Segments Ratio in Right-angled Triangle

Reply

sunken rock

#1 Mar 30, 2014, 3:15 pm

Incircle of a $A-$right-angled triangle ABC intersects its sides at D, E, F respectively. If H, K are the feet of perpendiculars from A to BC and from D to EF respectively, then $IH = IK\sqrt{2}$ (I is the incenter).

Best regards,
sunken rock

---

jayme

#2 Mar 30, 2014, 5:22 pm

Dear Mathlinkers,
an outline of the begining of my proof...

1. D' the midpoint of EF
2. Triangles IDH and ID'K are similar
3.....

Sincerely
Jean-Louis

---

sunken rock

#3 Apr 1, 2014, 10:21 am

> *jayme wrote:*
> Dear Mathlinkers,
> an outline of the begining of my proof...
>
> 1. D' the midpoint of EF
> 2. Triangles IDH and ID'K are similar
> 3.....

Well, I proved $\triangle AHI \sim \triangle DKI$, not entirely synthetic, I had to use metrical relations as well, hence I am eager to see the ending of your proof.

Best regards,
sunken rock

Luis González

#4 Apr 11, 2014, 11:28 am

Let U, V be the reflections of I on $DF, DE.$ Since A is reflection of I on $EF,$ then clearly $\triangle AUV \cong \triangle DEF$ are congruent with parallel sides and I is orthocenter of $\triangle AUV \Longrightarrow \widehat{UAV} = \widehat{EDF} = 45^\circ.$ Since D is circumcenter of $\triangle IUV,$ which has $\widehat{UIV} = 135^\circ,$ then $\triangle DUV$ is isosceles right at $D \Longrightarrow DK \perp UV$ is perpendicular bisector of $\overline{UV}.$ But $\widehat{UAH} = \widehat{ICH} = \widehat{IED} = \widehat{IAV},$ which implies that $J \equiv DK \cap AH$ is circumcenter of $\triangle AUV \Longrightarrow \triangle JUV$ is isosceles right at $J \Longrightarrow JUDV$ is a square with circumcircle passing through $H.$ Thus if L, M are the projections of A, U on $UV, VA,$ we have $AI \cdot IL = AM \cdot AV = AJ \cdot AH \Longrightarrow I, J, L, H$ are concyclic $\triangle AIH \sim \triangle AJL \Longrightarrow$ $\frac{IH}{LJ} = \frac{IH}{IK} = \frac{AI}{AJ} = \frac{AI}{AE} = \sqrt{2}.$

Quick Reply

High School Olympiads

Special angles

Reply

Source: Magazine

MexicOMM

#1 Apr 10, 2014, 10:40 pm

Let ABC be an acute triangle such that, the centroid G has the property that $\angle AGB=2\angle ACB$. Proof that $\angle ACB\ge 60$.

Luis González

#2 Apr 10, 2014, 11:06 pm • 1

Let H and O be the orthocenter and circumcenter of $\triangle ABC,$ lying inside $\triangle ABC,$ since it's acute.
$\angle AGB=2\angle ACB=\angle AOB \Longrightarrow G$ is on the circle $\odot(OAB).$ Now, since G is always between O and $H,$ it follows that H is outside $\odot(OAB) \Longrightarrow \angle AHB \le \angle AOB \Longrightarrow 180^\circ-\angle ACB \le 2\angle ACB \Longrightarrow \angle ACB \ge 60^\circ.$ Clearly, equality holds iff $G \equiv H,$ i.e. when $\triangle ABC$ is equilateral.

Quick Reply

High School Olympiads

Concurrency

Reply

Source: Turkey TST 2014 Day 2 Problem 5

emregirgin35

#1 Mar 12, 2014, 5:12 pm • 1

A circle ω cuts the sides BC, CA, AB of the triangle ABC at A_1 and A_2; B_1 and B_2; C_1 and C_2, respectively. Let P be the center of ω. A' is the circumcenter of the triangle A_1A_2P, B' is the circumcenter of the triangle B_1B_2P, C' is the circumcenter of the triangle C_1C_2P. Prove that AA', BB' and CC' concur.

IDMasterz

#2 Mar 12, 2014, 6:08 pm • 1

Consider the circles B_1B_2P, C_1C_2P. By radical axis they concur at A. So the perpendiculars from the $A'B'C'$ to ABC vice versa coincide, so the triangles are perspective.

Alternatively, consider the polar of A' as $\ell_{A'}$ and the polar of A as ℓ_A, inverse of A be A''. let they intersect at A_3. Let the the triangles formed by $\ell_{A'}, \ell_{B'}, \ell_{C'}$ be $A_4B_4C_4$ and note it is homothetic to ABC. Let the orthocentre of $A_4B_4C_4$ be H and consider the circles with diametres A_4A_3. They have radical centre H clearly, and note $PA'' \cdot PA_4 = PA'' \cdot PA \cdot k = PB'' \cdot PB \cdot k = r^2 \cdot k$ etc... where k is the homothety mapping $ABC \to A_4B_4C_4$, r is radius of (P), so P is on the radical centre of those circles, or they are co-axial with radical axis PH so done.

mathuz

#3 Mar 13, 2014, 12:54 am

very nice proof! Thank you, @IDMasterz 🙂

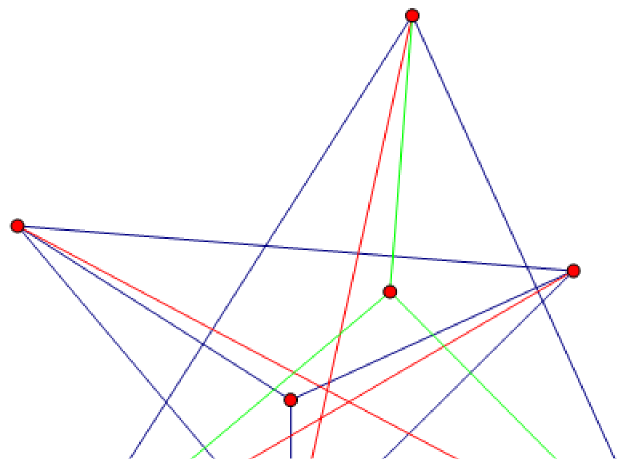
yunxiu

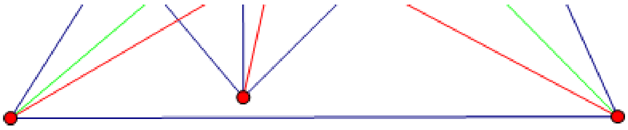
#4 Mar 13, 2014, 7:28 am

> *IDMasterz wrote:*
> So the perpendiculars from the $A'B'C'$ to ABC vice versa coincide, **so** the triangles are perspective.

It is wrong.

Attachments:





IDMasterz

#5 Mar 13, 2014, 12:59 pm • 1

Ahh, I understand your supposed counter example...

Read carefully before you claim something is wrong, because when "the perpendiculars coincide", I am referring to all of them meet at a point.

Thank you very much my dear, aops, math-friendly, good hearted, heavenly moyen, friend.

vlad1m1r

#6 Mar 21, 2014, 2:39 am

Can someone explain the last step in the solution, cause I couldn't understand it clearelly.

Tnx :)

IDMasterz

#7 Mar 23, 2014, 7:46 am

I showed both P, H are the radical centres of the circles, so the circles must be co-axial with radical axis PH.

anti-inequality

#8 Mar 23, 2014, 4:23 pm • 1

Let D, E, F be the foots of the perpendiculars from A, B, C to BC, CA, AB and Q, R, S be the midpoints of the segments $[A_1A_2], [B_1B_2], [C_1C_2]$, respectively. Obviously , $A' \in [PQ, B' \in [PR, C' \in [PS$.

Let the midpoint of $[PA_1]$ be M. Then triangles $PA'M$ and PA_2Q are similar, i.e. $PA'.PQ = PM.PA_2 = \frac{PA_1.PA_2}{2}$.

Similarly, we can find $PB'.PR = \frac{PB_1.PB_2}{2}$ and $PC'.PS = \frac{PC_1.PC_2}{2}$. So, $PA'.PQ = PB'.PR = PC'.PS$.

Let $BC = a, CA = b, AB = c, PQ = x, PR = y, PS = z$, and $PA' = \frac{k}{x}, PB' = \frac{k}{y}, PC' = \frac{k}{z}$.

By Ceva's Theorem, the problem is equivalent to prove $\frac{A(ABA')}{A(ACA')}.\frac{A(BCB')}{A(BAB')}.\frac{A(CAC')}{A(CBC')} = 1$.

Since $A(ABP) = \frac{zc}{2}, A(BPA') = \frac{BQ.k}{2x}, A(APA') = A(QPA') = \frac{QD.k}{2x}$, we get

$A(ABA') = \frac{z.c}{2} + \frac{BD.k}{2x} = \frac{zc}{2} + \frac{k.c.\cos B}{2x} = \frac{c}{2x}(zx + k\cos B)$. Similarly, we can find the other areas.

So, $\frac{A(ABA')}{A(ACA')}.\frac{A(BCB')}{A(BAB')}.\frac{A(CAC')}{A(CBC')} = 1$. and the proof is completed.

vittasko

#9 Mar 31, 2014, 11:32 pm • 1

> IDMasterz wrote:
>
> Consider the circles B_1B_2P, C_1C_2P. By radical axis they concur at A. So the perpendiculars from the $A'B'C'$ to ABC vice versa coincide, so the triangles are perspective.

Can anyone give us a reference about the result of the perspectivity of two orthologic triangles with their orthologic centers coincide with ?

I wonder about if it is a well known fact with also known proof. I have in mind only a reference in greek bibliography by a friend of mine **Nikos Kyriazis** who published a proof of this result in 1996.

Than you in advance, Kostas Vittas.

Luis González

#10 Apr 9, 2014, 11:25 pm • 5

Here is a strong generalization of the problem:

Let $\mathcal{C}$ be a conic with center K on the plane of $\triangle ABC$. $\triangle A_0B_0C_0$ is the polar triangle of $\triangle ABC$ WRT $\mathcal{C}$. Then any triangle $\triangle A_1B_1C_1$ homothethic to $\triangle A_0B_0C_0$ under a homothety with center K is perspective with $\triangle ABC$.

Proof: Throughout the proof we use polarity WRT $\mathcal{C}$. Let $Z \equiv KB_0 \cap AB$ and $Y \equiv KC_0 \cap AC$. Polars of Z, B_0, K concur at the line at infinite $\Longrightarrow$ polar τ_Z of Z passes through the pole C_0 of AB parallel to AC. Likewise, polar τ_Y of Y is the parallel to AB through B_0. If $U \equiv \tau_Y \cap \tau_Z$, then $\triangle AZY$ becomes the polar triangle of $\triangle UB_0C_0 \Longrightarrow K \equiv AU \cap ZB_0 \cap YC_0$ is the perspector of $\mathcal{C}$ WRT $\triangle UB_0C_0$, i.e. A, K, U are collinear $\Longrightarrow$ their polars B_0C_0, YZ and line at infinity concur $\Longrightarrow$ $YZ \parallel B_0C_0$ $(\star)$.

As $\triangle A_1B_1C_1$ varies, the series A_1, B_1, C_1 are similar inducing a homography $BB_1 \mapsto CC_1 \Longrightarrow J \equiv BB_1 \cap CC_1$ is on a fixed conic $\mathcal{K}$ through B, C. When $B_1 \equiv C_1 \equiv K$, then $J \equiv K$ and when $B_1 \equiv B_0, C_1 \equiv C_0$, then J becomes the perspector $P \equiv AA_0 \cap BB_0 \cap CC_0$ of $\mathcal{C}$ WRT $\triangle ABC$. When $B_1 \equiv Z$, then $C_1 \equiv Y$, due to $(\star)$, therefore $J \equiv A \Longrightarrow$ all J lie on unique conic $\mathcal{K}$ through A, B, C, P, K. By similar reasoning, $J' \equiv AA_1 \cap BB_1$ will lie on the same conic $\mathcal{K} \Longrightarrow$ $J \equiv J' \Longrightarrow J \equiv AA_1 \cap BB_1 \cap CC_1$, as desired.

IDMasterz

#11 Apr 10, 2014, 9:38 pm

> *vittasko wrote:*
>
> Can anyone give us a reference about the result of the perspectivity of two orthologic triangles with their orthologic centers coincide with ?
>
> I wonder about if it is a well known fact with also known proof. I have in mind only a reference in greek bibliography by a friend of mine **Nikos Kyriazis** who published a proof of this result in 1996.
>
> Than you in advance, Kostas Vittas.

Dear Mr Vittas,

I know what I used was a generate form of Sondat's theorem, which states that:

If two triangles are mutually orthologic and perspective, then the centres of orthology and perspectivity lie on a line.

Indeed, when the centres of orthology coincide, then the line always exists.

Kindest Regards,

Ivan

mefeke

#12 Sep 1, 2014, 12:45 am • 1

Same as **anti-inequality**'s solution:

Lemma:

$$\frac{[ABP]}{[ABA']} = \frac{[CBP]}{[CBC']}$$

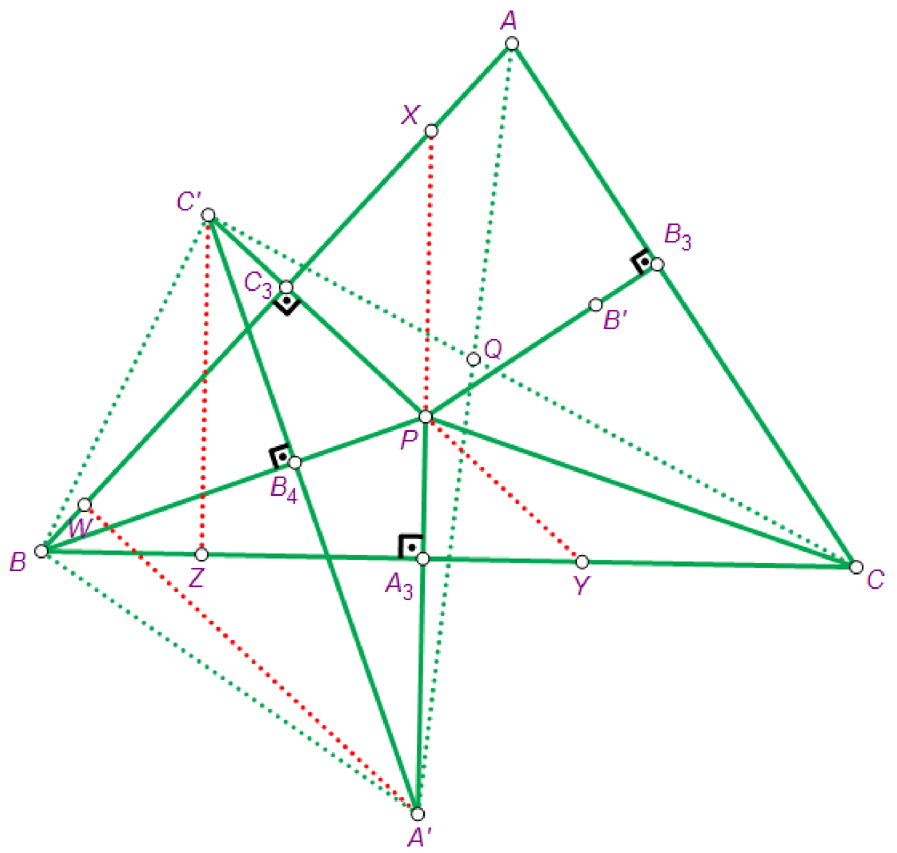
If our lemma is true, then it should be

$$\frac{[ABA']}{[ACA']} \cdot \frac{[ACC']}{[BCC']} \cdot \frac{[BCB']}{[ABB']} = \frac{[ABP]}{[ACP]} \cdot \frac{[ACP]}{[CBP]} \cdot \frac{[BCP]}{[ABP]} = 1.$$

$[ACA'] \quad [BCC'] \quad [ABB'] \qquad [ACP] \quad [CBP] \quad [ABP]$

So AA', BB', CC' will be concurrent.

Let's prove the lemma:



Let A_3 be the midpoint of A_1A_2. So $A_3 \in PA'$.
B_3, C_3 are defined similarly.

Let PA_3 and AB meet at X, PC_3 and BC meet at Y.
Let Z be the foot of altitude from C' to BC and W be the foot of altitude from A' to AB.

$A'C' \perp BP$ because BP is the radical axis of circles with center A' and C'.

P is the orthocenter of $\triangle BXY$. So $XY \perp BP$ and $A'C' \parallel XY$, that is $\dfrac{XP}{XA'} = \dfrac{YP}{YC'}$.

$$\frac{XP}{XA'} = \frac{PC_3}{A'W} = \frac{[ABP]}{[ABA']}$$

$$\frac{YP}{YC'} = \frac{PA_3}{C'Z} = \frac{[CBP]}{[CBC']}$$

$$\frac{[ABP]}{[ABA']} = \frac{[CBP]}{[CBC']}. \blacksquare$$

jayme

#13 Sep 1, 2014, 7:08 pm

Dear Mathlinkers,
you can see a synthetic proof concerninfg the perspectivity when the orthology centers coincide.
http://jl.ayme.pagesperso-orange.fr/ vol. 1 Le théorème de Sondat p. 1.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

How prove $G,H,T $ are collinear.

Reply

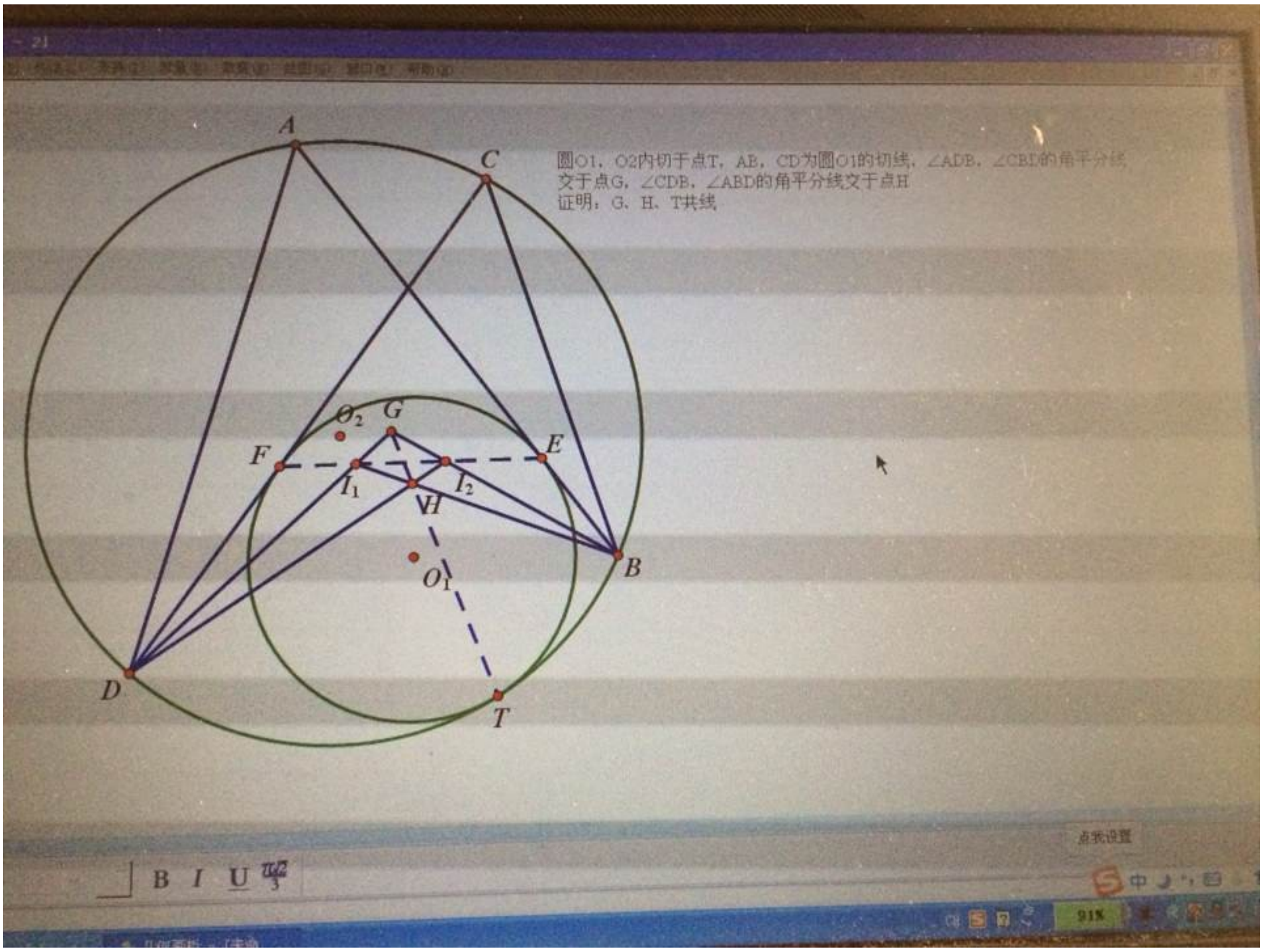
math1200

#1 Apr 2, 2014, 9:57 pm

Circle O_1 and O_2 are internally tangent at point T. AB and CD are tangents of circle O_1, the angle bisectors of Angle $\angle ADB$ and $\angle CBD$ intersects at point G, while the angle bisectors of angle $\angle CDB$ and $\angle ABD$ intersect at point H.

Prove that :
G,H,T are collinear.

Attachments:



Luis González

#2 Apr 9, 2014, 8:03 am

If $P \equiv AB \cap CD$ and $Q \equiv AD \cap BC$, then G and H are the incenters of $\triangle PBD$ and $\triangle QBD$. According to problem 3 circles with common tangency point, there is a circle (O_3) tangent to AD, BC and tangent to (O_1) internally at T, thus according to problem incenter of triangle, both TH and TG bisect $\angle BTD \Longrightarrow G,H,T$ are collinear.

Quick Reply

High School Olympiads

Two parallels

Reply

Source: own

jayme

#1 Apr 8, 2014, 2:35 pm

Dear Mathlinkers,

1. ABC a triangle
2. (O) the circumcircle of ABC
3. X, D the midpoints of the arch BC which contains, doesn't contains A
4. H the orthocenter of ABC
6. F the midpoint of HD
7. M the midpoint of BC
8. L the second point of intersection of the parallel to AC through X with (O)

Prove : FM is parallel to BL.

Sincerely
Jean-Louis

Luis González

#2 Apr 8, 2014, 8:16 pm • 2

Let N,K be the midpoints of AC,AB and P the foot of the A-altitude. 9-point circle $\odot(MNK)$ is image of (O) under homothety with center H and coefficient $\frac{1}{2} \Longrightarrow F$ is midpoint of the arc PM of $\odot(MNK)$. Since $\triangle MNK$ and $\triangle ABC$ are homothetic, then we deduce that $FM \parallel AX.$

On the other hand, let XL cut AB at $E. \widehat{EXA}=\widehat{CAX}=\widehat{EAX} \Longrightarrow \triangle EAX$ is E-isosceles, therefore the cyclic quadrilateral $ABLX$ is an isosceles trapezoid with bases $AX \parallel BL \Longrightarrow FM \parallel BL.$

Quick Reply

High School Olympiads

Stereometry and isogonal conjugate

Reply

osetekuas

#1 Apr 7, 2014, 9:26 pm

In tetrahedron $ABCD$ with inscribed sphere touching ABC at T and exscribed sphere against D (i.e. touching all planes of tetrahedron and plane ABC touching from other side than where D is) touching ABC at S, prove that $\angle ABT = \angle CBS$.

Luis González

#2 Apr 7, 2014, 9:52 pm

Similar problem was posted before; Poland MO 2003, if I'm not mistaken.

Let $\mathcal{C}$ be the cone with vertex D tangent to the insphere and D-exsphere of $ABCD$ (both spheres lie inside the cone). The planes DBC, DCA, DAB are clearly tangent to $\mathcal{C}$ along generatrices of $\mathcal{C} \Longrightarrow$ intersection ω of $\mathcal{C}$ with the plane ABC is a conic section inscribed in $\triangle ABC$ with foci T, S (Dandelin theorem) $\Longrightarrow \angle ABT = \angle CBS.$

randomusername

#3 Apr 8, 2014, 1:34 am

> *Luis González wrote:*
> ω is a conic section inscribed in $\triangle ABC$ with foci $T, S \Longrightarrow \angle ABT = \angle CBS.$

I'm not too well established in the study of conic sections - could you please explain this last step?

Luis González

#4 Apr 8, 2014, 4:20 am • 1

randomusername, it's a known property of the conic tangents. Assume the case where ω is an ellipse with foci $S, T,$ remaining cases are proved similarly. BP, BQ are tangents of ω, where $P, Q \in \omega$. Then $\angle PBS = \angle QBT,$ in other words BS, BT are isogonals WRT $BP, BQ.$

By the reflective property of conics (see post #4 at tangent line of a ellipse), BP and BQ bisect $\angle SPT$ and $\angle SQT$ externally. Hence, reflections M, N of S on BP, BQ lie on $PT, QT,$ respectively. $MT = PS + PT = QS + QT = NT \Longrightarrow T$ is on perpendicular bisector of $\overline{MN}$. But since $\overline{BP}, \overline{BQ}$ are perpendicular bisectors of $\overline{SM}, \overline{SN}$, then B is circumcenter of $\triangle SMN \Longrightarrow BT$ is perpendicular bisector of $\overline{MN}$, i.e. $BT \perp MN$ and this yields the desired isogonality.

Quick Reply

High School Olympiads

Circumscribed Pyramid

Reply

muscleman
#1 Apr 2, 2014, 8:44 pm

A circumscribed pyramid ABCDS is given. The opposite sidelines of its base meet at points P and Q in such a way that A and B lie on segments PD and PC respectively. The inscribed sphere touches faces ABS and BCS at points K and L. Prove that if PK and QL are complanar then the touching point of the sphere with the base lies on BD.

pi37
#2 Apr 7, 2014, 7:15 am

weird solution

Luis González
#3 Apr 7, 2014, 10:55 pm

Let $\mathcal{S}$ denote the insphere of $ABCDS$ touching the planes SCD, SDA and $ABCD$ at M, N, R. Plane $MNKL$ cuts $\mathcal{S}$ along a circle ω and SA, SB, SC, SD at $A', B', C', D' \Longrightarrow A'B'C'D'$ has incircle ω tangent to $B'C', C'D', D'A', A'B'$ at $L, M, N, K,$ thus by Newton's theorem $A'C', B'D', NL, MK$ concur at T.

Planes $p \equiv RNL$ and $q \equiv RMK$ are the polar planes of P and Q WRT $\mathcal{S} \Longrightarrow PQ$ and $RT \equiv p \cap q$ are conjugate lines WRT $\mathcal{S}$. If PK and QL are coplanar, PQ and KL are coplanar $\Longrightarrow$ their conjugate lines RT and SB WRT $\mathcal{S}$ are coplanar as well. Therefore projecting from S on the plane $ABCD$ takes $T \equiv A'C' \cap B'D'$ into $E \equiv AC \cap BD$ and TR into the line BER, i.e. $R \in BD$, as desired.

Quick Reply

High School Olympiads

tangent

Reply

Source: Unknown

Blitzkrieg97
#1 Apr 6, 2014, 11:05 pm

AD is the bisector of the triangle ABC. l is the line, which tangents circumcircles of triangles ADB and ADC in M and N, respectively. Show, that l line also tangents circle, which goes over midpoints of BD,DC,MN.

Luis González
#2 Apr 7, 2014, 12:35 am • 1

Let $(O_1),(O_2)$ denote the circumcircles of $\triangle ABD, \triangle ACD$. X,Y are the midpoints of $\overline{BD},\overline{DC}$ and Z is the midpoint of $\overline{MN}$ (intersection of MN with the radical axis AD).

$\angle BO_1D=2\angle BAD=\angle BAC=2\angle DAC=\angle DO_2C \Longrightarrow$ isosceles $\triangle O_1BD \sim \triangle O_2DC$ are similar, even homothetic $\Longrightarrow O_1D \parallel O_2C \Longrightarrow DC$ goes through the exsimilicenter of $(O_1) \sim (O_2)$, i.e. $H \equiv MN \cap O_1O_2 \cap BC$.

Let E be the midpoint of $\overline{AD}$ and F the midpoint of $\overline{O_1O_2}$ (intersection of perpendicular bisectors of XY and MN). Since $EX \parallel AB, EY \parallel AC$, it follows that $ED \perp O_1O_2$ bisect $\angle XEY \Longrightarrow F$ is midpoint of the circular arc $XEY \Longrightarrow HX \cdot HY=HE \cdot HF$. But since $ZF \perp MN$, we have $HZ^2=HE \cdot HF \Longrightarrow HZ^2=HX \cdot HY$ and the conclusion follows.

sunken rock
#3 Apr 7, 2014, 3:01 am • 1

Let O_1, O_2 be the circumcenters of $\triangle ABD, \triangle ADC$ respectively, and Q, R, S midpoints of $\overline{BD}, \overline{CD}, \overline{MN}$. $\angle BDO_1=90^\circ-\angle BAD=90^\circ-\angle DAC=\angle DCO_2 \implies DO_1 \parallel CO_2$, so the two circles are homothetic and $P\in BC\cap O_1O_2$ is the exsimilicenter of these circles, consequently $P\in MN$. Since P lies onto the perpendicular bisector of AD, the perpendicular at A onto AD intersects BC at D', symmetrical of D w.r.t. P, hence AD' is the external angle bisector of $\angle BAC$ and AP is tangent to the circle (ABC), consequently $PB\cdot PC=PA^2=PD^2\ (\ 1\ )$. Also as given, $PB\cdot PD=PM^2\ (\ 2\ )$ and $PD\cdot PC=PN^2\ (\ 3\ )$. Multiply (2) by (3) side by side and, with (1) get $PM\cdot PN=PD^2\ (\ 4\ )$.

We need to prove $PQ\cdot PR=PS^2\ (\ *\ )$, but $PQ=\frac{PB+PD}{2}, PR=\frac{PD+PC}{2}, PS=\frac{PM+PN}{2}$ which, substituted into $(*)$, with $(1), (2), (3), (4)$ getting an equality, hence $(*)$ was true.

Best regards,
sunken rock

Blitzkrieg97
#4 Apr 7, 2014, 7:40 pm

thank you :)

nima-amini
#5 Apr 9, 2014, 10:04 pm

it is russia 2000-2001

Quick Reply

High School Olympiads

Nice perpendicular

Reply

Source: China

fml1st_cqt

#1 Apr 3, 2014, 12:05 pm

Dear mathlinkers,
1. Let ABC be a triangle with (O) is its circumcircle.
2. H is its orthocenter, the internal bisector of $\angle BAC$ cuts (O) at D.
3. $E \in AB$, $OE \parallel AD$, F is midpoint of HD.
Prove that $EF \perp FC$.

jayme

#2 Apr 3, 2014, 8:55 pm

Dear Mathlinkers,
just a beginning of a synthetic proof

1. X the midpoint of the arc BC which contains A
2. B, O, X and E are concyclic
3....

Sincerely
Jean-Louis

Luis González

#3 Apr 6, 2014, 3:11 am • 3

Homothety with center H and coefficient $\frac{1}{2}$ takes (O) into the 9-point circle (N) of $\triangle ABC \Longrightarrow F \in (N)$ and H is also center of inversion that swaps (N) and (O). Hence if DH cuts (O) again at M, this inversion takes F to M and C to the foot T of the C-altitude $\Longrightarrow M, T, F, C$ are concyclic $(\star)$.

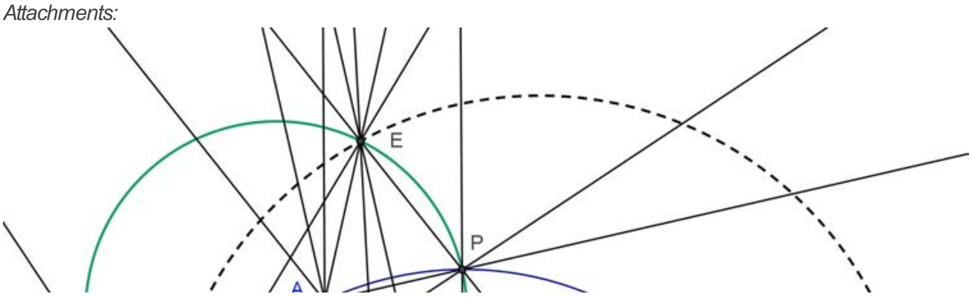
Let P be the midpoint of the arc BAC of (O) and OE cuts AC at L. Clearly $AEPL$ a rhombus. If DM cuts AP at U, then from $AH \parallel PD$, we get

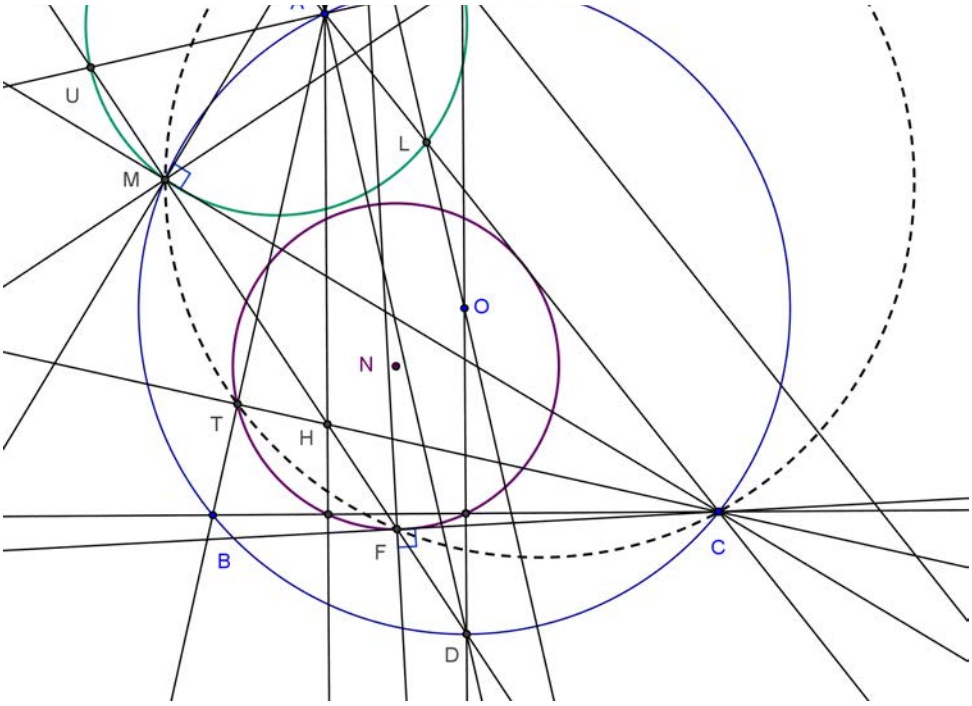
$$\frac{UA}{UP} = \frac{AH}{PD} = \frac{AH}{2R} = \cos \widehat{A} = -\cos \widehat{EPL} \Longrightarrow AP = (1 + \cos \widehat{EPL}) \cdot UP$$

But $\tan \frac{\widehat{EPL}}{2} = \frac{LE}{AP} \Longrightarrow \frac{LE}{UP} = \tan \frac{\widehat{EPL}}{2} \cdot (1 + \cos \widehat{EPL}) = \sin \widehat{EPL}$,

which means that PU is circumdiameter of the P-isosceles $\triangle PEL \Longrightarrow \widehat{PEU} = \widehat{PLU} = 90^\circ$. Since $\widehat{PMD} = 90^\circ$, then E, P, L, M, U are concyclic $\Longrightarrow \widehat{EMP} = \widehat{ELP} = \frac{\widehat{A}}{2} \Longrightarrow \widehat{EMC} = \widehat{EMP} + \widehat{PMC} = \frac{\widehat{A}}{2} + 90^\circ - \frac{\widehat{A}}{2} = 90^\circ \Longrightarrow E, M, T, C$ are concyclic. Together with $(\star)$, E, M, T, C, F are concyclic $\Longrightarrow \widehat{EFC} = \widehat{EMC} = 90^\circ$, i.e. $EF \perp FC$.

Attachments:





shinny98NT

#4 Apr 6, 2014, 12:23 pm

What a fabulous solution! It's nice from the additional points to the way to link every right angle together. From this, the problem that Mr Jean-Louis has just posted before is no longer a tough result. I really appreciate Mr Luis Gonzáles!!

IDMasterz

#5 Apr 7, 2014, 10:20 am

Suppose $\angle B > \angle C$. Let D' be the antipode of D, and A' be the antipode of A wrt $\odot ABC$. Let the perpendicular from D to AB be X and the midpoint of BC be M. Since the isogonal of a point is perpendicular to its simson line, $AD \perp XM \implies OE \perp DA' \parallel AD'$, thus $ED' \parallel AC$. Let $ED' \cap XM = L$, and the perpendicular from D' to AC be Y. By simple angle chasing, $EL = EX$, and due to $EA = ED'$, we have $D'L = AX$ which by corollary 1 here is equal to YC, so $CYD'L$ is a rectangle, or $\angle ELC = 90$. Hence, if the altitude from C to AB is Z, then $L \in \odot CZE$. Since $F \in (N)$ (the nine point circle), it follows by letting M_C be the midpoint of AB, and the converse of Reim's theorem on circles $\odot CZEL, (N)$ and lines EZ, ML, that due to $M_C M \parallel EL \implies$ the other intersection of $(N) \cap \odot CZEL \in ML$, which is clearly just F.

jayme

#6 Apr 9, 2014, 4:03 pm

Dear Mathlinkers,
can some one precises the origine of this problem : China....?
Very sincerely
Jean-Louis

sunken rock

#7 Apr 10, 2014, 2:00 am

Remark: $m(\widehat{FEC}) = \dfrac{m(\hat{A})}{2}$!.

Best regards,
sunken rock

jayme

#8 Apr 10, 2014, 3:49 pm

Dear Mathlinkersn,
I come back with my question :

can some one present the problem in chinese with a reference.
Thank in advance...
This problem is very interesting...
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Concurrent problem

Locked

Source: own .... ?

BlackSelena

#1 Apr 5, 2014, 12:18 am

Let triangle ABC with L, O are the Lemoine point, circumcenter respectively. D is the intersection of AL and BC. Let E, F on AB, AC such that DE, DF parallel to AC, AB respectively. Easy to see that B,E,F,C are concyclic, denote it center X. Construct Y, Z similarly. Prove that AX, BY, CZ are concurrent.
p/s: Can somebody prove that DX is parallel to LO?

Luis González

#2 Apr 5, 2014, 12:28 am

It's exactly a problem from China TST 2005 (P2 quiz 4).

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=558640
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=482207

High School Olympiads

Viet Nam TST 2014 day 1 problem 3

Reply

lehungvietbao
#1 Mar 31, 2014, 1:07 pm

Let ABC be triangle with $A<B<C$ and inscribed in a circle (O). On the minor arc ABC of (O) and does not contain point A, choose an arbitrary point D. Suppose CD meets AB at E and BD meets AC at F. Let O_1 be the incenter of triangle EBD touches with EB,ED and tangent to (O). Let O_2 be the incenter of triangle FCD, touches with FC,FD and tangent to (O).
a) M is a tangency point of O_1 with BE and N is a tangency point of O_2 with CF. Prove that the circle with diameter MN has a fixed point.
b) A line through M is parallel to CE meets AC at P, a line through N is parallel to BF meets AB at Q. Prove that the circumcircles of triangles $(AMP),(ANQ)$ are all tangent to a fixed circle.

shinny98NT
#2 Mar 31, 2014, 6:24 pm

You could try using Lyness-Sawayama Theorem (this time's much harder).
To prove the second question, we use the Inversion.

lehungvietbao
#3 Apr 4, 2014, 8:31 am

Anyone can post full solution for this problem ? :)

Luis González
#4 Apr 4, 2014, 9:58 am • 1

a) Let U,V be the tangency points of (O_1) and (O_2) with DE and DF, respectively. (O_1) is a Thebault circle of the cevian CD of $\triangle ABC$ externally tangent to its circumcircle (O), thus by extraversion of Sawayama's lemma UM goes through the A-excenter I_a of $\triangle ABC$ and similarly VN goes through I_a. Now, in any cyclic $ABDC$, the external bisectors of $\angle AEC$ and $\angle AFB$ are perpendicular (this is easy to prove via angle chase), thus UM and VN, being parallel to these bisectors, are perpendicular to each other, i.e. $\angle MI_aN=90^\circ \Longrightarrow$ all circles with diameter $\overline{MN}$ go through I_a.

b) Since $\angle PMU=\angle MUE=\angle EMU \Longrightarrow MU \equiv MI_a$ bisects $\angle AMP$ externally $\Longrightarrow I_a$ is also A-excenter of $\triangle AMP$, therefore $\triangle AMP, \triangle ABC$ have common A-mixtilinear excircle ω_A tangent to AB,AC at points S,T, such that $I_a \in ST \Longrightarrow \odot(AMP)$ always touches ω_A and similarly $\odot(ANQ)$ touches ω_A as well.

Quick Reply

High School Olympiads

Euler line and perspective

Reply

Stephen
#1 Apr 1, 2014, 4:20 pm

P, Q are two points in ABC which are in isogonal conjugate realationship. Prove that the following two statements are equivalent:

(1) PQ is parallel to Euler line of ABC

(2) 'The triangle formed by circumcenters of PAB, PBC, PCA' and ABC are perspective.

Luis González
#2 Apr 1, 2014, 6:28 pm

The proposition isn't complete, it should state that the triangle formed by circumcenters of PAB, PBC, PCA and ABC are perspective iff either PQ is parallel to the Euler line or one of the points P or Q lie on the circumcircle of ABC.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=269121
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=373375

Quick Reply

High School Olympiads

circles

Reply

andrejilievski

#1 Mar 31, 2014, 9:22 pm

Triangle ABC is given such that $AC > AB$. Let k be a circle touching AB and AC at M and N and intersects the circumcircle of ABC at P and Q. Let S, T be points on BA, AC respectively, such that $\frac{AS}{AB} = \frac{CN}{BM+CN}$ and $\frac{AT}{AC} = \frac{BM}{BM+CN}$ Prove that ST, BC, PQ intersect at one point

Luis González

#2 Apr 1, 2014, 9:13 am

Let $\{D, E\} \equiv k \cap BC, X \equiv MN \cap BC$ and $J \equiv BN \cap CM$. Since $A(M, N, J, X) = -1$, it follows that AJ is the polar of X WRT k. Hence if AJ cuts BC at Y, then X, Y are harmonic conjugates WRT B, C and $D, E \Longrightarrow$ $UX^2 = UY^2 = UB \cdot UC = UD \cdot UE$ (where U is the midpoint of XY), thus U has equal power WRT $\odot(ABC)$ and k $\Longrightarrow$ it's on their radical axis PQ. So, it suffices to prove that ST goes through U.

By Menelaus' theorem for $\triangle ABC, \overline{XMN}$, keeping in mind that $AM = AN$, we get $\frac{YB}{YC} = \frac{XB}{XC} = \frac{BM}{CN} \Longrightarrow \frac{BY}{YC} = \frac{BS}{SA}$ $\Longrightarrow YS \parallel AC$ and likewise $YT \parallel AB \Longrightarrow ASYT$ is parallelogram $\Longrightarrow V \equiv AY \cap ST$ is midpoint of ST. Since $A(S, T, Y, X) = -1$, then we deduce that $AX \parallel ST \Longrightarrow ST$ is Y-midline of $\triangle AXY$, hitting $\overline{XY}$ at its midpoint U, as desired.

Quick Reply

High School Olympiads

Equivalent areas

Reply

Source: maybe new?

sunken rock

#1 Mar 14, 2014, 11:28 pm

Let M, N be points onto the sides AB, CD respectively of the convex quadrilateral $ABCD$ such that $\frac{AM}{BM}=\frac{CN}{DN}$ and $P\in AN\cap DM, Q\in BN\cap CM.$
Prove that $[APD]+[BCQ]=[MQNP]$, where $[XYZT]$ means the area of $XYZT$.

Best regards,
sunken rock

sunken rock

#2 Mar 20, 2014, 9:52 pm

It seems it is not attactive! hint

Best regards,
sunken rock

Luis González

#3 Mar 25, 2014, 8:06 am

$$\frac{[AMD]}{[ABD]}=\frac{AM}{AB}=\frac{CN}{CD}=\frac{[CNB]}{[CDB]} \Longrightarrow$$

$$[AMD]+[CNB]=\frac{AM}{AB}\left([ABD]+[CDB]\right)=\frac{AM}{AB}\cdot [ABCD] \ \ (1).$$

Similarly we have $[BMC]+[DNA]=\frac{BM}{BA}\cdot [ABCD] \ \ (2).$

Adding the expressions (1) and (2) together gives

$$[AMD]+[CNB]+[BMC]+[DNA]=[ABCD] \Longrightarrow$$

$$[DNA]+[CNB]=[DMC] \Longrightarrow [APD]+[BCQ]=[MQNP].$$

sunken rock

#4 Mar 25, 2014, 4:14 pm

My solution was little bit uglier: Let $AM=a, MB=b, CN=ka, DN=kb$ and J, K, L the projections of A, M, B respectively onto CD. We can easily prove that $(a+b)MK=b.AJ+a.BL$ $(\star)$ or, multiplying by k, $k(a+b)MK=kb.AJ+ka.BL$ $(\ 1\ )$. The relation (1) means $[CMD]=[AND]+[BCN]$ $(\ 2\ )$, or $[MQNP]+[CNQ]+[NDP]=[BCQ]+[CNQ]+[APD]+[NDP]$, hence $[MQNP]=[BCQ]+[APD]$.

Note: relation $(\star)$ can be proved by extending AB, CD until they meet, and taking similar triangles.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Hard

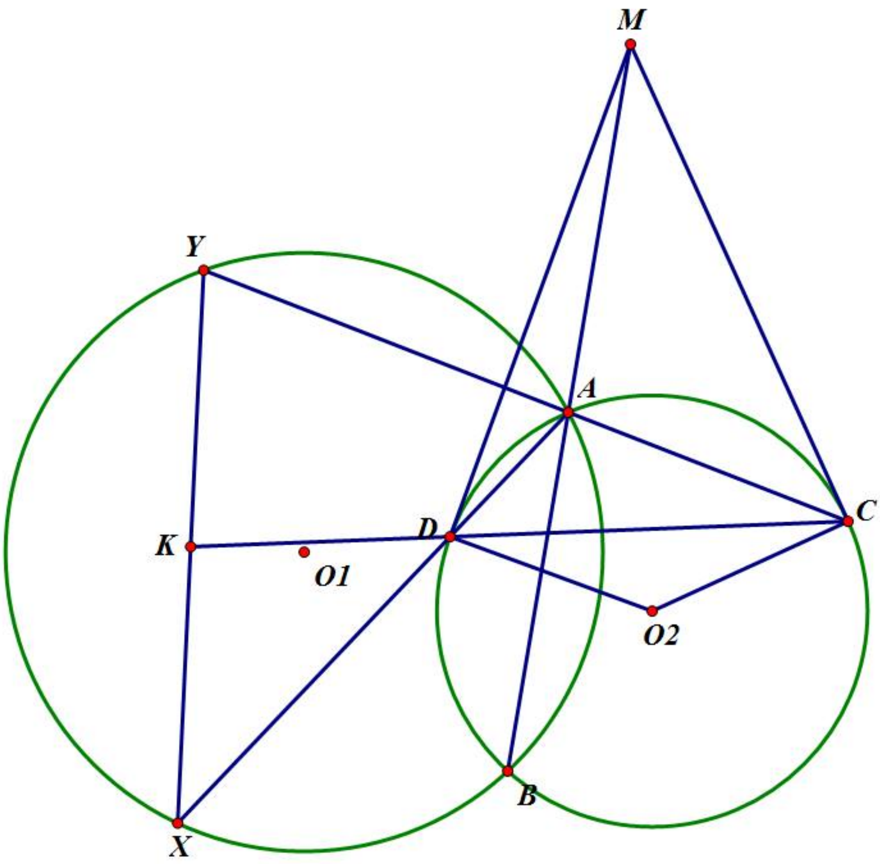
Reply

thomas99

#1 Mar 23, 2014, 10:27 pm

Let (O_1) and (O_2) meets at A and B.M $\in$ AB,D and C $\in$ (O_2) such that $O_2D \perp MD; O_2C \perp MC$.AD meets (O_1) at X ; AC meets (O_1) at Y and DC meets XY at K.Prove : $KY = KX$

Attachments:



Luis González

#2 Mar 24, 2014, 6:18 am • 1

Since $\angle YCB = \angle XDB$ and $\angle CYB = \angle DXB \Longrightarrow \triangle BCY \sim \triangle BDX \Longrightarrow \frac{CB}{DB} = \frac{CY}{DX}$. But since $ADBC$ is harmonic, we have $\frac{CB}{BD} = \frac{CA}{AD} \Longrightarrow \frac{CA}{CY} = \frac{AD}{DX}$. By Menelaus' theorem for $\triangle AXY, \overline{KDC}$, we get then

$$\frac{KY}{KX} = \frac{CY}{CA} \cdot \frac{DA}{DX} = \frac{CY}{CA} \cdot \frac{CA}{CY} = 1 \Longrightarrow KY = KX.$$

IDMasterz

#3 Mar 24, 2014, 12:35 pm

B is the miquel point of complete quad $XKYACD$, so $B \in \odot KXD$. So, $\angle XKB = \angle XDB = \angle ACB = \angle BMD$ where M is the midpoint of DC. Spiral similarity dictates $BDC \sim BXY \implies BMD \sim BKX$, so the result follows.

PROF65

#4 Mar 24, 2014, 10:32 pm

Let BC intersect C_1 at an other pt Z
MC // YZ
the harmonic bundle of lines(CD,CA,CB , CM) intersect YZ at K', Y,Z and the infinite pt then K' midpoint of YZ
it remain to prove that XZ // DC which is easy....

jayme

#5 Mar 25, 2014, 2:47 pm

Dear Mathlinkers,

1. ADBC is harmonic
2. DC is the D-symmedian of DAB
3. Z the second point of intersection of BD with (O1)
4. DC is the D-median of DXZ
5. According to the Reim's theorem YZ // DC
6. According to the midline theorem applied to XYZ, K is the midpoint of XY.

Sincerely
Jean-Louis

shinny98NT

#6 Mar 27, 2014, 7:53 pm

$ADBC$ is a harmonic quadrilateral. CD is a symmedian from C. Also, XY is the antiparallel of AB. From the features of the symmedian and the atiparallel, we have $KY=KX$.

Quick Reply

High School Olympiads

tangent circles

Locked

math4evernever

#1 Mar 24, 2014, 1:58 am

Let E be a point on the median CD of triangle ABC. Let S_1 be the circle passing through E and tangent to line AB at A, intersecting side AC again at M; let S_2 be the circle passing through E and tangent to line AB at B, intersecting side BC again at N. Prove that the circumcircle of triangle CMN is tangent to circles S_1 and S_2.

I also suppose that S_1 and S_2 are tangent at E, but I don't know if it is necessarily true...

Luis González

#2 Mar 24, 2014, 2:11 am

It's a problem from All-Russian Olympiads (2000). Use the search before posting contest problems.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=296438
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=514290

> math4evernever wrote:
> I also suppose that S_1 and S_2 are tangent at E, but I don't know if it is necessarily true

This is false in general. They are tangent iff DA=DB=DE.

High School Olympiads

Nice problems

Reply

Source: unknown

ThisIsART
#1 Mar 23, 2014, 10:18 am

Given C is a circle , L is a line tangent to C, and M is a point at L. Find all locus P so that: There are 2 points Q, R at L such that M midpoint QR and C is incircle of PQR.

Luis González
#2 Mar 23, 2014, 11:31 pm • 2

It's almost trivial; If X is the tangency point of the circle with the line L, then the locus of P is the line passing through the antipode of X on C and the reflection of X on M, in other words, all triangles PQR have commom P-Nagel cevian.

Stronger result: $\mathcal{C}$ is a fixed conic and ℓ is a fixed tangent of $\mathcal{C}$ at $X.$ Q,R vary on $\ell,$ such that $Q \mapsto R$ is an involution. Second tangents from Q,R to $\mathcal{C}$ meet at $P.$ Locus of P is a fixed line.

Proof: Denote O the center of $\mathcal{C}$ and U,V the tangency points of PQ,PR with $\mathcal{C},$ respectively. OQ,XU have conjugate directions WRT $\mathcal{C} \Longrightarrow XU \mapsto OQ$ is involutive and likewise $XV \mapsto OR$ is involutive. Since $OQ \mapsto OR$ is an involution, then $XU \mapsto XV$ is then an involutive homography taking $\mathcal{C}$ into itself $\Longrightarrow$ all lines UV go through a fixed point $K \Longrightarrow P$ runs on the polar κ of K WRT $\mathcal{C},$ clearly fixed.

ThisIsART
#3 May 24, 2014, 9:41 pm

Mmm i still do not understand sorry

yunxiu
#4 May 28, 2014, 8:25 am

1992IMO-P4
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=366410&sid=22e2a10f59dc9952eb115bca0b0b0457#p366410

ThisIsART
#5 May 28, 2014, 1:49 pm

Ok i am sorry
Pist doubles

Quick Reply

High School Olympiads

Kazakhstan national 2014, problem 2; 10 grades

Reply

rightways

#1 Mar 21, 2014, 12:44 am

Given triangle ABC with a circumscribed circle w. Points D and D_1 on AC are given such that a midpoint of AC is a midpoint of DD_1. BD and BD_1 intersects w again at points E and E_1. Show that all such EE_1 lines meet at a fixed point on a plane

Luis González

#2 Mar 21, 2014, 2:14 am

Valid for any arbitrary conic ω through A, B, C (not necessarily the circumcircle) and the fixed point is always on line AC.

As D varies on $AC, D \mapsto D_1$ is clearly an involution $\Longrightarrow BD \equiv BE \mapsto BD_1 \equiv BE_1$ is an involution $\Longrightarrow E \mapsto E_1$ is then an involutive homography carrying the conic ω into itself $\Longrightarrow$ all EE_1 pass through the fixed pole P of the involution. When $D \equiv A$, then obviously $D_1 \equiv C$, thus P is on AC.

mathuz

#3 Mar 21, 2014, 2:50 pm

let EE_1 and AC meet at point P. Then

$$\frac{PC}{PA}=(\frac{AB}{BC})^2.$$

Quick Reply

High School Olympiads

very nice problem 2

Reply

Source: vankhea

vankhea

#1 Mar 4, 2014, 9:01 pm

Let P, Q, R be points on the sides BC, CA, AB of triangle ΔABC. Let A', B', C' be reflection points of P, Q, R on A, B, C.
Prove that $[A'B'C'] = 6[ABC] + [PQR]$

Luis González

#2 Mar 5, 2014, 3:42 am

It's similar to the previous very nice problem; area chase using [QRB'C']=4[QRBC] and cyclic expressions.

[A'B'C']=[PQR]+[A'QR]+[B'RP]+[C'PQ]+[PB'C']+[QC'A']+[RA'B']

Substituting [PB'C']= [QRB'C']-[PQR]-([B'RP]+[C'PQ]) and cyclic expressions, we get

[A'B'C']=[QRB'C']+[RPC'A']+[PQA'B']-[A'QR]-[B'RP]-[C'PQ]-2[PQR] --->
[A'B'C']=4([QRBC]+[RPCA]+[PQAB])-[A'QR]-[B'RP]-[C'PQ]-2[PQR] (1)

On the other hand, we have [QRBC]=[PQR]+[BRP]+[CPQ] and adding the cyclic expressions together gives

[QRBC]+[RPCA]+[PQAB]=3[PQR]+2([ABC]-[PQR])=2[ABC]+[PQR] (2)

In the quadrilateral A'RPQ, we have [A'QR]=2[AQR]+[PQR]. Adding cyclic expressions together gives

[A'QR]+[B'RP]+[C'PQ]=2([ABC]-[PQR])+3[PQR]=2[ABC]+[PQR] (3)

Combining (1), (2) and (3), we get

[A'B'C']=4(2[ABC]+[PQR])-(2[ABC]+[PQR])-2[PQR]=6[ABC]+[PQR].

Quick Reply

High School Olympiads

Moldova TST triangle geometry

Reply

Source: Moldova TST 2014, First Day, Problem 3

TheBottle

#1 Mar 4, 2014, 10:55 pm

Let $\triangle ABC$ be an acute triangle and AD the bisector of the angle $\angle BAC$ with $D \in (BC)$. Let E and F denote feet of perpendiculars from D to AB and AC respectively. If $BF \cap CE = K$ and $\odot AKE \cap BF = L$ prove that $DL \perp BF$.

hyperbolictangent

#2 Mar 5, 2014, 12:09 am • 1

First, we'll show AK is perpendicular to BC so that $AK \cap BC = H$ is the foot of the altitude from A to BC. We'll use Ceva's Theorem; we want

$$\frac{AF}{FC} \times \frac{CH}{HB} \times \frac{BE}{EA} = 1$$

Triangles AED and AFD are clearly congruent, so $AF = EA$. Then it's enough to show

$$\frac{CH}{HB} = \frac{FC}{BE}$$

Of course

$$\frac{CH}{HB} = \frac{\tan\angle B}{\tan\angle C}$$

On the other hand, $FC = DF/\tan\angle C$ and $BE = \DE}{\tan{\angle{B}}}$ so that

$$\frac{FC}{BE} = \frac{\tan\angle B}{\tan\angle C}$$

By Ceva's theorem, the cevians from A, B, C to H, E, F are concurrent; in other words, AK, and therefore KH, is perpendicular to BC.

Now, consider the circles $(AEKL)$, $(AEFD)$, and (DKL). Their radical axes, pairwise, must concur. The radical axis of the first two is line AE, and the radical axis of the first and the third is KL. Both lines go through B, so B must be the radical center of the three circles. Thus the radical axis of $(AEFD)$ and (DKL), i.e. the line through D and the other intersection of these two circles, must go through B. It follows that the second intersection of these two circles must lie on BD. Yet the second intersection of $(AEFD)$ with BD is none other than H, since this second intersection X must satisfy $X \in BD$ and $\angle AXD + \angle AED = 180^\circ$. Then (DKL) intersects BD again H, so that $DHKL$ is cyclic. It follows that

$$\angle DLK = 180^\circ - \angle KHD = 180^\circ - \angle AHD = 180^\circ - 90^\circ = 90^\circ$$

Thus $DL \perp BF$, as desired.

Luis González

#3 Mar 5, 2014, 12:41 am

Circle $\odot(AEDF)$ with diameter $\overline{AD}$ cuts BC again at the projection P of A on it. Since D is clearly midpoint of the arc EPF, then PD, PA bisect $\widehat{EPF}$. Hence if $Q \equiv EF \cap BC, R \equiv EF \cap AP$, the pencil $P(E, F, R, Q) = -1$ is harmonic $\Longrightarrow (B, C, P, Q) = -1$. Thus from the complete quadrilateral $BCFE$, it follows that $K \equiv BF \cap CE$ is on AP. Now $BP \cdot BD = BE \cdot BA = BK \cdot BL \Longrightarrow PDLK$ is cyclic, so $\widehat{DLK} = \widehat{DPK} = 90^\circ$.

seledeur

#4 May 7, 2014, 2:29 am

> *hyperbolictangent wrote:*
> First, we'll show AK is perpendicular to BC so that $AK \cap BC = H$ is the foot of the altitude from A to BC. We'll use Ceva's Theorem; we want
>
> $$\frac{AF}{FC} \times \frac{CH}{HB} \times \frac{BE}{EA} = 1$$
>
> Triangles AED and AFD are clearly congruent, so $AF = EA$. Then it's enough to show
>
> $$\frac{CH}{HB} = \frac{FC}{BE}$$
>
> Of course
>
> $$\frac{CH}{HB} = \frac{\tan \angle B}{\tan \angle C}$$
>
> On the other hand, $FC = DF/\tan \angle C$ and $BE = \DE}{\tan{\angle{B}}$ so that
>
> $$\frac{FC}{BE} = \frac{\tan \angle B}{\tan \angle C}$$
>
> By Ceva's theorem, the cevians from A, B, C to H, E, F are concurrent; in other words, AK, and therefore KH, is perpendicular to BC.
>
> Now, consider the circles $(AEKL)$, $(AEFD)$, and (DKL). Their radical axes, pairwise, must concur. The radical axis of the first two is line AE, and the radical axis of the first and the third is KL. Both lines go through B, so B must be the radical center of the three circles. Thus the radical axis of $(AEFD)$ and (DKL), i.e. the line through D and the other intersection of these two circles, must go through B. It follows that the second intersection of these two circles must lie on BD. Yet the second intersection of $(AEFD)$ with BD is none other than H, since this second intersection X must satisfy $X \in BD$ and $\angle AXD + \angle AED = 180°$. Then (DKL) intersects BD again H, so that $DHKL$ is cyclic. It follows that
>
> $$\angle DLK = 180° - \angle KHD = 180° - \angle AHD = 180° - 90° = 90°$$
>
> Thus $DL \perp BF$, as desired.

Neat and very well explained ! Thank you.

Quick Reply

High School Olympiads

very nice problem

Reply

Source: vankhea

vankhea

#1 Mar 2, 2014, 9:26 pm

Let P,Q,R be points on sides BC,CA,AB such that P,Q,R collinear. Let X,Y,Z be reflection points of A,B,C on P,Q,R.
Prove that $[XYZ]=3[ABC]$

vankhea

#2 Mar 3, 2014, 8:03 pm

this one is harder than
Let P,Q,R be point on the sides BC,CA,AB respectively. Let X,Y,Z be reflection points of A,B,C on P,Q,R. Then we get: $[XYZ]=4[PQR]+3[ABC]$

Luis González

#3 Mar 4, 2014, 3:23 am

> vankhea wrote:
> Let P,Q,R be point on the sides BC,CA,AB respectively. Let X,Y,Z be reflection points of A,B,C on P,Q,R. Then we get: $[XYZ]=4[PQR]+3[ABC]$

The area chase is very figure-dependent, so we assume that C is outside XYZ, while A and B are inside. The remaining possibilities are treated similarly.

$4[ABPQ]=2\cdot AP\cdot BQ\cdot \sin\widehat{AP,BQ}=\tfrac{1}{2}AX\cdot BY\cdot\sin\widehat{AP,BQ}=[ABXY]$

But since $[ABC]=[AYC]=[BXC]$, then $[ABXY]=3[ABC]-[CXY] \Longrightarrow$

$4([ABC]-[CPQ])=3[ABC]-[CXY] \Longrightarrow 4[CPQ]-[ABC]=[CXY] \ (1).$

On the other hand, we have

$[CZY]=2[CYR]=4[CQR]+2[BCR] \ \ (2)$

$[CXZ]=2[CXR]=4[CPR]+2[ACR] \ \ (3).$

Adding the expressions (1), (2) and (3) together gives:

$([ABC]+2[BCR]+2[ACR])+4\left([CPR]+[CQR]-[CPQ]\right)=$

$=[CYZ]+[CXZ]-[CXY]=[XYZ] \Longrightarrow$

$3[ABC]+4[PQR]=[XYZ].$

Quick Reply

High School Olympiads

Orthopoles and fixed point

Reply

Source: own

livetolove212

#1 Dec 17, 2009, 3:32 pm

Given triangle ABC and an arbitray point P inside. Let l_1, l_2 be two lines through P such that $l_1 \perp l_2$. Let X_1, X_2 be two orthopoles of triangle ABC wrt l_1, l_2,respectively. When l_1, l_2 move, prove that X_1X_2 passes through a fixed point.

ma 29

#2 Dec 17, 2009, 10:05 pm

Denote points $M = AC \cap l_2, N = AB \cap l_2$.

A', C' be two points lie on l_1 such that $AA' \perp l_1, CC' \perp l_1$.

Let H, K be the orthocenters of ABC, AMN,respectively.

Call $S = AK \cap CH$.

We see that $A'X_1//AH, X_1C'//SH, A'C'//AS$,so triangles $A'X_1C', AHS$ are homothetic.

On the other hand ,by **Thales' theorem** we have:

$\frac{PA'}{PC'} = \frac{MA}{MC} = \frac{KA}{KS}$.

So triangles $A'X_1P, AHK$ are homothetic,we obtain $PX_1//HK$(Because that both $A'P//AK$ and $A'X_1//AH$).

But from the **Goormaghtigh's theorem** ,we have H, K, X_2 are collinear.

Therefore , $PX_1//HX_2(1)$

Similarly, $PX_2//HX_1(2)$

Since (1) and (2) then X_1X_2 passes through midpoint of PH which is a fixed point.

Thank to **Linh** ? for your nice result.

livetolove212

#3 Dec 18, 2009, 4:48 pm

Dear Mr.**Khanh**,
This result comes from two lemmas:
Lemma 1: Denote X_1, X_2 the orthopoles of two orthogonal lines through the orthocenter H wrt ΔABC. Then H is the midpoint of X_1X_2.
Lemma 2: Denote X_1, X_2 the orthopoles of two parallel lines l_1, l_2 then $X_1X_2 \perp l_1$ and X_1X_2 is equal to the distant of l_1, l_2.
Thanks you for your nice solution.

Luis González

#4 Mar 3, 2014, 9:39 am

Let H be the orthocenter of $\triangle ABC$ and A', B', C' the projections of P on BC, CA, AB. From the problem Orthopole lies on ellipse (post #2), we know that X_1, X_2 move on a ellipse $\mathcal{E}$ circumscribed to $\triangle A'B'C'$, further $A'X_1 \mapsto \ell_1$ and $A'X_2 \mapsto \ell_2$ are homographic. But since $\ell_1 \mapsto \ell_2$ is clearly an involution, then $X_1 \mapsto X_2$ is an involutive homography on $\mathcal{E} \Longrightarrow X_1X_2$ goes through a fixed point. Making $\ell_1 \equiv PA'$ and $\ell_1 \equiv PB'$, we figure out that the fixed point is none other than the center of $\mathcal{E}$, i.e. the midpoint of $\overline{PH}$.

Quick Reply

High School Olympiads

An old but hard result

shinny98NT

#1 Mar 2, 2014, 8:22 pm

Triangle ABC with H is the orthocenter and M is the midpoint of BC. D, E, F are the feet of altitudes from A, B, C on BC, AC, AB, respectively. MH cuts EF at K. Tangents at B and C meet at S. Prove that K, D, S are collinear.

From this result we can prove others nice problem about orthocenter H and midpoint of BC, such as:

1/ EF cuts BC at I. KD cuts circumscribe circle of triangle ABC at X and Y. Then IX and IY are tangents of circle.

2/ Assume Y is on the other side of A from the line BC. Then YA is the bisector of angle FYE.

Luis González

#2 Mar 2, 2014, 9:29 pm • 1

(O) is circumcircle of $\triangle ABC$ and L is antipode of A on (O). From parallelogram $BHCL,$ it follows that H,M,L are collinear, thus if LH cuts (O) again at $P,$ then $\angle APH \equiv \angle APL = 90^\circ \Longrightarrow P$ is also on circle $\odot(AEF)$ with diameter $\overline{AH} \Longrightarrow BC,EF$ and AP concur at radical center R of $(O),\odot(AEF)$ and the circle with diameter $\overline{BC}.$

EF cuts (O) at U,V and $\overline{AOL}$ is perpendicular bisector of $\overline{UV}$ cutting it at its midpoint $N.$ From cyclic $APKN,$ we get $RU \cdot RV = RP \cdot RA = RK \cdot RN \Longrightarrow (U,V,K,R) = -1 \Longrightarrow K$ is on the polar of R WRT $(O).$ But $(B,C,D,R) = -1 \Longrightarrow SD$ is the polar of R WRT $(O) \Longrightarrow K,D,S$ are collinear.

shinny98NT

#3 Mar 2, 2014, 9:57 pm

Thank you so much for a very short but sharp answer. I initially think that this can be proved by ordinary knowledge and method, such as Menelaus or Using Angle, and now I know the power of POLE AND POLAR in these simple structures. This is my first post, and I'm so impressed of the help and hospitality of people here. Thank you again.

High School Olympiads

Orthopole lies on ellipse

Reply

buratinogigle

#1 Sep 15, 2012, 11:36 pm • 1

Let ABC be a triangle, orthocenter H and a point P. Let $A'B'C'$ be pedal triangle of P. (E) is circumellipse of triangle $A'B'C'$ with center is midpoint of PH. Prove that orthopole of any line passing though P lies on (E).

Luis González

#2 Mar 1, 2014, 10:49 am • 3

Let τ be a variable line through P and R its orthopole WRT $\triangle ABC. O$ is the circumcenter of $\triangle ABC$ and OA, OB cut τ at $O_A, O_B,$ respectively. U, V are the projections of O_A on AC, AB and X, Z are the projections of O_B on $BC, BA.$

Intersection $A_0 \equiv B'C' \cap UV$ is on $A'R$ and likewise, $B_0 \equiv C'A' \cap XZ$ is on $B'R$ (see the 1st two paragraphs of the solution of radical axis and Simpson's lines are concurrent). Hence, as τ spins around $P, O_A \mapsto U$ is a perspectivity from the infinity point of $\perp AC$ and $U \mapsto A_0$ is a perspectivity from the infinite point of $BC \Longrightarrow O_A \mapsto A_0$ is a homography between OA and $B'C'$. Similarly, $O_B \mapsto B_0$ is a homography between OB and $C'A'$. But $O_A \mapsto O_B$ is perspectivity from $P \Longrightarrow A_0 \mapsto B_0$ is a homography between $B'C'$ and $C'A' \Longrightarrow$ pencils $A'A_0 \equiv A'R$ and $B'B_0 \equiv B'R$ are homographic $\Longrightarrow R$ moves on a conic $\mathcal{E}$ passing through B', C', which is clearly an ellipse, since R never goes to infinity for an ordinary τ. When $\tau \equiv PA'$, then $R \equiv A'$, i.e. $\mathcal{E}$ is circumscribed to $\triangle A'B'C'$.

Consider the position $\tau \parallel BC$ and let D be the projection of B on τ. Since $DB \parallel RH$ (both perpendicular to BC) and $DR \parallel BH$ (both perpendicular to AC), then $RHBD$ is parallelogram $\Longrightarrow RH = DB = PA' \Longrightarrow RPA'H$ is parallelogram $\Longrightarrow R$ is reflection of A' on the midpoint E of $\overline{PH}$. Likewise, R will coincide with the reflections of B' and C' on E, when $\tau \parallel CA$ and $\tau \parallel AB \Longrightarrow E$ is center of $\mathcal{E}$, i.e. R moves on ellipse $\mathcal{E}$ circumscribed to $\triangle A'B'C'$, whose center is the midpoint E of $\overline{PH}$.

Quick Reply

High School Olympiads

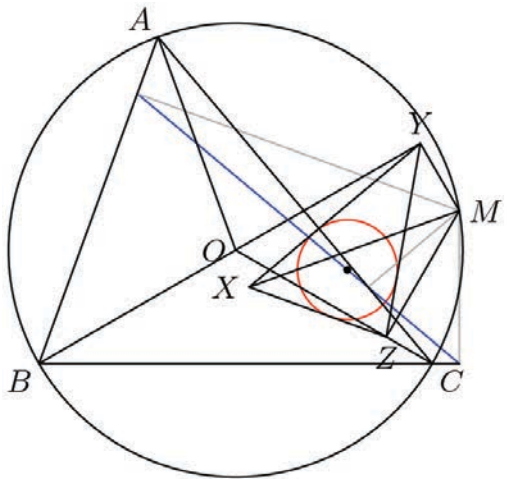
Incenter on Simson Line

Reply

nsato

#1 Jun 15, 2012, 6:08 am • 1

Let O be the circumcenter of acute triangle ABC, and let M be a point on the circumcircle of triangle ABC. Let X, Y, and Z be the projections of M onto OA, OB, and OC, respectively. Prove that the incenter of triangle XYZ lies on the Simson line of M. (Mongolia, 1996)



Find the locus of the incenter of triangle XYZ as M varies along the circumcircle. (I conjecture that the locus is an inconic whose center is O.)

[asy] import geometry; unitsize(3 cm); pair O, A, B, C, M, X, Y, Z; pair[] I, M; O = (0,0); A = dir(110); B = dir(210); C = dir(-30); for(int i = 0; i <= 4; ++i) { M[0] = circumcenter(A,B,C) + circumradius(A,B,C)*dir(360/5*i); X = (M[0] + reflect(O,A)*(M[0]))/2; Y = (M[0] + reflect(O,B)*(M[0]))/2; Z = (M[0] + reflect(O,C)*(M[0]))/2; I[i] = incenter(X,Y,Z); } conic inc = conic(I[0],I[1],I[2],I[3],I[4]); draw(inc,red); draw(Circle(O,1)); draw(A--B--C--cycle); label([aopsnowrap]"$$A$$"[/aopsnowrap], A, NW); label([aopsnowrap]"$$B$$"[/aopsnowrap], B, SW); label([aopsnowrap]"$$C$$"[/aopsnowrap], C, SE); dot([aopsnowrap]"$$O$$"[/aopsnowrap], O, NE); for(int i = 0; i <= 4; ++i) { dot(I[i]); } [/asy]

SnowEverywhere

#2 Jun 15, 2012, 9:04 am • 2

Here is an outline of a solution to the Mongolian problem. I am sorry for any omitted details.

Assume without the loss of generality that M lies on arc AC and let $m(a)$ denote the clockwise measure of an arc a around the circumcircle of ABC. Since X, Y and Z lie on a circle with diameter OM, homothetizing with centre M and ratio 2 sends X, Y and Z to points X', Y' and Z' on the circumcircle of triangle ABC. Note that lines OA, OB and OC are the perpendicular bisectors of MX', MY' and MZ', respectively. Therefore $m(MA) = m(AX')$, $m(MB) = m(BY')$ and $m(MC) = m(CZ')$. Now let P, Q and R denote the midpoints of arcs $Y'Z'$, $X'Z'$ and $X'Y'$ not containing X', Y' and Z', respectively. Angle chasing with the given directed arc lengths yields that PM is perpendicular to BC, QM is perpendicular to AC and RM is perpendicular to AB. Further angle chasing yields that ABC and PQR are congruent and of opposite orientation. This implies that they are reflections in a line perpendicular to AP, BQ and CR. Angle chasing with cyclic quadrilaterals implies that the simson line of M with respect to ABC is parallel to AP, BQ and CR. It is well known that this simson line homothetized with ratio 2 about M contains the orthocenter H of ABC. If I' denotes the orthocenter of PQR, then H and I' are reflections of one another in a line perpendicular to AP, BQ and CR, which implies that HI' is parallel to AP, BQ and CR. Therefore I' lies on this simson line homothetized with ratio 2 about M. However, since P, Q and R denote the midpoints of arcs $Y'Z'$, $X'Z'$ and $X'Y'$ not containing X', Y' and Z', respectively, I' is the incenter of $X'Y'Z'$.

Homothetizing with ratio $1/2$ yields the desired result.

RSM

#3 Jun 15, 2012, 2:58 pm • 1

As the solution of the first part has been posted already, I am posting a solution for the second part.

Note that, if I is the incenter of the moving triangle XYZ, then reflection of M on I(call it I') moves on the circle centered at O and passing through H where H is the orthocenter of ABC.

Also note that, I' and M move with same speed but in opposite directions. So I moves on a conic with center O. Call this conic Ω.

If we take $M = A, B, C$, then we get that the reflections of H on the sides of the medial triangle of ABC lies on Ω. Call this reflections as A', B', C'. Reflect A', B', C' on O to get A_1, B_1, C_1. Note that, $A_1B_1C_1$ is the cevian triangle of H'(isotomic point of H wrt ABC) wrt ABC.

We know that, for any point P and a triangle ABC there exists an inconic of ABC passing through the feet of the cevians through P and the center of the conic is the isotomocomplement of P wrt ABC.

Now, in this case, O is the isotomocomplement of H' wrt ABC. So the conic passing through A_1, B_1, C_1 with center O is an inconic of ABC. So done.

Luis González

#4 Feb 26, 2014, 11:08 pm

> *nsato wrote:*
>
> Let O be the circumcenter of acute triangle ABC, and let M be a point on the circumcircle of triangle ABC. Let X, Y, and Z be the projections of M onto OA, OB, and OC, respectively. Prove that the incenter of triangle XYZ lies on the Simson line of M. (Mongolia, 1996)

If D, E, F are the reflections of M on OA, OB, OC, then it is enough to show that the incenter J of $\triangle DEF$ is on the Steiner line τ of M.

Since B and C are the midpoints of the arcs ME and MF of (O), then line BC bisects both $\angle MBF$ and $\angle MCF \Longrightarrow U \equiv BF \cap CE$ is reflection of M on $BC \Longrightarrow U \in \tau$. Further EC and FB bisect $\angle MEF$ and $\angle MFE \Longrightarrow U$ is incenter of $\triangle MEF \Longrightarrow R \equiv MU \cap DJ$ is midpoint of the arc EF of (O). By similar reasoning, $V \equiv AF \cap CD \in \tau$ and $S \equiv MV \cap EJ$ is midpoint of the arc FD. Now by Pascal theorem for cyclic hexagon $DRMSEC$, the intersections $J \equiv DR \cap SE, U \equiv RM \cap EC$ and $V \equiv MS \cap CD$ are collinear, i.e. $J \in \tau$, as desired.

Luis González

#5 Feb 27, 2014, 12:12 am

> *nsato wrote:*
>
> Find the locus of the incenter of triangle XYZ as M varies along the circumcircle. (I conjecture that the locus is an inconic whose center is O.)

If R denotes the circumradius of $\triangle ABC$, then $YZ = R \cdot \sin 2\widehat{A}$, $ZX = R \cdot \sin 2\widehat{B}$ and $XY = R \cdot \sin 2\widehat{C} \Longrightarrow \triangle XYZ$ are all congruent $\Longrightarrow \triangle KYZ$ are all congruent (where K is incenter of XYZ). Hence according to problem Given Triangle And Locus, locus of K is an ellipse $\mathcal{O}$ with center O.

When M coincides with the antipodes A_0, B_0, C_0 of A, B, C, then K coincides with the projections of A_0, B_0, C_0 on BC, CA, AB, respectively, hence $\mathcal{O}$ is identical with the inconic with center O whose perspector is the retrocenter.

livetolove212

#6 Mar 1, 2014, 8:12 am

Hi Luis, this is another proof.
Let H_a, H_c be the projections of M on BC, AB; P, Q be the intersections of MH_c, MH_a and the circle with diameter OM.
Note that X, Y, Z, O, M are concyclic. Since $\angle YMP = \angle ABO = \angle BAO = \angle PMX$ we get P is the midpoint of arc YX. Similarly, Q is the midpoint of arc YZ.
Denote T the intersection of H_cX and H_aZ. We have:

$(TX,TZ) \equiv (TX,H_cM)+(H_cM,H_aM)+(H_aM,TZ) \equiv (AO,AM)+(BA,BC)+(CM,CO)$
$\equiv (BA,BC)+(AO,CO)+(CM,AM) \equiv (BA,BC)+(XO,ZO)+(BC,BA) \equiv (XO,ZO) \pmod{\pi}$
This means T lies on (OM). Applying Pascal theorem for 6 points P,M,Q,Z,X,T we get H_c,I,H_a are collinear.

Attachments:
Mongolia 1996.pdf (20kb)

Wolstenholme
#7 Aug 29, 2014, 7:01 pm

This problem is easily done with complex numbers. Let the homothety centered at M with ratio 2 send X,Y,Z to M_a,M_b,M_c respectively. Let A',B',C' be the reflections of M over BC,CA,AB respectively. It suffices to show that the I, the incenter of $\triangle M_aM_bM_c$, lies on the Steiner line of M - namely the line passing through A', B', and C'.

WLOG let the circumcircle of $\triangle ABC$ be the unit circle and let $A,B,C,M_a,M_b,M_c,A',B',C',I$ have complex coordinates $a,b,c,m_a,m_b,m_c,a',b',c',x$ respectively. We easily find that $m_a=\frac{a^2}{m}$ and $m_b=\frac{b^2}{m}$ and $m_c=\frac{c^2}{m}$ so the incenter of $\triangle M_aM_bM_c$ has coordinate $x=-\frac{1}{m}(ab+bc+ca)$.

Now $b'=a+c-\frac{ac}{m}$ and $c'=a+b-\frac{ab}{m}$ and we want to show that $\frac{x-b'}{\overline{x}-\overline{b'}}=\frac{b'-c'}{\overline{b'}-\overline{c'}}$. We have that $b'-c'=\frac{(b-c)(a-m)}{m}$ so $\frac{b'-c'}{\overline{b'}-\overline{c'}}=\frac{abc}{m}$. We have that $x-b'=\frac{(a+c)(b+m)}{m}$ and so $\frac{x-b'}{\overline{x}-\overline{b'}}=\frac{abc}{m}$ as well so we are done.

This post has been edited 1 time. Last edited by Wolstenholme, Sep 13, 2014, 10:56 pm

jayme
#8 Sep 11, 2014, 11:29 am

Dear Mathlinkers,
where can I find the original or link of the Mongolian Olympiad 1996?
Thank in advance.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Constant Area

Reply

BlackSelena

#1 Feb 25, 2014, 9:03 pm

Let (O) with AB is the constant diameter. C runs on (O). The excircles of A and B touch AB at M, N respectively. Let P, Q are the circumcenter of triangle ACM and triangle ABM respectively. Prove that the area S_{CPQ} is constant.

Luis González

#2 Feb 25, 2014, 9:40 pm

$S_{CPQ} = \frac{1}{16}|AB|^2 = \text{const}.$

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=334812. There was also a solution given by hxy09 in post #2, but the link there became stale.

Arab

#3 Feb 25, 2014, 10:17 pm

Let $BC=a, CA=b, AB=2R$ and denote by r the radius of the incircle of $\triangle ABC$,then $AN=BM=r$,and $OA\cdot OM=OB\cdot ON$, so O lies on the radical axis of the circumcirles of $\triangle ACM, \triangle BCN$,which is exactly OC,and therefore $OC\perp PQ$.

Moreover,$\triangle OAC\sim\triangle QNC$ due to $OA=OC, QN=QC$ and $\angle AOC=2\angle ABC=\angle NQC$,then $\triangle CAN\sim\triangle COQ$.Similarly,$\triangle CBM\sim\triangle COP$,and hence $OP=\frac{BM}{BC}\cdot OC=\frac{Rr}{a}, OQ=\frac{Rr}{b}$.Since $AC\perp BC, OP\perp AC, OQ\perp BC$,we get $OP\perp OQ$,so $PQ=Rr\cdot\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}=\frac{2R^2r}{ab}$.

Consequently,from $a+b=2(R+r)$ and $\frac{1}{2}r\cdot(a+b+2R)=r\cdot(2R+r)=[ABC]=\frac{1}{2}ab$,we obtain that $\frac{r\cdot(2R+r)}{ab}=\frac{1}{2}$,then $[CPQ]=\frac{1}{2}\cdot(OC\cdot PQ+OP\cdot OQ)=\frac{R^2r\cdot(2R+r)}{2ab}=\frac{1}{4}R^2$,as desired.

$Q.E.D.$

Quick Reply

High School Olympiads

Intersection of Simson Line(Own)

Reply

Arab

#1 Aug 28, 2012, 2:14 pm • 1

BC is a fixed chord of Γ the circumcircle of $\triangle ABC$. P, Q are on Γ such that $PQ \perp \ell_1$ the Simson Line of P wrt $\triangle ABC$. Let ℓ_2 be the Simson Line of Q wrt $\triangle ABC$ and ℓ_1 meets ℓ_2 at R. Find the locus of R, where all the movements are caused by A.

Luis González

#2 Feb 25, 2014, 8:50 am • 3

Let X, Z be the projections of P on $BC, AB,$ respectively and PX cuts Γ again at $S.$ From cyclic quadrilaterals $PXBZ$ and $PSAB,$ we get $\angle PXZ = \angle PBZ = \angle PSA \Longrightarrow AS \parallel XZ \equiv \ell_1 \Longrightarrow PQ$ is perpendicular to AS at $K.$

Intersection R of Simson lines of P, Q is orthopole of PQ WRT $\triangle ABC$ (see the lemma at Six orthopoles lie on a circle) $\Longrightarrow$ $KR \perp BC,$ i.e. $KR \parallel SXP.$ Therefore, if the perpendicular to ℓ_1 at R cuts PX at $M,$ then from parallelograms $KSXR$ and $KPMR,$ we get $PM = KR = SX = \text{const} \Longrightarrow M$ is fixed $\Longrightarrow R$ runs on circle with diameter $\overline{XM}.$

Quick Reply

High School Olympiads

Three nine-point circles

Reply

Sardor
#1 Feb 24, 2014, 8:22 pm

Let ABC be a triangle and let points P and Q lie on sides AB and AC, respectively. Let M and N be the midpoints of BP and CQ, respectively. Prove that the centers of the nine-point cicrles of triangles ABC,APQ,AMN are collinear.

Luis González
#2 Feb 24, 2014, 9:48 pm • 1

The property holds for all points M,N satisfying $\overline{MP}:\overline{MB}=\overline{NQ}:\overline{NC}=k.$

Denote O,O_1,O_2 the circumcenters of $\triangle ABC, \triangle APQ, \triangle AMN$ and H,H_1,H_2 the orthocenters of $\triangle ABC, \triangle APQ, \triangle AMN.$ Since $BH \parallel PH_1 \parallel MH_2$ and $CH \parallel QH_1 \parallel NH_2,$ it follows that H,H_1,H_2 are collinear, such that $\overline{H_2H_1}:\overline{H_2H}=k.$ Moreover, ratio of powers of M and N WRT (O) and (O_1) clearly equals $k \Longrightarrow (O_2)$ is coaxal with $(O),(O_1);$ its center O_2 lies on $OO_1,$ such that $\overline{O_2O_1}:\overline{O_2O}=k \Longrightarrow HH_1H_2$ is directly similar to $OO_1O_2 \Longrightarrow$ midpoints N,N_1,N_2 of OH,O_1H_1 and O_2H_2 (9-point centers of ABC,APQ and AMN) form then a figure similar to $HH_1H_2 \sim OO_1O_2 \Longrightarrow N,N_1,N_2$ are collinear.

P.S. See also centre of nine point circle for more solutions.

mathuz
#3 Feb 25, 2014, 1:23 pm

we have all lines have same Miquel's point. So easily get H,H_1,H_2 and O,O_1,O_2 are collinear. :)

Quick Reply

High School Olympiads

Geometry2014-02

Source: Old

Ligouras

#1 Feb 24, 2014, 4:09 am • 1

Let AD, BE, CF are the medians and the point G is the centroid of an triangle ABC.
Prove that

$$cotAGF + cotBGD + cotCGE = cotA + cotB + cotC.$$

Luis González

#2 Feb 24, 2014, 4:57 am • 3

If X is the reflection of G on D, then $BGCX$ is a parallelogram, thus $\widehat{AGF}=\widehat{CGX}, \widehat{BGD}=\widehat{GXC}$ and $\widehat{CGE}=\widehat{XCG}$. Moreover, $[CGX]=[GBC]=\frac{1}{3}[ABC]$. Hence in the $\triangle CGX$, we have

$$\cot\widehat{AGF}+\cot\widehat{BGD}+\cot\widehat{CGE}=\frac{GX^2+XC^2+GC^2}{4[CGX]}=$$

$$=\frac{GA^2+GB^2+GC^2}{4\cdot\frac{1}{3}[ABC]}=\frac{a^2+b^2+c^2}{4[ABC]}=\cot\widehat{A}+\cot\widehat{B}+\cot\widehat{C}.$$

Arab

#3 Feb 24, 2014, 3:13 pm • 1

This is a lemma of Romanian Masters In Mathematics 2012 Day1 Problem 2 official solution 1.

Ligouras

#4 Feb 24, 2014, 3:50 pm • 1

Problem n. 2

$$cotBAD + cotCBE + cotACF = cotCAD + cotABE + cotBCF$$

Arab

#5 Feb 24, 2014, 4:07 pm • 1

See page 3 of the attachment.

Attachments:

Romanian Masters In Mathematics 2012.pdf (62kb)

Ligouras

#6 Feb 24, 2014, 11:16 pm • 1

Nice $Luis$, Thanks!!

> *Luis González wrote:*
>
> If X is the reflection of G on D, then $BGCX$ is a parallelogram, thus $\widehat{AGF}=\widehat{CGX}, \widehat{BGD}=\widehat{GXC}$ and
> $\widehat{CGE}=\widehat{XCG}$. Moreover, $[CGX]=[GBC]=\frac{1}{3}[ABC]$. Hence in the $\triangle CGX$, we have

$\widehat{CGE} = \widehat{XCG}$. Moreover, $[CGX] = [GBC] = \frac{1}{3}[ABC]$. Hence in the $\triangle CGX$, we have

$$\cot \widehat{AGF} + \cot \widehat{BGD} + \cot \widehat{CGE} = \frac{GX^2 + XC^2 + GC^2}{4[CGX]} =$$

$$= \frac{GA^2 + GB^2 + GC^2}{4 \cdot \frac{1}{3}[ABC]} = \frac{a^2 + b^2 + c^2}{4[ABC]} = \cot \widehat{A} + \cot \widehat{B} + \cot \widehat{C}.$$

Ligouras

#7 Feb 26, 2014, 1:24 am

Dear $Arab$, then bring the solution in this space? you have a better solution? I thank you in advance

> *Arab wrote:*
> See page 3 of the attachment.

Quick Reply

High School Olympiads

Parallelogram

Locked

Source: Teacher

gobathegreat

#1 Feb 24, 2014, 2:33 am

Let ABC be acute scalene triangle and let H and O be its orthocenter and circumcenter, respectively. Let K be an arbitrary point inside triangle ABC different from O and H, and let L and M be points such that quadrilaterals $AKCL$ and $AKBM$ are parallelograms.If BL intersects CM at point N, and J is midpoint of HK, prove that quadrilateral $KONJ$ is parallelogram

Luis González

#2 Feb 24, 2014, 3:19 am

Source: Teacher ?. It's Indonesia National Science Olympiad 2010, Day 2 Problem 8 proposed by Raja Oktovin. Notations are even exactly the same. Kindly, use the search before posting contest problems.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=369036

High School Olympiads

Incenter and excenter

Locked

Source: Iran MO 2001 3rd round P5

Nazar_Serdyuk

#1 Feb 24, 2014, 12:42 am

In a triangle ABC, I and I_a denote the incenter and the excenter corresponding to side BC. Let A' and M respectively be the intersections of II_a with BC and the circumcircle of $\triangle ABC$, let N be the midpoint of arc MBA, and let S, T be the intersection points of rays NI and NI_a with the circumcircle of $\triangle ABC$. Prove that S, T and A' are collinear.

Luis González

#2 Feb 24, 2014, 12:57 am

Please, use the search before posting contests problems. This has been posted at least 8 times before!.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=6528
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=155875
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=205460
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=216901
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=298966
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=312564
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=412669

High School Olympiads

prove perpendicular

Reply

mad

#1 Feb 23, 2014, 9:25 pm

O is the center of $\bigcirc ABC$ and T is that of $\bigcirc AOC$. M is the mid-point of AC. On lines AB and BC there are points D and E, respectively, such that $\angle BDM = \angle BEM = \angle ABC$. Prove that $BT \perp DE$

Arab

#2 Feb 23, 2014, 9:54 pm

See here (the Problem 1 under Property 8).

Luis González

#3 Feb 23, 2014, 11:08 pm • 1

Let A', C' be the reflections of A, C across BC, BA, respectively. $\angle A'BC = \angle ABC = \angle BEM \Longrightarrow ME \parallel BA'$ and likewise $MD \parallel BC'$. By sine law for $\triangle MEC$ and $\triangle MDA$, we get

$$\frac{ME}{\sin \widehat{C}} = \frac{MC}{\sin \widehat{B}} = \frac{MA}{\sin \widehat{B}} = \frac{MD}{\sin \widehat{A}} \Longrightarrow \frac{ME}{MD} = \frac{\sin \widehat{C}}{\sin \widehat{A}} = \frac{AB}{CB} = \frac{BA'}{BC'}.$$

Hence, it follows that $\triangle MED$ and $\triangle BA'C'$ are homothetic $\Longrightarrow A'C' \parallel ED$. Thus, it suffices to prove that $BT \perp A'C'$, i.e. that the isogonal of BT WRT $\angle ABC$ passes through the circumcenter of $\triangle BA'C'$. This is a problem that has been posted before, e.g. Tuymaada 2009, Senior League, Second Day, Problem 3 and see also Reflections and two symmetric points (1st paragraph of the solution).

IDMasterz

#4 Feb 24, 2014, 1:14 pm • 1

Rephrase as follows:

Let ABC be a triangle, M be midpoint of BC. Let D be point on AC s.t. $\angle BAC = \angle MDA$ and define E on AC similarly. Let the circumcentre of BOC be O_A. Prove $AO_A \perp DE$.

To do this, note that if M_B, M_C are the midpoints of AC, AB respectively, and $MD \cap AB = X$ then XDA is isosceles, so XMM_C is isosceles. Hence, X is the intersection of the perpendicular bisector of MM_C, and similar for the other. They concur at ninepoint centre N, to letting the pedals on AB, AC be N_C, N_B, we have DE is the result of N_BN_C through a dilation about A with scale factor 2, hence $N_BN_C \parallel DE$. But, AO_A is isogonal to AN, so $AO_A \perp N_BN_C$.

Arab

#5 Feb 28, 2014, 6:08 pm

For more solutions,see here.

Quick Reply

High School Olympiads

very hard geometry

Reply

Source: own

vankhea

#1 Feb 21, 2014, 7:19 pm

Let O be circumcenter of triangle ΔABC. Let D,E,F be midpoints of BC,CA,AB. The rays DO,EO,FO meet circumcircle of ΔABC at X,Y,Z respectively. Let X',Y',Z' be midpoints of DX,EY,FZ respectively.

Prove that $[X'Y'Z']=\frac{1}{4}[DEF]$

triskaidecahedron

#2 Feb 21, 2014, 8:37 pm

Just for clarity, the line DO meets the circumcircle at X, where X is on the same side of BC as A.

vankhea

#3 Feb 21, 2014, 9:52 pm

Can you post your full solution please ?

Luis González

#4 Feb 22, 2014, 8:49 pm • 1

Let P,Q,R be the midpoints of $EF,FD,DE.$ Then PX' is D-midline of $\triangle DAX \Longrightarrow PX \parallel AX,$ thus since AX is external bisector of $\angle BAC,$ then PX' is external bisector of $\angle QPR.$ Similarly QY' and RZ' bisect $\angle RQP$ and $\angle QRP$ externally. If $M \equiv QY' \cap RZ', N \equiv RZ' \cap PX'$ and $L \equiv PX' \cap QY'$, then $\triangle PQR$ is orthic triangle of $\triangle MNL.$ Hence perpendicular bisector of $\overline{QR}$ is P-midline of $\triangle PDX,$ cutting both NL and PX' at their midpoint, i.e. P,X' are isotomic points WRT $NL.$ Similarly, Q,Y' and R,Z' are isotomic points WRT LM and $MN,$ respectively $\Longrightarrow$ $[X'Y'Z']=[PQR]=\tfrac{1}{4}[DEF].$

vankhea

#5 Feb 24, 2014, 7:55 pm

thanks you very much Luis González

Quick Reply

High School Olympiads

nice concurrent

Reply

DonaldLove
#1 Feb 21, 2014, 10:47 pm

triangle ABC. M is a point inside triangle. X,Y,Z are respectivly the euler center of triangle MBC,MCA,MAB. prove that the line through X perpendicular to AM, line through Y perpendicular to BM, line through Z perpendicular to CM are concurrent

fmasroor
#2 Feb 22, 2014, 3:33 am

Euler Center is the centroid?

Luis González
#3 Feb 22, 2014, 4:42 am • 1

The poster means the center of the 9-point circle, I believe.

Let M' be the isogonal conjugate of M WRT $\triangle ABC$ and $\triangle DEF, \triangle D'E'F'$ the pedal triangles of M, M' WRT $\triangle ABC$. Let ω be the pedal circle of M, M', i.e. commom circumcircle of $\triangle DEF, \triangle D'E'F'$. 9-point circles $(X), (Y), (Z)$ of $\triangle MBC, \triangle MCA, \triangle MAB$ and ω concur at the Poncelet point P of $ABCM$ (see post #2 at intersection of Simson lines /K.K 6.4 /2 and elsewhere).

If U is the midpoint of AM and DM cuts ω again at L, then P, U, L are collinear (see post #4 at Intersect on circle). Hence $\angle D'PU = \angle D'PL = \angle LDD' = 90^\circ$, i.e. $D'P \perp PU$, but $PU \perp YZ \Longrightarrow YZ \parallel D'P$. Similarly, we have $E'P \parallel ZX, F'P \parallel XY \Longrightarrow \triangle D'E'F'$ and $\triangle XYZ$ are parallelogic, being P a parallelogic center $\Longrightarrow$ parallels from X, Y, Z to $E'F', F'D', D'E'$ (perpendiculars from X,Y,Z to MA,MB,MC) concur at the another parallelogic center.

mathuz
#4 Feb 22, 2014, 6:04 am

the problem is equivalent to:
Let $ABCDEF$ is hexagon with opposite sides are parallel and equal. X, Y, Z, X', Y', Z' are circumcenter of the triangles $ABC, CDE, EFA, DEF, FAB, BCD$ respectively.
Then the line through X perpendicular to CD, the line through Y perpendicular to EF, the line through Z perpendicular to AB are collinear.

It's obvious! Because we have $Z'Y = ZY', Y'X = YX', X'Z = XZ$.

Arab
#5 Feb 22, 2014, 9:57 am

Moreover,X, Y, Z and the point of concurrency are concyclic (easy to prove by angle chasing).

Quick Reply

High School Olympiads

Radical Center

Reply

Iambosama

#1 Feb 21, 2014, 8:13 pm

Let triangle ABC with $(I), (O)$ are the incircle, circumcirle respectively. (I) touch BC, CA, AB at D, E, F respectively. Let M, N are the intersections of EF with (O); P, Q are the intersections of FD with (O). R, S are the intersections of DE with (O). X, Y, Z are the mid-point of BC, CA, AB respectively. H is the orthocenter of $\triangle DEF$
Prove that H is radical center of $(XMN), (YPQ), (ZRS)$

Luis González

#2 Feb 21, 2014, 8:50 pm

$MN \equiv EF$ cuts BC at the harmonic conjugate U of D WRT $B, C \Longrightarrow UD \cdot UX = UB \cdot UC = UM \cdot UN \Longrightarrow$ M, N, D, X are concyclic. Similarly, circles $\odot(YPQ)$ and $\odot(ZRS)$ pass through $E, F,$ resp. Now, it is the same as

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=253661
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=297000

Quick Reply

High School Olympiads

Concurrent problem

Reply

lambosama

#1 Nov 20, 2013, 12:34 am

Let E, F be the intersections of (O_1) and (O_2). I, S are 2 points lie on EF. Let IA, IB be the tagents of I to (O_1) and (O_2) respectively. IA, IB cut O_1O_2 at C, D respectively. SC cut (O_1) at M, N such that M is between N and S. SD cut (O_2) at Q, R such that Q is between S and R. Prove that AM, BQ, EF are concurrent. (By this similarly we have AN, BR, EF are concurrent)

fmasroor

#2 Nov 20, 2013, 3:18 am

I believe you are incorrect...

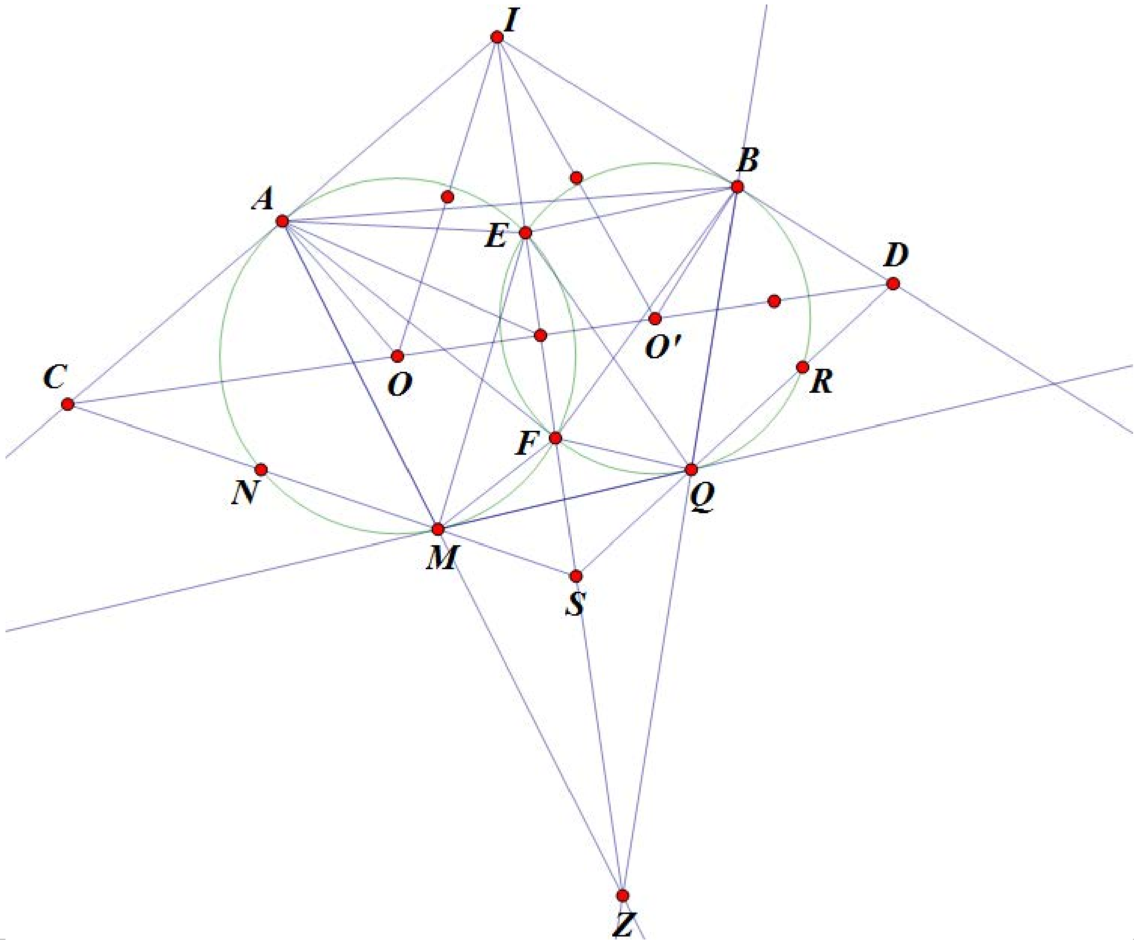
lambosama

#3 Nov 20, 2013, 10:23 am

> fmasroor wrote:
> I believe you are incorrect...

Is there sth wrong??

Attachments:



Luis González

#4 Feb 21, 2014, 10:57 am • 2

The property remains true even if the circles are disjoint, i.e. we only need I,S on their radical axis τ.

I is on radical axis τ of $(O_1),(O_2) \Longrightarrow IA=IB \Longrightarrow$ there is a circle ω tangent to IA,IB at A,B. A is exsimilicenter of $\omega \sim (O_1)$ and B is exsimilicenter of $\omega \sim (O_2)$, thus by Monge & d'Alembert theorem, AB always cuts O_1O_2 at the exsimilicenter H of $(O_1) \sim (O_2)$.

Animate the point I on τ. If AB cuts (O_2) again at A' and the tangent of (O_2) at A' cuts IB at J, then the line JA' is the homothetic image of IA and J runs on the polar h of H WRT $(O_2) \Longrightarrow JA' \mapsto JB$ is an involutive homology fixing the pencil H and h, therefore the correspondence $\mathbb{H}: IA \mapsto IB \equiv JB$ is homographic with a double serie on $\tau \Longrightarrow$ it is a homology fixing the pencil H and $\tau \Longrightarrow (O_1)$ and (O_2) are then homologic under $\mathbb{H} \Longrightarrow SC, SD$ are homologous under $\mathbb{H}$ and so are M,Q consequently $\Longrightarrow AM, BQ$ are homologous lines intersecting on τ.

fmasroor

#5 Feb 22, 2014, 3:31 am

Oh certainly there's a proof with simple radical axis that you can formulate? without homology and homography?

Quick Reply

High School Olympiads

when moving...

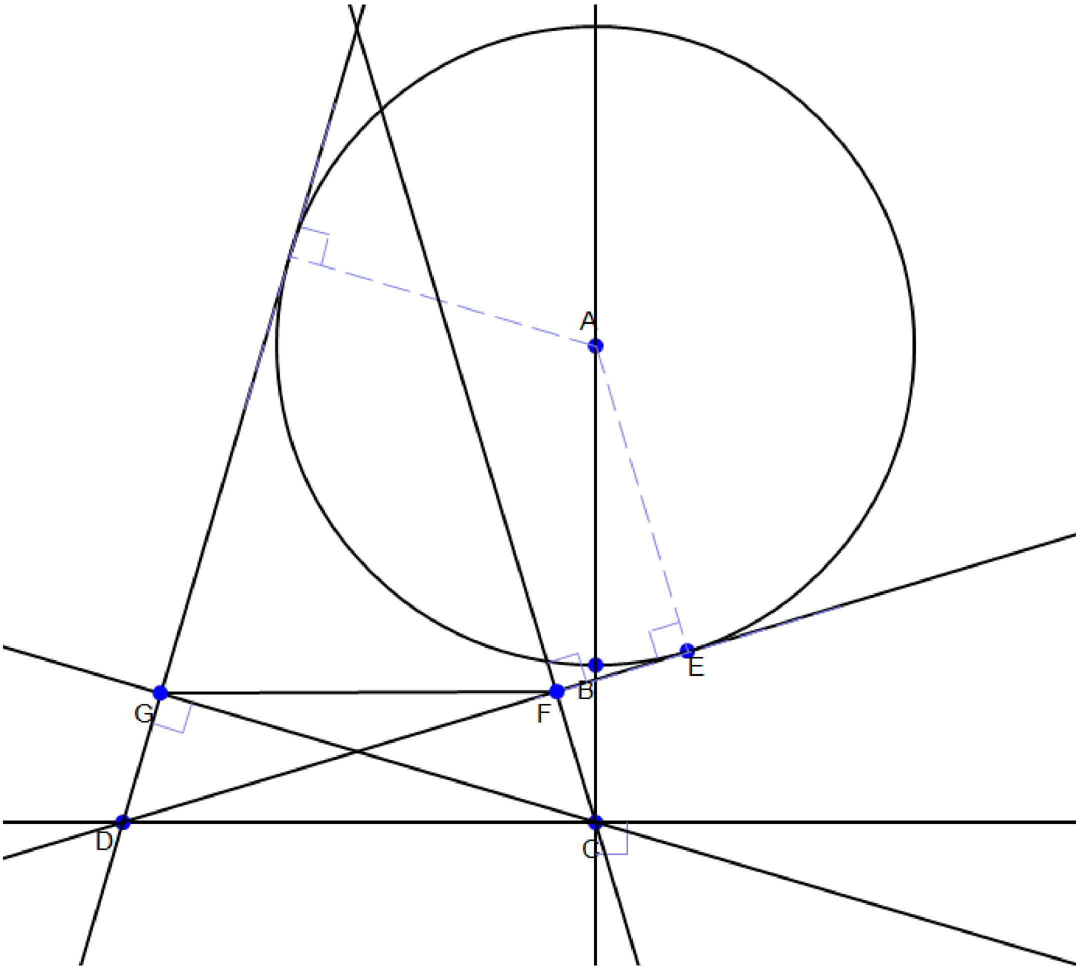
Locked

kazi

#1 Feb 12, 2014, 12:34 am

prove when D moving on the line CD ,GF passes trought a fixed point

Attachments:



Luis González

#2 Feb 12, 2014, 1:37 am

Kindly use the search before posting contest problems. It's IMO Shortlist 1994, G5

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=5198
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=294580

High School Olympiads

Bicentric Quadrilateral 2

Source: Me

juancarlos

#1 May 10, 2005, 11:48 pm • 1

Let $ABCD$ bicentric quadrilateral, I is the incenter, X, Y, Z, W are the touch points of the incircle (I) and AB, BC, CD, DA. Further E, F are the pedal points of the perpendiculars from I to BD, AC. Prove that $[XEY] = [WFX]$.

yetti

#2 May 11, 2005, 12:29 pm • 1

Let P be the common diagonal intersection of the bicentric quadrilateral $ABCD$ and its contact cyclic quadrilateral $XYZW$. The pentagons $AXIFW, BYIEX$ are both cyclic, because the angles $\angle AXI = \angle AFI = \angle AWI = 90^o$ spanning a circle with the diameter AI are right and the angles $\angle BYI = \angle BEI = \angle BXI = 90^o$ spanning a circle with the diameter BI are also right. Hence,

$$\angle FXW = \angle FAW \equiv \angle PAD, \quad \angle FWX = \angle FAX \equiv \angle PAB$$

$$\angle XFW = 180^o - (\angle FXW + \angle FWX) = 180^o - (\angle PAD + \angle PAB) =$$
$$= 180^o - \angle A = \angle C$$

$$\angle EYX = \angle EBX \equiv \angle PBA, \quad \angle EXY = \angle EBY \equiv \angle PBC$$

$$\angle YEX = 180^o - (\angle EYX + \angle EXY) = 180^o - (\angle PBA + \angle PBC) =$$
$$= 180^o - \angle B = \angle D$$

where we used the fact that the quadrilateral $ABCD$ is cyclic and $\angle A + \angle C = \angle B + \angle D = 180^o$. Let $r = IX = IY = ...$ be the quadrilateral inradius. From the isosceles triangles $\triangle WXI, \triangle XYI$, the segments WX, XY are equal to

$$WX = 2IX \cos \widehat{IXW} = 2r \cos \frac{\widehat{A}}{2}$$

$$XY = 2IY \cos \widehat{IYX} = 2r \cos \frac{\widehat{B}}{2}$$

Using the sine theorem for the triangle $\triangle FWX, \triangle EXY$, their remaining sides FW, FX, EX, EY are

$$FW = WX \frac{\sin \widehat{FXW}}{\sin \widehat{XFW}} = 2r \cos \frac{\widehat{A}}{2} \frac{\sin \widehat{PAD}}{\sin \widehat{C}}$$

$$FX = WX \frac{\sin \widehat{FWX}}{\sin \widehat{XFW}} = 2r \cos \frac{\widehat{A}}{2} \frac{\sin \widehat{PAB}}{\sin \widehat{C}}$$

$$EX = XY \frac{\sin \widehat{EYX}}{\sin \widehat{YEX}} = 2r \cos \frac{\widehat{B}}{2} \frac{\sin \widehat{PBA}}{\sin \widehat{D}}$$

$$FW = XY \frac{\sin \widehat{EXY}}{\sin \widehat{YEX}} = 2r \cos \frac{\widehat{B}}{2} \frac{\sin \widehat{PBC}}{\sin \widehat{D}}$$

The areas of the triangles $\triangle FWX, \triangle EXY$ are then

$$|\triangle FWX| = \frac{1}{2}\ FW\cdot FX\sin\widehat{C} = 2r^2\cos^2\frac{\widehat{A}}{2}\ \frac{\sin\widehat{PAD}\sin\widehat{PAB}}{\sin\widehat{C}} = r^2\ \frac{\sin\widehat{PAD}\sin\widehat{PAB}}{\tan\frac{\widehat{A}}{2}}$$

$$|\triangle EXY| = \frac{1}{2}\ EX\cdot EY\sin\widehat{D} = 2r^2\cos^2\frac{\widehat{B}}{2}\ \frac{\sin\widehat{PBA}\sin\widehat{PBC}}{\sin\widehat{D}} = r^2\ \frac{\sin\widehat{PBA}\sin\widehat{PBC}}{\tan\frac{\widehat{B}}{2}}$$

where we again used the fact that the quadrilateral $ABCD$ is cyclic, $\angle C = 180^o - \angle A, \angle D = 180^o - \angle B$ and

$$\sin\widehat{C} = \sin\widehat{A} = 2\sin\frac{\widehat{A}}{2}\cos\frac{\widehat{A}}{2},\quad \sin\widehat{D} = \sin\widehat{B} = 2\sin\frac{\widehat{B}}{2}\cos\frac{\widehat{B}}{2}$$

The ratio of areas of these 2 triangles is then

$$\frac{|\triangle FWX|}{|\triangle EXY|} = \frac{\sin\widehat{PAD}\sin\widehat{PAB}}{\sin\widehat{PBA}\sin\widehat{PBC}}\cdot\frac{\tan\frac{\widehat{B}}{2}}{\tan\frac{\widehat{A}}{2}} = \frac{\sin\widehat{PAB}}{\sin\widehat{PBA}}\cdot\frac{\tan\frac{\widehat{B}}{2}}{\tan\frac{\widehat{A}}{2}}$$

where we used equality of the angles $\angle PAD \equiv CAD = \angle DBC \equiv \angle PBC$ spanning the same arc CD of the quadrilateral circumcircle (O). To complete the proof, we have to show that

$$(?)\quad \frac{\sin\widehat{PAB}}{\sin\widehat{PBA}} = \frac{\tan\frac{\widehat{A}}{2}}{\tan\frac{\widehat{B}}{2}}$$

because then the ratio of areas of the 2 triangles will be equal to 1. Obviously,

$$\frac{BX}{AX} = \frac{r}{\tan\frac{\widehat{B}}{2}}\cdot\frac{\tan\frac{\widehat{A}}{2}}{r} = \frac{\tan\frac{\widehat{A}}{2}}{\tan\frac{\widehat{B}}{2}}$$

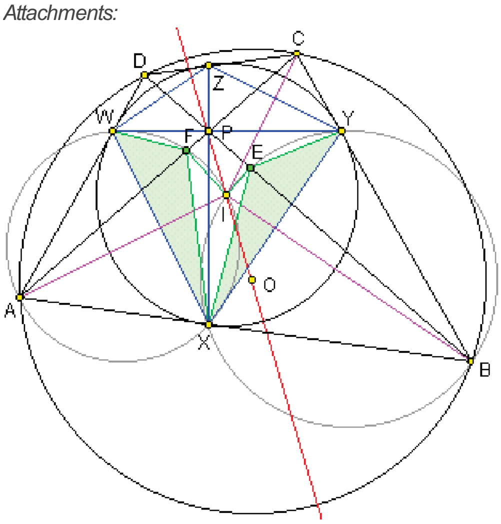
On the other hand, using the sine theorem for the triangle $\triangle ABP$, we get

$$\frac{\sin\widehat{PAB}}{\sin\widehat{PBA}} = \frac{BP}{AP} = \frac{BX}{AX}$$

The last equality is due to the fact that the diagonals of the contact quadrilateral of a bicentric quadrilateral are perpendicular to each other, i.e., $XZ \perp YW$ and that they bisect the angles formed by the diagonals of the bicentric quadrilateral. For a proof, see the problem Bicentric quadrilateral, steps 1, 2, and 3. Consequently, the line PX bisects the angle $\angle APB$ and divides the opposite side AB of the triangle $\triangle ABP$ in the ratio of the adjacent sides AP, BP. As a result

$$\frac{|\triangle FWX|}{|\triangle EXY|} = \frac{BX}{AX}\cdot\frac{AX}{BX} = 1$$

Attachments:



Luis González

#3 Feb 11, 2014, 8:37 pm • 1

From cyclic $IEXBY$ and $IFXAW$, we get $\widehat{EXY} = \widehat{EBY} = \widehat{FAW} = \widehat{FXW} \Longrightarrow \widehat{FXY} = \widehat{EXW}$. Moreover

$\widehat{BEX} = \widehat{BYX}$ and $\widehat{DEW} = \widehat{DZW} \implies \widehat{BEX} + \widehat{DEW} = 90^\circ - \frac{1}{2}\widehat{ABC} + 90^\circ - \frac{1}{2}\widehat{ADC} = 90^\circ \implies$ $\widehat{XEW} = 90^\circ$ and similarly $\widehat{XFY} = 90^\circ$. Therefore, the right $\triangle XFY$ and $\triangle XEW$ are similar $\frac{XF}{XE} = \frac{XY}{XW}$, i.e. $XF \cdot XW = XE \cdot XY$ and together with $\widehat{EXY} = \widehat{FXW}$, it follows that $[XEY] = [WFX]$.

jayme

#4 Feb 11, 2014, 8:45 pm • 3

Dear Mathlinkers,
just to remember that the beloved Juan Carlos Salazar of Venezuela was a great contributor to this site.
He died in South Africa the 30 Marz 2008.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

collinear problem

Reply

Shayanhas

#1 Feb 10, 2014, 8:32 pm

Let ABC be a triangle with incenter I and I is on the line l. A_1 is Symmetric of A with respect l and A_2 is on the BC such that $\angle AA_1A_2 = 90$. B_2,C_2 define similarly. Prove that A_2,B_2,C_2 are collinear and tangent to incircle.

Luis González

#2 Feb 10, 2014, 11:00 pm

Generalization: P is an arbitrary point on the plane of $\triangle ABC$ and ℓ is an arbitrary line through P. A_1 is the reflection of A on ℓ and $A_2 \in BC$, such that $\angle AA_1A_2 = 90^\circ$. Points B_2 and C_2 are defined cyclically. Then A_2, B_2, C_2 are collinear and this line touches the conic $\mathcal{C}$ inscribed in $\triangle ABC$ with center P.

ℓ is A-midline of $\triangle AA_1A_2 \Longrightarrow$ reflection X of A on P is on A_1A_2 and likewise, reflections Y and Z of B and C on P lie on B_1B_2 and $C_1C_2 \Longrightarrow XA_2 \parallel YB_2 \parallel ZC_2 \parallel \ell$. Thus, as ℓ spins around P the series A_2, B_2, C_2 are projective, inducing a homography between $BC, CA, AB \Longrightarrow \tau \equiv A_2B_2$ touches a fixed conic tangent to base lines BC, CA.

When A_2 is taken to $B, XZ \cap BC, XY \cap BC$ and ∞, respectively, then it's easy to see that τ coincides with $AB, XZ,$ XY and YZ, respectively. Hence τ touches the conic $\mathcal{C}$ inscribed in the hexagon bounded by BC, CA, AB, YZ, ZX, XY, whose center is none other that P. By similar reasoning B_2C_2 will touch the same conic $\mathcal{C} \Longrightarrow A_2, B_2, C_2$ are collinear and this line touches $\mathcal{C}$.

jayme

#3 Feb 11, 2014, 7:05 pm

Dear Mathlinkers,
for the first part of this problem, we can use a converse of the Gauss-Newton line theorem.
Sincerely
Jean-Louis

jayme

#4 Feb 11, 2014, 7:36 pm

Dear Mathlinkers,
for the second part, we can begin with the nice idea of Luis and finish with the Brianchon theorem.
Sincerely
Jean-Louis

TelvCohl

#5 Mar 3, 2015, 10:46 am

Another proof to the Generalization mentioned by Luis :

Let A^*, B^*, C^* be the reflection of A, B, C in P, respectively .
Let $A_\infty, B_\infty, C_\infty$ be the infinity point on BC, CA, AB, respectively .
Let $\mathcal{C}$ be a conic tangent to $BC, CA, AB, B^*C^*, C^*A^*, A^*B^*$ (P is the center of $\mathcal{C}$) .

Easy to see $A^* \in A_1A_2, B^* \in B_1B_2, C^* \in C_1C_2$.

From Brianchon theorem (for $B_2B_\infty C^*B^*C_\infty C_2$) we get B_2C_2 is tangent to $\mathcal{C}$.
Similarly we can prove C_2A_2 is tangent to $\mathcal{C} \Longrightarrow A_2, B_2, C_2$ are collinear at a line tangent to $\mathcal{C}$.

This post has been edited 1 time. Last edited by TelvCohl, Nov 6, 2015, 3:16 pm

Quick Reply

High School Olympiads

Tangent circle problem

Reply

daothanhoai

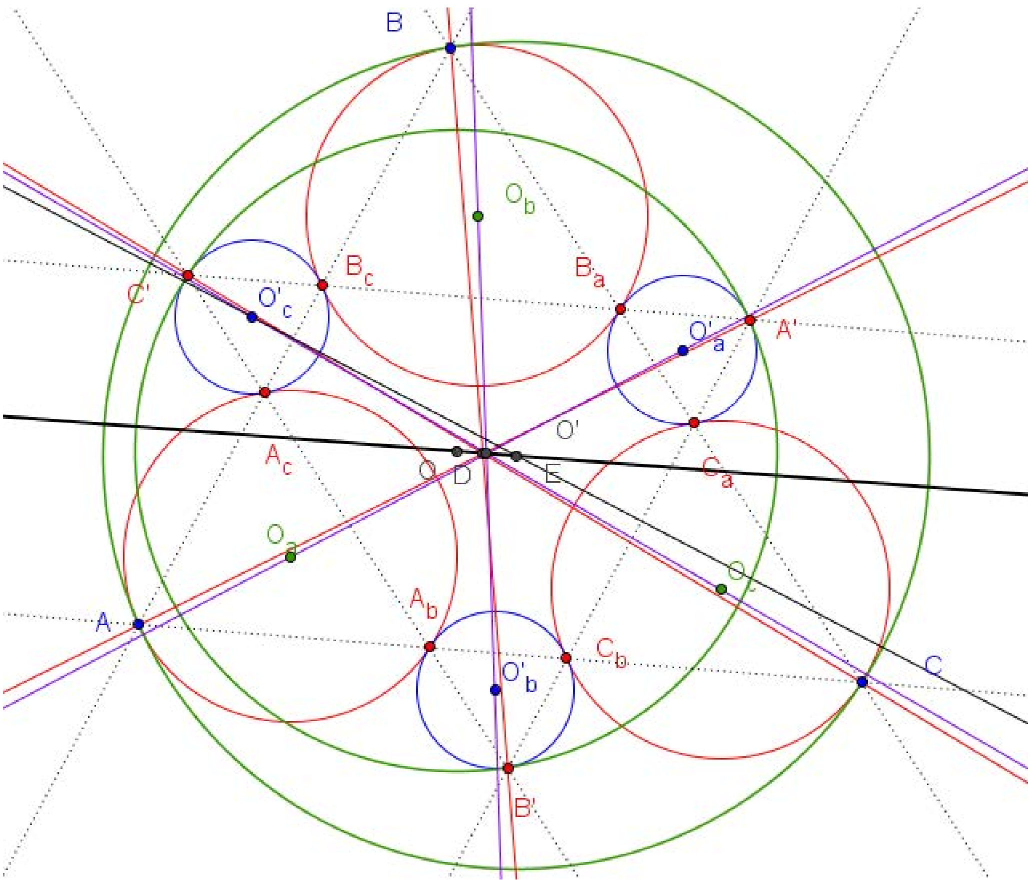
#1 Feb 9, 2014, 9:18 pm

Let ABC be a triangle. A_c, B_c lie on AB; B_a, C_a lie on BC; C_b, A_b lie on AC. Such that $(BB_aB_c), (CC_aC_b), (AA_bA_c)$ tangent with (ABC). $BcBa$ meets $CaCb$ at A'; $BcBa$ meets $AcAb$ at C'. $AcAb$ meets C_bC_a at B'.

1-$(C'B_cA_c)$ tangent with $(A'B'C'), (BBcBa), (AAcAb)$
2-BB', AA', CC' are concurrent at D
3-O_a, O'_a are center of (AA_cA_b) and $(A'B_aC_a)$ respectively. Denote $O_c, O'_c; O_b, O'_b$ cyclically then $O_aO'_a; O_bO'_b; O_cO'_c$ are concurrent at E.
4-O, O' are center of (ABC) and $(A'B'C')$. Then O, O', D, E collinear

http://www.geogebratube.org/student/m84377

Attachments:



Luis González

#2 Feb 10, 2014, 7:23 am

Since A is the exsimilicenter of $(O) \sim (O_a)$, then $BC \parallel A_bA_c \equiv B'C'$. Similarly $CA \parallel C'A'$ and $AB \parallel A'B'$. $\triangle ABC$ and $\triangle A'B'C'$ are then homothetic $\Longrightarrow AA', BB'$ and CC' concur at their homothetic center D, which is also insimilicenter of their circumcircles (O) and $(O)' \Longrightarrow D \in OO'$.

$\triangle B_cA_cC'$ and $\triangle A'B'C'$ are homothetic with center C', which is also exsimilicenter of their circumcircles $(O_c{}')$ and (O') $\Longrightarrow (O_c{}')$ and (O') are tangent at C' and similarly we prove that $(O_c{}')$ touches (O_a) and (O_b).

AA', BC and $OO_a{}'$ concur at the insimilicenter D of $\triangle ABC \sim \triangle A'C_aB_a$ and similarly AA', $B'C'$ and $O'O_a$ concur at

AA', BC and OO_a concur at the insimilicenter P of $\triangle ABC \sim \triangle A'C_aB_a$ and similarly AA', $B'C'$ and $O'O_a'$ concur at the insimilicenter Q of $\triangle A'B'C' \sim \triangle AA_cA_b$. Since AQ and $A'P$ are corresponding cevians of $\triangle AA_cA_b \sim \triangle A'C_aB_a$, then it follows that $O_aQO' \parallel O_a'PO$. Since $AO_aO \parallel A'O_a'O'$, then $OO_aO'O_a'$ is a parallelogram $\Longrightarrow O_aO_a'$ passes through the midpoint E of OO' and likewise, O_bO_b' and O_cO_c' pass through E. Thus O, O', D, E are collinear being E midpoint of $\overline{OO'}$.

daothanhoai

#3 Feb 10, 2014, 9:00 am

Thank to Mr Luis Gonzalez.

Other word:

Let two triangle ABC and $A_1B_1C_1$. Such that $AB//A1B1, BC//B1C1$; $AC//A_1C_1$. B_1C_1 meet AB, AC at A_c, A_b. Define B_c, B_a, C_a, C_b cyclically. Denote O, O_1 is center of circumcircle (ABC) and $(A_1B_1C_1)$. Denote Oa is center of circle (AA_bA_c), Define O_b, O_c cyclically. O_{a1} is center of $(A_1B_aC_a)$. Define O_{b1}, O_{c1} cyclically.
Prove that:

1-$(C_1B_cA_c)$ tangent with three circle $(A_1B_1C_1), (AA_cA_b), (BB_cB_a)$
2-$O_aO_{a1}, O_bO_{b1}, O_cO_{c1}$ are concurrent at D and D is midpoints of OO_1; when $A_1B_1C_1$ are median triangle then D is midpoints of center Nine point circle and Circumcircle

http://www.geogebratube.org/student/m84530

Quick Reply

High School Olympiads

Coaxial (well known?)

xiaotabuta
#1 Feb 9, 2014, 12:13 am

Hello !

Give triangle ABC, (O) is the circumcircle of ABC
D, E, F on BC, CA, AB respectively such that D, E , F is collinear
X is the foot of O on $\overline{D,E,F}$
Prove that (AXD), (BXE), (CXF) are coaxial (actually they have 2 common points)

Luis González
#2 Feb 9, 2014, 2:57 am • 3

This was posted before by livetolove212 (in 2010~2011), but I just realized that the whole thread was deleted and the reason is unknown to me. I even remember giving two solutions. Here is yet another solution:

Let $\odot(AXD), \odot(BXE), \odot(CXF)$ cut (O) again at P, Q, R. AP, BQ, CR cut $\tau \equiv DEF$ at D_0, E_0, F_0. Assume that τ cuts (O) at U, V, i.e. X is midpoint of $\overline{UV}$. Since $\overline{D_0D} \cdot \overline{D_0X} = \overline{D_0A} \cdot \overline{D_0P} = \overline{D_0U} \cdot \overline{D_0V}$, it follows that $(U, V, D, D_0) = -1 \Longrightarrow$ polar of D WRT (O) passes through D_0. Similarly, polars of E, F WRT (O) pass through E_0, F_0 $\Longrightarrow (D, E, F) \mapsto (D_0, E_0, F_0)$ is an involution. But if $Y \equiv BQ \cap CR$, then by Desargues involution theorem, τ cuts the opposite sidelines of the quadrangle $ABYC$ at pairs of point in involution, forcing $D_0 \in AYP$.

Now, powers of X and Y WRT $\odot(AXD), \odot(BXE), \odot(CXF)$ are 0 and $YA \cdot YP = YB \cdot YQ = YC \cdot YR \Longrightarrow$ they are coaxal with common radical axis XY.

xiaotabuta
#3 Feb 9, 2014, 2:42 pm

Sorry for being a dumb but what is "involution" and "Desargues involution theorem" ?

Luis González
#4 Feb 10, 2014, 1:07 am • 1

That's not a dumb question xiaotabuta. In general, involution is a proyectivity that interchanges elements (points on a line, rays in a pencil, etc). It can be proved that the involution on a line ℓ coincides with an inversion on ℓ.

Desargues involution theorem states that: Given a quadrangle $ABCD$, any line ℓ on its plane cuts its opposite sidelines $(AB, CD), (BC, DA)$ and (AC, BD) at pairs of points in involution. Furthermore, the pencil of conics through A, B, C, D also induces an involution on ℓ, but we didn't need this latter result.

High School Olympiads

Right triangle

Reply

Mirus

#1 Feb 8, 2014, 9:47 pm

ABC is a right triangle, $\angle C = 90^\circ$. D is a point on the ray AB such that B is between A and D, and $\angle BDC = 2\angle BCD$. Midpoint of DC is M. Prove that $\angle BMD = \angle AMC$.

Luis González

#2 Feb 9, 2014, 12:15 am • 1

Posted before with a different formulation:

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=423247
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=482414

Quick Reply

High School Olympiads

Three circumcircles have a second common point

Reply

Math-lover123
#1 Feb 6, 2014, 2:57 pm

Point O is the circumcenter and point H is the orthocenter in an acute non-isosceles triangle ABC. Circle ω_A is symmetric to the circumcircle of AOH with respect to AO. Circles ω_B and ω_C are defined similarly. Prove that circles ω_A,ω_B and ω_C have a common point, which lies on the circumcircle of ABC.

jayme
#2 Feb 6, 2014, 3:53 pm

Dear Mathlinkers,
just for history this result comes from Auguste Boutin in 1891.
For a synthetic proof, I have used the Schoute's theorem.
Sincerely
Jean-Louis

Luis González
#3 Feb 6, 2014, 11:59 pm

Let E,F be the midpoints of $CA,AB.$ $(A_1) \equiv \odot(AOH)$ cuts CA,AB at U,V and $(A_2) \equiv \omega_A$ cuts CA,AB at $R,S.$ (A_2) cuts circumcircle (O) again at $P.$

$\angle(HA,HO)=\angle(VA,VO)=\angle(SO,SA) \Longrightarrow \triangle OSV$ is O-isosceles with symmetry axis $OF \perp AB \Longrightarrow F$ is midpoint of $VS.$ Similarly, E is midpoint $UR \Longrightarrow \odot(OEAF)$ is midcircle of (A_1) and $(A_2),$ thus its center, midpoint of $OA,$ is midpoint of $A_1A_2 \Longrightarrow AA_1OA_2$ is parallelogram $\Longrightarrow AA_1 \parallel OA_2 \perp AP \Longrightarrow AP$ is tangent to $(A_1) \Longrightarrow$ $\angle PAB=\angle AOV=\angle AUV.$ But $UV \parallel OH,$ since AO,AH are isogonals WRT $\angle UAV,$ thus it follows that the isogonal of AP WRT $\angle BAC$ is parallel to $OH \Longrightarrow P \equiv X_{74}$ is then the isogonal conjugate of the Euler's infinite point. Similarly, ω_B and ω_C pass through $X_{74}.$

TuzelovDinmukhammet
#4 Feb 7, 2014, 9:10 am

let made an inversion centred at O and with radius OH. take B maps to B' and A', C' are similarly. let H_B be symetric to H with respect to BO and H_A,H_C are similarly. then ω_A maps to H_AA' others are similarly. after some angle calculating we get that $H_AA', H_BB' and H_CC'$ are concurrent and this point will be on circle centred at O and with radius OA'

Math-lover123
#5 Feb 7, 2014, 1:44 pm

My solution is much simpler:
Suppose that the second point of intersection of $\odot(ABC)$ and ω_A is T.
Then by easy angle-chasing we conclude that T also belongs to ω_B and ω_C.

Quick Reply

High School Olympiads

Concyclic

Reply

gobathegreat
#1 Feb 6, 2014, 12:52 am

Quadrilateral $ABCD$ is inscribed in circle such that AC is diameter of that circle and his diagonals are perpendicular at E. Extend DA and BA over A to points F and G, such that DG is parallel to BF. Foot of perpendicular from C on FG is H. Prove that B, E, F and H are concyclic

Luis González
#2 Feb 6, 2014, 8:28 am • 3

Since $CBHG$ is cyclic due to the right angles at $B, H,$ then $\angle BHF=\angle BCG.$ Hence $BEHF$ is cyclic $\Longleftrightarrow$ $\angle BEF=\angle BHF=\angle BCG,$ i.e. it is enough to prove that $\angle BEF=\angle BCG.$

As F varies, the pencils BF,DG having parallel rays are homographic $\Longrightarrow$ they induce a homography $\mathbf{H}_1 : AD \mapsto AB$ and if F^*,G^* vary along $AD,AB,$ such that $\angle BEF^*=\angle BCG^*,$ the pencils EF^*,CG^* are homographic, inducing a homography $\mathbf{H}_2 : AD \mapsto AB.$ Hence, all we need to prove is that they are identical.

When $F \equiv A,$ then G becomes infinite point of $\perp BC \Longrightarrow \angle BEF=\angle BCG=90^{\circ}.$ When $F \equiv D,$ then $G \equiv B \Longrightarrow$ $\angle BEF=\angle BCG=180^{\circ}$ and finally, when F is at infinity, then $G \equiv A \Longrightarrow \angle BEF=\angle BDG=\angle BCG.$ Hence $\mathbf{H}_1 \equiv \mathbf{H}_2,$ as desired.

Arab
#3 Feb 6, 2014, 12:20 pm

See China Girls Math Olympiad 2003.

mathuz
#4 Feb 7, 2014, 12:26 pm • 2

my proof:
Let point K be the symmetry point of G respect to the line CA. We have that $K\in AF$(line) and $BD \parallel GK$, $\triangle EKG$ is isosceles. Since $AD^2=AE\cdot AC=AG\cdot AF=AK\cdot AF$ we get $CFKE$ is cyclic. So since $CDHF$ is cyclic, we get that $\angle DHG=\angle DCF=90^{\circ}-\angle CFD=90^{\circ}-\angle KEA=\angle KEB=\angle GED \Rightarrow HEDG$ is cyclic.

Quick Reply

High School Olympiads

The extangents triangle

Reply

Source: From Mr Ye

Fang-jh

#1 Mar 28, 2010, 4:46 pm • 1

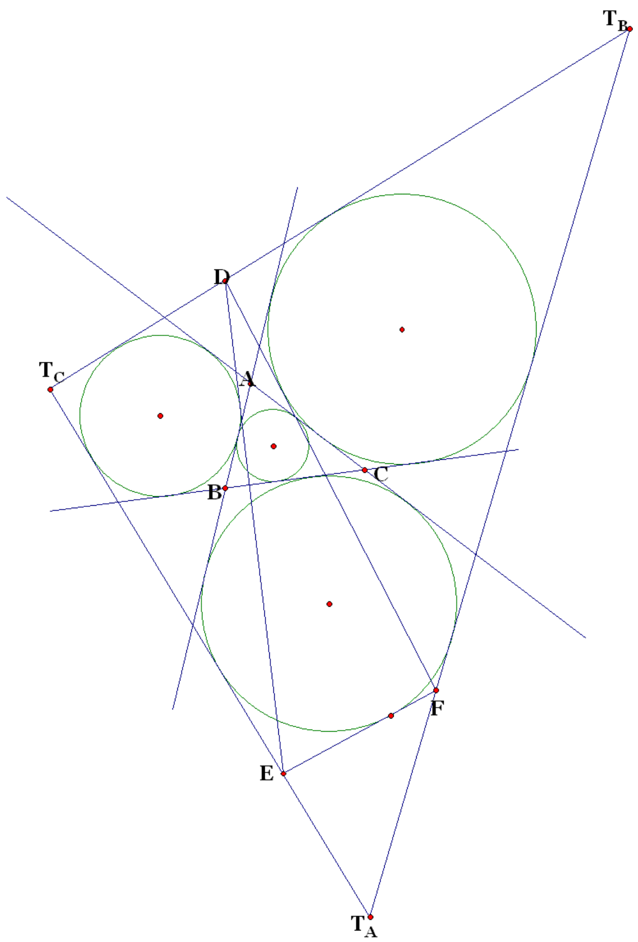
Let $T_AT_BT_C$ be the extangents triangle of triangle ABC. Let D be an arbitrary point on the side T_BT_C. The tangent (different from T_BT_C) from D to C-excircle intersects T_CT_A at E. The tangent (different from T_BT_C) from D to B-excircle intersects T_AT_B at F. Prove that the line EF is tangent to A-excircle.

Zhang Fangyu

#2 Mar 30, 2010, 5:24 pm

Is the following picture right?

Attachments:



Fang-jh

#3 Apr 26, 2010, 6:42 pm

An amazing proof! (from Mr.Li)
Denote by I_a, I_b, I_c the centers of $A-, B-, C-$ excircle, respectively. Let L be the intersection of DE with AB; N be the intersection of DF with CA.
Let l be the reflection of DE with respect to I_aI_c, which meets BC, CA at L', E', respectively.

Let l' be the reflection of DF wrt I_aI_b, which meets BC, AB at N', F', respectively.
Let I_cE' meet BC at D'.
Because
$\angle BD'I_c = \angle E'L'D' + \frac{1}{2}\angle AE'L' = \frac{1}{2}(\angle E'L'D' + \angle AE'L' + \angle DLA) = \frac{1}{2}(\angle ACB + \angle DLA) = \frac{1}{2}\angle T_CDL.$
Thus, two point D, D' are symmetric wrt I_bI_c.
Similarly, I_bF' also passes through D', that means three lines I_cE', BC, and I_bF' concur at D'.
By Pappus's hexagon theorem, we yield three points E', I_a, and F' are collinear.
On the other hand, since E, E' are symmetric wrt I_aI_c; F, F' are symmetric wrt I_aI_b.
thus $\angle T_CEI_a + \angle T_BFI_a = \angle AE'I_a + \angle AF'I_a = \pi - \angle CAB\ (i)$.
By (i), we get the desired result!

wangmengqi

#4 May 27, 2010, 1:57 pm

I can't understand this proof, can anyone help me?

Luis González

#5 Feb 4, 2014, 3:12 am • 3

Denote $(I_A), (I_B), (I_C)$ the excircles of $\triangle ABC$ againts $A, B, C.$ The mapping sending D to the intersection E of the tangent from D to (I_C) with T_AT_C is a homography between T_BT_C and T_CT_A and likewise the mapping sending D to F is a homography between T_BT_C and $T_AT_B \Longrightarrow E \mapsto F$ is then a homography between T_BT_A and $T_AT_C \Longrightarrow EF \equiv \tau$ envelopes a conic tangent to the base lines T_BT_A, T_AT_C as D varies.

When D coincides with intersections of AB, AC with $T_BT_C,$ then τ coincides with AB or $AC.$ When $D \equiv BC \cap T_BT_C,$ then $\tau \equiv BC \Longrightarrow \tau$ touches unique conic (I_A) inscribed in pentagon $BC, CA, AB, T_BT_A, T_AT_C.$

Quick Reply

High School Olympiads

Source: N.I.Beluhov

baysa

#1 Feb 25, 2013, 7:53 pm

Given a circle ω and point P outside of it. Draw a tangent PA and a secant PBC to circle ω. Points D, E lying on arc BC not containing point A and divide it into 3 equal arcs (points A, B, D, E, C are lying on the circle ω in this order). The line passing through point B perpendicular to line AD intersects AE at K. The line passing through point C perpendicular to line AE intersects AD at L. The lines PA, KL intersects at Q. Prove that A is midpoint of segment PQ.

yetti

#2 Mar 9, 2013, 12:03 pm

Let $[BC]=a, [AC]=b, [AB]=c$ and WLOG $c>b$. Let (P) be circle with radius $[PA]$, the A-Apollonius circle of $\triangle ABC$, cutting $[BC]$ at $F \Longrightarrow$

AF bisects $\measuredangle CAB=\widehat{A}$ and $\overline{FC}=\dfrac{ab}{b+c}$. Since radius of (P) is $[PF]=\dfrac{abc}{c^2-b^2} \Longrightarrow \overline{CP}=\overline{FP}-\overline{FC}=\dfrac{ab^2}{c^2-b^2}$.

Let KL cut BC, AC, AB at X, Y, Z. Let $M\in AL, N\in AK$ be midpoints of $[BK], [CL]$.

By Menelaus for $\triangle ABM$ cut by transversal $KLZ \Longrightarrow \dfrac{\overline{ZB}}{\overline{ZA}}=\dfrac{\overline{KB}}{\overline{KM}}\cdot\dfrac{\overline{LM}}{\overline{LA}}=-2\cdot\dfrac{c\cos\frac{\widehat{A}}{3}-b}{b}$ and likewise,

$\dfrac{\overline{YC}}{\overline{YA}}=-2\cdot\dfrac{b\cos\frac{\widehat{A}}{3}-c}{c}$.

By Menelaus for $\triangle ABC$ cut by transversal $KL\equiv XYZ \Longrightarrow \dfrac{\overline{XB}}{\overline{XC}}=\dfrac{\overline{ZB}}{\overline{ZA}}\cdot\dfrac{\overline{YA}}{\overline{YC}}=\dfrac{c}{b}\cdot\dfrac{c\cos\frac{\widehat{A}}{3}-b}{b\cos\frac{\widehat{A}}{3}-c}$.

Using $\overline{XC}=\overline{BC}-\overline{BX}=a+\overline{XB} \Longrightarrow \overline{XB}=-\dfrac{ac}{c^2-b^2}\cdot\dfrac{c\cos\frac{\widehat{A}}{3}-b}{\cos\frac{\widehat{A}}{3}}$ and likewise,

$\overline{XC}=-\dfrac{ab}{c^2-b^2}\cdot\dfrac{b\cos\frac{\widehat{A}}{3}-c}{\cos\frac{\widehat{A}}{3}}$.

Then $\overline{XP}=\overline{XC}+\overline{CP}=\dfrac{ab^2}{c^2-b^2}-\dfrac{ab}{c^2-b^2}\cdot\dfrac{b\cos\frac{\widehat{A}}{3}-c}{\cos\frac{\widehat{A}}{3}}=\dfrac{abc}{(c^2-b^2)\cos\frac{\widehat{A}}{3}} \Longrightarrow \dfrac{\overline{XB}}{\overline{XP}}=-\dfrac{c\cos\frac{\widehat{A}}{3}-b}{b}$.

By Menelaus for $\#\triangle ABP$ cut by transversal $KL\equiv XZQ \Longrightarrow \dfrac{\overline{QP}}{\overline{QA}}=\dfrac{\overline{ZB}}{\overline{ZA}}\cdot\dfrac{\overline{XP}}{\overline{XB}}=2 \Longrightarrow A$ is midpoint of $[PQ]$.

(Mar 09, 2013)

> *Luis González wrote:*
>
> ... Hence the problem is equivalent to http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=570916 for the triangle AKL.

(Jan 12, 2014)

Luis González

#3 Feb 3, 2014, 10:56 am

Since <BAL=<LAK and BK is perpendicular to AL, it follows that B is the reflection of K across AL and likewise, C is the reflection of L across AK. Hence the problem is equivalent to http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=570916 for the triangle AKL. In addition, if U is the intersection of BK with AL and V is the intersection of CL with AE, the line UV goes through the midpoint of AP.

Quick Reply

High School Olympiads

Externally tangent circles

Reply

Source: Romania TST 2009, Day6, Problem 1

Maxim Bogdan

#1 May 21, 2009, 3:27 pm • 1

In triangle ABC with $m(\angle C) = 2 \cdot m(\angle A)$, $(CD$ is the internal bisector of $\angle C$, $(D \in [AB])$. Let S be the center of the circle located on the same side
of line AC as B, which is tangent to AC and externally tangent to the circumcircles of $\triangle ACD$ and $\triangle BCD$, respectively. Prove that $\boxed{AB \perp CS.}$

pohoatza

#2 May 21, 2009, 4:23 pm

I've seen the problem in Crux Mathematicorum, but I don't remember exactly the reference. Can someone help?
Main idea of the solution I have in mind

FelixD

#3 May 21, 2009, 5:17 pm

What does $m(\angle C)$ mean?

Maxim Bogdan

#4 May 21, 2009, 5:19 pm

> FelixD wrote:
> What does $m(\angle C)$ mean?

It means the measure of angle C.

Luis González

#5 May 23, 2009, 1:49 am • 1

Let (S) be the circle tangent to AC, $(K_1) \equiv \odot(DAC)$ and $(K_2) \equiv \odot(DBC)$ externally. Since $\angle DAC = \angle BCD$, then ray CB is tangent to K_1 at C. Inversion with center C and power equal to the power of C WRT (S) takes (K_1) into the line k_1 passing through the inverse A' and tangent to (S). Similarly, (K_2) is taken into the line k_2 passing through B' and tangent to (S). Further, $k_1 \equiv \overline{A'D'}$ is parallel to CB', since CB is tangent to (K_1). k_1, k_2 meet at the inverse D' of D and since A, D, B are collinear, then AB is taken into the circumcircle ω of the isosceles trapezoid $A'D'B'C$. By conformity, angle between AB and AC equals the angle between ω and $A'C$. Thus, $\angle D'CA' = \angle A'D'C \Longrightarrow \triangle A'D'C$ and $\triangle D'B'A'$ are isosceles with apices A' and D'. Thus, S is the the center of $\omega \Longrightarrow AB \perp CS$.

leader

#6 Jan 29, 2014, 1:49 am • 1

Very nice problem.
Consider the composition P of symmetry wrt to line CD and inversion with center C and square radius $CA*CB$ now let CD cut circle ABC again at E. Then $P(D) = E$ $P(A) = B$ so circles ADC and BDC go to BE and AE. and line CA goes to CB so the circle which is externally tangent to circles ADC and BDC and line AC(circle k) goes to circle m which is tangent to segments BE, EA, BC. However $\angle BCE = \angle ECA = \angle BAC$ so $BC = BE = EA$ and the circumcenter of ABC point O is equidistant from BE, EA, BC. So the center of m is O. So now $P(S)$ is a point on CO and therefore

of ABC point O is equidistant from BE, EA, BC. So the center of m is O. So now $P(S)$ is a point on CO and therefore $\angle SCA = \angle BCO = 90 - \angle CAB$ so $CS \perp AB$.

sunken rock

#7 Jan 29, 2014, 9:53 pm • 1

Remark: **The circle S touches AC at the symmetrical of C about A!**

Best regards,
sunken rock

Luis González

#8 Feb 3, 2014, 6:58 am • 2

Here is another approach that also proves sunken rock's remark.

Let P, Q, R be the tangency points of (S) with $AC, \odot(DAC), \odot(BDC)$, respectively. D is clearly midpoint of the arc AQC and if $\odot(BCD)$ cuts AC again at E, then B is midpoint of the arc CRE. Inversion WRT (D, DC) swaps AC and $\odot(DAC)$ leaving (S) fixed and inversion WRT (B, BC) swaps AC and $\odot(BCE)$ leaving (S) fixed $\Longrightarrow P \in DQ$, $P \in BR$ and $DC^2 = DP \cdot DQ, BC^2 = BP \cdot BR \Longrightarrow B, D$ have equal powers WRT C and $(S) \Longrightarrow BDA$ is radical axis of C and $(S) \Longrightarrow CS \perp AB$ and AB bisects the tangent from C to (S), i.e. P is reflection of C on A.

Quick Reply

High School Olympiads

daothanhoai

#1 Jan 25, 2014, 7:05 pm

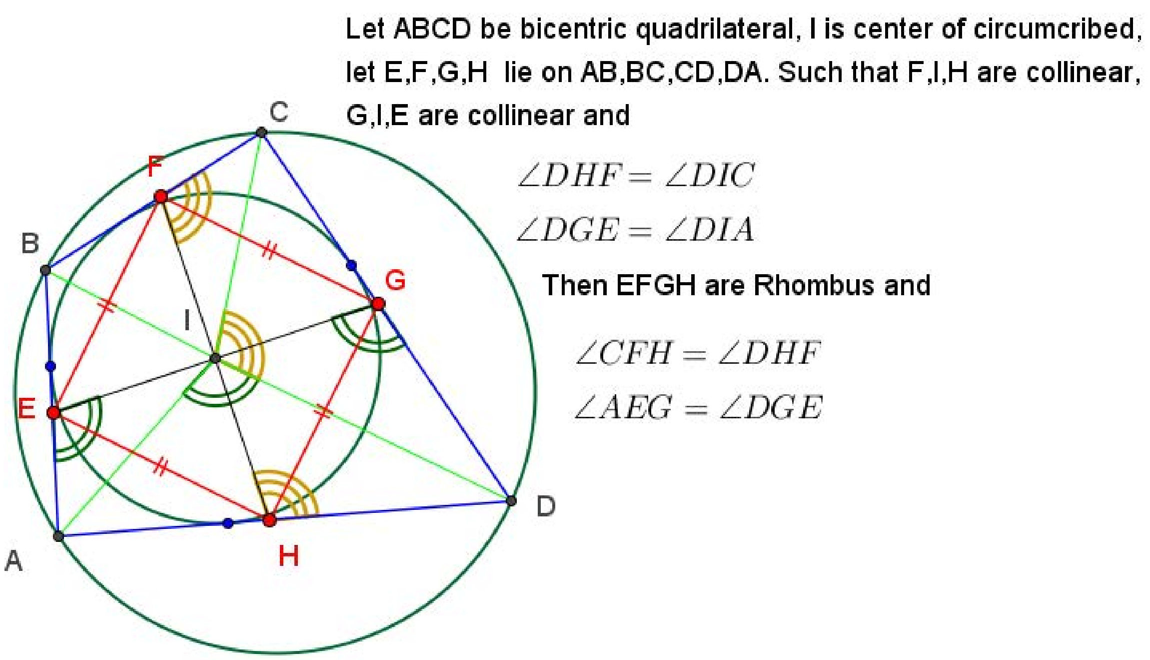
Dear Mathlinkers

Rhombus in Bicentric quadrilateral, please see problem in file ggb at here:

http://www.geogebratube.org/student/m79325

Best regards
Sincerely
Dao Thanh Oai

Attachments:



Luis González

#2 Feb 3, 2014, 12:29 am • 1

Let $P \equiv BC \cap DA$ and $Q \equiv AB \cap CD$. By angle chase, we have

$$\widehat{IQP} + \widehat{IPQ} = \tfrac{1}{2}(\widehat{AQD} + \widehat{CPD}) + \widehat{PQB} + \widehat{QPB} =$$

$$= \tfrac{1}{2}(\widehat{AQD} + \widehat{CPD}) + \widehat{PBA} = 180^\circ - \tfrac{1}{2}(\widehat{BCD} + \widehat{BAD}) - \widehat{CDA} + \widehat{PBA} =$$

$$= 180^\circ - \tfrac{1}{2}(\widehat{BCD} + \widehat{BAD}) = 180^\circ - 90^\circ = 90^\circ \Longrightarrow \widehat{PIQ} = 90^\circ.$$

If QI cuts BC, DA at F', H', then from the isosceles $\triangle PF'H'$, we get $\widehat{DH'F'} = \widehat{CF'H'} = 90^\circ + \tfrac{1}{2}\widehat{CPD} = \widehat{DIC}$ $\Longrightarrow H \equiv H'$ and $F \equiv F'$. Analogously, E, G are the intersections of PI with AB, CD. Hence I is midpoint of $\overline{FH}, \overline{EG}$ and $\widehat{EIF} \equiv \widehat{PIQ} = 90^\circ \Longrightarrow EFGH$ is a rhombus.

Quick Reply

High School Olympiads

Geometry problem 2

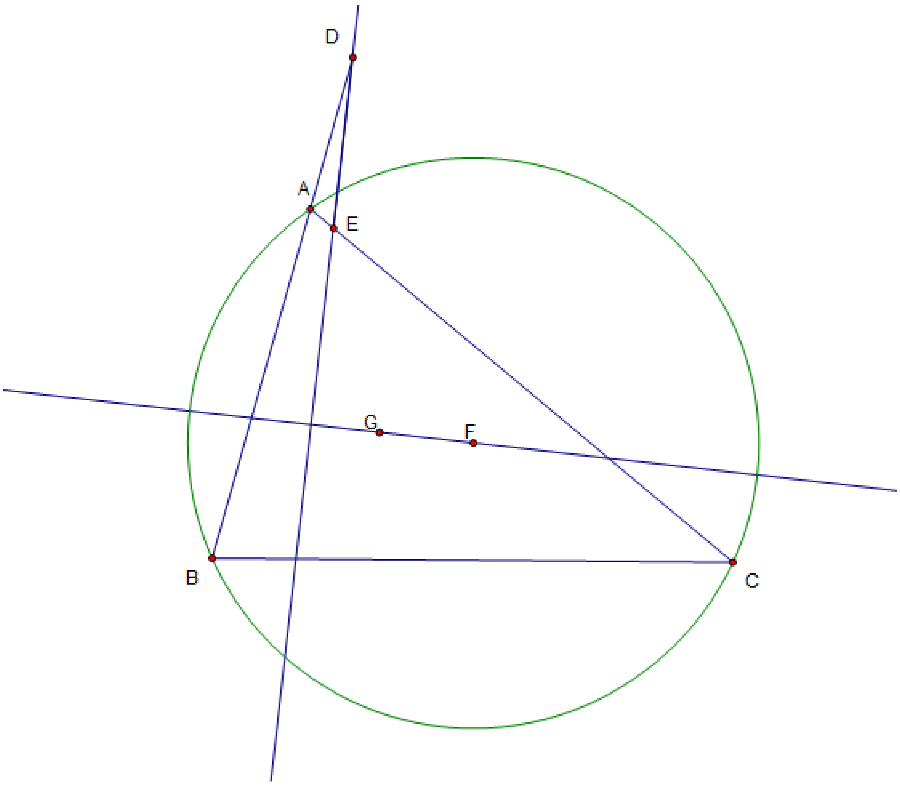
Locked

Parnpaniti

#1 Feb 2, 2014, 10:44 pm

F , G are circumcenter ,incenter of triangle ABC respevtive. D,E are point on AB ,AC respective. BD=BC=CE prove that GF is perpendicular with DE.

Attachments:



Luis González

#2 Feb 2, 2014, 11:05 pm

Firstly, give your topics meaningful subjects. Secondly, use the search before posting contest problems. Topic locked.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=14655
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=52596
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=182301
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=332588
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=321293

High School Olympiads

Concurrent with pole

Reply

Source: Own

buratinogigle

#1 Jan 31, 2014, 12:43 am

Let ABC be a triangle with mixtillinear incircles $(O_a),(O_b),(O_c)$. Let D be pole of O_bO_c with respect to (O_a). Similarly we have E,F. Prove that AD,BE,CF are concurrent.

Luis González

#2 Feb 1, 2014, 4:59 am • 1

I is the incenter of $\triangle ABC$. $(O_a),(O_b),(O_c)$ touch the circumcircle (O) at X,Y,Z. (O_a) touches AC,AB at U,V, resp and $(O_b),(O_c)$ touch BC at R,S, resp. From the internal tangencies of (O) with $(O_b),(O_c)$, it follows that YR,ZS bisect $\angle BYC$ and $\angle BZC \Longrightarrow M \equiv YR \cap ZS$ is midpoint of the arc BC of (O). Furthermore $MB^2=MC^2=MR \cdot MY=MS \cdot MZ \Longrightarrow YZSR$ is cyclic.

According to problem Concurrent, UIV,BC,MX concur at a point P. Since $\angle BIV=\angle AIB-90^\circ=\angle ICB$, then UV is tangent to $\odot(BIC)$ and since IR,IS are isogonals WRT $\angle BIC$ (due to <BIR=<CIS=90º), it follows that $\odot(ISR)$ is internally tangent to $\odot(BIC) \Longrightarrow PR \cdot PS=PI^2=PB \cdot PC=PX \cdot PM \Longrightarrow RSXM$ is cyclic $\Longrightarrow$ YZ,RS,XM concur at the radical center P of $(O),\odot(RSXM)$ and $\odot(YZSR)$, i.e. $P \in YZ$. But Y,Z are the exsimilicenters of $(O) \sim (O_b),(O) \sim (O_c) \Longrightarrow$ by Monge & d'Alembert theorem, P is the exsimilicenter of $(O_b) \sim (O_c)$, i.e. $P \in O_bO_c$.

Polars of O_b,O_c,P WRT (O_a) concur then at $D \Longrightarrow AD$ is polar of P WRT $(O_a) \Longrightarrow$ $A(V,U,D,P)=A(B,C,D,P)=-1$. Since P is on the orthopolar τ_I of I WRT $\triangle ABC$, then AD passes through the tripole of τ_I WRT $\triangle ABC$; the orthocorrespondent X_{57} of I. Similarly, BE and CF pass through X_{57}.

Quick Reply

High School Olympiads

Two nice parallels

Reply

Source: Own

jayme

#1 Jan 31, 2014, 7:36 pm

Dear Mathlinkers,

1. ABC a triangle
2. EAB, DAC two similar triangles outer ABC
3. J the point of intersection of BD and CE
4. U, V the orthocenters of EAB, DAC
5. X, Y the orthocenters of BJE, CJD.

Prove : UV is parallel to XY

Sincerely
Jean-Louis

Luis González

#2 Jan 31, 2014, 11:14 pm

Let M,N,L,K be the midpoints of $BC,CA,AB,DE,$ respectively and let R,S be the projections of A on $CD,BE,$ respectively. $\angle MNC=\angle BAC$ and $\angle CNR=2\angle CAR \Longrightarrow \angle MNR=\angle BAC+2\angle CAR.$ Similarly $\angle MLS=\angle BAC+2\angle BAS,$ but $\angle CAR=\angle BAS \Longrightarrow \angle MNR=\angle MLS.$ Since $MN=LA=LS$ and $ML=NA=NR,$ then it follows that $\triangle MNR \cong \triangle SLM \Longrightarrow MR=MS.$ Using the same reasoning for $\triangle ADE,$ we get $KR=KS \Longrightarrow MK$ is perpendicular bisector of $\overline{RS},$ i.e. $MK \perp RS.$

$\triangle EAB \sim \triangle DAC$ gives $\frac{AU}{US}=\frac{AV}{VR} \Longrightarrow UV \parallel RS \perp MK.$ Now, since MK and XY are the Newton line and Steiner line of $BDEC,$ respectively, we have $XY \perp MK \Longrightarrow UV \parallel XY.$

Quick Reply

High School Olympiads

Two perpendiculars

Reply

Source: own

jayme

#1 Jan 30, 2014, 7:56 pm

Dear Mathlinkers,

1. ABC a triangle
2. IBC, JCA, KAB three I, J, K-isoceles similar triangles, outer ABC
3. B'', C'' the feet of the A-altitudes of JCA, KAB

Prove : AI is perpendicular to B''C''.

Sincerely
Jean-Louis

Luis González

#2 Jan 30, 2014, 9:16 pm

Let $P \equiv BC'' \cap CB''$. Since $\angle ACB'' = \angle BCI$ and $\angle ABC'' = \angle CBI$, it follows that I and P are isogonal conjugates WRT $\triangle ABC \Longrightarrow AI, AP$ are isogonals WRT $\angle BAC$, but AB'', AC'' are isogonals WRT $\angle BAC \Longrightarrow AI, AP$ are isogonals WRT $\angle B''AC''$. Since AP is diameter of the circumcircle of $\triangle AB''C''$, due to right angles at B'', C'', then AI is A-altitude of $\triangle AB''C''$, i.e. $AI \perp B''C''$.

Quick Reply

High School Olympiads

Circumscribed quadrilateral 1

Reply

Source: own

livetolove212

#1 Jan 30, 2014, 12:48 am

Given circle (O). Let (O_1) and (O_2) be two circles that internally touch (O) at A and B respectively; d_1, d_2 be two tangents from A to (O_2), d_3, d_4 be two tangents from B to (O_1). Prove that d_1, d_2, d_3, d_4 form a circumscribed quadrilateral.

Luis González

#2 Jan 30, 2014, 8:08 am

A is the exsimilicenter of $(O) \sim (O_1)$ and B is the exsimilicenter of $(O) \sim (O_2) \Longrightarrow$ by Monge & d'Alembert theorem, AB passes through the exsimilicenter C of $(O_1) \sim (O_2)$. Consider the circle (O_3) tangent to lines d_1, d_2 and d_3, such that A is the exsimilicenter of $(O_2), (O_3)$, hence by Monge & d'Alembert theorem, $B \equiv CA \cap d_3$ is the exsimilicenter of $(O_1) \sim (O_3)$ $\Longrightarrow d_4$ is then the 2nd tangent from B to $(O_3) \Longrightarrow d_1, d_2, d_3, d_4$ touch (O_3).

livetolove212

#3 Jan 30, 2014, 8:45 am

Thanks Luis for your proof. That's the idea to creat this problem.

Quick Reply

High School Olympiads

Antipodes with respect to pedal circle

Reply

Source: known but with no synthetic proof

pohoatza

#1 Jan 30, 2014, 2:30 am

Let ABC be a triangle and let P be a point in its interior. Let D, E, F be the feet of the perpendiculars from P to BC, CA, AB, and let X, Y, Z be the antipodes of D, E, F with respect to the circumcircle of triangle DEF. Prove that the lines AX, BY, CZ are concurrent.

I know this has been on the forum before, but I don't remember having seen any nice proof for it. I would really like to see a short solution.

fmasroor

#2 Jan 30, 2014, 5:21 am

would a homothety centered at the isogonal conjugate with ratio -1/2 work?

Luis González

#3 Jan 30, 2014, 5:45 am

A more general result has been discussed before. See the links below, it's Pascal-Desargues madness.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=366219
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=431409

Quick Reply

High School Olympiads

Another right angle triangle

jayme
#1 Jan 28, 2014, 7:03 pm

Dear Mathlinkers,

1. ABC a triangle
2. B'AC, C'AB two similar triangles, outer ABC
3. (1), (2) the circumcircles of B'AC, C'BA
4. X the second point of intersection of (1) and (2)
5. (3) the circle passing through B, C, X
6. (4) the circle passing through B', C', X
7. Y the second point of intersection of (3) and (4).

Prove : the triangle XYA is X-right angle.

Sincerely
Jean-Louis

Luis González
#2 Jan 29, 2014, 2:14 am

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=557966, precisely the solution to problem 1. AX bisects <BXC and the circular arcs BXC and B'XC' meet at X and their common midpoint Y, i.e. XA and XY bisect <BXC, therefore <AXY is right.

jayme
#3 Jan 29, 2014, 10:46 am

Dear Mathlinkers,
this problem I have posted comes from IMO Shortlist... but I haven't the year?
Can some one precise this reference?
Sincerely
Jean-Louis

High School Olympiads

Perspective with the orthic triangle.

Reply

Source: Own.

mohohoho

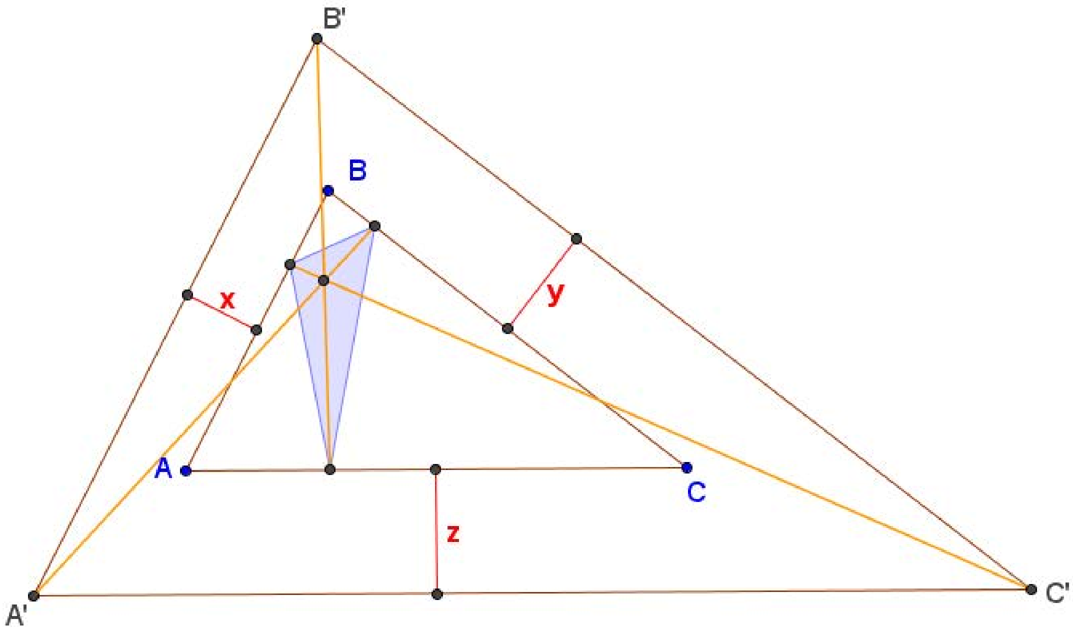
#1 Jan 25, 2014, 10:23 am • 1

Let ABC be a triangle. Let AB = a, BC = b, AC = c.
Let x, y, z be three distances from the sides of triangle ABC to the sides of triangle A'B'C', such that x/a = y/b = z/c.

Prove that the triangle A'B'C' is in perspective with the orthic triangle of ABC.

Emmanuel.

Attachments:



mohohoho

#2 Jan 26, 2014, 10:23 pm • 1

Any taker?

mohohoho

#3 Jan 27, 2014, 11:05 am

Sorry, Now I know that this configuration is associated with the Grebe Triangle (which is about squares erected on the sides). After all, "Own" only means: "I have come up with this by my own". Here is a link http://mathforum.org/kb/message.jspa?messageID=1071652

This post has been edited 1 time. Last edited by mohohoho, Feb 2, 2014, 8:47 pm

Luis González

#4 Jan 28, 2014, 8:52 am • 1

Let $A_0 \in BC, B_0 \in CA, C_0 \in AB$ be the vertices of the orthic triangle of $\triangle ABC$. B_0C_0 cuts $C'A'$, $A'B'$ at A_1, A_2;

C_0A_0 cuts $A'B', B'C'$ at B_1, B_2 and A_0B_0 cuts $B'C', C'A'$ at $C_1, C_2.$

Using that the distances from the symmedian point K of $\triangle ABC$ to its sidelines BC, CA, AB are proportional to $b, c, a,$ its follows that $\triangle A'B'C'$ and $\triangle ABC$ are homothethic through $K.$ Since AK bisects $\overline{B_0C_0}$ (symmedian line property), then $A'K$ bisects $\overline{A_1A_2} \Longrightarrow A_1, A_2$ are isotomic points WRT $B_0C_0.$ Similarly B_1, B_2 and C_1, C_2 are isotomic points WRT C_0A_0 and $A_0B_0.$ Hence by Carnot's theorem in $\triangle A_0B_0C_0,$ we deduce that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a same conic. By Pascal theorem for hexagon $A_1A_2B_1B_2C_1C_2,$ the intersections $A_1A_2 \cap B_2C_1, A_2B_1 \cap C_1C_2$ and $B_1B_2 \cap C_2A_1$ are collinear $\Longrightarrow \triangle A'B'C'$ and $\triangle A_0B_0C_0$ are perspective $\Longrightarrow A'A_0, B'B_0, C'B_0$ concur.

Remark: $A'A_0 \cap B'B_0 \cap C'B_0$ is a Kariya perspector of the orthic $\triangle A_0B_0C_0.$

mohohoho

#5 Jan 28, 2014, 9:45 am

Thank you for your solution, Luis.

Quick Reply

High School Olympiads

proposed by ...

Reply

Source: new

mathuz

#1 Jan 26, 2014, 3:54 pm

In a triangle ABC, (O)- circumcircle, (I_a)- excircle(opposite to angle BAC). Let P, Q are midpoints of the arcs BAC(including A), BC of the (O), respectively. Suppose that tangets from P to the circle (I_a) intersect the line BC at points X, Y and circles (PXY), (O) intersect at points P, R. Let $QR \cap BC = A'$.
Analoguously, defined B', C' ...
Prove that the circles $(AI_aA'), (BI_bB'), (CI_cC')$ are coaxal.

Arab

#2 Jan 27, 2014, 10:15 am • 1

Denote by A_1, B_1, C_1, B_2, C_2 the point of tangency of BC, CA, AB, YP, PX and ω the A-excircle of $\triangle ABC$ respectively. Let R_a be the circumradii of ω, and ω_1 is the circumcircle of $\triangle ABC$ and ω_2 is the circumcircle of $\triangle PXY$. Under the inversion $\mathcal{I}(I_a, R_a)$, we obtain that $\mathcal{I}(\omega_1) = \omega_3$ and $\mathcal{I}(\omega_2) = \omega_4$, where ω_3, ω_4 with centers N_1, N_2 are the nine point circles of $\triangle A_1B_1C_1, \triangle A_1B_2C_2$.

Let H_1 be the orthocenter of $\triangle A_1B_1C_1$ and H_2 be that of $\triangle A_1B_2C_2$, then $H_1A_1 \perp B_1C_1, H_2A_1 \perp B_2C_2$, hence $H_1A_1 \parallel AI_a$ and $H_2A_1 \parallel PI_a$. Denote by S, T the midpoints of B_1C_1, B_2C_2, it follows that S is the polar of PA with respect to ω, thus $S = B_1C_1 \cap B_2C_2$. Moreover, $H_1A_1 = 2I_aS, H_2A_1 = 2I_aT$, so $\triangle A_1H_1H_2 \sim \triangle I_aST$ and $N_1N_2 \parallel H_1H_2 \parallel ST$, therefore, $N_1N_2 \perp PI_a$. Combining the fact that $T = \mathcal{I}(P) = \mathcal{I}(\omega_1 \cap \omega_2) = \omega_3 \cap \omega_4$, we get that PI_a is the radical axis of ω_3, ω_4. Consequently, U the second point of intersection of ω_3, ω_4 lies on PI_a and $\mathcal{I}(U) = R$ imply that P, R, I_a are collinear. Denote by M the midpoint of BC. From $PQ \perp BC, PB \perp BQ, PR \perp RQ$ we get that P, M, R, A' are concyclic, so $I_aQ^2 = BQ^2 = QM \cdot QP = QR \cdot QA'$, and hence $AI_a \perp A'I_a$, thus O_1 the circumcenter of $\triangle AI_aA'$ is the midpoint of AA', and similarly define O_2, O_3.

Since $A'I_a^2 = A'R \cdot A'Q = A'B \cdot A'C$, we obtain that $\triangle BA'I_a \sim \triangle I_aA'C$, so $\frac{BA'}{A'C} = \frac{[BA'I_a]}{[I_aA'C]} = \frac{BI_a^2}{CI_a^2}$. From Menelaus' Theorem,and the equation $\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = \frac{AI_c^2}{BI_c^2} \cdot \frac{BI_a^2}{CI_a^2} \cdot \frac{CI_b^2}{AI_b^2} = 1$ (note that $\triangle BCI_a \sim \triangle CAI_b \sim \triangle ABI_c$), and hence A', B', C' are collinear. From Newton's Theorem, O_1, O_2, O_3 are collinear, as desired.

$Q.E.D.$

Luis González

#3 Jan 28, 2014, 1:19 am • 3

Actually P can be any point on (O) outside the A-excircle, not necessarily the midpoint of the arc BAC, as the proposition states. All circles $\odot(PXY)$ go through the tangency point of the A-mixtilinear excircle with (O) (this is an extraversion of Incircle & Mixtilinear circle, Mixtilinear incircles and somehow Poncelet's porism, etc). Hence R is the tangency point of the A-mixtilinear excircle with $(O) \Longrightarrow AI_a \perp A'I_a$ (this is again an extraversion of Concurrent), thus $\odot(AI_aA')$ goes through the foot H_a of the A-altitude. Similarly $\odot(BI_bB')$ and $\odot(CI_cC')$ go through the feet of the altitudes H_b, H_c on CA, AB.

If H, I denote the orthocenters of $\triangle ABC$ and $\triangle I_aI_bI_c$ (incenter of ABC), then we have $\overline{HA} \cdot \overline{HH_a} = \overline{HB} \cdot \overline{HH_b} = \overline{HC} \cdot \overline{HH_c}$ and $\overline{IA} \cdot \overline{II_a} = \overline{IB} \cdot \overline{II_b} = \overline{IC} \cdot \overline{II_c} \Longrightarrow \odot(AI_aA'), \odot(BI_bB')$ and $\odot(CI_cC')$ are coaxal with common radical axis HI.

mathuz

#4 Jan 28, 2014, 9:08 am • 1

very-very nice!
Thank you, Luis.

IDMasterz
#5 Jan 29, 2014, 12:30 pm

Nice Problem Mathuz... I solved it during school today, and I noticed like Luis the direct similarity with the other problem you gave me:

http://www.artofproblemsolving.com/Forum/blog.php?u=118092&b=94396

Indeed, using this result, it can easily be adjusted to the external case and then by Pascal's theorem we obtain that A' lies on the line through I_a perpendicular to AI_a. However, I just have this feeling that this problem admits to generalisation, further than the fact P can be any point on (O). Anyone up for finding it ?

mathuz
#6 Jan 31, 2014, 12:54 pm

OK , you are right IDMasterz!
My solution is same with your idea.
R is constant, it's exmixtilinear point.
But, its proof is harder!

Quick Reply

High School Olympiads

incircles and tangents problem

Reply

epsilon07

#1 Jan 26, 2014, 12:13 am

In a triangle ABC, D and E are points on [BC] such that $\angle BAD = \angle CAE$ M and N are the tangent points (on [BC]) of incircles of ABD and ACE, respectively.

if $MB = \frac{2}{3}, MD = \frac{2}{5}, NC = \frac{1}{3}$ then find the length of NE

Luis González

#2 Jan 27, 2014, 9:40 pm

According to G4 IMO Shortlist 1993, we have the relation:

$$\frac{1}{MB} + \frac{1}{MD} = \frac{1}{NC} + \frac{1}{NE} \Longrightarrow \frac{3}{2} + \frac{5}{2} = 3 + \frac{1}{NE} \Longrightarrow NE = 1.$$

Quick Reply

High School Olympiads

Tangential quadrilateral in quadrilateral(well known??)

Reply

daothanhoai

#1 Jan 24, 2014, 10:27 pm

Dear Mathlinkers

Let $ABCD$ be a any quadrilateral. E, F, G, H are intersection of angle bisector of four vertex $\angle A, \angle B, \angle C, \angle D$ (Show in the figure attachment. M,N,P,Q are intersection of four line through E, F, G, H and pependicular with AB, AD, DC, CB (show in the figure).

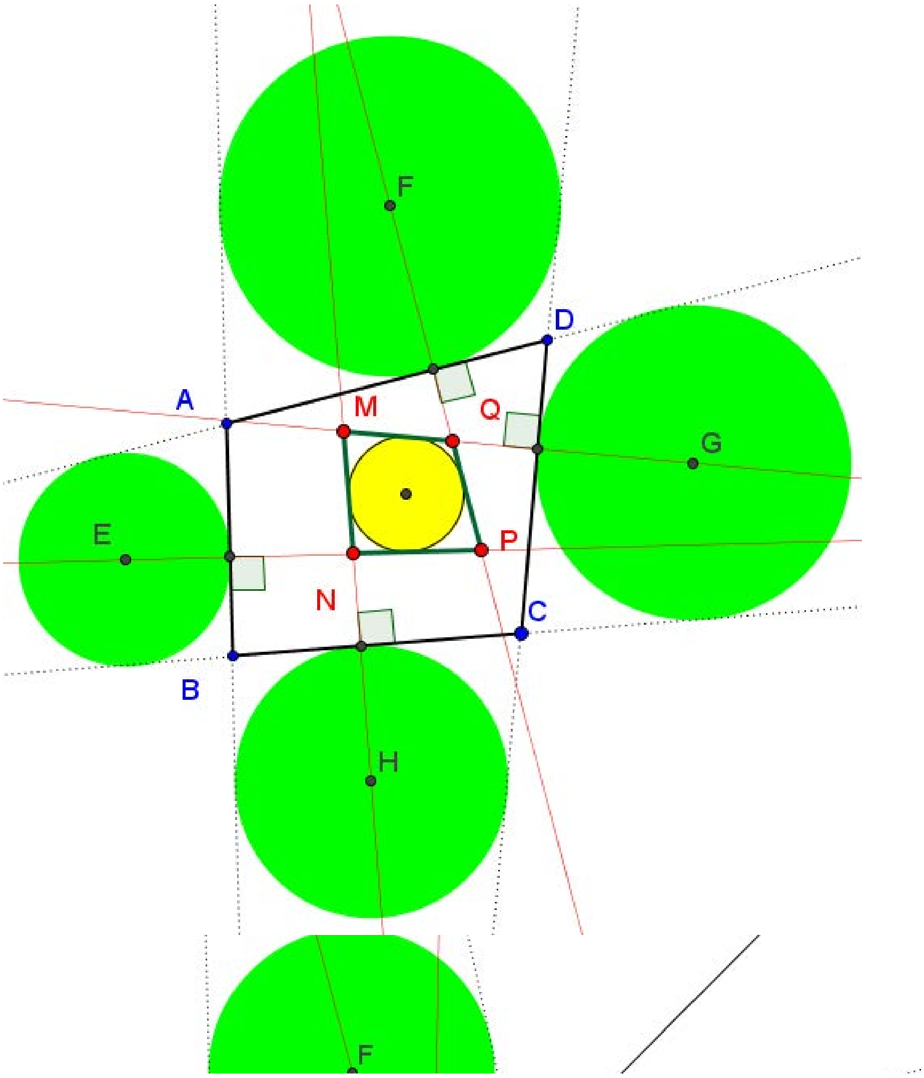
1-Prove that $MNPQ$ are tangential quadrilateral.

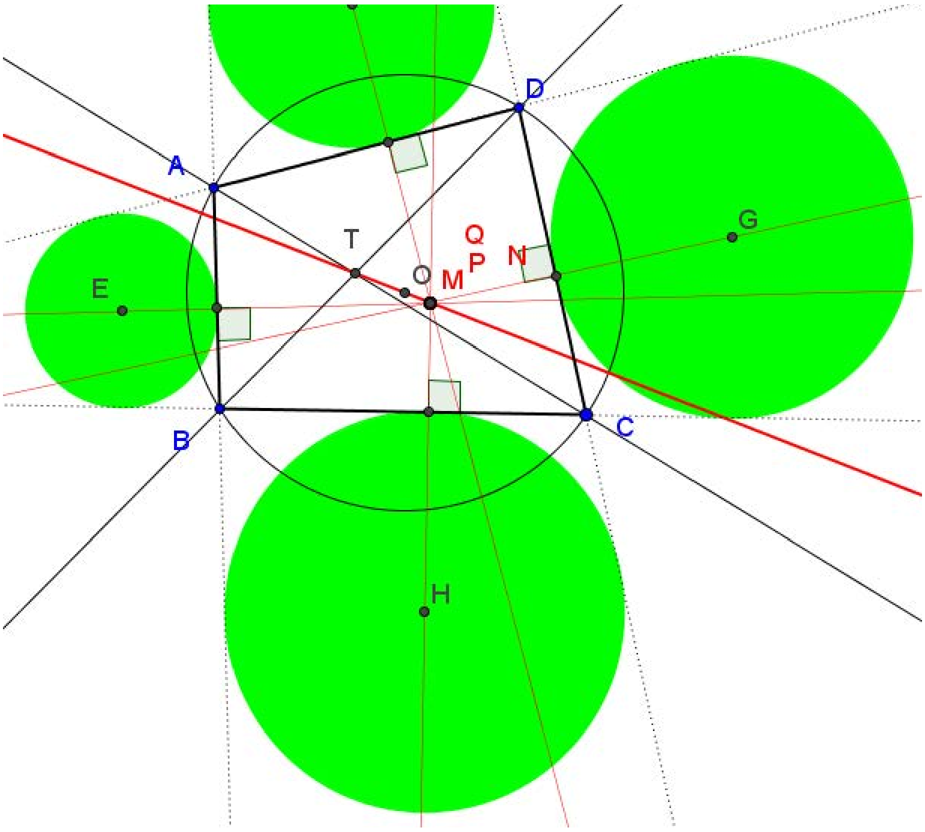
2-Prove that if $ABCD$ is cyclic, then $MNPQ$ is a point T, and center of circle $(ABCD)$,T and intersection of AC, BD are collinear

http://www.geogebratube.org/student/m79112

Best regards
Sincerely
Dao Thanh Oai

Attachments:





Luis González

#2 Jan 27, 2014, 1:58 am

1) Label $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}$ the angles of $ABCD.$ Then $\widehat{FAD} = 90^\circ - \frac{1}{2}\widehat{A}$ and $\widehat{FDA} = 90^\circ - \frac{1}{2}\widehat{D} \Longrightarrow \widehat{EFG} = \frac{1}{2}(\widehat{A} + \widehat{D}).$ Similarly we have $\widehat{EHG} = \frac{1}{2}(\widehat{C} + \widehat{B}) \Longrightarrow \widehat{EFG} + \widehat{EHG} = \frac{1}{2}(\widehat{A} + \widehat{B} + \widehat{C} + \widehat{D}) = 180^\circ \Longrightarrow EFGH$ is cyclic and let T be its circumcenter.

Since $PF \perp AD, PE \perp AB,$ we get $\widehat{PFA} = \widehat{PEA} = \frac{\widehat{A}}{2} \Longrightarrow \triangle PEF$ is P-isosceles $\Longrightarrow PT$ is perpendicular bisector of $\overline{EF} \Longrightarrow PT$ bisects $\widehat{NPQ}.$ Similarly, QT, MT, NT bisect $\widehat{PQM}, \widehat{QMN}, \widehat{MNP} \Longrightarrow MNPQ$ is tangential with incenter $T.$

2) Let $X \equiv BC \cap AD$ and WLOG assume that E is incenter of $\triangle XAB$ and G is X-excenter of $\triangle XDC.$ Then $\widehat{DGE} \equiv \widehat{DGX} = \frac{\widehat{C}}{2}.$ When $ABCD$ is cyclic, we have then $\widehat{DGE} = \widehat{FAD} = \frac{\widehat{C}}{2} \Longrightarrow ADGE$ is cyclic, i.e. AD is antiparallel to EG WRT $FE, FG \Longrightarrow$ perpendicular from F to AD goes through the circumcenter T of $\triangle FGE.$ Similarly, perpendiculars from G, H, E to DC, CB, BA pass through $T.$

From cyclic $ADGE,$ we get $XA \cdot XD = XE \cdot XG \Longrightarrow X$ has equal power WRT (O) and $\odot(EFGH) \equiv (T).$ Likewise, intersection $Y \equiv AB \cap DC$ has equal power WRT (O) and $(T) \Longrightarrow XY$ is radical axis of $(O), (T) \Longrightarrow OT \perp XY.$ But as XY is the polar of $Z \equiv AC \cap BD$ WRT $(O),$ we have $OZ \perp XY \Longrightarrow O, T, Z$ are collinear.

Quick Reply

High School Olympiads

circles thru feet of altitudes and antipodes

Reply

ThirdTimeLucky

#1 Jan 26, 2014, 4:31 pm

For a $\triangle ABC$ with circumcircle (O) let A_1 be the foot of the altitude from A to BC and let A_2 be the diametric opposite of A in (O). Let ω_A be the circumcircle of $\triangle AA_1A_2$ and similarly define ω_B, ω_C. Then show that $\omega_A, \omega_B, \omega_C$ have two common points.

Luis González

#2 Jan 26, 2014, 9:29 pm • 1

Here it's too easy to identify two points having equal powers WRT the 3 given circles, namely O and the orthocenter H. $\overline{OA} \cdot \overline{OA_2} = \overline{OB} \cdot \overline{OB_2} = \overline{OC} \cdot \overline{OC_2} = -R^2$ and by orthocenter property $\overline{HA} \cdot \overline{HA_1} = \overline{HB} \cdot \overline{HB_1} = \overline{HC} \cdot \overline{HC_1}$ $\Longrightarrow \omega_A, \omega_B$ and ω_C are coaxal with common radical axis OH. O is clearly inside these circles, thus they always intersect a two real points on OH.

sunken rock

#3 Jan 27, 2014, 6:45 am

Other idea:
Let M, N, P be the midpoints of BC, CA, AB respectively. The circumcenter Ω_A of $\triangle AA_1A_2$ is the intersection of the perpendicular bisectors of AA_1, AA_2, i.e. PN and the tangent at O to circle $\odot(APN)$, hence $\frac{\Omega_AP}{\Omega_AN} = \frac{OP^2}{ON^2}$ and other 2 similar relations which, multiplied give 1, or $\Omega_A, \Omega_B, \Omega_C$ are collinear.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Similar tangential quadrilateral in

Reply

daothanhoai

#1 Jan 24, 2014, 7:05 pm

Dear Mathlinkers

Let $ABCD$ be a tangential quadrilateral, AC meets BD at G. Construct a circle (G) center at G. N, M, P, Q are intersection of four polar line of $ABCD$ to (G). Easily show that $NMPQ$ are parallelogram. MQ meets NP at R.

Prove that R, G, and center of circumcribed $(ABCD)$ are collinear.

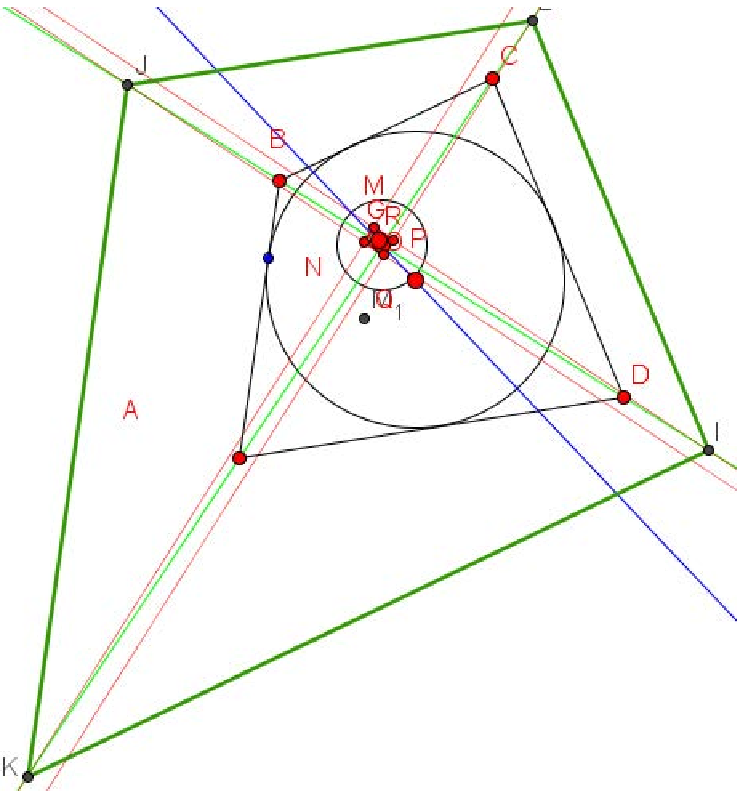
The polar of A, C meet BD at J, I ; The polar of B, D meet AC at K, L

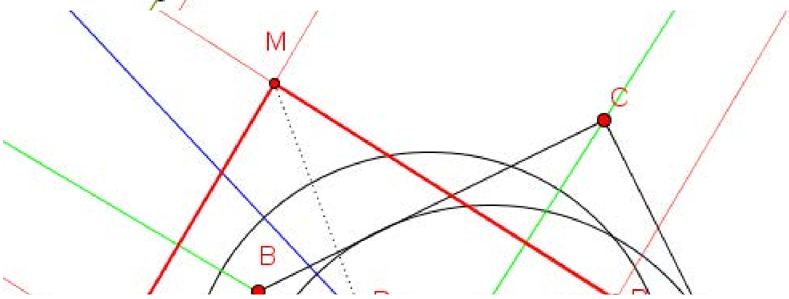
Prove that $KJLI$ are tangential quadrilateral similar $ABCD$ (but not homothetic)

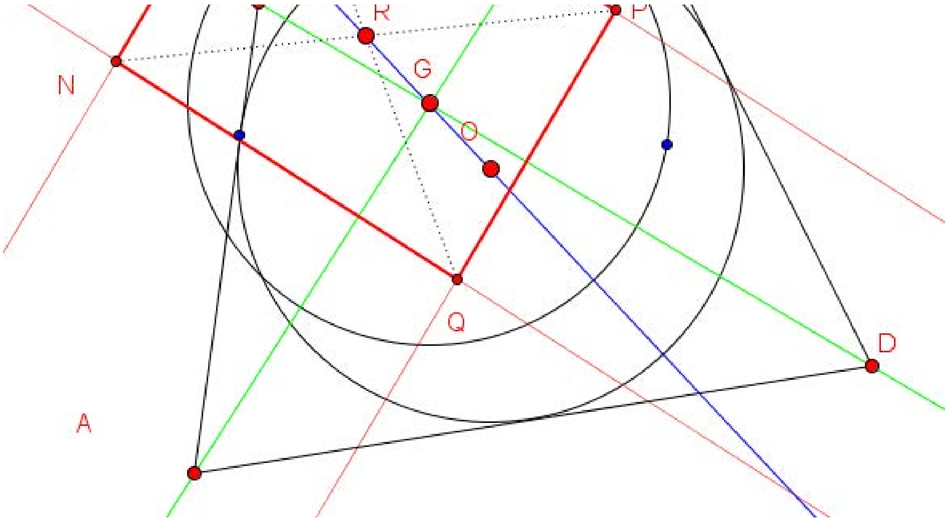
http://www.geogebratube.org/student/m79066

Best regards
Sincerely
Dao Thanh Oai

Attachments:







Luis González

#2 Jan 26, 2014, 2:29 am

N, M, P, Q are then the poles of AB, BC, CD, DA WRT $(G) \Longrightarrow PN$ and QM are the polars of $X \equiv AB \cap CD$ and $Y \equiv BC \cap DA$ WRT $(G) \Longrightarrow R \equiv PN \cap QM$ is the pole of XY WRT $(G) \Longrightarrow RG \perp XY$. But, as XY is the polar of G WRT (O), we have $OG \perp XY$ and therefore R, G, O are collinear.

The second proposition is not completely true. $KJLI$ is indeed tangential but not similar to $ABCD$ in general.

Let A', C' be the projections of A, C on BD and B', D' the projections of B, D on AC. Then AA', BB', CC', DD' are the polars of J, K, I, L WRT (G), i.e. A', B', C', D' are the inverses of J, K, I, L WRT (G). Hence, from cyclic $KJA'B'$, $ABA'B'$, we get $\angle JKG = \angle GA'B' = \angle GAB \Longrightarrow KJ \parallel BA$. Similarly $JL \parallel AD, LI \parallel CD, IK \parallel BC$, so $KJLI$ are all homothetic with center G.

Dilatate $KJLI$ carrying KJ onto AB, i.e. $K \equiv A, J \equiv B$ (note here that ABCD and ABLI are not similar). If (O) touches BC, DA, AB at U, V, W and UV cuts BL, AI at U', V', then $\triangle BUU'$ and $\triangle YUV$ are homothetic $\Longrightarrow \triangle BUU'$ is B-isosceles $\Longrightarrow BU' = BU = BW$. Similarly $AV' = AV = AW \Longrightarrow$ there is a circle (O') touching LB, IA, AB at U', V', W. By Newton's theorem (degenerate Brianchon theorem) in $ABCD$, the lines $BD \equiv BI, AC \equiv AL$ and $UV \equiv U'V'$ concur, thus by the converse of Newton's theorem in $ABLI$, it follows that LI touches (O'), i.e. $ABLI$ is tangential $\Longrightarrow$ the original $KJLI$ is tangential.

livetolove212

#3 Apr 26, 2015, 11:30 am

Let A_1, B_1, C_1, D_1 be the inverses of A, B, C, D wrt (G). The polars of A, B, C, D cuts AC, BD at A_2, B_2, C_2, D_2.
We know that the inversion center G take the vertices of $ABCD$ to the vertices of other circumscribed quadrilateral. Since $GA.GA_1 = GB.GB_1 = GC.GC_1 = GD.GD_1 = R^2_{(G)}$ then $A_1B_1C_1D_1$ is the circumscribed quadrilateral.
A_1, B_1 lies on (A_2B_2), similar with $B_1, C_1; C_1, D_1; D_1, A_1$.
Let $\angle A_1A_2B_1 = \angle A_1B_2B_1 = \angle B_1C_2C_1 = \angle A_1D_2D_1 = \alpha$. We have
$A_1D_1 + B_1C_1 = \sin\alpha(A_2D_2 + B_2C_2), A_1B_1 + C_1D_1 = \sin\alpha(A_2B_1 + C_2D_2)$. Therefore $A_1B_1C_1D_1$ is circumscribed quadrilateral iff $A_2B_2C_2D_2$ is circumscribed quadrilteral.

Remark: If $ABCD$ is a bicentric quadrilateral then $A_2B_2C_2D_2$ is also a bicentric quadrilateral.

Quick Reply

High School Olympiads

Iff

Locked

gobathegreat

#1 Jan 25, 2014, 2:33 am

In triangle ABC, ω is its circumcircle and O is the center of this circle. Points M and N lie on sides AB and AC respectively. ω and the circumcircle of triangle AMN intersect each other for the second time in Q. Let P be the intersection point of MN and BC. Prove that PQ is tangent to ω if and only if $OM = ON$.

Luis González

#2 Jan 25, 2014, 3:20 am

gobathegreat, please next time give your topics meaningful subjects and use the search before posting contest problems. If you already know the source, then check the Olympiad Resources page. It's Iran 3rd round 2011 P2 (geometry exam).

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=429223

High School Olympiads

Collinear

Locked

gobathegreat

#1 Jan 25, 2014, 2:28 am

Let $ABCD$ be a quadrilateral, and E be intersection points of AB,CD and AD,BC respectively. External bisectors of DAB and DCB intersect at P, external bisectors of ABC and ADC intersect at Q and external bisectors of AED and AFB intersect at R. Prove that P,Q,R are collinear

Luis González

#2 Jan 25, 2014, 2:54 am

This comes from Iran National Olympiad third round (2008). Use the search before posting contest problems.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=30921
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=225702
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=329544
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=349520

High School Olympiads

olympic

Reply

supama97
#1 Sep 13, 2012, 7:51 pm

Let acute triangle ABC, not isosceles, (O) is a circle pass through A,B,C. AD is a height of ABC. (A;AD) is a circle pass through D and cuts (O) at P,Q. PQ cuts AD at G. AO cuts BC at E and K is the midpoint of AD. Prove that
a) GE//OK
b) H is a orthocenter of ABC. M is a midpoint of BC. Prove that HM, GE, OD meet at a point

---

Moderator says: Please use LaTeX in your posts.
Let $\triangle ABC$ be acute, not isosceles. (O) is a circle passing through A, B, C. AD is a height of $\triangle ABC$. $(A; [AD])$ is a circle passing through D and cutting (O) at P, Q. PQ cuts AD at G. AO cuts BC at E and K is the midpoint of $[AD]$. Prove that
a) $GE \parallel OK$.
b) H is an orthocenter of $\triangle ABC$. M is a midpoint of $[BC]$. Prove that HM, GE, OD meet at a point.

yetti
#2 Sep 14, 2012, 8:01 am

(a) Inversion in circle $(A, [AD])$ takes line BC to circle (K) on diameter $[AD]$ and circumcircle (O) of $\triangle ABC$ to line PQ.
Intersections $B \equiv AB \cap BC \cap (O), C \equiv AC \cap BC \cap (O)$ go to intersections
$S \equiv AB \cap (K) \cap PQ, T \equiv AC \cap (K) \cap PQ$, resp. $\Longrightarrow G \equiv AD \cap PQ \equiv AD \cap ST$.
Let parallel to BC through G cut AB, AC at B', C', resp. Since $DG \perp B'C', DS \perp AB', DT \perp AC' \Longrightarrow SGT$ is Simson line of $\triangle AB'C'$ with pole $D \Longrightarrow D$ is on circumcircle (O') of $\triangle AB'C'$.
Since $\triangle AB'C' \sim \triangle ABC$, are centrally similar with similarity center A, having parallel sides, $O' \in AO$. Since $A, D \in (O'), E \in AO'$ and $AD \perp ED \Longrightarrow [AE]$ is diameter of (O').
Let AO cut (O) again at X. $\frac{AE}{AX} = \frac{AO'}{AO} = \frac{AG}{AD} \Longrightarrow \triangle AEG \sim \triangle AXD$ are also centrally similar with similarity center A $\Longrightarrow EG \parallel XD$. Line $OK \parallel XD$ is midline of $\triangle AXD \Longrightarrow EG \parallel OK$.

(b) Let AD cut (O) again at Y. $[AX]$ is diameter of $(O) \Longrightarrow XY \parallel (BC \equiv MD)$ (both $\perp ADY$). Common perpendicular bisector $OM \parallel AD$ of $[BC], [XY]$ cuts $[XY]$ at its midpoint $Z \Longrightarrow$
$[XY] = 2\ [ZY] = 2\ [MD]$. Combined with $XY \parallel MD \Longrightarrow XM$ cuts ADY at reflection of Y in $BC \equiv MD$, identical with orthocenter H of $\triangle ABC$.
Let $J \equiv EG \cap OD$. Opposite sides DJ, EX of trapezoid $DJEX$ with diagonals DE, JX meet at O. $OM \parallel AD$ is midline of $\triangle AXD$,
cutting trapezoid side $[XD]$ at its midpoint $N \Longrightarrow M \in DE$ is trapezoid diagonal intersection $\Longrightarrow M \in JX \Longrightarrow$ $J \in XMH \Longrightarrow J \equiv MH \cap EG \cap OD$.

Luis González
#3 Jan 25, 2014, 12:22 am

a) AO is perpendicular bisector of $\overline{PQ}$, cutting it at its midpoint N, and cuts (O) again at the reflection F of A on O. Inversion WRT circle (A, AD) swaps (O) and $PQ \Longrightarrow AD^2 = AN \cdot AF \Longrightarrow \odot(DNF)$ touches $AD \Longrightarrow \angle DFN = \angle GDN$. But from cyclic $DENG, \angle GDN = \angle GEN \Longrightarrow \angle DFN = \angle GEN \Longrightarrow EG \parallel FD \parallel OK$.

b) AD cuts (O) again at the reflection X of H on BC. Thus $\frac{DX}{AX} = \frac{DH}{AX} = \frac{EF}{AF} = \frac{DG}{AD} \Longrightarrow \frac{DH}{DG} = \frac{AX}{AD} = \frac{AF}{AE} = \frac{AD}{AG}$. If $S \equiv EG \cap OM$, then $OSM \parallel AGD$ gives $\frac{OM}{OS} = \frac{AD}{AG} = \frac{DH}{DG}$, which means that HM, OD and $GS \equiv GE$ concur.

Quick Reply

High School Olympiads

Beautiful geometry

Reply

Source: I'd like to know too..

Aiscrim
#1 Jan 22, 2014, 1:25 pm

In the convex quadrilateral $ABCD$, we have that $\widehat{ABC} = \widehat{ADC} = 135^\circ$. Let M, N be on AB and AD respectively, such that $\widehat{MCD} = \widehat{NCB} = 90^\circ$. If K is the second intersection point of the circles (AMN) and (ABD), show that $AK \perp KC$

Luis González
#2 Jan 23, 2014, 12:04 am • 1

Let CM, CN cut AD, AB at $P, Q,$ respectively and let Y, Z be the projections of C on $DP, BQ.$ Since $\angle PCD = \angle BCQ = 90^\circ$ and $\angle CDP = \angle CBQ = 45^\circ$, then $\triangle CDP$ and $\triangle CBQ$ are isosceles right with common apex $C \Longrightarrow Y, Z$ are midpoints of $\overline{PD}, \overline{QB}.$ Further $\angle MCD = \angle NCB = 90^\circ \Longrightarrow \angle DCN = \angle BCM \Longrightarrow \triangle CDP$ and $\triangle CQB$ are similar with corresponding cevians CY, CZ and $CN, CM \Longrightarrow \overline{YP} : \overline{YD} = \overline{ZQ} : \overline{ZB}$ and $\overline{NP} : \overline{ND} = \overline{MQ} : \overline{MB} \Longrightarrow$ circles $\odot(ABD), \odot(APQ), \odot(AYZ)$ and $\odot(AMN)$ concur at A and the center K of the spiral similarity that carries $\overline{PYND}$ into $\overline{QZMB},$ i.e. K is on circle with diameter $\overline{AC}$.

Luis González
#3 Jan 23, 2014, 12:10 am

Problem comes from Iran 2nd round (2002). For more solutions see

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=16899
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=36343
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=210284

Quick Reply

High School Olympiads

A circle through fixed point

Reply

daothanhoai

#1 Jan 20, 2014, 2:27 pm

Dear Mathlinkers

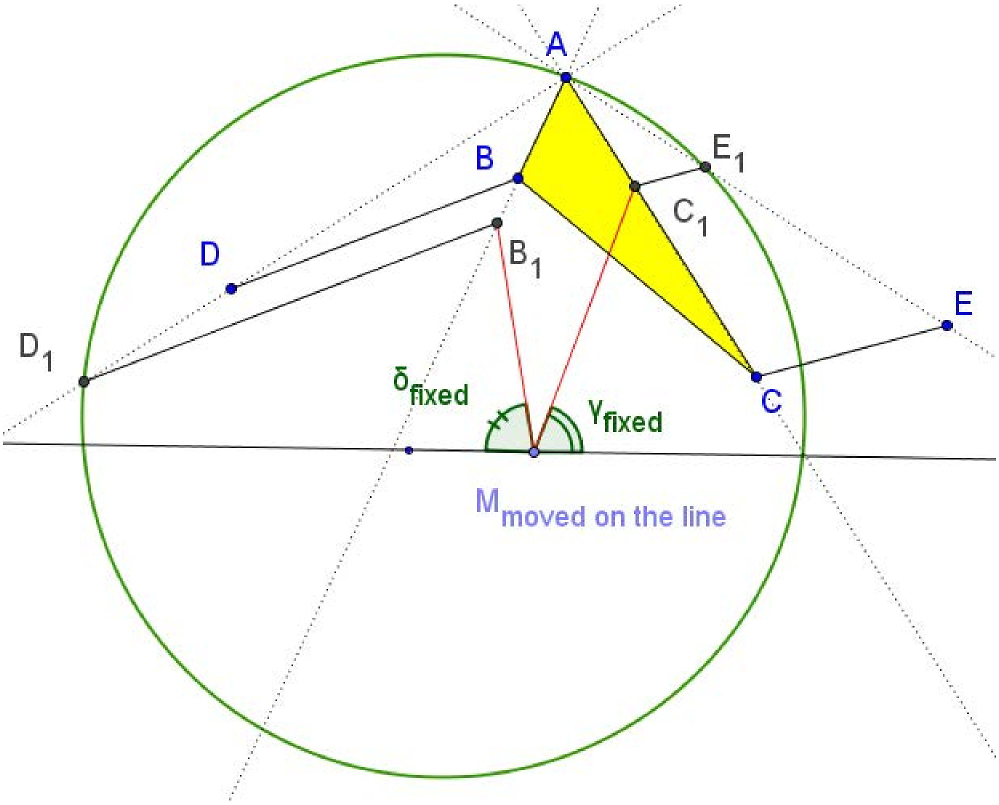
Let ABC be a triangle. Construct two triangle ADB, AEC on the two sides AB, AC. Let (d) on the plane, M is on (d)

Construct two lines through M, and these two lines meet AB, AC respectively at B_1, C_1, such that angle of three line are fixed when M moved on the line (d).

Denote D_1, E_1 respectively are on AD, AE such that $B_1D_1//BD$ and $C_1E_1//CE$. Prove that: When M moved on the line (d), the circle (AD_1E_1) through fixed point.

Best regards
Sincerely
Dao Thanh Oai

Attachments:



Luis González

#2 Jan 21, 2014, 2:43 am • 1

Since MC_1 and C_1E_1 have constant directions, then clearly $M \mapsto C_1$ is a perspectivity from d to AC and $C_1 \mapsto E_1$ is a perspectivity from AC to AE. Analogously, $M \mapsto B_1$ is a perspectivity from d to AB and $B_1 \mapsto D_1$ is a perspectivity from AB to $AD \Longrightarrow D_1 \mapsto E_1$ is a homography from AD to AE, mapping the infinite point of AD into the infinite point of AE, when M is at infinity $\Longrightarrow D_1E_1$ envelopes a fixed parabola $\mathcal{P}$ tangent to $AD, AE \Longrightarrow$ circles $\odot(AD_1E_1)$ go through the focus of $\mathcal{P}$.

Quick Reply

High School Olympiads

circumcentre, centroid and ratios

Reply

AndrewTom

#1 Jan 20, 2014, 1:46 am

The triangle $X_1X_2X_3$ has circumradius $R=1$ and centroid G. The line thorough X_1 and G meets the circumcircle again at Y_1 and the points Y_2 and Y_3 are defined correspondingly. Prove that

$$\frac{GX_1}{GY_1}+\frac{GX_2}{GY_2}+\frac{GX_3}{GY_3}=3.$$

Generalising to n points on the circle, prove that

$$\sum_{i=1}^{n}\frac{GX_i}{GY_i}=n.$$

nikoma

#2 Jan 20, 2014, 2:32 am • 1

Click to reveal hidden text

AndrewTom

#3 Jan 20, 2014, 3:03 am

Thanks nikoma.

Can this be done without barycentric coordinates?

sunken rock

#4 Jan 20, 2014, 7:18 am • 1

hint

Best regards,
sunken rock

Luis González

#5 Jan 20, 2014, 8:57 am • 1

We prove the general version for n points $X_1,X_2,X_3,\ldots,X_n$ lying on a circle $\omega\equiv(O,R)$. If $p(X,\omega)$ denotes the power of point X WRT ω, then we have

$$\frac{|GX_i|}{|GY_i|}=\frac{|GX_i|^2}{|GX_i|\cdot|GY_i|}=\frac{|GX_i|^2}{p(G,\omega)}\Longrightarrow\sum_{i=1}^{n}\frac{|GX_i|}{|GY_i|}=\frac{1}{p(G,\omega)}\cdot\sum_{i=1}^{n}|GX_i|^2\ \ (1).$$

Put equal masses m at $X_1,X_2,X_3,\ldots,X_n$, hence moments of inertia of $\{X_1,X_2,X_3,\ldots,X_n\}$ about G,O are then

$$\mathbf{M}_G=m\cdot\sum_{i=1}^{n}|GX_i|^2\ ,\ \mathbf{M}_O=m\cdot\sum_{i=1}^{n}|OX_i|^2=m\cdot n\cdot R^2.$$

But by Parallel axis theorem, we have $\mathbf{M}_O=\mathbf{M}_G+n\cdot m\cdot|OG|^2\Longrightarrow$

$$m \cdot n \cdot R^2 = m \cdot \sum_{i=1}^{n} |GX_i|^2 + n \cdot m \cdot |OG|^2 \Longrightarrow$$

$$\sum_{i=1}^{n} |GX_i|^2 = n \cdot (R^2 - |OG|^2) = n \cdot p(G, \omega) \quad (2).$$

Combining the expressions (1) and (2) gives $\sum_{i=1}^{n} \frac{|GX_i|}{|GY_i|} = n.$

Quick Reply

High School Olympiads

Radical axis something

Reply

fmasroor
#1 Jan 19, 2014, 7:28 am

I have already posted this first part in the high school preolympiad section but I have another part to add here. both parts are below:
1. Let P be a variable point on line BC of triangle ABC. The circle with diameter BP intersects the circumcircle of ACP again at Q. Prove that PQ passes through a fixed point as P moves.
2. Furthermore, let H be the orthocenter of ABP, and M the intersection of PQ with AC. Prove that, as P varies, MH passes through a fixed point.

Luis González
#2 Jan 19, 2014, 9:02 am

1. Let R be the 2nd intersection of PQ with the circle $\odot(ABQ)$. Since $\angle BQR=90^{\circ}$ and $\angle AQR=\angle ACB,$ we get $\angle AQB=90^{\circ}+\angle ACB=\text{const} \Longrightarrow \odot(ABQ)$ is fixed $\Longrightarrow PQ$ goes through the antipode R of B WRT $\odot(ABQ).$

2. H runs on the perpendicular h_A from A to BC and all lines PH go through the infinite point of $\perp AB \Longrightarrow H \mapsto P$ is a perspectivity from h_A to $BC.$ Since $P \mapsto M$ is another perspectivity from BC to $CA,$ then $H \mapsto M$ is homography from h_A to $AC.$ When P coincides with the intersection of BC with the perpendicular to AB at $A,$ we clearly have $H \equiv Q \equiv M,$ thus the homography $H \mapsto M$ is a perspectivity $\Longrightarrow MH$ passes through a fixed point.

fmasroor
#3 Jan 19, 2014, 9:54 am

1. Can you please explain why <AQR=<ACB?
2. What are perspectivity and homography?

Luis González
#4 Jan 19, 2014, 11:03 am

1. <AQR=<ACB is pretty obvious from cyclic quadrilateral ACPQ.

2. Two 1st category figures (points o a line, pencil of lines, pencil of planes, etc) are said to be homographic or projective if one of them can be obtained from the other by composition of projections and/or sections. Hence this is a bijection preserving cross-ratios and vice versa. A homography X --> X' from a line L to to another line L' is a perspectivity (XX' passing through a fixed point) iff the intersection point of L,L' is mapped into itself. Same happens in its dual planar figure; two pencils of lines are perspective iff the have a double ray.

fmasroor
#5 Jan 19, 2014, 8:15 pm

1. Oh yeah didnt see that
2. ok whatever you say... any papers introducing this topic?

Arab
#6 Jan 19, 2014, 8:30 pm

This fixed point lies on the line passing through B and perpendicular to BC.

Quick Reply

High School Olympiads

nice result geometry

Source: own

vankhea

#1 Dec 21, 2013, 7:50 am

Incircle of triangle ΔABC tangents to the sides BC, CA, AB at A', B', C' respectively.
X, Y, Z be midpoint of $B'C', C'A', A'B'$ respectively.
U, V, W be point on the rays AX, BY, CZ such that $UX = XA; VY = YB; WZ = ZC$.
Prove that $[A'B'C'] = [UVW]$ which $[KKK]$ be area of triangle ΔKKK.

vankhea

#2 Jan 17, 2014, 9:53 am

Generalization of this problem :
If a circle tangent to sides BC, CA, AB of a triangle ΔABC at D, E, F respectively and if X, Y, Z be reflection points of A, B, C on sides EF, FD, DE respectively then $[DEF] = [XYZ]$.

vankhea

#3 Jan 17, 2014, 10:11 am

Similarly problem :
If incircle of ΔABC tangent to the sides BC, CA, AB at D, E, F respectively and If excircle X_A, X_B, X_C of ΔABC tangent to the sides BC, CA, AB at P, Q, R respectively then $[DEF] = [PQR]$

vankhea

#4 Jan 17, 2014, 10:14 am

See more about area
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=571014

vankhea

#5 Jan 17, 2014, 10:18 am

Nice area about reflection points
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=568489

Luis González

#6 Jan 19, 2014, 7:15 am • 2

> *vankhea wrote:*
> Generalization of this problem:
> If a circle tangent to sides BC, CA, AB of a triangle ΔABC at D, E, F respectively and if X, Y, Z be reflection points of A, B, C on sides EF, FD, DE respectively then $[DEF] = [XYZ]$.

M, N, L denote the midpoints of $BC, CA, AB. I, S$ denote the incenters of $\triangle ABC, \triangle MNL$ and T denotes the orthocenter of $\triangle DEF$. A-, B- an C- excircle touch BC, CA, AB at D', E', F'.

Since N and L are also midpoints of $\overline{EE'}$ and $\overline{FF'}$, then it follows that parallels from E' and F' to $NS \parallel ET$ and $LS \parallel FT$, resp, cut ST at the reflection U of T on S. Hence if $B' \equiv IB \cap DF$ is midpoint of $\overline{FD}$, we have
$UF' = 2 \cdot SN - TE = BI = 2 \cdot IB' = YB' - IB' = IY$. Likewise $UE' = IZ$ which implies that

$UE' = 2 \cdot SN = YE = BI - 2 \cdot IB' = YB' = IB' = YI$. Likewise, $UF' = YZ$, which implies that $\triangle UE'F' \cong \triangle IYZ$. Similarly, $\triangle UF'D' \cong \triangle IZX$ and $\triangle UD'E' \cong \triangle IXY \Longrightarrow \triangle XYZ \cong \triangle D'E'F' \Longrightarrow [XYZ] = [D'E'F'] = [DEF]$. Last equality follows fom the fact that $\triangle DEF$ and $\triangle D'E'F'$, being cevian triangles of isotomic conjugates, have equal areas.

vankhea

#7 Jan 19, 2014, 4:31 pm

Thanks you very much Luis Gonzalez

buratinogigle

#8 Jan 19, 2014, 11:41 pm

> *vankhea wrote:*
>
> Generalization of this problem:
> If a circle tangent to sides BC, CA, AB of a triangle ΔABC at D, E, F respectively and if X, Y, Z be reflection points of A, B, C on sides EF, FD, DE respectively then $[DEF] = [XYZ]$.

If YZ, ZX, XY cut EF, FD, DE at K, L, M respectively then AK, BL, CM are concurrent.

Quick Reply

High School Olympiads

Intouch and extouch triangles are orthologic

Reply

Source: me

pohoatza
#1 Jan 17, 2014, 6:43 pm • 1

Let ABC be a triangle with circumcenter O, incircle Γ and excircles Γ_a, Γ_b, Γ_c. Let D, E, F be the tangency points of Γ with BC, CA, AB and let X, Y, Z be the tangency points of Γ_a, Γ_b, Γ_c with BC, CA, AB, respectively. Prove that:
a) The triangles DEF and XYZ are orthologic.
b) The orthology centers of DEF and XYZ are symmetrical with respect to O.

fmasroor
#2 Jan 18, 2014, 2:16 am • 1

What does orthologic mean?

fmasroor
#3 Jan 18, 2014, 2:49 am

Nvm got it. Anyway this admits an obvious generalization, if P and Q are points with the midpoint O, the circumcenter, then their pedal triangles are orthologic and their orthology centers share the same midpoint O.

fmasroor
#4 Jan 18, 2014, 4:07 am

I have a computational solution
Click to reveal hidden text

pohoatza
#5 Jan 18, 2014, 5:07 am

Very nice, there's also a synthetic proof for a). Let M, N, P be the midpoints of the arcs BAC, CBA, ACB of the circumcircle of ABC. Then, note that 1) triangles DEF and MNP are homothetic (angle chasing) and 2) points M, N, P lie on the perpendicular bisectors of YZ, ZX, XY, respectively (the Lemma from IMO 2013 Problem 3).

What about b) ?

Luis González
#6 Jan 18, 2014, 6:08 am • 2

> *fmasroor wrote:*
> Nvm got it. Anyway this admits an obvious generalization, if P and Q are points with the midpoint O, the circumcenter, then their pedal triangles are orthologic and their orthology centers share the same midpoint O.

In fact, any line PQ through O does the work; O does not need to be midpoint of PQ.

Let $\triangle P_1P_2P_3$ and $\triangle Q_1Q_2Q_3$ be the pedal triangles of P, Q WRT $\triangle ABC$; $P_1, Q_1 \in BC$, etc. Denote H_1 and H_2 the orthocenters of $\triangle P_1P_2P_3$ and $\triangle Q_1Q_2Q_3$. M, N, L denote the midpoints of BC, CA, AB.

Perpendiculars from A, B, C to P_2P_3, P_3P_1, P_1P_2 concur at the isogonal conjugate P^* of P WRT $\triangle ABC$, hence perpendiculars from M, N, L to P_2P_3, P_3P_1, P_1P_2 concur at the complement P_0 of P^* WRT $\triangle ABC \Longrightarrow$ perpendicular from Q_1 to P_2P_3 cuts H_1P_0 at R such that $\overline{P_0H_1} : \overline{P_0R} = \overline{MP_1} : \overline{MQ_1} = \overline{OP} : \overline{OQ}$. Similarly, perpendiculars from Q_2 and Q_3 to P_3P_1 and P_1P_2 hit H_1P_0 at the same point $R \Longrightarrow \triangle P_1P_2P_3$ and $\triangle Q_1Q_2Q_3$ are orthologic.

fmasroor

#7 Jan 18, 2014, 6:39 am

> pohoatza wrote:
> 2) points M, N, P lie on the perpendicular bisectors of YZ, ZX, XY, respectively (the Lemma from IMO 2013 Problem 3).

How exactly does this finish off your original problem though?

pohoatza

#8 Jan 18, 2014, 2:21 pm

Well, 2) just implies that MNP and XYZ are orthologic (the circumcenter of XYZ being an orthology center). So, since MNP and DEF are homothetic, it follows that DEF and XYZ are orthologic.

fmasroor

#9 Jan 18, 2014, 7:28 pm

Ah. Did not know about that theorem.

Quick Reply

High School Olympiads

Focus of Elipse tangent three sidelines at FaFbFc

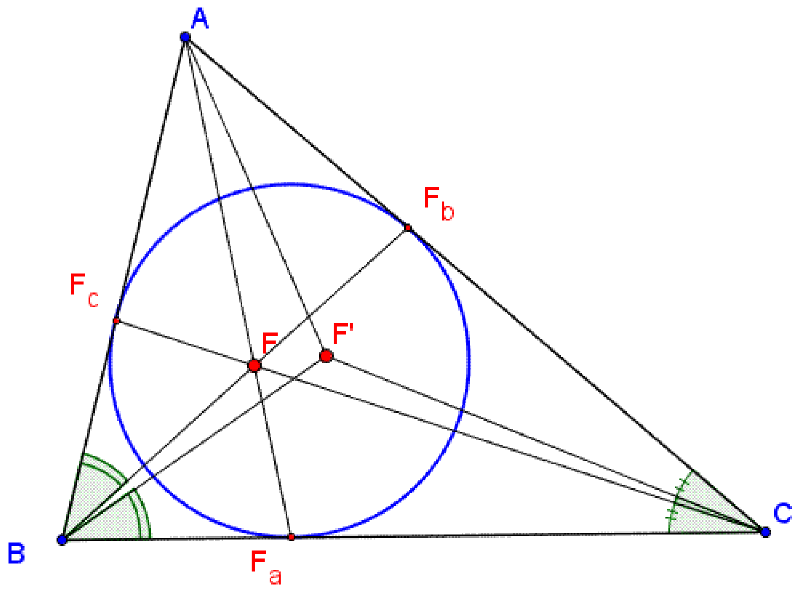
Reply

daothanhoai

#1 Jan 15, 2014, 10:19 am

Let ABC be a triangle, F is the first Femart point. Denote $\triangle F_aF_bF_c$ be the cevian triangle of F WRT $\triangle ABC$. Prove that Focus of Elipse tangent three sidelines at F_a, F_b, F_c is the first and isogonal conjugate of the first Fermat point

Attachments:



Luis González

#2 Jan 15, 2014, 11:01 am • 1

It is equivalent to show that the reflections of the isodymic point F' on BC,CA,AB lie on FA,FB,FC. Certainly, it has been discussed before.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=175130
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=304719
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=435239

daothanhoai

#3 Jan 15, 2014, 11:20 am

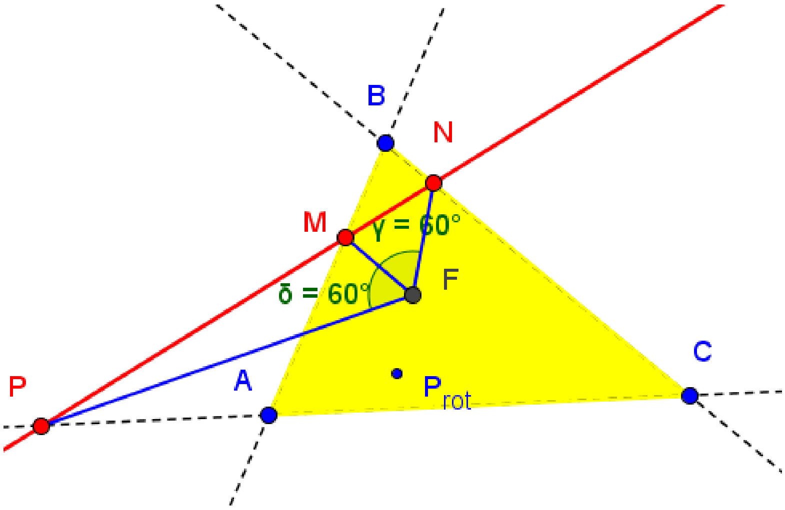
> *Luis González wrote:*
>
> It is equivalent to show that the reflections of the isodymic point F' on BC,CA,AB lie on FA,FB,FC. Certainly, it has been discussed before.
>
> http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=175130
> http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=304719
> http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=435239

Dear Mr Luis GonzaLez and Mathlinkers

Thank to you, prove that a line $\overline{NMP}$ tangents the Ellipse above

Define the line $\overline{NMP}$ following:

Let a triangle ABC and F is Fermat point of the triangle ABC. N, M, P are on BC, BA, AC and $\angle NFM = \angle MFP = \angle NFP = 60^0$ (Show in the figure attachment). Prove that N,M,P are collinear.



Synthetic proof N, M, P are collinear at here: http://www.cut-the-knot.org/m/Geometry/2x60.shtml

Best regards
Sincerely
Dao Thanh Oai

Luis González

#4 Jan 16, 2014, 10:38 am • 1

Let X, Y, Z, D be the projections of F on $BC, CA, AB, MN,$ respectively. Using cyclic quadrilaterals $FDNX, FDMZ$ and $FXBZ,$ we get

$$\angle XDZ = 180^\circ - (\angle NDX - \angle MDZ) = 180^\circ - (\angle NFX + \angle MFZ) =$$

$$= 180^\circ - (\angle XFZ - 60^\circ) = 60^\circ + \angle ABC = 180^\circ - \angle XYZ,$$

which means that $D \in \odot(XYZ).$ Hence, the line MN perpendicular to FD at D touches the conic with focus F and pedal circle $\odot(XYZ)$; the inconic $\mathcal{F}$ of $\triangle ABC$ with focus $F.$ By similar reasoning, MP is tangent to the same conic $\mathcal{F} \Longrightarrow$ M, N, P are collinear and $\overline{NMP}$ touches the inconic $\mathcal{F}.$

Quick Reply

High School Olympiads

Parallel tangent

Reply

ilovemath121

#1 Dec 19, 2013, 10:26 am

Let ABC be an acute triangle with circumcenter O, incenter I. IB, IC meet (O) at E, F ($E\neq B, F\neq C$). BO, CO meet (O) at K, L, resp. M, N be projections of K, L on OF, OE, res. Prove that one of two tangent of circle (E,EN) and circle (F,FM) parallel BC

Luis González

#2 Jan 15, 2014, 9:52 am • 1

We use standard triangle notation. $(O,R),(I,r)$ denote the circumcircle, incircle and $(I_a,r_a),(I_b,r_b),(I_c,r_c)$ denote the excircles against A,B,C. X,Y denote the projections of E,F on BC.

Clearly OE is perpendicular bisector of $\overline{AC}$ cutting it at its midpoint Q, hence from rectangle $AQNL$, it follows that O is midpoint of $\overline{QN} \Longrightarrow EN=OE+ON=R+OQ$. But since E is midpoint between I and the B-excenter I_b, we have $EX=\frac{1}{2}(r+r_b)$ and $EQ=\frac{1}{2}(r_b-r)$, which yields $EX-EQ=r$. As a result, $EN+EX=R+r+OQ+EQ=2R+r$. Similarly, we have $FM+FY=2R+r$. Hence if XE and YF cut $\odot(E,EN)$ and $\odot(F,FM)$ at U,V, (E between X,U and F between V,Y), we have $UX=EN+EX=2R+r=FM+FY=VY \Longrightarrow UVYX$ is rectangle $\Longrightarrow UV$ is common external tangent of $\odot(E,EN),\odot(F,FM)$ parallel to BC.

sunken rock

#3 Jan 15, 2014, 2:40 pm

Remark:
İt is well known the following problem:
İf E, F are respectively midpoints of the arcs $\overset{\frown}{AC}, \overset{\frown}{AB}$ of $\odot(ABC)$ which do not contain other vertex, then one of the common tangents to the circles: a) centered at E and tangent to AC and b) centered at F and tangent to AB is parallel to BC and contains the incenter of the circle.

The problem is valid for the excenters as well, the centers of the two circles being the midpoints of the arcs **containing** a vertex.

Now to the problem:
If $D\in KM\cap LN$, we can easily conclude from $Luis$' proof above that $D\in\odot(ABC)$, the required tangent is parallel to KL and passes through I_D. Since $KL\parallel BC$, we are done.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Coaxal Circles

Reply

Source: Turkey TST 2001 - P2

xeroxia

#1 Apr 4, 2013, 10:58 pm

A circle touches to diameter AB of a unit circle with center O at T where $OT > 1$. These circles intersect at two different points C and D. The circle through O, D, and C meet the line AB at P different from O. Show that

$$|PA| \cdot |PB| = \frac{|PT|^2}{|OT|^2}.$$

This post has been edited 1 time. Last edited by xeroxia, Jan 16, 2014, 2:13 pm

xeroxia

#2 Apr 4, 2013, 11:03 pm

Let perpendicular from P to OA meet the unit circle at S. Let ST meet the unit circle at U.
In fact, it is

$$\frac{PA \cdot PB}{AO^2} = \frac{PT^2}{OT^2} \Rightarrow \frac{SP}{AO} = \frac{PT}{OT} \Rightarrow SP \parallel OU.$$

See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=528123 for the rest.

thecmd999

#3 Jan 15, 2014, 5:37 am

Computational

Luis González

#4 Jan 15, 2014, 6:32 am • 1

Any line intersects a pencil of circles at pairs of points in involution, thus T is a double point of the involution $A \mapsto B, P \mapsto O \Longrightarrow (P, O, A, T) = (O, P, B, T) \Longrightarrow$

$$\frac{\overline{PA}}{\overline{PT}} \cdot \frac{\overline{OT}}{\overline{OA}} = \frac{\overline{OB}}{\overline{OT}} \cdot \frac{\overline{PT}}{\overline{PB}} \Longrightarrow \frac{\overline{PT}^2}{\overline{OT}^2} = \frac{\overline{PA} \cdot \overline{PB}}{\overline{OA} \cdot \overline{OB}} = \overline{PA} \cdot \overline{PB}.$$

Quick Reply

High School Olympiads

R=r_a iff D is on OI

Reply

Lumpa

#1 Jan 14, 2014, 6:24 pm

Prove (synthetically if possible) that if D is the projection of A on BC and R and r_a are the circumradius and the A-excircle radius, then $R=r_a \Leftrightarrow D \in OI$

Luis González

#2 Jan 15, 2014, 3:34 am • 1

Circumcircle (O) and incenter I of $\triangle ABC$ become 9-point circle and orthocenter of its excentral triangle $\triangle I_aI_bI_c$. Thus, midpoint P of the arc BC of (O) is also midpoint of $\overline{II_a}$. From $AD \parallel OP$, we deduce that I, O, D are collinear $\Longleftrightarrow$

$$\frac{OP}{AD}=\frac{IP}{IA} \Longleftrightarrow \frac{2R}{h_a}=\frac{II_a}{IA}=\frac{s}{s-a}-1=\frac{a}{s-a} \Longleftrightarrow R=\frac{a\cdot h_a}{2(s-a)}=r_a.$$

Quick Reply

High School Olympiads

Nagel's point

Reply

kazakhboy

#1 Jan 13, 2014, 11:45 pm

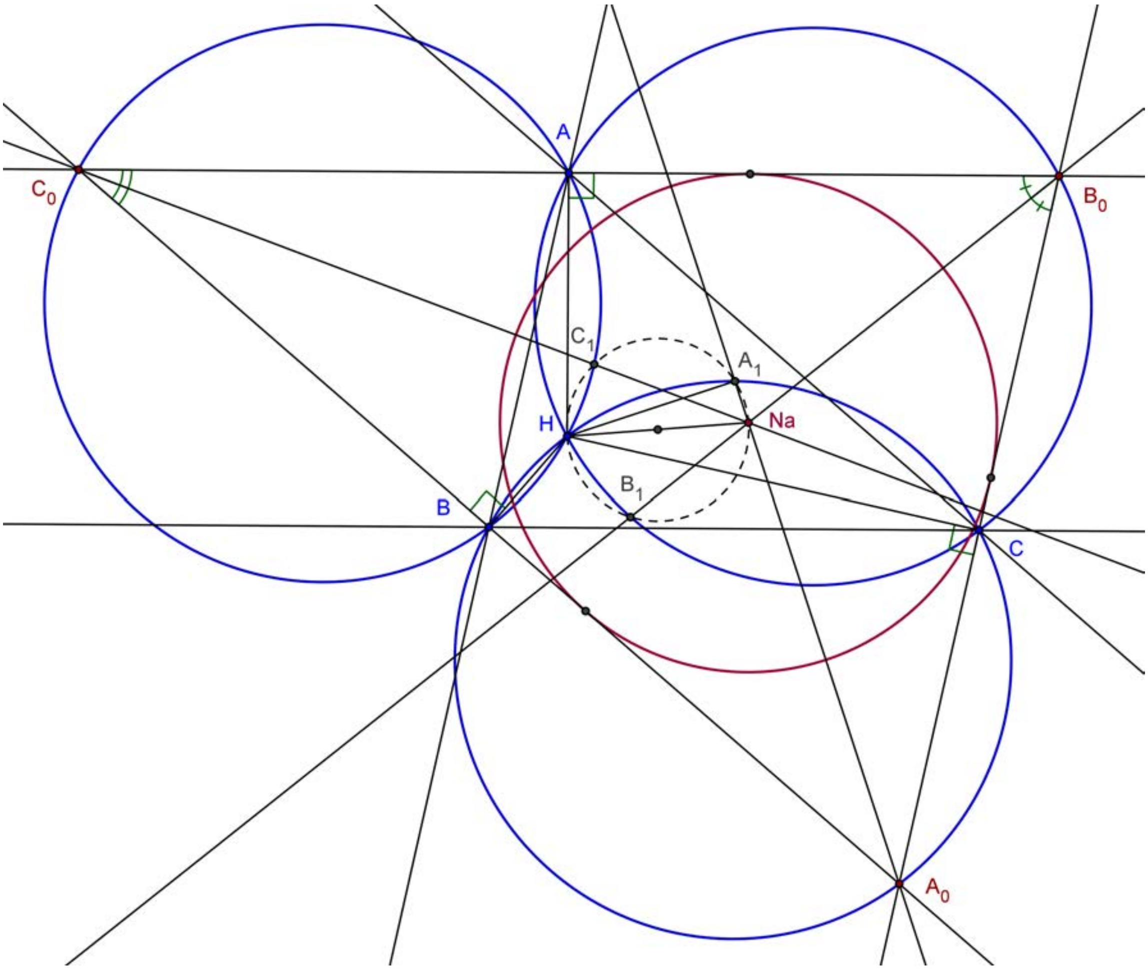
Let ABC acute-angle triangle with orthocentre H. The points A_1, B_1 and C_1 are the middles of arcs BHC,CHA and AHB of the circumcircles of the triangles BHC,AHC and AHB respectively. Prove that the circumcircle of the triangle $A_1B_1C_1$ contains the Nagel's point of the triangle ABC.

Luis González

#2 Jan 14, 2014, 2:50 am

See the diagram for a proof without words.

Attachments:



jayme

#3 Jan 14, 2014, 10:44 am

Dear Mathlinkers,
this is the answer that I was waiting for
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=50&t=570012
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

A triangle and four circle problem

Reply

daothanhoai
#1 Jan 11, 2014, 7:18 am

Dear Mathlinkers

Let ABC be a triangle, and any circle, construct three circle tangent two sides and tangent circle at A_1, B_1, C_1. Prove that AA_1, BB_1, CC_1 are concurrent

Best regards
Sincerely

Luis González
#2 Jan 11, 2014, 9:49 am

This is very figure-dependent. One circle and two distinct lines determine at most eight circles tangent to them, so it's important to give exact circles, otherwise the concurrency is not happening. For instance, assume that circle ω meets segments $\overline{BC}, \overline{CA}, \overline{AB}$. A circle ω_A touches rays AB, AC and touches ω externally at A_1. Circles ω_B and ω_C are defined similarly.

Let (I) be the incircle of $\triangle ABC$. A is the exsimilicenter of $(I) \sim \omega_A$ and A_1 is the insimilicenter of $\omega \sim \omega_A$. Thus, by Monge & d'Alembert theorem AA_1 passes through the insimilicenter S of $(I) \sim \omega$. Similarly, BB_1, CC_1 pass though S.

daothanhoai
#3 Jan 11, 2014, 11:48 am

Thank to Mr Luis González and Mathlinkers

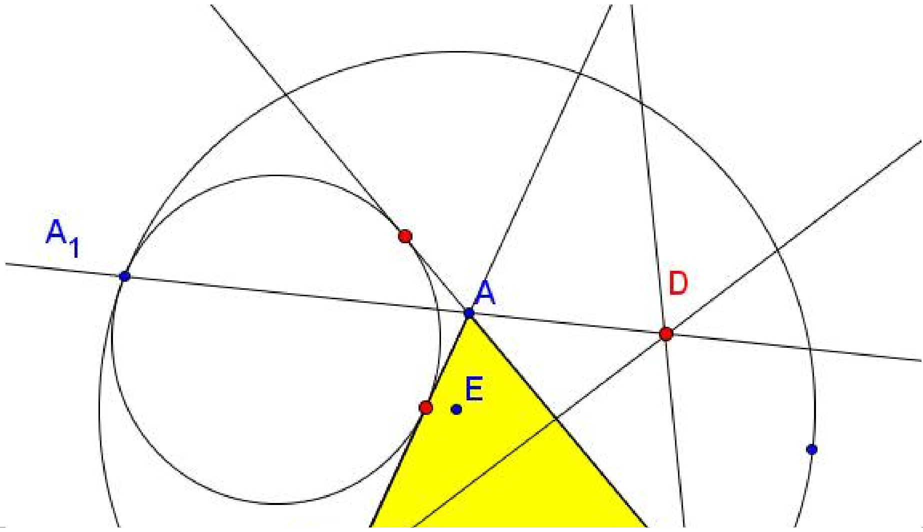
Please see the figure:

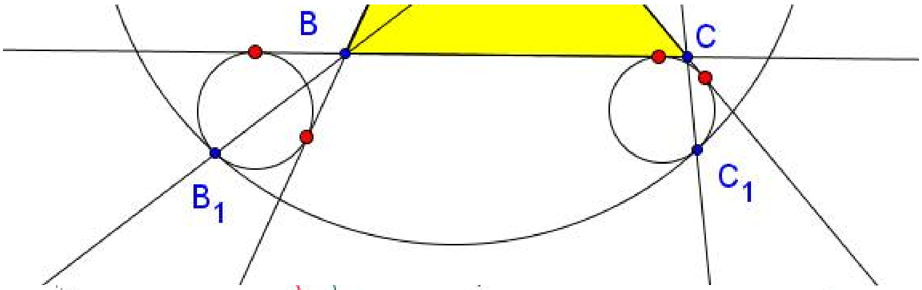
http://www.geogebratube.org/student/m68574

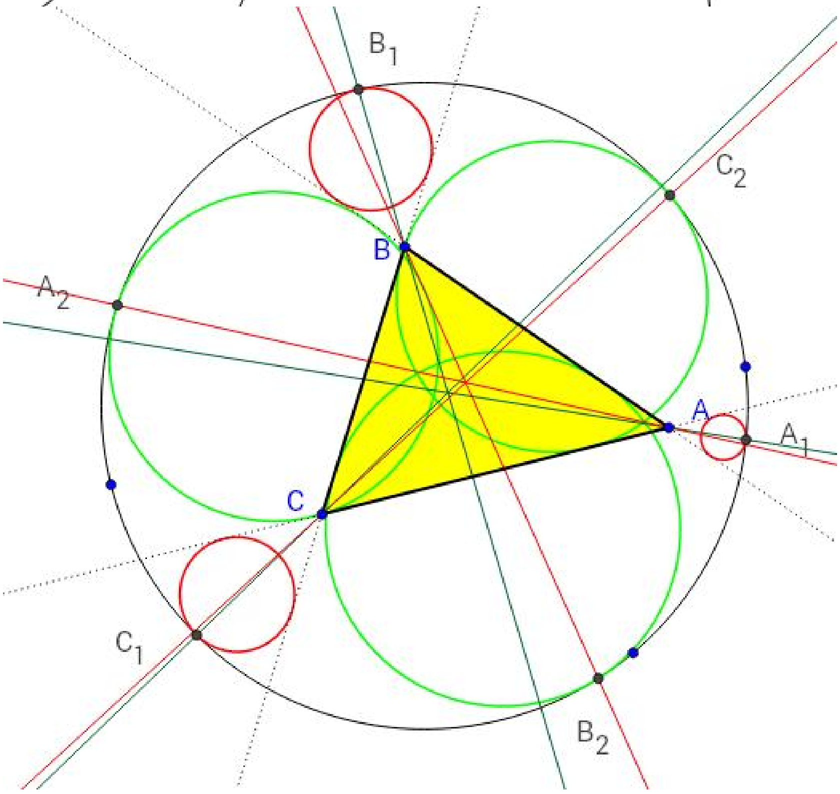
http://www.geogebratube.org/student/m68576

Best regards
Sincerely
Dao Thanh Oai

Attachments:







Luis González

#4 Jan 14, 2014, 2:04 am • 1

Ok, enough of these "who first discovered it" arguments that are polluting the threads. It seems that armpist just cannot let it go.

> *armpist wrote:*
>
> Why did you delete my post about Stanley Rabinowitz's paper
> in FG on exactly the same subject?
>
> Did you learn to do it in your electrical engineering classes in VN ?

I hope this is a kind of pestichian sarcasm, because I can hardly believe that you, having nearly 10 years in this forum, still have no idea how all this works. Only moderators and administrators have the ability to edit/delete any message.

For now on, further messages regarding this discussion or containing air of personal attacks will be removed.

Quick Reply

High School Olympiads

nice problem !

Reply

kazi

#1 Jan 9, 2014, 8:56 pm

Let ABC be a triangle and point K,L on AB,BC that AK=KL=LC. AL and KC meet each other on P. Draw a parallel line with bisector angle ABC from P that intersects AB on M. Now prove that AM=BC.

Luis González

#2 Jan 10, 2014, 10:17 am • 1

We only need $|AK|=|LC|$. If D is the point such that $ABCD$ is a parallelogram, it suffices to prove that P is on the internal bisector of $\angle CDA$.

When P varies, $K\mapsto L$ is clearly a projectivity (the series K and L are congruent) $\Longrightarrow$ pencils AL and CK are projective with double ray AC when $|AK|=|CL|=0 \Longrightarrow P$ describes a line. Intersections M,N of internal bisector of $\angle CDA$ with AB,BC verify $|AM|=|AD|=|CB|$ and $|CN|=|CD|=|AB| \Longrightarrow P\in DMN$.

kazi

#3 Jan 12, 2014, 1:08 am

thank you luis
I realy need an elementary solution.
any another solution?

Luis González

#4 Jan 12, 2014, 1:22 am • 1

Let Q be a point such that $APCQ$ is a parallelogram. Since $AL \parallel QC,$ we have $[QCL]=[QCA]$ and similarly $CK \parallel QA$ gives $[QAK]=[QCA] \Longrightarrow [QCL]=[QAK] \Longrightarrow \tfrac{1}{2}\text{dist}(Q,BC)\cdot |LC|=\tfrac{1}{2}\text{dist}(Q,BA)\cdot |AK| \Longrightarrow$ $\text{dist}(Q,BC)=\text{dist}(Q,BA)$, i.e. Q is equidistant from $BA,BC \Longrightarrow Q$ is on internal bisector of $\angle ABC \Longrightarrow$ reflection P of Q about midpoint of AC is then on internal bisector of $\angle CDA$.

kazi

#5 Jan 12, 2014, 2:06 pm

thank you luis! you are too useful in this forum ! in fact im new in geometry and I didnt think about using area . thank you again! you teach me the bests!

Quick Reply

High School Olympiads

Area calculating

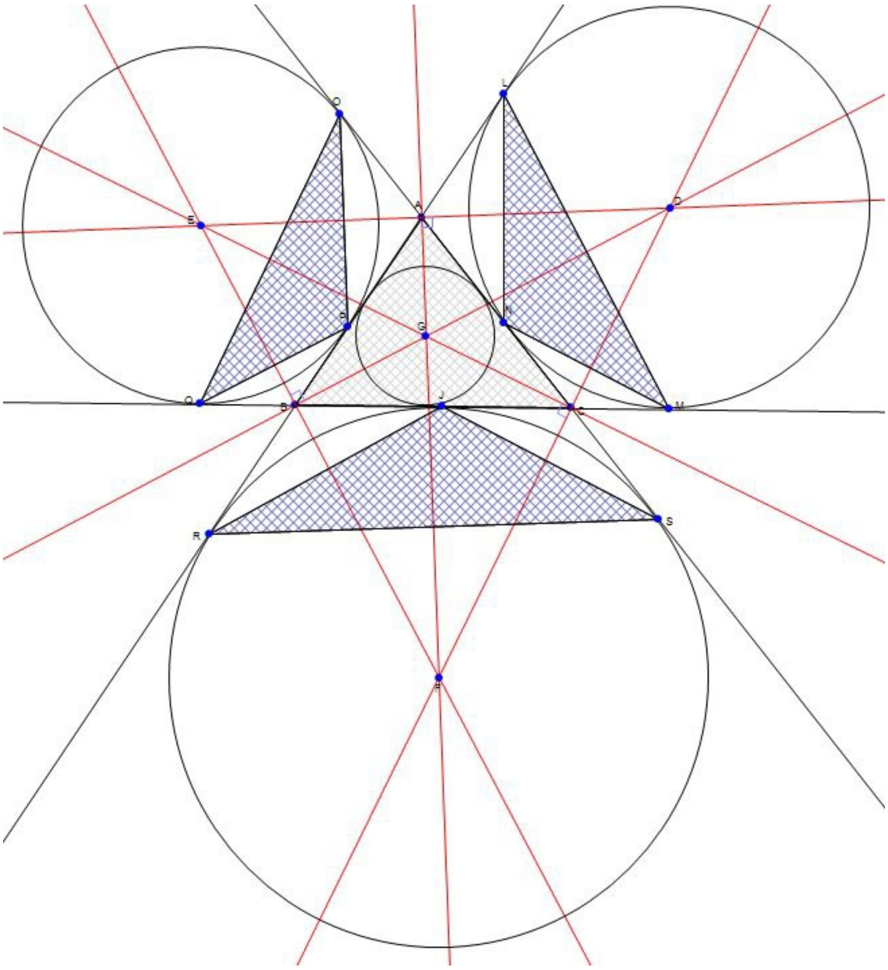
Reply

kazi

#1 Jan 9, 2014, 6:19 pm

red lines are bisectors of angles A ,b,c. prove the sum of the areas triangles that make with the intersection of the big circles with AB ,AC,BC(i mean blue triangles!) is equal with 2*(the area of ABC) + the area of the triangle that make with intersection of the incircle with AB ,AC,BC .

Attachments:



Luis González

#2 Jan 11, 2014, 11:12 pm • 3

Let (O,R) denote the circumcircle of $\triangle ABC$ and $(I,r),(I_A,r_A),(I_B,r_B),(I_C,r_C)$ its incircle and 3 excircles. Let S denote the area of $\triangle ABC$ and $S_I,S_{I_A},S_{I_B},S_{I_C}$ the areas of the pedal triangles of I,I_A,I_B,I_C WRT $\triangle ABC$, i.e its intouch and extouch triangles. By Euler's theorem we have

$$\frac{S_I}{S}=\frac{|p(I,(O))|}{4R^2}=\frac{R^2-IO^2}{4R^2}=\frac{R^2-(R^2-2Rr)}{4R^2}=\frac{r}{2R}.$$

Similarly, we find $\frac{S_{I_A}}{S}=\frac{r_A}{2R}$, $\frac{S_{I_B}}{S}=\frac{r_B}{2R}$, $\frac{S_{I_C}}{S}=\frac{r_C}{2R} \Longrightarrow$

$$\frac{S_{I_A}}{S}+\frac{S_{I_B}}{S}+\frac{S_{I_C}}{S}-\frac{S_I}{S}-2=\frac{r_A+r_B+r_C-r-4R}{4R}.$$

But $r_A+r_B+r_C=r+4R$ (well-known) $\Longrightarrow S_{I_A}+S_{I_B}+S_{I_C}=S_I+2S.$

Quick Reply

High School Olympiads

Prove that F,P,O are collinear

Source: Extension of All Russian 2012

PhantomLancer
#1 Jan 11, 2014, 7:33 pm

Let P be a point in angle bisector of A of ΔABC. C', B' are the reflections of P on AB, AC respectively. H is orthocenter of ΔABC and E the midpoint of AH. F is the reflection of E WRT $B'C'$. O is the circumcenter of ΔABC. Prove that O, P, F are collinear.

This extension was mentioned in the old topic of All Russian 2012. Can anyone post a solution? Thanks very much.

mathdebam
#2 Jan 11, 2014, 8:55 pm • 1

Please pardon me if I am wrong, but are u sure that the problem is al right ?
because I checked in geogebra and it seemed wrong. Plzz pardon me if I m wrong once again.

PhantomLancer
#3 Jan 11, 2014, 9:53 pm

Hi mathdebam,

Please check again if I had any typo. Thank you.
Image not found

Luis González
#4 Jan 11, 2014, 10:06 pm • 1

> *PhantomLancer wrote:*
> Let P be a point in angle bisector of A of ΔABC. C', B' are the reflections of P on AB, AC respectively. H is orthocenter of ΔABC and E the midpoint of AH. F is the reflection of E WRT $B'C'$. O is the circumcenter of ΔABC. Prove that O, P, F are collinear.

mathdebam is right, because your proposition specifies that C',B' are reflections of P on AB,AC, when they have to be projections instead. See the topic hard question!! (post #2).

PhantomLancer
#5 Jan 11, 2014, 10:12 pm

Thank you very much. I heard there was a simple computation solution. Never thought it would be such a tough problem.

High School Olympiads

Mabey Thebault have ovelooked

Reply

Source: Own?? I found when I construct Thebault circle

daothanhoai

#1 Jan 11, 2014, 7:16 am • 1

Dear Mathlinkers

Mabe Thebault have ovelooked (I don't know who discovered before)

Let ABC be a triangle, any P on the plane. Construct a circle tangent CP, AB circumcircle at C_A; Construct another circle tangent CP, AB circumcircle at C_B. Prove that C_AC_B through fixed point, when P moved

http://www.geogebratube.org/student/m68511

Best regards
Sincerely
Dao Thanh Oai

Luis González

#2 Jan 11, 2014, 12:32 pm • 4

Label $(J_A), (J_B)$ the Thebault circles of the cevian CP. (J_A) touches $\overline{AB}$ at X and the circumcircle (O) at C_A while (J_B) touches $\overline{AB}$ at Y and the circumcircle at C_B. From internal tangencies of $(O), (J_A)$ and $(O), (J_B)$, we deduce that C_AX, C_BY bisect $\angle AC_AB, \angle AC_BB \Longrightarrow M \equiv XC_A \cap YC_B$ is midpoint of the arc AB of (O).

By Sawayama's lemma, the incenter I of $\triangle ABC$ verifies $IX \perp IY \Longrightarrow$ pencils IX, IY are in involution $\Longrightarrow$ pencils $MX \equiv MC_A, MY \equiv MC_B$ are in involution $\Longrightarrow C_A \mapsto C_B$ is an involutive homography on $(O) \Longrightarrow$ all lines C_AC_B pass through the fixed pole of the involution. Making $A \equiv Y$ and $B \equiv X$, we figure out that the fixed point is the exsimilicenter X_{56} of $(I) \sim (O)$.

TelvCohl

#3 Oct 24, 2014, 11:30 pm • 4

My solution:

Let $D = CP \cap AB$ and O_1, O_2 be the center of these two Thebault circles .
Let O, I be the circumcenter, incenter of $\triangle ABC$ and $S = O_1O_2 \cap AB$.

By **Thebault theorem** we get O_1, I, O_2 are collinear,
so the exsimicenter of $((I), (O_1))$ and the exsimicenter of $((O_1), (O_2))$ is the same point S .
By D'Alembert theorem (for $((I), (O), (O_1))$) we get $S, C_A, X(56)$ are collinear . ... (1)
By D'Alembert theorem (for $((I), (O), (O_2))$) we get $S, C_B, X(56)$ are collinear . ... (2)
From (1) and (2) we get C_AC_B pass through a fixed point $X(56)$.

Q.E.D

buratinogigle

#4 Oct 25, 2014, 9:31 pm • 1

Circle (PC_AC_B) passes through two fix points, also.

TelvCohl

TelvCohl

#5 Oct 25, 2014, 11:53 pm • 2

> buratinogigle wrote:
> Circle (PC_AC_B) passes through two fix points, also.

I think you mean $\odot(DC_AC_B)$:)

My solution:

Let H be the projection of I on AB .
Let X, Y be the tangent point of $\odot(O_1), \odot(O_2)$ with BC and D' be the projection of D on O_1O_2 .

It's well known that C_A, C_B, X, Y are concyclic.
Since $\angle XIY = \angle O_2DO_1 = 90$ and $DO_1 \parallel IY, DO_2 \parallel IX$,
so $\angle YD'O_2 = \angle YDO_2 = \angle YXI$ (i.e. I, D', X, Y are concyclic) ,
hence we get $SC_A \cdot SC_B = SX \cdot SY = SI \cdot SD' = SH \cdot SD$. i.e. C_A, C_B, H, D are concyclic

From the original problem we get C_A, C_B, X_{56} are collinear ,
so $\odot(DC_AC_B)$ from an elliptic coaxial system with common radical axis $HX(56)$.
i.e. $\odot(DC_AC_B)$ pass through two fixed points when D varies on AB

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Apr 19, 2015, 1:38 pm

buratinogigle

#6 Oct 26, 2014, 11:02 am

Thank you for understand me :)!

Quick Reply

High School Olympiads

Fermat point and concurrent point

Reply

Source: Own??

daothanhoai

#1 Dec 22, 2013, 10:52 pm

Dear Mathlinkers

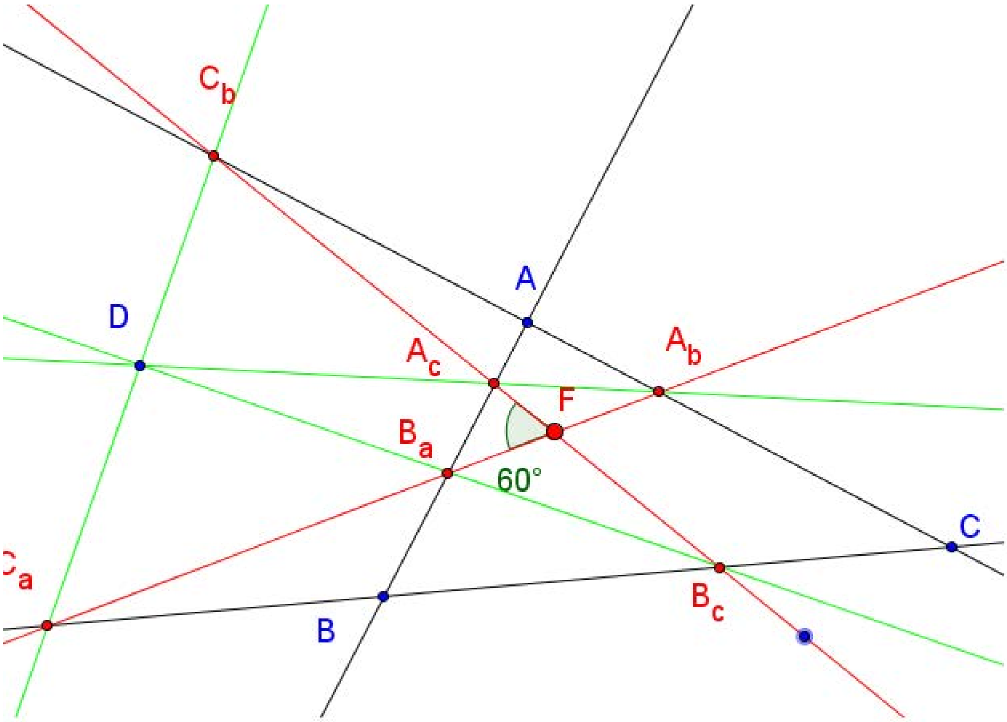
Let ABC be a triangle, Let F is the first or the secon Fermat point. Construc two line through F and angle of two lie is 60 degrees. Two lines meet three sideline at $A_b, A_c, B_c, B_a, C_a, C_b$ (show in the figure attachment). Prove that A_aA_b, B_aB_c, C_aC_b are concurrent.

Best regards

Sincerely

Dao Thanh Oai

Attachments:



Luis González

#2 Jan 11, 2014, 8:20 am • 1

We also show that the concurrency point D of the referred lines is on the trilinear polar of F WRT ABC.

Let $\triangle F_aF_bF_c$ be the cevian triangle of F WRT $\triangle ABC. X \equiv F_bF_c \cap BC$ and $Y \equiv F_cF_a \cap CA \Longrightarrow XY \equiv f$ is trilinear polar of F WRT $\triangle ABC,$ i.e. perspectrix of $\triangle ABC$ and $\triangle F_aF_bF_c.$

We use polarity WRT an abitrary circle (F) centered at $F.$ Polars of A,B,C bound an equilateral triangle since they are perpendicular to $FA,FB,FC,$ i.e. poles P,Q,R of BC,CA,AB are vertices of equilateral triangle. Polars of F_b,F_c are then the parallels through Q,R to $PR,PQ,$ respectively, meeting at the pole U of $F_bF_c.$ If V,W are the poles of $F_cF_a, F_aF_b,$ then $\triangle UVW$ is also equilateral, being the antimedial of $\triangle PQR.$ PU and QV are polars of X,Y meeting at the pole $O \equiv PU \cap QV$ of $f \equiv XY;$ the center of $\triangle PQR.$ Polars of A_b, B_a are two parallels through Q,R and polars of B_c, A_c

are two parallels through P, Q (these directions forming 120°, due to $\angle(A_bB_a, A_cB_c) = 60^\circ$) $\Longrightarrow$ poles M, L, K of A_bB_c, B_cB_a, A_cA_b form parallelogram $RKML$ circumscribed to $\triangle PQR$.

Since $\angle PLR = \angle POR = \angle ROQ = \angle RKQ = 60^\circ \pmod \pi$, then clearly the circles $\odot(PLR)$ and $\odot(RKQ)$ meet at R and O, which is then midpoint of their arcs PR, RQ. If LO cuts $\odot(RKQ)$ again at K', we have $\angle QK'O = 30^\circ = \angle RLO \Longrightarrow QK' \parallel RL \Longrightarrow K \equiv K' \Longrightarrow O, K, L$ are collinear $\Longrightarrow$ their polars f, A_bA_c, B_aB_c concur at D. By similar reasoning C_aC_b goes through $D \equiv f \cap A_bA_c \cap B_aB_c$.

Quick Reply

High School Olympiads

Find the locus of P

Reply

xiaotabuta

#1 Jan 10, 2014, 11:08 pm

Give an equilateral triangle ABC. Find the locus of the point P inside the triangle such that
$\angle PAB + \angle PBC + \angle PCA = 90^\circ$

Luis González

#2 Jan 11, 2014, 12:51 am

Posted several times before. Locus of P is the union of the three altitudes of ABC. More general, for an scalene triangle, the locus is the Mcay cubic of ABC (see the last link).

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=123822
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=141887
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=409326

Quick Reply

High School Olympiads

my problem13!!

Reply

Source: (O_o)

golmakani

#1 Jan 10, 2014, 1:59 am

let ABC be a triangle and w be the circumcircle and O be the circumcenter of that.
let M be the midpoint of BC and X be a point in w such $\widehat{XAC}=\widehat{MAB}$.
AX cross BC in Y.
if AO cross w in P and BP cross AX in Q and CP cross AM in R so prove that:
$PQRX$ is cyclic quadrilateral.

Luis González

#2 Jan 10, 2014, 8:11 am • 1

Let D,E be the reflections of A on $C,B.$ Let $K \equiv BP \cap AD$ and $S \equiv BQ \cap AM.$ According to Interesting geometry $X' \equiv ES \cap DR$ is on A-symmedian AY of $\triangle ADE.$

PQ and PR bisect $\angle RSX'$ and $\angle SRX'$ externally $\Longrightarrow P$ is X'-excenter of $\triangle X'RS \Longrightarrow X'P$ bisects $\angle DX'E$ externally $\Longrightarrow P \in \odot(DEX')$ is then midpoint of the arc $DX'E.$ But from K-isosceles $\triangle AEK,$ we have $\angle EKD=2\angle EAD=\angle EPD \Longrightarrow K \in \odot(PED).$ Therefore $\angle BPX'=\angle KEX'=\angle KDR=\angle CAM=\angle BAX' \Longrightarrow X' \in \odot(ABC) \Longrightarrow X \equiv X' \Longrightarrow Q$ is then R-excenter of $\triangle XRS \Longrightarrow R,X$ lie on circle with diameter $\overline{PQ}.$

Quick Reply

High School Olympiads

A Straightfoward Spacial Geometry Problem

Reply

Source: Problem 6, Brazilian MO, 1993

Davi Medeiros

#1 Jan 9, 2014, 5:13 pm

Let $P_1P_2\dots P_n$ a polygon inscribed on a circunference and contained in a plane α. Let Q be a point outside α. Consider, for each $i=1,2,\dots,n$, the plane β_i passing through P_i and perpendicular to QP_i. Prove that all the planes β_i intersect at one point.

vanstraelen

#2 Jan 10, 2014, 1:49 am

Let $P_i(r\cos t, r\sin t, 0)$ a point of the polygon in plane $\alpha=(x,y)$, parameter t.
Let $Q(a,b,c)$ with $c\neq 0$.
Numbers indicating the slope of $PQ:(r\cos t-a, r\sin t-b, -c)$.
Equation of the plane $\beta_i:(r\cos t-a)x+(r\sin t-b)y-cz=k$.
Calculation of k: point $P_i\in\beta_i$

$$(r\cos t-a)r\cos t+(r\sin t-b)r\sin t=k$$

$$k=r^2-ar\cos t-br\sin t$$

Equation of the plane β_i:

$$(r\cos t-a)x+(r\sin t-b)y-cz=r^2-ar\cos t-br\sin t$$

Rearranging:

$$r\cos t(x+a)+r\sin t(y+b)-ax-by-cz-r^2=0$$

This equation is independent of the parameter t if

$$x+a=0 \qquad y+b=0 \qquad -ax-by-cz-r^2=0$$

Solving, one point $W(-a,-b,\dfrac{a^2+b^2-r^2}{c})$.

Luis González

#3 Jan 10, 2014, 3:27 am • 1

Label ω the given circle on the plane α and P varies on it. Inversion with center Q and arbitrary power takes α into a spherical surface $\mathcal{S}$ through Q and the spherical surface containing Q,ω into a plane $\gamma \Longrightarrow$ inverse ω' of ω is a circle intersection of $\mathcal{S}\cap\gamma$. Thus the plane β orthogonal to QP at P goes to the spherical surface $\mathcal{O}$ with diameter $\overline{QP'}$, where $P'\in\omega'$ is the inverse of P. Since $\mathcal{O}$ can be obtained by revolving a semicircle about $\overline{QP'}$, then it always passes through the projection R of Q on γ, clearly fixed $\Longrightarrow$ all β go through the inverse of R under the referred inversion.

Quick Reply

High School Olympiads

Concurrent

Reply

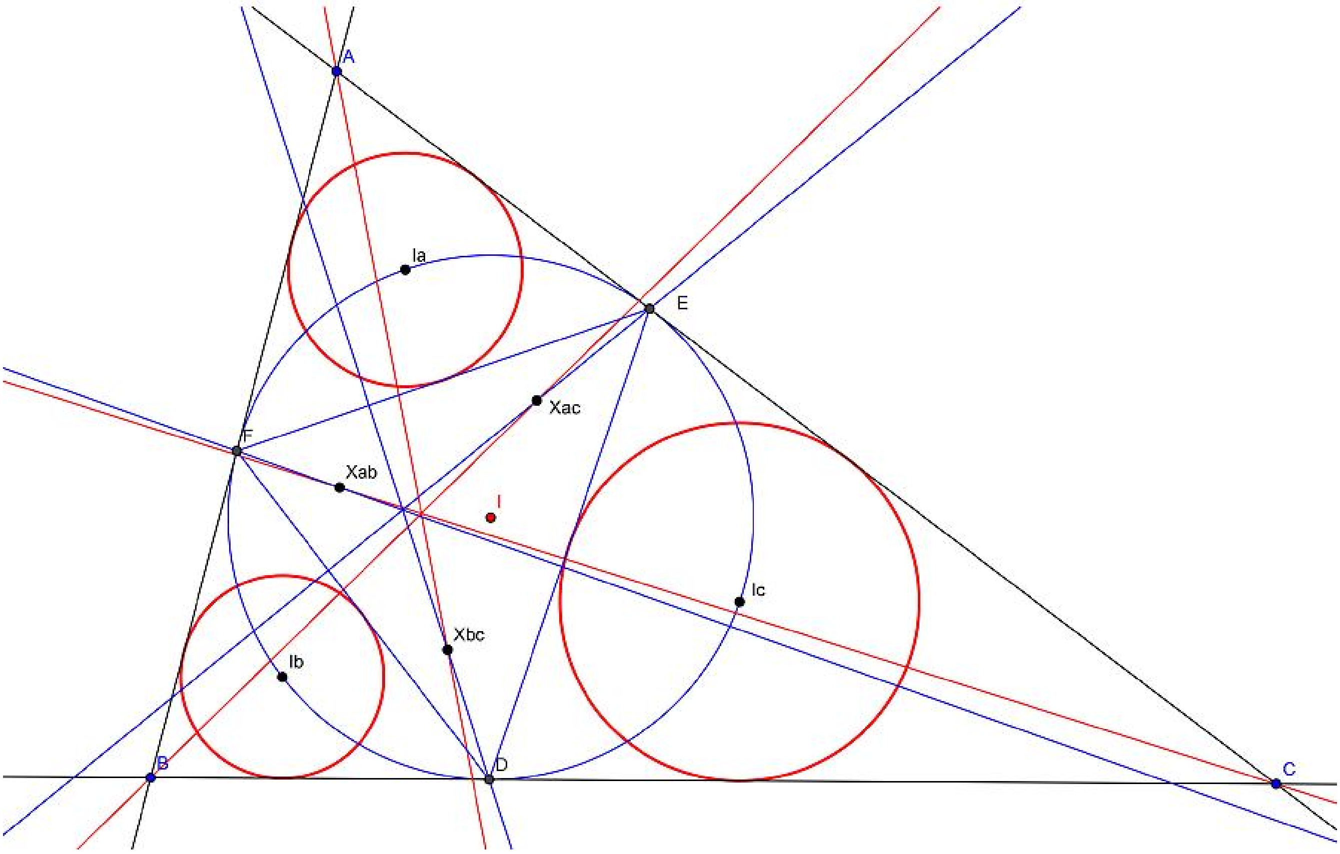
Source: myself

a00012025

#1 Jan 7, 2014, 10:30 pm

Let ABC be a triangle with incenter I, and the incircle of ABC intersects BC, CA, AB at D, E, F. Let the incenters of triangle AEF, BDF, CDE be Ia, Ib, Ic, and let interior common tangents of Ia, Ib intersect at Xab, and Xbc, Xac similiarly. Prove that:
(1) $AXbc, BXac, CXab$ are concurrent.
(2) $DXbc, EXac, FXab$ are concurrent.

Attachments:



Luis González

#2 Jan 9, 2014, 4:57 am • 1

Notation is unnecesarily nasty. Rename $X \equiv Xbc, Y \equiv Xca$ and $Z \equiv Xab.$

(1) Y is insimilicenter of $(I_c) \sim (I_a)$ and Z is insimilicenter $(I_a) \sim (I_b)$, hence by Monge & d'Alembert theorem YZ cuts BC at the exsimilicenter A_0 of $(I_b) \sim (I_c)$. Similarly, ZX and XY cut CA and AB at the eximilicenters B_0 and C_0 of $(I_c) \sim (I_a)$ and $(I_a) \sim (I_b)$. A_0, B_0, C_0 are collinear on a homothety axis of $(I_a), (I_b), (I_c)$, i.e. $\triangle ABC$ and $\triangle I_aI_bI_c$ are perspective through $\overline{A_0B_0C_0}$. By Desargues theorem AX, BY and CZ concur.

(2) Trivial angle chase reveals that I_a, I_b, I_c are the midpoints of the arcs EF, FD, DE of (I). Since $(I_b, I_c, X, A_0) = -1$, then it follows that DX is the polar of A_0 WRT (I). Similarly, EY and FZ are the polars of B_0 and C_0 WRT $(I) \Longrightarrow$ DX, EY, FZ concur at the pole of $\overline{A_0B_0C_0}$ WRT (I).

Quick Reply

High School Olympiads

Prove that DAB=Ia'AC

Reply

a00012025

#1 Jan 8, 2014, 10:45 pm

Let ABC be a triangle with incenter I and A-excenter I_a. Let D be reflection of I over BC, and I'_a be on segment OI_a such that $OI_a \times OI'_a = OA^2$. Prove that $\angle DAB = \angle I'_aAC$

Luis González

#2 Jan 8, 2014, 11:40 pm • 2

Let H be the orthocenter of $\triangle ABC$ and let the incircle (I) touch BC, CA, AB at $X, Y, Z. T, U$ denote the orthocenter and 9-point center of $\triangle XYZ.$ Midpoint M of $\overline{IA}$ is circumcenter of $\triangle IYZ \Longrightarrow XM$ is the X-cevian of the Kosnita point of $\triangle XYZ,$ isogonal conjugate of its 9-point center U (well-known) $\Longrightarrow XU, XM$ and XI, XT are pairs of isogonals WRT $\triangle XYZ \Longrightarrow \angle TXU = \angle IXM = \angle IDA = \angle HAD.$

$\triangle XYZ$ with 9-point center U and orthocenter T is similar to $\triangle I_aI_bI_c$ with 9-point center O and orthocenter $I \Longrightarrow \angle TXU = \angle II_aO.$ But $OA^2 = OI_a \cdot OI_a{}'$ gives $\angle II_aO = \angle OAI_a{}' \Longrightarrow \angle OAI_a{}' = \angle TXU = \angle HAD \Longrightarrow AD, AI_a{}'$ are isogonals WRT $\angle OAH,$ but AO, AH are isogonals WRT $\angle BAC \Longrightarrow AD, AI_a{}'$ are isogonals WRT $\angle BAC,$ or $\angle DAB = \angle I_a{}'AC.$

a00012025

#3 Jan 12, 2014, 8:39 pm

> Luis González wrote:
> $\Longrightarrow XM$ is the X-cevian of the Kosnita point of $\triangle XYZ,$ isogonal conjugate of its 9-point center U (well-known)

How to prove that the Kosnita point is isogonal conjugate of nine-point center? Thanks a lot.

jayme

#4 Jan 12, 2014, 8:45 pm • 1

Dear Mathlinkers,
for this result, you can see
http://perso.orange.fr/jl.ayme vol. 1 Le point de Kosnitza
Sincerely
Jean-Louis

Arab

#5 Jan 12, 2014, 8:50 pm • 1

See here also.

TelvCohl

#6 Oct 25, 2014, 1:29 am

Let Φ be the inversion $\mathbf{I}(A, \sqrt{AB \cdot AC})$ followed by reflection in AI. Since $B, I \stackrel{\Phi}{\longleftrightarrow} C, I_a$, so we get $BC \stackrel{\Phi}{\longleftrightarrow} \odot(ABC)$ and $I'_a \stackrel{\Phi}{\longleftrightarrow} D \Longrightarrow AD$ and AI'_a are isogonal conjugate WRT $\angle BAC$

This post has been edited 1 time. Last edited by TelvCohl, Dec 8, 2015, 11:02 pm

Quick Reply

High School Olympiads

They intersect on nine point circle! (hard with many lines!)

Reply

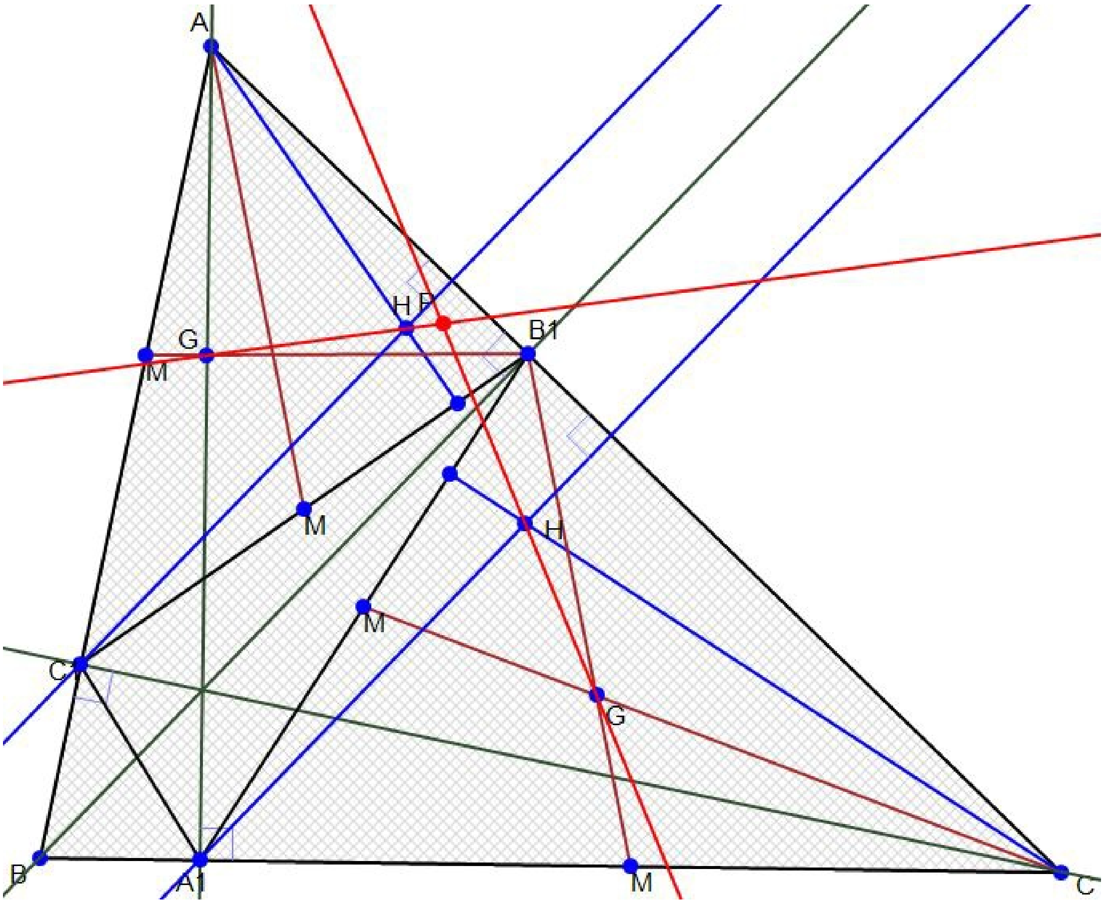
Source: famouse but I dont remember ...

kazi

#1 Jan 7, 2014, 8:10 pm

Let ABC be a triangle with the heights AA1 and BB1 and CC1. Now prove:
1) The euler lines AC1B1 and CA1B1 and BC1A1 meet each other on one point same as P on the nine point circle of ABC.
2) Prove one of the PA1 or PB1 or PC1 is the sum of two others.
The picture will be uploaded soon.

Attachments:



IDMasterz

#2 Jan 7, 2014, 8:26 pm

Well-known, posted a lot of times I think. Use spiral similarity.

kazi

#3 Jan 8, 2014, 8:13 pm

I need a help for part B please
I solved part A .

Luis González

#4 Jan 8, 2014, 10:12 pm • 1

Certainly, posted many times before. (1) Euler lines of AB1C1, BC1A1 and CA1B1 concur at the Feuerbach point of the antimedial triangle of A1B1C1, lying on its circumcircle. Hence (2) follows from Feuerbach point property.

(1) http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=370155
(2) http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=24959

Quick Reply

High School Olympiads

Geometric Con...

Reply

S.E.Louridas

#1 Jan 7, 2014, 11:37 pm

An $\angle xOy$ is ginen and given also an interior point P of $\angle xOy$.
Construct (*) a line segment AB with $A \in Ox, \quad B \in Oy, \quad P \in AB,$ such that the Area of the triangle OAB is k^2, when k is a given line segment.

(*) Geometric Constaction is better.

Luis González

#2 Jan 8, 2014, 6:15 am

Since $[OAB]=\frac{1}{2}OA\cdot OB\cdot \sin \widehat{AOB},$ then the problem boilds down to the construction of the line ℓ through P cutting the rays Ox,Oy at points $A,B,$ such that $OA\cdot OB$ equals a known quantity $d^2.$

The map $\mathbf{P}_1$ sending A to the intersection B_1 of PA with Oy is clearly projective and the map $\mathbf{P}_2$ sending A to a point B_2 on $Oy,$ such that $\overline{OA}\cdot \overline{OB_2}=d^2$ is also projective, being the composition of the axial symmetry across the angle bisector of $\angle xOy$ followed by involution with center O and power $d^2.$ Hence the points B solution are nothing but the double points of the projectivity $\mathbf{P}_1 \circ \mathbf{P}_2.$ Of course, if $\mathbf{P}_1 \circ \mathbf{P}_2$ turns out to be elliptic there are no solutions.

Construction of double points is quite easy by transforming the series $B_1 \mapsto B_2$ into circular series on arbitrary circle $\omega.$ The projective axis of this projectivity cuts ω at the double points.

P.S. See also the problem Triangle area for alternate construction.

Quick Reply

High School Olympiads

Prove that IP//OH

Reply

a00012025
#1 Jan 7, 2014, 8:45 pm

Let ABC triangle with incenter I, circumcenter O and orthocenter H. Let D,E,F be reflection of I over BC,CA,AB.
Prove that:
(1) AD,BE,CF are concurrent.
(2) Let AD,BE,CF be concurrent at P, prove that $IP//OH$

nima-amini
#2 Jan 7, 2014, 9:11 pm

1- use inversion whit center i and

$$r^2$$

IMI-Mathboy
#3 Jan 7, 2014, 9:54 pm

1) The lines through D, E and F paralell to BC, AC and AB intersects with each other at C_1, A_1 and B_1. if AB and CB intersect B_1C_1 and A_1B_1 at M and N. then $MD=NF$ hence it is easy by chevi!

IDMasterz
#4 Jan 7, 2014, 10:08 pm

For 1, multiple ways. Kariya's theorem, Sondats.

For 2, let ℓ be the perpspectrix of DEF with ABC. By Sondats theorem, $IP\perp\ell$ so it suffices to prove $OH\perp\ell$. In fact, this means it is parallel to the orthic axis. Unfortunately, so far, I only have a ratio bash proof.

Luis González
#5 Jan 8, 2014, 2:45 am • 1

The concurrency is a particular case of Jacobi's theorem, which has been posted many times before. Particularly, these lines AD,BE,CF concur at the Gray's point of ABC. As for the parallelism, it holds for all points whose reflection triangle is perspective with ABC (all points on the Neuberg cubic of ABC).

(1) http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=404816
(2) http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=269121

jayme
#6 Jan 8, 2014, 9:43 am • 1

Dear Mathlinkers,
for a synthetic proof
http://perso.orange.fr/jl.ayme vol. 2 La droite de Gray
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Concurrent with tangent circles

Source: Own

buratinogigle

#1 Jan 7, 2014, 2:43 pm

Let ABC be a triangle with altitude AD,BF,CE. (N) is Euler circle. The circle (K_a) passing through B,C and touches (N) at X such that X is inside triangle ABC. Similarly we have points Y,Z. Prove that DX,EY,FZ are concurrent.

Luis González

#2 Jan 7, 2014, 11:23 pm • 1

BC,EF and the common tangent of $(N),(K_a)$ are pairwise radical axis of (N), the circle with diameter $\overline{BC}$ and (K_a), concurring at their radical center $R \equiv EF \cap BC$. Hence in the $\triangle XEF$, we have the relation

$$\frac{RE}{RF}=\frac{XE^2}{XF^2} \Longrightarrow \frac{XE}{XF}=\sqrt{\frac{RE}{RF}}=\sqrt{\frac{DE}{DF}}.$$

Last equality follows from the fact that $DR \equiv BC$ bisects $\angle EDF$. Thus, multiplying the cyclic expressions together gives

$$\frac{XE}{XF}\cdot\frac{YF}{YD}\cdot\frac{ZD}{ZE}=\sqrt{\frac{DE}{DF}}\cdot\sqrt{\frac{EF}{ED}}\cdot\sqrt{\frac{FD}{FE}}=1,$$

which means that the diagonals DX,EY,FZ of the cyclic hexagon $FXEZDY$ concur.

High School Olympiads

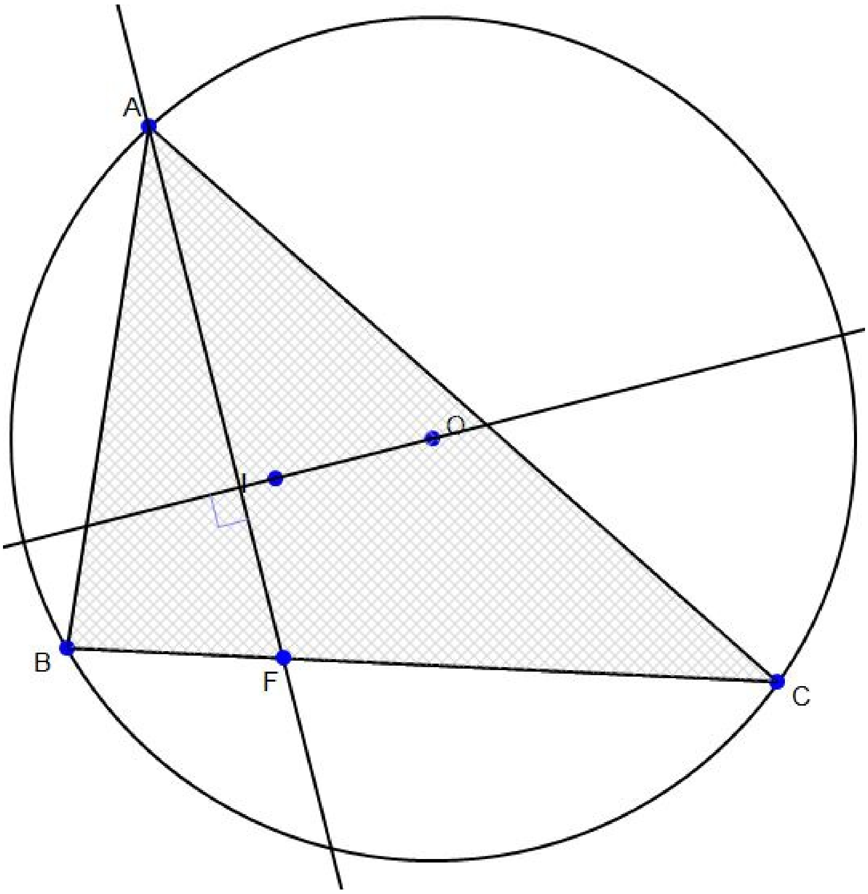
perpendicular to OI

Reply

kazi
#1 Jan 5, 2014, 5:23 pm • 3

Let ABC be a triangle with circumcenter O and incenter I. Draw a perpendicular line from A to OI that intersects BC on F. If the radius of the circumcircles ABF and ACF be K and M, prove that: |K-M|=|OI|
I hate these type of problems and I dont have any idea about them. Please give me a hint before writing your solution. Thank you

Attachments:



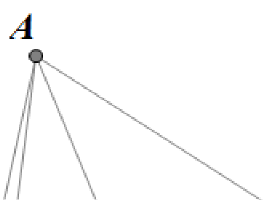
acupofmath
#2 Jan 6, 2014, 12:00 am • 1

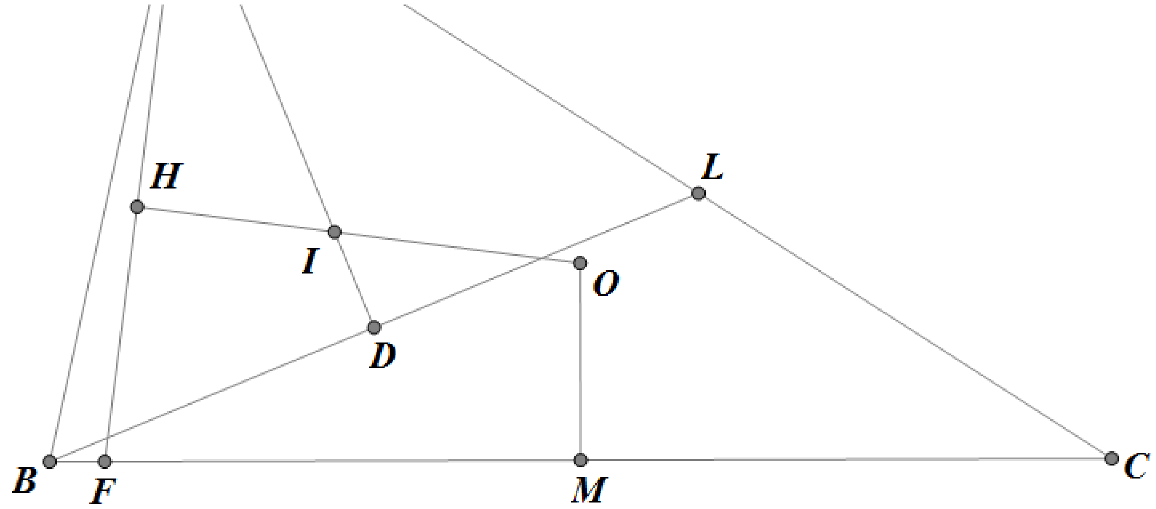
this picture is just a hint.

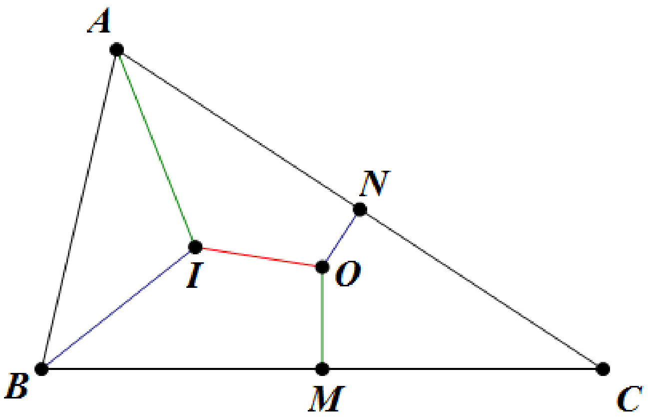
Click to reveal hidden text

Click to reveal hidden text

Attachments:

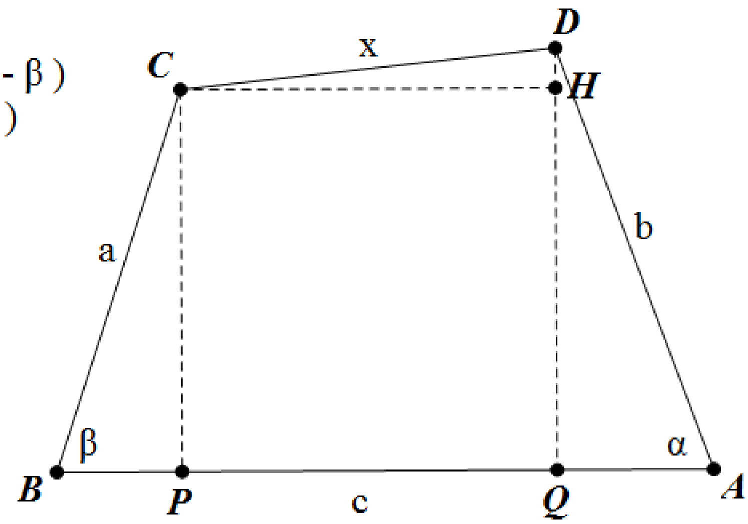






$$\text{show that : } OI = \frac{AC - AB}{2.\sin \angle IOM}$$

$$x^2 = CH^2 + DH^2$$
$$= a^2 + b^2 + c^2 + 2ab.\cos(\alpha - \beta)$$
$$- 2c(b.\cos\alpha + a.\cos\beta)$$



Luis González

#3 Jan 7, 2014, 12:51 pm • 5

WLOG assume that $AC > CB > BA$. Take points P, Q on rays $\overrightarrow{BA}, \overrightarrow{CA}$, respectively, such that $PB = BC = CQ$. From OI is perpendicular to MN and R'=OI, $PQ \perp OI \Longrightarrow AF \parallel PQ$ and the circumradius ϱ of $\triangle APQ$ equals OI.

Let PQ cut BC at R and let the parallel from F to BQ meet AC at D. By Pappus theorem for B, F, R and the infinite points of AB, BQ, QR, we deduce that $AB \parallel DR \Longrightarrow \frac{AD}{DC} = \frac{BR}{RC}$. But by Menelaus theorem for $\triangle ABC \cup \overline{PQR}$, using $PB = CQ$, gives $\frac{BR}{RC} = \frac{AQ}{AP} \Longrightarrow$

$$\frac{AD}{AQ} = \frac{DC}{AP} = \frac{FC}{AP} \Longrightarrow \frac{FC}{AP} = \frac{AQ + FB}{AQ} = 1 + \frac{FB}{AQ} \ (1).$$

For convenience denote ϱ_1, ϱ_2 the circumradii of $\triangle AFC, \triangle AFB$, respectively.

$$FC = 2\varrho_1 \cdot \sin \widehat{CAF} = 2\varrho_1 \cdot \sin \widehat{AQP} = 2\varrho_1 \cdot \frac{AP}{2\varrho} = \frac{\varrho_1}{\varrho} \cdot AP.$$

Similarly, we'll have $FB = \frac{\varrho_2}{\varrho} \cdot AQ \Longrightarrow$

$$\varrho_1 - \varrho_2 = \varrho\left(\frac{FC}{AP} - \frac{FB}{AQ}\right) = OI\left(\frac{FC}{AP} - \frac{FB}{AQ}\right) \ (2).$$

Combining (1) and (2) yields $\varrho_1 - \varrho_2 = OI$, as desired.

buratinogigle

#4 Jan 7, 2014, 10:23 pm

Very nice problem, here is a little extension

Let ABC be a triangle with circumcenter O, incenter I. A line which is perpendicular to OI, cuts BC, CA, AB at D, E, F, respectively. Let $(K, R_K), (L, R_L), (N, R_N)$ be circumcircles of triangles AEF, EDC, FDB, respectively. Prove that $OI + |R_L - R_N| = R_K$.

I think proof is similar to Luis.

kazi

#5 Jan 7, 2014, 11:47 pm

Outline of a solution:
let K be the intersection of the incircle with BC and T be the intersection of AI with circumcenter. we have
:$OI = (IT.sin\angle ITO)/sin\angle IOT(= IOM)$
so we must prove
:$IT.sin\angle ITO = (b-c)/2 \leftrightarrow IT.sin\angle(|B-(90-A/2)|) = IT.sin\angle(|b/2-c/2|) = IT.(sin\angle b/2.cos\angle c/2 - sin\angle c/2.cos\angle b/2$
we use BIC and K to write sin with IK and BK and CK . and then we reach the proof . sorry Im too slow in latex ...

Quick Reply

High School Olympiads

Centroid

Locked

Source: Bosnia and Herzegovina 2012 federal competition

gobathegreat

#1 Jan 6, 2014, 3:21 pm

Let ABC be acute triangle and T and O its centroid and circumcenter, respectively. Let D,E and F be circumcenters of triangles TAB, TBC and TAC. Prove that O is centroid of triangle DEF.

Luis González

#2 Jan 7, 2014, 9:43 am

Posted before. It's problem 2 from Greek TST 2009.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=499502
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=310743
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=228911
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=331763&start=560

High School Olympiads

A concurrent problem

Reply

xiaotabuta
#1 Jan 4, 2014, 11:18 pm

Let quadrilateral $ABCD$ with $(E),(F)$ are the incircles of triangle ABD,BCD respectively. Let M,N are the intersection of $(E),(F)$ with BD respectively. Prove that $BM=DN$ if and only if EF,AC,BD are concurrent.

Luis González
#2 Jan 5, 2014, 1:01 am

Assume that $BM=DN \Longrightarrow \frac{1}{2}(BA+BD-DA)=\frac{1}{2}(DC+DB-BC) \Longrightarrow BA+BC=DA+DC \Longrightarrow$ $ABCD$ is either A- or C- tangential. WLOG assume that it is A-tangential, i.e. there is a circle ω tangent to AB,AD and the extensions of BC,DC beyond C. $P\equiv EF\cap BD$ is insimilicenter of $(E)\sim(F)$, C is insimilicenter of $(F)\sim\omega$ and A is exsimilicenter of $(E)\sim\omega$. Thus, by Monge & d'Alembert theorem, A,P,C are collinear, i.e. AC,BD,EF concur.

The converse is proved with the same arguments. WLOG assume that CB,CD cut the rays AD,AB beyond D,B. Let ω be the circle tangent to AD and the extensions of CB,CD beyond C. The collinearity of A, the insimilicenter P of $(E)\sim(F)$ and the insimilicenter C of $(F)\sim\omega$ forces A to be the exsimilicenter of $(E)\sim\omega \Longrightarrow \omega$ also touches $AB \Longrightarrow$ $AB+BC=DA+DC \Longrightarrow BM=DN.$

mathuz
#3 Jan 5, 2014, 1:05 am

more original defenetion:
EF,AC,BD are concurrent if and only if $AB+BC=AD+DC$.
I have a synthetic proof, but now i am interesting about projective proof. Somebody can help me?

xiaotabuta
#4 Jan 5, 2014, 12:55 pm

> *Luis González wrote:*
> $AB+BC=AD+DC \Longrightarrow ABCD$ is either A- or C- tangential. WLOG assume that it is A-tangential, i.e. there is a circle ω tangent to AB,AD and the extensions of BC,DC beyond C.

Can you prove it?

Luis González
#5 Jan 6, 2014, 1:10 am

xiaotabuta, it is an extraversion of the so called Pitot theorem.

Let R,S,T,U be the tangency points of ω with AB,BC,CD,DA, respectively. Then $AB=AR-BS$ and $BC=BS-CT \Longrightarrow AB+BC=AR-CT$. Similarly, we have $AD+DC=AU-CS$, but $AU=AR$ and $CS=CT \Longrightarrow AB+BC=AD+DC$, as desired.

As for the converse, let ω' be the B-excircle of the triangle bounded by AB,BC,CD. 2nd tangent from A to ω' cuts CD at D' (it is either between C,D or in the extension of CD beyond D). Then $AB+BC=AD+DC=AD'+D'C$ forces $D\equiv D'$ by triangle inequality $\Longrightarrow \omega\equiv\omega' \Longrightarrow ABCD$ is A-tangential.

Quick Reply

High School Olympiads

Circumscribed quadrilateral

Reply

malilim

#1 Dec 30, 2013, 9:02 pm

$ABCD$ circumscribe (I), (I) intersects AD, BC at P and Q. AB intersects CD at S (A is in SB). (I_1) is the incircle of SAD, (I_1) intersects AD at K. (I_2) is the excircle of point S of triangle SBC, (I_2) intersects BC at L. M, N are midpoints of AD, BC. Suppose that S is belong to LK. Prove that I is belong to MN and $IA.IC = IB.ID$

Luis González

#2 Jan 4, 2014, 9:54 pm

In any triangle the line connecting the incenter with the midpoint of a side is parallel to the Nagel cevian issuing from the opposite vertex. Analogously, the line connecting an excenter with the midpoint of its corresponding side is parallel to the Gergonne cevian issuing from the opposite vertex. Hence, $IN \parallel SL, IM \parallel SK \Longrightarrow M, I, N$ lie on a parallel to $\overline{SKL}$.

Since P, Q are reflections of K, L on M, N, then $PQ \parallel MN \parallel KL$. If $T \equiv AD \cap BC$, then $\triangle TMN \sim \triangle TPQ$ is isosceles with symmetry axis TI. Now, assuming WLOG that (I) becomes incircle of $\triangle TDC$ and T-excircle of $\triangle TAB$, we have $\angle TNI = 90^\circ - \frac{1}{2}\angle ATB = \angle AIB$, which gives $\triangle IAB \sim \triangle NIB \Longrightarrow \frac{IB}{IA} = \frac{BN}{IN} = \frac{NC}{IN}$. Similarly, $\angle CNI = 90^\circ + \frac{1}{2}\angle DTC = \angle DIC$ gives $\triangle INC \sim \triangle DIC \Longrightarrow \frac{NC}{IN} = \frac{IC}{ID} \Longrightarrow \frac{IB}{IA} = \frac{IC}{ID}$.

Quick Reply

High School Olympiads

a quickie

Reply

jayme
#1 Jan 3, 2014, 6:12 pm

Dear Mathlinkers,

This problem is well known… but how can you prove it synthetically in four steps?

1.ABC a triangle
2 . H the orthoceter of ABC
3. Ba, B'a the A-inner (outer) bissectors of ABC ABC
4. U, V the feet of the perpendiculars through H wrt Ba, B'a
5. N the center of the Euler's circle of ABC.

Prove : UV goes through N.

Sincerely
Jean-Louis

Luis González
#2 Jan 4, 2014, 1:22 am

It has been posted many times before with short solutions. See the links below; the last one gives rise a generalization.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=256032
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=352738
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=378378

sunken rock
#3 Jan 4, 2014, 3:27 am

Let BD, CE be altitudes of ΔABC, M, N the midpoints of BC, AH respectively.
Clearly D, E belong to the circle $(VAUH)$ of center N, DE is the common chord of this circle and $(BCDE)$, so MN is the perpendicular bisector of DE.
AU being bisector of $\angle EAD \implies EU = UD$ and $M - U - N$ are collinear.
MN is a diameter of the 9 point circle, done.

Best regards,
sunken rock

jayme
#4 Jan 4, 2014, 11:52 am

Dear Mathlinkers,
yes, you are right... but for this known problem, the Lascases theorem has not been used. This was my idea...
Sincerely
Jean-Louis

utkarshgupta
#5 Jan 30, 2015, 7:55 pm

Is this the theorem being talked about
https://books.google.co.in/books?

https://books.google.co.in/books?
id=gPMlUd554EgC&pg=PA128&lpg=PA128&dq=lascases+theorem&source=bl&ots=XnS1jtqgvl&sig=ue9SoeZnSGs4mCTVHqAI
???

jayme

#6 Jan 30, 2015, 8:36 pm

Dear Mathlinkers,

see
http://jl.ayme.pagesperso-orange.fr/Docs/An%20unlikely%20concurrence.pdf p. 3-4

Sincerely
Jean-Louis

jayme

#7 Jan 30, 2015, 8:44 pm

Dear Mathlinkers,
finally your reference is working and this is OK.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

nice concurrent

Reply

DonaldLove

#1 Jan 2, 2014, 7:00 am

triangle ABC, circumscribed circle (O).D is the reflection of A through BC. E is the intersection of tangents line at B,C to (O). prove that DE, euler lines of triangle ABC, the tangent from A to (O) are concurrent

Luis González

#2 Jan 2, 2014, 12:43 pm

An equivalent version has been posted before; P3 of Romania TST 2009 and P2 of Mongolia TST (Nº2) 2011. Furthermore, it has been generalized.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=291602
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=346956
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=495103

jayme

#3 Jan 2, 2014, 12:44 pm

Dear Mathlinkers,
a proof of this result coming from Emelyanov can be seen on
http://perso.orange.fr/jl.ayme vol. 3 Le triangle sommital... p. 9-10
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Fixed point

Reply

leminhthang20081996

#1 Jan 2, 2014, 10:55 am

D is an arbitrary point on the side BC of a fixed triangle ABC. Draw $DE//AC, DF//AB$ (E is on AB, F is on AC). Prove that (AEF) pass through a fixed point.

Luis González

#2 Jan 2, 2014, 12:16 pm • 1

Posted before at Pencil of circles from parallelograms.

Another proof (projective): Since $EB:EA=DB:DC=FA:FC,$ it follows that $E \mapsto F$ is an affine homography, i.e. where the infinite point of AB goes to the infinite point of $AC \Longrightarrow EF$ envelopes a parabola $\mathcal{P}$ tangent to AB, AC at B, C (the images of A in the respective series). Therefore, circumcircles of the triangles bounded by 3 tangents AB, AC, EF of $\mathcal{P}$ pass though its fixed focus.

Quick Reply

High School Olympiads

Find two fixed circles

Reply

yumeidesu

#1 Dec 11, 2012, 11:50 pm

Let (O) is a circle and A, B is two distinct points in its, d is a line not intersect to the circle. Let C is an arbitrary point in the circle and assume that CA, CB intersect to d at D, E, respectively. Prove that the circle with the diameter DE is tangent to 2 fixed circles.

oneplusone

#2 Dec 25, 2012, 3:11 pm • 1

Drop perpendicular M from O to d. Let A', B' be reflection of A, B about OM, let $B'A, BA'$ intersect d at F, G. Then M is the midpoint FG. Let X be on OM such that $MX^2 = GA' \cdot GB$.

$\angle GBE = \angle A'AC = \angle FDA$ and similarly $\angle GEB = \angle FAD$, so $\triangle GBE \sim \triangle FDA$, which means $GE \cdot FD = GB \cdot FA = MX^2$. Now let D', E' be on d such that $MD' = FD$ and $ME' = GE$. (the direction has to be the same) Then $MD' \cdot ME' = MX^2$ which means the circle with diameter $D'E'$ passes through X, so the circle with diameter DE will be tangent to the fixed circle centered at X with radius MF, because $DD' = FM = EE'$. The second circle it tangents is just the reflection of the first about d.

stackrel37

#3 Jan 31, 2013, 8:21 pm

> yumeidesu wrote:
> Let (O) is a circle and A, B is two distinct points in its, d is a line not intersect to the circle. Let C is an arbitrary point in the circle and assume that CA, CB intersect to d at D, E, respectively. Prove that the circle with the diameter DE is tangent to 2 fixed circles.

Can you tell me what is the source of this problem? Thank you :)

yumeidesu

#4 Feb 20, 2013, 2:32 pm • 1

Dear stackrel37,

This problem are from a set of exercises to train VN team for IMO 2009. But the original form is easier than this one so much, I think this problem haven't be chosen for any competition. :)

Luis González

#5 Jan 2, 2014, 11:07 am

The property is still true for a general proyectivity $AC \equiv AD \mapsto BC \equiv BE$; not necessarily congruent pencils (C running on a circle through A,B). So the general version goes as follows:

$\mathbf{P} : D_i \mapsto E_i$ is a proyectivity on a line ℓ and denote Γ_i the circles with diameter $\overline{D_iE_i}$. Assume that three of these circles accept two circles ω_1, ω_2 either internally tangent or externally to them (they are clearly symmetric WRT ℓ), then all Γ_i envelop ω_1, ω_2. The proposed problem is an example of an elliptic projectivity, where ω_1, ω_2 always exist.

If $\overline{UV}$ is the diameter of ω_1 parallel to ℓ, then $UD_i \cap VE_i$ pass through the tangency point H_i of ω_1 and Γ_i (their center of similitude) $\Longrightarrow$ pencils $UH_i \equiv UD_i$ and $VH_i \equiv VE_i$ are congruent. Hence $(D_1, D_2, D_3, D_4) = (E_1, E_2, E_3, E_4)$ implies that $UD_4 \cap VE_4$ is on ω_1 and since $UV \parallel D_4E_4$, it follows that Γ_4 and ω_1 are either internally or externally tangent through $UD_4 \cap VE_4$, proving our claim.

External common tangents of ω_1, ω_2 cut then ℓ at the 2 limit points of the referred projectivity. These pairs of circles naturally meet ℓ according to whether $\mathbf{P}$ is elliptic, hyperbolic or parabolic.

Quick Reply

High School Olympiads

Right angled kite

yetti

#1 Dec 19, 2011, 6:04 pm • 2

$ABCD$ is a kite with right angles at vertices A, C and with incircle (I). Arbitrary tangent of (I) cuts AB, CD at X, Y, respectively. XI cuts DA at Z. Prove that $AY \perp BZ$.

tobash_co

#2 Dec 20, 2011, 11:32 am • 2

Drop perpendicular ZE to XY with E on XY. Then E is the reflection of A w.r.t. XZ, hence $ZE \perp EX$. Considering circumscribed $ADYX$, we have $YD - DA = YX - XA \Rightarrow$
$YC = YD - DC = YD - DA = YX - XA = YX - EX = YE$, therefore
$YZ^2 - ZE^2 = YE^2 = YC^2 = YB^2 - BC^2 \Rightarrow YZ^2 - AZ^2 = YB^2 - AB^2$, so $AY \perp BZ$, as desired.

Luis González

#3 Jan 1, 2014, 1:08 am • 1

This was pending in my favorites for so long. Here is a projective approach.

Obviously, the pencils $IX \equiv IZ$ and BZ are perspective and the pencils AY and BZ, with mutually perpendicular rays, are projective $\Longrightarrow \mathbf{P}_1 : X \mapsto Y$ is a projectivity. So, it suffices to show that $\mathbf{P}_1$ coincides with the projectivity $\mathbf{P}_2$ taking any X on AB into the intersection of CD with the second tangent from X to (I).

Clearly, $\mathbf{P}_1 : A \mapsto D, B \mapsto C$ and the tangency point M of (I) with AB (projection of I on AB) goes to $E \equiv AB \cap CD$. These are precisely homologous points under $\mathbf{P}_2$, because $\angle EIM = \angle BIC = \angle AID$. So $\mathbf{P}_1$ and $\mathbf{P}_2$, sharing 3 homologous points, coincide everywhere, i.e. $\mathbf{P}_1 \equiv \mathbf{P}_2$.

jayme

#4 Jan 1, 2014, 3:41 pm

Dear Yetti (V.) and Mathlinkers,
sorry, I have some difficulty to understand what is a kite? Can you make a figure if you have some time...
Sincerely
Jean-Louis

Lyub4o

#5 Jan 1, 2014, 7:24 pm

http://en.wikipedia.org/wiki/Kite_(geometry)

jayme

#6 Jan 1, 2014, 7:30 pm

Dear Mathlinkers,
thank for your help... in french this is a "cerf-volant".
Sincerely
Jean-Louis

High School Olympiads

Circumcircle of A_1B_1C_1 tangent to E-excircle of OEF

Reply

Source: Own

Math-lover123
#1 Jan 1, 2014, 12:16 am

In triangle ABC,D,E,F are midpoints of sides BC,CA,AB respectively.
A_1,B_1,C_1 are midpoints of segments EF,FD,DE respectively.
If O is the circumcenter of ABC then prove that the circumcircle of triangle $A_1B_1C_1$ is tangent to E-excircle of OEF.

Luis González
#2 Jan 1, 2014, 12:30 am

Ever heard of Feuerbach theorem ?.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=764
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=50&t=14927
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=211225

Math-lover123
#3 Jan 1, 2014, 1:49 am

Yes Luis I know but here is not easy to recognise Feuerbach theorem.

Quick Reply

High School Olympiads

Hexagon cyclic and two concurrent points

Reply

Source: Own?

daothanhoai

#1 Dec 27, 2013, 2:45 pm • 1

Dear Mathlinkers!

Let cyclic hexagon: $ABCDEF$. AF meets BC at G ; AF meets DE at I ; DE meets BC at H ;

HA meets GD at P ; HA meets IB at J ; HF meets GE at L ; HF meets IC at M ; GD meets IC at N ; GE meets IB at K.

Prove that:

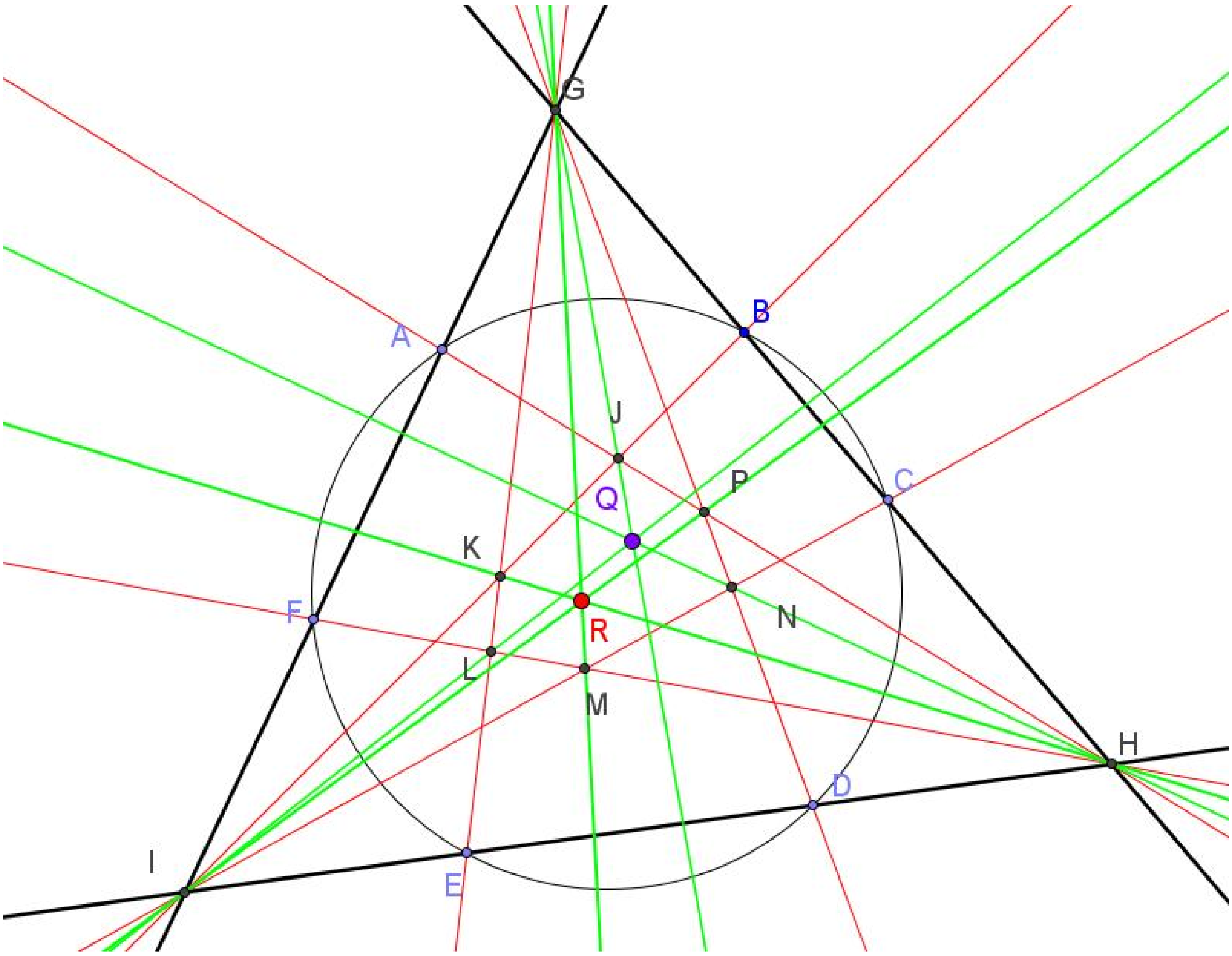
HN, GJ, IL are concurrent
HK, GM, IP are concurrent.

Best regards

Sincerely

Dao Thanh Oai

Attachments:



Luis González

#2 Dec 31, 2013, 9:50 pm

Lines IB, IC, GD, GE, HF, HA are tangent to a same conic $\mathcal{C}$ (see the problem Inscribe conic). Let $X_1, X_2, X_3, X_4, X_5, X_6$ be the tangency points of $\mathcal{C}$ with JP, PN, NM, ML, LK, KJ, respectively. Then HN, GJ and IL are the polars of $X_1X_4 \cap X_2X_3, X_1X_6 \cap X_2X_5$ and $X_3X_6 \cap X_4X_5$ WRT $\mathcal{C}$. But by Pascal theorem for the hexagon $X_1X_4X_5X_2X_3X_6$, the referred intersections lie on a line $\ell \Longrightarrow HN, GJ, IL$ concur at the pole of ℓ WRT $\mathcal{C}$. In the same way, we prove that HK, GM, IP concur as well.

vittasko

#3 Jan 6, 2014, 6:06 am • 2

These results and other two are also true, in some other configurations as follows :

Begining form a given triangle $\triangle\ IHG$ as in the above figure, we denote the pairs of points $D,\ E$ and $B,\ C$ and $A,\ F,$ on the side-segments $IH,\ HG,\ GI,$ respectively.

So, WHEN :

(a) - The points $A,\ B,\ C,\ D,\ E,\ F,$ lie on a circle (or on a conic).

(b) - The points $D,\ E,$ are isotomic conjugates with respect to the side-segment IH and similarly for the pairs of the points $B,\ C$ and $A,\ F.$

(c) - The lines $GD,\ GE,$ are isogonal conjugates with respect to the angle $\angle A$ and similarly for the pairs of the lines $IB,\ IC$ and $HA,\ HF.$

(d) - Three similar rectangls $IHXY,\ HGVU,\ GIWZ$ are erected on the side-segments $IH,\ HG,\ GI$ respectively and $D,\ E,$ are the points of intersection of the side-segment IH from the lines $GX,\ GY$ respecrively and similarly for the pairs of the points $B,\ C$ and $A,\ F.$

THEN :

(1) - The lines $GJ,\ IL,\ HN,$ are concurrent at one point so be it $Q.$

(2) - The lines $GM,\ IP,\ HK,$ are concurrent at one point so be it $R.$

(3) - The lines $JM,\ KN,\ LP,$ are concurrent at one point so be it $S.$

(4) - The points $Q,\ R,\ S,$ are collinear.

All these results are known and they can be proved by elementary ways and I will try to search for the references.

Kostas Vittas.

Quick Reply

High School Olympiads

incircles of triangles

Reply

malilim

#1 Dec 13, 2013, 2:32 pm

Let ABC is an acute triangle, P is in the triangle such that $AB+BP=AC+CP$, BP intersects AC at Y, CP intersects AB at Z. Prove that 2 common tangents of the incircles of APY, APZ and BC are concurrent

Luis González

#2 Dec 31, 2013, 6:40 am • 3

Denote (I_1) and (I_2) the incircles of $\triangle APZ$ and $\triangle APY$. $J_1 \equiv PI_2 \cap AI_1$ and $J_2 \equiv PI_1 \cap AI_2$ are then the A-excenters of $\triangle APB$ and $\triangle APC$, respectively. Denote (J_1) and (J_2) the A-excircles of $\triangle APB$ and $\triangle APC$.

$AB+BP=AC+CP \Longrightarrow$ there is a circle ω tangent to AB, AC and the rays PB, PC. Hence, B is exsimilicenter of $\omega \sim (J_1)$ and C is the exsimilicenter of $\omega \sim (J_2)$. By Monge & d'Alembert theorem, $K \equiv BC \cap J_1J_2$ is the exsimilicenter of $(J_1) \sim (J_2) \Longrightarrow A(J_1, J_2, P, K) = -1$. Therefore, from the complete quadrangle $I_1J_1J_2I_2$, it follows that $K \in I_1I_2 \Longrightarrow K$ is also exsimilicenter of $(I_1) \sim (I_2)$, i.e. external common tangents of (I_1), (I_2) meet on BC.

jayme

#3 Dec 31, 2013, 1:41 pm

Dear Mathlinkers,
This problem has something in common with the Urquhart theorem...
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

circumscribed quadrilateral

Locked

Iambosama

#1 Dec 31, 2013, 1:20 am

$ABCD$ is a cyclic quadrilateral. E is the intersection of AC and BD. Let $(I_a), (I_b), (I_c), (I_d)$ be the incircle of EAB, EBC, ECD, EDA respectively.
Prove that I_a, I_b, I_c, I_d is concyclic if and only if $ABCD$ is a circumscribed quadrilateral

Luis González

#2 Dec 31, 2013, 1:25 am

Please use the search before posting.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=21758
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=21768
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=64627
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=199861
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=228291

High School Olympiads

A figure like Droz-Farny theorem

Reply

jayme
#1 Dec 30, 2013, 6:02 pm

Dear Mathlinkers,

1. ABC a triangle,
2. H the orthocenter of ABC
3. L, M two perpendicular lines through H
4. X, Y the points of intersection of L, M wrt BC, CA
5. Z the point of intersection of the parallel to HX through A with the parallel to HY through B.

Prove : X, Y and Z are collinear.

Sincerely
Jean-Louis

Luis González
#2 Dec 30, 2013, 11:09 pm

Let U,V be the intersections of ℓ,m with $CA,CB,$ respectively. Let $\mathcal{P}$ be the parabola tangent to $BC,CA,AB,\ell.$ Its directrix τ is the Steiner line of the quadrangle bounded by $BC,CA,AB,\ell,$ passing through $H.$ Since any point on τ sees $\mathcal{P}$ at right angle, it follows that m is the second tangent from H to $\mathcal{P},$ thus $(A,Y,U,...) \mapsto (B,V,X,...)$ is a projectivity where the infinite point of CA goes to the infinite point of $CB,$ because $\mathcal{P}$ is tangent to the line at infinity. As a result, $\overline{AU}:\overline{AY}=\overline{BX}:\overline{BV} \Longrightarrow Z \in XY,$ such that $\overline{ZX}:\overline{ZY}=\overline{AU}:\overline{AY}.$

IMI-Mathboy
#3 Dec 30, 2013, 11:16 pm

Let P and Q be point of intersection of AZ,BH and BZ,AH. then P and Q are orthocenters of $\triangle AHY$ and $\triangle BHX$. We have also $\triangle AYH$, $\triangle BHX$ are similar. if M and N are intersection of BZ,HX and AZ,HY.Then we have $\frac{MZ}{MX}=\frac{NH}{MX}=\frac{HY}{HX}$ hence X,Y,Z are collinear :)

jayme
#4 Dec 31, 2013, 1:24 pm

Dear Mathlinkers,
I have solved this problem with a converse of the little Pappus theorem...
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Ratio proving

ptk_1411
#1 Dec 29, 2013, 11:02 pm

Let circles (O_1) and (O_2) intersect at A, B; common tangent EF ($E \in (O_1), F \in (O_2)$). An arbitrary line parallel to EF intersects $(O_1), (O_2)$ respectively at M, N such that M lies outsides (O_2) and N lies outsides (O_1). Prove that $EM/FN = EA/FA$.

Luis González
#2 Dec 30, 2013, 10:22 am

Let AM, AN cut EF at P, Q, respectively. From $PQ \parallel MN$ and the tangency of PQ with $(O_1) \equiv \odot(EMA)$ and $(O_2) \equiv \odot(FNA)$, we get

$$\frac{EM^2}{EA^2} = \frac{PM}{PA} = \frac{QN}{QA} = \frac{FN^2}{FA^2} \Longrightarrow \frac{EM}{FN} = \frac{EA}{FA}.$$

High School Olympiads

bisectors in a triangle

Reply

malilim

#1 Dec 27, 2013, 11:23 am

Let ABC is a triangle, AA_1, BB_1, CC_1 are the bisectors of the triangle, the perpendicular bisectors of AA_1, BB_1, CC_1 intersects AC, CB, AB at A_2, B_2, C_2 respectively. Prove that the area of $A_1B_1C_1$ is equal to the area of $A_2B_2C_2$

Luis González

#2 Dec 29, 2013, 8:53 pm

Posted before. See the 2nd reply in the 1st link for a generalization.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=546707
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=497205

Quick Reply

High School Olympiads

Fix point involving tangent lines

DonaldLove

#1 Dec 24, 2013, 5:27 pm

Given two circle $(O_1),(O_2)$ outside of each other with different radius. Let m n be two inside common tangent line of the two circles. M N is arbitrary point on m and n. The other tangent line fromM N to (O_1) meet at P. The other tangent line from M N to (O_2) meet at Q. Prove that PQ goes through a fixed point

Luis González

#2 Dec 29, 2013, 7:58 am

m touches $(O_1),(O_2)$ at S,Y and n touches $(O_1),(O_2)$ at V,R. NP touches (O_1) at U and MQ touches (O_2) at X.

$$MQ-MP=MY+QX-(MS-PU)=SY+QX+PU=$$

$$=NV-NR+QX+PU=NU+PU-NQ=NP-NQ.$$

This relation means that there exists a circle ω tangent to the lines MP,PN,NQ and QM. Hence, P is exsimilicenter of $(O_1)\sim\omega$ and Q is exsimilicenter of $(O_2)\sim\omega$. By Monge and d'Alembert theorem, PQ goes through the exsimilicenter of $(O_1),(O_2)$, obviously fixed.

High School Olympiads

Lemoine point

Reply

xiaotabuta

#1 Dec 20, 2013, 11:02 pm

Let triangle ABC inscribed (O). 2 tangents of (O) from B, C cut the tangent from A of (O) at B', C' respectively. Construct a parallelogram $C'B'CD$. Let M be the mid-point of BC. Prove that $ML \perp MD$ (with L is the Lemoine-point)

xiaotabuta

#2 Dec 21, 2013, 8:39 pm

Up!
Any idea please?

Luis González

#3 Dec 29, 2013, 2:09 am • 1

Let A' be the intersection of the tangents of (O) at B, C and let U, V be the midpoints of BB', CC'. D and E are then the reflections of B' and C' on V and U, respectively. UV is B'-midline of $\triangle DBB' \Longrightarrow DB \parallel UV$ and $UV = \frac{1}{2}BD$. Likewise, $CE \parallel UV$ and $UV = \frac{1}{2}CE \Longrightarrow \overline{BD}$ and $\overline{CE}$ are congruent and parallel $\Longrightarrow BECD$ is a parallelogram $\Longrightarrow M$ is also midpoint of $\overline{DE}$ $(\star)$.

Let the A'-excircle (J) of $\triangle A'B'C'$ touch $B'C', A'B'$ at X, Z. Then $DC = C'B' = CZ$ and $C'D = CB' = AB' = C'X \Longrightarrow C$ and C' have equal powers WRT D and $(J) \Longrightarrow CC'$ is radical axis of D and (J). Analogously, BB' is radical axis of E and $(J) \Longrightarrow L \equiv BB' \cap CC'$ is the radical center of $D, E, (J) \Longrightarrow L$ is on the radical axis of D, E; the perpendicular bisector of $\overline{DE}$. Together with $(\star)$, it follows that $LM \perp MD$.

BlackSelena

#4 Dec 31, 2013, 1:21 am

I believe that this is from IMO SL 2009

Quick Reply

High School Olympiads

Pedal

Reply

TrungK40PBC

#1 Dec 18, 2013, 11:30 pm

Problem Give triangle ABC with orthocentre H and a point Q. D, E, F are projections of Q on BC, CA, AB respectively. Let M, N, P are midpoints of HA, HB, HC; X, Y, Z are orthocentres of AEF, BFD, CDE. Prove that MX, NY, PZ are concurrent at the intersection of (DEF) and Euler circle wrt triangle ABC

Luis González

#2 Dec 28, 2013, 10:10 am • 1

Denote A_0, B_0, C_0 the midpoints of BC, CA, AB and O the circumcenter of $\triangle ABC$. Let V be the projection of A on QO, i.e. the 2nd intersection of circles $\odot(AB_0C_0)$ and $\odot(AEF) \Longrightarrow V$ is Miquel point of the quadrangle bounded by AB, AC, B_0C_0, EF. Since M is orthocenter of $\triangle AB_0C_0$, then MX is the Steiner line of V WRT the referred quadrangle $\Longrightarrow MX$ goes through the reflection U of V WRT B_0C_0. This is the orthopole of QO WRT $\triangle ABC$, which is then the intersection of $\odot(DEF)$ and the 9-point circle $\odot(A_0B_0C_0)$, other than the Poncelet point of $ABCQ$ (see the 1st Fontené theorem at Two Yango's problem. Similarly, NY and PZ go through U.

Quick Reply

High School Olympiads

I think this is a very famous problem but I cant find it

Reply

Source: on the search. Related to IMO problem 6 (2012?)

fmasroor
#1 Dec 21, 2013, 1:11 am

The reflections across the sides of ABC of a line tangent to its incircle bound another triangle. Prove that this triangle's incenter lies on ABC's circumcircle.

fmasroor
#2 Dec 21, 2013, 1:21 am

This also holds when the line is tangent to the ninepoint circle

IDMasterz
#3 Dec 21, 2013, 8:52 am

The other was around was an Iran problem... Interesting, appears this type of "transformation" is an involution hehe.

Luis González
#4 Dec 23, 2013, 3:49 am

> *fmasroor wrote:*
> The reflections across the sides of ABC of a line tangent to its incircle bound another triangle. Prove that this triangle's incenter lies on ABC's circumcircle.

> *fmasroor wrote:*
> This also holds when the line is tangent to the ninepoint circle

In fact, it holds for any line in the plane of ABC. See The incenter lies on circumcircle [Iran Second Round 95]

Quick Reply

High School Olympiads

A collinear problem

Reply

xiaotabuta

#1 Dec 10, 2013, 1:22 am

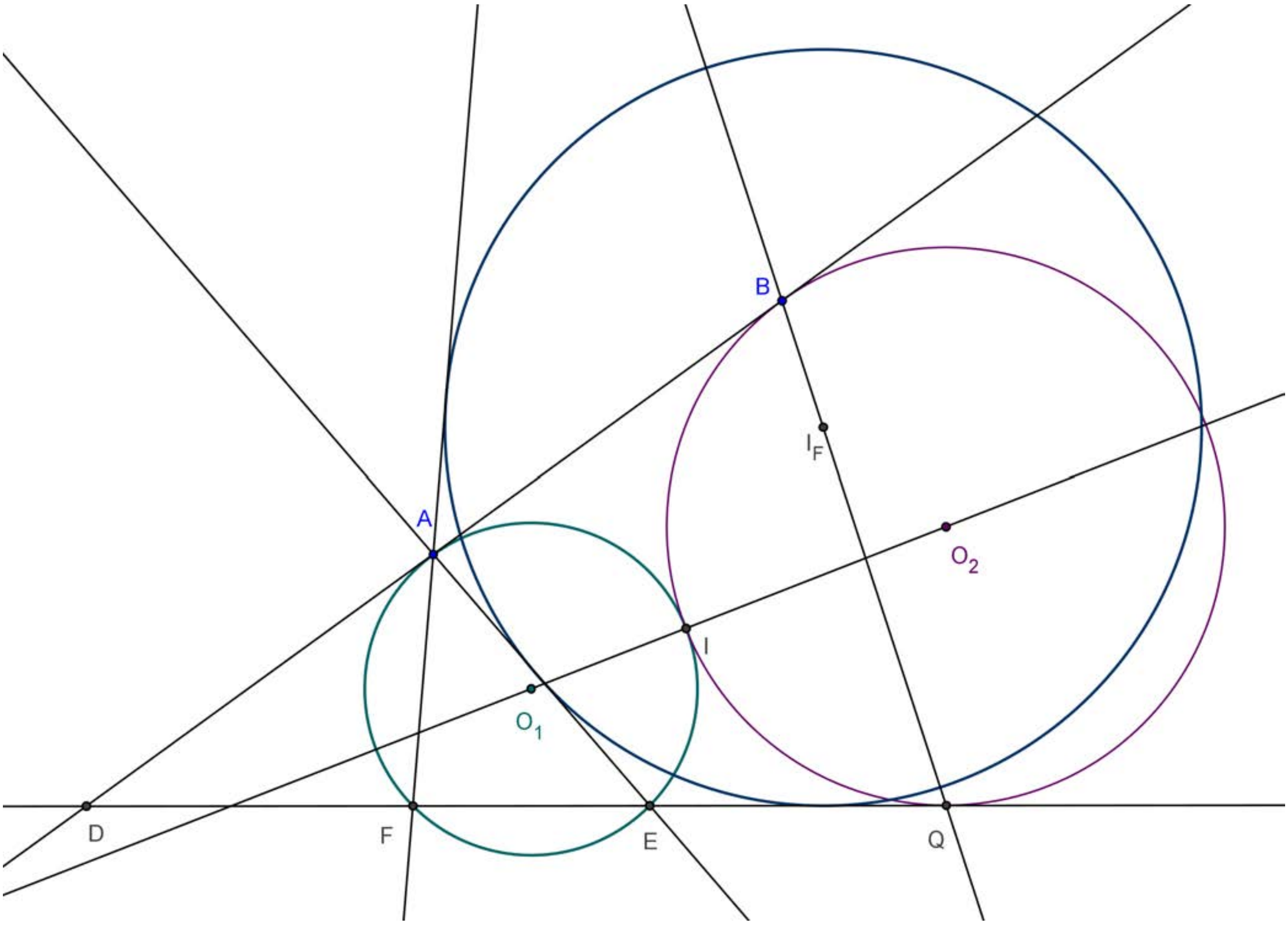
Let (O_1) and (O_2) touch at I. Δ is commond tangent of (O_1) and (O_2), (does not pass I) cut (O_1), (O_2) at A, B respectively. Asume that $R_{O_2} < R_{O_1}$, another tangent of (O_2) at K (K is at the another side as A, B w.r.t to O_1O_2) cut Δ at D, cut (O_1) at E, F respectively (E is between Q and F). Let I_F be the escribes circle angle F of $\triangle AEF$. Prove that Q, B, I_F is collinear

Luis González

#2 Dec 10, 2013, 2:58 am

The point Q is defined nowhere, but I believe K=Q. Then, this is an extraversion of the Sawayama's lemma; (O_2) is a Thebault circle of the cevian AD of AEF externally tangent to its circumcircle (O_1).

Attachments:



Quick Reply

High School Olympiads

a triangle, circles and concurrent lines

Locked

Source: Yufei Zhao, circles

gev_gev

#1 Dec 8, 2013, 1:52 am

Let Ω be the circumcircle of ABC. A circle ω is tangent to sides AB, AC and circle Ω at points X, Y and Z, respectively. Let M be the midpoint of the arc BC of Ω which does not contain A. Prove that lines XY, BC and ZM have a common point.

Luis González

#2 Dec 8, 2013, 2:17 am • 1

Posted before at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=151&t=145105

For a generalization see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=397123
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=399496

High School Olympiads

escribed circle

Reply

malilim

#1 Dec 6, 2013, 3:23 pm

let ABC is a triangle, (J) is the escribed circle of A, (O) is the circumcircle of ABC. (O) intersects (J) at P, Q. BY, CZ are bisectors of the triangle, YZ intersects (O) at D, E. prove that DP and EQ are tangents of (J)

Luis González

#2 Dec 8, 2013, 12:26 am • 2

From http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=385175, the common external tangents of $(O), (J)$ touch (O) at D, E. A' is a point on the arc PAQ of (O) and tangents from A' to (J) touch it at P', Q'. $A'P'$ and $A'Q'$ cut (O) again at B' and C'. By Poncelent porism for $(O) \cup (J)$, all $B'C'$ touch (J). When $A' \equiv D \equiv C'$, then $P' \equiv P \equiv B'$ $\Longrightarrow DP$ touches (J). Similarly EQ touches (J).

Quick Reply

High School Olympiads

distance to O equals R/2

Reply

DonaldLove

#1 Dec 3, 2013, 10:45 pm

triangle ABC, circumscribed circle (O). M1,M2 on (O) such that orthocenter H is on M1M2. A0 is the reflection of A through O. A0M1,A0M2 meet BC at A1,A2. Similarly we have B0,B1,B2 ; C0,C1,C2. Prove that centers of circles (A0A1A2),(B0B1B2),(C0C1C2) lie on the same line and distance of that line to O equals R/2.

P/s: I need to prove that the anti steiner point S of M1M2 and O lie on circle (A0A1A2)

Luis González

#2 Dec 4, 2013, 8:15 am

CH cuts (O) again at the reflection F of H on AB and M_1M_2 cuts AB at $Z \Longrightarrow FZ$ cuts (O) again at the anti-Steiner point S of M_1M_2. M_1M_2 cuts BC at K and circle $\odot(KSM_1)$ cuts BC again at Q. Angle chase (mod 180º) gives

$$\angle SM_1Q = \angle SKQ = \angle HKQ = \angle FHZ - \angle BCH =$$

$$= \angle SFC - \angle OAC = \angle SAC - \angle OAC = \angle SAA_0,$$

which means that M_1, Q, A_0 are collinear $\Longrightarrow Q \equiv A_1$. Hence S is the Miquel point of the complete $M_1M_2A_2A_1 \Longrightarrow S \in \odot(A_0A_1A_2)$. Further, A_1 is then midpoint of the arc SM_1 of $\odot(KSM_1) \Longrightarrow A_1M_1 = A_1S \Longrightarrow OA_1$ is perpendicular bisector of $SM_1 \Longrightarrow OA_1 \perp SM_1$. Similarly, $OA_2 \perp SM_2 \Longrightarrow \angle A_1OA_2 = \angle M_1SM_2 = \angle A_1A_0A_2$ (mod 180º) $\Longrightarrow O \in \odot(A_0A_1A_2)$.

Quick Reply

High School Olympiads

Nice result!

Source: Maybe own?

hongduc_cqt

#1 Dec 2, 2013, 6:28 pm

Given a triangle ABC. O is center of (ABC). H is orthocenter of $\triangle ABC$. $AH \cap BC = D$. K lies on segment AB such that $KH \parallel BC$. A line d through O and perpendicular with DK. Prove that d pass through midpoint of BD.

Luis González

#2 Dec 2, 2013, 10:07 pm • 1

Let M, P, R denote the midpoints of $\overline{BD}, \overline{BC}, \overline{AB}$, respectively. Then $KH \perp AD \parallel RM, DH \perp MP$ and $BH \perp AC \parallel RP$, which means that $\triangle KBD$ and $\triangle PMR$ are orthologic through $H \Longrightarrow$ perpendiculars to BD, BK through P, R meet at their second orthology center (these are the perpendicular bisectors of BC,AB meeting at O) $\Longrightarrow$ $MO \perp KD$, as desired.

Arab

#3 Dec 2, 2013, 10:15 pm

My solution is a little bit different. The one by **Luis** is really nice.
Let M be the midpoint of AB and N be that of BD, we obtain $MN \perp BD, OM \perp AB$, which implies that $\angle OMN = \angle KBD$. On the other hand, note that $\triangle ABD \sim \triangle CHD$, and hence $\frac{OM}{BK} = \frac{CH \cdot AD}{2AB \cdot DH} = \frac{CH \cdot MN}{AB \cdot DH} = \frac{MN}{BD}$, which follows that $\triangle OMN \sim \triangle KBD$. Consequently, $ON \perp KD$, as desired.
$Q.E.D.$

MMEEvN

#4 Dec 4, 2013, 1:18 pm

Let M be the midpoint of BD
$OD^2 - OK^2 - MD^2 + MK^2 = BK.KA - BD.DC - MD^2 + (BM^2 + BK^2 - 2BM.BKcosB$
$= BK.KA - BD.DC + BK^2 - 2BM.BKcosB = BK.BA - BD.DC - 2BM.BKcosB = BK.BA - BD.DC - BD.BKcosB = BK(BA-$
$)BAcosB.cosB) - BD.DC = BK.BAsin^2B - BD.DC = \frac{BA}{DA}.DH.BAsin^2B - \frac{DA}{DC}.DH.DC = DH((\frac{BA}{DA})BAsin^2B - DA) = 0$
(Since $DA = BAsinB$).Hence $OD^2 - OK^2 = DM^2 - MK^2$.Therefore $OM \perp KD$

High School Olympiads

Seven circles problem-II

Reply

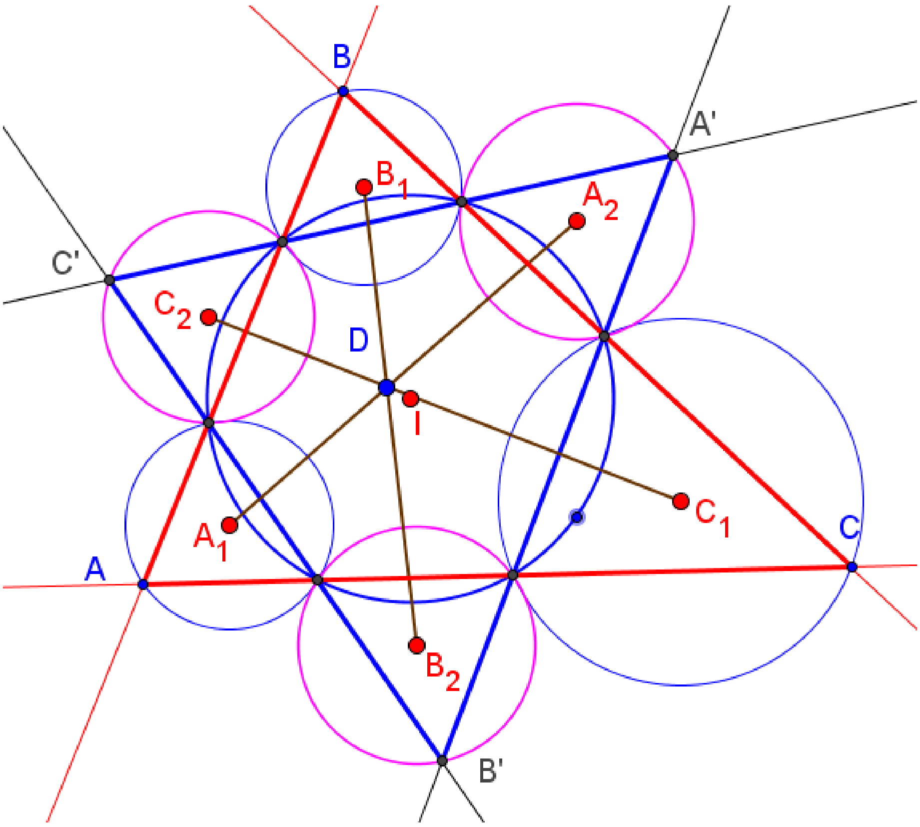
daothanhoai

#1 Sep 21, 2013, 3:33 pm

Seven circles problem-II

Let a triangle ABC. Construct circle with centre I (I is centre of incircle). The circle intersects AB, BC, CA at six points: M, N, P, Q, J, K. Three line NP, QJ, KM intersect at three points A', B', C'(respectively with A, B, C). Denoting $B_1, A_2, C_1, B_2, A_1, C_2$ are circumcircles of six circles $(BNP), (A'PQ), (CQJ), (B'JK), (AKM), (MNC')$. Prove that A_1A_2, B_1B_2, C_1C_2 are concurrent.

Attachments:



Luis González

#2 Dec 2, 2013, 6:12 am • 1

Let the incircle (I) touch BC, CA, AB at D, E, F and let $A_0 \equiv MQ \cap KP$, $B_0 \equiv KP \cap NJ$ and $C_0 \equiv NJ \cap QM$. We prove that A_1A_2, B_1B_2, C_1C_2 concur at a point X, which lies on the line connecting the Gergonne point of $\triangle ABC$ and the Kosnita point of $\triangle DEF$.

Clearly, $B'C' \parallel EF \parallel JN, C'A' \parallel FD \parallel MQ$ and $A'B' \parallel DE \parallel PK$. Thus $\triangle DEF \sim \triangle A_0KM$ are homothetic with center $A \Longrightarrow A, A_0, D$ are collinear. Likewise, B, B_0, E are collinear and C, C_0, F are collinear $\Longrightarrow$ $\triangle DEF \sim \triangle A_0B_0C_0$ are homothetic with center $G_e \equiv AD \cap BE \cap CF$, the Gergonne point of $\triangle ABC$.

On the other hand, let T denote the circumcenter of $\triangle A_0PQ$ and A_3 the circumcenter of $\triangle TPQ$. Since PA_0QA' is a parallelogram, then clearly A_2 is the reflection of T across $PQ \Longrightarrow A_0A_2$ and A_0A_3 are the cevians of the 9-point center and Kosnita point of $\triangle A_0PQ$ issuing from $A_0 \Longrightarrow$ they are isogonals WRT $\angle PA_0Q$. But since $A_0MA_1K \sim A_0PA_3Q$, then $\angle A_1A_0K = \angle A_3A_0Q = \angle A_2A_0P \Longrightarrow A_0, A_1, A_2$ are collinear. Similarly, B_0, B_1, B_2 are collinear and C_0, C_1, C_2 are collinear.

If A_4, B_4, C_4 denote the circumcenters of $\triangle AEF, \triangle BFD, \triangle CDE,$ then $DA_4 \parallel A_0A_1, EB_4 \parallel B_0B_1$ and $FC_4 \parallel C_0C_1. DA_4, EB_4, FC_4$ concur at the Kosnita point K_S of $\triangle DEF \Longrightarrow A_1A_2, B_1B_2, C_1C_2$ concur at Kosnita point X of $\triangle A_0B_0C_0,$ which is then on the line joining K_S with the homothetic center G_e of $\triangle DEF \sim \triangle A_0B_0C_0.$

daothanhoai

#4 Jan 10, 2014, 1:16 pm

2-Define $A_2, B_2, C_2, X_2, Y_2, Z_2$ respectively are midpoints of $A_1A, B_1B, C_1C, X_1X, Y_1Y, Z_1Z$. Prove that A_2X_2, B_2Y_2, C_2Z_2 are concurrent.

Quick Reply

High School Olympiads

Intersection of two circles

Reply

yumeidesu

#1 Nov 25, 2013, 11:57 am • 1

Let ABC be a triangle with the circumcircle (O) and incircle (I). On ray AB, AC, choose D, E such that $AD = AE = BC$. The circle (ADE) cut (O) at K. AI cuts (O) at P. Prove that three triangles KBD, KCE, POI are pairwise similar.

Luis González

#2 Dec 1, 2013, 1:31 am

Clearly K is the center of the spiral similarity that swaps $\triangle KBC$ with circumcircle (O) and $\triangle KDE$ with circumcircle (J) $\Longrightarrow \triangle KBD \sim \triangle KCE \sim \triangle KOJ.$ So, it suffices to show that $\triangle KOJ$ and $\triangle POI$ are either congruent or similar.

Since $\angle AJD = 2\angle AED = 2\left(90^\circ - \frac{1}{2}\angle A\right) = 180^\circ - \angle A = \angle BPC,$ it follows that the isosceles $\triangle AJD$ and $\triangle BPC,$ with equal bases $AD = BC,$ are congruent $\Longrightarrow JK = JA = PB = PI \Longrightarrow I, J$ are equidistant from the midpoint of $\overline{AP} \Longrightarrow OJ = OI.$ Since $OK = OP,$ then $\triangle POI \cong \triangle KOJ$ by SSS criterion.

Quick Reply

High School Olympiads

Another new result

Reply

Source: Own

jayme

#1 Nov 28, 2013, 5:47 pm • 1

Dear Mathlinkers

1. ABC a triangle
2. (O) the circumcircle of ABC
3. Na the Nagel point of ABC
4. A1B1C1 the Na-cevian triangle of ABC
5. (O1) the circumcircle of A1B1C1
6. Be* the isogonal of Be wrt ABC
7. M the midpoint of the arc BAC

Prove : ABe* is parallel to MO1

Sincerely
Jean-Louis

fmasroor

#2 Nov 28, 2013, 9:28 pm

What is Be? Bevan point?

jayme

#3 Nov 28, 2013, 10:04 pm

Dear Mathlinkers,
Be is the center of the excentral triangle of ABC.
Sincerely
Jean-Louis

Luis González

#4 Nov 30, 2013, 10:03 pm • 2

Since $BC_1=CB_1=s-a,$ then circles $(O) \equiv \odot(ABC)$ and $(O_2) \equiv \odot(AB_1C_1)$ meet at A and the center of the rotation N that swaps $\overline{BC_1}$ and $\overline{CB_1}$. Thus $NB_1=NC_1$ and $NB=NC \Longrightarrow N \equiv M$ is midpoint of the arc BAC and B_1AC_1. Since $\triangle A_1B_1C_1$ is pedal triangle of $B_e,$ it follows that O_1 is midpoint of $\overline{B_eB_e{}^*}$ and B_e is the antipode of A WRT $(O_2) \Longrightarrow O_1O_2$ is Be-midline of $\triangle AB_eB_e{}^* \Longrightarrow O_1O_2 \parallel AB_e{}^* \perp B_1C_1 \Longrightarrow O_1O_2$ is perpendicular bisector of $\overline{B_1C_1}$ $\Longrightarrow M \in O_1O_2 \Longrightarrow MO_1 \parallel AB_e{}^*.$

TelvCohl

#5 Oct 29, 2014, 9:25 pm

My solution:

Since $BC_1=CB_1, MB=MC, \angle MBC_1=\angle MCB_1$
so $\triangle MBC_1$ and $\triangle MCB_1$ are congruent, ie. $MC_1=MB_1$
hence we get MO_1 is the perpendicular bisector of B_1C_1 .
Since A, B_1, C_1, B_e are concyclic at (AB_e) ,
so $\angle B_e^*AB=\angle CAB_e=\angle B_1C_1B_e=\angle(MO_1, AB)$ (notice $B_1C_1 \perp MO_1, B_eC_1 \perp AB$) .
i.e. $AB_e^* \parallel MO_1$

Q.E.D

Quick Reply

High School Olympiads

constant ratio

Reply

DonaldLove

#1 Nov 28, 2013, 4:45 pm

Triangle ABC, orthocenter H. arbitrary point M. X,Y,Z are reflections of M through BC,CA,AB. XH meets YZ at T. Prove that $\frac{HX}{HT} = const$

Luis González

#2 Nov 30, 2013, 3:25 am

Let D, E, F be the projections of M on BC, CA, AB (midpoints of MX,MY,MZ) and θ denotes the angle between the directions AH and $YZ \parallel EF$. Then, we have

$$[HYZ] = [HYAZ] - [ZYA] = AH \cdot EF \cdot \sin\theta - \tfrac{1}{2} MA^2 \cdot \sin 2A \Longrightarrow$$

$$\frac{2[HYZ]}{\sin 2A} = \frac{2R \cdot EF \cdot \sin\theta}{\sin A} - MA^2 = MA \cdot (2R \cdot \sin\theta - MA).$$

Let AM cut the circumcircle (O, R) of $\triangle ABC$ again at K. Then $\measuredangle KCA = \measuredangle KCB + \measuredangle C = \measuredangle MEF + \measuredangle C = \theta \pmod{\pi} \Longrightarrow AK = 2R \cdot \sin\theta$.

$$\frac{2[HYZ]}{\sin 2A} = MA \cdot (AK - MA) = MA \cdot MK = |R^2 - MO^2|.$$

But by Euler's theorem, we get

$$\frac{[XYZ]}{[ABC]} = \frac{|R^2 - MO^2|}{R^2} \Longrightarrow \frac{[HYZ]}{[XYZ]} = \frac{R^2 \cdot \sin 2A}{2[ABC]} = \frac{[OBC]}{[ABC]} \Longrightarrow$$

$$\frac{HX}{HT} = \frac{TX}{TH} - 1 = \frac{[XYZ]}{[HYZ]} - 1 = \frac{[ABC]}{[OBC]} - 1 = \text{const}.$$

Arab

#3 Nov 30, 2013, 12:32 pm

I think it should be $\frac{[ABC]}{[OBC]} + 1$ or $\frac{[ABC]}{[OBC]} - 1$ depending on whether $\angle A > 90^\circ$.

Quick Reply

High School Olympiads

Incircle radius.

Reply

Iambosama

#1 Nov 30, 2013, 12:47 am

Give triangle ABC. M, N are 2 points on segment BC. Prove that $r_{ABN} = r_{ACM}$ if and only if $r_{ACN} = r_{ABM}$ where r is the incircle radius of the triangle.
I believe this is a well-khown problem

Luis González

#2 Nov 30, 2013, 1:18 am

Let X, Y, J, K denote the centers of the incircles of $\triangle ABM, \triangle ACN, \triangle ABN, \triangle ACM,$ respectively. In general, the lines BC, XY, JK concur (see the two references below). Hence, its follows that $r_{ABN} = r_{ACM} \Longleftrightarrow JK \parallel BC \Longleftrightarrow XY \parallel BC \Longleftrightarrow r_{ACN} = r_{ABM}$.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=113885
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=485516

Quick Reply

High School Olympiads

nice geometry 3

Locked

ilovemath121
#1 Nov 29, 2013, 9:26 am

Quadrilateral ABCD inscribe circle (O). AC intersects BD at X.
Circle (O1) contact XC,XD and contact (O) at Y ; circle (O2) contact AD, BC contact (O) at Y'. Prove that: Y=Y'

Luis González
#2 Nov 29, 2013, 9:40 am • 1

Why is it so difficult to use meaningful subjects ?. Subjects such as "nice geometry 1", "nice geometry 2", etc, do not describe the purpose of the problem. In addition, you did not even bother to check the second link that I posted in your latter problem nice geometry 2. Because of this, the topic is locked.

High School Olympiads

nice geometry 2

Reply

ilovemath121

#1 Nov 28, 2013, 4:57 pm

Quadrilateral ABCD inscribe circle (O). AC intersects BD at X. Circle (O1) contact XA,XB and contact (O) at Y. Circle (O2) contact XC,XD and contact (O) at Z. Prove that YZ,AD,BC are concurrent.

Luis González

#2 Nov 29, 2013, 8:58 am

Kindly, use meaningful subjects next time. This configuration has been discussed before.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=463508
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=474157

mathuz

#3 Dec 4, 2013, 1:37 pm

thank you, Luis.
I have seen the problem. But, without solution.
Poncelet's porism!
very easy solution.
Thanks to @armpist

Quick Reply

High School Olympiads

A circle through the Feuerbach's point

Reply

jayme
#1 Nov 27, 2013, 11:33 pm

Dear Mathlinkers

1. ABC a triangle
2. (O) the circumcircle of ABC
3. Na the Nagel point of ABC
4. A1B1C1 the Na-cevian triangle of ABC
5. (O1) the circumcircle of A1B1C1
6. Fe the Feuerbach point of ABC.

Prove : Fe is on (O1)

Sincerely
Jean-Louis

Luis González
#2 Nov 29, 2013, 6:11 am

Cevian circles of the incenter, orthocenter, Nagel point, Gergonne point, Mittenpunkt, Schiffler point, etc all pass through Fe. Why is this happening ?; they all lie on the Feuerbach circum-hyperbola of ABC, whose center is is Fe.

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=109112
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=416192 (proof)

Quick Reply

High School Olympiads

orthocenter lies on A_3A_5

DonaldLove

#1 Nov 19, 2013, 9:22 pm

triangle ABC, circumscribed circle (O). A_1 is on BC. A_2 is the reflection of A through O. A_1A_2 meets (O) at A_3. A_4 is the reflection of A through OA_1. line through A_4 perpendicular to BC meets (O) at A_5. prove that A_3A_5 go through the orthocenter of ABC.

Luis González

#2 Nov 20, 2013, 12:26 am • 1

Denote H the orthocenter of $\triangle ABC$. AH cuts (O) again at P. Since OA_1 bisects $\angle A_2OO_4$, then $OA_1 \parallel A_2A_4 \Longrightarrow$ pencils A_2A_4 and OA_1 are similar, thus they are projective. But since the pencils OA_1, A_2A_1 are perspective, then pencils A_2A_4 and A_2A_3 are projective $\Longrightarrow A_3 \mapsto A_4$ is a proyectivity. $A_4 \mapsto A_5$ is an involutive homography whose center is the infinite point of $\perp BC$, thus $A_3 \mapsto A_5$ is a proyectivity.

When A_1 is at infinity, then $A_3 \equiv P, A_5 \equiv A$ and when $A_1 \equiv OA_2 \cap BC$, then $A_3 \equiv A, A_5 \equiv P$. Therefore, it follows that $A_3 \mapsto A_5$ is an involutive homography on $(O) \Longrightarrow$ all A_3A_5 go through the fixed pole of the involution. When A_1 coincides with the midpoint D of BC, then $A_5 \equiv A_2$. Hence the fixed point is $H \equiv AP \cap DA_2$.

P.S. For other solutions see the topic Nice collinear.

High School Olympiads

Concurrent 1

Reply

jayme
#1 Nov 15, 2013, 6:30 pm

Dear Mathlinkers,
a very little problem for beginning...

1. ABC a triangle
2. H the orthocenter
3. (O) the circumcircle of ABC
4. A', B', C' the points of intersetion of OH wrt BC, CA, AB
5. A'', B'', C'' the circumtraces of AH, BH, CH on (O).

Prove : A'A'', B'B'' and C'C'' are concurrent

Sincerely
Jean-Louis

Luis González
#2 Nov 16, 2013, 3:13 am • 1

Jean Louis, I think that you're aware that this is a particular case of the following configuration:

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=327661
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=50&t=560673

jayme
#3 Nov 16, 2013, 11:31 am

Dear Luis and Mathlinkers,
Yes, of course…
Just to remember that this point of concur is the Euler reflexion point (X110)
and also that a little study has been put on

http://perso.orange.fr/jl.ayme , vol. 3, La P-transversale de Q, p. 8-12.

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

AD', IH, BC are concurrent

Locked

thiennhan97

#1 Nov 11, 2013, 7:49 pm

Given triangle ABC with (I) be incircle. (I) tangents BC, AC, AB at D, E, F respectively. H is orthocenter of triangle DEF. Let D' be symmetric with D through EF. Prove that AD', IH, BC are concurrent.

Luis González

#2 Nov 12, 2013, 12:21 am

Posted before at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=291602

For generalizations see
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=346956
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=425224
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=495103

High School Olympiads

Show that three lines in triangle are concurrent

Reply

Stefan4024
#1 Nov 10, 2013, 5:36 am

In a triangle ABC, a circle is inscribed with center in I. The inscribed circle touches sides BC,CA,AB in D,E,F respectively. Join the points C and F, B and E. Let Q and R be the point of intersection of the segments BE and CF with the inscribed circle respectively. Let M be the intersection point between the side BC and the line that passes through ER.

Show that MF,DE,QR are concurent

Luis González
#2 Nov 11, 2013, 2:25 am

Since $G \equiv BE \cap CF$ is always inside the incircle (I), then there exists a homology taking (I) into another circle whose center is the image of G. The projected $\triangle DEF$ and its tangential triangle $\triangle ABC$ clearly become equilateral with common center G. In this figure, by symmetry, M is the midpoint of $\overline{DC}$ and DE, DF trisect $\overline{QR}$ through U, V, i.e. U is midpoint of $\overline{VR}$. Since $DC \parallel VR$, then it follows that F, U, M are collinear, i.e. MF, DE and QR concur.

Stefan4024
#3 Nov 11, 2013, 7:26 am

Luis González Your proof is great, plain and simple, I understand every part of it, but one question for you, is it possible to construct that circle whose center is the image of G after the homology?

jayme
#4 Nov 11, 2013, 3:55 pm

Dear Mathlinkers,
we can also
1. EF, BC and QR are concurrent
2. Pascal theorem to the degenerated hexagon FDDEERF
and we are done...
Sincerely
Jean-Louis

Stefan4024
#5 Nov 11, 2013, 5:41 pm

@jayme The problem to show that EF,CB and QR are concurrent is differen from the one I've posted, but anyhow it's interesting one, but can you be more specific in you explanation. How to use the Pascal's Theorem?

Quick Reply

High School Olympiads

feet of altitude, bisector etc collinear

Reply

cause_im_batman
#1 Nov 3, 2013, 9:46 pm • 1

In triangle ABC, heights AA1 and BB1 and angle bisectors AA2 and BB2 are drawn. The inscribed circle is tangent to sides BC and AC at points A3 and B3, respectively. Prove that lines A1B1, A2B2 and A3B3 are concurrent.

mathreyes
#2 Nov 10, 2013, 9:42 pm

Does anyone know a proof for this?

Luis González
#3 Nov 10, 2013, 10:10 pm

It has been posted before, A_1B_1, A_2B_2 and A_3B_3 concur at the A-Pelletier point of ABC. One proof consists in proving the equality of the cross ratios (A_1, A_2, A_3, C) and (B_1, B_2, B_3, C). See Again again with two parallels (post #8).

Another proof: orthocenter $H \equiv AA_1 \cap BB_1$, incenter $I \equiv AA_2 \cap BB_2$ and Gergonne point $G_e \equiv AA_3 \cap BB_3$ lie on a same circumconic $\mathcal{H}$, the isogonal of OI WRT $\triangle ABC$, where O is the circumcenter. Hence CH, CI, CG_e cut AB at the poles C_1, C_2, C_3 of A_1B_1, A_2B_2, A_3B_3 WRT $\mathcal{H} \Longrightarrow A_1B_1, A_2B_2$ and A_3B_3 concur at the pole of AB WRT $\mathcal{H}$.

jayme
#4 Nov 11, 2013, 3:23 pm

Dear Mathlinkers,
for more, you can also see
http://perso.orange.fr/jl.ayme vol. 9 Le triangle reflechi p. 12
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Fixed line

Reply

Source: Own

buratinogigle

#1 Nov 8, 2013, 11:52 pm

Let ABC be a triangle and P is a fixed point. Q is a point on line BC. M,N are on line PA such that $QM\parallel PC, QN\parallel PB$. K,L are on QM,QN respectively such that $NK\parallel AB, ML\parallel AC$.

a) Prove that NK,ML,BC are concurrent.

b) Let BK cuts CL at R. Prove that R lies on a fixed line when Q moves on BC.

Luis González

#2 Nov 9, 2013, 4:15 am

a) Since $\angle MQD=\angle PCB, \angle NQD=\angle PBC, \angle NML=\angle PAC, \angle QML=\angle PCA, \angle QNK=\angle PBA$ and $\angle MNK=\angle PAB$, we have

$$\frac{\sin \widehat{MQD}}{\sin \widehat{NQD}}=\frac{\sin \widehat{PCB}}{\sin \widehat{PBC}} \ , \ \frac{\sin \widehat{NML}}{\sin \widehat{QML}}=\frac{\sin \widehat{PAC}}{\sin \widehat{PCA}} \ , \ \frac{\sin \widehat{QNK}}{\sin \widehat{MNK}}=\frac{\sin \widehat{PBA}}{\sin \widehat{PAB}}$$

$$\frac{\sin \widehat{MQD}}{\sin \widehat{NQD}} \cdot \frac{\sin \widehat{NML}}{\sin \widehat{QML}} \cdot \frac{\sin \widehat{QNK}}{\sin \widehat{MNK}}=\frac{\sin \widehat{PCB}}{\sin \widehat{PCA}} \cdot \frac{\sin \widehat{PAC}}{\sin \widehat{PAB}} \cdot \frac{\sin \widehat{PBA}}{\sin \widehat{PBC}}=1.$$

By the converse of trig Ceva theorem in $\triangle QMN$, the lines NK, ML and $QD \equiv BC$ concur.

Luis González

#3 Nov 9, 2013, 4:49 am

b) As Q varies, all $\triangle QMN$ with their cevian triangle $\triangle DLK$ are homothetic with center $D \equiv BC \cap AP$, thus K and L run on two lines τ, ℓ through D and all KL remain parallel $\Longrightarrow$ series K and L with base lines τ and ℓ are similar, thus they are perspective $\Longrightarrow$ pencils BK and CL are perspective. So, it suffices to check 3 positions of R.

Let the parallel through B to KL cut ℓ,τ at U,V and let the parallel through C to KL cut ℓ,τ at X,Y. When $K \equiv V$, or $L \equiv X$, we have $U \equiv R$ or $R \equiv Y$, respectively and when K is at infinity, R is the intersection of the parallels from B,C to DY,DU, respectively. By Pappus theorem for the hexagon $UBRCYD$, whose opposite sidelines are parallel, it follows that U,R,Y are collinear. Hence, all R fall on the line UY.

Quick Reply

High School Olympiads

A parallel to a perspectrix

Reply

Source: Own

jayme
#1 Nov 7, 2013, 4:36 pm

Dear Mathlinkers,

1. ABC a triangle
2. H the orthocenter of ABC
3. DEF the orthic triangle of ABC
4. U, V the points of intersection of the Euler line of ABC with AC, AB
5. M, N the points of intersection the points of the parallels to BC through U, V with resp. BH, CH.

Prouve : MN is parallel to the perspectrix of ABC and DEF.

Sincerely
Jean-Louis

Luis González
#2 Nov 8, 2013, 11:49 pm

First, we prove that the perspectrix of ABC and DEF is perpendicular to the Euler line (this is actually well-known). Let EF, FD, DE cut BC, CA, AB at X, Y, Z. If P is the midpoint of $\overline{BC}$, then from $(B, C, D, X) = -1$, we get $XB \cdot XC = XD \cdot XP \Longrightarrow X$ has equal power WRT the circumcircle (O) and the 9-point circle (O_9). Similarly, Y, Z have equal powers WRT (O) and $(O_9) \Longrightarrow XYZ$ is radical axis of $(O), (O_9) \Longrightarrow XYZ \perp OH$.

Let the perpendicular from A to UV cut BH, CH at M', N'. Since $HM' \perp AU$ and $AM' \perp HU$, then M' is the orthocenter of $\triangle AUH \Longrightarrow UM' \perp AH \Longrightarrow UM' \parallel BC \Longrightarrow M' \equiv M'$. Similarly $N \equiv N'$, thus $MN \perp UV \Longrightarrow MN \parallel XYZ$.

Quick Reply

High School Olympiads

geometry

Reply

elegant

#1 Sep 2, 2013, 2:34 pm

Given the quadrilateral $ABCD$, the inscribed circle (I), $A = 90^0$. BI intersects AD at M, DI intersects AB at N. Prove that : AC is perpendicular to MN

Novice_123

#2 Sep 2, 2013, 4:18 pm

We only need to prove that $AM^2 - AN^2 = CM^2 - CN^2$.
Take E of DC such that $DE = DA$, F on CB such that $BF = BA$.
Due to symmetrical properties, NE is perdendicular to CD, MF to perdendicular to BC and $NE = NA, MF = MA$.
$CM^2 - CN^2 = CF^2 + MF^2 - CE^2 - EN^2 = CF^2 - CE^2 + MF^2 - NE^2 = CF^2 - CE^2 + AM^2 - AN^2 = AM^2 - AN^2$
(since $CF = CB - BF = CB - BA = CD - AD = CD - DE = CE$).

Luis González

#3 Nov 8, 2013, 9:39 am • 1

Let $X \equiv AC \cap BD, Y \equiv AB \cap CD$ and $Z \equiv DA \cap BC. O$ denotes the midpoint of $\overline{MN}$.
$\angle INA = 90^\circ - \angle IDA = \frac{1}{2}\angle ADY = \angle AIY \Longrightarrow YI^2 = YA \cdot YN \Longrightarrow Y$ has equal power WRT the circle (O) with diameter $\overline{MN}$ and $I \Longrightarrow Y$ is on radical axis of $(O), I.$ Similarly Z is on radical axis of $(O), I \Longrightarrow YZ$ is radical axis of $(O), I \Longrightarrow YZ \perp OI.$ But since YZ is polar of X WRT (I), then $YZ \perp XI \Longrightarrow X \in OI.$

BD cuts YZ at the pole P of AC WRT $(I) \Longrightarrow IP \perp AC.$ Moreover, from the complete $BDYZ,$ the cross ratio (B, D, X, P) is harmonic $\Longrightarrow I(B, D, X, P) = I(M, N, O, P) = -1 \Longrightarrow MN \parallel IP \perp AC.$

Quick Reply

High School Olympiads

A nice midpoint

Reply

Source: own

jayme

#1 Nov 6, 2013, 7:22 pm

Dear Mathlinkers,
1. P a point
2. D the point of contact of the incircle of ABC with BC
3. U, V the incenters of the triangles ABP, ACP

Conclusion : we know that P, D, U and V are concyclic on (L)

4. U', V' the B-excenter of ABP, the C-excenter of ACP

Conclusion : we know that P, D, U and V are concyclic on (M)

5. X, Y the second points of intersection of AP with resp. (L), (M).

Prouve that A is the midpoint of XY.

Sincerely
Jean-Louis

Luis González

#2 Nov 6, 2013, 11:54 pm

Let the incircle (I) touch CA, AB at $E, F.$ Common external tangent τ of (U) and (V), different from BC, cuts AP at X'. Since $X'V, X'U$ bisect $\angle(AP, \ell)$ and PU, PV bisect $\angle(AP, BC)$, then $\angle UX'V$ and $\angle UPV$ are right $\Longrightarrow U, V, P, X'$ are concyclic $\Longrightarrow X$ and X' coincide. Now according to the problem Locus of intersection, we have $AX = AE = AF.$ In the same way, we will have $AY = AE = AF$, thus A is the midpoint of $\overline{XY}$.

Quick Reply

High School Olympiads

complete quad, cyclic property

Reply

cause_im_batman
#1 Nov 5, 2013, 6:48 pm

The quadrilateral ABCD is cyclic. AD and BC meet at P, AC and BD meet at R. The points X,Y,Z are the feet of the perpendiculars from D to AC,BC,PR. Prove that circle XYZ passes through M, midpoint of CD.

Luis González
#2 Nov 6, 2013, 3:25 am

Let (O) be the circumcircle of $ABCD. S \equiv AB \cap CD$ and N, L, K, G are the midpoints of $AB, PR, CP, CR,$ respectively. M, N, L are collinear on the Newton line of the complete $PARB.$

Since P, R are conjugate points WRT $(O),$ then the circle (L) with diameter $\overline{PR}$ is orthogonal to (O) and the circle $\odot(OMSN)$ whose diameter connects O with the pole S of PR WRT $(O) \Longrightarrow LR^2 = LM \cdot LN \Longrightarrow LR$ touches $\odot(RMN) \Longrightarrow \angle RMN = \angle LRN.$ But since $\triangle RAB$ and $\triangle RDC$ are similar with corresponding medians $RN, RM,$ we have $\angle GMR = \angle DRM = \angle ARN \Longrightarrow$
$\angle GML = \angle GMR - \angle RMN = \angle ARN - \angle LRN = \angle ARP = \angle GKL$ (mod 180°) $\Longrightarrow M \in \odot(GLK),$ i.e. 9-point circles of $\triangle CPR$ and $\triangle DCR$ meet at $G, M.$ Likewise, 9 point circle of $\triangle DPR$ goes through $M \Longrightarrow M$ is the Poncelet point of $CPDR \Longrightarrow$ pedal circle $\odot(XYZ)$ of D WRT $\triangle CPR$ goes through $M.$

Quick Reply

High School Olympiads

symmetric points in circle, concurrency

Reply

cause_im_batman

#1 Nov 3, 2013, 5:03 pm

Points A1 and A2 that lie inside a circle centered at O are symmetric through point O. Points P1, P2, Q1, Q2 lie on the circle such that rays A1P1 and A2P2 are parallel and in the same direction, and rays A1Q1 and A2Q2 are also parallel and in the same direction. Prove that lines P1Q2, P2Q1 and A1A2 are concurrent.

Luis González

#2 Nov 5, 2013, 8:44 pm • 1

Let A_2P_2 and A_2Q_2 cut the circle (O) again at X and Y, respectively. By obvious symmetry $A_1P_1A_2X$ is a parallelogram with diagonal intersection $O \equiv A_1A_2 \cap XP_1$ and $A_1Q_1A_2Y$ is a parallelogram with diagonal intersection $O \equiv A_1A_2 \cap YQ_1$. By Pascal theorem for the hexagon $Q_1P_2XP_1Q_2Y$, the intersections $Q_1P_2 \cap P_1Q_2$, $A_2 \equiv P_2X \cap Q_2Y$ and $O \equiv XP_1 \cap YQ_1$ are collinear $\Longrightarrow P_1Q_2, P_2Q_1$ and A_1A_2 are concurrent.

Quick Reply

High School Olympiads

triangle , concyclicity

Locked

cause_im_batman

#1 Nov 5, 2013, 7:05 pm

The points L,M,N are the midpoints of BC,CA,AB of triangle ABC. The points P and Q are on AB and BC. R and S are on AB and BC such that N is the midpoint PR and L is the midpoint of QS. Prove that if PS and QR meet at right angles at T, then T lies on circle LMN

Luis González

#2 Nov 5, 2013, 8:35 pm

Please, use the search before submitting a problem.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=364264
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=455772
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=492992

High School Olympiads

Circle tangent to the circumcircle of triangle BHC

Source: Own

Math-lover123

#1 Nov 1, 2013, 11:48 pm

In triangle ABC H,I are orthocenter and incenter respectively.
Incircle of ABC touches side BC at D.
E is a foot of perpendicular from H to the line AD.
M is the midpoint os segment AH.
Lines ME and AI intersect at F.
Prove that the circle centered at F and with radius FE is tangent to the circumcircle of triangle BHC.

Luis González

#2 Nov 4, 2013, 9:18 am

Let P, Q, R be the feet of the altitudes on BC, CA, AB. Obviously E is on circle $(M) \equiv \odot(AQR)$ with diameter $\overline{AH}$. Since $\triangle AQR \sim \triangle ABC$, then it follows that $AD \equiv AE$ is the isogonal of the A-Gergonne cevian of $\triangle AQR \Longrightarrow E$ is tangency point of the A-mixtilinear excircle of $\triangle AQR$ with $(M) \Longrightarrow (F, FE)$, with its center F on internal bisector of $\angle QAR$ and externally tangent to (M) at E, coincides then with the A-mixtilinear excircle of $\triangle AQR$.

Inversion with center A and power $\overline{AH} \cdot \overline{AP} = \overline{AQ} \cdot \overline{AC} = \overline{AR} \cdot \overline{AB}$ takes (M) into BC and $\odot(BHC)$ into the 9-point circle $\odot(PQR)$. By conformity (F, FE) goes to (I), which is certainly tangent to $\odot(PQR)$ by Feuerbach theorem, thus in the original figure, (F, FE) is tangent to $\odot(BHC)$.

Math-lover123

#3 Nov 4, 2013, 10:39 pm

> Luis González wrote:
> $\Longrightarrow E$ is tangency point of the A-mixtilinear excircle of $\triangle AQR$ with (M)

Can you prove this luis?

Luis González

#4 Nov 5, 2013, 2:17 am • 1

It has been posted many times before. I remember submitting a proof for the mixtilinear incircle case.

In any $\triangle ABC$, let D be the tangency point of its incircle (I) with BC. A-mixtilinear excircle (J) touches the circumcircle (O) at X. Let B', C' be the inverses of B, C under the inversion with center A that swaps (I) and (J). Thus $B'C'$ is the inverse of (O), tangent to (I) at the inverse X' of X. Since $B'C'$ is antiparallel to BC WRT AB, AC, then we deduce that $\triangle ABC \cong \triangle AC'B'$ are symmetric WRT $AI \Longrightarrow AD$ and $AX' \equiv AX$ are symmetric WRT AI, i.e. AD, AX are isogonals WRT $\angle BAC$.

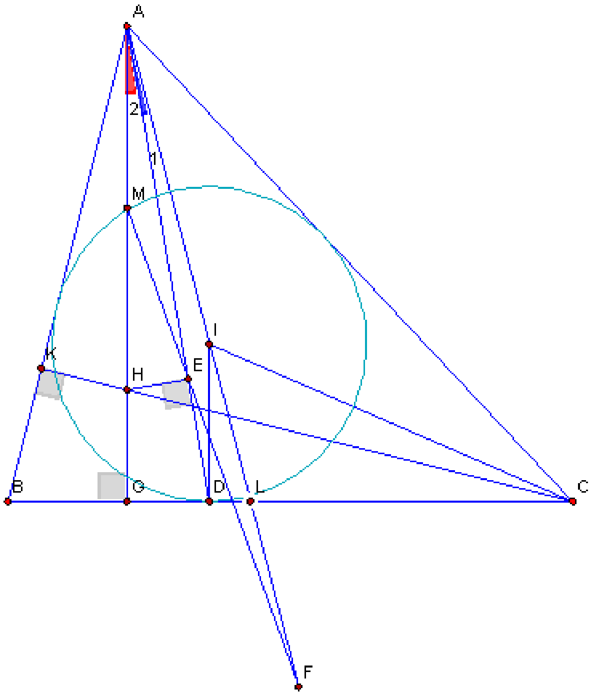
vslmat

#5 Nov 5, 2013, 11:10 pm

Another solution:
Notice that the distance from F to AB equals $AF.sin(A/2)$, so to get the desired, it suffices to show that $EF = AF.sin(A/2)$ (*). Denote the interior angles of ΔABC simply as $\angle A, \angle B, \angle C$. Let AG be the A-altitude in ΔABC, AL be the angle bisector of $\angle A$.
By sinus rule in ΔAEF we have $EF/sinA_1 = AF/sinA_2$ (see the diagram), but also

$ID/sinA_1 = AD/sin(B/2-C/2)$ and $sinA_2 = GD/AD$, so (*) is equivalent to
$ID.sin(B/2-C/2) = GD.sin(A/2)$.
Notice that $\angle GAL = B/2 - C/2$, thus easy to have $GD = AI.sin(B/2-C/2)$ and as $ID/AI = sin(A/2)$, (*) holds and the proof is completed.

Attachments:



Quick Reply

High School Olympiads

Triangle 36M418

Reply

Source: Geoffrey Kandall 2010

Ligouras
#1 Nov 4, 2013, 8:23 pm

E lies on the segment GQ and F lies on the segment GE. Let GEP be a triangle. A is on the sides PG and H lies on the segment GA. QA intersects PF at B, QA intersects PE at C, and QH intersects PE at D. Prove that

$$\frac{AB}{BC}\frac{CD}{DE}\frac{EF}{FG}\frac{GH}{HA}=1$$

Luis González
#2 Nov 4, 2013, 11:53 pm

Send the line PQ to infinite through central projection, the projected $A'C'E'G'$ is then a parallelogram and $F'B' \parallel A'G'$ and $D'H' \parallel A'C'$. Moreover, the product of the single ratios $(BAC)\cdot(DCE)\cdot(FEG)\cdot(HGA)$ is invariant $\Longrightarrow$

$$\frac{AB}{BC}\cdot\frac{CD}{DE}\cdot\frac{EF}{FG}\cdot\frac{GH}{HA}=\frac{A'B'}{B'C'}\cdot\frac{C'D'}{D'E'}\cdot\frac{E'F'}{F'G'}\cdot\frac{G'H'}{H'A'}.$$

From the parallelograms $A'B'F'G'$, $B'C'E'F'$, $C'D'H'A'$ and $D'E'G'H'$, we have $A'B'=F'G'$, $B'C'=E'F'$, $C'D'=H'A'$ and $D'E'=G'H'$. Hence, the RHS of the latter expression equals 1, as desired.

Ligouras
#3 Nov 5, 2013, 2:45 pm

intelligent and elegant solution ... thanks LUIS!!
is similar to the solution given by me.

the problem is open aspect and other solutions from Mathlinker

sunken rock
#4 Nov 5, 2013, 4:01 pm

@Ligouras:

Once you solve ***this one***, I'll show you another method!

Best regards,
sunken rock

sunken rock
#5 Nov 6, 2013, 12:57 am • 1

Now, thanks to **ThirdTimeLucky** I can present my solution.

Lemma:
(**Menelaos theorem for quadrilaterals**): Let $ABCD$ be a quadrilateral and M,N,P,Q the intersection of a line with its sides (we can have either 2 points on the sides and the remainder on the extensions of the other sides, or all points outside the quad).
Then the following relation does exist: $\frac{AM}{MB}\cdot\frac{BN}{CN}\cdot\frac{CP}{DP}\cdot\frac{DQ}{AQ}=1\ (*)$

Proof of lemma: A parallel to the line through B intersects AD, CD at R, S respectively. Then $\frac{AM}{MB}=\frac{AQ}{QR}, \frac{BN}{NC}=\frac{PS}{CP}$.
Substituting in $(*)$ we get $\frac{PS}{PD}=\frac{RQ}{QD}$, which is true, hence our assumption is true as well.

Now to problem:
Applying the **lemma** for quadrilateral $AGEC$ with the transversal $\overline{HDQ}$ we get: $\frac{AH}{GH} \cdot \frac{GQ}{QE} \cdot \frac{ED}{DC}=1\ (\ 1\ )$ and for the same quadrilateral with the transversal $\overline{FBP}$ we get $\frac{FG}{EF} \cdot \frac{PE}{PC} \cdot \frac{BC}{AB} \cdot \frac{AP}{PG}=1\ (\ 2\ )$.

Multiply the 2 previous relations side by side, take into account the result from **here** and we are done.

Best regards,
sunken rock

Quick Reply

High School Olympiads

daothanhoai

#1 Nov 1, 2013, 11:45 am

Dear Mathlinkers!

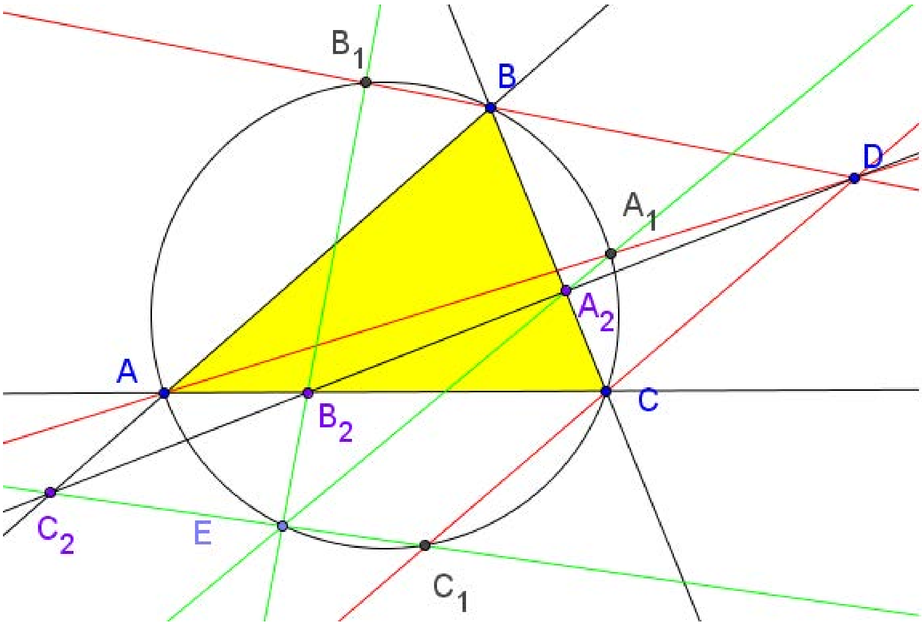
Let a conic through three point A, B, C*, and any point D on the plane.* DA, DB, DC *respectively intersect the conic at* A_1, B_1, C_1*. E is a point on the conic.* EA_1 *intersect* BC *at* A_2*,* EB_1 *intersect* AC *at* B_2*,* EC_1 *intersect* AB *at* C_2*. Then* A_2, B_2, C_2, D *are colinear.*

Converse: Let six points A, B, C, A_1, B_1, C_1 *are on a conic.* AA_1, BB_1, CC_1 *concurrent at* D*. Construct a line through* D *intersect* BC, CA, AB *at* A_2, B_2, C_2*. Then* A_1A_2, B_1B_2, C_1C_2 *are concurrent on the conic.*

Best regard

Dao Thanh Oai

Attachments:



A Generalization Simson's line,carnot theorem, Collings-Carnot theorem, Adam-Bliss-Ibid problem.pdf (194kb)

Luis González

#2 Nov 3, 2013, 10:25 pm

By Pascal theorem for the hexagon $ABCC_1EA_1,$ the intersections $C_2 \equiv AB \cap C_1E, A_2 \equiv BC \cap EA_1$ and $D \equiv CC_1 \cap A_1A$ are collinear. By Pascal theorem for the hexagon $BCAA_1EB_1,$ the intersections $A_2 \equiv BC \cap A_1E,$ $B_2 \equiv CA \cap EB_1$ and $D \equiv AA_1 \cap B_1B$ are collinear $\Longrightarrow A_2, B_2, C_2$ and D are collinear.

The converse is proved in the same way, in fact, it was posted before at The point of concurrency lies on the circumcircle.

fmasroor

#3 Nov 7, 2013, 9:17 am

Very cool thing here but I am unsure why D at infinity yields Simson lines...

Quick Reply

High School Olympiads

Concurrency-Three parabolas sharing common directrix.

Reply

Source: Own.

mohohoho

#1 Nov 3, 2013, 8:39 am

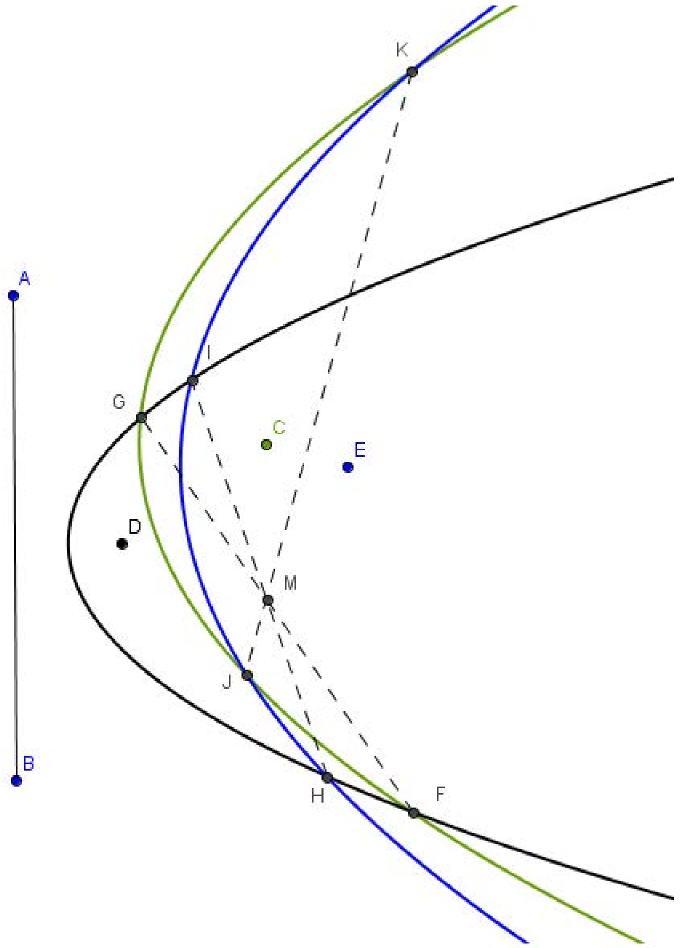
A beautiful result, indeed!

Prove the following statement:

If three parabolas share a common directrix and each pair intersect each other in two points, then, the lines joining the two intersection points of each pair of parabolas are concurrent.

By Emmanuel.

Attachments:



This post has been edited 1 time. Last edited by mohohoho, Nov 5, 2013, 6:50 am

Luis González

#2 Nov 3, 2013, 9:08 am • 3

We use the same notations given in diagram. ℓ denotes their common directrix and C, D, E are their respective foci. $KC = \text{dist}(K, \ell) = KE$ and similarly $JC = \text{dist}(J, \ell) = JE \Longrightarrow JK$ is the perpendicular bisector of $\overline{CE}$. Analogously, HI and GF are the perpendicular bisectors of $\overline{ED}$ and $\overline{DC} \Longrightarrow JK, HI$ and GF concur at the circumcenter of $\triangle CDE$.

mohohoho

#3 Nov 3, 2013, 9:14 am

Ha ha, it was pretty simple! :p

Quick Reply

High School Olympiads

Radical center on Kiepert hyperbola

Reply

daothanhoai

#1 Nov 2, 2013, 10:53 pm

Dear Mathlinkers

Let ABC be a triangle, construct isosceles similar triangle AC_0B, BA_0C, CB_0A (either all outward, or all inward), such that $\angle BAC_0 = \angle ABC_0 = \angle BCA_0 = \angle CBA_0 = \angle CAB_0 = \angle ACB_0 = \alpha$.

Then radical center of three circle ($(AC_0B), (BA_0C), (CB_0A)$) are on kiepert hyperbola.

Best regard

Dao Thanh Oai

Luis González

#2 Nov 3, 2013, 4:13 am • 1

WLOG assume that all triangles are erected outside $\triangle ABC,$ the remaining case is treated similarly. Let X be the 2nd intersection of $\odot(CAB_0)$ and $\odot(ABC_0).$ $\angle(XB,XA)=\angle AC_0B=\angle AB_0C=\angle(XA,XC) \Longrightarrow AX$ is internal bisector of $\angle BXC$ meeting the perpendicular bisector of $\overline{BC}$ at the midpoint A_1 of the arc BC of $\odot(BXC) \Longrightarrow$ $\angle A_1BC=\angle A_1CB=\angle BXA_1=\angle AC_0B=\pi-2\alpha \Longrightarrow$ radical axis AX of $\odot(CAB_0)$ and $\odot(ABC_0)$ goes through Kiepert perspetor $K(\pi-2\alpha).$

Likewise, radical axes of $\odot(BCA_0),\odot(CAB_0)$ and $\odot(ABC_0),\odot(BCA_0)$ go through $K(\pi-2\alpha) \Longrightarrow$ radical center of $\odot(BCA_0),\odot(CAB_0),\odot(ABC_0)$ is the Kiepert perspector $K(\pi-2\alpha)$ lying then on Kiepert hyperbola of $\triangle ABC.$

Quick Reply