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Unsolved Problems in Intuitive Mathematics

Volume I

Richard K. Guy

Unsolved Problems in Number Theory

Second Edition

With 18 figures



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Preface to the First Edition

To many laymen, mathematicians appear to be problem solvers, people who do "hard sums". Even inside the profession we classify ourselves as either theorists or problem solvers. Mathematics is kept alive, much more than by the activities of either class, by the appearance of a succession of unsolved problems, both from within mathematics itself and from the increasing number of disciplines where it is applied. Mathematics often owes more to those who ask questions than to those who answer them. The solution of a problem may stifle interest in the area around it. But "Fermat's Last Theorem", because it is not yet a theorem, has generated a great deal of "good" mathematics, whether goodness is judged by beauty, by depth or by applicability.

To pose good unsolved problems is a difficult art. The balance between triviality and hopeless unsolvability is delicate. There are many simply stated problems which experts tell us are unlikely to be solved in the next generation. But we have seen the Four Color Conjecture settled, even if we don't live long enough to learn the status of the Riemann and Goldbach hypotheses, of twin primes or Mersenne primes, or of odd perfect numbers. On the other hand, "unsolved" problems may not be unsolved at all, or may be much more tractable than was at first thought.

Among the many contributions made by Hungarian mathematician Erdős Pál, not least is the steady flow of well-posed problems. As if these were not incentive enough, he offers rewards for the first solution of many of them, at the same time giving his estimate of their difficulty. He has made many payments, from \$1.00 to \$1000.00.

One purpose of this book is to provide beginning researchers, and others who are more mature, but isolated from adequate mathematical stimulus, with a supply of easily understood, if not easily solved, problems which they can consider in varying depth, and by making occasional partial progress, gradually acquire the interest, confidence and persistence that are essential to successful research.

But the book has a much wider purpose. It is important for students and teachers of mathematics at all levels to realize that although they are not yet capable of research and may have no hopes or ambitions in that direction, there are plenty of unsolved problems that are well within their comprehension, some of which will be solved in their lifetime. Many amateurs have been attracted to the subject and many successful researchers first gained their confidence by examining problems in Euclidean geometry,

in number theory, and more recently in combinatorics and graph theory, where it is possible to understand questions and even to formulate them and obtain original results without a deep prior theoretical knowledge.

The idea for the book goes back some twenty years, when I was impressed by the circulation of lists of problems by the late Leo Moser and co-author Hallard Croft, and by the articles of Erdős. Croft agreed to let me help him amplify his collection into a book, and Erdős has repeatedly encouraged and prodded us. After some time, the Number Theory chapter swelled into a volume of its own, part of a series which will contain a volume on Geometry, Convexity and Analysis, written by Hallard T. Croft, and one on Combinatorics, Graphs and Games by the present writer.

References, sometimes extensive bibliographies, are collected at the end of each problem or article surveying a group of problems, to save the reader from turning pages. In order not to lose the advantage of having all references collected in one alphabetical list, we give an Index of Authors, from which particular papers can easily be located provided the author is not too prolific. Entries in this index and in the General Index and Glossary of Symbols are to problem numbers instead of page numbers.

Many people have looked at parts of drafts, corresponded and made helpful comments. Some of these were personal friends who are no longer with us: Harold Davenport, Hans Heilbronn, Louis Mordell, Leo Moser, Theodor Motzkin, Alfred Rényi and Paul Turán. Others are H. L. Abbott, J. W. S. Cassels, J. H. Conway, P. Erdős, Martin Gardner, R. L. Graham, H. Halberstam, D. H. and Emma Lehmer, A. M. Odlyzko, Carl Pomerance, A. Schinzel, J. L. Selfridge, N. J. A. Sloane, E. G. Straus, H. P. F. Swinnerton-Dyer and Hugh Williams. A grant from the National Research Council of Canada has facilitated contact with these and many others. The award of a Killam Resident Fellowship at the University of Calgary was especially helpful during the writing of a final draft. The technical typing was done by Karen McDermid, by Betty Teare and by Louise Guy, who also helped with the proof-reading. The staff of Springer-Verlag in New York has been courteous, competent and helpful.

In spite of all this help, many errors remain, for which I assume reluctant responsibility. In any case, if the book is to serve its purpose it will start becoming out of date from the moment it appears; it has been becoming out of date ever since its writing began. I would be glad to hear from readers. There must be many solutions and references and problems which I don't know about. I hope that people will avail themselves of this clearing house. A few good researchers thrive by rediscovering results for themselves, but many of us are disappointed when we find that our discoveries have been anticipated.

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Richard K. Guy

Preface to the Second Edition

Erdős recalls that Landau, at the International Congress in Cambridge in 1912, gave a talk about primes and mentioned four problems (see **A1**, **A5**, **C1** below) which were unattackable in the present state of science, and says that they still are. On the other hand, since the first edition of this book, some remarkable progress has been made. Fermat's last theorem (modulo some holes that are expected to be filled in), the Mordell conjecture, the infinitude of Carmichael numbers, and a host of other problems have been settled.

The book is perpetually out of date; not always the 1700 years of one statement in **D1** in the first edition, but at least a few months between yesterday's entries and your reading of the first copies off the press. To ease comparison with the first edition, the numbering of the sections is still the same. Problems which have been largely or completely answered are **B47**, **D2**, **D6**, **D8**, **D16**, **D26**, **D27**, **D28**, **E15**, **F15**, **F17** & **F28**. Related open questions have been appended in some cases, but in others they have become exercises, rather than problems.

Two of the author's many idiosyncrasies are mentioned here: the use of the ampersand (&) to denote joint work and remove any possible ambiguity from phrases such as '... follows from the work of Gauß and Erdős & Guy'; and the use of the notation

$\stackrel{?}{\dots}$

borrowed from the Hungarians, for a conjectural or hypothetical statement. This could have alleviated some anguish had it been used by the well intentioned but not very well advised author of an introductory calculus text. A student was having difficulty in finding the derivative of a product. Frustrated myself, I asked to see the student's text. He had highlighted a displayed formula stating that the derivative of a product was the product of the derivatives, without noting that the context was 'Why is ... not the right answer?'

The threatened volume on *Unsolved Problems in Geometry* has appeared, and is already due for reprinting or for a second edition.

It will be clear from the text how many have accepted my invitation to use this as a clearing house and how indebted I am to correspondents. Extensive though it is, the following list is far from complete, but I should at least offer my thanks to Harvey Abbott, Arthur Baragar, Paul Bateman, T. G. Berry, Andrew Bremner, John Brillhart, R. H. Buchholz, Duncan Buell, Joe Buhler, Mitchell Dickerman, Hugh Edgar, Paul Erdős,

Steven Finch, Aviezri Fraenkel, David Gale, Sol Golomb, Ron Graham, Sid Graham, Andrew Granville, Heiko Harborth, Roger Heath-Brown, Martin Helm, Gerd Hofmeister, Wilfrid Keller, Arnfried Kemnitz, Jenifer Lagarias, Jean Lagrange, John Leech, Dick & Emma Lehmer, Hendrik Lenstra, Hugh Montgomery, Peter Montgomery, Shigeru Nakamura, Richard Nowakowski, Andrew Odlyzko, Richard Pinch, Carl Pomerance, Aaron Potler, Herman te Riele, Raphael Robinson, Øystein Rødseth, K. R. S. Sastry, Andrzej Schinzel, Reese Scott, John Selfridge, Ernst Selmer, Jeffery Shallit, Neil Sloane, Stephane Vandemergel, Benne de Weger, Hugh Williams, Jeff Young and Don Zagier. I particularly miss the impeccable proof-reading, the encyclopedic knowledge of the literature, and the clarity and ingenuity of the mathematics of John Leech.

Thanks also to Andy Guy for setting up the electronic framework which has made both the author's and the publisher's task that much easier. The Natural Sciences and Engineering Research Council of Canada continue to support this and many other of the author's projects.

Calgary 94-01-08

Richard K. Guy

Contents

Preface to the First Edition	v
Preface to the Second Edition	vii
Glossary of Symbols	xiii
Introduction	1
A. Prime Numbers	3
A1. Prime values of quadratic functions. 4	
A2. Primes connected with factorials. 7	
A3. Mersenne primes. Repunits. Fermat numbers. Primes of shape $k \cdot 2^n + 2$. 8	
A4. The prime number race. 13	
A5. Arithmetic progressions of primes. 15	
A6. Consecutive primes in A.P. 17	
A7. Cunningham chains. 18	
A8. Gaps between primes. Twin primes. 19	
A9. Patterns of primes. 23	
A10. Gilbreath's conjecture. 25	
A11. Increasing and decreasing gaps. 26	
A12. Pseudoprimes. Euler pseudoprimes. Strong pseudoprimes. 26	
A13. Carmichael numbers. 30	
A14. "Good" primes and the prime number graph. 32	
A15. Congruent products of consecutive numbers. 33	
A16. Gaussian primes. Eisenstein-Jacobi primes. 33	
A17. Formulas for primes. 36	
A18. The Erdős-Selfridge classification of primes. 41	
A19. Values of n making $n - 2^k$ prime. Odd numbers not of the form $\pm p^a \pm 2^b$. 42	
B. Divisibility	44
B1. Perfect numbers. 44	
B2. Almost perfect, quasi-perfect, pseudoperfect, harmonic, weird, multiperfect and hyperperfect numbers. 45	
B3. Unitary perfect numbers. 53	
B4. Amicable numbers. 55	
B5. Quasi-amicable or betrothed numbers. 59	
B6. Aliquot sequences. 60	
B7. Aliquot cycles or sociable numbers. 62	
B8. Unitary aliquot sequences. 63	
B9. Superperfect numbers. 65	
B10. Untouchable numbers. 66	
B11. Solutions of $m\sigma(m) = n\sigma(n)$. 67	
B12. Analogs with $d(n)$, $\sigma_k(n)$. 67	
B13. Solutions of $\sigma(n) = \sigma(n+1)$. 68	
B14. Some irrational series. 69	
B15. Solutions	

- of $\sigma(q) + \sigma(r) = \sigma(q+r)$. 69 **B16.** Powerful numbers. 70
B17. Exponential-perfect numbers. 73 **B18.** Solutions of $d(n) = d(n+1)$. 73 **B19.** $(m, n+1)$ and $(m+1, n)$ with same set of prime factors. 75 **B20.** Cullen numbers. 77 **B21.** $k \cdot 2^n + 1$ composite for all n . 77 **B22.** Factorial n as the product of n large factors. 79
B23. Equal products of factorials. 79 **B24.** The largest set with no member dividing two others. 80 **B25.** Equal sums of geometric progressions with prime ratios. 81 **B26.** Densest set with no l pairwise coprime. 81 **B27.** The number of prime factors of $n+k$ which don't divide $n+i$, $0 \leq i < k$. 82 **B28.** Consecutive numbers with distinct prime factors. 83 **B29.** Is x determined by the prime divisors of $x+1$, $x+2, \dots, x+k$? 83 **B30.** A small set whose product is square. 84
B31. Binomial coefficients. 84 **B32.** Grimm's conjecture. 85
B33. Largest divisor of a binomial coefficient. 87 **B34.** If there's an i such that $n-i$ divides $\binom{n}{i}$. 89 **B35.** Products of consecutive numbers with the same prime factors. 89 **B36.** Euler's totient function. 90
B37. Does $\phi(n)$ properly divide $n-1$? 92 **B38.** Solutions of $\phi(m) = \sigma(n)$. 93 **B39.** Carmichael's conjecture. 94 **B40.** Gaps between totatives. 95 **B41.** Iterations of ϕ and σ . 96 **B42.** Behavior of $\phi(\sigma(n))$ and $\sigma(\phi(n))$. 99 **B43.** Alternating sums of factorials. 99
B44. Sums of factorials. 100 **B45.** Euler numbers. 101 **B46.** The largest prime factor of n . 101 **B47.** When does $2^a - 2^b$ divide $n^a - n^b$? 102 **B48.** Products taken over primes. 102 **B49.** Smith numbers. 103

C. Additive Number Theory 105
C1. Goldbach's conjecture. 105 **C2.** Sums of consecutive primes. 107
C3. Lucky numbers. 108 **C4.** Ulam numbers. 109 **C5.** Sums determining members of a set. 110 **C6.** Addition chains. Brauer chains. Hansen chains. 111 **C7.** The money-changing problem. 113 **C8.** Sets with distinct sums of subsets. 114 **C9.** Packing sums of pairs. 115
C10. Modular difference sets and error correcting codes. 118
C11. Three-subsets with distinct sums. 121 **C12.** The postage stamp problem. 123 **C13.** The corresponding modular covering problem. Harmonious labelling of graphs. 127 **C14.** Maximal sum-free sets. 128
C15. Maximal zero-sum-free sets. 129 **C16.** Nonaveraging sets. Nondividing sets. 131 **C17.** The minimum overlap problem. 132
C18. The n queens problem. 133 **C19.** Is a weakly independent sequence the finite union of strongly independent ones? 135 **C20.** Sums of squares. 136

D. Diophantine Equations 139
D1. Sums of like powers. Euler's conjecture. 139 **D2.** The Fermat problem. 144 **D3.** Figurate numbers. 146 **D4.** Sums of l kth powers. 150 **D5.** Sum of four cubes. 151 **D6.** An elementary solution

- of $x^2 = 2y^4 - 1$. 152 **D7.** Sum of consecutive powers made a power. 153 **D8.** A pyramidal diophantine equation. 154 **D9.** Difference of two powers. 155 **D10.** Exponential diophantine equations. 157
D11. Egyptian fractions. 158 **D12.** Markoff numbers. 166 **D13.** The equation $x^xy^y = z^z$. 168 **D14.** $a_i + b_j$ made squares. 169
D15. Numbers whose sums in pairs make squares. 170 **D16.** Triples with the same sum and same product. 171 **D17.** Product of blocks of consecutive integers not a power. 172 **D18.** Is there a perfect cuboid? Four squares whose sums in pairs are square. Four squares whose differences are square. 173 **D19.** Rational distances from the corners of a square. 181 **D20.** Six general points at rational distances. 185
D21. Triangles with integer sides, medians and area. 188
D22. Simplexes with rational contents. 190 **D23.** Some quartic equations. 191 **D24.** Sum equals product. 193 **D25.** Equations involving factorial n . 193 **D26.** Fibonacci numbers of various shapes. 194 **D27.** Congruent numbers. 195 **D28.** A reciprocal diophantine equation. 197

E. Sequences of Integers 199
E1. A thin sequence with all numbers equal to a member plus a prime. 199 **E2.** Density of a sequence with l.c.m. of each pair less than x . 200 **E3.** Density of integers with two comparable divisors. 201
E4. Sequence with no member dividing the product of r others. 201
E5. Sequence with members divisible by at least one of a given set. 202
E6. Sequence with sums of pairs not members of a given sequence. 203
E7. A series and a sequence involving primes. 203 **E8.** Sequence with no sum of a pair a square. 203 **E9.** Partitioning the integers into classes with numerous sums of pairs. 204 **E10.** Theorem of van der Waerden. Szemerédi's theorem. Partitioning the integers into classes; at least one contains an A.P. 204 **E11.** Schur's problem. Partitioning integers into sum-free classes. 209 **E12.** The modular version of Schur's problem. 211
E13. Partitioning into strongly sum-free classes. 213 **E14.** Rado's generalizations of van der Waerden's and Schur's problems. 213 **E15.** A recursion of Göbel. 214 **E16.** Collatz's sequence. 215
E17. Permutation sequences. 218 **E18.** Mahler's Z-numbers. 219
E19. Are the integer parts of the powers of a fraction infinitely often prime? 220 **E20.** Davenport-Schinzel sequences. 220 **E21.** Thue sequences. 222 **E22.** Cycles and sequences containing all permutations as subsequences. 224 **E23.** Covering the integers with A.P.s. 224
E24. Irrationality sequences. 225 **E25.** Silverman's sequence. 225
E26. Epstein's Put-or-Take-a-Square game. 226 **E27.** Max and mex sequences. 227 **E28.** B₂-sequences. 228 **E29.** Sequence with sums and products all in one of two classes. 229 **E30.** MacMahon's prime numbers of measurement. 230 **E31.** Three sequences of Hofstadter. 231 **E32.** B₂-sequences formed by the greedy algorithm. 232

E33. Sequences containing no monotone A.P.s. 237	E34. Happy numbers. 234	E35. The Kimberling shuffle. 235	E36. Klarner-Rado sequences. 237	E37. Mousetrap. 237	E38. Odd sequences. 238
F. None of the Above					
240					
F1. Gauß's lattice point problem. 240	F2. Lattice points with distinct distances. 241	F3. Lattice points, no four on a circle. 241	F4. The no-three-in-line problem. 242	F5. Quadratic residues. Schur's conjecture. 244	F6. Patterns of quadratic residues. 245
F7. A cubic analog of a Pell equation. 248					
F8. Quadratic residues whose differences are quadratic residues. 248					
F9. Primitive roots 248					
F10. Residues of powers of two. 249					
F11. Distribution of residues of factorials. 250					
F12. How often are a number and its inverse of opposite parity? 251					
F13. Covering systems of congruences. 251					
F14. Exact covering systems. 253					
F15. A problem of R. L. Graham. 256					
F16. Products of small prime powers dividing n . 256					
F17. Series associated with the ζ -function. 257					
F18. Size of the set of sums and products of a set. 258					
F19. Partitions into distinct primes with maximum product. 258					
F20. Continued fractions. 259					
F21. All partial quotients one or two. 259					
F22. Algebraic numbers with unbounded partial quotients. 260					
F23. Small differences between powers of 2 and 3. 261					
F24. Squares with just two different decimal digits. 262					
F25. The persistence of a number. 262					
F26. Expressing numbers using just ones. 263					
F27. Mahler's generalization of Farey series. 263					
F28. A determinant of value one. 265					
F29. Two congruences, one of which is always solvable. 266					
F30. A polynomial whose sums of pairs of values are all distinct. 266					
F31. An unusual digital problem. 266					
Index of Authors Cited					
268					
General Index					
280					

Glossary of Symbols

A.P.	arithmetic progression, $a, a+d, \dots, a+kd, \dots$	A5, A6, E10, E33
$a_1 \equiv a_2 \pmod{b}$	a_1 congruent to a_2 , modulo b ; $a_1 - a_2$ divisible by b .	A3, A4, A12, A15, B2, B4, B7, ...
$A(x)$	number of members of a sequence not exceeding x ; e.g. number of amicable numbers not exceeding x	B4, E1, E2, E4
c	a positive constant (not always the same!)	A1, A3, A8, A12, B4, B11, ...
d_n	difference between consecutive primes; $p_{n+1} - p_n$	A8, A10, A11
$d(n)$	the number of (positive) divisors of n ; $\sigma_0(n)$	B, B2, B8, B12, B18, ...
$d n$	d divides n ; n is a multiple of d ; there is an integer q such that $dq = n$	B, B17, B32, B37, B44, C20, D2, E16
$d \nmid n$	d does not divide n	B, B2, B25, E14, E16, ...
e	base of natural logarithms; 2.718281828459045...	A8, B22, B39, D12, ...
E_n	Euler numbers; coefficients in series for $\sec x$	B45
$\exp\{\cdot\}$	exponential function	A12, A19, B4, B36, B39, ...
F_n	Fermat numbers; $2^{2^n} + 1$	A3, A12

$f(x) \sim g(x)$	$f(x)/g(x) \rightarrow 1$ as $x \rightarrow \infty$. ($f, g > 0$)	A1, A3, A8, B33, B41, C1, C17, D7, E2, E30, F26
$f(x) = o(g(x))$	$f(x)/g(x) \rightarrow 0$ as $x \rightarrow \infty$. ($g > 0$)	A1, A18, A19, B4, C6, C9, C11, C16, C20, D4, D11, E2, E14, F1
$f(x) = O(g(x))$	{ there is a c such that $ f(x) < cg(x)$ for all sufficiently large x .	A19, B37, C8, C9, C10, C12, C16, D4, D12, E4, E8, E20, E30, F1, F2, F16
$f(x) \ll g(x)$	{ there is a $c > 0$ such that there are arbitrarily large x with $ f(x) \geq cg(x)$ ($g(x) > 0$).	A4, B4, B18, B32, B40, C9, C14, D11, E28, F4
$f(x) = \Omega(g(x))$	{ there are c_1, c_2 such that $c_1g(x) \leq f(x) \leq c_2g(x)$ ($g(x) > 0$) for all sufficiently large x .	D12, E25
$f(x) \asymp g(x)$	{ there are c_1, c_2 such that $c_1g(x) \leq f(x) \leq c_2g(x)$ ($g(x) > 0$)	B18
$f(x) = \Theta(g(x))$	{ there are c_1, c_2 such that $c_1g(x) \leq f(x) \leq c_2g(x)$ ($g(x) > 0$) for all sufficiently large x .	E20
i	square root of -1 ; $i^2 = -1$	A16
$\ln x$	natural logarithm of x	A1, A2, A3, A5, A8, A12, ...
(m, n)	g.c.d. (greatest common divisor) of m and n ; h.c.f. (highest common factor) of m and n	A, B3, B4, B5 B11, D2
$[m, n]$	l.c.m. (least common multiple) of m and n . Also the block of consecutive integers, $m, m+1, \dots, n$	B35, E2, F14 B24, B26, B32, C12, C16
$m \perp n$	m, n coprime; $(m, n) = 1$; m prime to n .	A, A4, B3, B4, B5, B11, D2
M_n	Mersenne numbers; $2^n - 1$	A3, B11, B38

$n!$	factorial n ; $1 \times 2 \times 3 \times \dots \times n$	A2, B12, B14, B22 B23, B43, ...
$!n$	$0! + 1! + 2! + \dots + (n-1)!$	B44
$\binom{n}{k}$	n choose k ; the binomial coefficient $n!/k!(n-k)!$	B31, B33, C10, D3
$\binom{p}{q}$	Legendre (or Jacobi) symbol	see F5 (A1, A12, F7)
$p^a \parallel n$	p^a divides n , but p^{a+1} does not divide n	B, B8, B37, F16
p_n	the n th prime, $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \dots$	A2, A5, A14, A17 E30
$P(n)$	largest prime factor of n	B32, B46
\mathbb{Q}	the field of rational numbers	D2, F7
$r_k(n)$	least number of numbers not exceeding n , which must contain a k -term A.P.	see E10
$s(n)$	sum of aliquot parts (divisors of n other than n) of n ; $\sigma(n) - n$	B, B1, B2, B8, B10, ...
$s^k(n)$	k th iterate of $s(n)$	B, B6, B7
$s^*(n)$	sum of unitary aliquot parts of n	B8
$S \cup T$	union of sets S and T	E7
$W(k, l)$	van der Waerden number	see E10
$\lfloor x \rfloor$	floor of x ; greatest integer not greater than x .	A1, A5, C7, C12, C15, ...
$\lceil x \rceil$	ceiling of x ; least integer not less than x .	B24
\mathbb{Z}	the integers $\dots, -2, -1, 0, 1, 2, \dots$	F14
\mathbb{Z}_n	the ring of integers, $0, 1, 2, \dots, n-1$ (modulo n)	E8
γ	Euler's constant; 0.577215664901532...	A8

ϵ	arbitrarily small positive constant.	A8, A18, A19, B4, B11, ...
ζ_p	p -th root of unity.	D2
$\zeta(s)$	Riemann zeta-function; $\sum_{n=1}^{\infty} (1/n^s)$	D2
π	ratio of circumference of circle to diameter; 3.141592653589793...	F1, F17
$\pi(x)$	number of primes not exceeding x	A17, E4
$\pi(x; a, b)$	number of primes not exceeding x and congruent to a modulo b	A4
\prod	product	A1, A2, A3, A8 A15, ...
$\sigma(n)$	sum of divisors of n ; $\sigma_1(n)$	B, B2, B5, B8, B9, ...
$\sigma_k(n)$	sum of k th powers of divisors of n	B, B12, B13, B14
$\sigma^k(n)$	k th iterate of $\sigma(n)$	B9
$\sigma^*(n)$	sum of unitary divisors of n	B8
Σ	sum	A5, A8, A12, B2 B14, ...
$\phi(n)$	Euler's totient function; number of numbers not exceeding n and prime to n	B8, B11, B36, B38, B39, ...
$\phi^k(n)$	k th iterate of $\phi(n)$	B41
ω	complex cube root of 1 $\omega^3 = 1, \omega \neq 1,$ $\omega^2 + \omega + 1 = 0$	A16
$\omega(n)$	number of distinct prime factors of n	B2, B8, B37
$\Omega(n)$	number of prime factors n , counting repetitions	B8
$\mathcal{C} \dots ?$	conjectural or hypothetical statement	A1, A9, B37, C6 E10, E28, F2, F18

Introduction

Number theory has fascinated both the amateur and the professional for a longer time than any other branch of mathematics, so that much of it is now of considerable technical difficulty. However, there are more unsolved problems than ever before, and though many of these are unlikely to be solved in the next generation, this probably won't deter people from trying. They are so numerous that they have already filled more than one volume: the present book is just a personal sample.

Some good sources of problems in number theory were listed in the Introduction to the first edition, some of which are repeated here, along with more recent references.

- Paul Erdős, Problems and results in combinatorial number theory III, *Springer Lecture Notes in Math.*, **626**(1977) 43–72; MR 57 #12442.
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- P. Erdős & R. L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*, Monographies de l'Enseignement Math. No. 28, Geneva, 1980.
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- Victor Klee & Stan Wagon, *Old and New Unsolved Problems in Plane Geometry and Number Theory*, Math. Assoc. of Amer. Dolciani Math. Expositions, **11**(1991).
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 Robert D. Silverman, A perspective on computational number theory, in Computers and Mathematics, *Notices Amer. Math. Soc.*, **38**(1991) 562–568.
 S. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.

Throughout this volume, “number” means natural number, i.e.,
 $0, 1, 2, \dots$

and c is an absolute positive constant, not necessarily taking the same value at each appearance. We use Donald Knuth’s now familiar “floor” ($\lfloor \cdot \rfloor$) and “ceiling” ($\lceil \cdot \rceil$) symbols for “the greatest integer not greater than” and “the least integer not less than.” A less familiar symbol may be “ $m \perp n$ ” for “ m is prime to n ” or “ $\gcd(m, n) = 1$.”

The book is partitioned, somewhat arbitrarily at times, into six sections:

- A. Prime numbers
- B. Divisibility
- C. Additive number theory
- D. Diophantine equations
- E. Sequences of integers
- F. None of the above.

A. Prime Numbers

We can partition the positive integers into three classes:
the unit 1

the primes $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \dots$

the composite numbers $4, 6, 8, 9, 10, 12, 14, 15, 16, \dots$

A number greater than 1 is **prime** if its only positive divisors are 1 and itself; otherwise it’s **composite**. Primes have interested mathematicians at least since Euclid, who showed that there are infinitely many.

Denote the n -th prime by p_n , e.g. $p_1 = 2, p_2 = 3, p_{50} = 523$; and the number of primes not greater than x by $\pi(x)$, e.g., $\pi(2) = 1, \pi(3^{\frac{1}{2}}) = 2, \pi(1000) = 168, \pi(4 \cdot 10^{16}) = 1075292778753150$. The greatest common divisor (\gcd) of m and n is denoted by (m, n) , e.g., $(36, 66) = 6, (14, 15) = 1, (1001, 1078) = 77$. If $(m, n) = 1$, we say that m and n are **coprime** and write $m \perp n$; for example $182 \perp 165$.

Dirichlet’s theorem tells us that there are infinitely many primes in any **arithmetic progression**,

$$a, \quad a + b, \quad a + 2b, \quad a + 3b, \quad \dots$$

provided $a \perp b$. An article, giving a survey of problems about primes and a number of further references, is

- A. Schinzel & W. Sierpiński, Sur certains hypothèses concernant les nombres premiers, *Acta Arith.*, **4**(1958) 185–208; erratum **5**(1959) 259; *MR 21 #4936*; and see **7**(1961) 1–8.

Table 7 (D 27) can be used as a table of primes < 1000 ; an entry 1, 3, 5 or 7 indicates a prime in that residue class (see A4) modulo 8.

The general problem of determining whether a large number is prime or composite, and in the latter case of determining its factors, has fascinated number theorists down the ages. With the advent of high speed computers, considerable advances have been made, and a special stimulus has recently been provided by the application to cryptanalysis. Some other references appear after Problem A3 and in the first edition of this book.

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- Carl Pomerance (editor), Cryptology and Computational Number Theory, *Proc. Symp. Appl. Math.*, **42** Amer. Math. Soc., Providence, 1990; MR **91k**: 11113.
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- R. Rivest, A. Shamir & L. Adleman, A method for obtaining digital signatures and public key cryptosystems, *Communications A.C.M.*, Feb. 1978.
- R. Solovay & V. Strassen, A fast Monte-Carlo test for primality, *SIAM J. Comput.*, **6**(1977) 84–85; erratum **7**(1978) 118; MR **57** #5885.
- Jonathan Sorenson, Counting the integers cyclotomic methods can factor, *Comput. Sci. Tech. Report*, **919**, Univ. of Wisconsin, Madison, March 1990.
- H. C. Williams & J. S. Judd, Some algorithms for prime testing using generalized Lehmer functions, *Math. Comput.*, **30**(1976) 867–886.

A1 Prime values of quadratic functions.

Are there infinitely many primes of the form $a^2 + 1$? Probably so, and in fact Hardy and Littlewood (their conjecture E) guessed that the number,

$P(n)$, of such primes less than n , was asymptotic to $c\sqrt{n}/\ln n$,

$$i \quad P(n) \sim c\sqrt{n}/\ln n \quad ?$$

i.e., that the ratio of $P(n)$ to $\sqrt{n}/\ln n$ tends to c as n tends to infinity. The constant c is

$$c = \prod \left\{ 1 - \frac{\left(\frac{-1}{p}\right)}{p-1} \right\} = \prod \left\{ 1 - \frac{(-1)^{(p-1)/2}}{p-1} \right\} \approx 1.3727$$

where $\left(\frac{-1}{p}\right)$ is the Legendre symbol (see F5) and the product is taken over all odd primes. They make similar conjectures, differing only in the value of c , for the number of primes represented by more general quadratic expressions. But we don’t know of any integer polynomial, of degree greater than one, for which it has been proved that it takes an infinity of prime values. Is there even one prime $a^2 + b$ for each $b > 0$? Sierpiński has shown that for every k there is a b such that there are more than k primes of the form $a^2 + b$.

Iwaniec has shown that there are infinitely many n for which $n^2 + 1$ is the product of at most two primes, and his results extend to other irreducible quadratics.

If $P(n)$ is the largest prime factor of n , Maurice Mignotte has shown that $P(a^2 + 1) \geq 17$ if $a \geq 240$. Note that $239^2 + 1 = 2 \cdot 13^4$ (yet another property of 239). It has been known for 50 years that $P(a^2 + 1) \rightarrow \infty$ with a .

Ulam and others noticed that the pattern formed by the prime numbers when the sequence of numbers is written in a “square spiral” seems to favor diagonals which correspond to certain “prime-rich” quadratic polynomials. For example the main diagonal of Figure 1 corresponds to Euler’s famous formula $n^2 + n + 41$.

There are some results for expressions (*not* polynomials!) of degree greater than 1, starting with that of Pyateckii-Šapiro, who proved that the number of primes of the form $\lfloor n^c \rfloor$ in the range $1 < n < x$ is $(1 + o(1))x/(1 + c)\ln x$ if $1 \leq c \leq \frac{12}{11}$. This range has been successively extended to $\frac{10}{9}$, $\frac{69}{62}$, $\frac{355}{362}$, $\frac{39}{34}$ and $\frac{15}{13}$ by Kolesnik, Graham and Leitmann independently, Heath-Brown, Kolesnik again, and by Liu & Rivat.

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G. H. Hardy & J. E. Littlewood, Some problems of ‘partitio numerorum’ III: on the expression of a number as a sum of primes, *Acta Math.*, **44**(1922) 1–70.

A. Prime Numbers

- 421 420 **419** 418 417 416 415 414 413 412 411 410 **409** 408 407 406 405 404 403 402
 422 347 346 345 344 343 342 341 340 339 338 **337** 336 335 334 333 332 331 330 401
 423 348 281 280 279 278 277 276 275 274 273 272 **271** 270 **269** 268 267 266 239 400
 424 **349** 282 **223** 221 220 219 218 217 216 215 214 213 212 211 210 265 328 399
 425 350 **283** 224 173 172 171 170 169 168 167 166 165 164 **163** 162 209 264 327 398
 426 351 284 225 174 131 130 129 128 **127** 126 125 124 123 122 121 161 208 **263** 326 **397**
 427 352 285 226 175 132 **97** 96 95 94 93 92 91 90 121 160 207 262 325 396
 428 **353** 286 **227** 176 133 98 71 70 69 68 **67** 66 **89** 120 150 205 261 324 395
 429 354 287 228 177 134 99 72 **53** 52 51 50 65 88 119 154 205 260 323 394
 430 355 288 **229** 178 135 100 73 **54** 43 42 49 64 87 118 **157** 204 259 322 393
 431 356 289 230 179 136 101 74 55 44 41 48 63 86 117 156 203 258 321 392
 432 357 290 231 180 **137** 102 75 56 45 46 **47** 62 85 116 154 202 **257** 320 391
 433 358 291 232 181 138 103 76 57 58 **59** 60 61 84 115 145 201 256 319 390
 434 **359** 292 **233** 182 **139** 104 77 78 **79** 80 81 82 83 114 153 200 255 318 **389**
 435 360 **293** 234 183 140 105 106 107 108 **109** 110 111 112 113 **152** **199** 254 317 388
 436 361 294 235 184 141 142 143 144 145 146 147 148 **149** 150 **151** 198 253 616 387
 437 362 295 236 185 186 187 188 189 190 191 192 **193** 194 197 252 315 386
 438 363 296 237 238 **239** 240 241 242 243 244 245 246 247 248 249 250 **251** 314 385
 439 364 297 298 299 300 301 302 303 304 305 306 **307** 308 309 310 **311** 312 **313** 384
 440 365 366 **367** 368 369 370 371 372 **373** 374 375 376 377 378 **379** 380 381 382 **383**

Figure 1. Primes (in bold) Form Diagonal Patterns.

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 G. A. Kolesnik, The distribution of primes in sequences of the form $[n^c]$, *Mat. Zametki*(2), **2**(1972) 117–128.
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 Daniel Shanks, On the conjecture of Hardy and Littlewood concerning the number of primes of the form $n^2 + a$, *Math. Comput.*, **14**(1960) 321–332.
 W. Sierpiński, Les binômes $x^2 + n$ et les nombres premiers, *Bull. Soc. Roy. Sci. Liège*, **33**(1964) 259–260.
 E. R. Stroka, Distribution of primes of the form $p = [n^c] = [t^d]$ in arithmetic progressions (Russian), *Zap. Nauchn. Semin. Leningrad Otdel. Mat. Inst. Steklova*, **121**(1983) 94–102; *Zbl.* **524**.10038.

A2. Primes connected with factorials.

A2 Primes connected with factorials.

Are there infinitely many primes of the form $n! \pm 1$ or of the form

$$X_k = 1 + \prod_{i=1}^k p_i$$

or of the form $X_k - 2$? Buhler, Crandall & Penk have shown that $n! + 1$ is prime for $n = 1, 2, 3, 11, 27, 37, 41, 73, 77, 116, 154, 320, 340, 399$ and 427; that $n! - 1$ is prime for $n = 3, 4, 6, 7, 12, 14, 30, 32, 33, 38, 94, 166, 324, 379$ and 469 and for no other $n < 546$; that X_k is prime for $p_k = 2, 3, 5, 7, 11, 31, 379, 1019, 1021$ and for no other $p_k < 3088$; and that $X_k - 2$ is prime for $p_k = 3, 5, 11, 13, 41, 89, 317, 337, 991, 1873$ and 2053, and is a probable prime (i.e. final tests had not been carried out) for $p_k = 2377$. Harvey Dubner has discovered the primes 872 \pm 1 and 1477 \pm 1 and shown that X_k is prime for $p_k = 3229, 4547$ and 4787. It is also prime for $p_k = 11549$ and 13649.

Let q_k be the least prime greater than X_k . Then R. F. Fortune conjectures that $q_k - X_k + 1$ is prime for all k . It is clear that it is not divisible by the first k primes, and Selfridge observes that the truth of the conjecture would follow from Schinzel's formulation of Cramer's conjecture, that for $x > 8$ there is always a prime between x and $x + (\ln x)^2$. Stan Wagon has calculated the first 100 fortunate primes:

$$\begin{array}{ccccccccccccccccc} 3 & 5 & 7 & 13 & 23 & 17 & 19 & 23 & 37 & 61 & 67 & 61 & 71 & 47 & 107 & 59 & 61 & 109 & 89 & 103 \\ 79 & 151 & 197 & 101 & 103 & 233 & 223 & 127 & 223 & 191 & 163 & 229 & 643 & 239 & 157 & 167 & 439 & 239 & 199 & 191 \\ 199 & 383 & 233 & 751 & 313 & 773 & 607 & 313 & 383 & 293 & 443 & 331 & 283 & 277 & 271 & 401 & 307 & 331 & 379 & 491 \\ 313 & 111 & 397 & 331 & 353 & 419 & 421 & 1883 & 547 & 1381 & 457 & 457 & 373 & 421 & 409 & 1061 & 523 & 499 & 619 & 727 \\ 457 & 509 & 439 & 911 & 461 & 823 & 613 & 617 & 1021 & 523 & 941 & 653 & 601 & 1877 & 607 & 631 & 733 & 757 & 877 & 641 \end{array}$$

under the assumption that the very large probable primes involved are genuine primes. The answers to the questions are probably “yes”, but it does not seem conceivable that such conjectures will come within reach either of computers or of analytical tools in the foreseeable future. Schinzel's conjecture has been attributed to Cramér, but Cramér conjectured (see reference at A8)

$$\limsup_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{(\ln p_n)^2} = 1 \quad ?$$

Schinzel notes that this doesn't imply the existence of a prime between x and $x + (\ln x)^2$, even for sufficiently large x .

More hopeful, but still difficult, is the following conjecture of Erdős and Stewart: are $1! + 1 = 2, 2! + 1 = 3, 3! + 1 = 7, 4! + 1 = 5^2, 5! + 1 = 11^2$ the only cases where $n! + 1 = p_k^b p_{k+1}^b$ and $p_{k-1} \leq n < p_k$? [Note that $(a, b) = (1, 0), (1, 1), (0, 1), (2, 0)$ and $(0, 2)$ in these five cases.]

Erdős also asks if there are infinitely many primes p for which $p - k!$ is composite for each k such that $1 \leq k < p$; for example, $p = 101$ and

$p = 211$. He suggests that it may be easier to show that there are infinitely many integers n ($l! < n \leq (l+1)!$) all of whose prime factors are greater than l , and for which all the numbers $n - k!$ ($1 \leq k \leq l$) are composite.

I. O. Angell & H. J. Godwin, Some factorizations of $10^n \pm 1$, *Math. Comput.*, **28**(1974) 307–308.

Alan Borning, Some results for $k! \pm 1$ and $2 \cdot 3 \cdot 5 \cdots p \pm 1$, *Math. Comput.*, **26**(1972) 567–570.

J. P. Buhler, R. E. Crandall & M. A. Penk, Primes of the form $n! \pm 1$ and $2 \cdot 3 \cdot 5 \cdots p \pm 1$, *Math. Comput.*, **38**(1982) 639–643; corrigendum, Wilfrid Keller, **40**(1983) 727; MR 83c:10006, 85b:11119.

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A3 Mersenne primes. Repunits. Fermat numbers. Primes of shape $k \cdot 2^n + 2$.

Primes of special form have been of perennial interest, especially the **Mersenne primes** $2^p - 1$. Here p is necessarily prime, but that is not a sufficient condition! $2^{11} - 1 = 2047 = 23 \cdot 89$. They are connected with perfect numbers (see B1).

The powerful Lucas-Lehmer test, in conjunction with successive generations of computers, and more sophisticated techniques in using them, continues to add to the list of primes p for which $2^p - 1$ is also prime:

2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, ...

The number of Mersenne primes is undoubtedly infinite, but proof is again hopelessly beyond reach. Suppose $M(x)$ is the number of primes $p \leq x$ for which $2^p - 1$ is prime. Find a convincing heuristic argument for the size of $M(x)$. Gillies gave one suggesting that $M(x) \sim c \ln x$. H. W. Lenstra, Pomerance and Wagstaff all believe this and in fact suggest that

$$\therefore M(x) \sim e^y \log x$$

where the log is to base 2.

The largest known prime is usually a Mersenne prime, but for a while the record was $391581 \cdot 2^{216193} - 1$, discovered by J. Brown, L.C. Noll, B. Parady, G. Smith, J. Smith & S. Zarantonello. In late March, 1992 this was beaten by the penultimate item in the above list, discovered by Slowinski & Gage.

D. H. Lehmer puts $S_1 = 4$, $S_{k+1} = S_k^2 - 2$, supposes that $2^p - 1$ is a Mersenne prime, notes that $S_{p-2} \equiv 2^{(p+1)/2}$ or $-2^{(p+1)/2} \pmod{2^p - 1}$ and asks: which?

Selfridge conjectures that if n is a prime of the form $2^k \pm 1$ or $2^{2k} \pm 3$, then $2^n - 1$ and $(2^n + 1)/3$ are either both prime or both composite. Moreover if both are prime, then n is of one of those forms. Is this an example of the Strong Law of Small Numbers? Dickson, on p. 28 of Vol. 1 of his *History*, says:

In a letter to Tannery (*L'Intermédiaire des math.*, **2**(1895) 317) Lucas stated that Mersenne (1644, 1647) implied that a necessary and sufficient condition that $2^p - 1$ be a prime is that p be a prime of one of the forms $2^{2n} + 1$, $2^{2n} \pm 3$, $2^{2n+1} - 1$. Tannery expressed his belief that the theorem was empirical and due to Frenicle, rather than to Fermat.

If p is a prime, is $2^p - 1$ always **squarefree** (does it never contain a repeated factor)? This seems to be another unanswerable question. It is safe to conjecture that the answer is "No!" This could be settled by computer if you were lucky. As D. H. Lehmer has said about various factoring methods, "Happiness is just around the corner". Selfridge puts the computational difficulties in perspective by proposing the problem: find fifty more numbers like 1093 and 3511. [Fermat's theorem tells us that if p is prime, then p divides $2^p - 2$; the primes 1093 and 3511 are the only ones less than $6 \cdot 10^9$ for which p^2 divides $2^p - 2$.] It is not known if there are infinitely many primes p for which p^2 divides $2^p - 2$. It is not even known if there are infinitely many p for which p^2 does not divide $2^p - 2$ — although this can be deduced from the very powerful "ABC conjecture" (see B17).

The so-called **repunits**, $(10^p - 1)/9$, are prime for $p = 2, 19, 23, 317$ and 1031. Repunits other than 1 are known never to be squares and Rotkiewicz has shown that they are not cubes. When are they squarefree? The primes 3, 487 and 56598313 are the only ones less than 2^{32} for which p^2 divides $10^p - 10$. Peter Montgomery lists cases where p^2 divides $a^{p-1} - 1$ for $a < 100$ and $p < 2^{32}$.

Selfridge asks if the sequence (in decimal notation)

1, 12, 123, 1234, 12345, 123456, 1234567, 12345678, 123456789, 12345678910, 123456789101, 1234567891011, ...

contains infinitely many primes. The question can also be asked for other

scales of notation; for example

$$12345610111213_7 = 131870666077_{10} \text{ is prime.}$$

Wagstaff observes that the only primes < 180 for which $(p^p - 1)/(p - 1)$ is prime are $p = 2, 3, 7, 19$ and 31 ; for $(p^p + 1)/(p + 1)$ they are $p = 3, 5, 17$ and 157 .

The Fermat numbers, $F_n = 2^{2^n} + 1$ are also of continuing interest; they are prime for $0 \leq n \leq 4$ and composite for $5 \leq n \leq 21$ and for many larger values of n . Hardy & Wright give a heuristic argument which suggests that only a finite number of them are prime. Selfridge would like to see this strengthened to support the conjecture that all the rest are composite.

It has been conjectured that the Fermat numbers are squarefree. It was verified by Gostin & McLaughlin that 82 of the 85 then known factors of the 71 known composite Fermat numbers (factors of F_m for $m = 3310, 4724$ and 6537 were *not* so tested) were not repeated. Wilfrid Keller and Hiromi Suyama have found several new factors of Fermat numbers. The table of 88 prime factors $k \cdot 2^n + 1$ or $2^{2^m} + 1$ given on p. ix of the Introduction of "the Cunningham project" (see the Brillhart *et al.* ref. below) expanded to 114 entries in the second edition, and has at least 150 at the time of writing. Lenstra, Leunstra, Manasse & Pollard have factored the ninth, and R. P. Brent the eleventh, Fermat numbers completely. People interested in factoring large numbers should make contact with Samuel S. Wagstaff.

Because of their special interest as potential factors of Fermat numbers, and because proofs of their primality are comparatively easy, numbers of the form $k \cdot 2^n + 1$ have received special attention, at least for small values of k . For example, large primes were found by Harvey Dubner and Wilfrid Keller, the record on 84-09-05 for a non-Mersenne prime being $(k, n) = (5, 23473)$ by Keller. Another of his discoveries, $(k, n) = (289, 18502)$ is amusing in that it may be written as $(18496, 18496)$, a Cullen prime (**B20**) and as $(17 \cdot 2^{9251})^2 + 1$, a prime of shape $a^2 + 1$ (**A1**).

As we mentioned, the record has since been beaten with a prime of shape $k \cdot 2^n - 1$ with $(k, n) = (391581, 216193)$. See also **B21**.

Hugh Williams has found, for $r = 3, 5, 7$ and 11 , all values of $n \leq 500$ for which $(r - 1)r^n - 1$ is prime:

$r = 3$	$n = 1, 2, 3, 7, 8, 12, 20, 23, 27, 35, 56, 62, 68, 131, 222, 384, 387$
$r = 5$	$n = 1, 3, 9, 13, 15, 25, 39, 69, 165, 171, 209, 339$
$r = 7$	$n = 1, 2, 7, 18, 55, 69, 87, 119, 141, 189, 249, 354$
$r = 11$	$n = 1, 3, 37, 119, 255, 355, 371, 497$

We are very unlikely to know for sure that the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \dots$$

A3. Mersenne primes. Repunits. Fermat numbers.

where $u_1 = u_2 = 1$ and $u_{n+1} = u_n + u_{n-1}$, contains infinitely many primes. (Hugh Williams has found the large prime Fibonacci number u_{2971} .) Similarly for the related Lucas sequence

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, \dots$$

and most other Lucas-Lehmer sequences (with $u_1 \perp u_2$) defined by second-order recurrence relations. However, Graham has shown that the sequence with

$$u_0 = 1786\ 772701\ 928802\ 632268\ 715130\ 455793$$

$$u_1 = 1059\ 683225\ 053915\ 111058\ 165141\ 686995$$

contains no primes at all! Knuth notes that Graham's numbers should have been given as

$$u_0 = 331\ 635635\ 998274\ 737472\ 200656\ 430763$$

$$u_1 = 1510\ 028911\ 088401\ 971189\ 590305\ 498785$$

and gives the smaller example

$$u_1 = 49463\ 435743\ 205655, u_2 = 62638\ 280004\ 239857$$

Raphael Robinson considers the Lucas sequence (sometimes called the Pell sequence) $u_0 = 0, u_1 = 1, u_{n+1} = 2u_n + u_{n-1}$ and defines the primitive part, L_n , by

$$u_n = \prod_{d|n} L_d$$

He notes that $L_7 = 13^2$ and $L_{30} = 31^2$ and asks if there is any larger n for which L_n is a square.

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A4. The prime number race.

A4 The prime number race.

A number a is said to be **congruent** to c , **modulo** a positive number b , written $a \equiv c \pmod{b}$, if b is a divisor of $a - c$. S. Chowla conjectured that if $a \perp b$, then there are infinitely many pairs of consecutive primes such that $p_n \equiv p_{n+1} \pmod{b}$. The case $b = 4$, $a = 1$ follows from a theorem of Littlewood. Bounds between which such consecutive primes occur have been given in this case, and for $b = 4$, $a = 3$ by Knapowski & Turán. Turán observed that it would be of interest (in connexion with the Riemann hypothesis, for example) to discover long sequences of consecutive primes $\equiv 1 \pmod{4}$. Den Haan found the nine primes

$$11593, 11597, 11617, 11621, 11633, 11657, 11677, 11681, 11689.$$

Four sequences of 10 such primes end at 373777, 495461, 509521 and 612217 and a sequence of 11 ends at 766373. Stephane Vandemergel has discovered no fewer than 16 consecutive primes of shape $4k + 1$; they are 207622000 + 273, 297, 301, 313, 321, 381, 409, 417, 421, 489, 501, 517, 537, 549, 553, 561.

Thirteen consecutive primes congruent to 3 mod 4 are 241000 + 603, 639, 643, 651, 663, 667, 679, 687, 691, 711, 727, 739 and 771.

If $p(b, a)$ is the least prime in the arithmetic progression $a + nb$, with $a \perp b$, then Linnik showed that there is a constant L , now called Linnik's **constant**, such that $p(b, a) \ll b^L$. Pan Cheng-Tung, Chen Jing-Run, Matti Jutila, Chen Jing-Run, Matti Jutila, S. Graham, Chen Jing-Run, Chen Jing-Run & Liu Jian-Min, and Wang Wei-Ji have successively improved the best known value of L to 5448, 777, 550, 168, 80, 36, 17, 13.5, and 8, and Heath-Brown has recently established the remarkable result $L \leq 5.5$.

Elliott & Halberstam have shown that

$$p(b, a) < \phi(b)(\ln b)^{1+\delta}$$

almost always.

In the other direction it is known (see the papers of Prachar, Schinzel and Pomerance) that, given a , there are infinitely many values of b for which

$$p(b, a) > \frac{cb \ln b \ln \ln b \ln \ln \ln b}{(\ln \ln b)^2}$$

where c is an absolute constant.

Turán was particularly interested in the **prime number race**. Let $\pi(n; a, b)$ be the number of primes $p \leq n$, $p \equiv a \pmod{b}$. Is it true that for every a and b with $a \perp b$, there are infinitely many values of n for which

$$\pi(n; a, b) > \pi(n : a_1, b)$$

for every $a_1 \not\equiv a \pmod{b}$? Knapowski & Turán settled special cases, but the general problem is wide open.

Chebyshev noted that $\pi(n; 1, 3) < \pi(n; 2, 3)$ and $\pi(n; 1, 4) \leq \pi(n; 3, 4)$ for small values of n . Leech, and independently Shanks & Wrench, discovered that the latter inequality is reversed for $n = 26861$ and Bays & Hudson that the former is reversed for two sets, each of more than 150 million integers, between $n = 608981813029$ and $n = 610968213796$.

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A5 Arithmetic progressions of primes.

How long can an arithmetic progression be which consists only of primes? Table 1 shows progressions of n primes, $a, a+d, \dots, a+(n-1)d$, discovered by James Fry, V.A. Golubev, Andrew Moran, Paul Pritchard, S.C. Root, W.N. Seredinskii, S. Weintraub and Jeff Young (see the first edition for earlier, smaller discoveries). Of course, the common difference must have every prime $p \leq n$ as a divisor (unless $n = a$). It is conjectured that n can be as large as you like. This would follow if it were possible to improve Szemerédi's theorem (see E10).

More generally, Erdős conjectures that if $\{a_i\}$ is any infinite sequence of integers for which $\sum 1/a_i$ is divergent, then the sequence contains arbitrarily long arithmetic progressions. He offers \$3000.00 for a proof or disproof of this conjecture.

Table 1. Long Arithmetic Progressions of Primes.

n	d	a	$a + (n - 1)d$	source
12	30030	23143	353473	G, 1958
13	510510	766439	6892559	S, 1965
14	2462460	46883579	78895559	
16	9699690	53297929	198793279	
16	223092870	2236133941	5582526991	R, 1969
17	87297210	3430751869	4827507229	W, 1977
18	71777060	4808316343	17010526363	P
19	4180566390	8297644387	83547839407	P
19	13608665070	244290205469	489246176729	F, Mar 1987
20	2007835830	803467381001	841616261771	F, Mar 1987
20	764355720	1140997291211	1286221049891	F, Mar 1987
20	18846497670	214861583621	572945039351	Y&F, 87-09-01
20	1140004565700	1845449006227	23505535754527	M&P, Nov 1990
20	19855265430	24845147147111	25222397190281	M&P, Nov 1990
21	1419763024680	142072321123	28537332814723	M&P, 90-11-30

Sierpiński defines $g(x)$ to be the maximum number of terms in a progression of primes not greater than x . The least x , $l(x)$, for which $g(x)$ takes the values

$$\begin{array}{ccccccccccccc} g(x) = 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \dots \\ \text{is} \\ l(x) = 1 & 2 & 3 & 7 & 23 & 29 & 157 & 1307 & 1669 & 1879 & 2089 & \dots \end{array}$$

Günter Löh has searched for arithmetic progressions of primes with first term q and length q . Examples are $(q, d) = (7, 150), (11, 1536160080)$ and $(13, 9918821194590)$.

Pomerance produces the "prime number graph" by plotting the points (n, p_n) and shows that for every k we can find k primes whose points are collinear.

Grosswald has shown that there are long arithmetic progressions consisting only of **almost primes**, in the following sense. There are infinitely many arithmetic progressions of k terms, each term being the product of at most r primes, where

$$r \leq [k \ln k + 0.892k + 1].$$

He has also shown that the Hardy-Littlewood estimate is of the right order of magnitude for 3-term arithmetic progressions of primes.

A5. Arithmetic progressions of primes.

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A6 Consecutive primes in A.P.

It has even been conjectured that there are arbitrarily long arithmetic progressions of consecutive primes, such as

$$251, 257, 263, 269 \quad \text{and} \quad 1741, 1747, 1753, 1759.$$

Jones, Lal & Blundon discovered the sequence $10^{10} + 24493 + 30k$ ($0 \leq k \leq 4$) of five consecutive primes, and Lander & Parkin, soon after, found six such primes, $121174811 + 30k$ ($0 \leq k \leq 5$). They also established that $9843019 + 30k$ ($0 \leq k \leq 4$) is the least progression of five terms, that there are 25 others less than $3 \cdot 10^8$, but no others of length six.

It is not known if there are infinitely many sets of three consecutive primes in arithmetic progression, but S. Chowla has demonstrated this without the restriction to consecutive primes.

Harry Nelson has collected the \$100.00 prize that Martin Gardner offered to the first discoverer of a 3×3 magic square whose nine entries are consecutive primes. These are *not* in arithmetic progression, of course. The central prime is 1480028171 and the others are this $\pm 12, \pm 18, \pm 30$ and ± 42 . He found more than 20 other such squares.

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A7 Cunningham chains.

A common method for proving that p is a prime involves the factorization of $p-1$. If $p-1 = 2q$, where q is another prime, the size of the problem has only been reduced by a factor of 2, so it's interesting to observe Cunningham chains of primes with each member one more than twice the previous one. D. H. Lehmer found just three such chains of 7 primes with least member $< 10^7$:

$$1122659, 2245319, 4490639, 8981279, 17962559, 35925119, 71850239$$

$$2164229, 4328459, 8656919, 17313839, 34627679, 59255359, 138510719 \\ 2329469, 4658939, 9317879, 18635759, 37271519, 74543039, 149086079$$

and two others with least members 10257809 and 10309889. The factorization of $p+1$ can also be used to prove that p is prime. Lehmer found seven

chains of length 7 based on $p+1=2q$. The first three had least members 16651, 67651 and 165901, but the second of these must be discarded, since the fifth member is $1082401 = 601 \cdot 1801$ (curiously, this is a divisor of $2^{25}-1$).

Lalout & Meeus found chains of length 8 of each kind, starting with 19099919 and 15514861, and these are the smallest of this length. Günter Löh has found many new chains: the least of length 9 start with 85864769 and 857095381; of length 10 with 26089808579 and 205528443121; of length 11 with 665043081119 and 1389122693971; of length 12 with 554688278429 and 216857744866621; and a chain of length 13 of the second kind starts with 758083947856951. A count of all chains of the first kind starting below 10^{11} and of length 6, 7, 8, 9, 10 gave the respective frequencies 19991, 2359, 257, 21, 2.

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A8 Gaps between primes. Twin primes.

There are many problems concerning the gaps between consecutive primes. Write $d_n = p_{n+1} - p_n$ so that $d_1 = 1$ and all other d_n are even. How large and how small can d_n be? Rankin has shown that

$$d_n > \frac{c \ln n \ln \ln n \ln \ln \ln \ln n}{(\ln \ln \ln n)^2}$$

for infinitely many n and Erdős offers \$5,000 for a proof or disproof that the constant c can be taken arbitrarily large. Rankin's best value is $c = e^\gamma$ where γ is Euler's constant: Maier & Pomerance have improved this by a factor $k \approx 1.31256$, the root of the equation $4/k - e^{-4/k} = 3$, and Pintz has made a further improvement.

A very famous conjecture is the Twin Prime Conjecture, that $d_n = 2$ infinitely often. If $n > 6$, are there always twin primes between n and $2n$? Conjecture B of Hardy and Littlewood (cf. A1) is that $P_k(n)$, the number of pairs of primes less than n and differing by an even number k , is given asymptotically by

$$P_k(n) \sim \frac{2cn}{(\ln n)^2} \prod \left(\frac{p-1}{p-2} \right)$$

where the product is taken over all odd prime divisors of k (and so is empty and taken to be 1 when k is a power of 2) and $c = \prod(1 - 1/(p-1)^2)$ taken over all odd primes, so that $2c \approx 1.32032$. If $\pi_{1,2}(n)$ is the number of primes p such that $p+2$ has at most two prime factors, then Fouvry & Grupp have shown that

$$\pi_{1,2}(n) \geq 0.71 \times \frac{2cn}{(\ln n)^2}$$

and 0.71 has been improved to 1.015 by Liu and then to 1.05 by Wu.

The large twin primes $9 \cdot 2^{211} \pm 1$ were discovered by the Lehmers and independently by Riesel. Crandall & Penk found twin primes with 64, 136, 154, 203 and 303 digits, Williams found $156 \cdot 5^{202} \pm 1$, Baillie $297 \cdot 2^{546} \pm 1$, Atkin & Rickert

$$694503810 \cdot 2^{2304} \pm 1 \quad \text{and} \quad 1159142985 \cdot 2^{2304} \pm 1$$

and in 1989 Brown, Noll, Parady, Smith, Smith & Zarantonello found

$$663777 \cdot 2^{7650} \pm 1, \quad 571305 \cdot 2^{7701} \pm 1, \quad 1706595 \cdot 2^{11235} \pm 1.$$

On 09:08:16 Harvey Dubner announced a new record with

$$2^{4025} \cdot 3 \cdot 5^{4020} \cdot 7 \cdot 11 \cdot 13 \cdot 79 \cdot 223 \pm 1,$$

numbers with 4030 decimal digits.

Richard Brent counted 224376048 primes p less than 10^{11} for which $p+2$ is also prime; about 9% more than predicted by Conjecture B.

Bombieri & Davenport have shown that

$$\liminf \frac{d_n}{\ln n} < \frac{2 + \sqrt{3}}{8} \approx 0.46650$$

(no doubt the real answer is zero; of course the truth of the Twin Prime Conjecture would imply this); G.Z. Pilt'yaï has improved the constant on the right to $(2\sqrt{2}-1)/4 \approx 0.45711$; Uchiyama to $(9 - \sqrt{3})/16 \approx 0.454256$; Huxley to $(4\sin\theta + 36)/(16\sin\theta) \approx 0.44254$, where $\theta + \sin\theta = \pi/4$, and later to 0.4394; and Helmut Maier to 0.248.

Huxley has also shown that

$$d_n < p_n^{7/12+\epsilon},$$

Heath-Brown & Iwaniec have improved the exponent to $11/20$; Mozzochi to 0.548; and Lou & Yao to $6/11$. Cramér proved, using the Riemann hypothesis, that

$$\sum_{n < x} d_n^2 < cx(\ln x)^4.$$

Erdős conjectures that the right-hand side should be $cx(\ln x)^2$, but thinks

that there is no hope of a proof. The Riemann hypothesis implies that $d_n < p_n^{1/2+\epsilon}$.

Dorin Andrica conjectures that, for all natural n ,

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1 \quad ?$$

Dan Grecu has verified this for $p_n < 10^6$. In *Amer. Math. Monthly*, 83(1976) 61, it is given as a difficult unsolved problem that

$$\lim_{n \rightarrow \infty} (\sqrt{p_{n+1}} - \sqrt{p_n}) = 0 \quad ?$$

If true, this implies Andrica's conjecture for large enough n , which is comparable with that of Cramér, mentioned in A2, and with the following one of Shanks, who has given a heuristic argument which supports the conjecture that if $p(g)$ is the first prime that follows a gap of g between consecutive primes, then $\ln p(g) \sim \sqrt{g}$. Record gaps between consecutive primes have been observed by Lehmer, Lander & Parkin, Brent, Weintraub, Young & Potler and others. Table 2 illustrates Shanks's conjecture.

Table 2. Earliest large gaps between consecutive primes.

g	$p(g)$	$(\ln p)^2$	$g/(\ln p)^2$
456	25056082543	573.33	0.7953
464	42652618807	599.09	0.7745
468	127976335139	654.09	0.7155
474	182226896713	672.29	0.7051
486	241160624629	686.90	0.7075
490	297501076289	697.95	0.7021
500	303371455741	698.98	0.7153
514	304599509051	699.19	0.7351
516	416608696337	715.85	0.7208
532	461690510543	721.36	0.7375
534	614487454057	736.80	0.7247
540	738832928467	746.84	0.7230
582	1346294311331	779.99	0.7462
588	1408695494197	782.53	0.7514
602	1968188557063	801.35	0.7512
652	2614941711251	817.52	0.7975
674	7177162612387	876.27	0.7692
716	13829048560417	915.53	0.7821
766	19581334193189	936.70	0.8178
778	42842283926129	985.24	0.7897
804	90874329412297	1033.01	0.7783

The last entry is an unpublished result of Aaron Potler & Jeff Young, communicated Aug. 1993.

Chen Jing-Run showed that, for x large enough, there is always a number with at most two prime factors in the interval $[x - x^\alpha, x]$ for any value of $\alpha \geq 0.477$. Halberstam, Heath-Brown & Richert (see reference at A5) showed that in such an interval with $\alpha = 0.455$ there are at least $x^\alpha/121 \ln x$ numbers with at most two prime factors, and Iwaniec & Laborde further reduced the exponent to $\alpha = 0.45$.

Victor Meally used the phrase **prime deserts**. He notes that below 373 the commonest gap is 2; below 467 there are 24 gaps of each of 2, 4 and 6; below 563 the commonest gap is 6, as it is between 10^{14} and $10^{14} + 10^8$ and probably also from 2 to 10^{14} . He asks: when does 30 take over as the commonest gap? Conway & Odlyzko call d a **champion** for x if it occurs most frequently as the difference between consecutive primes $\leq x$. There may be more than one champion for the same x : $C(135) = 4$, $C(100) = \{2, 4\}$. They suggest that the only champions are 4 and the prime factorials 2, 6, 30, 210, 2310, Do champions $\rightarrow \infty$? Does each prime p divide all champions for $x \geq x_0(p)$?

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A9. Patterns of primes.

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A9 Patterns of primes.

A conjecture more general than Chowla's (see A4) is that there are infinitely many sets of consecutive primes of any given pattern, provided that there are no congruence relations which rule them out. It seems likely, for example, that there are infinitely many triples of primes $\{6k - 1, 6k + 1, 6k + 5\}$ and $\{6k + 1, 6k + 5, 6k + 7\}$. This would be even harder to settle than the Twin Prime Conjecture, but its plausibility is of interest, since Hensley & Richards have shown that it is incompatible with

the well-known conjecture (also due to Hardy & Littlewood)

$$\text{LLL} \quad \pi(x+y) \leq \pi(x) + \pi(y) \quad ???$$

for all integers $x, y \geq 2$. We've put more queries than usual round this, since it is very likely to be false. Indeed, there's some hope of finding values of x and y which contradict it. However there's an alternative conjecture,

$$\text{L} \quad \pi(x+y) \leq \pi(x) + 2\pi(y/2) \quad ?$$

that the Hensley-Richards method doesn't comment on.

Montgomery & Vaughan showed that

$$\pi(x+y) - \pi(x) \leq 2y/\ln y$$

and Iwaniec observed that for each θ , $0 < \theta < 1$, there is an $\eta(\theta) > \theta$ such that

$$\pi(x+x^\theta) - \pi(x) < (2+\epsilon)x^\theta / (\eta(\theta) \ln x)$$

for sufficiently large x and he found that $\eta(\theta) = \frac{5}{3}\theta - \frac{2}{9}$ for $\theta > \frac{1}{3}$, and that $\eta(\theta) = (1+\theta)/2$ for $\theta > \frac{1}{2}$. Lou & Yao improve this in part by showing that $\eta(\theta) = (100\theta - 45)/11$ for $\frac{6}{11} < \theta \leq \frac{11}{20}$.

C. W. Trigg reported that in 1978 M. A. Penk found four primes p , $p+2$, $p+6$ and $p+8$ where

$$p = 802359150003121605557551380867519560344356971.$$

H. F. Smith noted that the pattern 11, 13, 17, 19, 23, 29, 31, 37 is repeated at least three times, starting with the primes 15760091, 25658841 and 93625991. In none of these cases is the number corresponding to 41 a prime, although $n-11, n-13, \dots, n-41$ are all primes for $n = 88830$ and 855750.

Leech gave as an unsolved problem to find 33 consecutive numbers greater than 11 which include 10 primes. In 1961 Herschel Smith found 20 such sets and also 5 examples of 37 consecutive numbers containing 11 primes. Smith writes that Selfridge noted some errors in his 1957 paper. Sten Säfholm found primes

$$\{n+11, \dots, n+43\} \quad \text{for } n = 33081664140$$

and rediscovered Smith's first three examples, that each of

$$\{n-11, \dots, n-43\} \quad \text{is prime for } n = 9853497780,$$

for $n = 21956291910$ and for $n = 22741837860$. Leech wondered why the latter sets seem to occur more readily than the former. My guess is that this is just a more complicated version of the prime number race (see A4)

A10. Gilbreath's conjecture.

and that with much more high-powered telescopes we'd see the balance being redressed (infinitely often). Dimitrios Betsis & Sten Säfholm have found many more patterns, culminating in $\{n+11, \dots, n+61\}$ for $n = 21817283854511250$ and $\{n-11, \dots, n-61\}$ for $n = 79287805466244270$.

Erdős asks, for each k , what is the smallest l for which $p_k, p_{k+1}, \dots, p_{k+l-1}$ is the only set of l consecutive primes with this pattern. E.g., the pattern 3, 5, 7 cannot occur again. The pattern 5, 7, 11, 13, 17 repeats at 101, 103, 107, 109, 113 and no doubt occurs infinitely often, but considerations mod 5 show that the pattern 5, 7, 11, 13, 17, 19 does not occur again. $(p_k, l) = (2, 2), (3, 3), (5, 6), \dots$

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A10 Gilbreath's conjecture.

Define d_n^k by $d_n^1 = d_n$ and $d_n^{k+1} = |d_{n+1}^k - d_n^k|$, that is, the successive absolute differences of the sequence of primes (Figure 2). N. L. Gilbreath

conjectured (and Hugh Williams notes that Proth long before claimed to have proved) that $d_k^k = 1$ for all k . This was verified for $k < 63419$ by Killgrove & Ralston. Odlyzko has checked it for primes up to $\pi(10^{13}) \approx 3 \cdot 10^{11}$; he only needed to examine the first 635 differences.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89
1	2	2	4	2	4	2	4	6	2	6	4	2	4	6	6	2	6	4	2	4	6	4	6
1	0	2	2	2	2	2	4	4	2	2	2	2	0	4	4	2	2	4	2	2	0	2	0
1	2	0	0	0	0	2	0	2	0	0	0	2	2	4	2	2	2	0	2	0	2	2	0
1	2	0	0	0	0	2	2	2	2	0	0	2	2	4	2	2	2	0	2	0	2	2	0
1	2	0	0	0	0	2	0	0	0	2	0	0	2	2	2	2	0	2	0	2	0	2	0
1	2	0	0	2	0	2	0	2	0	0	0	0	2	2	2	2	0	2	0	2	0	2	0
1	2	2	2	2	2	2	2	2	0	0	0	2	0	0	0	0	0	2	0	0	0	0	0
1	0	0	0	0	0	0	0	2	0	0	2	0	0	0	0	0	0	2	0	0	0	0	0

Figure 2. Successive Absolute Differences of the Sequence of Primes.

Hallard Croft and others have suggested that it has nothing to do with primes as such, but will be true for any sequence consisting of 2 and odd numbers, which doesn't increase too fast, or have too large gaps. Odlyzko discusses this.

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A11 Increasing and decreasing gaps.

Since the proportion of primes gradually decreases, albeit somewhat erratically, $d_m < d_{m+1}$ infinitely often, and Erdős & Turán have shown that the same is true for $d_n > d_{n+1}$. They have also shown that the values of n for which $d_n > d_{n+1}$ have positive lower density, but it is not known if there are infinitely many decreasing or increasing sets of three consecutive values of d_n . If there were not, then there is an n_0 so that for every i and $n > n_0$ we have $d_{n+2i} > d_{n+2i+1}$ and $d_{n+2i+1} < d_{n+2i+2}$. Erdős offers \$100.00 for a proof that such an n_0 does not exist. He and Turán could not even prove that for $k > k_0$, $(-1)^k(d_{k+r+1} - d_{k+r})$ can't always have the same sign.

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A12 Pseudoprimes. Euler pseudoprimes. Strong pseudoprimes.

Pomerance, Selfridge & Wagstaff call an odd composite n for which $a^{n-1} \equiv 1 \pmod{n}$ a **pseudoprime to base a** ($\text{psp}(a)$). This usage is introduced to avoid the clumsy “composite pseudoprime” which appears throughout the literature. Odd composite n which are $\text{psp}(a)$ for every a prime to n are **Carmichael numbers** (see A13). An odd composite n is an **Euler pseudoprime to base a** ($\text{epsp}(a)$) if $a \perp n$ and $a^{(n-1)/2} \equiv \left(\frac{a}{n}\right) \pmod{n}$, where $\left(\frac{a}{n}\right)$ is the Jacobi symbol (see F5). Finally, an odd composite n with $n-1 = d \cdot 2^s$, d odd, is a **strong pseudoprime to base a** ($\text{spsp}(a)$) if $a^d \equiv 1 \pmod{n}$ (otherwise $a^{d/2^s} \equiv -1 \pmod{n}$ for some r , $0 \leq r < s$). These definitions are illustrated by a Venn diagram (Figure 3) which displays the smallest member of each set.

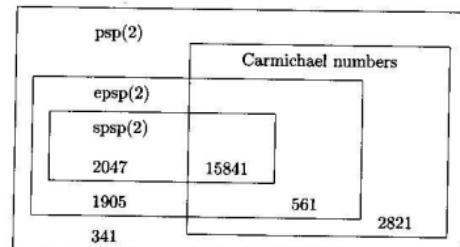


Figure 3. Relationships of Sets of psps with Least Element in Each Set.

The following values of $P_2(x)$, $E_2(x)$, $S_2(x)$ and $C(x)$ — the numbers of $P_2(x)$, $\text{epsp}(2)$, $\text{spsp}(2)$ and Carmichael numbers less than x , respectively — were given by Pomerance, Selfridge & Wagstaff:

x	10^3	10^4	10^5	10^6	10^7	10^8	10^9	10^{10}	$2.5 \cdot 10^{10}$
$P_2(x)$	3	22	78	245	750	2057	5597	14884	21853
$E_2(x)$	1	12	36	114	375	1071	2939	7706	11347
$S_2(x)$	0	5	16	46	162	488	1282	3291	4842
$C(x)$	1	7	16	43	105	255	646	1547	2163

Lehmer and Erdős showed that, for sufficiently large x ,

$$c_1 \ln x < P_2(x) < x \exp\{-c_2(\ln x \ln \ln x)^{1/2}\}$$

and Pomerance improved these bounds to

$$\exp\{(\ln x)^{5/14}\} < P_2(x) < x \exp\{(-\ln x \ln \ln \ln x)/2 \ln \ln x\}$$

and has a heuristic argument that the true estimate is the upper bound with the 2 omitted. The exponent 5/14 has been improved to 85/207 by Pomerance, using a result of Friedlander.

There are also examples of even numbers such that $2^n \equiv 2 \pmod{n}$. Lehmer found $161038 = 2 \cdot 73 \cdot 1103$ and Beeger showed that there are infinitely many. If F_n is the Fermat number $2^{2^n} + 1$, Cipolla showed that $F_{n_1} F_{n_2} \cdots F_{n_k}$ is ppsp(2) if $k > 1$ and $n_1 < n_2 < \dots < n_k < 2^{n_1}$.

If $P_n^{(a)}$ is the n -th ppsp(a), Szymiczek has shown that $\sum 1/P_n^{(a)}$ is convergent, while Mąkowski has shown that $\sum 1/\ln P_n^{(a)}$ is divergent. Rotkiewicz has a booklet on pseudoprimes which contains 58 problems and 20 conjectures.

For example, Problem #22 asks if there is a pseudoprime of form 2^{N-2} . Wayne McDaniel answers this affirmatively with $N = 465794$. Rotkiewicz has shown that the congruence $2^{n-2} \equiv 1 \pmod{n}$ has infinitely many composite solutions n . Shen Mok-Kong found five such less than a million, each of which ended in 7. McDaniel and Zhang Ming-Zhi have given the examples 73 · 48544121 and 524287 · 13264529 which each show that 3 is also a possible final digit.

Selfridge, Wagstaff & Pomerance offer \$500.00 + \$100.00 + \$20.00 for a composite $n \equiv 3$ or 7 mod 10 which divides both $2^n - 2$ and the Fibonacci number u_{n+1} (see A3) or \$20.00 + \$100.00 + \$500.00 for a proof that there is no such n .

Shen Mok-Kong has shown that there are infinitely many k such that $2^{n-k} \equiv 1 \pmod{n}$ has infinitely many composite solutions n , and Kiss & Phong have shown that this is so for all $k \geq 2$ and for all $a \geq 2$ in place of 2.

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A13 Carmichael numbers.

The Carmichael numbers ($\text{psp}(a)$ for all a prime to n , n composite) must be the product of at least three odd prime factors. As long ago as 1899 Korset had given a necessary and sufficient condition for n to be a Carmichael number; namely that n be squarefree and such that $(p-1)|(n-1)$ for each p that divides n . The smallest example is $561 = 3 \cdot 11 \cdot 17$. More generally, if $p = 6k+1$, $q = 12k+1$ and $r = 18k+1$ are each prime, then pqr is a Carmichael number. It seems certain that there are infinitely many such triples of primes, but beyond our means to prove it. Alford, Granville & Pomerance have shown (by a different method!) that there are infinitely many Carmichael numbers, in fact, for sufficiently large x , more than x^{β} of them less than x , where

$$\beta = \frac{5}{12} \left(1 - \frac{1}{2\sqrt{e}} \right) > 0.290306 > \frac{2}{7}$$

Erdős had conjectured that $(\ln C(x))/\ln x$ tends to 1 as x tends to infinity and he improved a result of Knödel to show that

$$C(x) < x \exp\{-c \ln x \ln \ln x / \ln \ln x\}.$$

Then Pomerance, Selfridge & Wagstaff (see A12) proved this with $c = 1 - \epsilon$ and give a heuristic argument supporting the conjecture that the reverse inequality holds with $c = 2 + \epsilon$.

They found 2163 Carmichael numbers $< 25 \cdot 10^9$ and Jaeschke finds 6075 more between that bound and 10^{12} ; seven of these have eight prime factors. Richard Pinch has counted

8241	19279	44706	105212	246683	such numbers
< 10^{12}	10^{13}	10^{14}	10^{15}	10^{16}	

A fair-sized specimen is

$$203745337604001 = 17 \cdot 37 \cdot 41 \cdot 131 \cdot 251 \cdot 571 \cdot 4159.$$

J. R. Hill found the large Carmichael number pqr where $p = 5 \cdot 10^{19} + 371$, $q = 2p - 1$ and $r = 1 + (p-1)(q+2)/433$. Wagstaff produced a 321-digit example, and Woods & Huenemann one of 432 digits. Dubner has continued to beat this and his own records, a 3710-digit specimen being $N = PQR$ where $P = 6M+1$, $Q = 12M+1$ and $R = 1 + (PQ-1)/X$ are primes given by $M = (TC-1)^4/4$, T the product of the odd primes up to 47, $C = 141847$, $A = 41$ and $X = 123165$. But Günter Löh & Wolfgang Niebuhr have developed new algorithms which completely eclipse these by producing a Carmichael number with no fewer than 1101518 prime factors, a number of 16142049 decimal digits!

Alford, Granville & Pomerance proved that there are infinitely many Carmichael numbers n with the stronger requirement that $p \nmid n$ implies $(p^2 - 1)|(n-1)$, but didn't know of any examples. Sid Graham found 18 such numbers, the smallest being

$$589398289990395334700037072001 = 29 \cdot 31 \cdot 37 \cdot 43 \cdot 53 \cdot 67 \cdot 79 \cdot 89 \cdot 97 \cdot 151 \cdot 181 \cdot 191 \cdot 419 \cdot 881 \cdot 883$$

Richard Pinch had already found the smallest of all:

$$44337288629441 = 17 \cdot 31 \cdot 41 \cdot 43 \cdot 89 \cdot 97 \cdot 167 \cdot 331$$

Graham found 17 other numbers that satisfy the slightly weaker condition $\frac{p^2-1}{2}|(n-1)$.

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- A14** “Good” primes and the prime number graph.
- Erdős and Straus called the prime p_n **good** if $p_i^n > p_{n-i}p_{n+1}$ for all i , $1 \leq i \leq n-1$; for example, 5, 11, 17 and 29. Pomerance used the “prime number graph” (see **A5**) to show that there are infinitely many good primes. He asks the following questions. Is it true that the set of n for which p_n is good has density 0? Are there infinitely many n with $p_np_{n+1} > p_{n-i}p_{n+i}$ for all i , $1 \leq i \leq n-1$? Are there infinitely many n with $p_n + p_{n+1} > p_{n-i} + p_{n+i}$ for all i , $1 \leq i \leq n-1$? Does the set of n for which $2p_n < p_{n-i} + p_{n+i}$ for all i , $1 \leq i \leq n-1$ have density 0? (Pomerance proved that there were infinitely many such n .) Is $\limsup\{\min_{0 < i < n}(p_{n-i} + p_{n+i}) - 2p_n\} = \infty$?

A15 Congruent products of consecutive numbers.

Erdős, in a letter dated 79-10-31, observes that $3 \cdot 4 \equiv 5 \cdot 6 \cdot 7 \equiv 1 \pmod{11}$ and asks for the least prime p such that there are integers a, k_1, k_2, k_3 and

$$\prod_{i=1}^{k_1} (a+i) \equiv \prod_{i=1}^{k_2} (a+k_1+i) \equiv \prod_{i=1}^{k_3} (a+k_1+k_2+i) \equiv 1 \pmod{p}.$$

He suggests that such primes p exist for any number of such congruent products.

Mąkowski sends examples corresponding to rows $n = 5$ and 6 in the table below [compare **F11**] and says that tables of indices can be used to find others. W. Narkiewicz also sends these examples, together with those in rows $n = 7, 8$ and 9 below. Landon Noll & Chuck Simmons generalize the problem slightly by asking for solutions of

$$q_1! \equiv q_2! \equiv \dots \equiv q_n! \pmod{p}$$

and they give the least prime p for which there is a solution with n terms.

n	p	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}
1	2	0										
2	2	0	1									
3	5	0	1	3								
4	17	0	1	5	11							
5	17	0	1	5	11	15						
6	23	0	1	4	8	11	21					
7	71	8	10	20	52	62	64	71				
8	599	29	51	123	184	251	290	501	540			
9	599	29	51	123	184	251	290	501	540	556		
10	3011	0	1	611	723	749	805	2205	2261	2287	2399	
11	3011	0	1	611	723	749	805	2205	2261	2287	2399	3009

Andrzej Mąkowski, On a number-theoretic problem of Erdős, *Elem. Math.*, **38** (1983) 101–102.

A16 Gaussian primes. Eisenstein-Jacobi primes.

Prime numbers can be defined in fields other than the rational field. In the complex number field they are called **Gaussian primes**. Many problems on ordinary primes can be reformulated for Gaussian primes.

Gaussian integers $a+bi$, where a, b are integers and $i^2 = -1$, behave like ordinary integers in the sense that there is **unique factorization** (apart from order, units $(\pm 1, \pm i)$ and associates; the associates of 7, for example, are 7, -7 , $7i$ and $-7i$). Primes of shape $4k-1$ (3, 7, 11, 19, 23,

... are still primes in the ring of Gaussian integers, but the other ordinary primes can be factored into Gaussian primes:

$$2 = (1+i)(1-i), \quad 5 = (2+i)(2-i) = -(2i-1)(2i+1), \text{ etc.}$$

$$13 = (2+3i)(2-3i), \quad 17 = (4+i)(4-i), \quad 29 = (5+2i)(5-2i), \dots$$

The Gaussian primes $\pm 1 \pm i, \pm 1 \pm 2i, \pm 2 \pm i, \pm 3, \pm 3i, \pm 2 \pm 3i, \pm 3 \pm 2i, \pm 4 \pm i, \pm 1 \pm 4i, \pm 5 \pm 2i, \pm 2 \pm 5i, \dots$ make a pleasing pattern (Figure 4) when drawn on an Argand diagram, which has been used for tiling floors and weaving tablecloths.

Motzkin and Gordon asked if one can "walk" from the origin to infinity using the Gaussian primes as "stepping stones" and taking steps of bounded length. Presumably not. Jordan & Rabung have shown that steps of length at least 4 are necessary.

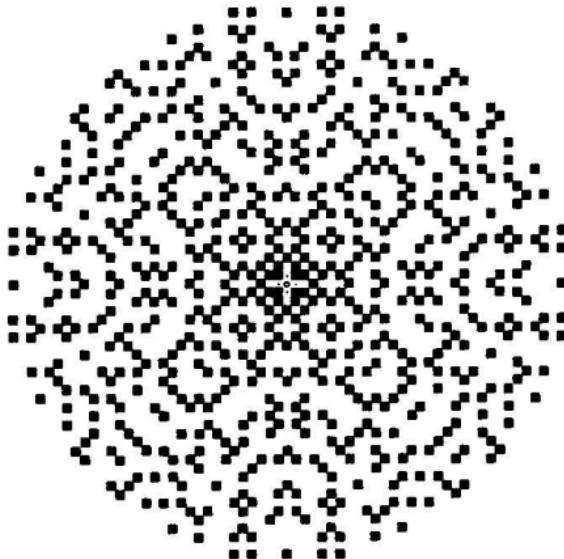


Figure 4. The Gaussian Primes with Norm Less Than 1000.

The **Eisenstein-Jacobi integers** $a + b\omega$, where a, b are integers and ω is a complex cube root of unity, $\omega^2 + \omega + 1 = 0$, also enjoy unique factorization. The primes again form a pattern (Figure 5), this time with hexagonal symmetry, because there are six units, $\pm 1, \pm \omega, \pm \omega^2$. The prime 2 and those of shape $6k - 1$ ($5, 11, 17, 23, 29, 41, \dots$) are still Eisenstein-Jacobi primes, but 3 and those of shape $6k + 1$ can be factored:

$$3 = (1-\omega)(1-\omega^2), \quad 7 = (2-\omega)(2-\omega^2), \quad 13 = (3-\omega)(3-\omega^2),$$

$$19 = (3-2\omega)(3-2\omega^2), \quad 31 = (5-\omega)(5-\omega^2), \quad 37 = (4-3\omega)(4-3\omega^2), \dots$$

John Leech asks for long arithmetic progressions of Gaussian primes and also of Eisenstein-Jacobi primes. He finds nine in Figure 4 and twelve in Figure 5. He later found the arithmetic progression

$$-8 - 13i, -3 - 8i, 2 - 3i, 7 + 2i, \dots, 37 + 32i$$

of ten Gaussian primes, the last three of which are outside Figure 4.

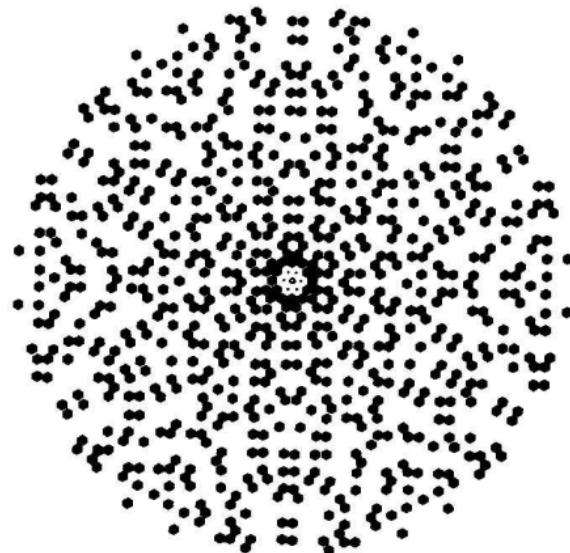


Figure 5. The Eisenstein-Jacobi Primes.

J. H. Jordan & J. R. Rabung, A conjecture of Paul Erdős concerning Gaussian primes, *Math. Comput.*, **24**(1970) 221–223.

A17 Formulas for primes.

Perhaps the philosopher's stone of number theory is a formula for p_n , or for $\pi(x)$, or for a necessary and sufficient condition for primality. Wilson's theorem seems to be unique (is Vantieghem's result, that $p > 2$ is prime if and only if $\prod_{d=1}^{p-1}(2^d - 1) \equiv p \pmod{2^p - 1}$ equivalent to it?); but even that is useless for computation. C. P. Willans and C. P. Wormell used it to give formulas which use only elementary functions, but which are too clumsy to print here. The Mann-Shanks algorithm is another curiosity, of even less practical value. Matiyasevich and other logicians have used Wilson's theorem and their solution of Hilbert's tenth problem to produce polynomials the positive part of whose range is exactly the set of primes.

Three theorems of Boris Stechkin may be worth recording. They are based on the function

$$S(n) = \#\left\{m : 2 \leq m \leq n, (m-1) \left\lfloor \frac{n(m-1)}{m} \right\rfloor \right\}$$

- (1) $n - 1$ is prime just if $S(n) = d(n)$, the number of divisors of n ,
- (2) $n \pm 1$ are twin primes just if $S(n) + S(n+1) = 2d(n)$.
- (3) $p < q$ odd primes implies $S(q) - S(q-1) + S(q-2) - \dots - S(p+1) = 0$.

The numerous papers on this topic vary widely in their sophistication and in their aim. It seems desirable to distinguish between

1. A formula for $\pi(x)$ as a function of x .
2. A formula for p_n as a function of n .
3. A necessary and sufficient condition for n to be prime.
4. A function that is prime for each member of its domain.
5. A function (the positive part of) whose range consists only of primes, or consists of all of the primes.
6. A function whose range contains a high density of primes.
7. A formula for the largest prime divisor of n .
8. A formula for the prime factors of n .
9. A formula for the smallest prime greater than n .
10. A formula for p_{n+1} in terms of p_1, p_2, \dots, p_n .
11. An algorithm for generating the primes. And so on

Examples of each can be found in the references. We have already mentioned (in A1) Euler's famous formula $n^2 + n + 41$. In some sense this is best possible, but quadratic expressions with positive discriminant can yield even longer sequences of prime values (though some of them may be negative). Gilbert Fung gives $47n^2 - 1701n + 10181, 0 \leq n \leq 42, \Delta = 979373$ and Russell Ruby $36n^2 - 810n + 2753, 0 \leq n \leq 44, \Delta = 2^2 3^2 7213$.

The first 1000 values of Euler's formula include 581 primes. Edgar Karst beats this with 598 values of $2n^2 - 199$ and in a 91-01-01 letter, Stephen Williams announces 602 prime values of $2n^2 - 1000n - 2609$. The corresponding numbers among the first 10000 values are 4148, 4373 and 4151. However, what is significant is not the actual density over the first so many values, which clearly has to tend to zero in all cases, but the **asymptotic** density, which, if we believe Hardy & Littlewood (see A1), is always $c\sqrt{n}/\ln n$, and the best that can be done is to make the value of c as large as possible. Shanks has calculated $c = 3.3197732$ for Euler's formula and $c = 3.6319998$ for a polynomial $x^2 + x + 27941$ found by Beeger. Fung & Williams (see reference at A1 and the references they give) have achieved $c = 5.0870883$ with the formula $x^2 + x + 132874279528931$. If Δ is the discriminant of the quadratic, then the Legendre symbol (see F5) $\left(\frac{\Delta}{p}\right)$ takes the value 1 for very few of the small primes, p .

Sierpiński observes that it follows from Fermat's theorem that if n is prime, then n divides

$$1^{n-1} + 2^{n-1} + \dots + (n-1)^{n-1} + 1.$$

Is the converse true? Giuga so conjectured and verified it for $n \leq 10^{1000}$ and Bedocchi verified it to $n \leq 10^{1700}$. Giuga observed that a counterexample would be a Carmichael number (A12, A13), that $p|n$ would imply that $(p-1)|(n-1)$ and that

$$\sum_{p|n} \frac{1}{p} - \frac{1}{n}$$

must be an integer, so that n contains at least eight distinct prime factors. An equivalent conjecture is that

$$nB_{n-1} \equiv -1 \pmod{n}$$

where the **Bernoulli numbers** B_k are the coefficients in the expansion of $x/(e^x - 1) = \sum_{k \geq 0} B_k x^k / k!$ (compare D2).

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A18 The Erdős-Selfridge classification of primes.

Erdős & Selfridge classify the primes as follows: p is in class 1 if the only prime divisors of $p+1$ are 2 or 3; and p is in class r if every prime factor of $p+1$ is in some class $\leq r-1$, with equality for at least one prime factor. For example:

class 1:	2 3 5 7 11 17 23 31 47 53 71 107 127 191 431 647 863 971 ...
class 2:	13 19 29 41 43 59 61 67 79 83 89 97 101 109 131 137 139 149 167 179 197 199 211 223 229 239 241 251 263 269 271 281 283 293 307 317 319 359 367 373
	377 383 419 439 449 461 467 493 503 509 557 563 577 587 593 599 619 641 643
	659 709 719 743 751 761 769 803 827 838 881 919 929 953 967 979 991 1019 ...
class 3:	37 103 113 151 157 163 173 181 193 227 233 257 277 311 331 337 347 353 379 389 397 401 409 421 457 463 467 487 491 521 523 541 547 571 601 607 613
	631 653 683 701 727 733 773 787 811 821 829 853 857 859 877 883 911 937 947
	983 997 1009 1013 1021 ...
class 4:	73 313 443 617 661 673 677 691 739 757 823 887 907 941 977 ...
class 5:	1321 1381 ...

It's easy to prove that the number of primes in class r , not exceeding n , is $o(n^{\epsilon})$ for every $\epsilon > 0$ and all r . Prove that there are infinitely many primes in each class. If $p_1^{(r)}$ denotes the least prime in class r , so that $p_1^{(1)} = 2$, $p_1^{(2)} = 13$, $p_1^{(3)} = 37$, $p_1^{(4)} = 73$ and $p_1^{(5)} = 1021$, then Erdős thought that $(p_1^{(r)})^{1/r} \rightarrow \infty$, while Selfridge thought it quite likely to be bounded.

A similar classification arises if $p + 1$ is replaced by $p - 1$:

class 1:	2 3 5 7 13 17 19 37 73 97 109 163 193 433 487 577 769 1153 ...
class 2:	11 29 31 41 43 53 61 71 79 101 103 113 127 131 137 149 151 157 181 191 197 211 223 229 239 241 251 257 271 281 293 307 313 337 379 389 401 409 421 439 443 449 459 491 521 541 547 571 593 601 613 631 641 647 653 673 677 701 751 757 761 773 811 877 883 911 919 937 963 971 1009 1021 ...
class 3:	23 59 67 83 89 107 173 191 227 233 263 311 317 331 349 355 367 373 383 397 419 431 463 479 503 509 523 563 569 587 607 617 619 661 683 727 733 739 743 787 809 821 823 853 859 881 887 907 929 947 977 983 991 1031 1033 ...
class 4:	47 139 167 179 269 277 347 461 467 499 599 643 691 709 797 827 829 839 857 863 967 997 1013 1019 ...
class 5:	283 359 557 659 941 ...
class 6:	719 1319 ...
class 7:	1439 ...

for which similar answers are to be expected. Are corresponding classes equally dense? There is a connexion with Cunningham chains (A7).

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A19 Values of n making $n - 2^k$ prime. Odd numbers not of the form $\pm p^a \pm 2^b$.

Erdős conjectures that 4, 7, 15, 21, 45, 75 and 105 are the only values of n for which $n - 2^k$ is prime for all k such that $2 \leq 2^k < n$. Mientka & Weitzenkamp have verified this for $n < 2^{44}$ and Uchiyama & Yorinaga have extended this to 2^{77} . Vaughan has proved that there are not too many such numbers, less than $x \exp\{-(\ln x)^c\}$ of them less than x , but he was unable to show that there were less than $x^{1-\epsilon}$.

Erdős also conjectures that for infinitely many n , all the integers $n - 2^k$, $1 \leq 2^k < n$ are squarefree (see also F13).

If we denote by $A(x)$ the number of $n \leq x$ for which all $n - 2^k$ are prime, $2 \leq 2^k < n$, then Hooley showed that the extended Riemann hypothesis implies that $A(x) = O(x^c)$ with an explicit $c < 1$, and Narkiewicz improved this to $c < \frac{1}{2}$.

Cohen & Selfridge ask for the least positive odd number *not* of the form $\pm p^a \pm 2^b$, where p is prime, $a \geq 0$, $b \geq 1$ and any choice of signs may be made. They observe that the number is greater than 2^{18} , but at most

$$\begin{aligned} 6120\ 669906072\ 7677809211\ 5601756625\ 4819576161- \\ 6319229817\ 3436854933\ 4512406741\ 7420946855\ 899326569. \end{aligned}$$

Crocker proved that there are infinitely many odd integers *not* of the form $2^k + 2^l + p$, where p is prime. Erdős suggests that there may be cx

A19. Values of n making $n - 2^k$ prime.

of them less than x , but can $> x^\epsilon$ be proved? Can we show that covering congruences (F13) do not help here? I.e., does $p+2^u+2^v$ (or $p+2^u+2^v+2^w$) meet every arithmetic progression? More generally, Erdős asks if, for each r , there are infinitely many odd integers not the sum of a prime and r or fewer powers of 2. Is their density positive? Do they contain an infinite arithmetic progression? In the opposite direction, Gallagher has proved that for every $\epsilon > 0$ there is a sufficiently large r so that the lower density of sums of primes with r powers of 2 is greater than $1 - \epsilon$.

Erdős also asks if there is an odd integer *not* of the form $2^k + s$ where s is squarefree.

Let $f(n)$ be the number of representations of n as a sum $2^k + p$, and let $\{a_i\}$ be the sequence of values of n for which $f(n) > 0$. Does the density of $\{a_i\}$ exist? Erdős showed that $f(n) > c \ln \ln n$ infinitely often, but could not decide if $f(n) = o(\ln n)$. He conjectures that $\limsup(a_{i+1} - a_i) < \infty$. This would follow if there are covering systems with arbitrarily large least moduli.

Carl Pomerance notes that for $n = 210$, $n - p$ is prime for all p , $n/2 < p < n$, and asked if there is any other such n . With help from Deshouillers, Granville & Narkiewicz he later answered this negatively.

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B. Divisibility

We will denote by $d(n)$ the number of positive divisors of n , by $\sigma(n)$ the sum of those divisors, and by $\sigma_k(n)$ the sum of their k th powers, so that $\sigma_0(n) = d(n)$ and $\sigma_1(n) = \sigma(n)$. We use $s(n)$ for the sum of the **aliquot parts** of n , i.e., the positive divisors of n other than n itself, so that $s(n) = \sigma(n) - n$. The number of distinct prime factors of n will be denoted by $\omega(n)$ and the total number, counting repetitions, by $\Omega(n)$.

Iteration of various arithmetic functions will be denoted, for example, by $s^k(n)$, which is defined by $s^0(n) = n$ and $s^{k+1}(n) = s(s^k(n))$ for $k \geq 0$.

We use the notation $d|n$ to mean that d divides n , and $e \nmid n$ to mean that e does not divide n . The notation $p^k \parallel n$ is used to imply that $p^k | n$ but $p^{k+1} \nmid n$. By $[m, n]$ we will mean the consecutive integers $m, m+1, \dots, n$.

B1 Perfect numbers.

A **perfect number** is one with $s(n) = n$. Euclid knew that $2^{p-1}(2^p - 1)$ was perfect if $2^p - 1$ is prime. For example, 6, 28, 496, ...; see the list of Mersenne primes in A3. Euler showed that these were the only even perfect numbers.

The existence or otherwise of odd perfect numbers is one of the more notorious unsolved problems of number theory. The lower bound for an odd perfect number has now been pushed to 10^{300} by Brent, Cohen & te Riele. Brandstein has shown that the largest prime factor is > 500000 and Hagis that the second largest is > 1000 . Cohen has shown that it contains a component (prime power divisor) $> 10^{20}$, and Sayers that there are at least 29 prime factors (not necessarily distinct).

Pomerance has shown that an odd perfect number with at most k distinct factors is less than

$$(4k)^{(4k)^{2^k}}$$

but Heath-Brown has much improved this by showing that if n is an odd number with $\sigma(n) = an$, then $n < (4d)^4$, where d is the denominator of a and k is the number of distinct prime factors of n . In particular, if n is an odd perfect number with k distinct prime factors, then $n < 4^4$.

B2 Almost perfect numbers.

John Leech asks for examples of spoof odd perfect numbers, like Descartes's

$$3^7 2^1 1^1 3^2 22021$$

which is perfect if you pretend that 22021 is prime.

For many earlier references, see the first edition of this book.

Michael S. Brandstein, New lower bound for a factor of an odd perfect number, #82T-10-240, *Abstracts Amer. Math. Soc.*, 3(1982) 257.

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B2 Almost perfect, quasi-perfect, pseudoperfect, harmonic, weird, multiperfect and hyperperfect numbers.

Perhaps because they were frustrated by their failure to disprove the existence of odd perfect numbers, numerous authors have defined a number of closely related concepts and produced a raft of problems, many of which seem no more tractable than the original.

For a perfect number, $\sigma(n) = 2n$. If $\sigma(n) < 2n$, n is called **deficient**. A problem in *Abacus* was to prove that every number $n > 3$ is the sum of two deficient numbers, or to find a number that was not. If $\sigma(n) > 2n$, then n is called **abundant**. If $\sigma(n) = 2n - 1$, n has been called **almost perfect**. Powers of 2 are almost perfect; it is not known if any other numbers are. If $\sigma(n) = 2n + 1$, n has been called **quasi-perfect**. Quasi-perfect numbers must be odd squares, but no one knows if there are any. Masao Kishore shows that $n > 10^{30}$ and that $\omega(n) \geq 6$. Hagis & Cohen have improved these results to $n > 10^{35}$ and $\omega(n) \geq 7$. Cattaneo originally claimed to

have proved that $3 \nmid n$, but Sierpiński and others have observed that his proof is fallacious. Kravitz, in a letter, makes a more general conjecture, that there are no numbers whose **abundance**, $\sigma(n) - 2n$, is an odd square. In this connexion Graeme Cohen writes that it is interesting that

$$\sigma(2^2 3^2 5^2) = 3(2^2 3^2 5^2) + 11^2$$

and that if $\sigma(n) = 2n + k^2$ with $n \perp k$, then $\omega(n) \geq 4$ and $n > 10^{20}$. He has also shown that if $k < 10^{10}$ then $\omega(n) \geq 6$, and that if $k < 44366047$ then n is primitive abundant (see below). Later, relaxing the condition $n \perp k$, he finds the solution

$$n = 2 \cdot 3^2 \cdot 238897^2, \quad k = 3^2 \cdot 23 \cdot 1999$$

and five solutions $n = 2^2 \cdot 7^2 \cdot p^2$, with

$$\begin{array}{cccccc} p = & 53 & 277 & 541 & 153941 & 358276277 \\ k = & 7 \cdot 29 & 5 \cdot 7 \cdot 23 & 5 \cdot 7 \cdot 43 & 5 \cdot 7 \cdot 103 \cdot 113 & 5 \cdot 7 \cdot 227 \cdot 229 \cdot 521 \end{array}$$

He verifies that the first of these last five is the smallest integer with odd square abundance. Sidney Kravitz has since sent two more solutions,

$$n = 2^3 \cdot 3^2 \cdot 1657^2, \quad k = 3 \cdot 11 \cdot 359,$$

$$n = 2^4 \cdot 31^2 \cdot 7992220179128893^2, \quad k = 44498798693247589.$$

In the latter, 31 divides k . Erdős asks for a characterization of the large numbers for which $|\sigma(n) - 2n| < C$ for some constant C . For example, $n = 2^m$; for other infinite families, see Makowski's two papers.

Wall, Crews & Johnson showed that the density of abundant numbers lies between 0.2441 and 0.2909. In an 83-08-17 letter Wall claimed to have narrowed these bounds to 0.24750 and 0.24893. Erdős asks if the density is irrational.

Sierpiński called a number **pseudoperfect** if it was the sum of *some* of its divisors; e.g., $20 = 1 + 4 + 5 + 10$. Erdős has shown that their density exists and says that presumably there are integers n which are not pseudoperfect, but for which $n = ab$ with a abundant and b having many prime factors; can b in fact have many factors $< a$?

For $n \geq 3$ Abbott lets $l = l(n)$ be the least integer for which there are n integers $1 \leq a_1 < a_2 < \dots < a_n = l$ such that $a_i | s = \sum a_i$ for each i (so that s is pseudoperfect). He can show that $l(n) > n^{c_1 \ln \ln n}$ for some $c_1 > 0$ and all $n \geq 3$ and that $l(n) < n^{c_2 \ln \ln n}$ for some $c_2 > 0$ and infinitely many n .

Call a number **primitive abundant** if it is abundant, but all its proper divisors are deficient, and **primitive pseudoperfect** if it is pseudoperfect, but none of its proper divisors are. If the harmonic mean of all the divisors of n is an integer, Pomerance called n a **harmonic number**.

B2. Almost perfect numbers.

A. & E. Zachariou call these "Ore numbers" and they call primitive pseudoperfect numbers "irreducible semiperfect". They note that every multiple of a pseudoperfect number is pseudoperfect and that the pseudoperfect numbers and the harmonic numbers both include the perfect numbers as a proper subset. The last result is due to Ore. All numbers $2^m p$ with $m \geq 1$ and p a prime between 2^m and 2^{m+1} are primitive pseudoperfect, but there are such numbers not of this form, e.g., 770. There are infinitely many primitive pseudoperfect numbers that are not harmonic numbers. The smallest odd primitive pseudoperfect number is 945. Erdős can show that the number of odd primitive pseudoperfect numbers is infinite.

García extended the list of harmonic numbers to include all 45 which are $< 10^7$, and he found more than 200 larger ones. The least one, apart from 1 and the perfect numbers, is 140. Are any of them squares, apart from 1? Are there infinitely many of them? If so, find upper and lower bounds on the number of them that are $< x$. Kanold has shown that their density is zero, and Pomerance that a harmonic number of the form $p^a q^b$ (p and q primes) is an even perfect number. If $n = p^a q^b r^c$ is harmonic, is it even?

Which values does the harmonic mean take? Presumably not 4, 12, 16, 18, 20, 22, ... ; does it take the value 23? Ore's own conjecture, that every harmonic number is even, implies that there are no odd perfect numbers!

Bateman, Erdős, Pomerance & Straus show that the set of n for which $\sigma(n)/d(n)$ is an integer has density 1, that the set for which $\sigma(n)/d(n)^2$ is an integer has density $\frac{1}{2}$, and that the number of rationals $r \leq x$ of the form $\sigma(n)/d(n)$ is $o(x)$. They ask for an asymptotic formula for

$$\frac{1}{x} \sum_{\substack{n \leq x \\ d(n) \mid \sigma(n)}} 1$$

where the sum is taken over those $n \leq x$ for which $d(n)$ does not divide $\sigma(n)$. They also note that the integers n for which $d(n)$ divides $s(n) = \sigma(n) - n$, have zero density, because for almost all n , $d(n)$ and $\sigma(n)$ are divisible by a high power of 2, while n is divisible only by a low power of 2.

Benkoski has called a number **weird** if it is abundant but not pseudoperfect. For example, 70 is not the sum of any subset of

$$1 + 2 + 5 + 7 + 10 + 14 + 35 = 74$$

There are 24 primitive weird numbers less than a million: 70, 836, 4030, 5830, 7192, ... Nonprimitive weird numbers include $70p$ with p prime and $p > \sigma(70) = 144$; $836p$ with $p = 421, 487, 491$, or p prime and ≥ 557 ; also 7192 · 31. Some large weird numbers were found by Kravitz, and Benkoski & Erdős showed that their density is positive. Here the open questions are: are there infinitely many primitive abundant numbers which are weird? Is every odd abundant number pseudoperfect (i.e., not weird)? Can $\sigma(n)/n$

be arbitrarily large for weird n ? Benkoski & Erdős conjecture “no” in answer to the last question and Erdős offers \$10 and \$25 respectively for solutions to the last two questions.

He also asks if there are extra-weird numbers n for which $\sigma(n) > 3n$, but n is not the sum of distinct divisors of n in two ways without repetitions. For example, 180 does not qualify, because although $\sigma(180) = 546$, $180=30+60+90$ and is the sum of all its other divisors except 6.

Numbers have been called **multiperfect**, **multiperfect** or **k -fold perfect** if $\sigma(n) = kn$ with k an integer. For example, ordinary perfect numbers are 2-fold perfect and 120 is 3-fold perfect. Dickson's *History* records a long interest in such numbers. Lehmer has remarked that if n is odd, then n is perfect just if $2n$ is triperfect.

Selfridge and others have observed that there are just six known 3-perfect numbers and they come from $2^h - 1$ for $h = 4, 6, 9, 10, 14, 15$. For example, the third one is illustrated by

$$\sigma(2^8 \cdot 7 \cdot 73 \cdot 37 \cdot 19 \cdot 5) = (2^9 - 1)(2^3)(37 \cdot 2)(19 \cdot 2)(5 \cdot 2^2)(2 \cdot 3).$$

It appears that there may be a similar explanation for the 36 known 4-perfect numbers, the last of which was published by Poulet as long ago as 1929.

For many years the largest known value of k was 8, for which Alan L. Brown gave three examples and Franqui & García two others. Stephen Gretton found numerous multiperfect numbers, including many fivefold, sixfold and sevenfold perfect numbers, and an eightfold, namely
 $2^{62} \cdot 3^{15} \cdot 5^9 \cdot 7^7 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23 \cdot 29 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43 \cdot 53 \cdot 61^2 \cdot 71^2 \cdot 73 \cdot 83 \cdot 89 \cdot 97^2 \cdot 127 \cdot 193 \cdot 283 \cdot 307 \cdot 317 \cdot 331 \cdot 337 \cdot 487 \cdot 521^2 \cdot 601 \cdot 1201 \cdot 1279 \cdot 2557 \cdot 3169 \cdot 5113 \cdot 92737 \cdot 649657$. This is believed to be the record for the smallest such.

In late 1992 and early 1993, half a dozen examples with $k = 9$ have already been found by Fred Helenius. The smallest is
 $2^{114} \cdot 3^{35} \cdot 5^{17} \cdot 7^{12} \cdot 11^4 \cdot 13^5 \cdot 17^3 \cdot 19^8 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 37^4 \cdot 41 \cdot 43 \cdot 47^2 \cdot 53 \cdot 61^2 \cdot 67 \cdot 71 \cdot 73 \cdot 79^2 \cdot 83^2 \cdot 89^2 \cdot 97 \cdot 103 \cdot 109 \cdot 127 \cdot 131^2 \cdot 151 \cdot 157 \cdot 167 \cdot 179^2 \cdot 197 \cdot 211 \cdot 227 \cdot 331 \cdot 347 \cdot 367 \cdot 379 \cdot 443 \cdot 523 \cdot 599 \cdot 709 \cdot 757 \cdot 829 \cdot 1151 \cdot 1699 \cdot 1789 \cdot 2003 \cdot 2179 \cdot 2999 \cdot 3221 \cdot 4271 \cdot 4357 \cdot 4603 \cdot 5167 \cdot 8011 \cdot 8647 \cdot 8713 \cdot 14951 \cdot 17293 \cdot 21467 \cdot 29989 \cdot 110563 \cdot 178481 \cdot 530713 \cdot 672827 \cdot 4036961 \cdot 218834597 \cdot 16148168401 \cdot 151871210317 \cdot 2646507710984041$

Can k be as large as we wish? Erdős conjectures that $k = o(\ln \ln n)$. It has even been suggested that there may be only finitely many k -perfect numbers with $k \geq 3$.

Rich Schroeppel is compiling as complete a list as possible of multiperfect numbers, and so would be a good checking point for those who believe that they have discovered new specimens. Since the previous three paragraphs were written, Shigeru Nakamura has drawn my attention to

the work of Motoji Yoshitake, who lists 3 5-perfect, 30 6-perfect, 35 7-perfect and 8 8-perfect numbers. $2+20+8+0$ of these are attributed to Carmichael, Mason or Cunningham. One of the 8-perfect numbers was given by Brown, and another can be derived from it by replacing $19^2 \cdot 127$ by $19^4 \cdot 151 \cdot 911$. This substitution, found by Cunningham in 1902, can be applied to the Carmichael-Mason table to give 50 multiperfect numbers. He also notes that Carmichael & Mason mistook 137561 = 151 · 911 and 485581 = 277 · 1753 for primes. In 1992 we knew of 700 k -perfect numbers with $k \geq 3$. In January, 1993, this number leapt to about 1150 from the discoveries of Fred Helenius which include 114 7-perfect, 327 8-perfect and two 9-perfect numbers. He continues to find dozens of new ones each month, so it is even more impossible to keep this section of the book up-to-date than it is to keep the rest; in March the total neared 1300; a postscript of a 93-09-08 letter from Schroepel gave 1526; by the time he mailed it next day it was 1605.

If n is an odd triperfect number, then McDaniel, Cohen, Kishore, Bugulov, Kishore, Cohen & Hagis, Reidlinger, and Kishore have respectively shown that $\omega(n) \geq 9, 9, 10, 11, 11, 11, 12$, and 12. Beck & Najar, Alexander, and Cohen & Hagis have shown that $n > 10^{50}, 10^{60}, 10^{70}$. Cohen & Hagis have shown that the largest prime factor of n is at least 100129 and that the second largest is at least 1009.

Shigeru Nakamura writes that Bugulov showed, in 1966, that odd k -perfect numbers contain at least ω distinct prime factors, where $(k, \omega) = (3, 11), (4, 21), (5, 54)$ [incorrectly stated in *MR 37* #5139 & *rNT A32-96*]. Nakamura claims to prove that for an even k -perfect number,

$$\omega > \max\{k^3/81 + \frac{5}{3}, k^5/2500 + 2.9, k^{10}/(14 \cdot 10^8) + 2.9999\}$$

and for an odd k -perfect number,

$$\omega > \max\{k^5/60 + \frac{47}{12}, k^5/50 - 20.8, 737k^{10}/10^9 + 11.5\}.$$

These improve the results of Cohen & Hendy and of Reidlinger; he also gives the improvements $(k, \omega) = (4, 23), (5, 56), (6, 142), (7, 373)$ to those of Bugulov.

Minoli & Bear say that n is **k -hyperperfect** if $n = 1 + k \sum d_i$, where the summation is taken over all proper divisors, $1 < d_i < n$, so that $k\sigma(n) = (k+1)n+k-1$. For example, 21, 2133 and 19521 are 2-hyperperfect and 325 is 3-hyperperfect. They conjecture that there are k -hyperperfect numbers for every k .

Ron Graham asks if $s(n) = \lfloor n/2 \rfloor$ implies that n is 2 or a power of 3.

Erdős lets $f(n)$ be the smallest integer for which $n = \sum_{i=1}^k d_i$ for some k , where $1 = d_1 < d_2 < \dots < d_k = f(n)$ is the increasing sequence of divisors of $f(n)$. Is $f(n) = o(n)$? Or is this true only for almost all n , with $\limsup f(n)/n = \infty$?

n 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28
 $f(n)$ 1 - 2 3 - 5 4 7 15 12 21 6 9 13 8 12 30 10 42 19 18 20 57 14 36 46 30 12

Erdős defines n_k to be the smallest integer for which if you partition the proper divisors of n_k into k classes, n_k will always be the sum of distinct divisors from the same class. Clearly $n_1 = 6$, but he is not even able to prove the existence of n_2 .

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B3 Unitary perfect numbers.

If d divides n and $d \perp n/d$, call d a **unitary divisor** of n . A number n which is the sum of its unitary divisors, apart from n itself, is a **unitary perfect number**. There are no odd unitary perfect numbers, and Subbarao conjectures that there are only a finite number of even ones. He, Carlitz & Erdős each offer \$10.00 for settling this question and Subbarao offers 10¢ for each new example. If $n = 2^a m$, where m is odd and has r distinct prime factors, then Subbarao and others have shown that, apart from $2 \cdot 3$, $2^2 \cdot 3 \cdot 5$, $2 \cdot 3^2 \cdot 5$ and $2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13$, there are no unitary perfect numbers with $a \leq 10$, or with $r \leq 6$. S. W. Graham has shown that the first and third are the only unitary perfect numbers of shape $2^a m$ with m odd and squarefree, and Jennifer DeBoer that the second is the only one of shape $2^a 3^2 m$ with $m \perp 6$ and squarefree.

Wall has found the unitary perfect number

$$2^{18} \cdot 3 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313$$

and shown that it is the fifth such. He can prove that any other unitary perfect number has an odd component greater than 2^{15} . Frey has shown that if $N = 2^m p_1^{a_1} \dots p_r^{a_r}$ is unitary perfect with $N \perp 3$, then $m > 144$, $r > 144$ and $N > 10^{440}$.

Peter Hagis investigates **unitary multiperfect numbers**: there are no odd ones. Write $\sigma^*(n)$ for the sum of the unitary divisors of n . If

$\sigma^*(n) = kn$ and n contains t distinct odd prime factors, then $k = 4$ or 6 implies $n > 10^{110}$, $t \geq 51$ and $2^{49}|n$; $k \geq 8$ implies $n > 10^{663}$ and $t \geq 247$; while k odd and $k \geq 5$ imply $n > 10^{461}$, $t \geq 166$ and $2^{166}|n$.

Cohen calls a divisor d of an integer n a **1-ary divisor** of n if $d \perp n/d$, and he calls d a **k -ary divisor** of n (for $k > 1$), and writes $d|_kn$, if the greatest common $(k-1)$ -ary divisor of d and n/d is 1 (written $d|(n/d)_{k-1} = 1$). In this notation $d|n$ and $d \parallel n$ are written $d|_0n$ and $d|_1n$. He also calls $p^\#$ an **infinitary divisor** of $p^\#(y > 0)$ if $p^\#|_{y-1}p^\#$. This gives rise to infinitary analogs of earlier concepts. Write $\sigma_\infty(n)$ for the sum of the infinitary divisors of n . He found 14 infinitary perfect numbers, i.e., with $\sigma_\infty(n) = kn$ and $k = 2$; 13 numbers with $k = 3$; 7 with $k = 4$; and two with $k = 5$. There are no odd ones, and he conjectures that there are no infinitary multiperfect numbers not divisible by 3.

Note that Suryanarayana (who also uses the term ‘ k -ary divisor’) and Alladi give different generalizations of unitary divisors.

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B4. Amicable numbers.

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B4 Amicable numbers.

Unequal numbers m, n are called **amicable** if each is the sum of the aliquot parts of the other, i.e., $\sigma(m) = \sigma(n) = m + n$. Several thousand such pairs are known. The smaller member, 220, of the smallest pair, occurs in *Genesis*, xxxii, 14, and amicable numbers intrigued the Greeks and Arabs and many others since. For their history see the articles of Lee & Madachy. The *Genesis* reference, from the King James Bible, is achieved by amalgamating 200 females and 20 males. Aviezri Fraenkel writes that in his Pentateuch, they occur at xxxii, 15, and gives the more convincing occurrences of 220 in *Ezra* viii, 20 and in *1 Chronicles* xv, 6; and of 284 in *Nehemiah* xi, 18. He notes that the three places are amicably related: all are connected to the tribe of Levi, whose name derives from the wish of Levi’s mother to be amicably related to his father (*Genesis* xxix, 34).

It is not known if there are infinitely many, but it is believed that there are. In fact Erdős conjectures that the number, $A(x)$, of such pairs with $m < n < x$ is at least $x^{1-\epsilon}$. He improved a result of Kanold to show that $A(x) = o(x)$ and his method can be used to obtain $A(x) \leq cx/\ln \ln \ln x$. Pomerance obtained the further improvement

$$A(x) \leq x \exp\{-c(\ln \ln x \ln \ln \ln \ln x)^{1/2}\}.$$

Erdős conjectured that $A(x) = o(x/(\ln x)^k)$ for every k whereupon Pomerance proved the stronger result

$$A(x) \leq x \exp\{-(\ln x)^{1/3}\}.$$

This implies that the sum of the reciprocals of the amicable numbers is finite, a fact not earlier known. He also notes that his proof can be modified to give the slightly stronger result

$$A(x) \ll x \exp\{-c(\ln x \ln \ln x)^{1/3}\}.$$

Herman te Riele has found all 1427 amicable pairs whose lesser members are less than 10^{10} . He remarks that the quantity $A(x)(\ln x)^3/x^{1/2}$ “remains

very close to 174.6", but I suspect that a much more powerful telescope would require the exponent 1/2 to be increased much nearer to 1. Moews & Moews have continued the complete search to beyond $2 \cdot 10^{11}$. Through the efforts of Battiatto and others, more than 40,000 pairs are known.

Some very large amicable pairs, with 32, 40, 81 and 152 decimal digits, discovered by te Riele, are mentioned by Kaplansky under "Mathematics" in the 1975 *Encyclopedia Britannica Yearbook*. The largest previously known had 25 decimal digits. More recently te Riele has constructed, from a "mother" list of 92 known amicable pairs, more than 2000 new pairs of sizes up to 38 decimal digits, and five pairs with from 239 to 282 digits. The largest amicable pair known in mid-1993 has 1041 decimal digits:

$$(2^9 p^{20} q_1 r s t u, 2^9 p^{20} q_2 v)$$

with $p = 5661346302015448219060051$; q_1, q_2 of shape $b c^{20} - 1$ with $b_1 = 5797874207198307252124352$, $b_2 = 5531348900141215019827200$, $c = 5661346302015448219060051$; and $r = 569$, $s = 5039$, $t = 1479911$, $u = 30636732851$; and $v = 136527918704382506064301$. It was found by Holger Wiethaus, a student at Dortmund in July 1988.

Elvin J. Lee has given half a dozen rules for amicable pairs of type $(2^n p q, 2^n r s)$ where p, q, r, s are primes of appropriate shape. E.g.,

$$p = 3 \cdot 2^{n-1} - 1, \quad q = 35 \cdot 2^{n+1} - 29, \quad r = 7 \cdot 2^{n-1} - 1, \quad s = 15 \cdot 2^{n+1} - 13,$$

but the simultaneous discovery of four such primes is a rare event.

Borho, Hoffman & te Riele have made considerable advances, both with proliferation of generalized Thabit rules, and with actual computation. Of the 1427 amicable pairs mentioned above, all but 17 have $m+n \equiv 0 \pmod{9}$. The smallest exception is Poulet's pair

$$2^4 \cdot 331 \cdot \begin{cases} 19 \cdot 6619 \\ 199 \cdot 661 \end{cases}$$

with $m+n \equiv 5 \pmod{9}$: te Riele gives the first examples

$$2^4 \cdot \begin{cases} 19^2 \cdot 103 \cdot 1627 \\ 3847 \cdot 16763 \end{cases} \quad \text{and} \quad 2^2 \cdot 19 \cdot \begin{cases} 13^2 \cdot 37 \cdot 43 \cdot 139 \\ 41 \cdot 151 \cdot 6709 \end{cases}$$

with m, n even, $m+n \equiv 3 \pmod{9}$.

It is not known if an amicable pair exists with m and n of opposite parity, or with $m \perp n$. Bratley & McKay conjectured that both members of all odd amicable pairs are divisible by 3, but Battiatto & Borho produced 15 counterexamples with from 36 to 73 decimal digits. In an 87-05-15 letter te Riele announced a 33-digit specimen

$$5 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19^3 \cdot 23 \cdot 37 \cdot 181 \left\{ \begin{array}{l} 101 \cdot 8643 \cdot 1947938229 \\ 365147 \cdot 47303071129 \end{array} \right.$$

B4. Amicable numbers.

Is this the smallest such pair? Is there an odd amicable pair with one member, but not both, divisible by 3?

An old conjecture of Charles Wall is that odd amicable pairs must be incongruent modulo 4. Its truth, he says, implies that there are no odd perfect numbers, so it may be more prudent to look for a counterexample than to try to prove it.

On p. 169 of *Mathematical Magic Show*, Vintage Books, 1978, Martin Gardner makes a conjecture about the digital roots of amicable numbers. Lee confirms this in part by showing that if $(2^n p q, 2^m s t u)$ is an amicable pair whose sum is not divisible by 9, then each number is congruent to 7, modulo 9.

Unitary amicable numbers have been studied by Peter Hagis and by Mariano García, who list 82 pairs.

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B5 Quasi-amicable or betrothed numbers.

García has called a pair of numbers (m, n) , $m < n$, **quasi-amicable** if

$$\sigma(m) = \sigma(n) = m + n + 1.$$

For example, (48,75), (140,195), (1575,1648), (1050,1925) and (2024,2295). Rufus Isaacs, noting that each of m and n is the sum of the *proper* divisors of the other (i.e., omitting 1 as well as the number itself) has much more appropriately named them **betrothed numbers**.

Mąkowski gave examples of betrothed numbers and also of **amicable triples**

$$\sigma(a) = \sigma(b) = \sigma(c) = a + b + c,$$

e.g., $2^2 3^{25} \cdot 11$, $2^5 3^7$, $2^2 3^{27} 1$. Similarly, in a 92-07-20 letter, Yasutoshi Kohmoto calls the set $\{a, b, c, d\}$ **quadr-amicable** if

$$\sigma(a) = \sigma(b) = \sigma(c) = \sigma(d) = a + b + c + d.$$

As examples which are not multiples of 3 he gives

$$a = x \cdot 173 \cdot 1933058921 \cdot 149 \cdot 103540742849 \quad b = x \cdot 173 \cdot 1933058921 \cdot 15531111427499$$

$$c = x \cdot 336352252427 \cdot 149 \cdot 103540742849 \quad d = x \cdot 336352252427 \cdot 15531111427499$$

where x is the product of

$$5^9 \cdot 7^2 \cdot 11^4 \cdot 17^2 \cdot 19 \cdot 29^2 \cdot 67 \cdot 71^2 \cdot 109 \cdot 131 \cdot 139 \cdot 179 \cdot 307 \cdot 431 \cdot 521 \cdot 653 \\ \cdot 1019 \cdot 1279 \cdot 2557 \cdot 3221 \cdot 5113 \cdot 5171 \cdot 6949$$

with a perfect number $2^{p-1}M_p$, $M_p = 2^p - 1$ being a Mersenne prime (see A3) with $p > 3$.

Hagis & Lord have found all 46 pairs of betrothed numbers with $m < 10^7$. All of them are of opposite parity. No pairs are known with m, n having the same parity. If there are such, then $m > 10^{10}$. If $m \perp n$, then mn contains at least four distinct prime factors, and if mn is odd, then mn contains at least 21 distinct prime factors.

Beck & Najar call such pairs *reduced amicable pairs*, and call numbers m, n such that

$$\sigma(m) = \sigma(n) = m + n - 1$$

augmented amicable pairs. They found 11 augmented amicable pairs. They found no reduced or augmented *unitary amicable* or *sociable numbers* (see B8) with $n < 10^5$.

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B6 Aliquot sequences.

Since some numbers are abundant and some deficient, it is natural to ask what happens when you iterate the function $s(n) = \sigma(n) - n$ and produce an **aliquot sequence**, $\{s^k(n)\}$, $k = 0, 1, 2, \dots$. Catalan and Dickson conjectured that all such sequences were bounded, but we now have heuristic arguments and experimental evidence that some sequences, perhaps almost all of those with n even, go to infinity. The smallest n for which there was ever doubt was 138, but D. H. Lehmer eventually showed that after reaching a maximum

$$s^{117}(138) = 179931895322 = 2 \cdot 61 \cdot 929 \cdot 1587569$$

the sequence terminated at $s^{117}(138) = 1$. The next value for which there continues to be real doubt is 276. A good deal of computation by Lehmer, subsequently assisted by Godwin, Selfridge, Wunderlich and others, pushed the calculation as far as $s^{469}(276)$, which was quoted in the first edition. Thomas Struppeck factored this term and computed two more iterates. Andy Guy wrote a PARI program which started from scratch and overnight verified all the earlier calculations and reached $s^{487}(276)$.

The first few sequences whose fate was unknown are the "Lehmer six" starting from 276, 552, 564, 660, 840 and 966. Our program has shown that the 840 sequence hit the prime $s^{746}(840) = 601$ having established a new record

$$\begin{aligned} s^{287}(840) &= 3463982260143725017429794136098072146586526240388 \\ &= 2^2 \cdot 64970467217 \cdot 6237379309797547 \cdot 2136965558478112990003 \end{aligned}$$

for the maximum of a terminating sequence. This has recently been beaten by Mitchell Dickerman who found that the 1248 sequence has length 1075 after reaching a maximum $s^{583}(1248) =$

$$1231636691923602991963829388638861714770651073275257065104 = 2^{4p}$$

of 58 digits. He has pursued the 276 sequence to its 628th term, which has

B6. Aliquot sequences.

65 decimal digits. Godwin investigated the fourteen main sequences starting between 1000 and 2000 whose outcome was unknown and discovered that the sequence 1848 terminated. We have found that those for 2580, 2850, 4488, 4830, 6792, 7752, 8862 and 9540 also terminate.

H. W. Lenstra has proved that it is possible to construct arbitrarily long monotonic increasing aliquot sequences. See the quadruple paper cited under **B41**. The last of the following references has a bibliography of 60 items concerning the iteration of number-theoretic functions.

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B7 Aliquot cycles or sociable numbers.

Poulet discovered two cycles of numbers, showing that $s^k(n)$ can have the periods 5 and 28, in addition to 1 and 2. For $k \equiv 0, 1, 2, 3, 4 \pmod{5}$, $s^k(12496)$ takes the values

$$\begin{aligned} 12496 &= 2^4 \cdot 11 \cdot 71, \quad 14288 = 2^4 \cdot 19 \cdot 47, \quad 15472 = 2^4 \cdot 967, \\ 14536 &= 2^3 \cdot 23 \cdot 79, \quad 14264 = 2^3 \cdot 1783. \end{aligned}$$

For $k \equiv 0, 1, \dots, 27 \pmod{28}$, $s^k(14316)$ takes the values

14316	19116	31704	47616	83328	177792	295488
629072	589786	294896	358336	418904	366556	274924
275444	243760	376736	381028	285778	152990	122410
97946	48976	45946	22976	22744	19916	17716

After a gap of over 50 years, and the advent of high-speed computing, Henri Cohen discovered nine cycles of period 4, and Borho, David and Root also discovered some. Recently Moews & Moews have made an exhaustive search for such cycles with greatest member less than 10^{10} . There are twenty-four: their smallest members are

1264460	7169104	46722700	330003580	2387776550	4424606020
2115324	18048976	81128632	498215416	2717495235	4823923384
2784580	18656380	174277820	1236402232	2879697304	5373457070
4938136	28158165	209524210	1799281330	3705771825	8653956136

Moews & Moews give five larger 4-cycles, and, in a 90-09-01 letter, another whose least member is:

$$2^6 \cdot 79 \cdot 1913 \cdot 226691 \cdot 207722852483$$

They also found an 8-cycle:

1095447416	1259477224	1156962296	1330251784
1221976136	1127671864	1245926216	1213138984

Ren Yuanhua had already found three of the 4-cycles and Achim Flammenkamp had also found many of them, as well as a second 8-cycle:

1276254780	2299401444	3071310364	2303482780
2629903076	2209210588	2223459332	1697298124

B8 Unitary aliquot sequences.

and a 9-cycle:

805984760	1268997640	1803863720	2308845400	3059220620
3367978564	2525983930	2301481286	1611969514	

Moews & Moews have continued their exhaustive search to uncover all cycles, of any length, whose member preceding the largest member is less than $3.6 \cdot 10^{10}$. There are three more 4-cycles, with least members

15837081520,	171616303220,	21669628904,
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and a 6-cycle, all of whose members are odd:

21548919483	$= 3^5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 19 \cdot 431$,	23625285957	$= 3^5 \cdot 7^2 \cdot 13 \cdot 19 \cdot 29 \cdot 277$,
24825443643	$= 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19 \cdot 20719$,	26762383557	$= 3^4 \cdot 7^2 \cdot 13 \cdot 19 \cdot 27299$,
25958284443	$= 3^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 167 \cdot 1427$,	2381697477	$= 3^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 218651$.

It has been conjectured that there are no 3-cycles. On the other hand it has been conjectured that for each k there are infinitely many k -cycles.

Walter Borho, Über die Fixpunkte der k -fach iterierten Teilersummenfunktion, *Mitt. Math. Gesellsch. Hamburg*, 9(1969) 34–48; MR 40 #7189.

Achim Flammenkamp, New sociable numbers, *Math. Comput.*, 56(1991) 871–873.

David Moews & Paul C. Moews, A search for aliquot cycles below 10^{10} , *Math. Comput.*, 57(1991) 849–855; MR 92e:11151.

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B8 Unitary aliquot sequences.

The ideas of aliquot sequence and aliquot cycle can be adapted to the case where only the **unitary** divisors are summed, leading to **unitary aliquot sequences** and **unitary sociable numbers**. We use $\sigma^*(n)$ and $s^*(n)$ for the analogs of $\sigma(n)$ and $s(n)$ when just the unitary divisors are summed (compare B3).

Are there unbounded unitary aliquot sequences? Here the balance is more delicate than in the ordinary aliquot sequence case. The only sequences which deserve serious consideration are those involving odd multiples of 6, which is a unitary perfect number as well as an ordinary one. Now the sequences tend to increase if $3|n$, but decrease when a higher power of 3 is present, and it is a moot point as to which situation will dominate. Once a term of a sequence is $6m$, with m odd, then $\sigma^*(6m)$ is an even multiple of 6, making $s^*(6m)$ an odd multiple of 6 again, except in the extremely rare case that m is 4 raised to an odd power.

te Riele pursued all unitary aliquot sequences for $n < 10^6$. The only one which did not terminate or become periodic was 89610. Later calculations

showed that this reached a maximum,

$$645\,856907\,610421\,353834 = 2 \cdot 3^2 \cdot 13 \cdot 19 \cdot 73 \cdot 653 \cdot 3047409443791$$

at its 568th term, and terminated at its 1129th.

One can hardly expect typical behavior until the expected number of prime factors is large. Since this number is $\ln \ln n$, such sequences are well beyond computer range. Of 80 sequences examined near 10^{12} , all have terminated or become periodic. One sequence exceeded 10^{23} .

Unitary amicable pairs and unitary sociable numbers may occur rather more frequently than their ordinary counterparts. Lal, Tiller & Summers found cycles of periods 1, 2, 3, 4, 5, 6, 14, 25, 39 and 65. Examples of unitary amicable pairs are (56430, 64530) and (1080150, 1291050), while (30, 42, 54) is a 3-cycle and

$$(1482, 1878, 1890, 2142, 2178)$$

is a 5-cycle.

Cohen (see B3 for definitions and a reference) finds 62 infinitary amicable pairs with smaller member less than a million, eight infinitary aliquot cycles of order 4 and three of order 6. The only other such cycle of order less than 17 and least member less than a million is of order 11:

$$\begin{aligned} 448800, 696864, 1124448, 1651584, 3636096, 6608784, \\ 5729136, 3736464, 2187696, 1572432, 895152. \end{aligned}$$

A type of aliquot sequence which can be unbounded has been suggested by David Penney & Carl Pomerance and is based on Dedekind's function: see B41.

Erdős, looking for a number-theoretic function whose iterates might be bounded, suggested defining $w(n) = n \sum 1/p_i^{\alpha_i}$ where $n = \prod p_i^{\alpha_i}$, and $W^k(n) = w(w^{k-1}(n))$. Note that $w(n) \perp n$. Can it be proved that $w^k(n)$, $k = 1, 2, \dots$, is bounded? Is $\{|w(n) : 1 \leq n \leq x|\} = o(x)$?

Erdős & Selfridge called a **barrier** for number-theoretic function $f(m)$ if, for all $m < n$, $m + f(m) \leq n$. Euler's ϕ -function (see B36) and $\sigma(m)$ increase too fast to have barriers, but does $\omega(m)$ have infinitely many barriers? The numbers 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 17, 18, 20, 24, 26, 28, 30, ..., are barriers for $\omega(m)$. Does $\Omega(m)$ have infinitely many barriers? Selfridge observes that 99840 is the largest barrier for $\Omega(m)$ that is $< 10^5$. Mąkowski observes that $n = 1$ is a barrier for every function, and that 2 is a barrier for every function $f(n)$ with $f(1) = 1$; in particular for $d(m)$, the number of divisors of m . The inequality

$$\max\{d(n-1) + n - 1, d(n-2) + n - 2\} \geq n + 2$$

holds for $n \geq 7$, but not for $n = 6$. But $d(n-1) + n - 1 \geq n + 1$ for $n \geq 3$, so $d(m)$ has no barriers ≥ 3 . Does

$$\max_{m < n} (m + d(m)) = n + 2$$

have infinitely many solutions? It is very doubtful. One solution is $n = 24$; the next larger is probably beyond computer range.

Paul Erdős, A mélange of simply posed conjectures with frustratingly elusive solutions, *Math. Mag.*, 52(1979) 67–70.

P. Erdős, Problems and results in number theory and graph theory, *Congressus Numerantium* 27, Proc. 9th Manitoba Conf. Numerical Math. Comput., 1979, 3–21.

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C. R. Wall, Topics related to the sum of unitary divisors of an integer, PhD thesis, Univ. of Tennessee, 1970.

B9 Superperfect numbers.

Suryanarayana defines **superperfect numbers** n by $\sigma^2(n) = 2n$, i.e., $\sigma(\sigma(n)) = 2n$. He and Kanold show that the even ones are just the numbers 2^{p-1} where $2^p - 1$ is a Mersenne prime. Are there any odd superperfect numbers? If so, Kanold shows that they are perfect squares, and Dandapat and others that n or $\sigma(n)$ is divisible by at least three distinct primes.

More generally, Bode defines **m -superperfect numbers** as numbers n for which $\sigma^m(n) = 2n$, and shows that for $m \geq 3$ there are no even m -superperfect numbers. He also shows that for $m = 2$ there is no superperfect number $< 10^{10}$. Hunsucker & Pomerance have raised this bound to 7×10^{24} and have unpublished results on the numbers of distinct prime factors of n and of $\sigma(n)$ if n is superperfect.

If $\sigma^2(n) = 2n+1$, it would be consistent with earlier terminology to call n quasi-superperfect. The Mersenne primes are such. Are there others? Are there “almost superperfect numbers” for which $\sigma^2(n) = 2n - 1$?

Erdős asks if $(\sigma^k(n))^{1/k}$ has a limit as $k \rightarrow \infty$. He conjectures that it is infinite for each $n > 1$.

Schinzel asks if $\liminf \sigma^k(n)/n < \infty$ for each k , as $n \rightarrow \infty$, and observes that it follows for $k = 2$ from a deep theorem of Rényi. Mąkowski & Schinzel give an elementary proof for $k = 2$ that the limit is 1. Helmut Maier has used sieve methods to prove the result for $k = 3$.

Dieter Bode, Über eine Verallgemeinerung der volkommnenen Zahlen, Dissertation, Braunschweig, 1971.

P. Erdős, Some remarks on the iterates of the ϕ and σ functions, *Colloq. Math.*, 17(1967) 195–202.

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A. Mąkowski & A. Schinzel, On the functions $\phi(n)$ and $\sigma(n)$, *Colloq. Math.*, 13(1964–65) 95–99.

A. Schinzel, Ungelöste Probleme Nr. 30, *Elem. Math.*, 14(1959) 60–61.

D. Suryanarayana, Super perfect numbers, *Elem. Math.*, 24(1969) 16–17; MR 39 #5706.

D. Suryanarayana, There is no superperfect number of the form p^{2a} , *Elem. Math.*, 28(1973) 148–150; MR 48 #8374.

B10 Untouchable numbers.

Erdős has proved that there are infinitely many n such that $s(x) = n$ has no solution. Alanel calls such n **untouchable**. In fact Erdős shows that the untouchable numbers have positive lower density. Here are the untouchable numbers less than 1000:

2	5	52	88	96	120	124	146	162	178	188	206	210	216	238	246
248	262	268	276	288	290	292	304	306	322	324	326	336	342	372	406
408	426	430	448	472	474	498	516	518	520	530	540	552	556	562	576
584	612	624	626	628	658	668	670	714	718	726	732	738	748	750	756
766	768	782	784	792	802	804	818	836	848	852	872	892	894	896	898
902	916	926	936	964	966	976	982	996							

In view of the plausibility of the Goldbach conjecture (C1), it seems likely that 5 is the only odd untouchable number since if $2n+1 = p+q+1$ with p and q prime, then $s(pq) = 2n+1$. Can this be proved independently?

B12 Analogs with $d(n)$, $\sigma_k(n)$.

Are there arbitrarily long sequences of consecutive even numbers which are untouchable? How large can the gaps between untouchable numbers be?

P. Erdős, Über die Zahlen der Form $\sigma(n) - n$ und $n - \phi(n)$, *Elem. Math.*, 28(1973) 83–86; MR 49 #2502.

Paul Erdős, Some unconventional problems in number theory, *Astérisque*, 61 (1979) 73–82; MR 81h:10001.

B11 Solutions of $m\sigma(m) = n\sigma(n)$.

Leo Moser has observed that while $n\phi(n)$ determines n uniquely, $n\sigma(n)$ does not. $\{\phi(n)\}$ is Euler's totient function; see B36.] For example, $m\sigma(m) = n\sigma(n)$ for $m = 12$, $n = 14$. The multiplicativity of $\sigma(n)$ now ensures an infinity of solutions, $m = 12q$, $n = 14q$, where $q \perp 42$. So Moser asked if there is an infinity of primitive solutions, in the sense that (m^*, n^*) is not a solution for any $m^* = m/d$, $n^* = n/d$, $d > 1$. The example we've given is the least of the set $m = 2p^{-1}(2^q - 1)$, $n = 2^q(2^p - 1)$, where $2^p - 1$, $2^q - 1$ are distinct Mersenne primes, so that only a finite number of such solutions is known. Another set of solutions is $m = 2^7 \cdot 3^4 \cdot 5^2 \cdot (2^p - 1)$, $n = 2^{p-1} \cdot 5^3 \cdot 17 \cdot 31$, where 2^{p-1} is a Mersenne prime other than 3 or 31; also $p = 5$ gives a primitive solution on deletion of the common factor 31. There are other solutions, such as $m = 2^4 \cdot 3 \cdot 5^3 \cdot 7$, $n = 2^{11} \cdot 5^2$ and $m = 2^9 \cdot 5$, $n = 2^3 \cdot 11 \cdot 31$. An example with $m \perp n$ is $m = 2^6 \cdot 5$, $n = 3^3 \cdot 7$. If $m\sigma(m) = n\sigma(n)$, is m/n bounded?

Erdős observes that if n is squarefree, then integers of the form $n\sigma(n)$ are distinct. He can also prove that the number of solutions of $m\sigma(m) = n\sigma(n)$ with $m < n < x$ is $cx + o(x)$. In answer to the question, are there three distinct numbers l , m , n such that $l\sigma(l) = m\sigma(m) = n\sigma(n)$, Mąkowski observes that for distinct Mersenne primes M_{p_i} , $1 \leq i \leq s$, we have $n_i\sigma(n_i)$ is constant for $n_i = A/M_{p_i}$, where $A = \prod_{j=1}^s M_{p_j}$. Is there an infinity of primitive solutions of the equation $\sigma(a)/a = \sigma(b)/b$? Without restricting the solutions to being primitive, Erdős can show that their number with $a < b < x$ is at least $cx + o(x)$; with the restriction $a \perp b$, no solution is known at all.

Erdős believes that the number of solutions of $x\sigma(x) = n$ is less than $n^{\epsilon}/\ln n$ for every $\epsilon > 0$, and says that the number may be less than $(\ln n)^c$.

P. Erdős, Remarks on number theory II: some problems on the σ function, *Acta Arith.*, 5(1959) 171–177; MR 21 #6348.

B12 Analogs with $d(n)$, $\sigma_k(n)$.

Analogous questions may be asked with $\sigma_k(n)$ in place of $\sigma(n)$, where $\sigma_k(n)$ is the sum of the k -th powers of the divisors of n . For example, are there

distinct numbers m and n such that $m\sigma_2(m) = n\sigma_2(n)$? For $k = 0$ we have $md(m) = nd(n)$ for $(m, n) = (18, 27), (24, 32), (56, 64)$ and $(192, 224)$. The last pair can be supplemented by 168 to give three distinct numbers such that $ld(l) = md(m) = nd(n)$. There are primitive solutions (m, n) of shape

$$m = 2^{pt-1}p, \quad n = 2^{pt-2^{tu}-1}$$

where p and $q = u + p \cdot 2^{tu}$ are primes, but it does not immediately follow that these are infinitely numerous. Many other solutions can be constructed; for example $(2^{70}, 2^{63} \cdot 71), (3^{10}, 3^{17} \cdot 5)$ and $(5^{51}, 5^{49} \cdot 13)$.

Bencze proves the inequalities

$$\frac{n^k + 1}{2} \geq \frac{\sigma_k(n)}{\sigma_{k-l}(n)} \geq \sqrt{n^l}$$

for $0 \leq l \leq k$ and gives no fewer than 60 applications.

Mihály Bencze, A contest problem and its application (Hungarian), *Mat. Lapok Ifjúsági Folyóirat (Románia)*, **91**(1986) 179–186.

B13 Solutions of $\sigma(n) = \sigma(n+1)$.

Sierpiński has asked if $\sigma(n) = \sigma(n+1)$ infinitely often. Hunsucker, Nebb & Stearns extended the tabulations of Mąkowski and of Mientka & Vogt and have found just 113 solutions

$$14, 206, 957, 1334, 1364, 1634, 2685, 2974, 4364, \dots$$

less than 10^7 . They also obtain statistics concerning the equation $\sigma(n) = \sigma(n+l)$, of which Mientka & Vogt had asked: for what l (if any) is there an infinity of solutions? They found many solutions if l is a factorial, but only two solutions for $l = 15$ and $l = 69$. They also ask whether, for each l and m , there is an n such that $\sigma(n) + m = \sigma(n+l)$.

One can ask corresponding questions for $\sigma_k(n)$, the sum of the k -th powers of the divisors of n . [For $k = 0$, see B15.] The only solution of $\sigma_2(n) = \sigma_2(n+1)$ is $n = 6$, since $\sigma_2(2n) > \sigma_2(2n+1)$ for $n > 7$ and $\sigma_2(2n) > 5n^2 > (\pi^2/8)(2n-1)^2 > \sigma_2(2n-1)$. Note that $\sigma_2(24) = \sigma_2(26)$; Erdős doubts that $\sigma_2(n) = \sigma_2(n+2)$ has infinitely many solutions, and thinks that $\sigma_3(n) = \sigma_3(n+2)$ has no solutions at all.

Richard K. Guy & Daniel Shanks, A constructed solution of $\sigma(n) = \sigma(n+1)$, *Fibonacci Quart.*, **12**(1974) 299; MR 50 #219.

John L. Hunsucker, Jack Nebb & Robert E. Stearns, Computational results concerning some equations involving $\sigma(n)$, *Math. Student*, **41**(1973) 285–289.

W. E. Mientka & R. L. Vogt, Computational results relating to problems concerning $\sigma(n)$, *Mat. Vesnik*, **7**(1970) 35–36.

B15. Solutions of $\sigma(q) + \sigma(r) = \sigma(q+r)$.

69

B14 Some irrational series.

Is $\sum_{n=1}^{\infty} (\sigma_k(n)/n!)$ irrational? It is for $k = 1$ and 2. Erdős established the irrationality of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1} = \sum_{n=1}^{\infty} \frac{d(n)}{2^n}$$

and Peter Borwein showed that

$$\sum_{n=1}^{\infty} \frac{1}{q^n + r} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{q^n + r}$$

are irrational if q is an integer other than 0, ± 1 and r is a rational other than 0 or $-q^n$.

Peter B. Borwein, On the irrationality of $\sum 1/(q^n + r)$, *J. Number Theory*, **37**(1991) 253–259.

Peter B. Borwein, On the irrationality of certain series, *Math. Proc. Cambridge Philos. Soc.*, **112**(1992) 141–146; MR 93g:11074.

P. Erdős, On arithmetical properties of Lambert series, *J. Indian Math. Soc. (N.S.)*, **12**(1948) 63–66.

P. Erdős, On the irrationality of certain series: problems and results, in *New Advances in Transcendence Theory*, Cambridge Univ. Press, 1988, pp. 102–109.

P. Erdős & M. Kac, Problem 4518, *Amer. Math. Monthly*, **60**(1953) 47. Solution R. Breusch, **61**(1954) 264–265.

B15 Solutions of $\sigma(q) + \sigma(r) = \sigma(q+r)$.

Max Rumney (*Eureka*, **26**(1963) 12) asked if the equation $\sigma(q) + \sigma(r) = \sigma(q+r)$ has infinitely many solutions which are primitive in a sense similar to that used in B11. If $q+r$ is prime, the only solution is $(q, r) = (1, 2)$. If $q+r = p^2$ where p is prime, then one of q and r , say q , is prime, and $r = 2^nk^2$ where $n \geq 1$ and k is odd. If $k = 1$, there is a solution if $p = 2^n - 1$ is a Mersenne prime and $q = p^2 - 2^n$ is prime; this is so for $n = 2, 3, 5, 7, 13$ and 19. For $k = 3$ there are no solutions, and none for $k = 5$ with $n < 189$. For $k = 7$, $n = 1$ and 3 give $(q, r, q+r) = (5231, 2 \cdot 7^2, 73^2)$ and $(213977, 2^3 \cdot 7^2, 463^2)$. Other solutions are $(k, n) = (11, 1), (11, 3), (19, 5), (25, 1), (25, 9), (49, 9), (53, 1), (97, 5), (107, 5), (131, 5), (137, 1), (149, 5), (257, 5), (277, 1), (313, 3)$ and $(421, 3)$. Solutions with $q+r = p^3$ and p prime are $\sigma(2) + \sigma(6) = \sigma(8)$ and

$$\sigma(11638687) + \sigma(2^2 \cdot 13 \cdot 1123) = \sigma(227^3)$$

Erdős asks how many solutions (not necessarily primitive) are there with $q+r < x$; is it $cx + o(x)$ or is it of higher order? If $s_1 < s_2 < \dots$ are

the numbers for which $\sigma(s_i) = \sigma(q) + \sigma(s_i - q)$ has a solution with $q < s_i$, what is the density of the sequence $\{s_i\}$?

M. Sugunamma, PhD thesis, Sri Venkataswara Univ., 1969.

B16 Powerful numbers.

Erdős & Szekeres studied numbers n such that if a prime p divides n , then p^i divides n where i is a given number greater than one. Golomb named these numbers **powerful** and exhibited infinitely many pairs of consecutive ones. In answer to his conjecture that 6 was not representable as the difference of two powerful numbers, Władysław Narkiewicz noted that $6 = 5^4 7^3 - 463^2$, and that there were infinitely many such representations. In fact in 1971 Richard P. Stanley (unpublished) used the theory of the Pell equation to show that every non-zero integer is the difference between two powerful numbers and that 1 is the difference between two non-square powerful numbers, each in infinitely many ways.

Erdős denotes by $u_1^{(k)} < u_2^{(k)} < \dots$ the integers all of whose prime factors have exponents $\geq k$; sometimes called **k -ful numbers**. He asks if the equation $u_{i+1}^{(2)} - u_i^{(2)} = 1$ has infinitely many solutions which do not come from Pell equations $x^2 - dy^2 = \pm 1$. Is there a constant c , such that the number of solutions with $u_i < x$ is less than $(\ln x)^c$? Does $u_{i+1}^{(3)} - u_i^{(3)} = 1$ have no solutions? Do the equations $u_{i+2}^{(2)} - u_{i+1}^{(2)} = 1$, $u_{i+1}^{(2)} - u_i^{(2)} = 1$ have no simultaneous solutions? And several other questions, some of which have been answered by Mąkowski.

For example, Mąkowski notes that $7^3 x^2 - 3^3 y^2 = 1$ has infinitely many solutions, and that this is not usually counted as a Pell equation. He also notes that

$$(2^{k+1} - 1)^k, \quad 2^k(2k+1-1)^k \quad \text{and} \quad (2^{k+1} - 1)^{k+1}$$

are k -ful numbers in A.P., and that if a_1, a_2, \dots, a_s are k -ful and in A.P. with common difference d then

$$a_1(a_s + d)^k, \quad a_2(a_s + d)^k, \quad \dots, \quad a_s(a_s + d)^k, \quad (a_s + d)^{k+1}$$

are $s+1$ such numbers. As

$$a^k(a^l + \dots + 1)^k + a^{k+1}(a^l + \dots + 1)^k + \dots + a^{k+l}(a^l + \dots + 1)^k = a^k(a^l + \dots + 1)^{k+1},$$

the sum of $l+1$ k -ful numbers can be k -ful. He says that these last two questions become difficult when we require that the numbers be relatively prime. However, Nitaj constructs three infinite families of solutions of $x + y = z$ in relatively prime 3-full numbers. A specific example is

$$17^3 \cdot 106219^3 + 2^7 \cdot 3^4 \cdot 5^3 \cdot 7^3 \cdot 2287^3 = 37^3 \cdot 197^3 \cdot 307^3$$

B16. Powerful numbers.

Heath-Brown has shown that every sufficiently large number is the sum of three powerful numbers; his proof would be much shortened if his conjecture could be proved that the quadratic form $x^2 + y^2 + 125z^2$ represents every sufficiently large $n \equiv 7 \pmod{8}$. Erdős suggested that this may follow from work of Duke and Iwaniec: in fact see the forthcoming paper by Moroz.

Are there only finitely many powerful numbers n such that $n^2 - 1$ is also powerful? (See the Granville reference at D2.)

Gerry Myerson notes that the following conjecture is still open. If p is an odd prime and u, v are the smallest positive integers such that $u^2 - pv^2 = 1$, then

$$l \quad p \nmid v \quad ?$$

This has been verified for $p \equiv 1 \pmod{4}$, $p < 6270713$ and for $p \equiv -1 \pmod{4}$, $p < 7679299$. The conjecture is false if p is not prime; Myerson believes that 46 and 430 are the two smallest counterexamples.

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B18. Solutions of $d(n) = d(n+1)$.

Yuan Ping-Zhi, On a conjecture of Golomb on powerful numbers (Chinese. English summary), *J. Math. Res. Exposition*, 9(1989) 453–456; MR 91c:11009.

B17. Exponential-perfect numbers.

If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, then Straus & Subbarao call d an **exponential divisor** (**e-divisor**) of n if $d|n$ and $d = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r}$ where $b_j|a_j$ ($1 \leq j \leq r$), and they call n **e-perfect** if $\sigma_e(n) = 2n$, where $\sigma_e(n)$ is the sum of the e-divisors of n . Some examples of e-perfect numbers are

$$\begin{aligned} 2^2 \cdot 3^2, \quad 2^2 \cdot 3^3 \cdot 5^2, \quad 2^3 \cdot 3^2 \cdot 5^2, \quad 2^4 \cdot 3^2 \cdot 11^2, \quad 2^4 \cdot 3^3 \cdot 5^2 \cdot 11^2, \\ 2^6 \cdot 3^2 \cdot 7^2 \cdot 13^2, \quad 2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 13^2, \quad 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 13^2, \quad 2^8 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 139^2 \\ 2^{19} \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 19^2 \cdot 37^2 \cdot 79^2 \cdot 109^2 \cdot 157^2 \cdot 313^2. \end{aligned}$$

If m is squarefree, $\sigma_e(m) = m$, so if n is e-perfect and m is squarefree with $m \perp n$, then mn is e-perfect. So it suffices to consider only powerful (B16) e-perfect numbers.

Straus & Subbarao show that there are no odd e-perfect numbers, in fact no odd n which satisfy $\sigma_e(n) = kn$ for any integer $k > 1$. They also show that for each r the number of (powerful) e-perfect numbers with r prime factors is finite, and that the same holds for e-multiperfect numbers ($k > 2$).

Is there an e-perfect number which is *not* divisible by 3?

Straus & Subbarao conjecture that there is only a finite number of e-perfect numbers *not* divisible by any given prime p .

Are there any e-multiperfect numbers?

E. G. Straus & M. V. Subbarao, On exponential divisors, *Duke Math. J.*, 41(1974) 465–471; MR 50 #2053.

M. V. Subbarao, On some arithmetic convolutions, *Proc. Conf. Kalamazoo MI, 1971, Springer Lecture Notes in Math.*, 251(1972) 247–271; MR 49 #2510.

M. V. Subbarao & D. Suryanarayana, Exponentially perfect and unitary perfect numbers, *Notices Amer. Math. Soc.*, 18(1971) 798.

B18. Solutions of $d(n) = d(n+1)$.

Claudia Spiro has proved that $d(n) = d(n + 5040)$ has infinitely many solutions and Heath-Brown used her ideas to show that there are infinitely many numbers n such that $d(n) = d(n+1)$, and Pinner has extended this to $d(n) = d(n+a)$ for any integer a . Many examples arise from pairs of consecutive numbers which are products of just two distinct primes, and

it has been conjectured that there is an infinity of *triples* of consecutive products of two primes, $n, n+1, n+2$. For example, $n = 33, 85, 93, 141, 201, 213, 217, 301, 393, 445, 633, 697, 921, \dots$. It is clearly not possible to have *four* such numbers, but it is possible to have longer sequences of consecutive numbers with the same number of divisors. For example,

$$d(242) = d(243) = d(244) = d(245) = 6 \quad \text{and}$$

$$d(40311) = d(40312) = d(40313) = d(40314) = d(40315) = 8.$$

How long can such sequences be? In an 87-07-16 letter Stephane Vandemergel sent the sequence of seven numbers: $171893 = 19 \cdot 83 \cdot 109$, $171894 = 2 \cdot 3 \cdot 28649$, $171895 = 5 \cdot 31 \cdot 1109$, $171896 = 2^3 \cdot 21487$, $171897 = 3 \cdot 11 \cdot 5209$, $171898 = 2 \cdot 61 \cdot 1409$, $171899 = 7 \cdot 13 \cdot 1889$, each with 8 divisors. In 1990, Ivo Düntsch & Roger Eggleton discovered several such sequences of 7 numbers, two of 8 and one of 9, each with 48 divisors; the last example starts at 17796126877482329126044, presumably not the smallest of its kind.

Erdős believes that there are sequences of length k for every k , but does not see how to give an upper bound for k in terms of n .

Erdős, Pomerance & Sárközy showed that the number of $n \leq x$ with $d(n) = d(n+1)$ is $\ll x/(\ln \ln x)^{1/2}$, and Hildebrand showed that this number is $\gg x/(\ln \ln x)^3$. The former authors also showed that the number of $n \leq x$ with the ratio $d(n)/d(n+1)$ in the set $\{2^{-3}, 2^{-2}, 2^{-1}, 1, 2, 2^2, 2^3\}$ is $\asymp x/(\ln \ln x)^{1/2}$.

Erdős showed that the density of numbers n with $d(n+1) > d(n)$ is $\frac{1}{2}$. This, with the above results, settles a conjecture of S. Chowla. Fabrykowski & Subbarao extend this to the case with $n+h$ in place of $n+1$.

Erdős also lets

$$1 = d_1 < d_2 < \cdots < d_r = n$$

be the set of all divisors of n , listed in order, defines

$$f(n) = \sum_{i=1}^{r-1} d_i/d_{i+1}$$

and asks us to prove that $\sum_{n=1}^x f(n) = (1 + o(1))x \ln x$.

Erdős & Mirsky ask for the largest k so that the numbers $d(n), d(n+1), \dots, d(n+k)$ are all distinct. They only have trivial bounds; probably $k = (\ln n)^c$.

P. Erdős, Problem P. 307, *Canad. Math. Bull.*, 24(1981) 252.

P. Erdős & L. Mirsky, The distribution of values of the divisor function $d(n)$, *Proc. London Math. Soc. (3)*, 2(1952) 257–271.

P. Erdős, C. Pomerance & A. Sárközy, On locally repeated values of certain arithmetic functions, II, *Acta Math. Hungarica*, 49(1987) 251–259; MR 88c:11008.

B19. $(m, n+1)$ and $(m+1, n)$ with same set of prime factors.

75

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B19 $(m, n+1)$ and $(m+1, n)$ with same set of prime factors.

Motzkin & Straus asked for all pairs of numbers m, n such that m and $n+1$ have the same set of distinct prime factors, and similarly for n and $m+1$. It was thought that such pairs were necessarily of the form $m = 2^k + 1$, $n = m^2 - 1$ ($k = 0, 1, 2, \dots$) until Conway observed that if $m = 5 \cdot 7$, $n+1 = 5^4 \cdot 7$, then $n = 2 \cdot 3^7$, $m+1 = 2^2 \cdot 3^2$. Are there others?

Similarly, Erdős asks if there are numbers m, n , $n < m$ other than $m = 2^k - 2$, $n = 2^k(2^k - 2)$ such that m and n have the same prime factors and similarly for $m+1, n+1$. Mąkowski found the pair $m = 3 \cdot 5^2, n = 3^5 \cdot 5$ for which $m+1 = 2^2 \cdot 19, n+1 = 2^6 \cdot 19$. Compare problem **B29**.

Pomerance has asked if there are any odd numbers $n > 1$ such that n and $\sigma(n)$ have the same prime factors. He conjectures that there are not.

The example $1 + 2 \cdot 3^7 = 5^4 \cdot 7$ in the first paragraph is of interest in connection with the “ABC conjecture”:

Many of the classical problems of number theory (Goldbach conjecture, twin primes, the Fermat problem, Waring's problem, the Catalan conjecture) owe their difficulty to a clash between multiplication and addition. Roughly, if there's an additive relation between three numbers, their prime factors can't all be small.

Suppose that $A + B = C$ with $\gcd(A, B, C) = 1$. Define the radical R to be the maximum squarefree integer dividing ABC and the power P by

$$P = \frac{\ln \max(|A|, |B|, |C|)}{\ln R}$$

then for a given η are there only finitely many triples $\{A, B, C\}$ with $P \geq \eta$? A stronger form of this conjecture is that $\limsup P = 1$; both forms of the conjecture seem to be hopelessly beyond reach. The example just given is the fifth in the list below.

Joe Kanapka, a student of Noam Elkies, has produced a list of all examples with $C < 2^{32}$ and $P > 1.2$. There are nearly 1000 of them. The “top ten” as far as I know are

P	A	B	C	author
1.629912	2	$3^{10} \cdot 10^9$	23^5	Reyssat
1.625991	11^2	$3^2 \cdot 5^6 \cdot 7^3$	$2^{21} \cdot 23$	de Weger (D10)
1.623490	$19 \cdot 1307$	$7 \cdot 29^2 \cdot 31^6$	$2^8 \cdot 3^{22} \cdot 5^4$	Browkin-Brzeziński
1.580756	283	$5^{11} \cdot 13^2$	$2^8 \cdot 3^8 \cdot 17^3$	Br-Br, Nitaj
1.567887	1	$2 \cdot 3^7$	$5^4 \cdot 7$	Lehmer (B29)
1.547075	7^3	3^{10}	$2^{11} \cdot 29$	de Weger
1.526999	$13 \cdot 19^6$	$2^{20} \cdot 5$	$3^{13} \cdot 11^2 \cdot 31$	Nitaj
1.502393	239	$5^6 \cdot 17^3$	$2^{10} \cdot 37^4$	Br-Br, Nitaj
1.497621	$5^2 \cdot 7937$	7^{13}	$2^{18} \cdot 3^7 \cdot 13^2$	de Weger
1.492432	$2^2 \cdot 11$	$3^2 \cdot 13^{10} \cdot 17 \cdot 151 \cdot 4423$	$5^9 \cdot 139^6$	Nitaj

Browkin & Brzeziński generalize the ABC-conjecture (which is their case $n = 3$) to an “ n -conjecture” on $a_1 + \dots + a_n = 0$ in coprime integers with non-vanishing subsums. With R and P defined analogously, they conjecture that $\limsup P = 2n - 5$. They prove that $\limsup P \geq 2n - 5$. They give a lot of examples for the ABC-conjecture with $P > 1.4$. Their method is to look for rational numbers approximating roots of integers (note that the best example above is connected to the good approximation $23/9$ for $109^{1/5}$). Abderrahmane Nitaj used a similar method. Some of these were found independently by Robert Styer (**D10**). The Catalan relation $1 + 2^3 = 3^2$ gives a comparatively poor $P \approx 1.22629$.

For connexions between the ABC conjecture and the Fermat problem, see the Granville references at **D2**. Indeed, if $A = a^p$, $B = b^p$, $C = c^p$ and the Fermat equation $A + B = C$ is satisfied, then the elliptic curve

$$y^2 = x(x - A)(x + B)$$

has discriminant $(4ABC)^2$.

Jerzy Browkin & Juliusz Brzeziński, Some remarks on the abc-conjecture, *Math. Comput.*, (to appear).

Noam D. Elkies, ABC implies Mordell, *Internat. Math. Res. Notices*, 1991 no. 7, 99–109; *MR 93d:11064*.

Serge Lang, Old and new conjectured diophantine inequalities, *Bull. Amer. Math. Soc.*, 23(1990) 37–75.

A. Mágkossy, On a problem of Erdős, *Enseignement Math.*(2), 14(1968) 193.

Abderrahmane Nitaj, 1993 preprint.

András Sárközy, On sums $a+b$ and numbers of the form $ab+1$ with many prime factors, *Österreichisch-Ungarisch-Slowakisches Kolloquium über Zahlentheorie* (Maria Trost, 1992), 141–154, *Grazer Math. Ber.*, 318 Karl-Franzens-Univ. Graz, 1993.

B21. $k \cdot 2^n + 1$ composite for all n .

C. L. Stewart & Yu Kun-Rui, On the abc conjecture, *Math. Ann.*, 291(1991) 225–230; *MR 92k:11037*.

R. Tijdeman, The number of solutions of Diophantine equations, in Number Theory, II (Budapest, 1987), *Colloq. Math. Soc. János Bolyai*, 51(1990) 671–696.

B20 Cullen numbers.

Some interest has been shown in the **Cullen numbers**, $n \cdot 2^n + 1$, which are all composite for $2 \leq n \leq 1000$, except for $n = 141$. This is probably a good example of the Strong Law of Small Numbers, because for small n , where the density of primes is large, the Cullen numbers are very likely to be composite because Fermat’s (little) theorem tells us that $(p-1)2^{p-1} + 1$ and $(p-2)2^{p-2} + 1$ are both divisible by p . Moreover, as John Conway observes, the Cullen numbers are divisible by $2n - 1$ if that is a prime of shape $8k \pm 3$. He asks if p and $p \cdot 2^p + 1$ can both be prime. Wilfrid Keller notes that Conway’s remark can be generalized as follows. Write $C_n = n \cdot 2^n + 1$, $W_n = n \cdot 2^n - 1$: then a prime p divides $C_{(p+1)/2}$ and $W_{(3p-1)/2}$ or it divides $C_{(3p-1)/2}$ and $W_{(p+1)/2}$ according as the Legendre symbol (see **F5**) $\left(\frac{2}{p}\right)$ is -1 or $+1$. Keller has found prime Cullen numbers with $n = 4713, 5795, 6611$ and 18496 . There are no others with $n \leq 30000$.

Riesel observes that the corresponding numbers $n \cdot 2^n - 1$ are prime for $n = 2, 3, 6, 30, 75, 81$ and 115 ; Jönsson for $n = 362$ and Keller for $n = 123, 249, 384, 462, 512$ (i.e. M_{521}), 751, 882, 5312, 7755, 9531, 12379, 15822 and 18885. Many of these were also found by Waldemar Gorzkowski. There are no others with $n \leq 20000$. In parallel with Conway’s question above, Keller notes that here 3, 751 and 12379 are primes.

Ingemar Jönsson, On certain primes of Mersenne-type, *Nordisk Tidskr. Informationsbehandling (BIT)*, 12 (1972) 117–118; *MR 47 #120*.

Wilfrid Keller, New Cullen primes, (92-11-20 preprint).

Hans Riesel, *Ett Bok om Primalt* (Swedish), Lund, 1968; supplement Stockholm, 1977; *MR 42 #4507, 58 #10681*.

B21. $k \cdot 2^n + 1$ composite for all n .

Let $N(x)$ be the number of odd positive integers k , not exceeding x , such that $k \cdot 2^n + 1$ is prime for no positive integer n . Sierpiński used covering congruences (see **F13**) to show that $N(x)$ tends to infinity with x . For example, if

$$k \equiv 1 \pmod{641 \cdot (2^{32} - 1)} \quad \text{and} \quad k \equiv -1 \pmod{6700417},$$

then every member of the sequence $k \cdot 2^n + 1$ ($n = 0, 1, 2, \dots$) is divisible by just one of the primes 3, 5, 17, 257, 641, 65537 or 6700417. He also noted that at least one of 3, 5, 7, 13, 17, 241 will always divide $k \cdot 2^n + 1$ for certain other values of k .

Erdős & Odlyzko have shown that

$$\left(\frac{1}{2} - c_1\right)x \geq N(x) \geq c_2x.$$

What is the least value of k such that $k \cdot 2^n + 1$ is composite for all values of n ? Selfridge discovered that one of 3, 5, 7, 13, 19, 37, 73 always divides 78557 $\cdot 2^n + 1$. He also noted that there is a prime of the form $k \cdot 2^n + 1$ for each $k < 383$ and Hugh Williams discovered the prime $383 \cdot 2^{6393} + 1$.

In the first edition we wrote that the determination of the least k may now be within computer reach, though Keller has expressed his doubts about this. Extensive calculations have been made by Baillie, Cormack & Williams, by Keller, and by Buell & Young. The answer seems almost certain to be $k = 78557$, but there remain the 35 possibilities

$$\begin{aligned} 4847 & 5297 & 5359 & 7013 & 10223 & 13787 & 19249 & 21181 & 22699 & 24737 \\ 25819 & 27653 & 27923 & 28433 & 33661 & 34999 & 39781 & 44131 & 46157 & 46187 \\ 46471 & 47897 & 48833 & 50693 & 54767 & 55459 & 59569 & 60443 & 60541 & 63017 \\ 65567 & 67607 & 69109 & 74191 & 74269 \end{aligned}$$

for none of which there is a prime with $n \leq 50000$.

A very full bibliography is appended to the thorough survey of the subject in the second of Keller's papers.

Robert Baillie, New primes of the form $k \cdot 2^n + 1$, *Math. Comput.*, **33**(1979) 1333–1336; *MR 80h:10009*.

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Wieb Bosma, Explicit primality criteria for $h \cdot 2^k \pm 1$, *Math. Comput.*, **61**(1993) 97–109.

D. A. Buell & J. Young, Some large primes and the Sierpiński problem, SRC Technical Report 88-004, Supercomputing Research Center, Lanham MD, May 1988.

G. V. Cormack & H. C. Williams, Some very large primes of the form $k \cdot 2^n + 1$, *Math. Comput.*, **35**(1980) 1419–1421; *MR 81i:10011*; corrigendum, Wilfrid Keller, **38**(1982) 335; *MR 82k:10011*.

Paul Erdős & Andrew M. Odlyzko, On the density of odd integers of the form $(p-1)2^{-n}$ and related questions, *J. Number Theory*, **11**(1979) 257–263; *MR 80i:10077*.

G. Jaeschke, On the smallest k such that all $k \cdot 2^N + 1$ are composite, *Math. Comput.*, **40**(1983) 381–384; *MR 84k:10006*; corrigendum, **45**(1985) 637; *MR 87b:11009*.

B22. Factorial n as the product of n large factors.

Wilfrid Keller, Factors of Fermat numbers and large primes of the form $k \cdot 2^n + 1$, *Math. Comput.*, **41**(1983) 661–673; *MR 85b:11119*; II (incomplete draft, 92-02-19).

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N. S. Mendelsohn, The equation $\phi(x) = k$, *Math. Mag.*, **49**(1976) 37–39; *MR 53 #252*.

Raphael M. Robinson, A report on primes of the form $k \cdot 2^n + 1$ and on factors of Fermat numbers, *Proc. Amer. Math. Soc.*, **9**(1958) 673–681; *MR 20 #3097*.

J. L. Selfridge, Solution of problem 4995, *Amer. Math. Monthly*, **70**(1963) 101.

W. Sierpiński, Sur un problème concernant les nombres $k \cdot 2^n + 1$, *Elem. Math.*, **15**(1960) 73–74; *MR 22 #7983*; corrigendum, **17**(1962) 85.

W. Sierpiński, *250 Problems in Elementary Number Theory*, Elsevier, New York, 1970, Problem 118, pp. 10 & 64.

R. G. Stanton & H. C. Williams, Further results on covering of the integers $1+k2^n$ by primes, *Combinatorial Math.* VIII, *Lecture Notes in Math.*, **884**, Springer-Verlag, Berlin–New York, 1980, 107–114.

B22 Factorial n as the product of n large factors.

Straus, Erdős & Selfridge have asked that $n!$ be expressed as the product of n factors, with the least one, l , as large as possible. For example, for $n = 56$, $l = 15$,

$$56! = 15 \cdot 16^3 \cdot 17^3 \cdot 18^8 \cdot 19^2 \cdot 20^{12} \cdot 21^9 \cdot 22^5 \cdot 23^2 \cdot 26^4 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53$$

Selfridge has two conjectures: (a) that, except for $n = 56$, $l \geq \lfloor 2n/7 \rfloor$; (b) that for $n \geq 300000$, $l \geq n/3$. If the latter is true, by how much can 300000 be reduced?

Straus was reputed to have shown that for $n > n_0 = n_0(\epsilon)$, $l > n/(e+\epsilon)$, but a proof was not found in his Nachlaß. It is clear from Stirling's formula that this is best possible. It is also clear that l is a monotonic, though not strictly monotonic, increasing function of n . On the other hand it does not take all integer values: for $n = 124$, 125, l is respectively 35 and 37. Erdős asks how large the gaps in the values of l can be, and can l be constant for arbitrarily long stretches?

Alladi & Grinstead write $n!$ as a product of prime powers each as large as $n^{\delta(n)}$ and let $\alpha(n) = \max \delta(n)$ and show that $\lim_{n \rightarrow \infty} \alpha(n) = e^{\epsilon-1} = \alpha$, say, where

$$c = \sum_{k=2}^{\infty} \frac{1}{k} \ln \frac{k}{k-1} \quad \text{so that} \quad \alpha = 0.809394020534\dots$$

K. Alladi & C. Grinstead, On the decomposition of $n!$ into prime powers, *J. Number Theory*, **9**(1977) 452–458; *MR 56 #11934*.

P. Erdős, Some problems in number theory, *Computers in Number Theory*, Academic Press, London & New York, 1971, 405–414.

B23 Equal products of factorials.

Suppose that $n! = a_1!a_2!\dots a_r!$, $r \geq 2$, $a_1 \geq a_2 \geq \dots \geq a_r \geq 2$. A trivial example is $a_1 = a_2! \dots a_r! - 1$, $n = a_2! \dots a_r!$! Dean Hickerson notes that the only nontrivial examples with $n \leq 410$ are $9! = 7!3!3!2!$, $10! = 7!6! = 7513!$ and $16! = 14515!2!$ and asks if there are any others. Jeffrey Shallit & Michael Easter have extended the search to $n = 18160$.

Erdős observes that if $P(n)$ is the largest prime factor of n and if it were known that $P(n(n+1))/\ln n$ tends to infinity with n , then it would follow that there are only finitely many nontrivial examples.

He & Graham have studied the equation $y^2 = a_1!a_2!\dots a_r!$. They define the set F_k to be those m for which there is a set of integers $m = a_1 > a_2 > \dots > a_r$ with $r \leq k$ which satisfies this equation for some y , and write D_k for $F_k - F_{k-1}$. They have various results, for example: for almost all primes p , $13p$ does not belong to F_5 ; and the least element of D_6 is 527. If $D_4(n)$ is the number of elements of D_4 which are $\leq n$, they do not know the order of growth of $D_4(n)$. They conjecture that $D_6(n) > cn$ but cannot prove this.

Earl Ecklund & Roger Eggleton, Prime factors of consecutive integers, *Amer. Math. Monthly*, **79**(1972) 1082–1089.

E. Ecklund, R. Eggleton, P. Erdős & J. L. Selfridge, on the prime factorization of binomial coefficients, *J. Austral. Math. Soc. Ser. A*, **26**(1978) 257–269; *MR 580:10009*.

P. Erdős, Problems and results on number theoretic properties of consecutive integers and related questions, *Congressus Numerantium XVI* (Proc. 5th Manitoba Conf. Numer. Math. 1975), 25–44.

P. Erdős & R. L. Graham, On products of factorials, *Bull. Inst. Math. Acad. Sinica, Taiwan*, **4**(1976) 337–355.

B24 The largest set with no member dividing two others.

Let $f(n)$ be the size of the largest subset of $[1, n]$ no member of which divides two others. Erdős asks how large can $f(n)$ be? By taking $[m+1, 3m+2]$ it is clear that one can have $\lceil 2n/3 \rceil$. D.J. Kleitman shows that $f(29) = 21$ by taking $[11, 30]$ and omitting 18, 24 and 30, which then allows the inclusion of 6, 8, 9 and 10. However, this example does not seem to generalize. In fact Lebensold has shown that if n is large, then

$$0.6725n \leq f(n) \leq 0.6736n.$$

Erdős also asks if $\lim f(n)/n$ is irrational.

Dually, one can ask for the largest number of numbers $\leq n$, with no number a multiple of any two others. Kleitman's example serves this pur-

pose also. More generally, Erdős asks for the largest number of numbers with no one divisible by k others, for $k > 2$. For $k = 1$, the answer is $\lceil n/2 \rceil$. For some related problems, see E2.

Driss Abouabdillah & Jean M. Turgeon, On a 1937 problem of Paul Erdős concerning certain finite sequences of integers none divisible by another, *Proc. 15th S.E. Conf. Combin. Graph Theory Comput.*, Baton Rouge, 1984, *Congr. Numer.*, **43**(1984) 19–22; *MR 86h:11020*.

P. Erdős, On a problem in elementary number theory and a combinatorial problem, *Math. Comput.*, (1964) 644–646; *MR 30 #1087*.

Kenneth Lebensold, A divisibility problem, *Studies in Appl. Math.*, **56**(1976–77) 291–294; *MR 58 #21639*.

Emma Lehmer, Solution to Problem 3820, *Amer. Math. Monthly*, **46**(1939) 240–241.

B25 Equal sums of geometric progressions with prime ratios.

Bateman asks if $31 = (2^5 - 1)/(2 - 1) = (5^3 - 1)/(5 - 1)$ is the only prime which is expressible in more than one way in the form $(p^r - 1)/(p - 1)$ where p is prime and $r \geq 3$ and $d \geq 1$ are integers. Trivially one has $7 = (2^3 - 1)/(2 - 1) = ((-3)^3 - 1)/(-3 - 1)$, but there are no others $< 10^{10}$. If the condition that p be prime is relaxed, the problem goes back to Goormaghtigh and we have the solution

$$8191 = (2^{13} - 1)/(2 - 1) = (90^3 - 1)/(90 - 1)$$

E. T. Parker observed that the very long proof by Feit & Thompson that every group of odd order is solvable would be shortened if it could be proved that $(p^q - 1)/(p - 1)$ never divides $(q^p - 1)/(q - 1)$ where p, q are distinct odd primes. In fact it has been conjectured that that these two expressions are relatively prime, but Nelson Stephens noticed that when $p = 17, q = 3313$ they have a common factor $2pq + 1 = 112643$. McKay has established that $p^2 + p + 1 \nmid 3^p - 1$ for $p < 53 \cdot 10^6$.

P. T. Bateman & R. M. Stemmler, Waring's problem for algebraic number fields and primes of the form $(p^r - 1)/(p^d - 1)$, *Illinois J. Math.*, **6**(1962) 142–156; *MR 25 #2059*.

Ted Chinburg & Melvin Henriksen, Sums of k th powers in the ring of polynomials with integer coefficients, *Bull. Amer. Math. Soc.*, **81**(1975) 107–110; *MR 51 #421*; *Acta Arith.*, **29**(1976) 227–250; *MR 53 #7942*.

A. Makowski & A. Schinzel, Sur l'équation indéterminée de R. Goormaghtigh, *Mathesis*, **68**(1959) 128–142; *MR 22 #9472*; **70**(1965) 94–96.

N. M. Stephens, On the Feit-Thompson conjecture, *Math. Comput.*, **25**(1971) 625; *MR 45 #6738*.

B26 Densest set with no l pairwise coprime.

Erdős asks what is the maximum k so that the integers a_i , $1 \leq a_1 < a_2 < \dots < a_k \leq n$ have no l among them which are pairwise relatively prime. He conjectures that this is the number of integers $\leq n$ which have one of the first $l-1$ primes as a divisor. He says that this is easy to prove for $l=2$ and not difficult for $l=3$; he offers \$10.00 for a general solution.

Dually one can ask for the largest subset of $[1, n]$ whose members have pairwise least common multiples not exceeding n . If $g(n)$ is the cardinality of such a maximal subset, then Erdős showed that

$$\frac{3}{2\sqrt{2}}n^{1/2} - 2 < g(n) \leq 2n^{1/2}$$

where the first inequality follows by taking the integers from 1 to $(n/2)^{1/2}$ together with the even integers from $(n/2)^{1/2}$ to $(2n)^{1/2}$. Choi improved the upper bound to $1.638n^{1/2}$.

S. L. G. Choi, The largest subset in $[1, n]$ whose integers have pairwise l.c.m. not exceeding n , *Mathematika*, **19**(1972) 221–230; **47** #8461.

S. L. G. Choi, On sequences containing at most three pairwise coprime integers, *Trans. Amer. Math. Soc.*, **183**(1973) 437–440; **48** #6052.

P. Erdős, Extremal problems in number theory, *Proc. Sympos. Pure Math. Amer. Math. Soc.*, **8**(1965) 181–189; *MR 30* #4740.

B27 The number of prime factors of $n+k$ which don't divide $n+i$, $0 \leq i < k$.

Erdős & Selfridge define $v(n; k)$ as the number of prime factors of $n+k$ which do not divide $n+i$ for $0 \leq i < k$, and $v_0(n)$ as the maximum of $v(n; k)$ taken over all $k \geq 0$. Does $v_0(n) \rightarrow \infty$ with n ? They show that $v_0(n) > 1$ for all n except 1, 2, 3, 4, 7, 8 and 16. More generally, define $v_l(n)$ as the maximum of $v(n; k)$ taken over $k \geq l$. Does $v_l(n) \rightarrow \infty$ with n ? They are unable to prove even that $v_1(n) = 1$ has only a finite number of solutions. Probably the greatest n for which $v_1(n) = 1$ is 330.

They also denote by $V(n; k)$ the number of primes p for which p^α is the highest power of p dividing $n+k$, but p^α does not divide $n+i$ for $0 \leq i < k$, and by $V_l(n)$ the maximum of $V(n; k)$ taken over $k \geq l$. Does $V_l(n) = 1$ have only a finite number of solutions? Perhaps $n = 80$ is the largest solution. What is the largest n such that $V_0(n) = 2$?

Some further problems are given in their paper.

P. Erdős & J. L. Selfridge, Some problems on the prime factors of consecutive integers, *Illinois J. Math.*, **11**(1967) 428–430.

A. Schinzel, Unsolved problem 31, *Elem. Math.*, **14**(1959) 82–83.

B29 Is x determined by the prime divisors of $x+1, x+2, \dots, x+k$? 83

B28 Consecutive numbers with distinct prime factors.

Selfridge asked: do there exist n consecutive integers, each having either two distinct prime factors less than n or a repeated prime factor less than n ? He gives two examples:

1. the numbers $a + 11 + i$ ($1 \leq i \leq n = 115$) where $a \equiv 0 \pmod{2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2}$ and $a + p \equiv 0 \pmod{p^2}$ for each prime p , $13 \leq p \leq 113$;
2. the numbers $a + 31 + i$ ($1 \leq i \leq n = 1329$) where $a + p \equiv 0 \pmod{p^2}$ for each prime p , $37 \leq p \leq 1327$ and $a \equiv 0 \pmod{2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29^2 \cdot 31^2}$.

It is harder to find examples of n consecutive numbers, each one divisible by two distinct primes less than n or by the square of a prime $< n/2$, though he believes that they could be found by computer.

This is related to the problem: find n consecutive integers, each having a composite common factor with the product of the other $n-1$. If the composite condition is relaxed, and one asks merely for a common factor greater than 1, then $2184 + i$ ($1 \leq i \leq n = 17$) is a famous example.

Alfred Brauer, On a property of k consecutive integers, *Bull. Amer. Math. Soc.*, **47**(1941) 328–331; *MR 2*, 248.

Ronald J. Evans, On blocks of N consecutive integers, *Amer. Math. Monthly*, **76**(1969) 48–49.

Ronald J. Evans, On N consecutive integers in an arithmetic progression, *Acta Sci. Math. Univ. Szeged*, **38**(1972) 295–296; *MR 47* #8408.

Heiko Harborth, Eine Eigenschaft aufeinanderfolgender Zahlen, *Arch. Math. (Basel)*, **21**(1970) 50–51; *MR 41* #6771.

Heiko Harborth, Sequenzen ganzer Zahlen, *Zahlentheorie (Tagung, Math. Forschungsinstitut Oberwolfach, 1970)* 59–66; *MR 51* #12775.

S. S. Pillai, On m consecutive integers I, *Proc. Indian Acad. Sci. Sect. A*, **11**(1940) 6–12; *MR 1*, 199; II **11**(1940) 73–80; *MR 1*, 291; III **13**(1941) 530–533; *MR 3*, 66; IV *Bull. Calcutta Math. Soc.*, **36**(1944) 99–101; *MR 6*, 170.

B29 Is x determined by the prime divisors of $x+1, x+2, \dots, x+k$?

Alan R. Woods asks if there is a positive integer k such that every x is uniquely determined by the (sets of) prime divisors of $x+1, x+2, \dots, x+k$. Perhaps $k = 3$?

For primes less than 23 there are four ambiguous cases for $k = 2$: $(x+1, x+2) = (2, 3)$ or $(8, 9)$; $(6, 7)$ or $(48, 49)$; $(14, 15)$ or $(224, 225)$; $(75, 76)$ or $(1215, 1216)$. The first three of these are members of the infinite family $(2^n - 2, 2^n - 1)$, $(2^n(2^n - 2), (2^n - 1)^2)$. Compare **B19**.

D. H. Lehmer, On a problem of Størmer, *Illinois J. Math.*, 8(1964) 57–79; MR 28 #2072.

B30 A small set whose product is square.

Erdős, Graham & Selfridge want us to find the least value of t_n so that the integers $n+1, n+2, \dots, n+t_n$ contain a subset the product of whose members with n is a square. The Thue-Siegel theorem implies that $t_n \rightarrow \infty$ with n , faster than a power of $\ln n$. Selfridge has shown that $t_n \leq \max(P(n), 3\sqrt{n})$, where $P(n)$ is the largest prime factor of n .

Alternatively, is it true that for every c there is an n_0 so that for every $n > n_0$ the products $\prod a_i$, taken over $n < a_1 < \dots < a_k < n + (\ln n)^c$ ($k = 1, 2, \dots$) are all distinct? They proved this for $c < 2$.

Selfridge conjectures that if n is not a square, and t is the next larger number than n such that nt is a square, then, unless $n = 8$ or 392, it is always possible to find r and s , $n < r < s < t$ such that trs is a square. E.g., if $n = 240 = 2^4 \cdot 3 \cdot 5$ then $t = 375 = 3 \cdot 5^3$ and we can find $r = 243 = 3^5$ and $s = 245 = 5 \cdot 7^2$.

P. Erdős & Jan Turk, Products of integers in short intervals, *Acta Arith.*, 44(1984) 147–174; MR 86d:11073.

B31 Binomial coefficients.

Earl Ecklund, Roger Eggleton, Erdős & Selfridge (see B23) write the binomial coefficient $\binom{n}{k} = n!/(k!(n-k)!)$ as a product UV in which every prime factor of U is at most k and every prime factor of V is greater than k . There are only finitely many cases with $n \geq 2k$ for which $U > V$. They determine all such cases except when $k = 3, 5$ or 7.

S. P. Khare lists all cases with $n \leq 551$: $k = 3, n = 8, 9, 10, 18, 82, 162$; $k = 5, n = 10, 12, 28$; and $k = 7, n = 21, 30, 54$.

Most binomial coefficients $\binom{n}{k}$ with $n \geq 2k$ have a prime factor $p \leq n/k$. After some computing with Lacampagne & Erdős, Selfridge conjectured that this inequality is true whenever $n > 17.125k$. A slightly stronger conjecture is that any such binomial coefficient has least prime factor $p \leq n/k$ or $p \leq 17$ with just 4 exceptions: $\binom{62}{6}, \binom{959}{56}, \binom{474}{66}, \binom{284}{28}$ for which $p = 19, 19, 23$ and 29 respectively.

These authors define the deficiency of the binomial coefficient $\binom{n+k}{k}$, $k \leq n$, as the number of i for which $b_i = 1$, where $n+i = a_ib_i$, $1 \leq i \leq k$, the prime factors of b_i are greater than k , and $\prod a_i = k!$ Then $\binom{44}{8}, \binom{76}{10}, \binom{174}{12}, \binom{239}{14}, \binom{5179}{14}, \binom{8433}{28}, \binom{8414}{42}$ and $\binom{96622}{42}$ each have deficiency 2; $\binom{46}{10}, \binom{10}{10}, \binom{241}{16}, \binom{2162}{25}, \binom{1119}{27}$ and $\binom{6459}{33}$ have deficiency 3; $\binom{11}{11}$ has deficiency 4; and $\binom{284}{28}$ has deficiency 9; and they conjecture that there

are no others with deficiency greater than 1. Are there only finitely many binomial coefficients with deficiency 1?

Erdős & Selfridge noted that if $n \geq 2k \geq 4$, then there is at least one value of i , $0 \leq i \leq k-1$, such that $n-i$ does not divide $\binom{n}{k}$, and asked for the least n_k for which there was only one such i . For example, $n_2 = 4$, $n_3 = 6$, $n_4 = 9$, $n_5 = 12$. $n_k \leq k!$ for $k \geq 3$.

Harry Ruderman asks for a proof or disproof that for every pair (p, q) of nonnegative integers there is a positive integer n such that

$$\frac{(2n-p)!}{n!(n+q)!}$$

is an integer.

A problem which has briefly baffled good mathematicians is: is $\binom{n}{r}$ ever prime to $\binom{n}{s}$, $0 < r < s \leq n/2$? The negative answer follows from the identity

$$\binom{n}{s} \binom{s}{r} = \binom{n}{r} \binom{n-r}{s-r}.$$

Erdős & Szekeres ask if the greatest prime factor of the g.c.d. is always greater than r ; the only counterexample with $r > 3$ that they noticed is

$$\gcd\left(\binom{28}{5}, \binom{28}{14}\right) = 2^3 \cdot 3^3 \cdot 5$$

Wolstenholme's theorem states that if n is a prime > 3 , then

$$\binom{2n-1}{n} \equiv 1 \pmod{n^3}.$$

James P. Jones asks if the converse is true. For other problems and results on the divisors of binomial coefficients, see B33.

D. F. Bailey, Two p^3 variations of Lucas's theorem, *J. Number Theory*, 35(1990) 208–215; MR 90f:11008.

Paul Erdős, C. B. Lacampagne & J. L. Selfridge, Estimates of the least prime factor of a binomial coefficient, *Math. Comput.*, 61(1993) 215–224; MR 93k:11013.

P. Erdős & J. L. Selfridge, Problem 6447, *Amer. Math. Monthly*, 90(1983) 710; 92(1985) 435–436.

P. Erdős & G. Szekeres, Some number theoretic problems on binomial coefficients, *Austral. Math. Soc. Gaz.*, 5(1978) 97–99; MR 80e:10010 is uninformative.

Richard J. McIntosh, A generalization of a congruential property of Lucas, *Amer. Math. Monthly*, 99(1992) 231–238.

Harry D. Ruderman, Problem 714, *Cruze Math.*, 8(1982) 48; 9(1983) 58.

David Segal, Problem E435, partial solution by H.W. Brinkman, *Amer. Math. Monthly*, 48(1941) 269–271.

B32 Grimm's conjecture.

Grimm has conjectured that if $n+1, n+2, \dots, n+k$ are all composite, then there are distinct primes p_i , such that $p_i|(n+j)$ for $1 \leq j \leq k$. For example

1802 1803 1804 1805 1806 1807 1808 1809 1810

are respectively divisible by

53 601 41 19 43 139 113 67 181

and

114 115 116 117 118 119 120 121 122 123 124 125 126

by

19 23 29 13 59 17 2 11 61 41 31 5 7

Ramachandra, Shorey & Tijdeman proved, under the hypothesis of Schinzel mentioned in A2, that there are only finitely many exceptions to Grimm's conjecture.

Erdős & Selfridge asked for an estimate of $f(n)$, the least number such that for each m there are distinct integers $a_1, a_2, \dots, a_{\pi(n)}$ in the interval $[m+1, m+f(n)]$ with $p_i|a_i$, where p_i is the i th prime. They and Pomerance show that, for large n ,

$$(3 - \epsilon)n \leq f(n) \ll n^{3/2}(\ln n)^{-1/2}$$

P. Erdős, Problems and results in combinatorial analysis and combinatorial number theory, in *Proc. 9th S.E. Conf. Combin. Graph Theory, Comput., Boca Raton, Congressus Numerantium XXI*, Utilitas Math. Winnipeg, 1978, 29–40.

P. Erdős & C. Pomerance, Matching the natural numbers up to n with distinct multiples in another interval, *Nederl. Akad. Wetensch. Proc. Ser. A*, 83 (= *Indag. Math.*, 42)(1980) 147–161; MR 81i:10053.

Paul Erdős & Carl Pomerance, An analogue of Grimm's problem of finding distinct prime factors of consecutive integers, *Utilitas Math.*, 24(1983) 45–46; MR 85b:11072.

P. Erdős & J. L. Selfridge, Some problems on the prime factors of consecutive integers II, in *Proc. Washington State Univ. Conf. Number Theory*, Pullman, 1971, 13–21.

C. A. Grimm, A conjecture on consecutive composite numbers, *Amer. Math. Monthly*, 76(1969) 1126–1128.

Michel Langevin, Plus grand facteur premier d'entiers en progression arithmétique, *Sém. Delange-Pisot-Poitou*, 18(1976/77) Théorie des nombres: Fasc. 1, Exp. No. 3, Paris, 1977; MR 81a:10011.

Carl Pomerance, Some number theoretic matching problems, in *Proc. Number Theory Conf.*, Queen's Univ., Kingston, 1979, 237–247.

Carl Pomerance & J. L. Selfridge, Proof of D.J. Newman's coprime mapping conjecture, *Mathematika*, 27(1980) 69–83; MR 81i:10008.

K. Ramachandra, T. N. Shorey & R. Tijdeman, On Grimm's problem relating to factorization of a block of consecutive integers, *J. reine angew. Math.*, 273(1975) 109–124.

B33 Largest divisor of a binomial coefficient.

B33 Largest divisor of a binomial coefficient.

What can one say about the largest divisor, less than n , of the binomial coefficient $\binom{n}{k} = n!/k!(n-k)!$? Erdős points out that it is easy to show that it is at least n/k and conjectures that there may be one between cn and n for any $c < 1$ and n sufficiently large. Marilyn Faulkner showed that if p is the least prime $> 2k$ and $n \geq p$, then $\binom{n}{k}$ has a prime divisor $\geq p$, except for $\binom{9}{2}$ and $\binom{10}{3}$. Earl Ecklund showed that if $n \geq 2k > 2$ then $\binom{n}{k}$ has a prime divisor $p \leq n/2$, except for $\binom{7}{3}$.

John Selfridge conjectures that if $n \geq k^2 - 1$, then, apart from the exception $\binom{62}{6}$, there is a prime divisor $\leq n/k$ of $\binom{n}{k}$. Among those binomial coefficients whose least prime factor p is $\geq n/k$ there may be only a finite number with $p \geq 13$, but there could be infinitely many with $p = 7$. That there are infinitely many with $p = 5$ was proved by Erdős, Lacampagne & Selfridge (B31).

A classical theorem, discovered independently by Sylvester and Schur, stated that the product of k consecutive integers, each greater than k , has a prime divisor greater than k . Leo Moser conjectured that the Sylvester-Schur theorem holds for primes $\equiv 1 \pmod{4}$, in the sense that for n sufficiently large (and $\geq 2k$), $\binom{n}{k}$ has a prime divisor $\equiv 1 \pmod{4}$ which is greater than k . However, Erdős does not think that this is true, but it may not be at all easy to settle. In this connexion John Leech notices that the fourteen integers 280213, ..., 280226 have no prime factor of the form $4m + 1 > 13$.

Thanks to Ira Gessel and John Conway, we can say that the generalization of the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$, requested in the first edition by Neil Sloane, is $\frac{(n,r)}{n}\binom{n}{r}$, which is always an integer (multiply by n and by r and Euclid knew that (n,r) is a linear combination of n and r). These are also known as generalized ballot numbers and they occur when enumerating certain lattice paths.

If $f(n)$ is the sum of the reciprocals of those primes $< n$ which do not divide $\binom{2n}{n}$, then Erdős, Graham, Ruzsa & Straus conjectured that there is an absolute constant c so that $f(n) < c$ for all n . Erdős also conjectured that $\binom{2n}{n}$ is never squarefree for $n > 4$. Since $4|\binom{2n}{n}$ unless $n = 2^k$, it suffices to consider

$$\binom{2^{k+1}}{2^k}.$$

Sárközy proved this for n sufficiently large and Sander has shown, in a precise sense, that binomial coefficients near the centre of the Pascal triangle are not squarefree. Granville & Ramaré completed Sárközy's proof by showing that $k > 300000$ was sufficiently large, and checking it computationally for $2 \leq k \leq 300000$. They also improved Sander's result by showing that there is a constant δ , $0 < \delta < 1$, such that if $\binom{n}{k}$ is squarefree

then k or $n - k$ must be $< n^\delta$ for sufficiently large n . They conjecture that k or $n - k$ must in fact be $< (\ln n)^{2-\delta}$, and that this is best possible in the sense that there are infinitely many squarefree $\binom{n}{k}$ with $\frac{1}{2}n > k > c(\ln n)^2$ for some $c > 0$. They prove such a result for $\frac{1}{2}n > k > \frac{1}{2}\ln n$. They show that there is a constant $\rho_k > 0$ such that the number of $n \leq N$ with $\binom{n}{k}$ squarefree is $\sim \rho_k N$. Since $\rho_k < c/k^2$ for some $c > 0$, they conjecture that there is a constant $\gamma > 0$ such that the number of squarefree entries in the first N rows of Pascal's triangle is $\sim \gamma N$.

Erdős has also conjectured that for $k > 8$, 2^k is not the sum of distinct powers of 3 [$2^8 = 3^5 + 3^2 + 3 + 1$]. If that's true, then for $k \geq 9$,

$$3 \left| \binom{2^{k+1}}{2^k} \right..$$

In answer to the question, is $\binom{342}{171}$ the largest $\binom{2^n}{n}$ which is not divisible by the square of an odd prime, Eugene Levine gave the examples $n = 784$ and 786. Erdős feels sure that there are no larger such n .

Denote by $e = e(n)$ the largest exponent such that, for some prime p , p^e divides $\binom{2^n}{n}$. It is not known whether $e \rightarrow \infty$ with n . On the other hand Erdős cannot disprove $e > c \ln n$.

Ron Graham offers \$100.00 for deciding if $(\binom{2^n}{n}, 105) = 1$ infinitely often. Kummer knew that n , when written in base 3 or 5 or 7, would have to have only the digits 0, 1 or 0, 1, 2 or 0, 1, 2, 3 respectively. H. Gupta & S. P. Khare found the 14 values 1, 10, 756, 757, 3160, 3186, 3187, 3250, 7560, 7561, 7651, 20007, 59548377, 59548401 of n less than 7^{10} , while Peter Montgomery, Khare and others found many larger values.

Erdős, Graham, Ruzsa & Straus showed that for any two primes p, q there are infinitely many n for which $(\binom{2^n}{n}, pq) = 1$. If $g(n)$ is the smallest odd prime factor of $\binom{2^n}{n}$, then $g(3160) = 13$ and $g(n) \leq 11$ for $3160 < n < 10^{10000}$.

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P. Erdős, A theorem of Sylvester and Schur, *J. London Math. Soc.*, 9(1934) 282–288.

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P. Erdős & R. L. Graham, On the prime factors of $\binom{n}{k}$, *Fibonacci Quart.*, 14(1976) 348–352.

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L. Moser, Insolubility of $\binom{2^n}{n} = \binom{2^a}{a} \binom{2^b}{b}$, *Canad. Math. Bull.*, 6(1963) 167–169. J. W. Sander, Prime power divisors of $\binom{2^n}{n}$, *J. Number Theory*, 39(1991) 65–74; MR 92i:11097.

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Renate Schneider & Hugh C. Williams, A method of tabulating the number-theoretic function $g(k)$, *Math. Comput.*, 59(1992) 251–257; MR 92k:11146.

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J. Sylvester, On arithmetical series, *Messenger of Math.*, 21(1892) 1–19, 87–120. W. Utz, A conjecture of Erdős concerning consecutive integers, *Amer. Math. Monthly*, 68(1961) 896–897.

B34 If there's an i such that $n - i$ divides $\binom{n}{k}$.

If $H_{k,n}$ is the proposition: there is an i , $0 \leq i < k$ such that $n - i$ divides $\binom{n}{k}$, then Erdős asked if $H_{k,n}$ is true for all k when $n \geq 2k$. Schinzel gave the counterexample $n = 99215$, $k = 15$. If H_k is the proposition: $H_{k,n}$ is true for all n , then Schinzel showed that H_k is false for $k = 15, 21, 22, 33, 35$ and thirteen other values of k . He showed that H_k is true for all other $k \leq 32$ and asked if there are infinitely many k , other than prime-powers, for which H_k is true: he conjectures not and later reported that it is true for $k = 34$, but for no other non-prime-powers between 34 and 201.

E. Burchack & J. Piekarczyk, P. 217, R. 1, *Colloq. Math.*, 10(1963) 365.

A. Schinzel, Sur un problème de P. Erdős, *Colloq. Math.*, 5(1957–58) 198–204.

B35 Products of consecutive numbers with the same prime factors.

Let $f(n)$ be the least integer such that at least one of the numbers $n, n+1, \dots, n+f(n)$ divides the product of the others. It is easy to see that

$f(k!) = k$ and $f(n) > k$ for $n > k!$ Erdős has also shown that

$$f(n) > \exp((\ln n)^{1/2-\epsilon})$$

for an infinity of values of n , but it seems difficult to find a good upper bound for $f(n)$.

Erdős asks if $(m+1)(m+2)\cdots(m+k)$ and $(n+1)(n+2)\cdots(n+l)$ with $k \geq l \geq 3$ can contain the same prime factors infinitely often. For example $(2 \cdot 3 \cdot 4 \cdot 5 \cdot 6) \cdot 7 \cdot 8 \cdot 9 \cdot 10$ and $14 \cdot 15 \cdot 16$ and $48 \cdot 49 \cdot 50$; also $(2 \cdot 3 \cdot 4 \cdot 5 \cdot 6) \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12$ and $98 \cdot 99 \cdot 100$. For $k = l \geq 3$ he conjectures that this happens only finitely many times.

If $L(n; k)$ is the l.c.m. of $n+1, n+2, \dots, n+k$, then Erdős conjectures that for $l > 1$, $n \geq m+k$, $L(m; k) = L(n; l)$ has only a finite number of solutions. Examples are $L(4; 3) = L(13; 2)$ and $L(3; 4) = L(19; 2)$. He asks if there are infinitely many n such that for all k ($1 \leq k < n$) we have $L(n; k) > L(n-k; k)$. What is the largest $k = k(n)$ for which this inequality can be reversed? He notes that it is easy to see that $k(n) = o(n)$, but he believes that much more is true. He expects that for every $\epsilon > 0$ and $n > n_0(\epsilon)$, $k(n) < n^{1/2+\epsilon}$ but cannot prove this.

P. Erdős, How many pairs of products of consecutive integers have the same prime factors? *Amer. Math. Monthly*, **87**(1980) 391–392.

B36 Euler's totient function.

Euler's totient function, $\phi(n)$, is the number of numbers not greater than n and prime to n . For example $\phi(1) = \phi(2) = 1$, $\phi(3) = \phi(4) = \phi(6) = 2$, $\phi(5) = \phi(8) = \phi(10) = \phi(12) = 4$, $\phi(7) = \phi(9) = 6$. Are there infinitely many pairs of consecutive numbers, $n, n+1$, such that $\phi(n) = \phi(n+1)$? For example, $n = 1, 3, 15, 104, 164, 194, 255, 495, 584, 975$. It is not even known if $|\phi(n+1) - \phi(n)| < n^\epsilon$ has an infinity of solutions for each $\epsilon > 0$. Baillie extended the work of others to find 306 solutions of $\phi(n) = \phi(n+1)$ below 10^8 and 85 between 10^8 and $2 \cdot 10^8$.

Schinzel conjectures that for every even k the equation $\phi(n+k) = \phi(n)$ has an infinity of solutions. He observes that the corresponding conjecture with k odd is implausible. For $k = 1$, the problem is that of the previous paragraph. For $k = 3$ he found only the solutions $n = 3$ and $n = 5$ in the range $n < 10^4$, and D.H. Lehmer extended this to 10^6 . Sierpiński has shown that $\phi(n+k) = \phi(n)$ has at least one solution for each value of k and Schinzel & Wakulicz have shown that there are at least two solutions for each $k < 2 \cdot 10^{28}$. Mąkowski has shown that $\phi(n+k) = 2\phi(n)$ has at least one solution for every k . For the equation $\phi(n+k) = 3\phi(n)$ see the solution to problem E 3215 in *Amer. Math. Monthly*, **96**(1989) 63–64.

Mąkowski (see reference at B5) also discusses the equation $\phi(x+k) = \phi(x) + \phi(k)$. J. Browkin showed that if $k = 3$, then there was no solution with $x < 37182142$.

Three curiosities are $\phi(5186) = \phi(5187) = \phi(5188) = 2^5 3^4$, $\phi(25930) = \phi(25935) = \phi(25940) = \phi(25942) = 2^7 3^4$ and $\phi(404471) = \phi(404473) = \phi(404477) = 2^8 3^2 5^2$.

Nontotients are positive even values of n for which $\phi(x) = n$ has no solution; for example, $n = 14, 26, 34, 38, 50, 62, 68, 74, 76, 86, 90, 94, 98$. The number, $\#(y)$, of these less than y has been calculated by the Lehmers.

y	10^3	10^4	$2 \cdot 10^4$	$3 \cdot 10^4$	$4 \cdot 10^4$	$5 \cdot 10^4$	$6 \cdot 10^4$	$7 \cdot 10^4$	$8 \cdot 10^4$	$9 \cdot 10^4$
#(y)	210	2627	5515	8458	11438	14439	17486	20536	23606	26663

Erdős & Hall have shown that the number, $\Phi(y) = y - \#(n)$, of n for which $\phi(x) = n$ has a solution is $ye^{f(y)/\ln y}$, where $f(y)$ lies between $c(\ln \ln y)^2$ and $c(\ln y)^{1/2}$. Maier & Pomerance more recently showed that the lower bound was correct, with $c \approx 0.8178$. Erdős conjectures that $\Phi(cy)/\Phi(y) \rightarrow c$, and that this, if true, may be the best substitute that one can find for an asymptotic formula for $\Phi(y)$.

Noncototients are positive values of n for which $x - \phi(x) = n$ has no solution; for example, $n = 10, 26, 34, 50, 52, 58, 86, 100$. Sierpiński and Erdős conjecture that there are infinitely many noncototients.

Erdős once asked if it was true that for every ϵ there is an n with $\phi(n) = m$, $m < \epsilon n$ and for no $t < n$ is $\phi(t) = m$; perhaps there are many such n .

Michael Ecker has asked for which values of x do each of the series $\sum_{n=1}^{\infty} \phi(n)/n^x$ and $\sum_{n=1}^{\infty} (-1)^{n+1} \phi(n)/n^x$ converge.

Robert Baillie, Table of $\phi(n) = \phi(n+1)$, *Math. Comput.*, **30**(1976) 189–190.
David Ballew, Janell Case & Robert N. Higgins, Table of $\phi(n) = \phi(n+1)$, *Math. Comput.*, **29**(1975) 329–330.

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P. Erdős & R. R. Hall, Distinct values of Euler's ϕ -function, *Mathematika*, **23**(1976) 1–3.

Patricia Jones, On the equation $\phi(x) + \phi(k) = \phi(x+k)$, *Fibonacci Quart.*, **28**(1990) 162–165; *MR 91e:11008*.

M. Lal & P. Gillard, On the equation $\phi(n) = \phi(n+k)$, *Math. Comput.*, **26**(1972) 579–582.

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 Masataka Yorinaga, Numerical investigation of some equations involving Euler's ϕ -function, *Math. J. Okayama Univ.*, **20**(1978) 51–58.

B37 Does $\phi(n)$ properly divide $n - 1$?

D. H. Lehmer has conjectured that there is no composite value of n such that $\phi(n)$ is a divisor of $n - 1$, i.e., that for no value of n is $\phi(n)$ a proper divisor of $n - 1$. Such an n must be a Carmichael number (**A13**). He showed that it would have to be the product of at least seven distinct primes, and Lieuwen has shown that if $3|n$, then $n > 5.5 \cdot 10^{571}$ and $\omega(n) \geq 212$; if the smallest prime factor of n is 5, then $\omega(n) \geq 11$; if the smallest prime factor of n is at least 7, then $\omega(n) \geq 13$. This supersedes and corrects the work of Schuh. Masao Kishore has shown that at least 13 primes are needed in any case, and Cohen & Hagis have improved this to 14. Siva Rama Prasad & Subbarao improve Lieuwen's 212 result to $\omega(n) \geq 1850$ and Hagis to $\omega(n) \geq 298848$. Siva Rama Prasad & Rangamma show that if $3|n$, n composite, $M\phi(n) = n - 1$, $M \neq 4$, then $\omega(n) \geq 5334$.

Pomerance has proved that the number of composite n less than x for which $\phi(n)|n - 1$ is

$$O(x^{1/2}(\ln x)^{3/4}(\ln \ln x)^{-1/2})$$

and Shan Zun improved the exponent $\frac{3}{4}$ to $\frac{1}{2}$.

Schinzel notes that if $n = p + 2p$, where p is prime, then $\phi(n) + 1$ divides n and asks if the converse is always true. Segal (see paper with Cohen) observes that Schinzel's question reduces to that of Lehmer, that it arises in group theory, and may have been raised by G. Hajós (see Miech's paper, though there it is attributed to Gordon).

If n is prime, it divides $\phi(n)d(n) + 2$. Is this true for any composite n other than $n = 4$? Subbarao also notes that if n is prime, then $n\sigma(n) \equiv 2 \pmod{\phi(n)}$, and also if $n = 4, 6$ or 22 ; is it true for infinitely many composite n ?

Subbarao has an analogous conjecture to Lehmer's, based on the function $\phi^*(n) = \prod(p^a - 1)$, where the product is taken over the maximal prime power divisors of n , $p^a||n$. He conjectures that $\phi^*(n)|(n-1)$ if and only if n is a power of a prime. He also has a 'dual' of Lehmer's conjecture, namely

B37. Does $\phi(n)$ properly divide $n - 1$?

that $\psi(n) \equiv 1 \pmod{n}$ only when n is a prime, where $\psi(n)$ is Dedekind's function (see **B41**).

Ron Graham makes the following conjecture

i. For all k there are infinitely many n such that $\phi(n)|(n - k)$?

He observes that it is true for $k = 0$, $k = 2^a$ ($a \geq 0$) and $k = 2^a3^b$ ($a, b > 0$) for example. Pomerance (see *Acta Arith.* paper quoted in **B2**) has treated Graham's problem. Victor Meally notes that $\phi(n)$ sometimes divides $n+1$, e.g., for $n = n_1 = 3 \cdot 5 \cdot 17 \cdot 353 \cdot 929$ and $n = n_1 \cdot 83623937$. [Note that $353 = 11 \cdot 2^5 + 1$, $929 = 29 \cdot 2^6 + 1$, $83623937 = 11 \cdot 29 \cdot 2^{18} + 1$ and $(353 - 2^8)(929 - 2^8) = 2^{16} - 2^8 + 1$.]

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B38 Solutions of $\phi(m) = \sigma(n)$.

Are there infinitely many pairs of numbers m, n such that $\phi(m) = \sigma(n)$? Since for p prime $\phi(p) = p - 1$ and $\sigma(p) = p + 1$ this question would be answered affirmatively if there were infinitely many twin primes (A7). Also if there were infinitely many Mersenne primes (A3) $M_p = 2^p - 1$, since $\sigma(M_p) = 2^p = \phi(2^{p+1})$. However there are many solutions other than these, sometimes displaying little noticeable pattern, e.g., $\phi(780) = 192 = \sigma(105)$.

Erdős remarks that the equation $\phi(x) = n!$ is solvable, and (apart from $n = 2$) $\sigma(y) = n!$ is probably solvable also. Charles R. Wall can show that $\psi(n) = n!$ is solvable for $n \neq 2$, where ψ is Dedekind's function (see B41).

Le Mao-Hua, A note on primes p with $\sigma(p^m) = z^n$, *Colloq. Math.*, **62**(1991) 193–196.

B39 Carmichael's conjecture.

Carmichael's conjecture. For every n it appears to be possible to find an m , not equal to n , such that $\phi(m) = \phi(n)$ and for a few years early in this century it was thought that Carmichael had proved this. Klee verified the conjecture for $\phi(n) < 10^{400}$, and for all n not divisible by $2^{42} \cdot 3^{47}$. Masai & Valette have raised the bound to 10^{10000} , and Schlafly & Wagon to $10^{1360000}$ and are proceeding to $10^{10000000}$. Pomerance has shown that if n is such that for every prime p for which $p - 1$ divides $\phi(n)$ we have p^2 divides n , then n is a counterexample. He can also show (unpublished) that if the first k primes $p \equiv 1 \pmod{q}$ (where q is prime) are all less than q^{k+1} , then there are no numbers n which satisfy his theorem. This also implies the truth of his conjecture that $p_k - 1 \prod_{i < k} p_i(p_i - 1)$. The truth of this last conjecture for all k also implies that there are no numbers n which satisfy his theorem.

Define the **multiplicity** of an integer as the number of times it occurs as a value of $\phi(n)$. For example, 6 has multiplicity 4 because $\phi(n) = 6$ for $n = 7, 9, 14, 18$ and no other values of n . The multiplicity may be zero (for any odd $n > 1$, and $n = 14, 26, 34, \dots$), but not, according to the Carmichael conjecture, equal to one. Sierpiński conjectured that all integers greater than 1 occur as multiplicities and Erdős has shown that if a multiplicity occurs once it occurs infinitely often. Schlafly & Wagon have found examples of all multiplicities from 2 through 65.

There are examples of even numbers n such that there is no odd number m such that $\phi(m) = \phi(n)$. Lorraine Foster has given $n = 33817088 = 2^9 \cdot 257^2$ as the least such.

Erdős proved that if $\phi(x) = k$ has exactly s solutions, then there are infinitely many other k for which there are exactly s solutions, and that $s > k^c$ for infinitely many k . If C is the least upper bound of those c for which this is true, then Wooldridge showed that $C \geq 3 - 2\sqrt{2} > 0.17157$.

B39. Carmichael's conjecture.

Pomerance used Hooley's improvement on the Brun–Titchmarsh theorem to improve this to $C \geq 1 - 625/512e > 0.55092$ and notes that further improvements by Iwaniec enable him to get $C > 0.55655$ so that $s > k^{5/9}$ for infinitely many k . Erdős conjectures that $C = 1$. In the other direction Pomerance also shows that

$$s < k \exp\{-(1 + o(1)) \ln k \ln \ln \ln k / \ln \ln k\}$$

and gives a heuristic argument to support the belief that this is best possible.

R. D. Carmichael, Note on Euler's ϕ -function, *Bull. Amer. Math. Soc.*, **28** (1922) 109–110; and see **13**(1907) 241–243.

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Aaron Schlafly & Stan Wagon, Carmichael's conjecture is valid below $10^{2,000,000}$, *Math. Comput.*, **50**.

M. V. Subbarao & L.-W. Yip, Carmichael's conjecture and some analogues, in *Théorie des Nombres (Québec, 1987)*, de Gruyter, Berlin–New York, 1989, 928–941 (and see *Canad. Math. Bull.*, **34**(1991) 401–404).

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K. R. Wooldridge, Values taken many times by Euler's phi-function, *Proc. Amer. Math. Soc.*, **76**(1979) 229–234; MR **80g:10008**.

B40 Gaps between totatives.

If $a_1 < a_2 < \dots < a_{\phi(n)}$ are the integers less than n and prime to it, then Erdős conjectured that $\sum(a_{i+1} - a_i)^2 < cn^2/\phi(n)$ and offered \$500.00 for a proof. Hooley showed that, for $1 \leq \alpha < 2$, $\sum(a_{i+1} - a_i)^\alpha \ll n(\phi(n))^{\alpha-1}$ and that $\sum(a_{i+1} - a_i)^2 \ll n(\ln \ln n)^2$, Vaughan established the conjecture "on the average" and he & Montgomery finally won the prize.

Jacobsthal asked what bounds can be placed on $J(n) = \max(a_{i+1} - a_i)$. Erdős asks if, for infinitely many x , there are two integers n_1, n_2 , $n_1 < n_2 < x$, $n_1 \perp n_2$, $J(n_1) > \ln x$, $J(n_2) > \ln x$.

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B41 Iterations of ϕ and σ .

There is a close relative to the sum of divisors and the sum of the unitary divisors function, which complements Euler's totient function and which is often named for Dedekind. If $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, denote by $\psi(n)$ the product $\prod p_i^{a_i-1}(p_i + 1)$, i.e., $\psi(n) = n \prod (1 + p^{-1})$, where the product is taken over the distinct prime divisors of n . It is easy to see that iteration of the function leads eventually to terms of the form $2^a 3^b$ where b is fixed and a increases by one in successive terms. Given any value of b there are infinitely many values of n which lead to such terms, for example, $\psi^k(2^a 3^b 7^c) = 2^{a+4k} 3^b 7^{c-k}$ ($0 \leq k \leq c$) and $\psi^k(2^a 3^b 7^c) = 2^{a+5k-c} 3^b$ ($k > c$).

David E. Penney & Pomerance, in an unpublished paper, show that there are values of n for which the iterates of the function $\psi(n) - n$ are unbounded as the number of iterations tends to infinity; the least such is $n = 318$.

If we average ψ with the ϕ -function, $\frac{1}{2}(\phi + \psi)$, and iterate, we produce sequences whose terms become constant whenever they are prime powers; for example $24, \frac{1}{2}(8+48) = 28, \frac{1}{2}(12+48) = 30, \frac{1}{2}(8+72) = 40, \frac{1}{2}(16+72) = 44, \frac{1}{2}(20+72) = 46, \frac{1}{2}(22+72) = 47, \frac{1}{2}(46+48) = 47, \dots$. Charles R. Wall gives examples where iteration leads to an unbounded sequence: start with 45, 48, ... or 50, 55, ... and continue 56, 60, 80, 88, 92, 94, 95, 96, ... ; each term after the 35th is the double of the last but seven!

We can also average the σ - and ϕ -functions, and iterate. Since $\phi(n)$ is always even for $n > 2$ and $\sigma(n)$ is odd when n is a square or twice a square,

B41. Iterations of ϕ and σ .

we will sometimes get a noninteger value. For example, 54, 69, 70, 84, 124, 142, 143, 144, 225 $\frac{1}{2}$; in this case we say that the sequence **fractures**. It is easy to show that $(\sigma(n) + \phi(n))/2 = n$ just if $n = 1$ or a prime, so sequences can become constant, for example, 60, 92, 106, 107, 107, Are there sequences which increase indefinitely without fracturing?

Of course, if we iterate the ϕ -function, it eventually arrives at 1. Call the least integer k for which $\phi^k(n) = 1$ the **class** of n .

k	n
1	2
2	3 4 6
3	5 7 8 9 10 12 14 18
4	11 13 15 16 19 20 21 22 24 26 ...
5	17 23 25 29 31 32 33 34 35 37 39 40 43 ...
6	41 47 51 53 55 59 61 64 65 67 68 69 71 73 ...
7	83 85 89 97 101 103 107 113 115 119 121 122 123 125 128 ...

The set of least values of the classes is $M = \{2, 3, 5, 11, 17, 41, 83, \dots\}$. Shapiro conjectured that M contained only prime values, but Mills found several composite members. If S is the union, for all k , of the members of class k which are $< 2^k$, then

$$S = \{3; 5; 7; 11, 13, 15; 17, 23, 25, 29, 31; 41, 47, 51, 53, 55, 59, 61; 83, 85, \dots\}$$

and Shapiro showed that the factors of an element of S is also in S . Catlin showed that if m is an odd element of M , then the factors of m are in M , and that there are infinitely many primes in M just if there are infinitely many odd numbers in M . Does S contain infinitely many odd numbers? Does M contain infinitely many odd numbers?

Pillai showed that the class, $k = k(n)$, of n satisfies

$$\left\lfloor \frac{\ln n}{\ln 3} \right\rfloor \leq k(n) \leq \left\lfloor \frac{\ln n}{\ln 2} \right\rfloor$$

and it's easy to see (look at $2^a 3^b$) that $k(n)/\ln n$ is dense in the interval $[1/\ln 3, 1/\ln 2]$. What is the average and normal behavior of $k(n)$? Erdős, Granville, Pomerance & Spiro conjecture that there is a constant α such that the normal order of $k(n)$ is $\alpha \ln n$ and prove this under the assumption of the Elliott-Halberstam conjecture. They also showed that the normal order of $\phi^h(n)/\phi^{h+1}(n)$ is $h e^\gamma \ln \ln \ln n$ for each positive integer h , where γ is Euler's constant. See their paper for many unsolved problems: for example, if $\sigma^k(n)$ is the k th iterate of the sum of divisors function, they are unable to prove or disprove any of the following statements.

$$i \quad \text{for every } n > 1, \sigma^{k+1}(n)/\sigma^k(n) \rightarrow 1 \text{ as } k \rightarrow \infty \quad ?$$

- i.* for every $n > 1$, $\sigma^{k+1}(n)/\sigma^k(n) \rightarrow \infty$ as $k \rightarrow \infty$?
- i.* for every $n > 1$, $(\sigma^k(n))^{1/k} \rightarrow \infty$ as $k \rightarrow \infty$?
- i.* for every $n > 1$, there is some k with $n|\sigma^k(n)$?
- i.* for every $n, m > 1$, there is some k with $m|\sigma^k(n)$?
- i.* for every $n, m > 1$, there are some k, l with $\sigma^k(m) = \sigma^l(n)$?

Miriam Hausman has characterized those integers n which are solutions of the equation $n = m\phi^k(n)$; they are mainly of the form $2^a 3^b$.

Finucane iterated the function $\phi(n) + 1$ and asked: in how many steps does one reach a prime? Also, given a prime p , what is the distribution of the values of n whose sequences end in p ? Are 5, 8, 10, 12 the only numbers which lead to 5? And 7, 9, 14, 15, 16, 18, 20, 24, 30 the only ones leading to 7?

Erdős similarly asked about the iteration of $\sigma(n) - 1$. Does it always end on a prime, or can it grow indefinitely? In none of the cases of iteration of $\sigma(n) - 1$, of $(\psi(n) + \phi(n))/2$, or of $(\phi(n) + \sigma(n))/2$ is he able to show that the growth is slower than exponential. For several results and conjectures, consult the quadruple paper cited below.

Atanassov defines some additive analogs of ϕ and σ , poses 17 questions and answers only three of them.

Krassimir T. Atanassov, New integer functions, related to ϕ and σ functions, *Bull. Number Theory Related Topics*, 11(1987) 3–26; MR 90j:11007.

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Charles R. Wall, Unbounded sequences of Euler-Dedekind means, *Amer. Math. Monthly*, 92(1985) 587.

B42 Behavior of $\phi(\sigma(n))$ and $\sigma(\phi(n))$.

Erdős asks us to prove that $\phi(n) > \phi(n - \phi(n))$ for almost all n , but that $\phi(n) < \phi(n - \phi(n))$ for infinitely many n .

Mąkowski & Schinzel prove that $\limsup \phi(\sigma(n))/n = \infty$,

$$\limsup \phi^2(n)/n = \frac{1}{2}, \quad \text{and} \quad \liminf \sigma(\phi(n))/n \leq \frac{1}{2} + \frac{1}{2^{34} \cdot 4}$$

and they ask if $\sigma(\phi(n))/n \geq \frac{1}{2}$ for all n . They point out that even $\inf \sigma(\phi(n))/n > 0$ is not proved, but Pomerance has since established this, using Brun's method.

John Selfridge, Fred Hoffman & Rich Schroeppel found 24 solutions of $\phi(\sigma(n)) = n$, namely

2^k for $k = 0, 1, 3, 7, 15 & 31; 2^2 \cdot 3; 2^8 \cdot 3^3; 2^{10} \cdot 3^3 \cdot 11^2; 2^{12} \cdot 3^3 \cdot 5 \cdot 7 \cdot 13; 2^4 \cdot 3 \cdot 5; 2^4 \cdot 3^2 \cdot 5; 2^9 \cdot 3 \cdot 5^2 \cdot 31; 2^9 \cdot 3^2 \cdot 5^2 \cdot 31; 2^5 \cdot 3^4 \cdot 5 \cdot 11; 2^8 \cdot 3^4 \cdot 5^2 \cdot 11; 2^8 \cdot 3^4 \cdot 5 \cdot 11; 2^8 \cdot 3^4 \cdot 5^2 \cdot 11; 2^5 \cdot 3^6 \cdot 7^2 \cdot 13; 2^6 \cdot 3^6 \cdot 7^2 \cdot 13; 2^{13} \cdot 3^7 \cdot 5 \cdot 7^2; 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7^2; 2^{21} \cdot 3^2 \cdot 5 \cdot 11^3 \cdot 31; 2^{21} \cdot 3^3 \cdot 5^2 \cdot 11^3 \cdot 31;$ and there are, of course, 24 corresponding solutions of $\sigma(\phi(n)) = m$. Are there others? An infinite number?

Colomb observes that if $q > 3$ and $p = 2q - 1$ are primes and $m \in \{2, 3, 8, 9, 15\}$, then $n = pm$ is a solution of $\phi(\sigma(n)) = \phi(n)$. Undoubtedly there are infinitely many such and undoubtedly no one will prove this in the foreseeable future. There are other solutions, 1, 3, 15, 45, ...; an infinite number? He gives the solutions 1, 87, 362, 1257, 1798, 5002, 9374 to $\sigma(\phi(n)) = \sigma(n)$. He also notes that if p and $(3^p - 1)/2$ are primes (e.g., $p = 3, 7, 13, 71, 103$), then $n = 3^{p-1}$ is a solution of $\sigma(\phi(n)) = \phi(\sigma(n))$; and shows that $\sigma(\phi(n)) - \phi(\sigma(n))$ is both positive and negative infinitely often and asks what is the proportion of each?

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343 Alternating sums of factorials.

The numbers

$$\begin{aligned} 3! - 2! + 1! &= 5, \\ 4! - 3! + 2! - 1! &= 19, \\ 5! - 4! + 3! - 2! + 1! &= 101, \\ 6! - 5! + 4! - 3! + 2! - 1! &= 619, \\ 7! - 6! + 5! - 4! + 3! - 2! + 1! &= 4421, \\ 8! - 7! + 6! - 5! + 4! - 3! + 2! - 1! &= 35899 \end{aligned}$$

and

are each prime. Are there infinitely many such? Here are the factors of $n! - (n-1)! + (n-2)! - + \dots - (-1)^n n!$ for the next few values of n :

n	A_n	n	A_n
9	79 · 4139	19	15578717622022981 (prime)
10	3301819 (prime)	20	8869 · 210101 · 1229743351
11	13 · 2816537	21	113 · 167 · 4511191 · 572926421
12	29 · 15254711	22	79 · 239 · 56947572104043899
13	47 · 1427 · 86249	23	85439 · 28999390945734779
14	211 · 1679 · 229751	24	12203 · 24281 · 2010359484638233
15	1226280710981 (prime)	25	59 · 555307 · 455254005662640637
16	53 · 6581 · 56470483	26	1657 · 234384986539153832538067
17	47 · 7148742955723	27	127 ² · 271 · 163 · 2065633479970130593
18	2683 · 2261044646593	28	61 · 221171 · 21820357757749410439949

The example $n = 27$ shows that these numbers are not necessarily squarefree. Wilfrid Keller has continued the calculations for $n \leq 335$; A_n is prime for $n = 41, 59, 61, 105$ and 160.

If there is a value of n such that $n+1$ divides A_n , then $n+1$ will divide A_m for all $m > n$, and there would be only a finite number of prime values. Wagstaff established that if there is such an n , it is larger than 46340.

344 Sums of factorials.

Kurepa defines $\ln = 0! + 1! + 2! + \dots + (n-1)!$ and asks if $\ln \not\equiv 0 \pmod{n}$ for all $n > 2$. Slavić used a computer to establish this for $3 \leq n \leq 1000$. The conjecture is that $(\ln, n!) = 2$. Wagstaff has extended the calculations and verified the conjecture for $n < 50000$, and Mijajlović for $n \leq 10^6$. He notes that for $K_n = [(n+1) - 1]! + 2! + \dots + n!$ we have $3|K_n$ for $n \geq 2$, K_n for $n \geq 5$ and $99|K_n$ for $n \geq 10$. Wilfrid Keller has since extended this and found no new divisibilities for K_n with $n < 10^6$. In a 91-03-21 letter, Reg. Bond offers an as yet unpublished proof of the conjecture.

It is also conjectured that, except that 2^2 divides 13, \ln is squarefree. Mijajlović has confirmed that $m^2 \nmid \ln$ for $m \leq 1223$.

B46 The largest prime factor of n .

L. Carlitz, A note on the left factorial function, *Math. Balkanika*, 5(1975) 37–42.
 Duro Kurepa, On some new left factorial propositions, *Math. Balkanika*, 4(1974) 383–386; MR 58 #10716.

Ž. Mijajlović, On some formulas involving \ln and the verification of the \ln -hypothesis by use of computers, *Publ. Inst. Math. (Beograd) (N.S.)* 47(61) (1990) 24–32; MR 92d:11134.

B45 Euler numbers.

The coefficients in the expansion of $\sec x = \sum E_n(ix)^n/n!$ are the **Euler numbers**, and arise in several combinatorial contexts. $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, $E_8 = 1385$, $E_{10} = -50521$, $E_{12} = 2702765$, $E_{14} = -199360981$, $E_{16} = 19391512145$, $E_{18} = -2404879675441$, Is it true that for any prime $p \equiv 1 \pmod{8}$, $E_{(p-1)/2} \not\equiv 0 \pmod{p^2}$? Is it true for $p \equiv 5 \pmod{8}$?

E. Lehmer, On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson, *Annals of Math.* 39(1938) 350–360; *Zbl.* 19, 5.

Barry J. Powell, Advanced problem 6325, *Amer. Math. Monthly*, 87(1980) 826.

B46 The largest prime factor of n .

Erdős denotes by $P(n)$ the largest prime factor of n and asks if there are infinitely many primes p such that $(p-1)/P(p-1) = 2^k$? Or $= 2^k \cdot 3^l$?

If $n > 2$, then $P(n)$, $P(n+1)$, $P(n+2)$ are all distinct. Show that each of the six permutations of {low, medium, high} occurs infinitely often, and that they occur with equal frequency. $2^k - 2$, $2^k - 1$, 2^k show that medium, high, low occur for infinitely many k because $P(2^k - 1) \rightarrow \infty$ as $k \rightarrow \infty$ by a theorem of Bang (or Mahler). To see that low, medium, high occurs infinitely often, ask if $p - 1$, p , $p+1$ works for p prime. No! Try $p^2 - 1$, p^2 , $p^2 + 1$. Maybe. If $P(p^2 + 1) < p$, try $p^4 - 1$, p^4 , $p^4 + 1$. Eventually, for each prime p , there will be a value of k such that $P(p^{2^k} + 1) > p$.

Selfridge settled the low, high, medium case with 2^k , $2^k + 1$, $2^k + 2$ and Tijdeman gave the following argument for medium, low, high: consider the possibilities $2^k - 1$, 2^k , $2^k + 1$; $2^{2k} - 1$, 2^{2k} , $2^{2k} + 1$; $2^{4k} - 1$, 2^{4k} , $2^{4k} + 1$;

P. Erdős & Carl Pomerance, On the largest prime factors of n and $n+1$, *Aequationes Math.*, 17(1978) 311–321; MR 58 #476.

B47 When does $2^a - 2^b$ divide $n^a - n^b$?

Selfridge notices that $2^2 - 2$ divides $n^2 - n$ for all n , that $2^{2^2} - 2^2$ divides $n^{2^2} - n^2$ and $2^{2^3} - 2^3$ divides $n^{2^3} - n^3$ and asks for what a and b does $2^a - 2^b$ divide $n^a - n^b$ for all n . The case $n = 3$ was proposed as E2468*, Amer. Math. Monthly, 81(1974) 405 by Harry Ruderman. In his solution (83(1976) 288–289) Bill Vélez omits $(b, a - b) = (0, 1)$ as trivial and gives 13 other solutions, (1,1), (1,2), (2,2), (3,2), (1,4), (2,4), (3,4), (4,4), (2,6), (3,6), (2,12), (3,12), (4,12). Remarks by Pomerance (84(1977) 59–60) show that results of Schinzel complete Vélez's solution. The problem was also solved by Sun Qi & Zhang Ming Zhi.

A. Schinzel, On primitive prime factors of $a^n - b^n$, Proc. Cambridge Philos. Soc., 58(1962) 555–562.

Sun Qi & Zhang Ming-Zhi, Pairs where $2^a - 2^b$ divides $n^a - n^b$ for all n , Proc. Amer. Math. Soc., 93(1985) 218–220; MR 86c:11004.

B48 Products taken over primes.

David Silverman noticed that if p_n is the n -th prime, then

$$\prod_{n=1}^m \frac{p_n + 1}{p_n - 1}$$

is an integer for $m = 1, 2, 3, 4$ and 8 and asked is it ever again an integer? Equivalently, as Mąkowski observes (reference at B16), for what $n = \prod_{r=1}^m p_r$ does $\phi(n)$ divide $\sigma(n)$? For example, if $\sigma(n) = 4\phi(n)$ then $2n$ is either perfect or abundant, $\sigma(2n) \geq 4n$.

Wagstaff asked for an elementary proof (e.g., without using properties of the Riemann ζ -function) that

$$\prod \frac{p^2 + 1}{p^2 - 1} = \frac{5}{2}$$

where the product is taken over all primes. It seems very unlikely that there is a proof which doesn't involve analytical methods. At first glance it might appear that the fractions might cancel, but none of the numerators are divisible by 3. Euler's proof is

$$\prod \frac{p^2 + 1}{p^2 - 1} = \prod \frac{p^2 - 1}{(p^2 - 1)^2} = \prod \frac{1 - p^{-4}}{(1 - p^{-2})^2} = \frac{\zeta^2(2)}{\zeta(4)} = \frac{(\pi^2/6)^2}{\pi^4/90} = \frac{5}{2}.$$

This uses $\sum n^{-k} = \prod (1 - p^{-k})^{-1}$ and $\sum n^{-2} = \pi^2/6$ and $\sum n^{-4} = \pi^4/90$. Wagstaff regards the first as elementary, but not the latter two. He would

like to see a direct proof of $2(\sum n^{-2})^2 = 5 \sum n^{-4}$ or of

$$4 \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=n+1}^{\infty} \frac{1}{m^2} = 3 \sum \frac{1}{n^4}$$

B49 Smith numbers.

Albert Wilansky named **Smith numbers** from his brother-in-law's telephone number

$$4937775 = 3 \cdot 5 \cdot 5 \cdot 65837,$$

the sum of whose digits is equal to the sum of the digits of its prime factors, and they soon caught the public fancy. Trivially, any prime is a Smith number: so are 4, 22, 27, 58, 85, 94, 121, Oltikar & Wayland gave the examples $3304(10^{17} - 1)/9$ and $2 \cdot 10^{45}(10^{317} - 1)/9$ and the race to find larger and larger Smith numbers was on. Yates has given

$$10^{3913210}(10^{1031} - 1)(10^{4594} + 3 \cdot 10^{2297} + 1)^{1476}$$

with 10694985 decimal digits, but has since beaten his own record with a 13614513-digit Smith number.

Stephen K. Doig, Math Whiz makes digital discovery, The Miami Herald, 1986-08-22; Coll. Math. J., 18(1987) 80.

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Ivars Peterson, In search of special Smiths, Science News, 86-08-16, p. 105.

A. Wilansky, Smith numbers, Two-Year Coll. Math. J., 13(1982) 21.

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Samuel Yates, Smith numbers congruent to 4 (mod 9), J. Recreational Math., 19(1987) 139–141.

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- Samuel Yates, Tracking titanic, in R. K. Guy & R. E. Woodrow (eds.), The Lighter Side of Mathematics, *Proc. Strens Mem. Conf., Calgary, 1986*, Spectrum Series, Math. Assoc. of America, Washington DC, 1994.

C. Additive Number Theory

C1 Goldbach's conjecture.

One of the most infamous problems is Goldbach's conjecture that every even number greater than 4 is expressible as the sum of two odd primes. Javier Echevarria has verified it up to 2^{32} , and Matti Sinisalo to 4×10^{11} . Vinogradov proved that every *odd* number greater than $3^{3^{15}}$ is the sum of three primes and Chen Jing-Run has shown that all large enough even numbers are the sum of a prime and the product of at most two primes. Chen & Wang have reduced the number $3^{3^{15}}$ to $e^{e^{11.503}}$.

“Conjecture A” of Hardy & Littlewood (cf. **A1**, **A8**) is that the number, $N_2(n)$, of representations of an even number n as the sum of two primes, is given asymptotically by

$$N_2(n) \sim \frac{2cn}{(\ln n)^2} \prod \left(\frac{p-1}{p-2} \right),$$

where, as in **A8**, $2c \approx 1.3203$ and the product is taken over all odd prime divisors of n .

Stein & Stein have calculated $N_2(n)$ for $n < 10^5$ and have found values of n for which $N_2(n) = k$ for all $k < 1911$. It is conjectured that $N_2(n)$ takes all positive integer values. They also verified the conjecture for $n < 10^8$. Granville, van de Lune & te Riele have extended this to $2 \cdot 10^{10}$.

Let $\phi(n)$ be Euler's totient function (**B36**) so that if p is prime, $\phi(p) = p - 1$. If the Goldbach conjecture is true, then there are, for each number m , prime numbers p, q , such that

$$\phi(p) + \phi(q) = 2m.$$

If we relax the condition that p and q be prime, then it should be easier to show that there are always numbers p and q satisfying this equation. Erdős & Leo Moser ask if this can be done.

Antonio Filz defined a **prime circle** of order $2m$ to be a circular permutation of the numbers from 1 to $2m$ with each adjacent pair summing to a prime. There is essentially only one prime circle for $m = 1, 2$ and 3;

two for $m = 4$ and 48 for $m = 5$. Are there prime circles for all m ? Give an asymptotic estimate of their number.

Similarly, Margaret Kenney proposed the prime pyramid

*						
1		2				
1	2	3				
1	2	3	4			
1	4	3	2	5	6	
1	4	3	2	5	6	7
1	—	—	—	—	—	7

in which row n contains the numbers $1, 2, \dots, n$, begins with 1, ends with n , and the sum of two consecutive entries is prime. How many ways are there of arranging the numbers in row n ? This problem was also proposed by Morris Wald; the solutions given are almost certain always to work, but a proof of this may be almost as difficult as proving the Goldbach conjecture itself. The slightly less restricted problem in which the end numbers are not prescribed was earlier asked by E. T. H. Wang.

Erdős asks if there are infinitely many primes p such that every even number $\leq p - 3$ can be expressed as the difference between two primes each $\leq p$. For example, $p = 13$: $10 = 13 - 3$, $8 = 11 - 3$, $6 = 11 - 5$, $4 = 7 - 3$, $2 = 5 - 3$.

J. Bohman & C.-E. Froberg, Numerical results on the Goldbach conjecture, *BIT*, 15(1975) 239–243.

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C2 Sums of consecutive primes.

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C2 Sums of consecutive primes.

Let $f(n)$ be the number of ways of representing n as the sum of (one or more) consecutive primes. For example

$$5 = 2 + 3 \quad \text{and} \quad 41 = 11 + 13 + 17 = 2 + 3 + 5 + 7 + 11 + 13$$

so that $F(5) = 2$ and $F(41) = 3$. Leo Moser has shown that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n=1}^x f(n) = \ln 2$$

and he asks: is $f(n) = 1$ infinitely often? Is $f(n) = k$ solvable for every k ? Do the numbers for which $f(n) = k$ have a density for every k ? Is $\limsup f(n) = \infty$?

Erdős asks if there is an infinite sequence of integers $1 < a_1 < a_2 < \dots$ such that $f(n)$, the number of solutions of $a_i + a_{i+1} + \dots + a_k = n$, tends to infinity with n . He notes that if we insist that $k > i$, then it is not even known if $f(n) > 0$ for all but finitely many n . If $a_i = i$, $f(n)$ is the number of odd divisors of n .

L. Moser, Notes on number theory III. On the sum of consecutive primes, *Canad. Math. Bull.*, **6**(1963) 159–161; *MR 28* #75.

C3 Lucky numbers.

Gardiner and others define **lucky numbers** by modifying the sieve of Eratosthenes in the following way. From the natural numbers strike out all even ones, leaving the odd numbers. Apart from 1, the first remaining number is 3. Strike out every third member (those of shape $6k - 1$) in the new sequence, leaving

$$1, 3, 7, 9, 13, 15, 19, 21, 25, 27, 31, 33, \dots$$

The next number remaining is 7. Strike out every seventh term (numbers $42k - 23, 42k - 3$) in this sequence. Next 9 remains: strike out every ninth term from what's left, and so on, until we are left with the lucky numbers

$$\begin{aligned} 1, 3, 7, 9, 13, 15, 21, 25, 31, 33, 37, 43, 49, 51, 63, 67, 69, \\ 73, 75, 79, 87, 93, 99, 105, 111, 115, 127, 129, 133, 135, 141, \\ 151, 159, 163, 169, 171, 189, 193, 195, 201, 205, 211, 219, 223, \\ 231, \dots \end{aligned}$$

Many questions arise concerning lucky numbers, parallel to the classical ones asked about primes. For example, if $L_2(n)$ is the number of solutions of $l + m = n$, where n is even and l and m are lucky, then Stein & Stein find values of n such that $L_2(n) = k$ for all $k \leq 1769$, and there is a corresponding conjecture to that made in C1.

W. E. Briggs, Prime-like sequences generated by a sieve process, *Duke Math. J.*, **30**(1963) 297–312; *MR 26* #6145.

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C4 Ulam numbers.

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 M. C. Wunderlich & W. E. Briggs, Second and third term approximations of sieve-generated sequences, *Illinois J. Math.*, **10**(1966) 694–700; *MR 34* #153.

C4 Ulam numbers.

Ulam constructed increasing sequences of positive integers by starting from arbitrary u_1 and u_2 and continuing with those numbers which can be expressed in just one way as the sum of two distinct earlier members of the sequence. Recamán asked some of the questions which arise in connexion with the **U-numbers** ($u_1 = 1, u_2 = 2$).

$$\begin{aligned} 1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, \\ 62, 69, 72, 77, 82, 87, 97, 99, 102, 106, 114, 126, 131, 138, 145, \\ 148, 155, 175, 177, 180, 182, 189, 197, 206, 209, 219, \dots \end{aligned}$$

(1) Can the sum of two consecutive U-numbers, apart from $1+2=3$, be a U-number?

(2) Are there infinitely many numbers

$$23, 25, 33, 35, 43, 45, 67, 92, 94, 96, \dots$$

which are *not* the sum of two U-numbers?

(3) (Ulam) Do the U-numbers have positive density?

(4) Are there infinitely many pairs

$$(1,2), (2,3), (3,4), (47,48), \dots$$

of consecutive U-numbers?

(5) Are there arbitrarily large gaps in the sequence of U-numbers?

In answer to Question 1, Frank Owens noticed that $u_{19} + u_{20} = 62 + 69 = 131 = u_{31}$. In answer to Question 4, Muller calculated 20000 terms and found no further examples. On the other hand, more than 60% of these terms differed from another by exactly 2.

David Zeitlin asked if the sequence of U-numbers is **complete** in the sense that every positive number is expressible as the sum of distinct members of the sequence. Stefan Burr notes that this is so since, after 2, each term is less than twice the preceding one.

The reader is warned that the name "U-numbers" was also used by Mahler in the theory of algebraic numbers in connexion with Alan Baker's characterization of "S-numbers" and "T-numbers".

More generally, one can define *s-additive sequences* which are constructed in the same way, except that each term is the sum of two earlier terms in exactly *s* ways, the U-numbers corresponding to *s* = 1. If *s* = 0 the sequence is constructed from numbers which are *not* the sum of two distinct earlier members. Compare problems C9, E28 and E32 below. More generally still there are (*s, t*)-*additive sequences* where each term has exactly *s* representations as the sum of *t* distinct earlier members. In this notation, the U-numbers are the (1,2)-sequence initiated by $u_1 = 1$, $u_2 = 2$. Steven Finch has experimented in this area and has a number of conjectures. For example, that the sequences initiated by (u_1, u_2) with $u_1 < u_2$ and $u_1 \perp u_2$ contain only finitely many even terms in the cases (a) $(u_1, u_2) = (2, u_2)$ for $u_2 \geq 5$, (b) $(4, u_2)$, (c) $(5, 6)$, (d) $u_1 \geq 6$ and even, and (e) $u_1 \geq 7$ odd with u_2 even; but infinitely many even terms otherwise.

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Marvin C. Wunderlich, The improbable behaviour of Ulam's summation sequence, in *Computers and Number Theory*, Academic Press, 1971, 249–257.

C5 Sums determining members of a set.

Leo Moser asked, and Selfridge, Straus and others largely settled, to what extent the sums of all the pairs of numbers in a set determine the set. They show that if the cardinality is not a power of two, then the members are determined. Suppose that y_1, y_2, \dots, y_s are the sums $x_i + x_j$ ($i \neq j$) of the

numbers x_1, x_2, \dots, x_{2^k} , so that $s = 2^{k-1}(2^k - 1)$. Are there more than two sets $\{x_i\}$ which give rise to the same set $\{y_j\}$? If $k = 3$ there may be three such sets, for example.

$$\{\pm 1, \pm 9, \pm 15, \pm 19\}, \quad \{\pm 2, \pm 6, \pm 12, \pm 22\}, \quad \{\pm 3, \pm 7, \pm 13, \pm 21\}$$

but there can't be more than three. For $k > 3$ the problem is open. The corresponding problem where sums of *triples* of elements of a set are given is also settled, except in two cases: do the sums of three distinct elements of $\{x_1, x_2, \dots, x_n\}$ determine the set if $n = 27$ or $n = 486$? The corresponding problem for sums of *four* distinct elements was settled by Ewell.

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J. L. Selfridge & E. G. Straus, On the determination of numbers by their sums of a fixed order, *Pacific J. Math.*, 8(1958) 847–856; MR 22 #4657.

C6 Addition chains. Brauer chains. Hansen chains.

An **addition chain** for n is a sequence $1 = a_0 < a_1 < \dots < a_r = n$ with each member after the zeroth the sum of two earlier, not necessarily distinct, members. For example

$1, 1+1, 2+2, 4+2, 6+2, 8+6$ and $1, 1+1, 2+2, 4+2, 4+4, 8+6$ are addition chains for 14 of length $r = 5$. The minimal length of an addition chain for n is denoted by $l(n)$.

The main unsolved problem is the Scholz conjecture

$$i \quad l(2^n - 1) \leq n - 1 + l(n) \quad ?$$

It has been proved for $n = 2^a, 2^a + 2^b, 2^a + 2^b + 2^c, 2^a + 2^b + 2^c + 2^d$ by Utz, Gioia et al. and Knuth, and demonstrated for $1 \leq n \leq 18$ by Knuth and Thurber. Brauer proved the conjecture for those n for which a shortest chain exists which is a **Brauer chain**, that is one in which each member uses the previous member as a summand. The second of the examples is not a Brauer chain, because the term $4+4$ does not use the summand 6. Such an n is called a **Brauer number**. Hansen proved that there are infinitely many non-Brauer numbers, but also that the Scholz conjecture still holds if n has a shortest chain which is a **Hansen chain**, that is one for which there is a subset H of the members such that each member of the chain uses the largest element of H which is less than the member. The second example is a Hansen chain, with $H = \{1, 2, 4, 8\}$. Knuth gives the example

$$1, 2, 4, 8, 16, 17, 32, 64, 128, 256, 512, 1024, 1041, 2082, 4164, 8328, 8345, 12509$$

of a Hansen chain ($H = \{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 1041, 2082, 4164, 8328, 8345\}$) for $n = 12509$ which is not a Brauer chain (32 does not use 17) and no such short Brauer chain exists for $n = 12509$.

Are there non-Hansen numbers?

It is clear that $l(2n) \leq l(n) + 1$. That strict inequality is possible was shown by Knuth with $l(382) = l(191) = 11$. The smallest even n with $l(2n) = l(n)$ is 13818, given by Thurber, who also noticed the odd adjacent pair 22453, 22455. Andrew Granville asks if there are n for which $l(4n) = l(2n) = l(n)$.

D. J. Newman considers a computer which costs 1 cent to perform each addition but nothing to perform multiplication. Then the addition chain for n costs maximally $(\log n)^{\frac{1}{2}+o(1)}$ instead of $\log n$, where \log is to base 2.

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C7. The money-changing problem.

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C7 The money-changing problem.

Given $n \geq 2$ integers $0 < a_1 < a_2 < \dots < a_n$ with $(a_1, a_2, \dots, a_n) = 1$, then $N = \sum_{i=1}^n a_i x_i$ has a solution in nonnegative integers x_i if N is large enough. The well known **coin problem** of Frobenius is to determine the greatest $N = g(a_1, a_2, \dots, a_n)$ for which there is no solution. Sylvester showed that $g(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1$ and that the number of non-representable numbers is $(a_1 - 1)(a_2 - 1)/2$. The case $n = 3$ was first solved explicitly by Selmer & Beyer, using a continued fraction algorithm. Their result was simplified by Rödseth and later by Greenberg. No general formulas are known for $n \geq 4$. Roberts found the value of g if the a_i are in arithmetic progression.

Upper bounds for g are also sought. In 1942 Brauer showed that $g(a_1, a_2, \dots, a_n) \leq \sum_{i=1}^n a_i(d_{i-1}/d_i - 1)$ where $d_i = (a_1, a_2, \dots, a_i)$. Erdős & Graham showed that

$$g(a_1, a_2, \dots, a_n) \leq 2a_{n-1}\lfloor a_n/n \rfloor - a_n$$

(which is best possible if $n = 2$ and a_2 is odd). They define

$$\gamma(n, t) = \max_{\{a_i\}} g(a_1, a_2, \dots, a_n)$$

where the maximum is taken over all $0 < a_1 < a_2 < \dots < a_n \leq t$ with $(a_1, a_2, \dots, a_n) = 1$. Their theorem shows that $\gamma(n, t) < 2t^2/n$ and they proved that $\gamma(n, t) \geq t^2/(n-1) - 5t$. Lewin showed that $\gamma(3, t) = \lfloor (t-2)^2/2 \rfloor - 1$ and generally that $g(a_1, a_2, \dots, a_n) \leq \lfloor (a_{n-1}-1)(a_n-2)/2 \rfloor - 1$ for $n \geq 3$.

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C8 Sets with distinct sums of subsets.

The set of integers $\{2^i : 0 \leq i \leq k\}$, of cardinality $k+1$, has the sums of all its 2^{k+1} subsets distinct. Erdős has asked for the maximum number, m , of positive integers $a_1 < a_2 < \dots < a_m \leq 2^k$, with all sums of subsets distinct. With Leo Moser he showed that $k+1 \leq m < k + \frac{1}{2} \log k + 2$ where the logarithm is to base 2. Noam Elkies improved the constant 2 on the right to $\frac{1}{2} \log \pi < 0.826$.

Conway & Guy have given a sequence, $u_0 = 0$, $u_1 = 1$, $u_{n+1} = 2u_n - u_{n-r}$ ($n \geq 1$) where r is the nearest integer to $\sqrt{2n}$, from which may be derived the set of $k+2$ integers

$$A = \{a_i = u_{k+2} - u_{k+2-i} : 1 \leq i \leq k+2\}.$$

They conjecture that this set has subsets with distinct sums (established by Mike Guy for $k \leq 40$ and by Fred Lunnon for $n \leq 79$). For $k \geq 21$, $u_{k+2} < 2^k$, so that $m \geq k+2$ for $k \geq 21$, since once a set with the desired cardinality is found, its cardinality may be increased by doubling the size of each member and adjoining the member 1 (or any odd number). Conway

C9 Packing sums of pairs.

& Guy conjecture that A gives, in essence, the best possible solution, $m = k+2$, to the problem, though Lunnon defines a class of generalized Conway-Guy sequences, some of which give a smaller limit (e.g., 0.220963) than that of $u_n/2^n$ (≈ 0.23512531). Erdős offers \$500.00 for a proof or disproof of $m = k + O(1)$.

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C9 Packing sums of pairs.

Suppose that m is the maximum number of integers $1 \leq a_1 < a_2 < \dots < a_m \leq n$ in a **Sidon sequence**, i.e., one in which the sums of pairs, $a_i + a_j$, are all different. It is known that

$$n^{1/2}(1-\epsilon) < m \leq n^{1/2} + n^{1/4} + 1.$$

The upper bound is due to Lindstrom, improving a result of Erdős & Turán. The lower bound is due to Singer. Erdős & Turán ask, is $m = n^{1/2} + O(1)$? Erdős offers \$500 for settling this question.

Cameron & Erdős ask for an estimate of $F(n)$, the number of Sidon sequences whose members are at most n . With m as above, it is not even known if $F(n)/2^m \rightarrow \infty$, only that the upper limit is infinite. They believe that $F(n) < n^{c\sqrt{n}}$. They would also like an estimate of the number of maximal Sidon sequences (those to which no further $a \leq n$ can be adjoined).

If $\{a_k\}$ continues as an infinite sequence, Erdős & Turán proved that $\limsup a_k/k^2 = \infty$ and gave a sequence with $\liminf a_k/k^2 < \infty$. Ajtai, Komlós & Szemerédi have shown that there is such a sequence with $a_k < ck^2/\ln n$.

Erdős & Rényi proved that there is a sequence satisfying $a_k < k^{2+\epsilon}$ for which the number of solutions of $a_i + a_j = t$ is $\leq c$.

Erdős notes that $\sum_{i=1}^x a_i^{-1/2} < c(\ln x)^{1/2}$ and asks if this is best possible. He asks if it is true that

$$\frac{1}{\ln x} \sum_{a_i+a_j \leq x} \frac{1}{a_i + a_j} \rightarrow 0$$

as $x \rightarrow \infty$ and suggests that perhaps

$$\sum_{a_i+a_j < x} \frac{1}{a_i + a_j} < c_1 \ln \ln x.$$

It is known that it can be $> c_2 \ln \ln x$.

Erdős also asks if a Sidon sequence $a_1 < a_2 < \dots < a_k$ can be prolonged to a perfect difference set (see C10), i.e.,

$$a_1 < a_2 < \dots < a_k < a_{k+1} < \dots < a_{p+1} = p^2 + p + 1$$

with the differences $a_u - a_v$, $1 \leq u, v \leq p+1$, $u \neq v$, representing every nonzero residue mod $p^2 + p + 1$ exactly once?

He could not even decide if it can be prolonged to

$$a_1 < a_2 < \dots < a_k < a_{k+1} < \dots < a_n, \quad a_n < (1 + o(1))n^2,$$

i.e., if it can be made as dense as possible asymptotically.

Let $a_1 < a_2 < \dots < a_n$ be any sequence of integers. Is it true that it contains a Sidon subsequence a_{i_1}, \dots, a_{i_m} with $m = (1 + o(1))n^{\frac{1}{2}}$? Komlós, Sulyok & Szemerédi (see E11) proved this with $m > cn^{\frac{1}{2}}$.

If $f(n)$ is the number of solutions of $n = a_i + a_j$, is there a sequence with

$$\lim f(n)/\ln n = c?$$

Erdős & Turán conjecture that if $f(n) > 0$ for all sufficiently large n , or if $a_k < ck^2$ for all k , then $\limsup f(n) = \infty$; Erdős also offers \$500 for settling this question.

Graham & Sloane rephrase the question in two more obviously packing forms:

Let $v_\alpha(k)$ [respectively $v_\beta(k)$] be the smallest v such that there is a k -element set $A = \{0 = a_1 < a_2 < \dots < a_k\}$ of integers with the property that the sums $a_i + a_j$ for $i < j$ [respectively $i \leq j$] belong to $[0, v]$ and

C9. Packing sums of pairs.

represent each element of $[0, v]$ at most once. The set A associated with v_β is often called a **B_2 -sequence** (compare E28).

They give the values of v_α and v_β displayed in Table 3 and note that the bounds

$$2k^2 - O(k^{3/2}) < v_\alpha, v_\beta < 2k^2 + O(k^{36/23})$$

follow from a modification of the Erdős-Turán argument.

Table 3. Values of v_α , v_β and Exemplary Sets.

k	$v_\alpha(k)$	Example of A	$v_\beta(k)$	Example of A
2	1	{0,1}	2	{0,1}
3	3	{0,1,2}	6	{0,1,3}
4	6	{0,1,2,4}	12	{0,1,4,6}
5	11	{0,1,2,4,7}	22	{0,1,4,9,11}
6	19	{0,1,2,4,7,12}	34	{0,1,4,10,12,17}
7	31	{0,1,2,4,8,13,18}	50	{0,1,4,10,18,23,25}
8	43	{0,1,2,4,8,14,19,24}	68	{0,1,4,9,15,22,32,34}
9	63	{0,1,2,4,8,15,24,29,34}	88	{0,1,5,12,25,27,35,41,44}
10	80	{0,1,2,4,8,15,24,29,34,46}	110	{0,1,6,10,23,26,34,41,53,55}

Cilleruelo has shown that there is a sequence $\{a_k\}$, $a_k \ll k^2$ such that the sums $a_i^2 + a_j^2$ are all different.

If $g(m)$ is the largest integer n such that every set of integers of size m contains a subset of size n whose pairwise sums are distinct, then Abbott has shown that $g(m) > cm^{1/2}$ for any constant $c < \frac{2}{25}$ and all sufficiently large m .

In 1956 Erdős proved the existence of a sequence S such that all sufficiently large integers n are represented between $c_1 \ln n$ and $c_2 \ln n$ times as the sum of two members of S , and more recently Erdős & Teli have obtained the corresponding result for the sum of k members of S .

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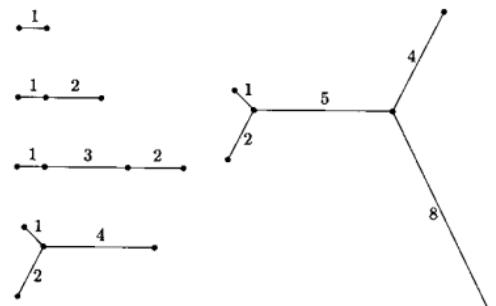
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C10 Modular difference sets and error correcting codes.

Singer's result, mentioned in C9, is based on **perfect difference sets**, i.e., a set of residues $a_1, a_2, \dots, a_{k+1} \pmod{n}$ such that every nonzero residue \pmod{n} can be expressed uniquely in the form $a_i - a_j$. For example, $\{1, 2, 4\} \pmod{7}$ and $\{1, 2, 5, 7\} \pmod{13}$. Perfect difference sets can exist only if $n = k^2 + k + 1$, and Singer proved that such a set exists whenever k is a prime power. Marshall Hall has shown that numerous non-prime-powers cannot serve as values of k and Evans & Mann that there is no such $k < 1600$ that is not a prime power. It is conjectured that no perfect difference set exists unless k is a prime power.

Can a given finite sequence, which contains no repeated differences, always be extended to form a perfect difference set?

Perfect difference sets may be used to make **Golomb rulers**. Subtract one from the elements of the difference set, e.g., $\{0, 1, 4, 6\}$ and take these as marks on a ruler of length 6, which can be used to measure all the lengths 1, 2, 3, 4, 5, 6. More generally we may look for less than perfect rulers of length n with $k+1$ marks $\{0, a_1, \dots, a_{k-1}, n\}$ subject to various conditions. E.g., (a) all $\binom{k+1}{2}$ distances distinct, (b) maximum number of distinct distances for given n and k , (c) all integer distances from 1 up to some maximum e to be measurable. We cannot satisfy all of these conditions if $k \geq 4$, but Leech has found examples of perfect 'joined' rulers. The trees



have edges with the lengths shown, and may be used to measure all lengths from 1 up to 1, 3, 6, 6, 15.

Gibbs & Slater, Herbert Taylor and Yang Yuan-Sheng have improved Leech's results for paths and for more general trees to

n	2	3	4	5	6	7	8	9	10	11	12
paths	1	3	6	9	13	18	24	29	37	45	(51)
trees	1	3	6	9	15	20	26	34	41	(48)	(55)

where the entries in parentheses are not necessarily best possible. There are connexions with the graceful labelling and harmonious labelling of graphs; see C13 and a possibly forthcoming combinatorics volume in this series.

Dean Hickerson asks for the maximum number n such that the integers $1 \leq a_1 < a_2 < \dots < a_m \leq n$ have differences $a_j - a_i$, $j > i$, among which the integer s occurs at most $2s$ times.

Graham & Sloane exhibit the problem of difference sets as the modular version of the packing problems of C9. They define $v_s(k)$ [respectively $v_s(k)$] as the smallest number v such that there exists a subset $A = \{0 = a_1 < a_2 < \dots < a_k\}$ of the integers \pmod{v} with the property that each

r can be written in at most one way as $r \equiv a_i + a_j \pmod{v}$ with $i < j$ [respectively $i \leq j$].

Their interest in v_γ is in its application to **error-correcting codes**. If $A(k, 2d, w)$ is the maximum number of binary vectors with w ones and $k-w$ zeros (**words of length k and weight w**) such that any two vectors differ in at least $2d$ places, then (for $d = 3$)

$$A(k, 6, w) \geq \binom{k}{w} / v_\gamma(k)$$

(and the result for general d uses sets for which all sums of $d - 1$ distinct elements are distinct modulo v).

They note that $A(k, 2d, w)$ has been studied by Erdős & Hanani, by Schönheim, and by Stanton, Kalbfleisch & Mullin in the context of extremal set theory. Let $D(t, k, v)$ be the maximum number of k -element subsets of a v -element set S such that every t -element subset of S is contained in at most one of the k -element subsets. Then $D(t, k, v) = A(v, 2k - 2t + 2, k)$.

The values of v_6 in Table 4 are from Baumert's Table 6.1 and those of v_γ from Graham & Sloane who give the following bounds

$$k^2 - O(k) < v_\gamma(k) < k^2 + O(k^{36/23}),$$

$$k^2 - k + 1 \leq v_6(k) < k^2 + O(k^{36/23}).$$

Equality holds on the left of the latter whenever $k - 1$ is a prime power.

Table 4. Values of v_γ , v_6 and Exemplary Sets.

k	$v_\gamma(k)$	Example of A	$v_6(k)$	Example of A
2	2	{0,1}	3	{0,1}
3	3	{0,1,2}	7	{0,1,3}
4	6	{0,1,2,4}	13	{0,1,3,9}
5	11	{0,1,2,4,7}	21	{0,1,4,14,16}
6	19	{0,1,2,4,7,12}	31	{0,1,3,8,12,18}
7	28	{0,1,2,4,8,15,20}	48	{0,1,3,15,20,38,42}
8	40	{0,1,5,7,9,20,23,35}	57	{0,1,3,13,32,36,43,52}
9	56	{0,1,2,4,7,13,24,32,42}	73	{0,1,3,7,15,31,36,54,63}
10	72	{0,1,2,4,7,13,23,31,39,59}	91	{0,1,3,9,27,49,56,61,77,81}

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C11 Three-subsets with distinct sums.

One can generalize the ideas of C9 and E28 and define a **B_h -sequence** to be one in which the sums of h terms are distinct. Bose & Chowla showed that if $A_h(n)$ is the largest cardinality of a B_h -sequence in $[1, n]$, then

$$A_h(n) \geq n^{1/h}(1 + o(1)).$$

In the opposite direction, Jia has shown that if $h = 2k$, then

$$A_h(n) \leq k^{1/2k}(k!)^{1/k}n^{1/h}(1 + o(1)).$$

For $h = 2k - 1$, Chen and Graham independently proved the bound

$$A_h(n) \leq (k!)^{2/(2k-1)} n^{1/h} (1 + o(1)).$$

For $h = 3$ Graham obtained the further small improvement

$$A_3(n) \leq (4 - \frac{1}{228})^{\frac{1}{3}} n^{\frac{1}{3}} (1 + o(1)).$$

In the infinite case, Erdős offers \$500.00 for a proof or disproof of

$$\liminf_{n \rightarrow \infty} \frac{A_h(n)}{n^{1/h}} = 0$$

For $h = 2$ this was proved by Erdős himself, and for $h = 4$ by Nash. The case $h = 6$ was treated by Jia, and more generally Chen showed that if $h = 2k$ was even, then

$$\liminf_{n \rightarrow \infty} A_h(n) \left(\frac{\ln n}{n} \right)^{\frac{1}{h}} < \infty$$

It remains an open problem to prove this when h odd.

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C12. The postage stamp problem.

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C12 The postage stamp problem.

The covering problem which is dual to the packing problem C9 goes back at least to Rohrbach. A popular form of it concerns the design of a set of integer denominations of postage stamp, $A_k = \{a_1, a_2, \dots, a_k\}$ with $1 = a_1 < a_2 < \dots < a_k$ to be used on envelopes with room for at most h stamps, so that all integer amounts of postage up to a given bound can be affixed. What is the smallest integer $N(h, A_k)$ which is *not* representable by a linear combination $\sum_{i=1}^k x_i a_i$ with $x_i \geq 0$ and $\sum_{i=1}^k x_i \leq h$? The number of consecutive possible amounts of postage, $n(h, A_k) = N(h, A_k) - 1$ is called the *h-range* (German: *h-Reichweite*) of A_k . In this context A_k is called an *additive basis of order h* or *h-basis*. At first the main interest was in the ‘global’ problem: Given h and k , find an *extremal basis* A_k^* with largest possible *h-range*, $n(h, k) = n(h, A_k^*) = \max_{A_k} n(h, A_k)$. More recently the ‘local’ aspect has come more into focus: Find $n(h, A_k)$ when h and a particular basis A_k are given.

The local problem is completely solved only for $k = 2$ and $k = 3$. Trivially $n(h, A_2) = (h + 3 - a_2)a_2 - 2$ for $h \geq a_2 - 2$. Rødseth developed a general method, based on a continued fraction algorithm, for determining $n(h, A_3)$. From this, Selmer derived explicit formulas covering (asymptotically) about 99% of all A_3 .

From the formula for $n(h, A_2)$ Stöhr concluded that

$$n(h, 2) = \lfloor (h^2 + 6h + 1)/4 \rfloor.$$

The global problem for $k = 3$ was solved by Hofmeister, who showed in particular that, for $h \geq 20$,

$$n(h, 3) = \frac{4}{3} \left(\frac{h}{3} \right)^3 + 6 \left(\frac{h}{3} \right)^2 + Ah + B$$

where A and B depend on the residue of h modulo 9. Mossige showed that

$$n(h, 4) \geq 2.008 \left(\frac{h}{4} \right)^4 + O(h^3)$$

and, together with Kirfel (so far unpublished) that this bound is sharp. Kirfel has also shown that the limit

$$c_k = \lim_{h \rightarrow \infty} \frac{n(h, k)}{(h/k)^k}$$

exists for all $k \geq 1$. Kolsdorf showed that

$$n(h, 5) \geq 3.06 \left(\frac{h}{5}\right)^5 + O(h^4).$$

Mrose showed that

$$n(h_1 + h_2, k_1 + k_2) \geq (n(h_1, k_1) + 1) \cdot (n(h_2, k_2) + 1) \quad (*)$$

for all positive integers $h - 1, h_2, k_1, k_2$ and deduced that if k_i ($i = 1, 2$) are fixed and

$$n(h, k_i) \geq \alpha_i \left(\frac{h}{k_i}\right)^{k_i} + O(h^{k_i-1})$$

then

$$n(h, k_1 + k_2) \geq \alpha_1 \alpha_2 \left(\frac{h}{k_1 + k_2}\right)^{k_1 + k_2} + O(h^{k_1 + k_2 - 1}).$$

Thus if x_i are fixed nonnegative integers with $k = \sum_{i=1}^5 x_i$ then

$$n(h, k) \geq (3.06)^{x_5} (2.008)^{x_4} \left(\frac{4}{3}\right)^{x_3} \left(\frac{h}{k}\right)^k + O(h^{k-1})$$

The best general upper bound for fixed k is due to Rødseth:

$$n(h, k) \leq \frac{(k-1)^{k-2}}{(k-2)!} \left(\frac{h}{k}\right)^k + O(h^{k-1}).$$

For fixed h , emphasis has been on the case $h = 2$. In 1937 Rohrbach showed that

$$c_1 \left(\frac{k}{2}\right)^2 + O(k) \leq n(2, k) \leq c_2 \left(\frac{k}{2}\right)^2 + O(k)$$

with $c_1 = 1$ and $c_2 = 1.9968$. After several improvements the best known results are $c_1 = \frac{8}{7}$ (Mrose) and $c_2 = 1.9208$ (Klotz). Windecker showed that

$$n(3, k) \geq \frac{4}{3} \left(\frac{k}{3}\right)^3 + \frac{16}{3} \left(\frac{k}{3}\right)^2 + O(k).$$

Again, with (*), if y_i are fixed nonnegative integers with $h = \sum_{i=1}^3 y_i$ then

$$n(h, k) \geq \left(\frac{4}{3}\right)^{y_3} \left(\frac{8}{7}\right)^{y_2} \left(\frac{k}{h}\right)^h + O(h^{h-1}).$$

Graham & Sloane (compare C9, C10) define $n_{\alpha(k)}$ [respectively $n_{\beta(k)}$] as the largest number n such that there is a k -element set $A = \{0 = a_1 < a_2 < \dots < a_k\}$ of the integers with the property that each r in $[1, n]$ can be written in at least one way as $r = a_i + a_j$ with $i < j$ [respectively $i \leq j$], so that their $n_{\beta(k)}$ is here written $n(2, k-1)$, and their $n_{\alpha(k)}$

corresponds to the problem of two stamps of different denominations, with a zero denomination included.

They give the values for $n_{\alpha(k)}$ and $n_{\beta(k)}$ in Table 5.

Table 5. Values for $n_{\alpha(k)}$ and $n_{\beta(k)}$ and Exemplary Sets.

k	$n_{\alpha(k)}$	Example of A	$n_{\beta(k)}$	Example of A
2	1	{0,1}	2	{0,1}
3	3	{0,1,2}	4	{0,1,2}
4	6	{0,1,2,4}	8	{0,1,3,4}
5	9	{0,1,2,3,6}	12	{0,1,3,5,6}
6	13	{0,1,2,3,6,10}	16	{0,1,3,5,7,8}
7	17	{0,1,2,3,4,8,13}	20	{0,1,2,5,8,9,10}
8	22	{0,1,2,3,4,8,13,18}	26	{0,1,2,5,8,11,12,13}
9	27	{0,1,2,3,4,5,10,16,22}	32	{0,1,2,5,8,11,14,15,16}
10	33	{0,1,2,3,4,5,10,16,22,28}	40	{0,1,3,4,9,11,16,17,19,20}
11	40	{0,1,2,4,5,6,10,13,20,27,34}	46	{0,1,2,3,7,11,15,19,21,22,24}
12	47	{0,1,2,3,6,10,14,18,21,22,23,24}	54	{0,1,2,3,7,11,15,19,23,25}
13	56	{0,1,2,4,6,7,12,14,17,21,30,39,48}	64	{0,1,3,4,9,11,16,21,23,28,29,31,32}
14	65	{0,1,2,4,6,7,12,14,17,21,30,39,48,57}	72	{0,1,3,4,9,11,16,20,25,27,32,33,35,36}

Tables for more general $n(h, k)$ were computed by Lunnon and extended by Mossige and recently by Challis.

Serious students of these problems will consult the encyclopaedic three volumes of Selmer, which contain 121 references. A useful summary, with 47 references, has been prepared by Djawadi & Hofmeister.

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C13. The corresponding modular covering problem.

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C13 The corresponding modular covering problem. Harmonious labelling of graphs.

Just as **C10** was the modular version of the packing problem **C9**, so we can propose the modular version of the corresponding covering problem **C12**.

Graham & Sloane complete their octad of definitions with $n_\gamma(k)$ [respectively $n_\delta(k)$] as the largest number n such that there is a subset $A = \{0 = a_1 < a_2 < \dots < a_k\}$ of the residue classes modulo n with the property that each r can be written in at least one way as $r \equiv a_i + a_j \pmod{n}$ with $i < j$ [respectively $i \leq j$].

They call a connected graph with v vertices and $e \geq v$ edges **harmonious** if there is a labelling of the vertices x with distinct labels $l(x)$ so that when an edge xy is labelled with $|l(x) - l(y)|$, the edge labels form a complete system of residues \pmod{e} . Trees (for which $e = v - 1$) are also called harmonious if just one vertex label is duplicated and the edge labels form a complete system $\pmod{v - 1}$. The connexion with the present problem is that $n_\gamma(v)$ is the greatest number of edges in any harmonious graph on v vertices.

For example, from Table 6 we note that $n_\gamma(5) = 9$ is attained by the set $\{0,1,2,4,7\}$ so that a maximum of 9 edges can occur in a harmonious graph on 5 vertices (Figure 6).

Table 6. Values of n_γ , n_δ and Exemplary Sets.

k	$n_\gamma(k)$	Example of A	$n_\delta(k)$	Example of A
2	1	—	3	$\{0,1\}$
3	3	$\{0,1,2\}$	5	$\{0,1,2\}$
4	6	$\{0,1,2,4\}$	9	$\{0,1,3,4\}$
5	9	$\{0,1,2,4,7\}$	13	$\{0,1,2,6,9\}$
6	13	$\{0,1,2,3,6,10\}$	19	$\{0,1,3,12,14,15\}$
7	17	$\{0,1,2,3,4,8,13\}$	21	$\{0,1,2,3,4,10,15\}$
8	24	$\{0,1,2,4,8,13,18,22\}$	30	$\{0,1,3,9,11,12,16,26\}$
9	30	$\{0,1,2,4,10,15,17,22,28\}$	35	$\{0,1,2,7,8,11,26,29,30\}$
10	36	$\{0,1,2,3,6,12,19,20,27,33\}$		

Graham & Sloane compare and contrast harmonious graphs with graceful graphs, which will be discussed in the graph theory chapter of a later volume in this series. A graph is **graceful** if, when the vertex labels are chosen from $[0, e]$ and the edge labels are calculated by $|l(x) - l(y)|$, the latter are all distinct (i.e., take the values $[1, e]$).

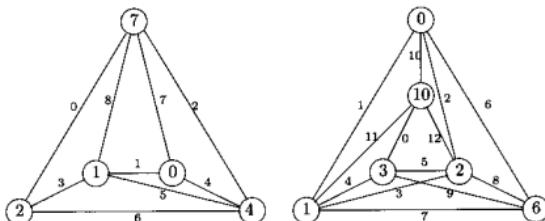


Figure 6. Maximal Harmonious Graphs.

Trees are conjectured to be both harmonious and graceful, but these are open questions. A cycle C_n is harmonious just if n is odd, and graceful just if $n \equiv 0$ or $3 \pmod{4}$. The friendship graph or windmill is harmonious just if $n \not\equiv 2 \pmod{4}$ and graceful just if $n \equiv 0$ or $1 \pmod{4}$. Fans and wheels are both harmonious and graceful, as is the Petersen graph. The graphs of the five Platonic solids are naturally graceful and one would expect them to be harmonious, but this is not so for the cube or octahedron. Joseph Gallian maintains a bibliography of graph labelling.

Joseph A. Gallian, A survey – recent results, conjectures and open problems in labeling graphs, *J. Graph Theory*, **13**(1989) 491–504; MR 90k:05132.

C14 Maximal sum-free sets.

Denote by $l(n)$ the largest l so that if a_1, a_2, \dots, a_m are any distinct natural numbers, one can always find l of them so that $a_{i_j} + a_{i_k} \neq a_m$ for $1 \leq j < k \leq l$, $1 \leq m \leq n$. Note that $j \neq k$, else the set $\{a_i = 2^i \mid 1 \leq i \leq n\}$ would imply that $l(n) = 0$. A remark of Klärner shows that $l(n) > c \ln n$. On the other hand, the set $\{2^i + 0, \pm 1 \mid 1 \leq i \leq s+1\}$ implies that $l(3s) < s+3$, so $l(n) < \frac{1}{3}n+3$. Selfridge extends this by using the set $\{(3m+t)2^{m-1} \mid -t < t < i, 1 \leq i \leq m\}$ to show that $l(m^2) < 2m$. Choi, using sieve methods, has further improved this to $l(n) \ll n^{0.4+e}$.

The problem can be generalized to ask if, for every l , there is an $n_0 = n_0(l)$ so that if $n > n_0$ and a_1, a_2, \dots, a_n are any n elements of a group with no product $a_{i_1}a_{i_2} = e$, the unit [here i_1, i_2 may be equal, so there is no a_i of order 1 or 2, and no a_i whose inverse is also an a_i] then there are l of the a_i such that $a_{i_j}a_{i_k} \neq a_m$, $1 \leq j < k \leq l$, $1 \leq m \leq n$. This has not even been proved for $l = 3$.

For the generalization to sets containing no solution of $a_1x_1 + \dots + a_kx_k = x_{k+1}$ see the papers of Funar and Moree.

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C15 Maximal zero-sum-free sets.

Erdős & Heilbronn asked for the largest number $k = k(m)$ of distinct residue classes, modulo m , so that no subset has sum zero. For example, the set

$$1, -2, 3, 4, 5, 6$$

shows that $k(20) \geq 6$, and in fact equality holds. The pattern of this example shows that

$$k \geq \lfloor (-1 + \sqrt{8m+9})/2 \rfloor \quad (m \geq 5)$$

Equality holds for $5 \leq m \leq 24$. However, Selfridge observes that if m is of the form $2(l^2 + l + 1)$, the set

$$1, 2, \dots, l-1, l, \frac{1}{2}m, \frac{1}{2}m+1, \dots, \frac{1}{2}m+l$$

implies that

$$k \geq 2l+1 = \sqrt{2m-3}$$

In fact he conjectures that, for any even m , this set or the set with l deleted always gives the best result. For example, $k(42) \geq 9$.

On the other hand, if p is a prime in the interval

$$\frac{1}{2}k(k+1) < p < \frac{1}{2}(k+1)(k+2)$$

he conjectures that $k(p) = k$, where the set can be simply

$$1, 2, \dots, k$$

The case $k(43) = 8$ was confirmed by Clement Lam, so k is not a monotonic function of m .

The only case where a better inequality is known than $k \geq \lfloor \sqrt{2m-3} \rfloor$ is $k(25) \geq \sqrt{50-1} = 7$, as is shown by the set 1, 6, 11, 16, 21, 5, 10. If m is of the form $25t(t+1)/2$ and odd, then it is possible to improve on the set 1, -2, 3, 4, ..., but if m is of that form and even, then the construction already given for m even is always better.

$I_s = \{(-1 + \sqrt{8m+9})/2\}$ for an infinity of values of m ?

For which values of m are there realizing sets none of whose members are prime to m ? For example, $m = 12: \{3, 4, 6, 10\}$ or $\{4, 6, 9, 10\}$. Is there a value of m for which all realizing sets are of this type?

Erdős & Heilbronn proved that if $a_1, a_2, \dots, a_k, k \geq 3(6p)^{1/2}$, are distinct residues mod p , where p is prime, then every residue mod p can be written in the form $\sum_{i=1}^k \epsilon_i a_i$, $\epsilon_i = 0$ or 1. They conjectured that the same holds for $k > 2\sqrt{p}$ and that this is best possible and Olsen proved this. They further conjectured that the number, s , of distinct residues of the form $a_i + a_j$, $1 \leq i < j \leq k$, is at least $\min\{p, 2k-3\}$. Partial results have been obtained by Mansfield, by Rødseth and by Freiman, Low & Pitman. Dias da Silva & Hamidoune gave a complete proof of the Erdős-Heilbronn conjecture, and in fact proved that if A^h denotes the set of all sums of h distinct elements of A , $A \subseteq \mathbb{Z}/p\mathbb{Z}$, $|A| = k$, then $|A^h| \geq \min\{p, hk - h^2 + 1\}$. Nathanson simplifies their proof and, with Ruzsa, shows that if $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$, $|A| = k > l = |B|$, then $|\{a+b : a \in A, b \in B, a \neq b\}| \geq \min\{p, k+l-2\}$.

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C16. Nonaveraging sets. Nondividing sets.

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C16 Nonaveraging sets. Nondividing sets.

A nonaveraging set A of integers $0 \leq a_1 < a_2 < \dots < a_n \leq x$ was defined by Erdős & Straus by the property that no a_i shall be the arithmetic mean of any subset of A with more than one element. Denote by $f(x)$ the maximum number of elements in such a set, and by $g(x)$ the maximum number of elements in a subset B of the integers $[0, x]$ such that no two distinct subsets of B have the same arithmetic mean, and by $h(x)$ the corresponding maximum where the subsets of B have different cardinality. Erdős & Straus and Abbott showed (by using Szemerédi's E10 result) that

$$\frac{1}{10} \log x + O(1) < \log f(x) < \frac{2}{3} \log x + O(1)$$

$$\frac{1}{2} \log x - 1 < g(x) < \log x + O(\ln \ln x)$$

$$\sqrt{\ln x} - 1 + O(1/\sqrt{\ln x}) < \log h(x) < 2 \log \ln x + O(1)$$

and conjecture that $f(x) = \exp(c\sqrt{\ln x}) = o(x^\epsilon)$ and that $h(x) = (1 + o(1))\log x$. [$\log x = (\ln x)/(\ln 2)$ is the logarithm to base 2.] Since then Abbott has improved the constant $\frac{1}{10}$ to $\frac{1}{5}$ and Bosznay to $\frac{1}{4}$ and Erdős & Sárközy have lowered the upper bound on $\log f(x)$ to $\frac{1}{2}(\log x + \log \ln x) + O(1)$.

Erdős originally asked for the maximum number, $k(x)$, of integers in $[0, x]$ so that no one divides the sum of any others. Such **nondividing sets** are obviously nonaveraging, so $k(x) \leq f(x)$. Straus showed that $k(x) \geq \max\{f(x/f(x)), f(\sqrt{x})\}$.

Abbott has shown that if $l(n)$ is the largest m such that *every* set of n integers contains a nonaveraging subset of size m , then $l(n) > n^{1/13-\epsilon}$.

Compare C14–16 with E10–14.

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C17 The minimum overlap problem.

Let $\{a_i\}$ be an arbitrary set of n distinct integers, $1 \leq a_i \leq 2n$, and $\{b_j\}$ be the complementary set $1 \leq b_j \leq 2n$, with $b_j \neq a_i$. M_k is the number of solutions of $a_i - b_j = k$ ($-2n < k < 2n$) and $M = \min \max_k M_k$, where the minimum is taken over all sequences $\{a_i\}$. Erdős proved that $M > n/4$; Scherk improved this to $M > (1 - 2^{-1/2})n$ and Swierczkowski to $M > (4 - \sqrt{6})n/5$. Leo Moser obtained the further improvements $M > \sqrt{2}(n-1)/4$ and $M > \sqrt{4 - \sqrt{15}}(n-1)$. In the other direction, Motzkin, Ralston & Selfridge obtained examples to show that $M < 2n/5$, contrary to Erdős's conjecture that $M = \frac{1}{2}n$. Is there a number c such that $M \sim cn$?

Leo Moser asks the corresponding question where the cardinality of $\{a_i\}$ is not n , but k , where $k = \lfloor \alpha n \rfloor$ for some real α , $0 < \alpha < 1$.

C18. The n queens problem.

A closely related problem is attributed to J. Czipszer: Let $A_k = \{a_1 + k, a_2 + k, \dots, a_n + k\}$ where $a_1 < a_2 < \dots < a_n$ are arbitrary integers and $k \geq 0$. Let M_k be the number of elements of A_k not in $A - 0$ and $M = \min_{A_k} \max_{0 < k \leq n} M_k$. Czipszer proved that $n/2 \leq M \leq 2n/3$ and conjectured that $M = 2n/3$. Katz & Schnitzer showed that $M > 0.6n$ for $n \geq 26$. Moser & Murdeshwar considered the continuous analog.

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C18 The n queens problem.

What is the minimum number of Queens which can be placed on an $n \times n$ chessboard so that every square is either occupied or attacked by a Queen? Gerge noted that, in graph theory language, this is the same as finding the minimum externally stable set for a graph on 64 vertices with two vertices joined just if they are on the same rank, file or diagonal. In his notation $\beta = 5$ for Queens (Figure 7a), 8 for Bishops (Figure 7b) and 12 for Knights (Figure 7c).

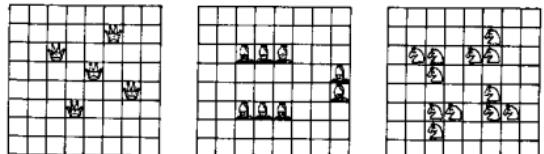


Figure 7. Minimum Covers of the Chessboard by Queens, Bishops and Knights.

Although there is no condition that a piece may not guard another piece, this condition is satisfied by the Queens and Bishops, but not by the Knights. Since in Chess a piece does not attack the square that it stands on, there are in fact two sets of problems. For example, Victor Meally notes

that, if queens are allowed to guard one another, 3 queens (at a6, c2, e4) suffice on a 6×6 board, and 4 on a 7×7 board.

For the Queens, Kraitchik gave the following table for an $n \times n$ board.

	5	6	7	8	9	10	11	12	13	14	15	16	17
number of Queens	3	4	5	5	5	5	6	7	8	9	9	9	9

Corresponding configurations for $n = 5, 6, 11$ are shown in Figure 8.

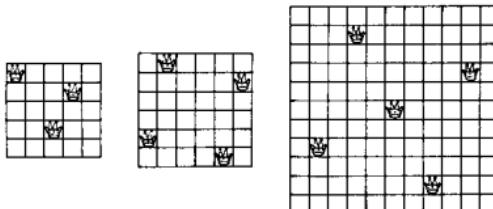


Figure 8. Queens Covering $n \times n$ Boards for $n = 5, 6$ and 11.

If we try to partition the numbers from 1 to $2n$ into n pairs a_i, b_i so that the $2n$ numbers $a_i \pm b_i$ fall one in each residue class modulo $2n$, then it is found to be impossible. Less restrictively, Shen & Shen asked that the $2n$ numbers $a_i \pm b_i$ be distinct. They gave examples for $n = 3$: 1, 5; 2, 3; 4, 6; for $n = 6$: 1, 10; 2, 6; 3, 9; 4, 11; 5, 8; 7, 12; and $n = 8$: 1, 10; 2, 14; 3, 16; 4, 11; 5, 9; 6, 12; 7, 15; 8, 13; and Selfridge showed that there was always a solution for $n \geq 3$. How many solutions are there for each n ?

If the condition $b_i = i$ ($1 \leq i \leq n$) is added, we have the reflecting Queens problem: place n queens on an $n \times n$ chessboard so that no two are on the same rank, file or diagonal, where, on a diagonal, we include reflexions in a mirror in the centre of the zero column (Fig. 9).

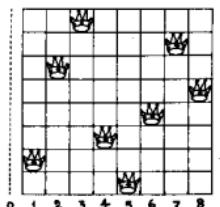


Figure 9. A Solution of the Reflecting Queens Problem.

C19. Weakly and strongly independent sequences.

We can again ask for the number of solutions for each n , both in the case where we distinguish between solutions obtained by rotation and reflexion and where we do not.

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C19 Is a weakly independent sequence the finite union of strongly independent ones?

Selfridge calls a set of positive integers $a_1 < a_2 < \dots < a_k$ independent if $\sum c_i a_i = 0$ (where the c_i are integers, not all 0) implies that at least one of the c_i is < -1 . By using the pigeonhole principle it is easy to show that if k positive integers are independent, then a_1 is at least 2^{k-1} . He offers \$10.00 for an answer to the question: is the set of k independent integers $a_i = 2^k - 2^{k-i}$ ($1 \leq i \leq k$) the only set with largest member less than 2^k ? It is the only such set with $a_1 = 2^{k-1}$.

Call a(n infinite) sequence $\{a_i\}$ of positive integers **weakly independent** if any relation $\sum \epsilon_i a_i$ with $\epsilon_i = 0$ or ± 1 and $\epsilon_i = 0$ except finitely often, implies $\epsilon_i = 0$ for all i , and call it **strongly independent** if the same is true with $\epsilon_i = 0, \pm 1$, or ± 2 . Richard Hall asks if every weakly independent sequence is a finite union of strongly independent sequences.

J. L. Selfridge, Problem 123, *Pi Mu Epsilon J.*, 3(1959-64) 118, 413-414.

C20 Sums of squares.

Paul Turán asks for a characterization of those positive integers n which can be expressed as the sum of four pairwise coprime squares, i.e., $n = x_1^2 + x_2^2 + x_3^2 + x_4^2$ with $x_i \perp x_j$ ($1 \leq i < j \leq 4$). Leech notes that at most one of the x_i is even, so that $n \equiv 3, 4$ or $7 \pmod{8}$. Similarly, numbers $n \equiv 2 \pmod{3}$ are not so representable.

Turán also conjectured that all positive integers can be represented as the sum of at most five pairwise coprime squares, but Mąkowski (see reference at B5) notes that numbers $4^k(24l+15)$ with $k \geq 2$ can't be represented in this way. There are arbitrarily large integers that are not representable as the sum of *exactly* five coprime squares, since $3n = x_1^2 + \dots + x_5^2$ implies that 3 divides two of the x_i . In fact, the product of two distinct primes of shape $24k+7$ cannot be the sum of fewer than ten pairwise coprime squares, and Leech asks if this is the record.

Apart from 256 examples, the largest of which is 1167, every number can be expressed as the sum of at most five *composite* numbers.

Chowla conjectures that every positive integer is the sum of at most four elements of the set $\{(p^2 - 1)/24\}$ where p is prime and $p \geq 5$. The smallest number which requires four such summands is 33.

Compare these problems with some earlier results of Wright, who showed, for example, that if $\lambda_1, \dots, \lambda_4$ were given real numbers with sum 1, then every n with sufficiently large odd factor is expressible as $n = m_1^2 + \dots + m_4^2$ with $|m_i^2 - \lambda_i n| = o(n)$. He has similar results for five or more squares, and for three squares (provided, of course, that n is not of the form $4_k(8l+7)$ in this last case).

Bohman, Fröberg & Riesel showed that there are just 31 numbers which can't be expressed as the sum of *distinct* squares, and that all numbers greater than 188 can be expressed as the sum of at most five distinct squares. Only two numbers, 124 and 188, require six distinct squares. The results are implicit in Gordon Pall's Theorem 3, and in Sprague's papers, which give the more general result for any powers. If an exact number of squares is required, then Halter-Koch showed that every integer > 412 and not divisible by 8 is a sum of four distinct nonzero squares, and that every odd integer > 157 is so representable. He also showed that every integer > 245 is the sum of five distinct nonzero squares, every integer > 333 is the

C20. Sums of squares.

sum of six such, every integer > 390 the sum of seven such, every integer > 462 the sum of eight such, and so on, up to all > 1036 being the sum of twelve such squares. Bateman, Hildebrand & Purdy have produced a sequel to Halter-Koch's paper.

Denote by s_n the largest integer that is *not* the sum of distinct n th powers of positive integers. Sprague showed that $s_2 = 128$; Graham reported that $s_3 = 12758$ and Lin used his method to obtain $s_4 = 5134240$. Cam Patterson used his sieve, and a result of Richert, to obtain $s_5 = 67898771$.

Štefan Porubský used a result of Cassels to give an affirmative answer to a question of R. E. Dressler: for each positive integer k , is every sufficiently large positive integer the sum of distinct k -th powers of primes?

One can also ask for every number to be expressible as the sum of as few as possible polygonal numbers of various kinds. For example there is Gauf's famous 1796-07-10 diary entry:

$$\text{ETPHKA!} \quad \text{num} = \Delta + \Delta + \Delta,$$

i.e., every number is expressible as the sum of three triangular numbers. For hexagonal numbers $r(2r-1)$, the answer is the same if you allow hexagonal numbers of negative rank, $r(2r+1)$, but if these are excluded, then 11 and 26 require six hexagonal numbers of positive rank to represent them. Is it possible that every sufficiently large number is expressible as the sum of three such numbers? Equivalently, is every sufficiently large number $8n+3$ expressible as the sum of three squares of numbers of shape $4r-1$, with r positive? The corresponding question for pentagonal numbers $\frac{1}{2}r(3r-1)$ is to ask if every sufficiently large number of shape $24n+3$ is expressible as the sum of three squares of numbers of shape $6r-1$?

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D. Diophantine Equations

“A subject which can be described briefly by saying that a great part of it is concerned with the discussion of the rational or integer solutions of a polynomial equation $f(x_1, x_2, \dots, x_n) = 0$, with integer coefficients. It is well known that for many centuries, no other topic has engaged the attention of so many mathematicians, both professional and amateur, or has resulted in so many published papers.”

This quotation from the preface of Mordell’s book, *Diophantine Equations*, Academic Press, London, 1969, indicates that in this section we shall have to be even more eclectic than elsewhere. If you’re interested in the subject, consult Mordell’s book, which is a thoroughgoing but readable account of what is known, together with a great number of unsolved problems. There are well-developed theories of rational points on algebraic curves, so we mainly confine ourselves to higher dimensions, for which standard methods have not yet been developed.

D1 Sums of like powers. Euler’s conjecture.

“It has seemed to many Geometers that this theorem [Fermat’s Last Theorem] may be generalized. Just as there do not exist two cubes whose sum or difference is a cube, it is certain that it is impossible to exhibit three biquadrates whose sum is a biquadrate, but that at least four biquadrates are needed if their sum is to be a biquadrate, although no one has been able up to the present to assign four such biquadrates. In the same manner it would seem to be impossible to exhibit four fifth powers whose sum is a fifth power, and similarly for higher powers.”

No advance was made on Euler’s statement until 1911 when R. Norrie assigned four such biquadrates:

$$30^4 + 120^4 + 272^4 + 315^4 = 353^4$$

Fifty-five years later Lander & Parkin gave a counter-example to Euler’s

more general conjecture:

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

It's now well known, since his discovery hit the national newspapers, that Noam Elkies has disproved Euler's conjecture for fourth powers. The infinite family of solutions, of which the first member is

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4,$$

comes from an elliptic curve given by $u = -5/8$ in the parametric solution of $x^4 - y^4 = z^4 + t^2$ given by Dem'janenko. The smallest solution,

$$95800^4 + 217519^4 + 414560^4 = 422481^4,$$

corresponding to $u = -9/20$, was subsequently found by Roger Frye. There are infinitely many values of u which give curves of positive rank, and further families of solutions.

The solution of Jan Kubiček, making the sum of three cubes a cube, coincides with that of F. Vieta (see Dickson's *History* [A17], Vol. 2, pp. 550–551).

Simcha Brudno asks the following questions. Is there a parametric solution to $a^3 + b^5 + c^8 + d^6 = e^5$? [The above solution is the only one with $e \leq 765$.] Is there a parametric solution to $a^4 + b^4 + c^4 + d^4 = e^4$? Are there counterexamples to Euler's conjecture with powers higher than the fifth? Is there a solution of $a^6 + b^6 + c^6 + d^6 + e^6 = f^6$? Although there are solutions of $a_1^s + \dots + a_{s-1}^s = b^s$ for $s = 4$ and 5, there is no known solution, even of $a_1^s + \dots + a_s^s = b^s$, for $n \geq 6$.

Parametric solutions are known for equal sums of equal numbers of like powers,

$$\sum_{i=1}^m a_i^s = \sum_{i=1}^m b_i^s$$

with $a_i > 0$, $b_i > 0$, for $2 \leq s \leq 4$ and $m = 2$ and for $s = 5, 6$ and $m = 3$. Can a solution be found for $s = 7$ and $m = 4$? For $s = 5$, $m = 2$, it is not known if there is any nontrivial solution of $a^5 + b^5 = c^5 + d^5$. Dick Lehmer once thought that there might be a solution with a sum of about 25 decimal digits, but a search by Blair Kelly III yielded no nontrivial solution with sum $\leq 1.02 \times 10^{26}$.

The "Hardy-Ramanujan number" 1729 = $1^3 + 12^3 = 9^3 + 10^3$ was first found by Bernard Frénicle de Bessy in 1657; Leech, in 1957, found

$$87539319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3$$

and Rosenstiel, Dardis & Rosenstiel recently found 6963472309248 =

$$2421^3 + 19083^3 = 5436^3 + 18948^3 = 10200^3 + 18072^3 = 13322^3 + 15530^3.$$

D1. Sums of like powers. Euler's conjecture.

Theorem 412 in Hardy & Wright shows that the number of such sums can be made arbitrarily large, but the least example is not known for five or more equal sums. If negative integers are allowed, then Randall Rathbun supplies the examples:

$$\begin{aligned} 6017193 &= 160^3 + 113^3 = 180^3 + 57^3 = 185^3 - (68)^3 \\ &= 209^3 - (146)^3 = 246^3 - (207)^3 \\ 1412774811 &= 963^3 + 804^3 = 1134^3 - (357)^3 = 1155^3 - (504)^3 \\ &= 1246^3 - (805)^3 = 2115^3 - (2004)^3 = 4746^3 - (4725)^3 \\ 11302198488 &= 1926^3 + 1608^3 = 1939^3 + 1589^3 = 2268^3 - (714)^3 = 2310^3 - (1008)^3 \\ &= 2492^3 - (1610)^3 = 4236^3 - (4008)^3 = 9492^3 - (9450)^3 \end{aligned}$$

Mordell and Mahler proved that the number of solutions of $n = x^3 + y^3$ can be $> c(\ln n)^\alpha$ and Silverman has improved their value of α from $\frac{1}{4}$ to $\frac{1}{3}$ and has shown that, if it is required that the pairs of cubes are to be mutually prime, then there is a constant c such that the number of such solutions is $< c^{r(n)}$, where $r(n)$ is the rank of the elliptic curve $x^3 + y^3 = n$. It is much harder to find these solutions. The largest number of representations so far found is three, by P. Vojta in 1983:

$$15170835645 = 517^3 + 2468^3 = 709^3 + 2456^3 = 1733^3 + 2152^3$$

but present-day insight and computing power may soon beat this. If negative integers are allowed, Randall Rathbun gives the cubefree example

$$16776487 = 220^3 + 183^3 = 255^3 + 58^3 = 256^3 + (-9)^3 = 292^3 + (-201)^3$$

The number 1729 is also the third Carmichael number (see A13) and Pomerance has observed that the second Carmichael number, 1105, is expressible as the sum of two squares in more ways than any smaller number. Granville invites the reader to make a corresponding statement about the first Carmichael number, 561.

Euler knew that 635318657 = $133^4 + 134^4 = 59^4 + 158^4$, and Leech showed this to be the smallest example. No one knows of three such equal sums.

A method is known for generating parametric solutions of $a^4 + b^4 = c^4 + d^4$ which will generate all published solutions from the trivial one $(\lambda, 1, \lambda, 1)$; it will only produce solutions of degree $6n+1$. Here, in answer to a question of Brudno, $6n+1$ need not be prime. Although degree 25 does not appear, 49 does. More recently, Ajai Choudhry has found a parametric solution of degree 25.

Swinnerton-Dyer has a second method for generating new solutions from old and can show that the two methods, together with the symmetries, generate all *nonsingular* parametric solutions, i.e., all solutions which correspond to points on curves with no singular points. [A point on a curve with homogeneous equation $F(x, y, z) = 0$ is *singular* just if $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$ there.] Moreover, the process is constructive in the sense that he can give

a finite procedure for finding all nonsingular solutions of given degree. All nonsingular solutions have odd degree, and all sufficiently large odd degrees do occur. Unfortunately, singular solutions do exist. Swinnerton-Dyer has a process for generating them, but has no reason to believe that it gives them all. The problem of describing them all needs completely new ideas. Some of the singular solutions have even degree and he conjectures (and could probably prove) that all sufficiently large even degrees occur in this way.

In the same sense, Andrew Bremner can find "all" parametric solutions of $a^6 + b^6 + c^6 = d^6 + e^6 + f^6$ which also satisfy the equations

$$\begin{aligned} a^2 + ad - d^2 &= f^2 + fc - c^2 \\ b^2 + be - e^2 &= d^2 + da - a^2 \\ c^2 + cf - f^2 &= e^2 + eb - b^2 \end{aligned}$$

(this is not such a restriction as might at first appear). Many solutions of $a^6 + b^6 + c^6 = d^6 + e^6 + f^6$ also satisfy $a^2 + b^2 + c^2 = d^2 + e^2 + f^2$ and all known simultaneous solutions (with appropriately chosen signs) of these two equations also satisfy the previous three equations, e.g., the smallest solution, found by Subba-Rao, $(a, b, c, d, e, f) = (3, 19, 22, -23, 10, -15)$. Is there a counter-example? Peter Montgomery has listed 18 equal sums of three sixth powers where the corresponding sums of squares are not equal. The least is

$$25^6 + 62^6 + 138^6 = 82^6 + 92^6 + 135^6.$$

Bremner can also find "all" parametric solutions of $a^5 + b^5 + c^5 = d^5 + e^5 + f^5$ which also satisfy $a + b + c = d + e + f$ and $a - b = d - e$.

Bob Scher calls a sum $\sum_{i=1}^m a_i^p = 0$ (where p is prime) "perfect" if, for any a_i , there is also a unique a'_i such that $a_i + a'_i \equiv 0 \pmod p$. If $a_i \equiv 0 \pmod p$, then $a'_i = a_i$. He shows that if $p = 3$ and $m < 9$ or if $p = 5$ and $m < 7$, then every such sum is perfect.

There has been some interest in the simultaneous equations

$$\begin{aligned} a^n + b^n + c^n &= d^n + e^n + f^n \\ a^n b^n c^n &= d^n e^n f^n \end{aligned}$$

For $n = 2$ the problem goes back at least to Bini, with partial solutions by Dubouis and Mathieu; a neat general solution has recently been given by John B. Kelly. Stephane Vandemergel has found 62 solutions for $n = 3$ and three solutions for $n = 4$: $(29, 66, 124; 22, 93, 116)$, $(54, 61, 196; 28, 122, 189)$ and $(19, 217, 657; 9, 511, 589)$. He notes that if $r^n + s^n = u^n + v^n$, then $(ru, su, v^2; rv, sv, u^2)$ is a solution, which shows that there are infinitely many solutions for $n \leq 4$. Most of his solutions are not of this form.

There is a considerable history (see Dickson [A17], II, Ch. 24) of the

D1. Sums of like powers. Euler's conjecture.

Tarry-Escott problem, and a book by Gloden on (systems of) **multigrade equations**:

$$\sum_{i=1}^l n_i^j = \sum_{i=1}^l m_i^j \quad (j = 1, \dots, k)$$

A spectacular example, with $k = 9$, $l = 10$, is due to Letac, with $(n_i; m_i) = (\pm 12, \pm 11881, \pm 20231, \pm 20885, \pm 23738; \pm 436, \pm 11857, \pm 20449, \pm 20667, \pm 23750)$.

Smyth has shown that this is a member of an infinite family of independent solutions.

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D2 The Fermat problem.

It seems that Fermat's Last Theorem, that the equation

$$x^p + y^p = z^p$$

is impossible in positive integers for an odd prime p , is at last indeed a theorem. But the proof is long and, at the time of writing, leans on the work of others. Indeed, as I write, a hole is reported to have been found,

D2. The Fermat problem.

to repair which may be possible, but “may take one or two years.” Ribet had shown that the theorem follows from the Taniyama-Weil conjectures on elliptic curves. For as much of the proof as our margin will hold, see **B19**. It raises philosophical problems, such as: are we running out of ‘comprehensible’ proofs and must we teach computers to check our results for us?

Kummer proved the theorem for all regular primes, where a prime p is **regular** if it doesn't divide h_p , the number of equivalence classes of ideals in the cyclotomic field $\mathbb{Q}(\zeta_p)$. He also showed that a prime was regular just if it did not divide the numerators of the Bernoulli numbers B_2, B_4, \dots, B_{p-3} , where

$$B_{2k} = (-1)^{k-1} \frac{(2k)!}{(2\pi)^{2k}} \zeta(2k)$$

($\zeta(2k) = \sum_{n=1}^{\infty} n^{-2k}$ and see **A17**). Of the 78497 primes less than 10^6 , 47627 are regular, agreeing well with their conjectured density, $e^{-1/2}$. However, it has not even been proved that there are infinitely many. On the other hand, Jensen has proved that there are infinitely many irregular primes. The prime $p = 16843$ divides B_{p-3} and Richard McIntosh, as well as Buhler, Crandall, Ernvall & Metsänkyö, observes that this is also true for $p = 2124679$.

Some of J. M. Gandhi's subsidiary problems, mentioned in the first edition, are covered by Wiles's method (if it holds up under scrutiny), which shows the unsolvability of

$$x^p + y^p + \gamma z^p = 0$$

if $p \geq 11$ and γ is a power of one of the primes 3, 5, 7, 11, 13, 17, 19, 23, 29, 53, 59,

For what integers c are there integer solutions of $x^4 + y^4 = cx^4$ with $x \perp y$ and $x > 1$? Leech gives a method for finding non-trivial solutions for any $c = a^4 + b^4$: the smallest he has found is $25^4 + 149^4 = 5906 \cdot 17^4$. Bremner & Morton show that 5906 is the least integer that is the sum of two rational fourth powers, but not the sum of two integer fourth powers.

Prove that $x^n + y^n = nlz^n$ has no integer solutions with $n > 2$. Erdős & Obláth showed that $x^p \pm y^p = n!$ has none with $p > 2$, and Erdős states that $x^4 + y^4 = n!$ has no solutions with $x \perp y$. Indeed, even without this last condition, Leech notes that for $n > 3$ there's a prime $\equiv 3 \pmod{4}$ in the interval $[n+1, 2n]$ and hence a simple (i.e. not repeated) such prime divisor of $n!$ for $n > 6$, so $n!$ is not even the sum of two squares for $n > 6$ (apart from $n = 0, 1, 2$, the only solution is $6! = 24^2 + 12^2$). [Compare **D25**.]

Granville's paper, in which the reviewer felt “a new powerful wave”, had already related the Fermat problem to numerous other conjectures, including the *ABC* conjecture (**B19**) and Erdős's powerful numbers conjecture (**B16**).

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D3 Figurate numbers.

Mordell, on p. 259 of his book, asks if the only integer solutions of

$$6y^2 = (x+1)(x^2 - x + 6)$$

are given by $x = -1, 0, 2, 7, 15$ and 74 ? By Theorem 1 of Mordell's Chapter 27 there are only finitely many. The equation arises from

$$y^2 = \binom{x}{0} + \binom{x}{1} + \binom{x}{2} + \binom{x}{3}.$$

Andrew Bremner gave the additional solution $(x, y) = (767, 8672)$ and showed that that is all, but the result was already given by Ljunggren in 1971.

Similarly, Martin Gardner took the figurate numbers: triangle, square, tetrahedron and square pyramid; and equated them in pairs. Of the six resulting problems, he noted that they were all solved except “triangle = square pyramid”, which leads to the equation

$$3(2y+1)^2 = 8x^3 + 12x^2 + 4x + 3.$$

The number of solutions is again finite. Are they all given by $x = -1, 0, 1, 5, 6$ and 85 ? Schinzel sends Avanesov's affirmative answer, rediscovered by Uchiyama.

The “triangle = tetrahedron” problem is a special case of a more general question about equality of binomial coefficients (see B31) — the only nontrivial examples of $\binom{n}{2} = \binom{m}{3}$ are $(m, n) = (10, 16), (22, 56)$ and $(36, 120)$. Are there nontrivial examples of $\binom{n}{2} = \binom{m}{4}$ other than $(10, 21)$?

The case “square pyramid = square” is Lucas's problem. Is $x = 24, y = 70$ the only nontrivial solution of the diophantine equation

$$y^2 = x(x+1)(2x+1)/6?$$

This was solved affirmatively by Watson, using elliptic functions, and by Ljunggren, using a Pell equation in a quadratic field. Mordell asked if there was an elementary proof, and affirmative answers have been given by Ma, by Xu & Cao, by Anglin and by Pintér.

The same equation in disguise is to ask if $(48, 140)$ is the unique nontrivial solution to the case “square = tetrahedron”, since the previous equation may be written

$$(2y)^2 = 2x(2x+1)(2x+2)/6,$$

though, as Peter Montgomery notes, this doesn't eliminate the possibility of an odd square. A more modern treatment is to put $12x = X - 6, 72y = Y$ and note that $Y^2 = X^3 - 36X$ is curve 576H2 in John Cremona's

tables. The point (12,36) (which gives an odd square) serves as a generator. There's an infinity of rational solutions, but the only nontrivial integer solution to the original problem is given by the point (294,5040).

More general than asking for the sum of the first n squares to be square, we can ask for the sum of any n consecutive squares to be square. If S is the set of n for which this is possible, then it is known that S is infinite, but has density zero, and that if n is a nonsquare member of S , then there are infinitely many solutions for such an n . If $N(x)$ is the number of members of S less than x , then the best that seems to be known is that

$$c\sqrt{x} < N(x) = O\left(\frac{x}{\ln x}\right).$$

The elements of S , $1 < n < 73$, and the corresponding least values of a for which the sum of n squares starting with a is square, are

n	2	11	23	24	26	33	47	49	50	59
a	3	18	7	1	25	7	539	25	7	22

More generally still, one can ask that the sums of the squares of the members of an arbitrary arithmetic progression should be square. K. R. S. Sastry notes that this can occur if the number of terms in the progression is square.

In answer to the question: which triangular numbers are the product of three consecutive integers, Tzanakis & de Weger gave the (only) answers 6, 120, 210, 990, 185136 and 258474216. Unfortunately, Mohanty's elementary proof of the same result is erroneous.

Other examples of elliptic curves were treated by Bremner & Tzanakis who showed that there are just 26 integer points on $y^2 = x^3 - 7x + 10$ and they also examined $y^2 = x^3 - bx + c$ for $(b, c) = (172, 505), (172, 820)$ and $(112, 2320)$.

There are infinitely many solutions of $\binom{a}{k} - \binom{b}{k} = c^k$ for $k = 3$. Are there any for $k = 4$? And is $(a, b, c) = (18, 12, 6)$ an isolated example for $k = 5$?

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D4 Sums of l k th powers.

Let $r_{k,l}(n)$ be the number of solutions of $n = \sum_{i=1}^l x_i^k$ in positive integers x_i . Hardy & Littlewood's Hypothesis K is that $\epsilon > 0$ implies that $r_{k,k}(n) = O(n^\epsilon)$. This is well-known for $k = 2$; in fact, for sufficiently large n ,

$$r_{2,2}(n) < n(1 + \epsilon) \ln 2 / \ln \ln n$$

and this does not hold if $\ln 2$ is replaced by anything smaller. Mahler disproved the hypothesis for $k = 3$ by showing that $r_{3,3} > c_1 n^{1/12}$ for infinitely many n .

Erdős thinks it possible that for all n , $r_{3,3} < c_2 n^{1/12}$ but nothing is known. Probably Hypothesis K fails for every $k \geq 3$, but also it's probable that $\sum_{n=1}^x (r_{k,k}(n))^2 < x^{1+\epsilon}$ for sufficiently large x .

S. Chowla proved that for $k \geq 5$, $r_{k,k}(n) \neq O(1)$ and, with Erdős, that for every $k \geq 2$ and for infinitely many n ,

$$r_{k,k} > \exp(c_k \ln n / \ln \ln n).$$

Mordell proved that $r_{3,2}(n) \neq O(1)$ and Mahler that $r_{3,2}(n) > (\ln n)^{1/4}$ for infinitely many n . No nontrivial upper bound for $r_{3,2}(n)$ is known. Jean Lagrange has shown that $\limsup r_{4,2}(n) \geq 2$ and that $\limsup r_{4,3}(n) = \infty$.

Another tough problem is to estimate $A_{k,l}(x)$, the number of $n \leq x$ which are expressible as the sum of l k -th powers. Landau showed that

$$A_{2,2}(x) = (c + o(1))x / (\ln x)^{1/2},$$

Erdős & Mahler proved that if $k > 2$, then $A_{k,2} > c_k x^{2/k}$, and Hooley that $A_{k,2} > (c_k + o(1))x^{2/k}$. It seems certain that if $l < k$, then $A_{k,l} > c_k x^{l/k}$ and that $A_{k,k} > x^{1-\epsilon}$ for every ϵ , but these have not been established.

It follows from the Chowla-Erdős result that for all k there is an n_k such that the number of solutions of $n_k = p^3 + q^3 + r^3$ is greater than k . No corresponding result is known for more than three summands.

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D5 Sum of four cubes.

Is every number the sum of four cubes? This has been proved for all numbers except possibly those of the form $9n \pm 4$.

More demanding is to ask if every number is the sum of four cubes with two of them equal. Specifically, is there a solution of $76 = x^3 + y^3 + 2z^3$? The other numbers less than 1000 which may still be in doubt are 148, 183, 230, 356, 418, 428, 445, 482, 491, 580, 671, 788, 931 and 967. A 70-01-20 letter from M. Lal to A. Mąkowski reports J. C. Littlejohn's observations that $253 = 0^3 + 5^3 + 2 \cdot 4^3$, $519 = 0^3 + 17^3 + 2(-13)^3$ and $734 = (-520)^3 + (-700)^3 + 2 \cdot 623^3$. Andrew Bremner tells me that $923 = 27512^3 + (-27517)^3 + 2 \cdot 1784^3$.

Are all numbers which are not of the form $9n \pm 4$ the sum of three cubes? Computer searches have found representations for all numbers less than 1000, except for

30	33	42	52	74	75	84	110	114
156	165	195	290	318	366	390	420	435
444	452	462	478	501	530	534	564	579
588	600	606	609	618	627	633	732	735
758	767	786	789	795	830	834	861	894
903	906	912	921	933	948	964	969	975

In a 93-05-25 email message Andrew Bremner tells me that

$$75 = 435203083^3 + (-435203231)^3 + 4381159^3$$

(which gives an imprimitive solution for 600) while Conn & Vaserstein have discovered that

$$84 = 41639611^3 + (-41531726)^3 + (-8241191)^3.$$

The equation $x^3 + y^3 + z^3 = 3$ has the solutions $(1, 1, 1)$ and $(4, 4, -5)$. Are there any others?

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D6 An elementary solution of $x^2 = 2y^4 - 1$.

Ljunggren has shown that the only solutions of $y^2 = 2x^4 - 1$ in positive integers are $(1,1)$ and $(239,13)$ but his proof is difficult. Mordell asks if it is possible to find a simple or elementary proof. Whether Steiner & Tzanakis have simplified the solution may be a matter of taste; they use the theory of linear forms in logarithms of algebraic numbers.

Ljunggren and others have made considerable investigations into equations of similar type. For references see the first edition. Cohn has considered the equation $y^2 = Dx^4 + 1$ for all $D \leq 400$. This will have rational solutions just when there are rational points on the curves $y^2 = x(x^2 - 4D)$ and $Y^2 = X(X^2 + 16D)$, which are nonsingular provided $\pm D$ is not square.

D7 Sum of consecutive powers made a power.

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D7 Sum of consecutive powers made a power.

Rufus Bowen conjectured that the equation

$$1^n + 2^n + \cdots + m^n = (m+1)^n$$

has no nontrivial solutions, and Leo Moser showed that there were none with $m \leq 10^{100000}$ and none with n odd. Zhou & Kang raised the bound to $m \leq 10^{2000000}$. Van de Lune & te Riele showed that the equation is almost never solvable. Note that $n \sim m \ln 2$.

Tijdeman observes that general results on the equation

$$1^n + 2^n + \cdots + k^n = m^n$$

do not appear to have any implications for the special equation.

Erdős proposed the problem to prove that if m, n are integers satisfying (K), then (L') and (M') are true, where

$$(K) \quad \left(1 - \frac{1}{m}\right)^n > \frac{1}{2} > \left(1 - \frac{1}{m-1}\right)^n,$$

$$(L') \quad 1^n + 2^n + \cdots + (m-2)^n < (m-1)^n,$$

$$(M') \quad 1^n + 2^n + \cdots + m^n > (m+1)^n,$$

and that (L) and (M) are each true infinitely often, where

$$(L) \quad 1^n + 2^n + \cdots + (m-1)^n < m^n,$$

$$(M) \quad 1^n + 2^n + \cdots + (m-1)^n > m^n.$$

Van de Lune proved that (K) implies (L') and Best & te Riele proved that (K) implies (M') and that (M) holds for at most $c \ln x$ values of $m \leq x$. Van de Lune & te Riele proved that (L) is true for almost all pairs (m, n) . Best & te Riele computed 33 pairs (m, n) for which (K) and (M) both hold, the smallest being

$$m = 1121626023352385, \quad n = 777451915729368.$$

Are there infinitely many such pairs?

More recently Pieter Moree, te Riele & Urbanowicz have shown that, in the original equation, n must be divisible by the l.c.m. of all integers up to 200, and that m is not divisible by any regular prime (see D2), nor by any irregular prime < 1000 .

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D8 A pyramidal diophantine equation.

Wunderlich asks for (a parametric representation of) *all* solutions of the equation $x^3 + y^3 + z^3 = x + y + z$. Bernstein, S. Chowla, Edgar, Fraenkel, Oppenheim, Segal, and Sierpiński have given solutions, some of them parametric, so there are certainly infinitely many. Eighty-eight of them have unknowns less than 13000. Bremner has effectively determined all parametric solutions.

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D9 Difference of two powers.

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D9 Difference of two powers.

Except that there remains a finite amount of computation, Tijdeman has settled the old conjecture of Catalan, that the only consecutive powers, higher than the first, are 2^3 and 3^2 . But this finite amount of computation is far beyond computer range and will not be achieved without some additional theoretical ideas. Langevin has deduced from Tijdeman's proof that if $n, n+1$ are powers, then $n < \exp \exp \exp \exp 730$, and Aaltonen & Inkeri have shown that $x^p - y^q = 1$ and $x, y > 2$ imply that $x, y > 10^{500}$. Mignotte has shown that $p < 1.21 \times 10^{26}$, $q < 1.31 \times 10^{18}$. If p and q are prime and $q \equiv 3 \pmod{4}$, these bounds can be reduced to 2.7×10^{24} and 1.23×10^{18} . There is no solution if $\min\{p, q\} = 2$ or 3. Glass & others have shown this to be true for $\min\{p, q\} \in \{5, 7, 11\}$. Mignotte shows that there are no solutions for $p = 19$ and that the only possibility for $p = 53$ is $q = 4889$.

Bennett has shown that $4 \leq N \leq k \cdot 3^k$ implies that

$$\left\| \left(\frac{N+1}{N} \right)^k \right\| > 3^{-k}$$

where $\|x\|$ is the distance from x to the nearest integer.

Leech asks if there are any solutions of $|a^m - b^n| < |a-b|$ with $m, n \geq 3$. With equality he notes $|5^3 - 2^7| = 5 - 2$ and $|13^3 - 3^7| = 13 - 3$. Are these all? are the shared exponents 3, 7 significant?

If $a_1 = 4, a_2 = 8, a_3 = 9, \dots$ is the sequence of powers higher than the first, Chudnowsky claims to have proved that $a_{n+1} - a_n$ tends to infinity with n . Erdős conjectures that $a_{n+1} - a_n > c'n^c$, but says that there is no present hope of proof.

Erdős asks if there are infinitely many numbers not of the form $x^k - y^l$, $k > 1, l > 1$.

Carl Rudnick denotes by $N(r)$ the number of positive solutions of $x^4 - y^4 = r$, and asks if $N(r)$ is bounded. Hansraj Gupta observes that Hardy & Wright (p. 201) give Swinnerton-Dyer's version of Euler's parametric solution of $x^4 - y^4 = u^4 - v^4$, which establishes that $N(r)$ is 0, 1 or 2 infinitely often. For example $133^4 - 59^4 = 158^4 - 134^4 = 300783360$. For an example with $N(r) = 3$, Zajta gives

$$401168^4 - 17228^4 = 415137^4 - 248289^4 = 421296^4 - 273588^4.$$

There can hardly be any doubt that $N(r)$ is bounded.

Hugh Edgar asks how many solutions (m, n) does $p^m - q^n = 2^h$ have, for primes p and q and h an integer? Examples are $3^2 - 2^3 = 2^0$; $3^3 - 5^2 = 2^1$; $5^3 - 11^2 = 2^2$; $5^2 - 3^2 = 2^4$; $3^4 - 7^2 = 2^5$; are there others? Andrzej Schinzel writes that work of Gelfond and of Rumsey & Posner implies that the equation has only finitely many solutions. Reese Scott goes a fair way towards settling the question. He observes that the finiteness of the number of solutions for given $(p, q, c(= 2^h))$ follows from a result of Pillai, and proves that this number is often at most one, with a small number of specifically listed exceptional cases, where it is two or possibly three.

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D10 Exponential diophantine equations.

Brenner & Foster pose the following general problem. Let $\{p_i\}$ be a finite set of primes and $\epsilon = \pm 1$. When can the exponential diophantine equation $\sum \epsilon_i p_i^{x_i} = 0$ be solved by elementary methods (e.g., by modular arithmetic)? More exactly, given p_i, ϵ_i , what criteria determine whether there exists a modulus M such that the given equation is equivalent to the congruence $\sum \epsilon_i p_i^{x_i} \equiv 0 \pmod{M}$? They solve many particular cases, mostly where the p_i are four in number and less than 108. In a few cases elementary methods avail, even if two of the primes are equal, but in general they do not. In fact, neither $3^a = 1 + 2^b + 2^c$ nor $2^a + 3^b = 2^c + 3^d$ can be reduced to a single congruence. Tijdeman notes that another approach to these purely exponential diophantine equations (which play a role in group theory) is by Baker's method (compare F23). This makes it possible to solve these last two equations.

Hugh Edgar asks if there is a solution, other than $1 + 3 + 3^2 + 3^3 + 3^4 = 11^2$, of the equation $1 + q + q^2 + \dots + q^{x-1} = p^y$ with p, q odd primes and $x \geq 5, y \geq 2$. An important breakthrough in this area is the paper of Reese Scott (see D9).

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D11 Egyptian fractions.

The Rhind papyrus is amongst the oldest written mathematics that has come down to us; it concerns the representation of rational numbers as the sum of unit fractions,

$$\frac{m}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_k}$$

This has suggested numerous problems, many of which are unsolved, and continues to suggest new problems, so the interest in Egyptian fractions is as great as it has ever been. Our bibliography shows only a fraction of what has been written. Bleicher has given a careful survey of the subject and draws attention to the various algorithms that have been proposed for constructing representations of the given type: the Fibonacci-Sylvester algorithm, Erdős's algorithm, Golomb's algorithm and two of his own, the Farey series algorithm and the continued fraction algorithm. See also the

extensive Section 4 of the collection of problems by Erdős & Graham mentioned at the beginning of this volume, and the bibliography obtainable from Paul Campbell.

Erdős & Straus conjectured that the equation

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

could be solved in positive integers for all $n > 1$. There is a good account of the problem in Mordell's book, where it is shown that the conjecture is true, except possibly in cases where n is congruent to $1^2, 11^2, 13^2, 17^2, 19^2$ or $23^2 \pmod{840}$. Several have worked on the problem, including Bernstein, Obláth, Rosati, Shapiro, Straus, Yamamoto, and Nicola Franceschini who has verified the conjecture for $n < 10^8$. Schinzel has observed that one can express

$$\frac{4}{at+b} = \frac{1}{x(t)} + \frac{1}{y(t)} + \frac{1}{z(t)}$$

with $x(t)$, $y(t)$, $z(t)$ integer polynomials in t with positive leading coefficients and $a \perp b$, only if b is not a quadratic residue of a .

Sierpiński made the corresponding conjecture concerning

$$\frac{5}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Palamà confirmed it for all $n \leq 922321$ and Stewart has extended this to $n \leq 1057438801$ and for all n not of the form $278460k + 1$.

Schinzel relaxed the condition that the integers x , y , z should be positive, replaced the numerators 4 and 5 by a general m and required the truth only for $n > n_m$. That n_m may be greater than m is exemplified by $n_{18} = 23$. The conjecture has been established for successively larger values of m by Schinzel, Sierpiński, Sedláček, Palamà and Stewart & Webb, who prove it for $m < 36$. Breusch and Stewart independently showed that if $m/n > 0$ and n is odd, then m/n is the sum of a finite number of reciprocals of odd integers. See also Graham's papers. Vaughan has shown that if $E_m(N)$ is the number of $n \leq N$ for which $m/n = 1/x + 1/y + 1/z$ has no solution, then

$$E_m(N) \ll N \exp\{-c(\ln N)^{2/3}\}$$

where c depends only on m . Hofmeister & Stoll have shown that if $F_m(N)$ is the number of $n \leq N$ for which $m/n = 1/x + 1/y$ has no solution, then

$$F_m(N) \ll N(\ln N)^{-1/\phi(m)}$$

where $\phi(m)$ is Euler's totient function (B36).

Hofmeister notes that this implies that $A_m(N)/N \rightarrow 1$ as $N \rightarrow \infty$ where $A_m(N)$ is the number of b , $1 \leq b \leq N$ for which there is a representation $m/b = 1/n_1 + 1/n_2$, so that almost all lattice points on the line $y = m$, $x \geq 1$ have such a representation. Paradoxically, Mittelbach lets $B(N)$ be the number of lattice points (a, b) , $1 \leq a \leq b \leq N$ for which there is a representation $a/b = 1/n_1 + 1/n_2$ and proves that $B(N)/\frac{1}{2}N(N+1) \rightarrow 0$. I.e., almost no lattice points in the triangle $(1,1)$, $(1,N)$, (N,N) have a representation for large N .

In contrast to the result of Breusch and Stewart, the following problem, asked by Stein, Selfridge, Graham and others, has not been solved. If m/n , a rational number (n odd), is expressed as $\sum 1/x_i$, where the x_i are successively chosen to be the least possible odd integers which leave a nonnegative remainder, is the sum always finite? For example,

$$\frac{2}{7} = \frac{1}{5} + \frac{1}{13} + \frac{1}{115} + \frac{1}{10465}$$

John Leech, in a 77-03-14 letter, asks what is known about sets of unequal odd integers whose reciprocals add to 1, such as

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{15} + \frac{1}{21} + \frac{1}{27} + \frac{1}{35} + \frac{1}{63} + \frac{1}{105} + \frac{1}{135} = 1$$

He says that you need at least nine in the set, while on the other hand the largest denominator must be at least 105. Notice the connexion with Sierpiński's pseudoperfect numbers (**B2**).

$$945 = 315 + 189 + 135 + 105 + 63 + 45 + 35 + 27 + 15 + 9 + 7$$

It is known that if n is odd, then m/n is always expressible as a sum of distinct odd unit fractions.

Tenenbaum & Yokota show that m/n can be expressed as the sum of r unit fractions with denominators $\leq 4n(\ln n)^2 \log_2 n$ where $r \leq (1+\epsilon) \ln n / \log_2 n$ but that $1+\epsilon$ cannot be replaced by $1-\epsilon$.

Victor Meally ordered the rationals a/b , $a \perp b$, between 0 and 1 by size of a/b and of a : $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{1}{5}, \frac{1}{6}, \frac{2}{5}, \frac{3}{4}, \dots$ and noted that $\frac{2}{3}, \frac{4}{5}$ and $\frac{8}{11}$ are the earliest members of the sequence that need 2, 3 and 4 unitary fractions to represent them. Which are the earliest that need 5? 6? 7? Stephane Vandemergel, in a 93-04-28 letter, states that $\frac{16}{17}$ requires 5 unitary fractions, and $\frac{77}{79}$ needs 6.

Barbeau expressed 1 as the sum of the reciprocals of 101 distinct positive integers, no one dividing another. Erdős & Graham showed that if n is squarefree, then m/n can always be written as a finite sum of reciprocals of squarefree integers each having exactly ω distinct prime factors, for $\omega \geq 3$. There are many cases in which ω can be taken as 2. For $m = n = 1$ at least 38 integers are required: Allan Johnson manages it with $\omega = 2$ and the 48 numbers

6	21	34	46	58	77	87	115	155	215	287	391
10	22	35	51	62	82	91	119	187	221	299	689
14	26	38	55	65	85	93	123	203	247	319	731
15	33	39	57	69	86	95	133	209	265	323	901

Is this the smallest possible set? Richard Stong also solved this problem, but used a larger set.

Erdős, in a 72-01-14 letter, sets $\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} = \frac{6}{5}$, where $b = [2, 3, \dots, n]$, the l.c.m. of $2, 3, \dots, n$. He observes that $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ and $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$ are such that $a \pm 1 \equiv 0 \pmod{b}$ and asks if this occurs again: he conjectures not. Is $a \perp b$ infinitely often?

If $\sum_{i=1}^t 1/x_i = 1$ with $x_1 < x_2 < x_3 < \dots$ distinct positive integers Erdős & Graham ask what is $m(t)$, the min max x_i , where the minimum is taken over all sets $\{x_i\}$. For example, $m(3) = 6$, $m(4) = 12$, $m(12) = 120$. Is $m(t) < ct$ for some constant c ? In this notation, is it possible to have $x_{i+1} - x_i \leq 2$ for all i ? Erdős conjectures that it is not and offers \$10.00 for a solution.

Erdős & Graham ask if it is true that any coloring of the integers with c colors gives a monochromatic solution of

$$\sum \frac{1}{x_i} = 1, \quad x_1 < x_2 < \dots \quad (\text{finite sum}).$$

This is open even for $c = 2$. If the answer is affirmative, let $f(c)$ be the smallest integer for which every c -coloring of the integers $1 \leq t \leq f(c)$ contains a monochromatic solution. Determine or estimate $f(c)$.

Erdős also asks that if

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_k} = 1, \quad x_1 < x_2 < \dots < x_k$$

and k is fixed, what is $\max x_i$? If k varies, what integers can be equal to x_k , the largest denominator? Not primes, and not several other integers; do the excluded integers have positive density? Density 1 even? Which integers can be x_k or x_{k-1} ? Which can be x_k or x_{k-1} or x_{k-2} ? Is $\liminf \frac{x_k}{x_1} > e$? It is trivial that the limit is $\geq e$. In fact perhaps it is infinite. If $m(k)$ is $\max(x_k)$ for each k , then Yokota improves a result of Erdős & Graham by proving that there is an increasing sequence of integers k for which $m(k)/k \leq (\ln \ln k)^3$. Is there a sequence of k such that $m(k)/k$ is bounded?

Erdős further asks if it is true that for every solution of

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_k} = 1,$$

$\max(x_{i+1} - x_i) \geq 3$? {2,3,6} shows that > 3 is not true but perhaps this is the only counterexample. Perhaps $\max(x_{i+1} - x_i) \leq c$ has only a

finite number of solutions. If the x_i are the union of r blocks of consecutive integers, the number of solutions is finite and depends only on r . That the sequence $\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \dots$ gives the max x_k as a function of r was proved by Curtiss.

If $N(n)$ is the set of integers that can be written as a sum of distinct reciprocals of integers $\leq n$, then Yokota shows that every natural number up to

$$\frac{\ln n}{2} \left(1 - \frac{2 \ln \ln n}{\ln n} \right)$$

is in $N(n)$, so that $\#N(n) \geq (\frac{1}{2} + o(1)) \ln n$. Erdős asks for an estimate for the size of the smallest integer not in $N(n)$, and of the largest in $N(n)$. How many integers < $\sum_{i=1}^n 1/i$ cannot be so expressed? Of the ones that can, what is their distribution? Do they come in bunches?

Given a sequence x_1, x_2, \dots of positive density, is there always a finite subset with $\sum 1/x_i = 1$? If $x_i < ci$ for all i , is there such a finite subset? Erdős again offers \$100 for a solution. If $\liminf x_i/i < \infty$, he strongly conjectures that the answer is negative and offers only \$5.00 for a solution.

Denote by $N(t)$ the number of solutions $\{x_1, x_2, \dots, x_t\}$ of $\sum 1/x_i = 1$ and by $M(t)$ the number of distinct solutions $x_1 \leq x_2 \leq \dots \leq x_t$. Singmaster calculated

t	1	2	3	4	5	6
$M(t)$	1	1	3	14	147	3462
$N(t)$	1	1	10	215	12231	2025462

and Erdős asked for an asymptotic formula for $M(t)$ or $N(t)$.

Graham has shown that if $n > 77$ we can partition $n = x_1 + x_2 + \dots + x_t$ into t distinct positive integers so that $\sum 1/x_i = 1$. More generally, that for any positive rational numbers α, β , there is an integer $r(\alpha, \beta)$, which we will take to be the least, such that any integer greater than r can be partitioned into distinct positive integers greater than β , whose reciprocals sum to α . Little is known about $r(\alpha, \beta)$, except that unpublished work of D. H. Lehmer shows that 77 cannot be partitioned in this way, so that $r(1, 1) = 77$.

Graham conjectures that for n sufficiently large (about 10^4 ?) we can similarly partition $n = x_1^2 + x_2^2 + \dots + x_t^2$ with $\sum 1/x_i = 1$. We can also ask for a decomposition $n = p(x_1) + p(x_2) + \dots + p(x_t)$ where $p(x)$ is any "reasonable" polynomial; for example $x^2 + x$ is unreasonable since it takes only even values.

In answer to a question of L.-S. Hahn: is there a set of integers, each having an immediate neighbor, the sum of whose reciprocals is an integer, Peter Montgomery gave the examples $\{1, 2, 7, 8, 13, 14, 39, 40, 76, 77, 285, 286\}$ and $\{2, 3, 4, 5, 6, 7, 9, 10, 17, 18, 34, 35, 84, 85\}$ whose reciprocals each add to 2.

L.-S. Hahn also asks: if the positive integers are partitioned into a finite number of sets in any way, is there always a set such that any positive rational number can be expressed as the sum of the reciprocals of a finite number of distinct members of it? Here it must be possible to choose the set, independent of the rational number. If this is not possible, then given any rational number, can one always choose a set with this property? Now the set can depend on the rational number.

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D12 Markoff numbers.

A diophantine equation which has excited a great deal of interest is

$$x^2 + y^2 + z^2 = 3xyz.$$

It obviously has what Cassels has called the singular solutions, $(1,1,1)$ and $(1,1,2)$ (with the usual definition, the variety has only the singular solution $(0,0,0)$). All solutions can be generated from these since the equation is a quadratic in each of the variables, so one integer solution leads to a second, and it can be shown that, apart from the singular solutions, all solutions have distinct values of x , y and z , so that each such solution is a **neighbor** of just three others (Figure 10). The numbers 1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, ... are called **Markoff numbers**. To avoid trivialities, assume that $0 < x \leq y \leq z$ (so that the inequalities are strict if $y \geq 2$). An outstanding problem is whether every Markoff number z defines a unique integer solution (x, y, z) . There are occasional claims to have proved that the Markoff numbers are unique in this sense, but so far proofs appear to be fallacious.

If $M(N)$ is the number of triples with $x \leq y \leq z \leq N$, then Zagier has shown that $M(N) = C(\ln N)^2 + O((\ln N)^{1+\epsilon})$ where $C \approx 0.180717105$, and calculations lead him to conjecture that the n th Markoff number, m_n , is

$\left(\frac{1}{3} + O(n^{-1/4+\epsilon})\right) A^{\sqrt{n}}$ where $A+e^{1/\sqrt{C}} \approx 10.5101504$. He has no results on distinctness, but can show that the problem is equivalent to the insolubility of a certain system of diophantine equations.

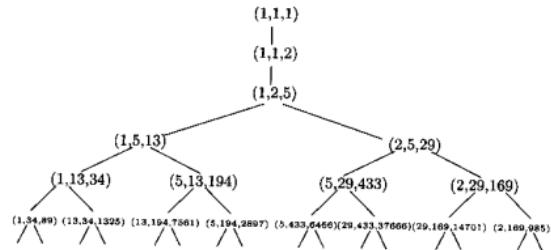


Figure 10. The Tree of Markoff Solutions.

Markoff's equation is a special case of the more general **Hurwitz equation**

$$x_1^2 + x_2^2 + \cdots + x_n^2 = ax_1x_2 \cdots x_n$$

for which there are no integer solutions if $a > n$, and for $a = n$ all integer solutions can be generated from $(1,1,\dots,1)$. For any a , $1 \leq a \leq n$, there is a finite set of solutions which generates all others. Baragar has shown that, for any g , there are infinitely many pairs (a, n) so that the equation requires at least g generators. Let $M(n, N)$ be the number of solutions of the Hurwitz equation with $a = n$ and each $|x_i| \leq N$, then Baragar has also shown that $M(n, N)$ grows like $C(\ln N)^{\alpha(n)+\epsilon}$ for all $\epsilon > 0$ and that $M(n, N) = \Omega((\ln N)^{\alpha(n)-\epsilon})$, where (Zagier) $\alpha(3) = 2$, but $\alpha(4)$ lies between 2.33 and 2.64 (later improved to $2.43 < \alpha(4) < 2.47$). Also that

$$\frac{2 \ln n}{\ln 4} \leq \alpha(n) \leq \frac{3 \ln n}{\ln 4}.$$

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D13 The equation $x^x y^y = z^z$.

Erdős asked for solutions of the equation $x^x y^y = z^z$, apart from the trivial ones $y = 1, x = z$. Chao Ko found an infinity of solutions, of which the first three are

$$\begin{array}{ccc} x & y & z \\ 12^6 & 6^8 & 2^{11}3^7 \\ 224^{14} & 112^{16} & 2^{68}7^{15} \\ 61440^{30} & 30720^{32} & 2^{357}15^{31} \end{array}$$

Did he find them all?

Claude Anderson conjectured that the equation $w^w x^x y^y = z^z$ has no solutions with $1 < w < x < y < z$, but Chao Ko & Sun Qi had earlier found an infinity of counterexamples to a generalization of the conjecture to any number of variables:

$$\begin{aligned} x_1 &= k^{n(n+1-2n-k)+2n}(k^n-1)^{2(k^n-1)} \\ x_2 &= k^{n(n+1-2n-k)}(k^n-1)^{2(k^n-1)+2} \end{aligned}$$

D14. $a_i + b_j$ made squares.

$$\begin{aligned} x_3 &= \dots = x_k = k^{k^n(k^{n+1}-2n-k)+n}(k^n-1)^{2(k^n-1)+1} \\ z &= k^{k^n(k^{n+1}-2n-k)+n+1} \end{aligned}$$

where, for $k \geq 3, n > 0$, and, for $k = 2, n > 1$. E.g., $w = 3^{12}2^6, x = 3^{13}2^5, y = 3^{14}2^4, z = 3^{14}2^5$. Ajai Choudhry also found a parametric solution for $k = 3$.

Chao Ko, Note on the diophantine equation $x^x y^y = z^z$, *J. Chinese Math. Soc.*, **2**(1940) 205–207; *MR 2*, 346.

Chao Ko & Sun Qi, On the equation $\prod_{i=1}^k x_i^{z_i}$, *J. Sichuan Univ.*, **2**(1964) 5–9.

W. H. Mills, An unsolved diophantine equation, *Report Inst. Theory of Numbers*, Boulder CO, 1959, 258–268.

D14 $a_i + b_j$ made squares.

Leo Moser asked for integers a_1, a_2, b_j ($1 \leq j \leq n$) such that the $2n$ numbers $a_i + b_j$ are all squares. This can be achieved by making $a_2 - a_1$ a sufficiently composite number; for example $a_1 = 0, a_2 = 2^{2n+1}, b_j = (2^{2n-j} - 2^{j-1})^2$.

John Leech observes that the extension to integers a_1, a_2, a_3, b_j ($1 \leq j \leq n$) is also solvable for any n . We may take a_1, a_2 to be $(x \pm y)^2; a_3$ can have an arbitrary value $x^2 + \lambda xy + y^2$, which can then be made square by putting $x = u^2 - v^2, y = 2uv + \lambda v^2$. Any values of u and v will give triads of squares with differences in this proportion. The problem to find values of u, v so that the scale factor is a rational square reduces to finding rational points on an elliptic curve; arbitrarily many rational values of b_j can then be simultaneously scaled to give integers a_1, a_2, a_3, b_j ($1 \leq j \leq n$) for any n . A much studied case is $\lambda = 0$, corresponding to sets of rational right-angled triangles of equal area. We can also specialize to fix $a_1 = b_1 = 0$. Provided that a_2, a_3 are squares p^2, q^2 such that q/p has a representation as the product of two distinct ratios $(u^2 - v^2)/2uv$, then it is again an elliptic curve problem to find rational squares $b_j = r_j^2$ such that both $p^2 + r_j^2$ and $q^2 + r_j^2$ are squares, and again we can rescale to find integers p, q, r_j ($1 \leq j \leq n$) for any n such that $p^2 + r_j^2$ and $q^2 + r_j^2$ are integer squares (cf. D20). For example, $13/6$ has representations $(u_1, v_1, u_2, v_2) = (9, 4, 5, 1)$ and $(8, 5, 9, 1)$ which yield

$$\begin{array}{cccccc} 0^2 & 351^2 & 650^2 & 1728^2 & 3200^2 \\ 720^2 & 801^2 & 970^2 & 1872^2 & 3280^2 \\ 1560^2 & 1599^2 & 1690^2 & 2328^2 & 3560^2 \end{array}$$

More generally we seek integers a_i ($1 \leq i \leq m$), b_j ($1 \leq j \leq n$). Jean Lagrange has produced the matrix

$$\begin{bmatrix} (54, 150, 111)^2 & (56, 150, 79)^2 & (72, 234, 177)^2 & (72, 186, 57)^2 \\ (6, 78, 96)^2 & (16, 48, 56)^2 & (48, 192, 168)^2 & (48, 120, 12)^2 \\ (54, 318, 384)^2 & (56, 312, 376)^2 & (72, 360, 408)^2 & (72, 312, 372)^2 \\ (6, -50, -96)^2 & (16, 0, -56)^2 & (48, 176, 168)^2 & (48, 104, -12)^2 \end{bmatrix}$$

of squares of quadratic forms, $(a, b, c) = au^2 + buv + cv^2$, which yields an infinity of solutions with $m = n = 4$. For example, $u = 2, v = 1$ gives

$$\begin{bmatrix} 627^2 & 603^2 & 933^2 & 717^2 \\ 276^2 & 216^2 & 744^2 & 444^2 \\ 1236^2 & 1224^2 & 1416^2 & 1284^2 \\ 172^2 & 8^2 & 712^2 & 388^2 \end{bmatrix}$$

Lagrange, in a letter dated 83-03-13, sends the matrices

$$\begin{bmatrix} 59^2 & 112^2 & 144^2 & 207^2 & 592^2 & 1351^2 & 4077^2 \\ 229^2 & 248^2 & 264^2 & 303^2 & 632^2 & 1369^2 & 4083^2 \\ 499^2 & 508^2 & 516^2 & 537^2 & 772^2 & 1439^2 & 4107^2 \end{bmatrix}$$

and

$$\begin{bmatrix} 18^2 & 234^2 & 346^2 & 514^2 \\ 282^2 & 366^2 & 446^2 & 586^2 \\ 477^2 & 531^2 & 589^2 & 701^2 \\ 1122^2 & 1146^2 & 1174^2 & 1234^2 \end{bmatrix}$$

In these examples, the a_i, b_j are not squares. If the a_i, b_j are themselves squares, then they provide configurations relevant to D20 (which see) where Lagrange & Leech have made considerable progress. Their triad and tetrad of squares a_i^2 ($i = 1, 2, 3$) and b_j^2 ($j = 1, 2, 3, 4$) with all $a_i^2 + b_j^2$ squares lead to a 4×5 array

$$\begin{bmatrix} 0^2 & 7422030^2 & 8947575^2 & 22276800^2 & 44142336^2 \\ 9282000^2 & 11184530^2 & 12892425^2 & 24132200^2 & 45107664^2 \\ 26822600^2 & 27830530^2 & 28275625^2 & 34867000^2 & 51652664^2 \\ 60386040^2 & 60840450^2 & 61045335^2 & 64364040^2 & 74799864^2 \end{bmatrix}$$

in the present problem.

D15 Numbers whose sums in pairs make squares.

Erdős & Leo Moser (and see earlier references) also asked the analogous question: are there, for every n , n distinct numbers such that the sum of any pair is a square? For $n = 3$ we can take

$$a_1 = \frac{1}{2}(q^2 + r^2 - p^2) \quad a_2 = \frac{1}{2}(r^2 + p^2 - q^2) \quad a_3 = \frac{1}{2}(p^2 + q^2 - r^2)$$

and for $n = 4$ we may augment these by taking s to be any number expressible as the sum of two squares in three distinct ways

D15. Numbers whose sums in pairs make squares.

$$s = u^2 + p^2 = v^2 + q^2 = w^2 + r^2 \text{ and } a_4 = s - \frac{1}{2}(p^2 + q^2 - r^2).$$

Jean Lagrange has given a quite general parametric solution for $n = 5$ and a simplification of it which appears to give a majority of all solutions. He tabulates the first 80 solutions, calculated by J.-L. Nicolas. The smallest is

$$-4878 \quad 4978 \quad 6903 \quad 12978 \quad 31122$$

and the smallest positive solution (at most one number can be negative) is

$$7442 \quad 28658 \quad 148583 \quad 177458 \quad 763442.$$

In a letter dated 72-05-19 he sends the following solution for $n = 6$:

$$-15863902 \quad 17798783 \quad 21126338 \quad 49064546 \quad 82221218 \quad 447422978$$

In fact the problem goes back to T. Baker who found five integers whose sums in pairs were squares, and C. Gill who found five whose sums in threes were squares.

Lagrange also found sets of n squares of which any $n - 1$ have their sum square. For $n = 3, 5$ and 8 , the smallest such are the squares of $(44, 117, 240)$, $(28, 64, 259, 392, 680)$ and $(79, 112, 204, 632, 896, 916, 1828, 2092)$.

Martin LaBar asked for a proof or disproof that a 3×3 magic square can be constructed from nine distinct integer squares. This requires that the nine quantities $x^2, y^2, z^2, y^2 + z^2 - x^2, z^2 + x^2 - y^2, x^2 + y^2 - z^2, 2x^2 - y^2, 2z^2 - x^2, 3x^2 - y^2 - z^2$ be distinct perfect squares. Not very likely, though it is not difficult to get four of the eight magic sums right.

T. Baker, *The Gentleman's Diary or Math. Repository*, London, 1839, 33–35, Question 1385.

C. Gill, *Application of the Angular Analysis to Indeterminate Problems of Degree 2*, New York, 1848, p. 60.

Martin LaBar, Problem 270, *Canad. Math. J.*, 15(1984) 69.

Jean Lagrange, Cinq nombres dont les sommes deux à deux sont des carrés, *Séminaire Delange-Pisot-Poitou (Théorie des nombres)* 12^e année, 20(1970–71) 10pp.

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Jean-Louis Nicolas, 6 nombres dont les sommes deux à deux sont des carrés, *Bull. Soc. Math. France*, Mém. No 49–50 (1977) 141–143; MR 58 #482.

A. R. Thatcher, A prize problem, *Math. Gaz.*, 61(1977) 64.

A. R. Thatcher, Five integers which sum in pairs to squares, *Math. Gaz.*, 62(1978) 25.

D16 Triples with the same sum and same product.

The problem to find as many different triples of positive integers as possible with the same sum and the same product has been solved by Schinzel: you can have arbitrarily many. In the interim Stephane Vandemergel found 13 triples each with sum 17116 and product $2^{10}3^35^27^211 \cdot 13 \cdot 19$. It may be of interest to ask for the smallest sums or products with each multiplicity. For example, for 4 triples, J. G. Mauldon finds the smallest common sum to be 118: (14,50,54), (15,40,63), (18,30,70), (21,25,72) and the smallest common product to be 25200: (6,56,75), (7,40,90), (9,28,100), (12,20,105).

Lorraine L. Foster & Gabriel Robins, Solution to Problem E2872, *Amer. Math. Monthly*, 89(1982) 499–500.

J. G. Mauldon, Problem E2872, *Amer. Math. Monthly*, 88(1981) 148.

D17 Product of blocks of consecutive integers not a power.

Erdős & Selfridge have proved that the product of consecutive integers is never a power, and the binomial coefficient $\binom{n}{k}$ (see B31) is never a power for $n \geq 2k \geq 8$. If $k = 2$, then $\binom{n}{2}$ is a square infinitely often, but Tijdeman's methods (see D9) will probably show that it is never a nontrivial higher power (for cubes and fourth powers, see Mordell's book), and that $k = 3$ never gives a power, apart from $n = 50$ (see D3).

Erdős & Graham ask if the product of two or more disjoint blocks of consecutive integers can be a power. Pomerance has noted that

$$\prod_{i=1}^4 (a_i - 1)a_i(a_i + 1)$$

is a square if $a_1 = 2^{n-1}$, $a_2 = 2^n$, $a_3 = 2^{n-1} - 1$, $a_4 = 2^{2n} - 1$, but Erdős & Graham suggest that if $l \geq 4$, then $\prod_{i=1}^k \prod_{j=1}^l (a_i + j)$ is a square on only a finite number of occasions.

K. R. S. Sastry notes that the product of the blocks $(n-1)n(n+1)$ and $(2n-2)(2n-1)2n$ is a square if $(n+1)(2n-1) = m^2$. This is equivalent to a Pell equation with an infinity of solutions. E.g. $n = 74$ gives

$$(73 \cdot 74 \cdot 75)(146 \cdot 147 \cdot 148) = 73^2 \cdot 74^2 \cdot 210^2$$

Erdős also asks if the product of (more than one) consecutive odd numbers is never a power (higher than the first)? Is the product of 4 consecutive members of an A.P. never a power? Euler showed that it cannot be a square. Fermat had shown that the members cannot be squares individually, while a nonsquare divisor must divide two distinct terms, either (a) 2 divides the first & third or the second & fourth, or (b) 3 divides the first

D18 Is there a perfect cuboid?

& fourth, or both. (a) alone is impossible mod 8, (b) alone is impossible mod 3, but we could have $6t^2, u^2, 2v^2, 3w^2$. But this implies $w^2 + t^2 = v^2$ and $w^2 + 4t^2 = u^2$, which can be disproved by descent – one Pythagorean ratio can't be twice another. For higher powers Leech notes that we can't have three cubes in A.P.

Sastry asks for which k can the product of four consecutive terms of an A.P. be a k -gonal number, where the r -th k -gonal number is $\frac{1}{2}r(r-1)r - (k-4)$.

Not for $k = 4$, as Euler showed, but Sastry finds solutions for all other k except 7, 14 and 37. Are these impossible?

P. Erdős, On consecutive integers, *Nieuw Arch. Wisk.*, 3(1955) 124–128.

P. Erdős & J. L. Selfridge, The product of consecutive integers is never a power, *Illinois J. Math.*, 19(1975) 292–301.

D18 Is there a perfect cuboid? Four squares whose sums in pairs are square. Four squares whose differences are square.

Is there a rational box? Our treatment of this notorious unsolved problem is owed almost entirely to John Leech. Does there exist a **perfect cuboid**, with integer edges x_i , face diagonals y_i and body diagonal z ; are there solutions of the simultaneous diophantine equations

$$(A) \quad x_{i+1}^2 + x_{i+2}^2 = y_i^2,$$

$$(B) \quad \sum x_i^2 = z^2 \quad ?$$

($i = 1, 2, 3$; and where necessary, subscripts are reduced modulo 3.)

Martin Gardner asked if any six of x_i, y_i, z could be integers. Here there are three problems: just the body diagonal z irrational; just one edge x_3 irrational; just one face diagonal y_1 irrational.

Problem 1. We require solutions to the three equations (A). Suppose such solutions have generators a_i, b_i where

$$x_{i+1} : x_{i+2} : y_i = 2a_i b_i : a_i^2 - b_i^2 : a_i^2 + b_i^2$$

Then we want integer solutions of

$$(C) \quad \prod \frac{a_i^2 - b_i^2}{2a_i b_i} = 1.$$

We can assume that the generator pairs have opposite parity and replace (C) by

$$(D) \quad \frac{a_1^2 - b_1^2}{2a_1 b_1} \cdot \frac{a_2^2 - b_2^2}{2a_2 b_2} = \frac{\alpha^2 - \beta^2}{2\alpha\beta}$$

An example is

$$(E) \quad \frac{6^2 - 5^2}{2 \cdot 6 \cdot 5} \cdot \frac{11^2 - 2^2}{2 \cdot 11 \cdot 2} = \frac{8^2 - 5^2}{2 \cdot 8 \cdot 5}$$

Kraitchik gave $241 + 18 - 2$ cuboids with odd edge less than a million. Lal & Blundon listed all cuboids obtainable from (D) with $a_1, b_1; \alpha, \beta \leq 70$, including the curious pair $(1008, 1100, 1155)$, $(1008, 1100, 12075)$. Leech has deposited a list of all solutions of (D) with two pairs of $a_1, b_1; a_2, b_2; \alpha, \beta \leq 376$.

Reversal of the cyclic order of subscripts in (C) leads to the derived cuboid: example (E) gives the least solution $(240, 44, 117)$, known to Euler, and the derived cuboid $(429, 2340, 880)$. Note that $240 \cdot 429 = 44 \cdot 2340 = 117 \cdot 880$.

Several parametric solutions are known: the simplest, also known to Euler, is

$$(F) \quad \alpha = 2(p^2 - q^2), \quad a_1 = 4pq, \quad b_1 = \beta = p^2 + q^2$$

For a_1, b_1 fixed, (D) is equivalent to the plane cubic curve

$$\frac{a_1^2 - b_1^2}{2a_1b_1} = \frac{u^2 - 1}{2u} \cdot \frac{2v}{v^2 - 1}$$

whose rational points are finitely generated, so Mordell tells us that one solution leads to an infinity. But not all rationals a_1/b_1 occur in solutions: $a_1/b_1 = 2$ is impossible, so there is no rational cuboid with a pair of edges in the ratio $3:4$.

Problem 2. Just an edge irrational. We want $x_1^2 + x_2^2 = y_3^2$ with $t + x_1^2, t + x_2^2, t + y_3^2$ all squares. This was proposed by "Mahatma" and readers gave $x_1 = 124, x_2 = 957, t = 13852800$. Bromhead extended this to a parametric solution. An infinity of solutions is given by

$$(G) \quad (x_1, x_2, y_3) = 2\xi\eta\zeta(\xi, \eta, \zeta), \quad t = \zeta^8 - 6\xi^2\eta^2\zeta^4 + \xi^4\eta^4$$

where (ξ, η, ζ) is a Pythagorean triple. The simplest such is $\xi = 5, \eta = 12; x_1 = 7800, x_2 = 18720; t = 211773121$. An earlier solution was given by Flood.

These are not all. We seek solutions of

$$(H) \quad x_1^2 + x_2^2 = y_3^2, \quad z^2 = x_1^2 + y_1^2 = x_2^2 + y_2^2$$

other than $z = y_3$ ($t = 0$). Leech found 100 primitive solutions with $z < 10^5$, 46 of which had $t > 0$. The generators for (H) satisfy

$$(I) \quad \frac{\alpha_1^2 + \beta_1^2}{2\alpha_1\beta_1} \cdot \frac{2\alpha_2\beta_2}{\alpha_2^2 + \beta_2^2} = \frac{a^2 - b^2}{2ab}$$

so solutions for fixed x_2/x_1 correspond to rational points on the cubic curve

$$(J) \quad x_1v(u^2 + 1) = x_2u(v^2 + 1)$$

The trivial solution $t = 0$ corresponds to several **ordinary points** which generate an infinity of solutions for each ratio x_2/x_1 . Solutions form cycles of four:

$$(K) \quad \begin{aligned} \zeta^2 &= \xi_1^2 + \eta_1^2 = \xi_2^2 + \eta_2^2 = \xi_3^2 + \eta_3^2 = \xi_4^2 + \eta_4^2, & \xi_1\xi_3 &= \xi_2\xi_4, \\ \xi_1^2 + \xi_2^2, \quad \xi_2^2 + \xi_3^2, \quad \xi_3^2 + \xi_4^2, \quad \xi_4^2 + \xi_1^2 & \text{ all squares,} \end{aligned}$$

corresponding to two pairs x_2/x_1 which correspond to two points collinear with the point $t = 0$ on (J). Conversely, such a pair of points corresponds to a cycle of four solutions.

Problem 3. Just one face diagonal irrational. There are two closely related problems: find three integers whose sums and differences are all squares; find three squares whose differences are squares. The cuboid form of the problem asks for integers satisfying

$$(L) \quad x_2^2 + y_2^2 = z^2, \quad x_1^2 + x_3^2 = y_2^2, \quad x_1^2 + x_2^2 = y_3^2$$

whose generators satisfy

$$(M) \quad \frac{\alpha_1^2 - \beta_1^2}{2\alpha_1\beta_1} \cdot \frac{\alpha_2^2 - \beta_2^2}{2\alpha_2\beta_2} = \frac{\alpha_3^2 + \beta_3^2}{2\alpha_3\beta_3}.$$

Write $u_i = (\alpha_i^2 - \beta_i^2)/4\alpha_i^2\beta_i^2$ and (M) becomes $u_1u_3 = 1 + u_2$, an equation investigated by many in the contexts of cycles and **frieze patterns**. Solutions occur in cycles of five! Leech listed 35 such with $\alpha_1, \beta_1, \alpha_2, \beta_2 \leq 50$ and deposited in UMT a list of all cycles with two pairs $\alpha_i, \beta_i \leq 376$. There is a close connexion with Napier's rules and the construction of rational spherical triangles.

Solutions to (L) are given by $x_2^2 = z^2 - y_2^2 = (p^2 - q^2)(r^2 - s^2)$, $x_3^2 = z^2 - y_3^2 = 4pqrs$, when the products of the numerators and denominators of $(p^2 - q^2)/2pq$ and $(r^2 - s^2)/2rs$ are each squares. Euler made p, q, r, s squares and found differences of fourth powers, e.g., $3^4 - 2^4, 9^4 - 7^4, 11^4 - 2^4$, whose products in pairs are squares. The first two give the second smallest solution $(117, 520, 756)$ of this type, whose cycle includes the smallest $(104, 153, 672)$, also known to Euler.

We can also express $z^2 = x_1^2 + y_2^2 = x_2^2 + y_3^2$ as the sum of two squares in two different ways: $x_3/x_1 = (\alpha_2^2 - \beta_2^2)/2\alpha_2\beta_2, x_2/x_1 = (\alpha_3^2 - \beta_3^2)/2\alpha_3\beta_3$ give $z^2 = 4(\alpha_2^2\alpha_3^2 + \beta_2^2\beta_3^2)/(\alpha_2^2\beta_3^2 + \alpha_3^2\beta_2^2)$. Euler made each factor square and found two rational right triangles of equal area $\frac{1}{2}\alpha_2\alpha_3\beta_2\beta_3$. Diophantus solved this with $\beta_2/\alpha_2 = (s+t)/2r, \beta_3/\alpha_3 = s/t$, where $r^2 = s^2 + st + t^2, s = l^2 - m^2, t = m^2 - n^2$. Put $(l, m, n) = (1, 2, -3)$ and we have a cycle

containing the third, fourth and fifth smallest of these cuboids. Leech found 89 with $z < 10^5$.

For fixed α_1/β_1 a nontrivial solution to (M) corresponds to an ordinary point, which generates an infinity of solutions, on the curve

$$(\alpha_1^2 - \beta_1^2)u(v^2 - 1) = 2\alpha_1\beta_1v(u^2 + 1).$$

The tangent at the point generates a cycle of special interest.

(M), like (D), but unlike (I), does not have nontrivial solutions for all ratios α/β ; e.g., there are none with $\alpha/\beta = 2$ or 3 and again no cuboids with edges in the ratio 3:4. Here we do not have "derived" cuboids.

Two other parametric forms for ratios $(p^2 - q^2)/2pq$ whose product and quotient are squares, are

$$p = 2m^2 \pm n^2, \quad r = m^2 \pm 2n^2, \quad q = s = m^2 \mp n^2.$$

Four squares whose sums in pairs are square. A solution of (C) gives three such squares; it may be portrayed as a trivalent vertex of a graph, the three edges joining it to nodes representing generator pairs for rational triangles. If such a pair occurs in one solution it occurs in infinitely many, so the valence of a node is infinite. We seek a subgraph homeomorphic to a tetrahedron K_4 whose vertices give four solutions of (C) and whose edges contain nodes corresponding to generator pairs common to pairs of solutions of (C). Lists of solutions have revealed no such subgraph; indeed, not even a closed circuit! Until a circuit is encountered, we need not distinguish between the pairs a, b and $a \pm b$.

So no example of four such squares is known. Construction of four squares with five square sums of pairs is straightforward.

A. R. Thatcher related the problem to the equation $y^2 = -x^8 + 35x^4 - 25$. The only integer solutions are $(\pm 1, \pm 3)$, but there may be a finite number of other rational solutions. Even if not, this does not preclude solutions to the original problem.

Four squares whose differences are square. This problem extends (M) analogously to the above extension of (C). A vertex is now pentavalent, adjacent to a cycle of five nodes or generator pairs. The cyclic order of the edges is important, but not the sense of rotation. A solution of (M) corresponds to three consecutive edges, and here we must distinguish between α, β and $\alpha \pm \beta$: the corresponding nodes are joined with double edges in Figure 11. The nodes are again of infinite valence. Four squares of the required type would correspond to a subgraph homeomorphic to K_6 , with six vertices and 15 nodes, one on each edge. The only cycle so far found is shown in Figure 11 and this will not serve as part of such a subgraph. Although a solution is unlikely, there do not appear to be any congruence conditions which forbid it.

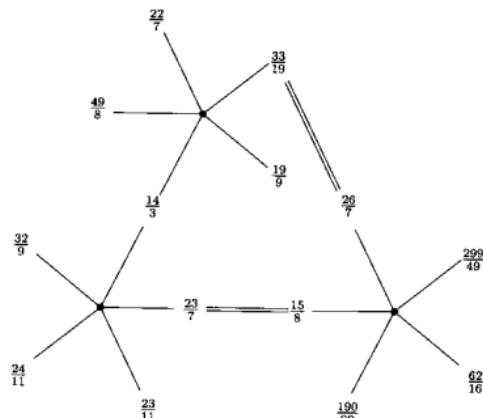


Figure 11. Three Cycles of Five Generator Pairs.

Though there is now no necessary connexion between the generator pairs α, β and $\alpha \pm \beta$, solutions of (M) containing both pairs do occur. Such a solution leads to a sequence of four squares with sums of two or three consecutive terms all square. How long can such a sequence be? For more than four we need a sequence of 5-cycles of solutions of (M) each containing adjacent generator pairs $\alpha, \beta; \bar{\alpha}, \bar{\beta}$ where $\alpha \pm \beta, \bar{\alpha} \pm \bar{\beta}$ belong one each to the neighboring cycles. Leech found the sequence

$(56, 31)(17, 6); (23, 11)(23, 7); (15, 8)(26, 7); (33, 19)(77, 19); (48, 29)(35, 4); (39, 31)(13, 9)$ where $23, 11 = 17 \pm 6$, etc., which gives a sequence of eight such squares. The squares of the edges of a perfect cuboid would form an infinite (periodic) sequence. Four nonzero squares with differences all square would lead to a sequence with terms three apart in constant ratio: an integer ratio would give an infinite sequence. Leech has since given the longer sequence

$$(14, 1)(224, 37); (261, 187)(155, 132); (287, 23)(23, 7); \\ (15, 8)(26, 7); (33, 19)(77, 19); (48, 29)(35, 4); (39, 31)(13, 9)$$

with "both ends surprisingly small" and asks "are they really ends?" He has no proof that a pair (a, b) can occur while $(a+b, a-b)$ does not. He has

some other sequences of the same length, but so far none longer. Randall Rathbun was unable to extend this sequence before (14,1) or after (13,9), but he was able to extend the earlier sequence to seven terms by prefixing (26767,2185)(87,25); or (940,693)(87,25); before (56,31).

The perfect rational cuboid. None of the known numerical solutions to problems 1, 2 and 3 gives a perfect cuboid, and many parametric solutions, for example (G), can be shown not to yield one. Spohn used Pocklington's work to show that *one* of the two mutually derived cuboids of (F) is not perfect and E. Z. Chein and Jean Lagrange have each shown that the derived cuboid is never perfect. On the other hand, no known parametric solution is complete so impossibility can't be proved from these alone. A solution of the problem in the previous section need not lead to a perfect cuboid. Korec showed that the least edge of a perfect rational cuboid must exceed 10^6 . Extensive searches, mainly by Randall Rathbun and Torbjorn Granlund, have shown that a perfect rational cuboid must have all its edges greater than 333750000. During the search, the results of which have been deposited in UMT, 6800 + 6380 + 6749 solutions of the three problems were found. Recently Korec has shown that the greatest edge is greater than 10^9 . Leech amplifies a result of Horst Bergmann to show that the product of the edges, face diagonals and body diagonal must be divisible by

$$2^8 \times 3^4 \times 5^3 \times 7 \times 11 \times 13 \times 17 \times 19 \times 29 \times 37.$$

Unsolved problems. Do three cycles of solutions of (M) exist whose graph is as in Figure 12? Here we've adopted John Leech's convention of writing the ratio of the sides as a fraction, e.g. $\frac{u_1}{x_1}$, where the same pair of generators belongs to both cycles (as the pair 14, 3 in Figure 11), but writing it as a ratio, e.g. $x_2 : x_3$, where a pair of generators belongs to one cycle, and their sum and difference to the other (as 15, 8 and 23, 7 in Figure 11).

Are there ratios $(p^2 - q^2)/2pq$, $(r^2 + s^2)/2rs$ with product and quotient both of the form $(m^2 - n^2)/2mn$? Is there a nontrivial solution of

$$(a^2c^2 - b^2d^2)(a^2d^2 - b^2c^2) = (a^2b^2 - c^2d^2)^2?$$

Such a solution would lead to a perfect cuboid. Is there a 5-cycle of solutions of (M) with

$$\alpha_1/\beta_1 = (p^2 - q^2)/2pq, \quad \alpha_2/\beta_2 = (r^2 + s^2)/2rs?$$

What circuits, if any, occur in the graph of solutions of (C)? What circuits occur in the graph of solutions of (M)? Are there cycles of solutions of (I) other than those of form (K)? What ratios besides 3/4 cannot occur as ratios of edges of cuboids in Problem 1? In Problem 3? Is there a

parallelepiped with all edges, face diagonals and body diagonals rational? Rathbun has found 41 pairs of primitive cuboids in which two edges of one equal two of the other. Twenty-one of these are pairs of solutions to Problem 1, with the body diagonal irrational; 13 are pairs of solutions to Problem 2; three are pairs of solutions to Problem 3, with a face diagonal irrational. Three are solutions to Problems 1 and 3, while the last is a solution to Problems 1 and 2.

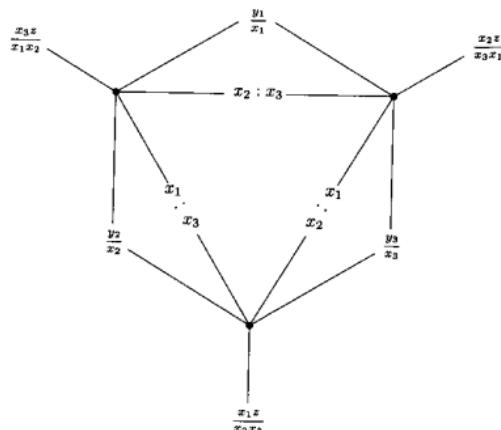


Figure 12. Are there three Cycles of Solutions of (M) like this?

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D19 Rational distances from the corners of a square.

Is there a point all of whose distances from the corners of the unit square are rational? It was earlier thought that there might not be any nontrivial example (i.e., an example not on the side of the square) of a point with three such rational distances, but John Conway & Mike Guy found an infinity of integer solutions of

$$(s^2 + b^2 - a^2)^2 + (s^2 + b^2 - c^2)^2 = (2bs)^2$$

where a, b, c are the distances of a point from three corners of a square of side s . There are relations between such solutions as shown in Figure 13.

For the fourth distance d to be an integer we also need $a^2 + c^2 = b^2 + d^2$. In the three-distance problem, one of s, a, b, c is divisible by 3, one by 4, and one by 5. In the four-distance problem, s is a multiple of 4 and a, b, c, d are odd (assuming that there is no common factor). If s is not a multiple of 3 (respectively 5) then two of a, b, c, d are divisible by 3 (resp. 5).

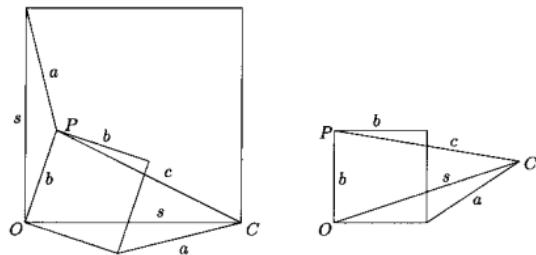


Figure 13. A Solution of the Three-Distance Problem, and its Inverse.

If the problem is generalized to a rational rectangle, then $a^2 + c^2 = b^2 + d^2$ is still required. This is the basis of a Martin Gardner puzzle [Mathematical Games, *Sci. Amer.* **210** #6 (June 1964), Problem 2, p.116]; see also the Dodge reference below. A similar problem with a square with an irrational side and one irrational distance occurs in Dudeney, Canterbury Puzzles, No. 66, pp. 107-109, 212-213. Gardner sends a copy of correspondence between Leslie J. Upton and J. A. H. Hunter. Hunter's 67-03-21 letter gives an infinity of solutions with the three distances in A.P.: $a = m^2 - 2mn + 2n^2$, $b = m^2 + 2n^2$, $c = m^2 + 2mn + 2n^2$ and $s^2 = 2m^2(m^2 + 4n^2)$ where s is an integer if $m = 2(u^2 + 2uv - v^2)$, $n = u^2 - 2uv - v^2$. For example, there is a point at distances 85, 99 and 113 from three consecutive corners of a square of side 140. It can be shown that the fourth distance is never rational in such solutions.

John Leech found points solving the three-distance problem which are dense in the plane. These include the Conway-Guy solutions, and the 'inverses' (in the sense of Fig. 13) of the Hunter solutions. But there are other solutions, and in some sense we now know "all" of them. Consider the more general problem of dissecting a rational square into rational triangles. It is known that at least four triangles are needed and there are just four candidate arrangements, the δ -configuration, the κ -configuration, the ν -configuration and the χ -configuration. The first two turn out to be "duals" and solutions are given by the rational points on an infinite family of elliptic curves. The first few hundred have been investigated by Bremner and Guy, who have also dealt similarly with the ν -configurations. The χ -configuration, i.e., the "four distance" problem, remains as an astonishingly hard nut to crack.

There are infinitely many solutions of the corresponding problem of integer distances a, b, c from the corners of an equilateral triangle of side t . In each of these one of a, b, c, t is divisible by 3, one by 5, one by 7 and one by 8. John Leech has sent us a neat and elementary proof of the fact that the points at rational distances from the vertices of *any* triangle with rational sides are dense in the plane of the triangle. This result was proved earlier by Almering; see reference at D21. Arnfried Kemnitz notes that $a = m^2 + n^2$, $b, c = m^2 \pm mn + n^2$ with $m = 2(u^2 - v^2)$, $n = u^2 + 4uv + v^2$ gives $t = 8(u^2 - v^2)(u^2 + uv + v^2)$ and an infinity of solutions in which the points are neither on the sides nor the circumcircle of the triangle. A computer search showed that (57,65,73,112) was the smallest such.

Thomas Berry writes the last displayed equation as

$$2(s^4 + b^4) + a^4 + c^4 = 2(s^2 + b^2)(a^2 + c^2)$$

and notes that this, and the corresponding equation

$$t^4 + a^4 + b^4 + c^4 = t^2 a^2 + t^2 b^2 + t^2 c^2 + b^2 c^2 + c^2 a^2 + a^2 b^2$$

for the equilateral triangle, both represent Kummer surfaces, i.e., quartic surfaces with just 16 singular points. They are not isomorphic, but are of the same special type, known as **tetrahedroids**.

There are the following consequences:

- A Kummer surface is not rational: there is no general parametric solution of either problem, in the sense that there are no polynomials (resp. rational functions) giving all integer (resp. rational) solutions.
- One-parameter families of solutions correspond to parametrizable curves on the surface. For example, the 16 conics (which always exist on a Kummer surface) give, in the equilateral triangle problem, points on the sides and circumcircle. The solution given by Arnfried Kemnitz corresponds to the plane section $b + c = 2a$.
- The elliptic curves used by Bremner & Guy to find the delta-lambda configurations form a pencil on the former surface, and since the surfaces are both tetrahedroids, there may be an elliptic pencil on the "equilateral triangle" surface, which allows an analogous attack.

Berry generalizes Almering's result by showing that if the squares on the sides of a triangle are rational and at least one side is rational, then the set of points at rational distances from all three vertices is dense in the plane of the triangle.

Jerry Bergum asks for what integers n do there exist positive integers x, y with $x \perp y$, x even, and $x^2 + y^2 = b^2$ & $x^2 + (y - nx)^2 = c^2$ both perfect squares. If $n = 2m(2m^2 + 1)$, then $x = 4m(4m^2 + 1)$, $y = mx + 1$

is a solution. There are no solutions if $n = \pm 1, \pm 2, \pm 4, \pm 11$ or $\pm p$ where $p \equiv 3 \pmod{4}$ and $p^2 + 4$ is prime, e.g., $\pm 3, \pm 7$. Bergum has several infinite families of values of n for which there are solutions, e.g., $n = 8t^2 \pm 4t + 2$ with $t > 0$. There are solutions with $n = \pm 5, \pm 6, \pm 8, \pm 9, \pm 14, \pm 19$. If $n = 8$, the least x for which there is a y is $x = 2996760 = 2^3 \cdot 3 \cdot 5 \cdot 13 \cdot 17 \cdot 113$ and if $n = 19$ the least x is 2410442371920 . One solution is $n = 5, x = 120, y = 391$ as may be seen from Fig. 15(b)!. The connexion between this problem and the original one is that (x, y) are the coordinates of P at distances b and c from the origin O and an adjacent corner of the square of side $s = nx$ where n is an integer.

Ron Evans notes that the problem may be stated: which integers n occur as the ratio base/height in integer-sided triangles? The sign of n is \pm according as the triangle is acute or obtuse (e.g., $n = -29, x = 120, y = 119$ is a solution). He also asks the dual problem: find every integer-sided triangle whose base divides its height. Here the height/base ratios 1 and 2 can't occur, but 3 can (e.g., base 4; sides 13, 15; height 12). If a ratio can occur, are there infinitely many primitive triangles for which it occurs? K. R. S. Sastry gives the triangles (3389, 21029, 24360) and (26921, 42041, 68880) in each of which the ratio base/height is 42, and (25, 26, 3) and (17, 113, 120) with ratios 1/8 and 8 (the third member of the triple is the base in each case).

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D20. Six general points at rational distances.

185

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D20 Six general points at rational distances.

The first edition of this book asked “are there six points in the plane, no three on a line, no four on a circle, all of whose mutual distances are rational?” Leech pointed out that one such configuration is obtained by fitting six copies of a triangle whose sides and medians are all rational. Such triangles were studied by Euler (see D21): the simplest has sides 68, 85, 87 and medians the halves of 158, 131, 127 [Fig. 14(a)]. Harborth & Kemnitz have shown that this triangle leads to the minimal configuration of six points, no three collinear, no four concyclic, at integer mutual distances. A related configuration is obtained by inversion in a concentric circle; the six triangles are then similar but no longer congruent. Can any such configuration be extended? Or are there any sets of more than six such points? Kemnitz has exhibited an unsymmetrical set of six points at integer distances, 13 of them distinct, the largest 319 [Fig. 14(b)].

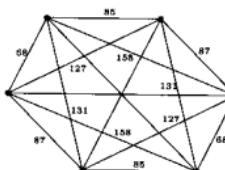


Figure 14(a). A triangle with rational medians, reflected in its centroid.

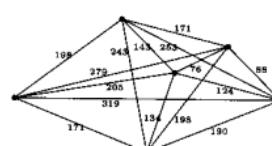


Figure 14(b). Six points at integer distances, 13 of them distinct.

There are two opposite extreme conjectures: (a) that there is a fixed number c such that any n points in a plane whose mutual distances are rational include at least $n - c$ which are collinear or concyclic, and (b) (ascribed to Besicovitch, but in 1959 he expressed the contrary opinion) that any polygon can be approximated arbitrarily closely by a polygon with all its sides and diagonals rational. If (a) is valid, what is the maximum value of c ?

If we have an infinite sequence of points $\{x_i\}$ with all distances rational, can we characterize the set of limit points? It was already known to Euler

that they could be dense on a circle. If the sequence is dense in the plane, Ulam conjectured that not all distances could be rational. Does it contain a dense subsequence with all distances irrational?

We can choose a straight line and two points at unit distance from it on a perpendicular line; then points on the first line at distances of the form $(u^2 - v^2)/2uv$ from the intersection form an infinite set of points with rational mutual distances. By inversion in a circle centred at one of the off-line points, we obtain a dense set of points on a circle, together with its centre, with rational mutual distances. This proves Conjecture (b) for cyclic polygons. Peebles (who quotes Huff) extended this to a straight line with four points at distances $\pm p, \pm q$ from it on a perpendicular line. If q/p has a representation as the product of two distinct ratios of the form $(u^2 - v^2)/2uv$, then it has an infinity of them, and there will be an infinity of points on the first line at rational distances from each other and from the four off-line points. Thus c is at least 4 in Conjecture (a). By inversion in a circle centred at one of the four off-line points, we obtain a dense set of points on a circle, together with its centre and a pair of points inverse in the circle, with rational mutual distances. Thus c is at least 3 for the circle in Conjecture (a). Are these the maximum values of c for infinite sets?

What finite sets surpass these values of c ? Leech gave an infinite family of sets of nine points, with no more than four on any line or circle, so $c = 5$ for these sets. They are based on solutions of the simultaneous Diophantine equations

$$x^2 + y^2 = \square, \quad x^2 + z^2 = \square, \quad x^2 + (y+z)^2 = \square, \quad x^2 + (y-z)^2 = \square;$$

most simply $x = 120, y = 209, z = 182$ [Fig. 15(b)].

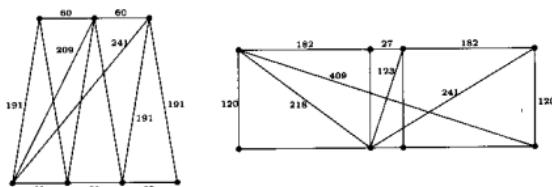


Figure 15 (a) & (b). Leech's Configurations of Points at Rational Distances.

Lagrange and Leech have given infinite families of pairs of triads of integers a_1, a_2, a_3 , and b_1, b_2, b_3 , such that the nine sums $a_i^2 + b_j^2$ are all squares. These lead to sets of 13 points on two perpendicular lines, comprising their intersection, points $(\pm a_i, 0)$ on one line and points $(0, \pm b_j)$

on the other, with integer mutual distances and no more than seven on a line or four on a circle, so $c = 6$ for these sets. The simplest example has $a_1 = 952, 1800, 3536$ and $b_1 = 960, 1785, 6630$. They also found an example in which one triad is extended to a tetrad; it has

$$\begin{aligned} a_1 &= 9282000, \quad 26822600, \quad 60386040, \\ b_1 &= 7422030, \quad 8947575, \quad 22276800, \quad 44142336, \end{aligned}$$

but these still give only $c = 6$. Leech has since found three further examples. Can a pair of tetrads be found with all 16 sums $a_i^2 + b_j^2$ square? If so, we would obtain a set of 17 points with integer mutual distances and $c = 8$.

Noll and Bell search for configurations with no three points collinear and no four concyclic, but using only lattice points. They call such configurations **N-clusters**. They, and independently William Kalsow & Bryan Rosenburg, found the 6-cluster $(0,0), (132,-720), (546,-272), (960,-720), (1155,540), (546,1120)$. They define the extent of an N-cluster to be the radius of the smallest circle, centred at one of the points, which contains them all. They find 91 nonequivalent prime 6-clusters of extent less than 20937, but no 7-clusters.

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D21 Triangles with integer sides, medians and area.

Is there a triangle with integer sides, medians and area? There are, in the literature, incorrect “proofs” of impossibility, but the problem remains open. It may be instructive to would-be solvers to find the fallacies in the arguments of Schubert and of Eggleston, referred to below. For some time it had been suspected that not even two rational medians were possible in a Heron triangle, but discoveries by Randall L. Rathbun, Arnfried Kemnitz and R. H. Buchholz have shown that they can occur. Even more recently, Rathbun has found infinitely many such, and it seems reasonable to conjecture that there are infinitely many infinite families, but this remains undemonstrated, as does the existence or impossibility of three rational medians. If we don’t require the area to be rational, there are many solutions. Euler gave a parametric solution of degree five,

$$a = 6\lambda^4 + 20\lambda^2 - 18, \quad b, c = \lambda^5 \pm \lambda^4 - 6\lambda^3 \pm 26\lambda^2 + 9\lambda \pm 9$$

with medians $-2\lambda^5 + 20\lambda^3 + 54\lambda, \pm\lambda^5 + 3\lambda^4 \pm 26\lambda^3 - 18\lambda^2 \pm 9\lambda + 27$. Recently George Cole has shown that, up to symmetry, there are just two parametric solutions of degree five, Euler’s and a new one. These have not so far yielded a non-degenerate triangle with rational area.

A host of problems arise from Pythagorean triangles. Eckert asked if there are two distinct Pythagorean triples whose products are equal, i.e., is there a solution of

$$xy(x^4 - y^4) = zw(z^4 - w^4)$$

in nonzero integers and Prothro asked if one product could be twice the other. More generally, Leech asked what small integers are the ratios of two

D21. Triangles with integer sides, medians and area.

such products? Trivially $xy(x^4 - y^4) = zyw(z^4 - w^4)$ when $x, y = z \pm w$. Since every product is divisible by $3 \cdot 4 \cdot 5$, many integers are possible, from $13 = \frac{5 \cdot 12 \cdot 13}{3 \cdot 4 \cdot 5}$ upwards. More subtly $11 = \frac{21 \cdot 220 \cdot 221}{13 \cdot 84 \cdot 85}$, and imprimitively

$$6 = \frac{6^3 \cdot 24 \cdot 143 \cdot 145}{135 \cdot 352 \cdot 377}.$$

Are these the smallest?

He also notes that one product is eight times the other on setting $x, y = z \pm w$.

How many primitive Pythagorean triangles can have the same area? A triple of such, with generators (77,38), (78,55) and (138,5) was found by Charles L. Shedd in 1945. In 1986 Rathbun found three more: (1610,869), (2002,1817), (2622,143) (2035,266), (3306,61), (3422,55) and (2201,1166), (2438,2035), (3565,198).

A fifth triple, (7238,2465), (9077,1122), (10434,731), was found independently on consecutive days by Dan Hoey and Rathbun. Is there an infinity of triples? Are there quadruples?

Sastry asks for Pythagorean triangles with a square and a triangle for legs and a pentagonal number, $\frac{1}{2}n(3n-1)$, for hypotenuse. Are there any nontrivial examples besides (3,4,5) and (105,100,145)? Does it help to allow pentagonal numbers of negative rank, $\frac{1}{2}n(3n+1)$?

Despairing of solving the problem of the rational box (D18), some people have investigated other polyhedra all of whose distances are integers. There are seven topologically different convex hexahedra, for example, and integer examples have been found by Harborth & Kemnitz (see D20) and by Peterson & Jordan. Sastry asked for solutions to the rational box problem, but using triangular numbers instead of squares. Charles Ashbacher gave the triangular numbers 66, 105, 105 whose sums of pairs and whose total are all triangular.

J. H. J. Almering, Heron problems, thesis, Amsterdam, 1950.

Ralph Heiner Buchholz, On triangles with rational altitudes, angle bisectors or medians, PhD thesis, Univ. of Newcastle, Australia, 1989.

George Raymond Cole, Triangles all of whose sides and medians are rational, PhD dissertation, Arizona State University, May 1991.

Eckert, J. Eckert, Problem 994, *Crux Mathematicorum*, **10**(1984) 318; comment 12(1986) 109.

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H. G. Eggleston, Isosceles triangles with integral sides and two integral medians, Note 2347, *Math. Gaz.*, **37**(1953) 208–209.

Albert Fässler, Multiple Pythagorean number triples, *Amer. Math. Monthly*, **98**(1991) 505–517; *MR 92d*: 11021.

Martin Gardner, Mathematical Games: Simple proofs of the Pythagorean theorem, and sundry other matters, *Sci. Amer.*, **211**#4 (Oct. 1964) 118–126.

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E. T. Prothro, *Amer. Math. Monthly* 95(1988) 31.

Randall L. Rathbun, Letter to the Editor, *Amer. Math. Monthly*, 99(1992) 283–284.

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H. Schubert, *Die Ganzahligkeit in der algebraischen Geometrie*, Leipzig, 1905, 1–16.

D22 Simplexes with rational contents.

Are there simplexes in any number of dimensions, all of whose contents (lengths, areas, volumes, hypervolumes) are rational? The answer is “yes” in two dimensions; there are infinitely many Heron triangles with rational sides and area. An example is a triangle of sides 13, 14, 15 which has area 84. The answer is also “yes” in three dimensions, but can all tetrahedra be approximated arbitrarily closely by such rational ones?

John Leech notes that four copies of an acute-angled Heron triangle will fit together to form such a tetrahedron, provided that the volume is made rational, and this is not difficult. E.g., three pairs of opposite edges of lengths 148, 195, 203. This is the smallest example: he finds the next few triples to be

$$(533, 875, 888), (1183, 1479, 1804), (2175, 2296, 2431), (1825, 2748, 2873),$$

$$(2180, 2639, 3111), (1887, 5215, 5512), (6409, 6625, 8484), (8619, 10136, 11275).$$

He also suggests examining references on p. 224 of Vol II of Dickson's *History* [A17]:

R. Güntsche, *Sitzungsber. Berlin Math. Gesell.*, 6(1907) 38–53.

R. Güntsche, *Archiv Math. Phys.*(3), 11(1907) 371.

E. Haentschel, *Sitzungsber. Berlin Math. Gesell.*, 12(1913) 101–108 & 17(1918) 37–39 (& cf. 14(1915) 371).

O. Schultz, *Ueber Tetraeder mit rationalen Masszahlen der Kantenlängen und des Volumen*, Halle, 1914, 292 pp.

Dickson appealed for a copy of this last. Did he ever get one? Does anyone know of a copy? Would they be willing to donate it, or offer it for sale, to the Strens Collection?

Leech also notes that this problem is answered positively in three dimensions by solutions to Problem 3 in D18 (find a box which is rational except

for one face diagonal). This problem was published as Problem 930 in *Crux Mathematicorum*, 10(1984) #3, p. 89, and the solution by the COPS (presumably an acronym for the Carleton (Ottawa) Problem Solvers) is:

Take a tetrahedron with a path of three mutually perpendicular edges, $a = p^2q^2 - r^2s^2$, $b = 2pqrs$, $c = p^2r^2 - q^2s^2$. Then $a^2 + b^2$, $b^2 + c^2$ are squares and $a^2 + b^2 + c^2 = (p^4 + s^4)(q^4 + r^4)$ is a square if

$$p^4 + s^4 = q^4 + r^4.$$

[John Leech notes “but not only if” and gives four casual examples,

$$(1^4 + 2^4)(2^4 + 13^4) = 697^2; \quad (1^4 + 2^4)(38^4 + 43^4) = 9673^2;$$

$$(1^4 + 2^4)(314^4 + 863^4) = 1275643^2; \quad (1^4 + 3^4)(9^4 + 437^4) = 1729298^2,$$

which imply further ones of type $(2^4 + 13^4)(38^4 + 43^4)$.]

This equation was solved by Euler. The solution mentioned in D9 is

$$p, q = x^7 + x^5y^2 - 2x^3y^4 \pm 3x^2y^5 + xy^6$$

$$r, s = x^6y \pm 3x^5y^2 - 2x^4y^3 + x^2y^5 + y^7$$

but this is not in any sense complete.

Buchholz found that the only rational tetrahedron with edge lengths ≤ 156 was that with edge lengths 117, 80, 53, 52, 51, 84, face areas 1800, 1890, 2016, 1170, and volume 18144. He also shows that a regular d -dimensional simplex with rational edge has rational d -dimensional volume just if d is of shape $4k(k+1)$ or $2k^2 - 1$.

Dove & Sumner relax the condition that the faces of a tetrahedron have rational area and find two tetrahedra with volume 3 and pairs of opposite edges (32,76) (33,70) (35,44) and (21,58) (32,76) (47,56). They ask if there are infinitely many tetrahedra with integer edges and the same integer volume. Is there such a tetrahedron with volume any given multiple of 3? They have examples from 3 to 99, except for 87.

The tetrahedron with a pair of opposite edges 896 and 990 and the other four edges each 1073, two face areas 436800 and 471240, and volume 62092800, is mentioned by Sierpiński and by Leitzmann.

Ralph Heiner Buchholz, Perfect pyramids, *Bull. Austral. Math. Soc.*, 45(1991) 353–368.

Kevin L. Dove & John L. Sumner, Tetrahedra with integer edges and integer volume, *Math. Mag.*, 65(1992) 104–111.

K. É. Kalyamanova, Rational tetrahedra (Russian), *Izv. Vyssh. Uchebn. Zaved. Mat.*, 1990 73–75; MR 92b:11014.

W. Leitzmann, Der pythagoreisch Lehrsatz, Leipzig, 1965, p. 91 [not in 1930 edition].

W. Sierpiński, *Pythagorean Triangles*, New York, 1962, p. 107.

D23 Some quartic equations.

Another of many unsolved diophantine equations is

$$(x^2 - 1)(y^2 - 1) = (z^2 - 1)^2$$

though Schinzel & Sierpiński have found all solutions for which $x - y = 2z$. Cao Zhen-Fu has shown that the only solutions satisfying $x - y = lz$ for other values of $l \leq 30$ are $|x| = |y|$ or $|z| = 1$, and Wang Yan Bin that these are the only solutions with $x - y = z^2 + 1$.

Kashihara has shown that all solutions of

$$(x^2 - 1)(y^2 - 1) = (z^2 - 1)$$

can be derived from the trivial solutions $(n, 1, 1)$ and $(1, n, 1)$.

For the equation $x^2 - 1 = y^2(z^2 - 1)$, Mignotte has shown that if z is large, then the greatest prime factor of y is at least $c \ln \ln y$.

Ron Graham has observed that the diophantine equations

$$2x^2(x^2 - 1) = 3(y^2 - 1) \quad \text{and} \quad (2x - 1)^2 = 2^n - 7$$

each have the solutions $x = 0, 1, 2, 3, 6$ and 91 . Is this merely an example of the Strong Law of Small Numbers? Evidently so! In a forthcoming paper Stroeker & de Weger find that Graham's equation has one pair and five quadruples of solutions. They note that it is unfair to count the solutions $x = 0$ and $x = 1$ of the Ramanujan-Nagell equation as separate, while 'forgetting' the solutions $x = -1, -2, -5, -90$.

Baragar has shown that the equation

$$x(x+1)y(y+1) = z(z+1),$$

studied by Katayama, is equivalent to a Markoff type equation (see D12)

$$x^2 + y^2 + z^2 = 2xyz + 5$$

and has counted the number of solutions of size less than N .

Cao Zhen-Fu, A generalization of the Schinzel-Sierpiński system of equations (Chinese; English summary), *J. Harbin Inst. Tech.*, **23**(1991) 9–14; MR 93b:11026.

Kenji Kashihara, The Diophantine equation $x^2 - 1 = (y^2 - 1)(z^2 - 1)$ (Japanese; English summary), *Res. Rep. Anan College Tech. No.* **26**(1990) 119–130; MR 91d:11025.

Shin-ichi Katayama & Kenji Kashihara, On the structure of the integer solutions of $z^2 = (x^2 - 1)(y^2 - 1) - a$, *J. Math. Tokushima Univ.*, **24**(1990) 1–11; MR 93c:11013.

Maurice Mignotte, A note on the equation $x^2 - 1 = y^2(z^2 - 1)$, *C. R. Math. Rep. Acad. Sci. Canada*, **13**(1991) 157–160; MR 92j:11026.

D25. Equations involving factorial n .

A. Schinzel & W. Sierpiński, Sur l'équation diophantienne $(x^2 - 1)(y^2 - 1) = [(y - x)/2]^2 - 1]^2$, *Elem. Math.*, **18**(1963) 132–133; MR 29 #1180.

Wang Yan-Bin, On the Diophantine equation $(x^2 - 1)(y^2 - 1) = (z^2 - 1)^2$ (Chinese; English summary), *Heilongjiang Daxue Ziran Kexue Xuebao*, **1989** no. 4 84–85; MR 91e:11028.

D24 Sum equals product.

For $k > 2$ the equation $a_1 a_2 \cdots a_k = a_1 + a_2 + \cdots + a_k$ has the solution $a_1 = 2, a_2 = k, a_3 = a_4 = \cdots = a_k = 1$. Schinzel showed that there is no other solution for $k = 6$ or $k = 24$. Misuzurewicz has shown that $k = 2, 3, 4, 6, 24, 114$ (misprinted as 144 in *Elem. Math.* and in the first edition of this book), 174 and 444 are the only $k < 1000$ for which there is exactly one solution.

This problem seems first to have been asked by Trost, arising from the solution of $a_1 a_2 \cdots a_k = a_1 + a_2 + \cdots + a_k = 1$ in rationals. For $k = 3$ this is due to Sierpiński; for $k > 3$ to Schinzel. Editorial comment in *Amer. Math. Monthly*, extended the result to $k \leq 10000$ and M. L. Brown gave necessary and sufficient conditions on k and extended the search to $k \leq 50000$.

M. L. Brown, On the diophantine equation $\sum X_i = \prod X_i$, *Math. Comput.*, **42**(1984) 239–240; MR 85d:11030.

M. Misuzurewicz, Ungelöste Probleme, *Elem. Math.*, **21**(1966) 90.

E. P. Starke & others, Solution to Problem E2262 [1970, 1008] & editorial comment, *Amer. Math. Monthly*, **78**(1971) 1021–1022.

E. Trost, Ungelöste Probleme, Nr. 14, *Elem. Math.*, **11**(1956) 134–135.

D25 Equations involving factorial n .

Are the only solutions of $n! + 1 = x^2$ given by $n = 4, 5$ and 7 ? Overholt has related this problem to a conjecture of Szpiro. Erdős & Obláth dealt with the equation $n! = x^p \pm y^p$ with $x \perp y$ and $p > 2$. For the case $p = 2$ with the plus sign, see Leech's remark at D2; and for the minus sign, split $n!$ into two even factors: $4! = 5^2 - 1^2 = 7^2 - 5^2; 5! = 11^2 - 1^2 = 13^2 - 7^2 = 17^2 - 13^2 = 31^2 - 29^2$. The number of solutions is $\frac{1}{2}d(n!/4)$.

Simmons notes that $n! = (m-1)m(m+1)$ for $(m, n) = (2, 3), (3, 4), (5, 5)$ and $(9, 6)$ and asks if there are other solutions. More generally he asks if there are any other solutions of $n! + x = x^k$. This is a variation on the question of asking for $n!$ to be the product of k consecutive integers in a nontrivial way ($k \neq n+1-j!$). Compare B23.

In a 93-05-07 letter to Ron Graham, Nobuhisa Abe states that $x(x+1) \cdots (x+k) = y^2 - 1$ has the unique solution $(x, y) = (2, 71)$ for $k = 5$ and no solutions for $k = 7$ or 11.

Berend & Osgood have shown that the set of n for which the equation $P(x) = n!$ has an integer solution x has density zero if $P(x)$ is a polynomial of degree ≥ 2 with integer coefficients.

Daniel Berend & Charles F. Osgood, On the equation $P(x) = n!$ and a question of Erdős, *J. Number Theory*, **42**(1992) 189–193; *MR 93e:11016*.

B. Brindza & P. Erdős, On some Diophantine problems involving powers and factorials, *J. Austral. Math. Soc. Ser. A* **51**(1991) 1–7; *MR 92i:11036*.

H. Brocard, Question 1532, *Nouv. Corresp. Math.*, **2**(1876) 287; *Nouv. Ann. Math.*(3) **4**(1885) 391.

P. Erdős & R. Obláth, Über diophantische Gleichungen der Form $n! = x^p \pm y^p$ und $n! \pm m! = x^p$, *Acta Szeged*, **8**(1937) 241–255.

M. Kraitchik, Recherches sur la Théorie des Nombres, tome 1, Gauthier-Villars, Paris, 1924, 38–41.

Marius Overholt, The Diophantine equation $n! + 1 = m^2$, *Bull. London Math. Soc.*, **25**(1993) 104; *MR 93m:11026*.

Richard M. Pollack & Harold N. Shapiro, The next to last case of a factorial diophantine equation, *Comm. Pure Appl. Math.*, **26**(1973) 313–325; *MR 50 #12915*.

Gustavus J. Simmons, A factorial conjecture, *J. Recreational Math.*, **1**(1968) 38.

D26 Fibonacci numbers of various shapes.

Stark asks which Fibonacci numbers (see A3) are half the difference or sum of two cubes. This is related to the problem of finding all complex quadratic fields of class number 2. Examples: $1 = \frac{1}{2}(1^3 + 1^3)$, $8 = \frac{1}{2}(2^3 + 2^3)$, $13 = \frac{1}{2}(3^3 - 1^3)$. Antoniadis has related all such fields to solutions of certain diophantine equations, and solved them all but two, which were later settled by de Weger.

Cohn showed that the only square Fibonacci numbers are 0, 1 and 144, and Luo Ming confirmed Vern Hoggatt's conjecture that the only triangular numbers, i.e. of the form $\frac{1}{2}m(m+1)$, are 0, 1, 3, 21 and 55, and later that the only such Lucas numbers are 1, 3 and 5778.

Jannis A. Antoniadis, Über die Kennzeichnung zweiklassiger imaginär-quadratischer Zahlkörper durch Lösungen diophantischer Gleichungen, *J. reine angew. Math.*, **339**(1983) 27–81; *MR 85g:11098*.

J. H. E. Cohn, On square Fibonacci numbers, *J. London Math. Soc.*, **39**(1964) 537–540; *MR 29 #1166*.

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D27 Congruent numbers.

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D27 Congruent numbers.

Congruent numbers are perhaps confusingly named; they are related to pythagorean triangles and have an ancient history. Several examples (5, 6, 14, the seventeen entries CA in Table 7, and ten more greater than 1000) are in an Arab manuscript more than a thousand years ago. But it is only for the last ten years, since the work of Tunnell, that we have a reasonably complete understanding of them. They are those integers a for which

$$x^2 + ay^2 = z^2 \quad \text{and} \quad x^2 - ay^2 = t^2$$

have simultaneous integer solutions. Part of the fascination is the often inordinate size of the smallest solutions. For example, $a = 101$ is a congruent number and Bastien gave the smallest solution:

$$x = 201524262949760001961 \quad y = 118171431852779451900$$

$$z = 2339148435306225006961 \quad t = 1628124370727269996961$$

and in spite of improved computing techniques and machines, it may still be some time before some other of the more recalcitrant examples are discovered. Some other large specimens, found by J. A. H. Hunter, M. R. Buckley and K. Gallyas, are given in the first edition of this book.

Congruent numbers are equivalently defined as those a for which there are solutions of the diophantine equation

$$x^4 - a^2y^4 = u^2$$

Table 7. Congruent (C) and Noncongruent (N) Numbers less than 1000. The entry for $a = 40c+r$ is in column c and row r .

Dickson's *History* gives many early references, including Leonardo of Pisa (Fibonacci); Genocchi; and Gérardin, who gave 7, 22, 41, 69, 77, the twenty Arabic examples and the forty-three entries CG in Table 7. We need consider only squarefree values of a ; of the 608 such that are less than 1000, 361 are congruent and 247 are noncongruent. It has long been conjectured that squarefree numbers are congruent if they are $\equiv 5, 6$ or $7 \pmod{8}$. This is now known to be true [modulo some widely believed conjectures concerning elliptic curves]. The entries C5, C7 and C6 in Table 7 are for primes $\equiv 5$ or $7 \pmod{8}$ and the doubles of primes $\equiv 3 \pmod{8}$. Bastien observed that the following are noncongruent: primes $\equiv 3 \pmod{8}$; products of two such primes; the doubles of primes $\equiv 5 \pmod{8}$; the doubles of the products of two such primes; and the doubles of primes $\equiv 9 \pmod{16}$; these are the respective entries N3, N9, NX, NL and N2 in Table 7. He gave some other noncongruent numbers (entries NB, though $a = 1$ is due to Fermat, and many others were known earlier, e.g. to Genocchi) and stated that a was noncongruent if it was a prime $\equiv 1 \pmod{8}$ with $a = b^2 + c^2$ and $b + c$ a nonresidue (see F5) of a . This accounts for several of the entries N1.

D27. Congruent numbers

Note that the entries 1, 3, 5 and 7 serve as a table of primes in these residue classes mod 8.

Entries C& and N& are from Alter, Curtz & Kubota, and CJ and NJ from Jean Lagrange's thesis.

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R. Alter, T. B. Curtz & K. K. Kubota, Remarks and results on congruent numbers, *Proc. 3rd S.E. Conf. Combin. Graph Theory Comput., Congr. Numer.* **6** (1972) 27–35; *MR 50* #2047.

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Paul Monsky, Three constructions of rational points on $Y^2 = X^3 \pm NX$, *Math. Z.*, **209**(1992) 445–462; *MR 93d*:11058.

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P. Serf, Congruent numbers and elliptic curves, in *Computational Number Theory* (Proc. Conf. Number Theory, Debrecen, 1989), de Gruyter, 1991, 227–238; *MR 93g*:11068.

M. Stephens, Congruence properties of congruent numbers, *Bull. London Math. Soc.*, **7**(1975) 182–184; *MR 52* #260.

Errol B. Tunnell, A classical diophantine problem and modular forms of weight $3/2$, *Invent. Math.* **72**(1983) 323–334; *MR 85d*:11046.

D28 A reciprocal diophantine equation.

Mordell asked for the integer solutions of

$$\frac{1}{w} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{wxyz} = 0.$$

Several papers have appeared, giving parametric families of solutions. For example, Takahiro Nagashima sends solutions to Mordell's equation: $(w, x, y, z) = (5, 3, 2, -1), (-7, -3, -2, 1), (31, -5, -3, 2), (1366, -15, 7, -13), (n+1, -n, -1, 1)$ and more generally $w = xyz + 1$ with

$$x = -2\epsilon h^3 - \epsilon h^2(n-3) + \epsilon h(n-1-2\epsilon) + 1,$$

$$y = 2\delta h^2 + \epsilon h(n-3) - \epsilon h(n-1) + 1,$$

$$z = -2\delta h^2 - \epsilon h(n-1) - 1,$$

where $\epsilon, \delta = \pm 1$ independently, but there seems to be no guarantee that these four two-parameter families give all solutions.

Zhang has shown how to obtain all solutions below a given bound, while Clelie Oursler and Judith Longyear have sent extensive analyses which each give a procedure for finding all solutions. That of Longyear extends to the equation $\sum(1/x_i) + \prod(1/x_i) = 0$ with $n(\geq 3)$ variables x_i in place of Mordell's $n = 4$.

Lawrence Brenton & Daniel S. Drucker, On the number of solutions of $\sum_{i=1}^n (1/x_i) + 1/(x_1 \cdots x_n) = 1$, *J. Number Theory*, 44(1993) 25–29.

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Zhang Ming-Zhi, On the diophantine equation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w} + \frac{1}{xyzw} = 0$, *Acta Math. Sinica (N.S.)*, 1(1985) 221–224; *MR 88a:11033*.

E. Sequences of Integers

Here we are mainly, but not entirely, concerned with infinite sequences; there is some overlap with sections C and A. An excellent text and source of problems is H. Halberstam & K. F. Roth, *Sequences*, 2nd edition, Springer-Verlag, New York, 1982. Other references are:

P. Erdős, A. Sárközi & E. Szemerédi, On divisibility properties of sequences of integers, in *Number Theory, Colloq. Math. Soc. János Bolyai*, 2, North-Holland, 1970, 35–49.

H. Ostmann, *Additive Zahlentheorie I, II*, Springer-Verlag, Heidelberg, 1956.

Carl Pomerance & András Sárközi, Combinatorial Number Theory, in R. Graham, M. Grötschel & L. Lovász (editors) *Handbook of Combinatorics*, North-Holland, Amsterdam, 1994.

A. Stöhr, Gelöst und ungelöst: Fragen über Basen der natürlichen Zahlenreihe I, II, *J. reine angew. Math.*, 194(1955) 40–65, 111–140; *MR 17*, 713.

Paul Turán (editor), *Number Theory and Analysis; a collection of papers in honor of Edmund Landau (1877–1938)*, Plenum Press, New York, 1969, contains several papers, by Erdős and others, on sequences of integers.

We will denote by $A = \{a_i\}$, $i = 1, 2, \dots$ a possibly infinite strictly increasing sequence of nonnegative integers. The number of a_i which do not exceed x is denoted by $A(x)$. By the density of a sequence we will mean $\lim A(x)/x$, if it exists.

E1 A thin sequence with all numbers equal to a member plus a prime.

Erdős offers \$50.00 for a solution of the problem: does there exist a sequence thin enough that $A(x) < c \ln x$, but with every sufficiently large integer expressible in the form $p + a_i$ where p is a prime?

For the analogous problem with squares in place of primes, Leo Moser showed that $A(x) > (1+c)\sqrt{x}$ for some $c > 0$, while Erdős showed that there was a sequence with $A(x) < c\sqrt{x}$. Moser's best value for c was 0.06, but this has been improved to 0.147 by Abbott, to 0.245 by Balasubramanian & Soundararajan, and to 0.273 by Cilleruelo. For the r th powers,

Cilleruelo obtains

$$A(x) > \frac{x^{1-\frac{1}{r}}}{\Gamma(2 - \frac{1}{r})\Gamma(1 + \frac{1}{r})}$$

For the problem with powers of two in place of primes, Ruzsa obtained the analog of Erdős's result, but it is not known if there is a constant $c > 0$ such that every sequence A for which every positive integer is representable in the form $a + 2^k$ has $A(x) > (1 + c)\log_2 x$.

H. L. Abbott, On the additive completion of sets of integers, *J. Number Theory*, 17(1983) 135–143.

R. Balasubramanian & K. Soundarajan, On the additive completion of squares, II, *J. Number Theory*, 40(1992) 127–129.

Javier Cilleruelo, The additive completion of k th powers, *J. Number Theory*, 44(1993) 237–243.

P. Erdős, Problèmes et résultats en additive number theory, *Colloque sur la Théorie des Nombres, Bruxelles*, 1955, 127–137, Masson, Paris, 1956.

L. Moser, On the additive completion of sets of integers, *Proc. Symp. Pure Math.*, 8(1965) Amer. Math. Soc., Providence RI, 175–180.

I. Ruzsa, On a problem of P. Erdős, *Canad. Math. Bull.*, 15(1972) 309–310.

E2 Density of a sequence with l.c.m. of each pair less than x .

What is the maximum value of $A(x)$ if the least common multiple $[a_i, a_j]$ of each pair of members of the sequence is at most x ? It is known that

$$(9x/8)^{1/2} \leq \max A(x) \leq (4x)^{1/2}.$$

The lower bound is obtained by taking all the numbers from 1 up to $\sqrt{x/2}$ and then the even numbers up to $\sqrt{2x}$.

And how many numbers less than x can we find with the greatest common divisor of any pair $< t$ for a given t ? If $t < n^{\frac{1}{2}+\epsilon}$, the number $\sim \pi(n)$, while if $t = n^{\frac{1}{2}+\epsilon}$, it is $\sim (1 + c')\pi(n)$.

Erdős also asks for bounds on $B(x)$, the smallest number so that any subset of $[1, x]$ of cardinality $B(x)$ always contains three members which have pairwise the same least common multiple. Perhaps $B(x) = o(x)$. Again, let $C(x)$ be the corresponding smallest cardinality, so that there are always three numbers with pairwise the same greatest common divisor. No doubt

$$\iota e^{c_1(\ln x)^{1/2}} < C(x) < e^{c_2(\ln x)^{1/2}} \quad ?$$

but the best that Erdős has proved is $C(x) < x^{3/4}$.

Given a sequence A , $a_1 < a_2 < \dots$, Erdős & Szemerédi denote by $F(A, x, k)$ the number of i for which the l.c.m. $[a_{i+1}, a_{i+2}, \dots, a_{i+k}] < x$, and ask if it is true that for every $\epsilon > 0$ there is a k for which $F(A, x, k) < x^\epsilon$?

E4 Sequence with no member dividing the product of r others.

They proved that $F(A, x, 3) < c_1 x^{1/3} \ln x$ for every A , and that there is an A for which $F(A, x, 3) > c_2 x^{1/3} \ln x$ for infinitely many x , but they don't know if there is an A for which this is true for all x .

Graham, Spencer & Witsenhausen ask how dense can a sequence of integers be so that $\{n, 2n, 3n\}$ never occur?

There is a good bibliography and many unsolved problems in this area in the paper of Erdős, Sárközy & Szemerédi. See also the references at **B24**.

P. Erdős, Problem, *Mat. Lapok*, 2(1951) 233.

P. Erdős & A. Sárközy, On the divisibility properties of sequences of integers, *Proc. London Math. Soc.* (3), 21(1970) 97–100; MR 42 #222.

P. Erdős, A. Sárközy & E. Szemerédi, On divisibility properties of sequences of integers, in *Number Theory Colloq. János Bolyai Math. Soc., Debrecen 1968*, North-Holland, Amsterdam (1970) 35–49; MR 43 #4790.

P. Erdős & E. Szemerédi, Remarks on a problem of the *American Mathematical Monthly*, *Mat. Lapok*, 28(1980) 121–124; MR 82c:10066.

R. L. Graham, J. H. Spencer & H. S. Witsenhausen, On extremal density theorems for linear forms, in H. Zassenhaus (ed.), *Number Theory and Algebra*, Academic Press, New York, 1977, 103–109; MR 58 #569.

E3 Density of integers with two comparable divisors.

Is it true that the density of those integers

6, 12, 15, 18, 20, 24, 28, 30, 35, 36, 40, 42, 45, 48, 54, 56, 60, 63, 66, 70, 72, ...

which have two divisors d_1, d_2 such that $d_1 < d_2 < 2d_1$, is one? Erdős has shown that the density exists. There is a connexion with covering congruences (**F13**). Since the first edition this has been solved affirmatively by Maier & Tenenbaum.

P. Erdős, On the density of some sequences of integers, *Bull. Amer. Math. Soc.*, 54(1948) 685–692; MR 10, 105.

Helmut Maier & G. Tenenbaum, On the set of divisors of an integer, *Invent. Math.*, 76(1984) 121–128; MR 86b:11057.

E4 Sequence with no member dividing the product of r others.

If no member of the sequence $\{a_i\}$ divides the product of r other terms, Erdős shows that

$$\pi(x) + c_1 x^{2/(r+1)} (\ln x)^{-2} < A(x) < \pi(x) + c_2 x^{2/(r+1)} (\ln x)^{-2}$$

where $\pi(x)$ is the number of primes $\leq x$. If, however, we suppose that the products of any number, not greater than r , of the a_i are distinct, what is $\max A(x)$? For $r \geq 3$, Erdős shows

$$\max A(x) < \pi(x) + O(x^{2/3+\epsilon}).$$

If $r = 1$, so that no term divides any other, the sequence is called primitive. Zhang has shown that for a primitive sequence whose members each contain at most four prime factors,

$$\sum_{a_i \leq n} \frac{1}{a_i \ln a_i} \leq \sum_{p \leq n} \frac{1}{p \ln p}$$

(and hence less than 1.64) for $n > 1$, the sums being taken over all members of the sequence up to n and all primes up to n .

P. Erdős, On sequences of integers no one of which divides the product of two others and on some related problems, *Inst. Math. Mec. Tomsk*, **2**(1938) 74–82.

P. Erdős, Extremal problems in number theory V (Hungarian), *Mat. Lapok*, **17**(1966) 135–155.

P. Erdős, On some applications of graph theory to number theory, *Publ. Ramanujan Inst.*, **1**(1969) 131–136.

P. Erdős & Zhang Zhen-Xiang, Upper bound of $\sum 1/(a_i \ln a_i)$ for primitive sequences, *Proc. Amer. Math. Soc.*, **117**(1993) 891–895.

Zhang Zhen-Xiang, On a conjecture of Erdős on the sum $\sum_{p \leq n} 1/(\ln p \log p)$, *J. Number Theory*, **39**(1991) 14–17; MR **92f**:11131.

Zhang Zhen-Xiang, On a problem of Erdős concerning primitive sequences, *Math. Comput.*, **60**(1993) 827–834; MR **93k**:11120.

E5 Sequence with members divisible by at least one of a given set.

Let $D(x)$ be the number of numbers not greater than x which are divisible by at least one a_i where $a_1 < a_2 < \dots < a_k \leq n$ is a finite sequence. Is $D(x)/x < 2D(n)/n$ for all $x > n$? The number 2 cannot be reduced: for example, $n = 2a_1 - 1$, $x = 2a_1 < a_2$. In the other direction it is known that for each $\epsilon > 0$ there is a sequence which does *not* satisfy the inequality $D(x)/x > \epsilon D(n)/n$.

A. S. Besicovitch, On the density of certain sequences, *Math. Ann.*, **110**(1934) 335–341.

P. Erdős, Note on sequences of integers no one of which is divisible by any other, *J. London Math. Soc.*, **10**(1935) 126–128.

E6 Sequence with sums of pairs not members of a given sequence.

Let $n_1 < n_2 < \dots$ be a sequence of integers such that $n_{i+1}/n_i \rightarrow 1$ as $i \rightarrow \infty$, and the $\{n_i\}$ are distributed uniformly mod d for every d ; i.e., the number $N(c, d; x)$ of the $n_i \leq x$ with $n_i \equiv c \pmod{d}$ is such that

$$N(c, d; x)/N(1, 1; x) \rightarrow 1/d \quad \text{as } x \rightarrow \infty$$

for each c , $0 \leq c < d$, and all d . If $a_1 < a_2 < \dots$ is an infinite sequence for which $a_j + a_k \neq n_i$ for any i, j, k then Erdős asks: is it true that the density of the a_j is less than $\frac{1}{2}$?

E7 A series and a sequence involving primes.

If p_n is the n th prime, Erdős asks if $\sum (-1)^n n/p_n$ converges. He notes that the series $\sum (-1)^n (n \ln n)/p_n$ diverges.

He also asks if, given three distinct primes and $a_1 < a_2 < a_3 < \dots$ are all the products of their powers arranged in increasing order, it is true infinitely often that a_i and a_{i+1} are both prime powers. And what if we use k primes or even infinitely many in place of three? Meyer & Tijdeman have asked a similar question for two finite sets S and T of primes with $a_1 < a_2 < a_3 < \dots$ formed from $S \cup T$. Are there infinitely many i for which a_i is a product of powers of primes from S , while a_{i+1} is a product of powers of primes from T ?

E8 Sequence with no sum of a pair a square.

Paul Erdős & David Silverman consider k integers $1 \leq a_1 < a_2 < \dots < a_k \leq n$ such that no sum $a_i + a_j$ is a square. Is it true that $k < n(1+\epsilon)/3$, or even that $k < n/3 + O(1)$? The integers $\equiv 1 \pmod{3}$ show that if this is true, then it is best possible. They suggest that the same question could be asked for other sequences instead of the squares.

Erdős & Graham added to their book at the proof stage that J. P. Marrias has discovered that the sum of any two integers

$$\equiv 1, 5, 9, 13, 14, 17, 21, 25, 26, 29, 30 \pmod{32}$$

is never a square mod 32, so k can be chosen to be at least $11n/32$. This is best possible for the modular version of the problem since Lagarias, Odlyzko & Shearer have shown that if $S \subseteq \mathbb{Z}_n$ and $S + S$ contains no square of \mathbb{Z}_n , then $|S| \leq 11n/32$.

J. C. Lagarias, A. M. Odlyzko & J. B. Shearer, On the density of sequences of integers the sum of no two of which is a square, I. Arithmetic progressions, *J. Combin. Theory Ser. A*, **33**(1982) 167–185; II. General sequences, *J. Combin. Theory Ser. A*, **34**(1983) 123–139; MR **85d**:11015ab.

E9 Partitioning the integers into classes with numerous sums of pairs.

The conjecture of K. F. Roth, that there exists an absolute constant c so that for every k there is an $n_0 = n_0(k)$ with the following property: For $n > n_0$, partition the integers not exceeding n into k classes $\{a_i^{(j)}\}$ ($1 \leq j \leq k$); then the number of distinct integers not exceeding n which can be written in the form $a_{i_1}^{(j_1)} + a_{i_2}^{(j_2)}$ for some j is greater than cn , has been confirmed by Erdős, Sárközy & Sós.

They also investigate the corresponding problem with products in place of sums, where the problem for $k = 2$ remains open.

P. Erdős & A. Sárközy, On a conjecture of Roth and some related problems, II, in R. A. Mollin (ed.) *Number Theory*, Proc. 1st Conf. Canad. Number Theory Assoc., Banff 1988, de Gruyter, 1990, 125–138.

P. Erdős, A. Sárközy & V. T. Sós, On a conjecture of Roth and some related problems, I, *Colloq. Math. Soc. János Bolyai* (1992).

E10 Theorem of van der Waerden. Szemerédi's theorem. Partitioning the integers into classes; at least one contains an A.P.

The well-known theorem of van der Waerden states that for every l there is a number $n(h, l)$ such that if the integers not exceeding $n(h, l)$ are partitioned into h classes, then at least one class contains an arithmetic progression (A.P.) containing $l + 1$ terms. More generally, given l_0, l_1, \dots, l_{h-1} , there is always a class V_i ($0 \leq i \leq h - 1$) containing an A.P. of $l_i + 1$ terms. Denote by $W(h, l)$, or more generally $W(h; l_0, l_1, \dots, l_{h-1})$, the least such $n(h, l)$.

Chvátal computed $W(2; 2, 2) = 9$, $W(2; 2, 3) = 18$, $W(2; 2, 4) = 22$, $W(2; 2, 5) = 32$ and $W(2; 2, 6) = 46$ and Beeler & O'Neil give $W(2; 2, 7) = 58$, $W(2; 2, 8) = 77$ and $W(2; 2, 9) = 97$. The values $W(2; 3, 3) = 35$ and $W(2; 3, 4) = 55$ were found by Chvátal and $W(2; 3, 5) = 73$ by Beeler & O'Neil. Stevens & Shantaram found $W(2; 4, 4) = 178$; Chvátal found $W(3; 2, 2, 2) = 27$ and Brown $W(3; 2, 2, 3) = 51$. Beeler & O'Neil also found $W(4; 2, 2, 2) = 76$.

Most proofs of van der Waerden's theorem give poor estimates for $W(h, l)$. Erdős & Rado showed that $W(h, l) > (2lh^l)^{\frac{1}{2}}$ and Moser, Schmidt, and Berlekamp successively improved this to

$$W(h, l) > lh^{c \ln h} \quad \text{and} \quad W(h, l) > h^{l+1-c\sqrt{(l+1)\ln(l+1)}}$$

Moser's bound has been improved for $l \geq 5$ by Abbott & Liu to

$$W(h, l) > h^{c_s(\ln h)^s}$$

where s is defined by $2^s \leq l < 2^{s+1}$, and Everts has shown that $W(h, l) > lh^l/4(l+1)^2$, a result which is sometimes better than Berlekamp's. For $h = 2$ Szabó has recently shown that $W(2, l) > 2^l/l^e$. All upper bounds were 'ackermanic' in size, until Shelah's proof reduced them to 'wosser' — for an explanation of these terms see the book by Graham, Rothschild & Spencer.

A closely related function, with $l+1 = k$, is the now famous $r_k(n)$, introduced long years ago by Erdős & Turán: the least r such that the sequence $1 \leq a_1 < a_2 < \dots < a_r \leq n$ of r numbers not exceeding n must contain a k -term A.P. The best bounds when $k = 3$ are due to Behrend, Roth, and Moser:

$$n \exp(-c_1 \sqrt{\ln n}) < r_3(n) < c_2 n / \ln \ln n$$

and for larger k Rankin showed that

$$r_k(n) > nl - c_s / (\ln n)^{s/(s+1)}$$

where s , much as before, is defined by $2^s < k \leq 2^{s+1}$.

A big breakthrough was Szemerédi's proof that $r_k(n) = o(n)$ for all k , but neither his proof, nor those of Furstenberg and of Katzenelson & Ornstein (see Thouvenot) give estimates for $r_k(n)$. Erdős conjectures that

$$i \quad r_k(n) = o(n(\ln n)^{-t}) \quad \text{for every } t \quad ?$$

This would imply that for every k there are k primes in A.P. See A5 for a potentially remunerative conjecture of Erdős, which, if true, would imply Szemerédi's theorem.

Another closely related problem was considered by Leo Moser, who wrote the integers in base three, $n = \sum a_i 3^i$ ($a_i = 0, 1$ or 2) and examined the mapping of n into lattice points (a_1, a_2, a_3, \dots) of infinite-dimensional Euclidean space. He called integers **collinear** if their images are collinear; e.g., $35 \rightarrow (2, 2, 0, 1, 0, \dots)$, $41 \rightarrow (2, 1, 1, 1, 0, \dots)$ and $47 \rightarrow (2, 0, 2, 1, 0, \dots)$ are collinear. He conjectured that every sequence of integers with no three collinear has density zero. If integers are collinear, they are in A.P., but not necessarily conversely (e.g., $16 \rightarrow (2, 1, 0, 0, \dots)$, $24 \rightarrow (0, 2, 2, 0, 0, \dots)$ and $32 \rightarrow (2, 1, 0, 1, 0, \dots)$ are not collinear) so truth of the conjecture would imply Roth's theorem that $r_3(n) = o(n)$.

If $f_3(n)$ is the largest number of lattice points with no three in line in the n -dimensional cube with three points in each edge, then Moser showed that $f_3(n) > c 3^n / \sqrt{n}$. It is easy to see that $f_3(n)/3^n$ tends to a limit; is it

zero? Chvátal improved the constant in Moser's result to $3/\sqrt{\pi}$ and found the values $f_3(1) = 2$, $f_3(2) = 6$, $f_3(3) = 16$. It is known that $f_3(4) \geq 43$.

More generally, if the n -dimensional cube has k points in each edge, Moser asked for an estimate of $f_k(n)$, the maximum number of lattice points with no k collinear. It is a theorem of Hales & Jewett, with applications to n -dimensional k -in-a-row (tic-tac-toe), that for sufficiently large n , any partition of the k^n lattice points into h classes has a class with k points in line. This implies van der Waerden's theorem on letting the point $(a_0, a_1, \dots, a_{n-1})$, $(0 \leq a_i \leq k-1)$ correspond to the base k expansion of the integer $\sum a_i k^i$. It is not known whether, for every c and sufficiently large n , it is possible to choose ck^n/\sqrt{n} points without including k in line. It is known for some c . Inequality (4) in Riddell's second paper quoted below implies that

$$f_k(n) > k^{n+1}/(2\pi e^3(k-1)n)^{\frac{1}{2}}$$

so that one can choose a "line-free" set of ck^n/\sqrt{n} points for some c . In the other direction he obtains $f_3(n) \leq 16 \cdot 3^{n-3}$. He acknowledges Leo Moser's inspiration in obtaining these results.

If you use the **greedy algorithm** to construct sequences not containing an A.P. you don't get a very dense sequence, but you do get some interesting ones. Odlyzko & Stanley construct the sequence $S(m)$ of positive integers with $a_0 = 0$, $a_1 = m$ and each subsequent a_{n+1} is the least number greater than a_n so that a_0, a_1, \dots, a_{n+1} does not contain a three-term A.P. For example

$$\begin{aligned} S(1): & 0, 1, 3, 4, 9, 10, 12, 13, 27, 28, 30, 31, 36, 37, 39, 40, 81, 82, 84, \\ & 85, 90, 91, 93, 94, 108, 109, 111, 112, 117, 118, 120, \dots \end{aligned}$$

$$\begin{aligned} S(4): & 0, 4, 5, 7, 11, 12, 16, 23, 26, 31, 33, 37, 38, 44, 49, 56, 73, 78, \\ & 80, 85, 95, 99, 106, 124, 128, 131, 136, 143, \dots \end{aligned}$$

If m is a power of three, or twice a power of three, then the members of the sequence are fairly easy to describe (write $S(1)$ in base 3), but for other values the sequences behave quite erratically. Their rates of growth seem to be similar, but this has yet to be proved.

The "simplest" such sequence containing no four-term A.P. is

$$\begin{aligned} 0, 1, 2, 4, 5, 7, 8, 9, 14, 15, 16, 18, 25, 26, 28, 29, 30, 33, 36, \\ 48, 49, 50, 52, 53, 55, 56, 57, 62, \dots \end{aligned}$$

Is there a simple description of this? How fast does it grow?

If we define the **span** of a set S to be $\max S - \min S$, what is the smallest span $sp(k, n)$ of a set of n integers containing no k -term A.P.? Zalman Usiskin gives the following values:

n	=	3	4	5	6	7	8	9	10	11	...
$sp(3, n)$	=	3	4	8	10	12	13	19	24	25	...
$sp(4, n)$	=	4	5	7	8	9	12	...			

Abbott notes that it follows from Szemerédi's theorem that for each $k \geq 3$, the sequence $\{sp(k, n+1) - sp(k, n)\}$ is unbounded, and asks if it contains a bounded subsequence.

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E11. Schur's problem. Partitioning integers into sum-free classes.

209

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E11 Schur's problem. Partitioning integers into sum-free classes.

Schur proved that if the integers less than n/e are partitioned into n classes in any way, then $x + y = z$ can be solved in integers within one class. Let $s(n)$ be the largest integer such that there exists a partition of the integers $[1, s(n)]$ into n classes with no solutions in any class. Abbott & Moser obtained the lower bound $s(n) > (89)^{n/4 - c \ln n}$ for some c and all sufficiently large n and Abbott & Hanson obtained $s(n) > c(89)^{n/4}$, improving Schur's own estimate of $s(n) \geq (3^n + 1)/2$. This last result is

in fact sharp for $n = 1, 2$ and 3 , but it is too low for larger values of n . The value $s(4) = 44$ was computed by Baumert: for example, the first 44 numbers may be split into four sum-free classes

$$\begin{aligned} \{1, 3, 5, 15, 17, 19, 26, 28, 40, 42, 44\}, & \quad \{2, 7, 8, 18, 21, 24, 27, 33, 37, 38, 43\}, \\ \{4, 6, 13, 20, 22, 23, 25, 30, 32, 39, 41\}, & \quad \{9, 10, 11, 12, 14, 16, 29, 31, 34, 35, 36\}. \end{aligned}$$

Later Fredricksen showed that $s(5) \geq 157$ (see E12 for his example) and this improves the lower bound for all subsequent Schur numbers: $s(n) \geq c(315)^{n/5}$ ($n > 5$).

Robert Irving has slightly improved Schur's upper bound from $\lfloor n/e \rfloor$ to $\lfloor n(e - \frac{1}{24}) \rfloor$. This result also appears in O'Sullivan's Ph.D. thesis (see E28). Eugene Levine says that this seems to be the best that can be deduced from Jon Folkman's result that the Ramsey number $R(3, 3, 3, 3) \leq 65$. Also, Schinzel notes that the result ascribed to Irving was attributed by the latter to Earl Glen Whitehead.

Denote by $v = \sigma(m, n)$ the least integer v such that any partition of $\{1, 2, \dots, v\}$ into n subsets has a part containing a_1, \dots, a_m (not necessarily distinct) which satisfy $a_1 + \dots + a_{m-1} = a_m$, i.e., $s(n) = \sigma(3, n)$. Beutelspacher & Brešovanský note that $\sigma(m, 1) = m - 1$ and $\sigma(2, n) = 1$ and prove that $\sigma(m, 2) = m^2 - m - 1$. They exhibit 3-sumfree 6- and 7-partitions that show that $\sigma(3, 6) \geq 476$ & $\sigma(3, 7) \geq 1430$. Hence $\sigma(3, n) \geq \frac{1}{2}(2859 \cdot 3^{n-7} + 1)$ for $n \geq 7$. They also define and investigate Schur numbers of arithmetic progressions.

E. & G. Szekeres and Schönheim have considered what Bill Sands calls an un-Schur problem. Call a partition of the integers $[1, n]$ into three classes **admissible** if there is no solution to $x+y = z$ with x, y, z in distinct classes. There is no admissible partition with the size of each class $\frac{1}{3}n$.

Is it true that if the integers are split into r classes, then some class contains three distinct integers x, y, z satisfying $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$? T. C. Brown has verified this for $r = 2$.

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E12. The modular version of Schur's problem.

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E12 The modular version of Schur's problem.

A similar problem to Schur's was considered by Abbott & Wang. Let $t(n)$ be the largest integer m so that there is a partition of the integers from 1 to m into n classes, with no solution to the congruence

$$x + y \equiv z \pmod{(m+1)}$$

in any class. Clearly $t(n) \leq s(n)$, where $s(n)$ is as in Schur's problem (E11), but for $n = 1, 2$ or 3 , we have equality, $t(1) = s(1) = 1$, $t(2) = s(2) = 4$, $t(3) = s(3) = 13$. Indeed, the only three partitions of $[1, 13]$ into three sets satisfying the sum-free condition,

$$\{1, 4, 10, 13\} \quad \{2, 3, 11, 12\} \quad \{5, 6, 8, 9\}$$

(with 7 in any of the three sets) all satisfy the seemingly more restrictive congruence-free condition, modulo 14, while Baumert's example (E11) shows only one failure: $33 + 33 \equiv 21 \pmod{45}$ in the second set. In fact

Baumert found 112 ways of partitioning [1,44] into four sum-free sets, and some of these are sum-free mod 45, so $t(4) = 44$. An example is

$$\{\pm 1, \pm 3, \pm 5, 15, \pm 17, \pm 19\}, \quad \{\pm 2, \pm 7, \pm 8, \pm 18, \pm 21\}$$

$$\{\pm 4, \pm 6, \pm 13, \pm 20, \pm 22, 30\}, \quad \{\pm 9, \pm 10, \pm 11, \pm 12, \pm 14, \pm 16\}.$$

Abbott & Wang obtained the inequality

$$f(n_1 + n_2) \geq 2f(n_1)f(n_2)$$

which holds for $f(n) = s(n) - \frac{1}{2}$ and leads to the same lower bound that Schur obtained for his problem, $t(n) \geq (3^n + 1)/2$. Indeed, they obtain evidence that $t(n) = s(n)$. Moreover, the example of Fredricksen

$$\pm\{1, 4, 10, 16, 21, 23, 28, 34, 40, 43, 45, 48, 54, 60\},$$

$$\pm\{2, 3, 8, 9, 14, 19, 20, 24, 25, 30, 31, 37, 42, 47, 52, 65, 70\},$$

$$\pm\{5, 11, 12, 13, 15, 29, 32, 33, 35, 36, 39, 53, 55, 56, 57, 59, 77, 79\},$$

$$\pm\{6, 7, 17, 18, 22, 26, 27, 38, 41, 46, 50, 51, 75\},$$

$$\pm\{44, 49, 58, 61, 62, 63, 64, 66, 67, 68, 69, 71, 72, 73, 74, 76, 78\},$$

which shows that $s(5) \geq 157$ is also sum-free mod 158 so that $t(5) \geq 157$ and $t(n) > c(315)^{n/5}$ as well.

Alon & Kleitman call a subset A of a commutative group **sum-free** if no sum of two elements of A is in A , $(A + A) \cap A = \emptyset$, and they show that every set of n nonzero elements of such a group contains a sum-free subset of cardinality $> \frac{2}{7}n$. That $\frac{2}{7}$ is best possible follows from a result of Rhemtulla & Street, though it can be improved for particular groups. They also show that any set of n nonzero integers contains a sum-free subset of cardinality $> \frac{1}{3}n$, where $\frac{1}{3}$ cannot be replaced by $\frac{12}{25}$. Füredi notes that the set $\{1,2,3,4,5,6,8,9,10,18\}$ shows that it cannot be replaced by $\frac{2}{5}$: is $\frac{1}{3}$ best possible?

Erdős lets $f(n)$ be the smallest integer for which the integers less than n can be partitioned into $f(n)$ classes so that n is not the sum of distinct members of the same class. For example, $f(11) = 2$, because of the partition $\{1, 3, 4, 5, 9\}, \{2, 6, 7, 8, 10\}$, but $f(12) = 3$. Erdős can prove $f(n) < n^{1/3}/\ln n$ but is unable to show that $f(n) > n^{1/3-\epsilon}$.

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E13 Partitioning into strongly sum-free classes.

Turán has shown that if the integers $[m, 5m + 3]$ are partitioned into two classes in any way, then in at least one of them the equation $x + y = z$ is solvable with $x \neq y$, and that this is not true for the integers $[m, 5m + 2]$. The uniqueness of the partition of $[m, 5m + 2]$ into two sum-free sets has been demonstrated by Znám.

Turán also considered the problem where x, y are not necessarily distinct. Define $s(m, n)$ as the least integer s such that however the interval $[m, m + s]$ is partitioned into n classes, one of them contains a solution of $x + y = z$. His result corresponding to the first problem is $s(m, 2) = 4m$. Clearly $s(1, n) = s(n) - 1$, where $s(n)$ is as in E11, and Irving's result implies that $s(m, n) \leq m[n!(e - \frac{1}{24}) - 1]$. Abbott & Znám (see E11) independently noted that $s(m, n) \geq 3s(m, n - 1) + m$ so that $s(m, n) \geq m(3^n - 1)/2$.

Abbott & Hanson call a class **strongly sum-free** if it contains no solution to either of the equations $x + y = z$ or $x + y + 1 = z$. They show that if $r(n)$ is the least r such that however $[1, r]$ is partitioned into n classes, one of them contains such a solution, then

$$r(m + n) \geq 2r(n)s(m) - r(n) - s(m) + 1.$$

They used this to improve the lower bound for $s(m, n)$; their method, with Fredericksen's example, now gives $s(m, n) > cm(315)^{n/5}$.

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E14 Rado's generalizations of van der Waerden's and Schur's problems.

Rado has considered a number of generalizations of van der Waerden's and Schur's problems. For example he shows that for any natural numbers a, b, c , there is a number u so that however the numbers $[1, u]$ are partitioned into two classes, there is a solution of $ax + by = cz$ in at least one of the classes. He gives a value for u , but, as in Schur's original problem, this is not best possible. For instance, with $2x + y = 5z$, the theorem gives $u = 20$, whereas it is true even for $u = 15$, though not for any smaller value of u : neither of the sets

$$\{1, 4, 5, 6, 9, 11, 14\} \quad \{2, 3, 7, 8, 10, 12, 13\}$$

contains a solution of $2x + y = 5z$. If we are allowed three sets, then 45 is the least value for u , since the three sets

$$\{1, 4, 5, 6, 9, 11, 14, 16, 19, 20, 21, 24, 26, 29, 31, 34, 36, 39, 41, 44\},$$

$$\{2, 3, 7, 8, 10, 12, 13, 15, 17, 18, 22, 23, 27, 28, 32, 33, 37, 38, 42, 43\}, \\ \{6, 7, 8, 9, 25, 30, 35, 40\}$$

contain all the numbers [1,44], even with 6, 7, 8 and 9 duplicated.

Rado called the equation $\sum a_i x_i = 0$, where the a_i are nonzero integers, ***n-fold regular*** if there is a number $u(n)$, which we can assume to be minimal, such that however the interval $[1, u(n)]$ is partitioned into n classes, at least one class contains a solution to the equation. He called it ***regular*** if it was *n*-fold regular for all n , and showed that an equation was regular just if $\sum a_j = 0$ for some subset of the a_i . For example, if $a_1 = a_2 = 1$ and $a_3 = -1$, we have Schur's original problem with $u(n) = s(n)$. Salié and Abbott considered the problem of finding lower bounds for $u(n)$; see E10 and E11 for references.

The example with $a_1 = 2$, $a_2 = 1$, $a_3 = -5$ is *not* regular, since, although we have seen that it is both 2-fold and 3-fold regular, it is not 4-fold regular. For, put every number $5l^4$, where $5 \nmid l$, into just one of four classes, according as k is even or odd, and $l \equiv \pm 1$ or $\pm 2 \pmod{5}$. It can be verified that none of these four classes contains a solution of $2x + y = 5z$.

Rado asked if there exist, for every k , equations which are k -regular, but not $(k+1)$ -regular.

For the equations $2x_1 + x_2 = 2x_3$ and $x_1 + x_2 + x_3 = 2x_4$, Salié, Abbott, and Abbott & Hanson obtained successively better lower bounds, culminating in $u(n) > c(12)^{n/3}$ and $c(10)^{n/3}$ respectively.

Vera Sós asks for the maximum size of subset of $[1, n]$ such that Rado's equation has no solution in the subset. For example if $a_1 = a_2 = 1$ and $a_3 = -2$, the answer is in the interval $[n \exp(-\sqrt{\ln n}), n/(\ln n)^\alpha]$. If $a_1 = a_2 = 1$ and $a_3 = a_4 = -1$, we have a Sidon set (compare C9) and the answer is $\approx \sqrt{n}$. If $a_1 = a_2 = 1$ and $a_3 = -1$, the answer is $n/2$. It is known more generally that the answer is $o(n)$ just if $x_1 = x_2 = \dots = 1$ is a solution of Rado's equation. Can the answer ever be comparable to n_α with $\frac{1}{2} < \alpha < 1$?

Compare problems E10–14 with C14–16.

Walter Deuber, Partitionen und lineare Gleichungssysteme, *Math. Z.*, **133** (1973) 109–123.

R. Rado, Studien zur Kombinatorik, *Math. Z.*, **36**(1933) 424–480.

E. R. Williams, M.Sc. thesis, Memorial University, 1967.

E15 A recursion of Göbel.

F. Göbel has remarked that the recursion $x_0 = 1$,

$$x_n = (1 + x_0^2 + x_1^2 + \dots + x_{n-1}^2)/n \quad n = 1, 2, \dots$$

[or, for $n > 0$, $(n+1)x_{n+1} = x_n(x_n + n)$] yields integers

$$x_1 = 2, 3, 5, 10, 28, 154, 3520, 1551880, 267593772160, \dots$$

for a long time, but Hendrik Lenstra found that x_{43} was not an integer!

The corresponding sequence with cubes in place of squares holds out as far as x_{89} . Henry Ibstedt has made extensive calculations, for various powers k and various initial values x_0 . The table shows the rank of the first noninteger member of the sequence

k	2	3	4	5	6	7	8	9	10	11
$x_1 = 2$	43	89	97	214	19	239	37	79	83	239
$x_1 = 3$	7	89	17	43	83	191	7	127	31	389
$x_1 = 4$	17	89	23	139	13	359	23	158	41	239
$x_1 = 5$	34	89	97	107	19	419	37	79	83	137
$x_1 = 6$	17	31	149	269	13	127	23	103	71	239
$x_1 = 7$	17	151	13	107	37	127	37	103	83	239
$x_1 = 8$	51	79	13	214	13	239	17	163	71	239
$x_1 = 9$	17	89	83	139	37	191	23	103	23	169
$x_1 = 10$	7	79	23	251	347	239	7	163	41	239
$x_1 = 11$	34	601	13	107	19	478	37	79	31	389

Raphael Robinson has observed that, in contrast to Göbel's sequence, the recurrence

$$x_n x_{n-k} = a x_{n-p} x_{n-k+p} + b x_{n-q} x_{n-k+q} + c x_{n-r} x_{n-k+r}$$

appears to generate integers from the starting values $x_0 = x_1 = \dots = x_k = 1$ for any integers $a \geq 0$, $b \geq 0$, $c \geq 0$, $p \geq 1$, $q \geq 1$, $r \geq 1$, k such that $p + q + r = k$.

David Gale, Mathematical Entertainments, *Math. Intelligencer*, **13**(1991) No. 1, 40–43.

Henry Ibstedt, Some sequences of large integers, *Fibonacci Quart.*, **28**(1990) 200–203; MR 91h:11011.

Janice L. Malouf, An integer sequence from a rational recursion, *Discrete Math.*, **110**(1992) 257–261.

Raphael M. Robinson, Periodicity of Somos sequences, *Proc. Amer. Math. Soc.*, **116**(1992) 613–619; MR 93a:11012.

Michael Somos, Problem 1470, *Croix Mathematicorum*, **15**(1989) 208.

E16 Collatz's sequence.

When he was a student, L. Collatz asked if the sequence defined by $a_{n+1} = a_n/2$ (a_n even), $a_{n+1} = 3a_n + 1$ (a_n odd) is tree-like in structure, apart

from the cycle 4, 2, 1, 4, ... (Figure 16) in the sense that, starting from any integer a_1 , there is a value of n for which $a_n = 1$. This has been verified for all $a_1 \leq 2 \cdot 10^{12}$ and for many larger numbers. Eliahou has shown that any nontrivial cycle has period at least 17087915.

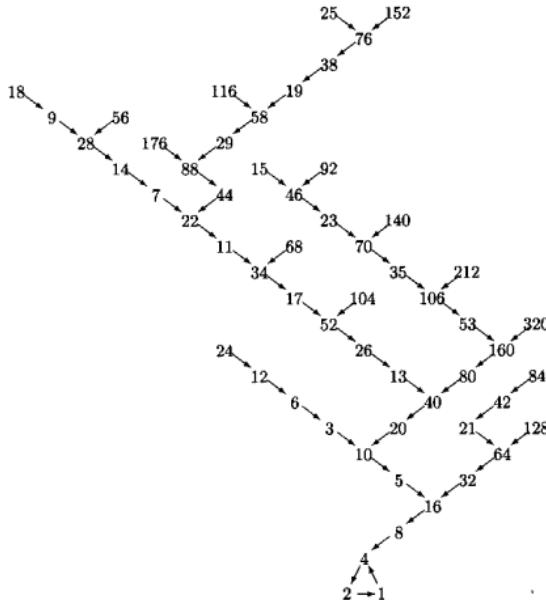


Figure 16. Is the Collatz Sequence Tree-like?

If $3a_n + 1$ is replaced by $3a_n - 1$ (or if we allow negative integers) then it seems likely that any sequence concludes with one of the cycles $\{1, 2\}$, $\{5, 14, 7, 20, 10\}$ or

$$\{17, 50, 25, 74, 37, 110, 55, 164, 82, 41, 122, 61, 182, 91, 271, 136, 68, 34\}.$$

This is true for all $a_1 \leq 10^8$.

David Kay and others define the sequence more generally by $a_{n+1} = a_n/p$ if $p|n$ and $a_{n+1} = a_nq + r$ if $p \nmid n$ and asks if there are numbers p ,

q, r for which the problem can be settled. For $(p, q, r) = (2, 5, 1)$ or $(2, 7, 1)$ it certainly seems very plausible that any sequence will increase rapidly, if erratically, but it seems to be just as hard to prove anything as it is for the original problem. The literature is enormous; would-be solvers are urged to study carefully the writings of Lagarias.

Define $f(n)$ to be the largest odd divisor of $3n + 1$. Zimian asked if

$$\prod_{i=1}^m n_i = \prod_{i=1}^m f(n_i)$$

holds for any (multi)set $\{n_i\}$ of integers $n_i > 1$. Erdős found that

$$65 \cdot 7 \cdot 7 \cdot 11 \cdot 11 \cdot 17 \cdot 17 \cdot 13 = 49 \cdot 11 \cdot 11 \cdot 17 \cdot 17 \cdot 13 \cdot 13 \cdot 5.$$

Call an integer n **self-contained** if n divides $f^k(n)$ for some $k \geq 1$. If this happens and if the Collatz sequence $n^* = f^k(n)/n$ reaches 1, then the set

$$\{n, f(n), \dots, f^{k-1}(n), n^*, f(n^*), \dots, 1\}$$

is a set such as the above. A computer search for $n \leq 10^4$ yielded five self-contained integers: 31, 83, 293, 347 and 671.

These mappings are not one-to-one; you can't retrace the history of a sequence, since there is often no unique inverse.

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E17 Permutation sequences.

The situation is different, though no more clear, in the case of **permutation sequences**. A simple example, probably the inverse of Collatz's original problem (see **E16** and Lagarias's article cited there), is

$$a_{n+1} = 3a_n/2 \quad (a_n \text{ even}), \quad a_{n+1} = \lfloor (3a_n + 1)/4 \rfloor \quad (a_n \text{ odd}),$$

or, perhaps more perspicuously,

$$2m \rightarrow 3m \quad 4m - 1 \rightarrow 3m - 1 \quad 4m + 1 \rightarrow 3m + 1$$

from which it is clear that the inverse operation works just as well. So the resulting structure consists only of disjoint cycles and doubly infinite chains. It is not known whether there is a finite or infinite number of each of these, nor even whether an infinite chain exists. It is conjectured that the only cycles are $\{1\}$, $\{2, 3\}$, $\{4, 6, 9, 7, 5\}$ and

$$\{44, 66, 99, 74, 111, 83, 62, 93, 70, 105, 79, 59\}.$$

Mike Guy, with the help of TITAN, showed that any other cycles have period greater than 320. What is the status of the sequence containing the number 8?

$$\dots, 97, 73, 55, 41, 31, 23, 17, 13, 10, 15, 11, 8, \\ 12, 18, 27, 20, 30, 45, 34, 51, 38, 57, 43, 32, 48, 72, \dots$$

Do the numbers 8, 14, 40, 64, 80, 82, 104, 136, 172, 184, 188, 242, 256, 274, 280, 296, 352, 368, 382, 386, 424, 472, 496, 526, 530, 608, 622, 638, 640, 652, 670, 688, 692, 712, 716, 752, 760, 782, 784, 800, 814, 824, 832, 860, 878, 904, 910, 932, 964, 980, ... each belong to a separate sequence?

There are some intriguing paradoxes: as you go “forward” you multiply by $3/2$ if the current number is even, and by about $3/4$ if it's odd — and get an erratic “pseudo-GP” of common ratio $3/\sqrt{8} \approx 1.060660172$. On the other hand, as you go “backward” you multiply by $2/3$ if the current number is a multiple of 3, and by about $4/3$ otherwise — a “pseudo-GP” of common ratio $3^{2/3}/3 \approx 1.058267368$. These two numbers should be reciprocal! We have a sort of discrete analog of an everywhere non-differentiable function. The “derivative” on the right is positive; that on the left is negative. Note that, when going “forwards” each successor of an even number is a multiple of 3 — half the numbers are multiples of three!

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David Gale, Mathematical Entertainments, *Math. Intelligencer*, **13**(1991) No. 3, 53–55.

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E18 Mahler's Z-numbers.

Mahler considered the following problem: given any real number α , let r_n be the fractional part of $\alpha(3/2)^n$. Do there exist **Z-numbers**, for which $0 \leq r_n < \frac{1}{2}$ for all n ? Probably not. Mahler shows that there is at most one between each pair of consecutive integers, and that, for x large enough, at most $x^{0.7}$ less than x . Flatto has improved on Mahler's results, but the problem remains unsolved.

A similar question is: is there a rational number r/s ($s \neq 1$) such that $\lfloor (\tau/s)^n \rfloor$ is odd for all n ? Tijdeman proved that for every odd integer $r > 3$, there are real numbers α such that the fractional part of $\alpha(\tau/2)^n$ is in $[0, \frac{1}{2})$ for all n .

Littlewood once remarked that it was not known that the fractional part of e^n did not tend to 0 as $n \rightarrow \infty$.

Leopold Flatto, *Z-numbers and β -transformations*, *Symbolic Dynamics and its Applications, Contemporary Math.*, 135, Amer. Math. Soc., 1992, 181–201.

K. Mahler, An unsolved problem on the powers of $3/2$, *J. Austral. Math. Soc.*, 8(1968) 313–321; *MR 37 #2694*.

R. Tijdeman, Note on Mahler's $\frac{3}{2}$ -problem, *Kongel. Norske Vidensk. Selsk. Skr.*, 16(1972) 1–4.

E19 Are the integer parts of the powers of a fraction infinitely often prime?

Forman & Shapiro have proved that infinitely many integers of the form $\lfloor (4/3)^n \rfloor$ and also of the form $\lfloor (3/2)^n \rfloor$ are composite. A. L. Whiteman conjectures that these two sequences also each contain infinitely many primes. The method appears not to work for other rationals.

W. Forman & H. N. Shapiro, An arithmetic property of certain rational powers, *Comm. Pure Appl. Math.*, 20(1967) 561–573; *MR 35 #2852*.

E20 Davenport-Schinzel sequences.

Form sequences from an alphabet $[1, n]$ of n letters such that there are no immediate repetitions $\dots aa \dots$ and no alternating subsequences $\dots a \dots b \dots a \dots b \dots$

of length greater than d . Denote by $N_d(n)$ the maximal length of any such sequence; then a sequence of this length is a **Davenport-Schinzel sequence**. The problem is to determine all D-S sequences, and in particular to find $N_d(n)$. We need only consider **normal** sequences in which the first appearance of an integer of the alphabet comes after the first appearance of every smaller one.

The sequences 12131323, 121212131313232323, and

1 2 1 3 1 4 1 ... 1 $\overline{n-1} \ 1 \ \overline{n-1} \ \overline{n-2} \dots 3 \ 2 \ n \ 2 \ n \ 3 \ n \dots n \ \overline{n-1} \ n$ show that $N_4(3) \geq 8$, $N_8(3) \geq 20$, and $N_4(n) \geq 5n - 8$. Davenport & Schinzel showed that $N_1(n) = 1$, $N_2(n) = n$, $N_3(n) = 2n - 1$; that $N_4(n) = O(n \ln n / \ln \ln n)$, lim $N_4(n)/n \geq 8$ and, with J. H. Conway, that $N_4(lm + 1) \geq 6lm - m - 5l + 2$, so that $N_4(n) = 5n - 8$ ($4 \leq n \leq 10$). Z. Kolba showed that $N_4(2m) \geq 11m - 13$ and Mills obtained the values of $N_4(n)$ for $n \leq 21$. For example, the sequence

abacadaeaafa fedcgbghbhgcicigdjgekekkgkjihflflhliljlk

(which, Günter Rote notes, was misprinted in the first edition) is part of the proof that $N_4(12) = 53$.

Roselle & Stanton fixed n rather than d and obtained $N_d(2) = d$, $N_d(3) = 2\lfloor 3d/2 \rfloor - 4$ ($d > 3$), $N_d(4) = 2\lfloor 3d/2 \rfloor + 3d - 13$ ($d > 4$) and $N_d(5) = 4\lfloor 3d/2 \rfloor + 4d - 27$ ($d > 5$), though Peterkin observed that this last parenthesis should be $d > 6$ since $N_6(5) = 34$. Roselle & Stanton also showed that normal D-S sequences of length $N_{2d+1}(5)$ are unique and that there are just two of length $N_{2d+1}(4)$ and $N_{2d}(5)$. Peterkin exhibited the 56 D-S sequences of length $N_5(6) = 29$ and showed that $N_5(n) \geq 7n - 13$ ($n > 5$) and $N_6(n) \geq 13n - 32$ ($n > 5$).

Table 8. Values of $N_d(n)$.

$d \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
2	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
3	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41
4	1	4	8	12	17	22	27	32	37	42	47	53	58	64	69	75	81	86	92	98	104
5	1	5	10	16	22	29															
6	1	6	14	23	34																
7	1	7	16	28	41																
8	1	8	20	35	53																
9	1	9	22	40	61																
10	1	10	26	47	73																

Rennie & Dobson gave an upper bound for $N_d(n)$ in the form

$$(nd - 3n - 2d + 7)N_d(n) \leq n(d - 3)N_d(n - 1) + 2n - d + 2 \quad (d > 3)$$

thus generalizing the result of Roselle & Stanton for $d = 4$.

Szemerédi showed that $N_d(n) < c_d n \log^* n$, where $\log^* n$ is a slow-growing function, the least number of iterations of the exponential function needed to exceed n . More recent work, mainly by Sharir, has shown that the order of $N_d(n)$ is $\Theta(n\alpha(n))$, where $\alpha(n)$ is the incredibly slow-growing

inverse of the Akermann function. See the paper of Agarwal, Sharir & Shor for precise details.

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E21 Thue sequences.

Thue showed that there are infinite sequences on 3 symbols which contain no two identically equal consecutive segments, and sequences on 2 symbols which contain no three identically equal consecutive segments, and many others have rediscovered these results.

If, instead of identically equal segments, we ask to avoid consecutive segments which are permutations of one another, Justin constructed a sequence on 2 symbols without five consecutive segments which are permutations of each other, and Pleasants constructed a sequence on 5 symbols without two such consecutive segments. Dekking has solved the (2,4) and (3,3) problems, but describes the (4,2) case as an interesting open problem. Is there a sequence on 4 symbols without consecutive segments which are permutations of each other?

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E22 Cycles and sequences containing all permutations as subsequences.

Hansraj Gupta asked, for $n \geq 2$, to find the least positive integer $m = m(n)$ for which a cycle a_1, a_2, \dots, a_m of positive integers, each $\leq n$, exists such that any given permutation of the first n natural numbers appears as a subsequence (not necessarily consecutive) of

$$a_j, a_{j+1}, \dots, a_1, a_2, \dots, a_{j-1}$$

for at least one j , $1 \leq j \leq m$. For example, for $n = 5$, such a cycle is $1, 2, 3, 4, 5, 4, 3, 2, 1, 5, 4, 5$, so that $m(5) \leq 12$. He conjectures that $m(n) \leq \lfloor n^2/2 \rfloor$.

Motzkin & Straus used the **ruler function** (exponent of the highest power of 2 which divides k), e.g., $n = 5$, $1 \leq k \leq 31$,

$$1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1,$$

but this doesn't make use of the cyclic options and gives only $m(n) \leq 2^n - 1$.

E23 Covering the integers with A.P.s.

If S is the union of n arithmetic progressions, each with common difference $\geq k$, where $k \leq n$, Crittenden & Vanden Eynden conjecture that S contains all positive integers whenever it contains those $\leq k2^{n-k+1}$. If this is true,

it's best possible. They have proved it for $k = 1$ and 2, and Simpson has proved it for $k = 3$.

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E24 Irrationality sequences.

Erdős & Straus called a sequence of positive integers $\{a_n\}$ an **irrationality sequence** if $\sum 1/a_n b_n$ is irrational for all integer sequences $\{b_n\}$. What are the irrationality sequences? Find some interesting ones. If $\limsup(\log_2 \ln a_n)/n > 1$, where the log is to base 2, then $\{a_n\}$ is an irrationality sequence. Notice that $\{n!\}$ is not an irrationality sequence, because $\sum 1/n!(n+2) = \frac{1}{2}$. Erdős has shown that $\{2^n\}$ is an irrationality sequence. The sequence 2, 3, 7, 43, 1807, ..., where $a_{n+1} = a_n^2 - a_n + 1$, is not an irrationality sequence, since we may take $b_n = 1$ and the sum of the reciprocals is 1, but what about the sequence of alternate terms, 2, 7, 1807, ...?

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E25 Silverman's sequence.

The sequence

$$1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 9, \dots$$

defined by $f(1) = 1$ and $f(n)$ as the number of occurrences of n in a nondecreasing sequence of integers was attributed to David Silverman in the first edition. It was given as a problem by Golomb, and solved by him, by van Lint, and by Marcus and Fine (see reference below): the asymptotic expression for the n -th term is indeed $\tau^{2-\tau} n^{\tau-1}$ where τ is the golden number $(1 + \sqrt{5})/2$. The error term $E(n)$ has been investigated by

Ilan Vardi, who conjectures that

$$E(n) = \Omega_{\pm} \left(\frac{n^{r-1}}{\ln n} \right)$$

where $E(n) = \Omega_{\pm}(g(n))$ means that there are constants c_1, c_2 such that $E(n) > c_1 g(n)$ and $E(n) < -c_2 g(n)$ are each true for infinitely many n , but he was unable to prove even that $|E(n)|$ is unbounded. He states some related unsolved problems.

Marshall Hall proved the existence of a sequence such that every positive integer occurs uniquely as the difference of two members of the sequence. For example,

$$1, 2, 4, 8, 16, 21, 42, 51, 102, 112, 224, 235, 470, 486, 972, 990, 1980, 2001, \dots$$

defined by $a_1 = 1$, $a_2 = 2$, $a_{2n+1} = 2a_{2n}$, $a_{2n+2} = a_{2n+1} + r_n$, where r_n is the least natural number which cannot be represented in the form $a_j - a_i$ with $1 \leq i < j \leq 2n+1$. There are other possible sequences, but the problem of finding the ones with smallest asymptotic growth remains open.

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E26 Epstein's Put-or-Take-a-Square game.

Richard Epstein's Put-or-Take-a-Square game is played with one heap of beans. Two players play alternately. A move is to add or subtract the largest perfect square number of beans that is in the heap. That is, the two players alternately name nonnegative integers a_n , where

$$a_{n+1} = a_n \pm \lfloor \sqrt{a_n} \rfloor^2,$$

the winner being the first to name zero. This is a loopy game and many numbers lead to a draw. For example, from 2 the next player will not take

1, allowing his opponent to win, so she goes to 3. Now to add 1 is a bad move, so her opponent goes back to 2. Similarly 6 leads to a draw with best play: 6, 10, 19!, 35, 60, 109!, 209!, 13!, 22!, 6, ..., where ! means a good move, not factorial! For example, 405 is a bad move after 209, since the next player can go to 5 which is a *P-position* (previous player winning). Similarly, from 60, it is bad to go to 11, an *N-position*, one in which the next player can win (by going to 20).

Do either the *P*-positions

$$0, 5, 20, 29, 45, 80, 101, 116, 135, 145, 165, 173, 236, 257, 397, 404, 445, 477, 540, 565, 580, 629, 666, 836, 845, 885, 909, 944, 949, 954, 975, 1125, 1177, \dots$$

or the *N*-positions

$$1, 4, 9, 11, 14, 16, 21, 25, 30, 36, 41, 44, 49, 52, 54, 64, 69, 71, 81, 84, 86, 92, 100, 105, 120, 121, 126, 136, 141, 144, 149, 164, 169, 174, 189, 196, 201, 208, 216, 225, 230, 245, 252, 254, 256, 261, \dots$$

have positive density?

E. R. Berlekamp, J. H. Conway & R. K. Guy, *Winning Ways for your Mathematical Plays*, Academic Press, London, 1982, Chapter 15.

E27 Max and mex sequences.

In his master's thesis, Roger Eggleton discussed **max sequences**, in which a given finite sequence a_0, a_1, \dots, a_n is extended by defining $a_{n+1} = \max_i(a_i + a_{n-i})$. One of the main results is that the first differences are ultimately periodic. For example, starting from 1, 4, 3, 2 we get 7, 8, 11, 12, 15, 16, ... with differences 3, -1, -1, 5, 1, 3, 1, 3, 1, What happens to **mex sequences**, where the **mex** of a set of nonnegative integers is the minimum excluded number, or least nonnegative integer which does not appear in the set? Now the sequence 1, 4, 3, 2 continues

$$0, 0, 0, 0, 5, 1, 1, 1, 1, 6, 2, 2, 0, 0, 0, 0, 0, 5, 1, 1, 1, 1, 1, 6, \dots$$

Are such sequences ultimately periodic?

A. S. Fraenkel is reminded of the sequence $a_i = \lfloor i\alpha \rfloor$, where α is any real number, which satisfies the inequalities

$$\max(a_{n-i} + a_i) \leq a_n \leq 1 + \min(a_{n-i} + a_i), \quad 1 \leq i < n, \quad n = 2, 3, \dots$$

E.g., $\alpha = \frac{1}{2}(1 + \sqrt{5})$ generates the sequence 1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, ... which complements that for $b_i = \lfloor i\beta \rfloor$, where $1/\alpha + 1/\beta = 1$. These **Beatty sequences** combine to form **Wythoff pairs**.

Motivation for this problem comes from the analysis of **octal games** using the Sprague-Grundy theory, where ordinary addition is replaced by **nim addition**, i.e., addition in base 2 without carry, or XOR. Now 1, 4, 3, 2 leads to

$$0, 0, 0, 0, 5, 1, 4, 1, 1, 1, 3, 6, 6, 6, 3, 0, 2, 2, 2, 7, 2, 4, 1, \dots$$

The behavior of such sequences remains a considerable mystery, clarification of which would lead to results about nim-like games.

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E28 B₂-sequences.

Call an infinite sequence $1 \leq a_1 < a_2 < \dots$ an **A-sequence** if no a_i is the sum of distinct members of the sequence other than a_i . Erdős proved that for every A-sequence, $\sum 1/a_i < 103$, and Levine and O'Sullivan improved this to 4. They also gave an A-sequence whose sum of reciprocals is > 2.035. Abbott, and later Zhang, have given the example

$$\{1, 2, 4, 8, 1 + 24t, 35950 + 24t : 1 \leq k \leq 55, 0 \leq t \leq 44\}$$

which improves this to 2.0648. A further block of terms will push this past 2.0649, but not as far as 2.065.

If $1 \leq a_1 < a_2 < \dots$ is a B₂-sequence (compare C9), i.e., a sequence where all the sums of pairs $a_i + a_j$ are different, what is the maximum of $\sum 1/a_i$? There are two problems, according as $i = j$ is permitted or not, but Erdős was unable to solve either of them.

The most obvious B₂-sequence is that obtained by the greedy algorithm (compare E10). Each term is the least integer greater than earlier terms which does not violate the distinctness of sums condition; $i = j$ is permitted:

$$1, 2, 4, 8, 13, 21, 31, 45, 66, 81, 97, 123, 148, 182, \dots$$

Mian & Chowla used this to show the existence of a B₂-sequence with $a_k \ll k^3$. If M is the maximum of $\sum 1/a_i$ over all B₂-sequences and S^* is the sum of the reciprocals of the Mian-Chowla sequence, then $M \geq S^* > 2.156$. But

Levine observes that if $t_n = n(n+1)/2$, then $M \leq \sum 1/(t_n + 1) < 2.374$, and asks if $M = S^*$? Zhang disproves this by showing that $S^* < 2.1596$ and $M > 2.1597$. The latter result is obtained by replacing the next term, 204, in the Mian-Chowla sequence, by 229, and then continuing with the greedy algorithm.

Let $a_1 < a_2 < \dots$ be an infinite sequence of integers for which all the triple sums $a_i + a_j + a_k$ are distinct. Erdős offers \$500.00 for a proof or disproof of an old conjecture of his, that $\lim a_n/n^3 = \infty$.

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E29 Sequence with sums and products all in one of two classes.

Partition the integers into two classes. Is it true that there is always a sequence $\{a_i\}$ so that all the sums $\sum \epsilon_i a_i$ and all the products $\prod a_i^{\epsilon_i}$ where the ϵ_i are 0 or 1 with all but a finite number zero, are in the same class? Hindman answered this question of Erdős negatively.

Is there a sequence $a_1 < a_2 < \dots$ so that all the sums $a_i + a_j$ and products $a_i a_j$ are in the same class? Graham proved that if we partition the integers [1,252] into two classes, there are four distinct numbers x , y , $x+y$ and xy all in the same class. Moreover, 252 is best possible. Hindman proved that if we partition the integers [2,990] into two classes, then one class always contains four distinct numbers x , y , $x+y$ and xy . No corresponding result is known for the integers ≥ 3 .

Hindman also proved that if we partition the integers into two classes, there is always an infinite sequence $\{a_i\}$ so that all the sums $a_i + a_j$,

($i = j$ permitted) are in the same class. On the other hand he found a decomposition into three classes so that no such infinite sequence exists.

J. Baumgartner, A short proof of Hindman's theorem, *J. Combin. Theory Ser. A*, **17**(1974) 384–386.

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E30 MacMahon's prime numbers of measurement

MacMahon's "prime numbers of measurement,"

$$1, 2, 4, 5, 8, 10, 14, 15, 16, 21, 22, 25, 26, 28, 33, 34, 35, 36, 38, 40, 42, \dots$$

are generated by excluding all the sums of two or more consecutive earlier members of the sequence.

If m_n is the n th member of the sequence, and M_n is the sum of the first n members, then George Andrews conjectures that

$$_ m_n \sim n(\ln n)/\ln \ln n \quad ? \quad \text{and} \quad _ M_n \sim n^2(\ln n)/\ln(\ln n)^2 \quad ?$$

and poses the following, presumably easier, problems: prove $\lim n^{-\Delta} m_n = 0$ for some $\Delta < 2$; prove $\lim m_n/n = \infty$; prove $m_n < p_n$ for every n , where p_n is the n th prime.

Jeff Lagarias suggests excluding only the sums of *two* or *three* consecutive earlier members, and asked if the resulting sequence

$$1, 2, 4, 5, 8, 10, 12, 14, 15, 16, 19, 20, 21, 24, 25, 27, 28, 32, 33, 34, \\ 37, 38, 40, 42, 43, 44, 46, 47, 48, 51, 53, 54, 56, 57, 58, 59, 61, \dots$$

has density $\frac{5}{6}$. Don Coppersmith has a better heuristic, suggesting that the answer is 'no.'

More generally, if $1 \leq a_1 < a_2 < \dots < a_k \leq n$ is a sequence in which no k is the sum of consecutive earlier members, then Pomerance found that $\max k \geq \lfloor \frac{n-3}{2} \rfloor$ and Róbert Freud later showed that $\max k \geq \frac{13}{36}n$. They notice, with Erdős, that $\max k \leq \frac{5}{6}n$, even if we only forbid sums of *two* consecutive earlier members. Coppersmith & Phillips have since shown that $\max k \geq \frac{13}{24}n - O(1)$ and they lower the upper bound to

$$\max k \leq (\frac{2}{3} - \epsilon)n + O(\ln n) \quad \text{with} \quad \epsilon = \frac{1}{896}.$$

Erdős asks if the lower density of the sequence is zero; perhaps

$$_ i \frac{1}{\ln x} \sum_{a_i < x} \frac{1}{a_i} \rightarrow 0 \quad ?$$

G. E. Andrews, MacMahon's prime numbers of measurement, *Amer. Math. Monthly*, **82**(1975) 922–923.

Don Coppersmith & Steven Phillips, On a question of Erdős on subsequence sums (preprint, Nov. 1992).

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P. A. MacMahon, The prime numbers of measurement on a scale, *Proc. Cambridge Philos. Soc.*, **21**(1923) 651–654.

Štefan Porubský, On MacMahon's segmented numbers and related sequences, *New Arch. Wisk.*(3) **25**(1977) 403–408; MR **58**:#5575.

N. J. A. Sloane, *A Handbook of Integer Sequences*, Academic Press, New York, 1973; sequences 363, 416, 1044.

E31 Three sequences of Hofstadter.

Doug Hofstadter has defined three intriguing sequences.

(a) $a_1 = a_2 = 1$ and $a_n = a_{n-a_{n-1}} + a_{n-a_{n-2}}$ for $n \geq 3$. What is the general behavior of this sequence?

$$1, 1, 2, 3, 4, 5, 6, 6, 6, 8, 8, 10, 9, 10, 11, 11, \\ 12, 12, 12, 12, 16, 14, 14, 16, 16, 16, 16, 20, 17, 17, \dots$$

Are there infinitely many integers 7, 13, 15, 18, ... that get missed out?

(b) $b_1 = 1$, $b_2 = 2$ and for $n \geq 3$, b_n is the least integer greater than b_{n-1} which can be expressed as the sum of two or more consecutive terms of the sequence, so it goes

$$1, 2, 3, 5, 6, 8, 10, 11, 14, 16, 17, 18, 19, 21, 22, 24, 25, 29, \\ 30, 32, 33, 34, 35, 37, 40, 41, 43, 45, 46, 47, 49, 51, \dots$$

This is a sort of dual of MacMahon's prime numbers of measurement (E30). How does the sequence grow?

(c) $c_1 = 2$, $c_2 = 3$, and when c_1, \dots, c_n are defined, form all possible expressions

$$c_i c_j - 1 \quad (1 \leq i < j \leq n) \quad \text{and append them to the sequence:} \\ 2, 3, 5, 9, 14, 17, 26, 27, 33, 41, 44, 50, 51, 53, 65, 69, 77, \\ 80, 81, 84, 87, 98, 99, 101, 105, 122, 125, 129, \dots$$

Does the result include almost all of the integers?

A similar sequence to the first of these three was given by Conway:

$$1, 1, 2, 2, 3, 4, 4, 5, 6, 7, 7, 8, 8, 8, 9, \dots$$

defined, for $n \geq 3$, by

$$a(n) = a(a(n-1)) + a(n-a(n-1)).$$

The difficult questions were answered in an entertaining paper by Mallows. Several identities have been obtained by Zeitlin. A variation is to define

$$b(n) = b(b(n-1)) + b(n-1-b(n-1)),$$

but this is related to the previous sequence by $b(n-1) = n - a(n)$. But if we write

$$c(n) = c(c(n-2)) + c(n-c(n-2))$$

then the increments become very irregular, and it is not even clear that $c(n)/n$ tends to a limit.

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Douglas R. Hofstadter, *Gödel, Escher, Bach*, Vintage Books, New York, 1980, p. 137.

Mark Kac, A history-dependent random sequence defined by Ulam, *Adv. in Appl. Math.*, 10(1989) 270-277; MR 91c:11042.

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E32 B₂-sequences formed by the greedy algorithm.

An old problem of Dickson is still unsolved. Given a set of k integers, $a_1 < a_2 < \dots < a_k$, define a_{n+1} for $n \geq k$ as the least integer greater than a_n which is not of the form $a_i + a_j$, $i, j \leq n$. Except for the prescribed section at the beginning of the sequence, these are sum-free sequences formed by the greedy algorithm (compare C9, C14, E10, E28).

Is the sequence of differences $a_{n+1} - a_n$ ultimately periodic?

Such sequences may take a long time before the periodicity appears. For example, even for $k = 2$, if we take $a_1 = 1$, $a_2 = 6$, the sequence is

$$1, 6, 8, 10, 13, 15, 17, 22, 24, 29, 31, 33, 36, 38, 40, 45, 47, 52, 54, 56, 59, 61, 63, 68, \dots$$

and one can be forgiven for not immediately recognizing the pattern. Try starting with the set {1,4,9,16,25}; after 82 irregular differences, it settles down to a period of length 224.

Queena (see reference at C4) considered the similar problem with $i < j \leq n$ in place of $i, j \leq n$. Such 0-additive sequences have also been conjectured to have ultimately periodic differences. Steven Finch has calculated $1\frac{1}{2}$ million terms of the sequence whose first 6 terms are given as {3, 4, 6, 9, 10, 17} without detecting any sign of ultimate periodicity of the differences.

Selmer tells me that Dickson's problem is the particular case $h = 2$ of Stöhr sequences : let $a_1 = 1$ and define a_{n+1} for $n \geq k$ to be the least integer greater than a_n which can not be written as the sum of at most h addends among a_1, a_2, \dots, a_n . Compare the h -bases of C12. In the great majority of cases, the sequence of differences $a_{n+1} - a_n$ turns out to be ultimately periodic, but there are a few of the examined cases where periodicity has not been established.

Neil J. Calkin, Sum-free sets and measure spaces, PhD thesis, Univ. of Waterloo, 1988.

Neil J. Calkin & Steven R. Finch, Necessary and sufficient conditions for a sum-free set to be ultimately periodic (preprint, 1993).

Peter J. Cameron, Portrait of a typical sum-free set, Surveys in Combinatorics 1987, London Math. Soc. Lecture Notes, 123(1987) Cambridge Univ. Press, 13-42.

L. E. Dickson, The converse of Waring's problem, *Bull. Amer. Math. Soc.*, 40 (1934) 711-714.

Steven R. Finch, Are 0-additive sequences always regular? *Amer. Math. Monthly*, 99(1992) 671-673.

Ernst S. Selmer, On Stöhr's recurrent h -bases for N , *Kgl. Norske Vid. Selsk. Skrifter*, 3(1986) 1-15.

Ernst S. Selmer & Svein Mossige, Stöhr sequences in the postage stamp problem, No. 32(Dec. 1984) Dept. Pure Math., Univ. Bergen, ISSN 0332-5407.

E33 Sequences containing no monotone A.P.s.

Erdős & Graham say that a sequence $\{a_i\}$ has a **monotone** A.P. of length k if there are subscripts $i_1 < i_2 < \dots < i_k$ such that the subsequence a_{i_j} , $1 \leq j \leq k$ is either an increasing or a decreasing A.P. If $M(n)$ is the number

of permutations of $[1, n]$ which have no monotone 3-term A.P., then Davis et al have shown that

$$M(n) \geq 2^{n-1} \quad M(2n-1) \leq (n!)^2 \quad M(2n) \leq (n+1)(n!)^2$$

They ask if $M(n)^{1/n}$ is bounded.

Davis et al have also shown that any permutation of (all) the positive integers must contain an increasing 3-term A.P., but there are permutations with no monotone 5-term A.P. It is not known whether a monotone 4-term A.P. must always occur.

If the positive integers are arranged as a doubly-infinite sequence then a monotone 3-term A.P. must still occur, but it's possible to prevent the occurrence of 4-term ones.

If all the integers are to be permuted then Tom Oddy has shown that no 7-term A.P. need occur in the singly-infinite case, but little else is known.

J. A. Davis, R. C. Entringer, R. L. Graham & G. J. Simmons, On permutations containing no long arithmetic progressions, *Acta Arith.*, 34(1977) 81–90; MR 58 #10705.

Tom Oddy, Solution to Problem E2440, *Amer. Math. Monthly*, 82(1975) 74.

E34 Happy numbers.

Reg Allenby's daughter came home from school in Britain with the concept of **happy numbers**. If you iterate the process of summing the squares of the decimal digits of a number, then it's easy to see that you either reach the cycle

$$4 \rightarrow 16 \rightarrow 37 \rightarrow 58 \rightarrow 89 \rightarrow 145 \rightarrow 42 \rightarrow 20 \rightarrow 4$$

or arrive at 1. In the latter case you started from a happy number. The first hundred happy numbers are

1	7	10	13	19	23	28	31	32	44	49	68	70	79	82	86	91	94	97	100
103	109	129	130	133	139	167	176	188	190	192	193	203	208	219	226	230	236	239	262
263	280	291	293	301	302	310	313	319	320	326	329	331	338	356	362	365	367	368	376
379	383	386	391	392	397	404	409	440	446	469	478	487	490	496	536	556	563	565	565
566	608	617	622	623	632	635	637	638	644	649	653	655	656	671	673	680	683	694	

It seems that about 1/7 of all numbers are happy, but what bounds on the density can be proved? How many consecutive happy numbers can you have? Can there be arbitrarily many? The first pair is 31, 32; the first triple is 1880, 1881, 1882 and an example of five consecutive happy numbers is 44488, 44489, 44490, 44491, 44492. Give bounds on the length of such a consecutive set in terms of the size of its members. How large can the gaps be in the sequence of happy numbers? We can define the height

E35. The Kimberling shuffle.

of a happy number to be the number of iterations needed to reach 1. For example, the least happy numbers of

height	0	1	2	3	4	5	6
are	1	10	13	23	19	7	356

Is 78999 the least happy number of height 7? Give bounds for the size of the least happy number of height h .

If we replace squares by cubes, then the situation is dominated, at least in base 10, by the fact that perfect cubes are congruent to 0 or $\pm 1 \pmod{9}$. The density of the corresponding numbers (1, 10, 100, 112, 121, 211, 778, ...) may be zero. Numbers $\equiv 0 \pmod{3}$ converge to 153 and numbers $\equiv 2 \pmod{3}$ converge to 371 or 407, so that interest is confined to numbers $\equiv 1 \pmod{3}$. These converge to 370, or to one of the 3-cycles (55, 250, 133) or (160, 217, 352), or to one of the 2-cycles (919, 1459) or (136, 244), or, just occasionally, to 1. What proportion goes to each?

And what of higher powers? And different bases?

Henry Ernest Dudeney, *536 Puzzles & Curious Problems* (edited Martin Gardner), Scribner's, New York, 1967, Problem 143, pp. 43, 258–259.

Joseph S. Madachy, *Mathematics on Vacation*, Scribner's, New York, 1966, pp. 163–165.

E35 The Kimberling shuffle.

Clark Kimberling considered the array:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
4	2	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
6	2	7	4	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
8	7	9	2	10	6	11	12	13	14	15	16	17	18	19	20	21	22	23
6	2	11	9	12	7	13	8	14	15	16	17	18	19	20	21	22	23	24
13	12	8	9	14	11	15	2	16	6	17	18	19	20	21	22	23	24	25
2	11	16	14	6	9	17	8	18	12	19	13	20	21	22	23	24	25	26
18	17	12	9	19	6	13	14	20	16	21	11	22	2	23	24	25	26	27

in which each row is obtained from the previous by boxing (and expelling) the main diagonal element, and then reading the first number after the box, the first before the box, the second after the box, the second before the box, and so on until all the initial numbers are read off, and then continuing

with all the remaining numbers (still in numerical order). Is every number eventually expelled?

The numbers 1 2 3 4 5 6 7 8 9 10
are expelled on rows 1 25 2 4 3 22 6 8 10 5

and the numbers 11 12 13 14 15 16 17 18 19 20
are expelled on rows 32 83 44 14 7 66 169 11 49595 9

and the numbers 40 68 106 147
are expelled on rows 93167 181393 270186 8765242

and the numbers 242 322 502 669
are expelled on rows 16509502 38293016 118850522 653494691

In the alternate reading from the right and left of the box, one could instead start from the left first, leading to the array

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
4	6	2	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
2	8	6	9	4	10	11	12	13	14	15	16	17	18	19	20	21	22	23
9	10	6	11	8	12	2	13	14	15	16	17	18	19	20	21	22	23	24
8	2	11	13	6	14	10	15	9	16	17	18	19	20	21	22	23	24	25
14	15	6	9	13	16	11	17	2	18	8	19	20	21	22	23	24	25	26
11	2	16	18	13	8	9	19	6	20	15	21	14	22	23	24	25	26	27
...

which displays a similar chaotic behavior.

In each array there is a small amount of pattern: observe knight's move arithmetic progressions of common difference 3. For example, in the original array, for each $y \geq 0$, the number $n + 3y$ is in position $2y + 1$ on row $x + y$, where, for each $t \geq 0$,

$$\begin{aligned} n &= 9 \cdot 2^t - 3t - 7, & 12 \cdot 2^t - 3t - 8, & \text{resp. } 15 \cdot 2^t - 3t - 9 \\ \text{when } x &= 3 \cdot 2^t - 1, & 4 \cdot 2^t - 1, & \text{resp. } 5 \cdot 2^t - 1 \end{aligned}$$

so that $n + 3y$ is expelled on row $2x - 1 = 2y + 1$. I.e.,

$$\begin{aligned} n &= 18 \cdot 2^t - 3t - 13, 24 \cdot 2^t - 3t - 14, \text{ resp. } 30 \cdot 2^t - 3t - 15 \\ \text{is expelled on row } &6 \cdot 2^t - 3, & 8 \cdot 2^t - 3, & \text{resp. } 10 \cdot 2^t - 3 \end{aligned}$$

for $t = -1, 0, 1, \dots$.

Clark Kimberling, Problem 1615, *Crux Mathematicorum*, 17#2(Feb 1991) 44.

E36 Klarner-Rado sequences.

The sequence

1, 2, 4, 5, 8, 9, 10, 14, 15, 16, 17, 18, 20, 26, 27, 28, 29, 30, 32, 33, 34, 36, 40, 44, 47, 50, 51, 52, 53, 54, 56, 57, 58, 60, 62, 63, 64, 66, 68, 72, 80, 83, 86, 87, 88, 89, 92, 93, 94, 98, 99, 100, 101, 102, 104, 105, 106, 108, 110, 111, 112, 114, 116, 120, 122, 123, 124, 126, 128, 132, 134, 136, ...
is the thinnest which contains 1, and whenever it contains x , also contains $2x$, $3x + 2$ and $6x + 3$. Does it have positive density?

Several questions of this type were asked in the paper:

David A. Klarner & Richard Rado, Arithmetic properties of certain recursively defined sets, *Pacific J. Math.*, 53(1974) 445–463.

In the review, MR 50 #9784, it was stated that a subsequent paper ("Sets generated by a linear operation", same J., to appear) settles many of the conjectures stated in this paper. Did it ever appear? See also:

David A. Klarner & Richard Rado, Linear combinations of sets of consecutive integers, *Amer. Math. Monthly*, 80(1973) 985–989.

David A. Klarner & Karel Post, Some fascinating integer sequences, *Discrete Math.*, 106/107(1992) 303–309; MR 93i:11031.

E37 Mousetrap.

Cayley introduced a permutation problem he called **Mousetrap** which is loosely based on the card game Treize. Suppose that the numbers $1, 2, \dots, n$ are written on cards, one to a card. After shuffling (permuting) the cards, start counting the deck from the top card down. If the number on the card does not equal the count, transfer the card to the bottom of the deck and continue counting. If the two are equal then set the card aside and start counting again from 1. The game is won if all the cards have been set aside, but lost if the count reaches $n + 1$. Cayley proposed two questions.

- For each n find all the winning permutations of $1, 2, \dots, n$.
- For each n find the number of permutations that eliminate precisely i cards for each i , $1 \leq i \leq n$.

A third question arose during our investigations. Consider a permutation for which every number is set aside. The list of numbers in the order that they were set aside is another permutation. Any permutation obtained in this way we call a **reformed** permutation.

- Characterize the reformed permutations.

The permutation 4213 is a winning permutation which gives rise to the permutation 2134; this in turn gives the reformed permutation 3214 which is not a winning permutation.

4. For a given n , what is the longest sequence of reformed permutations?

5. Are there sequences of arbitrary length? Are there any cycles other than

$$1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \dots \quad \text{and} \quad 12 \rightarrow 12 \rightarrow 12 \rightarrow 12 \dots ?$$

Modular Mousetrap. We can play Mousetrap, but instead of counting $n, n+1, \dots$, we can start again, $\dots, n, 1, 2, \dots$. Now at least as many cards get set aside. In fact if n is prime, then either the initial deck is a derangement, or all cards get set aside, so every sequence cycles or terminates in a derangement. The identity permutation $123\dots n$ will always form a 1-cycle and now there are also examples of nontrivial cycles.

6. Are there k -cycles for every k ? What is the least value of n which yields a k -cycle?

- A. Cayley, A Problem in Permutations, *Quart. Math. J.*, I(1857), 79.
- A. Cayley, On the Game of Mousetrap, *Quart. J. Pure Appl. Math.*, XV (1877), 8-10.
- A. Cayley, A Problem on Arrangements, *Proc. Roy. Soc. Edinburgh*, 9(1878) 338-342.
- A. Cayley, Note on Mr. Muir's Solution of a Problem of Arrangement, *Proc. Roy. Soc. Edinburgh*, 9(1878) 388-391.
- Richard K. Guy & Richard J. Nowakowski, Mousetrap, Proc. Erdős80 Keszthely Combin. Conf., 1993. [see also *Amer. Math. Monthly*, 101(1994).]
- T. Muir, On Professor Tait's Problem of Arrangement, *Proc. Royal Soc. Edinburgh*, 9(1878) 382-387.
- T. Muir, Additional Note on a Problem of Arrangement, *Proc. Royal Soc. Edinburgh*, 11(1882) 187-190.
- Adolf Steen, Some Formulae Respecting the Game of Mousetrap, *Quart. J. Pure Appl. Math.*, XV(1878), 230-241.
- Peter Guthrie Tait, *Scientific Papers*, vol. 1, Cambridge, 1898, 287.

E38 Odd sequences.

Call a sequence of n zeros and ones, $\{a_1, \dots, a_n\}$, odd if each of the n sums $\sum_{i=1}^{n-k} a_i a_{i+k}$, ($k = 0, 1, \dots, n-1$) is odd. For example, 1101 is odd. Pelikán conjectured that there were no odd sequences if $n \geq 5$, but Peter Alles showed that there are infinitely many: if \mathbf{o} is an odd sequence of length n and \mathbf{x} and \mathbf{z} are sequences of $n-1$ and $3n-2$ zeros, then $\mathbf{o}\mathbf{x}\mathbf{o}\mathbf{z}\mathbf{o}$ and $\mathbf{o}\mathbf{z}\mathbf{o}\mathbf{x}\mathbf{o}$ are odd sequences of length $7n-3$. For

$n =$	1	4	12	16	24	25	36	37	40	45
he finds	1	2	2	8	2	4	2	16	2	16

odd sequences and no others with $n \leq 50$. For example, 101011100011 and its reversal. He asks if n is always congruent to 0 or 1 modulo 4, and if, when there are odd sequences, they are always a power of two in number.

Peter Alles, On a conjecture of J. Pelikán, *J. Combin. Theory Ser. A*, 60 (1992) 312-313; *MR 93i:11028*.

J. Pelikán, Problem, in *Infinite and Finite Sets*, Vol. III (Keszthely, 1973), 1549, *Colloq. Math. Soc. János Bolyai*, 10, North-Holland, Amsterdam, 1975.

F. None of the Above

The first few problems in this miscellaneous section are about lattice points, whose Euclidean coordinates are integers. Most of them are two-dimensional problems, but some can be formulated in higher dimensions as well. Some interesting books are

- J. W. S. Cassels, *Introduction to the Geometry of Numbers*, Springer-Verlag, New York, 1972.
- L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*, Springer-Verlag, Berlin, 1953.
- J. Hammer, *Unsolved Problems Concerning Lattice Points*, Pitman, 1977.
- O. H. Keller, *Geometrie der Zahlen*, Enzyklopädie der Math. Wissenschaften 12, B. G. Teubner, Leipzig, 1954.
- C. G. Lekkerkerker, *Geometry of Numbers*, Bibliotheca Mathematica 8, Wolters-Noordhoff, Groningen; North-Holland, Amsterdam, 1969.
- C. A. Rogers, *Packing and Covering*, Cambridge Univ. Press, 1964.

F1 Gauß's lattice point problem.

A very difficult unsolved problem is **Gauß's problem**. How many lattice points are there inside the circle with centre at the origin and radius r ? If the answer is $\pi r^2 + h(r)$, then Hardy & Landau showed that $h(r)$ is not $o(r^{1/2}(\ln r)^{1/4})$. It is conjectured that $h(r) = O(r^{1/2+\epsilon})$. Iwaniec & Mozzochi have shown that $h(r) = O(r^{7/11+\epsilon})$, and the best that is known is $h(r) = O(r^{46/73+\epsilon})$, by Huxley.

One can ask analogous questions in three dimensions for the sphere and regular tetrahedron. For the rectangular tetrahedron of F22 see the paper of Lehmer and also that of Xu & Yau, but their suggested counterexample to Overhagen's upper bound for arbitrary convex bodies is incorrect.

Chen Jing-Run, The lattice points in a circle, *Sci. Sinica*, 12(1963) 633–649; MR 27 #4799.

Javier Cilleruelo, The distribution of the lattice points on circles, *J. Number Theory*, 43(1993) 198–202.

Andrew Granville, The lattice points of an n -dimensional tetrahedron, *Aequationes Math.*, 41(1991) 234–241; MR 92b:11070.

Martin Huxley, *Proc. London Math. Soc.* (to appear).

F3 Lattice points, no four on a circle.

Aleksandar Ivić, Large values of the error term in divisor problems and the mean square of the zeta-function, *Invent. Math.*, 71(1983) 513–520; MR 84i:10046.

H. Iwaniec & C. J. Mozzochi, On the divisor and circle problems, *J. Number Theory*, 29(1988) 60–93; MR 89g:11091.

D. H. Lehmer, The lattice points of an n -dimensional tetrahedron, *Duke Math. J.*, 7(1940) 341–353.

T. Overhagen, Zur Gitterpunktanzahl konvexer Körper im 3-dimensionalen euklidischen Raum, *Math. Ann.*, 216(1975) 217–224; MR 57 #281.

Xu Yijng & Stephen Yau S.-T., A sharp estimate of the number of integral points in a tetrahedron, *J. reine angew. Math.*, 423(1992) 199–219; MR 93d:11067.

F2 Lattice points with distinct distances.

What is the largest number k of lattice points (x, y) , $1 \leq x, y \leq n$, which can be chosen so that their $\binom{k}{2}$ mutual distances are all distinct? It is easy to see that $k \leq n$. This bound can be attained for $n \leq 7$, for example for $n = 7$ with the points $(1, 1), (1, 2), (2, 3), (3, 7), (4, 1), (6, 6)$ and $(7, 7)$, but not for any larger value of n . Erdős & Guy showed that

$$n^{2/3-\epsilon} < k < cn/(\ln n)^{1/4}$$

and they conjecture that

$$i \quad k < cn^{2/3}(\ln n)^{1/6} \quad ?$$

One can also ask for “saturated” configurations, containing a *minimum* number of points which determine distinct distances, but such that *no* lattice point may be added without duplicating a distance. Erdős observes that this needs at least $n^{2/3-\epsilon}$ lattice points. In one dimension he cannot improve on $O(n^{1/3})$ and suspects that $O(n^{1/2+\epsilon})$ is best possible.

P. Erdős & R. K. Guy, Distinct distances between lattice points, *Elem. Math.*, 25(1970) 121–123; MR 43 #7406.

F3 Lattice points, no four on a circle.

Erdős & Purdy ask how many of the n^2 lattice points (x, y) , $1 \leq x, y \leq n$ can you choose with no four of them on a circle. It is easy to show $n^{2/3-\epsilon}$, but more should be possible.

What is the smallest t so that you can choose t of the lattice points so that the $\binom{t}{2}$ lines that they determine contain all the n^2 lattice points? It is not hard to show $t > cn^{2/3}$ and Noga Alon has since obtained the bounds

$$cn^{d(d-1)/(2d-1)} \leq t(n, d) \leq Cn^{d(d-1)/(2d-1)} \ln n$$

for the problem in any number, d , of dimensions.

Noga Alon, Economical coverings of sets of lattice points, *Geom. Funct. Anal.*, 1(1991) 224–230; MR 92g:52017.

F4 The no-three-in-line problem.

Can $2n$ lattice points (x, y) ($1 \leq x, y \leq n$) be selected with no three in a straight line? This has been achieved for $2 \leq n \leq 32$ and for several larger even values of n . Guy & Kelly make four conjectures.

1. There are no configurations with the symmetry of a rectangle which do not have the full symmetry of the square.
2. The only configurations having the full symmetry of the square are those in Figure 17. The $n = 10$ configuration was first found by Acland-Hood. This conjecture has been verified by Flammenkamp for $n \leq 60$.
3. For large enough n , the answer to the initial question is "no," i.e., there are only finitely many solutions to the problem. The total number of configurations, not counting reflexions and rotations, is:

n	2	3	4	5	6	7	8	9	10	11	12	13
#	1	1	4	5	11	22	57	51	156	158	566	499

and the configurations with specific symmetries have been enumerated for larger values of n .

4. For large n we may select at most $(c + \epsilon)n$ lattice points with no three in line, where $3c^3 = 2\pi^2$, i.e. $c \approx 1.85$.

In the opposite direction, Erdős showed that if n is prime, it is possible to choose n points with no three in line, and Hall, Jackson, Sudbery & Wild have shown that for n large, $(\frac{3}{2} - \epsilon)n$ such points can be found.

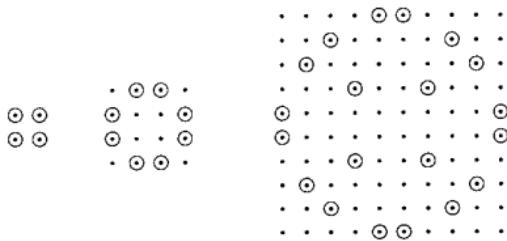


Figure 17. $2n$ Lattice Points, No Three in Line, $n = 2, 4, 10$.

F4. The no-three-in-line problem.

T. Thiele modifies Erdős's construction to show that one can find $(\frac{1}{4} - \epsilon)n$ points with no 3 in line and no 4 on a circle.

The no-three-in-line problem is a discrete analog of an old problem of Heilbronn. Place n (≥ 3) points in a disk (or square, or equilateral triangle) of unit area so as to maximize the smallest area of a triangle formed by three of the points. If we denote this maximum area by $\Delta(n)$, then Heilbronn originally conjectured that $\Delta(n) < c/n^2$, but Komlós, Pintz & Szemerédi disproved this by showing that $\Delta(n) > (\ln n)/n^2$. Roth showed that $\Delta(n) \ll 1/n(\ln \ln n)^{1/2}$; Schmidt improved this to $\Delta(n) \ll 1/n(\ln n)^{1/2}$ and Roth subsequently made the further improvement $\Delta(n) \ll 1/n^{\mu-\epsilon}$, first with $\mu = 2 - 2/\sqrt{5} > 1.1055$ and later with $\mu = (17 - \sqrt{65})/8 > 1.1172$.

Given $3n$ points in the unit square, $n \geq 2$, they determine n triangles in many ways. Choose the partition so as to minimize the sum of the areas, and let $a^*(n)$ be the maximum value of this minimum sum, taken over all configurations of $3n$ points. Then Odlyzko & Stolarsky show that $n^{-1/2} \ll a^*(n) \ll n^{-1/24}$. If the n triangles are required to be *area disjoint* it is not even clear that the sum of their areas tends to zero.

Acland-Hood, *Bull. Malayan Math. Soc.*, 0(1952-53) E11-12.

Michael A. Adena, Derek A. Holton & Patrick A. Kelly, Some thoughts on the no-three-in-line problem, *Proc. 2nd Austral. Conf. Combin. Math., Springer Lecture Notes*, 403(1974) 6-17; MR 50 #1890.

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C. E. Corzatt, Some extremal problems of number theory and geometry, PhD dissertation, Univ. of Illinois, Urbana, 1976.

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Hallard T. Croft, Kenneth J. Falconer & Richard K. Guy, *Unsolved Problems in Geometry*, Springer-Verlag, New York, 1991, §E5.

H. E. Dudeney, *The Tribune*, 1906-11-07.

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R. Goldstein, K. W. Heuer & D. Winter, Partition of S into n triples; solution to Problem 6316, *Amer. Math. Monthly*, 89(1982) 705-706.

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- Andrew M. Odlyzko, J. Pintz & Kenneth B. Stolarsky, Partitions of planar sets into small triangles, *Discrete Math.*, **57**(1985) 89–97; MR 87e:52007.
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F5 Quadratic residues. Schur's conjecture.

The **quadratic residues** of a prime p are the nonzero numbers r for which the congruence $r \equiv x^2 \pmod{p}$ has solutions. There are $\frac{1}{2}(p-1)$ of them in the interval $[1, p-1]$ and they are symmetrically distributed if p is of shape $4k+1$. If $p = 4k-1$, there are more quadratic residues in the interval $[1, 2k-1]$ than in $[2k, 4k-2]$, but all known proofs use Dirichlet's class-number formula. Is there an elementary proof?

For the first few values of d it is easy to remember which primes have d as a quadratic residue:

$$\begin{array}{ll} d = -1 & p = 4k+1 \\ d = -2 & p = 8k+1, 3 \\ d = -3 & p = 6k+1 \\ d = -5 & p = 20k+1, 3, 7, 9 \\ d = -6 & p = 24k+1, 5, 7, 11 \end{array} \quad \begin{array}{ll} d = 2 & p = 8k \pm 1 \\ d = 3 & p = 12k \pm 1 \\ d = 5 & p = 10k \pm 1 \\ d = 6 & p = 24k \pm 1, 5 \end{array}$$

However, it's just an example of the Strong Law of Small Numbers that in these small cases the residues are just those in the first half or the end quarters of the period, according to the sign of d . The **Legendre symbol**, $\left(\frac{a}{p}\right)$, is often used to indicate the quadratic character of a number a , $a \perp p$, relative to the prime p . Its value is ± 1 according as a is, or is not, a quadratic residue of p . For example, $\left(\frac{-1}{p}\right) = \pm 1$ according as

F6 Patterns of quadratic residues.

$p = 4k \pm 1$. The important properties of this symbol are that $\left(\frac{a}{p}\right) = \left(\frac{s}{p}\right)$ if $a \equiv c \pmod{p}$, and Gauss's famous **quadratic reciprocity law**, that for odd primes p and q , $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$ unless p and q are both $\equiv -1 \pmod{4}$, in which case $\left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right)$. These can be used for making quick verifications of the quadratic character of quite large numbers. For example,

$$\left(\frac{173}{211}\right) = \left(\frac{211}{173}\right) = \left(\frac{38}{173}\right) = \left(\frac{2}{173}\right) \left(\frac{19}{173}\right) = -\left(\frac{19}{173}\right) = -\left(\frac{173}{19}\right) = -\left(\frac{2}{19}\right) = +1$$

in fact $173 \equiv 54^2 \pmod{211}$.

A useful generalization of the Legendre symbol is the **Jacobi symbol** $\left(\frac{a}{b}\right)$ which is defined for $a \perp b$ and b any positive odd number, by the product

$$\prod \left(\frac{a}{p_i}\right)$$

of Legendre symbols, where $b = \prod p_i$ is the prime factorization of b , with repetitions counted. It has similar properties to the Legendre symbol, but note that, if b is not prime, then $\left(\frac{a}{b}\right) = +1$ does not necessarily imply that a is a quadratic residue of b .

If R (respectively N') is the maximum number of consecutive quadratic residues (respectively nonresidues) modulo an odd prime p , then A. Brauer showed that for $p \equiv 3 \pmod{4}$, $R = N' < \sqrt{p}$. On the other hand, if $p = 13$, then $N = 4 > \sqrt{13}$, since 5, 6, 7, 8 are all nonresidues of 13. Schur conjectured that $N < \sqrt{p}$ if p is large enough. Hudson proved Schur's conjecture; moreover, he believes that $p = 13$ is the only exception.

A. Brauer, Über die Verteilung der Potenzreste, *Math. Z.*, **35**(1932) 39–50; Zbl. 3, 339.

H. Davenport, *The Higher Arithmetic*, Fifth edition, Cambridge University Press, 1982, pp. 74–77.

Richard H. Hudson, On sequences of quadratic nonresidues, *J. Number Theory*, **3**(1971) 178–181; MR 43 #150.

Richard H. Hudson, On a conjecture of Issai Schur, *J. reine angew. Math.*, **289**(1977) 215–220; MR 58 #16481.

F6 Patterns of quadratic residues.

What patterns of quadratic residues are sure to occur? It is easy to see that a pair of neighboring ones always do, since at least one of 2, 5 and 10 is a residue, so that (1,2), (4,5) or (9,10) is such a pair. In the same way at least one of (1,3), (2,4) or (4,6) is a pair of residues differing by 2; (1,4) is a pair differing by 3; (1,5), (4,8), (6,10) or (12,16) is a pair differing by 4; and so on.

Suppose that each of $r, r+a, r+b$ is a quadratic residue modulo p . Emma Lehmer asks: for which pairs (a, b) will such a triplet occur for all sufficiently large p ? Denote by $\Omega(a, b)$ the least number such that a triplet is assured with $r \leq \Omega(a, b)$ for all $p > p(a, b)$, and write $\Omega(a, b) = \infty$ if there is no such finite number. For example, Emma Lehmer showed that $\Omega(1, 2) = \infty$, and more generally that $\Omega(a, b) = \infty$ if $(a, b) \equiv (1, 2) \pmod{3}$; or if $(a, b) \equiv (1, 3), (2, 3)$ or $(2, 4) \pmod{5}$; or if $(a, b) \equiv (1, 5), (2, 3)$ or $(4, 6) \pmod{7}$. Is $\Omega(a, b)$ finite in all other cases? Emma Lehmer conjectures that it is finite if a and b are squares. Of course, $\Omega(a, b) = 1$ if a, b are each one less than a square. As an example, let us see why $\Omega(5, 23) = 16$. If the triplets $(1, 6, 24)$ and $(4, 9, 27)$ are not all residues then 6 and 3 are not, and 2 must be a residue. If the triplets $(2, 7, 25)$ and $(13, 18, 36)$ are not all residues, then 7 and 13 must be nonresidues. Under these circumstances, $(r, r+5, r+23)$ are not all residues for $1 \leq r \leq 15$, but when $r = 16$, $(16, 21, 39)$ are residues.

Table 9 contains what are believed to be the (minimum) values of $\Omega(a, b)$. They provide good evidence for the conjectured finiteness in all cases except those already noted. Can an upper bound be obtained in terms of a and b ?

What about patterns of four residues, $r, r+a, r+b, r+c$? Of course these won't necessarily occur if any of the four subpatterns of three residues aren't forced to do so. We need examine only $(a, b, c) = (2, 5, 6), (1, 6, 7), (1, 4, 9), (5, 6, 9), (1, 6, 10), (1, 7, 10), \dots$ where $\Omega(a, b), \Omega(a, c), \Omega(b, c)$ and $\Omega(b-a, c-a)$ are each known to be finite. Some corresponding values of $\Omega(a, b, c)$ are $\Omega(1, 4, 9) = 357$ (Peter Montgomery corrects an error in the first edition), $\Omega(1, 4, 15) = 675$, and of course $\Omega(3, 8, 15) = 1$.

But although $\Omega(1, 6) = 24$, $\Omega(1, 7) = 38$, $\Omega(5, 6) = 49$ and $\Omega(6, 7) = 57$, it appears that $\Omega(1, 6, 7) = \infty$. In fact the pattern $r, r+a, r+b, r+c, r+d$, with $(a, b, c, d) = (1, 6, 7, 10)$ is such that, for each of the five subpatterns of four, $\Omega(1, 6, 7) = \Omega(1, 6, 10) = \Omega(1, 7, 10) = \Omega(5, 6, 9) = \Omega(6, 7, 10) = \infty$.

It is customary to define **k -th power residues** as numbers r for which $x^k \equiv r \pmod{p}$ has a solution, only with respect to those primes for which k divides $p-1$. In the same way that we remarked that every prime > 10 has a pair of consecutive quadratic residues not exceeding the pair $(9, 10)$, Hildebrand has shown that for each k there is a fixed bound, $\Lambda(k, 2)$ so that every sufficiently large prime has a pair of consecutive k -th power residues below this bound. There is no such bound for 3 consecutive quadratic or quartic, etc., residues. The argument consists in forcing primes of the form $3k+1$ to be residues and those of the form $3k+2$ to be nonresidues. Similarly by making 2 a residue and as many odd primes as necessary nonresidues, there is no such bound below which four consecutive k -th power residues must appear for any k . This leaves open only the question of three consecutive k -th powers for k odd. The case $k=3$ was solved by the Lehmers, Mills & Selfridge.

Table 9. Values of $\Omega(a, b)$ for $a < b \leq 25$.

a	$b = 4$	5	6	7	8	9	10	11	12	13	14
1	45	∞	24	38	∞	84	26	∞	∞	∞	∞
2	∞	25	20	∞	∞	70	30	∞	∞	∞	∞
3	174	39	∞	1	55	∞	∞	36	105	∞	∞
4	∞	∞	∞	91	36	∞	∞	∞	∞	∞	∞
5		49	∞	121	25	4	28	∞	∞	∞	∞
6			57	∞	33	30	24	∞	42	∞	∞
7				∞	75	74	44	∞	∞	∞	∞
8					66	∞	∞	∞	30	∞	∞
9						∞	54	42	66	∞	∞
10							60	85	∞	∞	∞
11								28	∞	119	∞
12									∞	∞	∞
13											∞
a	$b = 15$	16	17	18	19	20	21	22	23	24	25
1	77	35	∞	∞	15	∞	21	69	∞	∞	∞
2	54	∞	∞	25	98	∞	∞	∞	∞	∞	∞
3	1	∞	18	36	95	∞	∞	∞	1	51	∞
4	126	60	∞	38	168	90	∞	66	77	∞	∞
5	∞	∞	64	110	100	4	16	64	∞	∞	∞
6	60	36	38	∞	62	78	60	78	45	∞	∞
7	27	9	∞	∞	70	∞	42	40	45	∞	∞
8	1	∞	∞	77	48	∞	42	1	∞	∞	∞
9	57	66	∞	36	27	16	72	∞	21	119	∞
10	55	∞	32	102	77	∞	26	28	39	∞	∞
11	49	39	∞	∞	64	∞	25	∞	∞	∞	∞
12	∞	65	98	∞	36	4	∞	∞	90	∞	∞
13	42	∞	∞	36	∞	∞	∞	36	∞	∞	∞
14	35	∞	42	52	56	∞	64	81	∞	∞	∞
15	66	27	69	49	25	99	110	1	105	∞	∞
16			∞	102	169	95	50	56	∞	∞	∞
17						76	64	∞	∞	∞	∞
18							81	∞	40	185	144
19								33	∞	36	96
20								74	40	25	25
21									93	70	100
22										∞	98
23										∞	∞
24											63

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D. H. Lehmer, Emma Lehmer, W. H. Mills & J. L. Selfridge, Machine proof of a theorem on cubic residues, *Math. Comput.*, 16(1962) 407–415; MR 28 #5578.

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F7 A cubic analog of a Pell equation.

Hugh Williams observes that if $p \equiv 3 \pmod{4}$, then the equation $x^2 - py^2 = 2$ has integer solutions just if the congruence $w^2 \equiv 2 \pmod{p}$ does, i.e. just if $\left(\frac{2}{p}\right) = 1$, and asks for a cubic analog in the cases where $p \not\equiv \pm 1 \pmod{9}$: $x^3 + py^3 + p^2z^3 - 3pxyz = 3$ is solvable just if $w^3 \equiv 3 \pmod{p}$ is. Barrucand & Cohn have shown that this is true for $p \equiv 2$ or $5 \pmod{9}$. What about $p \equiv 4$ or $7 \pmod{9}$? This is a special case of a more general conjecture of Barrucand. If true, it would be useful in abbreviating the calculations needed to find the fundamental unit (regulator) of the cubic field $\mathbb{Q}(\sqrt[3]{p})$.

P.-A. Barrucand & Harvey Cohn, A rational genus, class number divisibility and unit theory for pure cubic fields, *J. Number Theory*, 2(1970) 7–21.

H. C. Williams, Improving the speed of calculating the regulator of certain pure cubic fields, *Math. Comput.*, 35(1980) 1423–1434.

F8 Quadratic residues whose differences are quadratic residues.

Gary Ebert asks us to find the largest collection of quadratic residues $r_i \pmod{p^n}$, given $p^n \equiv 1 \pmod{4}$, such that $r_i - r_j$ is a quadratic residue for all pairs (i, j) .

F9 Primitive roots

A primitive root, g , of a prime p is a number such that the residue classes of $g, g^2, \dots, g^{p-1} \equiv 1$ are all distinct. For example, 5 is a primitive root of 23 because

$$\begin{array}{cccccccccc} 5, & 5^2 \equiv 2, & 5^3 \equiv 10, & 4, & -3, & 8, & -6, & -7, & 11, & 9, & -1, \\ -5, & -2, & -10, & -4, & 3, & -8, & 6, & 7, & -11, & -9, & 1 \end{array}$$

all belong to different residue classes mod 23.

There is a famous conjecture of Artin, that for each integer $g \neq -1$, g not a square, there are infinitely many primes p with g as a primitive root. Hooley proved this assuming the extended Riemann hypothesis, and Gupta & Murty proved it unconditionally for infinitely many g . Heath-Brown

has proved the remarkable theorem that, but for at most two exceptional primes p_1, p_2 the following is true: For each prime p there are infinitely many primes q with p a primitive root of q . For example, there are infinitely many primes q with either 2 or 3 or 5 as a primitive root.

Erdős asks: if p is large enough, is there always a prime $q < p$ so that q is a primitive root of p ?

Given a prime $p > 3$, Brzolis asks if there is always a primitive root g of p and x ($0 < x < p$) such that $x \equiv g^x \pmod{p}$. If so, can g also be chosen so that $0 < g < p$ and $g \perp (p-1)$?

Vegh asks whether, for all primes $p > 61$, every integer can be expressed as the difference of two primitive roots of p . W. Narkiewicz notes that there is an affirmative answer for $p > 10^{19}$, so that this can, in theory, be answered by computer.

If p and $q = 4p^2 + 1$ are both primes, Gloria Gagola asks if 3 is a primitive root of q for all $p > 3$; is $p = 193$ the only odd prime for which 2 is not a primitive root of q ; is $p = 653$ the only prime for which 5 is neither a quadratic residue nor a primitive root of q ; and is there a number, perhaps a function of p (such as $2p-1$ for large p), which is always a primitive root of q ?

The Lehmers have checked that 6 is a primitive root for all primes p of the form $n^2 + 108$ for $p < 2 \cdot 10^8$.

Anton Dumitriu, Congruences du premier degré, *Rev. Roumaine Math. Pures Appl.*, 10(1965) 1201–1234.

Rajiv Gupta & Maruti Ram Murty, A remark on Artin's conjecture, *Invent. Math.*, 78(1984) 127–130; MR 86d:11003.

D. R. Heath-Brown, Artin's conjecture for primitive roots, *Quart. J. Math. Oxford Ser. (2)*, 37(1986) 27–38; MR 88e:11004.

C. Hooley, On Artin's conjecture, *J. reine angew. Math.*, 225(1967) 209–220; MR 34 #7445.

Leo Murata, On the magnitude of the least primitive root, *Astérisque No. 198-200*(1991) 253–257.

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F10 Residues of powers of two.

Graham asks about the residue of $2^n \pmod n$. There are no solutions of $2^n \equiv 1 \pmod n$ with $n > 1$. $2^n \equiv 2 \pmod n$ whenever n is a pseudoprime base 2 (see A12) or is a prime. The Lehmers have shown that the smallest solution of $2^n \equiv 3 \pmod n$ is $n = 4700063497 = 19 \cdot 47 \cdot 5263229$. Of course, n has to be composite, and it is not divisible by 2 or 3. In fact, Mąkowski (see reference at B5) notes that if $\left(\frac{2}{p}\right)$ and $\left(\frac{3}{p}\right)$ are of opposite sign, i.e. if $p = 24k \pm 7$ or ± 11 , then n is not divisible by p .

Rotkiewicz (compare A12) notes that if m satisfies $2^m \equiv 3 \pmod m$, then $n = 2^m - 1$ is a solution of $2^{n-2} \equiv 1 \pmod n$.

Benkoski asked if there was a solution to $2^n \equiv 4 \pmod n$ which didn't end in 7 when written in decimal notation. Zhang Ming-Zhi gave the solutions where $n \equiv 1$ or $3 \pmod{10}$ and asked if there were any with $n \equiv 9 \pmod{10}$.

Victor Meally reports that $2^n \equiv -1 \pmod n$ for $n = 3^k$ and $2^n \equiv -2 \pmod n$ for $n = 2, 6, 66, 946, \dots$. Schinzel notes that the existence of infinitely many n such that $2^n \equiv -2 \pmod n$ is proved in the remark to Exercise 4 on p. 235 of Sierpiński's *Elementary Theory of Numbers*, 2nd English Edition, 1987.

Zhang Ming-Zhi, A note on the congruence $2^{n-2} \equiv 1 \pmod n$ (Chinese. English summary), *Sichuan Daxue Xuebao*, 27 (1990) 130–131; MR 92b: 11003 (where the wrong Benkoski reference appears to be given).

F11 Distribution of residues of factorials.

What is the distribution of $1!, 2!, 3!, \dots, (p-1)!, p!$ modulo p ? About p/e of the residue classes are not represented. Here are the missing ones for the first few values of p :

$p = 2$ or 3 , none.	$p = 5$, $\{-2\}$.	$p = 7$, $\{-2, -3\}$.
$p = 11$, $\{-2, \pm 3, \pm 4\}$.	$p = 13$, $\{-3, 4, -5\}$.	
$p = 17$, $\{4, 5, -6, -7, -8\}$.	$p = 19$, $\{3, -5, -6, \pm 7, \pm 8\}$.	
$p = 23$, $\{-3, -4, -6, -7, -8, 10\}$.		
$p = 29$, $\{-2, -4, 7, -8, -9, -10, -11, -12, 13, -14\}$.		
$p = 31$, $\{\pm 3, 4, 8, \pm 10, 11, 12, 13, 14\}$.		
$p = 37$, $\{3, 4, \pm 5, -9, 10, 11, -14, \pm 15, -18\}$.		

Until we reach the last two entries we might be tempted to conjecture that there were always at least as many negative entries as positive ones. Are there infinitely many examples of each case? The value $p = 23$ is remarkable in that the only duplicates are ± 1 .

In answer to a question of Erdős, Rokowska & Schinzel show that if the residues of $2!, 3!, \dots, (p-1)!$ are all distinct, then the missing residue must

F13 Covering systems of congruences.

be that of $-\frac{p-1}{2}$, that $p \equiv 5 \pmod 8$, and that there are no such p with $5 < p \leq 1000$.

B. Rokowska & A. Schinzel, Sur une problème de M. Erdős, *Elem. Math.*, 15 (1960) 84–85.

R. Stauduhar, Problem 7, *Proc. Number Theory Conf.*, Boulder, 1963, p. 90.

F12 How often are a number and its inverse of opposite parity?

For each x ($0 < x < p$) where p is an odd prime, define \bar{x} by $x\bar{x} \equiv 1 \pmod p$ and $0 < \bar{x} < p$. Let N_p be the number of cases in which x and \bar{x} are of opposite parity. E.g., for $p = 13$, $(x, \bar{x}) = (1, 1), (2, 7), (3, 9), (4, 10), (5, 8), (6, 11), (12, 12)$ so $N_{13} = 6$. D. H. Lehmer asks us to find N_p or at least to say something nontrivial about it. $N_p \equiv 2$ or $0 \pmod 4$ according as $p \equiv \pm 1 \pmod 4$.

p	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61
N_p	0	2	0	4	6	10	4	12	18	4	14	18	20	16	30	32	30

F13 Covering systems of congruences.

A system of congruences $a_i \pmod{n_i}$ ($1 \leq i \leq k$) is called a **covering system** if every integer y satisfies $y \equiv a_i \pmod{n_i}$ for at least one value of i . For example: $0 \pmod 2$; $0 \pmod 3$; $1 \pmod 4$; $5 \pmod 6$; $7 \pmod{12}$. If $c = n_1 < n_2 < \dots < n_k$ then Erdős offers \$500.00 for a proof or disproof of the existence of covering systems with c arbitrarily large. Davenport & Erdős, and Fried, found systems with $c = 3$; Swift with $c = 6$; Selfridge with $c = 8$; Churchhouse with $c = 10$; Selfridge with $c = 14$; Krukenberg with $c = 18$; and Choi with $c = 20$.

Erdős offers \$25.00 for a proof of the nonexistence of covering systems with all moduli n_i odd, distinct, and greater than one; while Selfridge offers \$900.00 for an explicit example of such a system. Berger, Felzenbaum & Fraenkel showed that the l.c.m. of the moduli of such a system must contain at least six prime factors. More generally, “odd” could be replaced by “not divisible by the first r primes.” Simpson & Zeilberger showed that if the moduli are odd and squarefree then at least 18 primes are required.

Jim Jordan offers comparable rewards to those mentioned above for solutions to the analogous problems for Gaussian integers (A15).

Erdős noted that you can have a covering system with all moduli n_i distinct, squarefree, and greater than one by using the proper divisors of 210:

a_i	0	0	0	1	0	1	1	2	2	23	4	5	59	104
n_i	2	3	5	6	7	10	14	15	21	30	35	42	70	105

Krukenberg used 2 and squarefree numbers greater than 3. Selfridge asks if you can have such a system with $c \geq 3$ in place of $c = 2$. He observes that the n_i cannot all be squarefree with at most two prime factors, but the above example shows that you do not need more than three.

It is easy, but not trivial, to prove that, for a covering system with distinct moduli, $\sum_{i=1}^k 1/n_i > 1$. The sum can be arbitrarily close to 1 if $n_1 = 3$ or 4. Selfridge & Erdős conjecture that $\sum 1/n_i > 1 + c_{n_1}$ where $c_{n_1} \rightarrow \infty$ with n_1 .

Schinzel has asked for a covering system in which no modulus divides another. This would not exist if a covering with odd moduli does not exist.

Simpson calls a covering system **irredundant** if it ceases to cover the integers when one of the congruences is removed. He has shown that if the l.c.m. of the moduli of such a system is $\prod p_i^{a_i}$ then the system contains at least $\sum a_i(p_i - 1)$ congruences.

Erdős conjectures that all sequences of the form $d \cdot 2^k + 1$ ($k = 1, 2, \dots$), d fixed and odd, which contain no primes can be obtained from covering congruences (see B21 for examples). Equivalently, the least prime factors of members of such sequences are unbounded.

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R. J. Simpson & D. Zeilberger, Necessary conditions for distinct covering systems with squarefree moduli, *Acta Arith.*, **59**(1991) 59–70.

F14. Exact covering systems.

253

Zhang Ming-Zhi, A note on covering systems of residue classes, *Sichuan Daxue Xuebao* **26**(1989) 185–188; MR **92c**:11003.

Stefan Znám, A survey of covering systems of congruences, *Acta Math. Univ. Comen.*, **40–41**(1982) 59–79; MR **84e**:10004.

F14 Exact covering systems.

If a system of congruences is both covering and disjoint (each integer covered by just one congruence) it is called an **exact covering system**. Necessary, but not sufficient, conditions for a system to be exact are $\sum_{i=1}^k 1/n_i = 1$ and $(n_i, n_j) > 1$ for all i, j , where the notation is as in the first sentence of F13. There is a theorem, variously attributed to subsets of {Davenport, Mirsky, Newman, Rado}, that if a set of distinct numbers > 1 are the moduli of congruences, then either there is a number which is not in any of them or there is a number which is in more than one of them. The ingenious proof used a generating function and roots of unity. Combinatorial proofs were later given by Berger, Felzenbaum & Fraenkel and by Simpson.

Znám notes that $(n_1, n_2, \dots, n_k) > 1$ is *not*, as stated in the first edition, a necessary condition, as is evinced by the example $0 \pmod{6}$, $1 \pmod{10}$, $2 \pmod{15}$ and $3, 4, 5, 7, 8, 9, 10, 13, 14, 15, 16, 19, 20, 22, 23, 25, 26, 27, 28, 29 \pmod{30}$. He confirmed a conjecture of Mycielski by further proving that if p is the least prime divisor of n_k , then $n_k = n_{k-1} = \dots = n_{k-p+1}$. He conjectured that if there exist only pairs of equal moduli, then the moduli are all of the form $2^\alpha 3^\beta$, but later he and Burshtein & Schönheim and Joel Spencer each gave counter-examples, such as

$$\begin{array}{ccccccccccccc} 0, 1 & 2, 7 & 3, 8 & 13, 28 & 4, 9 & 14, 34 & 19, 39 & 59, 119 \\ \text{mod} & 5 & 10 & 15 & 30 & 20 & 40 & 60 & 120 \end{array}$$

Stein proved that if there is a single pair of equal moduli, the rest being distinct, then $n_i = 2^i$ ($1 \leq i \leq k-1$), $n_k = 2^{k-1}$. Znám proved analogously that if there is a triple of equal moduli, the rest being distinct, then $n_i = 2^i$ ($1 \leq i \leq k-3$), $n_{k-2} = n_{k-1} = n_k = 3 \cdot 2^{k-3}$. Beebe has extended Stein's result by showing that a system is exact just if

$$\sin \pi z = -2^{k-1} \prod_{i=1}^k \sin \frac{\pi}{n_i} (a_i - z).$$

Simpson extended work by Burshtein & Schönheim and showed that if the primes $p_1 < p_2 < \dots < p_t$ are those dividing the moduli of an exact covering system in which no modulus occurs more than N times, then

$$p_t \leq N \prod_{i=1}^{t-1} \frac{p_i}{p_i - 1}$$

The main outstanding problem is to characterize exact covering congruences.

Porubský asked if there is an "exactly m times covering system" which is not the union of m exact covering systems. More generally, call such a system **S reducible** if there is a partition $S = S_1 \cup S_2$ such that S_1 and S_2 are exactly l times and exactly $m-l$ times covering systems for some l , $0 < l < m$, and **irreducible** if there is no such partition. Zhang Ming-Zhi answers Porubský's question affirmatively by showing that for every $m > 1$ there is an irreducible exactly m times covering system. This had already been shown for $m = 2$ by S. L. G. Choi (Keszthely, 1973) and by Zeilberger, e.g.:

$$1(2); 0(3); 2(6); 0,4,6,8(10); 1,2,4,7,10,13(15); 5,11,12,22,23,29(30).$$

Infinite disjoint covering systems with all moduli distinct can exist. If the sum of the reciprocals of the moduli is 1, such systems exist with moduli $\{2, 2^2, 2^3, \dots\}$ and with sets of moduli of shape $2^a 3^b$. Fraenkel & Simpson conjecture that these are the only types. Lewis showed that the only possible exceptions had an infinite set of distinct primes dividing their moduli.

Questions can also be asked about covering systems of Beatty sequences (E27). Graham showed that if

$$\lfloor m\alpha_i + \beta_i \rfloor; \quad m \in \mathbb{Z}; \quad 1 \leq i \leq k$$

is such a system with $k > 2$ and at least one α_i irrational, then some two α_i must be equal. This is not so if all the α_i are rational, since

$$\left\lfloor m \frac{2^k - 1}{2^{k-i}} + 1 - 2^{i-1} \right\rfloor \quad (1 \leq i \leq k)$$

are exactly covering systems. Fraenkel conjectures that the only such systems with distinct α_i are of this form.

There are connexions with pseudoperfect numbers (B2) and with Egyptian fractions (D11).

John Beebe, Examples of infinite, incongruent exact covers, *Amer. Math. Monthly*, **95**(1988) 121–123; errata **97**(1990) 412; MR, **88g**:11013, **91a**:11013.

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F14. Exact covering systems.

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F15 A problem of R. L. Graham.

Szegedy won the prize that Graham offered for settling (affirmatively) the question: does $0 < a_1 < a_2 < \dots < a_n$ imply that $\max_{i,j} a_i/(a_i, a_j) \geq n$? His proof, and that of Zaharescu, is for n sufficiently large. Cheng & Pomerance have given the specific bound 10^{4275} but there is still a fair amount of ground to be covered.

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- Alexandru Zaharescu, On a conjecture of Graham, *J. Number Theory*, **27** (1987) 33–40; *MR 88k:11009*.

F16 Products of small prime powers dividing n .

Erdős defines $A(n, k)$ as $\prod p^a$ where the product is taken over primes p less than k with $p^a \mid n$ and asks: is

$$\max_n \min_{1 \leq i \leq k} A(n+i, k) = o(k) \quad ?$$

He remarks that it is easy to show that it is $O(k)$. Is

$$\min_n \max_{1 \leq i \leq k} A(n+i, k) > k^c$$

for every c and sufficiently large k ? Is

$$\sum_{i=1}^k \frac{1}{A(n+i, k)} > c \ln k \quad ?$$

F17 Series associated with the ζ -function.

Alf van der Poorten had asked for a proof that

$$\zeta(4) = \left[\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \right] = \frac{36}{17} \sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}}$$

before he and others showed that

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}} = \int_0^{\frac{\pi}{4}} \theta \left(\ln 2 \sin \frac{\theta}{2} \right)^2 d\theta = \frac{17\pi^4}{6480}.$$

It is also known that

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{2\pi\sqrt{3} + 9}{27}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} = \frac{\pi\sqrt{3}}{9},$$

$$\zeta(2) = \left[\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \right] = 3 \sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}},$$

$$2(\sin^{-1} x)^2 = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2 \binom{2n}{n}} \quad \text{and} \quad \zeta(3) = \left[\sum_{n=1}^{\infty} \frac{1}{n^3} \right] = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}$$

Some remarkable identities discovered by Gosper include

$$\sum_{k \geq 1} \frac{30k - 11}{4(2k-1)k^3 \binom{2k}{k}^2} = \zeta(3), \quad \sum_{n \geq 1} \frac{2^{-n}}{1+x^{2-n}} = \frac{1}{\ln x} + \frac{1}{1-x}$$

Louis Comtet, *Advanced Combinatorics*, D. Reidel, Dordrecht, 1974, p. 89.
 John A. Ewell, A new series representation for $\zeta(3)$, *Amer. Math. Monthly*, 97(1990) 219–220; MR 91d:11103.

R. William Gosper, Strip mining in the abandoned orefields of nineteenth century mathematics, *Computers in Mathematics* (Stanford CA, 1986), *Lecture Notes in Pure and Appl. Math.*, Dekker, New York, 125(1990) 261–284.

Leonard Levin, *Polylogarithms and Associated Functions*, North-Holland, 1981 [§7.62, and foreword by van der Poorten].

Alfred van der Poorten, A proof that Euler missed ... Apery's proof of the irrationality of $\zeta(3)$, *Math. Intelligencer*, 1(1979) 195–203.

Alfred J. van der Poorten, Some wonderful formulas ... an introduction to polylogarithms, *Proc. Number Theory Conf.*, Queen's Univ., Kingston, 1979, 269–286; MR 80i:10054.

F18 Size of the set of sums and products of a set.

If a_1, a_2, \dots, a_n are n numbers (not necessarily integers), how big is the set of their sums and products in pairs?

$$\iota \quad |\{a_i + a_j\} \cup \{a_i a_j\}| > n^{2-\epsilon} \quad ?$$

Erdős & Szemerédi have proved that the cardinality is greater than $n^{1+\epsilon_1}$ and less than $n^2 \exp(-c_2 \ln n / \ln \ln n)$.

P. Erdős, Some recent problems and results in graph theory, combinatorics and number theory, *Congress. Numer.*, 17 Proc. 7th S.E. Conf. Combin. Graph Theory, Comput., Boca Raton, 1976, 3–14 (esp. p. 11).

P. Erdős & E. Szemerédi, On sums and products of integers, *Studies in Pure Mathematics*, Birkhäuser, 1983, pp. 213–218; MR 86m:11011.

F19 Partitions into distinct primes with maximum product.

In the first edition we asked: if n is large and written in the form $n = a + b + c$, $0 < a < b < c$ in every possible way, are all the products abc distinct, but Leech observed that **D16** answers this negatively. See also Kelly's paper. The analogous problem of maximizing the l.c.m. in place of product was studied algorithmically by Drago.

J. Riddell & H. Taylor asked if, among the partitions of n into *distinct* primes, the one having the maximum *product* of parts is necessarily one of those with the maximum *number* of parts, but Selfridge answered this negatively with the example

$$\begin{aligned} 319 &= 2 + 3 + 5 + 7 + 11 + 13 + 17 + 23 + 29 + 31 + 37 + 41 + 47 + 53 \\ &= 3 + 5 + 11 + 13 + 17 + 19 + 23 + 29 + 31 + 37 + 41 + 43 + 47 \end{aligned}$$

F21 All partial quotients one or two.

but the partition with the smaller number of parts gives the largest possible product. Is this the least counterexample? Can the cardinalities of the two sets differ by an arbitrarily large amount?

Antonino Drago, Rules to find the partition of n with maximum l.c.m., *Atti Sem. Mat. Fis. Univ. Modena*, 16(1967) 286–298; MR 37 #180.

J. B. Kelly, Partitions with equal products, *Proc. Amer. Math. Soc.*, 15(1964) 987–990.

F20 Continued fractions.

A number x may be expressed as a **continued fraction**

$$x = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots}}}$$

which, out of kindness to the typesetter, is often written

$$x = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots}}}$$

When the numerators b_i are all 1 the continued fraction is called **simple**, and may be written

$$x = [a_0; a_1, a_2, a_3, \dots]$$

It may be finite or infinite, but if x is rational it is finite. In this case there are two possible forms, one of which has its *last partial quotient*, a_k , equal to 1:

$$\frac{7}{16} = [0; 2, 3, 2] = [0; 2, 3, 1, 1]$$

Zaremba conjectured that given any integer $m > 1$, there is an integer a , $0 < a < m$, $a \perp m$ such that the simple continued fraction $[0; a_1, \dots, a_k]$ for a/m has $a_i \leq B$ for $1 \leq i \leq k$ where B is a small absolute constant (say $B = 5$). He was only able to prove $a_i \leq C \ln m$.

T. W. Cusick, Zaremba's conjecture and sums of the divisor function, *Math. Comput.*, 61(1993) 171–176.

S. K. Zaremba, La méthode des "bons treillis" pour le calcul des intégrales multiples, in *Applications of Number Theory to Numerical Analysis (Proc. Symp. Univ. Montreal, 1971)* Academic Press, 1972, 93–119 esp. 69 & 76; MR 49 #8271.

F21 All partial quotients one or two.

Not every number n can be expressed as the sum of two positive integers $n = a + b$ so that the continued fraction for a/b has all its partial quotients

equal to 1 or 2. For 11, 17 and 19 we can have

$$\frac{4}{7} = [0; 1, 1, 2, 1], \quad \frac{5}{12} = [0; 2, 2, 2], \quad \text{and} \quad \frac{7}{12} = [0; 1, 1, 2, 2]$$

but 23 can't be so expressed. However Leo Moser conjectured that there is a constant c such that every n can be so expressed with the sum of the partial quotients, $\sum a_i < c \ln n$.

Bohuslav Divis asked for a proof that in any real quadratic field there is always an irrational number whose simple continued fraction expansion has all its partial quotients 1 or 2. He also asks the same question with 1 and 2 replaced by any pair of distinct positive integers.

F22 Algebraic numbers with unbounded partial quotients.

Is there an algebraic number of degree greater than two whose simple continued fraction has unbounded partial quotients? Does every such number have unbounded partial quotients? Ulam asked particularly about the number $\xi = 1/(\xi + y)$ where $y = 1/(1+y)$.

Littlewood observed that if θ has a continued fraction with bounded partial quotients a_n , then $\liminf n|\sin n\theta| \leq A(\theta)$, where $A(\theta)$ is not zero (though it is for almost all θ). He also asks if

$$\liminf n|\sin n\theta \sin n\phi| = 0$$

for all real θ and ϕ ? It is for almost all θ and ϕ . Cassels & Swinnerton-Dyer treat a dual problem and show incidentally that $\theta = 2^{1/3}$, $\phi = 4^{1/3}$ does not provide a counterexample. Davenport suggested that a computer might help with proving that

$$|(x\theta - y)(x\phi - z)| < \epsilon$$

has solutions for every θ , ϕ when, for example, $\epsilon = \frac{1}{10}$ or $\frac{1}{50}$.

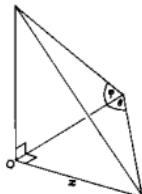


Figure 18. Rectangular Tetrahedron.

F23 Small differences between powers of 2 and 3.

J. W. S. Cassels & H. P. F. Swinnerton-Dyer, On the product of three homogeneous linear forms and indefinite ternary quadratic forms, *Philos. Trans. Roy. Soc. London Ser. A*, 248(1955) 73–96; MR 17, 14.

Harold Davenport, Note on irregularities of distribution, *Mathematika*, 3 (1956) 131–135; MR 19, 19.

John E. Littlewood, *Some Problems in Real and Complex Analysis*, Heath, Lexington MA, 1968, 19–20, Problems 5, 6.

F23 Small differences between powers of 2 and 3.

Problem 1 of Littlewood's book asks how small $3^n - 2^m$ can be in comparison with 2^m . He gives as an example

$$\frac{3^{12}}{2^{19}} = 1 + \frac{7153}{524288} \approx 1 + \frac{1}{73}$$

(the ratio of D^2 to E^3).

The first few convergents to the continued fraction (see F20)

$$1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{3 + \frac{1}{1 + \dots}}}}}$$

for $\log 3$ to the base 2 are

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{8}{5}, \frac{19}{12}, \frac{65}{41}, \frac{84}{53}, \dots$$

so Victor Meally observed that the octave may conveniently be partitioned into 12, 41 or 53 intervals, and that the system of temperament with 53 degrees is due to Nicolaus Mercator (1620–1687; not Gerhardus, 1512–1594, of map projection fame).

Ellison used the Gel'fond-Baker method to show that

$$|2^x - 3^y| > 2^x e^{-x/10} \quad \text{for } x > 27,$$

and Tijdeman used it to show that there is a $c \geq 1$ such that $|2^x - 3^y| > 2^x/x^c$.

Croft asks the corresponding question for $n! - 2^m$. The first few best approximations to $n!$ by powers of 2 are

$5!$	$20!$	$22!$	$24!$	$61!$	$63!$	$90!$
2^7	2^{81}	2^{70}	2^{79}	2^{278}	2^{290}	2^{459}
-1.34	+0.13	-0.10	+0.046	+0.023	-0.0017	-0.0007

where the third row is the percentage error in the exponent.

In the "Stellingen" that accompanied Benne de Weger's PhD thesis he observed that if the primes p_1, \dots, p_t are given, then there is an effectively

computable constant C , depending only on the p_i , such that for all n, k_1, \dots, k_t with $n! \neq p_1^{k_1} \cdots p_t^{k_t}$ it is true that

$$\left| n! - p_1^{k_1} \cdots p_t^{k_t} \right| > \exp(Cn/\ln n).$$

There is some experimental support for the conjecture that the right side could be replaced by $\exp(C'n\ln n)$. For a fixed m , the methods of his thesis will determine all solutions of

$$n! - p_1^{k_1} \cdots p_t^{k_t} = m.$$

Erdős believes the conjecture. He also observes that $n! = 2^a \pm 2^b$ only when $n = 1, 2, 3, 4$ and 5 .

F. Beukers, Fractional parts of powers of rationals, *Math. Proc. Cambridge Philos. Soc.*, **90**(1981) 13–20; *MR 83g:10028*.

A. K. Dubitskas, A lower bound on the value of $\|(3/2)^k\|$ (Russian), *Uspekhi Mat. Nauk*, **45**(1990) 153–154; translated in *Russian Math. Surveys*, **45**(1990) 163–164; *MR 91k:11058*.

W. J. Ellison, Recipes for solving diophantine problems by Baker's method, *Sém. Théorie Nombres*, 1970–71, Exp. No. 11, C.N.R.S. Toulouse, 1971.

R. Tijdeman, On integers with many small factors, *Compositio Math.*, **26** (1973) 319–330.

F24 Squares with just two different decimal digits.

Sin Hitotumatu asks for a proof or disproof that, apart from $10^{2n}, 4 \cdot 10^{2n}$ and $9 \cdot 10^{2n}$, there are only finitely many squares with just two different decimal digits, such as $38^2 = 1444$, $88^2 = 7744$, $109^2 = 11881$, $173^2 = 29929$, $212^2 = 44944$, $235^2 = 55225$ and $3114^2 = 9696996$.

F25 The persistence of a number.

In the sequence 679, 378, 168, 48, 32, 6, each term is the product of the decimal digits of the previous one. Neil Sloane defines the **persistence** of a number as the number of steps (five in the example) before the number collapses to a single digit. The smallest numbers with persistence 1, 2, ..., 11 are 10, 25, 39, 77, 679, 6758, 68889, 2677889, 268888999, 3778888999, 27777788888899. There is no number less than 10^{50} with persistence greater than 11. Sloane conjectures that there is a number d such that no number has persistence greater than d .

In base 2 the maximum persistence is 1. In base 3 the second term is zero or a power of 2. It is conjectured that all powers of 2 greater than 2^{15}

F27 Mahler's generalization of Farey series.

contain a zero when written in base 3. This is true up to 2^{500} . The truth of this conjecture would imply that the maximum persistence in base 3 is 3.

Sloane's general conjecture is that there is a number $d(b)$ such that the persistence in base b cannot exceed $d(b)$.

Erdős modifies the problem by letting $f(n)$ be the product of the *non-zero* decimal digits of n , and asks how fast one reaches a one-digit number, and for which numbers is the descent slowest. He says that it is easy to prove that $f(n) < n^{1-c}$, so that at most $c \ln \ln n$ steps are needed.

N. J. A. Sloane, The persistence of a number, *J. Recreational Math.*, **6**(1973) 97–98.

F26 Expressing numbers using just ones.

Let $f(n)$ be the least number of ones that can be used to represent n using ones and any number of + and \times signs (and parentheses). For example,

$$80 = (1 + 1 + 1 + 1 + 1) \times (1 + 1 + 1 + 1) \times (1 + 1 + 1 + 1)$$

so $f(80) \leq 13$. It can be shown that $f(3^k) = 3k$ and $3 \log_3 n \leq f(n) \leq 5 \log_3 n$ where the logs are to base 3. Does $f(n) \sim 3 \log_3 n$?

Daniel Rawsthorne has shown that $f(n) = 2a + 3b$ when n is of the form $2^a 3^b$ and not greater than 3^{10} . Is this true for larger such n ?

Is it always true that for a prime p , $f(p) = 1 + f(p-1)$? And that $f(2p) = \min\{2 + f(p), 1 + f(2p-1)\}$?

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K. Mahler & J. Popken, On a maximum problem in arithmetic (Dutch), *Nieuw Arch. Wiskunde*, **3** 1(1953) 1–15; *MR 14*, 852e.

Daniel A. Rawsthorne, How many 1's are needed? *Fibonacci Quart.*, **27**(1989) 14–17.

F27 Mahler's generalization of Farey series.

The **Farey series** of order n consists of all positive rational numbers in their lowest terms, with numerators and denominators not exceeding n , arranged in order of magnitude. For example, the Farey series of order 5 is

$$\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{4}, \frac{5}{3}, \frac{4}{3}, \frac{5}{2}, \frac{3}{2}, \frac{5}{3}, \frac{2}{1}, \frac{5}{2}, \frac{3}{1}, \frac{4}{1}, \frac{5}{1}$$

The determinant formed from the numerators and denominators of two adjacent fractions is -1 . Mahler regards the members of the sequence as positive real roots of linear equations whose coefficients have g.c.d. 1

and do not exceed n , and obtained the following apparent generalization to quadratic equations. List the coefficients (a, b, c) of the quadratic equations

$$ax^2 + bx + c = 0, a \geq 0, (a, b, c) = 1, b^2 \geq 4ac, \max\{a, |b|, |c|\} \leq n$$

which have positive real roots, in order of size of the roots. Then the third-order determinant (see F28) formed from any three consecutive rows of a, b, c appeared always to take the value 0 or ± 1 . In the first edition, Table 10 illustrated this for $n = 2$, with initial and final entries $(0, 1, 0)$ and $(0, 0, 1)$, corresponding to roots 0 and ∞ , just as the Farey series could include the terms $\frac{0}{1}$ and $\frac{1}{0}$. We also adopted a suggestion of Selfridge of duplicating rational roots to avoid trivial exceptions. The present Table 10 is an excerpt from the generalized series for $n = 3$. The entry in the last column is the value of the determinant formed from that row and its immediate neighbors.

Table 10. Segment of Generalized Farey Series of Order 3.

a	b	c	root	determinant
0	1	-1	1	
3	-1	-3	$(1 + \sqrt{37})/6$	0
3	-2	-2	$(1 + \sqrt{7})/3$	1
2	0	-3	$\sqrt{6}/2$	-1
3	-3	-1	$(3 + \sqrt{21})/6$	-1
2	-1	-2	$(1 + \sqrt{17})/4$	0
1	1	-3	$(\sqrt{13} - 1)/2$	-1
2	-2	-1	$(1 + \sqrt{3})/2$	1
3	-2	-3	$(1 + \sqrt{10})/3$	0
1	0	-2	$\sqrt{2}$	-1
3	-3	-2	$(3 + \sqrt{33})/6$	1
0	2	-3	$3/2$	

The conjecture was verified for $n \leq 5$, but Lambertus Hesterman of Canberra discovered counterexamples when $n = 7$; for example

a	b	c	root	determinant
2	-7	-7	$(7 + \sqrt{105})/4$	
1	-3	-6	$(3 + \sqrt{33})/2$	-2
1	-6	7	$3 + \sqrt{2}$	

Can this be rescued, or is it yet another example of the Strong Law of Small Numbers? Lewis Low proved that the absolute value of the determinant cannot exceed n . Can this bound be substantially reduced?

What can be said about the fourth-order determinants associated with cubic equations?

H. Brown & K. Mahler, A generalization of Farey sequences: some exploration via the computer. *J. Number Theory*, 3(1971) 364–370; MR 44 #3959.
 Lewis Low, Some lattice point problems, PhD thesis, Univ. of Adelaide, 1979;
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F28 A determinant of value one.

The third order determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$$

may be defined as $a_1(a_5a_9 - a_6a_8) - a_2(a_4a_9 - a_6a_7) + a_3(a_4a_8 - a_5a_7)$. Find whole numbers a_1, a_2, \dots, a_9 , none of them 0 or ± 1 , so that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = 1 = \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ a_4^2 & a_5^2 & a_6^2 \\ a_7^2 & a_8^2 & a_9^2 \end{vmatrix}$$

In the first edition we attributed this to Basil Gordon. It was asked by Molnar, who only required that $a_i \neq \pm 1$, and did not restrict the order of the determinants to 3. A topological significance was given, with references to Hilton. Solutions by Morris Newman, Peter Montgomery, Harry Applegate, Francis Coghlan and Kenneth Lau appeared, some of orders greater than 3, including several parametric families, for example

$$\begin{vmatrix} -8n^2 - 8n & 2n + 1 & 4n \\ -4n^2 - 4n & n + 1 & 2n + 1 \\ -4n^2 - 4n - 1 & n & 2n - 1 \end{vmatrix}$$

Richard McIntosh gave examples with a high proportion of Fibonacci numbers:

$$\begin{vmatrix} 1167 & 2 & 5 \\ 1698 & 3 & 8 \\ 2866 & 5 & 13 \end{vmatrix} \quad \begin{vmatrix} 610 & 5 & 13 \\ 1054 & 8 & 21 \\ 1665 & 13 & 34 \end{vmatrix}$$

Rudolf Wytek restricted himself to integers > 1 and in the closing days of 1987 used a computer to find

$$\begin{vmatrix} 2 & 3 & 2 & | & 2 & 3 & 5 & | & 2 & 3 & 6 & | & 5 & 7 & 6 & | & 8 & 7 & 8 & | & 10 & 7 & 12 \\ 4 & 2 & 3 & | & 3 & 2 & 3 & | & 3 & 2 & 3 & | & 6 & 4 & 7 & | & 12 & 11 & 7 & | & 4 & 2 & 7 \\ 9 & 6 & 7 & | & 9 & 5 & 7 & | & 17 & 11 & 16 & | & 17 & 16 & 20 & | & 17 & 15 & 16 & | & 17 & 12 & 20 \end{vmatrix}$$

the second of which had been found earlier by Kenneth Lau. The others were new and are not special cases of any of the parametric solutions. Solutions are evidently more numerous than might at first be thought.

Dănescu, Văjăitu & Zaharescu solve the problem for determinants of any order in which all the elements are $\geq k$ for any given k .

Will the problem extend to cubes?

Alexandru Dănescu, Viorel Văjăitu & Alexandru Zaharescu, Unimodular matrices whose components are squares of unimodular one.

P. J. Hilton, On the Grothendieck group of compact polyhedra, *Fundamenta Math.*, **61**(1967) 199–214.

P. J. Hilton, General Cohomology Theory & K-Theory, *L.M.S. Lecture Notes*, **1**, Cambridge University Press, 1971, p. 58.

E. A. Molnar, Relation between wedge cancellation and localization for complexes with two cells, *J. Pure Appl. Alg.*, **3**(1973) 141–158.

E. A. Molnar, A matrix problem, *Amer. Math. Monthly*, **81**(1974) 383–384; and see **82**(1975) 999–1000; **84**(1977) 809 and **94**(1987) 962.

Sadao Saito, Third-order determinant: E. A. Molnar's problem, *Acta Math. Sci.*, **8**(1988) 29–34; *MR 89j:15031*.

F29 Two congruences, one of which is always solvable.

Given a prime p , find pairs of functions $f(x)$, $g(x)$ such that one of the congruences $f(x) \equiv n$, $g(x) \equiv n \pmod{p}$ is solvable for all integers n . A trivial example is $f(x) = x^2$, $g(x) = ax^2$ where a is a quadratic nonresidue (**F5**) of the odd prime p . Mordell gives the further example $f(x) = 2x + dx^4$, $g(x) = x - 1/4dx^2$, where d is any integer prime to p and $1/z$ is defined as \bar{z} , where $z\bar{z} \equiv 1 \pmod{p}$.

F30 A polynomial whose sums of pairs of values are all distinct.

It was noted in **D1** that no nontrivial solution of $a^5 + b^5 = c^5 + d^5$ is known. In fact x^5 is a likely answer to the following unsolved problem of Erdős. Find a polynomial $P(x)$ such that all the sums $P(a) + P(b)$ ($0 \leq a < b$) are distinct.

F31 An unusual digital problem.

Express the integers in base 4, using the digits 0, 1, 2 and $\bar{1}$ ($= -1$). Let L be the set of integers which can be written in this way using the digits 0, 1 and $\bar{1}$, but not 2. Can every odd integer be written as the quotient of

F31. An unusual digital problem.

two elements of L ? Loxton & van der Poorten show that, given an odd k , there is indeed a multiplier m such that m and km are both in L , but their analysis is ineffective in the sense that they do not know how to estimate the smallest such m . It may be that there is an absolute constant C such that there is always a multiplier less than $|k|^C$. Examples requiring large multipliers are $k = 133 = 2011_4$, $m = 333 = 111\bar{1}_4$ and $k = 501 = 20\bar{1}11_4$, $m = 2739 = 1\bar{1}\bar{1}\bar{1}1\bar{1}_4$.

John Selfridge & Carole Lacampagne ask if every $k \equiv \pm 1 \pmod{3}$ can be written as the quotient of integers which can be represented in base 3 using the digits 1 and $\bar{1}$, but not 0. Experiments suggest that the answer is yes. If we allow the digits 0 and 1, but not 2, then which integers can be written as such a quotient?

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Index of Authors Cited

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- Altonen, M., D9
 Abbott, Harvey L., B2, C9, C14, D3, E1, E10, E11, E12, E28
 Abel, Ulrich, A17
 Abouabdillah, Driss, B24
 Acland-Hood, F4
 Adams, William W., A, A17
 Adena, Michael A., F4
 Adleman, Leonard M., A
 Agarwal, P. K., E20
 Aiello, Walter, C6
 Ajtai, Miklós, C9
 Alanen, Jack, B4, B6
 Alsaighi, Leon, B2
 van Albada, P. J., D11
 Alex, Leo J., D10
 Alexander, L. B., B2
 Alford, W. Red, A13
 Alladi Krishnaswami, B3, B22
 Alles, Peter, E38
 Allouche, J.-P., E16
 Almeling, J. H. J., D19, D21
 Alon, Noga, E12, F3
 Alter, Ronald, B37, D27
 Althoen, Stephen C., D3
 Altweig, M., D20
 Anderson, David Brent, F4
 Andrews, George E., C20, E30
 Ang, D. D., D20
 Anglin, W. S., D3
 Ankeny, Nesmith C., B16
 Ansari, A. R., A17
 Anshel, Michael, D11
 Antoniadis, Jannis A., D26
 Applegate, David, E16
 Archibald, Raymond Clare, A3
 Arno, Steven, A12
 Arshon, S., E21
 Artin, Emil, B16
 Artuhov, M. M., B2, B4
 Atanassov, Krassimir T., B41
 Atkin, A. Oliver L., A3, A8
 Aull, C. E., B2
 Avanesov, E. T., D3
 Babcock, W. C., C11
 Bailey, D. F., B31
 Baillie, Robert, A13, B21, B36
 Baker, T., D15
 Balasubramanian, R., E1
 Ball, W. W. Rouse, F4
 Ballieu, David, B36
 Balog, Antal, A9
 Bang, Theger, A17
 Baragar, Arthur, D12
 Baranov, Valery Ivanov, A17
 Barbeau, Edward J., D11
 Barbette, E., D3
 Barrucand, P.-A., F7
 Bastien, L., D27
 Bateman, Paul T., A3, A17, B2, B25, C20
 Battany, David M., D9
 Battiatto, S., B4
 Baumert, Leonard D., C10, E11
 Baumgartner, James E., E29
 Baxa, Christoph, A17
 Bays, Carter, A4
 Beach, B. D., B16
 Bear, Robert, B2
 Beatty, S., E27
 Beck, Walter E., B2, B5
 Bedocchi, E., A17
 Beeboe, John, F14
 Beckmanns, Laurent, D3, D11
 Beeger, N. G. W. H., A13
 Beeler, Michael D., E10
 Behrend, F. A., E10
 Bell, D. I., D20
 Benze, Mihály, B12
 Benkoski, S. J., B2
 Bennett, Michael, D9
 Berend, Daniel, D25
 Berge, Claude, C18
 Berger, Marc Aron, F13, F14
 Berlekamp, Elwyn Ralph, C7, E10, E26, E27
 Berman, Paul, C18
 Bernstein, Daniel J., E16
 Bernstein, Leon, D8, D11
 Berry, T. G., D19
 Besicovitch, Abram Samoilovitch, D20, E5
 Best, M. R., C10, D7
 Beukers, Frits, F23
 Beutelspacher, Albrecht, E11
 Beyer, Ö, C7
 Beyer, W. A., E31
 Bini, U., D1
 Birch, Bryan J., D1, D27
 Bleicher, M. N., D11
 Blundon, W. J., A6, D5, D18
 Bode, Dieter, B9
 Boesch, F. T., C10
 Bohman, Jan, C1, C20
 Bombieri, Enrico, A8
 Borho, Walter, B4, B7
 Borning, Alan, A2
 Borwein, Peter B., B14
 Rose, R. C., C8, C11
 Boshernitzan, Michael, E27
 Bosma, Wieb, A3, B21
 Boszay, Á. P., C16
 Boyarsky, A., D5
 Boyd, David, E16
 Brakemeier, W., C15
 Brandstein, Michael S., B1
 Bratley, P., B4
 Brauer, Alfred T., B28, C6, C7, F5, F6
 Braun, J., A17
 Braunholz, C. H., E21
 Bremner, Andrew, D1, D2, D3, D8, D18, D19
 Brenner, Joel L., D10
 Brent, Richard P., A3, A8, B1
 Brenton, Lawrence, D28
 Brestovansky, Walter, E11
 Breusch, Robert, B14, D11
 Briggs, William E., C3
 Brillhart, John David, A3
 Brindza, B., D25
 Brocard, H., D25
 Bromhead, H., D18
 Brouwer, Andries E., C10
 Brownkin, Jerzy, B19, E25
 Brown, Alan L., B2
 Brown, B. H., B4
 Brown, Ezra, B2, D18
 Brown, H., F27
 Brown, J. L., D11
 Brown, Martin Lawrence, D24
 Brown, Thomas Craig, E10, E21
 Brudno, Simcha, D1
 Bruen, A., C18
 Brzeziński, Juliusz, B19
 Buchholz, Ralph Heiner, D21, D22
 Buck, R. Creighton, A17
 Buell, Duncan A., A3, B21
 Bugulov, E. A., B2
 Buhrer, Joseph P., A2, D2
 Burbacka, E., B34
 Burnside, William, D18
 Burstein, N., F14
 Buschman, Robert G., C3
 Cadwell, J. H., A8
 Calkin, Neil J., E32
 Callan, David, B2
 Cameron, Peter J., E32
 Campbell, Paul J., D11
 Cao Zhen-Fu, D3, D9, D23, D28
 Carlitz, Leonard, B44
 Carmichael, Robert Daniel, A12, A13, B2, B39
 Case, Janell, B36
 Cassels, John William Scott, C20, D5, D12, D27, F, F22
 Catalán, E., B6
 Catlin, P. A., B41
 Cattaneo, Paolo, B2
 Cauchy, Augustin-Louis, C15
 Cayley, Arthur, E37
 Challis, M. F., C12
 Chan Wah-Keung, D28
 Chandra, Ashok K., E10
 Chang, S. J., D11
 Chein, E. Z., D18
 Chellali, M., D2
 Chen, Sheng, C11
 Chen Jing-Run, A4, A8, C1, F1

- Chen Wen-De, C11
 Cheng Yuanyou, F15
 Chinburg, Ted, B25
 Choi, S. L., B26, C14, E11, F13
 Choudhry, Ajai, D1
 Chowla, Sardom D., A6, B16, C9, D4, E28
 Churchhouse, R. F., F13
 Chvátal, Václav, E10
 Cilleruelo, Javier, C9, E1, F1
 Cipolla, M., A12
 Cockayne, Ernest T., C18
 Cohen, Fred, A19, F13
 Cohen, Graeme L., B1, B2, B3, B37
 Cohen, Robert, D11
 Cohn, Harvey, D12, F7
 Cohn, J. H. E., D6, D26
 Cole, George Raymond, D21
 Colman, W. J. A., D18
 Colquitt, W. N., A3
 Comtet, Louis, F17
 Conway, John Horton, A17, C7, C8, C10, D18, E17, E26, E27, E31, F26
 Cook, T. J., B3
 Coppersmith, Don, D2, E30
 Córdoba, Antonio, C9
 Cormack, G. V., B21
 van der Corput, J. G., C1
 Corzatt, C. E., F4
 Costello, Patrick, B4
 Cottrell, A., C6
 Coxeter, Harold Scott MacDonald, D18, F4
 Cragg, D., F4
 Cramér, Harald, A8
 Crandall, R. E., A2, D2, E16
 Cremona, John E., D3
 Crews, Philip L., B2
 Crittenden, Richard B., E23
 Crocker, R., A19
 Croft, Hallard T., F4
 Cross, J. T., B2
 Cucurezeanu, Ion, D3
 Cudakov, N. G., C1
 Curtis, Frank, C7
 Curtiss, D. R., D11
 Curtz, T. B., D27
 Cusick, Thomas W., D12, F20
 Dănescu, Alexandru, F28
 Dardis, J. A., D1
 Davenport, Harold, A8, C15, D4, E20, F5, F22
 Davidson, J. L., E16
 Davis, James A., E10, E33
 Daykin, David E., D20
 Dean, Richard A., E21
 DeBoer, Jennifer L., B3
 Dekking, F. M., E21, F31
 Delorme, Jean-Joël, D1
 Dem'janenko, V. A., D1
 Deshouillers, Jean-Marc, A19, C1
 Deuber, Walter, E14
 Devitt, John Stanley, B6
 Dewar, James, F14
 Diananda, Palahenedi Hewage, C14
 Dias da Silva, J. A., C15
 Dickson, Leonard Eugene, A17, B6, D12, D18, D27, E32
 Ding Xia-Xi, C1
 Dirksen, P. H., E20
 Dixmier, Jacques, C7
 Dixon, R., C18
 Djawadi, Mehdi, C12
 Dodge, Clayton W., D19
 Doig, Stephen K., B49
 Dolan, Stanley W., D5
 Dove, Kevin L., D22
 Downey, Peter J., E31
 Drago, Antonino, F19
 Draizin, David, B16
 Drucker, Daniel S., F28
 Dubitskas, A. K., F23
 Dubouloz, Harvey, A2, A3, A13
 Dubouis, D1
 Dudeney, Henry Ernest, D3, E34, F4
 Dudley, Underwood, A17
 Duke, W., B16
 Duparc, H. J. A., A13
 Ecker, Michael W., B36
 Eckert, Ernest J., D21
 Ekklund, Earl F., B23, B33
 Edgar, Hugh Maxwell Wallace, D8
 Edwards, Harold M., D2
 Eggleston, Harold Gordon, D21
 Eggleton, Roger Benjamin, B23, D19, E27
 Ehrman, John R., A3
 Eliahu, Shalom, E16
 Elkies, Noam D., B19, C8, D1
 Elliott, D. D., A17
 Elliott, Peter D. T. A., A5
 Ellis, David, A17
 Ellison, W. J., D5, F23
 Engel, Marcus, D18
 Entringer, Roger C., E10, E21, E33
 Erdős, Pál, A5, A6, A9, A11, A12, A13, A18, A19, B2, B4, B6, B8, B9, B10, B14, B16, B18, B21, B22, B23, B24, B26, B27, B30, B31

- B32, B33, B35, B36, B39, B40, B41, B42, B46, C3, C6, C7, C8, C9, C10, C12, C15, C16, C17, D2, D3, D4, D7, D11, D17, D25, E, E1, E2, E3, E4, E5, E9, E10, E11, E21, E24, E28, E31, F2, F13, F14, F18
 Ernvall, Reijo, A17, D2
 Escott, E. B., B4
 Estermann, Theodore, C1
 Evans, Ronald J., B28, C10, D19
 Evdokimov, A. A., E21
 Everett, C. J., E16
 Everts, F., E10
 Ewell, John A., C5, C20, F17
 Fabrykowski, J., B18
 Falconer, Kenneth J., F4
 Fassler, Albert, D21
 Faulkner, Marilyn, B33
 Fejes Tóth, László, F
 Feizenabaum, Alexander Gersh, F13, F14
 Fife, Earl Dennet, E21
 Filippini, P., E16
 Fliz, Antonio, C1
 Finch, Steven R., C4, E32
 Fine, Nathan J., D19, E25
 Finkelstein, Raphael, D3
 Flammenkamp, Achim, B7, F4
 Flatto, Leopold, E18
 Flood, P. W., D18
 Forbes, A., B5
 Forman, William, E19
 Forman, Robin, A17
 Foster, Lorraine L., B39, D10, D16
 Fourny, É., A8
 Fraenkel, Aviezri S., C5, D8, E27, F13, F14
 Franceschini, Nicola, D11
 Franqui, Benito, B2
 Fredricksen, Harold, E11, E12
 Freiman, G. A., C15
 Freud, Robert, C9, E30
 Frey, H. A. M., B3
 Friedlander, John B., A8, A9, A12
 Friedman, Charles N., D11
 Frobenius, G., D12
 Froberg, Carl-Erik, C1, C20
 Fuchs, W. H. J., C9
 Funar, Louis, C14
 Fung, Gilbert W., A1, A17
 Furedi, Zoltan, E20
 Furstenberg, H., E10
 Gale, David, E15, E17
 Gallagher, Patrick X., A19
 Gallian, Joseph A., C13
 Gamble, B., C18
 Gandhi, J. M., A17
 Garcia, Mariano, B2, B4
 Gardiner, Verna L., C3, D5
 Gardner, Martin, A1, A2, C8, D3, D18, D21, F4
 Gardy, D., E20
 Garner, Lynn E., E16
 Garrison, Betty, A17
 Gelfond, A. O., D9
 Genocchi, A., D27
 Gérardin, A., D1, D27
 Gerver, Joseph L., A5, E10
 Gibbs, Richard A., C10
 Giese, R. P., C6
 Gill, C., D15
 Gillard, P., B36
 Gillies, Donald B., A3
 Gilmer, Robert, B16
 Gioia, A. A., B4, C6
 Giuga, Giuseppe, A17
 Gloden, A., D1
 Godwin, Herbert James, A2, D27
 Goetgheluck, P., A17
 Golay, Marcel J. E., C10
 Goldfeld, Dorian, D11
 Goldstein, Richard, F4
 Goldston, Daniel L., A8, C1
 Golomb, Solomon W., A2, A17, B42, C18, D11
 Golubev, V. A., A5
 Goodstein, Reuben Louis, A17
 Gordon, Basil, C5
 Gordon, Daniel M., A12
 Gorzkowski, Waldemar, D4
 Gosper, R. William, E16, F17
 Gostin, Gary B., A3
 Gottschalk, W. H., E21
 Gouyon-Beauchamps, D., E20
 Gould, Henry W., A17
 Graham, Ronald Lewis, A3, B23, B33, C7, C9, C10, C20, D11, E2, E10, E25, E27, E30, E31, E33, F14
 Graham, Sidney W., A4, B3, C11
 Granlund, Torbjörn, D18
 Granville, Andrew, A4, A9, A13, A19, B33, B41, C1, C20, D2, F1
 Greenberg, Harold, C7
 Greenwell, Raymond N., E10
 Grimm, Charles A., B32
 Grinstead, Charles M., B22, D3
 Griswold, Ralph E., E31
 Grosswald, Emil, A5, C20
 Grupp, F., A8

- Gryte, D. G., D26
 Guillaume, D., A13
 Güntsche, R., D22
 Guo Zhi-Tang, D10
 Gupta, Rajiv, F9
 Gupta, Hansraj, B33, C8
 Gurak, S., A12
 Guy, Andrew William Peter, B6
 Guy, Michael John Thirian, F26
 Guy, Richard Kenneth, A, A3, A17, B6,
 B8, B13, C1, C7, C8, C20, D1,
 D19, E26, E27, E37, F2, F4, F4,
 F26
 Györy, K., D7
 Haentzschel, E., D22
 Haggis, Peter, A5, B1, B2, B3, B4, B5, B8,
 B37, B39
 Hahn, H. S., D11
 Hahn, Liang-Shin, D11
 Haight, J., F13
 Hajela, D., C11
 Hajnal, Péter, E20
 Hajóssy, G., E10
 Halberstam, Heini, A5, C9, E
 Hales, Alfred W., E10
 Hall, Marshall, C10, E25
 Hall, Richard R., B36, B41, F4
 Halter-Koch, Franz, C20
 Hamidoune, Yahya Ould, C15
 Hammer, Joseph, F
 Hämmerer, N., C12
 Hansche, B., C18
 Hansen, W., C6
 Hanson, Denis, D3, E10, E11
 Harborth, Heiko, B28, D3, D20, F4
 Hardy, B. E., B2
 Hardy, Godfrey Harold, A1, A17, D4
 Harman, Glyn, A8
 Harris, Vincent C., A17
 Hart, S., E20
 Härtter, E., A17
 Haugland, Jan Kristian, A8
 Hausman, Miriam, B41
 Hawkins, David, C3, E21
 Heath-Brown, D. Roger, A1, A4, A5, B16,
 B18, D5, F9
 Hebb, Kevin, C6
 Hedetniemi, Stephen T., C18
 Hedlund, Gustav A., E21
 Heilbron, Hans, C15
 Helm, Martin, C11
 Hendy, M. D., B2
 Henriksen, Melvin, B25
 Hensley, Douglas, A9
- Heppner, E., E16
 Herschfeld, Aaron, D9
 Herstein, I. N., E16
 Herzberg, Norman P., D12
 Heuer, Karl W., F4
 Higgins, Olga, A17
 Higgins, Robert N., B36
 Hildebrand, Adolf, A9, B18, C20, F6
 Hill, Jay Roderick, A13
 Hilton, Peter J., F28
 Hindman, Neil, E29
 Hoffmann, H., B4
 Hofmeister, Gerd, C12, D11
 Hofstadter, Douglas R., E31
 Hoggatt, Verner E., D26
 Holton, Derek A., F4
 Holzman, R., F14
 Hooley, Christopher, A19, B39, B40, F9
 Horn, Roger A., A17
 Hornbeck, B., B2
 Hudson, Richard H., A4, F5
 Hudson, W. H., D19
 Huennemann, Joel, A13
 Huff, G. B., C18, D20
 Hunsucker, John L., B9, B13
 Hurwitz, Adolf, D12
 Hurwley, Martin, A8, F1
 Ibstedt, Henry, E15
 Il'in, A. M., C6
 Inkeri, K., D2, D9
 Irving, R. W., E11
 Isenkrath, C., A17
 Ivic, Aleksandar, B16, F1
 Iwaniec, Henryk, A1, A8, A9, B16, B39,
 F1
 Jabotinsky, Eri, C3
 Jackson, D. E., E21
 Jackson, T. H., F4
 Jaeschke, Gerhard, A12, A13, B21
 Jerrard, R. P., B2
 Jewett, R. I., E10
 Jia Chao-Hua, A8, B16
 Jia Xing-De, C9, C11, C12
 Johnson, Allan William, D11
 Johnson, G. D., A3
 Johnson, Wells, B2, D2
 Joint, W. Howard, D18
 Jones, James P., A6, A17
 Jones, Patricia, B36
 Jönsson, Ingemar, B20
 Joo, I., A12, A13, D11
 Jordan, James H., A16, D21, F13
 Judd, J. S., A

- Justin, J., E21
 Jutila, Matti, A4
 Kac, Mark, B14, E31
 Kahn, Steven, A17
 Kalbfleisch, J. G., C10
 Kalyamanova, K. È., D22
 Kang Ji-Ding, D7
 Kanold, Hans-Joachim, B2, B4, B9
 Kaplansky, Irving, D1, E16
 Karst, Edgar, A5, A17
 Kashihara, Kenji, D23
 Katayama, Shin-ichi, D23
 Kato, H., C6
 Katz, M., C17
 Kay, David C., E16
 Keller, Ott-Heinrich, F
 Keller, Wilfrid, A3, A13, B20, B21
 Kellogg, O. D., D11
 Kelly, John B., D1, F19
 Kelly, Patrick A., F4
 Kemnitz, Arnfried, D19, D20
 Kemp, Paula A., F15
 Kendall, David G., B2
 Kenney, Margaret J., C1
 Khinchin, A. Y., E10
 Kilgrove, Raymond B., A10
 Kim Su-Hee, A12
 Kingsley, R. A., D26
 Kirfel, Christoph, C12, C12
 Kishore, Masao, B1, B2, B37
 Kiss, Péter, A12, E31
 Klamkin, Murray S., B2
 Klärner, David A., C18, E36
 Klee, Victor L., B36
 Klein, Rivka, F15
 Kleitman, Daniel J., E12
 Klotz, Walter, C12
 Klave, Torleiv, C11, F4
 Knapowski, Stanislaw, A4
 Knödel, W., A13
 Knopfmacher, John, A17
 Knuth, Donald Ervin, A3, C6
 Ko, Chao, D5, D11, D13
 Kobayashi, Masaki, A17
 Kolesnik, G. A., A1
 Kolodorf, H., C12
 Komjáth, Péter, E20
 Komlós, János, C9, E11, F4
 Korself, A., A13
 Koyama, Kenyi, D5
 Kraitchik, Maurice, A3, C18, D18, D25
 Krasikov, I., E16
 Kravitz, Sidney, A2, A3, B2
 Krückeberg, F., C9
- Krukenberg, C. E., F13
 Kubíček, Jan, D1
 Kuipers, Lauwers, A17
 Kurepa, Đuro, B44
 Laatsch, Richard, B2
 LaBar, Martin, D15
 Laborde, M., A8
 Lacampagne, Carole, B31, D3
 Lagarias, Jeffery C., A8, A17, E8, E16,
 E39
 Lagrange, Jean, D4, D15, D18, D20, D27
 Lal, Mohan, A6, B5, B8, B36, D5, D18
 Lalout, Claude, A7
 Lambel, Joachim, E27
 Lander, Leon J., D1
 Landman, Bruce M., E10
 Lang, Serge, B19
 Langevin, Michel, B32, D9
 Langmann, Klaus, A17
 Laub, Moshe, D3
 Lazarus, R. B., C3, D5
 Le Mao-Hua, B38, D9, D10
 Leibenson, Kenneth, B24
 Lee, Elvin J., B4
 Leech, John, A4, A9, C10, D1, D18, D20,
 E21
 Lehmer, Derrick Henry, A3, A7, A12, A13,
 A17, B6, B29, B37, F1, F6, F31
 Lehmer, Derrick Norman, D20
 Lehmer, Emma, B24, B45, D2, F6
 Leitmann, D., A1
 Lekkerkerker, Cornelius Gerrit, F
 Lenstra, Arjen K., A, A3
 Lenstra, Hendrik Willem, A, A3, B6
 Levin, Leonard, F17
 Levine, Eugene, E28
 Lewin, Mordechai, C7
 Lewis, Ethan, F14
 Li An-Ping, C11
 Li De-Lang, D11
 Li Xiao-Ming, C10
 Lieuwens, E., B37
 Lin Chio-Shih, E27, F14
 Lin Shen, C20, E27, F14
 Lind, D. A., D3
 Lindström, Bernt, C8, C9, C11
 Linnik, U. V., A4
 van Lint, Jacobus H., D11
 Lioen, Walter M., A3, D5
 Littlewood, John Edensor, A1, D4, F22
 Liu, A. C., E10
 Liu Hong-Quan, A1, A8, B16
 Liu Jian-Min, A4
 Liverance, Eric, A17

- Ljunggren, W., D3, D6
 Löh, Günter, A7, A13
 Lord, Graham, B5, B9
 Lossers, O. P., D3
 Lou Shi-Tuo, A8
 Low, Lewis, C15, F27
 Luxton, John H., F31
 Lucas, Edouard, D3
 van de Lune, J., C1, D7
 Lunnon, W. Fred, B4, C8, C12, E21
 Luo Ming, D26
 Lyness, Robert Cranston, D18
- Ma De-Gang, D3
 Macdonald, Sheila Oates, E20
 MacMahon, Percy A., E30
 MacWilliams, F. Jessie, C10
 MacLachy, Joseph S., B4, E34
 "Mahatma", D18
 Mahler, Kurt, D4, E18, F26, F27, F31
 Maier, Helmut, A8, A9, B9, B36, E3
 Mąkowski, Andrzej, A12, A15, B2, B5, B9,
 B16, B19, B25, B42, D5, D10
 Mallowin, Colin L., E31
 Malm, Donald E. G., A19
 Manasse, Mark S., A3
 Mann, Henry B., A17, C10, C15
 Mansfield, Richard, C15
 Marcus, Daniel, E25
 Markoff, A., D12
 Masai, Pierre, B39
 Mason, T. E., B2
 Massias, J.-P., A17
 Mathieu, D1
 Matiyasevich, Yuri V., A17
 Matthews, K. R., E16
 Mattics, L. E., C20, D3
 Mauldon, James G., D16, D19
 McDaniel, Wayne L., A12, B2, B16, B49
 McIntosh, Richard John, B31
 McKay, John H., B4
 McLaughlin, Philip B., A3
 Meeus, Jean, A7
 Mendelsohn, Nathan S., B21
 Mender France, Michel, F31
 Metropolis, N., C3
 Metsäkylä, T., D2
 Mian, Abdul Majid, E28
 Miech, R. J., B37
 Mientka, Walter E., A19, B13, E21
 Mignotte, Maurice, A1, D9, D23
 Mijalović, Ž., B44
 Miller, G. L., A
 Miller, Jeffrey Charles Percy, C10, D5
 Miller, Kathryn, B36
- Miller, V. S., A17
 Mills, William H., A17, B2, B41, D13, E20,
 F6
 Minoli, Daniel, B2
 Mirsky, Leon, B18, E11
 Misurierewicz, M., D24
 Mittelbach, F., D11
 Mo De-Ze, D10
 Moewe, David, B7
 Moewe, Paul C., B7
 Mohanty, Shreedhara Prasada, D3
 Möller, Herbert, E16
 Molin, Richard A., A17, B16
 Molnar, E. A., F28
 Monsky, Paul, D27
 Montgomery, Hugh L., A9, B40, C1
 Montgomery, Peter Lawrence, A, A3
 Moore, Eliakim Hastings, D18
 Moran, Andrew, A5
 Mordell, Louis Joel, B16, D6, D18, D20,
 D27
 Moree, Pieter, C14, D7
 Morikawa, Ryozo, F14
 Moroz, B. Z., B16
 Morse, Marston, E21
 Morton, Patrick, D2
 Moeser, Leo, A17, B33, C2, C12, C14, C17,
 D7, E1, E10, E11, E27
 Mossige, Svein, C12, E32
 Motzkin, Theodor S., C17
 Mozzochi, C. J., A8, F1
 Mrrose, Arnulf, C12
 Muir, Thomas, E37
 Müller, Helmut, E16
 Müller, P., C4
 Mullin, Ronald C., C10, E20
 Murata, Leo, F9
 Murdeshwar, M. G., C17
 Murty, Maruti Ram, F9
- Nagell, Trygve, D9, D11
 Nar, M., B18
 Najar, Rudolph M., B2, B5
 Nakamura, Shigeru, B2
 Namboodiripad, K. S., A17
 Narkiewicz, Włodysław, A19, C1
 Nash, John C. M., C11
 Nathanson, Melvyn B., C9, C12, C15
 Naur, Thorkil, A3
 Nebb, Jack, B13
 Neill, T. B. M., A17
 Nelson, Harry L., A6
 Newberry, R. S., B3
 Newman, David, E31
 Newman, Donald J., C6, D11

- Newman, Morris, F14
 Nicol, Charles A., B41, D2
 Nicolas, Jean-Louis, D15
 Niebuhr, Wolfgang, A13
 Nitaj, Abderrahmane, B19
 Niu Xue-Feng, C7
 Niven, Ivan, A17, B41, D11
 Noda, Kazunari, D27
 Noll, Landon Curt, D20
 Norrie, R., D1
 Novák, Bretislav, F14
 Nowakowski, Richard Joseph, E37
- Obálka, R., D2, D11, D25
 Odda, Tom, E33
 Odlyzko, Andrew M., A10, A17, B21, C10,
 E8, E10, F4
 Oertel, Philipp, F4
 Oltikar, Shan, B49
 Olsen, John E., C15
 Ondrejka, Rudolf, A3
 O'Neill, Patrick E., E10
 Oppenheim, Alexander, D8
 Ore, Oystein, A17, B2, B4
 Ortega Costa, Joaquín, A17
 Osgood, Charles F., D25
 Ostmann, H., E
 O'Sullivan, Joseph, E28
 Overhaugen, T., D25, F1
 Owings, J. C., D11
- Pajunen, Seppo, B2
 Palamá, G., D11
 Pall, Gordon, C20
 Pan Cheng-Dong, C1
 Pan Cheng-Tung, A4
 Papadimitriou, Makis, A17
 Parady, Bodo K., A8
 Parkin, T. R., A6, D1
 Patterson, Cameron Douglas, C20
 Paxson, G. Aaron, B6
 Peeples, W. D., D20
 Pelikán, József, E38
 Penk, M. A., A2
 Penney, David E., A2
 Peralta, René, F6
 Perelli, A., C1
 Perrine, Serge, D12
 Peterkin, C. R., E20
 Peterson, Blake E., D21
 Peterson, Evans, B49
 Phillips, Steven, E30
 Phong Bui-Ming, A12, A13
 Piekarzky, J., B34
 Pillai, S. Sivasankaranarayana, B28, D9
- Piltjai, G. Z., A8
 Pinch, Richard G. E., A13
 Pinner, C., B18
 Pintér, Ákos, D3
 Pintz, János, C1, F4
 Pitman, Jane, C15
 Plaksin, V. A., C20
 Pleasants, P. A. B., E21
 Pocklington, H. C., D18
 de Polignac, A., A19
 Pollack, Richard M., A3, D25
 Pollard, John M., A, C15
 Pomerance, Carl, A, A1, A4, A5, A8, A12,
 A13, B2, B4, B9, B18, B32, B36,
 B37, B39, B41, B42, B46, C1,
 D19, E, E10, F4, F15
- Popken, J., F26
 Porta, Horacio A., E16
 Porubský, Štefan, C20, E30, F14
 Posner, Edward C., D9
 Post, Karel, E36
 Potler, Aaron, A8
 Poulet, Paul, B2, B4, B6
 Pounder, J. R., C12
 Powell, Barry J., B45
 Prachar, K., A4
 Prellberg, Thomas, F4
 Pritchard, Paul, A5
 Prodinger, Helmut, E21
 Propp, James Gary, E10
 Proth, P., A10
 Prothro, E. T., D21
 Purdy, George B., C20
 Pyateckii-Sapiro, I. I., A1
- Queneau, Raymond, C4
- Rabung, John R., A16, E10
 Radó, Richard, E10, E14, E36
 Raitzin, Carlos, A17
 Ralston, Kenneth E., A10, C17
 Ramachandra, K., B32
 Ramaré, Olivier, B33
 Ramsey, L. Thomas, A5, E10
 Rangamma, M., B37
 Rankin, Robert Alasdair, E10
 Rassias, Themistocles Michael, E25
 Rathbun, Randall L., D18, D21
 Rawsthorne, Daniel A., E16, F26
 Recaman, Bernardo, C4
 Reddy, D. Ram, B3
 Regimbald, Stephen, A17
 Reidlinger, Herwig, B2
 Remak, Robert, D12
 Reunie, B. C., E20

- Rényi, Alfred, A6
 Rhemtulla, A. H., E12
 Ribenboim, Paulo, A, D2
 Ribet, Kenneth, D2
 Richards, Ian, A9
 Richert, Hans-Egon, A5, C20
 Rickert, N. W., A3, A8
 Rickert, U.-W., C15
 Riddell, James, C12, E10
 Rieger, G. J., B4
 te Riele, Herman J. J., A3, B1, B2, B4, B6, B8, C1, D5, D7
 Riesel, Hans, A, B20, C20
 Rivat, J., A1
 Rivest, R., A
 Robbins, Herbert E., E21
 Robbins, Neville, B2, D26
 Roberts, J. C., C7
 Roberts, S., D27
 Robin, G., A17
 Robina, Gabriel, D16
 Robinson, Raphael M., B21, E15
 Rödine, Arne, C12
 Radetsch, Øystein J., C7, C12, C15
 Rogers, Claude Ambrose, F
 Rohrbach, H., C12
 Rokowska, B., F11
 Root, S. C., A5
 Rosati, L. A., D11
 Roselle, David P., E20
 Rosenberger, Gerhard, D12
 Rosenstiel, C. R., D1
 Rosenstiel, E., D1
 Ross, P. M., C1
 Rosser, J. Barkley, A17
 Roth, Klaus F., C9, E, E10, F4
 Rothschild, Bruce L., E10
 Rotkiewicz, Andrzej, A3, A12, A13, D10
 Rubinstein, Michael, A17
 Ruderman, Harry D., B31
 Rumney, Max, B15
 Rumsey, Howard, D9
 Russell, W., D5
 Russza, Imre Z., B33, C9, C15, E1
 Ryavec, Charles, C15
 Saito, Sadao, D28, F28
 Salem, R., E10
 Saléi, H., E10
 Sander, J. W., B33, D11, E16, F15
 Sándor, József, B3, B37, B42
 Sárközy, András, A12, B18, B19, B33, C16, E, E2, E9
 Sastry, K. R. S., D21
 Satoh, Daishiro, A17
 Satyanarayana, M., B2
 Sawyer, Walter Warwick, D18
 Sayers, M. D., B1
 Scarowsky, W., D5
 Schatz, J. A., E21
 Scheidler, Renate, B33
 Schinzel, Andrzej, A, B9, B18, B25, B27, B34, B36, B42, B47, D5, D10, D11, D23, E20, F11, F13
 Schlaifly, Aaron, B39
 Schmidt, Peter Georg, B16
 Schmidt, Wolfgang M., E10, F4
 Schnitzer, F., C17
 Schoenfeld, Lowell, A17
 Scholz, Arnold, C6
 Schönheim, Johanan, C10, E11, F14
 Schrandt, R. G., E31
 Schroeppel, Rich, E16
 Schubert, H., D21
 Schulz, Fred., B37
 Schultz, O., D22
 Schur, Issai, B33, E11
 Schwarz, Hermann Amandus, D11
 Scott, Reese, D9
 Sebastian, J. D., C18
 Sedláček, Jiří, D11
 Segal, David, B31
 Segal, Sanford L., B37, D8
 Selfridge, John Lewis, A3, A12, A19, B6, B21, B23, B27, B31, B32, C5, C17, C18, C19, D1, D2, D17, F6, F13
 Selmer, Ernst S., C7, C12, E32
 Selvik, Björn Kristin, C12
 Sentance, W. A., B16
 Serf, P., D27
 Shallit, Jeffrey, E16
 Shamir, A., A1
 Shanks, Daniel, A, A1, A3, A4, A8, A17, B13, D2
 Shanaram, R., E10
 Shapiro, Harold N., B2, B41, D25, E19
 Sharir, Micha, E20
 Shearer, James B., E8
 Shelah, Saharon, E10
 Shen, Mok-Kong, A12, C18
 Shen Tsuen-Pao, C18
 Sheng, T. K., D20
 Shepherd, B., C18
 Shiu, Peter, B16, B18
 Shor, P., E20
 Shorey, T. N., B32, D10
 Siebert, Hartmut, A17
 Sierpiński, Waclaw, A, A1, A5, A17, B2, B18, B21, B36, D5, D8, D11,

- D18, D22, D23, E25
 Silverman, Joseph H., D1, D12
 Silverman, Robert D., A3
 Simmons, Gustavus D., D25, E10, E33
 Simpson, R. Jamie, E10, E23, F13, F14, F15
 Singer, J., C9
 Singer, M., A17
 Singmaster, David Breyer, D3, D11
 Sinisalo, Matti K., C1
 Sirota, E. R., A1
 Sitaramachandra Rao, R., B16
 Siva Rama Prasad, V., B3, B37
 Skinner, Christopher M., D10
 Slater, M., C18
 Slater, Peter J., C10
 Sloane, Neil J. A., C9, C10, F25
 Smith, Herschel F., A9
 Smith, Joel F., A8
 Smith, Paul, C8
 Smyth, C. J., D1
 Solovay, R., A
 Somos, Michael, E15
 Sompolski, R. W., D2
 Sonnag, Rolf, C6
 Sorenson, Jonathan, A
 Sős, Vera Turán, C9, E9
 Soundarajan, K., E1
 Spencer, D. C., E10
 Spencer, Joel H., E2, E10
 Spencer, P. H., C18
 Spiro, Claudia, B41
 Spohn, W. G., D18
 Sprague, Roland Percival, C20
 Srinivasan, B. R., A17
 Srinivasan, Seshaiah, F9
 Stanley, Richard P., E10
 Stanton, Ralph G., B21, C10, E20
 Stark, Harold M., D26
 Starke, Emory P., D24
 Stauduhar, Richard, F11
 Stechkin, Boris S., A17
 Steen, Adolf, E37
 Stein, M. L., C1
 Stein, P. R., C1, D5
 Stein, Sherman K., D11, F14
 Steinberg, Robert, C14
 Steiner, Ray P., D6, D26, E16
 Stemmler, Rosemarie M., B25
 Stemple, J. G., B4
 Stephens, A. J., B16, B25
 Stephens, Nelson M., D27
 Stevens, R. S., E10
 Stewart, Bonnie M., D11
 Stewart, C. L., B19
 Stöhr, Alfred, C12, E
 Stolarsky, Kenneth B., C6, E16, F4
 Stoll, Peter, D11
 Strassen, Volker, A
 Strauss, Ernst G., B2, B17, B33, C5, C6, C16, D11, E24
 Street, Anne Penfold, C14, E11, E12
 Stroeker, Roelof Jacobus, D11
 Styer, Robert, D10
 Subba-Rao, K., D1
 Subbarao, Matukumalli V., B2, B3, B17, B18, B37, B39, C6
 Sudbery, A., F4
 Suganamra, M., B15, C6
 Sulyok, Miklós, E11
 Summers, T., B8
 Summer, John L., D22
 Sun, Chi, D11
 Sun Qi, B47, D5, D13
 Suryanarayana, D., B2, B3, B9, B16, B17
 Suyama, Hiromi, A3
 Swierczkowski, S., C17
 Swinnerton-Dyer, Henry Peter Francis, D1, D9, F22
 Sylvester, James Joseph, B33, C7, D11
 Szabó, Zoltán István, E10
 Szalay, Michael, F9
 Szegedy, Márk, F15
 Szekeres, Esther, E11
 Szekeres, George, B16, B31, E11
 Szemerédi, Endre, C9, C15, D4, E, E2, E10, E11, E20, F4, F18
 Szymiczek, Kazimierz, A12
 Tait, Peter Guthrie, E37
 Tallman, M. H., D26
 Tanner, Jonathan W., D2
 Tanny, Stephen M., E31
 Taylor, Herbert, C10, C18
 Tee, Garry J., A17
 Temperley, Nicholas, B2
 Tempier, Mark, A2
 Tenenbaum, Gérard, D11, E3
 Terras, Riba, E16
 Tetali, Prasad, C9
 Teuffel, E., A17
 Thatcher, Alfred R., D15
 Thiele, Torsten, F4
 Thompson, John G., A17
 Thouvenot, J. P., E10
 Thue, Axel, E21
 Thurber, Edward G., C6
 Tijdemans, Robert, B19, B32, D7, D9, D10, D11, E18, F23
 Tiller, G., B8

- Tovey, Craig A., D3
 Trigg, Charles W., A9
 Trost, E., D24
 Trusov, Ju. D., D1
 Tsangaritis, P. G., A17
 Tunnell, Jerry D., B27
 Turán, Pál, A4, A5, A11, C9, E, E10
 Turgeon, Jean M., B24
 Turk, Jan, B30
 Tzanakis, Nikos, D3, D6
 Uchiyama, Saburō, A8, A19, D3
 Ulam, Stanislaw M., C3, C4, E31
 Urbanek, Friedrich J., E21
 Urbanowicz, Jerzy, D7
 Utz, W. R., B33, C6
 Vaidya, A. M., B4
 Väjäitö, Viorel, F28
 Valette, Alain, B39
 Vandenhende, Charles L., A17, B16, E23, F14
 van der Poorten, Alfred J., A13, D2, F17, F31
 Vandiver, Harold S., D2
 Vantieghem, E., A17
 Vardi, Ilan, C18, E16, E25
 Varnavides, P., C14
 Vaughan, Robert C., A9, A19, B40, C1, D11
 Vegh, Emanuel, F9
 Venturini, G., E16, E17
 Vinogradov, I. M., C1
 Vogt, R. L., B13
 Voorhoeve, M., D7
 Vucenic, W., C18
 Vulah, L. Ja., D12
 Wada, Hideo, A17, D27
 van der Waerden, B. L., E10
 Wagon, Stanley, B39, E16
 Wagstaff, Samuel S., A3, A12, A13, D2
 Wakulicz, A., B36
 Wald, Morris, C1
 Walker, David T., B16
 Wall, Charles R., B2, B2, B3, B8, B36, B41, D26
 Wall, David W., B37
 Wallis, Jennifer Seberry, E11
 Wallis, Walter Denis, E11
 Walsh, P. Gary, B16
 Wang Du-Zheng, D9
 Wang, Edward T. H., C1, C14, E12
 Wang Tian-Ze, A8, C1
 Wang Yan-Bin, D23
 Wang Yuan, C1
 Wang Wei, A4
 Ward, Morgan, D1
 Warren, L. J., B3
 Watson, George Neville, D3
 Watts, A. M., E16
 Webb, William A., D11
 Weber, J. M., B3
 de Weger, Benne M. M., D3, D10, D26
 Weinberger, Peter, A3
 Weintraub, Sol, A5
 Weiss, A., E16
 Weitzenkamp, Roger C., A19
 Welsh, L., A3
 Wen Zhang-Zeng, D28
 Werebrusow, A. S., D1
 Whitehead, Earl Glen, E11
 Wichmann, B., C10
 Wiens, Douglas, A17
 Wilansky, Albert, B49
 Wild, K., F4
 Wild, Herbert S., D11
 Williams, C. P., A17
 Williams, E. R., E14
 Williams, Hugh Cowie, A, A1, A3, A13, A17, B16, B21, B33, D2, D26, F7
 Winter, David, F4
 Winter, Dik, A3
 Wirsing, Edward, B2
 Wittsenhausen, Hans S., E2
 Witt, E., E10
 Woods, Dale, A13, B4
 Woodbridge, K. R., B39
 Woollett, M. F. C., D5
 Wormell, C. P., A17
 Wright, Edward Maitland, A17, C20
 Wu, Jie, A8
 Wu Yun-Fei, D10
 Wunderlich, Marvin C., B6, B8, C3, C4, D8
 Wyburn, C. T., C6
 Xie Sheng-Gang, A9
 Xu Yi-jing, F1
 Xu Z.-Y., D3
 Yamada, Masaji, E16
 Yamamoto, Koichi, D11
 Yao, Qi, A8
 Yates, Samuel, A3, B49
 Yau, Stephen S.-T., F1
 Yip L.-W., B39
 Yokota, Hisashi, D11
 Yorinaga, Masataka, A12, A13, A19, B36

- Yoshitake, Motoji, B2
 Young, Jeff, A3, A8, B21
 Yu Kun-Rui, B19
 Yuan Ping-Zhi, B16
 Zaccagnini, Alessandro, A8
 Zachariou, Andreas, B2
 Zachariou, Eleni, B2
 Zagier, Don B., D12
 Zaharescu, Alexandru, F15, F28
 Zajta, Aurel J., D1, D9
 Zarankiewicz, Kazimierz, A5
 Zarantonello, Sergio E., A8
 Zaremba, Stanisław Krystyn, F20
 Zarnke, C. R., B16
 Zay, Béla, E31
 Zeilberger, Doron, F13, F14
 Zeitlin, David, E31
 Zhang Ming-Zhi, A13, D28, F10, F13, F14
 Zhang Zhen-Xiang, B47, E4, E28
 Zhou Guo-Fu, D7
 Zhu Yi-Liang, C20
 Złotkowski, W., D10
 Znán, Štefan, E11, E13, E16, F13, F14
 Zun, Shan, B37
 Zwillinger, Dan, C1

General Index

Names appear here if their mention in the relevant section is unsupported by references. Single letter entries refer to the Introduction (pp. 1–2) and to the beginning of Sections A (pp. 3–4), E (p. 199) and F (p. 240).

A-sequence, E28
ABC conjecture, A3, B19, D2

Abe, Nobuhisa, D25
abundance, B2
abundant, B2
addition chain, C6
additive basis, C12
additive sequence, C4, E32
admissible partition, E11
Alanen, Jack, B10
algebraic number, F22

aliquot cycle, B7
 aliquot parts, B, B4
 aliquot sequence, B6

almost perfect, B2

amicable numbers, B4
 amicable triples, B5

Anderson, Claude, D13
 Andrica, Dorin, A8

Applegate, Harry, F28
 arithmetic progression, A, A5, A6, A19,
 E10, E23, E33

Ashbacher, Charles, D21
 associates, A16

asymptotic, A1
 asymptotic density, A17

*B*₂-sequence, C9, E28
 Baker's method, D10

Baker, Alan, C4
 ballot numbers, B33

Bang, Thøger, B46
 Baragar, Arthur, D23
 barrier, B8

basis, E32
 Beatty sequences, E27, F14

Benkoski, S. J., F10
 Bergmann, Horst, D18
 Bergum, Gerald E., D19
 Bernoulli numbers, A17, D2

betrothed numbers, B5
 Betris, Dimitrios, A9

binomial coefficient, B31, B33, D3, D17
 Bond, Reginald, B43

Bowen, Rufus, D7
 Brauer chain, C6

Brauer number, C6
 Brauer, Alfred, E10

Brzolis, Demetrios, F9
 Browkin, Jerzy, B36

Brun's method, B42
 Brun-Titchmarsh theorem, B39

Buckley, M. R., D27
 Burr, Stefan, C4

Cameron, Peter J., C9
 Carmichael number, A12, A13, B37, D1

Carmichael's conjecture, B39
 Catalan conjecture, B19, D9

Catalan numbers, B33
 chain, E17

champion, A8
 Choi, S. L. G., F14

Choudhry, Ajai, D13
 Chowla, Saradom D., A4, B18, D8

Chudnowsky, Gregory V., D9
 class-number formula, F5

cluster, D20
 coin problem, C7

Collatz, Lothar, E16
 collinear integers, E10

composite number, A

congruent, A4
 congruent number, D27
 continued fraction, D11, F20, F21, F22,
 F23

Conway, John Horton, A8, B19, B20, B33,
 D19, E20
 coprime, A

Coates array, C18
 Coughlan, Francis, F28
 covering congruences, B21, E3, F13, F14
 covering problem, C12, C13

covering system, F13
 Cramer's conjecture, A2
 Croft, Hailard T., A10, F23

cube, C13
 Cullen numbers, B20
 Cunningham chain, A7, A18
 cycle, D18, E17

Czipszer, J., C17
 factorial, A2, B22, B23, B43, D25, F11
 fan, C13

Farey series, D11, F27
 Feit, Walter, B25
 Fermat number, A3, A12, B21

Fermat problem, B19
 Fermat's (little) theorem, B20
 Fermat, Pierre de, D17, D27

Fibonacci number, A12, D26
 Fibonacci sequence, A3
 figurate number, D3

Finucane, Daniel M., B41
 Folkman, Jon, E11
 fortunate primes, A2

Fortune's conjecture, A2
 Frénicle de Bessy, Bernard, D1
 friendship graph, C13

frieze pattern, D18
 Frobenius, G., C7

Gagoia, Gloria, F9
 Gallyas, K., D27

Gandhi, J. M., D2
 Gardner, Martin, A6, B4, D19

Gaussian integer, A16, F13
 Gaussian prime, A16

Gaulf's problem, F1
 Gaulf, Carl Friedrich, F5

Gel'fond-Baker method, F23
 Gessel, Ira, B33

Gilmore's conjecture, A10
 Göbel, Fritz, E15

Godwin, Herbert James, B6
 Goldbach conjecture, B10, B19, C1

golden number, E25
 Golomb ruler, C10

Golomb, Solomon W., E25
 good primes, A14

Goormaghtigh, R., B25
 Gordon, Basil, A16, B37, F28

Erdős, Paul, A2, A15, B13, B15, B19, B34,
 B38, C1, C2, C11, D9, D13, D15

Gorskowski, Waldemar, B20
 graceful graph, C13

graceful labelling, C10
 Graham, Ronald Lewis, B2, B30, B37, D17, D23, D25, E8, E29, E33, F10, F15
 Graham, Sidney W., A13
 Granville, Andrew, B19
 greatest common divisor, A, E2
 Grecu, Dan, A8
 greedy algorithm, E10, E28, E32
 Grimm's conjecture, B32
 Gupta, Hansraj, D9, E22
 Guy, Michael John Thirian, D19, E17

Kaplansky, Irving, B4
 Keller, Wilfrid, B43
 Kelly, Blair, D1
 Kemnitz, Arnfried, D21
 Khare, S. P., B31, B33
 Kimberling shuffle, E35
 Klarnet, David, C14
 Klarner-Rado sequence, E36
 Kleitman, Daniel J., B24
 Kolha, Z., E20
 Kummer surface, D19
 Kummer, Ernst Eduard, B33, D2

Hajós, G., B37
 Hall, Richard R., C19
 Hansen chain, C6
 happy number, E34
 Harborth, Heiko, D21
 Hardy, Godfrey Harold, F1
 harmonic mean, B2
 harmonic number, B2
 harmonious graph, C13
 harmonious labelling, C10
 Heath-Brown, D. Roger, B1
 Heilbronn, Hans, F4
 Hellenius, Fred, B2
 Heron triangle, D22
 Hesterman, Lambertus, F27
 hexagonal number, C20
 hexahedra, D21
 Hickerson, Dean, B23, C10
 Hitotumatu, Sin, F24
 Hoey, Dan, D21
 Hoffman, Fred, B42
 Hooley, Christopher, D4
 Hunter, J. A. H., D19, D27
 Hurwitz equation, D12
 hyperperfect number, B2

independent, C19
 infinitary divisor, B3
 infinitary perfect number, B3
 irrationality sequence, E24
 irreducible semiperfect, B2
 irregular prime, D2, D7
 Isaacs, Rufus, B5

Jacobi symbol, A12, F5
 Jacobsthal, E., B40
 Jones, James P., B31

k -th power residues, F6
 Kaisow, William, D20
 Kanapka, Joe, B19

Lacampagne, Carole, F31
 Lagrange, Jean, D14
 Lam, Clement W. H., C15
 Landau, Edmund, D4, F1
 Lander, Leon J., A8
 lattice point, D20, E10, F1, F2, F3, F4
 Lau, Kenneth, F28
 least common multiple, B26, E2, F19
 Leech, John, A16, B33, C20, D2, D9, D11, D14, D17, D19, D21, D22, D25
 left factorial, B44
 Legendre symbol, F5
 Lehmer, Derrick Henry, A8, B2, B36, D1, D11, F9, F10, F12
 Lehner, Emma, B36, F9, F10
 Lenstra, Hendrik Willem, E15
 Leonardo of Pisa, D27
 Levine, Eugene, B33, E11
 Linnik's constant, A4
 van Lint, Jacobus H., E25
 Littlewood, John Edensor, A4, E18, F23
 Longyear, Judith, D28
 loopy game, E26
 Lucas number, D26
 Lucas's problem, D3
 Lucas-Lehmer sequence, A3
 Lucas-Lehmer test, A3
 lucky numbers, C3

MacMahon, Percy A., E31
 magic square, A6
 Mahler, Kurt, B46, C4, D1
 Makowski, Andrzej, B8, B11, B13, B48, C20, F10
 Markoff equation, D12, D23
 Markoff number, D12
 Marsias, J. P., E8
 mas sequence, E27
 McIntosh, Richard, F28
 McKay, John H., B25
 Meally, Victor, A8, B37, C18, D11, F10, F23

Mercator, Nicolaus, F23
 Mersenne prime, A3, B1, B5, B9, B11, B15, B38
 mex, E27
 mex sequence, E27
 minimum overlap problem, C17
 modular Mousetrap, E37
 monotone A.P., E33
 Montgomery, Peter Lawrence, B33, D1, D11, F6, F28
 Mordell, Louis Joel, D, D1, D3, D4, D11, D17, D28, F29
 Moser, Leo, B11, C1, C5, D14, D15, F21
 Motzkin, Theodor S., A16, B19, E22
 Mousetrap, E37
 multigrade equations, D1
 multiperfect, B2
 multiply perfect, B2

N -position, E26
 Nagashima, Takahiro, D28
 Napier's rules, D18
 Narkiewicz, Włodzimierz, A15, B16, F9
 Newman, Morris, F28
 nim addition, E27
 nim-like games, E27
 no-three-in-line problem, F4
 Noll, Landon Curt, A15
 nonaveraging set, C16
 noncototients, B36
 nondividing set, C16
 nontritontents, B36

octahedron, C13
 octal game, E27
 Odlyzko, Andrew, A8
 ordinary point, D18
 Ore number, B2
 Oursler, Cielie, D28
 Owens, Frank, C4

P -position, E26
 packing problem, C9
 paradox, E17
 Parker, Ernest T., B25
 Parkin, T. R., A8
 partial quotient, F20, F21, F22
 Pascal triangle, B33
 Pell equation, B16, D3, D17
 Pell sequence, A3
 Penney, David E., B8, B41
 pentagonal number, C20
 perfect cuboid, D18
 perfect difference set, C10
 perfect number, A3, B1, B5

permutation sequence, E17
 persistence, F25
 Petersen graph, C13
 Platonic solid, C13
 Pomerance, Carl, A3, B8, B19, B47, D17, E30
 postage stamp problem, C12
 power residue, F6
 powerful number, B16, D2
 prime circle, C1
 prime desert, A8
 prime factorial, A8
 prime number graph, A5, A14
 prime number race, A4
 prime numbers of measurement, E30
 prime pyramid, C1
 primitive abundant, B2
 primitive part, A3
 primitive pseudoperfect, B2
 primitive root, F9
 primitive sequence, E4
 pseudoperfect number, B2, D11, F14
 pseudoprime, A12, F10
 Purdy, George B., F3
 Pythagorean ratio, D17
 Pythagorean triple, D18, D21

quadratic reciprocity law, F5
 quadratic residue, F5, F6, F8, F9
 quadri-amicable, B5
 quasi-amicable, B5
 quasi-perfect, B2
 queens problem, C18

Ramanujan-Nagell equation, D23
 Ramsey number, E11
 Rathbun, Randall L., D1
 rational box, D18, D21
 rational simplex, D22
 recurrence relation, A3
 regular equation, E14
 regular prime, D2, D7
 Rényi, Alfred, B9
 repunit, A3
 residue class, A
 Rhind papyrus, D11
 Riddell, James, F19
 Riemann ζ -function, B48
 Riemann hypothesis, A4, A8, A19, F9
 Robinson, Raphael M., A3
 Rosenburg, Bryan, D20
 Rote, Günter, E20
 Roth, Klaus F., E9
 Rotkiewicz, Andrzej, F10
 Ruderman, Harry D., B47

- Rudnick, Carl, D9
ruler function, E22
- Säfholm, Sten, A9
Sastry, K. R. S., D3, D17, D19
Scher, Bob, D1
Scherk, Peter, C17
Schinzel's conjecture, A2
Schinzel, Andrzej, B37, D3, D9, D16, D24, E11
Schroeppel, Rich, B2, B42
Schur number, E11
Schur's conjecture, F5
Scott, Reese, D10
self-contained integer, E16
Selfridge, John Lewis, A2, A9, A18, B2, B8, B22, B28, B30, B33, B42, B46, B47, C14, C15, D11, F19, F22, F31
Shallit, Jeffrey, B23
Shapiro, Harold N., D11
Shedd, Charles L., D21
Sidon sequence, C9
Sierpiński, Waclaw, B13, D24, F10
sieve of Eratosthenes, C3
Silverman, David, B48, E8
Simmons, Chuck, A15
Singer, J., C10
singular point, D1
singular solution, D12
Slavíč, Dušan V., B44
Sloane, Neil J. A., B33
Smith numbers, B49
sociable numbers, B7
Sós, Vera Turán, E14
span, E10
sphere, F1
spherical triangle, D18
Spiro, Claudio, B18
Sprague-Grundy theory, E27
square pyramid, D3
squarefree, A3, B33, B44
Stanley, Richard P., B16
Stechkin, Boris S., A17
Stewart, Bonnie M., A2
Stirling's formula, B22
Stöhr sequence, E32
Stong, Richard, D11
Straus, Ernst G., B19, B22, E22
Stroeker, Roelof Jacobus, D23
Strong Law of Small Numbers, A3, B20, D23, F5, F27
strong pseudoprime, A12
strongly independent, C19
strongly sum-free, E13
- Struppeck, Thomas, B6
Styer, Robert, B19
Sulyok, Miklós, C9
sum-free, E12
sum-free sequence, E32
sum-free set, C14
superperfect numbers, B9
Szemerédi's theorem, A5, E10
Szeptyciak, Lucien, D25
table of primes, A
Tarry-Escott problem, D1
Taylor, Herbert, F19
tetrahedroid, D19
tetrahedron, D3, D22, F1
Thatcher, Alfred R., D18
Thompson, John G., B25
Thue sequence, E21
Thue-Siegel theorem, B30
Tijdeman, Robert, B46, E7
totient function, B11, B36, B41, B42, D11
triangular number, C20, D3, D21
triperfect number, B2
Turán, Pál, C20, E13
twin primes, A8, B19, B38
- U-numbers, C4
Ulam, Stanislas M., F22
unique factorization, A16
unit, A
unit fractions, D11
unitary aliquot sequence, B8
unitary amicable numbers, B4
unitary divisor, B3
unitary multiperfect, B3
unitary perfect number, B3
unitary sociable numbers, B8
units, A16
untouchable numbers, B10
Upton, Leslie J., D19
- Vandemergel, Stéphane, A4, B18, D1, D11, D16
Vélez, William Yslas, B47
Viète, François, D1
Vojta, P., D1
- van der Waerden's theorem, E10
Wagon, Stanley, A2
Wagstaff, Samuel S., B43, B48
Wall, Charles R., B38
Waring's problem, D4
weakly independent, C19
de Weger, Benno M. M., D23, F23
Weintraub, Sol, A8

- weird, B2
wheel, C13
Whiteman, Albert Leon, E19
Wiethaus, Holger, B4
Wiles, Andrew J., D2
Wilson's theorem, A17
Windedecker, C12
windmill, C13
Wolstenholme's theorem, B31
Woods, Alan R., B29
- Yang Yuan-Sheng, C10
Yuanhua, Ren, B7
- Z-number, E18
Zeitlin, David, C4
Zhang, Ming-Zhi, A12

Problem Books in Mathematics *(continued)*

Theorems and Problems in Functional Analysis
by A.A. Kirillov and A.D. Gvishiani

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