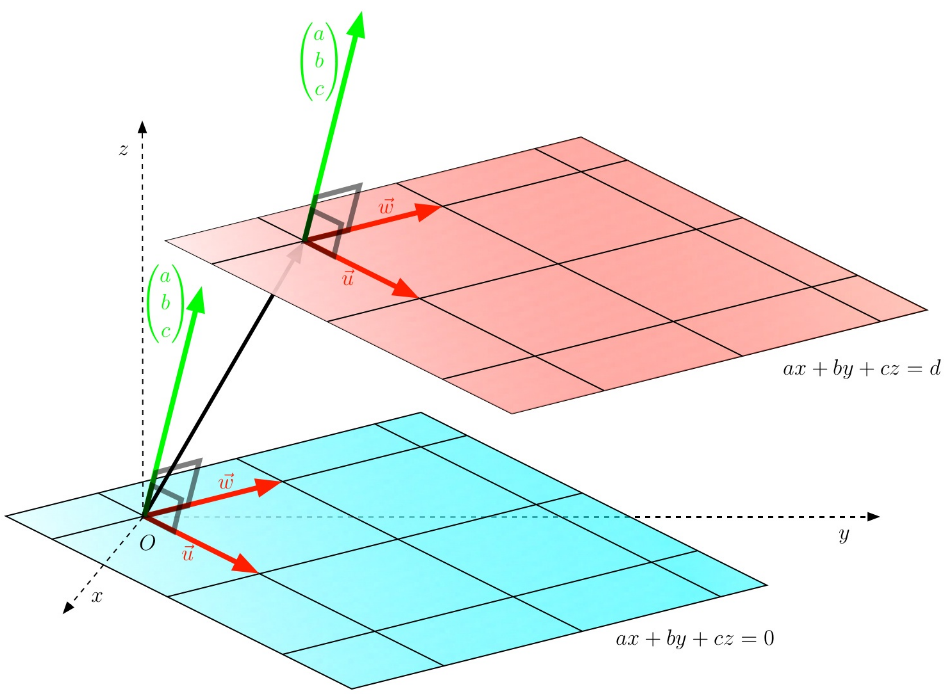


Figure 26: The original plane $ax + by + cz = d$ and the parallel plane through the origin, $ax + by + cz = 0$. The points with position vectors $\vec{u}$ and $\vec{w}$ lie on this plane through the origin.

4.5 Lines in space

Parametric forms of lines in $\mathbb{R}^3$ look just like parametric forms for lines in $\mathbb{R}^2$:

$$\vec{v}(t) = \vec{v}_0 + t\vec{w} \qquad t \in \mathbb{R}$$

the only difference is that all vectors involved have three components, instead of 2, as in Equation (3). Again, $\vec{v}_0$ is the position vector of an arbitrary point on the line, and $\vec{w}$ is an arbitrary displacement vector along the line. The same pictures as in Section 3.1 apply.

4.6 Intersections of planes in 3-space

First example

If we intersect two planes in $\mathbb{R}^3$, we expect the intersection to be a line.

$$\begin{array}{rcrcrcrl} 2x & - & 3y & - & 5z & = & 2 & (R_1) \\ 3x & - & 5y & + & 2z & = & -6 & (R_2) \end{array}$$

The two equations are labelled (R_1) and (R_2). Each of the two equations defines a plane, the intersection of the two planes consists of all vectors $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ satisfying *both* equations. The intersection is the set of solutions of the *system of two equations.*

To find a parametric form for the intersection, we proceed in two steps: first we eliminate one of the variables from the second equation, then we use substitution. Here is how it works.

Let us eliminate the term $3x$ from Equation (R_2). We do this by subtracting 3 times Equation (R_1) from 2 times Equation (R_2). The new equations are called (R_1') and (R_2'). The first equation remains unchanged: $(R_1') = (R_1)$, the second equation becomes $(R_2') = 2(R_2) - 3(R_1)$. We get an *equivalent system of equations.*

$$\begin{array}{rrcrcrcrl} (R_1) & 2x & - & 3y & - & 5z & = & 2 & (R_1') \\ 2(R_2) - 3(R_1) & & - & y & + & 19z & = & -18 & (R_2') \end{array}$$

On the right we have written the name of the equation, on the left the operation for obtaining it.

Now Equation (R_2) does not contain the variable x any more. We can use z as parameter:

$$z = t$$

we plug this into (R_2') and solve for y:

$$y = 18 + 19z = 18 + 19t$$

and then we plug the expression for z and y into (R_1') and get:

$$2x = 2 + 3y + 5z = 2 + 3(18 + 19t) + 5t = 2 + 54 + 57t + 5t = 56 + 62t$$

and

$$x = 28 + 31t$$

Now we have the sought after parametric form of the intersection:

$$\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 28 + 31t \\ 18 + 19t \\ t \end{pmatrix} = \begin{pmatrix} 28 \\ 18 \\ 0 \end{pmatrix} + t \begin{pmatrix} 31 \\ 19 \\ 1 \end{pmatrix}$$

The vector $\left(\begin{smallmatrix} 31 \\ 19 \\ 1 \end{smallmatrix}\right)$ is a displacement vector pointing along the intersection line. It points along both planes. So it is perpendicular to both normal vectors:

$$\begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 31 \\ 19 \\ 1 \end{pmatrix} = 62 - 57 - 5 = 0$$

$$\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 31 \\ 19 \\ 1 \end{pmatrix} = 93 - 95 + 2 = 0$$

In other words, $\left(\begin{smallmatrix} 31 \\ 19 \\ 1 \end{smallmatrix}\right)$ solves the associated homogeneous system:

$$\begin{array}{rcrcrcrl} 2x & - & 3y & - & 5z & = & 0 & \quad (H_1) \\ 3x & - & 5y & + & 2z & = & 0 & \quad (H_2) \end{array}$$

Of course, $\left(\begin{smallmatrix} 28 \\ 18 \\ 0 \end{smallmatrix}\right)$ is actually a point on the intersection line, and so it solves the original system consisting of (R_1) and (R_2). (It is a good idea to check that this is true!)

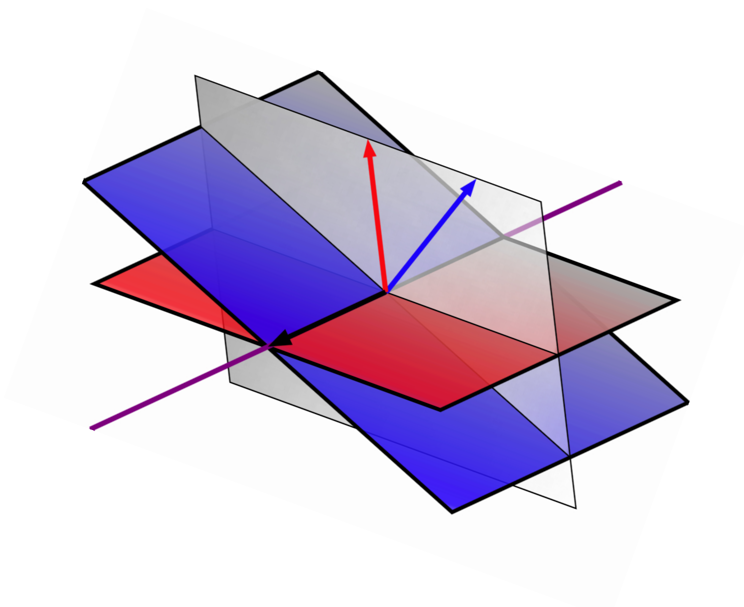


Figure 27: The red plane with red normal vector and the blue plane with blue normal vector intersect in the purple line. The black vector which points in the direction of the line is orthogonal to both normal vectors.

Second example: cannot use z as parameter

Let us do another example:

$$\begin{array}{rcrcrcrl} x & + & 2y & + & 4z & = & 3 & \quad (R_1) \\ x & + & 2y & + & 3z & = & 4 & \quad (R_2) \end{array}$$

We subtract (R_1) from (R_2) and obtain the equivalent system:

$$\begin{array}{rrcrcrcrl} (R_1) & x & + & 2y & + & 4z & = & 3 & \quad (R_1') \\ (R_2) - (R_1) & & & & - & z & = & 1 & \quad (R_2') \end{array}$$

This time we cannot use z as parameter, because

$$z = -1$$

and so z cannot vary at all. But we can use y as parameter. We have

$$y = t$$

and we can solve (R_1') for x in terms of the parameter:

$$x = 3 - 2y - 4z = 3 - 2t - 4(-1) = 7 - 2t$$

we get the parametric form of the intersection line as

$$\vec{v}(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 - 2t \\ t \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

Third example: planes do not intersect

Another example:

$$\begin{array}{rcrcrcrl} x & + & 2y & + & 3z & = & 3 & \quad (R_1) \\ -2x & - & 4y & - & 6z & = & -10 & \quad (R_2) \end{array}$$

Doing as before, we obtain

$$\begin{array}{rrcrcrcrl} (R_1) & x & + & 2y & + & 4z & = & 3 & \quad (R_1') \\ (R_2) + 2(R_1) & & & & & 0 & = & -4 & \quad (R_2') \end{array}$$

We reached the contradiction $0 = -4$. This means that the system of two equations has no solution (if there were a solution, we would run into this contradiction). Why does this happen? The two normal vectors are $\left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -2 \\ -4 \\ -6 \end{smallmatrix}\right)$. They point along the same line. Therefore, these two planes are parallel, and hence they don't intersect.

Fourth example: planes are equal

Final example:

$$\begin{array}{rcrcrcrl} x & + & 2y & + & 3z & = & 3 & \quad (R_1) \\ -2x & - & 4y & - & 6z & = & -6 & \quad (R_2) \end{array}$$

Doing as before, we obtain

$$\begin{array}{rrcrcrcrl} (R_1) & x & + & 2y & + & 4z & = & 3 & \quad (R_1') \\ (R_2) + 2(R_1) & & & & & 0 & = & 0 & \quad (R_2') \end{array}$$

This time we do not reach a contradiction, instead the second equation doesn't contain any information. The intersection is the plane given by (R_1').

Summary

We summarize our observations as follows:

If the normal vectors of the two planes are linearly independent, then the intersection is a line.

If the normal vectors of the two planes are linearly dependent, then the two planes are parallel. The intersection is either empty (if the two planes are parallel but different) or the intersection is a plane (if the two planes are parallel and equal)

4.7 Linear independence in $\mathbb{R}^3$

Three vectors in $\mathbb{R}^3$ are **linearly dependent** if one of them can be written as a linear combination of the other two.

Example 1 We have

$$\begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

so the three vectors $\left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 \\ 2 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 \\ -2 \\ 9 \end{smallmatrix}\right)$ are linearly dependent.

Example 2

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

so the three vectors $\left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 4 \\ 6 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$ are linearly dependent.

How can we determine if one vector is a linear combination of two others? Suppose we want to see if $\left(\begin{smallmatrix} 1 \\ 2 \\ 4 \end{smallmatrix}\right)$ is a linear combination of $\left(\begin{smallmatrix} 1 \\ -1 \\ 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ 0 \\ 3 \end{smallmatrix}\right)$. Suppose it can be done:

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = s \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

we get the following system of equations for s and t:

$$\begin{array}{rcrcl} s & + & t & = & 1 \\ -s & & & = & 2 \\ 2s & + & 3t & = & 4 \end{array}$$

We see that $s = -2$ (from the second equation), and $t = 1 - s = 1 + 2 = 3$ (from the first equation), which we plug into the third equation: $2(-2) + 3(3) = 4$, or $-4 + 9 = 4$, which is false. So it is impossible to find s and t without running into contradictions. We see that $\left(\begin{smallmatrix} 1 \\ 2 \\ 4 \end{smallmatrix}\right)$ is not a linear combination of $\left(\begin{smallmatrix} 1 \\ -1 \\ 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ 0 \\ 3 \end{smallmatrix}\right)$.

Another example: suppose we want to see if $\left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right)$ is a linear combination of $\left(\begin{smallmatrix} 2 \\ 4 \\ 6 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 3 \\ 6 \\ 9 \end{smallmatrix}\right)$. The vector equation

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = s \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + t \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

gives rise to the system of equations

$$\begin{array}{rcrcl} 2s & + & 3t & = & 1 \\ 4s & + & 6t & = & 2 \\ 6s & + & 9t & = & 3 \end{array}$$

A possible solution is $s = \frac{1}{2}, t = 0$. This solves all three equations. Another solution solving all three equations is $s = 0, t = \frac{1}{3}$. Thus we have

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}2\\4\\6\end{pmatrix} + 0\begin{pmatrix}3\\6\\9\end{pmatrix}$$

or

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = 0\begin{pmatrix}2\\4\\6\end{pmatrix} + \tfrac{1}{3}\begin{pmatrix}3\\6\\9\end{pmatrix}$$

At any rate, $\left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right)$ is a linear combination of $\left(\begin{smallmatrix}2\\4\\6\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}3\\6\\9\end{smallmatrix}\right)$.

Example 3 Find out if $\vec{v}_1 = \left(\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right)$, $\vec{v}_2 = \left(\begin{smallmatrix}1\\-2\\4\end{smallmatrix}\right)$, and $\vec{v}_3 = \left(\begin{smallmatrix}-2\\4\\-8\end{smallmatrix}\right)$ are linearly dependent or not. Let us try to write $\vec{v}_1$ as a linear combination of $\vec{v}_2$ and $\vec{v}_3$:

$$\begin{pmatrix}1\\2\\1\end{pmatrix} = s\begin{pmatrix}1\\-2\\4\end{pmatrix} + t\begin{pmatrix}-2\\4\\-8\end{pmatrix}$$

This gives

$$\begin{array}{rcrcl} s & - & 2t & = & 1\\ -2s & + & 4t & = & 2\\ 4s & - & 8t & = & 1 \end{array}$$

Solve the first equation for s in terms of t and plug the result into the second equation, you get a contradiction. So finding s and t is impossible, $\vec{v}_1$ is not a linear combination of $\vec{v}_2$ and $\vec{v}_3$. But we cannot conclude that the three vectors are independent. In fact, we see that $\vec{v}_3 = 0\vec{v}_1 - 2\vec{v}_2$:

$$\begin{pmatrix}-2\\4\\-8\end{pmatrix} = 0\begin{pmatrix}1\\2\\1\end{pmatrix} - 2\begin{pmatrix}1\\-2\\4\end{pmatrix}$$

So the three vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are linearly dependent.

Example 4 Find out if $\vec{u}_1 = \left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right)$, $\vec{u}_2 = \left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right)$, $\vec{u}_3 = \left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)$ are linearly dependent or independent. Trying to write $\vec{u}_1$ as a linear combination of $\vec{u}_2$ and $\vec{u}_3$ gives

$$\begin{pmatrix}0\\1\\1\end{pmatrix} = s\begin{pmatrix}1\\0\\1\end{pmatrix} + t\begin{pmatrix}1\\1\\0\end{pmatrix}$$

which says $t = 1$, $s = 1$ and $s + t = 0$, which is impossible. Trying to write $\vec{u}_2$ as a linear combination of $\vec{u}_1$ and $\vec{u}_3$ gives

$$\begin{pmatrix}1\\0\\1\end{pmatrix} = s\begin{pmatrix}0\\1\\1\end{pmatrix} + t\begin{pmatrix}1\\1\\0\end{pmatrix}$$

which gives the same contradiction. Finally, it's also impossible to write $\vec{u}_3$ as a linear combination of $\vec{u}_1$ and $\vec{u}_2$. So these three vectors are linearly independent.

To summarize: When we have three vectors, there are three questions asking "am I a linear combination of the other two?". For linear independence, we need to show that all three questions have "no" for an answer, so we have to show that three problems are impossible to solve. For linear dependence, it is enough to show that one of the three questions has "yes" for an answer, so it's enough if one of the three problems is solvable.

Later in the course we will learn more efficient methods to check for dependence/independence.

4.8 Geometric meaning

The **span** of three vectors is all linear combinations:

$$\text{span}(\vec{u}_1, \vec{u}_2, \vec{u}_3) = \text{all vectors } s\vec{u}_1 + t\vec{u}_2 + r\vec{u}_3, \text{ where } s, t, r \in \mathbb{R} \text{ are real numbers}$$

Theorem 3 *Three vectors in $\mathbb{R}^3$ are linearly dependent if and only if their span is contained in a plane through the origin. (This plane may or may not be determined by the three vectors.)*

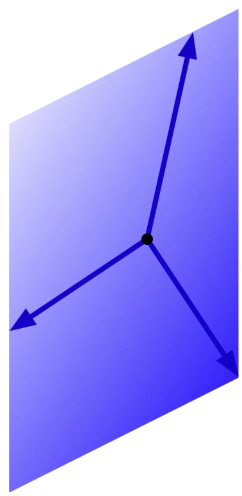


Figure 28: Three linearly dependent vectors. The vectors lie completely within a plane through the origin.

Let us explain why this theorem is true. The theorem actually says two things:

Claim 1. If three vectors are linearly dependent, then it's possible to find some plane containing all three of them. Why is this true? Well, if the three vectors are dependent, then it's possible to write one of them as a linear combination of the other two, say

$$\vec{u}_2 = 5\vec{u}_1 - 7\vec{u}_3$$

Once we have this, then if we have a linear combination of all three vectors, say

$$s\vec{u}_1 + t\vec{u}_2 + r\vec{u}_3$$

then we can write this a a linear combination of just two of them, by substituting:

$$\begin{aligned} s\vec{u}_1 + t\vec{u}_2 + r\vec{u}_3 &= s\vec{u}_1 + t(5\vec{u}_1 - 7\vec{u}_3) + r\vec{u}_3 \\ &= (s + 5t)\vec{u}_1 + (r - 7t)\vec{u}_3 \end{aligned}$$

This means that

$$\text{span}(\vec{u}_1, \vec{u}_2, \vec{u}_3) = \text{span}(\vec{u}_1, \vec{u}_3)$$

But the two vectors $\vec{u}_1$ and $\vec{u}_3$ span a plane through the origin, and so we see that, $\text{span}(\vec{u}_1, \vec{u}_2\vec{u}_3)$ is contained in that plane. It could also happen that $\vec{u}_1$ and $\vec{u}_3$ are linearly dependent, then $\operatorname{span}(\vec{u}_1, \vec{u}_3)$ would be a line. But a line is contained in many planes, so then $\operatorname{span}(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ would still be contained in *some* plane. (This is the case that the plane is not determined by the three vectors.)

The key fact is the following: if $\vec{u}_2$ is a linear combination of $\vec{u}_1$ and $\vec{u}_3$, then we can cross if off the list of three vectors without changing the span!

Claim 2. If the span of three vectors is contained in a plane, then the vectors are linearly dependent. Why is this true? We will have to do some case distinctions.

If $\vec{u}_1$ is zero, then the vectors are *obviously* linearly dependent: if $\vec{u}_1 = \left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right)$, then $\vec{u}_1 = 0\vec{u}_2 + 0\vec{u}_3$, no matter what $\vec{u}_2$ and $\vec{u}_3$ are. So $\vec{u}_1$ is a linear combination of the other two, and we have dependence, as required.

So let's now deal with the case that $\vec{u}_1$ is not zero. Then $\vec{u}_1$ spans a line through the origin. Again we'll look at two cases:

If $\vec{u}_2$ is contained within the line $\operatorname{span} u_1$, then $\vec{u}_2$ is a scalar multiple of $\vec{u}_1$: say $\vec{u}_2 = -26\vec{u}_1$. Then, again, we have linear dependence: $\vec{u}_2 = -26\vec{u}_1 + 0\vec{u}_3$.

So now we deal with the case that $\vec{u}_2$ is not in the line $\operatorname{span} u_1$. Then, $\vec{u}_1$ and $\vec{u}_2$ are linearly independent, and they span a plane. We are assuming that the span of all three vectors is contained in a plane, so this plane has to be the plane $\operatorname{span}(\vec{u}_1, \vec{u}_2)$. So we deduce that

$$\operatorname{span}(\vec{u}_1, \vec{u}_2, \vec{u}_3) = \operatorname{span}(\vec{u}_1, \vec{u}_2)$$

In particular, $\vec{u}_3 \in \operatorname{span}(\vec{u}_1, \vec{u}_2)$, so $\vec{u}_3$ is a linear combination of $\vec{u}_1$ and $\vec{u}_2$, an so, yet again, we concluded that the vectors are linearly dependent.

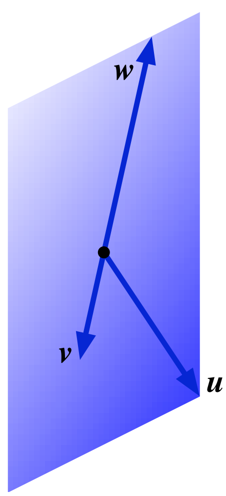


Figure 29: Another example of three linearly dependent vectors. In this case $\vec{v}$ is a linear combination of $\vec{w}$ and $\vec{u}$, but $\vec{u}$ is not a linear combination of $\vec{v}$ and $\vec{w}$.

4.9 Dimension

The **dimension** is the minimal number of parameters needed in parametric form.

Thus lines are one-dimensional, because we can use one parameter, planes are two-dimensional, because we need two parameters. All of $\mathbb{R}^3$ is 3-dimensional: we need three real numbers x, y, z to determine a vector $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$. At the other extreme: points are zero-dimensional.

We will now write down a very important theorem about dimension. For this we consider a subset $V \subset \mathbb{R}^3$. This V will be a point, a line, a plane, or all of $\mathbb{R}^3$. The theorem applies to all these cases!

Theorem 4 *We assume that V passes through the origin:*

(i) the dimension of V is the smallest number of vectors which span V,

(ii) the dimension of V is the largest number of linearly independent vectors inside V.

(The theorem is still true if V does not pass through the origin, as long as we make sure that we are talking about displacement vectors along V, rather than position vectors of points on V. See Figure 31.)

For the case that V is a line through the origin, this theorem says that

(i) we need at least one vector to span the line V,

(ii) two or more vectors contained in V are automatically linearly dependent.

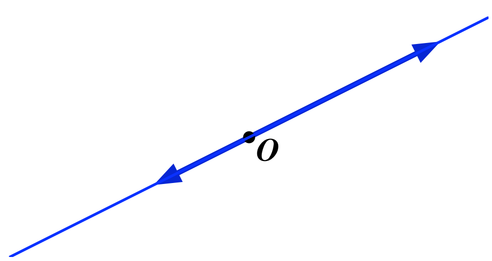


Figure 30: Two vectors in a line through the origin have to be linearly dependent.

For the case that V is a plane through the origin, the theorem says that

(i) we need at least two vectors to span the plane V. (If we take only one vector, we are only going to get a line.)

(ii) there cannot be more than two linearly independent vectors in the plane: three or more vectors in a plane are necessarily linearly dependent. (This was Claim 2 of Theorem 3.)

For the case of dimension 3, the theorem is about $V = \mathbb{R}^3$. In this case it says:

(i) we need at least three vectors to span $\mathbb{R}^3$. (With two vectors, we can get at most a plane.)

(ii) four or more vectors in $\mathbb{R}^3$ are automatically linearly dependent. (If three vectors already span $\mathbb{R}^3$, the fourth one will have to be a linear combination of the first three.)

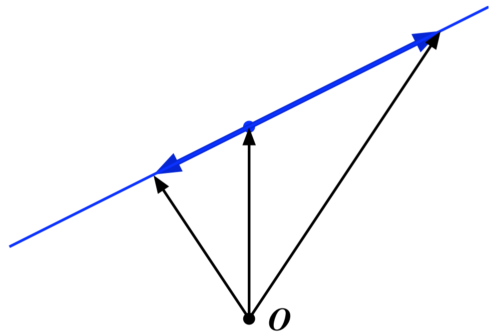


Figure 31: If the line does not pass through the origin, position vectors (black) point in all kinds of directions, and any two distinct one are linearly independent. But two displacement vectors along the line (blue) can't be independent, because the point in the same (or opposite) direction.

The theorem is also (somewhat) meaningful for the case of dimension zero: then V consists of the origin only. The theorem says:

(i) we don't need *any* vectors to span the origin. This is really a convention: the span of no vectors (the span of the empty set) is considered to be the origin.

(ii) even the zero vector all by itself is already a linearly dependent set of vectors. Again, you may consider this to be more of a convention, than a fact.

4.10 Intersection of three planes in $\mathbb{R}^3$

Usually, when we intersect three planes in $\mathbb{R}^3$, we expect to get a point as intersection:

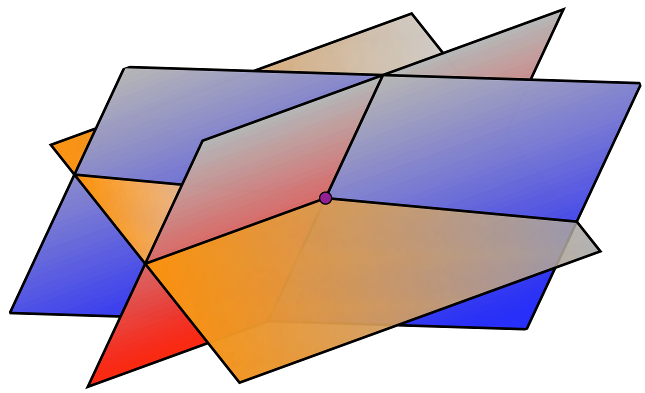


Figure 32: If the normal vectors are linearly independent, we get a unique intersection point.

Let us explain how to calculate the intersection point. Suppose the equations for the three planes are

$$\begin{array}{rcrcrcrl} x & - & y & + & 3z & = & 2 & (R_1) \\ 2x & - & y & + & 8z & = & 1 & (R_2) \\ -3x & + & 5y & - & 4z & = & -7 & (R_3) \end{array}$$

The first step is **forward elimination**. Try to eliminate the terms $2x$, $-3x$ and $5y$, which are in the lower left of the system of equations. First we deal with x:

$$\begin{array}{rrcrcrcrl} (R_1) & x & - & y & + & 3z & = & 2 & (R_1') \\ (R_2) - 2(R_1) & & & y & + & 2z & = & -3 & (R_2') \\ (R_3) + 3(R_1) & & & 2y & + & 5z & = & -1 & (R_3') \end{array}$$

Then we deal with y:

$$\begin{array}{rrcrcrcrl} (R_1') & x & - & y & + & 3z & = & 2 & (R_1'') \\ (R_2') & & & y & + & 2z & = & -3 & (R_2'') \\ (R_3') - 2(R_1') & & & & & z & = & 5 & (R_3'') \end{array}$$

The second step is **backward substitution**. Plug the value we found for z into the other two equations:

$$\begin{array}{rcrcrcrl} x & - & y & + & 3 \cdot 5 & = & 2 & (R_1''') \\ & & y & + & 2 \cdot 5 & = & -3 & (R_2''') \\ & & & & z & = & 5 & (R_3''') \end{array}$$

and simplify:

$$\begin{array}{rcrcrcrl} x & - & y & & & = & -13 & (R_1''') \\ & & y & & & = & -13 & (R_2''') \\ & & & & z & = & 5 & (R_3''') \end{array}$$

Then plug the value we found for y into the first equation:

$$\begin{array}{rcrcrcrl} x & + & 13 & & & = & -13 & (R_1'''') \\ & & y & & & = & -13 & (R_2'''') \\ & & & & z & = & 5 & (R_3'''') \end{array}$$

and simplify:

$$\begin{array}{rcrcrcrl} x & & & & & = & -26 & (R_1''''') \\ & & y & & & = & -13 & (R_2''''') \\ & & & & z & = & 5 & (R_3''''') \end{array}$$

So our intersection point is the point

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -26 \\ -13 \\ 5 \end{pmatrix}$$

It's always a good idea to check our result. Plug this point back into the three original equations:

$$\begin{array}{rcrcrcr} -26 & - & (-13) & + & 3 \cdot 5 & = & 2 \\ 2(-26) & - & (-13) & + & 8 \cdot 5 & = & 1 \\ -3(-26) & + & 5(-13) & - & 4 \cdot 5 & = & -7 \end{array}$$

and all three equations become true!

This method will always yield a unique point of intersection, if the three normal vectors (in our case $\left(\begin{smallmatrix} 1 \\ -1 \\ 3 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 2 \\ -1 \\ 8 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -3 \\ 5 \\ -4 \end{smallmatrix}\right)$) are linearly independent. If the method leads to contradictions, or doesn't uniquely determine the solution, it is because the normal vectors were linearly dependent.

Linearly dependent normal vectors

If the three normal vectors are linearly dependent, they are all contained in a plane. This means that the three planes we are intersecting are all perpendicular to the plane containing the three normal vectors. In this case, the intersection is either a line (the ‘open book’ situation, see Figure 33), or the intersection is empty (the ‘prism situation’, see Figures 34, 35 and 36).

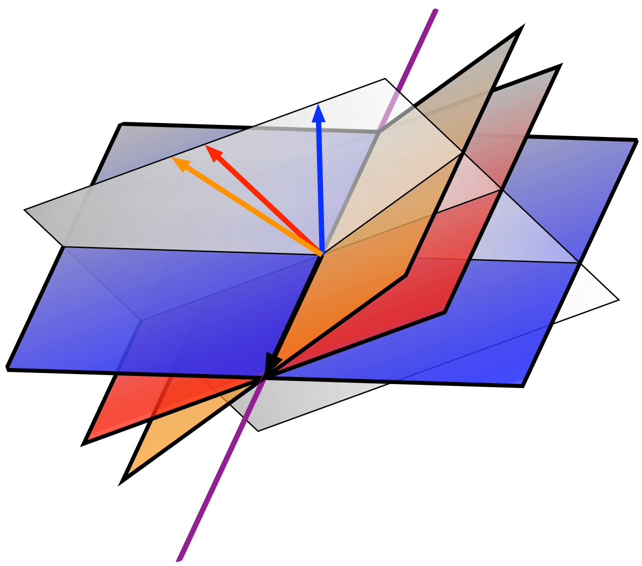


Figure 33: We are intersecting three planes: the blue, the red, and the orange plane. In this picture, the normal vectors are linearly dependent, so they are all in a plane (the grayish auxiliary plane). In this case the intersection is a line: the purple line. The intersection line is orthogonal to the plane containing the three normal vectors.

For a numerical example, let us consider the three equations

$$\begin{array}{rcrcrcrl} x & + & 2y & - & 4z & = & 3 & \quad (R_1) \\ 3x & - & y & + & 9z & = & 2 & \quad (R_2) \\ 2x & + & 2y & - & 2z & = & 4 & \quad (R_3) \end{array}$$

Forward eliminate x:

$$\begin{array}{rcrcrcrcrl} (R_1) & \quad x & + & 2y & - & 4z & = & 3 & \quad (R_1') \\ (R_2)-3(R_1) & & & -7y & + & 21z & = & -7 & \quad (R_2') \\ (R_3)-2(R_1) & & & -2y & + & 6z & = & -2 & \quad (R_3') \end{array}$$

Forward eliminate y:

$$\begin{array}{rcrcrcrcrl} (R_1') & \quad x & + & 2y & - & 4z & = & 3 & \quad (R_1'') \\ (R_2') & & & -7y & + & 21z & = & -7 & \quad (R_2'') \\ 7(R_3')-2(R_2') & & & & & 0 & = & 0 & \quad (R_3'') \end{array}$$

Simplify:

$$\begin{array}{rrcrcrcrl} (R_1'') & x & + & 2y & - & 4z & = & 3 & (R_1''') \\ -\frac{1}{7}(R_2'') & & & y & - & 3z & = & 1 & (R_2''') \\ (R_3'') & & & & & 0 & = & 0 & (R_3''') \end{array}$$

Use z as parameter:

$$z = t$$

Use (R_2''') to solve for y in terms of t:

$$y = 1 + 3t$$

Use (R_1''') to solve for x in terms of t:

$$x = 3 - 2(1 + 3t) + 4t = 1 - 2t$$

Write down the answer in vector form:

$$\vec{v}(t) = \begin{pmatrix} 1-2t \\ 1+3t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

We can also see numerically, that the three normal vectors are linearly dependent. Why did we get the parameter? Because it turned out that (R_3'') was zero. One of the equations "went away". Why did that happen? Because $7(R_3') - 2(R_2') = 0$. Plugging in the values $(R_3') = (R_3) - 2(R_1)$ and $(R_2') = (R_2) - 3(R_1)$, we see that

$$7\big((R_3) - 2(R_1)\big) - 2\big((R_2) - 3(R_1)\big) = 0$$

or, in other words,

$$-8(R_1) - 2(R_2) + 7(R_3) = 0$$

So we have the following *linear relation* among the normal vectors:

$$-8\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} - 2\begin{pmatrix} 3 \\ -1 \\ 9 \end{pmatrix} + 7\begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We can solve this for any of the three normal vectors to obtain, for example:

$$\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = \frac{7}{8}\begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 3 \\ -1 \\ 9 \end{pmatrix}$$

showing that the normal vectors are linearly dependent.

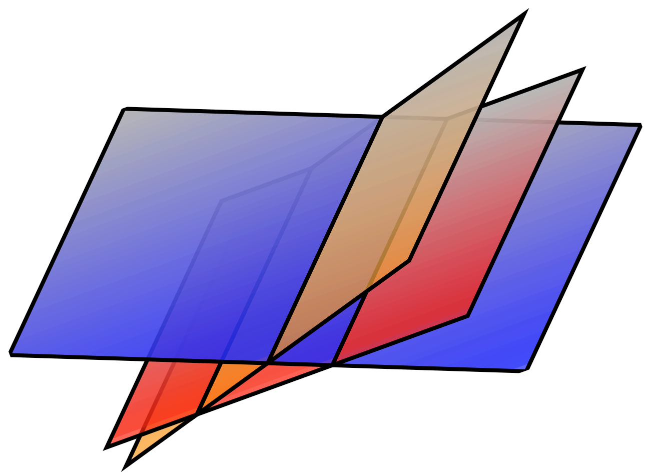


Figure 34: The normal vectors are the same as in Figure 33, but this time the intersection is empty. Each pair of planes intersect in a line, but these three lines do not intersect, because they are parallel to each other.

If we keep the same normal vectors, but change the numbers on the right hand side of the equations slightly, we get empty intersection:

$$\begin{array}{rcrcrcrl} x & + & 2y & - & 4z & = & 3 & (R_1) \\ 3x & - & y & + & 9z & = & 3 & (R_2) \\ 2x & + & 2y & - & 2z & = & 3 & (R_3) \end{array}$$

$$\begin{array}{rrcrcrcrl} (R_1) & x & + & 2y & - & 4z & = & 3 & (R_1') \\ (R_2)-3(R_1) & & & -7y & + & 21z & = & -6 & (R_2') \\ (R_3)-2(R_1) & & & -2y & + & 6z & = & -3 & (R_3') \end{array}$$

$$\begin{array}{rrcrcrcrl} (R_1') & x & + & 2y & - & 4z & = & 3 & (R_1'') \\ (R_2') & & & -7y & + & 21z & = & -7 & (R_2'') \\ 7(R_3')-2(R_2') & & & & & 0 & = & -9 & (R_3'') \end{array}$$

The contradiction $0 = -9$ shows that there are no values for x, y and z solving the system of equations: the intersection is empty. Note that this is kind of contradiction is only possible, because the normal vectors are linearly dependent. If they aren't linearly dependent, the left hand side would never turn into zero!

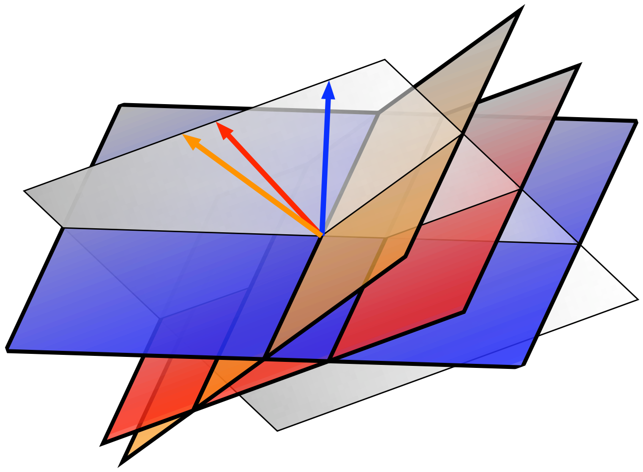


Figure 35: The same picture as in Figure 34, except for we have included the plane containing the three linearly dependent normal vectors. All three planes we are intersecting are perpendicular to this auxiliary plane..

If the three normal vectors lie on a line, then the three planes are parallel to each other. In this case, the intersection is either a plane (if all three planes are the same plane), or it is empty (if the planes are parallel, but not equal).

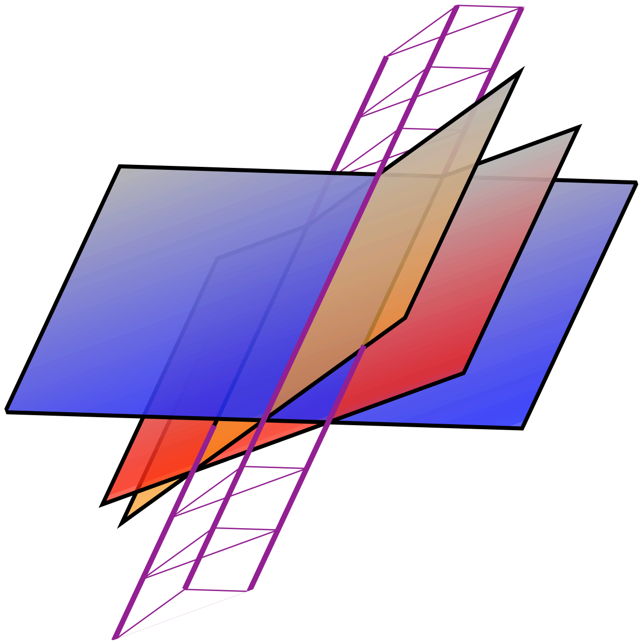


Figure 36: The same picture as in Figure 34, except for this time we included the three lines (in purple), where each pair of planes intersect. These lines are parallel and form the edges of an infinite prism. We have also indicated a few cross sections of this infinite prism

Summary

If the three normal vectors are linearly independent, the three planes intersect in a point.

If the three normal vectors are linearly dependent, then the intersection is either empty, or a line or a plane.

Say the normal vectors are $\vec{a}$, $\vec{b}$ and $\vec{c}$. Say the dimension of $\mathrm{span}(\vec{a}, \vec{b}, \vec{c})$ is d. Then the intersection of the three planes is either empty, or of dimension $3 - d$. The empty option can only happen if the three normal vectors are linearly dependent.

If you think about it, you can convince yourself that this last formulation is true for *any* number of planes ($n = 1, 2, 3, 4$ or higher):

> Say we intersect n planes in $\mathbb{R}^3$, with normal vectors $\vec{a}_1, \ldots, \vec{a}_n$, Say the dimension of $\text{span}(\vec{a}_1, \ldots, \vec{a}_n)$ is d. Then the intersection of the these planes is either empty, or of dimension $3 - d$. The empty option can only happen if the normal vectors are linearly dependent.

4.11 Exercises

Exercise 4.1 Find parametric forms for the planes defined by the equations

$$\begin{aligned} 2x + 3y + 5z &= 15 \\ 4y + 5z &= -3 \\ z &= 2 \end{aligned}$$

Exercise 4.2 Find an equation for the plane with parametric equation

$$\vec{v}(s,t) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

Exercise 4.3 Are the vectors $\vec{u} = \left(\begin{smallmatrix} 2 \\ 3 \\ 4 \end{smallmatrix}\right)$ and $\vec{w} = \left(\begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix}\right)$ linearly independent? Try to justify your answer.

Exercise 4.4 Are the vectors $\vec{u} = \left(\begin{smallmatrix} 2 \\ 3 \\ 4 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$ linearly independent? Justify your answer.

Exercise 4.5 Are the vectors $\vec{u} = \left(\begin{smallmatrix} -1 \\ -1 \\ -1 \end{smallmatrix}\right)$ and $\vec{w} = \left(\begin{smallmatrix} 13 \\ 13 \\ 13 \end{smallmatrix}\right)$ linearly independent? Justify your answer.