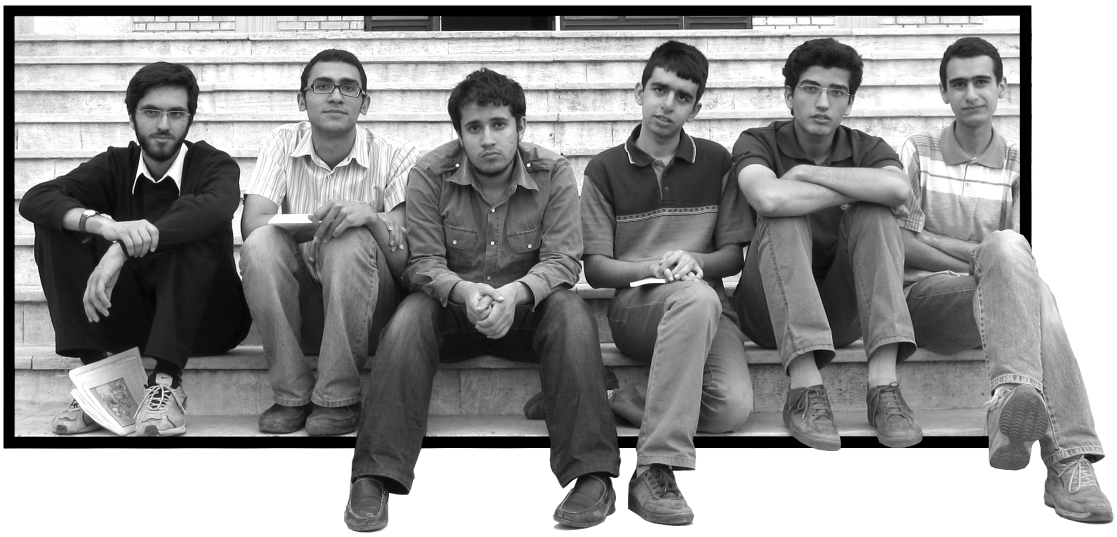


Iranian Team Members (from left to right):

- Hossein Dabirian
- Khashayar Khosravi
- Masoud Giveh-chi
- Sepehr Ghazi-Nezami
- Pouya Vahidi
- Seyyed Ehsan Azarm-sa

This booklet is prepared by Nima AhmadiPourAnari, AliAkbar Daemi, Ali Khezeli, Abbas Mehrabian, Omid Naghshineh, Morteza Saghafian, Erfan Salavati and Saeed Sarafraz.

26^{th}
Iranian Mathematical Olympiad
2008-2009

Contents

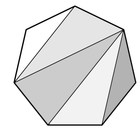
Problems **3**
First Round 4
Second Round 5
Third Round 9

Solutions **11**
First Round 12
Second Round 16
Third Round 25

Problems

First Round

1. In how many ways can one choose $n-3$ diagonals of a regular n-gon, so that no two have an intersection strictly inside the n-gon, and that no three form a triangle.



2. Let ABC be a triangle. Let I_a be the center of its A-excircle. Assume that the A-excircle touches AB and AC in B' and C', respectively. Let I_aB and I_aC intersect $B'C'$ in P and Q, respectively. Let M be the intersection of CP and BQ. Prove that the distance between M and the line[1] BC is equal to the inradius of ABC.

3. a, b, c and d are real numbers, and at least one of c, d is not zero. Let $f : \mathbb{R} \to \mathbb{R}$ be the function[2] defined by $f(x) = \frac{ax+b}{cx+d}$. Assume that $f(x) \neq x$ for every $x \in \mathbb{R}$. Prove that if there exists at least one a such that $f^{1387}(a) = a$, then for every x, for which f^{1387} is defined, we have $f^{1387}(x) = x$.

4. Let $a \in \mathbb{N}$ be such that for every $n \in \mathbb{N}$, $4(a^n + 1)$ be a perfect cube. Show that $a = 1$.

5. We want to choose some phone numbers for a new city. The phone numbers should consist of exactly ten digits, and 0 is not allowed as a digit in them. To make sure that different phone numbers are not confused with each other, we want every two phone numbers to either be different in at least two places, or have digits separated by at least 2 units, in at least one of the ten places.

   What is the maximum number of phone numbers that can be chosen, satisfying the constraints? In how many ways can one choose this amount of phone numbers?

6. Let ABC be a triangle and H be the foot of the altitude drawn from A. Let T, T' be the feet of the perpendicular lines drawn from H onto AB, AC, respectively. Let O be the circumcenter of ABC, and assume that $AC = 2OT$. Prove that $AB = 2OT'$.

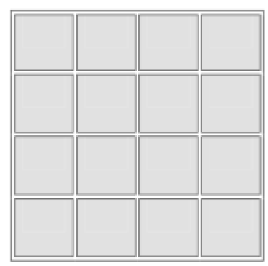
---

[1] not the line segment

[2] f might be undefined for some points of $\mathbb{R}$

Second Round

1. Police is trying to catch Caesar, the top most wanted criminal. Caesar lives on the streets of a square-shaped town, which consists of n vertical and n horizontal streets. Caesar moves freely along the streets, but he never leaves the streets. In each one of the following cases, how many police officers are needed to make sure that Caesar will be caught? Note that Caesar and police officers move continuously, and Caesar is caught when he passes through a point at which a police officer is present.

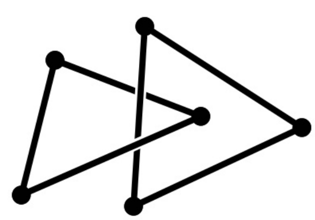



   (a) Police officers and Caesar move at the same speed, and police is always aware of Caesar's location by tracking his cellphone signals.

   (b) Police officers move at the same speed, but Caesar has virtually infinite speed. Caesar's cellphone is turned off, so police can't track him, even if Caesar passes through a street on which one of the police officers is standing.

   Try to find lower and upper bounds in case the exact answer is hard to find.

2. Consider six point in the 3-dimensional space. Assume that no three are collinear, and no four are coplanar. Draw a line segment between each two points. Prove that two triangles can be formed by these line segments, which have no intersections, but are entangled in each other.



3. (a) Prove that there exist two polynomials with integer coefficients, each having at least one coefficient bigger than 1387, whose product's coefficients are -1, 0 or 1.

   (b) Is there any polynomial multiple of $x^2 - 3x + 1$, whose coefficients are -1, 0 or 1?

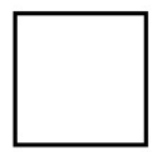
4. A figure S in the 2-dimensional plane is called algebraic, iff there exists a polynomial $P \in \mathbb{R}[x, y]$, whose zeroes in the plane, are exactly the points in S; more precisely

$$S = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}$$

For example the unit circle around the origin, is an algebraic figure, because it exactly consists of the zeroes of the polynomial $x^2 + y^2 - 1$.

Which of the following figures are algebraic?

   (a) An empty square



   (b) A semicircle with closed ends

5. (a) Let $RBR'B'$ be a convex quadrilateral in the 2-dimensional plane. We have colored the vertices R, R' red, and B, B' blue. There are some points strictly inside the quadrilateral. Among all of the points (including the quadrilateral vertices), no four are on a circle, except for possibly the four original quadrilateral vertices. The inner points are arbitrarily colored red or blue. Prove that exactly one of the following situations happens

   - There is a path from R to R', such that for every point on it, the nearest red point is strictly closer than the nearest blue point.
   - There is a path from B to B', such that for every point on it, the nearest blue point is strictly closer than the nearest red point.

A path of the first kind is called a red path, and path of the second kind is called a blue path.

Let n be natural number. Two persons are playing a game of $2n$ turns. They alternate turns, putting red and blue points inside the quadrilateral, such that the mentioned constraints (no four points being on a circle) still hold. The first person puts only red points and tries to make a red path appear at the end. The other person puts only blue points and tries to make a blue path appear at the end.

(b) Prove that if $RBR'B'$ is a rectangle, no matter what n is, the second player can always win.

(c) Try to find out which player can win, for other types of quadrilaterals.

6. Five research stations are operating on Mars. Is it always possible to divide the surface of Mars into five connected areas, all congruent to each other, such that each area contains exactly one station? Note that stations are points and Mars is assumed to be a complete sphere.

7. A simple graph is called self-intersecting iff it can be drawn in the 2-dimensional plane, drawing the edges as line segments and vertices as points, in such a way that each two edges intersect. Note that no edge can pass through a vertex except its endpoints.

   (a) For which natural numbers n, a cycle of length n is self-intersecting?

   (b) Prove that in a self-intersecting graph, the number of edges is no more than the number of vertices.

   (c) Specify all self-intersecting graphs.

8. The following is the ending part of an old mathematical script

The proof is all done, and the goal is reached, and that is what has been wanted.

This script is finished in a year that, when raised to the power of 13, becomes the following huge number

$$2581452668046920778582615126 63$$

Wise reader, be aware that if you have skills in the theory of numbers, this puzzle will be easy.

In what year has the script been finished?

Third Round

1. Let ABC be a triangle. A', B', C' are points lying on BC, CA, AB, respectively. The incenters of the triangles ABC and $A'B'C'$ are the same, and the inradius of $A'B'C'$ is half that of ABC. Prove that ABC is equilateral.

2. Let $a \in \mathbb{N}$ be fixed. Prove that there are infinitely many primes, each dividing $2^{2^n} + a$ for at least one $n \in \mathbb{N}$.

3. Let $a, b, c \in \mathbb{R}$ be positive numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{2 + a^2 + b^2} + \frac{1}{2 + b^2 + c^2} + \frac{1}{2 + c^2 + a^2} \leq \frac{3}{4}$$

4. Find all polynomials f with integers coefficients such that, for each prime $p \in \mathbb{N}$ and every arbitrary $u, v \in \mathbb{N}$, for which $p | uv - 1$, we have $p | f(u)f(v) - 1$.

5. Let ABC be a triangle. Let A', B', C' be the feet of altitudes drawn from A, B, C, respectively. Let P be the foot of the perpendicular line drawn from C' onto $A'B'$. Let Q be a point on $A'B'$ such that $QA = QB$. Prove that

$$\angle PBQ = \angle PAQ = \angle PC'C$$

6. We have a closed path passing through the points of an $n \times n$ lattice like $\{1, \ldots, n\}^2$. The path passes through all of the points, moving from a point to one of its four (possibly less) adjacent points. Prove that there are two points, adjacent in the lattice, that can not be reached from each other by moving less than a quarter of the path's length along the path. In other words the minimal length arc between them along the closed path, should be at least a quarter of the total path's length.

7. We have three groups of lines in the 2-dimensional plane. Each group consists of 11 different lines, all parallel to a group-unique direction. What is the maximum number of points through which three of these lines (one from each group) pass?

8. Find all polynomials $P \in \mathbb{R}[x, y]$ such that for all $x, y \in \mathbb{R}$, we have

$$P(x^2, y^2) = P\left(\frac{(x+y)^2}{2}, \frac{(x-y)^2}{2}\right)$$

9. Let ABC be a triangle. Assume that I is the incenter of ABC, and D, E, F are the points of tangency of the incircle with BC, CA, AB, respectively. Let M be the foot of the perpendicular line drawn from D onto EF. Assume that P is the midpoint of DM. Let H be the orthocenter of BIC. Prove that PH bisects EF.

10. Let ABC be a triangle. Assume that $AB \neq AC$. Let D be a point on BC such that $BA = BD$; assume that B lies between C and D. Let I_C be the center of ABC's C-excircle. Let T be the second intersection of CI_C with the circumcircle of ABC. Assume that $\angle TDI_C = \frac{\angle ABC + \angle ACB}{4}$. Find out the value of $\angle A$.

11. Let $n \in \mathbb{N}$. Prove that

$$3^{\frac{5^{2^n}-1}{2^{n+2}}} \equiv (-5)^{\frac{3^{2^n}-1}{2^{n+2}}} \pmod{2^{n+4}}$$

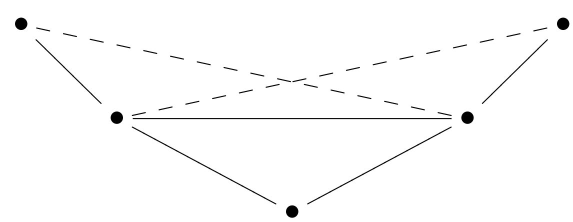
12. Let T be a subset of $\{1, 2, \ldots, n\}$. Assume that for every pair of different numbers $i, j \in T$, $2j$ is not divisible by i. Prove that

$$|T| \leq \frac{4}{9}n + \log_2 n + 2$$

Solutions

First Round

1. Using a simple induction, one can see that whenever $n-3$ diagonals of a convex n-gon are selected, such that they do not have any intersections inside the n-gon, the n-gon is divided into exactly $n-2$ triangles. One can simply consider one of the diagonals and divide the n-gon into two parts according to the diagonal, in order to complete the induction.
   The case $n = 3$ is easily dealt with. There is no diagonal to choose, and hence the answer is 1.
   When $n > 3$, no triangle can share 3 sides with the n-gon. However each triangle must at least share one side. We have $n-2$ triangles and n sides. Hence exactly two of the triangles will share two sides, and the rest will share one side, each. Now consider one of those two triangles. It can be any of the n triangles made from two adjacent sides of the n-gon.
   Now consider the third side of that triangle, the one which is a diagonal. The other triangle which uses that diagonal, can be any of the two triangles which use that diagonal and share a side with the n-gon. Just like the picture below.

   Considering the new diagonal again, another triangle uses it. It again has two possibilities. Continuing, each new diagonal has two possibilities. The first one had n possibilities. But we are counting everything twice, because there are two triangles which share two sides with the n-gon, and choosing any of them as the beginning triangle would yield the same result. Hence there are $n2^{n-4}$ divided by 2, or equivalently $n2^{n-5}$ possible ways.

2. Let D be the point where the A-excircle touches BC. Then $CD = CC'$,

because $I_aC'C$ and I_aDC are congruent, both having a right angle, two congruent angles, and two congruent sides. Let Q' be the foot of the perpendicular line drawn from B onto I_aC. Then $DCQ' \cong C'CQ'$, because $\angle Q'CD = \angle Q'CC'$, $Q'C = Q'C$ and $CD = CC'$. Hence $\angle C'Q'C = \angle DQ'C$. Since $\angle BQ'I_a$ and $\angle BDI_a$ are both right angles, $BDQ'I_a$ is cyclic. Hence $\angle DQ'C = \angle DBI_a$. Since BI_a is the angle bisector of $\angle DBB'$, we have $\angle DBI_a = \angle B'BI_a$. Because $\angle BB'I_a + \angle BQ'I_a = 90° + 90° = 180°$, $BB'I_aQ'$ is cyclic. Hence $\angle B'BI_a = \angle B'Q'I_a$. Finally, putting all things together, we have $\angle B'Q'I_a = \angle CQ'C'$. This proves that C', Q', B' are collinear. Hence Q' is the intersection of $B'C'$ and $C'I_a$. This proves that $Q = Q'$. Hence $\angle BQC = 90°$. Similarly $\angle CPB = 90°$.
Let I be the incenter of ABC. Then $CI \perp CI_a$, because they are the inner and outer bisectors of $\angle BCA$. We have $\angle BQC = 90°$ which shows that $BQ \perp CI_a$. Hence $BQ \parallel CI$. Similarly $CP \parallel BI$. This shows that $BMCI$ is a parallelogram. Hence the distance between M and BC is equal to the distance between I and BC which is equal to the inradius.

3. Since $f^{1387}(a) = a$ we have $f^{1388}(a) = f(a)$, because obviously $f(a)$ is defined (in fact $f^{1387}(a)$ is defined). If we let $b = f(a)$ then $f^{1387}(b) = b$. Since $f(a) \neq a$, a, b are two different solutions of $f^{1387}(x) = x$. Let $c = f(b)$. Then again $f^{1387}(c) = c$. Since $c = f(b)$, we have $c \neq b$. We want a, b, c to be three different numbers. The only pair of them that might possibly be equal is a, c. Assume that $a = c$. Then $f^2(a) = a$. So $f^4(a) = f^2(a) = a$ and continuing like this we get $f^{1388}(a) = a$. But $f^{1388}(a) = f(f^{1387}(a)) = f(a)$. Which is a contradiction, because $f(a) \neq a$. Hence a, b, c are all different solutions to $f(x) = x$.
The composition of two functions of the form $\frac{ax+b}{cx+d}$, is itself a function of the form $\frac{ax+b}{cx+d}$ (with different a, b, c, d). Hence f^{1387} is a function like $\frac{ex+f}{gx+h}$. The equation $\frac{ex+f}{gx+h} = x$ is equivalent to $gx^2 + (h-e)x - f = 0$ which is a polynomial equation of degree two, and hence can have at most two solutions unless it is totally zero, which means $g = f = 0$ and $e = h$ which means $f^{1387}(x) = x$ for every x.

4. $4(a^3+1)$ and $4(a^9+1)$ are perfect cubes. Hence their quotient $a^6 - a^3 + 1$ is also a perfect cube. If $a > 1$, then $a^6 - a^3 + 1 < (a^2)^3 = a^6$ and $a^6 - a^3 + 1 > a^6 - 3a^4 + 3a^2 - 1 = (a^2-1)^3$ because $3a^4 - a^3 - 3a^2 + 2 \geq$

$3 \cdot 2a^3 - a^3 - 3a^2 + 2 > 2a^3 + 2 > 0$. But $(a^2)^3$ and $(a^2-1)^3$ are two consecutive perfect cubes. So $a^6 - a^3 + 1$ which lies between them, can not be a perfect cube.

5. Let $S = \{1, 2, \ldots, 9\}$. Each phone number is a point in S^{10}. Two phone numbers are not confused with each other iff they have a distance greater than 1 in S^{10}. So the problem is to select the maximum number of non-adjacent points in the lattice S^{10}.
We use induction on n to show that the maximum number of non-adjacent points which can be selected from S^n is $(|S^n|+1)/2$ and there is only one way to do that. For $n = 0$, $|S^n| = 1$ and the claim is obviously true. Let $n > 0$. S^n is 9 copies of S^{n-1}; those that have 1 as their first coordinate, those that have 2 and so on. Let A_i be those elements that have i as their first coordinate. The maximum number of elements which can be chosen from A_1, A_2 is $(|A_1| + |A_2|)/2$ because if we pair elements of A_1, A_2 according to their last $n-1$ digits, from each pair at most one element can be chosen. Use the same argument for A_3, A_4, for A_5, A_6 and A_7, A_8. Finally use the induction hypothesis for A_9. This shows that the number of selected elements is at most $(|S^n| + 1)/2$. However we can select all elements that have an odd (or even according to the oddity of n) sum of digits; obviously no two elements will be adjacent. But these are $(|S^n| + 1)/2$ elements.
It remains to prove that there is only one way to do so. From the last argument it is obvious that the elements of A_9 are uniquely selected. But then the elements of A_8 are, because we can pair points of A_8, A_9 and from each pair one shall be chosen. Hence the selected elements of A_8 are exactly those that are not selected in A_9. Now put A_8, A_9 aside, and repeat the last argument for A_7, and then A_6. Continuing like this we see that there is only one way to select $(|S^n| + 1)/2$ non-adjacent elements.

6. Using the Stewart's theorem in OAB with the Cevian OT we get

$$R^2(AT + TB) = AB(OT^2 + TA \cdot TB) \quad \Longrightarrow \quad OT^2 = R^2 - TA \cdot TB$$

If we let $\theta = \angle ABC$ then $TB = BH \cdot \cos\theta = AH \cdot \cot\theta \cos\theta$ and $TA = AH \cdot \cos(90^\circ - \theta) = AH \cdot \sin\theta$. Hence $OT^2 = R^2 - AH^2 \cos^2\theta$. We have $AC = 2R\sin\theta$, hence $(AC/2)^2 = R^2 \sin^2\theta$. So $(AC/2) = OT$ iff $R^2 - AH^2\cos^2\theta = R^2\sin^2\theta$ or equivalently $AH = R$. The same is true for OT'. Since $AC/2 = OT$ we have $R = AH$ and hence $AB/2 = OT'$.

Second Round

1. (a) Two officers are sufficient. An officer standing in a row can align itself with Caesar horizontally, by mimicking his horizontal movements. If Caesar moves to the left, the officer moves to the left, and if Caesar moves to the right the officer moves to the right. In all other cases, the officer stands still. Because the town is bounded, after some time the officer will have the same x coordinate as Caesar, and this will always hold until the officer tries to do something else.

   The strategy is like this. The first officer moves to the first row and aligns itself with Caesar (horizontally). The second officer moves to the second row and aligns itself with Caesar. If Caesar is caught between the first and the second row, one of the officers can move towards him, finally capturing him. Now that we're sure Caesar is not between the first and the second row, the officer in the first row can move to the third row and align itself with Caesar. Again Caesar can not move beyond row 2, because an aligned officer is standing in there. If Caesar is caught between the second and the third row, using the same strategy as was previously stated, he will be caught. Now, the officer in the second row moves to the fourth row and so on. Continuing like this all rows will be swept, and Caesar will be finally caught.

   It's obvious that one officer is not sufficient. Caesar can wait until the officer arrives at a Manhattan distance of 0.5 at him. Then he can always move towards a point where the distance between him and the officer gets larger, because nowhere in the town is a dead end. So Caesar can maintain a Manhattan distance of 0.5 to the officer, thus avoiding being captured.

   (b) We will show that $n + 1$ officers are sufficient and $n - 1$ officers are not. So the exact amount is either n or $n + 1$.

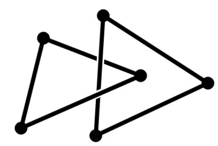
   $n + 1$ officers are sufficient. n officers can stand in the crosspoints of the first row, and the remaining officer can sweep the first row from left to right, making sure that Caesar is not in the first row. Then the n officers can move simultaneously to the crosspoints of the second row, and after that the remaining officer can sweep the second row. Continuing like this, all of the town will be swept and Caesar will be finally caught.

However $n-1$ officers are not sufficient. Call a row or a column dangerous if there is an officer which can reach that column or row by moving a distance of less than 0.25. Call all other rows and columns safe. Call a crosspoint safe iff both its row and its column are safe. An officer can make at most one row and at most one column dangerous. Hence if there are $n-1$ officers, there will always be at least one safe row and one safe column. Caesar can follow the strategy of staying in a safe crosspoint as long as it is safe. As soon as the crosspoint is not safe, he can move to a new safe crosspoint. This is true, because the row and column of its currently occupied point and the new safe point, are all free of officers, because these points are safe (or at least were before a short period). Caesar can move with an infinite speed along its current row to reach the column of the new safe point and then move to the new safe point. This way he will always be in a safe point and hence will never be captured.

2. Let's move one of the points like A continuously. We will move it in such a way that no three points will ever become collinear. Let B, C, D be three other points. While moving A, the line segment AD might pass through the line segment BC; this happens only when there is a brief moment in which the line segments have an intersection, and A, before and after the this moment, happens to be in different sides of the plane passing through B, C, D.
   While moving A, if AD passes through BC, and no other segment passes through any other one, then the number of pairs of disjoint and entangled triangles, will change by an even number. To prove this, let's name the other two points among the six as E, F. Clearly a pair of disjoint triangles in which either of AD or BC is not used, won't change state; i.e. they remain entangled or unentangled, just the same as their previous state. So the only pairs to consider are (ADE, BCF) and (ADF, BCE). Considering all of the possible situations one can see that they both change state. i.e. they become entangled if they were not and vice versa. So the number of entangled pairs changes by an even number.
   By moving points continuously like what was previously mentioned, we can reach any arrangement of points from any other one. Therefore it suffices to find an arrangement in which the number of entangled pairs

is odd.

But in the following picture, there is only one pair of entangled rectangles as can be checked by considering all pairs of disjoin rectangles.



3. (a) Let $f(x) = (x-1)^n$ and $g(x) = \prod_{i=0}^{n}(x^{2^i}-1)/(x-1) = \prod_{i=0}^{n}(x^{2^i-1}+\cdots+1)$. The coefficient of x in $g(x)$ is n. The coefficient of x^2 in $f(x)$ is $\binom{n}{2}$. When n is large enough both these numbers will be greater than 1387. The product $f(x)g(x) = \prod_{i=0}^{n}(x^{2^i}-1)$. Each number can be uniquely written as a sum of powers of 2. Hence in the product $\prod_{i=0}^{n}(x^{2^i}-1)$ there will be exactly 2^{n+1} different terms, each being a unique power of x. Since the coefficients of x^{2^i} and 1 are all ± 1, each term in the product will have a coefficient of ± 1, as was desired.

   (b) No, it's not possible. One of the roots of $x^2 - 3x + 1$ is $\alpha = \frac{3+\sqrt{5}}{2} > 2$. α is a root of every multiple of $x^2 - 3x + 1$. If this multiple is $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + 1$ then $p(\alpha) > \alpha^n - \alpha^{n-1} - \alpha^{n-2} - \cdots - 1$ because of the triangle inequality. But $\alpha^{n-1} + \cdots + 1 = (\alpha^n - 1)/(\alpha - 1) < \alpha^n - 1 < \alpha^n$. Hence $p(\alpha) > 0$ which is not possible. So no such multiple exists.

4. (a) No it's not. Suppose the contrary, and let $P(x, y)$ be the polynomial whose zeroes form the empty square. Assume that the square is the unit square whose vertices are $(0, 0), (0, 1), (1, 0), (1, 1)$. Let $f(x) = P(x, 0)$. Then $f(x)$ is a polynomial which is zero for every $0 < x < 1$. But a nonzero polynomial can have at most finitely many roots. So f must be identical to 0 which shows that the entire line $y = 0$ must be inside the figure which is not true. Hence the empty square is not algebraic.

   (b) No it's not. Suppose the contrary, and let $P(x, y)$ be the polynomial whose zeros form the semi-circle. The polynomial $Q(x, y) = x^2 + y^2 - 1$ is zero on the semi-circle. Divide P by Q according to the powers of x to obtain the remainder $R(x, y)$. R is zero on the semi-circle and the degree of x in R is at most 1. Let $R(x, y) = A(y)x + B(y)$ where A, B are polynomials in y. If for some y, one of A, B is not zero, then $R(x, y)$ can have at most one root with that y. But for $0 < y < 1$, $R(x, y)$ has two zeroes.

Hence $A(y), B(y)$ are identical to zero for $0 < y < 1$, but since they're polynomials they should be identical to 0 on the entire $\mathbb{R}$. Hence $R(x, y) = 0$, and so P is divisible by Q. But Q is zero on the entire circle, and so is P. So the other half of the circle must be inside the figure, which is not!

5. (a) Consider the Voronoi diagram of all points. Color each region of the Voronoi diagram with respect to the color of the point in the middle. So each region is either red or blue. Consider a vertex at infinity, and consider all rays in the diagram to be segments with ∞ as one endpoint.

   Each normal vertex in the diagram has degree 3, because having a degree of more than 4 means at least four of the original points must have been on the same circle (with the vertex as the center of the circle); which is not possible. The ∞ vertex has degree 4, because each ray in the Voronoi diagram corresponds to an edge in the convex hull of all points which is the original quadrilateral. Thus there are exactly four rays, each corresponding to the common border of the regions associated with consecutive vertices of the quadrilateral.

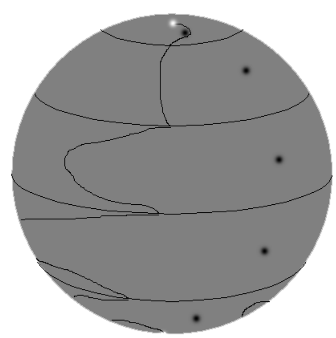
   First draw two blue rays from R, R' perpendicular to BB' but going out of the quadrilateral. The points on these rays are all parts of red regions because they are closer to R or R' than any other point. If there is a red path between R, R' these path along with the two mentioned rays divide the plane into two sides, with a red border. The same is true for a blue path. Hence if both a red path and a blue path exist, they must have an intersection. But the intersection must be either closer to a red point or a blue point, which is a contradiction. So both paths can not exist simultaneously.

   Now consider all edges of the Voronoi diagram which border regions of different color. There are four such edges coming out of ∞. Any other normal vertex, depending on whether the neighboring regions all have the same color or not, has degree 2 or 0. So there are exactly two disjoint cycles passing through ∞. Consider one of them that has the ray corresponding to RB. Moving along the cycle the right hand side regions always have the same color, and so do the left hand side regions. So starting from the ray corresponding to RB we can not arrive at the ray corresponding to $R'B'$, because that way the right hand side regions must have changed color somewhere along the cycle! So the final ray of the cycle is either the one corresponding to RB' or the one corresponding to BR'. Assume that it was the ray corresponding to RB'. Then there would be a blue path. One can move from B to a point near the ray corresponding to RB, then move along the cycle, keeping an infinitesimal distance from the cycle to be inside the blue regions, and finally move from a point on the ray corresponding to RB' directly to B'. In the other case, we would have a red path, using similar arguments.

(b) The second player can act like this. Whenever the first player puts a point somewhere like P, the second player can put a blue point on the line segment PR.
Let's prove that following this strategy, no red path will ever be created. Suppose that there is a red path starting from R and ending at R'. Move along this path. At some point, you must go out of the Voronoi region corresponding to R. If you're entering a region corresponding to $P \neq R'$, then you must be exactly on the perpendicular bisector of PR. But since there is a blue point on PR, the distance between you and the blue point will be less than the distance between you and P, which is a contradiction, because the point you're standing in right now must be part of the region corresponding to P.
The only case remaining is when $P = R'$. Note that until now, we have not used the assumption that $RBR'B'$ is a rectangle. If $P = R'$, then you must be standing on the perpendicular bisector of RR'. Because $RBR'B'$ is a rectangle, you're either closer to B or to B' (depending on what side of RR' you're standing) than you're to R, R'. This is again a contradiction. Hence following this strategy, there will be no red path, and hence there will always be a red path.

(c) We want to find some other quadrilaterals for which the second player, using the same strategy as mentioned in the previous section, can win. The only twist of the proof for the general quadrilateral is that the Voronoi regions of R, R' might have common border. However this is not true if $\angle R + \angle R' \leq 180°$. Let l be the perpendicular bisector of RR'. Let O be the circumcenter of RBR' and O' be the circumcenter of $RB'R'$. O, O' both lie on l. Any point on l is either on the same side of O as B is, or on the same side of O' as B' is. This is because $\angle B + \angle B' \geq 180°$ which shows that O' to same side of O as B does, because $O'B < O'B'$. Hence all points on l are parts of blue regions. So the Voronoi regions of R, R' can not have a common border.

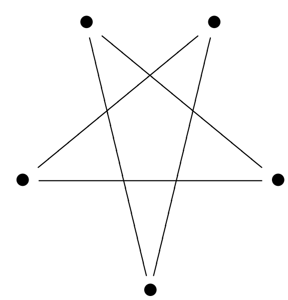
6. It is possible. In fact it is possible for every n research stations. There is a direction, to which one can never find a perpendicular plane passing through at least 2 research stations, because there are only finitely many research stations. We may also assume that no research stations are located at the poles of Mars along this direction. Now orient the sphere, so that the mentioned direction becomes the north-south direction. Consider the horizontal planes passing through north pole and south pole. Name them P_0 and P_n. Find $n-1$ other horizontal planes P_i $(1 \leq i \leq n-1)$, in such a way that the ith research station (sorted from top to bottom) be located between P_{i-1} and P_i. Now find a meridian which does not pass through any of the stations. Let this meridian intersect the plane P_i at A_i. Now consider a curve with monotonic latitude which goes from A_{i-1} to A_i lying completely between P_{i-1} and P_i. This curve when rotated about the polar axis by angles of $2\pi i/n$ and $2\pi(i+1)/n$ should form a region which contains P_i. Now paste all these curves together to form a curve which starts from the north pole and ends in the south pole. Rotate this curve by angles of $2\pi i/n$ about the polar axis to obtain n curves which divide the sphere's surface to n connected regions. All of these regions are congruent because they can be obtained from each other by rotations about the polar axis. It's easy to see that the ith station is located in the ith region. Hence the regions with the desired properties are obtained.



7. (a) For odd n's. For an even number n a cycle of length n can not be self-intersecting. Consider an edge. All other edges should intersect it. So the endpoints of each edge should be on different sides of the first edge. Let the vertices on the cycle be named

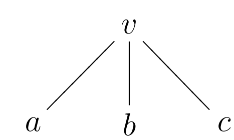
$v_1, v_2, \ldots, v_n$ in the same order as we traverse the cycle. Let v_1, v_2 be the endpoints of the first edge and let v_3 be on the side of v_1v_2 which we name $\triangle$ and let $\square$ be the name of the other side. Then v_4 should be on the $\square$ side. v_5 should be on the $\triangle$ end and so on. v_{2k} should be on the $\square$ side and v_{2k+1} should be on the $\triangle$ side. But then v_n is on the $\square$ side. Hence the segment v_nv_1 is on the $\square$ side and the segment v_2v_3 is on the $\triangle$ side. Hence these two can not have an intersection. So for even n, the graph C_n is not self-intersecting.

For odd n, just consider a regular n-gon, and draw two line segments from each vertex to the two opposite (farthest) vertices. This forms a star-shaped cycle which is isomorphic to C_n.

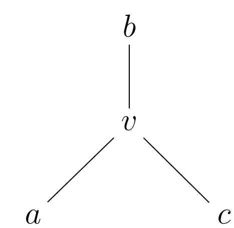


(b) We will prove that each vertex has at most two non-leaf neighbors. Suppose the contrary is true, and let v be a vertex with three non-leaf neighbors a, b, c. There are two possibilities for a, b, c. Either all of them can be fitted in a half-plane passing through v, or not.

In the first case, consider the one which lies between the other two, looking from v. For example let it be b. No line segment coming out from b can intersect both va and vc.



In the second case, no line segment coming out from any of a, b, c can intersect both of the other segments. For example any line segment coming out from a can intersect at most one of vb, vc.



Therefore existence of v, a, b, c yields a contradiction. So every vertex has at most two non-leaf neighbors. Now remove all leaves. Removing each leaf subtracts one from the vertices and one from the edges, and hence has no effect on the difference between those two. After doing so, the degree of each vertex will be at most 2. Hence half the sum of degrees which is the number of edges will be at most the number of vertices.

(c) Using the arguments of the previous section, such a graph, after removing all of its leaves, will become one where the degree of each vertex is 0, 1 or 2. Such a graph is a disjoint union of paths (possibly of length 0) and cycles. Any cycle, if it exists must be of odd length.

i. There is at least one cycle. This cycle must be of odd length. There is no edge between the vertices of the cycle, or otherwise a smaller cycle of even length would have been produced. No edge can exist having both endpoints outside the cycle, because using a side argument as in the first section, any vertex of that cycle must be present on both sides of that edge, which is impossible. So the graph is an odd cycle with some leaves coming out of its vertices, plus some alone vertices. Any graph of this kind is self-intersecting. One can first form the star-shaped cycle and then draw the leaves so that the edges pass through the center of the star; this way all edges will intersect.

ii. There is no cycle. By adding some edges and possible some vertices we can form an odd cycle containing all of the paths. The whole graph (including the leaves) will become a graph described in the first case, which is self-intersecting. But any subgraph of a self-intersecting graph is obviously self-intersecting. Hence any graph consisting of some disjoint paths with some leaves coming out of them is self-intersecting too.

8. Let $X^{13} = 2581452668046920778582615126663 = A$. Summing the digits of A, one can see that $3|A$. Hence $3|X$. Note that $\phi(10) = 4|13-1$, so $X \equiv X^{13} \equiv A \equiv 3 \pmod{10}$. Therefore $X \equiv 3 \pmod{30}$, because $3|A$.
Now note that $100^{13} = 10^{26} < A$. Hence $X > 100$. Also note that $200^{13} = 2^{13} \times 10^{26} > 8 \times 10^{29} > A$. So $X < 200$.
So far we have $X \equiv 3 \pmod{30}$ and $100 < X < 200$. The only possible numbers with these properties are $123, 153, 183$. We will prove that

$X = 183$. It's sufficient to prove that $X > 160$. But

$$160^{13} = (2^{13})^4 \times 10^{13} = 8192^4 \times 10^{13} < 10000^4 \times 10^{13} = 10^{29} < A$$

which completes the proof.

Third Round

1. Let I be the common incenter of triangles $ABC, A'B'C'$. Let r be the inradius of ABC, which is equal to the distance between I and BC. IA' is no less than this distance, hence $IA' \geq r$. Now let M be the foot of the perpendicular line drawn from I onto $A'B'$. IM is equal to the inradius of $A'B'C'$ which is $r/2$. So we have $IA' \geq 2IM$. Since $\angle IMA' = 90°$, we have $\angle IA'M \leq \arcsin(1/2) = 30°$. So $\angle B'A'C' = 2\angle IA'M \leq 60°$. The same is true for $\angle C'B'A', \angle A'C'B'$. However the sum of these angles is $180°$, so they all must be equal to $60°$. So $A'B'C'$ is equilateral. Since the inequalities should be equalities IA' should be perpendicular to BC, too. So A', B', C' are the points of tangency of the ABC's incircle with its sides. Since $BA' = BC'$ we have $\angle C'A'B = 90° - \angle ABC/2$; similarly $\angle B'A'C = 90° - \angle ACB/2$. So $\angle B'A'C' = (\angle B + \angle C)/2 = 60°$. Similarly $(\angle A + \angle B)/2 = 60°$ and $(\angle A + \angle C)/2 = 60°$. These three equalities yield $\angle A = \angle B = \angle C = 60°$, which completes the proof.

2. Suppose the contrary, and let $p_1, \ldots, p_k$ be all the primes which divide at least one of the numbers $2^{2^n} + a$. Consider the numbers $a^{2^i} + a$ for $1 \leq i \leq k+1$. These are finitely many numbers, and hence there is a number r for which none of them are divisible by p_j^r for any j. Since the numbers $2^{2^n} + a$ grow bigger and bigger, from some point on they will be greater than p^{rk} where p is $\max p_i$. By the pigeonhole principle, in the factorization of $2^{2^n} + a$ for such a large n, the power of one of p_j's will be greater than r. Now consider $k+1$ consecutive terms of this big enough numbers $2^{2^n} + a$. Since each one is divisible by p_j^r for some j, and there are only k such primes p_j, two of these numbers must be divisible by the same p_j^r, say p_1^r. So $p_1^r | 2^{2^n} + a$ and $p_1^r | 2^{2^{n+m}} + a$ where $0 < m \leq k$. We have

$$-a \equiv 2^{2^{n+m}} \equiv \left(2^{2^n}\right)^{2^m} \equiv a^{2^m} \pmod{p_1^r} \quad \Longrightarrow \quad p_1^r | a^{2^m} + a$$

which is a contradiction, because of the way we chose r.

3. By $\sum_{cyc}$ we mean a sum over all three similar terms in which a, b, c are permuted cyclically. We have to prove that

$$\sum_{cyc} \frac{1}{2 + a^2 + b^2} \leq \frac{3}{4}$$

Multiplying both sides by 2 and subtracting them from 3, the following equivalent inequality is obtained

$$\sum_{cyc} \frac{a^2+b^2}{2+a^2+b^2} \geq \frac{3}{2}$$

Using the Cauchy-Schwarz inequality we have

$$\left(\sum_{cyc} \frac{a^2+b^2}{2+a^2+b^2}\right)\left(\sum_{cyc} 2+a^2+b^2\right) \geq \left(\sum_{cyc} \sqrt{a^2+b^2}\right)^2$$

But we have

$$\left(\sum_{cyc} \sqrt{a^2+b^2}\right)^2 = 2\sum_{cyc} a^2 + 2\sum_{cyc} \sqrt{(a^2+b^2)(a^2+c^2)}$$

However note that using Cauchy-Schwarz again, we have

$$\sqrt{(a^2+b^2)(a^2+c^2)} \geq a^2+bc$$

Hence

$$\left(\sum_{cyc} \sqrt{a^2+b^2}\right)^2 \geq 2\sum_{cyc} a^2+2\sum_{cyc} a^2+2\sum_{cyc} ab = 3\sum_{cyc} a^2+(a+b+c)^2$$

The right hand side is equal to

$$9+3\sum_{cyc} a^2 = \frac{3}{2}(6+2\sum_{cyc} a^2) = \frac{3}{2}\sum_{cyc} 2+a^2+b^2$$

Putting all things together we obtain

$$\left(\sum_{cyc} \frac{a^2+b^2}{2+a^2+b^2}\right)\left(\sum_{cyc} 2+a^2+b^2\right) \geq \frac{3}{2}\left(\sum_{cyc} 2+a^2+b^2\right)$$

which proves the desired inequality.

4. Let $g(x) = x^n f(1/x)$ where n is the degree of f. g is a polynomial. Let x be a natural number. Let p be a prime bigger than x. Then x^{-1} modulo p exists. Let y be a natural number for which $xy \equiv 1 \pmod{p}$. Then it is easy to see that $f(x^{-1}) \equiv f(y) \pmod{p}$. Because $p|xy-1$ we have $p|f(x)f(y)-1$. So $x^n \equiv x^n f(x)f(y) \equiv x^n f(x)f(x^{-1}) \equiv f(x)g(x) \pmod{p}$. So we have

$$p|f(x)g(x) - x^n$$

But we can take p as large as we want, even larger than $|f(x)g(x) - x^n|$. Therefore $f(x)g(x) - x^n$ should be 0. Since f, g are polynomials $f(x)g(x) - x^n$ is identical to 0. Since the degree of f is n, g must be a constant and hence f must be of the form ax^n. But then $f(x)g(x) = a^2x^n = x^n$ which is only possible if $a = \pm 1$. So f must be of the form $\pm x^n$ for some n.
On the other hand every polynomial of the form $\pm x^n$ has the desired property, because

$$uv \equiv 1 \pmod{p} \quad \Longrightarrow \quad u^n v^n \equiv 1 \pmod{p}$$

5. Let M be the midpoint of AB. Then $MQ \perp AB$ because Q is on the perpendicular bisector of AB, having equal distances from A, B. $C'P \perp PQ$. Hence $MQPC'$ is cyclic. $MC'B'A'$ is circumscribed in the 9-point circle of ABC. $A'B'AB$ is also cyclic because both $\angle AA'B, \angle AB'B$ are right angles. Now let S be the intersection of the extensions of $A'B'$ and AB. Without loss of generality assume that A' lies between S, B'. Since $MQPC'$ is cyclic we have $SP \cdot SQ = SC' \cdot SM$. Since $MC'A'B'$ is cyclic we have $SC' \cdot SM = SA' \cdot SB'$. Finally because $A'B'AB$ is cyclic we have $SA' \cdot SB' = SA \cdot SB$. So $SP \cdot SQ = SA \cdot SB$ which shows that $ABPQ$ is cyclic. Because of this $\angle PBQ = \angle PAQ$.
$PC' \perp A'B'$ and $CC' \perp AB$. Hence we have $\angle PC'C = \angle A'SA$. Now consider the circumcircle of $ABPQ$. In this circle we have $\angle QSA = (\overset{\frown}{AQ} - \overset{\frown}{PB})/2$. But since Q is equidistant from A, B we have $\overset{\frown}{AQ} = \overset{\frown}{BQ}$. Hence $\angle QSA = (\overset{\frown}{BQ} - \overset{\frown}{BP})/2 = \overset{\frown}{PQ}/2 = \angle PBQ$. Thus $\angle PBQ = \angle QSA = \angle PC'C$.

6. We present two different solutions.
First solution: Draw the line segments between adjacent points of

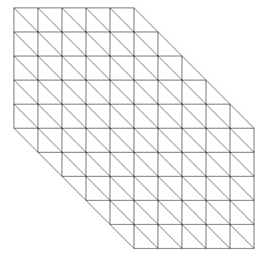
the lattice. Remove those segments which lie outside of the closed path (since the path is closed, it has both an inside and an outside). Now move the lattice points according to their position on the path to the vertices of a regular n^2-gon. Move the line segments previously drawn to the corresponding diagonals of the polygon. These diagonals do not intersect each other, because if two of them like ab and cd do, then c, d must be on different sides of ab. On the lattice, ab divides the path into two parts each having one of c, d. But c, d are adjacent points on lattice. This can not happen unless cd lies outside the closed path.
The closed path is divided into some quadrilaterals when the mentioned edges are drawn; so is the polygon, because the diagonals do not intersect each other. The center of the regular polygon must be inside one of the quadrilaterals, say $abcd$. One of the arcs ab, bc, cd, da must be at least a quarter of the total polygon's perimeter. For example let's assume it is ab. Then ab is a drawn diagonal of the polygon. Hence a, b must be adjacent in the lattice. The length of one of the arcs between a, b is at least a quarter of the total path's length. The length of the other one (which passes through c, d) is at least half of the path's length, because in the regular polygon, this arc contains the center. So a, b are the desired points.
Second solution: The area inside the closed path can be computed using the Pick's theorem. The number of points on the boundary is n^2 and the number of points inside is 0. Hence the area inside the path is $n^2/2 - 1$. Consider the cells in the lattice. Each cell is a square made from four adjacent points. There are $(n-1)^2$ cells, totally. The number of cells inside the closed path is equal to its area which is $n^2/2 - 1$. Consider each cell inside the path as a vertex of a graph, and draw an edge between two vertices if they share a side. The graph obtained will be a tree. It is connected obviously, because inside of a closed path is connected. It contains no cycles, because if it did, then the lattice points inside of that cycle and the lattice points outside of that cycle could not have been connected by the path. Hence the graph is a tree. Obviously each vertex of the graph has degree at most 4. Now we can use this lemma; in a tree with m vertices, each having a degree of at most k, there is an edge dividing the graph into two parts each having at least $(m-1)/k$ of the vertices. Using this lemma there are two adjacent cells which divide the inside of the path into two parts each having nearly a quarter of the cells. Let ab be the common side of these

two cells. Then by drawing ab, the closed path will be divided into two paths, each having an inner area of at least $(n^2/2-2)/4 = n^2/8-1/2$. Again using the Pick's theorem, each part of the path must have a perimeter of at least $2(n^2/8-1/2+1) = n^2/4+1$. So the distance between a, b along either one of the arcs is at least $n^2/4+1-1 = n^2/4$. So a, b are the desired pair.

Proof of the lemma: Choose the edge, that divides the graph into parts, the smallest of which is as large as possible. Let this edge be $e = (a, b)$, and let a be in the part with the smaller number of vertices. This part has fewer than $(m-1)/k$ vertices, or else there is nothing to prove. Let b be adjacent to the vertices $v_1, \ldots, v_d, a$. Hence $d+1 \leq k$. Let the component of the tree which contains v_i when we remove the edge (b, v_i) be V_i. Let A be the component of the tree containing a when we remove e. Then $|A|+\sum_i |V_i| = m-1$. And because $|A| < (m-1)/k$ and the number of V_i's is at most $k-1$, one of them should have at least $(m-1)/k$ vertices. Assume that $|V_1| > (m-1)/k$. Now consider the edge (b, v_1). It divides the tree into two parts, one of the being V_1 and the other one being $A \cup V_2 \cup \cdots \cup V_d \cup \{b\}$. If V_1 is the smaller part then V_1 has the desired number of vertices, and the result is achieved; however if the other part is the smaller one, it has at least one more vertex than A has, contradicting how we chose e. So the lemma is proved.

7. With an affine transformation we may assume that the groups of lines are parallel to the lines $x=0, y=0, x=y$. Sort the lines of the first groups according to their x, the lines of the second group according to their y and the lines of the last group according to their $x-y$, in ascending order. Let's call the lines of the first group, type x, the lines of the second group, type y, and the lines of the last group type $x-y$. Take the first line of type x and the first line of type y. Let p be their intersection. Obviously there is no special point (through which three lines pass) to the left or to the bottom of p. Each line of type $x-y$ intersects each of these two lines, once. However either the intersection with the x-line is to the left of p or the intersection with the y-line is to the bottom of p. So each line of type $x-y$ can pass through at most one special point which is located in either of the first x-line or the first y-line.

   Now remove the first x-line and the first y-line. At most 11 special points will be removed, because we have 11 lines of type $x-y$. Use the same arguments to remove the second x-line and the second y-line. Continue like this until the first 5 lines of type x and the first 5 lines of type y are removed. Till now, at most $11 \times 5 = 55$ special points are lost.



Now we have only 6 lines of type x and 6 lines of type y. These lines have at most $6 \times 6 = 36$ points of intersection. Hence there are at most 36 special points remaining. So in total, there are at most $11 \times 5 + 6 \times 6 = 91$ special points.
The following figure shows how 91 is possible.

Note: The number 11 can be easily replaced by n. This problem can also be stated in a more delicate manner. In the 3-dimensional space, we have three groups each consisting of n 2-dimensional planes parallel to one of the xy, yz, zx planes. The intersections of these planes form a $n \times n \times n$ semi-lattice. Now consider a plane not parallel to any of xy, yz, zx planes. At most how many points of the semi-lattice can be on this plane?
Taking the intersections of the plane groups with this plane, we obtain three groups of parallel lines. Thus the new problem statement is equivalent to the old one.

8. We have

$$\begin{aligned} P(y^2, x^2) &= P\left(\frac{(x+y)^2}{2}, \frac{(y-x)^2}{2}\right) \\ &= P\left(\frac{(x+y)^2}{2}, \frac{(x-y)^2}{2}\right) \\ &= P(x^2, y^2) \end{aligned}$$

But since x^2, y^2 are algebraically independent, we may replace them with x, y, to obtain $P(x,y) = P(y,x)$. This shows that P is a symmetric polynomial. Hence it can be written as a polynomial of $x+y, xy$. Let $P(x,y) = g(x+y, xy)$. Then

$$\begin{aligned} P(x^2, y^2) &= g(x^2+y^2, x^2y^2) = \\ P\left(\frac{(x+y)^2}{2}, \frac{(x-y)^2}{2}\right) &= g\left(x^2+y^2, \frac{(x^2-y^2)^2}{4}\right) \end{aligned}$$

The equality between g's is defined in terms of x^2, y^2. Since they are algebraically independent we can replace them by x, y. Thus

$$g(x+y, xy) = g\left(x+y, \frac{(x-y)^2}{4}\right)$$

Again since $x+y, xy$ are algebraically independent we can replace them with two independent variables u, v.

$$g(u,v) = g\left(u, \frac{u^2}{4} - v\right)$$

Now define $h(u,v) = g(u, u^2/8 - v)$. Then

$$\begin{aligned} h(u,v) &= g\left(u, \frac{u^2}{8} - v\right) = \\ g\left(u, \frac{u^2}{4} - \frac{u^2}{8} + v\right) &= g\left(u, \frac{u^2}{8} + v\right) = \\ h(u,-v) & \end{aligned}$$

Because $h(u,v) = h(u,-v)$, every occurrence of v in h is with an even degree. Hence h can be written as a polynomial of u, v^2. For example let $h(u,v) = Q_0(u, v^2)$. Then $g(u, u^2/8 - v) = Q_0(u, v^2)$. So $g(r,s) = Q_0(r, (s - r^2/8)^2)$. So $g(r,s)$ can be written as a polynomial of $r, s^2 - sr^2/4 + r^4/64$ or equivalently $r, s(r^2/4 - s)$; because we have r and we can eliminate $r^4/64$ from the second base element. So $g(r,s) = Q_1(r, s(r^2/4 - s))$. Now

$$P(x,y) = g(x+y, xy) = Q_1\left(x+y, xy\frac{(x-y)^2}{4}\right)$$

So $P(x,y)$ is a polynomial of $x+y, xy(x-y)^2$. The inverse is also true. If $P(x,y) = Q(x+y, xy(x-y)^2)$ then

$$P(x^2, y^2) = Q(x^2+y^2, x^2y^2(x^2-y^2)^2)$$

and

$$P\left(\frac{(x+y)^2}{2}, \frac{(x-y)^2}{2}\right) = Q(x^2+y^2, (x^2-y^2)^2(xy)^2)$$

It's easy to see that the right hand sides are equal. Hence the following relation holds

$$P(x^2, y^2) = P\left(\frac{(x+y)^2}{2}, \frac{(x-y)^2}{2}\right)$$

So the general answer is every polynomial of $x+y, xy(x-y)^2$.

9. Let Q be the midpoint of EF. QI would be the perpendicular bisector of EF. Let H' be the orthocenter of DEF. Clearly H' is located on DM, because DM is an altitude in DEF. I is the circumcenter of DEF. We will show that $IQ = DH'/2$. Although this is a well-know fact, one can prove it by considering the triangle $D'E'F'$ where D', E', F' are the midpoints of EF, FD, DE, respectively. This triangle can be mapped to DEF by a homothety with constant -2. I is the orthocenter of $D'E'F'$ because ID', IE', IF' are perpendicular to EF, FD, DE which in turn are parallel to $E'F', F'D', D'E'$. So under this homothety I is mapped to H'. D' which is identical to Q is mapped to D. So IQ is mapped to DH'. Therefore $DH' = 2IQ$ as mentioned before. Now note that $DP = DM/2$. Combining this with the last equation gives $IQ/DP = DH'/DM$.
   We have $BH \perp CI$. So $\angle HBC = 90° - \angle ICB = 90° - \angle ACB/2 = \angle ABC/2 + \angle BAC/2$. The quadrilateral $AEIF$ is cyclic having two right angles, so $\angle IFE = \angle IAE = \angle BAC/2$. $BDIF$ is also cyclic, so $\angle IFD = \angle IBD = \angle ABC/2$. So $\angle EFD = \angle EFI + \angle DFI = \angle ABC/2 + \angle BAC/2 = \angle HBC$. Similarly we have $\angle FED = \angle HCB$. So the two triangles DEF, HBC are similar, because they have equal angles. I is the orthocenter of HBC and H' is that of DEF. DM is an altitude of DEF, and HD is the corresponding altitude in HBC. Therefore $DH'/DM = HI/HD$.
   In conclusion, we have $IQ/DP = HI/HD$. Because IQ, DP are both perpendicular to EF, they are parallel to each other. So using the Thales's theorem, H, Q, P are collinear, proving the claim.

10. $AB = AD$, so $\angle BAD = \angle ADB = \angle ABC/2$. $\angle I_C AC = \angle BAC + (180° - \angle BAC)/2 = 90° + \angle BAC/2$ and $\angle ACI_C = \angle ACB/2$. So $\angle AI_C C = 180° - 90° - \angle BAC/2 - \angle ACB/2 = \angle ABC/2$. Hence $\angle ADC = \angle AI_C C$, which shows that $ACDI_C$ is cyclic. Therefore $\angle CDI_C = 180° - \angle I_C AC = 180° - 90° - \angle BAC/2 = \angle ABC/2 +$

$\angle ACB/2$. Hence DT bisects the angle $\angle I_C DC$. Just like the equality $\angle TI_C A = \angle ABC/2$ was achieved, we can achieve $\angle TI_C B = \angle BAC/2$. We have $\angle TBI_C = \angle ABI_C - \angle ABT = 90° - \angle ABC/2 - \angle ACB/2 = \angle BAC/2$. So $\angle TI_C B = \angle TBI_C$, which proves that $TB = TI_C$. T is equidistant from $BD, I_C D$ because it is located on the angle bisector of $\angle I_C DB$. So either we have $\angle TI_C D = \angle TBD$ or $\angle TI_C D + \angle TBD = 180°$. In the first case the triangles DTI_C, DTB would be congruent and hence $I_C DB$ would be isosceles. So $2\angle I_C BD + \angle BDI_C = 180° = 180° - \angle ABC + \angle ABC/2 + \angle ACB/2$ which yields $\angle ACB = \angle ABC$ which is impossible because $AB \neq AC$.
So the second case must happen. Hence $TI_C DB$ would be cyclic. So $\angle BTC = \angle I_C DB = \angle ABC/2 + \angle ACB/2 = 90° - \angle BAC/2$. But $\angle BTC = \angle BAC$. So we have

$$90° - \frac{\angle BAC}{2} = \angle BAC \quad \Longrightarrow \quad \angle BAC = 60°$$

11. First note that $\|3^{2^n} - 1\|_2 = 2 + \|2^n - 1\|_2 = 2 + n$ for $n \geq 1$. Therefore $\text{ord}_{2^n} 3 = 2^{n-2}$, because $2^n | 3^{2^{n-2}} - 1$ and $2^n \nmid 3^{2^{n-3}} - 1$. Therefore the powers of 3 form a subgroup of $\mathbb{Z}/2^n$ of order 2^{n-2}. Since the powers of 3 modulo 8 are $1, 3$, this subgroup consists of only elements of this form. But for $n \geq 3$, the number of elements equivalent to $1, 3$ modulo 8 are exactly 2^{n-2} which is equal to the size of the subgroup. Hence each number congruent to 1 or 3 modulo 8 can be written as a power of 3.
Now write -5 (which is congruent to 3 modulo 8) as a power of 3 modulo 2^{2n+4}. Let the exponent be k. Then $-5 \equiv 3^k$ for 2^l for every $l \leq 2n + 4$ including $l = n + 4$. Hence

$$(-5)^{\frac{3^{2^n}-1}{2^{n+2}}} \equiv 3^{k\frac{3^{2^n}-1}{2^{n+2}}} \pmod{2^{n+4}}$$

So it suffices to prove

$$3^{\frac{5^{2^n}-1}{2^{n+2}}} \equiv 3^{k\frac{3^{2^n}-1}{2^{n+2}}} \pmod{2^{n+4}}$$

Because each number raised to the power of 2^{n+2} will be 1 modulo 2^{n+4} is suffices to prove that

$$\frac{5^{2^n}-1}{2^{n+2}} \equiv k\frac{3^{2^n}-1}{2^{n+2}} \pmod{2^{n+2}}$$

Or equivalently

$$3^{k2^n} - 1 \equiv 5^{2^n} - 1 \stackrel{?}{\equiv} k(3^{2^n} - 1) \pmod{2^{2n+4}}$$

Moving all things to the left hand side and factoring $3^{2^n} - 1$ we get

$$(3^{2^n} - 1)(3^{(k-1)2^n} + 3^{(k-2)2^n} + \cdots + 1 - k) \stackrel{?}{\equiv} 0 \pmod{2^{2n+4}}$$

But the left hand side can be written as

$$(3^{2^n} - 1)(3^{(k-1)2^n} - 1 + 3^{(k-2)2^n} - 1 + \cdots + 1 - 1) = (3^{2^n} - 1)^2(\dots)$$

But $3^{2^n} - 1$ is divisible by 2^{n+2} and hence its square is divisible by 2^{2n+4}. The proof is complete.

12. Divide the elements of T into groups $A_0, \dots, A_k$ where k is the largest number such that $2^k \leq n$. We have $k = \lfloor \log_2 n \rfloor$. Put an element of T into A_i where i is the exponent of 2 in that element's factorization. So each element of A_i is of the form $2^i o$ where o is some odd number. Now let B_i be the set A_i divided by 2^i. So B_i's are sets of odd numbers. Clearly the elements of B_i are all less than or equal to $n/2^i$. For each i the elements of B_i, B_{i+1} can not be divisible by each other. Because if two elements are divisible, the corresponding elements in A_i, A_{i+1} would contradict the constraints imposed on T. Now write the elements of B_i, B_{i+1} in the form $3^j u$ where u is a number not divisible by 3. The u's obtained from all elements should be different because if two elements have the same u, like $3^j u, 3^{j'} u$, the one with the bigger exponent of 3 would be divisible by the other; a contradiction. So all u's obtained in this way must be different. These u's are odd numbers not divisible by 3. They are therefore $\equiv \pm 1$ modulo 6. The total amount of numbers congruent to ± 1 modulo 6 from 1 to m is at most $m/3 + 1$. This can be easily checked by considering all possible remainders of m when divided by 6. The u's obtained in our problem are all obtained from numbers in B_i, B_{i+1} which are between $1, n/2^i$. So there are at most $n/(3 \cdot 2^i) + 1$ such u's, in other words $|B_i| + |B_{i+1}| \leq n/(3 \cdot 2^i) + 1$. Writing this inequality for $i = 0, 2, 4, \dots$ and taking the sum results in

$$\sum_{i=0}^{2\lfloor (k-1)/2 \rfloor + 1} |B_i| \leq \lfloor \frac{k+1}{2} \rfloor + n \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \frac{1}{3 \cdot 4^i} \leq \frac{1 + \log_2 n}{2} + \frac{4}{9} n$$

The above sum takes care of all elements except for possibly the elements of B_k (if k is even). But B_k can have at most 1 element because all of its elements are less than or equal to $n/2^k$ where $k = \lfloor \log_2 n \rfloor$; so $n/2^k < 2$ and hence $|B_k| \leq 1$. So finally we have

$$|T| \leq \frac{\log_2 n}{2} + \frac{3}{2} + \frac{4}{9}n$$

which is just a little bit better than the inequality we wanted.