Problem Proposal: Flipping Bits in a String

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1 Problem

Let n be a positive integer. Given a random n-element sequence s of zeroes and ones, we perform the following operation repeatedly: if the sequence s has exactly k ones, then the k^{th} element of s is toggled from a 0 to a 1 or vice versa. For example, if $n = 3$ and the original sequence is $(0, 1, 0)$, then this operation results in the list of strings

$$(0, 1, 0) \to (1, 1, 0) \to (1, 0, 0) \to (0, 0, 0).$$

We see that it takes three steps for this process to reach the sequence consisting of all zeroes.

1. Prove that this process will eventually terminate regardless of what the initial string is. (Here we make the convention that the string of all zeroes terminates after zero steps.)
2. Determine, with proof, the average number of steps a random n-bit string will take for this process to stop.

2 Solution

As stated before, this solution is not 100% airtight, but I'm pretty certain it works. It also happens to be a bit clunky though, so if there's an easier solution I'd be happy to see it :)

We will solve both parts of this problem by the following observation. Consider a more substantial example, namely $n = 5$ and starting with the string 10111. Below is the sequence of moves, where the flipped bit after each step is in red.

10101	11110
10001	11100
11001	11000
11101	10000
11111	00000

Notice that this sequence consists of three phases: a downward phase of two zeroes, an upward phase of three ones, and a downward phase of five zeroes. The key idea is, using this, to assign this binary integer the subset $\{2, 3, 5\}$.

Define $\Phi : \mathcal{P}([n]) \to \{0, 1\}^n$ as follows. Given a subset $S = \{a_1, \ldots, a_k\} \subseteq [n]$ with $a_1 < a_2 < \cdots < a_k$, start with the binary string $00\ldots0$ and perform the following operations.

- Flip the bits in positions 1 through a_k to 1s;
- flip the bits in positions $a_k - a_{k-1}$ through $a_k - 1$ to 0s;
- flip the bits in positions $a_k - a_{k-1} + 1$ through $a_k - a_{k-1} + a_{k-2}$ to 1s;
- $\cdots$

This process eventually stops because our sequence is finite; call the result $\Phi(s)$.

Lemma 1. *After step t we have the bit sequence that comes from the subset $\{a_{k-t+1}, \ldots, a_k\}$.*

Proof. We proceed by induction on t. Our base case of $t = 0$ is vacuously true: at step zero our string $00\ldots0$ has zero ones, and so there is nothing to do.

For the induction step, assume the result holds true for $t - 1$, where $t \geq 1$ is an integer. Remark that after t steps, we have precisely $S := a_k - a_{k-1} + a_{k-2} - \cdots + (-1)^{t+1} a_{k-t+1}$ ones in our string; thus, at the beginning of this process we will flip the S^{th} bit of the string. Now we perform casework.

- Suppose first that t is odd, so that on the t^{th} step we add 1s to our string. Note that by the way we defined $\Phi(s)$, the S^{th} bit of this string is a 1; thus, the algorithm will change the 1 to a 0. This decreases the number of ones in our string by 1. As a result, the algorithm will continue down the row of 1s and change each one to a zero. At the end, we have precisely the string obtained after $t - 1$ steps, and so we may apply the inductive hypothesis to finish.
- Suppose next that t is even, so that on the t^{th} step we add 0s to our string. Note that by the way we defined $\Phi(s)$, the S^{th} bit of this string is a 0; thus, the algorithm will change the 0 to a 1. This increases the number of ones in our string by 1. As a result, the algorithm will continue up the row of 0s and change each one to a 1. At the end, we have precisely the string obtained after $t - 1$ steps, and so we may apply the inductive hypothesis to finish.

We have exhausted all possible cases, and so by induction we are done. $\square$

Lemma 2. *In fact Φ is a bijection.*

Proof. It suffices to prove injectivity, as then the claim follows by the fact that both sets have an equal number of members.

We'll show this by induction on the length n of the bit-string. The base case of $n = 1$ is easy, as the sets $\varnothing$ and $\{1\}$ yield the sequences 0 and 1 respectively. Now suppose that this transformation is injective on strings with n bits, and consider this function acting on strings with $n + 1$ bits. Denote by A and B the subsets of $\mathcal{P}([n + 1])$ consisting of all subsets without and with $n+1$ respectively. We know by the induction hypothesis Φ when restricted to A is bijective and, in particular, the $(n + 1)^{\text{st}}$ bit of any output string is a zero. Now suppose our subset S is in B. Remark that $\Phi(S)$ necessarily contains a 1 in the $(n + 1)^{\text{st}}$ bit, since after toggling all the bits to 1s on the first step we never touch the 1 again; thus, $\Phi[A] \cap \Phi[B] = \varnothing$, and so it suffices to show that the restriction of Φ to B is injective. Consider the following two operations on strings of bits on n digits:

- reversing the order of the digits, so that e.g. 1011 turns into 1101;
- flipping each bit from a 0 to a 1, so that e.g. 1101 turns into 0010.

Both of these transformations are bijective, so their composition is also bijective; denote the result of this transformation on a string s by $\bar{s}$. I claim that by the way we defined the transformation, if applying this process to $\{a_1, \ldots, a_{k-1}, n\}$ yields the string $s1$, then applying this process to $\{a_1, \ldots, a_{k-1}\}$ yields the string $\bar{s}$. Indeed, by reversing the order of the indexes in the process given above and swapping all zeroes for ones and vice versa, we obtain the same process as before:

- flip the bits in positions 1 through a_{k-1} to 1s;
- flip the bits in positions $a_{k-1} - a_{k-2}$ through $a_{k-2} - 1$ to 0s;
- $\cdots$

Thus injectivity of Φ on all subsets of $[n+1]$ containing $n+1$ is equivalent to injectivity of Φ on all subsets of $[n]$, which we know is true by the inductive hypothesis. $\square$

We can now solve both parts. Part (a) follows from surjectivity of Φ and the fact that every string in the image of Φ terminates under this algorithm. For (b) remark that the number of steps required for a string s to terminate is the sum of the elements of the set $\Phi(s)$; thus, the answer is the expected sum of elements of a random n-element subset of $[n]$, which is $\frac{1}{2}(1 + 2 + \cdots + n) = \boxed{\frac{n(n+1)}{4}}$. Yay.