High School Olympiads

A Nice Property regarding reflection triangle

Reply

Source: Me

RSM
#1 Jun 22, 2011, 12:53 am

In a triangle ABC, B', C' are the reflections of B, C on $AC, AB.AD, BE, CF$ are the altitudes of ABC. BE intersects DF at B_1 and CF intersects DE at C_1. AB', AC' intersects B_1C_1 at X, Y. Prove that X, Y, B', C' are cyclic.

skytin
#2 Jun 22, 2011, 2:29 pm

Angle C' BA = BCA = 180 - BHA were H is Orthocenter of BCA
C'BHA is cyclic
Let (YHC') intersect DA at points Z and A
Let O' is reflection of O wrt BA easy to see that O' is circumcenter of (C'BHA)
Not hard to prove that ZY is perpendicular to O'A , so to BO
We only need to prove that ZC is perpedicular to CO (Use radical center theorem)
Use homotety with center A
Let DE intersect B'A at point P and DF intersect C'A at point Q
angle BDQ = BAQ , so QBDA is cyclic , so BQ is perpendicular to QA , so BE = BQ and EF = FQ
Like the same EP = EF
So FP || BE and EQ || CF
Use lemma proposed here :
http://jl.ayme.pagesperso-orange.fr/Docs/Le%20theoreme%20de%20Feuerbach-Ayme.pdf
Page 27

So C_1B_1 = XY || PQ
Triangles ZXY and DPQ are homotetive with center A , so ZX || DE (DF is perp to BO , DE is perp to CO)

RSM
#3 Jun 22, 2011, 4:23 pm

I have added my proof to this problem in my blog. See **Property 4** here:-
http://www.artofproblemsolving.com/blog/53505

Luis González
#4 Jun 23, 2011, 12:11 am

According to Tuymaada 2009 Senior League Problem 3, the circumcenter U of $\triangle AB'C'$ lies on the A-cevian of the 9-point center N of $\triangle ABC$. Then if AN cuts the circumcircle (U) of $\triangle AB'C'$ again at P, it follows that $PB' \perp AX$. But it's well known that $AN \perp B_1C_1$, since B_1C_1 is the radical axis of the 9-point circle (N) and $\odot(HBC)$ (where H is the orthocenter of ABC). Therefore, $\angle AXY = \angle APB' = \angle AC'B' \Longrightarrow B', C', X, Y$ are concyclic.

RSM
#5 Jun 23, 2011, 12:51 am

@luis, actually in my proof to the problem that, circumcenter of $AB'C'$ lies on AN I used this problem as a lemma. So it would be better if you provide some solution which does not use this property. In my blog i have added such a proof. So I will expect some other proofs also.

Quick Reply

High School Olympiads

Problem (Own)

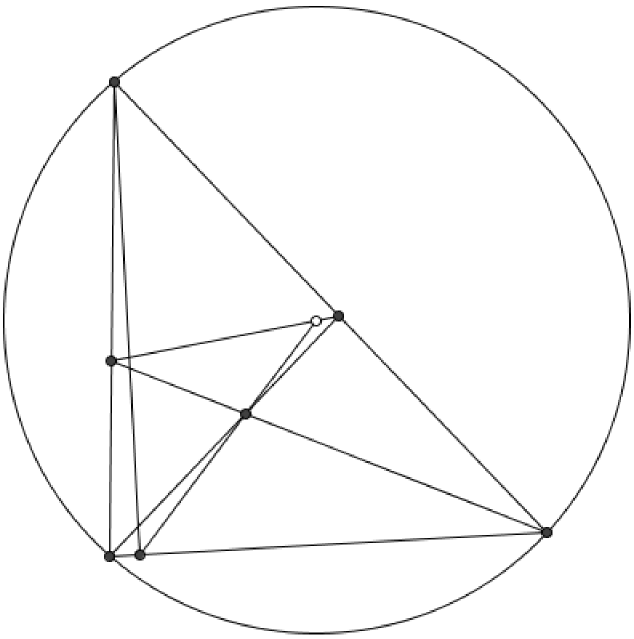
Reply

skytin

#1 Jun 20, 2011, 5:43 pm • 1

Given triangle ABC
I is incenter
AP and BQ are segments of angle bisectors
HC is height of ABC
O is circumcircle center
Given that I is on HO , prove that O is on PQ

Attachments:



skytin

#2 Jun 20, 2011, 5:47 pm

You can also find length HI if given r and R

malcolm

#3 Jun 21, 2011, 5:23 am • 1

I think we also need the assumption that $AC \neq BC$. Using barycentric coordinates, it is known that $P(0,b,c)$, $Q(a,0,c)$, $I(a,b,c)$, $O(a^2S_A, b^2S_B, c^2S_C)$, $H(S_BS_C, S_AS_C, 0)$ where $S_A = b^2+c^2-a^2$ and S_B, S_C are defined similarly. Hence

$$I \in HO \iff \begin{vmatrix} a^2S_A & b^2S_B & c^2S_C \\ a & b & c \\ S_BS_C & S_AS_C & 0 \end{vmatrix} = 0 \iff cS_C(b^2S_B^2 + acS_AS_C - a^2S_A^2 - bcS_BS_C) = 0$$

$\iff cS_C(bS_B - aS_A)(aS_A + bS_C - cS_C) = 0 \iff aS_A + bS_B = cS_C$, assuming $AC \neq BC$.

Now $O \in PQ \iff \begin{vmatrix} a^2S_A & b^2S_B & c^2S_C \\ 0 & b & c \\ a & 0 & c \end{vmatrix} = 0$

$\iff abc(aS_A + bS_B - cS_C) = 0 \iff aS_A + bS_B = cS_C \iff I \in HO$ as desired.

$\iff abc(aS_A + bS_B - cS_C) = 0 \iff aS_A + bS_B = cS_C \iff I \in HO$ as desired.

Luis González

#4 Jun 21, 2011, 10:20 pm • 1

Circumcircle (O) and incenter I of $\triangle ABC$ become 9-point circle and orthocenter of its excentral triangle $\triangle I_aI_bI_c$. Thus, midpoint D of the arc AB of (O) is also midpoint of II_c. If I, O, H are collinear, then we have

$$\frac{AH}{OD} = \frac{CI}{ID} \Longrightarrow \frac{2R}{h_c} = \frac{II_c}{CI} = \frac{s}{s-c} - 1 \Longrightarrow 2R = \frac{c \cdot h_c}{s-c} \Longrightarrow R = r_c \ (\star)$$

Circumcircle (U) of $\triangle II_aI_b$ is the image of (O) under the dilatation with center I_c and coefficient 2. Hence, its radius is $2R$ and its center U is the reflection of I_c about O. Then

$$p(O,(U)) = (2R)^2 - OI_c{}^2 = 4R^2 - (R^2 + 2R \cdot r_c) = 3R^2 - 2R \cdot r_c$$

Together with $(\star)$, it follows that $p(O,(U)) = 3R^2 - 2R^2 = R^2 = p(O,(O)) \Longrightarrow O$ has equal power to (O) and (U) $\Longrightarrow O$ lies on the radical axis PQ of $(O),(U)$.

skytin

#5 Jun 22, 2011, 3:35 pm

> luisgeometra wrote:
>
> Circumcircle (O) and incenter I of $\triangle ABC$ becomes 9-point circle and orthocenter of its excentral triangle $\triangle I_aI_bI_c$. Thus, midpoint D of the arc AB of (O) is also midpoint of II_c. If I, O, H are collinear, then we have
>
> $$\frac{AH}{OD} = \frac{CI}{ID} \Longrightarrow \frac{2R}{h_c} = \frac{II_c}{CI} = \frac{s}{s-c} - 1 \Longrightarrow 2R = \frac{c \cdot h_c}{s-c} \Longrightarrow R = r_c \ (\star)$$

Ok , now you can try to find length of HI if given r and R
Nice to see that this length is has same formula as in problem posted here :
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=412410
AC = 2*R

Quick Reply

High School Olympiads

Four concurrent lines !

Reply

Morleyiquel

#1 Jun 20, 2011, 4:10 pm

Circles w_1 and w_2 with centres O_1 and O_2 are externally tangent at point $D$and internally tangent to a circle w at points E and F respectively. Line t is the common tangent of w_1 and w_2 at D. Let AB be the diameter of w perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1,BO_2, EF and t are concurrent.

Luis González

#2 Jun 20, 2011, 10:03 pm

O is the center of $\omega.$ From the parallel radii $OA \parallel O_2D$ and $OB \parallel O_1D,$ it follows that DA, DB pass through the exsimilicenters F, E of $\omega \sim \omega_2$ and $\omega \sim \omega_1.$ Lines AE, BF, t concur at the orthocenter C of $\triangle ABD.$ Thus, D, E, C, F are concyclic on a circle Ω and ω_1, ω_2 are orthogonal to $\Omega.$ Let $T \equiv EF \cap t.$ BT is the polar of A WRT $\Omega \Longrightarrow$ Tangents of Ω through D, F intersect on $BT \Longrightarrow O_2 \in BT.$ Similarly, $O_1 \in AT.$ Thus, AO_1, BO_2, EF and t concur.

RSM

#3 Jun 21, 2011, 9:12 pm

Suppose, $T_E \cap T_F = R$. Note that t passes through R. E, F are the reflections of D on RO_1, RO_2. Also $EO_1 \cap FO_2 = O$. Midpoint of O_1O_2 is J. Note that AO_1, BO_2, OJ concurr. So we have to prove that OJ, EF, RD are concurrent.

So we can rephrase the problem as follows:-

Problem:-

In a triangle ABC, AD is the altitude. D_1 and D_2 are the reflections of D on AB, AC. $D_1D_2 \cap AD = X$ and $D_1B \cap D_2C = T$. Prove that P, A', X are collinear where A' is the mid-point of BC.

Proof:-

Suppose, O is the circumcenter of ABC. $AB \cap DD_1 = L, AC \cap DD_2 = M$. Note that $ALM \equiv ABC$. $AD \cap LM = K$ and $OA \cap BC = A_1$.

Note that $\frac{AX}{AD} = \frac{OA_1}{R} = \frac{OA'}{AD - OA'}$

Note that, A, O, T are collinear and T lies on $\odot BOC$. Suppose,

$TA' \cap AD = X'$. So $\frac{OA'}{AX'} = \frac{OT}{TA} = \frac{R^2}{OA_1(R + \frac{R^2}{OA_1})} = \frac{R}{OA_1 + R} = \frac{AD - OA'}{AD}$

So $\frac{AX'}{AD} = \frac{OA'}{AD - OA'}$

So $\frac{AX}{AD} = \frac{AX'}{AD}$

So $X \equiv X'$.

skytin

#4 Jul 28, 2011, 5:36 pm

Hint :
N is midpoint EF
BO_2 intersect AO_1 at point X , X is on EF and on (BAN)

Quick Reply

High School Olympiads

Interesting locus problem

Reply

RSM

#1 Jun 18, 2011, 7:35 pm

Given two circles (O_1) and (O_2) find the locus of the point P such that polar of P with respect to the two circles intersect on O_1O_2.

Luis González

#2 Jun 19, 2011, 11:32 pm

We assume that circles $(O_1), (O_2)$ do not intersect. Polars τ_1, τ_2 of P WRT $(O_1), (O_2)$ meet at a point U on O_1O_2 and let V the orthogonal projection of P onto $O_1O_2.$ Then PV is the polar of U WRT (O_1) and (O_2), respectively $\Longrightarrow P, U$ are conjugate points WRT $(O_1) \Longrightarrow$ circle $\odot(PUV)$ with diameter $\overline{PU}$ is orthogonal to $(O_1).$ Similarly, P, U are conjugate points WRT $(O_2) \Longrightarrow$ circle $\odot(PUV)$ with diameter $\overline{PU}$ is orthogonal to $(O_2).$ Hence, $\odot(PUV)$ is orthogonal to both $(O_1), (O_2) \Longrightarrow U, V$ are the two limiting points of $(O_1), (O_2).$ Thus, loci of P are the two perpendicular lines to O_1O_2 through $U, V.$

Quick Reply

High School Olympiads

Maybe famous collinearity

Reply

goodar2006

#1 Jun 19, 2011, 9:44 pm

in triangle ABC, BP and CQ are angle bisectors of angles B and C respectively. (P on AC and Q on AB). line l is the common external tangent to the circumcicle of ABC and it's A-excircle. l touches circumcircle of ABC at S. prove that P, Q and S are collinear.

Luis González

#2 Jun 19, 2011, 10:00 pm

Discussed on http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=385175. For instance, see post #5. If line PQ cuts the circumcircle (O) of ABC at S',S'', then have proved that S',S'' are identical with the tangency points of the common external tangents of (O),(Ia) with (O).

Quick Reply

High School Olympiads

tan A/2 ,tanB/2,..

Reply

Mathx

#1 Aug 17, 2005, 2:19 pm

prove in triangle ABC we have;

$$\tan^2(\frac{A}{2})+\tan^2(\frac{B}{2})+\tan^2(\frac{C}{2})=\frac{(4R+r)^2-2p^2}{p^2}$$

$(p=\frac{a+b+c}{2}$,R is radii of circumcircle,r is radii of incircle)

al.M.V.

#2 Aug 18, 2005, 2:48 am

Hint:

Click to reveal hidden text

Luis González

#3 Jun 19, 2011, 11:47 am

Let r_a, r_b, r_c be the exradii of $\triangle ABC$ against $A, B, C.$ Then we have

$$\tan\frac{A}{2}=\frac{r_a}{p}\ \ ,\ \ \tan\frac{B}{2}=\frac{r_b}{p}\ \ ,\ \ \tan\frac{C}{2}=\frac{r_c}{p}\ \Longrightarrow$$

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}=\frac{{r_a}^2+{r_b}^2+{r_c}^2}{p^2}$$

Squaring the identity $r_a+r_b+r_c=4R+r$ (Steiner theorem) yields

$${r_a}^2+{r_b}^2+{r_c}^2=(4R+r)^2-2(r_br_c+r_cr_a+r_ar_b)$$

But, on the other hand

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Longrightarrow\ r_br_c+r_cr_a+r_ar_b=\frac{r_ar_br_c}{r}=\frac{[\triangle ABC]^2}{r^2}=p^2$$

Therefore, ${r_a}^2+{r_b}^2+{r_c}^2=(4R+r)^2-2p^2\ \Longrightarrow$

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}=\frac{(4R+r)^2-2p^2}{p^2}$$

Quick Reply

High School Olympiads

The segments divide the parallelogram into four triangles

Source: Austrian Mathematical Olympiad 2003, Part 1, P4

Amir Hossein

#1 Jun 18, 2011, 2:38 pm

In a parallelogram $ABCD$, points E and F are the midpoints of AB and BC, respectively, and P is the intersection of EC and FD. Prove that the segments AP, BP, CP and DP divide the parallelogram into four triangles whose areas are in the ratio $1:2:3:4$.

Luis González

#2 Jun 19, 2011, 5:04 am

If DP cuts AB at Q, then $BQ=BA$. By Menelaus' theorem for $\triangle BEC$ cut by $\overline{PFQ}$, we get

$$\frac{PC}{EP}=\frac{QB}{QE}\cdot\frac{CF}{FB}=\frac{BA}{BA+\frac{1}{2}BA}=\frac{2}{3} \Longrightarrow \frac{PE}{CE}=\frac{3}{5}, \ \frac{PC}{EC}=\frac{2}{5}$$

$$\frac{|\triangle PAB|}{|\triangle ABC|}=\frac{PE}{CE}=\frac{3}{5}, \ \frac{|\triangle PBC|}{|\triangle ABC|}=\frac{1}{2}\cdot\frac{PC}{EC}=\frac{1}{5}, \ \frac{|\triangle PCD|}{|\triangle ABC|}=\frac{PC}{EC}=\frac{2}{5}$$

$$\frac{|\triangle PDA|}{|\triangle ABC|}=\frac{|ABCD|}{|\triangle ABC|}-\frac{|\triangle PAB|}{|\triangle ABC|}-\frac{|\triangle PBC|}{|\triangle ABC|}-\frac{|\triangle PCD|}{|\triangle ABC|}=2-\frac{6}{5}=\frac{4}{5}$$

$$\frac{|\triangle ABC|}{5}=\frac{|\triangle PDA|}{4}=\frac{|\triangle PAB|}{3}=\frac{|\triangle PCD|}{2}=\frac{|\triangle PBC|}{1}$$

High School Olympiads

Equal areas 3

Reply

Source: Macedonia

krenkovr

#1 Jun 18, 2011, 4:45 am

Let ABCD is convex quadrilateral. About triangles ABC, BCD, CDA and DAB describes circles k_1, k_2, k_3 and k_4. Let k_1 and k_3 intersect diagonal BD in points J and L, k_2 and k_4 intersect continuation of the diagonal AC in points F and H. Proof that area of quadrilateral ABCD is equal with area of quadrilateral FLHJ.

Luis González

#2 Jun 18, 2011, 6:04 am

Let $P \equiv AC \cap BD. J, L$ are the inverses of B, D under the inversion with center P and power $\overline{PA} \cdot \overline{PC}$ and F, H are the inverses of C, A under the inversion with center P and power $\overline{PB} \cdot \overline{PD}.$ Therefore

$$\frac{LJ}{BD} = \frac{PA \cdot PC}{PB \cdot PD} \ , \ \frac{FH}{AC} = \frac{PB \cdot PD}{PA \cdot PC} \Longrightarrow FH \cdot LJ = AC \cdot BD.$$

But if φ denotes the angle between $AC, BD,$ we have

$$2[ABCD] = AC \cdot BD \cdot \sin \varphi \ , \ 2[FLHJ] = JL \cdot FH \cdot \sin \varphi$$

$$\Longrightarrow \ [ABCD] = [FLHJ].$$

krenkovr

#3 Jun 18, 2011, 6:21 pm

Thank you luisgeometra. Can you see topic equal areas 2?

Quick Reply

High School Olympiads

Find segment (Own)

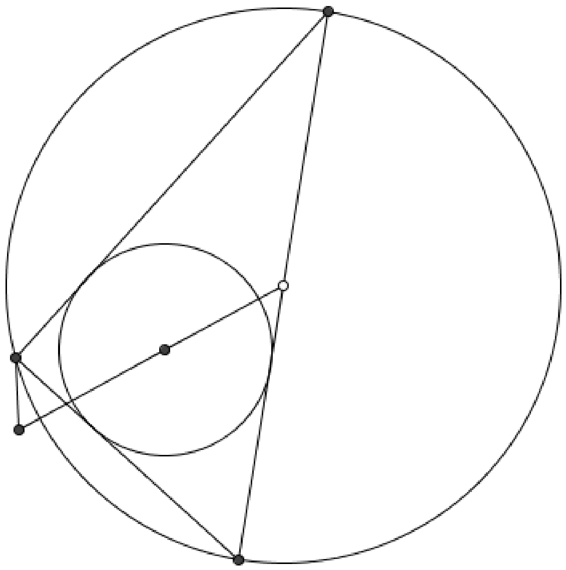
Reply

skytin

#1 Jun 17, 2011, 2:59 pm

Let given triangle ABC were angle ABC = 90
I is incenter
M is midpoint of AC
IM intersect exterior bisector of angle ABC at point X
Given r is radius of incircle , AC = a
Find XI

Attachments:



Luis González

#2 Jun 18, 2011, 1:08 am • 1

Let E, D be the midpoints of the arcs AC and ABC of the circumcircle of $\triangle ABC$. Incircle (I, r) touches AC at K and P is the orthogonal projection of B onto AC. If IM cuts BP at U, then we know that $BIKU$ is a parallelogram (which is true for any scalene triangle), thus $BU = IK = r$. From $\triangle XUB \sim \triangle XMD$, we get

$$\frac{BX}{BX+BD} = \frac{BU}{DM} = \frac{2r}{AC} \Longrightarrow BX = BD \cdot \frac{2r}{AC-2r}$$

Hence, by Pythagorean theorem for $\triangle BIX$ with hypotenuse IX, we get

$$IX^2 = BD^2 \cdot \frac{4r^2}{(AC-2r)^2} + 2r^2 = 2r^2 \cdot \frac{2BD^2 + (AC-2r)^2}{(AC-2r)^2} \quad (\star)$$

Since $EA = EC = EI = \frac{\sqrt{2}}{2}AC$, then $EB = EI + IB = \sqrt{2}\left(r + \frac{AC}{2}\right)$

Hence, by Pythagorean theorem for $\triangle DEB$ with hypotenuse $ED = AC$, we get

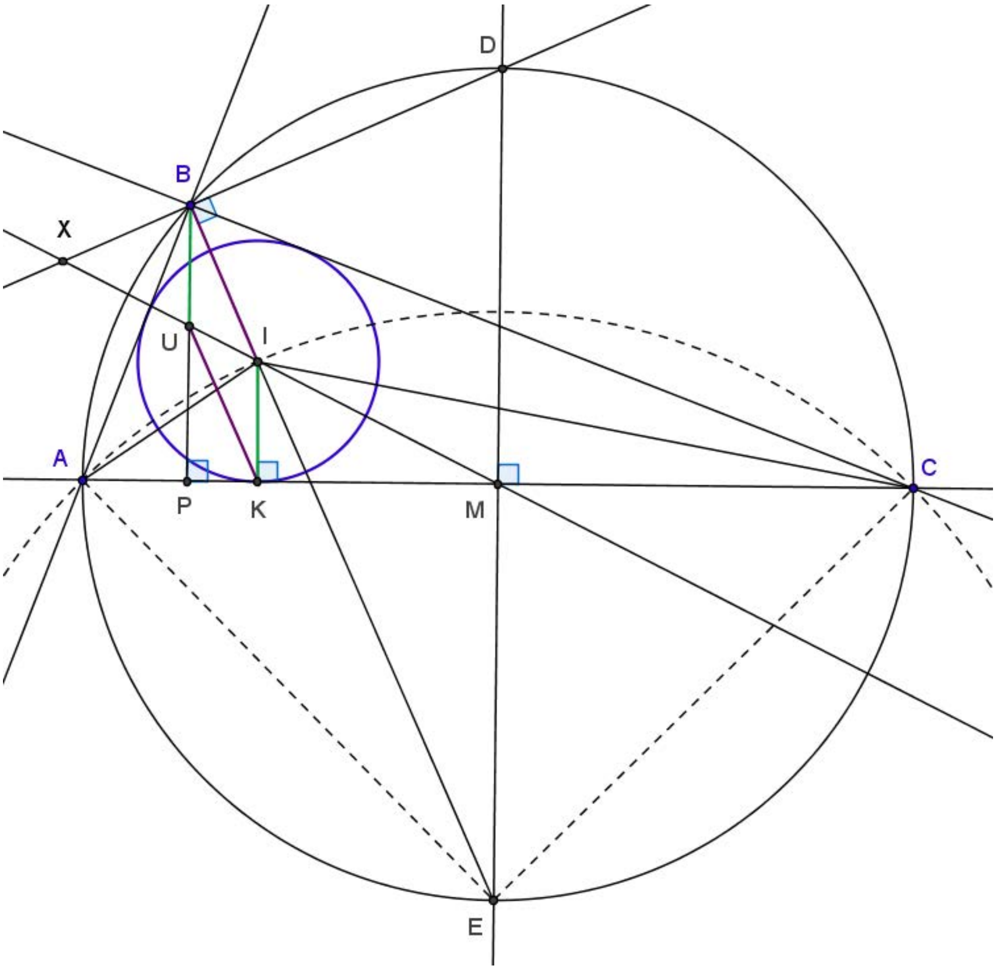
$$BD^2 = AC^2 - 2\left(r + \frac{AC}{2}\right)^2 = \frac{AC^2}{2} - 2r^2 - 2r \cdot AC$$

Substituting BD^2 from the latter equation into $(\star)$ yields

$$IX^2 = 2r^2 \cdot \frac{AC^2 - 4r^2 - 4r \cdot AC + AC^2 + 4r^2 - 4r \cdot AC}{(AC-2r)^2} \implies$$

$$IX = 2r \cdot \frac{\sqrt{AC(AC-4r)}}{AC-2r}$$

Attachments:



Virgil Nicula

#3 Jun 18, 2011, 7:27 pm • 1

PP (Skytin). Let ABC be a triangle with $B=90^\circ$ and the incircle $w=C(I,r)$. Denote the midpoint M of the side $[AC]$ and the point X for which $X\in MI$ and $BX\perp BI$. Ascertain the length of the segment $IX=f(b,r)$.

Proof. Supppose w.l.o.g. $a>c$. Observe that $\left\{\begin{array}{ccc} s=b+r & \iff & a+c=b+2r \\ ac=2sr & \iff & ac=2r(b+r) \end{array}\right\| \implies \boxed{a-c=\sqrt{b^2-4br-4r^2}}$. Remark that $\frac{b}{r}\ge 2\left(1+\sqrt{2}\right)$ (existence condition). Denote $\theta = m\left(\widehat{BMI}\right)$. Thus $\boxed{\tan\theta = \frac{2r}{a-c}}$, $AI = r\sqrt{2}$ and $m\left(\widehat{AIX}\right) = 45^\circ + C - \theta$.

Observe that $\cos\widehat{AIX} = \cos\left(45^\circ + C - \theta\right) = \frac{1}{\sqrt{2}}\cdot\left[\cos(C-\theta)-\sin(C-\theta)\right] =$

$\frac{1}{\sqrt{2}}\cdot\left(\cos C\cos\theta + \sin C\sin\theta - \sin C\cos\theta + \cos C\sin\theta\right) =$

$\frac{1}{b\sqrt{2}}\cdot\left(a\cdot\cos\theta + c\cdot\sin\theta - c\cdot\cos\theta + a\cdot\sin\theta\right) = \frac{1}{b\sqrt{2}}\cdot\left[(a+c)\cdot\sin\theta + (a-c)\cdot\cos\theta\right]$. Therefore,

$$IX = \frac{AI}{\cos\widehat{AIX}} = \frac{2br}{(a+c)\cdot\sin\theta + (a-c)\cdot\cos\theta} \iff IX = \frac{2br\cdot\sqrt{(a-c)^2+4r^2}}{2r(a+c)+(a-c)^2} \iff$$

$$IX = \frac{2br\sqrt{b(b-4r)}}{2r(b+2r)+(b^2-4br-4r^2)} \iff \boxed{IX = \frac{2r\sqrt{b(b-4r)}}{b-2r}} .$$

yetti

#4 Jun 20, 2011, 6:45 am • 1

Bisector BI of the right $\angle ABC$ cuts the circumcircle (M, R) of $\triangle ABC$ again at $E. EB = EI + IB = (R+r)\sqrt{2}$ and $CA = 2R. K$ is midpoint of $EB \Longrightarrow MK \perp EB \Longrightarrow MK \parallel BX. KI = EI - \frac{BE}{2} = \frac{(R-r)\sqrt{2}}{2} \Longrightarrow$
$IX = MI \cdot \frac{IB}{KI} = \sqrt{R^2 - 2rR} \cdot \frac{2r}{R-r} = \sqrt{CA(CA-4r)} \cdot \frac{2r}{CA-2r}$.

Quick Reply

High School Olympiads

prove that MP and MQ are tangent

Reply

sergei93

#1 Jun 17, 2011, 7:41 pm

Point M lies on diagonal BD of parallelogram $ABCD$. Line AM intersects side CD and line BC at points K and N, respectively. Let C_1 be the circle with center M and radius MA and C_2 be the circumcircle of triangle KCN. C_1, C_2 intersect at P and Q. Prove that MP and MQ are tangent to C_2.

Luis González

#2 Jun 17, 2011, 10:39 pm

$O \equiv AC \cap BD$ is the center of the parallelogram and E is the reflection of A about M. Since O is the midpoint of AC, then $CE \parallel DB$. P_∞ is the infinity point of the direction $CE \parallel DB$. Pencil $C(D, B, O, P_\infty)$ is harmonic $\Longrightarrow$ Cross ratio (K, N, A, E) is harmonic. Thus, by Newton's theorem, it follows that $MA^2 = ME^2 = MK \cdot MN \Longrightarrow$ Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are orthogonal, i.e. MP, MQ are tangent to $\mathcal{C}_2$.

sunken rock

#3 Jun 18, 2011, 12:44 am

$\triangle MKD \sim \triangle MAB$ and $\triangle AMD \sim \triangle NMB$ will give $\frac{KM}{AM} = \frac{MD}{BM} = \frac{AM}{MN}$, or $AM^2 = MK \cdot MN$, but $MP = MQ = AM$, done.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Pentagon.

Reply

Vladislao

#1 Jun 15, 2011, 5:23 am

Let $ABCDE$ be a pentagon such that there exist a point P inside the pentagon, in such a way that $AP \perp CD$, $BP \perp DE$, $CP \perp EA$, $DP \perp AB$. Show that $EP \perp BC$.

Note: This problem is supposed to be solved with euclidean geometry, but other solutions are also welcomed.

gustavoe

#2 Jun 15, 2011, 7:35 am • 2

Let P be the origin, and A, B, C, D, E the vectors representing the points. Then $A(C-D)=0, B(D-E)=0, C(E-A)=0, D(A-B)=0$ (we're taking dot product). Adding these four equations, we get $E(B-C)=0$, so $EP \perp BC$.

Vladislao

#3 Jun 15, 2011, 10:29 pm

Is this problem truly hard to be solved with Euclidean Geometry?

Luis González

#4 Jun 15, 2011, 11:49 pm

Let $Q \equiv AB \cap DE$. Perpendiculars $\ell_A, \ell_B, \ell_C, \ell_D$ from A, B, C, D to CD, DE, EA, AB concur at P. Perpendicular ℓ_Q from Q to BD passes through the orthocenter P of $\triangle QBD$, i.e. $P \equiv \ell_B \cap \ell_D \cap \ell_Q \ (\star)$. Triangles $\triangle BCD$ and $\triangle EQA$ are orthologic, one orthology center is $P \equiv \ell_B \cap \ell_C \cap \ell_D$. Thus, ℓ_A, ℓ_Q and the perpendicular ℓ_E from E to BC concur at the another orthology center. Together with $(\star)$, it follows that $P \in \ell_E$, i.e. $EP \perp BC$.

oneplusone

#5 Jun 18, 2011, 2:14 pm

Just use the fact that $AP \perp CD$ if and only if $CA^2-AD^2=CP^2-PD^2$. Add up the 4 equations to get the fifth one.

Quick Reply

High School Olympiads

Orthologic triangles

Reply

my_name_is_math
#1 Jun 13, 2011, 4:55 pm

Prove that the cevian triangles of the Gergone point and the cevian triangle of the Nagel point are orthologic.

Luis González
#2 Jun 13, 2011, 7:57 pm • 1

Incircle (I) and A-excircle (I_a) touch BC at $X_1, Y_1,$ respectively. Define (X_2, Y_2) and (X_3, Y_3) cyclically. D, E, F are the midpoints of $BC, CA, AB.$ Perpendicular from X_1 to X_2X_3 passes through the orthocenter X_{65} of $\triangle X_1X_2X_3$ and perpendicular from D to X_2X_3 becomes the angle bisector of $\angle EDF$ passing through the incenter X_{10} of $\triangle DEF.$ Since X_1, Y_1 are symmetrical about $D,$ then the perpendicular from Y_1 to X_2X_3 passes through the reflection of X_{65} about $X_{10}.$ Likewise, perpendiculars from Y_2, Y_3 to X_3X_1 and X_1X_2 pass through the reflection of X_{65} about $X_{10} \Longrightarrow \triangle X_1X_2X_3$ and $\triangle Y_1Y_2Y_3$ are orthologic.

Quick Reply

High School Olympiads

A circle is tangent to three circles (own)

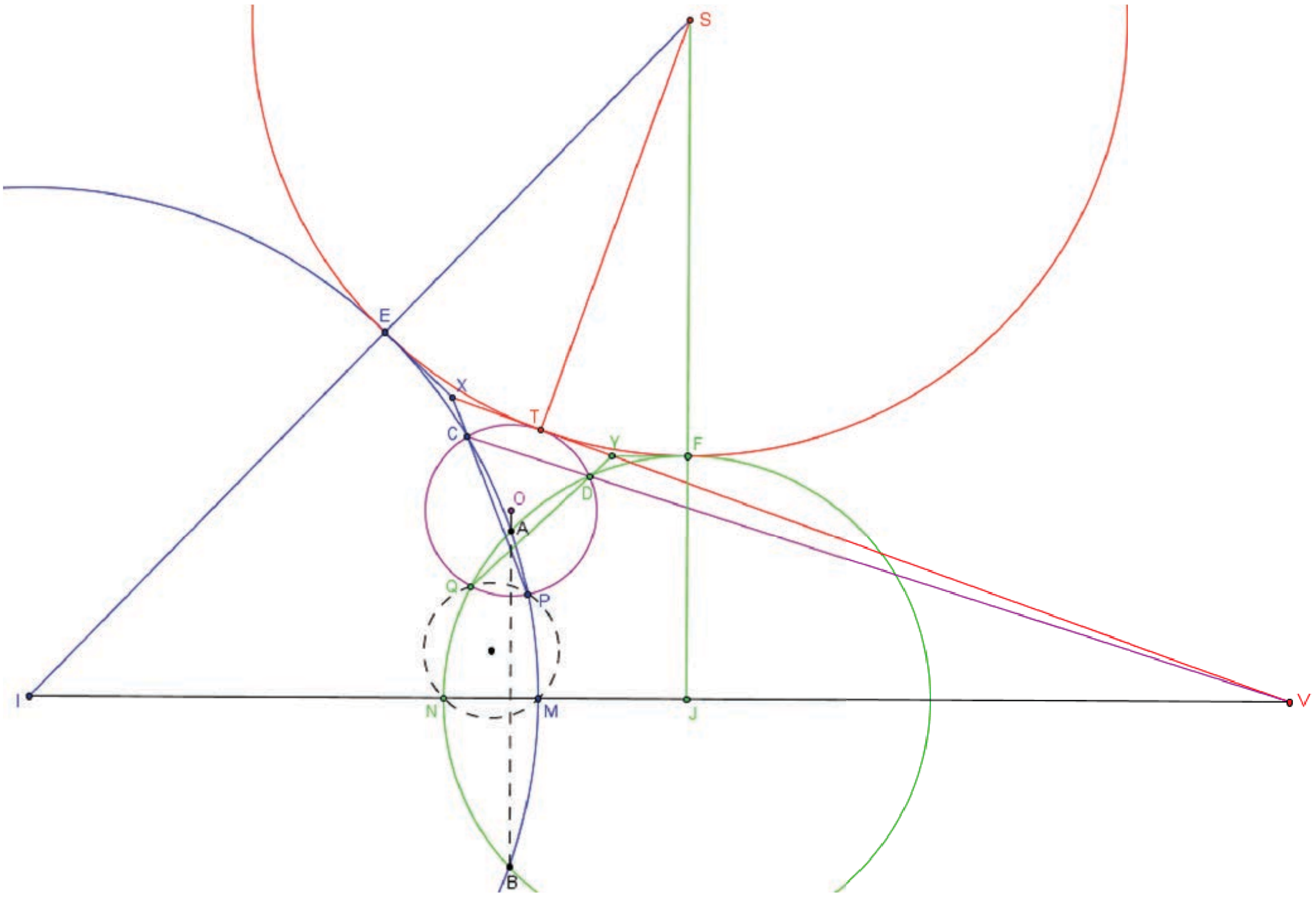
Reply

Petry

#1 Jun 10, 2011, 9:20 pm

Let $C\left(I\right)$ and $C\left(J\right)$ be two circles, $\left\{A,B\right\}=C\left(I\right)\cap C\left(J\right)$, $\left\{M\right\}=C\left(I\right)\cap\left(IJ\right)$ and $\left\{N\right\}=C\left(J\right)\cap\left(IJ\right)$.
A circle through M,N intersects again $C\left(I\right)$, $C\left(J\right)$ at P,Q respectively ($P\in\widehat{AM}$ and $Q\in\widehat{AN}$).
A circle $C\left(O\right)$, $O\in AB$, through P,Q intersects again $C\left(I\right)$, $C\left(J\right)$ at C,D respectively and $\left\{V\right\}=CD\cap IJ$.
VT is tangent to $C\left(O\right)$ at T ($T\notin\widehat{CPD}$), $\left\{X\right\}=PC\cap VT$ and $\left\{Y\right\}=QD\cap VT$.
XE is tangent to $C\left(I\right)$ at E ($E\notin\widehat{BAC}$), YF is tangent to $C\left(J\right)$ at F ($F\notin\widehat{BAD}$) and $\left\{S\right\}=IE\cap JF$.
Prove that the circle with center S and radius ST is tangent to the circles $C\left(I\right)$, $C\left(O\right)$, $C\left(J\right)$.

Attachments:



Luis González

#2 Jun 11, 2011, 12:19 am

$(M,N),(P,Q)$ and (D,C) are obviously inverse points under the direct inversion that swaps (I) and $(J)\Longrightarrow$
$V\equiv MN\cap PQ\cap CD$ is the center of the referred inversion. Since X and Y lie on the radical axes of $(I),\odot(PQD)$ and $(J),\odot(PQD)$, it follows that $XE=XT$ and $YF=YT$, i.e. circles $(X),(Y)$ with radii XE,YF are orthogonal to $(I),(J)$, respectively. Because of $VT^2=VC\cdot VD$, then T is a double point under the inversion. Therefore, due to conformity, the inverse of (Y) is the circle externally tangent to (Y) through T and orthogonal to $(I)\Longrightarrow(X)$ is the inverse of (Y), i.e. E,F are inverse points $\Longrightarrow$ Tangents of $(X),(Y)$ at E,F meet on the radical axis of $(X),(Y)$, i.e. the perpendicular to XY through T and the conclusion follows.

Quick Reply

High School Olympiads

another concurrency problem

Reply

Source: 0

sergei93

#1 Jun 10, 2011, 4:02 am

Let AD be one of the altitudes of $\triangle ABC$, and P an arbitrary point on it. The lines BP, CP meet AC, AB at M, N, respectively. Let MN meet AD at Q, and let F be an arbitrary point on AC. Let FQ intersect with line CN at E and with BC at T. Let MT intersect with CN at K. Prove that the lines FK, EM and CQ are concurrent.

Luis González

#2 Jun 10, 2011, 4:53 am • 1

Actually D can be any point on the sideline BC and $P \in AD$. Since $C(N, A, B, Q) = -1$, then $(E, F, Q, T) = -1 \Longrightarrow M(E, F, Q, T) = -1 \Longrightarrow (E, C, N, K) = -1$. Thus, in the complete quadrilateral $EQMC$, the lines EM, CQ and FK concur, as desired.

VHCR

#3 Jun 12, 2011, 1:50 am

> *Luis González wrote:*
> Thus, in the complete quadrilateral $EQMC$, the lines EM, CQ and FK concur, as desired.

Can you be more detailed in that part?

darij grinberg

#4 Jun 12, 2011, 10:14 pm

I assume that he uses the converse of the complete quadrilateral theorem. This converse states that if A, B, C, D are four points such that no three of them are collinear, and if we let $X = AB \cap CD$ and $Y = BC \cap DA$, and if we let $Z = XY \cap BD$, and if W is a point on the line XY such that $(X, Y, Z, W) = -1$, then $W \in AC$. This is, of course, equivalent to the complete quadrilateral theorem itself.

Quick Reply

High School Olympiads

Three collinear points (own)

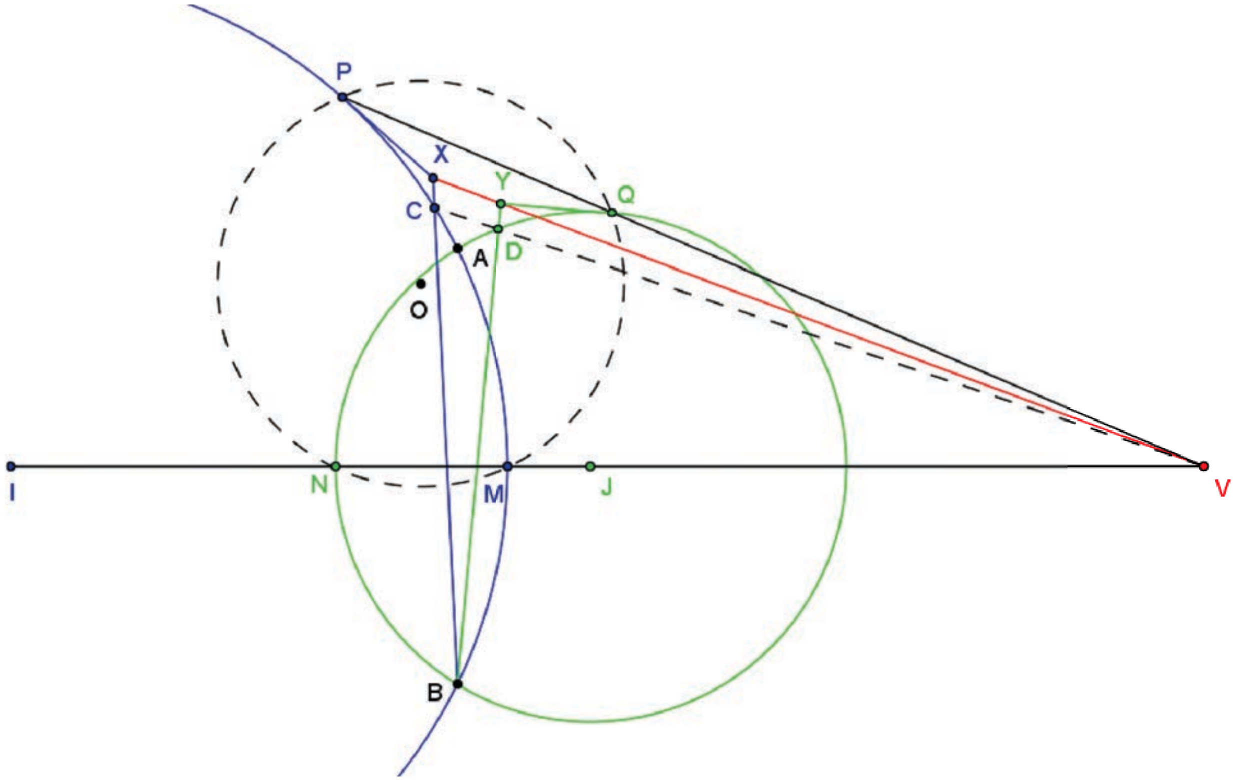
Reply

Petry

#1 Jun 10, 2011, 12:33 am

Let $C(I)$ and $C(J)$ be two circles, $\{A,B\} = C(I) \cap C(J)$, $\{M\} = C(I) \cap (IJ)$ and $\{N\} = C(J) \cap (IJ)$. A circle $C(O)$ through M, N intersects again $C(I), C(J)$ at P, Q respectively (A, P are on the same side of IJ) and $\{V\} = PQ \cap IJ$. A line through V intersects $C(I), C(J)$ at C, D respectively ($C \in \widehat{BMP}$, $D \in \widehat{BNQ}$). The tangent to $C(I)$ at P intersects BC at X and the tangent to $C(J)$ at Q intersects BD at Y.
Prove that X, Y, V are collinear points.

Attachments:



Luis González

#2 Jun 10, 2011, 2:59 am

Since M, N are inverse points under the direct inversion that takes $(I), (J)$ into each other, it follows that P, Q are also inverse points, i.e. $V \equiv MN \cap PQ$ is the center of the referred inversion. By conformity, we deduce that $\angle XPQ$ and $\angle YQP$ are equal and oppositely directed. Thus if $R \equiv PX \cap QY$, then $\triangle RPQ$ is isosceles with apex R, i.e. R has equal power to $(I), (J) \Longrightarrow R$ lies on the radical axis AB of $(I), (J)$. Hence, if PR, QR cut $(I), (J)$ again at E, F, then P, Q, E, F are concyclic. Since $\overline{XE} \cdot \overline{XP} = \overline{XC} \cdot \overline{XB}$ and $\overline{YF} \cdot \overline{YQ} = \overline{YD} \cdot \overline{YB}$, then XY is the radical axis of $\odot(BCD), \odot(PQE)$. Keeping in mind that P, Q, C, D are concyclic since P, Q and C, D are pairs of inverse points, then CD is the radical axis of $\odot(BDC), \odot(PQD)$ and PQ is the radical axis of $\odot(PQE), \odot(PQD) \Longrightarrow PQ, CD, XY$ concur at the radical center V of $\odot(PQD), \odot(PQE)$ and $\odot(BDC)$.

Quick Reply

High School Olympiads

incenter

Reply

mastergeo

#1 Jun 9, 2011, 9:49 pm

Let $\triangle ABC$ is a scalene triangle. D is an arbitrary point on BC. The incircle of triangle ABD is (I_1). The tangent line to (I_1) and parallel to BC cuts AD and AC at E, F. The incircle of triangle AEF is (I_2). Prove that:

$$r(I_1)+r(I_2)=r$$

, where $r(x)$ is the radius of the incircle of triangle x and r is the radius of the incenter of triangle ABC.

yetti

#2 Jun 10, 2011, 12:27 am • 1

h_a is the common A-altitude of $\triangle ABC, \triangle ABD, \triangle ADC$. (I_3) is incircle of $\triangle ADC \Longrightarrow$ $r(I_3)=r(I_2)\cdot\frac{h_a}{h_a-2r(I_1)}$ $(*)$ by similarity $\triangle ADC\sim\triangle AEF$ and $\frac{2}{h_a}=\frac{1}{r(I_1)}+\frac{1}{r(I_3)}-\frac{r}{r(I_1)r(I_3)}$ $(**)$ by An "Easy" Sangaku, The Nicest Sangaku, etc. Substitute $(*)$ into $(**)\Longrightarrow$ $2r(I_1)\left[r(I_1)+r(I_2)-r\right]=h_a\left[r(I_1)+r(I_2)-r\right]$. Since $2r(I_1)<h_a$, the conclusion follows.

Luis González

#3 Jun 10, 2011, 12:50 am • 1

Let (I_1) and (I_2) touch EF at M, N. Incircles $(I),(I_1),(I_3)$ of $\triangle ABC, \triangle ABD$ and $\triangle ADC$ touch BC at X, U, V. From the similarities $\triangle EMI_1\sim\triangle ENI_2$ and $\triangle ABC\cup D\sim\triangle(AB,AC,EF)\cup E$ we get

$$r_1+r_2=r_1\cdot\frac{EN+EM}{EM}=r_1\cdot\frac{DV+BU}{BU}$$

Substituting $DV=\frac{1}{2}(AD+DC-CA)$ and $BU=\frac{1}{2}(AB+BD-AD)$ gives

$$r_1+r_2=r_1\cdot\frac{BC+AB-CA}{2\cdot BU}=r_1\cdot\frac{BX}{BU}=r_1\cdot\frac{r}{r_1}=r.$$

Quick Reply

High School Olympiads

Concurrent

Reply

Headhunter

#1 Jun 9, 2011, 12:57 am

Hello.

Two triangles $\triangle ABC$, $\triangle PQR$ have a common incircle (I)
(I) touch BC, CA, AB, QR, RP, PQ at D, E, F, L, M, N respectively.
Show that PD, QE, RF are concurrent if and only if AL, BM, CN are concurrent.

I modified the name for beauty.

This post has been edited 1 time. Last edited by Headhunter, Jun 9, 2011, 7:36 am

Luis González

#2 Jun 9, 2011, 1:17 am • 1

Let $X \equiv MN \cap BC, Y \equiv NL \cap CA$ and $Z \equiv LM \cap AB.$ Lines MN and BC are the polars of P, D WRT $(I) \Longrightarrow X$ is the pole of PD WRT $(I).$ Likewise, Y, Z are the poles of QE, RF WRT $(I).$ Assume that PD, QE, RF concur $\Longrightarrow$ their poles X, Y, Z WRT (I) are collinear $\Longrightarrow \triangle ABC$ and $\triangle LMN$ are perspective through $\overline{XYZ},$ hence by Desargues theorem AL, BN, CN concur. The converse is immediate.

Quick Reply

High School Olympiads

midpoints and angle bisectors

Reply

Source: concurrence

prime04
#1 Jun 8, 2011, 4:23 pm

Let I be the incenter of $\triangle ABC$ and L, M, N be the midpoints of BC, CA, AB respectively. Let L_b, L_c be respectively the feet of perpendiculars from L to BI, CI. Also let M_c, M_a be the feet of perpendiculars from M to CI, AI and N_a, N_b be the feet of perpendiculars from N to AI, BI. Suppose P, Q, R are respectively the midpoints of L_bL_c, M_cM_a and N_aN_b.
Prove that LP, MQ and NR are concurrent.

This post has been edited 1 time. Last edited by prime04, Jun 22, 2012, 2:45 am

mahanmath
#2 Jun 8, 2011, 5:20 pm

Hint

Luis González
#3 Jun 9, 2011, 12:46 am • 1

According to this topic, the I-circumdiameter IA of $\triangle BIC$ is parallel to the L-median LP of $\triangle LL_bL_c$. Likewise, MQ and NR are respectively parallel to BI, CI. Consequently, LP, MQ and NR are the homologous lines of IA, IB and IC in the medial triangle $\triangle LMN$ of $\triangle ABC \Longrightarrow$ lines LP, MQ, NR concur at the incenter of $\triangle LMN$.

RSM
#4 Jun 9, 2011, 4:22 pm • 1

> *prime04 wrote:*
> Let I be the incenter of $\triangle ABC$ and L, M, N be the midpoints of BC, CA, AB respectively. Let L_b, L_c be respectively the feet of perpendiculars from L to BI, CI. Also let M_c, M_a be the feet of perpendiculars from M to CI, AI and N_a, N_b be the feet of perpendiculars from N to AI, CI. Suppose P, Q, R are respectively the midpoints of L_bL_c, M_cM_a and N_aN_b.
> Prove that LP, MQ and NR are concurrent.

I also proved that $LP \parallel AI$ but in a different way.
Draw $PX \perp LL_b, PY \perp LL_c$
Note that $\angle XPY = 90$
$$\frac{PX}{PY} = \frac{LL_c}{LL_b} = \frac{BI}{CI}$$
So PXY is homothetic with IBC.
So the line joining I and circumcenter of IBC is parallel to PL.
Note that the line joining I and circumcenter of IBC is II_a which is same as AI where I_a is a excenter of ABC.So $LP \parallel AI$.

Quick Reply

High School Olympiads

Mosp 2006 homework problem

paul1703

#1 Jun 7, 2011, 10:03 pm

Let ABC be a triangle with circumcenter O. Let A_1 be the midpoint of side BC. Ray AA_1 meets the circumcircle of triangle ABC again at A_2 (other than A). Let Q_a be the foot of the perpendicular from A_1 to line AO. Point P_a lies on line Q_aA_1 and on the tangent at A_2 to the circucmenter .Define points P_b and P_c analogously.
Prove that points P_a; P_b, and P_c lie on a line

Luis González

#2 Jun 7, 2011, 11:16 pm • 1

$G \equiv AA_1 \cap BB_1 \cap CC_1$ is the centroid of $\triangle ABC$. Let τ_A be the tangent of the circumcircle (O) through A. Since $A_1Q_a \perp AO$, then $A_1Q_a \parallel \tau_A \Longrightarrow A_1Q_a$ coincides with the image of τ_A under the homothety with center G and coefficient $-\frac{1}{2}$, which takes (O) into the 9-point circle $(N) \equiv \odot(A_1B_1C_1) \Longrightarrow P_aA_1$ is tangent to (N). Since $\triangle(\tau_A, AA_2, P_aA_2)$ is isosceles with base $\overline{AA_2}$, then $\triangle P_aA_1A_2$ is isosceles with legs $\overline{P_aA_1} = \overline{P_aA_2} \Longrightarrow P_a$ has equal power to $(N), (O)$. Similarly, P_b, P_c have equal powers to $(N), (O) \Longrightarrow P_a, P_b, P_c$ lie on the radical axis of $(N), (O)$.

Site Support

the font change

Reply

jeff10

#1 Jun 7, 2011, 5:35 am

Whenever I typed once, the font totally changed its size when I never toggled any settings. When I typed here, it was all fine. What was happening?

bluecarneal

#2 Jun 7, 2011, 6:02 am

It's happened twice to me in the past few minutes. I'm on IE.

Attachments:

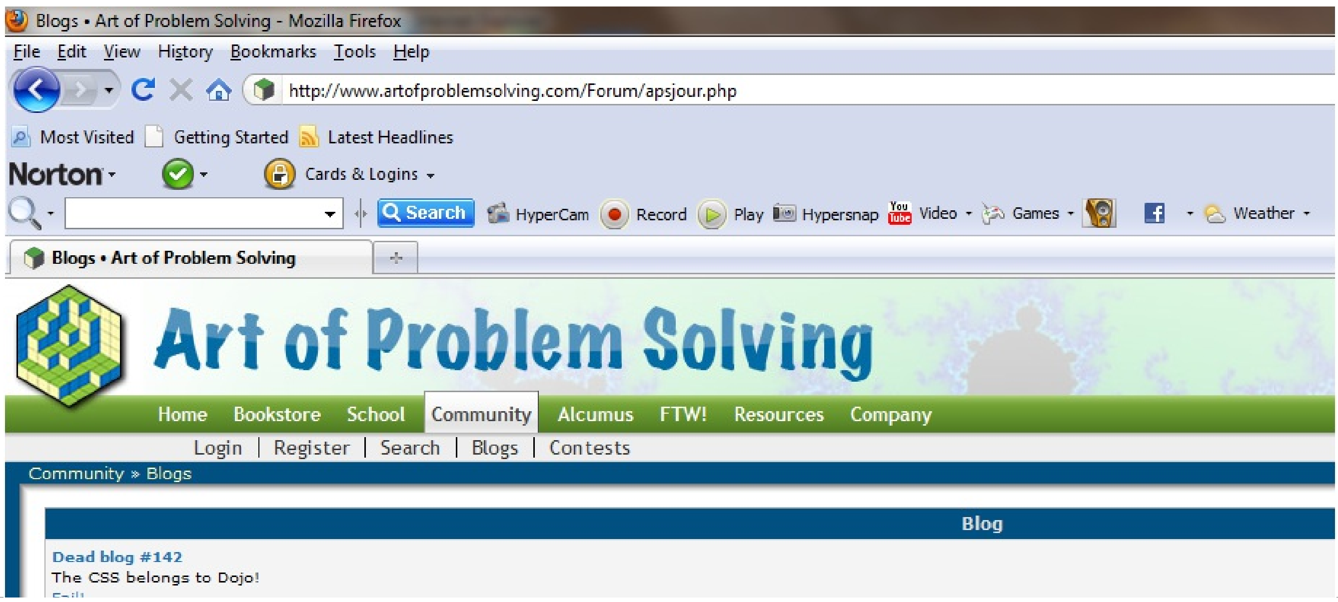
Blog	Owner	Entries	Views	Last Entry
Awesome Blog Encompassing Everything Awesome Your Portal to Awesomeness! served to you by PYTHON NUT CSS by: Dojo PNP points repealed!	PythonNut	41	894	35 minutes ago
The Life of a Hyperactive Nerd A blog about the not-so-typical teenager surnamed me. Contributor Rules *Most recent message:* What?	djmathman	41	608	Today, 6:57 pm

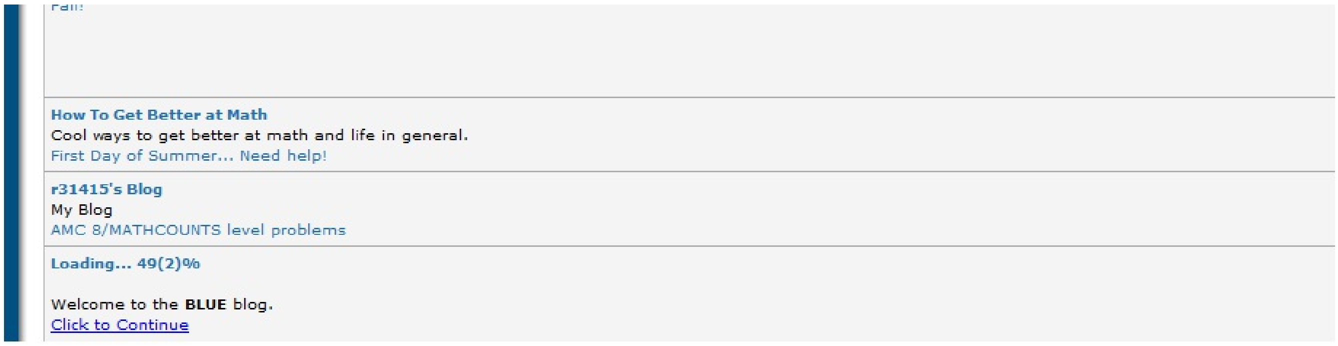
Spring

#3 Jun 7, 2011, 7:05 am

The font is still normal on Firefox though.

Attachments:





BOGTRO

#4 Jun 7, 2011, 8:56 am

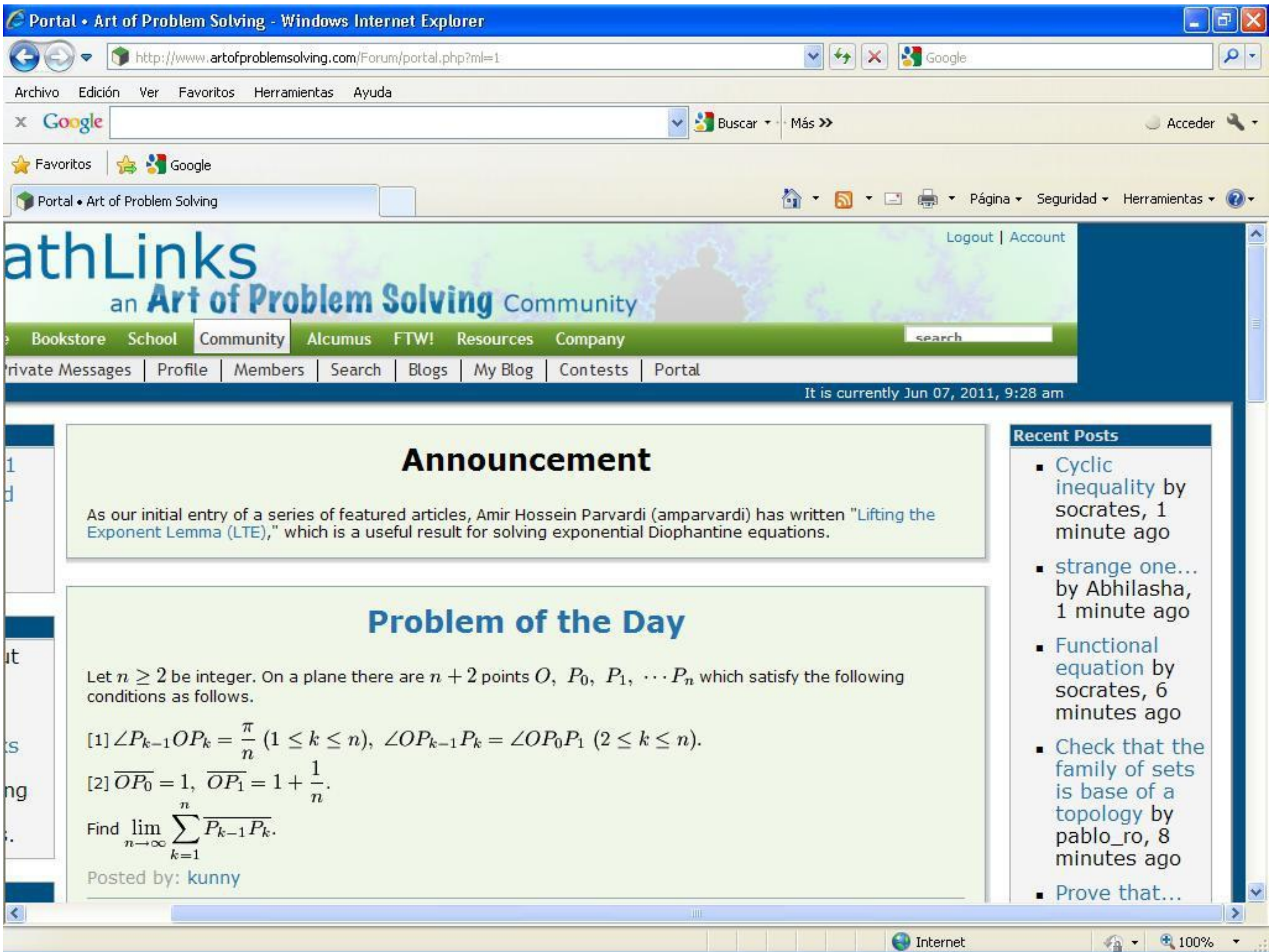
This has been occurring with me recently as well, on Firefox. Apparently it seems to be random and reverts back though.

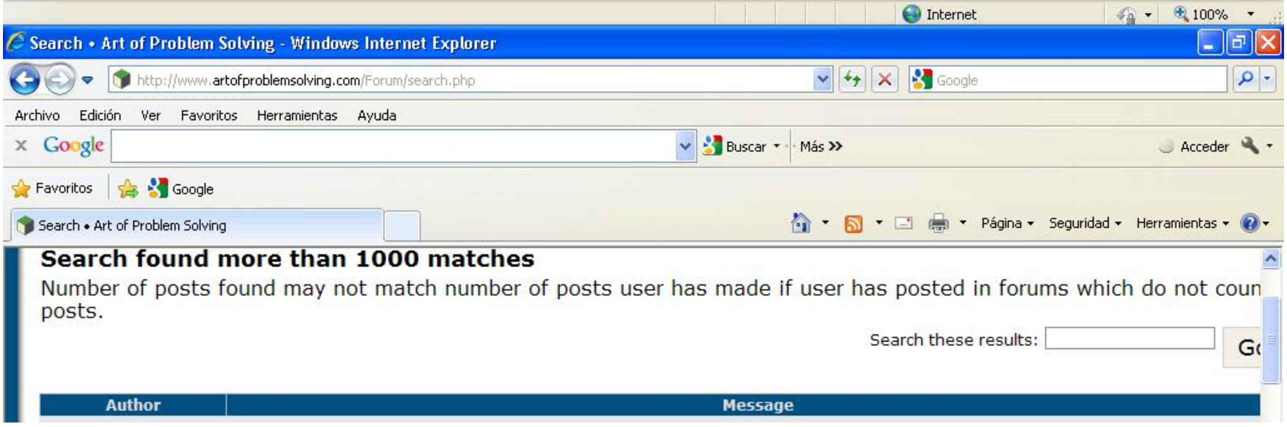
Luis González

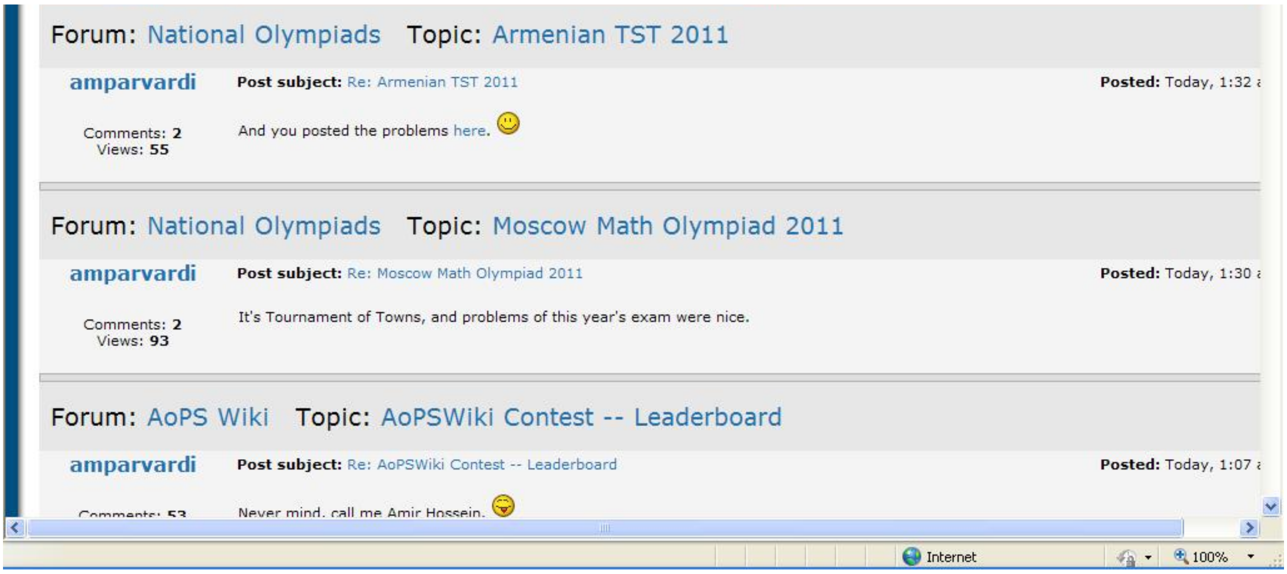
#5 Jun 7, 2011, 8:10 pm

I confirm the same problem with the font size using IE 8. But, it is still normal on Google Chrome and Firefox 4.0.

Attachments:






Forum: National Olympiads Topic: Armenian TST 2011

amparvardi | Post subject: Re: Armenian TST 2011 | Posted: Today, 1:32

Comments: 2
Views: 55

And you posted the problems here.

Forum: National Olympiads Topic: Moscow Math Olympiad 2011

amparvardi | Post subject: Re: Moscow Math Olympiad 2011 | Posted: Today, 1:30

Comments: 2
Views: 93

It's Tournament of Towns, and problems of this year's exam were nice.

Forum: AoPS Wiki Topic: AoPSWiki Contest -- Leaderboard

amparvardi | Post subject: Re: AoPSWiki Contest -- Leaderboard | Posted: Today, 1:07

Comments: 53

Never mind, call me Amir Hossein.

Internet 100%

BOGTRO

#6 Jun 7, 2011, 9:14 pm

Just occurred again. This has occurred with me in the past, but it was random and didn't think too much of it.

Can't upload screenshot because it's apparently too big.

QuantumTiger

#7 Jun 11, 2011, 8:41 am

Are you sure you aren't inadvertently doing Ctrl-plus or the equivalent on whatever browser you're on? On Chrome, there seems to be no problem.

phiReKaLk6781

#8 Jun 11, 2011, 11:55 am

I remember from back when I used Firefox that such things happen spontaneously at times for reasons I never figured out. Using the keyboard shortcuts or changing the zoom usually does restore the original sizes, though.

Quick Reply

High School Olympiads

Centroid

Reply

GeometryInqualitis
#1 Jun 6, 2011, 7:47 pm

Problem: Let ABCDEF be a convex hexagon. The points M, N, P, Q, R, S are chosen on the sides AB, BC, CD, DE, EF, FA respectively such that:

$$\frac{AM}{AB}=\frac{BN}{BC}=\frac{CP}{CD}=\frac{DQ}{DE}=\frac{ER}{EF}=\frac{FS}{FA}$$

Denote by G the centroid of triangle MPR and by G' the centroid of triangle NQS. Prove that lines GG' pass through a fixed point

Luis González
#2 Jun 6, 2011, 10:25 pm

In fact, the line GG' is fixed. δ_X denotes the oriented distance from X to the line AF and U,V denote the centroids of $\triangle ACE$ and $\triangle BDF$. Letting $\frac{MA}{MB}=k$, etc, we get

$$\delta_M=\frac{k\cdot\delta_B}{1+k}\ ,\ \delta_P=\frac{\delta_C+k\cdot\delta_D}{1+k}\ ,\ \delta_R=\frac{\delta_E}{1+k}$$

$$\Longrightarrow\ \delta_G=\frac{k\cdot\delta_B+k\cdot\delta_D+\delta_C+\delta_E}{3(1+k)}$$

Since $\delta_U=\dfrac{\delta_C+\delta_E}{3}\ ,\ \delta_V=\dfrac{\delta_B+\delta_D}{3}\Longrightarrow\ \delta_G=\dfrac{\delta_U+k\cdot\delta_V}{1+k}$

Using the same reasoning, i.e. projecting the same points on the line BC or DE yields that G lies on UV such that $\overline{GU}:\overline{GV}=k.$ By analogous reasoning we'll also have that G' lies on UV such that $\overline{G'V}:\overline{G'U}=k.$

ArefS
#3 Jun 6, 2011, 10:30 pm

Let G'',G''' be the centroids of the triangles ACE,BDF respectively.
we have:$3\overrightarrow{GG''}=\overrightarrow{MA}+\overrightarrow{PC}+\overrightarrow{RE}$ and $3\overrightarrow{GG'''}=\overrightarrow{MB}+\overrightarrow{PD}+\overrightarrow{RF}$.
it is obvious that there exists a real number t such that $\overrightarrow{MA}=t\overrightarrow{MB}$,$\overrightarrow{PC}=t\overrightarrow{PD}$,$\overrightarrow{RE}=t\overrightarrow{RF}$
so $GG''\parallel GG'''$ but GG'',GG''' have got a common point so G,G'',G''' are on a line. similarly G',G'',G''' are also on a line so GG' passes through a fixed point.

GeometryInqualitis
#4 Jun 7, 2011, 9:25 am

Denote by K the centroid of ABCDEF. Prove that KG=KG' and K lies on GG'

Quick Reply

High School Olympiads

pedal triangle

Reply

Source: 0

prime04

#1 Jun 5, 2011, 9:06 pm

Let P be a point inside triangle ABC and D, E, F be respectively the feet of perpendiculars from P to BC, CA, AB. Suppose BP meets EF at K and AP meets DF at L.
Prove that AB is tangent to the circumcircle of $\triangle DFE$ if and only if $KL \parallel AB$.

This post has been edited 3 times. Last edited by prime04, Feb 15, 2012, 12:03 am

Luis González

#2 Jun 6, 2011, 12:33 am • 1

Assume that $\odot(DEF)$ is tangent to $AB \Longrightarrow \angle DEF = \angle DFB.$ But since $PDBF$ is cyclic due to the right angles $\angle PFB, \angle PDB,$ it follows that $\angle DEF = \angle DPB \Longrightarrow EDPK$ is cyclic. Likewise, from $\angle EDF = \angle EFA$ and the cyclic quadrilateral $PEAF,$ we obtain that $DEPL$ is cyclic. Consequently, $EKPL$ is cyclic. Hence $\angle KLP = \angle FEP = \angle FAP \Longrightarrow KL \parallel AB.$ The converse is proved analogously.

Quick Reply

High School Olympiads

find angle and ratios of angles in acute triangle

Reply

Source: Albanian IMO 2011 TST

ridgers
#1 May 27, 2011, 1:02 pm

In the acute angle triangle ABC the point O is the center of the circumscribed circle and the lines OA, OB, OC intersect sides BC, CA, AB respectively in points M, N, P such that $\angle NMP = 90^o$.

(a) Find the ratios $\frac{\angle AMN}{\angle NMC}, \frac{\angle AMP}{\angle PMB}$.

(b) If any of the angles of the triangle ABC is 60^o, find the two other angles.

kalifornia001
#2 Jun 1, 2011, 1:42 am

could anyone solve this exercice?

Luis González
#3 Jun 1, 2011, 4:55 am • 1

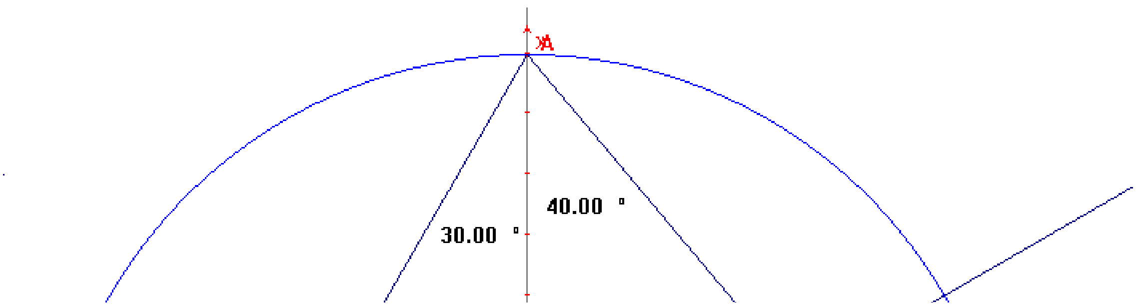
Part **(a)** can be solved without using any property of the circumcenter. This is, let O be an interior point whose cevian triangle $\triangle MNP$ is right at M $(\star)$. Lines MP and MN cut AC, AB at U, V. Since cross ratio (A, C, N, U) is harmonic and $MU \perp MN$, it follows that MN, MU bisect $\angle AMC$ internally and externally. Similarly, from the harmonic cross ratio (A, B, P, V) and $MV \perp MP$, we deduce that MP bisect $\angle AMB$ internally.

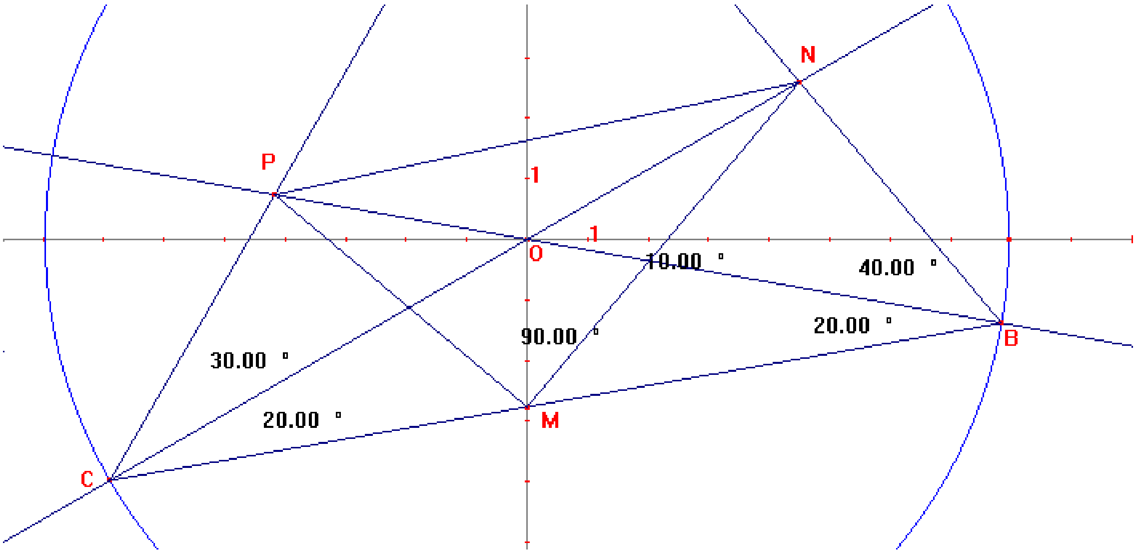
$(\star)$ Locus of point O whose cevian triangle $\triangle MNP$ is M-right, is a quartic $\mathcal{Q}$, but its isogonal conjugate WRT $\triangle ABC$ is a hyperbola $\mathcal{H}$, which passes though the feet of the internal and external angle bisectors of B, C and the isogonal conjugates of the B- and C- vertices of the antimedial triangle.

vanstraelen
#4 Jun 4, 2011, 2:18 am • 1

Construction.
Circle with midpoint O, see picture.
On the y-axis: point A.
Because $\angle ABC = 60^o$, $\angle AOC = 120^o \rightarrow$ point C.
$A(0,1), C(-\frac{\sqrt{3}}{2}, -\frac{1}{2}), B(\cos a, \sin a)$.
Calculating $M = BC \cap AO, N = AB \cap CO, P = AC \cap BO$.
Using this coordinates for $MN \perp MP \rightarrow$ equation in a, with solution $a : -10^o$
$\Longrightarrow \angle BAC = 70^o$ and $\angle ACB = 50^o$

Attachments:





Luis González

#5 Jun 5, 2011, 12:59 pm • 1

(b) Assume that $\angle B = 60^\circ$. We already know from part **(a)** that MP bisects $\angle AMB$ internally. Since $OPBM$ is cyclic, due to $\angle MOP = 120^\circ$, then it follows that P is the midpoint of the arc OB of $\odot(OMB)$, i.e. $\triangle POB$ is isosceles with apex P. Hence $\angle ABO = \angle POB \Longrightarrow$

$$180^\circ - 2\angle A = 90^\circ - \angle C = \angle A - 30^\circ \Longrightarrow \angle A = 70^\circ \Longrightarrow \angle C = 50^\circ$$

littletush

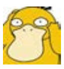
#6 Nov 6, 2011, 11:02 am

let $\angle BMP = x, \angle PMA = y, \angle AMN = z, \angle NMC = w$
then it's easy to prove $sinxsinz = sinysinw$
if $x > y$,then $w > z$
so $y + z < 90$,contradiction.
analagously we can reach a contradiction when $x < y$.
so $x = y, w = z$.

littletush

#7 Nov 6, 2011, 11:04 am

another solution
let MN intersects AB at Q
then B,P,A,Q form a harmonic sequence
and $\angle PMQ = 90$,so it's easy to prove that MP bisects $\angle BMA$.



Cassius

#8 Jun 2, 2012, 5:36 pm

(a) Simple angle chasing gives $\widehat{AOP} = \pi - 2\beta$, $\widehat{POB} = \pi - 2\alpha$, $\widehat{OAP} = \widehat{OBP} = \frac{\pi}{2} - \gamma$. Using the Sine theorem on triangles $\triangle APO$ and $\triangle PBO$ then equating the lenghts of OP give $\frac{AP}{BP} = \frac{\sin(2\beta)}{\sin(2\alpha)}$. Moreover, Let $Q := AB \cap NM$; then the Menelaus theorem in triangle $\triangle ABC$ with transversal MN gives $\frac{AQ}{BQ} = -\frac{\sin(2\beta)}{\sin(2\alpha)}$, so indeed we have $(ABPQ) = -1$. Since $PM \perp MQ$, PM is indeed a bisector of angle $\widehat{AMB}$ and so (by repeating the reasoning) the desired ratio equals 1.

(b) Again, suppose that $\beta = \frac{\pi}{3}$ (other cases are treated in the same way): angle chasing gives $\widehat{AMB} = \frac{\pi}{2} - \beta + \gamma$, $\widehat{AMP} = \frac{\pi}{2} - \gamma$, $\widehat{PMB} = \pi - 2\alpha$; using the previously established fact (MP bisector) and $\alpha + \gamma = \frac{2\pi}{3}$ allows us to establish easily that $\alpha = \frac{7\pi}{18}, \gamma = \frac{5\pi}{18}$.

Quick Reply

High School Olympiads

HM is perpendicular to a common chord

Reply

Source: Thailand, 2005

Vikernes
#1 Jun 3, 2011, 8:27 am

Let ABC an acute triangle so that $AB \neq AC$ and let H the orthocenter of ABC. Points D and E are in the sides AB, AC respectively so that D, H, E are collinear and $AE = AD$. Prove that if M is the midpoint of BC, then MH is perpendicular to the straight line that joins A with the second intersections (different of A) of the circumcircles of the triangles AED and ABC.

jgnr
#2 Jun 3, 2011, 9:34 am

=CWMO 2009 #3 (no solution though)

yetti
#3 Jun 3, 2011, 10:41 am • 1

> Vikernes wrote:
> Let ABC an acute triangle so that $AB \neq AC$ and let H the orthocenter of ABC. Points D and E are in the sides AB, AC respectively so that D, H, E are collinear and $AE = AD$. Prove that if M is the midpoint of BC, then MH is perpendicular to the straight line that joins A with the second intersections (different of A) of the circumcircles of the triangles AED and ABC.

$P \in AB, Q \in AC$ are feet of altitudes $CH, BH. Z \in \odot(ABC)$ is diametrically opposite of $A. BZCH$ is parallelogram with diagonal intersection $M. ZMH$ cuts $\odot(ABC)$ again at K and $ZMHK \perp AK. APHQK$ is cyclic, with circumcircle $\odot(APQ)$ on diameter $AH. \triangle AED$ is A-isosceles $\Longrightarrow \triangle BHD \sim \triangle CHE$ with corresponding H-altitudes HP, HQ are similar $\Longrightarrow \dfrac{p(D, \odot(APQ))}{p(D, \odot(ABC))} = \dfrac{\overline{DA} \cdot \overline{DP}}{\overline{DA} \cdot \overline{DB}} = \dfrac{\overline{DP}}{\overline{DB}} = \dfrac{\overline{EQ}}{\overline{EC}} = \dfrac{\overline{EA} \cdot \overline{EQ}}{\overline{EA} \cdot \overline{EC}} = \dfrac{p(E, \odot(APQ))}{p(E, \odot(ABC))} \Longrightarrow$ circles $\odot(ADE), \odot(APQ), \odot(ABC)$ are coaxal with common points A, K. (See the theorem at Constant value.)

Luis González
#4 Jun 3, 2011, 12:56 pm • 1

Generalization: A circle (M) passes through the vertices B, C of $\triangle ABC$ and cuts its sides $\overline{AB}, \overline{AC}$ again at P, Q. $H \equiv BQ \cap CP$ and D, E lie on $\overline{AB}, \overline{AC}$ such that $AD = AE$ and D, E, H are collinear. Then MH is perpendicular to the radical axis of $\odot(ABC)$ and $\odot(ADE)$.

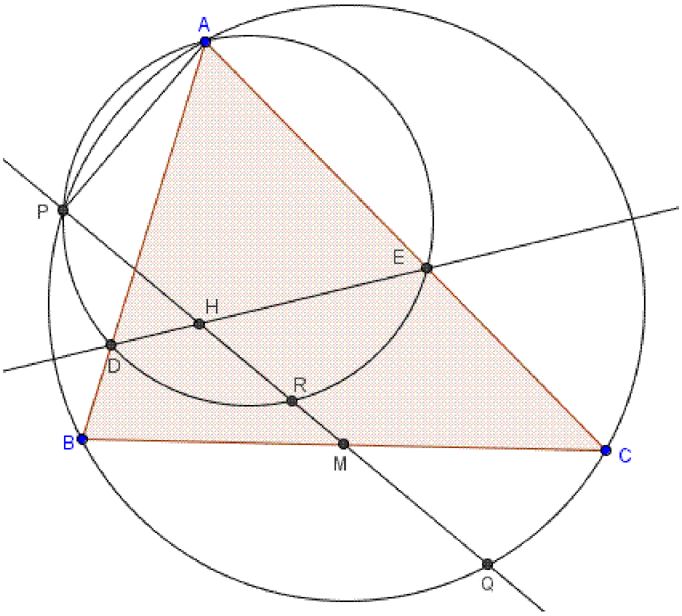
Let $F \equiv BC \cap PQ$. Since AF, AH are polars of H, F WRT (M), then $MH \perp AF$ (through U) and $AH \perp MF$ (through V), i.e. H is the orthocenter of $\triangle MAF$. Thus, $\overline{FU} \cdot \overline{FA} = \overline{FV} \cdot \overline{FM} = \overline{FB} \cdot \overline{FC} \Longrightarrow U$ lies on the circumcircle (O) of $\triangle ABC$. Since the pencil $A(D, E, H, U)$ is harmonic and $MHU \perp AU$, we deduce that UH, UA bisect $\angle DUE$ internally and externally. Thus, $AD = AE$ implies that A is the midpoint of the arc DUE of $\odot(DUE)$, i.e. A, D, E, U are concyclic and the conclusion follows.

jgnr
#5 Jun 3, 2011, 5:43 pm • 1

Let $(MH$ and $(HM$ meet the circumcircle of ABC at P and Q respectively. Let (MH) and circle (ADE) again at R. It is well-known that AQ is the diameter of circle (ABC), so $\angle APQ = 90^\circ$. From Russia 2000, we also know that AR is the diameter of (ADE). Since $\angle APR = 90^\circ$, then $APDE$ is cyclic. So P is the second intersection of (AED) and (ABC).

Since $MH \perp AP$, we are done.

Attachments:



sunken rock

#6 Jun 3, 2011, 6:03 pm

I shall use the same notations as yetti.
There are well known facts that HM intersects AW, the angle bisector of $\widehat{BAC}$ on the circumcircle of $\triangle ADE$, and the circumcircle of $\triangle ABC$ at Z, diametrically opposite to A.
Proof: obviously, $DE \perp AW$ (I do not find necessary to show again the proof that $BZCH$ is parallelogram), hence DE is the angle bisector of $\angle BHP$ and, as $\triangle BHP \sim CHQ$, we get $\frac{DP}{DB} = \frac{QE}{CE}$, that is, with $PH \parallel BZ \perp AB$ and $HQ \parallel CZ \perp AC$, we get that the perpendicular at D on AB and perpendicular at E on AC concur on HZ. Obviously, the angles at D and E being right, HZ intersects the circle $\odot ADE$ at the point diametrically opposite to A, i.e. on its angle bisector.
From these we get $AK \perp HZ$, done.

Note: the problem (lemma) I used has been posted before on this site (some Russian olympiad by 1990's).

Best regards,
sunken rock

jgnr

#7 Jun 3, 2011, 6:21 pm

> *sunken rock wrote:*
> Note: the problem (lemma) I used has been posted before on this site (some Russian olympiad by 1990's).

It's from Russia 2000, I used the same lemma. In fact, our proofs are identical

Quick Reply

High School Olympiads

\$aPA^2+bPB^2+cPC^2\$ is constant for P on incircle.

Locked

Source: ILL 1979-43

Goutham

#1 Jun 3, 2011, 12:11 am

Let a, b, c denote the lengths of the sides BC, CA, AB, respectively, of a triangle ABC. If P is any point on the circumference of the circle inscribed in the triangle, show that $aPA^2 + bPB^2 + cPC^2$ is constant.

Luis González

#2 Jun 3, 2011, 1:11 am

This problem has been discussed before several times. For instance, see the topics

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49&t=4197
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=260032
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=389137

High School Olympiads

Find the Locus

Reply

ryanstone

#1 Jun 1, 2011, 3:04 pm

Find the Locus of P lying inside an equilateral triangle ABC such that $\angle PAB+\angle PBC+\angle PCA=90^\circ$

teach1951

#2 Jun 2, 2011, 3:34 am

Since all the angles are 60 degrees, then P can lie on any of the angle bisectors. The segments from P to the other vertices creates an Isosceles triangle [geogebra]cc443b1c92ad8c054ace2d9642a80adc92c80aea[/geogebra] [geogebra]cbaace86bdb21d4534d673cb285ad770b6375675[/geogebra] This gives us one angle of 30 degrees another angle we can call x degrees and the third angle will be 60-x degrees. Their sum is therefore 90 degrees.

Pantum

#3 Jun 2, 2011, 8:01 am

Obviously, P is located in three on the midline.

Luis González

#4 Jun 2, 2011, 11:04 am • 3

Let us tackle this problem for a scalene triangle ABC. We'll prove that locus of points P satisfying the desired relation is a pivotal self-isogonal cubic, namely the M'Cay cubic.

PA, PB, PC cut the circumcircle (O) of $\triangle ABC$ again at A', B', C' and let X, Y, Z be the projections of P onto BC, CA, AB, respectively. $B'C'$ cut AC at U. Using directed angles (mod 180) we get

$$\angle AUC'=\angle CAB'+\angle AB'C'=\angle PBC+\angle PCA$$

$$\angle AYZ=90^\circ-\angle PYZ=90^\circ-\angle PAB$$

Since $\angle PAB+\angle PBC+\angle PCA=90^\circ$, it follows that $\angle AYZ=\angle AUC'\Longrightarrow B'C'\parallel YZ$. By similar reasoning we have $C'A'\parallel ZX$ and $A'B'\parallel XY\Longrightarrow \triangle A'B'C'$ and $\triangle XYZ$ are homothetic with homothetic center K lying on PP'. Thus, circumcenters O, V of $\triangle A'B'C'$ and $\triangle XYZ$ are collinear with the homothetic center K. Since the isogonal conjugate P^{-1} of P WRT $\triangle ABC$ is the reflection of P about V (well-known), then we deduce that PP^{-1} passes through O. Now, we use barycentric coordinates WRT $\triangle ABC$

$$O(a^2S_A:b^2S_B,c^2S_C)\ ,\ P(x:y:z)\ ,\ P^{-1}\left(\frac{a^2}{x}:\frac{b^2}{y}:\frac{c^2}{z}\right)$$

The collinearity of the latter points translates to:

$$a^2S_A(b^2z^2-c^2y^2)x+b^2S_B(c^2x^2-a^2z^2)y+c^2S_C(a^2y^2-b^2x^2)z=0$$

Which is the barycentric equation of the M'Cay cubic, i.e. the self-isogonal cubic with pivot O. If $\triangle ABC$ is equilateral (the proposed problem) then this cubic degenerates into

$$(z^2-y^2)x+(x^2-z^2)y+(y^2-x^2)z=0\Longleftrightarrow(x-y)(y-z)(z-x)=0$$

Thus, in such case, P lies on any of the three symmetry axes of the equilateral $\triangle ABC$.

Quick Reply

High School Olympiads

concurrent

Reply

huyhoang

#1 Jun 1, 2011, 2:54 pm • 1

Let ABC be triangle with circumcircle (O). A_0 is the midpoint of BC. AA_0 cuts the circumcircle (O) at A_1. A line through A which parallels to BC cuts (O) at A_2. B_1, B_2, C_1, C_2 are defined similarly. Prove that A_1A_2, B_1B_2, C_1C_2 concurrent at a point in the Euler line of triangle ABC.

Note: I have already proved that A_1A_2, B_1B_2, C_1C_2 are concurrent using pole and polar, but I couldn't prove the concurrent point is in the Euler line of triangle ABC.

ZetaX

#2 Jun 1, 2011, 3:22 pm

Does this look like an Article to you¿ -> Moved.

huyhoang

#3 Jun 1, 2011, 3:48 pm

Sorry but I don't know how to move, can you show me

Goutham

#4 Jun 1, 2011, 4:12 pm

> *huyhoang wrote:*
> Sorry but I don't know how to move, can you show me

You can't move it unless you are a moderator or an admin. ZetaX moved it already.

yetti

#5 Jun 1, 2011, 4:55 pm • 1

Tangents of (O) at A, B, C cut BC, CA, AB at X, Y, Z and tangents of (O) at A_1, B_1, C_1 cut BC, CA, AB at X_1, Y_1, Z_1. X, Y, Z are poles of A-, B-, C-symmedians of $\triangle ABC$ WRT (O). X_1, Y_1, Z_1. are reflections of X, Y, Z and A_1A_2, B_1B_2, C_1C_2 are reflections of the symmedians in the perpendicular bisectors of $BC, CA, AB \Longrightarrow X_1, Y_1, Z_1$ are poles of A_1A_2, B_1B_2, C_1C_2 WRT (O). X_1, Y_1, Z_1 are collinear and $X_1Y_1Z_1 \perp OH$ (see http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=23492) $\Longrightarrow$ the infinite point P_∞ of $X_1Y_1Z_1$ (identical with the direction of this line) is the pole of Euler line $OH \Longrightarrow$ polars $A_1A_2, B_1B_2, C_1C_2, OH$ of collinear points X_1, Y_1, Z_1, P_∞ concur.

huyhoang

#6 Jun 1, 2011, 8:07 pm

can you explain more? I couldn't understand the solution in the link

Luis González

#7 Jun 1, 2011, 11:22 pm • 2

The result is true for any point on the Euler line (not necessarily the centroid).

Proposition. P is an arbitray point on the Euler line e of $\triangle ABC.$ Lines AP, BP, CP cut its circumcircle (O) again at $A_1, B_1, C_1.$ Parallels from A, B, C to BC, CA, AB cut (O) again at $A_2, B_2, C_2.$ Then A_1A_2, B_1B_2, C_1C_2 concur on the Euler line e of $\triangle ABC.$

Normals to AA_1, BB_1, CC_1 through A_1, B_1, C_1 cut (O) again at $X, Y, Z \Longrightarrow AX, BY, CZ$ are circumdiameters of $\triangle ABC$ meeting at $O.$ By Pascal theorem for hexagon $AA_1YBB_1X,$ the intersections $P \equiv AA_1 \cap BB_1,$ $R \equiv XB_1 \cap YA_1$ and $O \equiv AX \cap BY$ are collinear. Since the antipodes X, Y, Z of A, B, C WRT (O) lie on the altitudes of the antimedial triangle $\triangle A^*B^*C^*$ of $\triangle ABC,$ it follows that XA_2, YB_2, ZC_2 intersect e at the orthocenter L of $\triangle A^*B^*C^*.$ Now, by Pascal theorem for hexagon $XA_2A_1YB_2B_1,$ the intersections $L \equiv XA_2 \cap YB_2,$ $Q \equiv A_1A_2 \cap B_1B_2$ and $R \equiv XB_1 \cap YA_1$ are collinear $\Longrightarrow e, A_1A_2, B_1B_2$ concur at $Q.$ Likewise, $Q \in C_1C_2.$

Quick Reply

High School Olympiads

Prove that

Reply

tuan119
#1 May 28, 2011, 2:29 pm

In the tetrahedron $OABC$, we denote by α, β, γ the measures of the angles $\angle BOC$, $\angle COA$, and $\angle AOB$, respectively.
Prove that:

$$36.V^2 = |OA|^2.|OB|^2.|OC|^2.\begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}$$

Luis González
#2 May 30, 2011, 1:27 am

For convenience let the unit sphere with center O cut the trihedron O_{ABC} into a spherical triangle XYZ with sides α, β, γ. $X \in OA, Y \in OB$ and $Z \in OC$. Let h be the spherical altitude issuing from Z. Then $\sin h = \sin\beta \cdot \sin X$, but by cosine theorem in XYZ (Bessel Formula) we have

$$\cos X = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma} \Longrightarrow \sin^2 X = 1 - \left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right)^2 \Longrightarrow$$

$$\sin(O_{XYZ}) = \sin\gamma \cdot \sin h = \sqrt{\sin^2\beta\sin^2\gamma - (\cos\alpha - \cos\beta\cos\gamma)^2}$$

Substituting $\sin^2\beta = 1 - \cos^2\beta$ and $\sin^2\gamma = 1 - \cos^2\gamma$ yields

$$\sin(O_{XYZ}) = \sin(O_{ABC}) = \sqrt{2\cos\alpha\cos\beta\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 1}$$

Since $V = \frac{1}{6}OA \cdot OB \cdot OC \cdot \sin(O_{ABC})$, then it follows that

$$36V^2 = OA^2 \cdot OB^2 \cdot OC^2 \cdot (2\cos\alpha\cos\beta\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 1)$$

P.S. The proof is substantially easier using vectors (scalar product). See here

tuan119
#3 May 30, 2011, 10:13 am

Thank **luis**!
and my solution: **(#5)**
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=386596

Quick Reply

High School Olympiads

In regular 30-gon some diagonals have a common point.

Reply

Markelov

#1 May 28, 2011, 2:13 am

A1..A30 is a regular triacontagon (30-gon). Prove, that diagonals 1..11, 8..18 and 10..27 intersect in one point. Moreover, 5-13, 6-14 and 9-22 intersects in the same point, so total 6 diagonals have the same point.

yetti

#2 May 28, 2011, 3:49 pm • 2

Label the regular 30-gon $P_1P_2...P_{30}$ with circumcircle (O, R). Label its side and diagonals $d_1 < d_2 < ... < d_{15}$. Obviously, $d_{15} = 2R = 2d_5$.

Let $X \equiv P_1P_{11} \cap P_8P_{18}$ and $X' \equiv P_5P_{13} \cap P_6P_{14}$ and $X'' \equiv P_9P_{22} \cap P_{10}P_{27}$.
By symmetry, X, X', X'' are all on the perpendicular bisector of diameter P_2P_{17}.
$\triangle OXP_8 \sim \triangle P_{23}P_{18}P_{15}$ are similar, having equal angles $\Longrightarrow OX = R \cdot \frac{d_5}{d_8}$.
$\triangle OX'P_6 \sim \triangle P_{21}P_{14}P_7$ are similar, having equal angles $\Longrightarrow OX' = R \cdot \frac{d_7}{d_{14}}$.
$\triangle OX''P_9 \sim \triangle P_{24}P_{22}P_{21}$ are similar, having equal angles $\Longrightarrow OX'' = R \cdot \frac{d_2}{d_3}$.

Trigonometry

From equilateral triangles with bases $d_{12}, d_{14} \Longrightarrow d_{12} = d_8 + d_2$ and $d_{14} = d_6 + d_4$.
By Ptolemy for $P_1P_3P_6P_9 \Longrightarrow d_5d_6 = d_2d_3 + d_3d_8 = d_3d_{12}$.
By Ptolemy for $P_1P_5P_8P_{13} \Longrightarrow d_7d_8 = d_4d_5 + d_3d_{12} = d_4d_5 + d_5d_6 = d_5d_{14}$.
$\Longrightarrow \frac{OX'}{OX} = \frac{d_7d_8}{d_5d_{14}} = 1 \Longrightarrow X' \equiv X$ are identical.
$\Longrightarrow$ Diagonals $P_1P_{11}, P_8P_{18}, P_5P_{13}, P_6P_{14}$ concur at X.

Missing Ptolemy ?

Let $Y \equiv P_4P_{21} \cap P_{15}P_{28}$ and $Y' \equiv P_7P_{23} \cap P_{12}P_{26}$.
By symmetry, Y, Y' are both on the perpendicular bisector of diameter P_2P_{17}.
$\triangle OYP_{28} \sim \triangle P_{17}P_{19}P_{28}$ are similar, having equal angles $\Longrightarrow OY = R \cdot \frac{d_2}{d_9}$.
$\triangle OY'P_{26} \sim \triangle P_{22}P_{23}P_{26}$ are similar, having equal angles $\Longrightarrow OY' = R \cdot \frac{d_1}{d_4}$.

Let $Q \equiv P_1P_{10} \cap P_4P_{13}$. $P_1OP_{13}Q$ is rhombus and $\triangle P_1P_4Q$ is isosceles $\Longrightarrow d_9 = d_3 + d_5$.
By Ptolemy for $P_1P_2P_3P_6 \Longrightarrow d_2d_4 = d_1d_3 + d_1d_5 = d_1d_9$.
$\Longrightarrow \frac{OY'}{OY} = \frac{d_1d_9}{d_2d_4} = 1 \Longrightarrow Y' \equiv Y$ are identical.
$\Longrightarrow$ Diagonals $P_4P_{21}, P_{15}P_{28}, P_7P_{23}, P_{12}P_{26}$ concur at Y.

$\Longrightarrow$ Diagonals $P_4P_{14}, P_{12}P_1, P_{23}P_9$ concur at isogonal conjugate Y^* of Y WRT $\triangle P_4P_{12}P_{23}$.
$\Longrightarrow$ Diagonals $P_6P_{14}, P_{11}P_1, P_{22}P_9$ concur at isogonal conjugate Y^{**} of Y^* WRT $\triangle P_1P_9P_{14}$.

$\Longrightarrow Y^{**} \equiv X$ and $Y^{**} \equiv X''$ are identical.
$\Longrightarrow$ Diagonals $P_1P_{11}, P_8P_{18}, P_5P_{13}, P_6P_{14}, P_9P_{22}, P_{10}P_{27}$ all concur at X.

Luis González

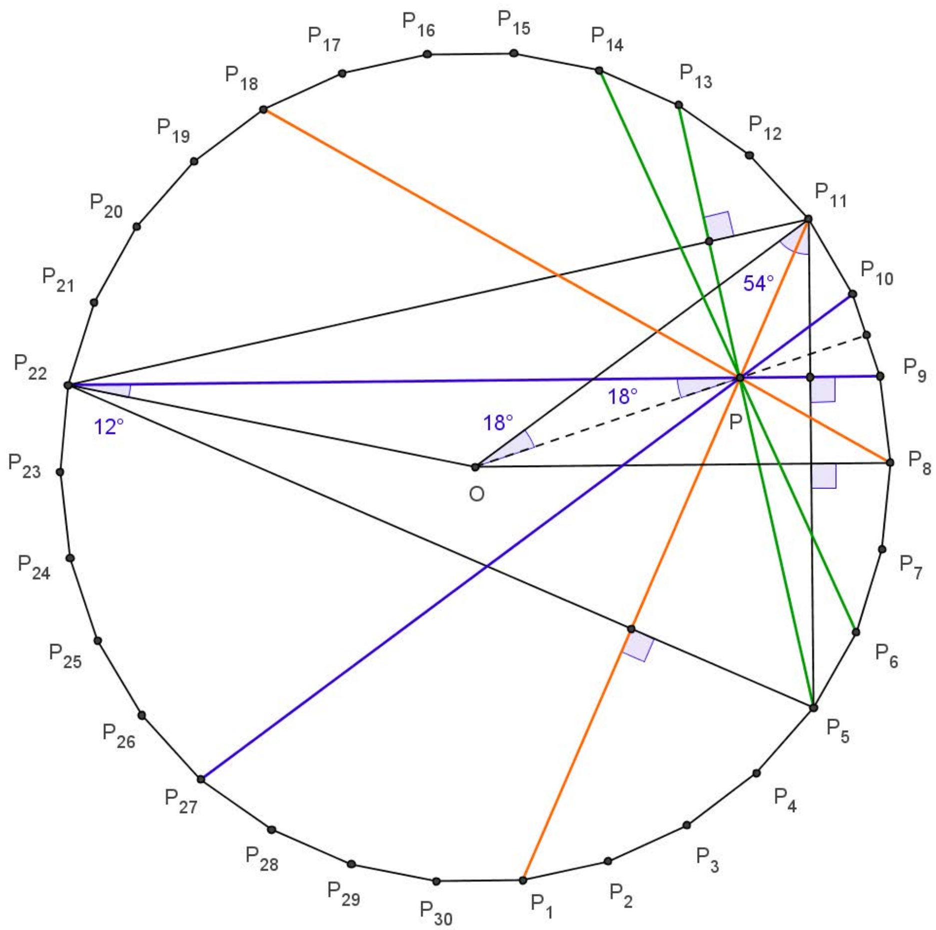
#3 May 29, 2011, 7:21 am

Working with the same initial notation of yetti, angle chasing reveals that $P_1P_{11} \perp P_5P_{22}, P_5P_{11} \perp P_9P_{22}$ and $P_5P_{13} \perp P_{11}P_{22}$. Hence, P_1P_{11}, P_5P_{13} and P_9P_{22} concur at the orthocenter P of $\triangle P_5P_{11}P_{22}$. Therefore, $PP_{22} = 2 \cdot \text{dist}(O, P_5P_{11}) = 2R \cdot \sin 54^\circ \Longrightarrow PP_{22} : R = \varphi$ is the golden number. Let $\angle OPP_{22} = \theta$.

$\dfrac{PP_{22}}{OP_{22}} = \dfrac{\sin(12^\circ + \theta)}{\sin \theta} = \varphi$ is strictly decreasing for $0^\circ < \theta < 180^\circ \Longrightarrow \theta = 18^\circ$

Consequently, $\angle POP_{11} = 18^\circ \Longrightarrow P$ lies on the perpendicular bisector of $\overline{P_9P_{10}}$. Now, since $P_6P_{14}, P_{10}P_{27}$ and P_8P_{18} are the reflections of P_5P_{13}, P_9P_{22} and P_1P_{11} about the perpendicular bisector of $\overline{P_9P_{10}}$, then it follows that $P_1P_{11}, P_8P_{18}, P_5P_{13}, P_6P_{14}, P_9P_{22}, P_{10}P_{27}$ concur at point P lying on the perpendicular bisector of $\overline{P_9P_{10}}$.

Attachments:



Quick Reply

High School Olympiads

two problems about cyclic quadrilateral

Reply

tamtoanls
#1 May 26, 2011, 12:35 pm • 1

1) The incircle of triangle ABC touches BC at H. The point $D \in BC$, (P) and (Q) are incircles of ABD and ACD. prove that PQHD is cyclic quadrilateral.
2) Given a cyclic quadrilateral ABCD,the lines AB and CD meet at P, AD and BC meet at Q. M,N are mid point of AC and BD. H is orthocentre of triangle MPQ. Prove that PQHN is cyclic quadrilateral

This post has been edited 1 time. Last edited by tamtoanls, May 26, 2011, 3:39 pm

mahanmath
#2 May 26, 2011, 1:05 pm

See a special case of $1)$ in
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=2067020&sid=da6ac280381672a8709f9efeb576d7c2#p2067020

Luis González
#3 May 26, 2011, 1:13 pm

> *Quote:*
> 1) The incircle of triangle ABC touches BC at H. The point $D \in BC$, (P) and (Q) are incircles of ABD and ACD. Prove that PQHD is cyclic quadrilateral.

Let circles (P) and (Q) touch BC at U, V. Internal common tangent of $(P), (Q)$, different from AD, cuts BC at T.

$$BT = BU + UT = BU + DV = \frac{1}{2}(AB + BD - AD) + \frac{1}{2}(AD + DC - CA)$$

$BT = \frac{1}{2}(AB + BC - CA) \Longrightarrow H \equiv T$. If $E \equiv PQ \cap AD$, then P, Q become the D- and H-excenter of $\triangle EHD$. Hence P, Q, H, D are obviously concyclic.

> *Quote:*
> 2) Given a cyclic quadrilateral ABCD, the lines AB and CD meet at P, AD and BC meet at Q. M,N are mid point of AC and BD. H is orthocentre of triangle MPQ. Prove that PQHN is cyclic quadrilateral

See problem a) of Concyclic and perpendicular segments

yetti
#4 May 26, 2011, 2:07 pm

> *tamtoanls wrote:*
> 2) Given a cyclic quadrilateral ABCD,the lines AB and CD meet at P, AD and BC meet at Q. M,N are mid point of AC and BD. H is orthocentre of triangle MPQ. Prove that PQHN is cyclic quadrilateral

$\triangle PAC \sim \triangle PBD$ with corresponding P-medians PM, PN
$\triangle QAC \sim \triangle QBD$ with corresponding Q-medians QM, QN
$\pi - \angle QNP = \angle PNB + \angle DNQ = \angle CMP + \angle QMC = \angle QMP = \pi - \angle QHP$
$\Longrightarrow PQNH$ is cyclic.

math_explorer

#5 May 26, 2011, 4:31 pm

@Luis: could you explain why $UT = DV$? (Is it well-known?)

gold46

#6 May 27, 2011, 6:05 am

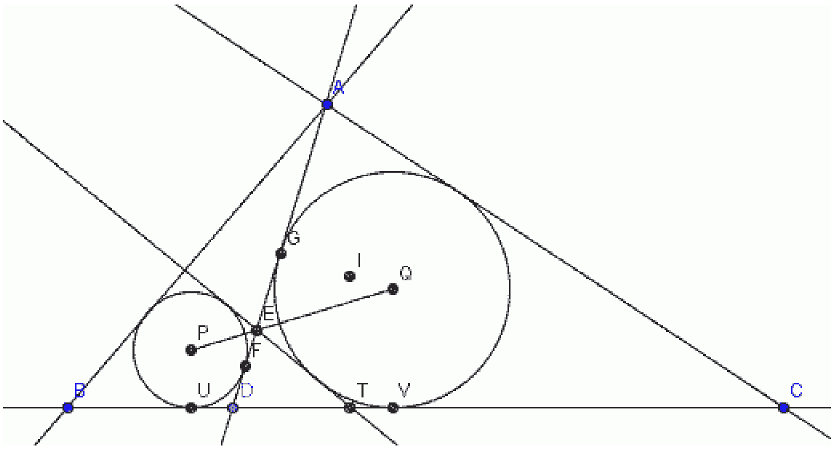
$UD = DE = DV + TV - UT \implies UT = DV$

math_explorer

#7 May 27, 2011, 9:23 am

Sorry I still don't understand. (I don't think $UD = DE$ either)
I made a diagram. Can somebody explain again? (and please take smaller steps than you would normally, I'm not good at geometry)

Attachments:



gold46

#8 May 27, 2011, 3:58 pm

oh sorry ! I didn't note E. You can put F instead of E. (DF=UD)

math_explorer

#9 May 27, 2011, 4:22 pm

Okay, I see it now, a lot of tangent-length-swapping including the two unlabeled tangency points where ET touches the circles. Thanks.

Quick Reply

High School Olympiads

Inequality

Reply

oneplusone
#1 May 21, 2011, 7:36 pm

Let P be a point in acute triangle ABC. Prove that

$$PA\cdot PB\sin\angle C+PB\cdot PC\sin\angle A+PC\cdot PA\sin\angle B\geq 2[ABC]$$

malcolm
#2 May 23, 2011, 7:17 am

Using $\sin\angle C=\frac{AB}{2R}, \frac{AB\cdot BC\cdot CA}{4R}=[ABC]$, where R is the circumradius of ABC, the inequality is equivalent to

$PA\cdot PB\cdot AB+PB\cdot PC\cdot BC+PC\cdot PA\cdot CA\geq 4R[ABC]$

or $\frac{PA\cdot PB}{BC\cdot CA}+\frac{PB\cdot PC}{CA\cdot AB}+\frac{PC\cdot PA}{AB\cdot BC}\geq\frac{4R[ABC]}{AB\cdot BC\cdot CA}=1$

Let a,b,c be the complex affixes of A,B,C respectively, and set P as the origin of the complex plane. Then we have

$\frac{PA\cdot PB}{BC\cdot CA}+\frac{PB\cdot PC}{CA\cdot AB}+\frac{PC\cdot PA}{AB\cdot BC}=\frac{|ab|}{|c-a||c-b|}+\frac{|bc|}{|a-b||a-c|}+\frac{|ca|}{|b-a||b-c|}$
$\geq\left|\frac{ab}{(c-a)(c-b)}+\frac{bc}{(a-b)(a-c)}+\frac{ca}{(b-a)(b-c)}\right|=1$ as desired.

Virgil Nicula
#3 May 26, 2011, 7:47 am

See **Proposed problem 3** from **here**.

Luis González
#4 May 26, 2011, 12:20 pm

Translate P and C by $\overrightarrow{AB}$ into Q and D. $PA=BQ, PC=QD, CD=AB$ and $DB=CA$. Thus, using Ptolemy's inequality in the quadrilaterals $BCDQ, PBQC$ yields

$BC\cdot PC+PA\cdot AB\geq CA\cdot QC\ ,\ PB\cdot QC+PC\cdot PA\geq BC\cdot AB$

Multiply first inequality by PB and the second one by CA, then adding them gives

$BC\cdot PB\cdot PC+CA\cdot PC\cdot PA+AB\cdot PA\cdot PB\geq BC\cdot CA\cdot AB\implies$

$\sin A\cdot PB\cdot PC+\sin B\cdot PC\cdot PA+\sin C\cdot PA\cdot PB\geq 2[\triangle ABC]$

Quick Reply

High School Olympiads

Circumcenters coincide

Reply

Source: LXII Polish Olympiad 2011, Problem 2

colosimo

#1 May 24, 2011, 8:04 pm • 1

The incircle of triangle ABC is tangent to BC, CA, AB at D, E, F respectively. Consider the triangle formed by the line joining the midpoints of AE, AF, the line joining the midpoints of BF, BD, and the line joining the midpoints of CD, CE. Prove that the circumcenter of this triangle coincides with the circumcenter of triangle ABC.

This post has been edited 1 time. Last edited by Amir Hossein, May 25, 2011, 12:07 am

Luis González

#2 May 24, 2011, 9:37 pm • 1

Assuming that "the center" is indeed the circumcenter, then this is a particular case of Bulgaria MO 2004 P1.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=5761
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=231881
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=380390

prime04

#3 May 25, 2011, 4:54 pm

Let $A_1, A_2, B_1, B_2, C_1, C_2$ be respectively the midpoints of AE, AF, BF, BD, CD, CE.
Let O be the circumcentre of $\triangle ABC$ and L, M, N be respectively the midpoints of BC, CA, AB.
Let A_2A_1 meets OM (produced ouside) in Q. Let X be the midpoint of A_2A_1, Y be the perpendicular from C to QC_1 and let C_1Q meet AC in Z.

Now, $AXMQ$ is cyclic $\Rightarrow \angle XAM = \angle XQM = \frac{1}{2}\angle A$. (Since $AA_2 = AA_1$)................(1)

Now, $A_2A_1||FE$ and $C_1C_2||DE \Rightarrow \angle XQC_1 = \angle FED = 90 - \frac{\angle B}{2}$.

$\Rightarrow \angle YQM = \frac{\angle C}{2}$

But $QMYC$ is cyclic $\Rightarrow \angle ZCY = \frac{\angle C}{2} \Rightarrow \angle C_1CY = \angle YQM = \frac{\angle C}{2}$.

Clearly, the triangles C_1CY and ZCY are congruent$\Rightarrow Z$ is the midpoint of $CE \Rightarrow Z = C_2$.
Hence, the lines A_2A_1,OM and C_1C_2 are concurrent.
Similarly,the lines A_1A_2, ON, B_2B_1 are concurrent and also the lines C_2C_1,OL, B_1B_2 are concurrent.
Let B_2B_1 meet A_1A_2 in R and let C_2C_1 meet B_1B_2 (Known that A_2A_1 and C_1C_2 meet in Q)

Observe that $\triangle PQR$~$\triangle DEF$

In $\triangle PQR, \angle PRQ = \angle DFE = \frac{1}{2}\angle POQ = 90 - \frac{\angle C}{2}$

Also, $\angle PQR = \angle DEE = \frac{1}{2}\angle POR = 90 - \frac{\angle B}{2}$

and $\angle RPQ = \angle FDE = \frac{1}{2}\angle ROQ = 90 - \frac{\angle A}{2}$

Hence O is the circumcenter of $\triangle PQR$ (by LEMMA).

$\Rightarrow QED$

LEMMA.
Let O be a point such that $\angle BOC = 2\angle BAC, \angle COA = 2\angle CBA$ and $\angle AOB = 2\angle ACB$. Then O is the circumcenter of $\triangle ABC$.

polya78

#4 Jan 6, 2012, 11:01 pm

Let I and O be the incenter and circumcenter respectively of $\triangle ABC$, and let $\triangle A'B'C'$ be the new triangle formed in the problem. Now $B'C' \perp AI$, so $B'C'$ is parallel to the external bisector of $\angle A$. So if I_A, I_B, I_C are the excenters of excircles J_A, J_B, J_C of $\triangle ABC$, then $\triangle A'B'C'$ and $\triangle I_AI_BI_C$ have parallel sides. Hence if A_1 and C_2 are the midpoints of AE and EC, we have that a dilation with a factor of 2 andcenter of homothecity E sends $\triangle A_1B'C_2$ into $\triangle AI_BC$.

If M is the foot of the perpendicular from B' to AC, then M is the midpoint of EE', where E' is the foot of the perpendicular from I_B to AC. But E' is the point of tangency of J_B and AC, which means that $AE = E'C = s - a$, where s is the semiperimeter of $\triangle ABC$. So M must be the midpoint of AC as well, which means that OB' is the perpendicular bisector of AC.

Now extend BI to hit the circumcircle (O) of $\triangle ABC$ at B", and construct similar points A" and C". Then B" bisects $\widehat{AC}$, so B" also lies on OB'. The angle between $B"C"$ and $AA"$ is equal to $1/2(\widehat{AB"} + \widehat{BC"} + \widehat{BA"}) = 90°$, so $B"C" \parallel B'C'$, and similarly for the other sides of $\triangle A"B"C"$ and $\triangle A'B'C'$. So $\triangle A"B"C"$ and $\triangle A'B'C'$ are homothetic with center O, which means that O is also the circumcircle of $\triangle A'B'C'$.

dinamometre1123

#5 Sep 3, 2012, 5:22 am • 1

Let be $A_1, A_2, B_1, B_2, C_1, C_2$ respectively the midpoints of AE, AF, BF, BD, CD, CE ; w is incenter of ABC and O its circumcenter.
A_1A_2 is radical axis of circles A and w , B_1B_2 is radical axis of B and w and C_1C_2 is radical axis of C and w (A, B and C are degenere circles)
A_1A_2 and B_1B_2 intersect at point C',A_1A_2 and C_1C_2 intersect at point B',C_1C_2 and B_1B_2 intersect at point A'
From the radical axis theorem A_1A_2, B_1B_2 and radical axis of A and B are concurent
so $C'A = C'B$, and also $B'A = B'C$, $A'B = A'C$
$OA = OB$ and $C'A = C'B => OC'$ and AB are perpendicular,and also OB' and AC,OA' and BC are perpendicular
$\angle OC'A' = \angle\frac{B}{2} = \angle OA'C'$ and $\angle OC'B' = \angle\frac{A}{2} = \angle OB'C'$ so O is circumcenter of $A'B'C'$

capu

#6 Mar 7, 2013, 7:42 am

Easy problem. Say the triangle XYZ is the one formed by the lines joining the midpoints. Say the circumcenter of ABC is O.

Since the line joining the midpoints of BD and BF is the radical axis of the incenter and B (as a circle of radius 0), then XB=XP, if XP is tangent to the incircle (since the power of a point from X to the incircle is the same as from X to B),

Similarly, XP=XB. Therefore XB=XP, and therefore OX is the perpendicular bisector of BC. Similarly, YO and ZO are the perpendicular bisectors of AC and AB.

Now say the midpoints of arcs BC, CA and AB are P, Q, R. Then note that OPX, OQY and ORZ are collinear. Also, say PQ intersects CB, CA at W, V. Then clearly CWV=CWQ=(BC/2)+(AC/2)=CVP=CVW, so that CVW is isosceles. Therefore VW is parallel to XY.

Therefore, PQR and XYZ are homothetic with center O. So clearly O is the circumcenter of XYZ.

Pedram-Safaei

#7 Mar 16, 2013, 9:19 pm

use radical axis as following:
we know that for every three circles the radical axis of each two of them are concurent.
now we can consider points A,B and incircle as three circles and discover that:the points of the triangle are on perpendicular bisectors of segments AB,BC,CA.
the remaining is obvious.

Quick Reply

High School Olympiads

Least perimeter of a triangle given an angle and inradius

Reply

Source: Canadian Mathematical Olympiad - 1980 - Problem 3.

BigSams

#1 May 17, 2011, 4:06 am

Among all triangles having (i) a fixed angle A and (ii) an inscribed circle of fixed radius r, determine which triangle has the least minimum perimeter.

Luis González

#2 May 23, 2011, 6:16 am

Consider the set of $\triangle ABC$ with constant inradius r and constant angle $\angle BAC = \alpha$. Using standard notations we get

$$[\triangle ABC] = \frac{a \cdot h_a}{2} = r \cdot s \Longrightarrow \ \ s = \frac{s-a}{1-\frac{2r}{h_a}} = \frac{r \cdot \cot \frac{\alpha}{2}}{1-\frac{2r}{h_a}}$$

Hence, perimeter $2s$ is minimum $\Longleftrightarrow h_a$ is maximum. If the incircle (I) touches BC at X and $\overrightarrow{AI}$ cuts (I) at D, then we have $AD = AI + IX = r + r\csc\frac{\alpha}{2} \geq AX \geq h_a$. Thus, $2s$ is minimum $\Longleftrightarrow D$ coincides with the foot of the A-altitude $\Longleftrightarrow \triangle ABC$ is A-isosceles with incircle r and apex angle α.

Rofler

#3 May 23, 2011, 6:58 am

Alternatively, the perimeter is twice the length of the tangent to the excircle. Therefore, the closer the excircle to the A, the smaller the perimeter will be. The closest the excircle can possible get is when the excircle is tangent to the incircle, which then forces the triangle to be isosceles so that it can be tangent to both circles at the point at which they touch.

Cheers,

Rofler

Quick Reply

High School Olympiads

Show that (AB+BC+CA)^2 ≤ 6(AD^2+BD^2+CD^2)

Reply

Amir Hossein

#1 May 22, 2011, 2:45 pm

In the tetrahedron $ABCD, \angle BDC = 90^o$ and the foot of the perpendicular from D to ABC is the intersection of the altitudes of ABC. Prove that:

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$

When do we have equality?

Luis González

#2 May 23, 2011, 5:01 am • 1

If the orthogonal projection of D on the plane ABC is the orthocenter of $\triangle ABC$, then $ABCD$ is orthocentric $\Longrightarrow$ Orthogonal projections of B, C on ADC and ADB are the orthocenters of the corresponding faces. Since $\angle BDC$ is right, then $\angle ADB$ and $\angle ADC = 90^\circ$ are also right. By Pythagorean theorem we obtain

$$AD^2 + DB^2 = AB^2 \ , \ AD^2 + CD^2 = CA^2 \ , \ BD^2 + CD^2 = BC^2$$

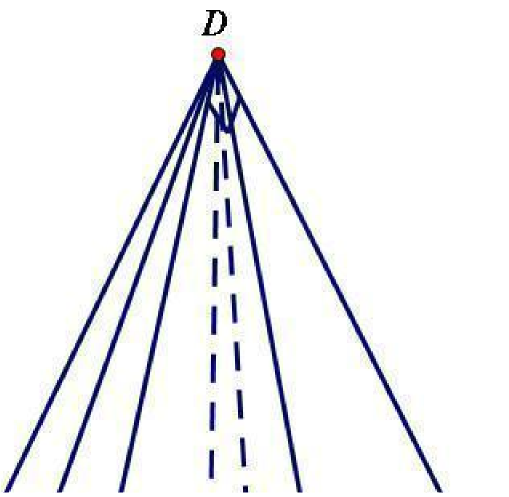
$$\Longrightarrow AD^2 + BD^2 + CD^2 = \frac{AB^2 + BC^2 + CA^2}{2}$$

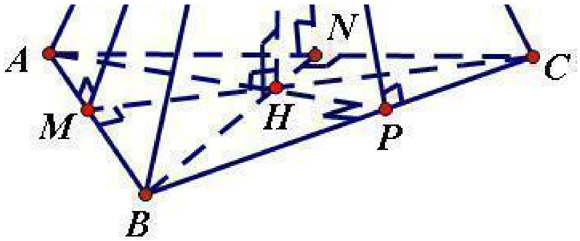
Thus, the desired inequality becomes $(BC + CA + AB)^2 \leq 3(BC^2 + CA^2 + AB^2)$, which follows from Cauchy-Schwarz. Thus, equality holds when $\triangle ABC$ is equilateral.

tuan119

#3 May 23, 2011, 12:23 pm • 1

> amparvardi wrote:
> In the tetrahedron $ABCD, \angle BDC = 90^o$ and the foot of the perpendicular from D to ABC is the intersection of the altitudes of ABC. Prove that:
>
> $$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$
>
> When do we have equality?





We have:
+) $AC \perp BN$ and $AC \perp DH \Rightarrow AC \perp (DBN) \Rightarrow AC \perp DB\,(1)$. Otherwise: $DB \perp DC$; (2)
Combined (1) and $(2) \Rightarrow DB \perp (DAC) \Rightarrow DB \perp DA \subset (DAC) \Rightarrow \widehat{BDA} = \frac{\pi}{2}$;
*)Similar proof as above: $\Rightarrow DC \perp (DAB) \Rightarrow DC \perp DA \subset (DAB) \Rightarrow \widehat{CDA} = \frac{\pi}{2}$;
+) $AB^2 + BC^2 + CA^2 \stackrel{Pythago}{\to} = (DA^2 + DB^2) + (DB^2 + DC^2) + (DA^2 + DC^2)$
$= 2(DA^2 + DB^2 + DC^2)$
$\iff \; 3(AB^2 + BC^2 + CA^2) = 6(DA^2 + DB^2 + DC^2) \geq (AB + BC + CA)^2$
$\iff \; 2(AB^2 + BC^2 + CA^2) \geq 2(AB.BC + BC.CA + CA.AB)$
$\iff \; (AB^2 + BC^2 + CA^2) \geq (AB.BC + BC.CA + CA.AB)$; (*) It's always right.
+ Equality holds when $AB = BC = CA$ (by $\mathbf{AM} - \mathbf{GM}$).

Quick Reply

High School Olympiads

angle

Locked

huyhoang

#1 May 22, 2011, 8:39 am

Let ABC be an acute angle with $\widehat{B}=60^{\circ}$ and I be the incenter of the triangle. E is a point on AB such that IE parallel to BC. F be a point on AC such that $AF=\frac{1}{3}AC$. Prove that $\widehat{AEF}=\frac{1}{2}\widehat{BAC}$.

Luis González

#2 May 22, 2011, 10:52 am • 1

Seriously, How many more times are we going to see this problem?

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=100938
http://www.artofproblemsolving.com/Forum/viewtopic.php?uid=56597&f=47&t=105421
http://www.artofproblemsolving.com/Forum/viewtopic.php?uid=56597&f=46&t=291507
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=313492
http://www.artofproblemsolving.com/Forum/viewtopic.php?uid=56597&f=46&t=318171
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=350773
http://www.artofproblemsolving.com/Forum/viewtopic.php?uid=56597&f=46&t=342389

High School Olympiads

parallel problem?

thanhnam2902

#1 May 20, 2011, 1:28 pm

Let $ABCD$ is a square, let AC meet BD at point O. Let G is a point lie on BC and H is a point lie on DC satisfy $\angle GOH = 45^0$. Let M is midpoint of AB. Prove that MG parallel to AH.

Luis González

#2 May 21, 2011, 9:33 am

AH cuts BC at E and P, Q are the projections of B, E on DE, DB. 9-point circle (K) of $\triangle DEB$, passing through P, Q, C, O, cuts $\overline{DE}, \overline{BE}$ again at their midpoints N, G. Clearly, $MG \parallel \overline{AHE}$, so we have to show $\angle GOH = 45^\circ$.

Perpendicular bisectors QG, CO of BE, BD meet at the circumcenter U of $\triangle DEB$, i.e. orthocenter of its medial $\triangle GON$. $\angle EPC = \angle DAC = 45^\circ$ implies that A, D, P, C are concyclic, but since the orthic triangle $\triangle PQC$ of $\triangle DEB$ is right at P, then we deduce that PQ passes through A. Let $F \equiv GQ \cap ON$ and $L \equiv DC \cap PQ$. Then $FNEG \sim ADEB$ are homothetic through E with homothety coefficient $\frac{1}{2} \Longrightarrow A, F, E, H$ are collinear. Since F is the midpoint of UQ and $CL \parallel UQ$, it follows that H is the midpoint of LC. Then from $\angle CPL = 90^\circ$, we get $HC = HL = HP \Longrightarrow H$ lies on the perpendicular bisector OK of CP. Consequently, $\angle GOH \equiv \angle GOK = 45^\circ$.

r1234

#3 May 21, 2011, 3:30 pm

I have another proof of this problem using simple geometry.First observe that the triangles BOG and DOH are similar.(Since $\angle OBG = \angle ODH = 45$ and $\angle BGO = \angle DOH$).Hence we get $BG/OB = OD/DH$ or,$BG.DH = OB^2 = AB^2/2$.Now look at the triangles ADH and BMG.We see that $\angle ADH = \angle MBG = 90$ and from the previous result we get $AD/BG = MB/DH$. hence triangles ADH and MBG are similar.Hence by angle chasing we get $\angle HAM + \angle AMG = 180$ indicating that MG and AH are parallel to each other.

Quick Reply

High School Olympiads

ABCD is a convex quadrilateral..

Locked

Source: Bulgaria national olympiad 1998

r1234

#1 May 20, 2011, 3:38 pm

Let ABCD is a convex quadrilateral such that $\angle DAB = \angle ABC$ and $AD = CD$. If M is the midpoint of BC and the line passing through D and M intersects AB at E then prove that $\angle DAC = \angle BEC$

Luis González

#2 May 20, 2011, 9:16 pm

Discussed before several times. For instance, see the following threads

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=326015
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=365705
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=369665

High School Olympiads

AP is perpendicular to PC

Reply

Source: China TST 2011 - Quiz 1 - D2 - P1

WakeUp

#1 May 19, 2011, 8:27 pm

Let one of the intersection points of two circles with centres O_1, O_2 be P. A common tangent touches the circles at A, B respectively. Let the perpendicular from A to the line BP meet O_1O_2 at C. Prove that $AP \perp PC$.

goodar2006

#2 May 19, 2011, 10:27 pm

It's somehow intresting to me that our teacher gave us this problem about 2 years ago! for solving it, extend PC to intersect the circle with center O_1...

Luis González

#3 May 19, 2011, 10:39 pm

Hmm, this is basically question 2 of 15 IBMO-Venezuela. See the following threads

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=83793
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=336494
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=288&t=80971

math154

#4 May 20, 2011, 8:27 am • 1

This solution assumes AB is the closer tangent to P, which I think the original problem states (it doesn't matter though).

Let $(O_1) \cap (O_2) = \{P, Q\}$, $M = AB \cap PQ$, and A' be the reflection of A over O_1. Clearly $\angle CPA = 90^\circ \iff \angle AA'C = \angle AA'P$. But $\angle AA'P = \angle BAP$, so it suffices to show $\triangle AA'C \sim \triangle BAP \iff \triangle AO_1C \sim \triangle BMP$ (equivalent since O_1, M are the midpoints of AA', BA respectively), which is just a simple angle chase.

RaleD

#5 May 28, 2011, 1:33 pm • 1

Click to reveal hidden text

yttrerium

#6 Jul 2, 2012, 2:14 pm

> RaleD wrote:
> Click to reveal hidden text

This solution by **RaleD** is a very nice solution. At first glance, I thought it is similar to mine. In fact, it's way simpler and nicer. In my opinion, it's very clever to use the radical axis.

My solution also uses something like the equation $CN * CA = CP^2$ to get a geometric relation.

Here is how it goes:
Let the line through C perpendicular to AB intersect AB at point S.
A direct calculation of side lengths (starting with $BC^2 - PC^2 = AB^2 - AP^2$) yields $AS \cdot AB = AP^2$
Now, it is easy to find that A, S, C, P are concyclic, which leads to $\angle APC = 90^\circ$

junioragd

#7 Jan 10, 2015, 11:57 pm

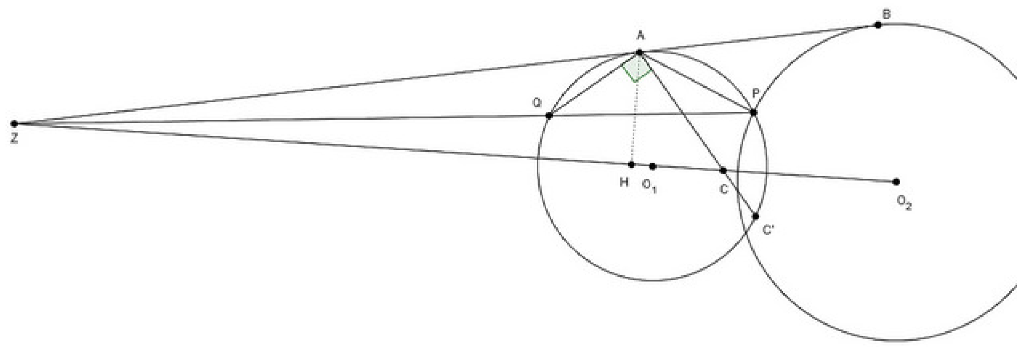
Denote Q the other intersection point of two circles,let M be the intersection point of PQ and AB and let O be the circumcenter of ABQ.Now,let C' be the intersection of the perpendicular from A to BP and the perpendicular from P to AP.Now,we have that AOM is similar to APC' so we get APM is similar to $AC'P$.Now,from an easy angle chase and using the previos similarity we get $QC'O$ is similar to QPB and from this we get $QC' = PC'$ so $C' = C$ and that is it.

Dukejukem

#8 May 26, 2015, 12:34 am

Here's another solution:

Define $Z \equiv AB \cap O_1O_2$ and let ZP, AC cut (O_1) for a second time at Q, C', respectively. Let H be the projection of A onto O_1O_2. Note that Z is the external center of homothety that maps $(O_2) \mapsto (O_1)$. It is clear that this homothety takes $B \mapsto A$ and $P \mapsto Q$, so $AC \perp BP \implies AC \perp AQ$. Hence, $\overline{QC'}$ is a diameter of $(O_1) \implies Q, C', O_1$ are collinear. Furthermore, note that $\angle ZHA = \angle ZAO_1 = 90^\circ \implies \triangle ZAH \sim \triangle ZO_1A \implies ZH \cdot ZO_1 = ZA^2 = ZP \cdot ZQ$, where the last step follows from Power of a Point. Therefore, H, O_1, P, Q are concyclic, so $\measuredangle PHC = \measuredangle PHO_1 = \measuredangle PQO_1 = \measuredangle PQC' = \measuredangle PAC' = \measuredangle PAC$, where the angles are directed. Hence, P, H, C, A are concyclic, so $\measuredangle APC = \measuredangle AHC = 90^\circ$, as desired. $\square$



Quick Reply

High School Olympiads

Symmedian point

Reply

jayme

#1 May 18, 2011, 9:05 pm

Dear Mathlinkers,
ABC a triangle, DEF the orthic triangle of ABC, K the symmedian point of ABC, Q the point of intersection of FK and AC, R the point of intersection of EK and AB.
Prove: QR / /EF.

Sincerely
Jean-Louis

Luis González

#2 May 19, 2011, 12:44 am

This is true for any point K on the A-symmedian of $\triangle ABC$ because AK coincides with the A-median of $\triangle AEF.$ This is, let M,N denote the midpoints of $BC,EF.$ Since $\triangle ABC$ and $\triangle AEF$ are similar, we have $\angle CAM=\angle FAN \Longrightarrow AN$ is the A-symmedian of $\triangle ABC.$ Hence, for any K on AN the intersection $EF \cap QR$ is the harmonic conjugate of N with respect to $E,F \Longrightarrow EF \cap QR$ is at infinity.

Quick Reply

High School Olympiads

Projections that concurr

Reply

hatchguy

#1 May 18, 2011, 5:52 am

Let ABC be a triangle and X a point in the plane. Let H be the orthocenter of ABC. The circle with diameter HX intersects AH and AX in A_1 and A_2. Define B_1, B_2, C_1 and C_2 in the same way. Prove that A_1A_2, B_1B_2 and C_1C_2 are concurrent.

Luis González

#2 May 18, 2011, 10:15 am

The same problem was already discussed in this topic. On the other hand, there is a nice generalization of this configuration due to Darij Grinberg. I haven't approached it yet.

Theorem. P, Q are two arbitrary points in the plane of $\triangle ABC$. ℓ_a, ℓ_b, ℓ_c are the perpendiculars dropped from Q to BC, CA, AB. X, Y, Z are the orthogonal projections of P on ℓ_a, ℓ_b, ℓ_c and X', Y', Z' are the orthogonal projections of Q on PA, PB, PC, respectively. Then the lines XX', YY', ZZ' concur.

sshago

#3 May 18, 2011, 12:14 pm

Let σ be the circle which contains A_1, A_2, B_1, B_2, C_1 and C_2. Moreover, we have $\angle XB_1H = 90 \Rightarrow XB_1 \parallel AC$ Similarly $XA_1 \parallel BC$ and $XC_1 \parallel AB$.
Since BB_2, AA_2, CC_2 meet in X, we have by trigonometric Ceva theorem with the triangle ABC:

$$\frac{\sin \angle ABB_2}{\sin \angle B_2BC} \cdot \frac{\sin \angle BCC_2}{\sin \angle C_2CA} \cdot \frac{\sin \angle CAA_2}{\sin \angle A_2AB} = 1.$$

(1)
Now, note that $\sin \angle B_2A_1C_1 = \sin \angle B_2XC_1$ (because they're the same arc in σ) and $\sin \angle B_2XC_1 = \sin \angle ABB_2$ (from $XC_1 \parallel AB$), and we conclude that $\sin \angle B_2A_1C_1 = \sin \angle ABB_2$. Similarly we obtain
$\sin \angle B_2BC = \sin \angle B_2C_1A_1, \sin \angle BCC_2 = \sin \angle A_1B_1C_2, \sin \angle C_2CA = \sin \angle C_2A_1B_1, \sin \angle CAA_2 = \sin \angle A_2C_1B_1$ and $\sin \angle A_2AB = \sin \angle A_2B_1C_1$
and sustituying these six equalities in (1), we obtain by trigonometric Ceva in triangle $A_1B_1C_1$ that A_1A_2, B_1B_2 and C_1C_2 concur and the problem ends.

Quick Reply

High School Olympiads

Tangential Cuadrylateral.

Reply

Vladislao

#1 May 17, 2011, 10:54 pm

Let $ABCD$ be a cuadrylateral such that $BC \parallel AD$ ($BC < AD$) and $AB = CD$. We know that $ABCD$ is a tangential cuadrylateral. Let I be the incenter of the cuadrylateral. Let P be the intersection between CI and AD. Let choose N in CD, in such a way that $PN \perp CD$. Show that $PN^2 = AD \cdot BC$.

Luis González

#2 May 18, 2011, 2:35 am

Incircle (I) of $ABCD$ touches AD, BC through their midpoints R, Q. Since $AD \parallel BC$, then internal bisector DI of $\angle(DA, DC)$ is perpendicular to the internal bisector CI of $\angle(CD, CB) \Longrightarrow \triangle DPC$ is isosceles with apex $D \Longrightarrow PN = \text{dist}(C, DA) = RQ$. Hence, by Pythagorean theorem for the right trapezoid $DRQC$ we get

$$PN^2 = DC^2 - (DR - CQ)^2 = \frac{1}{4}(AD + BC)^2 - \frac{1}{4}(AD - BC)^2 = AD \cdot BC.$$

Quick Reply

High School Olympiads

Prove that the radius of the circle..

Reply

Source: Indian IMO Training camp-2011

r1234

#1 May 17, 2011, 1:32 pm

Let ABC be a triangle each of whose angles is greater than 30°. Suppose a circle centered with P cuts segments BC in T,Q; CA in K,L and AB in M,N such that they are on a circle in counterclockwise direction in that order.Suppose further PQK, PLM, PNT are equilateral. Prove that:

$a)$ The radius of the circle is $\frac{2abc}{a^2+b^2+c^2+4\sqrt{3}S}$ where S is area.

$b) a\cdot AP=b\cdot BP=c\cdot PC.$

Luis González

#2 May 17, 2011, 11:43 pm • 3

a) From the cyclic quadrilateral $MLKN,$ we have $\angle AML=\angle AKN,$ but since $TQKN$ is an isosceles trapezoid with legs $QK=TN,$ then $NK \parallel BC \Longrightarrow \angle AML=\angle ACB,$ i.e. ML is antiparallel to BC WRT $AB,AC.$ Assuming that $LM,$ NT,QK separate P from the vertices $A,B,C,$ let X be the apex of the equilateral triangle $\triangle BCX$ constructed outside $\triangle ABC.$ Then $ALPM \sim ABXC$ (1) $\Longrightarrow \angle MAP=\angle CAX,$ i.e. AX,AP are isogonals WRT $\angle BAC.$ Hence, we deduce that P is the isogonal conjugate of the 1st Fermat point F of $\triangle ABC,$ i.e. the 1st Isodynamic point of $\triangle ABC.$ Since $F \in \odot(XBC),$ we have $\angle FBC=\angle PBA=\angle AXC \Longrightarrow \triangle ACX \sim \triangle APB$ (2). Thus, denoting ϱ the radius of the subject circle, from (1) and (2) we get

$$\frac{\varrho}{a}=\frac{AP}{AX} \ , \ \frac{AP}{b}=\frac{c}{AX} \ \Longrightarrow \ \varrho=\frac{abc}{AX^2}$$

By cosine law for $\triangle ACX,$ we get

$$AX^2=a^2+b^2-2ab\cos(C+60^\circ)=a^2+b^2-ab\cos C+\sqrt{3}ab\sin A$$

Substituting $\cos C=\frac{a^2+b^2-c^2}{2ab} \ , \ 2[\triangle ABC]=bc\sin A$ yields

$$AX^2=\tfrac{1}{2}(a^2+b^2+c^2)+2\sqrt{3}[\triangle ABC] \ \Longrightarrow \ \varrho=\frac{2abc}{a^2+b^2+c^2+4\sqrt{3}[\triangle ABC]}$$

Alternatively, if we assume that LM,NT,QK do not separate P from $A,B,C,$ then P is the 2nd Isodynamic point. Similarly, letting X be the apex of the equilateral triangle $\triangle BCX$ constructed inwardly, we have

$$AX^2=\tfrac{1}{2}(a^2+b^2+c^2)-2\sqrt{3}[\triangle ABC] \ \Longrightarrow \ \varrho=\frac{2abc}{a^2+b^2+c^2-4\sqrt{3}[\triangle ABC]}$$

b) Since the pedal triangle of P WRT $\triangle ABC$ is equilateral, then P is a common point of the 3 Apollonian circles of $\triangle ABC.$ Hence, we get $AP\cdot BC=BP\cdot CA=CP\cdot AB.$

r1234

#3 May 18, 2011, 2:15 pm

thank you luisgeometra

horizon

#4 Feb 5, 2012, 6:49 pm

P also has the property
$\angle APB=\angle C+60,\angle APC=\angle B+60,\angle BPC=\angle A+60$

Quick Reply

High School Olympiads

Prove perimeters the same

Reply

Source: All-Russian 2011

3333

#1 May 17, 2011, 7:12 am

Let ABC be an equilateral triangle. A point T is chosen on AC and on arcs AB and BC of the circumcircle of ABC, M and N are chosen respectively, so that MT is parallel to BC and NT is parallel to AB. Segments AN and MT intersect at point X, while CM and NT intersect in point Y. Prove that the perimeters of the polygons $AXYC$ and $XMBNY$ are the same.

Luis González

#2 May 17, 2011, 12:35 pm • 1

TM, TN cut BA, BC at E, D and the circumcircle of $\triangle ABC$ again at U, V, respectively. $UM \parallel CB$ and $VN \parallel AB$ imply that A, C are the midpoints of the arcs UVM, VUN. Thus, AC bisects $\angle MTN, \angle YCU$ and $\angle XAV \Longrightarrow TYCU$ and $TXAV$ are kites $\Longrightarrow CY = CU = BM$ and $AX = AV = BN$. Hence, to prove that $AXYC$ and $XMBNY$ have equal perimeters, it suffices to show that $MX + NY = AC$

$$MX + NY = TM - TU + TN - TV = TM - EM + TN - DN = TE + TD$$

But, from the parallelogram $TEBD$ and the equilateral $\triangle AET$ we have $TD = EB$ and $TE = EA$. Hence, $MX + NY = EB + EA = AB = AC$, as desired.

MBGO

#3 Dec 22, 2012, 10:54 pm • 1

Click to reveal hidden text

junioragd

#4 Jul 29, 2014, 10:41 pm

Let MT intersect circle of ABC again at R and NT intersects circle of ABC at P.Now,notice BN=AP(since AB//PN)=AX(since <ATX=<ATP=60 and <PAT=<TAX,since CP=CN,so ATX is congruent to APT).Similary,conclude BM=CR=CY,so we need to prove XM+YN=AC.Now,let S be the intersection of MT and AB and Q be the intersection of NT and BC.It is easy to see that TX=TP=QN and TR=TY=SM and from this we have that XM+YN=TS+TQ=TA+TC=AC,so we are finished

Quick Reply

High School Olympiads

Acute triangle

Reply

Source: All-Russian 2011

3333

#1 May 17, 2011, 7:06 am

Given is an acute angled triangle ABC. A circle going through B and the triangle's circumcenter, O, intersects BC and BA at points P and Q respectively. Prove that the intersection of the heights of the triangle POQ lies on line AC.

Luis González

#2 May 17, 2011, 10:57 am • 3

Circumcircles of $\triangle OAQ$ and $\triangle OCP$ meet at O and a point R lying on the sideline AC due to Miquel theorem. Hence, from the cyclic $PORC$ and $QORA$ we have

$$\angle ORP = \angle OCB = \angle OBC \ , \ \angle ORQ = \angle OAB = \angle OBA \Longrightarrow \angle PRQ = \angle B$$

Since $\angle OPR = \angle OCA = 90^\circ - \angle B,$ it follows that $PO \perp RQ.$ By similar reasoning, we have $QO \perp RP \Longrightarrow R$ is the orthocenter of $\triangle POQ.$

RSM

#3 Aug 1, 2011, 9:44 pm • 1

Suppose, $A'B'C'$ is the medial triangle of ABC. There exists a spiral similarity centered at O that takes $OA'C'$ to OPQ. Since $OA' \perp BC$ and $OB' \perp AC$, so image of B' will lie on AC and since B' is the orthocenter of $OA'C'$, so image of B' is the orthocenter of OPQ, so done.

Catalanfury

#4 Mar 22, 2012, 11:13 pm

> *luisgeometra wrote:*
> $\angle OCA = 90^\circ - \angle B,$ it follows that $PO \perp RQ.$

Why are those two angles equal, and how does it follow that PO and RQ are perpendicular?

MariusBocanu

#5 Oct 20, 2012, 11:50 am

Another easy solution. Denote X the orthocenter of $\triangle OPQ$. It's easy to see that $\widehat{QXP} = \widehat{ABC}$, and we have that $\widehat{CXP} + \widehat{PXQ} + \widehat{AXQ} = 180 - \widehat{C} - 180 + \widehat{BPO} + 90 - \widehat{B} + \widehat{B} + 180 - \widehat{A} - 180 + \widehat{BQO} + 90 - \widehat{B} = 180,$ so A, X, C are collinear.

Quick Reply

High School Olympiads

Parallelogram

Reply

Source: All-Russian 2011

3333

#1 May 17, 2011, 7:23 am

On side BC of parallelogram $ABCD$ (A is acute) lies point T so that triangle ATD is an acute triangle. Let O_1, O_2, and O_3 be the circumcenters of triangles ABT, DAT, and CDT respectively. Prove that the orthocenter of triangle $O_1O_2O_3$ lies on line AD.

Luis González

#2 May 17, 2011, 10:26 am • 1

(O_1) cuts AD again at P. Since $\angle ABT = \angle DPT = 180^\circ - \angle DCT$, it follows that $P \in (O_3) \Longrightarrow P$ is the reflection of T about O_1O_3. $\odot(PO_1O_3)$ cuts AD again at H. From the cyclic PHO_1O_3 we get $\angle HO_3O_1 = \angle APO_1 = 90^\circ - \angle ATP$. But $O_1O_2 \perp TA$ and $PT \perp O_1O_3$ imply then $\angle HO_3O_1 = 90^\circ - \angle O_2O_1O_3$. By similar reasoning, we have that $\angle HO_1O_3 = 90^\circ - \angle O_2O_3O_1$. Consequently, H coincides with the orthocenter of $\triangle O_1O_2O_3$.

NewAlbionAcademy

#3 Jun 5, 2013, 9:03 am

:::

sunken rock

#4 Jun 5, 2013, 10:49 am

@3333: is acute-angled, hence

Best regards,
sunken rock

sunken rock

#5 Nov 23, 2014, 5:07 pm

Almost similar to Luis: intersection of the circles O_1 and O_3, P, lies onto CD. Because $\angle ABT + \angle TCD = 180^\circ$ we get $\angle TO_1A = \angle TO_3D \implies \triangle ATO_1 \sim \triangle DTO_3$, i.e. $\frac{TO_1}{AT} = \frac{TO_3}{TD}$ and $\angle ATO_1 = \angle DTO_3 \implies \triangle O_1TO_3 \sim \triangle ATD$, hence $\angle TO_1O_3 = \angle TAD$ (1); by symmetry $\angle PO_1O_3 = \angle TO_1O_3$ (2).
If the circle $\odot(PO_1O_3)$ intersects 2-nd time CD at H we claim that H is the required orthocenter.
From the cyclic PO_1HO_3 we get $\angle PHO_3 = \angle PO_1O_3$ and, with (2) we see that $HO_3 \parallel AT \iff HO_3 \perp O_1O_2$.
Likewise $HO_1 \perp O_2O_3$, done.

Best regards,
sunken rock

TelvCohl

#6 Nov 23, 2014, 5:51 pm • 1

Approach with Steiner Line :)

jayme

#7 Nov 23, 2014, 5:59 pm

Dear Mathlinkers,
for the Steiner's line, you can have a look on

http://jl.ayme.pagesperso-orange.fr/ vol. 17 Symétrique d'une droite... p. 5...

Sincerely
Jean-Louis

anantmudgal09

#8 Apr 23, 2016, 1:23 am • 1

Very nice and easy

Indeed, we first observe by trivial angle chasing that points O_1, O_2, O_3, T are concyclic. Now, since the reflection of T in O_2O_1 is point A and of T in O_2O_3 is D, then line AD is the Steiner line of point T wrt $O_1O_2O_3$ and thus passes through the orthocenter of this triangle.

Remark

Quick Reply

High School Olympiads

Prove angle is right

Reply

Source: All-Russian 2011

3333

#1 May 17, 2011, 7:18 am

Given is an acute triangle ABC. Its heights BB_1 and CC_1 are extended past points B_1 and C_1. On these extensions, points P and Q are chosen, such that angle PAQ is right. Let AF be a height of triangle APQ. Prove that angle BFC is a right angle.

Luis González

#2 May 17, 2011, 9:14 am

Let $D \equiv BC \cap AF$ and H is the orthocenter of $\triangle ABC$. Quadrilaterals $PAFB_1$ and $QAFC_1$ are inscribed in circles with diameters $\overline{PA}, \overline{QA}$. Thus, $\angle DFB_1 = \angle APH$ and $\angle DFC_1 = \angle AQH$

$$\Longrightarrow \angle C_1FB_1 = \angle APH + \angle AQH = 360^\circ - 90^\circ - (180^\circ - \angle A) = 90^\circ + \angle A$$

Since $\angle C_1CB_1 = 90^\circ - \angle A$, then it follows that F, C, B_1, C_1 are concyclic $\Longrightarrow F$ lies on the circle with diameter $\overline{BC}$.

NewAlbionAcademy

#3 Jun 5, 2013, 6:03 am

Solution

Quick Reply

High School Olympiads

ABC is a triangle...

Reply

r1234

#1 May 16, 2011, 3:49 pm

In $\triangle ABC$ let I be its incenter . The line through the midpoint of AC and I intersects AB at X and the line through the midpoint of AB and I intersects AC at Y. Determine $\angle BAC$ if the area of $\triangle ABC$ equals the area of $\triangle AXY$.

Luis González

#2 May 16, 2011, 11:01 pm

We use barycentric coordinates WRT $\triangle ABC$. Coordinates of the incenter I and midpoints M, N of AC, AB are $I \equiv (a:b:c), M \equiv (1:0:1), N \equiv (1:1:0)$.

$$IM \equiv bx + (c-a)y - bz = 0 \Longrightarrow X \equiv IM \cap AB \equiv (a-c:b:0)$$

$$IN \equiv cx - cy + (b-a)z = 0 \Longrightarrow Y \equiv IN \cap AC \equiv (a-b:0:c)$$

$$\frac{|\triangle AXY|}{|\triangle ABC|} = \frac{1}{(a+b-c)(a+c-b)} \begin{pmatrix} 1 & 0 & 0 \\ a-c & b & 0 \\ a-b & 0 & c \end{pmatrix} = \frac{bc}{(a+b-c)(a+c-b)}$$

$$|\triangle ABC| = |\triangle AXY| \iff bc = (a+b-c)(a+c-b) \iff \angle BAC = 60^\circ$$

P.S. See also http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=261881

Quick Reply

High School Olympiads

isogonal center of circles(russian problem)

paul1703

#1 May 16, 2011, 4:00 pm

in the trapezoid $ABCD\ AD \parallel BC)$ we note $\{K\} = AB \cap CD$; and let P and Q be the center of the circumscribed circle of the triangles ABD, BCD.
prove that $\angle PKA \equiv \angle QKD$.

Luis González

#2 May 16, 2011, 10:22 pm

$\angle DQC = 2\angle DBC = 2\angle ADB = \angle APB$ implies that isosceles triangles $\triangle PAB$ and $\triangle QCD$ with apices P, Q are similar. Thus, $\angle KCQ = \angle KBP$ and

$$\frac{CQ}{BP} = \frac{CD}{BA} = \frac{CK}{BK} \Longrightarrow \triangle KCQ \sim \triangle KBP \text{ by SAS} \Longrightarrow \angle PKA = \angle QKD$$

High School Olympiads

Prove that!

Reply

tuan119

#1 May 16, 2011, 9:42 am

Let E be a point inside triangle ABC such that $\widehat{ABE}=\widehat{ACE}$. Let K and H be the feet of the perpendiculars from E to the internal and external bisectors, respectively, of angle $\widehat{BAC}$. Prove that the line KH passes throuth the midpoint of BC.

Luis González

#2 May 16, 2011, 11:00 am

Obviously, HK passes through the midpoint D of $AE.$ Isogonal conjugate E' of E lies on the perpendicular bisector of BC since $\angle E'BC=\angle E'CB$ and reflection F of E across the external bisector of $\angle BAC$ lies on $AE' \Longrightarrow HK$ is the E-midline of $\triangle EFE' \Longrightarrow HK$ passes through the midpoint of EE', i.e. center U of the pedal circle of E,E'. Let P,Q,R be the orthogonal projection of E onto $BC,CA,AB.$ Since $\angle EPR=\angle EBA$ and $\angle EPQ=\angle ECA,$ then $\angle EPR=\angle EPQ,$ i.e. BC bisects $\angle RPQ,$ so M is the midpoint of the arc RPQ of $(U).$ $HK \equiv \overline{DUM}$ is the perpendicular bisector of $QR.$

P.S. See also April's message in the topic Projections of orthocenter on bisectors.

Quick Reply

High School Olympiads

Three equal Chevians (Own?)

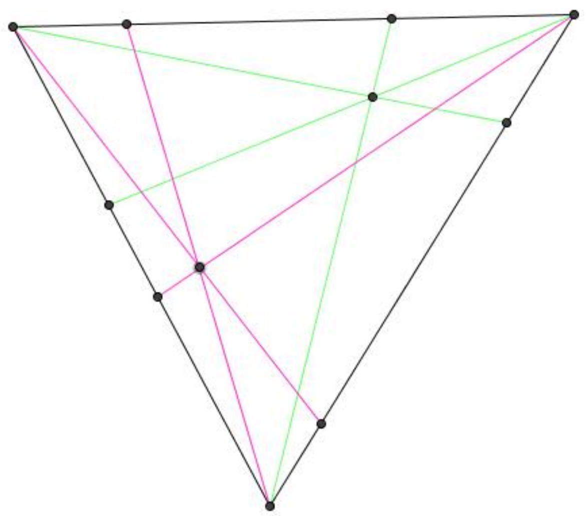
Reply

skytin

#1 May 14, 2011, 6:50 pm

Given triangle ABC and prove that there are exist two points P and Q such that all Chevians from them are equal

Attachments:



fedja

#2 May 14, 2011, 7:52 pm

You mean "given a triangle with no two sides of equal length"? :)

skytin

#3 May 14, 2011, 8:03 pm

ABC is not Equilateral triangle

This problem is Genralization of problem 9 from VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN :
http://www.geometry.ru/olimp/sharygin/2010/zaochsol-e.pdf

fedja

#4 May 15, 2011, 3:44 am

Actually, as I see it, you've got to add some extra assumptions (or clarifications). Indeed, if $AB < AC$ and A is obtuse, then every chevian from A is shorter than AC and every chevian from C is longer than AC, so you can never get true chevians (i.e., ones intersecting inside the triangle) with the desired property. One can figure such things out, of course, but it is always better to state problems correctly from the beginning... .

Luis González

#5 May 15, 2011, 11:45 am

We prove that the two Bickart points of ABC (foci of its Steiner circumellipse) satisfy the desired property, which answers this problem. On the other hand, proving that these are the unique points satisfying the property is much tougher.

U, V are the foci of the Steiner circumellipse $\mathcal{G}$ of $\triangle ABC$, whose center is its centroid G. Let $2d, 2e, 2f$ be the lengths of its minor axis, major axis and focal segment. We known that

$$e = \tfrac{1}{3}\sqrt{a^2+b^2+c^2+2\varrho}\ ,\ d = \tfrac{1}{3}\sqrt{a^2+b^2+c^2-2\varrho}\ ,\ \varrho^2 = \sum_{\text{cyclic}} a^4 - b^2c^2$$

Squaring $AV + AU = 2e$ and using median theorem in $\triangle AUV$ yields

$$2 \cdot AU \cdot AV = 4e^2 - 2GA^2 - 2f^2$$

Denote $\angle VAU = \lambda$. Then by cosine law for $\triangle AUV$, we get

$$\cos\lambda = \frac{AV^2 + AU^2 - UV^2}{2 \cdot AV \cdot AU} = \frac{GA^2 - f^2}{2e^2 - GA^2 - f^2} \Longrightarrow$$

$$\cos\frac{\lambda}{2} = \sqrt{\frac{1+\cos\lambda}{2}} = \sqrt{\frac{e^2 - f^2}{2e^2 - GA^2 - f^2}} = \sqrt{\frac{d^2}{2e^2 - GA^2 - f^2}}$$

Substituting $GA^2 = \frac{2}{9}(b^2+c^2) - \frac{1}{9}a^2$ and d, e, f into the latter equation yields

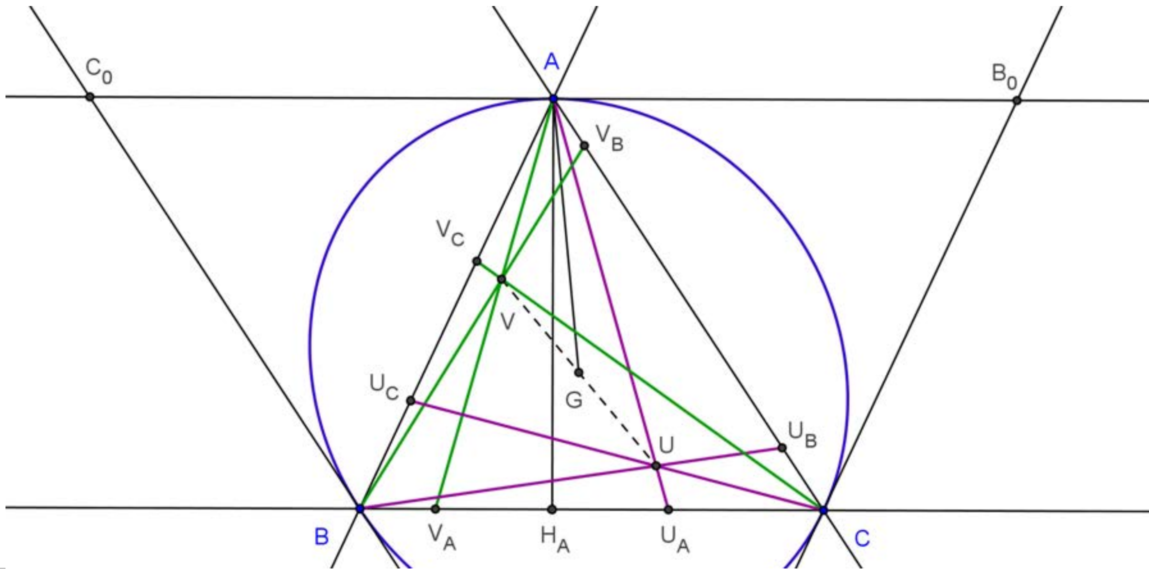
$$\cos\frac{\lambda}{2} = \sqrt{\frac{9d^2}{2(a^2+b^2+c^2-2\varrho) - 2(b^2+c^2) + a^2 - 4\varrho}} = \frac{\sqrt{3}d}{a} \quad (\star)$$

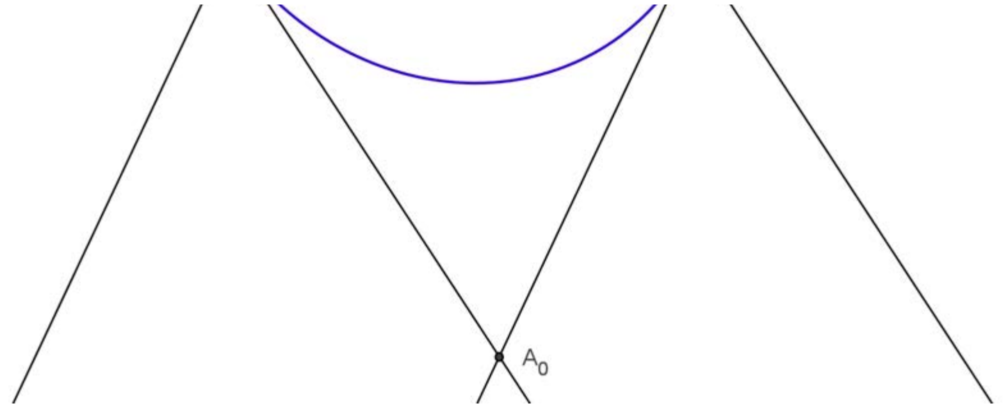
AU, AV cut BC at U_A, V_A and H_A is the foot of the A-altitude. Since the A-sideline B_0C_0 of the antimedial triangle of $\triangle ABC$ is tangent to $\mathcal{G}$, then AH_A is the normal to $\mathcal{G}$ through $A \Longrightarrow AH_A$ bisects $\angle U_AAV_A$, i.e. $\triangle AV_AU_A$ is isosceles with apex A. Thus, from $(\star)$ we get

$$AU_A = AV_A = AH_A \cdot \sec\frac{\lambda}{2} = \frac{2[\triangle ABC]}{a} \cdot \frac{a}{\sqrt{3}d} = \frac{2[\triangle ABC]}{\sqrt{3}d}$$

Obviously, this latter expression is independent of the chosen vertex, thus we conclude that the six cevians of $\triangle ABC$ passing through its two Bickart points are all congruent and the proof is completed.

Attachments:





skytin

#6 May 15, 2011, 2:23 pm

Thank for your solution Luisgeometra
My solution :
Use Lemma proposed in http://perso.orange.fr/jl.ayme
about two Equal segments
(Use names like in Luisgeometra Post)
So easy to see that A_0V is bissector of angle BVC , so ther exist circle a with center A_0 and tangent to CV and BV
like the same construct circles b and c
consruct second inner tangents to a , b , c
well known (you can prove it if you dont know) that they have same point U'
easy to see that A_0U' is angle bisseector of BU'C , so use Jean-Louis AYME
Post we get that Chevians frob B and C wich goes thru point U' are Equalent
like the same and A Chevian Equalent two
easy to see that a is tangent a is tangent to CU' , BU' , CV , BV , so U'C + CV = U'B + BV , so not hard to prove that angle U'A_0C = BA_0C like the same other angles , so U' is Isogonal conjugacy of points U' and V wrt A_0B_0C_0
angles C_0AV = U'AB_0 , so easy to see that all Chevians from U' and V are Equvalent
Easy to see that ellips with focusses U' and V and goes thru A , B , C is Steiner ellipse of A_0B_0C_0 , so there are exist two points P and Q such that all Chevians from them are equal . Done

TelvCohl

#7 Dec 18, 2015, 3:22 pm • 1

Lemma (well-known) **:** Let P be a point on the anticomplement of the A-internal bisector of $\triangle ABC$ WRT $\triangle ABC$ and let BP, CP cuts CA, AB at E, F, respectively. Then $CE = BF$.

Back to the main problem :

Let P be one of the focus of Steiner ellipse $\mathcal{E}$ of $\triangle ABC$. Let $\triangle A_1B_1C_1$ be the anticomplementary triangle of $\triangle ABC$ and let $\triangle P_aP_bP_c$ be the cevian triangle of P WRT $\triangle ABC$. Since C_1A_1, A_1B_1 is tangent to $\mathcal{E}$ at B, C, respectively, so we get PA_1 is the bisector of $\angle BPC$ (well-known) $\Longrightarrow$ the complement of A WRT $\triangle BPC$ lies on the P-internal bisector of $\triangle BPC \Longrightarrow$ $BP_b = CP_c$ (from **Lemma**). Similarly, we can prove $CP_c = AP_a$, so we conclude that $AP_a = BP_b = CP_c$.

P.S. I found another interesting property (with proof) as following **:** Let T be the image of the de Longchamps point of $\triangle ABC$ under the homothety $\mathbf{H}(P, -\frac{1}{2})$. Prove that $\triangle P_aP_bP_c$ is the pedal triangle of T WRT $\triangle ABC$.

Quick Reply

High School Olympiads

intersection of Simson lines /K.K 6.4 /2

Reply

sergei93

#1 May 15, 2011, 12:36 am

Let A, B, C, D be four points on a circle. Prove that the intersection of the Simson line of A with respect to $\triangle BCD$ with the Simson line of B with respect to $\triangle ACD$ is collinear with C and the orthocenter of $\triangle ABD$.

Luis González

#2 May 15, 2011, 3:24 am • 2

This follows from a more general configuration: Given four points in a plane with no three of them collinear, the pedal circles of each point with respect to the triangle formed by the other three concur at the Poncelet point of ABCD.

Label the 4 points $A, B, C, D.$ P, Q, R are the projections of C onto $AB, BD, DA.$ Let M, N, L be the midpoints of $AB, AC, AD.$ 9-point circles $\odot(MNP)$ and $\odot(MNL)$ of $\triangle ABC$ and $\triangle ADC$ intersect at N and the Poncelet point E of $ABCD.$ Thus

$$\angle REP = \angle REN + \angle PEN = \angle RLN + \angle PMN = \angle ADC + \angle ABC \pmod{\pi}$$

$$\angle RQP = \angle RQC + \angle PQC = \angle ADC + \angle ABC \pmod{\pi}$$

$\Longrightarrow \angle REP = \angle RQP \Longrightarrow E \in \odot(PQR).$ Similarly, pedal circles of A,B,D WRT the triangle formed by the three remaining vertices pass through the Poncelet point of ABCD. If A,B,C,D are concyclic, the Poncelet point of ABCD coincides with the center of symmetry of ABDC and the quadrilateral formed by the orthocenters of triangles BCD,CDA,DAB, ABC.

Quick Reply

High School Olympiads

Ratio of areas - triangle:quadrilateral

Reply

Source: Canadian Mathematical Olympiad - 1978 - Problem 4.

BigSams

#1 May 14, 2011, 8:12 am

The sides AD and BC of a convex quadrilateral $ABCD$ are extended to meet at E. Let H and G be the midpoints of BD and AC, respectively. Find the ratio of the area of the triangle EHG to that of the quadrilateral $ABCD$.

Luis González

#2 May 14, 2011, 10:48 am

$F \equiv AB \cap DC$ and let K be the midpoint of EF. Thus G, H, K are collinear on the Newton line of the convex $ABCD$. The area of the "boomerang" $EGFH$ is then twice the area of $\triangle EHG$. For convenience, let h_P denote the distance from a point P to the line EF and assume WLOG that H is between G and K.

$$[EGFH] = [GEF] - [HEF] = \frac{1}{2}EF(h_G - h_H) = \frac{1}{4}EF(h_C + h_A - h_B - h_D)$$

$$\implies 2[EGFH] = [CEF] + [AEF] - [BEF] - [DEF] = [ABCD]$$

$$\implies 2[EGFH] = 4[EHG] = [ABCD]$$

Quick Reply

High School Olympiads

Concurrency

Reply

r1234
#1 May 13, 2011, 3:19 pm

Let $\triangle A_1B_1C_1$ be the second orthic triangle of $\triangle ABC$.Prove that AA_1, BB_1 and CC_1 are concurrent on $Euler Line$.
[geogebra]8eb3c4e0fcad210bd0ac26103fb8710ca3525652[/geogebra]

jayme
#2 May 13, 2011, 4:05 pm

Dear Mathlinkers,
for the concurrency we can applied the cevian nests theorem.
See for example
http://perso.orange.fr/jl.ayme vol. 3
Sincerely
Jean-Louis

jayme
#3 May 13, 2011, 5:22 pm

Dear Mathlinkers,
If we reformulate the problem in this way
1. ABC a triangle
2. A'B'C' the orthic triangle of ABC
3. IaIbIc the excentral triangle of ABC
4. the line joining the center I of ABC (orthocentre of ABC) to the circumcircle O of ABC (the center of the Euler's circle of IaIbIc) is the Euler's line of IaIbIc.

We have to prove : IO passes through the perspector X of A'B'C' and IaIbIc.
Sincerely
Jean-Louis

Luis González
#4 May 13, 2011, 7:46 pm

Dear Jean Louis, that was my idea too. See Concurrent. Our perspector is

$$X_{24}\left(\frac{1}{S_A}-\frac{2S_A}{b^2c^2}:\frac{1}{S_B}-\frac{2S_B}{c^2a^2}:\frac{1}{S_C}-\frac{2S_C}{a^2b^2}\right)$$

Quick Reply

High School Olympiads

resembling pythagorean theorem

77ant

#1 Jul 29, 2009, 1:12 am

AB and AC are tangent to circle O at B and C, respectively, and D is a point on the major arc BC. Lines BD and AC meet at E, and lines DC and BA meet at F. If BF = a, CE = b, and EF = x, prove that $x^2 = a^2 + b^2 - ab$

Luis González

#2 May 12, 2011, 6:38 am

The proof is analogous to the one given in the other topic resembling pythagorean theorem apart from some sign changes. Tangent of (O) through D cuts BC at P. Then P, E, F are collinear on the Lemoine axis of $\triangle BDC$. Let $AB = AC = L$. By Menelaus' theorem for $\triangle ABC$ cut by PFE, we have

$$\frac{a}{L+a} \cdot \frac{L+b}{b} = \frac{PB}{PC} = \frac{DB^2}{DC^2} = \frac{a^2}{b^2} \cdot \frac{EB^2}{FC^2} \Longrightarrow \frac{EB^2}{FC^2} = \frac{b(L+b)}{a(L+a)} \quad (1)$$

On the other hand, by Stewart theorem for the cevians EB, FC in $\triangle AFE$, we have

$$EB^2 = \frac{x^2L + a(L+b)^2 - aL(L+a)}{L+a} \ , \ FC^2 = \frac{x^2L + b(L+a)^2 - bL(L+b)}{L+b}$$

$$\frac{EB^2}{FC^2} = \frac{L+b}{L+a} \cdot \frac{x^2L + a(L+b)^2 - aL(L+a)}{x^2L + b(L+a)^2 - bL(L+b)} \quad (2)$$

From (1) and (2) we have then

$$\frac{x^2L + a(L+b)^2 - aL(L+a)}{x^2L + b(L+a)^2 - bL(L+b)} = \frac{b}{a} \Longrightarrow x^2 = a^2 + b^2 - ab$$

Virgil Nicula

#3 May 13, 2011, 3:12 am

Nice problem and nice proof ! Remark that the relation $\boxed{x^2 = a^2 + b^2 - ab}$ is truly (analogous proof) for any point D which belongs to the circle (O) , on the minor arc BC or on the major arc BC .

High School Olympiads

resembling pythagorean theorem

Reply

77ant

#1 Jul 29, 2009, 1:14 am

AB and AC are tangent to circle O at B and C, respectively, and D is a point on the minor arc BC. Lines BD and AC meet at E, and lines CD and AB meet at F. If BF = a, CE = b, and EF = x, prove that $x^2 = a^2 + b^2 - ab$

Luis González

#2 May 12, 2011, 6:37 am

Tangent of (O) through D cuts BC at P. Thus, E,F,P are collinear on the Lemoine axis of $\triangle BDC$. Let $AB=AC=L$. By Menelaus' theorem for $\triangle ABC$ cut by $\overline{PFE}$, we have

$$\frac{a}{L-a} \cdot \frac{L-b}{b} = \frac{PB}{PC} = \frac{DB^2}{DC^2} = \frac{a^2}{b^2} \cdot \frac{EB^2}{FC^2} \Longrightarrow \frac{EB^2}{FC^2} = \frac{b(L-b)}{a(L-a)} \ \ (1)$$

On the other hand, by Stewart theorem for the cevians EB, FC in $\triangle AFE$, we have

$$EB^2 = \frac{x^2L + aL(L-a) - a(L-b)^2}{L-a} \ , \ FC^2 = \frac{x^2L + bL(L-b) - b(L-a)^2}{L-b}$$

$$\frac{EB^2}{FC^2} = \frac{L-b}{L-a} \cdot \frac{x^2L + aL(L-a) - a(L-b)^2}{x^2L + bL(L-b) - b(L-a)^2} \ \ (2)$$

From (1) and (2) we have then

$$\frac{x^2L + aL(L-a) - a(L-b)^2}{x^2L + bL(L-b) - b(L-a)^2} = \frac{b}{a} \Longrightarrow x^2 = a^2 + b^2 - ab$$

Quick Reply

High School Olympiads

Vietnam TST 2008

Reply

Source: Problem 5 - Day 2

April

#1 Apr 2, 2008, 10:11 am • 1

Let k is a given positive number. Let ABC be an acute triangle, $AB \neq BC \neq CA$, let O be its circumcenter, and AD, BE, CF be its internal angle bisectors. Points L, M, N lie on rays AD, BE, CF, respectively such that $\frac{AL}{AD} = \frac{BM}{BE} = \frac{CN}{CF} = k$. Denote by (O_1), (O_2), (O_3), respectively the circle passes through L and tangent to OA at A, passes through M and tangent to OB at B, passes through N and tangent to OC at C.

$1.$ When $k = \frac{1}{2}$, prove that circles (O_1), (O_2), (O_3) have two common points and the centroid G of triangle ABC lies on line connecting these common points.

$2.$ Find all k, for which circles (O_1), (O_2), (O_3) are coaxial.

Luis González

#2 May 11, 2011, 10:10 am • 1

Tangents to the circumcircle (O) through A, B, C cut BC, CA, AB at the centers of the the A-,B-,C- Apollonian circles O_A, O_B, O_C. Hence, perpendicular bisectors of AL, BM, CN cut AO_A, BO_B, CO_C at O_1, O_2, O_3, respectively. Isosceles $\triangle ALO_1$ and $\triangle ADO_A$ are similar with similarity coefficient k, a.s.o. Thus, $\frac{AO_1}{AO_A} = \frac{BO_2}{BO_B} = \frac{CO_3}{CO_C} = k$. Now, using barycentric coordinates WRT $\triangle ABC$, we have

$$O_A(0 : b^2 : -c^2) \ , \ O_1((b^2 - c^2)(1-k) : b^2k : -c^2k)$$

$$O_B(-a^2 : 0 : c^2) \ , \ O_2(-a^2k : (c^2 - a^2)(1-k) : c^2k)$$

$$O_C(a^2 : -b^2 : 0) \ , \ O_3(a^2k : -b^2k : (a^2 - b^2)(1-k))$$

$OA = OB = OC$ implies that O has equal power to $(O_1), (O_2), (O_3)$. Thus, $(O_1), (O_2), (O_3)$ are coaxal $\Longleftrightarrow$ O_1, O_2, O_3 are collinear $\Longleftrightarrow$

$$\begin{bmatrix} (b^2-c^2)(1-k) & b^2k & -c^2k \\ -a^2k & (c^2-a^2)(1-k) & c^2k \\ a^2k & -b^2k & (a^2-b^2)(1-k) \end{bmatrix} = 0$$

$$\Longleftrightarrow (k-1)(2k-1)(b^2-c^2)(c^2-a^2)(a^2-b^2) = 0$$

Since $\triangle ABC$ is not isosceles, $(O_1), (O_2), (O_3)$ are coaxal $\Longleftrightarrow$ either $k = 1$ or $k = \frac{1}{2}$. If $k = 1$, then $(O_1), (O_2), (O_3)$ coincide with the Apollonian circles $(O_A), (O_B), (O_C)$, whose common radical axis is the Brocard axis of $\triangle ABC$. If $k = \frac{1}{2}$, then O_1, O_2, O_3 are the midpoints of $AO_A, BO_B, CO_C \Longrightarrow O_1, O_2, O_3$ lie on the orthic axis of $\triangle ABC \Longrightarrow$ common radical axis of $(O_1), (O_2), (O_3)$ is the Euler line of $\triangle ABC$.

tuan119

#3 May 11, 2011, 11:13 am

You can draw pictures of items for everyone! .

Quick Reply

High School Olympiads

Korean problem

yunustuncbilek
#1 May 8, 2011, 8:12 pm

Let $\triangle ABC$ be a triangle and let L, M, N be points on BC, CA and AB, respectively. Let P, Q and R be the intersection points of the lines AL, BM and CN with the circumcircle of ABC, respectively. Prove that $(|AL| \div |LP|) + (|BM| \div |MQ|) + (|CN| \div |NR|) \geq 9$

tanquyen236
#2 May 9, 2011, 2:09 am

> yunustuncbilek wrote:
> Let $\triangle ABC$ be a triangle and let L, M, N be points on BC, CA and AB, respectively. Let P, Q and R be the intersection points of the lines AL, BM and CN with the circumcircle of ABC, respectively. Prove that $(|AL| \div |LP|) + (|BM| \div |MQ|) + (|CN| \div |NR|) \geq 9$

Apply the Stewart formula, we have easily that:
$BC.AL^2 = BL.AC^2 + CL.AB^2 - BL.CL.BC$
$\Rightarrow \dfrac{AL^2}{BL.CL} = \dfrac{AC^2}{CL.BC} + \dfrac{AB^2}{BL.BC} - 1$
But, on the one hand, by applying Cauchy-Schwarzt theorem we have:
$\dfrac{AC^2}{CL.BC} + \dfrac{AB^2}{BL.BC} \geq \dfrac{(AC+AB)^2}{BC^2}$
And on the other hand, $ABPL$ is a circle quadrilateral, so:
$AL.PL = BL.CL \Rightarrow \dfrac{AL}{LP} = \dfrac{AL^2}{AL.PL} = \dfrac{AL^2}{BL.CL}$
To sum up, we conclude that:
$\dfrac{AL}{LP} \geq \dfrac{(AC+AB)^2}{BC^2} - 1$
Likewise, we make up two other inequality and then sum all together, we infer that:
$\dfrac{AL}{LP} + \dfrac{BM}{MQ} + \dfrac{CN}{NR} \geq \dfrac{(AC+AB)^2}{BC^2} + \dfrac{(BC+BA)^2}{CA^2} + \dfrac{(CA+CB)^2}{AB^2} - 3$
However, we all know that:
$\dfrac{(AC+AB)^2}{BC^2} + \dfrac{(BC+BA)^2}{CA^2} + \dfrac{(CA+CB)^2}{AB^2} \geq \dfrac{1}{3}\left(\dfrac{AC+AB}{CB} + \dfrac{BC+BA}{CA} + \dfrac{CA+CB}{AB}\right)^2 \geq \dfrac{1}{3}.6^2 = 12$
In conclusion, we proved that:
$\dfrac{AL}{LP} + + \dfrac{BM}{MQ} + \dfrac{CN}{NR} \geq 9$

Virgil Nicula
#3 May 9, 2011, 8:50 pm

See **here**

Luis González
#4 May 10, 2011, 7:52 am

D, E are the midpoints of $\overline{BC}$ and the arc BC of the circumcircle (not containing A). h_a denotes the length of the A-altitude and let P' be the orthogonal projection of P onto BC. Since $ED \geq PP'$, we deduce that

$$\frac{AL}{LP} = \frac{h_a}{PP'} \geq \frac{h_a}{DE} = \frac{2h_a}{a \cdot \tan\frac{A}{2}} = \frac{(a+b+c)(b+c-a)}{a^2} = \left(\frac{b+c}{a}\right)^2 - 1$$

The cyclic sum yields:

$$\frac{AL}{LP} + \frac{BM}{MQ} + \frac{CN}{NR} \geq \left(\frac{b+c}{a}\right)^2 + \left(\frac{c+a}{b}\right)^2 + \left(\frac{a+b}{c}\right)^2 - 3 \ \ (1)$$

On the other hand, by RMS-AM and AM-HM, we have

$$3\sqrt{\frac{1}{3}\sum_{\text{cyclic}}\left(\frac{b+c}{a}\right)^2} \geq \sum_{\text{cyclic}} \frac{b+c}{a} = (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 3 \ \ (2)$$

$$\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \implies (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9 \ \ (3)$$

Adding (2) and (3) yields : $\left(\frac{b+c}{a}\right)^2 + \left(\frac{c+a}{b}\right)^2 + \left(\frac{a+b}{c}\right)^2 \geq 12 \ \ (4)$

Adding (1) and (4) yields : $\frac{AL}{LP} + \frac{BM}{MQ} + \frac{CN}{NR} \geq 9$

Quick Reply

High School Olympiads

inscribed quadrilateral

Reply

Source: Latvian Selection Problem 3

JustN

#1 May 8, 2011, 3:11 pm

Given quadrilateral $ABCD$.With incircle in it. AD is the biggest side of the quadriteral. AB and CD intersect in M. AC and BD intersect in N. Prove that:

$90 \le \angle AND \le 90 + \frac{1}{2}\angle AMD$

Luis González

#2 May 9, 2011, 9:59 pm

Incircle (I) of $ABCD$ touches BC, CD, DA, AB at $P, Q, R, S,$ respectively. $AD \ge CD$ and $RD = QD$ implies that $AR \ge CQ \Longrightarrow RS \ge PQ.$ Likewise, from $AD \ge AB$ and $AR = AS,$ we get $RQ \ge PS.$ Consequently, if $U \equiv RS \cap QP$ and $V \equiv RQ \cap SP,$ then P lies between U, Q and $V, S.$ Note that N, B, D, U are collinear on the polar of V WRT (I) and N, C, A, V are collinear on the polar of U WRT $(I) \Longrightarrow UN \perp IV, VN \perp IU,$ i.e. N is the orthocenter of $\triangle IUV.$ Since UN, VN cut the segments connecting their poles with $I,$ then either N is inside $\triangle IUV$ or $N \equiv I,$ i.e. $\triangle NUV$ is either N-obtuse or right (when U,V lie at infinite) $\Longrightarrow \angle UNV = \angle AND \ge 90^\circ.$ Since P lies inside $\triangle NUV,$ then $\angle UNV = \angle AND \le \angle UPV = \angle SPQ = 90^\circ + \tfrac{1}{2}\angle AMD.$

Quick Reply

High School Olympiads

ABC is an acute angled triangle..

Reply

r1234
#1 May 8, 2011, 2:50 pm

ABC is an acute angled triangle. A' is the midpoint of BC and P is any point on the median AA' such that $PA' = BA'$. The perpendicular from P to BC cuts BC at X.The perpendicular from X to PB cuts AB at Z and the perpendicular from X to PC cuts AC at Y. Show that the circle XYZ touches BC at X.

Luis González
#2 May 9, 2011, 2:03 am

Tangents of the circumcircle $\odot(ABC)$ through B, C meet at $T. Q$ is the isogonal conjugate of P WRT $\triangle ABC$, which lies on the A-symmedian AT. Since $\angle BPC = 90^\circ$, then it follows that $\angle BQC = 90^\circ + \angle BAC \Longrightarrow Q \in \odot(T, TB)$. $\triangle ABT$ and $\triangle ACA'$ are pseudo-similar, due to $\angle BAT = \angle CAA', \angle ABT = \pi - \angle ACB$, hence we have:

$$\frac{TB}{TA} = \frac{TQ}{TA} = \frac{A'C}{A'A} = \frac{A'P}{A'A} \Longrightarrow TA' \parallel PQ \Longrightarrow PQ \perp BC.$$

Hence, the pedal circle ω of (P, Q) WRT $\triangle ABC$ is tangent to BC through X. Further, Y, Z coincide with the orthogonal projections of Q onto AC, AB, i.e. $\odot(XYZ) \equiv \omega$.

tanquyen236
#3 May 9, 2011, 3:46 am

Suppose I, J are the inspections of PB, PC to AC, AB respectively and simultaneously let $M = A'$ just because of convenience in reading name of points . By Menelaus theorem, we have:

$$\frac{BM}{CM}.\frac{CI}{AI}.\frac{AJ}{BK} = 1 \Rightarrow \frac{CI}{AI} = \frac{BK}{AJ} \Rightarrow IJ//BC$$

$$\Rightarrow \frac{BJ}{CI} = \frac{AB}{AC}(*)$$

From the equality $PM = BM = CM$, we conclude that $\angle BPC = 90^o$. Simultaneously, let D, E are the projections of X to PC, PB respectively. Then, we will prove that $DE//YZ$.

Indeed, applying Sin theorem to $\Delta XBZ, \Delta XCY$, we have easily:

$$XZ = \frac{sin\angle ZBX}{sin\angle BZX}.XB = \frac{sin\angle ABC}{cos\angle PBJ}.XB = \frac{BJ.XB.sin\angle ABC}{PB}$$

$$XY = \frac{CI.XC.sin\angle ACB}{PC}$$

From the above two equality, we infer that:

$$\frac{XZ}{XY} = \frac{sin\angle ABC}{sin\angle ACB}.\frac{BJ}{CI}.\frac{XB}{XC}.\frac{PC}{PB} = \frac{AC}{AB}.\frac{BJ}{CI}.\frac{XB}{XC}.\frac{PC}{PB} = \frac{XB}{XC}.\frac{PC}{PB}$$ (thanks to $(*)$)

But we also have:

$$\frac{XB}{XC} = \frac{XB.XP}{XC.XP} = \frac{XE.PB}{XD.PC} \Rightarrow \frac{XB}{XC}.\frac{PC}{PB} = \frac{XE}{XD}$$

$$\Rightarrow \frac{XZ}{XY} = \frac{XE}{XD} \Rightarrow DE//YZ$$

$$\Rightarrow \angle ZYX = \angle EDX = \angle EPX = \angle BXZ$$

$\Rightarrow BC$ is a tangent of (XYZ) at X

Quick Reply

High School Olympiads

Find the measure of angle BID.

Reply

tuan119

#1 May 8, 2011, 8:53 pm

Let ABC be a right triangle with right angle at A and let AD be the angle bisector at A. Denote by M and N the bases of the altitudes from D onto AB and AC respectively, BN meets CM at K and AK meets DM at I. Find the measure of angle BID.

Luis González

#2 May 8, 2011, 10:39 pm

Let $\triangle ABC$ be any scalene triangle. E is the foot of the A-altitude and MN cuts the sideline BC at F. Since E lies on the circumcircle of $\triangle DMN$ and $DM=DN$, then it follows that $ED \equiv BC$ and EA bisects $\angle NEM$ externally and internally $\Longrightarrow E(M,N,A,F)$ is harmonic $\Longrightarrow (B,C,E,F)=-1$. Hence, $K \equiv BN \cap CM$ lies on $AE \Longrightarrow I \equiv DM \cap AK$ is the orthocenter of $\triangle ADB \Longrightarrow \angle BID = \frac{1}{2}\angle A \pmod{\pi}$

tuan119

#3 May 9, 2011, 9:32 am

Thank Mr. :)

Quick Reply

High School Olympiads

Ineq-G133 - Geometry

Ligouras

#1 May 7, 2011, 10:43 pm

For every triangle ABC, let D, E, F be a point located on segment BC, CA, AB, respectively.
Let O be the intersection of AD and FE.
Prove that:

$$\frac{AF}{AB \cdot CD} + \frac{AE}{AC \cdot DB} \geq \frac{4AO}{AD \cdot CB}$$

Luis González

#2 May 8, 2011, 5:12 am

By Cristea's theorem we have $CD \cdot \frac{FB}{AF} + DB \cdot \frac{EC}{AE} = BC \cdot \frac{OD}{AO}$

Substituting $FB = AB - AF$ and $EC = AC - AE$ yields

$$CD \cdot \frac{AB - AF}{AF} + DB \cdot \frac{AC - AE}{AE} = BC \cdot \frac{OD}{AO}$$

$$\frac{AB \cdot CD}{AF} + \frac{AC \cdot DB}{AE} - CD - DB = BC \cdot \frac{OD}{AO}$$

$$\frac{CD \cdot AB}{AF} + \frac{AC \cdot DB}{AE} = BC \cdot \frac{OD}{AO} + BC = BC \cdot \frac{AD}{AO}$$

By AM-HM we get $\frac{AF}{AB \cdot CD} + \frac{AE}{AC \cdot DB} \geq \frac{4}{\frac{CD \cdot AB}{AF} + \frac{AC \cdot DB}{AE}} = \frac{4AO}{AD \cdot BC}$

High School Olympiads

Two nice parallels (own)

Reply

jayme

#1 May 6, 2011, 8:51 pm

Dear Mathlinkers,

ABC a triangle,
(O) the circumcircle of ABC, O the center of (O),
Tb, Tc the tangents to (O) at B, C,
A* the point of intersection of Tb and Tc,
DEF the orthic triangle of ABC,
Y, Z the points of intersection of A*F and OB, A*E and OC.

Prove : YZ // EF.

Sincerely
Jean-Louis

Luis González

#2 May 7, 2011, 7:19 am

K is the symmedian point of $\triangle ABC$ and U, S are the orthocenter and Spieker point of $\triangle DEF$, i.e. midpoint of OU. We redefine Y, Z as the intersections of EK, FK with OB, OC, respectively. According to this topic, we know that $\frac{KE}{KY} = \frac{KF}{KZ} = \frac{KU}{KO}, YZ \parallel EF$. Since AK passes through the midpoint M of EF, then FY, EZ and AK concur at a point A'. Thus, we shall show that $A' \equiv A^*$. Since medial triangle of $\triangle DEF$ is homothetic to the tangential triangle of $\triangle ABC$ through K, we have

$$\frac{KA^*}{KM} = \frac{KO}{KS} \Longrightarrow \frac{MA^*}{MK} = \frac{OS}{KS}$$

From the concurrent cevians $A'M, EY, FZ$ in the triangle $\triangle A'EF$, we get

$$\frac{MK}{MA'} + \frac{YK}{YE} + \frac{ZK}{ZF} = 1 \Longrightarrow \frac{MK}{MA'} + 2 \cdot \frac{KO}{UO} = 1 \Longrightarrow$$

$$\frac{MA'}{MK} = \frac{UO}{UO - 2KO} = \frac{2OS}{2(OS - KO)} = \frac{OS}{KS} \Longrightarrow A' \equiv A^*$$

Quick Reply

High School Olympiads

Perpendicularity in a cyclic quad

Reply

Source: Balkan Mathematical Olympiad 2011. Problem 1.

frenchy

#1 May 6, 2011, 10:04 pm • 1

Let $ABCD$ be a cyclic quadrilateral which is not a trapezoid and whose diagonals meet at E. The midpoints of AB and CD are F and G respectively, and ℓ is the line through G parallel to AB. The feet of the perpendiculars from E onto the lines ℓ and CD are H and K, respectively. Prove that the lines EF and HK are perpendicular.

Luis González

#2 May 6, 2011, 11:12 pm • 1

Since ℓ is antiparallel to DC WRT $EC, ED,$ it follows that EH is the E-circumdiameter of $\triangle EDC.$ Let M be the orthogonal projection of E on HK and EM cuts DC at $T.$ From the cyclic quadrilateral $EKHG$ (due to its right angles <EKG and <EHG) we deduce that $\angle HEG=\angle HKG=\angle MKT=\angle KET.$ Since EK, EH are isogonals WRT $\angle DEC,$ then EM, EG are also isogonals WRT $\angle DEC.$ From $\triangle DEC \sim \triangle AEB,$ we have then $\angle FEB=\angle GEC=\angle MED \Longrightarrow$ F, E, M are collinear, i.e. $EF \perp HK.$

silouan

#3 May 7, 2011, 12:04 am • 6

Here is my solution to this nice problem.
Let EH meets AB at P. Then we have to prove that $\angle PEF+\angle EHK=90$.
But we have that $\angle EHK=\angle EGK=90-\angle KEG$. So we have to prove that $\angle KEG=\angle FEP$. But DEC and AEB are similar and the angles KEG, FEP are these that are formed by the median and the height to these similar triangles so they are equal and we are done.

Bugi

#4 May 7, 2011, 12:22 am • 2

Solution

MariusBocanu

#5 May 7, 2011, 2:30 pm

Let $U=HK\cap EF.$ The condition can be written as $\widehat{GKH}=\widehat{KEU}$,but G,H,K,E are cocyclic, so $\widehat{GKH}=\widehat{GEH}$.Denote $\widehat{GEH}=x, \widehat{HGE}=90-x$, denote $P=GE\cap AB$ so $\widehat{EPF}=90-x$ Denote $R=EK\cap AB$. We have to prove that $\widehat{REF}=x$ Denote $M=EF\cap DC$, so we have to prove that $\widehat{EMG}=\widehat{EPF}$, but we have $\widehat{PEF}=\widehat{MEG}$, so we have to prove that $\widehat{DGE}=\widehat{EFA}$, but this is well-known.(prove that $\widehat{AEF}=\widehat{DEG}$(use $\frac{AE}{EB}=\frac{sin\widehat{FEB}}{sin\widehat{AEF}}$.

manifestdestiny

#6 May 8, 2011, 5:29 am

Let $EF\cap l=X$. Then $\angle FXH=\angle AFE$.

We claim $\angle AFE=\angle EGD$. Indeed, $\triangle AEB\sim\triangle DEC$, and as F is the midpoint of AB, G the midpoint of CD, we must have $\triangle AFE\sim\triangle DGE$.

Now let $EX \cap KH = Y$. THen $EKGH$ is cyclic.
But $\angle EKH = \angle EGH$ and $\angle KEX = \angle GEH$, implying $\angle EYK = 90^o$ as desired.

salazar

#7 May 8, 2011, 5:24 pm

Does anyone know anything about the results?
Thank you!

aleksandar

#8 May 8, 2011, 7:38 pm

The medal cut-offs are known : 10 for bronze, 30 for gold, but I'm not sure if 17 or 21 are required for silver. Some of the countries posted their results, but the complete official results can't be seen on http://www.bmo2011.lbi.ro yet. I don't know what's wrong with the page.

matrix41

#9 May 9, 2011, 2:16 pm

Simson's line (E,KGH) (suppose EX' perpendicular to HK) + Proof $\angle BEF = \angle X'ED$ by Similarity
Done

TheIronChancellor

#10 May 9, 2011, 6:35 pm

> *matrix41 wrote:*
> Simson's line (E,KGH) (suppose EX' perpendicular to HK) + Proof $\angle BEF = \angle X'ED$ by Similarity
> Done

Well Done !

I had also notice it but my proof was little different : to prove that KH is the Simpson Line of TriangleKGH it is enough to prove that the steiner line of this triangle

is parallel to simpson line. This can be easily done if we recall the Brocard Lemma to help us, saying : "The quadrilateral ABCD is inscribed in the circle k with center O. Let
E = AB∩CD, F = AD∩BC, G = AC∩BD. Then O is the orthocenter of the triangle EFG. "

erfan_Ashorion

#11 Oct 20, 2011, 9:15 pm

oh yes!i proof some lemma!

lemma1:$\angle DEK = \angle HEC$

suppose EH intersect AB on P because l is parallel to AB $\angle BPE = 90$ so $\angle KEC = \angle PEB = \angle DEH$
so $\angle DEK = \angle HEC$!
end of proof of lemma 1.
suppose EF intersect HK at Q...!

lemma2:EQ is isogonal of EG wrt DE and EC

i proof that $\frac{\sin CEG}{\sin GED} = \frac{\sin DEQ}{\sin CEQ}$
proof of lemma 2:
$\frac{\sin CEG}{\sin GED} = \frac{ED}{EC}$
we know that $\angle DEQ = \angle FEB$ and $\angle DEC = \angle AEB$
$\frac{\sin BEF}{\sin FEA} = \frac{EA}{EB}$
know we must to proof $\frac{EA}{EB} = \frac{ED}{EC}$ and it is so easy that we can proof by $\triangle AEB \sim \triangle DEC$
end of proof of lemma 2.
know from lemma1 and lemma2 $\rightarrow \angle KEQ = \angle GEH$ so EQ is isogonal of EG wrt EK and EH know i need lemma 3

know from lemma1 and lemma2 $\to \angle KEQ = \angle GEH$ so EQ is isogonal of EG wrt EK and EH know i need lemma 3

lemma 3:theoream 2 from isogonal conjugation with respect to a traingle frome Darij grinberg

from lemma 3 and lemma 2 and lemaa 1 $\longrightarrow$ problem proof!

lemma 3:http://www.cip.ifi.lmu.de/~grinberg/Isogonal.zip

waver123

#12 Oct 20, 2011, 10:59 pm

easy angle chasing leads to EF bring the symmedian of triangle EDC .
Now some more easy angle chasing leads to the desired result.

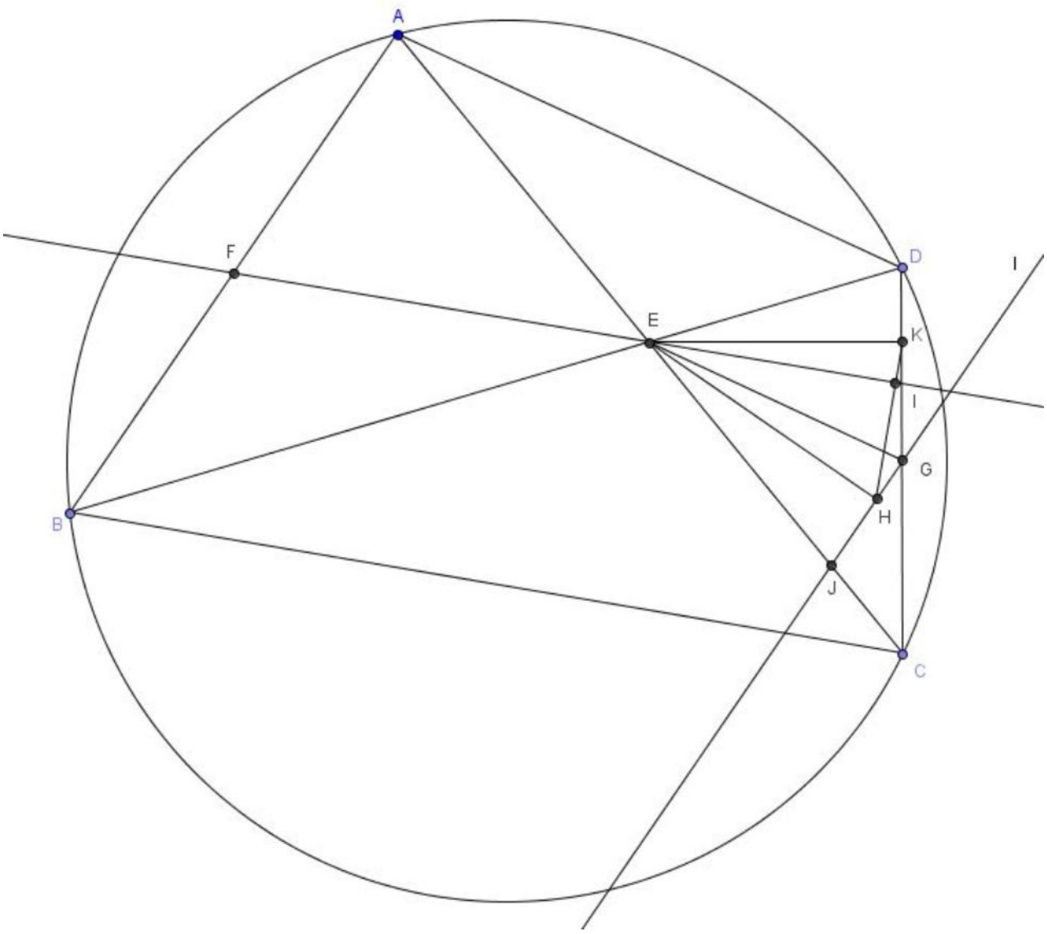
StefanS

#13 Apr 25, 2012, 10:24 pm • 1

My solution is probably the same as waver123's.

My solution

Attachments:



IstekOlympiadTeam

#14 Dec 4, 2015, 9:28 pm

My Solution:

Let's say that $\angle CAB = \angle CDE = \alpha,\ \angle ABD = \angle ACD = \beta,\ \angle DEG = \phi$ and $\angle AEF = \lambda$

In the triangle AEB we have:

$$\frac{sin\ \alpha}{sin\ \beta} = \frac{sin\ \lambda}{sin\ (\alpha + \beta + \lambda)}\ (\star)$$

and similarly in triangle DEC we have

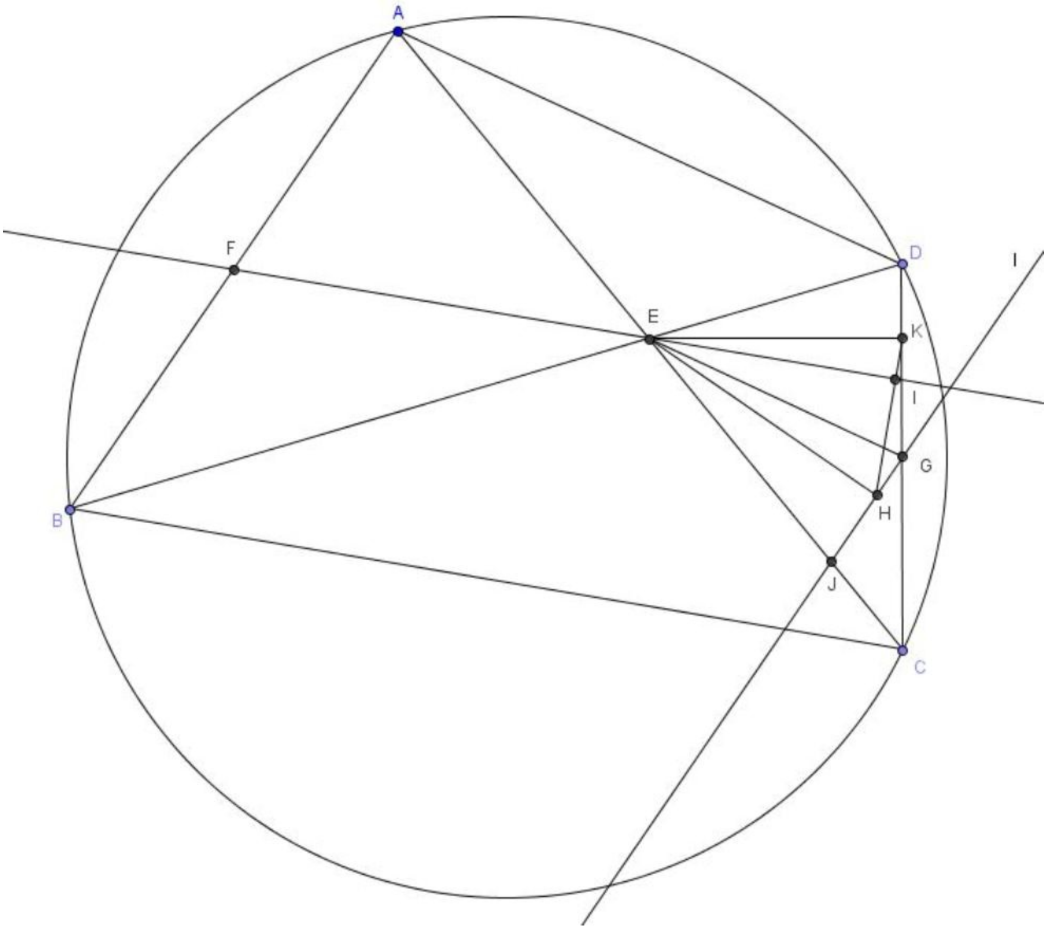
$$\frac{sin\ \alpha}{sin\ \beta} = \frac{sin\ \phi}{sin\ (\alpha+\beta+\phi)}\ (\star\star)$$

Combinig $(\star)$ and $(\star\star)$ we get

$$\frac{sin\ \phi}{sin\ (\alpha+\beta+\phi)} = \frac{sin\ \lambda}{sin\ (\alpha+\beta+\lambda)}$$

Solving the equation we get $\alpha = -\beta$ (Which is impossible) then $\phi = \lambda$ By angle Chasing we get $\angle EKT = \beta + \phi$ Where $T = HK \cap FE$ and since $\phi = \lambda$ we get $\angle KET = 90^\circ - \beta - \phi$ which follows $\angle ETK = 90^\circ$

Attachments:



This post has been edited 1 time. Last edited by IstekOlympiadTeam, Dec 4, 2015, 9:30 pm

hayoola

#15 Dec 5, 2015, 12:49 pm

let $l \cap AC = J$ and let $HK \cap EF = M$
we must prove that $\angle MEK = 90 - \angle EKM = 90 - \angle EGH$

$\angle EGH = \angle GEJ + \angle GJE$
$\angle GJE = \angle EAB = \angle EDK$
$\angle GEJ = \angle FEB = \angle DEM$
we know that $\angle EDK + \angle DEM = 90 - \angle EKM$
so we are done

henderson

#16 Apr 27, 2016, 1:07 am

Let $l \cap FE = J$ and $l \cap AC = L.$
Since $AB \parallel l,$ we have $\angle AFE = \angle EJH.$
Then from $\triangle AFE \sim \triangle DGE$ and cyclic quadrilateral $EKHG$ we have $\angle AFE = \angle DGE = \angle KHE = \angle LHE.$
So, $\angle EJH = \angle LHE.$
It means that $\triangle EJH \sim \triangle EHL.$ Since $\angle EHJ = 90^\circ,$ we get $\angle ELH = 90^\circ,$ which means that $EF \perp HK.$

This post has been edited 1 time. Last edited by henderson, Apr 27, 2016, 1:08 am

Quick Reply

+ Quick Reply

High School Olympiads

Concurrency

Locked

gilbert

#1 May 6, 2011, 4:07 pm

Hello! I am very interested in geometrical solution.
Given a triangle ABC, let A_1, B_1, and C_1 be the midpoints of BC, CA, and AB respecctively. Let P, Q, and R be the points of tangency of the incircle k with the sides BC, CA, and AB. Let P_1, Q_1, and R_1 be the midpoints of the arcs QR, RP, and PQ on which the points P, Q, and R divide the circle
k. Prove that the
lines A_1P_1, B_1Q_1, and C_1R_1 are concurrent.

Luis González

#2 May 6, 2011, 7:44 pm • 1

This problem has been discussed many times in the forum, e.g. see the following topics

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=20543
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=287975
http://www.artofproblemsolving.com/Forum/viewtopic.php?uid=56597&f=46&t=291388
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=376331
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=294728 (Generalization)

High School Olympiads

Problem

Reply

xenlulozo

#1 May 4, 2011, 7:40 pm

Let ΔABC is isosceles triangle in A, $\widehat{A}=\frac{\pi}{7}, AB=b, BC=a$. Prove: $a^5-4a^3b^2+3ab^4-b^5=0$

Luis González

#2 May 4, 2011, 11:39 pm • 2

Locate the points P,Q on AC,AB such that $CB=BP=PQ$. By easy angle chase we get that $\angle PQB=\frac{2\pi}{7} \Longrightarrow$ $\triangle QPA$ is Q-isosceles. Thus, $BP=PQ=QA=a$. Parallels from P,Q to BC cut AB,AC at S,T, respectively. Then $\triangle CBP$ and $\triangle QAT$ are congruent $\Longrightarrow PC=QT=x$, but $\triangle BCP$ and $\triangle ABC$ are similar

$$\Longrightarrow \frac{PC}{BC}=\frac{BC}{b} \Longrightarrow x=\frac{a^2}{b} \quad (1)$$

It's easy to see that $QTPS$ is an isosceles trapezoid with $PS=QS=y$. Then

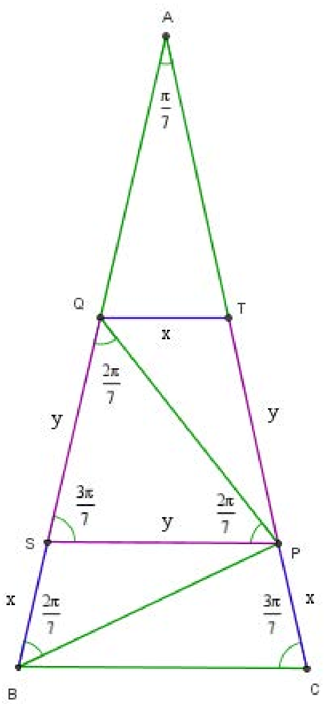
$$\frac{SP}{BC}=\frac{AS}{AB} \Longrightarrow \frac{y}{a}=\frac{y+a}{b} \Longrightarrow y=\frac{a^2}{b-a} \quad (2)$$

$$QS=TP=y \Longrightarrow TP+PC=AC-AT \Longrightarrow y+x=b-a \quad (3)$$

Combining $(1),(2),(3) \Longrightarrow \frac{a^2}{b-a}+\frac{a^2}{b}=b-a \Longrightarrow b^3+a^3-a^2b-2ab^2=0$

$$\Longrightarrow (b^3+a^3-a^2b-2ab^2)(a^2+ab-b^2)=0 \Longrightarrow a^5-b^5+3ab^4-4a^3b^2=0$$

Attachments:



Quick Reply

High School Olympiads

Ineq-G131 - Geometry

Ligouras

#1 May 3, 2011, 12:35 am

If r, R, O, I H are the inradius, the circumradius, the circumcenter, the incenter, and the orthocenter, respectively, of a triangle ABC, then

$$[OIH] \geq 4rR\sin\frac{|A-B|}{2}\sin\frac{|B-C|}{2}\sin\frac{|C-A|}{2}$$

Where the brackets denote the area of a triangle.

Luis González

#2 May 3, 2011, 11:11 am • 1

From Mollweide formulas we have $\sin\frac{|A-B|}{2}=\frac{|a-b|}{c}\cdot\cos\frac{C}{2}$, etc $\Longrightarrow$

$$\sin\frac{|A-B|}{2}\sin\frac{|B-C|}{2}\sin\frac{|C-A|}{2}=\frac{|(a-b)(b-c)(c-a)|}{abc}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

But, $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{1}{4}(\sin A+\sin B+\sin C)=\frac{a+b+c}{8R}\Longrightarrow$

$$\sin\frac{|A-B|}{2}\sin\frac{|B-C|}{2}\sin\frac{|C-A|}{2}=\frac{|(a-b)(b-c)(c-a)|}{abc}\cdot\frac{(a+b+c)}{8R} \quad (1)$$

From this topic we know that $[\triangle OIH]=\frac{|(a-b)(b-c)(c-a)|}{8r} \quad (2)$

Substituting (1) and (2) into the desired inequality gives:

$$\frac{|(a-b)(b-c)(c-a)|}{8r}\geq 4rR\cdot\frac{|(a-b)(b-c)(c-a)|}{abc}\cdot\frac{(a+b+c)}{8R}$$

$\frac{abc}{a+b+c}\geq 4r^2 \Longrightarrow R\geq 2r$, which is clearly true.

Ligouras

#3 May 3, 2011, 12:56 pm

very nice my brother, ***Luis Best Geometra*** thank

High School Olympiads

Concurrency triangles constructed outside a triangle

Reply

alphabeta1729
#1 May 2, 2011, 5:50 pm

Triangles BCD, CEA, AFB are constructed outside the triangle ABC such that $\angle ABF = \angle CBD$, $\angle BCD = \angle ACE$ and $\angle CAE = \angle BAF$. Prove that AD, BE, CF are concurrent.

PS: Please use synthetic geometry only

Luis González
#2 May 3, 2011, 9:40 am

Parallel from D to BC cuts AB, AC at P, Q, parallel from E to CA cuts BC, BA at R, S and parallel from F to AB cuts CA, CB at T, U.

$$\frac{PD}{DQ} = \frac{BD'}{D'C} \ , \ \frac{RE}{ES} = \frac{CE'}{E'A} \ , \ \frac{TF}{FU} = \frac{AF'}{F'B}$$

$$\Longrightarrow \frac{BD'}{D'C} \cdot \frac{CE'}{E'A} \cdot \frac{AF'}{F'B} = \frac{PD}{DQ} = \frac{RE}{ES} = \frac{TF}{FU}$$

Note that $\triangle AES \sim \triangle AFT, \triangle BDP \sim \triangle BFU$ and $\triangle CDQ \sim \triangle CER$. Thus

$$\frac{PD}{FU} = \frac{PB}{UB} \ , \ \frac{RE}{DQ} = \frac{RC}{QC} \ , \ \frac{TF}{ES} = \frac{TA}{SA}$$

$$\Longrightarrow \frac{BD'}{D'C} \cdot \frac{CE'}{E'A} \cdot \frac{AF'}{F'B} = \frac{PB}{QC} \cdot \frac{RC}{SA} \cdot \frac{TA}{UB} = \frac{c}{b} \cdot \frac{a}{c} \cdot \frac{b}{a} = 1$$

By the converse of Cevas' theorem we conclude that AD, BE, CF concur.

P.S. A nice and short proof was given by vittasko in this topic (see post #9).

jayme
#3 May 3, 2011, 7:29 pm

Dear Mathlinkers,
for a synthetic proof of Jacobi's theorem
http://perso.orange.fr/jl.ayme vol. 5 Le théorème de Jacobi
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Cyclic ABCD

Reply

HermitRecruit

#1 May 1, 2011, 3:00 pm

Given cyclic ABCD , M,N be midpoint of AB,CD ,respectively . P is on MN that AM/CN=MP/PN , AP=CP . if AD and BC intersect at point E . Prove that EAPC cyclic or EAPC is Kite .

Luis González

#2 May 2, 2011, 12:04 am

Assuming that the convex cyclic $ABCD$ is not a trapezoid, let the internal bisector of $\angle AEB$ intersect MN at P' and AB, DC at U, V. From $\triangle EAB \sim \triangle ECD$, we deduce that $\angle BUV=\angle CVU$, thus $\triangle P'UM$ and $\triangle P'VN$ are pseudo-similar (they have two equal angles and two supplementary angles). Hence, $\frac{P'M}{P'N}=\frac{UM}{VN}=\frac{AB}{DC} \Longrightarrow P' \equiv P'$. Consequently, if $PA=PC$, it follows that P is the midpoint of the arc AC of $\odot(EAC)$.

Quick Reply

High School Olympiads

Triangle ABC-A has 45 degree

Reply

HermitRecruit

#1 May 1, 2011, 10:57 am

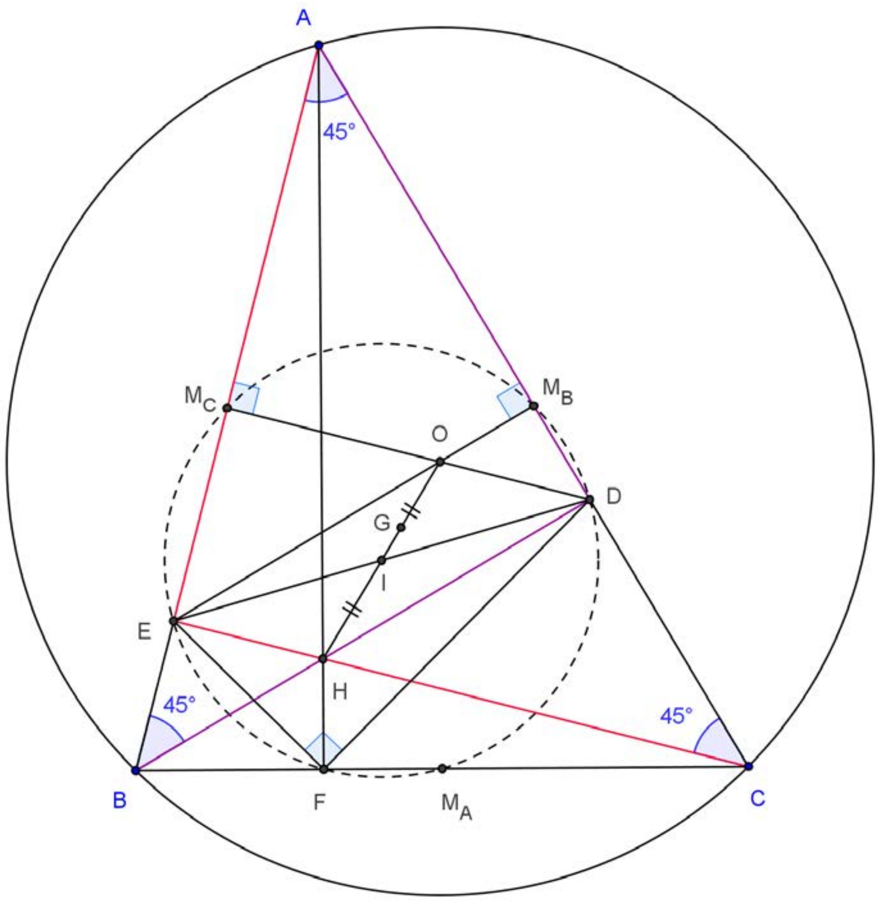
triangle ABC angle A has 45 degree . Altitude BD,CE intersect at H . I is midpoint of DE . Proof that H,I,G collinear [G is centroid]

Luis González

#2 May 1, 2011, 12:00 pm

Proof without words

Attachments:



HermitRecruit

#3 May 1, 2011, 12:21 pm

Thanks a lot for your kindness :')

HermitRecruit

#4 May 1, 2011, 12:40 pm

Could you please explain a little bit about "why G,O,H collinear"

Quick Reply

High School Olympiads

A conjecture with the symmedian point (own)

jayme

#1 Apr 29, 2011, 10:01 pm

Dear Mathlinkers,
ABC a triangle,
O the center of the circumcircle of ABC,
K the symmedian point of ABC,
DEF the orthic triangle of ABC,
Y the point of intersection of KE and BO,
Z the point of intersection of KF and CO.
Prove : YZ //EF.
Sincerely
Jean-Louis

Luis González

#2 Apr 30, 2011, 6:33 am

Since ABC is the excentral triangle of DEF and K,O become the Mittenpunkt and Bevan point of DEF, respectively, we can also restate the problem as follows

Proposition. $\triangle I_aI_bI_c$ is the excentral triangle of $\triangle ABC.$ A-cevian of the Mittenpunkt M of $\triangle ABC$ intersects the perpendicular from I_a to BC at $X.$ Similarly, we define Y,Z with respect to $B,C.$ Then $\triangle ABC$ and $\triangle XYZ$ are homothetic.

$\triangle A_0B_0C_0$ is the tangential triangle of the excentral $\triangle I_aI_bI_c.$ Medial triangle $\triangle M_aM_bM_c$ of $\triangle ABC$ and $\triangle A_0B_0C_0$ are obviously homothetic through $M \Longrightarrow$ incenters of $\triangle M_aM_bM_c$ and $\triangle A_0B_0C_0$ are collinear with the homothetic center $M,$ i.e. Spieker point $S,$ Mittenpunkt M and the Bevan point B_e of $\triangle ABC$ are collinear. Further, orthocenter H of $\triangle ABC$ also lies on $MB_e,$ since H,B_e are symmetric about $S.$ From $\triangle AHM \sim \triangle I_aB_eM,$ etc, we get

$$\frac{MA}{MX}=\frac{MB}{MY}=\frac{MC}{MZ}=\frac{MH}{MB_e} \Longrightarrow \triangle ABC \sim \triangle XYZ \text{ are homothetic.}$$

High School Olympiads

For an Olympiad again (own)

Reply

jayme

#1 Apr 28, 2011, 7:23 pm

Dear Mathlinkers,
ABCD a convex quadrilateral,
I the point of intersection of AC and BD,
J a point on the segment AC,
P and Q two lines passing through I,
P intersect AB, CD resp. at P, Q,
Q intersect BC, AD resp. at R, S,
AC intersect PS, QR resp. at M, N
the parallel to DJ passing through M intersect AD at M'
the parallel to DJ passing through N intersect CD at N'
M'N' intersect BC at T.
Prove without area or calculus that (A, C, I, T) = -1
Sincerely
Jean-Louis

jayme

#2 Apr 29, 2011, 11:16 am

Dear Mathlinkers,
sorry for two "typos"

ABCD a convex quadrilateral,
I the point of intersection of AC and BD,
J a point on the segment AC,
(P) and (Q) two lines passing through J,
(P) intersect AB, CD resp. at P, Q,
(Q) intersect BC, AD resp. at R, S,
AC intersect PS, QR resp. at M, N
the parallel to DJ passing through M intersect AD at M'
the parallel to DJ passing through N intersect CD at N'
M'N' intersect AC at T.
Prove that (A, C, I, T) = -1

Sincerely
Jean-Louis

Luis González

#3 Apr 29, 2011, 1:24 pm

Project the line through $AD \cap BC$ and $AB \cap DC$ to infinity. Denote projected points with subscript zero. $ABCD$ with diagonal intersection I goes to a paralellogram $A_0B_0C_0D_0$ with center I_0 and clearly $\triangle A_0P_0S_0$ and $\triangle C_0Q_0R_0$ are centrally similar through J_0. Since $I_0A_0 = I_0C_0$ and $\frac{J_0M_0}{J_0N_0} = \frac{A_0M_0}{C_0N_0}$, it follows that:

$$\left(\frac{I_0A_0}{I_0C_0} \cdot \frac{M_0C_0}{M_0A_0}\right) \cdot \left(\frac{J_0M_0}{J_0N_0} \cdot \frac{C_0N_0}{C_0M_0}\right) = (A_0, C_0, I_0, M_0) \cdot (M_0, N_0, J_0, C_0) = 1$$

Thus, $(A, C, I, M) \cdot (M, N, J, C) = 1 \Longrightarrow \frac{IA}{IC} \cdot \frac{CN}{JN} \cdot \frac{JM}{MA} = 1 \quad (1)$

By Menelaus theorem for $\triangle DAC$ cut by $\overline{TM'N'}$ we get:

$$\frac{TA}{TC} = \frac{N'D}{CN'} \cdot \frac{MA'}{DM'} = \frac{JN}{CN} \cdot \frac{MA}{JM} \quad (2)$$

From (1) and (2) we get $\frac{TA}{TC} = \frac{IA}{IC} \implies (A, C, I, T)$ is harmonic.

jayme

#4 Apr 29, 2011, 2:51 pm

Dear Luis and Mathlinkers,
thank you for your proof.
In my first message, I was asking for a proof without areas and calculus.
Dear Luis is this possible?
Thank in advance
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Circles and cocyclicity

Reply

momo1729

#1 Apr 28, 2011, 5:29 pm

Two circles C_1 and C_2 intersect in A and B. A line passing through B intersects C_1 in C and C_2 in D. Another line passing through B intersects C_1 in E and C_2 in F, (CF) intersects C_1 and C_2 in P and Q respectively. Make sure that in your diagram, $B, E, C, A, P \in C_1$ and $B, D, F, A, Q \in C_2$ in this order. Let M and N be the middles of the arcs BP and BQ respectively. Prove that if $CD = EF$, then the points C, F, M, N are cocylic in this order.

Luis González

#2 Apr 29, 2011, 11:19 am

$\angle AEB = \angle ACB, \angle ADB = \angle AFB$ and $CD = EF$ implies that $\triangle AEF$ and $\triangle ACD$ are congruent by ASA $\Longrightarrow$ $AE = AC$ and $AD = AF$. Consequently, $\triangle AEC$ and $\triangle AFD$ are similar A-isosceles triangles, which yields $\angle ABC = \angle ABF$, i.e. BA bisects $\angle FBC$ internally, thus CM, FN and BA concur at the incenter I of $\triangle BCF$. Hence, $\overline{IC} \cdot \overline{IM} = \overline{IB} \cdot \overline{IA} = \overline{IF} \cdot \overline{IN} \Longrightarrow C, F, M, N$ are concyclic.

Vo Duc Dien

#3 Dec 14, 2011, 5:47 am

This is problem 1 of Chinese MO 2010

http://www.artofproblemsolving.com/Forum/viewtopic.php?p=2100426&sid=409af24590687baf891d1562b71a79ed#p2100426

Vo Duc Dien

#4 Dec 14, 2011, 10:19 pm

Prove that MN||O1O2 where O1 and O2 are the centers of C1 and C2, respectively.

Quick Reply

High School Olympiads

conccurent circles(own)

paul1703

#1 Apr 29, 2011, 2:33 am

Let ABC be a triangle and A',B',C',O the midpoints of it's sides, the center of the circumscribed circle. Let w be a cicle of diameter R wich passes through O i.e(it is tangent to $C(ABC)$ and passes through O). Let the paralel lines to BC AC AB through O intersect the second time w at A'' B'' C'' prove that the circles $C(OA'A'')(OB'B'')(OC'C'')$; have a second comon point.

Luis González

#2 Apr 29, 2011, 6:18 am

Inversion with pole O and power R^2 carries the circumcircle (O) and the parallels τ_A,τ_B,τ_C to BC,CA,AB into themselves. A',B',C' are taken into the vertices A_0,B_0,C_0 of the tangential triangle and ω is taken into the tangent τ to (O) through P. Therefore, τ cuts τ_A,τ_B,τ_C at the inverses A_1,B_1,C_1 of A'',B'',C''.

Let P_A,P_B,P_C be the intersections of PA'',PB'',PC'' with BC,CA,AB, i.e. orthogonal projections of P onto BC,CA,AB. Since PA'',PB'',PC'' are the polars of A_1,B_1,C_1 WRT (O) and BC,CA,AB are the polars of A_0,B_0,C_0 WRT (O), we deduce that A_0A_1,B_0C_1,C_0C_1 are the polars of P_A,P_B,P_C WRT (O). Since P_A,P_B,P_C are collinear on the Simson line p of P WRT $\triangle ABC$, then their polars A_0A_1,B_0C_1,C_0C_1 concur at the pole of p WRT (O). Consequently, their inverses $\odot(OA'A''),\odot(OB'B''),\odot(OB'B'')$ under the referred inversion are coaxal.

High School Olympiads

Symmedian's problem

Reply

georgi111

#1 Apr 28, 2011, 7:43 pm • 1

It is given triangle with its symmedians AA1, BB1, CC1. If AA1=BB1=CC1 is it true that ABC is equilateral ?

Luis González

#2 Apr 28, 2011, 9:20 pm • 1

Let s_a, s_b, s_c and m_a, m_b, m_c denote the symmedians and medians issuing from $A, B, C.$ We shall prove that $s_b = s_c \Longleftrightarrow b = c,$ thus $s_a = s_b = s_c$ will imply that $a = b = c$ obviously. The following relations are well-known, i.e. easy to prove

$$s_a = \frac{2bc}{b^2+c^2} \cdot m_a \ \ , \ \ s_b = \frac{2ca}{c^2+a^2} \cdot m_b \ \ , \ \ s_c = \frac{2ab}{a^2+b^2} \cdot m_c$$

$$s_b = s_c \Longleftrightarrow \frac{c^2a^2}{(c^2+a^2)^2} \cdot (2(a^2+c^2)-b^2) = \frac{a^2b^2}{(a^2+b^2)^2} \cdot (2(a^2+b^2)-c^2)$$

$$\Longleftrightarrow \ (b^2-c^2)[2a^2(a^2+c^2)(a^2+b^2)+b^2c^2(2a^2+b^2+c^2)] = 0$$

Since $2a^2(a^2+c^2)(a^2+b^2)+b^2c^2(2a^2+b^2+c^2)] > 0,$ then $s_b = s_c \Longleftrightarrow b = c.$

Quick Reply

High School Olympiads

Point A outside a circle (O) with tangents AB,AC

Reply

Source: Vietnamese TST 2011 P2

Potla

#1 Apr 27, 2011, 7:44 pm

A is a point lying outside a circle (O). The tangents from A drawn to (O) meet the circle at $B, C.$ Let P, Q be points on the rays AB, AC respectively such that PQ is tangent to (O). The parallel lines drawn through P, Q parallel to $CA, BA,$ respectively meet BC at $E, F,$ respectively.
(a) Show that the straight lines EQ always pass through a fixed point $M,$ and FP always pass through a fixed point $N.$
(b) Show that $PM \cdot QN$ is constant.

Luis González

#2 Apr 27, 2011, 10:04 pm

Perpendicular to AO through O cuts AB, AC at $M, N.$ $\triangle AMN$ is isosceles with apex A and O is the A-excenter of $\triangle APQ,$ so $\angle OQN = 90^\circ - \frac{1}{2}\angle AQP.$

$$\angle QON = 180^\circ - (90^\circ - \tfrac{1}{2}\angle PAQ) - (90^\circ - \tfrac{1}{2}\angle AQP) = \angle MPO$$

Which implies that $\triangle OMP \sim \triangle QNO.$ Perpendicular to AN through Q cuts MN at $R.$ Then $\angle RQO = \angle AOP \Longrightarrow$ Cevians OA, QR are homologous in the similar triangles $\triangle OMP \sim \triangle QNO.$ Since $OC \parallel RQ,$ we deduce that

$$\frac{MP}{PA} = \frac{NO}{OR} = \frac{NC}{CQ}$$

Let $E' \equiv BC \cap QM.$ Since $\triangle QCE'$ and $\triangle MBE'$ are pseudo-similar, we have then

$$\frac{ME'}{E'Q} = \frac{MB}{CQ} = \frac{NC}{CQ} = \frac{MP}{PA} \Longrightarrow PE' \parallel AQ \Longrightarrow E \equiv E'$$

Hence, line EQ passes through the fixed $M.$ By similar reasoning, FP passes through the fixed $N.$ Now, from $\triangle OMP \sim \triangle QNO,$ we obtain

$$\frac{OM}{QN} = \frac{PM}{ON} \Longrightarrow PM \cdot QN = OM \cdot ON = ON^2 = \text{const}$$

vladimir92

#3 Apr 28, 2011, 5:18 am

Pretty same as the previous one.
Define M and N as the intersection of AB and AC with parallel to BC trough O. Easy to see that $OM = ON$ and $MB = NC$. From $\angle MBO = \angle NOA$ we get $MO^2 = MB \times MA$. It's not difficult to see that $\triangle OPM = \triangle QON$ hence:
$\frac{MP}{AM} = \frac{MP}{MO} \cdot \frac{MO}{MA} = \frac{ON}{NQ} \cdot \frac{MB}{MO} = \frac{NC}{NQ}$. Now let $E_1 \equiv (M_1N) \cap (PE)$. Then $\frac{NQ}{NC} = \frac{MA}{MP} = \frac{ME_1}{MN}$. Since CEE_1N is parallelograme, we get that $\frac{ME_1}{MN} = \frac{NQ}{EE_1}$ which implies that QE goes through the point M that is a fixed point.
Similary FP goes trough N that is also a fixed point. Second Part of the Problem is obvious.

Quick Reply

High School Olympiads

Semicircle and perpendicular

Reply

borislav_mirchev
#1 Apr 26, 2011, 3:53 am

It is given a semicircle k with diameter AB. From the point P are drawn the tangents PC and PD. E is the intersection point of the lines PC and AB. F is the intersection point of the lines PD and AB. Q is the intersection point of CF and DE. Prove that PQ is perpendicular to AB.

(I think the problem was posted here before, but I'm not sure)

Luis González
#2 Apr 27, 2011, 8:08 am • 1

Indeed, this configuration has been discussed before.

Let $R \equiv AC \cap BD$ and $H \equiv AD \cap BC. H$ is obviously the orthocenter of $\triangle RAB$, i.e. $HR \perp AB$. Inversion with pole R and power $\overline{RC} \cdot \overline{RA}$ carries the circumcircle ω of $RCHD$ into the sideline AB and k into itself. Since AB bisects k, then by conformity ω is orthogonal to $k \Longrightarrow P$ is the midpoint of $\overline{HR}$.

Leon
#3 Jun 25, 2012, 3:48 pm

> *Luis González wrote:*
> Since AB bisects k, then by conformity ω is orthogonal to $k \Longrightarrow P$ is the midpoint of $\overline{HR}$.

We must also prove that $Q \in HR$. How we can do ?

Quick Reply

High School Olympiads

Cyclic quadrilateral

Reply

vladimir92

#1 Apr 24, 2011, 10:53 pm

Let $\triangle ABC$ be a triangle with nine point center N. The reflection of B and C with repect to AC and AB are B_1 and C_1 respectively. The reflection of B_1 across AN is B_2. Prove that B_1 ,A , B_2 and C_1 are concyclic.

Luis González

#2 Apr 25, 2011, 2:00 am

This problem is basically the same as Tuymaada 2009 Senior League Problem 3. The line connecting A with the circumcenter O' of $\triangle AB_1C_1$ passes through the reflection D of the circumcenter O of $\triangle ABC$ about BC. Let H be the circumcenter of $\triangle ABC$. Since $AH=OD$ and $AH \parallel OD$, it follows that $AODH$ is a parallelogram $\Longrightarrow AO'$ bisects $\overline{OH}$, i.e. $N \in AO'$. Thus, reflections of B_1 and C_1 about AN lie on $\odot(AB_1C_1)$.

vladimir92

#3 Apr 25, 2011, 3:18 am

Hello luis.
Now I see that this problem is equivalent to the one discused on the link above since it's well-known that $AHO'O$ is a parallelograme. But I think it's more interesting/challenging to search for a solution without using that problem.
Thank's for your interest.

Quick Reply

High School Olympiads

Circles and triangles

Reply

Microtarx
#1 Apr 24, 2011, 2:48 am

A title problem:

Let Γ_1 and Γ_2 circles are externally tangent at point B. A line l is tangent to Γ_1 at point A and intersects Γ_2 in C and D. Let E be point of intersection of AB with Γ_2; F the midpoint of the arc CD that not contains E and H the point intersection of AF with Γ_2. Show that BF, CD and EH have a point in common.

Luis González
#2 Apr 24, 2011, 5:13 am

Let CB and DB cut Γ_1 again at P and Q. Since B is the insimilicenter of $\Gamma_1 \sim \Gamma_2$, it follows that $\overline{ACD}$ is parallel to $PQ \Longrightarrow$ Arcs AP, AQ of Γ_1 are equal $\Longrightarrow BA$ bisects the angle $\angle PBQ \equiv \angle CBD$ externally $\Longrightarrow E$ is the midpoint of the arc CBD of Γ_2. Thus, EF is the perpendicular bisector of $\overline{CD}$. Since $\angle EBF$ and $\angle FHE$ are right, then it follows that lines BF, CD and EH concur at the orthocenter of $\triangle AEF$.

Tales749
#3 Apr 24, 2011, 5:32 am

Let O_1 and O_2 be the centers of the circles Γ_1 and Γ_2 respectively. Let l_2 be the line tangent to Γ_1 and Γ_2 at the point B, and let $K = l_1 \cap l_2$, $G = AO_1 \cap BF, T = KB \cap EG$, $Q = l_2 \cap EF$. Clearly $AK = BK$. Then $\angle BAK = \angle ABK$. Since l_2 is tangent to Γ_1 then $\angle O_1AB = \angle GBT$, similarly $\angle EFB = \angle EBT$. Since l_1 is tangent to Γ_1 then $\angle O_1AB + \angle BAK = 90^\circ$ then $\angle KBF + \angle KBA = 90^\circ$ therefore $FB \perp AE$. Since $\angle EFB = \angle EBT = \angle KBA = \angle KAB$ then the quadrilateral $ABQF$ is cyclic because $\angle QFB = \angle QAB$, therefore $\angle ABF = \angle AQF = 90^\circ$.
Now, let P be the point of intersection of l_2 with FB, then clearly the point P is the orthocenter of the $\triangle AFE$. Since $FB \perp AE$ is clear that EF is a diameter of Γ_2 then $EH \perp AF$ then BF, CD and EH are concurrent at the point P

[geogebra]ce6b0c138cf98465b7e58cbc71d2edbb7df54192[/geogebra]

Quick Reply

High School Olympiads

Complete quadrilateral and a middle

Reply

borislav_mirchev
#1 Apr 23, 2011, 2:50 am

A quadrilateral $ABCD$ is inscribed in a circle k. E is the intersection point of the lines AB и CD and F is the intersection point of the lines AD and BC. If P is the second intersection point of the circumference of the triangle CEF and k, and M is the intersection point of the line AP and the segment EF prove that M is the middle of EF. (A is in a different half-plane with respect of the diagonal BD from E and F)

Luis González
#2 Apr 23, 2011, 8:22 am • 1

Let M be the midpoint of EF and MA cuts k again at P. Then we shall show that C, E, F and P are concyclic. O, R denote the center and radius of k, $K \equiv AC \cap BD$ and EK, FK cut OE, OF at U, V. Since EK, FK are the polars of F, E WRT k, it follows that FU, EV are perpendicular to OE, OF. Furthermore, $R^2 = OU \cdot OE = OV \cdot OF \Longrightarrow k$ and the circle (M) with diameter EF are orthogonal. Thus, power of M to k equals $ME^2 = MF^2$, i.e. $ME^2 = MF^2 = MA \cdot MP$, i.e. circles $\odot(PAE)$ and $\odot(PAF)$ are both tangent to EF through E, F. Consequently

$$\angle APF = \angle AFE \quad , \quad \angle APE = \angle AEF \Longrightarrow$$

$$\angle EPF = \angle AFE + \angle AEF = \angle BAD = \angle ECF \Longrightarrow P \in \odot(CEF).$$

borislav_mirchev
#3 Apr 24, 2011, 1:33 am

It is an excellent solution. Thank you for sharing it here.
Can the problem be solved without using polars?
Is it a well known statement?
What the level of difficulty of this problem is?

vladimir92
#4 Apr 25, 2011, 3:50 am • 1

> borislav_mirchev wrote:
> It is an excellent solution. Thank you for sharing it here.
> Can the problem be solved without using polars?
> Is it a well known statement?
> What the level of difficulty of this problem is?

Let AP intersect circumcircle of ΔEFC again at R.
We have $\angle PAE = \angle PCF = \angle PRF$ so $(AE) \parallel (RF)$. We also have: $\angle ARE = \angle PFE = \angle FAR$, hence $(AF) \parallel (RE)$ it follow that $AFRE$ is a parallelogram which implies that M is midpoint of FE.

borislav_mirchev
#5 Apr 25, 2011, 6:23 pm

It is an amazing solution, thank you very much for it!
Is the problem a well known statement?

vladimir92
#6 Apr 25, 2011, 6:41 pm • 1

I don't think so, I have never seen it before, and it's really beautifull.

borislav_mirchev
#7 Apr 25, 2011, 6:52 pm

Thank you again!
I (re)discovered it as a statement but most of the times I cannot solve the problems
I'm posting or have no time to do it.

Rijul saini
#8 Apr 28, 2011, 3:33 pm • 1

> *borislav_mirchev wrote:*
> A quadrilateral $ABCD$ is inscribed in a circle k. E is the intersection point of the lines AB и CD and F is the intersection point of the lines AD and BC. If P is the second intersection point of the circumference of the triangle CEF and k, and M is the intersection point of the line AP and the segment EF prove that M is the middle of EF. (A is in a different half-plane with respect of the diagonal BD from E and F)

I show that if P is the second intersection of the line joining A to the midpoint M of EF, then $P \in (CEF)$.
Now, let Y be the second intersection of the line through A parallel to EF. Therefore, the lines AY, AF, AM, AE form a harmonic pencil. Therefore, AY, AD, AP, AB also form a harmonic pencil, leading to $YDPB$ being a harmonic quadrilateral. Therefore, CY, CD, CX, CB form a harmonic pencil. Let $K = CX \cap EF, \ L = CY \cap EF$. Thus, have that $(EF, LK) = -1$. Now, since M is the midpoint of EF, therefore, $KE \cdot KF = KL \cdot KM$. Thus, it suffices to prove that C, P, M, L are concyclic.
For this, we have $\angle LMP = \angle YAP = \angle YCP$. Hence, we're done. :)

borislav_mirchev
#9 Apr 28, 2011, 7:52 pm

Thank you for the solution! It is interesting to see how much approaches this problem allows.

georgi111
#10 Apr 28, 2011, 8:56 pm

These article shows the fact mentioned in this problem as intermediate consequence ... http://www.irmo.ie/7.Lemoine.pdf. It is theorem 8 (Rigby) - The Lemoine point of triangle is centroid of its pedal triangle !!!

georgi111
#11 Apr 28, 2011, 9:03 pm

So things (re)discovered of sth ... dear Borislav does not apply to this problem particularly - you not invented something new as a fact it is just a well known fact. To use (re)discover is brave - it should be used carefully ;) because sometimes awkward situations like this appeared.

borislav_mirchev
#12 Apr 28, 2011, 11:17 pm • 1

Dear Georgi,
You are almost correct. I cannot say I'm the first person discovered a statement. It is the reason here I'm writing "(re)discovered", not "discovered".
I don't think the situation is awkward. I cannot know all the problems ever discovered. I don't think that there is a single person that knows them.
It is true that many times I don't spend enough time in spending the problems I'm posting but I think sometimes there are good problems posted by me. Even if one of them is a new problem it will be good and it is not a reason to not spend a good time in solving the problems.

Quick Reply

High School Olympiads

Again with incircle when A=90.

Reply

Source: Own.

Virgil Nicula

#1 Aug 6, 2008, 8:20 pm

Let $\triangle ABC$ with incenter I and A - excenter I_a . The B - excircle touches

AC at X. Denote $Y\in IX\cap BC$. Prove that $A=90^\circ\iff I_aY\perp AC$.

Luis González

#2 Apr 22, 2011, 11:19 pm

A-excircle (I_a) touches AC at D and I_aD cuts BC at $Z.$ Then $I_aY\perp AC\Longleftrightarrow Y\equiv Z.$ Let $V\equiv AI\cap BC.$ By Menelaus' theorem for $\triangle ACV$ cut by $\overline{ZDI_a}$ and $\overline{YXI},$ we get

$$\frac{ZC}{ZV}=\frac{I_aA}{VI_a}\cdot\frac{DC}{AD}\ \ ,\ \ \frac{YC}{YV}=\frac{IA}{VI}\cdot\frac{XC}{AX}$$

$$Y\equiv Z\Longleftrightarrow \frac{ZC}{ZV}=\frac{YC}{YV}\Longleftrightarrow \frac{I_aA}{I_aV}\cdot\frac{DC}{AD}=\frac{IA}{IV}\cdot\frac{XC}{AX}$$

Since $(A,V,I,I_a)=-1,$ then $\frac{I_aA}{I_aV}=\frac{IA}{IV}\Longrightarrow$

$$Y\equiv Z\Longleftrightarrow \frac{DC}{AD}=\frac{XC}{AX}\Longleftrightarrow \frac{s-b}{s}=\frac{s-a}{s-c}\Longleftrightarrow a^2=b^2+c^2.$$

Quick Reply

High School Olympiads

UG002 - Geometry

Reply

Ligouras

#1 Apr 20, 2011, 11:45 pm

Let ABC be a triangle with sides a, b, c and let P be any point in the plane but not on a side of the triangle.
If H_a, H_b, H_c are the orthocenters of triangles PBC, PCA and PAB, respectively and if AD, BE, CF are the altitudes of the triangle ABC and d, e, f are the sides of triangle DEF, prove that

$$[H_aH_bH_c] = \frac{abc}{2def}[DEF]$$

Where the brackets denote the area of a triangle.

Luis González

#2 Apr 21, 2011, 7:56 am • 1

Theorem. $ABCDEF$ is a hexagon (either convex or nonconvex) such that $AB \parallel DE, BC \parallel EF$ and $CD \parallel FA$. Then the triangles $\triangle ACE$ and $\triangle BDF$ have equal area.

Proof. For the sake of simplicity in the proof, we'll assume that $ABCDEF$ is convex. Let M, N, L be the reflections of D, F, B about the midpoints of CE, EA, AC. Then $CDEM, FANE$ and $BCLA$ are parallelograms $\Longrightarrow \triangle MNL$ is a triangle whose sides equal the difference of the opposite sides of $ABCDEF$. Furthermore, we have

$$[\triangle ACE] = [\triangle CME] + [\triangle ENA] + [\triangle ALC] + [\triangle MNL]$$

$$[\triangle ACE] = [ABCDEF] - [\triangle ACE] + [\triangle MNL] \implies$$

$$[ABCDEF] = 2[\triangle ACE] - [\triangle MNL]$$

Using the same reasoning for $\triangle BDF$, i.e. reflecting A, E, C across the midpoins of BF, FD, DB, we obtain a triangle $\triangle M'N'L'$ congruent to $\triangle MNL$ and

$$[ABCDEF] = 2[\triangle BDF] - [\triangle M'N'L'] \Longrightarrow [\triangle ACE] = [\triangle BDF]$$

Back to the problem, note that $H_cAH_bCH_aB$ is a hexagon whose opposite sides are parallel. Hence, from the above lemma, we get that $\triangle ABC$ and $\triangle H_aH_bH_c$ have equal area. Now, let R denote the circumradius of $\triangle ABC$

$$[\triangle H_aH_bH_c] = [\triangle ABC] = \frac{abc}{4R} \ , \ [\triangle DEF] = \frac{def}{2R}$$

$$\Longrightarrow [\triangle H_aH_bH_c] = \frac{abc}{2def} \cdot [\triangle DEF]$$

Ligouras

#3 Apr 21, 2011, 12:47 pm

Thanks Luis great Geometra.... 🙂

Quick Reply

High School Olympiads

Prove that PQ=2HP

Reply

Source: LXII Polish Olympiad 2011, Problem 5

mymath7
#1 Apr 20, 2011, 11:02 pm

In a tetrahedron $ABCD$, the four altitudes are concurrent at H. The line DH intersects the plane ABC at P and the circumsphere of $ABCD$ at $Q \neq D$. Prove that $PQ = 2HP$.

Luis González
#2 Apr 21, 2011, 12:52 am

It's well known that if $ABCD$ is orthocentric, then the feet of its altitudes are the orthocenters of the corresponding faces, i.e. P is the orthocenter of $\triangle ABC$. Let M, N be the orthogonal projections of B, C onto AC, AB. If CP cuts the circumcircle of $\triangle ABC$ again at R, then $\angle RBA = \angle RCA = \angle ABP$ implies that P, R are symmetric about AB. Thus, from the power of P WRT the circumsphere of $ABCD$, we get:

$$\overline{PQ} \cdot \overline{PD} = \overline{PR} \cdot \overline{PC} = 2 \cdot \overline{PM} \cdot \overline{PC} = 2 \cdot \overline{PB} \cdot \overline{PN} \ (1)$$

Let DP cut the circumcircle of $\triangle BDN$ again at S. Since H is the orthocenter of $\triangle BDN$, then H, S are symmetric about BN. From the power of P WRT the circumcircle of $\triangle BDN$, we have

$$\overline{PB} \cdot \overline{PN} = \overline{PD} \cdot \overline{PS} = -\overline{PD} \cdot \overline{PH} \ (2)$$

From (1) and (2) we get $\overline{PQ} \cdot \overline{PD} = -2 \cdot \overline{PD} \cdot \overline{PH} \Longrightarrow \ \overline{PQ} = -2 \cdot \overline{PH}$.

mymath7
#3 Apr 21, 2011, 5:36 am

Elegant solution, luisgeometra

littletush
#4 Oct 22, 2011, 5:07 pm

let BP intersects AC at T and the circumcircle of ABC at K.
point S is chosen such that PS=PY and S,P,T collinear.
hence
DP*PQ=BP*PK,
Pis the orthocenter of ABC
in triangle DBT,H is the orthocenter.so
$\angle DSP = \angle DTP = \angle BHP$
hence D,S,B,H concyclic
hence PD*PH=PB*PS=PB*PT=$\dfrac{PB*PK}{2}$
hence PQ=2PH
QED

Quick Reply

High School Olympiads

Cyclic quadrilateral

Reply

vladimir92

#1 Apr 19, 2011, 7:29 am • 1

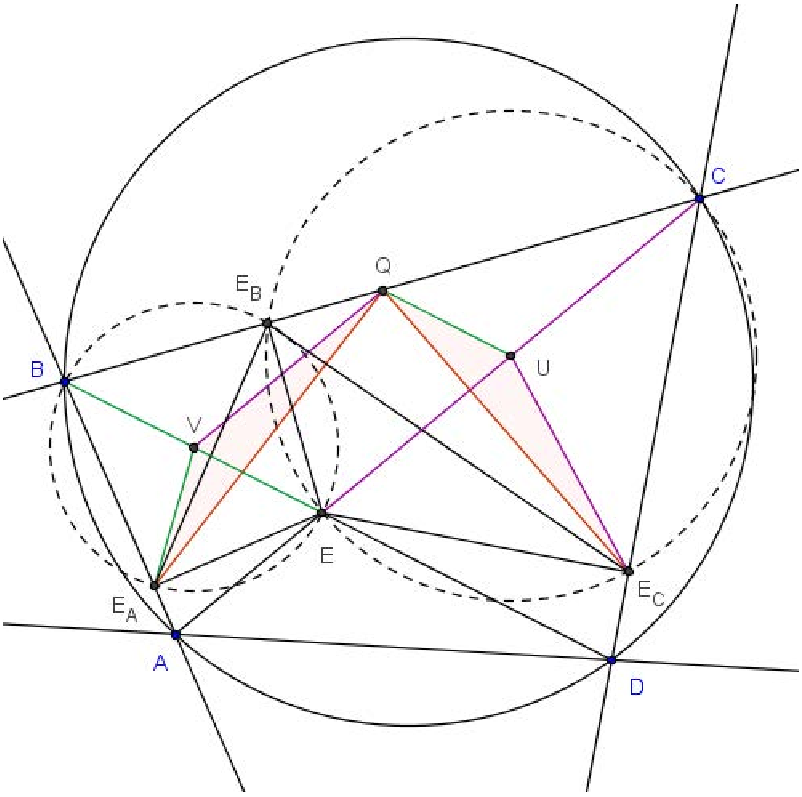
Problem *(Own?)*
Let $ABCD$ be a cyclic quadrilateral. Let diagonals AC and BD intersect at E. Denote by E_A , E_B and E_C the orthogonal projections of E into AB, BC and CD respectively. If Q is the midpoint of BC.
Show that E_A , E_B , E_C and Q are concyclic.

Luis González

#2 Apr 19, 2011, 9:05 pm • 1

This configuration have been discussed before. For instance, solutions given in 4 concyclic points and Perpendicular bisector work. This is: EE_B bisects $\angle E_AE_BE_C$ and if U, V denote the midpoints of EC, EB, then $\triangle VQE_A$ and $\triangle UE_CQ$ are congruent by SAS. Which implies that $QE_A = QE_C$, i.e. Q is the midpoint of the arc $E_AE_BE_C$.

Attachments:



vladimir92

#3 Apr 19, 2011, 10:33 pm

Dear Luis and Mathlinkers.
Thank's for you interest, and for the interesting link you gave above. Actually the following result is also true.
Problem.
On the same configuration, Let P and N be midpoints of AB and CD respectively, Then P , N , Q and E_B are concyclic.

This is a stronger fact which proves also the previous one. I'm sur you will like it.

N.B: What is exeptionally beautifull on those two problems (notably the second); is that they look easy, but they aren't so.

Luis González

#4 Apr 20, 2011, 10:22 pm

As for the 2nd result, since $UE=UC=UE_B$ and $VE=VB=VE_B$, we get

$$\frac{VE_B}{UE_B}=\frac{EB}{EC}=\frac{EA}{EC}=\frac{VP}{UN}$$

Since $\angle PVE_B=\angle AEB+2\angle BEE_B=\angle AED+2\angle ECB=\angle NUE_B$, then it follows that $\triangle PVE_B$ and $\triangle NUE_B$ are similar by SAS. Hence, $\angle PE_BV=\angle NE_BU \Longrightarrow \angle PE_BN=\angle VE_BU=\angle PQN.$ Therefore, P,N,Q and E_B are concyclic.

Quick Reply

High School Olympiads

4 circles and congruence

Reply

mousavi

#1 Apr 19, 2011, 8:37 pm

Each of circles S_1, S_2, S_3 is tangent to two sides of a triangle ABC and externally tangent to a circle S at A_1, B_1, C_1 respectively. Prove that the lines AA_1, BB_1, CC_1 meet in a point.

Luis González

#2 Apr 19, 2011, 9:31 pm • 1

We assume that S_1 is tangent to the rays $\overrightarrow{AB}, \overrightarrow{AC}$ and similarly S_2, S_3. The remaining configuration is treated analogously. (I) is the incircle of $\triangle ABC$ and U is the exsimilicenter of $(I) \sim S$. A is the exsimilicenter of $(I) \sim S_1$ and A_1 is the insimilicenter of $S \sim S_1$. Hence, by Monge & d'Alembert theorem, it follows that A, A_1, U are collinear, i.e. AA_1 passes through U. By similar reasoning, lines BB_1 and CC_1 pass through the exsimilicenter U of $(I) \sim S$.

Quick Reply

High School Olympiads

UG001 - Geometry

Reply

Ligouras

#1 Apr 18, 2011, 12:38 am

If d_1, d_2, d_3, d_4 be the distances of the nine-points centre from those of the inscribed and escribed circles and R be the circumradius of a triangle ABC, prove that

$$d_1^2+d_2^2+d_3^2+d_4^2=R^2[21-4(sin^2A+sin^2B+sin^2C)]$$

Luis González

#2 Apr 18, 2011, 10:29 pm

I, H, O, N are the incenter, orthocenter, circumcenter and 9 point center of $\triangle ABC.$ I_a, I_b, I_c are the three excenters of $\triangle ABC$ againts $A, B, C.$ Using the relation proved in the topic Nine point center, orthocenter (see post #2) in the excentral triangle $\triangle I_aI_bI_c$ with orthocenter I, 9-point center O and $P \equiv N$, we get

$$N{I_a}^2+N{I_b}^2+N{I_c}^2+NI^2=4\cdot NO^2+3\cdot(2R)^2=OH^2+12R^2$$

Substituting $OH^2=9R^2-4R^2(\sin^2A+\sin^2B+\sin^2C)$ gives

$$N{I_a}^2+N{I_b}^2+N{I_c}^2+NI^2=R^2[21-4(\sin^2A+\sin^2B+\sin^2C)]$$

Ligouras

#3 Apr 19, 2011, 1:25 am

Thanks Luis ***great*** Geometra....

Quick Reply

High School Olympiads

Circles and perpendicular

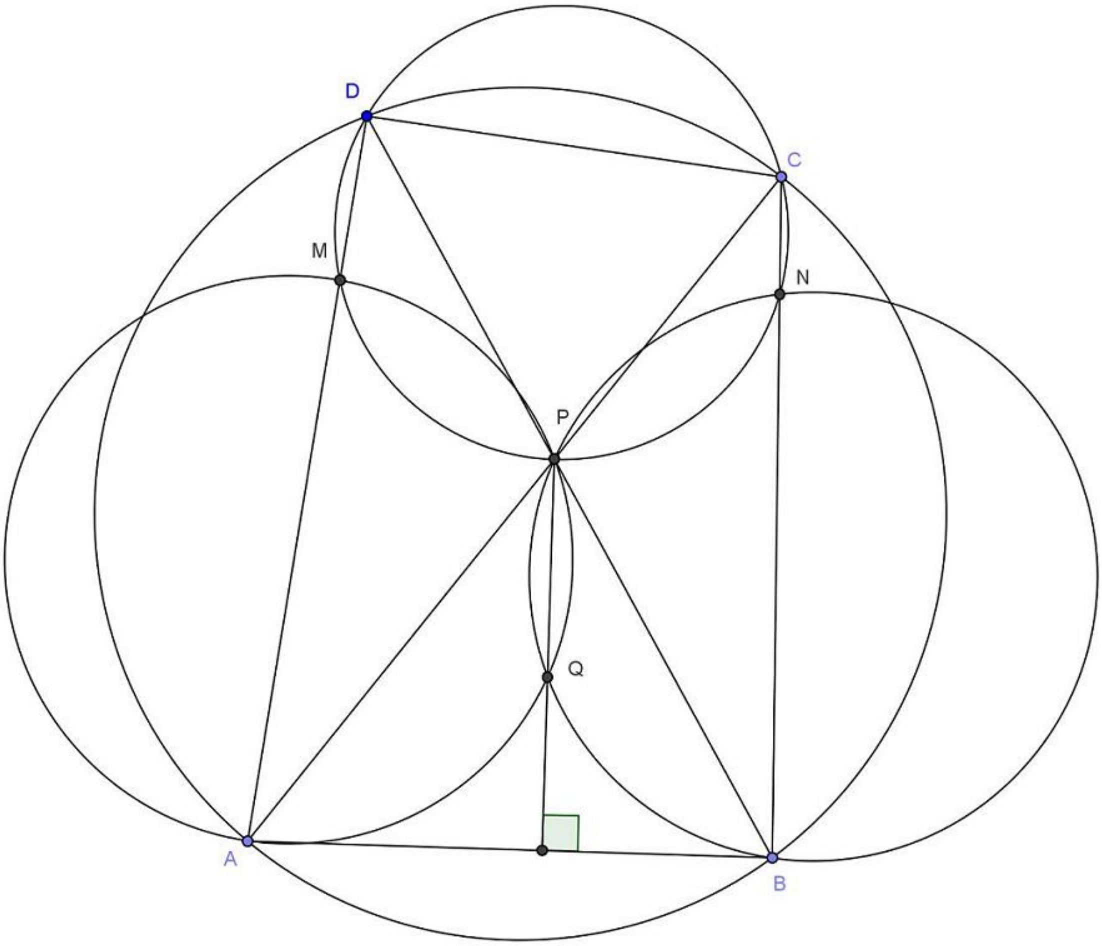
borislav_mirchev

#1 Apr 17, 2011, 1:26 am

It is given an inscribed quadrilateral $ABCD$. P is the intersection point of the diagonals. The circumference of the ΔCDP intersects the sides AD and BC at the points M and N respectively. The circumferences of the ΔAMP and ΔBNP intersects again at the point Q. Prove that $PQ \perp AB$.

I'm sorry if the problem is too easy for this section.

Attachments:



Luis González

#2 Apr 18, 2011, 12:16 am • 1

F is the circumcenter of $\triangle PDC$. Then $\angle DPF = \angle 90^\circ - \angle ACD = 90^\circ - \angle ABD$ implies that $PF \perp AB$. Therefore, it suffices to show that circles $\odot(AMP)$ and $\odot(BNP)$ intersect again on the line PF. Since PM, PN are antiparallel to CD with respect to AC, AD and BD, BC, it follows that

$$\frac{PM}{CD} = \frac{PA}{DA}, \ \frac{PN}{CD} = \frac{PB}{CB} \Longrightarrow PM = PN, \text{ due to } \frac{PA}{PB} = \frac{DA}{CB}.$$

Hence, PM, PN are symmetric about PF and $\angle PAD = \angle PBC$ implies that $\odot(AMP)$ and $\odot(BNP)$ are congruent, i.e. $\odot(AMP)$ and $\odot(BNP)$ are symmetric about $PF \Longrightarrow \odot(AMP)$ and $\odot(BNP)$ intersect again on PF.

borislav_mirchev

#3 Apr 18, 2011, 1:57 am

Dear, luisgeometra,
Thank you very much for the interesting solution. You can see two more solutions here.
http://dxdy.ru/topic44457.html

Quick Reply

High School Olympiads

GS parallel AM.

Reply

Virgil Nicula
#1 Sep 10, 2010, 12:30 am

Let ABC be a triangle. If G , S , M are the centroid, the symmedian point and the midpoint of $[BC]$ respectively then prove that $GS\parallel BC\iff b^2+c^2=2a^2\iff OG\perp AM$.

Mateescu Constantin
#2 Sep 10, 2010, 3:57 am • 1

The barycentric coordinates of G - centroid and S - symmedian point are : $\left\{\begin{array}{cc} G & (1:1:1) \\ S & (a^2:b^2:c^2)\end{array}\right.$.

Thus, the barycentric equation of the line GS is : $\boxed{GS\ :\ x\cdot (b^2-c^2)+y(c^2-a^2)+z(a^2-b^2)=0}$.

Therefore, $\boxed{GS\parallel BC}\iff c^2-a^2=a^2-b^2\iff \boxed{b^2+c^2=2a^2}$.

On the other hand, $\boxed{AM\perp OH}\iff AH^2+OM^2=HM^2+AO^2\iff$

$$\iff 4R^2\cdot\cos^2A+\frac{4R^2-a^2}{4}=\frac{8R^2\cdot (\cos^2B+\cos^2C)-a^2}{4}+R^2\iff$$

$$\iff 4\cos^2A=2\cos^2B+2\cos^2C\iff \boxed{2a^2=b^2+c^2}$$

Luis González
#3 Apr 15, 2011, 10:05 am

AM cuts the circumcircle (O) again at D. Then $OG\perp AM \Longleftrightarrow \overline{GD}=-\overline{GA}\Longleftrightarrow \overline{MD}=-\overline{MG}$. Thus, from the power of M with respect to (O), it follows that

$$OG\perp AM\Longleftrightarrow \overline{MG}\cdot\overline{MA}=\overline{MB}\cdot\overline{BC}\Longleftrightarrow \tfrac{1}{3}{m_a}^2=\tfrac{1}{4}a^2\Longleftrightarrow b^2+c^2=2a^2$$

U,V are the feet of the A- and B- symmedian. By Menelaus theorem for $\triangle AUC$ cut by $\overline{BSV}$

$$\frac{AS}{SU}=\frac{BC}{BU}\cdot\frac{VA}{CV}=\frac{b^2+c^2}{c^2}\cdot\frac{c^2}{a^2}=\frac{b^2+c^2}{a^2}$$

Hence, $GS\parallel BC\Longleftrightarrow \dfrac{AS}{SU}=\dfrac{AG}{GM}=2\Longleftrightarrow \dfrac{b^2+c^2}{a^2}=2\Longleftrightarrow b^2+c^2=2a^2$.

Consequently, $OG\perp AM\Longleftrightarrow GS\parallel BC\Longleftrightarrow b^2+c^2=2a^2$.

Quick Reply

High School Olympiads

hard problem

Reply

gauman

#1 Apr 13, 2011, 2:48 pm

Given a circle (O) and a cyclic quadrilateral $ABCD$ inscribed in (O). A point E in (O). 4 lines through A, B, C, D and perpendicular to EA, EB, EC, ED cut and form a quadrilateral $MNPQ$.
Prove that $S(OMQ) + S(ONP) = S(OMN) + S(OPQ)$, $S(ABC)$ is the area of the triangle ABC

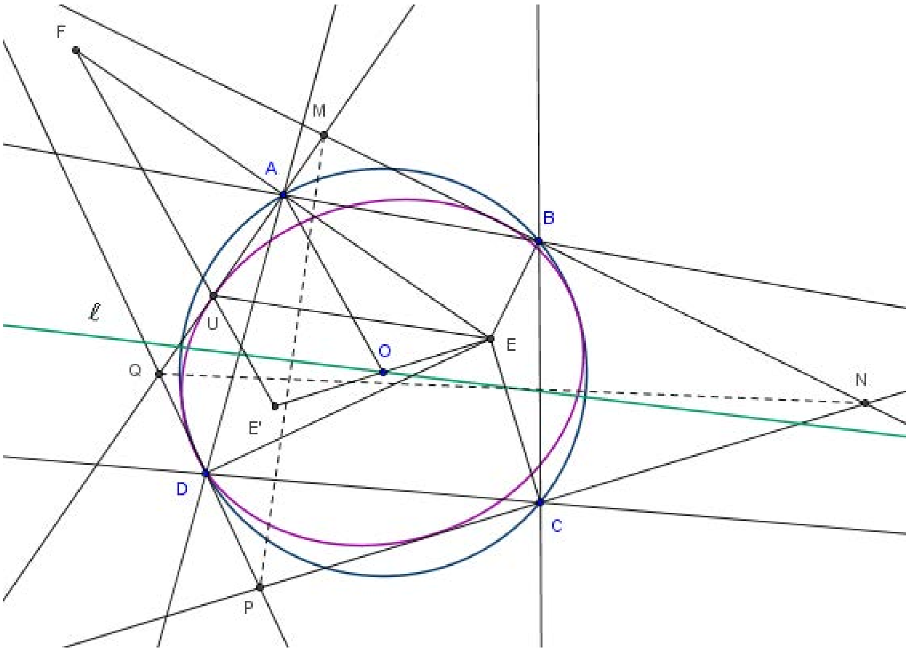
Luis González

#2 Apr 13, 2011, 10:23 pm

The result is still true for any E inside (O, ϱ). Let F and E' be the reflections of E about A and O, respectively. FE' cuts the perpendicular bisector MQ of EF at U. Since $UE = UF$, then we have $UE + UE' = FE' = 2\varrho \Longrightarrow U$ lies on the ellipse $\mathcal{E}$ with foci E, E' and pedal circle (O, ϱ). Since MQ bisects $\angle EUE'$ externally, then MQ is tangent to $\mathcal{E}$ through U. Likewise, MN, NP, PQ are tangent to $\mathcal{E}$. Therefore, the center O of $\mathcal{E}$ lies on the Newton line ℓ of the complete quadrangle $MNPQ$. Now, we know that any point O on the Newton line of a quadrangle $MNPQ$ satisfies

$$|\triangle OMQ| + |\triangle ONP| = |\triangle OMN| + |\triangle OPQ| = \frac{1}{2}|MNPQ|$$

Attachments:



gauman

#3 Apr 15, 2011, 6:56 pm

Can you do it without ellipse ,luisgeometra?

Quick Reply

High School Olympiads

A perspectrix perpendicular to OI (own)

Reply

Luis González

#1 Jan 1, 2011, 2:20 am • 4

$\triangle ABC$ is scalene with incenter I and circumcenter O. AI, BI, CI cut BC, CA, AB at D, E, F. Points X, Y, Z are chosen on the extension of segments AI, BI, CI, such that $\overline{IX} : \overline{IA} = \overline{IY} : \overline{IB} = \overline{IZ} : \overline{IC} = -1 : 2$. $\triangle XYZ$ and $\triangle DEF$ are perspective through I. Show that the perspectrix of $\triangle DEF, \triangle XYZ$ is perpendicular to OI at I.

Happy new year 2011 to all AoPS members

jayme

#2 Jan 1, 2011, 1:16 pm

Dear Luis and Mathlinkers,
This nice problem is for me a consequence of the Sondat's theorem (3) and a known result (2)...
A scheme of my proof after a white night
1. IaIbIc the excentral triangle of ABC
2. We know that OIa is perpendicular to EF and circularly
3. Sondat's theorem applied to ABC and IaIbIc : OI is perpendicular to the perspectrix of this two triangles
4. ABC, IaIbIc, DEF share the same perspectrix (to prove)
5. This perspectrix is parallel to the perspectrix of DEF and XYZ
And we are done
Happy New Year
Sincerely
Jean-Louis

Luis González

#3 Jan 1, 2011, 1:52 pm

Thanks for your interest dear Jean-Louis, not only we have to prove that the perspectrix is perpendicular to OI, but that it also passes through I, as the enunciation states. That's the tough part of the problem (I think).

jayme

#4 Jan 1, 2011, 4:15 pm

Dear Luis and Mathlinkers,
yes of course I omit this important result...
Sincerely
Jean-Louis

Luis González

#5 Apr 13, 2011, 3:54 am

Let the perpendicular τ to OI through I cuts EF, FD, DE at A', B', C'. Then we have to show that A', B', C' lie on YZ, ZX, XY, respectively. Incircle (I, r) touches BC, CA, AB at P, Q, R. For any point on τ the sum of its oriented distances to the sides BC, CA, AB equals $IP + IQ + IR$. For a proof, see the the lemma at Distance sum. If A_1, A_2, A_3 denote the orthogonal projections of A' onto BC, CA, AB, we have

$$\overline{A'A_1} + \overline{A'A_2} + \overline{A'A_3} = \overline{IP} + \overline{IQ} + \overline{IR} = 3r$$

But $A' \in EF$ implies that $\overline{A'A_2} + \overline{A'A_3} = \overline{A'A_1} \Longrightarrow \overline{A'A_1} = \frac{3}{2}\overline{IP}$

If $M \equiv IA' \cap BC$, then it follows that $\overline{IA'} : \overline{IM} = -1 : 2 \Longrightarrow A'$ lies on the image of BC under the homothety with center I and coefficient $-\frac{1}{2}$, i.e. $A' \in YZ$. Likewise, $B' \in ZX$ and $C' \in XY \Longrightarrow \tau$ is perspectrix of $\triangle DEF, \triangle XYZ$.

Quick Reply

High School Olympiads

Metric relation related to angle bisectors

Reply

alphabeta1729

#1 Apr 12, 2011, 8:55 pm

In a $\triangle ABC$ denote the in-center by I, in-radius by r and ex-radii by r_a, r_b and r_c.

Then prove that : $IA \cdot IB \cdot IC = (r_a - r) \cdot (r_b - r) \cdot (r_c - r)$

Luis González

#2 Apr 12, 2011, 10:06 pm

$$IA = \sqrt{\frac{bc(s-a)}{s}} \ , \ IB = \sqrt{\frac{ca(s-b)}{s}} \ , \ IC = \sqrt{\frac{ab(s-c)}{s}}$$

$$\Longrightarrow \ IA \cdot IB \cdot IC = \sqrt{\frac{a^2b^2c^2 \cdot (s-a)(s-b)(s-c)}{s^3}} = \frac{abc \cdot r}{s}$$

$$\frac{r_a - r}{r} = \frac{a}{s-a} \ , \ \frac{r_b - r}{r} = \frac{b}{s-b} \ , \ \frac{r_c - r}{r} = \frac{c}{s-c}$$

$$\Longrightarrow \ (r_a - r)(r_b - r)(r_c - r) = \frac{abc \cdot r^3}{(s-a)(s-b)(s-c)} = \frac{abc \cdot r}{s} = IA \cdot IB \cdot IC$$

Quick Reply

High School Olympiads

point [Feuerbach point of a triangle; FY + FZ = FX]

Source: friend

mathquark

#1 Jan 28, 2005, 10:09 pm

Let F be Feuerbach point of triangle ABC. X,Y,Z points are midpoints of [AB],[AC],[BC] line segments respectively. Show that |FY|+|FZ|=|FX|

Feuerbach

#2 Jan 29, 2005, 8:25 pm

Maybe you suppose, that C is the nearest vertex to F?..

PS I have to solve it!

darij grinberg

#3 Jan 29, 2005, 9:17 pm • 1

Yes, the problem indeed requires a condition: The side a has to be the middle side of triangle ABC, i. e. we must have either $c \leq a \leq b$ or $b \leq a \leq c$.

I rewrite the problem to make the notations more symmetric:

Problem. Let F be the Feuerbach point of a triangle ABC, and let A', B', C' be the midpoints of the segments BC, CA, AB, respectively. Assume that either $c \leq a \leq b$ or $b \leq a \leq c$, where a = BC, b = CA, c = AB are the sidelengths of triangle ABC. Prove that FB' + FC' = FA'.

I put my solution into a spoiler in order to give Feuerbach a chance to prove this property of his point for himself :

Solution

One *remark:* The problem states that the distances from the Feuerbach point F of triangle ABC to the midpoints A', B', C' of its sides BC, CA, AB have the property that one of these distances equals to the sum of the two others. This property remains true if we replace the Feuerbach point F by any of the three "exterior" Feuerbach points of triangle ABC (these are the points of tangency of the nine-point circle with the three excircles). This is clear by extraversion.

PS. I probably should also leave some comments on the other geometry topics, especially the one about the Musselman problem, but I fear I have no time now...

Darij

This post has been edited 2 times. Last edited by darij grinberg, May 18, 2005, 8:53 pm

Luis González

#4 Apr 12, 2011, 10:49 am

Let D,E,F be the midpoints of BC,CA,AB instead. F_e is the Feuerbach point of $\triangle ABC$ and assume WLOG that $b \geq a \geq c$. Incircle (I,r) touches BC at M. By Casey's chord theorem for (I) and (D) (with zero radius) tangent to the 9-point circle $(N,\frac{R}{2})$, we have:

$$F_eD^2=\frac{DM^2\cdot(\frac{R}{2})^2}{(\frac{R}{2}-r)(\frac{R}{2}-0)} \Longrightarrow F_eD=\sqrt{\frac{R}{R-2r}}\cdot\frac{(b-c)}{2}.$$

By similar reasoning, we have the expressions

$$F_eE = \sqrt{\frac{R}{R-2r}} \cdot \frac{(a-c)}{2} \ , \ F_eF = \sqrt{\frac{R}{R-2r}} \cdot \frac{(b-a)}{2}$$

$$\implies F_eE + F_eF = \sqrt{\frac{R}{R-2r}} \cdot \frac{b-c}{2} = F_eD.$$

Quick Reply

High School Olympiads

Croatian mathematical olympiad, day 1 problem 3

Reply

Matematika

#1 Apr 10, 2011, 10:02 pm

Triangle ABC is given with its centroid G and cicumcentre O is such that GO is perpendicular to AG. Let A' be the second intersection of AG with circumcircle of triangle ABC. Let D be the intersection of lines CA' and AB and E the intersection of lines BA' and AC. Prove that the circumcentre of triangle ADE is on the circumcircle of triangle ABC.

Luis González

#2 Apr 10, 2011, 11:49 pm

Since A' is a point on the A-median, it is clear that $ED \parallel BC$. If M is the midpoint of BC, then AM is also the A-median of $\triangle ADE$, i.e. $P \equiv AM \cap DE$ is the midpoint of DE. From the harmonic cross ratio (A, A', M, P), keeping in mind that $\overline{GA} = -\overline{GA'}$, we have $\frac{PA}{PA'} = -\frac{MA}{MA'} = \frac{MA}{MG} = 3 \Longrightarrow A'$ is centroid of $\triangle ADE$, i.e. B, C are midpoints of $AD, AE \Longrightarrow$ circumcenter of $\triangle ADE$ is the antipode of A WRT (O).

FantasyLover

#3 Apr 16, 2011, 4:16 am

Solution

littletush

#4 Oct 22, 2011, 4:50 pm

it's trivial that G is the migpointg of AA',let T be the midpoint of DE
then by Newton's line,G, M, T collinear
so $A, G, M, A'.T$ collinear
hence$BC \square DE$
so it suffices to prove $DE = 2BC$
which is trivial

Virgil Nicula

#5 Oct 23, 2011, 1:11 am

Remark. $GO \perp GA \iff 2a^2 = b^2 + c^2$.

bojler

#6 Nov 21, 2014, 1:37 am

Let X be intersection of AO and circumcircle of ABC and A_1, B_1, C_1 be the midpoints of $BC, AC, AB.H_{A,2}$ maps G to A' so $CA'||GB_1$ so B, C are midpoints of DA, AE and thus X is the circumcenter of ADE.

Quick Reply

High School Olympiads

A very beatiful problem about 3 concurrent lines

$f(e)$

efoski1687

#1 Jan 2, 2011, 7:40 pm

Let $\triangle\ ABC$ be a triangle, O be the circumcenter of $\triangle\ ABC$, N be the center of nine point circle of $\triangle\ ABC$ and X be the midpoint of the line segment ON. Let A', B', C' be the midpoints of the line segments BC, CA, AB, respectively. Let l_a, l_b, l_c be the lines through the points X and A', X and B', X and C', respectively. Finally, let H_b and H_c be feet of perpendiculars from B and C on the lines AC and BA, respectively. Prove that the perpendicular line from H_b on l_b, the perpendicular line from $H_c$on l_c and l_a are concurrent.

$f(e)$

efoski1687

#2 Jan 13, 2011, 1:21 am

I found some propositons that maybe important for solution:

The line the midpoint of the line segment OA lie on the line la.

$(x+1)^n$

abhinavzandubalm

#3 Jan 15, 2011, 9:51 am

I Think You Should Read The Book $Episodes\ in\ Nineteenth\ and\ Twentieth\ Century\ Euclidean\ Geometry$ By $Ross\ Honsberger$.

$f(e)$

efoski1687

#4 Jan 17, 2011, 2:21 pm

I have already read this book, but it's not enough to solve this problem. I'am looking for a synthetic proof. Actually one can solve this problem with barycentric coordinates.

Luis González

#5 Feb 8, 2011, 10:19 am • 2

Alternate formulation: $\triangle DEF$ is the incentral triangle of $\triangle ABC, D \in BC, E \in CA, F \in AB.$ Parallels from B, C to DF, DE meet at M and U is the midpoint of the arc BAC of the circumcircle $\odot(ABC)$. Then $UM \perp EF$.

Proof: Let P, Q be points on the extensions $\overrightarrow{AB}$ and $\overrightarrow{AC}$ such that $PB = BC = CQ$. ED cuts AB at K and the parallel from C to ED cuts AB, PQ at L, M'.

$$\frac{KL}{KA} = \frac{EC}{EA} \Longrightarrow KL = \frac{bc}{|b-a|} \cdot \frac{a}{c} = \frac{ab}{|b-a|} \Longrightarrow LP = \frac{a(a+c)}{|b-a|}$$

By Menelaus' theorem for $\triangle APQ$ cut by $\overline{CM'L}$, we get

$$\frac{QM'}{M'P} = \frac{LA}{LP} \cdot \frac{CQ}{AC} = \frac{a+c}{|b-a|} \cdot \frac{|b-a|}{a+c} \cdot \frac{b}{a} \cdot \frac{a}{b} = 1 \Longrightarrow M' \text{ is the midpoint of } PQ$$

Similarly, parallel from B to DF passes through the midpoint M' of PQ, i.e. $M \equiv M'$.

Since $\dfrac{AE}{EQ} = \dfrac{AF}{FP} = \dfrac{bc}{a(a+b+c)}$, it follows that $EF \parallel PQ$. $(\star)$

On the other hand, circles $\odot(ABC)$ and $\odot(APQ)$ meet at A and the center of the rotation taking the oriented segments $\overline{PB} \cong \overline{QC}$ into each other. Since $UB = UC,$ then U is the center of such rotation $\Longrightarrow UP = UQ \Longrightarrow UM$ is the perpendicular bisector of $PQ,$ i.e. $UM \perp PQ.$ Together with $(\star),$ we conclude that $UM \perp EF$ and the proof is completed.

$f(e)$

efoski1687

#6 Apr 10, 2011, 5:31 pm

I have read your solution, dear luis. Your solution is total false and your solution has nothing about the statement of my problem...
If anyone has a solution of this problem, can he/she post it. I have tried to solve for a lot hours but I could not.

skytin

#7 Apr 10, 2011, 8:36 pm

> *efoski1687 wrote:*
> I have read your solution, dear luis. Your solution is total false and your solution has nothing about the statement of my problem...
> If anyone has a solution of this problem, can he/she post it. I have tried to solve for a lot hours but I could not.

Wy did you say it ?
It's absolutely write solution
To get this formulation you need use this :
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=385175
see my post I_AO is perpendicular to XY

Luis González

#8 Apr 10, 2011, 10:28 pm

No efoski1687, that means that you neither read the proof nor tried to figure it out. If you don't understand something, then politely ask for clarifications through the forum or private message. The idea for the alternate formulation is to use that the cevians of the 9-point center N of ABC are perpendicular to the sidelines of the incentral triangle of its orthic triangle HaHbHc.

Quick Reply

High School Olympiads

X, Y, O1, O2 are concyclic

Reply

Source: Serbia Math Olympiad 2011

RaleD
#1 Apr 9, 2011, 2:13 am • 1

On sides AB, AC, BC are points M, X, Y, respectively, such that $AX = MX$; $BY = MY$. K, L are midpoints of AY and BX. O is circumcenter of ABC, O_1, O_2 are symmetric with O with respect to K and L. Prove that X, Y, O_1, O_2 are concyclic.

mahanmath
#2 Apr 9, 2011, 3:11 am

Remark. $OXYC$ is cyclic too .

hint

Luis González
#3 Apr 9, 2011, 9:43 am

O is the circumcenter of $\triangle ABC$, P is the foot of the C-altitude and D, E are the midpoints of BC, CA. Since $\triangle AEP$ and $\triangle BDP$ are isosceles with apices E, D, it follows that $PE \parallel MX$ and $PD \parallel MY$. Thus

$$\frac{EX}{XA} = \frac{PM}{MA}, \frac{DY}{YB} = \frac{PM}{MB} \Longrightarrow \frac{EX}{DY} = \frac{XA}{MA} \cdot \frac{MB}{YB} = \frac{EA}{DB} \cdot \frac{PB}{PA} = \frac{OE}{OD}$$

Thus, the right $\triangle OEX$ and $\triangle ODY$ are similar. As a result, $\angle XOY = \angle EOD$, i.e. C, X, Y, O are concyclic. Let S, T be the midpoints of MA, MB. If $\odot(MXY)$ cuts YT again at U, then $\angle OXY = \angle OCY = \angle MXS$ implies that

$$\angle OXS = \angle MXY = \angle MUT = \angle BUT \Longrightarrow OX \parallel BU$$

But $\angle AXU = \angle AXM + \angle MYU = 90^\circ - \angle A + \angle C \Longrightarrow UX \parallel BO$

Hence, $OBUX$ is a parallelogram with diagonal intersection $L \Longrightarrow U$ coincides with the reflection O_2 of O about $L \Longrightarrow O_2 \in \odot(MXY)$. By analogous reasoning, we'll have that $O_1 \in \odot(MXY)$.

JuanOrtiz
#4 Jan 1, 2014, 1:41 am

Call O_1=E and O_2=D. Look at the lines perpendicular to AB that pass through X, Y, O, K and L. The first two bisect AM and MB, respectively, so call the midpoint of AM F and the midpoint of MB G. The third line bisects AB. The fourth line bisects AG and the fifth line bisects BF. Using this we easily prove that the distance between the 1º and 4º line is the same as the distance between the 4º and 3º lines, and therefore D is on the first line. Analogously, E is on the second line.

Now we see OBEX and OADY are parallelograms. Therefore <DXE = 90 - <ABO = 90 - <BAO = <EYD and therefore XDEY is cyclic. The End.

JuanOrtiz
#5 Jan 1, 2014, 1:43 am

Also, M is also on this circle because <XMY = <BCA = 90 - <ABO = <YEX. So MXYDE is a cyclic pentagon.

Another remark, OXYC is cyclic, which is easy to see.

MMEEvN

#6 Jan 1, 2014, 1:05 pm • 1

.Let P be the midpoint of $AB \Rightarrow .PL=\frac{1}{2}XA=\frac{1}{2}MX$

.Similarly $PK=\frac{1}{2}MY$.Observe that

$\angle YMX=\angle A=\angle LPK \Rightarrow \Delta LPK \sim \Delta XMY \Rightarrow LK=\frac{1}{2}XY \Rightarrow O_1O_2=XY$ .Since $YOAO_1$ and OXO_2B are parallelograms $XO_2=OB=OA=O_1Y$ .Hence XYO_2O_1 is a isosceles trapezoid.Hence proved .

mathuz

#7 Jan 5, 2014, 12:51 am

my proof also similar to @MMEEnV 's proof! We will prove that XYO_2O_1 is isosceles trepazoid. We have that
$4KL^2=AX^2+XY^2+BY^2+AB^2-AY^2-BX^2=XY^2+AB^2-AM\cdot AB-BM\cdot AB=XY^2$.
So $O_1O_2=XY$ ...

Quick Reply

High School Olympiads

isosceles with circumcircle

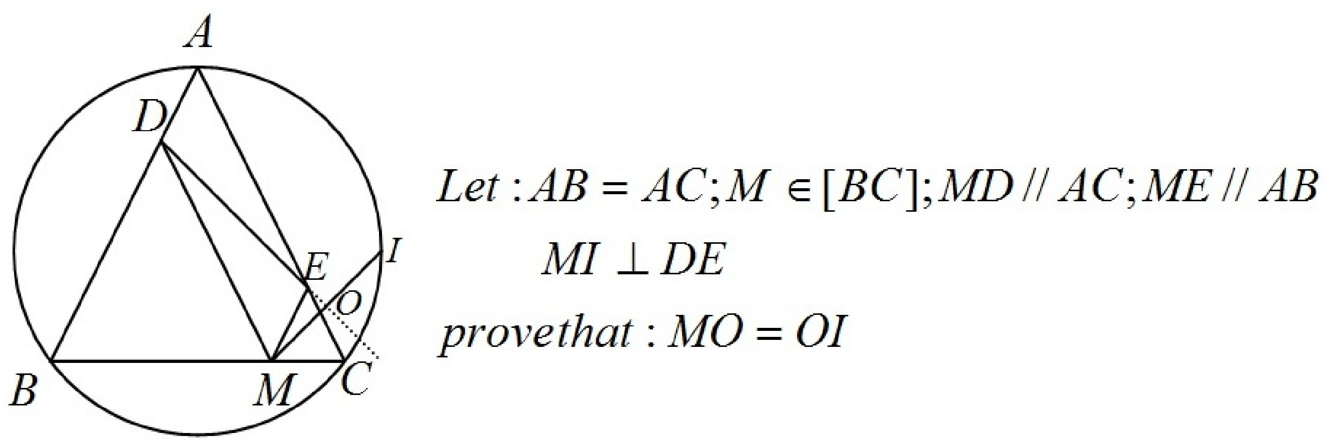
Reply

FongherHerjalearn

#1 Apr 8, 2011, 9:37 pm

please solve in the attachment

Attachments:



Luis González

#2 Apr 9, 2011, 12:09 am

Let P, Q be the midpoints of AC, AB. Since $AD + AE = AC = AB$, we deduce that $QD = PE = \frac{1}{2}|AE - AD|$. Thus, $\odot(APQ)$ and $(S) \equiv \odot(ADE)$ intersect at A and the center K of the rotation taking the oriented segments $\overline{QD} \cong \overline{PE}$ into each other. Since K, A are diametrically opposed in (S), it follows that K is the circumcenter of $\triangle ABC$ $\Longrightarrow SK$ is perpendicular bisector of $\overline{DE}$. Let (S) cut the circumcircle (K) of $\triangle ABC$ again at U. Then $AU \perp SK \Longrightarrow AU \parallel DE$. If AU cuts ME, MD at X, Y, we deduce that $\triangle ADE$ is the medial triangle of $\triangle MXY$, i.e. (S) is 9-point circle of $\triangle MXY \Longrightarrow U$ is the foot of the perpendicular from M to XY, i.e. DE is the perpendicular bisector of $\overline{MU} \Longrightarrow U \equiv I \Longrightarrow$ Reflection I of M across DE lies on $\odot(ABC)$.

Headhunter

#3 Apr 9, 2011, 12:41 am

MI is parallel to EJ and $DL = LJ \Longrightarrow \angle EMI = \angle EIM$

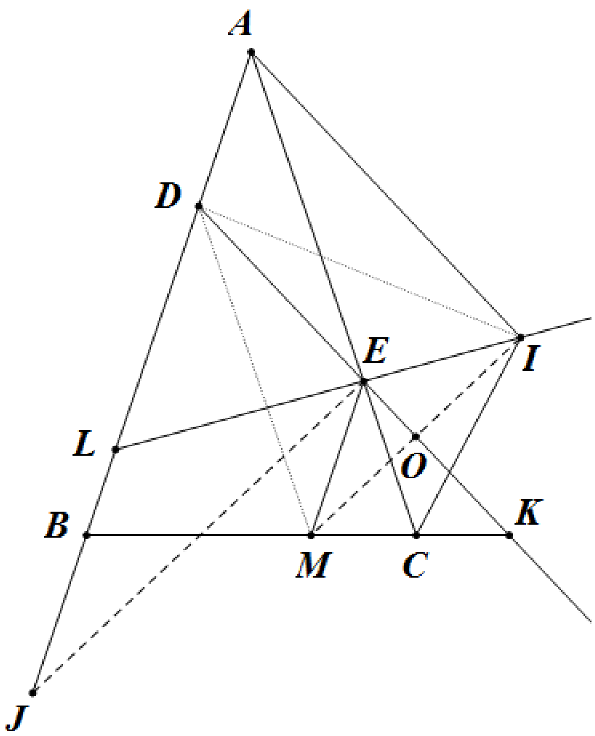
$\Longrightarrow MO = OI$, $EC = EM = EI$

$\triangle LDE$ is isosceles and $\dfrac{LE}{EI} = \dfrac{LD}{AD} \Longrightarrow DE$ is parallel to AI

$$\angle LAI = \frac{\angle LAE + \angle LEA}{2} = \frac{\angle MEI}{2} = \angle IMC + \angle MIC = \angle ICK$$

Thus, $\square ABCI$ is cyclic and then it's done.

Attachments:



Quick Reply

High School Olympiads

Incenter, Area

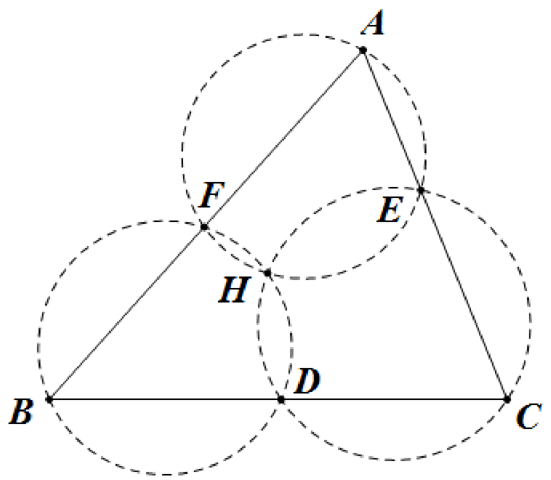
Headhunter

#1 Apr 7, 2011, 4:59 am

Hello.

D is on BC, where a circle (B,BD) cut AB at F and a circle (C,CD) cut CA at E
Two circumcircles of $\triangle BDF$, $\triangle CDE$ meet each other at H
Show that the area of $\triangle EFH$ is at max if and only if H is the incenter of $\triangle ABC$

Attachments:



Luis González

#2 Apr 8, 2011, 10:16 am

Since $\angle BHD=\angle BFD=90^\circ-\frac{1}{2}\angle B$ and $\angle CHD=\angle CED=90^\circ-\frac{1}{2}\angle C,$ it follows that B,C,H and the incenter I of $\triangle ABC$ are concyclic. Let AI cut BC at V and $\odot(IBC)$ again at $P.$ Since $\angle PBI$ is right, then PB is tangent to the circumcircle of the isosceles $\triangle BFD \Longrightarrow \angle HBP=\angle HDV=\angle VIH \pmod \pi).$ Hence, H is the Miquel point of $\triangle ABV \cup DIF,$ i.e. $I \in \odot(AFE).$ Incircle (I,r) touches BC,CA,AB at $M,N,L.$ WLOG assume that D lies on the segment $\overline{BM}.$ Since $\angle NIL=\angle EHF,$ then it suffices to show that $IN \cdot IL=r^2 \ge HE \cdot HF.$ Let U be the orthogonal projection of H on $BC.$ Since $\angle HIP=\angle HDB \le 90^\circ,$ we get that U is between B and $D.$ Thus

$$IH^2 \ge UM^2 \ge DM^2=FL^2=IF^2-r^2 \Longrightarrow r^2 \ge IF^2-IH^2 \ (\star)$$

Q lies on $\overrightarrow{FH},$ such that $HE=HQ,$ thus $\angle FHE=\angle FIE=2\angle FQE.$ Since $IE=IF,$ it follows that I is the center of $\odot(EFQ).$ Hence from the power of H to the circle $\odot(EFQ)$ we get $HF \cdot HQ=HF \cdot HE=IF^2-IH^2.$ Together with $(\star),$ we get $r^2 \ge HF \cdot HE \Longrightarrow |\triangle INL| \ge |\triangle HEF|,$ as desired.

High School Olympiads

Prove that

Reply

Pirkuliyev Rovsen

#1 Dec 20, 2010, 11:48 pm • 1

Let AA_1,BB_1 and CC_1 be the altitudes,AA_2,BB_2 and CC_2 be the median of an non-isosceles $\triangle ABC$. Prove that it's possible to organize triangle from stretch A_1A_2, B_1B_2 and C_1C_2.

Azerbaijan Land of Fire

Luis González

#2 Apr 7, 2011, 9:00 am • 1

Let the parallels from B_1 and C_2 to A_2B_2 and A_1C_1 intersect at P. Then we have

$$\angle PB_1C = \angle A_2B_2C = \angle BAC \ , \ \angle PC_2B = \angle A_1C_1B = \angle ACB \ \Longrightarrow$$

$$\angle C_2PB_1 = 360^\circ - (180^\circ - \angle BAC) - (180^\circ - \angle ACB) - \angle BAC = \angle ACB.$$

Since $\angle C_2A_1B_1 = \angle C_2B_2B_1 = \angle ACB$, then $\angle C_2PB_1 = \angle C_2A_1B_1 \Longrightarrow P$ lies on the 9-point circle $\odot(A_1B_1C_1)$ of $\triangle ABC$. Thus, the cyclic quadrilaterals $A_1C_1C_2P$ and $A_2B_2B_1P$ are isosceles trapezoids with legs $PA_1 = C_1C_2$ and $PA_2 = B_1B_2$. Therefore, the lengths of the segments $\overline{A_1A_2}, \overline{B_1B_2}, \overline{C_1C_2}$ form a triangle whose circumradius is half the circumradius of $\triangle ABC$.

Pirkuliyev Rovsen

#3 Apr 7, 2011, 7:08 pm

Thanks you

Quick Reply

High School Olympiads

prove collinear points

Reply

paul1703

#1 Apr 7, 2011, 1:03 am

Let ABC be a triangle and BB_1, CC_1 be the heights of triangle, and B_2C_2 their midpoints if Q is a point on BC and Q_1 the midpoint of the projections of Q on AB and AC, prove that C_2B_2, Q_1 are collinear.

Luis González

#2 Apr 7, 2011, 1:37 am

Let X,Y be orthogonal projections of Q on $AB, AC. Q_1$ is the midpoint of $\overline{XY}$. It suffices to show that the locus of Q_1 when Q varies along BC is a straight line ℓ. Define the rectangular reference where BC coincides with the y-axis and $Q \equiv (0,\varrho)$ is a variable point on it. Sidelines AB, AC are given by the equations $ax+by+c=0$ and $dx+ey+f=0$. Hence, coordinates of the orthogonal projections of $Q \equiv (0,\varrho)$ onto lines AB, AC are given by:

$$X\left(\frac{-ac-ab\varrho}{a^2+b^2}, \frac{a^2\varrho-bc}{a^2+b^2}\right) \ \ , \ \ Y\left(\frac{-df-de\varrho}{d^2+e^2}, \frac{d^2\varrho-ef}{d^2+e^2}\right)$$

Hence, coordinates of the midpoint $Q_1 \equiv (\overline{x},\overline{y})$ of $\overline{XY}$ are given by:

$$\overline{x}=-\frac{\varrho}{2}\left(\frac{ab}{a^2+b^2}+\frac{de}{d^2+e^2}\right)-\frac{1}{2}\left(\frac{ac}{a^2+b^2}+\frac{df}{d^2+e^2}\right)$$

$$\overline{y}=\frac{\varrho}{2}\left(\frac{a^2}{a^2+b^2}+\frac{d^2}{d^2+e^2}\right)-\frac{1}{2}\left(\frac{bc}{a^2+b^2}+\frac{ef}{d^2+e^2}\right)$$

This represents a parametric equation of a line ℓ. Note that with similar arguments we prove that: If Q moves on a fixed line ℓ, then the centroid of its pedal triangle moves on another line.

Quick Reply

High School Olympiads

3 lines Concurrent

Reply

vulalach

#1 Apr 5, 2011, 10:59 am

please help me solve this problems.

Let ABC be a triangle, O be the circumcenter of (ABC) , N be the center of nine point circle of A_1, B_1 and C_1 be the midpoint of the line segment BC, AC and AB. Let X be the midpoints of the line segments ON, respectively. Let d_a be the lines through the points X and A_1 , and d_t, and d_c , respectively. Finally, let H_b and H_c be feet of perpendiculars from B and C on the lines AC and AB , respectively. Prove that the perpendicular line from Hb on d_t , the perpendicular line from H_c on d_c and d_a are concurrent.

Luis González

#2 Apr 5, 2011, 12:16 pm

The same problem was discussed here. (see post #5). The idea for the alternate formulation is to use that the cevians of the 9-point center N are perpendicular to the sidelines of the incentral triangle of $\triangle H_aH_bH_c$.

Let H be the orthocenter of $\triangle ABC$ and $E \equiv HB \cap H_aH_c, F \equiv HC \cap H_aH_b$. Since $\overline{EH_a} \cdot \overline{EH_c} = \overline{EH} \cdot \overline{EB}$ and $\overline{FH_a} \cdot \overline{FH_b} = \overline{FH} \cdot \overline{FC}$, it follows that E, F have equal powers to (N) and $\odot(HBC) \Longrightarrow EF$ is the radical axis of $\odot(HBC), (N)$. Since N is the midpoint of the segment connecting A with the circumcenter of $\triangle HBC$ (reflection of O about BC), we deduce that $AN \perp EF$. Now, use this result for the excentral triangle of $\triangle ABC$.

Quick Reply

High School Olympiads

Prove an equality

Reply

PTMINH

#1 Apr 3, 2011, 3:25 pm

An acute triangle ABC have orthocenter H. A line d go through H. A', B', C' are perpendicular projections of A, B, C on d respectively. Prove that
$AA'^2.tanA + BB'^2.tanB + CC'^2.tanC = 2S(ABC)$
($S(ABC)$ is the area of ABC)

Luis González

#2 Apr 4, 2011, 12:12 pm

Let us represent the vertices of $\triangle ABC$ in a rectangular reference, such that $A \equiv (0,p), B \equiv (q,0), C \equiv (r,0)$. Thus, the coordinates of H are given by $H \equiv (0, -\frac{qr}{p})$.

$$\tan A = \frac{4|\triangle ABC|}{b^2+c^2-a^2} = \frac{4|\triangle ABC|}{2p^2+r^2+q^2-(q-r)^2} = \frac{2|\triangle ABC|}{p^2+qr}$$

$$\tan B = \frac{4|\triangle ABC|}{c^2+a^2-b^2} = \frac{4|\triangle ABC|}{q^2-r^2+(q-r)^2} = \frac{2|\triangle ABC|}{q^2-qr}$$

$$\tan C = \frac{4|\triangle ABC|}{a^2+b^2-c^2} = \frac{4|\triangle ABC|}{r^2-q^2+(q-r)^2} = \frac{2|\triangle ABC|}{r^2-qr}$$

m is the slope of an arbitrary line d through H. Thus, $d \equiv mx - y - \frac{qr}{p} = 0$.

If δ_A, δ_B and δ_C denote the distances from A, B, C to d, we get then

$$\delta_A{}^2 = \frac{(p+\frac{qr}{p})^2}{m^2+1} \ \ , \ \ \delta_B{}^2 = \frac{(mq-\frac{qr}{p})^2}{m^2+1} \ \ , \ \ \delta_C{}^2 = \frac{(mr-\frac{qr}{p})^2}{m^2+1} \Longrightarrow$$

$$\frac{m^2+1}{2|\triangle ABC|} \cdot \sum \tan A \cdot \delta_A{}^2 = \frac{(p+\frac{qr}{p})^2}{p^2+qr} + \frac{(mq-\frac{qr}{p})^2}{q^2-qr} + \frac{(mr-\frac{qr}{p})^2}{r^2-qr}$$

The RHS of the latter equation equals m^2+1 after simplifications. Therefore, we get

$$\tan A \cdot \delta_A{}^2 + \tan B \cdot \delta_B{}^2 + \tan C \cdot \delta_C{}^2 = 2|\triangle ABC|.$$

Quick Reply

High School Olympiads

construction

Reply

sergei93

#1 Apr 1, 2011, 10:04 am

Let $X_1, X_2, ..., X_5$ be five different points on the plane no three of which are collinear. Construct the pentagon $ABCDE$ such that the given points X_i are the midpoints of AB, BC, CD, DE and EA, respectively.

Try to make use of transformations.

Luis González

#2 Apr 1, 2011, 9:44 pm • 1

A is the double point under the composition $\mathcal{L} \equiv \mathcal{X}_1 \circ \mathcal{X}_2 \circ \mathcal{X}_3 \circ \mathcal{X}_4 \circ \mathcal{X}_5$, where $\mathcal{X}_i$ denotes the central symmetry about X_i. Since $\mathcal{X}_1 \circ \mathcal{X}_2$ and $\mathcal{X}_3 \circ \mathcal{X}_4$ are translations, then $\mathcal{L}$ is reduced to the composition of certain translation and $\mathcal{X}_5$. This is: pick an arbitrary point P and find its image P' under $\mathcal{X}_1 \circ \mathcal{X}_2 \circ \mathcal{X}_3 \circ \mathcal{X}_4$. Draw the parallel ℓ from X_5 to PP', then A, E lie on ℓ such that $\overline{AX_5} = \overline{EX_5} = \frac{1}{2}\overline{PP'}$. Now, the construction of the remaining vertices B, C, D is straightforward.

Quick Reply

High School Olympiads

Equal Angles

Headhunter

#1 Apr 1, 2011, 8:50 pm

Hello.

For $\triangle ABC$, D move on $\overline{CA}$ and E,F move on $\overline{BD}$, $\overline{BC}$, where $\angle BAE=\angle CAF$
P,Q are on $\overline{BC}$, $\overline{BD}$, where $\overline{EP}$,$\overline{FQ}$ are parallel to $\overline{DC}$, $\overline{CD}$ respectively.
Show that $\angle BAP=\angle CAQ$, *with* ***pure geometric ways***.

Please No Trig.

Luis González

#2 Apr 1, 2011, 9:27 pm • 2

Let $U \equiv AP \cap BD, V \equiv AF \cap BD$. Then $(B,D,U,V)=(B,C,P,F)$

$$\Longrightarrow \frac{BU}{UD} \cdot \frac{VD}{BV} = \frac{BP}{PC} \cdot \frac{FC}{BF}$$

Since $DC \parallel EP \parallel QF$, by Thales theorem we get $\frac{BQ}{QD}=\frac{BF}{FC}$, $\frac{BP}{PC}=\frac{BE}{ED}$

$$\Longrightarrow \frac{BU}{UD} \cdot \frac{BQ}{QD} = \frac{BE}{ED} \cdot \frac{BV}{VD}$$

Thus, by Steiner theorem, if AE, AV are isogonals with respect to $\angle BAD$, then AU, AQ are also isogonals with respect to $\angle BAD$, i.e. $\angle BAP=\angle CAQ$.

High School Olympiads

Geo. problem

Reply

Zeus93

#1 Mar 31, 2011, 6:12 pm

Triangle ABC with the incenter I. BI, CI intersect ABC's circumcircle at E, F respectively. M is mid-point of EF, (C) is a circle with diameter EF. IM intersect (C) at L, K and the arc BC of ABC's circumcircle (which don't go through A) at D. Prove that $\frac{DL}{IL}=\frac{DK}{IK}$

Luis González

#2 Apr 1, 2011, 7:53 am

Let P be the midpoint of the arc BC of the circumcircle (O) of $\triangle ABC$. It's well-known that AI, BI, CI are perpendicular to EF, FP, PE through $X, Y, Z,$ i.e. $I, \triangle XYZ$ are the orthocenter and orthic triangle of $\triangle PEF$. Inversion with center I and power $\overline{IX} \cdot \overline{IP}$ takes the 9-point circle $\odot(XYZ)$ of $\triangle PEF$ into its circumcirle (O). Thus, D is the inverse of $M \Longrightarrow$ X, M, P, D are concyclic, i.e. $MI \perp PD$. Hence, PD is the polar of I WRT the circle with diameter $EF \Longrightarrow$ Cross ratio (K, L, I, D) is harmonic $\Longrightarrow \frac{\overline{IL}}{\overline{IK}}=-\frac{\overline{DL}}{\overline{DK}}$.

Quick Reply

High School Olympiads

tangent circles conccurent lines

Reply

my_name_is_math

#1 Mar 30, 2011, 7:20 pm

Let ABC be a triangle. Circle k passes through points B and C. Circle w is tangent internally to k and also to sides AB and AC at T; P, and Q, respectively. Let M be midpoint of arc BC (containing T) of k. Prove that lines PQ, BC and MT are concurrent.

Luis González

#2 Mar 30, 2011, 9:00 pm

Let R, ϱ be the radii of $\mathcal{K}$ and ω, respectively. By Casey's chord theorem for $\omega, (B)$ and $\omega, (C)$, both $(B), (C)$ with zero radii and tangent to $\mathcal{K}$ through B, C, we obtain

$$TC^2 = \frac{CQ^2 \cdot R^2}{(R-\varrho)(R-0)} = \frac{CQ^2 \cdot R}{R-\varrho} \ , \ TB^2 = \frac{BP^2 \cdot R^2}{(R-\varrho)(R-0)} = \frac{BP^2 \cdot R}{R-\varrho}$$

$$\Longrightarrow \frac{TB}{TC} = \frac{BP}{CQ}$$

Let PQ cut BC at U. By Menelaus' theorem for $\triangle ABC$ cut by $\overline{UPQ}$, we have

$$\frac{UB}{UC} = \frac{BP}{AP} \cdot \frac{AQ}{CQ} = \frac{BP}{CQ} = \frac{TB}{TC}$$

Thus, by angle bisector theorem, U is the foot of the T-external bisector TM of $\triangle BTC$.

Luis González

#3 Mar 31, 2011, 12:29 am

Casey's chord theorem. Two circles $\Gamma_1(r_1)$ and $\Gamma_2(r_2)$ are internally/externally tangent to a circle $\Gamma(R)$ through A, B, respectively. The length δ_{12} of the common external tangent of Γ_1, Γ_2 is given by

$$\delta_{12} = \frac{AB}{R}\sqrt{(R \pm r_1)(R \pm r_2)}$$

The signs are chosen according to the type of tangencies. This is: (+) for external tangency and (-) for internal tangency.

Quick Reply

High School Olympiads

Intersecting lines

Reply

borislav_mirchev

#1 Mar 29, 2011, 2:04 am

It is given a triangle ABC.
M_1, M_2 and M_3 are the midpoints of the sides AB, BC and CA respectively.
P_1, P_2 and P_3 are the tangent point of incircle with the sides AB, BC, CA respectively.
A_1 is the intersection point of the lines P_1M_2 and P_2M_3.
B_1 is the intersection point of the lines P_2M_3 and P_3M_1.
C_1 is the intersection point of the lines P_3M_1 and P_1M_2.
Prove that the lines AA_1, BB_1 and CC_1 intersects at a common point.

Luis González

#2 Mar 29, 2011, 7:21 am • 1

The result is true for any conic section intersecting the sidelines AB at M_1, P_2, BC at M_2, P_2 and CA at M_3, P_3. By Pascal theorem for the nonconvex hexagon $M_1P_3M_3P_2M_2P_1$, the intersections $X \equiv M_1P_3 \cap P_2M_2, Y \equiv M_3P_3 \cap P_1M_2$ and $Z \equiv M_3P_2 \cap P_1M_1$ are collinear. Thus, the triangle $\triangle A_1B_1C_1$ bounded by M_1P_3,M_2P_1 and M_3P_2 is perspective with $\triangle ABC$ through $\overline{XYZ} \Longrightarrow AA_1, BB_1, CC_2$ concur.

P.S. In the proposed problem, the referred conic is the bicevian conic of X_2, X_7.

Quick Reply

High School Olympiads

Show that <MKN=90

Reply

Source: Iran Pre-Preparation Course Examination 1997, E3, P2

Amir Hossein
#1 Mar 29, 2011, 3:08 am

Two circles O, O' meet each other at points A, B. A line from A intersects the circle O at C and the circle O' at D (A is between C and D). Let M, N be the midpoints of the arcs BC, BD, respectively (not containing A), and let K be the midpoint of the segment CD. Show that $\angle KMN = 90^\circ$.

Luis González
#2 Mar 29, 2011, 5:30 am • 3

Let B' be the inverse of B under the inversion with center A and power $\overline{AC} \cdot \overline{AD}$. Circles $(O), (O')$ go to lines DB' and $CB' \Longrightarrow M' \equiv AM \cap DB'$ and $N' \equiv AN \cap CB'$ are the inverses of M, N, respectively. Let $U \equiv M'N' \cap CD$. Since the pencil $A(C, B, M, N)$ is harmonic, it follows that pencil $B'(M', N', A, U)$ is also harmonic $\Longrightarrow (C, D, A, U)$ is harmonic $\Longrightarrow \overline{AC} \cdot \overline{AD} = \overline{AK} \cdot \overline{AU}$, i.e. U is the inverse of A under the referred inversion. Since U, M', N' are collinear, then K, M, N, A are concyclic and the conclusion follows.

jgnr
#3 Mar 29, 2011, 5:01 pm • 1

Let L be the reflection of M accross K. Then $\triangle CKM \cong \triangle DKL$. We have $DL = CM = BM$ and $DN = BN$, and some angle chasing gives $\angle MBN = \angle LDN$. Thus $\triangle MBN \cong \triangle LDN$, and $MN = LN$. Since $MN = LN$ and $MK = KL$, then $NK \perp ML$, that is, $\angle KMN = 90^\circ$.

Virgil Nicula
#4 May 27, 2011, 6:48 am

@ Johan Gunardi. Why $\angle MBN = \angle LDN$?

jgnr
#5 May 27, 2011, 7:40 am

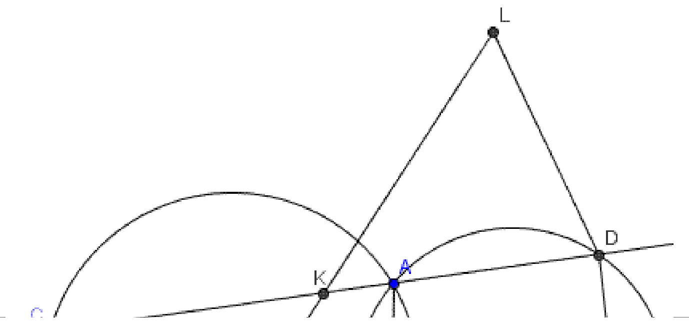
Probably there are other cases, but in my diagram it goes like this:

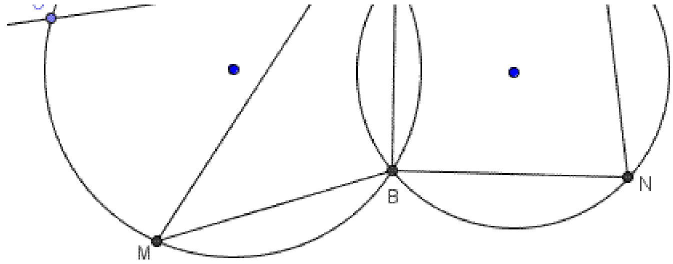
$$\angle MBN = 360^\circ - (\angle MBA + \angle ABN) = \angle ACM + \angle NDA = \angle KDL + \angle NDA = \angle NDL$$

Maybe we can use directed angle?

Anyway, I think there is a typo in the problem statement. It should be $\angle MKN = 90^\circ$.

Attachments:





Quick Reply

High School Olympiads

Ineq-G119 - Geometry

Reply

Ligouras
#1 Mar 27, 2011, 2:24 am

Let ABC be acute triangle. If the escribed circle which touches a is equal to the circum-circle, prove that

$$cosA \leq \frac{3}{4}$$

Luis González
#2 Mar 29, 2011, 1:19 am • 1

I is the incenter of $\triangle ABC$ and I_a, I_b, I_c are the three excenters againts $A, B, C.$ $\triangle ABC$ and (O) become orthic triangle and 9-point circle of $\triangle I_aI_bI_c.$ Let BI, CI cut CA, AB at $Y, Z.$ Since $YA \cdot YC = YI \cdot YI_b$ and $ZA \cdot ZB = ZI \cdot ZI_c,$ it follows that Y, Z have equal powers to (O) and $\odot(II_bI_c),$ i.e. YZ is the radical axis of $(O), \odot(II_bI_c).$ Let D be the circumcenter of $\triangle II_bI_c,$ i.e reflection of I_a across $O.$ Since $R = \varrho_a,$ we deduce that

$OI_a{}^2 = R^2 + 2R\varrho_a = 3R^2 \Longrightarrow \ 4R^2 - OI_a{}^2 = 4R^2 - OD^2 = R^2 \Longrightarrow O$ has equal power to $\odot(II_bI_c)$ and $(O),$ i.e. O lies on their radical axis $YZ.$

$$\Longrightarrow \ \text{d}(O, BC) = \text{d}(O, CA) + \text{d}(O, AB) \Longrightarrow \ R\cos A = R\cos B + R\cos C$$

But we know that $\cos A + \cos B + \cos C \leq \frac{3}{2} \Longrightarrow \ \cos A \leq \ \frac{3}{4}.$

BigSams
#3 Mar 29, 2011, 4:13 am

Hmmm I made some preliminary progress yesterday but it did not go anywhere
Now that a full solution has been given, maybe someone can complete this idea.

Let a, b, c be the sides of the triangle.

The circumradius is $\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$, and the exradius is $\sqrt{\frac{s(s-b)(s-c)}{(s-a)}}$.

Equating these two gives $\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} = \sqrt{\frac{s(s-b)(s-c)}{(s-a)}} \Longrightarrow \ abc = 4s(s-b)(s-c)$. So this is an identity that can be used.

Since it is an acute triangle, $a^2 + b^2 > c^2, b^2 + c^2 > a^2, c^2 + a^2 > b^2$.

Note that $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, so it needs to be proven that
$\frac{b^2 + c^2 - a^2}{2bc} \leq \frac{3}{4} \Longleftrightarrow 2b^2 + 2c^2 \leq 2a^2 + 3bc \Longleftrightarrow 7(s-b)(s-c) \geq s(s-a)$.

I was thinking that perhaps inequalities techniques - with Ravi Substitution if necessary - could algebraically crack it.

Rijul saini
#4 Mar 31, 2011, 2:00 am

" Ligouras wrote:

Ligouras wrote:

> Let ABC be acute triangle. If the escribed circle which touches a is equal to the circum-circle, prove that
>
> $cosA \leq \frac{3}{4}$

This bound is quite loose. I show that the best bound on $\cos A$ is $\sqrt{3}-1$ and which is tight.
Denote by r_a, H the exradius opposite A and the orthocentre of $\triangle ABC$. Now, it is well known that $r_a + AH = 2R + r$.
This straightforwardly implies that $2R\cos A = AH = R + r = R(\cos A + \cos B + \cos C)$ implying that
$\cos A = \cos B + \cos C$.
Now, $\cos A = \cos B + \cos C = 2\sin\frac{A}{2}\cdot\cos\frac{B-C}{2} \leq 2\sin\frac{A}{2}$, since $\sin\frac{A}{2}$ is positive.
If $\cos A$ is negative, then we're through. Else, it is positive, and we get
$\cos^2 A \leq 4\sin^2\frac{A}{2} = 2(1-\cos A) \implies (\cos A + 1)^2 \leq 3$
and therefore, we get

$$\boxed{\cos A \leq \sqrt{3}-1}$$

as desired. Equality holds when $A \approx 42.9414, B = C \approx 68.5292$.

Virgil Nicula
#5 Apr 5, 2011, 9:42 pm

See and **here**.

Quick Reply

High School Olympiads

Isogonal Conjugation Problems Request

Reply

Source: Darij Grinberg: www.cip.ifi.lmu.de/~grinberg/Isogonal.zip

Amir Hossein

#1 Mar 27, 2011, 12:12 am

Hello guys,

I recently read Darij's "Isogonal Conjugation wrt a triangle" article, and I want some problems to solve with this method. What do you suggest? Do you have any problems that can be solved with this method?

Thanks!

Luis González

#2 Mar 28, 2011, 12:58 am • 1

Another well-known property of isogonal conjugation: Isogonal conjugate of a line with respect to a triangle ABC is a conic passing through A,B,C and vice versa.

Proof: Let us consider barycentric coordinates WRT $\triangle ABC$. $px+qy+rz=0$ is the equation of a line ℓ. Isogonal conjugation $\mathcal{I}:(x:y:z)\mapsto\left(\frac{a^2}{x}:\frac{b^2}{y}:\frac{c^2}{z}\right)$ takes ℓ into a curve $\mathcal{L}$ with equation $\mathcal{L}\equiv pa^2yz+qb^2zx+rc^2xy=0$, i.e the equation of a homogeneous polynomial of secong degree.

Thus, $\mathcal{L}$ represents a conic passing through the vertices of $\triangle ABC$. Since the isogonal conjugate of the circumcircle (O) of $\triangle ABC$ is the line at infinity, we conclude that $\mathcal{L}$ is either hyperbola, parabola or ellipse if ℓ has either two, one or any common point with (O). For some applications, you may see the following threads

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=129699
http://www.artofproblemsolving.com/Forum/viewtopic.php?uid=56597&f=47&t=298124
http://www.artofproblemsolving.com/Forum/viewtopic.php?uid=56597&f=46&t=310440
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=351665
http://www.artofproblemsolving.com/Forum/viewtopic.php?uid=56597&f=46&t=369287
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=394491

Quick Reply

High School Olympiads

Find relation

Reply

skytin
#1 Mar 26, 2011, 11:14 pm

Given triangle ABC and A'B'C' such that CC' AA' BB' intersects at point L and C'L = LA' = LB'
G and G' are centroids of triangles ABC and A'B'C'
a) Prove that A'B'C' is Pedal triangle of point P
b) Find relation in wich line G'L devides segment PG

Luis González
#2 Mar 27, 2011, 12:16 pm • 2

(a) For convenience, let us restate the notations such that $\triangle A'B'C'$ is the anticevian triangle of the circumcenter O of $\triangle ABC$, then we shall show that perpendiculars to $B'C', C'A', A'B'$ through A, B, C concur. Let H, L be the orthocenter and the De Longchamps point X_{20} of $\triangle ABC$ (orthocenter of its antimedial triangle). $\triangle H_aH_bH_c$ and $\triangle L_aL_bL_c$ are their pedal triangles WRT $\triangle ABC$. Lines AL_a, BL_b, CL_c concur at the isotomic conjugate $R \equiv X_{69}$ of H WRT $\triangle ABC$. Since

$$\frac{HH_b}{HH_c} = \frac{CH_b}{BH_c} = \frac{AL_b}{AL_c} \Longrightarrow AH \text{ is the A-symmedian of } \triangle AL_bL_c$$

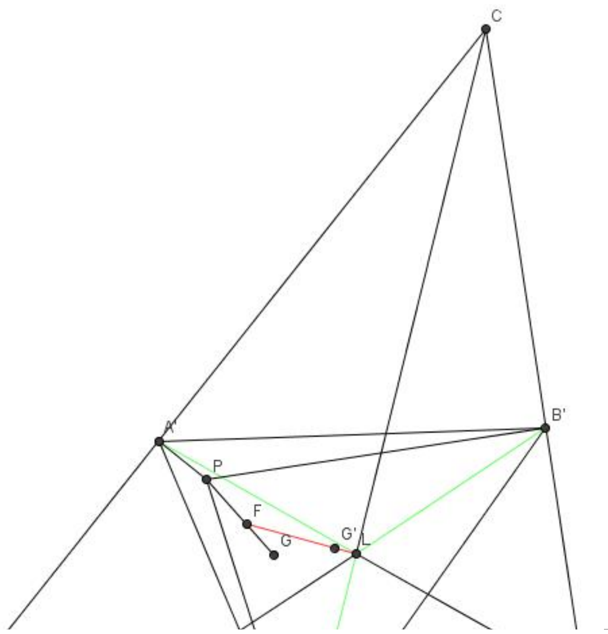
Thus, AO is the A-median of $\triangle AL_bL_c$. Likewise, BO, CO are the B- and C- medians of $\triangle BL_cL_a, \triangle CL_aL_b \Longrightarrow O$ is the isotomcomplement of R WRT $\triangle ABC$. Therefore, the anticevian triangle $\triangle A'B'C'$ of O is homothetic to the cevian triangle $\triangle L_aL_bL_c$ of $R \Longrightarrow$ Perpendiculars from A, B, C to $B'C', C'A', A'B$ concur at the isogonal conjugate X_{64} of X_{20} with respect to $\triangle ABC$.

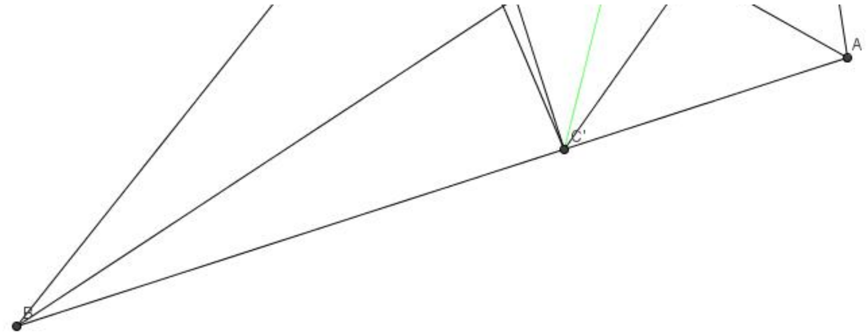
(b) My sketch gives $-\frac{2}{3}$, but I don't have a proof yet.

skytin
#3 Mar 27, 2011, 1:44 pm

Luis, try to prove (b) 🙂
Here is picture to problem (b)

Attachments:



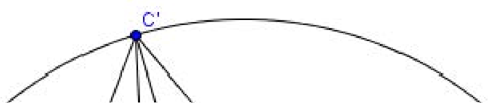


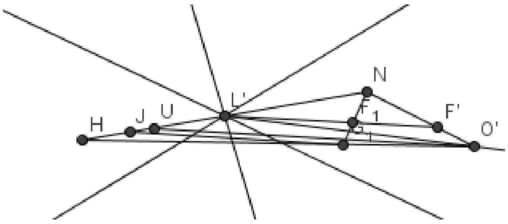
skytin

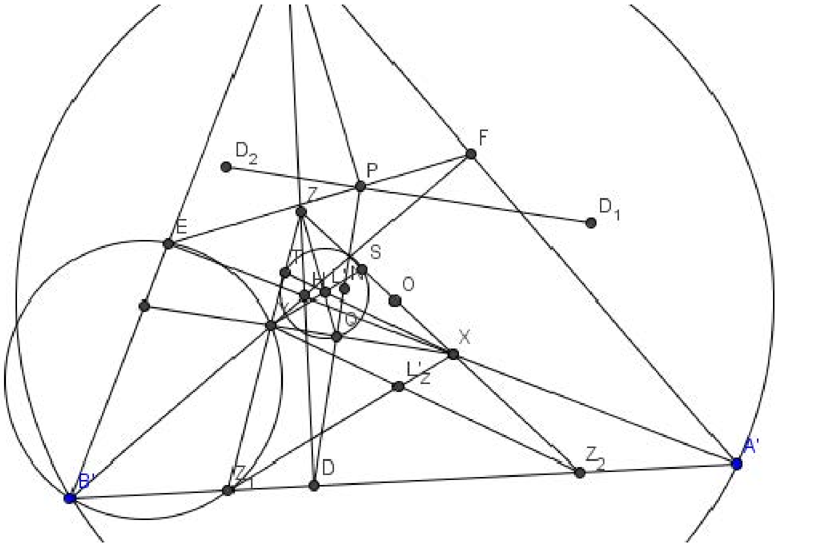
#4 Apr 23, 2011, 2:41 pm

Ok , my solution to a) and b)
(0) If given triangle A'B'C' then there exist only one triangle ABC were A'L B'L C'L goes thru A B C and A' B' C' are on BC , AC and AB
Let A'E DC' and B'F are altitudes of triangle A'B'C'
X , Y , Z are their midpoints (see picture)
Easy to see that triangle XDY ~ DB'E ~ A'DF ~ FEC' ~ A'B'C'
Let DQ is perpendicular from point D on XY
XDY ~ A'B'C' , so YQ/QX = B'D/DA'
Let PC' is perpendicular from C' on EF
Easy to see that DA'/B'D = EP/PF
Let make simmetry of point D wrt points X and Y and get points D_1 , D_2
Easy to see that DA = ED_1 and B'D = D_2F and B'A' || D_2F || ED_1 , so D_2F || ED_1 and D_2F/ED_1 = PF / EP , so P is on D_1D_2
And D_1P/PD_2 = XQ/QY , so Q is divides DP in two equal parts
Easy to see that QZ || PC'
If w is incircle wich is inscribed in medial triangle of DEF Not hard to prove that Q is tangent point of w and side of this triangle oposite to EF
QZ is perpedicular to this side , so QZ is goes thru center of w (point L')
Let TE and SF are perpendiculars from points E and F on YZ and XZ
Like the same lines XT , QZ and YS are intersected at point S'
Q , T and S are tangent points of w eith sides of medial triangle of DEF , so QTS is Homotetive to A'B'C' (TS make equal angles with sides DE and DF)
Use (0) , so XYZ is Homotetive to ABC
P is tangent point of Excircle DEF with side EF , so line DQ goes thru Nagel point of triangle DEF , like the same lines TE and SF goes thru this point , so lines DQ , TE and Sf meet at same point (a) done
Let lines XZ and YZ meets B'A' at points Z_1 an Z_2 (see picture)
Triangle YEZ ~ DB'E , so angle ZYE = Z_1B'E , so YZ_1B'E is cyclic , so angle YZ_1E = YB'E = FA'E and YEZ ~ FEC' , so YZ_1EZ ~ FA'EC'
So A'F/FC' = Z_1Y/YZ = XS/SZ , so Z_1X || YS
Like the same Z_2Y || XT
TS || B'A' , so if Z_2Y intersect Z_1X at point L'_Z then triangle Z_2Z_1L'_Z is Homotetive to L'TS , so Z_2L'_Z = Z_1L'_Z
YZ_1EZ ~ FA'EC' , so triangle A'ZZ_1E ~ ZEC' , so Z_1A'/ZC' = A'E/EC' like the same B'Z_2/ZC' = B'F/FC' and B'F/FC' = A'E/EC' , so Z_1A'/ZC' = B'Z_2/ZC' , so Z_1A' = B'Z_2, so B'Z_1 = Z_2A'
Let O is center of (A'B'C')
Easy to see that L'_ZO is perpendicular to B'A' and to TS
Let H is Orthocenter of triangle A'B'C'
Easy to see that is Isogonal Conjugacy of Nagel point of triangle DEF (point N) wrt triangle XYZ
Let U is midpoint of HL'
Let G_1 is centroid of triangle XYZ
Easy to see that XL'_ZYL' is Parallelogram , so if O' is midpoint of L'O then UO' is || to Euler line of A'B'C' and QTS
Let V is midpoint of XY then VO' || L'_ZO || HZ
Let make Homotety with center G_1 an k = -2
then easy to see that O' -> H
L' is midpoint of HN
Let Euler line intersect segment O'N at point F' then F'L' || O'U
Let L'F' intersect segment G_1N at point F_1
Let line parallel to F'L' from G_1 intersect HN at point J
HU/JU = HO'/G_1O' = 3 , so HJ = 2*JU and JL' = 4*JU and L'N = 6*JU , so G_1F_1/F_1N = JL'/L'N = 4/6 = 2/3 . (b) done

Attachments:







TelvCohl

#5 Nov 22, 2015, 10:00 pm • 2

Problem : Given a $\triangle ABC$ with circumcenter O. Let $\triangle A_1B_1C_1$ be the anticevian triangle of O WRT $\triangle ABC$. Let G, G_1 be the Centroid of $\triangle ABC, \triangle A_1B_1C_1$, respectively. Prove that **(a)** $\triangle ABC$ is a pedal triangle of P WRT $\triangle A_1B_1C_1$. **(b)** Let Q be the intersection of OG and PG_1, find the ratio $QP : QG_1$.

Proof :

Lemma : Given a $\triangle ABC$ and a point P. Let $\triangle DEF$ be the cevian triangle of P WRT $\triangle ABC$ and let D_1, E_1, F_1 be the midpoint of AD, BE, CF, resp. Let Q be the complement of P WRT $\triangle ABC$ and $\triangle XYZ$ be the anticevian triangle of Q WRT $\triangle ABC$. Then $\triangle D_1E_1F_1$ and $\triangle XYZ$ are homothetic.

Proof : Let R be the isotomic conjugate of P WRT $\triangle ABC$ and $\triangle R_aR_bR_c$ be the cevian triangle of R WRT $\triangle ABC$. Since E_1F_1 is the Newton line of the complete quadrilateral $\{\triangle ABC, EF\}$, so $E_1F_1 \parallel R_bR_c \parallel YZ$ ($\because Q$ is the isotomcomplement of R WRT $\triangle ABC$). Similarly, we can prove $F_1D_1 \parallel ZX$ and $D_1E_1 \parallel XY$, so $\triangle D_1E_1F_1$ and $\triangle XYZ$ are homothetic.

Back to the main problem :

I'll use some result proved at **[1]** Coaxal circles (post #5) and **[2]** Midpoint on Nagel line (Own) (post #4).

Let $\triangle DEF$ be the orthic triangle of $\triangle ABC$ and H be the orthocenter of $\triangle ABC$. Let X, Y, Z be the midpoint of AD, BE, CF, resp. Let N_a be the Nagel point of $\triangle DEF$ and let $\triangle D_1E_1F_1$ be the cevian triangle of N_a WRT $\triangle DEF$. Let D_2 be the intersection of DD_1 and YZ (define E_2 and F_2 similarly). Since X is the center of the spiral similarity of $EF_1 \mapsto E_1F$ (**[2]**), so notice $EF_1 = E_1F$ we get $\triangle XEF_1 \cong \triangle XE_1F \Longrightarrow XE = XE_1, XF = XF_1$. Similarly, we can prove $YF = YF_1, YD = YD_1$ and $ZD = ZD_1, ZE = ZE_1$, so YZ, ZX, XY is the perpendicular bisector of DD_1, EE_1, FF_1, respectively.

Since $\triangle D_2E_2F_2$ is the pedal triangle of N_a WRT $\triangle XYZ$, so E_2F_2, F_2D_2, D_2E_2 is perpendicular to AH, BH, CH, resp. ($\because H$ is the isogonal conjugate of N_a WRT $\triangle XYZ$ (**[1]**)), hence $\triangle D_2E_2F_2$ and $\triangle ABC$ are homothetic. On the other hand, from **Lemma** $\Longrightarrow \triangle XYZ$ and $\triangle A_1B_1C_1$ are homothetic, so we get $\triangle ABC \cup \triangle A_1B_1C_1$ and $\triangle D_2E_2F_2 \cup \triangle XYZ$ are homothetic **...** $(\bigstar) \Longrightarrow \triangle ABC$ is a pedal triangle of $\triangle A_1B_1C_1$ (**(a)** is proved).

Let K, G_2 be the Centroid of $\triangle XYZ, \triangle D_2E_2F_2$, respectively. From $(\bigstar) \Longrightarrow$ the circumcenter T of $\triangle D_2E_2F_2$ (midpoint of HN_a) lies on XD_2, YE_2 and ZF_2, so if $V \equiv TG_2 \cap N_aK$ then from $(\bigstar)$ we get $VN_a : VK = QP : QG_1$ **...** $(\star)$. Let $\mathbf{P}^K_{BC}, \mathbf{P}^K_{CA}, \mathbf{P}^K_{AB}$ be the perpendicular from K to BC, CA, AB, respectively. Let $\triangle M_aM_bM_c, \triangle M_dM_eM_f, \triangle M_xM_yM_z$ be the medial triangle of $\triangle ABC, \triangle DEF, \triangle XYZ$, respectively. Since T is the midpoint of the incenter H and the Nagel point N_a of $\triangle DEF$, so T is the Spieker point of $\triangle DEF \Longrightarrow TM_d \perp BC$. On the other hand, from ERIQ lemma we know M_x is the midpoint of M_aM_d, so if M is the midpoint of TO then $MM_x \perp BC \Longrightarrow -2 \cdot \text{dist}(M, \mathbf{P}^K_{BC}) = \text{dist}(H, \mathbf{P}^K_{BC})$.

Analogously, we can prove $-2 \cdot \text{dist}(M, \mathbf{P}^K_{CA}) = \text{dist}(H, \mathbf{P}^K_{CA})$ and $-2 \cdot \text{dist}(M, \mathbf{P}^K_{AB}) = \text{dist}(H, \mathbf{P}^K_{AB})$, so $KM : KH = 1 : -2 \Longrightarrow M$ is the complement of H WRT $\triangle XYZ$, hence combine $(\star)$ we conclude that

$$\frac{QP}{QG_1} = \frac{VN_a}{VK} = \frac{\text{dist}(N_a, TG_2)}{\text{dist}(K, TG_2)} = \frac{3}{2} \cdot \frac{\text{dist}(N_a, TG_2)}{\text{dist}(H, TG_2)} = -\frac{3}{2}$$

(**(b)** is proved).

Quick Reply

High School Olympiads

MP=MQ

Reply

jaydoubleuel
#1 Jan 26, 2011, 7:59 am

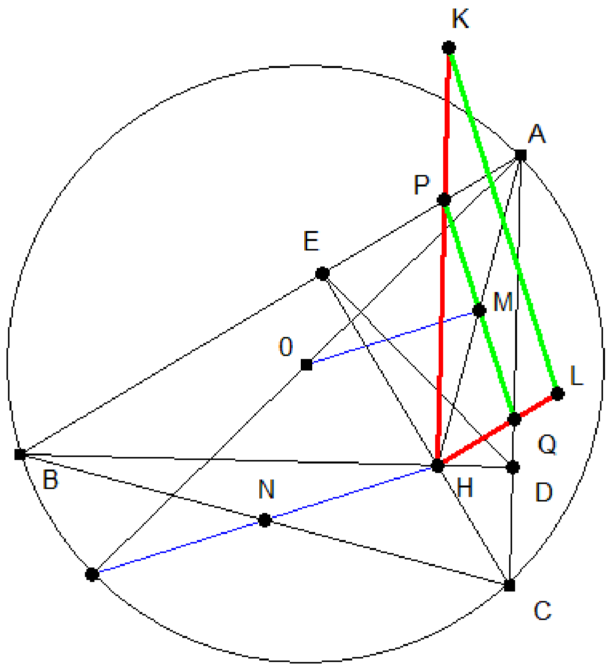
given a triangle $\triangle ABC$ with it's circumcenter O, orthocenter H,
M is the midpoint of AH and the perpendicular to OM at M meets AB, AC at P, Q
prove that $MP = MQ$

sunken rock
#2 Jan 26, 2011, 6:08 pm

Name $\{D\} \in AC \cap BH,\ \{E\} \in AB \cap CH$, N-midpoint of BC.
Take $HK \perp BH,\ KH = BH,\ \{P'\} \in [AB] \cap [KH],\ HL \perp CH, HL = CH, \{Q'\} \in [AC] \cap [HL]$; it is well known that $HN \perp KL$. As $KH \parallel AC,\ HL \parallel AB,\ AP'HQ'$ is a parallelogram and $P',\ M,\ Q'$ are collinear.
Now let's prove that $P'Q' \parallel KL$: $\triangle BHP' \sim \triangle CHQ'$, being right-angled and having $\angle HBA = \angle HCA$; from their similarity: $\frac{P'H}{BH} = \frac{Q'H}{CH} \iff \frac{P'H}{HK} = \frac{Q'H}{HL}$, or $P'Q' \parallel KL$.
With $NH \parallel OM$ we get $P'Q' \perp OM$, and $P' \equiv P,\ Q' \equiv Q$, done.

Best regards,
sunken rock

Attachments:



Luis González
#3 Mar 25, 2011, 11:06 pm

Parallels from H to AC, AB cut AB, AC at P', Q', i.e. $AP'HQ'$ is a paralellogram with diagonal intersection M.
$\angle HCA = \angle HBA$ and $\angle HP'B = \angle HQ'C = \angle BAC$ imply that $\triangle HP'B \sim \triangle HQ'C$. Hence:

$$\frac{HQ'}{\ } = \frac{P'A}{\ } = \frac{Q'C}{\ } \Longrightarrow \overline{P'A} \cdot \overline{P'B} = \overline{Q'C} \cdot \overline{Q'A}$$

$\overline{HP'} = \overline{Q'A} = \overline{P'B} \Longrightarrow P'A \cdot P'B = Q'C \cdot Q'A$

Therefore, P', Q' have equal power to the circumcircle (O) of $\triangle ABC \Longrightarrow OP' = OQ'$, i.e. OM is the perpendicular bisector of $\overline{P'Q'} \Longrightarrow P \equiv P'$ and $Q \equiv Q'$.

mathVNpro

#4 Mar 26, 2011, 4:09 am

> *jaydoubleuel wrote:*
>
> given a triangle $\triangle ABC$ with it's circumcenter O, orthocenter H,
> M is the midpoint of AH and the perpendicular to OM at M meets AB, AC at P, Q
> prove that $MP = MQ$

Let A' be the symmetry point of A with respect to O; then it is well-known result that HA' passes the midpoint M_a of BC. Now let Ax be a line through A which is parallel to PQ. Since $PQ \perp OM \Longrightarrow Ax$ also $\perp OM$. Further, since $OM \parallel A'H \equiv HM_a$. Hence $HM_a \perp Ax$. Let Hy be a line through H which is parallel to BC. We have $AM \equiv AH \perp Hx$, $AP \perp HC$, $AQ \perp HB$, and $HM_a \perp Ax$. Therefore, $(AP, AQ, AM, Ax) = (HC, HB, Hy, HM_a) = -1$, which implies that M is the midpoint of PQ. Our proof is completed then. $\square$

jgnr

#5 Mar 27, 2011, 7:35 am

Let D be the midpoint of BC, the line perpendicular to DH passing through H meets AB and AC at R and S respectively. It is well-known that $HR = HS$. Note that $MH = OD$ and $MH \parallel OD$, so $MHDO$ is a parallelogram, and $DH \parallel OM$. Thus $PQ \parallel RS$ and $\triangle APQ \sim \triangle ARS$. Since H is the midpoint of RS, then M is the midpoint of PQ.

Quick Reply

Site Support

Vertical alignment of latex formulas

Reply

Luis González
#1 Mar 22, 2011, 10:30 pm • 1

Vertical alignment of latex formulas is messed up in several threads after an odd behavior in the site on March 16th, 2011. I put a sample taken from different parts of the forum.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=150&t=394544
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=151&t=393842
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=151&t=394874
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=38&t=384521
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=383418
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=57&t=380720
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=67&t=385344
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=67&t=385187
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=296&t=395271

Edit: Now, the problem seems solved in their entirety.

Goutham
#2 Apr 30, 2011, 6:22 pm

And this: http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=404595

minsoens
#3 Jun 1, 2011, 11:33 pm

Also here and here

Goutham
#4 Jun 1, 2011, 11:36 pm

I have been noticing that the problem seems to hold only for google chrome as mentioned here.

minsoens
#5 Jun 1, 2011, 11:49 pm

I just checked Firefox and everything seems to be fine.

However, the problem persists with Google Chrome. I guess it has something to do with Latex formulas in quote blocks, for Chrome displays the Latex properly in all the topics mentioned by luisgeometria and Goutham but still not in the topics I mentioned.

Goutham
#6 Jun 1, 2011, 11:56 pm

There was a change in alignment in my link some time back. But it automatically got corrected. Also, note that the problem occurs only for a few topics. So, it is random but finite and may correct itself after sometime.

Quick Reply

High School Olympiads

OH=a-b

Reply

Mateescu Constantin
#1 Mar 22, 2011, 12:09 am

Let ABC be a triangle so that $A=96^\circ$ and $B=24^\circ$. Prove that : $\boxed{OH=a-b}$, where O and H represent the circumcenter and orthocenter of the given triangle.

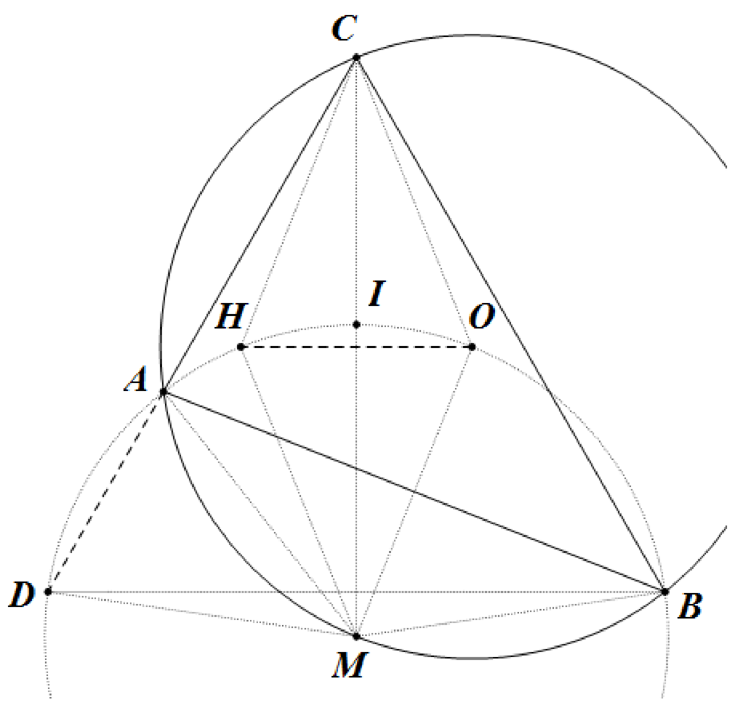
Luis González
#2 Mar 22, 2011, 2:17 am • 2

The relation is valid for any $\triangle ABC$ with $\angle C=60^\circ$. Since $\angle AHB=\angle AOB=120^\circ$ (mod 180), it follows that A,B,O,H are concyclic. $\odot(OAB)$ cuts CB again at E. Then $\triangle AEC$ is equilateral $\Longrightarrow$ $EB=|CB-CE|=|CB-CA|$. Since $\angle CEO=\angle OAB$ and $\angle OAB=\angle OBA=\angle HBC$, it follows that $OE \parallel BH \Longrightarrow OEBH$ is isosceles trapezoid with legs (or diagonals) $OH=EB=|CB-CA|$.

Headhunter
#3 Mar 22, 2011, 4:58 am • 2

$\angle C=60^\circ \implies A,H,I,O,B$ are cyclic , by angle chase. M is the midpoint of arc $\widehat{AB}$
$\angle C=60^\circ \implies \overrightarrow{OM}=\overrightarrow{CH}$ (***Servois***) and then $\square CHMO$ is rhombus. Reflect B in CM, which is D
$AD=a-b$, $MA=MB=MI=MH=MO=MD$ (***Mention***)
$\angle AMD=\angle DMC-\angle AMC=\angle A-\angle B=\angle HCO=\angle HMO$ by angle chase.
Thus, $\triangle MAD\cong\triangle MHO$ and then $OH=AD=a-b$

Attachments:



Luis González
#4 Mar 22, 2011, 8:48 pm • 3

However, there are other triangles (apart from those with <C=60) that satisfy OH=|a-b|.

$$OH^2 = 9R^2 - (a^2+b^2+c^2) \implies 9R^2 - (a^2+b^2+c^2) - (a-b)^2 = 0$$

$$\frac{9a^2b^2c^2}{(a+b+c)(b+c-a)(c+a+-b)(a+b-c)} - (a^2+b^2-c^2) - (a-b)^2 = 0$$

$$(a^2+b^2-c^2-ab)(2a^4+2b^4-4a^2b^2+3abc^2-a^2c^2-b^2c^2-c^4) = 0$$

Either $a^2+b^2-ab-c^2=0 \iff \angle C = 60^\circ$, or

$$c^4+c^2(a^2+b^2-3ab)-2(a^2-b^2)^2=0$$

Quick Reply

High School Olympiads

Tangent Line

Reply

Headhunter

#1 Mar 16, 2011, 1:35 pm

Hello.

$\triangle ABC$ is an acute isosceles triangle, where $AB = AC$
A point D is on AB , where $\triangle DBC$ is an acute triangle.
Let (O) be the incircle of $\triangle ADC$ and H be the orthocenter of $\triangle DBC$
(I) be the incircle of $\triangle HBC$ and (I) touch BC at E.
Show that IE is tangent to (O) , with ***no use of trigonometry***.

Luis González

#2 Mar 21, 2011, 6:17 am

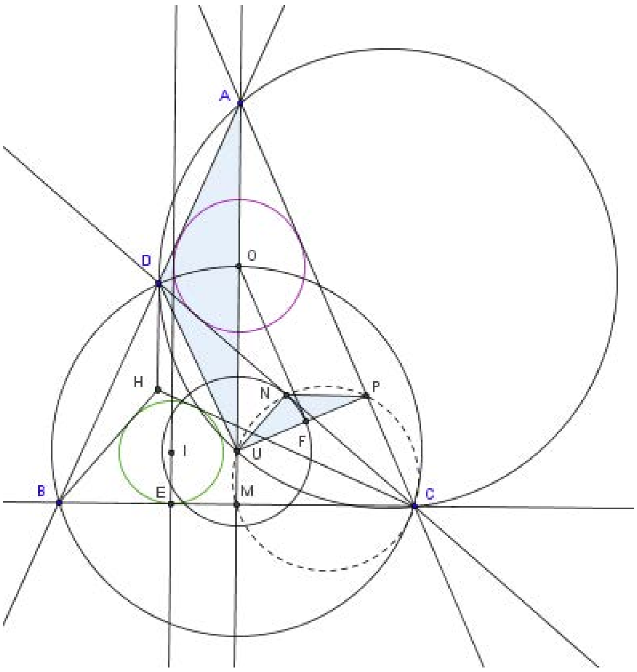
Let M, N be the midpoints of BC, DC and U the circumcenter of $\triangle BDC,$ i.e. midpoint of the arc DC of the circumcircle $\odot(ADC).$ Let P be the orthogonal projection of U onto $AC.$ Since quadrilateral $UNPC$ is cyclic on account of the right angles at $P, N,$ it follows that $\angle NUP = \angle NCP = \angle DUA$ and $\angle NPU = \angle NCU = \angle DAU \Longrightarrow \triangle UNP$ and $\triangle UDA$ are similar. If $\overline{UP}$ cuts (U, UN) at $F,$ then we have:

$$\frac{UN}{UP} = \frac{UF}{UP} = \frac{UD}{UA} = \frac{UO}{UA} \Longrightarrow OF \parallel AC.$$

Thus, tangent OF to (U, UN) is parallel to $AC.$ If r denotes the radius of $(O),$ we have

$r = \overline{UP} - \overline{UN} = \frac{1}{2}\overline{HC} - \frac{1}{2}\overline{HB} = \overline{EM} \Longrightarrow IE$ is tangent to $(O).$

Attachments:



Quick Reply

High School Olympiads

Three nice collinear points (own)

Reply

jayme

#1 Mar 18, 2011, 5:16 pm

Dear Mathlinkers,
ABC a triangle, A'B'C' the orthic triangle of ABC, (Ob), (Oc) the incircles of the resp. triangle BA'C', CA'B',
(T) the second external tangent of (Ob) and (Oc), DEF the contact triangle of ABC, D' the foot of the D-altitude of DEF,
U, V the points of contact of (Oc) wrt AC, (T).
Prove : U, V and D' are collinear.
Sincerely
Jean-Louis

vntbqpqh234

#2 Mar 18, 2011, 6:22 pm

can you explain that : what does that means?
the " orthic triangle" ? and "contact triangle"?

jayme

#3 Mar 18, 2011, 6:25 pm

orthic triangle, is the triangle determined by the feet of the three altitudes of ABC
conctac triangle, is the triangle determined by the feet of the perpendicular through the incenter on the sides of the triangle.
See also on Google...
Sincerely
Jean-Louis

Luis González

#4 Mar 19, 2011, 3:03 am

According to the topic Tangent and orthocenter, O_b, O_c are the orthocenters of $\triangle DBF, \triangle DCE,$ i.e. O_bO_c is antiparallel to BC with respect to IB, IC and the common external tangent τ of $(O_b), (O_b)$ (different from BC) passes through the orthocenter P of the intouch triangle $\triangle DEF$ (reflection of D across ObOc). Let $K \equiv ID \cap O_bO_c.$ From $PD' \perp O_bO_c,$ we get $\angle VPD' = \angle PDB = \angle IKO_c.$ Let (O_c) touch BC at $N.$ Then we have

$$\frac{DN}{PD'} = \frac{PV}{PD'} = \frac{ED \cdot \sin \frac{C}{2}}{ID \cdot \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{ID \cdot \cos \frac{C}{2} \sin \frac{C}{2}}{ID \cdot \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{\cos \frac{C}{2}}{\sin \frac{B}{2}} = \frac{KO_c}{KI}$$

Thus, $\triangle PVD' \sim \triangle KO_cI$ by SAS $\Longrightarrow \angle DD'V = \angle DIC = \angle DEU.$ But $EUDD'$ is cyclic on accout of the right angles $\angle EUD = \angle DD'E = 90^\circ,$ i.e. $\angle DD'U = \angle DEU \Longrightarrow$ Lines $D'V$ and $D'U$ are identical, i.e. D', U, V are collinear.

vntbqpqh234

#5 Mar 20, 2011, 6:08 am

That is my proof
Let (T) meets BC,CA,AB at P,M,N. have P,O_b,O_c collinear
Triangle $A'B'C$ similar ABC then $\frac{CU}{CD} = \frac{CB'}{CB} = cosC$
Hence D,O_c,U is collinear.Then $D'EUD$ is inscribed or $\angle AUD' = \angle D'DE = \angle(B/2)$(*)
Easy to see that triangle $A'O_bO_c$similar with triangle $A'BB'$ then
$\angle A'O_cO_b = \angle BAA' \Rightarrow \angle O_bO_cC + \angle CBO_b = \ pi \Rightarrow$ the $BCO_cO_b$is inscribed.
Then$\angle CPN = 2\angle CPO_b = \angle B - \angle C$ then $\angle PNB = ACB$
Have $NMCB$ is inscribed or $\angle NMC = \pi - \angle B$
Note that $\angle MUV = \frac{\pi - \angle NMC}{2}$ we have $\angle AUV = \angle(B/2)$(**)
When (*),(**) we have QED.

Quick Reply

High School Olympiads

The relation between area of a polygon and the pyramid

Source: IMO LongList 1982 - P13

Amir Hossein

#1 Mar 17, 2011, 1:22 am

A regular n-gonal truncated pyramid is circumscribed around a sphere. Denote the areas of the base and the lateral surfaces of the pyramid by S_1, S_2, and S, respectively. Let σ be the area of the polygon whose vertices are the tangential points of the sphere and the lateral faces of the pyramid. Prove that

$$\sigma S = 4S_1S_2\cos^2\frac{\pi}{n}.$$

Luis González

#2 Mar 18, 2011, 12:23 pm

$\mathcal{C}_1(r_1), \mathcal{C}_2(r_2)$ $(r_2 > r_1)$ are the incircles of the bases of the truncated pyramid. Let $\mathcal{C}_3(\varrho)$ be the circumcircle of the regular n-gon whose vertices are the tangency points of the subject sphere $\mathcal{E}$ with the lateral faces. Thus, $\mathcal{C}_1(r_1), \mathcal{C}_2(r_2)$ and $\mathcal{C}_3(\varrho)$ are obviously cross sections of a right cone with apex A circumscribed around $\mathcal{E}$. Arbitrary plane through the axis of the cone cuts $\mathcal{E}$ into a circle (I) and the bases $\mathcal{C}_2(r_2), \mathcal{C}_1(r_1)$, into the segments $BC = 2r_2, MN = 2r_1, (M \in AB$ and $N \in AC) \Longrightarrow$ $BCNM$ is an isosceles trapezoid with incircle (I). If (I) touches AB, AC at D, E, then $DE = 2\varrho, NE = r_1$ and $CE = r_2$. Therefore

$$DE = \frac{BC \cdot EN + MN \cdot EC}{NC} \Longrightarrow \varrho = \frac{2r_1r_2}{r_1 + r_2} \ (\star)$$

From the well-known formulae of the areas of regular polygons, we have:

$$\sigma = n\varrho^2\cos\frac{\pi}{n}\sin\frac{\pi}{n}$$

$$S_1 = n \cdot {r_1}^2\tan\frac{\pi}{n} \ , \ S_2 = n \cdot {r_2}^2\tan\frac{\pi}{2} \Longrightarrow S_1S_2 = n^2{r_1}^2{r_2}^2\tan^2\frac{\pi}{n}$$

Lateral faces of the truncated pyramid are congruent isosceles trapezoids with altitude $r_1 + r_2$, whose bases are the sides of the n-gons with incircles $\mathcal{C}_1(r_1), \mathcal{C}_2(r_2)$. Therefore

$$S = \frac{n}{2} \cdot (r_1 + r_2) \cdot \left(2r_1\tan\frac{\pi}{n} + 2r_2\tan\frac{\pi}{n}\right) = n(r_1 + r_2)^2\tan\frac{\pi}{n}$$

Substituting ϱ, r_1r_2 and $(r_1 + r_2)$ from the latter expressions into $(\star)$ yields:

$$\frac{\sigma}{n\cos\frac{\pi}{n}\sin\frac{\pi}{n}} = \frac{4S_1S_2}{n^2\tan^2\frac{\pi}{n}} \cdot \frac{n\tan\frac{\pi}{n}}{S} \Longrightarrow \sigma S = 4S_1S_2\cos^2\frac{\pi}{n}$$

High School Olympiads

Old and nice Geometry

Reply

vntbqpqh234

#1 Mar 16, 2011, 9:44 am

Triangle ABC is given. Points D and E are on line AB such that $D-A-B-E$, $AD=AC$ and $BE=BC$.Bisector of internal angles at A and B intersect BC,AC at P,Q, and circumcircle of ABC at M and N. Line which connects A with center of circumcircle of BME and the line which connects B with center of circumcircle of AND intersect at X. Prove that CX perpendicular PQ.

skytin

#2 Mar 17, 2011, 11:13 pm

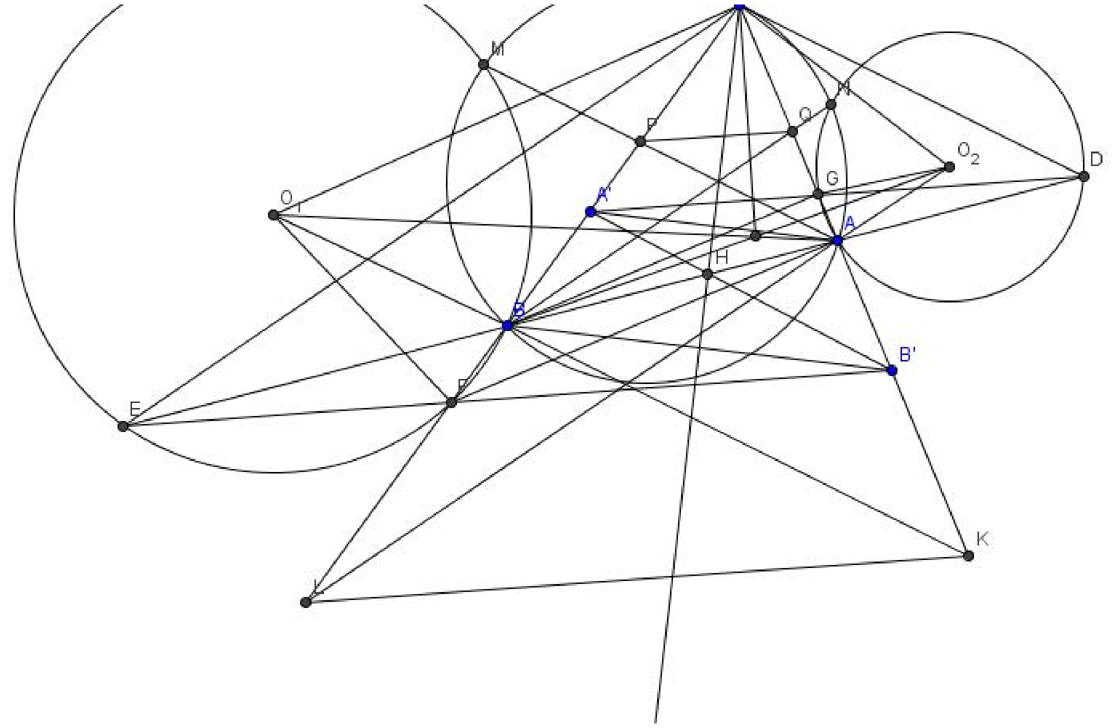
Let make simmetry of points B and A wrt bissector of angle BCA and get points B' and A' respectively
Easy to see that NC = AN , AC = AD and angle NCB = DAN , so angle NDA = NA'C , so DBA'N is cyclic , so N is Miquel point of lines AC , A'D , BA , BC , so let AC intersect A"D at point G , so G is on circle (DAN)
Like the same line EB" intersect (BEM) and BC at point F
BB' || AA' , B'A' , BA and bissector of angle BCA intersects at same point H
BH/HA = BC/AC = EB/AD , so EH/HD = BH/HA = B'H/HA' , so EB' || A'D
Use Lemma posed here :
http://jl.ayme.pagesperso-orange.fr/Docs/Le%20theoreme%20de%20Feuerbach-Ayme.pdf
Page 27
By Jean - Louis AYME

So FA || BG
So FC / BC = AC / GC
Easy to see that FO_1B ~ AGO_2 , so HC is angle bissector of O_1CO_2
Let line a goes thru point F and is || to height from C on A'G
Let line b goes thru point A and is || to height from C on A'G
Use Lemma posed here :
http://jl.ayme.pagesperso-orange.fr/Docs/Le%20theoreme%20de%20Jacobi.pdf
By Jean - Louis AYME

And also you can see my post here:
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=349056
So lines AO_1 , BO_2 and line thru C and || to a , b (height from C on A'G
) are intersects at one point
Let points L and K are on lines BC and AC and KA = LB = BA
Easy to see that KB || AP and LA || BQ
Use Lemma posed here :
http://jl.ayme.pagesperso-orange.fr/Docs/Le%20theoreme%20de%20Feuerbach-Ayme.pdf
Page 27
By Jean - Louis AYME

So LK || PQ
LB / A'C = BA/AC = BA/AD = KB/DC and KB || DC , so KLB ~ DA'C , so LK || A'D
So height from C on A'G is height frob C on PQ. Problem done

Attachments:



Luis González

#3 Mar 18, 2011, 1:25 am

Lemma. $\triangle ABC$ is acute with circumcenter O and 9-point center $N.$ D is the reflection of A across BC and P, Q are the circumcenters of $\triangle BOD, \triangle COD.$ Then $\angle QCB = \angle NAB$ and $\angle PBC = \angle NAC.$

Proof. Let H be the orthocenter of $\triangle ABC$ and O' the reflection O about $BC.$ Since $OO' = AH,$ then AO' bisects $OH,$ i.e. $N \in AO' \Longrightarrow DO$ is the reflection of AN about $BC \Longrightarrow \angle ODC = \angle NAC.$ Therefore, we have

$$\angle QCB = \angle QCO - \angle BCO = 90^\circ - \angle ODC - (90^\circ - \angle BAC) = \angle NAB \ \square$$

WLOG assume that $CA > CB.$ From $\angle NCD = \angle MCE = \frac{1}{2}(\angle B - \angle A)$ and $\frac{CN}{CM} = \frac{CD}{CE},$ it follows that $\triangle CND \sim \triangle CME$ $(\star)$ by SAS criterion. On the other hand, let U, V be the circumcenters of $\triangle DAN$ and $\triangle EBM.$ Lines CD, CE cut $(U), (V)$ again at $S, R.$ From $\angle CSN = \angle BAN = \angle CMR,$ it follows that $\triangle CSN \sim \triangle CMR.$ Further, since the isosceles $\triangle UNS$ and $\triangle VMR$ are also similar due to $(\star),$ then we deduce that the quadrilaterals $CSUN$ and $CMVR$ are similar $\Longrightarrow \angle UCN = \angle VCR,$ which implies that $\angle ACU = \angle BCV.$

Let I_a, I_b, I_c be the excenters of $\triangle ABC$ against $A, B, C.$ Circumcenter O of $\triangle ABC$ is 9-point center of $\triangle I_aI_bI_c.$ Using the previous lemma for $\triangle ACI_b, \triangle I_aCB$ with circumcenters N, M and reflections D, E of C across $AI_b, BI_a,$ we deduce that $\angle UAD = \angle OI_cA$ and $\angle VBE = \angle OI_cB,$ since $\triangle ACI_b \sim \triangle I_aCB \sim \triangle I_aI_cI_b.$ Therefore, reflections ℓ_a, ℓ_b of AU, BV about AI_b and BI_a are parallel to $OI_c.$ Now, since $\angle ACU = \angle BCV,$ by Jacobi's theorem we conclude that BU, AV and the parallel from C to $\ell_a \parallel \ell_b$ concur at $X \Longrightarrow CX \parallel OI_c,$ i.e. $CX \perp PQ.$

Quick Reply

High School Olympiads

Romania District Olympiad 2001 - Grade X

Reply

alex2008

#1 Mar 16, 2011, 8:10 pm • 1

Consider an inscriptible polygon $ABCDE$. Let H_1, H_2, H_3, H_4, H_5 be the orthocenters of the triangles ABC, BCD, CDE, DEA, EAB and let M_1, M_2, M_3, M_4, M_5 be the midpoints of DE, EA, AB, BC and CD, respectively. Prove that the lines $H_1M_1, H_2M_2, H_3M_3, H_4M_4, H_5M_5$ have a common point.

Dinu Serbanescu

Luis González

#2 Mar 16, 2011, 8:47 pm

Let (O) be the circumcircle of $ABCDE.$ Keeping in mind that, in any triangle the distance from the orthocenter to one vertex is twice the distance from the circumcenter to its opposite side, it follows that $AH_1 = DH_2 = 2OM_4.$ Since DH_2, AH_1 are perpendicular to $BC,$ then ADH_2H_1 is a parallelogram. Therefore, $H_1H_2 \parallel AD \parallel M_1M_2$ and $M_1M_2 = \frac{1}{2}H_1H_2.$ Similarly, $H_2H_3 \parallel M_2M_3$ and $M_2M_3 = \frac{1}{2}H_2H_3,$ etc. Hence, we conclude that pentagons $H_1H_2H_3H_4H_5$ and $M_1M_2M_3M_4M_5$ are homothetic with coefficient $-\frac{1}{2} \Longrightarrow$ Lines $H_1M_1, H_2M_2, H_3M_3, H_4M_4$ and H_5M_5 concur at their homothetic center.

Quick Reply

High School Olympiads

Concyclic points (own)

Reply

jayme
#1 Mar 15, 2011, 1:53 pm

Dear Mathlinkers,
ABC a triangle, A'B'C' the orthic triangle of ABC, (Ob), (Oc) the incircles of the resp. triangles BA'C', CA'B',
(T) the second common external tangent to (Ob) and (Oc),
M, N the points of intersection of T wrt AC, AB.
Prove : B, C, M and N are concyclic.
Sincerely
Jean-Louis

Luis González
#2 Mar 15, 2011, 8:14 pm

(I) touches BC, CA, AB at $X, Y, Z.$ From $\triangle ABC \sim \triangle A'BC' \sim \triangle A'B'C$ we get

$$\frac{BO_b}{BI} = \frac{A'C'}{AC} = \cos B \ , \ \frac{CO_c}{CI} = \frac{A'B'}{AB} = \cos C$$

Which implies that O_b and O_c are the orthocenters of $\triangle BXC$ and $\triangle CXY \Longrightarrow O_b, O_c$ are the reflections of I about the midpoints U, V of $XZ, XY,$ i.e. $UV \parallel O_bO_c.$ Since $BCVU$ is cyclic due to $IX^2 = IU \cdot IB = IV \cdot IC,$ then it follows that BCO_cO_b is also cyclic $\Longrightarrow \angle(O_bO_c, BC) = \frac{1}{2}(\angle B - \angle C) \Longrightarrow \angle(MN, BC) = \angle B - \angle C, (AC > AB).$ Hence, MN is antiparallel to $BC,$ i.e B, C, M, N are concyclic.

jayme
#3 Mar 15, 2011, 8:57 pm

Dear Luis and Mathlinkers,
thank for your intersting proof.
My proof which I have to write and put on my site is based on remarkable results which are already in some articles...This problem is a generalization of another one.
Sincerely, Dear Luis.
Jean-Louis

Quick Reply

High School Olympiads

Parabola And Segments

Reply

Headhunter

#1 Mar 15, 2011, 12:27 am

Hello.

Let α, β be two tangents of a parabola δ, which are symmetric to the axis of δ
Let γ be the tangent at the vertex of δ and consider an arbitrary tangent ζ of δ
ζ cut α, β, γ at A, B, C respectively. Show that $AC = BC$ by ***Only Euclidean*** geometric ways.

Luis González

#2 Mar 15, 2011, 1:57 am

Let $P \equiv \alpha \cap \beta$ and γ cuts α and β at $Q, R,$ respectively. Since the reflections of the focus F of δ about the tangents $\alpha, \beta, \gamma, \zeta$ lie on its directrix f, it follows that f is a Steiner line with pole F with respect to $\triangle PQR, \triangle QAC, \triangle PAB,$ i.e. F is the Miquel point of $\triangle PQR \cup \zeta.$ Since PF bisects $\angle APB$ (due to obvious symmetry) and $\angle FCA = \angle FQP = 90^\circ$, then it follows that FC is the perpendicular bisector of $AB,$ i.e. $\overline{CA} = -\overline{CB}.$

Alternate proof

Quick Reply

High School Olympiads

Line Through Centers

Reply

Headhunter

#1 Mar 14, 2011, 10:44 pm

Hello.

Let l be a line bisecting both of perimeter, area of triangle ABC
Let O,H,I,G be its circumcenter, orthocenter, incenter, centroid.
Does l pass through one of O,H,I,G certainly ?
No Coordinates system. I proved the existence of the bisecting line, but ...

Luis González

#2 Mar 14, 2011, 11:55 pm

If ℓ cuts AB, AC at $M,N,$ then $AM+AN=BM+CN+BC$ $(1).$ Let (I,r) be the incircle of $\triangle ABC$ and without loss of generality assume that I is inside $\triangle AMN.$ From $[\triangle AMN]=[\square BMNC],$ we get

$$[\triangle IAM]+[\triangle IAN]+[\triangle MIN]=[\triangle IBM]+[\triangle ICN]+[\triangle IBC]-[\triangle MIN]$$

$$2[\triangle MIN]+\tfrac{1}{2}r(AM+AN)=\tfrac{1}{2}r(BM+CN+BC) \ \ (2)$$

From (1) and $(2),$ it follows that $[\triangle MIN]=0 \Longrightarrow I \in \ell.$

Quick Reply

High School Olympiads

Ineq-G115 - Geometry

Reply

Ligouras

#1 Mar 14, 2011, 3:09 am

Let s be the semiperimeter o f an acute triangle ABC which is not equilateral, P a point inside ABC collinear with the incenter I and circumcenter O of ABC, and S, T, V projections of P on BC, CA and AB respectively.
Prove that

$$\frac{AV^2}{AB}+\frac{BS^2}{BC}+\frac{CT^2}{CA}\geq\frac{s}{2}$$

Luis González

#2 Mar 14, 2011, 3:56 am

Let $(x:y:z)$ be the normalized barycentric coordinates of P WRT $\triangle ABC$. Then we have

$$\overline{AV}=\frac{c^2y+zS_A}{c}\ ,\ \overline{BS}=\frac{a^2z+xS_B}{a}\ ,\ \overline{CT}=\frac{b^2x+yS_C}{b}$$

$$\overline{AV}+\overline{BS}+\overline{CT}=\frac{c^2y+zS_A}{c}+\frac{a^2z+xS_B}{a}+\frac{b^2x+yS_C}{b}$$

Comparing with the equation of IO, we get $\overline{AV}+\overline{BS}+\overline{CT}=\frac{a+b+c}{2}=s$

Now, from Cauchy-Schwartz inequality we deduce that

$$\frac{AV^2}{AB}+\frac{BS^2}{BC}+\frac{CT^2}{CA}\geq\frac{(AV+BS+CT)^2}{AB+BC+CA}=\frac{s}{2}$$

Quick Reply

High School Olympiads

Collinear points..

Reply

gilbert

#1 Mar 9, 2011, 8:37 pm • 1

Given quadrilateral $ABCD$ (BC isn't parallel to AD). O - the intersection point of the diagonals. S - the intersection point of AD and BC. There are points P and K on the AC and BD respectively such that:
$\frac{AP}{PC}=\frac{BO}{OD}$ and $\frac{DK}{KB}=\frac{CO}{OA}$.
Prove that points S,P,K are collinear.

This post has been edited 1 time. Last edited by gilbert, Mar 13, 2011, 8:14 pm

xeroxia

#2 Mar 13, 2011, 5:17 am

sketchpad says the problem is wrong.

vittasko

#3 Mar 13, 2011, 4:25 pm

I agree with **xeroxia**, that something is not correct in the notations of the propoded problem.

Kostas Vittas.

gilbert

#4 Mar 13, 2011, 8:18 pm

Yes, , the order of points is wrong (B and C mixed up). Now I have edited.

Luis González

#5 Mar 13, 2011, 11:52 pm • 2

SP cuts AB,DB at U,K'. By Menelaus' theorem for $\triangle BAD, \triangle ABC$ cut by AP

$$\frac{SA}{SD} \cdot \frac{DK'}{K'B} \cdot \frac{BU}{UA}=1 \ , \ \frac{SB}{SC} \cdot \frac{CP}{PA} \cdot \frac{AU}{UB}=1$$

$$\Longrightarrow \ \frac{SA}{SD} \cdot \frac{DK'}{K'B} \cdot \frac{SB}{SC} \cdot \frac{DO}{OB}=1. \ \ (1)$$

On the other hand, by Menelaus' theorem for $\triangle SBD$ and $\triangle SAC$ cut by AC,BD

$$\frac{CB}{SC} \cdot \frac{SA}{AD} \cdot \frac{DO}{OB}=1. \ \ (2) \ , \ \frac{DA}{DS} \cdot \frac{SB}{BC} \cdot \frac{CO}{OA}=1. \ \ (3)$$

Combining $(1),(2)$ and (3) yields $\frac{DK'}{K'B}=\frac{CO}{OA} \Longrightarrow K \equiv K'$, i.e. S,P,K are collinear.

Quick Reply

High School Olympiads

Own problem about triangle's altitude

truongtansang89

#1 Mar 13, 2011, 5:19 am

Let triangle ABC with circumcircle (O) with 3 altitudes AD, BE and CF. H is

the orthocenter of triangle ABC. I is the midpoint of AH and K is the midpoint

of EF. $CK \cap AH = L$ and $CI \cap EF = V$.

Prove that $VL \parallel BC$

Luis González

#2 Mar 13, 2011, 6:52 am • 1

Let $P \equiv AH \cap EF$ and L' is the orthogonal projection of V onto $AH.$ Since BP is the polar of C with respect to the circle with diameter $AH,$ it follows that $BP \perp CI,$ i.e. BP, IK, VL' concur at the orthocenter U of $\triangle PIV.$ But $P(I,V,U,C) = B(A,E,P,C) = -1$ implies that C, K, L' are collinear $\Longrightarrow L \equiv L',$ i.e. $VL \parallel BC.$

P.S. The result is still true if H is an arbitrary point on the plane ABC. For instance, if H is inside ABC then there exists a parallel projection taking ABC into an acute triangle A'BC whose orthocenter is the image of H.

High School Olympiads

Prove that <HMA=<GNS

Source: Serbia NMO 2010 problem 2

Goutham

#1 Mar 11, 2011, 12:46 pm • 1

In an acute-angled triangle ABC, M is the midpoint of side BC, and D, E and F the feet of the altitudes from A, B and C, respectively. Let H be the orthocenter of ΔABC, S the midpoint of AH, and G the intersection of FE and AH. If N is the intersection of the median AM and the circumcircle of ΔBCH, prove that $\angle HMA = \angle GNS$.

Proposed by Marko Djikic

Luis González

#2 Mar 11, 2011, 9:58 pm • 1

Inversion with center A and power $\overline{AH} \cdot \overline{AD}$ takes $\odot(AEF)$ into the sideline BC and $\odot(BHC)$ into the 9-point circle $\odot(DEF) \Longrightarrow M$ is the inverse of the second intersection of $\odot(AEF)$ and $\odot(BHC)$, i.e. $N \in \odot(AEF) \Longrightarrow \angle HNA$ is right. From the harmonic cross ratio (G, D, H, A), it follows that NA, NH bisect $\angle DNG$ externally and internally $\Longrightarrow$ $\odot(AEF)$ is N-Apollonius circle of $\triangle GND \Longrightarrow \odot(AEF) \perp \odot(GND)$. Hence, $\angle GNS = \angle HDN = \angle HMA$.

Virgil Nicula

#3 Mar 11, 2011, 11:44 pm • 2

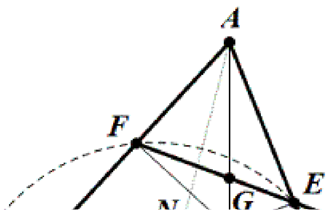
Denote $L \in EF \cap BC$. From an well-known property obtain that $LH \perp AM$. Denote $N' \in LH \cap AM$. Therefore, $AF \cdot AB = AH \cdot AD = AN' \cdot AM \implies AF \cdot AB = AN' \cdot AM \implies BFN'M$ is cyclically $\implies m\left(\widehat{BN'M}\right) = m\left(\widehat{BFM}\right) = B$. Also, $AE \cdot AC = AH \cdot AD = AN' \cdot AM \implies$ $AE \cdot AC = AN' \cdot AM \implies CEN'M$ is cyclically $\implies m\left(\widehat{CN'M}\right) = m\left(\widehat{CEM}\right) = C$. Therefore, $m\left(\widehat{BN'C}\right) = m\left(\widehat{BN'M}\right) + m\left(\widehat{CN'M}\right) = B + C = 180^\circ - A \implies BHN'C$ is cyclically $\implies N' \equiv N$. Since the division (A, G, H, D) is harmonically and $NH \perp NA$ obtain that $\widehat{HNG} \equiv \widehat{HND}$. Since the quadrilateral $DHNM$ is cyclically obtain that $\widehat{HND} \equiv \widehat{HMD}$. Thus, $\widehat{HNG} \equiv \widehat{HMD}$. In conclusion, $m\left(\widehat{HMA}\right) = 90^\circ - m\left(\widehat{DAM}\right) -$ $m\left(\widehat{DMH}\right) = 90^\circ - m\left(\widehat{ANS}\right) - m\left(\widehat{HND}\right) = 90^\circ - m\left(\widehat{ANS}\right) - m\left(\widehat{HNG}\right) = m\left(\widehat{GNS}\right) \implies \widehat{HMA} \equiv$ $\widehat{GNS}$. **Nice problem ! Thank you. See** the problem **PP6** from **here**.

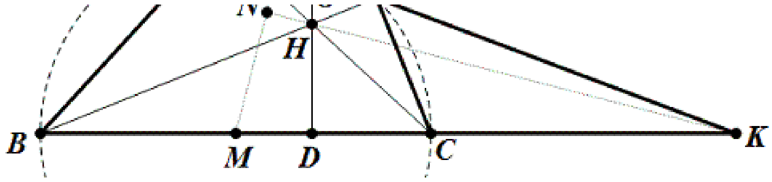
Headhunter

#4 Mar 12, 2011, 9:07 am • 2

$\square BCEF$ is cyclic. M is the orthocenter of $\triangle AHK$ (***Brocard***). $HK \cap AM = N$
Since (K,D/C,B)$= -1$ and $\square NMDH$ is cyclic , $KH.KN = KD.KM = KC.KB$
Then $\square BCHN$ is cyclic. N is that of the problem.
$\angle GNH = \angle HND$ since (D,G/H,A)$= -1$ and $\angle HNA = \pi/2$
Since $SN = SH, \angle SNH = \angle SHN$. $\angle SNH = \angle SNG + \angle GNH$ and
$\angle SHN = \angle HND + \angle HDN = \angle HND + \angle HMA$. Thus, $\angle SNG = \angle HMA$

Attachments:





littletush

#5 Nov 6, 2011, 11:09 am

let A' be the reflection of A in respect of M.
then B,H,C,A' cyclic
and HA' is the diameter.
hence $\angle HNM = 90$,obtaining B,H,N,C;F,H,N,E cyclic.
hence FE,HN,BC concurrent(at T).
and $\angle HNS = 90 - \angle SNA = 90 - \angle SAN = \angle AMD$
so it siffices to prove
$\angle GNH = \angle HMB$.
let Q be midpoint of EF
by MF=ME,we get $\angle MQG = 90$
yielding G,D,M,Q cyclic
then$TG * TQ = TD * TM = TH * TM$
G,Q,N,H cyclic
hence $\angle GNH = \angle HQF = \angle HMB$
QED

nawaites

#6 Jul 3, 2014, 1:14 am

Anyone give reason for $\angle HQF = \angle HMB$????

nawaites

#7 Jul 4, 2014, 12:42 pm

any help for little tush solution???????

Quick Reply

High School Olympiads

Concurrency

Reply

mastermindh

#1 Mar 11, 2011, 5:36 am

Let P be a point inside a triangle ABC such that
APB - ACB = APC - ABC: (these are angels)

Let D and E be the incenters of the triangles APB, APC respectively.
Show that AP, BD and CE meet at a point.

Please post as easy solution as possible...

truongtansang89

#2 Mar 11, 2011, 6:22 am

> *mastermindh wrote:*
> Let P be a point inside a triangle ABC such that
> APB - ACB = APC - ABC: (these are angels)
>
> Let D and E be the incenters of the triangles APB, APC respectively.
> Show that AP, BD and CE meet at a point.
>
> Please post as easy solution as possible...

Note that :

$\widehat{APC} = \widehat{ABC} + \widehat{BAP} + \widehat{BCP}$

$\widehat{APB} = \widehat{ACB} + \widehat{CAP} + \widehat{CBP}$

From $\widehat{APB} - \widehat{ACB} = \widehat{APC} - \widehat{ABC}$

$\Rightarrow \widehat{BAP} + \widehat{BCP} = \widehat{CAP} + \widehat{CBP}$

So, when we draw AP intersect (ABC) at K then $\widehat{KCP} = \widehat{KBP}$

Which gives us that (BPK) and (CPK) have the same circumcircle's radius.

Hence, $\dfrac{BP}{\sin \widehat{BKP}} = \dfrac{CP}{\sin \widehat{CKP}}$

$\Rightarrow \dfrac{BP}{CP} = \dfrac{\sin ACB}{\sin ABC} = \dfrac{AB}{AC}$

$\Rightarrow \dfrac{AB}{BP} = \dfrac{AC}{CP}$

The result is quite obvious then.

mavropnevma

#3 Mar 11, 2011, 6:30 am

Probably trigonometric Ceva's theorem may be of help.

Luis González

#4 Mar 11, 2011, 9:31 am

Let X, Y, Z be the orthogonal projections of P onto $BC, CA, AB.$ Then

$$\angle APB = \angle ACB + \angle XZY \ , \ \angle APC = \angle ABC + \angle XYZ$$

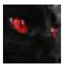
$$\angle APB - \angle ACB = \angle APC - \angle ABC \Longrightarrow \ \angle XZY = \angle XYZ$$

Therefore, pedal triangle $\triangle XYZ$ of P with respect to $\triangle ABC$ is isosceles with apex $X \Longrightarrow P$ lies on the A-Apollonius circle of $\triangle ABC \Longrightarrow \frac{AC}{PC} = \frac{AB}{PB}.$ By angle bisector theorem, it follows that bisectors BD, CE of $\angle ABP$ and $\angle ACP$ cut AP at the same point. For alternate proofs, see IMO 96 question #2.

Quick Reply

High School Olympiads

hard perpendicularity problem



paul1703

#1 Mar 10, 2011, 4:51 pm

Let ABC be a triangle with incenter I.Points M and N are the midpoints of AB and AC . And let D and E lie on AB and AC such that BD=CE=BC both on the same side of BC. line l1 passes through D and is perpendicular on IM. line l2 is defined analogously let P be the interesection of l1 and l2. prove that AP is perpendicular to BC.

hint

Luis González

#2 Mar 10, 2011, 10:53 pm

Carnot's theorem works well. Incircle (I,r) of $\triangle ABC$ touches the sides BC, CA, AB at $X, Y, Z,$ respectively. Then, we have $EY=BX$ and $DZ=CX,$ thus

$$EI^2=r^2+EY^2=r^2+(s-b)^2 \ \ , \ \ DI^2=r^2+DZ^2=r^2+(s-c)^2 \ \Longrightarrow$$

$$EI^2-EN^2+AN^2-AM^2+DM^2-DI^2=r^2+(s-b)^2-(a-\tfrac{1}{2}b)^2+\tfrac{1}{4}b^2-\tfrac{1}{4}c^2+(a-\tfrac{1}{2}c)^2-r^2-(s-c)^2=0$$

Perpendiculars dropped from E, D, A to IN, IM, MN are concurrent at P.

High School Olympiads

a problem about a circle

Reply

PTMINH

#1 Mar 10, 2011, 6:08 pm • 1

Given circle (O) with diameter AB. I is fixed on the segment AB, a line d is tangent to (O) at A. A line l through I intersects (O) at M, N (l is not fixed). Lines BM, BN intersect d at E, F respectively. K is intersection of two tangents to (O) which through E, F.
i) Prove that $AE.AF$ is constant
ii) Prove that K runs on a fixed line

mousavi

#2 Mar 10, 2011, 8:26 pm

i) $\frac{AE}{AB} = \frac{MA}{MB}$ and $\frac{AF}{AB} = \frac{AN}{BN}$

$\Longrightarrow AF.AE = (AB)^2.\frac{MA}{BN}.\frac{AN}{MB} = (AB)^2.\frac{AI}{IN}.\frac{IN}{IB} = (AB)^2.\frac{AI}{IB} = const$

Luis González

#3 Mar 10, 2011, 9:24 pm • 1

The results are true for any central conic $\mathcal{H}$ with vertices A, B and a fixed point I on its focal axis. For the first part, see reply #4 on the topic Constant product. Image of $\mathcal{H}$ under the homology $\mathcal{B}$ with center B and axis d is its pedal circle (O). Tangents from E, F to $\mathcal{H}$ (different from d) touch $\mathcal{H}$ at X, Y and BX, BY cut (O) again at the homologous X', Y' of X, Y. Since E, F are double points and tangencies are preserved under $\mathcal{B}$, it follows that EX' and FY' are tangent to $(O) \Longrightarrow K' \equiv EX' \cap FY'$ is the image of $K \equiv EX \cap FY$. Now, using the result of the topic locus for (O), the fixed tangent d through A and E, F varying along d such that $\overline{AE} \cdot \overline{AF} = \text{const}$, we conclude that locus of K' is a line ℓ' parallel to $d \Longrightarrow$ Locus of K is another line ℓ parallel to d i.e. image of ℓ' under $\mathcal{B}$.

Virgil Nicula

#4 Mar 10, 2011, 11:18 pm • 1

Using the second relation from **here** for $x := r$ obtain that in any triangle ABC exists the identity

$h_a = \frac{2r(s-b)(s-c)}{|r^2-(s-b)(s-c)|}$, where $2s = a+b+c$. Apply it to $\triangle EKF$ and obtain that

$h_k = \delta_{EF}(K)$ - distance of K to EF is constant because $(s-b)(s-c) := AE \cdot AF$ and $r := OA$.

In conclusion, the locus of K is a parallel line to d.

Similar proposed problem. Given are a circle w with diameter $[AB]$ and a fixed point $I \in AB$ so that $B \in (AI)$. Denote the line $d \equiv AA$ which is tangent to w at $A \in w$. For a mobile line l which pass through I denote $\{M, N\} = l \cap w$, $E \in BM \cap d$, $F \in BN \cap d$ and the intersection K of the tangents from E, F to w (different from d). Prove that $AE \cdot AF$ is constant and the locus of K is a fixed line (parallel with the fixed line d).

Indication. You can use the first relation from **here** for $x := r_b$ or $x := r_c$.

These problems appear and **here** (second and third).

Headhunter

#5 Mar 11, 2011, 1:45 am

Virgil Nicula , your relation is really amazing and so useful.

Virgil Nicula

#6 Mar 11, 2011, 1:55 am

Thanks. I used these "geometrical equations" for some old problems of this site. I don't remember where are.

Quick Reply

High School Olympiads

AK,BL,CN are concurrent

Reply

aspava
#1 Mar 7, 2011, 9:20 pm

Let the incentre of a triangle ABC be I.And construct equilateral triangles BCD,CAE,ABF on the sides.Call the intersection of DI,EI,FI with the sides is K,L,M.Show that AK,BL,CM concurrent

Luis González
#2 Mar 8, 2011, 1:32 am

If D,E,F are three points constructed outside $\triangle ABC,$ such that $\angle FAB=\angle EAC=\theta_a, \angle DBC=\angle FBA=\theta_b$ and $\angle ECA=\angle DCB=\theta_c,$ then the result is still true. We'll use barycentric coordinates with respect to $\triangle ABC.$ By Conway's notation, the coordinates of D,E,F are given by:

$$D \equiv (-a^2 : S_C+S_{\theta_c} : S_B+S_{\theta_b})$$

$$E \equiv (S_C+S_{\theta_c} : -b^2 : S_A+S_{\theta_a})$$

$$F \equiv (S_B+S_{\theta_b} : S_A+S_{\theta_a} : -c^2)$$

$$ID : \begin{pmatrix} -a^2 & S_C+S_{\theta_c} & S_B+S_{\theta_b} \\ a & b & c \\ x & y & z \end{pmatrix} = 0$$

$$\Longrightarrow \ K \equiv ID \cap BC \equiv \left(0 : \frac{1}{S_B+S_{\theta_b}+ca} : \frac{1}{S_C+S_{\theta_c}+ab}\right)$$

By exchange of elements we find the coordinates of the points L,M and we conclude that lines AK,BL,CM concur at a point U with barycentric coordinates

$$U\left(\frac{1}{S_A+S_{\theta_a}+bc} : \frac{1}{S_B+S_{\theta_b}+ca} : \frac{1}{S_C+S_{\theta_c}+ab}\right)$$

darij grinberg
#3 Mar 8, 2011, 10:56 pm • 2

More generally: Let ABC be a triangle. Let X, Y, Z be three points such that $\measuredangle XBC = \measuredangle ABZ$, $\measuredangle YCA = \measuredangle BCX$ and $\measuredangle ZAB = \measuredangle CAY$ (where all angles are directed angles modulo 180°). Let X', Y', Z' be three points such that $\measuredangle X'BC = \measuredangle ABZ'$, $\measuredangle Y'CA = \measuredangle BCX'$ and $\measuredangle Z'AB = \measuredangle CAY'$ (where all angles are directed angles modulo 180°).
Let the line XX' intersect the line BC at a point K, and similarly define two points L and M.
Then, the lines AK, BL and CM concur.

(Source: Hans Walser: Ein Schnittpunktsatz, Praxis der Mathematik 2/1991 pp. 70-71.)

For the proof, compute $\frac{BK}{KC}$ and the two analogous ratios using the sine law, and apply Ceva.

Luis González
#4 Mar 9, 2011, 2:07 am

Assume that quadrilaterals $BXCX',CYAY'$ and $AZBZ'$ are convex, the remaining configurations are treated analogously with appropriate signs. Using the relation found in the topic Prove the fraction on cotangents for $BXCX',CYAY'$ and $AZBZ'$ with diagonal intersections $K=BC \cap XX'$, $L=CA \cap YY'$ and $M=AB \cap ZZ'$, we get

$AZBZ'$ with diagonal intersections $K = BC \cap XX'$, $L = CA \cap YY'$ and $M = AB \cap ZZ'$, we get

$$\frac{\overline{BK}}{\overline{KC}} = \frac{\cot \widehat{CBX} + \cot \widehat{CBX'}}{\cot \widehat{BCX} + \cot \widehat{BCX'}} \quad (1)$$

$$\frac{\overline{CL}}{\overline{LA}} = \frac{\cot \widehat{BCX} + \cot \widehat{BCX'}}{\cot \widehat{CAY} + \cot \widehat{CAY'}} \quad (2)$$

$$\frac{\overline{AM}}{\overline{MB}} = \frac{\cot \widehat{CAY} + \cot \widehat{CAY'}}{\cot \widehat{CBX} + \cot \widehat{CBX'}} \quad (3)$$

Multiplying $(1), (2), (3)$ yields $\frac{\overline{BK}}{\overline{KC}} \cdot \frac{\overline{CL}}{\overline{LA}} \cdot \frac{\overline{AM}}{\overline{MB}} = 1 \Longrightarrow AK, BL, CM$ concur.

Quick Reply

High School Olympiads

aspava
#1 Mar 7, 2011, 6:58 pm

Let the center of the external circles of a triangle ABC be I_A, I_B, I_C.And construct a equilateral triangles BCD, CAE, ABF on the sides. Prove that DI_A, EI_B, FI_C concurrent

Luis González
#2 Mar 7, 2011, 8:05 pm • 2

Assume that equilateral triangles $\triangle BCD, \triangle CAE, \triangle ABF$ are constructed outside of $\triangle ABC.$ The remaining configuration is treated analogously. Construct equilateral triangles $\triangle I_BI_CD', \triangle I_CI_AE'$ and $\triangle I_AI_BF'$ inside of $\triangle I_AI_BI_C.$ Since BC and I_BI_C are antiparallel WRT $I_AB, I_AC,$ there exists a composition of an axial symmetry across the angle bisector of $\angle I_BI_AI_C$ and a homothety centered at I_A taking BC into $I_BI_C.$ Thus D is taken into $D' \Longrightarrow I_AD'$ is the isogonal of I_AD WRT $\angle I_BI_AI_C.$ Similarly, I_BE' and I_CF' are the isogonals of I_BE and I_CF meeting at the 2nd Fermat point of $\triangle I_AI_BI_C$ $\Longrightarrow I_AD, I_BE, I_CF$ concur at the 2nd Isodynamic point of $\triangle I_AI_BI_C.$

P.S. The same argument can be used for the configuration of similar isosceles triangles.

aspava
#3 Mar 7, 2011, 9:11 pm

is this best you can do?Can't you find easier solution

nsato
#4 Mar 7, 2011, 9:31 pm • 2

aspava, that is an extremely rude and ungrateful response. Luis has taken the time to solve for your problem. You should not criticize if you cannot even solve the problem yourself.

> *aspava wrote:*
> is this best you can do?Can't you find easier solution

aspava
#5 Mar 7, 2011, 9:39 pm

ı know and ı thank to him for this great solutıon, but ı believe that he can find nice solution am ı wrong?

darij grinberg
#6 Mar 8, 2011, 10:55 pm • 1

More generally: Let I_a, I_b, I_c be the centers of the excircles of a triangle ABC. Let X, Y, Z be three points such that $\measuredangle XBC = \measuredangle ABZ$, $\measuredangle YCA = \measuredangle BCX$ and $\measuredangle ZAB = \measuredangle CAY$ (where all angles are directed angles modulo 180°). Then, the lines I_aX, I_bY, I_cZ concur.

The proof is easy using the trigonometric form of Ceva's theorem.

Luis González
#7 Mar 9, 2011, 12:20 am

Construct point X' such that $\triangle I_bI_cX' \sim \triangle BCX,$ angles $\angle I_cX'I_b$ and $\angle BXC$ are directly congruent. Points Y', Z' are defined similarly. Since BC and I_bI_c are antiparallel WRT $I_aI_b, I_aI_c,$ then there exists the composition of an axial symmetry across the angle bisector of $\angle I_bI_aI_c$ and a homothety centered at A taking $\triangle BXC$ into $\triangle I_bI_cX' \Longrightarrow I_aX, I_aX'$ are isogonals WRT $\angle I_bI_aI_c.$ Likewise, I_bY', I_cZ' are the isogonals of $I_bY, I_cZ.$ By Jacobi's theorem in $\triangle I_aI_bI_c,$ the lines I_aX', I_bY', I_cZ' concur at a point $J \Longrightarrow I_aX, I_bY, I_cZ$ concur at the isogonal conjugate of J WRT $\triangle I_aI_bI_c,$ as desired.

Quick Reply

High School Olympiads

mop 2010

Reply

aspava

#1 Mar 8, 2011, 8:38 pm

Let the center of the external circles of a triangle ABC be D, E, F and for a chosen point P in the interior of triangle, if the incenters of BPC, CPA, APB called I_1, I_2, I_3. Show that the lines DI_1, EI_2, FI_3 have a common point

Luis González

#2 Mar 8, 2011, 11:13 pm

$(u : v : w)$ are the normalized barycentric coordinates of P with respect to $\triangle ABC$, this is $u + v + w = 1$. For the sake of ease when handling computations, we denote

$$|\overline{PA}| = \delta_A = \sqrt{2S_A vw + c^2v^2 + b^2w^2}$$

$$|\overline{PB}| = \delta_B = \sqrt{2S_B wu + a^2w + c^2u^2}$$

$$|\overline{PC}| = \delta_C = \sqrt{2S_C uv + b^2u^2 + a^2v^2}$$

Barycentric coordinates of I_1, I_2, I_3 with respect to $\triangle ABC$ are then

$$I_1 \equiv a \cdot (u : v : w) + \delta_C \cdot (0 : 1 : 0) + \delta_B \cdot (0 : 0 : 1) \equiv (au : av + \delta_C : aw + \delta_B)$$

$$I_2 \equiv (bu + \delta_C : bv : bw + \delta_A) \ , \ I_3 \equiv (cu + \delta_B : cv + \delta_A : cw)$$

Since $D(-a : b : c)$, $E(a : -b : c)$, $F(a : b : -c)$, then the barycentrics coordinates of DI_1, EI_2, FI_3 are then

$$DI_1 : [abw + b\delta_B - acv - c\delta_C \ , \ acu + a^2w + a\delta_B \ , \ -abu - a^2v - a\delta_C]$$

$$EI_1 : [-b^2w - b\delta_A - bcv \ , \ bcu + c\delta_C - abw - a\delta_A \ , \ abv + b^2u + b\delta_C]$$

$$FI_3 : [bcw + c^2v + cS_A \ , \ -c^2u - c\delta_B - acw \ , \ acv + a\delta_A - bcu - b\delta_B]$$

Lines DI_1, EI_1, FI_3 concur at a point $U(x_0 : y_0 : z_0)$ with barycentric coordinates:

$$x_0 = a(b\delta_B + c\delta_C - a\delta_A + 2ubc)$$

$$y_0 = b(c\delta_C + a\delta_A - b\delta_B + 2vca)$$

$$z_0 = c(a\delta_A + b\delta_B - c\delta_C + 2wab)$$

aspava

#3 Mar 8, 2011, 11:26 pm

this solution is great but not understandable for the new mathematicians,you are great but if you find an easier solution you will be magnificant

Quick Reply

High School Olympiads

unsolvable

Reply

aspava

#1 Mar 7, 2011, 6:42 pm

Let the center of the external circles of a triangle ABC be D, E, F and call the incenter I.If the circumcenters of the triangles EFI, FDI, DEI named O_1, O_2, O_3 respectively. Prove that the lines AO_1, BO_2, CO_3 have a common point.

Luis González

#2 Mar 7, 2011, 7:47 pm

Since I and $\triangle ABC$ are the incenter and orthic triangle of the excentral $\triangle DEF$, it follows that circumcircles (O_1),(O_2), (O_3) of $\triangle EFI, \triangle FDI, \triangle DEI$ are the reflections of the circumcircle (U) of $\triangle DEF$ across $EF, FD, DE \Longrightarrow O_1, O_2, O_3$ are the reflections of U across the external bisectors EF, FD, DE of $\angle A, \angle B, \angle C \Longrightarrow AO_1, BO_2, CO_3$ are the isogonals of AU, BU, CU meeting at the isogonal conjugate of U WRT $\triangle ABC$.

Quick Reply

High School Olympiads

Constant Product

Reply

Headhunter
#1 Mar 7, 2011, 7:38 am

Hello.

Let α be an ellipse : $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\ (a>b>0)$
Let A, B, C be $(-a,0)$, $(a,0)$, $(c,0)$ where $c\neq -a$
Let β be an arbitrary line through C and it cut α at M, N
$\overleftrightarrow{AM}$, $\overleftrightarrow{AN}$ meet the line tangent to α (at B), at two points P, Q

Show that $\overline{BP}\cdot\overline{BQ}$ is constant and find its value.

I want to see euclidean geometric ways with no use of coordinates system.

ThinkFlow
#2 Mar 7, 2011, 8:03 am

Solution

Headhunter
#3 Mar 7, 2011, 8:08 am

to ThinkFlow.

Many Thanks for the great solution.

Luis González
#4 Mar 7, 2011, 11:22 am

$\mathcal{H}$ is a central conic (either ellipse or hyperbola) with vertices $A,B.C$ is a fixed point on the line AB and a line through C cuts $\mathcal{H}$ at $M,N.AM,AN$ cut the tangent τ of $\mathcal{H}$ through B at $P,Q.$ Image of $\mathcal{H}$ under the homology $\mathcal{A}$ with center A and axis τ is the cirle tangent to $\mathcal{H}$ through A and passing through the double point $B,$ i.e. the pedal circle (O) of $\mathcal{E}.$ Thus, PA,PB cut (O) again at the homologous U,V of $M,N\Longrightarrow UV$ always passes through the fixed image D of C under $\mathcal{A}.$ Inversion through pole A with power AB^2 takes the pencil of lines UV into a pencil of coaxal cirles $\odot(APQ)$ passing through A and the inverse D' of $D.$ From power of B to $\odot(APQ)$ we get $\overline{BP}\cdot\overline{BQ}=\overline{BA}\cdot\overline{BD'}=\text{const}.$

Headhunter
#5 Mar 7, 2011, 11:26 am

to luisgeometra.

Thank you so much for the smart proof using inversion.

mavropnevma
#6 Mar 7, 2011, 1:32 pm

> *Headhunter wrote:*
> Hello.

Let α be an ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>b>0)$
Let A, B, C be $(-a,0)$, $(a,0)$, $(c,0)$ where $c\neq -a$
Let β be an arbitrary line through C and it cut α at M, N
$\overleftrightarrow{AM}$, $\overleftrightarrow{AN}$ meet the line tangent to α (at B), at two points P, Q

Show that $\overline{BP}\cdot\overline{BQ}$ is constant and find its value.

I want to see euclidean geometric ways with no use of coordinates system.

But **you are** using a coordinates system to state your problem - isn't that quite inconsistent?

Headhunter

#7 Mar 7, 2011, 4:14 pm

to mavropnevma.

It's just for descriptive convenience.

Quick Reply

High School Olympiads

Concur 3!

Reply

tdl

#1 Apr 30, 2008, 7:14 pm

Triangle ABC with bisector AD, BE, CF (D, E, F on BC, CA, AB respectively). Call D', E', F' are symetric point of D, E, F over midpoint of EF, FD, DE respectively. Prove that BE', CF', EF are concurrent at T_1, CF', AD', FD are concurrent at T_2, AD', BE', DE are concurrent at T_3 and $D'T_1, E'T_2, F'T_3$ are parallel!

Luis González

#2 Mar 6, 2011, 5:15 am

We'll use barycentric coordinates with respect to $\triangle ABC.$ Coordinates of the vertices D', E', F' of the antimedial triangle of $\triangle DEF$ are given by:

$$D' \equiv -\left(0 : \frac{b}{b+c} : \frac{c}{b+c}\right) + \left(\frac{a}{c+a} : 0 : \frac{c}{c+a}\right) + \left(\frac{a}{a+b} : \frac{b}{a+b} : 0\right)$$

$$D' \equiv (a(b+c)(2a+b+c) : b(c^2-a^2) : c(b^2-a^2))$$

$$E' \equiv (a(c^2-b^2) : b(c+a)(2b+a+c) : c(a^2-b^2))$$

$$F' \equiv (a(b^2-c^2) : b(a^2-c^2) : c(a+b)(2c+a+b))$$

$$T_1 \equiv BE' \cap CF' \equiv (a(b^2-c^2) : b(a^2-c^2) : c(b^2-a^2)) \Longrightarrow T_1 \in EF$$

$$D'T_1 : \begin{bmatrix} a(b+c)(2a+b+c) & b(c^2-a^2) & c(b^2-a^2) \\ a(b^2-c^2) & b(a^2-c^2) & c(b^2-a^2) \\ x & y & z \end{bmatrix} = 0$$

Infinity point of $D'T_1$ is indeed $X_{512} \equiv (a^2(b^2-c^2) : b^2(c^2-a^2) : c^2(a^2-b^2))$

In other words, Steiner circum-ellipse of $\triangle ABC$ cuts its circumcircle at A, B, C and the Steiner point $X_{99} \equiv S.$ Then direction $D'T_1 \parallel E'T_2 \parallel F'T_3$ coincides with the direction of the isogonals of $AS, BS, CS.$

Quick Reply

High School Olympiads

Ineq-G108 - Geometry

Reply

Ligouras

#1 Mar 4, 2011, 3:05 am • 1

L et P be a point lying on a given sphere.
Three mutually perpendicular rays from P intersect the sphere at points A, B, and C.
Let R be the radius of the sphere.
Prove that

$$\frac{1}{PA^2}+\frac{1}{PB^2}+\frac{1}{PC^2}\geq\frac{9}{4R^2}$$

Luis González

#2 Mar 5, 2011, 11:16 am • 1

Let O be the center of the sphere and M_A, M_B, M_c be the midpoints of PA, PB, PC, i.e. OM_A, OM_B, OM_c are perpendicular bisectors of PA, PB, PC. By Pythagorean theorem for $\triangle OPM_A, \triangle OPM_B$ and $\triangle OPM_C$ we get:

$$\tfrac{1}{4}PA^2=R^2-{OM_A}^2\ ,\ \tfrac{1}{4}PB^2=R^2-{OM_B}^2\ ,\ \tfrac{1}{4}PC^2=R^2-{OM_C}^2$$

$$\Longrightarrow\ PA^2+PB^2+PC^2=12R^2-4({OM_A}^2+{OM_B}^2+{OM_C}^2)\ (\star)$$

If O_A, O_B, O_C are the projections of O onto the planes PBC, PCA, PAB, then by Pythagorean theorem, we get:

$${OM_A}^2+{OM_B}^2+{OM_C}^2=2({OO_A}^2+{OO_B}^2+{OO_C}^2)=2R^2$$

Together with $(\star)$, we obtain $PA^2+PB^2+PC^2=4R^2$

Now, by AM-HM, we have $\frac{1}{PA^2}+\frac{1}{PB^2}+\frac{1}{PC^2}\geq\frac{9}{PA^2+PB^2+PC^2}=\frac{9}{4R^2}$

RSM

#3 Mar 5, 2011, 3:45 pm

My claim is at the optimum case PA=PB=PC
Suppose, it is not true.
Then choose two of PA,PB,PC which are not equal.
Suppose, chosen lengths are PA,PB.
Fix PC.
We have $\frac{1}{PA^2}+\frac{1}{PB^2}\geq\frac{2}{PA.PB}$
Again $\frac{2}{PA.PB}$ is minimum when PA.PB is maximum and it happens when PA=PB.
Now at the optimum case PA=PB=PC.
The proof follows.

Ligouras

#4 Mar 5, 2011, 5:07 pm

Correct and nice my friends!!!

Quick Reply

High School Olympiads

A variant of the parallel tangent theorem

Reply

jayme
#1 Mar 4, 2011, 6:19 pm

Dear Mathlinkers,
ABCD a cyclic quadrilateral,
(0) the circumcircle of ABCD,
(1), (2) the incircles of the triangles CAB, DAB,
d the second common external tangent of 1 and 2,
I, J the points of contact of d resp. with 1, 2.

Prove : IJ is parallel to CD.
Sincerely
Jean-Louis

wangmengqi
#2 Mar 4, 2011, 7:44 pm

dear jayme
can you tell what is the parallel tangent theorem?
thank you

jayme
#3 Mar 4, 2011, 8:04 pm

see for example
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=393179
Sincerely
Jean-Louis

Luis González
#4 Mar 4, 2011, 10:45 pm

Let $P \equiv AB \cap DC$. $(U), (V)$ are the incircles of $\triangle DAB, \triangle CAB$ and ℓ is the internal bisector of $\angle BPC$. Since U, V lie on the circle centered at the midpoint of the arc AB of (O) with radius $MA = MB$, it follows that UV is antiparallel to AB with respect to $UA, VB \Longrightarrow \angle(AB, UV) = \frac{1}{2}|\angle ABC - \angle BAD| = \frac{1}{2}\angle BPC$, i.e. $\ell \parallel UV$. Therefore, reflection of AB across UV (tangent IJ) and reflection of AB across ℓ (line CD) are parallel.

vittasko
#5 Mar 5, 2011, 6:02 am • 1

As **Luis** said before, the incenters $I_1,\ I_2$, of the triangles $\triangle BAC,\ \triangle BAD$ respectively, lie on the circle (M), centered at midpoint M of the arc AB not containing C, of the circumcircle (O) of the given cyclic quadrilateral $ABCD$, with radius $MA = MB$ as well.

Let $E,\ F$ be, the points of intersections of $BC,\ AD$ respectively, from the external common tangent line of $(I_1),\ (I_2)$, the other than AB, that passes through the point $P \equiv AB \cap I_1I_2$.

It is easy to show that the quadrilateral $ABEF$ is cyclic,

from $\angle AFP = \angle FAB - \angle FPA = 2\angle I_2AB - 2\angle I_2PA = 2\angle AI_2P = 2\angle ABI_1 = \angle B.$

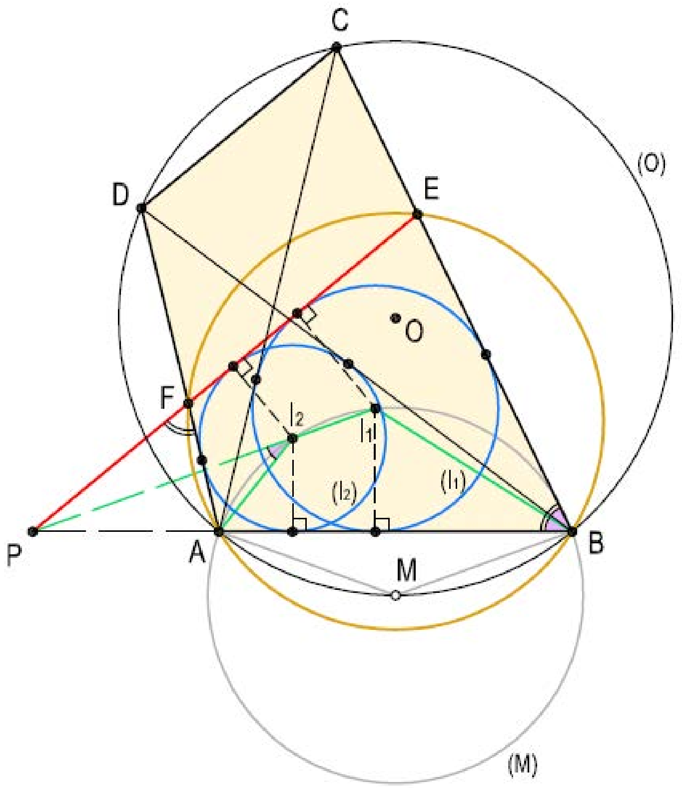
So, EF is an antiparallel line of AB, with respect to the line segments $AD,\ BC$.

But, from the given cyclic quadrilateral $ABCD$, we have that CD is also an antiparallel line of AB, with respect to $AD,\ BC$.

Hence, we conclude that $EF \parallel CD$ and the proof is completed.

Kostas Vittas.

Attachments:



This post has been edited 1 time. Last edited by vittasko, Mar 6, 2011, 2:52 am

jayme

#6 Mar 5, 2011, 3:38 pm

Dear Mathlinkers,
thank for your interest in this problem. In order to make a link with another situation,
my synthetic proof consists to use the Thébault's problem and to applie the parallel tangent theorem.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Equal Sum Of Angles

Reply

Headhunter
#1 Mar 4, 2011, 1:32 pm

Hello.

Let α be an ellipse with two foci F, G
Two lines m, n are tangent to α and pass through A
Two lines k, l are tangent to α and pass through B
m cut k at D and n cut l at E

Show that $\angle AEB + \angle BDA = \angle BFA + \angle BGA$

Only Euclidean Geometric Ways.

Luis González
#2 Mar 4, 2011, 8:23 pm • 1

Let lines AD and BE intersect at P. Simple angle chasing yields:

$$\angle BFA = \angle FBP + \angle FAP + \angle APB \ , \ \angle BGA = \angle GBP + \angle GAP + \angle APB$$

It is well-known that AF, AG are isogonal with respect to $\angle PAE$ and BF, BG are isogonal with respect to $\angle PBD$, i.e. $\angle FAP = \angle GAE, \angle GBP = \angle FBD$. Hence, adding the first two expressions gives:

$$\angle BFA + \angle BGA = 2\angle APB + \angle PAE + \angle PBD = \angle AEB + \angle BDA.$$

Quick Reply

High School Olympiads

Geometrical locus with distancies (own).

Virgil Nicula

#1 Mar 2, 2011, 9:36 pm

Proposed problem. Ascertain the geometrical locus Λ_k of the mobile interior point L w.r.t. acute given triangle ABC for which $\frac{\delta_{AB}(L)+\delta_{AC}(L)}{\delta_{BC}(L)}=k>0$ (constant) and show that the locus Λ_k pass through a fixed point F for any value of the constant k . I denoted the distance $\delta_d(X)$ of the point X to the line d .

This post has been edited 2 times. Last edited by Virgil Nicula, Mar 4, 2011, 1:10 am

Luis González

#2 Mar 3, 2011, 1:01 am

We can drop the constraint "L is inside ABC" if we use oriented distances instead. Let (α,β,γ) be the trilinear coordinates of the mobile point L with respect to $\triangle ABC$. Locus of points L satisfying the desired condition is a line ℓ with trilinear equation $\ell \equiv \beta+\gamma-k\alpha=0 \Longrightarrow \ell$ passes through the foot of the A-external bisector $(0,1,-1)$.

vittasko

#3 Mar 4, 2011, 12:25 am

• Let $D,\ E$ be, the points on the side-segments $AC,\ AB$ rspectively, such that $\frac{DP}{DQ}=\frac{ER}{ES}=k$, (1) where $k=\frac{\delta_{AB}(L)+\delta_{AC}(L)}{\delta_{BC}(L)}$ and $P,\ Q,$ are the orthogonal projections of $D,$ on $AB,\ BC$ respectively and $R,\ S,$ are the ones of $E,$ on $AC,\ BC,$ respectively.

We draw the external angle bisector of $\angle A,$ which intersects the line segment $DE,$ at point so be it $F.$

From $(1)\Longrightarrow \frac{ES}{DQ}=\frac{ER}{DP}=\frac{AE}{AD}=\frac{FE}{FD}$, (2) and so, because of $ES\parallel DQ,$ we conclude that the points $F,\ S,\ Q,$ are collinear.

That is, the line segment DE passes through the constant point $F,$ as the feet of the external angle bisector of $\angle A$ on the sideline BC of $\triangle ABC.$

• Let L be, an arbitrary point between $D,\ E$ and we will prove that $\frac{LX+LY}{LZ}=k,$ where $X,\ Y,\ Z,$ are the orthogonal projections of $L,$ on $AB,\ AC,\ BC,$ respectively.

We denote the point T in to the same side of DE as $Y,$ such that $DT\perp AC$ and $DT=DP$, (3) and let be the point $V\equiv RT\cap LY.$

From $(1),\ (3)\Longrightarrow \frac{DT}{DQ}=\frac{ER}{ES}$, (4) and so, we have the collinearity of the points $F,\ R,\ T.$

Also, from $YV\parallel DT\Longrightarrow \frac{YV}{DT}=\frac{RY}{RD}=\frac{EL}{ED}=\frac{LX}{DP}\Longrightarrow \frac{YV}{DT}=\frac{LX}{DP}\Longrightarrow YV=LX$, (5)

Hence, from (5) and from $\frac{LV}{LZ}=\frac{DT}{DQ}=\frac{DP}{DQ}=k$ we conclude that $\frac{LV}{LZ}=\frac{LY+YV}{LZ}=\frac{LX+LY}{LZ}=k$, (6)

• It is not difficult now to show that for any point K inwardly to $\triangle ABC,$ that doesn't lie on $DE,$ the relation $\frac{KX'+KY'}{KZ'}\neq k$