# High School Olympiads

## an easy problem of longlist 1983

Reply

**am2525**

#1 Jun 24, 2009, 6:59 pm

Let O be a point outside a given circle. Two lines OAB, OCD through O meet the circle at A,B,C,D, where A,C are the midpoints of OB,OD, respectively. Additionally, the acute angle θ between the lines is equal to the acute angle at which each line cuts the circle. Find cos θ and show that the tangents at A,D to the circle meet on the line BC.

**Luis González**

#2 Jun 25, 2009, 8:20 pm

It's enough to see that $ABDC$ is an isosceles trapezoid, thus the diagonal intersection $Q$ is collinear with the center $K$ of the circle and $O$. If tangents to the circle at $B$ and $C$ meet at $P$, then $PO$ is the polar of $Q$ WRT $(K) \Longrightarrow PO \parallel BD$. Note that $PB \parallel OD$ because of $\angle PBO = \angle BOD = \theta \Longrightarrow PBDO$ is a parallelogram and since $PD$ bisects $OB$, it follows that $P, A, D$ are collinear. Tangents $p_a, p_d$ to $(K)$ through $A, D$ are the polars of $A, D$ and $BC$ is the polar of $P$ WRT $(K)$. If $P, A, D$ are collinear, then $p_a, p_d, BC$ concur.

On the other hand, median $BC$ of $\triangle ODB$ is equal to its base $BD$. Then

$$BD^2 = \tfrac{1}{2}(BD^2 + OB^2) - \tfrac{1}{4}OB^2 \Longrightarrow OB^2 = 2BD^2$$

$$\Longrightarrow \cos\theta = 1 - \frac{BD^2}{2OB^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

Quick Reply

# High School Olympiads

## A problem about centroids and conics

Reply

**Source**: I guess hard and I need help

**Nebraska boy**
#1 Jun 18, 2009, 5:45 am

Is given a triangle ABC , P a point . AP,BP,CP cuts BC,CA,AB at A',B'C'. Demostrate that the centroids of PA'B,PA'C,PB'C,PB'A,PC'A,PC'B lie on a same conic if and oly if P lies on any of the median-lines of the triangle ABC. Also construct with ruler and compass the points P on the median AM such that this conic is a parabola.

I've tried to solve this problem by different ways , even by coordinate bashing but this is too hard for me , I think that this is the best place to post this topic. Thanks in advance

**jayme**
#2 Jun 18, 2009, 11:43 am

Dear Mathlinkers,
perhaps see

http://forumgeom.fau.edu/
Myakishev A., Woo P. Y., On the Circumcenters of Cevasix Configuration, Forum Geometricorum vol. 3 (2003) 57-63.
Nguyen M. H., Another Proof of van Lamoen's Theorem and its converse, Forum Geometricorum vol. 5 (2005) 127-132.

Ayme J.-L., http://perso.orange.fr/jl.ayme vol. 2 Le cercle de van Lamoen

Sincerely
Jean-Louis

**Nebraska boy**
#3 Jun 18, 2009, 6:53 pm

Thanks .It is a related problem because involves circumcenters but not centroids , it perhaps helps me to visualize an sketch

**Luis González**
#4 Jun 22, 2009, 1:55 am

Let us use barycentric coordinates with respect to $\triangle ABC$.

$P\ (u:v:w)\ ,\ A'\ (0:v:w)\ ,\ B'\ (u:0:w)\ ,\ C'\ (u:v:0)$

$A_c\ (3u^2+4uv+2wu+wv+v^2 : v(2u+2v+w) : (u+v)w)$

$A_b\ (3u^2+4uvw+w^2+2uv+vw : v(u+w) : w(2u+v+2w))$

$B_c\ (u(2u+2v+w) : u^2+4uv+wv+3v^2+2vw : (u+v)w)$

$B_a\ (u(v+w) : 3v^2+2uv+4wv+w^2+uw : w(u+2v+2w))$

$C_a\ (u(v+w) : u(u+2v+2w) : v^2+uv+4wv+3w^2+2uw)$

$C_b\ (u(2u+v+2w) : v(u+w) : u^2+uv+4uw+3w^2+2vw)$

These 6 points lie on a conic $\mathcal{C} \equiv Ax^2+By^2+Cz^2+2Exy+2Fyz+2Gxz=0 \iff$ the determinant of the 6x6 linear system formed by the coefficients of $\mathcal{C}$ equals zero. Alternatively, we also can use Pascal theorem to establish the collinearity of the intersection of the opposite sidelines of the hexagon $A_cA_b...C_b$. The first determinat $\Delta$ is given by

$$\Delta = 81uvw(v-w)(w-u)(u-v)(u+v)^2(v+w)^2(u+w)^2(u+v+w)^{12}$$

If $P\ (u:v:w)$ does not lie either on the sidelines of $\triangle ABC$, the sidelines of the antimedial triangle or the line at infinite, then the necessary condition for these six points to lie on a same conic is that

$$v-w=0\ ,\ w-u=0\ ,\ u-v=0$$

In other words, if $P$ lies on any of the three median lines of $\triangle ABC$.

Equation of the conic $\mathcal{C}$ when $P \in AM$ can be written as

$v(9u^3+58u^2v+122uv^2+72u^3)x^2+u(18u^3+87u^2v+118uv^2+38v^3)(y^2+z^2)-(9u^4+66u^3v+164uv^3+36v^4+175u^2v^2)x(y+z)-uyz(18u^3+75u^2v+98uv^2+34v^3)=0$

Its intersection with the line at infinite $x+y+z=0$ is

$$(2u(2u+v):-u(2u+v)-u\delta:-u(2u+v)+v\delta)$$

Where $\delta=\sqrt{-u(2u+v)(u^2+5uv+3v^2)}$

Thus, this conic is a parabola $\Longleftrightarrow \sqrt{-u(2u+v)(u^2+5uv+3v^2)}=0$

This yields three cases:

1) If $u=0, P\equiv M, \mathcal{C}$ is a degenerate parabola, which is the product of the parallel lines $x(2x-y-z)=0,$ i.e. the sideline $BC$ and the parallel to $BC$ passing through centroid $G$.

2) If $2u+v=0,$ the parabola corresponding to $(-1:2:2)$ degenerates into the product of the lines $10x+y-z=0$ and $13x+4y+4z=0,$ respectively.

3) If $u^2+5uv+3v^2=0,$ the roots of this quadratic equation give the wanted points

$$X\ (5+\sqrt{13}:-2:-2)\ ,\ Y\ (5-\sqrt{13}:-2:-2)$$

Further, they divide the median $AM$ in the ratios

$$\frac{\overline{XA}}{\overline{XM}}=-\frac{4}{5+\sqrt{13}}\ ,\ \frac{\overline{YA}}{\overline{YM}}=-\frac{4}{5-\sqrt{13}}$$

**yetti**

*#5 Jun 22, 2009, 5:28 pm*

Would not the calculation be much simpler for midpoints $K_b, K_c, L_c, L_a, M_a, M_b$ of the segments $BA', CA', CB', AB', AC', BC'$? Hexagon of these midpoints is centrally similar to the hexagon of the centroids with center $P$ and coefficient $\frac{3}{2}$. Barycentric of the midpoints can be written without calculation

$$\overrightarrow{K_b}=(w+u)(u+v)\ [0,w+2v,w],\ \overrightarrow{K_c}=(w+u)(u+v)\ [0,v,v+2w]$$

$$\overrightarrow{L_c}=(u+v)(v+w)\ [u,0,u+2w],\ \overrightarrow{L_a}=(u+v)(v+w)\ [w+2u,0,w]$$

$$\overrightarrow{M_a}=(v+w)(w+u)\ [v+2u,v,0],\ \overrightarrow{M_b}=(v+w)(w+u)\ [u,u+2v,0]$$

When the segment midpoints are on a conic, the intersections $A\equiv L_cL_a\cap M_bM_a, D\equiv K_bM_b\cap K_cL_c,$ $E\equiv K_bL_a\cap K_cM_a$ are collinear, which is equivalent to $(\overrightarrow{D}\times\overrightarrow{E})_x=0.$ I could not factor the last expression without error; $uvw(u+v)(v+w)(w+u)(v-w)$ factored out OK, but the remaining factor did not make sense.

**yetti**

*#6 Jun 23, 2009, 11:26 am*

Thanks to **Luis González** for big hint. By Carnot theorem, midpoints $M_a, M_b, K_c, K_b, L_c, L_a$ of $AC', BC', BA', CA', CB', AB'$ are on a conic $\Longleftrightarrow$

$$\overline{AM_a}\cdot\overline{AM_b}\cdot\overline{BK_b}\cdot\overline{BK_c}\cdot\overline{CL_c}\cdot\overline{CL_a}=\overline{AL_a}\cdot\overline{AL_c}\cdot\overline{CK_c}\cdot\overline{CK_b}\cdot\overline{BM_b}\cdot\overline{BM_a}$$

Continuing with directed line segments or barycentric coordinates $(u,v,w)$ of $P$ is almost the same. Assuming $P$ is not on triangle sidelines, i.e., none of $(u,v,w)$ is zero, substitute

$$\frac{\overline{AM_a}\cdot\overline{AM_b}}{\overline{BM_a}\cdot\overline{BM_b}}=\frac{v(u+2v)}{u(v+2u)},\ \frac{\overline{BK_b}\cdot\overline{BK_c}}{\overline{CK_b}\cdot\overline{CK_c}}=\frac{w(v+2w)}{v(w+2v)},\ \frac{\overline{CL_c}\cdot\overline{CL_a}}{\overline{AL_c}\cdot\overline{AL_a}}=\frac{u(w+2u)}{w(u+2w)}.$$

$$\Longrightarrow\ \frac{v(u+2v)}{u(v+2u)}\cdot\frac{w(v+2w)}{v(w+2v)}\cdot\frac{u(w+2u)}{w(u+2w)}=1,$$

$$(u+2v)(v+2w)(w+2u)=(v+2u)(w+2v)(u+2w).$$

Whatever's left, after multiplying this through and canceling equal terms, factors as

$$u^2w+uv^2+vw^2=v^2w+uw^2+u^2v$$

$$(u-v)(v-w)(w-u)=0,$$

which means at least 2 barycentric coordinates are equal $\Longleftrightarrow$ $P$ is on one of the median lines.

---

To find $P$, for which the conic is a parabola, parallel project $\triangle ABC$ to an isosceles right triangle, which takes midpoints to midpoints and parabolas to parabolas. That is, draw isosceles right triangle $\triangle A^*BC$ on the base $BC$; $AA^*$ is direction of the rays. Then rename the triangle back to $\triangle ABC$. Put origin at $A$ and y-axis along the A-median $AA'$ and WLOG, let $BC=2$. Let $P$ be the desired point and let $CP$ cut $AB$ at $C'$ with coordinates $[c,c]$. The parabola has symmetry axis $AA'$ and equation $y=kx^2+q$. It goes through midpoints $K_b=[\frac{1}{2},\ 1], M_a=[\frac{c}{2},\ \frac{c}{2}], M_b=[\frac{c+1}{2},\ \frac{c+1}{2}]$ of $BA', AC', BC'$. Plugging these coordinates to the parabola equation and eliminating $k,q$ yields $\dfrac{c-2}{c-1}=\dfrac{c^2-1}{c^2+2c}$ or $c^2-3c-1=0$, with roots $c=\dfrac{3\pm\sqrt{13}}{2}$. $CC'$ cuts the A-median at $P=[0,p], p=1+\dfrac{c-1}{c+1}=\dfrac{1\pm\sqrt{13}}{3}$ and $\dfrac{\overline{AP}}{\overline{AA'}}=p$. From $\dfrac{\overline{AC'}}{\overline{BC'}}=-\dfrac{c}{1-c}$, barycentic coordinates of $P$ are $(u,v,w)=(1-c,c,c)$.

**yetti**

*#7 Jun 26, 2009, 12:40 am*

I like the problem very much, because for any reasonable point inside any reasonable triangle, the 6 centroids are so close to one conic and yet, only points on medians will do. One theorem and two methods are used to put the points $P$ generating conics through the centroids on one of the medians. One more method is then used to find points $P$ generating parabolas:

1. central similarity
2. Carnot theorem (i.e., one of Carnot theorems, for there are at least three different Carnot theorems)
3. barycentric coordinates
4. parallel projection

Barycentric coordinates could be replaced by directed segments, but barycentric coordinates are actually simpler.

Quick Reply

# High School Olympiads

## Proof of Pythagorean theorem with inversion.

Reply

**skalgar**
*#1 Jun 20, 2009, 10:18 pm*

Well, is it possible to prove Pythagorean theorem with inversion? If I'm not mistaken, I have seen it somewhere on this forum, but I couldn't find it.

**Luis González**
*#2 Jun 20, 2009, 10:29 pm*

Further, we can prove Ptolemy's theorem by using inversion, which is basically applying an inversion with center in one of the vertices of the quadrilateral that carries the other three into a straight line. Using the formula that gives the inverse distances in terms of the primitive distances, we obtain the Ptolemy's relation. The Pythagorean theorem is just a particular case when the cyclic quadrilateral is a rectangle.

**skalgar**
*#3 Jun 20, 2009, 10:33 pm*

Thanks for the tip. I am quite familiar with the proof of Ptolemy's theorem with inversion.

Edit:
But still, is there a proof with inversion without application of Ptolemy's theorem?

**filosofimenos**
*#4 Jun 20, 2009, 11:24 pm*

I am not good at all at geometry but what Luis said, is that you can prove the Pythagorean Theorem by using the same method as the one you use to prove Ptolemey's Theorem. Instead of any quadrilateral, just pick a rectangle. You do not actually apply Ptolemey's Theorem anywhere...

Quick Reply

# High School Olympiads

## A nice problem with incenter



**Source**: Without using complex numbers

**SUPERMAN2**

*#1 Jun 20, 2009, 9:41 am*

The circle with the center $O$ is inscribed in a triangle $ABC$ and touches the sides $AB, BC, CA$ at $M, K, E$ respectively.Denote by $P$ the intersection of $MK$ and $AC$.Prove that $OP$ is perpendicular to $BE$.

Can we solve this problem by another method such as harmonic division?

**Luis González**

*#2 Jun 20, 2009, 10:44 am*

Although this is trivial by pole-polars, let us try something different.

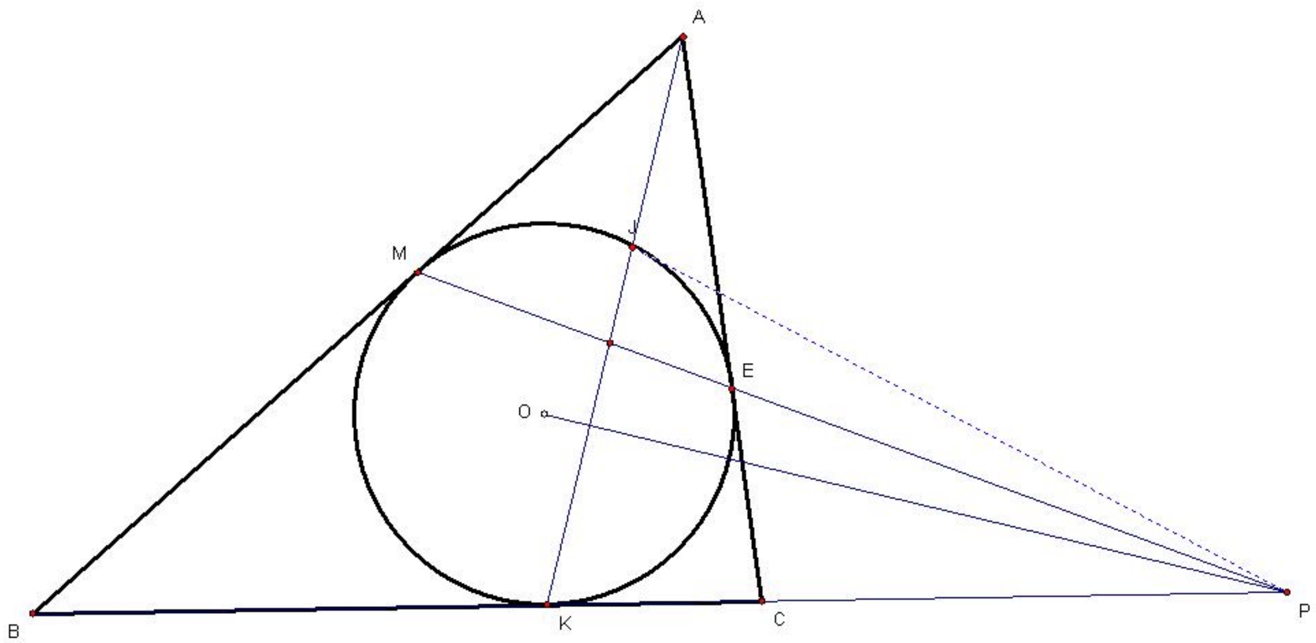
Let $Q$ be the projection of $O$ on $BE$; second intesection of the circle $\odot(OMBK)$ with diameter $OB$ and the circle with diameter $OE$. Now, $AC, MK, OQ$ are pairwise radical axes of $(O), \odot(OMBK)$ and $\odot(OQE)$ concurring at their radical center $P \Longrightarrow O, P, Q$ are collinear, or $OP \perp BE$, as desired.

**plane geometry**

*#3 Jun 20, 2009, 10:52 am*

pole and polar property
PJ is the tangent to the incircle

*Attachments:*




**dgreenb801**

*#4 Jun 20, 2009, 10:56 am*

This is my first problem I actually solved with polars, without help

MK is the polar of B wrt O since BM and BK are tangents to the circle, and AP is the polar of E wrt O, since E lies on the circle and AP is tangent to it at E, so P is the intersection of the two polars. Thus, BE is the polar of P, as Luis and plane geometry said.

EDIT: Here is another proof
Let Q be the perpendicular from O to BE. Since $\angle BMO = \angle BKO = \angle BQO = 90$, BKQOM is cyclic. Also, note that the circle with diamter OE passes through Q and is tangent to AC, since $OE \perp AC$. Thus, the radical axis of these two circles and the incircle must be concurrent, so MK, AC, and OQ are concurrent, so $PO \perp BE$.

**livetolove212**
*#5 Jun 20, 2009, 8:23 pm*

> *SUPERMAN2 wrote:*
> The circle with the center $O$ is inscribed in a triangle $ABC$ and touches the sides $AB, BC, CA$ at $M, K, E$ respectively.Denote by $P$ the intersection of $MK$ and $AC$.Prove that $OP$ is perpendicular to $BE$.
>
> Can we solve this problem by another method such as harmonic division?

Easy
Let $AK \cap (O) = \{J\}$
$JMKE$ is a harmonic quadrilateral then JP is the tangency of $(O) \rightarrow$ QED

**SUPERMAN2**
*#6 Jun 21, 2009, 9:54 am*

Thank you very much for your nice solutions.Complex number can also kill this problem easily but I still prefer the geometric method.

**Virgil Nicula**
*#7 Jun 22, 2009, 12:40 am*

> *SUPERMAN2 wrote:*
> Let $w = C(I, r)$ be the incircle of $\triangle ABC$ and $w$ touches the sides $AB, BC, CA$ at $M, K, E$
>
> respectively . Denote by $P$ the intersection of $MK$ and $AC$ . Prove that $PI \perp BE$ .

Very nice the **Dgreenb801**'s proof !

I"ll present a simple proof without complex numbers, radical axis, harmonical division (pole, polar) :

$PM \perp BI \iff PB^2 - PI^2 = MB^2 - r^2 \iff PB^2 - \left(PE^2 + r^2\right) = MB^2 - r^2 \iff$

$PB^2 - PE^2 = MB^2 \iff PB^2 - PE^2 = IB^2 - IE^2 \iff PI \perp BE$ .

See **here** a nice extension of this problem.

Quick Reply

# High School Olympiads

## Lines concurrent on incircle



**Source**: Very Nice.

**Moonmathpi496**
#1 Jun 19, 2009, 12:37 am

Let $D.E, F$ be the feet of the angle bisectors of angles $A, B, C$, respectively, of triangle $ABC$, and let $K_a, K_b, K_c$ be the points of contact of the tangents to the incircle of $ABC$ through $D, E, F$ (that is, the tangent lines not containing sides of the triangle).

Prove that the lines joining $ K_a, K_b, Kc_$ to the midpoints of $BC, CA.AB$ respectively, pass through a single point on the incircle of $ABC$.

**mathVNpro**
#2 Jun 19, 2009, 8:42 am

Hint

Back with a full solution.

Let $A_0, B_0, C_0$ be the tangency points of $(I)$ with the triangle sides $BC, CA, AB$ respectively, where $(I)$ is the incircle of $\triangle ABC$. Denote $M_a, M_b, M_c$ respectively by the midpoints of $BC, CA, AB$. We have
$(IB_0, IK_a) \equiv (IB_0, IA_0) + (IA_0, IK_a) \equiv (CB_0, CA_0) + (DA_0, DK_a) \equiv (CB_0, CA_0) + 2(DA_0, DA)$
$\equiv (CA, CB) - 2(DA, AC) - 2(CA, CD) \equiv (CA, CB) - (AB, AC) - 2(CA, CB) \equiv (BC, BA) \pmod{\pi}$.
Hence $(IB_0, IK_a) \equiv (BC, BA)$. With the same argument, we also have
$(IB_0, IK_c) \equiv (BA, BC) \implies (IB_0, IK_c) \equiv -(IB_0, IK_a) \pmod{\pi}$, which implies that $K_c \mapsto K_a$ through the reflection by $IB_0$. Therefore $K_aK_c \perp IB_0 \implies K_aK_c || M_aM_c$. Argue the same we also have $M_aM_b || K_aK_b$, $M_bM_c || K_bK_c$. Hence $\triangle M_aM_bM_c \sim \triangle K_aK_bK_c$. Moreover, $K_aM_a, K_bM_b, K_cM_c$ are concurrent at the point $F$ such that $\triangle M_aM_bM_c \mapsto \triangle K_aK_bK_c$ through a homothety with center $F$, let us denote this homothety by $\mathcal{H}_F$. Hence $\mathcal{H}_F : (M_aM_bM_c) \mapsto (K_aK_bK_c)$, or in other word, $(\mathcal{E}) \mapsto (I)$, where $(\mathcal{E})$ is the $9$- point circle wrt $\triangle ABC$. But we have already know that $(I)$ externally tangents to $(\mathcal{E})$, which implies that $F \equiv (I) \cap (\mathcal{E})$. Therefore, $K_cM_c, K_aM_a, K_bM_b$ are concurrent at a point $F \in (I)$. This $F$ is also known as the **Feurebach point** wrt $\triangle ABC$.
Our proof is completed then $\square$

*This post has been edited 5 times. Last edited by mathVNpro, Jun 20, 2009, 8:14 am*

**Moonmathpi496**
#3 Jun 19, 2009, 8:56 am

> mathVNpro wrote:
> $\triangle K_aK_bK_c \cong \triangle M_aM_bM_c$, where $M_a, M_b, M_c$ respectively are the midpoints of $BC, CA, AB$.

I believe that $\triangle K_aK_bK_c, \triangle M_aM_bM_c$ are centrally similar, not congruent. :(

**plane geometry**
#4 Jun 19, 2009, 1:03 pm

Actually,KcC,KaA,KbB,OI are concurrent

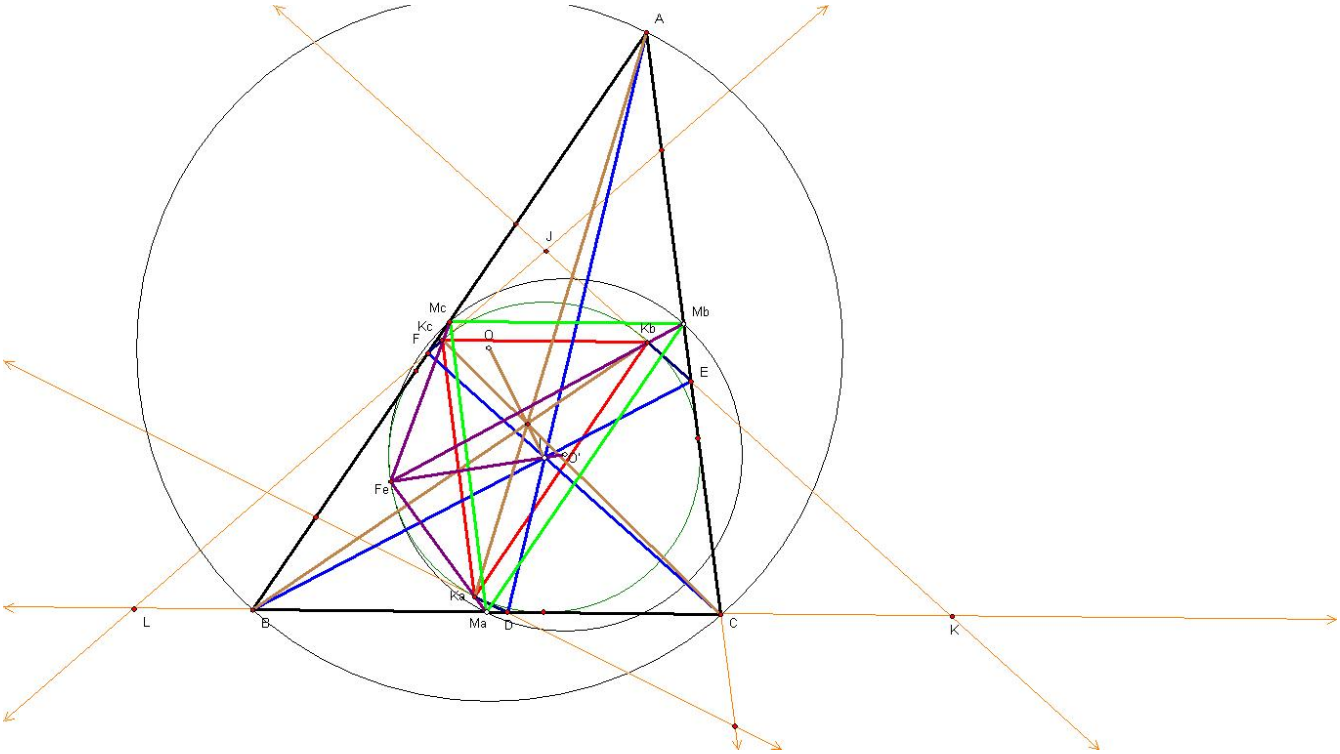
**plane geometry**
#5 Jun 19, 2009, 1:15 pm

Denote KcF∩BC=L , KbE∩BC=K

Denote KcI□BC=L , KbI□BC=K
L is the reflection of A with respect to CF K is the reflection of A with respect to BE
Thus JL=JK => KcKb//LK => KcKb//BC
Analogously, KcKa//CA KaKb//AB
□KcKaKb□□CAB
KcC,KaA,KbB,OI are concurrent at their inner similitude center

Back to our problem:
It is obvious,that □MaMbMc□□KcKaKb□□CAB with corresponding sides parallel
Thus MaKa MbKb McKc are concurrent at Fe
Therefore, IO pass through Fe
Notice the circumcenter of □MaMbMb is the nine-point circle of □ABC whose circumcenter is O’ => Fe is the Feurebach point

*Attachments:*




**Moonmathpi496**

*#6 Jun 19, 2009, 2:24 pm*

> *Quote:*
>
> Let $D.E, F$ be the feet of the angle bisectors of angles $A, B, C$, respectively, of triangle $ABC$, and let $K_a, K_b, K_c$ be the points of contact of the tangents to the incircle of $ABC$ through $D, E, F$ (that is, the tangent lines not containing sides of the triangle).
> Prove that the lines joining $ K_a, K_b, Kc_$ to the midpoints of $BC, CA.AB$ respectively, pass through a single point on the incircle of $ABC$.

Here goes my solution.
It is enough to prove that $\triangle K_aK_bK_c$ and $\triangle PQR$ are centrally similar. ($PQR$ is the medial triangle of $ABC$). Because the center of homothety $T$ of incircle and nine point circle is the Feuerbach point of $\triangle ABC$. So, If $\triangle K_aK_bK_c$ and $\triangle PQR$ are centrally similar, $T$ will be their center of homothety and the problem will follow.

$X, Y, Z$ be the points where the incircle touch the sides of the triangle $ABC$.
Now we have

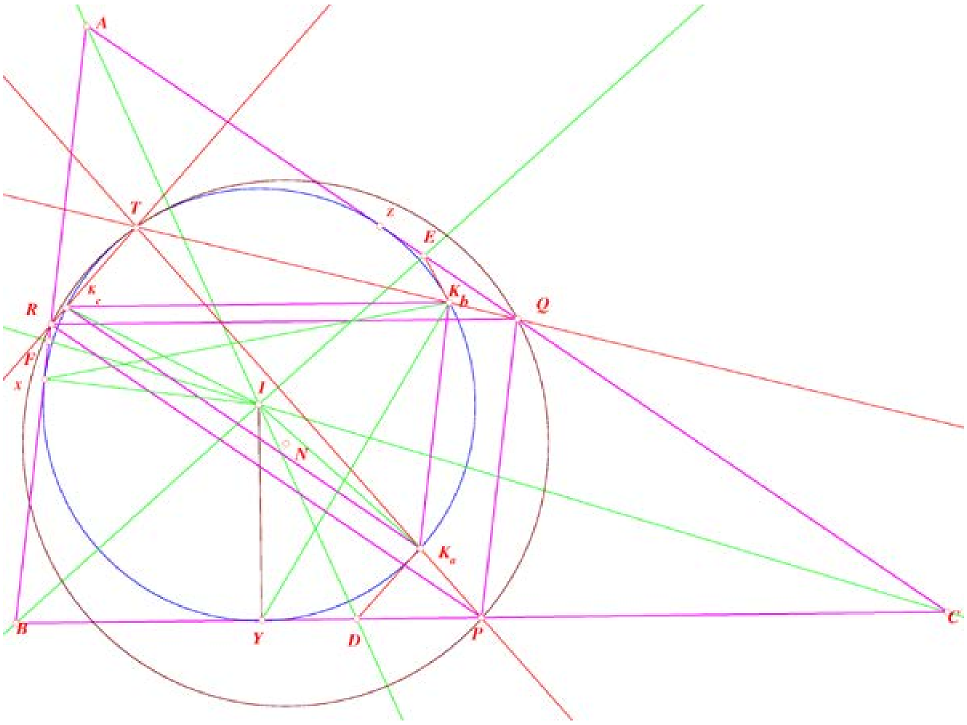
$$\begin{aligned} \angle K_cK_bK_a &= \angle K_cK_bX + \angle XK_bY + \angle YK_bK_a \\ &= \left(\frac{\pi}{2} - \frac{C}{2} - A\right) + \left(\frac{\pi}{2} - \frac{B}{2}\right) + \left(\frac{\pi}{2} - \frac{A}{2} - C\right) = B \end{aligned}$$

Analogously chasing other angles we can say that $\triangle K_aK_bK_c \sim \triangle PQR$.

Now we shall prove that, $K_cK_a \parallel RP$
We have $\measuredangle(IY, K_cK_a) = (\pi - A - 2C) + \left(\frac{\pi}{2} - B\right) = \frac{\pi}{2} - B$ also $\measuredangle(IY, RP) = \frac{\pi}{2} - B$.
So $K_cK_a \parallel RP$ and the proof is complete

So, $K_cK_a \parallel RP$ and the proof is complete.

*Attachments:*




**jayme**

*#7 Jun 19, 2009, 3:57 pm*

Dear Mathlinkers,
see also for example :
http://perso.orange.fr/jl.ayme vol. 4 Symétriques de (OI) par rapport aux côtés des triangles de contact et médian p. 10
Sincerely
Jean-Louis

**mathVNpro**

*#8 Jun 19, 2009, 4:46 pm*

**REMARK-** As **plane geometry** have said that $CK_c, AK_a, BK_b$ are concurrent is a special case of this problem:

**"Let $\triangle ABC$ with $(I)$ is its incircle. $A_0, B_0, C_0$ are the tangency point of $(I)$ with $BC, CA, AB$ respsectively. Let $A_1, B_1, C_1$ be the points on $BC, CA, AB$ such that $AA_1, BB_1, CC_1$ are concurrent. From $A_1, B_1, C_1$, let $A_2, B_2, C_2$ be the points on $(I)$ such that $AA_2, BB_2, CC_2$ are the second tangents from $A_1, B_1, C_1$ to $(I)$. Prove that: $AA_2, BB_2, CC_2$ are concurrent."**.

**Luis González**

*#9 Jun 20, 2009, 8:08 am*

Let $X, Y, Z$ be the tangency points of $(I)$ with $BC, CA, AB.$ It is clear that $YK_b \parallel ZX$ and $ZK_c \parallel YX.$ Therefore, $\angle K_cZX = \angle ZXY = \angle K_bYX \Longrightarrow K_cK_b$ is parallel to the tangent $BC$ to $(I)$ at $X.$ By similar reasoning, this means that $\triangle K_aK_bK_c$ and $\triangle ABC$ are homothetic $\Longrightarrow \triangle K_aK_bK_c$ and the medial triangle $\triangle M_aM_bM_c$ are homothetic. Therefore, $M_aK_a, M_bK_b, M_cK_c$ concur at their homothetic center, which is the exsimilicenter of their circumcircles $(I)$ and the nine-point circle $(N),$ i.e. the Feuerbach point of $\triangle ABC.$

Another approach

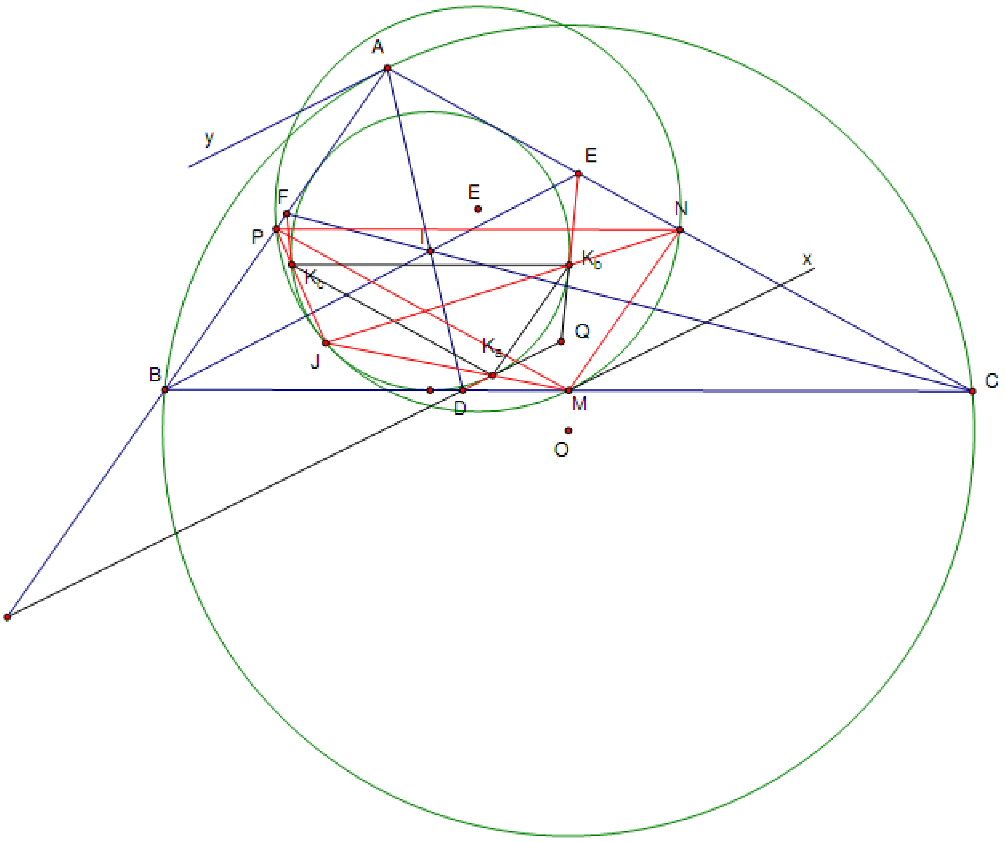
**livetolove212**

*#10 Jun 20, 2009, 8:14 pm*

Let (O) be the circumcircle of triangle ABC. Ay is the tangency of (O). Mx is the tangency of the nine-point circle (E) of triangle ABC. G is the centroid of triangle ABC.
By angle charsing, it's easy to prove that $DK_a//Ay$
The dilatation $D_G^{\frac{-1}{2}}: A \to M, B \to N, C \to P \Rightarrow \Delta ABC \to \Delta MNP$
Then $Ay//Mx$

Then $Ay//Mx$
So $DK_a//Mx$
Similarly for $EK_b, FK_c$ and we get $K_aK_b//MN, K_bK_c//NP, K_aK_c//MP$
And it's easy to prove that $DK_a, EK_b, FK_c$ are concurrent at J.
The Dilatation $D_J^{\frac{r}{r_E}}$: $M \to K_a, N \to K_b, P \to K_c$ then J lies on IE (J is the tangent of (I) and (E))
$\Rightarrow$ QED

*Attachments:*




Quick Reply

# High School Olympiads

## Cyclic quadrilateral with cosecant equality

Reply

Source: IMO LongList 1988, Spain 2, Problem 71 of ILL

**orl**

#1 Nov 10, 2005, 2:01 am

The quadrilateral $A_1A_2A_3A_4$ is cyclic, and its sides are $a_1 = A_1A_2, a_2 = A_2A_3, a_3 = A_3A_4$ and $a_4 = A_4A_1$. The respective circles with centres $I_i$ and radii $r_i$ are tangent externally to each side $a_i$ and to the sides $a_{i+1}$ and $a_{i-1}$ extended. $(a_0 = a_4)$. Show that

$$\prod_{i=1}^{4} \frac{a_i}{r_i} = 4 \cdot (\csc(A_1) + \csc(A_2))^2.$$

**Luis González**

#2 Jun 19, 2009, 12:26 am

$ABCD$ is cyclic with exradii $r_a, r_b, r_c, r_d$. We denote $AB = a, BC = b, CD = c, DA = d$. Let $X$ be the tangency point of $(I_a, r_a)$ with $AB$. Then we have

$$\tan\frac{A}{2} = \frac{AX}{r_a}\ ,\ \tan\frac{B}{2} = \frac{BX}{r_a} \Longrightarrow$$

$$\frac{a}{r_a} = \tan\frac{A}{2} + \tan\frac{B}{2} = \csc A + \csc B - \cot A - \cot B$$

Since the cosecant of opposite angles are equal and the cotangent of opposite angles have different signs, the cyclic product yields

$$\frac{abcd}{r_ar_br_cr_d} = [(\csc A + \csc B)^2 - (\cot A + \cot B)^2][(\csc A + \csc B)^2 - (\cot A - \cot B)^2]$$

$$\frac{abcd}{r_ar_br_cr_d} = 4[1 + \csc A(1 - \cos A\cos B)][1 + \csc A\csc B(1 + \cos A\cos B)]$$

$$\frac{abcd}{r_ar_br_cr_d} = 4[1 + 2\csc A\csc B + \csc^2 A\csc^2 B - (\csc^2 A - 1)(\csc^2 B - 1)]$$

$$\frac{abcd}{r_ar_br_cr_d} = 4(\csc^2 A + \csc^2 B + 2\csc A\csc B) = 4(\csc A + \csc B)^2.$$

Quick Reply

# Spain

## Problema!

Reply

**hucht**
#1 Aug 22, 2006, 7:52 am

A pedido del publico,m otro problema de geo

Dado $\triangle ABC$ y un triangulo $DEF$ equilatero inscrito en el, demostrar que $DE \geq \frac{2\sqrt{2}[ABC]}{\sqrt{a^2+b^2+c^2+4\sqrt{3}[ABC]}}$.

★ Capitán Mandarina

**conejita**
#2 Aug 22, 2006, 7:46 pm

Creo que ya me estoy acercando a la solucion, te explico:

En la IMO de 1961(creo) se demostro que a^2+b^2+c^2 >= 4(3^1/3)(ABC)
Entonces de ahi lo unico lo que tengo k demostrar se reduce a:
DE^2 >= (ABC)/(3^1/3)

Voy acercandome a la solucion o ya me desvie??

**hucht**
#3 Aug 22, 2006, 9:09 pm

> conejita wrote:
> Creo que ya me estoy acercando a la solucion, te explico:
>
> En la IMO de 1961(creo) se demostro que a^2+b^2+c^2 >= 4(3^1/3)(ABC)
> Entonces de ahi lo unico lo que tengo k demostrar se reduce a:
> DE^2 >= (ABC)/(3^1/3)
>
> Voy acercandome a la solucion o ya me desvie??

Creo que deberias hacer algo escencial, cuando un triangulo equilatero inscrito en uno dado tienen perimetro minimo??

tarea jeje

**conejita**
#4 Aug 23, 2006, 4:31 am

Pues yo se que en un triangulo cualquiera, el triangulo inscrito de menor perimetro es el ortico (o sea, el formado por los pies de las alturas). A eso te refieres???

**EDSON_**
#5 Aug 23, 2006, 1:05 pm

conejita lo que tu afirmas es correcto porque eso acabo de estudiarlo en el libro: MIRAR Y VER de Miguel Guzman, esta en el setimo ensayo titulado " El arte de mirar correctamente " pag 49 jajajaja inclusive esta su demostración setimo ensayo titulado "El arte de mirar correctamente" pag 49 jajajaja inclusive esta su demostracion

Si gustas agregame conejita soy de Lima Perú

edson_laura_galvez@hotmail.com

**José**

*#6 Aug 24, 2006, 12:12 am*

Por tu propia seguridad, poné que tu dirección es abc, donde a=edson_laura_galvez, b=@ y c=hotmail.com (Hay muchos bots que lo único que buscan en Internet son direcciones de mail)

**hucht**

*#7 Aug 26, 2006, 9:23 pm*

Ya que nadie lo resolvio lo are yo mismo

Empezaremos con algunos lemas y teoremas

Las coordenadas tripolares de un punto $P$ con respecto a $\triangle ABC$ es la terna $(x:y:z)$ tal que $\frac{x}{AP}=\frac{y}{BP}=\frac{z}{CP}$.

**Lema 1.** (Euler) Dado $\triangle ABC$ y $P=(x:y:z)$ en coordenadas tripolares reales (las distancias reales), entonces:

$(i)\sum(y^2+z^2-a^2)^2x^2=4x^2y^2z^2+\prod(y^2+z^2-a^2)$.
$(ii)\sum(-a^2+b^2+c^2)(y^2z^2+a^2x^2)=a^2b^2c^2+(a^2x^4+b^2y^4+c^2z^4)$

**Demostración.** Demostraremos $(i)$ dejando $(ii)$ para uds. Por el teorema de cosenos tenemos que $\cos\gamma=\frac{x^2+z^2-b^2}{2xz}$,
$\cos\alpha=\frac{y^2+z^2-a^2}{2yz}$ y
$\cos\beta=\frac{x^2+y^2-c^2}{2xy}$ pero de () tenemos que
$\cos^2\alpha+\cos^2\beta+cos^2\gamma=1+\cos\alpha\cos\beta\cos\gamma$. Reemplazando se sigue que $\sum(y^2+z^2-a^2)^2x^2=4x^2y^2z^2+\prod(y^2+z^2-a^2)$. ■

**Definición.** Dado $\triangle ABC$, sean $V_a, V_a'$ los puntos de interseccion de la $A-$bisectriz interior y exterior respectivamente con $\overline{BC}$. La circunferencia de diametro $\overline{V_aV_a'}$ es la $A-$*circunferencia de Apolonio* de $\triangle ABC$. Analogamente para $B$ y $C$.

**Teorema 2.** Las tres circunferencias de Apolonio tienen dos puntos en comun.

**Demostración.** Dado $\triangle ABC$, sean $J$ y $J'$ los puntos de intersección de la $B-$circunferencia de Apolonio y la $C-$circunferencia de Apolonio. De la definicion tenemos que $\frac{CJ}{JA}=\frac{a}{c}$ y $\frac{AJ}{JB}=\frac{b}{a}$ y multiplicando ambas expresiones tenemos que $\frac{CJ}{JA}.\frac{AJ}{JB}=\frac{a}{c}.\frac{b}{a}$ con lo que $\frac{CJ}{JB}=\frac{b}{c}$ y esto es, $J$ pertenece a la $A-$circunferencia de Apolonio de $\triangle ABC$ y an\'alogamente para $J'$. ■

Sean $J$ y $J'$ los puntos isodinamicos de $\triangle ABC$; esto es, el punto de interseccion de las circunferencias de Apolonio de $\triangle ABC$.

**Teorema 3.** Los tri\'angulos pedal de $J$ y $J'$ son equil\'ateros.

Para demostrarlo usaremos el siguiente

**Lema 4.** Las coordenadas tripolares de $J$ y $J'$ son iguales e iguales a $(bc : ca : ab)$.

**Demostración.** De la definición tenemos que $\dfrac{BJ}{JC} = \dfrac{c}{b}$, sea $\lambda$ tal que $BJ = \dfrac{ac}{\lambda}$ de ahi que $JC = \dfrac{ab}{\lambda}$. Ademas tenemos que $\dfrac{AJ}{JC} = \dfrac{c}{a}$ con lo que $JA = \dfrac{bc}{\lambda}$. Luego $J = (\dfrac{bc}{\lambda} : \dfrac{ca}{\lambda} : \dfrac{ab}{\lambda}) = (bc : ca : ab)$.
$\blacksquare$

Ahora pasaremos con la solucion del problema!

**Solución.** Dado $\triangle ABC$ sea $J_1J_2J_3$ el triangulo pedal de $J$ son respecto a $\triangle ABC$. Tenemos que $JJ_2 = AJ.sen\angle JAJ_2$ y ademas $\dfrac{J_2J_3}{JJ_2} = \dfrac{sen\angle JJ_3J_2}{sen\angle J_2JJ_3}$. Como $\square JJ_2AJ_3$ es c\'iclico tenemos que $m\angle JJ_3J_2 = m\angle JAJ_2$ y $m\angle J_{2}JJ_{3}+m\angle J_{2}AJ_{3}=180º$ con lo que $\dfrac{J_2J_3}{JJ_2} = \dfrac{sen\angle J_2AJ_3}{sen\angle JAJ_2}$,
es decir $J_2J_3 = JJ_2.\dfrac{sen\angle J_2AJ_3}{sen\angle JAJ_2}$.
Por el lema 4 tenemos que $J = (bc : ca : ab)$ en coordenadas tripolares,
de ahi que si $AJ = \dfrac{bc}{\lambda}$ entonces $J_2J_3 = AJ.senA = \dfrac{bc}{\lambda}.senA = \dfrac{4R^2}{\lambda}.senA.senB.senC$.
Analogamente tenemos que $J_3J_1 = \dfrac{4R^2}{\lambda}.senA.senB.senC$ y $J_1J_2 = \dfrac{4R^2}{\lambda}.senA.senB.senC$ con lo que $\triangle J_1J_2J_3$ es equil\'atero. $\blacksquare$

Ahora pasaremos a la solucion del problema:

Por el lema 1 tenemos:

$$\sum(-a^2+b^2+c^2)(\frac{c^2a^2}{\lambda^2}\frac{a^2b^2}{\lambda^2}+a^2\frac{b^2c^2}{\lambda^2}) = a^2b^2c^2+(a^2\frac{b^4c^4}{\lambda^4}+b^2\frac{c^4a^4}{\lambda^4}+c^2\frac{a^4b^4}{\lambda^4}).$$

Separando t\'erminos y dividiendo por $a^2b^2c^2$ tenemos que

$$\frac{1}{\lambda^4}\sum(-a^2+b^2+c^2)a^2+\frac{1}{\lambda^2}(a^2+b^2+c^2) = 1+\frac{1}{\lambda^4}(b^2c^2+c^2a^2+a^2b^2).$$

Esto es

$$(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)+(a^2+b^2+c^2)\lambda^2 = \lambda^4+(b^2c^2+c^2a^2+a^2b^2)$$

y reagrupando

$$\lambda^4-(a^2+b^2+c^2)\lambda^2+(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2) = 0$$

con lo que

$$2\lambda^2 = (a^2+b^2+c^2)+\sqrt{(a^2+b^2+c^2)^2-4(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2)}$$

y por el teorema de Heron
$2\lambda^2 = (a^2+b^2+c^2)+4\sqrt{3}[ABC]$ y de ahi que
$\sqrt{2}\lambda = \sqrt{(a^2+b^2+c^2)+4\sqrt{3}[ABC]}$. Como
$J_1J_2 = \dfrac{bc.senA}{\lambda} = \dfrac{2[ABC]}{\lambda}$ se concluye que

$$\frac{2\sqrt{2}[ABC]}{\sqrt{\quad}}.$$

$\sqrt{a^2+b^2+c^2+4\sqrt{3}[ABC]}$

Sean $P_1 \in \overline{BC}$, $P_2 \in \overline{CA}$, $P_3 \in \overline{AB}$ tal que $\triangle P_1P_2P_3$ es equil\'atero. La semejanza espiral $\varphi$ tal que $\varphi(J_i) = P_i$ para $i = 1, 2, 3$ con lo que se sigue que $P_iP_{i+1} \geq J_iJ_{i+1}$. Por lo tanto

$$P_iP_{i+1} \geq \frac{2\sqrt{2}[ABC]}{\sqrt{a^2+b^2+c^2+4\sqrt{3}[ABC]}};$$

esto es el lado de todo triangulo equilatero inscrito en $\triangle ABC$ es mayor o igual que

$$\frac{2\sqrt{2}[ABC]}{\sqrt{a^2+b^2+c^2+4\sqrt{3}[ABC]}}.$$

■

★ Capitán Mandarina!

**conejita**

#8 Aug 27, 2006, 5:05 am

Ola hucht, sinceramente kreo k tu solucion no es nada bonita, muy tecnica y poco ingeniosa. Le geometria siempre es un poco de las 2, yo kreo k debe de aber alguna solucion mediante simples cosas euclidianas. Si de casualidad sabes alguna asi, agradeceria k la pongas, xq sinceramente no entendi naa de tu solucion, y el problema en si se me ace interesante.
Grax

**hucht**

#9 Aug 27, 2006, 5:24 am

> *conejita wrote:*
> Ola hucht, sinceramente kreo k tu solucion no es nada bonita, muy tecnica y poco ingeniosa. Le geometria siempre es un poco de las 2, yo kreo k debe de aber alguna solucion mediante simples cosas euclidianas. Si de casualidad sabes alguna asi, agradeceria k la pongas, xq sinceramente no entendi naa de tu solucion, y el problema en si se me ace interesante.
> Grax

Sorry pero la geometria euclideana esta bien como reseña istorica pero como herramienta en resolucion de problemas no sirve de mucho. Quiza tu puedas buscar una solucion con geometria euclideana yo no tengo mucho tiempo para perder con exigencias inutiles exquisiteses, resuelvo los problemas como puedo, quiza no sean "beliisimas" mis soluciones pero tambien puedo encontrar de esas que les gustan a ustedes con "congruencia" y teorema de tales y pitagoras, cosas poco utiles
http://www.personal.us.es/rbarroso/trianguloscabri/sol/sol19chav.pdf - es mi solucion de un problema con geometria sintetica

Con todo, el mejor geometra que conozco (Darij Ginberg) tambien resuelve problemas asi, puedes verlo en donde demuestra dos teoremas de mi autoria usando miles de cosas http://de.geocities.com/darij_grinberg/isogonal.pdf

*This post has been edited 1 time. Last edited by hucht, Aug 27, 2006, 5:55 am*

**DarthTenebrus**

#10 Aug 27, 2006, 5:45 am

Yo creo que lo que dice hucth es correcto :

De hecho que para empezar en geometria los trazos y construcciones son buenos y muy bellos pero a medida que pasa el tiempo un joven olimpico no puede depender de herramienta tan simples para la solucion de problemas de geometria es decir . Debe aprender tambien las geometria euclidiana avanzada : Conjugacion armonica - Rectas Polares y la Inversion de Steiner . Tambien debe emplear para la maximizacion y minimizacion la tecnica de variacion parcial .

Pero eso no debe quedar alli tambien debe estar presente la geometria proyectiva y aun asi eso es poco .

En particular para mi mis mejores tecnicas en la geometria de esas que son detonates :

- Geometria vectiorial
-Geometria Analitica
-Geometria de los numeros complejos

Y Por ultimo porque no decirlo expresiones trigonometricas y todos los tipos de relacion de senos , cosenos y tangentes cuando

Y Por ultimo porque no decirlo expresiones trigonometricas y todos los tipos de relacion de senos , cosenos y tangentes cuando los angulos suman 180 grados osea el triangulo .

**José**

*#11 Aug 27, 2006, 5:47 am*

Creo que tenés razón, ya con la desigualdad a demostrar se ve que la solución no va a ser muy hermosa porque los números y letras son "feos".
Además, mientras que resuelvas el problema está bien, las soluciones elegantes son más para problemas sencillo y con datos enteros o exactos

**hucht**

*#12 Aug 27, 2006, 5:53 am*

> *DarthTenebrus wrote:*
> Yo creo que lo que dice hucth es correcto :
>
> De hecho que para empezar en geometria los trazos y construcciones son buenos y muy bellos pero a medida que pasa el tiempo un joven olimpico no puede depender de herramienta tan simples para la solucion de problemas de geometria es decir . Debe aprender tambien las geometria euclidiana avanzada : Conjugacion armonica - Rectas Polares y la Inversion de Steiner . Tambien debe emplear para la maximizacion y minimizacion la tecnica de variacion parcial .
>
> Pero eso no debe quedar alli tambien debe estar presente la geometria proyectiva y aun asi eso es poco .
>
> En particular para mi mis mejores tecnicas en la geometria de esas que son detonates :
>
> - Geometria vectiorial
> -Geometria Analitica
> -Geometria de los numeros complejos
>
> Y Por ultimo porque no decirlo expresiones trigonometricas y todos los tipos de relacion de senos , cosenos y tangentes cuando los angulos suman 180 grados osea el triangulo .

Yo tambien pienso lo mismo, mis tecnicas de resolucion las aprendi en el foro internacional, luego de mucho estudiar ahora puedo dominar:

Coordenadas baricentricas
Coordenadas Tripolares
Geometria con Numeros complejos
Semejanzas Espirales
Vectores
Relaciones metricas
Algunas congruencias
Proyectividades (entre ellas las colineaciones una colineacion particular es la polaridad)
Inversiones
Transformaciones Afines (Ahora estoy estudiando la Isogonologia)
Transformaciones asociadas al triangulo (Isogonales, Isotomicas, Isocriticas (es una que estoy inventando, mas informacion en el link de darij grinberg ya que los teoremas mios que el resolvio son parte de mi teoria ))

★ Capitán Mandarina

**conejita**

*#13 Aug 27, 2006, 8:47 am*

Weno, weno, perdonen la ofensa, kiza me molesto un poco, que la estuve intentando muxo tiempo, y no me salio nada!!!!
Ademas, yo todavia no domino muy bien otros temas, como son geometria vectorial , numeros complejos, y esos temas no euclidianos, k en realidad tienen razon, en muxisimos problemas son muy utiles!!

Disculpen el comentario!!

**DarthTenebrus**

*#14 Aug 27, 2006, 9:30 am*

NONO esperemos corregir algo la geometria sigue siendo euclideana lo nuevo entre comillas es las tecnicas de solucion a los problemas que por cierto no tiene mucho de nuevo .

**EDSON_**

*#15 Aug 27, 2006, 12:04 pm*

YO PIENSO QUE LAS SOLUCIONES DE GEOMETRIA MEDIANTE TRAZOS QUE AHORRAN OPERACIONES ES LA + BELLA ESO ES GEOMETRIA PURA POR SU ELEGANCIA Y POR LO VISTOSA QUE ES!!

MIENTRAS QUE LA GEOMETRIA QUE NOTO EN TU SOLUCION ES MAS TIRANDO PARA EL ALGEBRA OSEA DEBERIAN LLAMARLA GEOMETRIA ALGEBRAICA O ALGO ASI ,YO COMPARTO TU APRECIACION CONEJITA

Y A TI TE RESPETO HUCHT POR QUE TIENES HERRAMIENTAS DE SOBRA PARA ATACAR UN PROBLEMA MEN!! oe pero mi problemas pues x 2x 7x jejeje

**Claudio Espinoza**

*#16 Aug 31, 2006, 12:08 pm*

Hola conejita, el problema que propuso hucht pertenece a una shortlist IMO 93, aqui te paso el link, el problema esta resuelto de una manera mas entendible, es el problema ISA 1, esta en la parte 2, solo usa un poco de trigo y opera de manera conveniente.

http://www.mathlinks.ro/Forum/viewtopic.php?t=15580

Aunque al igual que la mayoría creo que mientras más herramientas tengamos mayores son las posibilidades de resolver un problema.

PD. Edson ya coloque la solucion de tu problema x,2x,7x.

**Luis González**

*#17 Jun 18, 2009, 7:36 am*

Es sabido que de todos los triángulos equiláteros circunscritos al triángulo $ABC$, aquel $\triangle XYZ$ que es homotético al triángulo de Napoleón externo, es el que tiene mayor área (lado). Por tanto, este tiene lado doble del lado $L$ del triangulo de Napoleón externo, A saber:

$$L=\sqrt{\tfrac{1}{6}(a^2+b^2+c^2)+\frac{2\sqrt{3}}{3}[ABC]}$$

Para cada triangulo equilátero $\Delta$ circunscrito a $\triangle ABC$ podemos asociar un triangulo $\Delta'$ homotético a éste e inscrito a $\triangle ABC$. Así pues, aplicando Teorema de *Gergonne-Ann* tenemos $[ABC]^2=[\Delta]\cdot[\Delta']$. Entonces el área (lado) $L'$ de $\Delta'$ será mínimo si el área (lado) de $\Delta$ es máximo. En otras palabras $[ABC]^2=\tfrac{3}{16}(L')^2(4L^2)$

Sustituyendo el valor de $L$ del lado del triángulo de Napoleón tenemos:

$$[ABC]^2=\tfrac{1}{8}(L')^2(a^2+b^2+c^2+4\sqrt{3}[ABC])$$

$$L'=\frac{2\sqrt{2}[ABC]}{\sqrt{a^2+b^2+c^2+4\sqrt{3}[ABC]}}$$

Todo equilátero $\triangle DEF$ inscrito a $\triangle ABC$ tiene pues lado mayor o igual que $L'$.

Quick Reply

# High School Olympiads

## Circumcenter-Incenter

Reply

**Xaenir**
#1 Jun 15, 2009, 9:36 pm

$I$ is the incenter of a non-isosceles triangle $\Delta ABC$ .If the incircle touches $BC, CA, AB$ at $A_1, B_1, C_1$ respectively, prove that the circumcenter of the triangles $\Delta AIA_1$,$\Delta BIB_1$,$\Delta CIC_1$ are collinear.

**mathVNpro**
#2 Jun 15, 2009, 9:46 pm

Let $A_0, B_0, C_0$ respectively be the midpoints of $B_1C_1, C_1A_1, A_1B_1$. Denote $r$ by the radii of $\triangle ABC$. Consider the inversion through pole $I$, power $r^2$, we have $\mathcal{I}(I,r) : A \mapsto A_0$, $B \mapsto B_0$, $C \mapsto C_0$, $A_1, B_1, C_1$ is mapped to itself. Hence $\mathcal{I}(I,r) : (AIA_1) \mapsto A_1A_0$, $(BIB_1) \mapsto B_1B_0$, $(CIC_1) \mapsto C_1C_0$. But $B_0B_1, C_0C_1, A_0A_1$ are the medians of $\triangle A_1B_1C_1 \implies B_0B_1, C_0C_1, A_0A_1$ are concurrent. Therefore $(AIA_1), (BIB_1), (CIC_1)$ are coaxal. The result is lead as follow.
Our proof is completed then.

**Xaenir**
#3 Jun 15, 2009, 9:52 pm

Nice proof...
And Inversion still keeps on amazing me...

**jayme**
#4 Jun 16, 2009, 3:55 pm

Dear Mathlinkers, the circumcircles are concurrent in a second point named "the Schröder's point".
Sincerely
Jean-Louis

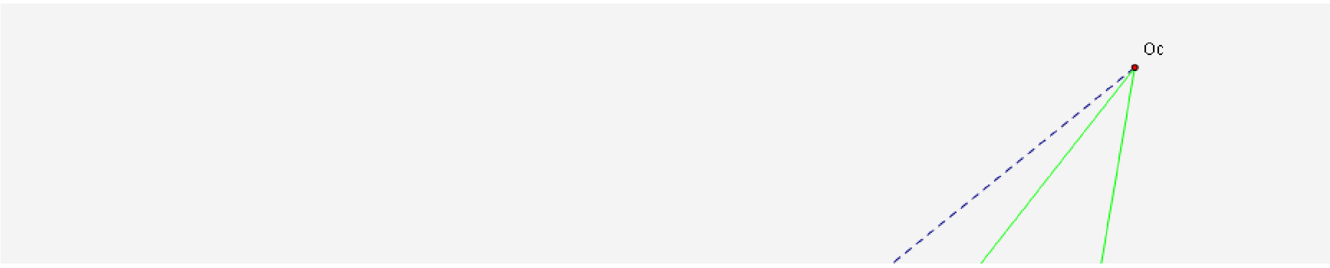
**hophinhan**
#5 Jun 16, 2009, 4:34 pm

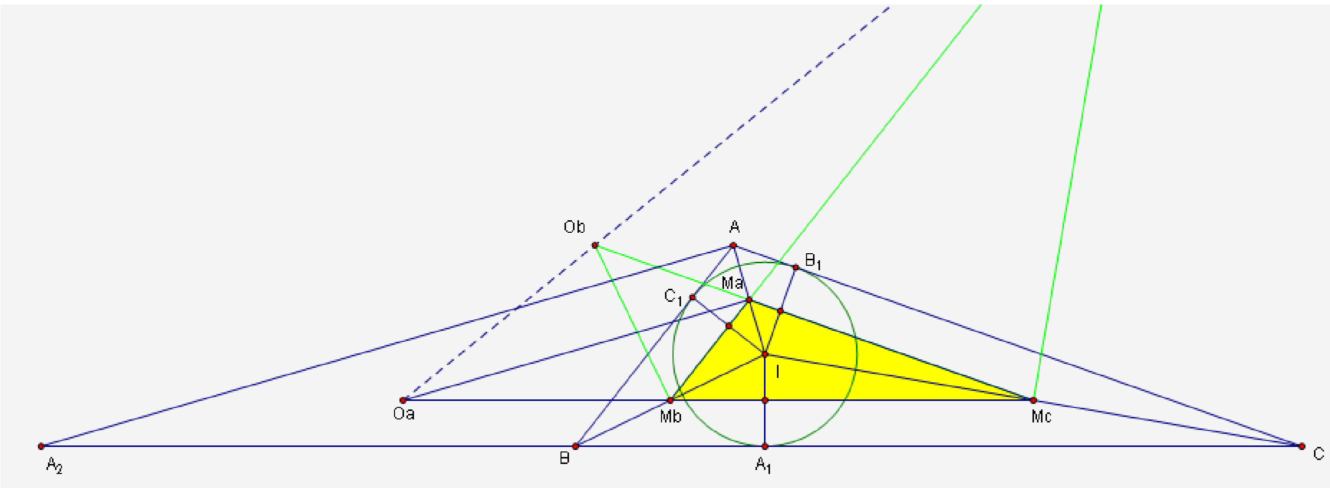
> *Xaenir wrote:*
> $I$ is the incenter of a non-isosceles triangle $\Delta ABC$ .If the incircle touches $BC, CA, AB$ at $A_1, B_1, C_1$ respectively, prove that the circumcenter of the triangles $\Delta AIA_1$,$\Delta BIB_1$,$\Delta CIC_1$ are collinear.

$$\frac{O_aM_b}{O_aM_c} = \frac{A_2B}{A_2C} = \frac{AB}{AC}$$

By Menelaus's theorem for $\triangle M_aM_bM_c$...we have done.

*Attachments:*






**Luis González**

#6 Jun 17, 2009, 4:38 am

Posted many times before, e.g; see An old problem, Incenter and 3 circles and elsewhere.

> jayme wrote:
> Dear Mathlinkers, the circumcircles are concurrent in a second point named "the Schröder's point".

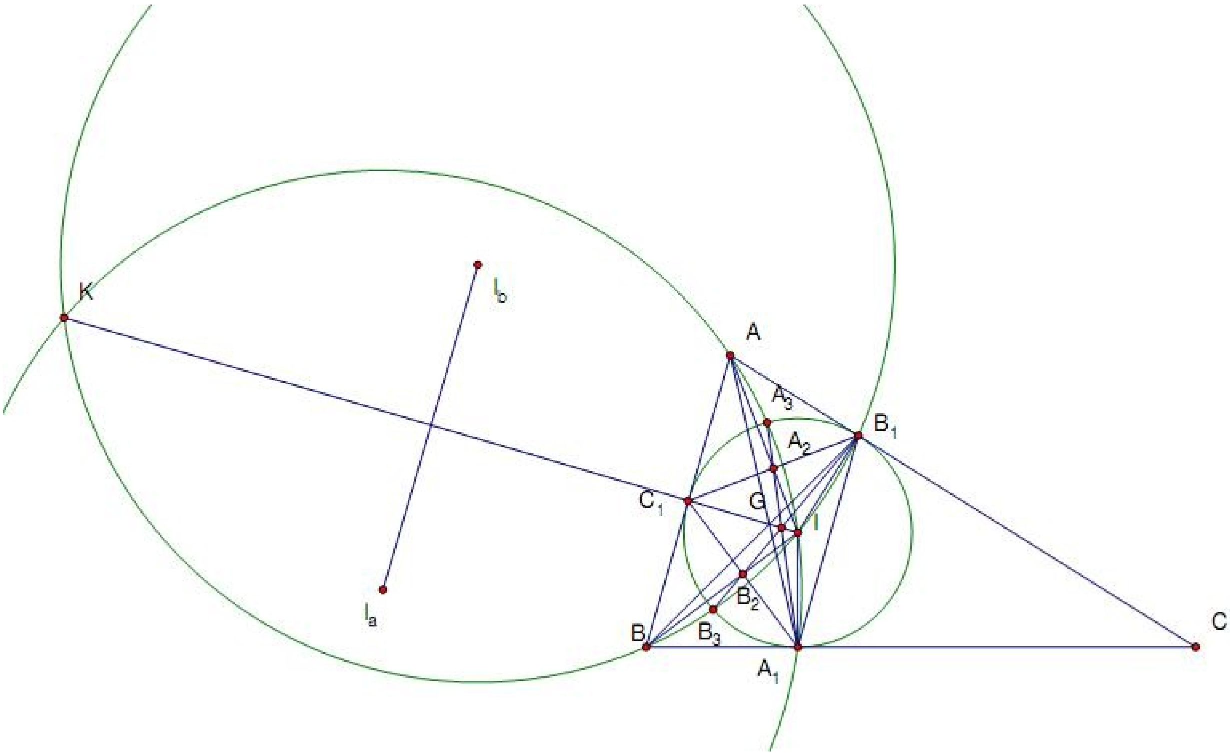
I found three more concurrent circles at the Schröder's point. See Circles through the Schroder point.

**livetolove212**

#7 Jun 17, 2009, 8:17 am

Let G be the centroid of triangle $A_1B_1C_1.A_1G, B_1G, C_1G$ cut $(I)$ at $A_3, B_3, C_3$, respectively.
We have $A_2A.A_2I = A_2C_1.A_2B_1 = A_2A_1.A_2A_3 \Rightarrow AA_1IA_3$ is cyclic.
Similarly $BB_1IB_3$ is cyclic
Moreover, $GA_1.GA_3 = GB_1.GB_3$ then G lies on the radical axis of $(I_a)$ and $(I_b)$
$\Rightarrow I_aI_b \perp IG$, similarly $I_bI_c \perp IG$ so $I_a, I_b, I_c$ are collinear.

*Attachments:*




Quick Reply

# High School Olympiads

## triangle, bisector, perpendicular lines

Reply

**Source**: maybe well-known

**birzhan**

#1 Jun 14, 2009, 7:28 pm

Given triangle $ABC$.From midpoints of $AB$ and $AC$ drawn lines perpendicular to bisectors of angles $C$ and $B$ respectively.Let these two lines intersects at $S$.prove that the line joining $S$ and midpoint of $BC$ is parallel to bisector of angle $A$.

**Luis González**

#2 Jun 15, 2009, 3:02 am

$M, N, L$ are the midpoints of $AB, AC, CB.$ Let $M'$ and $N'$ denote the feet of the angle bisectors of $\angle NML$ and $\angle MNL$ in the medial triangle $\triangle MNL.$ Clearly $MM'$ and $NN'$ are parallel to the angle bisectors of $C$ and $B,$ since $\triangle LNM$ and $\triangle ABC$ are centrally similar. Therefore, $SM \perp MM'$ and $SN \perp NN' \Longrightarrow S$ is the L-excenter of $\triangle LNM \Longrightarrow LS$ bisects $\angle NLM \Longrightarrow LS$ is parallel to the angle bisector of $A.$

Quick Reply

# High School Olympiads

## Tangents to incircle and hexagon area



**Grands**

#1 Jun 14, 2009, 4:03 am

In a triangle $ABC$ is inscribed a circle of radius $r$. The circumradius is $R$ and the semiperimeter is $p$. The tangents to the incircle, paraller to the sides of the triangle cut from it a hexagon of area $S$. Prove that

$$S=\frac{2r^2}{p}(r+4R)$$

I tried so many ways to manipulate the expression, but still don't get the result. The best that I got is:

$$S=\frac{2r^2}{p}\left(\frac{ap}{2r_a}+\frac{bp}{2r_b}+\frac{cp}{2r_c}\right)$$

and I know that $r_a+r_b+r_c=r+4R$

**Luis González**

#2 Jun 14, 2009, 5:54 am • 1

Let $X,Y$ be the intersections of the tangent of $(I)$ parallel to $BC$ with $AB,AC.$ $S_A$ stands for the area of $\triangle AXY.$ Since $\triangle AXY\sim\triangle ABC,$ the ratio between their areas equals the square ratio between their semiperimeters.

Since the semiperimeter of $\triangle AXY$ equals $p-a,$ it follows that

$$\frac{S_A}{[\triangle ABC]}=\frac{(p-a)^2}{p^2}\Longrightarrow\frac{S_A+S_B+S_C}{[\triangle ABC]}=\frac{(p-a)^2+(p-b)^2+(p-c)^2}{p^2}$$

$$\frac{[\triangle ABC]-S}{[\triangle ABC]}=\frac{(a^2+b^2+c^2-p^2)}{p^2}$$

$$1-\frac{S}{rp}=\frac{3p^2-2(4Rr+r^2+p^2)}{p^2}$$

$$\frac{S}{rp}=\frac{8Rr+2r^2}{p^2}\Longrightarrow S=\frac{2r^2(4R+r)}{p}$$

# High School Olympiads

## nice property of angles

Reply

**immodestius**

#1 Jun 13, 2009, 4:11 am

Let $ABC$ be a triangle. Its incircle with center $I$ touches the sides $BC, CA, AB$ in $D, E, F$. Denote the incenters of $BDIF$ and $CDIE$ by $X$ and $Y$. Prove that $2\cdot\angle XAY = \angle BAC$.

**Luis González**

#2 Jun 13, 2009, 6:58 am

The common internal tangent of the incircles of the tangential $DIFB$ and $DIEC$ (different from ID) goes through $A.$ For a proof see the topic Common tangent passes through A. If $K$ denote the intersection of this tangent with $BC,$ then the rays $AX$ and $AY$ bisect $\angle BAK$ and $\angle CAK$ internally $\Longrightarrow \angle BAC = 2\angle XAY,$ as desired.

Quick Reply

# High School Olympiads

## nice propertie about tangency of circunferences

Reply

**v235711**
#1 Jun 13, 2009, 5:06 am

The circunferences $k_1$ and $k_2$ intersect each other at points $X$ and $Y$ ,and are internally tangent to circunference $k$ at $P$ and $Q$ .$XY$ intersect $k$ at $A$ and $B$.let $PA$ and $QA$
intersect $k_1$ and $k_2$ at $C$ and $D$.prove that $CD$ is tangent to $k_1$ and $k_2$.

maybe I should´ve posted this stuff at highschool basics.Anyway , i wanted to show this to people that actually do math olympiad.this can be really useful someday.

**Luis González**
#2 Jun 13, 2009, 5:29 am

Since $A$ belows to the radical axis $XY$ of $k_1,k_2,$ the inversion with center $A$ and power equal to the power of $A$ WRT $k_1,k_2$ transforms $k_1,k_2$ into themselves and takes $Q\mapsto D$ and $P\mapsto C.$ Thus, the circle $\odot(APQ)$ is taken into the line $CD.$ Since $\odot(APQ)$ is tangent to $k_1,k_2,$ then it follows that $CD$ is tangent to $k_1,k_2$ as well.

**mathVNpro**
#4 Jun 13, 2009, 8:28 am

see also this for reference: http://www.mathlinks.ro/viewtopic.php?t=278931

**jayme**
#5 Jun 13, 2009, 3:59 pm • 1

Dear Mathlinkers,
main ideas...
0. Let Ta, Tc be the tangents to k, k1 at A, C
1. According to Monge (or d'Alembert)'s theorem, P, C, D, Q are concyclics
2. According to Reim's theorem, (CD) // Ta
3. The circle K and K1 being tangent at P, Ta // Tc...
Sincerely
Jean-Louis

Quick Reply

# High School Olympiads

## Three lines are concurrent



**Source**: MOSP 2007

**April**

#1 Sep 11, 2008, 7:29 am

Let $ABC$ be an acute triangle. Circle $\omega_{BC}$ has segment $BC$ as its diameter. Circle $\omega_A$ is tangent to lines $AB$ and $AC$ and is tangent externally to $\omega_{BC}$ at $A_1$. Points $B_1$ and $C_1$ are defined analogously. Prove that lines $AA_1$, $BB_1$, and $CC_1$ are concurrent.

I posted this problem in High school section long time ago, and received a hard solution by **Virgil Nicula**. But I just want to see a synthetic solution, so come on!!!

**yetti**

#2 Jun 12, 2009, 7:37 am

$(I)$ is incircle. $A$ is external similarity center of $\omega_A, (I)$ and $A_1$ is internal similarity center of $\omega_A, \omega_{BC} \Longrightarrow$ internal similarity center $J$ of $(I), \omega_{BC}$ is on $AA_1$. Let $A'$ be midpoint of $BC$, let $U, D, P \in BC$ be orthogonal projections of $A, I, J$ and let $AA_1J$ cut $BC$ at $X$. $J \in A'I$ and $A'I$ cuts $AD$ at its midpoint $M$ (well known). $\overline{A'D} = \frac{c-b}{2}$ and $\overline{JP} = \frac{a}{a+2r} \cdot \overline{ID} = \frac{ar}{a+2r}$. By Menelaus theorem for $\triangle ADX$ cut by the line $A'JM$,

$$\frac{\overline{A'D}}{\overline{A'X}} = \frac{\overline{MD}}{\overline{MA}} \cdot \frac{\overline{JA}}{\overline{JX}} = \frac{\overline{AU} - \overline{JP}}{\overline{JP}} = \frac{h_a(a+2r) - ar}{ar} = \frac{2pr + 2h_a r - ar}{ar} = \frac{b+c+2h_a}{a}$$

$$\overline{BX} = \overline{BA'} + \overline{A'X} = \frac{a}{2} + \frac{c-b}{2} \cdot \frac{a}{b+c+2h_a} = +\frac{a(c+h_a)}{b+c+2h_a}, \quad \overline{CX} = -\frac{a(b+h_a)}{b+c+2h_a}$$

$$\Longrightarrow \frac{\overline{BX}}{\overline{CX}} = -\frac{c+h_a}{b+h_a} = -\frac{ac+2pr}{ab+2pr}.$$

Similarly, if $K, L$ are internal similarity centers of $(I), \omega_{CA}$ and $(I), \omega_{AB}$ and if $BB_1K, CC_1L$ cut $CA, AB$ at $Y, Z$, then

$$\frac{\overline{BX}}{\overline{CX}} \cdot \frac{\overline{CY}}{\overline{AY}} \cdot \frac{\overline{AZ}}{\overline{BZ}} = -\frac{ac+2pr}{ab+2pr} \cdot \frac{ba+2pr}{bc+2pr} \cdot \frac{cb+2pr}{ca+2pr} = -1$$

**Luis González**

#3 Jun 12, 2009, 12:32 pm

Label $(K) \equiv \omega_A$. $X, Y$ are the projections of the midpoint $M$ of $BC$ on $AB, AC$. $P, Q$ are the tangency points of $(K)$ with $AB, AC$ and $B', C'$ are the projections of $A_1$ on $AB, AC$. In the trapezoid $XMKP$, we get

$$A_1B' = \frac{XM \cdot KA_1 + KP \cdot MA_1}{MK} = \frac{KA_1(XM + MA_1)}{MK} = \frac{KA_1(a+h_c)}{2MK}$$

Analogously, in the trapezoid $YMKQ$, we have $A_1C' = \frac{KA_1(a+h_b)}{2MK}$

$$\frac{A_1B'}{A_1C'} = \frac{a+h_c}{a+h_b} = \frac{1+\sin B}{1+\sin C} = \frac{\sin \angle BAA_1}{\sin \angle CAA_1}$$

Cyclically, we have the expressions

$$\frac{\sin\angle CBB_1}{\sin\angle ABB_1}=\frac{1+\sin C}{1+\sin A}\ ,\ \frac{\sin\angle ACC_1}{\sin\angle BCC_1}=\frac{1+\sin A}{1+\sin B}$$

$$\frac{\sin\angle BAA_1}{\sin\angle CAA_1}\cdot\frac{\sin\angle CBB_1}{\sin\angle ABB_1}\cdot\frac{\sin\angle ACC_1}{\sin\angle BCC_1}=\frac{1+\sin B}{1+\sin C}\cdot\frac{1+\sin C}{1+\sin A}\cdot\frac{1+\sin A}{1+\sin B}=1$$

$\Longrightarrow$ Lines $AA_1, BB_1, CC_1$ concur due to Ceva's theorem.

**Luis González**

*#4 Jun 12, 2009, 11:08 pm*

**Remark:** Let $D$ denote the intersection of the ray $AA_1$ with $BC$. Then we have

$$\frac{BD}{CD}=\frac{AB}{CA}\cdot\frac{\sin\angle BAA_1}{\sin\angle CAA_1}=\frac{AB}{CA}\cdot\frac{(1+\sin B)}{(1+\sin C)}$$

Then, using barycentric coordinates at this point, we obtain

$$D\equiv\left(0:\frac{b}{1+\sin B}:\frac{c}{1+\sin C}\right)\equiv\left(0:\frac{1}{ca+S}:\frac{1}{ab+S}\right)$$

By cyclic reasoning, it follows that $AA_1, BB_1, CC_1$ concur at Paasche point of $\triangle ABC$

$$X_{1123}\left(\frac{1}{bc+S}:\frac{1}{ca+S}:\frac{1}{ab+S}\right)$$

Quick Reply

# High School Olympiads

## Concyclic points

Reply

**stergiu**

#1 Sep 2, 2008, 2:35 pm

If $K$ is the projection of the orthocenter $H$ of an acute trianlge $ABC$ on the median $AM$ , prove that points

$B, C, K, H$ are concyclic. Moreever, if $BD, CE$ are altitudes in triangle $ABC$ , then lines $DE, BC, KH$

are concurent .

*Which basic olympiad problem reminds you the second question ?*

Babis

**¬[ƒ(Gabriel)³²¹º]¼**

#2 Sep 3, 2008, 6:10 am

Let A' be the symmetric of A wrt M and O the circumcenter of ABC. Let AO meet the circumcircle (O) on F. As know the circumcircle (O') of BHC (with center O') is the symemtric of (O) wrt BC. As know with a composition 2 symmetry with axes BC and MO, H go on F, then M is the midpoint of FH. Then AFA'H i sa parallelogram and $\overline{AF}=\overline{A'H}$, then $O'\in HA$, so $K\in (O')$.

The second part is a particolar case of this know problem:

Let ABCD a cyclic quad and E, F in it such that ABEF and EFCD are cyclic quads. Then AB, CD, EF are concurrent.

proof: Call $P:AB\cap CD$, call $\Gamma$ the circumcircle of ABC, $\Gamma_1$ of ABE and $\Gamma_2$ of CDE. Then P stay on the radical axes of $\Gamma_1$ and $\Gamma_2$ (EF) iff $pow_{\Gamma_1}(P)=pow_{\Gamma_2}(P)$, but $pow_{\Gamma_1}(P)=PA\cdot PB=pow_{\Gamma}(P)=PC\cdot PD=pow_{\Gamma_2}(P)$

**The QuattoMaster 6000**

#3 Sep 3, 2008, 8:07 am • 1

> stergiu wrote:
> If $K$ is the projection of the orthocenter $H$ of an acute trianlge $ABC$ on the median $AM$ , prove that points
>
> $B, C, K, H$ are concyclic. Moreever, if $BD, CE$ are altitudes in triangle $ABC$ , then lines $DE, BC, KH$
>
> are concurent .
>
> *Which basic olympiad problem reminds you the second question ?*
>
> Babis

Solution

**stergiu**

#4 Sep 3, 2008, 12:14 pm

Both are new solutions and I add them to my archiev. 🙂 .
Thank you very much.Of course , as you mentioned , the second part is based on the radical center of three circles.
The olympiad problem that I have in mind is this :

**Problem**

If $BD, CE$ are altitudes in triangle $ABC$ and line $DE$ meets line $BC$ at $R$, then $RH$ is perpendicular to the median $AM$ of $\triangle ABC$ .

**HINT**

We draw $HK \perp AM$ . Then , as in the problem above , $KH$ passes through $R$.
We have discussed this problem in this forum again and we have given many solutions.It has also apeared in CRUX..

**The QuattoMaster 6000**
#5 Sep 3, 2008, 12:25 pm • 1

> stergiu wrote:
> **Problem**
>
> If $BD, CE$ are altitudes in triangle $ABC$ and line $DE$ meets line $BC$ at $R$, then $RH$ is perpendicular to the median $AM$ of $\triangle ABC$ .

Ah, yes, thank you for pointing this hint to another solution out, dear **stergiu**!
Solution

**stergiu**
#6 Sep 4, 2008, 1:28 am

Thank you for the second solution!
Ah! For the proof of the lemma I suppose you use polars .Do you have any more elementary approach for it?
Babis

**The QuattoMaster 6000**
#7 Sep 4, 2008, 12:04 pm

> stergiu wrote:
> Do you have any more elementary approach for it?

I don't know polars, but here is my proof:
Proof

**stergiu**
#8 Sep 4, 2008, 1:50 pm

**To Quatto MAster 6000 only**

Click to reveal hidden text Babis

**dima ukraine**
#9 Sep 5, 2008, 8:25 pm

I think its not hard to use complex nubers for first part

**dgreenb801**
#10 Jun 12, 2009, 5:35 am

Here is my solution to the first part:
Let D be the foot of altitude from A to BC. Draw the circumcircle of $\triangle ABC$, extend AD to hit the circle at H', we know H' is the reflection of H over BC. So we need to show the reflection of K over BC also lies on the circumcircle, to show BHKC is cyclic. But since M is the midpoint of BC, by symmetry, this is equivalent to showing $KM = MP$, where P is the intersection of AM with the circle. By Power of a point, $MP \cdot AM = (\frac{a}{2})^2$, so $MP = \frac{a^2}{4AM}$.

Now, $\triangle HAK$ and $\triangle DAM$ are similar, so $\frac{AK}{AD} = \frac{AH}{AM}, or AK = \frac{AD \cdot AH}{AM}$. But $KM = AM - AK$, so
$KM = \frac{AM^2 - AD \cdot AH}{AM}$
So we have to show $a^2 = 4(AM^2 - AD \cdot AH)$. But
$AM^2 - AD \cdot AH = AM^2 - AD(AD - HD) = AM^2 - AD^2 + AD \cdot H'D = AM^2 - AD^2 + BD \cdot DC = DM^2 + (\frac{a}{2} - DM)(\frac{a}{2} + DM) = \frac{a^2}{4}$
We can use the radical axis theorem to finish the problem as Quatto did.

Here is a solution to the second problem (not second part of first problem) stergiu wrote:
Let K be the perpendicular from H to AM, and N the foot of the altitude from A to BC (passes through H). Then HKMN is cyclic ($\angle HKM = \angle HNM = 90$), AEHKD is cyclic ($\angle AEH = \angle AKH = \angle ADH = 90$), and EDMN is cyclic (they lie on the nine-point circle). Therefore, their radical axis, DE, HK, and MN concur, so R,H, and K are collinear, so $RH \perp AM$

*This post has been edited 1 time. Last edited by dgreenb801, Jun 12, 2009, 6:49 am*

**Luis González**

*#11 Jun 12, 2009, 8:35 am • 1*

Let $Q \equiv DE \cap BC.$ Line $AQ$ is the polar of $H$ WRT the semicircle $(M)$ with diameter $BC.$ Thus, $QH$ is the polar of $A$ WRT $(M) \Longrightarrow AM \perp QH.$ Let $K \equiv AM \cap QH.$ Note that $H$ is common orthocenter of $\triangle AQM$ and $\triangle ABC,$ thus $KH \cdot HQ = DH \cdot HB = CH \cdot HE.$ In other words, the inversion with center $H$ and power $KH \cdot HQ$ takes $K, B, C$ into the collinear points $D, E, Q \Longrightarrow B, C, H, K$ are concyclic, as desired.

Quick Reply

# High School Olympiads

## equal areas

Reply

**mathservant**

#1 Jun 11, 2009, 9:11 pm

Let ABCD be a convex quadrilateral and M,N,K,L be mid point of sides AB,BC,CD, and DA respectively.Let R and T points be mid point of AC and BD diagonals.If area(MRKT)=area(NTRL) show that one of diagonal of ABCD quadrilateral divide the area of ABCD quadrilateral equal two areas.

**Luis González**

#2 Jun 11, 2009, 10:57 pm

Let $Q \equiv AC \cap BD.$ Then it is clear that $MRKT$ and $NRLT$ are both parallelograms. If $[MRKT] = [NRLT]$, then $[\triangle MRT] = [\triangle NRT]$, which implies that $MN \parallel RT$, but $MN \parallel AC$. Hence, $R \equiv Q$. If $Q$ bisects the segment $DB$, then the distances from $D, B$ to $AC$ are equal $\Longrightarrow [\triangle ABC] = [\triangle CDA]$.

Quick Reply

# High School Olympiads

## Proof of Ono's inequality



**Source**: Geometric inequality in acute triangle

**skalgar**
#1 Jun 10, 2009, 11:35 pm

Do you know any proof for Ono's inequality that states:
For any acute triangle with area $A$

$$27(b^2+c^2-a^2)^2(c^2+a^2-b^2)^2(a^2+b^2-c^2)^2 \leq (4A)^6$$

**Luis González**
#2 Jun 11, 2009, 12:06 am

$$\cot A = \frac{b^2+c^2-a^2}{4A}\ ,\ \cot B = \frac{a^2+c^2-b^2}{4A}\ ,\ \cot C = \frac{a^2+b^2-c^2}{4A}$$

Then the inequality becomes:

$$\cot A \cdot \cot B \cdot \cot C \leq \tfrac{1}{9}\sqrt{3} \iff \tan A + \tan B + \tan C \geq 3\sqrt{3}$$

Which follows from Jensen's inequality for the fuction $\tan(\varphi), \varphi \in \left(0, \frac{\pi}{2}\right)$

**skalgar**
#3 Jun 11, 2009, 12:37 am

Thanks! However, is there any purely geometric proof?

**Lonesan**
#4 May 1, 2016, 11:43 pm

Area $A = \frac{\sqrt{xy+yz+zx}}{2}$, $a^2 = y+z$ and the others sides, for an acute-angled triangle. Replace and you will get something obvious. Greetings! :)

# High School Olympiads

all lines A_iB_i pass through P

Reply

Source: I.F.Sharygin contest 2009 - Correspondence round - Problem 4

**April**

#1 May 31, 2009, 6:29 am

Let $P$ and $Q$ be the common points of two circles. The ray with origin $Q$ reflects from the first circle in points $A_1$, $A_2$,... according to the rule "the angle of incidence is equal to the angle of reflection". Another ray with origin $Q$ reflects from the second circle in the points $B_1$, $B_2$,... in the same manner. Points $A_1$, $B_1$ and $P$ occurred to be collinear. Prove that all lines $A_iB_i$ pass through P.

**Luis González**

#2 Jun 10, 2009, 6:30 am • 1

$(O_1) \equiv (PQB_1), (O_2) \equiv (PQA_1)$ and let $A'_2 \equiv PB_2 \cap (O_2)$. Reflection $B_1B_2$ of $QB_1$ is symmetric WRT ray $B_1O_1$, i.e. $\triangle QB_1B_2$ is isosceles with base $QB_2$. Let $R$ be the second intersection of the ray $B_1Q$ with $(O_2)$. Since $\angle B_1QB_2 = \angle QRA_1$, it follows that $RA_1 \parallel QB_2$. Let $S \equiv B_1B_2 \cap RA_1$, note that $PA_1SB_2$ is cyclic because of $\angle PA_1R = \angle PB_2B_1 \Longrightarrow \angle B_2PA_1 = \angle B_2SR$. But $QRSB_2$ is an isosceles trapezoid with bases $QB_2 \parallel RS \Longrightarrow$ $\angle A'_2PA_1 = \angle A_1A'_2Q \Longrightarrow$ Chords $A_1Q$ and $A_1A'_2$ are equal. As a result, $A_1A'_2$ is the reflection of $QA_1$ about $O_2A_1 \Longrightarrow$ $A_1A_2$ and $A_1A'_2$ are identical, then $A_2B_2$ goes through $P$.

Let $A_2A_3$ be the reflection of $A_2A_1$ about $O_2A_2$ and $B_2B_3$ the reflection of $B_2B_1$ about $O_1B_2$. Since $QB_3B_1B_2$ and $QA_3A_1A_2$ are isosceles trapezoids, we have $\angle B_1QB_3 = \angle QB_1B_2 = \angle B_1PB_3$. Analogously, $\angle A_3PA_1 = \angle QA_1A_2$, but $\angle QB_1B_2 = \angle QA_1A_2$, hence $\angle A_3PA_1 = \angle B_3PB_1 \Longrightarrow A_3, B_3, P$ are collinear.

**armpist**

#3 Jun 13, 2009, 11:27 pm

The key point of this problem is the collinearity fact of A1,P and B1.
In physics sense it means that two material or immaterial points
rotate along the given circles in the same direction starting in point Q
then they meet again at Q.

By applying one of those facts that two reflections is equivalent to one
rotation (or something like that) to these two points with equal angular
velocities will make all Ai, P and Bi to be collinear.

M.T.

**vittasko**

#4 Jun 14, 2009, 4:40 pm • 1

Because of the collinearity of the points $A_1,\ P,\ B_1$, we conclude that the isosceles triangles $\triangle A_1QA_2\ \ (A_1Q = A_1A_2)$ and $\triangle B_1QB_2\ \ (B_1Q = B_1B_2)$ are similar, from $\angle QB_2B_1 = \angle QPA_1 = \angle QA_2A_1\,,(1)$

So, from $(1) \Longrightarrow \angle A_1PA_2 = \angle A_1QA_2 = \angle QA_2A_1 = \angle QB_2B_1 = \angle B_1QB_2 = \angle B_1PB_2\,,(2)$ and hence, we conclude that the points $A_2,\ P,\ B_2$, are collinear.

By the same way, from the collinearity of $A_2,\ P,\ B_2$, we conclude the collinearity of the points $A_3,\ P,\ B_3$ and so on.

That is, from the collinearity of the points $A_{i-1},\ P,\ B_{i-1}$, we conclude the collinearity of the points $A_i,\ P,\ B_i$ and the proof is completed.

Kostas Vittas.

*Attachments:*

t=279890.pdf (6kb)

Quick Reply

# Middle School Math

## simple geometry

Reply

**earth**

#1 Jun 8, 2009, 7:17 pm

O is any point inside triangle ABC.prove that
(AB+BC+AC)>(AO+BO+CO)

**PowerOfPi**

#2 Jun 8, 2009, 8:46 pm

Here is a picture:

[geogebra]dd37958872bfe94ac401f4ab1e3e1f7e3164fbe7[/geogebra]

If you just think about it, you will find that it is always true.

**batteredbutnotdefeated**

#3 Jun 9, 2009, 2:25 am

@powerofpi, that's not a proof...

See, geogebra doesn't work on my computer so just draw this out...

Draw any point inside triangle ABC. Then draw a line from the vertex that intersects the point "O". Then draw another line from another vertex(let's say B), that intersects our original line AO. Notice that no matter what kind of triangle it is that AB is always the hypotenuse as long as it's INSIDE the triangle. AO is just a leg.(A hypotenuse is obviously longer than a leg because of triangle inequality) Now just apply this to the other sides and we're done. Notice how any segment(AO, BO, CO), is always shorter than a median, and a median is always shorter than an actual side of triangle ABC.

**Luis González**

#4 Jun 9, 2009, 11:51 pm

Ray $BO$ cuts $AC$ at $X$. By triangle inequality in $\triangle ABX$ and $\triangle COX$, we get

$OX + XC > OC$ , $AB + AX > OB + OX$.

$OX + XC + AB + AX > OB + OC + OX \Longrightarrow AB + AC > OB + OC$.

Similarly we get $BA + BC > OA + OC$ , $CA + CB > OA + OB$.

Adding the three previous inequalities gives $AB + BC + CA > OA + OB + OC$.

Quick Reply

# High School Olympiads

A property realted to centroid and concyclic of points

Reply

**mathVNpro**

#1 Jun 9, 2009, 9:36 am

The below problem is a nice and beautiful property that **Mr. Dung Tran Nam** (nickname on mathlinks: **namdung**) found independently and has been proved by himself in 1983, which is also the year that he attended the IMO competition with Vietnam team:

**PROBLEM- Let $ABC$ be a triangle with $G$ is its centroid. Let $A_0, B_0, C_0$ respectively are the midpoints of $BC, CA, AB$. Prove that the circumcenters of triangles $AGC_0, AGB_0, BGA_0, BGC_0, CGA_0, CGB_0$, respectively are concyclic**.

**April**

#2 Jun 9, 2009, 10:14 am

http://www.mathlinks.ro/Forum/viewtopic.php?t=1536

**jayme**

#3 Jun 9, 2009, 10:45 am

Dear Mathlinkers,
for an entirely synthetic proof see also

http://perso.orange.fr/jl.ayme vol. 2 Le cercle de van Lamoen

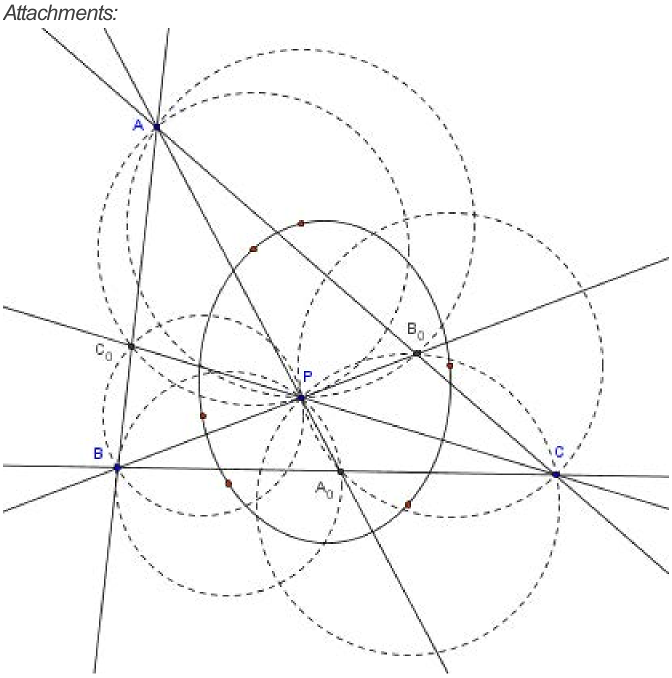
Next, I will put an additionnal historic note concerning Mr. Dung Tran Nam.
Sincerely
Jean-Louis

**Luis González**

#4 Jun 9, 2009, 10:53 am

For arbitrary P, those six circumcenters lie on a same conic.

*Attachments:*




Quick Reply

# High School Olympiads

## Parallel Lines



**babylon**

#1 Jun 8, 2009, 9:42 pm

The incircle $k$ of a non-isosceles $\triangle ABC$ touches the sides $AB, BC, CA$ at $C_1, A_1, B_1$ , respectively. If $M$ and $N$ are midpoints of $B_1C_1$ and $A_1C_1$ , respectively and $MN \cap BC = P$ prove that $C_1P||AA_1$.

**mathVNpro**

#2 Jun 8, 2009, 11:36 pm

Let $MN \cap BA \equiv M_a$, $A_1B_1 \cap AB \equiv D$. Since $M, N$ respectively are the midpoints of $C_1B_1, C_1A_1$. But $MN||A_1B_1$. Hence $M_a$ will be midpoint of $C_1D$. Now it is so obvious that $PM_a||A_1D \Longrightarrow \frac{\overline{PB}}{\overline{PA_1}} = \frac{\overline{M_aB}}{\overline{M_aD}}$. Note that $(DC_1BA) = -1$, $M_a$ is midpoint of $DC_1 \Longrightarrow M_aD^2 = \overline{M_aB} \cdot \overline{M_aA}$. Therefore

$$\frac{\overline{M_aB}}{\overline{M_aD}} = \frac{\overline{M_aD}}{\overline{M_aA}} = \frac{\overline{M_aB} - \overline{M_aD}}{\overline{M_aD} - \overline{M_aA}} = -\frac{\overline{DB}}{\overline{DA}} = \frac{\overline{C_1B}}{\overline{C_1A}} \Longrightarrow \frac{\overline{PB}}{\overline{PA_1}} = \frac{\overline{C_1B}}{\overline{C_1A}}$$, which implies $PC_1||AA_1$

Our proof is completed then.

*This post has been edited 7 times. Last edited by mathVNpro, Jun 9, 2009, 11:16 pm*

**Ahiles**

#3 Jun 9, 2009, 12:18 am

We have

$$d(C, MN) = d(C, A_1B_1) + d(A_1B_1, MN) =$$

$$CA_1 \cos\frac{C}{2} + \frac{1}{2}d(C_1, A_1B_1) = (p-c)\cdot\cos\frac{C}{2} + r\cos\frac{A}{2}\cos\frac{B}{2}$$

$$A_1D = CD - A_1C = \frac{d(C, MN)}{\cos\frac{C}{2}} - (p-c) = r\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}} =$$

$$\frac{\sqrt{p(p-a)(p-b)(p-c)}\cdot\sqrt{\frac{p(p-a)}{bc}}\cdot\sqrt{\frac{p(p-b)}{ac}}}{p\sqrt{\frac{p(p-c)}{ab}}} = \frac{(p-a)(p-b)}{c}$$

We need to prove

$$\frac{AC_1}{AB} = \frac{A_1D}{A_1B} \Longleftrightarrow \frac{p-a}{c} = \frac{\frac{(p-a)(p-b)}{c}}{p-b},$$

clearly true.

**Luis González**

#4 Jun 9, 2009, 6:25 am

We'll use barycentric coordinates with respect to $\triangle ABC$.

$I_c(a:b:-c)$ , $A_1(0:p-c:p-b)$

Line $B_1C_1$ meets $BC$ at $X_a(0:p-c:b-p)$

$I_cA_1 \equiv (c(p-c)+b(p-b))x - a(p-b)y - a(p-c)z = 0$

$Q \equiv AB \cap I_cA_1 \equiv (a(p-b) : c(p-c)+b(p-b) : 0)$

The incenter $I$ $(a:b:c)$ clearly satisfies the equation of the line $I_cA_1$

$X_aQ \equiv (p-b)(c(p-c)+b(p-b))x - a(p-b)^2y - a(p-c)(p-b)z = 0$

$\Longrightarrow Q, I, X_a$ are collinear.

$X_a, I_c$ and $P$ are the poles of $AA, MN$ and $I_cA_1$ WRT $(I) \Longrightarrow Q \equiv AB \cap I_cA_1$ is the pole of $PC_1$ WRT $(I)$. Since the line $X_aQ$ goes through $I$ then the polars of $X_a, Q$ meet at infinite $\Longrightarrow PC_1 \parallel AA_1$.

**Moonmathpi496**

#5 Jun 9, 2009, 1:29 pm

> *babylon wrote:*
> The incircle $k$ of a non-isosceles $\triangle ABC$ touches the sides $AB, BC, CA$ at $C_1, A_1, B_1$ , respectively. If $M$ and $N$ are midpoints of $B_1C_1$ and $A_1C_1$ , respectively and $MN \cap BC = P$ prove that $C_1P||AA_1$.

My idea is to slog with trigonometry! :)

Here it is enough to prove that, $\frac{AB}{AC_1}\frac{BA_1}{PA_1} \iff \frac{c}{s-a} = \frac{s-b}{PA_1}$

Now we calculate $PA_1$. Using sine law in $\triangle PNA_1$ we get, $\frac{PA_1}{\cos\frac{A}{2}} = \frac{\frac{C_1A_1}{2}}{\cos\frac{C}{2}}$

Also, using sine law in $\triangle BA_1C_1$ we get, $\frac{C_1A_1}{\sin B} = \frac{s-b}{\cos\frac{B}{2}} \iff C_1A_1 = 2(s-b)\sin\frac{B}{2}$

Finally, $PA_1 = \frac{(s-b)\cos\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$

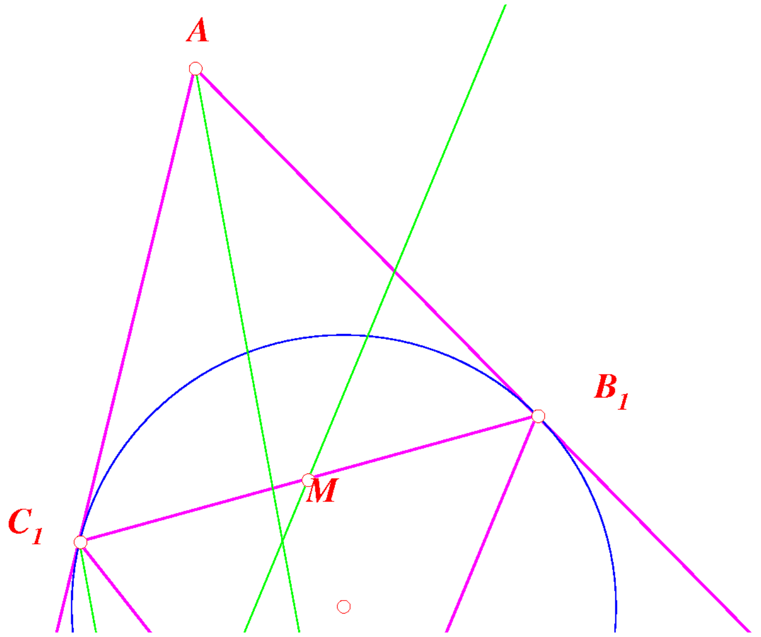
Now setting the values for half angles we get,

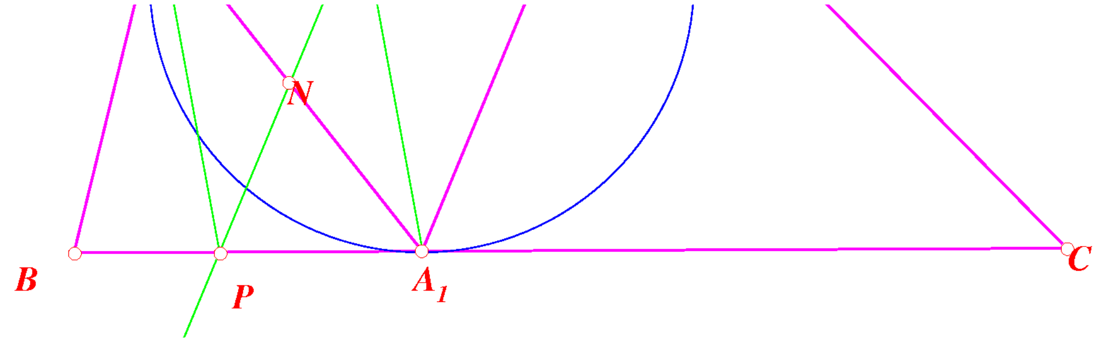
$$\frac{s-b}{PA_1} = \frac{\cos\frac{C}{2}}{\cos\frac{A}{2}\sin\frac{B}{2}} = \frac{\sqrt{\frac{s(s-c)}{ab}}}{\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}} = \frac{c}{s-a}$$

As desired.

EDIT: Fixed typo.

*Attachments:*






This post has been edited 1 time. Last edited by Moonmathpi496, Jun 9, 2009, 6:48 pm

**jayme**

#6 Jun 9, 2009, 6:25 pm

Dear Mathlinkers,
let L be the meet point of AA1 and MN.
For beginning synthetically this nice problem, we have to prove that C1L is parallel to BC and we are done.
Sincerely
Jean-Louis

**jayme**

#7 Jun 9, 2009, 6:53 pm

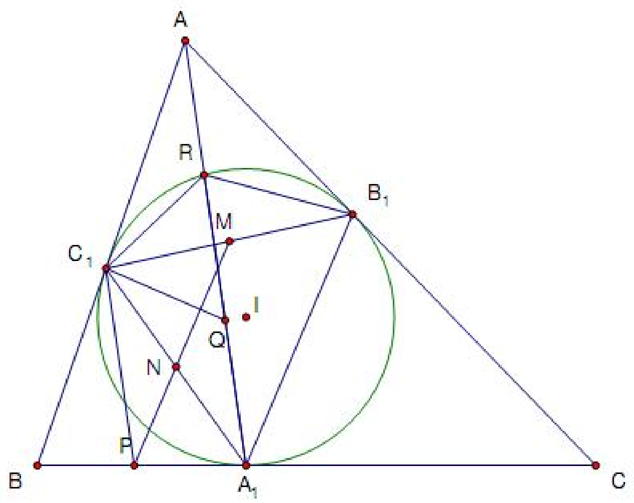
Dear Mathlinkers,
think to draw the parallel to BC through A...
Sincerely
Jean-Louis

**livetolove212**

#8 Jun 9, 2009, 8:13 pm

Let $AA_1 \cap (I) = \{R\}, Q$ be the midpoint of $RA_1$.
We have $\angle NA_1P = \angle C_1RQ$
But $PN//A_1B_1 \Rightarrow \angle NPA_1 = \angle B_1A_1C = \angle A_1C_1B_1$
Because quadrilateral $A_1C_1RB_1$ is harmonic, this result is well-known: $\angle A_1C_1B_1 = \angle RC_1Q$
Therefore, $\Delta PA_1N \sim \Delta C_1RQ \Rightarrow \Delta C_1PA_1$~$\Delta A_1C_1R$
$\Rightarrow \angle PC_1A_1 = \angle C_1A_1R \Rightarrow$ QED

*Attachments:*




**No Reason**

#9 Jun 9, 2009, 9:48 pm

The sketch of my proof (sorry because I don't have many time left)
Let $C_1P$ meet $A_1B_1$ at $R$.Using harmonic division,it is equivalent to prove that $(A_1R, A_1P, A_1C, A_1A)$ is harmonic,which is true since $AA_1, BB_1, CC_1$ are concurrent.

**mathVNpro**
#10 Jun 9, 2009, 11:03 pm

> No Reason wrote:
> The sketch of my proof (sorry because I don't have many time left)
> Let $C_1P$ meet $A_1B_1$ at $R$.Using harmonic division,it is equivalent to prove that $(A_1R, A_1P, A_1C, A_1A)$ is harmonic,which is true since $AA_1, BB_1, CC_1$ are concurrent.

Sorry for asking but $A_1P \equiv A_1C$, isn't it?

**No Reason**
#11 Jun 9, 2009, 11:46 pm

It's an typo,it must be $A_1C_1$ not $A_1C$

**bravado**
#12 Jun 10, 2009, 12:44 am

> jayme wrote:
> Dear Mathlinkers,
> let L be the meet point of AA1 and MN.
> For beginning synthetically this nice problem, we have to prove that C1L is parallel to BC and we are done.
> Sincerely
> Jean-Louis

Nice idea Jean-Louis! Let $\{T\} = AM \cap A_1B_1$, then $AT \perp BT$. Therefore

$$\frac{AC_1}{BC_1} = \frac{AM}{MT} = \frac{AL}{LA_1},$$

which proves that $C_1L||BC$ and the conclusion follows.

**jayme**
#13 Jun 10, 2009, 10:55 am

Dear Mathlinkers,
a purely synthetic proof:
1. let U, V the meetpoints of the parallel to BC through A with A1B1, A1C1
and L, X the meetpoints of the parallel to BC through C1 with AA1, A1B1
2. According to Boutin's theorem, A is the midpoint of UV ; it follows that L is the midpoint of C1X
(see for example, http://perso.orange.fr/jl.ayme vol. 1 A propos du théorème de Boutin)
3. According to the Thales's theorem (the line joining two midpoints in a triangle), L, M, N are collinear.
and we are done.
Sincerely
Jean-Louis

**dgreenb801**
#14 Nov 20, 2009, 5:25 am

Let $r$ be the inradius of $\triangle ABC$.
Note that
$\frac{BP}{PA_1} = \frac{[BNP]}{[PNA_1]} = \frac{BN \cdot NP \cdot sinBNP}{NP \cdot NA_1 \cdot sinPNA_1}$. But $\frac{BN}{NA_1} = \frac{BA_1}{r}$ and $\angle BNP = \angle MNI$, and $\angle PNA_1 = 90 - \angle MNI$. So
$\frac{BP}{PA_1} = \frac{BA_1}{r} \cdot \frac{sinINM}{cosINM} = \frac{BA_1}{r} \cdot tanINM = \frac{BA_1}{r} \cdot tanC_1AI* = \frac{BA_1}{r} \cdot \frac{r}{C_1A} = \frac{BA_1}{C_1A} = \frac{BC_1}{C_1A}$
Thus, $\frac{BP}{PA_1} = \frac{BC_1}{C_1A}$, so $C_1P \parallel AA_1$.

*This is true because $\triangle IC_1M \sim \triangle IC_1A$, so $IM \cdot IA = C_1I^2 = IN \cdot IB$, so $AMNB$ is cyclic by power of a point, so $\angle BAM + \angle MNB = 180$, so $\angle C_1AM = \angle MNI$.

**mathson**

#15 Nov 20, 2009, 12:29 pm

> livetolove212 wrote:
> Let $AA_1 \cap (I) = \{R\}, Q$ be the midpoint of $RA_1$.
> We have $\angle NA_1P = \angle C_1RQ$
> But $PN//A_1B_1 \Rightarrow \angle NPA_1 = \angle B_1A_1C = \angle A_1C_1B_1$
> Because quadrilateral $A_1C_1RB_1$ is harmonic, this result is well-known: $\angle A_1C_1B_1 = \angle RC_1Q$
> Therefore, $\Delta PA_1N \sim \Delta C_1RQ \Rightarrow \Delta C_1PA_1$~$\Delta A_1C_1R$
> $\Rightarrow \angle PC_1A_1 = \angle C_1A_1R \Rightarrow$ QED

Q: What does "Harmonic Quadrilateral" mean?

Quick Reply

# Spain

## Geometria

Reply

**wx69yz**

#1 Nov 2, 2006, 3:04 am

Hola gente, necesito demostrar el siguiente enunciado, espero me ayuden

Sea $ABC$ un triángulo cualquiera, Sea $O_c$ el excirculo q se opone a $C$, $O_b$ el excirculo q se opone a $B$.
$O_c$ y $O_b$ son tangentes a la recta $BC$ en $J$ y $F$, respectivamente, tangentes a la recta $AB$ en $Z$ y $G$, respectivamente, tangentes a $AC$ en $L$ y $Y$, respectivamente, y sea $l$ la cuarta tangente comun a $O_c$ y $O_b$.
Las rectas $FG$ y $JL$ se intersectan en $U$, las rectas $ZG$ y $YL$ intersectan a $l$ en $M$ y $N$ respectivamente.
Muestre q el circuncentro del triangulo $AMN$ se encuentra sobre la recta $AU$.

**Luis González**

#2 Jun 8, 2009, 10:58 am

Probaremos que $U$ está en la recta de la altura $AH$ de $\triangle ABC$. Así que supongamos $U \equiv JL \cap AH$ y $U' \equiv TG \cap AH$. Hay que probar que $U$ se confunde con $U'$. Aplicando Teorema de Menelao a la transversal $\overline{UAH}$ en $\triangle CLJ$ se tiene:

$$\frac{UA}{UH} \cdot \frac{JH}{JC} \cdot \frac{CL}{AL} = 1 \Longrightarrow \frac{UA}{UH} = \frac{AL}{JH}$$

Por teorema de Menelao a la transversal $\overline{U'AH}$ en $\triangle BGF$ se obtiene $\frac{U'A}{U'H} = \frac{AG}{FH}$

Por otro lado de la semejanza $\triangle O_bGA \sim \triangle O_cZA$ se tiene $\frac{O_bA}{O_cA} = \frac{AG}{AL}$

Las proyecciones ortogonales de $O_b, O_c, A$ sobre la recta $BC$ son $F, J, H$. Entonces por teorema de Thales

$$\frac{O_bA}{O_cA} = \frac{FH}{JH} \Longrightarrow \frac{FH}{JH} = \frac{AG}{AL} \Longrightarrow \frac{UA}{UH} = \frac{UA'}{U'H} \Longrightarrow U \equiv U'$$

Es claro que $\triangle AMN$ es simétrico a $\triangle ABC$ con respecto a la recta $O_bO_c$, por ende la altura $AH$ de $\triangle ABC$ es isogonal de la A-Altura del $\triangle AMN$ con respecto al ángulo comun $\angle BAC$. A saber, ésta isogonal $AU$ contiene el circuncentro de $\triangle AMN$.

Quick Reply

# High School Olympiads

## Uganda 1975 [when A-median, B-symmedian, C-altitude concur]

Reply

**Source**: Idi Amin's Personal collection and Lemoine

**pestich**

#1 Jan 14, 2005, 10:59 pm

If median passing thru vertex A, symedian passing thru vertex B, and altitude thru vertex C are concurrent, then this altitude is parallel to line connecting Brocard points of this triangle.

Col. Pestich.

**yetti**

#2 Jan 23, 2005, 6:36 pm

Problem is not stated correctly, it should be:
If the median passing through the vertex $A$, symedian through the vertex $B$, and the altitude passing through the vertex $C$ of a triangle $\triangle ABC$ are concurrent, then the altitude passing through the vertex $B$ (not $C$!) is parallel to the line connecting the two Brocard points of the triangle.

**Lemma:** A median through one vertex of a triangle, symmedian through another vertex and one of the Brocard rays from the remaining vertex of a triangle are concurrent.

Let $L, M, N$ be the midpoints of the sides $a = BC, b = CA, c = AB$. Let $U \in BC, V \in CA, W \in AB$ be feet of symmedians through the vertices $A, B, C$. Let $Z$ be the 2nd Brocard point of $\triangle ABC$, so that $\angle ZCB = \angle ZAC = \angle ZBA = \theta$ is the Brocard angle. Let $CZ$ cut $AB$ at $F$. Since $\angle ZBF = \angle ZBA = \angle ZCB = \theta \Longrightarrow AFB$ is tangent at $B$ of a circle passing through $Z, C \Longrightarrow FB^2 = FZ \cdot FC \Longrightarrow$ triangles $\triangle BCF \sim \triangle ZBF$ are similar $\Longrightarrow \frac{ZF}{BZ} = \frac{BF}{BC}$ and $\angle BZF = \angle B$ of $\triangle ABC$. Since $\angle ZAC = \angle ZCB = \theta \Longrightarrow BC$ is tangent at $C$ of a circle passing through $Z, A \Longrightarrow \angle FZA = \angle C$ of $\triangle ABC$. Using the sine theorem for the triangles $\triangle ABZ, \triangle AFZ$,

$$\frac{BZ}{AB} \cdot \frac{AF}{ZF} = \frac{\sin \widehat{FZA}}{\sin \widehat{BZA}} = \frac{\sin C}{\sin A} = \frac{c}{a}$$

Substituting for $ZF = BF \cdot \frac{BZ}{BC}$, we get

$$\frac{c}{a} = \frac{BZ}{AB} \cdot \frac{AF}{ZF} = \frac{AF}{BF} \cdot \frac{BC}{AB} = \frac{AF}{BF} \cdot \frac{a}{c} \Longrightarrow \frac{AF}{BF} = \frac{c^2}{a^2}$$

By Ceva's theorem,

$$\frac{\overline{BL}}{\overline{CL}} \cdot \frac{\overline{CV}}{\overline{AV}} \cdot \frac{\overline{AF}}{\overline{BF}} = -\frac{a^2}{c^2} \cdot \frac{c^2}{a^2} = -1$$

Consequently, median $AL$, symmedian $BV$ and Brocard ray $CZF$ are concurrent at a point $X$.

By the condition of the problem, the altitude through the vertex $C$ is identical with the Brocard ray $CZ \equiv CF$. Let $D, E$ be the feet of altitudes $AD, BE$ through the vertices $A, B$. From similarity of the triangles $\triangle CBF \sim \triangle ABD$, it immediately follows that the 1st Brocard point $Y$ lies on the altitude $AD$, i.e., this altitude is also a Brocard ray. Since $\angle BCF = \angle BAD = \theta$ (Brocard angle), the angle $\angle B = 90^\circ - \theta$.

Let $O$ be circumcenter of the triangle $\triangle ABC$. Since the angle $\angle BOM = \angle B$, the angle $\angle MBO = 90^\circ - \angle B = \theta$. Hence, the circumcenter $O$ lies on the Brocard ray $CY$ of the 1st Brocard point $Y$. The circumcenter $O$ also lies on the Brocard circle (together with the 2 Brocard points $Y, Z$ and the symmedian point). The triangle $\triangle YOZ$ is isosceles and the angle $\angle YOZ = 2\theta$ (double the Brocard angle). Hence, the angles $\angle OZY = \angle OYZ = \frac{180^\circ - 2\theta}{2} = 90^\circ - \theta = \angle B$. Since the angle $\angle YOM = \angle B$ as well, it follows that the perpendicular bisector $OM$ of the side $CA$ is tangent to the Brocard circle at the point $O$. Since the angles $\angle ZYO = \angle YOM$ are both equal to the angle $\angle B$, lines $ZY, OM$ cut by the transversal $YO$ are parallel. And of course, altitude $BE$ through the vertex $B$ to the side $CA$ is parallel to perpendicular bisector $OM$ of the side $CA$, i.e., also to the line $ZY$ connecting the 2 Brocard points $Z, Y$.

*This post has been edited 3 times. Last edited by yetti, Feb 9, 2005, 3:45 pm*

**darij grinberg**

*#3 Jan 23, 2005, 7:49 pm • 1*

> *yetti wrote:*
>
> Nice problem, unfortunately, it is not stated correctly. It should be:
>
> If the median passing through the vertex $A$, symedian through the vertex $B$, and the altitude passing through the vertex $C$ of a triangle $\triangle ABC$ are concurrent, then the altitude passing through the vertex $B$ (not $C$!) is parallel to the line connecting the two Brocard points of the triangle.

There is more to say: The line connecting the two Brocard points of the triangle ABC is the perpendicular to the side CA through the foot of the B-symmedian.

I have actually proved a more general result:

**Theorem.** Let the B-symmedian of a triangle ABC meet the side CA at a point E, and let W and W' be the two Brocard points of triangle ABC such that < WBC = < WCA = < WAB = w and < W'CB = < W'AC = < W'BA = w, where w is the Brocard angle of triangle ABC. Then, the following assertions are all equivalent:

**(1)** The A-median, the B-symmedian and the C-altitude of triangle ABC are concurrent.
**(2)** The point W' lies on the C-altitude of triangle ABC.
**(3)** The point W' lies on the perpendicular to the line CA at the point E.
**(4)** The C-median, the B-symmedian and the A-altitude of triangle ABC are concurrent.
**(5)** The point W lies on the A-altitude of triangle ABC.
**(6)** The point W lies on the perpendicular to the line CA at the point E.
**(7)** We have $\tan C \tan A = \tan^2 B$.
**(8)** We have w = 90° - B.
**(9)** We have $c^4 + a^4 = \left(c^2 + a^2\right) b^2$.
**(10)** The line WW' is perpendicular to the line CA.

*Proof.* What we need for solving the problem is that assertion **(1)** implies assertions **(3)** and **(6)**, but we will prove the equivalence of all ten assertions:

Let D be the midpoint of the side BC of triangle ABC, so that the line AD is the A-median of triangle ABC; let F be the point where the line CW' meets the side AB. Then, < FCB = < W'CB = w.

By the sum of angles in triangle BFC, we have < BFC = 180° - < FBC - < FCB = 180° - B - w. Now, the point W' lies on the C-altitude of triangle ABC if and only if $CW' \perp AB$, what is equivalent to < BFC = 90°, what is equivalent to 180° - B - w = 90° (according to < BFC = 180° - B - w), what simplifies to w = 90° - B. Thus, we have proven the equivalence **(2)** $\Longleftrightarrow$ **(8)**.

In post #2, Yetti proved that the symmedian from one vertex of a triangle, the median from another, and the line joining the third vertex with the appropriate Brocard point are concurrent. This fact can also be found with proof in Ross Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry (New Mathematical Library)*, Washington 1995, Chapter 10, §6. Anyway, applying this fact to triangle ABC, we see that the symmedian BE, the median AD, and the line CW' are concurrent. In other words, the lines AD, BE and CF are concurrent. Thus, the Ceva theorem yields $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$; but since the point D is the midpoint of the segment BC, we have $\frac{BD}{DC} = 1$, and thus we get $\frac{CE}{EA} \cdot \frac{AF}{FB} = 1$, so that $\frac{CE}{EA} = \frac{FB}{AF}$, or, equivalently, $\frac{CE}{EA} = \frac{BF}{FA}$, what immediately yields EF || BC after Thales.

From EF || BC, we conclude < W'FE = < W'CB. But since < W'CB = < W'AC, we have < W'FE = < W'AC; in other words, < W'FE = < W'AE. Thus, the points W', A, F, E lie on one circle, so that < AFW' + < AEW' = 180°. Hence, < AFW' = 90° holds if and only if < AEW' = 90°. But < AFW' = 90° is equivalent to $CW' \perp AB$, what is equivalent to the assertion that the point W' lies on the C-altitude of triangle ABC. On the other hand, < AEW' = 90° is equivalent to the assertion that the point W' lies on the perpendicular to the line CA at the point E. Thus, the point W' lies on the C-altitude of triangle ABC if and only if the point W' lies on the perpendicular to the line CA at the point E. So we have proven that **(2)** $\Longleftrightarrow$ **(3)**.

We know that in the triangle ABC, the B-symmedian BE, the A-median AD and the line CW' are concurrent. Hence, the A-median, the B-symmedian and the C-altitude of triangle ABC are concurrent if and only if the C-altitude coincides with the line CW'; but this obviously holds if and only if the point W' lies on the C-altitude of triangle ABC. Thus, we have proven the equivalence **(1)** $\Longleftrightarrow$ **(2)**.

Let T be the foot of the C-altitude of triangle ABC. Then, in the right-angled triangle CTA, we have $AT = CT \cdot \cot \measuredangle CAT = CT \cdot \cot A$. Similarly, the right-angled triangle BTC yields $TB = CT \cdot \cot \measuredangle TBC = CT \cdot \cot B$. Thus, $\frac{AT}{TB} = \frac{\cot A}{\cot B}$ and $c = AB = AT + TB = CT \cdot \cot A + CT \cdot \cot B = CT \cdot (\cot A + \cot B)$. If $\Delta$ is the area of triangle ABC, then, since $\text{area of a triangle} = \frac{1}{2} \cdot \text{ side } \cdot \text{ corresponding altitude}$, we have $\Delta = \frac{1}{2} \cdot c \cdot CT$, and thus $CT = \frac{2\Delta}{c}$. Thus, $c = CT \cdot (\cot A + \cot B) = \frac{2\Delta}{c} \cdot (\cot A + \cot B)$. Consequently, $c^2 = 2\Delta (\cot A + \cot B)$. Similarly, $a^2 = 2\Delta (\cot B + \cot C)$. Thus, $\frac{a^2}{c^2} = \frac{\cot B + \cot C}{\cot A + \cot B}$.

By the Ceva theorem, the A-median AD, the B-symmedian BE and the C-altitude CT of triangle ABC are concurrent if and only if $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AT}{TB} = 1$. But as we know, $\frac{BD}{DC} = 1$ and $\frac{AT}{TB} = \frac{\cot A}{\cot B}$, and since a symmedian of a triangle divides the opposite side in the ratio of the squares of the two adjacent sides, we have $\frac{CE}{EA} = \frac{a^2}{c^2}$, what, using $\frac{a^2}{c^2} = \frac{\cot B + \cot C}{\cot A + \cot B}$, becomes $\frac{CE}{EA} = \frac{\cot B + \cot C}{\cot A + \cot B}$. Hence, the equation $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AT}{TB} = 1$ rewrites as $1 \cdot \frac{\cot B + \cot C}{\cot A + \cot B} \cdot \frac{\cot A}{\cot B} = 1$. This is equivalent to $\frac{\cot A + \cot B}{\cot B + \cot C} = \frac{\cot A}{\cot B}$. By rearranging the terms, we rewrite this as $\frac{\cot A + \cot B}{\cot A} = \frac{\cot B + \cot C}{\cot B}$, or, equivalently, as $1 + \frac{\cot B}{\cot A} = 1 + \frac{\cot C}{\cot B}$. But this is clearly equivalent to $\frac{\cot B}{\cot A} = \frac{\cot C}{\cot B}$. Since $\cot x = 1/\tan x$ for any x, this becomes $\frac{1/\tan B}{1/\tan A} = \frac{1/\tan C}{1/\tan B}$, what simplifies to $\frac{\tan A}{\tan B} = \frac{\tan B}{\tan C}$, or, equivalently, to $\tan C \tan A = \tan^2 B$. Altogether, we see that the A-median AD, the B-symmedian BE and the C-altitude CT of triangle ABC are concurrent if and only if $\tan C \tan A = \tan^2 B$. Thus, the equivalence **(1)** $\Longleftrightarrow$ **(7)** is shown.

So we have proved the equivalences **(2)** $\Longleftrightarrow$ **(8)**, **(2)** $\Longleftrightarrow$ **(3)**, **(1)** $\Longleftrightarrow$ **(2)** and **(1)** $\Longleftrightarrow$ **(7)**. By a symmetric permutation of the vertices A, B, C of triangle ABC, these equivalences become **(5)** $\Longleftrightarrow$ **(8)**, **(5)** $\Longleftrightarrow$ **(6)**, **(4)** $\Longleftrightarrow$ **(5)** and **(4)** $\Longleftrightarrow$ **(7)**. Hence, the eight assertions **(1)**, **(2)**, **(3)**, **(4)**, **(5)**, **(6)**, **(7)**, **(8)** are all equivalent. In order to complete the proof of the theorem, it will be now enough to show that these assertions are equivalent to **(9)** and to **(10)**. We will do this by proving the equivalences **(7)** $\Longleftrightarrow$ **(9)** and **(7)** $\Longleftrightarrow$ **(10)**.

First let's show that **(7)** $\Longleftrightarrow$ **(9)**: As we showed above, $c^2 = 2\Delta (\cot A + \cot B)$ and $a^2 = 2\Delta (\cot B + \cot C)$. Similarly, $b^2 = 2\Delta (\cot C + \cot A)$. Hence,

$$c^2 + a^2 - b^2 = 2\Delta (\cot A + \cot B) + 2\Delta (\cot B + \cot C) - 2\Delta (\cot C + \cot A)$$
$$= 4\Delta \cot B.$$

Thus, $\cot B = \frac{c^2 + a^2 - b^2}{4\Delta}$, so that $\tan B = \frac{1}{\cot B} = \frac{4\Delta}{c^2 + a^2 - b^2}$. Similarly, $\tan C = \frac{4\Delta}{a^2 + b^2 - c^2}$ and $\tan A = \frac{4\Delta}{b^2 + c^2 - a^2}$. Hence, the equation $\tan C \tan A = \tan^2 B$ is equivalent to $\frac{4\Delta}{a^2 + b^2 - c^2} \cdot \frac{4\Delta}{b^2 + c^2 - a^2} = \left( \frac{4\Delta}{c^2 + a^2 - b^2} \right)^2$. This immediately simplifies to $(a^2 + b^2 - c^2)(b^2 + c^2 - a^2) = (c^2 + a^2 - b^2)^2$. But since $(a^2 + b^2 - c^2)(b^2 + c^2 - a^2) - (c^2 + a^2 - b^2)^2 = -2\left((c^4 + a^4) - (c^2 + a^2) b^2\right)$, this is equivalent to $c^4 + a^4 = (c^2 + a^2) b^2$. Thus, $\tan C \tan A = \tan^2 B$ is equivalent to $c^4 + a^4 = (c^2 + a^2) b^2$. This proves that **(7)** $\Longleftrightarrow$ **(9)**.

Remains to establish **(7)** $\Longleftrightarrow$ **(10)**. For that aim, we consider the points X, Y, Z where the tangents to the circumcircle of triangle ABC at the points A, B, C meet the lines BC, CA, AB, respectively. It is well-known that these points X, Y, Z lie on one line, which is called the Lemoine axis of triangle ABC and is perpendicular to the Brocard axis of triangle ABC. On the other hand, by the properties of the Brocard points, the Brocard axis of triangle ABC is perpendicular to the line WW' joining the two Brocard points W and W'. Hence, the line XYZ is parallel to the line WW'. Thus, the line WW' is perpendicular to the line CA if and only if the line XYZ is perpendicular to the line CA.

Let S be the orthogonal projection of the point Z on the line CA. Then, the line XYZ is perpendicular to the line CA if and only if S = Y. But S = Y is trivially equivalent to $\frac{AY}{CY}=\frac{AS}{CS}$. Now, by http://www.mathlinks.ro/Forum/viewtopic.php?t=6557 post #4 Lemma 2, we have $\frac{AY}{CY}=\frac{c^2}{a^2}$; in other words, $\frac{AY}{CY}=1:\frac{a^2}{c^2}=1:\frac{\cot B+\cot C}{\cot A+\cot B}=\frac{\cot A+\cot B}{\cot B+\cot C}$. On the other hand, in the right-angled triangle ASZ, we have $AS=ZS\cdot\cot\measuredangle ZAS=ZS\cdot\cot A$, and in the right-angled triangle CSZ, we have $CS=ZS\cdot\cot\measuredangle ZCS$. Since the line CZ is tangent to the circumcircle of triangle ABC at the point C, we have < ZCA = < ABC; in other words, < ZCS = B, and thus, $CS=ZS\cdot\cot B$. Hence, $\frac{AS}{CS}=\frac{ZS\cdot\cot A}{ZS\cdot\cot B}=\frac{\cot A}{\cot B}$. Altogether, this yields that the equation $\frac{AY}{CY}=\frac{AS}{CS}$ becomes $\frac{\cot A+\cot B}{\cot B+\cot C}=\frac{\cot A}{\cot B}$. As we already saw above, this is equivalent to $\tan C\tan A=\tan^2 B$.

Combining all these steps, we get a proof of the fact that the line WW' is perpendicular to the line CA if and only if $\tan C\tan A=\tan^2 B$. In other words, we have shown that **(7)** $\Longleftrightarrow$ **(10)**. This finally completes the proof of the theorem.

Darij

*This post has been edited 6 times. Last edited by darij grinberg, Jul 22, 2005, 6:01 am*

**yetti**

*#4 Jan 23, 2005, 8:52 pm*

I noticed the line connecting the Brocard points passing through the foot V of the symmedian BV through the vertex B to the line CA, wondered about it, but since I did not need it, I passed on.

yetti

**Luis González**

*#5 Jun 8, 2009, 3:32 am*

**Problem:** If median $AM_a$, symmedian $BK_b$ and altitude $CH_c$ of $\triangle ABC$ are concurrent, then the altitude $BH_b$ is parallel to the line connecting the two Brocard points $\Omega_1,\Omega_2$ of the $\triangle ABC$.

Let us use barycentric coordinates with respect to $\triangle ABC$.

$$AM_a\equiv y-z=0\ ,\ BK_b\equiv c^2x-a^2z=0\ ,\ CH_c\equiv S_Ax-S_By=0$$

These three lines are concurrent if and only if

$$\begin{bmatrix} 0 & 1 & -1 \\ c^2 & 0 & -a^2 \\ S_A & -S_B & 0 \end{bmatrix}=0\Longrightarrow a^2+c^4=a^2b^2+b^2c^2$$

Coordinates of the infinite point $B_\infty$ of $\perp AC$ are given by

$$(2a^2c^2-a^2b^2+b^2c^2-a^4-c^4:-2b^2(c^2-a^2):2a^2c^2-a^2b^2+b^2c^2+a^2+c^4)$$

Using the upper condition we get: $B_\infty(a^2(b^2-c^2):b^2(c^2-a^2):c^2(a^2-b^2))$

But note that the infinite point of the line connecting the two Brocard points $\Omega_1,\Omega_2$ is precisely $T_\infty(a^2(b^2-c^2):b^2(c^2-a^2):c^2(a^2-b^2))\Longrightarrow\Omega_1\Omega_2\parallel BH_b$.

*This post has been edited 1 time. Last edited by Luis González, Jun 8, 2009, 6:04 pm*

**Luis González**

*#6 Jun 8, 2009, 5:54 am*

> *yetti wrote:*
> **Lemma:** A median through one vertex of a triangle, symmedian through another vertex and one of the Brocard rays from the remaining vertex of a triangle are concurrent.

$$AM_a\equiv y-z=0\ ,\ BK_b\equiv c^2x-a^2z=0\ ,\ C\Omega_2\equiv c^2x-a^2y=0$$

$$\begin{bmatrix} 0 & 1 & -1 \\ c^2 & 0 & -a^2 \\ c^2 & -a^2 & 0 \end{bmatrix} = a^2c^2 - a^2c^2 = 0 \Longrightarrow AM_a \cap BK_b \cap C\Omega_2 \neq \emptyset$$

Quick Reply

# High School Olympiads

## Side triangle (2)

Reply

**Luis González**

#1 Jun 6, 2009, 1:14 am

Let $P$ and $Q$ be two isogonal conjugates WRT $\triangle ABC$ and let $\triangle P_aP_bP_c$ and $\triangle Q_aQ_bQ_c$ be the pedal triangles of $P,Q$ WRT $\triangle ABC.$ Show that the side triangle of $\triangle P_aP_bP_c$ and $\triangle Q_aQ_bQ_c$ is perspective with $\triangle ABC$ with perspector the infinite point of $\perp PQ.$

See this link for a definition of the side triangle.

**jayme**

#2 Jun 6, 2009, 11:19 am

Dear Luis and Mathlinkers,
thanks for evacating so nice situation.
The case with two pedal triangle coming from two isogonal points has been evocated in my article

http://perso.orange.fr/jl.ayme vol. 2 Les deux points de Schroeter, see my "commentaire p. 5". (the proof is the same)

Sincerely
Jean-Louis

**Luis González**

#3 Jun 6, 2009, 12:33 pm

Thanks dear Jean-Louis, your article is very nice. I note that this triangle has lots of projective properties. I have a different approach to this one. Define the points

$$M \equiv AQ \cap P_bP_c, \ \ N \equiv AP \cap Q_bQ_c, \ \ K \equiv AQ \cap Q_bQ_c, \ \ L \equiv AP \cap P_bP_c.$$

Let $\triangle A'B'C'$ be the side triangle of $\triangle P_aP_bP_c$ and $\triangle Q_aQ_bQ_c$. It is clear that $AM$ and $AN$ are perpendicular to $P_bP_c$ and $Q_cQ_b$. Thus, $A'$ is the orthocenter of $\triangle AKL$. Then from $\triangle AQQ_c \sim \triangle APP_b$ and $\triangle AKQ_c \sim \triangle ALP_b$, it follows that $KL \parallel PQ$ as a consequence of Thales theorem. Therefore, $AA' \perp PQ$ and cyclically $BB', CC'$ are perpendicular to $PQ$ $\Longrightarrow \triangle ABC$ and $\triangle A'B'C'$ are perspective with perspector at the infinite point of the direction $\perp PQ$.

**drmzjoseph**

#4 Oct 31, 2015, 2:21 pm • 1

Let $\triangle A'B'C'$ be the side triangle of $\triangle P_aP_bP_c$ and $\triangle Q_aQ_bQ_c$.. Let $O$ be the center of the pedal circle, if $J\equiv OP_b \cap QQ_b, K\equiv OP_c \cap QQ_c, X\equiv Q_bP_c \cap Q_cP_b$ Now we apply Pascal's Theorem on $P_c,P_b,Q_c,Q_b,J,K \Rightarrow X\in OQ$ now by Brocard's Theorem $OX \perp AA' \Rightarrow PQ \perp AA'$ analogously $PQ \perp BB'$ and $PQ \perp CC'$ this is sufficient.

**Another proof**

Let $\triangle A'B'C'$ be the side triangle of $\triangle P_aP_bP_c$ and $\triangle Q_aQ_bQ_c$., denote $\mathcal{P}(X)$ the powe of $X$ WRT the pedal circle of $\{Q,P\}$.Let $W$ be the center of $\odot(QQ_bQ_cA)$
Since
$AA'^2+QA'^2=2QW^2+2WA'^2=2\mathcal{P}(A')+QA^2 \Rightarrow QA'^2-QA^2=2\mathcal{P}(A')-AA'^2=PA'^2-PA^2 \Rightarrow PQ\perp AA'$ analogously $PQ \perp BB'$ and $PQ \perp CC'$ this is sufficient.

*This post has been edited 1 time. Last edited by drmzjoseph, Oct 31, 2015, 2:32 pm*

Quick Reply

# High School Olympiads

## Side triangle (1)

Reply

**Luis González**

#1 Jun 5, 2009, 12:42 am

$P,Q$ are two points in the plane of $\triangle ABC$. Let $\triangle P_aP_bP_c$ and $\triangle Q_aQ_bQ_c$ be the cevian triangles of $P,Q$ WRT $\triangle ABC$. Sidelines $P_bP_c$ and $Q_bQ_c$ meet at $A'$ and define cyclically the points $B',C'$. Triangle $\triangle A'B'C'$ is known as the 'side triangle' of $\triangle P_aP_bP_c$ and $\triangle Q_aQ_bQ_c$. Show that $\triangle A'B'C'$ and $\triangle ABC$ are perspective and the perspector $J$ is the intersection of the perspectrices of $\triangle P_aP_bP_c$ and $\triangle Q_aQ_bQ_c$ WRT $\triangle ABC$.

**jayme**

#2 Jun 5, 2009, 10:37 am

Dear Luis and Mathlinkers,
this situation has been initiated by Schroeter.
You will find a synthetic proof on my website: http://perso.orange.fr/jl.ayme vol. 2 Les deux points de Schroeter.

for the median and orthic triangle of a triangle see p. 4
for a generalization p. 6
for the position of the perspector p. 14

In my paper, I haven't studied the case of the orthocenter and Gergonne point.
Sincerely
Jean-Louis

**Luis González**

#3 Jun 5, 2009, 10:37 pm

Thanks for the references dear Jean-Louis, I found that proof very nice. Mine uses barycentric coordinates in order to identify J for particular situations.

Let $(u_1:v_1:w_1)$ and $(u_2:v_2:w_2)$ be the barycentric coordinates of $P,Q$

$P_a\ (0:v_1:w_1)\ ,\ P_b\ (u_1:0:w_1)\ ,\ P_c\ (u_1:v_1:0)$

$Q_a\ (0:v_2:w_2)\ ,\ Q_b\ (u_2:0:w_2)\ ,\ Q_c\ (u_2:v_2:0)$

Perspectrices $p,q$ of $\triangle P_aP_bP_c$ and $\triangle Q_aQ_bQ_c$ WRT $\triangle ABC$ are the trilinear polars of $P$ and $Q$. Namely

$$p \equiv \frac{x}{u_1}+\frac{y}{v_1}+\frac{z}{w_1}=0\ \ ,\ \ q \equiv \frac{x}{u_2}+\frac{y}{v_2}+\frac{z}{w_2}=0$$

$$p \cap q \equiv J\ (u_1u_2(w_1v_2-v_1w_2):v_1v_2(w_1u_2-u_1w_2):w_1w_2(v_1u_2-u_1v_2))$$

Lines $P_bP_c$ and $Q_bQ_c$ intersect at $A'$ with coordinates

$$A'\ (u_1u_2(v_1w_2-w_1v_2):v_1v_2(w_1u_2-u_1w_2):w_1w_2(v_1u_2-u_1v_2))$$

By cyclic exchange of notation, we get the coordinates of $B',C'$

$$B'\ (u_1u_2(w_1v_2-v_1w_2):v_1v_2(u_1w_2-w_1u_2):w_1w_2(v_1u_2-u_1v_2))$$

$$C'\ (u_1u_2(w_1v_2-v_1w_2):v_1v_2(w_1u_2-u_1w_2):w_1w_2(u_1v_2-v_1u_2))$$

Therefore, $\triangle ABC$ and $\triangle A'B'C'$ are perspective through

$J'\ (u_1u_2(w_1v_2-v_1w_2):v_1v_2(w_1u_2-u_1w_2):w_1w_2(v_1u_2-u_1v_2)) \Longrightarrow J\equiv J'$

**yetti**

#4 Jun 6, 2009, 12:01 pm

Lemma (well known): Let the cross ratio of 4 collinear points $A, B, C, D$ be harmonic, i.e., $\frac{\overline{AC}}{\overline{AD}}\cdot\frac{\overline{BD}}{\overline{BC}}=-1.$ Then $\frac{\overline{CA}}{\overline{CM}}=\frac{\overline{CD}}{\overline{CB}}\Longleftrightarrow M$ is midpoint of $AB.$

For example: $(M,m),(N,n)$ are circles with diameters $AB=2m, MD=2n,$ respectively. Their radical axis cuts their center line at $C,$ because then $\frac{\overline{AC}}{\overline{AD}}\cdot\frac{\overline{BD}}{\overline{BC}}=\frac{m+\frac{m^2}{2n}}{m+2n}\cdot\frac{2n-m}{\frac{m^2}{2n}-m}=-1\Longleftrightarrow$ powers of $C$ to $(M),(N)$ are equal, $\overline{CA}\cdot\overline{CB}=\overline{CM}\cdot\overline{CD}\Longleftrightarrow\frac{\overline{CA}}{\overline{CM}}=\frac{\overline{CD}}{\overline{CB}}.$

---

Let $X_a\equiv P_bP_c\cap BC, X_b, X_c$ defined cyclically, and let $Y_a\equiv Q_bQ_c\cap BC, Y_b, Y_c$ defined cyclically. $X_aX_bX_c$ is perspetrix of $\triangle P_aP_bP_c, \triangle ABC$ and $Y_aY_bY_c$ is perspetrix of $\triangle Q_aQ_bQ_c, \triangle ABC.$ Project one perspetrix to infinity, say $Y_aY_bY_c,$ and mark all projected points with asterisks. This makes projected $\triangle Q^*_aQ^*_bQ^*_c$ medial triangle of the projected $\triangle A^*B^*C^*$ and $Q^*$ its centroid. Now drop the asterisks in the projected notation.

$Q_bQ_c\parallel BC$ cuts $P_bP_c$ at $A'.$ Lines $PC, P_bX_a, AB, X_bP_a$ concurrent at $P_c$ form a harmonic pencil. $Q_b$ is midpoint of $AC$ $\Longrightarrow$ (by the lemma) $\frac{\overline{P_bC}}{\overline{P_bQ_b}}=\frac{\overline{P_bX_b}}{\overline{P_bA}}\Longrightarrow\triangle AA'Q_c\sim\triangle X_bX_aC$ are centrally similar with similarity center $P_b\Longrightarrow$ $AA'\parallel X_bX_a\equiv X_aX_bX_c.$ Similarly, $BB'\parallel X_aX_bX_c, CC'\parallel X_aX_bX_c\Longrightarrow\triangle ABC,\triangle A'B'C'$ are perspective, their perspector is the infinite point of $X_aX_bX_c,$ the intersection of $X_aX_bX_c$ with the line at infinity, identical with $Y_aY_bY_c.$

Quick Reply

# High School Olympiads

## Prove <BHF = <ABC

Reply

**babylon**
#1 Jun 5, 2009, 12:15 am

In a triangle $ABC$ with $AC = BC$, point $M$ on side $AB$ is such that $AM = 2MB$ , $F$ is the midpoint of $BC$ and $H$ the foot of the perpendicular from $M$ to $AF$. Prove that $\angle BHF = \angle ABC$.

**msecco**
#2 Jun 5, 2009, 1:42 am

Let $BC = 2a$ and $AB = 3x$.So, $AM = 2x$ and $BM = x$.
Let too $< FAB = y$ and $CBA = z$.
In order to prove that $< BHF =< ABC$, we will prove that the triangles $HFB$ and $BFA$ are similar.
For this, we will prove that $BF^2 = AF.FH$, wich will implies the result from the similary case $SAS$.
We have to prove that:

$a^2 = AF^2 - AF.AH$ $(1)$
By the Sine's Law in the triangle $AFB$, we get: $AF = \frac{a.sinz}{siny}$.
In the right triangle $AHM$, we have: $AH = 2x.cosy$.
By the Sine's Law in the triangle $AFB$, we have: $x = \frac{a.sin(y+z)}{3siny}$.
Hence, $(1)$ is equivalent to: $\frac{sin^2z - sin^2y}{sin^2y} = \frac{2sinz.cosy.sin(y+z)}{3sin^2y}$
This is equivalent to: $3(sin^2z - sin^2y) = 2sinz.cosy.sin(y+z)$
It's well-known and easy to demonstrate that $(sin^2z - sin^2y = sin(z+y).sin(z-y)$.
Then, we have to prove that: $3sin(z-y) = 2sinzcosy$, wich is equivalent to: $2sin(z-y) = sin(z+y)$, but this last is obtained easily from the Sine's Law in the triangle $AFC$.
This concludes the proof.

**dgreenb801**
#3 Jun 5, 2009, 2:50 am

Anybody have a pure-geo proof?

**Luis González**
#4 Jun 5, 2009, 6:34 am

Let $G$ be the centroid of $\triangle ABC$. $AF$ cuts the circumcircle $(O)$ again at $P$. Since $AM = 2MB$ and $AG = 2GF$, it follows that $GM \parallel BC$. If $N$ is the midpoint of $AB$, then $GN$ and $MH$ meet at the orthocenter $K$ of $\triangle AGM$, which coincides with the orthocenter of $\triangle ABC$, due to $GM \parallel BC$. Since $F$ is equidistant from $P$ and the projection $H$ of the othocenter $K$ onto $AF$, we deduce that $PBHC$ is a parallelogram $\Longrightarrow \angle BHF = \angle CPA$, but $\angle CPA = \angle ABC \Longrightarrow \angle BHF = \angle ABC$.

**livetolove212**
#5 Jun 5, 2009, 8:43 am

> *babylon wrote:*
> In a triangle $ABC$ with $AC = BC$, point $M$ on side $AB$ is such that $AM = 2MB$ , $F$ is the midpoint of $BC$ and $H$ the foot of the perpendicular
> from $M$ to $AF$. Prove that $\angle BHF = \angle ABC$.

from $M$ to $AF$. Prove that $\angle BHF = \angle ABC$.

Let G be the centroid of triangle ABC, I be the midpoint of AB.
We have $\angle GHM = \angle GIM = 90^o$ then $GMIH$ is cyclic, but $GM//FB$ so $FBIH$ is cyclic.
We get $\angle FHB = \angle FIB = \angle CAB = \angle CBA$

**mathVNpro**

#6 Jun 5, 2009, 9:59 am

Let $G$ be the centroid of $\triangle ABC$, $S$ be midpoint of $AB$. Let $X \equiv BH \cap GM, Y \equiv BH \cap GS$. It is so obvious that $(AMSB) = -1 \implies (GA, GM, GS, GB) = -1 \implies (HXYB) = -1$
$\implies (HXYB) = (AMSB) = (MASB) = -1$, which implies that $AX, GS, MH$ are concurrent. Therefore $AX \perp GM \implies HXMA$ are concyclic, which implies that $\angle FHB \equiv \angle GHX = \angle GMA$. But $GM||BC \implies \angle GMA = \angle CBA$, which lead to the result that $\angle FHB = \angle CBA$.
Our proof is completed then.

Quick Reply

# Spain

## Cuadrilateros ciclicos

Reply

**americoperu**

#1 Sep 30, 2006, 9:10 am

Este es el problema 2 del TST Cono Sur Peru 2006:

$AA_1$ y $BB_1$ son las alturas de un triangulo acutangulo no isosceles $ABC$. $A_0$ y $B_0$ son los puntos medios de $BC$ y $CA$, respectivamente. El segmento $A_1B_1$ corta al segmento $A_0B_0$ en $C'$. Probar que $CC'$ es perpendicular a la recta que une el ortocentro y el circuncentro de $ABC$.
Pongan su solucion

**jonfv**

#2 Oct 19, 2006, 3:22 am

H: ortocentrro; O: circuncentro; P : inters. de HC y A1B1; Q: inters. de OC y A0B0.
Tenemos las siguientes semejanzas de triangulos:
HA1C ~ OB0C
PA1C ~ QB0C
las cuales implican la semejanza:
HCO ~ PCQ
En particular, HC es paralela a PQ; y, como C' es el ortocentro de PCQ, obtenemos lo pedido.

**Luis González**

#3 Jun 4, 2009, 11:49 pm

El problema es un caso particular de la siguiente configuración:

**Teorema:** En un $\triangle ABC$, sean $P, Q$ dos puntos isogonales en él. $\triangle P_aP_bP_c$ es el triangulo pedal de $P$ y $\triangle Q_aQ_bQ_c$ es el triangulo pedal de $Q$. Sea $A' = P_aP_b \cap Q_aQ_b$ y definimos ciclicamente $B', C'$. Entonces $AA', BB', CC'$ son perpendiculares a $PQ$.

$$M = AQ \cap P_bP_c \ , \ N = AP \cap Q_bQ_c \ , \ K = AQ \cap Q_bQ_c \ , \ L = AP \cap P_bP_c$$

Es claro en vista que $AM$ y $AN$ son perpendiculares a $P_bP_c$ y $Q_cQ_b$ que $A'$ es ortocentro de $\triangle AKL$ y por la semejanza de los triangulos $\triangle AQQ_c \sim \triangle APP_b$ y $\triangle AKQ_c \sim \triangle ALP_b$ deducimos que $KL \parallel PQ$ como consecuencia del teorema de Thales. Por ende, $AA' \perp PQ$ y analogamente $BB', CC'$ son perpendiculares a $PQ$.

Quick Reply

# Spain

## Concurrencia

Reply

**M4RI0**

#1 May 26, 2007, 6:18 am

Primero consideremos una semicircunferencia de diámetro $AB$. Trazemos las perpendiculares por $A$ y por $B$ a $AB$, además trazemos una tangente por un punto cualquiera $D$ ($D$ diferente de $A$ y $B$) de la circunferencia, esta tangente intersecta a la perpendicular por $A$ en $X$ y a la perpendicular por $B$ en $Y$. Demostrar que $BX$, $AY$ y la perpendicular desde $D$ hacia $AB$ son concurrentes.

**tipe**

#2 May 26, 2007, 9:13 am

En resumen la solución sería:

Sea $Z$ el punto de intersección de $BX$ y $AY$. Tenemos que $\frac{BZ}{ZX} = \frac{BY}{AX} = \frac{YD}{XD}$ entonces $DZ$ es paralelo a $BY$, de donde $DZ$ es perpendicular a $AB$.

$Tipe$

**M4RI0**

#3 May 26, 2007, 9:29 am

Además $Z$ es el punto medio de la perpendicular desde $D$ hacia $AB$, este es uno de los lemas de Arquímides.

**S. E. Puelma Moya**

#4 Jun 15, 2008, 7:23 am

> M4RI0 wrote:
> Además $Z$ es el punto medio de la perpendicular desde $D$ hacia $AB$, este es uno de los lemas de Arquímides.

Creo que este problema es bueno para quienes quieren practicar con el teorema de Thales y semejanzas de triángulos como técnicas para resolver problemas (en un nivel iniciante). Para establecer que Z es punto medio, ya no son necesarias la semicircunferencia ni los ángulos rectos, por causa del siguiente resultado:

Sea ABCD un trapecio, cuyas diagonales se intersecan en un punto O, y cuyos lados paralelos (bases) $\overline{AB}$ y $\overline{CD}$ tienen longitudes a y b, respectivamente. Considere una recta paralela a estas bases, que pasa por el punto O e interseca a los lados $\overline{AD}$ y $\overline{BC}$ en los puntos $P$ y $Q$, respectivamente. Entonces

$$PO = QO = \frac{ab}{a+b}$$

Demostración

Click to reveal hidden text

**Luis González**

#5 Jun 4, 2009, 9:57 pm

Aparte $BX$ y $AY$ concurren en el punto medio $M$ del segmento de perpendicular desde $D$ a $AB$. Lo que también produce este interesante resultado:

Si $D$ es un punto variable en una semicircunferencia de diámetro $AB$ y la tangente por $D$ corta a las perpendiculares a $AB$ por $A$ y $B$ en $X, Y$, siendo $M \equiv AY \cap BX$, entonces el lugar geométrico de $M$ es una elipse que tiene el círculo de diámetro $AB$ como círculo pedal.

Quick Reply

# High School Olympiads

## a concurrency problem

Reply

Source: I.F.Sharygin contest 2009 - Correspondence round - Problem 17

**April**

#1 May 31, 2009, 8:35 am • 1

Given triangle $ABC$ and two points $X, Y$ not lying on its circumcircle. Let $A_1$, $B_1$, $C_1$ be the projections of $X$ to $BC$, $CA$, $AB$, and $A_2$, $B_2$, $C_2$ be the projections of $Y$. Prove that the perpendiculars from $A_1$, $B_1$, $C_1$ to $B_2C_2$, $C_2A_2$, $A_2B_2$, respectively, concur if and only if line $XY$ passes through the circumcenter of $ABC$.

**yetti**

#2 Jun 3, 2009, 3:22 pm • 1

Let $X', X'' \in XY$ be arbitrary. Let $B_1', B_1'' \in CA$ and $C_1', C_1'' \in AB$ be pedals of $X', X''$. Let $P_2, P_2', P_2'' \in C_2A_2$ be pedals of $B_1, B_1', B_1''$ and $Q_2, Q_2', Q_2'' \in A_2B_2$ pedals of $C_1, C_1', C_1''$. Let $D \equiv P_1P_2 \cap Q_1Q_2$, $D' \equiv P_1'P_2' \cap Q_1'Q_2'$, $D'' \equiv P_1''P_2'' \cap Q_1''Q_2''$. We have

$$\frac{\overline{P_2P_2'} \cdot \overline{P_2A_2}}{\overline{P_2P_2''} \cdot \overline{P_2A_2}} = \frac{\overline{P_2P_2'}}{\overline{P_2P_2''}} = \frac{\overline{P_1P_1'}}{\overline{P_1P_1''}} = \frac{\overline{XX'}}{\overline{XX''}} = \frac{\overline{Q_1Q_1'}}{\overline{Q_1Q_1''}} = \frac{\overline{Q_2Q_2'}}{\overline{Q_2Q_2''}} = \frac{\overline{Q_2Q_2'} \cdot \overline{Q_2A_2}}{\overline{Q_2Q_2''} \cdot \overline{Q_2A_2}}$$

It follows that circumcircles of $(A_2P_2DQ_2), (A_2P_2'D'Q_2'), (A_2P_2''D''Q_2'')$ are coaxal, their centers are collinear. But their centers are midpoints of $A_2D, A_2D', A_2D'' \Longrightarrow D, D', D''$ are also collinear; locus of $D$ is a line $h_D$.

Midpoints of $AY, BY, CY$ are circumcenters of quadrilaterals $(AC_2YB_2), (BA_2YC_2), (CB_2YA_2) \Longrightarrow$ perpendiculars to $B_2C_2, C_2A_2, A_2B_2$ from $A, B, C$ concur at the isogonal conjugate $Z$ of $Y$ WRT $\triangle ABC \Longrightarrow$ perpendiculars to $B_2C_2, C_2A_2, A_2B_2$ from midpoints $A', B', C'$ of $BC, CA, AB$ concur at the complement $Z'$ of $Z$.

Let $n_A, n_B, n_C$ be perpendiculars to $B_2C_2, C_2A_2, A_2B_2$ from $A_1, B_1, C_1$, respectively. Let $D \equiv n_B \cap n_C$, $E \equiv n_C \cap n_A, F \equiv n_A \cap n_B$. Then $D, E, F$ are on 3 fixed lines $h_D, h_E, h_F$, which are obviously concurrent at the orthocenter $H_2$ of the pedal $\triangle A_2B_2C_2$. Medial $\triangle A'B'C'$ is pedal triangle WRT circumcenter $O$ of $\triangle ABC. O \in XY \Longrightarrow$ $h_D, h_E, h_F$ are also concurrent at the complement $Z'$ of the isogonal conjugate $Z$ of $Y$. Consequently, $h_D \equiv h_E \equiv h_F$ are identical and $n_A, n_B, n_C$ concurrent for all $X \in YO$.

**mathVNpro**

#3 Jun 3, 2009, 9:10 pm

> Quote:
>
> $$\frac{\overline{P_2P_2'} \cdot \overline{P_2A_2}}{\overline{P_2P_2''} \cdot \overline{P_2A_2}} = \frac{\overline{P_2P_2'}}{\overline{P_2P_2''}} = \frac{\overline{P_1P_1'}}{\overline{P_1P_1''}} = \frac{\overline{XX'}}{\overline{XX''}} = \frac{\overline{Q_1Q_1'}}{\overline{Q_1Q_1''}} = \frac{\overline{Q_2Q_2'}}{\overline{Q_2Q_2''}} = \frac{\overline{Q_2Q_2'} \cdot \overline{Q_2A_2}}{\overline{Q_2Q_2''} \cdot \overline{Q_2A_2}}$$
>
> It follows that circumcircles of $(A_2P_2DQ_2), (A_2P_2'D'Q_2'), (A_2P_2''D''Q_2'')$ are coaxal, their centers are collinear. But their centers are midpoints of $A_2D, A_2D', A_2D'' \Longrightarrow D, D', D''$ are also collinear; locus of $D$ is a line $h_D$.

Dear **Yetti**, sorry for my silly question but can you tell me clearer about this property? I notice it very nice and useful way to prove circles are coaxal.
Thank you very much for your explanation.

**yetti**

#4 Jun 3, 2009, 9:25 pm

Dear mathVNpro, it is not a silly question. When powers of 2 points to 2 given circles are in the same ratio, these 2 points are on a circle coaxal with the 2 given circles.

See http://www.mathlinks.ro/viewtopic.php?t=121062 for a proof.
See http://www.mathlinks.ro/viewtopic.php?t=121088, http://www.mathlinks.ro/viewtopic.php?t=190473 for nice applications.

**mathVNpro**
*#5 Jun 3, 2009, 9:39 pm*

Dear **Mr. Yetti**, thank you very much for your sincere help that you gave me. I am truly thank you for your careful explanation that you gave. By the way, do you have any file or ebook or any document realted to the above formula.
Thank you in advance.

**yetti**
*#6 Jun 3, 2009, 10:10 pm*

Actually, I do. But even in MS Word format (rather than PDF), it is larger than allowd attachment size - I cannot post it or send it by PM, only by e-mail.

**mathVNpro**
*#7 Jun 3, 2009, 10:15 pm*

Dear **Yetti**, it will be such great if you send me that file. I will send you my e-mail address by private massage.
Thank you very very much in advance.

**Luis González**
*#8 Jun 3, 2009, 10:30 pm*

> *April wrote:*
> Given triangle $ABC$ and two points $X, Y$ not lying on its circumcircle. Let $A_1, B_1, C_1$ be the projections of $X$ to $BC$, $CA, AB$, and $A_2, B_2, C_2$ be the projections of $Y$. Prove that the perpendiculars from $A_1, B_1, C_1$ to $B_2C_2, C_2A_2$, $A_2B_2$, respectively, concur if and only if line $XY$ passes through the circumcenter of $ABC$.

This configuration calls for the 2nd Fontene theorem. What do you think Vladimir?

**yetti**
*#9 Jun 3, 2009, 10:45 pm*

This was the first thing I tried but I did not see a way. BTW, I did not prove "if and only if". I almost got it by contradiction, but could not complete the last step in a satisfactory way. Any idea ?

**yetti**
*#10 Jun 4, 2009, 12:36 pm*

OK, I got a simple contradiction for "if and only if."

> *yetti wrote:*
> Let $X', X'' \in XY$ be arbitrary. Let $B_1', B_1'' \in CA$ and $C_1', C_1'' \in AB$ be pedals of $X', X''$. Let $P_2, P_2', P_2'' \in C_2A_2$ be pedals of $B_1, B_1', B_1''$ and $Q_2, Q_2', Q_2'' \in A_2B_2$ pedals of $C_1, C_1', C_1''$. Let $D \equiv P_1P_2 \cap Q_1Q_2$, $D' \equiv P_1'P_2' \cap Q_1'Q_2', D'' \equiv P_1''P_2'' \cap Q_1''Q_2''$. We have
>
> $$\frac{\overline{P_2P_2'} \cdot \overline{P_2A_2}}{\overline{P_2P_2''} \cdot \overline{P_2A_2}} = \frac{\overline{P_2P_2'}}{\overline{P_2P_2''}} = \frac{\overline{P_1P_1'}}{\overline{P_1P_1''}} = \frac{\overline{XX'}}{\overline{XX''}} = \frac{\overline{Q_1Q_1'}}{\overline{Q_1Q_1''}} = \frac{\overline{Q_2Q_2'}}{\overline{Q_2Q_2''}} = \frac{\overline{Q_2Q_2'} \cdot \overline{Q_2A_2}}{\overline{Q_2Q_2''} \cdot \overline{Q_2A_2}}$$
>
> It follows that circumcircles of $(A_2P_2DQ_2), (A_2P_2'D'Q_2'), (A_2P_2''D''Q_2'')$ are coaxal, their centers are collinear. But their centers are midpoints of $A_2D, A_2D', A_2D'' \Longrightarrow D, D', D''$ are also collinear; locus of $D$ is a line $h_D$.
>
> Midpoints of $AY, BY, CY$ are circumcenters of quadrilaterals $(AC_2YB_2), (BA_2YC_2), (CB_2YA_2) \Longrightarrow$ perpendiculars to $B_2C_2, C_2A_2, A_2B_2$ from $A, B, C$ concur at the isogonal conjugate $Z$ of $Y$ WRT $\triangle ABC \Longrightarrow$ perpendiculars to $B_2C_2, C_2A_2, A_2B_2$ from midpoints $A', B', C'$ of $BC, CA, AB$ concur at the complement $Z'$ of $Z$.
>
> Let $n_A, n_B, n_C$ be perpendiculars to $B_2C_2, C_2A_2, A_2B_2$ from $A_1, B_1, C_1$, respectively. Let $D \equiv n_B \cap n_C$, $E \equiv n_C \cap n_A, F \equiv n_A \cap n_B$. Then $D, E, F$ are on 3 fixed lines $h_D, h_E, h_F$, which are obviously concurrent at the

orthocenter $H_2$ of the pedal $\triangle A_2B_2C_2$. Medial $\triangle A'B'C'$ is pedal triangle WRT circumcenter $O$ of $\triangle ABC$. $O \in XY$ $\Longrightarrow h_D, h_E, h_F$ are also concurrent at the complement $Z'$ of the isogonal conjugate $Z$ of $Y$. Consequently, $h_D \equiv h_E \equiv h_F$ are identical and $n_A, n_B, n_C$ concurrent for all $X \in YO$.

Assume $O \notin XY$ and perpendiculars $n_A, n_B, n_C$ to $B_2C_2, C_2A_2, A_2B_2$ from $A_1, B_1, C_1$ concur at $D \equiv E \equiv F$. This means that the 3 fixed lines $h_D, h_E, h_F$ through $H_2, D \equiv E \equiv F$ coincide. Let $X' \in XY$ be arbitrary and let $A_1' \in BC, B_1' \in CA, C_1' \in AB$ be its pedals. Since $h_D \equiv h_E \equiv h_F$, it follows that perpendiulars $n'_A, n'_B, n'_C$ to $B_2C_2, C_2A_2, A_2B_2$ from $A_1', B_1', C_1'$ also concur at $D' \in H_2D$. Consider now line $X'O$. Midpoints $A', B', C'$ of $BC, CA, AB$ are pedals of $O$ and perpendiculars $m_A, m_B, m_C$ to $B_2C_2, C_2A_2, A_2B_2$ from $A', B', C'$ concur at $Z'$. It follows that all points $X'' \in X'O$ also have the same property: perpendiculars $n''_A, n''_B, n''_C$ to $B_2C_2, C_2A_2, A_2B_2$ from the pedals of $X''$ concur at $D'' \in D'Z'$. Consequently, all points $X$ in the plane have the same property. This includes the triangle vertices $A, B, C$ in place of $X$. Pedal triangles of $A, B, C$ WRT $\triangle ABC$ degenerate to its altitudes $AA_1, BB_1, CC_1$. Let $X \equiv A$. Perpendiculars to $C_2A_2, A_2B_2$ from $B_1 \equiv C_1 \equiv A$ meet at $A$. Perpendicular to $B_2C_2$ from $A_1$ going through their intersection $A$ is identical with the A-altitude of $\triangle ABC \Longrightarrow (B_2C_2 \parallel BC) \perp AA_1$. Similarly, $C_2A_2 \parallel CA, A_2B_2 \parallel AB$ $\Longrightarrow Y \equiv O$, which is a contradiction.

Quick Reply

# High School Olympiads

## Two congruent triangles having same nine point circle

Reply

**Source**: ILL 1992/3

**Moonmathpi496**

#1 Jun 3, 2009, 11:28 am

Let $ABC$ be a triangle, $O$ its circumcenter, $S$ its centroid, and $H$ its orthocenter. Denote by $A_1$, $B_1$ and $C_1$ the centers of the circles circumscribed about the triangles $CHB, CHA$ and $AHB$ respectively.

Prove that the triangle $ABC$ is congruent to the triangle $A_1B_1C_1$ and that the nine-point circle of $ABC$ is also the nine-point circle of $A_1B_1C_1$.

**mathVNpro**

#2 Jun 3, 2009, 2:24 pm

Let $A_0, B_0, C_0$ respectively be the midpoints of $BC, CA, AB$. Consider the homothety with center $G$, ratio $k=-\frac{1}{2}$. We have $\mathcal{H}(G,k):\triangle ABC\mapsto\triangle A_0B_0C_0$, which implies that $\triangle ABC\sim\triangle A_0B_0C_0$. Now consider the homothety with center $O$, ratio $r=2$. We have $\mathcal{H}(O,r):\triangle A_0B_0C_0\mapsto\triangle A_1B_1C_1$, which implies that $\triangle A_1B_1C_1\cong\triangle A_0B_0C_0\Longrightarrow\triangle A_1B_1C_1\sim\triangle ABC$. Let $I$ be center of homothety that turns $\triangle ABC\mapsto\triangle A_1B_1C_1$. It is easy to see that $\mathcal{H}_I=\mathcal{H}_O\circ\mathcal{H}_G\Longrightarrow$ the ratio of this homothety is $-1$ and $I\in OG$- which is the $Euler$ line wrt $\triangle ABC$, further, $I$ is also the center of the $9$- point circle $(\mathcal{E})$ wrt $\triangle ABC$. Or in other word, $\mathcal{H}(I,-1)$ is a symmedian through $I$, let me denote it by $\mathcal{S}(I)$. It is so obvious that $\mathcal{S}(I):(\mathcal{E})\mapsto(\mathcal{E})$. Therefore $(\mathcal{E})$ is also the $9$- point circle wrt $\triangle A_1B_1C_1$.
Our proof is completed then.

**REMARK-** In my solution, I have considered $G\equiv S$.

*This post has been edited 3 times. Last edited by mathVNpro, Jun 3, 2009, 8:53 pm*

**Mathias_DK**

#3 Jun 3, 2009, 5:17 pm

> *Moonmathpi496 wrote:*
> Let $ABC$ be a triangle, $O$ its circumcenter, $S$ its centroid, and $H$ its orthocenter. Denote by $A_1$, $B_1$ and $C_1$ the centers of the circles circumscribed about the triangles $CHB, CHA$ and $AHB$ respectively.
>
> Prove that the triangle $ABC$ is congruent to the triangle $A_1B_1C_1$ and that the nine-point circle of $ABC$ is also the nine-point circle of $A_1B_1C_1$.

For any point, let the lowercase letter denote the complex number corresponoding to it. Let $O$ be the center of the complex plane.
Then $|a|=|b|=|c|=R$, where $R$ are the circumradius of the triangle.
Since $h-2o=a+b+c$ and $o=0$ we see: $h=a+b+c$.
Since the circumcenter of the triangle with points in $(-a,-b,c)$ must be $O$ (since $|-a|=|-b|=|c|=R$) the circumcenter of a triangle with points $(-a+a+b,-b+a+b,c+a+b)=(b,a,a+b+c)=(b,a,h)$ must be $a+b$. But then $c_1=a+b$ and likewise $a_1=b+c, b_1=a+c$.
Let $O_9$ be the nine point circle. It is wellknown that $2o_9=o+h=a+b+c$. Consider the function $f(z)=2o_9-z$, which is a reflection around $O_9$. Since $2o_9=a+b+c$ we see that $f(a)=a_1, f(b)=b_1, f(c)=c_1$. Hence $\triangle A_1B_1C_1$ is obtained by reflection $\triangle ABC$ around $O_9$. From here it is obvious that $\triangle ABC$ is congruent to $\triangle A_1B_1C_1$ and that it has the same nine point circle :)

**Moonmathpi496**

#4 Jun 3, 2009, 8:03 pm

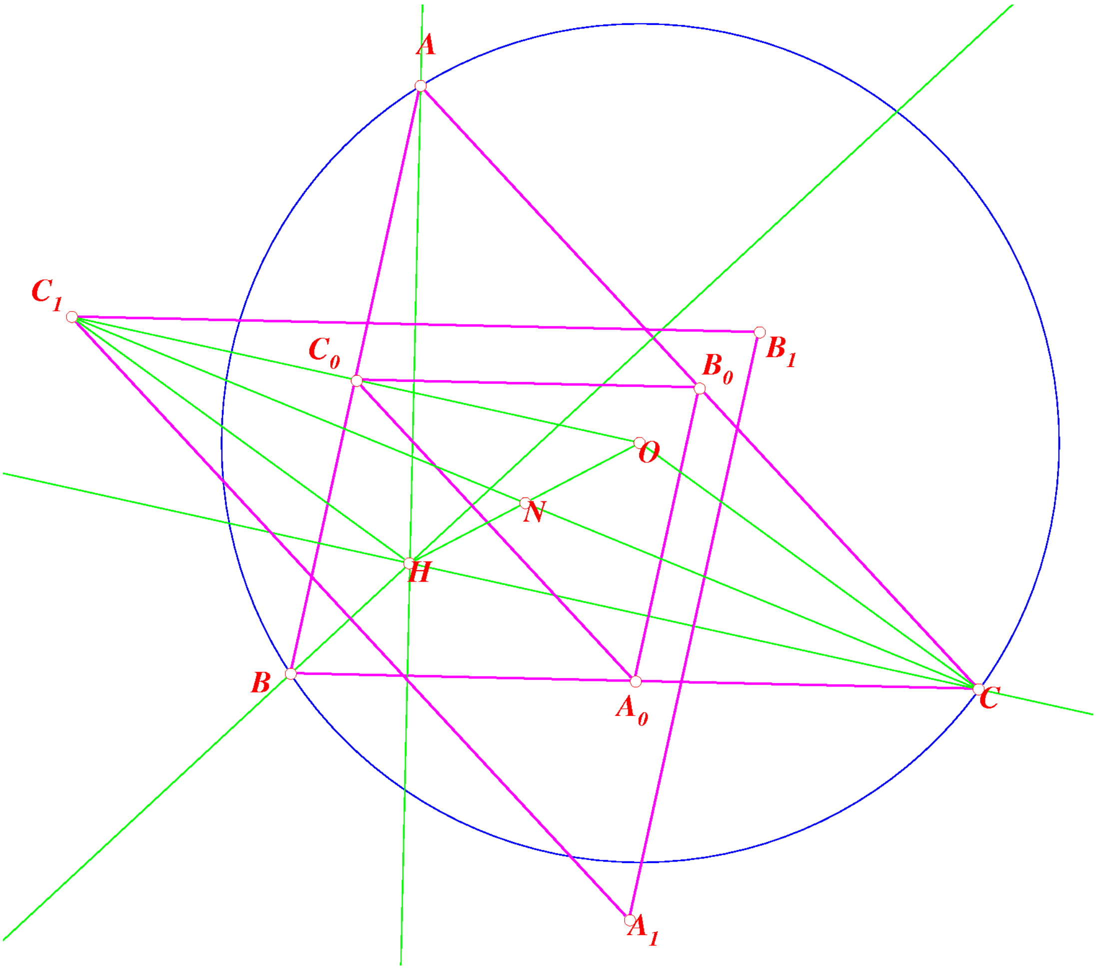
> Moonmathpi496 wrote:
> Let $ABC$ be a triangle, $O$ its circumcenter, $S$ its centroid, and $H$ its orthocenter. Denote by $A_1, B_1$ and $C_1$ the centers of the circles circumscribed about the triangles $CHB, CHA$ and $AHB$ respectively.
>
> Prove that the triangle $ABC$ is congruent to the triangle $A_1B_1C_1$ and that the nine-point circle of $ABC$ is also the nine-point circle of $A_1B_1C_1$.

My solution:
Let $\omega$ be the circumcircle of $\triangle ABC$. It is well known that the reflection of $H$ on $BC$ lies on $\omega$. This reflection maps $O \to A_1$. Similarly for $B_1$ and $C_1$. So, A homothety with center $O$ and ratio $\frac{1}{2}$ sends $\triangle A_1B_1C_1 \to \triangle A_0B_0C_0$. Where $A_0B_0C_0$ is the medial triangle of $\triangle ABC$. So, $\triangle A_1B_1C_1 \cong \triangle ABC$
Now it is clear the radius of the nine point circles of $\triangle ABC$ and $\triangle A_1B_1C_1$ are equal. So it is enough to show that the nine point center of the triangles are the same.

The nine point center $N$, of $\triangle ABC$ is the midpoint of segment $HO$. $O$ is the orthocenter of $\triangle A_0B_0C_0$, so it is also the orthocenter $\triangle A_1B_1C_1$. We shall show that $H$ is the circumcenter of $\triangle A_1B_1C_1$. We have $CH \parallel C_1O$ also, $CH = C_1O$. So $C_1HCO$ is a parallelogram and $C_1H = CO = R$. Analogously we can prove this for the other vertices. Which implies that $H$ is the circumcenter of $\triangle A_1B_1C_1$. From which the result follows.

$\mathbb{QED}$

*Attachments:*




**Luis González**

*#5 Jun 3, 2009, 8:32 pm*

Homothety $\mathcal{H}(G, -\frac{1}{2})$ takes $\triangle ABC$ into its medial triangle $\triangle A'B'C'$ and homothety $\mathcal{T}(O, 2)$ takes $\triangle A'B'C'$ into the triangle formed by the circumcenters of $\triangle HBC, \triangle HCA$ and $\triangle HAB$, i.e. reflections of $O$ about $BC, CA, AB$. The coefficient of composition $\mathcal{H} \circ \mathcal{T}$ is $-1$, which implies that $\triangle ABC$ and $\triangle A_1B_1C_1$ are centrally symmetric, and the circumcenter of one them is the orthocenter of the other $\Longrightarrow \triangle ABC$ and $\triangle A_1B_1C_1$ have the same nine-point circle.

**Moonmathpi496**

#6 Jun 3, 2009, 9:51 pm

Luis, does it follow directly from the homothety relations or should we prove like I did in my solution?
I mean when writing formal solution (like in a contests) how should we describe that the circumcenter of one , the orthocenter of the another ?

**Luis González**

#7 Jun 3, 2009, 10:08 pm

> Moonmathphi496 wrote:
> how should we describe that the circumcenter of one, the orthocenter of the another ?

Homothety between triangles takes orthocenters into orthocenters, centroids into centroids, circumcenters into circumcenters, etc. Note that the circumcenter of ABC is the orthocenter of its medial triangle.

**Agr_94_Math**

#8 Jun 4, 2009, 11:16 am

In Moonmathpi's solution, I do not understnad some things 1.why the proerty of reflection of orthocentres map $O$ to $A_1$.
2. Why should it be because of the above a homothety of centre $O$ and ratio $-1/2$ map $A_1B_1C_1$ to $A_0B_0C_0$?
3. WHy does 2. imply that the trinagle $ABC$ will be congruent to $A_1B_1C_1$? If the medial trinagle of a particular trinagle is homothetic to some trinagle by ratio $1/2$, does it mean the other tinagle is congruent to the original one?
I am sorry for these doubts but it will be really helpful if someone helps me out.
Thanks in advance. Please dont ignore this.

**mathVNpro**

#9 Jun 4, 2009, 4:00 pm

I think that the main idea in Moonmathpi's solution is the same as mine proof- the first one. You can see the answer for all you question in my solution.
Best regard

**Moonmathpi496**

#10 Jun 4, 2009, 4:23 pm

> Agr_94_Math wrote:
> In Moonmathpi's solution, I do not understnad some things 1.why the proerty of reflection of orthocentres map $O$ to $A_1$.
> 2. Why should it be because of the above a homothety of centre $O$ and ratio $-1/2$ map $A_1B_1C_1$ to $A_0B_0C_0$?
> 3. WHy does 2. imply that the trinagle $ABC$ will be congruent to $A_1B_1C_1$? If the medial trinagle of a particular trinagle is homothetic to some trinagle by ratio $1/2$, does it mean the other tinagle is congruent to the original one?
> I am sorry for these doubts but it will be really helpful if someone helps me out.
> Thanks in advance. Please dont ignore this.

Here are the answers to your questions:
1. Let the reflection of $H$ on $BC$ be $D$. Then It is very well known (or just prove that these two triangles are congruent as they have two equal angles and an equal side $BC$) that $D$ lies on $BC$. It implies that the circumcircle of $\triangle BHC$ and $\triangle BDC$ (and so of $\triangle ABC$ are equal and their center lies on midperpendicular of $BC$. So, $O \rightarrow A_1$.

2. It is just clear from the reflection. Just have a look at the diagram carefully.

3. Yes. Simply homothety transforms figures to similar figures and the only change (apart from displacement is the length of the sides). And here each side of medial triangle is half of the original and triangle. And then you know that of you double the side lengths you get a another one. So they are directly congruent.

I hope that I have successfully answered your questions. :)

**Moonmathpi496**

#11 Jun 4, 2009, 4:33 pm

> Luis González wrote:
> Homothecy between triangles takes orthocenters into orthocenters, centroids into centroids, circumcenters into

Homothecy between triangles takes orthocenters into orthocenters, centroids into centroids, circumcenters into circumcenters, etc. Note that the circumcenter of ABC is the orthocenter of its medial triangle.

Thanks I just forgot that. The last part of my solution is completely useless. It could certainly be made far shorter.

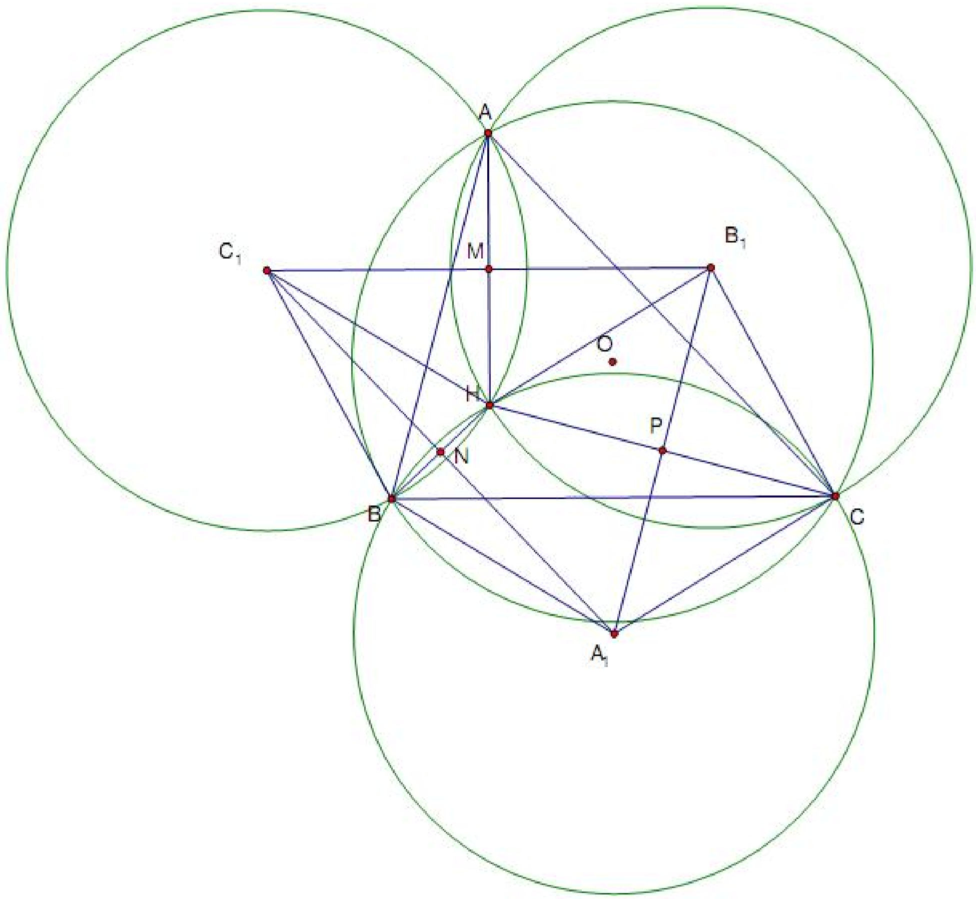
**livetolove212**

*#12 Jun 6, 2009, 3:01 pm*

> *Moonmathpi496 wrote:*
>
> Let $ABC$ be a triangle, $O$ its circumcenter, $S$ its centroid, and $H$ its orthocenter. Denote by $A_1, B_1$ and $C_1$ the centers of the circles circumscribed about the triangles $CHB, CHA$ and $AHB$ respectively.
>
> Prove that the triangle $ABC$ is congruent to the triangle $A_1B_1C_1$ and that the nine-point circle of $ABC$ is also the nine-point circle of $A_1B_1C_1$.

We have $2R_{AHB}=\frac{AH}{sin\angle ABH}=\frac{AH}{sin\angle ACH}=2R_{AHC}$
Similarly we get $R_{AHB}=R_{AHC}=R_{BHC}$
On the other hand, $\angle C_1BC+\angle B_1CB=\angle C_1BH+\angle HBC+\angle HCB+\angle B_1CH$
$=\angle HBA_1+\angle HCA_1+180^o-\angle BHC=360^o-\angle BHC-\angle BA_1C+180^o-\angle BHC=180^o$
Therefore $BC_1//CB_1$ and $BC_1=CB_1$ so $BCB_1C_1$ is a parallelogram $\Rightarrow BC=B_1C_1$
Similarly we get $AC=A_1C_1, AB=A_1B_1$ then $\Delta ABC=\Delta A_1B_1C_1$
But $M,N,P$ are the midpoints of $AH,BH,CH$, also are midpoint of $B_1C_1,C_1A_1,A_1B_1$ then $(MNP)$ is the 9 points circle of triangle $ABC$ and $A_1B_1C_1$

*Attachments:*




Quick Reply

# High School Olympiads

## Areas and Isogonal conjugate

Reply

**Ponclete**

#1 Jun 2, 2009, 6:40 pm

Given a triangle ABC and an arbitrary point P. Let A',B',C' be the reflection of the point P wrt BC,CA,AB, and Q be the isogonal conjugate wrt triangle A'B'C'. Let A",B",C" be the reflection of the point Q wrt B'C',C'A',A'B'.
Prove that [A'B'C']²=[ABC][A"B"C"],where [ABC] is area of triangle ABC.

**Luis González**

#2 Jun 3, 2009, 12:18 am

From the cyclic quadrilaterals $QA''B'C''$, $QB''A'C''$ and $PB'CA'$, we obtain

$\angle A''C''Q = \angle A''B'Q = \angle PB'A' = \angle PCA'$

$\angle B''C''Q = \angle B''A'Q = \angle PA'B' = \angle PCB' \Longrightarrow \angle ACB = \angle A''C''B''$

Mutatis mutandis $\angle ABC = \angle A''B''C'' \Longrightarrow \triangle ABC \sim \triangle A''B''C''$

Perpendicular lines through $B, C$ to $A'B'$ and $A'C'$ meet at the isogonal congugate $P'$ of $P$ WRT $\triangle ABC \Longrightarrow$ $\angle QC''B'' = \angle PCB$ and $\angle QB''C'' = \angle PBC$, which implies that $\triangle QB''C''$ and $\triangle PBC$ are homothetic, since $PC, PB$ are parallel to $QC'', QB''$. By similar reasoning, we conclude that $\triangle A''B''C''$ and $\triangle ABC$ are homothetic. Now, according to Gergonne-Ann theorem, $[\triangle A'B'C']^2 = [\triangle ABC] \cdot [\triangle A''B''C'']$.

Quick Reply

# Spain

## 1/PX = 1/PY + 1/PZ

Reply

**ilarrosa**

#1 Aug 27, 2005, 4:21 am

Este problema no es de una olimpiada, sino de las opsiciones para el ingreso en el cuerpo de profesores de enseñanza secundaria. Lo pongo aquí, porque creo que debe ser de ese nível.

"Una línea recta que pasa por el incentro de un triángulo ABC corta a los lados AB y AC en los puntos D y E respectivamente. Sea P el punto de intersección de BE y CD.
Si X, Y y Z son los respectivos pies de las perpendiculares desde P a BC, CA y AB, demuestra que:
1/PX = 1/PY + 1/PZ"

**Luis González**

#2 Jun 2, 2009, 9:48 am

Usamos coordenadas trilineales con respecto a $\triangle ABC$. Sean $(p,q,r)$ las coordenadas de un punto arbitrario $Q$. Así, la recta $\ell$ que pasa por $Q$ y $I \equiv (1:1:1)$ tiene ecuación $(r-q)\alpha+(p-r)\beta+(q-p)\gamma=0$. Entonces

$$D \equiv \ell \cap AB \equiv (r-p:r-q:0) \Longrightarrow \ CD \equiv (r-q)\alpha-(r-p)\beta=0$$

$$E \equiv \ell \cap AB \equiv (q-p:0:q-r) \Longrightarrow \ BE \equiv (q-r)\alpha-(q-p)\gamma=0$$

$$P \equiv BE \cap CD \equiv (-(p-r)(p-q):-(r-q)(p-q):-(p-r)(q-r))$$

La suma de los reciprocos de la segunda y tercera coordenada de $P$ es igual al reciproco de la primera. Siendo éstas directamente proporcionales a las distancias de $P$ a sus correspondientes lados, entonces el resultado se sigue.

Quick Reply

# High School Olympiads

## special tetrahedron; prove that S'A' = S'B' = S'C'

Reply

Source: All-Russian Olympiad 2006 finals, problem 11.6

**darij grinberg**

#1 May 6, 2006, 5:02 pm

Consider a tetrahedron $SABC$. The incircle of the triangle $ABC$ has the center $I$ and touches its sides $BC$, $CA$, $AB$ at the points $E$, $F$, $D$, respectively. Let $A'$, $B'$, $C'$ be the points on the segments $SA$, $SB$, $SC$ such that $AA' = AD$, $BB' = BE$, $CC' = CF$, and let $S'$ be the point diametrically opposite to the point $S$ on the circumsphere of the tetrahedron $SABC$. Assume that the line $SI$ is an altitude of the tetrahedron $SABC$. Show that $S'A' = S'B' = S'C'$.

**Luis González**

#2 Jun 2, 2009, 7:33 am

Let $\mathcal{T}_a, \mathcal{T}_b, \mathcal{T}_c$ be the spheres with centers $A, B, C$ passing through $(D, F, A'), (D, E, B'), (E, FC')$. $I$ is the radical center of the circles centered at $A, B, C$ passing through $(F, D), (D, E), (E, F)$. Therefore, the perpendicular line to the face $ABC$ through $I$ is the radical axis of $\mathcal{T}_a, \mathcal{T}_b, \mathcal{T}_c \Longrightarrow$ Powers of $S$ to $\mathcal{T}_a, \mathcal{T}_b, \mathcal{T}_c$ are equal to $k^2$

$SB^2 - (BB')^2 = SC^2 - (CC')^2 = SA^2 - (AA')^2 = k^2$

Power of $B$ with respect to the circumsphere $(O, R)$ is $R^2 - (B'O)^2 = BB' \cdot B'S$

Note that in the triangle $\triangle B'SS'$, the segment $B'O$ is the median issuing from $B'$. Thus

$(B'O)^2 = \frac{1}{2}(SB')^2 + \frac{1}{2}(S'B')^2 - \frac{1}{4}(2R)^2$

Substituting $(BO')^2$ from the previous expression yields:

$(S'B')^2 = 4R^2 - (SB')^2 - 2BB' \cdot B'S$

$(S'B')^2 = 4R^2 - SB^2 - (BB')^2 + 2BS \cdot BB' - 2BB' \cdot B'S$

$(S'B')^2 = 4R^2 - SB^2 - (BB')^2 + 2BB'(BS - B'S)$

$(S'B')^2 = 4R^2 - SB^2 + (BB')^2$

$(S'B')^2 = 4R^2 - k^2$

Since this latter expression is independent of the chosen vertex, we conclude that:

$S'A' = S'B' = S'C' = \sqrt{4R^2 - k^2}$

Quick Reply

# High School Olympiads

**Unidranix**
#1 Jun 1, 2009, 11:23 am

Given 3 line segment with the length respectively of $a, b, c$. With ruler and compasses, draw a line segment with the length of $x$ such that:
$\frac{1}{x^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

**stephencheng**
#2 Jun 1, 2009, 12:47 pm

> *Unidranix wrote:*
> Given 3 line segment with the length respectively of $a, b, c$. With ruler and compasses, draw a line segment with the length of $x$ such that:
> $\frac{1}{x^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

At first introduce a basic skill in this kind of construction.
[geogebra]0aae5d907bd8dab56b0254ef2080897cda2eb4c1[/geogebra]

Given Points $A, D, C$ we can find Point $B$ (by drawing perpendicular ($L_1$) to $AC$ at $D$ and using mid-point of $AC$ as center and $\frac{AC}{2}$ as radius to draw circle $C_1$ and then the intersection of $L_1$ and $C_1$ is $B$) .

Given Points $A, B, D$ we can find $C$ (by drawing perpendicular ($L_2$) to $AB$ at $B$ and then the intersection of $L_2$ and $AD$ is $C$).

And by similar triangle $pr = q^2$

**Solution:**
WLOG, we can assume that $a = 1$,

Then we can construct line segments of $b^2$ (set $p = 1$ and $q = b$) and $c^2$ (set $p = 1$ and $q = c$) .

So we can construct line segments of $\frac{1}{b^2}$ (set $p = b^2$ and $q = 1$) and $\frac{1}{c^2}$ (set $p = c^2$ and $q = 1$) and thus get a line segment of length $\frac{1}{x^2}$ and thus that of length $x^2$ (set $p = \frac{1}{x^2}$ and $q = 1$)

Then we can get a line segment of length $x$ by setting $p = 1$ and $r = x^2$.

**Luis González**
#3 Jun 2, 2009, 1:18 am

Use the the following property of the right-angled triangle:

If $z$ is the altitude on the hypotenuse of the right-angled triangle with legs $b, c$, then

$$\frac{1}{z^2} = \frac{1}{b^2} + \frac{1}{c^2}$$

Now, construct the right triangle with legs $b, c$, draw the altitude $z$ on the hypotenuse and construct the right triangle with legs $z, a$. The altitude of this triangle is the required $x$.

Quick Reply

# High School Olympiads

## Construct a bicentric quadrilateral



**Source**: I.F.Sharygin contest 2009 - Correspondence round - Problem 22

**April**
#1 May 31, 2009, 8:45 am

Construct a quadrilateral which is inscribed and circumscribed, given the radii of the respective circles and the angle between the diagonals of quadrilateral.

**Ahiles**
#2 Jun 1, 2009, 8:06 pm

Let $O$ be the center of the circumcircle. From $OI=\sqrt{R+r^2-r\sqrt{4R^2+r^2}}$, we can construct the point $I$ - the incenter.

If $P$ is the intersection of the diagonals, then $OP=\frac{2R^2d}{R^2+d^2}$ (proved here}). So, the point $P$ is also constructible. Points $P$, $I$, $O$ lie on a line in this order.

If $K$ and $L$ are midpoints of the diagonals $AC$ and $BD$, then $OL\perp BD$ and $OK\perp AC$, and we deduce, that quadrilateral $OLKP$ is inscribed in a circle $\omega$ with diameter $PO$ (this circle can be constructed). Points $K$, $L$ and $I$ lie on the Newton's line. We need to construct a line through $I$, which cuts $\omega$ at points $K$ and $L$, such that $\angle KPL=\alpha$, where $\alpha$ - the given angle betwwen diagonals. If $M$ is midpoint of $PO$, then $\angle KML=2\alpha$, and our problem is to construct the isosceles triangle $MKL$ with known vertex $M$, angle $\angle KML$, length of the side $KM$ and point $I$ on the side $KL$.

Apply Stewart in $\triangle KML$ for $MI$ and get

$$MI^2=\frac{MK^2\cdot IL+ML^2\cdot IK-IK\cdot IL\cdot KL}{KL}=MK^2-IK\cdot IL=MK^2-IK(KL-IK)=$$

$$=MK^2+IK^2-IK\cdot KL\Longleftrightarrow IK^2-IK\cdot KL-MI^2=0$$

We solve the quadratic $IK$. Note that $KL=2MK\sin\alpha$. Because the construction is possible, the equation has two roots. But just one satisfies us because $x_1\cdot x_2=-MI^2<0$. So we can find $IK$. We construct the circle $\Omega$ with the center $I$ and radius $IK$. Clearly, circles $\omega$ and $\Omega$ intersect at $K$ (there exist two such $K$s, we''l continue with one of them). The line $KI$ meets $\omega$ at $L$.

Line $PL$ meets the circumcircle at $B$ and $D$, but line $PK$ at points $A$ and $C$.

The construction is finished.

**Luis González**
#3 Jun 1, 2009, 9:55 pm

Let $A,B,C,D$ be the vertices of the wanted quadrilateral with incircle $(I)$ and circumcircle $(O)$. According to *Fuss relation* (mentioned by Ahiles), distance $IO$ is constructible, then we can place the incircle and the circumcircle. According to Poncelet porism, $O,I$ and diagonal intersection $P\equiv AC\cap BD$ are fixed for all quadrilaterals with incircle $(I)$ and circumcircle $(O)$ $\Longrightarrow P$ is constructible. Let $M,N$ the orthogonal projections of $O$ on $AC,BD$, it follows that $OM$ and $ON$ are perpendicular bisectors of $AC,AB$ and $MN\equiv n$ is the Newton line of the quadrangle $ABCD$ passing through its incenter $I$. Hence it suffices to draw the line $n$ passing through $I$ such that $\angle MPN$ is given. If $K$ is center of the circle with diameter $OP$, then central angle $\angle MKN$ is known as well $\Longrightarrow$ Isosceles triangle $\triangle MKN$ is constructible. Placing the constructed chord $MN$ in the circle $(K)$ passing through $I$ completes the quadrilateral $ABCD$ easily.

# Spain

una cosita de la linea de Simson

Reply

**Line**

#1 Jan 8, 2007, 4:01 am

sea $P$ un punto sobre el circuncirculo de un $\triangle ABC$, sean $L$, $M$ y $N$ las proyecciones ortogonales de $P$ sbre $\overline{BC}$, $\overline{AC}$ y $\overline{AB}$. Sean $\overline{PL}=l$, $\overline{PM}=m$, $\overline{PN}=n$ y $a$, $b$, $c$ los lados del triangulo inscrito, probar que $a\cdot m\cdot n=b\cdot l\cdot n+c\cdot l\cdot m$

sorry pero soy nuevo en esto

**conejita**

#2 Jan 8, 2007, 10:32 pm

Bueno, aqui te pongo mi solucion. (sorry por la ausencia de latex) jeje

Dividiendo entre lmn obtenemos: a/l= b/m+c/n.
Ahora, tenemos que (al)/2=(PBC) ---> l=2(PBC)/a
Analogamente tenemos m= 2(PAC)/b y n=2(PAB)/c
Sustituimos en lo que queremos mostrar, entonces:
a^2/(PBC)= b^2/(PAC) + c^2/(PAB)
Ahora, calculamos (PBC)= PB*PC(sen A)/2 , (PAC)= b*PC(sen ACP)/2, (PAB)= c*PB(sen ABP)/2
Sustituyendo obtenemos:

a^2/PB*PC(sen A) = b/PC(sen ACP) + c/PB(sen ABP)

Aora, (sen ACP)=(sen ABP) y ademas, por el teorema de Ptolomeo, el lado derecho de la igualdad queda:
a/(sen A)=PA/(sen ABP) lo cual sabemo que es cierto, pueso como ACPB es ciclico, entonces ambas proporciones son iguales a 2R (R= circunradio)
Y ya hemos demostrado lo que queriamos.

**Line**

#3 Jan 8, 2007, 10:44 pm

hola gracias.

podria ser sin trigo?

**Luis González**

#4 Jun 1, 2009, 10:41 am

**Proposición:** Sea $P$ un punto en el arco $BC$ de la circunferencia circunscrita del triángulo $\triangle ABC. L,M,N$ son las proyecciones de $P$ sobre $BC,CA,AB. PL=u,PM=v,PN=w.$ Mostrar que $avw=buw+cuv.$

Basta usar que si dos triángulos tienen un par de ángulos iguales o suplementarios entonces las razón entre sus áreas es igual a la razón entre el producto de los lados que conforman dichos ángulos. Aplicamos ello a los triángulos $\triangle PLN,\triangle PLM$ y $\triangle PNM$ respecto al triángulo $\triangle ABC.$ Luego la suma de las dos primeras áreas debe ser igual a la última ya que $M,L,N$ estan alineados en la recta de Simson de $P.$ Así

$$\frac{[\triangle PLN]}{[\triangle ABC]}=\frac{uw}{ac}=\frac{buw}{abc}\ ,\ \frac{[\triangle PML]}{[\triangle ABC]}=\frac{uv}{ab}=\frac{cuv}{abc}\ ,\ \frac{[\triangle PMN]}{[\triangle ABC]}=\frac{vw}{bc}=\frac{avw}{abc}$$

Entonces teniendo presente que $[\triangle PMN]=[\triangle PLN]+[\triangle PML]$, se llega pues a $avw=buw+cuv.$

Quick Reply

# Spain

## Incirculos iguales

Reply

**conejita**

#1 Sep 25, 2006, 3:47 am

Sea ABC un triangulo con CB=CA. Sobre CB tomamos un punto D tal que el incirculo de CBD y el excirculo de CAD (tangente a las prolongaciones de CD y CA) sean iguales (es decir del mismo radio). Demuestra que una cuarta parte de las alturas iguales(ya sea $Ha$ o $Hb$) es igual al radio de dichos circulos.

Por favor, este problema no me ha salido nunca!!! Necesito ayuda!!

**americoperu**

#2 Sep 25, 2006, 9:47 am

> *conejita wrote:*
> (...)Sobre CB tomamos un punto D tal que el incirculo de CBD y el excirculo (...)

No entiendo si $D$ esta en $CB$ -> $C$, $D$ y $B$ colineales??

**conejita**

#3 Sep 25, 2006, 7:23 pm

Una disculpa, deberia decir:
Sobre AB tomamos un punto D...

Ahora si esta bien el problema.
Grax

**aev5peru**

#4 Sep 26, 2006, 8:45 am

> *conejita wrote:*
> Una disculpa, deberia decir:
> Sobre AB tomamos un punto D...
>
> Ahora si esta bien el problema.
> Grax

jajja, de razon, yo estaba considerando sobre el lado, osea que en la region jajaja, bueno, entonces conejita corrige çchaufas saludos,.

**dhernandez**

#5 Sep 28, 2006, 3:40 am

> *conejita wrote:*
> Sea ABC un triangulo con CB=CA. Sobre CB tomamos un punto D tal que el incirculo de CBD y el excirculo de CAD (tangente a las prolongaciones de CD y CA) sean iguales (es decir del mismo radio). Demuestra que una cuarta parte de las alturas iguales(ya sea $Ha$ o $Hb$) es igual al radio de dichos circulos.

> Por favor, este problema no me ha salido nunca!!! Necesito ayuda!!

Has intentado con trigonometria?? Para aqui para el forum esta bien intentar siempre con geometria euclideana pero en un examen, este seria el tipico problema que me parece que si no me sale lindo rapidamente, blasfemaria un poco de senos y cosenos sobre el papel.
Afortunadamente, no hay necesidad de ello..

1- La idea está basada en un hecho bastante conocido y es que la suma de las perpendiculares trazadas desde un punto de la base de un triángulo isósceles hasta los lados es igual a la longitud de la altura. Con esto en mente

2- Sean $O_1$, $O_2$ los centros de las circunferencias, exinscrita en $CAD$ e inscrita en $CBD$ respectivamente. Sean $E$ y $F$ los puntos donde la perpendicular por $O_1$ corta a los rayos $CA$ y $CD$ respectivamente. Ahora $CEF$ es isosceles de base $EF$. Analogamente, sean $G$ y $H$ los puntos donde la perpendicular por $O_2$ corta a los rayos $CD$ y $CB$ respectivamente. $CGH$ isosceles base $GH$.

3- Ahora tenemos muchos triangulos isosceles como queremos. Vease que por (1) el diametro de los circulos es igual a las alturas con respecto a los lados iguales de los triangulos $CEF$ y $CGH$.

4- $O_1D = DO_2$. Esto es por igualdad de triangulos y utilizando el hecho de que los radios de los circulos son iguales y que ambos centros estan en la bisectriz del $\angle BDC$

5- Sea $M$ el punto donde se intersectan las rectas $EF$ y $AB$. Entonces se tiene $\angle O_1MD = \angle O_2CD$(¿por que?) luego $\triangle MDO_1 = \triangle CDO_2 \Rightarrow CD = DM$. Si $M'$ fuera el punto donde $GH$ intersecta a $AB$ entonces se tendría $\angle O_2M'D = \angle O_1CD$, $\angle M'DO_2 = \angle CDO_1$ y por (4) $O_1D = DO_2$, luego sería $\triangle DM'O_2 = \triangle DO_1C$ y $DM' = CD = DM$ y todo este lio es para esto. $GH$, $AB$ y $EF$ concurren en $M$

6- Considerese la circunferencia que circunscribe al $\triangle MGF$. Sea $O$ su centro. Entonces $\angle GOD = 2\angle GMD = \angle ACD$. Trácese la paralela por $F$ a $CA$ que corta a $AB$ en $N$. Entonces $\angle GFN = \angle ACD = \angle GOD$. Luego $OGNF$ puede inscribirse en una circunferencia. Luego $\angle NGF = \angle NOF = 2\angle DMF = \angle BCD$. Luego $GN \parallel BC$.

7- Finalmente, la distancia de $N$ a $AC$ es igual a la distancia de $F$ a $AC$(recuerdese $FN \parallel AC$) que es igual a $2r$ (3). Igualmente la distancia de $N$ a $BC$ es la distancia de $G$ a $BC$,tambien $2r$. Y la suma de las dos es la altura pues $N$ pertenece a $AB$. Luego $H = 4r$. Por cierto que $N$ es claramente el punto medio de $AB$. Una forma un poco enrevesada de llegar a el.

Si alguien encuentra una solucion mas rapida y que no utilice trigonometria me encantaria verla. Seguro que existe.

Daniel

**conejita**
*#6 Sep 29, 2006, 9:55 am*

Ola Daniel, disculpa pero no entiendo la construccion de las prependiculares que pasan por O1 y O2!!
Podrias explicarlo un poco mejor xfavor
Gracias

**Virgil Nicula**
*#7 Sep 29, 2006, 12:29 pm*

> *conejita - a very nice problem wrote:*
> Sea $ABC$ un triangulo con $AB = AC$. Sobre $BC$ tomamos un punto D tal que el incirculo de $ABD$ y el excirculo de $ACD$ (tangente a las prolongaciones de $AC$ y $AD$ ) sean iguales (es decir del mismo radio). Demuestra que una cuarta parte de las alturas iguales(ya sea $h_b$ o $h_c$) es igual al radio de dichos circulos.

**Proof 1 (metrical).** Denote the incircle $w_1 = C(I_1, \rho)$ of the triangle $ABD$ and the its tangent points : $P \in AB$ , $X \in BD$ , $U \in AD$ , $BX = x$ .

Denote the $A$- exincircle $w_2 = C(I_2, \rho)$ of the triangle $ACD$ and the its tangent points : $R \in AC$ , $Y \in CD$ , $V \in AD$ , $CY = y$ .

$$1. \blacktriangleright \ \triangle BXI_1 \sim \triangle I_2YC \Longrightarrow \frac{BX}{I_2Y} = \frac{XI_1}{YC} \Longrightarrow \boxed{xy = \rho^2} .$$

$$2. \blacktriangleright UV = AV - AU = AR - AP = (AC + CR) - (AB - BP) = CR + BP = CY + BX = x + y \Longrightarrow$$

$UV = x + y$ . But $UV = XY = BC - (BX + CY) = a - (x + y)$ . Thus, $x + y = a - (x + y)$, i.e. $\boxed{x + y = \frac{a}{2}}$ .

3. ► $x = \rho \cot \frac{B}{2}$ , $y = \rho \tan \frac{B}{2} \Longrightarrow x - y = \rho \left( \cot \frac{B}{2} - \tan \frac{B}{2} \right) \Longrightarrow \boxed{x - y = 2\rho \cot B}$ .

Therefore, $(x + y)^2 - (x - y)^2 = 4xy \Longrightarrow \frac{a^2}{4} - 4\rho^2 \cot^2 B = 4\rho^2 \Longrightarrow a^2 = 16\rho^2 \left(1 + \cot^2 B\right) \Longrightarrow 4\rho = a \sin B$ $\Longrightarrow \boxed{\boxed{4\rho = h_b}}$ .

**Remark.** From trigonometry I used only the well-known relation $\cot \frac{x}{2} - \tan \frac{x}{2} = 2 \cot x$ . Prove easily that $\frac{DB}{DC} = \frac{4b + a}{4b - a}$ .

**An interesting and very nice property.** Denote the incircle $w_3 = C(I', r')$ of the triangle $ACD$ .

From the my topic http://www.mathlinks.ro/Forum/viewtopic.php?t=50559 obtain :

$4\rho = h_b$ , $pr^2 + a\rho r' = pr(\rho + r')$ $\|\bullet 4 \Longrightarrow bh_b = 2S = 2pr$ , $4pr^2 + ah_b r' = pr(h_b + 4r')$ $\|\bullet b \Longrightarrow$

$4pr^2 b + a \cdot 2pr \cdot r' = pr \cdot 2pr + 4prr'b \Longrightarrow r' = \frac{r(2b - p)}{2b - a}$ , $2(2b - p) = 2b - a \Longrightarrow$ $\boxed{r' = \frac{r}{2}}$ $-$ great result !

*This post has been edited 1 time. Last edited by Virgil Nicula, Sep 30, 2006, 10:13 am*

**dhernandez**

*#8 Sep 29, 2006, 4:42 pm*

> *conejita wrote:*
> Ola Daniel, disculpa pero no entiendo la construccion de las prependiculares que pasan por O1 y O2!!
> Podrias explicarlo un poco mejor xfavor
> Gracias

Bueno, los rayos $CO_1$ y $CO_2$ son bisectrices de los angulos $\angle ACD$ y $\angle DCB$ respectivamente porque pasan por el ex/incentro de los triangulos en cuestion. Si ahora trazo una perpendicular por $O_1$ y tomo los puntos donde esta corta a los rayos $CA$ y $CD$ entonces ahora $CO_1$ no solo es bisectriz de ese triangulo nuevo sino tambien altura, luego tiene que ser isosceles.

Fijate que ahora el los centros de las circunferencias son puntos que pertenecen a la base de triangulos isosceles, y los radios son las distancias desde estos puntos hasta los lados respectivos. Luego, la suma de las distancias desde $O_1$ hasta $CA$ y $CD$, o sea $2r$ es igual a la altura desde $F$ del triangulo $CEF$.

Dime si te ayuda ahora

Daniel

**americoperu**

*#9 Sep 30, 2006, 9:04 am*

hola...en estos momentos no cuento con tiempo para hacer la solucion completa pero con esto seguro les saldra facil y sin usar trigonometria:
Como el triangulo es isosceles y las circunferencia inscrita y circuncrta tienen el mismo radio, una manera de relacionar los radios con la altura sobre los lados iguales seria utilizar areas: triangulos parciales y el total:
Siendo $r$, el radio de las circunferencias. Y $H$ pie de la perpendicular trazada de B sobre el lado AC.....$[XYZ] : areadelaregionXYZ$

$[CBD] = (\frac{CB + DB + CD}{2})r \ldots(i)$

$[ACD] = (\frac{AC + CD - AD}{2})r \ldots(ii)$

AC

$[ABC] = \frac{AC}{2} BH \ ...(iii)$
y tambien con $AC = CB$ se obtiene que: $BH = 4r$

**conejita**
*#10 Sep 30, 2006, 9:54 am*

Disculpa pero, no es tan obvia el resultado!!
Podrias poner completa tu solucion Americoperu??
Gracias :)

**Luis González**
*#11 Jun 1, 2009, 10:06 am*

**Proposición:** Sea $\triangle ABC$ un triángulo isosceles con $AB = AC = L$. $D$ es un punto en $BC$ tal que el incírculo de $\triangle ABD$ y el A-excírculo del $\triangle ADC$ son iguales a $r$. Demostrar que la altura $h$ sobre los lados iguales es igual a cuatro veces $r$.

Trazamos las perpendiculares $DP$ y $DL$ desde $D$ a $AB$ y $AC$, respectivamente. Usamos las formulas conocidas que relacionan el inradio y los exinradios con alturas.

$$DP = \frac{r(L + AD + BD)}{L} \ , \ DQ = \frac{r(AD + L - DC)}{L}$$

$$DP + DQ = \frac{r(2L + 2AD + BD - DC)}{L}$$

Pero es claro que $DP + DQ = h \implies h = \frac{r(2L + 2AD + BD - DC)}{L} \ , (*)$

Como son iguales estos dos círculos, seran iguales los segmentos de tangente desde $D$ a ambos, lo cual en términos de $\triangle ABD$ y $\triangle ACD$ implica $L + DC - AD = AD + BD - L$, es decir $2L = 2AD + BD - DC$.

De modo que en $(*)$ resulta $h = \frac{r(2L + 2L)}{L} = 4r$.

Quick Reply

# High School Olympiads

## A construction problem



**Source**: I.F.Sharygin contest 2009 - Correspondence round - Problem 13

**April**

#1 May 31, 2009, 8:29 am

In triangle $ABC$, one has marked the incenter, the foot of altitude from vertex $C$ and the center of the excircle tangent to side $AB$. After this, the triangle was erased. Restore it.

**Luis González**

#2 May 31, 2009, 10:03 pm

Let $A, B, C$ be the vertices of wanted triangle, given its incenter $I$, the C-excenter $I_c$ and the foot $H$ of the C-altitude. Let $V$ be the foot of its C-angle bisector. Since $(C, V, I, I_a) = -1 \Longrightarrow$ circle with diameter $\overline{CV}$ passing through $H$ and the circle with diameter $\overline{II_c}$ (which is known) are orthogonal. It remains to construct the cirle $\mathcal{Q}$ passing through $C, H, V,$ which belows to the orthogonal pencil defined by the cirle $(K)$ with diameter $II_c$ and whose center lies on $II_c$. Using that the power of $K$ to $\mathcal{Q}$ equals the square of the radius $KI$, then draw line $KH$ and construct point $P$ on the ray $KH$ such that $KP \cdot KH = KI^2$ $\Longrightarrow$ intersection of the perpendicular bisector of $\overline{PH}$ with the line $II_c$ gives the center of $\mathcal{Q}$. Once the circle $\mathcal{Q}$ is drawn, we get easily the poins $C, V$ and subsequently the line $VH$ whose intersections with $(K)$ are the desired vertices $A, B$.

**polya78**

#3 Jan 29, 2013, 10:23 pm

Let $D, I, J$ be the foot of the altitude, the incenter and excenter. Consider the line parallel to $AB$ and also tangent to the incircle and let its intersection with $CD$ be $E$. Then it is easy to see that $IE$=$ID$ and also $DJ \parallel IE$. So given, $D, I, J$, we can construct $E$. Then $C$ is simply $DE \cap IJ$. The rest of the construction follows easily.

**anantmudgal09**

#4 Mar 7, 2016, 6:31 am

This problem proves that you can't beat Russians when it comes to making sufficiently elegant yet decently hard problems. :)

Here is my solution.

*Construction*

Let $I, J, H$ be the given incenter, excenter, and feet of altitude. Now, we can construct the midpoint $M$ of $IJ$. Then, we can construct a point $Y$ on the ray $MH$ such that $MH.MY = MI^2$. We also construct the circle with diameter $IJ$ and call it $\Gamma$

Now, draw tangents from $Y$ to $\Gamma$ and construct the midline of these two tangents. Let this midline intersect $IJ$ at $X$. Draw circle with centre $X$ and radius $XY$ and let it meet $IJ$ again at $A, A'$ with $A$ farther from $M$ than $A'$. Then, draw a line perpendicular to $AH$ at $H$ and let it meet $\Gamma$ at $B, C$. We claim that triangle $ABC$ is the desired triangle.

*Proof*

Notice that $XY^2 = XI.XJ$ and so $(A, A'; I, J) = -1$ and also that $\angle AYA' = 90$ so we must have $A$ to be the requested vertex. Then, $B, C$ are also the requested ones.

# High School Olympiads

## A sphere is inscribed into a quadrangular pyramid

Reply

**Source**: I.F.Sharygin contest 2009 - Correspondence round - Problem 24

**April**

#1 May 31, 2009, 8:46 am

A sphere is inscribed into a quadrangular pyramid. The point of contact of the sphere with the base of the pyramid is projected to the edges of the base. Prove that these projections are concyclic.

**Luis González**

#2 May 31, 2009, 1:26 pm

Let $ABCD$ be the base of this pyramid and $E$ its opposite vertex. Let $X, Y, Z, W$ be the tangency points of its insphere $\omega$ with the faces $EAB, EBC, ECD, EDA$ and $P$ its tangency point with the base $ABCD$. It is easy to figure out that $\triangle EXB \cong \triangle EYB$, because of $BX = BY$ and $EX = EY$. *Mutatis mutandis* $\triangle EYC \cong \triangle EZC$, $\triangle EZC \cong \triangle EWD$ and $\triangle EWA \cong \triangle EXA$. Then we get

$$\angle CYB = 360^\circ - (\angle EZC + \angle EXB)$$

$$\angle AWD = 360^\circ - (\angle EZD + \angle EXA)$$

$$\angle CYB + \angle AWD = 720^\circ - (\angle EZC + \angle EXB + \angle EZD + \angle EXA)$$

$$\angle CYB + \angle AWD = \angle CZD + \angle AXB$$

Since $AX = AW = AP, BX = BY = BP, CY = CZ = CP$ and $DZ = DW = DP$ we get

$$\angle AXB = \angle APB, \ \angle BPC = \angle BYC$$

$$\angle CZD = \angle CPD, \ \angle AWD = \angle APD$$

$$\Longrightarrow \angle APD + \angle BPC = \angle CPD + \angle APB$$

$$\Longrightarrow \angle APD + \angle BPC = 180^\circ, \ (*)$$

Let $M, N, L, K$ be the orthogonal projections of $P$ onto the edges $AB, BC, CD, DA$. Since the quadrilaterals $PMBN$, $PNCL, PLDK, PKAM$ are all cyclic, it follows that

$$\angle APD = \angle KMA + \angle KLD, \ \angle BPC = \angle NMB + \angle NLC$$

Combining with $(*)$ yields $\angle KLM + \angle NMK = 180^\circ \Longrightarrow MNLK$ is cyclic.

Quick Reply

# High School Olympiads

## projections to the angle bisectors

Reply

Source: I.F.Sharygin contest 2009 - Correspondence round - Problem 7

**April**

#1 May 31, 2009, 6:33 am

Given triangle $ABC$. Points $M$, $N$ are the projections of $B$ and $C$ to the bisectors of angles $C$ and $B$ respectively. Prove that line $MN$ intersects sides $AC$ and $AB$ in their points of contact with the incircle of $ABC$.

*This post has been edited 1 time. Last edited by April, May 31, 2009, 8:25 am*

**Luis González**

#2 May 31, 2009, 7:05 am

Let $X, Y$ be the intersections of $MN$ with $AB, AC$, respectively. It suffices to see that quadrilateral $MNCB$ is cyclic, thus $\angle NMC = \angle NBC = \angle ABN \Longrightarrow MXIB$ is cyclic. Hence, $\angle IXB = \angle IMB = 90^\circ$. Mutatis mutandis $\angle IYC = 90^\circ \Longrightarrow X, Y$ are the tangency points of the incircle with $AB, AC$.

**jayme**

#3 May 31, 2009, 11:58 am

Dear Mathlinkers,
for more see for example
http://pagesperso-orange.fr/jl.ayme/vol4.html an unlikely concurrence p. 2-5
Sincerely
Jean-Louis

Quick Reply

# High School Olympiads

## show that DEF is an isosceles right triangle

Reply

**moldovan**

#1 May 30, 2009, 4:43 pm

Let $F$ be the midpoint of side $BC$ of triangle $ABC$. Construct isosceles right triangles $ABD$ and $ACE$ externally on sides $AB$ and $AC$ with the right angles at $D$ and $E$, respectively. Show that $DEF$ is an isosceles right triangle.

**jayme**

#2 May 30, 2009, 6:56 pm

Dear Mathlinkers,
complete the figure with the two square built on AB and AC externally...
Sincerely
Jean-Louis

**Luis González**

#3 May 30, 2009, 9:16 pm

Let $D$ and $E$ be the centers of the squares $ABXY$ and $ACPQ$. Then $\triangle ABQ$ and $\triangle ACY$ are congruent by SAS criterion, due to $AB = AY, AC = AQ$ and $\angle BAQ = \angle YAC$. Therefore, $BQ = YC$ and $\angle ACY = \angle AQB$. If $M \equiv YC \cap BQ$, then quadrilateral $AMCQ$ is cyclic $\Longrightarrow \angle QMC$ is right. Lines $FD$ and $FE$ are respectively parallel to $CY$ and $BQ$, since $D, E$ are midpoints of $BY, CQ$. Hence, $FD \perp FE \Longrightarrow \triangle DEF$ is isosceles right at $F$.

**livetolove212**

#4 Jun 6, 2009, 3:19 pm

I have a problem which use moldovan's problem to solve
Let $ABCD$ be a quadrilateral. Construct 4 squares $ABB_1A_1, BCC_1B_2, CC_2D_1D, DD_2A_2A$ externally $ABCD$. Let $X, Y, Z, T$ are the midpoints of $A_1A_2$, $B_1B_2$, $C_1C_2$, $D_1D_2$. Prove that $XZ = YT$ and $XZ \perp YT$

Quick Reply

# High School Olympiads

## Newton line goes through the circumcenter

Reply

**Luis González**

#1 May 26, 2009, 12:50 am • 1

Let $I$ be the incenter of $\triangle ABC$ and $\omega$ the circle passing through $A$ and tangent to $BC$ at the foot $W$ of the internal bisector of $\angle A$. Perpendiculars to $BC$ through $B, C$ cut external bisector of $\angle A$ at $B', C'$, respectively and the lines $CB'$ and $BC'$ meet the circle $\odot(IBC)$ again at $M, N$. Show that the perpendicular lines through $W$ to $CB'$ and $BC'$, the tangent to $\omega$ at $A$ and the radical axis of $\omega, \odot(IBC)$ bound a complete quadrangle whose Newton line passes through the circumcenter of $\triangle AMN$.

**yetti**

#2 May 29, 2009, 6:03 pm

Let $P \equiv BC' \cap CB'$. From trapezoid $BB'C'B$, $\frac{BP}{PC'} = \frac{BB'}{CC'} = \frac{AB}{CA} = \frac{BW}{WC} \Longrightarrow PW \perp BC$. Let $W' \equiv PW \cap B'C'$, then $P$ is midpoint of $WW'$. Let $AI$ cut circumcircle $(O)$ of $\triangle ABC$ again at $X$. Let $X' \equiv OX \cap B'C'$, then $\triangle AWW' \sim \triangle AXX'$ are centrally similar with center $A$. Their circumcircles $w \equiv (P), (O)$ are therefore internally tangent at $A$. Let $t \perp PA$ be their single common tangent at $A$.

Let $(X)$ be circumcircle of $\triangle IBC$. Let $BB'$ cut $(X)$ again at $B''$ and let $M \equiv B''W \cap CB'$. $W$ is diagonal intersection of $BMCB''$ and $B' \equiv BB'' \cap CM$ is on polar of $W$ WRT $(X) \Longrightarrow M \in (X)$ and $MW \perp CB'$. Similarly, if $N$ is foot of perpendicular from $W$ to $BC'$, then $N \in (X)$.

Let $(Q)$ be circle with diameter $PW$, then $M, N \in (Q)$. Let $(S)$ be circumcircle of $\triangle AMN$. $\{M, N\} \equiv (Q) \cap (S) \Longrightarrow P$ is inside of $(S)$. Inversion in $(X)$ takes $BC$ to $(O)$, $W$ to $A$ and $(Q)$ to $(S)$. Let $K$ be inversion image of $P$ in $(X)$, which lies on the radical axis $r$ of $(X), (P)$ and $PK \perp r$. Since $P \in (Q)$, it follows that $K \in (S)$. Since $A, K, M, N \in (S)$ are concyclic pedals of $P$ in the quadrilateral $(t, r, WM, WN)$, $(S)$ is pedal circle of an ellipse $\mathcal{E}$ with one focus $P$ inscribed in this quadrilateral. Projecting this ellipse to its pedal circle $(S)$ by a parallel projection keeps $S$ in place and takes $(t, r, WM, WN)$ with its Newton line $n$ to a quadrilateral tangential to $(S)$ and its Newton line $n'$. Since $S \in n'$, it follows that $S \in n$.

**Luis González**

#3 May 29, 2009, 6:20 pm

Thanks dear Vladimir for such an elegant proof. I knew that you were going to take this problem!. My solution is very similar, I'll post it next time. 🙂

**Luis González**

#4 May 29, 2009, 11:42 pm

Let $D, F$ be the points where $\omega$ cuts $AB, AC$. Simple angle chasing yields

$$\angle ADF = \angle AWF = \angle ABC \ , \ \angle AFD = \angle AWD = \angle ACB$$

Thus, $\triangle ADF \sim \triangle ABC$, which implies that $\omega$ and the circumcircle $(O)$ of $\triangle ABC$ are internally tangent through $A$. Antipode of $I$ WRT $(K) \equiv \odot(BIC)$ is the A-excenter $I_a$ and due to $(A, W, I, I_a) = -1$, it follows that $(K) \perp \omega$. As result, radical axis $p_a$ of $\omega, (K)$ is the polar of $K$ WRT $\omega$.

On the other hand, center $J$ of $\omega$ is the intersection of the lines $BC'$ and $CB'$. For a proof see the thread If and only if. Inversion WRT $(J)$ takes $(K)$ into itself due to the orthogonality. Thus, it takes $(O)$ into the circle $(O') \equiv \odot(AMN)$. Polars of $B, C$ with respect to $\omega$ are the perpendiculars $p_b, p_c$ through $W$ to $BC', CB'$ and tangent $\tau$ to $\omega$ at $A$ is the polar of $A$. Since $A, B, C, K$ lie on $(O)$, their polars $\tau, p_b, p_c, p_a$ are tangent to the polar conic $\mathcal{E}$ of $(O)$ WRT $\omega$, which in this case is an ellipse since $J$ lies inside $(O)$. The center of $\mathcal{E}$ is precisely the center $O'$ of the inverse circle of $(O)$ in the inversion WRT $(J)$, but according to Newton's theorem, the center of all inscribed conics in the quadrangle $\tau, p_b, p_c, p_a$ lies on its Newton line.

Quick Reply

# High School Olympiads

## The line PQ is parallel to BC.



**vittasko**

#1 May 28, 2009, 5:50 pm

**A triangle $\triangle ABC$ is given with circucircle $(O)$ and let $(K)$ be, an arbitrary circle taken as cord its side-segment $BC.$ Let be the point $D\equiv BC\cap AK,$ where $K$ is the center of $(K)$ and we denote as $E,\ F,$ the points of intersection of $(K),$ from the circles $(O_1),\ (O_2)$ with diameters $BD,\ CD,$ respectively. Prove that $PQ\parallel BC,$ where $P\equiv AB\cap DE$ and $Q\equiv AC\cap DF.$**

Kostas Vittas.

*Attachments:*
t=279094.pdf (11kb)

**livetolove212**

#2 May 28, 2009, 8:34 pm

> vittasko wrote:
> **A triangle $\triangle ABC$ is given with circucircle $(O)$ and let $(K)$ be, an arbitrary circle taken as cord its side-segment $BC.$ Let be the point $D\equiv BC\cap AK,$ where $K$ is the center of $(K)$ and we denote as $E,\ F,$ the points of intersection of $(K),$ from the circles $(O_1),\ (O_2)$ with diameters $BD,\ CD,$ respectively. Prove that $PQ\parallel BC,$ where $P\equiv AB\cap DE$ and $Q\equiv AC\cap DF.$**
>
> Kostas Vittas.

Let $J=DP\cap AO_1, R=DQ\cap AO_2$
Because $KO_1\perp BE$ and $\angle BED=90^o$ then $KO_1//DE$, similarly $KO_2//DF$
Using Thales's theorem we have $\frac{AJ}{AO_1}=\frac{AD}{AK}=\frac{AR}{AO_2}$
Then $JR//O_1O_2$
Applying Menelaus's theorem for triangles $JO_1D$ and $RDO_2$ we get $\frac{AJ}{AO_1}.\frac{BO_1}{BD}.\frac{PD}{PJ}=\frac{AR}{AO_2}.\frac{CO_2}{CD}.\frac{QR}{QD}$
$\Rightarrow \frac{PD}{PJ}=\frac{QR}{QD}\Rightarrow PQ//JR\rightarrow$ QED

**Luis González**

#3 May 28, 2009, 11:20 pm • 1

Since $BE$ and $CF$ are radical axes of $(K),(O_1)$ and $(K),(O_2)$, then $KO_1$ and $KO_2$ are respectively parallel to $DP$ and $DQ$ and go through the midpoints $M,N$ of $PB$ and $QC$. Hence, by Menelaus' theorem we obtain:

$$\frac{KD}{KA}=\frac{BM}{MA}=\frac{CN}{NA}\Longrightarrow MN\parallel BC$$

Therefore, $MN$ becomes the midline of the trapezoid $PQCB\Longrightarrow PQ\parallel BC$

# High School Olympiads

## Trigonometric relation 3

Reply

**Pain rinnegan**
#1 May 28, 2009, 2:39 am

Let be $ABC$ a triangle . Prove that :

$$(b+c)\sin\frac{A}{2}=a\sin\left(\frac{A}{2}+C\right)$$

**aleph0**
#2 May 28, 2009, 7:46 am

Solution

**Luis González**
#3 May 28, 2009, 8:46 am

Take a point $P$ on the ray $BA$ such that $AP=AC=c.\triangle APC$ is isosceles with $\angle APC=\angle ACP=\frac{1}{2}\angle A$ and notice that $\angle PCB=\angle C+\frac{1}{2}\angle A.$ Using sine law in the triangle $\triangle PBC,$ we get:

$$\frac{PB}{BC}=\frac{b+c}{a}=\frac{\sin(C+\frac{A}{2})}{\sin(\frac{A}{2})}\Longrightarrow(b+c)\cdot\sin\frac{A}{2}=a\cdot\sin\left(C+\frac{A}{2}\right).$$

Quick Reply

# Spain

## Olimpiada de mayo - nivel 2 - segunda problema

Reply

**tonypr**

#1 May 27, 2009, 10:11 pm

Sea ABCD un cuadrilátero convexo tal que el triángulo ABD es equilátero y el triángulo BCD es isósceles, con $\angle C = 90^\circ$. Si E es el punto medio del lado AD , calcula la medida del ángulo $\angle CED$.

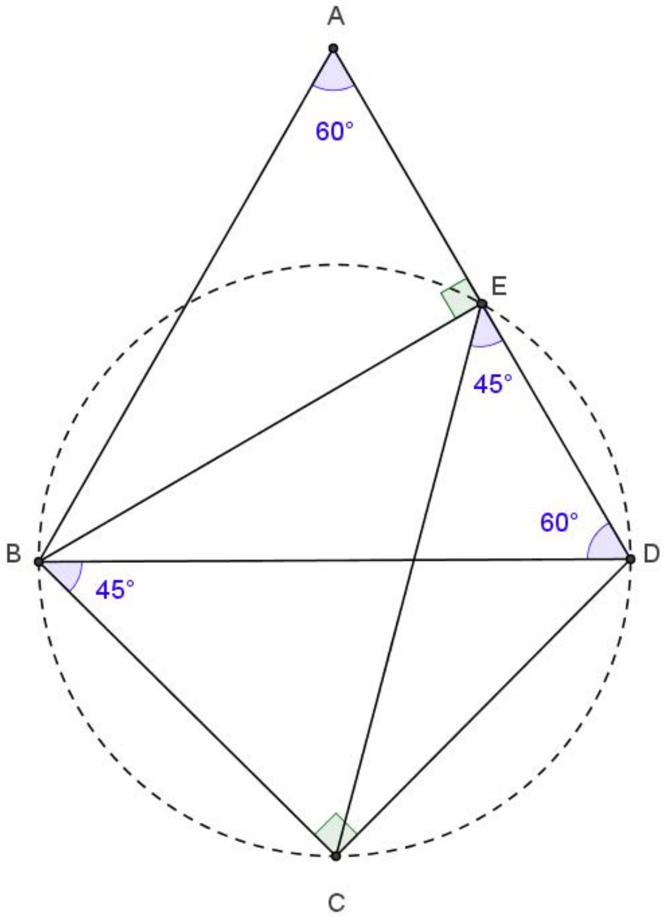
mi respuesta

**Luis González**

#2 May 28, 2009, 3:18 am

Creo que un diagrama será suficiente

*Attachments:*




**hatchguy**

#3 May 28, 2009, 8:09 am

soy pesimo en geometría y estaba buscando paralelismo, en lugar de cosas menos basicas como cuadrilateros ciclicos (por ser esta olimpiada para estudiantes de los primeros niveles en secundaria) cuando vi el cuadrilatero ciclico en mi casa me sentí demasiado mal..

Quick Reply

## High School Olympiads

### From problem of IMO 1994 proposal

Reply

**hophinhan**

#1 May 27, 2009, 5:20 pm • 1

The incircle of $\triangle ABC$ touches $BC, CA, AB$ at $D, E, F$, respectively.
Let $X$ be a point inside $\triangle ABC$ such that the incircle of $\triangle XBC$ touches $BC$ at $D$ also, and touches $CX$ and $XB$ at $Y$ and $Z$.
Prove that : $FX$ ; $EY$ and $AX$ are concurrent at a point lies on a line perpendicular $BC$ at $D$.

---

**livetolove212**

#2 May 27, 2009, 11:51 pm

> *hophinhan wrote:*
> The incircle of $\triangle ABC$ touches $BC, CA, AB$ at $D, E, F$, respectively.
> Let $X$ be a point inside $\triangle ABC$ such that the incircle of $\triangle XBC$ touches $BC$ at $D$ also, and touches $CX$ and $XB$ at $Y$ and $Z$.
> Prove that : $FX$ ; $EY$ and $AX$ are concurrent at a point lies on a line perpendicular $BC$ at $D$.

Let (I) and (I') be the incircles of triangles ABC and XBC, respectively.
Let FZ cut AX at K, EY cut AX at K'. We will prove $K \equiv K'$.
Applying Menelaus's theorem for the triangle ABX we have $\frac{FA}{FB}.\frac{ZB}{ZX}.\frac{KX}{KA} = 1$ or $\frac{KX}{KA} = \frac{ZX}{FA}(2)$
Similarly, $\frac{K'X}{K'A} = \frac{XY}{EA}(2)$
From $(1)(2)$ we get $\frac{KX}{KA} = \frac{K'X}{K'A}$ so $K' \equiv K$
Let $FZ$ cut $II'$ at $K_1$, $EY$ cut $II'$ at $K_2$, we will prove $K_1 \equiv K_2$
Let $I'Z \cap IF = \{M\}, I'Y \cap IE = \{L\}$
Since $BF = BZ$ and $\angle BFM = \angle BZM = 90^o$ we obtain $MF = MZ$, similarly $LE = LY$
Applying Menelaus's theorem for triangle MII' we have:
$\frac{FM}{FI}.\frac{K_1I}{K_1I'}.\frac{ZI'}{ZM}$ then $\frac{K_1I}{K_1I'} = \frac{FI}{ZI'} = \frac{r_{ABC}}{r_{XBC}}$
Similarly, $\frac{K_2I}{K_2I'} = \frac{r_{ABC}}{r_{XBC}}$ thus $K_1 \equiv K_2$
->QED

---

**mathVNpro**

#3 May 28, 2009, 1:06 am

**Another approach:**
First, let me restate the problem so that it can fits my solution:

**PROBLEM- Let $\triangle ABC$ with $(I)$ its incircle. $(I)$ tangents to $BC, CA, AB$ respectively at $A_0, B_0, C_0$. Let $X$ be the point inside $\triangle ABC$ such that the incircle $I_a$ of $\triangle XBC$ tangents $BC$ at $A_c$ also. Let $C_1, B_1$ be the tangency point of $(I_a)$ with $XB, XC$, respectively. Prove that: $C_0C_1, B_0B_1, AX$ are concurrent at $D$ and $D \in II_a$.**

*Proof*

Let $N \equiv C_0B_0 \cap BC$, $N' \equiv B_1C_1 \cap BC$.Then, it is very well- known that $(NA_0BC) = (N'A_0BC) = -1 \Longrightarrow N \equiv N'$. Thus $B_0C_0, B_1C_1, BC$ are concurrent. Hence by **Desgaurte theorem**, we conclude that $\triangle B_0B_1B, \triangle C_0C_1C$ are $2$ perspective triangles. But $A \equiv BB_0 \cap CC_0$, $X \equiv BB_1 \cap CC_1$, $D \equiv B_1B_0 \cap C_1C_0 \Longrightarrow A, X, D$ are collinear, which implies that $AX, C_0C_1, B_0B_1$ are concurrent at $D$.

Now, let $B_2 = I_aB_1 \cap IB_0$, $C_2 = I_aC_1 \cap IC_0$. It is very easy to note that $\{B_2, B_1, B_0, B\}$ and $\{C_2, C_1, C_0, C\}$ are sets of concyclic points. But $BB_0 = BB_1, CC_0 = CC_1 \Longrightarrow B_2B_0 = B_2B_1, C_2C_0 = C_2C_1$. Denote $(C_2), (B_2)$ be the circle with radii $C_2C_0, B_2B_0$, respectively. Then, $(C_2)$ internally tangents to $(I)$ at $C_0$ and externally tangents to $(I_a)$ at $C_1$. Or in other word, $C_0, C_1$ are the external center and internal center of homothety that turns $(C_2) \mapsto (I)$, $(I_a) \mapsto (C_2)$, respectively.
Hence the **Monge-d'Alembert circle theorem** now says that $C_1C_0$ passes the internal center of homothety wich turns $(I) \mapsto (I_a)$, let call that center $D'$. With the same argument, we also have $B_0B_1$ passes through $D'$.Hence $D' \equiv B_0B_1 \cap C_0C_1$, which implies $D' \equiv D$. Therefore $D, I, I_a$ are collinear. The result is lead as follow.
Our proof is completed then.

**REMARK-**
Click to reveal hidden text
*This post has been edited 9 times. Last edited by mathVNpro, May 28, 2009, 5:21 pm*

**Luis González**
*#4 May 28, 2009, 2:17 am • 1*

$FE$ and $ZY$ go through the harmonic conjugate $P$ of $D$ with respect to $(B, C)$. Then $\triangle AFE$ and $\triangle ZXY$ are perspective with perpectrix $BC \Longrightarrow AX, FZ$ and $EY$ concur at their perspector $J$. Note that $PE \cdot PF = PY \cdot PZ = PD^2$ implies that $FEYZ$ is a cyclic quadrangle $\Longrightarrow JY \cdot JE = JZ \cdot JF$. Also, tangents $(AC, XC)$ and $(AB, XB)$ through the pairs of points $(E, Y)$ and $(F, Z)$ meet at the radical axis $BC$ of both circles $\Longrightarrow J$ is the center of the inversion taking one circle into another $\Longrightarrow J$ lies on $ID$.

**hophinhan**
*#5 May 28, 2009, 8:38 am*

Thanks Linh ; Minh and Luis .... All solutions are very nice

**No Reason**
*#6 May 28, 2009, 4:15 pm*

Nice prolem,Nhạn!
Denote the points like my figure.
Let $J$ be the intersection of $IF$ and perpendicular line of $FM$ from $B$,$K$ be the intersection of $IE$ and perpendicular line of $EN$ from $C$.Easy to see that $(J)$ and $(K)$ are tangent to $(I)$ and $(I')$.
It's easy to that $FE, MN, BC$ are concurrent at a point $G$.Then $GD^2 = \overline{GF}.\overline{GE} = \overline{GM}.\overline{GN}$ then $F, E, M, N$ are concyclic then L lie on the radical axis of $(J)$ and $(K)$.
$AE^2 = AF^2$ and $XM^2 = XN^2$ so $A$ and $X$ also lie on the radical axis of $(J)$ and $(K)$.Hence $A, X, L$ are collinear.
I'' lie on the perpendicular bisector of MN so $I''M^2 = I''N^2$.
We have $I''N^2 - NC^2 + CD^2 - BD^2 + BM^2 - I''M^2 = CD^2 - BD^2 - CD^2 + BD^2 = 0$ then by Carnot's theorem,L lie on the line perpendicular to BC from D.

Image not found

Quick Reply

# High School Olympiads

## inequality with inradii

Reply

Source: Ukrainian journal contest, problem 348, by Vyacheslav Yasinskyy

**rogue**

#1 May 27, 2009, 7:44 pm

Let $G$ be the centroid of triangle $ABC$. Denote by $r, r_1, r_2$ and $r_3$ the inradii of triangles $ABC, GBC, GAC$ and $GAB$ respectively and by $p$ the semiperimeter of triangle $ABC$. Prove that $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\geq\frac{3}{r}+\frac{18}{p}$.

**Luis González**

#2 May 27, 2009, 10:05 pm

Centroid $G$ divides $\triangle ABC$ into three equivalent triangles $\triangle GBC, \triangle GCA, \triangle GAB$

$\frac{1}{3}[\triangle ABC]=r_1[\frac{a}{2}+\frac{1}{3}(m_b+m_c)]$

$\frac{1}{3}[\triangle ABC]=r_2[\frac{b}{2}+\frac{1}{3}(m_a+m_c)]$

$\frac{1}{3}[\triangle ABC]=r_3[\frac{c}{2}+\frac{1}{3}(m_a+m_b)]$

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{3p+2(m_a+m_b+m_c)}{[\triangle ABC]}=\frac{3}{r}+\frac{2(m_a+m_b+m_c)}{[\triangle ABC]}$$

Using $h_a+h_b+h_c\geq 9r$ and the fact that medians are greater-equal to altitudes

$$m_a+m_b+m_c\geq h_a+h_b+h_c\geq 9r \implies \frac{2(m_a+m_b+m_c)}{[\triangle ABC]}\geq\frac{18}{p}$$

$$\implies \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\geq\frac{3}{r}+\frac{18}{p}$$

Quick Reply

# High School Olympiads

## About incircle and excircle

Reply

**Source**: own

**livetolove212**

#1 May 24, 2009, 11:47 pm

Let AH be altitude of given triangle ABC. The incircle (I) touches three sides $BC, CA, AB$ at $D, E, F$, the excircle $(I_a)$ touches $BC, CA, AB$ at $M, S, N$, respectively. Show that $AH, ED, MN$ are concurrent, $AH, DF, MS$ are concurrent.

**Luis González**

#2 May 26, 2009, 10:39 am

Assume that $MN, DE$ meet $AH$ at $P, P'$, respectively. We shall prove $P \equiv P'$

By Menelaus' theorem for $\triangle AHB$ cut by the transversal $\overline{NMP}$, we get

$$\frac{NB}{NA} \cdot \frac{AP}{PH} \cdot \frac{MH}{MB} = 1 \Longrightarrow \frac{AP}{PH} = \frac{NA}{MH} = \frac{p}{MH} \quad (1)$$

By Menelaus' theorem for $\triangle AHC$ cut by the transversal $\overline{DP'E}$, we get

$$\frac{DH}{DC} \cdot \frac{CE}{EA} \cdot \frac{AP'}{P'H} = 1 \Longrightarrow \frac{AP'}{P'H} = \frac{EA}{DH} = \frac{p-a}{DH} \quad (2)$$

Antipode $D'$ of $D$ in the incircle is collinear with $A$ and $M$. Then from $\triangle D'MD \sim \triangle AMH$ we have:

$$\frac{MH - DH}{MH} = \frac{2r}{h_a} \Longrightarrow \frac{DH}{MH} = \frac{h_a - 2r}{h_a} = \frac{p-a}{p}$$

Therefore, from $(1)$ and $(2)$, it follow that $\frac{AP}{PH} = \frac{AP'}{P'H} \Longrightarrow P \equiv P'$

By similar reasoning, we prove that $AH, DF$ and $MS$ concur. $\square$

**livetolove212**

#3 May 26, 2009, 11:04 am

Another idea is using the radical center

**jayme**

#4 May 26, 2009, 7:02 pm

Dear Mathlinkers,
I know that the beloved Juan Carlos Salazar has worked on this situation
Salazar J. C., Mathlinks 08/01/2005.
an somebody give a more complete reference?
A prove without calculation will be nice.
Sincerely

**livetolove212**

#5 May 26, 2009, 7:31 pm

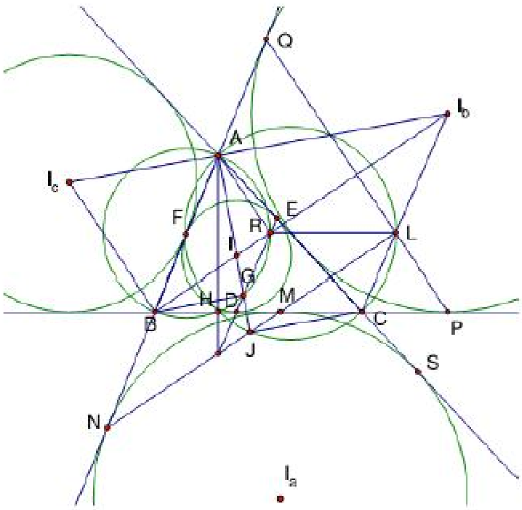
Here is my proof.

Attachments:

On the other hand, $\angle AGR = \angle ABR = \angle RBC = \angle BMN = \angle CML$
By comment 1, R, L lie on the median of triangle ABC agree to BC hence $RL//BC$
We claim $\angle CML = \angle MLR$
So $\angle AGR = \angle MLR$
Therefore GRLJ is a cyclic quadrilateral.
Applying the theorem about the radical center of the circumcircles of $AHGR, GRLJ, AHJL$ we get $AH, GR, JL$ are concurrent.
Or $AH, ED, MN$ are concurrent.
Similarly, AH, DF, MS are concurrent.□

Let AH be altitude of given triangle ABC. (I) touches 3 side $BC, CA, AB$ at $D, E, F$, $(I_a)$ touches $BC, CA, AB$ at $M, S, N$, respectively. Show that $AH, ED, MN$ are concurrent, $AH, DF, MS$ are concurrent.

**Proof.**




Let AI, BI cut ED at G, R, respectively, J be the intersection of AI and $MN$, $(I_b)$ touches $BC, AB$ at $P, Q$, respectively. $PQ \cap I_bC = L$.
Using problem 1 we get $\angle AJC = 90^o$ then the quadrilateral AJCL is cyclic the circle which has a diameter AC.
So $\angle JLC = \angle JAC$
On the other hand, $\angle CLP = \angle I_bAC$ thus $\angle JLP = \angle JLC + \angle CLP = \angle JAC + \angle I_bAC = 90^o$
Moreover, $MN//BI_b$ and $BI_b \perp PQ$ we obtain $MN \perp PQ$. But $J \in MN, JL \perp PQ$ therefore $M, N, L$ are collinear.
Using problem 1 again we have $\angle AGB = \angle ARB = 90^o = \angle AHB$ so the quadrilateral $AHGR$ is cyclic.

**vittasko**

#8 Jun 3, 2009, 12:33 am

Also, we can say that the points $A,\ D',\ M$ are collinear, because of $ID' \parallel I_aM$ and $\frac{AI}{AI_a} = \frac{IE}{I_aS} = \frac{ID'}{I_aM}$, applying the **Thales theorem**.

Kostas Vittas.

**muathuhanoi**

#9 Sep 24, 2011, 12:20 pm • 1

If $AB = AC$ , the result is obvious .
WLOG : $AB < AC$
$FD$ meets $AH$ at $K$
we'll prove: $KM||IC$

Apply Thales' theorem ,we get : $\frac{HD}{HM} = \frac{ID}{I_aM}$(1)
Beside , we have $\angle KDH= \angle MBI_a(=90-\frac{\angle{B}{2}) \Rightarrow {HK}{HD}=\frac{I_aM}{MB}$(2)
$\Rightarrow HKHM = \frac{ID}{I_aM}.\frac{I_aM}{MB} = \frac{ID}{MB} = \frac{ID}{CD}(ID = BM)$
And since $\frac{HK}{HM} = \frac{ID}{CD} \Rightarrow KM||IC$

From now on , we'll consider triangle $KDM$ and see that:
$AH, DE, DM$ are its altitude hence they will concurrent .
QED

**r1234**

#10 Sep 25, 2011, 1:32 pm

well, I solved the problem by lot of calculations but I think they are simple.So let's calculate..

Let $AX$ be the altitude and $AH, ED$ intersects at $X$.So now $AE = s - a, KD = bcosC - s + c$.Now in the triangle $ECD$ with triad $AX$ applying Menelaus we get $\frac{AE}{AC}.\frac{CK}{KD}.\frac{XD}{XE} = 1$.Putting the values of $QE, CK, KD$(easy to find out) we get $\frac{XD}{XE} = \frac{bcosC - s + c}{(s-a)\cosh}$

Now let $ED$ and $MN$ intersect at $X'$ and $MN$ cuts $AC$ at $T$.
$\frac{AT}{TC} = \frac{s}{s-b}$ gives $AT = \frac{bs}{2s-b}$ and $ET = \frac{2s(s-c)-ab}{2s-b}$.

Now in $\triangle EDC$ with triad $MN$ we get by Menelaus $\frac{ET}{TC}.\frac{CM}{MD}.\frac{X'D}{X'E} = 1$

Again putting the values of $ET, TC$ we get $\frac{X'D}{X'E} = \frac{b(b-c)}{a^2+b^2-c^2}$ and now its easy to show that $\frac{XD}{XE} = \frac{X'D}{X'E}$ and hence the result follows.

Quick Reply

# High School Olympiads

## Euler line and incenter

Reply

**Source**: Crux

**Moonmathpi496**
#1 May 25, 2009, 6:03 pm

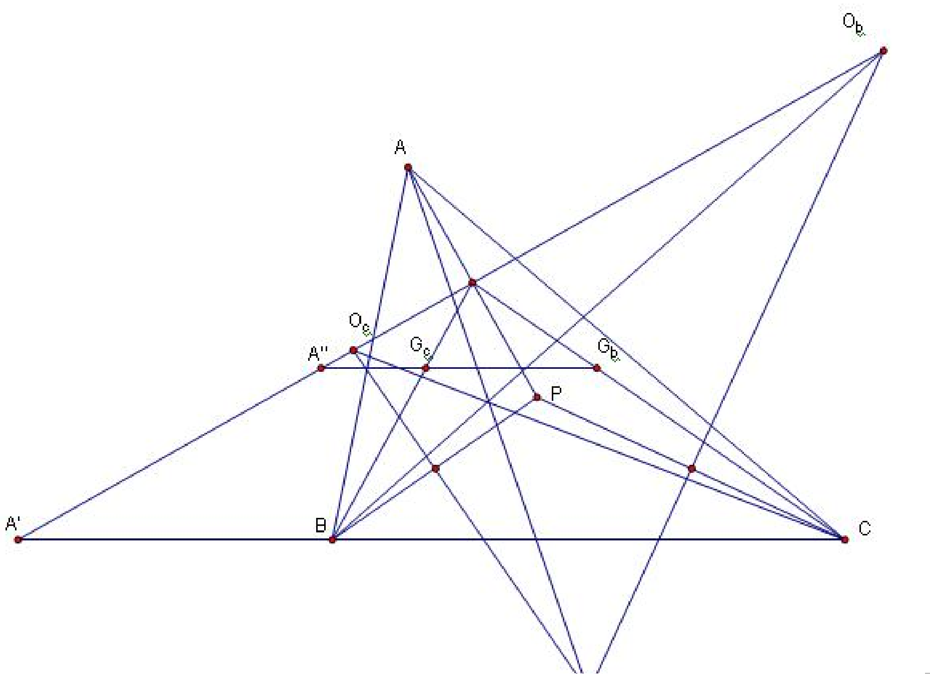
Prove that, the Euler line of the triangles $IBC, ICA, IAB$ are concurrent where $I$ is the incenter of $\triangle ABC$.

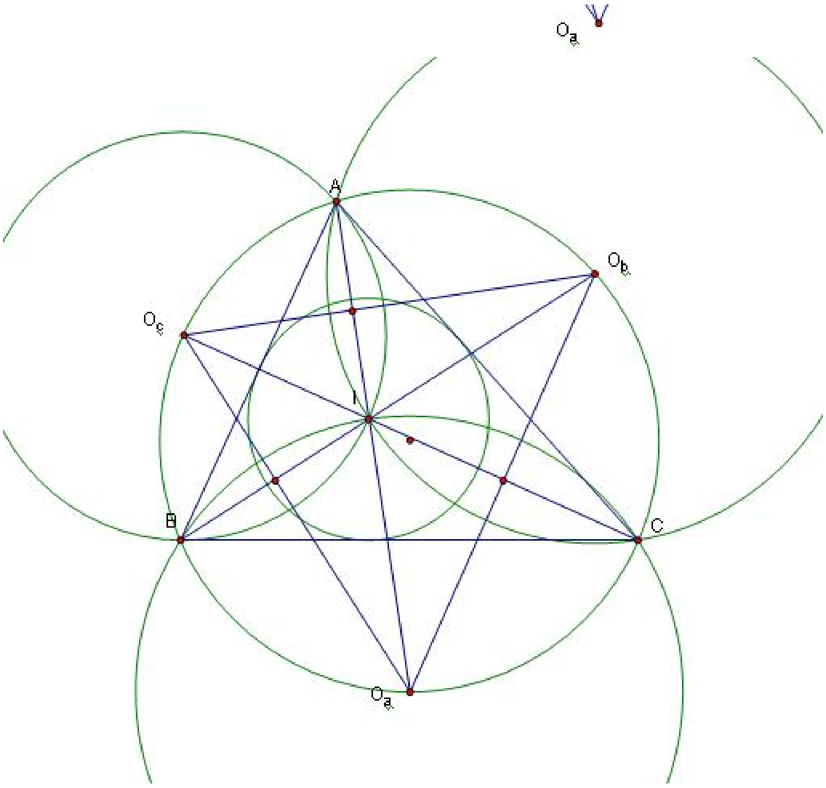
**livetolove212**
#2 May 25, 2009, 7:16 pm • 3

> Moonmathpi496 wrote:
> Prove that, the Euler line of the triangles $IBC, ICA, IAB$ are concurrent where $I$ is the incenter of $\triangle ABC$.

**Lemma:** Given a triangle ABC. P lies in triangle ABC. $O_a, O_b, O_c$ are the circumcenters of triangle $PBC, PCA, PAB$ then the Euler line of triangle $PBC, PAC, PAB$ are concurrent if and only if $AO_a, BO_b, CO_c$ are concurrent. (*)
**Proof:** (figure 1)
By **Desargues**'s theorem we get 3 Euler lines are concurrent if and only if
$(G_bG_c \cap O_bO_c), (G_cG_a \cap O_cO_a), (G_aG_b \cap O_aO_b)$ are collinear.
$\Leftrightarrow \frac{A''G_b}{A''G_c} \cdot \frac{B''G_c}{B''G_a} \cdot \frac{C''G_a}{C''G_b} = 1 (1)$
Because $G_bG_c // BC$ then $\frac{A''G_b}{A''G_c} = \frac{A'B}{A'C}$ ...
So $(1) \Leftrightarrow \frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1$
$\Leftrightarrow AO_a, BO_b, CO_c$ are concurrent (**Desargues**'s theorem)
Return to this problem:
Let $O_a, O_b, O_c$ are 3 circumcenter of triangle $BOC, COA, AOB$. It's easy to see that $O_a, O_b, O_c$ is the midpoint of arcs $BC, CA, AB$, respectively.
Then $AO_a, BO_b, CO_c$ are concurrent.
Applying lemma (*)->QED

*Attachments:*






*This post has been edited 1 time. Last edited by livetolove212, May 25, 2009, 10:46 pm*

**Luis González**

#3 May 25, 2009, 9:24 pm

Further, they concur on the Euler line of $\triangle ABC.$ We use barycentrics WRT ABC.

Coordinates of centroid $G_c$ of $\triangle ABI$ are $G_c(2a+b+c:a+2b+c:c)$

Coordinates of the orthocenter $H_c$ of $\triangle ABI$ are $H_c(b-c:a-c:c)$

$\Longrightarrow \ell_c \equiv H_cG_c \equiv c(b+c)x - c(a+c)y - (a^2-b^2)z = 0$

Cyclically, equations of Euler lines $\ell_a, \ell_b$ of $\triangle BCI$ and $\triangle CAI$ are given by

$\ell_a \equiv (b^2-c^2)x + a(c+a)y - a(b+a)z = 0$

$\ell_b \equiv -b(b+c)x + (c^2-a^2)y + b(a+b)z = 0$

Hence, $\ell_a, \ell_b, \ell_c$ concur at $X_{21}\left(\frac{a-b-c}{b+c} : \frac{b-a-c}{c+a} : \frac{c-a-b}{a+b}\right)$

Which is the Schiffler's point of $\triangle ABC$ lying on its Euler line.

**math_07**

#4 May 25, 2009, 10:38 pm

> *livetolove212 wrote:*
>
> By **Desargues**'s theorem we get 3 Euler lines are concurrent if and only if $(G_bG_c \cap O_bO_c), (G_cG_a \cap O_cO_a), (G_aG_b \cap O_aO_b)$ are collinear.
> $\Leftrightarrow \frac{A''G_b}{A''G_c}.\frac{B''G_c}{B''G_a}.\frac{C''G_a}{C''G_b} = 1(1)$

> *livetolove212 wrote:*
>
> So $(1) \Leftrightarrow \frac{A'B}{A'C}.\frac{B'C}{B'A}.\frac{C'A}{C'B} = 1$
> $\Leftrightarrow AO_a, BO_b, CO_c$ are concurrent (**Dersaques**'s theorem)

Could you please explain these two parts? Unfortunately I don't understand them

**livetolove212**

*#5 May 25, 2009, 10:45 pm*

I let A' be the intersection of BC and $O_bO_c$, $A''$ be the intersection of $G_bG_c$ and $O_bO_c$.

> *livetolove212 wrote:*
>
> By **Desargues**'s theorem we get 3 Euler lines are concurrent if and only if $(G_bG_c\cap O_bO_c),(G_cG_a\cap O_cO_a),(G_aG_b\cap O_aO_b)$ are collinear.
> $\Leftrightarrow \frac{A''G_b}{A''G_c}.\frac{B''G_c}{B''G_a}.\frac{C''G_a}{C''G_b}=1(1)$

This part is Menelaus's theorem for triangle $G_aG_bG_c$.

> *livetolove212 wrote:*
>
> So $(1)\Leftrightarrow \frac{A'B}{A'C}.\frac{B'C}{B'A}.\frac{C'A}{C'B}=1$
> $\Leftrightarrow AO_a,BO_b,CO_c$ are concurrent (**Dersaques**'s theorem)

Fom $\frac{A'B}{A'C}.\frac{B'C}{B'A}.\frac{C'A}{C'B}=1$ we get $A',B',C'$ are collinear then applying Dersargues's theorem we get $AO_a,BO_b,CO_c$ are concurrent.

**huymaick54**

*#6 Feb 4, 2010, 9:56 pm*

When P is altitude of triangle, your solution isn't right because $O_bO_c$ doesn't tough BC.
Can U slove it when $O_bO_c$//BC?

**livetolove212**

*#7 Feb 5, 2010, 3:44 pm*

No, my solution is still right if we note that $O_bO_c$ intersects $BC$ at the infinity point.

**huymaick54**

*#8 Feb 5, 2010, 10:35 pm*

Not right for high school students! Can U slove it for all school student?

Quick Reply

# High School Olympiads

## fun cyclic quadrilateral ABCD

Reply

Source: Art of Problem Solving: Volume 2

**modularmarc101**
#1 May 25, 2009, 5:29 am

Consider cyclic quadrilateral $ABCD$ with $AB = 6, BC = 7, CD = 8,$ and $AD = 9$. Find $(AC)^2$.

**Luis González**
#2 May 25, 2009, 6:12 am

Label $AB = a, BC = b, CD = c, DA = d, BD = m, AC = n.$ Using the 1st and 2nd Ptolemy's theorem we get

$$\frac{m^2}{n^2} = \frac{(ab+cd)^2}{(ad+bc)^2}, \ m^2n^2 = (ac+bd)^2 \Longrightarrow n^2 = \frac{(ac+bd)(ad+bc)}{ab+cd} = \frac{2035}{19}$$

**Rofler**
#3 May 25, 2009, 6:13 am

Let $\alpha$ be the angle at B. Then the angle at C is $180 - \alpha$.

By applying the cosine law twice,

$$AC^2 = 6^2 + 7^2 - 2*6*7cos\alpha$$
$$AC^2 = 8^2 + 9^2 + 2*8*9cos\alpha.$$

Equate, solve for $\alpha$, and plug it back in.

**modularmarc101**
#4 May 25, 2009, 6:22 am

Yeah that was the ideal solution.

Quick Reply

# High School Olympiads

## Angola 1971 [isotome of Lemoine axis perpendicular to Euler]



**Source**: UN

**pestich**

#1 Jan 13, 2005, 11:07 am

The centers of three circles isotomic with the Apollonian circles of triangle $ABC$ located on a line perpendicular to the Euler line of $ABC$.

**grobber**

#2 Jan 13, 2005, 3:31 pm

What's a circle isotomic to another circle? Is it the locus of the points isotomic to the points of the first circle? (I'm not sure that locus is a circle; I'm just asking)

**darij grinberg**

#3 Jan 14, 2005, 11:46 pm

Actually, the problem should be:

*The reflections of the centers of the three Apollonian circles of a triangle ABC in the midpoints of the respective sides BC, CA, AB lie on one line which is perpendicular to the Euler line of triangle ABC.*

Here is my **solution**:

First, we use a lemma which I showed in post #2 in the thread http://www.mathlinks.ro/Forum/viewtopic.php?t=16823 :

**Lemma 1.** Let P be a point in the plane of a triangle, and let the lines AP, BP, CP intersect the lines BC, CA, AB at the points X, Y, Z, respectively. Let the lines YZ, ZX, XY meet the lines BC, CA, AB at the points X', Y', Z', respectively. It is well-known that the points X', Y', Z' lie on one line, the so-called **tripolar** of the point P with respect to triangle ABC. Now, let the parallel to the line YZ through the point A intersect the line BC at A'. Then, the line X'Y'Z' (the tripolar of the point P with respect to triangle ABC) passes through the midpoint of the segment AA'.

Now, in the configuration of Lemma 1, we will show:

**Lemma 2.** If we consider not only the point A' where the parallel to the line YZ through the point A intersects the line BC, but also the points B' and C' where the parallels to the lines ZX and XY through the points B and C intersect the lines CA and AB, then the reflections A", B", C" of these points A', B', C' in the midpoints of the sides BC, CA, AB of triangle ABC lie on one line, and this line is the image of the line X'Y'Z' (the tripolar of the point P with respect to triangle ABC) under the homothety with center G and factor -2. Hereby, G is the centroid of triangle ABC.

*Proof.* Let T be the midpoint of the segment AA'; then, after Lemma 1, this point T lies on the line X'Y'Z'.

Let M be the midpoint of the side BC of triangle ABC. Then, the segment AM is a median of triangle ABC, and thus it passes through the centroid G of triangle ABC and is divided by G in the ratio AG : GM = 2 (with directed segments).

Since the point A" is the reflection of the point A' in the midpoint M of the side BC of triangle ABC, the point M is also the midpoint of the segment A'A". Hence, the segment AM is a median of triangle AA'A", and thus it passes through the centroid G' of triangle AA'A" and is divided by G' in the ratio AG' : G'M = 2 (with directed segments).

Now since both points G and G' lie on the segment AM and divide the segment AM in the same ratio AG : GM = 2 and AG' : G'M = 2, it follows that these points G and G' coincide. In other words, the centroid G of triangle ABC coincides with the centroid G' of triangle AA'A"; this yields that the point G is the centroid of triangle AA'A".

Now, since the point T is the midpoint of the side AA' of triangle AA'A", the line A"T is a median of triangle AA'A", and hence, it passes through the centroid G of triangle AA'A" and is divided by G in the ratio A"G : GT = 2. Hence, GA" : GT = - A"G : GT = -2. Consequently, the homothety with center G and factor -2 maps the point T to the point A". But since the point T lies on the line X'Y'Z', its image A" must therefore lie on the image of the line X'Y'Z' under the homothety with center G and factor -2. Similarly, the points B" and C" also lie on the image of the line X'Y'Z' under this homothety. Hence, the three points A", B", C" lie on one line, and this line is the image of the line X'Y'Z' under the homothety with center G and factor -2. Lemma 2 is proven.

Now, consider a particular case of all the above: In fact, let P be the orthocenter of triangle ABC. Then, the lines AP, BP, CP are the three altitudes of triangle ABC, and their points of intersection X, Y, Z with the lines BC, CA, AB are the feet of these altitudes. The triangle XYZ is the orthic triangle of triangle ABC; hence, by a well-known theorem, the sides YZ, ZX, XY of this orthic triangle are antiparallel to the sides BC, CA, AB of triangle ABC with respect to the triangle ABC. In particular, the line YZ is antiparallel to the side BC of triangle ABC. Thus, the line AA', being parallel to the line YZ, is also antiparallel to the side BC of triangle ABC. But an antiparallel to a side of a triangle through the opposite vertex is the tangent to the circumcircle of the triangle at that vertex; hence, the line AA', being antiparallel to the side BC of triangle ABC and passing through the opposite vertex A, must be the tangent to the circumcircle of triangle ABC at the vertex A. So, now we know what the point A' really is: It is the point where the tangent to the circumcircle of triangle ABC at the vertex A intersects the line BC. But it is well-known that this point is the center of the A-Apollonius circle of triangle ABC. Hence, we can conclude that the point A' is the center of the A-Apollonius circle of triangle ABC. Similarly, the point B' is the center of the B-Apollonius circle, and the point C' is the center of the C-Apollonius circle. And, of course, the points A", B", C" are the reflections of these centers A', B', C' in the midpoints of the sides BC, CA, AB of triangle ABC. Now the link to your problem is clear: Your problem just asks to prove that the points A", B", C" lie on one line which is perpendicular to the Euler line of triangle ABC. Actually, the fact that the points A", B", C" lie on one line now immediately follows from Lemma 2, so it remains to show that this line A"B"C" is perpendicular to the Euler line of triangle ABC.

In fact, Lemma 2 states that the line A"B"C" is the image of the line X'Y'Z' under a certain homothety. Since a homothety maps any line to a parallel line, it follows that the line A"B"C" is parallel to the line X'Y'Z'. Hence, instead of proving that the line A"B"C" is perpendicular to the Euler line of triangle ABC, it is enough to show that the line X'Y'Z' is perpendicular to the Euler line of triangle ABC. But remembering how we defined the points X', Y', Z' - namely, we defined them as the points of intersection of the lines YZ, ZX, XY with the lines BC, CA, AB, and the points X, Y, Z were the feet of the altitudes of triangle ABC -, we see that this directly follows from Lemma 1 of the posting [Schroeder6] in my "Schröder Points Database". And by the way, the line X'Y'Z' is called the **orthic axis** of triangle ABC.

■

Darij

*This post has been edited 1 time. Last edited by darij grinberg, Sep 13, 2009, 5:04 pm*

**Luis González**

*#4 May 24, 2009, 11:29 pm*

Let $O_a, O_b, O_c$ the centers of the A-,B- and C- Apollonius circles of $\triangle ABC$. These centers are the intersections of the tangents to the circumcircle $(O)$ at $A, B, C$ with $BC, CA, AB$, respectively. Using barycentric coordinates, we get

$$O_a(0 : b^2 : -c^2)\ ,\ O_b(-a^2 : 0 : c^2)\ ,\ O_c(a^2 : -b^2 : 0)$$

Reflections of $O_a, O_b, O_c$ about the midpoints of $BC, CA, AB$ are:

$$U_a(0 : -c^2 : b^2)\ ,\ U_b(c^2 : 0 : -a^2)\ ,\ U_c(-b^2 : a^2 : 0)$$

$U_a, U_b, U_c$ are collinear on the line $\mathcal{L} \equiv a^2x + b^2y + c^2z = 0$, which is the trilinear polar of the 3rd Brocard point of $\triangle ABC$, $X_{76}\left(\frac{1}{a^2} : \frac{1}{b^2} : \frac{1}{c^2}\right)$. The infinite point of $\perp \mathcal{L}$ is then $T_\infty(S_BS_A + S_CS_A - 2S_BS_C)$, which coincides the infinite point $X_{30}$ of the Euler line $e$ of $\triangle ABC \Longrightarrow e \perp \mathcal{L}$. In addition, $X_{858} \equiv \mathcal{L} \cap e$.

**yetti**

*#5 May 25, 2009, 7:53 am*

$O$ is circumcenter, $H$ orthocenter, $N$ 9-point circle center of $\triangle ABC$. Let $(U), (V), (W)$ be the A-, B-, C-Apollonius circles of $\triangle ABC$ and $(X), (Y), (Z)$ their reflections in the perpendicular bisectors of $BC, CA, AB$. Let $(U'), (V'), (W')$ be A'-, B'-, C'-Apollonius circles of its medial $\triangle A'B'C'$ and $(X'), (Y'), (Z')$ their reflections in the perpendicular bisectors of $B'C', C'A', A'B'$. Quadrilaterals $AU'A'X', BV'B'Y', CW'C'Z'$ are parallelograms $\Longrightarrow (X'), (Y'), (Z')$ are circles with diameters $AU, BV, CW$. Powers of $O$ to $(X'), (Y'), (Z')$ are all equal to $OA^2 = OB^2 = OC^2$ and powers $H$ to $(X'), (Y'), (Z')$ are all equal to half the power of $H$ to $(O) \Longrightarrow ONH \perp X'Y'Z'$ is radical axis of $(X'), (Y'), (Z') \Longrightarrow ONH \perp XYZ$ is radical axis of $(X), (Y), (Z)$.

EDIT: See also http://www.mathlinks.ro/viewtopic.php?t=278520 for another solution.

Quick Reply

# Spain

## Geometria chevere

Reply

**Leonardo**
#1 Feb 22, 2005, 12:35 pm

Dado un triangulo $\triangle ABC$, la tangente por $A$ al circuncirculo corta a $BC$ en $D$. La perpendicular a $BC$ por $B$ corta la mediatrez de $AB$ en E, y la perpendicular por $BC$ en $C$ corta a la mediatriz de $AC$ en $F$. Demuestre que $D$, $E$,$F$ son colineales.

**Leonardo**
#2 Mar 15, 2005, 11:52 am

A nadie se le ocurre una solucion?

**grobber**
#3 Mar 15, 2005, 12:49 pm

Try searching for it in the Geometry section. I'm sure it's been posted before.

**Leonardo**
#4 Mar 16, 2005, 7:59 am

its ok grobber, I already solved this problem, i just posted it to enrich the spanish comunity content

**Hector**
#5 Jun 13, 2005, 4:12 am

Esta Facil,

Tenemos el triangulo ABC, con angulos A, B, C => A+B+C=180
Notemos que angulo EBC= 90-B => BEC=B
por lo tanto
BE= AB/(2senB), idem FC=AC/(2senC)

Veamos que basta ver que EB/FC=DB/DC, para ver que D,F,E son colineales [un argumento seria EBD semejante con FCD, por razon-angulo-razon]

Por un lado EB/FC= *ABsenC / ACsenB*

Luego con Teo. Bisectriz generalizado, en ABC con AD tenemos: (Nota, como AD es tangente, DAB=C)
BD/DB= ABsenC / ACsen (A+C) =*ABsenC / ACsen (B)*

<u>Y acabamos</u>

[/hide][/list][/code]

**Luis González**
#6 May 24, 2009, 11:32 am

Supongamos que $D' \equiv EF \cap BC$ siendo $D$ distinto de $D'$. Por semejanza tenemos que:

$\frac{DC}{DB} = \frac{FC}{EB}$. Facilmente aplicando teorema del seno, reescribimos:

$DB \qquad EB$

$$\frac{DC}{DB} = \frac{FC}{EB} = \frac{b\sin B}{c\sin C}$$

Aplicando Ley del seno en los triángulos $\triangle AD'C$ y $\triangle AD'B$, teniendo presente que $\angle BAD' = \angle ACD'$, tenemos

$\dfrac{D'C}{D'B} = \dfrac{b\sin B}{c\sin C} \Longrightarrow D = D'$ y con esto demostramos la colinealidad de $E, F, D$

Quick Reply

# Spain

## Problema de geometría

Reply

**xXx--gm--xXx**

#1 May 22, 2009, 8:07 pm

Alguien sabe exactamente por qué se le da una rotación de 60 grados???? (apart del triándulo equilátero)

Gracias, apreciare mucho la ayuda

**Luis González**

#2 May 24, 2009, 10:04 am

Hola, trataría de ayudarte si explicases mejor lo que te inquieta. A que te refieres con rotación de 60 grados?, en que caso particular?. En el caso de la determinación del punto de Fermat del triángulo?. De ser el caso, puede verse el siguient link para una demostración http://centros5.pntic.mec.es/~marque12/matem/fermat.htm

Quick Reply

# High School Olympiads

## Kariya´s theorem

Reply

**Max D.R.**

#1 May 23, 2009, 11:12 pm

What is Kariya's theorem?
Help, please.

**mihai miculita**

#2 May 24, 2009, 1:08 am

Let D, E and F the points of contact of incircle (I; r) of triangle ABC with the sides BC, AC and AB, respectively. If $(I;\rho)\cap(ID=\{A'\};(I;\rho)\cap(IF=\{B'\};(I;\rho)\cap(IF=\{C'\}$, then lines AA', BB' and CC' concurrent!

*This post has been edited 1 time. Last edited by mihai miculita, May 24, 2009, 3:46 pm*

**Luis González**

#3 May 24, 2009, 3:36 am

**Kariya's theorem:** The homothetic circle of the incircle $(I,r)$ of $\triangle ABC$ through the homothety with center $I$ and factor $k$ meets the perpendicular rays from $I$ to $BC, CA, AB$ at $X,Y,Z.$ Then $\triangle XYZ$ and $\triangle ABC$ are perspective.

$U,V,W$ are the tangency points of $(I)$ with $BC,CA,AB.$ Using barycentric coordinates WRT $\triangle ABC$, we have

$I \equiv (a:b:c) \ , \ U \equiv (0:ab+S_C:ac+S_B)$

$V \equiv (ba+S_C:0:bc+S_A) \ , \ W \equiv (ab+S_C:bc+S_A:0)$

Using the barycentric equations of the homothety $\gamma(I,k)$, we get the coordinates of $X,Y,Z$ as

$X \equiv (-a^2(k+1):ab+kS_C:ac+kS_B)$

$Y \equiv (ab+kS_C:-b^2(k-1):bc+kS_A)$

$Z \equiv (ca+kS_B:bc+kS_A:-c^2(k-1))$

Therefore, $\triangle ABC$ and $\triangle XYZ$ are perspective through a point $J$

$$J\left(\frac{1}{bc+kS_A}:\frac{1}{ac+kS_B}:\frac{1}{ab+kS_C}\right).$$

**jayme**

#4 May 24, 2009, 2:54 pm

Dear Mathlinkers,
see for example : http://pagesperso-orange.fr/jl.ayme/vol2.html Le point de Gray (Kariya) p. 8-9.
Sincerely
Jean-Louis

**vittasko**

#5 May 24, 2009, 4:48 pm

Also, we can see this theorem, as a particular case of the Jacobi theorem ( or **Isogonic theorem** ).

( the line segments $AB'$, $AC'$, are isogonal with respect to the angle $\angle A$ and similarly for the pairs of the line segments, through the other vertices of a given triange $\triangle ABC$ ).

Kostas Vittas.

**Max D.R.**

*#6 May 30, 2009, 10:38 pm*

Thank you for the help to the persons who wrote answering my question.

Max D.R.

Quick Reply

# High School Olympiads

## Dedicate to COSMIN. MA=NC <==> TA/TC=SA/SC=DA/DC .

Reply

**Source**: An own generalization of ...t=277841 .

**Virgil Nicula**
#1 May 22, 2009, 8:40 am

Let $ABCD$ be a convex quadrilateral for which denote $\left|\begin{array}{c} E\in AB\cap CD\\\\ F\in AD\cap BC\end{array}\right|$. Define the points $\left|\begin{array}{c} N\in (FB\ ,\ FN=FA\\\\ M\in (EB\ ,\ EM=EC\\\\ Q\in AN\cap CM\ ,\ T\in DQ\cap AC\end{array}\right|$ and $\left|\begin{array}{c} P\in AD\cap CM\\\\ R\in CD\cap AN\\\\ S\in AC\cap PR\end{array}\right|$. Prove that $MA=NC\iff$ $\frac{TA}{TC}=\frac{DA}{DC}\iff\frac{SA}{SC}=\frac{DA}{DC}$, i.e. the rays $[DQ$ , $[DS$ are the bisectors of the angle $\widehat{ADC}$ .

See **here** a particular case when $ABCD$ is a parallelogram.

*This post has been edited 4 times. Last edited by Virgil Nicula, May 23, 2009, 4:19 am*

**Luis González**
#2 May 22, 2009, 10:23 am

By Menelaus' theorem for $\triangle FDC$ cut by transversal $\overline{ANR}$, we get

$$\frac{RC}{RD}\cdot\frac{DA}{FA}\cdot\frac{FN}{NC}=1\Longrightarrow\frac{RC}{RD}=\frac{NC}{DA}\ \ (1)$$

By Menelaus' theorem for $\triangle EAD$ cut by transversal $\overline{PMC}$, we get

$$\frac{PA}{PD}\cdot\frac{DC}{CE}\cdot\frac{EM}{AM}=1\Longrightarrow\frac{PA}{PD}=\frac{AM}{DC}\ \ (2)$$

By Menelaus' theorem for $\triangle DPR$ cut by transversal $\overline{ACS}$, we get

$$\frac{SC}{SA}\cdot\frac{PA}{PD}\cdot\frac{RD}{RC}=1\ \ (3)$$

Combining $(3)$ with $(1)$ and $(2)$ we get : $\frac{SC}{SA}\cdot\frac{MA}{DC}\cdot\frac{DA}{NC}=1$

Hence, it follows that $MA=NC\iff\frac{SA}{SC}=\frac{DA}{DC}$

**Virgil Nicula**
#3 May 23, 2009, 4:14 am

Thank you, **Luisgeometria !**

Your proof is identically with mine. See again the enunciation of my proposed problem because I added an interesting question to the conclusion.

**Luis González**
#4 May 23, 2009, 4:45 am

> Vigil Nicula wrote:
> the rays $[DQ, [DS$ are the bisectors of the angle $\widehat{ADC}$.

We already know that $\frac{SC}{SA} = \frac{DC}{DA} \Longleftrightarrow MA = NC$. Then it follows that $S$ is the foot of the external bisector of $\angle ADC$ in $\triangle ADC$. Thus, $T$ is the internal bisector of $\angle ADC$ since the pencil of rays $DA, DC, DS, DQ$ is harmonic $\Longrightarrow$ $(A, C, S, T)$ are harmonically separeted $\Longrightarrow DQ$ and $DS$ bisect $\angle ADC$ internally and externally.

**Virgil Nicula**
#5 May 23, 2009, 10:36 pm

**Here is the essential enunciation of my proposed problem and its proof. I like it very much.**

> Virgil Nicula wrote:
> Let $ABCD$ be a convex quadrilateral for which denote $\left|\begin{array}{c} E\in AB\cap CD\\\\ F\in AD\cap BC\end{array}\right|$.
>
> Define the points $\left|\begin{array}{c} N\in (FB\ ,\ FN=FA\\\\ M\in (EB\ ,\ EM=EC\end{array}\right|$ and $Q\in AN\cap CM$.
>
> Prove that $MA=NC\ \Longleftrightarrow$ the rays $[DQ$ is one of the the bisectors of the angle $\widehat{ADC}$.
>
> See **here** a particular case when $ABCD$ is a parallelogram.

**Proof.** Define the points $\left|\begin{array}{c} P\in MC\cap AD\\\\ R\in NA\cap CD\end{array}\right|$. Apply the **Menelaus' theorem** to the mentioned transversals/triangles :

$$\left|\begin{array}{ccc} \overline{MCP}/ADE & : & \frac{PD}{PA}\cdot\frac{MA}{ME}\cdot\frac{CE}{CD}=1\ \stackrel{(CE=ME)}{\Longrightarrow}\ \frac{PA}{PD}=\frac{MA}{CD}\ \ (1)\\\\ \overline{NAR}/CDF & : & \frac{RD}{RC}\cdot\frac{NC}{NF}\cdot\frac{AF}{AD}=1\ \stackrel{(AF=NF)}{\Longrightarrow}\ \frac{RD}{RC}=\frac{AD}{NC}\ \ (2)\end{array}\right|$$

. Define $T\in DQ\cap AC$ and apply the **Ceva's theorem** to $Q$ for $\triangle\ ACD\ :\ \frac{TA}{TC}\cdot\frac{RC}{RD}\cdot\frac{PD}{PA}=1\Longrightarrow\frac{TA}{TC}=\frac{PA}{PD}\cdot\frac{RD}{RC}\stackrel{(1)\wedge(2)}{\Longrightarrow}$ $\boxed{\frac{TA}{TC}=\frac{MA}{NC}\cdot\frac{DA}{DC}}\ (3)$.

In conclusion, $MA=NC\ \Longleftrightarrow\ \frac{TA}{TC}=\frac{DA}{DC}\ \Longleftrightarrow$ the rays $[DQ$ is one of the bisectors of the angle $\widehat{ADC}$.

**Remark.** Define the point $S\in PR\cap AC$ and apply the **Menelaus' theorem** to the pair $\overline{PRS}/ACD$ :

$$\frac{PD}{PA}\cdot\frac{SA}{SC}\cdot\frac{RC}{RD}=1\Longrightarrow\frac{SA}{SC}=\frac{PA}{PD}\cdot\frac{RD}{RC}\stackrel{(1)\wedge(2)}{\Longrightarrow}\frac{SA}{SC}=\frac{MA}{NC}\cdot\frac{DA}{DC}\ \stackrel{(3)}{\Longrightarrow}\frac{SA}{SC}=\frac{TA}{TC}.$$

In conclusion, $MA=NC\ \Longleftrightarrow$ the ray $[DS$ is another bisector of the angle $\widehat{ADC}$.

Quick Reply

# High School Olympiads

## easy problem (equilateral triangle)



**Source**: Irish Mathematical Olympiad 2008

**huang**
#1 May 21, 2009, 5:17 am

Point $P$ is inside an equilateral triangle $ABC$ such that $PA = 3$, $PB = 4$ and $PC = 5$. Find the area of the triangle.

**sunken rock**
#2 May 21, 2009, 2:27 pm

Rotate the triangle ABP 60 degs around B, A maps C and P goes to A'. The triangle BA'P is equilateral, while PCA' is the one constructed with the segments PA, PB and PC as sides.
Rotate the triangle BPC 60 degs around C, B maps A, P goes to B'; the triangle CPB' is equilateral, APB' the one constructed with the segments PA, PB and PC as sides.
Finally, rotate the triangle ACP 60 degs around A, C maps B', P goes to C'; the triangle APC' is equilateral, BPC' the one constructed with the segments PA, PB and PC as sides.

Thus the hexagon AC'CA'CB' has the area twice the one of ABC, as constructed. (*)
The triangles PA'C, PB'A and PC'B are equal, their area being 6 (they are right-angled, as their sides are 3-4-5). (**)
The triangles APC', BPA' and CPB' are equilateral, their sides being, of course, PA, PB and PC respectively, hence their total area will be [(PA^2 + PB^2 + PC^2)sqrt(3)]/4 = 12.5 sqrt(3)(***)

Hence 2S = 18 + 12.5 sqrt(3), that is, S = 9 + 6.25 sqrt(3).

Best regards,
sunken rock

**Luis González**
#3 May 22, 2009, 12:37 am

This could be one of those few applications of the tripolar coordinates.

In the equilateral triangle $\triangle ABC$, the tripolar coordinates of $P(x, y, z)$ satisfy the relation:

$$a^2(x^2+y^2+z^2)+x^2y^2+y^2z^2+x^2z^2-(x^4+y^4+z^4)-a^4=0$$

Substituting $(x, y, z) = (3, 4, 5)$ yields the biquartic equation

$$a^4-50a^2+193=0 \Longrightarrow a^2=25+12\sqrt{3} \Longrightarrow [\triangle ABC]=\frac{25}{4}\sqrt{3}+9.$$

**mihai miculita**
#4 May 22, 2009, 1:44 am

If $A_i; i=\overline{1;4}$ four coplanare points and $a_{ij}=|A_iA_j|,$ then have:

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} & 1 \\ a_{12} & 0 & a_{23} & a_{24} & 1 \\ a_{13} & a_{23} & 0 & a_{34} & 1 \\ a_{14} & a_{24} & a_{34} & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

If $a_{12}=a_{13}=a_{23}=a, a_{14}=3, a_{24}=4$ and $a_{34}=5,$ then:

$$\begin{vmatrix} 0 & a & a & 3 & 1 \\ a & 0 & a & 4 & 1 \\ a & a & 0 & 5 & 1 \end{vmatrix} = 0 \Rightarrow a = \cdots \Rightarrow S \ldots = \frac{a^2.\sqrt{3}}{} =$$

$$\begin{vmatrix} 3 & 4 & 5 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0 \Rightarrow a = \quad \Rightarrow S_{A_1A_2A_3} = \quad \frac{}{4} = \ldots$$

**mihai miculita**

*#5 May 23, 2009, 2:00 am*

Sorry, for my error...

$$\begin{vmatrix} 0 & a_{12}^2 & a_{13}^2 & a_{14}^2 & 1 \\ a_{12}^2 & 0 & a_{23}^2 & a_{24}^2 & 1 \\ a_{13}^2 & a_{23}^2 & 0 & a_{34}^2 & 1 \\ a_{14}^2 & a_{24}^2 & a_{34}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 0 & a^2 & a^2 & 3^2 & 1 \\ a^2 & 0 & a^2 & 4^2 & 1 \\ a^2 & a^2 & 0 & 5^2 & 1 \\ 3^2 & 4^2 & 5^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0 \ldots$$

Quick Reply

# High School Olympiads

## N,P,Q are collinear

Reply

Source: To all mathlinkers

**mathVNpro**

#1 May 21, 2009, 8:04 am

Let $(O_1), (O_2)$ be $2$ circles intersect each other at $P, Q$. Let $AB$ be the external tangent of $(O_1), (O_2)$, $AB$ is nearer $P$ than $Q$ and $A \in (O_2)$, $B \in (O_1)$. The line through $Q$ which is parrallel to $AB$ intersects $(O_1), (O_2)$, respectively, at $C, D$. Let $N$ be the second intersection of $(ACQ)$ and $(BDQ)$. Prove that: $N, P, Q$ are collinear.

**vittasko**

#2 May 21, 2009, 8:03 pm

Let be the points $A' \equiv (O'_2) \cap AB$ and $B' \equiv (O'_1) \cap AB$, where $(O'_1),\ (O'_2)$ are the circumcirles of the triangles $\triangle AQC,\ \triangle BQD$, respectively.

Because of the isosceles trapeziums $AQCB',\ BQDA'$, we conclude that their circumcenters $O'_1,\ O'_2$, lie on the line segments $O_1B,\ O_2A$, as the midperpendiculars of the cords $QC,\ QD$, respectively.

Because of now, $O'_2A \perp A'B$ and $O'_1B \perp AB'$, we have that $AA' = AB = BB'$ , $(1)$

Let be the point $M \equiv AB \cap PQ$ and then we have $MA = MB$ , $(2)$ as well ( from $(MA)^2 = (MP)\cdot(MQ) = (MB)^2$ ).

We denote the points $N \equiv (O'_1) \cap PQ$ and $N' \equiv (O'_2) \cap PQ$.

From the powers of $M$, with respect to the circles $(O'_1),\ (O'_2)$,

we have respectively that $(MN)\cdot(MQ) = (MA)\cdot MB'$ , $(3)$ and $(MN')\cdot(MQ) = (MB)\cdot(MA')$ , $(4)$

From $(1),\ (2),\ (3),\ (4) \Longrightarrow (MN)\cdot MQ) = (MN')\cdot(MQ) \Longrightarrow MN' = MN \Longrightarrow N' \equiv N$ and the proof is completed.

Kostas Vittas.

*Attachments:*
t=277929.pdf (15kb)

*This post has been edited 5 times. Last edited by vittasko, May 22, 2009, 4:08 pm*

**long14893**

#3 May 21, 2009, 8:20 pm

> *vittasko wrote:*
> Let be the points $A' \equiv (O_2) \cap AB$ and $B' \equiv (O_1) \cap AB$.

AB is the external tangent of $(O_1), (O_2)$ , so where is A' and B' ?

**mathVNpro**

#4 May 21, 2009, 9:52 pm

Last part of Vittako's proof can be seen with the "eye" of **radical axes**. Indeed:
Because $M$ is the midpoint of $AB$ and $BB' = AB = AA'$ hence $MA' = MB'$. Therefore $\overline{MA}.\overline{MB'} = \overline{MB}.\overline{MA'}$ $\Longrightarrow \mathcal{P}_{M/(ACQ)} = \mathcal{P}_{M/(BDQ)}$, which implies that $M$ belongs to the radical axes of $(ACQ), (BDQ) \Longrightarrow M \in QN$. The result is lead as follow...

result is lead as follow.

**Luis González**

#5 May 21, 2009, 10:53 pm

Inversion with center $Q$ and arbitrary power takes $(O_1)$ and $(O_2)$ into the lines $\ell_1, \ell_2$ meeting at the inverse $P'$ of $P$ and the common tangent $\overline{AB}$ into a circle $\omega$ passing through $Q$ and tangent to $\ell_1, \ell_2$ through $B', A'$. Circles $\odot(ACQ)$ and $\odot(BDQ)$ are taken into the lines $A'C'$ and $B'D'$, but $\omega$ is the incircle of the triangle $\triangle P'C'D'$ bounded by $\ell_1, \ell_2, CD$ due to $AB \parallel CD$. Since $C'A', D'B'$ and $P'Q$ concur at the Gergonne point of $\triangle P'C'D'$, then $\odot(ACQ), \odot(BDQ)$ and $PQ$ meet again on $PQ \Longrightarrow N, P, Q$ are collinear.

**vittasko**

#6 May 22, 2009, 4:02 pm

> *long14893 wrote:*
> ...AB is the external tangent of $(O_1), (O_2)$, so where is A' and B' ?

Thank you dear **long14893**, for pointing out of my mistakes. I have corrected them (I hope).

Best regards, Kostas Vittas.

Quick Reply

# High School Olympiads

## The line PQ, bisects the segment BC.

Reply

vittasko

#1 Dec 6, 2008, 3:42 pm

**A triangle $\triangle ABC$ is given with $AD$ its $A$-altitude and let $X,\ Y$ be, the intersection points of $AB,\ AC$ respectively, from the circle $(K)$ with diameter $AD$. We draw the tangent lines of $(K)$ at points $X,\ Y$, which intersect $BC$ at points $E,\ F$, respectively. Prove that the line segment $PQ$, where $P\equiv BY\cap CX$ and $Q\equiv EY\cap FX$, bisects the side-segment $BC$ of $\triangle ABC$.**

Kostas Vittas.

*Attachments:*

t=243720.pdf (4kb)

Petry

#2 Dec 7, 2008, 3:42 am • 1

Hello!

It's easy to prove that $EX=ED=EB$ and $FY=FD=FC$.

Let $M$ be the midpoint of the side $BC$.
$\{V'\}=XE\cap AM$ and $\{V"\}=YF\cap AM$. Let's prove that $V'=V"=V$.

$$\frac{V'A}{V'M}=\frac{XA}{XB}\cdot\frac{EB}{EM}=\frac{AD\cdot sin\left(\angle B\right)}{BD\cdot cos\left(\angle B\right)}\cdot\frac{\frac{BD}{2}}{\frac{BC}{2}-\frac{BD}{2}}=\frac{AD}{BD}\cdot\frac{BD}{CD}\cdot tg\left(\angle B\right)=$$
$$=\frac{AD}{CD}\cdot tg\left(\angle B\right)=\frac{AD^2}{BD\cdot CD}\Rightarrow\frac{V'A}{V'M}=\frac{AD^2}{BD\cdot CD}\ (1)$$

Similarly $\Rightarrow\frac{V"A}{V"M}=\frac{AD^2}{BD\cdot CD}$ (2)

(1),(2) $\Rightarrow V'=V"=V\Rightarrow M\in AV$

So $\frac{VA}{VM}=\frac{AD^2}{BD\cdot CD}=tg\left(\angle B\right)\cdot tg\left(\angle C\right)\Rightarrow\frac{MA}{MV}=\frac{cos\left(\angle A\right)}{cos\left(\angle B\right)\cdot cos\left(\angle C\right)}$ (*)

It's easy to prove that the triangle $\Delta VXY$ is isosceles $(VX=VY)$ and $\angle VXY=\angle VYX=\angle BAC$.

$$\{N\}=VQ\cap XY\Rightarrow\frac{NX}{NY}=\frac{EX}{EV}\cdot\frac{FV}{FY}=\frac{BD}{CD}\cdot\frac{sin\left(2\angle B\right)}{sin\left(2\angle C\right)}$$

$$\{D'\}=VN\cap BC\Rightarrow\frac{D'E}{D'F}=\frac{\frac{VD'\cdot sin\left(\angle XVN\right)}{sin\left(2\angle B\right)}}{\frac{VD'\cdot sin\left(\angle YVN\right)}{sin\left(2\angle C\right)}}=\frac{sin\left(\angle XVN\right)}{sin\left(\angle YVN\right)}\cdot\frac{sin\left(2\angle C\right)}{sin\left(2\angle B\right)}=$$
$$=\frac{NX}{NY}\cdot\frac{sin\left(2\angle C\right)}{sin\left(2\angle B\right)}=\frac{BD}{CD}=\frac{DE}{DF}\Rightarrow\frac{D'E}{D'F}=\frac{DE}{DF}\Rightarrow D'=D.$$

$$\frac{FV}{sin\left(2\angle B\right)}=\frac{EF}{sin\left(2\angle A\right)}\Rightarrow FV=\frac{BC}{2}\cdot\frac{sin\left(2\angle B\right)}{sin\left(2\angle A\right)}$$
$$\frac{QV}{QN}=\frac{XY}{XN}\cdot\frac{FV}{FY}=\frac{XY}{NX}\cdot\frac{BC\cdot sin\left(2\angle B\right)}{2\cdot sin\left(2\angle A\right)\cdot\frac{CD}{2}}=\frac{XY}{NX}\cdot\frac{BC}{CD}\cdot\frac{sin\left(2\angle B\right)}{sin\left(2\angle A\right)}=$$
$$=\frac{XY}{NX}\cdot\frac{\frac{AC\cdot sin\left(\angle A\right)}{sin\left(\angle B\right)}}{CD}\cdot\frac{2\cdot sin\left(\angle B\right)\cdot cos\left(\angle B\right)}{2\cdot sin\left(\angle A\right)\cdot cos\left(\angle A\right)}\Rightarrow\frac{QV}{QN}=\frac{XY}{NX}\cdot\frac{cos\left(\angle B\right)}{cos\left(\angle A\right)\cdot cos\left(\angle C\right)}\ (**)$$

$\{N'\}=AP\cap XY$

$$\frac{N'X}{} = \frac{XA\cdot sin\left(\angle BAP\right)}{} = \frac{sin\left(\angle B\right)}{}\ \frac{\frac{PB}{AB}}{} =$$

$$\frac{}{N'Y} = \frac{}{YA \cdot sin\left(\angle YAP\right)} = \frac{}{sin\left(\angle C\right)} \cdot \frac{}{\frac{PY}{YA}} =$$

$$= \frac{sin\left(\angle B\right)}{sin\left(\angle C\right)} \cdot \frac{YA}{AB} \cdot \frac{PB}{PY} = \frac{sin\left(\angle B\right)}{sin\left(\angle C\right)} \cdot \frac{AD \cdot sin\left(\angle C\right)}{\frac{AD}{sin(\angle B)}} \cdot \frac{XB}{XA} \cdot \frac{CA}{CY} =$$

$$= sin^2\left(\angle B\right) \cdot \frac{BD \cdot cos\left(\angle B\right)}{AD \cdot sin\left(\angle B\right)} \cdot \frac{\frac{AD}{sin(\angle C)}}{CD \cdot cos\left(\angle C\right)} = \frac{BD \cdot sin\left(2\angle B\right)}{CD \cdot sin\left(2\angle C\right)} = \frac{NX}{NY} \Rightarrow$$

$$\Rightarrow \frac{N'X}{N'Y} = \frac{NX}{NY} \Rightarrow N' = N.$$

$$\frac{PN}{PA} = \frac{NY \cdot sin\left(\angle BYX\right)}{AY \cdot sin\left(\angle BYA\right)} = \frac{NY}{AY} \cdot \frac{\frac{BX}{XY}}{\frac{AB}{AY}} = \frac{NY}{XY} \cdot \frac{XB}{AB} = \frac{NY}{XY} \cdot \frac{BD \cdot cos\left(\angle B\right)}{\frac{BD}{cos(\angle B)}} =$$

$$= \frac{NY}{XY} \cdot cos^2\left(\angle B\right) \Rightarrow \frac{PN}{PA} = \frac{NY}{XY} \cdot cos^2\left(\angle B\right)$$ (***)

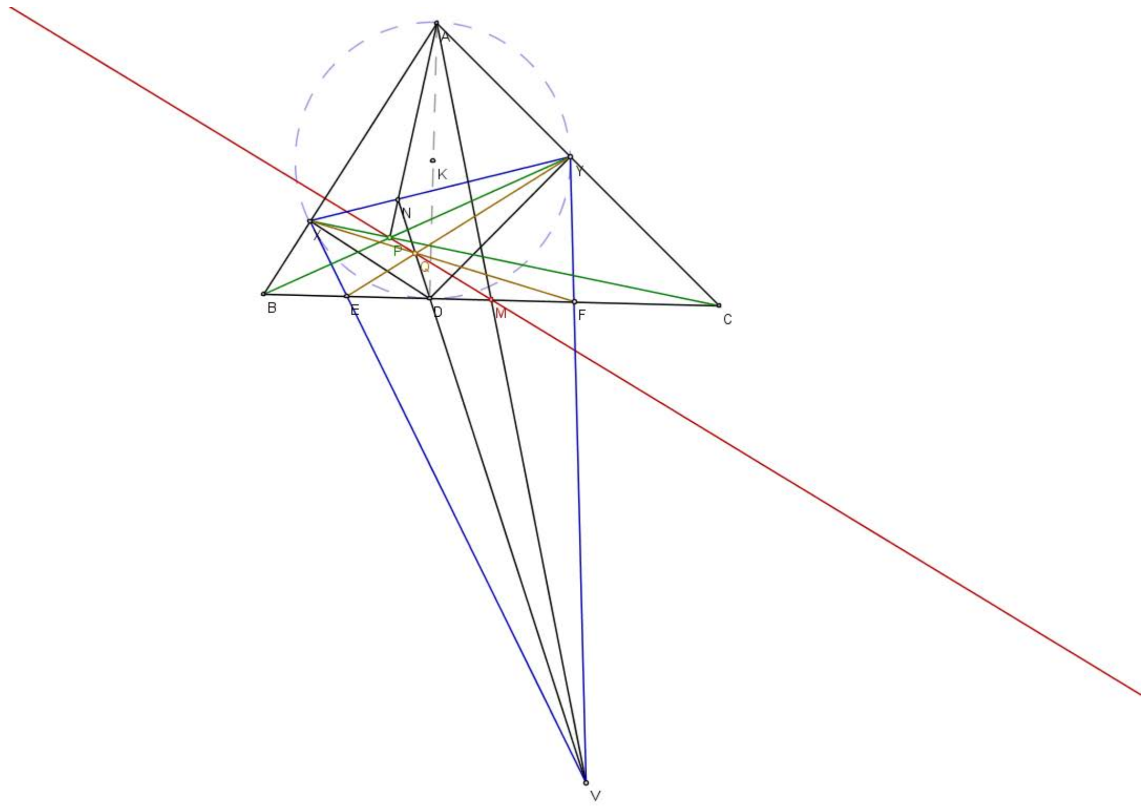
(*),(**),(***) $\Rightarrow \frac{MA}{MV} \cdot \frac{QV}{QN} \cdot \frac{PN}{PA} = \frac{cos\left(\angle A\right)}{cos\left(\angle B\right) \cdot cos\left(\angle C\right)} \cdot \frac{XY}{NX} \cdot \frac{cos\left(\angle B\right)}{cos\left(\angle A\right) \cdot cos\left(\angle C\right)} \cdot$

$$\cdot \frac{NY}{XY} \cdot cos^2\left(\angle B\right) = \frac{NY}{NX} \cdot \frac{cos^2\left(\angle B\right)}{cos^2\left(\angle C\right)} = \frac{CD \cdot sin\left(2\angle C\right)}{BD \cdot sin\left(2\angle B\right)} \cdot \frac{cos^2\left(\angle B\right)}{cos^2\left(\angle C\right)} =$$

$= \frac{CD}{BD} \cdot \frac{tg\left(\angle C\right)}{tg\left(\angle B\right)} = 1 \Rightarrow \frac{MA}{MV} \cdot \frac{QV}{QN} \cdot \frac{PN}{PA} = 1 \Rightarrow$ the points $P, Q, M$ are collinear. 🙂

Petrisor Neagoe

*Attachments:*




**Luis González**

*#3 May 21, 2009, 7:28 am • 1*

It's clear that $E, F$ are midpoints of $BD$ and $CD$, since the circles with diameters $BD$ and $CD$ are orthogonal to $k$. If $T \equiv EX \cap YF$, then $k$ is the T-excircle of $\triangle TEF$. $Q$ is then the T-Gergonne point of $\triangle TEF$ and the tangency point of the incircle of $\triangle TEF$ with $EF$ coincides with the midpoint $M$ of $BC$. Therefore, we can restate the problem as follows

**Problem:** Incircle $(I)$ of $\triangle ABC$ touches its side $BC$ at $K$. A-excircle touches the sidelines $BC, AB$ and $AC$ at $X, Y, Z$. Let $D, E$ be the reflections of $X$ about $C, B$ and $J$ the A-Gergonne point of $\triangle ABC$. If $P \equiv EZ \cap DY$, then $P, J, K$ are collinear.

Let us use barycentric coordinates with respect to $\triangle ABC$. Then we have

$$D\ (0 : -(p-b) : a+p-b)\ ,\ E\ (0 : a+p-c : -(p-c))$$

$Y\ (p-c:-p:0)\ ,\ Z\ (p-b:0:-p)$

A-Gergonne point has coordinates $J\ (-(p-b)(p-c):p(p-b):p(p-c))$

Then, lines $YD$ and $EZ$ intersect at $P$ with coordinates

$P\ (-a^2(p-b)(p-c):p(p-b)^2(a+p-c):p(p-c)^2(a+p-b))$

Hence, equation of line $JP$ is given by

$JP \equiv p((p-c)^2+(p-b)^2)x+y(p-c)(p-b)^2-z(p-b)(p-c)^2=0$

$K\ (0:p-c:p-b)$ satisfies the equation of $JP$ and the proof is completed.

**yetti**

*#4 May 22, 2009, 10:37 am • 1*

$AD \perp BC$ and $DY \perp AC \Longrightarrow \angle BCA = \angle DCY = \angle ADY = \angle AXY \Longrightarrow BCYX$ is cyclic. Tangents $XE, YF$ of $(K)$ meet at $T$. $AT$ is A-symmedian of $\triangle AYX$ and A-median of the oppositely similar $\triangle ABC$. It cuts $BC$ at its midpoint $M$.

The rest is a pure projective problem for lines. Arbitrary (convex) quadrilateral $AXTY$ is cut by arbitrary line $l$ and $B \equiv l \cap AX, C \equiv l \cap AY, E \equiv l \cap TX, F \equiv l \cap TY, M \equiv l \cap AT$. Define $P \equiv BY \cap CX, Q \equiv EY \cap FX$. Project $l$ to infinity. Projected quadrilaterals $A'X'T'Y' \cong P'Y'Q'X'$ become centrally congruent, because they have parallel corresponding sides $A'X' \parallel P'Y'$ (meet at $B'$), $X'T' \parallel Y'Q'$ (meet at $E'$), $T'Y' \parallel Q'X'$ (meet at $F'$), $Y'A' \parallel X'P'$ (meet at $C'$) and common corresponding diagonal $X'Y' \parallel Y'X'$. Consequently, their remaining diagonals $A'T' \parallel P'Q'$ are also parallel, which means that they meet at $M' \in l'$, so that the original lines $AT, PQ$ meet at $M \in l$.

Click to reveal hidden text

**vittasko**

*#5 May 23, 2009, 3:20 am • 1*

Thank you very much dear friends **Petrisor** and **Luis**, for your interesting solutions.

Really, very nice your proof dear **Vladimir**, I like it better.

I didn't see the quadrilaterals $AXTY,\ PYQX$ you mentioned, with their coincided diagonals $XY \equiv YX$ and let me to try an alternative approach inspired from your idea, applying the **Desarques theorem**.

**LEMMA. – Two quadrilaterals $ABCD,\ A'B'C'D'$ are given and suppose that the points of intersection of their homologous sidelines, $K \equiv AB \cap A'B'$ and $L \equiv BC \cap B'C'$ and $M \equiv CD \cap C'D'$ and $N \equiv AD \cap A'D'$, belong to the same line so be it $(e)$. Prove that if the point $P \equiv BD \cap B'D'$ lies on $(e)$ then the point $Q \equiv AC \cap A'C'$, lies also on this line.**

**PROOF.** – Because of the collinearity of the points $K,\ N,\ P$, based on the **Desarques theorem**, we conclude that the triangles $\triangle ABD,\ \triangle A'B'D'$ are perspective and so, we have the concurrency of $S \equiv AA' \cap BB' \cap DD'$.

Similarly, because of the collinearity of the points $M,\ P,\ L$, applying again the **Desarques theorem**, we conclude that the triangles $\triangle BCD,\ \triangle B'C'D'$ are perspective and then, the line segments $BB',\ CC',\ DD'$ are concurrent at the point $S$, as the point of intersection of $BB',\ DD'$.

Because of now, the concurrency of $S \equiv AA' \cap BB' \cap CC'$, the triangles $\triangle ABC,\ \triangle A'B'C'$ are perspective and then, the points $K \equiv AB \cap A'B'$ and $L \equiv BC \cap B'C'$ and $Q \equiv AC \cap A'C'$, are collinear.

That is, the point $Q$ as the **Lemma** states, lies on the line $(e)$ and the proof is completed.

**REMARKS.**

**(1)** – We can say the same words about the proof of the **Lemma**, in every configuration of the given quadrilaterals $ABCD,\ A'B'C'D'$ ( convex or non convex ) and their vertices, with the collinearity of the points as stated before ( $K \equiv AB \cap A'B'$ and $L \equiv$ ... ).

**(2)** – Also, the **Lemma** is true, in the particular case of the quadrilaterals $ABCE,\ A'B'C'E'$, with $BE \equiv B'E'$. We have again the result that the point $Q \equiv AC \cap A'C'$, lies on $(e)$.

• Return now into configuration of the proposed problem, after the result mentioned by **yetti**, that the line segment $AT$, where $T \equiv XE \cap FY$, passes through the midpoint $M$ of $BC$, we consider the quadrilaterals $AXTY$ (convex) and $PYQX$ (non convex), of which the points of intersection of their sidelines one per one, are collinear.

( $B \equiv AX \cap PY$ and $E \equiv XT \cap YQ$ and $F \equiv TY \cap QX$ and $C \equiv YA \cap XP$ )

Also, their diagonals $XY$ and $YX$, are coincided.

So, based on the above **Lemma**, we conclude that the point of intersection of their diagonals $AT$ and $PQ$, lies on the same line.

That is, the line segment $PQ$, passes through the midpoint $M$ of $BC$, as the point of intersection of $BC,\ AT$ and the proof of the proposed problem is completed.

Best regards, Kostas Vittas.

PS. I will post here later, the (not difficult I think) proof I have in mind, based on the **Double Ratio** theory.

*Attachments:*
t=243720(b).pdf (5kb)
t=243720(a).pdf (8kb)

*This post has been edited 2 times. Last edited by vittasko, May 23, 2009, 7:48 pm*

**yetti**
*#6 May 23, 2009, 7:53 am*

Dear Kostas, thanks a lot for your explanation. (If only I could use Desargues theorem this efficiently.)

I used very similar proof before at http://www.mathlinks.ro/viewtopic.php?t=78914; in this problem, quadrilaterals $ABCD$ and $A'B'C'D'$ of your general lemma are denoted $BB_5B_3B_4$ and $CC_5C_3C_4$. Intersections of their corresponding sides $A \equiv BB_5 \cap CC_5$ and $A \equiv B_3B_4 \cap C_3C_4$ are identical, their corresponding sides $BB_4, CC_4$ are collinar, one pair their corresponding diagonals meet at $I_a \equiv BB_3 \cap CC_3$, the other pair at $A_3 \equiv B_5B_4 \cap C_5C_4$. By the problem condition, $A, A_3, I_a$ are collinear; it follows that their remaining corresponding sides meet at $A_0 \equiv B_5B_3 \cap C_5C_3$ on the line $AA_3I_a$. I did it the same way - projecting $AA_3I_a$ to infinity; now I know how to do that entire problem with Desargues theorem.

**vittasko**
*#7 May 23, 2009, 9:33 pm*

I would like to present the solution I have in mind, in token of honour to my friends **Petry** and **Luisgeometria** and **yetti.**

• We see that the points $P,\ Q,\ M$, where $M$ is the midpoint of the side-segment $BC$, of the given triangle $\triangle ABC$, are the points of intersection of the rays, one per one, of the pencils $X.YCFM$ and $Y.XBEM$, which have the line segment $XY$, as their common ray.

( $P \equiv XC \cap YB,\ Q \equiv XF \cap YE,\ S \equiv XY \cap BC$ and $M \equiv XM \cap YM$ ).

So, it is enough to prove that these pencils have equal **double ratios** ( = **cross ratios** ) and then, we must prove that $(X.YCFM) = (Y.XBEM),(1)$

But $(X.YCFM) = (S,C,F,M),(2)$ and $(Y.XBEM) = (S,B,E,M),(3)$

From$(1),\ (2),\ (3)$, it is enough to prove that $(S,C,F,M) = (S,B,E,M),(4)$

From $(4) \Longrightarrow \frac{FS}{FC} \div \frac{MS}{MC} = \frac{ES}{EB} \div \frac{MS}{MB} \Longrightarrow \frac{FS}{FC} = \frac{ES}{EB},(5)$ ( because of $MB = MC$ ).

• It is easy to show that $E,\ F$, are the midpoints of the segments $BD,\ DC$ respectively and so, we have that $BE = EX = ED,(6)$ and $CF = FY = FD,(7)$

Because of now, $\angle SXE = 180^o - \angle EXY = 180^o - \angle SYF$, from the triangles $\triangle SBX,\ \triangle SFY$, with also $\angle BSX \equiv \angle FSY$, we conclude that $\frac{ES}{FS} = \frac{EX}{FY},(8)$

From $(6),\ (7),\ (8) \Longrightarrow \frac{ES}{FS} = \frac{EB}{FC} \Longrightarrow (5)$

So, the relation $(1)$ is true and then, we have the collinearity of the points $P,\ Q,\ M$.

That is the line segment $PQ$, passes through the midpoint $M$ of $BC$ and the proof is completed.

Kostas Vittas.

*Attachments:*
t=243720(c)pdf.pdf (5kb)

Quick Reply

# High School Math

## Regular polygons.

Reply

**bengal**

#1 May 20, 2009, 6:23 am

Here's an easy one I made: Let there be a right triangle with side lengths a, b, and c (with c being the hypotenuse). When you draw a polygon $P_A$ with side length "a" , a polygon $P_B$ with side length "b" , and a polygon $P_C$ with side length "c", with $P_A, P_B$, and $P_C$ being regular n-gons, prove that $[P_A] + [P_B] = [P_C]$.

**grn_trtle**

#2 May 20, 2009, 7:19 am

Solution

**Luis González**

#3 May 20, 2009, 7:36 am

**Generalization:** Given a right-angled triangle $\triangle ABC$ at $A$, if we construct three similar polygons $P_a, P_b, P_c$ on the sides $BC, CA, AB$, having $BC, CA, AB$ as homologous sides, then one of these polygons is equivalent to the sum of the other three.

The principle of equivalency states that if $a$ is a side of the polygon $P$, then there exists a triangle $p$ with side $a$ equivalent to the polygon $P$. Using this principle, we have three similar triangles $P_a, P_b, P_c$ (with the same area of the referred polygons) constructed on the sides of $\triangle ABC$. By Pythagorean theorem $a^2 = b^2 + c^2$, but since the triangles are similar we have:

$$\frac{P_a}{P_b} = \frac{a^2}{b^2}, \ \frac{P_a}{P_c} = \frac{a^2}{b^2} \Longrightarrow P_b + P_c = P_a.$$

Quick Reply

# High School Olympiads

## Simson lines and Euler circle



**Source**: Romania TST 2009, Day 3, Problem 2

**Maxim Bogdan**
#1 May 17, 2009, 10:54 pm

Prove that on the circumcircle of a triangle there exist exactly **3** points of whose **Simson lines** are tangent to the **Euler circle** of this triangle
and that points determine an equilateral triangle.

**Luis González**
#2 May 18, 2009, 1:23 am • 1

It's well-known that the envelope of the Simson lines of $\triangle ABC$ is the Steiner deltoid $\mathcal{S}$ tri-tangent to its Euler circle through $X, Y, Z$. Tangents of $\mathcal{S}$ at $X, Y, Z$ (Simson lines) from an equilateral triangle due to the central symmetry of $\mathcal{S}$. The poles of these Simson lines are the wanted points. Of course, such poles are not constructible (in general) since this equilateral triangle has the same orientation of Morley's triangle of $\triangle ABC$.

**Luis González**
#3 May 18, 2009, 4:34 am • 1

An alternate approach is using the transformation that I call Guzman's translation (sorry for such a random name) due to the spanish geometrician Miguel de Guzman.

Given a triangle $\triangle ABC$ and its circumcircle $(O)$. Let us define the transformation $\mathcal{H}(A, v)$ that takes $\triangle ABC$ into another $\triangle AB'C'$ with circumcircle $(O)$, such that $B'C' \parallel BC$. The vector $\vec{v}$ defines the translation that takes the line $BC$ into $B'C'$.

It's easy to prove that $\mathcal{H}$ has the following features:

1) Simson line $p$ with fixed pole $P$ on $(O)$ WRT $\triangle ABC$ is taken into another Simson line $p'$ with pole $P$ WRT $\triangle AB'C'$ by the translation $\vec{v}$. As a result, orientation $p \parallel p'$ is invariant.

2) Euler circle $\mathcal{E}$ of $\triangle ABC$ is taken into the Euler circle $\mathcal{E}'$ of $\triangle AB'C'$ under the translation $\vec{v}$, since the circumradius of $(O)$ is invariant and midpoints and feet of the altitudes from $A$ to $BC$ and $B'C'$ lie constantly on guidelines of the translation.

3) $\triangle ABC$ can be transformed into an equilateral triangle $\triangle A'B'C'$ through the composition of (at least) two translations. This can be easily proved by angle chase. Hence, $\mathcal{E}$ is taken into the incircle $\mathcal{E}'$ of $\triangle A'B'C'$ and Simson lines tangent to $\mathcal{E}$ into the sidelines of $\triangle A'B'C'$ yielding the conclusion.

Historical note: Miguel de Guzman used this transformation to study the Steiner's Deltoid of ABC. He showed synthetically that the orientation of the Steiner Deltoid is given by the sidelines of its Morley's triangle and its shape only depends of the circumcircle.

**yetti**
#4 May 18, 2009, 12:02 pm

See also http://www.mathlinks.ro/viewtopic.php?t=244550. If Simson lines $s_x, s_y, s_z$ with poles $X, Y, Z \in (O)$ from that problem (such that $s_x \parallel OX, s_y \parallel OY, s_z \parallel OZ$) concur at the 9-point circle center $N$, then Simson lines $s'_x \perp s_x, s'_y \perp s_y, s'_z \perp s_z$ with diametrically opposite poles $X', Y', Z' \in (O)$ are tangent to $(N)$.

**Luis González**
#5 May 18, 2009, 11:56 pm

Dear Maxim Bodgan and Vladimir, I'd like to attached the Guzman's investigation about the Steiner Deltoid. It shows several interesting features about it and the proof for all the assertions that I mentioned above.

http://nonio.mat.uc.pt/PENSAS_EN02/experdescgeomet/10deltoide/deltoideacademia/00delten.htm

**Luis González**

*#6 May 19, 2009, 4:35 am*

Perhaps a quick solution to this problem is using the result of very nice and interesting.

9-point center of $\triangle ABC$, which is incenter of the triangle $\triangle XYZ$ bounded by these three Simson lines, is equidistant from the orthocenter and circumcenter of $\triangle ABC$. This implies that $\triangle XYZ$ is equilateral.

**yetti**

*#7 May 19, 2009, 5:16 am*

You already have to know that (at least) 3 Simson lines are tangent to the 9-point circle. Even then, this only implies that the triangle formed by these 3 Simson lines has the middle angle of $60^\circ$.

**Luis González**

*#8 May 19, 2009, 7:13 am*

> *yetti wrote:*
>
> You already have to know that (at least) 3 Simson lines are tangent to the 9-point circle. Even then, this only implies that the triangle formed by these 3 Simson lines has the middle angle of 60

Indeed, disregard my latter comment.

**Stifler**

*#9 Sep 9, 2009, 7:37 pm*

**Solution.** Let $O$ (the circumcenter of $\triangle ABC$) be the origin of the complex plane, and let $a, b, c$ be the coordinates of $A, B$ and $C$, respectively.

Consider a point $X(x)$ lieing on the circumcircle of $\triangle ABC$ with the given property, $R$ be the projection of $X$ onto $BC$, and let $M$ be the midpoint of $[MH]$. Then $MR$ is the **Simson line** of point $X$ w.r.t $\triangle ABC$. It's clear that $MR$ is tangent to $\omega(\epsilon, \frac{R}{2})$ (**Euler circle** of triangle $ABC$) if and only if $\boxed{\epsilon M \perp MR}$

We have:

$$\boxed{\begin{array}{l} h = a+b+c \,;\; \epsilon = \dfrac{a+b+c}{2} \;;\; m = \frac{x+a+b+c}{2} \\[2ex] r = \dfrac{1}{2}\cdot\left(x+b+c-\dfrac{bc}{R^2}\cdot\overline{x}\right) \\[2ex] r \neq m \implies x \neq -\dfrac{bc}{a} \end{array}}$$

Now,

$\epsilon M \perp MR \iff \dfrac{m-\epsilon}{m-r} \in i\mathbb{R}^*$ i.e. $\dfrac{x}{a+\frac{bc}{R^2}\cdot\overline{x}} \in i\mathbb{R}^*$

Note that:

- $\dfrac{x}{a+\frac{bc}{R^2}\cdot\overline{x}} = \dfrac{x}{a+\frac{bc}{x}} = \dfrac{x^2}{ax+bc}$
- $\dfrac{x}{\dots} = \dfrac{\frac{1}{x}}{\dots} = \dfrac{\overline{abc}}{\dots}$

$a+\frac{bc}{R^2}\cdot\overline{x}$ $\frac{a}{R^2}+\frac{bc}{R^4}\cdot\overline{x}$ $\overline{x}\cdot(\overline{ax}+\overline{bc})$

Hence:

$$\frac{x}{a+\frac{bc}{R^2}\cdot\overline{x}}\in i\mathbb{R}^* \iff \frac{x}{a+\frac{bc}{R^2}\cdot\overline{x}}=-\overline{\frac{x}{a+\frac{bc}{R^2}\cdot\overline{x}}}$$

$$\iff \frac{x^2}{ax+bc}=-\frac{abc}{x(ax+bc)}\mid\cdot(ax+bc)\neq 0 \iff \boxed{x^3=-abc}$$

So, there are exactly three points $X_1(x_1),\ X_2(x_2),\ X_3(x_3)$ with the property mentioned in the statement and they form an **equilateral triangle** (because they are the *cubic roots* of a complex number). Thus the problem is solved.

Quick Reply

# High School Olympiads

## easy

Reply

**CoBa_c_Kacka**

#1 May 19, 2009, 1:39 am

Prove that for every triangle with incenter I and sides a, b and c it is true that $a.AI^2 + b.BI^2 + c.CI^2 = abc.$

**Luis González**

#2 May 19, 2009, 3:21 am

Use Leibniz formula for the incenter $I \equiv (a : b : c)$ and the circumcenter $O$ of $\triangle ABC$

$$BC \cdot R^2 + CA \cdot R^2 + AB \cdot R^2 = 2p \cdot IO^2 + BC \cdot IA^2 + CA \cdot IB^2 + AB \cdot IC^2$$

$$BC \cdot IA^2 + CA \cdot IB^2 + AB \cdot IC^2 = 2pR^2 - 2p(R^2 - 2Rr) = BC \cdot CA \cdot AB$$

Quick Reply

# Site Support

## Problems with latex

Reply

**Luis González**

#1 May 15, 2009, 11:10 am

Vertical alignment of Latex is messed up in several threads. For instance, see

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=272494
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=272398
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=271829
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=269313

*This post has been edited 2 times. Last edited by Luis González, Mar 24, 2012, 12:13 pm*

**bluecarneal**

#2 May 16, 2009, 8:08 pm

I'll probably post this somewhere else in this forum also, but I have also having difficulty with LaTeX. Specificall 2 digit exponets. They appear as the first digit as normal exponet size, but the 2 digit is normal number size.

**Kent Merryfield**

#3 May 16, 2009, 10:00 pm

See here.

And a mod should probably split this and bluecarneal's post off into either the test form or the $\LaTeX$ forum.

Learn to use {}.

x^{12} becomes $x^{12}$.

e^{-x^{2k}} becomes $e^{-x^{2k}}$.

**JBL**

#4 May 16, 2009, 10:29 pm

The LaTeX forum of which Kent speaks is http://www.artofproblemsolving.com/Forum/index.php?f=123 (123, very easy to remember); he also might have recommended the search feature http://www.artofproblemsolving.com/Forum/search.php (linked to from the top of the forum).

**Luis González**

#5 May 18, 2009, 7:10 am

This is also happening in other subfora. For example, see these threads

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=57381
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=36&t=273724
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=51&t=261698
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=57&t=184458

EDIT: The problem seems solved now.

*This post has been edited 2 times. Last edited by Luis González, Mar 24, 2012, 12:14 pm*

**jmerry**

*#6 May 18, 2009, 7:32 am*

This is about vertical alignments, so matrices and other tall bits of $\LaTeX$ show it best.

Some better examples:
The problem.
Normal display in post #2 and the rest of the first page. Bad alignment in post #26 and the rest of the second page.

**levans**

*#7 May 18, 2009, 7:49 am*

Hmm, I'm going to guess we may have had a small bug with the LaTeX stuff for a few days. Regenerating the images in that post 26 that jmerry mentioned and the topic luis mentioned made everything nice and clean.

**jmerry**

*#8 Jun 2, 2009, 5:19 am*

A new one, seen here: $(x-2)(x^2+2x+4)$. I think the vertical alignment is off as in the others from this thread, but it's the non-italic $x$ that really caught my eye.

*Quick Reply*

# High School Olympiads

## hard

Reply

**dragon001**

#1 May 16, 2009, 8:05 pm

In a triangle $ABC$,$BC$ í smallest edge.Choose $M$ and $N$ on $AB$ and $AC$ respectively such that $BM = BC$ and $CN = CB$
Prove that
$\frac{MN}{BC} = \sqrt{3 - 2(cosA + cosB + cosC)}$

**Ahiles**

#2 May 16, 2009, 9:16 pm

$$\sqrt{3 - 2\sum \cos A} = \sqrt{3 - 2\left(1 + \frac{r}{R}\right)} = \sqrt{1 - \frac{2r}{R}}$$

$$BN = 2BC \sin\frac{C}{2}$$

$$MN = \sqrt{BN^2 + BM^2 - 2BN \cdot BM \cos \angle MBN} =$$

$$BC\sqrt{1 + 4\sin^2\frac{C}{2} - 4\sin\frac{C}{2}\sin\left(B + \frac{C}{2}\right)}$$

So, we need to prove that

$$1 + 4\sin^2\frac{C}{2} - 4\sin\frac{C}{2}\sin\left(B + \frac{C}{2}\right) = 1 - \frac{2r}{R}$$

$$4\sin\frac{C}{2}\sin\left(B + \frac{C}{2}\right) - 4\sin^2\frac{C}{2} = \frac{2r}{R}$$

$$4\sin\frac{C}{2}\left(\sin B\cos\frac{C}{2} + \cos B\sin\frac{C}{2}\right) - 4\sin^2\frac{C}{2} = \frac{2r}{R}$$

$$2\sin C\sin B + 4\cos B\sin^2\frac{C}{2} - 4\sin^2\frac{C}{2} = \frac{2r}{R}$$

$$\sin C\sin B + 2\sin^2\frac{C}{2}(\cos B - 1) = \frac{r}{R}$$

$$\sin C\sin B - 4\sin^2\frac{C}{2}\sin^2\frac{B}{2} = \frac{r}{R}$$

$$\frac{bc}{4R^2} - 4\frac{(p-a)^2(p-c)(p-b)}{bca^2} = \frac{r}{R}$$

$$\frac{a^2b^2c^2 - 16R^2(p-a)^2(p-c)(p-b)}{4R} = \frac{bca^2S}{p} = \frac{b^2c^2a^3}{4Rp}$$

$$a^2b^2c^2p - 16R^2p(p-a)^2(p-b)(p-c) = b^2c^2a^3$$

$a^2b^2c^2p - 16R^2S^2(p-a) = b^2c^2a^3$

$a^2b^2c^2p - 16R^2\dfrac{a^2b^2c^2}{16R^2}(p-a) = b^2c^2a^3$

$p - (p-a) = a$

That's all....

**nayel**

*#3 May 16, 2009, 9:56 pm*

From the isosceles triangle $BCM$ we get $\angle BCM = \dfrac{\pi - B}{2}$ and $CM = 2BC\cos\dfrac{\pi - B}{2}$. We also have $CN = BC$ and $\angle MCN = C - \dfrac{\pi - B}{2} = \dfrac{C-A}{2}$. Thus we deduce from the law of cosines,

$$\begin{aligned} MN^2 &= CM^2 + CN^2 - 2CM\cdot CN\cos\angle MCN \\ &= 4BC^2\sin^2\frac{B}{2} + BC^2 - 4BC^2\sin\frac{B}{2}\cos\frac{C-A}{2} \\ &= BC^2\left(4\sin^2\frac{B}{2} + 1 - 2\sin\frac{B+C-A}{2} - 2\sin\frac{A+B-C}{2}\right) \\ &= BC^2\left(3 - 2 + 4\sin^2\frac{B}{2} - 2\sin\frac{\pi-2A}{2} - 2\sin\frac{\pi-2C}{2}\right) \\ &= BC^2\left(3 - 2\cos B - 2\cos A - 2\cos C\right). \end{aligned}$$

**Luis González**

*#4 May 17, 2009, 10:41 am*

**General problem.** Let $\triangle ABC$ be a scalene triangle and $BC$ is its shortest side. $M, N$ are two points on $AC, AB$ such that $BN = CM = BC$. Then $MN$ is perpendicular to $OI$, where $O, I$ are the circumcenter and incenter of $\triangle ABC$, and $\frac{MN}{BC} = \frac{OI}{R}$.

**Proof:** In the isosceles triangles $\triangle OAC$ and $\triangle OAB$, we have

$R^2 - OM^2 = AM\cdot MC,\ R^2 - ON^2 = AN\cdot BN$

$\Longrightarrow ON^2 - OM^2 = AM\cdot MC - AN\cdot BN$

$\Longrightarrow ON^2 - OM^2 = (CA - BC)BC - (AB - BC)BC = BC(CA - AB)$

Since $\triangle MIC$ and $\triangle MIB$ are isosceles, we have $IM = IB, IN = IC$. Then

$IN^2 - IM^2 = IC^2 - IB^2 = BC(CA - AB)$

$\Longrightarrow ON^2 - OM^2 = IN^2 - IM^2 \Longrightarrow\ IO \perp MN$

On the other hand, $\angle NBM = \angle OCI = 90^\circ - \frac{1}{2}\angle C - \angle A$. If $L \equiv CI \cap BM$, then it follows that $\angle NMB$ and $\angle OIC$ are supplementary, which implies that $\triangle IOC$ and $\triangle MNB$ are pseudo-similar, hence

$\dfrac{MN}{OI} = \dfrac{BN}{OC} \Longrightarrow \dfrac{MN}{BC} = \dfrac{OI}{R}.$

**livetolove212**

*#5 May 25, 2009, 10:31 pm*

> *dragon001 wrote:*
>
> In a triangle $ABC$,$BC$ í smallest edge.Choose $M$ and $N$ on $AB$ and $AC$ respectively such that $BM = BC$ and $CN = CB$
> Prove that
> $MN$

$$\frac{MN}{BC} = \sqrt{3 - 2(cosA + cosB + cosC)}$$

This problem is on Mathematical and Youth magazine and it can be solved easily by using vector. Don't need trigonometry.
To dragon001: You are bad

**mathVNpro**

*#6 May 26, 2009, 2:49 pm*

**Livetolove is right**, this topic must be deleted, the problem that has been proposed in this topic is the problem in **Mathematics and Youth magazine** in **Vietnam**, the latest number **(May/2009)**.

Quick Reply

# College Math

## Proof of the distance from the origin to a plane



**Source**: Application of the scalar equation of a plane

**catalan**

#1 May 16, 2009, 3:11 am

19. If *a*, *b*, and *c* are the x-intercept, the y-intercept, and the z-intercept of a plane, respectively, and *d* is the distance from the origin to the plane, show that:

$$\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

---

Hello there,

I am showing my work below. Could you please tell me where I've erred because my final step is different from the equation above?

Thank you.

---

I let:

a = (x, 0, 0)
b = (0, y, 0)
c = (0, 0, z)

Now:

First direction vector = [0, y, 0] - [x, 0, 0] = [-x, y, 0]

Second " " = [0, 0, z] - [0, y, 0] = [0, -y, z]

Therefore:

$\vec{\ell} = [x, 0, 0] + s[-x, y, 0] + t[0, -y, z]$

Conversion of $\vec{\ell}$ into scalar form:

$[-x, y, 0] \times [0, -y, z] = [yz, xz, xy]$.

$xyz + xyz + xyz + D = 0$

Subsitution of $(x, 0, 0)$ to find D:

$x^2yz + D = 0 \Rightarrow D = -x^2yz$

Since *d* = $\dfrac{|Ax + By + C + D|}{\sqrt{A^2 + B^2 + C^2}}$, and using (x, 0, 0):

*d* = $\dfrac{|xyz - x^2yz|}{\sqrt{y^2z^2 + x^2y^2 + x^2z^2}}$

*d* = $\dfrac{\sqrt{(xyz)^2 - (x^2yz)^2}}{}$

$$d = \frac{\ldots}{\sqrt{y^2z^2 + x^2y^2 + x^2z^2}}$$

$$d^2 = \frac{x^2y^2z^2 - (x^2yz)^2}{y^2z^2 + x^2y^2 + x^2z^2}$$

$$\frac{1}{d^2} = \frac{y^2z^2 + x^2y^2 + x^2z^2}{x^2y^2z^2 - x^4y^2z^2}$$

If I let a = x, b = y, and c = z, then the equation given by the question is:

$$\frac{1}{d^2} = \frac{y^2z^2 + x^2y^2 + x^2z^2}{x^2y^2z^2}$$

However, my equation has an additional $-x^4y^2z^2$.

**Kunihiko_Chikaya**

*#2 May 16, 2009, 4:25 am*

> catalan wrote:
>
> a = (x, 0, 0)
> b = (0, y, 0)
> c = (0, 0, z)

Why do you set like that? You can directly write the equation of the plane as $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

**Luis González**

*#3 May 17, 2009, 5:49 am*

We also can use pure geometry to prove this nice relation:

Let $A, B, C$ be the traces on the axes $x, y, z$. $\overline{OA} = a$ , $\overline{OB} = b$ , $\overline{OC} = c$

Volume of the tetrahedron $OABC$ is given by $[OABC] = \frac{1}{3}d \cdot [ABC]$

But, since trihedron $O(ABC)$ is orthogonal we have $[OABC] = \frac{1}{6}abc$

$$\Longrightarrow\ 2d \cdot [ABC] = abc\ \Longrightarrow\ \frac{1}{d^2} = \frac{4[ABC]^2}{a^2b^2c^2}$$

Plane $\alpha$ passing though $OA$ orthogonal to the plane $OBC$ cuts $BC$ at the foot $H$ of the altitude $AH = h$ from $A$ to the edge $BC$. By Pythagorean theorem we get then

$$[ABC]^2 = \tfrac{1}{4}h^2(b^2 + c^2),\ h^2 = a^2 + OX^2 = a^2 + \frac{b^2c^2}{b^2 + c^2}$$

$$\Longrightarrow\ [ABC]^2 = \tfrac{1}{4}(a^2b^2 + a^2c^2 + b^2c^2)$$

$$\Longrightarrow\ \frac{1}{d^2} = \frac{a^2b^2 + a^2c^2 + b^2c^2}{a^2b^2c^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

**Kunihiko_Chikaya**

*#4 May 17, 2009, 6:38 am*

For $A(a,\ 0,\ 0),\ B(0,\ b,\ 0),\ C(0,\ 0,\ c)\ (abc \neq 0)$, Decarte Gua's Theorem holds:

$$\boxed{(\triangle ABC)^2 = (\triangle OAB)^2 + (\triangle OBC)^2 + (\triangle OCA)^2}$$

**catalan**

*#5 May 20, 2009, 3:03 am*

Thank you for your responses!

Quick Reply

# High School Olympiads

## Apollonius Circle

Reply

**mathwizarddude**
#1 May 10, 2009, 10:48 am

How would you prove the theorem on Apollonius Circle in the following page

http://planetmath.org/encyclopedia/ApolloniusCircle.html

**mathwizarddude**
#2 May 16, 2009, 11:10 pm

Can anyone prove it?

**Differ**
#3 May 16, 2009, 11:31 pm

Try inversion. Cool stuff will happen.

**mathwizarddude**
#4 May 16, 2009, 11:55 pm

Could you explain how to do inversion? Thanks.

**Luis González**
#5 May 17, 2009, 3:09 am • 1

Inversion is not necessary.

Consider two fixed points $B, C$ and a point $A$ that belows to the locus, that is $\frac{AB}{AC} = k = \text{const}$. Internal and external bisectors of $\angle BAC$ meet the line $BC$ at two fixed points $X, Y,$ since by angle bisector theorem, we have that $\frac{XB}{XC} = \frac{YB}{YC} = \frac{AB}{AC} = k.$ Since the internal and the external bisector of $\angle A$ are perpendicular, then the locus of $A$ is the circumference with diameter $\overline{XY}$.

Quick Reply

# High School Olympiads

## 1/AM-1/AN=c, locus



**Source**: My own problem, difficult

**bilot4**

#1 May 17, 2009, 12:19 am

Let $xAy$ be a constant angle and $M, N$ points on $Ax, Ay$ respectively, such that $\frac{1}{AM} - \frac{1}{AN} = c$. Let also $P$ be the foot of the perpendicular from $A$ to $MN$. Find the locus of the points $P$, as $M, N$ "move" on $Ax, Ay$

**Luis González**

#2 May 17, 2009, 2:40 am

Line $MN$ cuts external bisector of $\angle xy$ at a fixed point $L$, since in $\triangle AMN$ we have

$\frac{2}{AL} = \left|\frac{1}{AM} - \frac{1}{AN}\right| \cdot \csc\frac{A}{2} \Longrightarrow P$ moves on the circle with diameter $\overline{AL}$.

**bilot4**

#3 May 17, 2009, 11:14 pm

In my solution, I use inversion in order to find that $MN$ cuts t he external bisector in a constant point!
Specifically, using inversion the inversion $(A, r^2)$, the points M, N go to points M', N' such that $AM' - AN'$ =constant. But we now that in this case, the circumcircle of $AM'N'$ passes through a constant point of the external bisector, so MN also passes through a constant point (the inverse of the previous one, wrt to the inversion that we did!

Quick Reply

# High School Olympiads

## collinear



**mathson**

#1 May 15, 2009, 11:44 pm • 2

In any acute triangle $ABC$, let $H$ be it's orthocenter. Let $O$ be the midpoint of $\overline{BC}$. Let $\Omega$ be a circle with center $O$ passes throw $B, C$. If $\overline{AN}, \overline{AM}$ tangent $\Omega$ in $M, N$. Prove that $M, N, H$ are collinear.

[geogebra]2c95f774b2381df91d3589518099e0a607f310ee[/geogebra]

**Luis González**

#2 May 16, 2009, 12:28 am

Let $D, E$ be the orthogonal projections of $B, C$ onto $AC, AB \Longrightarrow D, E \in \Omega$. If $ED$ cuts $BC$ at $P$, then the line pencil $A(B, C, H, P)$ is clearly harmonic, thus it follows that $AH \perp BC$ is the polar of $P$ with respect to $\Omega$. Therefore, the polar $\tau$ of $A$ with respect to $\Omega$ passes through $H \Longrightarrow MN \equiv \tau$, i.e. $H \in MN$.

**mathVNpro**

#3 May 16, 2009, 7:01 am

Here is another approach:
Consider the inversion through pole $A$, power $\mathcal{P}_{A/(O)} = k^2$. Let $D$ be the projection of $H$ onto $BC$. It is easy to note that $\mathcal{I}(A, k^2) : H \mapsto D, M \mapsto M, N \mapsto N$. But $\angle OMA = \angle ONA = \angle ODA = \frac{\pi}{2} \Longrightarrow \{A, M, N, O, D\}$ is a set of concyclic, which implies that $H, M, N$ are collinear.
Our proof is completed then.

**Ahiles**

#4 May 16, 2009, 6:58 pm

As $MN \perp AO$ we will just prove that $MH \perp AO$, or, equivalently

$$OM^2 - OH^2 = AM^2 - AH^2 \ (*)$$

But

$$BH^2 = AH^2 + a^2 - b^2$$
$$CH^2 = AH^2 + a^2 - c^2$$

$$OM^2 = \frac{a^2}{4}$$

$$AM^2 = AO^2 - OM^2 = \frac{2(b^2+c^2) - a^2}{4} - \frac{a^2}{4} = \frac{b^2+c^2-a^2}{2}$$

$$OH^2 = \frac{2BH^2 + 2CH^2 - a^2}{4} = \frac{4AH^2 + 4a^2 - 2b^2 - 2c^2 + a^2}{4} =$$

$$= AH^2 + \frac{3a^2 - 2b^2 - 2c^2}{4}$$

The relation $(*)$ is equivalent to

$$\frac{a^2}{4} - \left(AH^2 + \frac{3a^2 - 2b^2 - 2c^2}{4}\right) = \frac{b^2 + c^2 - a^2}{2} - AH^2$$

$$-\left(\frac{4a^2 - 2b^2 - 2c^2}{4}\right) = \frac{b^2 + c^2 - a^2}{2}$$

clearly true....

**Ashegh**

*#5 May 19, 2009, 4:41 am*

$AH \cap BC = H'$
$A, M, H', O, N$ lies on $\Omega$,then an inversion with center $A$,and radius of the power of $A$ to $\Omega$
tells us that the inverses,$M, H, N$ should be colinear!
(not return of Ashegh,yet :( )

**mathVNpro**

*#6 May 19, 2009, 3:36 pm*

> *Ashegh wrote:*
> $AH \cap BC = H'$
> $A, M, H', O, N$ lies on $\Omega$,then an inversion with center $A$,and radius of the power of $A$ to $\Omega$
> tells us that the inverses,$M, H, N$ should be colinear!
> (not return of Ashegh,yet :( )

This is exactly my solution above, you can see it, my friend :D

**Ashegh**

*#7 May 20, 2009, 4:11 am*

right,I usually don't see other solutions before posting my solution!
because I don't want to get other's Ideas about the problem,before thinking about it.
whish U luck ;)

**vittasko**

*#8 May 20, 2009, 5:43 am*

Let $D,\ E,\ F$ be, as the points of intersection of $BC,\ AC,\ AB$ respectively, from the line segments $AH,\ BH,\ CH,$ where $H$ is the orthocenter of the given triangle $\triangle ABC.$

The points $E,\ F,$ lie on the circle $(O)$ ( instead of $\Omega$ ), with diameter $BC,$ as well.

We denote as $K,$ the point of intersection of $AO,$ from the line segment $MN$ and from the right triangle $\triangle MAO$ with $\angle AMO = 90^o$ and $MK\perp AO,$ we have that $(AK)\cdot(AO) = (AM)^2$ , $(1)$

Let $K'$ be, the orthogonal projection of $H,$ on the line segment $AO$ and it is enough to prove that $K'\equiv K.$

Frome the cyclic quadrilaterals $K'ODH,\ HDBF, \Longrightarrow (AK')\cdot(AO) = (AH)\cdot(AD) = (AF)\cdot(AB) = (AM)^2$ , $(2)$

From $(1),\ (2) \Longrightarrow (AK)\cdot(AO) = (AK')\cdot(AO) \Longrightarrow AK' = AK \Longrightarrow K'\equiv K$ and the proof is completed.

Kostas Vittas.

*Attachments:*
t=276952.pdf (5kb)

**dgreenb801**

*#9 Sep 7, 2009, 8:53 pm*

> *Luis González wrote:*

> The polar of $H$ WRT $(O)$ goes through $A$

How do you prove this?

Quick Reply

# High School Olympiads

## Half of circle with diameter EF is tangent with BC,CA

Reply

Source: German MO 2009 481345

**tdl**

#1 May 15, 2009, 9:52 am

Let a triangle $ABC$. $E, F$ in segment $AB$ so that $E$ lie between $AF$ and half of circle with diameter $EF$ is tangent with $BC, CA$ at $G, H$. $HF$ cut $GE$ at $S$, $HE$ cut $FG$ at $T$. Prove that $C$ is midpoint of $ST$.

**plane geometry**

#2 May 15, 2009, 11:01 am

SEE
http://www.mathlinks.ro/viewtopic.php?p=1204373#1204373

**jayme**

#3 May 15, 2009, 2:16 pm

Dear Mathlinkers,
1. Let (0) be the circumcircle of EGH ; it passes through F
2. let Tg (=BC), Th (=AC) be the resp. tangents to (0) at G, H.
3. According to Pascal's theorem, SCT is the Pascal's line of the degenerated hexagon EG Tg BH ThE
4. Let (1) be the circle with diameter ST (it passes through G and H)
and Te the tangent to (0) at E
5 According to the Reim's theorem applied to (0) and (1), Te // ST
6. According to Bouton's theorem applied to EGH (see for example http://perso.orange.fr/jl.ayme vol. 1 A propos du théorème de Boutin), C is the midpoint of ST
Sincerely
Jean-Louis

**vittasko**

#4 May 15, 2009, 9:02 pm

Because of $EG\perp TF$ and $FH\perp TE$, we conclude that the point $S\equiv EG\cap FH$, is the orthocenter of the triangle $\triangle TEF$.

We denote the point $C'\equiv TS\cap AC$ and it is easy to show that $C'H = C'T$ , $(1)$

from $\angle C'HT = \angle EHA = \angle EFH = \angle ETC'\equiv\angle ETS$.

So, in the right triangle $\triangle HST$, with $\angle SHT = 90^{o}$, because of $(1)$, we conclude that $C'T = C'S$ , $(2)$

Let be the point $C''\equiv TS\cap BC$ and by the same way we can prove that $C''T = C''S$ , $(3)$

From $(2),\ (3)\Longrightarrow C'\equiv C''\equiv C$ and the proof is completed.

Kostas Vittas.

*Attachments:*
t=276882.pdf (5kb)

*This post has been edited 1 time. Last edited by vittasko, May 16, 2009, 1:26 am*

**Luis González**

#5 May 15, 2009, 11:10 pm

**Proof 1:** Let the tangent of the semicircle at $G$ cut $TS$ at $C'$. Since $S$ is clearly orthocenter of $\triangle TEF$, we get $\angle C'SG \equiv \angle TSG = \angle GFE = \angle C'GS \Longrightarrow C'$ is on perpendicular bisector of $\overline{TS} \Longrightarrow C'$ is midpoint of $\overline{TS}$. Similarly, we prove that the tangent of the semicircle at $H$ hits $\overline{TS}$ at its midpoint $C' \Longrightarrow C \equiv C'$.

**Proof 2:** Points $T, G, S, H$ are concyclic on a circle $\mathcal{Q}$, since $S$ is the orthocenter of $\triangle TEF$. The inversion with center $T$ and power $\overline{TH} \cdot \overline{TE}$ takes $\mathcal{Q}$ into the line $EF$ and the circumcirle $(O)$ of the quadrilateral $HGFE$ into itself. If $EF$ bisects $(O)$, then $\mathcal{Q}$ is orthogonal to $(O)$ due to conformity $\Longrightarrow C$ is center of $\mathcal{Q} \Longrightarrow C$ is midpoint of $\overline{TS}$.

**mathVNpro**

#6 Jun 14, 2009, 12:40 am

Let $X_a \equiv HG \cap EF$, it follows that $X_a$ is the pole of $TS$ wrt $(O)$. But the polar of $C$ which is $HG$ passes through $X_a$ hence $C \in TS$. Now it is easy to see that $S$ is the orthocenter of $\triangle TEF \Longrightarrow \angle THS = \angle TGS = 90^\circ$, which implies $T, H, S, G$ are concyclic $\Longrightarrow HG$ is the radical axes wrt $(O)$ and $(THSG)$. But $CG, CS$ are the tangents of $C$ to $(O)$ hence $CO$ is the perpendicular bisector of $HG$, $C$ lies on $1$ of the diameters of $(THSG)$, which immediately follows $C$ is the circumcenter of $(THGS)$, which implies that $C$ is the midpoint of $ST$.
Our proof is completed then.

Quick Reply

# High School Olympiads

## Prove AP/AQ = 3/2

Reply

**Source**: Argentina TST 2009

**uglysolutions**

#1 May 15, 2009, 8:06 am

Let $ABC$ be a triangle, $B_1$ the midpoint of side $AB$ and $C_1$ the midpoint of side $AC$. Let $P$ be the point of intersection ($\neq A$) of the circumcircles of triangles $ABC_1$ and $AB_1C$. Let $Q$ be the point of intersection ($\neq A$) of the line $AP$ and the circumcircle of triangle $AB_1C_1$.

Prove that $\frac{AP}{AQ} = \frac{3}{2}$.

**jayme**

#2 May 15, 2009, 3:10 pm

Dear Mathlinkers,
for the beginning AP is the A-symmedian of ABC.
Sincerely
Jean-Louis

**Luis González**

#3 May 15, 2009, 8:01 pm

Let $M, N, L$ be the midpoints of $CA, AB, BC$. $\mathcal{M}_1 \equiv \odot(ABM), \mathcal{M}_2 \equiv \odot(ACN)$ and $\mathcal{M}_3 \equiv \odot(AMN)$. Homothety $(A, \frac{3}{2})$ carries $\mathcal{M}_3$ into $\mathcal{M}'_3$, which passes through the intersections of $AC, AB$ with the parallels through $L$ to the medians $BM, CN$. Now, it suffices to show thaw $\mathcal{M}'_3, \mathcal{M}_1, \mathcal{M}_2$ are coaxal. Their barycentric equations are:

$$\mathcal{M}'_3 \equiv [0, \tfrac{1}{4}c^2, \tfrac{1}{4}b^2] \ , \ \mathcal{M}_1 \equiv [0, 0, \tfrac{1}{2}b^2] \ , \ \mathcal{M}_2 \equiv [0, \tfrac{1}{2}c^2, 0]$$

Radical axis of $\mathcal{M}_1, \mathcal{M}_2$ is the line $b^2z - c^2y = 0$ (A-symmedian) and the radical axis of $\mathcal{M}'_3, \mathcal{M}_2$ is the line $b^2z - c^2y = 0$ $\Longrightarrow \mathcal{M}'_3, \mathcal{M}_1, \mathcal{M}_2$ are coaxal, as desired.

**livetolove212**

#4 May 24, 2009, 1:38 pm • 1

> uglysolutions wrote:
> Let $ABC$ be a triangle, $B_1$ the midpoint of side $AB$ and $C_1$ the midpoint of side $AC$. Let $P$ be the point of intersection ($\neq A$) of the circumcircles of triangles $ABC_1$ and $AB_1C$. Let $Q$ be the point of intersection ($\neq A$) of the line $AP$ and the circumcircle of triangle $AB_1C_1$.
>
> Prove that $\frac{AP}{AQ} = \frac{3}{2}$.

Let $CB_1 \cap (AB_1C_1) = D, CP \cap (ABC_1) = L, BC \cap (ABC_1) = E, B_1C \cap BC_1 = O, S$ be a point on segment $C_1C$ such that $\frac{C_1S}{SC} = \frac{1}{3}$
Since $CC_1.CA = CE.CB = CD.CB_1$ we get $B_1DEB$ is a cyclic quadrilateral. Moreover $AB_1DC_1$ is a cyclic quadrilateral, so $EDC_1C$ is a cyclic quadrilateral.
Then $\angle DC_1E = \angle DCE(1)$
On the other side, $\angle BLP + \angle B_1CL = \angle BLP + \angle BAP = 180^o$ then $BL//B_1C$
Thus $\angle LC_1E = \angle LBC = \angle BCB_1(2)$

From (1) and (2) we have $\angle LC_1E = \angle DC_1E$ therefore $L, D, C_1$ are collinear.
Because $BL//OD$ then $\frac{C_1D}{C_1L} = \frac{C_1O}{C_1B} = \frac{1}{3} = \frac{C_1S}{C_1C}$
$\Rightarrow DS//LC$
We have $\angle DQA = \angle DB_1A = \angle CPA \Rightarrow DQ//PC$
So Q,D,S are collinear we get $QS//PC$
Hence $\frac{AP}{AQ} = \frac{AC}{AS} = \frac{3}{2}$

**Mathias_DK**
#5 May 27, 2009, 10:48 pm • 1

> uglysolutions wrote:
> Let $ABC$ be a triangle, $B_1$ the midpoint of side $AB$ and $C_1$ the midpoint of side $AC$. Let $P$ be the point of intersection ($\neq A$) of the circumcircles of triangles $ABC_1$ and $AB_1C$. Let $Q$ be the point of intersection ($\neq A$) of the line $AP$ and the circumcircle of triangle $AB_1C_1$.
>
> Prove that $\frac{AP}{AQ} = \frac{3}{2}$.

Inversion kills it :) Make an inversion with radius $r$ and center $A$. For every point $P$ let $P'$ denote the inversed point.
Then $\frac{AC_1'}{AC'} = \frac{AC}{AC_1} = 2$, and likewise with $B$, so $B', C'$ are midpoints of sides $A'B_1'$ and $A'C_1'$. The circles around $\triangle ABC_1$, $\triangle AB_1C$, and $\triangle AB_1C_1$ get transformed into lines passing through $B'C_1'$, $B_1'C$, and $B_1'C_1'$ respectively.
$B'C_1'$ and $B_1'C'$ intersect at $P'$, which must be the centroid of $\triangle AB_1'C_1'$. Then $Q'$ must be the intersection between $AP'$ and $B_1'C_1'$. And it is wellknown that $\frac{AQ'}{AP'} = \frac{3}{2}$.
But $\frac{AP}{AQ} = \frac{AQ'}{AP'} = \frac{3}{2}$. QED :)

**YOGRRR**
#6 May 28, 2009, 8:58 pm

Like Balkan's 2nd problem

**brianchung11**
#7 Jun 2, 2009, 11:12 am • 1

As $\angle B_1BP = \angle CC_1P$ and $\angle BB_1P = \angle C_1CP$ (from the two circles) so $\triangle B_1BP \sim \triangle CC_1P$ thus $\frac{PB_1}{PC} = \frac{BB_1}{C_1C} = \frac{AB}{AC}$, i.e. $AC \times B_1P = AB \times PC$

As $\angle BAQ = \angle B_1C_1Q$ and $\angle ABQ = \angle C_1B_1Q$ (from the two circles) so $\triangle B_1QC_1 \sim \triangle BQA$

As $\angle B_1C_1Q = \angle B_1AQ = \angle B_1CP$ and $\angle B_1QC_1 = \pi - \angle B_1AC_1 = \angle BPC$ so $\triangle B_1PC \sim \triangle B_1QC_1 \sim \triangle BQA$ thus $\frac{AB}{B_1C} = \frac{AQ}{PC}$ i.e. $B_1C = \frac{AB \times PC}{AQ}$

By Ptolemy, $AP = \frac{AB_1 \times PC + AC \times B_1P}{B_1C} = \frac{AB_1 \times PC + AB \times PC}{\frac{AB \times PC}{AQ}} = \frac{3}{2}AQ$

**abacadaea**
#8 Sep 7, 2010, 5:46 pm

Click to reveal hidden text

**jayme**
#9 Sep 7, 2010, 7:03 pm

Dear Mathlinkers,
0. Let Q the seond point of intersection of AP with the circumcircle of AB1C1

1. AP is the A-symmedian of ABC
2. the circumcircle of AB1C1 cut AQ in his midpoint like for example the circle going through A, B and tangent to AC at A...
3. Now we have to determine the nature of this last circle and we are done without any calculation...
Sincerely
Jean-Louis

**jayme**
#10 Sep 8, 2010, 7:32 pm

Dear Mathlinkers,
from where comes this nice problem?
Sincerely
jean-Louis

**muathuhanoi**
#11 Sep 24, 2011, 12:35 pm

There're many proofs but it seems none of them tries this approach .
Let $B', C'$ are centers of circumcircle $\triangle ABC_1$ , $\triangle ACB_1$ respectively
$AB_2, AC_2$ are diameter of $(B')$; $(C')$ respectively .$O$ is center of $(ABC)$
$\angle APB_2 = \angle APC_2 = 90 \Rightarrow B_2, P, C_2$ are collinear .
$\angle AB_1B' = \angle AB_1C_2 = 90 \Rightarrow B', B_1, C_2$ are collinear
(by similar way ) $C', C_1, B_2$ are collinear .
and since that ,tt's easy to see : $B'C_2, C'B_2$ meet each other at $O$.
But when we considered $\triangle AB_2C_2$ ,we knew : $B'C_2, C'B_2$ are medians of $AB_2C_2$
HEnce $O$ is centroid of $\triangle AB_2C_2 \Rightarrow \frac{AM}{AO} = \frac{3}{2}$(where $M$ is midpoint of $B_2C_2$
$OQ||MP(\perp AP) \Rightarrow \frac{AM}{AO} = \frac{AQ}{AP}$
Done

**avd96**
#12 Oct 12, 2012, 10:31 pm

> *brianchung11 wrote:*
> As $\angle B_1BP = \angle CC_1P$ and $\angle BB_1P = \angle C_1CP$ (from the two circles) so $\triangle B_1BP \sim \triangle CC_1P$ thus $\frac{PB_1}{PC} = \frac{BB_1}{C_1C} = \frac{AB}{AC}$, i.e. $AC \times B_1P = AB \times PC$
>
> As $\angle BAQ = \angle B_1C_1Q$ and $\angle ABQ = \angle C_1B_1Q$ (from the two circles) so $\triangle B_1QC_1 \sim \triangle BQA$
>
> As $\angle B_1C_1Q = \angle B_1AQ = \angle B_1CP$ and $\angle B_1QC_1 = \pi - \angle B_1AC_1 = \angle BPC$ so $\triangle B_1PC \sim \triangle B_1QC_1 \sim \triangle BQA$ thus $\frac{AB}{B_1C} = \frac{AQ}{PC}$ i.e. $B_1C = \frac{AB \times PC}{AQ}$
>
> By Ptolemy, $AP = \frac{AB_1 \times PC + AC \times B_1P}{B_1C} = \frac{AB_1 \times PC + AB \times PC}{\frac{AB \times PC}{AQ}} = \frac{3}{2}AQ$

can you explain why $\angle ABQ = \angle C_1B_1Q$ ?

Quick Reply

# High School Math

## Circle Chords Property



**triplebig**

#1 May 14, 2009, 8:34 am

In a circle with radius $R$, two perpendicular chords $\overline{AB}$ and $\overline{CD}$ intersect at $P$ . Show that $PA^2+PB^2+PC^2+PD^2=4R^2$.

I already got the natural solution, which uses the fact that $PA\cdot PB=PC\cdot PD$ . I was wondering if there was a quicker way to find this relation. I am thinking this is a problem where there is a creative catch that solves the problem, can anybody see it?

mod: Please proofread. Two problem obscuring typoes in one post is a bit much. Corrected.

**Rofler**

#2 May 14, 2009, 10:37 am

Let the center be $O$, and let the points be labeled $ADBC$ clockwise around the circle. The problem is equivelant to showing that $AC^2+BD^2=4R^2$ by simple pythagoras.

When 2 chords intersect, the angle between them is equal to half of the sum of the 2 arcs associated with this angle. In this case, the angle is $90$, so therefore, the sum of 2 of the opposing arcs is $180$ (and so is the sum of the other pair by the same argument).

We note that this is equivelant to saying that $\angle AOC+\angle BOD=180$. Rotate triangle $AOC$ counterclockwise until $OC$ coincides with $OB$, and let the images of $A$ and $C$ under this rotation be $A'$ and $C'$.

Then $C'=B$, and since the angles sum to 180, we must have $A'$ on the line $OD$. Furthermore, $A'$ lies on the circle, since it's distance from $O$ didn't change under rotation. Therefore, $A'D$ is a diamter.

So we finally have $AC^2+BD^2=A'C'^2+BD^2=A'B^2+BD^2=(2R)^2=4R^2$ since $A'D$ is a diamter.

Please draw this out, as you will not understand it just by looking at it. It is a very beautiful proof if I do say so myself (no trig).

**Luis González**

#3 May 14, 2009, 12:03 pm

Let the chords $AB,CD$ meet at $P.$ We shall show that $PA^2+PB^2+PC^2+PD^2$ is independent of $P.$ $O$ is the center of the given circle and $Q$ the second intersection of $DO$ with $(O).$ By Pythagorean theorem we get

$$PA^2+PB^2+PC^2+PD^2=\frac{1}{2}(DA^2+AC^2+CB^2+BD^2)$$

But since rays $DQ,DP$ are isogonal WRT $\angle ADB,$ it follows that $ACQB$ is an isosceles trapezoid with $CQ \parallel AB \Longrightarrow$ $AC=BQ$ and $CB=AQ.$ Thus, keeping in mind that $DQ$ is a diameter, again by Pythagorean theorem we have:

$$PA^2+PB^2+PC^2+PD^2=\frac{1}{2}(DA^2+AQ^2+BD^2+BQ^2)$$

$$PA^2+PB^2+PC^2+PD^2=\frac{1}{2}(DQ^2+DQ^2)=\frac{1}{2}(8R^2)=4R^2.$$

# High School Olympiads

## Properties of triangles-trigonomentry

Reply

**Source**: help please.(source=Rudiments of mathematics, P-I)

**Potla**
#1 May 12, 2009, 10:12 pm

I am trying the problem for a long time but I cannot solve it. Please help me ( ).

**Problem**

The radius of the circumcircle of $\Delta ABC$ with area $\Delta$ is $R$; and $D$ is the midpoint of $BC$, and $E$ abd $F$ are the feet of the perpendiculars from $D$ to $AB$ and $AC$. If $S$ is the area of $\Delta DEF$, prove that :

$R = \frac{a}{4}\sqrt{\frac{\Delta}{S}}$.

Click to reveal hidden text

**Potla**
#2 May 12, 2009, 11:03 pm

If you are trying to figure out how the image/diagram will be; then here it is:
(I typed the code for 30 minutes)

Please decode it .

```
draw((0,0)--(100,0));
draw((0,0)--(40,40));
draw((40,40)--(100,0));
dot((0,0));
dot((100,0));
dot((40,40));
label("A", (40,40), N);
label("B", (0,0), W);
label("C", (100,0), E);
dot((66,23));
dot((50,0));
dot((50,-6));
dot((27,25));
label("D, (50,0), N);
label("E", (27,25), S);
label("F", (66,23), S);
label("O", (50,-6), S);
draw((66,23)--(27,25));
draw((50,0)--(27,25));
draw((50,0)--(66,23));
draw(circle((50,-6), 49);
```

**Farenhajt**
#3 May 13, 2009, 3:35 am

We have $DE = \frac{h_c}{2}, DF = \frac{h_b}{2}$ and $\angle EDF = \pi - \alpha$ (either by noting that $AEDF$ is a cyclic quadrilateral, having two opposite angles right, or by arguing that
$\angle EDF = \pi - \angle EDB - \angle CDF = \pi - \left(\frac{\pi}{2} - \beta\right) - \left(\frac{\pi}{2} - \gamma\right) = \beta + \gamma = \pi - \alpha$)

Therefore $S = \frac{h_b}{2} \cdot \frac{h_c}{2} \cdot \frac{\sin\alpha}{2} = \frac{\Delta}{b} \cdot \frac{\Delta}{c} \cdot \frac{\Delta}{bc} = \frac{\Delta^3}{b^2c^2}$

Since $R = \dfrac{abc}{4\Delta}$, we get $bc = \dfrac{4R\Delta}{a}$. Plugging that in, we get

$$S = \frac{\Delta^3}{\frac{16R^2\Delta^2}{a^2}} = \frac{a^2\Delta}{16R^2}$$

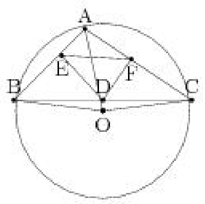
From there $R^2 = \dfrac{a^2\Delta}{16S} \iff R = \dfrac{a}{4}\sqrt{\dfrac{\Delta}{S}}$

**Potla**

*#4 May 13, 2009, 9:57 am*

By $\alpha, \beta, \gamma$ do you represent the respective angles of $\Delta DEF$?

*Attachments:*




Geometry Image 23

**Farenhajt**

*#5 May 13, 2009, 10:07 am*

No, the angles of $\triangle ABC$.

**Luis González**

*#6 May 13, 2009, 10:20 am*

This is a particular case of Euler's theorem.

$P$ is a point on the plane of $\triangle ABC$ with circumcircle $(O, R)$ and $X, Y, Z$ are the orhogonal projections of $P$ on the sidelines of $\triangle ABC$. Then we have

$$\frac{[\triangle XYZ]}{[\triangle ABC]} = \frac{p(P, (O))}{4R^2},$$

where $p(P, (O)) = |PO^2 - R^2|$ stands for the power of $P$ with respect to $(O, R)$.

Quick Reply

# High School Olympiads

## Bicentric quadrilateral 4

Reply

**Source**: Metric relation

**juancarlos**

#1 Jul 11, 2006, 9:07 pm

Let $ABCD$ be a bicentric quadrilateral with the circumcircle $(O,R)$ and the incenter $I$. If $P$ is the cut point of $AC$ and $BD$ besides $IO=d$. Prove that:$OP=\frac{2R^2d}{R^2+d^2}$

**yetti**

#2 Jul 15, 2006, 4:03 am • 1

Let $(I)$ be the quadrilateral incircle and $r$ the inradius. Since $O,I,P$ are collinear, $PO=PI+d$ and we can calculate $PI$ instead. Because of Poncelet's porism, it is sufficient to calculate $PI$ for a bicentric (isosceles) trapezoid. Let $AB \parallel CD$ be the trapezoid bases, $AB>CD,\ BC=DA.$ Let $E,F,G,H$ be the tangency points of $AB,BC,CD,DA$ with the incircle $(I).$ The tangency points $E,G$ are the midpoints of the bases $AB,CD.$ Let $M,N$ be the midpoints of the shoulder sides $BC,DA.$ Since $MN=\frac{AB+CD}{2}$ for a trapezoid and $AB+CD=BC+DA$ for a tangential quadrilateral, the trapezoid midline is equal to its shoulder sides, $MN=BC=DA.$ The incenter $I$ is the midpoint of the midline $MN$, the median $IM$ of the triangle $\triangle BCI$ is equal to half of its side $BC$, hence the angle $\angle BIC=90^\circ$ is right. Then the angles $\angle BIE+\angle CIG=90^\circ$ add up to a right angle, the right angle triangles $\triangle BIE\sim\triangle ICG$ are similar and $\frac{BE}{IE}=\frac{IG}{CG}$. By Pythagorean theorem,

$$BF^2=BE^2=OB^2-OE^2=OB^2-(IB-IO)^2=R^2-(r-d)^2$$

$$CF^2=CG^2=OC^2-OG^2=OC^2-(IC+IO)^2=R^2-(r+d)^2$$

$$\left(\frac{BE}{IE}\right)^2=\left(\frac{IG}{CG}\right)^2\ \Longrightarrow\ \frac{R^2-(r-d)^2}{r^2}=\frac{r^2}{R^2-(r+d)^2}$$

$$(R^2-r^2-d^2)^2-4r^2d^2=r^4\ \Longrightarrow\ r^2=\frac{(R^2-d^2)^2}{2(R^2+d^2)}$$

In addition, $\frac{PI}{GI}=\frac{FM}{CM}$, where $GI=r,\ FM=\frac{BF-CF}{2},\ CM=\frac{BF+CF}{2}$, so that

$$\frac{PI}{r}=\frac{BF-CF}{BF+CF}=\frac{(BF-CF)^2}{BF^2-CF^2}$$

$$BF^2-CF^2=4rd$$

$$(BF-CF)^2=2[R^2-(r^2+d^2)]^2-2\sqrt{[R^2-(r-d)^2][R^2-(r+d)^2]}=$$

$$=2[R^2-(r^2+d^2)]^2-2\sqrt{[R^2-(r^2+d^2)]^2-4r^2d^2}$$

Substituting for $r^2$ :

$$R^2-(r^2+d^2)=\frac{2(R^2+d^2)(R^2-d^2)-(R^2-d^2)^2}{2(R^2+d^2)}=$$

$$=\frac{(R^2+d^2)^2-4d^4}{2(R^2+d^2)}=\frac{(R^2+3d^2)(R^2-d^2)}{2(R^2+d^2)}$$

$$\sqrt{[R^2-(r^2+d^2)]^2-4r^2d^2}=\sqrt{\frac{(R^2+3d^2)^2(R^2-d^2)^2}{4(R^2+d^2)^2}-\frac{2d^2(R^2-d^2)^2}{R^2+d^2}}=$$

$$=\frac{R^2-d^2}{2(R^2+d^2)}\sqrt{(R^2+3d^2)^2-8d^2(R^2+d^2)}=\frac{(R^2-d^2)^2}{2(R^2+d^2)}$$

Substituting these expressions into $(BF-CF)^2$ and $\frac{PI}{r}$ :

$$(BF-CF)^2=\frac{(R^2+3d^2)(R^2-d^2)-(R^2-d^2)^2}{R^2+d^2}=\frac{4d^2(R^2-d^2)}{R^2+d^2}$$

$$\frac{PI}{r}=\frac{(BF-CF)^2}{BF^2-CF^2}=\frac{4d^2(R^2-d^2)}{4rd(R^2+d^2)}\implies PI=\frac{d(R^2-d^2)}{R^2+d^2}$$

$$PO=PI+d=\frac{d(R^2-d^2)+d(R^2+d^2)}{R^2+d^2}=\frac{2dR^2}{R^2+d^2}$$

**Luis González**

*#3 May 12, 2009, 3:58 am • 1*

Nice Vladimir's proof based on Poncelet's porism, I'll present an alternate approach.

Let $M\equiv AD\cap BC$ and $N\equiv AB\cap DC.$ Then $MN$ is the polar of $P$ with respect to $(O)$ and the polar of $P$ with respect to $(I)\implies P,I,O$ are collinear and $PIO\perp MN.$ If $X\equiv IO\cap MN,$ then $X$ is the inverse of $P$ under inversion WRT $(O)$ and $(I).$ Therefore

$OP\cdot OX=R^2$ and $IP\cdot IX=(OP-d)IX=r^2$

Since $OX-IX=d,$ we get

$$\frac{R^2}{OP}-\frac{r^2}{OP-d}=d\implies\frac{1}{r^2}=\frac{OP}{(R^2-OP\cdot d)(OP-d)}$$

From Fuss theorem we have the relation

$$\frac{1}{r^2}=\frac{1}{(R+d)^2}+\frac{1}{(R-d)^2}$$

$$\implies\frac{OP}{(R^2-OP\cdot d)(OP-d)}=\frac{1}{(R+d)^2}+\frac{1}{(R-d)^2}$$

Expanding the latter expression gives the quadratic equation

$$2d(R^2+d^2)OP^2-OP(R^4+d^4+6R^2d^2)+2dR^2(R^2+d^2)=0$$

This quadratic equation has two positive roots, namely

$OP=\frac{2R^2d}{R^2+d^2}$ and $OP=\frac{R^2+d^2}{2d},$ but since $R>OP>0$

$\implies OP=\frac{2R^2d}{R^2+d^2},$ as desired.

**yetti**

*#4 May 12, 2009, 7:05 am*

Thank you for the kind words. My proof is too long with too many formulas and I would definitely try something else now. Your proof is **really** nice.

Quick Reply

# High School Olympiads

## Iran TST 2009-Day2-P2

Reply

**khashi70**

#1 May 11, 2009, 7:56 pm

$ABC$ is a triangle and $AA'$ , $BB'$ and $CC'$ are three altitudes of this triangle . Let $P$ be the feet of perpendicular from $C'$ to $A'B'$ , and $Q$ is a point on $A'B'$ such that $QA = QB$ . Prove that :
$\angle PBQ = \angle PAQ = \angle PC'C$

**mathangel**

#2 May 11, 2009, 9:25 pm

$PC'$ is the angular bisector of $\angle APB$ and so the fact that $Q$ is the intersection of the bisector of $AB$ and the external angular bisector of $\angle APB$ would imply $AQPB$ is incriptible. This shows the first equality.

Let $M$ be the intersection of $PC'$ and the circle passing through $A, Q, P, B$. Then $\angle PBQ = \angle PAQ = \angle PMQ$. But we also have $\angle PMQ = \angle PCC'$ (notice that $MQ//CC'$). This completes the proof.

**Luis González**

#3 May 11, 2009, 9:55 pm • 1

You forgot to prove this

> *mathangel wrote:*
> $PC'$ is the angular bisector of $\angle APB$ and so the fact that $Q$ is the intersection of the bisector of $AB$ and the external angular bisector of $\angle APB$ would imply $AQPB$ is incriptible

It follows from $(A, B, C', D) = -1$, where $D \equiv A'B' \cap AB$.

**NNN**

#4 May 13, 2009, 9:37 pm

Let *QD* be the bisector of the triangle *AQB*. Let *H* be the orthocentre of triangle *ABC*. Let *AB* intersect *A'B'* in *M*.

*<A'C'B=<ACB*, because *AA'* and *CC'* are alptitudes.

Let *BB'* intersect *DQ* at *E*. Then *B'ADE* is circumcsribed(because *<AB'E=<ADE*=90 degrees), so *<EB'D=<EAD*, but *DQ* is bisector of *AB*,
so *<EAD=<EBD=90degrees-<BAC*. So *<DB'B=90degrees-<BAC*.
*A'HB'C* is circumscribed, so *<BB'A'=<C'CB=90degrees-<ABC*.
*<DB'A'=<DB'B+<BB'A'=90degrees-<BAC+90degrees-<ABC=<ACB.*

Then *<A'C'B=<ACB=<DB'A'* and *A'B'DC'* is circumscribed. So *MA*MB=MD*MC'=MA'*MB**.
*<QDM=<C'PM=90 degrees*, then triangles *MPC'* and *MDQ* are similar. So *MD*MC'=MP*MQ*. Then *MA*MB=MP*MQ* and *ABPQ* is circumscribed. Then *<APB'=<ABQ=<BAQ=<BPA'*. Then *<APC'=<C'PB*.

*PC'* is bisector of *<APB*. *QD* is bisector of *<AQB*. Let *PC'* intersect *QD* in *X*. Then *AXBQP* is circumscrtibed.

*<CC'B=<QDB*, so *CC'*||*QX*. So *<PC'C=<PXQ=<PBQ=<PAQ*, then *<PC'C=<PBQ=<PAQ*.

**Heebeen, Yang**

#5 Jun 7, 2009, 7:00 am

Using nine point circle and given 2 circles,
if use power of some point, it is easy to show that

**livetolove212**
#6 Jun 7, 2009, 9:39 am

> khashi70 wrote:
> $ABC$ is a triangle and $AA'$ , $BB'$ and $CC'$ are three altitudes of this triangle . Let $P$ be the feet of perpendicular from $C'$ to $A'B'$ , and $Q$ is a point on $A'B'$ such that $QA = QB$ . Prove that :
> $\angle PBQ = \angle PAQ = \angle PC'C$

Let I be the intersection of A'B' and AB.
We have $(ABC'I) = -1$ and $\angle C'PI = 90^o$ so $PC'$ is the bisector of $\angle APB$
Let $(ABP) \cap A'B' = \{Q'\}$
$\Rightarrow \angle BAQ' = \angle BPA' = \angle APQ' = \angle ABQ'$ so $Q'A = Q'B \Rightarrow Q' \equiv Q$
$\Rightarrow \angle PBQ = \angle PAQ$
On the other hand, $\angle PC'C = \angle C'IP = \angle ABP - \angle BPI = \angle ABP - \angle ABQ = \angle QBP$

**csl824**
#7 Jun 16, 2009, 8:03 pm

> Quote:
> the fact that is the intersection of the bisector of and the external angular bisector of would imply is incriptible.

why is this the case? :

**sayantanchakraborty**
#8 Oct 3, 2014, 8:56 pm

Let $J$ be the intersection point of $A'B'$ and $AB$ and drop $QX \perp AB$ with $X$ on $AB$.Then note that $X$ is the midpoint of $AB$ and since $AA',BB',CC'$ are concurrent we have $(J,C',A,B)=-1$.So $X$ being the midpoint of $AB$ and noting that $PC'XQ$ is cyclic( $\angle PC'X=\angle PQX=90^{\circ}$) we have $JA \cdot JB=JC' \cdot JX=JP \cdot JQ$ so $APQB$ is cyclic.So $\angle PAQ=\angle PBQ$.Next note that if $K=AQ \cap CC'$ then $\angle AKC=\angle AQX=\frac{1}{2}\angle AQB$.Also since $(J,C';A,B)=-1$ we see that $PJ,PC',PA,PB$ form a harmonic pencil.Now since $PC' \perp PJ$,$PC'$ must be the internal angle bisector of $\angle APB$.Thus $\angle APC'=\frac{1}{2}\angle APB=\frac{1}{2}\angle AQC$.So $\angle APC'=\angle AKC'=\frac{1}{2}\angle AQC \implies APKC'$ is cyclic.Thus $\angle PAQ=\angle PAK=\angle PC'K=\angle PC'C$ as desired.

**anantmudgal09**
#9 Oct 17, 2015, 6:42 am

Nice and elegant.

Here's my solution:
Let $A'B'$ meet $AB$ at the point $T$.(working in the projective plane we wont bother about $T$ being the point at infinity.)
Now, observe that $C',P,Q,M$ are con-cyclic because of the perpendicularity conditions and that $(B,A;C',T)=-1$.
This along with the fact that $M$ is the midpoint of $AB$ gives that $TC'.TM=TA.TB=TP.TQ$ which in turn proves that points $B,A,P,Q$ are con-cyclic.
Therefore, $\angle PBQ=\angle PAQ$ and since, $Q$ lies on the perpendicular bisector of $AB$, $PQ$ bisects $\angle APB$.
Now, $\angle PC'C=\frac{(\angle A'-\angle B')}{2}$ and $\angle PAQ=\frac{(\angle PBA-\angle PAB)}{2}$. Now, the rest is trivial angle-chase.

Quick Reply

# High School Olympiads

## Two circles and one incenter.

Reply

**Source**: Classic.

**Virgil Nicula**

#1 May 11, 2009, 1:46 am

Let $w(O)$ , $w'$ be two secant circles in $\{A, B\}$ so that $O \in w'$ . A line $d$ for which $O \in d$ , cut the circle $w$ in $\{X, Y\}$ , cut the line $AB$ in $C$ and cut the circle $w'$ in $Z$ so that $Y \in (CZ)$ . Prove that $Y$ is the incenter of the triangle $ABZ$ .

**Luis González**

#2 May 11, 2009, 2:59 am

It suffices to show that $ZO$ bisects $\angle AZB$ and $\angle AYB = 90^\circ + \frac{1}{2}\angle AZB.$ Indeed, from $AO = OB$ we deduce that $ZO$ bisects $\angle AZB$ and angle chase gives

$\angle AYB = 180^\circ - \angle AXB = 180^\circ - \frac{1}{2}\angle AOB = 180^\circ - \frac{1}{2}(180^\circ - \angle AZB)$

$\angle AYB = 90^\circ + \frac{1}{2}\angle AZB \Longrightarrow Y$ is incenter of $\triangle AZB$

**Virgil Nicula**

#3 May 11, 2009, 3:51 am

Thank you, **Luis** ! The your simple and nice proof confirms my opinion that this problem is for "Intermediate topics".

I found it as an application of the harmonical division. Thus, $CO \cdot CZ = CA \cdot CB = CX \cdot CY$, i.e. the relation $CO \cdot CZ = CX \cdot CY$

which characterizes in fact that $\{X, Y; C, T\}$ is a remarkable harmonical division. Therefore, $AY \perp AX \Longleftrightarrow \widehat{YAC} \equiv \widehat{YAZ}$ a.s.o.

**jayme**

#4 May 11, 2009, 11:05 am

Dear Virgil and Mathlinkers, has the circle with center O a name? I know only that Mention studied and discovered this circle... and the property mentioned by Virgil.
For me, I give it the name of Mention's circle.
At the same time, can someone tell me about another geometer known Mansion. Perhaps it is the same has Mention?
Any ideas?
Sincerely
Jean-Louis

**Luis González**

#5 May 11, 2009, 12:12 pm

> *Virgil Nicula wrote:*
> I found it as an application of the harmonical division

You're welcome dear Virgil, it is also great to know new and unexpected applications of harmonic division, but I believe this classic result of the circle passing through two vertices and the incenter of a triangle goes further back in time. Namely, when the celebrated 9-point circle came out.

> *jayme wrote:*
> Dear Virgil and Mathlinkers, has the circle with center O a name? I know only that Mention studied and discovered this circle and the property mentioned by Virgil.

It suffices to observe the orthocentric system formed by the incenter and 3 excenters.

Quick Reply

# High School Olympiads

## A concurrency from the past.

Reply

**Source**: QUANTUM 1995 (N. Sedrakian, S. Tkachov)

**vittasko**

#1 Apr 13, 2009, 3:52 am

**Let $\triangle A'BC,\ \triangle B'AC,\ \triangle C'AB$ be, three similar isosceles triangles with bases the side-segments $BC,\ AC,\ AB$ respectively of a given triangle $\triangle ABC$, erected outwardly to it. Prove that the line segments $AA',\ BB'\ CC'$, where $A',\ B',\ C'$ are the midpoints of the segments $EF,\ DF,\ DE$ respectively, are concurrent at one point.**

Kostas Vittas.

*Attachments:*

t=270785.pdf (3kb)

**Luis González**

#2 Apr 13, 2009, 7:47 am

Let $\phi$ be the angle base of these isosceles triangles. Using Conway's formula, the barycentric coordinates of $D,E,F$ are:

$$D \equiv (-a^2 : S_C + S_\phi : S_B + S_\phi)$$

$$E \equiv (S_C + S_\phi : -b^2 : S_A + S_\phi)$$

$$F \equiv (S_B + S_\phi : S_A + S_\phi : -c^2)$$

Since the sum of the coordinates of $D,E,F$ equals $2S_\phi$, we compute easily the coordinates of the midpoints of segments $DE,EF,FD$ as:

$$A' \equiv (a^2 + 2S_\phi : S_\phi - S_C : S_\phi - S_B)$$

$$B' \equiv (S_\phi - S_C : 2S_\phi + b^2 : S_\phi - S_A)$$

$$C' \equiv (S_\phi - S_B : S_\phi - S_A : 2S_\phi + c^2)$$

Therefore, $AA',BB',CC'$ concur at the inner Kiepert perspector relative to $\phi$

$$K \equiv \left(\frac{1}{S_A - S_\phi} : \frac{1}{S_B - S_\phi} : \frac{1}{S_C - S_\phi}\right)$$

**Luis González**

#3 May 11, 2009, 3:22 am

Reflect $\triangle BCD$ about sideline $BC$ into $\triangle BCD'$. Notice that $\angle FBD' = \angle ABC$, since $\angle FBD'$ is obtained by rotating $\angle ABC$ about $B$ with angle $+\phi$. Further, $\triangle BFD'$ and $\triangle BAC$ are similar with similarity coefficient $\frac{BF}{BA}$. But $\frac{BF}{BA} = \frac{AE}{AC} \Longrightarrow FD' = AE$ and analogously $ED' = AF$, which implies that $AFD'E$ is a parallelogram $\Longrightarrow$ Line $AA'$ goes through $D'$. By similar reasoning, we conclude that $AA', BB', CC'$ concur at the inner Kiepert perspector relative to $\phi$.

Quick Reply

# High School Olympiads

## Segment equality



**Source**: easy

**77ant**

#1 May 10, 2009, 1:43 am

given a triangle ABC, let r be its inradius (its incircle touches BC, CA, AB at D, E, F respectively)
Let the inradii of triangles AFE, BDF, CDE, DEF be r1, r2, r3, r4 .
Prove that r1 + r2 + r3 + r4 = 2r .

**Luis González**

#2 May 10, 2009, 7:32 am

$J_a, J_b, J_c$ are incenters of $\triangle AFE, \triangle BFD, \triangle CED.$ Quadrilaterals $DFJ_aE, EDJ_bF$ and $FEJ_cD$ are cyclic, since

$\angle DFE + \angle FJ_aE = 90^\circ + \frac{1}{2}\angle A + 90^\circ - \frac{1}{2}\angle A = 180^\circ$, etc.

Therefore, $J_a, J_b, J_c$ are midpoints of the arcs $EF, FD, DE$ of the incircle $(I).$ If $M, N, L$ denote the midpoints of $EF, FD, ED,$ by Carnot's theorem for $\triangle DEF,$ it follows that $IM + IN + IL = r + r_4.$ But $IM = r - r_1,$ $IN = r - r_2$ and $IL = r - r_3 \Longrightarrow r_1 + r_2 + r_3 + r_4 = 2r.$

# High School Olympiads

## Kazakshtan NO Geometry

Reply

**Source**: KAZAKHSTAN National Olympiad for 9 th grades

**Ulanbek_Kyzylorda KTL**

#1 May 9, 2009, 10:37 pm • 1

we have triangle ABC with AB<AC and there are heights from B and C to the sides AC and AB an intersects at points B' and C'.The line B'C' intersects with line CB at point P and AP intersects with line MH at K, where M is midpoint of BC and H is orthocenter. Prove that KM is bisector of angle B'KB.

**mathVNpro**

#2 May 9, 2009, 10:57 pm

Consider the inversion through pole $H$, power $\overline{HB}.\overline{HB'}=\overline{HC}.\overline{HC'}=k^2$. It is easy to notice that $I(H,k^2):M\mapsto K\implies K\in(AB'C')$. Hence, $\angle HKB'=\angle HAC=\angle HBC$. But it is a very well-known fact that $MB',MC'$ are the tangent of $M$ wrt $AB'C'$. Thus, $MB'^2=MC'^2=\overline{MH}.\overline{MK}$.
But $MC'=MB\implies MB^2=\overline{MH}.\overline{MK}$, which implies $MB$ is the tangent of $(KHB)\implies\angle HBC=\angle HKB$.
Therefore, $\angle HKB=\angle HKB'$, which implies that $KH$ or $KM$ is the bisector of $\angle BKB'$
Our proof is completed.

**Luis González**

#3 May 9, 2009, 11:09 pm • 1

Line $AP$ is the polar of $H$ with respect to the circle $\odot(B'C'BC)$ whose center is $M$. Therefore, $KH\perp AP\Longrightarrow$ pencil of lines $A(B,C,H,P)$ is harmonic $\Longrightarrow B,B',H$ and the projection of $B'$ onto $PA$ from $B$ are harmonically separeted. Since $KH\perp AP$, it follows that lines $KM$ and $AP$ bisects $\angle BKB'$ internally and externally.

**mathVNpro**

#4 May 9, 2009, 11:23 pm

> *luisgeometria wrote:*
> Line $AP$ is the polar of $H$ with respect to the circle $\odot(B'C'BC)$ whose center is $M$. Therefore, $KH\perp AP\Longrightarrow$ Line pencil $A(B,C,H,P)$ is harmonic $\Longrightarrow B,B',H$ and the projection of $B'$ onto $PA$ from $B$ are harmonically separeted. Since $KH\perp AP$, it follows that lines $KM$ and $AP$ bisects $\angle BKB'$ internally and externally.

Nice my friend

**lasha**

#5 May 13, 2009, 2:06 am

Hello! Here's my solution not using any non-olympiad methods or facts:
By the well known lemma, the lines $PH$ and $AM$ are perpendicular (Too easy to prove eg by counting $PM$, $PA$, $HM$ and $HA$). Consider the triangle $APM$: It's obvious that $AH$ and $PH$ are altitudes in this triangle. So, as the altitudes of any triangle concur, $MH$ is perpendicular to line $AP$, implying that $\angle MKA$ is right angle. Define by $A'$ the intersection point of lines $AH$ and $BC$. Than, the quadrilaterals $AKA'M$ and $AC'A'C$ are both cyclic and $KH\cdot HM=AH\cdot A'H=BH\cdot B'H$, implying $KH\cdot HM=BH\cdot B'H$. The last equality means that $KBMB'$ is also cyclic, but on the other hand, we have $B'M=BM=CM$. So, in circle, circumscribed on quadrilateral $KBMB'$, the chords $B'M$ and $BM$ are equal, giving $\angle BKM=\angle B'KM$, Q.E.D. Lasha Lakirbaia.

Quick Reply

# High School Math

## easy geo

Reply

**Ramchandran**
#1 May 7, 2009, 3:27 pm

Prove that the internal bisectors and the external angle bisector of the third angle meet the opposite sides in three collinear points.

**Ramchandran**
#2 May 9, 2009, 2:33 pm

anyone interested in this ?
it is quite easy but i want to see the approach of other people

**Luis González**
#3 May 9, 2009, 10:48 pm

Let $M, N$ the feet of the internal bisectors of $B, C$ and $L$ the foot of the external bisector of $A$. By angle bisector theorem we have

$$\frac{BN}{NA} = \frac{a}{b}, \ \frac{AM}{MC} = \frac{c}{a}, \ \frac{LC}{LB} = \frac{b}{c}$$

$\frac{LC}{LB} \cdot \frac{BN}{NA} \cdot \frac{AM}{MC} = \frac{b}{c} \cdot \frac{a}{b} \cdot \frac{c}{a} = 1 \Longrightarrow$ by Menelaus' theorem $M, N, L$ are collinear.

Quick Reply

# High School Olympiads

## Classic and beautiful Collinearity

Reply

Source: 0

**Tiger100**

#1 May 9, 2009, 5:43 am

**Show Synthetically that in any triangle the nine-point center , the two Vecten points and the symmedian point are collinear.**

*This post has been edited 1 time. Last edited by Tiger100, May 9, 2009, 7:57 am*

**mathVNpro**

#2 May 9, 2009, 6:45 am

> *Tiger100 wrote:*
> **the two Vecten points and the symmedian point are collinear.**

Can you tell me more about this??? I reaally don/t know what you mean.

**jayme**

#3 May 9, 2009, 11:08 am

Dear Mathlinkers,
1. to prove that the two Vecten points and the symmedian point are collinear, I use the same proof for the two Fermat points and the symmedian point are collinear ; see for example
http://perso.orange.fr/jl.ayme vol. 2 La fascinante figure de Cundy P. 13-15
2. to prove that the two Vecten points and the nine-points center are collinear, I use the Sondat's theorem... next on my website... see for example
http://perso.orange.fr/jl.ayme vol. 1 Le theoreme de Sondat
Sincerely
Jean-Louis

**vittasko**

#5 May 9, 2009, 12:51 pm

We can say that the line connecting the **Vecten points** of the given triangle $\triangle ABC$, passes through its **nine points center**, based on the more general **Lemma** which already has been proved in the topic http://www.mathlinks.ro/Forum/viewtopic.php?t=161643 (post #8#)

Propably, in this particular case of the proposed problem, it is possible to find another proof, simpler than the (difficult) one I presented there.

Kostas vittas.

**Luis González**

#6 May 9, 2009, 10:14 pm

Since the two Vecten points $V, V'$ are Kiepert perspectors, by Kierpert theorem we conclude that line $VV'$ goes through the symmedian point $K$ of $\triangle ABC$. Then it remains to prove that the nine-point center $N$ lies on $VV'$. Let $D, E$ be the centers of the squares constructed outwardly on $AB, AC$ and $F$ the center of the square constructed inwardly on $BC$. Note that $\angle DBF = \angle ABC$, since $\angle DBF$ has been obtained by rotating $\angle ABC$ by $+ \ 45^{\circ}$. Further, triangles $\triangle ABC$ and $\triangle DBF$ are similar with coefficient $\frac{BD}{BA}$, but $\frac{BD}{BA} = \frac{AE}{AC} \Longrightarrow AE = DF$. Likewise, we'll get $AD = EF \Longrightarrow AEFD$ is a parallelogram. Therefore, $AF$ goes through the midpoint of $DE$. As a result, medial triangle of the outer Vecten triangle is perspective with $\triangle ABC$ through the second Vecten point $V'$.

Consider again the squares $PQAB$ and $RSCA$ constructed outwardly on $AB, AC$. Note that $\triangle AQC$ and $\triangle BAR$ are congruent, since $AR = AC, QA = AB$ and $\angle QAR = \angle BAR$. Hence, we get $\angle ARB = \angle ACQ$. If $X \equiv CQ \cap BR$, then quadrilateral $ARCX$ is cyclic $\Longrightarrow RX \perp CQ$. If $M$ is midpoint of $BC$, then $DK \parallel QC$, because of $D$ is the midpoint of $BQ$ as well. Analogously, $ME \parallel BR$. If $AH$ is the A-altitude of $\triangle ABC$, then quadrilaterals $AECH$ and $ADBH$ are inscribed in circles with diameter $AC$ and $AB \Longrightarrow \angle EHD = \angle CAE + \angle BAD = 90^\circ$. Which implies that circle with diameter $DE$ goes through the midpoint $M$ of $BC$ and the foot of the A-altitude. Let $L$ be the center of the Vecten square constructed outwardly on $BC$ and let $A', B', C'$ be the midpoints of $ED, DL, LE$. Then, $\triangle A'B'C'$ and $\triangle ABC$ are perspective through $V'$ and orthologic through orthology centers the first Vecten point $V$ (obvious) and the nine-point center $N$, since $BC \perp A'N$ is radical axis of $\odot(DEMH), (N)$, etc. Hence, by Sondat's theorem for the orthologic triangles $\triangle ABC$ and $\triangle A'B'C'$, we conclude that $V', V, N$ are collinear.

This proof is dedicated to Jean-Louis and Kostas Vittas.

**vittasko**

*#7 May 11, 2009, 4:14 am*

Thank you very much dear **Luis** for the dedication of a nice proof based on the **Sondat theorem**.

I will post here next time, some comments about particular cases of perspective and simultaneously orthologic triangles, corresponded with our configuration ( similar isosceles triangles erected on the side-segments of a given triangle ), where we can easier to prove this strong theorem.

● Let us to present an approach of the collinearity of the points $V,\ V'$ (= **Vecten points** ) and $K$ ( = **symmedian point** ) of $\triangle ABC$, based on the **double ratio** (= **cross ratio** ) theory.

So, we denote as $E,\ E'$ the points of intersection, of the midperpendicular of the side-segment $AC$, from the circle $(Y)$ with diameter $AC$ and as $F,\ F'$ the points of intersection, of the midperpendicular of the side-segment $AB$, from the circle $(X)$ with diameter $AB$.

The points $V \equiv BE \cap CF,\ V' \equiv BE' \cap CF'$, are the outer and the inner **Vecten points** respectively, of $\triangle ABC$, as well.

Also, the point $K \equiv BB' \cap CC'$, where $B'$ is the point of intersection of the tangent lines of the circumcircle $(O)$ of $\triangle ABC$, at vertices $A,\ C$ and $C'$ is the one, at vertices $A,\ B$, is the **symmedian point** of $\triangle ABC$.

● Let be the points $P \equiv BC \cap EE'$ and $Q \equiv BC \cap FF'$.

We consider the pencils $A.B'EE'P,\ A.QF'FC'$ and it is easy to show that the angles formed by their homologous rays, are equal.

( $\angle B'AE = \angle QAF' = \angle B - 45^o$ and $\angle EAE' = \angle FAF' = 90^o$ and $\angle E'AP = \angle FAC' = \angle C - 45^o$ ).

So, we conclude that $(A.B'EE'P) = (A.QF'FC')$, $(1)$

From $(1) \Longrightarrow (B', E, E', P) = (Q, F', F, C') = (C', F, F', Q)$, $(2)$

But, $(B', E, E', P) = (B.B'EE'P)$, $(3)$ and $(C', F, F', Q) = (C.C'FF'Q)$, $(4)$

From $(2),\ (3),\ (4) \Longrightarrow (B.B'EE'P) = (C.C'FF'Q)$, $(5)$

From $(5)$ and because of the pencils $B.B'EE'P,\ C.C'FF'Q$, have the sideline $BC$ of $\triangle ABC$, as their common ray, we conclude that the points $K \equiv BB' \cap CC'$ and $V \equiv BE \cap CF$ and $V' \equiv BE' \cap CF'$, are collinear and the proof is completed.

● As a particular case of the **Lemma** which has been proved in the topic http://www.mathlinks.ro/Forum/viewtopic.php?t=161643 , we can say that the line $VV'$, also passes through the **nine point center** $N$, of $\triangle ABC$.

Best regards, .Kostas Vittas.

*Attachments:*
t=275742.pdf (8kb)

**vittasko**

*#8 May 11, 2009, 6:05 pm*

Let $E,\ F$ be the centers of two similar rectangles ( instead of squares ) with $\angle EAC = \angle FAB = \angle\omega$ erected on

Let $E,\ F$ be, the centers of two similar rectangles ( instead of squares ), with $\angle EAC = \angle FAB = \angle\omega$, erected on $AC,\ AB$ respectively, outwardly to $\triangle ABC$.

Similarly, we define the points $E',\ F'$, inwardly to $\triangle ABC$.

We can say the same words as before, about the proof of the result that the line connecting the points $T \equiv BE \cap CF$ and $T' \equiv BE' \cap CF'$ ( the outer and inner **Kiepert perspectors**, mentioned by **Luisgeometria**, post #7# ), passes through the **symmedian point** $K \equiv BB' \cap CC'$ of $\triangle ABC$.

( $\angle B'AE = \angle B - \angle\omega = \angle QAF'$ and $\angle E'AP = \angle C - \angle\omega = \angle FAC'$ and so on ).

Kostas Vittas.

**jayme**

#9 Apr 29, 2010, 4:28 pm

Dear Mathlinkers,
an article concerning the Vecten's figure and its developpement can be seeing on my site
http://perso.orange.fr/jl.ayme , la figure de Vecten vol. 5 p. 127
Sincerely
Jean-Louis

Quick Reply

# High School Math

## A bisector

Reply

**Pain rinnegan**

#1 Apr 26, 2009, 2:58 pm • 1

Let be $AM$ a bisector in the triangle $ABC$ , $M \in (BC)$ . Show that :

$AM^2 = AB \cdot AC - MB \cdot MC$

**Dr Sonnhard Graubner**

#2 Apr 26, 2009, 7:53 pm

hello, we have $\overline{MC} = \frac{ab}{b+c}$ and $\overline{MB} = \frac{ac}{b+c}$, inserting this in your equation we get
$\overline{AM}^2 = bc\left(1 - \frac{a^2}{(b+c)^2}\right)$ this is the equation for the bisector.
Sonnhard.

**Dr Sonnhard Graubner**

#3 Apr 26, 2009, 8:11 pm

hello, for the equation of an angle bisector see here
http://mathworld.wolfram.com/AngleBisector.html
Sonnhard.

**Luis González**

#4 May 8, 2009, 10:55 pm • 1

Ray $AM$ cuts the circumcircle of $\triangle ABC$ again at $P$. Since $AP$ is self-isogonal with respect to $\angle BAC$, it follows that $AP \cdot AM = AB \cdot AC \ (\star)$, but from power of $M$ with respect to the circumcircle $\odot(ABC)$ we get

$$BM \cdot MC = AM \cdot MP = AM(AP - AM) = AM \cdot AP - AM^2$$

Combining with $(\star)$ yields $AM^2 = AB \cdot AC - MB \cdot MC.$

Quick Reply

# High School Olympiads

## relation: d=(p-m)/(n-q)



**Source**: own

**Petry**

#1 Feb 1, 2009, 9:58 pm

Let $ABC$ be a triangle $M,P\in AB, N,Q\in AC, \{X\}=MN\cap PQ$ and $\{D\}=AX\cap BC$.
If $\frac{MB}{MA}=m, \frac{NC}{NA}=n, \frac{PB}{PA}=p, \frac{QC}{QA}=q$ and $\frac{DB}{DC}=d$ then prove $d=\frac{p-m}{n-q}$.

**Sashsiam_2**

#2 Feb 4, 2009, 2:39 am

Nice problem. It provides a way to determine the ratio DB/DC without all that Ceva's stuff.

Unfortunately, I couldn't solve it. I've tried many approaches, including extra parallels and area comparison, but nothing worked. **Petry**, can you please post your solution?

**Petry**

#3 Feb 5, 2009, 5:19 am

Hello!

My first post is incomplete. I am sorry.

Let $ABC$ be a triangle, $M,P\in(AB), N,Q\in(AC), \{X\}=MN\cap PQ, X\in Int\,(\Delta ABC)$ and $\{D\}=AX\cap BC$.
If $\frac{MB}{MA}=m, \frac{NC}{NA}=n, \frac{PB}{PA}=p, \frac{QC}{QA}=q$ and $\frac{DB}{DC}=d$ then prove $d=\frac{p-m}{n-q}$.

Solution:

$$\frac{DB}{DC}=\frac{AB\cdot sin\,(\angle BAD)}{AC\cdot sin\,(\angle CAD)}=\frac{AB}{AC}\cdot\frac{\frac{XM\cdot sin(\angle AXM)}{AM}}{\frac{XN\cdot sin(\angle AXN)}{AN}}=\frac{AB}{AC}\cdot\frac{AN}{AM}\cdot\frac{XM}{XN}=$$
$$=\frac{AB}{AC}\cdot\frac{AN}{AM}\cdot\frac{QA}{QN}\cdot\frac{PM}{PA}$$

So $\frac{DB}{DC}=\frac{AB}{AC}\cdot\frac{AN}{AM}\cdot\frac{QA}{QN}\cdot\frac{PM}{PA}$ (1)

$\frac{MB}{MA}=m\Rightarrow\frac{AB}{AM}=m+1$ (2)
$\frac{NC}{NA}=n\Rightarrow\frac{AN}{AC}=\frac{1}{n+1}$ (3)
(2),(3)$\Rightarrow\frac{AB}{AC}\cdot\frac{AN}{AM}=\frac{m+1}{n+1}$ (4)

$\frac{PM}{PA}=\frac{AM}{AP}-1$ (5)
$\frac{AM}{AB}=\frac{1}{m+1}$ and $\frac{AP}{AB}=\frac{1}{p+1}\Rightarrow\frac{AM}{AP}=\frac{p+1}{m+1}$ (6)
(5), (6) $\Rightarrow\frac{PM}{PA}=\frac{p-m}{m+1}$ (7)

$\frac{QA}{QN} = \frac{1}{1 - \frac{AN}{AQ}}$ (8)

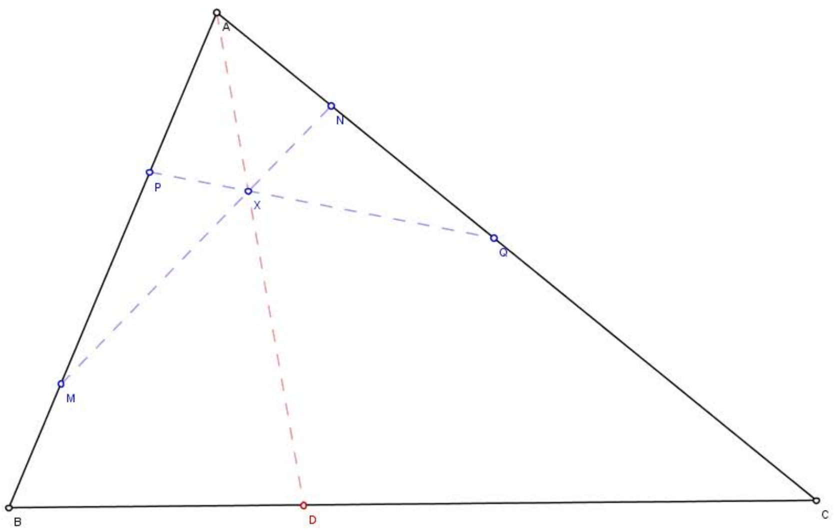
$\frac{AN}{AC} = \frac{1}{n+1}$ and $\frac{AQ}{AC} = \frac{1}{q+1} \Rightarrow \frac{AN}{AQ} = \frac{q+1}{n+1}$ (9)

(8), (9) $\Rightarrow \frac{QA}{QN} = \frac{n+1}{n-q}$ (10)

(1), (4), (7), (10) $\Rightarrow \frac{DB}{DC} = \frac{p-m}{n-q}$.

Best regards, Petrisor Neagoe

*Attachments:*




**Luis González**

*#4 May 7, 2009, 11:29 pm*

According to the given ratios, we express the barycentric coordinates of $M, N, P, Q$ as follows

$M\ (m:1:0)\ ,\ N\ (n:0:1)\ ,\ P\ (p:1:0)\ ,\ Q\ (q:0:1)$

Lines $MN \equiv x - ym - nz = 0$ and $PQ \equiv x - yp - qz = 0$ meet at

$X\ (mq - np : q - n : m - p)$

Hence, ray $AX$ cuts $BC$ at $D\ (0 : n - q : p - m) \Longrightarrow \frac{BD}{DC} = \frac{p-m}{n-q}$

**Virgil Nicula**

*#5 May 8, 2009, 9:29 pm*

> *Quote:*
>
> Let $ABC$ be a triangle . Given are the points $\{M, P\} \subset AB$ (sideline !) and $\{N, Q\} \subset AC$ (sideline !) .
>
> Denote $\left\|\begin{array}{c} X \in MN \cap PQ \\ D \in AX \cap BC \end{array}\right\|$ and $\frac{\overline{MB}}{\overline{MA}} = m\ ,\ \frac{\overline{NC}}{\overline{NA}} = n\ ,\ \frac{\overline{PB}}{\overline{PA}} = p\ ,\ \frac{\overline{QC}}{\overline{QA}} = q$ .
>
> Then there are the relations $\frac{\overline{DB}}{\overline{DC}} = \frac{m-p}{n-q}$ and $\frac{\overline{XD}}{\overline{XA}} = \frac{mq - np}{(m-n)-(p-q)}$ .

**Lemma (well-known !).** Consider in $\triangle ABC$ the points $D \in BC$ , $E \in CA$ , $F \in AB$

(sideline !) and $X \in EF \cap AD$ . Then $\frac{\overline{FB}}{\ldots}$ ... $\frac{\overline{EC}}{\ldots}$ ... $\frac{\overline{XD}}{\ldots}$ ...

(sidelines !) and $X \in EF \cap AD$ . Then $\frac{\overline{\phantom{AB}}}{FA} \cdot DC + \frac{\overline{\phantom{AC}}}{EA} \cdot BD = \frac{\overline{\phantom{AD}}}{XA} \cdot BC \quad (*)$ .

Proof.

**Remark.** Prove easily that the remarkable relation $(*)$ is equivalently with $\frac{\overline{AB}}{\overline{AF}} \cdot \overline{DC} + \frac{\overline{AC}}{\overline{AE}} \cdot \overline{BD} = \frac{\overline{AD}}{\overline{AX}} \cdot \overline{BC}$ .

**Proof of the proposed problem.** Apply the above lemma (I used always the oriented segments !) :

$$\left\| \begin{array}{ccc} \{M, X, N\} & \Longrightarrow & \frac{MB}{MA} \cdot DC + \frac{NC}{NA} \cdot BD = \frac{XD}{XA} \cdot BC \\ \{P, X, Q\} & \Longrightarrow & \frac{PB}{PA} \cdot DC + \frac{QC}{QA} \cdot BD = \frac{XD}{XA} \cdot BC \end{array} \right\| \Longrightarrow$$

$$\frac{XD}{XA} \cdot BC = m \cdot DC + n \cdot BD = p \cdot DC + q \cdot BD \Longrightarrow$$

$(n - q) \cdot BD = (p - m) \cdot DC \Longrightarrow \boxed{\frac{\overline{DB}}{\overline{DC}} = \frac{m - p}{n - q}}$ . Observe that $\frac{XD}{XA} \cdot BC = m \cdot DC + n \cdot BD \Longrightarrow$

$$\frac{XD}{XA} \cdot (BD + DC) = m \cdot DC + n \cdot BD \Longrightarrow \frac{XD}{XA}\left(\frac{BD}{DC} + 1\right) = m + n \cdot \frac{BD}{DC} \Longrightarrow$$

$$\frac{XD}{XA}\left(-\frac{m-p}{n-q} + 1\right) = m - n \cdot \frac{m-p}{n-q} \Longrightarrow \boxed{\frac{\overline{XD}}{\overline{XA}} = \frac{mq - np}{(m-n) - (p-q)}} .$$

**sunken rock**

#6 May 9, 2009, 9:07 pm

Take A as origin of a system of vectors. All single capital letters below mean the position vector from A (hence W means the vector AW). Given the problem conditions,
M = B/(m + 1), P = B/(p + 1), Q= C/(q + 1), R = C/(r + 1) ( 1 )
The equation of the line MN is given by the position vector of a moving point X from this line:
X = kM + (1 - k)N ( 2 ) while the equation of PQ is X = lP + (1 - l)Q ( 3 ).
Replace in (2) and (3) the relations (1) and, to get the position of the common point X of the lines MN and PQ, we have to equalize the coefficients of B and C: X[(n+1)(p+1) - (m+1)(q+1)] = (n-q)B + (p-m)C ( 4 ).
The equation of BC is given by the position vector of the moving point D: (d + 1)D = B + dC ( 5 ), while the equation of AX is given by a moving point D from AX: D = eX ( 6 ). The point common point of AX and BC will be found by equalizing the coefficients of B agardsnd C, from this we find d = (p - m)/(n - q), while $\boxed{\frac{\overline{XD}}{\overline{XA}} = \frac{mq - np}{(m-n) - (p-q)}}$ .

Best regards,
sunken rock

Quick Reply

# High School Olympiads

## Tangent to circumcircle

Reply

**Source**: China Team Training 2003

**Ichserious**
#1 May 6, 2009, 5:59 pm

In an acute triangle $ABC$, the tangents to its circumcircle at $A$ and $C$ intersect at $D$, the tangents to its circunmcircle at $C$ and $B$ intersect at $E$. $AC$ and $BD$ meet at $R$ while $AE$ and $BC$ meet at $P$. Let $Q$ and $S$ be the mid-points of $AP$ and $BR$ respectively. Prove that $\angle ABQ = \angle BAS$.

**sunken rock**
#2 May 6, 2009, 11:10 pm

Let a, b and c be the side lengths of the triangle; suppose, w.l.o.g a>b.
Obviously, AP and BR are symmedians and BP = ac^2/(b^2+c^2), AR = bc^2/(a^2+c^2)
Let E be the intersection of AS and BD; apply Menelaos to the triangle BCR with the transversal ASE and get BE=ac^2/(a^2+2c^2) ( 1 ).

Take D on (BC) so that BD = AD; from Stewart we get AD = BD = ac^2/(a^2+c^2-b^2) ( 2 ).

Call F the common point of BQ and AD, apply Menelaos to triangle ADP with the transversal BQF and get
AF/FD = BP/BD = (a^2 + c^2 - b^2)/(b^2 + c^2) ( 3 ).

From (1) and (2) we get BE/DE = AF/DF, hence EF || AB and ABEF isosceles trapezoid, therefore <BAE = <FBA; as S and Q lie on AE and BF respectively, the problem is solved.

Best regards,
sunken rock

**Luis González**
#3 May 7, 2009, 12:13 am

**Equivalent statement:** Let $M, N$ be the feet of the B-symmedian and C-symmedian of $\triangle ABC$. $D, E$ are the midpoints of $CN$ and $BM$. Then $\angle DBC = \angle ECB$.

Using barycentric coordinates with respect to $\triangle ABC$, we get

$M(a^2 : 0 : c^2)$ , $N(a^2 : b^2 : 0)$ , $D(a^2 : b^2 : a^2 + b^2)$ , $E(a^2 : a^2 + c^2 : c^2)$

$BD \equiv za^2 - x(a^2 + b^2) = 0$ and $CE \equiv a^2y - x(a^2 + c^2) = 0$ meet at the point

$P\ (a^2 : a^2 + c^2 : a^2 + b^2)$

Clearly, coordinates of $P$ satisfies the equation of the perpendicular bisector $\ell_a$ of $BC$, namely

$\ell_a \equiv (b^2 - c^2)x + a^2(y - z) = 0$

Thus, $\triangle PBC$ is P-isosceles $\Longrightarrow \angle DBC = \angle ECB$, as desired.

**Luis González**
#4 May 7, 2009, 10:34 am

There are some nice additional results about this configuration. $M, N, L$ are the feet of the symmedians issuing from $B, C, A$ and $D, E, F$ are the midpoints of $BM, CN, AL$.

1) If $\angle DBC = \angle ECB = \varphi$, then $\cot\varphi = \dfrac{3a^2+b^2+c^2}{4|\triangle ABC|}$.

2) If $M_a, M_b, M_c$ are the midpoints of $BC, CA, AB$, the lines $FM_a, EM_b, DM_c$ are concurrent at the complement of the third Brocard point $X_{76}$ of $\triangle ABC$.

3) $\triangle ABC$ and $\triangle FED$ are perspective through symmedian point $K$ (obvious) and their perspectrix coincides with the trilinear polar of the retrocenter $R \equiv X_{69}$ of $\triangle ABC$.

*This post has been edited 3 times. Last edited by Luis González, May 7, 2009, 9:46 pm*

**April**

*#5 May 7, 2009, 10:48 am*

Hallo Luis,

I am not good at barycentric coordinates but you can find some other solutions in these links: http://www.mathlinks.ro/viewtopic.php?t=207440 and http://www.mathlinks.ro/viewtopic.php?t=19806 .

Regards,
April

Quick Reply

# High School Olympiads

## Trigonometric condition

Reply

**Luis González**

#1 Apr 18, 2009, 10:51 pm

Show that Euler line of a nonequilateral $\triangle ABC$ is tangent to its incircle $(I)$, if and only if the following relation holds:

$$\sqrt{1-8\cos A\cos B\cos C}=\frac{|(\sin A-\sin B)(\sin B-\sin C)(\sin C-\sin A)|}{(1-\cos A)(1-\cos B)(1-\cos C)}$$

**yetti**

#2 May 4, 2009, 6:03 am

Standard notation: $(O,R)$ is the circumcircle, $(I,r)$ the incircle, $H$ the orthocenter, $a=BC, b=CA, c=AB$ and $p$ is the triangle semiperimeter. In addition, $K,L,M$ are midpoints of $BC,CA,AB$; $D,E,F$ are incircle tangency points with $BC,CA,AB$; $U,V,W$ are feet of the altitudes $AH,BH,CH$; the internal angle bisectors $AI,BI,CI$ cut the circumcircle again at $X,Y,Z$.

Power of $H$ to $(O)$ is $HO^2-R^2=2\ \overline{HU}\cdot\overline{HA}=-8R^2\cos A\cos B\cos C \Longrightarrow$ the LHS is equal to $\frac{HO}{R}$. Using the result of http://www.mathlinks.ro/viewtopic.php?t=256552 (also proved at http://www.mathlinks.ro/viewtopic.php?t=260863), the RHS is equal to

$$\frac{|(b-c)(c-a)(a-b)|}{8R^3\cdot\prod(1-\cos A)}=\frac{|\overline{KD}\cdot\overline{LE}\cdot\overline{MF}|}{KX\cdot LY\cdot MZ}=\frac{|\overline{DU}\cdot\overline{EV}\cdot\overline{FW}|}{r^3}=$$

$$=\frac{|(b-c)(p-a)\cdot(c-a)(p-b)\cdot(a-b)(p-c)|}{abc\cdot r^3}=$$

$$=\frac{|(b-c)(c-a)(a-b)|\cdot|\triangle ABC|^2}{4R\cdot|\triangle ABC|\cdot r^3\cdot p}=\frac{|(b-c)(c-a)(a-b)|}{4Rr^2}$$

$\Longrightarrow$ the given trigonometric condition is equivalent to $HO=\frac{|(b-c)(c-a)(a-b)|}{4r^2}$. The Euler line $OH$ is tangent to the incircle $\Longleftrightarrow |\triangle IOH|=\frac{r}{2}\cdot HO$. It is therefore sufficient to show that for arbitrary $\triangle ABC$, $|\triangle IOH|=\frac{|(b-c)(c-a)(a-b)|}{8r}$. WLOG, assume $a>b>c$. This area can be calculated from areas of the trapezoids $\square KOHU, \square KOID, \square DIHU$.

$$|\triangle IOH|=|\square KOHU|-|\square KOID|-|\square DIHU|=$$
$$=\tfrac{1}{2}(OK+HU)\cdot KU-\tfrac{1}{2}(OK+ID)\cdot KD-\tfrac{1}{2}(ID+HU)\cdot DU=$$
$$=\tfrac{1}{2}(OK\cdot DU+HU\cdot KD-ID\cdot KU)=$$

$$=\tfrac{1}{2}[R\cos A\cdot\frac{(b-c)(p-a)}{a}+2R\cos B\cos C\cdot\frac{b-c}{2}-R(\cos A+\cos B+\cos C-1)\cdot\frac{b^2-c^2}{2a}]=$$

$$=R\,\frac{b-c}{4a}((b+c-a)\cos A+2a\cos B\cos C-(b+c)(\cos A+\cos B+\cos C-1))=$$

$$=R\,\frac{b-c}{4a}(a(2\cos B\cos C-\cos A)-(b+c)(\cos B+\cos C-1)).$$

Substituting:

(1) $b\cos C + c\cos B = a,$

(2) $R(a\cos A + b\cos B + c\cos C) = 2|\triangle ABC|,$

(3) $2aR(\cos A + \cos B\cos C) = ah_a = 2|\triangle ABC| \implies$

$$|\triangle IOH| = R\,\frac{b-c}{4a}(b+c+a-2a(1+\cos A)) = R\,\frac{b-c}{2a}(\frac{|\triangle ABC|}{r} - \frac{a\sin A}{\tan\frac{A}{2}}) =$$

$$= \frac{b-c}{8r}(bc - 2a(p-a)) = \frac{|(b-c)(c-a)(a-b)|}{8r}$$

EDIT: I should have searched earlier. See http://www.mathlinks.ro/viewtopic.php?t=2878.

**Luis González**

#3 May 4, 2009, 8:22 am

Thanks for your very nice solution yetti!. I also used the result of the topic An IOH smart and interesting problem. Incircle $(I,r)$ is tangent to the Euler line $\iff |\triangle IOH| = \frac{1}{2}r\cdot OH.$ Hence

$$\frac{r^2\cdot OH^2}{4} = |\triangle IOH|^2 = \frac{(a-b)^2(b-c)^2(c-a)^2}{64r^2}$$

Now, we use $OH^2 = 9R^2 - (a^2+b^2+c^2),$ together with the identities

$$a = 2R\cdot\sin A\ ,\ \ b = 2R\cdot\sin B\ ,\ \ c = 2R\cdot\sin C$$

$$\frac{9}{4} - \sum\sin^2 A = \frac{R^4}{r^4}(\sin A - \sin B)^2\cdot(\sin B - \sin C)^2\cdot(\sin C - \sin A)^2 =$$

$$\frac{1}{4} - 2\cos A\cdot\cos B\cdot\cos C = \frac{R^4}{r^4}(\sin A - \sin B)^2(\sin B - \sin C)^2(\sin C - \sin A)^2$$

In the RHS we use the identity $\frac{r}{R} = 4\sqrt{\frac{1}{8}(1-\cos A)(1-\cos B)(1-\cos C)}$

After extracting the square root in both sides of the equation, we obtain

$$\sqrt{1 - 8\cos A\cos B\cos C} = \frac{|(\sin A - \sin B)(\sin B - \sin C)(\sin C - \sin A)|}{(1-\cos A)(1-\cos B)(1-\cos C)}$$

Quick Reply

# High School Olympiads

## Inversion of incircle to circumcircle



**Source**: tricky

**yetti**

#1 Jun 13, 2008, 12:18 pm

Let inversion in a circle $(M)$ take the incircle $(I)$ of a $\triangle ABC$ into its circumcircle $(O)$. It takes the triangle sidelines $a, b, c$ into circles $(P_a), (P_b), (P_c)$ with radii $R_a, R_b, R_c$ intersecting at the inversion center $M$ and pairwise on the incircle $(I)$ and they are tangent to the circumcircle $(O)$. $r, R$ are the triangle inradius and circumradius. Show that

$$\frac{1}{R_a}+\frac{1}{R_b}+\frac{1}{R_c}=\frac{2}{r}+\frac{2}{R}$$

**yetti**

#2 Apr 30, 2009, 3:30 pm

B.U.M.P. Any ideas ?

**Luis González**

#3 May 2, 2009, 10:00 am • 1

Let $k^2$ be the power of the inversion taking $(I)$ into $(O)$. It's center $M$ is the insimilicenter $X_{55}$ of $(I) \sim (O)$. Let $n_a, n_b, n_c$ the distances from $M$ to $BC, CA, AB$. By inversion properties we get $k^2=2n_a \cdot R_a=2n_b \cdot R_b=2n_c \cdot R_c$

$$\Longrightarrow \ \frac{1}{R_a}+\frac{1}{R_b}+\frac{1}{R_c}=\frac{2(n_a+n_b+n_c)}{k^2} \quad (1)$$

On the other hand, we have:

$$\frac{R}{r}=\frac{k^2}{r^2-IM^2} \ , \ \frac{OI}{IM}=\frac{R+r}{r} \Longrightarrow \frac{R}{r}=\frac{k^2}{r^2-\frac{r^2(R^2-2Rr)}{(R+r)^2}}$$

$$\Longrightarrow \ k^2=\frac{Rr(r^2+4Rr)}{(R+r)^2} \quad (2)$$

Assume that $\triangle ABC$ is acute, the remaining case is treated analogously with appropriate choice of signs. Let $d_a, d_b, d_c$ be the distances from $O$ to $BC, CA, AB$. If $(I)$ touches $BC$ at $X$ and $M_a$ is the midpoint of $BC$, then $OIXM_a$ is a right trapezoid and $n_a$ is a parallel to its bases

$$\Longrightarrow \ n_a=\frac{r \cdot OM+d_a \cdot IM}{IO}$$

Adding cyclic expressions, keeping in mind that $d_a+d_b+d_c=R+r$, we get:

$$n_a+n_b+n_c=\frac{3r \cdot OM+(R+r)IM}{IO}=3r+\frac{IM}{IO}(R-2r)$$

$$n_a+n_b+n_c=3r+r\frac{R-2r}{R+r}=\frac{r^2+4Rr}{R+r} \quad (3)$$

Combining $(1), (2), (3)$ we obtain:

$$\frac{1}{R_a}+\frac{1}{R_b}+\frac{1}{R_c}=\frac{2(r^2+4Rr)}{R+r}\frac{(R+r)^2}{Rr(r^2+4Rr)}=\frac{2(R+r)}{Rr}$$

$$\frac{1}{R_a}+\frac{1}{R_b}+\frac{1}{R_c}=2\left(\frac{1}{R}+\frac{1}{r}\right)$$

**yetti**

#4 May 3, 2009, 8:15 am • 1

Thanks for the solution. This is what I did:

Inversion center $M$ is similarity center of a circle and its image. As the incircle $(I)$ is inside the circumcircle $(O)$, $M$ is their internal similarity center. If $d=\overline{OI}$, then $\overline{OM}=\frac{Rd}{R+r}$, $\overline{MI}=\frac{rd}{R+r}$. Let the ray $(OI$ cut $(O),(I)$ at $U,V$. Using $d^2=R^2-2Rr$, power of inversion is

$$k^2=\overline{MU}\cdot\overline{MV}=(R-\overline{OM})(r+\overline{MI})=Rr-\overline{OM}\cdot\overline{MI}=Rr-\frac{Rrd^2}{(R+r)^2}=\frac{Rr^2(4R+r)}{(R+r)^2}.$$

Internal and external bisectors of $\angle A,\angle B,\angle C$ cut $BC,CA,AB$ at $D,E,F$ and $X,Y,Z$, respectively. Then $XEF,YFD,ZDE,XYZ$ are collinear. Let $p_a,p_b,p_c$ be directed distances of a point $P$ from $BC,CA,AB$. $p_a$ is positive when $P$ is on the same side of $BC$ as $A$ and negative otherwise; signs of $p_b,p_c$ are defined cyclically. When $P\in XEF$, then $p_b+p_c-p_a=0$ and cyclic exchange. This is well known, was posted before, the proof was algebraic, but I cannot find it. Anyway, let $P\in XEF$ and let perpendicular to $AID$ through $P$ cut $CA$ at $Q$. Let $q_c$ be directed distance of $Q$ from $AB$. Let parallel to $BC$ through $P$ cut $AB$ at $C'$ and let parallel to $AB$ through $Q$ cut $BC$ at $A'$. Let $C'P,A'Q$ meet at $S$. $\triangle ABX\sim\triangle QPS$ are centrally similar with center $E$, hence $BES$ are collinear, parallelogram $BA'SC'$ is a rhombus and $p_b+p_c=q_c=p_a$. On the other hand, if $P\in XYZ$, then $p_a+p_b+p_c=0$ (just a sign change) and the proof is exactly the same.

Let $I_a,I_b,I_c$ be excenters of the $\triangle ABC$ and $(O_e)$ circumcircle of the excentral $\triangle I_aI_bI_c$. $I$ is its orthocenter, $(O)$ its 9-point circle, $O_eOI$ its Euler line. $XYZ\perp O_eOI$ is radical axis of $(O_e),(O)$, let $K\equiv O_eOI\cap XYZ$. Let $P$ be arbitrary point, let parallel $p$ to $XYZ$ through $P$ cut $O_eOI$ at $K'$. Let the lines $IX,IY,IZ$ cut the parallel $p\parallel XYZ$ at $X',Y',Z'$. Parallels to $BC,CA,AB$ through $X',Y',Z'$, respectively, pairwise intersect at $A',B',C'$. Triangles $\triangle A'B'C'\sim\triangle ABC$ are centrally similar with similarity center $I$ and coefficient $\kappa=\frac{\overline{IK'}}{\overline{IK}}$. $A'X',B'Y',C'Z'$ are external bisectors of the angles $\angle A',\angle B',\angle C'$. Let $p'_a,p'_b,p'_c$ be directed distances of $P$ from $B'C',C'A',A'B'$. Since $P\in X'Y'Z'$, $p'_a+p'_b+p'_c=0$ and

$$p_a+p_b+p_c=p'_a+p'_b+p'_c+3r(1-\kappa)=3r(1-\kappa).$$

When $P\equiv K'\equiv M$ (the inversion center), which is inside the incircle $(I)$, the directed distances $n_a,n_b,n_c$ of $M$ from $BC,CA,AB$ are all positive and $n_a+n_b+n_c=3r(1-\mu)$, where $\mu=\frac{\overline{IM}}{\overline{IK}}=-\frac{\overline{MI}}{\overline{IK}}$. Powers of $K$ to $(O_e),(O)$ are equal,

$$(\overline{O_eO}+\overline{OK})^2-4R^2=\overline{OK}^2-R^2,$$

$$d^2+2d\cdot\overline{OK}=3R^2\implies\overline{IK}=\overline{OK}-\overline{OI}=\frac{3R^2-d^2}{2d}-d=\frac{3Rr}{d},$$

$$3r(1-\mu)=3r\left(1+\frac{rd}{R+r}\cdot\frac{d}{3Rr}\right)=\frac{r(4R+r)}{R+r}$$

$$\frac{1}{R_a}+\frac{1}{R_b}+\frac{1}{R_c}=\frac{2(n_a+n_b+n_c)}{k^2}=\frac{2(R+r)}{Rr}=\frac{2}{r}+\frac{2}{R}.$$

**Luis González**

#5 May 3, 2009, 11:26 am

> *yetti wrote:*
>
> Let $p_a,p_b,p_c$ be directed distances of a point $P$ from $BC,CA,AB$. $p_a$ is positive when $P$ is on the same side of $BC$ as $A$ and negative otherwise; signs of $p_b,p_c$ are defined cyclically. When $P\in XEF$, then $p_b+p_c-p_a=0$ and cyclic exchange. This is well known, was posted before, the proof was algebraic, but I cannot find it.

It also follows from the trilinear equation of $\overline{XEF}$, namely $\beta+\gamma-\alpha=0$.

Quick Reply

# High School Olympiads

## Locus of I,H

Reply

**Tiger100**

#1 May 2, 2009, 1:57 pm

It is given two circles T_1 and T_2 internally tangent at A , and T_1 is bigger than T_2. Avariable tangent to T_2 cuts T_1 at B,C.Find the locus of the incenter and orthocenter of the triangle ABC.

**Luis González**

#2 May 3, 2009, 12:10 am • 1

Thanks for posting this problem dear professor. I remember you proposed this problem to the class 4 years ago and no one (including me) was able to tackle the problem.

Let $V$ be the tangency point of $T_2$ with $BC$. It is known that $AV$ bisects $\angle BAC$. Then

$$\frac{AI}{IV} = \frac{CA + AB}{BC}$$

Ray $AV$ cuts $T_1$ at $P$. By Ptolemy's theorem for $ABPC$ we have

$$BC \cdot AP = CA \cdot PB + AB \cdot PC.$$

Since $PB = PC \Longrightarrow \frac{AP}{PB} = \frac{CA + AB}{BC} \Longrightarrow \frac{AI}{IV} = \frac{AP}{PB}.$

Notice that $\triangle PAB \sim \triangle PVB$ are similar because of $\angle VBP = \angle BAP$, thus we have $PB^2 = AP \cdot PV$. Combining this one with the previous expression we obtain

$\frac{IV^2}{AI^2} = \frac{PV}{AP}$ . But $PV = AP - AV$ and $IV = AV - AI$

$$\Longrightarrow \left(\frac{AW}{AI} - 1\right)^2 = 1 - \frac{AV}{AP}.$$

Ratio $\frac{AV}{AP} = \text{const}$ is the coefficient $k$ of the direct homothety taking $T_1$ into $T_2$. Therefore, locus of the incenter $I$ is the homothetic circumference of $T_2$ under the homothety with center $A$ and coefficient $\frac{1}{\sqrt{1-k}+1}$.

On the other hand, let $M$ be the midpoint of $\overline{BC}$ and $O$ the center of $T_1$ (circumcenter of ABC). Since $O$ is fixed and $OM \perp BC \Longrightarrow M$ describes the pedal curve of $O$ WRT $T_2$, i.e. a Pascal Limaçon $\mathcal{L}$ generated by $(O, T_2)$. Then, centroid $G$ of $\triangle ABC$ decribes the homothetic limaçon $\mathcal{L}'$ of $\mathcal{L}$ under the homothety $\Psi(A, \frac{2}{3})$. Since $H, O, G$ are collinear on the Euler line of $\triangle ABC$ and $\overline{OH} = 3\overline{OG}$, we conclude that locus of $H$ is the homothetic limaçon of $\mathcal{L}$ under the homothety $\Psi(P, 2)$. Where $P$ is a fixed point on the line $AO$.

**Luis González**

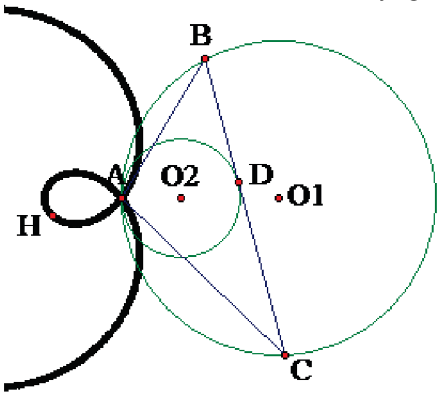
#3 May 3, 2009, 6:25 am • 1

**Remark:** Let $T(r')$ denote the circle described by the incenter $I$. The midpoint $N$ of $\overline{II_a}$ lies on the circle $T_1$. Then if $X, Y$ are the antipodes of $A$ WRT $T_1(R)$ and $T'(r')$ and $Q$ is the reflection of $Y$ about $X$, then the quadrilateral $II_aQY$ is a trapezoid with midline $NX \Longrightarrow QI_a \perp AI_a \Longrightarrow$ locus of the excenter $I_a$ of $\triangle ABC$ is another circle tangent to $T_1, T_2$ through $A$ with radius equal to $2R - r'$.

**tdl**

*#4 May 3, 2009, 8:33 am*

I wonder what locus of $H$ is? See my figure:




**Tiger100**

*#5 May 3, 2009, 1:34 pm*

Thanks luis,this is such a nice solution.and your remark as well.Mine is not that nice cuz I used polar coordinates

Quick Reply

# High School Olympiads

## Conic!! Conic!!

Reply

**mathVNpro**

#1 May 1, 2009, 5:57 pm

Let $P$ be a point in the plane of triangle $ABC$, not lying on the lines $AB, BC, CA$. Denote by $AA_b, A_c$ the intersections of the parrallels through $A$ to the line $PB, PC$ with $BC$. Define analogously $B_a, B_c, C_a, C_b$. Prove that $A_b, A_c, B_a, B_c, C_a, C_b$ lie on the same **conic**.

*(Proposed by Mihai Miculita)*

**Luis González**

#2 May 1, 2009, 9:56 pm

Let $(u:v:w)$ be the barycentric coordinates of $P$ WRT $\triangle ABC$. Then cevian $PB$ has equation $wx-uz=0$ and infinite point $P_\infty \equiv (-u:u+w:-w)$. Thus, parallel to $PB$ passing through $A$ has equation $wy+(u+w)z=0 \Longrightarrow$ feet of the cevians $AA_b, AA_c$ have coordinates

$$A_b \equiv (0:u+v:-w) \ , \ A_c \equiv (0:-v:u+v)$$

Similarly we get the coordinates of $B_a, B_c$ and $C_a, C_b$ as

$$B_a \equiv (v+w:0:-w) \ \ , \ \ B_c \equiv (-u:0:u+v)$$

$$C_a \equiv (v+w:-v:0) \ \ , \ \ C_b \equiv (-u:u+v:0)$$

$$\frac{AC_b}{C_bB} \cdot \frac{AC_a}{C_aB} \cdot \frac{CB_c}{B_cA} \cdot \frac{CB_a}{B_aA} \cdot \frac{BA_c}{A_cC} \cdot \frac{BA_b}{A_bC} = \frac{-uvw(u+w)(u+w)(u+w)}{uvw(u+w)(u+w)(u+w)} = 1$$

By Carnot's theorem we conclude that these 6 points lie on a same conic.

Quick Reply

# High School Olympiads

## Extremum problem.

Reply

**Source**: Classic.

**Virgil Nicula**

#1 Feb 25, 2008, 6:32 pm

> Quote:
> Let $w$ be the circumcircle of a given triangle $ABC$ and let $M \in w$ be a mobile point.
>
> Ascertain the maximum value of the product $p = MA \cdot MB \cdot MC$ .

**zaya_yc**

#2 Feb 27, 2008, 9:12 pm

$a \le c \le b$ $P \in AC$ --> $BM*AC = CM*AB + AM*BC$ ---->

$BM*AC = CM*AB + AM*BC \ge 2\sqrt{CM*AB*AM*BC}$ --->

$CM*AM \le \frac{(BM*AC)^2}{4AB*BC} = \frac{(b*BM)^2}{4ac}$ ---->

$P = MB \cdot MA \cdot MC \cdot \frac{(b*BM)^2}{4ac} = \frac{b^2BM^3}{4ac} \le \frac{b^2(2R)^3}{4ac} = \frac{2b^2R^3}{ac}$

$P_{max} = \frac{2b^2R^3}{ac}$

$BM = 2R$

$CM = \frac{bR}{c}$

$AM = \frac{bR}{a}$

**vittasko**

#3 Feb 29, 2008, 10:21 pm

If I am not mistaken dear **zaya_yc**, you mean that the antidiametric (= diametric opposite) point of $B,$ has the property as the problem states.

But I think this is not correct. We can see in my drawing, where $BC = \alpha = 1,90 < AB = c = 7,36 < AC = b = 7,48,$ that the midpoint $M$ of the arc $AC,$ not containing the vertex $B$ and another point $P,$ between $A$ and $M,$ have equal products as the problem states and bigger than the one with respect to the point $Q,$ where $BQ = 2R.$

That is we have $MA\cdot MB\cdot MC = PA\cdot PB\cdot PC > QA\cdot QB\cdot QC$ , $(1)$

I conjecture that the midpoint $N$ of the arc $PM,$ has the property of the maximum product as the problem states.

That is, I think $p_{max} = NA\cdot NB\cdot NC,$ but I have not in mind any proof about it.

Kostas Vittas

Kostas Vittas.

PS. I don't know the way to determine the point $P$, which is the key of the solution I thought, when the triangle $\triangle ABC$ is given. I made the configuration below, considered as fixed the point $P$, on a circle $(O)$ taken as cord the given segment $AC$, and the point $B$, is determined based on the problem Point's determination 6.

*Attachments:*
t=190749.pdf (6kb)

**Luis González**

*#4 Apr 30, 2009, 11:56 pm*

$2[\triangle MAB] = MA \cdot MB \cdot \sin C$

$2[\triangle MBC] = MB \cdot MC \cdot \sin A$

$2[\triangle MAC] = MA \cdot MC \cdot \sin B$

$\Longrightarrow 8[\triangle MAB][\triangle MBC][\triangle MAC] = (MA \cdot MB \cdot MC)^2 \cdot \sin A \cdot \sin B \cdot \sin C$

Let $(x : y : z)$ be the barycentrics of $M$ WRT $\triangle ABC.$ Dividing the latter equation into $[\triangle ABC]^3$ yields

$$\Longrightarrow 8xyz = \frac{(MA \cdot MB \cdot MC)^2 \cdot \sin A \cdot \sin B \cdot \sin C}{[\triangle ABC]^3}$$

Setting $\frac{1}{k} = \frac{\sin A \cdot \sin B \cdot \sin C}{8[\triangle ABC]^3} = \text{const}$

$$\Longrightarrow f(x, y, z) = \sqrt{kxyz} \ \ (1) \ \ , \ \ x + y + z = 1 \ \ (2)$$

Since $M \in \omega \Longrightarrow \ \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} = 0 \ \ (3)$

Now, we might use Lagrange multipliers on the fuction $f(x, y, z)$ subject to constraints $(2)$ and $(3)$, but the computations seem pretty heavy, so I will leave my proof at this point.

Quick Reply

# High School Olympiads

## Point's determination 6.

Reply

vittasko

#1 Feb 29, 2008, 4:56 am

**We draw a circle $(O)$, taken as cord the given segment $AB$ and let $M$ be, the midpoint of the arc $AB$. We define a fixed point $P$ on $(O)$, between $A,\ M$ and let $C$ be, a mobile point on the circle $(O)$. Determine the point $C$, in order to be $MA\cdot MB\cdot MC = PA\cdot PB\cdot PC.$**

Kostas Vittas.

PS. This not difficult problem is inspired from the Extremum problem, of **Virgil Nicula**.

*Attachments:*

t=191359.pdf (4kb)

Luis González

#2 Apr 30, 2009, 11:10 am

$\frac{PC}{MC}=\frac{MA\cdot MB}{PA\cdot PB}=k\Longrightarrow$ the wanted points $C$ satisfy $\frac{PC}{MC}=k.$

$\Longrightarrow C$ lies on the Apollonius circle relative to $P,M$ with ratio $k.$ Note that we do not need $M$ to be the midpoint of the arc $AB,$ unless you mean something else.

vittasko

#3 Apr 30, 2009, 5:31 pm

Thank you very much dear **Luis**, for your interest.

I have not in mind a different solution of this easy problem and as you said, it is not necessary to be the point $M,$ as the midpoint of the arc $AB.$

● This problem has been arisen, in trying to solve ( unfortunately without any success ) the Extremum problem, of **Virgil Nicula**.

I conjectured the wanted maximum product there, at the point $N,$ as the midpoint of the arc $PM$ ( please see the schema t=190749 there ).

So, if we can determine the point $P,$ such that $PA\cdot PB\cdot PC = MA\cdot MB\cdot MC,$ when the triangle $ABC$ is given ( consider the point $B$ in the schema t=190749, as the point $C$ in the schema t=191359 ) then, we can solve the Extremum problem.

Best regards, Kostas Vittas.

Quick Reply

# High School Olympiads

## Cyclic pentagon and product of distances

Reply

**Source**: Romania TST 2009. Day 2. Problem 3.

**Ahiles**
#1 Apr 21, 2009, 5:07 pm

Prove that pentagon $ABCDE$ is cyclic if and only if

$$\mathrm{d}(E,AB)\cdot\mathrm{d}(E,CD)=\mathrm{d}(E,AC)\cdot\mathrm{d}(E,BD)=\mathrm{d}(E,AD)\cdot\mathrm{d}(E,BC)$$

where $\mathrm{d}(X,YZ)$ denotes the distance from point $X$ ot the line $YZ$.

**Luis González**
#2 Apr 29, 2009, 7:44 am

I'll restate the problem with different notations.

**Problem:** $ABCDE$ is a pentagon and $X,Y,Z,W,P,Q$ are the orthogonal projections of $E$ onto $AB,CD,AC,BD,AD,BC$. Then $EX\cdot EY=EZ\cdot EW=EP\cdot EQ\Longleftrightarrow ABCDE$ is cyclic.

We assume that $ABCD$ is cyclic. If $EX\cdot EY=EP\cdot EQ\Longrightarrow\triangle EXP\sim\triangle EQY$ since $\angle XEP=\angle QEY$. Then $E$ is the center of the spiral similarity that takes $\triangle EXP$ into $\triangle EQY\Longrightarrow$ circles $\odot(EXP)$ and $\odot(EQY)$ meet on $QY$ and angle chase yields $\angle EBA=\angle ADE\Longrightarrow E\in\odot(ABCD)$. It remains to prove $EZ\cdot EW=EP\cdot EQ$, which folllows from the fact that $Y,W,Q$ are collinear on the Simson line with pole $E$ WRT $\triangle BCD$ and $Y,P,Z$ are collinear on the Simson line with pole $E$ WRT $\triangle ACD$. Then $\odot(YWP)$ and $\odot(YZQ)$ meet at $E\Longrightarrow E$ is the center of the spiral similarity taking $\triangle EZP$ into $\triangle EQW$, hence $EZ\cdot EW=EP\cdot EQ$.

**mumble**
#3 May 1, 2009, 5:06 pm

Besides this natural synthetic solution one could (not easily!) find the following one, which uses techniques of algebraic geometry. It involves some well-known properties of conics as well as some elements of complex geometry. I will write down the solution in different posts, for a better understanding.

**mumble**
#4 May 1, 2009, 5:31 pm

At first I will reformulate the problem:

*Prove that pentagon $A_0A_1A_2A_3A_4$ is cyclic iff the product* $ \mathrm{dist}\(A_4,A_iA_j)\cdot\mathrm{dist}(A_4,A_kA_l)$ *is the same for all choices of indices* $\{i,j,k,l\}=\{0,1,2,3\}$.

*This post has been edited 1 time. Last edited by mumble, May 1, 2009, 6:43 pm*

**mumble**
#5 May 1, 2009, 5:43 pm

**Solution.** In what follows, we denote by lowercases the complex coordinates of points denoted by uppercases.
The equation of the oriented line $A_jA_k$ is given by

$$f_{jk}(z,\bar{z})=\begin{vmatrix}a_j & \bar{a}_j & 1\\ a_k & \bar{a}_k & 1\\ z & \bar{z} & 1\end{vmatrix}=0$$

or $f_{jk}=(\bar{a}_j-\bar{a}_k)z-(a_j-a_k)\bar{z}+a_j\bar{a}_k-\bar{a}_ja_k=0.$

The distance from $A_4$ to $A_jA_k$ is $-\frac{J_{jk}(a_4, a_4)}{2\mathrm{i}|a_j - a_k|}$.
Next, it is not hard to verify that $f_{jk}((1-t)z + tz', (1-t)\bar{z} + t\bar{z}' = (1-t)f_{jk}(z, \bar{z}) + tf_{jk}(z', \bar{z}'), \forall t \in \mathbb{R}$ and $z, z' \in \mathbb{C}$.

*This post has been edited 1 time. Last edited by mumble, May 1, 2009, 6:44 pm*

**mumble**

*#6 May 1, 2009, 6:01 pm*

The condition in the statement is equivalent to the fact that $A_4$ lie on the following 2 conics passing through $A_0, A_1, A_2, A_3$:
$|a_0 - a_3| \cdot |a_1 - a_2| f_{01} f_{23} - |a_0 - a_1| \cdot |a_2 - a_3| f_{03} f_{12} = 0 \quad (1)$
and
$|a_0 - a_2| \cdot |a_1 - a_3| f_{01} f_{23} - |a_0 - a_1| \cdot |a_2 - a_3| f_{02} f_{13} = 0 \quad (2)$
This is possible iff the 2 conics coincide.
Both of the conics belong to the pencil of conics determined by $A_0, A_1, A_2, A_3$.
It is renowned that any two distinct conics of this pencil form a basis over $\mathbb{R}$ for the pencil; in particular, any two of the three (degenerate) conics: $f_{01}f_{23} = 0, f_{02}f_{13} = 0, f_{03}f_{12} = 0$.
These three are therefore linearly dependent over $\mathbb{R}$. Moreover, they satisfy, as we will show, the following Ptolemy-like relation:
$f_{01}f_{23} - f_{02}f_{13} + f_{03}f_{12} = 0 \quad (\star)$

*This post has been edited 1 time. Last edited by mumble, May 1, 2009, 6:21 pm*

**mumble**

*#7 May 1, 2009, 6:20 pm*

Assume $(\star)$ for the time being to rewrite (2) as:
$(|a_0 - a_2| \cdot |a_1 - a_3| - |a_0 - a_1| \cdot |a_2 - a_3|) f_{01} f_{23} - |a_0 - a_1| \cdot |a_2 - a_3| f_{03} f_{12} = 0 \quad (2')$
Now the two conics coincide iff:
$|a_0 - a_1| \cdot |a_2 - a_3| - |a_0 - a_2| \cdot |a_1 - a_3| + |a_0 - a_3| \cdot |a_1 - a_2| = 0.$
By **Ptolemy Theorem** the points $A_0, A_1, A_2, A_3$ are concyclic.

**mumble**

*#8 May 1, 2009, 6:43 pm*

It only remains to establish $(\star)$ for the proof to be complete.
Notice that $f = f_{01}f_{23} + f_{03}f_{12}$ vanishes at any point $z$ of the conic $f_{02}f_{13} = 0$: for example, if $z$ lies on the line $f_{02} = 0$ then $z = (1-t)a_0 + ta_2$, for some $t \in \mathbb{R}$ so that
$f(z, \bar{z}) = t(1-t)(f_{01}(a_2, \bar{a}_2) f_{23}(a_0, \bar{a}_0) + f_{03}(a_2, \bar{a}_2) f_{12}(a_0, \bar{a}_0) = 0.$
We used above the relation stated at the first step of the solution along with $(f_{01}(a_2, \bar{a}_2) = f_{12}(a_0, \bar{a}_0)$ and $f_{03}(a_2, \bar{a}_2) = -f_{23}(a_0, \bar{a}_0)$ which can be seen using determinants.
Consequently, one can find some $\alpha \in \mathbb{R}$ such that $f_{01}f_{23} + f_{03}f_{12} = \alpha f_{02}f_{13}$ and in a totally similar manner, one can find some $\beta \in \mathbb{R}$ such that $f_{01}f_{23} - f_{02}f_{13} = \beta f_{03}f_{12}$.
Combine the two to get $(\alpha - 1)f_{02}f_{13} - (\beta + 1)f_{03}f_{12} = 0$ and, furthermore, $\alpha = 1, \beta = -1$. This ends the proof.

**mumble**

*#9 May 3, 2009, 1:46 am*

I must say that this beautiful and surprising solution belongs to **Calin Popescu**.

---

Moderator EDIT: Keep "Disable BBCode in this post" **unchecked**, in order to compile $\LaTeX$ code.

**sunken rock**

*#10 May 3, 2009, 4:53 pm*

One can easily solve the problem using the following **Lemma:**

**Let AB be a chord and P a variable point in the plane of the circle (O). If M, K and L are the feet of the perpendiculars from P onto AB and onto the tangents at A and B to the circle, then the following relation holds:**

**PM^2 = PK*PL ( 1 )**

**iff P belongs to the circle (O).**

The proof comes directly from the similarity of the triangles PKM and PML: if P belongs to the circle, then the triangles are similar, from their similarity getting the relation (1).

from their similarity getting the relation (1).
If the relation (1) holds, then the triangles PKM and PML are similar (s.a.s., as if KA and LB meet at T, then PKTL is cyclic and, the bisectors of the angles <ATB and <KPL are parallel; PM being perpendicular onto AB is hence the angle bisector of <KPL).
But a simple angle chasing shows that these two triangles PKM and PML are similar to the triangle PAM and this compulsory brings P on the circumference ( BL is tangent to the circle (ABP), hence its center is the same with the one of (O) ).

Further, apply the lemma to (ABCD), supposed cyclic, and get (ABCDE).

Best regards,
sunken rock

**pohoatza**

*#11 May 3, 2009, 7:10 pm*

I haven't read your posts carefully, but I'm afraid **Luis** and **sunken rock** have only proved the weaker problem: Given a cyclic quadrilateral $ABCD$ and a point $E$ in plane, then this point $E$ lies on the circumcircle of $ABCDE$ if and only if $\mathrm{d}(E, AB)\cdot\mathrm{d}(E, CD) = \mathrm{d}(E, AC)\cdot\mathrm{d}(E, BD) = \mathrm{d}(E, AD)\cdot\mathrm{d}(E, BC)$.

The problem from the TST stated (and I don't see why this isn't clear from **Ahiles**' post): Let $A$, $B$, $C$, $D$, $E$ be five given points in plane (in general position if you like). Then they are *all* concyclic if and only if $\mathrm{d}(E, AB)\cdot\mathrm{d}(E, CD) = \mathrm{d}(E, AC)\cdot\mathrm{d}(E, BD) = \mathrm{d}(E, AD)\cdot\mathrm{d}(E, BC)$.

Actually, the part you missed (the identity implies the concyclicity of $A$, $B$, $C$, $D$) is the hardest part here, in my opinion.

**pohoatza**

*#12 May 4, 2009, 1:01 am • 2*

I proved it with (the converse of) Ptolemy's theorem. I hope it will pass as synthetic.

Let $k = \mathrm{d}(E, AB)\cdot\mathrm{d}(E, CD) = \mathrm{d}(E, AC)\cdot\mathrm{d}(E, BD) = \mathrm{d}(E, AD)\cdot\mathrm{d}(E, BC)$. The concyclicity of $A$, $B$, $C$, $D$ is equivalent with $AC\cdot BD = AB\cdot CD + BC\cdot DA$. Multiplying this by $k$, this rewrites as $|EAC|\cdot|EBD| = |EAB|\cdot|ECD| + |EBC|\cdot|EDA|$, where $|\mathcal{P}$ is the unsigned area of the convex polygon $\mathcal{P}$.
Expressing these areas differently, we get the new equivalent relation:
$\sin AEC \sin BED = \sin AEB \sin CED + \sin BEC \sin DEA$ (we have deleted the term $EA\cdot EB\cdot EC\cdot ED$).
Denote $\measuredangle AEB = x$, $\measuredangle BEC = y$, $\measuredangle CED = z$. In this case, the last relation can be rewritten as $\sin(x+y)\sin(y+z) = \sin x\sin z + \sin y\sin(x+y+z)$, which can be easily verified (being true for any angles with magnitudes $x$, $y$, $z$). Hence $ABCD$ is cyclic.

**mumble**

*#13 May 17, 2009, 12:12 am*

Throughout a review of the proof, I've noticed that one (essential ) thing misses, that is, the fact that our conic is indeed a circle.
I have reached that relation
$|a_0 - a_1|\cdot|a_2 - a_3| - |a_0 - a_2|\cdot|a_1 - a_3| + |a_0 - a_3|\cdot|a_1 - a_2| = 0$
which is equivalently to the cross ratio
$\dfrac{a_0 - a_1}{a_0 - a_3} : \dfrac{a_2 - a_1}{a_2 - a_3}$ being real and negative.
But this clearly holds. It is equivalent to the vanishing of the coefficient of $z^2$ (and $\bar{z}^2$ as well) in (1).
Hope everything's right now.

**ZODIACORACLE**

*#14 Sep 4, 2014, 10:20 pm*

Use inversion with center E!! Let some invertion with center E be I.
H1,H2,H3,H4,H5,H6 are feet of perpendicular from E to AB,CD,AC,BD,AD,BC respectively.So E,A,H1,H3,H5 E,B,H1,H4,H6 E,C,H2,H3,H6 E,D,H2,H4,H5 are concyclic. Let circle whose diameter is EA EB EC ED be C1,C2,C3,C4 respectively.
Now Ci(i=1,2,3,4) move to line. We call that line li. So l1 meets l2 l3 l4 at H1' H3' H5' respectivrey.
And l2 meets l3 l4 at H6' H4', l3 meets l4 at H2'.By EH1*EH2=EH3*EH4=EH5*EH6 we can get EH1'*EH2'=EH3'*EH4'=EH5'*EH6'.
If E is miquel point for 4 lines, we can prove that equation by similarity. If 4 lines are given, we can decide only one E, so E is miquel point. Now we can prove that A', B', C', D' are coliner by simson's theorem(A' B' C' D' are feet of perpendicular from E to l1,l2, l3, l4 because EA EB EC ED are diameters).
Therefore A,B,C,D,E are concyclic.

Quick Reply

+ Quick Reply

# High School Olympiads

## A problem about Fermat point.

Reply

**Source**: Concurency

**gb2124**
#1 Apr 28, 2009, 5:38 pm

Denote F by the Fermat point of the triangle ABC.Define Fa as the reflection of the side BC of the triangle.Similar we will have Fb,Fc.Prove that AFa,BFb,CFc are concurent.

**Agr_94_Math**
#2 Apr 28, 2009, 10:52 pm

A nice hint: When you draw the figure, join $AF_c, CF_a, AF_b$ to get three isiosceles triangles. Also join $AF_cB, BF_aC, AF_bC$.

**Luis González**
#3 Apr 29, 2009, 12:04 am

**Lemma.** If $P, Q$ are isogonal conjugates with respect to $\triangle ABC$, then the reflections of $Q$ across $BC, CA, AB$ form a triangle perspective with $\triangle ABC \Longleftrightarrow PQ$ is parallel to the Euler line of $\triangle ABC$.

The line joining the first Fermat point and the first isodynamic point has equation

$$\begin{bmatrix} a^2\sqrt{3}S_A + S & b^2\sqrt{3}S_B + S & c^2\sqrt{3}S_C + S \\ \frac{1}{\sqrt{3}S_A+S} & \frac{1}{\sqrt{3}S_B+S} & \frac{1}{\sqrt{3}S_C+S} \\ x & y & z \end{bmatrix} = 0$$

This line has infinite point $X_{30} : (S_AS_B + S_AS_C - 2S_BS_C)$, which coincides with the infinite point of the Euler line of $\triangle ABC$. Thus, reflections of the 1st Fermat point on $BC, CA, AB$ form a triangle perspective with $\triangle ABC$.

Quick Reply

# High School Olympiads

**Source**: Easy but nice!

**mathVNpro**

#1 Apr 24, 2009, 9:15 pm

Let $ABC$ be equilateral triangle inscribed in circumcircle $(O)$. Let $D$ be some point on the smaller arc of $BC$. Let $I_1, I_2, I_3$ be the incenters of triangle $DBC, DCA, DAB$, respectively. Prove that the line through $A, B, C$ perpendicular to $I_2I_3, I_3I_1, I_1I_2$ concurrent.

**mathVNpro**

#2 Apr 25, 2009, 10:29 am

**hint**: Why don't you use the **Carnot theorem** to prove it??

**sunken rock**

#3 Apr 25, 2009, 8:21 pm • 1

I shall try solving, before receiving full proof from its author...
First, let's re-name I_1, I_2 and I_3 as 1, 2 and 3 respectively.

It's known that the perpendiculars from the vertices of a triangle onto the sides of another triangle are concurrent iff the perpendiculars from the vertices of the 2nd triangle onto the sides of the first one are concurrent (is this Carnot's, or just a consequence?).

I shall show that the perpendiculars from 1 to BC, 2 to AC and 3 to AB concur at the Fermat's point of the triangle 123..

I name M and N the midpoints of the small arcs BD and CD, M' the point on the circle O such that B is the midpoint of the arc MM', and N' such as C is the midpoint of the arc NN'.
The points 1 and 3 lie on the circle C(M, BM) and, of course on the angle bisectors of <BAD and <BCD, while 1 and 2 lie on the circle C(N, NC), that is, 13M and 12N are equilateral triangles. The lines M3 and N2 pass through A and the angle <MAN = 30 degs (by construction), this making 13 and 12 perpendicular, or <M12 = <N13 = 150 degs, that is, AM12 and A31N are cyclic and <1M2 = <1A2 ( 1 ) and <1N3 = <1A3 ( 2 ), therefore M, 2 and N' are collinear, and so are N, 3 and M'.
Now it's easy to see that <MAC + <N'MA = 90 degs, that is, 2M and AC are perpendicular, and same are 3N and AB.
But 2M and 3N pass through the Fermat point of the triangle 123, hence the line joining 1 with Fermat point will be perpendicular to BC.

Best regards,
sunken rock

**mathVNpro**

#4 Apr 25, 2009, 9:52 pm

Very nice solution **sunken rock**. I cannot tell anything, just want to remind you all a very important outcome of **Carnot theorem**:
**"The perpendiculars from the vertices of a triangle onto the sides of another triangle are concurrent iff the perpendiculars from the vertices of the 2nd triangle onto the sides of the first one are concurrent"**

**Luis González**

#5 Apr 26, 2009, 10:59 pm • 1

If $I$ is incenter of $\triangle ABC,$ then according to Mikami and Kobayashi theorem, $II_1I_2I_3$ is a rectangle. Thus, it remains to prove that $\triangle II_2I_3$ and $\triangle ABC$ are orthologic. Let $X,Y$ be the projections of $I_2$ and $I_3$ onto $AB$ and $AC$ and $Q \equiv I_2X \cap I_3Y.$ Easy angle chasing gives that complete quadrangle $DI_2QI_3$ has excenter $A \Longrightarrow AX=AY \Longrightarrow$ Perpendiculars $I_2X$ and $I_3Y$ meet on the angle bisector of $\angle BAC.$ Hence, $\triangle ABC$ and $\triangle II_2I_3$ are orthologic $\Longrightarrow$ Perpendiculars from $A,B,C$ to the sidelines of $\triangle II_2I_3$ concur.

Quick Reply

# High School Olympiads

## Inequality involving coordinates.

Reply

**Source**: 0

**Luis González**

#1 Apr 26, 2009, 10:54 am

$P$ is an arbitrary point with normalized barycentric coordinates $(u:v:w)$ with respect to the acute triangle $\triangle ABC$ with centroid $G$ and Brocard angle $\omega$. Prove the inequality:

$$u \cdot \csc \widehat{BPC} + v \cdot \csc \widehat{CPA} + w \cdot \csc \widehat{APB} \geq \frac{1}{6}\cot\omega + \frac{\sqrt{3}}{2} - \frac{3}{4} \cdot \frac{PG^2}{[\triangle ABC]}$$

Figure out the coodinates of the point that fulfills the equality

Quick Reply

# High School Olympiads

## Incircle, Circumcircle and Concurrence

Reply

**Source**: not difficult but not bad

**77ant**

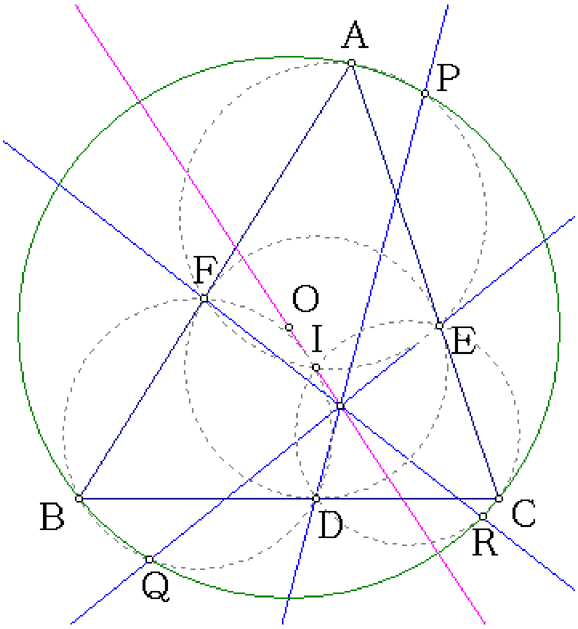
#1 Apr 25, 2009, 10:28 pm

For a triangle **ABC**, its incircle (**I**) touches **BC**,**CA**,**AB** at **D**,**E**,**F** respectively.
Let its circumcircle be (**O**) and the circumcircles of triangles **AEF**, **BDF**, **CDE** meet (**O**) at **P**, **Q**, **R** respectively.

(1) **PD**, **QE**, **RF** are concurrent (let the point of concurrence be **S**)
(2) **S** is on the line **OI**

Thank you for reading my post, in advance.

*Attachments:*




**Luis González**

#2 Apr 25, 2009, 11:53 pm • 1

**Lemma:** $H_a, H_b, H_c$ are the feet of the altitudes of $\triangle ABC$ and $P$ is any point on its plane. Then the circles $\odot(PAH_a)$, $\odot(PBH_b)$ and $\odot(PCH_c)$ meet again on $PH$.

Inversion WRT $(I)$ takes $\odot(AFE), \odot(CED), \odot(BDF)$ into the lines $FE, ED, DF$ and the circumcircle $(O)$ of $\triangle ABC$ into the 9-point circle $(O')$ of $\triangle DEF \Longrightarrow P, Q, R$ are taken into the altitude feet $X, Y, Z$ of $\triangle DEF$. Lines $PD, FR, QE$ are taken into the circles $\odot(DIX), \odot(EIY)$ and $\odot(FIZ)$. From the previous lemma, these latter circles meet on the Euler line $IO$ of $\triangle DEF$. Therefore, their inverse lines $PD$,$FR$ and $QE$ under the referred inversion concur on $IO$ as well.

*This post has been edited 1 time. Last edited by Luis González, Apr 25, 2009, 11:56 pm*

**plane geometry**

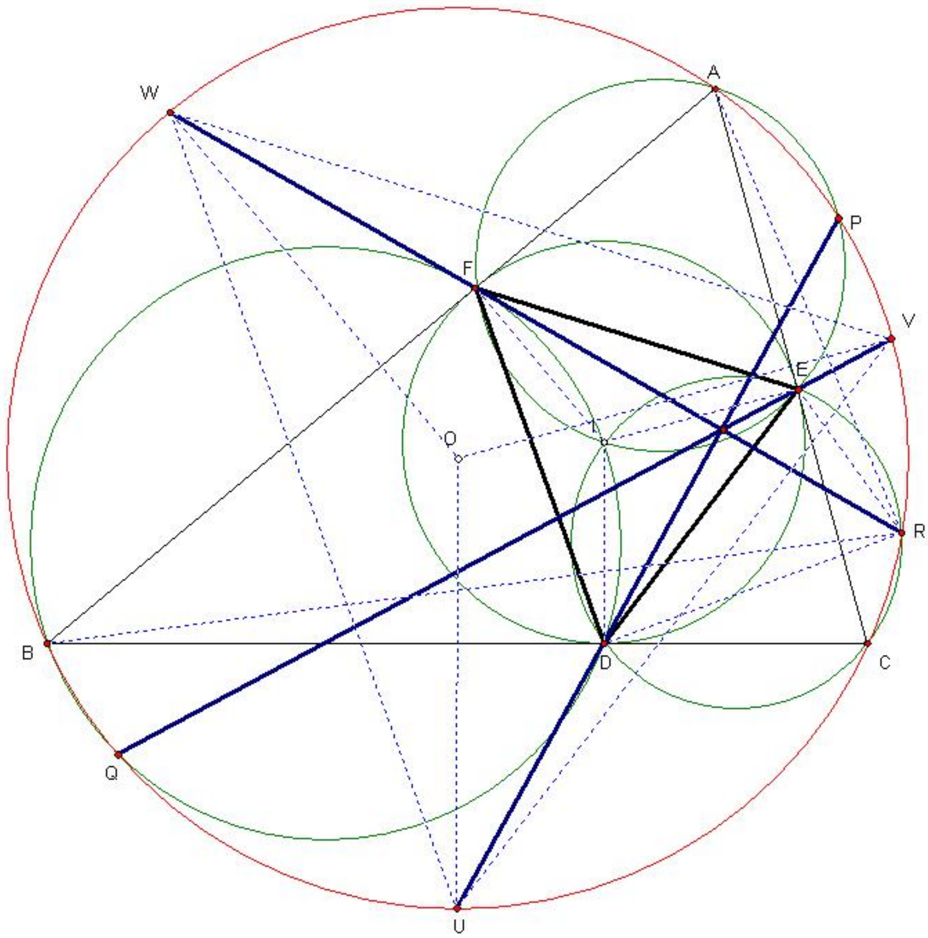
#3 Apr 25, 2009, 11:55 pm

1.triangle ARE and BRD are similar so RF bisects angleARB

2.denote U,V,W midpoints o fthe minor arc BC,CA,AB

then we have angle DEF and UVW are similar with corresponding sides parallel and the ratio is ID/OU=r/R
S lies on OI

*Attachments:*




**mathVNpro**

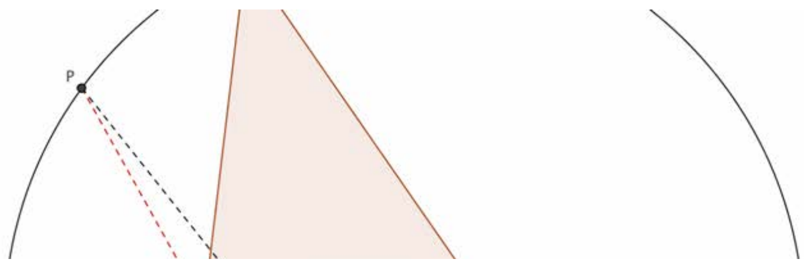
*#4 Apr 26, 2009, 10:26 pm*

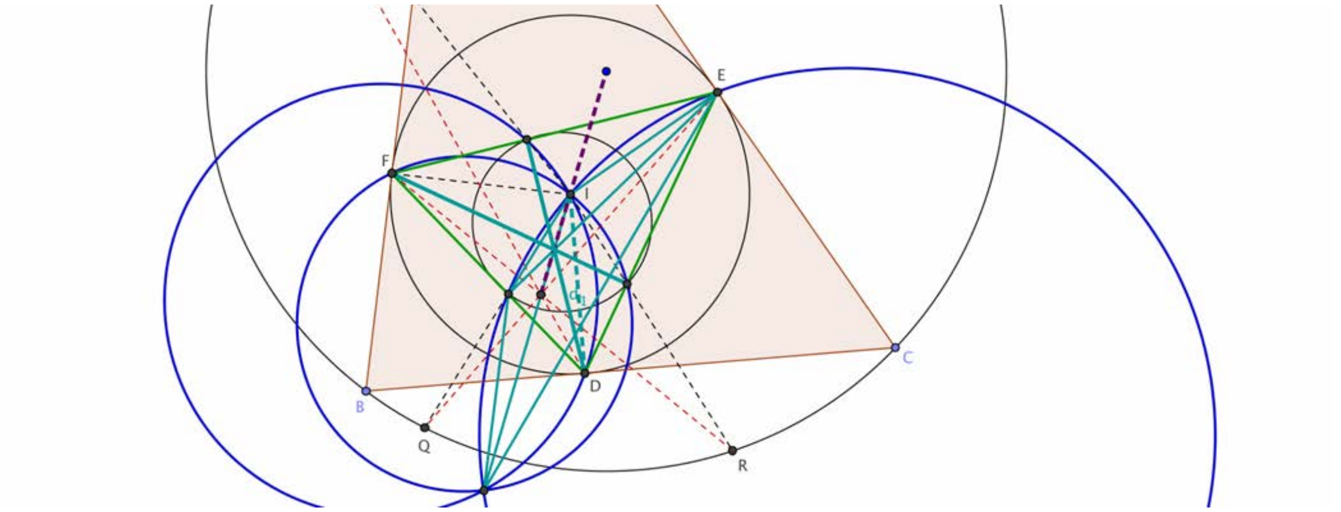
### Here is my solution (Using inversion )

Consider the inversion through pole $I$, power $r^2$, where $r$ is the radius of the incircle wrt $\triangle ABC$. Let $(C_1)$, $(C_2)$, $(C_3)$ be the circumcircles of $\triangle AEF$, $\triangle BFD$, $\triangle CDE$, respectively. It is obviously that $I(I,r): A \mapsto A_0$, $B \mapsto B_0$, $C \mapsto C_0$, where $A_0, B_0, C_0$ are the midpoints of $EF, DF, DE$, respectively, $D \mapsto D$, $E \mapsto E$, $F \mapsto F$. Hence, $I(I,r): (C_1) \mapsto EF$, $(C_2) \mapsto FD$, $(C_3) \mapsto ED$, $(O) \mapsto (A_0B_0C_0)$. Since, $P$ is the intersection of $(C_1)$ and $(O)$, therefore if $I(I,r): P \mapsto P'$, therefore, $P'$ is the intersection of $(A_0B_0C_0)$ and $EF$, arguing the same for $Q$ and $R$. Now, it is easy to notice that $(A_0B_0C_0)$ is the Euler circle wrt $\triangle DEF$, therefore, $DP', EQ', CR'$ are concurent, further, they are also the altitudes of $\triangle DEF$. Since the fact that $E, F, Q', R'$ are concyclic, hence, $\underline{E, F, Q, R \text{ are also concyclic}}$ (easy to notice this through the $I(I,r)$). Now, let $X$ be the intersection of $EF$ and $RQ$, hence, $\overline{XE}.\overline{XF} = \overline{XQ}.\overline{XR}$, which implies that $\mathcal{P}(X,(I)) = \mathcal{P}(X,(O))$. With the same arguement, we also get that $(P, F, D, R)$ and $(E, D, Q, P)$ are sets of concyclic points. Let $Y$, $Z$ define the same as $X$, respectively, wrt $(P, F, D, R)$ and $(E, D, Q, P)$. It is now easy to see that $X, Y, Z$, respectively, have the same power wrt $(I)$ and $(O)$. Therefore, $X, Y, Z$ are collinear. Hence, according to **Desgaurte theorem**, we get that $PD, QE, RF$ are concurrent.
$S$ is the concurrent point of these lines. It is easy to notice that $I(I,r): H_i \mapsto S$, where $H_i$ is the orthocenter wrt $\triangle DEF$. Therefore, $I, S, H_i$ are collinear. We need to prove that $I, O, H_i$ are collinear. But this is obviously true, you can see at this following link: **http://www.mathlinks.ro/viewtopic.php?t=152601**. Therefore, we will get the result of the whole problem
Which ends our proof

**@: Sorry if my solution is the same to any posts**

*Attachments:*






*This post has been edited 1 time. Last edited by mathVNpro, Apr 27, 2009, 5:38 pm*

**April**

*#5 Apr 27, 2009, 5:27 pm*

Posted before: http://www.mathlinks.ro/viewtopic.php?t=159997

**Mashimaru**

*#6 Apr 27, 2009, 5:30 pm*

First I will restate the problem:

**Problem:**
Let $(O)$ and $(I)$ in that order be the circumcenter and incenter of $\triangle ABC$. $(I)$ tangents to $BC, CA, AB$ at $A_0, B_0, C_0$. $(AB_0C_0), (BC_0A_0), (CA_0B_0)$ intersects $(O)$ again at $A_1, B_1, C_1$ respectively. Prove that $A_0A_1, B_0B_1, C_0C_1$ concur at a point on the Euler's line of $\triangle A_0B_0C_0$.

**Solution:** (inversion combined with angle chasing)
Denote by $A_3$ and $A_2$ the intersection of $IA$ and $IA_1$ with $B_0C_0$, we have $\widehat{AA_1A_2} = \widehat{AA_3A_2} = 90^o$ so $A, A_1, A_2, A_3$ are concyclic, which implies that $\overline{IA}.\overline{IA_3} = \overline{IA_1}.\overline{IA_2} = r^2$, where $r$ is the in-radius of $\triangle ABC$. Moreover: $I$ is the midpoint of arc $B_0C_0$ of circle $(AB_0C_0)$ so $A_1A_2$ is the angle bisector of $\widehat{B_0A_1C_0}$, but $\triangle A_1C_0B_0 \sim \triangle A_1BC$ so $A_1A_0$ is also the angle bisector of $\widehat{BA_1C_1}$. By this, $\widehat{A_1A_2C_0} = \widehat{A_1A_0B}$, which implies that $A_2$ is the foot of the altitude from $A_0$ of $\triangle A_0B_0C_0$.
Similarly, the images of $B_1, C_1$ through the inversion with center $I$ and power $r^2$ are the feet of altitudes of $\triangle A_0B_0C_0$.
But in $\triangle A_0B_0C_0$, three circles $(IA_0A_2), (IB_0B_2), (IC_0C_2)$ have a common point on its Euler's line, and so do $A_0A_1, B_0B_1, C_0C_1$, which ends the proof.

Quick Reply

# High School Olympiads

## Locus,locus,...no more:D

Reply

**mathVNpro**

#1 Apr 24, 2009, 9:32 pm

Let $ABC$ be an $A$-isosceles triangle. Let $M$ be a moving point on $[BC]$. The line through $M$ parallel to $AB, AC$, respectively, intersects $AC, AB$ at the point $X, Y$. Let $O_a$ be the circumcenter of triangle $AXY$. Prove that $O_a$ moves on the fixed line.

**Mashimaru**

#2 Apr 24, 2009, 11:17 pm

This is a very strange problem. I have found out the solution but it is now $12:15\text{am}$ so my mom won't allow me to stay up and post the solution (something from Fermat )

**Hint.** Let $BH_b, CH_c$ be the altitudes of $\triangle ABC$. Perpendicular bisectors of $AB, AC$ intersects $CH_c, BH_b$ at $K_c, K_b$. The locus of $O_a$ is segment $K_bK_c$.

**Comment.** After found, the locus $O_a$ is not very hard to prove but it is really a strange way to find out the locus of it.

**Luis González**

#3 Apr 24, 2009, 11:25 pm • 2

$AXMY$ is a parallelogram with $AX+AY=AB=k=\text{const}$. Let $M, N$ be the midpoints of segments $AB, AC$. Since $AM=AN=\frac{1}{2}k$, it follows that $XM=YN=\frac{1}{2}|AX-AY|$. If $O$ is circumcenter of $\triangle ABC$, then $O$ is the antipode of $A$ in the circle $\odot(AMN)$ and it's the center of the rotation taking the oriented segments $MX$ and $NY$ into each other $\Longrightarrow$ $O \equiv \odot(AXY) \cap \odot(AMN)$. Hence, locus of the circumcenter $O_a$ of $\triangle AXY$ is the perpendicular bisector of $\overline{AO}$.

**yetti**

#4 Apr 24, 2009, 11:43 pm

$M_0, X_0, Y_0$ are midpoints of $BC, AC, AB$. $M' \in BC$ is also arbitrary and parallels to $AB, AC$ through $M'$ cut $AC, AB$ at $X', Y'$. $\mathcal{P}_0, \mathcal{P}, \mathcal{P}'$ are circumcircles of $\triangle AX_0Y_0, \triangle AXY, \triangle AX'Y'$. Powers of $X_0, Y_0$ to $\mathcal{P}, \mathcal{P}'$ are in the same ratio:

$$\frac{p(X_0,\mathcal{P})}{p(X_0,\mathcal{P}')}=\frac{\overline{X_0X}\cdot\overline{X_0A}}{\overline{X_0X'}\cdot\overline{X_0A}}=\frac{\overline{X_0X}}{\overline{X_0X'}}=\frac{\overline{M_0M}}{\overline{M_0M'}}=\frac{\overline{Y_0Y}}{\overline{Y_0Y'}}=\frac{\overline{Y_0Y}\cdot\overline{Y_0A}}{\overline{Y_0Y'}\cdot\overline{Y_0A}}=\frac{p(Y_0,\mathcal{P})}{p(Y_0,\mathcal{P}')}$$

$\Longrightarrow$ circles $\mathcal{P}_0, \mathcal{P}, \mathcal{P}'$ are coaxal and their centers collinear.

*This post has been edited 2 times. Last edited by Luis González, Aug 11, 2015, 12:46 am*

**Luis González**

#7 Apr 25, 2009, 5:44 am

I have another approach for yetti's generalization.

**Proposition:** $D$ is a variable point on the side $\overline{BC}$ of $\triangle ABC$. Parallels through $D$ to $AC$ and $AB$ meet $AB$ and $AC$ at $X, Y$. Then, locus of the circumcenter of $\triangle AXY$ is a line perpendicular to the A-symmedian of $\triangle ABC$.

Let $M, N$ be the midpoints of $AB$ and $AC$. It's easy to prove that $\frac{XM}{NY}=\frac{c}{b} \Longrightarrow \odot(AXY)$ and $\odot(AMN)$ meet in the center $P$ of the spiral similarity that takes the oriented segments $\overline{XM}$ and $\overline{NY}$ into each other $\Longrightarrow \frac{PM}{PN}=\frac{c}{b} \Longrightarrow P$ is fixed on the arc $MN$ of $\odot(AMN)$ and lies on the A-Apollonius circle of $\triangle AMN$. Now, the conclusion follows.

**mathVNpro**

#8 Apr 25, 2009, 12:30 pm

nice work every body

Quick Reply

# High School Olympiads

## True or not

Reply

**greentreeroad**
#1 Apr 24, 2009, 8:44 pm

I thought of a statement but can't determine whether it is true.

Let ABC be a scalane triangle. Let D, E, F be points that its incircle touches BC, AC, AB respectively. Let AD intersect the incircle again at P. Are the three lines, BC, EF and the tangent line to the incircle at P, concurrent?

**Luis González**
#2 Apr 24, 2009, 8:50 pm

It is true and well-known indeed.

$(I)$ is the incircle of $\triangle ABC$. Line $BC$ is the polar of $D$ with respect to $(I)$. Line $EF$ is the polar of $A$ with respect to $(I)$ and $P$ is the pole of the tangent $\tau$ to $(I)$ through $P$. Since $A, P, D$ are collinear $\Longrightarrow EF, BC$ and $\tau$ concur.

**livetolove212**
#3 May 27, 2009, 9:56 am

> greentreeroad wrote:
> I thought of a statement but can't determine whether it is true.
>
> Let ABC be a scalane triangle. Let D, E, F be points that its incircle touches BC, AC, AB respectively. Let AD intersect the incircle again at P. Are the three lines, BC, EF and the tangent line to the incircle at P, concurrent?

$(FEPD) = -1$ then BC,EF and the tangent line to the incircle at P are concurrent.

**sunken rock**
#4 May 27, 2009, 3:08 pm

PFDE is a harmonic quad (the tangents at E and F to its circumcircle concur at A - on PD), hence the tangents at D and P concur on EF.

Best regards,
sunken rock

Quick Reply

# High School Olympiads

# Find the Ratio in triangle

Reply

**Source**: Please use triangle proportion

**triplebig**

#1 Apr 21, 2009, 7:09 am

In a triangle $ABC$ , let $\overline{BM}$ be the median, $\overline{BD}$ the internal bisector and $\overline{BN}$ the median simetric over the bisector.

Show that $\frac{AN}{CN} = \frac{c^2}{a^2}$

*This post has been edited 1 time. Last edited by triplebig, Apr 21, 2009, 4:49 pm*

**sunken rock**

#2 Apr 21, 2009, 10:32 am

There is a typo... the relation should be AN/CN = (c/a)^2, classic; use Steiner to prove, trivial.

Best regards,
sunken rock

**triplebig**

#3 Apr 21, 2009, 4:48 pm

Except I asked to use triangle proportion, so I don't think it's that trivial.

Thank you for identifying the typo, I fixed it.

**triplebig**

#4 Apr 24, 2009, 8:29 am

Does anybody know how to solve it this way?

**triplebig**

#5 Apr 24, 2009, 8:29 am

Does anybody know how to solve it this way?

**Luis González**

#6 Apr 24, 2009, 7:23 pm

Let $P$ be the second intersection of $AM$ with the circumcircle of $\triangle ABC.$ We have $\triangle ACN \sim \triangle ABP$ and $\triangle ABN \sim \triangle ACP.$ Therefore

$$\frac{CN}{PB} = \frac{CA}{AP} , \frac{PC}{BN} = \frac{AP}{AB} \Longrightarrow \frac{CN}{BN} = \frac{CA \cdot PB}{AB \cdot PC}$$

Since P lies in the reflection of the A-Apollonius circle about the perpendicular bisector of BC, we have then

$$\frac{PB}{PC} = \frac{CA}{AB} \Longrightarrow \frac{BN}{CN} = \frac{AB^2}{CA^2}.$$

**triplebig**

#7 Apr 24, 2009, 7:54 pm

Thank you once again for your solution though, I have a doubt, I would appreciate it if you could clarify how can I show that $P$ belongs to the apolonious circumference?

Quick Reply

# High School Olympiads

## Circumcenter

Reply

Source: In connection with http://www.mathlinks.ro/viewtopic.php?t=244262

**sunken rock**
#1 Apr 22, 2009, 1:12 am

Let M be a variable point on the side BC of the equilateral triangle ABC, I and J the incenters of the triangles ABM and AMC respectively, while K being the circumcenter of the triangle AIJ. Then MK and BC are perpendicular.

Best regards,
sunken rock

**Luis González**
#2 Apr 23, 2009, 9:12 am

Since $\angle AJM = \angle AIM = 120^\circ$ and $\triangle KIJ$ is equilateral, it follows that the pair of lines $JK, IK$ are the isogonals of $JA, IA$ with respect to $\triangle MIJ$, since they form equal and oppositely directed angles WRT $IJ, MJ$ and $IJ, MI$, respectively $\Longrightarrow A, K$ are isogonal conjugates in $\triangle MIJ \Longrightarrow MK, MA$ are isogonal rays $\Longrightarrow \angle IMK = \angle AMJ$. But $\angle AMJ = \angle JMC$ (MJ is bisector of <AMC) and $\angle IMK + \angle KMJ = 90^\circ \Longrightarrow \angle KMJ + \angle JMC = 90^\circ \Longrightarrow KM \perp BC$, as desired.

**sunken rock**
#3 Apr 23, 2009, 11:09 am

Well, Luis, I got an easier way to solve this: if B' and C' are circumcenters of triangles BMI and CJM respectively, then B' is the midpoint of arc BM of the circle (ABMJ) i.e. A, I and B' are collinear, C' the midpoint of arc CM of (AIMC), thence MK, B'J and C'I are concurrent and make 60 degs angles among them, that is, they are perpendicular on BC, CA and AB respectively.

Note: I hope the same, this to assist in solving that locus but, I do not see how, as yet!

Best regards,
sunken rock

**nsato**
#4 Apr 23, 2009, 10:50 pm

sunken rock, why are $MK$, $B'J$, and $C'I$ concurrent?

Edit: I see it now. Triangles $IJK$, $IB'M$, and $JC'N$ are all equilateral, so $MK$, $B'J$ and $C'I$ concur at the Fermat point of triangle $MIJ$.

Quick Reply

# High School Olympiads

## An old problem, but really nice:D

Reply

**mathVNpro**

#1 Apr 22, 2009, 10:54 am

Let $ABC$ be an acute triangle.Let $A_1, B_1, C_1$ be the projections of $A, B, C$ onto $BC, CA, AB$, respectively. $B_1C_1$ intersects $AA_1$ by the point $H_a$. Denote $K$ the midpoint of $AH$, where $H$ is the orthocenter of triangle $ABC$. Prove that: $H_a$ is the orthocenter of triangle $KBC$.

**mathVNpro**

#2 Apr 22, 2009, 11:41 am

It's OK if $K$ or $H_a$ is the orthocenter of triangle $H_aBC$, or $KBC$, respectively!!. I even can state that $B$ is the orthocenter of triangle $KH_aC$, they're all OK, my dear friend. . Anyway, very nice solution, after lunch, I will post my solution using harmonic bundle and Pascal+Desgaurter theorem.

**Luis González**

#3 Apr 22, 2009, 11:45 am

$\mathcal{K}$ is the circumcircle of the cyclic quadrangle $AC_1HB_1$. Then $BH_a$ is the polar of $C$ WRT $\mathcal{K}$ and $CH_a$ is the polar of $B$ WRT $\mathcal{K}$. Since the polar is perpendicular to the line connecting the center of the circle and the pole, it follows that the center $K$ of $\mathcal{K}$ is the orthocenter of $\triangle BCH_a \Longrightarrow H_a$ is the orthocenter of $\triangle KBC$.

**mathVNpro**

#4 Apr 22, 2009, 11:54 am

From $K$, let $xKy \perp AC, Kz \perp CC_1$. Let $X$ be the intersection of $Kz$ with $BB_1$, then it is easy to notice that $X$ is the midpoint of $HB$. Therefore, $(Kx, KX, KH, KB) = -1$. It is a very well- known result that $(A, H, H_a, A_1) = -1$. Therefore, $(CA, CH, CH_a, CA_1) = -1$. Hence, $(Kx, KX, KH, KB) = (CA, CH, CH_a, CA_1) = -1$, but $Kx \perp CA, KX \perp CH, CA_1 \perp KH$. Therefore, $CH_a \perp KB$, which leads to the fact that $H_a$ is the orthocenter of triangle $KBC$.
Our proof is completed.

**Mashimaru**

#5 Apr 23, 2009, 2:58 pm

There is a shorter solution with cross ratio:

Let $H_a$ be the orthocenter of $\triangle KBC$ then $H_a$ is a unique point lies on $AA_1$. We are to prove that $H_a, B_1, C_1$ are collinear.
Indeed:
$(BC_1, BH_a, BB_1, BC) = (AH_aHA_1) = (CC_1, CH_a, CB_1, CB)$, implies that $H_a, B_1, C_1$ are collinear and we are done.

**mathVNpro**

#6 Apr 23, 2009, 3:46 pm

I want to introduce you another solution of this problem (A little longer) :

**Solution**:
Let $(O)$ be the circumcircle of triangle $ABC$. Denote $A_2$ the intersection of $AA_1$ with $(O)$. It is obviously that $\overline{A_1A}.\overline{A_1A_2} = \overline{A_1B}.\overline{A_1C} = \overline{A_1H}.\overline{A_1A}$. But, $(A, H, H_a, A_1) = -1$, hence, we get, $\overline{A_1H}.\overline{A_1A} = \overline{A_1H_a}.\overline{A_1K}$.
Therefore, $A_1B.A_1C = A_1H_a.A_1K$.
Hence, $\frac{A_1B}{A_1K} = \frac{A_1K}{A_1C}$, which implies, triangle $A_1BK$ is similar to triangle $A_1H_aC$, which implies the fact that $CH_a \perp KB$.
Our proof is completed

Quick Reply

# High School Olympiads

## A parallel to the Euler's line

Reply

**Source**: an interesting result

**jayme**

#1 Apr 20, 2009, 5:39 pm

Dear Mathlinkers,
ABC a triangle, I the incenter of ABC, X, Y, Z the symetric of I wrt BC, CA, AB.
Prove
(1) AX, BY, CZ are concurrent
(2) the line passing through this point of concur and I is parallel to the Euler's line of ABC.
Sincerely
Jean-Louis

**mihai miculita**

#2 Apr 20, 2009, 7:17 pm

**1).**Using barycentric coordonates and denote by $T_a$, the fet of the perpendicular from I upon the side BC, then: $I(\frac{a}{a+b+c};\frac{b}{a+b+c};\frac{c}{a+b+c});T_a(0;\frac{a+b-c}{2a};\frac{a+c-b}{2a})$.

**2).** The point $T_a$ is midpoint of $[IX]\Rightarrow X(-\frac{a}{a+b+c};\frac{a^2+ab+b^2-c^2}{a(a+b+c)};\frac{a^2+ac+c^2-b^2}{a(a+b+c)})$.

Similarly: $Y(\frac{b^2+ab+a^2-c^2}{b(a+b+c)};-\frac{b}{a+b+c};\frac{b^2+bc+c^2-a^2}{a(a+b+c)})$ and $Z(\frac{c^2+ac+a^2-b^2}{c(a+b+c)};\frac{c^2+bc+b^2-a^2}{c(a+b+c)};-\frac{c}{a+b+c})$.

**3).**The equations of lines AX, BY, and CZ are, respectively:
$\frac{y}{a^2+ab+b^2-c^2}=\frac{z}{a^2+ac+c^2-b^2}$; $\frac{x}{b^2+ab+a^2-c^2}=\frac{z}{b^2+bc+c^2-a^2}$;
and $\frac{x}{c^2+ac+a^2-b^2}=\frac{y}{c^2+bc+b^2-a^2}$.

**4).**The three lines are concurrent in point $P(\frac{1}{b^2+bc+c^2-a^2};\frac{1}{a^2+ac+c^2-b^2};\frac{1}{a^2+ab+b^2-c^2})$.

**5).**The equation of Euler line is:

$$\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ a^2(b^2+c^2-a^2) & b^2(a^2+c^2-b^2) & c^2(a^2+b^2-c^2) \end{vmatrix}=0$$

and line PI have equation:

$$\begin{vmatrix} x & y & z \\ a & b & c \\ \frac{1}{b^2+bc+c^2-a^2} & \frac{1}{a^2+ac+c^2-b^2} & \frac{1}{a^2+ab+b^2-c^2} \end{vmatrix}=0.$$

**6).**To show that the two are parallel straight is enough to show that they have the same point at infinity.

*This post has been edited 4 times. Last edited by mihai miculita, Apr 20, 2009, 8:13 pm*

**Fang-jh**

#3 Apr 20, 2009, 7:21 pm

see http://perso.orange.fr/jl.ayme Volume 2 (2008) Le point de Gray et l'alignement Gra-I-X(500) and La droite de Gray,

**lym**

#4 Apr 20, 2009, 7:30 pm

Let $X,Y,Z$ be the symetric of P wrt $BC,CA,AB$□if $AX,BY,CZ$ are concurrent at $R$□then $PR||OH$ .

It has been totally solved by chinese famous expert zhong-hao Ye and his student with beaatiful primary Geometry□but I haven't seen their method□because I will research with myself when ihave time□**Fang-jh** knows that□you can ask him.

**jayme**

*#5 Apr 20, 2009, 7:51 pm*

Dear Fang-jh, lym and Mathlinkers,
what is the proof mentioned above by lym in order to renew my proof...
Sincerely
Jean-Louis

**lym**

*#6 Apr 21, 2009, 6:57 am*

Dear jayme and mathlinkers
If I'm not wrong□I think this problem what is $PR||OH$ was First discovered by **David** in November 2007 .
Then **fang-jh** put this problem to mathlinks□aboat the $P = I$□I had given a prove a half year ago□but it's not good .
You can see it on this website□(If you don't know chinese□you can use Google translation Tools)
http://www.aoshoo.com/bbs1/dispbbs.asp?boardid=43&id=10816&page=0&star=1

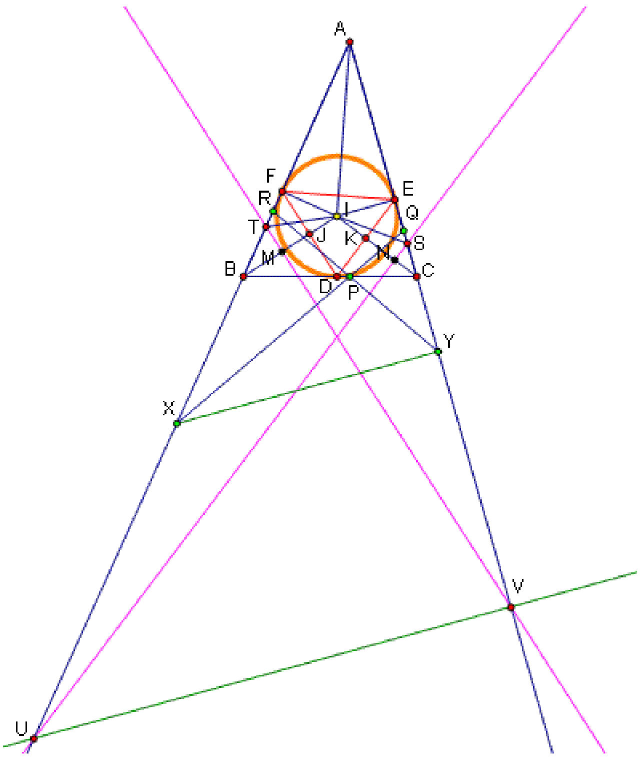
**lym**

*#7 Apr 22, 2009, 12:43 am*

I have found another way to prove this problem□we just should to prove $UV||XY$ .
Let $\square B > \square A, \square C > \square A, D, E, F$ be the Cut-off points of $\square I$ with $\Delta ABC$ . $P, Q, R$ are the vertical foot of $\Delta ABC$ .
$IB, IC$ intersect $DF, DE$ at $J, K$, and $IJ = JM, IK = KN, VM \square IB, UN \square IC, QP, RP$ intersect $AB, AC$ at $X, Y$ .
Let $\square C - \square A = 2m, \square B - \square A = 2n$ .

$(1) : AX/AY = sin\square AYP/sin\square AXP = sin2m/sin2n$ .
$(2) : \square ATI = \square BTM, \square AIT = 90 + n, \square AVT = m$, Then
$AV = AV/AT * AT/AI = sin(90 + n)/sinm = cosn/sinm$ .
Empathy□$AU = cosm/sinn$, so: $AU/AV = sin2m/sin2n$, so: $AU/AV = AX/AY$□Hence $UV||XY$ .

*Attachments:*




**Luis González**

*#8 Apr 22, 2009, 9:43 am*

**Kariya's theorem:** Incircle $(I, r)$ of $\triangle ABC$ touches $BC, CA, AB$ at $U, V, W$. Circle $(I, kr)$ (k>1) cuts the rays $IU, IV, IW$ at $X, Y, Z$ respectively. Then $AU, BV, CW$ concur.

**Proof:** We use barycentric coordinates with respect to $\triangle ABC$.

$I \equiv (a:b:c)$ , $U \equiv (0: ab+S_C : ac+S_B)$

$V \equiv (ba+S_C : 0 : bc+S_A)$ $W \equiv (ab+S_C : bc+S_A : 0)$

Using barycentric equations of the homothecy $\gamma(I,k)$ we obtain the coordinates of $X, Y, Z$ as

$X \equiv (-a^2(k+1) : ab+kS_C : ac+kS_B)$

$Y \equiv (ab+kS_C : -b^2(k-1) : bc+kS_A)$

$Z \equiv (ca+kS_B : bc+kS_A : -c^2(k-1))$

Therefore, $\triangle ABC$ and $\triangle XYZ$ are perspective through a point $J$

$$J \equiv \left(\frac{1}{bc+kS_A} : \frac{1}{ac+kS_B} : \frac{1}{ab+kS_C}\right)$$

In our case, $k=2$, hence the perspector is the point

$$X_{78} \equiv \left(\frac{1}{bc+2S_A} : \frac{1}{ac+2S_B} : \frac{1}{ab+2S_C}\right)$$

The line joining $I(a:b:c)$ and $X_{78}$ has equation

$$\begin{vmatrix} a & b & c \\ \frac{1}{bc+2S_A} & \frac{1}{ac+2S_B} & \frac{1}{ab+2S_C} \\ x & y & z \end{vmatrix} = 0$$

The infinite point of this line coincides with the infinite point of Euler line, namely

$X_{30} \equiv (S_AS_B + S_A + S_C - 2S_BS_C)$

**armpist**

*#9 Apr 26, 2009, 8:22 am*

The Gray point with the 3 ex-Gray points make up an orthocentric system.

Maybe with this fact there is no need to resort to tricks.

M.T.

**jayme**

*#10 Apr 26, 2009, 2:27 pm*

Dear Armpist and Mathlinkers,
yes, we have an orthocentric system, but how do you see the rest?
Nice idea with extraversion...
Sincerely
Jean-Louis

Quick Reply