Real Analysis

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DAY-ANALYSIS, OTIS*

§1 Reading

This is one of the most technical lectures in OTIS. Read as much of the following as you can:

- §8, §26, §28 of *Napkin* v1.5 (real analysis). Officially, there are some prerequisite chapters for these three chapters. In practice, you might be able to get away with not reading them.
- Lagrange Multipliers Done Correctly, from my website

§2 Lecture notes

§2.1 Order relations

Fact 2.1. If $A_1 \le A_2 \le \dots \le A_n \le A_1$ then $A_1 = \dots = A_n$.

Fact 2.2. If $A \le B + \varepsilon$ for all $\varepsilon > 0$ then $A \le B$.

Fact 2.3. $\mathbb{Q}$ is dense in $\mathbb{R}$: every nontrivial interval $[a, b]$ contains infinitely many rational numbers.

Fact 2.4. $\sup S$ and $\inf S$ are defined for any bounded sets.

§2.2 Limits and Convergence

In which I complain about people who assert $1 + 2 + 3 + \dots = -\frac{1}{12}$.

Definition 2.5. Consider a sequence $x_1, x_2, \dots$ of real or complex numbers.

- The sequence **converges** to a limit L if $|x_n - L| \to 0$.
- The sequence is **Cauchy** if $|x_m - x_n| \to 0$.

In what follows we will need the following two important properties of the real numbers.

Theorem 2.6 (Properties of $\mathbb{R}$)

We have:

- (Least upper bound) A increasing sequence of real numbers which is bounded above converges.
- (Completeness) Cauchy implies convergence over $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.7. Consider a series $\sum a_i$ of real or complex numbers. The series **converges** to a limit L if its partial sums converge to L.

In particular, sums that don't converge don't make sense at all. Note also that this depends on the order of the terms: if we permute the sequence, the limit might change! This is weird and bad since we would want "infinite addition" to be commutative, so we want a way to avoid this behavior. This is accomplished by using the so-called notion of absolute convergence.

Definition 2.8. If $\sum a_k$ converges, we say it **converges absolutely** if $\sum |a_k| < \infty$, and **converges conditionally** otherwise.

Example 2.9

Show that if $\sum_{k=1}^{\infty} |a_k| < \infty$ then $\sum_k a_k$ converges, but not conversely.

Walkthrough.

(a) Rewrite both the hypothesis and conclusion in terms of Cauchy convergence.

(b) Deduce the result using triangle inequality.

Theorem 2.10 (Permutation of terms okay for absolute convergence)

Absolutely convergent series are invariant under permutation of the terms (the sum will still converge, and the limit remains the same).

Thus, rearranging terms of an infinite sum is only okay if the sum converges absolutely. In particular, if $a_k \ge 0$ for all k, this automatically holds, since in that case conditional and absolute convergence coincide.

What do you think happens if a sequence is convergent but not absolutely? (Hint: it's not pretty.)

Contest example to get you used to all this:

Example 2.11 (Putnam 2016 B1)

Let $x_0, x_1, x_2, \dots$ be the sequence such that $x_0 = 1$ and for $n \ge 0$,

$$x_{n+1} = \log(e^{x_n} - x_n)$$

(as usual, log is the natural logarithm). Prove that the infinite series $x_0 + x_1 + \dots$ converges and determine its value.

Walkthrough. This is not hard, but does have details.

(a) Figure out the telescoping sum and use it to guess what the answer should be.

(b) Show that the limit of the partial sums converges to the answer you showed in (a).

Be careful here with (b); there are several steps, and most of the problem is understanding what is necessary to prove. When you think you have it, you should compare what you had to the official solution and see if you missed anything.

If you want some more practice convergence notions, then you might prove the following well-known fact.

> **Example 2.12** (Alternating series test)
> Let $a_0 \ge a_1 \ge a_2 \ge a_3 \ge \dots$ be a weakly decreasing sequence of nonnegative real numbers, and assume that $\lim_{n\to\infty} a_n = 0$. Show that the series $\sum_n (-1)^n a_n$ is convergent (it need not be absolutely convergent).

§2.3 How to use calculus correctly

The full important details are in the reading.

- Derivatives and/or Lagrange multipliers will detect *local* extrema over an open set.
- Compactness results guarantee the existence of *global* extrema over a compact (closed and bounded, for us) set.

> **Lemma 2.13** (Lemma 2.2 from LM reading; the big one)
> Let K be a compact set and $f \colon K \to \mathbb{R}$ be a continuous function. Then f has a **global** maximum over K – there exists a point $\mathbf{x} \in K$ such that $f(\mathbf{x}) \ge f(\mathbf{y})$ for any other point $\mathbf{y} \in K$.

Some Lagrange Multipliers practice follows. Note that the theory (statement of LM, compactness, etc.) is part of the reading, which in fact contains the same two examples.

> **Example 2.14**
> Prove that if $a+b+c=3$ for $a,b,c \ge 0$ then $abc \le 1$.

More serious example:

> **Example 2.15** (USAMO 2001/3)
> Let a,b,c be nonnegative real numbers such that $a^2+b^2+c^2+abc=4$. Show that
> $$0 \le ab+bc+ca-abc \le 2.$$

Walkthrough.

(a) Prove the much easier left-hand side inequality.

Thus we will focus on proving $ab + bc + ca - abc \le 2$. It's equivalent to show $f(a, b, c) = a^2+b^2+c^2+ab+bc+ca$ is at most 6, subject to constraint $g(a, b, c) = a^2+b^2+c^2+abc-3$.

(b) Define an open domain $U \in \mathbb{R}^3$ on which we can use Lagrange multipliers, whose closure $\overline{U}$ is compact. Argue that we can ignore the points not in $\overline{U}$.

(c) Show that f is at most 6 on $\overline{U} \setminus U$.

(d) Compute ∇f, ∇g, and verify $\nabla g \neq \mathbf{0}$ for any $(a, b, c) \in U$. Hence we may assume $\nabla f = \lambda \nabla g$. Check that $\lambda^{-1} \neq 0$.

We now turn to actually solving the equations.

(e) Show that we either have $a = b$ or $c = 2 - \lambda^{-1}$, and similarly. Reduce to the case where $2 - \lambda^{-1} = a = b \neq c$.

(f) Writing everything in terms of a, show that $c = \frac{3a^2-4a}{2-2a}$ and then that

$$\begin{aligned} 0 &= \left(3a^2-4a\right)^2 + a^2\left(3a^2-4a\right)(2-2a) + \left(2a^2-4\right)(2-2a)^2 \\ &= -6a^5 + 31a^4 - 48a^3 + 8a^2 + 32a - 16. \end{aligned}$$

(g) Solve the resulting degree five polynomial for a.

(h) Exactly one of the roots you found in the previous part will give a triple $p = (a, b, c) \in U$, which is the only critical point of f in U. Show that this critical point gives

$$f(p) = \frac{121 + 17\sqrt{17}}{32}.$$

(i) Verify that $f(p) < 6$.

(j) Repeat the boilerplate compactness/LM argument to establish that $f \le 6$ on $\overline{U}$.

§2.4 Asymptotic growth rates

If you have estimates on functions (using big-O-notation, say), then often you can make assertions about sufficiently large x or n, even without knowing the exact values. I'm not good at explaining exactly how this works, other than by example, but it should be pretty intuitive.

Example 2.16 (Putnam 2016 B2)

Define a positive integer n to be *squarish* if either n is itself a perfect square or the distance from n to the nearest perfect square is a perfect square. For example, 2016 is squarish, because the nearest perfect square to 2016 is $45^2 = 2025$ and $2025 - 2016 = 9$ is a perfect square.

For a positive integer N, let $S(N)$ be the number of squarish integers between 1 and N inclusive. Find positive constants α and β such that

$$\lim_{N\to\infty} \frac{S(N)}{N^\alpha} = \beta,$$

or show that no such constants exist.

Walkthrough.

(a) Show that
$$S(m^2+2m) = m + \sum_{k=1}^{m} 2\left\lfloor \sqrt{k} \right\rfloor$$

(b) Prove that $S(m^2+2m) = \frac{4}{3}m^{3/2} + O(m)$, by integration.

(c) Use this to estimate $S(x)$ for general x, within $O(\sqrt{x})$.

(d) Prove that $(\alpha, \beta) = \left(\frac{3}{4}, \frac{4}{3}\right)$ works.

§2.5 A few more useful facts

The following facts can also be useful.

- The inequality
$$e^x \ge 1 + x$$
holds for all x with equality only if $x = 0$.

- **Bernoulli inequality**: Let $r \ge 1$ be a real number and $t \ge -1$ be another real number. The
$$(1+t)^r \ge 1 + rt.$$
The inequality also holds for $r \le 0$, and holds with the reverse sign for $0 \le r \le 1$.

- If $f'(x) \ge 0$ for all f and $f(0) \ge 0$, then $f(x) \ge 0$ for all $x > 0$.

- Mean Value Theorem

- Intermediate Value Theorem

§3 Practice Problems

Instructions: Solve [36♣]. If you have time, solve [42♣]. Problems with red weights are mandatory.

The [9♣] problems are both really nice, so try to solve at least one.

> They're definitely of the "attack first, ask questions later" variety.
>
> Campaign dialogue in *Under the Burning Suns*, from *The Battle for Wesnoth*

[2♣] **Problem 1** (Math Prize 2019/3)**.** Say that a positive integer is *red* if it is a 2020th power, and *blue* if it is a 2019th power and not red. Determine whether the following statement is true or false: between every two red integers greater than $10^{100000000}$, there are at least 2019 blue integers.

[2♣] **Problem 2** (HMMT 2017)**.** Let $P(x), Q(x)$ be nonconstant polynomials in $\mathbb{R}[x]$. Prove that if

$$\lfloor P(y) \rfloor = \lfloor Q(y) \rfloor$$

for all real numbers y, then $P(x) = Q(x)$ for all real numbers x.

[2♣] **Problem 3** (Colin Tang)**.** Let γ be a closed loop in the plane. Show that there is a triangle ABC inscribed in γ which has maximal area, i.e. for any other triangle XYZ the area of XYZ is at most the area of ABC.

[2♣] **Problem 4** (HMMT 2017)**.** Does there exist a two-variable polynomial $P(x, y)$ with real number coefficients such that $P(x, y)$ is positive exactly when x and y are both positive?

[2♣] **Problem 5** (ELMO SL 2017, Michael Ma)**.** Let $0 < k < \frac{1}{2}$ be a real number and let a_0 and b_0 be arbitrary real numbers in $(0, 1)$. The sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are then defined recursively by

$$a_{n+1} = \frac{a_n + 1}{2} \qquad \text{and} \qquad b_{n+1} = b_n^k$$

for $n \geq 0$. Prove that $a_n < b_n$ for all sufficiently large n.

[3♣] **Problem 6** (ELMO 2018/5)**.** Let $a_1, a_2, \dots, a_m$ be a finite sequence of positive integers. Prove that there exist nonnegative integers b, c, and N such that

$$\left\lfloor \sum_{i=1}^{m} \sqrt{n + a_i} \right\rfloor = \left\lfloor \sqrt{bn + c} \right\rfloor$$

holds for all integers $n > N$.

[3♣] **Problem 7** (Iran 2001)**.** Determine whether there exists a sequence $a_1, a_2, \dots$ of nonnegative reals such that

$$a_n + a_{2n} + \dots = \frac{1}{n}$$

for every positive integer n.

[5♣] **Problem 8** (Pugh)**.** Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of real numbers. Assume a_n is monotonic and bounded, and moreover that $\sum_n b_n$ converges. Prove that $\sum_n a_n b_n$ converges. (Note that in both the hypothesis and statement, we do not have absolute convergence.)

[3♣] **Problem 9** (Putnam 2016 B6)**.** Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n+1}.$$

[5♣] **Problem 10** (Shortlist 2002)**.** Let $a_1, a_2, \dots$ be an infinite sequence of real numbers, for which there exists a real number c with $0 \le a_i \le c$ for all i, such that

$$|a_i - a_j| \ge \frac{1}{i+j} \quad \text{for all } i,\ j \text{ with } i \ne j.$$

Prove that $c \ge 1$.

[5♣] **Problem 11** (IMO 2016)**.** The equation

$$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

[5♣] **Problem 12** (USAMO 2015/6)**.** Consider $0 < \lambda < 1$, and let A be a multiset of positive integers. Let $A_n = \{a \in A : a \le n\}$. Assume that for every $n \in \mathbb{N}$, the multiset A_n contains at most $n\lambda$ numbers. Show that there are infinitely many $n \in \mathbb{N}$ for which the sum of the elements in A_n is at most $\frac{n(n+1)}{2}\lambda$.

[3♣] **Problem 13** (Shortlist 2001 A2)**.** Let $a_0, a_1, a_2, \dots$ be an arbitrary infinite sequence of positive real numbers. Show that the inequality

$$1 + a_n > a_{n-1}\sqrt[n]{2}$$

holds for infinitely many positive integers n.

[3♣] **Problem 14.** Prove that if P is a real polynomial of degree n, then the equation $P(t) = e^t$ has at most $n+1$ real solutions in t.

[5♣] **Problem 15** (Putnam 2016 B5)**.** Find all functions $f : (1,\infty) \to (1,\infty)$ with the following property: if $x, y \in (1,\infty)$ and $x^2 \le y \le x^3$, then

$$(f(x))^2 \le f(y) \le (f(x))^3.$$

[5♣] **Problem 16** (USMCA 2019/5)**.** The number 2019 is written on a blackboard. Every minute, if the number a is written on the board, Evan erases it and replaces it with a number chosen from the set

$$\{0, 1, 2, \dots, \lceil 2.01a \rceil\}$$

uniformly at random (here $\lceil \bullet \rceil$ is the ceiling function). Is there an integer N such that the board reads 0 after N steps with at least 99% probability?

[9♣] **Problem 17** (Shortlist 2012 A4)**.** Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

[9♣] **Problem 18** (Putnam 1996 B6)**.** Let (a_1, b_1), $\dots$, (a_n, b_n) be the vertices of a convex n-gon which contains the origin in its interior. Prove that there exist positive real numbers x and y such that

$$(0,0) = \sum_i x^{a_i} y^{b_i} \cdot (a_i, b_i).$$

[1♣] **Mini Survey.** At the end of your submission, answer the following questions.

(a) About how many hours did the problem set take?

(b) Name any problems that stood out (e.g. especially nice, instructive, boring, or unusually easy/hard for its placement).

Any other thoughts are welcome too. Examples: suggestions for new problems to add, things I could explain better in the notes, overall difficulty or usefulness of the unit.

§4 Solutions to the walkthroughs

§4.1 Solution 2.11, Putnam 2016 B1

The answer is $\boxed{e-1}$.

We begin by noting $x_{n+1} = \log(e^{x_n} - x_n) \ge \log 1 = 0$, owing to $e^t \ge 1+t$. So $x_n \ge 0$ for all n.

Next notice that

$$x_{n+1} = \log\left(e^{x_n} - x_n\right) < \log e^{x_n} = x_n.$$

So $(x_n)_n$ is strictly decreasing in addition to nonnegative. Thus it must converge to some limit L.

Third, observe that

$$x_n = e^{x_n} - e^{x_{n+1}} \implies x_0 + x_1 + \dots + x_n = e^{x_0} - e^{x_n} = e - e^{x_n} < e.$$

Since the partial sums are bounded by e, and $x_i \ge 0$, we conclude $L = 0$.

Finally, the limit of the partial sums is then

$$\lim_{n\to\infty} e - e^{x_n} = e - e^0 = e - 1.$$

> **Remark.** This problem has several pitfalls in it, and it may be hard for students without analysis experience to realize which steps are necessary to prove.
>
> There is another way to do the third step by appealing to *sequential continuity* which is more general. Let $f(t) = \log(e^t - t)$ which is continuous. Suppose the sequence $x_\bullet = (x_0, x_1, \dots)$ converges to L. Then applying sequential continuity means that $f(x_\bullet) = (f(x_0), f(x_1), \dots)$ converges to $f(L)$. But $f(x_\bullet)$ is just $(x_\bullet)$ with one fewer term, so that can only occur if $f(L) = L$, which implies $L = 0$.
>
> However, in that case the fourth step would require the same work anyways in order to compute the partial sums.

§4.2 Solution 2.12, Alternating series test

This is an application of Cauchy convergence, since one can show that

$$\left| \sum_{n=M}^{N} (-1)^n a_n \right| \le a_{\min(M,N)}.$$

Indeed, if M and N are even (for simplicity; other cases identical) then

$$\begin{aligned} a_M - a_{M+1} + a_{M+2} - \dots &= a_M - (a_{M+1} - a_{M+2}) - (a_{M+3} - a_{M+4}) \\ &\qquad - \dots - (a_{N-1} - a_N) \\ &\le a_M \\ a_M - a_{M+1} + a_{M+2} - \dots &= a_M - a_{M+1} + (a_{M+2} - a_{M+3}) + (a_{M+4} - a_{M+5}) \\ &\qquad + \dots + (a_{N-2} - a_{N+1}) + a_N \\ &\ge -a_{M+1}. \end{aligned}$$

In this way we see that the sequence of partial sums is Cauchy, hence converges to some limit.

Remark. Monotonicity cannot be dropped, even with $\lim_n a_n = 0$. As a concrete example the series

$$\frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}-1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}-1} - \frac{1}{\sqrt{4}+1} + \dots$$

has alternating signs and tends to zero, but the $2n$th partial sum is actually exactly $2\left(\frac{1}{1} + \dots + \frac{1}{n}\right)$ which diverges.

§4.3 Solution 2.15, USAMO 2001/3

The left-hand side of the inequality is trivial; just note that $\min\{a,b,c\} \le 1$. Hence, we focus on the right side. We use Lagrange Multipliers.

Define

$$U = \left\{(a,b,c) \mid a,b,c > 0 \text{ and } a^2+b^2+c^2 < 1000\right\}.$$

This is an intersection of open sets, so it is open. Its closure is

$$\overline{U} = \left\{(a,b,c) \mid a,b,c \ge 0 \text{ and } a^2+b^2+c^2 \le 1000\right\}.$$

Hence the constraint set

$$\overline{S} = \left\{\mathbf{x} \in \overline{U} : g(\overline{x}) = 3\right\}$$

is compact, where $g(a,b,c) = a^2+b^2+c^2+abc$.

Define

$$f(a,b,c) = a^2+b^2+c^2+ab+bc+ca.$$

It's equivalent to show that $f \le 6$ subject to g. Over $\overline{S}$, it must achieve a global maximum. Now we consider two cases.

If $\mathbf{x}$ lies on the boundary, that means one of the components is zero (since $a^2+b^2+c^2 = 1000$ is clearly impossible). WLOG $c = 0$, then we wish to show $a^2+b^2+ab \le 6$ for $a^2+b^2 = 4$, which is trivial.

Now for the interior U, we may use the method of Lagrange Multipliers. Consider a local maximum $\mathbf{x} \in U$. Compute

$$\nabla f = \langle 2a+b+c, 2b+c+a, 2c+a+b \rangle$$

and

$$\nabla g = \langle 2a+bc, 2b+ca, 2c+ab \rangle.$$

Of course, $\nabla g \neq \mathbf{0}$ everywhere, so introducing our multiplier yields

$$\langle 2a+b+c, a+2b+c, a+b+2c \rangle = \lambda \langle 2a+bc, 2b+ca, 2c+ab \rangle.$$

Note that $\lambda \neq 0$ since $a,b,c > 0$. Subtracting $2a+b+c = \lambda(2a+bc)$ from $a+2b+c = \lambda(2b+ca)$ implies that

$$(a-b)([2\lambda-1] - \lambda c) = 0.$$

We can derive similar equations for the others. Hence, we have three cases.

1. If $a = b = c$, then $a = b = c = 1$, and this satisfies $f(1,1,1) \le 6$.

2. If a, b, c are pairwise distinct, then we derive $a = b = c = 2 - \lambda^{-1}$, contradiction.

3. Now suppose that $a = b \neq c$. That means $\lambda = \frac{1}{2-a}$ (of course our conditions force $c < 2$). Now

$$2a + 2c = a + b + 2c = \lambda(2c + ab) = \frac{1}{2-a}(2c + a^2)$$

which implies

$$4a + 4c - 2a^2 - 2ac = 2c + a^2$$

meaning (with the additional note that $a \neq 1$) we have

$$c = \frac{3a^2 - 4a}{2 - 2a}.$$

Note that at this point, $c > 0$ forces $1 < a < \frac{4}{3}$.

The constraint $a^2 + b^2 + c^2 + abc = 4 \iff c^2 + a^2c + (2a^2 - 4) = 0$ now gives

$$\left(3a^2 - 4a\right)^2 + a^2\left(3a^2 - 4a\right)(2 - 2a) + \left(2a^2 - 4\right)(2 - 2a)^2 = 0.$$

Before expanding this, it is prudent to see if it has any rational roots.

A quick inspection finds that $a = 2$ is such a root (precisely, $16 - 32 + 16 = 0$; or even $(a, b, c) = (2, 2, -2)$). Now, we can expand and try to factor:

$$\begin{aligned} 0 &= -6a^5 + 31a^4 - 48a^3 + 8a^2 + 32a - 16 \\ &= (a-2)(-6a^4 + 19a^3 - 10a^2 - 12a + 8) \\ &= (a-2)^2\left(-6a^3 + 7a^2 + 4a - 4\right) \\ &= (a-2)^2(2 - 3a)(2a^2 - a - 2). \end{aligned}$$

The only root a in the interval $\left(1, \frac{4}{3}\right)$ is $a = \frac{1}{4}\left(1 + \sqrt{17}\right)$. To finish, write

$$c = \frac{a(3a-4)}{2-2a} = \frac{1}{8}\left(7 - \sqrt{17}\right)$$

and

$$f(a, b, c) = 3a^2 + 2ac + c^2 = \frac{1}{32}\left(121 + 17\sqrt{17}\right).$$

This is the last critical point, so we're done once we check this is less than 6. This follows from the inequality $17^3 < (6 \cdot 32 - 121)^2$; in fact, we actually have

$$\frac{1}{32}\left(121 + 17\sqrt{17}\right) \approx 5.97165.$$

This completes the solution.

> **Remark.** Equality holds for the upper bound if $(a, b, c) = (1, 1, 1)$ or $(a, b, c) = (\sqrt{2}, \sqrt{2}, 0)$ and permutations. The lower bound is achieved if $(a, b, c) = (2, 0, 0)$ and permutations.

§4.4 Solution 2.16, Putnam 2016 B2

Just an outline: one can count directly to show that

$$S(m^2 + 2m) = m + \sum_{k=1}^{m} 2\left\lfloor \sqrt{k} \right\rfloor$$

since the interval $[k^2, k^2+2k]$ has exactly $1 + 2\left\lfloor\sqrt{k}\right\rfloor$ squarish numbers. Now,

$$\sum_{k=1}^{m} \sqrt{k} = O(m) + \int_0^m \sqrt{x}\,dx = \frac{2}{3}m^{3/2} + O(m).$$

So from all this we obtain

$$S(m^2+2m) = \frac{4}{3}m^{3/2} + O(m).$$

Now for general x, if $m = \lfloor\sqrt{x}\rfloor$ then $S(m^2+2m)$ and $S(x)$ differ by $O(m)$, so

$$S(x) = O(\sqrt{x}) + \frac{4}{3}(\sqrt{x})^{3/2} = \frac{4}{3}x^{3/4} + O(\sqrt{x}).$$

Hence the answer

$$\boxed{(\alpha, \beta) = \left(\frac{3}{4}, \frac{4}{3}\right)}.$$