High School Olympiads

3 circles concurrent at Feuerbach point

Reply

Source: (generalization from right triangle)

yetti

#1 Sep 15, 2009, 2:47 am

Let $\triangle ABC$ be a scalene triangle with 9-point circle (N) and incircle (I), touching BC, CA, AB at $D, E, F,$ respectively. Let H be orthocenter of the reference triangle $\triangle ABC$ and H' orthocenter of its intouch triangle $\triangle DEF.$ Let altitudes of these 2 triangles pairwise intersect at $D' \equiv AH \cap DH', E' \equiv BH \cap EH', F' \equiv CH \cap FH'.$ Show that the 3 circles with diameters DD', EE', FF' concur at the Feuerbach point $Fe \equiv (N) \cap (I)$ of the triangle $\triangle ABC.$

This is generalization of the problem A circle through the Feuerbach's point.

Luis González

#2 Sep 15, 2009, 6:38 am

Nice generalization yetti!. My proof won't be very different from the one that I gave here.

Reflection n of BC about the angle bisector AV_a is the inverse of the nine-point circle (N) under the inversion with center M (midpoint of BC) and power $MD^2 \Longrightarrow F'_e \equiv n \cap (I)$ is the inverse of F_e. Let H_a be the foot of the A-altitude and S the midpoint of $IV_a,$ which is the center of the circle $\odot(F'_eV_aD)$ by obvious symmetry.

Since $DD' \parallel AV_a,$ it follows that $\angle D'DI = \angle DIV_a = \angle IDS.$ This implies that circles (S) and $(K_a) \equiv \odot(H_aDD')$ form equal and oppositely directed angles with the double circle (I). Since the inversion preserves angles, D is double and V_a, H_a are inverse points, it follows that circle (S) is identical with the inverse of (K_a). F'_e is common point of $(I), (S), n,$ then in the primitive figure, F_e must be common point of $(I), (K_a), (N).$

Petry

#3 Sep 16, 2009, 2:28 am

Hello!

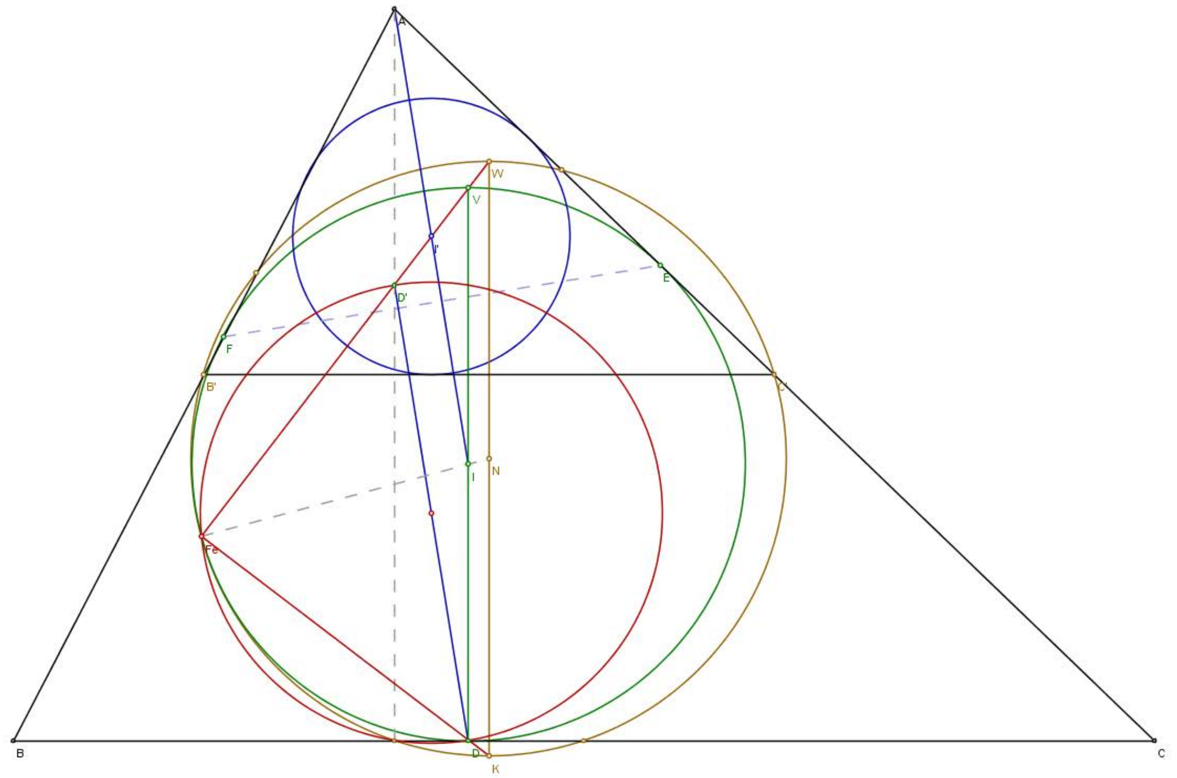
r is the inradius of the triangle ABC and R is the circumradius of the triangle ABC.
B' is the midpoint of AB, C' is the midpoint of AC, I' is the incenter of the triangle $AB'C'$. It's easy to prove that I' is the midpoint of AI.
$\{D, V\} = DI \cap (I)$ and the perpendicular to the line BC through the point N intersects the circle (N) at the points K, W ($AK > AN$, $AW < AN$).
The points N, I, F_e are collinear, $IV || NW$ and $\dfrac{IF_e}{NF_e} = \dfrac{IV}{NW} = \dfrac{2r}{R} \Rightarrow$ the points W, V, F_e are collinear. (1)
Similarly, the points K, D, F_e are collinear and it's easy to see that $DF_e \perp VF_e$. (2)

The proposed problem posted here http://www.artofproblemsolving.com/Forum/viewtopic.php?t=284806 $\Rightarrow I' \in F_eV$. (3)
$AD' \perp BC$ and $ID \perp BC \Rightarrow AD' || ID$. (4)
$DD' \perp EF$ and $IA \perp EF \Rightarrow DD' || IA$. (5)
(4),(5) $\Rightarrow AD'DI$ is a parallelogram $\Rightarrow AD' = ID = IV = r$.
$AD'IV$ is a parallelogram and I' is the midpoint of $AI \Rightarrow I'$ is the midpoint of $D'V$. (6)
(1),(3),(6) $\Rightarrow$ the points F_e, D', I', V, W are collinear. (7)

(2),(7) $\Rightarrow DF_e \perp D'F_e$. So, the point F_e lies on the circle with diameter DD'.
Similarly, F_e lies on the circles with diameters EE', FF'.

Best regards, Petrisor Neagoe

Attachments:



yetti

#4 Sep 16, 2009, 9:50 am

Thank you, my friends, for nice proofs of the proposed problem. In my opinion, if reflection in a line is part of synthetic geometry, so is reflection in a circle (inversion). Would there be a shorter proof of the problem referenced by **Petry**? I went for a different parallelogram in my solution:

Internal angle bisectors AI, BI, CI cut the triangle circumcircle (O) again at $X, Y, Z.$ Let A', B', C' be midpoints of $BC, CA, AB.$ Internal bisector of the angle $\angle C''A'B'$ cuts the 9-point circle (N) again at $X'.$ Let $XU, X'U' \perp BC$ be diameters of $(O), (N).$ Feuerbach point Fe is the external similarity center of $(I), (N) \Longrightarrow Fe, D, U'$ are collinear $\Longrightarrow$ $\angle DFeA' = \angle U'FeA' = \angle U'X'A' = \angle UXA.$

Let X'', Y'', Z'' be reflections of X, Y, Z in $BC, CA, AB. X$ is circumcenter of $\triangle IBC$; midpoint N_a of IX'' is its 9-point circle center. Let the 9-point circle (N_a), passing through $A', D,$ cut its I-altitude ID at $K \Longrightarrow KI = A'X$ and $A'XIK$ is a parallelogram $\Longrightarrow \angle DKA' = \angle IKA' = \angle A'XI = \angle UXA \Longrightarrow \angle DFeA' = \angle DKA'$ and $Fe \in (N_a).$

$\triangle X''Y''Z'' \sim \triangle XYZ$ is Fuhrmann triangle and its circumcircle (O'') is Fuhrmann circle of $\triangle ABC,$ with diameter $HM,$ where M is Nagel point, the anticomplement of the incenter $I.$ The Fuhrmann circle cuts the altitudes AH, BH, CH at $A'', B'', C'',$ such that $AA'' = BB'' = CC'' = 2ID = 2r$ (8-point circle!). In addition, $X''A'', Y''B'', Z''C''$ concur at $I.$ For a proof, see the general case of P-Hagge circle at http://www.mathlinks.ro/viewtopic.php?t=285239. Proof for I-Hagge circle (or Fuhrmann circle) is substantially simpler, on account of the incenter I being its own isogonal conjugate. In particular, no supporting lemma is neccessary.

Since $ID \parallel AD'$ (both $\perp BC$) and $AI \parallel DD'$ (both $\perp EF$) $\Longrightarrow AIDD'$ is a parallelogram and $AD' = ID \Longrightarrow$ $D'A'' = ID$ and $IDA''D'$ is a parallelogram $\Longrightarrow$ its diagonals IA'', DD' cut each other in half at $Q_a \Longrightarrow X'', N_a, I, Q_a$ are collinear. Let (Q_a) be circle with diameter $DD'.$ As a result, the circles $(I), (N_a), (Q_a),$ concurrent at $D,$ are coaxal and also concurrent at the other intersection Fe of $(I), (N_a).$ Similarly, circles with diameters EE', FF' also pass through $Fe. \square$

jayme

#5 Sep 16, 2009, 5:25 pm

Dear Vladimir, Luis and Mathlinkers,
very nice generalization and interesting proofs.
I think that another approach can be obtain after looking on my figure that you can see on:
http://perso.orange.fr/jl.ayme vol. 3 the cross-cevian point p. 28.
by considering the pivot theorem...
I have to write the proof of my result and your generalization.
Sincerely
Jean-Louis

Petry

#6 Oct 2, 2009, 12:45 am

yetti wrote: "Would there be a shorter proof of the problem referenced by **Petry**?"
Finally, I found an easy solution. See here , the message #7.

jayme

#7 Oct 3, 2009, 2:00 pm

Dear Mathlinkers,
a synthetic proof of this result can be see
on: http://perso.orange.fr/jl.ayme vol. 5 Le théorème de Feuerbach-Ayme p. 16
Sincerely
Jean-Louis

shoki

#8 May 19, 2010, 1:44 pm

there is also a solution here

skytin

#9 May 20, 2010, 10:32 pm

I have very easy solution:
Let AX is altitude from A on BC let Y is intersection of EF and DD' and let Z is intersection of line EF and CB . D'YXZ is cyclic so angle XD'D = XZY = C + A/2 - 90 . let point Fe=F', let point M is midpoint of BC , let point N is midpoint of AB , line F'D is bissector of angle XF'M (it's Archimedes lemm for incircle and 9 points circle) and angle XF'M = XNM and easy to prove that angle XNM = 180 - 2B - A and angle XF'D = XNM/2 = 90 - B - A/2 . 180 = A + B + C so C + A/2 - 90 = 90 - B - A/2 so XD'F'D is cyclic so 3 circles with diameters DD' ... concur at the Feuerbach point

Quick Reply

High School Math

A problem of locus?Help me!

foraus
#1 Sep 13, 2009, 9:23 pm

Two circles (O;R) and (O';r) are tangent internally at a point A (R>r).A line tangent to (O') at a point D intersects (O) at points B and C.(I) is the inscribed circle of triangle ABC.Find the locus of I.

foraus
#2 Sep 14, 2009, 6:41 pm

> foraus wrote:
> Two circles (O;R) and (O';r) are tangent internally at a point A (R>r).A line tangent to (O') at a point D intersects (O) at points B and C.(I) is the inscribed circle of triangle ABC.Find the locus of I.

No one helps me?:(.Is it too hard?:(

This post has been edited 1 time. Last edited by foraus, Sep 15, 2009, 3:55 pm

Luis González
#3 Sep 14, 2009, 10:03 pm

I'll restate the problem as follows:

Problem: Two circles T_1 and T_2 are internally tangent at A and T_1 is bigger than T_2. A variable tangent of T_2 cuts T_1 at B, C. Then the locus of the incenter of $\triangle ABC$ is another circle tangent to T_1, T_2 through A.

Proof. Let V be the tangency point of T_2 with BC. It is known that AV bisects $\angle BAC$.

$$\frac{AI}{IV} = \frac{CA+AB}{BC}.$$

If the ray AV cuts T_1 at P, then by Ptolemy's theorem for $ABPC$, we have

$$BC \cdot AP = CA \cdot PB + AB \cdot PC.$$

Since $PB = PC \Longrightarrow \frac{AP}{PB} = \frac{CA+AB}{BC} \Longrightarrow \frac{AI}{IV} = \frac{AP}{PB}$.

Note that $\triangle PAB \sim \triangle PVB$ are similar because of $\angle VBP = \angle BAP$, thus we have $PB^2 = AP \cdot PV$. Combining this one with the previous expression yields

$$\frac{IV^2}{AI^2} = \frac{PV}{AP}$$, but $PV = AP - AV$ and $IV = AV - AI$

$$\Longrightarrow \left(\frac{AW}{AI} - 1\right)^2 = 1 - \frac{AV}{AP}.$$

Ratio $\frac{AV}{AP} =$ const is the coefficient k of the direct homothety taking T_1 into T_2. Therefore, locus of the incenter I is the homothetic circumference of T_2 under the homothety with center A and coefficient $\frac{1}{\sqrt{1-k}+1}$.

High School Olympiads

Ibero Team Selection Test

Reply

Source: Ibero Puerto Rican Team Selection 2009

modularmarc101

#1 Sep 12, 2009, 9:52 am

1. A positive integer is called *good* if it can be written as the sum of two distinct integer squares. A positive integer is called *better* if it can be written in at least two was as the sum of two integer squares. A positive integer is called *best* if it can be written in at least four ways as the sum of two distinct integer squares.

a) Prove that the product of two *good* numbers is *good*.

b) Prove that 5 is *good*, 2005 is *better*, and 2005^2 is *best*.

2. In each box of a 1×2009 grid, we place either a 0 or a 1, such that the sum of any 90 consecutive boxes is 65. Determine all possible values of the sum of the 2009 boxes in the grid.

3. Show that if h_A, h_B, and h_C are the altitudes of $\triangle ABC$, and r is the radius of the incircle, then

$h_A + h_B + h_C \geq 9r$

4. The point M is chosen inside parallelogram $ABCD$. Show that $\angle MAB$ is congruent to $\angle MCB$, if and only if $\angle MBA$ and $\angle MDA$ are congruent.

5. The weird mean of two numbers a and b is defined as $\sqrt{\frac{2a^2+3b^2}{5}}$. 2009 positive integers are placed around a circle such that each number is equal to the the weird mean of the two numbers beside it. Show that these 2009 numbers must be equal.

This post has been edited 1 time. Last edited by modularmarc101, Sep 12, 2009, 8:17 pm

Dr Sonnhard Graubner

#2 Sep 12, 2009, 7:48 pm

hello, for 3) i think it must be
$h_A + h_B + h_C \geq 9r$
Sonnhard.

mavropnevma

#3 Sep 12, 2009, 7:54 pm

2. Denote by a the word made by the first 29 boxes in the grid, and by b the word made by the next 61 boxes in the grid. Then the word representing the grid is $g = (ab)^{22}a$. Since the sum of the numbers in a may be any integer value between 0 and 29, the sum of the numbers in the grid may be any integer value between $22 \cdot 65 = 1430$ and $22 \cdot 65 + 29 = 1459$.

5. Expression $\sqrt{\frac{2a^2+3b^2}{5}}$ is called weird mean for a good reason: indeed $\min(a,b) \leq \sqrt{\frac{2a^2+3b^2}{5}} \leq \max(a,b)$ (in effect it is a pondered quadratic mean).
Take x (one of) the least number(s) written on the circle, and a, b its neighbors, so $x \leq a$ and $x \leq b$. According with the above it follows $x = a = b$. Continue around the circle to get all numbers equal to x.

modularmarc101

#4 Sep 12, 2009, 8:18 pm

Thanks Sonnhard

enndb0x

#5 Sep 12, 2009, 8:45 pm

solution to problem 3

Mewto55555

#6 Sep 13, 2009, 11:15 pm

1a is a direct consequence of the Brahmagupta-Fibonacci Identity.

Bugi

#7 Sep 13, 2009, 11:20 pm

4

Luis González

#8 Sep 14, 2009, 1:02 am

> *modularmarc101 wrote:*
> 4. The point M is chosen inside parallelogram $ABCD.$ Show that $\angle MAB$ is congruent to $\angle MCB,$ if and only if $\angle MBA$ and $\angle MDA$ are congruent.

Let $X \equiv AM \cap CD$ and $Y \equiv CM \cap AD.$ Note that quadrilateral $ACXY$ is cyclic $\Longleftrightarrow \angle DAX = \angle DCY \Longleftrightarrow \angle YXM = \angle MCA$ and $\angle XYM = \angle MAC.$ Since $\angle MXD = \angle MAB$ and $\angle MYD = \angle MCB,$ it follows that $MYDX$ and $MABC$ are similar $\Longleftrightarrow \angle MBA = \angle MDA.$

Quick Reply

High School Olympiads

Iran(3rd round)2009

Source: Problem 3 Geometry

shoki

#1 Sep 13, 2009, 6:28 pm

3-There is given a trapezoid $ABCD$ in the plane with $BC||AD$.We know that the angle bisectors of the angles of the trapezoid are concurrent at O.Let T be the intersection of the diagonals AC,BD.Let Q be on CD such that $\angle OQD=90^\circ$.Prove that if the circumcircle of the triangle OTQ intersects CD again at P then $TP||AD$.

Luis González

#2 Sep 13, 2009, 11:54 pm

The trapezoid has incircle (O) and X,Y,Q are the tangency point of (O) with $BC,AD,CD,$ respectively. By Newton's theorem, the diagonal intersection T lies on $XY,$ which is perpendicular to its bases. Since the quadrilateral $PQTO$ is cyclic, it follows that $\angle OQP=\angle PTO=90^{\circ} \Longrightarrow TP \perp XY \Longrightarrow TP \parallel AD,$ as desired.

High School Olympiads

Iran(3rd round)2009

Reply

Source: Problem 1 Geometry

shoki
#1 Sep 13, 2009, 6:25 pm

1-Let $\triangle ABC$ be a triangle and (O) its circumcircle. D is the midpoint of arc BC which doesn't contain A. We draw a circle W that is tangent internally to (O) at D and tangent to BC.We draw the tangent AT from A to circle W.P is taken on AB such that $AP=AT$.P and T are at the same side wrt A.PROVE $\angle APD=90^\circ$.

Luis González
#2 Sep 13, 2009, 11:01 pm

Let X,Y be the orthogonal projections of D on AB,AC and B' the reflection of B about $AD.$ Then it follows that B' lies on the circle centered at D that goes through $B,C,$ due to $\angle BB'C=90^\circ+\frac{1}{2}\angle A.$ Hence, DY is perpendicular bisector of $B'C.$ WLOG assume that $AC>AB \Longrightarrow B'Y=CY=\frac{1}{2}(AC-AB).$ Therefore

$$AX=AY=AB'+B'Y=AB+\tfrac{1}{2}(AC-AB)=\tfrac{1}{2}(AB+AC) \ \ (1)$$

On the other hand, by Casey's theorem for $(A,B,C,\omega),$ all tangent to $(O),$ we have

$$AT\cdot BC=AC\cdot BM+AB\cdot CM=\tfrac{1}{2}BC(AB+AC)$$

$$\Longrightarrow AP=AT=\tfrac{1}{2}(AB+AC) \ \ (2)$$

From (1) and $(2),$ we conclude that X and P are identical $\Longrightarrow \angle APD=90^\circ.$

mathVNpro
#3 Sep 13, 2009, 11:11 pm

Let M be midpoint of BC and D_1,D_2 respectively be the projections of D onto AB, AC. Then by *Simpson line theorem*, we have D_1,M,D_2 are collinear. Let $X\equiv AD\cap D_1D_2$. In the other hand, since AD is the internal bisector wrt $\angle BAC$ of $\triangle ABC$. Therefore, $AD\perp D_1D_2$ at X, which implies $\angle MXD=90^\circ$. Thus $X\in(W)$. Moreover, we have $AD_1^2=\overline{AX}.\overline{AD}=\mathcal{P}_{A/(W)}=AT^2\Longrightarrow AD_1=AT$, which implies that $D_1\equiv P$, which leads to the result of the problem. $\square$

livetolove212
#4 Sep 14, 2009, 11:41 am

Let M be the midpoint of BC.
$\frac{AD}{CD}=\frac{AT}{CM}\Rightarrow\frac{AD}{AP}=\frac{CD}{CM}$ and $\angle MCD=\angle DAB\Rightarrow\angle APD=\angle CMD=90^o$

Moonmathpi496
#5 Feb 7, 2010, 12:34 am

> *shoki wrote:*
> 1-Let $\triangle ABC$ be a triangle and (O) its circumcircle. D is the midpoint of arc BC which doesn't contain A. We draw a circle W that is tangent internally to (O) at D and tangent to BC.We draw the tangent AT from A to circle W.P is taken on AB such that $AP=AT$.P and T are at the same side wrt A.PROVE $\angle APD=90^\circ$.

Complete trig bashing!
Let $DP' \perp AB$. Then it is enough to prove that $AP' = AP = AT$.
We have $AP' = AD\cos\frac{B}{2}$. Sine law in $\triangle ABD$ gives,

$$\begin{aligned}AD &= 2R\sin(B + A/2)\\ \Longrightarrow AP' &= R\cdot 2\sin(B+A/2)\cos\frac{B}{2} = R(\sin C + \sin B) = \frac{b+c}{2}\end{aligned}$$

Let M be the midpoint of BC. Now from power of point, $AT^2 = AW^2 - MW^2$
From Cosine law in $\triangle AMD$,

$$\begin{aligned}AW^2 &= AM^2 + MW^2 - 2AM\cdot MW\cos\angle AMD\\ \Longleftrightarrow AT^2 &= AW^2 - MW^2 = AM^2 + 2AM\cdot MW\cos\angle AMH'\end{aligned}$$

Here $AH' \perp MO$. We can see that $H'M = AH = AM\cos\angle AMH'$
We know that $AM^2 = \frac{2b^2+2c^2-a^2}{4} = \frac{b^2+c^2}{4} + \frac{b^2+c^2-a^2}{4} = \frac{b^2+c^2}{4} + \frac{bc\cos A}{2}$.
Also, $AH = b\sin C$, and $MD = \frac{a}{2}\tan\frac{A}{2}$
So,

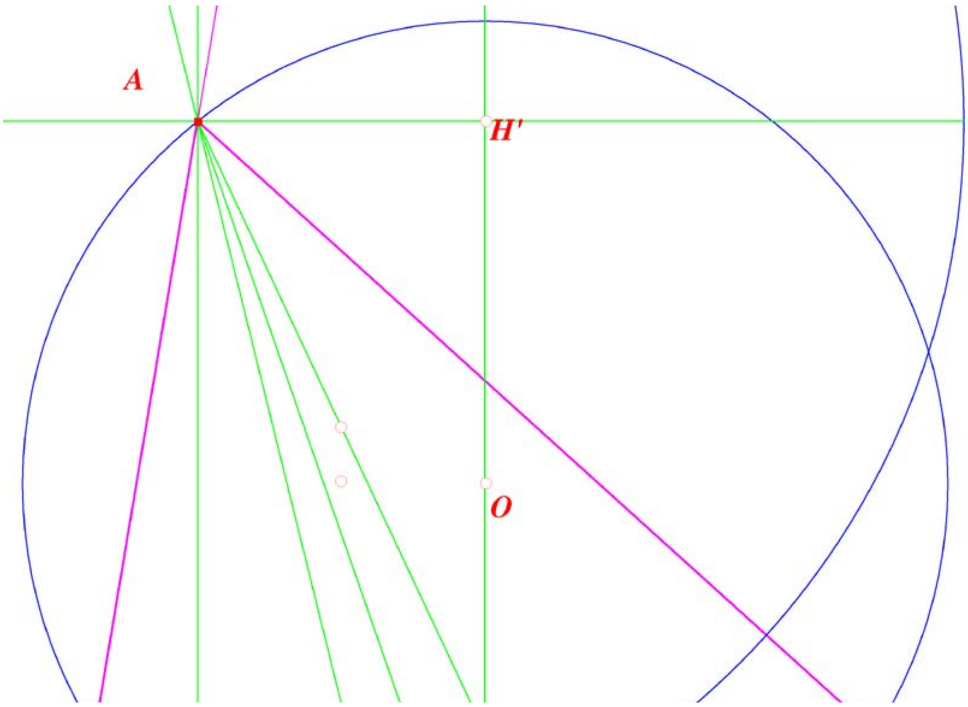
$$\begin{aligned}2AM\cdot MW\cos\angle AMH' =& AH\cdot MD\\ =&\frac{1}{2}ab\sin C\tan\frac{A}{2}\\ =&\frac{bc}{2}\sin A\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}\\ =&\frac{bc}{2}\cdot 2\sin^2\frac{A}{2}\end{aligned}$$

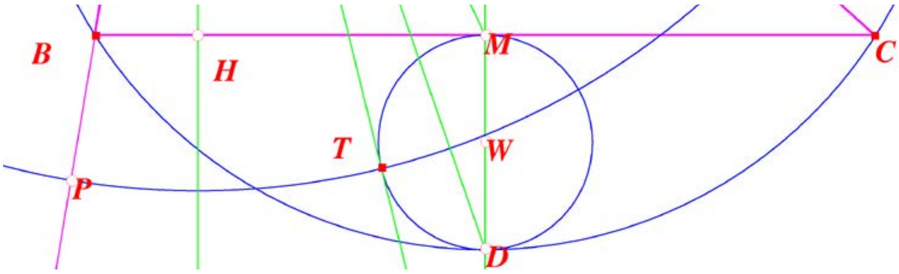
At last,

$$\begin{aligned}AP^2 =&\frac{b^2+c^2}{4} + \frac{bc\cos A}{2} + \frac{bc}{2}\cdot 2\sin^2\frac{A}{2}\\ =&\frac{1}{4}\left(b^2+c^2+2bc(\cos A + 2\sin^2 A/2)\right)\\ =&\frac{(b+c)^2}{4}\end{aligned}$$

So, $AP = AP' = \frac{b+c}{2}$, and we are done!

Attachments:





dgreenb801

#6 Feb 7, 2010, 9:43 pm

> livetolove212 wrote:
> $$\frac{AD}{CD}=\frac{AT}{CM}$$

Why?

Quick Reply

High School Olympiads

plane geometry 2

Reply

Source: symmetric point

Dimitris X

#1 Sep 13, 2009, 3:23 am

In a triangle ABC,we consider points B', C' on AB, AC such that $BB' = CC'$.Let D be the intersection of lines BC' and $B'C$.Prove that the symmetric point of D with respect to the midpoint M of BC lies on the angle bisector of $\angle A$.

Luis González

#2 Sep 13, 2009, 7:17 am

It suffices to show that D lies on the anticomplement of the bisector of $\angle BAC$, i.e. the A_C bisector of the anticomplementary triangle $\triangle A_0B_0C_0$. Let $k = BB' = CC'$. Using barycentric coordinates we have

$$B' \ (k : c-k : 0) \ , \ C' \ (k : 0 : b-k) \ , \ BC' \cap CB' \equiv D \ (k : c-k : b-k)$$

Eliminating k from the coordinates of D, the locus of D is $(b-c)x + by - cz = 0$, which is the anticomplement of the A-angle bisector $cy - bz = 0$.

ma 29

#3 Sep 17, 2009, 5:00 am

Denote by$P$ the symmetric point of Awrt$M$.
$BC' \cap CP = S$.
We have :
$\frac{DB}{DS} = \frac{BB'}{CS} = \frac{CC'}{CS} = \frac{PB}{PS}$
$=>$ Dlies on the angle bisector of the angle BPS.
$=>$ Elies on the angle bisector of the angle BAC.
:)

lajanugen

#4 Feb 14, 2010, 7:30 pm

Let D' be the reflection of D across M
Since $BC'//D'C$ and $CB'//D'B$, we have $[BD'B'] = [BD'C] = [D'CC']$ and hence, D' lies on the bisector of $\angle BAC$

Quick Reply

Poland

geometria - trojkat

Reply

Megus

#1 Feb 5, 2006, 10:29 pm • 1

Jest dany trojkat ABC. Rysujemy linie laczaca punkty w ktorych dwusieczne katow ABC i ACB przecinaja przeciwlegle sciany trojkata. Przez punkt przeciecie tej prostej z dwusieczna kata BAC, rysujemy prosta rownolegla do BC. Niech ta prosta przecina AB w M i AC w N. Udowodnij, ze $2MN = BM + CN$

P.S. Zapodalem prosciutka geometrie, bo w koncu forum trzeba rozruszac

Cinek

#2 Apr 13, 2006, 9:23 pm

a to podpowiedz z czego tu korzystać bo z moją geometrią na poziomie gimnazjum trudno cokolwiek zauważyć

buli_

#3 Jul 12, 2006, 10:39 pm

odświeżam dość stary wątek, bo może Cinek dalej nad tym myśli można wszystko przeliczyć korzystając z tw. o dwusiecznej i z podobieństwa. A może jest jakiś ładniejszy sposób?

M4RI0

#4 Jul 17, 2006, 6:21 am

Could someone translate the problem to english :

TomciO

#5 Jul 17, 2006, 11:14 am

Sure.
There is a given triangle ABC. We draw a line connecting points in which angle bisectors of $\angle ABC$ and $\angle ACB$ intersect the opposite sides of triangle. From a point of intersection of this line and angle bisector of $\angle BAC$ we draw line which is parallel to line BC. This line intersects AB in M and AC in N. Prove that: $2MN = BM + CN$

Luis González

#6 Sep 12, 2009, 9:16 pm • 1

> Quote:
>
> In $\triangle ABC$, let V_a, V_b, V_c be the feet of the internal angle bisectors of $\widehat{A}, \widehat{B}, \widehat{C}$. Let P be the intersection of AV_a with V_bV_c. The parallel to BC through P cuts AB, AC at M, N. Then $2 \cdot MN = BM + CN$.

We use trilinear coordinates with respect to $\triangle ABC$. Thus

$$V_a(0:1:1)\ ,\ V_b(1:0:1)\ ,\ V_c(1:1:0)$$

Hence $V_bV_c \equiv \alpha - \beta - \gamma = 0$, which means that the line V_bV_c is the locus of the points whose sum of oriented distances to AB, AC equals the oriented distance to BC. Therefore if X, Y, Z denote the orthogonal projections of P onto BC, CA, AB, we have $PY + PZ = PX$, but $PY = PZ \Longrightarrow PX = 2 \cdot PZ$.

$[BMNC] = [PBM] + [PCN] + [PBC] = \frac{1}{2}PY \cdot (BM + NC) + \frac{1}{2}PX \cdot BC.$

But the area of trapezoid $BMNC$ is $[BMNC] = \frac{1}{2}PX \cdot (MN + BC) \implies$

$PX \cdot (MN + BC) = PY \cdot (BM + NC) + PX \cdot BC \implies$

$PX \cdot MN = PY \cdot (BM + CN) \implies 2 \cdot MN = BM + CN.$

Quick Reply

High School Math

tricky geometry!

Reply

BaBaK Ghalebi

#1 Sep 12, 2009, 2:02 am

we have a square with side length a,we draw 4 circles with radius a from each of the 4 vertices of the square,calculate the area of the intersection of these 4 circles.

P.S:I'm not sure if this is the right place to post this problem,so sorry if it isn't...

Luis González

#2 Sep 12, 2009, 2:24 am

Denote by A,B,C,D the vertices of the square with side length R. Circles $(C),(D)$ centered at C,D with radii R meet inside $ABCD$ at P. Likewise, $Q\equiv(D)\cap(A), R\equiv(A)\cap(B)$ and $S\equiv(B)\cap(C)$. Clearly, $\triangle ABR$ is equilateral with side length R, thus $\angle RAB=60^\circ\Longrightarrow\angle DAR=30^\circ$ and due to obvious symmetry, $\angle BAQ=30^\circ$. Thus, the region S overlapped by $(A),(B),(C),(D)$ is the sum of the square $PQRS$ and 4 circular segments with radii R and central angle equal to 30°.

$$S=R^2\left(2-2\cos\frac{\pi}{6}\right)+4\left[\frac{R^2}{2}\left(\frac{\pi}{6}-\sin\frac{\pi}{6}\right)\right]=R^2\left(1-\sqrt{3}+\frac{\pi}{3}\right).$$

Quick Reply

Spain

Un lugar geométrico del circuncentro

Luis González

#1 Apr 25, 2009, 8:13 am

D es un punto variable en el lado $\overline{BC}$ de un $\triangle ABC.$ Por D se trazan las paralelas a los lados AB y AC que intersectan a AC y AB en $X, Y,$ respectivamente. Demostrar que el lugar geométrico del circuncentro de $\triangle AXY$ es una recta que es perpendicular a la A-simediana del $\triangle ABC.$

Luis González

#2 Sep 11, 2009, 6:00 am

Si M y N sin puntos medios de AB y $AC,$ con semejanza de triángulos es facil probar que $\frac{MX}{NY}=\frac{c}{b}.$ Las circunferencias $\odot(AMN)$ y $\odot(AXY)$ se cortan pues en el centro de la semejanza espiral que lleva $\overrightarrow{MX}$ a $\overrightarrow{NY}.$ Entonces $\frac{PM}{PN}=\frac{AM}{AN}=\frac{c}{b} \Longrightarrow P$ es fijo y pertenece entonces a la A-circunferencia de Apolonio del triángulo $\triangle AMN.$

Expandiendo con la homotecia de centro en A y coeficiente $2,$ se ve que P es punto medio de $AQ,$ donde Q es el punto de corte de la A-circunferencia de Apolonio de $\triangle ABC$ con su circunferencia circunscrita, la cual no es mas que la recta de la A-simediana de $\triangle ABC.$ En definitiva, el lugar del circuncentro de $\triangle AXY$ es la mediatriz de $AP.$

Attachments:

Lugar geométrico, circuncentro.pdf (19kb)

High School Olympiads

A circle through the Feuerbach's point

Reply

Source: own

jayme

#1 Sep 9, 2009, 4:16 pm

Dear Mathlinkers,
let ABC be a rectangular triangle at A,
A' the foot of the A-altitude on BC,
J, K the incenters of the triangles AA'B, AA'C
and Fe the Feuerbach's point of ABC.
The circle with diameter JK passes through Fe.
Any reference? I haven't see thsi result in the litterature.
Sincerely
Jean-Louis

Luis González

#2 Sep 10, 2009, 7:32 am

Let X, Y denote the tangency points of the incircle (I) and the A-excircle (I_a) with BC. W denotes the foot of the angle bisector of $\angle BAC$. Since the midpoint M of BC is also midpoint of XY and the cross ratio (A', W, X, Y) is harmonic, it follows from Newton's theorem that $MX^2 = MY^2 = MW \cdot MA'$. Consequently A' and W are inverse points under the inversion with center M and power $MX^2 = MY^2$. Such inversion takes (I) and (I_a) into themselves and the nine-point circle (N) of $\triangle ABC$ into the common internal tangent n of $(I), (I_a)$ different from the sideline BC, which is the reflection of BC about AW. Then $F'_e \equiv n \cap (I)$ is the image of F_e under the referred inversion.

From the topic Incircles and parallel segments, we know that the circumference (V) with diameter JK goes through A', X and the triangle $\triangle KXJ$ isosceles and right at X, thus XV bisects $\angle KXJ$. Further, $XK \parallel AB$ and $XJ \parallel AC$, i.e. angles $\angle BAC$ and $\angle KXJ$ have parallel sides $\Longrightarrow$ their bisectors are parallel as well $\Longrightarrow VX \parallel AW$.

Let S be the midpoint of IW, which is obviously by symmetry the center of the circle $\odot(XWF'_e)$. We have $\angle SXI = \angle IXV$ due to $VX \parallel IW$. In other words, angles between circles (V) and $(S) \equiv (XWF'_e)$ with (I) are equal but oppositely directed. Since the inversion preserves angles between curves, the point X is double and A' and W are inverse images, it follows that (S) is identical with the inverse image of (V) under the inversion. F'_e is common point of $(I), (S), (n)$, then in the primitive figure, F_e must be common point of $(I), (V), (N)$.

Attachments:

right-angled triangle-Feuerbach point.pdf (23kb)

jayme

#3 Sep 13, 2009, 6:38 pm

Dear Luis and Mathlinkers,
I appreciate your proof requiring inversion.
Why not a synthetic one without inversion?
A new challenge...
Sincerely
Jean-Louis

livetolove212

#4 Sep 14, 2009, 12:39 am

Let L be the incenter of triangle $AA'B$.
A well-known result: I is the orthocenter of triangle ALK.
$\Rightarrow \angle IAK = \angle ILK = \angle KCB = \angle LAA'$.

Therefore the circumcenter H of triangle ALK lies on AA'.
Since $\angle LAK = 45^o$ then $\angle LHK = 90^o$. We get $AH = HK = DK = ID = r = AR$
But AK is the bisector of angle HAR then $AHKR$ is a rhombus.
$\Rightarrow RK = r$.
Let P be the intersection of BI and AC. Construct a line which passes through P and parallel to RK, it cuts KM at S.
We will show that RS is a bisector of angle KRM
$\Leftrightarrow \frac{RK}{RM} = \frac{KS}{SM} = \frac{RP}{PM}$
$\Leftrightarrow \frac{r}{RM} = \frac{r - RP}{RP} = \frac{r}{RP} - 1(*)$
Note that $r = \frac{b+c-a}{2}, RM = \frac{b}{2} - \frac{b+c-a}{2} = \frac{a-c}{2}$ and
$RP = AP - AR = \frac{bc}{c+a} - \frac{b+c-a}{2} = \frac{(a+c-b)(a-c)}{2(a+c)}$
So $(*) \Leftrightarrow \frac{b+c-a}{a-c} + 1 = \frac{(c+a)(b+c-a)}{a+c-b}$
$\Leftrightarrow a^2 = b^2 + c^2$ (right!)
We obtain $\angle PSR = \angle PRS = 1/2\angle B$. Therefore PS is a tangency of (I).
Thus MS passes through Fe. Similarly QL passes through Fe and $\angle LFeK = \angle MFeQ = 90^o$ hence Fe lies on (LK)

Attachments:
picture3.pdf (6kb)

jayme
#5 Sep 14, 2009, 2:25 pm

Dear Livetolove and Mathlinkers,
nice metric proof...
Why not a synthetic one without calculus?
A new possible challenge...
Sincerely
Jean-Louis

yetti
#6 Sep 15, 2009, 3:47 am

Let the incircle (I) of $\triangle ABC$ touch BC, CA, AB at D, E, F. $BI \perp AK, CI \perp AJ \Longrightarrow I$ is orthocenter of $\triangle AJK$ and $AI \perp JK$. $\angle KAA' = \frac{1}{2}\angle ABC = \frac{1}{2}(90^\circ - \angle BCA) = \angle IAJ \Longrightarrow AI, AA'$ are isogonals WRT the angle $\angle KAJ$ $\Longrightarrow$ circumcenter P of $\triangle AJK$ is on AA'. Since $\angle KAJ = 45^\circ \Longrightarrow \triangle PJK$ is isosceles right. Let (Q) be its circumcircle on diameter JK; then $A', D \in (Q)$. (This is true for any $\triangle ABC$ and for any $A' \in BC$, well known and posted before). From the right angle $\angle DA'P$, DP is diameter of (Q) and $JDKP$ is a square. Since $JK \perp AI$, it follows that $DP \parallel AI$ and $DP \perp EF$. By the problem 3 circles concurrent at Feuerbach point, the circle (Q) with diameter DP goes through the Feuerbach point Fe of $\triangle ABC$.

At the time of this post, the problem referenced above is to be solved. I do have a synthetic proof.

shoki
#7 Sep 15, 2009, 6:28 am

i'll use the following lemma:
Lemma- ABC is a right angled triangle at A and H_a its projection on BC. Let K, J be the incenters of AH_aC, AH_aB.Let C', B' be the midpoints of AB, AC.then $C'J, B'K$ intersection lies on the circle with diameter JK and also on the circle with diameter $B'C'$(The nine-point circle).
Let S be the intersection point mentioned then we have
$\Delta AH_aB \sim \Delta AH_aC \Longrightarrow \angle JC'A = \angle KB'C, \angle A'B'C = \angle A'C'A = 90^\circ \Longrightarrow \angle A'C'J = \angle A'B'K \Longrightarrow$ C'SA'B' is concyclic thus$\angle JSK = \angle C'SB' = \angle C'A'B' = 90^\circ$
Done.
Also i will use this well-known property :Let I be the incenter and D its projection on BC. then we have $\angle IDJ = \angle B, \angle KDI = \angle C$.it has a synthetical proof.
Let T_c, T_b be the intersections of AJ, CK and AK, BJ respectively.Let X, L be the projections of I on AB, AC respectively.Let the line parallel to BC and passing through I intersects $C'J, B'K$ at T and T' respectively.Let K', J' be the midpoints of JI, KI respectively.we have the following properties:
1-$\angle JAK = 45^\circ$,
2-I is the orthocenter of ΔAJK which implies the perpendicularity of CK, AJ at T_c and similarly for BJ, AK at T_b,
4-T_c, T_b, B', C' are collinear,

5-$IT_c = T_cJ, IT_b = T_bK$,
6-$JD = DI = DK$,
7-X, T_c, D are collinear,Similarly for D, T_b, L,
8-$\angle T_cDT_b = \angle XDL = 45^\circ$,
9-$DI = T_cT_b$,we can prove this by the fact that $\Delta DJ'I = \Delta J'T_cT_b$.
10-$TI||T_cT_b$,$T'I||T_cT_b$.
11-$DK' = K'X, \angle K'T_bD = \angle K'DT_b = 45^\circ \implies \Delta DT_bX \text{is right - angled isocele}$ and similarly for ΔT_cDL.
12-T_cKT_bX, JT_bLT_c are parallellograms.
13-we'll prove $TT_c||JT_b$.We must prove $\frac{C'T}{TJ} = \frac{C'T_c}{T_cT_b}$.
but we have $C'T_b||TI \implies \frac{C'T}{TJ} = \frac{T_bI}{IJ}$ and also $XT_c||AT_b \implies \frac{C'T_c}{T_cT_b} = \frac{C'X}{XA}$.So we must prove the equality $\frac{T_bI}{IJ} = \frac{C'X}{XA}$.
Let O be the projection of C' on AJ.Let $P = AJ \cap XT_b$.If we prove $C'P$ is the angle bisector of $B'C'A$ then we'll have $\Delta C'OA \sim \Delta JKT_b$.
So we should only show that $C'P \perp AK$.but it's known that $AC' = C'T_b, AP = PT_b$ so both C', P lie on the perpendicular bisector of AT_b.But we have $\angle OC'P = \angle C'PX = 180^\circ - ((\angle B/2) + (90^\circ + \angle C/2)) = 45^\circ$.in this way we get: $C'P$ is the angle bisector of $AC'B'$ thus
$\frac{C'X}{XA} = \frac{OP}{PA}, \Delta C'OA \sim \Delta JKT_b, \angle OC'P = \angle IKT_b = 45^\circ, \implies \frac{OP}{PA} = \frac{T_bI}{IJ}$.
14-$TT_c||JT_b \implies TT_c||IT_b$,Similarly for $T'T_b$ and IT_c.
By (10) and (9) we can concude that TT_cT_bI and $T_cT_bT'I$ are parallellograms $\implies T_bT_c = DI = IT = IT'$.But we have $\angle TST' = 90^\circ \implies IS = IT = IT' = ID \implies \text{S is on the nine - point circle and the incenter thus } S = F_e$
DONE!

jayme
#8 Sep 16, 2009, 10:29 am

Dear Mathlinkers,
a file concerning my result has been open on a French site :
http://www.les-mathematiques.net/phorum/read.php?17,538526
Sincerely
Jean-Louis

jayme
#9 Oct 3, 2009, 1:56 pm

Dear Mathlinkers,
a synthetic proof of this result can be see
on: http://perso.orange.fr/jl.ayme vol. 5 Le théorème de Feuerbach-Ayme
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

angle (equal)

Reply

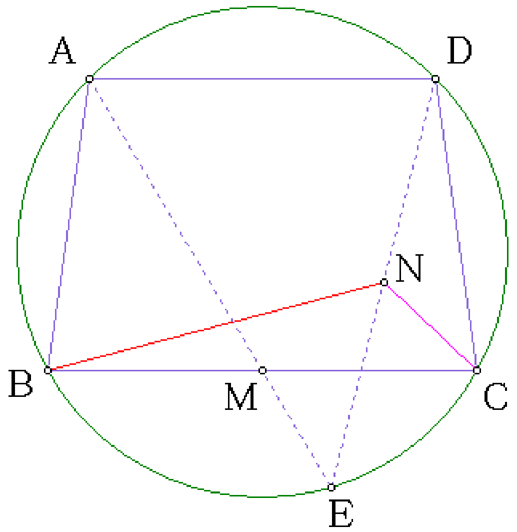
77ant
#1 Sep 9, 2009, 11:30 pm

Dear everyone.

for cyclic trapezoid ABCD (AD//BC)
M=midpoint of BC
E=AM$\cap$circumcircle
N=midpoint of DE

prove that $\angle BNE = \angle CNE$

Attachments:



Luis González
#2 Sep 10, 2009, 1:56 am

Let O be the circumcenter of the trapezoid $ABCD.$ Axial symmetry about the perpendicular bisector of BC takes $\triangle ABC$ into $\triangle DCB$ and therefore the chord AE into its corresponding DE' passing through M such that $BE' = CE,$ which implies that DE' and DE are isogonal rays with respect to $\angle BDC \Longrightarrow DE$ is the D-symmedian of $\triangle DBC.$ As a result, perpendicular bisector ON of $DE,$ tangent to (O) at D and BC concur at the center Q of the D-Apollonius circle of $\triangle DBC.$ If DE cuts BC at $F,$ then from the harmonic cross ratio (B,C,F,Q) and $NF \perp NQ,$ it follows that NQ and NE bisect $\angle BNC.$

Virgil Nicula
#3 Sep 12, 2009, 7:51 am

Nice problem!

> 77ant wrote:
> Let $ABCD$ be a trapezoid inscribed in a circle $w=C(O)$ so that $AD \parallel BC$. Denote the midpoint M of $[BC]$,
>
> the second intersection E of AM with w and the midpoint N of $[DE]$. Prove that $\widehat{BNE} \equiv \widehat{CNE}$.

Proof (similarly with **Luisgeometria**'s). Denote $S\in BC\cap DE$. Observe that $\widehat{BDS}\equiv\widehat{BDE}\equiv\widehat{BAE}\equiv\widehat{BAM}\equiv\widehat{CDM}$, i.e. $\widehat{BDS}\equiv\widehat{CDM}$.

Thus, $[DS$ is the D-symmedian in $\triangle BDC$. Denote $L\in DD\cap BC$. Is well-known that $\frac{SB}{SC}=\frac{LB}{LC}=\left(\frac{DB}{DC}\right)^2$, i.e. the points S , L

are harmonical conjugate w.r.t. B , C . Therefore, the line DS is the polar of L w.r.t. the circle w $\Longrightarrow L\in ON\cap EE$. In conclusion, from

an well-known property obtain $NL\perp NS$ $\Longrightarrow$ $\widehat{SNB}\equiv\widehat{SNC}$, i.e. $\widehat{BNE}\equiv\widehat{CNE}$.

This post has been edited 4 times. Last edited by Virgil Nicula, Sep 12, 2009, 11:51 pm

vittasko

#4 Sep 12, 2009, 7:04 pm

● Let be the points $E'\equiv (O)\cap DM$ and $K\equiv AE'\cap DE$ and we have $KM\perp BC,$ as well.

So, the line segment BC is the polar of $K,$ with respect to the circumcircle (O) of the given trapezium $ABCD.$

Let be the point $Q\equiv ON\cap BC$ and it is easy to show that the line segment DE is the polar of $Q,$ with respect to $(O),$ because BC as the polar of K with respect to $(O),$ passes through the point Q and the line segment DE as the line perpendicular to $QO,$ passes through the point $K.$

● Hence, we conclude that the points $B,\ P\equiv BC\cap DE,\ C,\ Q,$ are in harmonic conjugation and then, the pencil $N.BPCQ,$ is in harmonic conjugation too.

Because of now $NQ\perp NP,$ we conclude that the line segment $DE\equiv NP,$ bisects the angle $\angle BNC$ and the proof is completed.

Kostas Vittas.

Attachments:
t=300161.pdf (5kb)

Virgil Nicula

#5 Sep 12, 2009, 11:52 pm

Another proof. Denote $S\in BC\cap DE$ and $T\in BB\cap CC$. Observe that $\widehat{BDS}\equiv\widehat{BDE}\equiv\widehat{BAE}\equiv\widehat{BAM}\equiv\widehat{CDM}$, i.e. $\widehat{BDS}\equiv\widehat{CDM}$.

Therefore, the ray $[DS$ is the D-symmedian in $\triangle BDC$. From an well-known property obtain $T\in DS$. Since the points $B,$ C , N belong

to the circle with the diameter $[OT]$ obtain $TB=TC$ $\Longrightarrow$ $\widehat{TNB}\equiv\widehat{TNC}$, i.e. $\widehat{BNE}\equiv\widehat{CNE}$.

Quick Reply

High School Olympiads

Concurrency involving the incircle and A-excircle

Reply

Luis González

#1 Sep 9, 2009, 12:34 am

The incircle (I) of $\triangle ABC$ is tangent to BC, BA at X_a, X_c. The A-excircle (I_a) is tangent to BC at Y_a. Line X_aX_c meets I_aY_a at E and draw the circle with diameter EY_a. Tangents from B, C to the latter circle (different from BC) meet at D. Let $F \equiv X_aX_c \cap AI$ and $J \equiv (I) \cap AY_a$, J is the closer point to BC. Show that AD, FJ and BC concur.

Attachments:

Incircle, concurrency.pdf (22kb)

yetti

#2 Sep 14, 2009, 5:40 am

Let (K) be circle with diameter Y_aE, the incircle of $\triangle DBC$, and let (K_d) be its D-excircle, tangent to BC at X_a. As $X_aB - X_aC = AB - AC = DC - DB = Y_aC - Y_aB = Y_aX_a$, D, A are on hyperbola $\mathcal{H}$ with foci B, C and vertices Y_a, X_a. Angle bisectors AII_a, DKK_d are hyperbola tangents at A, D and perpendiculars IX_aK_d, KY_aI_a to BC are hyperbola tangents at X_a, Y_a. Quadrilateral II_aKK_d (convex or reflex) is tangential to $\mathcal{H}$ and AY_aDX_a is its contact quadrilateral; they have the same diagonal intersection $S \equiv IK \cap I_aK_d \equiv AD \cap Y_aX_a \equiv AD \cap BC$. This intersection is the common similarity center of the incircles $(I), (K)$ and the excircles $(I_a), (K_d)$, external when $AB > AC$ and internal when $AB < AC$.

Let M_a be the common midpoint of BC, Y_aX_a. Angle $\angle AJX_a$ is right on account of A being the external similarity center of the circles $(I), (I_a) \Longrightarrow$ angle $\angle X_aJY_a$ is right $\Longrightarrow M_aJ = M_aX_a \Longrightarrow M_aJ$ is tangent of (I) at J. On the other hand, $M_aF \parallel AB$ is C-midline of $\triangle ABC \Longrightarrow$ isosceles $\triangle FM_aX_a \sim \triangle X_cBX_a$ are centrally similar with center $X_a \Longrightarrow M_aF = M_aX_a = M_aY_a \Longrightarrow$ angle $\angle X_aFY_a$ is right $\Longrightarrow$ angle $\angle Y_aFE$ is right and $F \in (K) \Longrightarrow M_aF$ is tangent of (K) at F. From the isosceles $\triangle M_aFJ$, the line FJ forms angles of equal size with the incircles $(I), (K)$, with the same signs when $AB > AC$ and with the opposite signs when $AB < AC \Longrightarrow FJ$ goes through their similarity center S.

Quick Reply

High School Olympiads

cyclic

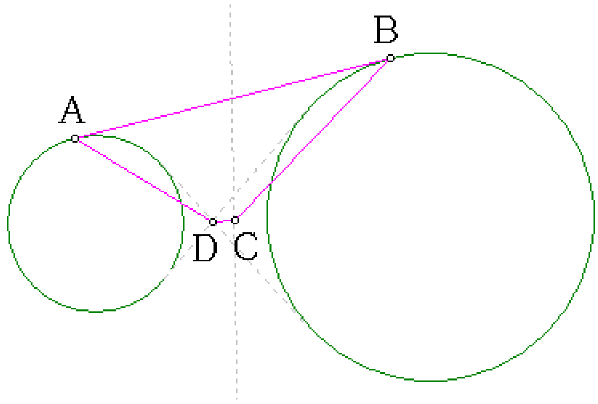
Reply

77ant
#1 Sep 7, 2009, 1:03 am

in two non intersecting circles, one of common external tangents touches them at A, B.
then, their radical axe meets line through two centers at C.
let their insimilar center be D.
prove that A,B,C,D are cyclic.

I'm afraid of not being good post.

Attachments:



shoki
#2 Sep 7, 2009, 3:28 am

Let the circle that A lies on it have the center named O. we must only prove $\Delta OAD \sim \Delta MCB$.where M is the midpoint of AB and it lies on the radical axe obviously.For this we should only prove $\frac{OA}{MC}=\frac{OD}{MB}$.the rest is solved easily by computing.

Luis González
#3 Sep 8, 2009, 12:35 am

Let O_1, O_2 denote the centers of the given circles, $A \in (O_1)$ and $B \in (O_2)$. Radical axis of $(O_1), (O_2)$ goes through the midpoint M of the common external tangent $\overline{AB}$. Let $E \equiv AB \cap O_1O_2$ be the exsimilicenter of $(O_1), (O_2)$. Since cross ratio (O_1, O_2, D, E) is harmonic, its orthogonal projection (A, B, L, E) on the external tangent is also harmonic $\Longrightarrow$ $EA \cdot EB = EL \cdot EM$. But the quadrilateral $LMCD$ is cyclic on account of the right angles $\angle DLM$ and $\angle MCD$, then $EL \cdot EM = ED \cdot EC \Longrightarrow EA \cdot EB = ED \cdot EC \Longrightarrow$ Quadrilateral $ABCD$ is cyclic.

plane geometry
#4 Sep 8, 2009, 5:43 am

denote E is the intersection of AB and CD
ED/EO2=EB=EH=2R1/(R1+R2), where R1,R2 is the radii of O1 and O2

Quick Reply

High School Olympiads

orthogonal circles

77ant

#1 Sep 7, 2009, 1:16 am

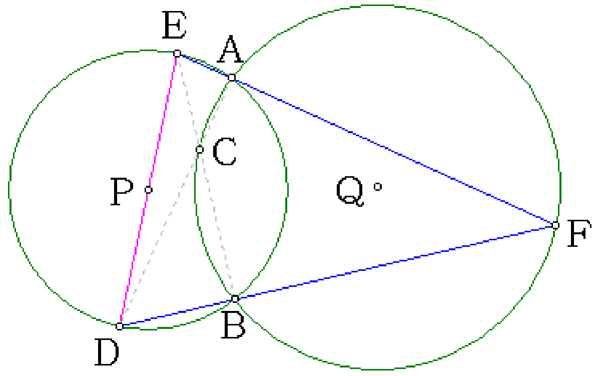
two orthogonal circles meet each other at A, B.
for an arbitrary point C on arc AB (small arc), let AC, BC meet one of them at D, E respectively.
EA meets DB at F.

prove that

(1) ED is a diameter of one of circles (i.e. circle(P))
(2) F is on one of circles (i.e. circle(Q))

thanks for reading.
I'm afraid of not being good post.

Attachments:



Luis González

#2 Sep 7, 2009, 2:08 am

1) Negative inversion with center C and power $\overline{CB} \cdot \overline{CE}$ takes (P) into itself and the circle (Q) into the line ED. Since the inversion preserves angles, the angle between the line ED and (P) is also right $\Longrightarrow ED$ goes through P.

2) The fact that $F \equiv EA \cap DB$ lies on (Q) follows from the right angles $\angle EAD$ and $\angle EBD$. The quadrilateral $ACBF$ is cyclic and F is the antipode of C WRT (Q).

vittasko

#3 Sep 7, 2009, 6:00 pm

We draw the line through the point A and perpendicular to AC, which intersects the circle (P) at point so be it E and let be the points $D \equiv (P) \cap AC$ and $F \equiv (Q) \cap AE$, as the antidiametric (= diametric opposite) points of $E,\ C$, respectively.

Becase of the line segment PA tangents to (Q) at point A, we have that $\angle AFZ = \angle PAD = \angle ADZ$, (1) where $Z \equiv DE \cap CF$.

From (1) we conclude that the quadrilateral $AFDZ$ is cyclic and so, we have $\angle DZF = \angle DAF = 90^o \Longrightarrow FZ \perp DE$, (2)

So, the point C is the orthocenter of the triangle $\triangle DEF$ and then we have $EC \perp DF$ and let be the point $B' \equiv DF \cap EC$.

• Because of now $\angle CB'F = 90^o = \angle CB'D = \angle EB'D$, we conclude that the point B' lies on the circle (P) and simultaneously on the cirle (Q).

Hence, we conclude that $B' \equiv B \equiv (P) \cap (Q)$ and the proof is completed.

Kostas Vittas.

Attachments:
t=299716.pdf (9kb)

77ant

#4 Sep 8, 2009, 10:26 pm

Dear luisgeometria and vittasko

Thank you for interests. Your nice answers help my study a lot.

Quick Reply

High School Olympiads

Incircles and parallel segments

Reply

stergiu
#1 Sep 7, 2009, 12:31 am

In a right angled triangle ABC ($\angle A = 90$) we draw the altitude AH. If I, K are the incenters of triangles

ABC, AHC respectively and the circle (I) touches side BC at P, prove that $KP \parallel AB$

Babis

Luis González
#2 Sep 7, 2009, 1:37 am

D, E are the tangency points of the incircles $(K), (J)$ of $\triangle AHC, \triangle AHB$ with BC

$PD = PC - DC = \frac{1}{2}(BC + AC - AB - AC - HC + AH)$

$PD = \frac{1}{2}(BC - AB - HC + AH)$

On the other hand, we have:

$2JE + AB = BH + AH \Longrightarrow JE = \frac{1}{2}(BH + AH - AB)$

$JE = \frac{1}{2}(BC - AB - HC + AH) = PD$

Similarly, we have $KD = PE \Longrightarrow \triangle KPD \cong \triangle PJE.$ Then it follows that $\angle KPJ = 90^\circ$, which implies that $KPHJ$ is cyclic $\Longrightarrow \angle KPD = \angle KJH.$

Since $\triangle AHC \sim \triangle BHA$ and $\triangle KHD \sim \triangle JHE \Longrightarrow \frac{KD}{JE} = \frac{AC}{AB} = \frac{HK}{HJ}$

Thus, $\triangle ABC \sim \triangle HJK \Longrightarrow \angle KJH = \angle ABC = \angle KPD \Longrightarrow KP \parallel AB.$

vittasko
#3 Sep 8, 2009, 1:54 am

We denote the points $E \equiv (O) \cap CI$ and $F \equiv (O) \cap EP$ and $K \equiv CI \cap AF$ and we will prove that K is the incenter of the right triangle $\triangle DAC$, where AD (instead of AH) is the altitude of the given right triangle $\triangle ABC$ and (O) its circumcircle.

The quadrilateral $BPIE$ is cyclic, because of $\angle CEB = 90^o = \angle BPI$ and so, we have that $\angle IBP = \angle CEF = \angle CAF$, (1)

Because of now, $\angle CAD = 90^o - \angle C = \angle B$ and $\angle IBP = \frac{\angle B}{2}$ we conclude that $\angle CAF = \frac{CAD}{2}$, (2) and then, the point $K \equiv CI \cap AF$, is the incenter of $\triangle DAC$.

● It is easy to Show now, that the quadrilateral $CKPF$ is cyclic, because of $\angle KFP = \angle KCA = \angle KCP$ and so, we have $\angle KPC = \angle KFC$, (3)

But, $\angle KFC \equiv \angle AFC = \angle B$, (4)

From $(3),\ (4) \Longrightarrow \angle KPC = \angle B \Longrightarrow KP \parallel AB$ and the proof is completed.

Kostas Vittas.

Attachments:

t=299708.pdf (10kb)

livetolove212

#4 Sep 13, 2009, 12:47 pm

Let J be the incenter of triangle AHB.
This result is well-known: $\Delta JHK \sim \Delta BAC$
Let P' be a point such that JKP' is a right isoscele triangle. Since $JHP'K$ is cyclic we get
$\angle KHP' = \angle KJP' = 45^o = \angle KHC$
Therefore P' lies on BC.
$\angle JIK = 135^o = 180^o - 1/2\angle JP'K$.
So $P'I = P'K = P'J$
Note that $AC = AL$ and CK is the bisector of angle ACB thus $KI \perp AJ$, similar we obtain I is the orthocenter of triangle AJK.
$\angle P'IJ = \angle SIK = \angle C/2 + 45^o$
$\Rightarrow \angle JP'I = \angle B$
But $\angle KP'C = \angle KJH = \angle B$ hence $\angle IP'C = 90^o \Rightarrow P' \equiv P \Rightarrow KP//AB$

Attachments:

picture1.pdf (5kb)

jayme

#5 Sep 13, 2009, 6:47 pm

Dear Stergiu and Mathlinkers,
I think that this problem is a consequence of
http://www.mathlinks.ro/viewtopic.php?t=300127.
Sincerely
Jean-Louis

zool007

#6 Sep 14, 2009, 12:42 pm

note that $CK = (c(b^2) \div (a))/((a+b+c)sin(C/2))$ and $CP = (bc \div (a+b+c))/tan(C/2)$
$CK/CP = b/(acos(C/2)) = sinKPC/sinCKP = sinKPC/sin((45 - B/2 + KPC) = (sinB/cos(45 - B/2))$
let $x = KPC$
hence $cos(x + 45 - 3B/2) = cos(45 + B/2 - x)$
which implies that $B = x$
therefore $KP//AB$

ma 29

#7 Sep 15, 2009, 10:43 am

(K)touches AC at$M$.
$MK \cap BC = P'$
We have :
$\frac{CP'}{CB} = \frac{CM}{CA} = \frac{CA+CH-AH}{2.CA} => CP' = \frac{CB.CA+CB.CH-CB.AH}{2.CA} = \frac{CB.CA+CA^2-CA.AB}{2.CA} = \frac{CB+CA-AB}{2} = CP => P' \equiv P$
the proof is completed. :)

Virgil Nicula

#8 Sep 16, 2009, 12:27 am

> *stergiu wrote:*
>
> Let ABC be an A-right triangle with the incircle w . Denote the point $P\in BC\cap w$, $H\in BC$
>
> for which $AH\perp BC$ and the incircle $\theta = C(K)$ of $\triangle\ AHC$. Prove that $KP\parallel AB$.

Proof. Denote $L\in AC\cap\theta$. Observe that $\triangle ABC\stackrel{(*)}{\sim}\triangle HAC\implies\boxed{\frac{LA}{LC}=\frac{PB}{PC}}$ because the points L , P are analogously in a similarity $(*)\implies PL\parallel AB\implies PL\perp AC$. Since $KL\perp AC$ obtain $K\in PL$, i.e. $KP\parallel AB$.

> Quote:
> **An easy extension.** Let ABC be a triangle with the incircle w . Suppose $a>b$. Denote the point $P\in BC\cap w$,
> $H\in (BC)$ for which $\widehat{HAC}\equiv\widehat{ABC}$ and the incircle θ of $\triangle\ AHC$, $L\in AC\cap\theta$. Prove that $PL\parallel AB$.

jayme
#9 Oct 3, 2009, 1:57 pm

Dear Mathlinkers,
a synthetic proof of this result can be see
on: http://perso.orange.fr/jl.ayme vol. 5 Le théorème de Feuerbach-Ayme p. 19
Sincerely
Jean-Louis

sunken rock
#10 Dec 13, 2009, 11:22 pm

See that K and I are analogous in the similarity $\triangle AHC\sim\triangle BAC$, hence $\frac{CK}{CI}=\frac{AC}{BC}$ (1).
If $D=CI\cap AB$, then $\frac{CI}{CD}=\frac{a+b}{a+b+c}$ (2), from (1) & (2) we get $\frac{CK}{CD}=\frac{b\cdot(a+b)}{a\cdot(a+b+c)}$ (3), but $\frac{PC}{BC}=\frac{s-c}{a}$ (4),
where $s=\frac{a+b+c}{2}$, from (3) & (4) we see that $\frac{CK}{CD}=\frac{CP}{BC}$, i.e. $PK\parallel AB$.

Best regards,
sunken rock

Quick Reply

High School Olympiads

easy geometry

Reply

aadil

#1 Sep 6, 2009, 5:45 pm

a point p inside an equilateral triangle of side s has distances 3,5,4 from the vertice A,B,C respectively.find s?

Dr Sonnhard Graubner

#2 Sep 6, 2009, 8:51 pm

hello, let E, F, D be the orthopols form the point P to the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ of the given triangle, then we have to solve the following equationsystem
$u^2+x^2=9$
$(s-u)^2+x^2=25$
$v^2+y^2=25$
$(s-v)^2+y^2=16$
$w^2+z^2=16$
$z^2+(s-w)^2=9$
$x+y+z=\frac{\sqrt{3}}{2}s$
with $u=\overline{AE}$, $v=\overline{BF}$, $=\overline{CD}$ $x=\overline{PE}$,$y=\overline{PF}$,$z=\overline{PD}$ and $s=\overline{AB}=\overline{BC}=\overline{CA}$.
Sonnhard.

Luis González

#3 Sep 6, 2009, 10:38 pm

We use tripolar coordinates with respect to $\triangle ABC$.

In the equilateral triangle $\triangle ABC$, the tripolar coordinates of $P(x,y,z)$ satisfy the relation:

$$a^2(x^2+y^2+z^2)+x^2y^2+y^2z^2+x^2z^2-(x^4+y^4+z^4)-a^4=0$$

Substituting $(x,y,z)=(3,4,5)$ yields the biquartic equation

$$a^4-50a^2+193=0 \Longrightarrow \ a^2=25+12\sqrt{3} \Longrightarrow [\triangle ABC]=\tfrac{25}{4}\sqrt{3}+9.$$

sunken rock

#4 Mar 13, 2012, 2:04 am

Let $PA=a, PB=b, PC=c$;rotate $\triangle ABP$ about B with an angle $\alpha=60^\circ$ so that A goes to C and P to a point X and $\triangle BPX$ is equilateral, rotate then $\triangle BCP$ of 60° about C so that B goes to A and P to Y, with $\triangle CPY$ equilateral, and $\triangle ACP$ of 60° about A so that C goes to B and P to Z and $\triangle APZ$ is equilateral.
The hexagon $AZBXCY$ has area twice the area of the $\triangle ABC$ and, joining P with its vertices, we get three congruent triangles $\triangle APY=\triangle PZB=\triangle XCP$ and three equilateral triangles of sides a, b, c respectively.
Consequently, if l is the side of the $\triangle ABC$, and s the area of $\triangle APY$ we get $\frac{l^2\sqrt{3}}{2}=3\cdot s+\frac{\sqrt{3}}{4}\cdot(a^2+b^2+c^2)$. For the general case, the area s is calculated by Heron Formula, but in our case is about a right-angled triangle, of sides $3, 4, 5$, hence its area is 6.

Best regards,
sunken rock

Quick Reply

High School Olympiads

an inequality regarding 4 points in space

Reply

Agr_94_Math
#1 Sep 5, 2009, 9:52 am

For any 4 points in space A, B, C, D , Prove that $AC^2 + BD^2 + AD^2 + BC^2 \geq AB^2 + CD^2$.

Kunihiko_Chikaya
#2 Sep 5, 2009, 10:18 am

Let $\overrightarrow{AB} = \overrightarrow{b},\ \overrightarrow{AC} = \overrightarrow{c},\ \overrightarrow{AD} = \overrightarrow{d},$

$AC^2 + BD^2 + AD^2 + BC^2 = |\overrightarrow{c}|^2 + |\overrightarrow{d} - \overrightarrow{b}|^2 + |\overrightarrow{d}|^2 + |\overrightarrow{c} - \overrightarrow{b}|^2$

$= 2(|\overrightarrow{b}|^2 + |\overrightarrow{c}|^2 + |\overrightarrow{d}|^2) - 2\overrightarrow{b}\cdot(\overrightarrow{c} + \overrightarrow{d})$

$AB^2 + CD^2 = |\overrightarrow{b}|^2 + |\overrightarrow{c}|^2 + |\overrightarrow{d}|^2 - 2\overrightarrow{c}\cdot\overrightarrow{d}$

$\therefore AC^2 + BD^2 + AD^2 + BC^2 - (AB^2 + CD^2)$

$= |\overrightarrow{c}|^2 - 2(\overrightarrow{b} - \overrightarrow{d})\cdot\overrightarrow{c} + |\overrightarrow{b} - \overrightarrow{d}|^2$

$= |\overrightarrow{b} - \overrightarrow{c} - \overrightarrow{d}|^2 \geq 0\ Q.E.D.$

Luis González
#3 Sep 5, 2009, 10:35 am

If M, N are the midpoints of AB and $DC,$ according to a Casey's theorem we have

$AC^2 + BD^2 + BC^2 + AD^2 = AB^2 + CD^2 + 4MN^2$

The inequality obviously follows from $MN \geq 0$

Agr_94_Math
#4 Sep 5, 2009, 6:46 pm

luisgeometria: Shouldn't it be M, N midpoints of AC, BD?
In the website, it is given that the points are the midpoints of the diagnols of the polygon.?

Luis González
#5 Sep 5, 2009, 10:11 pm

Yes, MN is the segment connecting the midpoints of the diagonals, but indeed the relation holds for both complex and simple quadrilaterals. We could state that AB,DC are diagonals of the complex quadrilateral ADBC or AC,BD are diagonals of the convex/concave quadrilateral ABCD.

Quick Reply

High School Olympiads

A problem of cyclic quadrilateral of orthocenter

Reply

Source: using analysis geometry will be better

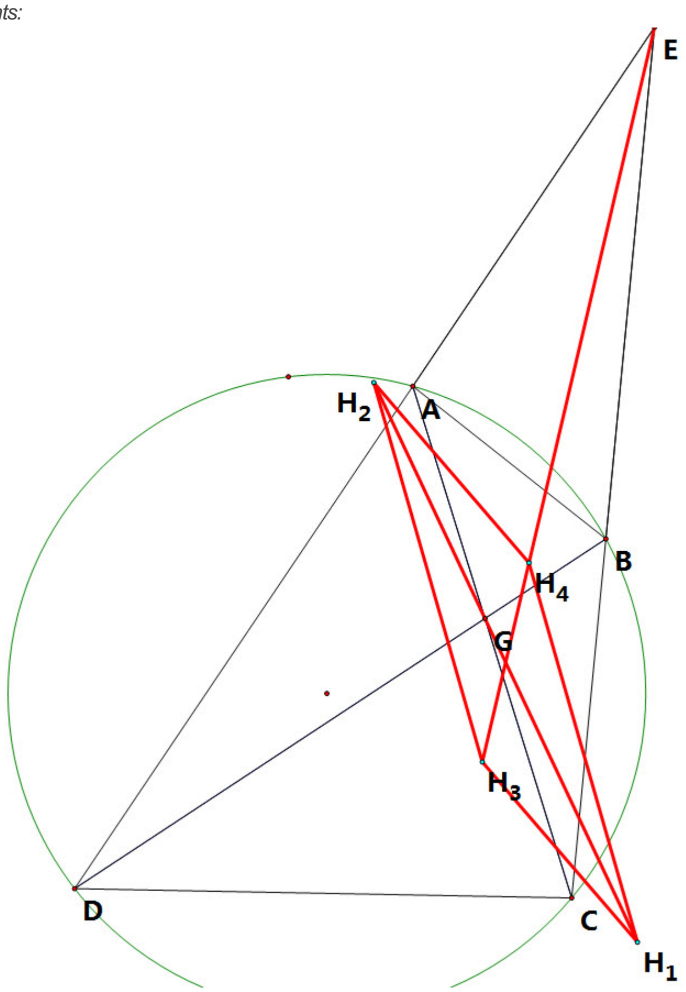
admire9898

#1 Sep 3, 2009, 6:37 pm

Given a cyclic quadrilateral ABCD□AD intersect BC at E□AB intersect CD at F□AC intersect BD at G. H1□H2□H3□H4 is the orthocenter of ECD□EAB□GCD□GAB separately□Try to proof□H1H3H2H4 is a parallelogram.

the picture is uploaded
I tried hard for using analytic geometry. but failed.
If you have some effective way to solve this by using analytic geometry,then will be better!(I am seeking...)

Attachments:



This post has been edited 1 time. Last edited by admire9898, Sep 4, 2009, 4:40 pm

livetolove212

#2 Sep 4, 2009, 7:59 am

#2 Sep 4, 2009, 7:59 am

Solution

Attachments:

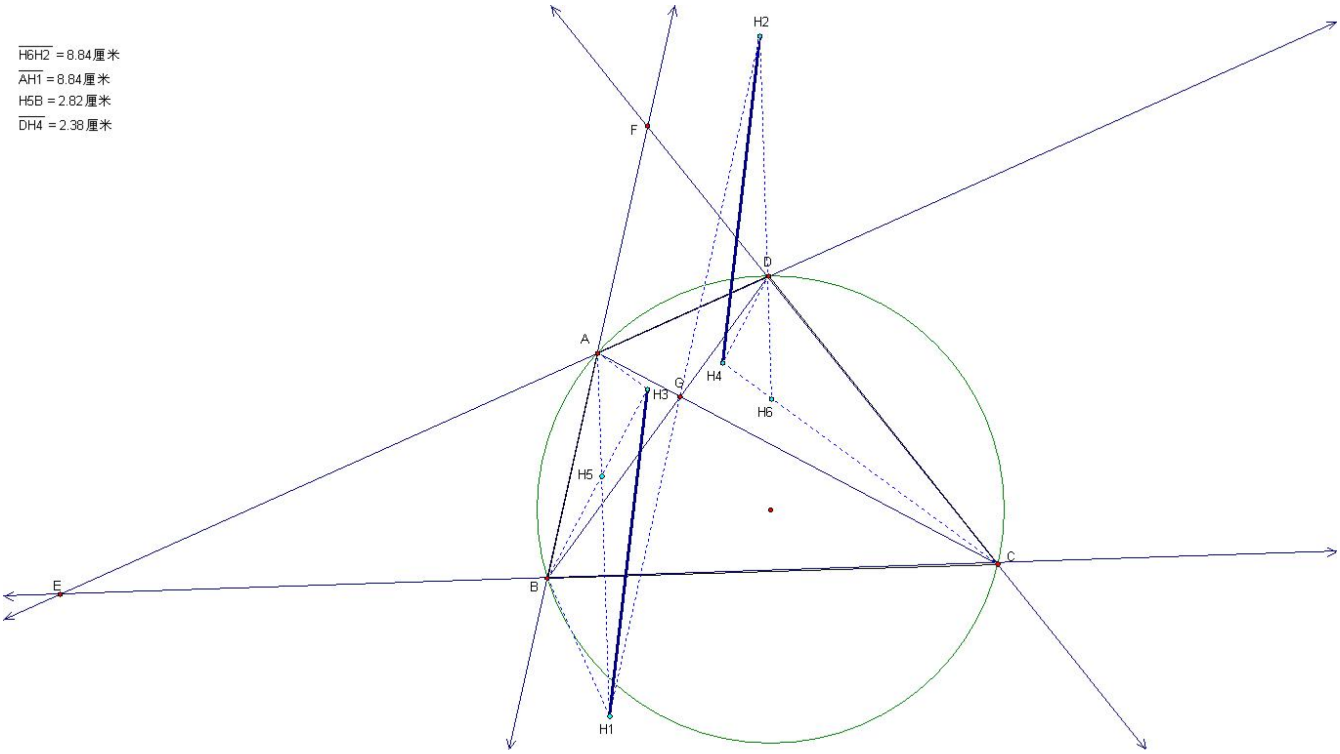
picture29.pdf (31kb)
picture30.pdf (22kb)
picture31.pdf (32kb)

plane geometry

#3 Sep 4, 2009, 8:47 am

http://www.aoshoo.com/bbs1/dispbbs.asp?boardid=43&Id=17110
if you know chinese ,see the proof I mentioned in floor2,
sorry, I have little time recently, I will translate my solution when I have time

Attachments:



Luis González

#4 Sep 5, 2009, 9:48 am

Lemma. Steiner line of a cyclic quadrangle $ABCD$ with circumcircle (O) goes through its diagonal intersection P.

Proof. Let M, N, L denote the midpoints of AD, AC, BD. $\triangle MNL$ is pedal triangle of O WRT $\triangle APD$, thus the perpendiculars from D and A to ML and MN meet at the isogonal conjugate H_1 of O WRT $\triangle PAD$, which becomes orthocenter of the triangle bounded by the lines AB, AD, DC, since $MN \parallel DC$ and $ML \parallel AB$. Then it follows that lines PH_1, PO are isogonals WRT the angle between the diagonals AC, BD. In exactly the same way, we get that the orthocenter H_2 of the triangle bounded by the lines AB, BC, CD is the isogonal conjugate of O WRT $\triangle PBC \Longrightarrow$ lines PH_1 and PH_2 are identical $\Longrightarrow$ Steiner line of the cyclic quadrangle $ABCD$ goes through P.

Let X, Y, Z, W be the orthocenters of $\triangle EAC, \triangle EBD, \triangle AGD, \triangle BGC$. They are collinear on the Steiner line of the complete quadrangle $EAGB$ and clearly H_2YH_1X and H_3WH_4Z are both parallelograms. From the previous lemma, the Steiner line H_1H_2 of the cyclic quadrangle $ABCD$ goes through G and since $GW \parallel H_1Y$ and $GZ \parallel H_1X$, it follows that $\triangle GWZ$ and $\triangle H_1YX$ are centrally similar through the center S of the parallelogram H_2YH_1X. Hence they have H_1S as their common median on ZW and $XY \Longrightarrow XY$ bisects H_3H_4 and $H_1H_2 \Longrightarrow H_1H_3H_2H_4$ is a parallelogram.

Quick Reply

High School Olympiads

iran(5-th round)1998

Reply

Source: it has an easy solution

shoki

#1 Sep 2, 2009, 3:02 am

The circle (C) and two points on it N, M are given. l_1 is tangent to C at point M and l_2 tangent to C at point N.Two lines x and y passe through M and N,respectively.Line x intersects l_2 at P_1 and line y intersects l_1 at P_2.If P_1P_2 intersects MN at D and the tangent at the second point of intersection of x with the circle C intersects l_1 at E and the same thing for line y intersects l_2 at F,then prove the collinearity of D, E, F.

Luis González

#2 Sep 2, 2009, 3:34 am

Let B, C be the second intersections of the lines x, y with $\mathcal{C}$. Let P_0 denote the diagonal intersection of the cyclic quadrilateral $MNBC$, $A \equiv l_1 \cap l_2$ and $L \equiv MN \cap AP_0$. From the concurrent cevians P_1M, P_2N, AL in the $\triangle AP_1P_2$, it follows that cross ratio (M, N, L, D) is harmonic. Suppose $D' \equiv BC \cap MN$. Then AP_0 is the polar of D' WRT $\mathcal{C} \Longrightarrow$ Cross ratio (N, M, L, D') is harmonic $\Longrightarrow D \equiv D'$. D, E, F are poles of the concurrent lines $AP_0, MB, NC \Longrightarrow D, E, F$ are collinear on the polar of P_0 WRT $\mathcal{C}$.

Quick Reply

High School Olympiads

Nine-point circle's center

Reply

Source: Own

livetolove212

#1 Aug 29, 2009, 8:33 pm

Given triangle ABC with circumcircle (O). Let A_1, A_2, A_3 be the projections of A, B, C onto BC, CA, AB, respectively, X, Y, Z be the midpoints of $BC, CA, AB. XY \cap (O) = \{A_2, A_3\}$, similar for B_2, B_3, C_2, C_3. Prove that **Nine-point circle**'s center of triangle ABC is the radical center of three circles $(A_1A_2A_3)$, $(B_1B_2B_3)$ and $(C_1C_2C_3)$

Luis González

#2 Aug 31, 2009, 11:50 am

$\triangle AA_2A_3$ is the reflection of $\triangle A_1A_2A_3$ about YZ. Since O is the circumcenter of $\triangle AA_2A_3$, then the circumcenter O_a of $\triangle A_1A_2A_3$ is the reflection of O about YZ. Likewise, circumcenters O_b and O_c of $\triangle B_1B_2B_3$ and $\triangle C_1C_2C_3$ are the reflections of O about ZX and XY, respectively. $\triangle O_aO_bO_c$ has parallel sidelines to the orthic triangle of $\triangle XYZ$, since they are centrally similar with similarity center O and coefficient 2. Thus, radical axis ℓ_c of $(O_a), (O_b)$ goes through Z, due to $ZB_2 \cdot ZB_3 = ZA_2 \cdot ZA_3$ and it is perpendicular to the line O_aO_b antiparallel to XY WRT $ZY, ZX \Longrightarrow \ell_c$ is the isogonal of ZO WRT $\angle XZY$. Similarly, ℓ_a, ℓ_b go through the isogonal conjugate of O in the medial triangle $\triangle XYZ$, i.e. the nine-point center N of $\triangle ABC \Longrightarrow N$ is the radical center of the circles $(O_a), (O_b), (O_c)$.

Quick Reply

High School Olympiads

Help me!!

Reply

cphy7415
#1 Aug 30, 2009, 9:24 pm

There are two chords AB,CD that are perpendicular to each other in a circle that has a radius of R.
Prove AC^+CB^+BD^+DA^ is constant.

p.s. I want to know what kind of process I have to think to find a strategy to solve this problem.

livetolove212
#2 Aug 30, 2009, 9:33 pm

You can draw a diameter AE then prove that $CBED$ is a trapezoid. Prove that $AC^2+BD^2=AC^2+CE^2=4R^2$ ...

cphy7415
#3 Aug 30, 2009, 9:42 pm

> livetolove212 wrote:
> You can draw a diameter AE then prove that $CBED$ is a trapezoid. Prove that $AC^2+BD^2=AC^2+CE^2=4R^2$ ...

I'm not quite sure how to do that. I think I have to use law of cosine but don't know how. I am thinking of making a relationship between the radius and each of the side of the quadrilateral.

Luis González
#4 Aug 30, 2009, 9:52 pm

Let the chords AB,CD meet at $P.$ We shall show that $PA^2+PB^2+PC^2+PD^2$ is independent of $P.$ O is the center of the given circle and Q the second intersection of DO with $(O).$ By Pythagorean theorem we get

$$PA^2+PB^2+PC^2+PD^2=\frac{1}{2}(DA^2+AC^2+CB^2+BD^2)$$

But since rays DQ,DP are isogonal WRT $\angle ADB,$ it follows that $ACQB$ is an isosceles trapezoid with $CQ \parallel AB \Longrightarrow AC=BQ$ and $CB=AQ.$ Thus, keeping in mind that DQ is a diameter, again by Pythagorean theorem we have:

$$PA^2+PB^2+PC^2+PD^2=\frac{1}{2}(DA^2+AQ^2+BD^2+BQ^2)$$

$$PA^2+PB^2+PC^2+PD^2=\frac{1}{2}(DQ^2+DQ^2)=\frac{1}{2}(8R^2)=4R^2.$$

Quick Reply

High School Olympiads

Radical center and Euler line

Source: Own

livetolove212

#1 Aug 29, 2009, 4:15 pm

Given triangle ABC. Denote A', B', C' the midpoints of BC, CA, AB, respectively. Prove that the radical center of $(A, AA'), (B, BB'), (C, CC')$ lies on **Euler line** of triangle ABC.

Luis González

#2 Aug 29, 2009, 11:55 pm

Let H, O, N denote the orthocenter, circumcenter and nine-point center of $\triangle ABC.$ Let P be the midpoint of $\overline{HN}.$ By Stewart's Theorem for the cevian AP in $\triangle AHO$

$$AP^2 = \frac{1}{4}R^2 + \frac{3}{4}AH^2 - \frac{3}{16}OH^2 = \frac{1}{4}R^2 + \frac{3}{4}(2OA')^2 - \frac{3}{16}OH^2$$

$$AP^2 = \frac{1}{4}R^2 + 3(R^2 - \frac{1}{4}a^2) - \frac{3}{16}OH^2 = \frac{13}{4}R^2 - \frac{3}{4}a^2 - \frac{3}{16}OH^2$$

Let $\mathcal{P}_a$ denote the power of P with respect to $(A, AA').$ Then

$$\mathcal{P}_a = AA'^2 - AP^2 = \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2 - \frac{13}{4}R^2 + \frac{3}{4}a^2 + \frac{3}{16}OH^2$$

$$\Longrightarrow \ \mathcal{P}_a = \frac{1}{2}(a^2 + b^2 + c^2) - \frac{13}{4}R^2 + \frac{3}{16}OH^2$$

Since this expression is symmetric in terms of (a, b, c), we conclude that midpoint P of $\overline{HN}$ has equal power with respect to circles $(A, AA'), (B, BB'), (C, CC')$.

High School Olympiads

BCIJOH are contained in a conic!

Reply

Source: Doo Sung Park

Parkdoosung

#1 Aug 28, 2009, 3:59 pm

Consider a triangle ABC. Let I, J, O, H be the incenter, A-excenter, circumcenter, and orthocenter respectively. Prove that B, C, I, J, O, H are contained in a conic.

mihai miculita

#2 Aug 28, 2009, 5:20 pm

Using barycentric coordonates, we:
$B(0;1;0), C(0;0;1); I(a;b;c), J(-a;b;c); O(\sin 2A;\sin 2B;\sin 2C); H(\tan A;\tan B;\tan C)$.
The point B, C, I, J, O, H are contained in the conic with equation:
$Mx^2+Ny^2+Pz^2+Qxy+Rxz+Syz=0 \Leftrightarrow$

$$0=\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ a^2 & b^2 & c^2 & ab & ac & bc \\ a^2 & b^2 & c^2 & -ab & -ac & bc \\ \sin^2 2A & \sin^2 2B & \sin^2 2C & \sin 2A.\sin 2B & \sin 2A.\sin 2C & \sin 2B.\sin 2C \\ \tan^2 A & \tan^2 B & \tan^2 C & \tan A.\tan B & \tan A.\tan C & \tan B.\tan C \end{vmatrix}$$

$$\Leftrightarrow \begin{vmatrix} a^2 & ab & ac & bc \\ a^2 & -ab & -ac & bc \\ \sin^2 2A & \sin 2A.\sin 2B & \sin 2A.\sin 2C & \sin 2B.\sin 2C \\ \tan^2 A & \tan A.\tan B & \tan A.\tan C & \tan B.\tan C \end{vmatrix}=0 \Leftrightarrow$$

$$\begin{vmatrix} a^2 & ab & ac & bc \\ a^2 & 0 & 0 & bc \\ \sin^2 2A & \sin 2A.\sin 2B & \sin 2A.\sin 2C & \sin 2B.\sin 2C \\ \tan^2 A & \tan A.\tan B & \tan A.\tan C & \tan B.\tan C \end{vmatrix}=0 \Leftrightarrow$$

$$\Leftrightarrow a^2.\begin{vmatrix} ab & ac & bc \\ \sin 2A.\sin 2B & \sin 2A.\sin 2C & \sin 2B.\sin 2C \\ \tan A.\tan B & \tan A.\tan C & \tan B.\tan C \end{vmatrix}=$$

$$=bc.\begin{vmatrix} a^2 & ab & ac \\ \sin^2 2A & \sin 2A.\sin 2B & \sin 2A.\sin 2C \\ \tan^2 A & \tan A.\tan B & \tan A.\tan C \end{vmatrix} \Leftrightarrow \ldots$$

Bugi

#3 Aug 28, 2009, 5:51 pm

Hint for a synthetical solution

Luis González

#4 Aug 29, 2009, 6:48 am

Let J be the antipode of I WRT $\odot(IBC)$. Its isogonal conjugate J_∞ WRT $\triangle IBC$ is the infinity point of the altitude AH_a, since the isogonal CJ_∞ of CJ WRT $\angle C$ is orthogonal to BC. Let O', H' denote the isogonal conjugates of O, H WRT $\triangle IBC$. Then, by simple angle chase we have

$$\angle H'BC = 90^\circ - \angle C - \angle IBC = 90^\circ - \angle C - \tfrac{1}{2}\angle B = \tfrac{1}{2}|\angle A - \angle C| \quad (1)$$

Analogously, $\angle H'CB = \frac{1}{2}|\angle A - \angle B| \ \ (2)$

$\angle O'BC = \angle IBC - \angle OBC = \frac{1}{2}\angle B - (90^\circ - \angle A) = \frac{1}{2}|\angle A - \angle C| \ \ (3)$

Analogously, $\angle O'CB = \frac{1}{2}|\angle A - \angle B| \ \ (4)$

From $(1), (2), (3), (4)$ we get $\angle O'BC = \angle H'BC$ and $\angle O'CB = \angle H'CB \Longrightarrow BO'$ and CO' are the reflections of BH' and CH' about the sideline $BC \Longrightarrow O'$ is reflection of H' about $BC \Longrightarrow O'H' \perp BC$. Since O', H' and J_∞ are collinear, then isogonal conjugation WRT $\triangle IBC$ takes O', H', J_∞ into O, H, J lying on a circumconic of $\triangle IBC$.

Quick Reply

High School Olympiads

2 angle are equal

Source: Romania TST 2001

SUPERMAN2

#1 Aug 28, 2009, 10:25 pm

Let ABC be an acute triangle and O be its circumcenter.The tangents at A and B to (O) intersect the tangent at C at D and E respectively.AE meets BC at P and BD meets AC at R.Let S,Q be the midpoints of BR,AP respectively.Prove that $\angle ABQ=\angle BAS$.

Luis González

#2 Aug 28, 2009, 10:33 pm

Equivalent formulation: Let M,N be the feet of the B-symmedian and C-symmedian of $\triangle ABC. D,E$ are the midpoints of CN and $BM.$ Then $\angle DBC=\angle ECB.$

Using barycentric coordinates with respect to $\triangle ABC,$ we get

$M(a^2:0:c^2)\ ,\ N(a^2:b^2:0)\ ,\ D(a^2:b^2:a^2+b^2)\ ,\ E(a^2:a^2+c^2:c^2)$

$BD\equiv za^2-x(a^2+b^2)=0$ and $CE\equiv a^2y-x(a^2+c^2)=0$ meet at the point

$P\ (a^2:a^2+c^2:a^2+b^2)$

Clearly, coordinates of P satisfy the equation of the perpendicular bisector ℓ_a of $BC,$ namely

$\ell_a\equiv(b^2-c^2)x+a^2(y-z)=0$

Thus, $\triangle PBC$ is P-isosceles $\Longrightarrow\angle DBC=\angle ECB,$ as desired.

High School Olympiads

triangle ABC with /_C=120

Reply

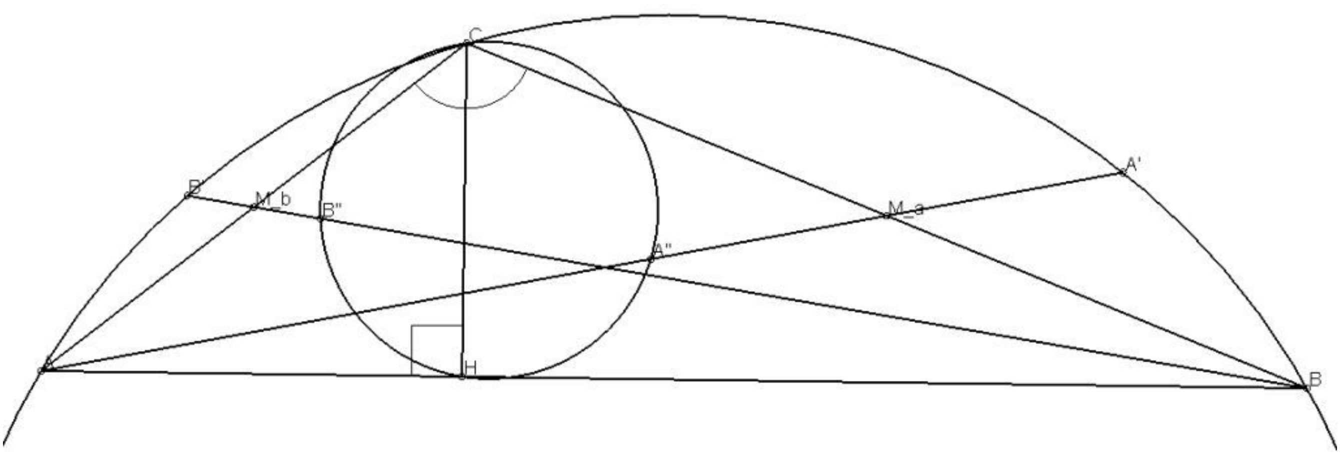
Source: hard

Nazar_Serdyuk

#1 Aug 25, 2009, 11:22 pm

Given a triangle ABC with $\angle C = 120^{\circ}$ and circumcircle w. M_a, M_b are midpoints of CB, CA respectively, AM_a, BM_b meet w again at A', B' respectively, A'', B'' are reflections of A', B' on M_a, M_b respectively, H is a foot of perpendicular from C to AB. Prove that points C, A'', B'', H lie on a circle.

Attachments:



saif

#2 Aug 26, 2009, 4:07 am

hello it's not to hard
see that CB"AB' and CA"BA' are parallelogram
and it's very easy to prove that AMbB"H and BMaA"H are cocyclic
now easily we can prove that B"CA"+B"HA"=180°
sorry am tired i can't write the full solution

cosinator

#3 Aug 26, 2009, 4:34 am

> *saif wrote:*
> sorry am tired i can't write the full solution

Could you write up a full solution later? Some of the things you say are easy, I cannot see. I would greatly appreciate an elaboration.

Luis González

#4 Aug 26, 2009, 5:06 am

Lemma. In any triangle $\triangle ABC$, the orthocenter H, its othogonal projection on the median AM and the vertices B, C are concyclic.

Proof. Let D, E be the feet of the altitudes from B, C and $Q \equiv DE \cap BC.$ Line AQ is polar of H WRT the circle (M) with diameter $BC \Longrightarrow QH$ is polar of A WRT $(M) \Longrightarrow AM \perp QH.$ Let $K \equiv AM \cap QH.$ Note that H is common orthocenter of $\triangle AQM$ and $\triangle ABC,$ thus $KH \cdot HQ = DH \cdot HB = CH \cdot HE.$ Hence, inversion with center H and power $KH \cdot HQ$ takes K, B, C into the collinear points $Q, D, E \Longrightarrow B, C, H, K$ are concyclic. Thus if AM cuts the circumcircle of $\triangle ABC$ again at $L,$ then $KCLB$ is a parallelogram, due to $\angle BKC = \angle BLC = 180^\circ - \angle A.$

Using the lemma, we get that A'' and B'' are the orthogonal projections of the orthocenter onto the medians AM_a and $BM_b,$ since the quadrilaterals $CA'BA''$ and $CB'AB''$ are parallelograms. Let H_c denote the foot of the altitude from C and H the orhocenter of $\triangle ABC.$ By angle chase, we have:

$\angle B''H_cA'' = \angle HAM_a + \angle HBM_b = 180^\circ - 2\angle AHB + \angle CBM_b + \angle CAM_a$

$\angle B''H_cA'' = 180^\circ - 120^\circ + \angle CBM_b + \angle CAM_a$

$\angle B''H_cA'' = 60^\circ + (120^\circ - \angle B''CA'') = 180^\circ - \angle B''CA'' \Longrightarrow CA''H_cB''$ is a cyclic quadrilateral.

Nazar_Serdyuk

#5 Aug 26, 2009, 2:32 pm

Your solution is much more easier then mine. I proved that these points lie on a circle with diameter CL where L is a foot of bisector of $\angle ACB$.
Now I see that the problem is not hard.

Luis González

#6 Aug 27, 2009, 6:19 am

I did not notice that the foot L of the bisector of $\angle ACB$ also lies on this circle. Using the same notations, we can prove it easily by angle chase

$\angle A''CL = \angle BCL - \angle BCA'' = 60^\circ - \angle CBA' = 60^\circ - \angle CAA' \quad (1)$

From the cyclic quadrilateral $HA''H_cA$ we have

$\angle CH_cA'' = \angle HAM_a = \angle CAA' + \angle HAC = \angle CAA' + 30^\circ \quad (2)$

From (1) and (2) we get $\angle A''CL + \angle CH_cA'' = 90^\circ$ and the conclusion follows.

---

saif

#7 Aug 27, 2009, 7:32 pm

dear cosinator i think that you're meaning $AM_bB"H$ and $BM_aA"H$ are cocyclic
it's easy because
$\angle M_bHA = \angle M_bAH = \angle CB'B = \angle M_bB"A$
and by a simple angle chase we can get the result.

Luis González

#8 Aug 27, 2009, 10:37 pm

> saif wrote:
> $\angle M_bHA = \angle M_bAH = \angle CB'B = \angle M_bB"A$

Dear Saif, could you justify this step?. I'm not seeing where <C=120 is used.

---

saif

#9 Aug 28, 2009, 12:13 am

the condition $\angle C = 120^o$ we will use it later in our angle chase to prove that $\angle B"CA" + \angle B"HA" = 180^o$ but

$\angle M_bHA = \angle M_bAH = \angle CB'B = \angle M_bB''A$ to prove that $AM_bB"H$ is cyclic I think it's clear
now ill prove that $\angle B"CA" + \angle B"HA" = 180^o$

we have $\angle M_bHM_a = 120^o$
so to prove the result $\angle B"CA" + \angle B"HA" = 180^o$ we need to prove
$\angle M_bCB'' + \angle M_bHB'' + \angle M_aCA'' + \angle M_aHB'' = 60^o$
it's easy because $\angle M_bCB'' = \angle CBB'' = \angle B''AM_b$
and $\angle M_bHB'' = \angle M_bAB'' = \angle B'CA = \angle B'BA$ so
$\angle M_bCB'' + \angle M_bHB'' = \angle B$
and $\angle M_aCA'' + \angle M_aHB'' = \angle A$ ....

Virgil Nicula
#10 Sep 6, 2009, 9:01 am

> *Quote:*
> **Lemma.** In a triangle ABC the orthocenter H , its othogonal projection on the median AM and the vertices B, C are concyclic.

A nice and short proof. Denote $E \in BH \cap AC$ and $K \in AM$, $HK \perp AM$. Prove easily that E and K belong to the circle w with diameter $[AH]$ and ME is tangent to w . Thus, $ME^2 = MK \cdot MA$, i.e. $MC^2 = MK \cdot MA \implies MCK \sim MAC \implies \widehat{MKC} \equiv \widehat{BCA}$, i.e. $m(\widehat{HKC}) + m(\widehat{HAC}) = (90^\circ + C) + (90^\circ - C) = 180^\circ$, i.e. $BHKC$ is cyclic.

licu-an
#11 Sep 12, 2009, 7:38 pm

virgil, you killed the problem outright, your solution is very simple and direct.

Quick Reply

High School Olympiads

Two circles

Reply

jgnr

#1 Aug 26, 2009, 3:54 pm

Circles Γ_1 and Γ_2 intersect at P and Q. The common tangent of Γ_1 and Γ_2 touch the circles at A and B respectively. C is an arbitrary point on ray AB. Let CP intersect Γ_1 at D. Point E is on CD such that BE is parallel to AD. Let PQ intersect the common tangent at R. Prove that $\angle RED = 90^\circ$.

edit: sorry, the problem is wrong. additional information is needed: $\angle RQC = 90^\circ$.

plane geometry

#2 Aug 26, 2009, 6:55 pm

thanks for the attachment
, I will have a try

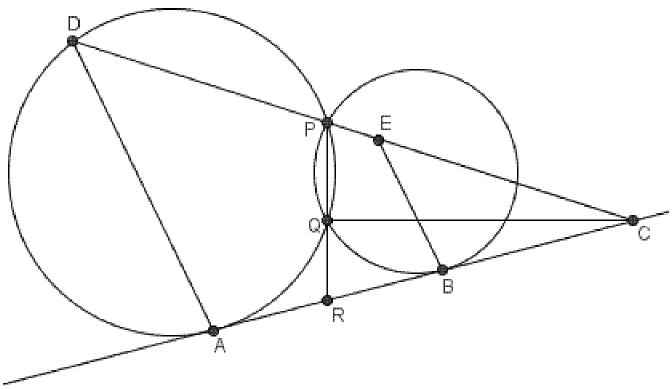
This post has been edited 2 times. Last edited by plane geometry, Aug 26, 2009, 7:08 pm

jgnr

#3 Aug 26, 2009, 7:04 pm

see attachment

Attachments:

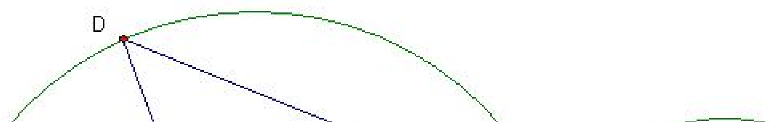


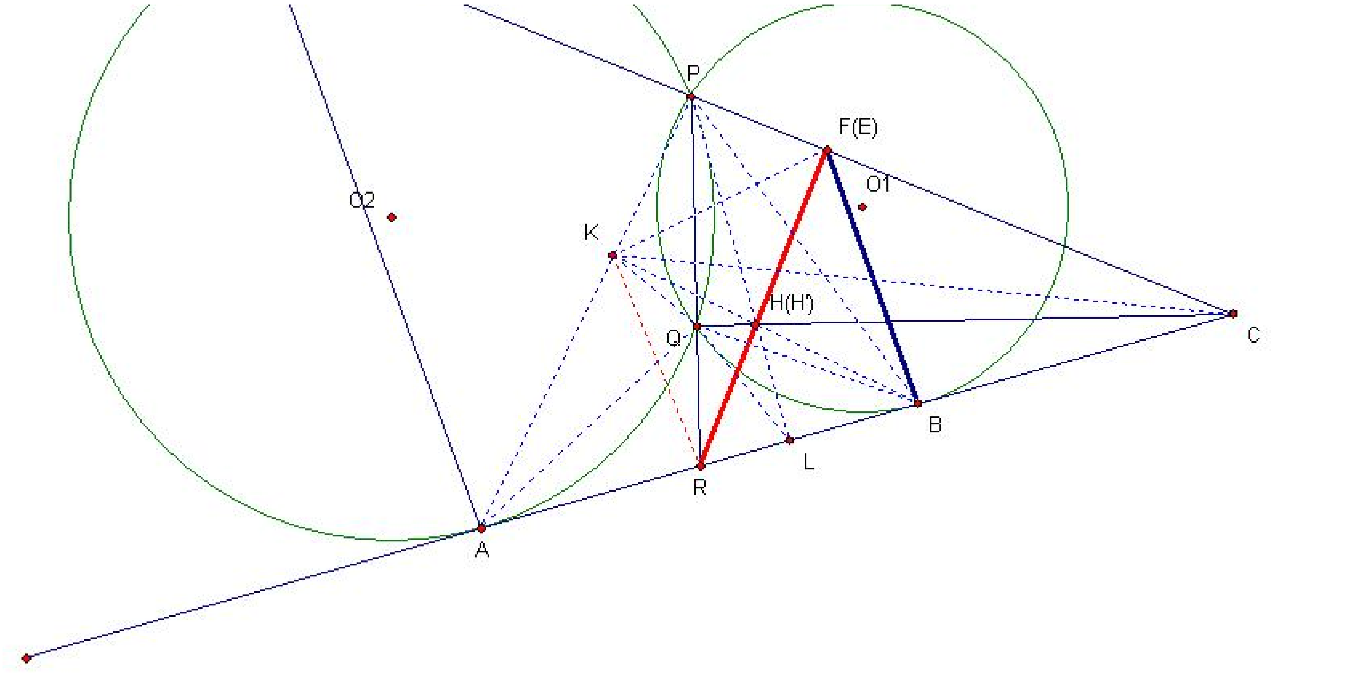
plane geometry

#4 Aug 26, 2009, 10:15 pm

Denote H' is the orthocenter of □PRC , H is the orthocenter of □PBA
K is the foot of altitude through B to PA , F is the foot of altitude through R to PC
Then we have B,H,K are collinear and R,H',F are collinear
RA^2=RB^2=RQ*RP => R is the midpoint of AB => RK^2= RQ*RP => □KRQ=□RPK=□QAR => K,Q,R,A are concyclic =>
PQ*PR=PK*PA=PH*PL=PH'*PL
=> H=H' => BH*HK=PH*LH=RH*RF => K,R,B,F are concyclic =>
□ADP=□KAR=□AKR=90-□RKB=90-□RFB=□BFC => BF//AD => F=E
Done!

Attachments:





Luis González

#5 Aug 27, 2009, 3:59 am

Let E' and S be the projections of R and P on PC and AB. PS, RE' and CQ concur at the orthocenter H of $\triangle PRC$. Since $RA^2 = RB^2 = RQ \cdot RP = RS \cdot RC$, it follows by Newton's theorem that cross ratio (A, B, S, C) is harmonic $\Longrightarrow$ $CB \cdot CA = CS \cdot CR$, but since $CS \cdot CR = CE' \cdot E'P \Longrightarrow CB \cdot CA = CE' \cdot CP \Longrightarrow PE'BA$ is cyclic. Then $\angle CE'B = \angle PAB = \angle CDA \Longrightarrow BE' \parallel AD \Longrightarrow E \equiv E'$.

● Another interesting result about this configuration is that, if T is the second intersection of the ray DR with Γ_1, then circles $\odot(RTA)$ and $\odot(HRB)$ are congruent and externally tangent.

Quick Reply

High School Olympiads

nice

Reply

Source: APMO

am2525

#1 Aug 23, 2009, 6:23 pm

Let ABC be a triangle with median AM and angle bisector AN. Draw the perpendicular to line NA through N, hitting lines MA and BA at Q and P, respectively. Also let O be the point where the perpendicular to line BA through P meets line AN. Prove that QO and BC are perpendicular .

Luis González

#2 Aug 24, 2009, 10:00 am

Let L be the midpoint of the arc BC of the circumcircle (O). $S \equiv OP \cap LB$. Since $\angle RPO = \angle CAL = \angle CBS$, it follows that $BNSP$ is cyclic $\Longrightarrow \angle BNS = 90^\circ$. Let $R \equiv PN \cap AC$. In the isosceles triangle $\triangle ARP$ we have:

$$\frac{QR}{PQ} = \frac{c}{b} \Longrightarrow \frac{PR}{PQ} = \frac{b+c}{b} \Longrightarrow \frac{PN}{PQ} = \frac{b+c}{2b} \Longrightarrow \frac{PO\cos\frac{A}{2}}{PQ} = \frac{b+c}{2b}$$

$$\Longrightarrow \frac{PO}{PQ} = \frac{b+c}{2b\cos\frac{A}{2}} \ . (1)$$

Notice that, since $\triangle ABN \sim \triangle PSN$, we have $\frac{PS}{PN} = \frac{c}{AN} = \frac{b+c}{2b\cos\frac{A}{2}} \ . (2)$

From (1) and (2), $\frac{PS}{PN} = \frac{PO}{PQ} \Longrightarrow OQ \parallel NS \Longrightarrow OQ \perp BC.$

mr.danh

#3 Feb 7, 2010, 4:28 pm

Another solution

Quick Reply

High School Olympiads

Geometric inequality relating to

Reply

Source: intersection of angle bisecotrs and the circumcircle

Agr_94_Math
#1 Aug 21, 2009, 11:38 pm

In a triangle ABC, the angle bisectors of angles CAB, ABC, ACB intersect the circumcircle at P, Q, R respectively. Prove that $AP + BQ + CR > AB + BC + AC$.

Luis González
#2 Aug 22, 2009, 12:36 am

Let $PB = PC = L$. By Ptolemy's theorem for cyclic quadrilateral $ABPC$ we have

$$AB \cdot L + AC \cdot L = AP \cdot BC \Longrightarrow AP = \frac{L(AB + AC)}{BC}$$

On the other hand, by triangle inequality we get $PB + PC > BC \Longrightarrow 2L > BC$

Therefore, $AP > \frac{1}{2}(AB + AC)$. The cyclic sum yields:

$$AP + BQ + CR > \frac{1}{2}(2AB + 2BC + 2CA) > AB + BC + CA.$$

Quick Reply

High School Olympiads

Geometrical ineq.

Reply

Mateescu Constantin

#1 Aug 20, 2009, 11:13 pm

Let M be a point inside the equilateral triangle ABC with side lenght a.

Prove that $MA+MB+MC<2a$.

Marius Mainea

Luis González

#2 Aug 21, 2009, 12:17 am

We prove a more general result:

Lemma: M is a point inside $\triangle ABC$ whose shortest side is BC. Then we have that $b+c>MA+MB+MC$.

Draw the parallel to BC passing through M that cuts AC and AB at X, Y, respectively. Draw the altitude AH and WLOG assume that M lies inside $\triangle AHB$. We have $YA>MA$ (1) and since $AC>CB \Longrightarrow AX>XY$, due to the similarity $\triangle ABC \sim \triangle AXY$. Thus, $AX+XC=b>XY+XC$ (2)

By triangle inequality $MX+XC>MC$, $MY+YB>MB$. Adding these two inequalities gives

$CX+XY+YB>MB+MC$ (3)

Adding $(1), (2), (3)$ together yields

$b+YA+YB+XC+XY>MA+MB+MC+XY+XC$

$\Longrightarrow b+c>MA+MB+MC.$

Quick Reply

High School Olympiads

concyclic

Source: russia mo

am2525

#1 Aug 19, 2009, 8:00 pm

Let BB_1 and CC_1 be the altitudes of an acute-angled nonisosceles triangle ABC. The bisector of the acute angles between lines BB_1 and CC_1 intersects sides AB and AC at P and Q, respectively. Let H be the orthocenter of triangle ABC and let M be the midpoint of BC. Let the bisector of BAC intersect HM at R. Prove that quadrilateral $PAQR$ is cyclic.

Luis González

#2 Aug 20, 2009, 7:13 am

Let L be the midpoint of the arc BC of the circumcircle (O). Then $\triangle ARH \sim \triangle LRM$

$$\frac{RM}{RH}=\frac{ML}{AH}=\frac{BC\tan\frac{A}{2}}{2BC\cot A}=\frac{1}{2}\tan A\cdot\tan\frac{A}{2}\quad(1)$$

On the other hand, let K be the orthogonal projection of M onto AB.

$$PK=\tfrac{1}{2}BC_1-PC_1\Longrightarrow\frac{PK}{PC_1}=\frac{BC_1}{2PC_1}-1=\frac{BC\cos B}{2HC_1\tan\frac{A}{2}}-1$$

$$\frac{PK}{PC_1}=\frac{BC\cos B}{2AH\cos B\tan\frac{A}{2}}-1=\frac{\tan A}{2\tan\frac{A}{2}}-1=\frac{1}{2}\tan A\cdot\tan\frac{A}{2}\quad(2)$$

From (1) and (2) we have $\frac{RM}{RH}=\frac{PK}{PC_1}\Longrightarrow RP\perp AB$. Analogously $RQ\perp AC$. Hence, P and Q lie on the circumference with diameter AR.

High School Olympiads

easy

Reply

Source: russia mo

am2525

#1 Aug 19, 2009, 7:59 pm

Let E be a point on the median CD of triangle ABC.
Let S1 be the circle passing through E and tangent to line AB at A,
intersecting side AC again at M; let S2 be the circle passing through
E and tangent to line AB at B, intersecting side BC again at N Prove that the circumcircle of triangle CMN is tangent to circles S1 AND S2

Luis González

#2 Aug 19, 2009, 10:45 pm

Since D has equal power $\frac{1}{4}AB^2$ with respect to (S_1) and (S_2), it follows that CD is the radical axis of $(S_1), (S_2)$. Inversion with center C and power equal to the power of C WRT (S_1) and (S_2) takes (S_1) and (S_2) into themselves and M, N into A, B, respectively. Therefore, circle $\odot(CMN)$ is taken into the common external tangent AB of (S_1) and (S_2). Then by conformity, $\odot(CMN)$ is tangent to $(S_1), (S_2)$.

Quick Reply

High School Olympiads

Three circles externally tangent - M, N, and L are collinear

Reply

Source: Moldova TST 2002 - E1 - P3

am2525

#1 Aug 19, 2009, 7:51 pm

The circles W_1, W_2, W_3 in the plane are pairwise externally tangent to each other. Let P_1 be the point of tangency between circles W_1 and W_3, and let P_2 be the point of tangency between circles W_2 and W_3. A and B, both different from P_1 and P_2, are points on W_3 such that AB is a diameter of W_3. Line AP_1 intersects W_1 again at X, line BP_2 intersects W_2 again at Y, and lines AP_2 and BP_1 intersect at Z. Prove that X, Y, and Z are collinear.

Luis González

#2 Aug 19, 2009, 10:07 pm

Let B', A' denote the 2nd intersections of BP_1 and AP_2 with ω_1 and ω_2. Since P_1 and P_2 are insimilicenters of $\omega_1 \sim \omega_3$ and $\omega_2 \sim \omega_3$, it follows that $AB \parallel XB' \parallel YA'$. If a, b denote the tangents of ω_3 at A, B and a', b' the tangents of ω_2, ω_1 at A', B', we have $a \parallel a', b \parallel b'$. Since $a \parallel b$, then $a' \parallel b' \Longrightarrow b'$ is image of a' through the negative homothety that takes ω_1 into $\omega_2 \Longrightarrow A', B', P_3$ are collinear. $B'X, A'Y$ are diameters of ω_1 and $\omega_2 \Longrightarrow X, Y$ lie on perpendicular to $A'B'$ through P_3. From cyclic quadrilateral ABP_2P_1, we get

$$\angle ABP_1 = \angle BB'X = \angle AP_2P_1 \ , \ \angle BAP_2 = \angle AA'Y = \angle BP_1P_2$$

On the other hand, from the cyclic quadrilaterals P_1P_3XB', P_2YP_3A' we get $\angle P_1P_3Y = \angle BB'X, \angle P_2P_3Y = \angle AA'Y$ $\Longrightarrow \angle P_2P_3P_1 = \angle AP_2P_1 + \angle BP_1P_2 \Longrightarrow ZP_2P_3P_1$ is cyclic. Since $\angle P_2P_3Y = \angle ZP_1P_2$, then YX is identical with the diagonal ZP_3 of the quadrilateral $ZP_2P_3P_1$, i.e. X, Y, Z are collinear.

Quick Reply

Spain

TST PERU - IMO 2007 [Dia 2- P02]

Reply

cuenca

#1 May 11, 2007, 6:00 am • 1

En un triangulo $ABC, CA \neq CB$, los puntos A_1 y B_1 son los puntos de tangencia de las circunferencias exinscritas relativas a los lados CB y CA, respectivamente, e I es el incentro. La recta CI intersecta a la circunferencia circunscrita al triangulo ABC en el punto P. La recta que pasa por P y es perpendicular a CP intersecta a la recta AB en el punto Q. Pruebe que las rectas QI y A_1B_1 son paralelas.

Luis González

#2 Aug 19, 2009, 10:29 am

> Quote:
> En un triangulo $ABC, AB \neq AC$, los puntos B_1 y C_1 son los puntos de tangencia de las circunferencias exinscritas relativas a los lados AC y AB, respectivamente e I es el incentro. La recta AI intersecta a la circunferencia circunscrita al triángulo ABC en el punto P. La recta que pasa por P y es perpendicular a AP intersecta a la recta BC en el punto Q. Pruebe que las rectas IQ y B_1C_1 son paralelas.

Usemos coordenadas baricentricas con respecto a $\triangle ABC$. Entonces:

$$B_1 = (p-a:0:p-c)\ ,\ C_1 = (p-a:p-b:0).$$

Luego la recta B_1C_1 tiene por punto del infinito

$$T_\infty = ((p-a)(b-c):b(p-b):-c(p-c))$$

Siendo P es punto medio del arco BC del circuncirculo, este tiene coordenadas

$$P = (-a^2:b(b+c):c(b+c))$$

La recta $\mathcal{L}_a$ que pasa por P perpendicular a AP es paralela a la bisectriz exterior de $\angle BAC$ por ello tienen el mismo punto del infinito $(c-b:b:-c)$. Su ecuación es entonces:

$\mathcal{L}_a \equiv bc(b+c)x - cS_Cy - bS_Bz = 0$. Esta corta a BC en $Q = (0:bS_B:-cS_C)$

$\Longrightarrow\ IQ \equiv abcx - cS_Cy - bS_Bz = 0$. Las coordenadas de su punto del infinito son:

$R_\infty = (cS_C - bS_B : b(S_B + ac) : -c(S_C + ab))$. Esta la podemos reescribir teniendo presente las identidades:

$$cS_C - bS_B = 2p(p-a)(b-c)\ ,\ S_B + ac = 2p(p-b)\ ,\ S_C + ab = 2p(p-c)$$

$$R_\infty = ((p-a)(b-c):b(p-b):-c(p-c)) \Longrightarrow\ T_\infty \equiv R_\infty \Longrightarrow\ IQ \parallel B_1C_1$$

Quick Reply

High School Olympiads

geometry

Reply

shoki

#1 Aug 19, 2009, 12:53 am

Let ABC be an acute triangle and (O) be its circumcircle. The tangents from A and B to (O) meet the tangent from C to (O) in points D and E respectively. Let AE meet BC at P and BD meet AC at R. Points Q and S are respectively the midpoints of AP and BR. Prove that <ABQ = <BAS.

Luis González

#2 Aug 19, 2009, 3:08 am

This "infamous" problem has been posted before. For instance, see

http://www.artofproblemsolving.com/viewtopic.php?t=275380
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=207440
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=19806

Quick Reply

Spain

Tres ejes radicales concurrentes

Reply

Luis González

#1 Jun 16, 2009, 10:26 am

Sea P un punto del plano de $\triangle ABC$ y sea $\triangle A'B'C'$ el triángulo ceviano de P con respecto a $\triangle ABC.$ Las circunferencias $\odot(ABB')$ y $\odot(ACC')$ se cortan en A y $X.$ Definimos igualmente Y,Z con respecto a B y $C.$ Mostrar que AX,BY,CZ concurren en el complemento del isogonal de P respecto a $\triangle ABC.$

Luis González

#2 Aug 18, 2009, 5:16 am

Véase aqui una solución del problema con coordenadas baricentricas http://www.artofproblemsolving.com/viewtopic.php?t=282732. También se discuten varias cuestiones interesantes sobre esta configuración.

Quick Reply

High School Olympiads

tangent

Reply

Source: nice

stvs_f

#1 Aug 16, 2009, 11:36 pm

let o is circumcircle of ABC triangle and <B=90. ω is tangent to AB,BC and o.
prove that : Radius of ω=Diagonal of incircle of ABC.

Luis González

#2 Aug 17, 2009, 1:21 am

Let X,Y denote the tangency points of $\omega \equiv (V)$ with $BC,BA.$ It's well-known that the polar of vertex B WRT the B-mixtilinear incircle (V) (the line XY), goes through the incenter I of $\triangle ABC.$ If $\angle ABC=90^{\circ} \Longrightarrow I$ is the center of the rectangle $BXVY.$ Thus, it follows that the radius of (V) is twice the inradius of $\triangle ABC.$

Quick Reply

High School Olympiads

inversion,diffficult

Reply

bye

#1 Aug 15, 2009, 10:20 am

denotefive circles c1,c2,c3,c4,c5.if we take c5 as the inversin circle, then c1 and c4 are inversly similar,c2 and c3 are also inversly similar.prove:there exists a circle tangents to c1,c2,c3,c4.
(how to use Casey theorem to get the result?)

bye

#2 Aug 16, 2009, 4:15 pm

i wonder if i've described clearly.
it's easy to calculate the tangents between c1 and c4,c2 and c3.
but i don't know how to deal with that of c1 and c3,respectively.
in fact,it's a conclusion in Modern Euclidean Geometry.as a **direct** deduction of Casey theorem.[/b]

yetti

#3 Aug 16, 2009, 8:00 pm

I don't think this has much in common with Casey's theorem. Suppose $\mathcal C_1, \mathcal C_2$ both intersect $\mathcal C_5$. Since inversion preserves angles between curves, the circles $\mathcal C_1, \mathcal C_4$ form equal angles with the inversion circle $\mathcal C_5$. The same holds for the circles $\mathcal C_2, \mathcal C_3$ WRT $\mathcal C_5$. Invert the figure $\mathcal C_1 \cup \mathcal C_2 \cup \mathcal C_3 \cup \mathcal C_4 \cup \mathcal C_5$ in arbitrary circle $\mathcal O$ centered on $\mathcal C_5$. This inversion takes the previous inversion circle $\mathcal C_5$ into a line c_5 and the remaining circles $\mathcal C_1, \mathcal C_2, \mathcal C_3, \mathcal C_4$ into some other circles $\mathcal C'_1, \mathcal C'_2, \mathcal C'_3, \mathcal C'_4$. Again, inversion preserves angles between curves, therefore the circles $\mathcal C'_1, \mathcal C'_4$ form equal angles with the line c_5 and the same holds for the circles $\mathcal C'_2, \mathcal C'_3$ WRT c_5. This means that $\mathcal C'_1, \mathcal C'_4$ are symmetrical WRT the line c_5 and $\mathcal C'_2, \mathcal C'_3$ are also symmetrical WRT c_5. By symmetry, four circles $\mathcal K'_1, \mathcal K'_2, \mathcal K'_3, \mathcal K'_4$ exist, all centered on the line c_5 (perpendicular to this line), each tangent to all $\mathcal C'_1, \mathcal C'_2, \mathcal C'_3, \mathcal C'_4$. If $\mathcal K_1, \mathcal K_2, \mathcal K_3, \mathcal K_4$ are inversion images of $\mathcal K'_1, \mathcal K'_2, \mathcal K'_3, \mathcal K'_4$ in $\mathcal O$, then each of $\mathcal K_1, \mathcal K_2, \mathcal K_3, \mathcal K_4$ is tangent to all $\mathcal C_1, \mathcal C_2, \mathcal C_3, \mathcal C_4$.

Luis González

#4 Aug 16, 2009, 11:52 pm

Nice solution Vladimir, I've got this approach:

Let P_1, P_4 denote the variable tangency points of a circle $\mathcal T_i$ with $\mathcal C_1$ and $\mathcal C_4$. Let us consider the positive inversion WRT $\mathcal C_5$, although the result is still true for both positive or negative inversion WRT $\mathcal C_5$. Pairs P_1, P_4, lying on a same side of the line O_1O_4, connecting the centers of $\mathcal C_1, \mathcal C_4$ are inverse under such positive inversion, due to the conformity, or equivalently P_1P_4 passes through the center O_5 of $\mathcal C_5$. Hence, we have a set of four circles $\mathcal T_i, i = 1, 2, 3, 4$, where a pair is internally and externally tangent to $\mathcal C_2$ for each pair P_1, P_4 lying on different sides of O_1O_4. This inversion maps $\mathcal C_1 \mapsto \mathcal C_4, \mathcal C_2 \mapsto \mathcal C_3$ and takes $\mathcal T_i$ into itself. If $\mathcal T_i$ is tangent to $\mathcal C_2$, then $\mathcal T_i$ is tangent to the inverse image $\mathcal C_3$ as well.

yetti

#5 Aug 17, 2009, 2:57 am

I used the following lemma before, without posting a proof:

Lemma: Let (C) be arbitary circle, let (O) be arbitrary circle with center $O \in (C)$ and c their radical axis. Let $\mathbf R_{(O)}, \mathbf R_{(C)}, \mathbf R_c$ be reflections in $(O), (C), c,$ respectively. Obviously, $c = \mathbf R_{(O)}((C))$. Let A be arbitrary point and B its reflection (inversion image) in (C). Then

$$B = \mathbf R_{(C)}(A) = \mathbf R_{(O)} \circ \mathbf R_c \circ \mathbf R_{(O)}(A) = \mathbf R^{-1}_{(O)} \circ \mathbf R_c \circ \mathbf R_{(O)}(A).$$

Proof: Let A' be inversion image of the point A in (O), let B' be reflection of the point A' in the line c, and let B be inversion image of the point B' in the circle (O). By the basic properties of inversion in (O), the quadrilateral $AA'B'B$ is cyclic, let (P) be its circumcircle. Since (P) passes through A and its inversion image A' in $(O) \Longrightarrow (P) \perp (O)$. Since (P) is centered on the perpendicular bisector c of $A'B' \Longrightarrow (P) \perp c$. Since $(P), (C)$ are inversion images of $(P), c$ in $(O) \Longrightarrow (P) \perp (C)$. Let polars of A, B WRT (O) (perpendiculars to OA, OB at A', B') intersect at X. The quadrilateral $OA'B'X$ is cyclic on account of right angles at A', B', let (Q) be its circumcircle on diameter OX. Its center Q, the midpoint of OX, is also on the perpendicular bisector c of $A'B'$. Let (Q) cut perpendicular OC to c again at C'. Then $XC' \perp OC$ and c is also perpendicular bisector of the segment $OC' \Longrightarrow$ the line XC' is polar of the point C WRT (O). Since the polars of A, B, C concur at X, these 3 points are collinear. But $A, B \in (P)$ and inversion in (C) takes $(P) \perp (C)$ to itself $\Longrightarrow$ the point B is the inversion image of the point A in the circle (C). $\square$

Applying the lemma to $\mathcal{C}_5$ as the circle (C) and all points $A \in \mathcal{C}_1 \cup \mathcal{C}_2$, (hence $B \in \mathcal{C}_4 \cup \mathcal{C}_3$), the inversion images $\mathcal{C}'_1, \mathcal{C}'_4$ of $\mathcal{C}_1, \mathcal{C}_4$ in (O) are symmetrical with respect to the radical axis c_5 of $(O), \mathcal{C}_5$. So are the inversion images $\mathcal{C}'_2, \mathcal{C}'_3$ of $\mathcal{C}_2, \mathcal{C}_3$ in (O). Regardless of $\mathcal{C}_1, \mathcal{C}_2$ intersecting $\mathcal{C}_5$ or not.

Spain

Reto de la semana #5

Reply

Pascual2005

#1 Sep 18, 2006, 8:36 am • 1

1. Sea ABC un triángulo isósceles con $BC = AC$. Sea P un punto de su circunferencia circunscrita situado en el arco AB que no contiene a C. Sea D el pie de la perpendicular trazada desde C a la recta PB. Demostrar que $PA + PB = 2PD$

(Problema cambiado)

2. Sean a, b enteros positivos tales que $a|b^2, b^2|a^3, a^3|b^4 \dots$ y asi sucesivamente. Pruebe que $a = b$.

(Problema Cambiado)

3. Halle todos los polinomios $P(n)$ tales que existen complejos $z_1, z_2 \dots, z_k$ tales que $\sum_{i=1}^{k} z_i^m = 0$ si y sólo si $m \neq P(n)$ para todo n.

4. Se tienen un número finito de segmentos en una linea de modo que entre cualesquier $m + 1$ segmentos hay dos que se intersectan. Pruebe que se pueden seleccionar m puntos en la linea de modo que cualesquier segmento contiene al menos uno de estos puntos.

This post has been edited 1 time. Last edited by Pascual2005, Sep 19, 2006, 9:47 am

conejita

#2 Sep 18, 2006, 9:07 pm

Ola Pascual, esta vez si voa entrar al reto, ya me salieron dos problemas!!!
Donde pongo las soluciones???

Grax

R.G.A.M.

#3 Sep 19, 2006, 1:35 am

¿Existe algún material útil para iniciarse con polinomios? En mi país hemos dado una breve noción acerca de cómo trabajar con ellos, mas resulta insuficiente.
Por supuesto no quiero que la información necesaria para trabajar con ellos esté servida, pero me agradaría contar con un buen material de consulta.

Pascual2005

#4 Sep 19, 2006, 2:04 am

Las soluciones deben enviarlas a mi correo: pascual2k1@hotmail.com

Ahora estoy corrigiendo las soluciones de la cuarta edición, entonces tenganme paciencia.

Respecto a lo de polinomios, puedes buscar en internet hay muchas elcturas, inclusive aqui en el foro colgue un pdf de polinomios. Sino un buen libro puede ser polynomials de Barbeau.

Suerte

hucht

#5 Sep 19, 2006, 5:49 am

> *Pascual2005 wrote:*
> 1. Sea ABC un triángulo isósceles con $BC = AC$. Sea P un punto de su circunferencia circunscrita situado en el arco AB que no contiene a C. Sea D el pie de la perpendicular trazada desde C a la recta PB. Demostrar que $PA + PB = 2PD$

Por fin encontre donde postean el reto de la semana! ... lastima que el problema de geo sea tan facil

Por Ptolomeo $AC.PB + CB.AP = AB.CP$ con lo que $AC(AP + PB) = AB.CP$. Es obvio que $AB = 2AC.cos\angle CAB$ y $PD = APcos\angle CPB$. Desde que $m\angle CAB = m\angle CPB$ la conclusion es inmediata.

---

conejita

#6 Sep 19, 2006, 7:34 am

Aki mi solucion:
Supongamos que $a > b$, entonces como son enteros positivos tenemos que $a^n > b^n$ de aqui que la division dea^{n-1}/b^n es un entero menor que a. Pero ahora, dado que a es un entero positivo, entonces entre 1 y a existen $a - 1$ diferentes enteros, los cuales es claro que son finitos.
Ahora, debido a que es fnito, en algun momento, alguna de esas divisiones va a tener que ser igual a otra, supongamos que si $1 \leq k < a$ y que $a^{n-1}/b^n = a^{m-1}/b^m = k$ de ai deducimos que como a es entero positivo, entonces $a = b$ $CONTRADICCION$
Ahora, si empezamos suponiendo que $b > a$ usamos un argumento similar, solo que ahora usaremos las divisiones b^{n-1}/a^n y de ahi llegamos nuevamente a que $a = b$ otra $contradiccion$.
De manera que la unica forma es que a=b.

Pascual2005

#7 Sep 19, 2006, 8:48 am

NO CHICOS! no leyeron las reglas? los problemas no los deben discutir, en vista de que ya dieron soluciones a estos, los cambiaré! la idea es que me envien sus soluciones y dentro de una semana ya los pueden discutir,e sto es para ponerlo cierto toque competitivo al asunto.

Pascual2005

#8 Sep 19, 2006, 8:51 am

1. Pruebe que la ecuación $2x^2 - 3x = 3y^2$ tiene infinitas soluciones enteras.

2. Sea ABC un triángulo para el cual existe un punto F tal que $\angle AFB = \angle BFC = \angle CFA$. Las líneas BF y CF intersectan a los lados
AC y AB en los puntos D y E, respectivamente. Demostrar que

$$AB + AC \geq 4DE.$$

---

aev5peru

#9 Sep 19, 2006, 10:03 am

BUENO YA QUE NO RESPETARON LAS REGLAS, Y PUBLICARON SUS SOLUCIONES, Y ESOS PROBLEMAS NO VALEN PUBLICARE MIS SOLUCIONES DEL 1 YA Q EL PROBLEMA 2 LO ISE = QUE CONEJITA
PROBLEMA 1..
PROLONGAMOS PB HASTA E TALQUE $BD = DE$, ENTONCES TENEMOS $AC = CB = CE$, SOLO FALTARIA PROBAR QUE $< AEP =< ABP$.
UNIMOS $AE \Rightarrow < CAE =< CEA$, SEA $< AEP = b$, POR EL ISOCELES $(BCE), < CBE =< CEA + b$, AHORA POR EL INSCRIPTIBLE $ABCD, < CAP =< CBE$, CON LO CUAL SE DEMUESTRA QUE $< EAP =< AEP$, POR LO TANTO ISOCELES,$AP + PB = 2PD$

This post has been edited 1 time. Last edited by aev5peru, Sep 23, 2006, 9:25 am

R.G.A.M.
#10 Sep 23, 2006, 7:56 am

Pascual2005, ¿podrías facilitarme la dirección del archivo .pdf de poliinomios que colgaste?

Muchas gracias.

conejita
#11 Oct 6, 2006, 10:08 pm

Pascual, ya vas a poner las soluciones de los problemas del reto??
Me interesa la del punto de Fermat....
GRAX

Pascual2005
#12 Oct 8, 2006, 10:09 pm

ya pueden comentar los problemas, como es usual si nadie resuelve alguno tratare de poner mi solución. Animo y espero 4 bellas soluciones!

Chen241290
#13 Oct 19, 2006, 8:38 am

Aun no cuelguen las soluciones, si me sale o no el 2, el domingo envio mis soluciones.

Pascual2005
#14 Nov 6, 2006, 11:35 am

En el segundo la idea es que P es un polinomio con coeficientes enteros y m, n son enteros. Chen, puedes colgar tus soluciones por favor ya que segun el mail que recibi las tienes todas?

Chen241290
#15 Nov 17, 2006, 8:41 am

1.- Una solucion es $x_1 = -3, y_1 = 3$ Construímos las sucesiones x_n y y_n
Así: $x_{k+1} = 5x_k - 6y_k - 3$ $y_{k+1} = 3 - 4x_k + 5y_k$ por induccion x_n es siempre negativo y y_n es siempre positivo, entonces $x_{k+1} < x_k$ y $y_{k+1} > y_k$ Por induccion tbn podemos demostar que (x_n, y_n) es una solucion del sistema, luego tenemos infintas soluciones distintas.

2.-Hay que considerar que F esta dentro de ABC, sino se puede encontrar un contraejemplo.
Sean: s=FE, t=FD. del triangulo ABF el angulo BAF es menor que 60, entonces en el triangulo AFE AF>FE, es decir x>s, analogamente x>t.
$AB + AC \geq 2\sqrt{AB.AC}$. Luego basta probar que $AB.AC \geq 4DE^2$
Para el triangulo AFE, Fb es bisectriz exterior entonces AB es como x, y BE es como s, luego AB/AE=x/(x-s). AE del teorema de cosenos es la raiz cuadrada de x^2-xs+s^2={x-s}^2+(x-s)s+s^2\ge 3{\frac{(x-s)+s}{2}}^2. Luego $AE \geq \frac{x\sqrt{3}}{2}$, entonces $AB \geq \frac{x^2\sqrt{3}}{2(x-s)}$.
Analogamente $AC \geq \frac{x^2\sqrt{3}}{2(x-t)}$.
Entonces $AB.AC \geq \frac{3x^4}{4(x-s)(x-t)}$. Ademas del teorema de cosenos, $DE^2 = s^2 + st + t^2$
Sea g(x)=$\frac{x^4}{(x-s)(x-t)}$

Si s=t entonces $(x-2s)^2 \geq 0$, luego $x^2 \geq 4s(x-s)$
Entonces $x^4 \geq 16s^2(x-s)^2$ ademas DE^2=3s^2
entonces $AB.AC \geq 3/4.g(x) \geq 4DE^2$.
Si s y t son distintos:
Sean: a=$\sqrt{s^2+st+t^2}$, 2b=s+t.
Si $x \leq a+b$:
x>s,x>t, sumando: 2x>2b.
Luego $0, entonces $(x-b)^2 < a^2$.

Pero la sgte desigualdad, despues de dividir todo entre a^2-3b^2(que es ((s-t)^2)/4>0) es equivalente a la anterior.
$(x-s)(x-t)a^2 \leq (x-b)^2 3b^2$
Analogamente al caso s=t, podemos demostrar que
$x^4 \geq 16b^2(x-b)^2$, el termino de la derecha es mayor o igual que 16/3. $(x-s)(x-t)a^2$. Luego $AB.AC \geq 3/4g(x) \geq 4$
Si $x \geq a+b$:
g'(x)=(x^3)/2(x-s)(x-t))^2 . (4x^2-12xb+32b^2-8a^2).
El termino de la izquierda es positivo pues x>a. y a>s,t,0
El termino de la derecha es (2x-3b)^2+23b^2-8a^2
Como $x \geq a+b$: $2x-3b \geq 2a-b$ a>s,a>t, entonces 2a>2b, luego 2a-b=a+(a-b) es positivo. luego $(2x-3b)^2 \geq (2a-$
Luego el termino de la derecha mencionado, es mayor o igual que 24b^2-4ab-4a^2=4(3b+a)(2b-a). (s+t)^2>a^2, entonces 2b>a.
Luego g'(x)>0, mientras $x \geq a+b$
entonces es creciente en el intervalo de extremos a+b e infinito positivo.
Luego $g(x) \geq g(a+b); x \geq a+b$
Notar que $g(a+b) = \dfrac{a+b^4}{3b^2}$
pero $(a+b)^2 \geq 4ab$ entonces $(a+b)^4 \geq 16a^2b^2$.
Luego $g(a+b) \geq 16/3.a^2$.
$AB.AC \geq 3/4.g(x) \geq 3/4.g(a+b) \geq 4a^2 = 4DE^2$.

3.- una cosa mas, para que no esté 0^0, los complejos son no nulos, m es natural, o solo diferente de cero?

4.- Considerando intervalos cerrados a los segmentos
Por induccion:
para m=1. cada 2 segmentos se intersecan, entonces tomemos el segmento con extremo izquierdo, lo mayor posible., los dema
Supong. cierta la afirmacion para m=n:
Entonces Si tenemos que entre cualesquiera n+2 segmentos hay 2 que se intersecan. Sea s el segmento con mayor extremo iz

Luis González
#16 Aug 16, 2009, 9:02 am

> Pascual2005 wrote:
> 1. Sea ABC un triángulo isósceles con $BC = AC$. Sea P un punto de su circunferencia circunscrita situado en el arco AB que no contiene a C. Sea D el pie de la perpendicular trazada desde C a la recta PB. Demostrar que $PA + PB = 2PD$.

Sea $CA = CB = L$. Aplicando teorema de Ptolomeo al cuadrilátero cíclico $CAPB$ resulta

$$PA \cdot L + PB \cdot L = PC \cdot AB \Longrightarrow \ PA + PB = \frac{PC \cdot AB}{L} \ (*)$$

Por otro lado si H es el pie de la perpendicular de C a AB (punto medio de AB) se tiene que

$$\angle CAH = \angle CPD \Longrightarrow \ \triangle CPD \sim \triangle CAH \Longrightarrow \ \frac{2PD}{AB} = \frac{PC}{L}$$

Combinando esta ultima con $(*)$ se obtiene $PA + PB = 2PD$.

BlAcK_CaT
#17 Aug 8, 2010, 6:36 am

> Pascual2005 wrote:
> 2. Sean a, b enteros positivos tales que $a|b^2, b^2|a^3, a^3|b^4 \ldots$ y asi sucesivamente. Pruebe que $a = b$.

Sea $c(n)$ la cantidad de divisores primos de n. Primero, es sencillo ver que $c(n^k) = c(n)$ para todo $k \in \mathbb{N}$. Como cada divisor primo de a divide a b^2, entonces b^2 contiene todos los divisores primos de a. Esto implica que $c(a) \leq c(b^2) = c(b)$. Con el mismo argumento, obtenemos que $c(a) = c(a^3) \geq c(b^2) = c(b)$ y por ende $c(a) = c(b)$. Es más, a, b tienen los mismos divisores primos.

Escribamos $a = \prod_{k=1}^{r} p_k^{\alpha_k}$ y $b = \prod_{k=1}^{r} p_k^{\beta_k}$ donde $p_i < p_j$ siempre que $i < j$ y cada α_k, β_k es un natural. Para cada n natural, tenemos que $a^{2n-1}|b^{2n}$, esto implica que $(1-\dfrac{1}{2n})\alpha_k \leq \beta_k$. Si $n \to \infty$, tenemos que $1-\dfrac{1}{2n} \to 1$ y por ende $\alpha_k \leq \beta_k$

para todo $1 \leq k \leq r$. Por otra parte, como $b^{2n}|a^{2n+1}$, tenemos que $(1+\frac{1}{2n})\alpha_k \geq \beta_k$. Esta vez, si $n \to \infty$, tenemos que $1+\frac{1}{2n} \to 1$ y esto significa que $a_k \geq \beta_k$. Por lo tanto $\alpha_k = \beta_k$ para cada $k \in \{1, 2, ..., r\}$, de donde se deduce que $a = b$.

High School Olympiads

Find angles of ABC if < BDE = 24° and < CED = 18°

Reply

Source: IMO Longlist 1992 problem JPN 2, also: The Special Mathematic High School in Belgrade

grobber

#1 Nov 22, 2003, 6:51 pm • 1

Let D and E be points on the sides CA and AB of a triangle ABC such that the lines BD and CE are the interior angle bisectors of the angles ABC and BCA, respectively. Given that $\angle BDE = 24^\circ$ and $\angle CED = 18^\circ$, determine the angles $\angle A$, $\angle B$, $\angle C$ of triangle ABC.

grobber

#2 Aug 23, 2004, 8:09 am

The answer is A=96° (this is easy to obtain right from the beginning, the harder thing is getting the values of B and C) and B=60°, C=24° (I think). I'm still working on a solution.

Line

#3 Jan 14, 2007, 11:13 pm

does anybody have a solution?

(with an image for better understanding)

Altheman

#4 Jan 15, 2007, 1:49 am

i get the nasty trig system:

$$(\sin 18 + \frac{C}{2})(\sin 96 + \sin B) = (\sin 24 + \frac{B}{2})(\sin 96 + \sin C)$$
$$B + C = 84$$

it is pretty clear that the solution (B,C)=(60,24), all that remains to prove is that there is only one solution, and this should not be so bad...

Line

#5 Jan 15, 2007, 1:50 am

ohhh man!!!, very nasty, but thanks anyway!!!

anyone has a solution without trig.?

yetti

#6 Jan 15, 2007, 5:54 pm • 1

> Altheman wrote:
> ...it is pretty clear that the solution (B,C)=(60,24), ...

That's because you did not try.

$\angle A$

$\frac{}{2} + 90^\circ = \angle BIC = 180^\circ - (\angle BDE + \angle CED) = 138^\circ$

Then $\angle A = 96^\circ, \angle B + \angle C = 84^\circ, \angle B,\ \angle C$ are both acute. By the sine theorem for the $\triangle BDE,\ \triangle CED$,

$$\frac{\sin\frac{B}{2}}{\sin 24^\circ} = \frac{DE}{BE}, \quad \frac{\sin\frac{C}{2}}{\sin 18^\circ} = \frac{DE}{CD},$$

$$\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}} \cdot \frac{\sin 18^\circ}{\sin 24^\circ} = \frac{CD}{BE} = \frac{ab}{c+a} \cdot \frac{a+b}{ca} = \frac{\sin B}{\sin C} \cdot \frac{\sin 96^\circ + \sin B}{\sin 96^\circ + \sin C}$$

$$\frac{\cos\frac{C}{2}\sin 18^\circ}{\cos\frac{B}{2}\sin 24^\circ} = \frac{\sin\left(48^\circ + \frac{B}{2}\right)\cos\left(48^\circ - \frac{B}{2}\right)}{\sin\left(48^\circ + \frac{C}{2}\right)\cos\left(48^\circ - \frac{C}{2}\right)} = \frac{\cos\left(42^\circ - \frac{B}{2}\right)\cos\left(48^\circ - \frac{B}{2}\right)}{\cos\left(42^\circ - \frac{C}{2}\right)\cos\left(48^\circ - \frac{C}{2}\right)}$$

Using $\frac{\angle B}{2} = 42^\circ - \frac{C}{2}$,

$$\frac{\sin 18^\circ}{\sin 24^\circ} = \frac{\cos\left(48^\circ - \frac{B}{2}\right)}{\cos\left(48^\circ - \frac{C}{2}\right)} = \frac{\cos\left(\frac{C}{2} + 6^\circ\right)}{\cos\left(\frac{C}{2} - 48^\circ\right)}$$

$$\sin\left(\frac{C}{2} - 30^\circ\right) + \sin\left(66^\circ - \frac{C}{2}\right) = \sin\left(\frac{C}{2} + 30^\circ\right) + \sin\left(18^\circ - \frac{C}{2}\right)$$

$$\cos\frac{C}{2} = 2\sin 30^\circ\cos\frac{C}{2} = 2\cos\left(42^\circ - \frac{C}{2}\right)\sin 24^\circ = 2\sin 24^\circ\sin\left(\frac{C}{2} + 48^\circ\right) =$$

$$= 2\sin 24^\circ\left(\sin\frac{C}{2}\cos 48^\circ + \cos\frac{C}{2}\sin 48^\circ\right) =$$

$$= (\sin 72^\circ - \sin 24^\circ)\sin\frac{C}{2} - (\cos 72^\circ - \cos 24^\circ)\cos\frac{C}{2}$$

$$\tan\frac{C}{2} = \frac{1 + \cos 72^\circ - \cos 24^\circ}{\sin 72^\circ - \sin 24^\circ} = \frac{1 - \cos 72^\circ}{\sin 72^\circ} = \tan 36^\circ$$

because

$$\frac{1 + \cos 72^\circ - \cos 24^\circ}{\sin 72^\circ - \sin 24^\circ} = \frac{1 - \cos 72^\circ}{\sin 72^\circ} \iff$$

$$2\sin 72^\circ\cos 72^\circ = \sin 72^\circ\cos 24^\circ + \sin 24^\circ\cos 72^\circ - \sin 24^\circ \iff$$

$$\sin 144^\circ = \sin 96^\circ - \sin 24^\circ = 2\cos 60^\circ\sin 36^\circ = \sin 36^\circ$$

As a result, $C = 72^\circ,\ B = 12^\circ$.

Check: Let DE meet BC at F. By Menelaus theorem,

$$\frac{DC}{DA} \cdot \frac{EA}{EB} \cdot \frac{FB}{FC} = \frac{a}{c} \cdot \frac{b}{a} \cdot \frac{FC + a}{FC} = 1, \quad FC = \frac{ab}{c-b}$$

$$CE = \frac{\sqrt{ab[(a+b)^2 - c^2]}}{a+b} = \frac{\sqrt{2a^2b^2(1+\cos C)}}{a+b} = \frac{2ab\cos\frac{C}{2}}{a+b}$$

$$\frac{CE}{FC} = \frac{2ab\cos\frac{C}{2}}{a+b} \cdot \frac{c-b}{ab} = 2\cos\frac{C}{2} \cdot \frac{\sin C - \sin B}{\sin A + \sin B} =$$

$$= 2\cos\frac{C}{2} \cdot \frac{\cos\frac{C+B}{2}\sin\frac{C-B}{2}}{\sin\frac{A+B}{2}\cos\frac{A-B}{2}} = \frac{2\sin\frac{C-B}{2}\cos\frac{C+B}{2}}{\cos\frac{A-B}{2}}$$

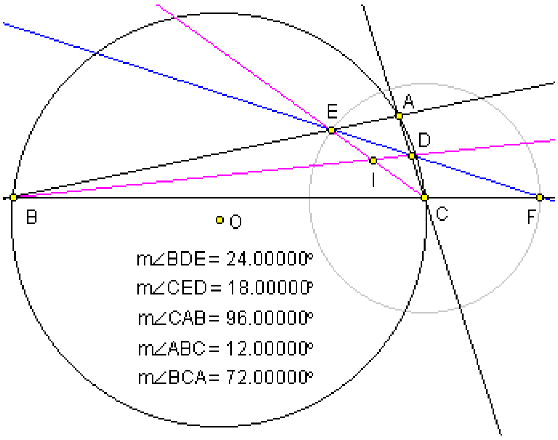
Assuming $\angle C = 72^\circ,\ \angle B = 12^\circ$,

$$\frac{CE}{FC} = \frac{2\sin 30^\circ\cos 42^\circ}{\cos 42^\circ} = 1$$

The triangle $\triangle CEF$ is then isosceles, $2\angle CED = \angle CEF + \angle CFE = \angle BCE = \frac{\angle C}{2}$, and $\angle CED = \frac{\angle C}{4} = 18^\circ$.

The other given angle is then $\angle BDE = \angle DBF + \angle FBD = \frac{\angle B}{2} + \angle CED = 24^\circ$.

Attachments:



Luis González

#7 Aug 15, 2009, 10:48 am • 2

$I \equiv BD \cap CE$ is the incenter of $\triangle ABC$.

$\angle BAC = 2(\angle EID - 90^\circ) = 2(180^\circ - 24^\circ - 18^\circ - 90^\circ) \Longrightarrow \angle BAC = 96^\circ$

Let P, Q be the intersections of the internal bisector of $\angle BAI$ with DE, DB. Then

$\angle BAQ = \angle QAI = \angle EDI = 24^\circ \Longrightarrow$ quadrilaterals $AEQD, APID$ are cyclic

$\Longrightarrow \angle DEQ = \angle DAQ = 72^\circ$, $\angle QIP = \angle DAQ = 72^\circ$

$\angle QEI = \angle DEQ - \angle DEI = 72^\circ - 18^\circ = 54^\circ$, $\angle EIP = \angle QIP - 42^\circ = 30^\circ$

If R is the the reflection of P about EI, we have $\angle PEI = \angle REI = 18^\circ$. Since $\angle EIP = 30^\circ$, then $\triangle PRI$ is equilateral $\Longrightarrow \angle PRI = \angle RPI = 60^\circ$

On the other hand, $\angle ESI = \angle EQI + \angle QES = 84^\circ + 36^\circ = 120^\circ$, which implies that $PRSI$ is cyclic $\Longrightarrow$ $\angle RPS = \angle RIS = 42^\circ - 30^\circ = 12^\circ \Longrightarrow \angle EPS = 84^\circ$. This means that $\triangle QES \cong \triangle PES \Longrightarrow EP = EQ \Longrightarrow$ $\triangle EPQ$ is isosceles with base $PQ \Longrightarrow \angle EQP = 90^\circ - 36^\circ = 54^\circ$

Since $EQIA$ is cyclic, we have $\angle AIE = \angle EQP = 54^\circ$

$\Longrightarrow \angle AIB = 42^\circ + 30^\circ + 54^\circ = 126^\circ$

$\angle ACB = 2(\angle AIB - 90^\circ) = 252^\circ - 180^\circ = 72^\circ \Longrightarrow \angle ABC = 12^\circ$.

biomathematics

#8 Feb 28, 2015, 9:25 pm

@Luis, $EQIA$ is not cyclic.

Quick Reply

High School Olympiads

collinear-sharigin

Reply

stvs_f

#1 Aug 14, 2009, 11:41 am

let ABC be a triangle such that M_a, M_b, M_c are midpoints of BC,AC anc AB.
A_1, A_2 are 2points on BC that$A_1M_a = A_2M_a$
same as $(A_1, A_2), (B_1, B_2)$are on AC and $(C_1, C_2)$are onAB.
if $G, G_1, G_2,$ be centroid of $ABC, A_1B_1C_1, A_2B_2C_2$:prove that:
G, G_1, G_2 are collinear and$S_{A_1B_1C_1} = S_{A_2B_2C_2}$

livetolove212

#2 Aug 14, 2009, 1:01 pm

We have $3\vec{G_2G} = \sum \vec{G_2A}, 3\vec{G_1G} = \sum \vec{G_1A}$

$\vec{G_2A_2} = \frac{BA_2}{BC}.\vec{G_2C} + \frac{CA_2}{BC}.\vec{G_2B}$

Similarly we get $\vec{0} = \sum \vec{G_2A_2} = \sum(\frac{BA_2}{BC} + \frac{AB_2}{AC})\vec{G_2C}$

$\sum(\frac{BA_1}{BC} + \frac{AB_1}{AC})\vec{G_1C} = \vec{0}$

$\Leftrightarrow \vec{G_1G_2}.\sum(\frac{BA_1}{BC} + \frac{CA_1}{BC}) + \sum(\frac{BA_1}{BC} + \frac{AB_1}{AC})\vec{G_2C} = \vec{0}$

$\Leftrightarrow 3\vec{G_1G_2} + (2 - \frac{BA_2}{BC} - \frac{AB_2}{AC})\vec{G_2C} = \vec{0}$

$\Leftrightarrow 3\vec{G_1G_2} + 2\sum \vec{G_2C} = \vec{0}$

Or $3\vec{G_1G_2} = -6\vec{G_2G}$ therefore G, G_1, G_2 are collinear. Moreover, G is the midpoint of G_1G_2

Let $\frac{AB_2}{AC} = x, \frac{CA_2}{BC} = y, \frac{BC_2}{AB} = z.$

We can easy calculate $S_{A_2B_2C_2} = (1 - x - y - z + xy + yz + zx)S_{ABC}$

Note that $\frac{CB_1}{AC} = x, \frac{BA_1}{BC} = y, \frac{AC_1}{AB} = z$

Similarly we get $S_{A_1B_1C_1} = S_{A_2B_2C_2}$

Luis González

#3 Aug 14, 2009, 9:54 pm

Let $\delta(P)$ denote the oriented distance from point P to the line G_1G_2. We know that G lies on $G_1G_2 \Longleftrightarrow$ Sum of the oriented distances from A, B, C to G_1G_2 equals zero.

$\delta(A_1) - \delta(B_1) - \delta(C_1) = 0\ ,\ \delta(A_2) + \delta(C_2) - \delta(B_2) = 0$

$\delta(A_1) + \delta(A_2) = \delta(B_1) + \delta(B_2) + \delta(C_1) - \delta(C_2).$

On the other hand, we have

$\delta(B_1) + \delta(B_2) = \delta(A) - \delta(C)$

$\delta(C_1) - \delta(C_2) = \delta(A) - \delta(B)$

$\delta(A_1) + \delta(A_2) = \delta(B) + \delta(C)$

$\Longrightarrow \delta(B) + \delta(C) = \delta(A) - \delta(B) - \delta(C) + \delta(A)$

$\Longrightarrow \delta(B) + \delta(C) - \delta(A) = 0 \Longrightarrow G \in G_1G_2$

$\Longrightarrow v(B) + v(C) - v(A) = 0 \Longrightarrow G \in G_1G_2$.

Further, G_1, G_2, G are collinear such that G is the midpoint of $\overline{G_1G_2}$.

Areal coordinates of A_1, B_1, C_1 WRT $\triangle ABC$ are given by

$$A_1\ (0 : v_1 : w_1)\ ,\ B_1\ (u_2 : 0 : w_2)\ ,\ C_1\ (u_3 : v_3 : 0)$$

$$\frac{|\triangle A_1B_1C_1|}{|\triangle ABC|} = u_2v_3w_1 + u_3v_1w_2 = \frac{AC_1 \cdot BA_1 \cdot CB_1 + AB_1 \cdot CA_1 \cdot BC_1}{BC \cdot CA \cdot AB}$$

Note that the latter ratio remains invariant after switching the areal coordinates of A_1, B_1, C_1 with the areal coordinates of A_2, B_2, C_2. Thus, $|\triangle A_1B_1C_1| = |\triangle A_2B_2C_2|$.

Quick Reply

High School Olympiads

Two circles and geometric locus

Reply

xxxxtt

#1 Aug 14, 2009, 2:37 pm

We have two secant circles whics intersect in points A and B.
A variable line passing through the point A cuts the circles in points M and N.
Find the geometric locus of the midpoint of the segment MN!

Luis González

#2 Aug 14, 2009, 2:48 pm

(O_1, r_1) and (O_2, r_2) are the given circles. Let L be the midpoint of MN. Powers of L to $(O_1), (O_2)$ are given by:

$$LO_1^{\ 2} - r_1^{\ 2} = \frac{1}{2} MN \cdot AL \ , \ r_2^{\ 2} - LO_2^{\ 2} = \frac{1}{2} MN \cdot AL$$

$$\Longrightarrow \ LO_1^{\ 2} + LO_2^{\ 2} = r_1^{\ 2} + r_2^{\ 2} = \text{const}$$

Locus of L is a circle centered at the midpoint of O_1O_2 and coaxal with $(O_1), (O_2)$.

DangChienbn

#3 Aug 14, 2009, 2:52 pm

> *xxxxtt wrote:*
> We have two secant circles whics intersect in points A and B.
> A variable line passing through the point A cuts the circles in points M and N.
> Find the geometric locus of the midpoint of the segment MN!

Oh, we set AO_1 intersects (O_1) at P and AO_2 intersects (O_2) at Q
Denote H is the midpoint of PQ and the geometric locus of midpoint of the segment MN is the circumcircle of triagle ABH

Luis González

#4 Aug 14, 2009, 3:26 pm

> *DangChienbn wrote:*
> Oh, we set AO_1 intersects (O_1) at P and AO_2 intersects (O_2) at Q. Denote H is the midpoint of PQ and the geometric locus of midpoint of the segment MN is the circumcircle of triagle ABH

Sorry dear DangChienbn, you haven't offered any proof.

livetolove212

#5 Aug 14, 2009, 3:46 pm

> *luisgeometria wrote:*
> Sorry dear **DangChienbn** you haven't offered any proof

It's easy to see that IH is the midline parallel of trapezoid $PMNQ$ and note that $\angle PMA = 90^o$ then I lies on (AH)

Quick Reply

High School Olympiads

Orthocenter and concurrent lines

Reply

Source: Nice

KDS

#1 Aug 14, 2009, 10:02 am

Let ABC be an acute triangle and the points A_1, B_1, C_1 are the feets of the altitudes from A, B, C respectively.
A circle passes through A_1, B_1 and touches the smaller arc AB of the circumcenter of $\triangle ABC$ in point C_2.
Points A_2, B_2 are defined analogously.
Prove that A_1A_2, B_1B_2, C_1C_2 have a common point,which lies on the Euler line of $\triangle ABC$

Luis González

#2 Aug 14, 2009, 3:12 pm

Let $\mathcal{T}_c$ be tangent of the circumcircle (O) at C_2. $\mathcal{T}_c$ is radical axis of $(K_c) \equiv \odot(A_1B_1C_2)$ and (O), A_1B_1 is radical axis of (K_c) and $\odot(A_1B_1AB)$, AB is radical axis of (O) and $\odot(A_1B_1AB)$. Therefore $\mathcal{T}_c, AB$ and A_1B_1 concur at the radical center P_c of $(O), (K_c), \odot(A_1B_1AB)$. Since cross ratio (P_c, C_1, A, B) is harmonic, it follows that C_2C_1 is the polar of P_c with respect to (O). Mutatis mutandis, B_2B_1 and A_2A_1 are the polars of P_b and P_a. Since P_a, P_b, P_c are collinear on the orthic axis τ of $\triangle ABC$, then A_1A_2, B_1B_2, C_1C_2 concur at the pole of τ WRT (O), which lies on the perpendicular to τ passing through O, i.e. the Euler line of $\triangle ABC$ and the proof is completed.

Quick Reply

High School Olympiads

Perpendicular segments

Reply

Source: solution without polars, if possible

stergiu

#1 Aug 14, 2009, 12:01 am

Let I be the incenter of triangle ABC. Let K, L, M be the points of tangency of the incircle of ABC with BC , CA, AB, respectively. The line which passes through B and is parallel to MK meets line KL at the point S.Line IB meets line MK at R. Prove that IS is perpendicular to RC.

Babis

Luis González

#2 Aug 14, 2009, 5:03 am

Perphaps any other proof without using polars would be equivalent.

Let r denote the parallel line to MK from $B \Longrightarrow R$ is the pole of r WRT (I). Polar of S WRT (I), which passes through the intersection C of the tangents of (I) at K, L, goes through the pole R of $r \Longrightarrow RC$ is therefore the polar of S WRT (I) and consequently $RC \perp IS$.

Quick Reply

High School Math

triangle

Reply

kurt.math

#1 Aug 13, 2009, 10:27 am

In triangle ABC, $\angle ABC = 138^\circ$, and $\angle ACB = 24^\circ$. Point D is on AC so that $\angle BDC = 60^\circ$ and point E is on AB so that $\angle ADE = 60^\circ$. If $\angle DEC = x^\circ$, find x

Luis González

#2 Aug 13, 2009, 9:16 pm

By angle chase we obtain $\angle DBC = 96^\circ, \angle ABD = 42^\circ$. In $\triangle CBD,$ the external bisectors of $\angle CBD$ and $\angle BDC$ form acute angles of 42° and 60° with the sideline $BD \Longrightarrow BE$ and DE are identical with the external bisectors of $\angle CBD$ and $\angle BDC \Longrightarrow E$ is the C-excenter of $\triangle CBD \Longrightarrow CE$ is the internal bisector of $\angle BCA \Longrightarrow \angle ECA = 12^\circ \Longrightarrow$ $\angle DEC = 180^\circ - 120^\circ - 12^\circ = 48^\circ$.

Quick Reply

Spain

Problema hermoso

Reply

Dave Vather

#1 Jul 26, 2006, 5:06 am

Les paso este problema, trae 2 dificultades, pero en escencia es el mismo. No vean la versión fácil a menos que no les salga la versión difícil.
Versión difícil
Sea ABC un triángulo y P un punto en su interior. Se construye el punto M fuera del triángulo de manera que $<ABC=<CBM$ y que $<ACP=<BCM$ Supóngase que el área del triángulo ABC es la misma que del cuadrilátero PCMB. Demostrar que BMC es semejante a ACP

Versión Fácil:
Click to reveal hidden text

Luis González

#2 Aug 12, 2009, 11:46 pm

Como $\angle ABC=\angle CBM,$ la simetría axial respecto a BC lleva M a un punto Q sobre AB tal que CQ es isogonal de CP respecto a $\angle C,$ ya que $\angle ACP=\angle BCM=\angle QCB.$ Entonces $[PCMB]=[ABC]$ implica:

$$[PBC]+[BQC]=[APB]+[APC]+[PBC]$$

$$[BQC]=[APB]+[APC]=[ABC]-[PBC]=[BQC]+[AQC]-[BPC]$$

$$[AQC]=[BPC] \Longrightarrow AC\cdot QC\sin\angle ACQ=PC\cdot BC\sin\angle PCB$$

Pero como CP y CQ son rayos isogonales con respecto al ángulo $C,$ entonces $\angle ACQ=\angle PCB.$ Así

$$\frac{QC}{PC}=\frac{BC}{AC} \Longrightarrow \ \triangle CMB\cong\triangle CQB\sim\triangle CPA.$$

mathteam

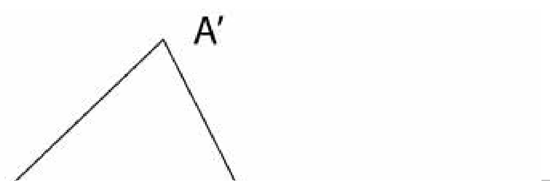
#3 Dec 16, 2009, 6:37 am

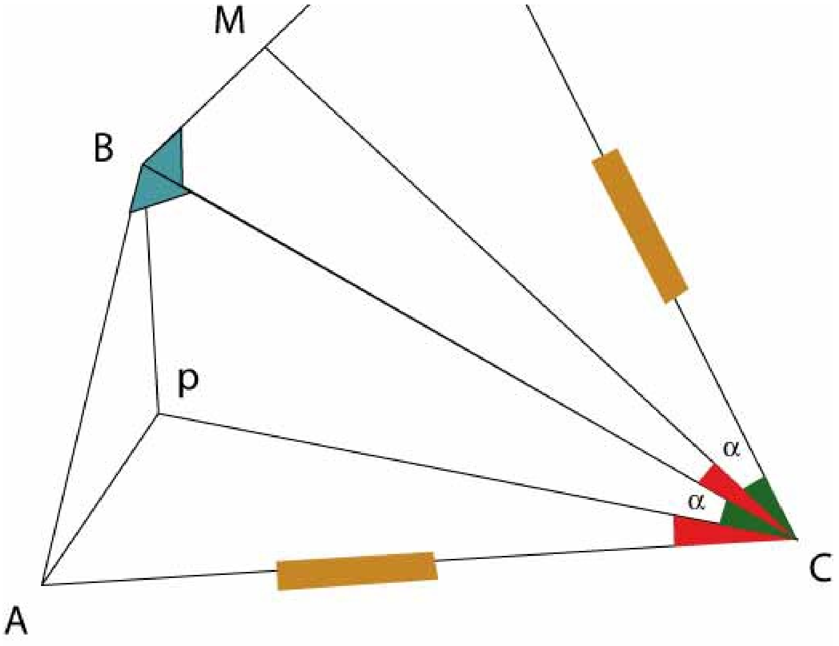
Mi solución es muy parecida sólo que uso la idea clave hacia afuera como se podrá observar en la figura adjunta:
Dado que $\Delta ABC\cong\Delta A'BC$, lo que hace que $[ABC]=[A'BC]$ luego $[BPCM]=[A'BC]$ por lo que $[BCP]=[MA'C]$ luego como ambos triángulos poseen un ángulo igual se usa la propiedad de áreas
$\frac{[BCP]}{[MA'C]}=\frac{PCxBC}{A'CxMC}=(\frac{PC}{AC})x(\frac{BC}{MC})=1 \Longrightarrow \ \Delta PCA\sim\Delta MCB$

Nota:
La propiedad mencionada puede ser usada para áreas de triángulos con ángulos iguales o suplementarios, más aún este teorema se puede generalizar para volumenes de piramides cuyo ángulo solido sea el mismo.

Centro Juvenil de Entrenamiento Olímpico
http://www.cjeo-peru.com

Attachments:





This post has been edited 1 time. Last edited by mathteam, Dec 16, 2009, 11:22 am

mathteam

#4 Dec 16, 2009, 11:20 am

Enunciado:

Sea ABC un triángulo y P un punto en su interior. La paralela a BC por P intersecta a AB en E y al circuncírculo de APC en D. Supóngase que el área del triángulo ABC es la misma que la suma de las áreas de AED y PBC. Demostrar que EAD es semejante a ACP

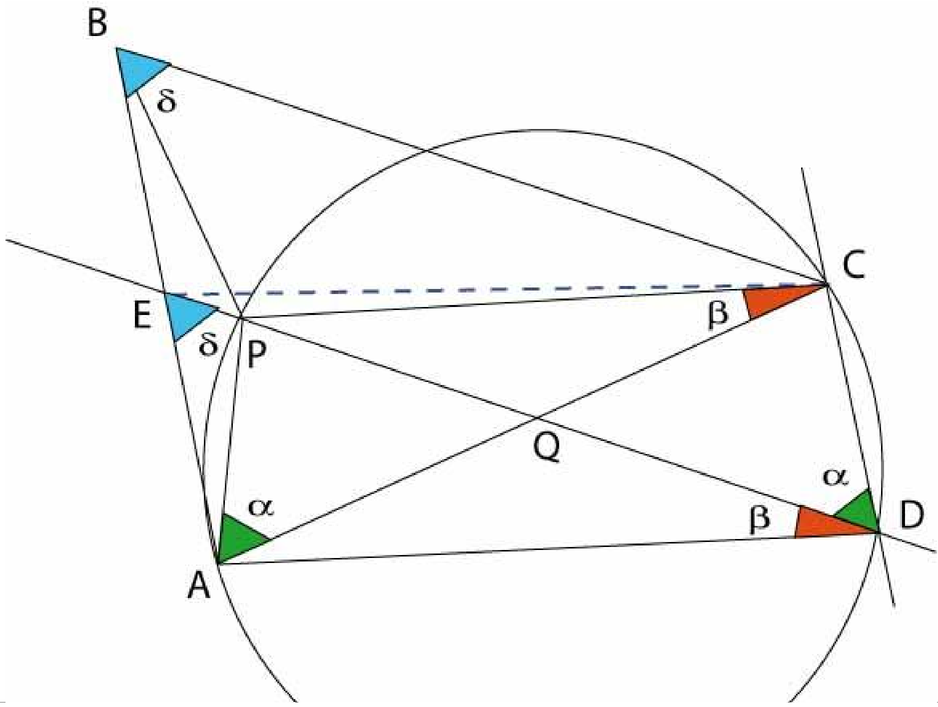
Solución

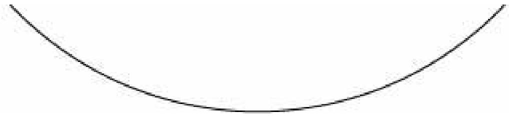
Como ED//BC entonces $[BEC]=[BPC]$ y siendo que $[ABC]=[AED]+[PBC]$
Tenemos que $[BEC]+[AEQ]+[EQC]=[AEQ]+[QAD]+[BEC]$
por lo que: $[EQC]=[AQD]$
Así de esta forma EA//CD por tanto $\angle AED=\angle EDC$ y puesto que APCD es conciclico concluimos que $\Delta EAD \sim \Delta APC$

Nota:
A propio criterio personal me ha gustado más este ejercicio que el de la versión dificil, más aún me parece muy educativo.

Centro Juvenil de Entrenamiento Olímpico
http://www.cjeo-peru.com

Attachments:





Quick Reply

Spain

Reto #5 del 2006

JOTAJOTARIM
#1 Mar 4, 2006, 6:40 am

NIVEL BASICO

Problema 1 [Geometria]
P es un punto dentro de una esfera. Tres rayos mutuamente perpendiculares intersecan a la esfera en los puntos U, V y W. Q denota el vértice diagonalmente opuesto a P en el paralelepípedo determinado por PU, PV, PW. Encontrar los lugares de Q para todos los posibles conjuntos de tales rayos desde P. (IMO 1987)

Problema 2 [Teoría de Números]
Dados $a, b, cin\mathbb{Z}^+$. Mostrar que:
$9|a^2+b^2+c^2 \Rightarrow 9|a^2-b^2$ o $9|b^2-c^2$ o $9|a^2-c^2$.

Problema 3 [Inecuaciones]
Si $a, b, c, d > 0$, entonces al menos una de las siguientes desigualdades es incorrecta:

$$a+b<c+d, \qquad (a+b)(c+d)<ab+cd, \qquad (a+b)cd<ab(c+d)$$

Los Organizadores

....
Versión PDF y LaTeX del Reto #5 del 2006

Attachments:
Reto5.tex (3kb)
Reto5.pdf (46kb)

carlosbr
#2 Mar 5, 2006, 7:40 am

Por cuestiones de tiempo ... delegue la tarea de la publicacion del reto #5 del 2006 a JOTAJOTARIM ...
(Juan Neyra)

Ahora tienen las preguntas listas en version pdf y tex, para una mejor descarga ..
esperamos sus soluciones ..

Carlos Bravo
Lima - Peru

Pascual2005
#3 Mar 5, 2006, 9:45 pm

1.

El comite de seleccion de problemas de una olimpiada matematica prepara algunos formatos de la olimpiada. Cada formato tiene 4 problemas, escogidos de una lista de n problemas, y dos formatos distintos tienen maximo un problema en comun.

a) Si $n = 14$, determine el maximo numero de formatos que se pueden hacer.

b) Halle el minimo n tal que es posible hacer 10 variantes.

2.

Sean I_A e I_B los excentros de ABC opuestos a A y a B respectivamente. Sea P un punto en el circuncirculo de ABC. Pruebe que el punto medio del segmento cuyos extremos son los circuncentros de los triangulos I_ACP y I_BCP coincide con el circuncentro de ABC.

3.

Sean $x_1, x_2..., x_n$ reales positivos y t un real positivo tal que se tiene que:

$$\frac{1}{t+x_1} + \frac{1}{t+x_2} + ...\frac{1}{t+x_n} = \frac{1}{t}$$

Entonces se cumple que:

$$\sqrt[n]{x_1 x_2 x_3 ... x_n} \geq (n-1)t$$

jasamper88

#4 Mar 22, 2006, 4:13 am

Que son las variantes en el problema 1??

M4RI0

#5 Mar 23, 2006, 4:04 am

> *jasamper88 wrote:*
>
> Que son las variantes en el problema 1??

Son los problemas que se escogen de la lista.

tipe

#6 Mar 23, 2006, 12:37 pm

Son los formatos de 4 problemas cada uno.
$Tipe$

Luis González

#7 Jul 25, 2009, 10:27 pm

> *JOTAJOTARIM wrote:*
>
> **Problema 1:** P es un punto en una superficie esférica. Tres rayos mutuamente perpendiculares intersecan a la esfera en los puntos U, V y $W. Q$ denota el vértice diagonalmente opuesto a P en el paralelepípedo determinado por $PU, PV, PW.$ Encontrar los lugares de Q para todos los posibles conjuntos de tales rayos desde $P.$ (IMO 1987)

Sea O el centro de la esfera y M, N, L las proyecciones de O en las cuerdas $PU, PV, PW.$ Es claro que M, N, L son puntos medios de $PU, PV, PW.$ Aplicando teorema de Pitagoras en los triángulos rectángulos OMP, ONP y OLP con hipotenusa común $OP = r$ tenemos:

$\frac{1}{4}PU^2 = r^2 - OM^2$, $\frac{1}{4}PV^2 = r^2 - ON^2$, $\frac{1}{4}PW^2 = r^2 - OL^2$

$PU^2 + PV^2 + PW^2 = PQ^2 = 12r^2 - 4(OM^2 + ON^2 + OL^2) = 4r^2$

El lugar geométrico de Q es pues la superficie esférica de centro P y radio $2r$.

Luis González

#8 Aug 12, 2009, 7:11 am

> *Pascual2005 wrote:*
>
> Sean I_A e I_B los excentros de ABC opuestos a A y a B, respectivamente. Sea P un punto en el circuncirculo de ABC. Pruebe que el punto medio del segmento cuyos extremos son los circuncentros de los triangulos I_ACP y I_BCP coincide con el circuncentro de ABC.

Sean (O_a), (O_b) los circuncírculos de $\triangle I_aCP, \triangle I_bCP$ y I_c el tercer excentro de $\triangle ABC$. Como $\triangle ABC$ es el triángulo órtico de su triangulo excentral $\triangle I_aI_bI_c$, entonces el circuncírculo (O) de $\triangle ABC$ es el círculo de Feuerbach de $\triangle I_aI_bI_c$ $\Longrightarrow$ (O) corta a I_bI_c en C y su punto medio C'. C' tiene igual potencia respecto a (O_a) y (O_b) ya que $CC' \cdot C'I_b = CC' \cdot C'I_a$, por ende C' ha de estar en una circunferencia coaxial con (O_a) y (O_b) cuyo centro es el punto medio de O_aO_b. Ésta circunferencia evidentemente es identica a (O) por pasar por C, C' y P.

Quick Reply

High School Olympiads

Fixed point

Reply

ll931110

#1 Aug 11, 2009, 9:39 am

Given a circle (O) and a line d which has not common points with (O). Let M be a point on d. From M, we draw tangents MA and MB of (O). Let H be the projection of O to d. Let P, Q be the projections of H to MA, MB.

Show that: PQ passes through a fixed point while M runs on d
(Sorry for my bad English)

shoki

#2 Aug 11, 2009, 12:15 pm

if PQ cuts OH at S then S is the fix point and we have
OS= ((OH)/(2)) + ((R^2)/(2*OH))
R is the radius of (O).

Luis González

#3 Aug 11, 2009, 1:36 pm

Since $\angle MHO$ is right, then H lies on the circumcircle of $\triangle MAB \Longrightarrow$ projection R of H onto AB is collinear with P, Q on the Simson line with pole H WRT $\triangle MAB.$ AB goes through the fixed pole D of d WRT $(O),$ thus R moves on the circle with diameter $DH.$ Let $V \equiv OH \cap PQ.$ From the cyclic quadrilaterals $AMHO$ and $AQHR,$ we get $\angle MOH = \angle MAH = \angle QRH,$ which implies that $\angle VDR = \angle VRD \Longrightarrow \triangle DVR$ is isosceles with apex $V \Longrightarrow V$ is midpoint of $DH.$

Quick Reply

High School Olympiads

THREE CIRCLEs

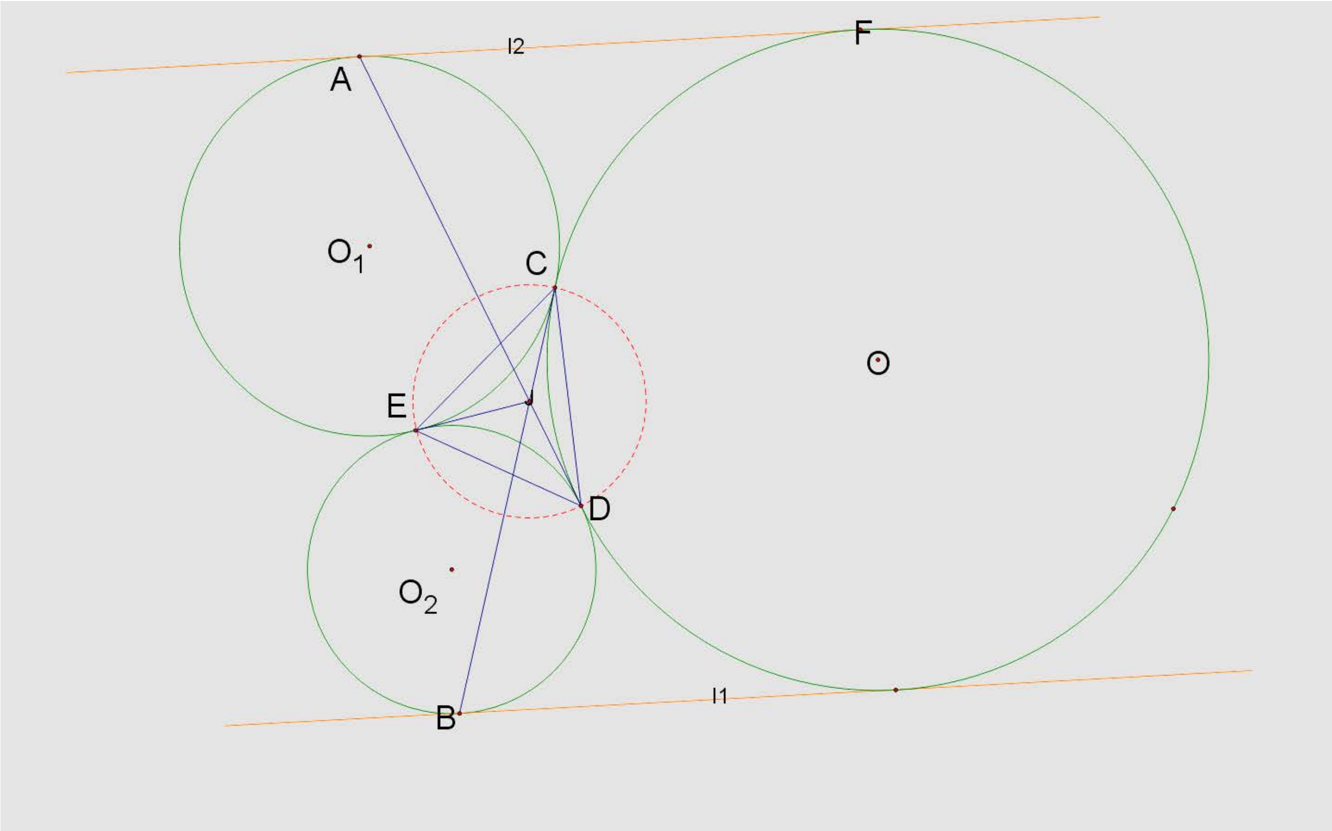
Source: nice(seems to be an IMO shortlist problem)

CCMath1

#1 Aug 11, 2009, 12:59 am

□O1 is tagent to l2 and □O2□□O□ □O2 is tangent to □O1□□Oand l1□□O is tangent to □O1□ □O2 and l1□l2□BC□DA meet at J□Prove that J is circumcentre of □ECD. :

Attachments:



Luis González

#2 Aug 11, 2009, 5:29 am

Inversion with center A and power equal to the power of A WRT (O) takes (O) into itself and (O_1) and l_1 into each other $\Longrightarrow$ Circle (O_2) is double, which means that AD is the radical axis of $(O),(O_2)$. By similar reasoning, considering the inversion with center B and power equal to the power of B WRT (O), we prove that CB is the radical axis of $(O),(O_1)$. Then J the is radical center of $(O),(O_1),(O_2) \Longrightarrow$ Tangents to (O) at C,D and the common internal tangent of $(O_1),(O_2)$ concur at J $\Longrightarrow J$ is equidistant from C,D,E.

High School Olympiads

Geometric inequality with h,r,R

Reply

Source: KÃ¼rschÃ¡k Competition 1978/3

Moonmathpi496
#1 Aug 10, 2009, 7:29 pm

A triangle has inradius r and circumradius R. Its longest altitude has length H. Show that if the triangle does not have an obtuse angle, then $H \geq r + R$. When does equality hold?

Luis González
#2 Aug 11, 2009, 12:16 am

Lemma. $\triangle ABC$ is acute with altitudes $h_a, h_b, h_c,$ such that $h_a \geq h_b \geq h_c$. Let P be a point on $\overline{BC}$ and X, Y the feet of the perpendiculars from P to AB, AC. Then

$$h_b \geq PX + PY \geq h_c$$

Proof. Let M and N be the projections of P onto the altitudes $h_b \equiv BH_b$ and $h_c \equiv CH_c$. Hence, $\angle XPB = 90^\circ - \angle B$ and $\angle H_bBC = 90^\circ - \angle C$. Since $\angle C \geq \angle B$, due to $AB \geq AC \Longrightarrow \angle XPB \geq \angle H_bBC$.

Let $\triangle PM'B$ be the reflection of $\triangle PMB$ about the perpendicular bisector of PB. M' lies on the semicircle with diameter PB and PM' is an internal ray of $\angle XPB$, due to $\angle XPB \geq \angle H_bBC \geq \angle M'PB$. Then chord PM' is greater than the chord PX, i.e. $PM' = MB \geq PX$. Since H_bMPY is a rectangle, we have then

$$MB + PY \geq PX + PY \ , \ PY = MH_b \Longrightarrow MB + MH_b = h_b \geq PX + PY.$$

Similarly, we get $PX + PY \geq h_c \Longrightarrow h_b \geq PX + PY \geq h_c$.

Let O be the circumcenter of $\triangle ABC$. WLOG assume $h_a \geq h_b \geq h_c$. Let the parallel to BC through O intersects AB, AC at B', C' and the altitude h_a at D. Let M_a, M_b, M_c be the projections of O on BC, AB, AC. Since $\triangle AB'C' \sim \triangle ABC$, we use the above lemma for $\triangle AB'C'$ with shortest altitude AD. Thus

$$AD \geq OM_a + OM_b \Longrightarrow AD + OM_a = h_a \geq OM_a + OM_b + OM_c.$$

By Carnot theorem for the acute triangle $\triangle ABC$, we get then

$h_a \geq OM_a + OM_b + OM_c = R + r$. Equality holds when $\triangle ABC$ is equilateral.

Rijul saini
#3 Jun 6, 2010, 9:40 pm

> Moonmathpi496 wrote:
> A triangle has inradius r and circumradius R. Its longest altitude has length H. Show that if the triangle does not have an obtuse angle, then $H \geq r + R$. When does equality hold?

It is equivalent to INMO 2008 Problem 5

Quick Reply

High School Olympiads

CGI is right iff GM || AB

Reply

Source: I.F.Sharygin 2009 - Final round - Problem 10.6

April

#1 Aug 3, 2009, 9:55 am

Let M, I be the centroid and the incenter of triangle ABC, A_1 and B_1 be the touching points of the incircle with sides BC and AC, G be the common point of lines AA_1 and BB_1. Prove that angle CGI is right if and only if $GM \parallel AB$.

Author: A.Zaslavsky

yetti

#2 Aug 10, 2009, 8:24 am

Let CG cut A_1B_1 at D and AB at C_1. G is Gergonne point $\Longrightarrow C_1$ is tangency point of the incircle (I) with AB. Let (I) cut CG again at J. From quadrilateral CA_1GB_1, the cross ratio $(C, G, D, C_1) = \frac{\overline{CC_1}}{\overline{CD}} \cdot \frac{\overline{GD}}{\overline{GC_1}} = -1$ is harmonic. A_1B_1 is polar of C WRT $(I) \Longrightarrow$ the cross ratio $(C, D, J, C_1) = \frac{\overline{CC_1}}{\overline{CJ}} \cdot \frac{\overline{DJ}}{\overline{DC_1}} = -1$ is also harmonic.

$\Longrightarrow$: Assume $GM \parallel AB$. From the harmonic cross ratio $(C, G, D, C_1) = -1$ and $\frac{\overline{GC}}{\overline{GC_1}} = -2 \Longrightarrow D$ is midpoint of CC_1 and $\frac{\overline{DC_1}}{\overline{GD}} = -2$. On the other hand, from $\frac{\overline{CJ}}{\overline{DJ}} = -\frac{\overline{CC_1}}{\overline{DC_1}} = -2 \Longrightarrow D$ is also midpoint of GJ. It follows that G is midpoint of JC_1, which means that the angle $\angle CGI = \angle JGI$ is right.

$\Longleftarrow$: Assume $\angle CGI$ is right $\Longrightarrow G$ is midpoint of JC_1. From the harmonic cross ratio $(C, D, J, C_1) = -1$, we then have $\frac{\overline{CC_1}}{\overline{DC_1}} = \frac{\overline{GJ}}{\overline{GD}} = -\frac{\overline{GC_1}}{\overline{GD}}$. Comparing this with the harmonic cross ratio $(C, G, D, C_1) = -1$, we get $\overline{DC_1} = \overline{CD}$, which means that D is midpoint of CC_1. From the harmonic cross ratio $(C, G, D, C_1) = -1$ then follows that $\frac{\overline{GC}}{\overline{GC_1}} = -2$ or $GM \parallel AB$.

Luis González

#3 Aug 10, 2009, 10:02 pm

Since A_1, B_1 lie on the circle with diameter $\overline{IC}$, the statement is equivalent to prove that Gergonne point G lies on the circle (J) with diameter $IC \Longleftrightarrow MG \parallel AB$. Let C_1 denote the tangency point of (I) with AB. Assume that G lies on (J). AB is the polar of intersection $P \equiv IC \cap A_1B_1$ WRT (J), thus $JP \perp AB \Longrightarrow JP \parallel IC_1$. Since J is midpoint of IC, then P is midpoint of CC_1. From the harmonic division $(C, G, P, C_1) = -1$, we get $CG = 2GC_1 \Longrightarrow MG \parallel AB$.

The opposite statement is immediatly true, this is: If $MG \parallel AB$, from the harmonic division $(C, G, P, C_1) = -1$, P is midpoint of $CC_1 \Longrightarrow JP \perp AB$. Since P is the pole of AB WRT (J), it follows that $G \in (J)$.

Virgil Nicula

#4 Aug 11, 2009, 6:56 am

Caution ! I"ll use another notations.

> Quote:
>
> Let ABC be a triangle with the centroid G and the incircle $C(\mathrm{I}, r)$ which touches given triangle in $D \in (BC)$, $E \in (CA)$

$F \in (AB)$. Denote the **Gergogne's point** $\Gamma \in AD \cap BE \cap CF$. Prove that $\boxed{G\Gamma \parallel BC \iff I\Gamma \perp AD}$.

Proof (metric). Denote $\left\|\begin{array}{c} 2p = a+b+c \\ p-a = x > 0 \\ p-b = y > 0 \\ p-c = z > 0 \end{array}\right\|$. From $IA^2 - ID^2 = IA^2 - IE^2 = AE^2 = (p-a)^2$ obtain $\boxed{IA^2 - ID^2 = x^2}$ (1).

Aubel's relation : $\frac{\Gamma A}{\Gamma D} = \frac{FA}{FB} + \frac{EA}{EC} = \frac{p-a}{p-b} + \frac{p-a}{p-c} = \frac{x}{y} + \frac{x}{z} = \frac{x(y+z)}{yz}$, i.e. $\boxed{\frac{\Gamma A}{x(y+z)} = \frac{\Gamma D}{yz} = \frac{AD}{x(y+z)+yz}}$ (2).

Stewart's relation : $AD^2 \cdot (y+z) + yz(y+z) = z(x+y)^2 + y(x+z)^2$, i.e $\boxed{AD^2(y+z) = x^2(y+z) + 4xyz}$ (3).

Therefore, $\Gamma A^2 - \Gamma D^2 \overset{(2)}{=} \frac{x^2(y+z)^2 - y^2z^2}{[x(y+z)+yz]^2} \cdot AD^2 \overset{(3)}{\iff}$ $\boxed{\Gamma A^2 - \Gamma D^2 = \frac{x(y+z)-yz}{x(y+z)+yz} \cdot \frac{x^2(y+z)+4xyz}{y+z}}$ (4).

▶ $G\Gamma \parallel BC \iff \Gamma A = 2 \cdot \Gamma D \overset{(2)}{\iff} x(y+z) = 2yz$, i.e. $\boxed{G\Gamma \parallel BC \iff x(y+z) = 2yz}$ (5).

▶ $\Gamma I \perp AD \iff \Gamma A^2 - \Gamma D^2 = IA^2 - ID^2 \overset{(1)\wedge(4)}{=} \frac{x(y+z)-yz}{x(y+z)+yz} \cdot \frac{x^2(y+z)+4xyz}{y+z} = x^2 \iff$

$\frac{x(y+z)-yz}{x(y+z)+yz} = \frac{x^2(y+z)}{x^2(y+z)+4xyz} \iff \frac{x(y+z)}{yz} = \frac{2x^2(y+z)+4xyz}{4xyz}$, i.e. $\boxed{\Gamma I \perp AD \iff x(y+z) = 2yz}$ (6).

From $(5) \wedge (6)$ obtain the chain of equivalencies $\boxed{G\Gamma \parallel BC \iff a(p-a) = 2(p-b)(p-c) \iff I\Gamma \perp AD}$.

Remark. Exists a triangle ABC for which $G\Gamma \parallel BC$, i.e. $a(p-a) = 2(p-b)(p-c)$?! The answer is **YES**. Indeed, show easily

that $a(p-a) = 2(p-b)(p-c) \iff f(\underline{a}) \equiv 2\underline{a}^2 - (b+c)\underline{a} - (b-c)^2 = 0$. Observe that $a > 0$ and $\frac{b+c}{4} \le a$. Suppose

w.l.o.g. $b < c$. Thus $\begin{array}{c} f(b) = c(b-c) < 0 \\ f(c) = b(c-b) > 0 \\ f(c-b) = -2b(c-b) < 0 \end{array} \implies \max\{c-b, b\} < a < c$ and $A < 90^\circ$, i.e. $\boxed{b < a < c < a+b}$.

If $b = c$, then $f(a) = 0 \iff 2a^2 - 2ab = 0$, i.e. $a = b = c$ and in this case $G \equiv I \equiv \Gamma$.

Quick Reply

High School Olympiads

A configuration with several properties

Reply

Luis González

#1 Aug 10, 2009, 4:59 am

$\triangle ABC$ is an equilateral triangle, M is the midpoint of BC and let P be a point on the circumference with center M and radius $MA.$ Show the following propositions:

• $B, C,$ the centroid and orthocenter of $\triangle PBC$ are concyclic on a circle $\mathcal{K}.$

• Envelope of the Euler line of $\triangle PBC$ while P varies on (M, MA) is a circle.

yetti

#2 Aug 10, 2009, 1:46 pm

Let O, H, G be circumcenter, orthocenter and centroid of $\triangle PBC.$ Let PG cut (O) again at $Q.$
$\frac{3}{4}BC^2 = MA^2 = MP^2 = \frac{1}{4}(2CP^2 + 2PB^2 - BC^2) \Longrightarrow CP^2 + PB^2 = 2BC^2.$
By Steiner theorem, moment of inertia of P, B, C WRT O is
$3R^2 = OP^2 + OB^2 + OC^2 = GP^2 + GB^2 + GC^2 + 3GO^2 =$
$\frac{1}{3}(BC^2 + CP^2 + PB^2) + 3GO^2 = BC^2 + 3GO^2.$
Power of G to (O) is then $\overline{GP} \cdot \overline{GQ} = R^2 - GO^2 = -\frac{1}{3}BC^2 = -GP^2 \Longrightarrow$
$\overline{GQ} = -\overline{GP}, G$ is midpoint of $PQ \Longrightarrow OG \perp MG,$ where $MG = \frac{1}{3}MP = \frac{1}{3}MA = \text{const} \Longrightarrow OG$ is tangent to circle with center M and radius $MG.$
G is midpoint of $PQ \Longrightarrow M$ is midpoint of $GQ \Longrightarrow B, C, H, G$ are all on reflection of (O) in $BC.$

Luis González

#3 Aug 10, 2009, 7:52 pm

Thanks for your solution Vladimir, your proof is quite shorter than mine. I proved here an equivalent statement, which is: If B, C, H, G are concyclic, then locus of P is the circumference with center M and radius $MA.$ The perpendicularity of the Euler line of $\triangle PBC$ and PG also follows easily from the latter result.

Quick Reply

Spain

???

Reply

mhuarancca
#1 Sep 4, 2006, 7:20 am

bueno un problema en el cual quiero ver varias soluciones

Dos cirfunferencias secante C_1 y C_2 de radis r_1 y r_2 se cortan en los puntos A y B por B se traza una recta variable que corta de nuevo a C_1 y C_2 en dos puntos que llamarenos P_r y Q_r respectivamente.
Demuestre la siguiente propiedad : Existe un punto M (sugerencia : que depende solo de C_1 y C_2) , tal que la mediatriz del segmento P_rQ_r pasa por M.

conejita
#2 Sep 4, 2006, 7:18 pm

Oye, yo tengouna solucion de ese ejercicio, fue una tarea que nos marcaron aki en Mexico. No tengo tiempo aora para poner toda la solucion. Apenas tenga tiempo yo lo pongo!!

Luis González
#3 Aug 10, 2009, 12:33 am

Proposición: Sean C_1 y C_2 dos circunferencias secantes en A, B cuyos centros son O_1, O_2. Una recta variable que pasa por A corta a C_1 y C_2 respectivamente en P, Q. Demostrar la mediatriz de PQ pasa por un punto fijo M.

Sea N el punto medio de PQ. Las potencias de N respecto a C_1 y C_2 son respectivamente $NA \cdot NP = \frac{1}{2}AN \cdot PQ$ y $NA \cdot NQ = \frac{1}{2}AN \cdot PQ$. Es decir que N tiene igual potencia respecto a ambas. En otras palabras el lugar geométrico de M es una circunferencia $\mathcal{K}$ coaxial con C_1 y C_2 cuyo centro es el punto medio de O_1O_2. La perpendicular a PQ por N (mediatriz de PQ) cortará pues a $\mathcal{K}$ en el antipodal M de A en ella, obviamente fijo.

Quick Reply

High School Olympiads

Parabolas

Reply

happyme
#1 Aug 7, 2009, 9:15 am

Let F and V denote the focus and vertex, respectively, of the parabola $x^2 = 4py$. If PQ is a focal chord of the parabola, show that:

$$PF \cdot FQ = VF \cdot PQ$$

Luis González
#2 Aug 9, 2009, 11:35 pm

Let T be the intersection of the focal axis FV with the directrix τ and let M, N be the projections of P, Q onto τ. In the right trapezoid $PQNM$, we have:

$$FT = \frac{PM \cdot FQ + QN \cdot FP}{PQ}.$$

But since $FT = 2 \cdot VF$ and $PM = FP,\ QN = FQ$, it follows that

$$2 \cdot VF = \frac{PF \cdot FQ + PF \cdot FQ}{PQ} \Longrightarrow PF \cdot FQ = VF \cdot PQ.$$

mathwizarddude
#3 Aug 10, 2009, 4:10 am

Since the parabola is of the form $x^2 = 4py$, the focus F is at $(0, p)$, vertex V $(0, 0)$, directrix eqn is $y = -p$. Then P, Q should be $(-2p, p)$, $(2p, p)$ since PQ is the focal chord (parallel to the directrix and passing through the focus), and $PF \cdot FQ = 2p \cdot 2p = 4p^2$ while $VF \cdot PQ = p \cdot 4p = 4p^2$...

mathwizarddude
#4 Aug 11, 2009, 11:58 pm

Oh I was informed that my proof is wrong, what's the definition of focal chord then? (I researched that before and it returned the previous understanding, but maybe I misread).

Quick Reply

High School Olympiads

Fixed angle, fixed point

Source: Turkey NMO 2000 - Problem 5

xeroxia

#1 Aug 6, 2009, 4:39 am

The ray $[OA$, the ray $[OB$, and a positive number k are given. Let P be a point on $[OA$ and Q be a point on $[OB$ such that $OP + OQ = k$. Prove that the circumcircles of $\triangle OPQ$ pass through a fixed point.

Luis González

#2 Aug 9, 2009, 1:50 am

Let M,N be two points on the rays OA, OB such that $OM = ON = \frac{1}{2}k.$ Then

$\frac{1}{2}k - MP + \frac{1}{2}k + NQ = k \implies MP = NQ.$

Thus, circles $\odot(OMN)$ and $\odot(OPQ)$ meet at the center K of the rotation that takes the oriented segments MP and NQ into each other. But K is the antipode of O WRT the circle $\odot(OMN)$ because of $KM = KN.$ Then K is fixed.

xeroxia

#3 Aug 15, 2009, 6:15 am

Let $OM' = ON' = k/2$ and $OM'' + ON'' = k$. Let P' be the intersection of $(OM'N')$ and $(OM''N'')$. $\angle M''P'M' = N''P'N'$ and $\angle OM''P' + \angle ON''P' = 180^\circ$. Also we know $M'M'' = NN''$. Use sinus theorem, then $M'P' = P'N'$. $OM'N'$ is fixed, so its angle bisector and its circumcircle, so their intersection point is fixed. So all (OMN)s pass through a point on angle bisector of $\angle MON$.

xeroxia

#4 Aug 15, 2009, 6:26 am

We will use ptolemy. Let P be the intersection point on angle bisector of $\angle MON$ and (OMN). Use ptolemy. $(OM + ON).MP = MN.OP$. Use sine law at $\triangle MPN$, then $\frac{MN}{sin(180^\circ - \alpha)} = \frac{MP}{sin(\alpha/2)}$. Combine two equations. $OP = \frac{k}{2cos(\alpha/2)}$ which is fixed. The line OP is fixed, $|OP|$ is fixed, so P is fixed. All (OMN)s pass through the fixed point P.

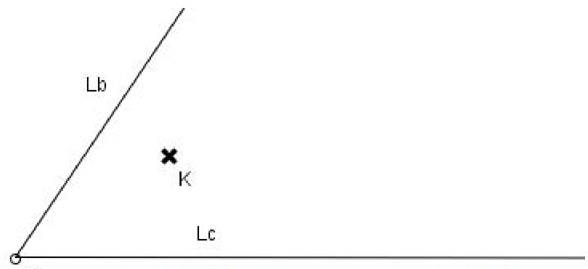
High School Olympiads

Ib Ic K

Reply

jrrbc
#1 Aug 8, 2009, 7:45 pm

Attachments:



Given the sidelines **Lb** and **Lc** and the *Lemoine point* **K**, find the vertices of the triangle ABC [*Lemoine point* is the point of concurrence of the symmedians (reflections of medians in corresponding angle bisectors)]

Luis González
#2 Aug 8, 2009, 9:39 pm

A-symmedian AK is the locus of points such that their ratio of distances to $\mathcal{L}_b$ and $\mathcal{L}_c$ equals $\frac{b}{c}$. Hence, projecting the symmedian point on $\mathcal{L}_b, \mathcal{L}_c$, or any point on AK, pertims to find such ratio. The construction of $\triangle AB'C'$ homothetic to $\triangle ABC$ with B', C' on the rays $\mathcal{L}_b$ and $\mathcal{L}_c$ follows easily. Symmedian point K' of $\triangle AB'C'$ is then constructible and since K, K' are homologous, then draw parallels through K to $K'B'$ and $K'C'$ intersecting the sidelines $\mathcal{L}_b$ and $\mathcal{L}_c$ at the wanted vertices B, C.

jrrbc
#3 Aug 9, 2009, 1:44 am

i constructed the median by A, then with locus the side B'C' with median point on median.
So i have the triangle homothetic AB'C', then i constructed the k' of this triangle and then the triangle ABC

Quick Reply

High School Olympiads

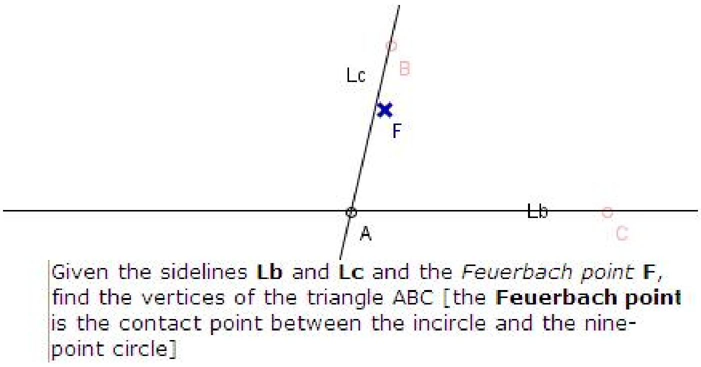
Ib Ic F

Reply

jrrbc

#1 Aug 8, 2009, 3:18 am

Attachments:



Luis González

#2 Aug 8, 2009, 10:57 am

Draw the incircle (I) passing though F and tangent to the sidelines $\mathcal{L}_b$ and $\mathcal{L}_c$. This is the Apollonius problem (line, line, point). However, it is equivalent to the easier case (point, point, line) by reflecting F on the internal bisector of A. This yields two solutions but consider the one according to the sketch. Let Y, Z be tangency points of (I) with AC, AB. It is well-known that the reflections of the diacentral line IO about the sidelines of the intouch triangle concur at its orthopole, i.e. the Feuerbach point F. Hence, construct the reflection F' of F on $ZY \Longrightarrow F'I$ is the diacentral line $d \equiv IO$. Since F is the orthopole of d, construct the projection A' of A on $d \Longrightarrow FA'$ is perpendicular to BC. Construct the tangent to the incircle (at the opposite arc of ZFY) perpendicular to FA' to complete the $\triangle ABC$.

jrrbc

#3 Aug 8, 2009, 7:38 pm

thanks

Quick Reply

High School Olympiads

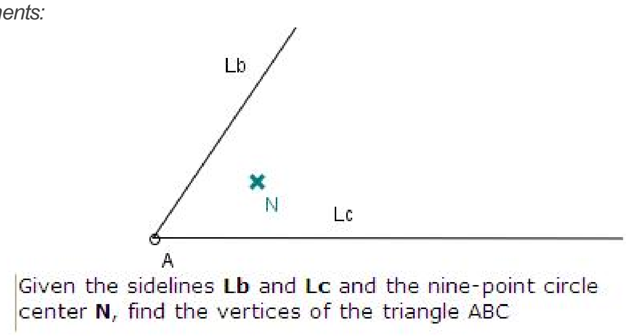
Ib Ic N

Reply

jrrbc

#1 Aug 8, 2009, 4:43 am

Attachments:



Luis González

#2 Aug 8, 2009, 6:50 am

Let M_a, M_b, M_c be the unknown midpoints on $BC, CA, AB.$ N is circumcenter of the medial triangle $\triangle M_aM_bM_c$ and the angle $\angle M_bNM_c = 2\angle A$ is known (assume that A is acute). Therefore, the isosceles $\triangle NM_cM_b$ is constructible. This is, perform the rotation with center N and magnitude $2\angle A$ (regardless of the orientation) and carry one of the sidelines, say $\mathcal{L}_b,$ in this movement and cut the other sideline $\mathcal{L}_c$ at $M_c.$ In the same way we get $M_b.$ Parallels to $\mathcal{L}_b$ and $\mathcal{L}_c$ from M_b, M_c meet at $M_a.$ The construction of $\triangle ABC$ follows straightforwardly from its medial triangle.

jrrbc

#3 Aug 8, 2009, 7:58 pm

Perform the rotation with center N and magnitude $2\angle A$

rotation of what ?

jrrbc

#4 Aug 9, 2009, 5:41 am

rotation the two sides of the angle

thanks

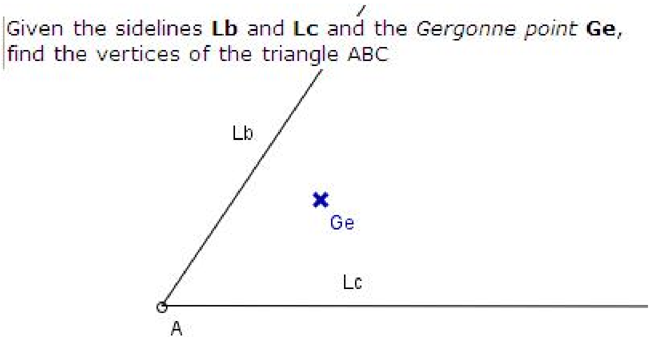
Quick Reply

High School Olympiads

Ib Ic Ge

jrrbc
#1 Aug 4, 2009, 4:46 am

Attachments:



Luis González
#2 Aug 8, 2009, 1:37 am

$\mathcal{L}_b, \mathcal{L}_c$ denote the given sidelines meeting at the known vertex $A.$ Let X, Y, Z be the unknown tangency points of (I) with $BC, CA, AB.$ $\mathcal{L}_b$, $\mathcal{L}_c$ and the Gergonne ray AG_e are double under any direct homothety with center $A.$ Thus, draw an arbitrary incircle (I') tangent to $\mathcal{L}_b, \mathcal{L}_c,$ the ray AG_e cuts (I') at the homologous X' of X under such homothety with center A that takes $(I) \mapsto (I').$ Tangent of (I') at X' is then parallel to BC and cuts $\mathcal{L}_b, \mathcal{L}_c$ at the homologous B', C' of $B, C.$ Gergonne point G'_e of $\triangle AB'C'$ is constructible. Since this homothety maps G_e into G'_e, then draw parallels trough G_e to $B'G'_e$ and $C'G'_e$. These meet $\mathcal{L}_b$ and $\mathcal{L}_c$ at Y, Z and $B \equiv \mathcal{L}_c \cap YG_e, C \equiv \mathcal{L}_b \cap ZG_e.$

jrrbc
#3 Aug 8, 2009, 2:29 am

thanks

High School Olympiads

a problem of geometry

Source: help

duythuc_lqd
#1 Aug 7, 2009, 2:50 pm

Give an acute-angled triangle ABC. Bisector in triangle is AD. $(O_1),(O_2)$ is circumcircle of triangle ABD and ADC. Two tangent of (O_1) and (O_2) intersect AD at P,Q. Prove that:$PQ^2 = AB.AC$

Luis González
#2 Aug 7, 2009, 6:12 pm

Assume that $PQ^2 = AB \cdot AC$ and let r_1 and r_2 the circumradii of $\triangle ABD, \triangle ACD$

$$BD = 2r_1 \cdot \sin \frac{A}{2} \ \ (1) \ \ , \ \ CD = 2r_2 \cdot \sin \frac{A}{2} \ \ (2)$$

$$AD^2 = AB \cdot AC - BD \cdot CD \Longrightarrow \ PQ^2 = AD^2 + BD \cdot CD$$

$$\Longrightarrow (AD + 2PA)^2 = AD^2 + BD \cdot CD \Longrightarrow 4PA^2 + 4AD \cdot PA = BD \cdot CD$$

$$\Longrightarrow 4PA^2 + 4(PD - PA) \cdot PA = BD \cdot CD \Longrightarrow 4PD \cdot PA = BD \cdot CD$$

But $PD \cdot PA$ is the power of P to both circles, hence $XY^2 = BD \cdot CD,$ where $\overline{XY}$ is their common tangent segment. Now we express XY in terms of radii r_1, r_2 by using the formulas (1) and (2)

$$(O_1O_2)^2 - (r_1 - r_2)^2 = 4r_1r_2 \cdot \sin^2 \frac{A}{2}$$

But note that $\angle O_1DO_2 = \angle BAC.$ Hence, by cosine law in $\triangle DO_1O_2,$ we obtain

$$r_1{}^2 + r_2{}^2 - 2r_1r_2 \cdot \cos A - r_1{}^2 - r_2{}^2 + 2r_1r_2 = 4r_1r_2 \cdot \sin^2 \frac{A}{2}$$

Therefore, $2\sin^2 \frac{A}{2} = 1 - \cos A,$ which is true and the proof is completed.

High School Olympiads

Bisectors and incenter

Reply

binaj

#1 Aug 6, 2009, 8:26 pm

ABC is a triangle which is inscribed in circle O. Bisector of angle ABC meet AC at F and O at G .
Bisector of BAC meet BC at D and O at E. Let P be a common point of lines GE and FD. Prove that AB || PI (I is a incenter of triangle ABC)

Luis González

#2 Aug 7, 2009, 12:30 am

Since I is orthocenter of its circuncevian triangle, it follows that DE is the perpendicular bisector of AI. The tangent $\mathcal{T}_a$ to (O) through A is then the reflection of the parallel $\mathcal{L}_a$ to BC passing through I about DE. Hence, the problem is equivalent to show that $V_bV_c, \mathcal{T}_a$ and $\mathcal{L}_a$ concur. Using barycentric coordinates, we have

$V_bV_c \equiv bcx-acy-abz=0 \ , \ \mathcal{T}_a \equiv c^2y+b^2z=0.$

$\mathcal{L}_a$ passes through $I(a:b:c)$ and the infinite point of BC $(0:1:-1)$

$\Longrightarrow \mathcal{L}_a \equiv (b+c)x-ay-az=0$

Therefore, $V_bV_c, \mathcal{T}_a, \mathcal{L}_a$ concur at $P \ (a(c-b):-b^2:c^2)$.

livetolove212

#3 Aug 12, 2009, 7:29 pm

Let $EG \cap AB=\{L\}$
Appying **Menelaus's** theorem for triangle IEG with line PDF we have:
$\frac{PE}{PG}.\frac{FG}{FI}.\frac{DI}{DE}=1$
$\Rightarrow \frac{PE}{PG}=\frac{FI}{FG}.\frac{DE}{DI}=\frac{r}{AG.sin\angle\frac{B}{2}}.\frac{BE.sin\angle\frac{A}{2}}{r}=\frac{BE.sin\angle\frac{A}{2}}{AG.sin\angle\frac{B}{2}}$
$\Rightarrow 1+\frac{EG}{PE}=\frac{AG.sin\angle\frac{B}{2}}{BE.sin\angle\frac{A}{2}}$
On the other side, $\frac{LE}{LG}=\frac{NE}{MG}=\frac{AE.sin\angle\frac{A}{2}}{BG.sin\angle\frac{B}{2}}$
$\Rightarrow 1+\frac{EG}{LE}=\frac{BG.sin\angle\frac{B}{2}}{AE.sin\angle\frac{A}{2}}$
$\Rightarrow \frac{PE}{LE}=(\frac{BG.sin\angle\frac{B}{2}}{AE.sin\angle\frac{A}{2}}-1):(\frac{AG.sin\angle\frac{B}{2}}{BE.sin\angle\frac{A}{2}}-1)=\frac{BE}{AE}.\frac{BG.sin\angle\frac{B}{2}-AE.sin\angle\frac{A}{2}}{AG.sin\angle\frac{B}{2}-BE.sin\angle\frac{A}{2}}$
But $\frac{BE}{AE}=\frac{EI}{AE}$ so $IP//AB$ iff $AG.sin\angle\frac{B}{2}-BE.sin\angle\frac{A}{2}=BG.sin\angle\frac{B}{2}-AE.sin\angle\frac{A}{2}$
$\Leftrightarrow AG.sin\angle\frac{B}{2}+AE.sin\angle\frac{A}{2}=BG.sin\angle\frac{B}{2}+BE.sin\angle\frac{A}{2}$
$\Leftrightarrow IG.sin\angle\frac{B}{2}+(AI+IE).sin\angle\frac{A}{2}=(BI+IG).sin\angle\frac{B}{2}+EI.sin\angle\frac{A}{2}$
$\Leftrightarrow AI.sin\angle\frac{A}{2}=BI.sin\angle\frac{B}{2}=r$ (true!)
We are done!

Attachments:
picture17.pdf (10kb)

zool007

#4 Aug 19, 2009, 5:27 pm

is there a non-trigonometric proof?

limes123

#5 Mar 12, 2010, 3:48 am

Assume $AC < BC$ ($AC = BC$ is obvious). Apply Pascal Theorem to degenerate hexagon $ACCBGE$ to get that P is intersection of GE and tangent to O at C. We see that $PC = PI$ and $\angle PIC = \angle ICP = \angle B + \angle\frac{C}{2}$ and $\angle CIG = \angle\frac{B}{2} + \angle\frac{C}{2}$ hence $\angle GIP = \angle\frac{B}{2} = \angle GBA$.

jayme

#6 Mar 12, 2010, 11:47 am

Dear Mathlinkers,
1. let (O) the circumcircle of ABC, Y the second point of intersection of CI with (O), X the meet point of YE and BC, Tc the tangent to (O) at C.
2. According to a special case of Pascal's theorem applied to ACTcBGEA, Tc goes through P.
3. According to Pascal's theorem applied to CYEGBCTc, I, X and P are collinear.
4. Let Ty be the tangent to (O) at Y; we have Ty // AB.
5. According to a special case of Pascal's theorem applied to CYTyEABC, IX is parallel to AB
and we are done...
Sincerely
Jean-Louis

Quick Reply

Spain

Probl 5 (XX OMM)

conejita

#1 Nov 28, 2006, 9:06 pm

Aki el problema 5:

Sea ABC un triangulo acutangulo, y AD, BE, CF sus alturas. La circunferencia con diametro AD corta a los lados ABy AC en M y N respectivamente. Sean P y Q los puntos de interseccion de AD con EF y MN respectivamente. Demuestra que Q es el punto medio de PD.

[/img]

M4RI0

#2 Nov 29, 2006, 3:23 am

Sea R el punto de interseccion de FC y MN, T la intereccion de BE y MN, con el cuadrilatero inscriptible $RDCN$ tenemos $\measuredangle DRC = 90^\circ$, en consecuencia $MDRF$ es un rectángulo, de lo cual: $MD = FH + HR$

DMN, HRT, HFE son semejantes: $\frac{FH}{PH} = \frac{HR}{HQ} = \frac{MD}{QD}$

Luego: $PH + HQ = QD \Rightarrow \boxed{PQ = QD}$

Nota: H es el ortocentro de ABC

Luis González

#3 Aug 6, 2009, 9:39 am

Se puede plantear un resultado más general:

Sea X un punto del plano de $\triangle ABC$ y $\triangle DEF$ su triángulo ceviano. $D \in BC, E \in CA$ y $F \in AB.$ Las paralelas por D a BE y CF cortan a AC, AB en $N, M.$ AD corta a EF y MN en $P, Q.$ Entonces Q es punto medio de $PD.$

Note que $\triangle DMN \sim \triangle XFE$ son homotéticos con centro de homotecia A y coeficiente $\frac{AD}{AX}$, por tanto $\frac{QD}{PX} = \frac{AD}{AX}$.

Por teorema de Thales se tiene $\frac{PQ}{AP} = \frac{EN}{AE} = \frac{XD}{AX}$

Combinado con la anterior resulta:

$\frac{PQ}{QD} = \frac{AP}{PX} \cdot \frac{XD}{AD}$. Pero como $(A, X, P, D) = -1 \Longrightarrow \frac{AP}{PX} = \frac{AD}{XD}$

Lo cual implica $PQ = QD$ como se queria probar.

High School Olympiads

Concurrent and excircles

Source: Own

livetolove212

#1 Aug 2, 2009, 3:50 pm

Let $(I_a), (I_b), (I_c)$ be the excircles of triangle ABC.(I_a) touches AB, AC at C'_a, B'_a, similar to B'_c, A'_c, C'_b, A'_b. Let O_a, O_b, O_c be circumcenters of triangle $AB'_cC'_b, BC'_aA'_c, CA'_bB'_a$. l_a be a line which passes through I_a and perpendicular to O_bO_c. Simiar to l_b, l_c. Prove that l_a, l_b, l_c are concurrent.

yetti

#2 Aug 4, 2009, 12:18 pm

Let (O) to the circumcircle and (I) incircle of the $\triangle ABC$. Incircle (I) touches BC, CA, AB at D, E, F. Excircles $(I_a), (I_b), (I_c)$ touch BC, CA, AB at D_a, F_b, F_c. Internal angle bisectors AI, BI, CI cut (O) again at X, Y, Z. Perpendicular bisectors OX, OY, OZ of BC, CA, AB cut these at their midpoints A', B', C' and the circumcircle (O) again at X', Y', Z'.

Lemma 1: Perpendicular bisectors of AD, BE, CF cut segments $A'X', B'Y', C'Z'$ at their midpoints P_a, P_b, P_c. Similarly, perpendicular bisectors of AB'_c, BC'_a, CA'_b cut sgments $A'X, B'Y, C'Z$ at their midpoints O_a, O_b, O_c.

Parallel of BC through A cuts (O) again at A''. By Ptolemy for $AA''BC, \overline{AA''} = \dfrac{c^2-b^2}{a}$. Let ID cut AA'' at D'' and tangent of (O) at X' at X''. $\overline{X'X''} = \overline{A'D} = \frac{1}{2}\overline{D_aD} = \frac{1}{2}(c-b)$.

$$-\overline{D''A} \cdot \overline{D''A''} = \tfrac{1}{4}(\overline{A''A} - \overline{D_aD}) \cdot (\overline{A''A} - \overline{D_aD}) =$$
$$= \tfrac{1}{4}\left(\frac{(c^2-b^2)^2}{a^2} - (c-b)^2\right) = \frac{p(p-a)(c-b)^2}{a^2}$$

$$-\overline{D''D} \cdot \overline{D''X''} = \overline{DD''} \cdot (\overline{A'X'} - \overline{DD''}) = h_a \cdot \left(\frac{r_b + r_c}{2} - h_a\right) =$$
$$= \frac{2pr}{a}\left(\frac{pr}{2(p-b)} + \frac{pr}{2(p-c)} - \frac{2pr}{a}\right) = \frac{p^2r^2}{a^2} \cdot \frac{(c-b)^2}{(p-b)(p-c)}$$

$\Longrightarrow \overline{D''A} \cdot \overline{D''A''} = \overline{D''D} \cdot \overline{D''X''} \Longrightarrow AX''A''D$ is cyclic and $D_a \in \odot(AX''A''D)$ by symmetry. Its circumcenter is the common midpoint P_a of D_aX'' and $A'X'$. Therefore, P_a is circumcenter of $\triangle ADD_a$ and similarly, O_a is circumcenter of $\triangle AB'_cC'_b$. $\square$

Lemma 2: Nagel lines AD_a, BE_b, CF_c of $\triangle ABC$ are perpendicular to P_bP_c, P_cP_a, P_aP_b. Similarly, Gergonne lines AD, BE, CF of $\triangle ABC$ are perpendicular to O_bO_c, O_cO_a, O_aO_b.

Let (P_b) be circle with center P_b passing through A, C and let (P_c) be circle with center P_c passing through A, B. Let $(P_b), (P_c)$ cut BC again at D_b, D_c, respectively. Powers of B, E to (P_b) are both equal to $-(p-a)(p-c)$ and likewise, powers of C, F to (P_c) are both equal to $-(p-a)(p-b) \Longrightarrow$

$$\overline{D_aC} \cdot \overline{D_aD_b} = \overline{D_aC} \cdot (\overline{CB} + \overline{BD_b} - \overline{CD_a}) = -(p-b)\left(a + \frac{(p-a)(p-c)}{a} - (p-b)\right) = -\frac{(p-b)(p-c)}{a}$$
$$\overline{D_aB} \cdot \overline{D_aD_c} = \overline{D_aB} \cdot (\overline{BC} + \overline{CD_c} - \overline{BD_a}) = -(p-c)\left(a + \frac{(p-a)(p-b)}{a} - (p-c)\right) = -\frac{(p-b)(p-c)}{a}$$

$\Longrightarrow AD_a$ is radical axis of $(P_b), (P_c) \Longrightarrow AD_a \perp P_bP_c$. $\square$

Lemma 3: I-symmedians of the in/excentral triangles $\triangle II_bI_c, \triangle II_bI_c, \triangle II_bI_c$ are parallel to Nagel lines AD_a, BE_b, CF_c of their common orthic triangle $\triangle ABC$. Similarly, I_a, I_b, I_c-symmedians of the excentral triangle $\triangle I_aI_bI_c$ are parallel to

Gergonne lines AD, BE, CF of its orthic triangle $\triangle ABC$.

Let D' be midpoint of $AD.$ Newton line $A'D'$ of degenerated tangential quadrilateral $ABDC$ with diagonals AD, BC goes through its incenter I.. Therefore $(A'I \equiv AD') \parallel AD_a.$ is midline of $\triangle AD_aD. BC$ is antiparallel of I_bI_c WRT $\angle I_cII_b.$ The I-symmedian of $\triangle II_bI_c$ has to cut this antiparallel in half $\Longrightarrow A'I$ is the I-symmedian of this triangle.

Midpoints O_a, O_b, O_c of $A'X, B'Y, C'Z$ from lemma 1 are circumcenters of the $\triangle AB'_cC'_b, \triangle BC'_aA'_c, \triangle CA'_bB'_a,$ respectively. By lemmas 2, 3, perpendiculars l_a, l_b, l_c to O_bO_c, O_cO_a, O_aO_b through I_a, I_b, I_c are symmedians of the excentral triangle $\triangle I_aI_bI_c,$ concurrent at its symmedian point. $\square$

livetolove212

#3 Aug 4, 2009, 12:59 pm

Dear Yetti,
Thanks you for your solution, but I think that using radical center can be shorter

Luis González

#4 Aug 5, 2009, 2:17 am

Let Y_a, Y_b, Y_c be the tangency points of $(I_a), (I_b), (I_c)$ with $BC, CA, AB.$ Note that circles $(O_a), (O_b)$ and (O_c) are the reflections of the circles $\odot(AY_cY_b), \odot(BY_aY_c)$ and $\odot(CY_aY_b)$ about the external bisectors of $A, B, C.$ Circles $\odot(BY_aY_c),$ $\odot(CY_aY_b)$ obviously pass through the Bevan point B_e of $\triangle ABC,$ namely the concurrency point of the perpendiculars from I_a, I_b, I_c to $BC, CA, AB.$ Therefore, I_a has equal power WRT $(O_b), (O_c) \Longrightarrow \ell_a$ is radical axis of $(O_b), (O_c).$ Mutatis mutandis, ℓ_b, ℓ_c are radical axes of the pairs $(O_a), (O_c)$ and $(O_a), (O_b)$ respectively $\Longrightarrow \ell_a, \ell_b, \ell_c$ meet at the radical center of $(O_a), (O_b), (O_c)$ (Mittenpunkt of ABC).

Quick Reply

High School Olympiads

Maybe know one

Reply

linker

#1 Aug 3, 2009, 8:45 pm

Let ABC be a triangle and M,N points on AB respectively AC such that $\frac{BM}{MA}=x$ and $\frac{CN}{NA}=y$. Prove that MN passes through the incenter I of the triangle ABC if and only if $a=bx+cy$, where a,b,c are the lenghts of the sides of the triangle.

PS: Also try to prove simillarly for G(centroid) or O(circumcenter) instead of I (the incenter).

Luis González

#2 Aug 3, 2009, 10:08 pm

Cristea's theorem: D,E,F are three points on the sides CB,BA,AC of $\triangle ABC$ and $M\in AD.$ Then the transversal EF goes through M if and only if

$$\overline{DC}\cdot\frac{\overline{EB}}{\overline{EA}}+\overline{BD}\cdot\frac{\overline{FC}}{\overline{FA}}=\overline{BC}\cdot\frac{\overline{MD}}{\overline{MA}}$$

Quick Reply

High School Olympiads

a side equals to one third of triangle's perimeter

Reply

Source: I.F.Sharygin 2009 - Final round - Problem 9.1

April

#1 Aug 3, 2009, 9:28 am • 1

The midpoint of triangle's side and the base of the altitude to this side are symmetric w.r.t. the touching point of this side with the incircle. Prove that this side equals one third of triangle's perimeter.

Author: A.Blinkov, Y.Blinkov

Luis González

#2 Aug 3, 2009, 11:24 am

Let H,M,X be the foot of the A-altitude, midpoint of BC and tangency point of the incircle (I) with $BC.$ If H,M are symmetric WRT $X,$ then the tangency point of (I) with the 9-point circle is the antipode of X on $(I).$ Thus, Feuerbach point F_e satisfies the barycentric equation of the A-Nagel ray $AY \equiv (p-c)y-(p-b)z=0$

$$F_e \ ((p-a)(b-c)^2:(p-b)(a-c)^2:(p-c)(a-b)^2)$$

$$\Longrightarrow (p-c)(a-c)^2(p-b)-(p-b)(a-b)^2(p-c)=0$$

$$\Longrightarrow 2a=\frac{b^2-c^2}{b-c} \Longrightarrow \ 3a=a+b+c$$

Quick Reply

High School Olympiads

TRAN THAI HUNG
#1 Jul 30, 2009, 12:14 pm

Let H is the orthocentre of ABC. G is the intersection point of the medians of ABC.
Assume that B,C is the fixed points and H, G, B, C are on hte same circle.Find the locus of A

poohthewinnie
#2 Aug 1, 2009, 1:24 pm

Hmm, first I attacked this with the euler line and ptolemy - of course, I had no progress.

The easiest way to go is probably using the angles. So here goes.

Let BHGC be concyclic(In that order)

and Let H[C] be the foot of the perpendicular on AB. M[C] be the median of AB. and same goes with H **and M, both on CA.**

then

angle HBG=HCG,

and that is the same with

angle H[C] C M[C]=H B M

which is equivalent to

angle A M[C] C=A M[A] C

thus A, M[C], M[A], C are concyclic.

but since

segment M[C]M[A] // CA

angle A=B M[C] M[A]= angle C

hence BA= BC, and the locus of A is the circle that is formed with radius BC=a and the center B.

Luis González
#3 Aug 3, 2009, 1:08 am

Let $\angle BGC=\theta_a \Longrightarrow \cot \theta_a=\frac{GB^2+GC^2-a^2}{4[\triangle GBC]}.$

Now, we use the well-known identities

$GB^2=\frac{2}{9}(a^2+c^2)-\frac{1}{9}b^2 \ , \ GC^2=\frac{2}{9}(a^2+b^2)-\frac{1}{9}c^2 \Longrightarrow$

$\cot \theta_a=\frac{\frac{2}{9}(2a^2+c^2+b^2)-\frac{1}{9}(b^2+c^2)-a^2}{\quad}=\frac{b^2+c^2-5a^2}{\quad}$

$4[\triangle GBC]$ $36[\triangle GBC]$

B,C,H,G are concyclic $\Longleftrightarrow \cot\theta_a=\cot(\pi-A)=-\cot A=\dfrac{a^2-b^2-c^2}{4[\triangle ABC]}$

Therefore, keeping in mind that $[\triangle ABC]=3[\triangle BGC],$ we obtain

$$\frac{a^2-b^2-c^2}{4[\triangle ABC]}=\frac{b^2+c^2-5a^2}{12[\triangle ABC]}\Longrightarrow b^2+c^2=2a^2\Longrightarrow AM=\tfrac{1}{2}\sqrt{3}a=\text{const}.$$

Thus, if we construct the equilateral triangle $\triangle PBC,$ then $MP=MA.$ The locus of A is then the circle centered at the midpoint M of BC with radius $\overline{MP}.$

TRAN THAI HUNG

#4 Aug 4, 2009, 11:18 am

Welldone, I like your proof very much. And I use transformation.

Quick Reply

High School Olympiads

Collinear

Reply

Source: nice

jayme
#1 Aug 1, 2009, 10:39 am

Dear Mathlinkers,
let ABC be a triangle, A'B'C' the orthic triangle of ABC, A"B"C" the Euler's triangle of ABC, X the meet point of B"C' and C"B', O the center of the circumcircle of ABC and N the center of the Euler's circle of ABC.
Prove : X, A and N are collinear.
Sincerely
Jean-Louis

Luis González
#2 Aug 1, 2009, 11:02 pm

Let K be the circumcenter of $\triangle AB'C'$ lying on $AH,$ since rays AO,AH are isogonal WRT $\angle BAC.$ Let K_0 be the reflection of K about $B'C'.$ Note that $\angle HC'B''=\angle HB'C'' \Longrightarrow$ Rays $C'X,C'K_0$ and $B'X,B'K_0$ form equal and oppositely directed angles with respect to the pairs of rays $C'B',C'A$ and $B'C',B'A \Longrightarrow X,K_0$ are isogonal conjugates WRT $\triangle AB'C' \Longrightarrow$ Rays AK_0 and AX are isogonal WRT $\angle BAC.$ Since $B'C'$ is antiparallel to $BC,$ the axial symmetry about the internal bisector of $\angle BAC$ takes $B'C'$ into $B''C''$ parallel to $BC,$ K into K'' on the ray AO and K_0 into $K_0{}'$ on the ray $AX.$ Then homothety with center $A,$ that takes $B''C''$ into $BC,$ sends K'' into O and carries $K_0{}'$ into the reflection O_a of O about the sideline $BC.$ Since $AH=OO_a \Longrightarrow AHO_aO$ is a parallelogram.

jayme
#3 Aug 2, 2009, 11:52 am

Dear Luis,
why do you haven't thought at the Pascal's theorem?
Sincerely
Jean-Louis

livetolove212
#4 Aug 2, 2009, 2:48 pm

Dear Jayme,
Yes Pascal's theorem is very useful if we let B_1,C_1 be the midpoint of $AC,AB.$

Quick Reply

High School Olympiads

Geometry(own)

Reply

Source: really interesting and easy

TRAN THAI HUNG

#1 Jul 30, 2009, 12:09 pm

Let G is the intersection point of the medians of ABC. (I) is the inscribed circle.
Prove that
[B,C,I,G are on the same circle] is equivalent to $AB^2 + BC^2 + AC^2 = 3AB.AC$

Luis González

#2 Jul 31, 2009, 7:55 am • 1

We use barycentric coordinates with respect to $\triangle ABC$. The circle passing through $B \equiv (0:1:0), C \equiv (0:0:1)$ and $G \equiv (1:1:1)$ has equation

$(O') \equiv a^2yz + b^2xz + c^2xy - \frac{1}{3}(a^2+b^2+c^2)(x+y+z)x = 0$

B, C, I, G are concyclic $\Longleftrightarrow I \equiv (a:b:c)$ satisfies the equation of (O'). Then

$a^2bc + b^2ac + c^2ab + \frac{1}{3}(a^2+b^2+c^2)(a+b+c)a = 0$

$abc(a+b+c) - \frac{1}{3}(a^2+b^2+c^2)(a+b+c)a = 0 \Longrightarrow 3bc = a^2+b^2+c^2.$

TRAN THAI HUNG

#3 Jul 31, 2009, 12:25 pm

I also use coordinate(with axis Ox, Oy) to solve this problem but Sorry, I wonder what barycentric coordinate is. And how can you have 3 variables x,y,z on a plane.

Quick Reply

High School Olympiads

Geometry (own),quite hard!

Reply

Source: delicated to everyone

TRAN THAI HUNG

#1 Apr 16, 2009, 10:31 am

Let ABC is a triangle, G is the intersection of the medians. (O) is the outcircle. Prove that **[B,C,O,G is on a same circle]** $\Longleftrightarrow AB^4 + AC^4 = BC^4 + AB^2 \cdot AC^2$

TRAN THAI HUNG

#2 Jul 30, 2009, 11:55 am

Sorry, (O) is the circumscribed circle of ABC.
No one solves my problem :(.It's really interesting.

Luis González

#3 Jul 30, 2009, 6:45 pm • 1

We use barycentric coordinates with respect to $\triangle ABC$. The equation of the circle that goes through $O \equiv (a^2S_A : b^2S_B : c^2S_C), B \equiv (0:1:0)$ and $C \equiv (0:0:1)$ is

$$(O_a) \equiv a^2yz + b^2zx + c^2xy - \frac{b^2c^2}{2S_A}x(x+y+z) = 0$$

G, O, B, C are concyclic $\Longleftrightarrow G \equiv (1:1:1)$ satisfies the equation of (O_a). Then

$$2S_A(a^2+b^2+c^2) - 3b^2c^2 = (b^2+c^2-a^2)(a^2+b^2+c^2) - 3b^2c^2 = 0$$

$$a^2b^2 + a^2c^2 - a^4 + b^4 + b^2c^2 - a^2b^2 + b^2c^2 + c^4 - a^2c^2 - 3b^2c^2 = 0$$

$$b^4 + c^4 = a^4 + b^2c^2.$$

Altheman

#4 Jul 31, 2009, 4:32 am

> *luisgeometra wrote:*
> The equation of the circle that goes through $O \equiv (a^2S_A : b^2S_B : c^2S_C), B \equiv (0:1:0)$ and $C \equiv (0:0:1)$ is
>
> $$(O_a) \equiv a^2yz + b^2zx + c^2xy - \frac{b^2c^2}{2S_A}x(x+y+z) = 0$$

This is really nice, how do you get this?

TRAN THAI HUNG

#5 Aug 4, 2009, 5:06 pm

To luisgeometra
How can we write the equation of a circle which goes through three points in barycentric coordinate? :D

Quick Reply

High School Olympiads

collinearity

77ant

#1 Jul 29, 2009, 1:05 am

For a triangle ABC with incenter I and circumcenter O. Circles D, E, and F are congruent circles passing through point G. Circle D is tangent to AB and AC, circle E is tangent to AB and BC, and circle F is tangent to AC and BC. Prove that O, G, and I are collinear.

livetolove212

#2 Jul 29, 2009, 9:41 am

Let I_a, I_b, I_c be excenters of triangle ABC. We know that OI is **Euler's line** of triangle $I_aI_bI_c$
On the other side, $(A'), (B'), (C')$ concur at G so G is the circumcenter of triangle $A'B'C'$
Note that $A'B'//AB, A'C'//AC$ and AI is the bisector of angle BAC so $A'I$ is the bisector of angle $B'A'C'$
Similarly we get I is the incenter of triangle $A'B'C'$.
Let I'_a, I'_b, I'_c is the excenter of triangle $A'B'C'$. It's easy to see that $I'_aI'_b//I_aI_b, I'_bI'_c//I_bI_c, I'_cI'a//I_cI_a$ then **Euler's line** of two triangles $I_aI_bI_c$ and $I'_aI'_bI'_c$ are parallel, but they have common point I so G, I, O are collinear.
We complete the proof.

Attachments:
picture11.pdf (6kb)

plane geometry

#3 Jul 29, 2009, 6:37 pm

> *livetolove212 wrote:*
> Let I_a, I_b, I_c be excenters of triangle ABC. We know that OI is **Euler's line** of triangle $I_aI_bI_c$
> On the other side, $(A'), (B'), (C')$ concur at G so G is the circumcenter of triangle $A'B'C'$
> Note that $A'B'//AB, A'C'//AC$ and AI is the bisector of angle BAC so $A'I$ is the bisector of angle $B'A'C'$
> Similarly we get I is the incenter of triangle $A'B'C'$.
> Let I'_a, I'_b, I'_c is the excenter of triangle $A'B'C'$. It's easy to see that $I'_aI'_b//I_aI_b, I'_bI'_c//I_bI_c, I'_cI'a//I_cI_a$ then **Euler's line** of two triangles $I_aI_bI_c$ and $I'_aI'_bI'_c$ are parallel, but they have common point I so G, I, O are collinear.
> We complete the proof.

dear **livetolove212**
why G is the circumcenter of triangle $A'B'C'$?

livetolove212

#4 Jul 29, 2009, 7:08 pm

Because $(A'), (B'), (C')$ are congruent and pass through G.

plane geometry

#5 Jul 29, 2009, 7:20 pm

> *livetolove212 wrote:*
> Because $(A'), (B'), (C')$ are congruent and pass through G.

I am sorry ,I did not notice the condition "D, E, and F are congruent circles"!

Luis González

#6 Jul 30, 2009, 12:13 am

> *77ant wrote:*
>
> For a triangle ABC with incenter I and circumcenter O. Circles D, E, and F are congruent circles passing through point G. Circle D is tangent to AB and AC, circle E is tangent to AB and BC, and circle F is tangent to AC and BC. Prove that O, G, and I are collinear.

Since D, E, F are equidistant from the respective pairs of sides of $\triangle ABC,$ it follows that $\triangle DEF$ and $\triangle ABC$ are homothetic with homothetic center $I.$ X is obviously circumcenter of $\triangle DEF.$ Hence O, X and the homothetic center I are collinear. In fact, the concurrency point X of the circles $(D),(E),(F)$ is the insimilicenter of $(I) \sim (O).$

Quick Reply

High School Olympiads

resembling pythagorean theorem

Reply

77ant

#1 Jul 29, 2009, 1:11 am

ABCD is an inscribed quadrilateral in a circle O. The extensions of sides AB and DC meet at point E, and the extensions of sides AD and BC meet at point F. EG and FH are tangent to the circle O. If EG = a, FH = b and EF = c, prove that $a^2 + b^2 = c^2$

Luis González

#2 Jul 29, 2009, 5:38 am

EG, EI, FH are the tangent segments from E, F to the circle (O). IG is the radical axis of the circles (E, EG) and (O). Since F lies on their radical axis (polar of E WRT (O)), then it has equal power with respect to (O) and (E, EG)

$$FB \cdot FC = FH^2 = EF^2 - EG^2 \implies EF^2 = EG^2 + FH^2$$

Quick Reply

High School Olympiads

concurrency

Reply

77ant

#1 Jul 29, 2009, 1:04 am

For a quadrilateral ABCD with perpendicular diagonals intersecting at E. EF, EG, EH and EM are perpendicular to AB, BC, CD, and AD, respectively. Prove that lines FG, AC, and MH are concurrent.

Luis González

#2 Jul 29, 2009, 4:49 am

It is enough to see that $FGHM$ is cyclic due to

$$\angle FGH = \angle ABE + \angle ECD \ , \ \angle FMH = \angle EAB + \angle EDC \ \Longrightarrow$$

$$\angle FGH + \angle FMH = \angle ABE + \angle ECD + \angle EAB + \angle EDC = 90^\circ + 90^\circ = 180^\circ$$

$FGCA$ is also cyclic due to $\angle BFG = \angle BEG = \angle BCA \Longrightarrow FG$ and AC are antiparallel WRT rays BA, BC. Similarly $MHCA$ is cyclic $\Longrightarrow AC, FG, MH$ concur at the radical center of $\odot(FGHM), \odot(FGCA)$ and $\odot(MHCA)$

Quick Reply

High School Olympiads

intersection

Reply

Source: Ukraine 2005 grade 11

moldovan

#1 Jul 28, 2009, 9:29 pm

In an acute-angled triangle ABC, ω is the circumcircle and O its center, ω_1 the circumcircle of triangle AOC, and OQ the diameter of ω_1. Let M and N be points on the lines AQ and AC respectively such that the quadrilateral $AMBN$ is a parallelogram. Prove that the lines MN and BQ intersect on ω_1.

Luis González

#2 Jul 29, 2009, 3:57 am

Let L be the midpoint of AB and P the second intersection of $\odot(AOC)$ with BQ. $\angle APQ = \angle BNA = \angle ABC$. If $R \equiv BQ \cap AC$ and $D \equiv BN \cap AP$, then it follows that $PDNR$ is cyclic. Notice that PQ bisects $\angle APC$, since Q bisects the arc AC of $\odot(AOC) \Longrightarrow \angle BPC = \pi - \angle ABC = \angle BNC \Longrightarrow BPNC$ is cyclic, thus $\angle NPR = \angle ACB$. But since $PDNR$ is cyclic, it follows that $\angle NPR = \angle NDR = \angle ACB = \angle ABN \Longrightarrow DR \parallel BA$. Therefore, cevian NP of $\triangle BNA$ passes through the midpoint L of $AB \Longrightarrow M$ lies on NP as desired.

Quick Reply

Spain

Reto de la semana 3

Pascual2005

#1 Aug 10, 2006, 1:42 am

Reto de la semana #3

1. Sea $a_n=[n\sqrt{2}]+[n\sqrt{3}]$, pruebe que esta sucesion contiene infinitos números pares e infinitos números impares.

2. Sea a_n una secuencia tal que $m-n|a_m-a_n$ y existe un polinomio q tal que $a_n<q(n)$. pruebe que existe un polinomio con coeficientes racionales P tal que $P(n)=a_n$

3. Se tiene un cuadrilatero inscribible y circunscribible a la vez, demuestre que el punto de corte de sus diagonales, y los centros de los circulos inscritos y circunscritos son colineales.

4. Sean a,b,c reales positivos, pruebe que

$$\frac{(a^2+b^2)}{(a+b)^2}\frac{(b^2+c^2)}{(b+c)^2}\frac{(c^2+a^2)}{(c+a)^2}\geq\frac{3(a^2+b^2+c^2)}{8(a+b+c)^2}$$

This post has been edited 1 time. Last edited by Pascual2005, Aug 29, 2006, 6:53 pm

Pascual2005

#2 Aug 16, 2006, 4:27 am

Pilas, el plazo se vence esta semana y no hay soluciones aun!!!!

espero 14 soluciones proximamente!!!

Tony2006

#3 Aug 16, 2006, 5:34 am

soy Juan Antonio Ríos Briceño de Mérida, Yucatán y no eh participado en los retos de la semana, todavia puedo participar?

Pascual2005

#4 Aug 16, 2006, 5:51 am

por supuesto, quien quiera participar puede hacerlo en cualquier momento enviandome sus soluciones!!!

jasamper88

#5 Aug 22, 2006, 8:26 am

Pido perdon por no haber mandado nada la semana pasada cuando tocaba mandar soluciones.No tuve mucho tiempo pues acababa de volver a Bogota y tenia que organizar todo nuevamente.
Veo que nadie posteo nada por lo que me pregunto si se puede extender este reto hasta el viernes este reto y ahora si definitivamente mandar soluciones.
No se como será en los otros paises de latino america, pero nosotros aca en Colombia empezamos de nuevo el colegio en esta epoca. Se acabaron las vacaciones y hay que volver a trabajar duro para la Ibero y la IMO que viene. Espero respuesta. Saludes a todos.

Pascual2005

#6 Aug 22, 2006, 9:18 am

Petición aceptada!

El viernes se cerrará este cuestionario y se colgará uno nuevo.

Chen241290

#7 Aug 24, 2006, 7:36 am

Yo ya envie 2 soluciones, no llegaron? que yo recuerde escribí bien el correo de Pascual.
Ah, mejor pregunto aqui: la 2 no debería decir que P es de coef. racionales. Con enteros tengo un contraejemplo:

$$a_n = \frac{n^4 + n^2}{2}$$

Pascual2005

#8 Aug 24, 2006, 10:41 am

Es verdad no deben ser enteros... Si recibi tus soluciones, el plazo ser´´a hasta el proximo viernes...

Pascual2005

#9 Aug 27, 2006, 2:26 am

Gracias a quienes participaron, sobretodo a Chen y Israel Diaz

Mañana colgare sus soluciones

Pascual2005

#10 Aug 29, 2006, 8:02 pm

Soluciones Chen:

Sean $b_n = [n\sqrt{2}], c_n = [n\sqrt{3}]$
$a_n = b_n + c_n$
Supongamos que existen finitos numeros de cierta paridad, entonces existe N tal que a_n es de cierta paridad(distinta a la anterior), $\forall n \geN.$
Sea $m \geN$ Entonces a_m es de cierta paridad, y a_{m+1} también.
$b_{m+1} = [(m+1)\sqrt{2}] \Rightarrow b_{m+1} > (m+1)\sqrt{2} - 1 > m\sqrt{2} \geq b_m$
luego: $b_{m+1} \leq (m+1)\sqrt{2} < (m\sqrt{2} - 1) + 3 < b_m + 3$
entonces $b_{m}+1 \leb_{m+1}\leb_{m}+2$
Analogamente: $c_{m}+1 \lec_{m+1}\lec_{m}+2$
Entonces $a_{m}+2 \lea_{m+1}\lea_{m}+4$
como $a_{m+1} - a_m$ es par, a_{m+1} es $a_m + 2$ o $a_m + 4$
Si $a_{m+1} = a_m + 2$, $b_{m+1} = b_m + 1$ y $c_{m+1} = c_m + 1$
Si $a_{m+1} = a_m + 4$, $b_{m+1} = b_m + 2$ y $c_{m+1} = c_m + 2$
Luego, $b_{m+1}-c_{m+1}=b_{m}-c_{m}, \forall m \geN$
Entonces $b_{N+5} - c_{N+5} = b_N - c_N$
$b_{N+5} - b_N = c_{n+5} - c_N$
$b_{N+5} \leq (N+5)\sqrt{2} < b_N + 8 \Rightarrow b_{N+5} - b_N \leq 7$
$c_{N+5} > (N+5)\sqrt{3} - 1 > c_N + 7 \Rightarrow c_{n+5} - c_N \geq 8$
(contradiccion). Por lo tanto en tal sucesion existen infinitos pares e infinitos impares.

Chen envió una solución del 4 enla que afirma que: $3\sum_{sym} a^6b^2 \geq 6\sum_{sym} a^6bc$ esto es calramente falso...

3.

Lema.- Dado r natural: $[r, ..., r+s] \geq \frac{r(r+1)...(r+s)}{1!2!...s!}$; para todo natural s. ($[a, b]$ es MCM de a y b, y (a, b) es MCD de a y b)
Dem: Por induccion:
para $s = 1$ es cierto, pues r y $r+1$ son coprimos.
Supongamos que para $s = t$ es cierto, veamos para $s = t+1$:
$[r, ..., r+t+1] = [a, r+t+1]; a = [r, ..., r+t] = \frac{a(r+t+1)}{\ldots} \geq \frac{r(r+1)...(r+t)}{\ldots}\frac{r+t+1}{\ldots}$; por

$\ldots (a, r+t+1) \ldots 1!2!\ldots r+t! \ldots (a, r+t+1)$, por
hipotesis inductiva.
Si demostramos que $(a, r+t+1) = d \leq (t+1)!$ completamos la induccion:
d divide a $r+t+1$ y tambien a a(de aqui, d divide a $r(r+1)...(r+t)$.
$r(r+1)...(r+t) \equiv (-(t+1))(-t)...(-1)$ (mod $r+t+1$).
Entonces existe k entero tal que: $r(r+1)...(r+t) = (r+t+1)k$+(-1)^{t+1}$(t+1)!$
entonces d divide a $(t+1)!$, luego $d \leq (t+1)!$ y la induccion queda completada.

En el problema:
Si m es el grado de el polinomio q.
Dados $a_1, a_2, \ldots, a_{m+1}$. Existe un polinomio p de coef racionales y de grado a lo mas m, tal que $p(i) = a_i$; $1 \leq i \leq m+1$ (basta tomar el polinomio de Lagrange)
Sea N el MCM de los denominadores de los coef. de p.
Sea el polinomio P, cuya regla de corresp. es $P(x) = Np(x)$ (polinomio de coef. enteros)
Sea el polinomio Q, cuya regla de corresp. es $Q(x) = Nq(x)$
$Na_{r+m+1} \equiv Na_i \equiv P(i) \equiv P(r+m+1)$ (mod r+m+1-i) ; $\forall 1 \leq i \leq m+1$
entonces $[r, ..., r+m]$ divide a $Na_{r+m+1} - P(r+m+1)$
Luego
$-[r, ...r+m] < -P(r+m+1) < Na_{r+m+1} - P(r+m+1) < Q(r+m+1) - P(r+m+1) < [r, ..., r+m]$
para todo natural r mayor que T. Pues del lema podemos decir que ese mcm crece mas que un polinomio de grado $m+1$. y como P y $Q-P$ son de grado a lo mas m, lo anterior es cierto para un r suficientemente grande, es decir existe T tal que lo de arriba es cierto para $r > T$.
Se concluye que el termino del centro es cero(es un multiplo de las cotas de los extremos)
Sea $S = T+m+1$. entonces $Na_n = P(n)$, para todo $n > S$
entonces $a_n = p(n)$, para todo $n > S$
Por último:
$Na_i \equiv Na_{n+i} \equiv P(n+i) \equiv P(i)$ (mod n) para todo $n > S$, e i:natural
entonces $Na_i - P(i)$ tiene infinitos divisores, por lo que es cero. De alli: $a_i = p(i)$, para todo natural i.

Tony2006
#11 Aug 31, 2006, 5:18 am

pascual puedo postear la solucion del problema de geometria?
jeje

Chen241290
#12 Aug 31, 2006, 7:16 am

las sumas que yo puse no eran sumas simetricas. Eran otro tipo de sumas. Lo aclaré en mi solucion.

Luis González
#13 Jul 28, 2009, 8:41 am

> *Pascual2005 wrote:*
> 3. Se tiene un cuadrilatero inscribible y circunscribible a la vez, demuestre que el punto de corte de sus diagonales, y los centros de los círculos inscritos y circunscritos son colineales.

Sea $ABCD$ un cuadrilátero bicentrico con circuncírculo (O) e incírculo (I). (I) es tangente a AB, BC, CD, DA en X, Y, Z, W. Sean $M \equiv DA \cap CB, N \equiv AB \cap DC$. Denotemos P el punto de corte de las diagonales. La recta MN es la polar de P respecto a (O). Por otro lado XZ y YW se cortan en P (Teorema de Newton), entonces las polares de M, N respecto a (I) pasan por P, lo que implica que P es el polo de MN respecto a (I). Ahora como el polo está en la recta perpendicular a su polar por el centro de la circunferencia, resulta $PO \perp MN, PI \perp MN \Longrightarrow O, I, P$ están alineados.

Quick Reply

Spain

Reto #4 del 2006

Reply

carlosbr

#1 Feb 26, 2006, 2:44 am

NIVEL BASICO

Problema 1 [Geometria]
Demostrar que, si $ABCD$ es un cuadrilatero inscrito, la suma de los radios de las circunferencias inscritas en los triangulos ABC y ACD es igual a la suma de los radios de las circunferencias inscritas en los triangulos BCD y BDA.
[Sugerido por:*Los Organizadores*]

Problema 2 [Inecuaciones]
Sean $a, b, c > 0$. Mostrar que:
$\frac{a+b+c}{abc} \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$.
[Sugerido por:*Carlos Bravo*]

Problema 3 [Combinatorio]
Considerar todos $2^n - 1$ subconjuntos no vacios apartir del conjunto $\{1, 2, \cdots, n\}$. Para cada subconjunto, encontramos el producto de los rec\'iprocos de cada uno de sus elementos. Encontrar la suma de todos estos productos.
[Sugerido por:*Los Organizadores*]

Disfruten de los problemas :)
Los Organizadores

This post has been edited 1 time. Last edited by carlosbr, May 12, 2006, 11:37 am

Pascual2005

#2 Feb 26, 2006, 8:11 am

NIVEL AVANZADO

Problema 1
Hallar el menor entero t, tal que existen enteros $x_1, x_2 ... x_t$ tales que:

$$x_1^3 + x_2^3 ... + x_t^3 = 2002^{2002}$$

Problema 2
Sea P un polinomio con coeficientes enteros. Pruebe que si n es un entero positivo impar y existen enteros $x_1, x_2 ... x_n$ tales que $x_2 = P(x_1), x_3 = P(x_2) ... x_n = P(x_{n-1}), x_1 = P(x_n)$ entonces $x_1 = x_2 ... = x_n$

Problema 3
Sea S un conjunto de n puntos en el plano no 3 colineales. Pruebe que existe un conjunto de $2n - 5$ puntos tales que todo triangulo con vertices en S contiene uno de estos puntos.

Disfruten de los problemas :D

Los Organizadores

This post has been edited 1 time. Last edited by Pascual2005, Mar 1, 2006, 2:29 am

carlosbr

#3 Feb 26, 2006, 9:10 pm

Aki ..
la version en pdf ....
de los problemas ...
de esta semana : RETO #4.

Carlos Bravo
Lima - Peru

Attachments:
LISTA_DESAFIO_SEMANA_24FEB_03MAR.pdf (54kb)

Pascual2005

#4 Mar 3, 2006, 5:58 pm

Esta muy dificil o que? esperocomentarios y sugerencias porque el numero de soluciones esta empezando a tender a cero.

hucht

#5 Mar 3, 2006, 7:50 pm

> *carlosbr wrote:*
> **NIVEL BASICO**
>
> **Problema 1** [Geometria]
> Demostrar que, si $ABCD$ es un cuadrilatero inscrito, la suma de los radios de las circunferencias inscritas en los triangulos ABC y ACD es igual a la suma de los radios de las circunferencias inscritas en los triangulos BCD y BDA.
> [Sugerido por:*Los Organizadores*]
>
> **Problema 2** [Inecuaciones]
> Sean $a, b, c > 0$. Mostrar que:
> $\frac{a+b+c}{abc} \le \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$.
> [Sugerido por:*Carlos Bravo*]

Solucion del Problema 1.
Enunciaremos el teorema general:

Definicion. Dado un poligono $P_1P_2...P_n$ una triangulacion es el conjunto de triangulos determinados al unir el vertice P_k con los demas. Es obvio que resultan $n-2$ triangulos.

Teorema 1. Sea $P_1P_2...P_n$ un polígono inscrito en una circunferencia. La suma de los inradios de los $n-2$ triángulos formados por una triangulación es independiente del vértice escogido.

Para la demostracion usaremos el lema:

Lema 2. (Carnot) Dado $\triangle ABC$ de circuncentro O inradio r y circunradio R, sean M_a, M_b, M_c los puntos medios de los lados BC, AC, AB respectivamente, entonces:
i. $OM_a + OM_b + OM_c = R + r$ si O y C están en el mismo semiplano con respecto a la recta AB
ii. $OM_a + OM_b - OM_c = R + r$ si O y C están en diferentes semiplanos con respecto a la recta AB

Luego: Sean T_i los $n-2$ triángulos formados en la triangulación, de inradio r_i, radio M_i el punto medio de $P{}_iP{}_i_+_1$ y sea D_i el punto medio de $P{}_1P{}_i_+_2$. Así, aplicando el teorema de carnot en el triangulo $P{}_1P{}_iP{}_i_+_1$ tenemos:

$OM_1 + OM_2 - OD_1 = R + r_1$
$OD_1 + OM_3 - OD_2 = R + r_2$

$OD_2 + OM_4 - OD_3 = R + r_3$
:
$OM_{n-1} + OM_n - OD_{n-2} = R + r_{n-2}$

$\sum_{i=1}^{\(n}\(OM_i =\sum_{i=1}^{\(n-2}\(r_i + (n-2)R$

luego:

$\sum_{i=1}^{\(n-2}\(r_i =\sum_{i=1}^{\(n}\(OM_i - (n-2)R = constante$

Desde que la distancia del centro a los puntos medios de los lados del poligono es constante y $(n-2)R$ es constante. □

Image not found

Ahora bien, sea $ABCD$ un cuadrilatero inscrito, la $D - triangulacion$ es $[ABD, BCD]$ y la $A - triangulacion$ es $[ABC, ACD]$. de ahi que, por el teorema 1 para $n = 4$ la suma de los inradios de los triangulos ABC y ACD es igual a la suma de los inradios de los triangulos BCD y BDA. □

Solucion del problema 2.
Como la suma es cerrada en el conjunto $\math P_0$ (Positivos union 0) entonces

$(1/a - 1/b)^2 + (1/b - 1/c)^2 + (1/a - 1/c)^2 \geq 0$

asi: $1/a^2 + 1/b^2 + 1/c^2 \geq 1/ab + 1/bc + 1/ac$

y se sigue que $\dfrac{a+b+c}{abc} \leq \dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}$.

PD. Coloquen problemas de geometria del nivel superior

Atte.

$Jose^/Carlos$

---

federico...
#6 Mar 4, 2006, 4:20 am

Ahi dejo el 3 de nivel superior, todavia no tengo Latex asi que lo pongo en word

Attachments:
Reto4.doc (27kb)

Luis González
#7 Jul 28, 2009, 6:56 am

> *carlosbr wrote:*
> Demostrar que, si $ABCD$ es un cuadrilátero inscrito, la suma de los radios de las circunferencias inscritas en los triángulos ABC y ACD es igual a la suma de los radios de las circunferencias inscritas en los triángulos BCD y BDA.

Este es el llamado segundo teorema Japonés, cuyo caso mas general para n-gonos es el primer teorema Japonés o Teorema de Mikami y Kobayashi. Hay numerosas demostraciones pare este teorema, la solución mas sencilla y tal vez la mas elegante es usando el teorema de Carnot reiteradamente en cada triángulo que resulta de la triangulación, tal cual como fué probado por hutch.

Quick Reply

High School Olympiads

conjugate points

Reply

Source: properties

77ant

#1 Jul 26, 2009, 11:58 pm

Hi, everyone

Prove the following:

if P and Q be a pair of conjugate points for a circle to which they are external,

(i) PQ^2 is equal to the sum of the squares of the tangents from P and Q

(ii) PQ is twice the tangent from the middle point of PQ

(iii) $PU\cdot UQ$ is equal to the square of the tangent from U, U being the foot of the perpendicular from the center of the circle on PQ

(iv) the circle on PQ as diameter is orthogonal to the given circle

Thanks

yetti

#2 Jul 27, 2009, 4:05 am

P, Q are conjugate WRT circle (O) and external $\Longrightarrow$ a convex $ABCD$ inscribed in (O, r) exists, such that $P \equiv AB \cap CD, Q \equiv BC \cap DA.$ Let $R \equiv AC \cap BD. PQ, QR, RP$ are polars of R, P, Q WRT $(O) \Longrightarrow$ $PO \perp QR, QO \perp RP \Longrightarrow O$ is orthocenter of $\triangle PQR$ and $R \in OU.$ Let $V \equiv PO \cap QR, W \equiv QO \cap RP$ be feet of the first 2 altitudes. Let (X) be circle with diameter $PQ.$

(iv) Inversion in (O) takes $P, Q \in (X)$ to $V, W \in (X) \Longrightarrow$ it takes (X) to itself, $(X) \perp (O).$

(iii) Inversion in (O) takes R to $U \Longrightarrow RO \cdot UO = r^2.$ Power of U to circumcircle (S) of $\triangle PQR$ is $p(U, (S)) = -PU \cdot UQ = -UR \cdot UO = -(UO - RO) \cdot UO = r^2 - UO^2$ $\Longrightarrow$ power of U to (O) is $p(U, (O)) = UO^2 - r^2 = PU \cdot UQ.$

(ii) $(X) \perp (O) \Longrightarrow$ tangent length from X to (O) is then equal to the radius $\frac{1}{2}PQ$ of $(X).$

(i) O is orthocenter of $\triangle PQR \Longrightarrow R$ is orthocenter of $\triangle PQO.$ By basic properties of inversion in $(O),$ $VW = PQ \cdot \frac{r^2}{OP \cdot OQ}.$ $\triangle UVW$ is orthic triangle of $\triangle PQO \Longrightarrow VW = PQ \cos \widehat{QOP} \Longrightarrow$ $\cos \widehat{QOP} = \frac{r^2}{OP \cdot OQ}.$ By cosine theorem for $\triangle PQO,$ $PQ^2 = OP^2 + OQ^2 - 2OP \cdot OQ \cdot \cos \widehat{QOP} = OP^2 - r^2 + OQ^2 - r^2 = p(P, (O)) + p(Q, (O))$

Luis González

#3 Jul 27, 2009, 7:28 am

This simple figure has lots of interesting things, thus I'd like to highlight other properties besides the ones mentioned by 77ant. Let PA, PB, QC be the tangent segments from P, Q to the circle $(O).$ AB is radical axis of the circles (P, PA) and $(O).$ Since Q lies on their radical axis, then it has equal power with respect to (O) and (P, PA)

$QB \cdot QA = PQ^2 - PB^2 = QC^2 \Longrightarrow PQ^2 = PB^2 + QC^2$

The remaining properties follow directly from the fact that the circle with diameter $\overline{PQ}$ belows to the orthogonal pencil defined by the axis $PQ \equiv r$ and the circle $(O),$ thus the intersection points P, Q are homologous through an involution over r whose center is obviously the foot U of the perpendicular line from O to $r.$ Double points (if they exist) are at most the two intersections of $r \cap (O).$ Such inversion whose radius equals the radius of the line involution takes the circle (O) into itself.

Furtermore, if we project these two conjugate points from the pole R of $r,$ we obtain the polars p, q of P, Q WRT $(O),$ each one passing though the pole of the another $\Longrightarrow \triangle PQR$ is autopolar WRT $(O) \Longrightarrow p, q$ are conjugate lines WRT $(O).$ Thus they form an involution.

Quick Reply

High School Olympiads

segment equality

Reply

Source: related to polar

77ant
#1 Jul 26, 2009, 11:48 pm

Hi, everyone

Prove the following:
if a, b, p be the polars of the points A, B, P for a circle whose center is O,

$$\frac{(P,a)}{(P,b)}\div\frac{(A,p)}{(B,p)}=\frac{(O,a)}{(O,b)}=\frac{(B,a)}{(A,b)}$$

p.s. (A,p)=distance from A to p

Thanks

Luis González
#2 Jul 27, 2009, 12:54 am

By Salmon's theorem we have the following relations:

$$\frac{PO}{AO}=\frac{(P,a)}{(A,p)},\ \frac{PO}{OB}=\frac{(P,b)}{(B,b)},\ \frac{OB}{OA}=\frac{(B,a)}{(A,b)}$$

$$\Longrightarrow \frac{(P,a)}{(A,p)}\cdot\frac{(B,p)}{(P,b)}=\frac{OB}{OA}=\frac{(B,a)}{(A,b)}$$

Since the polar of a point is the inverse under (O,r^2) of the circe whose diameter is the perpendicular dropped from O to its pole, we have $r^2=(O,a)\cdot OA=(O,b)\cdot OB$

$$\Longrightarrow \frac{(P,a)}{(A,p)}\cdot\frac{(B,p)}{(P,b)}=\frac{(O,a)}{(O,b)}=\frac{(B,a)}{(A,b)}$$

77ant
#3 Jul 27, 2009, 12:57 am

Thank you, luisgeometria
The theorem seems so beautiful. I have not seen it

Quick Reply

Spain

Otro problema de Geometria

Reply

Leonardo

#1 Feb 23, 2005, 5:25 am

Este no es un problema como tal, sino que reduje un problema que tengo a demostrar esto.
Dado el triangulo $\triangle ABC$, sea I su incentro y sean D, E, F las reflexiones de I a travez de BC, CA, AB respectivamente.
Demuestre que AD, BE, CF concurren.

grobber

#2 Feb 23, 2005, 1:55 pm

This is a particular case of the following:

http://www.xtec.es/~qcastell/ttw/ttweng/definicions/d_Kariya_p.html

Leonardo

#3 Feb 23, 2005, 10:02 pm

Ya, que chevere esta eso, pero aun no logro demostrar que el caso particular sigue sirviendo.

salamitrosky

#4 May 14, 2005, 6:42 am

Hola Leo, cómo andás tantos meses?, te cuento algo de tu problema. Hace un tiempo Carlos me mandó una generalización, supongo que debe ser la que grobber comenta (no me fijé). Pero a su vez ese problema que pienso es un caso particular de éste (mmm ahora que reflexiono debe ser éste el problema que grobber dice): Dado un punto P, sean D, E y F los pies de las perpendiculares a BC, CA y AB, respectivamente. Sea X un punto en PD, la perpendicular de PC por X intersecta a PE en Y; la perpendicular a AP por Y intersecta a PF en Z. Bueno, no cuesta nada ver que XZ es también perpendicular a BP. Demostrar que XYZ es perspectivo con ABC.
Bueno, creo que era así, de última después lo arreglo, pero casi seguro que era así. Tengo la solución pasada en .jpg, ¿puede mandar acá?, un gran saludo amigo (seguís en ee.uu.?)

Luis González

#5 Jul 26, 2009, 5:05 am

Como ya han mencionado previamente este es un caso particular del llamado Teorema de Kariya

Teorema: En un $\triangle ABC, (I, r)$ es su incírculo y $\triangle X_aX_bX_c$ su triángulo de contacto interno. La circunferencia (I, kr) corta a los rayos IX_a, IX_b, IX_c en $D, E, F.$ Entonces $\triangle ABC$ y $\triangle DEF$ son perspectivos y ortológicos.

La demostración es relativamente simple usando el Teorema de Jacobi, que de hecho es un caso mas general que este teorema, o bien utilizando las ecuaciones en coordenadas baricentricas de la homotecia con centro I y coeficiente k

La expresión general del perspector de Kariya es:

$$J\left(\frac{1}{bc+kS_A}:\frac{1}{ac+kS_B}:\frac{1}{ab+kS_C}\right)$$

En este caso en particular como D, E, F son las reflecciones de I en los lados del $\triangle ABC,$ entonces $k=2,$ resultando que las rectas AD, BE, CF concurren en el conocido punto de Gray X_{79} de $\triangle ABC.$

$$X_{79}\left(\frac{1}{bc+2S_A}:\frac{1}{ac+2S_B}:\frac{1}{ab+2S_c}\right)$$

Quick Reply

High School Olympiads

compute the angle

Reply

Source: Ukraine 2005 grade 8

moldovan

#1 Jul 24, 2009, 11:15 pm

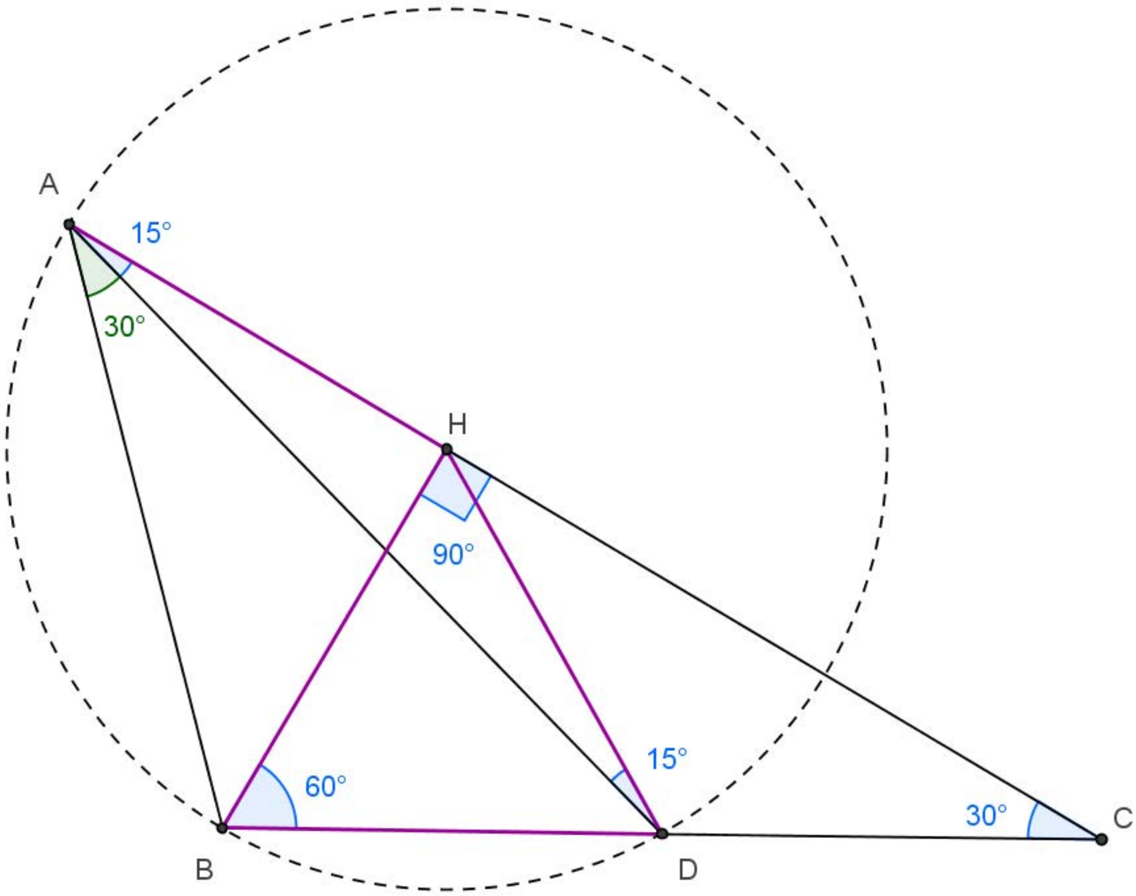
Let AD be the median of a triangle ABC. Suppose that $\angle ADB = 45^\circ$ and $\angle ACB = 30^\circ$. Compute $\angle BAD$.

Luis González

#2 Jul 25, 2009, 12:58 am

Proof without words

Attachments:



Quick Reply

High School Olympiads

zig-zag in circle

Reply

hollandman

#1 Jul 21, 2009, 3:01 pm

Let A, B, C, D, E be points on a circle such that $\angle ABC = \angle BCD = \angle CDE = 45^\circ$. Prove that the region above the zig-zag line $ABCDE$ and the region below it have equal areas.

Luis González

#2 Jul 24, 2009, 12:28 pm

Let $X \equiv CB \cap AE, Y = CD \cap AE.$ Then $\triangle CAE$ is right with $CA = CE = L.$ Now, we shall prove that the area of $\triangle CXY$ equals the sum of the areas of $\triangle ABX$ and $\triangle EDY.$ Since these triangles are similar, because of $AB \parallel CY$ and $ED \parallel CX,$ we shall show $XY^2 = AX^2 + EY^2.$ $\triangle CXY \sim \triangle AYC \sim \triangle EXC$ yields

$$\frac{AY}{L} = \frac{L}{EX} \Longrightarrow AY \cdot EX = L^2 = \frac{1}{2}AE^2$$

$$2(AX + XY)(EY + XY) = (XY + AX + EY)^2 \Longrightarrow XY^2 = AX^2 + EY^2.$$

Quick Reply

High School Olympiads

Hyperbola

Reply

Source: constant angle (not so easy)

77ant

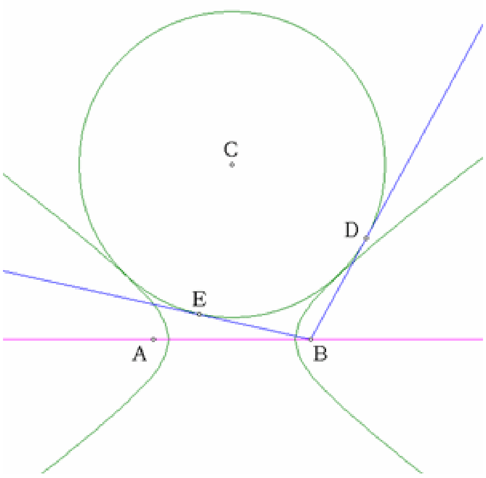
#1 Jul 23, 2009, 12:26 am

Hi, everyone.

There is a hyperbola with two foci A and B, and infinite many circles (with center C) tangent to the hyperbola. draw two tangent lines BD, BE from a focus B.
Prove that $\angle DBE$ is always constant.

with synthetic proofs (no analytic, no trigonometric)

Attachments:



yetti

#2 Jul 23, 2009, 1:40 am

By symmetry, circle (C) is tangent to the hyperbola at 2 points S, T symmetrical WRT the hyperbola minor axis. $ST \parallel AB$ and $AS = BT \Longrightarrow ASTB$ is isosceles trapezoid, cyclic, let (P) be its circumcircle centered on the minor axis. The hyperbola tangent at T is also tangent of (C). This tangent bisects $\angle ATB \Longrightarrow$ it cuts (P) again at X on the minor axis. Angles $\angle XTC$ is right $\Longrightarrow C \in (P)$. Then BC bisects $\angle SBT$. Let $Q \equiv BC \cap ST \Longrightarrow \frac{SB}{SQ} = \frac{TB}{TQ}$. Since the hyperbola is locus of points with equal ratios of distances from the focus B and corresponding directrix b, it follows that $Q \in b \Longrightarrow \frac{TB}{TQ} = \epsilon = \text{const}$ is equal to the hyperbola excentricity for any (C). Triangles $\triangle BTC \sim \triangle TQC$ are similar, having equal angles $\Longrightarrow \frac{BC}{TC} = \frac{TB}{TQ} = \epsilon$. This means that the figure $(C) \cup B$ remains similar for any (C) and $\angle DBE = \text{const}$.

Click to reveal hidden text

77ant

#3 Jul 23, 2009, 1:57 am

Thanks, yetti

I am unable to take my eyes off your great solution!

Luis González

#4 Jul 23, 2009, 2:29 am

By symmetry C lies on the perpendicular bisector of AB and the normal line to the hyperbola at P. In other words, C lies on the circumcircle of $\triangle PAB$ being the midpoint of the arc APB. We must prove that ratio beetwen the distance BC and the radius $CD=CP=r$ is constant. Indeed, by Ptolemy's theorem for the cyclic quadrilateral $ABPC$, keeping in mind that $CB=CA$, we have

$$r \cdot AB + PB \cdot CB = CB \cdot AP \Longrightarrow \frac{r}{CB} = \frac{AP-PB}{AB} = \text{const}.$$

yetti

#5 Jul 23, 2009, 7:43 am

> armpist wrote:
>
> ...

Your previous post was spam and your next post was a lie.

Quick Reply

High School Olympiads

A concurrent problem

Reply

ll931110

#1 Jul 10, 2009, 7:02 pm

Let ABC be a triangle. D, E, F are in BC, CA, AB, respectively such that AD, BE, CF are concurrent. Denote A_1, B_1, C_1 be midpoints of BC, CA, AB and D_1, E_1, F_1 be midpoints of EF, FD, DE, respectively.

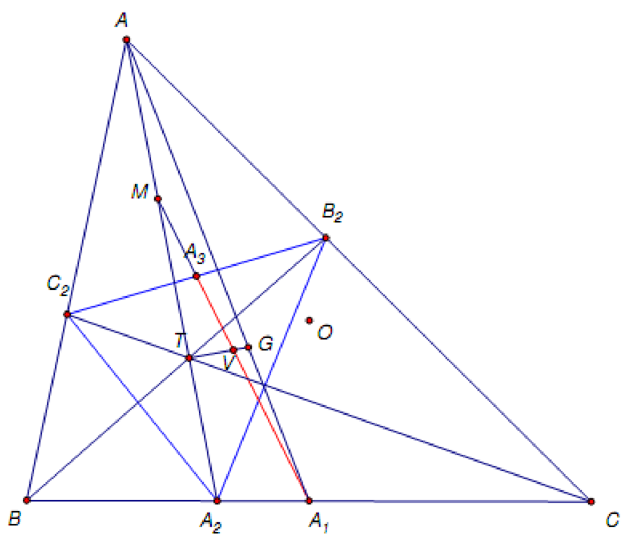
Show that: A_1D_1, B_1E_1, C_1F_1 are concurrent./.

livetolove212

#2 Jul 10, 2009, 7:42 pm

First, I will change the names of points such that available to my solution
Let T be the intersection of AA_2, BB_2, CC_2, G be the centroid of triangle ABC.
Applying **Gauss**'s line, we have M, A_3, A_1 are collinear.
Applying **Menelaus**'s theorem for triangle ATG with line (MVA_1) we get:
$\frac{VG}{VT}.\frac{MT}{MA}.\frac{A_1A}{A_1G}=1$
Then $\frac{VG}{VT}=\frac{1}{3}$
Similarly we obtain B_1B_3, C_1C_3 also pass through V.
And we are done.

Attachments:



mathVNpro

#3 Jul 10, 2009, 11:30 pm

First, let me restate the problem so that it may fits my solution.

PROBLEM-Let ABC be a triangle and A_0, B_0, C_0 respectively be the point such that AA_0, BB_0, CC_0 are concurrent at J. Let M_a, M_b, M_c respectively be the midpoints of BC, CA, AB; P_a, P_b, P_c respectively are the midpoints of B_0C_0, C_0A_0, A_0B_0. Prove that M_aP_a, M_bP_b, M_cP_c are concurrent.

Solution

Let N_a, N_b, N_c respectively be the midpoints of JA, JB, JC. Let H_1, H_2 respectively be the orthocenters of $\triangle C_0BJ$,

$\triangle B_0CJ$. Let C_1, J_1, B_1 respetively be the projections of H_1 onto BJ, BC_0, JC_0. Let B_2, C_2, J_2 be the projections of H_2 onto CJ, JB_0, CB_0. It is followed then $H_1C_1.H_1C_0 = H_1B_1.H_1B = H_1J_1.H_1J$ and $H_2C_2.H_1C = H_2B_2.H_2B_0 = H_2J_2.H_2J$, which implies that $\mathcal{P}_{H_1/(M_a)} = \mathcal{P}_{H_1/(N_a)} = \mathcal{P}_{H_1/(P_a)}$, where $(M_a), (N_a), (P_a)$ repsetively are the circles with diameters BC, AJ, B_0C_0. Simliarly for H_2. Thus H_1H_2 are the radical axes of $(M_a), (N_a), (P_a) \Longrightarrow M_a, N_a, P_a$ are collinear. With the same argument, we also have (M_b, N_b, P_b), (M_c, N_c, P_c) are collinear. Let d_a be the line which contains M_a, N_a, P_a, define the same for d_b, d_c.

Consider the homothety with center J, ratio $k_1 = 2$. We obtain $\mathcal{H}(J, k_1) : N_a \mapsto A$, $N_b \mapsto B$, $N_c \mapsto C$. Therefore $\mathcal{H}(J, k_1) : \triangle N_aN_cN_c \mapsto \triangle ABC$. Now consider the homothety with center G- the centroid of $\triangle ABC$, ratio $k_2 = -\frac{1}{2}$, we obtain $\mathcal{H}(G, k_2) : A \mapsto M_a$, $B \mapsto M_b$, $C \mapsto M_c$. Hence $\mathcal{H}(G, k_2) : \triangle ABC \mapsto \triangle M_aM_bM_c$. Therefore $\mathcal{H}(G, k_2) \circ \mathcal{H}(J, k_1) : \triangle N_aN_bN_c \mapsto M_aM_bM_c$. But $\mathcal{H}(G, k_2) \circ \mathcal{H}(J, k_1) = \mathcal{H}(G', -1)$. Hence M_aN_a, M_bN_b, M_cN_c are concurrent $\Longrightarrow d_a, d_b, d_c$ are concurrent at G', which implies to the result of the problem. $\square$

Attachments:

concurrent.pdf (27kb)

Luis González

#4 Jul 12, 2009, 1:36 am

There is another interesting result about this configuration.

Proposition. If P is the concurrency point of of the cevians AD, BE, CF, then the lines AD_1, BE_1, CF_1 concur in the center of the inconic $\mathcal{K}$ with perspector P.

Proof. The concurrency of AD_1, BE_1, CF_1 follows immediatly by the Cevian Nest Theorem. Using barycentric coordinates with respect $\triangle ABC$, we have

$$P\ (u : v : w)\ ,\ D\ (0 : v : w)\ ,\ E\ (u : 0 : w)\ ,\ F\ (u : v : 0)$$

$$D_1 \left(\frac{2u + w + v}{wv} : \frac{1}{u} + \frac{1}{w} : \frac{1}{u} + \frac{1}{v} \right)$$

$$E_1 \left(\frac{1}{v} + \frac{1}{w} : \frac{u + 2v + w}{uw} : \frac{1}{u} + \frac{1}{v} \right)$$

$$F_1 \left(\frac{1}{v} + \frac{1}{w} : \frac{1}{u} + \frac{1}{w} : \frac{u + v + 2w}{uv} \right)$$

Therefore, triangles $\triangle ABC$ and $\triangle D_1E_1F_1$ are perspective through

$$U \left(\frac{1}{v} + \frac{1}{w} : \frac{1}{u} + \frac{1}{w} : \frac{1}{u} + \frac{1}{v} \right)$$

Complement of the isotomic conjugate of P, i.e. center of the inconic $\mathcal{K}$ with perspector P.

thaithuan_GC

#5 Jul 13, 2009, 9:13 pm

> *livetolove212 wrote:*
>
> First, I will change the names of points such that available to my solution
> Let T be the intersection of AA_2, BB_2, CC_2, G be the centroid of triangle ABC.
> Applying **Gauss**'s line, we have M, A_3, A_1 are collinear.
> Applying **Menelaus**'s theorem for triangle ATG with line (MVA_1) we get:
> $\frac{VG}{VT} \cdot \frac{MT}{MA} \cdot \frac{A_1A}{A_1G} = 1$
> Then $\frac{VG}{VT} = \frac{1}{3}$
> Similarly we obtain B_1B_3, C_1C_3 also pass through V.
> And we are done.

It's not your problem. It has been posted at http://www.mathlinks.ro/viewtopic.php?t=214231 .

vittasko

#6 Jul 13, 2009, 9:38 pm

See also at the topic Concurrent lines II, of **April**.

Kostas Vittas.

mathVNpro

#7 Jul 13, 2009, 9:51 pm

> *vittasko wrote:*
>
> See also at the topic Concurrent lines II, of **April**.
> Kostas Vittas.

> *vittasko wrote:*
>
> An interesting result in this configuration, is that the points $P, Q, G,$ where G is the centroid of $\triangle ABC,$ are collinear such that $PQ = 3(QG),$ mentioned by **Little Gauss,** in a **gemath's** problem at http://www.mathlinks.ro/Forum/viewtopic.php?t=132528
> Kostas Vittas.

The above property that you mentioned above is obvious in my proof for this problem at this topic. Since $\mathcal{H}(G, k_2) \circ \mathcal{H}(J, k_1) = \mathcal{H}(G', -1)$, where G' is the concurrent point of d_a, d_b, d_c. Hence G', J, G are concurrent

livetolove212

#8 Jul 14, 2009, 3:57 pm

> *thaithuan_GC wrote:*
>
> It's not your problem. It has been posted at http://www.mathlinks.ro/viewtopic.php?t=214231 .

Sorry I don't know it was posted on Mathlinks before. Last week I discovered it and then my friend **II931110** post at here

livetolove212

#9 Jul 22, 2009, 2:43 pm

More interesting result: Denote G_a, G_b, G_c the centroid of TBC, TCA, TAB. Prove that $AG_a, BG_b, CG_c, A_1A_3, B_1B_3, C_1C_3$ are concurrent.

Luis González

#10 Jul 22, 2009, 11:49 pm

> *livetolove212 wrote:*
>
> More interesting result: Denote G_a, G_b, G_c the centroid of $TBC, TCA, TAB.$ Prove that $AG_a, BG_b, CG_c, A_1A_3,$ B_1B_3, C_1C_3 are concurrent.

Using the same notation of your figure above. The centroid G_a of $\triangle TBC$ has barycentric coordinates $G_a(u : u+2v+w : u+v+2w).$ Thus AG_a passes through $V\ (2u+v+w : u+2v+w : u+v+2w)$ and this point satisfies the equation of the Newton line of the quadrangle $TB_2AC_2,$ namely

$$\mathcal{N}_a \equiv (w-v)x + (2u+v+w)(y-z) = 0$$

Thus, $AG_a, BG_b, CG_c, \mathcal{N}_a, \mathcal{N}_b, \mathcal{N}_c$ concur at the complement of the complement of $T.$

vittasko

#11 Jul 23, 2009, 2:24 am

> *livetolove212 wrote:*
>
> More interesting result: Denote G_a, G_b, G_c the centroid of TBC, TCA, TAB. Prove that

More interesting result: Denote G_a, G_b, G_c the centroid of TBC, TCA, TAB. Prove that $AG_a, BG_b, CG_c, A_1A_3, B_1B_3, C_1C_3$ are concurrent.

• Because of $TG_b = 2G_bB_1$ and $TG_c = 2G_cC_1$, we conclude that the lines through $G_b,\ G_c$ and parallel to $AC,\ AB$ respectively, intersect the line segment AT, at the same point so be it A', such that $TA' = 2A'A$.

It is easy to show now, that the points $B' \equiv TB \cap A'G_c$ and G_a and $C' \equiv TC \cap A'G_b$, are collinear and $B'C' \parallel BC$.

Because of $\triangle G_aG_bG_c$ is the medial triangle of $\triangle A'B'C'$, we have that $G_bG_c \parallel B'C' \Longrightarrow G_bG_c \parallel BC$ and similarly, $G_aG_b \parallel AB$ and $G_aG_c \parallel AC$.

So, the triangles $\triangle G_aG_bG_c,\ \triangle ABC$ are perspective and based on the **Desarques theorem**, we conclude that the line segments $AG_a,\ BG_b,\ CG_c$ are concurrent at one point so be it P.

• We will prove now, that the line segment A_1A_3, where $A_1,\ A_3$ are the midpoints of the segments $BC,\ B_2C_2$ respectively, passes through the point P.

Because of $G_aG_b \parallel AB$, based on the **Thales theorem**, we have that $\dfrac{PA}{PG_a} = \dfrac{AB}{G_aG_b} = 3\,,(1)$ because of $AB = \dfrac{3}{2}A'B'$ and $G_aG_b = \dfrac{1}{2}A'B'$

Let be the point $M \equiv AT \cap A_1P$ and applying the **Menelaos theorem**, in triangle $\triangle ATG_a$ with transversal A_1PM, we have that $\dfrac{MA}{MT} \cdot \dfrac{A_1T}{A_1G_a} \cdot \dfrac{PG_a}{PA} = 1 \Longrightarrow MA = MT\,,(2)$

• Because of now, the complete quadrilateral AB_2TC_2BC, we conclude that the line segment $A_1P \equiv A_1M$, connecting the midpoints $A_1,\ M$ of $BC,\ AT$ respectively, based on the **Newton theorem**, passes through the midpoint A_3 of B_2C_2.

That is, the line segment A_1A_3 as the problem states, passes through the point $P \equiv AG_a \cap BG_b \cap CG_c$ and similarly for the line segments B_1B_3 and C_1C_3.

Hence, the line segments $AG_a,\ BG_b,\ CG_c,\ A_1A_3,\ B_1B_3,\ C_1C_3$, are concurrent at point P and the proof is completed.

Kostas Vittas.

livetolove212

#12 Jul 23, 2009, 6:57 am

Here is my solution:
Let M be the midpoint of BC, G be the centroid of triangle ABC, $AG_a \cap TG = \{V\}$
Applying **Menelaus**'s theorem: $\dfrac{G_aT}{G_aM}.\dfrac{AM}{AG}.\dfrac{VG}{VT} = 1$
$\Rightarrow \dfrac{VG}{VT} = \dfrac{1}{3}$...

Quick Reply

High School Olympiads

Perpendicular to bases drawn through P meets line BQ at K

Reply

Source: Tuymaada 2009, Junior League, Second Day, Problem 2

orl
#1 Jul 20, 2009, 5:46 am

M is the midpoint of base BC in a trapezoid $ABCD$. A point P is chosen on the base AD. The line PM meets the line CD at a point Q such that C lies between Q and D. The perpendicular to the bases drawn through P meets the line BQ at K. Prove that $\angle QBC = \angle KDA$.

Proposed by S. Berlov

Luis González
#2 Jul 20, 2009, 6:17 am

Such an easy problem

Let $N \equiv QB \cap AD.$ Then QM is the Q-median of $\triangle QBC.$ Since $BC \parallel ND,$ then P is midpoint of $ND.$ The triangle $\triangle NDK$ is isosceles. So if $R \equiv KD \cap BC,$ then $\triangle KBR$ is also isosceles $\Longrightarrow \angle QBC = \angle BRK = \angle KDA.$

hvaz
#3 Jun 19, 2010, 7:27 am

Let $B' = QB$ and $R = KD$.
Notice that we only need to prove that $\triangle KB'D$ is isosceles, because if this happens, $\triangle KBM$ is also isosceles(because $B'D//BC$), and so $\angle KBR = \angle QBC = \angle KRB = \angle CRD = \angle RDA = \angle KDA$.
But $BC//B'D$, and so there're an homothety that goes from $\triangle QBC$ to $\triangle QND$, and if M is the midpoint of BC, P is the midpoint of $B'D$, what implies that $\triangle KND$ is isosceles.

Q.E.D.

Tharun270
#4 Jun 19, 2010, 10:59 am

Angle ADK=Angle DMC=Angle KXM=Angle KBM.

This is just one situation. X is the point of intersection between DK and BC.

sunken rock
#5 Jun 19, 2010, 9:18 pm

From **Luis's** solution we get KP external bisector of angle $\angle QKD$, hence the parallel to the bases of trapezoid through K is the internal angle bisector of $\angle QKD$, done.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Circumcenters, DAX = DAH

Reply

Source: USA TST 2009 #2

MellowMelon

#1 Jul 19, 2009, 4:40 am

Let ABC be an acute triangle. Point D lies on side BC. Let O_B, O_C be the circumcenters of triangles ABD and ACD, respectively. Suppose that the points B, C, O_B, O_C lies on a circle centered at X. Let H be the orthocenter of triangle ABC. Prove that $\angle DAX = \angle DAH$.

Zuming Feng.

This post has been edited 1 time. Last edited by MellowMelon, Jul 19, 2009, 11:49 am

limes123

#2 Jul 19, 2009, 7:18 am

Let $BO_B \cap CO_C = Y$. We have $\angle CBO_B = 90^\circ - \angle DAB$ and $BCO_C = 90^\circ - \angle DAC$ hence $\angle BYC = 180^\circ - (90^\circ - \angle DAB) - (BCO_C = 90^\circ - \angle DAC = \angle BAC$ which means that point Y lies on the circumcircle of triangle ABC (1). A simple angle chase shows that $\angle BAO_B = \angle CAO_C$ which implies $\angle O_BAO_C = \angle BAC$ and point Y lies on the circumcircle of triangle AO_BO_C (2). From (1) and (2) we have AY - radical axis of circles (ABC) and (AO_BO_C), O_BO_C - radical axis of circles (BCO_CO_B) and (AO_BO_C) and BC - radical axis of (ABC) and (BCO_CO_B) and these three lines are concurrent in point R. Let $AH \cap BC = T$ and $O_BO_C \cap AD = Q$. We know that AD is radical axis of circles (ABD) and (ACD) hence $O_BO_C \perp QR$ and $AT \perp RT$ which implies that $RTQA$ is cyclic quadrilateral and $\angle DAH = \angle RAT = \angle DRQ$ (3). We know that $AX \perp AY$ (4) (it was posted on the forum, I'll write proof later). A simple angle chase shows that RBO_BA is cyclic quadrilateral and $BO_B = AO_B$ hence QR is angle bisector of $\angle DRA$ which combined with $AD \perp QR$ implies $AR = DR$ and $\angle RDA = \angle DAR$ (5). Let $AX \cap QR = S$. Using (4) and (5) we get $\angle RDA = \angle DAR = \angle RSA$ (because $\triangle RQA$ and $\triangle RAS$ are similar) hence $RDSA$ is cyclic. Using this and (3) we get $\angle DAX = \angle DAS = \angle DRS = \angle TRQ = \angle TAQ = \angle HAD$ qed.

We can also prove that BO_C, CO_B and AX are concurrent.

yetti

#3 Jul 19, 2009, 4:06 pm

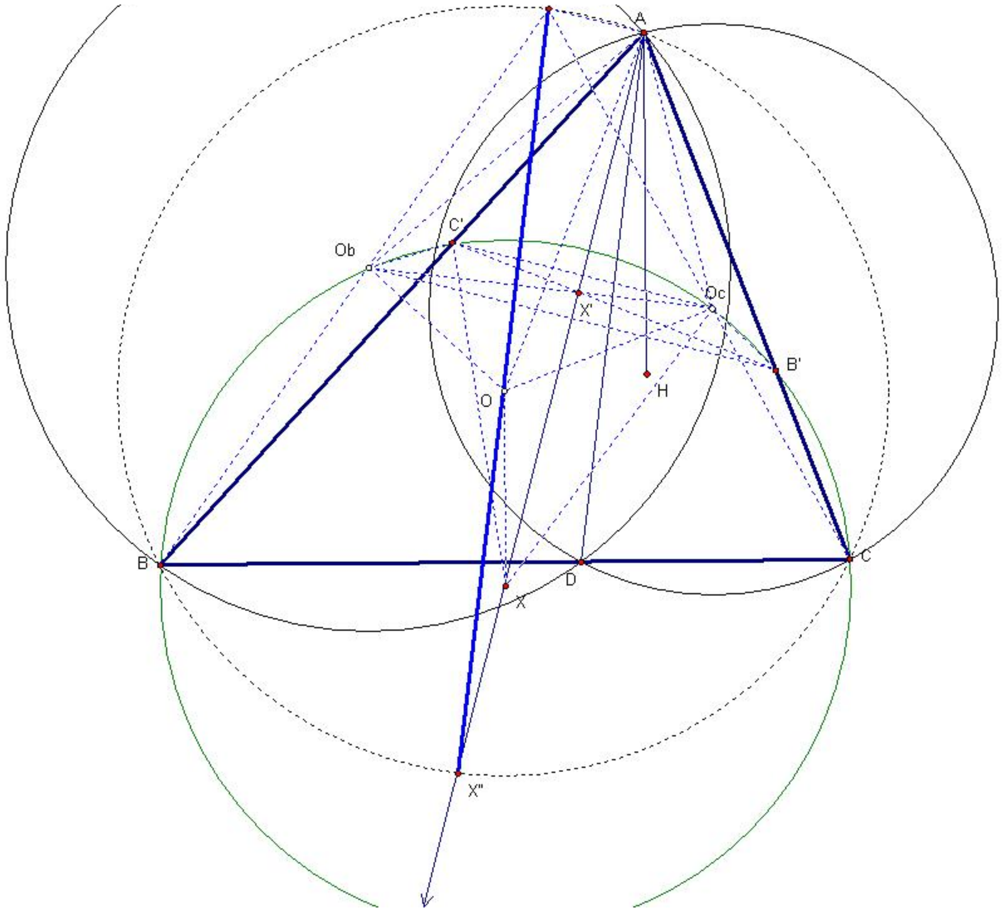
Let $F \in BC$ be foot of AH. Let A', B', C', D', F' be midpoints of BC, CA, AB, AD, AF. O_BO_C is perpendicular bisector of $AD \Longrightarrow \angle AO_BO_C = \angle ABD = \angle ABC$ and $\angle AO_CO_B = \angle ACD = \angle ACB \Longrightarrow$ $\triangle AO_BO_C \sim \triangle ABC \sim \triangle AC'B'$ for any $D \in BC$. Let K, M be midpoints of $O_BO_C, C'B'$ and $AD'KL, AF'MN$ rectangles; KL, MN are perpendicular bisectors of $O_BO_C, C'B'$, respectively. Since D', F' are A-altitude feet of $\triangle AO_BO_C \sim \triangle AC'B'$, these 2 rectangles are similar $\Longrightarrow \triangle ALN \sim \triangle AD'F'$ are similar by SAS and $LN \perp (AN \parallel B'C')$. Consequently, foot L of perpendicular AL to KL from A is on midparallel MN of $AH \parallel OA'$. This means that the perpendicular bisector KL of O_BO_C is tangent to a parabola with focus A, vertex tangent MN and directrix OA'. If KL cuts the directrix OA' at X, the parabola tangent KL bisects the angle $\angle OXA$. Since $AD \parallel KL \Longrightarrow AD$ bisects the angle $\angle XAH$.
This is true regardless of BCO_CO_B being cyclic or not. But if it happens to be cyclic, the intersection X of the perpendicular bisectors OA', KL of BC, O_BO_C is its circumcenter.

plane geometry

#4 Jul 19, 2009, 4:32 pm

I will post the solution later in mind

Attachments:



This post has been edited 1 time. Last edited by plane geometry, Jul 19, 2009, 5:17 pm

plane geometry

#5 Jul 19, 2009, 5:17 pm

Denote X' is the intersection of B'C' and ObOc
□ABOb□□ACOc => □ObBC'=□OcCB' => C'OcB'Ob is an isosceles trapezoid
=> ObOc=B'C'
□AC'B'□□ACB □AOcOb□□ACB =>
B'C'/BC=AB'/AC= ObOc/BC=AOc/AC => AB'=AOc
Analogously, we have AOb=AC'
AX' is the bisector of □C'AOc
On the other hand, we have AX is the bisector of □C'AOc
Thus A,X,X' are collinear
By angle chasing □A'AB+□C'AC=90 which indicates A',O,X''are collinear
Therefore □OX''A=□OAX=□ABA'=□DAH (1)
Notice □OAX=□X'AC'-□OAB=□X'Oc-□OcAD=□DAX (2)
Form (1)(2) we get the result

Zhang Fangyu

#6 Jul 19, 2009, 7:32 pm

> *plane geometry wrote:*
>
> Denote X' is the intersection of $B'C'$ and O_bO_c
> $\triangle ABO_b \sim \triangle ACOc \Rightarrow \angle O_bBC' = \angle O_cCB' \Rightarrow C'O_cB'O_b$is an isosceles trapezoid
> $\Rightarrow O_bO_c = B'C'$
> $\triangle AC'B' \sim \triangle ACB \triangle AO_cO_b \sim \triangle ACB \Rightarrow \dfrac{B'C'}{BC} = \dfrac{AB}{AC} = \dfrac{O_bO_c}{BC} = \dfrac{AO_c}{AC} \Rightarrow AB = AO_c$
> Analogously, we have $AO_b = AC'$
> AX' is the bisector of $\angle C'AO_c$
> On the other hand, we have AX is the bisector of $\angle C'AO_c$
> Thus A, X, X' are collinear
> By angle chasing $\angle A'AB + \angle C'AC = 90^0$ which indicates A', O, X'' are collinear
> Therefore$\angle OX''A + \angle OAX + \angle ABA' + \angle DAH(1)$
> Notice $\angle OAX = \angle X'AC' - \angle OAB = \angle X'AO_c - \angle O_cAD = \angle DAX(2)$
> From$(1)(2)$ we get the result

Very nice solution! Dear plane geometry

mathVNpro
#7 Jul 19, 2009, 11:56 pm

> *MellowMelon wrote:*
> Let ABC be an acute triangle. Point D lies on side BC. Let O_B, O_C be the circumcenters of triangles ABD and ACD, respectively. Suppose that the points B, C, O_B, O_C lies on a circle centered at X. Let H be the orthocenter of triangle ABC. Prove that $\angle DAX = \angle DAH$.
>
> *Zuming Feng.*

Dear mathlinkers! How can we construct D on BC such that O_B, O_C, B, C are on a circle where O_B, O_C is the circumcenters of triangles ABD and ACD, respectively?

Luis González
#8 Jul 20, 2009, 12:24 am

> *mathVNpro wrote:*
> Dear mathlinkers! How can we construct D on BC such that O_B, O_C, B, C are on a circle where O_B, O_C is the circumcenters of triangles ABD and ACD, respectively?

The point D is not constructible (in general). Calculate the angles $\angle CAD$ and $\angle BAD$ in terms of the angles of $\triangle ABC$, then realize that the construction is equivalent to the angle trisection.

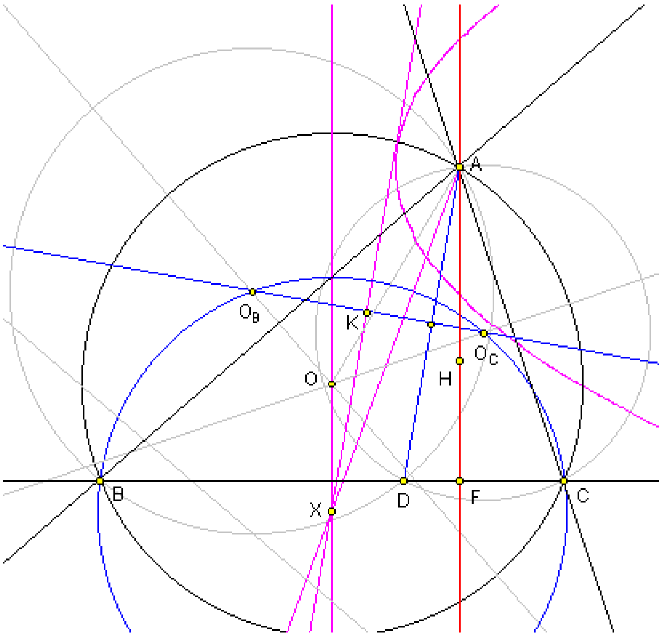
yetti
#9 Jul 20, 2009, 3:23 am

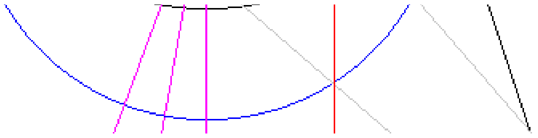
> *Luis González wrote:*
> ...the construction is equivalent to the angle trisection.

Calulate angle $\theta = \angle DAH$:
$\angle O_BBC = \theta + \angle ABC$ and $\angle BCO_C = \angle BCA - \theta$,
$\angle O_CO_BB = \pi - (\angle ABC + 2\theta)$ and $\angle CO_CO_B = \pi - (\angle BCA - 2\theta)$.
BCO_CO_B is cyclic $\Longleftrightarrow \angle O_BBC + \angle CO_CO_B = \pi \Longleftrightarrow \theta = \frac{1}{3}(\angle BCA - \angle ABC)$.

Even though D is not constructible for a given $\triangle ABC$, you can still construct some $\triangle ABC$ with BCO_CO_B cyclic, given $\triangle ABD$. (If $F \in BD$ is A-altitude foot, pick D, so that $\angle ABD + 3\angle DAF < \frac{\pi}{2}$.)

Attachments:



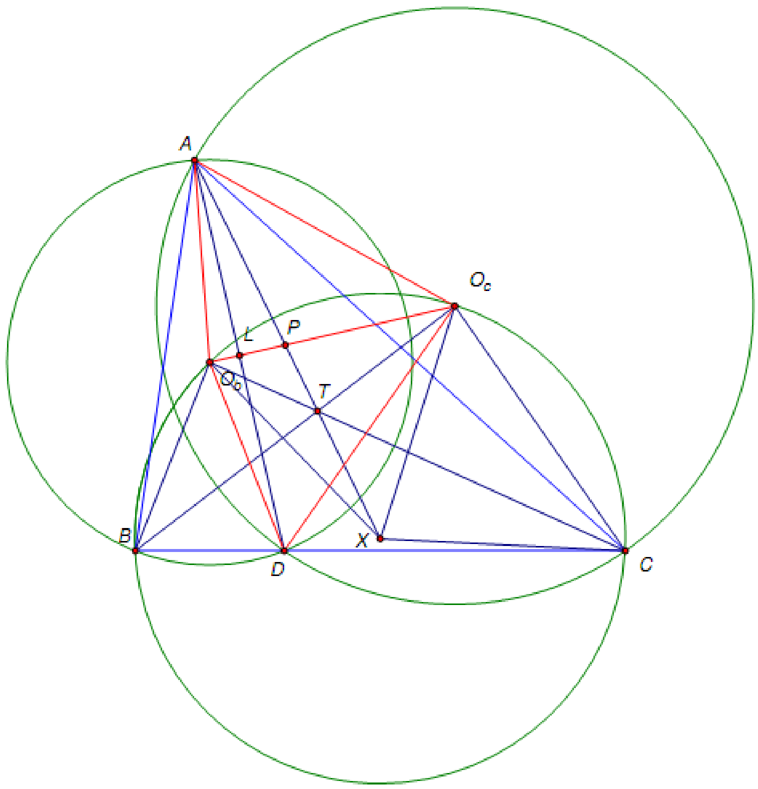


livetolove212

#10 Jul 20, 2009, 7:37 am

We know that AX is the bisector of $\angle BAO_c$ and $\Delta AO_bO_c \sim \Delta ABC$
On the other side, denote T the intersection of BO_c and CO_b.
We have $\frac{O_bT}{O_cT}=\frac{BO_b}{CO_c}$
$\Rightarrow \frac{O_bT}{TC}=\frac{R_b}{R_c}.\frac{O_cT}{TC}=\frac{R_b}{R_c}.\frac{O_bO_c}{BC}=\frac{R_b}{AC}=\frac{AO_b}{AC}$
So AT is the bisector of $\angle O_bAC$ or $\angle BAO_c$
We obtain AX, BO_b, CO_c are concurrent.
Applying Tri-Ceva theorem for triangle O_bO_cX we have:
$\frac{sin\angle O_cXT}{sin\angle O_bXT}.\frac{sin\angle XO_bT}{sin\angle O_cO_bT}.\frac{sin\angle O_bO_cT}{sin\angle XO_cT}=1$
Note that $XO_b=XO_c$ so $\frac{sin\angle O_cXT}{sin\angle O_bXT}=\frac{PO_c}{PO_b}$
And $\frac{sin\angle XO_bT}{sin\angle O_cO_bT}.\frac{sin\angle O_bO_cT}{sin\angle XO_cT}=\frac{sin\angle TCB}{sin\angle TBC}.\frac{cos\angle O_bBD}{cos\angle O_cCD}=\frac{TB}{TC}.\frac{cos\angle O_bBD}{cos\angle O_cCD}$
$=\frac{R_b}{R_c}.\frac{cos\angle O_bBD}{cos\angle O_cCD}=\frac{BD}{CD}$
Therefore $\frac{PO_c}{PO_b}=\frac{CD}{BD}$
We get $\Delta ABD \sim \Delta AO_bP$ then $\angle BAD=\angle O_bAP \Rightarrow \angle DAX=\angle BAO_b=\angle DAH$

Attachments:



mathVNpro

#11 Sep 3, 2009, 2:46 pm

> *Quote:*
>
> Let ABC be an acute triangle. Point D lies on side BC. Let O_B, O_C be the circumcenters of triangles ABD and ACD, respectively. Suppose that the points B, C, O_B, O_C lies on a circle centered at X. Let H be the orthocenter of triangle ABC. Prove that $\angle DAX = \angle DAH$.

Zuming Feng.

We have $\angle AO_BB = 2(180^\circ - \angle ADB) = 2\angle ADC = \angle AO_CC$. In the other hand, since $\triangle AO_BB$ and $\triangle AO_CC$ are all issoceles triangles respectively with respect to vertexes O_B, O_C. Hence it exists a spiral similarity through center A, let denote this transformation by f_A such that $f_A : \triangle AO_BB \mapsto \triangle AO_CC$ $(*)$. Thus $f_A : O_B \mapsto O_C$ and $B \mapsto C$. Therefore, it also exists a spiral similarity through A, let denote this by g_a such that $g_A : O_b \mapsto B, O_c \mapsto C \Longrightarrow O_BO_C \mapsto BC$, which implies that $\triangle AO_BO_C \mapsto \triangle ABC$ $(**)$. From $(*)$ and $(**)$, we have $(O_BO_C, BC) = (AO_c, AC)$ and $(O_BB, O_CC) = (AB, AC)$. Thus, if $P \equiv BO_B \cap CO_C$ and $Q \equiv O_BO_C \cap BC$ then we have $\{A, O_C, C, Q\}$ and $\{P, B, C, A\}$ are two sets of concyclic points, which leads to the fact that A is the *Miquel point* with respect to the complete quadrilateral (PB, PC, QO_B, QB). Note that OO_BO_CC is concyclic $\Longrightarrow A, P, Q$ are collinear. Further, if $G \equiv CO_B \cap BO_C$ then PQ is the polar of G with respect to $(O_A) \equiv (BO_BO_CC)$, O, A, G are collinear and $\overline{OGA} \perp PQ$ at $A \Longrightarrow \angle PAO_A = \angle O_AAQ = 90^\circ$.

In the other hand, we have $\angle DAH_A = \angle DAC - \angle H_AAC = \angle DAC - (90^\circ - \angle ACD)$. Also, $\angle DAO_A = \angle PAD - \angle PAO_A = \angle PAD - 90^\circ$. Therefore, in order to prove that $\angle DAH_A = \angle DAO_A$, we need to prove that $\angle PAD = \angle DAC + \angle ACD = 180^\circ - \angle ADQ = 180^\circ - \angle QAD$, which equivalent to $\angle ADQ = \angle QAD$, which is obviously true since $\overline{O_BO_CQ}$ bisects AD and $Q \equiv PA \cap O_BO_C$. Our proof is completed then. $\square$

Attachments:

USA_TST09.pdf (20kb)

math154

#12 May 23, 2011, 9:58 am

WLOG $B \ge C$.

First, by trivial angle chasing, $\angle BO_BO_C = B + 2\angle BAD$ while $\angle O_BBC = 90^\circ - \angle BAD$, so because BO_BO_CC is cyclic, $\angle BAD = 90^\circ - \dfrac{2B+C}{3}$ and $\angle CAD = 90^\circ - \dfrac{B+2C}{3}$. We find that $\angle DAH = \angle DAB - \angle HAB = \dfrac{B-C}{3}$, so it remains to show that $\angle DAX = \dfrac{B-C}{3}$. (*)

Note that $\angle O_BXC = 2\angle O_BBC = \dfrac{4B+2C}{3}$ and $\angle O_CXB = 2\angle O_CCB = \dfrac{2B+4C}{3}$.

However, $\angle O_BAC = \angle O_BAD + \angle CAD = 180^\circ - \dfrac{4B+2C}{3}$, so by symmetry, $ABXO_C$ and $ACXO_B$ are cyclic.

Finally,

$$\begin{aligned}\angle DAX + \angle DAB = \angle XAB = \angle XO_CB \\ = \angle BO_BO_C - 90^\circ = 90^\circ - \frac{B+2C}{3} = \angle DAB + \frac{B-C}{3},\end{aligned}$$

so we're done by (*).

subham1729

#13 Apr 25, 2013, 12:53 pm • 1

Suppose radius of $\odot BAD = r_1$ and of $\odot CAD = r_2$ and $\angle DAB = \theta$ now so $AD = 2r_1 SinB = 2r_2 Sinc$ also $\angle O_BDO_C = \pi - (B+C) = A$ and that implies O_BDO_C is similar to ABC and so $\angle O_CO_BD = B, \angle O_BO_CD = C$ now also note $B + 2\theta + C + \dfrac{B+\theta}{2} = \pi \implies \theta = \dfrac{\pi}{2} - \dfrac{2B+C}{3}$.Now suppose $\angle XBC = \alpha$ then $aSin(\alpha) = r_1 Sin(\dfrac{2B+C}{3} - \alpha)$ also $c = 2r_1 Cos(\dfrac{B-C}{3})$, using this we get $\dfrac{Sin(B-\alpha)}{Sin(C-\alpha)} = \dfrac{Cos(A-B)}{Cos(A-C)}$ now apply ceva with AX, BX, CX and note $\dfrac{Sin\angle XAC}{Sin\angle XAB} = \dfrac{Cos(A-B)}{Cos(A-C)} = \dfrac{Cos(\frac{C+2B}{3})}{Cos(\frac{B+2C}{3})}$ and so obviously $\angle XAB = \dfrac{\pi}{2} - \dfrac{B+2C}{3}, \angle XAC = \dfrac{\pi}{2} - \dfrac{2B+C}{3}$,and hence $\angle XAD = \dfrac{B-C}{3}$ and which is indeed $\angle DAH$,so done!

v_Enhance

#14 Apr 28, 2013, 10:26 am

Without loss of generality $AC > AB$. It is easy to verify via angle chasing that $\angle AO_BB = AO_CC$. Since O_BO_CCB is cyclic, it follows that A is the Miquel point of O_BO_CCB. Therefore, AO_CXB is cyclic.

Set $x = \angle BAD$, $y = \angle CAD$. Then

$\angle BO_BO_C = \angle BO_BD + \angle D_OBC = 2x + B \implies \angle BXC = 360 - 4x - 2B \implies \angle BAX = \angle BO_CX = 2x + B - 90.$

On the other hand, $\angle BAH = 90 - B$. From here it is easy to derive that $\angle HAD = x + B - 90 = \angle XAD$, as desired.

BBAI

#15 Apr 30, 2013, 1:23 am

> *v_Enhance wrote:*
>
> Without loss of generality $AC > AB$. It is easy to verify via angle chasing that $\angle AO_BB = AO_CC$. Since O_BO_CCB is cyclic, it follows that A is the Miquel point of O_BO_CCB. Therefore, AO_CXB is cyclic.
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>
> On the other hand, $\angle BAH = 90 - B$. From here it is easy to derive that $\angle HAD = x + B - 90 = \angle XAD$, as desired.

How AO_cXB is cyclic? Is it a property..

math154

#16 Apr 30, 2013, 1:29 am

Yes; see derpy.me/cyclicquad (Fact 12; thanks to Zhero for making this link)

Edit: made link clearer; also http://yufeizhao.com/olympiad/cyclic_quad.pdf

Linked the other one as well. -- mod

This post has been edited 2 times. Last edited by math154, Apr 30, 2013, 3:21 am

BBAI

#17 Apr 30, 2013, 1:48 am

The link isn't working .Paste it in URL Form

IDMasterz

#18 Feb 5, 2014, 2:23 pm

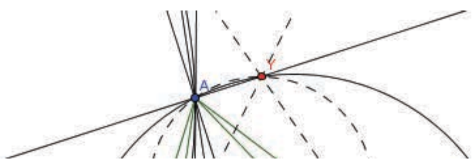
Note that $AO_BO_C \sim ABC$ by simple angle chasing, so let $Y = O_BO_C \cap BC$ and $Z = AD \cap O_BO_C$. Notice that by miquel's theorem, we have $Y \in ABO_B$ and ACO_C so since O_BO_CBC are concyclic, we have by radical axis theorem that BO_B, CO_C, AY concur, so A is miquel point of O_BO_CCB. Hence, $XA \perp AY$. If we let the midpoint of O_BO_C be M then $\angle DAX = \angle AOM = \angle AYM$ and clearly $\angle HAD = \angle MYD$ (Let $AH \cap BC = E$, then $AYEM$ are concyclic). So, we must show YM bisects $\angle AYE$, or $AM = MQ$ which follows since M is the circumcentre of AED.

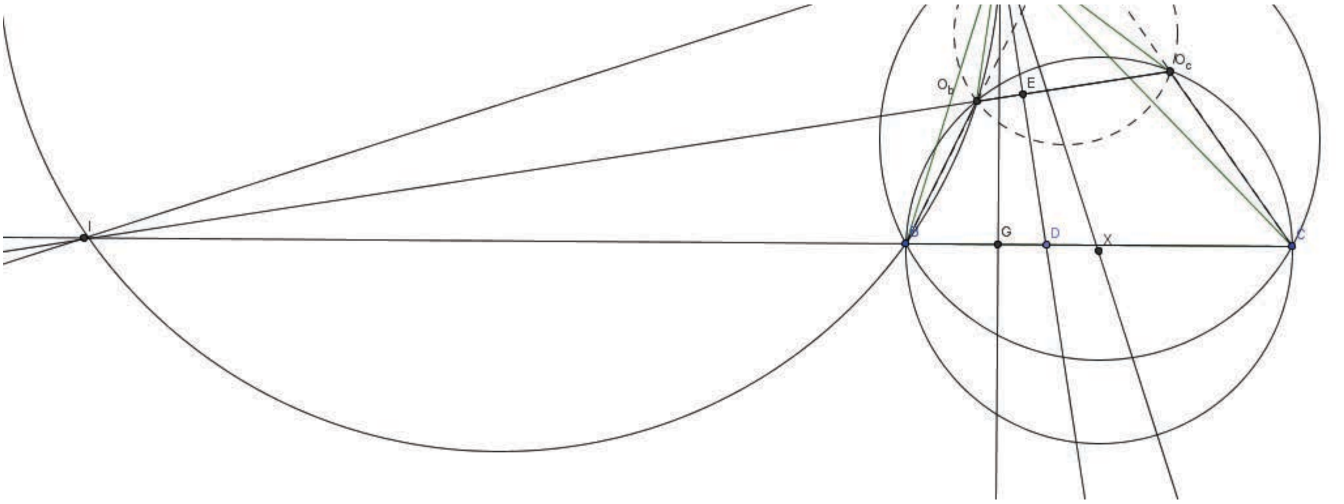
Mahi

#19 Jun 2, 2014, 10:16 pm

Sorry for a little glitch in the diagram ($AHG \perp BC, G = AH \cap BC$).
Let $BO_B \cap CO_C = Y$.
We have $AO_BO_CY, ABCY, BO_BO_CC$ concylic by a little angle chasing, and thus A is the miquel point of cyclic quadrilateral BO_BO_CC (by http://yufeizhao.com/olympiad/cyclic_quad.pdf)
So, we have $XA \perp YA$.
Now, let $BC \cap O_CO_B = I$. Then, AO_BBI is cyclic, and $AO_B = BO_B$ implies line IO_BO_C is the bisector of $\angle AIB$.
As $AH \perp IB, AD \perp IO_B$ and $AX \perp IA$, AD is the bisector of $\angle HAD$, and thus $\angle DAX = \angle DAH$.

Attachments:





sayantanchakraborty

#20 Dec 12, 2014, 11:30 am

Let J be the circumcenter of BO_BO_CC and O be the circumcenter of $\triangle ABC$.Note that $\angle AO_BO_C=\angle AOO_C=B$ so A,O_B,O,O_C lie on a circle.Also by some angle chasing we obtain that BO_BO_CC is cyclic yeilds that $\angle ADC=90+\frac{B-C}{3}$.Next note that $BO_B\cap CO_C=K$ lie on the circumcircle of ABC as well as AO_BO_C.Let r_1,r_2,R denote the circumradii of AO_BO_C,BO_BO_CC and ABC respectively.

$\angle BOJ=\angle BKO_C=A$ and $\angle OBJ=\angle OBC-\angle JBC=\angle BO_CC-\angle BKO_C=\angle KBO_C$.Thus $\triangle OBJ\sim\triangle KBO_C\implies\frac{OJ}{R}=\frac{KO_C}{KB}=\frac{O_BO_C}{BC}=\frac{r_1}{R}\implies OJ=r_1=\frac{R}{2sin\angle ADC}=\frac{R}{2cos\frac{(B-C)}{3}}$.

Extend AD and OJ so that they meet at X.We now get

$$\frac{AO}{AX}=\frac{OJ}{JX}$$
$$\Leftrightarrow\frac{OX}{OJ}=\frac{AX}{AO}+1$$
$$\Leftrightarrow\frac{2cos\frac{(B-C)}{3}\cdot R}{\frac{R}{2cos\frac{(B-C)}{3}}}=\frac{sin(B-C)}{sin\frac{(B-C)}{3}}+1$$
$$\Leftrightarrow 4cos^2\frac{(B-C)}{3}=4-4sin^2\frac{(B-C)}{3}$$
$$\Leftrightarrow cos^2\frac{(B-C)}{3}+sin^2\frac{(B-C)}{3}=1$$

which is an obvious identity.Therefore AJ bisects $\angle DAO$ and consequently $\angle DAJ=\angle DAH=\frac{B-C}{3}$.

Dukejukem

#21 Mar 23, 2015, 6:12 am

Notice that $\triangle AO_BB$ and $\triangle AO_CC$ are both isoceles. Furthermore, since O_B and O_C are circumcenters, we know that $\measuredangle AO_BB=2\measuredangle ADB=2\measuredangle ADC=\measuredangle AO_CC$ where the angles are directed. It follows that $\triangle AO_BB\sim\triangle AO_CC$ with similar orientation. Therefore, A is the center of spiral similarity that sends $\overline{O_BB}$ to $\overline{O_CC}$, so A is the Miquel point of cyclic quadrilateral $BO_BO_CC.$ Hence, it is well-known (and can be proven using harmonic divisions) that A is the inverse of $Y\equiv BO_C\cap CO_B$ with respect to $\odot(BO_BO_CC).$ Then since C,Y,O_B are collinear, under inversion in $\odot(BO_BO_CC)$, it follows that X,O_B,A,C are concyclic. Then because O_B is the circumcenter of $\triangle ABD$ and the points X,O_B,A,C are concyclic, we have

$$\measuredangle DAX=\measuredangle O_BAX-\measuredangle O_BAD=\measuredangle O_BCX-(90^\circ-\measuredangle DBA)=\measuredangle O_BCX+\measuredangle DBA-90^\circ.$$

Meanwhile, since $AH\perp BC$, we have $\measuredangle HAD=90^\circ-\measuredangle ADB.$ Therefore,

$$\measuredangle DAX=\measuredangle HAD\iff\measuredangle O_BCX+\measuredangle DBA+\measuredangle ADB=180^\circ=0.$$

Since the sum of the directed angles in $\triangle ABD$ is equal to 0, we need only show that $\measuredangle O_BCX=\measuredangle BAD.$ But because X is the circumcenter of $\odot(BO_BO_CC)$, we have

$$\measuredangle O_BCX=90^\circ-\measuredangle CBO_B=90^\circ-\measuredangle DBO_B=\measuredangle BAD.$$

$\square$

JuanOrtiz

#22 Jun 2, 2015, 3:24 am

A is Miquel point of BC(OB)(OC). Done

hayoola

#23 Nov 24, 2015, 10:42 pm

Let W be the circumcircle of triangle ABC.and let W_1be the circumcircle of triangle AO_bO_c.and let W_2 be the circumcircle of BO_bO_cC.and let $T = W \cap W_1$ By the known lemma we know that lines AT, O_bO_c, BC are concurrent. Let M be their interestion.

$$\measuredangle O_bBA = \measuredangle O_cCA$$

.

So we find that $T = O_bB \cap O_cC$ so we find that A is the Miquel point of $BC(OB)(OC)$.

By the known lemma if K be the circumcenter of BCO_bO_c we find that

$MA \perp KA$

By angle casing we find that MAO_cC is cyclice.

Now we know that angles$KAD = AMO_c = ACO_c = ABO_b = DAH$

K6160

#24 Apr 4, 2016, 11:30 am

We have that $\angle AO_BO_C = \dfrac{1}{2}\angle AO_BD = \angle B$ and likewise $\angle AO_CO_B = \angle C$. So A is the center of spiral similarity sending AO_BO_C to ABC and is hence the Miquel point of BCO_CO_B. It is well known that AO_BXC is cyclic. So

$$\angle XAC = \angle XO_BC = 90 - \angle O_BBC = \angle BAD \implies \angle XAD = \angle DAC - \angle BAD.$$

Also, $\angle BAH = 90 - \angle B = 2\angle BAD - \angle DAC$, which follows from $\angle BO_BC = \angle O_CCB$. Therefore, $\angle DAH = \angle DAC - \angle BAD$ and the result follows. $\square$

Quick Reply

High School Olympiads

Help me

every

#1 Jul 19, 2009, 9:23 am

In a triagle ABC,BC is the smallest edge.Choose M and N on AB and Ac respectively such that BM=BC and CN=CB.Prove that $\frac{MN}{BC}=\sqrt{3-2(cosA+cosB+cosC)}$

Luis González

#2 Jul 19, 2009, 11:02 am

Proposition. Let $\triangle ABC$ be a scalene triangle and BC is its shortest side. M,N are two points on $AC,AB,$ such that $BN=CM=BC.$ Then $MN\perp OI,$ where O,I are the circumcenter and incenter of $\triangle ABC$ and $\frac{MN}{BC}=\frac{OI}{R}.$

Proof. In the isosceles triangles $\triangle OAC$ and $\triangle OAB,$ we have

$$R^2-OM^2=AM\cdot MC,\ \ R^2-ON^2=AN\cdot BN$$

$$\Longrightarrow ON^2-OM^2=AM\cdot MC-AN\cdot BN$$

$$\Longrightarrow ON^2-OM^2=(CA-BC)BC-(AB-BC)BC=BC(CA-AB)$$

Since $\triangle MIC$ and $\triangle MIB$ are isosceles, we have $IM=IB, IN=IC.$ Then

$$IN^2-IM^2=IC^2-IB^2=BC(CA-AB)$$

$$\Longrightarrow ON^2-OM^2=IN^2-IM^2 \Longrightarrow \ IO\perp MN.$$

On the other hand, $\angle NBM=\angle OCI=90^\circ-\frac{1}{2}\angle C-\angle A.$ If $L\equiv CI\cap BM,$ then it follows that $\angle NMB$ and $\angle OIC$ are supplementary, which implies that $\triangle IOC$ and $\triangle MNB$ are pseudo-similar, hence

$$\frac{MN}{OI}=\frac{BN}{OC}\Longrightarrow \frac{MN}{BC}=\frac{OI}{R}.$$

Quick Reply

High School Olympiads

Concurrency

Reply

Ponclete

#1 Jul 19, 2009, 8:46 am

A convex quadrilateral $ABCD$ stisfy $\angle B = \angle C = \angle D$. Let E, F be the projectives of A, C to BC, DA and M be midpoint of BD. prove that E,M,F are collinear.

livetolove212

#2 Jul 19, 2009, 9:18 am

Let O be the intersection of AD and BC.
E, M, F are collinear iff $\frac{FO}{FD}.\frac{MD}{MB}.\frac{EB}{EO} = 1$ (Menelaus's theorem)
$\Leftrightarrow \frac{FO}{FD} = \frac{EO}{EB}$
$\Leftrightarrow \frac{FO}{EO} = \frac{FD}{EB} = \frac{CF}{AE} (\Delta DFC \sim \Delta BEA)$
But $CFAE$ is cyclic quadrilateral then $\frac{FO}{EO} = \frac{CF}{AE}$
So $(*)$ is true.
We are done!

livetolove212

#3 Jul 19, 2009, 9:28 am

This problem only need $\angle B = \angle D$. But if $\angle B = \angle C = \angle D$ then we have another collinear:
Let H, I be the orthocenter and circumcenter of triangle BCD. Prove that A, H, I are collinear.
Click to reveal hidden text

Luis González

#4 Jul 19, 2009, 10:43 am

Let $M' \equiv EF \cap BD, X \equiv BC \cap AD$. By Menelaus' theorem for $\triangle XBD$ cut by the transversal $\overline{EM'F}$, we have

$$\frac{EB}{EX} \cdot \frac{XF}{FD} \cdot \frac{DM'}{M'B} = 1$$

But since $\triangle EAB \sim \triangle FCD$ and CF, EA are antiparallel WRT the rays XA, XB

$$\frac{EB}{FD} = \frac{EA}{FC} = \frac{EX}{XF} \Longrightarrow M'D = M'B \Longrightarrow M \equiv M'.$$

As Linh pointed out, in this case we only need $\angle B = \angle D$. If $\angle B = \angle C = \angle D$, then A lies on the Euler line of $\triangle CBD$. Let $Y \equiv DC \cap AB$ and N, L the midpoints of CD, CB. XN and YL are perpendicular bisectors of CD, CB. XD, XN, XC meet at X and YD, YL, YB meet at $Y \Longrightarrow$ Intersections $XN \cap YM$ (circumcenter of CBD), $BN \cap DL$ (centroid of CBD) and $A \equiv XD \cap YB$ are collinear, due to Pappus theorem.

Quick Reply

Spain

El reto de La semana. Semana 3.

Reply

Pascual2005

#1 Oct 29, 2005, 12:01 am

Aqui estan los problemas, son 6 sin nivel asi que piensenlos todos:

1. **(corregido)** Sea I el incentro del $\triangle ABC$ con $AB < BC$, donde M es el punto medio de AC y N punto medio del arco ABC de la circunferencia circunscrita del triangulo. Probar que: $\angle IMA = \angle INB$.

2. **(corregido)** Sea $\triangle ABC$ un triangulo y sea $\triangle PQR$ otro triangulo **semejante al inicial** con P en BC, Q en AC y R en AB. Pruebe que el ortocentro de $\triangle PQR$ es el circuncentro de $\triangle ABC$.

3. Sea T_1 y T_2 dos circulos en el plano tal que no se tocan asi mismos. Una tangente externa a los circulos tocan a T_1 en A y a T_2 en C , y una tangente interna toca a las circunferencias T_1 en B y a T_2 en D.
Probar que la interseccion de AB y CD cae en la linea que resulta de unir los centros de las circunferencias.

4. Sea $\triangle ABC$ un triangulo y D el pie de la altura trazada desde A . Sean E y F sobre la linea que pasa por D tal que AE es perpendicular a BC y AF es perpendicular a CF, y E y F son diferentes desde D . Sean M y N los puntos medios de los segmentos BC y EF , respectivamente. Probar que: AN es perpendicular a MN.

5. Sean T_1 y T_2 dos circunferencias que se tocan en A y B. Sea PQ una recta por A tal que P esta en T_1 y Q esta en T_2.
Hallar el lugar geometrico del punto medio de PQ al variar esta recta.

6. Los circulos C_1 y C_2 estan dentro de otro circulo grande C y son tangentes a el en M y N respectivamente. C_1 pasa por el centro de C_2. La cuerda comun de C_1 y C_2 al extenderse corta a C en A y B. Las lineas MA y MB cortan a C_1 en E y F. Pruebe que EF es tangente a C_2.

This post has been edited 1 time. Last edited by Pascual2005, Oct 31, 2005, 1:12 am

fabycv

#2 Oct 30, 2005, 5:41 am

Ahora si!!!.

en la mañana imprimi esto y no tenia la correccion en el prob 02
la semejanza entre $\triangle ABC$ y $\triangle PQR$ salva el problema

la k de proporcionalidad no puede ser 1 (semejanza), en este caso hablariamos de congruencia entre triangulos, ...
entonces esta bien..

FABIOLA

fabycv

#3 Oct 30, 2005, 5:42 am

Cuando ponen la 6??

y cual es el tema de la siguiente semana??

ojala sea combiinatorio

Fabiola

manuel

#4 Oct 30, 2005, 6:40 am

siiii tiren combinatoria la semana que viene.

Pascual2005

#5 Oct 31, 2005, 1:01 am

si la semana que viene es combinatoria, pero no cualquier combinatoria, ya van a ver...

manuel

#6 Oct 31, 2005, 7:18 am

jaja no cualquier combinatoria, suena bueno........

mcphisto

#7 Oct 31, 2005, 7:22 am

bueno entonces estaremos esperando los problemas de combinatoria(me muero de impaciencia!) 10/10

carlosbr

#8 Oct 31, 2005, 9:33 am

Saludo el entusiasmo de varios de los participantes de esta comunidad ....

pero recuerden que ya existe un grupo de problemas de esta semana
GEOMETRIA ....
pues a intentarlas .....
y a enviar sus soluciones..

Gracias ...

Carlos Bravo
Lima -PERU

Pascual2005

#9 Nov 2, 2005, 7:20 pm

Soluciones hasta el viernes 4 de noviembre.

Pascual2005

#10 Nov 11, 2005, 6:51 am

ya pueden discutirlos

Gracias a este tipo que resolvio todo y me ahorro mucho trabajo!!! Felicitaciones >Chen!

Attachments:
1.doc (247kb)

carlosbr

#11 Nov 12, 2005, 9:55 am

Muy buen trabajo Chen ..

espero que sigamos recibiendo su aporte en esta comunidad ..

extiendo este saludo a todos los chicos de los talleres del local de Huanquica (Lima)
y esperamos una mayor participacion de alumnos y entrenadores. ..

Gracias

Carlos Bravo
Lima -PERU

Luis González
#12 Jul 11, 2009, 9:33 am

> Pascual2005 wrote:
> 1. Sea I el incentro del $\triangle ABC$ con $AB < BC$, donde M es el punto medio de AC y N punto medio del arco ABC de la circunferencia circunscrita del triangulo. Probar que: $\angle IMA = \angle INB$.

La mediatriz MN de AC corta la bisectriz BI en P, estando P en la circunferencia circunscrita (O). Es sabido que la circunferencia $\mathcal{K}$ con centro P que pasa por A, C pasa también por el incentro I. Ahora considerando la inversión respecto a $\mathcal{K}$, ésta trasforma $M \to N$ y como I es un punto doble, entonces la circunferencia $\mathcal{K}' \equiv (IMN)$ es invariable, es decir $\mathcal{K}'$ es ortogonal al círculo doble $\mathcal{K}$. Por consiguiente, PI es tangente a $\mathcal{K}'$ en $I \Longrightarrow \angle NIB = \angle IMN$. De aqui deducimos que $\angle INB = \angle IMA$ en vista que $\angle NBI$ y $\angle NMA$ son rectos.

Luis González
#13 Jul 16, 2009, 12:43 am

> Pascual2005 wrote:
> 2. Sea $\triangle ABC$ un triángulo y sea $\triangle PQR$ otro triángulo **semejante al inicial** con P en BC, Q en AC y R en AB. Pruebe que el ortocentro de $\triangle PQR$ es el circuncentro de $\triangle ABC$.

Las circunferencias $\odot(ARQ), \odot(BPR)$ y $\odot(CQP)$ concurren en el Punto de Miquel M. Luego por relaciones angulares en cuadrilateros cíclicos tenemos:

$\angle BMC = \angle BMP + \angle CMP = \angle PQC = \angle BAC + \angle RPQ = 2\angle BAC$. Analogamente se tiene $\angle AMC = 2\angle ABC \Longrightarrow M$ es circuncentro de $\triangle ABC$.

Si PM corta RQ en H, entonces se tendrá

$\angle RHP = \angle RQP + \angle MPQ = \angle RQP + \angle MCA = 90^\circ \Longrightarrow PM \perp RQ$. Mutatis mutandi $RM \perp PQ \Longrightarrow M$ es ortocentro de $\triangle PQR$.

Luis González
#14 Jul 17, 2009, 12:32 pm

> Pascual2005 wrote:
> 3. Sea T_1 y T_2 dos círculos en el plano tal que no se tocan asi mismos. Una tangente externa a los círculos tocan a T_1 en A y a T_2 en C y una tangente interna toca a las circunferencias T_1 en B y a T_2 en D. Probar que la interseccion de AB y CD cae en la linea que resulta de unir los centros de las circunferencias.

Si denotamos O_1, O_2 los centros de las circunferencias T_1, T_2 y $E \equiv BD \cap AC$, $P \equiv AB \cap CD$, el resultado es directo aplicando el teorema degenerado de Pappus. Note que $EO_1 \parallel CD, EO_2 \parallel AB, AO_1 \parallel CO_2$. CD, EO_1 se cortan en el punto del infinito U de CD. EO_2, AB se cortan en el punto del infinito V de AB. AO_1, CO_2 se cortan en el punto del infinito W del gradiente perpendicular a AC. U, V, W están en la recta del infinito y A, E, C están alineados, así P, O_1, O_2 son colineales.

Luis González
#15 Jul 17, 2009, 11:16 pm

> Pascual2005 wrote:
> 4. Sea $\triangle ABC$ un triangulo y D el pie de la altura trazada desde A. Sean E y F sobre la linea que pasa por D tal que AE es perpendicular a ED y AF es perpendicular a FC y E y F son diferentes desde D. Sean M y N los puntos medios de los segmentos BC y EF, respectivamente. Probar que: AN es perpendicular a MN.

Como los cuadriláteros $AEBD$ y $ADFC$ son cíclicos, resulta que $\angle AEF = \angle ABC$ y $\angle AFE = \angle ACB \Longrightarrow$ $\triangle ABC \sim \triangle AEF.$ A es pues centro de la semejanza espiral con ángulo de giro $\angle EDB = \angle AEB = \angle CAF.$ Como los puntos medios de BC y EF son homologos en la semejanza, entonces el ángulo $\angle ANM$ es igual al ángulo de giro, por tanto $\angle ANM = \angle CDF \Longrightarrow ANDM$ es cíclico $\Longrightarrow \angle ANM = \angle ADM = 90^\circ.$

Luis González

#16 Jul 18, 2009, 10:10 am

> *Pascual2005 wrote:*
> 5. Sean T_1 y T_2 dos circunferencias que se tocan en A y $B.$ Sea PQ una recta por A tal que P esta en T_1 y Q esta en $T_2.$ Hallar el lugar geométrico del punto medio de PQ al variar ésta recta.

Denotemos $T_1 \equiv (O_1, r_1), T_2 \equiv (O_2, r_2)$ y M punto medio de $PQ.$ Las potencias de M respecto a T_1 y T_2 son

$$\frac{1}{2}PQ \cdot AM = (r_1)^2 - (MO_1)^2 \ , \ \frac{1}{2}PQ \cdot AM = (MO_2)^2 - (r_2)^2$$

$$\Longrightarrow \ (MO_1)^2 + (MO_2)^2 = (r_1)^2 + (r_2)^2$$

El lugar geométrico de M es pues la circunferencia de centro en el punto medio de O_1O_2 coaxial con $T_1, T_2.$

Luis González

#17 Jul 19, 2009, 2:38 am

> *Pascual2005 wrote:*
> 6. Los círculos C_1 y C_2 están dentro de otro círculo grande C y son tangentes a él en M, N respectivamente. C_1 pasa por el centro de $C_2.$ La cuerda común de C_1 y C_2 al extenderse corta a C en A y $B.$ MA y MB cortan a C_1 en E y $F.$ Pruebe que EF es tangente a $C_2.$

Denotemos O_1, O_2 los centros de C_1, C_2, PQ la cuerda común y G el segundo corte de AN con $C_2.$ M es centro de la homotecia positiva que transforma C_1 en $C,$ por consiguiente $AB \parallel EF.$ Entonces los arcos EP y FQ son iguales y como los arcos O_2P y O_2Q también son iguales, deducimos pues que MO_2 es bisectriz de $\angle EMF.$ Por otra parte, la inversión con centro A y potencia igual a la que tiene A respecto a $C_1, C_2,$ deja a éstas invariables y transforma a C en una tangente común que pasa por los puntos inversos E, G de $M, N.$ Por tanto, EG es tangente a C_1 en $E \Longrightarrow$

$$\angle GEO_2 = \angle EMO_2 = \angle FMO_2 = \angle FEO_2$$

$\Longrightarrow EF$ es la simétrica de EG respecto a $EO_2 \Longrightarrow EF$ es tangente a $C_2.$

Quick Reply

High School Olympiads

Equal angles.

Reply

Source: Nice and easy.

Virgil Nicula

#1 Jul 16, 2009, 8:41 am

Let ABC be a triangle ($b > c$) with the incenter I . Denote the midpoint M of the side $[BC]$, $D\in BC$

so that $AD\perp BC$, $S\in MI\cap AD$ and $X\in MI\cap AB$. Prove that $A=2B\ \iff\ \widehat{SDI}\equiv\widehat{SDX}$.

This post has been edited 2 times. Last edited by Virgil Nicula, Jul 17, 2009, 2:39 am

Virgil Nicula

#2 Jul 16, 2009, 9:42 am

Sorry, I corrected above the initial enunciation.

Luis González

#3 Jul 16, 2009, 11:12 pm

Let V be the foot of the A-angle bisector and let the perpendicular bisector of BC meet the arc BC of the circumcircle at N. Perpendicular line to BC through B meets AN at B'. If $\angle A=2\angle B$, then $\angle CBN=\angle ABC \Longrightarrow BV$ and BB' bisect $\angle ABN$ internally and externally $\Longrightarrow$ Division (A,N,V,B') is harmonic and so is its orthogonal projection onto BC, namely (D,M,V,B). Pencil $A(D,M,V,B)$ is harmonic $\Longrightarrow (S,M,I,X)$ is harmonic and since $AD\perp BC$, then AD and BC bisect $\angle XDI$ internally and externally $\Longrightarrow \angle SDI=\angle SDX$. The converse is proved with the same arguments.

Virgil Nicula

#4 Jul 17, 2009, 2:39 am

> *Virgil Nicula wrote:*
> Let ABC be a triangle ($b > c$) with the incenter I . Denote the midpoint M of the side $[BC]$, $D\in BC$
>
> so that $AD\perp BC$, $S\in MI\cap AD$ and $X\in MI\cap AB$. Prove that $A=2B\ \iff\ \widehat{SDI}\equiv\widehat{SDX}$.

Proof. I'll use also the **harmonical division**. Denote $L\in AI\cap BC$. Since $LC=\frac{ab}{b+c}$ obtain $A=2B\iff\widehat{CAL}\equiv\widehat{CBA}\iff b^2=a\cdot LC\iff a^2=b(b+c)$. Thus, $\boxed{A=2B\iff a^2=b(b+c)}$.

Ascertain easily $LD=\frac{2p(p-a)(b-c)}{a(b+c)}$, $LM=\frac{a(b-c)}{2(b+c)}$, $BD=\frac{a^2+c^2-b^2}{2a}$. Observe that $\widehat{SDI}\equiv\widehat{SDX}\iff$ the division $\{X,S,I,M\}$ is harmonically $\iff\{B,D,L,M\}$ is harmonically $\iff\frac{LD}{LM}=\frac{BD}{BM}\iff$ $4p(p-a)=a^2+c^2-b^2\iff(b+c)^2-a^2=a^2+c^2-b^2$, i.e. $\boxed{\widehat{SDI}\equiv\widehat{SDX}\ \iff\ a^2=b(b+c)}$.

Quick Reply

High School Olympiads

MaTaN

Reply

jrrbc

#1 Jul 16, 2009, 3:31 am

Attachments:

Find the vertices of the triangle ABC given the center **N** of the nine-points circle, the midpoint **Ma** of the side BC and the foot **Ta** of the internal bisector of angle A

Luis González

#2 Jul 16, 2009, 5:01 am

Draw the nine-point circle (N, NM_a) intersecting the line T_aM_a again at the foot H_a of the A-altitude. Let X_a, Y_a be the unknown tangency points of the incircle (I) and the A-exincircle (I_a) with BC. Since cross ratio (A, T_a, I, I_a) is harmonic, then so is the orthogonal projection on BC, namely (H_a, T_a, X_a, Y_a). M_a is also midpoint of X_aY_a, thus by Newton's theorem, we have $M_aY_a{}^2 = M_aT_a \cdot M_aH_a \Longrightarrow$ segment $\varrho = M_aY_a$ is constructible. Circle (M_a, ϱ) meets M_aT_a at X_a, Y_a, such that X_a lies between H_a, T_a. Construct the incircle (I) tangent to M_aT_a at X_a and internally tangent to (N) in the upper halfplane. Draw the angle bisector of $\angle BAC$ passing though I, T_a. This cuts the perpendicular line to M_aH_a through H_a at vertex A. Tangents from A to (I) complete $\triangle ABC$.

jrrbc

#3 Jul 16, 2009, 7:47 am

thanks

Quick Reply

High School Olympiads

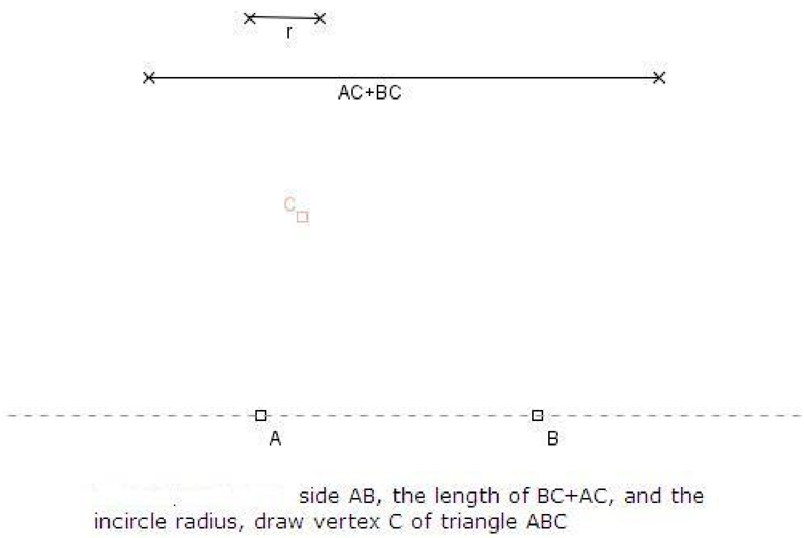
r AC+BC [AB]

Reply

jrrbc

#1 Jul 15, 2009, 1:52 am

Attachments:



livetolove212

#2 Jul 15, 2009, 7:06 am

Click to reveal hidden text

Luis González

#3 Jul 15, 2009, 9:46 am

Based on given data, the segments $(p-c\ ,p)$ can be constructed. Then the construction of $\triangle A'B'C'$ congruent to $\triangle ABC$ follows. Incircle (I') touches $A'C'$, $B'C'$ at X,Y. The quadrilateral $C'XI'Y$ is then constructible. On the rays $C'X$ and $C'Y$ pick the points M,N, such that $C'M=C'N=p \Longrightarrow$ C'-excircle (I'_c) is tangent to the sidelines $C'X$ and $C'Y$ at M,N $\Longrightarrow$ common internal tangents of (I'), (I'_c) complete the two possible $\triangle A'B'C'$. From this we get the desired sides $CA=C'A'$ and $CB=C'B'$. Thus the construction of $\triangle ABC$ follows straightforwardly.

jrrbc

#4 Jul 15, 2009, 10:10 pm

how is the intersection I' with ellipse construction?

Quick Reply

High School Olympiads

a problem of Geometry (incenter)□Help□

Reply

Source: some difficult

admire9898

#1 Jul 14, 2009, 9:19 pm

P is a point that in the triangle ABC, □APB-□ACB=□APC-□ABC , and D is the incenter of APB.E is the incenter of APC.
Proof that AP,BD,CE all intersect at one point

I'm puzzled at how to use the condition

Luis González

#2 Jul 14, 2009, 9:54 pm

Let X,Y,Z be the orthogonal projections of P onto $BC,CA,AB.$ Then

$$\angle APB=\angle ACB+\angle XZY\ ,\ \angle APC=\angle ABC+\angle XYZ$$

$$\angle APB-\angle ACB=\angle APC-\angle ABC \Longrightarrow \ \angle XZY=\angle XYZ$$

Therefore, pedal triangle $\triangle XYZ$ of P with respect to $\triangle ABC$ is isosceles with apex $X \Longrightarrow P$ lies on the A-Apollonius circle of $\triangle ABC \Longrightarrow \frac{AC}{PC}=\frac{AB}{PB}$. By angle bisector theorem, it follows that bisectors BD,CE of $\angle ABP$ and $\angle ACP$ cut AP at the same point.

prime04

#3 Jul 14, 2009, 10:12 pm

Attachments:

Let AP meet the circumcirle at T.

Clearly,$\angle PBT=\angle APB-\angle PTB=\angle APB-\angle ACB$...............(1)

Similarly,,$\angle PCT=\angle APC-\angle PTC=\angle APC-\angle ABC$..........(2)

But it is given that:$\angle APB-\angle ACB=\angle APC-\angle ABC$

hence we have $\angle PBT=\angle PCT$...(3)

Now:

In triangle PBT:

$$\frac{PT}{\sin(\angle PBT)}=\frac{PB}{\sin(\angle C)}$$

In triangle PCT:

$$\frac{PT}{\sin(\angle PCT)}=\frac{PC}{\sin(\angle B)}$$

From the above equations: $\frac{BP}{BA}=\frac{CP}{CA}$..............................(4)

Thus angle bisectors BD and CE of triangles APB and APC divide AP in the same ratio implies AP,BD,CE are concurrent

admire9898

#4 Jul 15, 2009, 3:44 pm

Thanks for the prefect solusions! 😂

Quick Reply

High School Olympiads

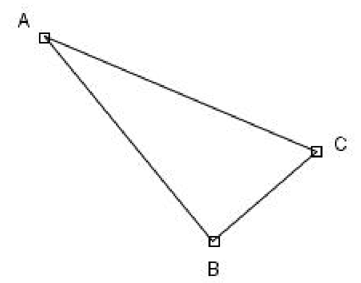
equilateral triangle

Reply

jrrbc
#1 Jul 14, 2009, 7:19 pm

Attachments:

Circumscribe triangle ABC with the equilateral triangle of maximum size.



Luis González
#2 Jul 14, 2009, 9:45 pm

The desired triangle $\triangle XYZ$ is centrally similar to the outer Napoleon triangle of $\triangle ABC$. This is, construct outwardly on the sides of $\triangle XYZ$ the equilateral triangles $\triangle BCP, \triangle CAQ$ and $\triangle ABR$. Centers O_1, O_2, O_3 of these triangles form the outer Napoleon triangle. Through A, B, C draw parallels to O_2O_3, O_3O_1, O_1O_2, respectively. These three lines bound $\triangle XYZ$.

Proof: Let $\triangle X_0Y_0Z_0$ be any equilateral triangle circumscribed to $\triangle ABC \Longrightarrow X_0 \in (O_1), Y_0 \in (O_2)$ and $Z_0 \in (O_3)$. If M, N are the midpoints of AY_0, AZ_0, then from the right trapezoid O_2O_3NM, we have $O_2O_3 \geq MN = \frac{1}{2}Y_0Z_0 \Longrightarrow 2 \cdot O_2O_3 = YZ \geq Y_0Z_0 \Longrightarrow [XYZ] \geq [X_0Y_0Z_0]$.

jrrbc
#3 Jul 15, 2009, 1:29 am

thanks

Quick Reply

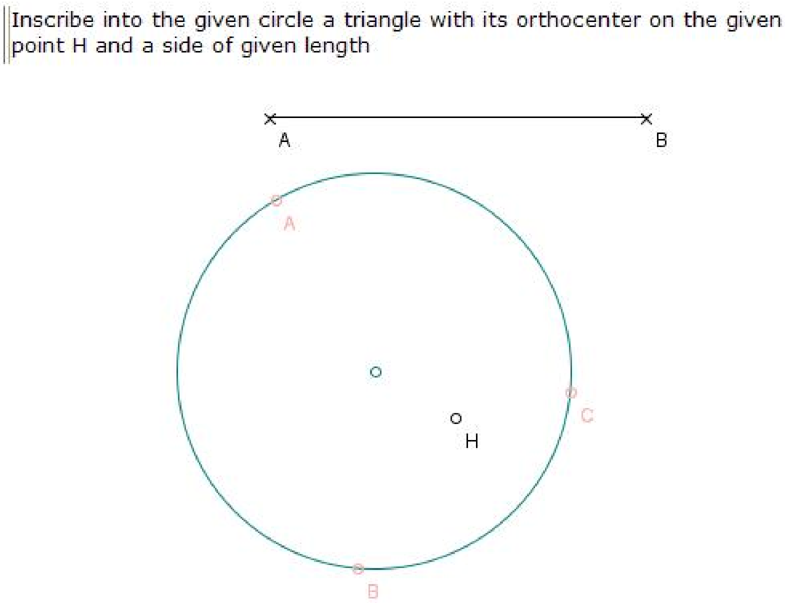
High School Olympiads

triangle

Reply

jrrbc
#1 Jul 14, 2009, 2:27 am

Attachments:



Luis González
#2 Jul 14, 2009, 8:32 am

Circle passing through A,B,H is congruent to the given circumcircle (O), thus we can construct a triangle $\triangle A'B'C'$ congruent to $\triangle ABC$. This is, construct a chord $A'B'$ equal to AB on (O) and reflect (O) about $A'B'$ into $(O') \Longrightarrow (O')$ contains the orthocenter H' of $\triangle A'B'C'$. Circle (O,OH) cuts (O') at the possible orthocenters H'. The construction of $\triangle A'B'C'$ is then straightforward. Now carry A',B',C' in the rotation with center O, taking H' into H, to get the desired A,B,C.

jrrbc
#3 Jul 14, 2009, 6:43 pm

thanks

Quick Reply

High School Olympiads

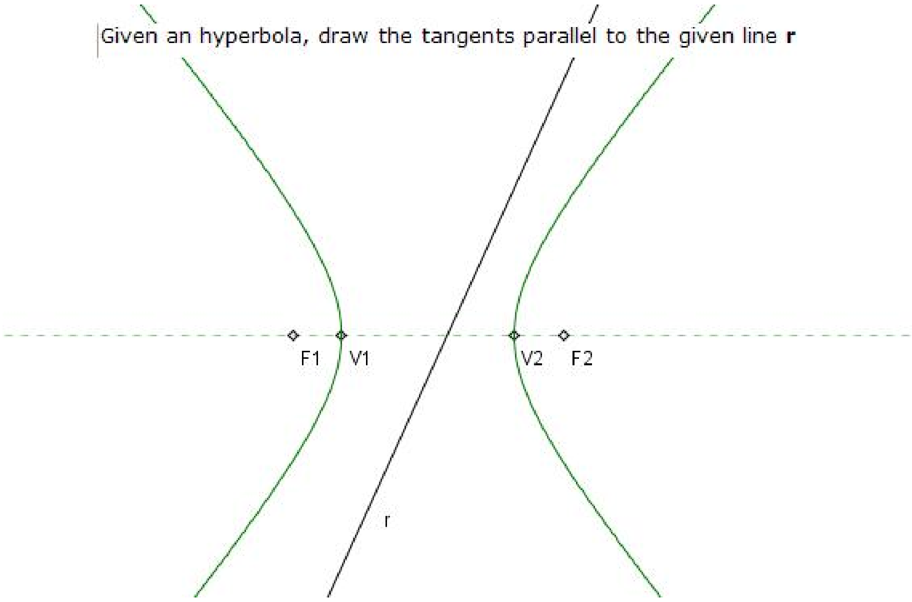
tangents

Reply

jrrbc

#1 Jul 14, 2009, 2:22 am

Attachments:



Luis González

#2 Jul 14, 2009, 7:27 am

Let P be one of the unknown tangency points, say on the left branch of the hyperbola. Its corresponding tangent τ bisects $\angle F_1PF_2$ internally. Let X be the reflection of F_1 about $\tau \Longrightarrow X$ lies on PF_2. Note that F_2X is the lenght of the major axis, therefore $\triangle XF_1F_2$ is constructible, since the sides $F_1F_2, F_2X = V_1V_2$ and the angle $\angle XF_1F_2 = 90^\circ - \varphi$ are known, where φ is the acute angle formed by the given direction r of the tangent and the focal axis F_1F_2. Such construction permits to find the vertex $X \Longrightarrow \tau$ is the perpendicular bisector of F_1X. Reflect τ about the center of the hyperbola to get the another solution.

jrrbc

#3 Jul 14, 2009, 7:08 pm

thanks

Quick Reply

High School Olympiads

Perpendicular with symmedian

Source: own

The QuattoMaster 6000

#1 Jul 14, 2009, 5:43 am • 1

Given triangle ABC, with circumcenter O, let the A-symmedian hit BC at S. Let the parallel through S to AO hit the A-altitude at X. Prove that $OX \perp AS$.

Luis González

#2 Jul 14, 2009, 6:51 am • 1

Let the A-symmedian cut the circumcircle again at T. Tangents to (O) at A, T meet at a point V on BC, i.e. the center of the A-Apollonius circle of $\triangle ABC$. Let $OY \perp AS$ and define $X' \equiv OY \cap AH_a$. From the cyclic quadrilaterals AYH_aV and YSH_aX' we get $\angle YSX' = \angle YH_aA = \angle YVA = \angle OAY \Longrightarrow OA \parallel SX' \Longrightarrow X \equiv X'$.

TelvCohl

#3 Feb 18, 2015, 9:21 pm • 1

My solution:

Let Z be the projection of X on AS .
Let the tangent of $\odot(ABC)$ through A intersect BC at Y .

It suffices to prove $O \in XZ$.

From $SX \parallel AO \Longrightarrow SX \perp AY$,
so X is the orthocenter of $\triangle AYS \Longrightarrow Y \in XZ$.

Since AS is the polar of Y WRT $\odot(O)$,
so we get $YO \perp AS \Longrightarrow O, X, Y, Z$ are collinear .

Q.E.D

buratinogigle

#4 Feb 19, 2015, 12:33 am • 1

An extension

Let O be circumcenter of triangle ABC. P is a point on OA. (K) is circumcircle of triangle PBC. PQ is diameter of (K). AQ cuts BC at S. T is a point such that $AT \perp BC$ and $ST \parallel AO$. Prove that PT and AS intersect on (K).

drmzjoseph

#5 Feb 19, 2015, 1:58 pm • 1

> buratinogigle wrote:
>
> An extension
>
> Let O be circumcenter of triangle ABC. P is a point on OA. (K) is circumcircle of triangle PBC. PQ is diameter of (K). AQ cuts BC at S. T is a point such that $AT \perp BC$ and $ST \parallel AO$. Prove that PT and AS intersect on (K).

Reformulation

Let O be circumcenter of triangle ABC. S is a point on BC; T is a point such that $AT \perp BC$ and $ST \parallel AO$; P a point on AO sucht that $PT \perp AS$ and $PT \cap AS \doteq F$. Prove that B, C, F and P are cyclic

Proof

$BC \cap TP \doteq Z$ and T is orthocenter of ΔAZS then $TS \perp AZ$ ie. $AO \perp AZ \Rightarrow ZP.ZF \doteq AZ^2$ and AZ is tangent to circumcircle of ΔABC ie. $ZB.ZC \doteq AZ^2$
$ZP.ZF \doteq ZB.ZC \doteq AZ^2 \Rightarrow B, C, F$ and P are cyclic.
Done!

TelvCohl

#6 Feb 19, 2015, 4:35 pm • 1

> buratinogigle wrote:
>
> An extension
>
> Let O be circumcenter of triangle ABC. P is a point on OA. (K) is circumcircle of triangle PBC. PQ is diameter of (K). AQ cuts BC at S. T is a point such that $AT \perp BC$ and $ST \parallel AO$. Prove that PT and AS intersect on (K).

My solution:

Lemma:

Let O be the circumcenter of $\triangle ABC$ and P be a point on AO .
Let Q be the antipode of P in $\odot(PBC)$ and M be the projection of P on BC .

Then AM, AQ are isogonal conjugate of $\angle BAC$.

Proof:

Let R be the isogonal conjugate of Q WRT $\triangle ABC$.
Let Y, Z be the projection of P on AC, AB, respectively .
Let B^*, C^* be the projection of R on AC, AB, respectively .

It suffices to prove A, M, R are collinear .

Since PM, PQ are isogonal conjugate of $\angle BPC$,
so we get $\angle C^*BR = \angle QBC = \angle BPM = \angle BZM \Longrightarrow MZ \parallel RB$.
Similarly we can prove $MY \parallel RC$.

From $P \in AO$ we get $YZ \parallel CB$,
so $\triangle MYZ$ and $\triangle RCB$ are homothetic with center $A \Longrightarrow R \in AM$.

Back to the main problem:

Let the tangent of $\odot(ABC)$ through A intersect BC at X .
Let Y be the projection of T on AS and M be the projection of P on BC .

From $ST \parallel AO \Longrightarrow ST \perp AX$,
so T is the orthocenter of $\triangle AXS \Longrightarrow Y \in XT$.
Since A, P, M, X lie on a circle with diameter XP ,
so combine with the lemma we get $\angle MXY = \angle TAS = \angle MAP = \angle MXP$,
hence P, T, X, Y are collinear $\Longrightarrow \angle QYP = 90^\circ$. i.e. $Y \equiv AS \cap PT \in \odot(K)$

Q.E.D

Quick Reply

High School Olympiads

locus

Source: Austria 1999

moldovan

#1 Jul 12, 2009, 1:42 am

Let ϵ be a plane and k_1, k_2, k_3 be spheres on the same side of ϵ. The spheres k_1, k_2, k_3 touch the plane at points T_1, T_2, T_3, respectively, and k_2 touches k_1 at S_1 and k_3 at S_3. Prove that the lines S_1T_1 and S_3T_3 intersect on the sphere k_2. Describe the locus of the intersection point.

Luis González

#2 Jul 12, 2009, 3:35 am

Consider the diametral section α of the spheres k_2, k_3 containing their centers O_2, O_3 and the right trapezoid $O_3O_2T_2T_3$. Since S_3 is the insimilicenter of $k_3 \sim k_2$, then T_3S_3 goes through the antipode P of T_2 on the sphere k_2. Similarly, by considering the diametral section β of the sphere k_1, k_2 orthogonal to e, then T_1S_1 goes through P. Since

$$\frac{(T_2T_1)^2}{(T_2T_3)^2} = \frac{4r_1r_2}{4r_2r_3} = \frac{r_1}{r_3} = \text{const}$$

It follows that $T_2 \in e$ moves on the Apollonian circle $\mathcal{C}$ of the segment T_1T_3 with ratio $\sqrt{\frac{r_1}{r_3}}$. As a result, P moves on the surface of the right cylinder with base $\mathcal{C}$ on the plane e. On the other hand, in the right trapezoid $PO_3T_3T_2$, we have:

$PO_3{}^2 = 4r_2r_3 + (2r_2 - r_3)^2 = 4r_2{}^2 + r_3{}^2$. Similarly, $PO_1{}^2 = 4r_2{}^2 + r_1{}^2$

$\Longrightarrow PO_3{}^2 - PO_1{}^2 = r_3{}^2 - r_1{}^2 = \text{const} \Longrightarrow P$ lies on the radical plane π of k_2, k_3. Thus, locus of P is a Ellipse $\mathcal{E}$, i.e. intersection of the right cylinder $\mathcal{C}$ with the plane π.

High School Olympiads

maximum area

Reply

Source: Austria 1989

moldovan

#1 Jul 10, 2009, 9:37 pm

We are given a circle k and nonparallel tangents t_1, t_2 at points P_1, P_2 on k, respectively. Lines t_1 and t_2 meet at A_0. For a point A_3 on the smaller arc P_1P_2, the tangent t_3 to k at P_3 meets t_1 at A_1 and t_2 at A_2. How must P_3 be chosen so that the triangle $A_0A_1A_2$ has maximum area?

Luis González

#2 Jul 10, 2009, 11:57 pm

Let us change the notation in order to use the standard triangle notation.

Problem: (I_a, r_a) is a fixed circle and b, c are fixed tangents to (I_a) through N, M meeting at $A.$ Variable tangent τ to (I_a) through the small arc MN meets AN, AM at $C, B.$ Construct tangent τ such that $\triangle ABC$ has maximum area.

Using the well-know relation $[\triangle ABC] = r_a(p-a),$ keeping in mind that $AN = AM = p = \text{const},$ we deduce that $[\triangle ABC]$ is maximum $\Longleftrightarrow a$ has minimum lenght. Let h_a be the length of altitude issuing from $A.$ We have then

$$a = \frac{2r_a p}{h_a + 2r_a} \Longrightarrow \ a \text{ is mimum} \Longleftrightarrow \ h_a \text{ is maximum.}$$

If BC cuts AI_a at P' and P is the midpoint of the arc $MN,$ then $AP \geq AP' \geq h_a \Longrightarrow h_a$ has maximum lenght $AP \Longrightarrow \tau$ forms the isosceles triangle $\triangle ABC$ with apex $A.$

Virgil Nicula

#3 Jul 11, 2009, 9:37 am

> moldovan wrote:
>
> We are given a circle w and nonparallel tangents AE , AF at points $\{E, F\} \subset w$. For a point mobile M on the smaller arc EF
>
> the tangent to w at M meets AE at B and AF at C . How must M be chosen so that the triangle ABC has maximum area ?

Proof. I'll use the standard notations for the triangle ABC with the A-exincircle $w = C(I_a, r_a)$. Therefore, $AE = AF = p$ (constant)

and $\left\|\begin{array}{c} a = (p-b) + (p-c) \\ \\ p - b = r_a \tan\frac{C}{2} \\ \\ p - c = r_a \tan\frac{B}{2} \end{array}\right\| \Longrightarrow \boxed{a = r_a\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)}$. In conclusion,

$S = [ABC] = r_a(p-a)\ -\ \text{maximum} \iff$

$a\ -\ \text{minimum} \iff \left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)\ -\ \text{minimum}$, where $\left(\frac{B}{2} + \frac{C}{2}\right) = 90^\circ - \frac{A}{2}$ (constant) $\iff B = C$ $\iff BC \parallel EF$.

Quick Reply

High School Olympiads

compute the distance

Reply

Source: Austria 1990

moldovan

#1 Jul 10, 2009, 10:14 pm

A convex pentagon $ABCDE$ is inscribed in a circle. The distances of A from the lines BC, CD, DE are a, b, c, respectively. Compute the distance of A from the line BE.

Luis González

#2 Jul 10, 2009, 11:14 pm

Lemma: $ABCD$ is cyclic and P is a point lying on its circumcircle. Let X, Y, Z, V the orthogonal projections of P on sides AD, AB, BC, CD. Then $PX \cdot PZ = PY \cdot PV$.

$$\frac{PX}{PY} = \frac{\sin \widehat{PAX}}{\sin \widehat{PAB}} , \frac{PZ}{PV} = \frac{\sin \widehat{PCB}}{\sin \widehat{PCD}}$$

$\angle PAB = \angle PCD$ and $\angle PAX = \angle PCB$ implies that $PX \cdot PZ = PY \cdot PV$.

Quick Reply

Spain

libros para preparacion de olimpiadas

Reply

lambert

#1 Jul 10, 2009, 8:58 am

bueno soy de colombia y la cosa es que acabo de comenzar ing.quimica y pues me entere de las olimpiadas de matematica universitaria y me gustaria presentarme para las del proximo año pero no manejo los temas que se requieren para la resolucion de los problemas:Calculo,teoria de numeros,algebra abstracta,algebra lineal,geometria. entonces queria pedirles me ayudaran aconsejandome libros sobre cada uno de los temas que deo estudiar para comenzar a prepararme.hable con un profesor y me recomendaba mas que estudiara librosn de analisis pues es mas riguroso que uno de calculo ahi si pues lo dejo a su opinion.en todo caso les agradezco cualquier ayuda

Luis González

#2 Jul 10, 2009, 10:03 am

Hola Lambert, personalmente creo que una de las mejores maneras de mejorar es participando activamente en los foros de olimpiadas de AoPS, donde se han posteado numerosisimos problemas de las IMO. Alli podrás apreciar la gran variadad de soluciones y herramientas disponibles para la resolución de problemas. En cuanto a geometría, mi experiencia me indica que la esencial para las olimpiadas es la sintética, es decir, aquella que no involucra el analisis puramente algebraico para las resoluciones sino que ésta emplea recursos puramente geométricos como: relaciones angulares, razones simples y dobles, transformaciones en el plano, etc.

Tengo un material muy bueno (en ingles) sobre las diversas tecnicas que se emplean para resolver los problemas geométricos de nivel de olimpiadas. Es pesado para adjuntarlo por aqui. Yo con mucho gusto te lo enviaré a algun correo que me indiques a través de un mensaje privado.

Quick Reply

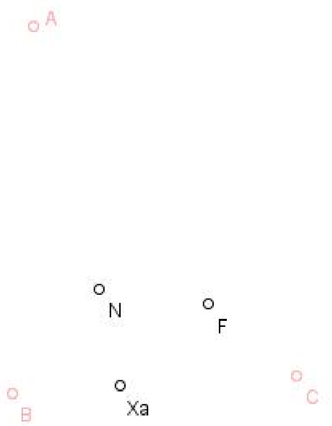
High School Olympiads

NFXa

Reply

jrrbc
#1 Jul 9, 2009, 5:20 am

Attachments:



Given F, N, & Xa, the Feuerbach point, the nine-point center and an intouch point, find the vertices of triangle ABC.

Luis González
#2 Jul 9, 2009, 8:09 am

Perpendicular bisector of FX_a meets the line FN at the incenter I. Draw the incircle (I) with center I and radius IF and the nine-point circle (N) with center N and radius NF. Tangent to (I) through X_a meets (N) at the midpoint M_a and the foot of the A-altitude H_a (decide which are M_a and H_a according to your sketch). Construct the reflection Y_a of X_a about M_a and the reflection J_a of X_a about I. Vertex A is the intersection of J_aY_a with the perpendicular to H_aX_a through H_a. Draw the two tangents from A to (I) to complete $\triangle ABC$.

jrrbc
#3 Jul 9, 2009, 11:43 am

thanks

Quick Reply

High School Olympiads

constructible

Reply

Source: Austria 1987

moldovan
#1 Jul 9, 2009, 12:38 am

The sides a, b and the bisector of the included angle γ of a triangle are given. Determine necessary and sufficient conditions for such triangles to be constructible and show how to reconstruct the triangle.

Luis González
#2 Jul 9, 2009, 3:50 am

Let $CV = w_c$ be the length of the angle bisector of $\angle C.$ Take the point M on ray BC such that $AC = CM = b.$ Then $\triangle AMB \sim \triangle WCB \Longrightarrow AM = \frac{w_c(a+b)}{a}.$

The length of this segment is constuctible, therefore isosceles triangle $\triangle ACM$ with known sides is constructible. Construct B on the ray MC such that $CB = a,$ then $\triangle ABC$ is completed. The necessary condition for $\triangle ABC$ to be constructible is that $\triangle ACM$ is also constructible, that is $AC + CM > AM \Longrightarrow \frac{2ab}{a+b} > w_c.$

Quick Reply

High School Olympiads

locus

Source: Ireland 1993

moldovan
#1 Jun 29, 2009, 4:57 pm

A line l is tangent to a circle S at A. For any points B, C on l on opposite sides of A, let the other tangents from B and C to S intersect at a point P. If B, C vary on l so that the product $AB \cdot AC$ is constant, find the locus of P.

Luis González
#2 Jul 8, 2009, 10:08 am

Let us rewrite the problem in order to use the common triangle notation

Problem: Let (I, r) be a fixed circle tangent to a fixed line $\mathcal{L}$ at X. B, C are points on $\mathcal{L}$ such that $X \in \overline{BC}$ and $BX \cdot CX = k^2 = \text{const}$. Tangents from B, C to (I) different from $\mathcal{L}$ meet at A. Find the locus of A.

Since $BX = p - b\ ,\ CX = p - c \Longrightarrow (p-b)(p-c) = k^2 = \text{const}$

From the identity $r = \sqrt{\frac{(p-a)(p-b)(p-c)}{p}}$ we get

$$r^2 = \frac{(p-a)k^2}{p} = \left(1 - \frac{a}{p}\right)k^2 \Longrightarrow \frac{p}{a} = \frac{k^2}{k^2 - r^2}$$

$$a \cdot h_a = 2rp \Longrightarrow h_a = \frac{2rp}{a} = \frac{2r \cdot k^2}{k^2 - r^2} = \text{const}$$

$\Longrightarrow$ A-altitude has constant length, hence the locus of A is a parallel line to BC.

High School Olympiads

collinear points (maybe posted before)

Reply

Source: Austria 1985

moldovan

#1 Jul 8, 2009, 2:18 am

A line meets the lines containing sides BC, CA, AB of a triangle ABC at A_1, B_1, C_1, respectively. Points A_2, B_2, C_2 are symmetric to A_1, B_1, C_1 with respect to the midpoints of BC, CA, AB, respectively. Prove that A_2, B_2, and C_2 are collinear.

Luis González

#2 Jul 8, 2009, 5:56 am

If A_1' is the harmonic conjugate of A_1 WRT BC, then the harmonic conjugate A_2' of its reflection A_2 about the midpoint M_a is clearly the reflection of A_1' about M_a. By similar reasoning, AA_2', BB_2', CC_2' concur at the isotomic conjugate U' of the tripole U of the transversal $\overline{A_1B_1C_1}$. Then, A_2, B_2, C_2 are collinear on the trilinear polar of U'.

mathVNpro

#3 Jul 8, 2009, 1:09 pm

Let M, N, P respectively be the midpoint of BC, CA, AB. Since A_1, B_1, C_1 are collinear, we obtain:

$$\frac{A_1B}{A_1C}\cdot\frac{B_1C}{B_1A}\cdot\frac{C_1A}{C_1B}=1$$

Since A_2, B_2, C_2 respectively are the symmentric points of A_1, B_1, C_1 wrt M, N, P, therefore, we obtain:

$\frac{A_2B}{A_2C}\cdot\frac{B_2C}{B_2A}\cdot\frac{C_2A}{C_2B}=1$, which follows that A_2, B_2, C_2 are collinear. $\square$

Quick Reply

High School Olympiads

fixed point

Reply

Source: Ireland 2007

moldovan

#1 Jul 7, 2009, 3:31 am

The point P is a fixed point on a circle and Q is a fixed point on a line. The point R is a variable point on the circle such that $P, Q,$ and R are not collinear. The circle through $P, Q,$ and R meets the line again at V. Show that the line VR passes through a fixed point.

yetti

#2 Jul 7, 2009, 10:35 am

Let l be the fixed line and $\mathcal K$ the fixed circle. Inversion with center P and arbitraty power takes l into a fixed circle $\mathcal L'$ through P and $\mathcal K$ through P into a fixed line k'. The fixed point $Q \in l$ goes to a fixed point $Q' \in \mathcal L'$ and the variable point $R \in \mathcal K$ goes to a variable point $R' \in k'$. Circumcircle $\odot(PQR)$ goes to the line $Q'R'$ cutting $\mathcal L'$ again at the image V' of V, and line VR goes to the circumcircle $\odot(PV'R')$. Let S be its center. $\angle PSR' = 2\angle PV'R' = 2\angle PV'Q' = \text{const} \Longrightarrow$ the isosceles triangle $\triangle PSR'$ with the fixed vertex P and $R' \in k'$ remains similar for all $R' \in k' \Longrightarrow$ locus of S is a fixed line $s \Longrightarrow$ circles $\odot(PV'R')$ intersecting at P form a pencil, they also intersect at the reflection W' of P in $s \Longrightarrow$ lines VR go through the fixed image W of W'.

Luis González

#3 Jul 7, 2009, 11:53 pm

Denote k the fixed circle and k' the variable circle (PQR). Let M be the second intersection of PQ with k. The parallel line through M to the fixed line cuts k again in the fixed point S. By Reim's theorem applied to k, k' with common chord PR, the points R, S, V are collinear. The line VR goes through the fixed point S

Quick Reply

High School Olympiads

another collinearity

Reply

nsato

#1 Feb 25, 2009, 10:46 am

(Inspired by http://www.artofproblemsolving.com/Forum/viewtopic.php?t=260225)

Let I be the incentre of triangle ABC, and let M_1, M_2, M_3 be the midpoints of the sides. Let Q_1 be the point on M_2M_3 such that $Q_1I \perp AI$, and define points Q_2, Q_3 similarly. Show that Q_1, Q_2, and Q_3 are collinear.

Click to reveal hidden text

yetti

#2 Feb 25, 2009, 11:30 pm

Incircle (I) touches BC, CA, AB at $D, E, F.$ (I_a) is the A-excircle, tangent to BC at $D_a.$ Midline $M_2M_3,$ angle bisector $BI,$ and the intouch triangle sideline DE concur at a point P_a [1]. Their poles WRT (I) are therefore collinear. C is the pole of $DE,$ direction perpendicular to BI is the pole of $BI,$ therefore the pole of M_2M_3 WRT (I) is intersection K_a of perpendicular to BI through C with the line $DI \perp M_2M_3.$ M_1 is also midpoint of $DD_a \Longrightarrow BD_a = CD \Longrightarrow$ the right $\triangle BD_aI_a \cong \triangle CDK_a,$ having parallel sides, are centrally congruent with similarity center M_1 and coefficient -1 $\Longrightarrow M_1$ is midpoint of $I_aK_a.$

Let G, M be centroid and Nagel point of $\triangle ABC$ and J incenter of the medial $\triangle M_1M_2M_3.$ Since I is the Nagel point of the medial triangle, the points G, I, J, M are collinear and M is reflection of I in $J.$ ($\overline{GM} = -2\overline{GI}$ and $\overline{GI} = -2\overline{GJ} \Longrightarrow \overline{JM} = -\overline{JI}.$) K_a is the pole of $M_2M_3,$ direction perpendicular to $IQ_1,$ i.e., parallel to IA is the pole of $IQ_1,$ therefore parallel q_1 to IA through K_a is polar of $Q_1.$ As q_1 is reflection of IA in $M_1,$ midparallel of $IA \parallel q_1$ is the internal bisector of the angle $\angle M_3M_1M_2,$ passing through the medial triangle incenter $J.$ Consequently, q_1 goes through reflection M of I in $J.$ Similarly, polars q_2, q_3 of Q_2, Q_3 WRT (I) go through $M,$ therefore their poles Q_1, Q_2, Q_3 are collinear and the line $Q_1Q_2Q_3$ is the polar of M WRT $(I).$

Reference:
[1] http://www.mathlinks.ro/viewtopic.php?t=255001

Luis González

#3 Feb 26, 2009, 6:41 am

We can also use barycentric coordinates WRT $\triangle ABC.$ Orthocenter O_3 of $\triangle IAB$ is the pole of the C-midline M_1M_2 WRT the incircle $(I).$ Barycentric coordinates of O_3 are given by $O_3(b-c : a-c : c),$ thus polar τ_c of Q_3 WRT (I) passes through O_3 orthogonally to $IQ_3,$ i.e. parallel to the C-angle bisector $bx - ay = 0.$ Hence, equation of τ_c is

$$\tau_c \equiv (a^2+ab-ac)x + (-b^2+bc-ab)y + (a^2-b^2+bc-ac)z = 0$$

We verify that the Nagel point $X_8(b+c-a : a+c-b : a+b-c)$ satisfies the equation of $\tau_c.$ By analogous reasoning, we conclude that Q_1, Q_2, Q_3 lie on the polar of X_8 WRT (I) and the proof is completed.

gb2124

#4 Jul 5, 2009, 9:43 am

Why the barycentric coordinate of orthocenter of IAB is (b-c:a-c:c)

Luis González

#5 Jul 5, 2009, 10:30 am

There are different ways to find the the coordinates of O_3, e.g intersecting the line joining $I(a:b:c)$ with $I(0:p-c:p-b)$

There are different ways to find the the coordinates of O_3, e.g intersecting the line joining $I(a : b : c)$ with $J(0 : p - c : p - b)$ and the line ℓ passing through $A(1:0:0)$ and the infinite point P_∞ of the gradient $\perp IB$, namely

$$P_\infty(aS_B + a^2c : aS_A - cS_C : -ac^2 - cS_B) \Longrightarrow$$

$$IJ \equiv (bc^2 + cS_A)x - (ac^2 + cS_B)y + (bS_B - aS_A)z = 0$$

$$\ell \equiv (ac^2 + cS_B)y + (aS_A - cS_C)z = 0$$

By solving these two latter equations we get $O_3 \ (b - c : a - c : c)$.

gb2124

#6 Jul 7, 2009, 6:18 pm

Sorry I don't quite understand the "infinite point of the perpendicular gradient"
What is it ?What is the gradient of a line in barycentric?
I feel quite confused.

Luis González

#7 Jul 7, 2009, 6:25 pm

Infinite point of any perpendicular line to $px + qy + rz = 0$ is given by

$$(S_B(r-p) - S_C(p-q) : S_C(p-q) - S_A(q-r) : S_A(q-r) - S_C(r-p))$$

$$S_A = \frac{b^2 + c^2 - a^2}{2} \ \ , \ \ S_B = \frac{c^2 + a^2 - b^2}{2} \ \ , \ \ S_C = \frac{a^2 + b^2 - c^2}{2}$$

gb2124

#8 Jul 7, 2009, 6:50 pm • 1

Thank you, I think I got to know it.
There is really a lot for me to learn in the barycentric world.
I wonder if there is some useful materials to help me learn the barycentric method?

Quick Reply

Spain

TST PERU - IMO 2007 [Dia 1- P01]

Reply

carlosbr

#1 May 6, 2007, 4:56 am • 2

Pregunta 01:
Sabado 05 de mayo

Sea P un punto interior de un semicirculo de diametro AB (el angulo $\angle APB$ es obtuso). La circunferencia inscrita al triangulo ABP es tangente a los lados AP y BP en los puntos M y N, respectivamente. La recta MN corta a la semicircunferencia en los puntos X e Y. Pruebe que la medida del arco $\widehat{XY}$ es igual a la medida del angulo $\angle APB$.

$\LaTeX ed$ by carlosbr

Jutaro

#2 May 7, 2007, 9:35 am • 1

Probaremos primero que BX es la bisectriz de $\angle ABP$. Pongamos $\angle BAP = 2\alpha$, $\angle ABP = 2\theta$. Entonces, si I es el incentro del triángulo ABP, $\angle AIB = \pi - \alpha - \theta$. Por otra parte, tenemos que $\angle AXB = \pi/2$, dado que AB es diámetro, y también $\angle AMI = \pi/2$. Luego el cuadrilátero $AIMX$ es cíclico, de donde $\angle AIX = \angle AMX$. Pero como $\angle APB = \pi - 2\alpha - 2\theta$, se sigue que $\angle PMN = \alpha + \theta$, y en consecuencia $\angle PMN = \angle AMX = \angle AIX = \alpha + \theta$. Luego $\angle AIX + \angle AIB = \pi$, esto es, B, I, X están alineados; así que X está sobre la bisectriz de $\angle ABP$. Esto implica que $\angle PBX = \angle XBA = \theta$. Pero si O es el centro del semicírculo, tenemos que $\angle AOX = 2\angle XBA = 2\theta = \angle PBA$. Por tanto BP y OX son paralelas.

De manera semejante podemos probar que AP y OY son paralelas; luego el cuadrilátero determinado por las rectas OX, OY, AP, BP es un paralelogramo, y podemos concluir que $\angle APB = \angle XOY$, como queríamos demostrar.

M4RI0

#3 May 7, 2007, 11:42 am • 1

Me parece que para decir que el cuadrilatero $AIMX$ es cíclico luego que tienes $\measuredangle AXB = \frac{\pi}{2}$ y $\measuredangle AMI = \frac{\pi}{2}$ estas asumiendo que BX corta a AI en I.

O sera q estoy con sueño , podrias responderme esa duda :

Jutaro

#4 May 8, 2007, 3:35 am • 1

ahhh si, tenes razon... estoy asumiendo lo q queria probar perdon x el post vacio, voy a tratar de remediar ese "pequeño" detalle mas tarde xq no estoy en mi casa ahorita! gracias x el comentario

cuenca

#5 May 11, 2007, 6:21 am

mi solucion:
Sean A_1, B_1 los pies de las perpendiculares de A, B a la recta MN, sea $2a =< APB$ entonces $< NBB_1 = a =< MAA_1$, se sabe que siendo $AB = 2R, BP = l, AP = t$ y sea p el semiperimetro del triangulo ABP, entonces $NB = p - t$ entonces $BB_1 = (p - t)cosa$ tambien $AA_1 = (p - l)cosa$ sea K el pie de la perpendicular de

entonces $NB = p - t$, entonces $BB_1 = (p - t)cosa$, tambien $AA_1 = (p - l)cosa$, sea K el pie de la perpendicular de O(centro de la semicircunferencia) a MN, luego $OK = \dfrac{(p-l)cosa + (p-t)cosa}{2} = Rcosa$, siendo que R es el radio de la semicircunferencia, luego $OY = R$, en el triangulo OYK, se tiene $< YOK = a$, luego $< XOY = 2a =< APB$

Luis González

#6 Jul 6, 2009, 10:11 pm

Proposición: Sea $\triangle ABC$ un triángulo obtusángulo. El incírculo (I) es tangente a BC, CA, AB en $D, E, F,$ y la recta EF corta a la semicircunferencia $\overline{BC}$ en $X, Y.$ Si M es punto medio de $BC,$ pruebe que $\angle XMY = \angle BAC.$

Sean T, P, Q, K las proyecciones ortogonales de A, B, C, M en la recta $XY,$ respectivamente.

De las semejanzas $\triangle BPF \sim \triangle ATF$ y $\triangle CQE \sim \triangle ATE$ obtenemos:

$$\frac{BP}{AT} = \frac{p-b}{p-a}\ ,\ \frac{CQ}{AT} = \frac{p-c}{p-a} \implies BP + CQ = 2MK = \frac{BC \cdot AT}{(p-a)}$$

En otras palabras, $\dfrac{MK}{AT} = \dfrac{MX}{AE} \Longrightarrow$ Los triángulos isósceles $\triangle AFE$ y $\triangle MPX$ son semejantes.

Por consiguiente $\angle XMY = \angle BAC,$ como se deseaba probar.

Quick Reply

High School Olympiads

easy exercise

Reply

Source: Ireland 2001

moldovan

#1 Jul 6, 2009, 1:31 am

In an acute-angled triangle ABC, D is the foot of the altitude from A, and P a point on segment AD. The lines BP and CP meet AC and AB at E and F respectively. Prove that AD bisects the angle EDF.

Luis González

#2 Jul 6, 2009, 2:29 am

This is a particular case of the following fact:

P is a point on the plane of $\triangle ABC$. Lines PA, PB, PC cut BC, CA, AB at D, E, F. Define $U \equiv EF \cap AD$ and let V be the orthogonal projection of U onto BC. Then lines VU, BC bisect $\angle FVE$.

mathVNpro

#3 Jul 6, 2009, 8:01 am

Let $H_a \equiv EF \cap BC$. It is well- known that $(H_aDBC) = -1$. Let $G_a \equiv EF \cap AD \Longrightarrow (H_aG_aEF) = -1$. Hence $D(H_aG_aEF) = -1$, but $DH_a \perp DG_a$, thus DG_a is the internal bisector of $\angle EDF$ $\square$

sunken rock

#4 Aug 4, 2010, 11:31 pm

Let DF and DE intersect the parallel through A to BC at X and Y respectively and, as $XY \parallel BC \perp AD$, we need to prove $AX = AY$.

But $\frac{AX}{BD} = \frac{AF}{BF}$ and $\frac{AY}{DC} = \frac{AE}{CE}$, or, dividing side by side the two equalities: $\frac{AX}{AY} = \frac{AF \cdot BD}{BF} \cdot \frac{CE}{AE \cdot DC}$, but, as from Ceva for $\triangle ABC$ with P, the right side of the last equality is 1, hence our goal has been achieved.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Help me with barycentric coordinates

Reply

gb2124
#1 Jul 5, 2009, 9:14 am

How to identify the infinity point by the barycentric equation???
And what is the Conway's formula ?

Luis González
#2 Jul 5, 2009, 10:47 am

The line $px+qy+rz=0$ has infinite point $P_{\infty}(q-r:r-p:p-q)$.

The Conway formula gives the barycentric coordinates of a point P, for instance, defined by the balanced angles $\angle CBP=\phi$, $\angle BCP=\theta \Longrightarrow P \equiv (-a^2:S_C+S_\theta:S_B+S_\phi)$.

Keeping in mind the general Conway notation $S_\theta=S\cot\theta$.

Quick Reply

High School Olympiads

nice exercise (maybe posted before)

Reply

Source: Ireland 1998

moldovan

#1 Jul 4, 2009, 9:50 pm

The distances from a point P inside an equilateral triangle to the vertices of the triangle are $3, 4$, and 5. Find the area of the triangle.

Luis González

#2 Jul 4, 2009, 10:55 pm

Assume that $PA = 5, PB = 3$ and $PC = 4$. Let Q be the homologous of P under the rotation with center B and rotational angle 60° counterclockwise. Then $\triangle QBC \cong \triangle PBA$ and $\triangle BPQ$ is equilateral. Consequently, $QC = 5$ and $PQ = 3 \Longrightarrow \triangle PQC$ is right at $P \Longrightarrow \angle BPC = 150^\circ$. Hence, by cosine law in $\triangle PBC$ we get

$$AB^2 = 4^2 + 3^2 - 2 \cdot 3 \cdot 4 \cdot \cos 150^\circ = 25 + 12\sqrt{3} \Longrightarrow |\triangle ABC| = \frac{25\sqrt{3}}{4} + 9$$

Quick Reply

High School Olympiads

ratio along a side of a triangle

Reply

CatalystOfNostalgia
#1 Jul 4, 2009, 3:46 am

In triangle non-isosceles ABC, let AB=c, BC=a, CA=b. The A-altitude, angle bisector, and median meet BC at D, E, M, respectively. Prove that $\frac{DM}{ME}=\frac{(b+c)^2}{a^2}$.

Looking for a synthetic solution; it's pretty easy to just compute the lengths, but it's somewhat ugly and I think it requires the law of cosines. Also I know if a pretty contrived solution that involves taking MI and intersecting it with AD, but perhaps there is something simpler?

Luis González
#2 Jul 4, 2009, 1:20 pm

Perpendicular bisector of BC and angle bisector AE meet at the midpoint N of the arc BC and $\frac{DM}{ME}=\frac{AN}{EN}$. Power of E to (O) is $EB\cdot EC=AE\cdot EN$ and since AN is self-isogonal WRT $\angle A$, it follows that $AN\cdot AE=AB\cdot AC$

$$\Longrightarrow \frac{DM}{ME}=\frac{AN}{NE}=\frac{AE\cdot AB\cdot AC}{BE\cdot EC\cdot AE}=\frac{AB\cdot AC}{BE\cdot EC}$$

Using angle bisector theorem we get:

$$BE=\frac{ac}{b+c}\ ,\ CE=\frac{ab}{b+c}\Longrightarrow \frac{DM}{ME}=\frac{bc(b+c)^2}{a^2bc}=\frac{(b+c)^2}{a^2}$$

Quick Reply

High School Olympiads

Inspired by Ramanujan

Source: own

jayme

#1 Jul 2, 2009, 3:17 pm

Dear Mathlinkers,
let ABC be a triangle rectangular at A, (0) the circumcircle of ABC,
(1) the circle centered at B and cutting the segment BC at Q,
(2) the circle centered at C and cutting the segment BC at P,
D the second point of intersection of AQ with (0),
(3) the circle passing through A, P, D and
E the second point of intersection of BC with (3).
Prove that PQ = 2.CE.
Sincerely

vittasko

#2 Jul 3, 2009, 2:29 am

Dear **Jean Louis**, I guess that you mean the circle centered at B with radius BA and similarly, the circle centered at $C,$ with radius $CA.$

So, it is enough to to prove that $(PQ)^2 = 2(BP)(QC)$, (1) which is well known result, in the configuration of a right triangle $\triangle ABC,$ with $\angle A = 90^o$ and two points on the side-segment $BC,$ such that $BQ = BA$ and $CP = CA.$

If I am not mistaken, I will post here later the proof of (1) I have in mind, based on the fact that the circles (B) (centered at B with radius BA) and (C) (centered at C with radius CA), are orthogonal each other and then, the points $F,\ P,\ Q,\ Z,$ where $F\equiv (B)\cap BC$ (the other than Q) and $Z\equiv (C)\cap BC$ (the other than P), are in harmonic conjugation.

Kostas Vittas.

This post has been edited 1 time. Last edited by vittasko, Jul 4, 2009, 12:45 am

jayme

#3 Jul 3, 2009, 10:40 am

Dear Kostas,
sorry for my forgetfullness... You are right.
Is it possible to prove first the result of my problem, the to deduct the relation (of Ramanujan) you have mentionned?
Sincerely
Jean-Louis

vittasko

#4 Jul 4, 2009, 12:09 am

Let me dear **Jean Louis** to restate the notations of your nice problem.

Let $\triangle ABC$ be a right triangle with $\angle A = 90^o$ and circumcircle $(O).$ We denote the points $P,\ Q$ on the side-segment $BC,$ such that $BQ = BA$ and $CP = CA$ and let be the point $D\equiv (O)\cap AQ.$ The circumcircle (O_1) of the triangle $\triangle APD,$ intersects the sideline BC of $\triangle ABC,$ at point so be it $E.$ Prove that $PQ = 2(CE).$

PROOF. - We draw the circles $(B),\ (C),$ centered at verices $B,\ C$ of the given triangle $\triangle ABC,$ with radii $BA = BQ$ and $CA = CP$ respectively and we denote the points $F\equiv (B)\cap BC$ (the other than Q) and $Z\equiv (C)\cap BC$ (the other than P).

From the powers of the point Q, with respect to the circles $(O),\ (O_1)$, we have $(PQ)(QE) = (AQ)(QD) = (BQ)(QC)$, (1)

From $(1) \Longrightarrow (PQ)(QC + CE) = (BP + PQ)(QC) \Longrightarrow (PQ)(CE) = (BP)(QC)$, (2)

From (2), in order to be $PQ = 2(CE)$ it is enough to prove $(PQ)^2 = 2(BP)(QC)$, (3) (I didn't know that this relation, is a result by the famous **Ramanujian**).

From $(3) \Longrightarrow (PQ)^2 = 2(BQ - PQ)(PC - PQ) \Longrightarrow (PQ)^2 + 2(BQ)(PC) = 2(PQ)(BQ + PC)$, (4)

We will prove that the relation (4) is true.

• Because of BA tangents to the circle (C) at point A, we have that $(BP)(BZ) = (BA)^2 = (BQ)^2 = (BF)^2$, (5)

From (5), based on the **Newton theorem**, we conclude that the points $F,\ P,\ Q,\ Z$, are in harmonic conjugation as well.

So, we have that $\dfrac{QP}{QZ} = \dfrac{FP}{FZ} \Longrightarrow (PQ)(FZ) = (FP)(QZ)$, (6)

From $(6) \Longrightarrow (PQ)[2(BQ) + 2(PC) - PQ] = [2(BQ) - PQ][2(PC) - PQ] \Longrightarrow$
$4(PQ)(BQ + PC) = 4(BQ)(PC) + 2(PQ)^2$, (7)

From $(7) \Longrightarrow (PQ)^2 + 2(BQ)(PC) = 2(PQ)(BQ + PC)$ and the proof is completed.

> *jayme wrote:*
> ... Is it possible to prove first the result of my problem, the to deduct the relation of Ramanujan?

Although I have the proof of some results, probably useful, however I don't know if I can prove first the relation $PQ = 2(CE)$ of the proposed problem.

Best regards, Kostas Vittas.

Attachments:
t=286472.pdf (8kb)

Luis González

#5 Jul 4, 2009, 3:44 am

Notice that $BP \cdot CQ = (a - b)(a - c) = a^2 + bc - ac - ab$ and

$PQ = a - (a - b) - (a - c) = b + c - a$

$\Longrightarrow PQ^2 = b^2 + c^2 + a^2 + 2bc - 2ac - 2ab$

Since $a^2 = b^2 + c^2 \Longrightarrow PQ^2 = 2(a^2 + bc - ac - ab)$

But $BQ \cdot QC = (a - b)(a - c) = a^2 + bc - ab - ac \Longrightarrow PQ^2 = 2BP \cdot QC\ (*)$

Inversion with center Q and power $BQ \cdot QC$ maps $C \mapsto B$ and $E \mapsto P$. Therefore

$$CE = BP \cdot \frac{BQ \cdot QC}{BQ \cdot PQ} \Longrightarrow CE \cdot PQ = BP \cdot QC$$

Then combinig with $(*)$ yields $CE \cdot PQ = \frac{1}{2} PQ^2 \Longrightarrow PQ = 2CE.$

Virgil Nicula

#6 Jul 4, 2009, 3:58 am

> *Virgil Nicula wrote:*
> Let ABC be a triangle for which $a > b$, $a > c$. Consider the points $\{P, Q\} \subset [BC]$ so that $PC = b$, $BQ = c$.
>
> The line AQ cut again the circumcircle of $\triangle ABC$ in D and the line BC cut again the circumcircle of $\triangle APD$ in E .
>
> Prove that $\boxed{\dfrac{PQ}{CE} = 2 + \dfrac{b^2 + c^2 - a^2}{(a-b)(a-c)}}$ and $A = 90^\circ \iff PQ = 2 \cdot CE$.

$$\boxed{\frac{PQ}{CE} = 2 + \frac{b^2+c^2-a^2}{(a-b)(a-c)}}$$

Proof. Using the power of a point w.r.t. a circle obtain : $QB \cdot QC = QA \cdot QD = QP \cdot QE$. Since $QB = c$,

$QC = a - c$, $QP = b + c - a$ results $QE = \frac{c(a-c)}{b+c-a}$ and $CE = QE - QC \implies CE = \frac{(a-b)(a-c)}{b+c-a}$.

Therefore, $\frac{PQ}{CE} = \frac{(b+c-a)^2}{(a-b)(a-c)} = 2 + \frac{b^2+c^2-a^2}{(a-b)(a-c)}$ a.s.o.

Remarks.

1 ▶ If $A = 120^\circ$, then $PQ < CE$.

2 ▶ A very interesting and nice algebraic identity (**own** ?!) with many applications :

$\boxed{\sum \frac{b^2+c^2-a^2}{(a-b)(a-c)} = -2}$, where $\{a, b, c\} \subset \mathbb{C}$ and $a \ne b \ne c \ne a$.

3 ▶ Prove "easily" that in generally :

> Virgil Nicula wrote:
> Let ABC be a triangle for which $a \ne b \ne c \ne a$. Consider the points $P \in (CB$, $Q \in (BC$ so that $CP = b$, $BQ = c$.
>
> The line AQ cut again the circumcircle of $\triangle ABC$ in D and the line BC cut again the circumcircle of $\triangle APD$ in E . Prove that
>
> $\boxed{\frac{\overline{PQ}}{\overline{CE}} = 2 + \frac{b^2+c^2-a^2}{(a-b)(a-c)}}$, where $X(x)$, $\overline{XY} = y - x$ - an analytical geometry on the orientated line $d = BC$
> .

4 ▶ Prove similarly that :

> Virgil Nicula wrote:
> Let ABC be a triangle . Consider the points $\{P, Q\} \subset BC$ so that $C \in (BP)$, $B \in (CQ)$ and $CP = b$, $BQ = c$.
>
> The line AQ cut again the circumcircle of $\triangle ABC$ in D and the line BC cut again the circumcircle of $\triangle APD$ in E .
>
> Prove that $\boxed{\frac{PQ}{CE} = 2 + \frac{b^2+c^2-a^2}{(a+b)(a+c)}}$. Is this **Ramanujan**'s problem ?!

This post has been edited 4 times. Last edited by Virgil Nicula, Jul 5, 2009, 1:31 am

jayme

#7 Jul 4, 2009, 6:05 pm

Dear Kostas, Luis, Virgil and Mathlinkers,
thank you for all your nice proofs.
The result involves by Kostas comes from Ramanujan (Bernd, B.C. (1994): Ramanujan's Noteboooks, Part IV. Springer -Verlag).
For my problem, I had in mind the Haruki's result : BP.CQ/PQ = CE... which comes from a more general case.
Sincerely
Jean-Louis

Virgil Nicula

#8 Jul 5, 2009, 5:42 am

> Virgil Nicula wrote:
> Let ABC be a triangle for which $a > b$, $a > c$. Consider the points $\{P, Q\} \subset [BC]$ so that $PC = b$, $BQ = c$.
>
> The line AQ cut again the circumcircle of $\triangle ABC$ in D and the line BC cut again the circumcircle of $\triangle APD$ in E

The line AQ cut again the circumcircle of $\triangle ABC$ in D and the line BC cut again the circumcircle of $\triangle APD$ in E.

Prove that $\boxed{\frac{PQ}{CE}=2+\frac{b^2+c^2-a^2}{(a-b)(a-c)}}$ and $A=90^{\circ}\iff PQ=2\cdot CE$.

Proof. Using the power of a point w.r.t. a circle obtain : $QB\cdot QC=QA\cdot QD=QP\cdot QE$. Since $QB=c$,

$QC=a-c$, $QP=b+c-a$ results $QE=\frac{c(a-c)}{b+c-a}$ and $CE=QE-QC\implies CE=\frac{(a-b)(a-c)}{b+c-a}$.

Therefore, $\frac{PQ}{CE}=\frac{(b+c-a)^2}{(a-b)(a-c)}=2+\frac{b^2+c^2-a^2}{(a-b)(a-c)}$ a.s.o.

Remarks.

1 ▶ If $A=120^{\circ}$, then $PQ<CE$.

2 ▶ A very interesting and nice algebraic identity (**own** ?!) with many applications :

$\boxed{\sum\frac{b^2+c^2-a^2}{(a-b)(a-c)}=-2}$, where $\{a,b,c\}\subset\mathbb{C}$ and $a\ne b\ne c\ne a$.

3 ▶ Prove "easily" that in generally :

> *Virgil Nicula wrote:*
> Let ABC be a triangle for which $a\ne b\ne c\ne a$. Consider the points $P\in(CB$, $Q\in(BC$ so that $CP=b$, $BQ=c$.
>
> The line AQ cut again the circumcircle of $\triangle ABC$ in D and the line BC cut again the circumcircle of $\triangle APD$ in E. Prove that
>
> $\boxed{\frac{\overline{PQ}}{\overline{CE}}=2+\frac{b^2+c^2-a^2}{(a-b)(a-c)}}$, where $X(x)$, $\overline{XY}=y-x$ - an analytical geometry on the orientated line $d=BC$
> .

4 ▶ Prove similarly that :

> *Virgil Nicula wrote:*
> Let ABC be a triangle . Consider the points $\{P,Q\}\subset BC$ so that $C\in(BP)$, $B\in(CQ)$ and $CP=b$, $BQ=c$.
>
> The line AQ cut again the circumcircle of $\triangle ABC$ in D and the line BC cut again the circumcircle of $\triangle APD$ in E.
>
> Prove that $\boxed{\frac{PQ}{CE}=2+\frac{b^2+c^2-a^2}{(a+b)(a+c)}}$. Is this **Ramanujan**'s problem ?!

5 ▶ Prove similarly that :

> *Virgil Nicula wrote:*
> Let ABC be a triangle and consider the points $\{P,Q\}\subset(BC)$ so that $BQ^2+CP^2=BC^2$. The line AQ cut again
>
> the circumcircle of $\triangle ABC$ in D and the line BC cut again the circumcircle of $\triangle APD$ in E. Prove that $PQ=2\cdot CE$.

6 ▶ Prove similarly that :

> *Virgil Nicula wrote:*
> Let ABC be a A-right triangle and consider the points $\{X,Y,Z,T\}\subset BC$ (line !) so that
>
> $\left\|\begin{array}{ccccc} B\in(XY) & , & Y\in(BC) & , & BX=BY=AB\\ C\in(ZT) & , & Z\in(BC) & , & CZ=CT=AC\end{array}\right\|$. Prove that $\widehat{XAZ}\equiv\widehat{ZAY}\equiv\widehat{YAT}$.

$$\| \; C \in (ZT) \quad , \quad Z \in (BC) \quad , \quad CZ = CT = AC \; \|$$

jayme

#9 Jul 5, 2009, 11:59 am

Dear Virgil and Mathlinkers,
your last message is very interesting.
The Ramanujan's problem after my source is:

let ABC be a triangulo rectangular at A. On the side BC take to points P and Q so that BQ=BA and CP=CA.
Prove: $PQ^2=2BP.QC$.

Sincerely
Jean-Louis

jayme

#10 Jul 9, 2009, 1:48 pm

Dear Mathlinkers,
my proof of Ramanujan's problem:
according to Haruki's theorem, BP.CQ = PQ.CE
See email : http://www.mathlinks.ro/Forum/viewtopic.php?p=1555284#1555284
PQ = 2.CE
and we are done
Sincerely
Jean-Louis

armpist

#11 Jul 18, 2009, 10:24 pm

It seems to me that Ramanujan came up with this identity PQ^2 = 2*BP * CQ

immediately after reading Propositions 5 thru 8

of Book II of the Elements. That one, after all, is concerned with Geometric Algebra.

It is hard to tell which of the above Propositions affected Srini the most.

If it turns out that his is a new Proposition, then maybe we can squeeze it right between

Props. 8 and 9 in the same Book II on the account of it's great importance.

And it will be another confirmation of Fermat's notion that those bearded Greeks

did not know everything.

It is also worth while investigating the "2r" connection of this problem with the Sangakus

see http://www.mathlinks.ro/Forum/viewtopic.php?search_id=1150708027&t=264276

http://www.mathlinks.ro/Forum/viewtopic.php?search_id=1150708027&t=270926

Historical note:

Young Srini failed math tests to make the Indian team to IMO two years in a row.
He managed to do Geometry problems, yet Number Theory tasks proved to be beyond
his abilities.

Also, his last name points towards Armenian origin Ramanujian. It is interesting that Armenian
mathematicians haven't yet claimed him to be their own.

M.T.

Attachments:

Srini.doc (24kb)

jayme

#12 Jul 19, 2009, 10:52 am

Dear Mark,
thank you for your historical note and references.
You attached figure suggest an area approch...
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Interesting relation

Reply

Source: Me

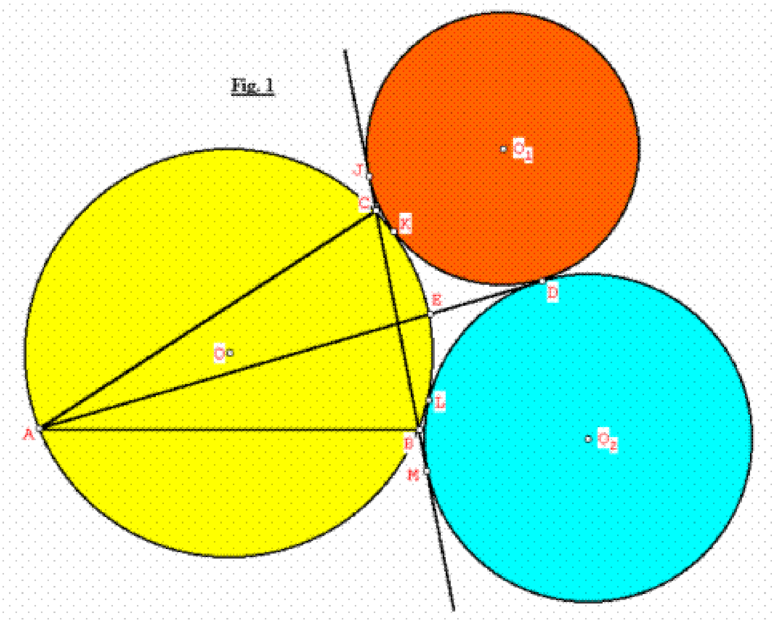
juancarlos

#1 Apr 27, 2005, 5:18 am

Drawn two external tangent circles (O_1, r_1) and (O_2, r_2) also tangents to the line BC and circumcircle (O) of ABC triangle such that the common internal tangent of (O_1) and (O_2) passes trough A. Similarly drawn two external tangent circles (O_3, r_3) and (O_4, r_4) and tangents to the line AC and (O) with common internal tangent of (O_3) and (O_4) passes trough B and also drawn the external tangent circles (O_5, r_5) and (O_6, r_6) and tangents to the line AB and (O) with common internal tangent of (O_5) and (O_6) passes trough C. If r is the inradius of ABC, prove that: $\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} + \frac{1}{r_5} + \frac{1}{r_6}$

See Fig.1, drawing the circles (O_1, r_1) and (O_2, r_2):

Attachments:



This post has been edited 1 time. Last edited by juancarlos, Jun 18, 2005, 5:16 am

jhaussmann5

#2 Apr 28, 2005, 5:27 pm

Can you give us a hint?

juancarlos

#3 Apr 29, 2005, 2:57 am

In Fig. 1, D is excenter.

juancarlos

#4 Jan 26, 2006, 5:33 am

Please see:
http://www.personal.us.es/rbarroso/trianguloscabri/sol/sol293sal.htm

Luis González

#5 Jul 3, 2009, 11:52 pm

From Thebault/Sawayama theorem $D \equiv I_a$ is the A-excenter. Letting P, Q, Y be the tangency points of $(O_1), (O_2), (I_a)$ with BC, then in the trapezoid $O_1O_2QP, I_aY = r_a$ is the harmonic mean between its bases $\Longrightarrow$

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{r_a}$$

Thus, the cyclic sum yields:

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} + \frac{1}{r_5} + \frac{1}{r_6} = 2\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right) = \frac{2}{r}$$

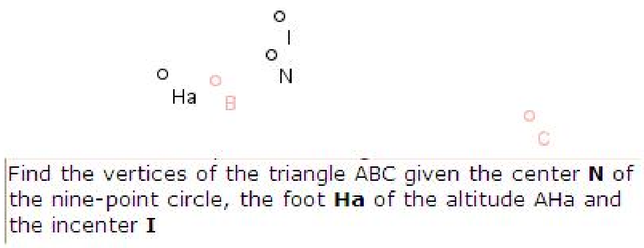
Quick Reply

High School Olympiads

HaNI

jrrbc
#1 Jul 2, 2009, 4:06 am

Attachments:



Luis González
#2 Jul 2, 2009, 5:59 am

Draw the 9-point circle with center N and radius $NH_a = \frac{1}{2}R$. Draw the incircle with center I internally tangent to (N). Tangent from H_a to (I) meets (N) again at the midpoint M_a of BC (we have two possibilities). Perpendicular line to the chord M_aH_a through M_a is the perpendicular bisector $\mathcal{L}_a$ of BC. Since the distance $IO = \sqrt{R^2 - 2Rr}$ is known, the intersections of the circle $(I, \sqrt{R^2 - 2Rr})$ with $\mathcal{L}_a$ gives the possible circumcenters O. The circle (O, R) meets the line M_aH_a at B, C and the perpendicular to M_aH_a trough H_a at A.

jrrbc
#3 Jul 2, 2009, 7:31 am

please the IO construction

Luis González
#4 Jul 2, 2009, 9:26 am

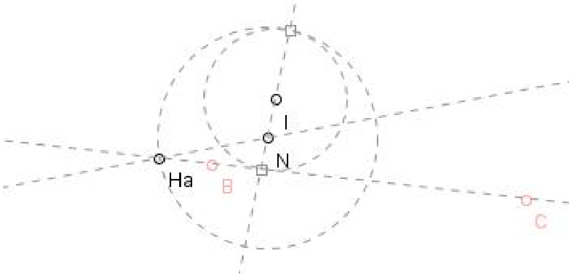
It is a basic ruler-compass construction

On an arbitray line $\mathcal{L}$ take three points X, Y, Z such that $XY = 2r$ and $YZ = R$. Draw a semicircle with diameter XZ. Perpendicular through Y meets the semicircumference at M, $MY = z$. Now, construct a right-angled triangle with cathetus z and hypotenuse R. The other cathetus measures IO.

jrrbc
#5 Jul 2, 2009, 10:51 pm

is this the tangent circles construction ?

Attachments:

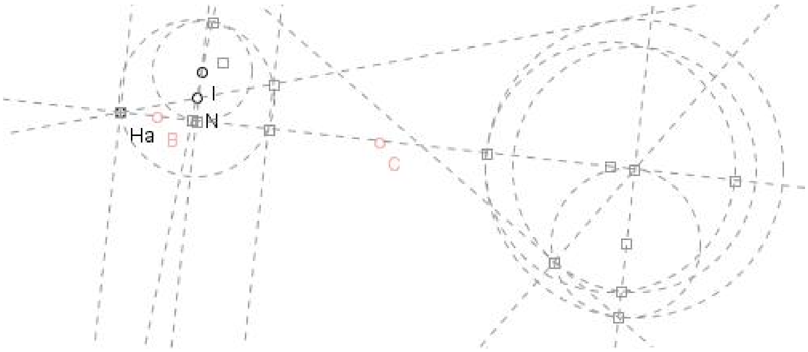


jrrbc

#6 Jul 2, 2009, 11:54 pm

the construction is not ok

Attachments:



Luis González

#7 Jul 3, 2009, 12:15 am

I really don't see what isn't clear in my resolution. The incircle and nine-point circle are internally tangent (celebrated Feuerbach theorem). Ray NI cuts (N) at Fe (Feuerbach point). (I,IFe) is the incircle of ABC. Tangent from Ha to (I) cuts (N) again at Ma (choose Ma according to your sketch). Finally make sure that you constructed the segment IO well.

yetti

#8 Jul 6, 2009, 1:08 am

Circle (N) with center N and radius NH_a is 9-point circle. Ray $\overrightarrow{NI}$ cuts (N) at the Feuerbach point F. Circle (I) with center I and radius IF is the incircle. Let a, a' be tangents to (I) from H_a, with tangency points D, D', leading to 2 solutions. Consider only one tangent a, cutting (N) again at M_a. Perpendicular m_a to a at M_a is the perpendicular bisector of BC and perpendicular h_a to a at H_a is the A-altitude line. Parallel $p \parallel a$ through I cuts h_a at K. DK cuts m_a at $X \in (O)$ on the triangle circumcircle. IX is internal bisector of $\angle CAB$, cuttng h_a at A. Tangents to (I) from A cut a at B, C.

jrrbc

#9 Jul 6, 2009, 3:23 am

thanks

Quick Reply

High School Olympiads

easy geometry

Reply

Source: Ireland 1996

moldovan

#1 Jul 2, 2009, 12:18 am

Let F be the midpoint of the side BC of a triangle ABC. Isosceles right-angled triangles ABD and ACE are constructed externally on AB and AC with the right angles at D and E. Prove that the triangle DEF is right-angled and isosceles.

Luis González

#2 Jul 2, 2009, 12:31 am

Let D and E be the centers of the squares $ABXY$ and $ACPQ$. Then $\triangle ABQ$ and $\triangle ACY$ are congruent by SAS criterion, due to $AB = AY, AC = AQ$ and $\angle BAQ = \angle YAC$. Therefore, $BQ = YC$ and $\angle ACY = \angle AQB$. If $M \equiv YC \cap BQ$, then quadrilateral $AMCQ$ is cyclic $\Longrightarrow \angle QMC$ is right. Lines FD and FE are respectively parallel to CY and BQ since D, E are midpoints of BY, CQ. Hence $FD \perp FE \Longrightarrow \triangle DEF$ is isosceles right at F.

dgreenb801

#3 Jul 4, 2009, 11:22 pm

Let M and N be the midpoints of AB and AC. Then $DM = NF = AB/2$ and $EN = MF = AC/2$, and $\angle DF = \angle DMB + \angle BFM = 90 + \angle A = \angle ENF$, so $\triangle DMF$ is congruent to $\triangle ENF$, so $DF = FE$. Also we know $\angle MFN = \angle A$ and $\angle MFD + \angle NFE = \angle MFD + \angle MDF = 180 - \angle DMF = 180 - (90 + \angle A) = 90 - \angle A$, so $\angle DFE = 90$.

mathVNpro

#4 Jul 4, 2009, 11:48 pm

We have $\mathcal{R}_E^{-90^\circ} : C \mapsto A, \mathcal{R}_D^{-90^\circ} : A \mapsto B$. Therefore $\mathcal{R}_D^{-90^\circ} \circ \mathcal{R}_E^{-90^\circ} : C \mapsto B$. But $\mathcal{R}_D^{-90^\circ} \circ \mathcal{R}_E^{-90^\circ} = \mathcal{R}_M^{-180^\circ}$. Therefore $M \equiv F$ which is the midpoint of BC. but $\angle DFE = \angle DEF = \dfrac{90^\circ}{2} = 45^\circ$. Hence $\triangle DEF$ is right-angled and isosceles $\square$.

Quick Reply

Spain

Geometria

Reply

juancarlos

#1 May 25, 2006, 2:48 am

En el triángulo ABC el incirculo y el excirculo opuesto al vertice A determinan los puntos de tangencia $(D,E),(M,J),(N,K)$ sobre BC,AC y AB respectivamente. Si DP y EQ son perpendiculares a MN y JK, probar que $<BAP=<CAQ$.

M4RI0

#2 May 25, 2006, 6:14 am

P y Q son puntos arbitrarios :

juancarlos

#3 May 25, 2006, 8:12 pm

Favor considerar P sobre MN y Q sobre JK

José

#4 May 31, 2006, 5:34 am

alguna sugerencia? y el gráfico?

Ashegh

#5 May 31, 2006, 2:30 pm

im curies to solve it. could any body plz trans late it?

thanks my dears.

senioritta....

skuge

#6 Jun 13, 2006, 8:39 am

> juancarlos wrote:
> En el triángulo ABC el incirculo y el excirculo opuesto al vertice A determinan los puntos de tangencia $(D,E),(M,J),(N,K)$ sobre BC,AC y AB respectivamente. Si DP y EQ son perpendiculares a MN y JK, probar que $<BAP=<CAQ$.

Trazo la perpendicular a BC que es tangente al excirculo en H y corta a la bizecriz interna por A en S. Trazo la perpendicular a JK por H y esta cora al excirculo en E'.
I es el centro del excirculo. AI es bisectriz entonces es perpendicular a JK. AI es paralela a HE'.
De lo anterior $\angle AIE'=\angle IE'H$ como IE'H es isoseles $\angle IE'H=\angle IHE'$.
Otra vez por ser paralelas $\angle IHE'=\angle HIS$
$\angle AIE'=\angle HIS$
Reflejo H sobre IA y obtengo H' sobre el excirculo. que es colinear con Q y E. (Esto no es fácil de ver, pero E es opuesto diametralmente a H y H' es opuesto a E' porque $\angle AIE'=\angle HIS$)
Ahora hago una homotesia de toda la figura superior sobre A de manera que el incirculo caiga sobre el excírculo.

Ahora hago una homotesia de toda la figura superior sobre A de manera que el incírculo caiga sobre el exírculo.
en esta homotesia es claro que P se vuelve el corte de HE' y JK (Este punto lo llamaremos P')
ahora reflejo P' sobre AI y obtengo Q (por lo que probé anteriormente).
$< BAP =< BAP'$ y $< BAP' =< CAQ$
=> $< BAP =< CAQ$
QED

No está muy riguroso y puede tener errores, pero creo que está completo.
Cualquier sugerencia bienvenida.

---

felipeochoa
#7 Jun 15, 2006, 9:47 am

> *Ashegh wrote:*
> im curies to solve it. could any body plz trans late it?

In the triangle ABC the incircle and the excircle opposite vertex A are tangent at $(D, E), (M, J), (N, K)$ on $BC, AC,$ and $AB,$ respectively. If DP and EQ are perpendicular to MN and JK, prove that $\angle BAP = \angle CAQ$

Ashegh
#8 Jun 23, 2006, 11:30 pm

thanks my dear.

did any body solve it by inversion?

if not i can send my solution.

thanks again for translating...

José
#9 Jun 24, 2006, 3:06 am

why do you love so much inversion and projection?????

Yes, post your solution...

Ashegh
#10 Jun 24, 2006, 4:11 pm

> *José wrote:*
> why do you love so much inversion and projection?????
>
> Yes, post your solution...

dear jose, first i should say that in this time about 9 hour later,

the game betwin argentina and mexico will begin,

and i hope we(argentina) will win...

and second, hmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmm.

dear jose, i dont know why i love inversion and projection.

may be, because they are incredible.

Luis González
#11 Jul 1, 2009, 6:01 am

Definamos las coordenadas baricentricas con respecto a $\triangle ABC$.

$D(0:p-c:p-b)$, $M(p-c:0:p-a)$, $N(p-b:p-a:0)$

$E(0:p-b:p-c)$, $K(p-c:-p:0)$, $J(p-b:0:-p)$

$MN \equiv (p-a)x-(p-b)y-(p-c)z=0,\ KJ \equiv px+(p-c)y+(p-b)z=0$

La rectas $\mathcal{L}_1$ y $\mathcal{L}_2$ que pasan por D, E perpendiculares a MN y KJ tienen por punto del infinito $(b+c:-b:-c)$ que es el de la bisectriz de $\angle A$. Por tanto sus ecuaciones son:

$\mathcal{L}_1 \equiv (c(p-c)-b(p-b))x-(b+c)(p-b)y+(b+c)(p-c)z=0$

$\mathcal{L}_2 \equiv (b(p-c)-c(p-b))x+(b+c)(p-c)y-(b+c)(p-b)z=0$

$\mathcal{L}_1$ y $\mathcal{L}_2$ intersectan MN y KJ en

$P\ (***:b(p-a)(p-c):c(p-b)(p-a))$, $Q\ (***:b(p-b):c(p-c))$

En otras palabras, los rayos AP y AQ pasan por

$X_{57}=(a(p-b)(p-c):b(p-a)(p-c):c(p-b)(p-a))$

$X_9=(a(p-a):b(p-b):c(p-c))$

Estos puntos son isogonales y en consecuencia $\angle BAP=\angle CAQ$

Click to reveal hidden text

Quick Reply

High School Olympiads

equilateral triangle

Reply

Source: Ireland 1994

moldovan

#1 Jun 30, 2009, 12:56 am

Let A, B, C be collinear points on the plane with B between A and C. Equilateral triangles ABD, BCE, CAF are constructed with D, E on one side of the line AC and F on the other side. Prove that the centroids of the triangles are the vertices of an equilateral triangle, and that the centroid of this triangle lies on the line AC.

Luis González

#2 Jun 30, 2009, 6:21 am

Let $(O_1) \equiv (ADB), (O_2) \equiv (BCE)$ and $O \equiv (O_1) \cap (O_2)$. It is enough to see that $\angle AOC = 120^\circ \Longrightarrow O$ lies on the circumcircle (O_3) of $\triangle CAF$. OA, OB and OC are radical axes of $(O_1) \cup (O_3), (O_1) \cup (O_2)$ and $(O_2) \cup (O_3) \Longrightarrow$ $OA \perp O_1O_3, OB \perp O_1O_2$ and $OC \perp O_2O_3$. As a result, we have $\angle O_2O_1O_3 = \angle AOB = 60^\circ$ and $\angle O_1O_2O_3 = \angle BOC = 60^\circ \Longrightarrow \triangle O_1O_2O_3$ is equilateral.

Click to reveal hidden text

Centroid of $\triangle O_1O_2O_3$ lies on the line $AC \Longleftrightarrow$ Sum of the oriented distances from O_1, O_2, O_3 to AC equals zero. Letting M, N, L be the midpoints of AB, BC, AC

$$O_1M = \frac{1}{3}DM \ , \ O_2N = \frac{1}{3}EN \ , \ O_3L = \frac{1}{3}FL \ \Longrightarrow$$

$$O_1M + O_2N - O_3L = \frac{1}{3}(DM + EN - FL) = \frac{1}{6}\sqrt{3}(AC - AC) = 0$$

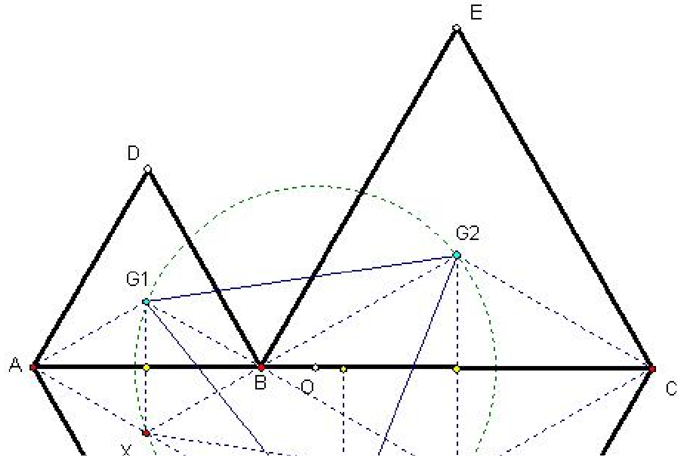
$\Longrightarrow$ The centroid of $\triangle O_1O_2O_3$ lies on the line AC, as desired.

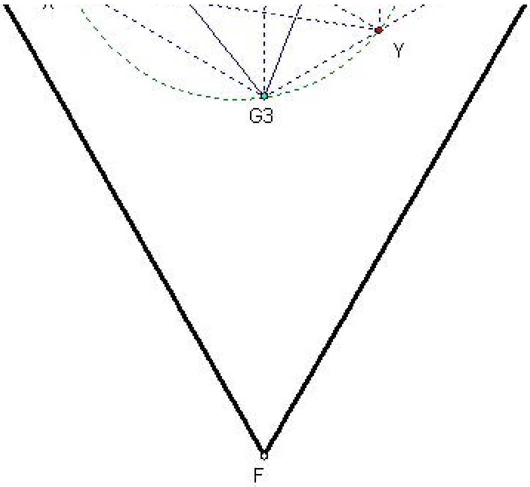
plane geometry

#3 Jun 30, 2009, 8:40 am

using cos law
we acquire G1G2=G2G3=G1G3
we can also prove G1AX G2CY are both equilateral triangles
G1G2G3XY are coocyclic further, G1G2YX is an isosceles trapezoid thus the circumcenter lies on the perpendicular bisector of XG1,YG2
done

Attachments:





mathVNpro

#4 Jun 30, 2009, 10:55 am

Denote G_1, G_2, G_3 respectively by the centroid of $\triangle DAB$, $\triangle EBC$, $\triangle FAC$. Consider the rotation around $G_i, i \in \{1,2\}$ with the angle 120°, we have $\mathcal{R}_{G_1}^{120^\circ} : A \mapsto B$. Similarly, we also have $\mathcal{R}_{G_2}^{120^\circ} : B \mapsto C \Longrightarrow \mathcal{R}_{G_1}^{120^\circ} \circ \mathcal{R}_{G_2}^{120^\circ} : A \mapsto C$. But $\mathcal{R}_{G_1}^{120^\circ} \circ \mathcal{R}_{G_2}^{120^\circ} = \mathcal{R}_{G}^{240^\circ} = \mathcal{R}_{G}^{-120^\circ}$. Therefore $\mathcal{R}_{G}^{-120^\circ} : A \mapsto C \Longrightarrow G \equiv G_3$. But we also have $\angle GG_1G_2 = \angle GG_2G_1 = \dfrac{120^\circ}{2} = 60^\circ \Longrightarrow \triangle G_3G_1G_2$ is an equilateral triangle.

Now, let $\mathcal{Z}\left(B, -120^\circ, \dfrac{BE}{BA}\right) : A \mapsto E, D \mapsto C, \triangle BDA \mapsto \triangle BCE$. Hence $G_1 \mapsto G_2$. Therefore, $\angle G_1BG_2 = 120^\circ$. Let the circumcircle (O) of $\triangle G_1BG_2$ intersects AC at G_4 as the second point. We have $\angle G_1OG_2 = G_1BG_2 = 120^\circ$, but $\angle G_4G_1G_2 = \angle G_2BG_4 \equiv \angle G_2BC = 60^\circ$, $\angle G_4G_2G_1 = \angle G_1BA = 60^\circ$. Then it is followed that through the rotation with center G_4, angle -120°, we have $\mathcal{R}_{G_4}^{-120^\circ} : G_1 \mapsto G_2$. Hence G_4 must be the centroid of $\triangle G_1G_2G_3$. The result is lead as follow. $\square$

Virgil Nicula

#5 Jul 2, 2009, 8:03 am

> *Napoleon wrote:*
>
> Let A , B , C be three points on the plane. Equilateral triangles ABD , BCE , CAF are constructed outside of given triangle.
>
> Prove that $AE = BF = CD$, $AE \cap BF \cap CD \neq \emptyset$, the centroids G_c , G_a , G_b of the triangles ABD , BCE , CAF
>
> are the vertices of an equilateral triangle and the triangles ABC and $G_aG_bG_c$ have a common centroid G and ... another properties.
>
> Indication.

Particular case. When $B \in (AC)$ obtain the proposed problem in which the common centroid of the triangles ABC

(degenerated !) and $G_aG_bG_c$ is the point $G \in (BM)$, where M is the midpoint of $[AC]$ and $BG = 2 \cdot GM$.

Quick Reply

High School Olympiads

configuration

Reply

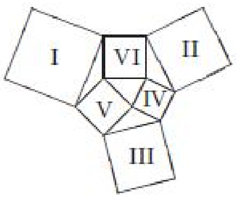
Source: Netherlands 1992

moldovan

#1 Jun 28, 2009, 4:43 pm

Consider the configuration of six squares as shown on the picture. Prove that the sum of the area of the three outer squares (I, II and III) equals three times the sum of the areas of the three inner squares (IV, V and VI).

Attachments:



EcstaticPotter

#2 Jun 28, 2009, 11:47 pm

use the law of cosines repeatedly, and the fact that cos (180-x) = - cos (x)

Luis González

#3 Jun 29, 2009, 6:04 am • 1

Problem: On the sides of $\triangle ABC,$ we construct outwardly the squares $ABED, CHFB, ALKC.$ Let S_1, S_2, S_3 be the areas of the squares with side length $LD, EF, HK.$ Prove that

$$[ABED]+[CHFB]+[ALKC]=\frac{1}{3}(S_1+S_2+S_3)$$

Solution: Let B' be the reflection of B about $A.$ Then it is easy to see that $\triangle ADL$ and $\triangle AB'C$ are congruent. Thus if M is the midpoint of $BC,$ we have $DL=B'C=2AM.$ This means that DL, FE, HK are twice the corresponding medians m_a, m_b, m_c of $\triangle ABC.$ From ${m_a}^2+{m_b}^2+{m_c}^2=\frac{3}{4}(a^2+b^2+c^2),$ we deduce then the desired relation.

jayme

#4 Oct 24, 2014, 2:38 pm • 1

Dear Mathlinkers,
you can see
http://jl.ayme.pagesperso-orange.fr/ vol. 5 la figure de Vecten p. 104-106

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Source: IMO Shortlist 1993, Indonesia 1

orl

#1 Mar 26, 2006, 4:23 am

The vertices D, E, F of an equilateral triangle lie on the sides BC, CA, AB respectively of a triangle ABC. If a, b, c are the respective lengths of these sides, and S the area of ABC, prove that

$$DE \geq \frac{2 \cdot \sqrt{2} \cdot S}{\sqrt{a^2 + b^2 + c^2 + 4 \cdot \sqrt{3} \cdot S}}.$$

tweedledum

#2 Apr 8, 2006, 3:33 am

does anyone know how to solve this problem??
because i've tried a lot and i couldn't solve it...

yetti

#3 Apr 14, 2006, 2:33 pm

The pedal triangle of the 1st isodynamic point Q (the intersection of the 3 Apollonius circles inside the triangle $\triangle ABC$) is equilateral. For example: Let D, E, F be the feet of normals from Q to BC, CA, AB. The A-Apollonius circle is a locus of points with the ratio of distances $\frac{AB}{AC}$ from the triangle vertices B, C and similarly for the B- and C-Apollonius circles. Hence $\frac{QB}{QC} = \frac{AB}{AC}, \frac{QC}{QA} = \frac{BC}{BA}, \frac{QA}{QB} = \frac{CA}{CB}$. The quadrilateral AEQF is cyclic with the right angles at the vertices E, F, hence its circumradius is $R_A = \frac{AQ}{2}$, so that $EF = 2R_A \sin A = AQ \sin A$ and similarly $FD = BQ \sin B, DE = CQ \sin C$. Consequently,

$$\frac{DE}{EF} = \frac{CQ \sin C}{AQ \sin A} = \frac{BC \sin C}{AB \sin A} = 1, \quad DE = EF$$

and similarly EF = FD. From the cyclic quadrilaterals AEQF, BFQD,

$$\angle AQB = \angle AQF + \angle BQF = \angle AEF + \angle BDF =$$

$$= 180^\circ - \angle A - \angle AFE + 180^\circ - \angle B - \angle BFD =$$

$$= 180^\circ - (\angle A + \angle B) + 180^\circ - (\angle AFE + \angle BFD) = \angle C + 60^\circ$$

and similarly, $\angle BQC = \angle A + 60^\circ, \angle CQA = \angle B + 60^\circ$. Let e = DE = EF = FD be the side length of the equilateral pedal triangle $\triangle DEF$. The area S of the triangle $\triangle ABC$ with circumradius R is

$$S = \frac{1}{2}[AQ \cdot BQ\ \sin(C + 60^\circ) + BQ \cdot CQ\ \sin(A + 60^\circ) + CQ \cdot AQ\ \sin(B + 60^\circ)] =$$

$$= \frac{e^2}{2}[\frac{\sin(C + 60^\circ)}{\sin A \sin B} + \frac{\sin(A + 60^\circ)}{\sin B \sin C} + \frac{\sin(B + 60^\circ)}{\sin C \sin A}] =$$

$$= \frac{4R^3 e^2}{abc}[\sin A \sin(A + 60^\circ) + \sin B \sin(B + 60^\circ) + \sin C \sin(C + 60^\circ)] =$$

$$= \frac{R^2 e^2}{S}[\frac{1}{2}(\sin^2 A + \sin^2 B + \sin^2 C) + \frac{\sqrt{3}}{2}(\sin A \cos A + \sin B \cos B + \sin C \cos C)] =$$

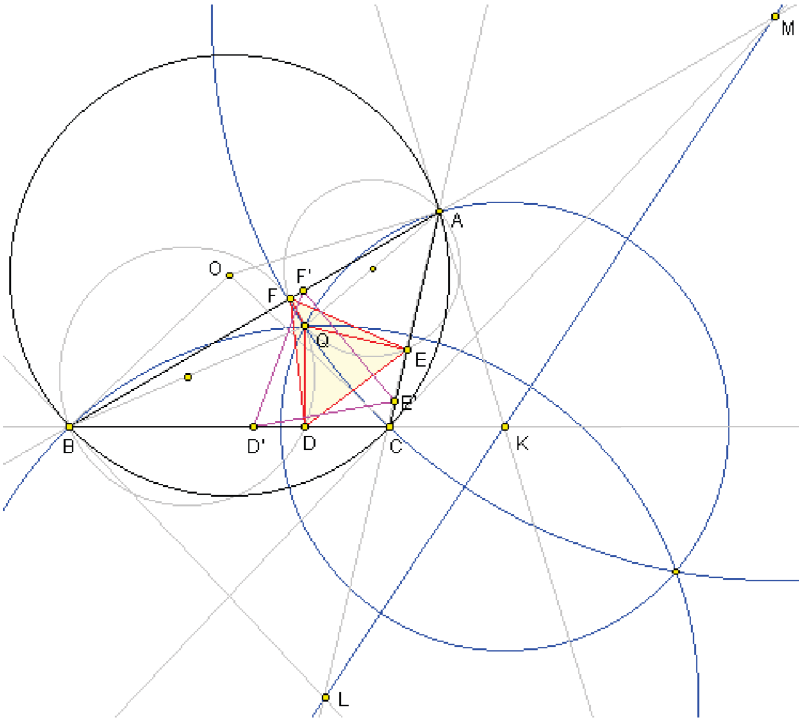
$$= \frac{e^2}{8S}[a^2 + b^2 + c^2 + \frac{4\sqrt{3}R^2}{2}\ (\sin 2A + \sin 2B + \sin 2C)] =$$

$$= \frac{e^2}{8S}(a^2 + b^2 + c^2 + 4S\sqrt{3})$$

$$e = \frac{2S\sqrt{2}}{\sqrt{a^2 + b^2 + c^2 + 4S\sqrt{3}}}$$

Thus the expression on the right side of the inequality in question is precisely the side length of the equilateral pedal triangle $\triangle DEF$ of the 1st isodynamic point Q. Any other equilateral triangle $\triangle D'E'F'$ inscribed in the triangle $\triangle ABC$, so that $D' \in BC,\ E' \in CA,\ F' \in AB$ is obviously obtained from the equilateral pedal triangle $\triangle DEF$ by a spiral similarity with the center Q and similarity coefficient greater than 1, hence its side e' = D'E' is greater than the side e = DE.

Attachments:



Albanian Eagle

#4 Aug 1, 2006, 5:54 pm

if M is the fermat point of ABC then the inequality reduces to :
$\sum |DE| \cdot |MC| \geq 2S$
this is kind of obvious because the product of the diagonals of a quadrilateral is $\geq$ twice its area...

yetti

#5 Aug 4, 2006, 8:28 pm

Not true in general, namely if the 1st Fermat point M is outside the triangle $\triangle ABC$, say, if $\angle C > 120^\circ$. Then the inequality is reduced to

$$|EF| \cdot |MA| + |FD| \cdot |MB| - |DE| \cdot |MC| \geq 2S$$

and the "obvious" argument fails.

The 1st isodynamic point Q is then also outside the triangle $\triangle ABC$ (inside its circumcircle) and the angle $\angle AQB$, the convex one, is equal to

$$\angle AQB = \angle AQF + \angle BQF = 180^\circ - \angle AEF + 180^\circ - \angle BDF =$$

$$\angle A + \angle AFE + \angle B + \angle BFD = 180^\circ - \angle C + 120^\circ = 360^\circ - (\angle C + 60^\circ)$$

so that the concave $\angle AQB = \angle C + 60^\circ$, as before. The area S of the triangle $\triangle ABC$ is still

$$S = \frac{1}{2}\left[AQ \cdot BQ \sin(C + 60^\circ) + BQ \cdot CQ \sin(A + 60^\circ) + CQ \cdot AQ \sin(B + 60^\circ)\right] =$$

$$... = \frac{e^2}{8S}\left(a^2 + b^2 + c^2 + 4S\sqrt{3}\right)$$

etc.

Luis González

#6 Jun 28, 2009, 5:27 am

It is known that among all the equilateral triangles circumscribed in $\triangle ABC$, the triangle $\triangle XYZ$ homothetic to the outer Napoleon triangle of $\triangle ABC$ has the maximum area (side length). Hence, its side is twice the measure of the side L of the outer Napoleon triangle.

$$L = \sqrt{\tfrac{1}{6}(a^2 + b^2 + c^2) + \frac{2\sqrt{3}}{3}[\triangle ABC]}.$$

For every circumcribed equilateral triangle Δ, we can associate an equilateral triangle Δ' homothetic to Δ and inscribed in $\triangle ABC$. Thus, by Gergonne-Ann theorem, it follows that $[\triangle ABC]^2 = [\Delta][\Delta']$. The area (side L') of Δ' will be minimum if the area (side) of Δ is maximum. In other words, $[\triangle ABC]^2 = \frac{3}{16}(L')^2(4L^2)$.

Substituting the value of the side length L of the Napoleon triangle yields:

$$[\triangle ABC]^2 = \tfrac{1}{8}(L')^2(a^2 + b^2 + c^2 + 4\sqrt{3}[\triangle ABC])$$

$$\Longrightarrow L' = \frac{2\sqrt{2}[\triangle ABC]}{\sqrt{a^2 + b^2 + c^2 + 4\sqrt{3}[\triangle ABC]}}.$$

SUPERMAN2

#7 Jun 28, 2009, 5:42 pm

Here is my solution:
Denote by T the Toricelly point of triangle ABC.Let $DE = EF = FD = x$
Appying Ptolemy's theorem

$$S = [AEFT] + [CDTE] + [BDTF] \le \frac{1}{2}(TA.EF + TC.DE + TB.FD) = \frac{x(TA + TB + TC)}{2}$$

Hence $x \ge \dfrac{2S}{TA + TB + TC} = \dfrac{2S\sqrt{2}}{\sqrt{a^2 + b^2 + c^2 + 4S\sqrt{3}}}$

Luis González

#8 Jun 28, 2009, 11:48 pm

> SUPERMAN2 wrote:
> $S = [AEFT] + [CDTE] + [BDTF]\ldots$

This only works when the Fermat point lies inside ABC. See post #5

Quick Reply

High School Olympiads

maximum area

Reply

Source: Canada 1981

moldovan

#1 Jun 23, 2009, 6:23 pm

A line l is tangent to a circle of radius r at a point P. From a variable point R on the circle, a perpendicular RQ is drawn to l with $Q \in l$. Determine the maximum area of triangle PQR.

Luis González

#2 Jun 28, 2009, 4:19 am

Let O be the center of the circle and M the midpoint of the chord $\overline{PR}$. Note easily that $\triangle PQR \sim \triangle OMP$.

$$\frac{[\triangle PQR]}{[\triangle OMP]} = \frac{PR^2}{OP^2} \Longrightarrow [\triangle PQR] = [\triangle POR] \cdot \frac{PR^2}{2r^2}$$

Let $\angle POR = \theta$. Then we have

$[\triangle POR] = \frac{1}{2}r^2 \sin\theta \ , \ PR^2 = 2r^2(1-\cos\theta)$

$[\triangle PQR] = \frac{1}{2}r^2(\sin\theta - \sin\theta\cos\theta)$

$[\triangle PQR]$ is maximum $\Longleftrightarrow f(\theta) = \sin\theta - \sin\theta\cos\theta$ is maximum.

Note that $\dfrac{\partial f(\theta)}{\partial\theta} = \cos\theta - 2\cos^2\theta + 1$

Therefore, $\dfrac{\partial f(\theta)}{\partial\theta} = 0$ yields $\Longrightarrow \ \cos\theta = -\frac{1}{2}$ and $\cos\theta = 1$

As a result, $[\triangle PQR]$ is maximum when $\theta = 120^\circ \ , \ (180^\circ \geq \theta \geq 0^\circ)$

Then, the maximum area is given by $[\triangle PQR] = \dfrac{3\sqrt{3}}{8}r^2$

Quick Reply

High School Olympiads

Meets at the midpoint

Reply

Unidranix
#1 Jun 18, 2009, 7:23 am

Let ABC be a triangle ($\hat{A} = 90^o$). Draw $Ax \parallel BC$. Let $D \in Ax$ such that $BD = BA$. Draw $AH \perp BD$. Prove that CD meets AH at the midpoint of AH.

Edit: $AH \perp BD$

This post has been edited 1 time. Last edited by Unidranix, Jun 18, 2009, 3:48 pm

dvtvd
#2 Jun 18, 2009, 10:17 am • 1

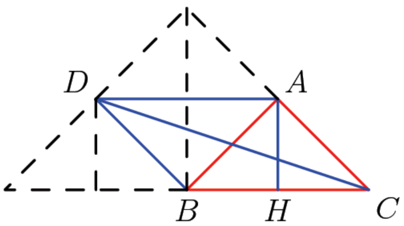
I think that's not correct

Unidranix
#3 Jun 18, 2009, 3:08 pm

NO. IT'S TRUE.
Can you tell me the reason why you said it's not correct, dvtvd?

Kouichi Nakagawa
#4 Jun 18, 2009, 3:43 pm

This problem is incorrect.
For example,



Unidranix
#5 Jun 18, 2009, 3:47 pm

Sorry
I have edited the problem: $AH \perp BD$

Unidranix
#6 Jun 21, 2009, 6:21 am

Now the problem is true, can someone help me ?

dgreenb801
#7 Jun 21, 2009, 9:49 am

Nice problem!
We know $\angle BDA = \angle DAB = \angle B$. Let CD intersect AB at X.
By Menelaus, we have to show $\frac{DH}{DB} \cdot \frac{BX}{XA} = 1$.
We know $DH = DA \cdot cosB$ and $DB = \frac{DA}{2cosB}$, $so \frac{DH}{DB} = 2cos^2B$
Also $\frac{AX}{XB} = \frac{[ADC]}{[DBC]} = \frac{AD \cdot ACsin(90+B)}{BD \cdot BCsin(180-B)}$. Write everything in terms of AB, this cancels to $2cos^2B$, which finishes the proof.

My method is messy, but I believe there is a nice proof by taking the midpoint of a side and using the orthocenter (haven't found it, just have a feeling).

This post has been edited 1 time. Last edited by dgreenb801, Jun 22, 2009, 3:23 am

panos_lo
#8 Jun 21, 2009, 10:57 pm

I'd like to post my solution, which I found after a lot of time.Correct me if I am mistaken

Suppose that CD meets AH at M. We draw the circle (B,BD), which will be refered as C1. If E is the second point that DH meets C1 we have:<EBC=<EDA
<CBA=<BAD, <BAD=<EDA. Clearly, the triangles EBC,CAB are equal. Thus, as CA is tangent to C1, CE is also tangent to C1. So CB is perpedincular to AE and CB meets AE in its midpoint K. If CD C1 again at P we have: CA^2=CP*CD
CA^2=CK*CB. So:CP*CD=CK*CB and P,B,K,D lie on the same circle. So <KPD=<EBC=<ABC=<ADB. <HAE=90-<HEA=<ADB, as DE is a diameter of of C1. So <KMP=<KAP=<EDP and KM is obviously parallel to ED. Thus, KM//EH and as K is the midpoint of AE, the proof is completed.

Unidranix
#9 Jun 22, 2009, 8:53 am

panos_lo, your solution is very nice but it'll be nicer if you use $latex$
And thank you dgreenb801 too

Luis González
#10 Jun 27, 2009, 9:21 am

Let $M \equiv AH \cap DC$ and $N \equiv AB \cap DC$. By Menelaus' theorem for $\triangle AHB$ cut by line $\overline{DMN}$, we get

$$\frac{AM}{MH} = \frac{AB}{DH} \cdot \frac{NA}{BN}$$

But since $\triangle ADN \sim \triangle BCN$ and $\triangle ADH \sim \triangle CBA$, we obtain

$$\frac{NA}{BN} = \frac{AD}{BC} = \frac{DH}{AB} \Longrightarrow \frac{AM}{MH} = \frac{AB}{DH} \cdot \frac{DH}{AB} = 1 \Longrightarrow AM = MH.$$

Unidranix
#11 Jun 27, 2009, 10:17 am

luis, thanks for your very nice solution

Quick Reply

High School Olympiads

Six circles tangent to the nine-point circle

Luis González

#1 Jun 22, 2009, 6:09 am

Let $\triangle ABC$ be an acute-angled triangle with circumcircle (O, R) and nine-point circle (N). Let R_a, R_b, R_c be the radii of the circles internally tangent to (O) at A, B, C and internally tangent to (N). Let r_a, r_b, r_c be the radii of the circles internally tangent to (O) at A, B, C and externally tangent to (N). Show the following relations:

1) $\dfrac{1}{R_a}+\dfrac{1}{R_b}+\dfrac{1}{R_c}=\dfrac{4}{R}$

2) $\dfrac{1}{r_a}+\dfrac{1}{r_b}+\dfrac{1}{r_c}=\dfrac{1}{R}(\sec A\sec B\sec C+4)$

yetti

#2 Jun 26, 2009, 12:22 pm

Lemma: Let (R, ϱ) be a circle with a chord UV. Let (P, ϱ_1) and (Q, ϱ_2) be circles internally tangent to (R) at U, V and externally tangent to each other at W. Inversion with center U and power ϱ^2 takes $(R), (P)$ into 2 parallel lines $r \parallel p$, perpendicular to UR. (Q) goes to a circle (Q') tangent to r, p at V', W', with radius

$$\varrho_2'=\frac{V'W'}{2}=\frac{1}{2}\left(\frac{\varrho}{2}-\frac{\varrho^2}{2\varrho_1}\right)=\frac{\varrho(\varrho-\varrho_1)}{4\varrho_1}$$

(Q') cuts UV at $V' \in UV$ and again at T'. Since $V'W'$ is diameter of (Q'), $W'T' \perp V'T'$. Power of U to (Q') is

$$UV'\cdot UT'=UV'\cdot(UV'+V'T')=UV'\cdot\left(UV'+V'W'\cdot\frac{UV}{2\cdot UR}\right)=$$

$$=\frac{\varrho^2}{UV}\cdot\left(\frac{\varrho^2}{UV}+2\varrho_2'\cdot\frac{UV}{2\varrho}\right)=\varrho^2\cdot\left(\frac{\varrho^2}{UV^2}+\frac{\varrho_2'}{\varrho}\right)$$

U is homothety center of $(Q), (Q')$ with coefficient $\dfrac{\varrho^2}{UV'\cdot UT'}$, hence

$$\boxed{\frac{1}{\varrho_2}=\frac{1}{\varrho_2'}\cdot\frac{UV'\cdot UT'}{\varrho^2}=\frac{1}{\varrho}+\frac{1}{\varrho_2'}\cdot\frac{\varrho^2}{UV^2}=\frac{1}{\varrho}+\frac{4\varrho\varrho_1}{(\varrho-\varrho_1)\cdot UV^2}}$$

Problem: Let H be orthocenter and D, E, F feet of altitudes AH, BH, CH. Let AH, BH, CH cut (O) again at X, Y, Z. Midpoints K, L, M of AH, BH, CH are on 9-point circle (N). Let DN, EN, FN cut AO, BO, CO at O_a, O_b, O_c. $AO \parallel KN \Longrightarrow \angle ADO_a=\angle KDN=\angle DKN=\angle DAO=\angle DAO_a \Longrightarrow$ circle (O_a) with radius $R_a=O_aD=O_aA$ is internally tangent to both $(O), (N)$ at A, D, respectively. Similarly for the circles $(O_b), (O_c)$ with radii R_b, R_c. A is similarity center of $(O_a), (O) \Longrightarrow R_a=R\cdot\dfrac{AD}{AX}$ and likewise for R_b, R_c.

$$\frac{1}{R_a}+\frac{1}{R_b}+\frac{1}{R_c}=\frac{1}{R}\left(\frac{AX}{AD}+\frac{BY}{BE}+\frac{CZ}{CF}\right)=\frac{1}{R}\left(3+\frac{HD}{AD}+\frac{HE}{BE}+\frac{HF}{CF}\right)$$

$$=\frac{1}{R}\left(3+\frac{[HBC]+[HCA]+[HAB]}{[ABC]}\right)\Longrightarrow\boxed{\frac{1}{R_a}+\frac{1}{R_b}+\frac{1}{R_c}=\frac{4}{R}}$$

Let A', B', C' be midpoints of BC, CA, AB. Let $(P_a), (P_b), (P_c)$ be circles internally tangent to (O) at A, B, C and externally tangent to (N). Obviously, (P_a) is internally tangent to (O_a) at A, etc. Applying the lemma with $(R),(P),(Q),UV \longrightarrow (O_a),(N),(P_a),AD$, etc.,

$$\frac{1}{r_a}=\frac{1}{R_a}+\frac{4R_aR}{(2R_a-R)\cdot AD^2}=\frac{1}{R_a}+\frac{4R}{(2AD-AX)\cdot AD}=\frac{1}{R_a}+\frac{4R}{AH\cdot AD}$$

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{4}{R}+4R\left(\frac{1}{AH\cdot AD}+\frac{1}{BH\cdot BE}+\frac{1}{CH\cdot CF}\right)=$$

$$=\frac{4}{R}+\frac{2R}{[ABC]}\left(\frac{a}{AH}+\frac{b}{BH}+\frac{c}{CH}\right)=$$

$$=\frac{4}{R}+\frac{8R^2}{[ABC]}\cdot\frac{[HBC]+[HCA]+[HAB]}{AH\cdot BH\cdot CH}\implies$$

$$\boxed{\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{4}{R}\left(1+\frac{2R^3}{AH\cdot BH\cdot CH}\right)=\frac{4}{R}\left(1+\frac{R^3}{4\cdot OA'\cdot OB'\cdot OC'}\right)}$$

Luis González

#3 Jun 26, 2009, 11:09 pm

Thanks for your nice solution, this is what I did:

Let H_a,H_b,H_c the feet of the altitudes and P_a,P_b,P_c the second intersections of the corresponding altitudes with the circumcircle (O). Consider the circle (O'_a) internally tangent to (N) at H_a passing through A. Negative inversion with center H and power $\overline{HH_a}\cdot\overline{HA}$ takes (N) into (O) and (O'_a) into itself $\Longrightarrow$ (O) is internally tangent to (O'_a), which implies that (O'_a) is identical with (O_a,R_a). Thus cevians of the circumcenter are diameters of $(O_a),(O_b),(O_c)$. Let D_a,D_b,D_c be the feet of the cevians of the circumcenter. Since rays AP_a and AD_a are isogonal WRT $\angle BAC$, it follows that:

$$AB\cdot AC=AD_a\cdot AP_a \implies \frac{1}{AD_a}=\frac{1}{2R_a}=\frac{BC\cdot AP_a}{BC\cdot CA\cdot AB}$$

$$\frac{1}{AD_a}=\frac{2[ABP_aC]}{BC\cdot CA\cdot AB}=\frac{2[\triangle ABC]+2[\triangle HBC]}{BC\cdot CA\cdot AB}$$

The cyclic sum yields

$$\frac{1}{R_a}+\frac{1}{R_b}+\frac{1}{R_c}=\frac{16[\triangle ABC]}{BC\cdot CA\cdot AB}=\frac{4}{R}$$

Inversion with center A and power ${k_a}^2$ equal to the power of A WRT (N) takes (O_a,R_a) and (K_a,r_a) into two parallel lines τ and k tangent to (N). If d_a and ${d_a}'$ denote the distances from A to k and τ, by inversion properties we have

$d_a\cdot 2r_a={d_a}'\cdot 2R_a={k_a}^2$. Since $R=d_a-{d_a}' \Longrightarrow \dfrac{1}{r_a}-\dfrac{1}{R_a}=\dfrac{2R}{{k_a}^2}$

The cyclic sum yields

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}-\frac{4}{R}=2R\left(\frac{1}{{k_a}^2}+\frac{1}{{k_b}^2}+\frac{1}{{k_c}^2}\right)$$

Powers of A,B,C with respect to (N) are given by

$${k_a}^2=[\triangle ABC]\cot A\ \ ,\ \ {k_b}^2=[\triangle ABC]\cot B\ \ ,\ \ {k_c}^2=[\triangle ABC]\cot C$$

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{2R}{[\triangle ABC]}\cdot(\tan A+\tan B+\tan C)+\frac{4}{R}$$

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{R}\cdot\frac{\tan A\cdot\tan B\cdot\tan C}{\sin A\cdot\sin B\cdot\sin C}+\frac{4}{R}$$

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{R}(\sec A\sec B\sec C+4)$$

High School Olympiads

Inequality in a convex quadrilateral

Source: hard

Moonmathpi496
#1 Jun 18, 2009, 10:05 pm

A convex quadrilateral $ABCD$ is inscribed in a circle whose center O is inside the quadrilateral. Let $MNPQ$ be the quadrilateral whose vertices are the projections of the intersection point of the diagonals AC and BD onto the sides of $ABCD$. Provethat $2[MNPQ] \le [ABCD]$.

plane geometry
#2 Jun 23, 2009, 5:25 pm

Is there a synthetical proof?

Moonmathpi496
#3 Jun 26, 2009, 7:34 am

> *plane geometry wrote:*
> Is there a synthetical proof?

Most probably the answer is no. :(

Luis González
#4 Jun 26, 2009, 7:56 am

Let $S \equiv AC \cap BD$ and k^2 denotes the power of S WRT (O). By Euler's theorem for pedal triangles $\triangle SMN, \triangle SNP$, $\triangle SPQ, \triangle SQM$ of S WRT $\triangle ABC, \triangle BCD, \triangle CDA, \triangle DAB$, we get

$[\triangle SMN] = \frac{k^2}{4R^2} \cdot [\triangle ABC]$. Then the cyclic sum yields:

$$[MNPQ] = \frac{k^2}{4R^2} \cdot [ABCD + ABCD] = \frac{k^2}{2R^2} \cdot [ABCD]$$

k^2 is maximum $\Longleftrightarrow S \equiv O$, i.e. $k^2 = R^2 \Longrightarrow [ABCD] \ge 2[MNPQ]$.