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A Note on Hölder's Inequality

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Abstract

In this note an easy method has been introduced to prove Holder's Inequality.

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1 Introduction

In 1888 Rogers proved the Inequality we will be considering in this Note which has been proved later in 1889 by Holder and named after him even though some Mathematicians prefer to call it Rogers Inequality.

In 1994 Hovenier [2] proved sharpening Cauchy's Inequality; and in 1995 Abramovich, Mond, and Pecaric [1] generalized the result of Hovenier to Holder's Inequality. Finally, it is vital to mention that Holder's Inequality is used to prove Minkowski's Inequality.

In this Note we will give an easier proof of Holder's Inequality.

2 Preliminary Notes

We will first consider the following Cauchy's Inequality.

2.1 Cauchy's Inequality

Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ are complex vectors. In addition, let $1 \prec p \prec \infty$; where
$\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left|\sum_{j=1}^{n} a_j b_j\right| \leq (\sum_{j=1}^{n} |a_j|^p)^{\frac{1}{p}} (\sum_{j=1}^{n} |b_j|^q)^{\frac{1}{q}} \tag{1}$$

In (1) equality holds if and only if the sequences $\{|a_j|^p\}$ and $\{|b_j|^q\}$ for $j = 1, 2, ..., n$ are proportional and $\arg(a_j bj)$ is independent of j.

3 Main Results

These are the main results of the paper.

Lemma 3.1 *If $a \succ 0$, $b \prec 1$, $s + t = 1$, $s \succ 0$, and $t \succ 0$. Then*

$$(1-a)^s (1-b)^t \leq 1 - a^s b^t \tag{2}$$

Proof: We will prove the above Lemma by two Methods.
Method 1.
Consider $F(a) = (1-a)^s (1-b)^t + a^s b^t$; where b is fixed. Then
$F^{'}(a) = 0$ implies that $(\frac{1}{a} - 1)^{1-s} = (\frac{1}{b} - 1)^t$.
Which gives that $a = b$.
Now
$[F^{''}(a)]_{a=b} = s(s-1)[(1-a)^{-1} + a^{-1}] \prec 0$.
This means that F attains its maximum at $a = b$, and in turns we have
$[F(a)]_{a=b} = 1 - a + 1 = 1$.
Hence $F(a) \leq 1$ which gives inequality (2).

Method 2.

Consider $f(x) = \frac{x^s - 1}{x - 1}$; where $x \succ 0$, $s \prec 1$, $s + t = 1$, $s \succ 0$, and $t \succ 0$.
Observe that f is decreasing with $\lim_{x \to 1^-} f(x) = s$.
Therefore $f(x) \geq s$.

Let $x = \frac{a}{b}$; and $a \prec b$. (if $a \succ b$, take $x = \frac{b}{a}$). Therefore
$a^s b^t \leq as + bt$.
Now, replace a and b by $1-a$ and $1-b$ respectively we obtain

$$\begin{aligned}(1-a)^s(1-b)^t &\leq s(1-a)+t(1-b)\\ &= 1 - sa - tb\\ &\leq 1 - a^s b^t.\end{aligned}$$

We will now proof Holder's Inequality.

3.2 Holder's Inequality

$$\left|\sum_{j=1}^{n-1} a_j b_j\right| \leq (\sum_{j=1}^{n-1} |a_j|^p)^{\frac{1}{p}} (\sum_{j=1}^{n-1} |b_j|^q)^{\frac{1}{q}} \tag{3}$$

Proof:

We want to prove that the above inequality (3) is true for $j = 1, 2, ..., n$.
Using the above Lemma we have

$$\begin{aligned}\left|\sum_{j=1}^{n} a_j b_j\right| &\leq \left|\sum_{j=1}^{n-1} a_j b_j\right| + |a_n b_n|\\ &\leq (\sum_{j=1}^{n-1} |a_j|^p)^{\frac{1}{p}} (\sum_{j=1}^{n-1} |b_j|^q)^{\frac{1}{q}} + |a_n|\,|b_n|\\ &= (\sum_{j=1}^{n} |a_j|^p - |a_n|^p)^{\frac{1}{p}} (\sum_{j=1}^{n} |b_j|^q - |b_n|^q)^{\frac{1}{q}} + |a_n|\,|b_n|\\ &= (\sum_{j=1}^{n} |a_j|^p)^{\frac{1}{p}} (\sum_{j=1}^{n} |b_j|^q)^{\frac{1}{q}} \left(1 - \frac{|a_n|^p}{\sum_{j=1}^{n}|a_j|^p}\right)^{\frac{1}{p}} \left(1 - \frac{|b_n|^q}{\sum_{j=1}^{n}|b_j|^q}\right)^{\frac{1}{q}}\\ &\quad + |a_n|\,|b_n|\\ &\leq (\sum_{j=1}^{n} |a_j|^p)^{\frac{1}{p}} (\sum_{j=1}^{n} |b_j|^q)^{\frac{1}{q}} \left(1 - \frac{|a_n||b_n|}{\left(\sum_{j=1}^{n}|a_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n}|b_j|^q\right)^{\frac{1}{q}}}\right)\\ &\quad + |a_n|\,|b_n|\\ &= (\sum_{j=1}^{n} |a_j|^p)^{\frac{1}{p}} (\sum_{j=1}^{n} |b_j|^q)^{\frac{1}{q}}\end{aligned}$$

References

[1] Abramovich, S., Mond, B. and Pecaric, J.E., Sharpening Holder's Inequality, *J. Math. Anal. Appl.* **196** (1995), 1131 - 1134.

[2] Hovenier, J.W., Sharpening Cauchy's Inequality, *J. Math. Anal. Appl.* **186** (1994), 156 - 160.

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